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Distribution of values of a real function
Means, moments, and symmetry

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For a distribution function D we define its ***absolute*** and ***signed moments*** of order $k \in \mathbf{R}$, which generalise in a natural way the Hamburger moments of orders an even and an odd natural number. Similarly, for a real function h we define its ***absolute*** and ***signed asymptotic means*** of order $k \in \mathbf{R}$. We show that if the means exist on an infinite and bounded set of values of k, then they exist on an interval I and coincide on I° with the moments of $D = D_h$, the distribution function of the values of h, which is shown to exist (in the sense of Wintner). We also give a sufficient condition for D_h to be symmetric. These results apply to a class of functions h that contain in particular error terms related to the Euler phi function and to the sigma divisor function. A further application on a certain class of converging trigonometrical series implies in particular classical results of A. Wintner establishing the existence for such functions of a distribution function as well as Hamburger moments of arbitrarily large orders. The remainder term of the prime number theorem belongs to this class provided the Riemann hypothesis holds, and the distribution function of its values is shown to be "almost" symmetric.

0. Introduction and statement of the results.

Let $h : [0,\infty) \longrightarrow \mathbf{R}$ be measurable and define, if they exist, its *absolute asymptotic mean of order k*,

$$m_k = m_k(h) := \lim_{x\to\infty} \frac{1}{x} \int_0^x |h(t)|^k \, dt \,, \tag{0.1}$$

and its *signed asymptotic mean of order k*,

$$\mu_k = \mu_k(h) := \lim_{x\to\infty} \frac{1}{x} \int_0^x \operatorname{sgn}(h(t))|h(t)|^k \, dt \,, \tag{0.2}$$

where $\operatorname{sgn}(x)$ is $\frac{x}{|x|}$ if $x \neq 0$ and 0 if $x = 0$, and where k is a positive real number. We note that the standard *asymptotic mean of order k*,

$$M_k = M_k(h) := \lim_{x\to\infty} \frac{1}{x} \int_0^x (h(t))^k \, dt \,, \tag{0.3}$$

which suffers the deficiency of not being defined for nonintegral k if h takes negative values, is m_k if k is even and μ_k if k is odd.

Now consider

$$D_0(u) = D_{0,h}(u) \ := \ \lim_{x\to\infty} \frac{1}{x}\mu\{t \in [0,x], h(t) \le u\} \ , \tag{0.4}$$

where μ is the Lebesgue measure, and let

$$D_0(u^+) \ := \ \lim_{\substack{v\searrow u \\ v\in E}} D_0(v) \ ,$$

where E is the set of values for which D_0 exists; define $D_0(u^-)$ similarly. If, *and only if*, D_0 exists almost everywhere, we say that D *exists*, where $D : \mathbf{R} \longrightarrow [0,1]$ with

$$D(u) = D_h(u) \ := \ \frac{1}{2}(D_0(u^+) + D_0(u^-)) \ , \tag{0.5}$$

and we call D, slightly abusively, the *distribution function of the values of h*. We note that D_0, where it exists, and thence D, are non decreasing. It follows that D is continuous almost everywhere, and with (0.5) that D_0 exists and coincides with D at least wherever D is continuous.

Hence, provided we also have $\lim_{u\to-\infty} D(u) = \lim_{u\to\infty}(1 - D(u)) = 0$, to say that D_0 exists almost everywhere is equivalent to say, in the terminology of Wintner [10, p.537], that *h possesses an asymptotic distribution function.*

We also note that D is *normalized*, i.e.

$$D(u) \ = \ \frac{1}{2}(D(u^+) + D(u^-)) \tag{0.6}$$

for every real number u.

If D exists we define for k positive, if they exist, its *absolute moment of order k*, the Stieltjes integral

$$\beta_k \ := \ \int_{-\infty}^{\infty} |s|^k \, dD(s) \ , \tag{0.7}$$

and its *signed moment of order k*,

$$\gamma_k \ := \ \int_{-\infty}^{\infty} \operatorname{sgn}(s)|s|^k \, dD(s) \ , \tag{0.8}$$

and we remark, as for the means, that the standard *Hamburger moment of order k*,

$$\alpha_k \ := \ \int_{-\infty}^{\infty} s^k \, dD(s) \ , \tag{0.9}$$

defined when k is a natural number, is β_k when k is even and γ_k when k is odd.

We clearly have

Lemma 1.

(a) If β_ℓ exists, then β_k exists for every positive number $k \le \ell$.

(b) β_ℓ exists if and only if γ_ℓ exists.

One can construct examples showing that neither (a) nor (b) remains true when we replace the absolute and signed moments by absolute and signed means. However, under the assumption that D exists we establish in Section 1 the equivalence of moments and means.

Theorem 1. *Let $D = D_h$ exist.*
If m_ℓ exists, then β_ℓ exists. Moreover, m_k and μ_k exist for every positive number $k < \ell$ and

$$\beta_k = m_k \qquad \text{and} \qquad \gamma_k = \mu_k \ . \tag{0.10}$$

We note that this result is best possible in the sense that μ_ℓ doesn't necessarily exist, and that even if it does the relations (0.10) for $k = \ell$ don't necessarily hold.

The preliminary assumption of Theorem 1 that D_h exists appears however as a serious handicap. It is indeed in general much more difficult to decide by direct considerations whether D_h exists than to obtain information on the means of h. In Section 2 we prove

Theorem 2. *If $m_\ell = m_\ell(h)$ and $\mu_\ell = \mu_\ell(h)$ both exist for every $\ell \in L$, where L is an infinite bounded set of positive real numbers, then they also exist for every $\ell \in (0, \sup L)$, and $D = D_h$ exists.*

Corollary. *If $H : [0,\infty) \longrightarrow \mathbf{C}$ is such that $m_\ell(H)$ exists for every $\ell \in L$, where L is an infinite bounded set of real numbers, then $m_\ell(H)$ also exists for every ℓ in $(0, \sup L)$.*
(Indeed, Theorem 2 applies to $h = |H|$, since $m_\ell(h) = \mu_\ell(h)$ for every $\ell \in L$.)

As a corollary to the proof of Theorem 2 we further obtain

Theorem 3. *If, in addition to the hypotheses of Theorem 2, $\mu_\ell = 0$ for every $\ell \in L$, then D is* symmetric, *that is*

$$D(u) = 1 - D(-u) \tag{0.11}$$

for every real number u.

In Section 3 we show that Theorems 2 and 3 are applicable to a certain class of functions h, which in particular contains two error terms H and E related to the Euler phi function and to the divisor sigma function. I established earlier [7] the truth of (0.11) for Δ_h, the distribution function of the values taken at integers by $h = H$ or E, and it implies (0.11) for D_h. But this proof suffers the shortcoming of using the existence of Δ_h: although known, it was established by Erdös and Shapiro [4] for H through an ad hoc method, extendable to the case $h = E$ but difficult to generalise further.

In Section 4 we apply, combined with a recent result of Kueh [5], Theorems 2 and 3 to a class of functions represented by certain converging trigonometrical series and to which, under the assumption of the Riemann hypothesis,

$$h_0(u) \ := \ \frac{\psi_0(x) - x \ + \ \log\left(2\pi\sqrt{1-x^{-2}}\right)}{\sqrt{x}} \qquad (x = e^u) \tag{0.12}$$

belongs, where

$$\psi_0(x) \ := \ \sum_{p^m \leq x}{}' \ \log p \ , \tag{0.13}$$

and where $\sum'$ in (0.13) means that the last term of the sum is $\frac{1}{2}\log x$ if x is of the form p^m. We obtain for h_0 the existence of the means or moments $m_k = \beta_k$ and $\mu_k = \gamma_k$ of all orders $k > 0$, the existence of the distribution function D_h (first shown by Wintner [10]), and an "explicit" representation

$$h_0(u) \;=\; h_1(u) \;+\; h_2(u)\;, \tag{0.14}$$

where h_1 remains bounded, and where D_{h_2} exists and is symmetric. The symmetry of D_{h_2} becomes meaningful on recalling another result of Wintner's: the spectrum of D_h is infinite [11].

1. Proof of Theorem 1.

We first prove the existence of β_ℓ. By virtue of Lemma 1 we shall thus have at our disposal the existence of β_k and γ_k for every $k \le \ell$.

Suppose on the contrary that β_ℓ doesn't exist, and take a constant $C > m_\ell$: then there is a positive number A such that

$$\int_{-A}^{A} |s|^\ell \, dD(s) \;\ge\; C\;. \tag{1.1}$$

Now let $D_x : \mathbf{R} \longrightarrow [0,1]$ for $x > 0$ be defined by

$$D_x(u) \;:=\; \frac{1}{x}\mu\{t \in [0,x], h(t) \le u\}\;. \tag{1.2}$$

We have

$$m_\ell \;=\; \lim_{N\to\infty} \frac{1}{N}\int_0^N |h(t)|^\ell \, dt \;=\; \lim_{N\to\infty}\int_{-\infty}^{\infty} |s|^\ell \, dD_N(s)$$

and since by hypothesis $D_N(s) \to D(s)$ as $N \to \infty$ on every point s of continuity of D, the second theorem of Helly [8, p.xiii] and (1.1) ensure that

$$m_\ell \;\ge\; \lim_{N\to\infty}\int_{-A}^{A} |s|^\ell \, dD_N(s) \;=\; \int_{-A}^{A} |s|^\ell \, dD(s) \;\ge\; C\;,$$

a contradiction.

Now we prove the existence of m_k and show that $\beta_k = m_k$ for all $k < \ell$. The existence of μ_k and the relation $\gamma_k = \mu_k$ are established similarly. Let $x_N \to \infty$ as $N \to \infty$ and $\Delta_N := D_{x_N}$. If B is an arbitrary positive number we have

$$\frac{1}{x_N}\int_0^{x_N} |h(t)|^k \, dt \;=\; \int_{-\infty}^{\infty} |s|^k \, d\Delta_N(s)$$

$$= \int_{-\infty}^{\infty} |s|^k \,(d\Delta_N(s) - dD(s)) + \int_{-\infty}^{\infty} |s|^k \, dD(s)$$

$$= \Big(\int_{-\infty}^{-B} + \int_{B}^{\infty}\Big)|s|^k d\Delta_N(s) - \Big(\int_{-\infty}^{-B} + \int_{B}^{\infty}\Big)|s|^k dD(s) + \int_{-B}^{B} |s|^k (d\Delta_N(s) - dD(s)) + \beta_k$$

$$=: \quad \mathrm{I}_k \quad + \quad \mathrm{II} \quad + \quad \mathrm{III} \quad + \quad \beta_k \ , \tag{1.3}$$

say. Let $\epsilon > 0$: there is a $B_0 = B_0(\epsilon)$ such that if $B \geq B_0$,

$$|\mathrm{II}| \ < \ \epsilon \ , \tag{1.4}$$

since β_k exists. Now $\Delta_N \to D$ as $N \to \infty$ on every point of continuity of D, and thus by the second theorem of Helly there is a $N_0 = N_0(B)$ such that if $N \geq N_0$,

$$|\mathrm{III}| \ < \ \epsilon \, . \tag{1.5}$$

Hence, if B is large, and if N is sufficiently large with respect to B, we have from (1.3), (1.4) and (1.5)

$$\frac{1}{x_N} \int_0^{x_N} |h(t)|^k \, dt \ = \ \mathrm{I}_k \ + \ \epsilon_{B,N} \ + \ \beta_k \ , \tag{1.6}$$

where $|\epsilon_{B,N}| < 2\epsilon$. With possibly other values of B_0 and N_0, (1.6) remains true if we replace k by ℓ since β_ℓ exists. And by the existence of m_ℓ, the left side remains bounded as $N \to \infty$, by the positive constant K, say. So for each B large enough, and for each N sufficiently large with respect to B,

$$\mathrm{I}_\ell \ \leq \ K + \beta_\ell + 2\epsilon \, .$$

But

$$\tag{1.7}$$

$$\mathrm{I}_\ell \ \geq \ |B|^{\ell-k} \, \mathrm{I}_k \ .$$

Thus, for some $B_1 = B_1(\epsilon)$, for each $B \geq B_1$ and each $N \geq N_1 = N_1(B)$ we have

$$\left| \frac{1}{x_N} \int_0^{x_N} |h(t)|^k \, dt \ - \ \beta_k \right| \ < \ 3\epsilon \, . \tag{1.8}$$

Now (1.8) does not depend on the parameter B, and is obtained for an arbitrary sequence $x_N \to \infty$: letting $\epsilon \to \infty$ yields $m_k = \beta_k$. □

2. Proofs of Theorems 2 and 3.

We first introduce some notation. For real functions $F_1, F_2, F_3, \ldots$, and F continuous almost everywhere, we say that the F_i *converge substancially* to F, noted $F_i \to_s F$, if $F_i(u) \to F(u)$ as $i \to \infty$ for each point of continuity u of F. And if the real function G is also continuous almost everywhere, we say it is *substancially equivalent* to F, noted $G \equiv_s F$, if G coincides with F on each point of continuity of both F and G. Note that if in addition F and G are monotonic, then F is continuous exactly where G is.

In order to prove Theorem 2 we show that there is a normalized distribution function D such that for any sequence $x_N \to \infty$ $(N \to \infty)$ we have $\Delta_N := D_{x_N} \to_s D$. We first need an auxiliary result.

Lemma 2. *If m_ℓ exists for some $\ell > 0$, then*

$$\lim_{s\to\infty} \varliminf_{x\to\infty} D_x(s) = 1 \tag{2.1}$$

and

$$\lim_{s\to-\infty} \varlimsup_{x\to\infty} D_x(s) = 0 . \tag{2.2}$$

Proof. Suppose for instance that (2.1) is not true: then there is a positive constant $C < 1$ such that for sequences $x_i \to \infty$ and $s_i \to \infty$ as $i \to \infty$ we have

$$\mu\{t \in [0, x_i], h(t) > s_i\} \geq (1 - C)x_i .$$

But then,

$$\frac{1}{x_i}\int_0^{x_i} |h(t)|^\ell \, dt \geq (1 - C)s_i^\ell \to \infty \quad (i \to \infty) ,$$

a contradiction to the existence of m_ℓ. □

We proceed to prove Theorem 2. Let the sequence $x_N \to \infty$ be fixed. By the first theorem of Helly [8, p.xiii] there is a subsequence $\Delta_{N_i} \to_s D$, where D is a certain distribution function which we may suppose is normalized. Moreover, either the whole sequence Δ_N converges substancially to D, or there is another subsequence $\Delta_{N_j} \to_s D_1$, where D_1 is a distribution function with $D_1 \not\equiv_s D$.

We show that the latter case is impossible. Let $\ell, \ell' \in L$ with $\ell < \ell'$. We have

$$m_\ell = \lim_{i\to\infty} \int_{-\infty}^{\infty} |s|^\ell \, d\Delta_{N_i}(s) = \lim_{j\to\infty} \int_{-\infty}^{\infty} |s|^\ell \, d\Delta_{N_j}(s) . \tag{2.3}$$

Let now $\epsilon > 0$. Then there is $M_0 = M_0(\epsilon)$ such that if $M \geq M_0$, we have for all i

$$\int_M^\infty |s|^\ell \, d\overline{\Delta}_i(s) < \epsilon , \qquad \text{where} \qquad \overline{\Delta}_i(s) := \Delta_{N_i}(s) - \Delta_{N_i}(-s) . \tag{2.4}$$

Indeed if it were not the case there would for each $\delta > 0$ exist a subsequence $i_k \to \infty$ and a sequence $M_k \to \infty$ as $k \to \infty$ such that

$$\int_{M_k}^\infty |s|^{\ell'} \, d\overline{\Delta}_{i_k}(s) \geq M_k^{\ell'-\ell} \epsilon ;$$

but this would contradict the existence of

$$m_{\ell'} = \lim_{i\to\infty} \int_{-\infty}^{\infty} |s|^{\ell'} \, d\Delta_{N_i}(s) .$$

Thus, as $M \to \infty$,

$$\int_{-M}^{M} |s|^\ell \, d\Delta_{N_i(s)} \longrightarrow \int_{-\infty}^{\infty} |s|^\ell \, d\Delta_{N_i}(s)$$

uniformly in i, and consequently, by Helly's second theorem,

$$m_\ell = \lim_{i\to\infty} \int_{-\infty}^{\infty} |s|^\ell \, d\Delta_{N_i}(s) = \int_{-\infty}^{\infty} |s|^\ell \, dD(s) . \tag{2.5}$$

We obtain similarly

$$\mu_\ell = \int_{-\infty}^{\infty} \operatorname{sgn}(s)|s|^\ell \, dD(s) . \tag{2.6}$$

Hence

$$M^+ := \frac{1}{2}(m_\ell + \mu_\ell) = \int_0^{\infty} s^\ell \, d(D(s) - 1) \tag{2.7}$$

and

$$M^- := \frac{1}{2}(m_\ell - \mu_\ell) = \int_0^{\infty} s^\ell \, d(-D(-s)) , \tag{2.8}$$

where Lemma 2 ensures that

$$\lim_{s\to\infty}(D(s) - 1) = \lim_{s\to\infty} D(-s) = 0 . \tag{2.9}$$

Now the same process applied to the sequence N_j yields

$$M^+ = \int_0^{\infty} s^\ell \, d(D_1(s) - 1) \tag{2.10}$$

and

$$M^- = \int_0^{\infty} s^\ell \, d(-D_1(-s)) . \tag{2.11}$$

Consequently, the proof that $D_1 \not\equiv_s D$ is impossible, and thus that $\Delta_N \to_s D$, will be complete if we can show

Lemma 3. *Let $F : [0,\infty) \longrightarrow \mathbf{R}$ be normalized on $(0,\infty)$, and of bounded variation on every bounded interval. Suppose that*

$$f(\ell) := \int_0^{\infty} x^\ell \, dF(x) = 0 \tag{2.12}$$

on a bounded sequence $\ell = \ell_i$ $(i = 1, 2, ...)$ of distinct real numbers. If

$$\inf_{x>0} |F(x)| = 0 , \tag{2.13}$$

then $F \equiv 0$.

We first conclude the proof of Theorem 2. If $y_N \to \infty$ is another sequence, we apply the argument above to a sequence $z_N \to \infty$ containing the x_N and the y_N as subsequences: since $D_{x_N} \to_s D$, we have $D_{z_N} \to_s D$, whence in particular $D_{y_N} \to_s D$. □

Proof of Lemma 3. Let $e^{-y} = x$, $\alpha(y) = \alpha(-\log x) = F(x)$, and $\beta(y) = \alpha(y) - \alpha(0)$; then

$$f(\ell) = -\int_{-\infty}^{\infty} e^{-y\ell}\, d\beta(y) \tag{2.14}$$

is a bilateral Laplace-Stieltjes integral that converges for $\ell = u + iv$, $u \in (s, S)$, where $s = \inf \ell_i$ and $S = \sup \ell_i$. The complex function $f(\ell)$ is analytic in the strip $u \in (s, S)$, and since it vanishes on a set having a finite point of accumulation, it must be identically zero on the whole strip. Moreover, the function β is normalized since F is, and in the strong sense that it also satisfies $\beta(0) = 0$. Thus, by the uniqueness of the representation of a function by the bilateral Laplace-Stieltjes transform of a normalized function [9, Ch.VI.6] we have $\beta \equiv 0$, i.e $\alpha \equiv \alpha(0)$, whence by (2.13) $\alpha \equiv 0$ and $F \equiv 0$, as claimed. □

Proof of Theorem 3. If in addition $\mu_{\ell_i} = 0$ for all i, we see from (2.7) and (2.8) that

$$\int_0^{\infty} s^{\ell}\, d(D(-s) + D(s) - 1) = 0 \tag{2.15}$$

for $\ell \in L$. An appeal to Lemma 3 completes the proof. □

3. Application to functions of the class $C_z(\alpha, f)$.

Let $h : [1, \infty) \longrightarrow \mathbf{R}$ satisfy, for a function $z = z(x)$ slowly increasing to infinity,

$$h(x) = \sum_{n \le x} \frac{\alpha(n)}{n} f\left(\frac{x}{n}\right) = \sum_{n \le z} \frac{\alpha(n)}{n} f\left(\frac{x}{n}\right) + K(h) + o(1), \tag{3.1}$$

where the real sequence α is $K \in \mathbf{R}$ on average, i.e.

$$\sum_{n \le x} \alpha(n) = Kx + o(x), \tag{3.2}$$

where the real function f of bounded variation is periodic with period T, and 0 on average, i.e.

$$\int_0^T f(t)\, dt = 0, \tag{3.3}$$

and where

$$K(h) := K \int_1^{\infty} \frac{f(u)}{u}\, du. \tag{3.4}$$

Then we say that $h \in C_z(\alpha, f)$. For such a function h we know [6] that m_k exists for each $k > 0$. In this Section we prove

Theorem 4. *If $h \in C_z(\alpha, f)$, then μ_k exists for each $k > 0$. And thus D_h exists.*

Theorem 5. *If $h \in C_z(\alpha, f)$ with f odd almost everywhere, and $h^*(x) := h(x) - K(h)$, then*

$$\mu_k^* := \mu_k(h^*) = 0 \tag{3.5}$$

for each $k > 0$, and D_{h^*} is symmetric.

Corollary. *The functions*

$$H(x) \ := \ \sum_{n\le x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x \ , \tag{3.6}$$

$$E(x) \ := \ \sum_{n\le x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6}x + \frac{1}{2}\log x + \frac{\gamma}{2} + 1 \ , \tag{3.7}$$

and

$$G_{-1,k}(x) \ := \ \sum_{n\le\sqrt{x}} \frac{1}{n}\psi_k\left(\frac{x}{n}\right) \quad (k = 1,2,\ldots) \ , \tag{3.8}$$

where ϕ is Euler's function, σ the sum-of-divisors function, and $\psi_k(y)$ the k-th Bernoulli polynomial of argument the fractional part of y, all possess means or moments $m_k = \beta_k$ and $\mu_k = \gamma_k$ of all positive orders k, and a distribution function. Moreover, the distribution function D_{h^*} defined in Theorem 5 is symmetric in the cases of H, E, and $G_{-1,2k+1}$ $(k = 1,2,\ldots)$.

This corollary follows, with Theorems 2 through 5, from the fact [6] that $H \in C_z(-\mu,\psi_1)$, $E \in C_z(-1,\psi_1)$ and $G_{-1,k} \in C_z(-1,\psi_k)$.

Proof of Theorem 4. By virtue of Theorem 2 we may suppose that k is of the form $\frac{n}{m}$ where $(n,m) = 1$ and n,m are both odd. We set, for $y = y(x) = o(x)$ $(x \to \infty)$,

$$h(x,y) \ := \ \sum_{n\le y} \frac{\alpha(n)}{n} f\left(\frac{x}{n}\right) + K(h) \ , \tag{3.9}$$

and we note that, for our choice of k, if a is an arbitrary real number we have

$$\operatorname{sgn}(a)|a|^k \ = \ a^k,$$

if by convention a^k is the (unique) m-th real root of a^n. Now by (3.1) if

$$\mu_{k,z} \ := \ \lim_{x\to\infty} \frac{1}{x}\int_1^x (h(u,z(u)))^k \, du \tag{3.10}$$

exists, then $\mu_k = \mu_{k,z}$. Also, for each $N \ge 1$ the function $(h(x,N))^k$ is periodic, and thus $\mu_{k,N}$ exists. We write

$$\int_1^x (h(u,z))^k \, du \ =: \ \int_1^x (h(u,N))^k \, du \ + x\, R_{k,N}(x) \ = \ \mu_{k,N}\, x + x\, R_{k,N}(x) + o(x) \ , \tag{3.11}$$

and we prove in Lemma 4 below that

$$\lim_{N\to\infty} \left(\overline{\lim_{x\to\infty}} \, |R_{k,N}| \right) \ = \ 0 \ . \tag{3.12}$$

Finally, from the fact that the sequence $\{\mu_{k,N}\}_{N=1}^{\infty}$ is bounded (see Codecà's [3, (4.6)]) we easily conclude (see the end of the proof of Theorem 6 below). □

Lemma 4. *With the notation of (3.9) we have*

$$\lim_{N\to\infty}\left(\overline{\lim_{x\to\infty}}\,\frac{1}{x}\int_1^x |(h(u,z))^k-(h(u,N))^k|\,du\right) = 0 . \tag{3.13}$$

Proof. An application of the Cauchy-Schwarz inequality for integrals yields

$$\int_1^x \left|(h(u,z))^k-(h(u,N))^k\right|\,du \le \sqrt{\alpha\beta} , \tag{3.14}$$

where, as in the Section 4 of [6], $\beta = O(x)$, and where

$$\alpha = \int_1^x \left|(h(u,z))^{\frac{1}{m}}-(h(u,N))^{\frac{1}{m}}\right|^2 du . \tag{3.15}$$

Now since $\mu := \frac{1}{m} \le 1$, if a and b are arbitrary real numbers we easily see that $|a^\mu - b^\mu| \le 2|a-b|^\mu$, whence

$$\alpha \le 4\int_1^x |h(u,z)-h(u,N)|^{2\mu}\,du . \tag{3.16}$$

And from a theorem of Codecà's we know that for every $\nu > 0$ we have [3, (5.5)]

$$\lim_{N\to\infty}\left(\overline{\lim_{x\to\infty}}\,\frac{1}{x}\int_1^x |h(u,z)-h(u,N)|^\nu\,du\right) = 0 , \tag{3.17}$$

whence (3.13). □

Proof of Theorem 5. For $k = \frac{n}{m}$, where $(n,m) = 1$ and n, m are odd, we have by the proof of Theorem 4,

$$\mu_k^* = \lim_{n\to\infty} \mu_{k,N}^* , \tag{3.18}$$

where

$$\mu_{k,N}^* := \lim_{x\to\infty}\frac{1}{x}\int_1^x (h^*(u,N))^k\,du \tag{3.19}$$

and where, for $x \in \mathbf{R}$,

$$h^*(x,N) := \sum_{n\le N}\frac{\alpha(n)}{n} f\left(\frac{x}{n}\right) . \tag{3.20}$$

Now f is odd almost everywhere and periodic, and so is thus (for $N \ge 1$ fixed) the function $(h^*(x,N))^k$. Hence, if S_N denotes $\sup_x |h^*(x,N)|$ and P_N the period of $h^*(x,N)$, we have for $x \ge 1$

$$\left|\int_1^x (h^*(u,N))^k\,du\right| \le P_N S_N^k , \tag{3.21}$$

whence (3.5) in view of (3.18). And the symmetry of D_{h^*} follows from Theorem 3. □

4. Application to trigonometrical series and the prime number theorem.

Let the series

$$h(u) \ := \ \sum_{\gamma \in S} c(\gamma)\, e^{i\gamma u} \tag{4.1}$$

be convergent for every real number u, where $S \subset \mathbf{R}$, and where the complex coefficients $c(\gamma)$ satisfy

$$f(t) \ := \sum_{t<\gamma\leq t+1} c(\gamma) \ \in \ L^p(-\infty,\infty) \tag{4.2}$$

for every $p > 1$. By establishing in [5] the B^k almost periodicity (in the sense of Besicovitch [1]) of h for every $k \geq 2$, Kueh obtains the existence of $m_k = m_k(h)$ for $k \geq 2$.

Remark 1. It then follows from the corollary to Theorem 2 that in fact m_k exists for every positive k.

Remark 2. If we assume that (4.2) is satisfied only for $p > p_0$ where $1 < p_0 < 2$, then Kueh's method with Remark 1 yield the existence of m_k for $0 < k < \frac{p_0}{p_0-1}$.

Here we shall restrict our interest to the *real* functions h satisfying (4.1) and (4.2), and show that their signed means μ_k also exist for every $k = \frac{m}{n} \geq 2$, where $(m,n) = 1$ and m, n are odd. With Theorem 2 we shall thus obtain

Theorem 6. *If the function h is real and satisfies (4.1) and (4.2), then it possesses a distribution function, and the means or moments*

$$m_k \ = \ \beta_k \qquad \textit{and} \qquad \mu_k \ = \gamma_k$$

exist for every positive order k.

Corollary. *The remainder term of the prime number theorem h_0 in (0.12) possesses, under the assumption of the Riemann hypothesis, a distribution function, as well as absolute and signed means or moments of every positive order, and a representation of the form (0.14).*

Proof of the Corollary. From the Riemann-von Mangoldt "explicit" formula

$$\psi_0(x) \ = \ x - \sum_{\rho} \frac{x^\rho}{\rho} - \log\left(2\pi\sqrt{1-x^{-2}}\right) \quad (x > 1)\ , \tag{4.3}$$

where ρ runs over the non-trivial zeros of the Riemann zeta-function, we have under the assumption of the Riemann hypothesis

$$h_0(u) \ = \ - \sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ \gamma>0}} \frac{1}{\frac{1}{4}+\gamma^2}\cos\gamma u \ - \sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ \gamma>0}} \frac{2\gamma}{\frac{1}{4}+\gamma^2}\sin\gamma u \ =: \ h_1(u) \ + \ h_2(u)\ . \tag{4.4}$$

The boundedness of h_1 and (4.2) for $h = h_0$ both follow from another Riemann-von Mangoldt formula,

$$N(T) = \frac{T}{2\pi}\left(\log\left(\frac{T}{2\pi}\right) - 1\right) + O(\log T) , \tag{4.5}$$

where $N(T)$ denotes the number of zeros $\sigma + it$ of the Riemann zeta-function in the region $0 < \sigma < 1$, $0 < t \leq T$. As for the symmetry of D_{h_2}, it is an easy consequence of the proof of Theorem 6 below with the fact that the function

$$h_2(u,N) := \sum_{0<|\gamma|\leq N} \frac{2\gamma}{\frac{1}{4}+\gamma^2} \sin \gamma u \tag{4.6}$$

is odd for every N.

Remark 3. As mentioned in Section 0, relation (0.14) becomes meaningful in view of the fact (Wintner [11]) that the spectrum of D_h is infinite. We further note that although Wintner also obtains in [11] an asymptotic estimate on the growth of $\alpha_{2k} = \beta_{2k}$ as the integer $k \to \infty$, it cannot ensure, as he points it out himself, that the (Hamburger) momenta determine their distribution function uniquely: thus it cannot ensure either that D_{h_2} is symmetric (see [8, Ch.II]).

Proof of Theorem 6. We set

$$h(u,N) := \sum_{0\leq|\gamma|<N} c(\gamma)\, e^{i\gamma u} , \tag{4.7}$$

we note as in Section 3 that for our choice of k we have $\operatorname{sgn}(a)|a|^k = a^k$ if a is any real number, and we remark that for every N the signed mean

$$\mu_{k,N} := \lim_{x\to\infty} \frac{1}{x}\int_0^x (h(u,N))^k\, du \tag{4.8}$$

exists for every positive order k since $(h(u,N))^k$ is an *almost periodic function* (in the sense of Bohr [2]). Let now, as in (3.11),

$$\int_0^x (h(u))^k\, du =: \int_0^x (h(u,N))^k\, du + x\,R_{k,N}(x) = \mu_{k,N}\, x + x\,R_{k,N}(x) + o(x) . \tag{4.9}$$

We have

Lemma 5. *The sequence $R_{k,N}$ of (4.9) satisfies*

$$\lim_{n\to\infty}\left(\overline{\lim_{x\to\infty}}\, |R_{k,N}(x)|\right) = 0 .$$

Proof. By Hölder's inequality, if $p^{-1}+q^{-1}=1,\ p>1,\ q>1,$ we have

$$\int_0^x \left|(h(u))^k - (h(u,N))^k\right| du \leq \alpha^{\frac{1}{p}}\beta^{\frac{1}{q}} , \tag{4.10}$$

where

$$\beta := \int_0^x \left(\sum_{\ell=0}^{n-1} |h(u)|^{\frac{\ell}{m}} |h(u,N)|^{\frac{n-1-\ell}{m}} \right)^q du \tag{4.11}$$

and where, as in (3.16),

$$\alpha := \int_0^x \left| (h(u))^{\frac{1}{m}} - (h(u,N))^{\frac{1}{m}} \right|^p du \ \le\ 2^p \int_0^x |h(u) - h(u,N)|^{\frac{p}{m}} du \ . \tag{4.12}$$

If $\lambda := \frac{p}{m} \ge 2$ and $\epsilon > 0$, it follows from Kueh's proof [5] of the B^λ almost periodicity of h that there is N_0 such that if $N \ge N_0$ and x is large enough,

$$\alpha < \epsilon x \ . \tag{4.13}$$

As for the other factor, we have

$$\beta = O(x) \ . \tag{4.14}$$

Indeed, for every $\kappa > 0$ the relation

$$\int_0^x |h(u,N)|^\kappa du = O_\kappa(x) \tag{4.15}$$

holds with an implied constant independant of N, since it does for $\kappa^* = \max(\kappa, 2)$ by the existence of m_{κ^*} and the B^{κ^*} almost periodicity of h (see Kueh's proof [5, p.4]). □

Finally, in order to conclude the proof of the theorem it is sufficient to note that the sequence $\{\mu_{k,N}\}$, being bounded by (4.15), contains a subsequence $\{\mu_{k,N_i}\}$ converging to some real value μ_k^*. Indeed, by (4.9) and Lemma 5, we must then have $\mu_k = \mu_k^*$ (and in fact $\lim_{N\to\infty} \mu_{k,N} = \mu_k$). □

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