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TRACING OF PSEUDOTRAJECTORIES OF DYNAMICAL SYSTEMS AND STABILITY OF PROLONGATIONS OF ORBITS

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We investigate properties of dynamical systems associated with the approximation of pseudotrajectories of a dynamical system by its trajectories. According to modern terminology, a property of this sort is called the "property of tracing pseudotrajectories" (also known in the English literature as the "shadowing property"). We prove that dynamical systems given by mappings of a compact set into itself and possessing this property are systems with stable prolongation of orbits. We construct examples of mappings of an interval into itself that prove that the inverse statement is not true, i.e., that dynamical systems with stable prolongation of orbits may not possess the property of tracing pseudotrajectories.

1. Introduction

The principal question considered in the present paper is the following: What is the relationship between the property of dynamical systems to trace pseudotrajectories by "real" trajectories (see, e.g., [1], [2, Chap. 1, Sec. 3, Chap. 2, Appendix], and [3]) and the property of dynamical systems to possess a stable prolongation of orbits (see, e.g., [4–6] and [7, Chap. 3, Sec. 4])? The fact that a dynamical system possesses the property of tracing pseudotrajectories (for brevity, we shall say "a system with the property of tracing pseudotrajectories" instead of "a system possessing the property of tracing pseudotrajectories by its "real" trajectories") means the possibility of approximation of an arbitrary pseudotrajectory of a dynamical system by a certain trajectory of this system, i.e., the motion along an arbitrary pseudotrajectory occurs arbitrarily close to the motion along certain "real" trajectories. The fact that a system has a stable prolongation of orbits means the following: For any initial point, every point of the phase space attainable by some pseudotrajectory originating at this initial point is also attainable by a certain "real" trajectory that originates in a sufficiently small neighborhood of this initial point.

Thus, the definition of systems with the property of tracing pseudotrajectories and the definition of systems with stable prolongation of orbits are based on the concept of *pseudotrajectory* (ε-trajectory) and its closeness to a "real" trajectory. However, in the definition of a prolongation of an orbit, a "trajectory" is regarded as an unordered set on which the order of passing from one point to another is not indicated (i.e., the "motion" along a trajectory with time is not defined). On the other hand, in the definition of the property of tracing pseudotrajectories, a "trajectory" is regarded as an ordered set on which a linear order (determining the "motion" along a trajectory [8]) is given. In view of this, the property of tracing pseudotrajectories requires stronger restrictions on a system than those imposed by the property of stability of prolongations of orbits. Moreover, the set of systems with the property of tracing pseudotrajectories is narrower than the set of systems with stable prolongation of orbits. In what follows, we discuss this problem in more detail.

Also note the following important question: How typical are dynamical systems with the property of tracing pseudotrajectories? As is known [6], mappings with stable prolongation of orbits are typical. Namely, the set of such mappings forms a set of the second category in the set of all C^r-smooth mappings of a manifold into itself for any $r \geq 0$. The clarification of the correlation between the set of systems with stable prolongation of orbits and the set of systems with the property of tracing pseudotrajectories can also give an answer to this question.

2. Definitions and Notation

Let X be a compact set in $\mathbb{R}^m$ and let $\mathfrak{M} = C^r(X, X)$ be the space of C^r-smooth mappings $f\colon X \to X$, $r \geq 0$. We use the following notation: d denotes a metric in X (e.g., the Euclidean metric), ρ denotes the C^r-metric

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in the space of mappings $\mathfrak{M}$, $U_\varepsilon(x) = \{\tilde{x} \in X \mid d(x, \tilde{x}) < \varepsilon\}$ is the ε-neighborhood of a point $x \in X$, $\mathfrak{U}_\varepsilon(f) = \{\tilde{f} \in \mathfrak{M} \mid \rho(f, \tilde{f}) < \varepsilon\}$ is the ε-neighborhood of a mapping $f \in \mathfrak{M}$,

$$O(x,f) = \bigcup_{i=0}^{\infty} f^i(x)$$

denotes the f-orbit of a point $x \in X$, f^i is the ith iteration of a mapping f, and

$$\mathcal{T}(x,f) = \{x, f(x), f^2(x), \ldots, f^i(x), \ldots,\} = \{f^i(x)\}_{i=0}^{\infty}$$

denotes the f-trajectory of a point $x \in X$, i.e., a linearly ordered set $O(x,f)$.

Remark 1. As indicated above, we distinguish between the concept of "orbit" and the concept of "trajectory" of a point: An "orbit" is regarded as an unordered set, and a "trajectory" is regarded as an ordered set. The concept of space–time variation, namely, the motion along a trajectory with time, is essential in our investigation.

Following [2, Chap. 1, Sec. 3; Chap. 2, Appendix], [7, Chap. 3, Sec. 2], and [9-11], we introduce the notion of (ε, f)-trajectory.

Definition 1. *A sequence of points* $\{x_i\}_{i=0}^{n}$, $n < \infty$, *is called an* (ε, f)*-trajectory if* $d(x_{i+1}, f(x_i)) < \varepsilon$ *for* $0 \le i \le n-1$.

Remark 2. Up to now, we have used the term "pseudotrajectory" in the same sense as the term "(ε, f)-trajectory." Below, for the most part, we use the latter, more informative, notion.

Definition 2. *We say that an* (ε, f)*-trajectory* $\{x_i\}_{i=0}^{n}$, $n < \infty$, *is* δ*-traced by an f-trajectory of a point* $x \in X$ [*or an f-trajectory of a point* $x \in X$ δ*-traces an* (ε, f)*-trajectory* $\{x_i\}_{i=0}^{n}$, $n < \infty$] *if* $d(x_i, f^i(x)) < \delta$ *for* $0 \le i \le n$.

Definition 3. *A mapping* f *is called a mapping with the property of tracing pseudotrajectories if, for any* $\delta > 0$, *there exists* $\varepsilon = \varepsilon(\delta, f) > 0$ *such that, for any* (ε, f)*-trajectory, there exists an* f*-trajectory that* δ*-traces this* (ε, f)*-trajectory.*

It is also advisable to consider dynamical systems with weaker conditions on tracing of pseudotrajectories.

Definition 4. *A mapping* f *is called a mapping with nonuniform tracing of pseudotrajectories if, for any point* $x_0 \in X$ *and any* $\delta > 0$, *there exists* $\varepsilon = \varepsilon(\delta, f, x_0) > 0$ *such that, for any* (ε, f)*-trajectory originating at the point* x_0, *there exists an f-trajectory that* δ*-traces this* (ε, f)*-trajectory.*

Following [4–6] and [7, Chap. 3, Sec. 4], we present the definitions of prolongations of orbits.

Definition 5. *The prolongation of an f-orbit of a point* $x \in X$ *with respect to the initial point is defined as the set*

$$D(x,f) = \bigcap_{\varepsilon > 0} D_\varepsilon(x, f),$$

where

$$D_\varepsilon(x,f) = \overline{\bigcup_{\tilde{x}\in U_\varepsilon(x)} \bigcup_{i\ge 0} f^i(\tilde{x})} = \overline{\bigcup_{\tilde{x}\in U_\varepsilon(x)} O(\tilde{x}, f)}.$$

The prolongation of an f-orbit of a point $x \in X$ with respect to a dynamical system is defined as the set

$$P(x,f) = \bigcap_{\varepsilon>0} P_\varepsilon(x,f),$$

where

$$P_\varepsilon(x,f) = \overline{\bigcup_{\tilde{f}\in \mathfrak{U}_\varepsilon(f)} \bigcup_{i\ge 0} \tilde{f}^i(x)} = \overline{\bigcup_{\tilde{f}\in \mathfrak{U}_\varepsilon(x)} O(x, \tilde{f})}.$$

Definition 6. *We say that an f-orbit of a point $x \in X$ has a stable prolongation if $D(x,f) = P(x,f)$.*

Remark 3. The fact that an f-orbit of a point $x \in X$ has a stable prolongation means that the f-orbit is stable under constantly acting perturbations (see, e.g., [5]).

Definition 7. *A mapping f is called a mapping with stable prolongation of orbits if every f-orbit has a stable prolongation.*

3. Principal Results

Assume that $\mathfrak{N}$ is the subset of $\mathfrak{M}$ consisting of all mappings with stable prolongations of orbits, $\mathfrak{N}_{\text{trac}}$ is the subset of $\mathfrak{M}$ consisting of all mappings with tracing of pseudotrajectories, and $\mathfrak{N}^*_{\text{trac}}$ is the subset of $\mathfrak{M}$ consisting of all mappings with nonuniform tracing of pseudotrajectories.

Consider the problem of correlation between the sets $\mathfrak{N}$, $\mathfrak{N}_{\text{trac}}$, and $\mathfrak{N}^*_{\text{trac}}$.

Theorem 1. *The following chain of strict inclusions is true:*

$$\mathfrak{N}_{\text{trac}} \subset \mathfrak{N}^*_{\text{trac}} \subset \mathfrak{N}. \tag{1}$$

We prove Theorem 1 in several steps.

Theorem 1A. *The following strict inclusion is true:*

$$\mathfrak{N}_{\text{trac}} \subset \mathfrak{N}^*_{\text{trac}}. \tag{2}$$

Proof. The inclusion $\mathfrak{N}_{\text{trac}} \subseteq \mathfrak{N}^*_{\text{trac}}$ follows directly from the definitions. The fact that (2) is strict is proved by the following example:

Example 1. *The mapping of a closed interval of the real line into itself with stable prolongation of orbits that does not possess the property of tracing pseudotrajectories but possesses the property of nonuniform tracing pseudotrajectories.*

Let $f(x) = x + g(x)$, where $x \in I = [-1, 1]$,

$$g(x) = \begin{cases} x^{r+2}(1-x) & \text{for } x \in [0,1]; \\ \dfrac{1}{2C}\exp\left(\dfrac{1}{x}\right)\sin\left(\dfrac{1}{x}\right) & \text{for } x \in [-1,0), \end{cases}$$

and

$$C = \sup_{x \in [-1,0)} \left\{ \frac{1}{x^2}\exp\left(\frac{1}{x}\right) \right\}.$$

The mapping f is $(r+1)$-differentiable because $f'(\pm 0) = 1$ and $f^{(i)}(\pm 0) = 0$ for $1 < i \le r+1$ at the joining point $x = 0$ of two analytic parts.

On the interval $[0, 1]$, the mapping f has one attracting point $x = 1$ because $x < f(x) < 1$ for $x \in (0, 1)$ and, hence, $f^n(x) \to 1$ as $n \to \infty$.

On the interval $[-1, 0)$, the mapping f is monotonically increasing because

$$f'(x) = 1 - \frac{1}{2Cx^2}\exp\left(\frac{1}{x}\right)\left(\sin\frac{1}{x} + \cos\frac{1}{x}\right) > 0,$$

and has the fixed points $x_k = -1/k\pi$, $k = 1, 2, \dots$.

At the points $a_k = -1/(2k\pi)$, $k = 1, 2, \dots$, we have $f'(a_k) > 1$, while, at the points $b_k = -1/((2k+1)\pi)$, $k = 0, 1, \dots$, we have $f'(b_k) < 1$. Consequently, f has alternating repelling a_k and attracting b_k fixed points, which are concentrated toward the point 0; $f(x) > x$ for $x \in (a_k, b_k)$, and $f(x) < x$ for $x \in (b_k, a_{k+1})$, $k = 1, 2, \dots$.

The mapping f possesses the following properties:

I. The mapping f is a mapping with stable prolongation of orbits. In particular, $D(x,f) = P(x,f) = O(x,f) \cup \{b_{k-1}\}$ for $x \in (a_{k-1}, a_k)$, $k = 1, 2, \dots$, $D(a_k, f) = P(a_k, f) = [b_{k-1}, b_k]$ for $k = 1, 2, \dots$, $D(0,f) = P(0,f) = [0, 1]$, and $D(x,f) = P(x,f) = O(x,f) \cup \{1\}$ for $x \in (0, 1]$.

II. The mapping f does not possess the property of tracing pseudotrajectories. This means that there exists $\delta > 0$ such that, for any ε, one can find an (ε, f)-trajectory originating at a certain point $x_0 \in X$ and such that the f-trajectories of all points $x \in U_\delta(x_0)$ do not δ-trace this (ε, f)-trajectory.

Indeed, let us fix an arbitrarily small $\delta > 0$. One can find k_0 such that $|a_{k+1} - a_k| < \delta$ for $k \ge k_0$. We set $\varepsilon = a_{k_0+1} - a_{k_0}$.

We choose an arbitrary point x_0 from the interval $(-\gamma, 0)$, where $\gamma = \min\{\delta, \varepsilon\}/2$. The f-trajectory $\mathcal{T}(x,f)$ of any point $x \in U_\delta(x_0)$ lies in the interval $(-\delta, 0)$. At the same time, the sequence of points $y_0 = x_0$, $y_1 = \gamma/2$, $y_i = f^{i-1}(y_1)$, for $i > 1$, is an (ε, f)-trajectory because $d(y_1, f(y_0)) < 2\gamma = \min\{\delta, \varepsilon\} \le \varepsilon$. But it cannot be δ-traced by the f-trajectories of all points $x \in U_\delta(y_0)$ because $\mathcal{T}(x,f) \subset (-\delta, 0)$, and $y_i \in (0, 1)$ for $i \ge 1$.

III. The mapping f possesses the property of nonuniform tracing pseudotrajectories. To prove this fact, it suffices, for arbitrary fixed x_0 and $\delta > 0$, to choose $\varepsilon = \varepsilon(\delta, f, x_0) > 0$ so that $\varepsilon < \min\{x_0 - a_k, a_{k+1} - x_0\}/2$ for $x_0 \in (a_k, a_{k+1})$. Then the (ε, f)-trajectory lies in the interval $[a_k + \varepsilon, a_{k+1} - \varepsilon]$ and is δ-traced by a certain f-trajectory.

Now consider mappings with the property of nonuniform tracing pseudotrajectories.

Theorem 1B. *If* $f \in \mathfrak{N}^*_{\text{trac}}$, *then* $f \in \mathfrak{N}$.

Proof. Let us show that $D(x_0, f) = P(x_0, f)$ for an arbitrary point $x_0 \in X$.
We prove this theorem in several steps.

Lemma 1. *For arbitrary* $f \in \mathfrak{M}$ *and* $x \in X$, *the inclusion* $D(x,f) \subseteq P(x,f)$ *is true.*

Let us show that, for every point $x \in X$ and any $\varepsilon > 0$, there exists $\eta > 0$ such that $P_\varepsilon(x,f) \supseteq D_\eta(x,f)$.

1. Since the mapping f is continuous (moreover, it is C^r-smooth) on the compact set X, it is uniformly continuous, i.e., for any $\gamma > 0$, there exists $\beta = \beta(\gamma) > 0$ such that if $d(x', x'') < \beta$, then $d(f(x'), f(x'')) < \gamma$ for any points $x', x'' \in X$.

2. Let x_0 be an arbitrary point from X. Let us construct a family of special mappings close to f and different from f only in a sufficiently small neighborhood of the point x_0.

We choose and fix an arbitrary $\gamma > 0$. From the condition of uniform continuity of f, we choose the corresponding $\beta = \beta(\gamma) > 0$ and construct, for every point $\tilde{x} \in U_\beta(x_0)$, the following special mapping:

$$f_{\tilde{x},\beta}(x) = \begin{cases} f(x) & \text{for } x \notin U_\beta(x_0); \\ f(\tilde{x}) & \text{for } x = x_0; \\ f_\mu(x) & \text{for } x \in U_\beta(x_0) \setminus \{x_0\}, \end{cases}$$

where $f_\mu(x) = f_\mu(x, x_0, \beta)$ is a function that realizes joining with the corresponding C^r-smoothness. An example of the construction of the function $f_\mu(x)$ is given in [9].

We present the principal steps of the construction of such a function. Let $(U_\beta(x_0), F)$ be a chart such that $F(x_0) = 0$ and $U_\beta(x_0) \to \mathbb{R}^m$ (this is possible because $X \subset \mathbb{R}^m$). Let $\eta > 0$ be such that $U_{3\eta}(0) \subset F(U_\beta(x_0))$. Let us construct a C^r-retraction of the set $F(U_\beta(x_0))$ given by the relation $q_h(y) = y + h\mu_\eta(\|y\|)$, where $h \in \mathbb{R}^m$,

$$\|y\| = \left(\sum_{i=1}^{m} y_i^2\right)^{1/2}.$$

is the standard Euclidean norm in $\mathbb{R}^m$, and $\mu_\eta(\tau)$ is a C^∞-smooth Urysohn function such that

$$\mu_\eta(\tau) = \begin{cases} 1 & \text{for } \tau \le \eta; \\ 0 < \mu_\eta(\tau) < 1 & \text{for } \eta < \tau < 2\eta; \\ 0 & \text{for } \tau \ge 2\eta. \end{cases}$$

The mapping q_h linearly depends on h and is a C^r-homeomorphism close to the identical one. Further, we introduce the C^r-homeomorphism

$$Q_h = \begin{cases} id & \text{for } x \in X \setminus U_\beta(x_0); \\ F^{-1} \circ q_h \circ F & \text{for } x \in U_\beta(x_0), \end{cases}$$

whose order of closeness to the identical homeomorphism is determined by the order of smallness of $\|h\|$. And, finally, the mapping $f_\mu \big|_{x \in U_\beta(x_0) \setminus \{x_0\}} = Q_h \circ f \circ Q_h^{-1}$ is the required mapping that realizes a C^r-smooth joining.

3. By using the special mappings constructed above, we show that the prolongation of an f-orbit of a point x_0 with respect to the initial point is contained in the prolongation of an f-orbit of the point x_0 with respect to the dynamical system.

First, the mappings $f_{\bar{x},\beta}$ belong to $\mathfrak{U}_\gamma(f)$ and, second, they possess the property

$$O(\bar{x}, f) \subseteq \bigcup_{i \in \mathcal{J}_{\bar{x},\beta}} O(x_0, f_{\bar{x}_i,\beta}),$$

where $\mathcal{J}_{\bar{x},\beta}$ is the set of the numbers of iterations of the mapping $f_{\bar{x},\beta}$ such that $f^i_{\bar{x},\beta}(\bar{x}) = \bar{x}_i \in U_\beta(x_0)$ for $i \geq 0$ and, moreover, $\bar{x}_0 = \bar{x}$. (Note that if $f^i_{\bar{x},\beta}(\bar{x}) \notin U_\beta(x_0)$ for any $i > 0$, then $\mathcal{T}(\bar{x}, f) = \mathcal{T}(x_0, f_{\bar{x}_i,\beta})$, which follows from the construction of the mapping $f_{\bar{x},\beta}$.)

By using the definition of $D_\beta(x_0, f)$ and the inclusion presented above, we obtain

$$\overline{\bigcup_{\bar{x} \in U_\beta(x_0)} O(\bar{x}, f)} \subseteq \overline{\bigcup_{\bar{x} \in U_\beta(x_0)} \left(\bigcup_{k \in \mathcal{J}_{\bar{x},\beta}} O(x_0, f_{\bar{x}_k,\beta}) \right)} = \overline{\bigcup_{\bar{x} \in U_\beta(x_0)} O(x_0, f_{\bar{x}_k,\beta})}.$$

However, since $f_{\bar{x},\beta} \in \mathfrak{U}_\gamma(f)$ for any $\bar{x} \in U_\beta(x_0)$, we get

$$\overline{\bigcup_{\bar{x} \in U_\beta(x_0)} O(\bar{x}, f_{\bar{x}_k,\beta})} \subseteq \overline{\bigcup_{\tilde{f} \in \mathfrak{U}_\gamma(f)} O(x_0, \tilde{f})} = P_\gamma(x_0, f).$$

Then

$$P(x_0, f) = \bigcap_{\gamma > 0} P_\gamma(x_0, f) \supseteq \bigcap_{\beta = \beta(\gamma)} D_\beta(x_0, f) \supseteq \bigcap_{\eta > 0} D_\eta(x_0, f) = D(x_0, f).$$

We have thus shown that $D(x_0, f) \subseteq P(x_0, f)$. The lemma is proved.

Lemma 2. *If* $f \in \mathfrak{N}^*_{\text{trac}}$, *then* $D(x, f) \supseteq P(x, f)$ *for any point* $x \in X$.

Let x_0 be an arbitrary point of X.

1. Let us fix an arbitrary $\delta > 0$. For this $\delta > 0$ and a point $x_0 \in X$, according to the definition of the property of nonuniform tracing pseudotrajectories, one can find a number $\varepsilon = \varepsilon(\delta, f, x_0) > 0$ such that, for every (ε, f)-trajectory originating at the point x_0, there exists an f-trajectory that δ-traces this (ε, f)-trajectory.

2. Let us show that an arbitrary (ε, f)-trajectory originating at the point x_0 is contained in the δ-neighborhood of the f-orbits of points from the δ-neighborhood of the point x_0.

Let $\{x_i\}_{i=0}^{n}$, $n < \infty$, be an arbitrary (ε, f)-trajectory originating at the point x_0. Since f is a mapping with nonuniform tracing of pseudotrajectories, there exists a point $x \in X$ such that $x_i \in U_\delta(f^i(x))$ for all $i \geq 0$. In this case, $x_0 \in U_\delta(x)$ and, hence, $x \in U_\delta(x_0)$. Then

$$\bigcup_{i=0}^{n}\{x_i\} \subseteq \bigcup_{i=0}^{n} U_\delta(f^i(x)) \subseteq U_\delta(O(x, f)),$$

i.e.,

$$\bigcup_{i=0}^{n}\{x_i\} \subseteq U_\delta(O(x, f))$$

for a certain point $x \in U_\delta(x_0)$. Then, all the more, we have

$$\bigcup_{i=0}^{n}\{x_i\} \subseteq U_\delta\left(\bigcup_{\tilde{x} \in U_\delta(x_0)} O(\tilde{x}, f)\right).$$

3. Let us construct a family of mappings ε-close to f and different from f only in a sufficiently small neighborhood of an (ε, f)-trajectory.

Denote by $\mathcal{A}_\varepsilon(x_0)$ the set of all (ε, f)-trajectories originating at a point x_0. Let $\alpha = \{x_i\}_{i=0}^{n}$ be an arbitrary element of the set $\mathcal{A}_\varepsilon(x_0)$. We construct a mapping $g_{\alpha,\varepsilon}$ in the following way:

$$g_{\alpha,\varepsilon}(x) = \begin{cases} f(x) & \text{for } x \in X \setminus \bigcup_{i=0}^{n} U_\varepsilon(f(x_i)); \\ x_{i+1} & \text{for } x = x_i,\ 0 \leq i \leq n-1; \\ f_\mu(x) & \text{for } x \in \left\{\bigcup_{i=0}^{n} U_\varepsilon(f(x_i))\right\} \setminus \{x_i\}_{i=0}^{n}, \end{cases}$$

where $f_\mu(x)$ is the function that realizes joining with the corresponding C^r-smoothness constructed by analogy with the proof of Lemma 1.

4. By using the special mappings $g_{\alpha,\varepsilon}$ constructed above, we show that the prolongation of the f-orbit of the point x_0 with respect to the dynamical system is contained in the prolongation of the f-orbit of the point x_0 with respect to the initial point.

By construction, $g_{\alpha,\varepsilon} \in \mathfrak{U}_\varepsilon(f)$ because f has the largest deviation from $g_{\alpha,\varepsilon}$ at the points of the (ε, f)-trajectory; for these points, by the definition of (ε, f)-trajectory, we have $x_{i+1} \in U_\varepsilon(f(x_i))$.

Then

$$\bigcup_{\alpha \in \mathcal{A}_\varepsilon(x_0)} O(x_0, g_{\alpha,\varepsilon}) \subseteq U_\delta\left(\bigcup_{\tilde{x} \in U_\delta(x_0)} O(\tilde{x}, f)\right) \subseteq U_\delta\left(\overline{\bigcup_{\tilde{x} \in U_\delta(x_0)} O(\tilde{x}, f)}\right) = U_\delta(D_\delta(x_0, f)).$$

For any (ε, f)-trajectory, one can construct the corresponding mapping $g_{\alpha,\varepsilon} \in \mathfrak{U}_\varepsilon(f)$ and, for each $\tilde{f} \in \mathfrak{U}_\varepsilon(f)$, one can choose an (ε, f)-trajectory that is the trajectory of the mapping $\tilde{f}$.

Thus, having analyzed all (ε, f)-trajectories originating at the point x_0, we obtain all mappings $\tilde{f} \in \mathfrak{U}_\varepsilon(f)$. Then we get

$$\bigcup_{\tilde{f} \in \mathfrak{U}_\varepsilon(f)} O(x_0, \tilde{f}) \subseteq U_\delta(D_\delta(x_0, f)).$$

Further,

$$P_\varepsilon(x_0, f) = \overline{\bigcup_{\tilde{f} \in \mathfrak{U}_\varepsilon(f)} O(x_0, \tilde{f})} \subseteq \overline{U_\delta(D_\delta(x_0, f))}.$$

Hence,

$$\bigcap_{\delta > 0} \overline{U_\delta(D_\delta(x_0, f))} \supseteq \bigcap_{\varepsilon = \varepsilon(\delta, f, x_0)} P_\varepsilon(x_0, f) \supseteq \bigcap_{\eta > 0} P_\eta(x_0, f) = P(x_0, f).$$

Note that the sequence $\overline{U_\delta(D_\delta(x_0, f))}$ is such that if $\delta' < \delta''$, then $\overline{U_{\delta'}(D_{\delta'}(x_0, f))} \subset \overline{U_{\delta''}(D_{\delta''}(x_0, f))}$ (i.e., it is a sequence of imbedded sets). By passing to the limit as $\delta \to 0$, we get

$$\bigcap_{\delta > 0} \overline{U_\delta(D_\delta(x_0, f))} = \bigcap_{\delta > 0} D_\delta(x_0, f) = D(x_0, f).$$

Hence, we have the inclusion $D(x_0, f) \supseteq P(x_0, f)$. Lemma 2 is proved.

The inclusions $P(x_0, f) \supseteq D(x_0, f)$ and $D(x_0, f) \supseteq P(x_0, f)$ imply the equality $D(x_0, f) = P(x_0, f)$ for an arbitrary point $x_0 \in X$.

Thus, we have established that the mapping f is a mapping with stable prolongation of orbits. Theorem 1B is proved.

Theorem 1C. *The following strict inclusion is true:*

$$\mathfrak{N}^*_{\text{trac}} \subset \mathfrak{N}. \tag{3}$$

This statement is proved by the following example:

Example 2. A mapping of a closed interval of the real line into itself with stable prolongation of orbits that does not possess the property of nonuniform tracing pseudotrajectories.

Let $I = [-1, 1]$ and let $f(x) = -\sin 2\pi x$.

The mapping $f\colon I \to I$ possesses the following properties:

I. The mapping f is a mapping with stable prolongation of orbits because $D(x, f) = P(x, f) = [-1, 1]$ for all $x \in [-1, 1]$.

II. Let us take an arbitrary $\delta < 1/2$. On the interval $[0, \delta]$, we choose an arbitrary point x_1. Let a point

$x_0 \in [-1, -1+\delta]$ be the preimage of the point x_1. We set $\varepsilon = \delta/2$. Let $y_1 \in [x_1 - \varepsilon, 0]$. Let us construct an (ε, f)-trajectory $\{y_i\}_{i=0}^n$, $n < \infty$, as follows: $y_0 = x_0$ and $y_i = f^{i-1}(y_1)$ for $i > 1$. Then, for $i > 2$ and all $x \in U_\delta(x_0)$, the points $f^i(x)$ and y_i lie in the intervals $(-1, 0)$ and $(0, 1)$, respectively, in the case where i is even, or in the intervals $(0, 1)$ and $(-1, 0)$, respectively, in the case where i is odd. Hence, the (ε, f)-trajectory $\{y_i\}_{i=0}^n$ is not δ-traced by any f-trajectory of any point $x \in U_\delta(y_0)$.

Remark 4. It should be noted that the mapping $g = f^2$ is not a mapping with stable prolongation of orbits because $D(x, g) = [-1, 0]$ for all $x \in [-1, 0)$, $D(x, g) = [0, 1]$ for all $x \in (0, 1]$, and $P(x, g) = [-1, 1]$ for all $x \in [-1, 1]$.

Combining Theorems 1A–1C, we obtain the proof of Theorem 1.

4. Conclusion

Let us formulate several unsolved problems.

The mapping constructed in Example 2 does not possess the property of tracing pseudotrajectories but is a mapping with stable prolongation of orbits. Moreover, the mapping given by its second iteration is not a mapping with stable prolongation of orbits. It is possible that the "loss of the stability of prolongation of orbits when passing to the iterations of mappings" is a property not typical of the majority of systems. This reasoning leads to the formulation of the following questions:

Denote $\mathfrak{N}^\infty = \{ f \in \mathfrak{N} \mid g = f^k \in \mathfrak{N} \text{ for any } k \in \mathbb{Z}^+\}$.

Question 1. *Is it true that the set* $\mathfrak{N}^\infty$ *is a set of the second category in the space of mappings* $\mathfrak{M}$*?*

Question 2. *Is it true that* $\mathfrak{N}^\infty = \mathfrak{N}^*_{\text{trac}}$*?*

If the answer to these questions is positive, then we receive a positive answer to the question concerning the typicalness of mappings possessing the property of tracing pseudotrajectories posed at the beginning of the paper.

Question 3. *Is the set* $\mathfrak{N}_{\text{trac}}$ *or, at least, the set* $\mathfrak{N}^*_{\text{trac}}$, *a set of the second category in the space of mappings* $\mathfrak{M}$*?*

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