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Essential Sequences Over an Ideal and Essential Cograde

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1. Introduction

In [6] the concepts of essential prime divisors, essential sequences, and the essential grade of an ideal I in a Noetherian ring were introduced and therein it was shown that these concepts are an excellent analogue of, respectively, associated primes, R-sequences, and the standard grade of I, in the classical theory, and also of, respectively, asymptotic prime divisors, asymptotic sequences, and the asymptotic grade of I, in the asymptotic theory. It turns out that essential sequences and essential grade yield useful information concerning all the prime divisors of zero in the completion of a local ring. Thus these concepts should prove useful in many future research papers.

Asymptotic prime divisors, asymptotic sequences, and asymptotic grade have recently been useful in several research papers. And in [5] it was shown that asymptotic sequences over an ideal I in a Noetherian ring R and the asymptotic cograde of I (when R is local) have some useful properties, and several bounds on this cograde were established in [5]. The main purpose of this paper is to show that, similarly, essential sequences over I behave nicely when passing to certain rings related to R and that the essential cograde of I is well defined (when R is local) and satisfies certain rather natural inequalities.

Section 2 contains the definitions and a list of the basic facts concerning essential prime divisors, essential sequences, and essential grade that are needed in the remainder of the paper. It also contains several results showing that essential sequences over an ideal I in a Noetherian ring R behave nicely with respect to passing to certain rings related to R. In Section 3 it is shown that the essential cograde of I is well defined when R is local and likewise behaves nicely when passing to the same type of related rings. In Section 4 we briefly

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show the relationships between the essential-properties and the corresponding asymptotic-properties. In Section 5 we give two upper bounds and two lower bounds on the essential cograde of I, and in Sect. 6 we use this cograde to characterize two classes of local rings.

For the most part our arguments are elementary, and the results in this paper are analogous to those in [5] for the asymptotic-theory (but no knowledge of the asymptotic-theory is required, except in the auxiliary Sect. 4). It is hoped that the results in this paper will be as useful in developing the essential-theory as those in [5] are in developing the asymptotic-theory.

2. Essential Sequences Over an Ideal

All rings in this paper are commutative with identity, and they will usually be Noetherian. The terminology is generally standard, and when R is a local ring R^* will always be used to denote the *completion* of R in its natural topology.

We begin with several definitions which will be used throughout this paper.

(2.1) **Definition.** Let I be an ideal in a Noetherian ring R and let $b_1, \ldots, b_d$ be nonunits in R. Then:

(2.1.1) $P \in \operatorname{Spec} R$ is an *essential prime divisor* of I in case $I \subseteq P$ and there exists $z \in \operatorname{Ass}(R_P)^*$ such that $I(R_P)^* + z$ is $P(R_P)^*$-primary.

(2.1.2) $E(I) = \{P \in \operatorname{Spec} R;\ P$ is an essential prime divisor of $I\}$, $A^*(I) = \{P \in \operatorname{Spec} R;\ P \in \operatorname{Ass} R/I^n$ for all large $n\}$, and $\hat{A}^*(I) = \{P \in \operatorname{Spec} R;\ P \in \operatorname{Ass} R/(I^n)_a$ for all large $n\}$, where $(I^n)_a$ is the integral closure in R of I^n.

(2.1.3) b_1 is *essentially prime* to I in case $(I, b_1)R \neq R$ and $b_1 \notin \bigcup E(I)$. $b_1, \ldots, b_d$ are an *essential sequence over* I in case b_i is essentially prime to $(I, b_1, \ldots, b_{i-1})R$ for $i = 1, \ldots, d$. An essential sequence over (0) is simply called an *essential sequence in* R.

(2.1.4) The *essential grade* of I, denoted $\operatorname{egd}(I)$, is the length of an essential sequence maximal with respect to coming from I.

(2.1.5) If R is local, then the *essential cograde* of I, denoted $\operatorname{ecogd}(I)$, is the length of a maximal essential sequence over I.

The concepts of essential sequence over I and $\operatorname{ecogd}(I)$ are new to this paper. But the other concepts defined in (2.1) have previously been studied and a number of their properties have been determined. In what follows we will need to use several of these properties, so (2.2) contains a list of those that are most often used below.

(2.2) *Remark.* Let I be an ideal in a Noetherian ring R. Then the following hold:

(2.2.1) The sets $\operatorname{Ass} R/I^n$ and $\operatorname{Ass} R/(I^n)_a$ are stable for all large n, by [1] and [11, (2.7)], so $A^*(I)$ and $\hat{A}^*(I)$ are well defined finite sets of prime ideals. Also, $E(I) \subseteq A^*(I)$ and $\hat{A}^*(I) \subseteq A^*(I)$, by [6, (3.3.1)] and [11, (2.8)], so $E(I)$ is also a finite subset of $\operatorname{Spec} R$.

(2.2.2) It is clear from the definition that each minimal prime divisor of I is in $E(I)$.

(2.2.3) [6, (3.3.4)]. If $P \in \operatorname{Spec} R$ is a minimal prime divisor of $I+z$, where $z \in \operatorname{Ass} R$, then $P \in E(I)$. In particular, $E((0)) = \operatorname{Ass} R$.

(2.2.4) [6, (3.3.2)]. If $I \subseteq P \in \operatorname{Spec} R$ and S is a multiplicatively closed set in R such that $P_S \neq R_S$, then $P \in E(I)$ if and only if $P_S \in E(I_S)$.

(2.2.5) [6, (3.6)]. If $P \in \operatorname{Spec} R$, then $P \in E(I)$ if and only if there exists $z \in \operatorname{Ass} R$ such that $z \subseteq P$ and $P/z \in E((I+z)/z)$.

(2.2.6) [6, (3.7)]. If A is a faithfully flat Noetherian extension ring of R, then the following hold: (a) If $P^* \in E(IA)$, then $P^* \cap R \in E(I)$; and, (b) If $P \in E(I)$ and P^* is a minimal prime divisor of PA, then $P^* \in E(IA)$ and $P^* \cap R = P$.

(2.2.7) [6, (3.9)]. If B is a finite integral extension ring of R, then the following hold: (a) If $P \in E(I)$, then there exists $P' \in E(IB)$ such that $P' \cap R = P$; and, (b) If $z \in \operatorname{Ass} B$ implies $z \cap R \in \operatorname{Ass} R$, then if $P' \in E(IB)$, then $P' \cap R \in E(I)$.

(2.2.8) It follows immediately from (2.2.2) that if $b_1, \ldots, b_d$ are an essential sequence over I, then height $(I, b_1, \ldots, b_d)R \geqq$ height $I+d$. Therefore, if $b_1, \ldots, b_d$ are an essential sequence in R, then height $(b_1, \ldots, b_d)R = d$, by the Generalized Principal Ideal Theorem.

(2.2.9) It follows readily from (2.1.3) that $b_1, \ldots, b_d$ are a maximal essential sequence over I if and only if they are an essential sequence over I and $M \in E((I, b_1, \ldots, b_d)R)$ for all maximal ideals M in R containing $(I, b_1, \ldots, b_d)R$.

(2.2.10) [6, (5.3)]. $\operatorname{egd}(I)$ is unambiguously defined; in fact, $\operatorname{egd}(I) = \min\{\text{height } (I(R_P)^* + z)/z;\ I \subseteq P \in \operatorname{Spec} R \text{ and } z \in \operatorname{Ass}(R_P)^*\}$.

In the remainder of this section it will be shown that essential sequences over I behave nicely when passing to certain rings related to R. The first of these results, (2.3), concerns passing to localizations.

(2.3) **Theorem.** *Let I be an ideal in a Noetherian ring R and let $b_1, \ldots, b_d$ be nonunits in R. Then the following hold:*

(2.3.1) *If $b_1, \ldots, b_d$ are an essential sequence over I and S is a multiplicatively closed set in R such that $(I, b_1, \ldots, b_d)R_S \neq R_S$, then the images of $b_1, \ldots, b_d$ in R_S are an essential sequence over I_S. The converse holds if $P_S \neq R_S$ for all $P \in \bigcup\{E((I, b_1, \ldots, b_i)R);\ i = 0, 1, \ldots, d-1\}$.*

(2.3.2) *If $b_1, \ldots, b_d$ are a maximal essential sequence over I, then for each maximal ideal M in R that contains $(I, b_1, \ldots, b_d)R$ it holds that the images in R_M of $b_1, \ldots, b_d$ are a maximal essential sequence over I_M. The converse holds if the b_i are contained in the Jacobson radical of R.*

Proof. (2.3.1) follows immediately from (2.2.4), and the first statement in (2.3.2) follows from (2.3.1) and (2.2.9). Also, if it is shown that $b_1, \ldots, b_d$ are an essential sequence over I, then the last statement in (2.3.2) follows from (2.2.4) and (2.2.9). So suppose they are not an essential sequence over I, so there exists $i \in \{1, \ldots, d\}$ such that $b_i \in P \in E((I, b_1, \ldots, b_{i-1})R)$. Let M be a maximal ideal in R

that contains P. Then $(I, b_1, \ldots, b_d)R \subseteq M$, since the b_j are in the Jacobson radical of R, and the image of b_i in R_M, is in $P_M \in E((I, b_1, \ldots, b_{i-1})R_M$, and this contradicts the hypothesis. Therefore $b_1, \ldots, b_d$ are a maximal essential sequence over I, q.e.d.

We next consider passing to R/z with $z \in \operatorname{Ass} R$.

(2.4) **Theorem.** *Let I be an ideal in a Noetherian ring R and let $b_1, \ldots, b_d$ be nonunits in R. Then the following hold:*

(2.4.1) *$b_1, \ldots, b_d$ are an essential sequence over I if and only if their images in R/z are an essential sequence over $(I+z)/z$ for all $z \in \operatorname{Ass} R$.*

(2.4.2) *$b_1, \ldots, b_d$ are a maximal essential sequence over I if and only if their images in R/z are an essential sequence over $(I+z)/z$ for all $z \in \operatorname{Ass} R$ and for all maximal ideals M in R containing $(I, b_1, \ldots, b_d)R$ there exists $z \in \operatorname{Ass} R$ such that $z \subseteq M$ and $M/z \in E(((I, b_1, \ldots, b_d)R + z)/z)$.*

Proof. (2.4.1) follows readily from (2.2.5), and (2.4.2) follows from (2.2.5) and (2.2.9), q.e.d.

We next consider faithfully flat extension rings.

(2.5) **Theorem.** *Let $R \subseteq A$ be Noetherian rings such that A is a faithfully flat R-module, let I be an ideal in R, and let $b_1, \ldots, b_d$ be nonunits in R. Then the following hold:*

(2.5.1) *$b_1, \ldots, b_d$ are an essential sequence over I if and only if they are an essential sequence over IA.*

(2.5.2) *If $R \subseteq A$ satisfy the Theorem of Transition, then $b_1, \ldots, b_d$ are a maximal essential sequence over I if and only if they are a maximal essential sequence over IA.*

Proof. (2.5.1) follows readily from (2.2.6), and (2.5.2) follows from (2.2.6) and (2.2.9), q.e.d.

In (2.6) we consider finite integral extension rings.

(2.6) **Theorem.** *Let B be a finite integral extension ring of a Noetherian ring R, let I be an ideal in R, and let $b_1, \ldots, b_d$ be nonunits in R. Then the following hold:*

(2.6.1) *If $b_1, \ldots, b_d$ are an essential sequence over IB, then they are an essential sequence over I.*

(2.6.2) *If $z \in \operatorname{Ass} B$ implies $z \cap R \in \operatorname{Ass} R$, then $b_1, \ldots, b_d$ are an essential sequence over I if and only if they are an essential sequence over IB.*

(2.6.3) *If $z \in \operatorname{Ass} B$ implies $z \cap R \in \operatorname{Ass} R$, then $b_1, \ldots, b_d$ are a maximal essential sequence over I if and only if they are an essential sequence over IB and for each maximal ideal M in R that contains $(I, b_1, \ldots, b_d)R$ there exists a maximal ideal N in B such that $N \cap R = M$ and $N \in E((I, b_1, \ldots, b_d)B)$.*

Proof. (2.6.1) and (2.6.2) follow immediately from (2.2.7), and (2.6.3) follows from (2.2.7) and (2.2.9), q.e.d

In (2.9) we consider essential sequences over I and an indeterminate. To prove (2.9) we need the following useful result concerning essential prime divisors and an indeterminate.

(2.7) **Lemma.** *Let I be an ideal in a Noetherian ring R and let X be an indeterminate. Then $E((I,X)R[X])=\{(P,X)R[X]; P\in E(I)\}$.*

Proof. Let $P\in E(I)$, let $N=(P,X)R[X]$, and let $C=R[X]_N$. Also, let $A=R_P$ and $B=A^*[X]$. Then NB is a prime ideal and C is a dense subspace of B_{NP}, by [9, Lemma 3.2]. Also, by hypothesis there exists $z\in \operatorname{Ass} A^*$ such that IA^*+z is PA^*-primary. Then $(B_{NP})^*/z(B_{NP})^*=C^*/zC^*=$(say) D is such that $D/XD=A^*/z$, so $ND/XD=PA^*/z$ is a minimal prime divisor of $(IA^*+z)/z$, so it follows that NC^* is a minimal prime divisor of $(I,X,w)C^*$ for some minimal prime divisor w of zC^*. Then $w\in \operatorname{Ass} C^*$, since $zB_{NP}\in \operatorname{Ass} B_{NP}$ and $C^*=(B_{NP})^*$, so $N\in E((I,X)R[X])$.

For the converse, let $N\in E((I,X)R[X])$ and let $P=N\cap R$, so $N=(P,X)R[X]$. Let A, B, and C be as in the preceding paragraph. Then if $w\in \operatorname{Ass} C^*$, then $z=w\cap A^*\in \operatorname{Ass} A^*$, since $z=((w\cap B_{NP})\cap A^*[X])\cap A^*$. And w is a minimal prime divisor of zC^*, since B_{NP}/zB_{NP} is analytically unramified and unmixed, by [10, (6.5)]. Therefore the proof that $P\in E(I)$ is similar to that given in the previous paragraph, q.e.d.

(2.8) Remark. Let I be an ideal in a Noetherian ring R, let M be a maximal ideal in R containing I, and let $N=(M,f)R[X]$ be a maximal ideal in $R[X]$. It may be assumed that f is a monic polynomial. Then $N\in E((I,f)R[X])$ if and only if $M\in E(I)$.

Proof. Since $R[f]\cong R[X]$, it follows from (2.7) that $M\in E(I)$ if and only if $(M,f)R[f]\in E((I,f)R[f])$. Now $R[X]$ is integral over $R[f]$, since f is monic, and $z\in \operatorname{Ass} R[X]$ implies $z\cap R[f]\in \operatorname{Ass} R[f]$. Also, N is the only prime ideal in $R[X]$ that lies over $(M,f)R[f]$. Therefore $(M,f)R[f]\in E((I,f)R[f])$ if and only if $N\in E((I,f)R[X])$, by (2.2.7), q.e.d.

(2.9) **Theorem.** *Let I be an ideal in a Noetherian ring R and let $b_1,\dots,b_d$ be nonunits in R. Then the following hold:*

(2.9.1) *The following are equivalent:*

(a) *$b_1,\dots,b_d$ are an essential sequence over I.*

(b) *$b_1,\dots,b_i,X,b_{i+1},\dots,b_d$ are an essential sequence over $IR[X]$ for some $i=0,1,\dots,d$.*

(c) *(b) holds for every $i=0,1,\dots,d$.*

(2.9.2) *The following are equivalent:*

(a) *$b_1,\dots,b_d$ are a maximal essential sequence over I.*

(b) *$b_1,\dots,b_i,X,b_{i+1},\dots,b_d$ are a maximal essential sequence over $IR[X]$ for some $i=0,1,\dots,d$.*

(c) *(b) holds for every $i=0,1,\dots,d$.*

Proof. (2.9.1) For $j=0,1,\dots,i$, $E((I,b_1,\dots,b_j)R[X])=\{PR[X]; P\in E((I,b_1,\dots,b_j)R)\}$, by (2.2.6) (and since, for an ideal J in R, the prime divisors of

$JR[X]$ and the $PR[X]$ with P a prime divisor of J). Also, it is clear that X is not in any prime divisor of $(I, b_1, \ldots, b_i)R[X]$, and, for

$$\begin{aligned} k=0,1,\ldots,d-i, E((I, b_1, \ldots, b_i, X, b_{i+1}, \ldots, b_{i+k})R[X]) \\ =\{(P,X)R[X]; P\in E((I, b_1, \ldots, b_{i+k})R)\}, \end{aligned}$$

by (2.7). Therefore it follows that (2.9.1)(a)–(c) are equivalent.

(2.9.2) follows immediately from (2.9.1) and (2.2.9), since the maximal ideals in $R[X]$ containing $(I,X)R[X]$ are the ideals $(M,X)R[X]$ with M a maximal ideal in R containing I, q.e.d.

We next consider essential sequences over ideals with the same radical.

(2.10) **Theorem.** *Let I and J be ideals in a Noetherian ring R such that* $\operatorname{Rad} I = \operatorname{Rad} J$ *and let $b_1, \ldots, b_d$ be nonunits in R. Then the following hold:*

(2.10.1) *$b_1, \ldots, b_d$ are an essential sequence over I if and only if they are an essential sequence over J.*

(2.10.2) *$b_1, \ldots, b_d$ are a maximal essential sequence over I if and only if they are a maximal essential sequence over J.*

Proof. $\operatorname{Rad}(I, b_1, \ldots, b_i)R = \operatorname{Rad}(J, b_1, \ldots, b_i)R$ for $i=0,1,\ldots,d$, since $\operatorname{Rad} I = \operatorname{Rad} J$, so $E((I, b_1, \ldots, b_i)R) = E((J, b_1, \ldots, b_i)R)$ for $i=0,1,\ldots,d$, by [6, (3.3.5)], so (2.10.1) follows immediately from the definition, (2.1.3). And (2.10.2) follows immediately from (2.10.1) and (2.2.9), q.e.d.

(2.11) gives some additional basic information concerning essential sequences over an ideal.

(2.11) *Remark.* Let I be an ideal in a Noetherian ring R and let $b_1, \ldots, b_d$ be nonunits in R. Then the following hold:

(2.11.1) The following statements are equivalent:

(a) $b_1, \ldots, b_d$ are an essential sequence over I;

(b) $b_1^{n_1}, \ldots, b_d^{n_d}$ are an essential sequence over I for some positive integers n_i; (c) (b) holds for all positive integers n_i.

(2.11.2) *The following statements are equivalent:*

(a) $b_1, \ldots, b_d$ are an essential sequence over I; (b) There exists i $(0 \leqq i < d)$ such that $b_1, \ldots, b_i$ are an essential sequence over I and $b_{i+1}, \ldots, b_d$ are an essential sequence over $(I, b_1, \ldots, b_i)R$; (c) (b) holds for all i $(i=0,1,\ldots,d-1)$.

Proof. (2.11.1) follows readily from the definition, (2.1.3), and the fact that $\operatorname{Rad}(I, b_1, \ldots, b_i)R = \operatorname{Rad}(I, b_1^{n_1}, \ldots, b_i^{n_i})R$ for $i=0,1,\ldots,d$. And (2.11.2) is clear by the definition, (2.1.3), q.e.d.

This section will be closed with another fact concerning essential prime divisors that will be useful in Sect. 3.

(2.12) **Lemma.** *Let (R, M) be a local ring such that* $\operatorname{Ass} R^*$ *has exactly one element and let I be an ideal in R. Then $M \in E(I)$ if and only if I is M-primary.*

Proof. If I is M-primary, then $M \in E(I)$, by (2.2.2). And, if $M \in E(I)$, then there exists $z \in \operatorname{Ass} R^*$ such that $IR^* + z$ is MR^*-primary, by (2.1.1). Therefore, since every prime ideal in R^* that contains IR^* must contain z, by hypothesis, it follows that IR^* is MR^*-primary, so I is M-primary, q.e.d.

3. Essential Cograde

In this section we show that the essential cograde of an ideal in a local ring is unambigously defined and behaves well when passing to certain related local rings. We begin with an important special case.

(3.1) **Lemma.** *If (R, M) is a complete local ring that has only one prime divisor of zero and if I is an ideal in R, then* $\operatorname{egd}(I) = \operatorname{height} I$ *and* $\operatorname{ecogd}(I) = \operatorname{depth} I$, *so* $\operatorname{egd}(I) + \operatorname{ecogd}(I) = \operatorname{altitude} R$.

Proof. Let $\operatorname{height} I = h$ and $\operatorname{depth} I = d$. Then since R is unmixed, [6, (6.1)] says that $\operatorname{egd}(I) = h$. Also, $h + d = \operatorname{altitude} R$, since R is catenary. If $h = \operatorname{altitude} R$, then I is M-primary, so $M \in E(I)$, hence no element in M is essentially prime to I, and so $\operatorname{ecogd}(I) = 0$, as desired. If $h < \operatorname{altitude} R$, then $M \notin E(I)$, by (2.12), so there exists b_1 in M which is essentially prime to I. Then, since R is catenary, it follows that $\operatorname{height}(I, b_1)R = \operatorname{height} I + 1 = h + 1$ and $\operatorname{depth}(I, b_1)R = \operatorname{depth} I - 1 = d - 1$. If $h + 1 < \operatorname{altitude} R$, that is, if $(I, b_1)R$ is not M-primary, then the preceding may be repeated with $(I, b_1)R$ in place of I, so it follows that this process will terminate after $d = \operatorname{altitude} R - h$ steps. Thus there exist $b_1, \ldots, b_d$ in M which are an essential sequence over I. Finally, if $c_1, \ldots, c_e$ are a maximal essential sequence over I, then necessarily $M \in E((I, c_1, \ldots, c_e)R)$, so M is a minimal prime divisor of $(I, c_1, \ldots, c_e)R$, by (2.12). Also, $\operatorname{height}\,(I, c_1, \ldots, c_e)R = \operatorname{height}\, I + e$, since c_i is not in any minimal prime divisor of $(I, c_1, \ldots, c_{i-1})R$ for $i = 1, \ldots, e$ (by 2.2.2)) and since R is caternary. Therefore $\operatorname{height}\, I + e = \operatorname{altitude} R = \operatorname{height}\, I + \operatorname{depth}\, I$, so $e = \operatorname{depth} I = d$, hence $\operatorname{ecogd}(I) = \operatorname{depth} I$ and it follows that $\operatorname{egd}(I) + \operatorname{ecogd}(I) = \operatorname{altitude} R$, q.e.d.

We now show that $\operatorname{ecogd}(I)$ is unambiguously defined for ideals in local rings.

(3.2) **Theorem.** *If I is an ideal in a local ring (R, M), then any two maximal essential sequences over I have the same length. In fact,* $\operatorname{ecogd}(I) = \min\{\operatorname{depth}(IR^* + z);\ z \in \operatorname{Ass} R^*\} = \min\{\operatorname{altitude}\ R^*/z\text{-}\operatorname{height}\ (IR^* + z)/z;\ z \in \operatorname{Ass} R^*\}$.

Proof. Let $b_1, \ldots, b_d$ in M be a maximal essential sequence over I. Then $b_1, \ldots, b_d$ are a maximal essential sequence over IR^*, by (2.5.2), so their images in R^*/z are an essential sequence over $(IR^* + z)/z$ for all $z \in \operatorname{Ass} R^*$ and these images are a maximal essential sequence over $(IR^* + z)/z$ for some such z, by (2.4.2). Therefore, since the essential cograde of an ideal is unambiguously defined in a complete local domain, by (3.1), it follows that $\operatorname{ecogd}((IR^* + z)/z) \geqq d$ for all $z \in \operatorname{Ass} R^*$ and equality holds for some such z. Now $\operatorname{ecogd}((IR^* + z)/z) = \operatorname{depth}(IR^* + z)/z = \operatorname{depth} IR^* + z$, by (3.1), so it follows that $\operatorname{ecogd}(I)$ is unambiguously defined and is equal to $\min\{\operatorname{depth}(IR^* + z);$

$z \in \operatorname{Ass} R^*\}$. Finally, the last equality follows from the fact that depth $IR^* + z = \operatorname{depth}(IR^* + z)/z$ and height $(IR^* + z)/z + \operatorname{depth}(IR^* + z)/z = \operatorname{altitude} R^*/z$, q.e.d.

(3.3) **Corollary.** *If I and J are ideals in a local ring R, then the following hold:*

(3.3.1) *If $I \subseteq J$, then* $\operatorname{ecogd}(I) \geqq \operatorname{ecogd}(J)$.
(3.3.2) *If* $\operatorname{Rad} I = \operatorname{Rad} J$, *then* $\operatorname{ecogd}(I) = \operatorname{ecogd}(J)$.

Proof. (3.3.1) $I \subseteq J$ implies that depth $IR^* + z \geqq$ depth $JR^* + z$ for all $z \in \operatorname{Ass} R^*$, so $\operatorname{ecogd}(I) \geqq \operatorname{ecogd}(J)$, by (3.2).

For (3.3.2), if $\operatorname{Rad} I = \operatorname{Rad} J$, then there exist positive integers m and n such that $I^m R^* + z \subseteq JR^* + z$ and $J^n R^* + z \subseteq IR^* + z$ for all $z \in \operatorname{Ass} R^*$. Thus it follows that depth $IR^* + z =$ depth $JR^* + z$, and so (3.3.2) follows from (3.2), q.e.d.

The final result in this section shows that $\operatorname{ecogd}(I)$ behaves nicely when passing to certain related local rings.

(3.4) **Corollary.** *Let $I \subseteq P$ be ideals in a Noetherian ring R such that P is prime. Then the following hold:*

(3.4.1) $\operatorname{ecogd}(I_P) = \min\{\operatorname{ecogd}((I_P + z)/z);\ z \in \operatorname{Ass} R_P\}$.
(3.4.2) $\operatorname{ecogd}(I_P) = \operatorname{ecogd}(I(R_P)^*)$.
(3.4.3) *If A is a faithfully flat Noetherian extension ring of R and if $P^* \in \operatorname{Spec} A$ lies over P, then* $\operatorname{ecogd}(I_P) \leqq \operatorname{ecogd}(I(A^*_{P^*}))$ *and equality holds if P^* is a minimal prime divisor of PA^*.*
(3.4.4) *If B is a finite integral extension ring of R such that $z \in \operatorname{Ass} B$ implies $z \cap R \in \operatorname{Ass} R$, then* $\operatorname{ecogd}(I_P) \leqq \operatorname{ecogd}(IB_{P'})$ *for all prime ideals P' in B that lie over P and equality holds for some such P'.*
(3.4.5) $\operatorname{ecogd}(I_P) = \operatorname{ecogd}((I_P, X)R[X]_{(P,X)})$.

Proof. (3.4.1)–(3.4.5) follow immediately from (3.2) and, respectively, (2.4.2), (2.5.2), (2.5), (2.6.3), and (2.9.2), q.e.d.

Concerning (3.4), it follows from (3.4.1) and (3.4.2) that if R is local, then $\operatorname{ecogd}(I) = \operatorname{ecogd}((IR^* + z)/z)$ for some $z \in \operatorname{Ass} R^*$. And it was shown in [6, (5.6)] that $\operatorname{egd}(I) = \operatorname{egd}((IR^* + w)/w)$ for some $w \in \operatorname{Ass} R^*$. It would be nice if z and w could always be chosen to be equal, for then $\operatorname{egd}(I) + \operatorname{ecogd}(I) = \operatorname{height}(IR^* + z)/z + \operatorname{depth}(IR^* + z)/z = \operatorname{depth} z$ for some $z \in \operatorname{Ass} R^*$, by (3.1). Unfortunately this does not hold. For example, let R be a complete local ring with $\operatorname{Ass} R = \{z, w\}$ such that $z + w$ is primary for the maximal ideal M in R and depth $z =$ depth $w > 1$. Then for all ideals I and R it holds that $\operatorname{ecogd}(I) = 0$, by (3.2), and $\operatorname{egd}(I) = \operatorname{height} I$, by [6, (6.1)], since R is unmixed.

4. A Comparison of Asymptotic Properties and Essential Properties

In Section 5 we give a pair of upper bounds and lower bounds on the essential cograde of an ideal I. A number of bounds on the asymptotic cograde of I were given in [5], so to relate these results, in (4.1) we show the close relationship between certain asymptotic properties and the corresponding essential properties of an ideal in a large class of Noetherian rings. Then in (4.2) it is shown that in general no such comparisons exist.

For (4.1) we need the following definitions: elements $b_1, \ldots, b_d$ in a Noetherian ring R are an *asymptotic sequence over* an ideal I in R in case $(I, b_1, \ldots, b_d)R \neq R$ and $b_i \notin \bigcup \hat{A}((I, b_1, \ldots, b_{i-1})R)$ for $i=1, \ldots, d$. If $I=(0)$, then the elements $b_1, \ldots, b_d$ are called as an *asymptotic sequence in R*. The *asymptotic grade* of I, denoted, $\operatorname{agd}(I)$, is the length of an asymptotic sequence maximal with respect to coming from I, and if R is local, the the *asymptotic cograde* of I, denoted $\operatorname{acogd}(I)$, is the length of a maximal asymptotic sequence over I. It is shown in [12] and [4] that these concepts are well defined, and several bounds on $\operatorname{acogd}(I)$ are given in [5].

(4.1) **Proposition.** *Let R be a Noetherian ring such that $(R_M)^*$ has no imbedded prime divisors of zero for all maximal ideals M in R. Then the following hold for all ideals I in R:*

(4.1.1) $E(I) \subseteq \hat{A}^*(I)$.

(4.1.2) *Elements $b_1, \ldots, b_d$ are an essential sequence in R if and only if they are an asymptotic sequence in R.*

(4.1.3) $\operatorname{egd}(I) = \operatorname{agd}(I)$.

(4.1.4) *If $b_1, \ldots, b_d$ are an asymptotic sequence over I, then they are an essential sequence over I.*

(4.1.5) $\operatorname{ecogd}(I) \geqq \operatorname{acogd}(I)$ *if R is local.*

Proof. For (4.1.1) let $P \in E(I)$ and let $L = R_P$ and $M = PL$. Then $M \in E(IL)$, by (2.2.4), so there exists $z \in \operatorname{Ass} L^*$ such that $IL^* + z$ is ML^*-primary, by (2.1.1). Then ML^*/z is a minimal prime divisor of $(IL^* + z)/z$, so $ML^*/z \in \hat{A}^*((IL^* + z)/z)$. Therefore $ML^* \in \hat{A}^*(IL^*)$, by [11, (6.3)], since z is a minimal prime ideal, so $M \in \hat{A}^*(IL)$, by [11, (6.8)], hence $P \in \hat{A}^*(I)$, by [12, (3.3.2)].

It is shown in [12, (3.6)] that $b_1, \ldots, b_d$ are an asymptotic sequence in R if and only if for $i=1, \ldots, d$ and for every prime ideal P in R that contains $b_1, \ldots, b_i$ it holds that the images in R_P of $b_1, \ldots, b_i$ are an asymptotic sequence, and it follows from [6, (3.3.2)] that a similar statement holds for essential sequences, so to prove (4.1.2) it may be assumed that R is local. Then it is shown in [6, (5.4)] (resp., [12, (2.6)]) that $b_1, \ldots, b_d$ are an essential (resp., asymptotic) sequence in R if and only if their images in R^*/z are a subset of a system of parameters for all (resp., for all minimal) $z \in \operatorname{Ass} R^*$, so the hypothesis implies that (4.1.2) holds.

(4.1.3) is clear by (4.1.2), (4.1.4) is clear by (4.1.1) and (4.1.5) is clear by (4.1.4), q.e.d.

Concerning (4.1.5), it is possible that $\operatorname{ecogd}(I) > \operatorname{acogd}(I)$. For example, let R be a complete regular local domain such that altitude $R = d \geqq 3$, let $P = (b_1, \ldots, b_{d-1})R$ and $p = b_d R$, where $b_1, \ldots, b_d$ are a regular system of parameters in R, and let $I = p \cap P$. Then $\operatorname{ecogd}(I) = \operatorname{depth} I = d-1$, by (3.1), and $\operatorname{acogd}(I) = d - l(I)$, by [4], and $l(I) = d-1$, since $l(I) \geqq l(I_P) = \operatorname{height} P = d-1$, by [5, (2.5)] and [8, Theorem 1, p. 154], and since $l(I) \leqq d-1$ (since $I = p(P:p) = pP$ is generated by $d-1$ elements).

(4.2) shows that none of the statements in (4.1) hold in general local rings.

(4.2) *Remark.* (4.2.1) An essential sequence in a Noetherian ring R is always an asymptotic sequence in R, but not conversely, by [6, (4.2.6)], so $\operatorname{egd}(I) \leqq \operatorname{agd}(I)$ always holds. However, the inequality can hold, for if z is an imbedded prime divisor of zero in R, then $\operatorname{egd}(z)=0$, by (2.2.3), and $\operatorname{agd}(z)>0$, by [12, (2.3.1)].

(4.2.2) If (R,M) is as in [2, Proposition 3.3], then R is a local domain of altitude two such that R is quasi-unmixed and there exists a depth one prime divisor of zero in R^*. Therefore $MR^* \in E(IR^*)$ for all nonzero ideals I in R, by (2.2.3), so $M \in E(IR^*)$, by (2.2.6). But if $b \in M$ and $b \neq 0$, then $M \notin \hat{A}(bR)$, by [12, (2.3.6)], so $\hat{A}^*(bR) \subset E(bR)$.

(4.2.3) With (R,M) as in (4.2.2), $\operatorname{agd}(M)=2$, by [12, (4.1)], since R is quasi-unmixed, and $\operatorname{egd}(M)=1$, by (2.2.10), so if $0 \neq b \in M$, then $\operatorname{ecogd}(bR)=0$ and $\operatorname{acogd}(bR)=1$, and so $\operatorname{acogd}(bR) > \operatorname{ecogd}(bR)$.

5. Some Bounds on ecogd (I)

In this section we give two upper bounds and two lower bounds on the essential cograde of an ideal in a local ring R. Our first pair of bounds follow immediately from (3.2).

(5.1) **Theorem.** *If I is an ideal in a local ring R, then the following hold:*
(5.1.1) $\operatorname{ecogd}(I) \leqq \operatorname{depth} I$.
(5.1.2) $\operatorname{ecogd}(I) \geqq \min\{\operatorname{depth} P^*;\ P^* \in E(IR^*)\}$.

Proof. For (5.1.1) note that $\operatorname{depth} I = \operatorname{depth} IR^* \geqq \min\{\operatorname{depth} IR^* + z;\ z \in \operatorname{Ass} R^*\} = \operatorname{ecogd}(I)$, by (3.2).

For (5.1.2), by (3.2) let z and P^* be prime ideals in R^* such that $z \in \operatorname{Ass} R^*$ and P^* is a minimal prime divisor of $IR^* + z$ such that $\operatorname{depth} P^* = \operatorname{ecogd}(I)$. Then $P^* \in E(IR^*)$, by (2.2.3), so $\operatorname{ecogd}(I) = \operatorname{depth} P^* \geqq \min\{\operatorname{depth} Q^*;\ Q^* \in E(IR^*)\}$, q.e.d.

The following lemma will be helpful in determining another upper bound on ecogd (I).

(5.2) **Lemma.** *If $I \subseteq J \subseteq P$ are ideals in a Noetherian ring R such that P is prime, and if $P \in E(I)$, then $P \in E(J)$.*

Proof. If $P \in E(I)$, then there exists $z \in \operatorname{Ass}(R_P)^*$ such that $I(R_P)^* + z$ is $P(R_P)^*$-primary. Then clearly $J(R_P)^* + z$ is $P(R_P)^*$-primary, so $P \in E(J)$, by (2.1.1), q.e.d.

(5.3) **Theorem.** *Let I be an ideal in a local ring (R,M) and let $x_1, \dots, x_h$ be an essential sequence in I. Then there exists a maximal essential sequence over I, say $b_1, \dots, b_d$, such that $x_1, \dots, x_h, b_1, \dots, b_d$ are an essential sequence in R. Therefore* $\operatorname{ecogd}(I) \leqq \operatorname{egd}(M) - \operatorname{egd}(I)$.

Proof. Let $X = (x_1, \dots, x_h)R$ and let $d = \operatorname{ecogd}(I)$. It may clearly be assumed that $d>0$, so $M \notin E(I)$, and so $M \notin E(X)$, by (5.2). Therefore there exists $b_1 \in M - ((\bigcup E(I)) \cup (\bigcup E(X)))$. Then b_1 is an essential sequence over M and $x_1, \dots, x_h, b_1$ are an essential sequence in $(I, b_1)R$. Also $\operatorname{ecogd}((I, b_1)R) = d-1$, since it readily follows from (2.1.3) and (3.2) that $b_1, b_2, \ldots, b_d$ are a maximal essential sequence over I if and only if $b_2, \ldots, b_d$ are a maximal essential sequence over $(I, b_1)R$. Therefore the first statement follows by induction on d, and the second statement is then clear, q.e.d.

We need the following lemma to prove the final lower bound on $\operatorname{ecogd}(I)$.

(5.4) **Lemma.** *Let I be an ideal in a Noetherian ring R and assume that n is such that $\operatorname{Ass} R/I^{n+k} = \operatorname{Ass} R/I^n$ for all $k \geqq 0$. If A is a Noetherian ring such that A is a flat R-module and $IA \neq A$, then $\operatorname{Ass} A/I^{n+k}A = \operatorname{Ass} A/I^n A$ for all $k \geqq 0$.*

Proof. Let $P^* \in \operatorname{Ass} A/I^{n+k}A$ for some $k \geqq 0$, and let $P = P^* \cap R$. Then $P \in \operatorname{Ass} R/I^{n+k}$, by [7, (18.11)], so $P \in \operatorname{Ass} R/I^n$, by hypothesis, and so $P^* \in \operatorname{Ass} A/I^n A$, by [7, (18.11)], q.e.d.

(5.5) gives a useful lower bound on $\operatorname{ecogd}(I)$.

(5.5) **Theorem.** *Let I be an ideal in a local ring (R, M) and let n be large enough that $\operatorname{Ass} R/I^{n+k} = \operatorname{Ass} R/I^n$ for all $k \geqq 0$. Then if $b_1, \ldots, b_d$ in M are such that their images in R/I^n are an essential sequence, then $b_1, \ldots, b_d$ are an essential sequence over I^n. Therefore $\operatorname{ecogd}(I) \geqq \operatorname{egd}(M/I^n)$ for all large n.*

Proof. Assume that the images of $b_1, \ldots, b_d$ in R/I^n are an essential sequence and suppose that $b_1, \ldots, b_d$ are not an essential sequence over I^n. Therefore choose the first i $(0 \leqq i < d)$ such that there exists $P \in E((I^n, b_1, \ldots, b_i)R)$ with $b_{i+1} \in P$. Then $P_P \in E((I^n, b_1, \ldots, b_i)R_P)$, by (2.2.4), and the images of $b_1, \ldots, b_{i+1}$ in $R_P/I^n R_P$ are an essential sequence, by [6, (4.2.2)]. Also, $\operatorname{Ass} R_P/I^{n+k}R_P = \operatorname{Ass} R_P/I^n R_P$ for all $k \geqq 0$, by (5.4), so it may be assumed that R is local with maximal ideal P.

Now $PR^* \in E((I^n, b_1, \ldots, b_i)R^*)$, by (2.2.6), and the images of $b_1, \ldots, b_{i+1}$ in $R^*/I^n R^* \cong (R/I^n)^*$ are an essential sequence, by [6, (4.2.3)], so by (5.4) it may be assumed that R is a complete local ring. Then there exists $z \in \operatorname{Ass} R$ such that $(I^n, b_1, \ldots, b_i)R + z$ is P-primary, by hypothesis and (2.1.1). Let q be a minimal prime divisor of $I^n + z$. Then $q \in E(I^n)$, by (2.2.2), so $q \in A^*(I^n) = \{p \in \operatorname{Spec} R;\ p$ is a prime divisor of I^k for all large $k\}$, by (2.2.1), so $q \in \operatorname{Ass} R/I^n$, by the choice of n. Also, $(q, b_1, \ldots, b_i)R$ is P-primary, so $(q, b_1, \ldots, b_i)R/I^n$ is P/I^n-primary and q/I^n is a prime divisor of zero, hence $P/I^n \in E((b_1 + I^n, \ldots, b_i + I^n)(R/I^n))$, by (2.2.3). But $b_{i+1} \in P$, so $b_{i+1} + I^n \in P/I^n$, and this contradicts the hypothesis. Therefore $b_1, \ldots, b_d$ are an essential sequence over I^n, q.e.d.

Concerning (5.5), the following should be noted. (a) If b in R is such that $b + I$ is regular in R/I, then $b_1, \ldots, b_d, b$ need not be an essential sequence in R. This follows, since some of the essential prime divisors of I may not be prime divisors of I; that is, $E(I) \subseteq A^*(I)$, by (2.2.1), but possibly $E(I) \nsubseteq \operatorname{Ass} R/I$. (b) If $b_1, \ldots, b_d, b$ are an essential sequence over I and $B = (b_1, \ldots, b_d)R$, then $b + B$ need not be an essential sequence over $(I + B)/B$. A specific example of this is given in [6, (7.1)] with $I = (0)$.

In [5, (4.5)] it was shown that the opposite inequality in (5.5) holds for the asymptotic case: that is, $\operatorname{acogd}(I) \leqq \operatorname{agd}(M/I)$. And in [5, (7.4)] is was shown that $\operatorname{agd}(M/I^n) \geqq \operatorname{acogd}(I) \geqq \operatorname{grade}(M/I^n)$ for all large n. Therefore, if n is large and $(R/I^n)^*$ has no imbedded prime divisors of zero, then (3.3.2), (5.5), (4.1.3), and [5, (7.4)] imply that $\operatorname{ecogd}(I^n)=\operatorname{ecogd}(I)\geqq\operatorname{egd}(M/I^n)=\operatorname{agd}(M/I^n)\geqq\operatorname{acogd}(I)\geqq\operatorname{grade}(M/I^n)$.

6. Essential Cograde and Unmixedness

In this section we use essential cograde to characterize unmixed local rings and another (related) class of local rings. We begin with the unmixed case.

(6.1) **Theorem.** *The following statements are equivalent for a local ring* (R, M):
(6.1.1) *R is unmixed.*
(6.1.2) $\operatorname{egd}(I)=\operatorname{height} I$ *for all ideals I in R.*
(6.1.3) $\operatorname{egd}(M)=\operatorname{height} M$.
(6.1.4) $\operatorname{ecogd}((0))=\operatorname{depth}(0)$.
(6.1.5) $\operatorname{ecogd}(X)=\operatorname{depth} X$ *for all ideals X generated by an essential sequence in R.*

Proof. (6.1.1)-(6.1.3) are equivalent by [6, (6.1)].

Assume that (6.1.2) holds and let X be an ideal generated by an essential sequence $x_1,\dots,x_h$. Then $x_1,\dots,x_h$ can be extended to a maximal essential sequence $x_1,\dots,x_h,x_{h+1},\dots,x_a$ in R, so $\operatorname{ecogd}(X)=a-h$. Also, $h=\operatorname{height} X$, by (2.2.8), and (6.1.2) implies that $a=\operatorname{height} M$, so $\operatorname{height} X+\operatorname{depth} X\leqq a$. Therefore, since $\operatorname{ecogd}(X)\leqq\operatorname{depth} X$, by (5.1.1), we have $a-h=\operatorname{ecogd}(X)\leqq\operatorname{depth} X\leqq a-\operatorname{height} X=a-h$, so (6.1.2) $\Rightarrow$ (6.1.5).

(6.1.5) $\Rightarrow$ (6.1.4), since (0) is generated by the empty essential sequence, and (6.1.4) $\Rightarrow$ (6.1.3), since the definitions imply that a maximal essential sequence over (0) is simply a maximal essential sequence in R, q.e.d.

The results analogous to (6.1) for Cohen-Macaulay local rings (classical theory) and for quasi-unmixed local rings (asymptotic theory) are also true. The equivalence of the analogous first three statements for Cohen-Macaulay (resp., quasi-unmixed) local rings is given in [3, Definition, p. 95, and Theorem 136] (resp., [12, (4.1)]), and the proof of the equivalence of the analogous first and last two statements is similar to that given to prove (6.1).

We now begin considering another class of local rings.

(6.2) **Lemma.** *Let R be a local ring and consider the following statements:*
(6.2.1) $\operatorname{ecogd}(I)=\operatorname{depth} I$ *for all ideals I in R.*
(6.2.2) $\operatorname{egd}(I)+\operatorname{ecogd}(I)=\operatorname{altitude} R$ *for all ideals I in R.*
(6.2.3) (6.2.2) *holds for all depth one prime ideals I in R.*
(6.2.4) *R is unmixed and there exists a unique prime divisor of zero in R.*
Then (6.2.1) $\Leftrightarrow$ (6.2.2) $\Leftrightarrow$ (6.2.3) $\Rightarrow$ (6.2.4).

Proof. If (6.2.1) holds, then R is unmixed, by (6.1.5) $\Rightarrow$ (6.1.1). Suppose z and w are distinct prime divisors of zero in R, so $\operatorname{depth} w=\operatorname{depth} z$, by unmixedness. Let w^* be a (minimal) prime divisor of wR^*, so $\operatorname{depth} w^*=\operatorname{depth} w=\operatorname{depth} z$. Then $\operatorname{ecogd}(z)\leqq\operatorname{depth} zR^*+w^*$, by (3.2), and clearly $\operatorname{depth} zR^*+w^*<\operatorname{depth} w^*=\operatorname{depth} z$, and this contradicts (6.2.1), so (6.2.1) $\Rightarrow$ (6.2.4).

Also, if (6.2.1) holds, then R is unmixed, as just noted, so $\operatorname{egd}(I)=\operatorname{height} I$, by (6.1.1) $\Rightarrow$ (6.1.2). Therefore, since an unmixed local ring is catenary, (6.2.1) $\Rightarrow$ (6.2.2).

It is clear that (6.2.2) $\Rightarrow$ (6.2.3).

Finally, assume that (6.2.3) holds and suppose that (6.2.1) fails for some ideal I in R. Choose such an I such that I has minimal depth amongst all such ideals in R and let $d = \operatorname{depth} I$. Then $d > 0$, by (5.1.1). Also, if $d = 1$, then $\operatorname{ecogd}(I) = 0$, so $\operatorname{ecogd}(P) = 0$ for each depth one prime ideal P in R containing I, by (3.3.1), and this contradicts (6.2.3). Therefore $d > 1$, so there exist infinitely many prime ideals P in R such that $I \subset P$ and depth $P = d - 1$. Let $\mathscr{P}$ be the set of these prime ideals. Now there exists $z^* \in \operatorname{Ass} R^*$ such that $\operatorname{ecogd}(I) = \operatorname{depth} IR^* + z^*$, by (3.2). Let $\mathscr{P}^* = \{P^* \in \operatorname{Spec} R^*; P^*$ is a minimal prime divisor of $PR^* + z^*$ with $P \in \mathscr{P}\}$. Then $\mathscr{P}^*$ is an infinite set and depth $P^* = d - 1$ for all $P^* \in \mathscr{P}^*$, as will now be shown. Namely, if $P \in \mathscr{P}$, then by the choice of d we have $d - 1 = \operatorname{depth} P = \operatorname{ecogd}(P)$, and $\operatorname{ecogd}(P) \leqq \operatorname{depth} P^*$ (where $P^* \in \mathscr{P}^*$ is a minimal prime divisor of $PR^* + z^*$), by (3.2), and depth $P^* \leqq \operatorname{depth} PR^* = \operatorname{depth} P = d - 1$. Therefore P^* is a minimal prime divisor of PR^*, and hence there are infinitely many such P^*, since there are infinitely many such P. Therefore, since $d - 1 = \operatorname{depth} P^* \leqq \operatorname{depth} IR^* + z^* = \operatorname{ecogd}(I) < \operatorname{depth} I = d$, it follows that $IR^* + z^*$ has infinitely many minimal prime divisors, and this is a contradiction, so (6.2.3) $\Rightarrow$ (6.2.1), q.e.d.

We can now characterize the second class of local rings using $\operatorname{ecogd}(I)$.

(6.3) **Proposition.** *The following statements are equivalent for a complete local ring R;*

(6.3.1) $\operatorname{ecogd}(I) = \operatorname{depth} I$ *for all ideals I in R.*

(6.3.2) $\operatorname{egd}(I) + \operatorname{ecogd}(I) = \operatorname{altitude} R$ *for all ideals I in R.*

(6.3.3) (6.3.2) *holds for all depth one prime ideals I in R.*

(6.3.4) *R has a unique prime divisor of zero.*

Proof. By (6.2) it need only be shown that (6.3.4) $\Rightarrow$ (6.3.1), and this was shown in (3.1), q.e.d.

Concerning (6.3), it is not true for an arbitrary local ring R that $\operatorname{ecogd}(I) = \operatorname{depth} I$ for all ideals I in R implies that R^* has a unique prime divisor of zero, since every local domain of altitude one satisfies this hypothesis.

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