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CONSTRUCTIVELY COMPLETE FINITE SETS

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1. Introduction

For the constructive study of metric spaces, it is helpful to investigate each classical proposition for numerical content. Classically, any finite subset of a metric space is complete, but this is not constructively true. Here we study this problem on the real line. To identify precisely the nonconstructivities involved requires a splitting of Markov's Principle into two separate principles. One of these has arisen previously in connection with constructive continuity problems [5].

We follow here the strict constructive approach of ERRETT BISHOP [1], a discussion of which may be found in [7].

The nonconstructivities of the classical result that every finite subset of a metric space is complete are sufficiently well demonstrated by considering two-element sets of real numbers. In this context, two questions arise: *Does every two-valued Cauchy sequence of real numbers converge to one of its values? Given two nonequal real numbers, if every Cauchy sequence with these two values converges to one of the values, must these two real numbers be distinct?* Together, affirmative answers to these statements yield the Limited Principle of Existence (LPE): *Any two nonequal real numbers are distinct* (this is also known as "Markov's Principle").

The concepts "nonequal" and "distinct", which are classically equivalent, have here different meanings. Two real numbers a and b are *nonequal* if it is contradictory that $a = b$; they are *distinct* if $|a - b| > 0$ has a constructive proof. Considering a single real number c, when $c \geqq 0$ but c is not equal to 0, we say that c is *almost positive*, and write $c \gg 0$. A much stronger condition is that c be *positive*, written $c > 0$; this requires the explicit construction of a rational approximation q, within $1/n$ of c, with $q > 1/n$. The weaker condition $c \gg 0$ is equivalent to the double negation $\neg\neg(c > 0)$ of the stronger. These conditions are discussed further in [5, sect. 3]. Between these two conditions is the "pseudo-positive" condition, which arose in [5, 3.9] in connection with constructive continuity problems. A real number c is *pseudo-positive* if for any real number x, either $x \ll c$ or $x \gg 0$ (pseudo-positive numbers were formerly called *almost separating numbers*). It is clear that

$$c > 0 \Rightarrow c \text{ is pseudo-positive} \Rightarrow c \gg 0.$$

The converses of these implications constitute the following nonconstructive principles which are shown here to be related to the problems concerning complete finite sets.

The Lesser Limited Principle of Existence (LLPE): *Every almost positive real number is pseudo-positive.*

[1]) The author is indebted to FRED RICHMAN for several valuable suggestions.

The Weak Limited Principle of Existence (WLPE): *Every pseudo-positive real number is positive.*

WLPE is discussed in section 16 of [5] (the principle WLPE was previously termed the Almost Separating Principle (ASP)). LPE states that every almost positive real number is positive; it is equivalent to LLPE and WLPE combined.

The basic constructive properties of real numbers and metric spaces are found in Chapters 2 and 4 of [1] or [2]. Brouwerian counterexamples, decision sequences, and the nonconstructive omniscience principles, such as LPO, are also described there, and in section 2 of [5]. A subset F of a metric space (X, ϱ) which may be expressed in the form $F \equiv \{x_1, x_2, \ldots, x_n\}$ is said to be a *finitely enumerable set of at most n elements* (this term, due to RAY MINES, seems a significant improvement over the previously used term *subfinite*). If in addition $\varrho(x_i, x_j) \gg 0$ for $i \neq j$, then F is said to be a *finite set of n elements*. Finally, if $\varrho(x_i, x_j) > 0$ for $i \neq j$, then F is said to be *metrically discrete*.

2. Basic lemmata

Lemma 2.1. *If $a, b \in \mathsf{R}$ and $\{a, b\}$ is complete, then either $a \leqq b$ or $b \leqq a$.*

Lemma 2.2. *Any finite, metrically discrete, subset F of a metric space X, and all subsets of F, are complete.*

Lemma 2.3. *Let $a, b \in \mathsf{R}$ and define $c \equiv |a - b|$. Then $F \equiv \{a, b\}$ is complete if and only if $G \equiv \{0, c\}$ is complete.*

Proof. Necessity. Let F be complete; by Lemma 2.1, we may assume $a \leqq b$. Then $c = b - a$ and G is a translation of F.

Sufficiency. Let G be complete, and let $\{x_n\}$ be a Cauchy sequence in F with limit $x \in \mathsf{R}$. Then $\{|x_n - a|\}$ is a Cauchy sequence in G, and it follows that its limit $|x - a|$ is in G; thus either $|x - a| = 0$ or $|x - a| = c$, and it suffices to consider the latter case. Let n be any integer; either $c < 1/n$ or $c > 0$. In the first case, $|x - b| \leqq \leqq |x - a| + |a - b| = 2c < 2/n$. In the second case, F is metrically discrete, and thus complete. Since $|x - a| > 0$, it follows that $x = b$. Thus in either case we have $|x - b| < 2/n$, and it follows that $x = b$.

Lemma 2.4. *Let $c \in \mathsf{R}$ with $c \gg 0$. Then $G \equiv \{0, c\}$ is complete if and only if c is pseudo-positive.*

Proof. Sufficiency. Let $\{x_n\}$ be a Cauchy sequence in G with limit $x \in \mathsf{R}$. Thus $0 \leqq x \leqq c$. By hypothesis, either $x < c$ or $x \gg 0$. In the first case, suppose $x > 0$. Then $x_n > 0$ eventually, so $x_n = c$ eventually, and $x = c$, a contradiction; hence $x = 0$. Similarly, in the second case $x = c$.

Necessity. Let $x \in \mathsf{R}$, and construct a decision sequence $\{a_n\}$ so that $c < 1/n$ when $a_n = 0$, and $c > 0$ when $a_n = 1$. Now construct a sequence $\{y_n\}$ in G as follows. If $a_n = 0$, define $y_n \equiv 0$. If $a_n = 1$, first examine a_{n-1}. If $a_{n-1} = 0$, determine $x < c$ or $x > 0$, and define $y_n \equiv 0$ or $y_n \equiv c$ accordingly. However, if $a_{n-1} = 1$, then define $y_n \equiv y_{n-1}$. To show that $\{y_n\}$ is a Cauchy sequence, let $n < m$. We need consider only the case in which $y_n = 0$ and $y_m = c$. Then $a_n = 0$, because $\{y_n\}$ becomes constant as soon as $a_n = 1$. Thus $c < 1/n$ and $|y_n - y_m| < 1/n$. Hence $\{y_n\}$ converges to a point of G. If $y_n \to c$, suppose $x \leqq 0$. Then all $y_n = 0$, so $y_n \to 0$, contradicting the hypothesis $c > 0$. Hence $x > 0$. In the case $y_n \to 0$, suppose $x \geqq c$. For any n, if $a_n = 0$, then $y_n = 0 > c - 1/n$. On the other hand, if $a_n = 1$, then $y_n = c$. Thus $c - 1/n < < y_n \leqq c$ for all n, and it follows that $y_n \to c$, contradicting $c > 0$; hence $x < c$. Thus c is pseudo-positive.

Lemma 2.5. *Let S be a set of real numbers such that every subset of the form $\{a, b\}$ is complete. Then every finitely enumerable subset of S is complete.*

Proof. Let F be a finitely enumerable subset of S with at most n elements. By induction, assume that $n > 2$ and that every finitely enumerable subset of F with at most $n - 1$ elements is complete. Then F may be written as $F = \{a_1, a_2, \ldots, a_{n-2}, b, c\}$. By Lemma 2.1, we may assume that $a_1 \leqq a_2 \leqq \ldots \leqq a_{n-2} \leqq b \leqq c$. Let $\{x_n\}$ be a Cauchy sequence in F. Define $y_n \equiv x_n \wedge b$; thus $\{y_n\}$ is a Cauchy sequence in $\{a_1, a_2, \ldots, a_{n-2}, b\}$ and converges to one of these points. Similarly, $z_n \equiv x_n \vee b$ defines a sequence convergent in $\{b, c\}$. Since for any real numbers x and b, we have $x + b = (x \wedge b) + (x \vee b)$, it follows that $x_n = y_n + z_n - b$ for all n. If $z_n \to b$, then $x_n \to \lim y_n$. In the case $z_n \to c$, define $x \equiv \lim x_n$, and define a sequence $\{w_n\}$ by $w_n \equiv x_n$ unless $x_n = b$, in which case $w_n \equiv c$. To show that $w_n \to x$, let $\varepsilon > 0$ and choose N so that $|x_n - x_m| < \varepsilon/2$ for $n, m \geqq N$. Let $n \geqq N$ and suppose $|w_n - x| > \varepsilon$; since $|x_n - x| \leqq \varepsilon/2$ it follows that $x_n = b < c - \varepsilon/2$. Thus $x_m \leqq b$ for $m \geqq N$, and $z_m \to b$, a contradiction. Hence $|w_n - x| \leqq \varepsilon$. This shows that $w_n \to x$; from the induction hypothesis it follows that $x \in F$.

3. Characterizations of pseudo-positive numbers

The splitting of LPE into the two principles LLPE and WLPE depends on the introduction of the concept of pseudo-positive number; it will be useful to have alternative characterizations of this concept.

Theorem 3.1. *Let $c \in \mathsf{R}$ with $c > 0$. Then the following are equivalent:*

(a) *c is pseudo-positive.*

(b) *If x is any real number with $0 \leqq x \leqq c$, then there exists a real number t such that $0 \leqq t \leqq 1$ and $x = tc$.*

Proof. That $(a) \Rightarrow (b)$ is proved in [5, Cor. 16.4], using the theory of reliefs. Now assume (b), let $y \in \mathsf{R}$, and define $x \equiv 0 \vee y \wedge c$. Then $0 \leqq x \leqq c$ so there exists a real number t such that $0 \leqq t \leqq 1$ and $x = tc$. Either $t < 1$ or $t > 0$. In the first case, it follows from [5, 3.8] that $x = tc < c$. Suppose $y \geqq c$; then $x = c$, a contradiction. Hence $y < c$. Similarly, in the second case it follows that $y > 0$. Thus c is pseudo-positive.

Most of the nonconstructive omniscience principles, while having convenient expressions in terms of real numbers, either arose from problems involving decision sequences, or at least are easily so expressed. In contrast, WLPE arose in terms of real numbers, and has no obvious expression in terms of decision sequences. The next characterization of pseudo-positive numbers will provide such an expression.

Definitions. A decision sequence $\{b_n\}$ will be said to be *dependent on* a decision sequence $\{a_n\}$ if $b_n < b_{n+1}$ implies $a_n < a_{n+1}$. A decision sequence $\{a_n\}$ is said to be *decided* if either all $a_n = 0$ or some $a_n = 1$. It is *weakly decided* if either all $a_n = 0$ or not all $a_n = 0$. If $c \in \mathsf{R}$ with $c \geqq 0$, then a decision sequence $\{a_n\}$ is said to *represent* the number c if $a_n = 0$ implies $c < 1/n$, while $a_n = 1$ implies $c > 0$.

Theorem 3.2. *Let $c \in \mathsf{R}$ with $c \gg 0$. Then the following are equivalent:*

(a) *c is pseudo-positive.*

(b) *If $\{a_n\}$ is a decision sequence which represents c, then any decision sequence which is dependent on $\{a_n\}$ is weakly decided.*

Proof. (a) $\Rightarrow$ (b). Let $\{a_n\}$ represent c, and let $\{b_n\}$ be dependent on $\{a_n\}$. To show that $\{b_n c\}$ is a Cauchy sequence, let $n < m$. It suffices to consider the case $b_n = 0$ and $b_m = 1$; then $a_n = 0$, so $|b_n c - b_m c| = c < 1/n$. Let x be the limit of $\{b_n c\}$; either $x \lessdot c$ or $x \gg 0$. In the first case, if some $b_n = 1$ then $x = c$, a contradiction; hence all $b_n = 0$. In the second case, all $b_n = 0$ implies $x = 0$, which is contradictory.

(b) $\Rightarrow$ (a). Construct a decision sequence which represents c, and let $x \in \mathsf{R}$; then construct a decision sequence $\{b_n\}$ as follows. Define $b_1 \equiv 0$ and for $n > 1$ define $b_n \equiv b_{n-1}$ unless $a_{n-1} < a_n$; in this event, decide $x \lessdot c$ or $x \gg 0$, and define $b_n \equiv 0$ or $b_n \equiv 1$ accordingly. Thus $\{b_n\}$ is dependent on $\{a_n\}$, and hence it is weakly decided. If all $b_n = 0$, suppose $x \geqq c$. Suppose further that $c > 0$ and choose k so that $c > 1/k$; then $a_k = 1$, and it follows that some $b_n = 1$, a contradiction. Hence $c = 0$, a contradiction; it follows that $x \lessdot c$. If not all $b_n = 0$, suppose $x \leqq 0$; then all $b_n = 0$, a contradiction, and thus $x \gg 0$.

4. Main results

The first result shows that the classical theorem which states that finite sets of real numbers are complete is nonconstructive. The second treats finitely enumerable sets in a similar manner. The third result shows the nonconstructivity of the converse to Lemma 2.2.

Theorem 4.1. *The following are equivalent:*

(a) *Every finite subset of R is complete.*

(b) *Every finite two-element subset of R is complete.*

(c) *Every ordered finite two-element subset of R is complete.*

(d) *For any $c \in \mathsf{R}$ with $c \gg 0$, the set $\{0, c\}$ is complete.*

(e) *Every almost positive real number is pseudo-positive* (LLPE).

(f) *Let $c \in \mathsf{R}$ with $c \gg 0$. Then for any real number x with $0 \leqq x \leqq c$, there exists a real number t with $0 \leqq t \leqq 1$ such that $x = tc$.*

(g) *For any $x \in \mathsf{R}$, if $|x| \gg 0$, then either $x \gg 0$ or $x \lessdot 0$.*

(h) *For any $x \in \mathsf{R}$, if $|x| \gg 0$, then either $|x| = x$ or $|x| = -x$.*

(j) *For any decision sequence $\{a_n\}$ such that not all $a_n = 0$, either the first integer such that $a_n = 1$ (if any) is even, or the first integer such that $a_n = 1$ (if any) is odd.*

Proof. The equivalence of (b), (c), (d), and (e) follows from Lemmata 2.3 and 2.4. The equivalence of (g), (h), and (j) is immediate. The equivalence of (e) and (f) follows from Theorem 3.1. Lemma 2.5 shows that (b) $\Rightarrow$ (a).

(e) $\Rightarrow$ (g). Let $|x| > 0$. By (e), either $x > 0$ or $x < |x|$. In the second case, if $x \geqq 0$ then $|x| = x$, a contradiction; hence $x < 0$.

(g) $\Rightarrow$ (e). Let $c > 0$, and let $x \in \mathsf{R}$. Construct a decision sequence $\{a_n\}$ such that $c < 1/n$ when $a_n = 0$ and $c > 0$ when $a_n = 1$. Construct a sequence $\{y_n\}$ as follows. Define $y_n \equiv 0$ unless $a_{n-1} < a_n$; in this event, determine $x > 0$ or $x < c$, and define $y_n \equiv 1/2^n$ or $y_n \equiv -1/2^n$ accordingly. Define $y \equiv \sum_n y_n$. Suppose $y = 0$; then all $a_n = 0$ and $c = 0$, a contradiction. Hence $|y| > 0$. By (g), either $y > 0$ or $y < 0$. In the first case, suppose $x \leqq 0$; then all $y_n \leqq 0$ and thus $y \leqq 0$, a contradiction. Hence $x > 0$. Similarly, in the second case $x < c$. Thus c is pseudo-positive.

Remarks. In intuitionistic mathematics, LLPE takes the form "the virtual order of the continuum is a pseudo-order", and a Brouwerian-type counterexample is given using free-choice sequences [3; 8.1.1, p. 121]. Furthermore, the fan theorem is used to show that LLPE, in the form of condition (g) above, is contradictory [3; 8.1.2, Thm. 2]. However, these methods, using free-choice sequences and the fan theorem, do not apply in the Bishop-type strict constructive framework of the present paper.

In 8.1.1 and 8.1.2 of [3], different forms of LLPE were used; it was apparently not known that they are equivalent, as is now shown by Theorem 4.1 above.

The next result establishes the nonconstructivity of the classical result on the completeness of finitely enumerable sets. To facilitate comparison with Theorem 4.1, it includes a few previously known conditions.

Theorem 4.2. *The following are equivalent:*

(a) *Every finitely enumerable subset of* R *is complete.*

(b) *Every finitely enumerable, at most two-element, subset of* R *is complete.*

(c) *Every finitely enumerable, at most two-element, ordered subset of* R *is complete.*

(d) *For any* $c \in \mathsf{R}$ *with* $c \geqq 0$, *the set* $\{0, c\}$ *is complete.*

(e) *If c is any real number with $c \geqq 0$, then for any real number x, either "$c > 0 \Rightarrow x > 0$" or "$c > 0 \Rightarrow x < c$"* (cf. [5, 10.12]).

(f) *If c and x are real numbers with $0 \leqq x \leqq c$, then there exists a real number t such that $0 \leqq t \leqq 1$ and $x = tc$* (cf. [5; 10.11, 16.4]).

(g) *For any real number x, either $x \leqq 0$ or $x \geqq 0$.*

(h) *For any real number x, either $|x| = x$ or $|x| = -x$.*

(j) *For any decision sequence $\{a_n\}$, either the first integer such that $a_n = 1$ (if any) is even, or the first integer such that $a_n = 1$ (if any) is odd* (LLPO).

(k) *For any real numbers a and b, either $a \leqq b$ or $b \leqq a$.*

(m) *For any real numbers x and y, if $xy = 0$, then either $x = 0$ or $y = 0$.*

(n) *For any real numbers a and b, there exists a real number p such that either $a = pb$ or $b = pa$* (cf. [6, 1.3]).

Proof. It is well-known that (g), (h), (j), (k), and (m) are equivalent. The equivalence of (b), (c), and (d) follows from Lemma 2.3. That (b) implies (a) follows from Lemma 2.5. That (b) implies (k) follows from Lemma 2.1.

(k) $\Rightarrow$ (d). Let $\{x_n\}$ be a Cauchy sequence in $F \equiv \{0, c\}$ with $x_n \to x \in \mathrm{R}$. Either $x \geqq c/2$ or $x \leqq c/2$. In the first case, suppose $x < c$; then $0 < x < c$ and F is metrically discrete, hence complete, a contradiction. Thus $x = c$. Similarly, in the second case, $x = 0$.

(k) $\Rightarrow$ (e). Either $x \geqq c/2$ or $x \leqq c/2$. In the first case, $c > 0 \Rightarrow x > 0$, while in the second case $c > 0 \Rightarrow x < c$.

(e) $\Rightarrow$ (g). Let $x \in \mathrm{R}$ and define $c \equiv |x|$. Either $|x| > 0 \Rightarrow x > 0$ or $|x| > 0 \Rightarrow x < |x|$. In the first case, $x < 0$ is contradictory, so $x \geqq 0$. In the second case, $x > 0$ is contradictory, so $x \leqq 0$.

(k) $\Rightarrow$ (f). Let $0 \leqq x \leqq c$. Define a sequence $\{t_n\}$ in $[0, 1]$ as follows. Decide $x \leqq c/2$ or $x \geqq c/2$ and define $t_1 \equiv 0$ or $t_1 \equiv 1/2$ accordingly. If $t_1 = 0$, decide $x \leqq c/4$ or $x \geqq c/4$ and define $t_2 \equiv 0$ or $t_2 \equiv 1/4$ accordingly. If $t_1 = 1/2$, decide $x \leqq 3c/4$ or $x \geqq 3c/4$ and define $t_2 \equiv 1/2$ or $t_2 \equiv 3/4$ accordingly. This process continues by induction, resulting in a Cauchy sequence converging to a real number t such that $x = tc$.

(f) $\Rightarrow$ (h). Let $x \in \mathrm{R}$; then $0 \leqq |x| - x \leqq 2|x|$. Construct t so that $0 \leqq t \leqq 1$ and $|x| - x = 2t|x|$; thus $(1 - 2t)\,|x| = x$. Either $t < 1$ or $t > 0$. In the first case, suppose $|x| - x > 0$. Then $x < |x|$, so either $0 < |x|$ or $0 > x$, and it follows that $|x| > 0$. Thus either $|x| = x$ or $|x| = -x$, but the first possibility is ruled out. Hence $|x| = -x > 0$ so $(1 - 2t)\,(-x) = x$ and $t = 1$, a contradiction; thus $|x| = x$. Similarly, in the second case $|x| = -x$.

(f) $\Rightarrow$ (n). Since (f) $\Rightarrow$ (g), we may assume $a \geqq 0$ and $b \geqq 0$. Since (f) $\Rightarrow$ (k), we may assume $0 \leqq a \leqq b$. Thus there exists t such that $0 \leqq t \leqq 1$ and $a = tb$.

(n) $\Rightarrow$ (f). Let $0 \leqq x \leqq c$ and construct p so that $x = pc$ or $c = px$. In the first case, define $t \equiv 0 \vee p \wedge 1$. Then $tc = 0 \vee pc \wedge c = 0 \vee x \wedge c = x$. In the second case, either $p < 1$ or $p > 0$. In the first subcase, $c = px \leqq x$; hence $x = c$ and $t \equiv 1$ suffices. In the second subcase, define $t \equiv (1/p) \wedge 1$; then $0 \leqq t \leqq 1$ and $tc = (c/p) \wedge c = = x \wedge c = x$.

Theorem 4.3. *The following are equivalent:*

(a) *If a finite, two-element, set $\{a, b\}$ of real numbers is complete, then $|a - b| > 0$.*

(b) *Every pseudo-positive real number is positive* (WLPE).

(c) *If $c \in \mathrm{R}$ with $c \gg 0$, and for any x with $0 \leqq x \leqq c$ there exists t such that $0 \leqq t \leqq 1$ and $x = tc$, then $c > 0$.*

(d) *If $\{a_n\}$ is any decision sequence such that not all $a_n = 0$, and any decision sequence dependent on $\{a_n\}$ is weakly decided, then some $a_n = 1$.*

(e) *Every real-valued function on the closed unit interval, which is nondecreasing and approximates intermediate values, is continuous* (LCP).

Proof. The equivalence of (a) and (b) follows from Lemmata 2.3 and 2.4. The equivalence of (c) and (d) with WLPE follows from the results in section 3. The equivalence of LCP with WLPE is shown in [5, Thm. 16.5].

5. Comparison of LLPE and LLPO

It is clear from a comparison of similar conditions in Theorems 4.1 and 4.2, that LLPO $\Rightarrow$ LLPE. Although this comparison strongly indicates that the converse is not true, no conclusive evidence is available. By combining another principle with LLPE, we do obtain LLPO, in fact the even stronger principle, the Weak Limited Principle of Omniscience (WLPO). In terms of real numbers, WLPO states that *for any real number* $x \geqq 0$, *either* $x = 0$ *or* $x > 0$; see [5; 2.4, 2.6]. The added principle is a weakened version of the principle PID introduced in [4] in connection with denumerability problems (the principle PID was previously termed the Principle of Finite Possibility (PFP)). In terms of real numbers, PID states that *for any real number* $x \geqq 0$ *there exists a real number* $y \geqq 0$ *such that* $x = 0 \Leftrightarrow y > 0$.

The Weak Principle of Inverse Decision (WPID): *For any real number x there exists a real number y such that* $x = 0 \Leftrightarrow y > 0$.

Proposition 5.1. LLPE + WPID $\Leftrightarrow$ WLPO.

Proof. Assume the left-hand side and let $x \geqq 0$. By WPID, construct y such that $x = 0 \Leftrightarrow y > 0$. Suppose $x = y$; then both $x > 0$ and $x = 0$ lead to contradictions, so it follows from [5, Cor. 12.10] that $x = y$ is contradictory. Hence $|x - y| > 0$. By LLPE, either $x - y > 0$ or $x - y < 0$. In the first case, $x > 0$, while in the second case, $x = 0$. Thus WLPO obtains. The converse is clear.

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