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On the configurations of even unimodular lattices of rank 48

By

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1. Introduction. Let Γ_{8k} ($k \geqq 1$) be the genus consisting of all isomorphic classes of positive definite even unimodular lattices of rank $8k$. Let L be an element of Γ_{8k}. An element x of L is called a $2m$-vector if it satisfies $(x,x) = 2m$, where $(,)$ is the metric attached to L. We let $\mathscr{L}_{2m}(L)$ denote the sublattice of L generated by all $2m$-vectors in L, and $\mathscr{L}_{2m_1+2m_2}(L)$ the sublattice of L generated by all $2m_1$-vectors and $2m_2$-vectors in L. Let $a(2t, L)$ be the number of $2t$-vectors in L for positive integer t. For a sublattice L_1 of L, the rank of L_1, which is denoted by rank L_1, is defined to be the maximal number of linearly independent vectors over $\mathbb{Q}$ in L_1, where $\mathbb{Q}$ is the field of rational numbers.

We use the notations A_n, D_n and E_n to denote the root lattices, i.e. the irreducible lattices generated by 2-vectors in them of the indicated rank n.

The main purpose of this paper is to prove

Theorem 1. *Let L be an element of Γ_{48}. If it hold that $a(2, L) = a(4, L) = 0$, then we have*

$$\mathscr{L}_6(L) = L.$$

and

Theorem 2. *Let L be an element of Γ_{48}. Assume that either $a(2, L) > 0$ or $a(4, L) > 0$ holds. Then we have,*

(i) *when* rank $\mathscr{L}_2(L) \geqq 3$, *then it holds*
rank $\mathscr{L}_4(L) = 48$,

(ii) *when* rank $\mathscr{L}_2(L) = 2$ *and* $\mathscr{L}_2(L) \cong A_2$ *(isomorphic), then either*
rank $\mathscr{L}_4(L) = 48$ *or*
rank $\mathscr{L}_6(L) = 48$ *holds,*

(iii) *when* rank $\mathscr{L}_2(L) = 2$ *and* $\mathscr{L}_2(L) \cong A_1 \oplus A_1$, *then it holds*
rank $\mathscr{L}_4(L) = 48$,

(iv) *when* rank $\mathscr{L}_2(L) = 1$, *then it holds either*
rank $\mathscr{L}_4(L) = 48$ *or*
rank $\mathscr{L}_6(L) = 48$,

(v) *when* rank $\mathscr{L}_2(L) = 0$, *then it holds*
rank $\mathscr{L}_4(L) = 48$.

In the first version of this paper, the form of Theorem 1 is weaker than that of the present version. We have refined it. Note that Theorem 1 is also stated in [8] without proof. Theorems 1 and 2 include the Theorem in [5] as a special result.

2. Preliminary results. Let L be an element in Γ_{8k}, then theta-series of degree 1 attached to L is defined by

$$\vartheta(\mathbf{z}, L) = \sum_{x \in L} \mathbf{e}((x, x)\mathbf{z}),$$

where $\mathbf{z}$ is the variable of the upper-half plane H and $\mathbf{e}(\cdot) = \exp(\pi i \cdot)$. Theta-series with spherical function P_v of degree v attached to L is defined by (Conf. [1], [5], [6])

$$\vartheta(\mathbf{z}, P_v, L) = \sum_{x \in L} \{P_v(x; \alpha)\}\, \mathbf{e}((x, x)\mathbf{z}),$$

where α is a vector in $L \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Z}$ is the ring of rational integers. If we use the finite set

$$\Lambda_{2t}(L) = \{x \in L \,|\, (x, x) = 2t\},$$

then we have

$$\begin{aligned} \vartheta(\mathbf{z}, L) &= 1 + \sum_{t=1}^{\infty} \sum_{x \in \Lambda_{2t}(L)} \mathbf{e}((x, x)\mathbf{z}) \\ &= \sum_{t=0}^{\infty} a(2t, L)\, \mathbf{e}(2t\mathbf{z}) \quad \text{and} \\ \vartheta(\mathbf{z}, P_v, L) &= \sum_{t=1}^{\infty} \sum_{x \in \Lambda_{2t}(L)} \{P_v(x; \alpha)\}\, \mathbf{e}((x, x)\mathbf{z}). \end{aligned}$$

Here we give the precise forms of the spherical function $P_v(x; \alpha)$ of degree ($v = 6, 8$), which are not given in [1]:

$$\begin{aligned} P_6(x; \alpha) = (x, \alpha)^6 &- 15/(f + 8) \quad (x, \alpha)^4 (x, x) (\alpha, \alpha) \\ &+ 45/(f + 8)(f + 6) \quad (x, \alpha)^2 (x, x)^2 (\alpha, \alpha)^2 \\ &- 15/(f + 8)(f + 6)(f + 4) \quad (x, x)^3 (\alpha, \alpha)^3, \\ P_8(x; \alpha) = (x, \alpha)^8 &- 28/(f + 12) \quad (x, \alpha)^6 (x, x) (\alpha, \alpha) \\ &+ 210/(f + 12)(f + 10) \quad (x, \alpha)^4 (x, x)^2 (\alpha, \alpha)^2 \\ &- 420/(f + 12)(f + 10)(f + 8) \quad (x, \alpha)^2 (x, x)^3 (\alpha, \alpha)^3 \\ &+ 105/(f + 12)(f + 10)(f + 8)(f + 6) \quad (x, x)^4 (\alpha, \alpha)^4, \end{aligned}$$

where f is the rank of the lattice L.

Let $\mathbf{M}(1, k)$ (resp. $\mathbf{S}(1, k)$) be the linear space of modular (resp. cusp) forms of degree 1 and weight k belonging to $SL(2, \mathbb{Z})$.

From now on, we specify the lattice L to be in Γ_{48}. Then it is known that

$$\vartheta(\mathbf{z}, L) \in \mathbf{M}(1, 24), \quad \vartheta(\mathbf{z}, P_v, L) \in \mathbf{S}(1, 24 + v) \tag{1}$$

and

$$\dim \mathbf{M}(1, 24) = 3, \quad \dim \mathbf{S}(1, 26) = 1, \quad \dim \mathbf{S}(1, 24 + v) = 2 \qquad v = 4, 6, 8. \tag{2}$$

We choose $\mathbf{E}_4^6(\mathbf{z})$, $\mathbf{E}_4^3(\mathbf{z})\,\Delta_{12}(\mathbf{z})$, $\Delta_{12}^2(\mathbf{z})$ (resp. $\mathbf{E}_{14}(\mathbf{z})\,\Delta_{12}(\mathbf{z})$) as the basis (resp. base) for $\mathbf{M}(1, 24)$ (resp. $\mathbf{S}(1, 26)$), where $\mathbf{E}_k(\mathbf{z})$ is the normalized Eisenstein series of weight k and $\Delta_{12}(\mathbf{z})$ is the normalized cusp form of weight 12. Their Fourier expansions can be easily calculated. By the above facts, we get

$$\vartheta(\mathbf{z}, P_2, L) = c_1\,\mathbf{E}_{14}(\mathbf{z})\,\Delta_{12}(\mathbf{z}) \tag{3}$$

and

$$\vartheta(\mathbf{z}, L) = c_2\,\mathbf{E}_4^6(\mathbf{z}) + c_3\,\mathbf{E}_4^3(\mathbf{z})\,\Delta_{12}(\mathbf{z}) + c_4\,\Delta_{12}^2(\mathbf{z}), \tag{4}$$

with suitable constants c_1, c_2, c_3, c_4 in $\mathbb{C}$, the field of complex numbers.

From (3), we have the relations

$$\sum_{x\in\Lambda_2(L)} \{(x,\alpha)^2 - (\alpha,\alpha)(x,x)/48\} = c_1, \tag{5}$$

$$\sum_{y\in\Lambda_4(L)} \{(y,\alpha)^2 - (\alpha,\alpha)(y,y)/48\} = -48\,c_1 \tag{6}$$

and

$$\sum_{w\in\Lambda_6(L)} \{(w,\alpha)^2 - (\alpha,\alpha)(w,w)/48\} = -195\,804\,c_1. \tag{7}$$

By comparing the Fourier coefficients in the both sides of (4), we get

$$c_2 = 1,$$

$$a(2, L) = 1\,440 + c_3, \tag{8}$$

$$a(4, L) = 876\,960 + 696\,c_3 + c_4 \tag{9}$$

and

$$a(6, L) = 292\,072\,320 + 162\,252\,c_3 - 48\,c_4. \tag{10}$$

By noting

$$\sum_{x\in\Lambda_{2t}(L)} (x,x) = 2t\;a(2t, L) \qquad \text{for } t \geqq 1,$$

we see that the relations (5) and (6) (resp. (5) and (7)) can be unified to

$$\begin{aligned}\sum_{y\in\Lambda_4(L)} (y,\alpha)^2 - (\alpha,\alpha)\,a(4,L)/12 \\ = -48 \sum_{x\in\Lambda_2(L)} (x,\alpha)^2 + 2(\alpha,\alpha)\,a(2,L)\end{aligned} \tag{11}$$

and

$$\begin{aligned}\sum_{w\in\Lambda_6(L)} (w,\alpha)^2 - (\alpha,\alpha)\,a(6,L)/8 \\ = -195\,804 \sum_{x\in\Lambda_2(L)} (x,\alpha)^2 + 16\,317(\alpha,\alpha)\,a(2,L)/2\end{aligned} \tag{12}$$

respectively.

Throughout the following Props. 1–3, we suppose that L is an element of Γ_{48}. First we prove

Proposition 1. *If* $a(2, L) = 0$ *holds, then we have either*

$$\operatorname{rank} \mathscr{L}_4(L) = 48 \quad or$$
$$\operatorname{rank} \mathscr{L}_6(L) = 48.$$

P r o o f. Since it hold that $a(2, L) = 0$ and $\Lambda_2(L) = \emptyset$ (empty set), the Eqs. (9) and (10) become

(13) $$\sum_{y \in \Lambda_4(L)} (y, \alpha)^2 = (\alpha, \alpha)\, a(4, L)/12$$

and

(14) $$\sum_{w \in \Lambda_6(L)} (w, \alpha)^2 = (\alpha, \alpha)\, a(6, L)/8$$

respectively.

First we assume that $a(4, L) \neq 0$. If it holds that $\operatorname{rank} \mathscr{L}_4(L) < 48$, then we can find a non-zero vector $\alpha \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ so that $(y, \alpha) = 0$ for any $y \in \Lambda_4(L)$. Thus the left-hand side of (13) equals zero, whereas the right-hand side is not zero. This is a contradiction, so that we must have

$$\operatorname{rank} \mathscr{L}_4(L) = 48.$$

Next we assume that $a(2, L) = a(4, L) = 0$. Then by (8), (9) and (10), we get

$$a(6, L) = 52\,416\,000.$$

If it holds that $\operatorname{rank} \mathscr{L}_6(L) < 48$, then we can find a non-zero vector $\alpha \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ so that $(w, \alpha) = 0$ for any $w \in \Lambda_6(L)$. This leads to a contradiction on the both sides in (14), so that we must have

$$\operatorname{rank} \mathscr{L}_6(L) = 48. \qquad \square$$

Proposition 2. *If* $a(4, L) = 0$ *holds, then we have either*

$$\operatorname{rank} \mathscr{L}_2(L) = 48 \quad or$$
$$\operatorname{rank} \mathscr{L}_6(L) = 48.$$

P r o o f. From the assumption $a(4, L) = 0$ and the formula (11), we obtain

(15) $$\sum_{x \in \Lambda_2(L)} (x, \alpha)^2 = (\alpha, \alpha)\, a(2, L)/24.$$

Combining (12) with (15), we obtain

(16) $$\sum_{w \in \Lambda_6(L)} (w, \alpha)^2 = (\alpha, \alpha)\, a(6, L)/8.$$

If $a(2, L) > 0$, then we can conclude that

$$\operatorname{rank} \mathscr{L}_2(L) = 48,$$

using a similar reasoning to that of the proof of Prop. 1.

If $a(2, L) = 0$ and $a(4, L) = 0$, then we should have $a(6, L) > 0$, because $\vartheta(\mathbf{z}, L)$ vanishes when $a(2, L) = a(4, L) = a(6, L) = 0$. Then by virtue of (16), we can conclude that

$$\operatorname{rank} \mathscr{L}_6(L) = 48. \qquad \square$$

Proposition 3. *If either $a(2, L) > 0$ or $a(4, L) > 0$ holds, then we have*

$$\operatorname{rank} \mathscr{L}_{2+4}(L) = 48.$$

Proof. We rewrite the formula (11) to the form

$$\sum_{y \in \Lambda_4(L)} (y, \alpha)^2 + 48 \sum_{x \in \Lambda_2(L)} (x, \alpha)^2 = (\alpha, \alpha)\,[a(4, L)/12 + 2a(2, L)]. \tag{17}$$

Suppose it holds that

$$\operatorname{rank} \mathscr{L}_{2+4}(L) < 48,$$

then we can find a non-zero vector $\alpha \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ so that $(y, \alpha) = 0$ for any $y \in \Lambda_4(L)$ and $(x, \alpha) = 0$ for any $x \in \Lambda_2(L)$. Hence the left-hand side of (17) is zero, and the right-hand side of (17) is positive, so that we must have

$$\operatorname{rank} \mathscr{L}_{2+4}(L) = 48. \qquad \square$$

Lemma 1. *If the linearly independent 2-vectors $x_1, \ldots, x_r$ $(r \geqq 3)$ form an irreducible (or reducible), lattice, then from them we can make r linearly independent 4-vectors.*

Proof. In [3], we have proved that the lattice L generated by 2-vectors $x_1, \ldots, x_r$ has basis consisting of 2-vectors (Prop. 2–2) and that such L can be decomposed into an orthogonal sum of some of A_n, D_n, E_6, E_7 and E_8 (admitting the repetitions). We can easily show that

(*) $\quad A_n$ $(n \geqq 3)$ (resp. D_n, E_n) contains n linearly independent 4-vectors.

For instance, let $u_1, u_2, \ldots, u_n$ be linearly independent 2-vectors in A_n satisfying $(u_1, u_2) = (u_2, u_3) = \cdots = (u_{n-1}, u_n) = -1$ and $(u_i, u_j) = 0$ for $|i - j| \geqq 2$, then $u_1, \ldots, u_n$ are the basis of A_n and $u_1 + u_3, \ldots, u_1 + u_n$, $u_1 - u_3$ and $u_1 + 2u_2 + u_3$ are linearly independent 4-vectors. For the cases D_n, E_6, E_7 or E_8, we can similarly show the above fact (*). When L is irreducible the proof is already over.

When L is reducible, then the combinations $u + v$, where u and v are 2-vectors taken from different irreducible components, form a desired system – we may pick up from some of them when the number of the members exceeds r –. $\square$

3. Proofs of Theorems 1 and 2.

Proof of Theorem 1. Let L be an element of Γ_{48} satisfying $a(2, L) = a(4, L) = 0$. Two vectors x_1 and x_2 in L are said to be equivalent mod $\mathscr{L}_6(L)$ if they satisfy $x_1 - x_2 \in \mathscr{L}_6(L)$. This is an equivalence relation on L, and we divide L into equivalence classes.

We shall show that each equivalence class is represented by a 6-vector. This implies that $\mathscr{L}_6(L) = L$.

Let K be any one of the equivalence classes, and x_0 be a minimal representative of K, that is, a non-zero vector satisfying

$$(x_0, x_0) \leqq (y, y) \quad \forall y \in K.$$

Suppose that x_0 is not a 6-vector, then it holds that $(x_0, x_0) = 2m$ for some $m \geqq 4$. Furthermore, we see that

$$|(x_0, y)| \leqq 3 \quad \text{for any } y \in \Lambda_6(L). \tag{18}$$

For, if $(x_0, y) > 3$, then $z = x_0 - y$ satisfies $(z, z) < (x_0, x_0)$, contradicting to the minimality of x_0 in K.

We introduce similar quantities to those given in [7], namely:

$$N_k(x_0) = \text{the cardinality of } \{y \in \Lambda_6(L) \mid (x_0, y) = k\} \quad 0 \leqq k \leqq 3.$$

We see that

$$\sum_{x \in \Lambda_6(L_0)} (x, x_0)^{2a} = 2N_1(x_0) + 2^{2a+1} N_2(x_0) + 2 \cdot 3^{2a} N_3(x_0) \quad \text{for } a \geqq 1. \tag{19}$$

The conditions (1), (2) and $a(2, L) = a(4, L) = 0$ leads to

$$\vartheta(\mathbf{z}, P_v, L) = 0 \quad \text{for } v = 2, 4, 6, 8.$$

This implies

$$\sum_{x \in \Lambda_{2t}(L)} P_v(x; \alpha) = 0 \qquad t \geqq 1, \ v = 2, 4, 6, 8. \tag{20}$$

If we take $\alpha = x_0$ and $t = 3$, then by virtue of (20) and the explicit forms of P_v we obtain

$$\begin{aligned} \sum_{x \in \Lambda_6(L)} (x, x_0)^2 &= 1/48 \sum_{x \in \Lambda_6(L)} (x, x)(x_0, x_0) \\ &= 6\,552\,000\,(x_0, x_0), \end{aligned} \tag{21}$$

$$\sum_{x \in \Lambda_6(L)} (x, x_0)^4 = 2\,358\,720\,(x_0, x_0)^2, \tag{22}$$

$$\sum_{x \in \Lambda_6(L)} (x, x_0)^6 = 1\,360\,800\,(x_0, x_0)^3 \quad \text{and} \tag{23}$$

$$\sum_{x \in \Lambda_6(L)} (x, x_0)^8 = 1\,058\,400\,(x_0, x_0)^4. \tag{24}$$

From (19), (21), (22), (23), we have

$$\begin{aligned} N_1(x_0) &= 28\,350\,(x_0, x_0)^3 - 638\,820\,(x_0, x_0)^2 + 4\,914\,000\,(x_0, x_0), \\ N_2(x_0) &= -\,11\,340\,(x_0, x_0)^3 + 196\,560\,(x_0, x_0)^2 - 491\,400\,(x_0, x_0), \\ N_3(x_0) &= 1\,890\,\ (x_0, x_0)^3 - 16\,380\,(x_0, x_0)^2 + 36\,400\,(x_0, x_0). \end{aligned}$$

We substitute the above equations into (19) with $a = 4$, then by (24) we get

$$\begin{aligned} 1\,058400\,(x_0, x_0)^4 = 19\,051\,200\,(x_0, x_0)^3 - 115\,577\,280\,(x_0, x_0)^2 \\ + 235\,872\,000\,(x_0, x_0). \end{aligned}$$

This equation does not hold for $(x_0, x_0) \geqq 8$, so that we reach a contradiction. And we must have $(x_0, x_0) \leqq 6$. This completes the proof of Theorem 1.

Proof of Theorem 2. First we treat the case (i). By Prop. 3, we can find 2-vectors $x_1, x_2, x_3, \ldots, x_r$ $(r \geqq 3)$ and 4-vectors $y_1, \ldots, y_s$ such that $r + s = 48$ and $x_1, \ldots, x_r$, $y_1, \ldots, y_s$ are linearly independent over $\mathbb{Q}$.

By Lemma 1, we can make r 4-vectors $w_1, \ldots, w_r$ which are linearly equivalent to $x_1, \ldots, x_r$ over $\mathbb{Q}$. Then $w_1, \ldots, w_r$ and $y_1, \ldots, y_s$ are the desired system, and this implies

$$\operatorname{rank} \mathscr{L}_4(L) = 48.$$

Next we prove (ii). By Prop. 3, we can find 2-vectors x_1, x_2 and 4-vectors $y_1, y_2, \ldots, y_{46}$ that are linearly independent over $\mathbb{Q}$. We may suppose that

$$(x_1, x_2) = 1 \quad \text{and} \tag{25}$$

$$|(x_i, y_j)| \leqq 2 \quad \text{with} \quad i = 1, 2 \quad 1 \leqq j \leqq 46. \tag{26}$$

The equality (25) is derivable from the assumption that $\mathscr{L}_2(L) \cong A_2$, and (26) is derivable from the inequality

$$(x_i, y_j)^2 \leqq (x_i, x_i)\,(y_j, y_j) = 8.$$

There are four subcases (a) $\sim$ (d) to treat separately.

(a) When $(x_1, y_j) = (x_2, y_j) = 0$ hold for $1 \leqq j \leqq 46$. Then we can verify that $x_1 - y_1$, $x_1 - y_2, \ldots, x_1 - y_{46}$, $x_1 + x_2$ and $x_1 - 2x_2$ are linearly independent 6-vectors, so that we have

$$\operatorname{rank} \mathscr{L}_6(L) = 48.$$

(b) When it hold that $(x_1, y_j) = 0$ for all $1 \leqq j \leqq 46$ and $(x_2, y_k) = \pm 1$ for some $1 \leqq k \leqq 46$. We may assume that $(x_2, y_k) = 1$, for if $(x_2, y_k) = -1$ we use $-y_k$ instead of y_k. Then we can verify that $y_1, y_2, \ldots, y_{46}$, $-x_1 + x_2 - y_k$ and $x_2 - y_k$ are linearly independent 4-vectors over $\mathbb{Q}$, so that we have

$$\operatorname{rank} \mathscr{L}_4(L) = 48.$$

(c) When $(x_1, y_j) = \pm 1$ and $(x_2, y_k) = \pm 1$ hold for some $1 \leqq j, k \leqq 46$. Then we can verify that $y_1, \ldots, y_{46}$, $x_1 - (x_1, y_j)\, y_j$ and $x_2 - (x_2, y_k)\, y_k$ are linearly independent 4-vectors over $\mathbb{Q}$, so that we have

$$\operatorname{rank} \mathscr{L}_4(L) = 48.$$

(d) When either $(x_1, y_k) = \pm 2$ or $(x_2, y_k) = \pm 2$ holds for some $1 \leqq k \leqq 46$. If $(x_1, y_k) = \pm 2$, then x_1, x_2 and $x_1 - (x_1, y_k)\, y_k$ are linearly independent 2-vectors, so that we have $\operatorname{rank} \mathscr{L}_2(L) \geqq 3$, contrary to the assumption of (ii). The case $(x_2, y_k) = \pm 2$ is equally treated. This completes the proof of (ii).

Proof of (iii). By Prop. 3, we can find 2-vectors x_1 and x_2 and 4-vectors $y_1, \ldots, y_{46}$, linearly independent over $\mathbb{Q}$. By our assumption, it holds that $(x_1, x_2) = 0$. It is easy to see that $x_1 + x_2$, $x_1 - x_2$, $y_1, \ldots, y_{46}$ are linearly independent 4-vectors over $\mathbb{Q}$, so that we have

$$\operatorname{rank} \mathscr{L}_4(L) = 48.$$

Proof of (iv). By Prop. 3, we can find a 2-vector x_1 and 4-vectors $y_1, \ldots, y_{47}$ in L which are linearly independent over $\mathbb{Q}$. We may assume that $|(x_1, y_j)| \leqq 1$ for all $1 \leqq j \leqq 17$. There are two subcases (a) and (b) to treat separately.

(a) When it holds that $(x_1, y_j) = 0$ for all $1 \leqq j \leqq 47$, then $-x_1 + y_1$, $x_1 + y_j$, $1 \leqq j \leqq 47$, are linearly independent 6-vectors, so that we have

$$\operatorname{rank} \mathscr{L}_6(L) = 48.$$

(b) When it holds that $(x_1, y_k) = \pm 1$ for some $1 \leqq k \leqq 47$, then $y_1, \ldots, y_{47}$ and $x_1 - (x_1, y_k)\, y_k$ are linearly independent 4-vectors. Hence we have

$$\operatorname{rank} \mathscr{L}_4(L) = 48.$$

Proof of (v). In the case, we know $a(2, L) = 0$ and $a(4, L) > 0$ by the assumption of this theorem. And the assertion follows from the proof of Prop. 1. □

Remark 1. In Theorems 1 and 2, we have enumerated all possible lattices in Γ_{48}, but it is not clear whether each possibility actually occurs.

Remark 2. We can give some examples of lattices in Γ_{48} characterized in Theorems 1 and 2.

(i) The lattice $L_1 \in \Gamma_{48}$ with the properties $\mathscr{L}_2(L_1) = \emptyset$ and rank $\mathscr{L}_4(L_1) = 48$ is given by an orthogonal sum of two copies of Leech lattice of rank 24.

(ii) The lattices $L_2 \in \Gamma_{48}$ with the properties $\mathscr{L}_2(L_2) = \mathscr{L}_4(L_2) = \emptyset$ and $\mathscr{L}_6(L_2) = L_2$ are given by the lattices constructed from Pless code or quadratic residue code. (Conf. [2]).

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