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RESEARCH ARTICLE

TK-operator Semigroups for Cryptogroups*

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1. Introduction

Completely regular semigroups, which are unions of their subgroups, are very important and have been studied intensely. *Cryptogroups* are completely regular semigroups on which the Green relation $\mathcal{H}$ is a congruence. It is well known that any cryptogroup is a band of groups and conversely (see [5]).

The most efficient method of studying congruences on regular semigroups is the kernel-trace approach. This method proved quite successful for inverse semigroups (see [6]). Though the analysis for general regular semigroups encounters considerable difficulties (see [2] and [3]), this approach has been established and helpful for the studies of lattices of some completely regular semigroup varieties (see [4]).

For a regular semigroup S, denote by $E(S)$ the set of idempotents in S and by $\mathcal{C}(S)$ the *lattice of congruences* on S. For any ρ in $\mathcal{C}(S)$,

$$\ker\rho = \{a \in S \mid (\exists\, e \in E(S))\, a\rho e\}, \qquad \operatorname{tr}\rho = \rho|_{E(S)}$$

are the *kernel* and *trace* of ρ, respectively. The fundamental result here is that ρ is uniquely determined by the congruence pair $(\ker\rho, \operatorname{tr}\rho)$. By ρK and ρk [resp. ρT and ρt] we denote the greatest and the least congruences on S having the same kernel [resp. trace] as ρ. Thus we obtain four *operators* on $\mathcal{C}(S)$, which are denoted by K, k, T and t. Let $\Gamma = \{K, k, T, t\}$, we denote by Γ^+ and Γ^*, respectively, the free semigroup and free monoid generated by Γ. For any $\rho \in \mathcal{C}(S)$, if we act on ρ by Γ^*, we can obtain a *network of congruences* ρ, ρK, ρk, ρT, $\rho t, \dots$, ordered by inclusion. Considering networks for all congruences on S, we may figure out the lattice of congruences on S. The research of networks of congruences is a further development of the kernel-trace approach. Several works have been completed, see [7] and [8] for details. In the case of *Clifford semigroups* (semilattices of groups) and *completely simple semigroups*, Petrich determined in [7] and [8] the semigroups generated by these four operators with relations valid in all networks of these kinds of semigroups and thus characterized the entire networks of congruences. These successful works are due to the relatively precise descriptions of congruences on Clifford and completely simple semigroups. We shall manage to do the same work as in [7] and [8] in this paper for cryptogroups, and confirm Petrich's guess that the same analysis of congruences on more general classes of semigroups could be made.

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The special relations among cryptogroup congruences, band congruences and idempotent separating congruences on regular semigroups obtained in [1] give a characterization of cryptogroups in terms of congruences. Section 2 presents this information and other neccessary notation and terminology. Using the result for cryptogroups stated above, we obtain in Section 3 the conclusions about the operators similar to that for Clifford semigroups in [7]. As in [7], we detemine the semigroup generated by operators Γ with relations valid in all networks on general cryptogroups. Furthermore, we obtain the semigroup generated by these operators for E-unitary cryptogroups. In Section 4, we investigate further the case of completely simple semigroups. The semigroups generated by these operators for some special classes of completely simple semigroups are offered. In each of these cases, we present an example of a completely simple semigroup.

2. Preliminaries

We shall use the notation and terminology of [2] and [7]. Let S be a regular semigroup. Recall from [2] that for any congruence ρ on S, ρ is uniquely determined by its kernel and trace in the following way. An equivalence ξ on $E(S)$ is normal if $\xi = \operatorname{tr}\xi^*$, where ξ^* is the congruence on S generated by ξ. A subset N of S is normal if $N = \ker \pi_N$, where π_N is the greatest congruence on S which saturates N.

A pair (N, ξ) is a *congruence pair* for S, if the following conditions are satisfied.

(1) N is a normal subset of S,

(2) ξ is a normal equivalence on $E(S)$,

(3) $N \subseteq \ker(\mathcal{L}\xi\mathcal{L}\xi\mathcal{L} \cap \mathcal{R}\xi\mathcal{R}\xi\mathcal{R})^0$,

(4) $\xi \subseteq \operatorname{tr}\pi_N$,

where γ^0 is the greatest congruence on S contained in γ. In such a case, $\rho_{(N,\xi)}$ is defined by

$$\rho_{(N,\xi)} = \pi_N \cap (\mathcal{L}\xi\mathcal{L}\xi\mathcal{L} \cap \mathcal{R}\xi\mathcal{R}\xi\mathcal{R})^0 .$$

Result 1. [2, Theorem 2.13] *Let S be a regular semigroup. If (N, ξ) is congruence pair for S, then $\rho_{(N,\xi)}$ is the unique congruence ρ on S for which* $\ker\rho = N$, $\operatorname{tr}\rho = \xi$. *Conversely, if ρ is a congruence on S, then* $(\ker\rho, \operatorname{tr}\rho)$ *is a congruence pair for S and $\rho = \rho_{(\ker\rho, \operatorname{tr}\rho)}$.* ■

Denote by $Cp(S)$ the poset of all congruence pairs under the componentwise inclusion order. Then the mappings

$$\rho \longmapsto (\ker\rho, \operatorname{tr}\rho), \qquad (N, \xi) \longmapsto \rho_{(N,\xi)}$$

are mutually inverse isomorphisms between $\mathcal{C}(S)$ and $Cp(S)$. In view of this fact, if we want to act on $\rho_{(N,\xi)}$ by operators, we sometimes act on its corresponding congruence pair (N, ξ) in an obvious way. For any congruence ρ on a regular semigroup, define

$$K_\rho = \{\gamma \in \mathcal{C}(S) \mid \ker\gamma = \ker\rho\}, \qquad T_\rho = \{\gamma \in \mathcal{C}(S) \mid \operatorname{tr}\gamma = \operatorname{tr}\rho\}.$$

Result 2. [2, Theorem 3.2] *Let ρ be a congruence on a regular semigroup S. Then K_ρ and T_ρ are intervals of $\mathcal{C}(S)$ such that*

$$K_\rho = [\rho k, \rho K], \qquad T_\rho = [\rho t, \rho T],$$

where $\rho k = \{(x, x^2) \mid x \in \ker\rho\}^*$, $\rho K = \pi_{\ker\rho}$, $\rho t = \xi^*$ *and* $\rho T = (\mathcal{L}\xi\mathcal{L}\xi\mathcal{L} \cap \mathcal{R}\xi\mathcal{R}\xi\mathcal{R})^0$, $\xi = \operatorname{tr}\rho$. *If* $\rho, \gamma \in \mathcal{C}(S)$, *then*

$$\operatorname{tr}\rho \subseteq \operatorname{tr}\gamma \implies \rho t \subseteq \gamma t, \quad \rho T \subseteq \gamma T,$$
$$\ker\rho \subseteq \ker\gamma \implies \rho k \subseteq \gamma k. \qquad \blacksquare$$

For $\rho \in \mathcal{C}(S)$, ρ is called an *idempotent separating congruence* if for any $e, f \in E(S)$, $e\rho f$ implies $e = f$; ρ is called *idempotent pure* if for $e \in E(S)$, $a \in S$, $e\rho a$ implies $a \in E(S)$. Denote the universal relation and the equality relation on the set S by $\omega = \omega_S$ and $\varepsilon = \varepsilon_S$, respectively. Then $\sigma = \omega t$ [resp. $\beta = \omega k$] is the *least group* [resp. *band*] *congruence* on S, and $\mu = \varepsilon T$ [resp. $\tau = \varepsilon K$] is the *greatest idempotent separating* [resp. *idempotent pure*] *congruence* on S. A regular semigroup S is *E-unitary* if $E(S)$ is an unitary subset of S, that is, for any $e \in E(S)$ and $s \in S$, $se \in E(S)$ implies $s \in E(S)$. S is *E-disjunctive* if $\tau = \varepsilon$ in $\mathcal{C}(S)$; while $\pi = \sigma k$ is the *least E-unitary congruence* on S.

A *completely simple semigroup* is a special cryptogroup. It is well known that a completely simple semigroup S can be represented as a Rees matrix semigroup over a group G with a sandwich matrix P and denoted by $S = \mathcal{M}(I, G, \Lambda; P)$. In order to obtain precise descriptions of $\rho K, \rho k, \rho T$ and ρt for a congruence ρ on a completely simple semigroup, we need a characterization of congruences on regular semigroups with Q-inverse transversals developed in [9].

An inverse subsemigroup S° of a regular semigroup S is called a *Q-inverse transversal* of S, if S° contains a unique inverse a° for any $a \in S$ and $S^\circ \subseteq S^\circ S S^\circ$. In this case, let

$$E = \{e \in E(S) \mid e = ee^\circ\}, \qquad F = \{g \in E(S) \mid g = g^\circ g\},$$

then E and F are a left and a right normal band, respectively. For any $x \in S$, $xx^\circ \in E$ and $x^\circ x \in F$. If we let $E^\circ = E(S^\circ)$, then $E^\circ \subseteq E \cap F$.

Let S be a regular semigroup with a Q-inverse transversal S°. Let η be a congruence on S°, τ_E and τ_F be congruences on E and F, respectively. If they satisfy the following conditions, (τ_E, η, τ_F) is called a *congruence triple* for S.

(T1) $\eta|_{E^\circ} = \tau_E|_{E^\circ} = \tau_F|_{E^\circ}$,

(T2) $(\forall\, e, f \in E, g \in F)\quad e\tau_E f \implies ge\eta gf$,
$(\forall\, g, h \in F, e \in E)\quad g\tau_F h \implies ge\eta he$.

Define a relation $\rho_{(\tau_E, \eta, \tau_F)}$ on S by the following rule

$$x\rho_{(\tau_E,\eta,\tau_F)}y \iff xx^\circ\tau_E yy^\circ,\ x^\circ\eta y^\circ,\ x^\circ x\tau_F y^\circ y.$$

Then $\rho = \rho_{(\tau_E,\eta,\tau_F)}$ is the unique congruence on S such that $\rho|_{S^\circ} = \eta$, $\rho|_E = \tau_E$ and $\rho|_F = \tau_F$, and every congruence on S can be obtained in this way.

For a completely simple semigroup $S = \mathcal{M}(I, G, \Lambda; P)$, we may assume that I and Λ have a common element 1. Set $S^\circ = \{(1, a, 1) \mid a \in G\}$, then $S^\circ \cong G$ and S° is a Q-inverse transversal on S. For any $(i, a, \lambda) \in S$, $(i, a, \lambda)^\circ = (1, p_{\lambda 1}^{-1} a^{-1} p_{1i}^{-1}, 1)$. We have $E = \{(i, p_{1i}^{-1}, 1) \mid i \in I\}$ and $F = \{(1, p_{\lambda 1}^{-1}, \lambda) \mid \lambda \in \Lambda\}$. It is clear that E and F are left and right zero semigroups, respectively. Since E° has the identity of S° as its only element, the condition (T1) is satisfied trivially. Hence (τ_E, η, τ_F) is a congruence triple if and only if it satisfies (T2). From Theorems 4.4 and 4.8 in [9], for $\rho = \rho_{(\tau_E, \eta, \tau_F)}$, $\rho' = \rho_{(\tau'_E, \eta', \tau'_F)} \in \mathcal{C}(S)$,

$$\ker \rho = \ker \rho' \iff \eta = \eta',$$
$$\operatorname{tr} \rho = \operatorname{tr} \rho' \iff \tau_E = \tau'_E,\ \tau_F = \tau'_F.$$

Summarizing Theorems 4.4 and 4.8 in [9], we have following result which will be used in examples.

For a congruence triple (τ_E, η, τ_F), we define the following relations on S°, I and Λ respectively,

$$\tau_t^\circ = \left\{(p_{\lambda 1}^{-1} p_{\lambda i} p_{1i}^{-1},\ p_{\mu 1}^{-1} p_{\mu j} p_{1j}^{-1}) \mid (i, p_{1i}^{-1}, 1)\tau_E(j, p_{1j}^{-1}, 1), (1, p_{\lambda 1}^{-1}, \lambda)\tau_F(1, p_{\mu 1}^{-1}, \mu)\right\}^*,$$
$$(i, p_{1i}^{-1}, 1)\kappa_E(\eta)(j, p_{1j}^{-1}, 1) \iff (\forall\, \lambda \in \Lambda)\, p_{\lambda i} p_{1i}^{-1}\, \eta\, p_{\lambda j} p_{1j}^{-1},$$
$$(1, p_{\lambda 1}^{-1}, \lambda)\kappa_F(\eta)(1, p_{\mu 1}^{-1}, \mu) \iff (\forall\, i \in I)\, p_{\lambda 1}^{-1} p_{\lambda i}\, \eta\, p_{\mu 1}^{-1} p_{\mu i}.$$

Result 3. *Let $S = \mathcal{M}(I, G, \Lambda; P)$ be a completely simple semigroup. For any $\rho_{(\tau_E, \eta, \tau_F)} \in \mathcal{C}(S)$, we have*

$$\rho_{(\tau_E,\eta,\tau_F)}K = \rho_{(\kappa_E(\eta),\eta,\kappa_F(\eta))}, \qquad \rho_{(\tau_E,\eta,\tau_F)}k = \rho_{(\varepsilon_E,\eta,\varepsilon_F)},$$
$$\rho_{(\tau_E,\eta,\tau_F)}T = \rho_{(\tau_E,\omega_{S^\circ},\tau_F)}, \qquad \rho_{(\tau_E,\eta,\tau_F)}t = \rho_{(\tau_E,\tau_t^\circ,\tau_F)}. \qquad \blacksquare$$

Result 4. [1, Corollary 3] *For a regular semigroup S, the following statements are equivalent.*

(1) *S is a cryptogroup;*

(2) $\mu = \beta$;

(3) *For every $\rho \in \mathcal{C}(S)$, $\rho T = \rho \vee \beta$;*

(4) *For every $\rho \in \mathcal{C}(S)$, $\rho k = \rho \wedge \mu$.* $\blacksquare$

The following lemma is essential.

Lemma 2.1. *For a regular semigroup S, the following statements are equivalent.*

(1) *S is a cryptogroup;*

(2) *For any $\rho \in \mathcal{C}(S)$, ρT is a band congruence;*

(3) *For any $\rho \in \mathcal{C}(S)$, ρk is an idempotent separating congruence.*

Proof. (1) $\Leftrightarrow$ (2) follows from Result 4.

(1) $\Rightarrow$ (3). From Result 4, $\rho k = \rho \wedge \mu \subseteq \mu$, so ρk is an idempotent separating congruence.

(3) $\Rightarrow$ (1). We have $\rho k \subseteq \mu$, so $\rho k \subseteq \rho \wedge \mu$. But $\rho \wedge \mu \subseteq \rho k$, so $\rho k = \rho \wedge \mu$. Hence by Result 4, S is a cryptogroup.

■

The following lemma is immediate from Lemma 2.1.

Lemma 2.2. *For a cryptogroup S and any congruence pair (N, ξ) for S,*

$$(N,\xi)K = (N, N\kappa), \qquad (N,\xi)k = (N,\varepsilon),$$
$$(N,\xi)T = (S,\xi), \qquad (N,\xi)t = (\xi\theta,\xi),$$

where $N\kappa = \operatorname{tr}(\rho_{(N,\xi)}K)$, $\xi\theta = \ker(\rho_{(N,\xi)}t)$. ■

Lemma 2.1 indicates that for any congruence ρ on a cryptogroup S, ρT and ρk fall into the min network of ω. So we may obtain the min network of ω for a cryptogroup as follows.

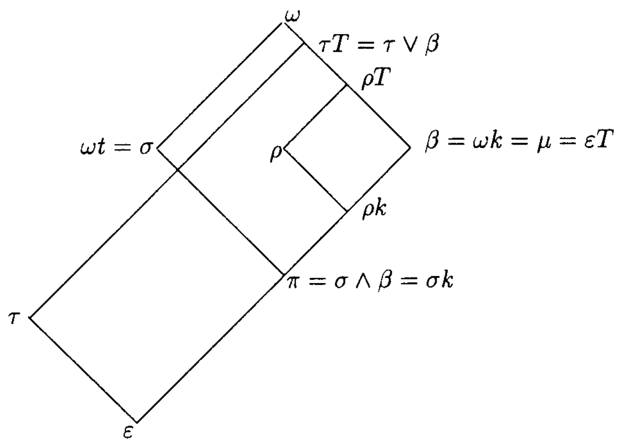

Diagram 1.

Furthermore, we have

Lemma 2.3. *Let S be a regular semigroup. Then S is a cryptogroup if and only if $Tk = kT$ and Tk is a constant operator.*

Proof. Let S be a cryptogroup. Then by Lemma 2.1, for any $\rho \in \mathcal{C}(S)$, $\rho Tk = \beta$ and $\rho kT = \beta$. Hence $Tk = kT$ and they are constant operators. Conversely, assume that $Tk = kT$ and Tk is constant operator for a regular semigroup S. Then for any $\rho \in \mathcal{C}(S)$, $\rho Tk = \omega Tk = \beta$. Thus $\ker(\rho T) = \ker\beta = \ker\omega = S$. Hence, ρT is a band congruence on S. From Lemma 2.1, S is a cryptogroup. ■

3. TK-operator semigroups of the class of cryptogroups

From now on we assume that S stands for a cryptogroup. In this section, we present the semigroup Γ^+/Σ^* for the relations Σ valid for all cryptogroups by a system of representatives for the congruence Σ^* on Γ^+, and prove that Γ^+/Σ^* is just the TK-operator semigroup of the class of cryptogroups.

Lemma 3.1. (1) *For any group congruence ρ on S, $\rho K = \rho$, so $\sigma K = \sigma$.*

(2) *For any band congruence ρ on S, $\rho T = \rho$, so $\beta T = \beta$.*

(3) *For any idempotent pure congruence ρ on S, $\rho t = \rho$, so $\tau t = \tau$.*

Proof. (1) If ρ is a group congruence, then $\operatorname{tr}\rho = \omega|_{E(S)}$. Since

$$(\ker\rho,\ \operatorname{tr}\rho)K = (\ker\rho, \omega|_{E(S)})K = (\ker\rho, \omega|_{E(S)})\,,$$

we have $\rho K = \rho$.

(2) If ρ is a band congruence, then $\ker\rho = S$. Since

$$(\ker\rho,\ \operatorname{tr}\rho)T = (S,\ \operatorname{tr}\rho)T = (S,\ \operatorname{tr}\rho) = (\ker\rho,\ \operatorname{tr}\rho)\,,$$

we have $\rho T = \rho$.

(3) If ρ is an idempotent pure congruence on S, then $\ker\rho = E(S)$. Since

$$(\ker\rho,\ \operatorname{tr}\rho)t = (E(S),\ \operatorname{tr}\rho)t = (E(S),\ \operatorname{tr}\rho) = (\ker\rho,\ \operatorname{tr}\rho)\,,$$

we have $\rho t = \rho$. ■

Different from [7], the following lemma about the constant operators is obtained from the characterization of cryptogroups in terms of congruences (Lemma 2.1).

Lemma 3.2. *Let $\rho \in \mathcal{C}(S)$.*

(1) $\rho TK = \omega$,

(2) $\rho TKt = \sigma$,

(3) $\rho Tk = \rho kT = \beta$,

(4) $\rho TKtk = \sigma \wedge \beta = \pi$,

(5) $\rho kt = \varepsilon$,

(6) $\rho ktK = \tau$,

(7) $\rho ktKT = \tau T = \tau \vee \beta$.

Proof. We note that by Lemma 2.1, ρT is a band congruence and ρk is an idempotent separating congruence. These facts and Lemma 2.3 imply (1) – (7). ■

As in [7], we drop ρ in the expressions in Lemm 3.2, and use the constant values for the corresponding constant operators.

$$\begin{array}{llll} \varepsilon = kt, & \tau = ktK, & \tau\vee\beta = ktKT, & \beta = kT, \\ \omega = TK, & \sigma = TKt, & \sigma\wedge\beta = TKtk. & \end{array}$$

Also we set
$$\Delta = \{\varepsilon, \sigma, \beta, \tau, \sigma\wedge\beta, \tau\vee\beta, \omega\}.$$
In the sequel, we sometimes regard the elements in Δ as congruences.

From Lemma 3.1, for any element $U \in \Delta$, UK, Uk, UT and Ut all belong to Δ.

The following lemma is similar to Lemma 3 in [7].

Lemma 3.3. *Let ξ, ξ' be two normal equivalences on $E(S)$.*

(1) $\xi \subseteq \xi' \Longrightarrow \xi\theta \subseteq \xi'\theta$.

(2) $\xi \subseteq \xi\theta\kappa$.

(3) $\xi\theta = \xi\theta\kappa\theta$.

Proof. (1) From $\xi^* \subseteq \xi'^*$ and Result 2, we have $\xi^* t \subseteq \xi'^* t$. Since $\operatorname{tr}\xi^* = \xi$, $\operatorname{tr}\xi'^* = \xi'$, $(\ker\xi^*, \xi)$ and $(\ker\xi'^*, \xi')$ are congruence pairs, and thus the above inclusion implies $(\xi\theta, \xi) \subseteq (\xi'\theta, \xi')$. Hence $\xi\theta \subseteq \xi'\theta$.

(2) As $(\ker\xi^*, \xi)$ is a congruence pair, we have
$$(\ker\xi^*, \xi)t = (\xi\theta, \xi), \qquad (\ker\xi^*, \xi)tK = (\xi\theta, \xi)K = (\xi\theta, \xi\theta\kappa)\,.$$
Thus $(\ker\xi^*, \xi)t \leq (\ker\xi^*, \xi)tK$ implies $\xi \subseteq \xi\theta\kappa$.

(3) By (1), $(\xi\theta, \xi)$ is a congruence pair. We have
$$(\xi\theta, \xi)tK = (\xi\theta, \xi)K = (\xi\theta, \xi\theta\kappa)\,,$$
$$(\xi\theta, \xi)tKt = (\xi\theta, \xi\theta\kappa)t = (\xi\theta\kappa\theta, \xi\theta\kappa)\,.$$
Hence $\xi\theta\kappa\theta \subseteq \xi\theta$. By (2), $\xi \subseteq \xi\theta\kappa$, and thus by (1) $\xi\theta \subseteq \xi\theta\kappa\theta$. Therefore the equality holds. ∎

We give the relations Σ satisfied in all networks of congruences on cryptogroups.

Lemma 3.4. *Operators Γ satisfy the following relations Σ.*

(1) $K^2 = kK = K$, $\quad k^2 = Kk = k$,
$t^2 = Tt = t$, $\quad T^2 = tT = T$.

(2) $KTK = TKT = TK$,
$tkt = ktk = kt$.

(3) $tKt = tK$.

(4) $kT = Tk$.

Proof. (1) This follows immediately from the definition of operators Γ.

(2) From Lemma 3.2, we have $TK = \omega$ and $kt = \varepsilon$ and hence the equalities follow immediately.

(3) For $\rho = \rho_{(N,\xi)} \in \mathcal{C}(S)$, we have

$$(N,\xi)tKt = (\xi\theta,\xi)Kt = (\xi,\xi\theta\kappa)t = (\xi\theta\kappa\theta,\xi\theta\kappa)\,,$$
$$(N,\xi)tK = (\xi\theta,\xi\theta\kappa)\,.$$

By Lemma 3.3 (3), the above equalities yield $\rho tK = \rho tKt$.

(4) This follows from Lemma 2.3. ■

For convenience, we regard Σ as relations on Γ^+ in an obvious way. So Σ^* is a congruence on Γ^+ generated by Σ, while Γ^+/Σ^* is the semigroup generated by Γ together with the relations Σ. We denote the elements of Γ^+/Σ^* exactly by the representatives for the congruence classes of Σ^* and remember that the product of operators satisfies the relations Σ. We have a similar theorem as in [7].

Theorem 3.5. *We have*

$$\begin{array}{rcllllllll} \Gamma^+/\Sigma^* & = & \{\,K, & KT, & Kt, & KtK, & Ktk, & KtKT, & k, & \\ & & t, & tk, & tK, & & tKT, & & T\,\} \cup \Delta\,. \end{array}$$

The multiplication is subject to the relation Σ. The elements in Δ are constant operators and hence are right zeros of Γ^+/Σ^*, and Δ is a minimal ideal of Γ^+/Σ^*.

Proof. From the relations in Σ, we can deduce that for any element U in Δ, $U\Gamma \subseteq \Delta$. This implies that any word beginning with T or k is Σ^*-related to a word in Δ. So we need only consider the words beginning with K and t. If we note that the elements in Δ, which are all products of operators beginning with T or k, are right zeros of Γ^+/Σ^*, we are left with the words beginning with K or t and ending with T or k if there are any. From the relations Σ in Lemma 3.4, it is not difficulty to see that any word in Γ^+ is Σ^*-related to a word in the statement of the theorem. Example 1 in [7] shows that all the elements presented in the above are distinct. From the min network of ω (Diagram 1) the action on Δ (whose elements are regarded as congruences) by Γ^+/Σ^* is transitive. Hence Δ is a minimal ideal of Γ^+/Σ^*. ■

Definition 1. Let S be a regular semigroup, the semigroup $\Gamma(S)$ generated by the operators Γ on $\mathcal{C}(S)$ is called the TK-operator semigroup of S. For a class $\mathcal{A}$ of regular semigroups, if there exists a semigroup A in $\mathcal{A}$ such that for any semigroup S in $\mathcal{A}$, $\Gamma(S)$ is a homomorphic image of $\Gamma(A)$, then $\Gamma(A)$ is called the TK-operator semigroup of $\mathcal{A}$. ■

From [7], there exists a Clifford semigroup S such that $\Gamma(S) \cong \Gamma^+/\Sigma^*$. Thus from Theorem 3.5, we have the following result.

Theorem 3.6. *The TK-operator semigroup of the class of cryptogroups is Γ^+/Σ^*.* ■

Now we determine the TK-operator semigroup of the class of E-unitary cryptogroups. From Theorem 3.6, this TK-operator semigroup is a homomorphic image of Γ^+/Σ^*. We give a charaterization of E-unitary regular semigroups below.

Lemma 3.7. [1, Corollary 5] *For a regular semigroup S, the following statements are equivalent.*

(1) *S is E-unitary.*

(2) $\sigma = \tau$.

(3) *For every $\rho \in \mathcal{C}(S)$, $\rho t = \rho \wedge \tau$.*

(4) *For every $\rho \in \mathcal{C}(S)$, ρt is an idempotent pure congruence.*

(5) *Every idempotent pure congruence on S is E-unitary.*

(6) *There exists an idempotent pure E-unitary congruence on S.* ■

The next lemma exhibits some additional relations between operators for E-unitary cryptogroups.

Lemma 3.8. *Let S be an E-unitary cryptogroup. Then the operators Γ satisfy the following relations Σ_1.*

$$\sigma = \tau = tK = KtK\,,$$

$$\omega = \tau \vee \beta = KtKT = tKT\,,$$

$$\varepsilon = \sigma \wedge \beta = tk = Ktk\,.$$

Proof. From Lemma 3.7, $\sigma = \tau$ and ρt is an idempotent pure congruence for every $\rho \in \mathcal{C}(S)$. So by Lemma 3.1, for any $\rho \in \mathcal{C}(S)$, $\sigma = \tau = \rho tK = \rho KtK$. Multiplying both sides of the equalities on the right by T, we have $\omega = \tau \vee \beta = KtKT = tKT$. Multiplying both sides of the same equalities on the right by k, we get $\varepsilon = \pi = tk = Ktk$. ■

It is worthy to note that for cryptogroups, the three sets of equalities in Lemma 3.8 are equivalent. We have the following conclusion if the relations Σ_1 are considered to be a relation Ψ on Γ^+/Σ^*.

Lemma 3.9. *The relation Ψ is a congruence on Γ^+/Σ^*.*

Proof. All the non-singleton classes of Ψ are

$$\{\tau, \sigma, tK, KtK\}, \quad \{\omega, \tau \vee \beta, KtKT, tKT\}, \quad \{\varepsilon, \sigma \wedge \beta, tk, Ktk\}.$$

It is sufficient to prove that products by any operator in Γ on the left and the right of any of the above classes are contained in a class of Ψ. This is a routine matter. ■

Example 1. Let $S = \mathbf{Z}_2 \times Y_2$, where $Y_2 = \{0, 1\}$ is a semilattice. Then S is an E-unitary cryptogroup. Let $I = \mathbf{Z}_2 \times \{0\}$, then I is an ideal of S and S/ρ_I is not E-unitary. ■

The above example makes the following proposition have sense.

Proposition 3.10. *Let S be an E-unitary cryptogroup. If not all congruences on S are E-unitary, then*

$$\begin{aligned}\Gamma(S) &\cong (\Gamma^+/\Sigma^*)/\Psi \\ &\cong \{K, KT, Kt, k, t, T\} \cup \{\varepsilon, \sigma, \beta, \omega\},\end{aligned}$$

where the last set is endowed with the product being subject to the relations Σ and Σ_1.

Proof. By Lemma 3.8, as an element of $\Gamma(S)$, any element in Γ^+/Σ^* coincides with an element in the above set. We prove below that the elements of the above set are different.

It is clear $\varepsilon \neq \omega$ and $\sigma \neq \beta$. If $\varepsilon = \sigma$ or $\varepsilon = \beta$, then S is a group or a band, and so every congruence on S is E-unitary, contradicting our hypothesis. So $\varepsilon \neq \sigma$ and $\varepsilon \neq \beta$. If $\sigma = \omega$, then $\varepsilon = \tau k = \sigma k = \omega k = \beta$, and so S is a band. Again this is a contradiction, so $\sigma \neq \omega$. If $\beta = \omega$, then $\varepsilon = \beta t = \omega t = \sigma$, and so S is a group, contradicting the hypothesis. Thus $\beta \neq \omega$. Therefore $\varepsilon, \sigma, \beta, \omega$ are different.

It is obvious that K, k, T and t are distinct. Since $\varepsilon KT = \omega$, $\varepsilon Kt = \sigma$, we have $KT \neq Kt$. Similarly, we can verify that KT and Kt are different from each of K, k, T, t. Let ρ be a congruence on S which is not E-unitary. Of course, ρ is not a group congruence, so $\rho KT \neq \omega$, $\rho Kt \neq \sigma$, and hence $KT \neq \omega$, $Kt \neq \sigma$. It is not difficult to show that KT as well as Kt are different from any constant operator. From Theorems 3.5 and 3.6, Lemma 3.8, we have

$$\Gamma(S) \cong \{K, KT, Kt, k, t, T\} \cup \{\varepsilon, \sigma, \beta, \omega\}.$$

From Lemma 3.9, $(\Gamma^+/\Sigma^*)/\Psi$ is isomorphic to the above semigroup. ■

Now we can present the result for E-unitary semigroups.

Theorem 3.11. *$\Gamma^+/(\Sigma \cup \Sigma_1)^*$ is the TK-operator semigroup of the class of E-unitary cryptogroups and*

$$\begin{aligned}(\Gamma^+/\Sigma^*)/\Psi &\cong \Gamma^+/(\Sigma \cup \Sigma_1)^* \\ &= \{K, KT, Kt, k, t, T\} \cup \{\varepsilon, \sigma, \beta, \omega\}.\end{aligned}$$

Proof. As in Theorem 3.5, we denote elements of $\Gamma^+/(\Sigma \cup \Sigma_1)^*$ by the representatives for the congruence classes of $(\Sigma \cup \Sigma_1)^*$. It is not difficult to show that all the elements of $\Gamma^+/(\Sigma \cup \Sigma_1)^*$ are $\{K, KT, Kt, k, t, T\} \cup \{\varepsilon, \sigma, \beta, \omega\}$. From Example 1 and Proposition 3.10, we claim that as elements of $(\Sigma \cup \Sigma_1)^*$, all the elements in the above set are different. Thus the proof is completed. ■

Remark 3.12. Further analysis on the semigroup Γ^+/Σ^* may give more understanding of the concept of TK-operator semigroups. Consider the relation Υ on Γ^+/Σ^* with the following non-singleton classes,

$$\{Kt, KtK,\}, \ \{KT, KtKT\}, \ \{\sigma, \tau\}, \ \{\omega, \tau \vee \beta\}, \ \{\varepsilon, \tau \wedge \beta\}.$$

It may be proved that for any $J \in \Gamma$, multiplying every class above on the left and right by J yields a set contained in a class of Υ. So Υ is a congruence on Γ^+/Σ^*. Clearly

$$(\Gamma^+/\Sigma^*)/\Upsilon \cong \{K, KT, Kt, Ktk, k, t, tk, tK, tKT, T\} \cup \{\varepsilon, \sigma, \beta, \omega\}.$$

If there is a cryptogroup S such that $\Gamma(S) \cong (\Gamma^+/\Sigma^*)/\Upsilon$, then S must be E-unitary. By Theorem 3.11, $\Gamma(S)$ is a homomorphic image of $(\Gamma^+/\Sigma^*)/\Psi$. But $|(\Gamma^+/\Sigma^*)/\Psi| = 10$ and $|(\Gamma^+/\Sigma^*)/\Upsilon| = 14$, this is a contradiction. So there is no cryptogroup S such that $\Gamma(S) \cong (\Gamma^+/\Sigma^*)/\Upsilon$. Thus we may claim that a homomorphic image of Γ^+/Σ^* need not to be the TK-operator semigroup of some cryptogroup. This observation gives an explanation for Definition 1 and Theorem 3.6. ∎

4. *TK*-operator semigroups of completely simple semigroups

In this section, we shall deal with E-unitary, E-disjunctive and group congruence-free completely simple semigroups, respectively. We shall determine the TK-operator semigroups of these special completely simple semigroups. Examples will be provided to support our discussion.

Petrich determined the TK-operator semigroup of the class of completely simple semigroups in [8]. Here, we interpret this TK-operator semigroup as a homomorphic image of that of the class of cryptogroups. The additional relations satisfied by the operators for completely simple semigroups are presented as follows.

Lemma 4.1. *Let S be a completely simple semigroup. The following relations hold on $\mathcal{C}(S)$, and will be denoted by Θ.*

$$Kt = KtK, \qquad KT = KtKT.$$

If Θ is viewed as a relation on Γ^+/Σ^*, then Θ is a congruence on Γ^+/Σ^*.

Proof. The proof of the first assertion may be found in the proof of Lemma 6 in [8], and the second may be shown as in the proof of Lemma 3.8. ∎

We note that for a cryptogroup S, the relation $KT = KtKT$ holds if and only if $Kt = KtK$ holds from Lemma 3.4. The relation $Kt = KtK$ is viewed as a relation on Γ^+ and is denoted by M. The first assertion of the following theorem may be obtained from Theorem 1 and Corollary 1 in [8]. The second follows from Theorem 3.6 and Lemma 4.1.

Theorem 4.2. *Suppose that $\mathcal{M}$ is the class of completely simple semigroups, then the TK-operator semigroup of $\mathcal{M}$ is*

$$\begin{aligned}\Gamma^+/(\Sigma \cup M)^* \;=\; \{&K,\; KT,\; Kt,\; Ktk,\; k,\\ &t,\; tk,\; tK,\; tKT,\; T\} \cup \Delta,\end{aligned}$$

where the multiplication is subject to the relations Σ and M. Further, we have $(\Gamma^+/\Sigma^*)/\Theta \cong \Gamma^+/(\Sigma \cup M)^*$. ∎

The TK-operator semigroup of the class of E-unitary completely simple semigroups S is easy to obtain. From Lemmas 3.8 and 4.1, if S is an E-unitary completely simple semigroup, then the following additional relations M_1 hold on $\mathcal{C}(S)$.

$$\sigma = \tau = tK = Kt = KtK\,,$$
$$\omega = \tau \vee \beta = KT = KtKT = tKT\,,$$
$$\varepsilon = \sigma \wedge \beta = tk = Ktk\,.$$

Thus all the elements of $\Gamma(S)$ fall into the set $\{K, k, t, T\} \cup \{\varepsilon, \sigma, \beta, \omega\}$. It is known that a completely simple semigroup is E-unitary if and only if it is a rectangular group. Let $Q = I \times G \times \Lambda$ be a rectangular group, if $|I| \neq 1$, $|\Lambda| \neq 1$ and $|G| \neq 1$, then all these elements are different from each other on $\mathcal{C}(Q)$. If we regard M_1 as a relation Θ_1 on Γ^+/Σ^*, then it may be shown as in Lemma 3.8 that Θ_1 is a congruence on Γ^+/Σ^*. We have the following theorem whose proof is omitted.

Theorem 4.3. *The TK-operator semigroup of the class of E-unitary completely simple semigroups (rectangular groups) is*

$$\begin{aligned}(\Gamma^+/\Sigma^*)/\Theta_1 &\cong \Gamma^+/(\Sigma \cup M_1)^* \\ &= \{K, k, t, T\} \cup \{\varepsilon, \sigma, \beta, \omega\}\,,\end{aligned}$$

where the multiplication is subject to the relations Σ and M_1. ∎

Now we turn to E-disjunctive completely simple semigroups. We have the following lemma to describe E-disjunctive cryptogroups.

Lemma 4.4. *Let S be a cryptogroup.*

(1) *S is E-disjunctive if and only if $\tau \subseteq \beta$.*

(2) *If $\tau \vee \beta = \pi$, then S is E-disjunctive.*

Proof. (1) Obvious.

(2) If $\tau \vee \beta = \pi$, that is $\tau T = \pi$, then by acting by t on both sides, we obtain $\tau t = \varepsilon$. By Lemma 3.1, we have $\tau = \tau t = \varepsilon$. Hence S is E-disjunctive. ∎

By Lemma 4.4, $\tau \vee \beta = \beta$ for E-disjunctive cryptogroups. So for an E-disjunctive completely simple semigroup S, we have the following additional relations M_2 holding on $\mathcal{C}(S)$.

$$\varepsilon = \tau\,, \quad \beta = \tau \vee \beta\,,$$
$$Kt = KtK\,, \quad KT = KtKT\,.$$

If we regard M_2 as a relation Θ_2 on Γ^+/Σ^*, it may be shown as before that Θ_2 is a congruence.

We need the following two lemmas for Example 2. The first indicates the consequences of the difference between σ and τ.

Lemma 4.5. *Let S be a cryptogroup. If S is not E-unitary, then*

(1) $\varepsilon \neq \pi$, $\varepsilon \neq \sigma$, $\varepsilon \neq \beta$, $\varepsilon \neq \tau \vee \beta$, $\varepsilon \neq \omega$.

(2) $\tau \neq \pi$, $\tau \neq \sigma$, $\tau \neq \beta$, $\tau \neq \tau \vee \beta$, $\tau \neq \omega$.

(3) $\pi \neq \sigma$, $\pi \neq \omega$.

(4) $\beta \neq \sigma$, $\beta \neq \omega$.

(5) $\tau \vee \beta \neq \omega$.

Proof. We show that if any of these inequalities does not hold, then by Lemma 3.7, S must be E-unitary.

(1) If one of $\varepsilon = \beta$, $\varepsilon = \sigma$ and $\varepsilon = \omega$ holds, then S is respectively a band, group and the single element semigroup, of course it is E-unitary. If $\varepsilon = \pi$, then $\varepsilon K = \pi K$, that is $\tau = \sigma$, and so by Lemma 3.7, S is E-unitary. If $\varepsilon = \tau \vee \beta$, acting on both sides by Kt gives $\sigma = \tau$, and hence S is E-unitary.

(2) If $\tau = \pi$ or $\tau = \beta$, acting by K on both sides yields $\tau = \sigma$ or $\tau = \omega$, so S is E-unitary by Lemma 3.7. If $\tau = \tau \vee \beta$, acting on both sides by Kt, we have $\tau = \sigma$ and so S is E-unitary.

(3) If $\pi = \sigma$ on $\pi = \omega$, acting by tK on both sides yields $\tau = \sigma$, and thus S is E-unitary.

(4) If $\beta = \sigma$ or $\beta = \omega$, acting by tK on both sides gives $\tau = \sigma$, and so S is E-unitary.

(5) If $\tau \vee \beta = \omega$, acting on both sides by t, we get $\tau = \sigma$ and so S is E-unitary. ∎

Lemma 4.6. *Let $\mathcal{M}(I, G, \Lambda; P)$ be a completely simple semigroup, $1 \in I \cap \Lambda$.*

(1) *If the elements in the 1-row of the matrix P are all equal, then for $i, j \in I$, $(i, p_{1i}^{-1}, 1)\kappa_E(\varepsilon)(j, p_{1j}^{-1}, 1)$ if and only if the i-column and the j-column of the matrix P are the same.*

(2) *For a congruence η on S°, $(\omega_E, \eta, \omega_F)$ is a congruence triple if and only if for all $i \in I$, $\lambda \in \Lambda$, the products $(1, p_{\lambda 1}^{-1}, \lambda)(i, p_{1i}^{-1}, 1)$ are in the same class of the congruence η.*

Proof. (1) From Result 3 and the hypothesis, we have

$$\begin{aligned}(i, p_{1i}^{-1}, 1)\kappa_E(\varepsilon)(j, p_{1j}^{-1}, 1) &\iff (\forall \lambda \in \Lambda)(1, p_{\lambda 1}^{-1}, \lambda)(i, p_{1i}^{-1}, 1) = (1, p_{\lambda 1}^{-1}, \lambda)(j, p_{1j}^{-1}, 1)\\ &\iff (\forall \lambda \in \Lambda) p_{\lambda 1}^{-1} p_{\lambda i} p_{1i}^{-1} = p_{\lambda 1}^{-1} p_{\lambda j} p_{1j}^{-1}\\ &\iff (\forall \lambda \in \Lambda) p_{\lambda i} = p_{\lambda j}\,.\end{aligned}$$

(2) If $(\omega_E, \eta, \omega_F)$ is a congruence triple, let $\rho = \rho_{(\omega_E, \eta, \omega_F)}$, we have $(1, p_{\lambda 1}^{-1}, \lambda)\,\rho(1, p_{\mu 1}^{-1}, \mu)$ and $(i, p_{1i}^{-1}, 1)\rho(j, p_{1j}^{-1}, 1)$ for any $i, j \in I$ and $\lambda, \mu \in \Lambda$. So, $(1, p_{\lambda 1}^{-1}, \lambda)\,(i, p_{1i}^{-1}, 1)\rho(1, p_{\mu 1}^{-1}, \mu)(j, p_{1j}^{-1}, 1)$ as required. If we assume conversely, let $(i, p_{1i}^{-1}, 1)$, $(j, p_{1j}^{-1}, 1) \in E$. Then for any $(1, p_{\lambda 1}^{-1}, \lambda) \in F$, $(1, p_{\lambda 1}^{-1}, \lambda)(i, p_{1i}^{-1}, 1)\eta(1, p_{\lambda 1}^{-1}, \lambda)(j, p_{1j}^{-1}, 1)$, and the dual implication is also true. Thus $(\omega_E, \eta, \omega_F)$ satisfies the condition (T2) for it to be a congruence triple. ∎

Example 2. Let $G = \mathbf{Z}_8$. Set $I = \Lambda = \{1, 2, 3\}$, and

$$P = \begin{pmatrix} \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{4} & \bar{0} \\ \bar{0} & \bar{0} & \bar{4} \end{pmatrix}.$$

Let $S = \mathcal{M}(I, G, \Lambda; P)$, then S is a completely simple semigroup with a Q-inverse transversal $S^\circ = \{(1, \bar{k}, 1) \mid \bar{k} \in G\}$. For any $(i, p_{1i}^{-1}, 1) \in E$ and $(1, p_{\lambda 1}^{-1}, \lambda) \in F$, we have

$$(1, p_{\lambda 1}^{-1}, \lambda)(i, p_{1i}^{-1}, 1) = \begin{cases} (1, \bar{4}, 1), & \text{if } \lambda = i = 2 \quad \text{or} \quad \lambda = i = 3, \\ (1, \bar{0}, 1), & \text{if otherwise.} \end{cases}$$

From Result 3, we have

$$\sigma = \rho_{(\omega_E, \omega_{S^\circ t}, \omega_F)}, \qquad \tau = \rho_{(\kappa_E(\varepsilon), \varepsilon_{S^\circ}, \kappa_F(\varepsilon))}.$$

The non-trivial congruences on S° are η and ζ, where the η-classes are $\{(1, \bar{0}, 1), (1, \bar{2}, 1), (1, \bar{4}, 1), (1, \bar{6}, 1)\}$ and $\{(1, \bar{1}, 1), (1, \bar{3}, 1), (1, \bar{5}, 1), (1, \bar{7}, 1)\}$, and the ζ-classes are $\{(1, \bar{0}, 1), (1, \bar{4}, 1)\}$, $\{(1, \bar{2}, 1), (1, \bar{6}, 1)\}$, $\{(1, \bar{1}, 1), (1, \bar{5}, 1)\}$, $\{(1, \bar{3}, 1), (1, \bar{7}, 1)\}$. Then by Lemma 4.6. (2), $(\omega_E, \zeta, \omega_F)$ is a congruence triple; while $(\omega_E, \varepsilon_{S^\circ}, \omega_F)$ is not a congruence triple. Thus $\sigma = \rho_{(\omega_E, \zeta, \omega_F)}$. It is clear from Lemma 4.6. (1) and its dual that $\tau = \varepsilon$, so S is an E-disjunctive completely simple semigroup. We have $\sigma \neq \tau$, $\sigma \neq \omega$ and consequently $\pi \neq \beta$. Let τ_E be a congruence on E with only one non-singleton class $\{(1, p_{11}^{-1}, 1), (2, p_{12}^{-1}, 1)\}$, and τ_F be a congruence on F with only one non-singleton class $\{(1, p_{21}^{-1}, 2), (1, p_{31}^{-1}, 3)\}$. It is clear that (τ_E, ζ, τ_F) is a congruence triple. We have

$$\rho_{(\tau_E, \zeta, \tau_F)} t = \rho_{(\varepsilon_E, \zeta, \varepsilon_F)}, \qquad \rho_{(\tau_E, \zeta, \tau_F)} tK = \rho_{(\omega_E, \zeta, \omega_F)}.$$

So we have $t \neq tK$, and equivalently $T \neq tKT$ by Lemma 3.4. (1) and (3).

We have the following description of the TK-operator semigroup of E-disjunctive completely simple semigroups.

Theorem 4.7. *The TK-operator semigroup of the class of E-disjunctive completely simple semigroups is*

$$\begin{aligned} (\Gamma^+/\Sigma^*)/\Theta_2 &\cong \Gamma^+/(\Sigma \cup M_2)^* \\ &= \{K,\ KT,\ Kt,\ Ktk,\ k, \\ &\qquad t,\ tk,\ tK,\ tKT,\ T\} \cup \{\varepsilon, \sigma, \beta, \sigma \wedge \beta, \omega\}, \end{aligned}$$

where the last set is endowed with the multiplication being subject to the relations Σ and M_2.

Proof. It may be proved as in Theorem 3.5 that $\Gamma^+/(\Sigma \cup M_2)^*$ is just the above set with the multiplication being subject to the relations Σ and M_2. It can be verified easily that $(\Gamma^+/\Sigma^*)/\Theta_2 \cong \Gamma^+/(\Sigma \cup M_2)^*$. By the discussion after Lemma 4.4, for any E-disjunctive completely simple semigroup Q, any element in $\Gamma(Q)$ falls into the above set. For the completely simple semigroup S in Example 2, $\sigma \neq \tau$, $\sigma \neq \omega$,

$\pi \neq \beta$. We also have $\sigma \neq \pi$ and $\omega \neq \beta$, for otherwise they would imply $\tau = \varepsilon$. So $\varepsilon, \sigma, \beta, \sigma \wedge \beta, \omega$ are different. For any operator except $\varepsilon, \sigma, \beta, \sigma \wedge \beta, \omega$, we may find at least two congruences in $\{\varepsilon, \sigma, \beta, \sigma\wedge\beta, \omega\}$ such that the actions of this chosen operator on these two congruences are different. Hence any element except $\varepsilon, \sigma, \beta, \sigma \wedge \beta, \omega$ is not constant operator. For any pair of non-constant operators except the pairs (t, tK) and (T, tKT), we can find a congruence in $\{\varepsilon, \sigma, \beta, \sigma \wedge \beta, \omega\}$ such that the actions of these two operators on this congruence are different. By Example 2, $t \neq tK$ and $T \neq tKT$ on $\mathcal{C}(S)$. Hence as elements of $\Gamma(S)$, all the operators presented above are different. Therefore

$$\Gamma(S) = \{\, K,\ KT,\ Kt,\ Ktk,\ k,\ t,\ tk,\ tK,\ tKT,\ T\} \ \cup\ \{\varepsilon,\ \sigma,\ \beta,\ \sigma \wedge \beta,\ \omega\}\,.$$

Consequently, $\Gamma(S)$ is the TK-operator semigroup of the class of E-disjunctive completely simple semigroups. ■

Finally, we discuss the group congruence-free completely simple semigroups. A regular semigroup is called group congruence-free if $\sigma = \omega$ or equivalently $\pi = \beta$. For a group congruence-free completely simple semigroup S, we have the following additional relations M_3 holding on $\mathcal{C}(S)$.

$$\sigma = \omega\,, \qquad \sigma \wedge \beta = \beta\,, \qquad Kt = KtK\,, \qquad KT = KtKT\,.$$

If we regard M_3 as a relation Θ_3 on Γ^+/Σ^*, we can prove that Θ_3 is a congruence. The following example plays the same role in our discussion as the preceding examples.

Example 3. Let $G = \mathbf{Z}_8$. Set $I = \Lambda = \{1,2,3\}$, and

$$P = \begin{pmatrix} \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{2} & \bar{3} \\ \bar{1} & \bar{2} & \bar{3} \end{pmatrix}$$

Let $S = \mathcal{M}(I, G, \Lambda; P)$. Then S is a completely simple semigroup with a Q-inverse transversal $S^\circ = \{(1, \bar{k}, 1) \mid \bar{k} \in G\}$. For any $(i, p_{1i}^{-1}, 1) \in E$ and $(1, p_{\lambda 1}^{-1}, \lambda) \in F$, we have

$$(1, p_{\lambda 1}^{-1}, \lambda)(i, p_{1i}^{-1}, 1) \;=\; \begin{cases} (1, \bar{0}, 1), & \text{if} \quad i = 1 \quad \text{or} \quad \lambda = 1\,, \\ (1, \bar{1}, 1), & \text{if} \quad (\lambda, i) = (2,2) \quad \text{or} \quad (3,2)\,, \\ (1, \bar{2}, 1), & \text{if} \quad (\lambda, i) = (2,3) \quad \text{or} \quad (3,3)\,. \end{cases}$$

Let η and ζ be the congruences on S° given in Example 2. By Lemma 4.6. (2), $\sigma = \rho_{(\omega_E, \omega_{S^\circ}, \omega_F)} = \omega$. So $\pi = \beta$. Hence S is group congruence free. By Lemma 4.6. (1), $\kappa_E(\varepsilon) = \varepsilon_E$, and some calculation shows that $\kappa_F(\varepsilon)$ has only one non-singleton class $\{(1, p_{21}^{-1}, 2), (1, p_{31}^{-1}, 3)\}$. We have $\tau = \rho_{(\varepsilon_E, \varepsilon_{S^\circ}, \kappa_F(\varepsilon))}$. It is clear that $\tau \neq \varepsilon$ and $\tau \neq \sigma$. From Lemmas 4.4 and 4.5, $\varepsilon, \beta, \tau, \tau \vee \beta, \omega$ are different from each other. Let τ_E be a congruence on E with only one non-singleton class $\{(1, p_{11}^{-1}, 1), (3, p_{13}^{-1}, 1)\}$, and τ_F be a congruence on F with only one non-singleton class $\{(1, p_{21}^{-1}, 2), (1, p_{31}^{-1}, 3)\}$. It is not difficult to see that (τ_E, η, τ_F) is a congruence triple for S. Clearly $\kappa_E(\eta) \neq \omega_E$, so $\kappa_E(\eta) = \tau_E$. Similarly $\kappa_F(\eta) = \tau_F$. Thus we have

$$\rho_{(\tau_E, \eta, \tau_F)} K = \rho_{(\kappa_E(\eta), \eta, \kappa_F(\eta))} = \rho_{(\tau_E, \eta, \tau_F)}\,.$$

Since $(\tau_E, \varepsilon_{S^\circ}, \tau_F)$ is not a congruence triple, while (τ_E, ζ, τ_F) is a congruence triple, we have

$$\rho_{(\tau_E,\eta,\tau_F)}Kt = \rho_{(\tau_E,\eta,\tau_F)}t = \rho_{(\tau_E,\zeta,\tau_F)}\,.$$

Thus $K \neq Kt$. As $Kt = KtK$, we have $K \neq KtK$, and equivalently $k \neq Ktk$.

As in preceding discussion, we obtain the following assertion for group congruence-free completely simple semigroups.

Theorem 4.8. *The TK-operator semigroup of the class of group congruence-free completely simple semigroups is*

$$\begin{aligned}(\Gamma^+/\Sigma^*)/\Theta_3 &\cong \Gamma^+/(\Sigma \cup M_3)^* \\ &= \{K,\ KT,\ Kt,\ Ktk,\ k,\\ &\qquad t,\ tk,\ tK,\ tKT,\ T\} \cup \{\varepsilon, \tau, \beta, \tau \vee \beta, \omega\},\end{aligned}$$

where the last set is endowed with the multiplication being subject to the relations Σ and M_3.

Proof. Similarly to the proof of the preceding theorems, we need to prove that for the completely simple semigroup S presented in Example 3, as the elements of $\Gamma(S)$, all the elements in the above set are different. We can prove as before that the elements not occurring in $\{\varepsilon, \tau, \beta, \tau \vee \beta, \omega\}$ are not constant operators. By checking the actions of the non-constant operators on $\{\varepsilon, \tau, \beta, \tau \vee \beta, \omega\}$, we claim that the actions of any pair of operators are different except the following three pairs, (K, Kt), (K, KtK) and (k, Ktk). As indicated in Example 3, the two operators in any of these three pairs are different. Hence all the operators presented above are different from each other for S. Therefore, $\Gamma(S)$ is the desired TK-operator semigroup. ∎

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