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Estimation of very narrow bounds to the behavior of nonlinear incompressible elastic composites

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Abstract Variational bounds for the effective behavior of nonlinear composites are improved by incorporating more-detailed morphological information. Such bounds, which are obtained from the generalized Hashin–Shtrikman variational principles, make use of a reference material with the same microstructure as the nonlinear composite. The geometrical information is contained in the effective properties of the reference material, which are explicitly present in the analytical formulae of the nonlinear bounds. In this paper, the variational approach is combined with estimates for the effective properties of the reference composite via the asymptotic homogenization method (AHM), and applied to a hexagonally periodic fiber-reinforced incompressible nonlinear elastic composite, significantly improving some recent results.

Keywords Nonlinear composites · Variational bounds · Asymptotic homogenization · Effective properties

1 Introduction

One of several attempts at estimating the effective behavior of nonlinear composites, which have arisen in material science, deals with the construction of variational bounds from generalized nonlinear Hashin–Shtrikman principles. Such generalized variational principles rely on the introduction of a reference material which is commonly chosen to be linear and to present the same microgeometrical structure as the nonlinear composite (an approach introduced by Ponte Castañeda in [7]). With such a selection of reference material, the nonlinear bounds explicitly depend on its effective properties, which are usually estimated using available linear bounds

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(such as Hashin–Shtrikman bounds and recently those by Bruno [2]). The resulting bounds yield a significant improvement over previous techniques. However, in an important number of cases such as periodic structures the improvement provided by this approach is not sufficient, as accurate estimations of the effective properties of the reference material strongly depend on the geometrical characteristics of the material, so even narrow linear bounds (such as those in [2] that were obtained for general random composites) can degenerate in cases such as quasi-percolation or when applied to the nonlinear bounds. In this case, an alternative consists of estimating such effective properties by means of a direct approach such as the asymptotic homogenization method (AHM) [1]. AHM is a rigorous mathematical tool dealing with the system of partial differential equations with rapidly oscillating coefficients which model the problem. The AHM has been successfully applied to cases of linear elasticity and piezoelectricity (for instance, [4]). In this paper, with the purpose of illustrating how estimations of nonlinear behavior are improved by incorporating more-accurate geometrical information, we present particular results obtained by applying an integrated approach. This approach combines the variational and asymptotic procedures described above [6], to hexagonally periodic fiber-reinforced incompressible nonlinear elastic composites. For example, consider a linear matrix (quantities denoted by subscript m) containing nonlinear fibers (denoted by subscript f) with phase stress potentials

$$U_m(\sigma) = \frac{\sigma_e^2}{6\mu_m} \tag{1.1}$$

and

$$U_f(\sigma) = \frac{\sigma_e^2}{6\mu_f} + \frac{\varepsilon_0\sigma_0}{n+1}\left(\frac{\sigma_e}{\sigma_0}\right)^{n+1}, \tag{1.2}$$

where μ_m and μ_f are the phase shear moduli, σ_e is the equivalent stress, n is the hardening exponent, and σ_0 and ε_0 are some reference or initial stress and strain fields. The nonlinearity considered here (Eq. (1.2)) corresponds to the incompressible case of the Ramberg–Osgood model of the J_2 deformation theory of plasticity [5,8]. By neglecting elastic effects in Eq. (1.2), the resulting power law [3,9] characterizes the high-temperature creep of metals, the Newtonian viscous material and the von Mises rigid ideally plastic material correspond to $n = 1$ and $n \to \infty$, respectively [8]. In this work, Sect. 2 deals with the statement of the problem which will serve as an example for illustrating our integrated procedure; Sect. 3 is devoted to presenting the specialization of nonlinear variational bounds relevant to the problem at hand; in Sect. 4, explicit formulae for the estimations of the overall behavior of the reference material are summarized; numerical calculations for the illustrative example are presented in Sect. 5. Sections 3 and 4 are also complemented with Appendices A, B and C, in which details on the derivation of the improved bounds and the AHM approximation of the effective properties of the reference material are described.

2 Problem statement

Consider an incompressible nonlinear elastic fiber-reinforced composite in which fibers are inserted into the matrix following a hexagonal scheme of periodicity in a macroscopic or global $OX_1X_2X_3$ Cartesian coordinate system (CCS). The directions of fibers are parallel to the OX_3 axis and their cross sections are circular. The periodic cell occupies a domain Ω containing the origin of a microscopic or local Ox_1x_2 CCS, so it is a two-dimensional situation (see Fig. 1). Both phase materials are isotropic and perfectly bonded across the interface. From now on, the analysis will be restricted to the periodic cell, as it is possible to extend the solution periodically to the whole composite.

The constitutive behavior of such a composite is characterized via the piecewise-convex stress potential

$$U(\sigma, x) = U_m(\sigma)1_m(x) + U_f(\sigma)1_f(x), \tag{2.1}$$

where $1_m(x)$ and $1_f(x)$ stand for phase-characteristic functions. With Cauchy's geometrical law

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2.2}$$

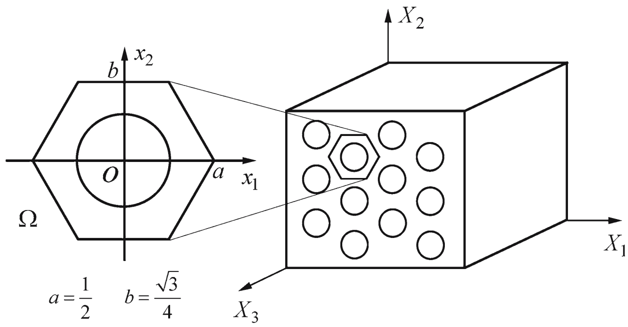

Fig. 1 The hexagonal periodic cell and its distribution in the composite

relating the strain ε to the displacement u and $i, j = \overline{1, 3}$, the usual nonlinear constitutive relation,

$$\varepsilon_{ij} = \frac{\partial U}{\partial \sigma_{ij}}, \tag{2.3}$$

holds. Equilibrium is also required, so

$$\sigma_{ij,j} = 0, \tag{2.4}$$

which is completed with certain boundary conditions given below, so the problem is well posed. Here, the summation convention over subscript j is understood. In addition, continuity of displacements and stresses across the matrix–fiber interface is assumed. The effective behavior of such a composite can be studied by means of the principle of minimum complementary energy

$$\hat{U}(\bar{\sigma}) = \frac{1}{|\Omega|} \inf_{\sigma \in S(\bar{\sigma})} \int_{\Omega} U(\sigma, x)\mathrm{d}x, \tag{2.5}$$

where $S(\bar{\sigma})$ is the admissibility set of Ω-periodic statically self-equilibrated stress fields (Eq. (2.4)) with mean value $\bar{\sigma}$. The effective stress potential $\hat{U}(\bar{\sigma})$ defined by Eq. (2.5) describes the overall behavior of the composite. Similarly, the effective strain potential $\hat{U}^*(\bar{\varepsilon})$ is defined via the minimum energy principle

$$\hat{U}^*(\bar{\varepsilon}) = \frac{1}{|\Omega|} \inf_{\varepsilon \in S^*(\bar{\varepsilon})} \int_{\Omega} U^*(\varepsilon, x)\mathrm{d}x, \tag{2.6}$$

where $S^*(\bar{\varepsilon})$ is the set of Ω-periodic kinematically admissible strain fields (Eq. (2.2)) with a prescribed value $\bar{\varepsilon}$ on the boundary. The superscript * represents the Legendre transformation. Now, as the formal solution of (2.5) is practically impossible to find, the problem at hand is therefore addressed as estimating the effective behavior of the nonlinear composite described above, by applying some variational bounding procedures for the effective stress potential. Although the statement presented here is consistent with any material response characterized by a piecewise-convex stress potential (Eq. (2.1)), only the phase stress potentials (1.1) and (1.2) will be considered hereafter.

3 Variational bounds

Nowadays, one of the most successful techniques in estimating the effective behavior of nonlinear composites consists of the derivation of variational bounds starting from generalized nonlinear Hashin–Shtrikman principles. In this section, relevant classical and improved nonlinear bounds are outlined.

3.1 Elementary bounds

The first results on estimating $\hat{U}(\bar{\sigma})$ are the classical elementary bounds, which can be easily attained by evaluating the mean stress and strain fields $\bar{\sigma}$ and $\bar{\varepsilon}$ into Eqs. (2.5) and (2.6), respectively. Note that no micro-geometrical information is included in the definitions of such bounds, so application to all types of composites (with prescribed phase potentials and volume fractions c_m and c_f) is feasible. For this reason, elementary bounds are commonly wide apart and do not provide useful estimations in practice beyond control purposes. After normalizing by Eq. (1.1), the particular expressions of those bounds in the present case are the following. The elementary upper bound is

$$\frac{\hat{U}}{U_m}(\bar{\sigma}) \leq c_m + c_f\left(\frac{\mu_m}{\mu_f} + \frac{2}{n+1}\left(\frac{3\varepsilon_0\mu_m}{\sigma_0}\right)\left(\frac{\bar{\sigma}_e}{\sigma_0}\right)^{n-1}\right), \tag{3.1}$$

while the elementary lower bound is

$$\frac{\hat{U}}{U_m}(\bar{\sigma}) \geq \frac{1}{c_m}\left(1 - c_f t\right)^2 + c_f\left(\frac{\mu_m}{\mu_f}t^2 + \frac{2}{n+1}\left(\frac{3\varepsilon_0\mu_m}{\sigma_0}\right)\left(\frac{\bar{\sigma}_e}{\sigma_0}\right)^{n-1} t^{n+1}\right), \tag{3.2}$$

where the real parameter t satisfies the polynomial condition

$$c_m\left(\frac{3\mu_m\varepsilon_0}{\sigma_0}\right)\left(\frac{\bar{\sigma}_e}{\sigma_0}\right)^{n-1} t^n + \left(c_f + c_m\frac{\mu_m}{\mu_f}\right)t - 1 = 0. \tag{3.3}$$

Note the complexity in calculating the values of Eq. (3.2), as it depends on the solution of the condition (3.3). As far as we have seen, it is not usual for an analytical formula for this bound, which involves the calculation of $\left\{\overline{U^*}\right\}^*(\bar{\sigma})$, to be explicitly given (for instance in [9]). It is usually replaced by the harmonic mean of the phase stress potentials weighted with the phase concentrations as in the linear case. However, when nonlinearity is considered, there are cases in which the harmonic mean does not behave as a lower bound.

3.2 Bounds incorporating morphological information

Because of the reasons presented in the previous sections, and despite the generality of the elementary bounds, it is imperative to improve them. In this case, improvement is achieved by including microstructural information about the composite via the introduction of an incompressible linear elastic reference material with the same geometry as the nonlinear composite. Moreover, such a reference material is chosen to have the same matrix as the nonlinear composite, and linear fibers characterized by a shear modulus μ_0. The effective shear response of this material is described by the modulus $\hat{\mu}_0$. From this idea, and by using the Legendre duality and the minimum complementary energy principle with the potentials contrast, the Ponte Castañeda-type lower bound [7] is regained. In our special case, after normalizing by Eq. (1.1), this bound reads

$$\frac{\hat{U}}{U_m}(\bar{\sigma}) \geq \max_{0<z\leq\frac{\mu_f}{\mu_m}}\left\{\frac{1}{m(z)} - \frac{n-1}{n+1}c_f\left(\frac{\bar{\sigma}_e}{\sigma_0}\right)^{-2}\left(\frac{3\mu_m\varepsilon_0}{\sigma_0}\right)^{-\frac{2}{n-1}}\left(\frac{1}{z} - \frac{\mu_m}{\mu_f}\right)^{\frac{n+1}{n-1}}\right\}, \tag{3.4}$$

where $z = \frac{\mu_0}{\mu_m}$ and $m(z) = \frac{\hat{\mu}_0}{\mu_m}(z)$ is the matrix-normalized effective shear modulus of the reference material. Notice that the argument of the maximum in the right-hand side of Eq. (3.4) defines a class of lower bounds depending on the parameter z, which characterizes the reference material. So, in order to select the reference material which produces the best lower bound, it is necessary to perform the optimization process indicated on the right-hand side of Eq. (3.4). Also notice that, before maximization is performed, an estimation of $m(z)$ is required. The derivation of this bound is described in detail in Appendix A.

It is not possible to construct an upper bound following the scheme used for the lower bound. However, an upper bound has been derived in [9] for a case consistent with the one considered in this work, that is, a linear matrix containing nonlinear inclusions. Derivation of such an upper bound depends on the solution of a linear thermoelastic problem [10], from which it is inferred that the reference material consists of a linear matrix containing inclusions with zero mechanical energy that are under the influence of certain thermal field. Then, the upper bound is obtained by applying the Legendre transformation. For our special case, after normalizing by Eq. (1.1), this bound reads

$$\frac{\hat{U}}{U_m}(\bar{\sigma}) \leq \min_{t\geq 0}\left\{\frac{1}{m(0)} + 2\left(1-\frac{1}{m(0)}\right)t + \left(\frac{1}{m(0)} - 1 + c_f\left(\frac{\mu_m}{\mu_f}-1\right)\right)t^2 \right.$$
$$\left. + c_f\frac{2}{n+1}\left(\frac{3\mu_m\varepsilon_0}{\sigma_0}\right)\left(\frac{\bar{\sigma}_e}{\sigma_0}\right)^{n-1} t^{n+1}\right\}. \tag{3.5}$$

Again, the argument of the minimum in the right-hand side of Eq. (3.5) defines a class of upper bounds and an estimation of the effective property $m(0)$ of the reference material is needed. In this case, the selection of the best upper bound follows by minimizing with respect to the parameter t, defined by $\sigma = t\bar{\sigma}$.

Note that, because of the type of nonlinearity considered here, Eqs. (3.4) and (3.5) become Eqs. (20) and (21) in [9] as $\frac{\mu_m}{\mu_f} \to 0$, a fact used for control. Details on the derivation of this bound and the estimation of $m(z)$ are described in Appendix B and the next section, respectively.

4 Estimating the effective behavior of the reference material

The bounds (3.4) and (3.5) have no practical realization unless estimations of $m(z)$ are available, so this section is devoted to discussing some of them. As said above, $m(z)$ refers to a linear composite, which is an exhaustively studied situation. Here, the bounds calculated by Bruno [2] and the AHM approximation in [4], which are most relevant to the present problem, are commented on.

4.1 Linear bounds

In the literature, the most frequent way to estimate the effective behavior of the reference material is to use available linear bounds for the relevant effective properties to the case in study. In general, such bounds, as stand-alone tools, describe the effective behavior of linear composites quite well, even in the presence of a highly random spatial distribution of inclusions or heterogeneities. On the other hand, when those bounds are applied to estimate the response of the reference material contained in a nonlinear bound, the results, although improved, occasionally fail to describe accurately the effective behavior of the nonlinear composite, even in the case of a periodic structure. Such problems occur, for instance, for concentrations near the percolation value. Nonetheless, as those bounds are widely used, we present them here to show how our approach succeeds in obtaining better outcomes. In this work (as in [9]), the bounds for $m(z)$ developed by Bruno [2], are employed.

In our special case, parameters and formulae relevant to these bounds are $q = 2\sqrt{\frac{c_f}{\pi}\sin\left(\frac{\pi}{3}\right)}$, $s_\pm = \frac{1}{2}\left(1 \pm q^2\right)$, $\delta = s_+ - s_-$,

$$B_1(z) = 1 - \frac{c_f}{\frac{1}{1-z} - \frac{c_m}{2}} \tag{4.1}$$

and

$$B_2(z) = \frac{1-(1-z)s_+}{1-(1-z)s_-}\left(1 + \frac{\left(1-\frac{c_f}{\delta}\right)^2}{\left(1-\frac{c_f}{\delta}\right)\frac{1-(1-z)s_+}{1-(1-z)\delta} + \frac{c_f}{\delta^2}\left(\frac{c_m}{2} - s_-\right)}\right). \tag{4.2}$$

Then, the bounds read

$$\begin{cases} B_2(z) \leq m(z) \leq B_1(z) & \text{if } 0 \leq z < 1, \\ B_1(z) \leq m(z) \leq B_2(z) & \text{if } z \geq 1. \end{cases} \tag{4.3}$$

4.2 AHM approximation

A different approach, of great applicability in periodically structured linear composites, consists of the application of a direct methodology such as the asymptotic homogenization method (AHM) [1]. The AHM produces accurate estimates which have been proven to always lie between relevant linear bounds such as those mentioned above (see, for instance, [6]). In [6], as will be done here, it is illustrated that the application of the AHM estimates to nonlinear bounds produces very narrow bounds even for percolation cases. This results in a significant improvement over the use of known linear bounds, giving an accurate approximation of the effective nonlinear behavior. In this work, the AHM estimation consistent with the geometry considered here was obtained in [4]. Normalization of such an approximation [Eq. (3.15) in [4]] reads

$$m(z) = \frac{1 - c_f \chi(z) - V_1^T M^{-1} V_2 \chi^2(z)}{1 + c_f \chi(z) - V_1^T M^{-1} V_2 \chi^2(z)}, \tag{4.4}$$

where $\chi(z) = \frac{1-z}{1+z}$, and V_1, V_2 and M are infinite-order vectors and a matrix, respectively, which depend on powers of the fiber radius. (Further details on the derivation of $m(z)$ are given in Appendix C.) As the fiber radius is less than 1, the truncation of V_1, V_2 and M needed in practical calculations does not significantly affect the results. In fact, truncation to the second order yields good results which do not change appreciably with truncation to higher orders. In Fig. 2, the upper and lower bounds calculated according to Bruno (labeled "UB" and "LB", respectively) are compared to the AHM prediction (labeled "AHM") against the fiber concentration c_f for $z = 23.45$ and $z = 0$. Note that the AHM prediction (4.4) always lies between the bounds (4.3) calculated according to Bruno [2].

5 Numerical examples

In this section, calculations involving the variation of different relevant parameters related to the nonlinear character of the studied composite are presented. This is done with the purpose of illustrating the efficacy of our integrated approach which combines variational and asymptotic techniques of homogenization.

Several numerical experiments involving the elementary bounds (3.1) and (3.2) (labeled "EB" in Figs. 3, 4 and 5) and the improved nonlinear bounds (3.4) and (3.5) were carried out. Specifically, the integrated approach occurs when the AHM prediction of $m(z)$ (Eq. (4.4)) is used in those improved bounds (in Figs. 3, 4 and 5, this combination is labeled as "IB+H"), instead of the application of the linear bounds (4.3) (this combination is labeled in Figs. 3, 4 and 5 as "IB+B"). All the calculations were carried out for $\frac{3\mu_m \varepsilon_0}{\sigma_0} = 1$ taking $\log_2\left(\frac{\bar{\sigma}_e}{\sigma_0}\right)$, c_f and n as free parameters in Figs. 3, 4 and 5, respectively. Other relevant data for these calculations presented here are displayed as part of those figures. Note that the integrated approach "IB + H" defines very narrow ranges of material responses, so the effective behavior is obtained with a satisfactory degree of accuracy, while bounds "IB+B" are widely separated as they do not contain accurate morphological information for the geometry considered here. Nevertheless, it must be said that bounds "IB + B" are the best estimations for composites exhibiting a matrix reinforced with randomly distributed fibers with the prescribed phase potentials (1.1) and (1.2). The elementary bound (3.1) is omitted from Fig. 5 on purpose as it is a wide distance apart from all the other bounds.

6 Conclusions

In this work, a hexagonally periodic fiber-reinforced incompressible nonlinear elastic composite is studied by means of nonlinear variational bounds. These bounds explicitly depend on the effective shear modulus of a reference material, as it is taken to be linear with the same microstructure as the nonlinear composite. Such effective modulus is approximated via the linear bounds (4.3) calculated according to Bruno and the AHM exact estimation (4.4). Our numerical experiments have shown that the combination of the nonlinear bounds (3.4) and (3.5) with the AHM estimation (4.4) yields much better results than those values obtained by combining those nonlinear bounds with the linear bounds (4.3). Using the AHM prediction in the nonlinear bounds results in very narrow ranges of material responses, so the effective behavior of the composite studied is much more accurately obtained, illustrating that significant improvements are attained by including more-detailed morphological information.

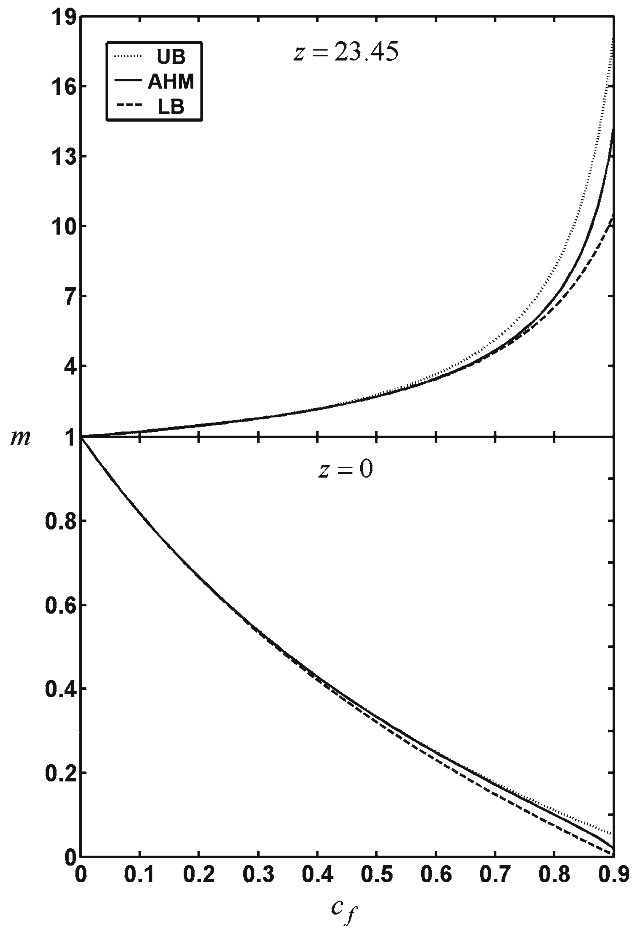

Fig. 2 Upper (UB) and lower (LB) bounds calculated according to Bruno [2] and the AHM estimate for the matrix-normalized effective shear moduli $m(z)$ of two ($z = 23.45$ and $z = 0$) linear composites containing hexagonally distributed fibers plotted against fiber concentration c_f up to quasi-percolation

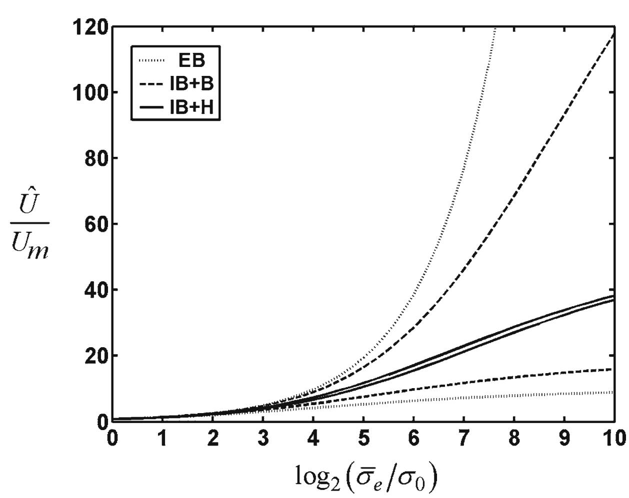

Fig. 3 Upper and lower bounds for the matrix-normalized effective stress potential $\frac{\hat{U}}{U_m}(\bar{\sigma})$ of a linear matrix containing nonlinear fibers plotted against the normalization parameter $\log_2\left(\frac{\bar{\sigma}_e}{\sigma_0}\right)$ for $c_f = 0.9$ (quasi-percolation) and $n = 2$. EB: elementary bounds; IB + B: improved nonlinear bounds with the linear bounds calculated according to Bruno [2]; IB + H: improved nonlinear bounds with the AHM estimate

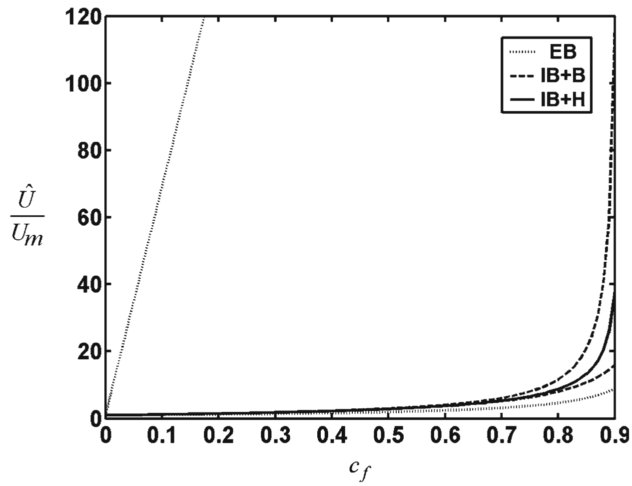

Fig. 4 Upper and lower bounds for the matrix-normalized effective stress potential $\frac{\hat{U}}{U_m}(\bar{\sigma})$ of a linear matrix containing nonlinear fibers plotted against the fiber concentration c_f up to quasi-percolation for $\log_2\left(\frac{\bar{\sigma}_e}{\sigma_0}\right) = 10$ and $n = 2$. EB: elementary bounds; IB+B: improved nonlinear bounds with the linear bounds calculated according to Bruno [2]; IB+H: improved nonlinear bounds with the AHM estimate

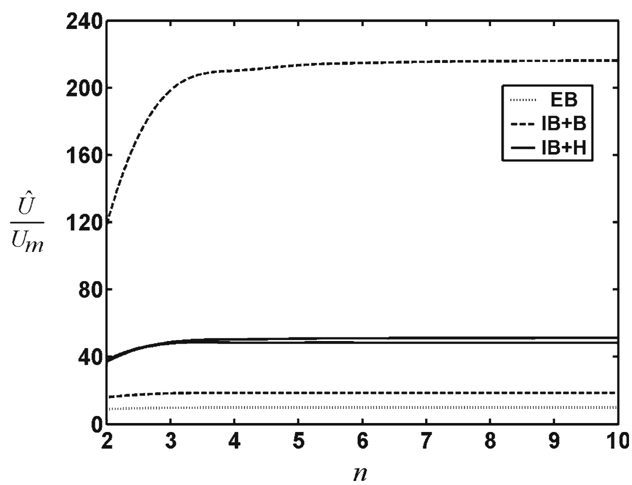

Fig. 5 Upper and lower bounds for the matrix-normalized effective stress potential $\frac{\hat{U}}{U_m}(\bar{\sigma})$ of a linear matrix containing nonlinear fibers plotted against the hardening exponent n for $\log_2\left(\frac{\bar{\sigma}_e}{\sigma_0}\right) = 10$ and $c_f = 0.9$ (quasi-percolation). EB: elementary bounds; IB+B: improved nonlinear bounds with the linear bounds calculated according to Bruno [2]; IB+H: improved nonlinear bounds with the AHM estimate

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Appendix A. Some ideas on the derivation of the lower bound (3.4)

Let $U_0(\sigma, x)$ be the stress potential describing the constitutive behavior of a linear reference material exhibiting the same microstructure as the nonlinear composite. Consider now the application of the Legendre transformation to the contrast of potentials $U(\sigma, x)$ and $U_0(\sigma, x)$, that is,

$$(U - U_0)^*(s, x) = \sup_{\sigma}\left\{s_{ij}\sigma_{ij} - (U - U_0)(\sigma, x)\right\} \tag{A.1}$$

for every field s. After some manipulations and by applying the principle of minimum complementary energy (2.5), Eq. (A.1) becomes the variational lower bound

$$\hat{U}(\bar{\sigma}) \geq \frac{1}{|\Omega|} \inf_{\sigma \in S(\bar{\sigma})} \int_{\Omega} \left\{ s_{ij}\sigma_{ij} + U_0(\sigma, x) - (U - U_0)^*(s, x) \right\} \mathrm{d}x. \tag{A.2}$$

Such general inequality will not be useful, unless the shapes of the fields and potentials are specified. In this work, the relevant selection is $s \equiv 0$ and, as mentioned above, the reference material is chosen here to have a matrix described by the same stress potential $U_m(\sigma)$ as the matrix of the nonlinear composite and linear fibers which can be characterized by the stress potential

$$U_{0f}(\sigma) = \frac{\sigma_e^2}{6\mu_0}, \tag{A.3}$$

where μ_0 is the shear modulus of the fibers material. With these considerations and after some algebra, the lower bound becomes

$$\hat{U}(\bar{\sigma}) \geq \hat{U}_0(\bar{\sigma}) + c_f \min_{\sigma} \big(U_f - U_{0f} \big)(\sigma), \tag{A.4}$$

where $\hat{U}_0(\bar{\sigma})$ is the effective stress potential of the linear reference material with shear response described by the effective modulus $\hat{\mu}_0$. In our particular case, $U_f(\sigma)$ and $U_{0f}(\sigma)$ are given by Eqs. (1.2) and (A.3), respectively. Then, by normalizing Eq. (A.4) by $U_m(\bar{\sigma})$ (Eq. (1.1)) and recalling that $z = \frac{\mu_0}{\mu_m}$ and $m(z) = \frac{\hat{\mu}_0}{\mu_m}(z)$, the lower bound (3.4) follows from maximizing the resulting expression with respect to z.

Appendix B. Some ideas on the derivation of the upper bound (3.5)

In order to derive an upper bound, the procedure adopted here differs from that in Appendix A. First, consider the Legendre transformation of the nonlinear stress potential of the fiber phase, from which the inequality

$$U_f^*(\varepsilon) \geq \varepsilon_{ij}\sigma_{ij} - U_f(\sigma) \tag{B.1}$$

is obtained. Substitution of Eq. (B.1) into the principle of minimum energy (2.6) yields a lower bound for the effective strain potential, that is,

$$\hat{U}^*(\bar{\varepsilon}) \geq \frac{1}{|\Omega|} \inf_{\varepsilon \in S^*(\bar{\varepsilon})} \int_{\Omega} \left\{ U_m^*(\varepsilon) 1_m(x) + \varepsilon_{ij}\sigma_{ij} 1_f(x) \right\} \mathrm{d}x - c_f U_f(\sigma). \tag{B.2}$$

The first term on the right-hand side of Eq. (B.2) is equivalent to finding the effective strain energy of a linear thermoelastic composite whose fibers are under a thermal field σ and have zero mechanical energy. These fibers are embedded in a matrix of the same material as the matrix of the nonlinear composite. Such a thermoelastic composite will be considered here as the reference material with an effective response described by the potential $\hat{U}_0(\bar{\sigma})$. By using the solution to such a problem provided in [10] and then, by applying the Legendre transformation to the resulting expression, the upper bound

$$\hat{U}(\bar{\sigma}) \leq \hat{U}_0(\bar{\sigma}) + 2\Big(U_m(\bar{\sigma}) - \hat{U}_0(\bar{\sigma}) \Big) t + \Big(\hat{U}_0(\bar{\sigma}) - \big(1 + c_f\big) U_m(\bar{\sigma}) \Big) t^2 + c_f U_f(t\bar{\sigma}) \tag{B.3}$$

is induced. In Eq. (B.3), recall that $\sigma = t\bar{\sigma}$ so, in our particular case, with $U_m(\bar{\sigma})$ and $U_f(\bar{\sigma})$ given by Eqs. (1.1) and (1.2), respectively, and normalizing by $U_m(\bar{\sigma})$, the upper bound (3.5) is obtained by minimizing with respect to parameter t.

Appendix C. Some ideas on the derivation of the effective shear modulus (4.4) via the AHM

The bounds presented in this paper were derived considering the variational formulation of nonlinear incompressibility stated by the principle of minimum energy (2.5) and (2.6), while the differential formulation given by the equilibrium equation (2.4) is used explicitly only to obtain the effective shear modulus of the linear reference material. In the case of linear incompressibility, Eq. (2.4) becomes

$$\left(\mu(x,z)u_{i,j}\right)_{,j}=0, \tag{C.1}$$

where the local variable x is related to the global variable X in planes perpendicular to the OX_3 axis through a small geometric parameter α, and $z=\frac{\mu_0}{\mu_m}$. Equation (C.1) is satisfied everywhere outside the matrix–fiber interface and the Ω-periodic rapidly oscillating coefficient $\mu(x,z)=\mu_m\left(1_m(x)+z1_f(x)\right)$ is the local formulation of the constitutive shear modulus. Continuity conditions for displacements and stresses across the matrix–fiber interface and suitable boundary conditions are assumed, so the problem will be well posed.

Homogenization is achieved here by the application of the AHM [1], so a solution is sought to satisfy a two-scale asymptotic expansion in powers of α, that is, $u=u^{(0)}(X,x)+u^{(1)}(X,x)\alpha+O\left(\alpha^2\right)$, where $u^{(k)}(X,x)$ are Ω-periodic functions in the local variable, for which local and global variables are assumed to be independent. By substituting the asymptotic expansion into Eq. (C.1) and the continuity conditions of stresses and displacements, applying the chain rule, and by comparing similar powers of α, a sequence of equations is obtained from which an $u^{(0)}(X)$ independent of the local variable, and an $u^{(1)}(X,x)$ expressed as the direct product of the gradient of $u^{(0)}(X)$ and an Ω-periodic function $N(x)$ are found. Then, homogenization of (C.1) is achieved as

$$\left(\hat{\mu}_0(z)u^{(0)}_{i,j}\right)_{,j}=0, \tag{C.2}$$

where the effective shear modulus $\hat{\mu}_0(z)$ is given by

$$\hat{\mu}_0(z)=\mu_m\left(c_m+c_fz\right)+\frac{1}{|\Omega|}\int_{\Omega}\mu(x,z)N_{,1}\mathrm{d}x. \tag{C.3}$$

The local function $N(x)$ can be obtained by considering the two-dimensional Laplacian equation $N_{,jj}=0$ subject to perfect contact conditions. The solution is found among doubly periodic harmonic functions with periods $\omega_1=(1,0)$ and $\omega_2=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ (as a hexagonal geometry is considered) with particular phase components involving the quasi-periodic Weierstrass zeta function and its derivatives [4]. By substituting such phase components of $N(x)$ into Eq. (C.3) and the perfect contact conditions, and from the application of Green's theorem, the effective shear modulus is found to be related to the solution of an infinite system of equations depending on the fiber radius, from which it is obtained that

$$\hat{\mu}_0(z)=\mu_m\left(1-\frac{2c_f\chi(z)}{1+c_f\chi(z)-V_1^TM^{-1}V_2\chi^2(z)}\right), \tag{C.4}$$

where the infinite-order vectors $V_1=(V_{1s})$, $V_2=(V_{2t})$ and the matrix $M=(M_{st})$ are given by

$$V_{1s}=R^{12s}\eta_{1\,6s-1}\quad\text{and}\quad V_{2t}=\eta_{6t-1\,1}, \tag{C.5}$$

and

$$M_{ts}=\delta_{ts}-\chi^2(z)R^{12s}\sum_{i=1}^{\infty}R^{12i}\eta_{6t-1\,6i+1}\eta_{6i+1\,6s-1}, \tag{C.6}$$

with R is the fiber radius, δ_{ts} is the Kronecker delta, and

$$\eta_{kl}=-\frac{(k+l-1)!}{l!(k-1)!}\sum_{r_1,r_2}\frac{1}{(r_1\omega_1+r_2\omega_2)^{k+l}} \tag{C.7}$$

for $k+l\geq 2$ and $(r_1,r_2)\in\mathbb{Z}^2-\{(0,0)\}$. Finally, by recalling that $m(z)=\frac{\hat{\mu}_0}{\mu_m}(z)$ and after a little algebra, Eq. (C.4) becomes the AHM approximation (4.4). A much more detailed derivation of the effective shear modulus (4.4) can be found in pages 1451–1453 of [4].

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