Archimedes and the Elements: *Proposal for a Revised Chronological Ordering of the Archimedean Corpus*

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Summary. With the rediscovery by J. L. HEIBERG of the manuscript of ARCHIMEDES' *Method* and its publication by him in 1907, a suspicion held for centuries by mathematicians and mathematical historians was at last corroborated by direct evidence: ARCHIMEDES had indeed applied in his discovery of geometric theorems a heuristic technique of analysis, although the highly formal synthetic proofs given in their demonstration concealed this fact.[1] Disappointingly little

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[1] J. L. HEIBERG, "Neue Archimedeshandschrift," pp. 300–301. Many of the geometers of the 17th century complained of the opaqueness of the ancient formal methods: TORRICELLI was convinced that the ancients possessed heuristic methods which they deliberately concealed (*Opere*, I, 1919, p. 140); DESCARTES sometimes suspected they had an analytic method for the discovery of new theorems (Rule IV in *Regulae ad directionem ingenii*, 1629; *cf.* NEWTON, *Mathematical Papers*, ed. D. T. WHITESIDE, IV, p. 277). For discussions of this 17th-century attitude, see C. BOYER, *Calculus*, ch. IV and D. T. WHITESIDE, *Mathematical Thought*, ch. VIII and IX.

effort has since been made to search for other indications of growth in the composition of ARCHIMEDES' treatises. In particular, no one has attempted to exploit the relative chronology determined by J. TORELLI in 1792 for a major part of the corpus. Despite the potential of even this partial ordering for a deeper understanding of ARCHIMEDES' work, no subsequent edition, translation or commentary has retained it as the organizing principle. In the present re-examination of this question of Archimedean chronology, I will (1) show how tenuous are the foundations of the orderings now in standard usage; (2) present a chronological order which differs from the standard accepted sequence in certain important respects, notably, in the position of the *Dimension of the Circle* and the *Method;* and (3) distinguish an "early group" of studies from the "mature group", employing as my criterion the predominantly Euclidean character of the techniques used in the former. I believe that the results of my inquiry have broader implications, as in reinterpreting as Archimedean certain contributions preserved by such later authors as HERO and PAPPUS.

I. Traditional views on the order of the works

Most of the extant treatises of ARCHIMEDES derive from three independent manuscript codices.[2] One is the prototype in Greek (no longer surviving) from which several copies were made in the fifteenth and sixteenth centuries (HEIBERG designated this source *A*). A second is the Latin translation made by WILLIAM of Moerbeke in the thirteenth century (HEIBERG's *B*). The third (*C*) is the now famous Greek palimpsest, a tenth-century manuscript, partially erased and over-written a few centuries later, whose Archimedean provenance thereafter went unsuspected until HEIBERG identified it in 1906. The codices present the separate treatises in differing orders (see Table 1), none according fully with the sequence of composition of the works, even to the degree that this can be established through the explicit evidence of the texts. Following the start made by TORELLI in 1792, HEIBERG had in 1879 attempted to determine the relative chronology of the treatises then known to him.[3] But in setting them out in his ensuing editions of ARCHIMEDES he chose to retain the traditional order in the principal manuscripts, based on the prototype *A*, and then tacked on the few remaining works and fragments preserved in other sources (see Table 2). HEIBERG's ordering has been adopted in all subsequent editions and translations, notably those by T. L. HEATH, P. VER EECKE, E. J. DIJKSTERHUIS and C. MUGLER. Indeed, VER EECKE pronounced it to be of all possible orderings "le plus rationnel."[4] What began as merely a philological concern to keep strictly to the sequence of the manuscript

[2] For a detailed summary of the character of the codices, see J. L. HEIBERG, *Archimedes*, III, "Prolegomena critica." These are reviewed by HEATH, *Archimedes*, ch. II and *HGM*, *II*, pp. 25–27, and by DIJKSTERHUIS, *Archimedes*, ch. II.

[3] J. L. HEIBERG, *Quaestiones Archimedeae*, pp. 10–12, based on J. TORELLI, *Archimedis Opera*, Oxford, 1792, p. xiii.

[4] VER EECKE, *Archimède*, p. xxxi. HEATH follows the ordering of HEIBERG's first edition (1880–81); DIJKSTERHUIS follows that of the revised edition (1910–13), with a few changes in order for convenience of exposition (*e.g.*, he places the discussion of the *Method* before that of *QP*).

Table 1: Order of the Archimedean works in the principal manuscript codices

A (Greek prototype, lost; 9th century)	*B* (MOERBEKE'S Latin; 13th century)	*C* (Greek palimpsest; 10th century)
Sphere and Cylinder I, II	Spiral Lines	... Plane Equilibria II
Dimension of the Circle	Plane Equilibria	Floating Bodies I, II
Conoids and Spheroids	Quadrature of the Parabola	Method ...
Spiral Lines	Dimension of the Circle	... Spiral Lines
Plane Equilibria I, II	Sphere and Cylinder I, II	Sphere and Cylinder I, II ...
Sand Reckoner	EUTOCIUS' commentaries on Sphere and Cylinder I, II	... Dimension of the Circle
Quadrature of the Parabola	Conoids and Spheroids	Stomachion
EUTOCIUS' commentaries on Sphere and Cylinder I, II, Dimension of the Circle, and Plane Equilibria I, II	EUTOCIUS' commentaries on Plane Equilibria I, II	
	Floating Bodies I, II	

Table 2. Order of the treatises in HEIBERG'S second edition (1910–15)

Sphere and Cylinder I, II
Dimension of the Circle
Conoids and Spheroids
Spiral Lines
Plane Equilibria I, II
Sand Reckoner
Quadrature of the Parabola
Floating Bodies I, II
Stomachion
Method
Book of Lemmas
Cattle Problem
Fragments
EUTOCIUS' commentaries on Sphere and Cylinder I, II, Dimension of the Circle, and Plane Equilibria I, II

Table 3. Chronological order proposed by HEATH (1921)

Plane Equilibria I
Quadrature of the Parabola
Plane Equilibria II
Method
Sphere and Cylinder I, II
Spiral Lines
Conoids and Spheroids
Floating Bodies I, II
Dimension of the Circle
Sand Reckoner

sources has thus since given rise to the astonishing view that this ordering has intrinsic rational merit, despite such patent incongruities as the placing of the *Sand Reckoner* and the *Quadrature of the Parabola* and others to be discussed below.[5]

[5] G. SARTON, *History of Science*, II, p. 22 appears to have discerned a principle of ordering by decreasing length! So much for the "rationality" of the ordering of the treatises in the MSS.

Whenever the question of the chronological sequence of ARCHIMEDES' works is raised – as in the recent editions of MUGLER and DIJKSTERHUIS, or in such reference works as R. TATON's *Histoire générale des sciences* or the *Dictionary of Scientific Biography* – the scheme shown in Table 3 is provided. This derives from that given in HEATH's standard *History of Greek Mathematics* (1921), which slightly modifies that in his earlier English paraphrase of ARCHIMEDES (1897).[6] HEATH in his turn relied on HEIBERG'S dissertation (1879), while HEIBERG drew his ordering in large part from TORELLI (1792).[7]

What is this ordering based on? In the cases of the five treatises addressed to DOSITHEUS, each of which bears a prefatory letter, sufficient cross-citations are made between them to establish the order of their composition. ARCHIMEDES opens the preface to the *Quadrature of the Parabola* thus:

> Having heard that Conon, who never failed us in friendship, was dead, but that you had become an acquaintance of Conon and were familiar with geometry, ... we have taken it in hand to send to you, as we had determined to write to Conon, a certain geometric theorem.[8]

From this it is clear that ARCHIMEDES had previously had no direct communication with DOSITHEUS. In the preface to *Sphere and Cylinder I* he begins thus:

> We earlier wrote and sent to you some things studied by us with proof, that every segment bounded by a line and a section of a right-angled cone is four-thirds the triangle having the same base as the segment and equal height.[9]

As the theorem stated here is the one proved in *Quadrature of the Parabola*, we infer *QP* to be the earlier communication. In the same way the preface to *Sphere and Cylinder II* enunciates four of the theorems contained in *SC I* with the indication that their proofs have already been sent, while other theorems "on the spirals and on the conoids" are promised in due course.[10] In the preface to *Spiral Lines* ARCHIMEDES includes a list of all the theorems in *SC II*, indicating that their proofs have been previously sent; and he adds the statements of definitions of conoids (paraboloids of revolution) and of four theorems, but says the proofs have not yet been communicated.[11] These latter theorems appear in *Conoids and Spheroids*, together with results on the volumes of spheroids and "obtuse-angled" conoids (ellipsoids and hyperboloids of revolution, respectively) which, according to the preface of that work, were recent discoveries of his.[12] Thus, the sequence of

[6] HEATH, *HGM*, *II*, p. 22; *Archimedes*, p. xxxi. DIJKSTERHUIS, *Archimedes*, pp. 45–47. MUGLER, *Archimède*, I, pp. xvii–iii. J. ITARD in R. TATON, *Histoire générale des sciences*, Paris, 1957, I, pp. 328 f. M. CLAGETT in *DSB*, "Archimedes," I, p. 214. Such a list could be significantly extended if articles and more general works were brought into account.

[7] See note 3.

[8] ARCHIMEDES (HEIBERG), II, p. 262. All translations from ARCHIMEDES are my own, based on the revised text by HEIBERG (1910–13).

[9] Arch. (Heib.), I, p. 2.

[10] *Ibid.*, pp. 168–170.

[11] *Ibid.*, II, pp. 2–12.

[12] *Ibid.*, I, p. 246.

the five treatises is determined to be (in advancing order of composition): $QP \rightarrow SC\ I \rightarrow SC\ II \rightarrow SL \rightarrow CS$. Moreover, in the preface to *SL* ARCHIMEDES remarks that "many years have passed since the death of Conon."[13] We cannot therefore be far wrong in assigning a period of from five to ten years for the completion of this sequence. While, as we shall discuss below, much of this material involved giving formal demonstrations of results discovered earlier by ARCHIMEDES and communicated by him without proof to CONON (as he makes clear in the preface to *SL*), he made a number of new discoveries during this period (see the preface to *CS*) and also certain advances in overcoming difficulties which arose in the course of his formalizing the proofs.

This much appears sure. But what of the remaining works? To place these in sequence HEIBERG sought instances where ARCHIMEDES assumes as already proven theorems which in fact appear elsewhere in the corpus. Thus, as the *Quadrature of the Parabola* requires several of the mechanical theorems proved in *Plane Equilibria I*, one infers that *QP* must follow *PE I* in order of composition; similarly, *PE II* requires in its argument theorems on parabolas and the area of parabolic segments which are to be found in *QP*. Again, as *Floating Bodies II* employs theorems on the volume of paraboloidal segments, one infers that *FB I* and *II* must follow *Conoids and Spheroids*. In the *Sand Reckoner* (I, 19) ARCHIMEDES asserts as "having been proved by us" that the circumference of the circle is less than $3\frac{1}{7}$ of its diameter; since the proof is given in the third proposition of *Dimension of the Circle*, we may conclude that *SR* is to be placed after *DC*. Such is HEIBERG's argument. Yet, valid as this criterion might seem, we should take care not to apply it incautiously; for the sequence of composition of several of ARCHIMEDES' works was demonstrably more complicated than HEIBERG allowed. For instance, if *PE II* intervened in the sequence between *QP* and *SC I*, as HEIBERG claims, why does ARCHIMEDES omit mention of it in the preface to *SC I*, where he recalls his previous communication of theorems on the parabola? It is at least possible that the technique assumed in *PE II* was based on an alternative treatment of the parabola, of which the extant *QP* was a later reworking. Indeed, we shall discover reasons for preferring this view to the one advanced by HEIBERG. Further, in the case of the two works, *Dimension of the Circle* and *Method*, HEIBERG's conclusions are especially tenuous and warrant careful re-examination.

HEATH, relying on HEIBERG, has set the *Dimension of the Circle* ninth by order of composition in his sequence of ten Archimedean works, and this authoritative judgment has been unanimously accepted ever since. But on what is it based? HEIBERG, like TORELLI before him, had recognized that there exists an affinity between *DC* and *SC I*. In particular, in *DC*, prop. 1 ARCHIMEDES assumes, without providing any proof, that the sequence of polygons inscribed in a circle comes to approach arbitrarily closely in area to the bounding circle, as the number of their sides is successively doubled; the same assumption, again without proof, is made in *SC I*, 6. Moreover, axioms and lemmas on the comparative lengths of convex arcs are required, but are not explicitly stated, in the proof in *DC*, 1 of the convergence of the sequence of circumscribed polygons; these same items are stated and proved in the opening parts of *SC I*. TORELLI observed further that the principal results

[13] *Ibid.*, II, p. 2.

proved in *SC I* and *II* (*e.g.*, the measurements of areas and volumes of spheres and spherical segments) have no practical utility without some numerical estimate such as that derived in *DC*, 3 (sc. $3\frac{10}{71} < \pi < 3\frac{1}{7}$). On such grounds, then, *DC* was placed after *SC I* and *II* in order of composition. Yet HEIBERG was apparently not fully convinced of the force of this argument. So he set *DC* at the *end* of his own list, with a question-mark (?) to indicate that its placement in the series was uncertain. HEATH, in his edition of 1897, removed the question-mark, but added the comment:

> with regard to [the *Dimension of the Circle*], no more is certain than that it was written after [*On the Sphere and Cylinder I, II*] and before [the *Sand-reckoner*].[14]

In his later *History* (1921) HEATH omitted this qualifying remark, with the result that editors and historians have been misled ever since.

HEIBERG and ZEUTHEN sought to establish that the *Method* occupied a relatively early position in the chronological order.[15] As ARCHIMEDES refers at the end of prop. 1 to a geometric proof "which we discovered and published earlier," and which would appear to be that of *QP*, 24, the *Method*, they argued, is presumably subsequent to *QP* as well as to *PE I*, on whose mechanical theorems both works heavily rely.[16] But allusions to other Archimedean works in the *Method* are not so straightforwardly identified. A comment made at the end of *M*, 2 – on the manner of deducing the surface of the sphere as a corollary to its volume – reverses the order of the corresponding theorems in *SC I* (33 and 34, respectively).[17] Moreover, the study of the volume of the sphere in *SC I*, 34 shows no trace of the division into parallel plane sections adopted in *M*, 2. ZEUTHEN infers that the *Method* was produced and communicated before *SC I*. He proposes further that it was the *Method* on which ARCHIMEDES based the theorems listed without proof in the communication to CONON (the list reproduced in the preface to *Spiral Lines*), and whose formal demonstrations were later provided in the treatises sent to DOSITHEUS. Nevertheless, ZEUTHEN expresses strong reservations as to whether his conjecture is necessarily correct, and adds a short statement on the possibility of holding the contrary view: namely, that the *Method* might have been a late work describing the heuristic background to the proofs of theorems already published (such as those communicated to DOSITHEUS) on the measurement of parabolic segments, spheres, conoids and spheroids. HEATH chose to accept the view that the *Method* is an early work, but he transmitted neither the arguments for this view, nor ZEUTHEN's reservations about it. With few exceptions, later scholars have adopted HEATH's view without comment.[18]

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[14] HEATH, *Archimedes*, p. xxxii.

[15] This chronological placement for the *Method* is argued by ZEUTHEN, "Neue Schrift des Archimedes," pp. 359–363, in his commentary on HEIBERG's translation of the work in the same article. His argument is examined in detail by F. ARENDT ("Zu Archimedes," pp. 289–296) who, pursuing some terminological leads opened by T. KIERBOE earlier ("Terminologie des Archimedes," pp. 38–40), rejects ZEUTHEN's conclusion and settles instead upon a very late placement for the *Method*.

[16] Arch. (Heib.), II, p. 438. On difficulties in the text of this passage, see note 40 below.

[17] See notes 31 and 72 and the associated text below.

In the discussion which follows I will argue that the *Dimension of the Circle* was an early composition – perhaps, indeed, the earliest – in the sequence of the extant works of ARCHIMEDES. I will also undertake, by extending some observations made by F. ARENDT, to establish the lateness of the *Method*. But more is at issue here than the mere chronological ordering of a set of writings. For the proposed order will yield a new criterion for distinguishing early works from late ones in the Archimedean corpus: namely, the greater application of standard Euclidean techniques in the earlier works, as against the increasing employment of ARCHIMEDES' own techniques in the later ones. We shall thus obtain, should I be right, a real sense of ARCHIMEDES' growth and development as a mathematician, and shall come to oppose the prevalent ahistorical attitude which assumes for the whole corpus of ARCHIMEDES' works, apart from the *Method*, an ever-present and uniformly rigorous formal technique.[19]

II. Studies of the circle and related figures: early works

1. Placing the *Dimension of the Circle*

The claim that the *Dimension of the Circle* was a late work should strike us at once as implausible. For of all ARCHIMEDES' geometrical treatises, this is the easiest to penetrate, barely going beyond *Elements* XII, 2 in its technical demands. A confirmation of this is its frequent citation in antiquity, for instance, by HERO, PTOLEMY, PAPPUS, THEON and PROCLUS.[20] Like *Sphere and Cylinder* and the *Method*, it remained in circulation as a standard work for centuries after ARCHIMEDES, as we infer from the references in HERO's *Metrica;* also like them, the extant text has been stripped of its original Doric dialect, a fact which has plausibly been taken to reflect its wide and frequent use in mathematical instruction. It was one of but five Archimedean works still available to the commentator EUTOCIUS in the sixth century and was familiar to mathematicians in the medieval period, both in Islam and in Latin Europe.[21]

The arguments by TORELLI and HEIBERG, reviewed above, establish a certain association of *DC* and the opening theorems of *SC I*, but these are compatible equally well with the composition of *DC* before *SC I* as after it. For instance, in both

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[18] Of the writers listed in note 6 above, ITARD and MUGLER (following ITARD) set the *Method* last in order; all the others accept the early placement.

[19] HEATH, *HGM, II,* p. 20: "The treatises are, without exception, monuments of mathematical exposition ... ." Of course, he recognizes the *Method* to be just such an exception to this judgment (p. 21).

[20] HERO, *Metrica,* I, 25 and 37. PTOLEMY, *Syntaxis,* VI, 7 (ed. J. L. HEIBERG, p. 513). PAPPUS, *In Ptolemaeum* (ed. A. ROME), pp. 253–260 and *Collection* (ed. F. HULTSCH), pp. 312, 336ff. PROCLUS, *In Euclidem* (ed. G. FRIEDLEIN), p. 423.

[21] On the works still available to EUTOCIUS, see HEIBERG, *Archimedes,* III, p. xcii. On the works available in the Middle Ages, see M. CLAGETT, "Archimedes," *DSB,* I, pp. 223–229.

DC, 1 and *SC I*, 6, the convergence of the inscribed polygons to the circle is accepted without proof. But in *SC I*, 6 ARCHIMEDES supports this by explicit citation of the *Elements*, and, in fact, the needed demonstration appears in XII, 2. Presumably, ARCHIMEDES could do the same in *DC*, 1. Further, ARCHIMEDES applies in both theorems the fact that the perimeter of any circumscribed polygon is greater than the circumference of the circle. This is established as a theorem in *SCI*, 1, based on a special axiom:

> Of two lines lying in the same plane and having the same extreme points and convex in the same direction, such that one is wholly included between the other and the straight line connecting the same points, the included line is the lesser.[22]

As *DC*, 1 likewise requires *SC I*, 1 and this axiom, must we conclude that *DC* was composed after *SCI* in order that such an appeal would be possible? Not necessarily, as the following two considerations indicate: (a) The tract on the *Dimension of the Circle* as now extant is patently a fragment from a larger study of the circle. PAPPUS, for instance, cites from an Archimedean work he calls *On the Circumference of the Circle* a theorem on the area of any sector of a circle; but this is not included in our version of *DC*.[23] We thus do not know how many of the unproven steps in the proofs of *DC*, as extant now, were, in fact, justified by other theorems in the original version. It is quite possible that the requisite axiom was stated in the complete version. The assumption of the earlier composition of *DC* need not conflict with the restatement of such axioms and lemmas in *SC I*. For ARCHIMEDES does sometimes repeat important axioms as required in his treatises.[24] Further, as an early work, *DC* may plausibly be supposed to have been addressed to CONON, while *SC I* was sent to DOSITHEUS. This situation might have recommended to ARCHIMEDES that he repeat those lemmas and axioms possibly unfamiliar to the new correspondent. (b) Since we do not possess *DC* in its original form, we cannot know for certain whether it was composed as a fully systematic and formal treatment of the type of the other extant treatises. Indeed, if the study of the circle was early, we might expect that ARCHIMEDES' mastery of formal proof was less secure in *DC* than in later works. An assertion like "the perimeter of the circumscribed polygon is greater than the circumference of the circle" (*SC I*, 1) might at first be accepted as obvious, the need for a proof based on some more general principle being perceived only later.[25] We shall propose further support for this latter alternative in what follows.

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[22] Arch. (Heib.), I, p. 8. I have rendered this passage somewhat freely.

[23] PAPPUS, *In Ptolemaeum*, pp. 256, 258n.; HERO, *Metrica*, I, 37. That *DC*, as now extant, is a fragment of a more extensive treatment by ARCHIMEDES of the circle is generally agreed; *cf.* HEATH, *HGM II*, p. 50 and DIJKSTERHUIS, p. 222.

[24] The "Archimedean axiom" on convergence is stated three times, *viz.* in the prefaces to *QP*, *SC I* and *SL*. It sometimes happens that whole theorems are repeated: the lemma on the summation of square terms is proved in *SL*, 10 and again before *CS*, 2; the equilibrium of parabolic segments on a balance is proved in *PE II*, 1, although this is merely a special case of *PE I*, 6.

[25] When THEON introduces this result into his presentation of isoperimetric figures, he says that "Archimedes assumes (λαμβάνει) this in *Sphere and Cylinder*" (*In Ptolemaeum*, ed. A. ROME, p. 359). In precisely the same context, PAPPUS says that the same

These observations thus establish the affinity of *DC* to the opening part of *SC I* and the possibility that *DC* was the earlier composition. But a closer examination of the proofs of *DC*, 1 and *SC I*, 6 reveals the more primitive technical level of the former. In particular, ARCHIMEDES' method of establishing convergence, both for inscribed and circumscribed polygons, in *DC*, 1 is just that employed in the *Elements* (XII, 2), whereas a far more sophisticated technique is used in *SC I*, 6. This contrast, which we will elaborate on below, strongly suggests that these theorems in *SC I* were a refinement upon the earlier treatment of the circle in *DC*.

As DIJKSTERHUIS has observed, ARCHIMEDES applies the "method of exhaustion" (or "indirect method of limits") in three different forms:[26]

(a) Under the "approximation" method one inscribes within a curved figure a sequence of polygons and establishes that the difference in content between the curved figure and the polygons becomes less than any preassigned magnitude when the polygons are taken with sufficiently many sides. Usually, each polygon in the sequence is constructed as having double the number of sides of its predecessor and convergence is assured via the principle of successive bisection of the difference (*Elements* X, 1). Of course, the same technique will establish the convergence of a sequence of circumscribed polygons downward to the limiting curve.

(b) Under the "compression method – difference form" the curvilinear figure is enclosed by sequences of circumscribed and inscribed rectilinear figures, such that the difference between the bounding figures can be made less than any finite preassigned magnitude. Thus, the enclosed curve *a fortiori* differs from terms of both sequences by less than that amount. This is the manner used in *QP*, 16; in the area theorems of *SL* (21, 22, 23); in *CS*, 19, 20; and in *M*, 15.[27]

(c) Under the "compression method – ratio form" polygonal figures may be constructed to bound a curvilinear figure above and below, such that the ratio of the polygons is closer to unity than any preassigned ratio. It follows *a fortiori* that polygons may be found to approach the curve arbitrarily closely, both in ratio and in difference. ARCHIMEDES uses this method in *SC I*, 2–6 and in the tangent theorems in *SL*.[28]

The "approximation method" is nothing other than the Eudoxean-Euclidean method of limits, employed throughout *Elements* XII. DIJKSTERHUIS asserts that

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result "is supposed (ὑπόκειται) by ARCHIMEDES in *Sphere and Cylinder*" (*Collection V*, 2, ed. HULTSCH, p. 312). Since this result is *proved* as a *theorem* in *SC I*, 1, both HULTSCH and ROME have elected to delete the verbs as interpolations. We shall see, however, that a version of *SC I* different from that now extant was in circulation in late antiquity, and that in it such results could well have appeared as unproven preliminary assumptions rather than as theorems. These statements by PAPPUS and THEON, as transmitted, may thus have been based on the alternative version of *SC I*.

[26] DIJKSTERHUIS, *Archimedes*, pp. 130–133. One may observe that the Greeks never adopted a term to designate the technique we commonly call the "method of exhaustion" (a phrase coined in the 17th century).

[27] These theorems are used to establish *SL*, 24–26 and *CS*, 21–22, 25–26, 27–30.

[28] *SC I*, 2–6 are ancillary to the proofs of *SC I*, 13–14, 33–34, 42 and 44. A limiting method based on ratios is implicit in the constructions through *neuses*, or "inclinations" of a sliding ruler, in *SL*, 5–9, which are employed in the proofs of the theorems on the tangents to the spiral in *SL*, 18–20.

ARCHIMEDES uses it only once, in the "geometric proof" of the area of the parabolic segment, *QP*, 20 (the convergence lemma for *QP*, 24); he contends that ARCHIMEDES uses the "compression-by-difference" method in *DC*, 1.[29] But in this DIJKSTERHUIS is mistaken. For in *DC*, 1 ARCHIMEDES does not enclose the circle by simultaneously converging sequences of inscribed and circumscribed regular polygons. Rather, he treats separately the inscribed and circumscribed cases, establishing convergence for each by means of an "approximation" method. For the inscribed polygons, he defers (implicitly) to *Elements* XII, 2, just as he does for the inscribed case in *SC I*, 6. For the circumscribed case of *DC*, 1, however, he presents a proof relying on the bisection criterion of *Elements* X, 1: he shows that each doubling of the number of sides of the circumscribed polygons reduces the difference from the area of the circle by more than half.[30] This contrasts with the approach in *SC I*, 6, where the "compression-by-ratio" method is used: that is, two similar regular polygons may be constructed, one inscribed in the circle, the other circumscribing the circle, such that the ratio of their sides is closer to unity than a preassigned ratio (via *SC I*, 3); it follows that their areas too may be made arbitrarily close to each other (*SC I*, 5), whence the circle may be circumscribed by a regular polygon differing from it by an arbitrarily small magnitude (*SC I*, 6). Thus, the comparison between *DC*, 1 and *SC I*, 6 reveals how closely ARCHIMEDES adheres to the Eudoxean-Euclidean method of establishing convergence in the former, while making full use of his own adaptation, a new convergence method of the type employed throughout his later works, in the latter. This is, I maintain, a strong recommendation for assigning to *DC* an earlier date of composition than that of *SC I*.[30a] (This conclusion gains additional support through a comparison of the methods of proof used in *SC I*, 6 and *DC*, 3. See *Note 1, added in proof.*)

This conclusion is affirmed explicitly in a remark ARCHIMEDES makes after prop. 2 in the *Method:*

> Once it was perceived (1) that every sphere is four times the cone having as its base the greatest circle and as its height the line drawn from the center of the sphere, the thought occurred (2) that the surface of every sphere is four times the circle greatest of those in the sphere; for there was the apprehension both (3) that every circle is equal to the triangle having as its base the circumference of the circle and as its height the line drawn from the center, and also (4) that every

[29] DIJKSTERHUIS, p. 132.

[30] See Appendix, section 1.

[30a] PAPPUS and HERO attribute to ARCHIMEDES a theorem on the area of the sector of a circle (see note 23). PAPPUS preserves a proof of this theorem, in which the technique of convergence parallels that in *SC I*, 6: notably, the polygon which circumscribes the sector is constructed as similar to the inscribed polygon; thus, the convergence argument can be significantly abridged (*In Ptolemaeum*, p. 260). Unfortunately, the argument is faulty: the fact that the inscribed polygon differs from the sector by less than a given magnitude does not guarantee that the same is true of the circumscribing polygon similar to it. It is possible that PAPPUS has committed this error in the course of drafting a proof of the theorem along the lines of *SC I*, 6. Alternatively, he has reproduced an early Archimedean version in which the proof was not completely formalized. If the latter view is correct, PAPPUS' text preserves a link between the technique in *DC*, 1 and that in *SC I*, 6.

> sphere is equal to the cone having as its base the surface of the sphere and as its height the line drawn from the center of the sphere.[31]

Theorem (3) is the expression for the area of the circle proved in *DC*, 1. ARCHIMEDES thus explains how the analogous expression for the volume of the sphere (4) enabled him to recognize the connection between the measurement of the volume of the sphere (1) and that of its surface (2). Theorems (1) and (2) are, respectively, the 34th and 33rd propositions in *SC I*, the results toward whose proofs most of the theorems in that book are directed. In other words, the result in *DC*, 1 was implicit in the preliminary thinking-through of the organization of *SC I*. It is also important to observe that ARCHIMEDES leaves no impression that (3) or (4) was a recent discovery at the time he was exploring heuristically the measurement of the sphere (1), or that the "mechanical method" by which he studied (1), as laid out in *Method*, prop. 2, was in any way responsible for his perception of (3) and (4).

We thus have presented two arguments to support the claim that the *Dimension of the Circle* preceded *Sphere and Cylinder I* in composition: (a) the convergence argument in *DC*, 1 follows that of the *Elements*, whereas *SC I*, 6 employs a method characteristic of ARCHIMEDES' most sophisticated works; (b) the result proved in *DC*, 1 was already familiar to ARCHIMEDES at the time of his heuristic study of the sphere (as in *M*, 2), preliminary to the formal organization of *SC I*. We will now argue, further, that *DC* predated not only *SC I*, but also the earlier *Quadrature of the Parabola*.

As before, one important indication is the manner of establishing convergence. In *QP* ARCHIMEDES first presents the necessary theorems on the parabola (prop. 1–5) and on the mechanical properties of rectilinear figures (prop. 6–13). He next constructs rectilinear figures which circumscribe or are inscribed in the given parabolic segment and establishes, by means of a mechanical argument, inequalities between these rectilinear figures and a specified triangle (prop. 14–15). These culminate in the theorem that the parabolic segment is four-thirds the triangle having the same base and height (prop. 17). The critical convergence-argument appears in prop. 16 where the "compression-by-difference" method is used. In its use of this method, *QP*, 16 thus manifests the more advanced stage of ARCHIMEDES' technique, in contrast with *DC*, 1, which adheres to the technique of the *Elements*.

Another feature of ARCHIMEDES' convergence method in *QP*, 16 is its application of the principle of continuity in a form different from the Euclidean bisection-principle (X, 1) to which appeal is made in *DC*, 1. The "Archimedean axiom" receives the following formulation in the preface to *QP*:

> Of unequal areas the excess by which the greater exceeds the lesser can, when added to itself exceed any preassigned finite area.[32]

This axiom is important for ARCHIMEDES. He will state it again (in a form extended to cover the cases of lines and solids, as well as areas) in the prefaces to *Sphere and Cylinder I* and to *Spiral Lines*. But it is interesting to note ARCHIMEDES' defensive

[31] Arch. (Heib.), II, p. 446. We shall return to this passage when we come to date the *Method*; see note 72 and the associated text.

[32] *Ibid*., p. 264. See HEIBERG's note, p. 265n.

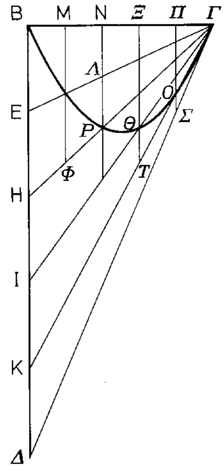


Fig. 1

tone as he admits to the need for this axiom (or "lemma") in the preface to *QP:* he recalls how earlier geometers had attempted the quadrature of the circle, but required lemmas "not readily admitted"; he states the axiom precisely – apparently viewing it as an innovation – yet insists that the need for such a special axiom does not diminish the value of his proofs, for earlier geometers (comparison with the preface to *SC I* indicates that EUDOXUS is intended) made use of a very similar assumption. The latter assumption can be identified as the bisection-principle of *Elements* X, 1 on the grounds that ARCHIMEDES names four theorems which required it and these all happen to appear in *Elements* XII and make appeal to *X*, 1.[33] What is odd about this is that the bisection-principle was perfectly adequate for ARCHIMEDES' purposes in *QP*.

Let us review how the "Archimedean axiom" is employed in *QP*, 16. ARCHIMEDES intends to show that the given parabolic segment equals a rectilinear area Z, one-third the triangle BΓΔ enclosing the segment (as in Fig. 1) – where ΓΔ is tangent to the segment, BΔ parallel to its diameter, and BΓ perpendicular to the diameter.[34] If the areas are not equal, then let it first be assumed that the segment is the greater. By the "axiom" the excess of the segment over the area Z may be added to itself sufficiently many times until it exceeds triangle BΓΔ. The line BΔ is next divided into the same number of equal parts, so that the whole triangle BΓΔ has been divided into smaller triangles each equal to BΓE. Moreover, BΓE is less than the difference between the area of the segment and Z. Two rectilinear figures may be marked off, ΓΣOTΘ... EB circumscribing the segment, the other in similar fashion inscribed in the segment. The difference between these bounding figures is the triangle ΓΣO together with the trapezia OΘ, ΘP, *etc.*, and their aggregate equals the area of triangle BΓE and is thus less than the difference between the segment and Z.

[33] See HEIBERG's note, *ibid.*, p. 265n. The theorems named are XII, 2, 18, 7 and 10. X, 1 appears in the proofs of XII, 2, 16, 5 and 10.

[34] Here and in all other diagrams I follow the lettering in HEIBERG's text.

It follows that Z together with triangle ΒΓΕ is less than the segment, or that Z is less than the inscribed rectilinear figure. Since triangle ΒΓΔ is three times Z, it is less than three times the inscribed figure. But from *QP*, 14–15 it is known that triangle ΒΓΔ must be greater than three times the inscribed figure. In like manner, the assumption that the segment is less than Z leads to contradiction. Thus, the segment equals Z, one-third triangle ΒΓΔ.

The axiom thus serves to guide the construction of circumscribing and inscribed rectilinear figures whose difference shall be less than a preassigned amount, namely, the hypothesized difference between the segment and Z. But this could be done with equal facility by means of X, 1. One first constructs any pair of bounding figures (for instance, that shown in Fig. 1); as seen, their difference equals triangle ΒΓΕ. If, next, the number of divisions is doubled, by bisecting each of the intervals along the lines ΒΔ and ΒΓ, and the new pair of bounding figures marked off, their difference will be the triangle whose base is ΒΓ and whose height is half of ΒΕ; this triangle thus equals half triangle ΒΕΓ which was equal to the difference between the initial pair of bounding figures. Thus, this procedure of doubling the number of divisions reduces the difference between the bounding figures by one-half with each application. By X, 1, then, the bounding figures may be taken to differ by less than any preassigned amount and *a fortiori* the segment will differ from each bounding figure by less than this same amount.

In view of this, why did not ARCHIMEDES appeal to the bisection-principle in *QP*, 16? That he did not do so is especially surprising in that he appears to view the need to postulate the new axiom as an embarrassment and a potential point of criticism against his proofs. Did he perhaps perceive some subtle but important difference between these two forms of the lemma? If so, the use of the bisection-principle in *DC*, 1 might be taken as a further confirmation of its having been composed before *QP*. For, just as *QP*, 16 can be established via X, 1, so also can *DC*, 1 be effected by means of direct appeal to the Archimedean axiom – although the proof would require some major changes along the lines of *SC I*, 6. But the issue is rather more complicated than this. I will thus defer the fuller discussion of it until section III. 2 below.

We have noticed a third indication of the priority of *DC*. In the *Method* (at the end of prop. 2) ARCHIMEDES indicates that *DC*, 1 was already available when he began heuristically to investigate the content of *SC I*. Assuming that there was no major gap in time between this mechanical study of the sphere (*M*, 2) and the mechanical study of the area of parabolic segments (*M*, 1), we may place *DC*, 1 at least as early as ARCHIMEDES' heuristic study of the parabola, on which the formal version in *QP*, 16 was closely patterned. This corroborates our principal argument: that because the more advanced technique of convergence is applied in *QP*, 16, we may assign to it a date later than that of *DC*, 1.

2. Parabolic segments

The *Quadrature of the Parabola* presents two different proofs of the theorem on the area of parabolic segments: the first ARCHIMEDES calls "mechanical," the second "geometric." Now, the former – completed in *QP*, 16 – closely resembles the other "mechanical" treatment given in the *Method* (prop. 1). In the latter,

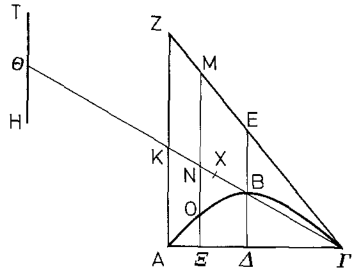


Fig. 2

ARCHIMEDES conceives the median of the enclosing triangle AΓZ to be extended a distance ΘK equal to KΓ (Fig. 2); these lengths are viewed as the equal arms of a balance, with fulcrum at K. If, now, any line MΞ in triangle AΓZ is taken, parallel to the diameter BΔ of the parabolic segment ABΓ, the proportion MΞ : OΞ = ΘK : NK follows, as a property of the parabola (*cf. QP*, 5). In the context of the balance, this means that if line OΞ in the parabola is transferred to Θ, it will be in equilibrium with line MΞ of the triangle suspended in position at N. As the same relation holds for all lines in the triangle and all lines in the parabolic segment, we infer by filling out the figures that the area of the parabolic segment, suspended at Θ is in equilibrium with the triangle, suspended in position. Since the center of gravity of the triangle is at X, one-third the length of KΓ (*cf. PE I*, 14), it follows that (segment) : (triangle) = KX : ΘK = 1 : 3 (via *PE I*, 6).

The "mechanical" treatment in *QP* develops along the same lines, as a comparison of the diagrams associated with *M*, 1 and *QP*, 16 (Figs. 2 and 1, respectively) reveals. In both, the parabolic segment and the enclosing triangle are divided into parallel components, but in *M*, 1 these are lines, in *QP*, 16 narrow trapezia. In both, these components are conceived to be in equilibrium, as if weighed on a balance, via the same property of the parabola (*QP*, 5). In *M*, 1 this leads directly to the desired conclusion – (segment):(triangle) = 1:3. But in *QP*, 14–15 inequalities result: namely, (inscribed figure):(triangle) < 1:3 and (circumscribed figure):(triangle) > 1:3. As we have seen, the final equality is obtained via the indirect argument presented in *QP*, 16. It thus seems clear that ARCHIMEDES has recast the treatment in *M*, 1 to produce the version in *QP*, 14–17. The reason is clear: as ARCHIMEDES asserts in the preface to the *Method*, and several times later in that work, the mechanical method does not constitute a *proof* of the theorem so studied; it does provide a "certain conviction" of the truth of the theorem, and is certainly a valuable instrument for the discovery of theorems and their proofs. But always the result of the inquiry must be established in a formally accepted "geometric" manner before it can be considered *proven*. Clearly, the introduction of indivisible components of magnitude – lines whose aggregate fill out an area, for instance – was a formally objectionable feature of the argument in *M*, 1. It is by way of eliminating the indivisibles that ARCHIMEDES produced the version of *QP*, 14–17.

Nevertheless, this version is still mechanical: the notions of the balance, of equilibrium and of center of gravity are crucial for the argument in *QP*, 14–15. We

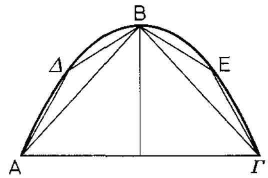


Fig. 3

should thus expect that in producing the fully "geometric" proof, ARCHIMEDES would seek to abolish the mechanical aspects of the treatment at hand, but otherwise retain its basic structure. This would accord with ARCHIMEDES' opinion that the mechanical method had its value for the discovery of the proofs of theorems; indeed, the formal proofs of the volume of the segment of the paraboloid (*CS*, 19–22) and of the "hoof" (*M*, 15) do proceed analogously to their heuristic treatments (*M*, 4 and 14, respectively). A formal proof of the segment of the parabola can be provided on the model of *QP*, 14–17 in several different ways. For instance, the area of the bounding rectilinear figures can be divided into ensembles of triangles increasing as the successive square integers. By means of the appropriate summation expression (*e.g.*, that proved in *SL*, 10) the inequalities in *QP*, 14–15 may be derived.[35] Just such a procedure is employed in the evaluation of the areas bounded by arcs of the spiral in *SL*, 24–26.

But the geometric proof actually provided by ARCHIMEDES is nothing of this kind. The division of the parabola is not into parallel sections, as in *QP*, 14–16, but into inscribed polygonal arcs the number of whose sides is successively doubled (see Fig. 3). This is remarkably close to the manner of approximating the circle employed in *DC*, 1 and *Elements* XII, 2. ARCHIMEDES shows that each doubling of the number of sides reduces the difference between the areas of the polygon and the segment by more than half (*QP*, 20), so that X, 1 applies to guarantee convergence. ARCHIMEDES thus adopts here the "approximation" method of the *Elements* instead of the "compression" method used in *QP*, 16. In fact, circumscribed polygons are not introduced at all, thus complicating the upper-bound argument in *QP*, 24. Further, the Euclidean lemma on convergence (X, 1) is used, rather than the "Archimedean axiom," whose necessity had been the occasion for ARCHIMEDES of the careful – one might say, pained – remarks in the preface to *QP*.

The anomalies of the geometric proof do not end with this. In further respects the set of theorems *QP*, 18–24 takes on the guise of an appendix, independent of the rest of *QP*. For instance, the terms "vertex", "base" and "height" of parabolic segments are defined just before *QP*, 18, even though all three terms have already been used in *QP*, 17.[36] Otherwise, these terms do not appear in the earlier part of the treatise and the lines they designate are conceived differently: in the later section, one begins with an area bounded by a straight line and a curve (the parabola will be a special case); the line is the "base" of the segment, while the

[35] See Appendix, section 2.

[36] Arch. (Heib.), II, p. 300. See HEIBERG's comment, *ibid.*, p. 299n (reproduced by HEATH, *Archimedes*, p. 246n).

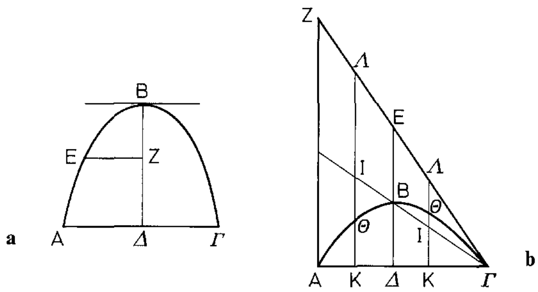


Fig. 4

"height" is the greatest of the lines drawn from the curve perpendicular to the base; the "vertex" is the point of the curve from which the height is drawn. In *QP*, 18 these are applied to the parabola: the line drawn from its vertex parallel to the diameter of the parabola will bisect the base; moreover, the proof establishes that the tangent at the vertex is parallel to the base. In the opening portion of *QP* the approach is different: the starting-points are the parabola, a point on it, and the diameter or a line parallel to it drawn from that point; it is then shown that any line in the parabola parallel to the tangent at the given point is bisected by the diameter or line parallel to it (*QP*, 1). Over all, however, the vertex of the segment and the tangent to it play no role in the first proof of the area theorem (*QP*, 14–17); they are central to the conception of the second proof (*QP*, 18–24). HEIBERG notes the oddity that the proof of *QP*, 17 employs the converse of *QP*, 18 (*i.e.*, the line drawn parallel to the diameter of the segment and bisecting the base meets the parabolic arc at its vertex). This disarray of the formal structure of the treatise around prop. 17 and 18 appears to follow from the converse attitude toward the terms defining the parabolic segment. Moreover, *QP*, 17 has the task of establishing the theorem on the area of the parabolic segment in a form unnatural to the structure of the inquiry leading to it. Throughout the mechanical treatment, both in *QP*, 14–16 and in *M*, 1 the area of the segment has been compared with that of the enclosing triangle (BΓΔ in Fig. 1; AΓZ in Fig. 2). In the geometric proof, the comparison is between the segment and the enclosed triangle ABΓ. Only in *QP*, 17 does the former treatment alter the reference to the inscribed triangle and so require the notion of the vertex of the segment. (See *Note 2, added in proof.*)

Furthermore, the mechanical and the geometric proofs develop around two different properties of the parabola. The geometric version makes key use of the proportionality $B\Delta : BZ = A\Delta^2 : EZ^2$ (Fig. 4a); this result had been stated in prop. 3 of *QP*, but no proof was provided, since "it is demonstrated in the Conic Elements" (that is, presumably, of ARISTAEUS or EUCLID). But in the mechanical versions a different proportionality is basic – $K\Theta : \Lambda\Theta = AK : K\Gamma$ (Fig. 4b); this result appears as *QP*, 5 and depends on *QP*, 4, for both of which full proofs are given. It thus appears that the technique of conics required for the mechanical treatment was somewhat more advanced than that needed for the geometric proof, the former introducing lemmas not available to ARCHIMEDES from the compilations on conics then in use.

Further, ARCHIMEDES' summation of the geometric progression in *QP*, 23 is noteworthy; for, instead of adapting the closely related Euclidean proof (IX, 35), he produces a completely new one. EUCLID's theorem establishes that for a finite sequence of integers A, B, C, D in continued proportion (the number of terms chosen, of course, is indifferent), $(B-A):A=(D-A):(A+B+C)$. The proof makes admirably efficient use of proportions. Since $A:B=B:C=C:D$, it follows *separando* that $(B-A):A=(C-B):B=(D-C):C$ (*Elements* VII, 11). By VII, 12, the sum of all the numerators and the sum of all the denominators will be in the same ratio; that is, $(D-A):(A+B+C)=(B-A):A$, which was to be proved. This may be adapted to cover all magnitudes, not just integers, in increasing or decreasing geometric progression; one merely replaces appeal to V, 17 to justify the proportion *separando*, and appeal to V, 12 to justify the proportion of the sums. In the case of *QP*, 23, the magnitudes A, B, C, D form a decreasing geometric progression, where $B:A=C:B=D:C=1:4$. It follows that $3:4=(A-D):(A+B+C)$, whence $A+B+C+D+\frac{1}{3}D=\frac{4}{3}A$, as claimed in *QP*, 23. But ARCHIMEDES does not attempt to employ the Euclidean method. The technique he adopts is, in fact, far less elegant than the Euclidean. In effect, he observes that $(B+\frac{1}{3}B)+(C+\frac{1}{3}C)+(D+\frac{1}{3}D)=\frac{1}{3}A+\frac{1}{3}B+\frac{1}{3}C$; thus, $A+B+C+D+\frac{1}{3}D=\frac{4}{3}A$. Why has ARCHIMEDES chosen to introduce this alternative proof of the special case? For it is common practice in his works to defer to prior works for proofs of lemmas required for his own theorems. He does this, as we have seen, in the opening to *QP*, where three properties of the parabola are asserted on the basis of proofs to be found in the "Elements of Conics."

One may take another unusual fact into consideration: that the theorem IX, 35 is more general than the purposes of EUCLID require – only the sum of the sequence of integers 1, 2, 4, 8, ... is employed in the proof of the "perfect number theorem" (IX, 36); moreover, both these theorems are, in respect of the refinement of their proof technique, notably out of place in the series of simple theorems on odd and even integers which precedes. One may suspect that the Euclidean theorems IX, 35 and 36, as now extant, were replacements, perhaps by EUCLID, of earlier, more elementary proofs, in the style of the proofs of IX, 21–34; indeed, this view has been proposed by BECKER in his examination of the ancient theory of the odd and even[36a]. ARCHIMEDES' not following the technique of IX, 35 in his proof of *QP*, 23 might then result from his not having in hand the Euclidean proof – for instance, from his studying arithmetic theory in its pre-Euclidean versions. This would further suggest that the drafting of the proof of the parabola was an early effort by ARCHIMEDES; for later in his career, he would surely have come upon the Euclidean edition and recognized its relevance for the proofs of his theorem.[36b]

For these reasons, then, the geometric proof of the area of the parabolic segment (*QP*, 18–24) has the appearance of an appendix to the mechanical treatment which constitutes the much larger part of the *Quadrature of the Parabola*. Moreover, each of the tokens of discontinuity of exposition between the two sections places the geometric treatment at the less advanced technical level, whether in the lemmas on

[36a] BECKER, *Quellen und Studien*, 1934, 3, pp. 539–544.

[36b] We shall encounter a similar instance of ARCHIMEDES' duplicating results in the Euclidean number theory in the *Sand Reckoner*. See section II.5 below.

conics employed or in the manner of establishing convergence. We saw, further, that *QP*, 17, the conclusion of the mechanical portion, serves to bring that treatment into conformity with the formulation of the area-theorem basic to the geometric part. This would appear to justify the inference that the geometric proof actually *preceded* the mechanical treatment chronologically. Such a view is admittedly odd, for it seems to reverse the logical order—heuristic inquiry developing into formal proof. But in what follows we shall discover further signs of its likelihood.

3. Circular segments

ARCHIMEDES' geometric proof of the area of the parabolic segment may be compared with the treatment of the circular segment by HERO of Alexandria. Whereas in *QP*, 24 ARCHIMEDES proves that

> any segment bounded by a straight line and the section of a right-angled cone is four-thirds the triangle having the same base and equal height,

in his *Metrica* (I, 27–29, 32) HERO shows that

> any segment of the circle is greater than four-thirds the triangle having the same base and equal height.

The two theorems receive closely analogous demonstrations.

HERO opens with the lemma (I, 27) that in any finite sequence of magnitudes A, B, C, D, each one-fourth the term preceding, the sum of all but the first together with one-third the last equals one-third the first. In effect, he argues that since $\frac{1}{3}A = B + \frac{1}{3}B$, $\frac{1}{3}B = C + \frac{1}{3}C$, and $\frac{1}{3}C = D + \frac{1}{3}D$, it follows that $\frac{1}{3}A = B + C + D + \frac{1}{3}D$. This is the same result proved by ARCHIMEDES in *QP*, 23 in essentially the same way. Their common failure to apply a technique comparable to that of *Elements* IX, 35 is especially noteworthy.

HERO next shows (I, 28) that in the circular segment ΑΒΓ, ΒΔ:ΕΖ < 4:3 (Fig. 5a). Analogously, ARCHIMEDES establishes in *QP*, 19 that for the parabolic arc ΑΒΓ, ΒΔ:ΕΖ = 4:3 (Fig. 5b)[37]. The proofs differ in that HERO uses the property of the circle —that a perpendicular to the diameter is the mean proportional between the

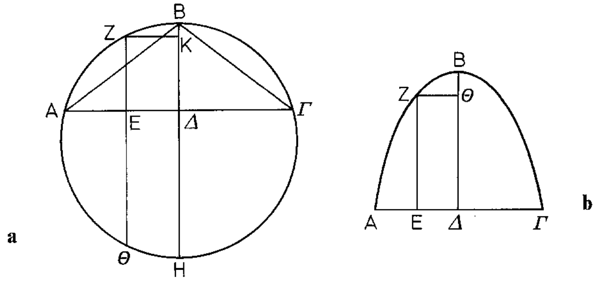


Fig. 5

[37] Note that the letterings in the two diagrams, which here so closely conform, are faithful to the texts of *QP* and the *Metrica;* they are not the result of editorial alteration.

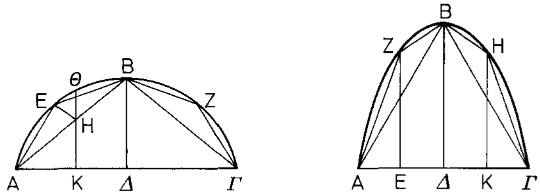
a  b

Fig. 6

segments of the diameter cut off—to obtain that ΔB:KB>4:1. ARCHIMEDES appeals to the property of the parabolic segment stated in *QP*, 3—that ordinates have the same ratio as the squares of the corresponding abscissas—to obtain that BΔ:ΘB=4:1. In both proofs, the line AΔ equals twice the line EΔ.

In I, 29 HERO shows that if the arcs AB and BΓ are bisected at E and Z, respectively, the triangle ABΓ is less than four times the sum of triangles AEB and BZΓ (Fig. 6a). Analogously, ARCHIMEDES shows in *QP*, 21 that each triangle AZB or BHΓ equals one-eighth triangle ABΓ (Fig. 6b).[38]

To establish the inequality for the circular segment (I, 32) HERO posits a decreasing sequence of terms by successively dividing triangle ABΓ (Fig. 6a) by four, "until one-third the last has become less than K," where K is the excess of triangles AEB, BZΓ over one-fourth triangle ABΓ (*cf.* I, 29). By successively bisecting the arc ABΓ, he can thus construct an inscribed polygon AEBZΓ which exceeds four-thirds triangle ABΓ. It follows *a fortiori* that the circular segment ABΓ exceeds that same amount. Unlike HERO, ARCHIMEDES must employ an indirect argument in *QP*, 24 to establish the equality of the parabolic segment and four-thirds the triangle ABΓ (Fig. 6b). His treatment of the case where the segment is assumed to be less than four-thirds the triangle is analogous to HERO'S proof in I, 32. ARCHIMEDES successively divides triangle ABΓ by four, "until the last has become less than the excess" by which four-thirds triangle ABΓ exceeds the segment.[38a]

[38] HERO introduces the auxiliary point Θ (Fig. 6a) such that AK=KΔ and line ΘHK is parallel to BΔ. Its construction is thus analogous to that of point Z in ARCHIMEDES' proof of *QP*, 21 (Fig. 6b).

[38a] Note the striking resemblance between the quoted Archimedean phrase (*ἕως κα γένηται τὸ ἔσχατον ἔλασσον τᾶς ὑπεροχᾶς*) and that just quoted above from HERO (*ἕως οὗ τὸ τοῦ ἐσχάτου τρίτον ἔλαττον γένηται τοῦ* K). Another stylistic feature links this proof with ARCHIMEDES. HERO represents the sum of the triangles AEB, BZΓ by ΘK (*i.e.*, Θ+K); but only a few steps *after* doing this does he explain that Θ is to be taken equal to one-fourth triangle ABΓ. Thus, in this somewhat confusing way he inverts the order of the construction of the terms in his argument. Now, there is no equivalent place in ARCHIMEDES' proof of *QP*, 24 where such a step is called for. But there is one in the proof of *PE I*, 7. There, ARCHIMEDES expresses a given weight as AB; only much later in the proof does he define A as satisfying conditions of closeness to AB and commensurability with another given weight Γ. (See Appendix 5. ARCHIMEDES' terms AB, A are there represented by *A*, *A'*, respectively.) The fact that HERO's proof accords in this way with an unusual stylistic usage by ARCHIMEDES, even though there is no instance of this feature in the text of *QP*, HERO's alleged model, reinforces our view that HERO is merely reproducing an Archimedean original. The stylistic link between HERO's proof and *PE I*, 7 would thus follow from the fact that ARCHIMEDES was their common author.

This leads to the construction of an inscribed polygon which exceeds the segment. As this is impossible, neither can four-thirds triangle ΑΒΓ exceed the segment, as assumed by hypothesis.

The striking similarity of these two theorems is evident. HERO further sharpens the comparison by attaching at the end of I, 32 the prescription for measuring the area of a segment when the curve is not a circle, but a parabola; he then states the theorem on the parabola, precisely as worded in *QP*, 24, and explicitly attributes it to ARCHIMEDES. No wonder that HEATH has noted of HERO'S procedure that it "is exactly modelled on ARCHIMEDES' quadrature of a segment of a parabola."[39] But is it so clear that the treatment of the circular segment is patterned on that of the parabola? Might it not be the other way around? Further consideration reveals the latter view to be the more plausible.

When HERO cites the Archimedean theorem on the parabola, he appears to be quoting his source. For instance, he follows the archaic terminology "section of a right-angled cone," but adds the clarification "that is, the parabola." The wording is identical with that in *QP*, 24. But he says that "ARCHIMEDES has shown [this] in the *Method*."[40] Now, the alleged model of HERO'S treatment of the circular segment is the geometric proof in *QP*, 18–24 and this does not appear in the *Method*. Although he cites the *Dimension of the Circle*, the *Sphere and Cylinder* and the *Method* several times, HERO nowhere alludes explicitly to any other Archimedean work.[41] Thus, it

[39] *HGM II*, p. 330.

[40] *Metrica* I, 32. HERO makes the very same assertion in I, 35. In the former, the word rendered "shown" is ἔδειξε; in the latter, it is ἀπέδειξεν, the regular word for "proved." Is it possible that HERO could have viewed the treatment in the *Method* as a *proof*, despite ARCHIMEDES' consistent assertion in that work that the mechanical method has only heuristic, not demonstrative, force? Probably; for HERO's purposes in producing a work suitable for the practical uses of geometry contrast with the highly formal ideal motivating the work of ARCHIMEDES. But in this context, we may consider a point raised by HEIBERG. In the light of ARCHIMEDES' remark after *M*, 1: "seeing that this has not been proved [by these means], but suspecting that the conclusion is true, we will search out and order the geometric proof published earlier" (II, p. 438), HEIBERG infers that a geometric proof, in the style of *QP*, 24, was appended to the *Method*, but that these pages are now lost (II, pp. 439 n, 507 n). Later reconsideration, however, led him to doubt this hypothesis, as the number of folios missing from Codex "*C*" does not allow sufficient space for such a proof (III, pp. iv–v, lxxxviii). ARENDT has proposed emendations to the text here so that it no longer carries the promise to append the proof ("Zu Archimedes," pp. 293–4). But we may note that a future tense – as here, τάξομεν, "we shall order" – frequently has a potential, rather than a temporal force in mathematical writing. For instance, ARCHIMEDES says "similarly we shall prove (δείξομεν)" in *SC I*, 6, by which he indicates only that the proof *can* be supplied along lines analogous to those of a proof just given; thus, we might better translate this by "we *may* prove," or "we *can* prove." (See HEIBERG's first index in *Archimedes*, vol. III for dozens of instances of δείξομεν, δειχθήσεται and similar expressions, many of which have this potential sense.) Accordingly, the text from the *Method* can be accepted unaltered without our being compelled to adopt the hypothesis that a geometric proof of *M*, 1 has been lost.

[41] Unless the treatise *On Plinthides and Cylinders* mentioned in *Metrica* I, 25 was an Archimedean work, as HERO indicates. HEIBERG conjectures that the reference might be to a larger version of *DC* (*Archimedes*, II, p. 542). See also my remarks in "Archimedes and the Measurement of the Circle," pp. 123, 133.

appears that HERO himself did not create the proof on the circular segments, but merely lifted it from a source. This is not surprising, for little of substance in the *Metrica* is original with HERO; indeed, HERO never makes pretense of orginality.

Shall we then assume that some pre-Heronian editor had produced this proof by way of extending the Archimedean treatment in *QP*, 18–24? There is difficulty with such an assumption. It was surely a mark of ingenuity to realize that one could use a parabolic approximation to obtain an estimate for the area of the circular segment. But given this insight together with possession of the Archimedean proof to serve as a model, our hypothetical editor would appear to have managed the new proof with amazing imperceptiveness. Instead of following out the whole sequence of Archimedean theorems, he ought surely have observed from a comparison of *QP*, 19 with the analogous result for circular segments he himself has worked out (as in I, 28) that if a circular segment and a parabolic segment have the same base and vertex, the circular segment wholly contains the parabolic segment. From this, the claimed inequality for the area of the circular segment follows at once, making dispensable the whole remaining part of the argument (*i.e.*, I, 27, 29, 32). Certainly, any post-Archimedean commentator clever enough to perceive the relation between the circular and parabolic segments in the first place would have gone on to perceive this more efficient way of exploiting the relation. It thus becomes more plausible to suppose that HERO'S source was a pre-Archimedean writer, or perhaps ARCHIMEDES himself. In fact, it is possible that HERO copied the proof from ARCHIMEDES' *Dimension of the Circle.* As mentioned above, that work, as now extant, is a fragment. HERO draws from a work of that name not only the three propositions we know of it (expressed, however, in slightly different terms), but an additional theorem: that any circular sector equals half the rectangle whose base and height equal, respectively, the arc and radius of the sector.[42] The theorem on the segment of the circle is associated with these other parts of *DC* not only by content, but also by method. The various theorems divide the areas in similar ways and all make appeal to the bisection-principle of convergence (X, 1). Moreover, ARCHIMEDES' treatment of the parabola in *QP*, 18–24 is well understood as a modification and refinement of an earlier treatment of circular segments.[43] (See *Note 3, added in proof.*)

Under this view the following remark by ARCHIMEDES in the preface to the *Quadrature of the Parabola* is especially interesting:

> of those who earlier devoted themselves to geometry some endeavored to prove that it is possible to find a rectilinear figure equal to a given circle and to a given segment of a circle, and afterward they tried to square the area contained by the section of the whole cone and a straight line[44]; but they assumed lemmas not

[42] See note 23. It is also possible that the refined limits to the value of the ratio of the circumference of a circle to its diameter which are attributed to ARCHIMEDES by HERO in *Metrica* I, 25 appeared in such an extended work; see my article cited in note 41 above.

[43] His following the model of a prior study of the circular segments may also explain why ARCHIMEDES does not introduce circumscribed figures into his proof of *QP*, 24, although we should have expected him to do so from formal considerations, as well as to be in line with the pattern set by *Elements* XII, 2. See Appendix, section 3.

[44] The meaning of "segment of the whole cone" is in doubt. HEIBERG (*Archimedes,*

> readily admitted, so that these things were criticized by most as not being discovered by them. But the area contained by a straight line and the segment of a section of a right-angled cone, we know of none of those earlier who has undertaken to square this, which is precisely what has now been found by us.[45]

Thus, ARCHIMEDES' prior study of efforts to evaluate the circle and its segments led him to the successful quadrature of the parabolic segment. In line with our discussion above, we may infer that ARCHIMEDES' own treatments of the circle, as in *DC*, 1 and 3, as well as the approximation of the segment preserved by HERO are to be classed among those earlier efforts, and the quadrature of the parabola to which he refers is the version now extant as *QP*, 18–24. That this sets ARCHIMEDES himself among the "earlier geometers" is not a difficulty. For ARCHIMEDES wishes clearly to distinguish his present study of the parabola from those related studies of the circle which are viewed as failing their purpose. The Heronian treatment of circular segments yields only an approximation to the area; similarly, *DC*, 3 provides only upper and lower bounds on the ratio of the circumference and diameter of the circle. Moreover, *DC*, 1 squares the circle only on the assumption that a straight line equal to the circumference of the circle can be produced; this is certainly the type of assumption which would occasion criticisms of *DC*, 1 as a solution to the quadrature problem. An analogous situation is cited by ARCHIMEDES in the preface to the *Method*. Reviewing his prior theorems on volumes, he says

> those figures, namely the conoids and spheroids and their segments, were compared in size to figures of cones and cylinders, but none of those was found to be equal to a solid figure bounded by planes, while each of these two figures

II, p. 263n), and HEATH following him (*Archimedes*, p. 233n), both conjecture that the measure of the ellipse is intended, although there are textual difficulties with this – on this view, presumably, "whole (ὅλου) cone" would be a corruption of "acute-angled (ὀξυγωνίου) cone." But let me suggest an alternative explanation. We have already seen that ARCHIMEDES' theorem on the area of the circle (*DC*, 1) is at once extendable to the areas of circular sectors. It is easy to see how the area of the surface of an isosceles cone can be reduced to that of a circular sector, by imagining the cone to be cut open and rolled out flat. In fact, HERO obtains the area of the cone in precisely this manner in *Metrica* I, 37 (where, also, ARCHIMEDES' theorem on the circular sector is cited). Moreover, ARCHIMEDES provides the formal proof of this measure of the cone as *SC I*, 14. It strikes me as more natural to suppose that *this* theorem should have been the outgrowth of prior studies of the measurement of circles and circular segments (as mentioned in *QP*, preface), than to assume that the measure of the ellipse should have been next attempted. On this view, the word "whole" would have been introduced to distinguish this kind of "section" from the sectioning which produces the parabola. Or again, "whole" might be a corruption for "right" (ὀρθοῦ) or "isosceles" (ἰσοσκελοῦς), terms which appear in *SC I*, 13, 14. The phrase "and a straight line" (καὶ εὐθείας) would then be a copyist's error or misinterpretation, either of καὶ ἐπιπέδου ("and a plane") or ἐπιπέδῳ ("by a plane"), as in *SC I*, 16. The phrase should thus be taken to mean "the area bounded by the section of the whole (right/isosceles) cone and/by a plane" and so conform with the phrasing in *SC I*.

[45] Arch. (Heib.), II, pp. 262–264.

[given special examination in the *Method*] is bounded by planes and surfaces of cylinders and is found to be equal to solid figures bounded by planes.[46]

As in the *Method*, so also in the *Quadrature of the Parabola*, the ability to equate a curvilinear with a rectilinear figure was deemed especially noteworthy.

To summarize the preceding, I have argued that the following studies may be viewed as an early Archimedean group, unified in their content and technique: (a) the *Dimension of the Circle* (including the theorem on sectors, cited by HERO and PAPPUS); (b) the theorem on the area of circular segments (preserved by HERO); and (c) the "geometric" proof of the area of the parabolic segment (*QP*, 18–24). I propose now to argue that a number of other studies may be associated with this group: (i) the two treatises *On Plane Equilibria;* (ii) the *Sand Reckoner;* and (iii) a version of the studies presented by PAPPUS on the surface of the sphere, on the areas bounded by spiral lines, and on isoperimetric plane figures.

4. The two books *On Plane Equilibria*

The second book *On Plane Equilibria* is devoted to determining the centers of gravity of parabolic segments and truncated segments. HEIBERG placed the composition of this work after that of *QP*, since *PE II* assumes the area of parabolic segments. But *PE II* requires only those results which appear in *QP*, 18–24 and the three lemmas *QP*, 1–3 drawn form the "Elements of Conics." The manner of approximating the parabolic segment by means of inscribed polygons and the use of indirect arguments based on the bisection-principle (X, 1) are characteristic of both treatments. None of the more refined aspects in evidence in *QP*, 14–17 are to be found in *PE II*. It seems particularly odd that in his proof of such a mechanical property of the parabolic segment as its center of gravity, ARCHIMEDES should have found no place for the techniques in his own mechanical proof of the segment's area. In fact, one can modify the proof in *PE II* along the lines of that mechanical proof to some advantage.[47] Assuming, then, that affinities in content and method are a valid criterion, we may associate *PE II* with *QP*, 18–24, viewing both as prior to the mechanical investigations in *QP*, 14–17.

The first book *On Plane Equilibria* is an effort to axiomatize several of the important results in elementary geometric statics. It proves the principle of the balance (prop. 6–7), the position of the center of gravity of the parallelogram (prop.

[46] Arch. (Heib.), II, p. 428. The first of these solids (sometimes called the "hoof," although ARCHIMEDES does not give it a name) is formed from the intersection of a cylinder by a plane through a diameter of its base; the second is formed by the intersection of two cylinders whose axes cut each other at right angles. HERO (*Metrica* II, 14–15) considers both of these solids and says of the second that it can be applied in cases where inlets or windows are cut into vaults. We may here detect an instance in which a practical problem might have influenced the direction of ARCHIMEDES' geometric researches (see note 112). It also enables us neatly to associate the work of DIONYSODORUS (late 3rd century) with ARCHIMEDES' work in the *Method*, since HERO presents DIONYSODORUS' measurement of the torus in the context of architectural applications of such solids (*Metrica* II, 13; *cf.* I. THOMAS, "Dionysodorus," *DSB*).

[47] See Appendix, section 4.

9–10), of the triangle (prop. 13–14) and of the trapezium (prop. 15). These results are applied many times in other Archimedean works, such as *PE II*, *QP* (prop. 6–15), the two books on *Floating Bodies* and the *Method*. However, as a number of additional mechanical theorems, not found in *PE I*, are assumed without proof in these other works, it has usually been supposed that *PE I* is an excerpt from a more comprehensive treatment, perhaps that called the *Mechanica* (in *QP*, 6 and 10) or the *Elements of Mechanics* (*FB II*, 2).[48] At any rate, the fact that the results established in *PE I* are needed in *PE II* and the inelegance of the former work, certainly by comparison with the other formal Archimedean treatises, have led to a general acceptance of HEIBERG's judgment of *PE I* as a youthful work by ARCHIMEDES. Certainly from the point of view of convergence technique, *PE I* fits in with the other works we have set in the "early group" – applying only the Euclidean approximation method and the principle of bisection. In fact, the technique of proportions employed here appears to be pre-Euclidean[49]; this further supports viewing it as an early work.[50]

5. The *Sand Reckoner*

Placing the *Sand Reckoner* in the context of the other works has been difficult, since internal references to results established in other Archimedean works are so

[48] W. STEIN, "Schwerpunkt," pp. 224–233, 243–244; DIJKSTERHUIS, p. 296. HEATH writes: "It is possible that there was originally a larger work by Archimedes *On Equilibriums* of which the surviving books *On Plane Equilibriums* formed only a part" (*HGM II*, pp. 23f.). Again, he says of the determination of the center of gravity of the cone, assumed without proof in the *Method*, that it "may have been solved ... in a larger mechanical work of which the extant books *On Plane Equilibriums* formed only a part" (*Archimedes, Supplement*, p. 15n). Now, most of ARCHIMEDES' citations of works, as the *Mechanica*, the *Elements of Mechanics*, and even the *Equilibria* (as in *QP*, 6 and 10; *FB II*, 2 and *M*, 1), relate to theorems found in *PE I* (8, 14, 15). But the work which must have contained the theorem on the center of gravity of the cone and of other solids such as paraboloids (as assumed in *FB II*, 2) was of a very different nature from the one conceived by HEATH. See notes 113 and 114 below.

[49] See Appendix, section 5.

[50] J. L. BERGGREN adduces an argument to this effect, based on biographical accounts on ARCHIMEDES ("Spurious Theorems," p. 89). According to PLUTARCH, ARCHIMEDES supervised the construction of defensive and offensive siege equipment for King HIERON of Syracuse, although, as it happened, these were never used in the defense of the city in HIERON's lifetime, as he reigned in peace. BERGGREN infers that these devices were built near the time of the alliance of Syracuse with Rome (263 B.C.), when ARCHIMEDES was around 30. The argument is not as tight as BERGGREN might wish, however, since vicious fighting was waged on Sicilian soil throughout the First Punic War (264–241 B.C.) between Roman and Carthaginian forces. Yet his conclusion may be sustained via the following observation: HIERON, as ally and provider for Rome, supplied offensive engines right from the start of this period (see M. I. FINLEY, *Ancient Sicily*, N.Y., 1968, p. 115). We may suppose that ARCHIMEDES was engaged in practical mechanical activities soon after the alliance was settled. Of course, the drafting of *PE I* might have occurred at any time during the period 263–241 B.C., corresponding to his being between 25 and 45 years of age. But on other grounds we have seen that the earlier date is to be preferred.

few – in fact, there is just one. In order to supply an upper bound for the perimeter of an inscribed polygon, ARCHIMEDES uses the fact that the circumference of the circle is less than $3\frac{1}{7}$ times its diameter, "for you know this has been proved by us." [51] As this is the bound verified in *DC*, 3, HEIBERG has assigned to *SR* a later date of composition than *DC*. Yet this may presume too specific a source for ARCHIMEDES' comment. Later, when he requires a lower bound, he does not draw from *DC*, 3, but merely employs the value 3. [52] It is thus possible, for instance, that *SR* was composed at a time when the studies in *DC* were still incomplete. Unlike the formal works addressed to DOSITHEUS at Alexandria, the *Sand Reckoner* is communicated to King GELON in ARCHIMEDES' home-city of Syracuse. While the reasoning is rigorous throughout, the expository style is less formal, and the second-person form of address appears in all parts, not just the opening remarks. [53] Thus, some results might be accepted as proved on the basis of concurrent oral communication, rather than formal writings. The remaining technical steps, both arithmetic and geometric, are either proved in *SR* itself or, if assumed without proof, can be referred to pre-Archimedean works, such as the *Elements*. Among the more advanced of these latter steps is the theorem on the inequality between the ratio of arcs and the ratio of the corresponding chords, as well as the inequality between the ratios of arcs and tangents. But the same results had earlier been assumed by ARISTARCHUS and proofs could be drawn from such works as the Euclidean *Optics* or the compilations in spherics. [54]

The difficulty posed by the absence of technical references to the other Archimedean works may, in fact, be the key to the solution. For, if the *Sand Reckoner* was produced before those works, such references would of course have been impossible. Admittedly, *SR* is fundamentally a popularizing effort. While the argument is managed with rigor and skill, no serious mathematical result is established, and even the system of expressing large numbers which ARCHIMEDES sets forth and applies does not appear to be intended for actual scientific use. As popularization, the work would naturally operate on a fairly elementary level and put a premium on the avoidance of technically demanding materials. But even granting this, there are opportunities for bringing in results from the other treatises. For instance – as ARCHIMEDES wishes to show how to express a number so large

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[51] Arch. (Heib.), II, p. 230.

[52] *Ibid.*, p. 234. HEIBERG incorrectly takes this as an allusion to *DC*, 3.

[53] ARCHIMEDES restricts second-person forms to the prefaces – the bodies of the treatises are always in highly formal language, employing impersonal forms or a didactic first-person plural. (A partial exception is the *Method* where a less formal language appears in notes after prop. 1 and 2.)

[54] That is, the geometric equivalent of $\tan\alpha/\tan\beta > \alpha/\beta > \sin\alpha/\sin\beta$, for angles $\alpha > \beta$. This is assumed in *SR* (Arch. (Heib.), II, p. 232) and is assumed by ARISTARCHUS in *Sizes and Distances*, prop. 4 (ed. HEATH, pp. 364–370). The lower inequality appears with proof in PTOLEMY, *Syntaxis*, I, 10. The upper inequality is proved in EUCLID's *Optics* (prop. 8), in THEON's commentary on PTOLEMY (ed. ROME, p. 358) and in a scholium to the *Spherics* (THEODOSIUS, III, prop. 11). THEON's source is the treatise on isoperimetric figures by ZENODORUS (see note 64 below). One may suppose, from context, that ARCHIMEDES' knowledge of these inequalities derived from the tradition of spherical geometry, as was also likely the case for ARISTARCHUS.

that it exceeds the number of grains of sand which could fill the universe (where the huge dimensions of ARISTARCHUS' heliocentric universe are adopted), wouldn't the most direct procedure be to estimate the number of grains in some standard volume, say a cubic dactyl or a cubic stade, then multiply this by the volume of the universe calculated in these units? This would occasion the introduction of several results from his own works, in particular, the representation of the volume of the sphere in relation to that of the circumscribing cylinder in *SC I* – a result of which he was known to have been so proud that he commanded the configuration to be engraved on his tombstone.[55] But, on the contrary, a rather different strategy is followed. A small spherical volume (a "poppy-seed") is assumed to have a diameter no greater than 1/40 finger-breadth and to contain no fewer than 10,000 grains of sand. By assigning a value to the diameter of the cosmos, ARCHIMEDES can express the ratio of the diameters of the two spheres; he then introduces the theorem that spheres are in the triplicate ratio of their diameters (*Elements* XII, 18) in order to assign a number to the ratio of volumes. Multiplying this result by 10,000 yields the desired upper bound on the number of grains of sand which would fill the universe. It is thus clear that ARCHIMEDES has framed his computation around a Euclidean theorem, when minor alteration of the argument could have made it a demonstration piece of his own results on the volume of the sphere. This missed opportunity is all the more surprising in that elsewhere in the *Sand Reckoner* ARCHIMEDES takes pains to point out his own contributions: he describes in detail the construction of a sighting instrument by which he modified ARISTARCHUS' values for the angular diameter of the sun (ARCHIMEDES/HEIBERG, II, pp. 222–226); he cites his own proof that $3\frac{1}{7}$ is an upper bound on the ratio of the circumference to the diameter of the circle (*ibid.*, p. 230); and the whole work is in fact an exhibition of the system of naming large numbers which was his own invention (*ibid.*, pp. 216, 236). In the same vein, ARCHIMEDES mentions his father's estimate of the ratio of the diameters of the sun and moon, along with EUDOXUS' and ARISTARCHUS' estimates. It is thus puzzling that ARCHIMEDES should call attention to his own contributions in these ways, yet in the case of the measurement of the sphere let pass by the chance of applying some of his most accessible and interesting theorems. On the other hand, this anomaly disappears at once on the assumption that the *Sand Reckoner* was presented before ARCHIMEDES had worked out those theorems.

In the course of setting out his arithmetical system, ARCHIMEDES proves a theorem on numbers in continued proportion, the equivalent of what we should write: $a^n \times a^m = a^{n+m}$, for positive integers a, n, m (*ibid.*, p. 242). Now, this same fact is asserted in a porism to *Elements* IX, 11 as being "manifest" in light of the theorem there proved. Why has ARCHIMEDES proved again this straightforward result, when his procedure in the *Sand Reckoner* is generally to assume what has been proved in earlier works (*e.g.*, the inequalities of the ratios of chords and tangents, discussed in the next section)? The inference that ARCHIMEDES drew his basic arithmetic technique from texts different from the Euclidean theory, to the extent that he lacked parts of what are now extant in *Elements* IX, recalls an observation

[55] This information derives from CICERO and PLUTARCH; see HEATH, *Archimedes*, p. xviii; and DIJKSTERHUIS, p. 32.

we have made in connection with *QP*, 23. There too ARCHIMEDES proved a result which should have followed as an immediate consequence of a Euclidean theorem (IX, 35). We thus appear to have, in this failure to cite *Elements* IX, a connection between *SR* and the "geometric proof" in *QP*. The early dating we have argued for the latter thus supports our argument here for an early dating of *SR*.

In other respects, the *Sand Reckoner* seems better associated with the science of the earlier part of the third century than of the later part. For instance, in presenting the relative dimensions of the earth, moon and sun, ARCHIMEDES refers to values proposed by EUDOXUS, by ARISTARCHUS, and by his own father, the astronomer PHIDIAS.[56] Such results were accessible to anyone from about 270 B.C. on. Did no one after that time re-examine these estimates until HIPPARCHUS, over a century later? Or again, ARCHIMEDES cites an estimate of the circumference of the earth as 300,000 stades. This value is generally attributed to DICAEARCHUS of Messina (c. 300 B.C.).[57] Certainly, the smaller estimate by ERATOSTHENES of 250,000 (or 252,000) stades was produced late in ARCHIMEDES' lifetime (say, 230 B.C. or later). But did no one in the years between venture a recomputation to which ARCHIMEDES might refer? As the scientific and geometric references made in *SR* were all available from the early part of the third century, it would appear fitting to place its production early in ARCHIMEDES' career. Considering that this view makes it in principle impossible that references to late technical advances appear in *SR*, I believe that a somewhat greater burden of proof be imposed on anyone proposing that the work is a late one; certainly, he should be expected to account for the absence of any such technical references.

Assuming a relatively early date for the *Sand Reckoner*, we may suggest the occasion of its presentation. "King" GELON II of Syracuse never actually reigned in his own right, as his father, HIERON II, outlived him by less than a year. But at some point HIERON had named GELON co-regent; in this and other ways HIERON was embellishing his own kingship in imitation of the PTOLEMIES at Alexandria. For instance, PTOLEMY I appointed his son, then 23 years of age, as co-regent in 285 B.C.[58] Following this example, HIERON would have named GELON co-regent in his early twenties, that is, around 250 to 245 B.C. ARCHIMEDES was then about

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[56] Arch. (Heib.), II, p. 220. On the measurement of distances of sun and moon in antiquity, see HEATH, *Aristarchus*, ch. IV.

[57] W. W. TARN, p. 302. On ancient estimates of the size of the earth, see HEATH, *loc. cit.*

[58] On HIERON's diverse imitations of the PTOLEMIES, see FINLEY, *Ancient Sicily*, p. 112. On PTOLEMY II, see *Pauly Wissowa.* It appears that PTOLEMY III (b.c. 285 B.C.) served as co-regent beginning 267/6 B.C. On GELON, see H. BERVE, "König Hieron II," *Abhandlungen der Bayerischen Akademie der Wissenschaften (phil.-hist. kl.)*, Munich, N.S. 47, 1959, pp. 39–41. BERVE assigns GELON's elevation to the co-regency to "not long before 240 B.C." One may suppose that, given the hostilities raging in Sicily until 241 B.C. and the fact that HIERON's rule was based on his usurpation of the royal title in 269, HIERON must have been eager to confirm the dynastic succession as soon as possible; indeed, this ambition is evident in his choice of the name GELON for his son, in reminiscence of the great Syracusan tyrants GELON I and HIERON I of the 5th century B.C. As GELON was born in the early 260's his appointment as co-regent would seem to fall in the early or middle 240's B.C.

forty, according to the traditional chronology. As ARCHIMEDES' tone in the *Sand Reckoner* assumes a certain knowledgeability in geometry on GELON's part, it seems plausible to infer that ARCHIMEDES had participated in guiding the heir's studies and was perhaps ready to take on a larger responsibility for them. This, too, would be in keeping with the Ptolemaic practice of appointing the most eminent scholars, the heads of the Library at Alexandria, as royal tutors.[59] At any rate, some such occasion must have induced ARCHIMEDES to write the *Sand Reckoner* and dedicate it to GELON. For, as already observed, that work has no serious scientific objective.

6. Isoperimetric figures, segments of the sphere, and spirals

One further item in *SR* is of interest. When ARCHIMEDES requires a lower bound for the ratio of the inscribed polygon of 1000 sides and the diameter of the circle, he adopts the value 3,

> for it has been proved that of every circle the diameter is less than the third part of the perimeter of any [inscribed] polygon which is equilateral and has a greater number of sides than the hexagon inscribed in the circle.[60]

Thus, instead of adopting the technique of *DC*, 3, he appeals to a special case of a theorem to this effect: that of regular polygons inscribed in the same circle, the one having the greater number of sides has the greater perimeter. While no proof of this theorem has survived from antiquity, one may be provided on the basis of the same inequality between the ratios of arcs and chords which ARCHIMEDES has already introduced earlier in *SR*.[61] But this theorem has a strong resemblance to one of the basic results in the theory of isoperimetric plane figures, that of regular polygons of equal perimeter, the one with the greater number of sides encloses the greater area. The proof of this employs the other inequality, between ratios of arcs and tangents, also used in *SR*.[62] ARCHIMEDES' theorem on regular polygons may thus point to an awareness by him of some isoperimetrical results. This is supported by the appearance of a far more advanced result of this type in *SC II*, 9: that of all spherical segments of equal surface, that which is a hemisphere encloses the greatest volume. While *SC II*, as now extant, was composed at a much later date than the studies which we are here examining, much of it was based on prior work. For instance, the statement of a false theorem on isosuperficial segments was among the list of theorems without proofs which ARCHIMEDES addressed to CONON prior to the communication of *QP*.[63] As the latter can be dated to around 240 B.C., we can assign to this time – not much later than that already proposed for *SR* – a strong background and interest in isoperimetric studies by ARCHIMEDES. To be sure, the collection of theorems on isoperimetric plane figures which PAPPUS and THEON

[59] Tarn, p. 270.

[60] Arch. (Heib.), II, p. 234.

[61] See note 54 and Appendix, section 6.

[62] This theorem is proved by PAPPUS (ed. ROME, p. 312) and by THEON (ed. ROME, p. 359), the latter drawing explicitly from ZENODORUS. See Appendix, section 6.

[63] *SL*, preface (Arch. (Heib.), II, pp. 2–4); see note 124 and the associated text. On the dating of CONON's death, see note 132.

have transmitted derives from ZENODORUS, a mathematician-astronomer of the early second century B.C.[64] But this need not conflict with assigning a substantially earlier origin for these studies. Technically, they require nothing beyond the *Elements* and ARCHIMEDES' theorems on the circle. For instance, the important result that the circle encloses a greater area than does any regular polygon having equal perimeter makes such critical use of *DC*, 1 that PAPPUS and THEON (both, perhaps, following ZENODORUS) produce here complete versions of the proof of *DC*, 1.[65] Further, SIMPLICIUS remarks that the maximal properties of both circle and sphere "have been proved both before Aristotle ... and by Archimedes and by Zenodorus more broadly."[66] While any pre-Archimedean statements of these results must largely have been founded on intuition, we may suppose they received at ARCHIMEDES' time their first demonstrations, these later being elaborated by ZENODORUS. I thus propose that some form of the isoperimetrical studies is to be assigned to the "early group" of Archimedean works.

I believe we can associate another series of studies to this group: a preliminary form of the results on the surface of the sphere whose formal treatment is extant as *SC I* and *II*. PAPPUS provides a lengthy discussion which amounts, as he says, to an alternative treatment of *SC I*.[67] We might at first suppose that he has composed these theorems himself, producing a commentary on the Archimedean treatise. But this is not so. For when he states the theorem on the surface of a spherical segment, it is in this form:

> of every segment of a sphere the convex surface is equal to the circle of which the line from the center equals the line from the pole (*πόλος*) of the segment.[68]

In ARCHIMEDES this result is presented in two theorems, one for the case of a segment less than the hemisphere (*SC I*, 42), the other for a segment greater than the hemisphere (43); in addition, the special case of the surface of the whole sphere receives a separate proof (33). To cite the first of these:

> of every segment of a sphere less than the hemisphere the surface is equal to the circle of which the line from the center equals the line from the vertex (*κορυφή*) of the segment drawn to the circumference of the circle that is the base of the segment of the sphere.[69]

Certainly PAPPUS is not quoting ARCHIMEDES *verbatim*, especially in view of the discrepancy between "pole" and "vertex". Perhaps his wording is an abridgment of ARCHIMEDES'. But we may compare HERO's statement of the same theorem:

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[64] See G. J. TOOMER, *Diocles*, p. 2.

[65] PAPPUS (ed. ROME), pp. 312–316; THEON (ed. ROME), pp. 360–364.

[66] *In Aristotelis de Caelo*, ed. J. L. HEIBERG, Berlin, 1894, p. 412 (commentary on *de Caelo* 287a23). For discussion of this passage, see W. MÜLLER, "Das isoperimetrische Problem im Altertum," *Sudhoffs Archiv*, 1963, 37, pp. 44f and W. SCHMIDT, "Zur Geschichte der Isoperimetrie im Altertum," *Bibliotheca Mathematica*, 1901, 2_3, pp. 5–8. Both raise the possibility, in the light of SIMPLICIUS' remark, that ARCHIMEDES contributed more to isoperimetric studies *per se* than is contained in *DC* and *SC I*.

[67] PAPPUS, *Collection* V, 20–43.

[68] *Ibid.*, 30 (ed. HULTSCH, p. 382).

[69] Arch. (Heib.), I, p. 156.

> Archimedes himself has proved in the book *On the Sphere and Cylinder* that of every segment of a sphere the surface is equal to the circle of which the line from the center equals the line from the pole (πόλος) of the base of the segment.[70]

It is clear that, despite the explicit reference to *Sphere and Cylinder*, HERO's wording conforms to PAPPUS', not ARCHIMEDES'. Now, of course, HERO, living two centuries or more before PAPPUS, could not draw his expression of the theorem from PAPPUS. Nor could PAPPUS draw his from HERO, since PAPPUS provides not only the statement, but also the full proof, which is not to be found in HERO. It thus appears that both HERO and PAPPUS had access to a common pre-Heronian source presenting alternative proofs to the results in *SC I*.

The general approach employed by PAPPUS in the study of the sphere is close to that of ARCHIMEDES. The sphere is bounded above or below by solids consisting of frusta of cones and the major theorems are established by indirect proofs based on inequalities relating to the surfaces of these solids. But there are significant differences. In PAPPUS the inscribed and circumscribed cases are always treated separately, in the style of the Euclidean "approximation" method; but, as we saw, ARCHIMEDES employs a "compression-via-ratio" method in *SC I*. Further, ARCHIMEDES applies his own form of the convergence-axiom, whereas PAPPUS applies the bisection-principle. We thus encounter a situation similar to that in connection with HERO's measurement of circular segments: if PAPPUS' source was the work of a post-Archimedean commentator, why did he choose to remove those features of elegance which had made ARCHIMEDES' proofs so efficient? This puzzle, coupled with the fact that HERO cites the same source as if it *were* a writing by ARCHIMEDES, encourages us to view it as ARCHIMEDES' work, a version of *SC I* fashioned in the manner of the "early group," in effect, a preliminary draft of the formal treatise now extant.[71]

Adopting this view, we may answer a problem which puzzled ZEUTHEN.[72] In a passage, quoted above (page 220), from the *Method*, prop. 2, ARCHIMEDES describes how his discovery of the volume of the sphere led him to realize how to measure its surface. The key insight – which we labelled (4) – is that the sphere may be viewed as being equal in volume to a cone whose base is equal in area to the surface of the sphere and whose height is the radius of the sphere. But this manner of treating the theorem on surface-area as a consequence of that on the volume reverses the order

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[70] *Metrica*, I, 39. The words "of the base" do not here make sense and have probably entered the text by mistake; they do not appear in HERO's subsequent discussion of this theorem.

[71] In respect of terminology, the treatment in PAPPUS fits an early dating. The term πόλος is well-established in the spherical tradition, as in AUTOLYCUS (before 300 B.C.). When the context involves ratios or equalities of the second degree, this treatment, as a rule, employs the old "power" terminology: "such-and-such lines are equal *in power* (δυνάμει)", whereas later writers say "the *squares* (τετράγωνα) on such-and-such lines are equal." The "power"-terminology is common in the pre-Euclidean period (see my *Evolution of the Euclidean Elements*, ch. III, sect. I), but was removed by EUCLID from the *Elements*, save in Book X and several places in XIII. ARCHIMEDES and APOLLONIUS still recognize this terminology, but apply it sparingly (see HEIBERG's index, *Archimedes*, III, s.v. δύναμις and δύναμαι). (See p. 264 and note 124a).

[72] "Neue Schrift des Archimedes," pp. 359–363.

in which they are presented in *SC I:* the former theorem is prop. 33, the latter is prop. 34, and the analogous theorems for the areas and volumes of spherical sectors are prop. 42–43 and prop. 44, respectively. Moreover, in *SC I* theorems 33 and 34 receive entirely independent demonstrations; indeed, theorem (4) is neither stated nor proved in *SC I*, although it may be established either as a special case of prop. 44 or as a corollary to props. 33 and 34.[73] Further, the theorems on volume betray no trace of the "mechanical" approach, in which the sphere is divided into parallel circular sections. For these reasons, ZEUTHEN inferred that at the time of writing the *Method* ARCHIMEDES had not yet worked out the procedure adopted in *SC I;* and on this basis, he places the *Method* between *QP* and *SC I* in order of composition. He admits, this still does not explain why ARCHIMEDES deviated so far from the heuristic method in producing the formal proofs; indeed, ZEUTHEN deems the order adopted in *SC I* to be pedagogically perverse, as it gives priority to what is in his view the more difficult problem of surface measurement. We will take up the chronological placement of the *Method* below. But our claim that PAPPUS' source was an early Archimedean treatment of the theorems on the sphere can resolve ZEUTHEN's question. As we have set them out, the theorems in ARCHIMEDES' "early group" all concern *areas:* of the circle, its sectors and segments, parabolic segments and surfaces of cones. Both in content and method, the theorems preserved by PAPPUS on the surface of the sphere and its segments fall into the same grouping. If at this stage ARCHIMEDES discovered what the volume of the sphere is, for instance, through the heuristic argument of *Method*, 2, he could thus recognize how the two facts were related via the result expressed in theorem (4). Given the prior proof of the surface of the sphere, one need only prove (4) to obtain a proof of the volume of the sphere. This is precisely what is done in the version preserved by PAPPUS: the proof of (4), in conjunction with the theorem on the surface of the sphere, leads to the volume of the sphere as a corollary, in the form stated in *SC I*, 34 and *M*, 2.[74] Subsequently, the whole treatment might be revised by means of more sophisticated formal methods, without, however, major changes in the order of the principal results. Under this view, then, the discrepancies between the order of theorems in *SC I* and the order indicated in the note after *M*, 2 followed from ARCHIMEDES' recognition of how to apply the discovery of the volume of the sphere to already established results in the most efficient way.

Our view helps account for another peculiar fact: although both treatises on the sphere – the work of ARCHIMEDES and the version preserved by PAPPUS – bear the title *On the Sphere and Cylinder*, neither assigns a role to the direct comparison of the sphere and its circumscribing cylinder. In both works, the result that the volumes are in the ratio of 3:2, as are the areas, is given as a corollary to other expressions for the volume and area of the sphere.[74a] Why should this result, which ARCHIMEDES states prominently in the prefaces to *SC I* and *II*, be so peripheral to the structure of ARCHIMEDES' study? The answer may be found in our view that the

[73] ARCHIMEDES does not prove the sphere-cone relation in form (4); but he establishes analogous results for solids bounded by conical frusta and approximating the sphere (*SC I*, 26 and 31).

[74] *Collection* V, 40–41.

[74a] *SC I*, 34 cor. (Arch./Heib., I, p. 130); *Collection* V, 43 (ed. HULTSCH, pp. 408–410).

theorem on the volume of the sphere, as presented in *Method*, prop. 2, was discovered *after* the theorem on the surface of the sphere had been completed in the form preserved by PAPPUS. In the *Method*, ARCHIMEDES shows that the ratio of the volumes of the sphere and cylinder is 2:3 by means of a relation among the circular sections which constitute the cylinder, the sphere and an associated cone. Having discovered this ratio between the volumes, ARCHIMEDES could recognize that the same ratio holds between the surfaces, as a consequence of the earlier theorem on the surface of the sphere. But the proof of that theorem makes no reference to the cylinder. Thus, when ARCHIMEDES came to formalize his treatment of the volume of the sphere on the pattern of the earlier proof of the theorem on its surface, the relation of sphere and cylinder would not enter into the demonstration, but acquire instead the status of a corollary.

There is one further study which I propose to associate with the "early group." This is the set of theorems, preserved by PAPPUS, on the area enclosed by the plane and spherical forms of the Archimedean spiral. As I have discussed this material elsewhere, I will not go into detail here.[75] PAPPUS presents a theorem which "Archimedes proved using a certain remarkable conception," namely, the theorem on the area bounded by a single turn of the spiral. But the method in PAPPUS differs radically from that extant as *Spiral Lines*, prop. 24. Actually, PAPPUS' version is but the sketch of a proof. No formal convergence argument is given, but the closest analogue in the extant works appears to be *Method*, prop. 15. Nevertheless, as PAPPUS cites it, no bisection-process is applied; the area is divided into any number of integral parts. The inscribed, but not the circumscribed, bounding figures are presented, but it is difficult to imagine how the formal convergence argument could have been managed by an "approximation" rather than a "compression" technique. Now, PAPPUS also says that this same theorem was "proposed by Conon," so that his source appears to be based ultimately on a communication by ARCHIMEDES to CONON, in which such information would be found. PAPPUS' presentation of the surface bounded by the spiral on the sphere is stylistically a twin of that of the plane spiral.[76] In it, ARCHIMEDES' theorem on the surface of spherical segments is a necessary lemma, but in the appearance of such terms as the "pole" of the segment, it may be viewed as developing from the alternative version of *Sphere and Cylinder*, rather than *SC I* as now extant. These studies on the spiral may well be transitional – before *QP* in composition, but showing signs of some of the techniques characteristic of the formal writings after *QP*. Alternatively, it may be that ARCHIMEDES' thought in its less formal stages was characterized by such features as we find in PAPPUS' versions of the theorems on spirals. What would mark off the early works from the later ones would then be ARCHIMEDES' perception of how to advance beyond the strictly Euclidean manner of arguing convergence, and to provide a formal argument closer to the spirit of his preliminary thought.

To summarize, we have presented reasons – based on the content and the technique of proof, as well as other indications, in works by ARCHIMEDES and others

[75] See my "Archimedes and the Spiral" in *Historia Mathematica*; it is based on PAPPUS, *Collection* IV, 21–25.

[76] *Collection* IV, 35.

– to view the following as an "early group" of Archimedean writings: *Dimension of the Circle*, *Sand Reckoner*, *Quadrature of the Parabola* (prop. 18–24), and *Plane Equilibria I* and *II*. In addition, theorems on the sector and segment of the circle may be associated with *DC*, theorems on isoperimetric plane figures with *DC* and *SR*; and preliminary versions of the principal theorems in *Sphere and Cylinder* and *Spiral Lines* may be identified as more advanced efforts, placed late in the group. In all, they fall within a period of about twenty years, roughly from ARCHIMEDES' 25[th] to his 45[th] year, ending around 240 B.C. with the death of CONON and the commencement of formal correspondence with DOSITHEUS.[77] We may now take up the mature works and the genesis of the techniques which characterize them.

III. Heuristic and formal techniques in the mature works

We will take the five formal treatises addressed to DOSITHEUS as definitive of ARCHIMEDES' mature work. In this section, we will inquire into the origins of two technical features important for these works: the "mechanical" method which provided the heuristic background to several key theorems; and the "Archimedean axiom" on continuity and its relation to the corresponding Euclidean principle. We will also seek to place the remaining Archimedean works in their chronological position with respect to the basic five, arguing that the writings on hydrostatics and the mechanical properties of solids and the *Method* should be viewed as late works.

1. The "mechanical" method

We have already illustrated ARCHIMEDES' heuristic "mechanical" method in our discussion of his quadrature of the parabola. As presented in the *Method*, prop. 1, that method entails a sort of weighing procedure of the indivisible elements which constitute the measured magnitude (*e.g.*, the parallel line segments which fill out a plane area). It is frequently supposed that ARCHIMEDES' use of indivisibles was inspired by the atomistic conception of matter; most notably, S. LURIA has developed the thesis of a Democritean system of geometric analysis based on indivisibles and has surveyed the whole series of ARCHIMEDES' works as an elaboration upon atomist notions.[78] A difficulty with this view, as LURIA came to recognize, was that the primary testimony to such a Democritean theory in geometry – namely, a remark in ARCHIMEDES' *Method* – in fact indicates that ARCHIMEDES had but scant knowledge at best of DEMOCRITUS' methods. ARCHIMEDES says merely that DEMOCRITUS probably deserved some credit for the discovery of the theorems on the volume of the cone and the pyramid, as he was

[77] On ARCHIMEDES' biography, see HEATH, *Archimedes*, ch. I; DIJKSTERHUIS, ch. I; and below, note 137 and associated text.

[78] S. LURIA, *Archimedes*, Moscow, 1945. The thesis of a Democritean geometrical atomism, which included the study of geometric theorems via indivisibles, was developed by him in "Infinitesimaltheorie der antiken Atomisten" (1933). The view that DEMOCRITUS' theory was a source of ARCHIMEDES' technique in the *Method* had gained some acceptance even before LURIA (*cf.* HEATH, *Archimedes*, *Supplement*, pp. 10–11). We shall see here that this thesis is highly unlikely. Indeed, the assumption of the existence of a Democritean geometry-via-indivisibles has, at best, very tenuous support.

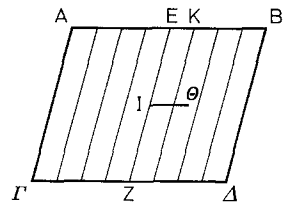

Fig. 7

first to state them, although without proof; by contrast, it was EUDOXUS who provided the first formal proofs. In the preface to *Sphere and Cylinder I* he makes the far stronger assertion that all the geometers before EUDOXUS were ignorant of these properties, "nor were these perceived by a single one of them."[79] It is thus evident that, whatever DEMOCRITUS' method might have been, ARCHIMEDES' knowledge of it was belated and indistinct.

In lieu, however, of conjecturing that it was an external importation, we can argue that the notion and method of indivisibles arose as an internal development within ARCHIMEDES' thought, to be converted by him into a viable mathematical method. Let us consider a typical result, one where ARCHIMEDES employs the standard Euclidean "exhaustion" technique in a work we have already seen to be early: namely, the determination of the center of gravity of a parallelogram in *Plane Equilibria I*, 9. ARCHIMEDES here asserts that the center of gravity lies on the line joining the mid-points of opposite sides. If we assume the contrary, the center of gravity must be displaced a certain distance off this line. In Fig. 7, the parallelogram is ΑΒΔΓ, the line joining opposite mid-points is ΕΖ, off which lies the center of gravity Θ. The opposite sides are then successively bisected until (via implicit appeal to the Euclidean principle of bisection, *Elements* X, 1) a remainder ΕΚ is left which is less than the distance ΙΘ measuring the displacement of the center of gravity from the median line. The points of division are used to divide the area into an ensemble of congruent parallelograms by lines parallel to the median. From *PE I*, 4 it is known that the center of gravity of two equal magnitudes is the midpoint of the lines joining their respective centers of gravity. In the present theorem, the small parallelograms may be taken two-by-two, starting from the outermost pair and proceeding toward the center. By congruence, the center of each pair will be the same point, which will therefore be the center of gravity of the whole original parallelogram (*PE I*, 5 cor.). The center of gravity of the innermost pair must thus be the point Θ. But this is impossible, since Θ lies wholly outside the convex figure formed by this pair of parallelograms (post. 7). This contradiction establishes that the center of gravity of the parallelogram lies on the median.

This theorem illustrates how application of the standard exhaustion technique in problems on centers of gravity leads naturally to the subdivision of areas into narrow parallel rectilinear elements — the same is evident also in prop. 13 on the

[79] These passages will be re-examined below in connection with the dating of the *Method*.

center of gravity of the triangle. Now, if these elements were extremely narrow, from an intuitive point of view they would differ but imperceptively from lines. Treating them as such, one obtains an informal version of the theorem which avoids the indirect argument.[80] In the case of prop. 9, one sees that each line in the parallelogram is exactly balanced by the line situated an equal distance on the other side of the median. Thus, the two halves, which are the aggregates of all the lines, are in balance about a point on the median; that is, the center of gravity of the whole parallelogram lies on this line. Once ARCHIMEDES came to conceive theorems like prop. 9 in this way, he could begin to investigate in similar fashion the properties of a wide variety of other plane and solid figures.

If this view is correct, *Plane Equilibria I* becomes of interest, not only as the earliest of ARCHIMEDES' mechanical writings, but also as an indicator of the origins of the heuristic "mechanical" technique elaborated in the *Method*. It is thus important to take note of the criticisms raised against the structure and proof technique of *PE I*; for, pressed to the extreme, these can be viewed as an argument against the authenticity of the work, as J.L. BERGGREN does in a recent study.[81] ARCHIMEDES' proof of the principle of the balance (for the case of commensurable weights, in prop. 6) has occasioned a huge scholarly commentary. But that one or more additional axioms must be articulated, if the proof is to succeed, need hardly compel denial of the Archimedean origin of what is admittedly the first effort to formalize the principles of mechanics. Again, the treatment of the incommensurable case (prop. 7) is defective; clearly, a portion of the proof has been lost in the transmission of the text. But the loss can be remedied without difficulty.[82] If the opening part of the book develops around the notion of point-of-balance, while the later part elaborates the notion of center of gravity, this discrepancy may stem from the work's being composed in separate stages; we are not compelled to reject one or the other part as spurious. Indeed, the appearance of alternative proofs to props. 10 and 13 – which BERGGREN takes as a further mark of inauthenticity of the work – would seem to suggest just such a composition by stages.[83] BERGGREN raises

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[80] In the 17th century it was often held that the methods based on indivisibles were merely a simplification of the formal "exhaustion" method. See G. VIVANTI, "Note sur l'histoire de l'infiniment petit," *Bibliotheca Mathematica*, $\mathbf{8}_2$, 1894, pp. 1–12 and BOYER, *Calculus*, ch. IV. and WHITESIDE, *Mathematical Though*, ch. IX (esp. p. 347n).

[81] "Spurious Theorems," pp. 91–103. (Our discussion takes up several of BERGGREN's points.) Other writers have presented defenses or reconstructions of *PE I* to make its formal and logical structure satisfactory against such criticisms: W. STEIN, "Schwerpunkt;" G. GOE, "Archimedes' Theory of the Lever and Mach's Criticism," *Studies in the History and Philosophy of Science*, 1972, **2**, pp. 329–345; O. SCHMIDT, "Axioms for the Archimedean Theory of Equilibrium," *Centaurus*, 1975, **19**, pp. 1–35. For a review of the criticism of *PE I*, especially as it concerns the proof of the balance-principle (prop. 6), see DIJKSTERHUIS, pp. 289–304.

[82] See Appendix, section 5.

[83] We have already argued that the appearance of *QP*, 18–24 as an alternative proof to *QP*, 14–17 introduces a discontinuity in the structure of *QP* as a whole. Further, there is an alternative proof after *SC II*, 8, and *SL*, 10 reappears as a lemma before *CS*, 2; the theorems in *SC II* are in the form of "analysis and synthesis," in effect, a type of duplicate proof. Thus, the appearance of alternative proofs need not argue against the authenticity of *PE I*, as BERGGREN charges. In this regard, the relation of *PE II*, 1 to *PE I*, 6–7 is of

doubts about theorems 11 and 12. But they are not superfluous; they serve for the alternative proof of prop. 13, thus determining anew the center of gravity of the triangle via a technique based on the properties of similar triangles. The origin of the work as a compilation would thus explain the inconsistencies in its overall structure. Moreover, as BERGGREN admits with others that these mechanical studies are early ones,[84] it seems hardly appropriate to use the formal precision of such a mature work as *Spiral Lines* as a measure of the level of sophistication to be expected in *PE I*. Certain evidence of an incomplete mastery of formal techniques is bound to be present in an early work and thus does not in itself argue against authenticity.

In my view, two facts allay any doubts concerning the Archimedean authorship of *PE I*. First, EUTOCIUS expresses no reservations of any such sort, even though he includes in his commentary the discussion of technical points in the proof of prop. 7 and the alternative proof of prop. 13 – that is, two of the theorems deemed suspect in the argument above. Second, the work is composed in the Doric dialect. Of the extant works by ancient mathematical authors, those of ARCHIMEDES are unique in the use of this dialect. Whenever a work of his was judged suitable for general use, as in school instruction – this being the case for the *Dimension of the Circle*, the *Sphere and Cylinder I* and *II*, and the *Method* – the Doric forms were systematically edited out.[85] It would thus be remarkable that *PE I*, assumed to be a compilation of Archimedean and non-Archimedean materials for school use, should have been composed in Doric. Indeed, when EUTOCIUS relates how he chanced to come upon a mathematical fragment in Doric in an old book, he was at once alert to the possibility of its being an Archimedean writing, a suspicion subsequently confirmed.[86] For these reasons, we are surely justified in continuing

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interest. The former, establishing the center of gravity of a system comprising two parabolic segments, ought to be merely a special case of the latter; instead, ARCHIMEDES provides an independent proof, based on *PE I*, 10, the center of gravity of a parallelogram. Here, the possibility of a different approach to the proof of the principal theorem on center of gravity, and that in the special context of the parabolic segments, the subject-matter of *PE II*, may have been justification enough for its presentation. One should note further an odd feature which links *PE II*, 1 with *PE I*. The fact actually proved in that proposition is the *converse* of what is enunciated: *i.e.*, ARCHIMEDES asserts that the center of gravity of the system divides the line between the centers of gravity of the two segments inversely as their areas; yet in the proof, he starts from the assumption of that point of division and shows it is the center of gravity of the system. This virtual conflation of the principle of the balance (*i.e.*, inverse proportionality → equilibrium) with its converse is characteristic of *PE I* as well: *PE I*, 8, 13 and 15 apply not *PE I*, 6–7, but the converse. In view of the unquestioned authenticity of *PE II*, this link confirms the acceptance of *PE I* as genuine.

[84] See note 50.

[85] On the school use of *PE I*, see BERGGREN, p. 91 and DIJKSTERHUIS, pp. 33–36.

[86] Commentary on *SC II*, 4; Arch. (Heib.), III, pp. 130–132. The recovered fragment, reproduced by EUTOCIUS, gives the construction of a step assumed in that theorem, but whose demonstration ARCHIMEDES explicitly promised to append to the book. Incidentally, this is also an instance of how, even late in antiquity, scholars could come upon portions of early mathematical works which were not clearly identified by author or title. PAPPUS' versions of the sphere and of the spiral might in similar fashion have

to accept *PE I* as a genuine work by ARCHIMEDES, and to use it as an index of the early stages of ARCHIMEDES' mathematical thought.

We have seen how studies like those in *PE I* might have led ARCHIMEDES to the technique presented in the *Method*. It is important to recognize that for him what distinguished this technique was not its application of indivisibles, but its mechanical character, as, indeed, his calling it "a certain mechanical method" indicates. He has nothing to say about what the indivisibles are or in what way they can sum to a finite magnitude, even though earlier natural philosophers had agonized over such questions and had revealed how fraught with logical difficulties the notion of the mathematical indivisible was.[87] Doubtless, ARCHIMEDES viewed their use merely as a simplification of the usual "exhaustion" method.[88] Of course, the appearance of indivisibles in a proof is formally objectionable; but their replacement by finite magnitudes in an indirect version of the same proof is straightforward. The proof of the area of the parabolic segment in *QP*, 14–17 is based on the version in *M*, 1, but avoids the use of indivisibles. Nevertheless, both treatments are mechanical and thus *both* are unsatisfactory as formal demonstrations. For one of the formal criteria of a proof is that only those assumptions are introduced which are in the nature of the things studied; the proof of a purely geometric property of a figure, such as its area, must not appeal to extrinsic properties, like weight or center of gravity.[89]

DIJKSTERHUIS has argued, to the contrary, that it was the use of indivisibles, and only this, which made the "mechanical" technique of the *Method* invalid as proof.[90] But two facts, which he does not take into account, act against his view. ARCHIMEDES never assigns to the "mechanical method" the status of proof. By this method, as he says in the *Method*, one "studies" the properties of figures, these "become apparent" or one derives "some conviction of their truth"; he consciously avoids terms indicating "proof" in this connection. But he does the same in the preface to *Quadrature of the Parabola:* the theorem has been "found" or "studied" by means of mechanics, but it is "proved" by means of geometry. If the mechanical method, as presented in *QP*, were formally secure, why should ARCHIMEDES

depended on such rediscoveries – that is, in old works of unspecified authorship. Missing the introductory parts of the *Collection*, we do not know the general nature of his sources.

[87] See, in particular, ARISTOTLE's objections to indivisibles, *Physics* VI, 1–3, and the Peripatetic tract *On Indivisible Lines*.

[88] See note 80 above.

[89] *Cf.* ARISTOTLE, *Posterior Analytics* I, 6–7 (HEATH, *Mathematics in Aristotle*, Oxford, 1949, pp. 44–46). When two sciences are in a priority relationship (*e.g.*, geometry as prior to optics, or arithmetic as prior to music theory), one may prove theorems in the latter by means of principles in the former, but not conversely. What determines priority is the nature of the things studied. For instance, in mechanics the figures are assumed to have both magnitude and weight, whereas in geometry they have only magnitude; hence, geometry is prior to mechanics.

[90] *Archimedes*, pp. 318–319. His contention that mechanics might be converted into a formal axiomatic system (as *PE I* has the aim of doing) does not make a mechanical proof like that of *QP*, 14–16 formally acceptable; for it still faces the objection raised in the light of the preceding note: *QP*, 14–16 proves a geometric theorem via the principles of a posterior science (*i.e.*, mechanics).

appear reluctant to speak directly of a "mechanical proof"? Moreover, in none of the four treatises subsequently sent to DOSITHEUS does ARCHIMEDES speak of or apply any form of the mechanical method. This is especially striking in the case of *Conoids and Spheroids* where the principal theorems are proved in a way which retains the basic structure of the heuristic treatments in the *Method*. We may suspect that in sending *QP* to DOSITHEUS – as in sending the *Method* to ERATOSTHENES – ARCHIMEDES wished to communicate not only a set of particular results, but also an impression of his basic heuristic method. But not knowing DOSITHEUS personally (the preface to *QP* makes this explicit), ARCHIMEDES could not know for sure what his reaction would be. As a mathematician, DOSITHEUS would welcome learning of a new method for the discovery of theorems; but as a purist on formal matters, he might attack the proofs for their use of mechanical assumptions. To cover himself, ARCHIMEDES could present the mechanical version in its most rigorous form, avoiding the indivisibles which would be certain to provoke the formalist; but he appended the geometric proof (prop. 18–24), thus to assure acceptance of the theorem, even if the mechanical proof of it was rejected. To judge from ARCHIMEDES' subsequent correspondence with DOSITHEUS, his worst fears on gaining a sympathetic reaction to his mechanical method appear to have been borne out.

We argued above that the theorem on the area of parabolic segments and the geometric proof of it as in *QP*, 18–24 were early results. Accordingly, what was new in the communication to DOSITHEUS must not have been the theorem itself, but the manner of studying it – namely the mechanical method. Moreover, the list of theorems without proof forwarded earlier to CONON must likewise have had the purpose of informing him of the new method and of a set of results newly discovered by its means. [91] Interestingly, neither the theorem on the parabola nor those on the sphere from *SC I* were in this list; this would appear to confirm our assigning their study to an earlier period. However, there is a passage from the preface to *Quadrature of the Parabola* which must be examined with great care. For it appears to leave a quite different impression. ARCHIMEDES writes to DOSITHEUS as follows:

> ... we have taken it in hand to send to you, as we had determined to write to Conon, a certain geometric theorem which had not formerly been studied, but had now been studied by us, one we first have found out by means of mechanics (1) and afterward have exhibited also by means of geometry (2).... We do not know of any earlier geometer who undertook to square the area contained by a straight line and the segment of the section of a right-angled cone, which is precisely what has now been found out by us; for it is proved that every segment bounded by a straight line and a section of a right-angled cone is four-thirds the triangle having the same base and height equal to the segment (3).... So we have written up and are sending to you the proofs of it, first as it has been studied by means of mechanics (4), then after these also as it is proven by means of geometry (5). [92]

[91] The list is presented in the preface to *SL*; *cf.* ARCHIMEDES' partial statement of its contents in *SC II*, preface.

[92] Arch. (Heib.), II, pp. 262–266.

Taken together, statements (1) and (2) would seem to indicate that ARCHIMEDES originally discovered the area of the parabolic segment via the mechanical method, then went on to work out the geometric proof. Confirmation of this may be had in the fact that the word I render "find out" (*εὑρίσκω*) regularly means "discover", "devise" or "invent": HEATH, for instance, often translates it as "find the solution". A related point is made in the preface to the *Method*. Describing the mechanical method to ERATOSTHENES, ARCHIMEDES says

> ... I have become convinced that it is no less useful as well toward the proofs of the theorems themselves. For also some of the things which first became clear to me mechanically (6) were later demonstrated geometrically (7), since investigation in this manner is separate from proof .... So we are writing first what was indeed first to become clear by means of mechanics (8) [namely, the theorem on the area of the parabolic segment], then after this each of the things studied in the same manner.[93]

Here, the manifestly chronological separation between (6) and (7) would appear to mirror a similar contrast between (1) and (2).

Under our view, however, the contrast between (1) and (2) cannot be a chronological one. For we have argued that the geometric form of the proof of the area of the parabolic segment (as in *QP*, 18–24) was an early result, antedating the introduction of the mechanical method, the method itself having been discovered not long before the death of CONON and the sending of *QP* to DOSITHEUS. In other words, we regard both of the mechanical methods – that employing indivisibles in *M*, 1, and that employing the indirect argument in *QP*, 14–17 – as having been worked out later than the geometric proof in *QP*, 18–24.

Is this view compatible with the above passages? I believe it is. First, let us note that after (3) ARCHIMEDES introduces his version of the axiom of continuity, indicating how former geometers had employed a similar lemma, so that its assumption here ought not be viewed as a criticism of his own proof. But the axiom is used only in the mechanical proof (*QP*, 16), not the geometric (*QP*, 24). This being the case, his concern over the axiom is puzzling, for the acceptance of the geometric proof will be assured even without it. Perhaps he wished to guarantee the acceptance of the mechanical version, or at least forestall criticism of it as far as its convergence technique was concerned. But, then, he could just as easily have introduced the bisection-criterion (X, 1) in place of the new axiom, as we have seen. We may explain this anomaly on the assumption that ARCHIMEDES originally intended to – and perhaps did – supply a fully geometrized version of the proof along the lines of *QP*, 14–16. Such a proof would present difficulties. It would require, for instance, summations which ARCHIMEDES provides in *Spiral Lines*, but might not yet have worked out at the time of writing *QP*.[94] Alternatively, the text of *QP* may indeed have included a geometric proof in this form, which later editors, regarding it as unnecessarily complicated, replaced with the simpler (and, as we have argued, earlier) version now extant. In this way, the geometric version (as reconstructed) would indeed

[93] *Ibid.*, pp. 428–430.

[94] See Appendix, section 2.

follow the heuristic and mechanical versions in order of time. But the version in *M*, 1 would not then be the original manner of the discovery of the theorem itself, but only of this particular form of its proof.

Second, we may observe that (1) and (2) need not stand in the same sharp chronological contrast as do (6) and (7). Rather, taken in conjunction with (4) and (5), they may be ARCHIMEDES' statement of the sequential order of exposition adopted in the treatise, and not an assertion about the chronological order of their discovery. Indeed, the word ἔπειτα, which I render as "afterward" in (2), need not entail order in time as the term ὕστερον, rendered as "later" in (7), does. We may compare this with another passage from the *Method*. Having completed the mechanical treatment of the theorem on the parabola, ARCHIMEDES says,

> This has not of course been proved by the things we have just said, but it has made some kind of impression that the conclusion is true. Whence, seeing that this has not been proved, but suspecting the conclusion to be true, we may search out and arrange in order the geometric proof, such as the one published before.[95]

Here, there can be no doubt. ARCHIMEDES is making a general prescription about the use of the mechanical method; namely, that any claim made on the basis of its application must be followed by a formal proof. Of course, the real value of the method lies in its utility for the discovery of new results; but in the present instance – as also, I claim, in the preface to *QP* – ARCHIMEDES speaks of his manner of exposition: after contriving a mechanical treatment, he must go on to add a rigorous geometric proof. A suitable one is here available from geometric results already proven.

Third, DIJKSTERHUIS' view that the mechanical method (in the form of *QP*, 16, where indivisibles are avoided) is itself to be taken as being formally valid seems untenable. Were it true, there would no longer be need for any geometric proof (such as *QP*, 24) in the *Quadrature of the Parabola*. Moreover, the remarks in the *Method* would then have no relevance to the remarks in the preface to *QP*, since *M*, 1 and *QP*, 14–16 would not have the same informal status. In our view, however, even if ARCHIMEDES himself was comfortable with the argument in *QP*, 14–16, he must have anticipated – and doubtless correctly so – that no strict formalist would accept it; in this way, the remarks in the *Method* would indeed also explain the inclusion of the geometric proof in *QP*. We have proposed that the demonstration in *QP*, 24 might not have been the one originally intended for that work, but that ARCHIMEDES, failing to work out a geometric proof along the lines of *QP*, 16, let suffice an earlier treatment in the Euclidean style. Whatever the precise historical facts were in this matter, our view is at the very least consistent with ARCHIMEDES' remarks in the preface to *QP* and, as we saw above, serves well to explain discrepancies in the body of that work.[96]

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[95] Arch. (Heib.), II, p. 438. On alleged difficulties in the text, see note 40 above.

[96] It will be noted that line (8) of the passage is not a difficulty; for it merely asserts that of those theorems which came to be treated by the mechanical method, the theorem on the parabolic segment was the first. It does not say that that theorem itself was originally discovered in this fashion.

2. Archimedes' lemma on continuity

A question still unresolved concerns the Archimedean lemma on continuity and its relation to the Euclidean principle of bisection. In the preface to the *Quadrature of the Parabola*, where the lemma makes its first appearance, ARCHIMEDES formulates it as follows:

> of unequal areas the excess by which the greater exceeds the lesser can, when added [sufficiently many times] to itself, exceed any preassigned finite area.[97]

As we have seen, this lemma is employed in *QP*, 16; is restated in virtually identical terms in the prefaces to *SC I* and *SL*; and is applied directly in the proofs of *SC I*, 2 and *SL*, 1 and 4. Each of the last-named propositions is in its turn basic to the convergence arguments in its respective treatise (particularly, those of *SC I*, 3 and *SL*, 18–20). Further, ARCHIMEDES says in the preface to *QP* that the lemma serves the same function which a similar lemma did in the principal exhaustion proofs of *Elements* XII. By this he evidently has in mind EUCLID's principle of bisection in X, 1, namely,

> if two unequal magnitudes be set out, and from the greater more than its half have been subtracted, and from the remainder more than its half, and this shall have occurred continually [lit. always], there will [eventually] remain a certain magnitude which will be less than the smaller of the magnitudes set out.[98]

In the *Elements*, the proof of X, 1 makes use of a more general principle, V, Definition 4:

> magnitudes are said to have a ratio to each other if they can, on being multiplied, be made to exceed each other.[99]

While there are evident differences in the enunciations of this Euclidean definition and the Archimedean lemma, their basic sense appears to be the same, and their role in convergence arguments is in practice equivalent – the Archimedean lemma being introduced into proofs directly, the Euclidean principle implicitly via X, 1.

Given this equivalence with EUCLID's principle, why has ARCHIMEDES considered it necessary to provide a new formulation of it? One might conjecture that the principle of bisection, despite its prominent role in the Euclidean theory of irrational lines (in Book X) and in the evaluation by "exhaustion" of areas and volumes (in Book XII), originally had the status of an axiom, or unproved lemma, in the *Elements* – and that ARCHIMEDES was the first to articulate the more general lemma (co-ordinate with V, Def. 4) by which X, 1 could be proved. But this can hardly be true, since the principle in V, Def. 4 is needed in the *Elements* for other theorems also, *e.g.*, V, 8. Moreover, both principles – that of X, 1 and that of V, Def. 4 – are to be found earlier in

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[97] Arch. (Heib.), II, p. 264; *cf.* also the prefaces to *SC I* (*ibid.*, I, p. 8) and to *SL* (*ibid.*, II, p. 12), where the axiom is stated in closely parallel language.

[98] EUCLID (ed. HEIBERG), III, p. 4.

[99] *Ibid.*, II, p. 2.

ARISTOTLE's discussion of the infinite and the infinitely divisible,[100] so that we may reasonably conclude that the analogous notions presented in the *Elements* were already well known in ARISTOTLE's time, presumably through the efforts of EUDOXUS and his followers.

Alternative answers to this question seek to detect distinctions of varying degrees of subtlety between the Archimedean and Euclidean versions of the lemma. J. HJELMSLEV, for instance, has argued that ARCHIMEDES saw the need for a new axiom in studying more advanced problems involving curves – the drawing of tangents to the spiral, for example, and the measurement of spherical surfaces.[101] But we do not find this persuasive. ARCHIMEDES first introduces and applies his lemma in the context of a problem (namely, the measurement of the *area* of the parabolic segment in *QP*) of the same type as those in *Elements* XII. Further, as DIJKSTERHUIS has observed, there is nothing in the nature of the problems in *SC I* and *SL* which demands any reformulation of the two Euclidean principles.

DIJKSTERHUIS proposes an alternative explanation:[102] the Euclidean and Archimedean lemmas are indeed distinct – ARCHIMEDES compares the *difference* of given magnitudes with a finite magnitude of the same type, whereas EUCLID compares, not the difference, but the given magnitudes themselves, one to the other. Now, where such a distinction is significant is in the generation of indivisible or infinitesimal quantities, as by the abstraction of a line from its endpoints, or the subtraction of a curvilinear from a rectilinear angle to leave as remainder the infinitely small "horn-angle." Through his experience of the mechanical method which utilizes indivisibles, ARCHIMEDES would recognize these as an exception to the Euclidean principle, and so introduce his new axiom as a supplement to the Euclidean. Indeed, as most applications of X, 1 in the *Elements* involve taking the difference of given magnitudes and comparing that against another finite magnitude, the Archimedean axiom acts also as a correction to the Euclidean.

But DIJKSTERHUIS' explanation will not do either. For it does not tally with ARCHIMEDES' actual application of the axiom. As we saw, *QP*, 16 can be treated as readily by the bisection-principle as by ARCHIMEDES' axiom. The same holds for the applications in *SL*, 4 and *SC I*, 2. Indeed, the former – which shows how to construct a line segment intermediate in length between a given line and a given circumference – is proved by PAPPUS according to the bisection-method, in one of the theorems in the set of alternative proofs to *Sphere and Cylinder* which we discussed above.[103] But in *SC I*, 3 – the construction of circumscribed

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[100] *Physics*, 206b7; 266b7. See the discussion by HEATH, *Mathematics in Aristotle*, pp. 106–110; and in my *Evolution of the Euclidean Elements*, pp. 272, 285, 290. The latter passage may be rendered thus: "adding continually onto the finite I will exceed any bounded [magnitude] (1), and subtracting I will likewise fall short (2)." While the phrasing is loose, context makes the meaning clear: (1) expresses the principle in *Elements* V, Def. 4, while (2) expresses that in X, 1.

[101] "Eudoxus' Axiom and Archimedes' Lemma," pp. 2–11. His interpretation is criticized by DIJKSTERHUIS, p. 148n.

[102] *Archimedes*, pp. 146–149.

[103] *Collection* V, 29; *cf.* also *Elements* XII, 16. *SL*, 4 is important for the proofs of the tangent theorems to the spiral in *SL*, 18–20.

and inscribed polygons whose perimeters have a ratio closer to unity than a preassigned ratio – the bisection-principle is used, even though the Archimedean axiom would be as convenient. Again, the bisection method is used in the convergence arguments on the area bounded by the spiral (*SL*, 21–23) and on the volume of conoids (*CS*, 19–20), although application of the Archimedean axiom, on the model of *QP*, 16, would do as well. As *SC I*, 3 and *SL*, 21–23 all entail the division of circular arcs, the bisection-principle might presumably be preferred on grounds of constructibility, since the multisection of an arc is not generally possible by Euclidean devices. But this consideration does not apply to *CS*, 19–20. Moreover, in contrast with some modern theories on proof-technique, ARCHIMEDES and EUCLID seem not to have imposed constructibility as a formal requirement in the acceptance of convergence demonstrations.[104]

One should note an important respect in which the Euclidean and the Archimedean lemmas on continuity differ. The Euclidean principle in V, Def. 4 provides a *definition*, specifying when magnitudes may be said to possess a ratio to each other. Yet to be applied, it must be supplemented by an *assumption* – that given magnitudes, be they lines, areas or solids, possess the property of being able to exceed each other after finite multiplication. This assumption is implicit in all Euclidean applications of the definition, *e.g.*, in V, 8 and in X, 1. By contrast, ARCHIMEDES makes this same assumption explicit through his statement of the lemma. Conceivably, then, ARCHIMEDES' recognition that the definition was insufficient even for the purpose of the Euclidean theorems in which it appears may have induced him to state the required supplementary lemma.[104a] This view draws on the logical distinction between the definition and the lemma and on the formal necessity for such a lemma. But it meets difficulties in the texts. If ARCHIMEDES intended merely to supply the lemma needed to implement the Euclidean definition, we should expect the wording of the lemma to mirror that of the definition; yet it does not. The definition speaks of the *multiplication* of one magnitude until it exceeds another; ARCHIMEDES' lemma speaks of the continued *addition* of the *difference* between two magnitudes to itself until it exceeds a third. Indeed, the model for the wording of ARCHIMEDES' lemma is neither V, Def. 4 nor X, 1, but rather the applications of these principles in the exhaustion proofs of *Elements* XII.[104b] An account of

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[104] *Cf.* O. BECKER, "Eudoxos-Studien IV," *Quellen und Studien*, 1933, 2:B, pp. 369–387 on the application of such non-constructive assumptions as the existence of the fourth proportional to three given magnitudes in the convergence proofs of *Elements* XII. A treatment of this issue appears in my article, "Archimedes' *Neusis*-Constructions in *Spiral Lines*," *Centaurus*, 1978, **22**, pp. 101–122.

[104a] This distinction between the Euclidean and Archimedean principles was brought to my attention during the discussion period following my presentation on ARCHIMEDES to the seminar on the history of mathematics at the Courant Institute of New York University on December 1, 1977.

[104b] The bisection-principle is stated in a variety of ways in ancient mathematical texts. The major difference is that in many applications, especially those found in ARCHIMEDES' works, the process entails *exact* bisection at each step. In the Euclidean theorem X, 1, by contrast, it is required that *more than* the half is removed at each step; the case of exact bisection is stated in a corollary. EUCLID's approach is odd, in that exact bisection is the stronger condition; what he proves in X, 1 ought logically to follow as a

ARCHIMEDES' lemma ought to account for its phrasing, as well as for the manner of its application in the various Archimedean theorems.

Thus, the situation appears far more complex than any of the above views of the difference between the Euclidean and Archimedean axioms would yet suggest. I believe an explanation more in the spirit of the first one above will suffice – except that we must suppose, not a failing in EUCLID's edition of the *Elements*, but rather ARCHIMEDES' not having access to the *Elements* as we know them. As mentioned in connection with *PE I*, 7, ARCHIMEDES' treatment of proportions is quite different from what we should expect, assuming his technique was based on Book V. But the only overt application of the Euclidean notion of proportion, V, Def. 5, by ARCHIMEDES occurs in *SL*, 1. By contrast, he reveals in numerous ways familiarity with a source on the "exhaustion" method virtually identical with Book XII as now extant.[104c] Now, the bisection-principle is critical for the convergence arguments in this book (as in XII, 2, 5,

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corollary to it. ARCHIMEDES' usual expression of the principle is in this form: "halving the magnitude continually, we shall leave a magnitude less than the preassigned." It appears in basically this form in *SC I*, 3; *CS*, 19; *SL*, 21; *PE I*, 10, 13. PAPPUS' expression in the proof of the sector-theorem (see note 30a) is also in this form. The somewhat altered form used in *QP*, 24 and HERO's theorem on segments (see note 38a) is echoed in PAPPUS' version of *DC*, 1 (*Collection* V, 3; ed. HULTSCH, p. 315). An expression manifestly modeled after X, 1 appears in the corollary to *QP*, 20. I suspect that the appearances of the principle in the proofs of *Elements* XII were the primary model, and that variations, including the form in X, 1, developed from this. I intend to elaborate this point in the paper cited in Appendix 5.

[104c] One such indication, as we have seen, is the close resemblance between *DC*, 1 and *QP*, 24 and the Euclidean circle-quadrature in XII, 2. There are further agreements on idiosyncratic details. For instance, it happens eight times in Book XII (*e.g.*, in XII, 2) that the theorem on the alternation of ratios (V, 16) is used in the following way: given $A>C$ and $A:B=C:D$, *alternando* $A:C=B:D$, so that $B>D$. But the conclusion $B>D$ ought to follow at once from V, 14. It would appear that EUCLID's source for Book XII drew its technique of proportions from a treatment in some respects different from Book V. (This anomaly in Book XII has been indicated to me by MALCOLM BROWN (Brooklyn College) on the basis of his computerized concordance of the *Elements*.) But ARCHIMEDES too makes the same unnecessary appeal to alternation, as in *SC I*, 13, 34, 44; *SL*, 18, 19; and *PE II*, 1, 8. Moreover, when ARCHIMEDES cites theorems now extant in Book XII, his wording is virtually *verbatim* that in EUCLID (*cf.* the prefaces to *QP* and *SC I*, and the lemmas preceding *SC I*, 17). There is one striking exception to this: in *QP*, preface, ARCHIMEDES mentions the theorem that "circles have to each other the duplicate ratio (διπλασίονα λόγον) of the diameters;" the Euclidean phrasing is "circles are to each other as the squares (τετράγωνα) on the diameters" (XII, 2). Now, ARCHIMEDES' usage is consistent with that employed throughout the *Elements* (*e.g.*, "duplicate ratio": V, Def. 9, VI, 19; "triplicate ratio": V, Def. 10, XI, 33, XII, 12, 18). It is thus EUCLID who is inconsistent in XII, 2. One might suppose that EUCLID had restored an earlier version of the theorem on the circle. But this seems unlikely, as the older wording was "circles have in power (δυνάμει) the ratio of their diameters" (see note 71), as we learn from EUDEMUS (apparently quoting HIPPOCRATES). Rather, it would appear that the older "power"-terminology was supplanted by that of "duplicate ratio" in the pre-Euclidean redactions of Books V, VI, XI and XII; and that EUCLID himself altered the wording of XII, 2, perhaps for pedagogical reasons. In maintaining usage of "duplicate ratio", ARCHIMEDES so betrays reliance on a pre-Euclidean form of Book XII.

10, 11, 12, 16). But its proof is given not here, but as prop. 1 of Book X, the theory of irrational lines. Moreover, the lemma on which it is based (V, Def. 4) is among the preliminaries to still a different work, the theory of proportion. If ARCHIMEDES' early study was based on an exposition of the "exhaustion" method close in form to Book XII, but on treatments of the theories of proportion and incommensurability different from Books V and X, it would be easily possible that the bisection-principle was known to him as a lemma whose proof did not appear in his sources. In his early studies on the circle (*DC*, 1), the parabola (*QP*, 18–24) and related figures, he could employ the principle without misgiving, as its assumption was firmly established in the textbook tradition he knew. But at the time of writing *QP*, his situation has changed. He indicates in the preface to *QP* an anxiety that his proofs might not meet the formal standards of his new correspondent DOSITHEUS. Imposing on himself a tougher standard, he could thus recognize the need to justify the crucial convergence-lemma, and from his effort to provide the proof would come to articulate the more general "Archimedean axiom" which he enunciates in the preface. Indeed, the wording he gives it is in significant respects closer to that it receives in such applications as in XII, 2 – *e.g.*, in its emphasis on the "excess by which one magnitude exceeds another" instead of on the given magnitudes themselves – than it is to X, 1 or V, Def. 4. That is, ARCHIMEDES' axiom appears to be framed after the form of the applications of the bisection-principle, rather than after the Euclidean expressions of its statement and proof. By the later time of writing *SL*, however, ARCHIMEDES has come to possess the Euclidean version of proportion theory, for he applies V, Def. 5 in *SL*, 1. He would thus now recognize that the convergence-principle had been satisfactorily proved in the tradition of elementary geometry, on the basis of an assumption (V, Def. 4) equivalent to his own "lemma." It would thus no longer be an issue which of the two formulations to apply – the decision would be a matter of convenience.[105] Indeed, the special appropriateness of the bisection-form for the study of the area of the circle and the force of the Euclidean model of proofs in *Elements* XII seem to have weighted this decision in favor of that form of the lemma, even where ARCHIMEDES' axiom might be more efficient, as in *CS*, 19. It is X, 1 which is applied by later geometers, not the Archimedean axiom.[106]

I thus maintain that ARCHIMEDES' applications of the lemma reveal no substantial differences from the principle of bisection; but that his articulation of the alternative form and his applications of it in *QP*, *SC I* and *SL* are to be understood as having a historical explanation, rather than as being effected for technical or formal reasons.

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[105] One might suppose that the condition of bisection restricts the applicability of X, 1, in comparison with the Archimedean form. But the Euclidean principle can be extended through a slight modification, so to minimize this distinction between the two principles; see Appendix, section 7.

[106] See, for instance, DIOCLES, *Burning Mirrors*, prop. 15 (ed. TOOMER, pp. 108, 174); oddly, the Archimedean form would be more convenient for DIOCLES' purposes. In PAPPUS, all convergence arguments use the bisection-principle; but as our view takes these materials to be based on early Archimedean treatments, the non-appearance of the Archimedean axiom is expected.

3. Observations

A few remarks may be made in connection with the prefaces to the treatises following *QP*. These will assist our later discussion of the placing of the *Method*.

(a) In *SC I* ARCHIMEDES first mentions his previous communication of the proof of *QP*, 17 (or 24). He goes on: "but later, theorems worthy of note having occurred (ὑποπεσόντων) to us, we worked out (πεπραγματεύμεθα) their proofs. They are as follows."[107] He then lists in order *SC I*, 33, 42–43, 34 (corollary). In the preface to *SC II*, the same theorems are listed as having been sent already; but he there adds *SC I*, 44 at the end of the list. Now, these lists parallel the order of discovery we have proposed for ARCHIMEDES' early studies of the sphere: proofs on the surface of the sphere (*cf.* prop. 33) and its segments (42–43), followed by an independent proof of their volumes (34, 44) based on the preliminary heuristic study by means of mechanics. But *SC I* leaves the impression that this work was then fairly recent. From *SC II* we can infer otherwise. For he begins its preface thus: "You previously wrote me to send you the proofs of the problems whose statements I had sent to CONON. But it happens that most of them are proved via the theorems whose proofs I previously sent to you"[108] – these latter theorems are then listed in order as *SC I*, 33, 42–43, 34 (cor.), 44. It is thus clear that ARCHIMEDES had discovered and, presumably, proved in some form the principal results in *SC I* prior to sending the list to CONON. Moreover, these results were not themselves on the list to CONON (as we learn from the preface to *SL*), so that they must have been first worked out some considerable time before CONON's death.

In view of this, the above translation from *SC I* cannot be correct. An alternative rendering, which in fact is closer to the usual meanings of the words set in parentheses above, is as follows:

> but later, theorems worthy of note having succumbed, we labored over their proofs.[109]

That is, his efforts between the sendings of *QP* and *SC I* were over problems with the proofs in the latter, not the discovery of its major results. This is in keeping with our view that the proofs in *SC I* followed after a reworking of earlier proofs (as preserved in PAPPUS), during a period when ARCHIMEDES was becoming especially sensitive to matters of formal technique.

(b) In the preface to *Spiral Lines*, ARCHIMEDES provides what appears to be the complete list of the theorems stated in the earlier letter to CONON. These consist of all the theorems in *SC II;* the definition of the paraboloid and four theorems relating to the measurement of its volume (*CS*, 11, 21, 12, 24); and, finally, four theorems on spirals: the area enclosed by a single revolution (*SL*, 24), the tangent drawn to it at the end of the first revolution (18), and two

[107] Arch. (Heib.), I, p. 2.

[108] *Ibid.*, p. 168.

[109] According to LIDDELL-SCOTT-JONES, *Greek Lexicon*, ὑποπίπτω regularly means "fall down," "sink," or "succumb;" the sense of "occur (as to the mind)" is only rarely attested, one of the passages cited being *SC I*, preface.

theorems on areas bounded by arcs of the spiral (27, 28). When, subsequently, he sends the proofs on conoids (*CS*), additional proofs on hyperboloids and ellipsoids, not mentioned in the earlier preface to *SL*, are included. ARCHIMEDES explains in the opening lines of the preface to *CS*:

> I have written and am sending to you in this book the proofs of those remaining theorems which you do not have in the books previously sent, and of other theorems subsequently found out besides. Although I had already many times taken up their investigation previously, I was at a loss for their discovery, which seemed somehow elusive. For this reason the propositions themselves were not published along with the others. But afterward I became more attentive to them and found out what I had been at a loss over. What remained of the previous theorems were propositions on the right-angled conoid; those now found out besides pertain to the obtuse-angled conoid and the spheroids.[110]

Here, ARCHIMEDES speaks of the theorems as evading discovery. Thus, it was a difficulty in the *discovery* of the properties of hyperboloids and spheroids – not just a difficulty in their *proofs* – which led him earlier to withhold enunciations of these properties. This indicates that ARCHIMEDES was actively engaged in the discovery of new results in this period and that the heuristic studies, by means of the mechanical method, were not without some difficulty. This is reflected in the *Method* itself, where the measurements of segments of hyperboloids and spheroids entail rather complicated arguments and are presented late in the order of theorems. An earlier indication that, conversely, ARCHIMEDES might forward the statement of a theorem whose proof was not complete occurs in the preface to *SL*. There, he indicates that two of the theorems in *SC II* (prop. 8 and 9) were so stated as to be false in the original list to CONON. We may thus suppose that at the earlier time he as yet lacked the full proof of the two theorems but did not on that account withhold their enunciation.

(c) At the end of the preface to *CS*, having just completed the enunciations of ten representative theorems touching on the whole content of the work, ARCHIMEDES adds: "after the stated theorems have been proved, many theorems and problems are solved." He then poses three theorems on similar spheroids and equal spheroids, and a problem: to cut a spheroid or conoid by a plane parallel to a given plane, such that the segment produced equals a given cone, cylinder or sphere.[111] In the tradition of his earlier works, his posing of such problems suggests that formal proofs would be prepared for a future communication. If such was the case, however, the treatise containing these materials has not survived. It may also be noted that these theorems and problems parallel those presented on the sphere in *SC II*. We may suppose that ARCHIMEDES also applied himself to the examination of the surfaces of conoids and spheroids, again on the model of *SC I* and *II*. But any efforts in this direction were bound to be frustrated, as the requisite techniques go far beyond the scope of ancient methods.

[110] Arch. (Heib.), I, p. 246.
[111] *Ibid.*, p. 258.

(d) The treatises *On Floating Bodies I* and *II* bear no prefatory letters, so we have no direct information on the circumstances which occasioned them. The second proves theorems on the stability of floating bodies, shaped as paraboloidal segments, under a variety of conditions. As it requires theorems on the volumes of such segments, proved as *CS*, 21–24, and on the cross-sections of such solids, proved as *CS*, 11, it is commonly recognized that *FB II* must follow *CS* in order of composition. Moreover, as the basic principles of hydrostatics, such as that of specific weight, are provided in *FB I*, that work must precede *FB II*. The close affinity of the two works argues for placing them together in order.[112]

(e) In *FB II*, 2 ARCHIMEDES assumes the position of the center of gravity of a paraboloidal segment, with the explanation: "for this has been proved in the *Equilibria* (ἐν ταῖς Ἰσορροπίαις)."[113] No formal treatise bearing a proof of this proposition is extant. The heuristic version appears in the *Method* as prop. 5; but ARCHIMEDES can hardly be referring to this work in *FB II*, 2, since by his own insistence in the preface to *M* theorems established by the mechanical method alone cannot be considered as *proved* or as acceptable for formal applications. We must thus posit the preparation by ARCHIMEDES of a formal treatise on the centers of gravity of solids, composed between *CS* and *FB I* and *II*.[114] From other parts of the *Method* we can infer that the work included determinations of the centers of gravity of the cone and of segments of spheres, conoids and spheroids.

The points raised in this section will have their bearing in what follows on the placement of the *Method* in the sequence of ARCHIMEDES' writings.

4. Placing the *Method*

The major part of the *Method* is devoted to illustrating the use of the mechanical method in the study of geometric figures. Most of its propositions treat of theorems whose formal demonstrations also appear in the Archimedean

[112] In *FB I* ARCHIMEDES proves the principle of specific weight, whose discovery is immortalized in the "*heureka*" anecdote of the crown and the bath (in VITRUVIUS; see DIJKSTERHUIS, pp. 18–21). Similarly, a practical background to the studies of *FB II*, on the stability of floating paraboloids, may lie in ARCHIMEDES' participation in the construction of HIERON'S great transport ships. An account of the greatest of these, the *Syracusia*, has been preserved (*cf.* SARTON, II, pp. 122–125) and ARCHIMEDES is expressly associated with its construction. The story is that it was so large, no port save Alexandria could accommodate it; so HIERON presented it as a gift to PTOLEMY. Now, it was PTOLEMY IV (reigned 221–205 B.C.) who earned a reputation for the building of ships. We may have in this a confirmation of the late dating assigned to *FB II*, this work being an outgrowth of such activities late in ARCHIMEDES' career – *i.e.*, after 230 B.C. (For an indication of architectural applications associable with the *Method*, see note 46 above.)

[113] Arch. (Heib.), II, p. 350.

[114] This conclusion has also been reached by ARENDT, "Zu Archimedes," p. 301; *cf.* HEIBERG, *Archimedes*, II, p. 548. I discuss the nature of this work and reconstruct proofs of the center of gravity of the cone and of the paraboloid in "Archimedes' Lost Treatise on Centers of Gravity of Solids," *Mathematical Intelligencer* (Springer Verlag), **1**(2), 1978.

corpus, as the summary presented in Table 4 indicates. One finds here results from *QP*, *SC I* and *II*, *CS* and the lost *Equilibria* – that is, from all the works we have styled the "mature group," save *SL* and *FB I* and *II*, whose subject matters are not well adapted to a mechanical treatment of the Archimedean type.

The prior publication of a formal proof is expressly stated in the instance of only one theorem, *M*, 1. But in the preface to the *Method* ARCHIMEDES refers to "theorems found out earlier [in which] we compared the figures of spheroids and their segments with figures of cones and cylinders." The impression seems clear enough that he assumes on the part of his correspondent, ERATOSTHENES, a familiarity with those results and intends to illustrate the heuristic mechanical method, which he knows ERATOSTHENES has not encountered, by means of some of those theorems. ARCHIMEDES could not have expected a reader, ignorant of the formal theorems which he had previously sent, to accept any of the results presented in the *Method;* for he there unequivocally denies that the mechanical method can have the logical status of rigorous demonstration. In the case of the two new theorems on solids formed from cylinders (*cf. M*, 12–15), he says that only their enunciations had been communicated earlier; for these alone a heuristic treatment is followed by the presentation of formal geometric proofs. Thus, it seems clear that ARCHIMEDES is sending out the *Method* some time after the treatises in which its theorems had been proved formally. We must so conclude that the *Method* is the last of his extant writings.

HEATH, followed unquestioningly by many later scholars, asserts his belief that the *Method* is an early work, preceded only by *PE I*, *QP* and *PE II* in order of composition.[115] But he provides neither justification nor defense for this view. Are we to assume that ARCHIMEDES discovered, and communicated through the *Method*, the mechanical treatments of all these theorems long before the publication of any of the formal proofs? But this view would make the *Method* serve as an invitation for ERATOSTHENES and his colleagues to search out adequate formal proofs; yet whenever ARCHIMEDES had this purpose in mind he forwarded merely the statements of the theorems, as in the letter to CONON and the previous letter to ERATOSTHENES. His express purpose in the *Method* was different: to invite his readers to apply the mechanical technique to discover *new* theorems, a goal which would have been hindered, had he expected his reader at the same time to busy himself with working out formal proofs of the theorems which appear in the *Method.* Are we then to assume that ARCHIMEDES possessed formal proofs of these theorems already, and that the *Method* was, in essence, merely an announcement of forthcoming treatises where they would be supplied, namely, *SC I* and *II*, *CS* and the *Equilibria*? Such an announcement indeed was issued – but in his letter to CONON. As we saw in (b) of the preceding section, however, that list contained a different set of theorems from those in *M*. How shall we account for this discrepancy? Alternatively, are we to assume that ARCHIMEDES had already published formal proofs, in a series of treatises on the volume and center of gravity of spheres, conoids and spheroids and their segments, different from those proofs later addressed to DOSITHEUS? This is the view adopted by HEIBERG and ZEUTHEN,[116] whose

[115] *HGM II*, p. 22.

[116] See note 15.

Table 4. The theorems in the *Method* and their equivalents in other Archimedean works

Prop. of *Method*	Equivalent	Content	Comparison of the treatments
1	*QP*, 17	Area of parabolic segment	Similar subdivision of areas and use of mechanical propositions in *M*, 1 and in *QP*, 14–16.
1	*QP*, 24	Area of parabolic segment	*QP*, 24 employs the technique of the Euclidean quadrature of the circle (*Elements* XII, 2).
2	*SC I*, 34	Volume of the sphere	No trace of mechanical approach in *SC I*, 34.
3	*CS*, 27	Volume of spheroid	*M*, 3 extends to spheriods the analysis given in *M*, 2 to the sphere. Both *M*, 3 and *CS*, 27 examine, by means of comparable subdivisions of volumes, the case of the segment made by a plane through the center of the spheroid. The more general case is stated in *M*, 8.
4	*CS*, 21	Volume of segment of paraboloid	Subdivision of volumes closely related.
5	None extant	Center of gravity of paraboloidal segment	Result assumed in *FB II*, 2 with a citation there of a work called the *Equilibria*. Presumably, that work, now lost, presented the formal proof.
6	None extant	Center of gravity of hemisphere	Presumably, a formal proof appeared in the *Equilibria*.
7	*SC II*, 2	Volume of segment of sphere	No trace of mechanical approach in *SC II*, 2.
8	*CS*, 29, 31	Volume of spheroidal segment	Only the enunciation is given in *M*, 8. The formal proofs in *CS* suggest a prior mechanical treatment.

Note: In all propositions *M*, 1–11 the treatment is heuristic, employing indivisibles in the mechanical manner. In all of the equivalent propositions cited the treatment is strictly formal, in some cases bearing traces of a prior mechanical treatment, in others not.

judgment HEATH merely transmits without proper attribution. But in this view, what purpose can be assigned for the later series of writings to DOSITHEUS? For there, many of the same results are proved—as in the case of the theorems on conoids and spheroids in *CS*—in ways hinting that they had initially been worked out by means of a mechanical method. Finally, how shall we be able to trust ARCHIMEDES' conflicting statement in the preface to *CS* that many of those theorems were then recent discoveries of his?

For these reasons, the view that the *Method* was a late work is surely

Table 4: (continued)

Prop. of *Method*	Equivalent	Content	Comparison of the treatments
9	None extant	Center of gravity of segment of sphere	Presumably, a formal proof appeared in the *Equilibria.*
10	None extant	Center of gravity of segment of spheroid	Enunciation only in *M*, 10. Formal proof probably appeared in the lost *Equilibria.*
11	*CS*, 25	Volume of segment of hyperboloid	Enunciation only in *M*, 11. Proof in *CS*, 25 indicates the manner of a prior mechanical treatment.
11	None extant	Center of gravity of hyperboloidal segment	Enunciation only in *M*, 11. Proof probably appeared in the lost *Equilibria.* ARCHIMEDES adds that many other examples of the mechanical method could be given, but that it has been sufficiently illustrated by the ones already presented.
12–13		Volume of oblique section of cylinder by plane through the diameter of its base	One of the two new theorems stated in the preface to *M*. *M*, 12–13 provide the heuristic mechanical treatment.
14		Same as *M*, 12–13	A heuristic treatment employing indivisibles, but no mechanical assumptions.
15		Same as *M*, 14	The formal proof, based on *M*, 14, without use of indivisibles.
16?		Volume of solid formed by intersection of two cylinders	The end of the *Method* is lost. Presumably, mechanical and formal treatments were given of the second theorem on solids, stated in the preface.

preferable to the view that it was early. The case against the latter view has already been advanced by F. ARENDT in 1914, expanding upon a prior argument of T. KIERBOE.[117] ARENDT's points may be stated briefly as follows:

(i) The *Method* employs terminology newly introduced in *CS; e.g.*, the term "axis" for solids of revolution, and, in particular, the use of this term, instead of "height," in the case of spherical segments.

[117] KIERBOE, "Terminologie des Archimedes," pp. 38–40; ARENDT, "Zu Archimedes," pp. 289–296.

(ii) Some of the theorems in the *Method*—namely, those on the volume of spheroids and hyperboloids—were, as indicated in the preface to *CS*, recent discoveries near to the time of composition of that work.

(iii) In the preface to the *Method*, the phrase "theorems found out (εὑρημένων) earlier [about] the conoids and spheroids and their segments" ought to indicate that formal proofs of these theorems had already appeared.

(iv) There appears with one of the lemmas preliminary to the body of the *Method* the remark that "we use also in the previously written book on *Conoids* the following theorem [namely, *CS*, 1]."[118] HEIBERG had noted that the phrasing is irregular and treated the line as an interpolation. But ARENDT proposes a minor emendation of word-order to obtain a more coherent reading: "we use also from among the things previously written in the book on *Conoids* the following theorem." This would then be a direct citation of *CS* as an earlier writing than the *Method*.

(v) In the preface to the *Method* ARCHIMEDES attributes to DEMOCRITUS "no small share" in the discovery of the theorems on the volume of the cone and the pyramid, which he was first to "assert without proof," but EUDOXUS was first to prove.[119] But in the preface to *SC I*, in connection with these same theorems, he asserts that "of the many noteworthy geometers before EUDOXUS these properties of the figures were recognized by not a single one."[120] This discrepancy is far too strong to explain away as marking the strictness of ARCHIMEDES' conception of demonstration, as ZEUTHEN attempts to do.[121] For in the *Method* ARCHIMEDES allows that the mechanical method does indeed yield "some signification that the conclusion is true." HEIBERG's suggestion that ARCHIMEDES does not place DEMOCRITUS among the geometers will not do either; for DEMOCRITUS had produced a large number of works on mathematics and astronomy.[122] At any rate, once the theorem had been stated, on whatever grounds, it would thereby come into the *recognition*, albeit not necessarily the formally established knowledge, of geometers. ARCHIMEDES, after all, observes that the mere awareness of what the conclusion is greatly assists finding the proof.[123] ARENDT's explanation of the discrepancy thus seems to be preferred: that *SC I* was written before *M* and that in the time between writing them ARCHIMEDES happened upon information about DEMOCRITUS which caused him to change his judgment on EUDOXUS' absolute priority in the discovery of these theorems.

On the whole, ARENDT's points seem to be well taken and his conclusion persuasive that the *Method* is a late work. Nevertheless, certain modifications in his argument may be recommended.

In (ii) ARENDT weakens his point by translating "theorem" in the preface to

[118] Arch. (Heib.), II, p. 434.

[119] *Ibid.*, p. 430.

[120] *Ibid.*, I, pp. 2–4.

[121] "Neue Schrift des Archimedes," p. 344.

[122] "Neue Archimedesschrift," p. 300. On DEMOCRITUS' mathematical writings, see H. DIELS & W. KRANZ, *Fragmente der Vorsokratiker* (6. ed.), II, 68A33 and B11; and HEATH, *HGM I*, pp. 176–181.

[123] Preface to the *Method*, Arch. (Heib.) II, pp. 428–430.

CS as synonomous with "proof." He maintains that ARCHIMEDES held to the principle never to publish even the statement of a theorem unless he had worked out the complete formal proof beforehand. But we observed in the previous section that two of the theorems later proved in *SC II* had been stated falsely in the earlier letter to CONON. ARCHIMEDES' moral on this, in the preface to *SL*, is that "those who claim to solve all [of the problems], but do not produce any of their proofs, [beware lest] they be refuted as having admitted to solve things that are impossible."[124] Now, the sense of this appears to be that ARCHIMEDES *deliberately* introduced the two false theorems as a potential snare for his Alexandrian colleagues; there is, in other words, a certain malice in his intent. But this does not fit well the fact that the theorems at issue had been sent years earlier in a letter to CONON, a man whose mathematical expertise ARCHIMEDES has just lauded in the highest terms in the same preface here. Why should he have wished to lure his friend into such a humiliating situation? Rather, I take ARCHIMEDES' statement on the false theorems to be a face-saving ploy to cover for an earlier error occasioned through an over-hasty communication of theorems not completely worked out. As he has already remarked in the preface, and here with unmistakable bitterness, that "in the many years since CONON's death no one has taken up a single one of these problems," ARCHIMEDES appears to be building a case in his defense for the slip. Agreeing that ARCHIMEDES might well communicate the statement of a result before having worked out the formal demonstration, we have in the later preface to *CS* an indication that the theorems on spheroids and hyperboloids were recent discoveries. As these theorems are treated in the *Method*, that work must be viewed as posterior to *CS*.

There are difficulties with point (iii) of ARENDT's argument. Unfortunately, his assertion rests on his claim that ARCHIMEDES never employs εὑρίσκειν ("discover," "find out," "resolve") save where a proven result is involved. But we have seen that this seems not to be the case. For instance, in the preface to *QP* he speaks of the "theorem" on the parabolic segment as "first *found out* by means of mechanics"; but he has consistently denied that a mechanical investigation amounts to proof. When he later speaks of the same theorems as "*studied* by means of mechanics'" we see that there is no such sharp division in the contexts of εὑρισκειν and θεωρεῖν as ARENDT insists upon. Moreover, ARENDT's conclusion that formal proofs of the theorems in *CS* preceded the writing of *M* does not answer ZEUTHEN's supposition of the prior existence of an alternative version of *CS*.

In connection with point (iv) we may observe that even if ARENDT's emendation is disallowed – indeed, even if the line is wholly rejected, as HEIBERG recommends – ARENDT's view still holds. For the theorem at issue receives a full proof in *CS*, 1, while it is assumed as already proven for the purposes of the *Method*. Were it the case that *M* preceded *CS*, we should expect that ARCHIMEDES could assume the theorem without proof in *CS* just as he does in *M*. The Archimedean treatises are not intended to be self-sufficient expositions; ARCHIMEDES time and again applies results understood to be

[124] Arch. (Heib.), II, pp. 2–4; compare HEATH's translation in *Archimedes*, p. 151.

proved elsewhere, whether in his own earlier writings or in those of others. We thus appear to have in this theorem a sign of the priority of *CS* to *M* in order of composition.

There is another feature of terminology, not recognized by ARENDT, which confirms a late placement for the *Method:* the absence from it of instances of δύναμις ("power") in expressions involving second-order terms. For example, to express of lines A, B, C, D the relation $A:B=C^2:D^2$, one says according to this manner that "A and B have the same ratio in length which C and D have in power." This form is commonly found in pre-Euclidean writers, such as HIPPOCRATES, and in mathematical passages in PLATO and ARISTOTLE. But EUCLID and later writers like APOLLONIUS as a rule avoid such expressions, preferring instead dictions like "as A is to B, so is the square on C to the square on D."[124a] ARCHIMEDES appears intermediate in this trend. As HEIBERG's index indicates, usages of δύναμις and the correlative verb δύνασθαι ("to be equal in power") are frequent in certain works, like *SC I*, whereas the alternative expressions are more common in others, like *SL* and *CS*. This co-ordinates rather well with our chronology: the older "power" terminology is more strongly marked in such works of the early group as *DC*, *PE II* and *QP*—as also in *SC I*, which we indicated existed in an early version within this group; by contrast, this terminology tapers off, as a rule, in the later works. Indeed, where "power" expressions appear in the late group, they often occur in citations of early theorems, like *QP*, 3 and *Elements* XII, 2 (*e.g.*, in *SL*, 25, 26 and *CS*, 21, 22). This change is striking in *CS*, 24 where ARCHIMEDES proves that segments of the paraboloid have the same ratio "as the *squares* on their axes;" in the preface to *CS* he states this result in the form, the segments have "the *duplicate* (διπλασίονα) *ratio* of their axes." But in the preface to *SL* another form is used: that the segments have the ratio "which the lines drawn parallel to the axis ... have to each other *in power*." As we have already noted, the preface to *SL* reproduces the statements of theorems forwarded in a list to CONON much earlier and, in general, the wording of these accords with that used in the extant proofs, as in *SC II* and *SL*. This discrepancy with the wording of *CS*, 24, then, points to a change of style, in which the "power" terminology gave way to the alternative expressions.

It can thus be argued as a sign of late composition that the *Method* employs the "square" terminology exclusively, even though the "powers" forms might be appropriate at numerous places. In particular, ARCHIMEDES' handling of the properties

[124a] Related discussions and references appear in notes 71 and 104c. In EUCLID and APOLLONIUS, the "power" forms survive for the most part in definitions of terms: for instance, "lines are commensurable *in power* (δυνάμει) when the squares on them are measured by the same area" (X, Def. 2); the *latus rectum* is defined as the line against which the ordinates *become equal in power* (δύνανται) to the abscissas—that is, the rectangle produced by the *latus rectum* and any ordinate equals the square on the corresponding abscissa (*Conics*, I, 11; *cf. CS*, 3). But their operations with these terms invariably follow the "square" terminology. Only in *Elements* and XIII are "power" forms frequent. PAPPUS often employs "power" terminology, as in *Collection* V. But, without doubt, this reflects his dependence on early Archimedean writings (see note 71 and associated text).

of the parabola contrasts sharply with his approach in other works. Usually, he adopts a ratio-form of the ordinate-abscissa relation as the principal property of this curve; that is, with reference to Fig. 4a, $\mathrm{B\Delta : BZ = A\Delta^2 : EZ^2}$. In *QP*, 3 this is stated, "as BΔ is to BZ in length, so is AΔ to EZ in power." Precisely this form appears in *PE II*, 10, even though that context recommends the "square" terminology.[124b] Moreover, ARCHIMEDES does not provide the proof for *QP*, 3, but refers the reader to the "*Conic Elements*" for that purpose. In using this expression for the property ARCHIMEDES thus appears to follow an early source of theorems on the conic sections. By contrast, in the *Method* the "square" forms are used. For instance, in *M*, 4 and 5 these forms occur, although the corresponding passages in *CS*, 21, 22 adopt the "power" forms, reminiscent of *QP*, 3. Likewise, the "square" form appears in *M*, 14. It is especially noteworthy here that ARCHIMEDES *derives* the ratio-form of the parabola from its area-form (namely, the geometric form of the standard Cartesian expression, $ay = x^2$). Now, the ratio-form is very easily obtained from the construction of the parabola as the section of a cone; moreover, ARCHIMEDES regularly uses analogous ratio-forms for the ellipse and hyperbola as well.[124c] Presumably, then, the early studies of the conics assigned a prior role to the ratio-form. However, APOLLONIUS takes a different approach: he first establishes the area-form from the construction (*Conics* I, 11) and from this in turn produces the ratio-form (I, 20). Clearly, this parallels the procedure followed by ARCHIMEDES in *M*, 14. That is, ARCHIMEDES' treatment of the parabola, both in method and in terminology, is different from that typifying his other works, but conforms with that adopted later by APOLLONIUS.

On the basis of these observations, then, we claim that one's initial impression of the *Method* is, in fact, correct: that it is a retrospective work surveying a large number of ARCHIMEDES' previously published theorems with the intent of revealing the heuristic method by which they were initially worked out.[125] The *Method* thus falls after the publication of the treatises in which the formal proofs of its theorems are given. This sets *M* after *CS* and the subsequent (no longer extant) *Equilibria*. Its relation to *Floating Bodies* cannot be established directly. To be sure, no theorems from that work appear in *M*; but this need not be significant, as the theorems in *FB II*, like those in *SL*, do not lend themselves to a treatment of the type presented in *M*. I favor assigning *FB* the earlier position. It is a natural sequel to the studies on volumes and centers of gravity of conoids — an area from which ARCHIMEDES could have drawn many more examples, as he says in *M*, 11. But the new theorem presented in the *Method*, on the volume of the cylindrical section, is effected without mechanical

[124b] In this theorem, ARCHIMEDES introduces ratios of areas and ratios of solids; *e.g.*, $A^3 : B^3$, expressed as "the ratio of the cube on *A* to the cube on *B*." One would expect him, remaining consistent, to express ratios like $A^2 : B^2$ in terms of the squares; but in these cases he adheres to the "power" terminology instead.

[124c] For a summary of the properties of the conic sections used by ARCHIMEDES, see HEATH, *Archimedes*, pp. lii–iv.

[125] Here I assume, in view of previous remarks, the qualification about the more complex relationship between the heuristic arguments on the parabola and the sphere in *M*, 1 and 2 and the corresponding formal treatments in *QP* and *SC I*.

assumptions, both in the heuristic and the formal treatments (*M*, 14–15). I take this as a possible sign that ARCHIMEDES' researches were moving away from mechanics, perhaps toward a broader application of indivisibles in the study of the volumes of geometric figures.[126] But it must be admitted that the placement of *M* and *FB II* relative to each other is not certain.[127]

In setting the *Method* among ARCHIMEDES' latest works we of course do not mean that its theorems or the methods of examining these are products of his late thought. Certainly *M*, 1–4 had already been worked out before the sending of the list of theorems to CONON, while *M*, 8 and 11 were available well before

[126] The solids studied here by ARCHIMEDES were reintroduced in the 17th century by KEPLER and CAVALIERI, who of course were unaware of the existence or contents of the *Method*. It seems to me not unlikely that ARCHIMEDES advanced to some degree his geometric studies in the direction initiated in the *Method* and, in fact, pursued by the early modern geometers.

[127] ARENDT and KIERBOE favor assigning to *FB* the later position; but their attempts to use terminology to confirm this view are not persuasive. First, against the observation of ARENDT (p. 290), the *Method* does not go "one step further" than *FB II* in its use of the term "axis;" the two works must use the term in precisely the same way. Earliest uses of the term, as in *Elements* XI and XII, refer to the fixed line about which a solid of revolution (*e.g.*, cone or cylinder) is generated. In the preface to *CS*, ARCHIMEDES applies the term not only to such solids, but also to their segments, so that an asymmetrical solid, like the segment of a paraboloid, might be said to have an "axis." This broadening of the term is done by analogy with usage of the term "diameter" already established in the study of the plane conics. In the *Method*, however, "axis" denotes also the line through a prism which joins the centers of gravity of its opposite parallel faces. Which sense—that of *CS* or that of *M*—applies to the uses in *FB II* of "axis" for segments of paraboloids? While no explicit definition appears there, *FB II* must implicitly recognize the sense adopted in *M*. For *FB II* requires the determination of the center of gravity of the paraboloid segment, which in its turn requires the position of the center of gravity of the prism and the cylinder. These are theorems which were included in the prior *Equilibria*, as cited in *FB II*, 2. (For details on these proofs, see my paper cited in note 114.) Use of the term "axis," then, cannot help us assign relative positions to *FB II* and *M*, although ARENDT's major argument—using the term to assign *M* a position posterior to *CS*—remains valid. In a second related argument, ARENDT uses the term "diameter" to justify placing *PE II* near these three late treatises in time of composition (ARENDT, pp. 296–7; cf. 298–9, 301). But here he is merely mistaken. ARCHIMEDES does not define the *diameter* of a conoid in *CS*, pref.; he defines its *axis*, by analogy with a definition of diameter, already familiar, presumably, in the study of conics. Thus, the term "diameter of the segment" of a parabola, appearing in *PE II*, 10, need not place that work after *CS*. Further, the appearances of Apollonian terminology—in particular, the "ordinates" of the parabolic curve—in *PE II*, 10, as in *M*, 1 and 4 (Arch.-Heib., II, 206, 436, 456), strongly suggest the hand of an interpolator; for in each case the sentence repeats information about lines and points in terms different from those of their original introduction in the same proposition. (Note also that the section from *PE II*, 10—*ibid.*, p. 206, lines 7–11—contains seven instances of Ionic readings. HEIBERG always takes these as signs of interpolation and substitutes Doric readings in his text, as here. But I propose that in this case the entire passage has been interpolated to explicate a step in ARCHIMEDES' argument.) Indeed, save for this one suspected section, *PE II* conforms in terminology as well as technique with the "early" group of writings, antedating *QP*, as we have argued.

the writing of *CS*. Only the studies of the two new solids in *M*, 12–15 can be considered contemporaneous with the date of the compilation and sending of the *Method* as a single work.

A final question about the *Method* may be raised: why did ARCHIMEDES address it to ERATOSTHENES? It is sometimes said of DOSITHEUS, ARCHIMEDES' earlier correspondent, that his calibre as a mathematician is attested by the honor ARCHIMEDES conferred on him in dedicating so many of his formal works to him.[128] But this entirely misconstrues the nature and intent of the prefatory letters. In addressing his works to DOSITHEUS ARCHIMEDES wished merely to reach the attention of the Alexandrian scientific community at large. In the preface to *SC I* he says, "it will now be open to those competent [in geometry] to examine these things; ... deeming it well to transmit these things to those familiar with mathematics, we have written out and send you the proofs, which it will be open to those who apply themselves to mathematics to examine."[129] In the preface to *SL* he says that by sending enunciations of his theorems in advance, but delaying transmission of the proofs, he hopes to encourage others to engage in their study.[130] Thus, the treatises are open communications, not dedications to DOSITHEUS. It is also clear that ARCHIMEDES has no expectation that DOSITHEUS himself will contribute to the study of these problems. For he spares no praise of CONON's mathematical abilities and the contributions he might have made had he lived to study these problems (*cf.* the prefaces to *QP*, *SC I* and *SL*); he speaks well of ERATOSTHENES' talents (preface to *M*) and expresses the hope that he might make profitable use of the mechanical method in his own researches; but of DOSITHEUS he says nothing more than "I have heard you knew Conon and are familiar with geometry" (preface to *QP*).[131] So much for DOSITHEUS' mathematical acumen.

It thus appears that up to the time of CONON's death, ARCHIMEDES communicated with him, and through him with the other mathematicians at Alexandria. In writing next to DOSITHEUS, he wished to maintain contact with those scholars; as ARCHIMEDES had no particular regard for DOSITHEUS' talents, the choice of correspondent must have rested on DOSITHEUS' official position at Alexandria. Now, ERATOSTHENES held one of the highest offices there, head of the Library. Given, moreover, ARCHIMEDES' regard for ERATOSTHENES as a mathematician, we should expect that if the choice were ARCHIMEDES' to make, he would prefer ERATOSTHENES over DOSITHEUS as correspondent. This suggests that it was the departure of DOSITHEUS and the arrival of ERATOSTHENES which occasioned the change of correspondent. This conforms with our placement of the *Method* after the writings addressed to DOSITHEUS. As the first of these was *QP*, sent soon after CONON's death, not before c. 245 B.C.[132] and up to seven major communications intervened

[128] G. SARTON, II, pp. 71n, 73.

[129] Arch. (Heib.), I, p. 4.

[130] *Ibid.*, II, p. 2.

[131] *Ibid.*, p. 262.

[132] An estimate of CONON's date of death may be obtained through an epigram which the poet CALLIMACHUS dedicated to PTOLEMY III and which survives in a Latin

between it and the *Method*, it seems plausible to argue that ERATOSTHENES came to Alexandria some fifteen years later, that is, around 230 B.C. This is feasible, in that ERATOSTHENES was then about 45 and his duties as new Librarian presumably included service as tutor to the royal heir, then aged 15. ERATOSTHENES' own study of the duplication of the cube appears to date from that time; for in the epigram dedicated to PTOLEMY on this subject, ERATOSTHENES speaks of the young PTOLEMY as a "youth." Thus, ARCHIMEDES might well have gained at this time a favorable impression of ERATOSTHENES' potential in geometry and so judged it fitting to assist his efforts by sending the details of the mechanical method to him. This view accords also with what is known of the political situation in Alexandria at that time.[133] I believe, then, that my placing of the *Method* as a late Archimedean work supports a date of c. 230 B.C. both for its composition and for ERATOSTHENES' assumption of the office of Librarian. If I am correct, the conventional view that he took up this position earlier, perhaps as early as 245 B.C., needs to be re-examined.[134]

IV. Summary and conclusions: implications of the new chronology

The results we have argued on the relative chronology of the work of ARCHIMEDES are presented in Table 5. A comparison of it against the chronology most commonly accepted to now, that given by HEATH, reveals several significant differences. As we showed, HEATH's placing of the *Dimension of the Circle* near the end of the list resulted, in effect, from a transcriptional oversight on his part. We have argued, to the contrary, that *DC* ought to be set among

paraphrase by CATULLUS. In it, CONON is responsible for the dedication of a newly articulated constellation, the "Coma Berenices," to PTOLEMY. Now, the "tress of Queen BERENICE" refers to the incident in which the Queen was said to have offered her hair to the gods for the safe return of PTOLEMY from the Syrian campaign. As the war was waged in 246–245 B.C., we possess a *terminus post quem* for the death of CONON. For details on this, see the articles on CONON in *Pauly Wissowa*, the *Oxford Classical Dictionary* and the *Dictionary of Scientific Biography*.

[133] For accounts of the biography of ERATOSTHENES, see the reference works cited in the preceding note. TARN (p. 278) observes that ERATOSTHENES' appointment as Librarian may have been linked with struggles between Cyrenian and native Alexandrian factions at Court, the former of which centered around Queen BERENICE. ERATOSTHENES was from Cyrene and he displaced APOLLONIUS ("of Rhodes"), a native Alexandrian; DOSITHEUS was a native, from Pelusium near the Egypt-Sinai border. As the Crown Prince neared age 15, the appointment of the Librarian, who would serve also as Royal Tutor, must have become a political issue of some moment; ERATOSTHENES' election doubtless led to a general reshuffling of the officials of the Museum and Library. It is perhaps pertinent to note that one of PTOLEMY IV's first acts upon succeeding as King in 221 B.C. was to execute BERENICE and her closest associates.

[134] This view seems to be based on little more than the surmise that ERATOSTHENES would enter the office near the time of the accession of PTOLEMY III (247 B.C.). Altering the chronology on this point may also modify certain conventional views about the vicious dispute between the poets CALLIMACHUS and APOLLONIUS of Rhodes, and the causes of the departure of the latter from Alexandria and his eventual return as Librarian. For ERATOSTHENES was to be his successor in that post.

Table 5. The rearranged chronological order now proposed and its relation to that of HEATH

A. The early group		Chronological order proposed by HEATH (1921)
Archimedean treatises	Related studies in later authors	
Dimension of the Circle Sand-Reckoner Quadrature of the Parabola, 18–24 Plane Equilibria I Plane Equilibria II	Isoperimetric studies (ZENODORUS) Segment of the circle (HERO) Sphere and cylinder (PAPPUS) Spiral lines (PAPPUS)	Plane Equilibria I Quadrature of the Parabola Plane Equilibria II Method Sphere and Cylinder I, II Spiral Lines Conoids and Spheroids Floating Bodies I, II Dimension of the Circle Sand Reckoner
B. The mature group		
Quadrature of the Parabola, 4–17 Sphere and Cylinder I Sphere and Cylinder II Spiral Lines Conoids and Spheroids Equilibria (lost) Floating Bodies I, II Method		

ARCHIMEDES' earliest efforts. By contrast, the *Method*, which HEATH, following HEIBERG and ZEUTHEN, placed relatively early, appears to be much better viewed as a very late work, as we have proposed, extending arguments by ARENDT and KIERBOE. The geometric proof in the *Quadrature of the Parabola* (prop. 18–24) is closely associable with the technique of *DC*, 1, whereas it does not well conform with the mechanical treatment of the parabola (*QP*, 14–17) or with the organization of *QP* as a whole. Yet *QP*, 18–24 contains all the geometric material necessary for the determination of the center of gravity of parabolic segments in *PE II*, so that we may obtain a modified and more convenient view of the relation of *QP*, *PE I* and *II*. The *Sand Reckoner* cannot be placed with certainty; the present ordering of it as second reflects the fact that the single explicit reference it makes to a result as "having been proved by us" is to a result that is, in fact, proved in *DC*, 3. At any rate, the presence of many scientific and mathematical allusions known from work early in the third century B.C., together with the complete absence of references to any late contributions, encourages an early placement of this writing. The content and placement of the lost work *Equilibria*, on the determination of centers of gravity of solids, can be deduced from references in *Floating Bodies II* and the *Method*. It is difficult to decide whether *FB II* or *M* should occur last; for reasons of content, I have favored assigning that position to the latter.[135]

[135] I have not entered shorter works and fragments in the list. As the *Cattle Problem* is addressed to ERATOSTHENES, we may view it as late. Most of the other fragments

There is an important difference in the sense we can give to chronological position in the "early group" and in the "mature." Most of the writings classed as "mature" are placeable through explicit remarks in the prefaces or through technical cross-references in the proofs. What we thus have is the order of composition of the works, or, more precisely still, the order of their publication via communication to Alexandria. In some very important respects, this order need not parallel the order of discovery of the results proved. For instance, the *Method* contains results proved in *QP*, *SC I* and *II*, *CS* and the lost *Equilibria*, yet treats these in a manner closer to the original manner of discovery than any of those earlier writings does.[135a] As a letter addressed to CONON before the sending of *QP* had contained the statements without proof of the principal results in *SC II*, *SL* and *CS*, those treatises present theorems which were not then new discoveries (an important exception being the results on ellipsoids and hyperboloids in *CS*). We may assume, however, that the formal proofs were the product of research done closer to the time of the sending of the treatises in which they appear. We have argued, further, that large parts of *SC I* and *SL* were produced in versions earlier than those extant, and that such were the sources of PAPPUS' discussion of ARCHIMEDES' studies of the sphere and the spirals in the *Collection*.

The ordering of the items in the "early group," by contrast, is less chronological than logical. Their placement is justified through considerations of the development of content and technical methods. In particular, where the subject-matter of a work or the manner of proof—as in the manner of establishing convergence—is close to that in the *Elements*, the work will be classed earlier. This is the feature that links *QP*, 18–24 with *DC*, 1, and both in turn with *Elements* XII, 2. We have discovered, however, a variety of additional indicators confirming such associations. It should be recognized that the items in this group are of a nature different from that of the mature treatises. For instance, *DC* and *QP*, 18–24 are fragments of larger works whose content, purpose and degree of formalization are not known. *PE I* may be a compilation of two or three different approaches to the geometric proof of mechanical theorems. Recognizing the fragmentary character of what has survived from this group, we have seen that results given in other writers, HERO, PAPPUS and THEON, may derive from portions of those or other Archimedean works from this period. These are indicated in Table 5 as the "related works", and include contributions to the study of circles, spheres, spirals and the isoperimetric properties of plane figures. The context in PAPPUS suggests that his source on the spirals was a

appear to conform with our account of the works of the early group in terms of subject matter and methods. These include the rule for the area of a triangle (preserved in HERO, *Metrica* I, 8), the description of the thirteen semi-regular solids (preserved in PAPPUS, *Collection* V, 19), the figures called "arbelos" and "salinon" (in the *Book of Lemmas*) and the *Stomachion*. Two studies—the trisection of the angle, as implicit in *Book of Lemmas*, prop. 8, and the construction of the inscribed heptagon, preserved in Arabic—make key use of the *neusis*-technique of construction. As the same is true of the study of the tangents in *SL*, in that a set of required lemmas is established via *neusis* in *SL*, 5–9, we may have cause for relating these studies chronologically as well.

[135a] This is understood with the qualification stated in note 125.

communication to CONON. Such may be the case of the originals from which the other materials in the "early group" derived. On the other hand, as none of these writings bears a preface or any indication that one ever existed, it is possible that ARCHIMEDES was himself residing and studying at Alexandria when these were written. The dedication of the *Sand Reckoner* to GELON could then be explained as a communication from Alexandria back to Syracuse, in this case being a somewhat popularized report to give an impression of the nature and quality of ARCHIMEDES' work.

The detection of such an early group of studies helps resolve a puzzle raised by ARENDT on the biography of ARCHIMEDES, namely, the time of the appearance of his first mathematical communications.[136] We know on good evidence that ARCHIMEDES perished during the sack of Syracuse by the Romans in 212 B.C. One may establish his date of birth – through the testimony of the late verse-biographer JOANNES TZETZES (12th century A.D.) that ARCHIMEDES was 75 when he died – thus having been born in 287 B.C. Now, from what we said earlier, CONON, ARCHIMEDES' friend and colleague at Alexandria, died not before 245 B.C.[137] As this is noted in the preface to *QP*, we may date that work no earlier than 245 B.C. This would make ARCHIMEDES no younger than 42, and without doubt rather older than this, at the time of his first communication of mathematical results. ARENDT viewed as highly implausible that ARCHIMEDES should not have made his work known earlier than this. We might add that as ARCHIMEDES' father PHIDIAS was an astronomer, we should expect ARCHIMEDES to have been introduced into mathematical studies at a young age; this would reinforce ARENDT's remark. His suspicions thus raised, and doubting for good reasons the general reliability of TZETZES, ARENDT urged assigning ARCHIMEDES a date of birth later than that accepted by about ten years or more. Now, the situation of a precocious mathematician deferring his formal publication until relatively late in life is not unheard of, although it is highly unusual.[138] But ARENDT has assumed HEIBERG's view on the general chronology of ARCHIMEDES' work. Our modifications of his scheme have indicated that a rather large number of researches can be assigned to the early period of ARCHIMEDES' career, and that some of this work, such as the *Dimension of the Circle* in an augmented version, had been issued publicly. Thus, the anomaly of ARCHIMEDES' being in his 40's at the time of sending *QP* is resolved.

The new chronology also reverses an impression left by that of HEATH. The mechanical writings no longer have a prominent place in ARCHIMEDES' early work and appear to have had a more restricted role in the development of his thought.[139] His earliest work, under the influence of his father and CONON, astronomers both, includes, as we should expect, efforts in astronomy and

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[136] "Zu Archimedes," p. 303.

[137] See note 132. For a survey of the evidence on ARCHIMEDES' biography, see DIJKSTERHUIS, ch. I, esp. pp. 9, 26–32.

[138] ISAAC NEWTON is the notorious example.

[139] BERGGREN, for instance, drawing upon HEATH's ordering of the works, has been led to emphasize the role of the mechanical investigations for the whole of the Archimedean corpus ("Spurious Theorems," p. 90).

geometry. ARCHIMEDES studied "exhaustion" methods from a work closely resembling *Elements* XII, undoubtedly drew his knowledge of solid geometry from textbooks in spherics, and knew of the work of EUDOXUS and ARISTARCHUS. The early studies on the circle, parabola and sphere show no traces of the mechanical heuristic background. But in various ways ARCHIMEDES came to be drawn to studies in mechanics: he constructed instruments and devices, like the astronomical sighting device described in *SR* and a planetarium; he was involved in the construction of the siege engines to defend Syracuse – as BERGGREN has observed, probably around age 30; we may suspect that his participation later in the design of HIERON'S great ships inspired his inquiries into hydrostatics. But these mechanical ventures became for him invitations to pursue purely geometric studies: the attempted formalization of the principles of statics in *PE I*, the determinations of centers of gravity in *PE II*, and the lost *Equilibria*, and the solution of hydrostatical problems in *FB I* and *II*. Nevertheless, he came to realize that a certain intuitive appeal to mechanical notions could serve toward geometric discoveries, just as geometry could serve toward proofs of the mechanical principles. His "mechanical method" seems to have been recent near the time of CONON'S death, for only about then was ARCHIMEDES ready to transmit the first discovery he made by it, namely what we view to be an alternative treatment of the area of the parabola, as given in *QP*, 14–17. This method continued to characterize his heuristic work in geometry throughout the "mature group" of treatises, a period we have estimated at some 15 years or so in duration. Yet in the work he devotes to an exposition of this very technique, the *Method* sent to ERATOSTHENES, ARCHIMEDES persists in denying any apodictic value to it; and the new results presented in that work, as in *M*, 14–15, make no application of mechanical assumptions at all. The formal geometric proof (prop. 15) is based on a second indivisibilist treatment (prop. 14) which departs from the mechanical-indivisibilist treatment (props. 12–13) which had just preceded it. The version in props. 14–15 thus provides another example against the view of DIJKSTERHUIS that it was only the use of indivisibles, not the application of mechanical principles, which made the treatments in the *Method* formally unacceptable. As ARCHIMEDES' closing theorems in the *Method* – which we view to be among his last geometric efforts – involve the purely geometric manipulation of indivisibles, it is intriguing to consider to what degree ARCHIMEDES could – or perhaps did – make additional discoveries by these means, as KEPLER and CAVALIERI were to do centuries later. We thus see that the mechanical aspects of ARCHIMEDES' work were subordinate to the geometric aim and that the ancient biographers may have been correct after all in noting the disdain with which ARCHIMEDES viewed the investigations of mechanics *per se.*[140]

[140] According to PLUTARCH, ARCHIMEDES "regarded as sordid and ignoble the construction of instruments; ... he only strove after those things which, in their beauty and excellence, remain beyond all contact with the common needs of life" (cited by DIJKSTERHUIS, p. 13; passage reproduced in I. THOMAS, *HGM II*, p. 30). PAPPUS reports a tradition stemming from CARPUS to the same effect (*Collection* VIII; *cf.* VER EECKE, pp. 813–814). This is certainly an exaggerated view. For, even passing by the large range of his mechanical achievements and their great repute in antiquity, one observes that

The principal value of the new chronology, I believe, lies in the insight one can derive in the development of ARCHIMEDES' geometric technique. There are at least five respects in which the mature group reveal a systematically higher technical level than the early writings.

(a) In the early group convergence arguments are always managed according to the "approximation" form of the method of "exhaustion," as is characteristic of *Elements* XII. In the mature works, a method of "compression," either by ratio or by difference, is always employed. In fact, in the early group circumscribed figures do not appear at all, save in *DC*, 1 and the versions PAPPUS gives of the studies of the sphere and spiral. Of the studies in the early group, the only one in which a compression technique might suggest itself as a significantly more efficient approach is the theorem on the spiral. Unfortunately, the proof as PAPPUS presents it is only partially formalized, so that we cannot tell whether a convergence argument was then worked out for it or even intended. However, the result of such a formalization would have been close in form to the arguments in *SL*, 24 and *CS*, 19. It is thus possible that ARCHIMEDES hit upon the compression technique in the course of his studies of the spirals, despite the fact that the germ of this idea is already present in his treatments of the circle (*DC*, 1) and of the sphere.

(b) The early studies are based on a pre-Euclidean concept of proportion – involving application of the bisection-principle of convergence – as the proof of *PE I*, 7 reveals. Only in the proof of *SL*, 1 do we find an express application of the definitions in *Elements* V. We have seen how it was possible that use of the older proportion technique, in the context of the mechanical investigations in *PE I*, led to the realization of the mechanical heuristic method, described in the *Method*.

(c) That mechanical method, so important as the background to works in the mature group, such as *QP*, *CS* and the lost *Equilibria*, cannot be construed as the heuristic background to any proof in the early group. The fact that the method is likewise irrelevant to the proofs in *SC I* and *SL* – despite the fact that proofs of the mechanical type are possible, at least in the case of *SC I*, as *M*, 2 indicates – may be cited as support for our assigning to the early group the studies of the sphere and the spiral, in the versions preserved in PAPPUS.

(d) A special form of the convergence-lemma, the so-called "Archimedean axiom," is introduced in the mature group, whereas the Euclidean bisection-principle is always used in the early group. The issue is vexed, however, since ARCHIMEDES also uses the bisection-principle frequently in the mature group. I have argued that the two forms of the principle are technically equivalent in

ARCHIMEDES' geometry was also readily susceptible of practical applications. HERO's *Metrica* makes this abundantly clear; and ARCHIMEDES' contemporary biographer, HERACLIDES, noted of the *Dimension of the Circle* its necessity for practical matters (EUTOCIUS' commentary on *DC* in Arch.-Heib., III, p. 228). Indeed, PAPPUS relays an alternative view on ARCHIMEDES, emphasizing his extraordinary combination of the practical and theoretical. Nevertheless, the small representation of practical mechanics in the written tradition (ARCHIMEDES left only one work of this type, the *Spheropoeia*) puzzled the later commentators. And their impression that ARCHIMEDES' true interests lay in the area of pure geometry may be a credible judgment.

their suitability for the theorems investigated by ARCHIMEDES, and that his formulation of the principle is explainable by his lacking the Euclidean proportion theory on which justification of the lemma depends. Once he came into possession of that theory, as indicated in (b) above, he could recognize the equivalent applicability of the two forms of the principle. His choice of one form or the other would thus be guided by the special demands of each theorem studied. There seems to be a preference for the bisection-form where the division of angles is required, as in *SC I*, 3 and *SL*, 4 and 24. But sometimes the bisection-form is used where ARCHIMEDES' alternative formulation would seem to be the better choice, as in *CS*, 19. (It would be interesting to know which version was adopted in the proof in *M*, 15; unfortunately, the manuscript has been found illegible in the passages where the convergence argument must be given.) The ultimate phasing out of the Archimedean form, if we can impose that interpretation on the use of the bisection-form in *SL*, 24 and *CS*, 19, goes beyond ARCHIMEDES. For no proofs have survived from later geometers in which the axiom of ARCHIMEDES is used. Nevertheless, the axiom appears to have been important for a stage of ARCHIMEDES' thought; and the role it plays in *QP*, 16 may point to an association of the axiom with the introduction of the mechanical method.

(e) The mature group of writings reveal a great concern with formal precision in proof. This is evidenced in the care ARCHIMEDES takes to introduce the new axiom in *QP*, his clear separation of heuristic and demonstrative techniques in the *Method*, and of course his meticulous organization of all the theorems in this group. In contrast, the level of mastery of formal details is much lower in the early group. To be sure, this impression may be due merely to the fragmentary survival of the early materials. It is possible, for instance, that the full treatise on the *Dimension of the Circle*, from which the extant *DC* was excerpted, and which is quoted by HERO and PAPPUS, may have included statements of many of the lemmas and assumptions requisite for *DC*, 1; that is, that it was a work much in the formal manner of *SC I*. But this seems to us unlikely. Among the early writings, the one clear attempt to axiomatize a subject-matter, *PE I*, was unsuccessful in a number of important respects. Moreover, it is possible to see the commencement of the mature group as a real turning-point in ARCHIMEDES' concern with formal matters. Having lost a sympathetic correspondent in the death of CONON, and being compelled to communicate with the Alexandrian mathematicians through DOSITHEUS (a man he knew only by hearsay), ARCHIMEDES might well have been anxious over the reception of his work. His remarks in the prefaces reveal just such concern over the judgment of those scholars: when he presents the mechanical treatment in *QP*, 14–16, he still makes sure to append a formally acceptable geometric version as well; in the preface to *SL* he deplores the Alexandrians' inattention to the advancement of geometry through new discoveries. Indeed, it would appear that DOSITHEUS' interests were more scholastic than geometric. For despite his persistent demands to see the proofs of theorems whose statements ARCHIMEDES had already forwarded, where is there any evidence that he had worked out proofs for himself or that he had advanced the subject by new discoveries or methods of investigation?[141] Only in the work addressed to ERATOSTHENES, the *Method*, does ARCHIMEDES bring up the topic of the mechanical method again, expressing the hope that it may be applied profitably as a research tool.

In these ways, I believe, the new chronology here argued will be able to extend and reshape our understanding of the nature and development of ARCHIMEDES' thought. The present study is at best only a first step in the re-examination to be undertaken of the Archimedean works. Others have addressed the question of the chronological sequence in ARCHIMEDES' writings by considering changes in terminological usage. The approach which I have taken here, however, has been different, in that it is founded on the techniques employed by ARCHIMEDES. In deciding where to place the *Dimension of the Circle* within the sequence of works, I hit upon a criterion which appeared quite plausible in itself and which might be adopted in examining other works: that ARCHIMEDES' reliance on Euclidean techniques must have manifested itself more strongly in his earlier studies. I believe this intrinsic type of criterion has the advantage over any external, purely philological one, that it penetrates more closely to the things we really wish to know: namely, defining more precisely the nature of ARCHIMEDES' debt to his predecessors and separating out the character of his own particular contributions to geometry and exact science. At the very least, I have exposed inconsistencies and implausibilities in the standard chronology and have indicated what value a reasonable working solution to this issue can have.

The notions of the Socratic PLATO and the Platonizing ARISTOTLE, however controversial they have proved to be in detail, have nevertheless transformed the modern study of classical philosophy by introducing the possibility of genetic and motivational hypotheses and interpretations. The same advantage, I maintain, may accrue to the student of ancient science by viewing ARCHIMEDES' works in their relation to the *Elements*, so to perceive the differences between the early Archimedean writings and the late ones. Thereby we may at last recog-

[141] We may cite two instances of the scholastic character of Alexandrian mathematics. (a) In the preface to *Conics* IV (*cf.* HEATH, *HGM II*, pp. 130–131), APOLLONIUS recalls how NICOTELES had been swift to attack the proofs of CONON, yet had made no attempts to correct them. As a mathematician, APOLLONIUS found much of use in CONON's work, in contrast with NICOTELES who judged it useless. (b) In similar vein, in a passage in PAPPUS the pre-occupation over first-principles of many geometers is contrasted with the writer's own interest in exploring new methods and making new discoveries (*Collection* VII, 41–42; ed HULTSCH, pp. 680–682); to make his point, the writer proposes a theorem on the measurement of volumes by means of centers of gravity (it is an equivalent of the GULDIN-rule) and notes its use for the alternative proofs of familiar results, like those in *Elements* XII, and the discovery of new ones. As this principle may be associated with DIONYSODORUS' treatment of the torus, it is possible to place this passage in PAPPUS in the context of mathematics in the late 3rd century (see I. THOMAS, "Dionysodorus," in *DSB*). This would serve as another indicator of the schism among geometers at that time: CONON, ARCHIMEDES and APOLLONIUS giving greater weight to the active and creative work in mathematics over against the restrictive formalism of such as NICOTELES and, perhaps, DOSITHEUS.

nize the shallowness and poverty of the prevailing, effectively ahistorical attitude toward ARCHIMEDES, which assumes in all but a few of his works a uniform standard of rigor and sophistication. ARCHIMEDES' genius for formal precision in mathematics remains unchallenged. But we can now begin to recognize the steps by which he attained that level of precision in his mature works, having first mastered and then outgrown the technical methods of his precursors.

V. Appendix: Some important theorems with outlines of their proofs

1. Euclidean and Archimedean versions of the quadrature of the circle

In *DC*, 1 ARCHIMEDES establishes an expression for the area of the circle: that it equals the right triangle whose legs equal, respectively, the radius and the circumference of the circle. The method of proof follows closely the pattern of *Elements* XII, 2 (that circles are as the squares on their diameters).

We will write A for the area of the given circle, r for its radius and c for its circumference. The corresponding terms for inscribed regular polygons will be P_i, r_i and p_i; and for circumscribed polygons P_c, r_c and p_c. We will denote by T the area of the triangle of the theorem; *i.e.*, $T=\frac{1}{2}rc$. The theorem thus claims that $T=A$.

The proof is indirect. Suppose first $T<A$. There is a finite magnitude k such that $A-T>k$. Then, as shown in Fig. 8a, we inscribe a regular polygon (say, a square) in the circle and we note that the difference between the circle and the square is a finite magnitude, say, d. We next bisect each arc of the circle, beginning from AC bisected at B, so that the inscribed octagon is obtained after all the chords AB, BC, *etc.* have been drawn. If rectangle $ADEC$ is erected on AC, we observe that triangle ABC is one-half its area, whence the same triangle is greater than half the segment ABC. It follows that the difference between the circle and the inscribed octagon is less than $\frac{1}{2}d$. Continuing this procedure, we have that each doubling of the number of sides reduces the difference between circle and polygon by more than half. By X, 1 (see section 7 below) we will eventually obtain a remainder less than any preassigned finite magnitude. Let this process thus result in a polygon P_i which differs from the circle by less than

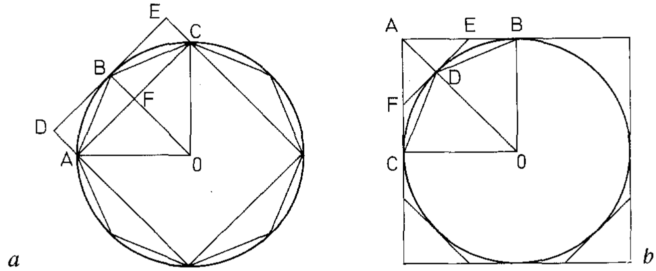

Fig. 8

k. Since $p_i < c$, it follows that $P_i = \frac{1}{2} r_i p_i < \frac{1}{2} r c = T$; while $A - P_i < k < A - T$, that is, $T < P_i$. This contradiction establishes that A cannot be greater than T.

ARCHIMEDES does not actually provide the full details of the bisection argument. But as the same argument is required in his proof of *SC I*, 6, and he there justifies the step by the remark, "for this has been handed down in the *Elements*," I have presumed the same here and so filled out the argument from XII, 2. In *QP*, 20, the convergence argument for the quadrature of the parabolic segment, ARCHIMEDES provides a thoroughly analogous argument.

It must now be shown that $T > A$ is impossible. We suppose $T - A > k$. Circumscribe about the circle a regular polygon (say, the square, as in Fig. 8b); let their difference be d. Bisect the arc BC at D and draw the tangent EDF. Triangle ADE is greater than triangle EDB. (ARCHIMEDES omits the proof, but it is supplied as a lemma by THEON (ed. ROME, pp. 361f); it follows since the two triangles have the same altitude, while $AE > EB$.) Thus, triangle AEF is greater than the half of the two triangles ABD, ADC; whence, *a fortiori*, it is greater than the half of $ABDC$ bounded by the circular arc BDC and the right lines BA, AC. Doing this for each remaining vertex of the circumscribed square, we have that the difference between the circumscribed octagon and the circle is less than $\frac{1}{2} d$. With each doubling of the number of sides the difference is in the same way reduced by more than one-half. By X, 1 a remainder will be obtained eventually less than any preassigned magnitude. Let P_c be a circumscribed polygon so obtained that $P_c - A$ is less than k. Thus, $P_c < T$. But as P_c is circumscribed about the circle, $p_c > c$. (This is assumed in *DC*, 1; it is proved as *SC I*, 1 on the basis of ARCHIMEDES' postulate on convex arcs.) Thus, $P_c = \frac{1}{2} r p_c > \frac{1}{2} r c = T$. This contradiction establishes that T cannot be greater than A; nor can it be less. Therefore, $T = A$.

The manner of establishing convergence in *DC*, 1, as in *QP*, 20 is entirely in the pattern of XII, 2. The bisection-principle is used exclusively and the circumscribed case is handled separately from the inscribed (*i.e.*, the method is by "approximation," rather than by "compression"). As EUCLID's theorem involves the ratios of figures, he can avail himself of a logical symmetry which makes the circumscribed case dispensable. As this was not open to ARCHIMEDES, he was compelled to go beyond the Euclidean model in the important respect of articulating the axioms required for the comparison of arc-lengths. But in his actual handling of the convergence, even in the circumscribed case, he adheres to the Euclidean procedure.

In *SC I* an entirely different strategy is devised to establish the convergence of the polygonal sequences to the circle. The general method is of "compression via ratio."

In *SC I*, 3 ARCHIMEDES shows how to construct for a given circle polygons P_c, P_i, such that for any given ratio of magnitudes a, b (a the greater), $p_c : p_i < a : b$. He first takes lines c, d such that $c : d < a : b$ and constructs the right triangle GHI, with $GH = d$, $GI = c$ (Fig. 9a); we set angle $HGI = \theta$. He then bisects the quadrant of the given circle in succession until an angle α remains which is less than θ. (It is interesting that a bisection-process is used here. ARCHIMEDES might just as easily have chosen a submultiple of the quadrant-angle less than θ, justifying this on the basis of the "Archimedean axiom" stated in the preface to *SC I* and

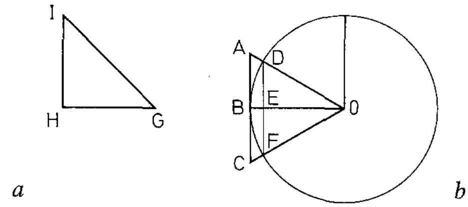

Fig. 9

employed in such a way, for the case of areas, in *QP*, 16. But the use of the bisection-principle in the subdivision of angles may reflect a concern for constructibility, at least where the *Elements* had established a firm precedent.) Then, with α as central angle, the similar triangles OAB and ODE are drawn (as in Fig. 9b). As $\alpha<\theta$, $OD:OE<GI:GH$, so that (by similar triangles and the fact that $OD=OB$) $AB:DE<c:d$. Completing the polygons whose half-sides are, respectively, AB and DE, we have $p_c:p_i<c:d<a:b$, as claimed.

From this it follows that the areas too may be found as to have a ratio less than any preassigned. Given a, b as above, we take e their mean proportional, *i.e.*, $(a:e)^2=a:b$. We find P_c, P_i such that $p_c:p_i<a:e$. Then, $P_c:P_i=(p_c:p_i)^2<(a:e)^2=a:b$.

In *SC I*, 6 it is shown how to find polygons which differ from the circle by less than any preassigned amount. Let the preassigned difference be d and the area of the circle be C. Since for all inscribed polygons, $P_i<C$, $(C+d):C<(P_i+d):P_i$. Applying the previous result, we may find polygons such that $P_c:P_i<(C+d):C$; whence, $P_c<P_i+d$. Since the circle is contained between P_c and P_i, it follows that it differs from either polygon by less than d.

With the above result, one could complete the proof of *DC*, 1. In this way, the Euclidean features of "approximation" and "bisection" could be eliminated. Thus, it should be clear, against those who assign *DC* a place *after SC I*, that if ARCHIMEDES could make appeal to *SC I*, 6 in his proof of *DC*, 1, the manner of proof would have been far different from that actually used. We have, accordingly, seen in this a sign of the prior composition of *DC*, 1.

Similar considerations apply to the proof of *DC*, 3. (See *Note 1, added in proof*.)

2. Area of the parabolic segment via summation

In *QP*, 24 ARCHIMEDES abandons the model of the mechanical treatment of the area of the parabolic segment (*QP*, 14–16) to provide a geometric proof in the Euclidean manner, instead. But the mechanical method could have been geometrized in a variety of ways. The one shown here is based on the quadrature of the spirals in *SL*, 24.

The parabolic segment is contained between circumscribed and inscribed rectilinear figures (C_n, I_n, respectively), where the lines AB and BF have each been divided into n-many equal parts, each equal to AD, BC, respectively (Fig. 10; *cf.* Fig. 1). We set triangle $ABF=\lambda$, so that triangle $ABC=\lambda/n$ and

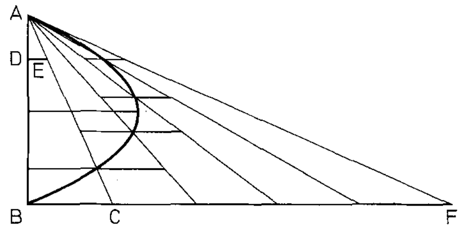

Fig. 10

$ADE = k = \lambda/n^3$. We may conceive each rectilinear figure as an aggregate of triangles, each with vertex at A, whose altitudes and bases increase arithmetically. Thus, $I_n = k + 4k + 9k + \cdots + (n-1)^2 k$ and $C_n = k + 4k + 9k + \cdots + n^2 k$. By the summation-formula established in *SL*, 10 it follows that $I_n < \frac{1}{3}n^3 k = \frac{1}{3}\lambda$ and $C_n > \frac{1}{3}n^3 k = \frac{1}{3}\lambda$. Thus, both areas $\frac{1}{3}\lambda$ and P (the area of the segment) are bounded by C_n, I_n, so that their difference d is less than $C_n - I_n$. Moreover, as $C_n - I_n = \lambda/n$, we may choose n large enough so that this latter difference is less than any preassigned amount, say d. We thus have the contradiction $d = |P - \frac{1}{3}k| < C_n - I_n < d$. It follows that $P = \frac{1}{3}\lambda$.

The convergence argument here is that given in *QP*, 16; the summation-argument replaces the mechanical argument of *QP*, 14–15. As ARCHIMEDES uses an analogous summation-argument in the quadrature of the spirals (*SL*, 24), there is a question why he did not choose to do the same here. In the light of our view of the earlier origin of the "geometric" proof (*QP*, 24), an explanation may be proposed: that, already possessing this version, ARCHIMEDES had no need to geometrize the mechanical version in order to produce a formally acceptable proof. Moreover, as *SL* was dispatched rather later than *QP*, it might well be that the summation-expression introduced as *SL*, 10 had not yet been worked out at the time of *QP*. Indeed, ARCHIMEDES admits in the preface to *CS* (next after *SL*) that certain theorems on hyperboloids and ellipsoids had given him difficulty; these theorems (*CS*, 25–30) rely on a summation (*SL*, 2) based on a lemma identical with *SL*, 10. We may thus understand that ARCHIMEDES, possessing a Euclidean-style geometric proof of the area of the parabolic segment, but not yet able to effect the complete geometrization of *QP*, 14–16, introduced the discontinuity of exposition in *QP* out of practical necessity.

3. Area of the parabolic segment via compression

In *QP*, 24 ARCHIMEDES effects the geometric proof of the area of the parabola in strict analogy with the Euclidean quadrature of the circle (see section 1). But a more efficient treatment was available to him, had he introduced his "compression" technique instead of the Euclidean "approximation".

In Fig. 11 the parabolic segment ABC has base AC, vertex B and diameter BG. If the tangents at A and C are drawn, they will meet at E such that EBG is a straight line and $EB = BG$ (*QP*, 2; *cf.* 17). Also, $AG = GC$ (*QP*, 1 and 18). It

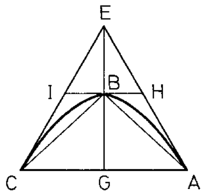

Fig. 11

follows that triangle EBC is equal to BCG and $AEB = ABG$. If the tangent at B is drawn to meet EC at I and AE at H, $BI = \frac{1}{2}CG$ and $BH = \frac{1}{2}AG$ (since HBI is parallel to AGC and $EB = BG$). Thus, triangles AHB and CIB are each equal to half of ABG, CBG. If now the segment is bounded by inscribed and circumscribed rectilinear figures via a process of successive doubling (as illustrated in Fig. 3), we have that $I_{n+1} = k + (\frac{1}{4})k + (\frac{1}{4})^2 k + \cdots + (\frac{1}{4})^m k$ and $C_{n+2} = k + (\frac{1}{4})k + (\frac{1}{4})^2 k + \cdots + (\frac{1}{4})^m k + \frac{1}{2}(\frac{1}{4})^m k$, where k equals the area of triangle ABC, and the number of sides $n = 2^{m+1}$. (For the inscribed figure, the base of the segment adds one side; for the circumscribed figure there is yet another side added, as the sides from A and C are but half-tangents.) Since $C_{n+2} - I_{n+1} = \frac{1}{2}(\frac{1}{4})^m k$, each doubling of the number of divisions of the arc ABC reduces the difference by three-quarters; X, 1 applies to ensure convergence. (In *QP*, 20 ARCHIMEDES introduces a bisection argument, strictly in the manner of XII, 2 and *DC*, 1 and clearly less indicative of the actual rate of convergence.) From *QP*, 23 it is known that $\frac{4}{3}k = k + (\frac{1}{4})k + (\frac{1}{4})^2 k + \cdots + (\frac{1}{4})^m k + \frac{1}{3}(\frac{1}{4})^m k$. Thus, C_{n+2} and I_{n+1} are bounds for both $\frac{4}{3}k$ and P, the area of the segment. The equality of these areas follows from an indirect argument, as already given in section 2.

4. The center of gravity of the parabolic segment

In his determination of the center of gravity of the parabolic segment in *PE II*, ARCHIMEDES subdivides the parabola precisely in the manner of the "geometric" proof of *QP*, 18–24. It is noteworthy that he should do this, rather than develop upon the manner of the "mechanical" treatment of *QP*, 14–16, since in fact, the argument developed in *PE II* can easily be adapted to provide such a proof.

Let us begin by enclosing the segment between two rectilinear figures, each an aggregate of triangles with vertices at A, as shown in Fig. 10. Set k_c as the center of gravity of the circumscribed figure, k_i as that of the inscribed figure, and k as that of the segment. From inspection it follows that k_i is nearer to A and to AB the base of the segment, than is k_c. We denote by T the trapezium formed by the rays from A through the points k_c, k_i and by the lines through those points parallel to the diameter of the segment. By an argument analogous to *PE II*, 5 it follows that k lies within this trapezium.

Let any two parabolic segments be taken and subdivided into the same number of parts, as in Fig. 10. Then the points k_c, k_i in the one and the corresponding points k'_c, k'_i in the other will be similarly situated in relation to the base and diameter of their respective segments. This follows by an argument analogous to that of *PE II*, 3, in that the triangles of the bounding figures in each progress in the same proportion, 1:4:9:... of successive square integers, while the triangles are disposed similarly in relation to the whole.

It can next be established that for any preassigned length a subdivision of the segment may be found such that the distance between k_c and k_i is less than this length, on the pattern of *PE II*, 6. This demonstration does not make use of the bisection-criterion of X, 1, but develops a contradiction from the assumption that the distance between k_c and k_i (or that between k and k_i) shall always exceed a certain finite length. That is, the technique used is in the spirit of the convergence arguments using the "Archimedean axiom," such as *QP*, 16.

It follows that k lies on the diameter of the segment. For suppose it lies below by a certain distance. Construct bounding figures such that the distance between k_c and k_i is less than that distance; and so, the whole trapezium they define lies below the diameter. If, now, the segment is inverted and subdivided into the same number of parts, the corresponding trapezium formed by k_c', k_i' will lie below the diameter (that is, away from B), while the center of gravity of the segment, k, will lie above it. This contradicts the result that k must lie within the trapezium. (ARCHIMEDES establishes this result in *PE II*, 3–4 in a very different manner, utilizing the symmetry inherent in the "geometric" subdivision of the parabola – in this, there is an evident analogy with his determination of the center of gravity of the triangle in *PE I*, 13. The alternative "mechanical" subdivision possesses no such symmetry in relation to the diameter of the segment.)

One may now show that the centers of gravity k, k' of two parabolic segments divide the respective diameters in the same ratio. The proof may follow the pattern of *PE II*, 7, based on an indirect argument much like the one just used to show that the center of gravity lies on the diameter.

Finally, one may show that the center of gravity divides the diameter into segments having the ratio 3:2 through an essentially algebraic manipulation, as ARCHIMEDES does in *PE II*, 8.

As sketched here, the two versions are closely comparable, that actually presented by ARCHIMEDES being in some respects the more economical. Nevertheless, it is striking that he should not maintain the "mechanical" subdivision of the segment, had that been fundamental for his prior studies of the area of the segment. But in two respects we can see how even in his "geometric" version technical considerations encouraged the modification of the Euclidean approach: (a) To establish the convergence of the k_i to k (in *PE II*, 6) the Euclidean argument must be radically altered before successive bisection of the remainders can be assured. Rather than do this, ARCHIMEDES introduces a new approach, based on a proportionality, one reminiscent of the use of the "Archimedean axiom" in *QP*, 16. (b) In conceiving the area of the segment not as approximated by sequences of triangles in the Euclidean manner, but as parallel lamina bisected by the diameter, ARCHIMEDES invokes a symmetry which greatly simplifies the proof that k lies on the diameter. Moreover, it is just such a subdivision of areas and volumes into parallel elements which is characteristic of the "mechanical" method. I take both observations as encouraging evidence that the mechanical works – and *PE II* in particular – are transitional in the sequence of ARCHIMEDES' development: like *DC*, 1 and *QP*, 24, they are strongly fashioned in the Euclidean geometric mould; and yet their special circumstances call for the introduction of new modes and techniques – notably, the parallel laminar sectionings characteristic of the mechanical method, and the criterion for convergence in its more general Archimedean form – which were to become widely applied in the mature works.

These remarks on the possibility of an alternative treatment of the determination of the center of gravity of the parabolic segment should not diminish our admiration for the excellence of ARCHIMEDES' mathematical argumentation in *PE II* as it is. For if one set out to solve this problem straightforwardly – that is, in a manner analogous to its familiar formulation in terms of a definite integral – it would require an equivalent to the evaluation of $\int y^4 \, dy$, that is, to the summation of successive fourth-order terms. To be sure, the Greek geometers sometimes introduced fourth- and higher-order entities, such as in HERO's measurement of the area of a triangle whose sides are given (*Metrica* I, 8) and PAPPUS' discussion of the general $2n$-line locus (*Collection* VII, ed. HULTSCH, p. 680). But such expressions were judged improper, and a correct formal treatment would be extremely cumbersome. In the present instance, ARCHIMEDES has finessed these difficulties cleverly; through the astute manipulation of the properties of similarly disposed configurations of rectilinear figures, he has shown how to solve this problem of centers of gravity within the compass of Euclidean analytic methods.

5. Archimedes' proof of the principle of the balance

In *PE I*, 6 ARCHIMEDES establishes that two commensurable weights A, B suspended at distances a, b respectively, will be in equilibrium if $A:B=b:a$. Passing over the controversies on the validity of his proof and the adequacy of his axioms (these are surveyed in detail by DIJKSTERHUIS, *Archimedes*, pp. 291–304), let us turn to the treatment of the incommensurable case, given in *PE I*, 7. As the text of the proof of this appears incomplete, I will closely paraphrase it, indicating within square brackets what must be supplied.

> Let A, B be incommensurable magnitudes, respectively suspended at distances a, b, such that $A:B=b:a$. It is claimed that they are in equilibrium. If not, let it be assumed that A at distance a is too great to balance B at distance b. Remove from A an amount such that the remainder A' at a is still too great to balance B at b, and A' is commensurable with B. Since $A':B<b:a$ and A' and B are commensurable, it follows that A' at a will not balance B at b, [but that A' will be too small to balance B. For if A' is suspended at the distance a' such that $A':B=b:a'$, it will balance B (by *PE I*, 6); since $b:a'<b:a$, it follows $a'>a$; thus, A' at the shorter distance a must be too small to balance B at b (by Postulate 1). But this is impossible, since A' was constructed to be too great to balance B.] For the same reasons, B [at b] cannot be too great to balance A [at a. Therefore, A, B are in equilibrium when suspended at the distances a, b, respectively.]

This proof has been attacked as invalid and unsalvageable, save through additional axioms (DIJKSTERHUIS, pp. 305–6; BERGGREN, p. 97). In point of fact, the proof is sound, as supplemented above, and requires nothing more than appeal to the commensurable case (*PE I*, 6) and to Axiom 1. But there *is* something unusual about the argument: ARCHIMEDES assumes that the magnitude A' may be constructed, as still to overbalance B and also to be commensurable with it. In a paper in progress I trace the sources from which ARCHIMEDES could draw to back up this assumption; it emerges that he is operating within a technique of proportions differing from the Euclidean.

To appreciate this, let us construct a Euclidean proof of this same theorem. One should note the important part played by the introduction of equimultiples of the given magnitudes, in order to invoke the definition of unequal ratio (V, Definition 7).

> Let A, B be incommensurable magnitudes, suspended at distances a, b respectively, such that $A:B=b:a$. It is claimed they are in equilibrium. If not, let A at a overbalance B at b. Then, if A' at a balances B at b, A' will be less than A (*PE I*, 2). Thus, $A':B<b:a$. By the definition of unequal ratio (V, Def. 7), there are integers m, n such that $mA'<nB$, while $mb \geqq na$. Now, from an argument analogous to the commensurable case in *PE I*, 6, we may establish that if A', B are in equilibrium at a, b, then mA', nB are in equilibrium at na, mb. But mA' is the lesser weight and na is the lesser distance. Therefore, they cannot be in equilibrium (by *PE I*, 3). This contradiction establishes that A, B must be in equilibrium when suspended at a, b.

That ARCHIMEDES did not invoke the Euclidean definition, despite the effectiveness of a proof along these lines, but instead appealed to what (at least to us) is an obscure lemma on the construction of commensurable magnitudes is, I believe, a *prima facie* reason for assuming that he was working within a tradition of proportion theory different from the Euclidean one of *Elements* V. This assumption is further confirmed by the fact that ARCHIMEDES *does* apply the Euclidean definition elsewhere in his works, namely in *SL*, 1 – that is, in a work we have argued to be much later in date of composition than *PE I*. The existence of one such pre-Euclidean proportion theory, based on *anthyphairesis*, the "Euclidean" division algorithm, has long been argued (see my *Evolution of the Euclidean Elements*, ch. VIII, sect. II and App. B). But the view I am in progress of elaborating links ARCHIMEDES' technique in *PE I*, 7 not with this anthyphairetic theory, but with a different one, built on the bisection-principle of convergence and attributable to EUDOXUS himself.

6. A theorem on isoperimetric figures

ARCHIMEDES assumes as proven a theorem on inscribed polygons (*SR*, II, 2; ed. HEIBERG, II, p. 234) which resembles one of the basic lemmas in the theory of isoperimetric figures. We will state both theorems and sketch their proofs.

The Archimedean theorem is this: let n be the number of sides of a regular

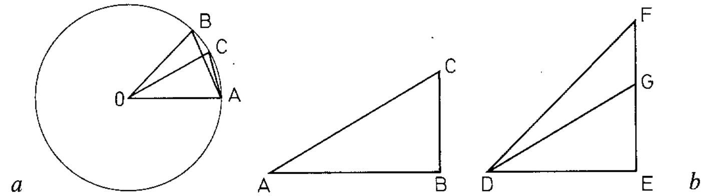

Fig. 12

polygon P_n inscribed in a circle; then if $n>m$, the perimeter $p_n>p_m$. In Fig. 12a let chord AB be the side of P_m, subtending angle α at the center O of the circle; chord AC is the side of P_n, subtending angle β at O. From an inequality between the ratios of chords and angles, assumed earlier in *SR* (see note 54), it follows that $\alpha:\beta>AB:AC$; that is, $n:m>s_m:s_n$, or $ns_n>ms_m$. Thus, $p_n>p_m$.

The related isoperimetric lemma is this: if two regular polygons have equal perimeter, that having the greater number of sides contains the greater area. In Fig. 12b let right-triangle ABC be an element of the regular polygon of n-many sides; that is, $CB=\frac{1}{2}s_n$; let $AB=r_n$ its apothem. Similarly, in triangle DEF, let FE be half the side of the regular polygon of m-many sides, $DE=r_m$ be its apothem. Let us assume $n>m$. We define s'_n such that $s'_n:r_m=s_n:r_n$, and mark off $EG=s'_n$. Taking angle FDE as α, and angle $GDE(=CAB)$ as β, we have by the tangent-inequality corresponding to the chord-inequality used above that $s_m:s'_n>\alpha:\beta=n:m$. Substituting for s'_n and noting that $ns_n=ms_m$ (since $p_n=p_m$) result in $r_n>r_m$. Thus, $P_n=\frac{1}{2}p_n r_n>\frac{1}{2}p_m r_m=P_m$, as claimed.

From this lemma, combined with DC, 1, it is then proved that the circle encloses a greater area than does any regular polygon having its perimeter equal to the circumference of the circle. In this way, the importance of ARCHIMEDES' studies for the theory of isoperimetric figures is clear. But the appearance of the related theorem in *SR* suggests that ARCHIMEDES' contribution to that theory was likely to have been more direct than is usually assumed.

A further remarkable feature of ARCHIMEDES' theorem in *SR* is that results from DC and $SC\,I$ would well suffice for establishing what he requires: a lower bound for the inscribed polygon of 1000 sides, *e.g.*, $p_{1000}>3d$. For, from $SC\,I$, 3 (see sect. 1) he can assume that the ratio $s'_n:s_n$ decreases with increasing n—here, s'_n is a side of the regular polygon of n-many sides circumscribed about the circle, while s_n is the side of the corresponding inscribed polygon. Moreover, from DC, 3 it is known that $s'_{96}:s_{96}<3\frac{1}{7}:3\frac{10}{71}$. A lower bound may then be calculated, *e.g.*, $p_{1000}>3\frac{1}{8}d$. Here, again, the puzzle of ARCHIMEDES' not citing results from the extant works is most easily explained on the view that those results, such as $SC\,I$, 3, had not yet been worked out. Moreover, ARCHIMEDES' choice is a mark of his interest in work closely associable with the study of isoperimetric figures.

7. An extension of the bisection-principle X, 1

In X, 1 it is established that, given two unequal magnitudes A, B (A the greater), if from A more than its half is removed, and from the remainder more than its half, and so on, eventually a remainder is left which is less than B. We will show here that the theorem holds, even if parts smaller than half are removed, as long as the proportional parts exceed some constant finite fraction.

First, let us sketch the Euclidean proof. By V, Def. 4 there is a finite multiple $C=nB$ which exceeds A. Let the sequence A, A_1, A_2, *etc.*, be formed such that from each term A_n more than its half is removed to produce A_{n+1}. Let the sequence C, C_1, C_2, *etc.*, be formed, where subtraction from C_n of a part equal to B yields C_{n+1}. Then, for all $k<n$, $C_k>A_k$, as A_k results from the removal of a greater fractional part from a smaller term. Thus, $C_{n-1}=B>A_{n-1}$, establishing the theorem.

In the more general case, we form the sequence A, A_1, A_2, *etc.*, where removal of more than the m^{th} part of each term A_k produces the next term. Set $B'=B/(m-1)$ and take a finite multiple $C=nB'>A$. The same reasoning as above ensures that for all $k<n-m+1$, $C_k>A_k$. Thus, $A_{n-m+1}<C_{n-m+1}=nB'-(n-m+1)B'=(m-1)B'$. Substituting for B' results in $A_{n-m+1}<B$, as desired.

In practice, if the sequence A, A_1, A_2, *etc.* converges more slowly than successive bisection, a subsequence may be selected which converges more rapidly. For instance, if it is assured only that more than the third part is removed at each step, then every two applications of the procedure guarantee bisection; for $A_{k+2}<\frac{2}{3}(\frac{2}{3}A_k)<\frac{1}{2}A_k$. Similarly, if successive removal of only more than the fourth part is assured, then every three applications guarantee bisection; if the fifth part, then every four applications, and so on. What one should recognize from this is that the condition of bisection in X, 1 is not an essential technical restriction on its applicability – and when ARISTOTLE says, "in a finite magnitude, if one takes a definite part, and takes besides another part *in the same ratio*, [and one does this continually,] one will not traverse the finite magnitude" (*Physics* 206b7), he may be reflecting an awareness among geometers of his time of this extendability of the bisection-principle. Thus, we must look for reasons other than technical necessity to account for ARCHIMEDES' formulation and application of the convergence-axiom as given in *QP*, *SC I* and *SL*. As indicated, we have proposed an explanation based on ARCHIMEDES' possession of a theory of proportion different from *Elements* V (see sect. 5); so that, lacking the Euclidean definitions (notably, V, Def. 4), he was compelled to articulate for himself the equivalent axioms needed to establish the bisection-principle.

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Key to abbreviations of Archimedean titles:

DC = *Dimension of the Circle*
SR = *Sand Reckoner*
PE I, II = *Plane Equilibria I, II*

QP = *Quadrature of the Parabola*
SC I, II = *Sphere and Cylinder I, II*
SL = *Spiral Lines*
CS = *Conoids and Spheroids*
FB I, II = *Floating Bodies I, II*
M = *Method*

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Notes Added in Proof

1. *(to pp. 220, 278)* The alternative proofs given in Appendix 1 pose problems for those who would propose to date *DC* after *SCI*: (a) The convergence argument required for *DC*, 1 is actually given in *SCI*, 6. Why, then, did ARCHIMEDES provide one at all in *DC*, 1? (b) In providing that argument, why did ARCHIMEDES choose to ignore the more efficient technique of *SCI*, 6, returning instead to the Euclidean techniques of "approximation" and "bisection"?

Similar puzzles arise with regard to ARCHIMEDES' handling of *DC*, 3. There he establishes that $3\frac{10}{71} < \pi < 3\frac{1}{7}$, the lower bound by means of a sequence of inscribed regular polygons, the upper bound by means of circumscribed polygons. As the term we denote π appears in ARCHIMEDES always as the ratio of the circumference to the diameter of the circle, a lemma is needed to the effect that the *circumferences* of circles are proportional to their diameters. In *DC*, 1 ARCHIMEDES has produced an expression for the *areas* of circles; conceivably, the original version of *DC*, 2 effected the step from areas to circumferences, but the extant version is not satisfactory. We find the lemma in PAPPUS (*Collection* V, 11) with a proof exactly suited to the structure of *DC*. Paraphrasing in modern notation, we denote by C_1, C_2 the areas of two circles whose respective diameters are d_1, d_2 and whose perimeters are p_1, p_2. Then, $C_1 : C_2 = d_1^2 : d_2^2$ (via XII, 2), while $4C_1 = d_1 p_1$ and $4C_2 = d_2 p_2$ (via *DC*, 1). Thus, $d_1 p_1 : d_2 p_2 = d_1^2 : d_2^2$; whence $d_1 p_1 : d_1^2 = d_2 p_2 : d_2^2$ or $p_1 : \mathrm{d}_1 = p_2 : d_2$. Thus, $p_1 : p_2 = d_1 : d_2$, as claimed.

PAPPUS indicates that an alternative proof is possible without the assumption of *DC*, 1: "for the similar polygons inscribed in and circumscribed about the circles have the same ratio as the radii of the circles." He thus alludes to a proof dispensing entirely with *DC*, 1, appealing instead to the convergence argument in *SCI*, 5 (*cf.* note 30a). But ARCHIMEDES shows no awareness of such an alternative in his treatment of *DC*, 3. The proportionality of circumferences and diameters is assumed there, apparently as a corollary to *DC*, 1.

Other features of ARCHIMEDES' method in *DC*, 3 sharpen the contrasts separating *DC* from *SCI*. To derive the upper bound on π, ARCHIMEDES computes an upper bound on the perimeter of the regular polygon of 96 sides which circumscribes the circle; this polygon is obtained after four successive halvings of the central angle subtended by each side of the circumscribed hexagon. He thus needs a rule for obtaining from the side and diameter of a given circumscribed regular polygon the corresponding dimensions of the polygon having twice as many sides; he proves this as a lemma: with reference to triangles FOC and AOC (Fig. 8b), $CO:FC=AO+OC:AC$. The procedure for the lower bound is analogous, depending on a rule for the sequence of inscribed polygons. ARCHIMEDES provides a full proof of the lemma that $BG:AB=CG+AG:AC$, where AOG is the diameter, AC the side of the given inscribed polygon and AB the side of the polygon having twice as many sides (*cf.* Fig. 8a; note that point G is not drawn in that diagram).

A remarkable aspect of ARCHIMEDES' procedure here is that he gives separate and *independent* proofs of these two computing rules, even though they are one and the same rule. This latter is evident from the figure of *SC I*, 3 which relates the sides of the similar circumscribed and inscribed polygons (Fig. 9b). Since $OB:AB=OE:DE$, the rule for computing the sides AC of the circumscribed polygons leads at once to that for the sides DF of the inscribed polygons. It is difficult to imagine why ARCHIMEDES would have provided two different proofs of the half-angle rule, had he already worked out the results presented in *SC I*, 3 and 5.

Recognizing this equivalence, one may perceive a significant abridgment of the computation of bounds. Let us assume, for instance, ARCHIMEDES' value $153:4673\frac{1}{2}$ as an upper bound on $AB:BO$ for the circumscribed 96-gon (Fig. 9b) – from this it follows *a fortiori* that $3\frac{1}{7}$ ($>96\cdot 153/4673\frac{1}{2}$) is an upper bound on π. By "HERO's rule" for square roots, we may assign to AO the value 4676 ($=4673\frac{1}{2}+2\frac{1}{2}$). Thus, 153:4676 will be an estimate for $AB:AO=DE:DO$, from which the ratio of the perimeter of the *inscribed* 96-gon to its diameter follows. Now, this will be an *upper* bound on that ratio, while the computation seeks a *lower* bound on π. (In fact, $96\cdot 153/4676$ *is* a lower bound on π. But this has not yet been established by the computation as it stands, since rounding-off at intermediate steps has been made in the wrong direction.) If the accumulated error incurred through upward rounding-off in the course of the computation of the circumscribed 96-gon can be estimated, we may adjust the bound on the inscribed polygon accordingly. Estimating that error to amount to less than 5 parts in 70,000 of the final results, for instance, we can conclude that $3\frac{10}{71}$ is a lower bound on π. This is the value ARCHIMEDES actually obtains by his independent computation of the inscribed 96-gon. Even adopting a larger estimate of the total error, say 10 parts in 70,000, we will obtain a usable lower bound (here, $3\frac{9}{64}$). But neither in the computations presented in *DC*, 3 nor in the figures related to a more refined computation, reported by HERO, is there any evidence that ARCHIMEDES attempted to derive both upper and lower bounds through a single calculation. (See my article, "Archimedes and the Measurement of the Circle," pp. 126–129.) He thus misses an insight which would substantially ease the computational efforts of later mathematicians like TSU CH'UNG-CHIH (5th century A.D.) and LUDOLPH of Ceulen (1596; 1615).

Thus, ARCHIMEDES' manner of estimating the ratio of the circumference and diameter of the circle belies any claim that he composed *DC* after *SC I*. First, the application of the Euclidean "approximation" technique of convergence in *DC*, 1 would be puzzling, on the assumption that he had earlier worked out the "compression" approach of *SC I*, 6; indeed, the convergence argument in *DC*, 1 would be strictly unnecessary, given the result in *SC I*. Second, the estimates in

DC, 3 require a theorem on the ratio of the *circumferences* of circles, not their areas – and this could be obtained directly through the result established in *SC I*, 5. That ARCHIMEDES chooses first to derive an expression for the *area* of the circle (in *DC*, 1) – apparently wishing to deduce the proportionality of circumferences from that of areas (XII, 2) – points to his adherence, in the closest degree possible, to the model of the treatment of the circle in the *Elements*. Third, the computational procedure adopted to obtain the bounds in *DC*, 3, including the unnecessary independent proof of the half-angle rule for inscribed polygons, betrays no sign of the relation of the sides of the circumscribed and inscribed polygons proved in *SC I*, 3 and 5.

Maintaining *DC* to be a much earlier composition than *SC I*, we can readily understand these features: the treatment in *SC I*, 1–6 is a reworking of the measurement of the circle based on convergence methods more sophisticated than those available to ARCHIMEDES when he had produced *DC*. By contrast, those who support a later dating for *DC*, even as an appendix to *SC I*, must confront major difficulties in explaining ARCHIMEDES' procedure.

2. *(to p. 226)* Another stylistic feature differentiates the two presentations of the theorem on parabolic segments in *Quadrature of the Parabola*. In the usual formal manner, each of the theorems in the "geometric" treatment (*QP*, 1–3, 18–24) is enunciated first in *general* form, before the elements of the specific diagram are introduced. By contrast, all but the last of the theorems in the "mechanical" treatment (*QP*, 4–17) are enunciated with reference to specific constructions. For instance, *QP*, 16 begins: "Let there next be a segment ΒΘΓ bounded by a line and a section of a right-angled cone..." While this usage finds its echo in *M*, 1, ARCHIMEDES elsewhere always adopts the general enunciation, as in the "geometric" part of *QP*. For instance, *QP*, 20 begins: "If in a segment bounded by a line and a section of a right-angled cone a triangle is inscribed..." Our view of the separate composition of the two parts of *QP* is thus supported by what would otherwise be an arbitrary shift in style.

3. *(to p. 231)* Since completing this paper, I came upon an item expressly confirming my argument for the Archimedean provenance of the theorem, preserved by HERO, on the area of circular segments. A scholium to the Heronian *Geometrica* asserts the following:

> Archimedes proves that any segment of the circle is greater than four-thirds the triangle having the same base and equal height. (*Heronis Opera*, ed. HEIBERG, V, p. 229)

Thus, the scholastic ascribes directly to ARCHIMEDES the theorem on segments, whose proof in HERO so closely follows the model of *DC*, 1 and *QP*, 18–24. Of course, evidence drawn from scholia must always be handled cautiously, since we but rarely know the source of the scholiast's information. But here, as we have seen, the attribution is intrinsically credible. Moreover, we can detect in this case what the source was. For the scholiast reproduces HERO's statement of the theorem *verbatim* as in *Metrica* I, 32 (*cf.* our p. 18); he indicates also that this manner of estimating circular segments applies only for those cases where the base of the segment is greater than triple its height–just as HERO indicates at the end of *Metrica* I, 31. It thus appears that the scholiast has annotated his text of the *Geometrica* on the basis of a text of the *Metrica* and that the attribution to ARCHIMEDES in I, 32 has since dropped out of the tradition, as represented by the unique manuscript now extant. Just as in III, 17 and 23 (where excerpts of proofs from *SC II*, 3 and 4 are given), so also in I, 32 HERO must have reproduced a proof in explication of a theorem ascribed explicitly to ARCHIMEDES.

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