MASSIVE GAUGE FIELDS

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The main results of the quantum theory of vector gauge fields with nonvanishing mass are discussed. Aspects considered are the gauge-invariant formulation of the theory, the renormalization problem, and the relationship to the theory of massless gauge fields.

1. Introduction

The concept of massive gauge fields, which was first clearly formulated by Sakurai [1], plays an important role in elementary particle physics. It is employed in currently popular methods like the method of effective Lagrangians, field algebra, and the hypothesis of vector dominance. However, all these methods are semiphenomenological in nature, this being due to a considerable extent to the unsatisfactory state of the quantum theory of vector fields. This has led to the appearance in recent years of a large number of papers devoted to the quantum theory of massive gauge fields. These investigations have also been partly stimulated by the great successes achieved in the quantization of gauge fields with vanishing mass [2-4]. Although the most optimistic hopes placed in the theory of massive gauge fields have not yet been justified, it has been possible to achieve certain successes in the analysis of divergences and also in questions of the gauge-invariant formulation of the theory and the passage to the limit of vanishing mass. It is therefore an appropriate moment to summarize some of the initial successes.

In investigations devoted to massive gauge fields, different methods are employed but the results obtained overlap to a considerable extent. We shall dispense with a description of all the methods employed and attempt merely to give an exposition of the main results from a unified point of view. In the author's opinion, the majority of the results can be derived most simply by means of the method of functional integration, and we shall therefore use this method in general.

In the second section we discuss in detail the gauge-invariant formulation of the theory of a massive non-Abelian gauge field. In the third section we analyze ultraviolet divergences and in the fourth the question of the transition to vanishing mass. In the last section we discuss various possibilities for the future and possible applications of the theory.

2. Gauge-Invariant Formulation of the Theory of a Massive Vector Field

It is well known that the equations of the electromagnetic field interacting with charged fields $\Psi(x)$ are invariant under gauge transformations:

$$\Psi(x)\to e^{ig\varphi(x)}\Psi(x),\quad \overline{\Psi}(x)\to e^{-ig\varphi(x)}\overline{\Psi}(x),\quad A_\mu\to A_\mu+\frac{\partial\varphi}{\partial x^\mu}. \tag{1}$$

Yang and Mills [5] generalized the concept of gauge invariance to the group SU_2 (an arbitrary compact group was considered by Utiyama [6], Glashow and Gell-Mann [7], and Schwinger [8]). In brief, their argument is this. The conservation of isospin is due to the invariance of the Lagrangian under isotopic rotations with coordinate-independent parameters, i.e., under simultaneous transformations of the fields at all space—time points. However, in a local theory it is natural to require invariance under independent transformations of the field at different space—time points, since "fixing the phase of a proton at Berkeley, we are

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not obliged to fix it in the same manner at Dubna." The validity of this hypothesis is confirmed by the electromagnetic interaction, whose Lagrangian is invariant under the local gauge transformation (1).

The requirement that the Lagrangian of charged fields be invariant under the gauge transformations

$$\delta\Psi(x) = \lambda T^a \alpha^a(x)\Psi(x) \tag{2}$$

(here $\alpha^a(x)$ are infinitesimally small functions and the matrices T realize a representation of some compact group G) means that one must introduce a "compensating" vector field $B_\mu^a(x)$, the analog of the photon in the case of an Abelian gauge group, this transforming according to the law

$$\delta B_\mu^a(x) = \lambda t^{abc} B_\mu^b \alpha^c + \frac{\partial \alpha^a}{\partial x^\mu}, \tag{3}$$

where t^{abc} are the structure constants of the group:

$$[T^a, T^b] = t^{abc} T^c. \tag{4}$$

For what follows it is convenient to introduce matrix functions $B_\mu(x)$ with values in the Lie algebra of the group:

$$B_\mu(x) = t^a B_\mu^a(x), \tag{5}$$

where t^a are the group generators:

$$(t^a)^{bc} = t^{bac},\ [t^a, t^b] = t^{abc}t^c,\ \mathrm{Tr}(t^a t^b) = -2\delta^{ab}. \tag{6}$$

The finite transformation corresponding to the infinitesimal transformation (3) has the following form in terms of these matrices:

$$B_\mu \to \Omega B_\mu \Omega^{-1} - \frac{1}{\lambda}\Omega \partial_\mu \Omega^{-1}, \tag{7}$$

where $\Omega(x)$ for each x is an element of G. In particular, in the exponential parametrization

$$\Omega(x) = \exp\{\lambda t^a \alpha^a(x)\}. \tag{8}$$

A Lagrangian of the charged fields that is invariant under the transformation (2) and (3) is constructed by replacing the usual derivative by the covariant derivative

$$D_\mu = (\partial_\mu - \lambda T^a B_\mu^a). \tag{9}$$

The invariant Lagrangian of the "free" Yang–Mills fields can also, as in electrodynamics, be expressed in terms of field strengths:

$$F_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + \lambda[B_\mu, B_\nu], \tag{10}$$

which transform in accordance with the law

$$F_{\mu\nu} \to \Omega F_{\mu\nu} \Omega^{-1}, \tag{11}$$

and have the form

$$\mathscr{L} = +\frac{1}{8}\mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu}. \tag{12}$$

In contrast to electrodynamics, the "free" Lagrangian (12) leads to a nontrivial S matrix and contains all the characteristic features of the theory. As a rule, we shall therefore consider a self-interacting gauge field and omit the terms that describe the interaction with the other fields. This interaction can be taken into account in a trivial manner. For the sake of brevity, we shall also restrict the treatment to the gauge field that is in practice the most important – the Yang–Mills field. In this case, the group G is the isotopic group and the structure constants t^{abc} form the completely antisymmetric tensor ε^{abc}. All the arguments can be directly generalized to an arbitrary compact group.

Sakurai [1] suggested that the requirement of local gauge invariance should be regarded as a fundamental physical principle, in accordance with which each conservation law must correspond to a gauge or "compensating" field, the corresponding interaction being described by a Lagrangian that is invariant under transformations of the type (2) and (3). This principle establishes a natural and elegant relationship between the symmetry properties and the dynamics.

A serious shortcoming of Sakurai's hypothesis is the circumstance that the Lagrangian (12) describes a massless field, whereas all the known vector particles except the photon have a nonvanishing mass. In principle, one could hope to obtain a mass as a result of spontaneous symmetry violation. Such a possibility is undoubtedly attractive, but no real successes have yet been achieved in this direction. We shall therefore investigate another possibility, in which the gauge invariance is "deliberately" violated by the addition of a mass term to the Lagrangian (12). Such a theory retains much in common with the theory of a massless field. The mass term does not violate the universality of the interaction and the integral of the zeroth component of the current source of the vector field remains a generator of the corresponding group. It can be shown that a modification of the formalism enables one to restore the gauge invariance of the theory and thus reconcile Sakurai's hypothesis with the existence of vector mesons of nonvanishing mass. Although the gauge-invariant formulation is somewhat artificial in this case, it is helpful for the investigation of the relationship between the massive and massless theories and greatly simplifies the analysis of the divergences.

For an Abelian group, a gauge-invariant theory of a massive vector field was already constructed by Stueckelberg [9] (a different possibility has been discussed by Ogievetskii and Polubarinov [10]). In the Stueckelberg formalism, the gauge invariance is ensured by the introduction of an auxiliary scalar field transforming in accordance with the law

$$B \to B + m\Lambda. \tag{13}$$

The gauge-invariant Lagrangian has the form

$$\mathscr{L} = -\frac{1}{4} f_{\mu\nu} f_{\mu\nu} - \frac{m^2}{2}\left(A_\mu - \frac{1}{m}\frac{\partial B}{\partial x^\mu}\right)^2 + \bar{\Psi}[\gamma_\mu(\partial_\mu - eA_\mu) + M]\Psi. \tag{14}$$

The vector field is not described by a five-component quantity (A_μ, B), but, as in electrodynamics, the gauge invariance leads to a reduction in the number of degrees of freedom by two and therefore there remain three degrees of freedom.

A generalization of the Stueckelberg formalism to the case of non-Abelian groups was apparently first proposed by Kunimasa and Goto [11]. They noted that, without violating the gauge invariance, one could add to the Lagrangian (12) the mass term

$$+\frac{m^2}{4}\operatorname{Tr}(B_\mu - L_\mu)^2, \tag{15}$$

where the vector function L_μ has the form

$$\lambda L_\mu = \Omega(\varphi)\frac{\partial \Omega^{-1}(\varphi)}{\partial x^\mu}. \tag{16}$$

The orthogonal matrices $\Omega[\varphi(x)]$ for each x belong to the group of rotations, the scalar fields $\varphi(x)$ playing the role of local coordinates on the group. Under the transformations

$$\Omega(x) \to \tilde{\Omega}\Omega(x), \tag{17}$$

the L_μ transform in the same manner as the gauge fields,

$$L_\mu \to \tilde{\Omega} L_\mu \tilde{\Omega}^{-1} - \frac{1}{\lambda}\tilde{\Omega}\partial^\mu \tilde{\Omega}^{-1}, \tag{18}$$

and this ensures the desired gauge invariance. As in the Stueckelberg formalism for the case of an Abelian group, the vector field is described by a five-component quantity (B_μ, φ). However, in contrast to the Stueckelberg case, the scalar field φ enters the Lagrangian nonlinearly. Nevertheless, as we shall immediately show, the quantization of such a Lagrangian leads to the same S matrix as the usual, nongauge-invariant formalism.

The Lagrangian under consideration,

$$\mathscr{L} = \frac{1}{8}\operatorname{Tr}\{F_{\mu\nu}F_{\mu\nu} + 2m^2(B_\mu - L_\mu)^2\}, \tag{19}$$

is singular, since the equations

$$\frac{\partial \mathscr{L}}{\partial_0 B_\mu} = p_\mu \tag{20}$$

cannot be solved for $\partial_0 B_\mu$. The standard procedure of quantization cannot therefore be applied directly to the Lagrangian. Referring the reader who is interested in the general theory of quantization of singular Lagrangians to Faddeev's paper [12], we shall here content ourselves with the procedure for the Lagrangian (19).

We first of all set up the canonical formalism. At this stage it is convenient to return to the coordinate form of notation. The Lagrangian (19) can be written in the form

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu}{}^a F_{\mu\nu}{}^a - \frac{m^2}{2}(B_\mu{}^a B_\mu{}^a - 2B_\mu{}^a f_{am}\partial_\mu\varphi^m + \partial_\mu\varphi^m g_{mn}\partial_\mu\varphi^n), \tag{21}$$

where the matrices f_{am} and g_{mn} are defined by the equations

$$L_{a\mu} = f_{an}\partial_\mu\varphi^n; \quad g_{mn} = (f^T)_{mc} f_{cn}. \tag{22}$$

One can show (see, for example, [13]) that the matrix g_{mn} has the meaning of a two-sided invariant metric on the group of rotations.

The momenta conjugate to the fields B_μ^a and φ^n are

$$p_i{}^a = \frac{\partial\mathscr{L}}{\partial_0 B_i{}^a} = F_{i0}{}^a, \quad p_0{}^a = 0, \tag{23}$$

$$\pi_n = \frac{\partial\mathscr{L}}{\partial_0\varphi^n} = m^2 B_0{}^b f_{bn} - m^2\partial_0\varphi^m g_{mn}. \tag{24}$$

The action, expressed in terms of the variables B_μ^a, p_i^a, π_n, φ^n, has the form

$$S = \int\Big\{p_i{}^a\dot B_i{}^a + \pi_n\dot\varphi^n - \frac{1}{2}p_i{}^a p_i{}^a + (2m^2)^{-1}\pi_m g^{mn}\pi_n - \frac{1}{4}F_{mn}{}^a F_{mn}{}^a$$
$$+ \frac{m^2}{2}(B_m{}^a - L_m{}^a)^2 + B_0{}^a(\partial_i p_i{}^a + \lambda\varepsilon^{abc}p_i{}^b B_i{}^c - (f^T)^{an}\pi_n)\Big\}dx$$
$$(g_{km}g^{mn} = \delta_k{}^n, \ (f^T)_{an}(f^T)^{nb} = \delta_a{}^b). \tag{25}$$

It can be seen that, because of the vanishing of the conjugate momentum, B_0^a is not a canonical variable but plays the role of a Lagrangian multiplier. Varying the action (25) with respect to B_0^a, we obtain the constraint equation

$$\partial_i p_i{}^a + \lambda\varepsilon^{abc}p_i{}^b B_i{}^c - (f^T)^{an}\pi_r = 0. \tag{26}$$

To carry out the canonical quantization, it is first necessary, having solved the constraint equation, to express the action in terms of the independent canonical variables. As the latter, we take the three-dimensionally transverse components of the vectors p_i^a and B_i^a and the scalars π_n and φ^n; this corresponds to a decomposition of the vector field into components with unit and vanishing helicity. Substituting the decomposition

$$p_i{}^a = p_{i\perp}{}^a + \partial_i\xi^a \tag{27}$$

into the condition (26), we obtain an equation that enables us to express ξ^a in terms of the remaining canonical variables:

$$N^{ab}\xi^b \equiv \nabla\xi^a + \lambda\varepsilon^{abc}\partial_i\xi^b B_i{}^c = (f^T)^{ad}\pi_d - \lambda\varepsilon^{abc}p_{i\perp}{}^b B_i{}^c. \tag{28}$$

In this theory, only quantities that are invariant under gauge transformations are observable; it follows that in the determination of the fields B_μ and φ there is an ambiguity that can be eliminated by the imposition of an additional gauge-fixing condition. We shall use this freedom to eliminate the longitudinal part of the field B_i by imposing the condition

$$\partial_i B_i{}^a(x) = 0. \tag{29}$$

Thus, in terms of the independent canonical variables, the action can be written in the form

$$S = \int\Big\{p_{i\perp}{}^a\dot B_{i\perp}{}^a + \pi_a\dot\varphi^a - \frac{1}{2}p_{i\perp}{}^a p_{i\perp}{}^a - \partial_i\xi^a\partial_i\xi^a + (2m^2)^{-1}\pi_a g^{ab}\pi_b - \frac{1}{4}F_{mn}{}^a F_{mn}{}^a + \frac{m^2}{2}(B_{i\perp}{}^a - L_i{}^a)^2\Big\}dx, \tag{30}$$

where ξ^a is a solution of Eq. (28).

For the quantization, we can now use the method of functional integration. In this method, of course, the transition amplitude between the states $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$ is given by the Feynman functional integral

$$\langle \mathrm{out}|\mathrm{in}\rangle = \int \exp\left\{i\int [p\dot{q} - H]\,dt\right\} dp\,dq, \tag{31}$$

where the paths q(t) as $t \to \pm\infty$ are determined by the states $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$. In our case

$$\langle \mathrm{out}|\mathrm{in}\rangle = \int \exp\left\{i\int S(p, B, \pi, \varphi)\,dx)\right\}\prod_x \delta(\partial_i B_i)\,\delta(\partial_i p_i)\,dB_i\,dp_i\,d\varphi\,d\pi. \tag{32}$$

This integral can be rewritten in the form

$$\begin{aligned}\langle \mathrm{out}|\mathrm{in}\rangle = \int \exp\Big\{ i\int\Big[& p_i^a\dot{B}_i^a + \pi_a\dot{\varphi}^a - \frac{1}{2}p_i^a p_i^a + (2m^2)^{-1}\pi_a g^{ab}\pi_b \\ & - \frac{1}{4}F_{mn}{}^a F_{mn}{}^a + \frac{m^2}{2}(B_i^a - L_i^a)^2 + B_0^a(\partial_i p_i^a + \lambda\varepsilon^{abc}p_i^b B_i^c \\ & - (f^T)^{ad}\pi_d)\Big]dx\Big\}\det N\prod_x \delta(\partial_i B_i)\,dB_i\,dB_0\,dp_i\,d\pi\,d\varphi,\end{aligned} \tag{33}$$

where the operator N is defined by Eq. (28); indeed, we can integrate over B_0 in Eq. (33) and obtain

$$\delta[\partial_i p_i^a + \lambda\varepsilon^{abc}p_i^b B_i^c - (f^T)^{ad}\pi_d].$$

The integration with respect to the measure Πdp_i can be replaced by integration with respect to the measure

$$\Pi\delta(\partial_i p_i)\,dp_i d\xi.$$

After this, the integral with respect to ξ is removed by a δ-function. The Jacobian that then arises cancels against det N, and we return to formula (32).

In Eq. (33), the integrals with respect to p_i and π are Gaussian and can be completely calculated. Making the calculations, we obtain the final expression for the S matrix in the Coulomb gauge:

$$\langle \mathrm{out}|\mathrm{in}\rangle = \int \exp\left\{i\int\left[\frac{\mathrm{Tr}}{8}F_{\mu\nu}F_{\mu\nu} + \frac{m^2}{4}\mathrm{Tr}(B_\mu - L_\mu)^2\right]dx\right\}\ \det N\sqrt{\det g}\prod_x \delta(\partial_i B_i)\,dB\,d\varphi. \tag{34}$$

A characteristic feature of this expression is the fact that the measure of integration is not equal to the product of the field differentials but contains the additional factors det N and $\sqrt{\det g}$. The first factor is due to the singularity of the Lagrangian and is also characteristic of gauge theories with vanishing mass. The factor $\sqrt{\det g}$ is due to the presence of derivatives in the interaction Lagrangian of the field φ. Similar factors appear in all theories in which the interaction Lagrangian contains at least two derivatives; in particular, in nonlinear chiral theories [13]. One can show that the expression $\sqrt{\det g}\,d\varphi$ is an invariant measure on the group of rotations.

The expansion of the integral (34) in a perturbation theory series generates a noncovariant diagrammatic technique and the propagation function of the fields B_i has the form

$$\overline{B_i B_j} \sim (\delta^{ij} - k^i k^j k^{-2})\,(k^2 - m^2)^{-1}.$$

The expansion in a perturbation theory series of the factors det N and $\sqrt{\det g}$ generates additional vertices whose explicit form we shall discuss below.

A shortcoming of formula (34) is the absence of explicit relativistic invariance. To obtain a relativistically invariant diagrammatic technique, we shall employ the following device proposed in [2]. We introduce a functional $\Delta(B)$ defined by the condition

$$\Delta(B)\int\prod_x \delta(\partial_\mu B_\mu^a)\,d\Omega = 1, \tag{35}$$

where $d\Omega$ is an invariant measure on the group of rotations. The functional $\Delta(B)$ is gauge invariant in the sense that $\Delta(B^\Omega) = \Delta(B)$. This follows directly from the invariance of the measure of integration in formula (35).

We multiply the integral (34) by the constant factor (35) and make a group change of variables:

$$B_\mu \to \tilde{\Omega} B_\mu \tilde{\Omega}^{-1} - \frac{1}{\lambda} \tilde{\Omega} \partial_\mu \tilde{\Omega}^{-1}, \quad \Omega \to \tilde{\Omega}\Omega. \tag{36}$$

We now note that the factor $\det N$ is the value on the surface $\partial_i B_i = 0$ of a gauge-invariant functional $\Delta^k(B)$ defined by an equation analogous to (36):

$$\Delta^k(B) \int \prod_x \delta(\partial_i B_i^{\Omega}) d\Omega = 1. \tag{37}$$

Indeed, since only the neighborhood of the identity element makes a contribution to the integral (37) on the surface $\partial_i B_i = 0$, the integral can be written in the form

$$\int \prod_x \delta(\nabla u + \lambda [\partial_i u, B_i]) du. \tag{38}$$

This integral can readily be calculated and is equal to $(\det N)^{-1}$. Consequently,

$$\det N \delta(\partial_i B_i) = \Delta^k(B) \delta(\partial_i B_i). \tag{39}$$

Under the group change of variables (36), the Lagrangian and the factor $\Delta^k(B)$ do not change and the integral (34) transforms as follows:

$$\langle \text{out} | \text{in} \rangle = \int \exp \{ i \mathscr{L}(x) dx \} \Delta^k(B) \Delta(B) \prod_x \delta(\partial_\mu B_\mu) \delta(\partial_i B_i^{\tilde{\Omega}}) dB_i \, d\Omega \, d\tilde{\Omega}. \tag{40}$$

The integral with respect to $\tilde{\Omega}$ is removed by a δ-function and, with allowance for Eq. (37), we obtain an expression for the S matrix in the transverse gauge:

$$\langle \text{out} | \text{in} \rangle = \int \exp \left\{ i \operatorname{Tr} \int \left[\frac{1}{8} F_{\mu\nu} F_{\mu\nu} + \frac{m^2}{4} (B_\mu - L_\mu)^2 \right] dx \right\} \Delta(B) \prod_x \delta(\partial_\mu B_\mu) d\Omega \, dB. \tag{41}$$

At the same time it is convenient to take the independent dynamical variables that determine the asymptotic states as the four-dimensionally transverse components of the field B_μ. Then the field φ does not participate at all in the asymptotic states and one can integrate explicitly with respect to this field. The expression (41) for the S matrix of the massive Yang–Mills field was obtained in [14, 15] (see also [16]).

It is not difficult to prove the equivalence of this expression to the ordinary expression for the S matrix obtained from the noninvariant Lagrangian

$$\mathscr{L} = \operatorname{Tr} \left\{ \frac{1}{8} F_{\mu\nu} F_{\mu\nu} + \frac{m^2}{4} B_\mu B_\mu \right\}. \tag{42}$$

For, the dependence on φ in the argument of the exponential function can be annihilated by the group change of variables

$$B_\mu \to \Omega B_\mu \Omega^{-1} - \frac{1}{\lambda} \Omega \partial_\mu \Omega^{-1}. \tag{43}$$

Under this transformation $\delta(\partial_\mu B_\mu)$ goes over into $\delta(\partial_\mu B_\mu^{\Omega})$. The integration with respect to Ω is removed by a δ-function and we finally arrive at the standard expression for the S matrix of the massive Yang–Mills field:

$$\langle \text{out} | \text{in} \rangle = \int \exp \left\{ i \operatorname{Tr} \int \left[\frac{1}{8} F_{\mu\nu} F_{\mu\nu} + \frac{m^2}{4} B_\mu B_\mu \right] dx \right\} \prod_x dB. \tag{44}$$

We emphasize that we are here concerned only with equality of the matrix elements on the mass shell. The Green's functions corresponding to (41) and (44) are, of course, different.

Let us now analyze the diagrammatic technique generated by the expansion of the functional (41) in a perturbation theory series.* Because of the presence under the integral of $\delta(\partial_\mu B_\mu)$ in free Green's

*The Feynman rules for the massive Yang–Mills field are described in detail in the appendix.

functions of the fields B_μ

$$D_{\mu\nu}{}^{c} \sim \frac{g^{\mu\nu} - k^{-2}k^{\mu}k^{\nu}}{k^2 - m^2}. \tag{45}$$

The expansion of the functions L_μ in a perturbation theory series generates an infinite number of vertices describing the interaction of the field φ with the field B_μ and with itself. Their explicit form depends on the choice of the parametrization of the matrices Ω. The Green's functions of the scalar fields φ have the usual form

$$D^{cab} \sim \delta^{ab}\frac{1}{k^2}. \tag{46}$$

Apart from the vertices already mentioned, the perturbation series also contains additional vertices due to the presence of the factors $\Delta(B)$ and $\sqrt{\det g}$.

The explicit form of the functional $\Delta(B)$ can be obtained by the following considerations [2]. Under natural boundary conditions, the equation $\partial_\mu B_\mu^\Omega = 0$, where $\partial_\mu B_\mu = 0$, has a unique solution $\Omega(x) = 1$. Therefore, only the neighborhood of the identity element in fact makes a contribution to the integral (35):

$$\Delta^{-1}(B) = \int \prod_x \delta(\Box\varphi + \lambda[\partial_\mu\varphi, B_\mu])\, d\varphi. \tag{47}$$

It follows from this that the functional $\Delta(B)$ is equal to the determinant of the operator M:

$$M\varphi \equiv \Box\varphi + \lambda[\partial_\mu\varphi, B_\mu]. \tag{48}$$

Separating out the trivial constant factor $\det\Box$, we can write the determinant of M in the form

$$\mathrm{const}\,\Delta(B) = \mathrm{const}\det M = \exp\{\mathrm{Tr}\ln(1 - \lambda[B_\mu, \partial_\mu\Box^{-1}])\}$$

$$= \exp\left\{-\sum_{n=2}^{\infty}\frac{\lambda^n}{n}\int dx_1 \ldots dx_n\, \mathrm{Tr}[B_{\mu_1}(x_1)\ldots B_{\mu_n}(x_n)]\, \partial_{\mu_1}D^c(x_1 - x_2)\ldots \partial_{\mu_n}D^c(x_n - x_1)\right\}. \tag{49}$$

From the point of view of the diagrammatic technique, $\Delta(B)$ is a sum of closed cycles with respect to which a scalar fermion of vanishing mass propagates.

The quantity $\sqrt{\det g}$ can also be represented in the form of an addition to the action. The functions $g_{ab}(x)$ are the diagonal matrix elements of the "matrix" g in the x spaces:

$$\langle x, a|g|x', b\rangle = \delta(x - x')g_{ab}(x). \tag{50}$$

Therefore,

$$\sqrt{\det g} = \exp\left\{\frac{1}{2}\mathrm{Tr}\ln g\right\} = \exp\left\{\frac{1}{2}\delta(0)\int dx\, \mathrm{Tr}\ln g(x)\right\}. \tag{51}$$

Thus, an additional term $\sim\delta(0)$ appears in the effective Lagrangian. This term is important to ensure invariance of the theory. Essentially, it is a counter term that compensates the noninvariant divergences that arise as a result of the pairings

$$\partial_\mu\overbrace{\varphi^a g_{ab}(x)\partial_\mu\varphi^b}(x). \tag{52}$$

[of course, one must introduce an intermediate regularization to give a meaning to the expressions (51) and (52)].

The described diagrammatic technique has an explicitly relativistically invariant form and does not lead to the appearance of terms that depend on the normals. As is well known, in the usual approach one has the problem of the so-called normal terms [17-19]. If the interaction Lagrangian contains derivatives or vector fields, the interaction Hamiltonian in the Dirac picture is not equal to the Lagrangian with the opposite sign but contains additional noncovariant contact terms. At the same time, the Green's function of the vector field contains the noncovariant term

$$(D_{\mu\nu}{}^{c})_D = \langle T_D B_\mu(x) B_\nu(y)\rangle = \frac{1}{(2\pi)^4}\int e^{ik(x-y)}\left\{\frac{g^{\mu\nu} - k^\mu k^\nu m^{-2}}{k^2 - m^2} + \frac{\delta_{\mu 0}\delta_{\nu 0}}{m^2}\right\} d^4k. \tag{53}$$

The subscript D means that we have the Dyson T product, i.e., the product is defined by multiplication by θ-functions. By some very cumbersome and, in our view, insufficiently convincing calculations a number of authors [17, 18, 20] have succeeded in showing that the additional contact terms in the Hamiltonian are compensated by noncovariant additions to the Green's function and that the following equation holds:

$$T_D\left(\exp\left\{i\int H_{\text{int}}(x)\,dx\right\}\right)=T_w\left(\exp\left\{i\int \mathscr{L}_{\text{int}}(x)\,dx\right\}\right), \tag{54}$$

where T_W means that the T product is defined by the Wick theorem with the convolutions (53) but without the noncovariant terms. The appearance of normal-dependent terms can also be avoided by choosing an appropriate regularization (see, for example, [21]). Thus, if one performs a regularization of the Pauli–Villars type,

$$B_\mu^R = B_\mu + \frac{1}{m}\partial_\mu\varphi, \tag{55}$$

where the field φ corresponds to an indefinite metric in the state space, the difference between T_W and T_D vanishes. It is readily shown that

$$\langle T_w B_\mu^R(x) B_\nu^R(y)\rangle = \langle T_D B_\mu^R(x) B_\nu^R(y)\rangle. \tag{56}$$

Accordingly, the difference between the Hamiltonian and the interaction Lagrangian disappears and Eq. (54) is then satisfied identically. However, if the action is defined in this manner one faces the problem of the uniqueness of the results, which depends explicitly on the chosen regularization.

The approach we have described enables us to avoid completely the problem of normal terms, since, proceeding from the canonical and, consequently, unitary form for the S matrix, we have obtained the explicitly covariant expression (41).

To conclude this section, we shall show that in the theory of the massive Yang–Mills field, as in electrodynamics, there is invariance under "operator" gauge transformations, i.e., the S matrix (41) does not depend on the choice of the longitudinal part of the Green's function. A similar result for the Yang–Mills field with vanishing mass is obtained in [3, 12, 22, 23]. Here we shall use the method proposed in [23].

We introduce the functional $\Delta^c(B)$ defined by the equation

$$\Delta^c(B)\int\prod_x \delta[\partial_\mu B_\mu^a(x) - c(x)]\,d\Omega = 1. \tag{57}$$

In the same way as in the transition from the Coulomb to the Lorentz gauge, we multiply the integral (41) by the constant factor (57) and make the group change of variables (36). We then obtain

$$\langle \text{out}|\text{in}\rangle = \int \exp\left\{i\int \mathscr{L}(x)\,dx\right\}\Delta^c(B)\prod_x \delta[\partial_\mu B_\mu - c(x)]\,dB\,d\Omega. \tag{58}$$

The functional $\Delta^c(B)$ on the surface $\partial_\mu B_\mu = c(x)$ is calculated in exactly the same way as the functional $\Delta(B)$ on the surface $\partial_\mu B_\mu = 0$. It is easy to see that

$$\Delta^c(B)\,|_{\partial_\mu B_\mu = c} = \Delta(B)\,|_{\partial_\mu B_\mu = 0} = \det M. \tag{59}$$

Therefore, multiplying (58) by $\exp\{i\int c^2(x)dx(2\alpha)^{-1}\}$, where α is an arbitrary constant, and integrating the result with respect to c(x), we obtain

$$\langle \text{out}|\text{in}\rangle = \int \exp\left\{i\int\left[\mathscr{L}(x) + \frac{1}{2\alpha}(\partial_\mu B_\mu^a)^2\right]dx\right\}\det M\prod_x dB\,d\Omega_0. \tag{60}$$

The gauge of the Green's function is fixed by the choice of the constant α. In particular, $\alpha = 0$ corresponds to the transverse gauge; $\alpha = 1$ to the Feynman gauge. Since α in Eq. (60) is any number, it follows that the S matrix is independent of the choice of the longitudinal part of the Green's function. At this point we end our discussion of the gauge invariance of the Yang–Mills field with nonvanishing mass and turn to the analysis of divergences.

3. Analysis of Divergences

In the standard formulation of the theory of the massive Yang–Mills field, one proceeds from the gauge-noninvariant Lagrangian (42). The propagator corresponding to this Lagrangian has the form

$$D_{\mu\nu}{}^{c} \sim \frac{g^{\mu\nu} - k^{\mu}k^{\nu}m^{-2}}{k^2 - m^2} \tag{61}$$

and it tends to a constant as $k \to \infty$. As a result a formal calculation of the degrees of divergence leads to the conclusion that there is an infinite number of primitively diverging diagrams – the theory is unrenormalizable.

The divergence index of a diagram consisting of N_3 triple vertices, N_4 fourfold vertices, and L external lines is

$$2N_3 + 4N_4 - 2L + 4. \tag{62}$$

Consequently, any diagram that contains at least one closed loop diverges. However, this estimate is too high and, because of the symmetry of the theory, the real divergence degree is reduced. We recall that in the case of an Abelian gauge group the longitudinal part of the propagator (61) makes no contribution at all to the elements of the S matrix. As a result, the interaction of a neutral vector field with a conserved current can be renormalized. The Yang–Mills field is a direct generalization of a neutral vector theory and one could expect that a similar effect obtains in this case. However, a direct calculation of the lowest orders of perturbation theory shows [24] that in the Yang–Mills theory the longitudinal part of the propagator gives a nonvanishing contribution. Attempts have been made [25, 26] to generalize this analysis to diagrams of arbitrary order. However, the generalization is incorrect [in particular, it does not take into account diagrams generated by the expansion of $\Delta(B)$]. In this connection, the question has recently been reconsidered in [27, 28].

The reason for the difference in the behavior of Abelian and non-Abelian gauge theories can be explained as follows. In the case of an Abelian field the n-th term in the expansion of the S matrix can be represented in the form

$$S_n = \frac{i^n}{n!} \int T[j_{\mu_1}(x_1) \ldots j_{\mu_n}(x_n)]\, T[U_{\mu_1}(x_1) + \partial_{\mu_1}\varphi(x_1) \ldots U_{\mu_n}(x_n) + \partial_{\mu_n}\varphi(x_n)]\, dx. \tag{63}$$

Here we have split the vector field into longitudinal and transverse (in the four-dimensional sense) components. The convolutions of the fields U_μ are determined by formula (45). The contribution of the longitudinal can be made to vanish by integrating by parts. We then obtain: first, terms containing $\partial_\mu j_\mu(x)$; secondly, equal-time current commutators that arise because of the differentiation of the θ-functions. The terms of the first type vanish because of the current conservation; the terms of the second type vanish since the gauge group is Abelian. Neither of these arguments is applicable to the Yang–Mills field. Since the current source of the vector field itself depends on this field in the case under consideration, the separation out of the longitudinal part violates the current conservation. Secondly, the gauge group is non-Abelian and the equal-time commutators make a nonvanishing contribution. Nevertheless, the symmetry of the theory results in partial compensation of the terms that arise as a result of integration by parts of formulas of the type (63).

The perturbation theory series is analyzed in detail in [27, 28]. It is shown that diagrams containing not more than one closed loop can be described by the following effective S matrix:

$$S_{\text{эф}} = T \exp\Big\{ i \int \Big[-\frac{\lambda}{2} \varepsilon^{abc} f_{\mu\nu}{}^{a} U_\mu{}^{b} U_\nu{}^{c} - \frac{\lambda^2}{4} \varepsilon^{abc}\varepsilon^{ade} U_\mu{}^{b} U_\nu{}^{c} U_\mu{}^{d} U_\nu{}^{e} \Big] dx$$

$$- \frac{1}{2} \int \sum_{n=2}^{\infty} \frac{\lambda^n}{n} \varepsilon^{abc}\varepsilon^{cde} \ldots \varepsilon^{uva} U_{\mu_1}{}^{b}(x_1) \ldots U_{\mu_n}{}^{v}(x_n)\, \partial_{\mu_1} D^c(x_1 - x_2) \ldots \partial_{\mu_n} D^c(x_n - x_1)\, dx_1 \ldots dx_n \Big\}. \tag{64}$$

Here $D^c(x)$ is the Feynman Green's function of the d'Alembert operator. Since the convolutions of the fields $U_\mu(x)$ defined by Eq. (45) decrease for large k as k^{-2}, the S matrix (64) corresponds to a renormalizable theory. In contrast to the formal calculation, the single-loop diagrams do not generate any new divergences apart from the usual renormalizations of the mass, charge, and wave function.

Equation (64) can readily be obtained by means of the representation (41) we have found for the S matrix. To this end we use any specific parametrization of the matrices Ω. For example, in the Weinberg parametrization [33]

$$L_\mu = \frac{1}{(1+\lambda\varphi)}\partial_\mu\varphi\frac{1}{(1-\lambda\varphi)}. \tag{65}$$

The effective Lagrangian that describes the interaction of the scalar field φ with the field B_μ has the form

$$\tilde{\mathscr{L}} = \frac{m^2}{2}\mathrm{Tr}\left\{-B_\nu\frac{1}{(1+\lambda\varphi)}\partial_\nu\varphi\frac{1}{(1-\lambda\varphi)}+\frac{1}{2}\frac{\partial_\mu\varphi\partial_\mu\varphi}{(1+\lambda^2\varphi)}\right\}. \tag{66}$$

We substitute this expression into Eq. (41) for the S matrix and restrict ourselves to a few of the first terms of the expansion of $\tilde{\mathscr{L}}$ with respect to λ. Then the integral with respect to $d\Omega$ takes the form

$$\int \exp\left\{i\int \mathrm{Tr}\frac{m^2}{4}\left[\frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\mu}+\lambda[\varphi, B_\mu]\partial_\mu\varphi+O(\lambda^2\varphi^3)\right]dx\right\}\prod_x d\varphi. \tag{67}$$

The terms that have not been written down explicitly describe vertices from which more than two scalar lines emanate. Such vertices make a contribution to only those diagrams that contain at least two closed loops. Therefore, in the single-loop approximation, they can be omitted and the integral (67) can then be calculated to the end. In this approximation (67) can be written in the form of a Gaussian integral:

$$\int \exp\left\{c\int \varphi(x)M\varphi(x)\,dx\right\}\prod_x d\varphi,$$

where the operator M was defined by formula (48). To within a constant factor, this integral is equal to $(\det M)^{-1/2}$. As we have already seen, the functional $\Delta(B)$ is equal to the determinant of the same operator M. Thus, in the adopted approximation, $(\det M)^{-1/2}$ cancels partially with $\Delta(B)$ and the S matrix takes the form

$$\langle \mathrm{in}|\mathrm{out}\rangle = \int \exp\left\{i\int \mathrm{Tr}\left[\frac{1}{8}F_{\mu\nu}F_{\mu\nu}+\frac{m^2}{4}B_\mu B_\mu\right]dx+\frac{1}{2}\ln\Delta\right\}\prod_x \delta(\partial_\mu B_\mu)\,dB. \tag{68}$$

The expansion of $\ln\Delta$ with respect to the fields B_μ is determined by formula (49). Substituting this expansion into (68), we see that the latter is equal to (64).

The conclusion that the S matrix of the massive Yang–Mills field is renormalizable in the single-loop approximation is obviously unaffected by inclusion of an interaction with other fields. Since the complete Lagrangian is, as before, invariant under the gauge transformations (36) augmented by corresponding transformations of the fields Ψ, allowance for this interaction leads only to the appearance in (68) of additional terms of the form $j_\mu^a B_\mu^a$, that do not destroy the renormalizability.

If we do not restrict ourselves to the single-loop approximation, we must take into account the following terms in the expansion of (66). The expansion of the integrand in (66) generates an infinite number of vertices with an ever increasing number of scalar lines, which is characteristic of unrenormalizable theories. However, it could be that there is a further compensation between the diagrams generated by the expansion of the first and the second terms in (66), this leading to an additional suppression of singularities. The possibility of such compensation is due to the existence of the passage to the limit of a vanishing mass. It is shown in [27] that if there exists (to within logarithmic infrared divergences) a limit of the elements of the S matrix of the massive Yang–Mills field as $m \to 0$, this S matrix can be renormalized and the equivalent Feynman rules are determined in all orders by (64). The proof is based on the fact that the following terms of the expansion of the Lagrangian (66) in a series in the fields φ depend in a singular manner on the mass. For example, in the Weinberg parametrization*

$$\begin{aligned}\tilde{\mathscr{L}} = \frac{B_\mu{}^a}{(1+\lambda^2m^{-2}\varphi^2)^2}\Big[&(1-\lambda^2m^{-2}\varphi^2)\partial_\mu\varphi^a+2\lambda\varepsilon^{abc}\varphi^b\partial_\mu\varphi^c\\ &+2\lambda^2(1+\lambda^2m^{-2}\varphi^2)\varphi^a(\varphi^b\partial_\mu\varphi^b)+\frac{1}{2}\frac{(\partial_\mu\varphi^a\partial_\mu\varphi^a)}{(1+\lambda^2m^{-2}\varphi^2)^2}\Big].\end{aligned} \tag{69}$$

*In contrast to formula (66), we have here adopted a natural normalization of the fields φ, this corresponding to the substitution $\varphi \to m^{-1}\varphi$.

At the same time, all the remaining terms and also the corresponding propagators are not singular for m = 0. Therefore, for the existence of the limit of the S matrix for a mass that tends to zero it is necessary that the total contribution of the diagrams generated by the additional singular terms should vanish.

Since the theory of the massless Yang–Mills field exists and is renormalizable, it is possible that the desired limit exists. It is also known that the S matrix for the electromagnetic field can be obtained as the limit of the S matrix of a neutral massive vector field interacting with a conserved current. However, the situation is much more complicated for the Yang–Mills field. It is shown in [14, 15] that the matrix elements in the Yang–Mills theory are discontinuous as $m \to 0$. Referring a more detailed discussion of this question to the following section, we mention only that, because of the absence of a passage to the limit, the theory is, in the usual sense, unrenormalizable. To give a final answer to this question, it is necessary to formulate the procedure by an invariant regularization. We already encounter the need for such a regularization in considering the simplest two-loop diagrams shown in Figs. 1 and 2 (the wavy line stands for the propagation function of the vector particles; the dashed line for that of the scalar particles with vanishing mass). In [28, 30] the imaginary part of the diagram (1) is calculated. It has the polynomial growth in the momenta characteristic of unrenormalizable theories (this is obvious even from dimensional considerations since the diagram (1) contains the factor m^{-2}). In [30] this is taken as the basis for the deduction that the theory is unrenormalizable. However, this does not take into account the competing diagram (2), which could, in principle, compensate the contribution of the diagram (1). To see whether this is the case, it is necessary to calculate both diagrams using an invariant regularization; unfortunately, a procedure for doing this has not yet been formulated.

In addition, all calculations have hitherto been made with some specific parametrization of the matrices Ω. It is natural to ask whether the results obtained are due to the use of this specific parametrization. This possibility cannot in principle be excluded, and we shall therefore describe below an invariant, parametrization-independent method of calculating the integral with respect to $d\Omega$ [31]. This method may also be useful in an attempt to go beyond the framework of perturbation theory.

The integral we require has the form

$$I(B)=\int \exp\left\{i\int \operatorname{Tr}\frac{m^2}{4}[-2L_\mu B_\mu + L_\mu L_\mu]\,dx\right\}\prod_x d\Omega. \tag{70}$$

We introduce a generating functional $Z(\eta_\mu)$ defined by

$$Z(\eta_\mu)=\int \exp\left\{i\int\left[-\frac{m^2}{2}L_\mu{}^a L_\mu{}^a + m\eta_\mu{}^a L_\mu{}^a\right]dx\right\}\prod_x d\Omega. \tag{71}$$

Here I(B) is expressed in terms of $Z(\eta_\mu)$ as follows:

$$I(B)=\sum_{n=2}^{\infty}\frac{m^n}{n!}\int dx_1\ldots dx_n B_{\mu_1}^{a_1}(x_1)\ldots B_{\mu_n}^{a_n}(x_n)\left.\frac{\delta^n Z}{\delta\eta_{\mu_1}^{a_1}(x_1)\ldots\delta\eta_{\mu_n}^{a_n}(x_n)}\right|_{\eta=0}. \tag{72}$$

Thus, to calculate the integral, it is sufficient to find the functional Z. This functional satisfies the system of Ward identities

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial Z}{\partial\eta_\mu{}^a(x)}\right)-i\frac{\partial\eta_\mu{}^a}{\partial x^\mu}Z=-\frac{\lambda}{m}\varepsilon^{abc}\eta_\mu{}^b(x)\frac{\partial Z}{\partial\eta_\mu{}^c(x)}. \tag{73}$$

To obtain these identities, we make an infinitesimally small group shift $\delta\Omega = \xi\Omega$ in the integral (70). The variation of Z under this transformation is

$$\delta Z=\int\exp\left\{i\int\left[-\frac{m^2}{2}L_\mu{}^a L_\mu{}^a + mL_\mu{}^a\eta_\mu{}^a\right]dx\right\}\left\{m^2\int\left[-L_\mu{}^a\partial_\mu\xi^a+\frac{1}{m}\eta_\mu{}^a\partial_\mu\xi^a+\frac{\lambda}{m}\varepsilon^{abc}\xi^a\eta_\mu{}^b L_\mu{}^c\right]dy\right\}\prod_x d\Omega. \tag{74}$$

Equating the coefficient of the arbitrary function ξ to zero, we obtain (73).

Apart from the Ward identities (73), the variational derivatives of Z satisfy the relation

$$\partial_\mu\left(\frac{\partial Z}{\partial\eta_\nu{}^a(x)}\right)-\partial_\nu\left(\frac{\partial Z}{\partial\eta_\mu{}^a(x)}\right)=i\frac{\lambda}{m}\varepsilon^{abc}\frac{\partial^2 Z}{\partial\eta_\mu{}^b(x)\partial\eta_\nu{}^c(x)}, \tag{75}$$

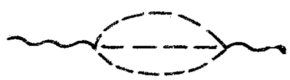


Fig. 1

Fig. 2

which follows directly from the definition of the vectors L_μ^a. This relation reflects the fact that the covariant curl of the vector L_μ^a vanishes. The relations (73) and (75) can be combined in the single equation

$$i\frac{\partial Z}{\partial\eta_\mu{}^a(x)}=\int\frac{\partial D_c(x-y)}{\partial x^\mu}\left\{\frac{\partial\eta_\nu{}^a}{\partial y^\nu}Z-i\frac{\lambda}{m}\varepsilon^{abc}\eta_\nu{}^b(y)\frac{\partial Z}{\partial\eta_\nu{}^c(y)}\right\}dy-\frac{\lambda}{m}\int\frac{\partial D_c(x-y)}{\partial y^\mu}\varepsilon^{abc}\frac{\partial^2 Z}{\partial\eta_\mu{}^b(y)\partial\eta_\nu{}^c(y)}dy. \tag{76}$$

Here $D^c(x)$ is the Feynman Green's function of the d'Alembert operator (as usual the choice of the rules for avoiding the pole is determined by the possibility of a transition to a Euclidean metric) [32, 33].

Equation (76) enables us to find $\partial Z/\partial\eta_\nu^a(x)$ in the form of a series in the coupling constant. The result is explicitly independent of the parametrization and automatically takes into account the fact that the integration is with respect to an invariant measure. It is easy to show that in the single-loop approximation the expression (68) for the S matrix again follows from (76). A formal calculation indicates the presence of divergences of higher degree in the more complicated diagrams. The question of their possible compensation has not yet been investigated since an adequate renormalization procedure has not been formulated for Eq. (76). However, since there is no limiting transition in the mass, such a possibility is improbable.

4. Passage to the Limit of a Vanishing Mass

The approximate S matrix (68) describing the Yang–Mills field in the single-loop approximation obviously admits a passage to the limit $m \to 0$. The limit expression has the form

$$S=\int\exp\left\{\int i\mathscr{L}^0(x)\,dx+\frac{1}{2}\ln\Delta\right\}\prod_x\delta(\partial_\mu B_\mu)\,dB, \tag{77}$$

where $\mathscr{L}^0$ is defined by Eq. (12). At the same time, if we set $m = 0$ directly in Eq. (41) we should arrive at the expression for the S matrix of the massless Yang–Mills field in the form obtained by Popov and Faddeev [2],

$$S=\int\exp\left\{\int i\mathscr{L}^0(x)\,dx+\ln\Delta\right\}\prod_x\delta(\partial_\mu B_\mu)\,dB. \tag{78}$$

It can be seen that even in the single-loop approximation Eqs. (77) and (78) lead to different results, since in the first case $\ln\Delta$ enters with the coefficient 1/2 but in the second with the coefficient 1.

At the first glance a paradoxical situation arises: the S matrix (77) is unitary, or at least to the fourth order for any arbitrarily small m (since all the scattering diagrams to fourth order inclusive are single-loop) but it does not coincide for $m = 0$ with the S matrix (78), which, as is shown in [2], is also unitary. This paradox is unaffected by the behavior of the omitted terms of higher order, since the unitarity condition must be satisfied in each order of perturbation theory.

However, this apparent contradiction can be resolved quite simply. Particles with vanishing and arbitrarily small mass have a different number of degrees of freedom – a particle with vanishing mass has two polarization states, whereas one with a finite mass has three. In the case of a neutral vector field, all the matrix elements that are nondiagonal in the number of longitudinal photons vanish as $m \to 0$. However, this is not so for the Yang–Mills field and in the limit we obtain an S matrix that, besides the transversely polarized "photons," also describes scalar particles of vanishing mass.

To explain what we have said, let us consider the matrix element of a transition from the transverse state $|l\rangle$ to a state $|a_3, n\rangle$ containing at least one longitudinal quantum:

$$\langle l|S|a_3{}^b,n\rangle=\int\left\langle l\left|\frac{\delta S}{\delta B_\mu{}^a(x)}\right|n\right\rangle[B_\mu{}^a(x),a_3{}^b(\mathbf{k})]\,d^4x. \tag{79}$$

Here $B_\mu(k)$ can be decomposed in the usual manner into components with definite polarizations:

$$B_i(\mathbf{k}) = e_1^i a_1 + e_2^i a_2 + \frac{k_i k_0}{|\mathbf{k}|m} a_3; \quad B_0(\mathbf{k}) = \frac{|\mathbf{k}|}{m} a_3. \tag{80}$$

The unit vectors e_1 and e_2 are orthogonal to each other and the vector k. The coefficients of the operators a_3 can be represented in the form

$$\begin{gathered} \frac{\mathbf{k}k_0}{|\mathbf{k}|m} = \frac{\mathbf{k}}{m} + \frac{\mathbf{k}m^2}{2|\mathbf{k}|^2 m} + O(m^3), \\ \frac{|\mathbf{k}|}{m} = \frac{k_0}{m} - \frac{m^2}{2k_0 m} + O(m^3). \end{gathered} \tag{81}$$

Using the representation, we rewrite the matrix element (79) as follows:

$$\langle l|S|a_3, n\rangle = \int d^4x \left\{ \left\langle l \left| \partial_\mu \left(\frac{\delta S}{\delta B_\mu(x)} \right) \right| n \right\rangle \frac{e^{ikx}}{2mk_0} + \int \left\langle l \left| \frac{\delta S}{\delta B_\mu(x)} \right| n \right\rangle e^{ikx} O_\mu(m)\, d^4x. \right. \tag{82}$$

The first term, containing m^{-1}, disappears because of the current conservation. If the state $|n\rangle$ in its turn contains longitudinal quanta, one can once again apply the procedure that leads to (82). Then the coefficients that are singular in m appear with expressions of the type

$$\partial_{\mu_i} \left(\frac{\delta^n S}{\delta B_{\mu_1}(x_1) \ldots \delta B_{\mu_n}(x_n)} \right). \tag{83}$$

In the neutral theory all these terms vanish by virtue of the Ward identities. However, in the Yang–Mills theory the corresponding Ward identities have a more complicated form. For example, the divergence of the vertex function is equal to

$$\partial_\rho \Gamma^{abc}_{\mu\nu\rho}(x, y, z) = \frac{\lambda}{m} \varepsilon^{abc} \{ G_{\mu\nu}(x-y)\delta(x-z) - G_{\mu\nu}(x-z)\delta(x-y) \}. \tag{84}$$

It can be seen that the divergence of the Green's functions for $n > 2$ does not vanish and the second term in Eq. (82) does not disappear. In particular, the probability of a transition into two longitudinal quanta is finite in the limit $m \to 0$. It is this that explains the difference between formulas (77) and (78). Since only the two-particle intermediate state makes a contribution to the unitarity condition in the single-loop approximation, one must, in order to obtain a unitary S matrix describing only transversely polarized quanta, subtract from the S matrix (77) the contribution of the two-particle longitudinal state; it is then exactly equal to the S matrix (78).

Let us illustrate this effect for the example of the diagrams of lowest order. In the unitarity condition written down for the case $m \neq 0$:

$$\langle l|S_2 + S_2^+|l\rangle = \sum_n \langle l|S_1^+|n\rangle\langle n|S_1|l\rangle \tag{85}$$

summation is understood over three polarizations on the right-hand side. A direct calculation readily shows that a state containing a single longitudinal quantum makes a vanishing contribution to this sum, whereas a state with two longitudinal quanta gives a contribution, nonvanishing in the limit $m \to 0$, equal to

$$\frac{\lambda}{2} \varepsilon^{ijk} \varepsilon^{ljk} \left\langle l \left| \int d^4x d^4y\, \partial_\mu D_0^-(x-y)\, \partial_\nu D_0^-(x-y) : b_\mu^i(x) b_\nu^l(y) : \right| l \right\rangle. \tag{86}$$

For the unitarity condition to remain valid in the limit $m \to 0$ when there is summation over only the transverse states it is necessary to subtract from S_2 an expression whose imaginary part exactly compensates the contribution of (86). It is readily seen that the expression (86) defines precisely the imaginary part (with the opposite sign) of order λ^2 in the expansion of $(1/2)\ln\Delta$. Consequently, elimination of the longitudinal states leads to the replacement of the coefficient $1/2$ of $\ln\Delta$ by 1 and we finally arrive at the Feynman rules for the massless case obtained in [2-4]. Our arguments can be directly generalized to any diagram containing a single closed loop [14, 30, 34, 35, 36].

Hitherto we have considered processes in which not more than two longitudinal quanta participate. It can be shown by a direct calculation that the probability of a transition to a state with three longitudinal quanta diverges in the limit $m \to 0$ [30].

Some authors [22, 30] have conjectured that the limit as $m \to 0$ exists outside the framework of perturbation theory. Of course, perturbation theory is not a physically adequate method for considering the S matrix (41) for small m since the effective coupling constant λ/m tends to infinity as $m \to 0$. However, the question of the existence of a limit outside the framework of perturbation theory is, in our opinion, rather scholastic at the present time; for there do not as yet exist sufficiently effective methods for calculating the S matrix other than by perturbation theory.

It is important to emphasize that we are concerned with the existence of a limit of the matrix elements and not the Lagrangian. In the effective Lagrangian in formula (41) there is no obstacle to setting $m = 0$ and we then obtain the theory of the massless Yang–Mills field; however, this is in no way an argument in favor of the existence of a limit of the matrix elements as $m \to 0$.

An analogous effect with regard to the absence of a zero-mass limit is also observed in gravitational theory [35, 37]. In gravitation, this effect already arises in the classical theory. Recently, Vainshtein [38], considering the classical Einstein equations, argued that for the exact solution of these equations a limit exists as $m \to 0$. However, the question of the existence of an exact solution in the quantum theory of gravitation and in the Yang–Mills theory remains open.

5. Conclusions

We conclude with a discussion of the possible further development and applications of the theory of the massive Yang–Mills field. As we have seen, the theory behaves as a renormalizable theory in the single-loop approximation but for more complicated diagrams this has not yet been proved. In this connection attempts have been made [39] to apply the Efimor–Fradkin method [10, 41] to this theory. In this method, as is well known, the test of renormalizability is the asymptotic behavior of the interaction Lagrangian

$$\mathscr{L}_{\text{int}} \leqslant M^4, \tag{87}$$

the scalar field being ascribed the asymptotic behavior M, the spinor field $M^{3/2}$, etc. In [39] this test is applied to the Lagrangian (69) that describes the part of the interaction of the Yang–Mills field that is nonrenormalizable according to perturbation theory. According to the estimate of [39], this Lagrangian has growth order M^4 and therefore generates a finite number of primitively divergent diagrams. However, the treatment given in [39] is too crude. In particular, no allowance is made for the complications due to the presence of derivatives in the interaction Lagrangian. In the literature there are also contradictory assertions [42] that even by means of the Efimov–Fradkin method the divergences cannot be eliminated correctly in theories of this type. This question must be investigated further. In addition, in this approach there remain the usual difficulties of the Efimov–Fradkin method due to the ambiguity of summation and the transition to the pseudo-Euclidean region.

An unpleasant property is also the dependence of the estimate of the renormalizability of the theory on the parametrization. It is easy to find a parametrization in which $\mathscr{L}_{\text{int}} > M^4$, indicating nonrenormalizability. In this connection, it would seem to be more appropriate to work not with the parametrization-dependent Lagrangian (69) but with the invariant equation (76), in which allowance is also made for the fact that the integration in formula (41) for the S matrix is with respect to an invariant measure.

The last and, indeed, most serious objection is that the solutions obtained by the Efimov–Fradkin method do not in general have the symmetry built into the original Lagrangian.

We are therefore more inclined to a conservative approach – construction of the S matrix by perturbation theory with the introduction of the requisite number of counter terms [43] to eliminate divergences and allowance for the Ward identities relating the renormalization constants of the different vertices. For example, the generating functional for the Green's function defined by the Lagrangian (42),

$$Z(\eta_\mu) = \int \exp i\left\{\operatorname{Tr}\int\left[\frac{1}{8}F_{\mu\nu}F_{\mu\nu} + \frac{m^2}{4}B_\mu B_\mu + \frac{1}{2}B_\mu\eta_\mu\right]dx\right\}dB, \tag{88}$$

satisfies a system of Ward identities analogous to (73):

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial Z}{\partial \eta_\mu^a(x)}\right) - i\frac{\partial \eta_\mu^a}{\partial x^\mu}Z = -\frac{\lambda}{m}\varepsilon^{abc}\eta_\mu^b(x)\frac{\partial Z}{\partial \eta_\mu^c(x)}. \tag{89}$$

It follows from the relations (89), in particular, that there is no renormalization of the longitudinal part of the Green's function and in addition the identities (84) hold, these relating the renormalization constants of the vertex function and the Green's function. The Ward identities for the Green's functions determined by the gauge-invariant functional (41) can be obtained by the method proposed in [44].

The symmetry properties of the theory are automatically taken into account in such an approach. Of course, its applicability is restricted and it cannot pretend to a description of high-energy processes. However, at energies that are not high this approach has proved to be very sensible and is actually used in practical calculations. Thus, Basdevant and Zinn-Justin [45] in a Padé approximants approach have considered the process of low-energy $\pi-\pi$-scattering, assuming that the ρ-meson is described by the Yang–Mills field. They have shown that even the single-loop approximation, in which there is no need to introduce any additional counter terms beyond the usual renormalization constants, leads to reasonable agreement with the experimental data.

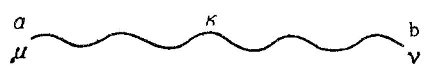
Another possible field of application of the massive Yang–Mills field is in the theory of weak interactions. This possibility, first noted by Salam and Ward [46], is discussed, for example, in [47]. If the intermediate vector meson is described by a Yang–Mills field, then, as follows from the results of §2, it is possible to calculate the radiative corrections to the lepton processes right up to fourth order without introducing any but the ordinary counter terms. This accuracy will undoubtedly be matched by the experimental possibilities in the near future. In contrast to the generally adopted scheme, the Yang–Mills scheme of the weak interactions does not, even when allowance is made for the lowest radiative corrections, lead to an inadmissibly rapid growth of the cross section with the energy (strictly speaking, because of the presence of the axial current and the inequality of the lepton masses the arguments of §2 are not directly applicable to the weak-interaction Lagrangian. However, it is easy to show that in the lowest orders the additional terms that arise in the effective Lagrangian do not generate new divergences).

We have here pointed out only two of the numerous possible applications of the theory of the massive Yang–Mills field. These examples show that even in its present imperfect form this theory is a useful instrument in the physics of elementary particles.

APPENDIX

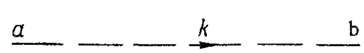
In this appendix we describe the Feynman rules for the massive Yang–Mills field in the transverse gauge. In general, these rules depend on the parametrization of the matrices Ω. However, in the single-loop approximation this dependence disappears and in any parametrization the Feynman rules have the following form.

Propagation function of a vector particle with momentum k:



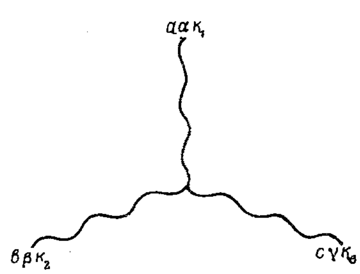
$$\frac{\delta^{ab}}{k^2-m^2+i\varepsilon}\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right),$$

propagation function of scalar fermion with momentum k:



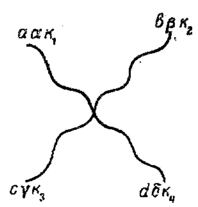
$$\frac{\delta^{ab}}{k^2+i\varepsilon},$$

triple vector vertex:



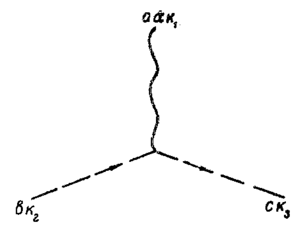
$$i\lambda\varepsilon^{abc}\{\delta_{\beta\gamma}(k_3-k_2)_\alpha+\delta_{\gamma\alpha}(k_1-k_3)_\beta+\delta_{\alpha\beta}(k_2-k_1)_\gamma\}.$$

fourfold vector vertex:



$$\lambda^2\{\varepsilon^{eac}\varepsilon^{ebd}(\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha\delta}\delta_{\gamma\beta})+\varepsilon^{ead}\varepsilon^{ebc}(\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha\gamma}\delta_{\beta\delta})+\varepsilon^{eab}\varepsilon^{ecd}(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma})\},$$

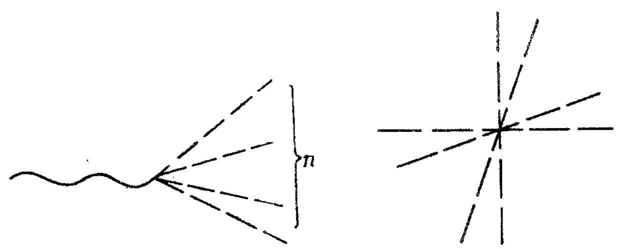
vertex describing the interaction of a scalar fermion with vector particle,



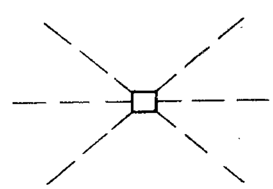
$$i\lambda\varepsilon^{abc}k_3{}^{\alpha}.$$

Each vertex makes a contribution $(2\pi)^4 i$. Each internal line makes a contribution $(2\pi)^{-4} i$. Each closed scalar loop adds a factor $(-1/2)$.

In the higher approximations (two and more loops) there are vertices of the form



with an arbitrary number of scalar lines. Their explicit form depends on the parametrization of the matrices Ω. For each concrete parametrization, for example, for the parametrization (66), they can be readily found. In addition, because of the decomposition of the invariant measure we have vertices of the form



where $\square$ stands for $\delta(0)$.

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