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Analysis of the equations of motion of linearized controlled structures

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Abstract. The linearized equations of motion of controlled structures possess coefficient matrices that lack the familiar properties of symmetry and definiteness. A method is developed for the efficient analysis of linearized controlled structures. This constructive method utilizes equivalence tranformations in Lagrangian coordinates and does not require conversion of the equations of motion to first-order forms. Compared with the state-space approach, this method can offer substantial reduction in computational effort and ample physical insight. However, it is often necessary to draw upon some type of decoupling approximation for fast solution. Many numerical techniques involve discretized equations resembling those of linearized controlled structures. These numerical techniques can also be greatly streamlined if the method of equivalence transformations is incorporated.

Keywords. Linearized control structures; equivalence transformation; classical modal analysis; Lagrangian coordinates.

1. Introduction

A time-honoured procedure for the approximate solution of nonlinear mechanical systems is linearization, or in some cases piecewise linearization. The linearized equation of motion of an n-degree-of-freedom controlled structure can be written as

$$A\ddot{q} + B\dot{q} + Cq = f(t), \tag{1}$$

where A, B and C are arbitrary square matrices of order n. Embedded in these real coefficient matrices are the control gains, and as a result they need not possess any of the familiar properties of symmetry or definiteness. The Lagrangian coordinate q and the generalized excitation $f(t)$ are n-dimensional vectors. The various ways in which equations of the above type can arise have been discussed by Soong (1990) and other authors. These equations tend to manifest on a widespread scale in recent years, owing mostly to the increasing use of control devices in structures. Development of a fast method for the analysis of such equations is thus much deserving.

In theory, it is always possible and in fact customary to investigate a linearized controlled structure with the state-space approach, whereby the second-order equation

(1) is recast into a first-order system of dimension $2n$. After conversion into a first-order form, a large variety of numerical techniques are then available for subsequent analysis. But the state-space method has never appealed to structural engineers. An inordinate amount of computational effort is usually given as a reason. More importantly, there is serious absence of physical insight in tackling a first-order equation recast from the Lagrangian formulation. The reluctance to employ state-space techniques has led to the development of a whole array of special techniques for analysing subclasses of linearized controlled structures in the Lagrangian coordinate q. Some of these developments have been reported by Huseyin & Leipholz (1973), Fawzy & Bishop (1976), Inman (1983), and Caughey & Ma (1993).

Equations of the type (1) are coincident with those governing linear nonconservative vibrations, which have drawn relatively scant attention in classical theory. To be sure, it would be preferable if classical modal analysis could somehow be modified to treat systems with nonsymmetric coefficients. That might indeed be assumed in some earlier investigations. A literature survey in the testing of aircraft flutter and in the stressing of ship hulls, for example, reveals rather occasional use of terms such as logarithmic decrements, modes, or natural frequencies. But the meanings of these terms are not clear. When does a linearized controlled structure possess classical normal modes? What are the natural frequencies of a system with nonsymmetric coefficients? A theoretical basis, on which these concepts can be properly explored, is greatly desired.

Were system (1) symmetric and definite, the coefficient of acceleration would be a positive–defnite mass matrix M, and the coefficients of velocity and displacement would respectively be positive–semidefinite damping and stiffness matrices D and K. In this case, (1) would take the familiar form

$$M\ddot{q} + D\dot{q} + Kq = f(t). \tag{2}$$

In linear nonconservative vibrations, the difference between the matrices B and D is sometimes accounted for by gyroscopic forces, and that between C and K by circulatory forces (Müller & Schiehlen 1985). Hence $B = 0$ implies an undamped nongyroscopic structure, and so on. While this type of terminology may not be applicable to linearized controlled structures, it will be convenient to adhere to such terminology. Applications in which A is not symmetric are given, for example, by Schmitz (1973) as well as Soom & Kim (1983).

The purpose of this article is to expound a fast method for the analysis of linearized controlled structures. This constructive method utilizes equivalence transformations in Lagrangian coordinates, and does not involve conversion of the equations of motion to first-order forms. As will be evident, the method represents a direct extension of classical modal analysis. The same method can certainly be applied to the analysis of linear nonconservative vibrations. The organization of this article is as follows. In § 2, it is shown that a controlled structure in which $B = 0$ can be completely decoupled by an equivalence transformation. In addition, the meaning of natural frequencies will be clarified. Equivalence transformation is applied to a general linearized structure in § 3. There, comparison is made with classical modal analysis of viscously damped systems. An illustrative example is given in § 4, in which practical implications of the method are also discussed. A summary of findings is provided in § 5. The terms, *linearized controlled structures* and *linear nonconservative vibrations*, will be used interchangeably in exposition.

2. Undamped nongyroscopic structures

Consider a linearized controlled structure in which $B = 0$. The equation of motion is

$$A\ddot{q} + Cq = f(t). \tag{3}$$

This system can be decoupled if and only if there exist two nonsingular matrices U and V such that VAU, VCU are diagonal. Two square matrices P and Q, related by $P = VQU$, are said to be connected by an equivalence transformation. An equivalence transformation between P and Q preserves the rank of the matrices. If $V = U^{-1}$, the equivalence transformation is called a similarity transformation. In the event that $V = U^T$, the equivalence transformation is a congruence transformation. The classical modal transformation is an example of congruence transformation. A congruence transformation is also a similarity transformation if U is an orthogonal matrix. Equivalence transformations that are neither similarity nor congruence transformations are rarely used in structural analysis. Nevertheless, in what follows, it will be shown that a linearized controlled structure in which $B = 0$ can be decoupled by equivalence transformation in practically every situation.

Let u be a column vector of order n and α be a scalar constant. If

$$q = u\,e^{\alpha t}, \tag{4}$$

is a homogeneous or complementary solution to (3), the generalized eigenvalue problem

$$Cu = \lambda\, Au \tag{5}$$

must be satisfied, where $\lambda = -\alpha^2$. Eigenvalue problems of this kind were traditionally addressed in the abstract theory of matrix pencils. Emphasis was usually placed on symmetric and definite pencils. As a consequence, results applicable to the above eigenvalue problem are scattered and rather incomplete. An extension based on the presentation of Zurmühl & Falk (1984) will be made here. Associated with eigenvalue problem (5) is the adjoint eigenvalue problem

$$C^T v = \lambda A^T v. \tag{6}$$

As (5) and (6) lead to the same characteristic determinant, the corresponding eigenvalues are identical.

At this point, the two assumptions underlying this investigation must be reviewed. First, it is required that A is nonsingular. This is a common assumption made in earlier investigations. Technically speaking, there is no practical loss of generality in accommodating this assumption. Should the coefficient matrix A be singular, at least one acceleration term can be removed from the formulation. A linearized controlled structure is termed degnerate if its coefficient of acceleration A is singular. The eigenvalue problem (5) possesses n eigenvalues if and only if A is nonsingular. Thus, the implicit assumption that a system is not degenerate ensures the existence of a full set of eigenvalues. The second requirement presumes independence of the eigenvectors associated with eigenvalue problem (5). An eigenvalue problem is termed defective if it does not possess a full complement of independent eigenvectors. Experience indicates that an eigenvalue problem (5) possessing physical significance is invariably not defective. And the assumption that there is a full complement of independent eigenvectors can be made without fail in almost every application. A sufficient condition under which problem (5) is not defective is for the associated eigenvalues to be distinct. However, this is only a sufficient and not a necessary condition. In addition, the eigenvalue problem (5) is not defective if and only if its adjoint problem (6) is not defective. Henceforth, without practical loss of generality, it will be assumed that a linearized controlled structure is not degenerate or defective.

Corresponding to each eigenvalue λ_i, an eigenvector u_i can be found such that

$$Cu_i = \lambda_i Au_i. \tag{7}$$

Likewise, a solution to the adjoint eigenvalue problem associated with the eigenvalue λ_j is expressed by

$$C^T v_j = \lambda_j A^T v_j. \tag{8}$$

Note that each column vector u_i or v_j is undetermined to the extent of an arbitrary multiplicative constant. Transpose the above equation to obtain

$$v_j^T C = \lambda_j v_j^T A. \tag{9}$$

Premultiply (7) by v_j^T and postmultiply (9) by u_i. It follows by subtraction that

$$(\lambda_i - \lambda_j) v_j^T A u_i = 0. \tag{10}$$

Provided $\lambda_i \neq \lambda_j$, the eigenvectors u_i and v_j are orthogonal, with A playing the role of a weighting matrix. Thus, the two sets of column vectors u_i and v_j are biorthogonal with respect to A if the corresponding eigenvalues are distinct. An extension to include repeated eigenvalues has been made by Ma & Caughey (1995).

Based on the above clarification, a biorthogonality relation between the eigenvectors u_i and v_j holds whether or not there are repeated eigenvalues. Normalization of these eigenvectors leads to

$$v_j^T A u_i = \delta_{ij}, \quad i, j = 1, 2, \ldots, n. \tag{11}$$

Note that each eigenvector u_i and its adjoint eigenvector v_i are still determined within an arbitrary multiplicative constant. If the multiplier for u_i is a_i and that for v_i is b_i, the above equation determines only the product $a_i b_i$. In other words, the choice of either a_i or b_i separately is still arbitrary. The above relation implies, in addition, that

$$v_j^T C u_i = \lambda_i \delta_{ij}, \quad i, j = 1, 2, \ldots, n. \tag{12}$$

Define the following square matrices of order n by

$$U = [u_1, u_2, \ldots, u_n], \tag{13}$$

$$V = [v_1, v_2, \ldots, v_n]^T, \tag{14}$$

$$\Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]. \tag{15}$$

The biorthogonality relations (11) and (12) can now be expressed in a compact form:

$$VAU = I, \tag{16}$$

$$VCU = \Lambda. \tag{17}$$

Let $q = Up$. Equation (3) may be simplified to

$$\ddot{p} + \Lambda p = Vf(t), \tag{18}$$

which represents a completely decoupled system. Needless to emphasize, the solution of a decoupled system is immediate. The following statement has been established.

Theorem 1. *A linearized controlled structure that is not degenerate or defective and in which $B = 0$ can always be decoupled by equivalence transformation.*

The decoupling equivalence transformation is defined by two adjoint eigenvalue problems (5) and (6). It has been proved by Ma & Caughey (1995) that any other equivalence transformation that decouples system (3) must be derivable from these eigenvalue problems. As explained earlier, the decoupling transformation is uniquely determined within arbitrary multiplicative constants in the eigenvectors u_i and v_j if the associated eigenvalues are distinct. The complex eigenvectors u_i and v_j may be termed modes and adjoint modes respectively. In the case of repeated eigenvalues, the decoupling transformation is not unique.

For a symmetric and definite system, (5) takes the form

$$Ku = \lambda M u, \tag{19}$$

leading to classical normal modes and natural frequencies (Meirovitch 1967). Due to symmetry of M and K, (5) and (6) are identical. The modes and adjoint modes are equal and are real. Solution of only the eigenvalue problem (5) is sufficient for determining the modes. Thus, *the decoupling equivalence transformation reduces to classical modal transformation if the coefficient matrices possess symmetry and definiteness.* The method of equivalence transformation represents a direct extension of classical modal analysis, and *the lack of symmetry in a system only approximately doubles the computational effort.*

The method of equivalence transformations offers physical insight. This can be briefly demonstrated by clarification of the meaning of natural frequencies of a linearized controlled structure. According to theorem 1, it is legitimate to write the equation of an undamped nongyroscopic structure in the form (18). For free vibration, in which $f(t) = 0$, the coordinate p admits a harmonic solution if and only if $\lambda_i > 0$ ($i = 1, 2, \ldots, n$). Let $\lambda_i = \omega_i^2$, where each ω_i is real and positive. A component p_i of the vector p then has the solution

$$p_i = c_i \cos(\omega_i t - \phi_i), \quad i = 1, 2, \ldots, n. \tag{20}$$

The real constants c_i and ϕ_i can be identified as amplitude and phase angle. Bearing in mind that $q = Up$, one obtains

$$\mathbf{q} = \sum_{i=1}^{n} c_i u_i \cos(\omega_i t - \phi_i). \tag{21}$$

Validity of the following statement has therefore been demonstrated.

Theorem 2. *The components of a linearized controlled structure in which $B = 0$ can perform harmonic vibration with identical frequency if the associated eigenvalue problem (5) possesses positive eigenvalues.*

The natural frequencies of free vibration are simply the square roots of these positive eigenvalues. It remains to determine if there is any feature peculiar to the harmonic vibration of a controlled structure. In a symmetric and definite system, the modes u_i are real. Thus (21) implies that in harmonic motion with a specified natural frequency ω_i, all components also vibrate *identical phase angle* ϕ_i. That need not be true for a controlled structure, whose eigenvectors u_i are generally complex. As is evident from (21), these complex eigenvectors induce phase differences among components of the response vector. Therefore, at a specified frequency ω_i, *the components of an undamped nongyroscopic structure usually vibrate with different phase angles.* Amplitudes of vibration of the components are proportional to the magnitudes of elements in u_i. For this reason, each complex eigenvector u_i still determines a mode shape, in analogy to classical modal vibration. The general response in free vibration is then the superposition of n harmonic motions of this kind.

3. General linearized structures

In classical modal analysis of a viscously damped system, the modal matrix diagonalizes the mass matrix M and stiffness matrix K but, in general, will not diagonalize the damping matrix D. Caughey & O'Kelly (1965) showed that D can be diagonalized by the modal matrix if and only if $M^{-1}D$ and $M^{-1}K$ commute in multiplication. In other words,

$$DM^{-1}K = KM^{-1}D \tag{22}$$

is a necessary and sufficient condition for the modal transformation to completely decouple a damped symmetric and definite system. Condition (22) is not usually satisfied. A common procedure in this case is to ignore the off-diagonal elements in the transformed damping matrix, especially when these elements are small. This procedure is termed the decoupling approximation in damping, which allows modal transformation to decouple an entire system. In addition, an iterative scheme can be used to improve the accuracy of approximate solution if the off-diagonal elements in the transformed damping matrix are not small. A discussion of this aspect was given by Udwadia & Esfandiari (1990) as well as Hwang & Ma (1993). Practically speaking, there is no objection to using the decoupling approximation in damping. But, as reviewed by Park *et al* (1994), regorous analysis of errors committed by such an approximation has not been reported in the open literature.

A similar situation arises in the analysis of linearized controlled structures. Referring to (1), an equivalence transformation diagonalizes A and C. As before, let this equivalence transformation be defined by the two nonsingular matrices U and V. Recall $q = Up$. Equation (1) may be expressed in the form

$$\ddot{p} + VBU\dot{p} + \Lambda p = Vf(t). \tag{23}$$

The transformed coefficient matrix of velocity VBU is in general not diagonal. That means the entire controlled structure is not decoupled by equivalence transformation. Clearly, an extension to the decoupling approximation in damping may be proposed: the off-diagonal elements in VBU can be ignored if they are small. Intuitively, the errors of approximation should be small if the off-diagonal elements in VBU are small. In addition, an iterative scheme, similar to the one for symmetric and definite systems, may be applied to improve the accuracy of approximate solution if the off-diagonal elements in VBU are not small. These approximate techniques have immense practical implications. However, as in classical modal analysis, these are also points of speculation. To continue to dwell upon them would merely be distracting.

It is possible to further refine the method of equivalence transformation. The coefficient matrices A and C in a controlled structure need not have the same divine rights as the symmetric matrices M and K. Sometimes it is more convenient to apply an equivalence transformation to diagonalize A and B instead. Afterwards, the entire structure may be decoupled through approximating the transformed coefficient matrix of displacement by a diagonal matrix. This digression will not be pursued. Henceforth, it will be assumed that an equivalence transformation will be chosen primarily to diagonalize A and C, in a fashion expounded earlier. An upshot at this stage in the following statement.

Theorem 3. *The linearized controlled structure (1) can be decoupled by an equivalence transformation if and only if the matrices $A^{-1}B$ and $A^{-1}C$ commute in multiplication.*

In other words, this extension of criterion (22) asserts that

$$BA^{-1}C = CA^{-1}B. \tag{24}$$

is a necessary and sufficient condition for an equivalence transformation to completely decouple linearized controlled structures. To prove this, it will be easier if condition (24) is first recast in a more transparent form. From (16), observe that

$$A^{-1} = UV. \tag{25}$$

Condition (24) is therefore equivalent to

$$BUVC = CUVB. \tag{26}$$

Premultiply the above equation by V and postmultiply by U. This gives

$$VBUVCU = VCUVBU. \tag{27}$$

Let $S = VBU$. It follows, on substitution of (17) into the above equation, that

$$S\Lambda = \Lambda S. \tag{28}$$

Conditions (24) and (28) are equivalent, satisfaction of one implies satisfaction of the other.

If the equivalence transformation defined by U and V decouples system (1), the matrix VBU must be diagonal. Since diagonal matrices commute in multiplication, condition (28) is satisfied. Condition (24) is therefore also satisfied.

On the other hand, assume condition (24) is valid. Satisfaction of condition (28) implies, on expansion, that

$$\lambda_j s_{ij} = \lambda_i s_{ij}, \quad i,j = 1,2,\ldots,n, \tag{29}$$

where s_{ij} is the ijth element of S. If all eigenvalues λ_i are distinct, the above equation implies that $s_{ij} = 0$ when $i \neq j$. That means S is diagonal, and the equivalence transformation which diagonalizes A and C decouples the entire system. The case of repeated eigenvalues can be treated similarly and the observation that S is diagonal remains valid (Ma & Caughey 1995). This completes the demonstration that condition (24) is both necessary and sufficient for system (1) to be decoupled by an equivalence transformation.

4. Discussion and example

The exposition of linear nonconservative vibrations by Inman (1983) provides an inspiring account for this investigation. This article also draws upon a remarkable study by Fawzy & Bishop (1976). There, under more restrictive assumptions, the authors attempted to derive orthogonality relations involving all three matrices A, B and C. As a result, their orthogonality relations contained the eigenvalues and became rather impractical. In view of theorem 3, an equivalence transformation cannot be constructed to diagonalize A, B and C in every case.

There are three practical implications of the method developed herein. First, this constructive method represents a direct extension of classical modal analysis and only approximately doubles the computational effort required of modal transformation. While the method is exact if $B = 0$, it is necessary to use some type of decoupling approximation for a general linearized structure. Extensive simulations have indicated that the method is substantially more efficient than the state-space approach. Reduction in computational effort, particularly in the case of large-scale systems, is indeed very attractive. Second, this method appears to possess *ample physical insight*, more of which has yet to be uncovered. For instance, in clarification of theorem 2, it has been pointed out that the complex eigenvectors u_i determine the mode shapes. Third, the method of equivalence transformations can be used to streamline computational algorithms based upon the method of weighted residuals. Many numerical techniques in this family of algorithms generate the same type of equations as (1). As an example, the collocation method leads to equations resembling that of a controlled structure with $B = 0$. These numerical techniques can be greatly streamlined by utilizing equivalence transformations.

As shown by Ma & Caughey (1995), any equivalence transformation that decouples system (3) must be derivable from the adjoint eigenvalue problems (5) and (6). It follows that a general theory of decoupling by equivalence transformations has been presented. Equivalence transformations are already the most general nonsingular linear transformations. It can therefore be stated that *no nonsingular linear transformation will ever diagonalize A, B and C simultaneously every time*. Further research to find universal decoupling transformations will not be necessary. Equation (23) is the simplest representation of a linearized controlled structure in Lagrangian coordinate.

Example. A linearized controlled structure, whose equation of motion has the form (1), is defined by

$$A = \begin{bmatrix} 7 & 1 & 2 \\ -1 & 7 & 0 \\ 1 & -1 & 6 \end{bmatrix}, \tag{30}$$

$$B = \begin{bmatrix} 2{\cdot}2577 & -0{\cdot}7016 & -0{\cdot}0567 \\ 0{\cdot}7305 & 4{\cdot}4257 & 0{\cdot}9294 \\ 1{\cdot}4657 & 0{\cdot}8812 & 3{\cdot}9978 \end{bmatrix}, \tag{31}$$

$$C = \begin{bmatrix} 15{\cdot}7158 & 9{\cdot}0316 & 10{\cdot}7895 \\ -35{\cdot}0000 & -17{\cdot}0000 & -24{\cdot}5000 \\ 27{\cdot}7368 & 11{\cdot}4737 & 19{\cdot}8421 \end{bmatrix}. \tag{32}$$

Solution of two adjoint eigenvalue problems (5) and (6) yields, after normalization,

$$U = \begin{bmatrix} -0{\cdot}2136 & -0{\cdot}5502 & -0{\cdot}7315 \\ 1{\cdot}0000 & -0{\cdot}2893 & 0{\cdot}0614 \\ -0{\cdot}9580 & 1{\cdot}0000 & 1{\cdot}0000 \end{bmatrix}, \tag{33}$$

$$V = \begin{bmatrix} -0{\cdot}3869 & -0{\cdot}1728 & -0{\cdot}1887 \\ -1{\cdot}2311 & -0{\cdot}9129 & -0{\cdot}5196 \\ 0{\cdot}8396 & 0{\cdot}7751 & 0{\cdot}5124 \end{bmatrix}. \tag{34}$$

With this equivalence transformation, the system can be reduced to the form (23), where

$$\Lambda = \mathrm{diag}[1{\cdot}9333, 0{\cdot}2198, 0{\cdot}0495]. \tag{35}$$

The natural frequencies are simply the square roots of the positive eigenvalues and are 0·22, 0·47 and 1·39 radians per second. All modes u_i, which are columns of U, are real in this case. Hence, if the coefficient matrix B were absent, all components of the system could perform harmonic vibration with identical phase angle at each natural frequency. The system still does not possess classical normal modes because the eigenvectors u_i are not orthogonal to each other with respect to A. The adjoint modes v_j constitute the rows of V. In addition to complementing biorthogonality relations, the role of the adjoint modes is to modify the generalized excitation $f(t)$.

An examination of the transformed coefficient matrix of velocity reveals that

$$VBU = \mathrm{diag}[0{\cdot}4685, 0{\cdot}5109, 0{\cdot}5733], \tag{36}$$

and therefore the system has been completely decoupled. As a result, this system can be regarded as composing of three independent single-degree-of-freedom systems. According to theorem 3, condition (24) must be satisfied. This can be verified by a simple calculation. In most applications, explicit verification of condition (24) at the start of computation is not advisable. An equivalence transformation should simply be applied to diagonalize the coefficient matrices A and C. Afterwards, one can examine the transformed coefficient matrix of velocity VBU to determine if the use of decoupling approximation would require a corrective scheme. Although the coefficient matrices A, B and C in this system can be diagonalized by equivalence transformation, they cannot be diagonalized by similarity transformation. Among other things, A itself is not diagonalizable by similarity transformation because there is only one eigenvector $[1, 1, -1]^T$ associated with the repeated eigenvalue 6. In addition, the coefficient matrices A, B and C cannot be simultaneously reduced to symmetric forms because A itself is not similar to a real symmetric matrix (Inman 1983). From this discussion, the power and generality of equivalence transformations are clear.

5. Conclusions

A time-honoured procedure for the approximate solution of nonlinear mechanical systems is linearization, or in some cases piecewise linearization. Development of an efficient method for analyzing and interpreting the equations of motion of linearized controlled structures is timely due to increasing use of control devices in structures in recent years. The method expounded in this article represents a direct extension of classical modal analysis. This constructive method utilizes equivalence transformations in Lagrangian coordinates, and does not require conversion of the equations of motion to first-order forms. It is assumed, without practical loss of generality, that a linearized structure is not degenerate or defective. That means the coefficient of acceleration A is nonsingular, and the eigenvalue problem (5) possesses a full complement of independent eigenvectors. The major results, summarized in the following, are applicable for any generalized excitation $f(t)$.

(1) A controlled structure in which $B = 0$ can always be decoupled by equivalence transformation. Compared with classical modal transformation, the lack of symmetry in a structure only approximately doubles the computational effort.
(2) In free vibration, all components of an undamped nongyroscopic structure can perform harmonic vibration with identical frequency if the associated eigenvalue problem (5) possesses positive eigenvalues. The natural frequencies are simply the square roots of these positive eigenvalues, and the mode shapes can be determined from the corresponding complex eigenvectors. Unlike classical modal vibration, the system components generally vibrate with different phase angles.
(3) The linearized controlled structure (1) can be decoupled by an equivalence transformation if and only if the matrices $A^{-1}B$ and $A^{-1}C$ commute in multiplication. Similar to classical modal analysis of a viscously damped system, it is often necessary to draw upon some type of decoupling approximation for fast solution.

Compared with the state-space approach, the method expounded herein offers substantial reduction in computational effort and ample physical insight. The method of equivalence transformations is applicable to nonsymmetric systems of any order, just as classical modal analysis is applicable to symmetric systems of arbitrary order. The tremendous power of equivalence transformation reflects its role as mathematically the most general nonsingular linear transformation. In addition, many numerical techniques involve equations resembling those of linearized controlled structures. These numerical algorithms will economize on both core memory and computing time if the method of equivalence transformations is incorporated. Among other things, it is hoped that the present paper would point to directions along which further research efforts can be profitably made. It appears feasible, for example, to examine stability and other qualitative features of a controlled structure with equivalence transformations. Analysis of errors committed by the application of decoupling approximation to a controlled structure is also worthwhile in a subsequent course of investigation.

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