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Discretization of asymptotic line parametrizations using hyperboloid surface patches

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Abstract Two-dimensional affine A-nets in 3-space are quadrilateral meshes that discretize surfaces parametrized along asymptotic lines. The defining property of A-nets is planarity of vertex stars, hence elementary quadrilaterals of a generic A-net are skew. The present article deals with the extension of A-nets to differentiable surfaces, by gluing hyperboloid surface patches into the skew quadrilaterals. The obtained surfaces, named "hyperbolic nets", are a novel, piecewise smooth discretization of surfaces parametrized along asymptotic lines. A simply connected affine A-net can be extended to a hyperbolic net if all quadrilateral strips are "equi-twisted". The geometric condition of equi-twist implies the combinatorial property, that all inner vertices of the A-net have to be of even degree. If an A-net can be extended to a hyperbolic net, then there exists a 1-parameter family of such extensions. It is briefly explained how the generation of hyperbolic nets can be implemented on a computer. We use a projective model of Plücker line geometry in order to describe A-nets and hyperboloids.

Keywords Discrete differential geometry · Discrete asymptotic line parametrization · A-nets · Hyperboloids · Projective geometry · Plücker line geometry

Mathematics Subject Classification (2000) 51M30 · 53A05 · 65D17

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1 Introduction

The present paper deals with the discretization of surfaces in 3-space that are parametrized along asymptotic lines. Usually, the discretization of parametrized surfaces within discrete differential geometry (DDG) leads to quadrilateral nets, often also called quadrilateral meshes. Compared with, e.g., discrete triangulated surfaces, quadrilateral nets do not only discretize smooth surfaces understood as (point) sets, but also reflect the combinatorial structure of the parametrizations to be discretized. This generalizes to nets of arbitrary dimension, for which elementary 2-cells are quadrilaterals. Therefore, quadrilateral nets of various dimension are omnipresent in DDG as discretizations of all kinds of parametrized geometries. While unspecified quadrilateral nets discretize arbitrary parametrizations, the discretization of distinguished types of parametrizations yields quadrilateral nets with special geometric properties. One of the most fundamental examples is the discretization of conjugate parametrizations by quadrilateral nets with planar faces. Discretizing more specific conjugate parametrizations then yields planar quadrilateral nets with additional properties. For instance, different discretizations of curvature line parametrizations, where the latter may be characterized as orthogonal conjugate parametrizations, have led to the notions of circular and conical nets [3,10,19,21].[1]

Curvature line parametrizations of smooth surfaces exist (at least locally) around non-umbilic points and are essentially unique, i.e., unique up to reparametrization of the parameter lines. In perfect analogy, one has unique asymptotic line parametrizations around hyperbolic points of surfaces, that is, around points of negative Gaussian curvature. Indeed, the description of curvature lines in Lie sphere geometry is analogous to the description of asymptotic lines in Plücker line geometry, see, e.g., [16]. This deep connection becomes apparent, when describing surfaces in Lie and in Plücker geometry in terms of their *contact elements*. In Lie geometry, a contact element consists of all oriented spheres touching in a point, while in Plücker geometry it is composed of all lines in a plane through a point in that plane. In both cases it is convenient to imagine a contact element as a plane containing a distinguished point, see Fig. 1. Curvature lines are characterized in Lie geometry by the fact that "infinitesimally close contact elements along a curvature line" through a point p share a sphere, which is the principal curvature sphere of that curvature line in p. The corresponding statement in Plücker geometry says that infinitesimally close contact elements along an asymptotic line through a point p share a line, which is the tangent to that asymptotic line in p. However, as asymptotic line parametrizations are not conjugate parametrizations, they are not modelled by quadrilateral nets with planar faces. Instead, asymptotic line parametrizations are properly discretized by quadrilateral nets with planar vertex stars, that is, nets for which every vertex is coplanar with its next neighbors as shown in Fig. 2. We use the terminology of [8], calling discrete nets with planar quadrilaterals *Q-nets* and discrete nets with planar vertex stars *A-nets*. Q-nets and A-nets as discretizations of conjugate and asymptotic line parametrizations were already introduced in [27].

For our purpose, the description of A-nets in the projective model of Plücker line geometry is essential [12]. In this setting, A-nets appear as discrete congruences of isotropic lines, that is, lines that are completely contained in the Plücker quadric. Since isotropic lines in the Plücker quadric represent contact elements in $\mathbb{R}\mathrm{P}^3$, this means that A-nets in $\mathbb{R}\mathrm{P}^3$ are understood as configurations of contact elements, the latter corresponding to the planar vertex stars with distinguished central vertices. Indeed, the description of A-nets in terms of contact elements

[1] Circular and conical nets are unified within the framework of principle contact element nets [7,26]. The latter are not point maps anymore but take contact elements as values at vertices of the domain.

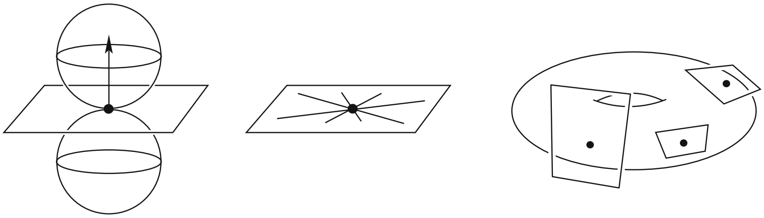
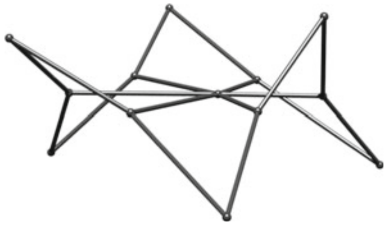


Fig. 1 From *left* to *right*: contact element in Lie geometry, contact element in Plücker geometry, surface composed of contact elements

Fig. 2 Edges adjacent to a vertex of a discrete A-net lie in a 2-plane



is a literal discretization of the aforementioned characterizing property of asymptotic lines in Plücker geometry.[2]

Asymptotic line parametrizations, smooth and discrete, are preserved by projective transformations, i.e., they are objects of projective geometry. Often, an A-net in $\mathbb{R}\mathrm{P}^3$ is described as a parametrized set of its vertices, but this description does not contain information about edges, i.e., line segments connecting adjacent vertices. However, if we speak about an A-net as a quadrilateral mesh, we have distinguished edges in mind. Therefore, we use the term *affine A-net* to reflect the idea that, starting out in the general projective setting, choosing an ideal plane at infity determines finite edges connecting adjacent vertices of a projective A-net.

Various aspects of smooth asymptotic line parametrizations have been discretized using A-nets. For example, the discretization of surfaces of constant negative Gaussian curvature as special A-nets, nowadays often called K-surfaces, can be found in [28,30]. Much later, in context of the connections between geometry and integrability, Bobenko and Pinkall [4] established the relation between K-surfaces and the discrete sine-Gordon equation that was set down by Hirota [13]. For a special instance of this relation, see, for example, [14] on discrete Amsler-surfaces. Discrete indefinite affine spheres [6] are an example for the discretization of a certain class of smooth A-nets within affine differential geometry. The discrete Lelieuvre representation of A-nets and the related discrete Moutard equations are, for instance, treated in [17,23].

In the setting of discrete A-nets, one also obtains a convenient discretization of the class of transformations that is associated with smooth A-surfaces. In general, the class of associated transformations and related permutability theorems are an essential aspect of specific integrable surface parametrizations. In the case of smooth A-nets, the associated transformations are called *Weingarten transformations.* Two smooth A-surfaces are said to be Weingarten transforms of each other if the line connecting corresponding points is the intersection of the

[2] In the same spirit, so-called "principle contact element nets" are a literal discretization of the characterizing property of curvature lines in Lie geometry, cf. [8].

tangent planes to the two surfaces at these points. This relation carries over naturally to the setting of discrete A-nets [11,12,22].

In the present article, we show how to extend discrete affine A-nets to piecewise smooth surfaces, called *hyperbolic nets*, which are a novel discretization of surfaces parametrized along asymptotic lines. The extension of A-nets to hyperbolic nets uses surface patches taken from hyperboloids, where a hyperboloid in our sense is a doubly ruled quadric, i.e., a one-sheeted hyperboloid or a hyperbolic paraboloid from the affine viewpoint. Those *hyperboloid patches* are inserted into the skew quadrilaterals of an A-net, such that a patch is bounded by the edges of the corresponding quadrilateral. In particular, the edges of a supporting quadrilateral are asymptotic lines of the inserted patch. Furthermore, two patches meeting in a common edge are required to have coinciding tangent planes along this edge. Therefore, at each point of the compound surface one has a well defined tangent plane that varies continuously when moving from one patch to an adjacent patch. In this sense, we call hyperbolic nets "piecewise smooth C^1-surfaces" although two edge adjacent patches may form a cusp. The discretization of asymptotic line parametrized surfaces as hyperbolic nets is very similar to the discretization of curvature line parametrized surfaces by cyclidic nets, which were introduced in [2].

There is only little work about surfaces composed of hyperboloid patches. Discrete affine minimal surfaces, which are special A-nets, have been extended by hyperboloid patches in [9]. Starting with the discrete Lelieuvre representation of A-nets, Craizer et al. construct a bilinear extension of the considered discrete surfaces. This yields continuous surfaces composed of hyperboloid patches that are adapted to the underlying discrete A-net. It turns out that, in the very special situation of discrete affine minimal surfaces, this extension indeed yields a piecewise smooth C^1-surface, i.e., a hyperbolic net in our sense. However, for general A-nets the extension yields only continuous surfaces, which do not posses unique tangent planes along the joints of adjacent hyperboloid patches.

It is natural to look for applications of hyperbolic nets in the context of Computer Aided Geometric Design and architectural geometry. The latter is an emerging field of applied mathematics, which provides the architecture community with sophisticated geometric knowledge to tackle diverse problems. Focussing also on important aspects such as efficient manufacturing, providing intuitive control of available degrees of freedom, and similar issues, many results in DDG have already been applied in an architectural context. We expect that the present article will also find application in this area, as there is growing interest in the extension of discrete support structures to surfaces that are built from curved panels [5,21,24]. Indeed, our contribution has already been recognized in that field [18,29].

Structure of the article and main results Section 2 introduces our notations and the needed background. In particular, a projective model of Plücker line geometry is introduced and the description of discrete A-nets in this model is discussed. In Sect. 3, aiming at the extension of A-nets to hyperbolic nets, we introduce *pre-hyperbolic nets* as an intermediate step. While hyperbolic nets contain a hyperboloid patch for each elementary quadrilateral, pre-hyperbolic nets contain the whole supporting hyperboloid. The main result of Sect. 3 is that any simply connected A-net with interior vertices of even degree can be extended to a pre-hyperbolic net. In this case, there exists a 1-parameter family of pre-hyperbolic nets. One particular extension is determined by the choice of one initial hyperboloid, which is then propagated to all other quadrilaterals. The essential step is to show the consistency of this propagation. Since we are dealing with simply connected topology only, there are no global closure conditions for the evolution. Hence, we have to show consistency of the propagation around a single inner vertex only. It turns out that a necessary and sufficient condition for the consistency is even

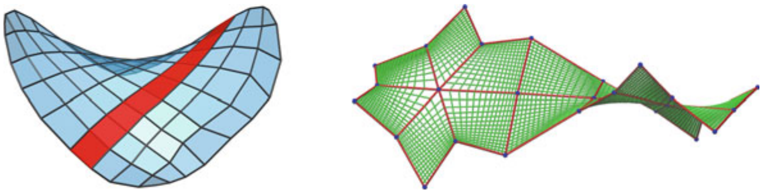


Fig. 3 An equi-twisted quadrilateral strip (*left*) and a hyperbolic net (*right*)

vertex degree. In Sect. 4 we turn to discrete affine A-nets, i.e., A-nets whose vertices are connected by finite edges, and describe their extension to hyperbolic nets. The characterizing property of simply connected affine A-nets that can be extended to hyperbolic nets turns out to be "equi-twist" of quadrilateral strips, which automatically implies that inner vertices are of even degree. Figure 3 shows examples for an equi-twisted A-net and a hyperbolic net. At the end of Sect. 4, we comment on the computer implementation of our results. The Appendix provides some calculations in coordinates.

2 Background

In this section, first we introduce the basic concepts and notations for projective geometry and Plücker line geometry. Afterwards, we discuss discrete A-nets and hyperboloids in the projective model of Plücker geometry.[3]

2.1 Projective geometry and Plücker line geometry background

Classical Plücker geometry is the geometry of lines in the projective 3-space $\mathbb{R}\mathrm{P}^3 = \mathrm{P}(\mathbb{R}^4)$. In the projective model of Plücker geometry, those lines are represented as points on the 4-dimensional Plücker quadric $\mathscr{Q}^{3,3}$. This 4-dimensional quadric is embedded in the 5-dimensional projective space $\mathbb{R}\mathrm{P}^{3,3} = \mathrm{P}(\mathbb{R}^{3,3})$, where $\mathbb{R}^{3,3}$ is $\mathbb{R}^6$ equipped with an inner product defining the Plücker quadric. For an introduction to projective geometry and quadrics in projective spaces, see, e.g., [1].

Notation For the readers convenience we use different fonts to distinguish the projective spaces $\mathbb{R}\mathrm{P}^3$ and $\mathbb{R}\mathrm{P}^{3,3}$. For objects in $\mathbb{R}\mathrm{P}^3$ we use bold font and denote, for example, points $\mathsf{x} \in \mathbb{R}\mathrm{P}^3$ and lines $\mathsf{h} \subset \mathbb{R}\mathrm{P}^3$. Objects in $\mathbb{R}\mathrm{P}^{3,3}$ are written in normal math font, e.g., the Plücker representative of a line $\mathsf{h} \subset \mathbb{R}\mathrm{P}^3$ is denoted $h \in \mathscr{Q}^{3,3} \subset \mathbb{R}\mathrm{P}^{3,3}$. Homogeneous coordinates of both, $\mathbb{R}\mathrm{P}^3$ and $\mathbb{R}\mathrm{P}^{3,3}$, are marked with a hat. We write for example $\mathsf{x} = [\widehat{\mathsf{x}}] = \mathrm{P}(\mathbb{R}\widehat{\mathsf{x}}) \in \mathbb{R}\mathrm{P}^3$ and $h = [\widehat{h}] = \mathrm{P}(\mathbb{R}\widehat{h}) \in \mathbb{R}\mathrm{P}^{3,3}$. More general, for a projective subspace U, the corresponding linear subspace of homogeneous coordinates is denoted by $\widehat{U}$.

The inclusion minimal projective subspace containing projective subspaces $U_i = \mathrm{P}(\widehat{U}_i)$, $i = 1, \ldots, n$ is the *projective span* of the U_i:

$$\mathrm{inc}[U_1, \ldots, U_n] := \mathrm{P}(\mathrm{span}(\widehat{U}_1, \ldots, \widehat{U}_n)).$$

As mentioned before, the projective space $\mathbb{R}\mathrm{P}^{3,3}$ is equipped with the Plücker quadric $\mathscr{Q}^{3,3}$. Polar subspaces with respect to $\mathscr{Q}^{3,3}$ are denoted by

[3] The description of hyperboloids in the projective model of Plücker geometry is completely analogous to the description of Dupin cyclides in the projective model of Lie geometry, which is one manifestation of Lie's famous line-sphere correspondence.

$$\mathrm{pol}[U_1,\ldots,U_n] := \mathrm{pol}[\mathrm{inc}[U_1,\ldots,U_n]] = \mathrm{P}(\mathrm{span}(\widehat{U}_1,\ldots,\widehat{U}_n)^{\perp}),$$

where $\perp$ denotes the orthogonal complement with respect to the inner product on $\mathbb{R}^{3,3}$ defining the Plücker quadric.

Projective model of Plücker geometry There exist a lot of books dealing with the projective model of Plücker geometry. A classical reference for details on Plücker coordinates and the Plücker inner product is [16]. For a modern treatment see, e.g., [8,25].

The *Plücker inner product* is a symmetric bilinear form $\langle\cdot,\cdot\rangle : \mathbb{R}^6 \times \mathbb{R}^6 \to \mathbb{R}$ of signature $(+++---)$. We denote $\mathbb{R}^6$ equipped with this product by $\mathbb{R}^{3,3}$ and write the corresponding null-vectors in $\mathbb{R}^{3,3}$ as

$$\mathbb{L}^{3,3} := \left\{\widehat{v} \in \mathbb{R}^{3,3} \mid \langle\widehat{v},\widehat{v}\rangle = 0\right\}.$$

Projectivization of $\mathbb{L}^{3,3}$ yields the *Plücker quadric*

$$\mathcal{Q}^{3,3} := \mathrm{P}(\mathbb{L}^{3,3}) \subset \mathrm{P}(\mathbb{R}^{3,3}) =: \mathbb{R}\mathrm{P}^{3,3}.$$

An element $h = [\widehat{h}] \in \mathbb{R}\mathrm{P}^{3,3}$ is contained in $\mathcal{Q}^{3,3}$ if and only if $\widehat{h} \in \mathbb{R}^{3,3}$ are homogeneous Plücker coordinates of a line $\mathsf{h} \subset \mathbb{R}\mathrm{P}^3$. Two lines $\mathsf{h}_1, \mathsf{h}_2$ in $\mathbb{R}\mathrm{P}^3$ intersect if and only if their representatives $h_1, h_2 \in \mathcal{Q}^{3,3}$ are polar with respect to the Plücker quadric, i.e., $\langle\widehat{h}_1,\widehat{h}_2\rangle = 0$.

Remark 1 The projective model of Plücker geometry is often formulated in the language of exterior algebra, see, for instance, [8,12,25]. One advantage of the exterior algebra formulation is the immediate presence of Plücker line coordinates and easy calculation with them. On the other hand, the formulation chosen for the present article is more elementary and points out the deep relation between Plücker line geometry and Lie sphere geometry, cf. [2].[4] Benefiting from both descriptions, in the Appendix we briefly translate between the two languages and use the exterior algebra formulation to carry out some calculations. In the main text we are able to avoid the introduction of explicit line coordinates in $\mathbb{R}^{3,3}$, since for our considerations only the existence of an inner product with the mentioned properties is relevant. In particular, we only need to describe polarity with respect to the Plücker quadric and make general statements about the signature of the Plücker product restricted to projective subspaces.

Two lines $\mathsf{h}_1, \mathsf{h}_2 \subset \mathbb{R}\mathrm{P}^3$ intersecting in a point $\mathsf{x} \in \mathbb{R}\mathrm{P}^3$ span a *contact element*, which consists of all those lines through x that lie in the plane spanned by h_1 and h_2. In the projective model of Plücker geometry this contact element is the projective line $L = \mathrm{inc}[h_1, h_2]$. The fact $L \subset \mathcal{Q}^{3,3}$ is equivalent to $\langle\widehat{h}_1,\widehat{h}_2\rangle = 0$ and shows that indeed each point on L represents a line in $\mathbb{R}\mathrm{P}^3$. Lines contained in a quadric are commonly called *isotropic lines* and we define

$$\mathcal{L}^{3,3} = \left\{\text{Isotropic lines in } \mathbb{R}\mathrm{P}^{3,3}\right\} \cong \left\{\text{Contact elements of } \mathbb{R}\mathrm{P}^3\right\}.$$

It is convenient to think of a contact element in $\mathbb{R}\mathrm{P}^3$ as a plane P with a distinguished point x (see Fig. 4) and write (x, P) for $L \in \mathcal{L}^{3,3}$.

The *signature of a projective subspace* $U = \mathrm{P}(\widehat{U}) \subset \mathbb{R}\mathrm{P}^{3,3}$ is the signature of $\langle\cdot,\cdot\rangle$ restricted to $\widehat{U}$. In particular, the signature of a contact element, i.e., an isotropic line, is (00).

Remark 2 Two skew lines in $\mathbb{R}\mathrm{P}^3$ do not belong to a common contact element and hence define a line of signature $(+-)$ in $\mathbb{R}\mathrm{P}^{3,3}$. In particular, the three pairs of opposite edges

[4] The projective model of (3-dimensional) Lie geometry can be realized in $\mathrm{P}(\mathbb{R}^{4,2})$, i.e., using an inner product of signature (4,2) instead one of signature (3,3) on $\mathbb{R}^6$.

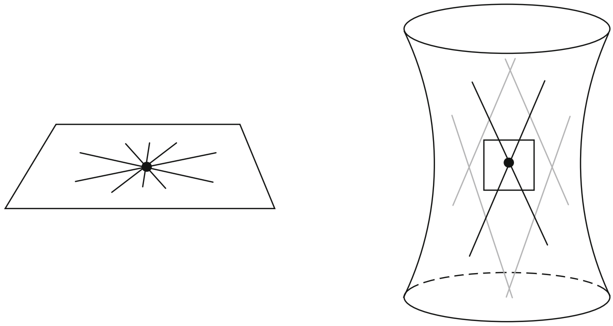


Fig. 4 *Left* A contact element consists of all lines in a plane passing through a distinguished point. *Right* For a hyperboloid $\mathscr{H}$, two asymptotic lines of different families intersect in a unique point and span the corresponding contact element of $\mathscr{H}$ (the supporting plane being tangent to $\mathscr{H}$)

of a non-planar tetrahedron in $\mathbb{R}\mathrm{P}^3$ yield three lines of signature $(+-)$. These three lines toghether span the whole $\mathbb{R}\mathrm{P}^{3,3}$. Correspondingly, one can use the six lines supporting the tetrahedron as a basis for Plücker line geometry and write Plücker coordinates with respect to this "coordinate tetrahedron".

2.2 Hyperboloids in Plücker geometry

A large part of this article is concerned with projective geometry, so we use the term *hyperboloid* for an arbitrary doubly ruled quadric in $\mathbb{R}\mathrm{P}^3$. In particular, a hyperboloid of one sheet and a hyperbolic paraboloid, being different in affine geometry, are indistinguishable in projective geometry. With respect to an affine chart of $\mathbb{R}\mathrm{P}^3$, one may say that a hyperbolic paraboloid is a hyperboloid of one sheet that is tangent to the ideal plane at infinity. Indeed, looking at the intersection of a hyperboloid with the ideal plane, an affine hyperboloid of one sheet intersects it in a non-degenerate conic, while a hyperbolic paraboloid intersects it in two intersecting lines, i.e., in a certain degenerate conic (see, e.g., [1]).

An important statement about quadrics in $\mathbb{R}\mathrm{P}^3$ is, that any three mutually skew lines determine a unique hyperboloid. From the previous considerations it follows, that in a fixed affine chart this hyperboloid is a hyperbolic paraboloid, if and only if the three intersection points of the considered lines with the ideal plane are collinear.

The following description of hyperboloids in the projective model of Plücker geometry can be found, for example, in [16].

Theorem 1 *Hyperboloids in* $\mathbb{R}\mathrm{P}^3$ *are in bijection with polar decompositions of* $\mathbb{R}\mathrm{P}^{3,3}$ *into two projective planes of signatures* $(++-)$ *and* $(+--)$.

Usually we denote the two (disjoint) planes corresponding to a hyperboloid by $P^{(1)}$ and $P^{(2)}$. The conic sections $\mathscr{Q}^{3,3} \cap P^{(1)}$ and $\mathscr{Q}^{3,3} \cap P^{(2)}$ describe of the two families of rulings of a hyperboloid. Two lines $h^{(1)} \in P^{(1)}$ and $h^{(2)} \in P^{(2)}$, one of each family, define a unique contact element of the hyperboloid (cf. Figs. 4, 5).

Definition 1 (*Hyperbolic family of lines/regulus/ruling*) Denote

$$\mathscr{P}_{\text{hyp}} = \left\{\text{Planes in } \mathbb{R}\mathrm{P}^{3,3} \text{ of signature } (++-) \text{ or } (+--)\right\}.$$

A 1-parameter family of lines corresponding to a conic section $P \cap \mathscr{Q}^{3,3}$ with $P \in \mathscr{P}_{\text{hyp}}$ is called a *hyperbolic family of lines* (cf. Fig. 5). We also use the term *regulus* for a

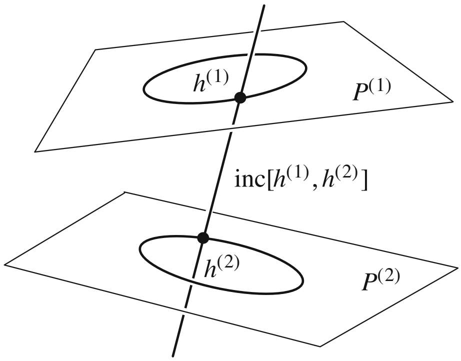


Fig. 5 Polar planes $P^{(1)}$, $P^{(2)}$ of signatures $(++-)$ and $(+--)$ describe a hyperboloid $\mathscr{H} \subset \mathbb{R}\mathrm{P}^3$: Intersections $\mathscr{H}^{(i)} = \mathscr{Q}^{3,3} \cap P^{(i)}$ are two non-degenerate conics, each conic corresponding to one regulus of $\mathscr{H}$. Each point of the hyperboloid is obtained as intersection point of two lines $\mathsf{h}^{(1)}$ and $\mathsf{h}^{(2)}$, where $h^{(i)} \in \mathscr{H}^{(i)}$. The isotropic line $\mathrm{inc}[h^{(1)}, h^{(2)}] \subset \mathscr{Q}^{3,3}$ is the corresponding contact element of $\mathscr{H}$ at $\mathsf{x} = \mathsf{h}^{(1)} \cap \mathsf{h}^{(2)}$

hyperbolic family of lines. A line of a regulus is called a *ruling* or simply *asymptotic line* of the corresponding hyperboloid.

Remark 3 The complementary signatures $(++-)$ and $(+--)$ in Definition 1 reflect the two possible orientations of hyperbolic families of lines in $\mathbb{R}\mathrm{P}^3$, cf. [16]. In particular, the two reguli of a hyperboloid in $\mathbb{R}\mathrm{P}^3$ are of opposite orientation, according to Theorem 1. The different orientations of reguli will become important in the context of hyperbolic nets, so a more detailed discussion is given in Sect. 4.

2.3 Discrete A-nets

Two dimensional discrete A-nets are a discretization of smooth surfaces, which are parametrized along asymptotic lines, going back to Sauer [27]. We use two equivalent definitions of A-nets, i.e., one in terms of the vertices and another one in terms of contact elements (cf. [8,12]). In order to describe general combinatorics of asymptotic lines, we need the notion of a *quad-graph*.

Definition 2 (*Quad-graph/vertex star*) A *quad-graph* is a strongly regular polytopal cell decomposition of a surface, such that all faces are quadrilaterals. We write $\mathscr{D} = (V, E, F)$, where V is the set of vertices (0-cells), E is the set of edges (1-cells), and F is the set of faces (2-cells) of the quad-graph.

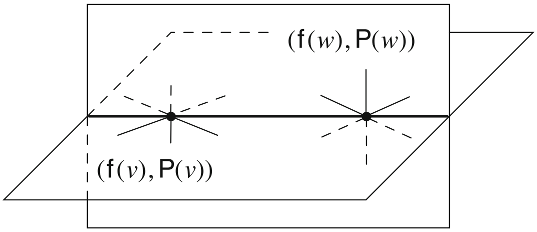
Let $v \in V$ be a vertex of $\mathscr{D}$. The *vertex star* of v is the set of all vertices that are adjacent to v, including v itself.

Remark 4 A cell decomposition of a surface is called *regular* if for each face the boundary consists of pairwise distinct edges and pairwise distinct vertices. A regular cell decomposition is called *strongly regular* if in addition the intersection of two faces is either empty, a single vertex, or the closure of an edge.

Definition 3 (*Discrete A-net*) Let $\mathscr{D}$ be a quad-graph with vertices V.

- *Vertex description of A-nets:* Let $\mathsf{f} : V \to \mathbb{R}\mathrm{P}^3$. Then f is a discrete A-net if all vertex stars are planar, cf. Fig. 2 from the introduction.
- *Contact element description of A-nets:* Let $L : V \to \mathscr{L}^{3,3} \subset \mathbb{R}\mathrm{P}^{3,3}$. Then L defines the contact elements of a discrete A-net if adjacent isotropic lines intersect, i.e., if L is a so-called discrete congruence of (isotropic) lines in $\mathbb{R}\mathrm{P}^{3,3}$. In other words: contact elements, seen as 1-parameter families of lines in $\mathbb{R}\mathrm{P}^3$, associated with adjacent vertices share a line.

Fig. 6 Discrete asymptotic lines of an A-net are the lines shared by adjacent contact elements



Equivalence of vertex and contact element description of A-nets Recall, that contact elements in $\mathbb{R}P^3$ can be written as a pair (x, P) of a point and a plane, see Sect. 2.1. The two descriptions of A-nets are related as follows: Starting with the vertex description, one obtains distinguished contact elements for each vertex by considering the (planar) vertex stars with given central vertex. For adjacent vertices v and w of the underlying quad graph $\mathscr{D}$, denote $\mathsf{P}(v)$ and $\mathsf{P}(w)$ the planes that contain the respective vertices $\mathsf{f}(v)$ and $\mathsf{f}(w)$. This yields $L(v) = (\mathsf{f}(v), \mathsf{P}(v))$ and $L(w) = (\mathsf{f}(w), \mathsf{P}(w))$ in $\mathscr{L}^{3,3}$. The line l supporting the edge $(\mathsf{f}(v), \mathsf{f}(w))$ in $\mathbb{R}P^3$ is exactly the intersection $l = L(v) \cap L(w) \in \mathscr{Q}^{3,3}$, cf. Fig. 6. Naturally, we associate l to the corresponding edge (v, w) of $\mathscr{D}$. Those lines common to adjacent contact elements are called *discrete asymptotic lines* of the A-net. Representatives of the discrete asymptotic lines in $\mathbb{R}P^{3,3}$ constitute the so-called *focal net* of the discrete line congruence L.[5]

Affine A-nets Note that the definition of A-nets in the projective setting does not contain information about edges, i.e., line segments connecting adjacent vertices. However, considering an A-net in an affine part $\mathbb{R}^3 \subset \mathbb{R}P^3$, there is a distinguished plane at infinity, and it is natural to connect the vertices of the A-net with finite edges as in Fig. 2. We call such A-nets equipped with finite edges *affine A-nets* and refer to the supporting discrete asymptotic lines also as *extended edges*.

Genericity assumption We assume that the A-nets under consideration are generic, i.e., elementary quadrilaterals are skew and there is a unique supporting plane for each vertex star. In particular, the four isotropic lines in $\mathbb{R}P^{3,3}$ that correspond to the contact elements of an elementary quadrilateral, span a 3-space of signature $(++--)$, cf. Remark 2. The restriction of the Plücker quadric to this space describes all lines in $\mathbb{R}P^3$ that intersect the two diagonals of the initial skew quadrilateral.

Notation for discrete maps A special instance of a quad-graph is (a piece of) $\mathbb{Z}^2$. In this case, it is convenient to represent shifts in lattice directions by lower indices. For $z = (m, n) \in \mathbb{Z}^2$ and a map φ on $\mathbb{Z}^2$ we write

$$\varphi_1(z) := \varphi(m+1, n),\ \varphi_{11}(z) := \varphi(m+2, n),\ \varphi_2(z) := \varphi(m, n+1), \quad \text{etc.}$$

However, also for a general quad-graph $\mathscr{D}$ we use indices as shift operators when talking about local properties if the quad-graph can be identified locally with $\mathbb{Z}^2$.

Usually we omit the argument for discrete maps and write

$$\varphi = \varphi(z), \quad \varphi_1 = \varphi_1(z) \quad \text{etc.}$$

[5] In the book [8], the definition of a focal net is given for regular lattices $\mathbb{Z}^m$. In this case there exist distinguished global coordinate directions and the authors define multiple focal nets of a discrete line congruence, each focal net associated with a certain coordinate direction.

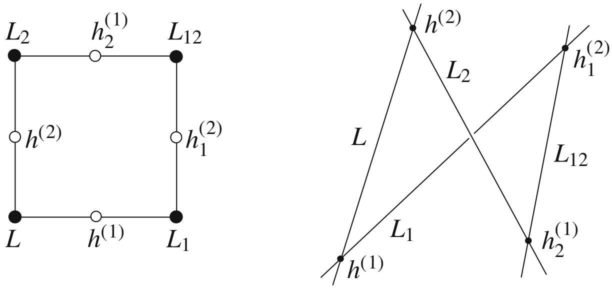


Fig. 7 Notation for a discrete congruence of lines L and the corresponding focal net h, combinatorially and geometrically

The structure of A-nets as isotropic line congruences and the notation we use is shown in Fig. 7. Upper indices are used to associate vertices of the focal net with (local) net directions.

3 Pre-hyperbolic nets

In this section, we show how to attach hyperboloids to the elementary quadrilaterals of a discrete A-net, such that hyperboloids attached to edge-adjacent quadrilaterals are tangent along a common asymptotic line. These common asymptotic lines are discrete asymptotic lines of the supporting A-net. One obtains so-called *pre-hyperbolic nets*, where this name is chosen, because one may see pre-hyperbolic nets as an intermediate step towards hyperbolic nets. According to the two equivalent descriptions of discrete A-nets, we give two descriptions of pre-hyperbolic nets, i.e., one in $\mathbb{R}\mathrm{P}^3$ and another one in $\mathbb{R}\mathrm{P}^{3,3}$ in terms of Pücker geometry. Then we show that an elementary quadrilateral of an A-net admits a 1-parameter family of adapted hyperboloids, i.e., hyperboloids for which the extended edges of the quadrilateral are rulings, and discuss the extension of A-nets to pre-hyperbolic nets. Subsequently, we prove the central Theorem 2, which states that a simply connected A-net allows for such an extension if and only if all interior vertices are of even degree and that in this case there exists a 1-parameter family of adapted pre-hyperbolic nets. The motivation to introduce pre-hyperbolic nets is two-fold. On the one hand, they provide a convenient way to prove the consistency of the extension of A-nets to hyperbolic nets later on, since every hyperbolic net is the restriction of a pre-hyperbolic net. On the other hand, they are interesting structures in their own right, especially in their Plücker geometric description.[6]

We start with the geometric description of pre-hyperbolic nets in $\mathbb{R}\mathrm{P}^3$: A generic discrete A-net with hyperboloids associated with elementary quadrilaterals is called a *pre-hyperbolic net*, if

(i) the four discrete asymptotic lines associated with edges of an elementary quadrilateral of the A-net are asymptotic lines of the corresponding hyperboloid, and
(ii) hyperboloids belonging to edge-adjacent quadrilaterals have coinciding contact elements along the common (discrete) asymptotic line, say h, i.e., the hyperboloids are tangent along h.

The translation of (i) and (ii) into the projective model of Plücker geometry reads

[6] In exactly the same spirit one could introduce pre-cyclidic nets in [2] and use them to prove the consistency of the propagation of Dupin cyclides.

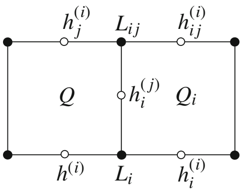


Fig. 8 Lines and contact elements of edge-adjacent quadrilaterals of an A-net

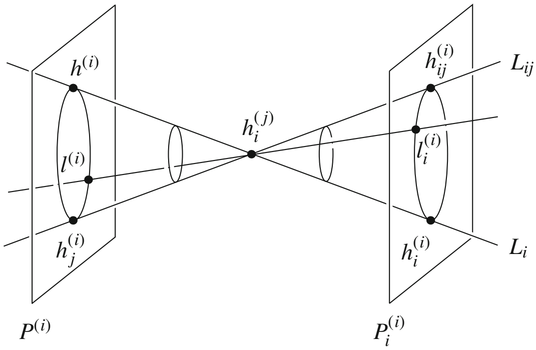


Fig. 9 Planes $P^{(i)}$ and $P_i^{(i)}$ of adjacent hyperboloids $\mathscr{H}$ and $\mathscr{H}_i$ that are tangent along the common asymptotic line $h_i^{(j)}$ $(i \neq j)$

(i) The four discrete asymptotic lines $\mathsf{h}^{(1)}, \mathsf{h}_2^{(1)}$ and $\mathsf{h}^{(2)}, \mathsf{h}_1^{(2)}$ of an elementary quadrilateral correspond to four points $h^{(1)}, h_2^{(1)}$ and $h^{(2)}, h_1^{(2)}$ on the Plücker quadric $\mathscr{Q}^{3,3}$, cf. Fig. 7. The hyperboloid $\mathscr{H} \subset \mathbb{R}\mathrm{P}^3$ associated with the quadrilateral corresponds to two polar planes $P^{(1)}$ and $P^{(2)}$ in $\mathbb{R}\mathrm{P}^{3,3}$ (cf. Theorem 1). Hence, the extended edges of the quadrilateral are asymptotic lines of the hyperboloid $\mathscr{H}$ if the points $h^{(1)}, h_2^{(1)}$ lie in the plane $P^{(1)}$ and the points $h^{(2)}, h_1^{(2)}$ lie in the plane $P^{(2)}$.

(ii) Let $\mathscr{H}$ and $\mathscr{H}_i$ be the two hyperboloids associated with edge-adjacent quadrilaterals Q and Q_i (cf. Fig. 8) and denote the corresponding planes $P^{(1)}, P^{(2)}$ and $P_i^{(1)}, P_i^{(2)}$, respectively. Then $\mathscr{H}$ and $\mathscr{H}_i$ are tangent along the common edge $h_i^{(j)}$, $i \neq j$, if all contact elements of the two hyperboloids along this edge coincide, which means $\mathrm{inc}[h_i^{(j)}, P^{(i)}] = \mathrm{inc}[h_i^{(j)}, P_i^{(i)}]$. In particular, if $l^{(i)} \in P^{(i)} \cap \mathscr{Q}^{3,3}$ and $l_i^{(i)} \in P_i^{(i)} \cap \mathscr{Q}^{3,3}$ are asymtotic lines of $\mathscr{H}$ and $\mathscr{H}_i$ through one point on the common edge, then the contact elements $\mathrm{inc}[h_i^{(j)}, l^{(i)}]$ and $\mathrm{inc}[h_i^{(j)}, l_i^{(i)}]$ coincide, cf. Fig. 9.

Definition 4 (*Pre-hyperbolic net/adapted hyperboloids/C^1-condition*) A generic discrete congruence of isotropic lines L in $\mathscr{Q}^{3,3}$, defined on a quad-graph $\mathscr{D}$, together with polar 2-planes $P^{(1)}, P^{(2)} \in \mathscr{P}_{\mathrm{hyp}}$ associated with elementary quadrilaterals of $\mathscr{D}$ is a *pre-hyperbolic net* if

(i) $h^{(1)}, h_2^{(1)} \in P^{(1)}$ and $h^{(2)}, h_1^{(2)} \in P^{(2)}$, and

(ii) for hyperboloids associated with adjacent quadrilaterals Q, Q_i of $\mathscr{D}$ as in Fig. 8, the subspaces $\mathrm{inc}[h_i^{(j)}, P^{(i)}]$ and $\mathrm{inc}[h_i^{(j)}, P_i^{(i)}]$ of $\mathbb{R}\mathrm{P}^{3,3}$ coincide for $i \neq j$, cf. Fig. 9.

Hyperboloids that satisfy property (i) are called *adapted* to the corresponding skew quadrilateral. Property (ii) is called the C^1-*condition* or C^1-*property* for adjacent hyperboloids.

Remark 5 Property (ii) in Definition 4 implies that corresponding reguli of adjacent hyperboloids have the same orientation. If the signature of $P^{(i)}$ is $(++-)$, then the 3-space

$\mathrm{inc}[h_i^{(j)}, P^{(i)}]$ is of (degenerate) signature $(++-0)$, cf. Fig. 9. Therefore, as $P_i^{(i)}$ does not contain $h_i^{(j)}$, the plane $P_i^{(i)}$ is also of signature $(++-)$. Since the two reguli of a single hyperboloid have complementary signature, also the signatures of the planes $P^{(j)}$ and $P_i^{(j)}$ coincide.

Extending discrete A-nets to pre-hyperbolic nets Let $Q = (L, L_1, L_{12}, L_2)$ be an elementary quadrilateral of an A-net. We define

$$\mathrm{inc}[Q] := \mathrm{inc}[L, L_1, L_{12}, L_2],$$
$$H[Q] := \mathrm{pol}[L, L_1, L_{12}, L_2] = \mathrm{pol}[Q].$$

Usually we write H, H_i for $H[Q], H[Q_i]$, etc.

Proposition 1 *Let Q be an elementary quadrilateral of a generic A-net that is defined on a quad-graph $\mathscr{D}$. Then:*

(i) The space $\mathrm{inc}[Q] \subset \mathbb{R}\mathrm{P}^{3,3}$ *is 3-dimensional and of signature* $(++--)$. *The space* $H[Q] \subset \mathbb{R}\mathrm{P}^{3,3}$ *is a projective line of signature* $(+-)$.
(ii) The lines $H = \mathrm{pol}[Q]$ *associated with elementary quadrilaterals Q constitute a discrete line congruence on the dual graph of $\mathscr{D}$.*

Proof *(i)* An elementary quadrilateral of a generic A-net is a non-planar quadrilateral in $\mathbb{R}\mathrm{P}^3$. According to Remark 2, the pairs $h^{(2)}, h_1^{(2)}$ and $h^{(1)}, h_2^{(1)}$, as well as the diagonals g_1, g_2 each span lines of signature $(+-)$ in $\mathbb{R}\mathrm{P}^{3,3}$. Thus, the line $H[Q] = \mathrm{inc}[g_1, g_2]$ has signature $(+-)$ and its polar space $\mathrm{inc}[Q] = \mathrm{inc}[h^{(2)}, h_1^{(2)}, h^{(1)}, h_2^{(1)}]$ has signature $(++--)$.
(ii) It remains to show that for edge-adjacent quadrilaterals Q and Q_i the lines H and H_i intersect. This follows from the observation, that L_i and L_{ij}, $i \neq j$, are both contained in each of the projective spaces $\mathrm{inc}[Q]$ and $\mathrm{inc}[Q_i]$, cf. Fig. 8. Therefore $H, H_i \subset \mathrm{pol}[L_i, L_{ij}]$, and since $\mathrm{pol}[L_i, L_{ij}]$ is a projective plane, the lines H and H_i intersect. □

A skew quadrilateral admits a 1-parameter family of adapted hyperboloids, as indicated in Fig. 11. These hyperboloids can be described in terms of Plücker geometry, as explained in the following

Lemma 1 *For an elementary quadrilateral Q of a generic A-net, there exists a 1-parameter family of adapted hyperboloids. Each such hyperboloid is uniquely determined by the choice of two labeled points $q^{(1)}, q^{(2)} \in H[Q]$ that are polar (see Figs. 10, 11).*

Proof For each point $q \in H = H[Q]$ that does not lie on the Plücker quadric, there is a unique polar point on H due to the signature $(+-)$ of H. Each pair of polar points $q^{(1)}, q^{(2)} \in H \setminus \mathscr{Q}^{3,3}$ determines a pair of polar planes $P^{(i)} := \mathrm{inc}[h^{(i)}, h_j^{(i)}, q^{(i)}]$ in $\mathscr{P}_{\mathrm{hyp}}$. These planes define a unique hyperboloid with prescribed asymptotic lines, cf. Fig. 10. Conversely, any hyperboloid $\mathscr{H}$ that contains the given discrete asymptotic lines can be constructed in this way, since associated polar planes $P^{(i)} \in \mathscr{P}_{\mathrm{hyp}}$ intersect H in polar points $q^{(i)}$.

Remark 6 There are two special choices for $q \in H$, namely the isotropic points in $H \cap \mathscr{Q}^{3,3}$. In this case, q is one of the two diagonals of the non-planar quadrilateral in $\mathbb{R}\mathrm{P}^3$. Furthermore, q is self polar and does not yield a decomposition of $\mathbb{R}\mathrm{P}^{3,3}$ into two disjoint planes in $\mathscr{P}_{\mathrm{hyp}}$

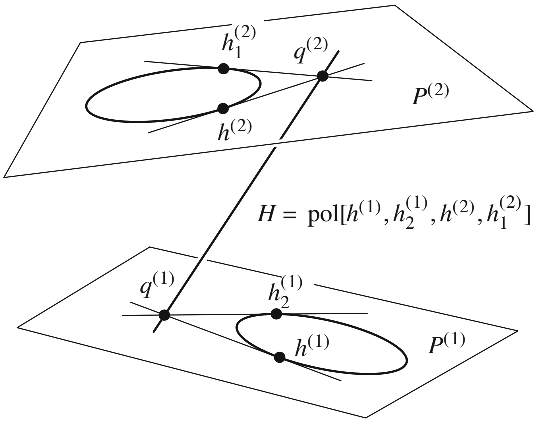


Fig. 10 Configuration in $\mathbb{R}\mathrm{P}^{3,3}$, which describes the extension of an elementary quadrilateral of a discrete A-net to a hyperboloid. Given representatives $h^{(1)}, h_2^{(1)}, h^{(2)}, h_1^{(2)} \in \mathcal{Q}^{3,3}$ of the discrete asymptotic lines, the extension is performed by choosing polar points $q^{(1)}, q^{(2)} \in H$

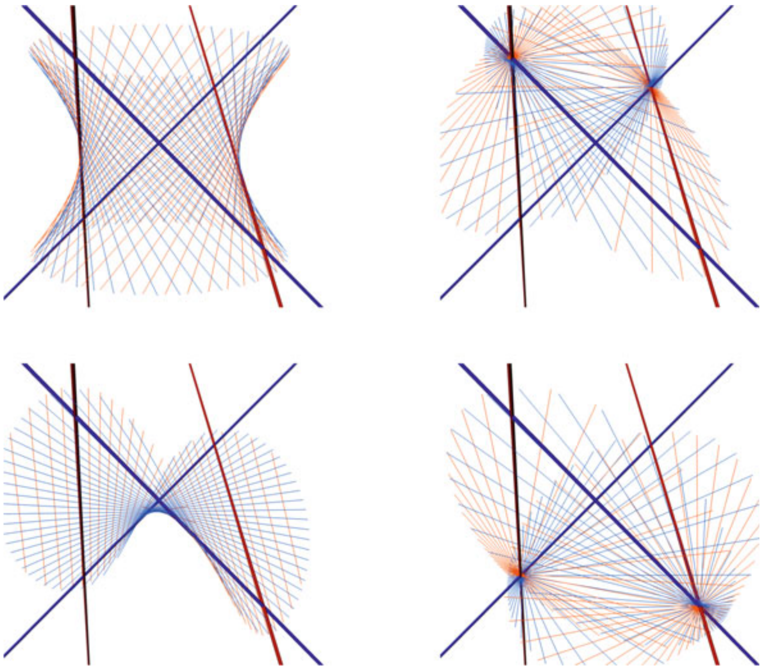


Fig. 11 The 1-parameter family of adapted hyperboloids supporting an elementary quadrilateral of an A-net: Two non-degenerate hyperboloids (*left*) and the two degenerate cases (*right*, cf. Remark 6)

as required by Theorem 1. Hence, there is no hyperboloid associated with $q = q^{(1)} = q^{(2)}$. Instead, this choice corresponds to one of the two limiting cases of hyperboloids $\mathscr{H}$ for which $\mathsf{h}^{(1)}, \mathsf{h}_2^{(1)}, \mathsf{h}^{(2)}, \mathsf{h}_1^{(2)}$ are asymptotic lines. The hyperboloid degenerates to two planes in $\mathbb{R}\mathrm{P}^3$ that contain those lines and intersect in the diagonal q (see Fig. 11 right). If we considering the planes $P^{(1)} = \mathrm{inc}[h^{(1)}, h_2^{(1)}, q]$ and $P^{(2)} = \mathrm{inc}[h^{(2)}, h_1^{(2)}, q]$ as in the generic case, each intersection $P^{(i)} \cap \mathcal{Q}^{3,3}$ with the Plücker quadric is a degenerate conic. Each degenerate conic consists of two isotropic lines, i.e., two contact elements. The planes of these contact

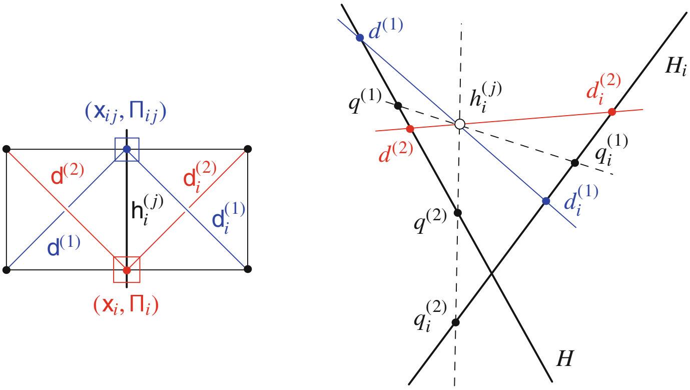


Fig. 12 *Left* the polar line pol$[Q]$ of a skew quadrilateral is spanned by its diagonals, i.e., $H = \mathrm{pol}[Q] = \mathrm{inc}[d^{(1)}, d^{(2)}]$ and $H_i = \mathrm{inc}[d_i^{(1)}, d_i^{(2)}]$. Intersecting diagonals span contact elements at the common vertices of the quadrilaterals, namely $\mathrm{inc}[d^{(1)}, d_2^{(1)}] \sim (\mathsf{x}_{ij}, \Pi_{ij})$ and $\mathrm{inc}[d^{(2)}, d_1^{(2)}] \sim (\mathsf{x}_i, \Pi_i)$. These contact elements share the discrete asymptotic line $\mathsf{h}_i^{(j)}$. *Right* for a pre-hyperbolic net, pairs $(q^{(1)}, q^{(2)})$ and $(q_i^{(1)}, q_i^{(2)})$ of polar points that describe adjacent hyperboloids are related by the projection through the common asymptotic line $h_i^{(j)}$

elements are exactly the planes in $\mathbb{R}\mathrm{P}^3$ intersecting in q, and the points of the contact elements are the vertices of the quadrilateral that are adjacent to q.

The C^1-condition for adjacent hyperboloids in a pre-hyperbolic net determines the whole net if one initial hyperboloid is given. In particular, there exists a 1-parameter family of pre-hyperbolic nets for a generic discrete A-net. To prove the consistency of the evolution, we first describe the propagation of a hyperboloid from one quadrilateral Q to an adjacent quadrilateral Q_i in the setting of Lemma 1.

Lemma 2 *Let Q and Q_i be edge-adjacent quadrilaterals of a pre-hyperbolic net. The pairs $(q^{(1)}, q^{(2)})$ and $(q_i^{(1)}, q_i^{(2)})$ of polar points that describe the corresponding adapted hyperboloids are related by the projection through the common asymptotic line $h_i^{(j)}$, cf. Fig. 12.*

Proof The lines $H = \mathrm{pol}[Q]$ and $H_i = \mathrm{pol}[Q_i]$ intersect according to Proposition 1(ii). Moreover, H and H_i are spanned by the diagonals of the skew quadrilaterals, i.e., with respect to the notation of Fig. 12, $H = \mathrm{inc}[d^{(1)}, d^{(2)}]$ and $H_i = \mathrm{inc}[d_i^{(1)}, d_i^{(2)}]$. Therefore, the plane $\mathrm{inc}[H, H_i]$ contains $h_i^{(j)} = \mathrm{inc}[d^{(1)}, d_i^{(1)}] \cap \mathrm{inc}[d^{(2)}, d_i^{(2)}]$. Denote by τ the projection between H and H_i through $h_i^{(j)}$. We have already established

$$\tau(d^{(1)}) = d_i^{(1)}, \quad \tau(d^{(2)}) = d_i^{(2)}. \tag{1}$$

Now consider the projection through $h_i^{(j)}$ between the planes $P^{(i)}$ and $P_i^{(i)}$, which identifies the conics $\mathscr{C}^{(i)}$ and $\mathscr{C}_i^{(i)}$ (cf. Fig. 9). This reveals

$$\tau(q^{(i)}) = q_i^{(i)}, \tag{2}$$

because $q^{(i)}$ is the pole of the line $\mathrm{inc}[h^{(i)}, h_j^{(i)}]$ in the plane $P^{(i)}$ with respect to the conic $\mathscr{C}^{(i)}$ and analogous for $q_i^{(i)}$ (cf. Fig. 10). Moreover, since $H \cap \mathscr{Q}^{3,3} = \{d^{(1)}, d^{(2)}\}$ and $q^{(1)}$ and $q^{(2)}$ are polar points, we find for the cross-ratio of those four points

$$\mathrm{cr}(d^{(1)}, q^{(1)}, d^{(2)}, q^{(2)}) = -1$$

and analogously

$$\mathrm{cr}(d_i^{(1)}, q_i^{(1)}, d_i^{(2)}, q_i^{(2)}) = -1.$$

As τ is projective transformation, it preserves the cross-ratio and therefore (1) and (2) together with the cross-ratio equations imply

$$\tau(q^{(j)}) = q_i^{(j)}.$$

□

We are now ready to proof the main theorem about pre-hyperbolic nets.

Theorem 2 *For a simply connected generic A-net with all interior vertices of even degree, there exists a 1-parameter family of pre-hyperbolic nets. Each such net is determined by the choice of one initial hyperboloid.*

Proof Lemma 1 shows that for each elementary quadrilateral of an A-net, there is a 1-parameter family of adapted hyperboloids. Hence, there is a 1-parameter freedom for the choice of one initial hyperboloid. Lemma 2 explains how this hyperboloid determines the four neighboring hyperboloids associated with edge-adjacent quadrilaterals by projecting the pair $(q^{(1)}, q^{(2)})$ of polar points through the focal net of the line congruence. It remains to show, that this evolution is consistent, i.e., that the propagation of hyperboloids to all quadrilaterals of the net is independent of the particular chosen path. Since the quad-graph is assumed to be simply connected, there are no non-trivial cycles and we have to consider consistency of the evolution around single inner vertices only.

First, we consider a regular vertex of degree four as in Fig. 13. We start with the two points $q^{(1)}$ and $q^{(2)}$ on $H = \mathrm{pol}[Q]$, which determine the hyperboloid $\mathscr{H}$ associated with the lower left quadrilateral. By Lemma 2, the projections of $(q^{(1)}, q^{(2)})$ onto H_1 and H_2 through the projection centers $h_1^{(2)}$ and $h_2^{(1)}$ yield the adjacent hyperboloids $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Consistency for all initial data means that the composed map that propagates the pair $(q^{(1)}, q^{(2)})$ around the central vertex is the identity. We have to check two properties. Firstly, it has to hold

$$\tau_2^{(1)} \circ \tau_{12}^{(2)} \circ \tau_{12}^{(1)} \circ \tau_1^{(2)} = \mathrm{id} \quad \Leftrightarrow \quad \tau_{12}^{(1)} \circ \tau_1^{(2)} = \tau_{12}^{(2)} \circ \tau_2^{(1)}, \tag{3}$$

where lower and upper indices correspond to the center of projection. The maps $\tau_2^{(1)}, \tau_{12}^{(2)}, \tau_{12}^{(1)}, \tau_1^{(2)}$ are understood as involutions. For example the projection $\tau_1^{(2)} : H \to H_1$ through $h_1^{(2)}$ as a map in $\mathbb{R}\mathrm{P}^{3,3}$ is inverted by exchanging domain and co-domain (cf. Fig. 12). Secondly, we have to prove that the labeling of q's is preserved.

To prove (3), first note that by definition of the lines H, H_1, H_{12}, H_2, the space $V = \mathrm{inc}[H, H_1, H_{12}, H_2]$ is contained in the 3-space $\mathrm{pol}[L_{12}] \supset L_{12}$. Moreover, $\mathrm{inc}[H, H_1] \cap L_{12} = h_1^{(2)}$ and analogous for the other H-lines, cf. Fig. 12. Now, since the intersection points $h_1^{(2)}, h_{12}^{(1)}, h_{12}^{(2)}, h_2^{(1)}$ of the line L_{12} with the planes $\mathrm{inc}[H, H_1]$, $\mathrm{inc}[H_1, H_{12}]$, $\mathrm{inc}[H_{12}, H_2]$, and $\mathrm{inc}[H_2, H]$ are distinct, the space V has to be at least 3-dimensional. Therefore $V = \mathrm{pol}[L_{12}]$. As a consequence, we can restrict ourselves to the 3-space V and understand the projections $\tau_2^{(1)}, \tau_{12}^{(2)}, \tau_{12}^{(1)}, \tau_1^{(2)}$ as one and the same projection τ through L_{12}, just acting on different (co-)domains contained in V. This proves (3).

So far, exactly the same argumentation holds for arbitrary degree of the central vertex and corresponding identities analogous to (3). However, the labeling of q's is only preserved

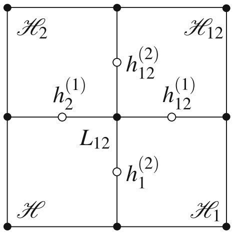


Fig. 13 All four projection centers $h_1^{(2)}, h_{12}^{(1)}, h_{12}^{(2)}, h_2^{(1)}$ lie on the line L_{12}

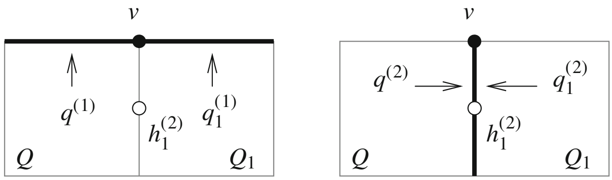


Fig. 14 Propagation of the labeling of q's, represented by edges adjacent to the central vertex, with respect to the map $\tau_1^{(2)}$

for even vertex degree: Given a point $q \in H = \mathrm{pol}[Q]$, one can understand the labeling $q \to q^{(i)}$ as association of q to an edge that is adjacent to the central vertex, say v, since the two edges of Q adjacent to v are natural representatives of the two reguli of any admissible hyperboloid. Now, if q is propagated $q \mapsto q_i = \tau_i^{(j)}(q), i \neq j$, then the labeling, i.e., allocation of q to an edge adjacent to v, has to be propagated accordingly as indicated in Fig. 14. It follows, that the labeling of q's is propagated consistently, if and only if v is of even degree. $\square$

Remark 7 A simply connected quad-graph is either a disc or a sphere, in particular it is orientable. Moreover, if all interior vertices are of even degree, it has to be a disc. This follows from the Euler characteristic, which yields that for a strongly regular cell-decomposition of a sphere the sum of the number of triangles and the number of vertices of degree 3 is at least 8. Using this fact, one can also show easily, that the considered quad-graphs do not contain self-intersecting or closed quadrilateral strips (see Definition 7) and that two different quadrilateral strips intersect at most once. This statement about quad-strips shows for example, that there exist rhombic embeddings of the considered graphs [20], which in turn plays an important role in the analysis of Cauchy problems for 3D consistent 2D quad equations to be imposed on the faces of a quad-graph, see, e.g., [8, Chap. 6].

4 Hyperbolic nets

This section describes the extension of affine A-nets to hyperbolic nets. An A-net in $\mathbb{RP}^3$ is turned into an affine A-net by choosing an affine chart of $\mathbb{RP}^3$ and equipping the A-net with finite edges that connect adjacent vertices, see Sect. 2.3. The elementary quadrilaterals of affine A-nets are therefore skew quadrilaterals in $\mathbb{R}^3$. Extending an affine A-net to a hyperbolic net amounts to fitting hyperboloid surface patches into the skew quadrilaterals in such a way, that edge-adjacent patches have coinciding tangent planes along the common edge. We refer to this relation as the C^1*-property for adjacent hyperboloid patches* (cf. Definition 4). If (locally) the extension of an affine A-net to a hyperbolic net is possible, one adapted surface patch determines its next neighbours according to the C^1-property. Starting with one initial hyperboloid patch, the consistency of this propagation is guaranteed by the consistency of the propagation of hyperboloids in the context of pre-hyperbolic nets. However, it turns out that not all affine A-nets that allow for an extension to pre-hyperbolic nets can also be extended to hyperbolic nets. The additional characteristic property is shown to be "equi-twist" of quadrilateral strips.

We discuss the orientation of reguli and introduce the related twist of skew quadrilaterals in Sect. 4.1. In Sect. 4.2, we first investigate hyperboloid surface patches, focussing on patches bounded by a prescribed skew quadrilateral, before introducing hyperbolic nets and establishing their connection with affine A-nets with equi-twisted quadrilateral strips. The result is Theorem 3 about the extension of affine A-nets to hyperbolic nets. We finish with some comments on the computer implementation of our results.

4.1 Orientation of reguli and twist of skew quadrilaterals

Orientation of reguli Any three skew lines in $\mathbb{R}P^3$ span a regulus. The corresponding plane in $\mathscr{P}_{\text{hyp}}$ is either of signature $(++-)$ or of signature $(+--)$. The different signatures correspond to the two possible orientations of a regulus, so an orientation in $\mathbb{R}P^3$ is defined for three skew lines. Now, consider a hyperboloid $\mathscr{H}$ in $\mathbb{R}P^3$ that is described by two polar planes $P^{(1)}$ and $P^{(2)}$ in $\mathbb{R}P^{3,3}$, both being spanned by representatives of arbitrary three lines of the corresponding reguli of $\mathscr{H}$. Assume $P^{(1)}$ is of signature $(++-)$ and $P^{(2)}$ is of signature $(+--)$. The signature of a plane P is the signature of the Plücker product restricted to the 3-space $\widehat{P}$ of homogeneous coordinates. With respect to a particular basis of $\widehat{P}$,, the restriction $\langle\cdot,\cdot\rangle\big|_{\widehat{P}}$ is represented by a symmetric 3×3 matrix M_P, cf. the Appendix. Due to Sylvester's law of inertia, the signature of M_P is independent of the chosen basis and coincides with the signature of $\langle\cdot,\cdot\rangle\big|_{\widehat{P}}$. As the determinant of a matrix is the product of its eigenvalues, we have

$$\det(M_{P^{(1)}}) < 0 \quad \text{and} \quad \det(M_{P^{(2)}}) > 0.$$

According to the sign of the determinant, the regulus $P^{(1)} \cap \mathscr{Q}^{3,3}$ is said to be of negative orientation, while $P^{(2)} \cap \mathscr{Q}^{3,3}$ is said to be of positive orientation. For the calculation of the orientation of a regulus (or, equivalently, the orientation of three skew lines) in coordinates, see the Appendix.

Remark 8 There are different choices of Plücker line coordinates and the orientation of a regulus depends on the particular choice. Referring to a fixed affine chart, one can identify the two possible orientations of reguli with clockwise and counter-clockwise screw motions in space. However, this interpretation depends on the choice of the affine chart. While for fixed Plücker coordinates the orientation of a regulus is well defined, in different affine charts the same regulus may appear rotating either clockwise or counter-clockwise.

Now recall the extension of an elementary quadrilateral Q of an A-net to a hyperboloid, as discussed in Sect. 3. By Lemma 1, there exists a 1-parameter family of different hyperboloids adapted to Q (see Fig. 11). The 1-parameter family of adapted hyperboloids corresponds to the 1-parameter family of polar points $q, q^* \in H[Q] = \text{pol}[h^{(1)}, h_2^{(1)}, h^{(2)}, h_1^{(2)}]$. We ignore the degenerate case $q = q^*$, which is equivalent to $\langle\widehat{q},\widehat{q}\rangle = 0$ (see Remark 6). In the generic case, $\langle\widehat{q},\widehat{q}\rangle$ and $\langle\widehat{q}^*,\widehat{q}^*\rangle$ have opposite signs. Such a generic pair of polar points yields two different hyperboloids, depending on the labeling of the points, i.e., either $q^{(1)} = q$ and $q^{(2)} = q^*$ or the other way around. The essential difference between the labelings is the following: If we change the labeling, this interchanges the signatures of $P^{(1)} = \text{inc}[h^{(1)}, h_2^{(1)}, q^{(1)}]$ and

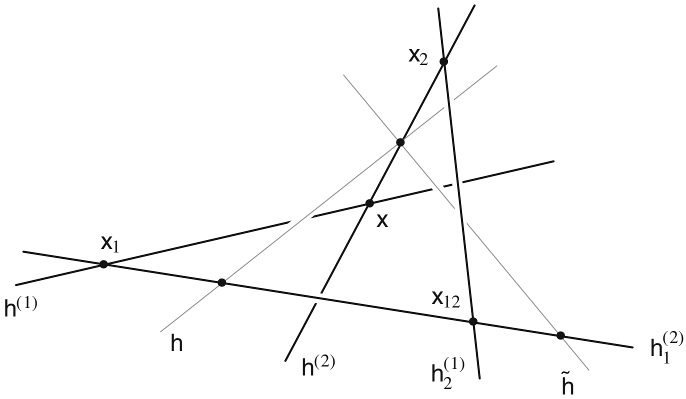


Fig. 15 With respect to a fixed affine chart, the orientation of a regulus can be referred to as a clockwise or counter-clockwise screw motion. The regulus spanned by $\mathsf{h}^{(1)}$, h, $\mathsf{h}_2^{(1)}$ is oriented clockwise, while the regulus spanned by $\mathsf{h}^{(1)}$, $\tilde{\mathsf{h}}$, $\mathsf{h}_2^{(1)}$ is oriented counter-clockwise

$P^{(2)} = \mathrm{inc}[h^{(2)}, h_1^{(2)}, q^{(2)}]$, i.e., the orientations of the hyperbolic families of lines $P^{(1)} \cap \mathscr{Q}^{3,3}$ and $P^{(2)} \cap \mathscr{Q}^{3,3}$ are interchanged.

As mentioned in Remark 8, in a fixed affine chart, the orientation of a regulus can be described as a clockwise or counter-clockwise screw motion. We will now explain this geometric interpretation in the context of the extension of a skew quadrilateral to an adapted hyperboloid. Extending the pair of lines $\mathsf{h}^{(1)}$ and $\mathsf{h}_2^{(1)}$ to a regulus of an adapted hyperboloid corresponds to choosing a third line h that is skew to $\mathsf{h}^{(1)}$ and $\mathsf{h}_2^{(1)}$, but which intersects $\mathsf{h}^{(2)}$ and $\mathsf{h}_1^{(2)}$, cf. Fig. 15. Referring to the affine chart used in Fig. 15, the regulus $\mathscr{H}^{(1)} = \mathrm{inc}[h^{(1)}, h, h_2^{(1)}] \cap \mathscr{Q}^{3,3}$ is oriented clockwise. This reflects how the lines of $\mathscr{H}^{(1)}$ twist around any line of the complementary regulus $\mathscr{H}^{(2)} = \mathrm{pol}[h^{(1)}, h, h_2^{(1)}] \cap \mathscr{Q}^{3,3}$. If one considers the regulus $\tilde{\mathscr{H}}^{(1)} = \mathrm{inc}[h^{(1)}, \tilde{h}, h_2^{(1)}] \cap \mathscr{Q}^{3,3}$, which is spanned by the skew lines $\mathsf{h}^{(1)}$, $\tilde{\mathsf{h}}$, and $\mathsf{h}_2^{(1)}$ instead, then $\tilde{\mathscr{H}}^{(1)}$ is oriented counter-clockwise.

Twist of skew quadrilaterals A skew quadrilateral in $\mathbb{R}\mathrm{P}^3$ consists of four points in general position which are connected by edges, i.e., line segments. The notion of orientation for reguli induces a natural orientation for a pair of opposite edges of a skew quadrilateral. Any skew quadrilateral in $\mathbb{R}\mathrm{P}^3$ can be seen as a skew quadrilateral in $\mathbb{R}^3$, i.e., there is an affine chart such that all four edges of the quadrilateral are finite. Accordingly, consider Fig. 15 in affine $\mathbb{R}^3$, such that there is a unique skew quadrilateral with vertices $\mathsf{x}, \mathsf{x}_1, \mathsf{x}_2$, and x_{12} and finite edges connecting them. The *twist of the opposite edge pair* $(\mathsf{x}, \mathsf{x}_1)$ and $(\mathsf{x}_2, \mathsf{x}_{12})$ is defined as the orientation of any regulus that is spanned by $\mathsf{h}^{(1)}, \mathsf{h}_2^{(1)}$, and a third line h, where h is skew to $\mathsf{h}^{(1)}, \mathsf{h}_2^{(1)}$ but intersects the complementary lines $\mathsf{h}^{(2)}$ and $\mathsf{h}_1^{(2)}$ in the edges of Q. So the twist of an opposite edge pair can be either positive, or negative, where the two edge pairs of a quadrilateral have complementary twist. Intuitively it is clear, that the twist does not depend on the particular choice of the line h; a proof using coordinates can be found in the Appendix. With respect to the affine chart of Fig. 15, we may say that the edge pair $(\mathsf{x}, \mathsf{x}_1)$, $(\mathsf{x}_2, \mathsf{x}_{12})$ turns clockwise, while the edge pair $(\mathsf{x}, \mathsf{x}_2)$, $(\mathsf{x}_1, \mathsf{x}_{12})$ turns counter-clockwise (see Remark 8).

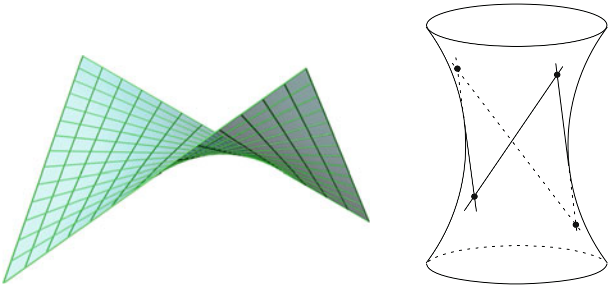


Fig. 16 *Left* a finite hyperboloid patch. *Right* a finite skew quadrilateral on a hyperboloid that does not bound a hyperboloid patch

4.2 Hyperbolic nets

In Sect. 3, it was shown that simply connected A-nets with all interior vertices of even degree can be extended to pre-hyperbolic nets. We will now characterize those affine A-nets that can be extended to piecewise smooth surfaces by attaching hyperboloid patches to the skew quadrilaterals, such that adjacent patches satisfy the C^1-property. The resulting surfaces are called hyperbolic nets. We start with the formal definition of hyperboloid patches and the extension of skew quadrilaterals to such patches, before defining hyperbolic nets.

Definition 5 (*Hyperboloid patch*) A *hyperboloid patch* is a (parametrized) surface patch, obtained by restricting a (global) asymptotic line parametrization of a hyperboloid to a closed rectangle.

Geometrically, a hyperboloid patch is a piece of a hyperboloid cut out along four asymptotic lines, two per regulus, cf. Fig. 16. Conversely, four asymptotic lines of a hyperboloid cut the hyperboloid into four patches, since a hyperboloid in $\mathbb{R}\mathrm{P}^3$ is a torus, topologically.[7] If we choose a plane at infinity, then this plane intersects some, or all of the patches. More precisely, in the affine setting, a hyperboloid is cut by four asymptotic lines into either four infinite patches or into three infinite patches and one finite patch.

According to Lemma 1, for a skew quadrilateral Q there exists a 1-parameter family of adapted hyperboloids. The crucial observation is, that an adapted hyperboloid can be restricted to a hyperboloid patch bounded by Q if and only if the twists of the opposite edge pairs of Q coincide with the corresponding orientations of the reguli of $\mathscr{H}$, cf. Fig. 16. The reason for that is, that there is a patch bounded by Q if and only if asymptotic lines of $\mathscr{H}$ that intersect one edge of Q also intersect the opposite edge of Q. If the twist of one opposite edge pair coincides with the orientation of the supporting regulus, this is obviously also the case for the other edge pair. Note that, for a hyperboloid patch, any two asymptotic lines of different families intersect in a unique point of the patch. Therefore, if the edges of Q are finite, then any patch bounded by Q is also finite, cf. Fig. 17.

We summarize the above in the following Lemma 3.

Lemma 3 *Let $\mathscr{H}$ be a hyperboloid and Q be a skew quadrilateral contained in $\mathscr{H}$. The hyperboloid can be restricted to a hyperboloid patch that is bounded by Q if and only if for each opposite edge pair of Q the twist coincides with the orientation of the corresponding*

[7] In our setting, a natural way to obtain a homeomorphism between $S^1 \times S^1$ and the contact elements of a hyperboloid is to parametrize each of the corresponding hyperbolic families of lines over S^1 and use the structure explained in Fig. 5.

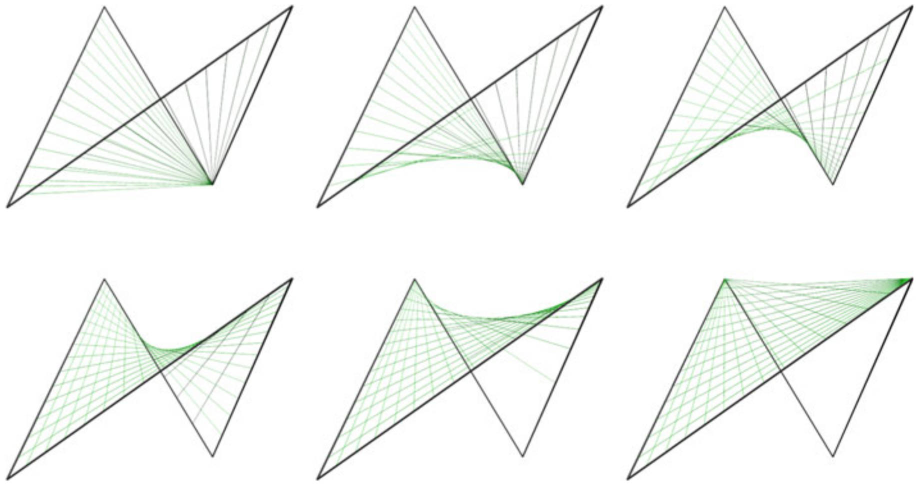


Fig. 17 Hyperboloid patches adapted to a skew quadrilateral

regulus of $\mathscr{H}$. If twist and orientation coincide for one pair, they coincide for the other pair as well. The restriction to a patch is unique if it exists. With respect to an affine chart, the obtained patch is finite if and only if Q is finite.

Finally, we arrive at the definition of our main object of interest.

Definition 6 (*Hyperbolic net*) A generic discrete affine A-net with hyperboloid patches associated with the elementary quadrilaterals is a *hyperbolic net* if

(i) the edges of the quadrilaterals are the bounding asymptotic lines of the hyperboloid patches, and
(ii) the surface composed of the hyperboloid patches is a piecewise smooth C^1-surface in the sense that the compound surface has a unique tangent plane in each point and that the tangent plane varies continuously with the point.

In other words, a hyperbolic net consists of a supporting A-net with hyperboloid patches adapted to elementary quadrilaterals, such that edge-adjacent patches satisfy the C^1-property, cf. Fig. 18.

As mentioned before, there exist affine A-nets that cannot be extended to hyperbolic nets, even though there exists a 1-parameter family of pre-hyperbolic nets. The characterizing property for affine A-nets to be extendable to hyperbolic nets is captured in the following

Definition 7 (*Quad-strip/equi-twisted strips*) A sequence $\{Q_i\}_{i=1,\dots,n}$ of quadrilaterals in a quadrilateral mesh, such that the edges $l_i = Q_{i-1} \cap Q_i$ and $r_i = Q_i \cap Q_{i+1}$ are opposite edges in Q_i, is called a *strip* (cf. Fig. 19). A strip of an A-net is said to be *equi-twisted* if the edge pairs (l_i, r_i) have the same twist for all skew quadrilaterals along the strip (see Sect. 4.1).

Definition 7 is motivated by the propagation of the hyperboloids for pre-hyperbolic nets, since the signatures of corresponding reguli are preserved by the propagation (see Remark 5). We obtain the following characterization of those affine A-nets that can be extended to hyperbolic nets.

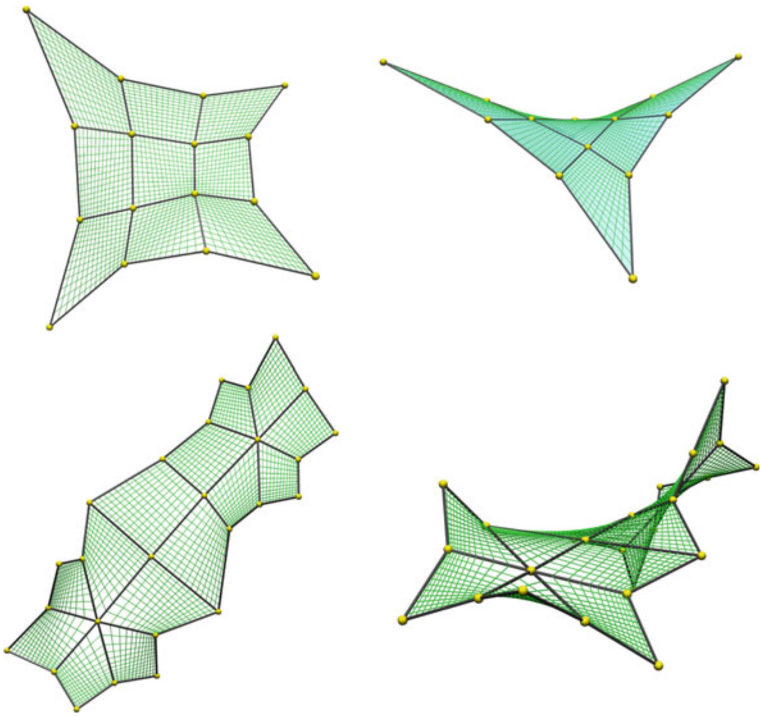
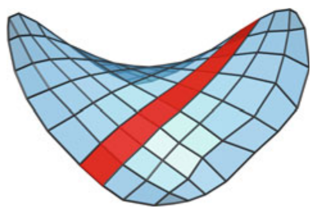


Fig. 18 Two examples of hyperbolic nets. The *top one* is of regular $\mathbb{Z}^2$ combinatorics, while the *bottom one* has two degree six vertices

Fig. 19 An equi-twisted strip of an A-net



Theorem 3 *A simply connected generic affine A-net can be extended to a hyperbolic net if and only if all strips of the A-net are equi-twisted. In this case there exists a 1-parameter family of adapted hyperbolic nets. Each such net is determined by the choice of one initial hyperboloid patch that is bounded by one arbitrary skew quadrilateral of the A-net.*

Proof First observe, that in an A-net with equi-twisted strips all interior vertices necessarily have even vertex degree, because the two pairs of opposite edges of a skew quadrilateral have opposite twists. The rest follows from Theorem 2 together with Lemma 3 and Remark 5.

Hyperbolic nets with cusps As mentioned before, two edge-adjacent hyperboloid patches may form a cusp, also if they satisfy the C^1-property, cf. Fig. 20. One can easily characterize this case in terms of the supporting quadrilaterals. Denote those quadrilaterals by Q_1 and Q_2 and call the common edge e. Further, let x be one vertex of e and denote by e_i the second edge of Q_i that is adjacent to x. All three edges e, e_1, e_2 are contained in the vertex plane P

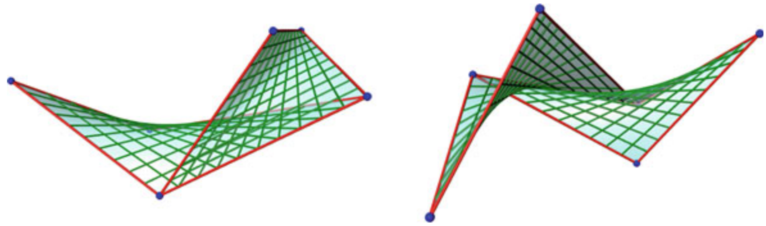


Fig. 20 Two views of hyperbolic patches that satisfy the C^1-condition but form a cusp

of x. The two patches form a cusp if and only the edges e_1 and e_2 lie on the same side of e in the plane P.

Computer implementation The generation of hyperbolic nets has been implemented in java, using jReality [15]. Starting with an arbitrary quadrilateral mesh of suitable combinatorics as input, the first step is to turn the mesh into an A-net. This is achieved by a variational approach, in which the non-planarity of vertex stars is minimized. The corresponding functional $\mathscr{S}$ is a sum of volumes of tetrahedra. The sum covers all possible tetrahedra Δ that are spanned by vertices of the mesh, under the restriction that for each tetrahedron the four vertices have to be contained in a single vertex star

$$\mathscr{S} = \sum_{v \in V} \sum_{v_1,\ldots,v_4 \in \mathrm{star}(v)} \mathrm{Vol}(\Delta(v_1, v_2, v_3, v_4)).$$

In our approach we don't incorporate automatic optimization for equi-twisted quadrilateral strips, but obtain this property by choosing appropriate initial data.

To extend the obtained A-net to a hyperbolic net, we have to choose an initial hyperboloid patch for some quadrilateral. As described in the previous section, we define the initial patch by choosing a point $q \in \mathbb{R}\mathrm{P}^{3,3}$ on the line spanned by the representatives of the two diagonals of the initial skew quadrilateral. The initial patch is then propagated automatically to all other quadrilaterals and one obtains a hyperbolic net.

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5 Appendix

In contrast to the main text, using the projective model of Plücker geometry in $\mathbb{R}\mathrm{P}^{3,3}$, we will now briefly introduce an equivalent projective model that is formulated in the language of exterior algebra. One advantage of this latter model is, that the exterior algebra formulation makes it easier to introduce explicit line coordinates and to do concrete calculations with points and lines. Again, we will only present what is necessary for our purpose. A more detailed introduction to the exterior algebra model can be found, for example, in [25] or [8].

Using exterior algebra, Plücker line coordinates arise in a very natural way if a basis of the space $\mathbb{R}^4$ of homogeneous coordinates for $\mathbb{R}\mathrm{P}^3$ has been chosen. The space of line coordinates in the exterior algebra description is $\Lambda^2(\mathbb{R}^4)$, so the projective space containing the Plücker quadric becomes $\mathrm{P}(\Lambda^2(\mathbb{R}^4))$. The wedge product on $\mathbb{R}^4$ gives Plücker coordinates of lines, i.e., the line h spanned by two points $\mathsf{x}, \mathsf{y} \in \mathbb{R}\mathrm{P}^3$ has Plücker coordinates

$$h = \widehat{\mathsf{x}} \wedge \widehat{\mathsf{y}} \in \Lambda^2(\mathbb{R}^4).$$

Scaling homogenous coordinates $\widehat{\mathsf{x}}, \widehat{\mathsf{y}}$ by non-zero factors results in a corresponding scaling of h.

The wedge product on $\Lambda^2(\mathbb{R}^4)$ is exactly the Plücker product (after canonical identification of $\Lambda^4(\mathbb{R}^4)$ with $\mathbb{R}$). Lines $\mathsf{h} = \mathrm{inc}[\mathsf{x}, \mathsf{y}]$ and $\mathsf{l} = \mathrm{inc}[\mathsf{u}, \mathsf{v}]$ in $\mathbb{RP}^3$ intersect, if and only if

$$\begin{aligned} h \wedge l = (\widehat{\mathsf{x}} \wedge \widehat{\mathsf{y}}) \wedge (\widehat{\mathsf{u}} \wedge \widehat{\mathsf{v}}) &= 0 \quad \text{in } \Lambda^2(\mathbb{R}^4) \\ \iff \langle \widehat{h}, \widehat{l} \rangle &= 0 \quad \text{in } \mathbb{R}^{3,3}. \end{aligned}$$

In particular, for $h \in \Lambda^2(\mathbb{R}^4)$ the condition $h \wedge h = 0$ is equivalent to h being decomposable. This means that there exist $\widehat{\mathsf{x}}, \widehat{\mathsf{y}} \in \mathbb{R}^4$, such that $h = \widehat{\mathsf{x}} \wedge \widehat{\mathsf{y}}$, i.e., h are Plücker coordinates of a line.

Orientation of reguli As explained in Sect. 4.1, three skew lines $\mathsf{h}_0, \mathsf{h}_1, \mathsf{h}_2$ in $\mathbb{RP}^3$ (or $\mathbb{R}^3$) always span a regulus of positive or negative orientation. Let $P = \mathrm{P}(\widehat{P})$ be the plane in $\mathrm{P}(\Lambda^2(\mathbb{R}^4))$ spanned by the representatives $[h_0]$, $[h_1]$, $[h_2]$, and denote by

$$M_P = \begin{pmatrix} 0 & h_0 \wedge h_1 & h_0 \wedge h_2 \\ h_0 \wedge h_1 & 0 & h_1 \wedge h_2 \\ h_0 \wedge h_2 & h_1 \wedge h_2 & 0 \end{pmatrix}$$

the matrix representation of the Plücker product restricted to $\widehat{P}$ with respect to the basis h_0, h_1, h_2. The orientation of $\mathsf{h}_0, \mathsf{h}_1, \mathsf{h}_2$ is the sign of the determinant of M_P. Since the determinant of a matrix is the product of its eigenvalues, we have

$$\text{signature of } P \text{ is } \begin{Bmatrix} (++-) \\ (+--) \end{Bmatrix} \iff \det(M_P) \begin{Bmatrix} < \\ > \end{Bmatrix} 0$$

and one computes

$$\det(M_P) = 2(h_0 \wedge h_1)(h_0 \wedge h_2)(h_1 \wedge h_2).$$

Note, that the sign of $\det(M_P)$ is invariant under rescaling $h_i \mapsto \lambda_i h_i,\ \lambda_i \in \mathbb{R} \setminus \{0\}$, of line coordinates. After all the orientation of a regulus only depends on the particular choice of Plücker coordinates on the space of lines in $\mathbb{RP}^3$, which in the present case are fixed by choosing a basis of $\mathbb{R}^4$.

Twist of skew quadrilaterals In Sect. 4.1, the twist of a pair of opposite edges in a skew quadrilateral $Q = (\mathsf{x}, \mathsf{x}_1, \mathsf{x}_{12}, \mathsf{x}_2)$ is defined. For example the twist of the edge pair $(\mathsf{x}, \mathsf{x}_1)$ and $(\mathsf{x}_2, \mathsf{x}_{12})$ is the orientation of any regulus spanned by the lines $\mathsf{h}^{(1)} \supset (\mathsf{x}, \mathsf{x}_1)$, $\mathsf{h}_2^{(1)} \supset (\mathsf{x}_2, \mathsf{x}_{12})$, and a line $\mathsf{g}^{(1)}$ that is skew to $\mathsf{h}^{(1)}$ and $\mathsf{h}_2^{(1)}$, but that intersects the two remaining edges $(\mathsf{x}, \mathsf{x}_2)$, $(\mathsf{x}_1, \mathsf{x}_{12})$ in the interior. We will now show by computation that the twist of an edge pair is well defined, i.e., that it is independent of the particular choice of such a line $\mathsf{g}^{(1)}$. For ease of notation, we write $Q = (\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d})$ and denote the three lines in question $\mathsf{h}_0, \mathsf{h}_1, \mathsf{h}_2$.

Choose an affine chart of $\mathbb{RP}^3$ by normalizing homogeneous coordinates to lie in an affine hyperplane of $\mathbb{R}^4$, such that all edges of Q are finite. Points on the edges are then described by convex combinations of the (normalized) homogeneous coordinates $\widehat{\mathsf{a}}, \widehat{\mathsf{b}}, \widehat{\mathsf{c}}, \widehat{\mathsf{d}}$ of vertices. Let h_0 be the line supporting the edge (a, b) and h_1 be the line supporting the edge (d, c). (We use this order for the vertices on the edges to indicate that in the quadrilateral $(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d})$, the vertex a is connected to d and b is connected to c.) The third line h_2 is spanned by two points $[\alpha\widehat{\mathsf{a}} + (1-\alpha)\widehat{\mathsf{d}}]$ and $[\beta\widehat{\mathsf{b}} + (1-\beta)\widehat{\mathsf{c}}]$, where $0 < \alpha, \beta < 1$. The Plücker coordinates of the three lines are given by

$$\begin{aligned} h_0 &= \widehat{\mathsf{a}} \wedge \widehat{\mathsf{b}}, \\ h_1 &= \widehat{\mathsf{d}} \wedge \widehat{\mathsf{c}}, \\ h_2 &= (\alpha\widehat{\mathsf{a}} + (1-\alpha)\widehat{\mathsf{d}}) \wedge (\beta\widehat{\mathsf{b}} + (1-\beta)\widehat{\mathsf{c}}) \\ &= \alpha\beta\, \widehat{\mathsf{a}} \wedge \widehat{\mathsf{b}} + \alpha(1-\beta)\, \widehat{\mathsf{a}} \wedge \widehat{\mathsf{c}} + (1-\alpha)\beta\, \widehat{\mathsf{d}} \wedge \widehat{\mathsf{b}} + (1-\alpha)(1-\beta)\, \widehat{\mathsf{d}} \wedge \widehat{\mathsf{c}}. \end{aligned}$$

According to the previous consideration about orientation of reguli, the orientation of $\mathsf{h}_0, \mathsf{h}_1, \mathsf{h}_2$ is the sign of the determinant

$$2(h_0 \wedge h_1)(h_0 \wedge h_2)(h_1 \wedge h_2) = \alpha\beta(1-\alpha)(1-\beta)(\widehat{\mathsf{a}} \wedge \widehat{\mathsf{b}} \wedge \widehat{\mathsf{d}} \wedge \widehat{\mathsf{c}})^3$$

Since $0 < \alpha, \beta < 1$, the sign of the expression is exactly the sign of the product $\widehat{\mathsf{a}} \wedge \widehat{\mathsf{b}} \wedge \widehat{\mathsf{d}} \wedge \widehat{\mathsf{c}}$, and therefore it is independent of the particular choice of the line h_2. We say that the edges (a, b) and (d, c) have *positive resp. negative twist* depending on this sign.

Note, that for fixed Plücker coordinates the twist is independent of the way of calculation, since the Plücker coordinates for lines in $\mathbb{R}\mathrm{P}^3$ do not depend on the choice of an affine chart. In particular, the sign of $\det(M_P)$ does not change under homogenous rescaling of line coordinates.

The same computation for the lines, say, g_0 and g_1 spanned by the other two opposite edges (a, d) and (b, c), and another line g_2 inbetween, yields that the twist of the second edge pair (a, d), (b, c) is given by the sign of

$$\widehat{\mathsf{a}} \wedge \widehat{\mathsf{d}} \wedge \widehat{\mathsf{b}} \wedge \widehat{\mathsf{c}} = -(\widehat{\mathsf{a}} \wedge \widehat{\mathsf{b}} \wedge \widehat{\mathsf{d}} \wedge \widehat{\mathsf{c}}).$$

Hence the two pairs of opposite edges have opposite twists. This coincides with the observation that the two reguli of a hyperboloid have opposite orientations.

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