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Insider trading and risk aversion ☆

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Abstract

This paper is a continuous time version of Holden and Subrahmanyam (Economics Letters 44 (1994) 181). The paper extends Kyle (Econometrica 53 (1985) 1315) by introducing risk aversion on the side of the monopolist informed trader and allows for the liquidity traders instantaneous demand to depend on cost of trading, as well as on the risk of the stock. The main result of the paper is that, in equilibrium, the price pressure decreases with time regardless of the elasticity of the liquidity demand function. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Kyle (1985) models a sequential batch market with a strategic, informed trader who submits market orders to competitive, risk-neutral market makers. His orders are aggregated with liquidity traders' orders, so the market makers observe only the cumulative market order imbalance. This anonymity of trade allows the informed trader to exploit his information. The market makers offset their losses to the informed trader by charging the liquidity traders a premium for immediacy. This premium is measured by the price pressure, λ, or the market depth, which is the reciprocal of λ.

The informed trader, in the Kyle model, runs the risk that profitable trading opportunities will be lost as the liquidity traders shift prices. This risk is ignored

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when the informed trader is risk neutral. Consequently, as was pointed out by Kyle (1985), a time varying price pressure would allow the informed trader to use profitable destabilization schemes. In contrast, the risk liquidity traders impose on the informed trader matters when the informed trader is risk averse. This risk motivates the risk-averse informed trader to trade more intensely at the early stages, thus releasing more information than a risk neutral informed trader would. Early release of information reduces the information asymmetry and allows the market makers to decrease λ with time as they become more confident about the prices they set.

We show that the rate at which the market depth (reciprocal of λ) increases with time is equal to the product of the informed trader's coefficient of risk aversion and the instantaneous volatility of liquidity trading. Equivalently, the depth at time t is equal to some constant plus the product of the coefficient of risk aversion and the cumulative volatility. Since the latter quantity can be approximated by a series of lagged instantaneous volatilities, our model shows that depth is determined by past volatility of trade: the higher the volatility of trading is, the higher is the future depth. In view of our previous discussion, this prediction is natural. The higher the volatility of liquidity trading, the greater is the informed trader's intensity of trade, the faster information is incorporated into prices, and the faster λ decreases (and depth increases) with time.

We also consider in our model elastic liquidity demand functions from a certain class. This class includes liquidity demand functions that are sensitive to λ, as in Admati and Pfleiderer (1988), and to the level of risk of the asset, i.e., the conditional variance of the asset, as in Massoud and Bernhardt (1999). Unlike these papers, however, the liquidity demand function is not endogenous.

Holden and Subrahmanyam (1994) study a model similar to ours in a discrete time setting with an inelastic liquidity demand function.[1] Our work is also an extension of Back (1992), who considers only the case of a risk neutral, informed trader in a continuous time setting. He shows that for general distributions of the liquidation value, λ has to be a martingale. Back and Pedersen (1998) show that, under the assumption of normality, λ is a martingale when the risk neutral, informed trader learns his information through time. Back, Cao and Willard (2000) study the risk neutral case with N informed traders. In their model, λ first decreases with time and later increases at an increasing rate until the information is publicly revealed. Massoud and Bernhardt (1999) provide a numerical solution to a three-period model with an endogenous liquidity demand function and a risk neutral informed trader. In their model, the motivation for liquidity trading is hedging liquidity shocks. Massoud and Bernhardt show that for different levels of risk aversion of the liquidity traders, λ exhibits different time patterns. In particular, both increasing and decreasing patterns of λ are consistent with their finite period

[1] Holden and Subrahmanyam (1994) also model the case of N traders with identical information. They suggest that as in the risk neutral case (see Holden and Subrahmanyam, 1992; Back et al., 2000) there is no linear equilibrium with long lived information. Therefore, we do not consider this case here.

model.[2] We show, in this paper, that a linear equilibrium does not exist in the continuous time analogue of the liquidity demand functions they consider. Moreover, in contrast with their results, λ is decreasing with time even when liquidity demand is elastic, though the rate at which λ decreases does depend on the form of the liquidity demand function.

Mendelson and Tunca (2000) also model informed trading with endogenized liquidity trading. In their model, the informed trader is risk neutral and the motivation for trade of the risk averse liquidity traders is shocks to their individual beliefs that make them value the asset above or below the market price. Mendelson and Tunca show that when the market is open continuously, the rate at which λ changes over time equals the interest rate. Thus, the rate of change is independent of the elasticity of the liquidity demand. In particular, λ is constant when interest rate is set to zero.

The paper is organized as follows. Section 2 describes the model and the equilibrium. Section 3 presents the comparative statics. Section 4 discusses the empirical implications of the model, and Section 5 concludes with a brief summary.

2. The model and equilibrium

We consider a market with one risky asset and one non-risky asset (numéraire). The price of the risky asset, at time one, is denoted by v, and is normally distributed with mean p_0, and variance ϕ^2.

The market is open continuously and traders submit their orders to competitive, risk neutral market makers. There are an unspecified number of liquidity (noise) traders who submit orders and we denote their cumulative order flow at time t by z_t. At time zero, a single risk-averse, informed trader learns the realization of v. He trades to maximize his expected utility and we assume that his utility function has the form $-\exp(-Aw)$, where A is the coefficient of risk aversion. We denote by x_t the amount of the risky asset held by the informed trader at time t, and we let $y_t = x_t + z_t$ be the cumulative market order flow at time t.

Anonymity of trades implies that the cumulative market order flow, y, is the only information available to the market makers. Consequently, competition among market makers implies that at time t, the market is cleared at a price, p_t, equal to $E[\tilde{v}|(y_s)_{s\leqslant t}]$.

We are looking for an equilibrium in which: (i) the price rule is given by

$$p_t = p_0 + \int_0^t \lambda(s)\,\mathrm{d}y_s \tag{1}$$

[2] Notice that in a three-period model, the liquidity trader faces an enormous execution risk that is positively correlated with the liquidation value. This risk causes the risk-averse liquidity trader to over-rebalance his portfolio. In contrast, execution risk does not exist in the continuous time framework.

for some deterministic function $\lambda(\cdot)$, and (ii) the informed trader's optimal strategy has the form

$$\mathrm{d}x = \beta(t)(\tilde{v} - p_t)\,\mathrm{d}t \tag{2}$$

for some deterministic function $\beta(t)$. As in Kyle (1985), $\lambda(t)$ is defined as the price pressure and $\beta(t)$ is defined as the informed trader's intensity of trade.

To allow the model to accommodate elastic liquidity demand functions, we assume that the instantaneous demand of liquidity trades is given by

$$\mathrm{d}z_t = \sigma(\lambda(t), \Sigma(t), t)\,\mathrm{d}B_t, \tag{3}$$

where B_t is a standard Brownian motion, σ is a strictly positive function, and $\Sigma(t)$ is the conditional variance of $\tilde{v}$ given $(y_s)_{s\leqslant t}$. Notice that we do not allow σ to depend on the price, p_t.[3] However, we do know that under the assumption that they can trade only once, liquidity traders with mean variance preferences do not condition their demand on the current price. Furthermore, only this case has been considered in the finite-period literature. Massoud and Bernhardt (1999) endogenize the liquidity traders' demand function and show that σ is increasing with Σ and decreasing with λ. Liquidity demand increases with the conditional variance, Σ, because the higher it is, the greater is the need to hedge positions in the stock. The demand decreases with λ, because it represents a transaction cost (see also Admati and Pfleiderer (1988)). In what follows, we do not impose any restriction on the function σ apart from being strictly positive.

Under the price rule (1), the informed trader, simply by monitoring the price process, can learn the realization of B. We denote by $\mathbf{F} := \{\mathscr{F}_t \mid 0 \leqslant t \leqslant 1\}$ the augmentation of the filtration generated by v and the process B_t, and we require that the informed trader's strategy be a semimartingale with respect to $\mathbf{F}$.

We define the informed trader's wealth at time t, denoted by w_t^x, as the amount of numéraire plus vx_t. By the self-financing condition, and the rule for integration by parts for semimartingales, the wealth process evolves according to[4]

$$w_1^x = w_t^x + \int_t^1 (v - p_{s-}^y)\,\mathrm{d}x_t - ([p^y, x]_1 - [p^y, x]_t). \tag{4}$$

We denote by χ the class of semimartingales with respect to $\mathbf{F}$ that satisfies $w^x \geqslant -M$ for some positive constant M. The meaning of this constraint is that we allow the trader to buy only up to a certain limit if the price is higher than v (or sell even if it is lower than v).[5]

A Linear Equilibrium is defined as a pair $(\lambda(\cdot), \beta(\cdot))$ such that

(i) Given the trading strategy $x_t = \int_0^t \beta(u)(\tilde{v} - p_u)\,\mathrm{d}u$, the price rule is competitive, i.e.,

[3] Even for the risk neutral case, a linear equilibrium does not exist when σ depends on p_t.

[4] Under the price rule (1), for every strategy that is a semimartingale with respect to $\mathbf{F}$, the price p is a semimartingale as well.

[5] This restriction plays the role of ruling out doubling strategies.

$$p_0 + \int_0^t \lambda(s)\,\mathrm{d}y_s = E[v|(y_s)_{s\leqslant t}]. \tag{5}$$

(ii) Given $\lambda(\cdot)$, the trading strategy $\int_0^t \beta(u)(\tilde{v} - p_u)\,\mathrm{d}u$ maximizes

$$E[-\exp(-A(w))] \tag{6}$$

over all processes x in χ.

The next theorem shows that to construct a linear equilibrium, we have to solve a boundary-value problem.

Theorem 1. *Let $\lambda(t)$ and $\Sigma(t)$ be a solution to the system of differential equations*:

$$\left(\frac{1}{\lambda(t)}\right)' = A\sigma^2(\lambda(t), \Sigma(t), t) \tag{7}$$

$$\Sigma'(t) = -\lambda^2(t)\sigma^2(\lambda(t), \Sigma(t), t) \tag{8}$$

with the two boundary conditions: $\Sigma(0) = \phi^2$ *and* $\Sigma(1) = 0$. *Set*

$$\beta(t) = \frac{\lambda(t)\sigma_t^2}{\Sigma(t)}. \tag{9}$$

The pair $(\lambda(\cdot), \beta(\cdot))$ *forms a linear equilibrium in which the function* $\Sigma(t)$ *is the conditional variance of* $\tilde{v}$ *given* $\mathcal{F}_t^y$.

The differential equation (7) is a local condition for the existence of an optimal finite solution to the informed trader's problem. In particular, when $A = 0$, which represents the risk neutral case, the price pressure, λ, is constant as in the original Kyle model. Indeed, unless the price pressure is constant, a risk neutral trader will concentrate all his trade at the time in which the cost of trading, as measured here by λ, is the lowest, and a finite solution to the problem does not exist. In contrast, when the trader is risk averse, the cost of trading is not his sole concern. Indeed, although equation (7) shows that the cost of trading decreases with time, the trader does not find it optimal to postpone his trade. The fear that future profitable trading opportunities will be lost as liquidity traders shift prices deters the informed trader from postponing his trade. Eq. (7) shows that when the reciprocal of $\lambda(t)$ increases at a rate that is equal to the product of the coefficient of risk aversion, A, and the instantaneous variance of liquidity demand order imbalance, σ, the two effects offset each other.

Eq. (8), with its boundary conditions, and Eq. (9) appear in every continuous time variant of the Kyle model. The boundary condition $\Sigma(0) = \phi$ is natural, and the boundary condition $\Sigma(1) = 0$ simply states that the informed trader never finds it optimal to leave money on the table; i.e., his optimal strategy must reveal all his information prior to the public announcement. Eq. (9) shows that the ratio of informed to uniformed trading; i.e., $\beta(t)/\sigma_t^2$, which measures the extent of the adverse selection problem is equal to the ratio $\lambda(t)/\Sigma(t)$. To understand this relation, notice that Σ measures the confidence market makers have in the price they set. For a given Σ, increasing the ratio β/σ^2 results in a higher price impact of order flow; i.e., higher $\lambda(t)$.

The conditions for equations (7)–(8) are given at the boundaries. The standard method to solving this type of boundary-value problem is the shooting method. We start with a guess for the value of $\lambda(1)$. We then begin solving the differential equations (7) and (8) from time 1 backward toward time 0 with the initial conditions $\Sigma(1) = 0$ and $\lambda(1)$ equal to our guess. We accept the solution if the value $\Sigma(0)$ is equal to ϕ^2. If $\Sigma(0)$ is not equal to ϕ^2, we try another guess.

Our main theorem is a standard verification theorem. We still need to verify existence of the equilibrium by showing that a solution to the system of equations (7)–(8) exists. For example, we can verify that a linear equilibrium does not exist in the continuous time analogue of Massoud and Bernhardt (1999). Indeed, in the continuous time analogue of that model,

$$\sigma(\lambda, \Sigma) = \frac{\eta\Sigma(t)}{2(2\lambda(t) + \eta\Sigma(t))}, \tag{10}$$

where η is the level of risk aversion of the liquidity trader. This demand function is increasing with Σ and decreasing with λ. The motivation for liquidity trading in this model is endowment shocks. Thus, the higher Σ is, the greater the need to hedge. Moreover, for $\Sigma = 0$, the liquidity demand is zero. Since, in equilibrium, the informed trader does not leave money on the table, the boundary condition $\Sigma(1) = 0$ holds. Thus, an instant before the public announcement is made, the liquidity traders' demand is not sufficiently large enough to keep the market open, and thus a linear equilibrium does not exist. Indeed, we can show that a solution to equations (7) and (8) that satisfies the boundary condition $\Sigma(1) = 0$ fails to exist. However, as the level of liquidity traders' risk aversion approaches infinity, the liquidity demand function becomes inelastic. In that case, as we point out below in Corollary (1), a linear equilibrium does exist.

Mendelson and Tunca (2000) consider another example where the liquidity demand function is an increasing function of Σ. In their model, the initial endowment of the risk averse liquidity trader is zero, and his motive for trade is a shock to his valuation of the stock. Thus, the lower Σ is, the more a liquidity trader is willing to bet on his private valuation. Mendelson and Tunca's (2000) model allows for an information structure similar to the one found in Back and Pedersen (1998). However, when we restrict their information structure to the one discussed here, the liquidity demand function is given by

$$\sigma(\lambda, \Sigma) = \frac{1}{2\lambda(t) + \eta\Sigma(t)}, \tag{11}$$

where again, η is the level of risk aversion of the liquidity traders. Notice also that liquidity demand is decreasing with the cost of trading. Mendelson and Tunca show that a necessary and sufficient condition for the existence of equilibrium is $\phi^2 \leqslant 1/4$ and $\rho > 0$, where ρ is the level of risk aversion of the liquidity traders. In Lemma (A.1), found in Appendix A, we show the same condition is also valid for the risk-averse case considered here. Interestingly, this condition is only stated in terms of the level of information asymmetry, and it does not depend on the level of risk aversion of the informed trader, nor the level of risk aversion of the liquidity traders, as long as it is strictly positive.

3. Comparative statics

In this section, we examine the effect risk aversion has on equilibrium outcomes, and then study the effect the elasticity of liquidity demand has. We first note that one property is shared by all equilibria: λ decreases with time, regardless of the form of the liquidity demand function. Indeed, from Eq. (7) we have $\lambda'(t) = -A\lambda^2\sigma^2$. Also, the relation $\lambda(t) = \lambda(0) + A\Sigma(t)$ follows from Theorem (1). Thus, the rate at which $\lambda(t)$ decreases is proportional to the rate at which information is incorporated into prices, which is a very appealing result: The lower the information asymmetry at time t, the lower $\lambda(t)$ is. In the following corollary, we report the effect the informed trader's coefficient of risk-aversion has on equilibrium outcomes.

Corollary 1. *Let the instantaneous liquidity trades' demand be* $\mathrm{d}z_t = \sigma(t)\,\mathrm{d}B_t$. *Then, when we increase the coefficient of risk aversion* A, *the following cross sectional results hold*:

(i) $\lambda(0)$ *increases*;
(ii) $\lambda(1)$ *decreases*;
(iii) *The aggregate execution cost of the liquidity traders decreases.*

The first two results are intuitive. As the informed trader becomes more risk averse, he trades more aggressively at early stages. This results in a severe, adverse-selection problem, thus we expect and find a high price impact at the beginning. This early intensive trading, however, releases information to the market, and allows the market makers to use a lower price pressure in the future. Thus, again, we expect and find that cross sectionally, $\lambda(1)$ decreases with A. Part (iii) verifies the Holden and Subrahmanyam (1994) conjecture that insider trading may be less problematic than was suggested by Kyle (1985). Notice that this result does not depend on the pattern of liquidity trading. Even if most of the liquidity trading takes place at early stages, when cost of trading is higher, still the aggregated cost of execution is lower than in the risk neutral case.

Next, we study the effect that elasticity of liquidity demand has on equilibrium outcomes. To that end, we consider a family of instantaneous liquidity demand functions, $\mathrm{d}z_t = 1/\lambda^c\,\mathrm{d}B_t$, where c is the elasticity. Indeed,

$$\frac{\mathrm{d}(\sigma\,\mathrm{d}B)}{\mathrm{d}\lambda}\frac{\lambda}{\sigma\,\mathrm{d}B} = c. \tag{12}$$

Fig. 1 shows several members of this family of liquidity functions. To simplify the discussion, we assume that the informed trader is risk neutral; i.e., $A = 0$. Thus, the only exogenous parameter is ϕ, the level of information asymmetry at time zero. In the following cross-sectional analysis, we treat ϕ as if it were a variable. From

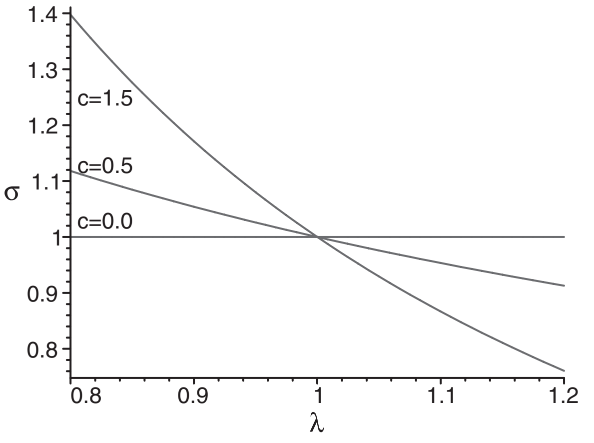

Fig. 1. A family of liquidity demand functions with different levels of elasticity.

Theorem (1), we have[6]

$$\lambda(\phi) = \phi^{1/(1-c)},$$
$$\beta(\phi) = \frac{\phi^{1/(c-1)}}{(1-t)},$$
$$\sigma(\phi) = \phi^{c/(c-1)},$$
$$\Sigma(\phi) = \phi^2(1-t).$$

We see that information is incorporated into prices at the same rate regardless of the elasticity of the liquidity demand function. We also notice that for ϕ equals one, all the equilibria are identical. Fig. 2 illustrates the effect c has on lambda. In particular, we see that the equilibrium outcomes and liquidity demand are identical when $\phi = 1$. By holding the equilibrium outcomes fixed, we can isolate the effects the elasticity of liquidity demand has. Thus, we compare the equilibria when we perturb ϕ around one.

For $\phi = 1$, the following results hold:

$$\lambda'(1) = \frac{1}{1-c},$$
$$\beta'(1) = \frac{1}{(c-1)(1-t)},$$
$$\sigma'(1) = \frac{c}{c-1}.$$

[6] Equilibrium fails to exist when $c = 1$. This case corresponds to the failure of equilibrium in Mendelson and Tunca (2000) when the liquidity traders are risk neutral.

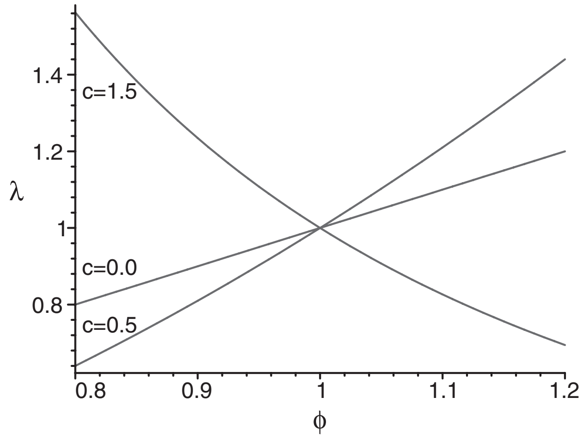

Fig. 2. The Price pressure, λ, as a function of information asymmetry for different levels of elasticity.

That is, when elasticity is low ($c<1$), λ increases with ϕ, which is also the case when the liquidity demand is inelastic. Interestingly, when elasticity is very high ($c>1$), increasing ϕ in fact decreases λ. The reason is that with very high elasticity, the liquidity demand decreases sharply with the cost of trading, λ. In that case, the informed trader correlates his trade with the liquidity traders so that in equilibrium, λ is low to ensure liquidity traders will not leave the market.

4. Empirical implications

The continuous time variants of the Kyle model provide strong implications with regard to the behavior of the price pressure, λ, prior to predetermined public announcements, such as earning announcements. Kyle (1985) and Back (1992) show that under the assumption of a monopolistic informed trader, λ has to remain constant, or at least be a martingale. Holden and Subrahmanyam (1992) show that when they consider the case of N informed traders with identical information, λ has to be zero prior to the announcement. In contrast, Back et al. (2000) show that under the assumption that information is not identically distributed among the informed traders, λ increases at an increasing rate prior to the announcement.[7] Holden and Subrahmanyam (1994) show that when the informed trader is risk averse, a decreasing price pressure emerges. Massoud and Bernhardt (1999) show in a three period model that the shape of λ is sensitive to the elasticity of the liquidity demand function. This paper shows that, in a continuous time model with a risk-averse informed trader, the liquidity demand function only determines the rate at

[7] In fact, they show that the price pressure does not increase over the entire trading period, but after some date it increases at an increasing rate until the announcement.

which λ decreases. In particular, λ decreases whenever the informed trader is risk-averse.

Lee et al. (1993), Brooks (1994), and Krinsky and Lee (1996) report that prior to earning announcements, the adverse selection component of the spread is higher than in a benchmark period. However, these papers do not report the dynamics of λ or the dynamics of the adverse selection component of the spread within the pre-disclosure period. Thus, these papers do not provide a test of the predictions made by the continuous time variants of the Kyle model. Notice, however, that Massoud and Bernhardt (1999) argue that their finite period model is consistent with these empirical findings.

The current study also has implications about the time series relation between the price pressure and volatility. Eq. (7) states the rate at which the market depth ($1/\lambda$) increases is proportional to the instantaneous volatility of liquidity trading. Consequently, $1/\lambda$ can be approximated by a linear combination of lagged instantaneous volatilities. Thus, if we agree to view the volatility of trade in the Kyle model as a measure of volume, as for example Admati and Pfleiderer (1988) do, then our model predicts that high volume periods are followed by higher depth. We notice that this result contrasts with the findings of Lee et al. (1993). They show that following high volume periods, spreads widen and depths fall. An explanation that can reconcile the results found in this paper and the empirical findings is that the adverse selection literature (including the current work) assumes that at any point in time, the liquidity suppliers (market makers and traders who submit limit orders) provide the liquidity traders with the best possible execution price; i.e., they do not extract rents. In practice, every market order that is paired-off with limit orders causes a temporary reduction in market liquidity (depth) for the period of time that it takes new limit orders to replace those that were executed. Thus, during high volume periods, market makers plausibly face less competition from the limit-order book, which causes market liquidity to decrease. Indeed, Lee et al. (1993) find high volume before and after earning announcements, and low liquidity before and after earning announcements. Clearly, low liquidity after the announcements have been made cannot be attributed to asymmetric information.

5. Conclusion

This paper presents a variant of the Kyle model that accommodates a risk averse, informed trader and elastic liquidity demand functions. In the Kyle model with a risk neutral informed trader, the only role of liquidity traders is to provide the informed trader with camouflage. In the original Kyle model, the price pressure, λ, is constant. In contrast, this paper models a risk-averse, informed trader and allows for an elastic liquidity demand function. The main result is that λ decreases with time because the informed trader cares about the effects that liquidity trading has on future prices. Additionally, a high elasticity of liquidity demand can actually result in a lower overall cost of trading.

Appendix A

Proof of Theorem 1. Let $\lambda(t)$ and $\Sigma(t)$ be a solution to the system (7)–(8), and define $\beta(t)$ as in (9). Let $\sigma_z(t) = \sigma(\Sigma(t), \lambda(t), t)$. Then, the instantaneous liquidity traders' demand is given by $\mathrm{d}z = \sigma_z(t)\,\mathrm{d}B_t$, and from the theory of Kalman Filter (see Kallianpur, 1980, p. 269), it follows that the price rule $\mathrm{d}p = \lambda(t)\,\mathrm{d}y$ is competitive, and that $\Sigma(t)$ is the conditional variance of $\tilde{v}$ given the past realization of y. To prove that the strategy $\int_0^t \beta(u)(\tilde{v} - p_u)\,\mathrm{d}u$ is optimal in χ, we define the function

$$V(t,p) = \frac{(p-v)^2}{2\lambda(t)} + \int_t^1 \frac{\lambda(s)\sigma_z^2(s)}{2}\,\mathrm{d}s. \tag{A.1}$$

It follows that the function $V(t,p)$ satisfies

$$V_p = \frac{(p-v)}{\lambda(t)}, \tag{A.2}$$

$$V_{pp}\lambda^2(t) = \frac{1}{\lambda(t)}, \tag{A.3}$$

and (from (7))

$$V_t + \tfrac{1}{2}V_{pp}\lambda^2(t)\sigma_z^2(t) - \tfrac{1}{2}A\sigma_z^2(t)(p-v)^2 = 0. \tag{A.4}$$

Let w_0 be the informed trader's wealth at time zero, let x be an arbitrary feasible strategy, and let p and w be the price process and the wealth process associated with x; i.e. p and w satisfy (1) with the boundary condition p_0 and (4) with the boundary condition w_0, respectively. We will show that $E - \exp(-Aw_1) \leqslant -\exp(-A(w_0 + V(0,p_0))$, with equality when the strategy x_t is $\int_0^t \beta(t)(\tilde{v} - p_t)\,\mathrm{d}t$. By Ito's rule

$$\begin{aligned}
&- \mathrm{e}^{-A(w_1+V(1,p_1))} + \mathrm{e}^{-A(w_0+V(0,p_0))} \\
&\quad = \int_0^1 A\mathrm{e}^{-A(w_t+V)}(\mathrm{d}w_t + V_t\,\mathrm{d}t + V_p\,\mathrm{d}p_t) \\
&\qquad + \int_0^1 -A^2\mathrm{e}^{-A(w_t+V)}\left(\frac{1}{2}\mathrm{d}[w,w]_t^c + V_p\,\mathrm{d}[p,w]_t^c + \frac{1}{2}V_p^2\,\mathrm{d}[p,p]_t^c\right) \\
&\qquad + \int_0^1 A\mathrm{e}^{-A(w_t+V)}V_{pp}\,\mathrm{d}[p,p]_t^c \\
&\qquad + \sum_{0\leqslant t\leqslant 1} -\Delta\mathrm{e}^{-A(w_t+V)} + A\mathrm{e}^{-A(w_t+V)}\Delta(w_t+V) \\
&\quad = \int_0^1 A\mathrm{e}^{-A(w_t+V)}((v-p)\,\mathrm{d}x - \lambda\,\mathrm{d}[x,y]_t^c + V_t\,\mathrm{d}t + V_p\lambda\,\mathrm{d}y) \\
&\qquad - \int_0^1 A^2\mathrm{e}^{-A(w_t+V)}\left(\frac{1}{2}(v-p)^2\,\mathrm{d}[x,x]_t^c + V_p\lambda(v-p)\,\mathrm{d}[x,y]_t^c\right) \\
&\qquad - \int_0^1 \frac{1}{2}A\mathrm{e}^{-A(w_t+V)}\lambda^2(AV_p^2 - V_{pp})\,\mathrm{d}[y,y]_t^c
\end{aligned}$$

$$+\sum_{0\leqslant t\leqslant 1} -\Delta e^{-A(w_t+V)} + Ae^{-A(w_t+V)}\Delta(w_t+V).$$

Using (A.2), (A.3) and (A.3) and the bilinear property of the quadratic variation, we derive

$$\begin{aligned}
&-e^{-A(w_1+V(1,p_1))}+e^{-A(w_0+V(0,p_0))}\\
&\quad=\int_0^1 Ae^{-A(w_t+V)}V_p\lambda\sigma_z\,\mathrm{d}B-\int_0^1\frac{1}{2}Ae^{-A(w_t+V)}\lambda\,\mathrm{d}[x,x]_t^c\\
&\qquad+\sum_{0\leqslant t\leqslant 1} -\Delta e^{-A(w_t+V)} + Ae^{-A(w_t+V)}\Delta(w_t+V).
\end{aligned}$$

We add the expression $-\exp(-Aw_1)+\exp(-A(w_1+V(1,p_1))$ to both sides:

$$\begin{aligned}
&-e^{-Aw_1}+e^{A(w_0+V(0,p_0))}\\
&\quad=-\exp(-Aw_1)+\exp(-A(w_1+V(1,p_1))\\
&\qquad+\int_0^1 Ae^{-A(w_t+V)}(p-v)\sigma_z\,\mathrm{d}B-\int_0^1\frac{1}{2}Ae^{-A(w_t+V)}\lambda\,\mathrm{d}[x,x]_t^c\\
&\qquad+\sum_{0\leqslant t\leqslant 1} -\Delta e^{-A(w_t+V)} + Ae^{-A(w_t+V)}\Delta(w_t+V).
\end{aligned}$$

We now show that the expectation of the left hand side is non-positive, and zero for bounded variation strategies such that $p_1=v$. From the definition of the function $V(t,p)$, we know that the term $\exp(-Aw_1)-\exp(-A(w_1+V(1,p_1))$ is non-positive and zero only if $p_1=v$. A sufficient condition for the integral $\int_0^1\exp(-A(w_t+V(t,p_t)))(p-v)\sigma_z\,\mathrm{d}B$ to be a martingale is $E\int_0^1\exp(2A(w_t+V(t,p_t)))(p-v)^2\sigma_z^2\,\mathrm{d}t<\infty$. We use the inequality $\exp(-x)x\leqslant\exp(-1)$ which holds for all x, and the fact that $\lambda(t)>0$, to conclude that $A\exp(2A(w_t+V(t,p_t)))(p-v)^2\sigma_z^2<\exp(2AM-1)\lambda\sigma_z^2$, where the constant M is the lower bound for the agent's wealth, that exists by the feasibility of the strategy. Because $\lambda(t)$ and $\sigma_z(t)$ are finite, the stochastic integral is indeed a martingale, and so its expected value is zero. The bounded variation integral is negative if x contains unbounded variation term, and zero if otherwise. Also, because of the concavity of the function $-\exp(\cdot)$, the terms in the summation are non-positive for each jump in x.

The strategy $\int_0^t\beta(t)(\tilde{v}-p_t)\,\mathrm{d}t$ is continuous, and it has bounded variation, and we have shown that when the trader follows it, then $\Sigma(1)=0$ (and so $p_1=v$). We, therefore, conclude that this strategy is optimal. □

Proof of Corollary 1. Let $S(t)=\int_0^t\sigma^2(\lambda(u),\Sigma(u),u)\,\mathrm{d}u$. From Theorem (1), we have

$$\lambda(t)=\frac{\phi(\phi AS(1)+\sqrt{S(1)(\phi^2A^2S(1)+4)})}{2S(1)+A^2\phi^2S(t)S(1)+A\phi S(t)\sqrt{S(1)(\phi^2A^2S(1)+4)}},$$

$$\beta(t)=\frac{\sigma^2(\lambda(t),\Sigma(t),t)(\phi AS(1)+\sqrt{S(1)(\phi^2A^2S(1)+4)})}{2\phi(S(1)-S(t))},$$

$$\Sigma(t) = \frac{2\phi^2(S(1) - S(t))}{2S(1) + A^2\phi^2 S(t)S(1) + A\phi S(t)\sqrt{S(1)(\phi^2 A^2 S(1) + 4)}}.$$

In particular,

$$\lambda(0) = \frac{\phi(\phi A S(1) + \sqrt{S(1)(\phi^2 A^2 S(1) + 4)})}{2S(1)},$$

$$\lambda(1) = \frac{\phi(\phi A S(1) + \sqrt{S(1)(\phi^2 A^2 S(1) + 4)})}{2S(1) + A^2\phi^2 S^2(1) + A\phi S(1)\sqrt{S(1)(\phi^2 A^2 S(1) + 4)}}.$$

Parts (i) and (ii) then follow from the assumption that σ depends only on t.

We follow Back and Pedersen (1998) and define the aggregate execution cost of liquidity traders as $\int_0^1 \lambda\sigma^2\,\mathrm{d}t$. From (7), the aggregate execution cost is equal to

$$-\frac{1}{A}\int_0^1 \frac{\lambda'}{\lambda}\,\mathrm{d}t = \frac{1}{A}\log\left(\frac{\lambda(0)}{\lambda(1)}\right).$$

That the above is decreasing with A is straightforward to verify and proves part (iii).

Lemma A.1. *Let*

$$\sigma(\lambda, \Sigma) = \frac{1}{2\lambda(t) + \eta\Sigma(t)}.$$

Then, there is a linear equilibrium if and only if $\phi^2 = \Sigma(0) \leqslant 1/4$.

Let σ have the above form, and consider system (7)–(8) with the boundary condition $\Sigma(1) = 0$ and $\lambda(1) = \lambda_1$. The standard conditions for existence and uniqueness of solution hold. Moreover, the solution is continuous with respect to the initial value λ_1. To emphasize the dependence of the solution on λ_1, we denote the solution by $\lambda(t, \lambda_1)$ and $\Sigma(t, \lambda_1)$. Clearly, if we can find λ_1 so that $\Sigma(0, \lambda_1) = \phi^2$, then we can construct a linear equilibrium.

We know that for every scalar λ_1, we have $\lambda(t, \lambda_1) = A\Sigma(t, \lambda_1) + \lambda_1$. Thus, from (8) and the boundary condition $\Sigma(1) = 0$, we have

$$\Sigma(0, \lambda_1) = \int_0^1 \left(\frac{A\Sigma(t, \lambda_1) + \lambda_1}{2\lambda_1 + \Sigma(t, \lambda_1)(2A + \eta)}\right)^2 \mathrm{d}t \leqslant \left(\frac{\lambda_1}{2\lambda_1}\right)^2 = \frac{1}{4}.$$

This proves that ϕ^2 has to be smaller or equal to 1/4.

To show that the condition is sufficient, we note that the integral above is a continuous function of the initial condition λ_1. Moreover, for $\lambda_1 = 0$, the integral vanishes, and for λ_1 equal to infinity, it equals 1/4. Thus, for any $\phi^2 \leqslant 1/4$, there exists λ_1 such that $\Sigma(0, \lambda_1) = \phi^2$. $\square$

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