Isomorphic Factorizations VII. Regular Graphs and Tournaments

N. C. Wormald
LOUISIANA STATE UNIVERSITY
BATON ROUGE, LOUISIANA 70803

ABSTRACT

It is shown using enumeration results that for $r > 2t$, almost all labeled r-regular graphs cannot be factorized into $t \geqslant 2$ isomorphic subgraphs. However, no examples of such nonfactorizable graphs are known which satisfy the obvious divisibility condition that the number of edges is divisible by t. Similar observations hold for regular tournaments ($t \geqslant 2$) and for r-regular digraphs ($r > t \geqslant 2$).

A graph G with q lines is *divisible by t* if its line set can be partitioned into t subsets such that the subgraphs (called *factors*) of G induced by the subsets are all isomorphic. Such a partition is an *isomorphic factorisation* of G *into t parts*. The obvious necessary condition $t \mid q$ is called the *divisibility condition* for G and t. If the divisibility condition is satisfied for G and t but G is not divisible by t, we say G is *t-irrational*. Otherwise G is *t-rational*. G is *rational* if it is t-rational for all $t \geqslant 2$, and G is *irrational* otherwise, in which case it is t-irrational for some t dividing q. In the first [4] and fifth [6] papers in this series, it was shown that all complete graphs and complete directed graphs are rational. Also, Quinn has proved the equipartite conjecture of [5] by showing that all complete multipartite graphs in which all parts are of the same cardinality are rational. Alspach has suggested that perhaps these results are special cases of a much more general result: perhaps all vertex-transitive graphs are rational. This could also be true of vertex-transitive digraphs. No counterexamples are known in either case. One might be tempted to extend this idea to cover all regular graphs and digraphs, since regularity seems to be conducive to isomorphic factorization. (In a regular digraph, all points have the same indegrees and outdegrees.) However, it is our purpose in this paper to show that there are regular graphs and digraphs,

Journal of Graph Theory, Vol. 8 (1984) 117–122
 CCC 0364-9024/84/010117-06$04.00

and also regular tournaments, which are irrational, even though no examples of such irrational graphs are known.

Henceforth, the points of all graphs and digraphs are labeled. By saying that *almost all* graphs with property S have property T, we mean that for those values of n such that n-point graphs with property S exist, the proportion of these sharing property T approaches 1 as $n \to \infty$. We have occasion to use Stirling's formula in the form $n! = n^{n(1+o(1))}$ as $n \to \infty$.

An *r-regular* graph (or digraph) is one in which all points have degree (or indegree and outdegree) precisely r. Clearly all 1-regular graphs are rational. Our only other positive result in this direction is the following.

Theorem 1. (i) All 2-regular graphs are rational.
(ii) All 1-regular digraphs are rational.

Proof. Firstly we consider (i) with $t = 2$. Let G be 2-regular with an even number of lines; then G is the disjoint union of cycles of lengths at least 3. We define as follows the factors F_1 and F_2 in an isomorphic factorization of G into 2 parts. The lines in each even cycle of G are assigned alternately to F_1 and F_2, so that the lines within each of F_1 and F_2 are disjoint. In each odd cycle, two adjacent lines are assigned to either F_1 or F_2 and the rest are assigned alternately, so that one factor receives only disjoint lines and the other receives disjoint lines together with a disjoint copy of a path of length 2. Since G has an even number of odd cycles, it can clearly be arranged that F_1 and F_2 are isomorphic.

To show (i) in case $t \geq 3$, we assign the lines of a 2-regular G to factors $F_1, \ldots, F_t$ in a slightly different way. If C is a cycle of G of length $mt + k$, $0 \leq k < t$, choose k of $F_1, \ldots, F_t$ to be designated as *special* factors for C. As $t \geq 3$, it is easy to assign the lines of C to $F_1, \ldots, F_t$ in such a way that each special factor receives $m + 1$ disjoint lines and all the others receive m disjoint lines each. If t divides the number of lines of G, it can be arranged that the number of cycles C for which a given factor F_i is special is the same for each i. Then all factors are isomorphic, which establishes (i).

If D is a 1-regular digraph then its underlying multigraph G is 2-regular with perhaps some multiple edges. This multigraph G can be factorized isomorphically into t parts whenever t divides the number of lines of G (or arcs of D) using the factorizations described above, with a 2-cycle treated as a cycle of even length. This factorisation induces an isomorphic factorization of D, which serves to establish (ii). ∎

In the next two theorems, some very rough counting yields useful bounds on the proportions of graphs and digraphs divisible by t. We denote $p(p - 1)/2$ by E throughout.

Theorem 2. For r and t fixed such that $t \geq 2$ and $r \geq 2t + 1$, almost all r-regular graphs are not divisible by t.

Proof. From Bender and Canfield [2], the number $h = h(r, p)$ of r-regular graphs with p points is at least $c(rp/e)^{rp/2}(r!)^{-p}$ for some constant c depending only on r, which is $p^{(1+o(1))pr/2}$ as $p \to \infty$. Here p is even if r is odd. However, if $g = g(p, q, t)$ denotes the number of graphs with p points and q lines divisible by t, then

$$g(p, q, t) \leqslant (p!)^{t-1}\binom{E}{q/t}.$$

Here the binomial coefficient is the number of choices for one of the factors in an isomorphic factorization of a (p, q)-graph into t parts, and the factorials bound the numbers of choices of isomorphisms from this factor onto the $t - 1$ others. For r-regular graphs $q = rp/2$, and we have

$$\begin{aligned} g(p, rp/2, t) &\leqslant p^{p(t-1)}E^{rp/2t}/(rp/2t)! \\ &= p^{p(t-1)+rp/t-(rp/2t)(1+o(1))} \\ &= p^{p(t-1+r/2t)(1+o(1))} \end{aligned}$$

as $p \to \infty$. Hence $g/h \leqslant p^{p(t-1)(1-r/2t)(1+o(1))} = o(1)$ as $p \to \infty$, provided $r > 2t \geqslant 4$. The theorem follows. ∎

Theorem 3. For fixed t and r such that $2 \leqslant t < r$, almost all r-regular digraphs are not divisible by t.

Proof. By Bender [1], the number $d = d(r, p)$ of r-regular digraphs on p points is at least $c(pr)!(r!)^{-2p}$ for some constant c depending only on r, which is $p^{(1+o(1))pr}$. However, if $f = f(p, q, t)$ denotes the number of digraphs with p points and q arcs divisible by t, then

$$f(p, q, t) \leqslant (p!)^{t-1}\binom{2E}{q/t}.$$

This is established the same way as the bound on g in the proof of Theorem 2. For r-regular digraphs $q = rp$, and

$$\begin{aligned} f(p, rp, t) &\leqslant p^{p(t-1)}E^{rp/t}/(rp/t)! \\ &\leqslant p^{p(t-1+r/t)(1+o(1))} \end{aligned}$$

as $p \to \infty$. Hence $f/d \leqslant p^{p(t-1)(1-r/t)(1+o(1))} = o(1)$ as $p \to \infty$, provided $r > t \geqslant 2$. The theorem follows. ∎

The result of Theorem 3 is also valid for oriented graphs, that is, digraphs in which symmetric pairs of arcs are prohibited. This is because the number of r-regular oriented graphs on p points is easily determined asymptotically using the methods in [8], and is at least $cd(r, p)$ for a positive constant c.

We deal with tournaments next. In this case it is necessary to use a more refined argument than that used to bound g and f in Theorems 2 and 3.

Theorem 4. For fixed $t \geq 2$, almost all regular tournaments are not divisible by t.

Proof. By Spencer [7], the number R_p of regular tournaments with p points (p odd) satisfies

$$R_p = 2^E\left((1 + o(1))\left(\frac{2}{\pi p}\right)^{1/2}\right)^p = (2 + o(1))^E$$

as $p \to \infty$. However, if $a = a(p, t)$ denotes the number of regular tournaments on p points divisible by t, we can bound a as follows.

Suppose T is a regular tournament divisible by t. Then there is a factorization of T into oriented graphs $S_1, \ldots, S_t$, together with permutations $\sigma_2, \ldots, \sigma_t$ of the point set of T, such that σ_i induces an isomorphism from S_1 to S_i for $2 \leq i \leq t$. The number of possibilities for $\sigma_2, \ldots, \sigma_t$ is at most $(p!)^{t-1}$. If these permutations are specified, we can obtain an appropriate upper bound on the number of possibilities for F_1 by bounding the number of choices for the arcs of S_1 in order, and then by dividing by $(E/t)!$ since S_1 has E/t arcs. The first arc can be chosen in $2E$ ways. Each arc chosen for S_1 is such that its images under the permutations $\sigma_2, \ldots, \sigma_t$ are arcs in $S_2, \ldots, S_t$, respectively. In particular all these images therefore join distinct unordered pairs of points which are then ineligible to be chosen later as the sites of arcs of S_1. Thus, the number of ordered choices of the arcs of S_1 is at most $2E(2E - 2t) \ldots (2t)$. It follows that

$$\begin{aligned} a(p, t) &\leq (p!)^{t-1}\, 2E(2E - 2t) \cdots (2t)/(E/t)! \\ &\leq (p!)^{t-1}(2t)^{E/t} \\ &\leq ((2t)^{1/t} p^{2(t-1)/(p-1)})^E \\ &\leq ((2t)^{1/t}(1 + o(1)))^E. \\ &= o\,(R_p) \end{aligned}$$

as $p \to \infty$ provided $t \geq 3$.

For $t = 2$ the argument is modified. In this case S_1 is an orientation of a self-complementary graph G. The number of possibilities for G is $(2 + o(1))^{E/4}$ (see Harary and Palmer [3, p. 208]). The number of orientations of G is $2^{E/2}$, and the number of possibilities for σ_2 is at most $p! = (1 + o(1))^E$. Hence

$$a(p, 2) \leqslant (2 + o(1))^{3E/4}$$
$$= o(R_p). \qquad \blacksquare$$

The derivation of the upper bound on $a(p, t)$ in the proof of Theorem 4 also yields $g(p, q, t) \leqslant t^{q/t}(p!)^{t-1}\binom{E/t}{q/t}$ and $f(p, q, t) \leqslant t^{q/t}(p!)^{t-1}\binom{2E/t}{q/t}$. But these improvements are not sufficient to settle even the cases $r = 2t$ for graphs and $r = t$ for digraphs. Indeed, it seems that for substantial improvements one must do better than merely estimate the total number of isomorphic factorizations of graphs into t parts, as we have been doing.

From Theorems 2, 3, and 4 it follows that there are t-irrational r-regular graphs for all r and t such that $r \geqslant 2t + 1 \geqslant 5$, t-irrational regular tournaments for all $t \geqslant 2$, and t-irrational r-regular digraphs for all $r > t \geqslant 2$. By the same method, many other classes of graphs can be shown to contain irrational graphs. However, the regular graphs are of special interest for the reasons mentioned above, particularly the lack of examples of irrational regular graphs. All cubic graphs with 8 or fewer points are rational, as is the Petersen graph.

Open problems. Many problems suggest themselves. Here are some.

(1) Find an irrational regular graph.
(2) Find an irrational regular tournament.
(3) Find an irrational regular digraph.
(4) Are all 3-regular or 4-regular graphs rational?
(5) Are all 2-regular digraphs rational?
(6) For each $r \geqslant 3$, is there some $t(r)$ such that all r-regular graphs are $t(r)$-rational? What about r-regular digraphs with $r \geqslant 2$?

One might also try to examine similar problems in which the value of t depends on the number of points in the graph, although this is usually outside the scope of the present series of papers. For instance, for what values of $t \geqslant 2$ are all regular tournaments on p points divisible by t? By Theorem 4, such t must be "large" for large p. Kelly's conjecture that every regular tournament is decomposable into directed Hamilton cycles would imply that $t = (p - 1)/2$ is such a value for all odd p.

ACKNOWLEDGMENT

Research supported by National Science Foundation Grant MCS-8101555, and by the Australian Department of Science and Technology under the

Queen Elizabeth II Fellowships Scheme. Current address: Mathematics Department, University of Newcastle, N.S.W., 2308, Australia.

References

[1] E. A. Bender, The asymptotic number of non-negative integer matrices with given row and column sums. *Discrete Math.* 10 (1974) 217–223.
[2] E. A. Bender and E. R. Canfield, The asymptotic number of labelled graphs with given degree sequences, *J. Combinatorial Theory Ser. A* 24 (1978) 296–307.
[3] F. Harary and E. M. Palmer, *Graphical Enumeration.* Academic, New York (1973).
[4] F. Harary, R. W. Robinson and N. C. Wormald, Isomorphic factorisations I: Complete graphs. *Trans. Amer. Math. Soc.* 242 (1978) 243–260.
[5] F. Harary, R. W. Robinson and N. C. Wormald, Isomorphic factorisations III: Complete multipartite graphs, *Combinatorial Mathematics,* Proceedings of the International Conference on Combinatorial Theory (Canberra). Lecture Notes No. 686, Springer-Verlag, Berlin (1978) 47–54.
[6] F. Harary, R. W. Robinson, and N. C. Wormald, Isomorphic factorisations V: Directed graphs. *Mathematika* 25 (1978) 279–285.
[7] J. H. Spencer, Random regular tournaments. *Period. Math. Hungar.* 5 (1974) 105–120.
[8] N. C. Wormald, The asymptotic distribution of short cycles in random regular graphs. *J. Combinatorial Theory Ser. B* 31 (1981) 156–167.