STABILITY ESTIMATES FOR THE SOLUTIONS OF A CERTAIN ULTRAPARABOLIC EQUATION

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In the domain $Q=\{(x, t, \tau): x \in \mathbf{R}^n, x_j>0, j=1, \ldots, n, t \in \mathbf{R}^m, t_j>0, j=1, \ldots, m, \tau \in \mathbf{R}^p\}$ we consider the equation

$$\sum_{j=1}^{m} u_{t_j}+\sum_{j=1}^{p} u_{\tau_j}=\sum_{j=1}^{n} u_{x_j x_j}. \tag{1}$$

We consider the problem of restoring in Q the solution of Eq. (1), satisfying the conditions

$$u|_{t_j=0}=0, \quad j=1, \ldots, m \tag{2}$$

if the values of the solution for $x_j=a_j, a_j \geqslant 0, \sum_{j=1}^{n} a_j>0, j=1, \ldots, n$, are known:

$$u|_{x_j=a_j}=f_j\left(x^{(j)}, t, \tau\right), \ x^{(j)}=\left(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\right). \tag{3}$$

As it is known (see [1-4]), problems of this type are ill-posed in the sense that the smallness of $u|_{x_q=a_q}, a_q>0, 1 \leqslant q \leqslant n$, in some norm does not imply always the smallness of $u(x, t, \tau)$ for $0 \leqslant x_q<a_q$ in the same norm.

Assuming that the solution of Eq. (1) under the conditions (2) and (3) exists and belongs to a definite class, one has, firstly, to obtain a theorem characterizing the conditional stability and, secondly, to construct an approximate solution from the values of the solution at $x_j = a_j$, $j = 1,\ldots,n$, known with a certain error.

By virtue of Holmgren's theorem, problem (1)-(3) has at most one solution in the class of bounded functions.

We rewrite Eq. (1) in the form

$$\partial_l u=\sum_{j=1}^{n} u_{x_j x_j}, \tag{4}$$

where $\partial_l u=\sum_{j=1}^{m} u_{t_j}+\sum_{j=1}^{p} u_{\tau_j}$. By a solution of this equation we shall mean a function having second-order derivatives with respect to x_j, $j = 1,\ldots,n$, and first-order derivative in the direction of the vector $\ell = (1,\ldots,1)$ at each point of the domain Q, and satisfying Eq. (4). We denote $\mathbf{R}_j^{n-1}=\{(x^{(j)}): x^{(j)}=(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)\}$, $j=1, \ldots, n$. For $j = 1,\ldots,n$ we extend the function $u(x, t, \tau)|_{x_j=0}$ by zero into the space $\mathbf{R}_j^{n-1} \times \mathbf{R}^{m+p}=\{(x^{(j)}, t, \tau): x^{(j)} \in \mathbf{R}_j^{n-1}, \ t \in \mathbf{R}^m, \ \tau \in \mathbf{R}^p\}$. We have the following representation of the solution of Eq. (4) under the condition (2):

$$u(x, t, \tau)=\sum_{j=1}^{n} u_j(x, t, \tau),$$

where $u_j(x, t, \tau)=\frac{x_j}{(4\pi)^{n/2}} \int_0^{t_1} \int_{\mathbf{R}_j^{n-1}}(t_1-\eta)^{-n/2-1} \exp\left(-\frac{x_j^2+|x^{(j)}-\zeta^{(j)}|^2}{4(t_1-\eta)}\right) \varphi_j(\zeta^{(j)}, \eta, t^{(1)}-t_1+\eta, \tau-t_1+\eta)\, d\zeta^{(j)} d\eta$,

$\varphi_j(x^{(j)}, t, \tau)=u(x, t, \tau)|_{x_j=0}$, $x_q \geqslant 0$, $q=1, \ldots, j-1, j+1, \ldots, n$, $\varphi_j(x^{(j)}, t, \tau)$ is even with respect to each of the components $x^{(j)}$ relative to zero, $t^{(1)}-t_1+\eta=(t_2-t_1+\eta, \ldots, t_m-t_1+\eta)$, $\tau-t_1+\eta=(\tau_1-t_1+\eta, \ldots, \tau_p-t_1+\eta)$.

We note that $u_j(x, t, \tau) \to 0$ for $x_q \to 0$, $q = 1,\ldots,j-1, j+1,\ldots,n$, $u_{x_j} = u_{jx_j}$ for $x_j = 0$, $j = 1,\ldots,n$.

We obtain estimates for the conditional stability of the solution of the problem (2)-(4). Let δ be a parameter, defining uniquely a line ℓ, parallel to the vector **1**. We denote

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$$\|u\|^2_{L_2}(x_j,\delta) = \int_l \underbrace{\int_0^\infty \dots \int_0^\infty}_{n-1} |u(x,t,\tau)|^2 dx^{(j)}\,dl. \tag{5}$$

LEMMA. Assume that

$$\|u_j\|_{L_2}(x_j,\delta) < \varepsilon, \quad \varepsilon > 0, \quad a_j > 0, \tag{6}$$

$$\|u_{x_j}\|_{L_2}(x_j = 0, \delta) < E, \quad E > 0 \tag{7}$$

hold uniformly relative to δ (or lines ℓ). Then

$$\|u_j\|_{L_2}(x_j,\delta) \leqslant (\sqrt{2}\,E)^{1-x_j/a_j}\varepsilon^{x_j/a_j}\left(a_j^{-1}\ln\frac{2E^2}{\varepsilon^2}\right)^{-1+x_j/a_j}(1+o(1)) \tag{8}$$

for $\varepsilon/E \to 0$, $0 < x_j < a_j$,

$$\|u\|_{L_2}(x_j = 0,\delta) \leqslant \sqrt{2}a_jE\left(\ln\frac{a_jE/\varepsilon}{\ln\frac{\sqrt{2}\,E}{\varepsilon}}\right)^{-1}\left(1+o\left(\left(\ln\frac{\sqrt{2}\,E/\varepsilon}{\ln\frac{\sqrt{2}\,E}{\varepsilon}}\right)^{-1}\right)\right) \tag{9}$$

for ε/E → 0; there exists C > 0 such that

$$\sup_{(x^{(j)},t,\tau)}|u_j(x,t,\tau)| \leqslant C\left(-a_j^{-1}\ln\left(\frac{\varepsilon}{\sqrt{2}\,Ea_j}\ln\frac{2E^2}{\varepsilon^2}\right)\right)^{n+1}\left(a_j^{-1}\ln\frac{2E^2}{\varepsilon^2}\right)^{-1}\left(\frac{\varepsilon}{\sqrt{2}\,Ea_j}\ln\frac{2E^2}{\varepsilon^2}\right)^{\frac{x_j}{a_j}+\left(\ln\left(\frac{\varepsilon}{Ea_j\sqrt{2}}\ln\frac{2E^2}{\varepsilon^2}\right)\right)^{-1}}(1+o(1)) \tag{10}$$

for $\varepsilon/E \to 0$, $x_j > -a_j\left(\ln\left(\frac{\varepsilon}{\sqrt{2}\,Ea_j}\ln\frac{2E^2}{\varepsilon^2}\right)\right)^{-1}$.

The estimates (8) and (9) are uniform relative to δ (or lines ℓ).

Proof. For each fixed parameter δ, we can assume that $u_j(x, t, \tau)$ depends only on x and t_1. Let $\hat{u}_j(x_j, \xi^{(j)}, s, \delta)$ be the Fourier transform of the function $u_j(x, t, \tau)$ with respect to $x^{(j)}$ and t_1 for a fixed δ. By the theorem on the convolution of two functions, from (5) we obtain

$$\hat{u}_j(x_j,\xi^{(j)},s,\delta) = \exp(-x_j\sqrt{|\xi^{(j)}|^2 - is})\,\hat{\varphi}_j(\xi^{(j)},s,\delta),$$

where $\exp\left(-x_j\sqrt{|\xi^{(j)}|^2 - is}\right) = \exp\left(-\frac{x_j}{\sqrt{2}}\sqrt{|\xi^{(j)}|^2+\sqrt{|\xi^{(j)}|^4+s^2}} + ix_js\left(2|\xi^{(j)}|^2+2\sqrt{|\xi^{(j)}|^4+s^2}\right)^{-1/2}\right)$, i is the imaginary unit. According to Parseval's equality, we have

$$\|u_{x_j}\|^2_{L_2}(x_j = 0,\delta) = 2^{1-n}\int_{\mathbf{R}^n}(|\xi^{(j)}|^4+s^2)^{1/2}|\hat{\varphi}_j(\xi^{(j)},s,\delta)|^2 d\xi^{(j)}ds,$$

$$\|u_j\|^2_{L_2}(x_j,\delta) = 2^{1-n}\int_{\mathbf{R}^n}\exp\left(-x_j\sqrt{2|\xi^{(j)}|^2+2\sqrt{|\xi^{(j)}|^4+s^2}}\right)|\hat{\varphi}_j(\xi^{(j)},s,\delta)|^2 d\xi^{(j)}ds.$$

In the sequel we need a function of the form (see [1, p. 13])

$$\mu = \Phi(r^{-2}\exp(-br)) = r^{-2}\exp(-b_1r), \quad 0 < b < b_1,$$

which has the following properties:

1) $\Phi(\lambda)/\lambda$ increases monotonially with respect to λ,

2) $\mu = \Phi(\lambda)$ is a convex function,

3) $\lambda = \Phi^{-1}(\mu) = \mu^{b/b_1}\left(b_1^{-1}\ln\frac{1}{\mu}\right)^{2\left(-1+\frac{b}{b_1}\right)}(1+o(1))$ when μ → 0.

We denote $r = \sqrt{2|\xi^{(j)}|^2+2\sqrt{|\xi^{(j)}|^4+s^2}}$, $b = \sqrt{2}\,x_j$, $b_1 = \sqrt{2}\,a_j$, $\lambda = r^{-2}\exp(-br)$, $\mu = r^{-2}\exp(-b_1r)$. Then

$$\|u_j\|^2_{L_2}(x_j,\delta) = \int_{\mathbf{R}^n}\exp(-br)|\hat{\varphi}_j(\xi^{(j)},s,\delta)|^2 d\xi^{(j)}ds.$$

By virtue of Jensen's inequality and condition (6), we have

$$\Phi\left(\|u_j\|^2_{L_2}(x_j,\delta)\Big/\int_{\mathbf{R}^n} r^2|\widehat{\varphi}_j(\xi^{(j)},s,\delta)|^2 d\xi^{(j)}ds\right) \leqslant \int_{\mathbf{R}^n}\Phi(\lambda) r^2|\widehat{\varphi}_j(\xi^{(j)},s,\delta)|^2 d\xi^{(j)}ds\Big/\int_{\mathbf{R}^n} r^2|\widehat{\varphi}_j(\xi^{(j)},s,\delta)|^2 d\xi^{(j)}ds =$$

$$= \|u_j\|^2_{L_2}(a_j,\delta)\Big/\int_{\mathbf{R}^n} r^2|\widehat{\varphi}_j(\xi^{(j)},s,\delta)|^2 d\xi^{(j)}ds \leqslant \varepsilon^2\Big/\int_{\mathbf{R}^n} r^2|\widehat{\varphi}_j(\xi^{(j)},s,\delta)|^2 d\xi^{(j)}ds. \tag{11}$$

From (7) there follows the estimate

$$\int_{\mathbf{R}^n} r^2|\widehat{\varphi}_j(\xi^{(j)},s,\delta)|^2 d\xi^{(j)}ds < 2E^2.$$

Then from the properties 1, 3 of the function $\mu = \Phi(\lambda)$ and the inequality (11) it is easy to derive the desired estimate (8). The estimate (9) is proved in a similar manner. In this case one makes use of a function of the form (see [1, p. 8]) $\mu = \Phi(\lambda) = \lambda\exp(-b_1\sqrt{2/\lambda})$, $b_1 > 0$. The estimate (10) is a consequence of the estimate (8).

With the aid of the lemma one proves the following

THEOREM. Let $u(x,t,\tau)|_{x_j=0} \geqslant 0$, $\|u\|_{L_2}(a_j,\delta) < \varepsilon$, $\|u_{x_j}\|_{L_2}(x_j=0,\delta) < E$, $\varepsilon > 0$, $E > 0$, $j = 1,\ldots,n$, uniformly with respect to δ. Then

$$\|u\|_{L_2}(x_j\ \delta) \leqslant (\sqrt{2}E)^{1-\frac{x_j}{a_j}}\varepsilon^{\frac{x_j}{a_j}}\left(a_j^{-1}\ln\frac{2E^2}{\varepsilon^2}\right)^{-1+\frac{x_j}{a_j}}(1+o(1))$$

for $\varepsilon/E \to 0$, $0 < x_j < a_j$, $j = 1,\ldots,n$,

$$\|u\|_{L_2}(x_j=0,\delta) \leqslant \sqrt{2}\,a_jE\left(\ln\frac{a_jE/\varepsilon}{\ln\frac{\sqrt{2}E}{\varepsilon}}\right)^{-1}\left(1+o\left(\left(\ln\frac{\sqrt{2}E/\varepsilon}{\ln\frac{\sqrt{2}E}{\varepsilon}}\right)^{-1}\right)\right)$$

for $\varepsilon/E \to 0$, $j = 1,\ldots,n$; there exists $C > 0$ such that

$$\sup_{(x^{(j)},t,\tau)}|u(x,t,\tau)| \leqslant C\left(-a_j^{-1}\ln\left(\frac{\varepsilon}{\sqrt{2}Ea_j}\ln\frac{2E^2}{\varepsilon^2}\right)\right)^{n+1}\left(a_j^{-1}\ln\frac{2E^2}{\varepsilon^2}\right)^{-1}\left(\frac{\varepsilon}{\sqrt{2}Ea_j}\ln\frac{2E^2}{\varepsilon^2}\right)^{\frac{x_j}{a_j}+\left(\ln\left(\frac{\varepsilon}{\sqrt{2}Ea_j}\ln\frac{2E^2}{\varepsilon^2}\right)\right)^{-1}}(1+o(1))$$

for $\varepsilon/E \to 0$, $j = 1,\ldots,n$, $x_j > -a_j\left(\ln\left(\frac{\varepsilon}{\sqrt{2}Ea_j}\ln\frac{2E^2}{\varepsilon^2}\right)\right)^{-1}$. (The estimates are uniform with respect to δ.)

We give an explicit representation of an approximate solution $u(x, t, \tau)$ of the problem (2)-(4) in the case when $a_j = 0$, $j = 1,\ldots,k-1,k+1,\ldots,n$, $a_k > 0$, $1 \leqslant k \leqslant n$. Assume that with some error $\varepsilon > 0$ we know the values $f_j(x^{(j)},t,\tau)$ of the exact solution $u_T(x, t, \tau)$ for $x_j = a_j$, $j = 1,\ldots,n$:

$$\int_l\underbrace{\int_0^\infty\ldots\int_0^\infty}_{n-1}|u_T(x,t,\tau)|_{x_j=a_j} - f_j(x^{(j)},t,\tau)|^2 dx^{(j)}dl < \varepsilon^2,\ \varepsilon > 0$$

uniformly with respect to the lines ℓ. We extend $f_j(x^{(j)}, t, \tau)$, $j = 1,\ldots,n$, x_q, $q = 1,\ldots, j-1, j+1,\ldots,n$, in an odd manner with respect to $x_q = 0$, and by zero with respect to t_q, $q = 1,\ldots,m$, in $\mathbf{R}_j^{n-1}\times\mathbf{R}^{m+p}$. We construct the solution in the form

$$u(x,t,\tau) = \sum_{j=1}^n u_j(x,t,\tau),$$

$$u_j(x,t,\tau) = \frac{x_j}{(4\pi)^{n/2}}\int_0^{t_1}\int_{\mathbf{R}_j^{n-1}}(t_1-\eta)^{-\frac{n}{2}-1}\exp\left(-\frac{x_j^2+|x^{(j)}-\zeta^{(j)}|^2}{4(t_1-\eta)}\right)f_j(\zeta^{(j)},\eta,t^{(1)}-t_1+\eta,\tau-t_1+\eta)\,d\zeta^{(j)}d\eta$$

for $j = 1,\ldots,k-1, k+1,\ldots,n$,

$$u_k(x,t,\tau) = \frac{x_k}{(4\pi)^{n/2}}\int_0^{t_1}\int_{\mathbf{R}_k^{n-1}}(t_1-\eta)^{-\frac{n}{2}-1}\exp\left(-\frac{x_k^2+|x^{(k)}-\zeta^{(k)}|^2}{4(t_1-\eta)}\right)\psi_k(\zeta^{(k)},\eta,t^{(1)}-t_1+\eta,\tau-t_1+\eta)\,d\zeta^{(k)}d\eta,$$

where $\psi_k(x^{(k)}, t, \tau)$ has to be determined. We set $\Phi_k(x^{(k)}, t, \tau) = f_k(x^{(k)}, t, \tau) - \sum_{\substack{j=1 \\ j \neq k}}^{n} u_j(x, t, \tau)|_{x_k = a_k}$. Assume that the inequality

$$\frac{2^{1-n}}{E^2} \int\limits_{\mathbf{R}^n} \frac{\sqrt{|\xi^{(k)}|^4 + s^2}\, |\Phi_k(\xi^{(k)}, s, \delta)|^2 d\xi^{(k)} ds}{\exp\left(-a_k \sqrt{2|\xi^{(k)}|^2 + 2\sqrt{|\xi^{(k)}|^4 + s^2}}\right) + \left(\frac{\varepsilon}{E}\right)^2 \sqrt{|\xi^{(k)}|^4 + s^2}} < 1$$

is satisfied uniformly with respect to δ. By the least square method we construct

$$\psi_k(x^{(k)}, t, \tau) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbf{R}^n} \frac{\Phi_k(\xi^{(k)}, s, \delta) \exp\left(-a_k \sqrt{|\xi^{(k)}|^2 - is} - i(\xi^{(k)}, x^{(k)}) - it_1 s\right) d\xi^{(k)} ds}{\exp\left(-a_k \sqrt{2|\xi^{(k)}|^2 + 2\sqrt{|\xi^{(k)}|^4 + s^2}}\right) + \left(\frac{\varepsilon}{E}\right)^2 \sqrt{|\xi^{(k)}|^4 + s^2}}.$$

It is easy to show that $u(x, t, \tau) = \sum_{j=1}^{n} u_j(x, t, \tau)$ satisfies Eq. (1), condition (2) and

$$\int\limits_{l} \underbrace{\int\limits_0^\infty \ldots \int\limits_0^\infty}_{n-1} |u_T(x, t, \tau) - u(x, t, \tau)|^2 dx^{(j)} dt < C(\varepsilon), \quad C(\varepsilon) \to 0, \ \varepsilon \to 0,$$

$$j = 1, \ldots, n.$$

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CARLEMAN'S FORMULA FOR FUNCTIONS OF MATRICES

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In the theory of functions of one complex variable, the classical Carleman formula [1] and its generalization obtained in [2, 3]

$$f(z) = \lim_{m \to \infty} \frac{1}{2\pi i} \int\limits_M \frac{f(\zeta)}{\zeta - z} \left[\frac{\varphi(\zeta)}{\varphi(z)}\right]^m d\zeta \tag{1}$$

are well known; here f(z) is a function of the Hardy class $H^1(D)$, D is a bounded, simply connected domain in $\mathbf{C}^1$ with a piecewise smooth boundary ∂D, the set $M \subset \partial M$ has positive one-dimensional Lebesgue measure, $\varphi(z) = \exp \psi(z)$, while $\psi(z)$ is holomorphic in D and is constructed with the aid of the solution of an appropriate Dirichlet problem for a harmonic function [2].

Multidimensional analogues of formula (1) are given in [4] (also there see the reference regarding this topic).

In this paper we obtain an analogue of formula (1) for functions of matrices.

Let $\tau_0 = \{w \in \mathbf{C}^{n^2}: ww^* < I\}$ be the "generalized unit circle," where w is a square $n \times n$ matrix, $w^* = \bar{w}^T$, I is the identity matrix, and the notation $ww^* < I$ means that the Hermitian matrix $I - ww^*$ is positive definite.

We denote by $G \subset \mathbf{C}^{n^2}$ a bounded set of $n \times n$ matrices w. Then, by Gershgorin's theorem [5], there exists some bounded, simply connected domain $D \subset \mathbf{C}^1$ with a piecewise smooth boundary, containing all the eigenvalues of $w \in G$.

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