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Infinitesimal generators and the Loewner equation on complete hyperbolic manifolds

Leandro Arosio · Filippo Bracci

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Abstract We characterize infinitesimal generators on complete hyperbolic complex manifolds without any regularity assumption on the Kobayashi distance. This allows to prove a general Loewner type equation with regularity of any order $d \in [1, +\infty]$. Finally, based on these results, we focus on some open problems naturally arising.

Keywords Loewner equation · Complete hyperbolic manifolds

1 Introduction

The classical theory of Ch. Loewner has been used and generalized in many aspects. We refer the reader to the book [13], and the recent survey papers [3,5] for an updated account.

In the papers [7,8], the second named author with M. D. Contreras and S. Díaz-Madrigal developed a general theory of Loewner type both on the unit disc and on complex (Kobayashi) complete hyperbolic manifolds, which relates evolution families to Herglotz vector fields, via the Loewner ODE. Next, in [11] Contreras et al.

L. Arosio: Titolare di una Borsa della Fondazione Roma—Terzo Settore bandita dall'Istituto Nazionale di Alta Matematica.

L. Arosio
Istituto Nazionale di Alta Matematica "Francesco Severi",
Città Universitaria, Piazzale Aldo Moro 5, 00185 Rome, Italy
e-mail: arosio@altamatematica.it

F. Bracci (✉)
Dipartimento Di Matematica, Università di Roma "Tor Vergata",
Via Della Ricerca Scientifica 1, 00133 Rome, Italy
e-mail: fbracci@mat.uniroma2.it

fitted the Loewner PDE in the picture, and, in [4], the authors with H. Hamada and G. Kohr extended (with different methods) such results to complex complete hyperbolic manifolds.

However, the Loewner ODE theory required some technical hypotheses which, although satisfied in the most interesting cases (such as the unit ball of $\mathbb{C}^q$), made the theory a bit artificial in its generality. The aim of the present paper is exactly to show that such hypotheses are redundant.

In order to properly state the results, we need to give some notions. First, we recall that, given a complex manifold M, a holomorphic vector field H on M is said an *infinitesimal generator* provided the Cauchy problem

$$\begin{cases} \dot{z}(t) = H(z(t)), \\ z(0) = z_0 \end{cases}$$

has a solution $z : [0, +\infty) \to M$ for all $z_0 \in M$.

Infinitesimal generators are called this way because they generate continuous semigroups of holomorphic self-maps, see, e.g., [2,20] for details. In [6] it is shown that if $D \subset \mathbb{C}^q$ is a bounded strongly convex domain with smooth boundary then a holomorphic vector field H is an infinitesimal generator if and only if

$$(dk_D)_{(z,w)}(H(z), H(w)) \leq 0 \quad \forall z, w \in D, z \neq w,$$

where $k_D : D \times D \to \mathbb{R}^+$ is the Kobayashi distance of D (see [16] or [1] for definition and properties), which is known to be C^∞ outside the diagonal by Lempert [17].

As a first result we prove that the same characterization of infinitesimal generators holds in general. After having shown that the Kobayashi (pseudo)distance k_M on a complex manifold M is locally Lipschitz—and denoting by dk_M its Dini directional derivative, which coincides a.e. with the usual differential—we prove the following:

Theorem 1.1 *Let M be a complete hyperbolic complex manifold with Kobayashi distance k_M, and let H be an holomorphic vector field on M. Then the following are equivalent:*

a) *there exists $\varepsilon > 0$ and a family of holomorphic mappings $(f_t : M \to M)_{t\in[0,\varepsilon)}$ such that $f_0(z) = z$ and*

$$\frac{\partial}{\partial t} f_t(z)|_{t=0} = \lim_{t\to 0^+} \frac{f_t(z) - z}{t} = H(z),$$

b) *for every $z \neq w \in M$ one has*

$$(dk_M)_{(z,w)}(H(z), H(w)) \leq 0,$$

c) *H is an infinitesimal generator.*

This theorem allows to extend the so called "product formula" of Reich and Shoikhet [19, Theorem 3] to complete hyperbolic manifolds and to prove that the set of infinitesimal generators of a complete hyperbolic complex manifold form a (closed) real cone (see Sect. 2).

With Theorem 1.1 in mind, we can give the following definition (cfr. [8, Definition 3]):

Definition 1.2 Let M be a complex manifold endowed with an Hermitian metric $\|.\|$, and let $d \in [1, +\infty]$. A *weak holomorphic vector field* of order $d \geq 1$ on M is a mapping

$$G\colon M \times [0, +\infty) \to TM$$

satisfying

WHVF1. for all $z \in M$ the map $t \mapsto G(z, t)$ is measurable,
WHVF2. for all $t \geq 0$ the map $z \mapsto G(z, t)$ is a holomorphic vector field,
WHVF3. for any compact set $K \subset M$ and for any $T > 0$ there exists a non-negative function $c_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$\|G(z, t)\| \leq c_{K,T}(t), \quad z \in K, 0 \leq t \leq T.$$

A *Herglotz vector field* of order $d \geq 1$ on M is a weak holomorphic vector field of order $d \geq 1$ such that $M \ni z \mapsto G(z, t)$ is an infinitesimal generator for a.e. fixed $t \in [0, +\infty)$.

Now we are going to define evolution families:

Definition 1.3 Let M be a complex manifold endowed with an Hermitian metric and let d_M denote the associated integrated distance. A family $(\varphi_{s,t})_{0\leq s\leq t<+\infty}$ of holomorphic self-maps of M is an *evolution family of order* $d \in [1, +\infty]$ (in short, an *L^d-evolution family*) if

EF1. $\varphi_{s,s} = \mathsf{id}_M$,
EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < +\infty$,
EF3. for any compact subset $K \subset M$ and for any $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that for all $0 \leq s \leq u \leq t \leq T$ and for all $z \in K$,

$$d_M(\varphi_{s,u}(z), \varphi_{s,t}(z)) \leq \int_u^t k_{K,T}(\xi)d\xi.$$

In [8] it has been proved the following result:

Theorem 1.4 [8] *Let M be a complete hyperbolic manifold with Kobayashi distance k_M. Assume that $k_M \in C^1(M \times M \backslash Diag)$.*

(1) *Let $G(z, t)$ be a Herglotz vector field of order $d \in [1, +\infty]$ on M. Then there exists a unique evolution family $(\varphi_{s,t})$ of order d on M such that*

$$\frac{\partial \varphi_{s,t}(z)}{\partial t} = G(\varphi_{s,t}(z), t) \quad a.e.\ t \in [s, +\infty). \tag{1.1}$$

(2) *Let* $(\varphi_{s,t})$ *be an evolution family of order* $+\infty$ *on* M. *Then there exists a Herglotz vector field* $G(z,t)$ *of order* $+\infty$ *which satisfies* (1.1). *Moreover, if* $G'(z,t)$ *is another weak holomorphic vector field which satisfies* (1.1) *then* $G(z,t) = G'(z,t)$ *for all* $z \in M$ *and a.e.* $t \geq 0$.

In case $M = \mathbb{D}$ the unit disc in $\mathbb{C}$, in [7] it is proved that the second part of the previous theorem holds even when the evolution family has order $d \in [1, +\infty]$, giving rise to a Herglotz vector field of the same order. Such a result is based on the Berkson-Porta formula for infinitesimal generators, a tool which is not available in higher dimensions. In [14] the same is proved for the unit ball of $\mathbb{C}^q$, using the Loewner PDE defined in [4].

In this paper we prove (see Propositions 3.1, 3.2) the following result:

Theorem 1.5 *Let* M *be a complete hyperbolic manifold.*

(1) *Let* $G(z,t)$ *be a Herglotz vector field of order* $d \in [1, +\infty]$ *on* M. *Then there exists a unique evolution family* $(\varphi_{s,t})$ *of order* d *on* M *which satisfies* (1.1).
(2) *Let* $(\varphi_{s,t})$ *be an evolution family of order* $d \in [1, +\infty]$ *on* M. *Then there exists a Herglotz vector field* $G(z,t)$ *of order* d *which satisfies* (1.1). *Moreover, if* $G'(z,t)$ *is another weak holomorphic vector field which satisfies* (1.1) *then* $G(z,t) = G'(z,t)$ *for all* $z \in M$ *and a.e.* $t \geq 0$.

We end up the paper with a section of natural open problems deriving from what we explained before.

2 Infinitesimal generators on complete hyperbolic manifolds

2.1 Regularity of the Kobayashi distance

We first recall some definition from analysis. A function on a manifold is said to be *locally Lipschitz* if it is locally Lipschitz on one—and hence any—chart. We show that the Kobayashi pseudodistance on a complex manifold is locally Lipschitz. First we recall the following estimate (for a proof, see, e.g. [1]).

Lemma 2.1 *Let* M *be a complex manifold,* $U \subset M$ *an open domain and* $\psi : U \to \mathbb{B}^q$ *a biholomorphism from* U *to the open ball* $\mathbb{B}^q \subset \mathbb{C}^q$. *Then for every compact subset* $K \subset U$ *there exists* $C_K > 0$ *such that*

$$k_M(z,w) \leq C_K \|\psi(z) - \psi(w)\|, \quad z, w \in K.$$

Proposition 2.2 *Let* M *be a complex manifold. Then the Kobayashi pseudodistance*

$$k_M : M \times M \to \mathbb{R}^+$$

is locally Lipschitz.

Proof Let $\psi\colon U \to \mathbb{B}^q$ and $\varphi\colon V \to \mathbb{B}^q$ be two biholomorphic coordinate charts. We show that the mapping

$$k_M \circ (\psi^{-1}, \varphi^{-1})\colon \mathbb{B}^q \times \mathbb{B}^q \to \mathbb{R}^+$$

is locally Lipschitz. Let $K \subset U$ and $H \subset V$ be two compact subsets. Let $(z, w), (z', w')$ be in $K \times H$. Then by the triangular inequality and Lemma 2.1 we have

$$\begin{aligned}
|k_M(z, w) - k_M(z', w')| &= |k_M(z, w) - k_M(z', w) + k_M(z', w) - k_M(z', w')| \\
&\leq |k_M(z, w) - k_M(z', w)| + |k_M(z', w) - k_M(z', w')| \\
&\leq k_M(z, z') + k_M(w, w') \leq C_K \|\psi(z) - \psi(z')\| + C_H \|\varphi(w) - \varphi(w')\|,
\end{aligned}$$

and we are done. $\square$

Definition 2.3 Let f be a real-valued function defined on an interval $I = [t_0, a)$. The (lower) *Dini derivative* is defined as

$$\underline{\mathrm{D}} f(t) := \liminf_{h \to 0+} \frac{f(t+h) - f(t)}{h}.$$

Let Ω be a domain in $\mathbb{R}^q$, and let $f\colon \Omega \to \mathbb{R}$ be a locally Lipschitz function. If $x \in \Omega$ and $v \in \mathbb{R}^q$, then the *directional Dini derivative* is defined as

$$\underline{\mathrm{D}} f(x, v) := \liminf_{h \to 0+} \frac{f(x+hv) - f(x)}{h}.$$

Lemma 2.4 *Let Ω be a domain in $\mathbb{R}^q$, and let $f\colon \Omega \to \mathbb{R}$ be a locally Lipschitz function. Let $\gamma\colon [0, \varepsilon) \to \Omega$ be a mapping such that $\gamma(0) = x$ and $\frac{d}{dt}\gamma(0) = v$. Then*

$$\underline{D} f(\gamma(0)) = \underline{D} f(x, v).$$

Proof We claim that $\|f(\gamma(t)) - f(x+tv)\| = o(t)$. Indeed there exist $C > 0$ such that

$$\|f(\gamma(t)) - f(x+tv)\| \leq C \|\gamma(t) - (x+tv)\|,$$

and the claim follows since $\frac{d}{dt}\gamma(0) = v$.

Thus

$$\begin{aligned}
\liminf_{t \to 0+} \frac{f(\gamma(t)) - f(x)}{t} &= \liminf_{t \to 0+} \frac{f(\gamma(t)) - f(x+tv) + f(x+tv) - f(x)}{t} \\
&= \lim_{t \to 0+} \frac{f(\gamma(t)) - f(x+tv)}{t} + \liminf_{t \to 0+} \frac{f(x+tv) - f(x)}{t} \\
&= \liminf_{t \to 0+} \frac{f(x+tv) - f(x)}{t}.
\end{aligned}$$

$\square$

If $f : M \to \mathbb{R}$ is locally Lipschitz then, outside a set of zero measure (with respect to the Lebesgue measure on M), the differential df_z exists and coincides with the Dini partial derivative $\underline{\mathrm{D}}f(z, \cdot)$. Therefore, it is natural to simply denote $df_z := \underline{\mathrm{D}}f(z, \cdot)$.

In particular, for what we have seen, the Dini partial derivative dk_M of the Kobayashi distance is well defined for any complex manifold M.

2.2 Characterization of infinitesimal generators

Now we are in good shape to give the proof of our first result:

Proof of Theorem 1.1 a) $\Longrightarrow$ b) Let $z \neq w \in M$ and define $\gamma_z(t) := f_t(z)$ and $\gamma_w(t) := f_t(w)$ for all $t \in [0, \varepsilon)$. Since holomorphic self-mappings of M contract the Kobayashi distance, the function $t \mapsto k_M(\gamma_z(t), \gamma_w(t))$ is non-increasing, thus taking the liminf of the incremental ratio as $t \to 0^+$ we get

$$dk_M((z, w), (H(z), H(w))) \leq 0.$$

b) $\Longrightarrow$ c) Let $z \in M$ and let $f_t(z)$ be the maximal solution with escaping time $I(z) > 0$ which solves the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} f_t(z) = H(f_t(z)), & t \in [0, I(z)) \\ f_0(z) = z. \end{cases} \tag{2.1}$$

Fix $z \neq w \in M$. Let $J := [0, I(z)) \cap [0, I(w))$ and define the continuous real valued function

$$h(t) := k_M(f_t(z), f_t(w)).$$

Since h is continuous and $\underline{\mathrm{D}}h(t) \leq 0$ for all $t \in J$, one has that h is non-increasing. If it were $I(z) < I(w)$, since M is complete hyperbolic and

$$\{f_t(w)\}_{t \in [0, I(z)]} \subset\subset M,$$

we would have

$$+\infty = \limsup_{t \to I(z)} k_M(f_t(z), f_t(w)) \leq k_M(z, w) < +\infty,$$

a contradiction. Therefore $I(z) \geq I(w)$, and similarly $I(w) \geq I(z)$.

Since the Cauchy problem is autonomous, this implies that $I(z) = +\infty$ for all $z \in X$, which proves c).

c) $\Longrightarrow$ a) follows from the holomorphic flow-box theorem (see for example [15]). □

2.3 The product formula

For a bounded convex domain of a complex Banach space, the following “product formula” is proved in [19, Theorem 3].

Proposition 2.5 *Let M be a complete hyperbolic complex manifold, and let H be an holomorphic vector field on M. Suppose there exists $\lambda > 0$ and a family of holomorphic mappings $(f_t \colon M \to M)_{t\in[0,\lambda)}$ converging uniformly on compact subsets to $f_0(z) = z$ as $t \to 0+$, and such that for all $w \in M$ one has, in a local coordinate chart $w \in W \to \mathbb{C}^q$,*

$$\lim_{t\to 0+} \frac{f_t(z) - z}{h} = H(z)$$

uniformly on compacta of W. Then H is an infinitesimal generator and the semigroup (φ_t) associated to H satisfies the following “product formula”:

$$\varphi_t = \lim_{m\to\infty} (f_{t/m})^{\circ m},$$

where the limit is uniform on compacta of M.

Proof Fix $w \in M$. Let $\psi \colon U \to \mathbb{B}^q$ be a biholomorphic coordinate chart relatively compact in M and centered at w, i.e., $\psi(w) = 0$. Let $\frac{1}{2}U := \psi^{-1}(\frac{1}{2}\mathbb{B}^q)$. In the following, as customary, we identify U and $\mathbb{B}^q$ via ψ without mentioning it anymore.

By Theorem 1.1, the vector field H is an infinitesimal generator. Let (φ_t) be the associated semigroup. Since $U \subset\subset M$, there exists $C \geq 0$ such that $\|H(z)\| \leq C$ for all $z \in U$. Also, by continuity, there exists $\mu > 0$ such that for all $0 \leq t < \mu$ the mapping φ_t sends $\frac{1}{2}U$ in U, and thus

$$\|\varphi_t(z) - z\| \leq \int_0^t \|H(\varphi_\xi(z))\| d\xi \leq Ct, \quad z \in \tfrac{1}{2}U,\ 0 \leq t \leq \mu. \tag{2.2}$$

By Lemma 2.1 and (2.2) there exists $\mathcal{L} \geq 0$ such that

$$k_M(\varphi_t(z), z) \leq \mathcal{L}t, \quad z \in \tfrac{1}{2}U,\ 0 \leq t < \mu.$$

Thus for all $0 \leq \tau < \mu$ and all $z \in \frac{1}{2}U$, $\ell \in \mathbb{N}$

$$k_M(\varphi_\tau^{\circ\ell}(z), z) \leq \sum_{j=0}^{\ell} k_M(\varphi_\tau^{\circ(j+1)}(z), \varphi_\tau^{\circ j}(z)) \leq \ell k_M(\varphi_\tau(z), z) \leq \ell\mathcal{L}\tau.$$

Fix $t > 0$. Next, fix $r > 0$ such that the Kobayashi ball $B_M(0, t\mathcal{L} + r)$ of center 0 and radius $t\mathcal{L} + r$ is contained in $\frac{1}{2}U$. Let $m \in \mathbb{N}$ be such that $t/m \leq \mu$. Then for all $z \in B_M(0, r)$,

$$k_M(\varphi_{t/m}^{\circ\ell}(z), 0) \leq k_M(\varphi_{t/m}^{\circ\ell}(z), z) + k_M(z, 0) \leq \ell\mathcal{L}(t/m) + r,$$

hence

$$\{\varphi_{t/m}^{\circ\ell}(z) : 0 \leq \ell \leq m-1\} \subset B_M(0, t\mathcal{L} + r).$$

By (2.2), the family $\frac{1}{h}(\varphi_h(z) - z)$ is bounded on $\frac{1}{2}U$ and thus converges uniformly to $H(z)$ as $h \to 0+$. Up to shrinking U if necessary, by hypothesis the same holds for $\frac{1}{h}(f_h(z) - z)$. Hence for each $\varepsilon > 0$ there exists $\eta > 0$ such that

$$k_M(\varphi_s(y), f_s(y)) \leq s\varepsilon, \quad y \in \tfrac{1}{2}U, \ 0 \leq s \leq \eta.$$

Take $N \geq 0$ so large that $s := \frac{t}{m} \leq \eta$ for all $m > N$. Then for all $z \in B_M(0, r)$ we have for $m > N$,

$$\begin{aligned} k_M(f_{t/m}^{\circ m}(z), \varphi_t(z)) &= k_M(f_{t/m}^{\circ m}(z), \varphi_{t/m}^{\circ m}(z)) \\ &\leq \sum_{k=1}^{m} k_M(f_s^{\circ(k-1)}(f_s(\varphi_s^{\circ(m-k)}(z))), f_s^{\circ(k-1)}(\varphi_s(\varphi_s^{\circ(m-k)}(z)))) \\ &\leq \sum_{k=1}^{m} k_M(f_s(y_k), \varphi_s(y_k)) \leq t\varepsilon, \end{aligned}$$

where $y_k := \varphi_s^{\circ(m-k)}(z) \in B_M(0, t\mathcal{L} + r) \subset \frac{1}{2}U$. Thus $f_{t/m}^{\circ m}(z) \to \varphi_t(z)$ uniformly on $B_M(0, r)$. Since local uniform convergence implies uniform convergence on compacta, the result follows. □

For a bounded convex domain of a complex Banach space, the following corollary is proved in [19, Corollary 4], and its proof is an easy consequence of Theorem 1.1 and Proposition 2.5.

Corollary 2.6 *Let M be a complete hyperbolic complex manifold. Let H_1 and H_2 be two infinitesimal generators on M with associated semigroups (φ_t) and (ψ_t), respectively. Then the holomorphic vector field $H_1 + H_2$ is an infinitesimal generator and the associated semigroup (η_t) satisfies*

$$\eta_t = \lim_{m \to +\infty} (\varphi_{t/m} \circ \psi_{t/m})^{\circ m},$$

where the limit is uniform on compacta of M.

We have also:

Corollary 2.7 *Let M be a complete hyperbolic complex manifold. The set of infinitesimal generators on M form a closed convex cone with vertex in* 0.

3 The Loewner equation on complete hyperbolic manifolds

Proposition 3.1 *Let M be a complete hyperbolic manifold. Let $d \geq 1$ and let $G(z,t)$ be a Herglotz vector field of order d. Then there exists a unique evolution family $(\varphi_{s,t})$ of order d which solves the Cauchy problem* (1.1).

Proof For $z \in M$ and $s \in \mathbb{R}^+$ let $\varphi_{s,t}(z)$ be the maximal solution with escaping time $I(s,z) > 0$ which solves the Cauchy problem (1.1). Fix $z \neq w \in M$. Let

$$J := [s, I(s,z)) \cap [s, I(s,w))$$

and let

$$h(t) := k_M(\varphi_{s,t}(z), \varphi_{s,t}(w)).$$

The function $h(t)$ is locally absolutely continuous since by Lemma 2.2 the Kobayashi distance k_M is locally Lipschitz. Thus $h(t)$ is differentiable for a.e. $t \in J$. By Theorem 1.1 for a.e. $t \in \mathbb{R}^+$ the holomorphic vector field $G(z,t)$ satisfies b), thus $\underline{\mathrm{D}}h(t) \leq 0$ for a.e. $t \in J$, which implies

$$\frac{d}{dt}h(t) \leq 0, \quad \text{for a.e. } t \in J,$$

hence h is non-increasing. If $I(s,z) < I(s,w)$, since M is complete hyperbolic and

$$\{\varphi_{s,t}(w)\}_{t\in[s,I(s,z)]} \subset\subset M,$$

we have

$$+\infty = \limsup_{t\to I(s,z)} k_M(\varphi_{s,t}(z), \varphi_{s,t}(w)) \leq k_M(z,w) < +\infty,$$

a contradiction. Then $I(s,z) \geq I(s,w)$, and similarly $I(s,w) \geq I(s,z)$.

The proof proceeds now as in Steps 2–6 of [8, Proposition 1]. □

Proposition 3.2 *Let M be a complete hyperbolic manifold of dimension $q \geq 1$. Then for any evolution family $(\varphi_{s,t})$ of order $d \geq 1$ in M there exists a Herglotz vector field of order $d \geq 1$ which satisfies* (1.1)*. Moreover, if $G'(z,t)$ is another weak holomorphic vector field which satisfies* (1.1) *then $G(z,t) = G'(z,t)$ for all $z \in M$ and a.e. $t \geq 0$.*

Proof The proof is similar to that of [8, Proposition 2] and we only describe the main differences. Let U be a chart in M, let $U' \subset\subset U$ be an open set, and let $T > 0$. There exists $n(T,U') > 0$ such that for all $n \in \mathbb{N}$, $n \geq n(T,U')$, $t \in [0,T]$ it holds

$$\varphi_{t,t+\frac{1}{n}}(U') \subset U.$$

We define locally in U',

$$G_{n,s}(z) := n(\varphi_{s,s+\frac{1}{n}}(z) - z), \quad t \in [0,T],\ n \geq n(U',T).$$

Let $k_T := k_{\overline{U'},T+1} \in L^d([0,T+1],\mathbb{R})$ be the non-negative function given by EF3. We extend k_T to all of $\mathbb{R}$ by setting zero outside the interval $[0,T+1]$. Then for $0 \leq s \leq T$ and every $n \in \mathbb{N}$, $n \geq n(T,U')$ and all $z \in U'$,

$$n\|\varphi_{s,s+\frac{1}{n}}(z) - z\| = n\|\varphi_{s,s+\frac{1}{n}}(z) - \varphi_{s,s}(z)\| \leq n \int_s^{s+1/n} k_T(\xi)d\xi \leq \mathsf{Max}_{k_T}(s),$$

where

$$\mathsf{Max}_{k_T}(s) := \sup\left\{\frac{1}{|I|}\int_I k_T(\xi)d\xi : \ I \text{ is a closed interval of the real line and } s \in I\right\}$$

is the so-called maximal function associated to k_T. Since $k \in L^1(\mathbb{R},\mathbb{R})$, by Hardy–Littlewood maximal theorem there exists a subset $N(T) \subset [0,+\infty)$ of zero measure such that $\mathsf{Max}_{k_T}(s) < +\infty$ for every $s \in [0,T]\backslash N(T)$. Let

$$N := \bigcup_{T\in\mathbb{N}} N(T).$$

Note that N is a set of measure zero in $[0,+\infty)$. Then for all $T \in \mathbb{N}$ there exists $C(T,U') > 0$ such that for all $s \in [0,T]\backslash N$, $z \in U'$ and $n \geq n(T,U')$

$$\sup_n \|G_{n,s}(z)\| \leq C(T,U'). \tag{3.1}$$

For any $T \in \mathbb{N}\backslash\{0\}$ let

$$\Theta_T := \left\{f : U' \to \mathbb{C}^q \text{ holomorphic} : \sup_{z\in U'} \|f(z)\| \leq C(T,U')\right\}.$$

Then Θ_T has a natural structure of a complete metric space. Note that, by (3.1), $\{G_{n,s}\} \subset \Theta_T$ for $s \in [T-1,T)$. Let

$$\Gamma_T : [T-1,T) \to \Theta_T,\ s \mapsto \Gamma(s) = \begin{cases} \mathsf{ac}(G_{n,s}) & s \notin N, \\ \{0\} & s \in N, \end{cases}$$

where $\mathsf{ac}(G_{n,s})$ denotes the accumulation points of the sequence Θ_T in the metric space Θ_T. Note that the multifunction Γ_T is well-defined and, since Θ_T is a metric space, $\Gamma_T(s)$ is a closed subset of Θ_T for every $s \in [T-1,T)$. By (3.1) the family $\{G_{n,s}\}_n$ is uniformly bounded, thus it has accumulation points in Θ_T, so that $\Gamma_T(s)$ is not empty for any $s \geq 0$.

Now we are going to apply the following result:

Theorem 3.3 ([10, Theorem III.30, p. 80]) *Let (Ω, Σ, μ) be a positive σ-finite complete measure space, $[X, d]$ a separable and complete metric space and Γ a multifunction from Ω to the subsets of X. Assume that:*

(i) *For every $\omega \in \Omega$, $\Gamma(\omega)$ is a closed non-empty subset of X.*
(ii) *For every $x \in X$ and every $r > 0$, $\{\omega \in \Omega : \Gamma(\omega) \cap B(x, r) \neq \emptyset\} \in \Sigma$. (As usual, $B(x, r)$ denotes the open unit ball in X with center x and radius r).*

Then Γ admits a measurable selector $\sigma : \Omega \longrightarrow X$; namely, for every $\omega \in \Omega$, we have $\sigma(\omega) \in \Gamma(\omega)$ and the inverse image by σ of any borelian in X belongs to Σ.

We already saw that Γ_T satisfies hypothesis (i) of Theorem 3.3.

In order to check condition (ii) in Theorem 3.3 for Γ_T, one can argue similar to the proof of [7, Theorem 6.2], thus we omit such details.

Therefore, the multifunction Γ_T satisfies the hypotheses of Theorem 3.3. Thus there exists a measurable selector $\sigma_T : [T, T+1) \to \Theta_T \subset \mathsf{Hol}(U', \mathbb{C}^q)$ for Γ_T. We define $G : U' \times [0, +\infty) \to \mathbb{C}^q$ by

$$G(z, s) := \sigma_T[s](z),$$

for $z \in U'$ and $s \in [T, T+1)$, $T \in \mathbb{N}\backslash\{0\}$. Hence, for every $s \in [0, +\infty)\backslash N$ there exists a strictly increasing sequence $\{n_k(s)\} \subset \mathbb{N}$ such that, for all $z \in U'$,

$$G(z, s) := \lim_{k\to\infty} G_{n_k(s),s}(z),$$

and the convergence is uniform on U'. Then, by construction, $G(z, s)$ is a weak holomorphic vector field on U'. Moreover, for any $s \in [0, T+1]\backslash N$ and for all $z \in U'$ it follows

$$\|G_{n_k(s),s}\| \leq n_k(s) \int_s^{s+1/n_k(s)} k_T(\xi) d\xi.$$

Passing to the limit for $k \to \infty$, we obtain that for almost all $s \in [0, T+1]$ it holds

$$\|G(z, s)\| \leq k_T(s),$$

proving that G has order d.

Now we can argue as in the proof of [8, Proposition 2], see the steps from 6’ to 8’ pp. 959–960, and we see that $G(z, t)$ is a weak holomorphic vector field of order d which satisfies (1.1). It is then a Herglotz vector field from a) of Theorem 1.1. $\square$

4 Open problems

4.1 On the definition of evolution families

Let $(\varphi_{s,t})$ with $0 \leq s \leq t < +\infty$ be a family of holomorphic self-maps of a complete hyperbolic manifold M with satisfies EF1 and EF2 of Definition 1.3.

Question 1: Are there simple conditions which guarantee that EF3 holds?

In other words, what (if any) are the simplest conditions that imply that $(\varphi_{s,t})$ is an evolution family? For instance, in [7] it is proved that a family $(\varphi_{s,t})_{0\leq s\leq t}$ of holomorphic self-maps of the unit disc $\mathbb{D}$ satisfying EF1, EF2 and

EF3'. for any $z \in \mathbb{D}$ and for any $T > 0$ there exists a non-negative function $k_{z,T} \in L^d([0,T],\mathbb{R})$ such that for all $0 \leq s \leq u \leq t \leq T$

$$d_M(\varphi_{s,u}(z), \varphi_{s,t}(z)) \leq \int_u^t k_{z,T}(\xi)d\xi.$$

solves (1.1) for a given L^d-Herglotz vector field G on $\mathbb{D}$. Hence, *a posteriori*, from Theorem 1.5, it follows that $(\varphi_{s,t})$ is a L^d-evolution family in the sense of Definition 1.3 (in fact, this can be proved directly using distortion theorems).

If $M = D \subset\subset \mathbb{C}^q$ is a bounded convex domain, a simpler condition can be stated as follows:

Proposition 4.1 *Let $D \subset\subset \mathbb{C}^q$ be a bounded convex domain, $d \in [1, +\infty]$. Let $(\varphi_{s,t})_{0\leq s\leq t}$ be a family of holomorphic self-maps of D satisfying EF1 and EF2. Then $(\varphi_{s,t})$ is an L^d-evolution family if and only if for any $T > 0$ there exists a non-negative function $k_T \in L^d([0,T],\mathbb{R})$ such that for all $0 \leq s \leq t \leq T$*

$$\|\varphi_{s,t}(0)\| + \|d(\varphi_{s,t})_0 - \mathsf{id}\| \leq \int_s^t k_T(\xi)d\xi. \tag{4.1}$$

The proof is based on some type of distortion theorem for infinitesimal generators. Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a continuous linear operator. As customary, we define

$$V(A) := \sup\{|\langle A(v), v\rangle| : \|v\| = 1\}.$$

The following result is proved in its generality for the case of Banach spaces in [9]. In case $G(0) = 0$ and $V(T) > 0$ it is proved in [12].

Lemma 4.2 *Let $D \subset \mathbb{C}^n$ be a bounded convex domain. Let $G(z) = G(0) + Tz + \sum_{j\geq 2} Q_j(z)$ be an infinitesimal generator in D. Then for all $z \in D$*

$$\|G(z)\| \leq 5\|G(0)\| + \frac{4\|z\|}{(1-\|z\|)^2}V(T). \tag{4.2}$$

Proof of Proposition 4.1 One direction is clear. As for the other, assume that (4.1) holds and fix $T > 0$. Let $G_{n,s}(z) := n(\varphi_{s,s+\frac{1}{n}}(z) - z)$ for $n \in \mathbb{N}$ and $z \in D$. By [18, Proposition 4.3], $G_{n,s}(z)$ is an infinitesimal generator in D. Hence Lemma 4.2 and (4.1) implies (3.1). From here we can argue similarly to the proof of Proposition 3.2 and obtain an L^d-Herglotz vector field $G(z,s)$ which satisfies (1.1). Then by Theorem 1.5 the family $(\varphi_{s,t})$ is an L^d-evolution family. □

4.2 Relations between semigroups and evolution families

Let $G(z,t)$ be a Herglotz vector field on a bounded convex domain D with associated evolution family $(\varphi_{s,t})$. Hence, by (1.1), for a.e. $s \geq 0$, it holds

$$G(z,s) = \lim_{t \to s^+} \frac{\varphi_{s,t}(z) - z}{t - s}.$$

In particular, if $(\phi_r^s)_{r \geq 0}$ is the semigroup associated to the infinitesimal generator $z \mapsto G(z,s)$, the product formula implies that

$$\phi_r^s = \lim_{m \to \infty} (\varphi_{s,s+\frac{r}{m}})^{\circ m}$$

uniformly on compacta of D. Thus, there is a link between the families of semigroups generators by $G(\cdot, s)$ when $s \geq 0$ and the evolution family given by (1.1). Thus we have the following natural philosophical question:

Question 2: What are the relations between the dynamics of the families of semigroups (ϕ_r^s) and the dynamics of the evolution family $(\varphi_{s,t})$?

The previous question is open even for the case of the unit disc $\mathbb{D} \subset \mathbb{C}$.

4.3 The embedding problem

Let $f : M \to M$ be an univalent self-map of a complete hyperbolic manifold.

Question 3: When does there exists an evolution family $(\varphi_{s,t})$ order $d \geq 1$ such that $f = \varphi_{0,1}$?

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