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ON THE TRIANGULAR FORM OF A POLYNOMIAL MATRIX AND ITS INVARIANTS WITH RESPECT TO THE SEMISCALAR EQUIVALENCE

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We investigate the structure of polynomial matrices in connection with their reducibility by semiscalar-equivalent transformations to simpler forms. We obtain a system of invariants for one class of matrices with respect to semiscalar equivalence. On this basis, we establish the almost canonical form of a matrix with respect to semiscalar equivalence.

Introduction

The semiscalar equivalence of polynomial matrices was first introduced and studied in [3, 4]. In particular, it was proved there that a polynomial matrix over a field of complex numbers can be reduced by semiscalar-equivalent transformations to the triangular form with invariant multipliers on the main diagonal. Later, the same triangular form was obtained in [9]. A similar result for the so-called PS-equivalence was also established in [11] for polynomial matrices over an arbitrary infinite field. It is worth noting that a more general result concerning finite sets of polynomial matrices over infinite fields and certain finite fields was obtained in [5]. The author of this work solved the problem of the reduction of separate classes of polynomial matrices over a field of complex numbers to the canonical form with respect to semiscalar equivalence [6, 7]. In the present paper, we study the mentioned equivalence of polynomial matrices over an arbitrary field provided that the elementary divisors of these matrices are different degrees of a linear polynomial and refer to the same characteristic root. As is well known, the problem of semiscalar equivalence is tightly bound with the problem of pairs of matrices. This problem was studied in numerous works (see, e.g., [1, 2, 12]). Many classification problems, in particular, the problem of equivalence of spatial matrices [10], can be reduced to the problem mentioned above.

1. Definitions and Auxiliary Propositions

Let F be an arbitrary field. Consider a polynomial matrix $N(x) \in M(n, F[x])$ and write it in the form of a matrix polynomial

$$N(x) = N_0 x^s + N_1 x^{s-1} + \ldots + N_{s-1} x + N_s ,$$

where the matrix N_0 is nonzero. Then, as usual, the number s is called a degree, $N_0 x^s$ the highest term, and the matrix N_0 is the highest coefficient of the matrix polynomial, or polynomial matrix $N(x)$. The minimum degree of a (nonzero) monomial among all monomials that form a given polynomial matrix is called its lowest degree and denoted by $\operatorname{codeg} N(x)$. The monomial of minimal degree and its coefficient are called the

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lowest term and the lowest coefficient of the polynomial matrix, respectively. We use the same terminology for usual (from the ring $F[x]$) polynomials. The lowest degree of a zero matrix by definition is equal to symbol $+\infty$. We denote by h the number of pairwise different invariant multipliers of the matrix $N(x)$, $2 \le h \le n$ (the case $h=1$ is trivial). In what follows, we assume that the characteristic polynomial $\det N(x)$ has the form $\det N(x) = a(x-\alpha)^{s_0}$, $s_0 > 0$. Without loss of generality, we can take $a=1$, $\alpha = 0$. According to Proposition 1 in [8], the matrix $N(x)$ by the semiscalar-equivalent transformation $N(x) \to CN(x)Q(x)$, $C \in GL(n,F)$, $Q(x) \in GL(n,F[x])$ is reduced to the block-triangular form

$$A(x) = \begin{Vmatrix} E_{\ell_1}x^{s_1} & & & \\ A_{21}(x) & E_{\ell_2}x^{s_2} & & \\ \cdots & \cdots & \cdots & \\ A_{h1}(x) & A_{h2}(x) & \ldots & E_{\ell_h}x^{s_h} \end{Vmatrix}, \tag{1}$$

where E_{ℓ_i} is the identity matrix of order ℓ_i, $A_{uv}(x) \in M(\ell_u, \ell_v, F[x])$, $\deg A_{uv}(x) < s_u$, $\operatorname{codeg} A_{uv}(x) > s_v$, $s_{u-1} < s_u$, $u = 2,\ldots,h$, $v < u$. It is clear that the matrix $A(x)$ is determined ambiguously. Let $N(x)$ be reduced to the matrix

$$B(x) = \begin{Vmatrix} E_{\ell_1}x^{s_1} & & & \\ B_{21}(x) & E_{\ell_2}x^{s_2} & & \\ \cdots & \cdots & \cdots & \\ B_{h1}(x) & B_{h2}(x) & \cdots & E_{\ell_h}x^{s_h} \end{Vmatrix}, \tag{2}$$

where $\deg B_{uv}(x) < s_u$ and $\operatorname{codeg} B_{uv}(x) > s_v$, $u = 2,\ldots,h$, $v < u$. Then the following proposition is true:

Proposition 1. *Let the matrices $A(x)$ and $B(x)$ of the form (1) and (2) be semiscalar-equivalent, i.e.,*

$$SA(x) = B(x)R(x), \tag{3}$$

where $S \in GL(n,F)$ and $R(x) \in GL(n,F[x])$. Then, in the block matrices $S = \left\| S_{ij} \right\|_1^h$, $S_{ij} \in M(\ell_i, \ell_j, F)$, and $R(x) = \left\| R_{ij}(x) \right\|_1^h$, $R_{ij}(x) \in M(\ell_i, \ell_j, F[x])$, we have $S_{uv} = 0$, $u = 2,\ldots,h$, $v < u$, i.e., the matrix S is upper block-triangular:

$$S = \begin{Vmatrix} S_{11} & S_{12} & \ldots & S_{1h} \\ 0 & S_{22} & \ldots & S_{2h} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & S_{hh} \end{Vmatrix}, \tag{4}$$

and $\operatorname{codeg} R_{uv}(x) \ge s_u - s_v$.

Proof. Comparison of the lowest degrees of blocks in the positions $(1,u)$, $u=2,\ldots,h$, on both sides of equality (3) shows that $\operatorname{codeg} R_{1u}(x)\ge s_u-s_1$. On this basis, comparing blocks in the positions $(2,u)$, $u=3,\ldots,h$, on both sides of equality (3), we obtain $\operatorname{codeg} R_{2u}(x)\ge s_u-s_2$. Continuing further in such a way, we finally arrive at the inequality $\operatorname{codeg} R_{h-1,h}(x)\ge s_h-s_{h-1}$.

By using relation (3), we can write for an arbitrary pair of subscripts v,u, $v<u$,

$$S_{uv}x^{s_u}+\sum_{m=v+1}^{h}S_{um}A_{mv}(x)=\sum_{k=1}^{v-1}B_{uk}(x)R_{kv}(x)+\sum_{\ell=v}^{u-1}B_{u\ell}(x)R_{\ell v}(x)+x^{s_u}R_{uv}(x). \tag{5}$$

Since, in equality (5), $\operatorname{codeg} B_{uk}(x)>s_k$ and $\operatorname{codeg} R_{kv}(x)\ge s_v-s_k$ according to proved above, we see that $\operatorname{codeg}(B_{uk}(x)R_{kv}(x))>s_v$. In addition, $\operatorname{codeg} A_{mv}(x)$, $\operatorname{codeg} B_{u\ell}(x)$, and s_u are greater than s_v. Hence, the lowest degree of all terms of equality (5), except the first on the left-hand side, is greater than s_v. Therefore, $S_{uv}=0$.

2. Invariants of a Polynomial Matrix with Respect to the Semiscalar Equivalence: General Case

Theorem 1. *Semiscalar-equivalent matrices of the form (1) have the same greatest common divisors (with the highest coefficient* 1*) of minors of order* k_i $(k_i=1,\ldots,\ell_{h-i+1}+\ldots+\ell_h)$ *located in the last* i $(i=1,2,\ldots,h)$ *block rows.*

Proof of this theorem for $i=1,2,\ldots,h-1$ follows from the equality

$$\left\|\begin{matrix} S_{h-i+1,h-i+1} & \cdots & S_{h-i+1,h}\\ & \ddots & \vdots\\ 0 & & S_{hh}\end{matrix}\right\|\left\|\begin{matrix} A_{h-i+1,1}(x) & \cdots & A_{h-i+1,h-i}(x) & E_{\ell_{h-i+1}}x^{s_{h-i+1}} & & 0\\ \vdots & & \vdots & \vdots & \ddots & \\ A_{h1}(x) & \cdots & A_{h,h-i}(x) & A_{h,h-i+1}(x) & \cdots & E_{\ell_h}x^{s_h}\end{matrix}\right\|$$

$$=\left\|\begin{matrix} B_{h-i+1,1}(x) & \cdots & B_{h-i+1,h-i}(x) & E_{\ell_{h-i+1}}x^{s_{h-i+1}} & & 0\\ \vdots & & \vdots & \vdots & \ddots & \\ B_{h1}(x) & \cdots & B_{h,h-i}(x) & B_{h,h-i+1}(x) & \cdots & E_{\ell_h}x^{s_h}\end{matrix}\right\|R(x),$$

written on the basis of relation (3) with regard for Proposition 1. Here, one should keep in mind that the blocks $S_{h-i+1,h-i+1},\ldots,S_{hh}$ are nonsingular, and the matrix $R(x)$ is invertible. For the case $i=h$, the proof is obvious.

Corollary 1. *The invariant multipliers of the submatrix consisting of* i $(i=1,2,\ldots,h)$ *last block rows of matrix (1) are invariants of the matrix* $N(x)$ *with respect to semiscalar equivalence.*

The lowest degree of the last block row of matrix (1) coincides with the degree of its first invariant multiplier and, hence, is an invariant of the matrix $N(x)$ with respect to semiscalar equivalence. However, other block rows of this matrix have no such a property.

Example 1. In the semiscalar-equivalent matrices

$$\begin{Vmatrix} 1 & 0 & 0 \\ x^2+x^3 & x^4 & 0 \\ x^3 & 0 & x^5 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 1 & 0 & 0 \\ x^3 & x^4 & 0 \\ x^3 & 0 & x^5 \end{Vmatrix},$$

the lowest degrees of the second rows are 2 and 3, respectively.

To decrease the ambiguity of a matrix of the form (1), it is necessary to increase the lowest degrees of all its block rows because the degrees of these rows are bounded from above by the degrees of the corresponding diagonal elements (invariant multipliers). In connection with this, there arises a question: is it possible to increase the lowest degree of a single block row without decreasing the lowest degrees of other rows? The following theorem gives a positive answer to this question:

Theorem 2. *In the class $\{CA(x)Q(x)\}$ of semiscalar-equivalent matrices of the form (1), there exists a matrix in which each of the block rows has the maximum possible lowest degree for this class.*

Proof. If the matrices $A(x)$ and $B(x)$ of the form (1) and (2) are semiscalar-equivalent, then, using the obvious equality (3) and Proposition 1, we can write

$$S_{hh}\begin{Vmatrix} A_{h1}(x) & \ldots & A_{h,h-1}(x) & E_{\ell_h}x^{s_h} \end{Vmatrix} = \begin{Vmatrix} B_{h1}(x) & \ldots & B_{h,h-1}(x) & E_{\ell_h}x^{s_h} \end{Vmatrix} R(x).$$

In view of invertibility of the matrices S_{hh} and $R(x)$, we establish that

$$\operatorname{codeg}\begin{Vmatrix} A_{h1}(x) & \ldots & A_{h,h-1}(x) & E_{\ell_h}x^{s_h} \end{Vmatrix} = \operatorname{codeg}\begin{Vmatrix} B_{h1}(x) & \ldots & B_{h,h-1}(x) & E_{\ell_h}x^{s_h} \end{Vmatrix}.$$

Thus, the lowest degree of the last block row of the matrix $A(x)$ is already maximum possible. Suppose, by induction, that each of the last k, $1 \le k < h-1$, block rows in this matrix has the maximum possible lowest degree, but the $(h-k)$th block row does not have it. Also let the block row with number $h-k$ have the maximum possible lowest degree in the matrix $B(x)$ of the form (2) from the set $\{CA(x)Q(x)\}$. Then, in view of invertibility of the free term of submatrix $\left\| R_{ij}(x) \right\|_1^{h-k}$ of the transforming matrix $R(x)$, equality (3) implies that

$$\operatorname{codeg}\left(\begin{Vmatrix} S_{h-k,h-k} & \ldots & S_{h-k,h} \end{Vmatrix}\begin{Vmatrix} A_{h-k,1}(x) & \ldots & A_{h-k,h-k-1}(x) & E_{\ell_{h-k}}x^{s_{h-k}} \\ \ldots & \ldots & \ldots & \ldots \\ A_{h1}(x) & \ldots & A_{h,h-k-1}(x) & A_{h,h-k}(x) \end{Vmatrix}\right)$$

$$= \operatorname{codeg} \left\| B_{h-k,1}(x) \quad \dots \quad B_{h-k,h-k-1}(x) \quad E_{\ell_{h-k}} x^{s_{h-k}} \right\|. \tag{6}$$

In the matrix S of the form (4) from equality (3), we now replace all superdiagonal blocks, except the $(h-k)$th block row, by zero blocks and denote the matrix obtained in this way by S'. According to Proposition 2 from [8], with the help of this nonsingular matrix S' and a certain invertible matrix $R'(x)$, we can pass from the matrix $A(x)$ to another matrix $A'(x)$ such that

$$S'A(x) = A'(x)R'(x), \tag{7}$$

where

$$A'(x) = \left\| \begin{matrix} E_{\ell_1} x^{s_1} & & & \\ A'_{21}(x) & E_{\ell_2} x^{s_2} & & \\ \dots & \dots & \dots & \\ A'_{h1}(x) & A'_{h2}(x) & \cdots & E_{\ell_h} x^{s_h} \end{matrix} \right\|,$$

$$R'(x) = \left\| R'_{ij}(x) \right\|_1^h \in GL(n, F[x]).$$

Proceeding from relation (7), by analogy with equality (6), we can write

$$\operatorname{codeg} \left(\left\| S_{h-k,h-k} \quad \dots \quad S_{h-k,h} \right\| \left\| \begin{matrix} A_{h-k,1}(x) & \dots & A_{h-k,h-k-1}(x) & E_{\ell_{h-k}} x^{s_{h-k}} \\ \dots & \dots & \dots & \dots \\ A_{h1}(x) & \dots & A_{h,h-k-1}(x) & A_{h,h-k}(x) \end{matrix} \right\| \right)$$

$$= \operatorname{codeg} \left\| A'_{h-k,1}(x) \quad \dots \quad A'_{h-k,h-k-1}(x) \quad E_{\ell_{h-k}} x^{s_{h-k}} \right\|. \tag{8}$$

As follows from equalities (6) and (8), the lowest degree of the $(h-k)$th block row in the matrix $A'(x)$ is maximum possible, and this is also true for $B(x)$. Here, the lowest degrees of the last k block rows of the matrix $A'(x)$ remain the same as in $A(x)$.

Note that semiscalar-equivalent matrices of the form (1) have the same greatest common divisors of each block column. They are the invariant multipliers $x^{s_1}, \dots, x^{s_h}$. In this context, the following corollary will be interesting:

Corollary 2. *Semiscalar-equivalent matrices of the form (1) with the maximum possible lowest degrees of the block rows whose existence follows from Theorem 2 have the same greatest common divisors of elements of each block row and, hence, the same location of the block rows, which contain only one nonzero (diagonal) block.*

3. Invariants of a Polynomial Matrix with Respect to Semiscalar Equivalence: the Case of the Absence of Identical Invariant Multipliers

Consider in more detail the case where the initial matrix $N(x)$ does not contain identical invariant multipliers. Then the matrix $A(x)$ from (1) will have the form

$$A(x) = \begin{Vmatrix} x^{s_1} & & & \\ a_{21}(x) & x^{s_2} & & \\ \cdots & \cdots & \cdots & \\ a_{n1}(x) & a_{n2}(x) & \cdots & x^{s_n} \end{Vmatrix}, \tag{9}$$

where $\deg a_{uv}(x) < s_u$, $\operatorname{codeg} a_{uv}(x) > s_v$, $s_{u-1} < s_u$, $u = 2,\ldots,n$, $v < u$. Suppose that, according to Theorem 2, the lowest degrees of rows in this matrix are maximum possible.

An element $a_{uv}(x) \neq 0$ is called a key element of the u th row of matrix (9), if $\operatorname{codeg} a_{uv}(x) = \operatorname{codeg} \begin{Vmatrix} a_{u1}(x) & \ldots & a_{u,u-1}(x) \end{Vmatrix}$ and $\operatorname{codeg} a_{uv}(x) < \operatorname{codeg} a_{uw}(x)$ for $w > v$ ($2 \le u \le n$, $v < u$). In the case where all nondiagonal elements of the u th row are equal to zero, the diagonal element x^{s_u} of this row is called a key element.

Proposition 2. *In the class $\{CA(x)Q(x)\}$ of semiscalar-equivalent matrices of the form (9) with the maximum possible lowest degrees of rows, there exists a matrix where the key elements of the rows of identical lowest degrees are located in different columns.*

Proof. Suppose that, in matrix (9), the key elements $a_{kq}(x)$ and $a_{pq}(x)$, $k < p$, of the k th and p th rows, respectively, belong to the same q th column, have identical lowest degrees, and a_{kq0}, a_{pq0} are their lowest coefficients. We construct a matrix S, which differs from the identity matrix E_n only by the element $s_{kp} \neq 0$ in the position (k, p), which satisfies the condition $a_{kq0} + s_{kp}a_{pq0} = 0$. With the help of this matrix, based on Proposition 2 from [8], we can pass from the matrix $A(x)$ to the semiscalar-equivalent matrix $B(x)$ of the form

$$B(x) = \begin{Vmatrix} x^{s_1} & & & \\ b_{21}(x) & x^{s_2} & & \\ \cdots & \cdots & \cdots & \\ b_{n1}(x) & b_{n2}(x) & \cdots & x^{s_n} \end{Vmatrix}, \tag{10}$$

where $\deg b_{uv}(x) < s_u$, $\operatorname{codeg} b_{uv}(x) > s_v$, $s_{u-1} < s_u$, $u = 2,\ldots,n$, $v < u$. Here, for a certain invertible matrix $R(x)$, equality (3) must be satisfied. We represent this equality in the form

$$S^{-1}B(x) = A(x)R^{-1}(x)\,. \tag{11}$$

We now show that the lowest degrees of rows in the obtained matrix $B(x)$ do not change as compared with $A(x)$, and, moreover, the positions of the key elements of rows, except the k th row, also remain invariable. The key element of the k th row of the matrix $B(x)$ will be located to the left of the q th position of this row.

Denoting $R^{-1}(x) = \left\| t_{ij}(x) \right\|_1^n$, we can write according to (11)

$$b_{uv}(x) = a_{u1}(x)t_{1v}(x) + \ldots + a_{u,u-1}(x)t_{u-1,v}(x) + x^{s_u} t_{uv}(x), \qquad v = 1,\ldots,n-1, \qquad u \neq k, \tag{12}$$

$$b_{kw}(x) - s_{kp} b_{pw}(x) = a_{k1}(x)t_{1w}(x) + \ldots + a_{k,k-1}(x)t_{k-1,w}(x) + x^{s_k} t_{kw}(x), \qquad w = 1,\ldots,k-1. \tag{13}$$

Equalities (12) and (13) imply that

$$\operatorname{codeg} b_{uv}(x) \geq \operatorname{codeg} \left\| a_{u1}(x) \quad \ldots \quad a_{u,u-1}(x) \right\|, \qquad u = 2,\ldots,n, \qquad v < u.$$

Since the lowest degrees of rows of the matrix $A(x)$ are maximum possible, we have

$$\operatorname{codeg} \left\| b_{u1}(x) \quad \ldots \quad b_{u,u-1}(x) \right\| = \operatorname{codeg} \left\| a_{u1}(x) \quad \ldots \quad a_{u,u-1}(x) \right\|, \qquad u = 2,\ldots,n.$$

As follows from Proposition 1, $\operatorname{codeg} t_{vu}(x) > 0$ for $v = 1,\ldots,n-1$, $v < u$, and, hence, $\operatorname{codeg} t_{vv}(x) = 0$. Therefore, if $a_{uh}(x)$ is the key element of the u th row of the matrix $A(x)$, then $b_{uh}(x)$ is the key element of the u th row of the matrix $B(x)$ for $u \neq k$. Furthermore, the lowest coefficients of all corresponding key elements of rows, except the k th rows, coincide in the matrices $A(x)$ and $B(x)$. Taking into account this fact and the relation $a_{kq0} + s_{kp} a_{pq0} = 0$, we obtain from equalities (13) that $\operatorname{codeg} b_{k\ell}(x) > \operatorname{codeg} a_{kq}(x)$ for all $\ell \geq q$. Hence, the key element in the k th row of the matrix $B(x)$ is located to the left of the position (k, q).

Thus, beginning from the last right column of the matrix $A(x)$, which contains the key elements of identical lowest degrees, moving by columns to the left, and repeatedly applying the technique described above, we can obtain, after a countable number of steps, a matrix where the key elements of identical lowest degrees will be located in different columns.

We say that the k th and p th rows $\bar{a}_k(x)$ and $\bar{a}_p(x)$ of a matrix satisfying the conditions of Proposition 2 form inversion if $\operatorname{codeg} \bar{a}_k(x) > \operatorname{codeg} \bar{a}_p(x)$ or $\operatorname{codeg} \bar{a}_k(x) = \operatorname{codeg} \bar{a}_p(x)$ for $k < p$, but the key elements $a_{kv}(x)$ and $a_{pw}(x)$ of these rows are located so that $v < w$.

Proposition 3. *In the semiscalar-equivalent matrices $A(x)$ and $B(x)$, which satisfy the conditions of Proposition 2, the positions of key elements of the corresponding rows and their lowest degrees coincide. The transformation matrix S of the upper triangular form in relation (3) has a zero element in the position (k, p) if the k th and p th rows of the matrices $A(x)$ and $B(x)$ form inversion.*

Proof. The invariance of the lowest degrees of key elements is not doubtful because they are equal to the lowest degrees of the corresponding rows, which are maximum possible.

Suppose that the matrices $A(x)$ and $B(x)$ have the form (9) and (10), respectively, and the element $b_{nh_n}(x)$ is a key element in the n th row of the matrix $B(x)$. Using equality (3), where, according to Proposition 1, the transformation matrix $S = \| s_{ij} \|_1^n$ has the upper triangular form ($s_{uv} = 0$, $u > v$), and $R(x) = \| r_{ij}(x) \|_1^n$, we write

$$s_{nn} a_{nq}(x) = b_{n1}(x) r_{1q}(x) + \ldots + b_{n,n-1}(x) r_{n-1,q}(x) + x^{s_n} r_{nq}(x), \qquad q = 1, \ldots, n-1. \tag{14}$$

Since the h_n th term for $q = h_n < n$ on the right-hand side of equality (14) is of the lowest degree $\operatorname{codeg} b_{nh_n}(x)$, and the lowest degrees of all other terms exceed $\operatorname{codeg} b_{nh_n}(x)$, we obtain $\operatorname{codeg} a_{nh_n}(x) = \operatorname{codeg} b_{nh_n}(x)$. In view of the fact that, for all $q > h_n$, the lowest degree of the right-hand side of equality (14) exceeds $\operatorname{codeg} b_{nh_n}(x)$, we have the inequality $\operatorname{codeg} a_{nh_n}(x) < \operatorname{codeg} a_{nq}(x)$. Hence, $a_{nh_n}(x)$ is the key element of the last row of the matrix $A(x)$ if $h_n < n$. For the case $h_n = n$, this proposition follows from Corollary 2.

Suppose, by induction, that the key elements in the last m, $1 \le m < n-1$, rows of the matrices $A(x)$ and $B(x)$ occupy identical positions. Let also $s_{kp} = 0$ in the matrix S for $n - m < k < p \le n$ if the k th and p th rows of the matrix $A(x)$ [and, hence, $B(x)$] form inversion. Then, based on equality (3), we can write

$$s_{n-m,n-m} a_{n-m,q}(x) + \ldots + s_{n-m,n} a_{nq}(x) = b_{n-m,1}(x) r_{1q}(x) + \ldots$$

$$\ldots + b_{n-m,n-m-1}(x) r_{n-m-1,q}(x) + x^{s_{n-m}} r_{n-m,q}(x), \qquad q = 1, \ldots, n-m-1. \tag{15}$$

If the $(n-m)$ th row of the matrix $B(x)$ [or $A(x)$] does not form inversion with each of its last m rows, and $b_{n-m,h_{n-m}}(x)$ is the key element of this row, $h_{n-m} < n - m$, then, for $q = h_{n-m}$, equalities (15) imply that

$$\operatorname{codeg} a_{n-m,h_{n-m}}(x) = \operatorname{codeg} b_{n-m,h_{n-m}}(x).$$

Further, for all $q > h_{n-m}$, we have from equalities (15)

$$\operatorname{codeg} a_{n-m,h_{n-m}}(x) < \operatorname{codeg} a_{n-m,q}(x).$$

Thus, $a_{n-m,h_{n-m}}(x)$ is the key element of the $(n-m)$ th row of the matrix $A(x)$. For the case $h_{n-m} = n - m$, our proposition follows from Corollary 2.

Now let the $(n-m)$ th row of the matrix $A(x)$ form inversion with a certain row from the last m rows of the matrix. Suppose that $((i_1, j_1), \ldots, (i_\ell, j_\ell)$ is a sequence of positions of the key elements in all rows of this sort ordered by increasing the lowest degrees and, in the case of their equality, by decreasing the second components ($n-m<i_1,\ldots,i_\ell \le n$, $1 \le j_1,\ldots,j_\ell \le n-m$). Taking, in turn, $q = j_1,\ldots,q=j_\ell$ in (15), we obtain $s_{n-m,i_1} = \ldots = s_{n-m,i_\ell} = 0$. If $b_{n-m,h_{n-m}}(x)$ is the key element of the $(n-m)$th row of the matrix $B(x)$, then, taking $q = h_{n-m}$, we find from (15) that

$$\operatorname{codeg} a_{n-m,h_{n-m}}(x) = \operatorname{codeg} b_{n-m,h_{n-m}}(x),$$

and , for all $q > h_{n-m}$, equalities (15) yield

$$\operatorname{codeg} a_{n-m,h_{n-m}}(x) < \operatorname{codeg} a_{n-m,q}(x).$$

Thus, in this case, $a_{n-m,h_{n-m}}(x)$ is the key element of the $(n-m)$th row of the matrix $A(x)$ as well.

For the case $h_{n-m} = n-m$, the proof of our proposition follows from Corollary 2.

Proposition 4. *Suppose that the matrix $A(x)$ satisfies the conditions of Proposition 2, and S is an arbitrary upper triangular nonsingular matrix of the same order over a field F with zero elements in all positions (k, p) if the k th and p th rows of the matrix $A(x)$ form inversion. Then the product $SA(x)$ is right-equivalent to a matrix satisfying the conditions of Proposition 2.*

Proof. The right equivalence of the product $SA(x)$ to a matrix $B(x)$ of the form (10) is guaranteed by Proposition 2 from [8]. If $\|0 \ \ldots \ 0 \ s_{uu} \ \ldots \ s_{un}\|$, $\bar{a}_u(x)$, and $\bar{b}_u(x)$ are the u th rows, $u = 2,\ldots,n$, of the matrices S, $A(x)$, and $B(x)$, respectively, then, for a certain invertible matrix $R(x)$, the equality

$$s_{uu}\bar{a}_u(x) + \ldots + s_{un}\bar{a}_n(x) = \bar{b}_u(x)R(x) \tag{16}$$

must be true. If $\operatorname{codeg}\bar{a}_v(x) < \operatorname{codeg}\bar{a}_u(x)$, $u < v$, then the left-hand side of (16) does not contain the term $s_{uv}\bar{a}_v(x)$ $(s_{uv} = 0)$, and, hence, $\operatorname{codeg}\bar{a}_u(x) \le \operatorname{codeg}\bar{b}_u(x)$. However, strict inequality is here impossible in view of the maximality of the lowest degree of the row $\bar{a}_u(x)$. Therefore, we have actually the following equality: $\operatorname{codeg}\bar{a}_u(x) = \operatorname{codeg}\bar{b}_u(x)$.

As follows from equality (16) for $u = n$, the positions of key elements of the last rows of the matrices $A(x)$ and $B(x)$ coincide. Suppose, by induction, that, in the last m, $1 \le m < n-1$, rows of these matrices, the positions of the corresponding key elements coincide, but, in the $(n-m)$th rows, there is no such coincidence. Let k and p be the numbers of key elements in the $(n-m)$th rows of the matrices $A(x)$ and $B(x)$, respectively, where $k < p$. Comparing the p th elements on the left- and right-hand sides of equality (16) for $u = n-m$, we arrive at the inequality of their lowest degrees, but this is impossible. By analogy, we can obtain the inequality of the lowest degrees of the k th elements of rows on the left- and right-hand sides of equality (16) in the case $k > p$. Therefore, in the $(n-m)$th rows of the matrices $A(x)$ and $B(x)$, the key elements occupy identical positions.

Recall that, for a square matrix $G = \| g_{ij} \|_1^n$, the matrix $G_* = \| G_{ij} \|_1^n$, where G_{ij} is the algebraic adjunct of an element g_{ji} of the matrix G, is called adjugate (or adjoint) to the matrix G.

Theorem 3. *In the class* $\{CA(x)Q(x)\}$ *of semiscalar-equivalent matrices satisfying the conditions of Proposition 2, there exists a matrix such that each column of the matrix adjugate to it has the maximum possible lowest degree for this class.*

Proof. First, we note that the first columns of the matrices adjugate to matrices from the class $\{CA(x)Q(x)\}$ satisfying the conditions of Proposition 2 have identical lowest degrees and, hence, are maximum possible. It is easy to prove this property by passing in equality (3) written for any two matrices of this sort $A(x)$ and $B(x)$ to the adjugate matrices and equating the lowest degrees of the first columns on the left- and right-hand sides of obtained equality. Suppose, by induction, that each of the first m, $1 \le m < n-1$, columns of the matrix $A_*(x)$ and the $(m+1)$th column of the matrix $B_*(x)$ adjugate to a certain matrix $B(x)$ from the class $\{CA(x)Q(x)\}$ have the maximum possible lowest degrees. Based on the upper triangular matrix S from relation (3), which connects the matrices $A(x)$ and $B(x)$, we construct a matrix S_1 so that its adjugate matrix S_{1*} has elements on the main diagonal and in the $(m+1)$th column identical with those of the matrix S_*, whereas all its remaining elements are equal to zero. Since, according to Proposition 3, the matrix S has a zero element in the position (k, p), $k < p$, if the kth and pth rows of the matrix $A(x)$ form inversion, the matrix S_1 has the same property, as is easy to verify. Proposition 4 provides the existence of a matrix $A_1(x)$ satisfying the conditions of Proposition 2, and, hence, for a certain invertible matrix $R_1(x)$, we can write

$$S_1 A(x) = A_1(x) R_1(x).$$

By using the obvious equality

$$A_*(x) S_{1*} = R_{1*}(x) A_{1*}(x)$$

and taking into account the form of the matrix S_{1*} and the invertibility of the matrix $R_{1*}(x)$, we establish that the lowest degrees of the first m columns of the matrix $A_*(x)$ coincide with the lowest degrees of the corresponding columns of the matrix $A_{1*}(x)$. Based on the equality

$$A_*(x) S_* = R_*(x) B_*(x),$$

we also conclude that the lowest degrees of the $(m+1)$th columns of the matrices $A_{1*}(x)$ and $B_*(x)$ coincide.

At first sight, it seems that the statement of Theorem 3 is trivial. However, the next example shows that this is not true:

Example 2. The semiscalar-equivalent matrices

$$A(x) = \left\| \begin{matrix} 1 & 0 & 0 \\ x^2 & x^5 & 0 \\ x & x^6 & x^7 \end{matrix} \right\|,$$

$$B(x) = \left\| \begin{matrix} 1 & 0 & 0 \\ x^2 & x^5 & 0 \\ x - x^3 + x^5 & 0 & x^7 \end{matrix} \right\|$$

have the maximum possible lowest degrees of all rows but the second columns of the matrices adjugate to them have different lowest degrees.

In what follows, we assume that the matrix $A(x)$ of the form (9) satisfies the conditions of Theorem 3, i.e., each its row and each column of the matrix $A_*(x)$ adjugate to $A(x)$ have the maximum possible lowest degrees. An element $a_{uv*}(x) \neq 0$ ($2 \le u \le n$, $v < u$) is called a key element of the v th column of the adjugate matrix $A_*(x)$ if

$$\operatorname{codeg} a_{uv*}(x) = \operatorname{codeg} \left\| a_{v+1,v*}(x) \quad \ldots \quad a_{nv*}(x) \right\|^{\top}$$

and

$$\operatorname{codeg} a_{uv*}(x) < \operatorname{codeg} a_{wv*}(x)$$

for $w < u$. Here, symbol "$\top$" means the operation of transposition. In the case where all subdiagonal elements of a column of the matrix $A_*(x)$ are equal to zero, we consider the diagonal element of this column as the key element.

Proposition 5. *Let $\{CA(x)Q(x)\}$ be the class of semiscalar-equivalent matrices satisfying the conditions of Theorem 3. In this class, there exists a matrix $B(x)$ such that each pair of the key elements of columns of the matrix $B_*(x)$ of identical lowest degrees is not located in the same row if the corresponding (with the same numbers) pair of rows of the matrix $B(x)$ does not form inversion.*

Proof. It is sufficient to prove this proposition for an arbitrary pair of the key elements of the columns of identical lowest degrees. Let, in the matrix $A_*(x)$, the key elements $a_{qk*}(x)$ and $a_{qp*}(x)$ of the k th and p th, $k < p$, columns, respectively, have identical lowest degrees and be located in the same q th row. We denote by a_{qk0} and a_{qp0} the lowest coefficients of these elements and construct a matrix S_2 so that its adjugate matrix S_{2*} differs from the identity matrix E_n only by one nonzero element s_{kp*} satisfying the relation $a_{qp0} + s_{kp*} a_{qk0} = 0$. According to Proposition 4, with the help of the matrix S_2 and a certain invertible matrix $R_2(x)$, one can construct for $A(x)$ a matrix $A_2(x)$ satisfying the conditions of Proposition 2:

$$S_2 A(x) = A_2(x) R_2(x).$$

Here, as follows from Proposition 3, the positions of the key elements of rows of the matrix $A_2(x)$ and their lowest degrees will not change as compared with the matrix $A(x)$. If one passes in this equality to adjugate matrices, it is easy to show that, in the matrix $A_{2*}(x)$, the lowest degrees of the key elements of all columns remain invariable. The positions of all key elements of the columns of this matrix also do not change, except the position of the key element of the p th column. In the matrix $A_{2*}(x)$, this element will occupy a position lower than the q th row.

Let the matrix $A(x)$ of the form (9) satisfy the conditions of Proposition 5. This means that its rows have the maximum possible lowest degrees, and no two key elements of the columns of identical lowest degrees are located in the same column. In addition, the columns of the adjugate matrix $A_*(x)$ have the maximum possible lowest degrees, and the key elements of arbitrary two columns of identical lowest degrees are located in different rows if the corresponding rows of the matrix $A(x)$ do not form inversion.

We say that the k th and p th columns of the matrix $A_*(x)$ form inversion if, for $k < p$, the lowest degree of the k th column is smaller than that of the p th column, or these degrees are equal, but the key element of the k th column is located higher than the key element of the p th column.

Proposition 6. *Let the matrices $A(x)$ and $B(x)$ satisfying the conditions of Proposition 5 be semiscalar-equivalent. Then, in these matrices and the matrices $A_*(x)$ and $B_*(x)$ adjugate to them, the positions and the lowest degrees of the corresponding key elements of rows and columns coincide. The transforming upper triangular matrix S in relation (3) has a zero element in the position (k, p), $k < p$, if the k th and p th rows of the matrix $A(x)$ or the k th and p th columns of the matrix $A_*(x)$ form inversion.*

Proof. The part of this proposition concerning the key elements of rows of the matrices $A(x)$ and $B(x)$ was substantiated in Proposition 3. The remaining part can be proved similarly.

Proposition 7. *Suppose that the matrix $A(x)$ satisfies the conditions of Proposition 5, and S is an upper triangular nonsingular matrix of the same order over a field F with zero elements in all positions (k, p) if the k th and p th rows of the matrix $A(x)$ or the k th and p th columns of $A_*(x)$ form inversion. Then the product $SA(x)$ is right-equivalent to the matrix satisfying the conditions of Proposition 5.*

Proof. In view of Proposition 4, the product $SA(x)$ is right-equivalent to the matrix $B(x)$ satisfying the conditions of Proposition 2. By analogy with the proof of Proposition 4, one can show that the matrix $B(x)$ satisfies the conditions of Proposition 5.

Proposition 8. *The set of matrices S of structure described in Proposition 7 forms a subgroup in the multiplicative group of nonsingular upper triangular matrices of fixed order.*

Proof. Let M be the set of matrices S of order n whose structure is described in Proposition 7. If $S_1, S_2 \in M$, then, according to Proposition 7,

$$S_1 A(x) = B_1(x) R_1(x), \tag{17}$$

where $B_1(x)$ is a matrix satisfying the conditions of Proposition 5, and $R_1(x) \in GL(n, F[x])$. Premultiplying both sides of equality (17) by S_2, we obtain

$$S_2 S_1 A(x) = S_2 B_1(x) R_1(x). \tag{18}$$

Since, according to Proposition 6, the positions and lowest degrees of the key elements of rows of the matrices $A(x)$ and $B_1(x)$ as well as the positions and lowest degrees of the key elements of columns of the matrices $A_*(x)$ and $B_{1*}(x)$ coincide, we can apply Proposition 7 to the product $S_2 B_1(x)$. Hence, we obtain

$$S_2 B_1(x) = B_2(x) R_2(x), \tag{19}$$

where the matrix $B_2(x)$ satisfies the conditions of Proposition 5, and $R_2(x) \in GL(n, F[x])$. Substituting (19) in (18), we find

$$S_2 S_1 A(x) = B_2(x) R_2(x) R_1(x).$$

Since $R_2(x) R_1(x) \in GL(n, F[x])$, we can obtain from Proposition 6 that $S_2 S_1 \in M$. It is clear that the identity matrix E_n belongs to M and, writing relation (17) in the form

$$S_1^{-1} B_1(x) = A(x) R_1^{-1}(x)$$

and using Proposition 6, we establish that $S_1^{-1} \in M$.

Thus, every class $\{CA(x)Q(x)\}$ of semiscalar-equivalent matrices is associated with a subgroup of the multiplicative group of nonsingular matrices over the field F of upper triangular matrices of the same order. Furthermore, different subgroups correspond to matrices that are not semiscalar-equivalent.

As follows from Proposition 6, the positions of key elements of the matrices $A(x)$ and $A_*(x)$ as well as their lowest degrees are invariants of the initial matrix $N(x)$ with respect to semiscalar equivalence. However, the system of these invariants is incomplete because, generally speaking, the matrix $A(x)$ is determined ambiguously. In addition, the less the number of inversions formed by the rows and columns of the matrices $A(x)$ and $A_*(x)$, the greater is this ambiguity. The total number of inversions in these matrices is a measure of "canonicity" of the matrix $A(x)$. It is interesting to isolate a class of matrices such that the matrix $A(x)$ in it will be canonical or almost canonical. For this purpose, we consider the case where every couple of rows in the matrix $A(x)$ of the form (9) satisfying the conditions of Proposition 5 or the corresponding pair of columns in its adjugate matrix $A_*(x)$ form inversion. Then every nonzero element of the first column of the matrix $A(x)$ is a key element of the corresponding row. Taking the lowest degrees of the elements $a_{i+1,1}(x)$, $i = 1, \ldots, n-1$, of the first column of the matrix $A(x)$ and the degrees s_{i+1} of its diagonal elements, we construct two sequences:

$$p_1, \dots, p_{n-1}, \tag{20}$$

$$q_1, \dots, q_{n-1}, \tag{21}$$

where $p_1 = \operatorname{codeg} a_{i+1,i}(x) + \operatorname{codeg} a_{n1}(x) - s_1$, $q_i = s_{i+1}$. Sequence (21) is a numerical sequence. It may happen that the first term of sequence (20) is $+\infty$. This is true in the case where $a_{21}(x) \equiv 0$ for the element of the matrix $A(x)$.

Theorem 4. *If each term of sequence (20) is not smaller than the corresponding term of sequence (21), then the matrix $A(x)$ is determined accurate to diagonal similarity. If some terms of sequence (20) are smaller than the corresponding terms of sequence (21), and m is the smallest number of such term $(p_m < q_m)$, then the matrix $A(x)$ is semiscalar-equivalent to the matrix $B(x)$ where there is no a monomial of degree p_m in the $(m+1)$th element of the first column. The matrix $B(x)$ is determined accurate to diagonal similarity.*

Proof. We first assume that, in the semiscalar-equivalent matrices $A(x)$ and $B(x)$ of the form (9) and (10), each term of sequence (20) is not smaller than the corresponding term of sequence (21). According to Proposition 6, the left transforming matrix S in relation (3) has the form

$$S = \begin{Vmatrix} s_{11} & 0 & \dots & 0 & s_{1n} \\ & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & & & & s_{nn} \end{Vmatrix}.$$

Therefore, comparing elements in the positions $(2,1)$ on the left- and right-hand sides of equality (3), we write

$$s_{22}a_{21}(x) - s_{11}b_{21}(x) - s_{1n}a_{n1}(x)\tilde{b}_{21}(x) \equiv 0 \qquad (\operatorname{mod} x^{s_2}),$$

where $x^{s_1}\tilde{b}_{21}(x) = b_{21}(x)$. As follows from here, $s_{22}a_{21}(x) = s_{11}b_{21}(x)$. Further, comparing elements in the positions $(3,1)$, we recursively obtain $s_{33}a_{31}(x) = s_{11}b_{21}(x)$. For elements in the positions $(3,2)$, we have

$$s_{33}a_{32}(x) - s_{22}b_{32}(x) + s_{1n}a_{n2}(x)\begin{Vmatrix} \tilde{b}_{21}(x) & 1 \\ \tilde{b}_{31}(x) & \tilde{b}_{32}(x) \end{Vmatrix} \equiv 0 \qquad (\operatorname{mod} x^{s_3}),$$

where $x^{s_1}\tilde{b}_{31}(x) = b_{31}(x)$ and $x^{s_2}\tilde{b}_{32}(x) = b_{32}(x)$. From this congruence, we recursively find $s_{33}a_{32}(x) = s_{22}b_{32}(x)$ an so on. Hence,

$$\operatorname{diag}(s_{11},\ldots,s_{nn})A(x) = B(x)\operatorname{diag}(s_{11},\ldots,s_{nn}).$$

On the other hand, if $p_\ell \geq q_\ell$, $\ell = 1,\ldots,m-1$, $m < n$, and $p_m < q_m$ in sequences (20) and (21), we construct a matrix

$$S_0 = \left\| \begin{matrix} 1 & 0 & \ldots & 0 & s_{1n} \\ & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & & & & 1 \end{matrix} \right\|,$$

where the element s_{12} satisfies the relation

$$s_{1n}a_{m+1,1r}a_{n1t} - a_{m+1,1p_m} = 0.$$

Here, $a_{m+1,1r}$, a_{n1t} are the lowest coefficients of the elements $a_{m+1,1}(x)$, $a_{n1}(x)$, respectively, and $a_{m+1,1p_m}$ is the coefficient of monomial of degree p_m of the element $a_{m+1,1}(x)$ of the matrix $A(x)$. According to Proposition 7, the product $S_0A(x)$ is right-equivalent to the matrix $B(x)$ satisfying the conditions of Proposition 5. One can easily make sure that, in the matrix $B(x)$, the element in the position $(m+1,1)$ does not contain a monomial of degree p_m.

Suppose that m is the minimal subscript for which the inequality $p_\ell \geq q_\ell$ is violated and, in the semiscalar-equivalent matrices $A(x)$ and $B(x)$, the $(m+1)$th elements of the first columns do not contain a monomial of degree p_m. According to Proposition 6, all nondiagonal elements of the matrix S in equality (3), except the element in the position $(1,n)$, are equal to zero. Thus, by using equality (3) and comparing elements in the positions $(m+1,1)$, we conclude that the matrix S is in fact diagonal. Simple considerations lead to the equality $S = R(x)$. Therefore, the matrices $A(x)$ and $B(x)$ are diagonal similar.

The theorem is proved.

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