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THERMOELASTOPLASTIC BENDING OF COMPLEXLY REINFORCED PLATES

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A problem on the transverse-longitudinal bending of reinforced plates of variable thickness under a thermal-force loading is formulated. A qualitative analysis of the problem is carried out, and a way of its linearization is indicated. Calculations of isotropic and metal composite plates subjected to the transverse or transverse-longitudinal bending showed that their bearing capacity in the elastoplastic bending is a number of times (occasionally, by an order of magnitude) greater than in the elastic bending. The heating of the plates sharply decreases their resistance to the bending in the case of elasticity and affects it only slightly in the case of elastoplasticity. In the elastoplastic bending, the bearing capacity of the metal composite plate a number of times exceeds that of isotropic plates made of conventional structural metal alloys.

Introduction

The numerous studies known at present from the literature on the problems of deformation of thin-walled composite structures (see, for example, [1-4]), as a rule, are dedicated to investigations of the linear thermoelastic behavior of anisotropic plates and shells. Only some studies, for example, [5, 6] consider the problem of elastoplastic deformation of structures with simplest types of anisotropy (homogeneous orthotropic materials whose integral mechanical characteristics are known from experiments). However, as an analysis of reference data [7] shows, for many modern fibrous compositions (carbon- and boron-magnesium, boron-aluminum, magnesium-steel, magnesium-tungsten, etc.), the ultimate elastic strain of reinforcing fibers is several times (sometimes, by an order of magnitude) greater than that of the binder matrix. Therefore, for such compositions, the bearing capacity of force elements (fibers) in linearly elastic bending is utilized insignificantly (sometimes, only by several percents), which entails a low bearing capacity of the reinforced plate and an inefficient utilization of the high-strength reinforcement. Therefore, of topical importance is the investigation of thermoelastoplastic transverse-longitudinal bending of complexly reinforced plates, when the bearing capacity of fibers can be utilized more completely than in the case of linearly elastic bending, which will allow one to raise the bearing capacity of the structure at a fixed consumption of phase materials or to lower their consumption at a fixed loading level. The present investigation is dedicated to the formulation and analysis of this problem.

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1. Initial Equations and Boundary Conditions

Let us consider the elastoplastic transverse-longitudinal bending of a Kirchhoff plate of variable thickness $2H$, consisting of an isotropic matrix reinforced with thin homogeneous fibers of constant or variable cross section. (The limits of applicability of the Kirchhoff theory in solving the problems of bending are investigated in [8] for anisotropic plates and in [9] for anisotropic plates, where it is shown that, for compositions with metal binders, especially for metal composites, the Kirchhoff theory is quite applicable. The plate is regular and quasi-homogeneous across its thickness, and the deflections are small. The loading of the plate is assumed quasi-static and simple, therefore, the nonlinearly elastic or plastic behavior of phase materials is described by relationships of the deformation theory of plasticity [10, 11]. (By virtue of the last assumption, in the present study, we consider in fact metal composites and composites with a metal binder and elastic brittle fibers: carbon- and boron-aluminum, carbon- and boron-magnesium, etc.)

The plate is referred to a rectangular Cartesian system of coordinates $x_1 x_2 z$; the $x_1 x_2$ $(z = 0)$ plane is made coincident with the plate midplane before bending, and the z axis is perpendicular to this plane. The plate is reinforced with N families of fibers (possibly of various physical nature), which are stacked in planes parallel to the $x_1 x_2$ plane (plane reinforcement) or on surfaces the distances between which along the z axis vary in proportion to $H(x_1, x_2)$ (three-dimensional reinforcement).

It is assumed that the deviation T of temperature from its value in the natural state of the plate is known and is distributed linearly in the transverse direction z:

$$T(x_1, x_2, z) = T_0(x_1, x_2) + zT_z(x_1, x_2), \quad |z| \le H(x_1, x_2), \quad (x_1, x_2) \in G, \tag{1.1}$$

where T_0 and T_z are known functions and G is the region occupied by the plate in the plan. (Hereafter, for brevity, we will simply call it the "temperature T of the structure," meaning by this its deviation from the natural state.)

To formulate the problem of thermoelastoplastic bending of reinforced plates, we will first determine the relationship between the membrane strains e_{ij}, the parameters of curvature κ_{ij} of the plate midplane, the temperature (T_0, T_z), and reinforcement parameters on the one hand and the internal force factors (moments M_{ij} and membrane forces F_{ij}) on the other. These factors are determined in terms of the average stresses σ_{ij} in the composition by the equalities [1-3]

$$F_{ij}(x_1, x_2) = \int_{-H}^{H} \sigma_{ij}(x_1, x_2, z)dz, \quad M_{ij}(x_1, x_2) = \int_{-H}^{H} \sigma_{ij}(x_1, x_2, z)zdz, \quad i, j = 1, 2, \tag{1.2}$$

where σ_{ij} are connected with the stresses σ_{ij}^0 and σ_k in the matrix and the reinforcement of a kth family, respectively, by the relations (according to the model of a reinforced layer suggested in [12])

$$\begin{gathered} \sigma_{ij}(x_1, x_2, z) = a\sigma_{ij}^0(x_1, x_2, z) + \sum_k \sigma_k(x_1, x_2, z)\omega_k l_{ki} l_{kj} \quad (i, j = 1, 2), \\ l_{k1} = \cos\psi_k(x_1, x_2), \quad l_{k2} = \sin\psi_k(x_1, x_2), \quad a(x_1, x_2) = 1 - \sum_k \omega_k(x_1, x_2). \end{gathered} \tag{1.3}$$

Here, ω_k and ψ_k are the intensity and the angle (reckoned from the x_1 direction) of reinforcement with fibers of the kth family. In addition, the physical restrictions

$$0 \le \omega_k \quad (k = 1, 2, ..., N), \quad \sum_k \omega_k \le \omega_* = \text{const} \le 1 \tag{1.4}$$

must be fulfilled, where ω_* is the ultimate total intensity of reinforcement; the summation is carried out from 1 to N if no limits are specified. {In [13], a rather detailed comparison is made between the calculated rigidities and strength characteristics of unidirectional and cross-reinforced composites with a metal binder and the experimental data available. In determining the strength properties of the composites, the plastic deformation of phase materials was taken into account. The calculations showed that model (1.3), within the framework of which a uniaxial stress state is realized in the fibers, agreed well with experiments.}

By substituting the stresses σ_{ij} from Eq. (1.3) into Eqs. (1.2), we obtain the expressions for membrane forces and moments

$$F_{ij} = aF_{ij}^0 + \sum_k F_k \omega_k l_{ki} l_{kj}, \quad M_{ij} = aM_{ij}^0 + \sum_k M_k \omega_k l_{ki} l_{kj}, \tag{1.5}$$

where

$$F_{ij}^0(x_1, x_2) = \int_{-H}^{H} \sigma_{ij}^0(x_1, x_2, z)dz,$$

$$M_{ij}^0(x_1, x_2) = \int_{-H}^{H} \sigma_{ij}^0(x_1, x_2, z)zdz \quad (i, j = 1, 2), \tag{1.6}$$

$$F_k(x_1, x_2) = \int_{-H}^{H} \sigma_k(x_1, x_2, z)dz,$$

$$M_k(x_1, x_2) = \int_{-H}^{H} \sigma_k(x_1, x_2, z)zdz \quad (k = 1, 2, \ldots, N).$$

Let us assume that the tension and compression diagrams for all phase materials coincide and exhibit a linear hardening. Then, the relation between the stress σ_k and the mechanical strain component ε_k in the reinforcement of a *k*th family, owing to the uniaxial stress state in the fibers [12], has the form [10]

$$\sigma_k = \begin{cases} E_k \varepsilon_k, & |\varepsilon_k| \le \varepsilon_{sk} = \sigma_{sk}/E_k; \\ \operatorname{sgn}(\varepsilon_k)\sigma_{sk} + E_{sk}[\varepsilon_k - \operatorname{sgn}(\varepsilon_k)\varepsilon_{sk}], & \varepsilon_{sk} < |\varepsilon_k| \le \varepsilon_{yk}. \end{cases} \tag{1.7}$$

Here, with account of Eq. (1.1) (see [3, 12]),

$$\varepsilon_k = \varepsilon_{11}\cos^2\psi_k + \varepsilon_{22}\sin^2\psi_k + \varepsilon_{12}\sin 2\psi_k - \alpha_k T \equiv e_k + z\kappa_k,$$

$$k = 1, 2, \ldots, N, \tag{1.8}$$

$$e_k(x_1, x_2) \equiv e_{11}\cos^2\psi_k + e_{22}\sin^2\psi_k + e_{12}\sin 2\psi_k - \alpha_k T_0, \tag{1.9}$$

$$\kappa_k(x_1, x_2) \equiv \kappa_{11}\cos^2\psi_k + \kappa_{22}\sin^2\psi_k + \kappa_{12}\sin 2\psi_k - \alpha_k T_z, \ k = 1, 2, \ldots, N,$$

where ε_{ij} are the total strains of the Kirchhoff plate, which are determined by the relations [1]

$$\varepsilon_{ij} = e_{ij} + z\kappa_{ij}, \quad e_{ij} = \frac{u_{i,j} + u_{j,i}}{2}, \quad \kappa_{ij} = -w_{,ij}, \quad |z| \le H, \ i, j = 1, 2, \tag{1.10}$$

and w and u_i are the deflection and displacement of midplane points of the plate in the x_i direction, respectively; E_k, E_{sk}, and σ_{sk} are the elastic and hardening moduli and the yield point of the material of fibers of the *k*th family, respectively; ε_{sk} and ε_{yk} are the ultimate elastic strain and the strain corresponding to the ultimate strength σ_{yk} of the material of fibers of the *k*th family; α_k is the coefficient of linear thermal expansion of fibers of the *k*th family; the comma before a superscript means partial differentiation with respect to the corresponding coordinate.

The boundary between the elastic and plastic zones in the reinforcement of a kth family is determined by the equality $|\varepsilon_k| = \varepsilon_{sk} > 0$, from which follows the equation

$$\varepsilon_k^2 \equiv (e_k + z\kappa_k)^2 = e_k^2 + 2e_k\kappa_k z + \kappa_k^2 z^2 = \varepsilon_{sk}^2 \tag{1.11}$$

determining the z coordinate of boundaries between the elastic and plastic zones in the reinforcement of the kth family. The discriminant of this equation is

$$D_k = e_k^2\kappa_k^2 - \kappa_k^2(e_k^2 - \varepsilon_{sk}^2) = \varepsilon_{sk}^2\kappa_k^2 \geq 0, \quad k = 1, 2, \ldots, N, \tag{1.12}$$

where $D_k = 0$ only at $\kappa_k = 0$ (absence of the mechanical component of the flexural strain of fibers of the kth family), since $\varepsilon_{sk} > 0$. But, at $\kappa_k = 0$, the left-hand side of the last equality in (1.11) does not depend on z at all, therefore, along the entire z axis, passing through the point $(x_1, x_2) \in G$, either an elastic (if $e_k^2 \leq \varepsilon_{sk}^2$) or a plastic (if $e_k^2 > \varepsilon_{sk}^2$) state exists in the fibers of the kth family. If $\kappa_k \neq 0$ (mechanical bending of fibers of the kth family), then $D_k > 0$, $\kappa_k^2 > 0$, and the z coordinates of the boundaries between the elastic and plastic zones are determined by the equalities

$$z_1 \equiv h_k^* = \frac{-e_k\kappa_k - \sqrt{D_k}}{\kappa_k^2}, \quad z_2 \equiv H_k^* = \frac{-e_k\kappa_k + \sqrt{D_k}}{\kappa_k^2}, \tag{1.13}$$

where

$$\sqrt{D_k} = \varepsilon_{sk}|\kappa_k| > 0, \quad |\kappa_k| > 0, \quad k = 1, 2, \ldots, N. \tag{1.14}$$

However, since the values of h_k^* and H_k^* calculated by Eqs. (1.13) and (1.14) can lie outside the plate $|z| \leq H$, the z coordinates of boundaries between the elastic and plastic zones in the reinforcement of the kth family finally are

$$h_k(x_1, x_2) = \begin{cases} H & \text{for } h_k^* \geq H, \\ h_k^* & \text{for } -H < h_k^* < H, \\ -H & \text{for } h_k^* \leq -H, \end{cases} \quad H_k(x_1, x_2) = \begin{cases} H & \text{for } H_k^* \geq H, \\ H_k^* & \text{for } -H < H_k^* < H, \\ -H & \text{for } H_k^* \leq -H. \end{cases} \tag{1.15}$$

From these equalities, we obtain the upper plastic zone in the reinforcement of the kth family in the case $H_k < z \leq H$, the elastic zone in the reinforcement in the case $h_k \leq z \leq H_k$, and the upper plastic zone in the fibers of the kth family in the case $-H \leq z < h_k$.

Using Eqs. (1.7)-(1.9) and (1.15), we calculate the integrals F_k and M_k in Eqs. (1.6) without account of thermosensitivity of the material of fibers:

$$\begin{aligned} F_k = {} & [E_k(H_k - h_k) + E_{sk}(2H - H_k + h_k)](e_{ii}l_{ki}^2 + e_{jj}l_{kj}^2 + e_{ij}l_{ki}l_{kj} + e_{ji}l_{kj}l_{ki}) \\ & + 0.5(H_k^2 - h_k^2)(E_k - E_{sk})(\kappa_{ii}l_{ki}^2 + \kappa_{jj}l_{kj}^2 + \kappa_{ij}l_{ki}l_{kj} + \kappa_{ji}l_{kj}l_{ki}) \\ & + (\sigma_{sk} - E_{sk}\varepsilon_{sk})[\operatorname{sgn}(e_k + H\kappa_k)(H - H_k) + \operatorname{sgn}(e_k - H\kappa_k)(h_k + H)] \\ & - \alpha_k\{[E_k(H_k - h_k) + E_{sk}(2H - H_k + h_k)]T_0 + 0.5(H_k^2 - h_k^2)(E_k - E_{sk})T_z\}, \end{aligned} \tag{1.16}$$

$$M_k = 0.5(H_k^2 - h_k^2)(E_k - E_{sk})(e_{ii}l_{ki}^2 + e_{jj}l_{kj}^2 + e_{ij}l_{ki}l_{kj} + e_{ji}l_{kj}l_{ki})$$

$$+\frac{1}{3}[E_k(H_k^3-h_k^3)+E_{sk}(2H^3-H_k^3+h_k^3)](\kappa_{ii}l_{ki}^2+\kappa_{jj}l_{kj}^2+\kappa_{ij}l_{ki}l_{kj}+\kappa_{ji}l_{kj}l_{ki})$$

$$+0.5(\sigma_{sk}-E_{sk}\varepsilon_{sk})[(H^2-H_k^2)\operatorname{sgn}(e_k+H\kappa_k)+\operatorname{sgn}(e_k-H\kappa_k)(h_k^2-H^2)]$$

$$-\alpha_k\left\{0.5(H_k^2-h_k^2)(E_k-E_{sk})T_0+\frac{1}{3}[E_k(H_k^3-h_k^3)+E_{sk}(2H^3-H_k^3+h_k^3)]T_z\right\},$$

[the quantities e_k and κ_k are described by Eqs. (1.9)].

Thus, Eq. (1.16), with account of Eqs. (1.12)-(1.15) and (1.9), determine the functions F_k and M_k in terms of the membrane strains e_{ij}, the parameters of curvature of the plate midplane κ_{ij} $(i, j = 1, 2)$, the temperature (T_0, T_z), and the reinforcement angles ψ_k $(1 \le k \le N)$.

For calculating the integrals F_{ij}^0 and M_{ij}^0 in Eqs. (1.6), we must determine the stresses σ_{ij}^0 in the binder. (The stresses σ_{ij}^0 for the elastic and plastic zones in the binder are calculated according to the procedure described in [11] for the case of transverse bending of homogeneous isotropic plates, with some corrections necessary for realization of the transverse-longitudinal bending examined in the present study, and with account of thermal action. In this approach, we disregard the normal σ_{zz}^0 and the transverse tangential σ_{1z}^0, σ_{2z}^0 stresses in the binder compared with the tangential stresses σ_{ij}^0 $(i, j = 1, 2)$. In [9], it is shown that, for compositions with a metal binder, such a neglect is allowable for plates with a relative thicknesses of 0.017 to 0.04. At some fastening conditions of the plates, their relative thicknesses can be even twofold greater than the mentioned ones.)

The Duhamel–Neumann relations in the elastic zone of the isotropic binder of a Kirchhoff plate have the form

$$\sigma_{ii}^0 = Ea_1(\varepsilon_{ii}+\nu\varepsilon_{jj}-\alpha T/a_2), \quad \sigma_{ij}^0 = Ea_2\varepsilon_{ij}, \quad j=3-i, \quad i=1, 2, \tag{1.17}$$

where

$$a_1 = \frac{1}{1-\nu^2}, \quad a_2 = \frac{1}{1+\nu}, \tag{1.18}$$

and E, ν, and α are the elastic modulus, the Poisson ratio, and the coefficient of linear thermal expansion of the binder material, respectively.

The relationship between the stresses σ_{ij}^0, strains ε_{ij}, and temperature T in the plastic zones of the binder, without account of the mechanical component of dilatation (triaxial compression), has the form [10, 11]

$$\sigma_{ii}^0 = \frac{4\sigma_*}{3\varepsilon_*}(\varepsilon_{ii}+0.5\varepsilon_{jj}-1.5\alpha T), \quad \sigma_{ij}^0 = \frac{2\sigma_*}{3\varepsilon_*}\varepsilon_{ij}, \quad j=3-i, \quad i=1, 2, \tag{1.19}$$

where σ_* and ε_* are the intensities of stresses and mechanical components of strains in the matrix, which are connected by the strain diagram

$$\sigma_*(\varepsilon_*) = \begin{cases} 3G_0\varepsilon_*, & \varepsilon_* \le \varepsilon_s = \sigma_s/(3G_0), \\ \sigma_s + E_s(\varepsilon_*-\varepsilon_s), & \varepsilon_s < \varepsilon_* \le \varepsilon_y. \end{cases} \tag{1.20}$$

Here, $G_0 = a_2E/2$ is the shear modulus of the binder; σ_s and E_s are the yield point and the hardening modulus of the binder on the strain diagram, respectively; ε_s and ε_y are the values of strain intensity corresponding to the yield point σ_y and to the ultimate strength σ_s of the binder on the strain diagram. {In the elastic zone, relations (1.19) coincide with (1.17) if $\nu = 1/2$ (the dilatation is absent). The account of the mechanical component of dilatation in the plastic zones leads to significant technical difficulties; in particular, this does not allow us to derive explicit expressions for F_{ij}^0 and M_{ij}^0 in terms of e_{ij}, κ_{ij}, T_0, and T_z, but practically does not affect the calculation results for reinforced bent plates [14].}

In the elastic zone of the binder of a Kirchhoff plate, the intensity of mechanical strains is given by [11]

$$\varepsilon_{0*} = \frac{2}{\sqrt{3}}\sqrt{A(\varepsilon_{*1}^2 - B\varepsilon_{*1}\varepsilon_{*2} + \varepsilon_{*2}^2) + \varepsilon_{12}^2}, \tag{1.21}$$

where

$$\varepsilon_{*i} = \varepsilon_{ii} - \alpha T \quad (i = 1, 2), \quad A = \frac{1 - \nu + \nu^2}{3(1-\nu)^2}, \quad B = \frac{1 - 4\nu + \nu^2}{1 - \nu + \nu^2}. \tag{1.22}$$

Here, ε_{*i} are the mechanical components of the linear strains ε_{ii}. In the plastic zones of the binder, the intensity of mechanical strain components without account of the mechanical component of dilatation is [11]

$$\varepsilon_{*} = \frac{2}{\sqrt{3}}\sqrt{\varepsilon_{*1}^2 + \varepsilon_{*1}\varepsilon_{*2} + \varepsilon_{*2}^2 + \varepsilon_{12}^2}. \tag{1.23}$$

In the absence of the mechanical component of dilatation in the elastic zone ($\nu = 1/2$), Eqs. (1.21) and (1.23) coincide.

According to the strain diagram of the binder (1.20), the boundaries between the elastic and plastic zones in the binder are determined by the equality $\varepsilon_{0*} = \varepsilon_s$ or $\varepsilon_{0*}^2 = \varepsilon_s^2$, wherefrom, with account of Eqs. (1.21), (1.22), (1.10), and (1.1), we have

$$\begin{aligned} &A[(e_{*1} + z\kappa_{*1})^2 - B(e_{*1} + z\kappa_{*1})(e_{*2} + z\kappa_{*2}) + (e_{*2} + z\kappa_{*2})^2] \\ &\quad + (e_{12} + z\kappa_{12})^2 = 3\varepsilon_s^2/4 = \sigma_s^2\big/(12G_0^2) > 0, \end{aligned} \tag{1.24}$$

where

$$e_{*i} = e_{ii} - \alpha T_0, \quad \kappa_{*i} = \kappa_{ii} - \alpha T_z, \quad i = 1, 2. \tag{1.25}$$

Removing the parentheses in Eqs. (1.24), we come to the equation

$$A_0 z^2 + 2B_0 z + C_0 = 0, \tag{1.26}$$

where

$$\begin{aligned} A_0 &= A(\kappa_{*1}^2 - B\kappa_{*1}\kappa_{*2} + \kappa_{*2}^2) + \kappa_{12}^2 \geq 0, \\ B_0 &= A\left[e_{*1}\kappa_{*1} - \frac{B(e_{*1}\kappa_{*2} + e_{*2}\kappa_{*1})}{2} + e_{*2}\kappa_{*2}\right] + e_{12}\kappa_{12}, \\ C_0 &= A(e_{*1}^2 - Be_{*1}e_{*2} + e_{*2}^2) + e_{12}^2 - 3\varepsilon_s^2/4. \end{aligned} \tag{1.27}$$

Equation (1.26) [or (1.24)] can be regarded as a quadratic equation in z, which determines the z coordinates of boundaries between the elastic and plastic zones in the binder. The discriminant of this equation is

$$D_0 = B_0^2 - A_0 C_0. \tag{1.28}$$

Since the function $A_0(x_1, x_2)z^2$ is proportional to the squared intensity of mechanical components of flexural strains, $A_0 = 0$ only if $\kappa_{*1} = \kappa_{*2} = \kappa_{12} = 0$ (in the absence of the mechanical component of bending); if at least one of the functions κ_{*1}, κ_{*2}, and κ_{12} at a point (x_1, x_2) differs from zero (the mechanical components of bending or twisting are present), $A_0 > 0$.

In the absence of mechanical bending, it follows from Eq. (1.27) that $A_0 = B_0 = 0$, and the left-hand side of Eq. (1.26) does not depend on z at all. Therefore, along the entire z axis, passing through the point (x_1, x_2), either elasticity (at $C_0 \le 0$) or plasticity (at $C_0 > 0$) arises in the binder. If $A_0 > 0$ (mechanical bending), at $D_0 > 0$, the z coordinates of boundaries between the elastic and plastic zones in the binder are found from the equalities

$$z_1 \equiv h_0^* = \frac{-B_0 - \sqrt{D_0}}{A_0}, \quad z_2 \equiv H_0^* = \frac{-B_0 + \sqrt{D_0}}{A_0}, \tag{1.29}$$

where Eq. (1.28) must be taken into account. However, the values of h_0^* and H_0^* calculated by formulas (1.29) can lie outside the plate $|z| \le H$, therefore, the z coordinates of boundaries between the elastic and plastic zones in the binder finally are

$$h_0(x_1, x_2) = \begin{cases} H & \text{for } h_0^* \ge H, \\ h_0^* & \text{for } -H < h_0^* < H, \\ -H & \text{for } h_0^* \le -H, \end{cases} \quad H_0(x_1, x_2) = \begin{cases} H & \text{for } H_0^* \ge H, \\ H_0^* & \text{for } -H < H_0^* < H, \\ -H & \text{for } H_0^* \le -H. \end{cases} \tag{1.30}$$

From Eqs. (1.30), we obtain the upper plastic zone in the binder in the case $H_0 < z \le H$, the elastic zone in the binder in the case $h_0 \le z \le H_0$, and the upper plastic zone in the binder in the case $-H \le z < h_0$. If $A_0 = 0$ and $C_0 \le 0$ (elasticity along the entire axis), we have to assume formally that

$$h_0(x_1, x_2) = -H, \quad H_0(x_1, x_2) = H \quad (A_0 = 0,\ C_0 \le 0); \tag{1.31}$$

if $A_0 = 0$ and $C_0 > 0$ or $A_0 > 0$ and $D_0 < 0$ (plasticity along the z axis),

$$h_0(x_1, x_2) = H_0(x_1, x_2) = -H \quad (A_0 = 0,\ C_0 > 0 \ \text{ or } \ A_0 > 0,\ D_0 < 0). \tag{1.32}$$

Thus, in the elastic zone of the binder, relations (1.17), with account of Eqs. (1.1), (1.10), (1.22), and (1.25), take the form

$$\begin{gathered} \sigma_{ii}^0 = Ea_1(e_{*i} + \nu e_{*j}) + zEa_1(\kappa_{*i} + \nu\kappa_{*j}), \\ \sigma_{ij}^0 = Ea_2(e_{ij} + z\kappa_{ij}), \quad j = 3 - i, \quad i = 1, 2. \end{gathered} \tag{1.33}$$

Using Eqs. (1.19), (1.20), and (1.22), the stresses in the plastic zones of the binder can be represented as

$$\begin{gathered} \sigma_{ii}^0 = \frac{2\sigma_*}{3\varepsilon_*}(2\varepsilon_{*i} + \varepsilon_{*j}) = \frac{2}{3}E_s(2\varepsilon_{*i} + \varepsilon_{*j}) + \frac{2(\sigma_s - E_s\varepsilon_s)}{3\varepsilon_*}(2\varepsilon_{*i} + \varepsilon_{*j}), \\ \sigma_{ij}^0 = \frac{2\sigma_*}{3\varepsilon_*}\varepsilon_{ij} = \frac{2}{3}E_s\varepsilon_{ij} + \frac{2(\sigma_s - E_s\varepsilon_s)}{3\varepsilon_*}\varepsilon_{ij}, \quad j = 3 - i, \quad i = 1, 2. \end{gathered} \tag{1.34}$$

The value of ε_* is found from Eq. (1.23), which, with account of Eqs. (1.22), (1.25), (1.1), and (1.10), can be brought to the form

$$\varepsilon_* = \sqrt{\kappa_*^2 z^2 + e_\kappa z + e_*^2}, \tag{1.35}$$

$$\kappa_*^2 = \frac{4(\kappa_{*1}^2 + \kappa_{*1}\kappa_{*2} + \kappa_{*2}^2 + \kappa_{12}^2)}{3}, \quad e_*^2 = \frac{4(e_{*1}^2 + e_{*1}e_{*2} + e_{*2}^2 + e_{12}^2)}{3}, \tag{1.36}$$

$$e_\kappa = 8\left(e_{*1}\kappa_{*1} + \frac{e_{*1}\kappa_{*2} + e_{*2}\kappa_{*1}}{2} + e_{*2}\kappa_{*2} + e_{12}\kappa_{12} \right)\Big/3.$$

Here, e_*^2 is the squared intensity of the mechanical components of membrane strains (in the absence of the mechanical component of dilatation); $\kappa_*^2 z^2$ is the squared intensity of mechanical components of bending strains [at $\nu = 1/2$, the equalities $\kappa_*^2 = 4A_0/3$ and $e_*^2 = 4C_0/3 + \varepsilon_s^2$ are valid, see Eqs. (1.27)]. Hence, $e_*^2 \ge 0$ and $\kappa_*^2 \ge 0$; in this case, $e_*^2 = 0$ and $\kappa_*^2 = 0$ only at $e_{*1} = e_{*2} = e_{12} = 0$ and $\kappa_{*1} = \kappa_{*2} = \kappa_{12} = 0$, respectively. If $\kappa_*^2 = 0 = 0$, we have from Eqs. (1.36) that $e_\kappa = 0$, while from Eq. (1.35) it follows that the strain intensity ε_* in Eqs. (1.34) does not depend on z; at $e_*^2 = 0$, we have $e_\kappa = 0$, and from Eq. (1.35) it follows that $\varepsilon_* = |z|\kappa_*$ ($\kappa_* > 0$), which corresponds to the case of transverse bending of the plate. Since the strain intensity ε_* at $e_*^2 > 0$* and $\kappa_*^2 > 0$ (transverse-longitudinal bending) is always greater than zero ($\varepsilon_* = 0$ only at $e_*^2 = \kappa_*^2 = 0$, i.e., in the absence of mechanical deformation), it follows from Eq. (1.35) that the expression under the radical sign is also greater than zero, and thus we have

$$e_\kappa^2 < 4\kappa_*^2 e_*^2 \quad (\kappa_*^2 > 0, \quad e_*^2 > 0). \tag{1.37}$$

From Eqs. (1.34), with account of Eqs. (1.35), (1.1), (1.10), (1.22), and (1.25), we come to the final expressions for stresses in the plastic zones of the binder

$$\sigma_{ii}^0 = \frac{2}{3}E_s[(2e_{*i} + e_{*j}) + z(2\kappa_{*i} + \kappa_{*j})] + \frac{2(\sigma_s - E_s\varepsilon_s)}{3\sqrt{\kappa_*^2 z^2 + e_\kappa z + e_*^2}}[(2e_{*i} + e_{*j}) + z(2\kappa_{*i} + \kappa_{*j})],$$

$$\sigma_{ij}^0 = \frac{2}{3}E_s(e_{ij} + z\kappa_{ij}) + \frac{2(\sigma_s - E_s\varepsilon_s)}{3\sqrt{\kappa_*^2 z^2 + e_\kappa z + e_*^2}}(e_{ij} + z\kappa_{ij}), \quad j = 3 - i, \quad i = 1, 2. \tag{1.38}$$

Using Eqs. (1.33) and (1.38), we calculate the integrals F_{ij}^0 and M_{ij}^0 in Eqs. (1.6)

$$F_{ii}^0 = \left\{ Ea_1(H_0 - h_0) + \frac{4}{3}E_s(2H - H_0 + h_0) + \frac{4}{3}(\sigma_s - E_s\varepsilon_s)[f(H, H_0) \right.$$

$$\left. + f(h_0, -H)] \right\} e_{ii} + \left\{ Ea_1\nu(H_0 - h_0) + \frac{2}{3}E_s(2H - H_0 + h_0) + \frac{2}{3}(\sigma_s - E_s\varepsilon_s)[f(H, H_0) + f(h_0, -H)] \right\} e_{jj}$$

$$+ \left\{ (H_0^2 - h_0^2)\left(\frac{Ea_1}{2} - \frac{2E_s}{3}\right) + \frac{4}{3}(\sigma_s - E_s\varepsilon_s)[f_z(H, H_0) + f_z(h_0, -H)] \right\} \kappa_{ii}$$

$$+ \left\{ (H_0^2 - h_0^2)\left(\frac{Ea_1\nu}{2} - \frac{E_s}{3}\right) + \frac{2}{3}(\sigma_s - E_s\varepsilon_s)[f_z(H, H_0) + f_z(h_0, -H)] \right\} \kappa_{jj}$$

$$- \alpha\left(\left\{ \frac{Ea_1(H_0 - h_0)}{a_2} + 2E_s(2H - H_0 + h_0) \quad + 2(\sigma_s - E_s\varepsilon_s)[f(H, H_0) + f(h_0, -H)] \right\} T_0 \right.$$

$$\left. + \left\{ (H_0^2 - h_0^2)\left(\frac{Ea_1}{2a_2} - E_s\right) + 2(\sigma_s - E_s\varepsilon_s)[f_z(H, H_0) + f_z(h_0, -H)] \right\} T_z \right),$$

$$F_{ij}^0 = \left\{ Ea_2(H_0 - h_0) + \frac{2}{3}E_s(2H - H_0 + h_0) + \frac{2}{3}(\sigma_s - E_s\varepsilon_s)[f(H, H_0) \right.$$

$$+f(h_0,-H)]\Big\}e_{ij}+\left\{(H_0^2-h_0^2)\left(\frac{Ea_2}{2}-\frac{E_s}{3}\right)+\frac{2}{3}(\sigma_s-E_s\varepsilon_s)[f_z(H,H_0)+f_z(h_0,-H)]\right\}\kappa_{ij}, \tag{1.39}$$

$$M_{ii}^0=\left\{(H_0^2-h_0^2)\left(\frac{Ea_1}{2}-\frac{2E_s}{3}\right)+\frac{4}{3}(\sigma_s-E_s\varepsilon_s)[f_z(H,H_0)+f_z(h_0,-H)]\right\}e_{ii}$$

$$+\left\{(H_0^2-h_0^2)\left(\frac{Ea_1\nu}{2}-\frac{E_s}{3}\right)+\frac{2}{3}(\sigma_s-E_s\varepsilon_s)[f_z(H,H_0)+f_z(h_0,-H)]\right\}e_{jj}$$

$$+\left\{\frac{Ea_1(H_0^3-h_0^3)}{3}+\frac{4}{9}E_s(2H^3-H_0^3+h_0^3)+\frac{4}{3}(\sigma_s-E_s\varepsilon_s)[f_{zz}(H,H_0)+f_{zz}(h_0,-H)]\right\}\kappa_{ii}$$

$$+\left\{\frac{Ea_1\nu(H_0^3-h_0^3)}{3}+\frac{2}{9}E_s(2H^3-H_0^3+h_0^3)+\frac{2}{3}(\sigma_s-E_s\varepsilon_s)[f_{zz}(H,H_0)+f_{zz}(h_0,-H)]\right\}\kappa_{jj}$$

$$-\alpha\left(\left\{(H_0^2-h_0^2)\left(\frac{Ea_1}{2a_2}-E_s\right)+2(\sigma_s-E_s\varepsilon_s)[f_z(H,H_0)+f_z(h_0,-H)]\right\}T_0\right.$$

$$\left.+\left\{\frac{Ea_1(H_0^3-h_0^3)}{3a_2}+\frac{2}{3}E_s(2H^3-H_0^3+h_0^3)+2(\sigma_s-E_s\varepsilon_s)[f_{zz}(H,H_0)+f_{zz}(h_0,-H)]\right\}T_z\right), \tag{1.39}$$

$$M_{ij}^0=\left\{(H_0^2-h_0^2)\left(\frac{Ea_2}{2}-\frac{E_s}{3}\right)+\frac{2}{3}(\sigma_s-E_s\varepsilon_s)[f_z(H,H_0)+f_z(h_0,-H)]\right\}e_{ij}$$

$$+\left\{\frac{Ea_2(H_0^3-h_0^3)}{3}+\frac{2}{9}E_s(2H^3-H_0^3+h_0^3)+\frac{2}{3}(\sigma_s-E_s\varepsilon_s)[f_{zz}(H,H_0)+f_{zz}(h_0,-H)]\right\}\kappa_{ij},\ j=3-i,\quad i=1,2,$$

where

$$f(b,c)=\int_c^b\frac{dz}{\sqrt{\kappa_*^2z^2+e_\kappa z+e_*^2}},\quad f_z(b,c)=\int_c^b\frac{zdz}{\sqrt{\kappa_*^2z^2+e_\kappa z+e_*^2}},$$
$$f_{zz}(b,c)=\int_c^b\frac{z^2dz}{\sqrt{\kappa_*^2z^2+e_\kappa z+e_*^2}}. \tag{1.40}$$

If the transverse-longitudinal bending ($\kappa_*^2>0$ and $e_*^2>0$) is realized at a point $(x_1,x_2)\in G$ and inequalities (1.37) are fulfilled, integrals (1.40) have the form [15]

$$f(b,c)=\frac{1}{\kappa_*}\left(\sinh^{-1}\frac{2\kappa_*^2b+e_\kappa}{\sqrt{4\kappa_*^2e_*^2-e_\kappa^2}}-\sinh^{-1}\frac{2\kappa_*^2c+e_\kappa}{\sqrt{4\kappa_*^2e_*^2-e_\kappa^2}}\right),$$

$$f_z(b,c)=\frac{1}{\kappa_*^2}\left[\sqrt{\kappa_*^2b^2+e_\kappa b+e_*^2}-\sqrt{\kappa_*^2c^2+e_\kappa c+e_*^2}-0.5e_\kappa f(b,c)\right], \tag{1.41}$$

$$f_{zz}(b,c)=\frac{1}{4\kappa_*^4}\Bigg[(2\kappa_*^2 b-3e_\kappa)\sqrt{\kappa_*^2 b^2+e_\kappa b+e_*^2}$$

$$-(2\kappa_*^2 c-3e_\kappa)\sqrt{\kappa_*^2 c^2+e_\kappa c+e_*^2}+0.5(3e_\kappa^2-4\kappa_*^2 e_*^2)f(b,c)\Bigg],$$

$$\sinh^{-1}(y)=\ln\left(y+\sqrt{y^2+1}\right)\quad (e_*>0,\ \kappa_*>0).$$

If the transverse bending ($\kappa_*^2>0,\ e_*^2=e_\kappa=0$) is realized at a point $(x_1,x_2)\in G$, then $H_0=-h_0>0$ [see Eqs. (1.26)-(1.30)], and expressions (1.39) contain the cofactors [see Eqs. (1.40)]

$$f(H,H_0)+f(h_0,-H)=\int_{H_0}^{H}\frac{dz}{|z|\kappa_*}+\int_{-H}^{-H_0}\frac{dz}{|z|\kappa_*}=\frac{2}{\kappa_*}\int_{H_0}^{H}\frac{dz}{z}=\frac{2}{\kappa_*}\ln\left(\frac{H}{H_0}\right),$$

$$f_z(H,H_0)+f_z(h_0,-H)=\int_{H_0}^{H}\frac{zdz}{|z|\kappa_*}+\int_{-H}^{-H_0}\frac{zdz}{|z|\kappa_*}=0, \tag{1.42}$$

$$f_{zz}(H,H_0)+f_{zz}(h_0,-H)=\int_{H_0}^{H}\frac{z^2dz}{|z|\kappa_*}+\int_{-H}^{-H_0}\frac{z^2dz}{|z|\kappa_*}=\frac{2}{\kappa_*}\int_{H_0}^{H}zdz=\frac{1}{\kappa_*}(H^2-H_0^2),$$

$$\kappa_*^2>0,\quad e_*^2=e_\kappa=0,\ H_0=-h_0>0$$

(this case is considered in detail in [16]). For the case where the generalized plane stress state ($(\kappa_*^2=e_\kappa=0,\ e_*^2>0)$ is realized at a point $(x_1,x_2)\in G$, we have $H_0=H$ and $h_0=-H$ at $e_*^2\le\varepsilon_s^2$ (the binder is in the linearly elastic state), and consequently

$$f(H,H_0)+f(h_0,-H)=f_z(H,H_0)+f_z(h_0,-H)$$
$$=f_{zz}(H,H_0)+f_{zz}(h_0,-H)=0 \tag{1.43}$$

$$(0<e_*^2\le\varepsilon_s^2,\ \kappa_*^2=e_\kappa=0,\ H_0=H,\ h_0=-H),$$

in (1.39). At $e_*^2>\varepsilon_s^2>0$ (the binder in the plastic state), we have $H_0=-H$ and $h_0=-H$, and therefore

$$f(H,H_0)+f(h_0,-H)=\int_{-H}^{H}\frac{dz}{e_*}=\frac{2H}{e_*},$$

$$f_z(H,H_0)+f_z(h_0,-H)=\int_{-H}^{H}\frac{zdz}{e_*}=0, \tag{1.44}$$

$$f_{zz}(H,H_0)+f_{zz}(h_0,-H)=\int_{-H}^{H}\frac{z^2dz}{e_*}=\frac{2H^3}{3e_*}\quad (e_*^2>\varepsilon_s^2>0,\ \kappa_*^2=e_\kappa=0)$$

in (1.39).

Thus, relations (1.39) determine the functions F_{ij}^0 and M_{ij}^0 in terms of the membrane strains e_{ij}, the parameters of curvature κ_{ij} (i, j = 1, 2) of the plate midplane, and the temperature (T_0, T_z).

Now, we substitute the functions F_{ij}^0, M_{ij}^0, F_k, and M_k from Eqs. (1.16) and (1.39) into Eq. (1.5) and find the internal force factors F_{ij} and M_{ij} arising in the reinforced plate in thermoelastoplastic bending

$$F_{ij} = \sum_{m=1}^{2}\sum_{l=1}^{2}(A_{ijml}e_{ml} + B_{ijml}\kappa_{ml}) + E_{ij},$$

$$M_{ij} = \sum_{m=1}^{2}\sum_{l=1}^{2}(B_{ijml}e_{ml} + C_{ijml}\kappa_{ml}) + D_{ij} \quad (i,\, j = 1,\, 2), \tag{1.45}$$

where

$$A_{iiii} = a\left\{Ea_1(H_0 - h_0) + \frac{4}{3}E_{\mathrm{s}}(2H - H_0 + h_0)\right.$$

$$\left. + \frac{4}{3}(\sigma_{\mathrm{s}} - E_{\mathrm{s}}\varepsilon_{\mathrm{s}})[f(H,H_0) + f(h_0,-H)]\right\}$$

$$+ \sum_k \omega_k l_{ki}^4 [E_k(H_k - h_k) + E_{\mathrm{s}k}(2H - H_k + h_k)],$$

$$A_{iijj} = a\left\{Ea_1\nu(H_0 - h_0) + \frac{2}{3}E_{\mathrm{s}}(2H - H_0 + h_0)\right. \tag{1.46}$$

$$\left. + \frac{2}{3}(\sigma_{\mathrm{s}} - E_{\mathrm{s}}\varepsilon_{\mathrm{s}})[f(H,H_0) + f(h_0,-H)]\right\}$$

$$+ \sum_k \omega_k l_{ki}^2 l_{kj}^2 [E_k(H_k - h_k) + E_{\mathrm{s}k}(2H - H_k + h_k)],$$

$$A_{iiij} = \sum_k \omega_k l_{ki}^3 l_{kj} [E_k(H_k - h_k) + E_{\mathrm{s}k}(2H - H_k + h_k)],$$

$$A_{ijij} = a\left\{Ea_2\frac{H_0 - h_0}{2} + \frac{1}{3}E_{\mathrm{s}}(2H - H_0 + h_0)\right.$$

$$\left. + \frac{1}{3}(\sigma_{\mathrm{s}} - E_{\mathrm{s}}\varepsilon_{\mathrm{s}})[f(H,H_0) + f(h_0,-H)]\right\}$$

$$+ \sum_k \omega_k l_{ki}^2 l_{kj}^2 [E_k(H_k - h_k) + E_{\mathrm{s}k}(2H - H_k + h_k)],$$

$$B_{iiii} = a\left\{(H_0^2 - h_0^2)\left(\frac{Ea_1}{2} - \frac{2E_{\mathrm{s}}}{3}\right) + \frac{4}{3}(\sigma_{\mathrm{s}} - E_{\mathrm{s}}\varepsilon_{\mathrm{s}})[f_z(H,H_0) + f_z(h_0,-H)]\right\}$$

$$+ 0.5\sum_k \omega_k l_{ki}^4 (E_k - E_{\mathrm{s}k})(H_k^2 - h_k^2),$$

$$B_{iijj} = a\left\{(H_0^2 - h_0^2)\left(\frac{Ea_1\nu}{2} - \frac{E_{\mathrm{s}}}{3}\right) + \frac{2}{3}(\sigma_{\mathrm{s}} - E_{\mathrm{s}}\varepsilon_{\mathrm{s}})[f_z(H,H_0) + f_z(h_0,-H)]\right\}$$

$$+0.5\sum_k \omega_k l_{ki}^2 l_{kj}^2 (E_k - E_{sk})(H_k^2 - h_k^2),$$

$$B_{iiij} = 0.5\sum_k \omega_k l_{ki}^3 l_{kj} (E_k - E_{sk})(H_k^2 - h_k^2),$$

$$B_{ijij} = a\left\{0.5(H_0^2 - h_0^2)\left(\frac{Ea_2}{2} - \frac{E_s}{3}\right) + \frac{1}{3}(\sigma_s - E_s\varepsilon_s)[f_z(H, H_0) + f_z(h_0, -H)]\right\}$$

$$+0.5\sum_k \omega_k l_{ki}^2 l_{kj}^2 (E_k - E_{sk})(H_k^2 - h_k^2),$$

$$C_{iiii} = \frac{a}{3}\left\{Ea_1(H_0^3 - h_0^3) + \frac{4}{3}E_s(2H^3 - H_0^3 + h_0^3)\right.$$

$$\left. +4(\sigma_s - E_s\varepsilon_s)[f_{zz}(H, H_0) + f_{zz}(h_0, -H)]\right\}$$

$$+\frac{1}{3}\sum_k \omega_k l_{ki}^4 [E_k(H_k^3 - h_k^3) + E_{sk}(2H^3 - H_k^3 + h_k^3)],$$

$$C_{iijj} = \frac{a}{3}\left\{Ea_1\nu(H_0^3 - h_0^3) + \frac{2}{3}E_s(2H^3 - H_0^3 + h_0^3)\right.$$

$$\left. +2(\sigma_s - E_s\varepsilon_s)[f_{zz}(H, H_0) + f_{zz}(h_0, -H)]\right\}$$

$$+\frac{1}{3}\sum_k \omega_k l_{ki}^2 l_{kj}^2 [E_k(H_k^3 - h_k^3) + E_{sk}(2H^3 - H_k^3 + h_k^3)],$$

$$C_{iiij} = \frac{1}{3}\sum_k \omega_k l_{ki}^3 l_{kj} [E_k(H_k^3 - h_k^3) + E_{sk}(2H^3 - H_k^3 + h_k^3)], \tag{1.46}$$

$$C_{ijij} = \frac{a}{6}\left\{Ea_2(H_0^3 - h_0^3) + \frac{2}{3}E_s(2H^3 - H_0^3 + h_0^3)\right.$$

$$\left. +2(\sigma_s - E_s\varepsilon_s)[f_{zz}(H, H_0) + f_{zz}(h_0, -H)]\right\}$$

$$+\frac{1}{3}\sum_k \omega_k l_{ki}^2 l_{kj}^2 [E_k(H_k^3 - h_k^3) + E_{sk}(2H^3 - H_k^3 + h_k^3)],$$

$$E_{ii} = -\alpha a\left(\left\{Ea_1\frac{H_0 - h_0}{2a_2} + 2E_s(2H - H_0 + h_0)\right.\right.$$

$$\left. +2(\sigma_s - E_s\varepsilon_s)[f(H, H_0) + f(h_0, -H)]\right\}T_0$$

$$\left. +\left\{(H_0^2 - h_0^2)\left(\frac{Ea_1}{2a_2} - E_s\right) + 2(\sigma_s - E_s\varepsilon_s)[f_z(H, H_0) + f_z(h_0, -H)]\right\}T_z\right)$$

$$+\sum_k \omega_k l_{ki}^2\Big((\sigma_{sk}-E_{sk}\varepsilon_{sk})[\operatorname{sgn}(e_k+H\kappa_k)(H-H_k)$$

$$+\operatorname{sgn}(e_k-H\kappa_k)(h_k+H)]-\alpha_k\{[E_k(H_k-h_k)+E_{sk}(2H-H_k+h_k)]T_0$$

$$+0.5(H_k^2-h_k^2)(E_k-E_{sk})T_z)\}\Big),$$

$$E_{ij}=\sum_k \omega_k l_{ki} l_{kj}\Big((\sigma_{sk}-E_{sk}\varepsilon_{sk})[\operatorname{sgn}(e_k+H\kappa_k)(H-H_k)$$

$$+\operatorname{sgn}(e_k-H\kappa_k)(h_k+H)]-\alpha_k\{[E_k(H_k-h_k)+E_{sk}(2H-H_k+h_k)]T_0$$

$$+0.5(H_k^2-h_k^2)(E_k-E_{sk})T_z)\}\Big),$$

$$D_{ii}=-\alpha a\left(\left\{(H_0^2-h_0^2)\left(\frac{Ea_1}{2a_2}-E_s\right)+2(\sigma_s-E_s\varepsilon_s)[f_z(H,H_0)+f_z(h_0,-H)]\right\}T_0+\left\{Ea_1\frac{H_0^3-h_0^3}{3a_2}+\frac{2}{3}E_s(2H^3-H_0^3+h_0^3)\right.\right.$$

$$\left.\left.+2(\sigma_s-E_s\varepsilon_s)[f_{zz}(H,H_0)+f_{zz}(h_0,-H)]\right\}T_z\right)$$

$$+\sum_k \omega_k l_{ki}^2\Big(0.5(\sigma_{sk}-E_{sk}\varepsilon_{sk})[(H^2-H_k^2)\operatorname{sgn}(e_k+H\kappa_k)$$

$$+\operatorname{sgn}(e_k-H\kappa_k)(h_k^2-H^2)]-\alpha_k\Big\{0.5(H_k^2-h_k^2)(E_k-E_{sk})T_0$$

$$+\frac{1}{3}[E_k(H_k^3-h_k^3)+E_{sk}(2H^3-H_k^3+h_k^3)]T_z\Big\}\Big),\qquad(1.46)$$

$$D_{ij}=\sum_k \omega_k l_{ki} l_{kj}\Big(0.5(\sigma_{sk}-E_{sk}\varepsilon_{sk})[(H^2-H_k^2)\operatorname{sgn}(e_k+H\kappa_k)$$

$$+\operatorname{sgn}(e_k-H\kappa_k)(h_k^2-H^2)]-\alpha_k\Big\{0.5(H_k^2-h_k^2)(E_k-E_{sk})T_0$$

$$+\frac{1}{3}[E_k(H_k^3-h_k^3)+E_{sk}(2H^3-H_k^3+h_k^3)]T_z\Big\}\Big),\quad j=3-i,\ i=1,2,$$

$$A_{iijj}=A_{jjii},\quad A_{iiij}=A_{ijii}=A_{jiii}=A_{iiji},$$
$$A_{ijij}=A_{ijji}=A_{jiij}=A_{jiji},\quad j=3-i,\ i=1,2.\qquad(1.47)$$

Equalities similar to Eqs. (1.47) are also valid for the coefficients B_{ijml} and C_{ijml}.

Thus, Eqs. (1.45), with account of Eqs. (1.46), (1.47), (1.41)-(1.44), (1.36), (1.25)-(1.32), (1.9), and (1.13)-(1.15), determine the internal force factors F_{ij} and M_{ij} in terms of the membrane strains e_{ij}, the parameters of curvature of the plate midplane κ_{ij} ($i, j = 1, 2$), the temperature (T_0, T_z), and the reinforcement parameters ψ_k and ω_k ($1 \le k \le N$). These relationships are valid not only in Cartesian but also in any curvilinear orthogonal system of coordinates in which the z axis is rectilinear. Although the above-mentioned equalities were derived without account of thermosensitivity of phase materials, they also hold true in the case where the thermosensitivity of substructural elements of the composition is taken into account if the temperature T is constant across the thickness of the structure ($T_z \equiv 0$ or $|HT_z| \ll |T_0|$). In this case, the physicomechanical characteristics of phase materials of the composition depend on the temperature T_0, i.e., they are known functions of the variables x_1 and x_2.

To complete the formulation of the problem on the elastoplastic transverse-longitudinal bending of reinforced plates, Eqs. (1.10), (1.45), and (1.46) must be combined with the well-known equilibrium equations [1, 2]

$$F_{i1,1} + F_{i2,2} = -p_i(x_1, x_2, \omega), \quad M_{i1,1} + M_{i2,2} + m_i(x_1, x_2, \omega) = F_{zi} \quad (i = 1, 2),$$
$$F_{z1,1} + F_{z2,2} = -p_z(x_1, x_2, \omega), \quad \omega = \{\omega_1, \omega_2, \ldots, \omega_N\}, \tag{1.48}$$

where F_{zi} are the shear forces; p_i and p_z are the reduced external distributed loads in the x_i and z directions, respectively, which depend on ω_k if the mass loads are taken into account; m_i are the reduced external distributed bending moments.

In the case of a thin plate 3D-reinforced with continuous fibers of constant cross section, the connection between the reinforcement parameters is approximately determined by the equality [17]

$$(\omega_k H \cos\psi_k)_{,1} + (\omega_k H \sin\psi_k)_{,2} = 0, \quad k = 1, 2, .., N, \tag{1.49}$$

which, in the case of plane reinforcement, is reduced to the exact equality following from Eq. (1.49) at $H = 1$.

Let the region G occupied by the plate in the plan be restricted by a contour Γ. Then, on one part of this contour (designated by Γ_p), we can assign the static boundary conditions for the bending moment [1, 2]

$$M_{11} n_1^2 + M_{22} n_2^2 + 2M_{12} n_1 n_2 = M_n, \quad n_1 = \cos\beta, \ n_2 = \sin\beta, \ (x_1, x_2) \in \Gamma_p, \tag{1.50}$$

for the reduced transverse Kirchhoff force

$$F_{z1} n_1 + F_{z2} n_2 + \partial_\tau (M_{n\tau}) = F_{nz}, \quad (x_1, x_2) \in \Gamma_p,$$
$$M_{n\tau} = (M_{22} - M_{11}) n_1 n_2 + M_{12}(n_1^2 - n_2^2), \ \partial_\tau (M_{n\tau}) = -n_2 M_{n\tau,1} + n_1 M_{n\tau,2} \tag{1.51}$$

and for the membrane forces

$$F_{11} n_1^2 + F_{22} n_2^2 + 2F_{12} n_1 n_2 = F_n, \quad (F_{22} - F_{11}) n_1 n_2 + F_{12}(n_1^2 - n_2^2) = F_\tau. \tag{1.52}$$

On the other part of the contour (designated by Γ_u), we can set the kinematic conditions

$$w(\Gamma_u) = w_0, \quad w_{,1} n_1 + w_{,2} n_2 = \theta_n, \quad u_i(\Gamma_u) = u_{0i} \quad (i = 1, 2), \ (x_1, x_2) \in \Gamma_u, \tag{1.53}$$

where M_n and F_{nz} are the bending moment and the reduced transverse Kirchhoff force given on the part of the contour Γ_p; F_n and F_τ are the membrane forces normal and tangential to the contour Γ_p, respectively; w_0 and u_{0i} are the deflection and the displacements of points of the plate midplane along the x_i directions on the contour Γ_u; θ_n is the derivative of the deflection along the external normal to the contour, specified by an angle β; ∂_τ is the derivative along the contour. (On the contour Γ, mixed boundary conditions from Eqs. (1.50)-(1.53), for example, the conditions of simple support, can also be given.)

On the part of the contour Γ (designated by Γ_k) where the fibers of a kth family, having constant cross sections, enter the region G, we must assign the boundary conditions for the intensity of reinforcement [17]

$$\omega_k(\Gamma_k) = \omega_{0k}, \quad k = 1, 2, .., N, \tag{1.54}$$

where ω_{0k} are functions given on the contour Γ_k.

The solution to the problem on thermoelastoplastic bending of the reinforced plate must satisfy the strength restrictions [3, 10-12]

$$\sigma_*(x_1, x_2, \pm H) \le \sigma_0^*, \quad -\sigma_k^- \le \sigma_k(x_1, x_2, \pm H) \le \sigma_k^+,$$

$$\sigma_0^* > 0, \quad \sigma_k^\pm > 0, \quad 1 \le k \le N, \tag{1.55}$$

where σ_0^* is the ultimate strength of the binder, which is equal to the yield point σ_s in thermoelastic bending or to the ultimate strength σ_y in thermoelastoplastic bending; σ_k^- and σ_k^+ are the ultimate strengths of fibers of a kth family in compression and tension, respectively (under the action of compressive loads, some mode of buckling instability of the fibers can arise; therefore, $\sigma_k^- \ne \sigma_k^+$ in the general case).

2. System of Resolving Equations

To obtain a system of resolving equations for the elastoplastic transverse-longitudinal bending of reinforced plates and the corresponding static boundary conditions in deflections and displacements, we have to substitute the expressions of e_{ij} and κ_{ij} from Eqs. (1.10) into Eqs. (1.45), insert the resulting relations into equilibrium equations (1.48) and boundary conditions (1.50)-(1.52), and exclude the transverse forces F_{zi} from consideration. Then the equilibrium equations take the form

$$\sum_{j=1}^{2}\left\{\sum_{m=1}^{2}\sum_{l=1}^{2}\left[\frac{1}{2}A_{ijml}(u_{m,l}+u_{l,m})-B_{ijml}w_{,ml}\right]+E_{ij}\right\}_{,j}=-p_i(x_1,x_2,\omega)$$

$$(i=1,2), \tag{2.1}$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\left\{\sum_{m=1}^{2}\sum_{l=1}^{2}\left[\frac{1}{2}B_{ijml}(u_{m,l}+u_{l,m})-C_{ijml}w_{,ml}\right]+D_{ij}\right\}_{,ij}=-p_z-m_{1,1}-m_{2,2},$$

and the static boundary conditions are transformed to the following forms: for the bending moment

$$(B_{1111}\cos^2\beta+B_{1122}\sin^2\beta+B_{1112}\sin 2\beta)u_{1,1}+(B_{2211}\cos^2\beta+B_{2222}\sin^2\beta$$

$$+B_{2212}\sin 2\beta)u_{2,2}+(B_{1211}\cos^2\beta+B_{1222}\sin^2\beta+B_{1212}\sin 2\beta)(u_{1,2}+u_{2,1})$$

$$-(C_{1111}\cos^2\beta+C_{1122}\sin^2\beta+C_{1112}\sin 2\beta)w_{,11}-(C_{2211}\cos^2\beta$$

$$+C_{2222}\sin^2\beta+C_{2212}\sin 2\beta)w_{,22}-2(C_{1211}\cos^2\beta+C_{1222}\sin^2\beta$$

$$+C_{1212}\sin 2\beta)w_{,12}+D_{11}\cos^2\beta+D_{22}\sin^2\beta+D_{12}\sin 2\beta=M_n,$$

$$(x_1,x_2)\in\Gamma_p \tag{2.2}$$

for the reduced Kirchhoff force

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\left\{\sum_{m=1}^{2}\sum_{l=1}^{2}\left[\frac{1}{2}B_{ijml}(u_{m,l}+u_{l,m})-C_{ijml}w_{,ml}\right]+D_{ij}\right\}_{,j}n_i$$

$$+\frac{1}{2}\partial_\tau \{[(B_{1122}-B_{1111})\sin 2\beta + 2B_{1112}\cos 2\beta]u_{1,1} + [(B_{2222}-B_{2211})\sin 2\beta$$

$$+2B_{2212}\cos 2\beta]u_{2,2} + [(B_{1222}-B_{1211})\sin 2\beta + 2B_{1212}\cos 2\beta](u_{1,2}+u_{2,1})$$

$$-[(C_{1122}-C_{1111})\sin 2\beta + 2C_{1112}\cos 2\beta]w_{,11} - [(C_{2222}-C_{2211})\sin 2\beta$$

$$+2C_{2212}\cos 2\beta]w_{,22} - 2[(C_{1222}-C_{1211})\sin 2\beta + 2C_{1212}\cos 2\beta]w_{,12}$$

$$+(D_{22}-D_{11})\sin 2\beta + 2D_{12}\cos 2\beta\} = F_{nz} - m_1 n_1 - m_2 n_2, \quad (x_1, x_2) \in \Gamma_p, \tag{2.3}$$

and for the membrane forces

$$(A_{1111}\cos^2\beta + A_{1122}\sin^2\beta + A_{1112}\sin 2\beta)u_{1,1} + (A_{2211}\cos^2\beta + A_{2222}\sin^2\beta$$

$$+A_{2212}\sin 2\beta)u_{2,2} + (A_{1211}\cos^2\beta + A_{1222}\sin^2\beta + A_{1212}\sin 2\beta)(u_{1,2}+u_{2,1})$$

$$-(B_{1111}\cos^2\beta + B_{1122}\sin^2\beta + B_{1112}\sin 2\beta)w_{,11} - (B_{2211}\cos^2\beta$$

$$+B_{2222}\sin^2\beta + B_{2212}\sin 2\beta)w_{,22} - 2(B_{1211}\cos^2\beta + B_{1222}\sin^2\beta$$

$$+B_{1212}\sin 2\beta)w_{,12} + E_{11}\cos^2\beta + E_{22}\sin^2\beta + E_{12}\sin 2\beta = F_n, \ (x_1, x_2) \in \Gamma_p, \tag{2.4}$$

$$[(A_{1122}-A_{1111})\sin 2\beta + 2A_{1112}\cos 2\beta]u_{1,1} + [(A_{2222}-A_{2211})\sin 2\beta$$

$$+2A_{2212}\cos 2\beta]u_{2,2} + [(A_{1222}-A_{1211})\sin 2\beta + 2A_{1212}\cos 2\beta](u_{1,2}+u_{2,1})$$

$$-[(B_{1122}-B_{1111})\sin 2\beta + 2B_{1112}\cos 2\beta]w_{,11} - [(B_{2222}-B_{2211})\sin 2\beta$$

$$+2B_{2212}\cos 2\beta]w_{,22} - 2[(B_{1222}-B_{1211})\sin 2\beta + 2B_{1212}\cos 2\beta]w_{,12}$$

$$+(E_{22}-E_{11})\sin 2\beta + 2E_{12}\cos 2\beta = F_\tau, \quad (x_1, x_2) \in \Gamma_p.$$

Kinematic boundary conditions (1.53) remain the same.

As follows from Eqs. (1.46) with account of Eqs. (1.47), (1.41)-(1.44), (1.36), (1.25)-(1.32), (1.9), and (1.13)-(1.15), the coefficients A_{ijml}, B_{ijml}, C_{ijml}, D_{ij}, and E_{ij} in Eqs. (2.1) and boundary conditions (2.2)-(2.4) nonlinearly depend on the plate thickness H, the derivatives with respect to the displacements and deflection $u_{m,l}$ and $w_{,ml}$, the temperatures T_0 and T_z, and the reinforcement parameters ψ_k and ω_k. For linearly elastic bending, we must assume that $H_k = H_0 = -h_k = -h_0 = H$ in Eq. (1.46). Then, from Eqs. (2.1)-(2.4), we obtain equilibrium equations and static boundary conditions linear in u_i and w. Since $B_{ijml} = 0$ [see Eq. (1.46)], the problem splits into two independent subproblems:

- determination of the displacements u_i, $i = 1, 2$ [the first two equations of (2.1), static boundary conditions (2.4), and the last two kinematic conditions of (1.53)] and
- determination of the deflection w [the third equation of (2.1), static boundary conditions (2.2) and (2.3), and the first two kinematic conditions of (1.53)].

$B_{ijml} \neq 0$ in the nonlinearly elastic or elastoplastic transverse-longitudinal bending, and the problems of determining the displacements u_i and the deflection w are coupled. In this case, Eqs. (2.1)-(2.4) in their form (not in essence) are similar to the equilibrium equations and boundary conditions of the linearly elastic bending of layered anisotropic plates inhomogeneous across their thickness, with their structure asymmetric about the midplane [18].

If the thickness $2H$ of a plate and the reinforcement trajectories (i.e., the angles ψ_k) are given, the initial boundary-value problem (1.49), (1.54) determines the intensity ω_k of reinforcement with fibers of a kth family. (The initial boundary-value problems for the linear partial differential equations of the first order (1.49) are well investigated [19], therefore, we will not consider this question in more detail; we will only point out that Eq. (1.49) has real characteristics coinciding with the reinforcement trajectories of the kth family.) If the functions ψ_k, ω_k, and H are known (given) and satisfy physical restrictions (1.4), temperature field (1.1) in the plate can be determined (the corresponding boundary-value problems are formulated and analyzed, for example, in [20]), and then the three equilibrium equations (2.1) will be closed with respect to the three functions u_1, u_2, and w. To this system of quasi-linear (in the case of elastoplastic bending) or linear (in the case of linearly elastic bending) differential equations of elliptic type there correspond nonlinear or linear boundary conditions (1.53) and (2.2)-(2.4).

The basic difficulty arising in solving the formulated problem on the thermoelastoplastic bending of reinforced plates consists in its strong nonlinearity. This problem can be linearized by the following iterative process, which is an adaptation of the method of variable parameters of elasticity [10] as applied to bent plates. Let us assume that the coefficients A_{ijml}, B_{ijml}, C_{ijml}, D_{ij}, and E_{ij} in Eqs. (2.1)-(2.4) are known from the solution on the previous iteration [on the first iteration, a thermoelastic bending is assumed, i.e., $h_0 = h_k = -H$, $H_0 = H_k = H$, $1 \le k \le N$, see Eq. (1.46)]. Then, approximate values of the deflection w and displacements u_i can be determined by solving the linear boundary-value problem (1.53), (2.1)-(2.4) with variable coefficients. Using the approximations of the functions w and u_i ($i = 1, 2$) obtained from this boundary-value problem and employing Eqs. (1.9)-(1.15) and (1.27)-(1.30), we determine approximations for the z coordinates of boundaries between the elastic and plastic zones in composition phases, and then — from Eqs. (1.46) — new approximations for the coefficients A_{ijml}, B_{ijml}, C_{ijml}, D_{ij}, E_{ij}, etc., until the iterative process converges with the accuracy required. The convergence of the method of variable elastic parameters is proved in [10, 11].

Since, on each iteration of the process, we have to solve a linear boundary-value problem that is formally similar to the problem of thermoelastic bending of layered anisotropic and heterogeneous plates with layers of variable thickness, for integrating this boundary-value problem, we can use the known and well-developed finite-difference schemes, the methods of finite elements, or other approximate variational methods (see, for example, [1], etc.). We will not dwell on this question here.

3. Calculation Results and Discussion

As an example, let us consider the thermoelastoplastic bending of circular plates of a constant thickness $2H = 0.04$ m bounded by edges of radii $r_0 = 0.25$ m and $r_1 = 1$ m. The inner edges r_0 of the plates are rigidly fixed ($\theta_n = 0$, $w_0 = u_{0i} = 0$, $i = 1$, 2), but the outer edges r_1 are load-free ($M_n = 0$, $F_n = F_\tau = F_{nz} = 0$). The plates are subjected to a uniformly distributed transverse load $p_z = -p = \text{const}$ ($p_1 = p_2 = 0$, $m_1 = m_2 = 0$). The structures can operate either in the natural ($T \equiv 0$) or a heated ($T = \text{const} > 0$) state.

The loading level p is selected such that at least in one of the phase materials the plastic state ($p = p_{\text{s}} > 0$) (thermoelastic bending) is reached or the stress state reaches the ultimate strength ($p = p_{\text{y}} > 0$) (thermoelastoplastic bending).

Let us consider plates of three types: two isotropic plates made of an ML12 magnesium alloy and a 40X structural steel (annealed), respectively, and one metal-composite plate made of an ML12 alloy and reinforced radially and circumferentially with a U8A steel wire ($d = 80$ μm), which corresponds to reinforcement along the directions of main stresses and strains. The physicomechanical characteristics of the phase materials are presented in Table 1.

Let us assume that the reinforcing fibers do not break inside the plate and have a constant diameter d. Then, the density of reinforcement with fibers of the radial family varies according to the law [17]

$$\omega_1(r) = r_0 \omega_{01}/r, \quad \omega_{01} = \omega_1(r_0) = \text{const} \quad (r_0 \le r \le r_1), \tag{3.1}$$

where r is the polar radius. The density of reinforcement $\omega_2(r)$ with fibers of the circumferential family may be given arbitrarily [17], provided physical restrictions (1.4) are fulfilled. Therefore, we assign the quantity ω_2 in the form

$$\omega_2(r) = \omega_{01}(1 - r_0/r),$$

TABLE 1. Physicomechanical Characteristics of Phase Materials [7, 10]

Material	E, GPa	σ_s, MPa	σ_y, MPa	δ, %	ν	$\alpha \cdot 10^6$, K^{-1}
ML12 alloy	44.0	140.0	250.0	8.0	0.31	28.0
40X steel	205.0	428.0	700.0	25.0	0.3	15.0
U8A wire	200.0	3968.0	4408.0	8.2	0.3	15.0

TABLE 2. Ultimate Strength of Bent Plates

Material of a plate	T = 0°C		T = 80°C	
	p_s, kPa	p_y, kPa	p_s, kPa	p_y, kPa
ML12 alloy	84.2	273.0	12.1	260.0
40X steel	257.0	933.0	90.5	927.0
Metal-matrix composite (ML12/U8A)	212.0	1822.0	132.0	1819.0

which, with account of Eq. (3.1), ensures a constant total reinforcement density of the composite plate ($\omega_1 + \omega_2 = \omega_{01}$). In our calculations, we assume that ω_{01} = 0.6, which in practice corresponds to the reinforcement density close to the maximum admissible one.

Table 2 presents the ultimate values of p_s and p_y for the plates examined. The calculations were carried out for the temperature of natural state of the structure (T = 0°C) and for heating by T = 80°C. In the first and second cases, the transverse and the transverse-longitudinal bending are realized in the plate. The values of p_s and p_y are characterized by the occurrence of the ultimate (elastic and plastic) stress state in the binder on the lower (at T = 80°C) or on both (at T = 0°C) faces of the plate ($|z| = H$) in the neighborhood of the fixed inner edge $r = r_0$.

Figure 1a shows the profile of a plate and the traces of the surfaces demarcating the plastic and elastic zones in the phase materials at the instant of initial failure of the binder ($p = p_y$) at the temperature of natural state of the structure (T = 0°C). Since, at T = 0°C the transverse bending is realized, the curves with identical numbers are symmetric about the trace of the plate midplane (z = 0).

Figure 1b shows similar curves for the heated (T = 80°C) plates. Since the transverse-longitudinal bending is realized in such structures, the curves with identical numbers are not symmetric about the straight line z = 0.

The sharp converging of the elastoplastic boundaries at $r = r_0$ shown in Fig. 1 indicates that, by the moment of initial failure of all the plates considered, the stress state in the phase materials is close to the appearance of a plastic hinge in the neighborhood of the fixed edge.

A comparison of the values for T = 0 and 80°C presented in Table 2 shows that, in the elastic case, even an insignificant heating of the plates sharply (6.96 times for the structure with the ML12 alloy, 2.84 times with the 40X steel, and 1.61 times for the metal-composite plate) reduces its resistance to bending. But a comparison of the values of p_y allows us to conclude that, in the inelastic case, the heating of the structure affects its bending strength only slightly (in the example considered, the values of p_y at T = 80°C are lower than at T = 0°C only by 4.8% for the plate of the ML12 alloy, by 0.6% with the 40X steel, and by 0.2% for the metal-composite structure). As follows from a comparison between the values of p_s and p_y, the bearing capacity of the plates in elastoplastic bending is several times (in heating, by an order of magnitude and more) higher than in the elastic deformation, since the bearing capacity of the material in the first case is utilized completely.

From all the plates considered, the greatest bearing capacity was found for the isotropic plate made of 40X steel (see the values of p_s at T = 0°C) in elastic bending and for the composite plate in all the other cases. The last fact is explained by the

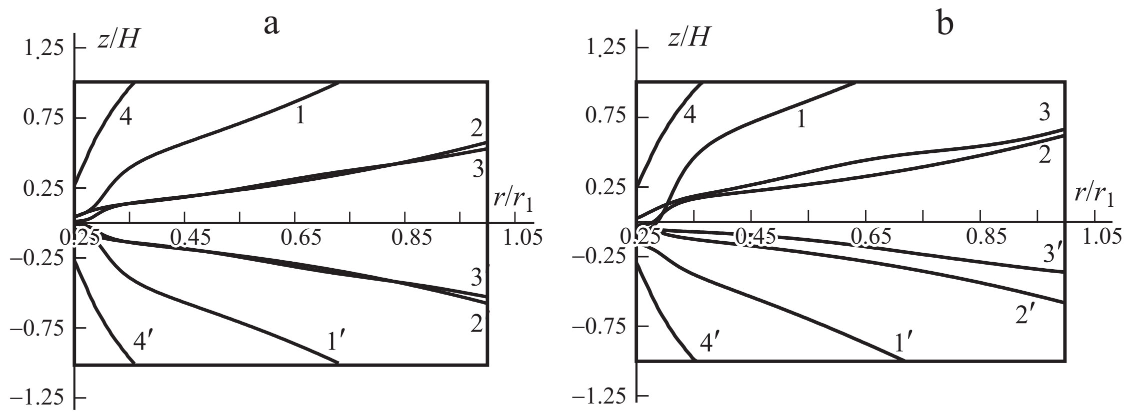

Fig. 1. Elastoplastic limits in phases of the composite in the lateral (a) and transverse-longitudinal bending (heating) (b): 1-4 — the boundary between the upper zone of plasticity and the elastic domain (the elastic domain lies between the curves with the same numbers). Lines 1 and 1′ define the elastoplastic limits in the isotropic plate made of an ML12 alloy; lines 2 and 2′ — for the 40X steel; 3 and 3′ — in the binder of the metal composite plate; 4 and 4′ — in reinforcing fibers of the radial family of the composite structure (the fibers of the circumferential family remain elastic).

high yield point and ultimate strength of the U8A reinforcing wire, whose yield point, for example, is almost tenfold higher than that of the 40X structural steel (compare the values of σ_s in Table 1). In elastic bending, the bearing capacity of the metal-composite plate is slightly lower than that of the steel one, since, in this case, the bearing capacity of fibers is utilized insignificantly. Indeed, it follows from the data in Table 1 that the ultimate elastic strain $\varepsilon_s = \sigma_s / E$ of the binder (ML12 alloy) in the composite plate is 6.24 times smaller than that of the U8A reinforcing wire. Therefore, by the moment of appearance of initial plastic strains in the binder, the stresses in the fibers do not exceed 16% of the yield point. Such a low degree of utilization of the bearing capacity of reinforcement is responsible for the low ultimate elastic resistance of the composite plate compared with that of the steel structure in elastic bending.

In the case of thermoelastic bending ($T = 80°$C), as a result of heating the structure and the significant (almost twofold) difference in the coefficients of linear thermal expansion of the binder and reinforcement (compare the values of α in Table 1), redistribution of stresses between the binder and fibers occurs. Therefore, the bearing capacity of the reinforcement in heating is utilized more completely than at $T = 0°$C, which ensures the higher bearing capacity of the metal-composite plate compared with that of the steel one. (In more detail, this effect is investigated in [21].)

Conclusions

The analysis of bending of reinforced and isotropic plates performed shows that the account of elastoplastic deformation of phase materials allows one to raise the bearing capacity of the structures several times (sometimes by an order of magnitude and more) compared with that in the case of elastic bending. The bearing capacity of metal-composite plates in thermoelastoplastic bending is a number of times higher than that of similar structures made of traditional structural metal alloys.

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