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Application of an idea of Voronoĭ to lattice packing

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Abstract The geometric variant of a criterion of Voronoĭ says, a lattice packing of balls in $\mathbb{E}^d$ has (locally) maximum density if and only if it is eutactic and perfect. This article deals with refinements of Voronoĭ's result and extensions to lattice packings of smooth convex bodies. Versions of eutaxy and perfection are used to characterize lattices with semi-stationary, stationary, maximum and ultra-maximum lattice packing density, where ultra-maximality is a sharper version of maximality. Surprisingly, for balls, the lattice packings with maximum density have ultra-maximum density. To make the picture more complete, for $d = 2, 3$, we specify the lattices that provide lattice packings of balls with maximum properties. These lattices are related to Bravais types. Finally, similar results of a duality type are given.

Keywords Voronoĭ type result · Lattice packing of balls · Lattice packing of convex bodies · Maximum density · Maximum lattice · Perfect lattice · Eutactic lattice

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1 Introduction

Voronoĭ's [47] classical criterion on positive definite quadratic forms characterizes the extreme forms by the properties of eutaxy and perfection. Equivalently, a lattice packing of balls has locally maximum density if and only if it is eutactic and perfect. Alternative proofs are due to Kneser [32] and Barnes [2], see also the short proofs in the books of the author [19] and Schürmann [42]. For $d = 2, 3$, Minkowski [35] specified necessary conditions for lattice packings of convex bodies of globally maximum density. A different approach to the densest lattice packing problem is the effective algorithm of Betke and Henk [9] for

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convex 3-polytopes. This article is the first attempt to extend Voronoĭ's criterion for lattice packing of balls to lattice packing of convex bodies.

The first part deals with lattice packings of smooth convex bodies. Using more general Voronoĭ type properties of the packing lattices, lattice packings with refined (local) maximum properties of the density are characterized as follows: The density is semi-stationary if and only if the lattice is semi-eutactic. There is no lattice with stationary density. The density is ultra-maximum if and only if the lattice is eutactic and perfect. (Theorem 1). The new notion of ultra-maximality is stronger than maximality. See Sect. 2 for the definition. In the ultra-maximum case, the number of minimum points, the kissing number of the lattice, is at least $2d^2$ (Corollary 1). For balls, the Voronoĭ type and the extremality notions are slightly different and similar results hold. Surprisingly, the cases of maximum and ultra-maximum density coincide (Theorem 2). These results extend and refine Voronoĭ's criterion. Using results of Bergé and Martinet [8] and Bavard [6], the Bravais classes of those lattices in $\mathbb{E}^2$ and $\mathbb{E}^3$ are specified for which the ball packing density is semi-stationary, maximum and ultra-maximum. In higher dimensions, there are only finitely many similarity classes of lattices such that the density of the corresponding ball packings have maximum properties (Corollary 3).

The second part contains related results of a duality character for the product of the density of a lattice packing of a smooth and strictly convex body and the density of the packing of the polar body with the polar lattice: The product of the densities is semi-stationary if and only if the lattice is dual semi-eutactic. Lattice packings with stationary product of densities exist and are characterized. The product of the densities is ultra-maximum if and only if the lattice is dual eutactic and dual perfect (Theorem 3). There are similar results for balls. In addition, lattice packings of balls with maximum and ultra-maximum product of densities coincide (Theorem 4). The duality results for balls refine work of Bergé and Martinet [7]. The article concludes with a short section on open problems.

For related, more precise results on planar packing and covering with solid circles, see the author [27]. The translation of the results for lattice packing of balls into the language of positive definite quadratic forms is left to the reader, a minor exception being the remark after Corollary 3.

The idea underlying the proofs is to identify lattices in $\mathbb{E}^d$ with points in $\mathbb{E}^{\frac{1}{2}d(d+1)}$ if packings of balls are considered, and in $\mathbb{E}^{d^2}$, in case of convex bodies. (Balls are treated differently because of their high symmetry.) The lattice packing problems then are translated into transparent geometric problems on convex polytopes or polyhedral convex cones in $\mathbb{E}^{\frac{1}{2}d(d+1)}$ or in $\mathbb{E}^{d^2}$, which, sometimes, are easier to solve. In essence, this idea is due to Voronoĭ [47–49]. Similar ideas in the context of projective geometry can be traced back to Monge, Plücker and Klein. For pertinent results and references to numerous applications of this idea to positive definite quadratic forms and lattice packings of balls, zeta functions, closed geodesics on the Riemannian manifolds of a Teichmüller space, and John type and minimum position problems in the asymptotic theory of normed spaces, see the articles [5–8], [11,25,36], [30,38–41], [17,20,29] and the surveys [22,26].

Considering the results of this article, the question arises to give Voronoĭ type criteria for lattice coverings with balls or general convex bodies of (locally) minimum density. Covering results are more difficult to prove than corresponding packing results, see [24] for an explanation. A first result of this type for balls is due to the joint efforts of Barnes and Dickson [3], Delone, Dolbilin, Ryshkov and Shtogrin [12] and Schürmann and Vallentin [44]. For an approach to this problem in the spirit of this article, see the author [24].

For information on the geometric theory of positive definite quadratic forms and the geometry of numbers, see [10,19,28,34,37,42,50].

Let the symbols tr, dim, bd, relint, $\text{relint}_{\mathcal{S}}$, $\text{relbd}_{\mathcal{S}}$, pos, lin, conv, $\|\cdot\|$, $\cdot$, V, B^d, S^{d-1}, ^{T}, ${}^\perp$ stand for trace, dimension, boundary, interior relative to the affine hull, interior and boundary relative to the linear subspace $\mathcal{S}$, positive (nonnegative), linear and convex hull, Euclidean norm, inner product, volume, unit ball and unit sphere in Euclidean d-space $\mathbb{E}^d$, transposition, and orthogonal complement.

2 Maximum properties of the lattice packing density

In this section, relations between maximum properties of the density $\delta(C, \cdot)$ in the neighborhood of a given lattice L and Voronoĭ type properties of the lattice are studied.

Lattice packing of general convex bodies C and the Euclidean unit ball B^d are treated slightly different. The concepts of maximality and the Voronoĭ type properties for general C are defined in the context of $\mathbb{E}^{d^2}$, the corresponding concepts and properties for B^d in the context of $\mathbb{E}^{\frac{1}{2}d(d+1)}$. For this reason, the results for C and B^d differ to some extent.

Before stating the results and presenting the proofs, we collect

2.1 Basic notions

Let C be a *convex body*, that is, a compact, convex subset of $\mathbb{E}^d$ with nonempty interior. We assume that C is of class $\mathcal{C}^1$, that is, the body C is smooth, and symmetric in the origin o. The latter is no essential restriction, see, e.g., [19], p. 443. Let $\|\cdot\|_C$ denote the norm on $\mathbb{E}^d$ with unit ball C. Let L be a *lattice*, that is, the set of all integer linear combinations of d linearly independent vectors in $\mathbb{E}^d$. The volume of the parallelepiped generated by these vectors is the *determinant* $d(L)$ of L. The quantity

$$\varrho(C, L) = \min\left\{\frac{1}{2}\,\|l\|_C : l \in L\backslash\{o\}\right\}$$

is called the *packing radius* of C with respect to L. The set of *minimum points* of L (with respect to C) then is given by $M = \{l \in L : \|l\|_C = 2\varrho(C, L)\}$. The translates $\varrho(C, L)\,C + l : l \in L$ do not overlap pairwise and thus form a *lattice packing* of $\varrho(C, L)\,C$. Its *density* is

$$\delta(C, L) = \frac{\varrho(C, L)^d\, V(C)}{d(L)}.$$

It is the local maxima of the density of $\delta(B^d, \cdot)$ which Voronoĭ studied in his fundamental memoir [47]. Note that

(1) δ does not change, if L is replaced by a positive multiple of it.

The *kissing number* of the lattice packing $\{\varrho(C, L)C + l : l \in L\}$ is the number of translates of the form $\varrho(C, L)C + l$ which touch the body $\varrho(C, L)C$. It is easy to see that it equals the number of minimum points of L.

Identify a (real) $d \times d$ matrix A with a point of $\mathbb{E}^{d^2}$ or of $\mathbb{E}^{\frac{1}{2}d(d+1)}$ when we deal with balls and thus assume that A is symmetric. For $d \times d$ matrices $A = (a_{ik})$, $B = (b_{ik})$, the *inner product* and the *norm* are defined by $A \cdot B = \sum a_{ik}b_{ik}$ and $\|A\| = (\sum a_{ik}^2)^{\frac{1}{2}}$. The symbols $\|\cdot\|$ and $\cdot$ denote also the usual norm and inner product on $\mathbb{E}^d$. Let O, I be the $d \times d$ zero and unit matrix. For $x, y \in \mathbb{E}^d$, the *tensor product* of x and y is the $d \times d$ matrix $x \otimes y = x\,y^T = (x_i y_k) \in \mathbb{E}^{d^2}$. For $l \in \mathbb{E}^d \setminus \{o\}$, let u_l be the exterior unit normal vector of the convex body $\|l\|_C C$ at its boundary point l and $n_l = l/l \cdot u_l$.

$$\mathcal{S} = \left\{A \in \mathbb{E}^{d^2} : \mathrm{tr} A = A \cdot I = 0\right\} = I^{\perp}$$

is a linear subspace of $\mathbb{E}^{d^2}$ of codimension 1.

2.2 Maximum and Voronoĭ type properties

The maximum and Voronoĭ type properties which will be defined below split into two types, one for general convex bodies and one for Euclidean balls. While the definitions are quite natural, the situation arises that the density for B^d may be ultra-maximum at a lattice L, yet not ultra-maximum if B^d is considered as a general convex body. Since in the following we clearly distinguish between results dealing with general C and results dealing with B^d, this should cause no ambiguity.

In order to study the local behavior of $\delta(C, \cdot)$ at a lattice L, one has to compare $\delta\big(C, (I + A)L\big)$ with $\delta(C, L)$, where A ranges over a neighborhood of O in $\mathbb{E}^{d^2}$. Since

$$\frac{d(I + A)}{\mathrm{tr}(I + A)} = I + B \text{ where } B \in \mathcal{S} \text{ and } \delta\big(C, (I + A)L\big) = \delta\big(C, (I + B)L\big),$$

by (1), the local behavior of $\delta(C, \cdot)$ at L is already visible if A ranges over a neighborhood of O in the subspace $\mathcal{S}$ of $\mathbb{E}^{d^2}$. This will be the viewpoint in the following.

The density $\delta(C, \cdot)$ of the packing $\big\{\varrho(C, L)C + l : l \in L\big\}$ is *(locally) semi-stationary, stationary, maximum*,or *ultra-maximum* at L, if

$$\frac{\delta\big(C, (I + A)L\big)}{\delta(C, L)} \begin{Bmatrix} \leq 1 + o(\|A\|) \\ = 1 + o(\|A\|) \\ \leq 1 \\ \leq 1 - \mathrm{const}\|A\| \end{Bmatrix} \text{ as } A \to O,\ A \in \mathcal{S},$$

where an inequality holds as $A \to O$ if it holds for all A for which $\|A\|$ is sufficiently small. The symbol const stands for a positive constant. If one of the symbols const, $o(\cdot)$ or $O(\cdot)$, appears several times in the same context, it may be different each time.

For a convex body, the appropriate versions, respectively, refinements of the classical notions of eutaxy and perfection, are as follows: The lattice L is *semi-eutactic, eutactic* or *perfect* with respect to C, if

$$I = \sum_{l \in M} \lambda_l\, u_l \otimes n_l \text{ with suitable } \lambda_l \begin{Bmatrix} \geq 0 \\ > 0 \end{Bmatrix},$$
$$\text{resp. } \mathbb{E}^{d^2} = \mathrm{lin}\{u_l \otimes n_l : l \in M\}.$$

Note that we do not require that this representation is unique. It may happen that I has a representation which makes L eutactic, and a different representation, which makes it semi-eutactic.

2.3 Relations between maximum and Voronoĭ type properties

2.3.1 The general case

Theorem 1 *The following statements hold:*

(i) $\delta(C, \cdot)$ *is semi-stationary at* $L \Leftrightarrow L$ *is semi-eutactic with respect to* C.
(ii) $\delta(C, \cdot)$ *is not stationary at any lattice.*
(iii) $\delta(C, \cdot)$ *is ultra-maximum at* $L \Leftrightarrow L$ *is eutactic and perfect with respect to* C.

A result of Swinnerton-Dyer [45] says, the kissing number of a lattice packing of maximum density of a convex body is at least $d(d+1)$. A consequence of Theorem 1(iii) is the following sharper estimate for ultra-maximality:

Corollary 1 *If $\delta(C, \cdot)$ is ultra-maximum at L, then the kissing number of L is at least $2d^2$.*

For comments, see Sect. 4.

2.3.2 The case of balls

If $C = B^d$, then, due to the invariance of B^d with respect to orthogonal transformations, it is customary to replace $\mathbb{E}^{d^2}$ by $\mathbb{E}^{\frac{1}{2}d(d+1)}$, to use only symmetric $d \times d$ matrices, and instead of the subspace $\mathcal{S}$, the subspace

$$\mathcal{T} = \left\{A \in \mathbb{E}^{\frac{1}{2}d(d+1)} : \mathrm{tr}A = A \cdot I = 0\right\} = I^{\perp}$$

of $\mathbb{E}^{\frac{1}{2}d(d+1)}$. Note also that then $u_l = n_l = l/\|l\|$. Lattice packings of B^d with *(locally) semi-stationary, stationary, maximum* and *ultra-maximum* density are defined correspondingly. This holds also for the concepts of *semi-eutactic, eutactic* and *perfect* lattices with respect to B^d. These Voronoĭ type notions are the classical ones and, in addition, semi-eutaxy.

With these remarks in mind, we have the following equivalences, the last being Voronoĭ's criterion. The formulation for positive definite quadratic forms is left to the reader.

Theorem 2 *The following statements hold:*

(i) *$\delta(B^d, \cdot)$ is semi-stationary at L $\Leftrightarrow$ L is semi-eutactic with respect to B^d.*
(ii) *$\delta(B^d, \cdot)$ is not stationary at any lattice.*
(iii) *$\delta(B^d, \cdot)$ is ultra-maximum at L $\Leftrightarrow$ L is eutactic and perfect with respect to B^d $\Leftrightarrow$ $\delta(B^d, \cdot)$ is maximum at L.*

That maximum density, actually, is ultra-maximum was a surprise. Equally surprising was that the proof is short, simple and makes use only of elementary tools.

An immediate consequence of Theorem 2(iii) is the following result due to Korkin and Zolotarev [33].

Corollary 2 *If $\delta(B^d, \cdot)$ is maximum at L, then the kissing number of L is at least $d(d+1)$.*

Bergé and Martinet [8], Batut [4] and Bavard [6] described for B^d the semi-eutactic and eutactic lattices in dimensions $d = 2, 3, 4, 5$ and showed that in general dimensions, there are only finitely many similarity classes of such lattices; see also [1]. A scrutiny of their proofs and Theorem 2 together yield the next remark.

Corollary 3 *In $\mathbb{E}^2$ and $\mathbb{E}^3$, it is precisely the following lattices at which $\delta(B^d, \cdot)$ is semi-stationary, respectively, maximum or ultra-maximum:*

$\mathbb{E}^2$: *tp*	*square*	$\mathbb{E}^3$: *cP*	*cubic primitive*
hp	*hexagonal*	*shP*	*special hexagonal primitive*
		cF	*cubic face centered*
		cI	*cubic body centered*
		stI	*special tetragonal body centered*
$\mathbb{E}^2$: *hp*	*hexagonal*	$\mathbb{E}^3$: *cF*	*cubic face centered.*

In general dimensions, there are only finitely many similarity classes of lattices with semi-stationary, maximum or ultra-maximum density.

The Bravais types are a common classification of lattices in crystallography and tp,...,tI are their usual symbols. Each of the Bravais types tp, hp, cP, cF, cI consists of one similarity class of lattices, the Bravais types hP, tI of more than one. An s is added to the symbol if a special similarity class in the Bravais type is meant. These similarity classes of lattices correspond to the classes of positive definite quadratic forms which are equivalent to multiples of the following forms:

tp	x^2+y^2	cP	$x^2+y^2+z^2$
hp	x^2+xy+y^2	shP	$x^2+y^2+z^2+xy$
		cF	$x^2+y^2+z^2+xy+yz+zx$
		cI	$x^2+y^2+z^2+\frac{2}{3}(xy+yz-zx)$
		stI	$x^2+y^2+z^2+xy+\frac{1}{2}yz+zx.$

See, e.g., Erdös, Gruber and Hammer [14] and Engel [13].

Given a lattice L, it may be difficult to find out whether it is eutactic and perfect with respect to B^d. In some cases, simple sufficient conditions are helpful. To state such conditions, we need the following concepts. A *spherical n-design* is a subset $M=\{\pm l_1,\dots,\pm l_k\}$ of the unit sphere S^{d-1}, such that for any polynomial $p:\mathbb{E}^d\to\mathbb{R}$ of degree at most n holds,

$$\int_{S^{d-1}} p(u)\,d\sigma(u)=\frac{1}{2k}\sum_{l\in M}p(l),$$

where σ is the usual normalized rotation invariant area measure on S^{d-1}. Venkov [46] showed that an o-symmetric subset M of S^{d-1} is a spherical n-design if and only if

$$\sum_{l\in M}(l\cdot x)^n=\text{const}\|x\|^n \text{ for } \quad x\in\mathbb{E}^d.$$

Let M be the set of minimum points of L with respect to B^d. A voluminous literature shows that for many lattices, the set of minimum points (properly scaled) is a spherical 4-design. For results and references, see Venkov [46]. The *symmetry* or *automorphism group* $\mathcal{A}=\mathcal{A}(L)$ of L consists of all orthogonal transformations of $\mathbb{E}^d$, which fix the origin o and map L onto itself.

The following statements are special cases of results of Bergé and Martinet [7] and Gruber [25].

Proposition 1 *The following implications on L and B^d hold:*

(i) *If M is a spherical 4-design, then L is eutactic and perfect.*
(ii) *If L is perfect and $\mathcal{A}$ transitive on M, then L is eutactic.*

2.4 Proofs of the theorems

The connection between Voronoĭ type properties of a lattice L and local maximum properties of the density $\delta(C,\cdot)$ in a neighborhood of L can roughly be described as follows: The different Voronoĭ type properties of L are related to the property that a certain convex polytope $\mathcal{P}$ in $\mathcal{S}$ is proper or not, and to the position of $\mathcal{P}$ relative to the origin. (The origin is an exterior point, a point on the relative boundary or in the relative interior of $\mathcal{P}$.) These simple geometric properties turn out to be equivalent to different upper and lower estimates of the density which, in turn, are equivalent to local maximum properties of $\delta(C, \cdot)$.

We first put together simple tools and then present several lemmas. The lemmas contain rather more information than is needed for the proofs of the theorems. The reason for this is to clarify the geometric meaning of the Voronoĭ type notions of semi-eutaxy, etc.

The following identities hold:

$$\begin{aligned}
&x \otimes x \cdot y \otimes z = (x \cdot y)(x \cdot z),\\
&I \cdot x \otimes y = x \cdot y,\\
&A \cdot x \otimes y = Ay \cdot x = x^T Ay,\\
&\text{for } x, y, z \in \mathbb{E}^d \text{ and } A \in \mathbb{E}^{d^2},\\
&\det(I + A) = 1 + \mathrm{tr}A + \tfrac{1}{2}\left((\mathrm{tr}A)^2 - A \cdot A^T\right) + O(\|A\|^3)\\
&\text{as } A \to O, A \in \mathbb{E}^{d^2}.
\end{aligned}$$

To see the last identity, note that

$$\det(I + A) = \begin{vmatrix} 1 + a_{11} & a_{12} & a_{13} & \cdots & a_{1d} \\ a_{21} & 1 + a_{22} & a_{23} & \cdots & a_{2d} \\ a_{31} & a_{32} & 1 + a_{33} & \cdots & a_{3d} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{d1} & a_{d2} & a_{d3} & \cdots & 1 + a_{dd} \end{vmatrix}$$

$$\begin{aligned}
&= 1 + \mathrm{tr}\, A + \sum_{i<j} a_{ii}a_{jj} - \sum_{i<j} a_{ij}a_{ji} + O(\|A\|^3)\\
&= 1 + \mathrm{tr}\, A + \frac{1}{2}\Big(\sum_i a_{ii}\Big)^2 - \frac{1}{2}\sum_i a_{ii}^2 - \frac{1}{2}\sum_{i \neq j} a_{ij}a_{ji} + O(\|A\|^3)\\
&= 1 + \mathrm{tr}\, A + \frac{1}{2}\big((\mathrm{tr}\, A)^2 - A \cdot A^T\big) + O(\|A\|^3) \text{ as } A \to O, A \in \mathbb{E}^{d^2}.
\end{aligned}$$

A particular case of this identity is as follows:

$$\det(I + A)^{-1} = \left(1 - \frac{1}{2}\|A\|^2 + O\left(\|A\|^3\right)\right)^{-1} = 1 + \frac{1}{2}\|A\|^2 + O\left(\|A\|^3\right)$$
$$\text{as } A \to O, A \in \mathcal{T}.$$

Next,

$$\begin{aligned}
&\|l + Al\|_C = \|l\|_C\big(1 + A \cdot u_l \otimes n_l + o(\|A\|)\big)\\
&\text{as } A \to O,\ A \in \mathcal{S},\ l \in M, \text{ where } o(\cdot) \text{ depends only on } C, L.
\end{aligned}$$

This is a consequence of the assumption that $C \in \mathcal{C}^1$ and an easy geometric argument:

$$\begin{aligned}
&\|l + Al\|_C = \|l\|_C \frac{l \cdot u_l + Al \cdot u_l + o(\|Al\|)}{l \cdot u_l} = \|l\|_C\big(1 + A \cdot u_l \otimes n_l + o(\|A\|)\big)\\
&\text{as } A \to O,\ A \in \mathcal{S},\ l \in M, \text{ where } o(\cdot) \text{ depends only on } C.
\end{aligned}$$

Having shown this, the definition of the packing radius yields the equalities,

$$\begin{aligned}
\varrho\big(C, (I + A)L\big)^d &= \tfrac{1}{2^d} \min\left\{\|l + Al\|_C : l \in M\right\}^d\\
&= \tfrac{1}{2^d} \min\left\{\|l\|_C\big(1 + A \cdot u_l \otimes n_l + o(\|A\|)\big) : l \in M\right\}^d\\
&= \varrho(C, L)^d \left(1 + d \min\{A \cdot u_l \otimes n_l : l \in M\} + o(\|A\|)\right) \text{ as } A \to O, A \in \mathcal{S}.
\end{aligned}$$

The definition of δ, together with these equalities, now implies the identity,

$$\begin{aligned}\delta\big(C,(I+A)L\big) &= \frac{\varrho\big(C,(I+A)L\big)^d\, V(C)}{d\big((I+A)L\big)}\\ &= \frac{\varrho(C,L)^d\, V(C)}{d(L)}\ \frac{1+d\min\{A\cdot u_l\otimes n_l : l\in M\}+o(\|A\|)}{1-\frac{1}{2}A\cdot A^T+O(\|A\|^3)}\\ &= \delta(C,L)\big(1+d\min\{A\cdot u_l\otimes n_l : l\in M\}+o(\|A\|)\big) \text{ as } A\to O,\ A\in\mathcal{S}.\end{aligned}$$

If $C=B^d,\ A\in\mathcal{T},\ A\cdot A^T=\|A\|^2$, then $u_l=n_l$, and we have the following:

$$\begin{aligned}\|n_l+An_l\|^d &= \big((n_l+An_l)^2\big)^{\frac{d}{2}} = \big(1+2A\cdot n_l\otimes n_l+(An_l)^2\big)^{\frac{d}{2}}\\ &= 1+dA\cdot n_l\otimes n_l+O(\|A\|^2)\\ &\qquad \text{as } A\to O,\ A\in\mathcal{T},\ \text{ where } O(\|A\|^2)\ge 0,\end{aligned}$$

and thus,

$$\begin{aligned}&\delta\big(B^d,(I+A)L\big)=\delta(B^d,L)\big(1+d\min\{A\cdot n_l\otimes n_l : l\in M\}+O(\|A\|^2)\big)\\ &\text{as } A\to O,\ A\in\mathcal{T},\ \text{ where } O(\|A\|^2)>0.\end{aligned}$$

If $\mathcal{E}$ is a linear subspace of $\mathbb{E}^{d^2}$ or $\mathbb{E}^{\frac{1}{2}d(d+1)}$, then ${}^{\mathcal{E}}$ is the orthogonal projection of $\mathbb{E}^{d^2}$, resp. $\mathbb{E}^{\frac{1}{2}d(d+1)}$ onto $\mathcal{E}$. Note that

$$\begin{aligned}&A\cdot u_l\otimes n_l = A\cdot(u_l\otimes n_l)^{\mathcal{S}} \text{ for } A\in\mathcal{S},\ l\in M,\\ &A\cdot u_l\otimes n_l = A^{\mathcal{E}}\cdot(u_l\otimes n_l)^{\mathcal{S}} \text{ for } A\in\mathcal{S}, l\in M,\ (u_l\otimes n_l)^{\mathcal{S}}\in\mathcal{E},\ \mathcal{E} \text{ linear subspace of } \mathcal{S}.\end{aligned}$$

Let $\mathcal{P}$ be a convex polytope in $\mathcal{S}$ such that $O\in\mathcal{P}$. The *(inner) normal cone* $\mathcal{N}$ of $\mathcal{P}$ at O is defined by

$$\mathcal{N}=\{N\in\mathcal{S} : N\cdot P\ge 0 \text{ for } P\in\mathcal{P}\}.$$

It is a closed polyhedral convex cone in $\mathcal{S}$ with apex O. The equality $\mathcal{N}=\{O\}$ holds precisely in case when O is an interior point of $\mathcal{P}$ relative to $\mathcal{S}$. Convexity arguments yield the next remarks.

Let $\mathcal{P}$ be a convex polytope in $\mathcal{S}$ with $O\in\mathcal{P}$. Then, the following statement (i) holds:

(i) If $\mathcal{N}$ is the normal cone of $\mathcal{P}$ at O in $\mathcal{S}$ with apex O, then holds

$$\min\{A\cdot P : P\in\mathcal{P}\}\left\{\begin{array}{ll} =0 & \text{for } A\in\mathcal{N}\\ <0 & \text{for } A\in\mathcal{S}\setminus\mathcal{N}\end{array}\right\}.$$

Moreover, the following statements (ii) and (iii) are equivalent:

(ii) $\mathcal{P}$ is proper in $\mathcal{S}$ with $O\in\operatorname{relbd}_{\mathcal{S}}\mathcal{P}$.
(iii) There is a pointed closed convex cone $\mathcal{N}$ in $\mathcal{S}$ with apex O such that

$$\min\{A\cdot P : P\in\mathcal{P}\}\left\{\begin{array}{ll} =0 & \text{for } A\in\mathcal{N}\\ <0 & \text{for } A\in\mathcal{S}\setminus\mathcal{N}\end{array}\right\}.$$

Recall that

$$I=\sum_{l\in M}\lambda_l\, u_l\otimes n_l \text{ with suitable } \lambda_l\ge 0.$$

Lemma 1 *The following statements* (i)-(iii) *on* $\delta(C,\cdot)$ *and* L *are equivalent:*

(i) *L is semi-eutactic with respect to C.*
(ii) *$\delta(C,\cdot)$ is semi-stationary at L.*

(iii) $\dfrac{\delta\big(C, (I+A)L\big)}{\delta(C, L)} \leq 1 - \text{const}\|A^{\mathcal{E}}\| + o(\|A\|)$, *as* $A \to O,\ A \in \mathcal{S}$,
where $\mathcal{E} = \text{lin}\{(u_l \otimes n_l)^{\mathcal{S}} : l \in M, \lambda_l > 0\} \subseteq \mathcal{S}$ *and* $\dim \mathcal{E} \geq d - 1$.
Moreover,

(iv) $\delta(C, \cdot)$ *is not stationary at any lattice.*

In case $\mathcal{E} = \mathcal{S}$, we may write A instead of $A^{\mathcal{E}}$ and omit $o(\|A\|)$ in (iii). Then, $\delta(C, \cdot)$ is even ultra-maximum at L. The statement (iii) gives a more precise idea of the meaning of semi-stationarity in the present context.

There are lattices at which the density is maximum and thus, a fortiori, semi-stationary. Lemma 1 then shows that semi-eutactic lattices exist. An example of a semi-eutactic lattice in $\mathbb{E}^2$ which is not eutactic is given in [27].

Proof of Lemma 1 (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i):

$$
\begin{aligned}
&L \text{ is semi-eutactic}\\
\Rightarrow\ &\tfrac{1}{d} I = \sum_{l \in M} \lambda_l\, u_l \otimes n_l, \quad \text{where } \lambda_l \geq 0\\
\Rightarrow\ &\tfrac{1}{d} I = \sum_{l \in E} \lambda_l\, u_l \otimes n_l,\ \lambda_l > 0,\\
&\text{where } E = \{l \in M : \lambda_l > 0\},\ \sum_{l \in E} \lambda_l = \sum_{l \in E} \lambda_l\, u_l \otimes n_l \cdot I = \tfrac{1}{d} I \cdot I = 1\\
\Rightarrow\ &O = \big(\tfrac{1}{d} I\big)^{\mathcal{S}} = \sum_{l \in E} \lambda_l\, (u_l \otimes n_l)^{\mathcal{S}},\ \lambda_l > 0,\ \sum_{l \in E} \lambda_l = 1\\
\Rightarrow\ &O \in \text{relint}_{\mathcal{E}} \text{conv}\{(u_l \otimes n_l)^{\mathcal{S}} : l \in E\},\\
&\text{where } \mathcal{E} = \text{lin}\{(u_l \otimes n_l)^{\mathcal{S}} : l \in E\} \subseteq \mathcal{S}\\
\Rightarrow\ &\min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in E\} < 0 \text{ for } A \in \mathcal{E}\\
\Rightarrow\ &\min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in E\} \leq -\text{const}\|A\| \text{ for } A \in \mathcal{E}\\
\Rightarrow\ &\min\{A^{\mathcal{E}} \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in E\} \leq -\text{const}\|A^{\mathcal{E}}\| \text{ for } A \in \mathcal{S}\\
\Rightarrow\ &\min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in E\} \leq -\text{const}\|A^{\mathcal{E}}\| \text{ for } A \in \mathcal{S}\\
\Rightarrow\ &\min\{A \cdot u_l \otimes n_l : l \in E\} \leq -\text{const}\|A^{\mathcal{E}}\| \text{ for } A \in \mathcal{S}\\
\Rightarrow\ &\frac{\delta\big(C, (I+A)L\big)}{\delta(C, L)} = 1 + d \min\{A \cdot u_l \otimes n_l : l \in M\} + o(\|A\|)\\
&\leq 1 + d \min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in E\} + o(\|A\|)\\
&\leq 1 - \text{const}\|A^{\mathcal{E}}\| + o(\|A\|) \text{ as } A \to O,\ A \in \mathcal{S}\\
\Rightarrow\ &\frac{\delta\big(C, (I+A)L\big)}{\delta(C, L)} \leq 1 + o(\|A\|) \text{ as } A \to O,\ A \in \mathcal{S}\\
\Rightarrow\ &\delta(C, \cdot) \text{ is semi-stationary at } L\\
\Rightarrow\ &\frac{\delta\big(C, (I+A)L\big)}{\delta(C, L)} = 1 + d \min\{A \cdot u_l \otimes n_l : l \in M\} + o(\|A\|)\\
&\leq 1 + o(\|A\|) \text{ as } A \to O,\ A \in \mathcal{S}\\
\Rightarrow\ &\min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in M\} = \min\{A \cdot u_l \otimes n_l : l \in M\} \leq 0 \text{ for } A \in \mathcal{S}\\
\Rightarrow\ &O \in \text{conv}\{(u_l \otimes n_l)^{\mathcal{S}} : l \in M\}\\
\Rightarrow\ &O = \big(\tfrac{1}{d} I\big)^{\mathcal{S}} = \sum_{l \in M} \lambda_l\, (u_l \otimes n_l)^{\mathcal{S}},\ \lambda_l \geq 0,\ \sum_{l \in M} \lambda_l = 1\\
\Rightarrow\ &\tfrac{1}{d} I = \sum_{l \in M} \lambda_l\, u_l \otimes n_l, \ \text{where } \lambda_l \geq 0,\ \sum_{l \in M} \lambda_l = 1\\
\Rightarrow\ &L \text{ is semi-eutactic.}
\end{aligned}
$$

It remains to show that $\dim \mathcal{E} \geq d - 1$. Since L is semi-eutactic, one may represent I in the following form, where u_i is the normal unit vector of the body $\|l_i\|_C C$ at its boundary point l_i and $n_i = l_i / l_i \cdot u_i$:

$$I = \sum_{i=1}^{p} \mu_i \, u_i \otimes n_i \text{ with } l_i \in E, \ \mu_i \in \mathbb{R},$$

and linearly independent tensor products $u_1 \otimes n_1, \dots, u_p \otimes n_p$. If $p \geq d$, then among the orthogonal projections of these tensor products in $\mathbb{E}^{d^2}$ into the subspace $\mathcal{S}$ of $\mathbb{E}^{d^2}$ of codimension 1, there are at least $d-1$ linearly independent vectors. Thus, $\dim \mathcal{E} \geq d-1$. If $p < d$, choose $a \in \mathbb{E}^d \backslash \{o\}$ such that $a \cdot u_1 = \dots = a \cdot u_p = 0$. Then,

$$0 < a^2 = a \otimes a \cdot I = \sum_{i=1}^{p} \mu_i \, a \otimes a \cdot u_i \otimes n_i = \sum_{i=1}^{p} \mu_i \, (a \cdot u_i)(a \cdot n_i) = 0,$$

a contradiction. Thus, $p < d$ cannot occur, and the proof that $\dim \mathcal{E} \geq d-1$ is complete.

(iv): Finally, to prove (iv), assume the contrary. Then, there is a lattice L such that $\delta(C, \cdot)$ is stationary and, thus, semi-stationary at L. Noting that $\dim \mathcal{E} > 0$, we may choose $A \in \mathcal{E} \backslash \{O\}$. Then, $\|A^{\mathcal{E}}\| = \|A\| > 0$, and by (iii),

$$\frac{\delta\big(C, (I + \tau A)L\big)}{\delta(C, L)} \leq 1 - \tau \mathrm{const} \|A\| + o(\tau \|A\|) \text{ as } \tau \to 0+ \ (A = A^{\mathcal{E}}),$$

a contradiction to the assumed stationarity of $\delta(C, \cdot)$. □

Lemma 2 *The following statements on $\delta(C, \cdot)$ and L are equivalent:*

(i) *L is eutactic with respect to C.*

(ii)

$$\frac{\delta\big(C, (I + A)L\big)}{\delta(C, L)} \left\{ \begin{array}{l} \leq 1 - \mathrm{const} \|A^{\mathcal{M}}\| + o(\|A\|) \\ \geq 1 - \mathrm{const} \|A^{\mathcal{M}}\| + o(\|A\|) \end{array} \right\}, \textit{as } A \to O, \ A \in \mathcal{S},$$

where $\mathcal{M} = \mathrm{lin}\{(u_l \otimes n_l)^{\mathcal{S}} : l \in M\} \subseteq \mathcal{S}$ *and* $\dim \mathcal{M} \geq d-1$.

These statements clearly imply semi-stationarity of the density. It is well-known that for B^d, there are lattices which are eutactic in the usual sense. This follows, for example, from Voronoĭ's criterion. Such lattices are also eutactic lattices with respect to the smooth convex body $C = B^d$ in the sense of our definition for convex bodies. Hence, convex bodies with eutactic lattices do exist.

Proof of Lemma 2 (i)⇔(ii):

$$\begin{aligned}
& L \text{ is eutactic} \\
\Leftrightarrow\ & \tfrac{1}{d} I = \sum_{l \in M} \lambda_l \, u_l \otimes n_l, \quad \text{where } \lambda_l > 0, \sum_{l \in M} \lambda_l = \tfrac{1}{d} I \cdot I = 1 \\
\Leftrightarrow\ & O = \left(\tfrac{1}{d} I\right)^{\mathcal{S}} = \sum_{l \in M} \lambda_l \, (u_l \otimes n_l)^{\mathcal{S}}, \ \lambda_l > 0, \sum_{l \in M} \lambda_l = 1 \\
\Leftrightarrow\ & O \in \mathrm{relint}\, \mathrm{conv}\{(u_l \otimes n_l)^{\mathcal{S}} : l \in M\} \\
\Leftrightarrow\ & -\mathrm{const}\|A\| \leq \min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in M\} \leq -\mathrm{const}\|A\| \text{ for } A \in \mathcal{M} \\
\Leftrightarrow\ & -\mathrm{const}\|A^{\mathcal{M}}\| \leq \min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in M\} \leq -\mathrm{const}\|A^{\mathcal{M}}\| \text{ for } A \in \mathcal{S} \\
\Leftrightarrow\ & \frac{\delta\big(C, (I + A)L\big)}{\delta(C, L)} = 1 + d \min\{A \cdot u_l \otimes n_l : l \in M\} + o(\|A\|) \\
& = 1 + d \min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in M\} + o(\|A\|) \\
& \qquad \left\{ \begin{array}{l} \leq 1 - \mathrm{const}\|A^{\mathcal{M}}\| + o(\|A\|) \\ \geq 1 - \mathrm{const}\|A^{\mathcal{M}}\| + o(\|A\|) \end{array} \right\} \text{ as } A \to O, \ A \in \mathcal{S}.
\end{aligned}$$

The bound for $\dim \mathcal{M}$ follows from Lemma 1 since in the present case $\mathcal{E} = \mathcal{M}$. □

Lemma 3 *The following statements on $\delta(C,\cdot)$ and L are equivalent:*

(i) *L is perfect with respect to C.*

(ii) $$\frac{\delta\big(C,(I+A)L\big)}{\delta(C,L)}\begin{Bmatrix}\le 1-\mathrm{const}\|A\|+A\cdot V\\ \ge 1-\mathrm{const}\|A\|+A\cdot V\end{Bmatrix}, \textit{as } A\to O,\ A\in\mathcal{S},$$
where $V=\frac{d}{2k}\sum_{l\in M}u_l\otimes n_l$.

Example A perfect lattice has no maximum property, unless it is semi-eutactic or eutactic. To see that there are smooth bodies for which there are perfect lattices in the sense of the present definition, take a smooth convex body C with a lattice L (for example, B^8 and the root lattice E_8, or B^{24} and the Leech lattice L^{24}, suitably normalized), such that for the set of minimum points holds $M\subseteq \mathrm{bd}\,2C$, and in M, there are d pairwise disjoint d-tuples of linearly independent vectors, say

$$\{l_{11},\dots,l_{1d}\},\dots,\{l_{d1},\dots,l_{dd}\}.$$

(The remark that E_8 and L^{24} have this property is due to Schürmann [43].) Let

$$\{u_{11},\dots,u_{1d}\},\dots,\{u_{d1},\dots,u_{dd}\}$$

be the corresponding exterior normal unit vectors of $\mathrm{bd}\,C$. Then,

$$p(x_{11},\dots,x_{1d},\dots,x_{d1},\dots,x_{dd})=\det(x_{11}\otimes l_{11},\dots,x_{dd}\otimes l_{dd})$$

is a polynomial of degree d^2 in $x_{11},\dots,x_{dd}$. (Write the $d\times d$ matrices $x_{ik}\otimes l_{ik}$ as $d^2\times 1$ columns.) If $p(u_{11},\dots,u_{dd})\neq 0$, then the lattice L is perfect with respect to C. If this is not the case, note that for

$$x_{11}=\ldots=x_{1d}=b_1,\dots,x_{d1}=\ldots=x_{dd}=b_d,$$

where $\{b_1,\dots,b_d\}$ is the standard basis of $\mathbb{E}^d$, the polynomial p is $\neq 0$. Hence, it is not identically 0, and there are points $x_{11},\dots,x_{dd}$ arbitrarily close to $u_{11},\dots,u_{dd}$, respectively, with $p(x_{11},\dots,x_{dd})\neq 0$. Thus, one may change C close to $l_{11},\dots,l_{dd}$ as to obtain a smooth o-symmetric body D with M as its set of minimum vectors with respect to L and such that $\{x_{11},\dots,x_{dd}\}$ is a set of corresponding exterior normal vectors. Normalizing the sets of vectors $\{l_{11},\dots,l_{dd}\}$ and $\{x_{11},\dots,x_{dd}\}$ suitably, then shows that L is perfect with respect to D.

Proof of Lemma 3 The proof makes use of a translation argument.

(i)⇔(ii):

$$\begin{aligned}
&L \text{ is perfect}\\
\Leftrightarrow\ &\mathbb{E}^{d^2}=\mathrm{lin}\{u_l\otimes n_l: l\in M\}\\
\Leftrightarrow\ &\mathrm{relint}_{\mathcal{S}}\mathrm{conv}\{(u_l\otimes n_l)^{\mathcal{S}}: l\in M\}\neq\emptyset \text{ (note that } I\cdot u_l\otimes n_l=1 \text{ for each } l\in M)\\
\Leftrightarrow\ &O\in \mathrm{relint}_{\mathcal{S}}\mathrm{conv}\{(u_l\otimes n_l)^{\mathcal{S}}-U: l\in M\},\ \text{where } U=\frac{1}{2k}\sum_{l\in M}(u_l\otimes n_l)^{\mathcal{S}}\\
\Leftrightarrow\ &-\mathrm{const}\|A\|\le \min\{A\cdot((u_l\otimes n_l)^{\mathcal{S}}-U): l\in M\}\le -\mathrm{const}\|A\| \text{ for } A\in\mathcal{S}\\
\Leftrightarrow\ &\frac{\delta\big(C,(I+A)L\big)}{\delta(C,L)}=1+d\min\{A\cdot(u_l\otimes n_l)^{\mathcal{S}}: l\in M\}+o(\|A\|)\\
&=1+d\min\{A\cdot((u_l\otimes n_l)^{\mathcal{S}}-U): l\in M\}+d\,A\cdot U+o(\|A\|)\\
&\begin{Bmatrix}\le 1-\mathrm{const}\|A\|\\ \ge 1-\mathrm{const}\|A\|\end{Bmatrix}+d\,A\cdot U \text{ as } A\to O,\ A\in\mathcal{S}.
\end{aligned}$$

□

While the problem of characterizing lattices at which the density is maximum remains open, the following result yields a characterization of the stronger notion of ultra-maximality.

Lemma 4 *The following statements on* $\delta(C,\cdot)$ *and* L *are equivalent:*

(i) *L is eutactic and perfect with respect to C.*
(ii) *$\delta(C,\cdot)$ is ultra-maximum at L.*
(iii) $\dfrac{\delta\big(C,(I+A)L\big)}{\delta(CL)}\left\{\begin{matrix}\le 1-\text{const}\|A\|\\ \ge 1-\text{const}\|A\|\end{matrix}\right\}$ *as* $A\to O,\ A\in\mathcal{S}$.

The question whether there are convex bodies such that the density is ultra-maximum at certain lattices remains open, but see the discussion in Sect. 4.

Proof of Lemma 4 Let $\mathcal{M}$ be as in Lemma 2. The lattice L then is perfect if and only if $\mathcal{M}=\mathcal{S}$. In this case, one may omit $\mathcal{M}$ and then also the $o(\cdot)$-symbols in Lemma 2(ii). Hence, Lemma 2 yields the implications (i)⇒(iii)⇒(ii) for the statements of Lemma 4. It remains to show that

(ii)⇒(i):

$$\begin{aligned}
&\delta(C,\cdot)\text{ is ultra-maximum at }L\\
\Leftrightarrow\ &\frac{\delta\big(C,(I+A)L\big)}{\delta(C,L)}=1+d\min\{A\cdot(u_l\otimes n_l)^{\mathcal{S}}:l\in M\}+o(\|A\|)\\
&\le 1-\text{const}\|A\|\text{ as }A\to O,\ A\in\mathcal{S}\\
\Leftrightarrow\ & d\min\{A\cdot(u_l\otimes n_l)^{\mathcal{S}}:l\in M\}+o(\|A\|)\le-\text{const}\|A\|\text{ as }A\to O,\ A\in\mathcal{S}\\
\Leftrightarrow\ &\min\{A\cdot(u_l\otimes n_l)^{\mathcal{S}}:l\in M\}<0\text{ for }A\in\mathcal{S}\backslash\{O\}\\
\Leftrightarrow\ & O\in\operatorname{relint}_{\mathcal{S}}\operatorname{conv}\{(u_l\otimes n_l)^{\mathcal{S}}:l\in M\},\ \mathcal{M}=\mathcal{S}\\
\Leftrightarrow\ &\tfrac{1}{d}I=\sum_{l\in M}\lambda_l\,u_l\otimes n_l,\ \text{where }\lambda_l>0\text{ and }\mathbb{E}^{d^2}=\operatorname{lin}\{(u_l\otimes n_l):l\in M\}\\
\Leftrightarrow\ & L\text{ is eutactic and perfect},\ \mathcal{M}=\mathcal{S}.
\end{aligned}$$

□

Proof of Theorem 1 Theorem 1 is an immediate consequence of Lemmas 1 and 4. □

Proof of Theorem 2 Slight modifications of the proofs of Lemmas 1 to 4 show that these lemmas and, therefore, also Theorem 1 continue to hold for balls. This yields Theorem 2 on noting that the last equivalence in Theorem 2 is Voronoĭ's criterion. □

3 Maximum properties of the product of the density and its polar

This section contains relations between dual maximum properties of $\delta(C,\cdot)\,\delta(C^*,\cdot^*)$ and dual Voronoĭ type properties of L. Similar results hold for the weighted sum

$$\frac{\delta(C,\cdot)}{\delta(C,L)}+\frac{\delta(C^*,\cdot^*)}{\delta(C^*,L^*)}.$$

In analogy to Sect. 2, the dual maximum and Voronoĭ type properties for general C are defined in the context of $\mathbb{E}^{d^2}$, while those for B^d are defined in the context of $\mathbb{E}^{\frac{1}{2}d(d+1)}$.

3.1 Dual maximum and Voronoĭ type properties

Let C be an o-symmetric, smooth and strictly convex body and L a lattice in $\mathbb{E}^d$. The *polar body* C^* of C and the *polar lattice* L^* of L are defined by

$$C^*=\{y\in\mathbb{E}^d:x\cdot y\le 1\text{ for }x\in C\},\ L^*=\{m\in\mathbb{E}^d:l\cdot m\in\mathbb{Z}\text{ for }l\in L\}.$$

It is clear what is meant when we say that the product $\delta(C,\cdot)\,\delta(C^*,\cdot^*)$ is *(locally dual) semi-stationary, stationary, maximum* or *ultra-maximum* at L. The lattice L is *(dual) semi-eutactic*, *eutactic* or *perfect* with respect to C if

$$\sum_{l\in M}\lambda_l\, u_l\otimes n_l=\sum_{m\in M^*}\mu_m\, p_m\otimes v_m\neq O\ \text{ with suitable }\lambda_l,\mu_m\begin{Bmatrix}\geq 0\\ >0\end{Bmatrix},$$
$$\text{resp. }\mathbb{E}^{d^2}=\operatorname{lin}\bigl(\{u_l\otimes n_l: l\in M\}\cup\{p_m\otimes v_m: m\in M^*\}\bigr).$$

Here, M and M^* are the sets of minimum points of L with respect to C and of L^* with respect to C^*. The vectors n_l, u_l and p_m, v_m are assigned to $l\in M$ and $m\in M^*$, respectively, as in Sect. 2. Note that the order of the factors of the tensor products on the right side is reversed.

For the corresponding notions for B^d replace $\mathbb{E}^{d^2}$ by $\mathbb{E}^{\frac{1}{2}d(d+1)}$. Note that in this case, $u_l=n_l=l/\|l\|,\ v_m=p_m=m/\|m\|$.

3.2 Relations between dual maximum and dual Voronoĭ type properties

3.2.1 The general case

Theorem 3 *The following statements hold:*

(i) $\delta(C,\cdot)\,\delta(C^*,\cdot^*)$ *is dual semi-stationary at* $L\Leftrightarrow L$ *is dual semi-eutactic with respect to* C.
(ii) $\delta(C,\cdot)\,\delta(C^*,\cdot^*)$ *is dual stationary at* $L\Leftrightarrow M=\{\pm l\},\ M^*=\{\pm m\}$ *and* $u_l\otimes n_l=p_m\otimes v_m$.
(iii) $\delta(C,\cdot)\,\delta(C^*,\cdot^*)$ *is dual ultra-maximum at* $L\Leftrightarrow L$ *is dual eutactic and dual perfect with respect to* C.

3.2.2 The case of balls

Theorem 3 continues to hold with slight modifications also for $C=B^d$. The second equivalence in (iii) has been proved by Bergé and Martinet [7]. Also, for these results, the formulation of the versions for positive definite quadratic forms is left to the reader.

Theorem 4 *The following statements hold:*

(i) $\delta(B^d,\cdot)\,\delta(B^d,\cdot^*)$ *is dual semi-stationary at* $L\Leftrightarrow L$ *is dual semi-eutactic with respect to* B^d.
(ii) $\delta(B^d,\cdot)\,\delta(B^d,\cdot^*)$ *is dual stationary at* $L\Leftrightarrow M=\{\pm l\},\ M^*=\{\pm m\}$ *and* l,m *are linearly dependent.*
(iii) $\delta(B^d,\cdot)\,\delta(B^d,\cdot^*)$ *is dual ultra-maximum at* $L\Leftrightarrow L$ *is dual eutactic and dual perfect with respect to* $B^d\Leftrightarrow\delta(B^d,\cdot)\,\delta(B^d,\cdot^*)$ *is dual maximum at* L.

3.3 Proofs of the duality theorems

The background of the duality results in this section is similar to the earlier one: The different dual Voronoĭ type properties of L are related to the properties that two convex polytopes $\mathcal{P},\mathcal{Q}$ in $\mathcal{S}$ span or do not span the subspace $\mathcal{S}$ of $\mathbb{E}^{d^2}$, have intersecting relative interiors, or are singletons and coincide. These properties are equivalent to different maximum properties of the product of the densities.

First, some tools are put together. From

$$\begin{aligned}
&\|(I+A)l\|_C = \|l+Al\|_C = \|l\|_C\big(1 + A\cdot u_l\otimes n_l + o(\|A\|)\big),\\
&\|(I+A)^{-T}m\|_{C^*} = \|m - A^T m + A^{T2}m - + \cdots\|_{C^*}\\
&\quad = \|m\|_{C^*}\big(1 - A^T p_m\cdot v_m + o(\|A\|)\big) = \|m\|_{C^*}\big(1 - A\cdot p_m\otimes v_m + o(\|A\|)\big)\\
&\quad \text{as } A\to O,\ A\in\mathcal{S},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\varrho\big(C,(I+A)L\big) = \varrho(C,L)\big(1+\min\{A\cdot u_l\otimes n_l : l\in M\} + o(\|A\|)\big)\\
&\varrho\big(C^*,\big((I+A)L\big)^*\big) = \varrho(C^*,L^*)\big(1+\min\{-A\cdot p_m\otimes v_m : m\in M^*\}\\
&\qquad + o(\|A\|)\big) \text{ as } A\to O,\ A\in\mathcal{S}.
\end{aligned}$$

This together with the definition of δ and the equality

$$\det(I+A) = 1 - \frac{1}{2}A\cdot A^T + o(\|A\|^3) \text{ as } A\to O,\ A\in\mathcal{S}$$

yields

$$\begin{aligned}
\frac{\delta\big(C,(I+A)L\big)}{\delta(C,L)} &= \big(1+\min\{A\cdot u_l\otimes n_l : l\in M\} + o(\|A\|)\big)^d\\
&\quad\times\det(I+A)^{-1},\\
\frac{\delta\big(C^*,\big((I+A)L\big)^*\big)}{\delta(C^*,L^*)} &= \big(1+\min\{-A\cdot p_m\otimes v_m : m\in M^*\} + o(\|A\|)\big)^d\\
&\quad\times\det\big((I+A)^{-T}\big)^{-1} \text{ as } A\to O, A\in\mathcal{S},
\end{aligned}$$

and thus,

$$\begin{aligned}
(2)\quad &\frac{\delta\big(C,(I+A)L\big)\,\delta\big(C^*,\big((I+A)L\big)^*\big)}{\delta(C,L)\,\delta(C^*,L^*)}\\
&= 1 + d\min\{A\cdot u_l\otimes n_l : l\in M\} + d\min\{-A\cdot p_m\otimes v_m : m\in M^*\} + o(\|A\|)\\
&\quad \text{as } A\to O, A\in\mathcal{S}.
\end{aligned}$$

Lemma 5 *The following properties of $\delta(C,\cdot)\,\delta(C^*,\cdot^*)$ and L are equivalent, where $\mathcal{E}$ and $\mathcal{E}^*$ are linear subspaces of $\mathcal{S}$ to be defined in the proof:*

(i) *L is dual semi-eutactic with respect to C.*
(ii) *$\delta(C,\cdot)\,\delta(C^*,\cdot^*)$ is dual semi-stationary at L.*
(iii)

$$\frac{\delta\big(C,(I+A)L\big)\,\delta\big(C^*,\big((I+A)L\big)^*\big)}{\delta(C,L)\,\delta(C^*,L^*)} \le 1 - \text{const}\|A^{\mathcal{E}}\| - \text{const}\|A^{\mathcal{E}^*}\| + o(\|A\|),$$
as $A\to O,\ A\in\mathcal{S}$.

Each lattice for which $\delta(C,\cdot)\,\delta(C^*,\cdot^*)$ attains its absolute maximum is an example of a dual semi-eutactic lattice.

Proof of Lemma 5 $(i) \Rightarrow$(iii) $\Rightarrow$(ii) $\Rightarrow$(i):

$$
\begin{aligned}
& L \text{ is dual semi-eutactic} \\
\Rightarrow & \sum_{l\in E} \lambda_l\, u_l \otimes n_l = \sum_{m\in E^*} \mu_m\, p_m \otimes v_m \neq O, \\
& \text{where } E = \{l \in M : \lambda_l > 0\},\ E^* = \{m \in M^* : \mu_m > 0\} \\
& \sum_{l\in E} \lambda_l = \sum_{l\in E} \lambda_l\, I \cdot u_l \otimes n_l = \sum_{m\in E^*} \mu_m\, I \cdot p_m \otimes v_m = \sum_{m\in E^*} \mu_m. \\
& \text{Thus we may assume that } \sum_{l\in E} \lambda_l = \sum_{m\in E^*} \mu_m = 1 \\
\Rightarrow & U \in \operatorname{relint}\operatorname{conv}\{(u_l \otimes n_l)^{\mathcal{S}} : l \in E\} \cap \operatorname{relint}\operatorname{conv}\{(p_m \otimes v_m)^{\mathcal{S}} : m \in E^*\} \\
& \text{where } U = \sum_{l\in E} \lambda_l\, (u_l \otimes n_l)^{\mathcal{S}} = \sum_{m\in E^*} \mu_m\, (p_m \otimes v_m)^{\mathcal{S}} \in \mathcal{S} \\
\Rightarrow & O \in \operatorname{relint}_{\mathcal{E}}\operatorname{conv}\{(u_l \otimes n_l)^{\mathcal{S}} - U : l \in E\} \cap \operatorname{relint}_{\mathcal{E}^*}\operatorname{conv}\{(p_m \otimes v_m)^{\mathcal{S}} - U : m \in E^*\} \\
& \text{where } \mathcal{E} = \operatorname{lin}\{(u_l \otimes n_l)^{\mathcal{S}} - U : l \in E\},\ \mathcal{E}^* = \operatorname{lin}\{(p_m \otimes v_m)^{\mathcal{S}} - U : m \in E^*\} \\
\Rightarrow & \min\{A \cdot ((u_l \otimes n_l)^{\mathcal{S}} - U) : l \in E\} \leq -\text{const}\|A\| \text{ for } A \in \mathcal{E} \\
& \min\{A \cdot (-(p_m \otimes v_m)^{\mathcal{S}} + U) : m \in E^*\} \leq -\text{const}\|A\| \text{ for } A \in \mathcal{E}^* \\
\Rightarrow & \min\{A \cdot ((u_l \otimes n_l)^{\mathcal{S}} - U) : l \in E\} \leq -\text{const}\|A^{\mathcal{E}}\| \text{ for } A \in \mathcal{S} \\
& \min\{A \cdot (-(p_m \otimes v_m)^{\mathcal{S}} + U) : m \in E^*\} \leq -\text{const}\|A^{\mathcal{E}^*}\| \text{ for } A \in \mathcal{S} \\
\Rightarrow & \frac{\delta\big(C, (I+A)L\big)\, \delta\big(C^*, \big((I+A)L\big)^*\big)}{\delta(C, L)\, \delta(C^*, L^*)} \\
& \quad = 1 + d\min\{A \cdot u_l \otimes n_l : l \in M\} + d\min\{-A \cdot p_m \otimes v_m : m \in M^*\} + o(\|A\|) \\
& \quad \leq 1 + d\min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in E\} + d\min\{-A \cdot (p_m \otimes v_m)^{\mathcal{S}} : m \in E^*\} + o(\|A\|) \\
& \quad = 1 + d\min\{A \cdot ((u_l \otimes n_l)^{\mathcal{S}} - U) : l \in E\} \\
& \qquad + d\min\{-A \cdot ((p_m \otimes v_m)^{\mathcal{S}} - U) : m \in E^*\} + o(\|A\|) \\
& \quad \leq 1 - \text{const}\|A^{\mathcal{E}}\| - \text{const}\|A^{\mathcal{E}^*}\| + o(\|A\|) \leq 1 + o(\|A\|) \\
& \quad \text{as } A \to O,\ A \in \mathcal{S}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow & \ \delta(C, \cdot)\, \delta(C, \cdot^*) \text{ is dual semi-stationary at } L \\
\Rightarrow & \ \frac{\delta\big(C, (I+A)L\big)\, \delta\big(C^*, \big((I+A)L\big)^*\big)}{\delta(C, L)\, \delta(C^*, L^*)} \\
& \quad = 1 + d\min\{A \cdot u_l \otimes n_l : l \in M\} + d\min\{-A \cdot p_m \otimes v_m : m \in M^*\} + o(\|A\|) \\
& \quad \leq 1 + o(\|A\|) \text{ as } A \to O,\ A \in \mathcal{S} \\
\Rightarrow & \ \min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in M\} + \min\{-A \cdot (p_m \otimes v_m)^{\mathcal{S}} : m \in M^*\} \leq 0 \text{ for } A \in \mathcal{S} \\
\Rightarrow & \ \min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in M\} \leq \max\{A \cdot (p_m \otimes v_m)^{\mathcal{S}} : m \in M^*\} \text{ for } A \in \mathcal{S} \\
\Rightarrow & \ \operatorname{conv}\{(u_l \otimes n_l)^{\mathcal{S}} : l \in M\} \cap \operatorname{conv}\{(p_m \otimes v_m)^{\mathcal{S}} : m \in M^*\} \neq \emptyset \\
\Rightarrow & \ \sum_{l\in M} \lambda_l\, (u_l \otimes n_l)^{\mathcal{S}} = \sum_{m\in M^*} \mu_m\, (p_m \otimes v_m)^{\mathcal{S}} \\
& \quad \text{where } \lambda_l,\ \mu_m \geq 0,\ \sum_{l\in M} \lambda_l = \sum_{m\in M^*} \mu_m = 1 \\
\Rightarrow & \ L \text{ is dual semi-eutactic.}
\end{aligned}
$$

□

In contrast to the case of $\delta(C, \cdot)$, there are lattices for which the product $\delta(C, \cdot)\, \delta(C^*, \cdot^*)$ is dual stationary. To see this, we first show the following characterization of stationarity:

Lemma 6 *The following properties* (i) *and* (ii) *of* $\delta(C, \cdot)\, \delta(C^*, \cdot^*)$ *and* L *are equivalent:*

(i) $\delta(C, \cdot)\, \delta(C^*, \cdot^*)$ *is dual stationary at* L.
(ii) $M = \{\pm l\},\ M^* = \{\pm m\}$ *and* $u_l \otimes n_l = p_m \otimes v_m$.

Example An example of a lattice at which the product of the densities is dual stationary can be obtained as follows. We begin with a simple property of polarity:

Let L be a lattice in $\mathbb{E}^2$ with $d(L) = 1$. Then, L^* is obtained from L by a rotation about o through the angle $\frac{\pi}{2}$.

Let

$$L = B\mathbb{Z}^2, \text{ where } B = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \text{ is a basis matrix of } L, \det B = \pm 1.$$

Then,

$$\begin{aligned} L^* = B^{-T}\mathbb{Z}^2 &= \begin{pmatrix} s & -r \\ -q & p \end{pmatrix}^T \mathbb{Z}^2 = \begin{pmatrix} s & -q \\ -r & p \end{pmatrix} \mathbb{Z}^2 = \begin{pmatrix} -q & -s \\ p & r \end{pmatrix} \mathbb{Z}^2 \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & r \\ q & s \end{pmatrix} \mathbb{Z}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} L, \end{aligned}$$

where we have used the fact that, with a basis matrix, its negative and any matrix which is obtained from it by a permutation of the columns or by replacing a column by its negative is also a basis matrix of L. The next needed property is the following:

Let L, M be lattices in orthogonal subspaces of $\mathbb{E}^d$. Then, $L \oplus M$ is a lattice in $\mathbb{E}^d$ and $(L \oplus M)^* = L^* \oplus M^*$.

The proof is simple:

$$\begin{aligned} (L \oplus M)^* &= \{p + q : (p + q) \cdot (l + m) \in \mathbb{Z} \quad \text{for all } l + m \in L \oplus M\} \\ &= \{p + q : p \cdot l + q \cdot m \in \mathbb{Z} \quad \text{for all } l \in L, m \in M\} \\ &= \{p \in \operatorname{lin} L : p \cdot l \in \mathbb{Z} \quad \text{for all } l \in L\} \\ &\quad + \{q \in \operatorname{lin} M : q \cdot m \in \mathbb{Z} \quad \text{for all } m \in M\} \\ &= L^* + M^* = L^* \oplus M^*. \end{aligned}$$

Let $C = B^3$, and let L be the lattice with basis

$$\left\{ \left(\frac{\sqrt{2}}{\sqrt[4]{3}}, 0, 0 \right), \left(\frac{1}{2} \frac{\sqrt{2}}{\sqrt[4]{3}}, \frac{\sqrt[4]{3}}{\sqrt{2}}, 0 \right), (0, 0, 1) \right\}.$$

Then, $d(L) = 1$, and L is the direct sum of the lattices with the following bases:

$$\left\{ \left(\frac{\sqrt{2}}{\sqrt[4]{3}}, 0, 0 \right), \left(\frac{1}{2} \frac{\sqrt{2}}{\sqrt[4]{3}}, \frac{\sqrt[4]{3}}{\sqrt{2}}, 0 \right) \right\}, \text{ resp. } \{(0, 0, 1)\}.$$

By the above properties, the lattice L^* is obtained from the lattice L by a rotation about the x_3-axis through the angle $\frac{\pi}{2}$. Thus, the sets of minimum vectors of L and L^* both are equal to the set $\{\pm(0, 0, 1)\}$. Now, apply Lemma 6.

Proof of Lemma 6 (i)$\Leftrightarrow$(ii):

$$\begin{aligned} &\delta(C, \cdot)\, \delta(C, \cdot^*) \text{ is dual stationary at } L \\ \Leftrightarrow\ & \frac{\delta\big(C, (I + A)L\big)\, \delta\big(C^*, \big((I + A)L\big)^*\big)}{\delta(C, L)\, \delta(C^*, L^*)} \\ &\quad = 1 + d \min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in M\} \\ &\qquad + d \min\{-A \cdot (p_m \otimes v_m)^{\mathcal{S}} : m \in M^*\} + o(\|A\|) \\ &\quad = 1 + o(\|A\|) \text{ as } A \to O, A \in \mathcal{S} \\ \Leftrightarrow\ & \min\{A \cdot (u_l \otimes n_l)^{\mathcal{S}} : l \in M\} = \max\{A \cdot (p_m \otimes v_m)^{\mathcal{S}} : m \in M^*\} \text{ for } A \in \mathcal{S} \\ \Leftrightarrow\ & \operatorname{conv}\{(u_l \otimes n_l)^{\mathcal{S}} : l \in M\} = \operatorname{conv}\{(p_m \otimes v_m)^{\mathcal{S}} : m \in M^*\} \text{ are singletons} \\ \Leftrightarrow\ & M = \{\pm l\}, M^* = \{\pm m\}, \text{ say, where } (u_l \otimes n_l)^{\mathcal{S}} = (p_m \otimes v_m)^{\mathcal{S}} \\ \Leftrightarrow\ & M = \{\pm l\}, M^* = \{\pm m\}, u_l \otimes n_l = p_m \otimes v_m. \end{aligned}$$

□

Lemma 7 *The following properties of* $\delta(C,\cdot)\,\delta(C^*,\cdot^*)$ *and* L *are equivalent, where* $\mathcal{M}$ *and* $\mathcal{M}^*$ *are linear subspaces of* $\mathcal{S}$ *to be defined in the proof:*

(i) L *is dual eutactic with respect to* C.

(ii)

$$\frac{\delta\big(C,(I+A)L\big)\,\delta\big(C^*,\big((I+A)L\big)^*\big)}{\delta(C,L)\,\delta(C^*,L^*)} \le 1-\mathrm{const}\|A^{\mathcal{M}}\|-\mathrm{const}\|A^{\mathcal{M}^*}\|+o(\|A\|) \textit{as } A\to O,\ A\in\mathcal{S}.$$

(iii)

$$\frac{\delta\big(C,(I+A)L\big)\,\delta\big(C^*,\big((I+A)L\big)^*\big)}{\delta(C,L)\,\delta(C^*,L^*)} \left\{\begin{array}{l}\le 1-\mathrm{const}\|A^{\mathcal{M}}\|-\mathrm{const}\|A^{\mathcal{M}^*}\|+o(\|A\|)\\ \ge 1-\mathrm{const}\|A^{\mathcal{M}}\|-\mathrm{const}\|A^{\mathcal{M}^*}\|+o(\|A\|)\end{array}\right\}$$

as $A\to O,\ A\in\mathcal{S}$.

Proof of Lemma 7 Since the proof is similar to that of the previous lemma, several details are omitted.

(i)⇒(iii)⇒(ii)⇒(i):

$$\begin{aligned}
&L \text{ is dual eutactic}\\
&\dots\\
\Rightarrow\ & U\in\operatorname{relint}\operatorname{conv}\{(u_l\otimes n_l)^{\mathcal{S}}:l\in M\}\cap\operatorname{relint}\operatorname{conv}\{(p_m\otimes v_m)^{\mathcal{S}}:m\in M^*\}\\
& O\in\operatorname{relint}_{\mathcal{M}}\operatorname{conv}\{(u_l\otimes n_l)^{\mathcal{S}}-U:l\in M\}\cap\operatorname{relint}_{\mathcal{M}^*}\operatorname{conv}\{(p_m\otimes v_m)^{\mathcal{S}}-U:m\in M^*\},\\
&\quad \text{where } \mathcal{M}=\operatorname{lin}\{(u_l\otimes n_l)^{\mathcal{S}}-U:l\in M\},\ \mathcal{M}^*=\operatorname{lin}\{(p_m\otimes v_m)^{\mathcal{S}}-U:m\in M^*\}\\
&\dots\\
\Rightarrow\ & -\mathrm{const}\|A^{\mathcal{M}}\|\le\min\{A\cdot((u_l\otimes n_l)^{\mathcal{S}}-U):l\in M\}\le-\mathrm{const}\|A^{\mathcal{M}}\|,\\
& -\mathrm{const}\|A^{\mathcal{M}^*}\|\le\min\{-A\cdot(p_m\otimes v_m)^{\mathcal{S}}-U\}:m\in M^*\}\le-\mathrm{const}\|A^{\mathcal{M}^*}\|\\
&\quad \text{for } A\in\mathcal{S}\\
\Rightarrow\ & \frac{\delta\big(C,(I+A)L\big)\,\delta\big(C^*,\big((I+A)L\big)^*\big)}{\delta(C,L)\,\delta(C^*,L^*)}\\
&\quad =1+d\min\{A\cdot(u_l\otimes n_l)^{\mathcal{S}}:l\in M\}+d\min\{-A\cdot(p_m\otimes v_m)^{\mathcal{S}}:m\in M^*\}+o(\|A\|)\\
&\quad =1+d\min\{A\cdot((u_l\otimes n_l)^{\mathcal{S}}-U):l\in M\}\\
&\qquad +d\min\{-A\cdot((p_m\otimes v_m)^{\mathcal{S}}-U):m\in M^*\}+o(\|A\|)\\
&\quad \left\{\begin{array}{l}\le 1-\mathrm{const}\|A^{\mathcal{M}}\|-\mathrm{const}\|A^{\mathcal{M}^*}\|+o(\|A\|)\\ \ge 1-\mathrm{const}\|A^{\mathcal{M}}\|-\mathrm{const}\|A^{\mathcal{M}^*}\|+o(\|A\|)\end{array}\right\} \text{ as } A\to O,A\in\mathcal{S}\\
\Rightarrow\ & \frac{\delta\big(C,(I+A)L\big)\,\delta\big(C^*,\big((I+A)L\big)^*\big)}{\delta(C,L)\,\delta(C^*,L^*)}\\
&\quad =1+d\min\{A\cdot(u_l\otimes n_l)^{\mathcal{S}}:l\in M\}+d\min\{-A\cdot(p_m\otimes v_m)^{\mathcal{S}}:m\in M^*\}+o(\|A\|)\\
&\quad \le 1-\mathrm{const}\|A^{\mathcal{M}}\|-\mathrm{const}\|A^{\mathcal{M}^*}\|+o(\|A\|) \text{ as } A\to O,\ A\in\mathcal{S}\\
\Rightarrow\ & \min\{A\cdot(u_l\otimes n_l)^{\mathcal{S}}:l\in M\}+\min\{-A\cdot(p_m\otimes v_m)^{\mathcal{S}}:m\in M^*\}<0 \text{ for } A\in\mathcal{M}\backslash\{O\}\\
\Rightarrow\ & \operatorname{relintconv}\{(u_l\otimes n_l)^{\mathcal{S}}:l\in M\}\cap\operatorname{relint}\{(p_m\otimes v_m)^{\mathcal{S}}:m\in M^*\}\neq\emptyset\\
&\dots\\
\Rightarrow\ & L \text{ is dual eutactic.}
\end{aligned}$$

□

Lemma 8 *The following properties of $\delta(C,\cdot)\,\delta(C^*,\cdot^*)$ and L are equivalent:*

(i) *L is dual eutactic and dual perfect with respect to C.*
(ii) *$\delta(C,\cdot)\,\delta(C^*,\cdot^*)$ is dual ultra-maximum at L.*
(iii) $$\frac{\delta\big(C,(I+A)L\big)\,\delta\big(C^*,\big((I+A)L\big)^*\big)}{\delta(C,L)\,\delta(C^*,L^*)}\left\{\begin{matrix}\le 1-\mathrm{const}\|A\|\\ \ge 1-\mathrm{const}\|A\|\end{matrix}\right\} \text{ as } A\to O,\ A\in\mathcal{S}.$$

As for $\delta(C,\cdot)$, the question remains open whether there are convex bodies C with dual eutactic and dual perfect lattices. See Sect. 4.

Proof of Lemma 8 Let $\mathcal{M},\mathcal{M}^*$ be as in the proof of Lemma 7.
(i)⇒(iii)⇒(ii)⇒(i):

$$\begin{aligned}
&L \text{ is dual eutactic and dual perfect}\\
\Rightarrow\ &\mathcal{M}+\mathcal{M}^*=\mathcal{S}\\
&\dots\\
\Rightarrow\ &\frac{\delta\big(C,(I+A)L\big)\,\delta\big(C^*,\big((I+A)L\big)^*\big)}{\delta(C,L)\,\delta(C^*,L^*)}\\
&\dots\\
&\quad\left\{\begin{matrix}\le 1-\mathrm{const}\|A^{\mathcal{M}}\|-\mathrm{const}\|A^{\mathcal{M}^*}\|+o(\|A\|)\le 1-\mathrm{const}\|A\|\\ \ge 1-\mathrm{const}\|A^{\mathcal{M}}\|-\mathrm{const}\|A^{\mathcal{M}^*}\|+o(\|A\|)\ge 1-\mathrm{const}\|A\|\end{matrix}\right\}\\
&\quad\text{as } A\to O,\ A\in\mathcal{S}\\
\Rightarrow\ &\frac{\delta(C,(I+A)L)\,\delta(C^*,((I+A)L)^*)}{\delta(C,L)\,\delta(C^*,L^*)}\le 1-\mathrm{const}\|A\| \text{ as } A\to O,\ A\in\mathcal{S}\\
\Rightarrow\ &\frac{\delta\big(C,(I+A)L\big)\,\delta\big(C^*,\big((I+A)L\big)^*\big)}{\delta(C,L)\,\delta(C^*,L^*)}\\
&\quad=1+d\min\{A\cdot(u_l\otimes n_l)^{\mathcal{S}}:l\in M\}\\
&\qquad+d\min\{-A\cdot(p_m\otimes v_m)^{\mathcal{S}}:m\in M^*\}+o(\|A\|)\\
&\quad\le 1-\mathrm{const}\|A\| \text{ as } A\to O,\ A\in\mathcal{S}\\
\Rightarrow\ &\min\{A\cdot(u_l\otimes n_l)^{\mathcal{S}}:l\in M\}+\min\{-A\cdot(p_m\otimes v_m)^{\mathcal{S}}:m\in M^*\}<0 \text{ for } A\in\mathcal{S}\backslash\{O\}\\
&\dots\\
\Rightarrow\ &\sum_{l\in M}\lambda_l\,u_l\otimes n_l=\sum_{m\in M^*}\mu_m\,p_m\otimes v_m,\ \lambda_l,\mu_m>0,\ \sum_{l\in M}\lambda_l=\sum_{m\in M^*}\mu_m=1\\
&\mathcal{M}+\mathcal{M}^*=\mathcal{S}\\
\Rightarrow\ &L \text{ is dual eutactic and dual perfect.}
\end{aligned}$$

□

Proof of Theorem 3 Theorem 3 is an immediate consequence of Lemmas 5, 6, and 8. □

Proof of Theorem 4 For balls, Lemmas 5, 6 and 8 hold with slight modifications. This together with the extension of Voronoĭ's criterion to the duality case by Bergé and Martinet [7] yields Theorem 4. □

4 Open problems

The results in this article raise a series of problems. We state four, all related to the density $\delta(C,\cdot)$, and leave it to the reader to formulate the corresponding duality problems.

4.1 Maximum and ultra-maximum density

When, in the following, we speak of a *generic* o-symmetric convex body, this is meant in the *Baire category sense.* A result of Klee [31] and the author [15] says that a generic o-symmetric convex body is smooth and strictly convex. See [18] for information on Baire categories in the context of convex geometry.

By an estimate of Minkowski [35], the kissing number of any lattice packing for o-symmetric, strictly convex C is at most $2^{d+1} - 2$. Now, noting that $2^{d+1} - 2 < 2d^2$ for $d = 2, 3, 4$, Lemma 4 implies that o-symmetric, smooth and strictly convex bodies C with ultra-maximum lattice packing density exist, if at all, only in dimensions $d \geq 5$. This explains the difficulty to specify examples of ultra-maximum lattice packings. In spite of this, we believe that there are examples in abundance and formulate this as follows:

Problem 1 *Show that in all sufficiently high dimensions, for a generic o-symmetric convex body C, the lattices at which $\delta(C, \cdot)$ is maximum, respectively, ultra-maximum, coincide.*

4.2 Characterization of packings with maximum density

The answer to Problem 1 may or may not be affirmative. In any case, there are convex bodies with maximum but not ultra-maximum lattice packing density. One example is the lattice packings of B^d for which the density is maximum. By Theorem 2, these lattice packings have ultra-maximum density in the sense of the definition for balls. In the sense of the definitions of maximality, respectively, ultra-maximality for convex bodies, these lattice packings have only maximum but not ultra-maximum density (as can be shown). Therefore, the following problem makes sense.

Problem 2 *Given an o-symmetric convex body, characterize the lattices at which the density is maximum but not ultra-maximum.*

The margin between lattice packings of an o-symmetric convex body of maximum, respectively, ultra-maximum density, seems to be small. It is unclear whether simple Voronoĭ type concepts can distinguish between these two properties. For this reason, a solution of Problem 2 might be difficult to obtain.

4.3 Kissing numbers of packings with global maximum density

What is the situation in the special case of lattices which provide lattice packings of global maximum density? The author [16] has shown that for a generic o-symmetric convex body, the kissing number of any lattice packing of global maximum density is at most $2d^2$. By Corollary 1, the kissing number of such a lattice packing is at least $2d^2$ if, in addition, the lattice packing has ultra-maximum density. Finally, an estimate of Swinnerton-Dyer [45] implies that a lattice which provides a packing of global maximum density has kissing number at least $d(d+1)$. The question arises, where between $d(d+1)$ and $2d^2$ is the kissing number of packings of global maximum density of a generic convex body? These remarks lead to the following problem.

Problem 3 *Show that, in all sufficiently high dimensions, for a generic o-symmetric convex body C, a lattice L at which the density $\delta(C, \cdot)$ has a global maximum has the following properties:*

(i) *L is eutactic and perfect with respect to C, and therefore, the corresponding packing has ultra-maximum density at L.*
(ii) *L has kissing number $2d^2$.*

4.4 Uniqueness of packings with maximum and ultra-maximum density

The problem whether a (generic) convex body has a unique lattice packing of global maximum density seems to have escaped attention so far. First, pertinent results are as follows: If in $\mathbb{E}^d$ for a generic o-symmetric convex body all lattice packings of maximum density are connected, then for a generic convex body, the lattice packing of maximum density is unique. This is the case in $\mathbb{E}^d$, $d = 2, 3$. For general d, there is a constant $a(d)$ such that for generic C the number of different lattice packings of C of globally maximum density is at most $a(d)$. See Gruber [23]. We conclude with the following, more general question.

Problem 4 *How many lattice packings of maximum, respectively, ultra-maximum density, does a generic o-symmetric convex body admit? One, a bounded number, or infinitely many?*

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