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PERIODIC GALERKIN FINITE ELEMENT METHOD OF UNSTEADY PERIODIC FLOW OF VISCOUS FLUID

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SUMMARY

Finite element analysis of unsteady flow of two-dimensional incompressible viscous fluid is presented employing stream function as a field variable. The Galerkin approach is applied in the procedure of discretizing in time. Using trigonometric series as the interpolation function, the non-linear algebraic simultaneous equation system is derived. Several numerical examples show the adaptability of the formulation.

INTRODUCTION

In recent years, a number of finite element methods have been presented in the field of steady and unsteady flow analysis of viscous fluid. Huebner[1] reviewed the finite element methods of flow problems. Steady flow analysis considering non-linear convection has been discussed by Zienkiewicz,[2] Tong,[3,4] Oden and Wellford,[5] Taylor and Hood,[6,7] Olson,[8,9] Gartling and Becker,[10] Nickel *et al.*[11] Lieber *et al.*[12] and Kawahara *et al.*[13–15] In unsteady flow analysis, several numerical methods have been presented. Guymon,[16] Argyris and Mareczek,[17] Usuki and Kudo,[18] Baker[19–21] discussed the method taking velocity and pressure as unknown variables. Oden and Wellford[5] presented the numerical integration method, which is originated from the Runge–Kutta method. Kawahara *et al.*[22] employed the perturbation method for numerical time integration. Taylor and Hood[6] reported that the 'mid-difference method' in time is successful in both velocity-pressure and vorticity-stream function formulations. The research by Cheng[23] is the vorticity-stream function method discretized by the central-difference method in time. Bratanow, *et al.*[24–26] discussed vorticity-stream function formulation using the explicit difference method for integration with respect to time. Their numerical results include the flow at Reynolds number of 10^6 employing 10^{-6} units. Carlo and Piva[27] presented the mixed finite element method of thermal fluid flow.

In this paper, finite element analysis of two-dimensional unsteady flow of incompressible viscous fluid is presented taking stream function as an unknown variable. To consider the boundary condition of surface force, a variational equation is first formulated in terms of velocity and pressure. By introducing the definition equation of stream function into the velocity-pressure variational equation, the governing variational equation is derived. Assuming that the motion of flow is periodic, the variational Galerkin approach is applied to the integration with respect to time. In this case, trigonometric function series is used as the interpolation function in time. To solve the non-linear simultaneous equation system derived by the periodic Galerkin procedure, the Newton–Raphson method is applied. Several numerical

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results are discussed to show the validity of the formulation. A part of this research has been presented in Kawahara[28] and Kawahara and Kaneko.[29]

BASIC EQUATIONS

Throughout this paper, equations are described by using indicial notation and usual summation convention with repeated indices. For simplification, this paper restricts the analysis within two-dimensional flow of incompressible viscous fluid. Spatial description with rectangular co-ordinate system X_i $(i = 1, 2)$ is employed. Notation $(\ )_{,i}$ means partial differentiation with respect to X_i. Symbols δ_{ij} and ε_{ij} are Kronecker's delta and Edington's epsilon function respectively. With the use of conservation of linear momentum, the equation of motion:

$$\rho\frac{\partial u_i}{\partial t}+\rho u_j u_{i,j}-\tau_{ij,j}=\rho\hat{f}_i \tag{1}$$

is obtained, where u_i and $\hat{f}_i$ mean velocity and body force and ρ is density. The constitutive equation for stress τ_{ij} is written in the form as:

$$\tau_{ij}=-p\delta_{ij}+\mu(u_{i,j}+u_{j,i}) \tag{2}$$

where p denotes pressure and μ is viscosity coefficient. In the derivation of equation (2), incompressible and Newtonian fluid is assumed. The equation of continuity is expressed as:

$$u_{i,i}=0 \tag{3}$$

Regarding boundary condition, the velocity is assumed to be prescribed on boundary S_1, i.e.,

$$u_i=\hat{u}_i \qquad \text{on } S_1 \tag{4}$$

surface force S_i to be on boundary S_2, i.e.,

$$S_i=\tau_{ij}n_j=\hat{S}_i \qquad \text{on } S_2 \tag{5}$$

where superposed $\hat{}$ means prescribed values on the boundary and n_j are the components of the unit normals to the boundary surface. The boundaries S_1 and S_2 are assumed as:

$$\begin{aligned} S_1\cup S_2&=S\\ S_1\cap S_2&=\phi \end{aligned} \tag{6}$$

where S means the whole boundary surface of the flow field to be analysed and ϕ is the null set.

Let u_i^* be the weighting function, the value of which is arbitrary except on boundary S_1, where it takes the value zero. Multiplying both sides of equation (1) by u_i^*, integrating over the whole volume V and using Green's theorem, the following variational equation is obtained.

$$\int_V\left(\rho u_i^*\frac{\partial u_i}{\partial t}\right)\mathrm{d}V+\int_V(\rho u_i^* u_j u_{i,j})\,\mathrm{d}V+\int_V(u_{i,j}^*\tau_{ij})\,\mathrm{d}V=\int_V(\rho u_i^*\hat{f}_i)\,\mathrm{d}V+\int_{S_2}(u_i^*\tau_{ij}n_j)\,\mathrm{d}S \tag{7}$$

Introducing equation (2) into equation (7), using equation (5) and rearranging the terms, the variational equation can be transformed into the following form.

$$\int_V\left(\rho u_i^*\frac{\partial u_i}{\partial t}\right)\mathrm{d}V+\int_V(\rho u_i^* u_j u_{i,j})\,\mathrm{d}V-\int_V(u_{i,i}^* p)\,\mathrm{d}V+\int_V\mu(u_{i,j}^* u_{i,j})\,\mathrm{d}V+\int_V\mu(u_{i,j}^* u_{j,i})\,\mathrm{d}V=\hat{\Gamma} \tag{8}$$

where

$$\hat{\Gamma} = \int_V (\rho u_i^* \hat{f}_i)\, \mathrm{d}V + \int_{S_2} (u_i^* \hat{S}_i)\, \mathrm{d}S \tag{9}$$

The variational equation (8) considers the surface force condition on boundary S_2 as the natural boundary condition. Other boundary conditions can also be discussed by similar formulation. (*c.f.* Kawahara *et al.*[22]). By using equation (3), a variational form of equation of continuity is obtained as:

$$\int_V (p^* u_{i,i})\, \mathrm{d}V = 0 \tag{10}$$

where p^* is the weighting function.

FINITE ELEMENT ANALYSIS

A number of finite element procedures are available based on equations (8) and (10). If a velocity which automatically satisfies the equation of continuity is employed in the formulation, a numerically stable finite element method is obtained. For this purpose, it is convenient to use stream function as unknown variables. Let ψ be the stream function:

$$u_i = \varepsilon_{ij} \psi_{,j} \tag{11}$$

Substituting equation (11) into equations (8) and (9) and rearranging the terms, the variational equation is reformulated in the following form.

$$\int_V \rho(\varepsilon_{ik}\varepsilon_{il}\psi^*_{,k}\dot{\psi}_{,l})\, \mathrm{d}V + \int_V \rho(\varepsilon_{ik}\varepsilon_{jl}\varepsilon_{im}\psi^*_{,k}\psi_{,l}\psi_{,mj})\, \mathrm{d}V$$

$$+ \int_V \mu\varepsilon_{ik}\varepsilon_{il}(\psi^*_{,kj}\psi_{,lj})\, \mathrm{d}V + \int_V \mu\varepsilon_{ik}\varepsilon_{jl}(\psi^*_{,kj}\psi_{,li})\, \mathrm{d}V = \hat{\Gamma} \tag{12}$$

where

$$\hat{\Gamma} = \int_V \rho(\varepsilon_{ik}\psi^*_{,k}\hat{f}_i)\, \mathrm{d}V + \int_{S_2} (\varepsilon_{ik}\psi^*_{,k}\hat{S}_i)\, \mathrm{d}S \tag{13}$$

Assume that the flow field of interest is divided into small regions called finite elements. The approximate representation of both trial and weighting functions of stream function is expressed by the form:

$$\psi = \Phi_\alpha \psi_\alpha \tag{14}$$

$$\psi^* = \Phi_\alpha \psi^*_\alpha \tag{15}$$

where Φ_α denotes interpolation function and ψ_α and ψ^*_α are nodal values of the stream function and corresponding weighting function of the flow field respectively. Substituting equations (14) and (15) into equations (12) and (13) and considering the arbitrariness of ψ^*_α yields a set of finite element governing equations:

$$M_{\alpha\beta}\frac{\partial \psi_\beta}{\partial t} + K_{\alpha\beta\gamma}\psi_\beta\psi_\gamma + S_{\alpha\beta}\psi_\beta = \hat{\Omega}_\alpha \tag{16}$$

where

$$M_{\alpha\beta}=\int_V \rho(\varepsilon_{ik}\varepsilon_{il}\Phi_{\alpha,k}\Phi_{\beta,l})\,\mathrm{d}V$$

$$K_{\alpha\beta\gamma}=\int_V \rho(\varepsilon_{ik}\varepsilon_{jl}\varepsilon_{im}\Phi_{\alpha,k}\Phi_{\beta,l}\Phi_{\gamma,mj})\,\mathrm{d}V$$

$$S_{\alpha\beta}=\int_V \mu(\varepsilon_{ik}\varepsilon_{il}\Phi_{\alpha,kj}\Phi_{\beta,lj})\,\mathrm{d}V+\int_V \mu(\varepsilon_{ik}\varepsilon_{jl}\Phi_{\alpha,kj}\Phi_{\beta,li})\,\mathrm{d}V$$

$$\hat{\Omega}_\alpha=\int_V \rho(\varepsilon_{ik}\Phi_{\alpha,k}\hat{f}_i)\,\mathrm{d}V+\int_{S_2}(\varepsilon_{ik}\Phi_{\alpha,k}\hat{S}_i)\,\mathrm{d}S$$

Superposing equation (16) of each finite element for the whole flow field leads to the governing equation in the following form.

$$F_\alpha\equiv M_{\alpha\beta}\frac{\partial\psi_\beta}{\partial t}+K_{\alpha\beta\gamma}\psi_\beta\psi_\gamma+S_{\alpha\beta}\psi_\beta-\hat{\Omega}_\alpha=0 \tag{17}$$

For shape function, incomplete polynomial of the third degree is employed. The function is based on the polynomial function:

$$\begin{aligned}\psi&=a_1+a_2x+a_3y+a_4x^2+a_5xy+a_6y^2\\&\quad+a_7x^3+a_8x^2y+a_9xy^2+a_{10}y^3\end{aligned} \tag{18}$$

where a_1–a_{10} are real constants, which are determined by the nodal value ψ_α:

$$\psi_\alpha=[\psi_1,\psi_{1,x},\psi_{1,y},\psi_2,\psi_{2,x},\psi_{2,y},\psi_3,\psi_{3,x},\psi_{3,y}] \tag{19}$$

and an additional compatibility condition.

PERIODIC GALERKIN METHOD

It is necessary to introduce a discretization procedure in time to solve the governing equation (17). Finite difference methods or finite element methods based on polynomials of time co-ordinate are commonly used in the conventional analysis. In this paper, the Galerkin approach in time is employed, which is called the periodic Galerkin method since the method is based on the fact that the motion of flow is periodic. The method was originally introduced by Urabe.[30,31] It is assumed that the given function $\hat{\Omega}_\alpha$ is expressed by the sum of trigonometric function series having period ω as follows.

$$\hat{\Omega}_\alpha=\hat{\Omega}_\alpha^{(0)}+\sum_{n=1}^{N}\hat{\Omega}_{\alpha s}^{(n)}\sin(n\omega t)+\sum_{m=1}^{N}\hat{\Omega}_{\alpha c}^{(m)}\cos(m\omega t) \tag{20}$$

On the basis of equation (17), the variational equation in time can be written in the following form.

$$\int_0^{2\pi/\omega}\left(\psi_\alpha^*M_{\alpha\beta}\frac{\partial\psi_\beta}{\partial t}\right)\mathrm{d}t+\int_0^{2\pi/\omega}(\psi_\alpha^*K_{\alpha\beta\gamma}\psi_\beta\psi_\gamma)\,\mathrm{d}t+\int_0^{2\pi/\omega}(\psi_\alpha^*S_{\alpha\beta}\psi_\beta)\,\mathrm{d}t-\int_0^{2\pi/\omega}(\psi_\alpha^*\hat{\Omega}_\alpha)\,\mathrm{d}t=0 \tag{21}$$

The trial function is assumed to be expressed by:

$$\psi_\beta = \psi_\beta^{(0)} + \sum_{m=1}^{N} a_\beta^{(m)} \sin m\omega t + \sum_{n=1}^{N} b_\beta^{(n)} \cos n\omega t \tag{22}$$

where $\psi_\beta^{(0)}, a_\beta^{(1)}, a_\beta^{(2)}, \ldots, a_\beta^{(N)}, b_\beta^{(1)}, b_\beta^{(2)}, \ldots, b_\beta^{(N)}$ are the unknown vectors to be determined. In a similar manner, the weighting function is put in the form as:

$$\psi_\beta^* = \psi_\beta^{*(0)} + \sum_{k=1}^{N} a_\beta^{*(k)} \sin k\omega t + \sum_{l=1}^{N} b_\beta^{*(1)} \cos l\omega t \tag{23}$$

Substituting equation (23) into equation (21), rearranging the terms and using the arbitrariness of the weighting vectors $\psi_\beta^{*(0)}, a_\beta^{*(1)}, a_\beta^{*(2)}, \ldots, a_\beta^{*(N)}, b_\beta^{*(1)}, b_\beta^{*(2)}, \ldots, b^{*(N)}$ lead to the following weighted residual equations.

$$\int_0^{2\pi/\omega} F_\alpha(\psi_\beta^{(0)})\, dt = 0 \tag{24}$$

$$\int_0^{2\pi/\omega} \sin k\omega t \,.\, F_\alpha(\psi_\beta^{(0)}, a_\beta^{(m)}, b_\beta^{(n)})\, dt = 0,\ k = 1, 2, \ldots, N \tag{25}$$

$$\int_0^{2\pi/\omega} \cos l\omega t \,.\, F_\alpha(\psi_\beta^{(0)}, a_\beta^{(m)}, b_\beta^{(n)})\, dt = 0,\ l = 1, 2, \ldots, N \tag{26}$$

Using equation (22) with equations (24)–(26) and integrating term by term, the discretized non-linear simultaneous equation system for $m, n = 1, 2, \ldots, N$ can be derived as follows. From equation (24), the steady solution is obtained as:

$$K_{\alpha\beta\gamma}\psi_\beta^{(0)}\psi_\gamma^{(0)} + S_{\alpha\beta}\psi_\beta^{(0)} = \hat{\Omega}_\alpha^{(0)} \tag{27}$$

From equations (25) and (26),

$$\begin{aligned}
&-\omega M_{\alpha\beta} n b_\beta^{(n)} \delta_{nk} + S_{\alpha\beta} a_\beta^{(m)} \delta_{mk} + K_{\alpha\beta\gamma}[\psi_\beta^{(0)} a_\gamma^{(i)} \delta_{ik} + a_\beta^{(m)} \psi_\gamma^{(0)} \delta_{mk} \\
&\quad + \tfrac{1}{2} a_\beta^{(m)} b_\gamma^{(j)} \delta_{(m+j)k} + \tfrac{1}{2} a_\beta^{(m)} b_\gamma^{(j)} \delta_{(m-j)k} - \tfrac{1}{2} a_\beta^{(m)} b_\gamma^{(j)} \delta_{(j-m)k} \\
&\quad + \tfrac{1}{2} b_\beta^{(n)} b_\gamma^{(i)} \delta_{(n+i)k} + \tfrac{1}{2} b_\beta^{(n)} a_\gamma^{(j)} \delta_{(j-n)k} - \tfrac{1}{2} b_\beta^{(n)} a_\gamma^{(j)} \delta_{(n-j)k}] \\
&\quad - \hat{\Omega}_{\alpha s}^{(m)} \delta_{mk} = 0 \qquad (k = 1, 2, \ldots, N)
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\omega M_{\alpha\beta} m a_\beta^{(m)} \delta_{ml} + S_{\alpha\beta} b_\beta^{(n)} \delta_{nl} + K_{\alpha\beta\gamma}[\psi_\beta^{(0)} b_\gamma^{(j)} \delta_{jl} + b_\beta^{(n)} \psi_\gamma^{(0)} \delta_{nl} \\
&\quad + \tfrac{1}{2} a_\beta^{(m)} a_\gamma^{(j)} \delta_{(j-m)l} + \tfrac{1}{2} a_\beta^{(m)} a_\gamma^{(j)} \delta_{(m-j)l} - \tfrac{1}{2} a_\beta^{(m)} a_\gamma^{(j)} \delta_{(m+j)l} \\
&\quad + \tfrac{1}{2} b_\beta^{(n)} b_\gamma^{(j)} \delta_{(n+j)l} + \tfrac{1}{2} b_\beta^{(n)} b_\gamma^{(j)} \delta_{(j-n)l} + \tfrac{1}{2} b_\beta^{(n)} b_\gamma^{(j)} \delta_{(n-j)l}] \\
&\quad - \hat{\Omega}_{\alpha c}^{(n)} \delta_{nl} = 0 \qquad (l = 1, 2, \ldots, N)
\end{aligned} \tag{29}$$

For example, if N is taken to be 1, then the following two equations are derived.

$$-\omega M_{\alpha\beta} b_\beta^{(1)} + S_{\alpha\beta} a_\beta^{(1)} + K_{\alpha\beta\gamma}\psi_\beta^{(0)} a_\gamma^{(1)} + K_{\alpha\beta\gamma} a_\beta^{(1)} \psi_\gamma^{(0)} - \hat{\Omega}_{\alpha s}^{(1)} = 0 \tag{30}$$

$$\omega M_{\alpha\beta} a_\beta^{(1)} + S_{\alpha\beta} b_\beta^{(1)} + K_{\alpha\beta\gamma}\psi_\beta^{(0)} b_\gamma^{(1)} + K_{\alpha\beta\gamma} b_\beta^{(1)} \psi_\gamma^{(0)} - \hat{\Omega}_{\alpha c}^{(1)} = 0 \tag{31}$$

Also, if N is taken to be 2, then the following four equations are obtained.

$$\begin{aligned}
&-\omega M_{\alpha\beta} b_\beta^{(1)} + S_{\alpha\beta} a_\beta^{(1)} + K_{\alpha\beta\gamma}[\psi_\beta^{(0)} a_\gamma^{(1)} + a_\beta^{(1)} \psi_\gamma^{(0)} + \tfrac{1}{2} a_\beta^{(2)} b_\gamma^{(1)} - \tfrac{1}{2} a_\beta^{(1)} b_\gamma^{(2)} \\
&\quad + \tfrac{1}{2} b_\beta^{(1)} a_\gamma^{(2)} - \tfrac{1}{2} b_\beta^{(2)} a_\gamma^{(1)}] - \hat{\Omega}_{\alpha s}^{(1)} = 0
\end{aligned} \tag{32}$$

$$-2\omega M_{\alpha\beta}b_{\beta}^{(2)}+S_{\alpha\beta}a_{\beta}^{(2)}+K_{\alpha\beta\gamma}[\psi_{\beta}^{(0)}a_{\gamma}^{(2)}+a_{\beta}^{(2)}\psi_{\gamma}^{(0)}+\tfrac{1}{2}a_{\beta}^{(1)}b_{\gamma}^{(1)}+\tfrac{1}{2}b_{\beta}^{(1)}a_{\gamma}^{(1)}]$$
$$-\hat{\Omega}_{\alpha s}^{(2)}=0 \tag{33}$$

$$\omega M_{\alpha\beta}a_{\beta}^{(1)}+S_{\alpha\beta}b_{\beta}^{(1)}+K_{\alpha\beta\gamma}[\psi_{\beta}^{(0)}b_{\gamma}^{(1)}+b_{\beta}^{(1)}\psi_{\gamma}^{(0)}+\tfrac{1}{2}a_{\beta}^{(1)}a_{\gamma}^{(2)}+\tfrac{1}{2}a_{\beta}^{(2)}a_{\gamma}^{(1)}$$
$$+\tfrac{1}{2}b_{\beta}^{(1)}b_{\gamma}^{(2)}+\tfrac{1}{2}b_{\beta}^{(2)}b_{\gamma}^{(1)}]-\hat{\Omega}_{\alpha c}^{(1)}=0 \tag{34}$$

$$2\omega M_{\alpha\beta}a_{\beta}^{(2)}+S_{\alpha\beta}b_{\beta}^{(2)}+K_{\alpha\beta\gamma}[\psi_{\beta}^{(0)}b_{\gamma}^{(2)}+b_{\beta}^{(2)}\psi_{\gamma}^{(0)}-\tfrac{1}{2}a_{\beta}^{(1)}a_{\gamma}^{(1)}+\tfrac{1}{2}b_{\beta}^{(1)}b_{\gamma}^{(1)}]$$
$$-\hat{\Omega}_{\alpha c}^{(2)}=0 \tag{35}$$

Equations (28) and (29) are a non-linear simultaneous equation system. To solve the non-linear equations, the Newton–Raphson method was employed in the numerical examples shown in this paper.

NUMERICAL EXAMPLES

To illustrate the adaptability of the present periodic Galerkin finite element method, several numerical examples are given. Figures 1–7 show the numerical results of flow in a cavity. Reynolds numbers are shown in the figures by $R_e = du_0/\nu$ where d and u_0 are the fundamental units of length and velocity respectively and ν is dynamic viscosity μ/ρ. The hatched boundary in the figures indicates rigid wall on which both components of velocity are taken to be zero.

The first example is the computation of the steady flow comparing the results obtained by the present finite element method and by the finite difference method of Marshall and Van Spiegel.[32] Figure 1 shows the computed stream function of steady flow at Reynolds number of 400. Solid contour lines illustrate the computed stream function by the present finite element method and dotted lines by the finite difference method. The boundary conditions used in this computation are as follows:

$$\begin{array}{llll} A-B: & \psi=0, & \psi_{,x}=0, & \psi_{,y}=0 \\ B-C: & \psi=0, & \psi_{,x}=0, & \psi_{,y}=0 \\ C-D: & \psi=0, & \psi_{,x}=0 & \psi_{,y}=0 \\ D-A: & \psi=0, & \psi_{,x}=0, & \psi_{,y}=-u_0 \end{array}$$

The computed stream function by the present finite element method using only 52 nodal points (i.e., 156 degrees-of-freedom) are comparable to the result obtained by the finite difference method, in which 40×40 mesh points have been used.

The second example is the comparison between the computed results obtained by the governing equations taking $N=1$, $N=2$ and $N=3$. Figure 2 shows the finite element idealization, the computed stream function and velocity at Reynolds number of 500. Arrowed lines show the computed velocity. Solid contour lines are the computed stream function. These stream lines are calculated by the interpolation relation, i.e., equation (14), using the computed nodal values of stream function. Figures 3–5 show the computed velocities at nodal number of 18 versus time calculated by the governing equations of $N=1$, N = 2 and $N=3$. The computed velocities in Figures 3–5 are the results of the case where $\hat{u}_0$ is chosen to be:

$$\hat{u}_0=-500+100\sin t$$
$$\hat{u}_0=100+50\sin t$$
$$\hat{u}_0=-100+100\sin t$$

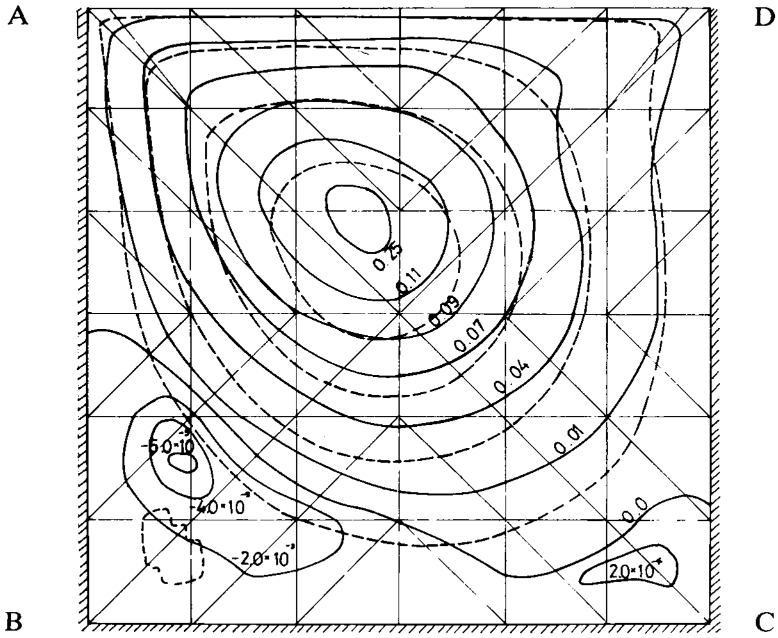

Figure 1. Computed stream function at Reynolds number of 400

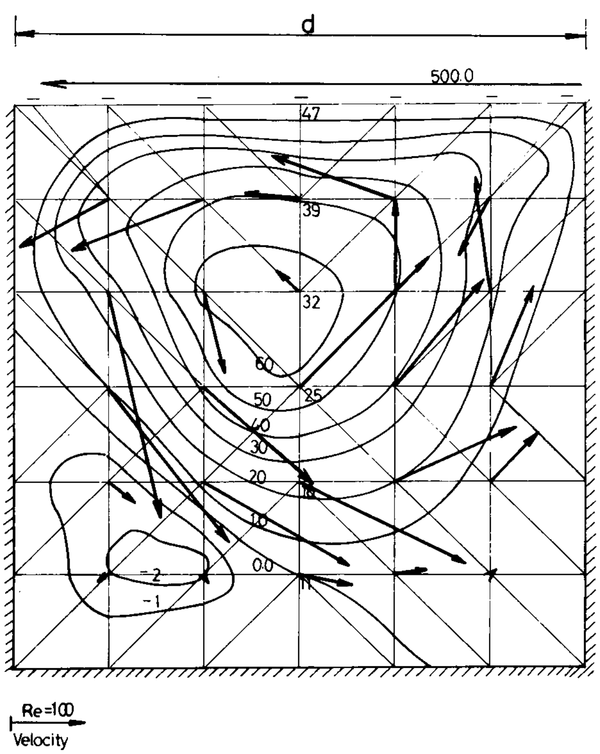

Figure 2. Computed steady flow stream function and velocity

Comparing the computed values in these three figures, it is observed that the computed velocities by the governing equations $N=2$ and $N=3$ resulted in almost the same values.

The third example is the comparison of the numerical results obtained by the periodic Galerkin method with the results by the conventional implicit method for time integration. During short time increment Δt, assume that

$$\frac{\partial \psi_\beta}{\partial t} \fallingdotseq \frac{1}{\Delta t}(\psi_\beta^{(n+1)} - \psi_\beta^{(n)})$$

$$K_{\alpha\beta\gamma}\psi_\beta\psi_\gamma \fallingdotseq \tfrac{1}{2}K_{\alpha\beta\gamma}[\{(1-\theta)\psi_\beta^{(n+1)} + \theta\psi_\beta^{(n)}\}\psi_\gamma^{(n)} + \{1-\theta)\psi_\gamma^{(n+1)} + \theta\psi_\gamma^{(n)}\}]\psi_\beta^{(n)}$$

$$S_{\alpha\beta}\psi_\beta \fallingdotseq S_{\alpha\beta}\{(1-\theta)\psi_\beta^{(n+1)} + \theta\psi_\beta^{(n)}\}$$

where $\psi_\beta^{(n)}$ is the value of ψ_β at $(n+1)$th time point, and θ is a constant $0 \leqslant \theta < 1$. Introducing these results into equation (17) and arranging the terms, the following implicit method is derived.

$$\left[\frac{1}{\Delta t}M_{\alpha\beta} + \frac{1-\theta}{2}(K_{\alpha\beta\gamma} + K_{\alpha\gamma\beta})\psi_\gamma^{(n)} + S_{\alpha\beta}\right]\psi_\beta^{(n+1)} = \Omega_\alpha^{(n)} + \frac{1}{\Delta t}M_{\alpha\beta}\psi_\beta^{(n)} - \theta K_{\alpha\beta\gamma}\psi_\beta^{(n)}\psi_\gamma^{(n)} - \theta S_{\alpha\beta}\psi_\beta^{(n)}$$

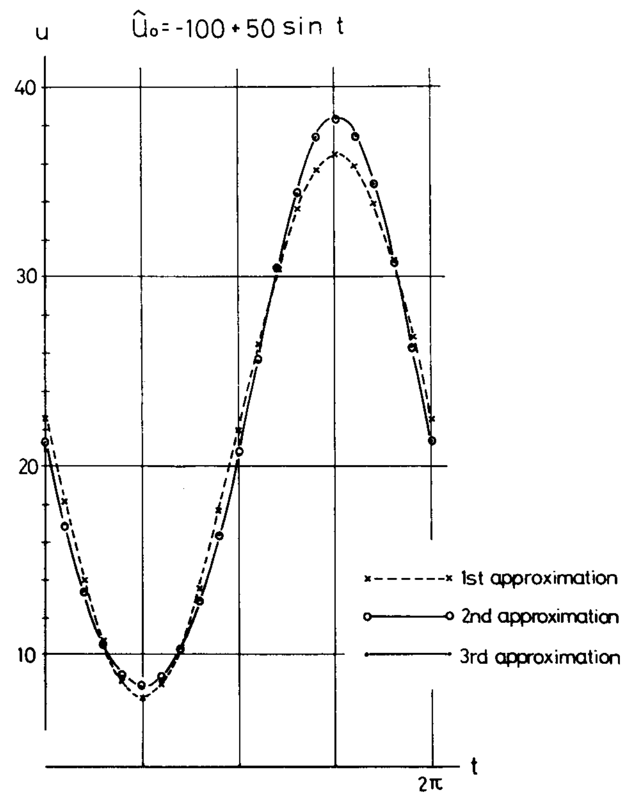

Figure 3. Computed velocities at nodal number 18

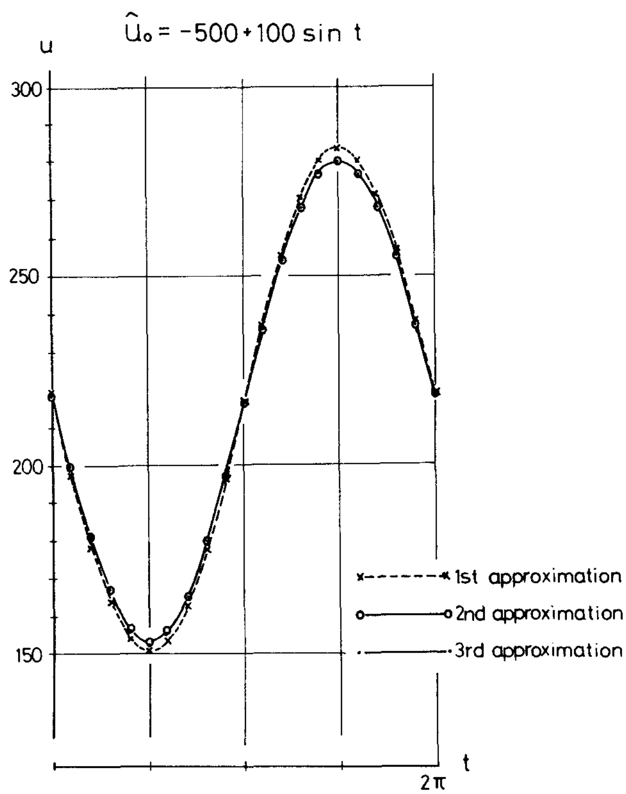

Figure 4. Computed velocities at nodal number 18

Figure 6 is the comparison of velocities obtained by the present method with the results by the conventional implicit method with $\theta = 0{\cdot}5$ at time $t = \pi$. The given function $\hat{u}_0$ is chosen to be:

$$\hat{u}_0 = -20 + 10 \sin t$$

The results computed by both the methods are well in agreement. Computed velocities versus time diagrams at nodes numbered 11, 18 and 39 are illustrated in Figure 7.

CONCLUSION

In this paper, an unsteady two-dimensional flow analysis of incompressible viscous fluid by the finite element method assuming the motion of flow is periodic has been presented. The method is characterized by the following.

(i) The method is based on the variational equation in which stream function is taken as a field variable. In the formulation, the condition of surface force on the boundary is considered.

(ii) Polynomials of the third degree are employed for the interpolation function of stream function. The numerical results by the present method are comparable to the results by the finite difference method in the steady flow analysis. The total numbers of degrees-of-freedom in the computation of the present method is substantially less than that of the finite difference method.

(iii) In the unsteady flow analysis, the periodic Galerkin method is employed. As the interpolation function approximating time, trigonometric function series is used to save the

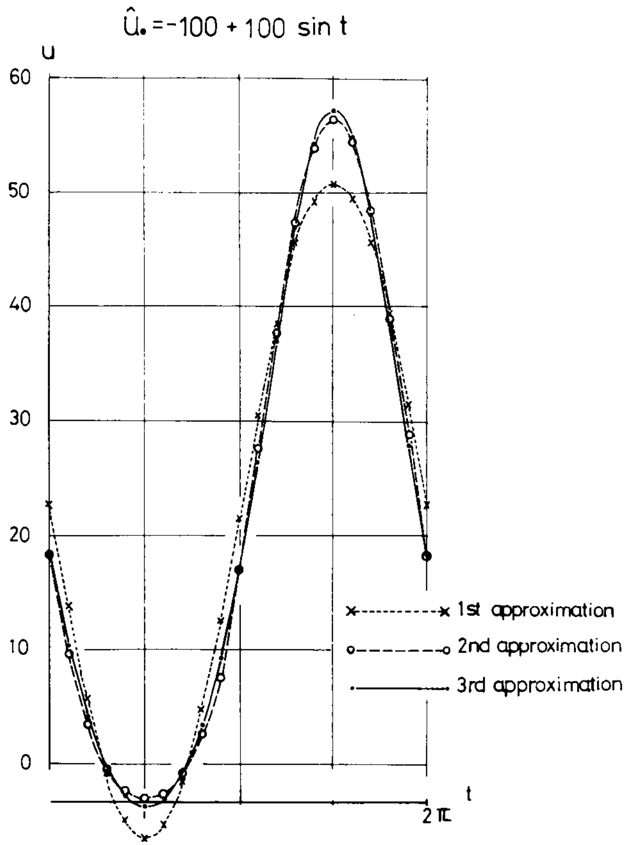

Figure 5. Computed velocity at nodal number 18

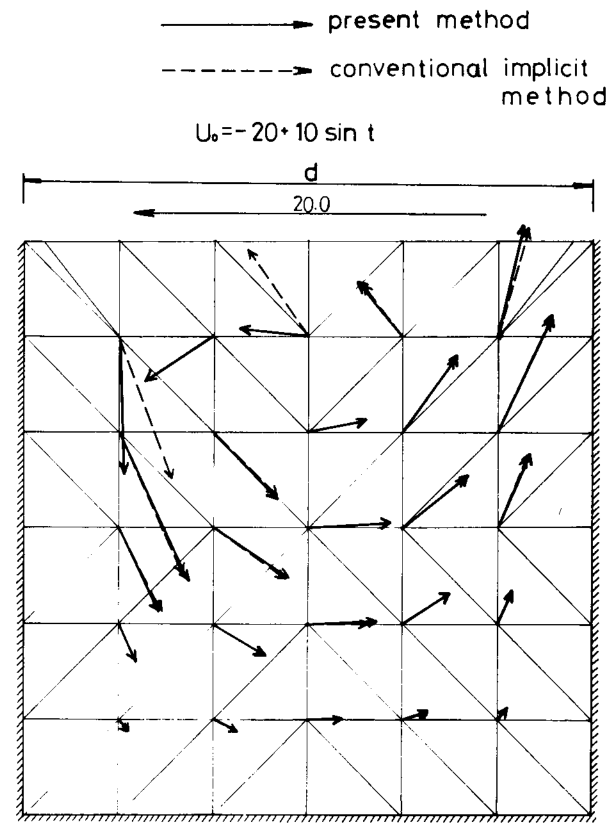

Figure 6. Computed velocity by the present method and conventional implicit method

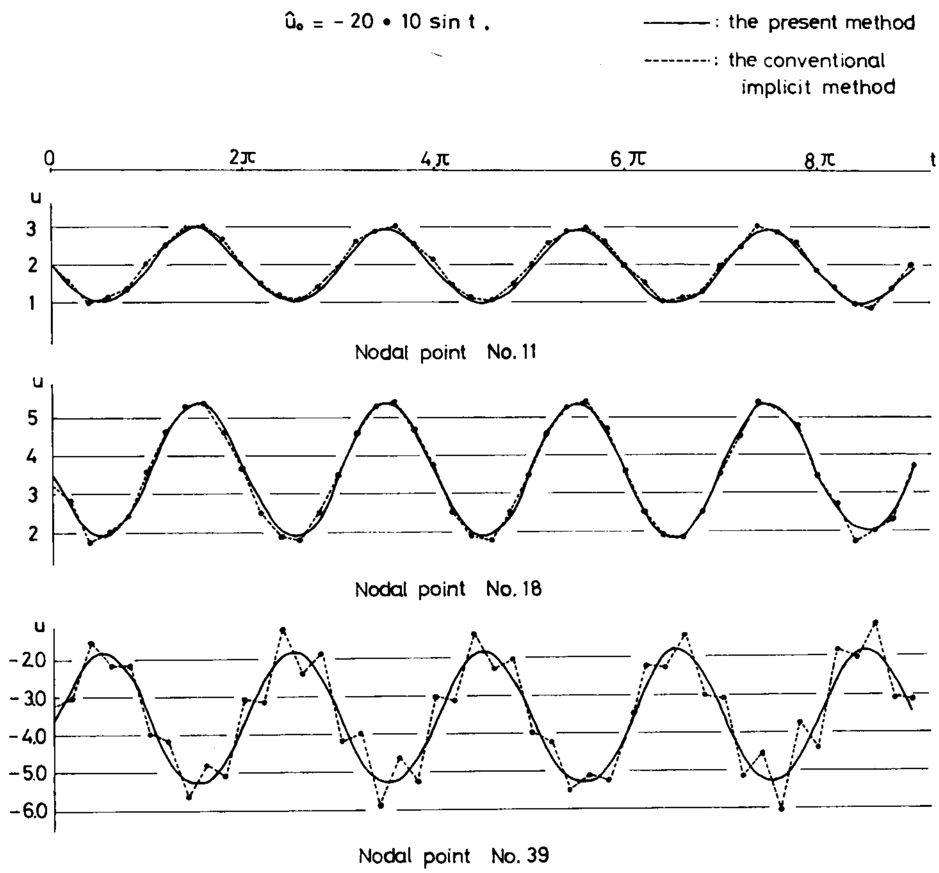

Figure 7. The computed velocity by the present method and the conventional implicit method

computational time. The numerical results obtained by the present method are well in agreement with the results by the conventional implicit method.

In the selection of variational equation, the present method employed the variational formulation in terms of velocity and pressure as the basic equation. By introducing the definition equation of stream function into this variational equation, finite element basis is derived. Because the discretization procedure of velocity which automatically satisfies the equation of continuity is employed, a numerically stable finite element method is obtained.

To solve a periodic flow problem, the implicit difference in time based on step by step procedure is commonly used in the conventional finite element analysis. The present method is characterized by the fact that a periodic flow problem is solved by similar means to a steady flow problem. The final governing equation of this method results in a large scale, but independent of time, non-linear simultaneous equation system. The method need not require the computation to continue until a certain stationary state is obtained.

To solve the non-linear simultaneous equation system, the Newton–Raphson method is employed. In the author's numerical experience, the computation of the flow at Reynolds number 1,000 has already been performed in a steady flow analysis and of the flow at Reynolds number 700 in a periodic flow analysis.

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