An Extension of the Birkhoff Integrability for Multifunctions

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Abstract. A comparison between a set-valued Gould type and simple Birkhoff integrals of $bf(X)$-valued multifunctions with respect to a non-negative set function is given. Relationships among them and Mc Shane multivalued integrability is given under suitable assumptions.

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1. Introduction

After Aumann, set-valued integrals have been defined and studied by many authors using different techniques: see, for example, [8,11–13,16–18,30,31,35,38] for the Birkhoff and Gould multivalued integrability, [2–4,6,7,9,10,22–26] for the Kurzweil–Henstock, Mc Shane, Pettis integrability and their relations. These notions have shown to be useful when modelling some theories in different fields as optimal control and mathematical economics; see, for example, [1,15,20,27]. Moreover, the theory of set-valued functions has interesting and important applications also in many theoretical or practical domains such as statistics, fixed point theory, stochastic processes, information sciences and optimization. Integrals of multifunctions can be used as an aggregation tool when dealing with a large amount of information fusing and with data mining problems such as programming and classification. In decision making theory, if the available information about the states of nature is given by means of a monotone measure, the integral of the payoff function (or multifunction) gives an average value that could be used to find the best solution for multi-criteria decision making.

A comparison between these integrals in the vector-valued case was given, for example, in [9,10,19,21,28,39,40,42–44]. Parallel to this study, a comparison in the multivalued case was obtained in [7,26,36,37].

In this paper, this field of research is continued and a new definition of the Birkhoff multivalued integrability is given, when the hyperspace is that of bounded subsets of a Banach space X $(b(X))$. A similar definition was introduced also in [17]: the definition adopted here is simpler and (at least formally) weaker; later (Definition 3.2) the differences between them will be described more precisely. Moreover, a comparison of this integral, called Birkhoff simple, is given, with Gould and Mc Shane integrability.

After some preliminaries in Sects. 2 and 3 regarding, respectively, set functions and the properties of Gould and Birkhoff simple multivalued integral, in Sects. 3.1 and 3.2 a comparison between Gould and Birkhoff simple integrability is stated both in countably additive case and under the weaker condition of countable null-null-additivity. While in Sect. 4, a comparison is obtained with the Mc Shane multivalued integral when the set function μ is pointwise non-atomic, countably additive and the multifunctions F take value in $b(X)$.

2. Preliminaries

Let T be an abstract, nonvoid set, $\mathcal{A}$ an algebra of subsets of T and $\mu : \mathcal{A} \to \mathbb{R}_0^+$ a non-negative set function with $\mu(\emptyset) = 0$. In this paper, the set functions μ considered are mainly finitely additive or countably additive; for all unexplained definitions on set functions, see, for example, [16].

Definition 2.1 [32]. Let $\mu : \mathcal{A} \to \mathbb{R}_0^+$ be a non-negative set function, with $\mu(\emptyset) = 0$. *The variation* $\overline{\mu}$ *of* μ *is the set function* $\overline{\mu} : \mathcal{P}(T) \to \mathbb{R}_0^+$ *defined by* $\overline{\mu}(E) = \sup\{\sum_{i=1}^n \mu(A_i)\}$, *where the supremum is extended over all families of pairwise disjoint sets* $\{A_i\}_{i=1,\ldots,n} \subset \mathcal{A}$, *with* $A_i \subseteq E$, *for every* $i \in \{1, 2, \ldots, n\}$. μ *is said to be of finite variation on* $\mathcal{A}$ *if* $\overline{\mu}(T) < \infty$.

For the properties of $\overline{\mu}$, see, for example, [16, Remark 2.3].

Definition 2.2. Let $\mu : \mathcal{A} \to [0, \infty)$ be a non-negative set function. A set $A \in \mathcal{A}$ is said to be an *atom* of μ if $\mu(A) > 0$ and for every $B \in \mathcal{A}$, with $B \subset A$, it is $\mu(B) = 0$ or $\mu(A \backslash B) = 0$. μ is said to be *finitely purely atomic* if there is a finite family $(A_i)_{i=1,\ldots,n}$ of pairwise disjoint atoms of μ so that $T = \cup_{i=1}^n A_i$;

- *null-additive* if $\mu(A \cup B) = \mu(A)$, for every $A, B \in \mathcal{A}$, with $\mu(B) = 0$;
- *null-null-additive* if $\mu(A \cup B) = 0$, for every $A, B \in \mathcal{A}$, with $\mu(A) = \mu(B) = 0$;
- σ-*null-null-additive* if $\mu(\cup_{n=1}^\infty A_n) = 0$, for every $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, with $\mu(A_n) = 0$;
- *continuous from below* if $\lim_n \mu(A_n) = \mu(\bigcup_n A_n)$ whenever $(A_n)_n$ is an increasing sequence in $\mathcal{A}$.

A finite (*countable*, respectively) *partition of* T is a finite (countable, respectively) family of non-empty sets $P = \{A_i\}_{i=1,\ldots,n}$ ($\{A_n\}_{n \in \mathbb{N}}$, respectively) $\subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset, i \neq j$ and $\cup_{i=1}^n A_i = T$ ($\cup_{n \in \mathbb{N}} A_n = T$, respectively).

If P and P' are two finite (or countable) partitions of T, then P' is said to be *finer than* P, denoted by $P \leq P'$ (or $P' \geq P$), if every set of P' is included in some set of P.

The *common refinement* of two finite or countable partitions $P = \{A_i\}$ and $P' = \{B_j\}$ is the partition $P \wedge P' = \{A_i \cap B_j : A_i \cap B_j \neq \emptyset\}$. Obviously, $P \wedge P' \geq P$ and $P \wedge P' \geq P'$. We denote by $\mathcal{P}$ the class of all finite partitions of T.

Proposition 2.3 [16, Proposition 2.7, Corollary 2.8]. *Suppose* $\mu : \mathcal{A} \to \mathbb{R}_0^+$ *is null-additive and monotone and* $A \in \mathcal{A}$ *is an atom of* μ.

2.3 (i) *If* $\{B_i\}_{i=1}^n$ *is a partition of* A, *then there exists only one* $i_0 \in \{1, \ldots, n\}$ *so that* $\mu(B_{i_0}) = \mu(A)$ *and* $\mu(B_i) = 0$, *for every* $i = \{1, \ldots, n\}$, *with* $i \neq i_0$.

2.3 (ii) *Suppose* μ *is* σ-*null-null-additive. If* $\{B_n\}_{n \in \mathbb{N}}$ *is a partition of* A, *then there exists only one* $n_0 \in \mathbb{N}$ *so that* $\mu(B_{n_0}) = \mu(A)$, B_{n_0} *is an atom of* μ *and* $\mu(B_n) = 0$, *for every* $n \in \mathbb{N} \backslash \{n_0\}$.

2.3 (iii) *If* $A \in \mathcal{A}$ *is an atom of* μ, *then* $\overline{\mu}(A) = \mu(A)$.

The next definitions will be useful later. The Gould integral will be introduced first.

Definition 2.4 [38, Definition 3.6]. Suppose $F : T \to b(X)$ is a multifunction and $\mu : \mathcal{A} \to \mathbb{R}_0^+$ is any set function defined on the algebra $\mathcal{A}$.

2.4 (i) *F is said to be Gould integrable (on T) relative to* μ *if there exists a set* $I \in bf(X)$ *such that for every* $\varepsilon > 0$, *there is a finite partition* P_ε *of* T *such that all the sets of* P_ε *have finite measure and, moreover, for every partition of* T, $P := \{A_i\}_{i=1}^n$, *with* $P \geq P_\varepsilon$ *and every* $t_i \in A_i, i \in \{1, 2, \ldots, n\}$, *it holds*

$$h\left(\sum_{i=1}^n F(t_i)\mu(A_i), I\right) < \varepsilon.$$

The set I *is called the Gould integral of* F *on* T *with respect to* μ, *denoted by* $(G) \int_T F d\mu$;

2.4 (ii) *If* $B \in \mathcal{A}, F$ *is said to be Gould integrable on* B *if the restriction* $F_{|B}$ *of* F *to* B *is Gould integrable on* $(B, \mathcal{A}_B, \mu_B)$.

Next, the notion of *integrable set function* is given.

Definition 2.5. The set function μ is said to be *integrable* if the (real) constant function 1 is Gould integrable (see [16, Definition 4.5]) with respect to μ.

In other words, μ is integrable if and only if there exists in $\mathbb{R}_0^+$ the limit $\lim_> s(\mu, P)$, where P runs along all finite partitions of T, $P = \{A_1, \ldots, A_k\}$, ordered by refinement, and $s(\mu, P) := \sum_{i=1}^k \mu(A_i)$.

If μ is integrable, then every *restriction* μ_A is, for $A \in \mathcal{A}$, where $\mu_A(B) := \mu(A \cap B)$ for every $B \in \mathcal{A}$, and the integral is an additive set function on $\mathcal{A}$. This function will be denoted by $\int_A \mu$ or also $\Psi(A)$.

Remark 2.6. The following facts are easy to prove, for a non-negative, integrable, increasing set function μ (see also [8, Definition 4.5] for a similar concept):

2.6 (1) Fix $\varepsilon > 0$ and let $P_\varepsilon \equiv \{E_1, \ldots, E_n\}$ be any partition of T such that

$$\left| \Psi(T) - \sum_{i=1}^{K} \mu(C_i) \right| \le \varepsilon$$

for every partition $\{C_1, \ldots, C_K\}$ finer than P_ε. Such a partition P_ε will be called an ε-approximated partition for μ in T. It turns out that for every set $A \in \mathcal{A}$, the partition $P_{\varepsilon,A} := \{E_1 \cap A, E_2 \cap A, \ldots, E_n \cap A\}$ is a 2ε-approximated partition for μ_A.

2.6 (2) A kind of *Henstock lemma* holds, for Ψ: for every $\varepsilon > 0$ there exists a partition P_ε of T, such that for every finer partition $P \equiv \{C_1, \ldots, C_K\}$ and every set $A \in \mathcal{A}$:

$$\sum_i |\Psi(C_i \cap A) - \mu(C_i \cap A)| \le \varepsilon.$$

In the following proposition a result of countable additivity of Ψ is proved, under suitable hypotheses.

Proposition 2.7. *Let $\mu : \mathcal{A} \to \mathbb{R}_0^+$ be any increasing set function, satisfying $\mu(\emptyset) = 0$, and assume that μ is continuous from below. Then, if μ is integrable, the mapping $A \mapsto \Psi(A) = \int_A d\mu$ is countably additive.*

Proof. Since Ψ is additive and non-negative, it is sufficient to prove its continuity from below. So, let $(A_k)_n$ denote any increasing sequence in $\mathcal{A}$, with union A, and fix $\varepsilon > 0$. Then, there exists an $\varepsilon/2$-approximated partition of T, denoted by $P_\varepsilon := \{E_1, \ldots, E_n\}$. Thanks to the item 2.6 (1), the partition $\{E_1 \cap A_k, E_2 \cap A_k, \ldots, E_n \cap A_k\}$ is an ε-approximated partition of A_k for all k. Now, let k_0 be an integer large enough that

$$\mu(E_i \cap A) \le \mu(E_i \cap A_k) + \frac{\varepsilon}{n}$$

for all $i \in \{1, \ldots, n\}$ and all $k \ge k_0$. Then

$$\Psi(A) - \varepsilon \le \sum_{i=1}^{n} \mu(A \cap E_i) \le \sum_{i=1}^{n} \mu(A_k \cap E_i) + \varepsilon \le \Psi(A_k) + 2\varepsilon,$$

for all $k \ge k_0$. This implies that $\Psi(A) \le \lim_k \Psi(A_k)$. The converse inequality is trivial, by monotonicity of Ψ. $\qquad\square$

In what follows, let $(X, \|\cdot\|)$ be a Banach space with the origin 0, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X, and consider its subfamilies:

- $b(X)$ the family of nonvoid bounded subsets of X,
- $bf(X)$ the family of closed, bounded, nonvoid subsets of X,
- $bcf(X)$ the family of all bounded, closed, convex, nonvoid subsets of X,
- $ck(X), (cwk(X))$ the family of all (weakly) compact and convex, nonvoid subsets of X.

Let $h(A, B) = \max\{e(A, B), e(B, A)\}$, for every non-empty subset A, B of X, where

$$e(A, B) = \sup_{x \in A} d(x, B)$$

is *the excess* of A over B and d is the metric induced by the norm of X. On $bcf(X)$, h becomes a metric, called the *Hausdorff metric*. For every $A \in \mathcal{P}_0(X)$, denote

$$|A| = h(A, \{0\}) = \sup_{x \in A} \|x\|.$$

For every $A \subseteq X$, the symbols $\overline{A}, \oplus$ denote the norm closure of A and the norm closure of the Minkowski addition, respectively.

When the symbol $\sum$ is used for multifunctions this means also the norm closure of the addition. For the properties of the Hausdorff metric h; see, for example, [14, 33, 38]. Some of these are now recalled:

Proposition 2.8. *The following properties hold for every $\alpha, \beta \in \mathbb{R}, A, B, C, D \in b(X)$:*

2.8 (i) $h(\alpha A, \beta A) \leq |\alpha - \beta| \cdot |A|$;

2.8 (ii) $h(A \oplus B, C \oplus D) \leq h(A, C) + h(B, D)$;

2.8 (iii) $h(A \oplus C, B \oplus C) \leq h(A, B)$;

2.8 (iv) $h(A, B) \leq |A| + |B|$;

Lemma 2.9. *Fix $A, B, C \in b(X)$. Then*

$$|h(A \oplus B, C) - h(A, C)| \leq |B|.$$

Proof. It is

$$h(A \oplus B, C) = h(A \oplus B, C \oplus \{0\}) \leq h(A, C) + |B|;$$
$$h(A, C) \leq h(A, A \oplus B) + h(A \oplus B, C) \leq |B| + h(A \oplus B, C),$$

so the assertion follows. □

Recall that $ck(X)$ $(cwk(X))$ denotes the family of all convex and (weakly) compact subsets of X. Thanks to the Rådström embedding theorem, $ck(X)$ $(cwk(X))$ endowed with the Hausdorff distance h is a complete metric space that can be isometrically embedded into a Banach space:

Lemma 2.10 ([34, Theorems 5.6 and 5.7]). *Let $S = cwk(X)$ or $ck(X)$. Then there exist a compact Hausdorff space Ω and an isometry $j : S \to C(\Omega)$, endowed with the* sup*-norm, such that*

2.10 (a) $j(\alpha A + \beta C) = \alpha j(A) + \beta j(C)$ *for all $A, C \in S$ and $\alpha, \beta \in \mathbb{R}^+$,*

2.10 (b) $h(A, C) = \|j(A) - j(C)\|_\infty$ *for every $A, C \in S$,*

2.10 (c) $j(S)$ *is norm closed in $C(\Omega)$,*

2.10 (d) $j(co(A \cup C)) = \max\{j(A), j(C)\}$, *for all $A, C \in S$.*

Definition 2.11. A multifunction $F : T \to b(X)$ is called *bounded* if there exists $M \in \mathbb{R}_0^+$ such that $|F(t)| \leq M$, for every $t \in T$.

Definition 2.12. Let $\upsilon : \mathcal{A} \to \mathbb{R}_0^+$ be a non-negative set function and let $F : T \to \mathcal{P}_0(X)$ be a multifunction.

2.12 (i) F is called υ-*totally measurable* (*on* T) if for every $\varepsilon > 0$ there exists a finite family $\{A_i\}_{i \in \{0,\dots,n\}}$ of non-empty pairwise disjoint subsets of T, such that $\bigcup_{i=0}^{n} A_i = T$, $\upsilon(A_0) < \varepsilon$ and $\sup_{t,s \in A_i} h(F(t), F(s)) = osc(F, A_i) < \varepsilon$, for every $i \in \{1, \dots, n\}$.

2.12 (ii) F is called υ-*totally measurable on* $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is υ-totally measurable on $(B, \mathcal{A}_B, \upsilon_B)$, where $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$ and $\upsilon_B = \upsilon|_{\mathcal{A}_B}$.

3. A Birkhoff Simple Integral

In the sequel, suppose $\mu : \mathcal{A} \to \mathbb{R}_0^+$ is a non-negative set function with $\mu(T) > 0$ and the cardinality of T is greater than $\aleph_0$. The Birkhoff multivalued integral that will be extended is the following:

Definition 3.1 [11, Definition 2.1]. Let $F : T \to cwk(X)$ be a multifunction and μ a non-negative countably additive measure defined on a σ-algebra $\mathcal{A}$. F is Birkhoff integrable if the single-valued function $j \circ F : T \to C(\Omega)$ is Birkhoff integrable. (Observe that the Authors proved that the definition is independent of the embedding used, in the paper they used $l_\infty(B_{X^*})$.)

To this aim a slight modification of [17, Definition 3.1] is proposed:

Definition 3.2. Let $F : T \to b(X)$ be a multifunction and let $\mathcal{C}$ be a non-empty subfamily of $b(X)$. Let also $\mu : \mathcal{A} \to \mathbb{R}_0^+$ be a non-negative set function with $\mu(T) > 0$, defined in a σ-algebra $\mathcal{A}$. F is called *Birkhoff simple integrable in $\mathcal{C}$ on T* if there exists a closed set $I \in \mathcal{C}$ with the following property: for every $\varepsilon > 0$, there exists a countable partition $P_\varepsilon \subset \mathcal{C}$ of T so that for every countable partition $P = \{A_n\}_{n \in \mathbb{N}}$ of T, with $P \geq P_\varepsilon$ and every $t_n \in A_n, n \in \mathbb{N}$, one has

$$\limsup_n h\left(\sum_{i=1}^{n} F(t_i)\mu(A_i), I\right) \leq \varepsilon. \tag{1}$$

The set I is called the *Birkhoff simple integral in $\mathcal{C}$ of F on T with respect to μ* and is denoted by $(B_s) \int_T F \mathrm{d}\mu$.

The simple Birkhoff integrability/integral of F on a set $A \in \mathcal{A}$ is defined in the usual manner.

Remark 3.3. Observe that, in [17], the definition does not make use of the $\limsup$ operation, but requires instead that for every $\varepsilon > 0$ there exist a countable partition P_ε and a suitable integer n_ε such that

$$h\left(\sum_{i=1}^{n} F(t_i)\mu(A_i), I\right) \leq \varepsilon$$

holds, as soon as $P = \{A_n\}_{n \in \mathbb{N}}$ is finer than P_ε and $n \geq n_\varepsilon$. So the Definition 3.2 looks more intuitive, and at least formally weaker, since it avoids the condition of uniformity about the integer n_ε.

Corollary 3.4. *Let $F : T \to cwk(X)$ be a multifunction. Then F is Birkhoff simple integrable in $cwk(X)$ if and only if $j \circ F$ is Birkhoff simple integrable.*

Proof. It follows directly from Lemma 2.10 since $j(cwk(X))$ is norm closed in $C(\Omega)$ and j is norm preserving and additive, namely

$$h\left(\sum_{i=1}^{n} F(t_i)\mu(A_i), I\right) = \left\|\sum_{i=1}^{n} j \circ F(t_i)\mu(A_i) - j(I)\right\|_{\infty}.$$

□

Observe, moreover, that the Birkhoff simple integral is additive, in fact:

Theorem 3.5. *If μ is finitely additive, and $F : T \to b(X)$ is Birkhoff simple integrable with respect to $\mathcal{C}$ on a pair of disjoint sets $A, B \in \mathcal{A}$, then F is Birkhoff simple integrable in $\mathcal{C}$ on $A \cup B$ too and*

$$(B_s) \int_{A \cup B} F d\mu = (B_s) \int_A F d\mu \oplus (B_s) \int_B F d\mu.$$

Proof. The proof works as in [17, Theorem 4.1]. □

To prove that Definition 3.2 is an extension of Definition 3.1 when μ is a non-negative countably additive measure and $\mathcal{C}$ is $cwk(X)$ or $ck(X)$, a comparison with the Gould integral is needed. This result will be obtained for the case of finite or σ-finite non-negative, (countably) additive measure.

3.1. The Case of Finite Measure μ

The first step is to compare the Gould and simple Birkhoff integrability in the case of μ finite. The notion of Gould integral has been introduced in Definition 2.4. In case F takes values in one of the spaces $ck(X)$ or $cwk(X)$, thanks to completeness of these spaces with respect to the Hausdorff distance, the Gould integral, if it exists, is in the same space $ck(X)$ or $cwk(X)$, respectively. Indeed

Proposition 3.6. *Assume that $F : T \to cwk(X)$ is Gould integrable, then the integral I is in $cwk(X)$. The same holds for $ck(X)$-valued mappings.*

Proof. Only the case of $cwk(X)$-valued functions will be discussed, the other one is quite similar and is contained in [38, Remark 3.7].

Let I denote the Gould integral of F, and fix arbitrarily $k \in \mathbb{N}$. Then there exists a finite partition P_k of T, $P_k := \{E_1^k, \ldots, E_{n_k}^k\}$, such that, for every finer partition $P := \{B_1, \ldots, B_m\}$ and every choice of points $t_j \in B_j, j = 1, \ldots, m$, it holds

$$h\left(\sum_{j=1}^{m} F(t_j)\mu(B_j), I\right) \leq \frac{1}{k}.$$

In particular, as soon as points $\tau_1 \in E_1^k, \ldots, \tau_{n_k} \in E_{n_k}^k$ are fixed, one has

$$h\left(\sum_{i=1}^{n_k} F(\tau_i)\mu(E_i^k), I\right) \leq \frac{1}{k}.$$

Now, setting $s_k := \sum_{i=1}^{n_k} F(\tau_i)\mu(E_i^k)$, the elements s_k are in $cwk(X)$, and clearly they form a Cauchy sequence with respect to the Hausdorff distance.

So, by completeness of $(cwk(X), h)$, there exists an element $I_0 \in cwk(X)$ such that $\lim_k s_k = I_0$ in that space. Hence, for each $\varepsilon > 0$ there exists an integer k_ε such that $h(s_k, I_0) \le \varepsilon$ for every $k \ge k_\varepsilon$.

The element I_0 is the Gould integral of F. Indeed, as soon as $\varepsilon > 0$ is fixed, choose any integer k^* larger than $k_\varepsilon \vee \frac{1}{\varepsilon}$. Then, for every partition $P := \{B_1, \ldots, B_m\}$ finer than $P_{k^*} := \{E_1^{k^*}, \ldots, E_{n_{k^*}}^{k^*}\}$, and every choice of points $t_j \in B_j, j = 1, \ldots, m$, it holds

$$h\left(\sum_{j=1}^m F(t_j)\mu(B_j), I_0\right) \le h\left(\sum_{j=1}^m F(t_j)\mu(B_j), I\right) + h(I, s_{k^*}) + h(s_{k^*}, I_0)$$

$$\le \frac{2}{k^*} + \varepsilon \le 3\varepsilon.$$

This clearly suffices to prove the assertion. $\qquad\square$

Remark 3.7. Observe that when the mapping F is single valued, this definition is the same as *RLF-integrability* of [40, Definition 13]. Moreover, the notion of RLF-integrability can be adopted for the case of $cwk(X)$- or $ck(X)$-valued maps, as in Corollary 3.4, thanks to the Rådström embedding theorem (Lemma 2.10) since, by Proposition 3.6, the integral lives in the same hyperspace, $cwk(X)$ or $ck(X)$. Indeed, since the embedding j of $ck(X)$ or $cwk(X)$ is an isometry, it is clear that

$$h\left(\sum_{i=1}^n F(t_i)\mu(A_i), I\right) = \left\|\sum_{i=1}^n j(F(t_i))\mu(A_i) - j(I)\right\|_\infty$$

for all partitions $\{A_n : n \in \mathbb{N}\}$ and choices $t_n \in A_n$, so it follows that any $cwk(X)$- or $ck(X)$-valued mapping F is Gould integrable if and only if $j(F)$ is as a single-valued function (as was done in Corollary 3.4 for the Birkhoff simple integral).

Theorem 3.8. *Suppose $\mathcal{A}$ is a σ-algebra, $\mu : \mathcal{A} \to [0, \infty)$ is a countably additive measure and $F : T \to b(X)$ is a bounded multifunction. Then F is Birkhoff simple integrable if and only if it is Gould integrable and*

$$(B_s)\int_T F \mathrm{d}\mu = (G)\int_T F \mathrm{d}\mu.$$

Proof. Assume that F is simple-Birkoff integrable, with integral I, and fix $\varepsilon > 0$. Then there exists a countable partition $P^1 := \{E_j : j \in \mathbb{N}\}$ such that for every countable partition P^2 finer than P^1, $P^2 = \{B_j, j \in \mathbb{N}\}$, and every choice of the points $t_j \in B_j$ it is

$$\limsup_n h\left(\sum_{i=1}^n F(t_i)\mu(B_i), I\right) \le \frac{\varepsilon}{4}. \tag{2}$$

Let M be any majorant for $\sup_{t \in T} h(F(t), \{0\})$. Then, let N be any integer such that $\sum_{j>N} \mu(E_j) \le \frac{\varepsilon}{4M}$, and denote by A the union of all the sets

$E_j, j > N$. Clearly, for every (countable) partition of A, say $\{A_i, \ i \in \mathbb{N}\}$, and every choice of points $\tau_i \in A_i$ for all $i \in \mathbb{N}$, one has

$$\sum_i h(F(\tau_i)\mu(A_i), \{0\}) \le \frac{\varepsilon}{4}; \tag{3}$$

$$h\left(\sum_{i=1}^{N} F(t_i)\mu(E_i), I\right) \le \frac{\varepsilon}{2}. \tag{4}$$

In particular, for $E_i, i \ge N$ it holds

$$h\left(\sum_{i=1}^{N+n} F(t_i)\mu(E_i), \sum_{i=1}^{N} F(t_i)\mu(E_i)\right) \le \frac{\varepsilon}{4} \tag{5}$$

for every integer n, and every choice of the points $t_i \in E_i$, since the elements $E_j, \ j > N$, form a partition of A.

Let $P_\varepsilon := \{E_1, E_2, \ldots, E_N, A\}$. For every finite partition $P'' := \{G_1, \ldots, G_l\}$ of $T \backslash A$, finer than $\{E_1, \ldots, E_N\}$, $i = 1, \ldots, l$, consider the partition $P^* := \{G_1, \ldots, G_l, E_j, j \ge N+1\} = \{C_i, i \in \mathbb{N}\}$, finer than P^1 and any choice of the points $t''_i \in G_i, \tau_k \in E_k, k > N$: then there exists $n_1 \ge \max\{N, l\}$ such that for every $n \ge n_1$ it is

$$h\left(\sum_{i=1}^{l} F(t''_i)\mu(G_i) \oplus \sum_{i=l+1}^{n} F(\tau_i)\mu(E_i), I\right) \le \frac{\varepsilon}{2}$$

Then, by Lemma 2.9 and (3) it is

$$h\left(\sum_{i=1}^{l} F(t''_i)\mu(G_i), I\right) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{4}. \tag{6}$$

So, if P' is any finite partition, finer than P_ε, $P' = \{H_1, .., H_u\}$, for every choice of the points $\tau_i \in H_i$, $i = 1, \ldots, u$, one clearly has

$$\sum_{i=1}^{u} F(\tau_i)\mu(H_i) = \sum_{H_i \subset T \backslash A} F(\tau_i)\mu(H_i) \oplus \sum_{H_i \subset A} F(\tau_i)\mu(H_i),$$

and

$$h\left(\sum_{H_i \subset T \backslash A} F(\tau_i)\mu(H_i), I\right) \le \frac{3}{4}\varepsilon$$

by (6) and

$$h\left(\sum_{H_i \subset A} F(\tau_i)\mu(H_i), \{0\}\right) \le \frac{\varepsilon}{4}$$

by (3). Then clearly one can conclude:

$$h\left(\sum_{i=1}^{u} F(\tau_i)\mu(H_i), I\right) \le \varepsilon.$$

Now, the converse implication will be proved. So, assume that F is Gould integrable, with integral I: then for every $\varepsilon > 0$ a finite partition P_ε exists, such that, for every finite finer partition $P' := \{B_1, \ldots, B_m\}$, and every choice of the points $t_i \in B_i$, $i = 1, \ldots, m$, one has

$$h\left(\sum_i F(t_i)\mu(B_i), I\right) \leq \varepsilon/2.$$

The partition P_ε is suitable also for simple Birkhoff integrability. Indeed, denote $P_\varepsilon := \{E_1, \ldots, E_n\}$, and pick any countable finer partition P', $P' := \{B_1, B_2, \ldots\}$, together with a choice of tags $\tau_j \in B_j$, $j \in \mathbb{N}$. By countable additivity, there exists an integer j_0 large enough that $\mu(\bigcup_{j>j_0} B_j) \leq \varepsilon(4M)^{-1}$. Therefore, for every choice of the points $t_j \in B_j$ one has

$$\sum_{j>j_0} h(F(t_j)\mu(B_j), \{0\}) \leq \frac{\varepsilon}{4}. \tag{7}$$

Denote by Q the union of all the sets B_j, with $j > j_0$. Consider now the following finite partition P'', finer than P_ε: P'' is obtained by taking for each index i the elements of the type B_j, $j \leq j_0$ that are contained in E_i, (with the corresponding tag τ_j) and also $E_i \cap Q$, with any tag. Now, since P'' is finer than P_ε,

$$h(\Sigma(F, P''), I) \leq \varepsilon/2;$$

then, from (7) and Lemma 2.9 one clearly has

$$h\left(\sum_{j=1}^{J} F(\tau_j)\mu(B_j), I\right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \leq \varepsilon,$$

as soon as $J \geq j_0$. This also means that

$$\limsup_J h\left(\sum_{j=1}^{J} F(\tau_j)\mu(B_j), I\right) \leq \varepsilon$$

and, therefore, the assertion. $\qquad\square$

In the next result, another relationship between Gould and simple Birkhoff integrabilities on atoms is presented. This is done in weaker hypothesis, i.e. just assuming that μ is σ-null-null-additive and monotone.

Theorem 3.9. *Suppose $F : T \to b(X)$ is fixed, $\mu : \mathcal{A} \to \mathbb{R}_0^+$ is σ-null-null-additive and monotone and $A \in \mathcal{A}$ is an atom of μ. Then F is Birkhoff simple integrable on A if and only if F is Gould integrable on A and, moreover,*

$$(G)\int_A F\mathrm{d}\mu = (B_s)\int_A F\mathrm{d}\mu.$$

Proof. Let $\varepsilon > 0$ be arbitrary and denote $I = (B_s) \int_A F \mathrm{d}\mu$. There exist a countable partition $P_\varepsilon := \{B_n\}_{n \in \mathbb{N}}$ of A and an integer $n_0 \in \mathbb{N}$ so that for every $t_n \in B_n$ one has

$$h\left(\sum_{k=1}^{n} F(t_k)\mu(B_k), I\right) < \varepsilon, \qquad \text{for every } n \geq n_0. \tag{8}$$

Since $A = \cup_{n=1}^{\infty} B_n$ is an atom of μ, by Proposition 2.3(ii) it can be assumed without loss of generality that $\mu(B_1) > 0$, B_1 is an atom of μ, $\mu(B_1) = \mu(A)$ and $\mu(B_n) = 0$, for every $n \geq 2$.

Therefore, $\sum_{k=1}^{n} F(t_k)\mu(B_k) = F(t_1)\mu(B_1)$, so (8) becomes

$$h(F(t_1)\mu(B_1), I) < \varepsilon. \tag{9}$$

So let $\widetilde{P}_\varepsilon = \{B_1, \cup_{n=2}^{\infty} B_n\}$ be a finite partition of A.

Consider an arbitrary partition $\widetilde{P} = \{C_i\}_{i=1}^{m}$ of A, so that $\widetilde{P} \geq \widetilde{P}_\varepsilon$ and arbitrary $t_i \in C_i, i = 1 \ldots m$.

Then $m \geq 2$ and $B_1 = \cup_{i=1}^{l} C_i$, $\cup_{n=2}^{\infty} B_n = \cup_{i=l+1}^{m} C_i$, for some $l \in \{1, \ldots, m-1\}$.

Since μ is σ-null-null-additive, then $\mu(\cup_{n=2}^{\infty} B_n) = 0$, so, by the monotonicity of μ it results $\mu(C_i) = 0$, for every $i \in \{l+1, \ldots, m\}$. Then,

$$\sum_{i=1}^{m} F(t_i)\mu(C_i) = \sum_{i=1}^{l} F(t_i)\mu(C_i). \tag{10}$$

On the other hand, because B_1 is an atom of μ and $\{C_i\}_{i=1,\ldots,l}$ is a partition of B_1, by Proposition 2.3(i), it is possible to assume, without loss of generality, that $\mu(C_1) = \mu(B_1)$ and $\mu(C_i) = 0$, for every $i \in \{2, \ldots, l\}$. From (10) it follows

$$\sum_{i=1}^{m} F(t_i)\mu(C_i) = F(t_1)\mu(B_1),$$

which yields from (9) that

$$h\left(\sum_{i=1}^{m} F(t_i)\mu(C_i), I\right) = h(F(t_1)\mu(B_1), I) < \varepsilon.$$

So, F is Gould m integrable on A and

$$(G)\int_A F \mathrm{d}\mu = (B_s)\int_A F \mathrm{d}\mu.$$

Viceversa suppose that F is Gould integrable on A, namely there exists a set $I_A \in bf(X)$ such that for every $\varepsilon > 0$, there is a finite partition $P_{\varepsilon,A}$ of A such that for every partition of A, $P := \{A_i\}_{i=1}^{n}$, with $P \geq P_{\varepsilon,A}$ and every $t_i \in A_i, i \in \{1, 2, \ldots, n\}$, it holds

$$h\left(\sum_{i=1}^{n} F(t_i)\mu(A_i), I_A\right) < \varepsilon.$$

As before, since A is an atom by Proposition 2.3(i) it can be assumed without loss of generality that A_1 is an atom, $\mu(A_1) = \mu(A)$ and $\mu(A_n) = 0$, for every

$n \geq 2$. Therefore, $\sum_{k=1}^{n} F(t_k)\mu(A_k) = F(t_1)\mu(A_1)$, so the previous inequality becomes

$$h(F(t_1)\mu(A_1), I_A) < \varepsilon.$$

Consider now an arbitrary countable partition $P_{\varepsilon,A}^c$ of A $P_{\varepsilon,A}^c \geq \{A_1, \cup_{j \geq 2} A_j\}$. Then, for every $P \geq P_{\varepsilon,A}^c$, let $\mathbb{M}_p$ the set of indexes of elements of P which cover A_1. Since A_1 is an atom, one can assume, without loss of generality, that $\mathbb{M}_p$ consists of just one element m_0. So,

$$\sup_{k \geq m_0} h \left(\sum_{i=1}^{k} F(t_i)\mu(A_i), I \right) \leq \varepsilon,$$

and this proves that F is Birkhoff simple integrable on A. $\qquad\square$

The two previous theorems extend analogous results in [16] obtained by some of the authors when F is a scalar Birkhoff integrable function.

Now, to prove that Birkhoff simple integrability extends Definition 3.1, the case $cwk(X)$-valued is considered and a single-valued result is stated:

Theorem 3.10. *Let $(T, \mathcal{A}, \mu)$ be a finite countably additive measure space, and $f : T \to X$ be any bounded function. Then the following are equivalent:*

3.10(1) *f is Gould integrable;*
3.10(2) *f is Birkhoff integrable.*

Proof. It is a consequence of [40, Theorems 15 and 16], when dealing with single-valued functions, see also [16, Theorem 5.1]. $\qquad\square$

Remark 3.11. Clearly, the same equivalence is valid also for $ck(X)$-or $cwk(X)$-valued bounded multifunctions F thanks to Proposition 3.6 and Lemma 2.10 since $j \circ F$ is Gould integrable if and only if F is by Remark 3.7.

An easy consequence of the Henstock Lemma 2.6(2) is the following result that will be useful later.

Proposition 3.12. *Let $F : T \to cwk(X)$ be any bounded multifunction, and assume that μ is finite, non-negative, monotone and integrable, with integral function Ψ. Then F is simple-Birkhoff (respectively, Gould) integrable with respect to μ if and only if it is simple-Birkhoff (respectively. Gould) integrable with respect to Ψ, with the same integral.*

Proof. The proof will be given only for the simple Birkhoff integrability: the other case is quite similar.

Let M denote any positive upper bound for $\sup_{t \in T} |F(t)|$. Fix ε and let P_ε be any ε-approximated partition of T for the Henstock lemma [see 2.6(2)]. Now, assume that F is simple-Birkhoff integrable with respect to μ (with integral J), and fix any countable partition P_ε', finer than P_ε and such that for every finer partition $P'' := \{B_1, B_2, \ldots\}$ and every choice of the tags $t_i \in B_i$, it holds

$$\limsup_n h \left(\sum_{i=1}^{n} F(t_i)\mu(B_i), J \right) \leq \varepsilon.$$

Now, since P'' is finer than P_ε, thanks to the Henstock lemma 2.6(2), for every n it holds

$$\sum_{i=1}^{n} h(F(t_i)\mu(B_i), F(t_i)\Psi(B_i)) \le M\varepsilon$$

(indeed, the pairwise disjoint sets $B_1, \ldots, B_n$ can be seen as part of a finite partition finer than P_ε). So,

$$\limsup_{n} h\left(\sum_{i=1}^{n} F(t_i)\Psi(B_i), J\right) \le (1+M)\varepsilon.$$

This suffices to prove that F is simple Birkhoff integrable with respect to Ψ.

A perfectly similar argument shows the converse implication. $\qquad\square$

Now the main result of this subsection can be achieved as a consequence of Theorems 3.8 and 3.10. Namely at least for $cwk(X)$- or $ck(X)$-valued bounded functions and when μ is finite and countably additive, there is equivalence among Birkhoff, simple Birkhoff and Gould integrability.

Theorem 3.13. *Let $\mu : T \to \mathbb{R}_0^+$ be any finite monotone, integrable set function, continuous from below. Then, for every bounded function $F : T \to cwk(X)$, simple Birkhoff integrability with respect to μ is equivalent to Birkhoff and Gould integrability.*

Proof. Let Ψ denote the integral function of μ. Since F is bounded, Gould or (simple)-Birkhoff integrability of F with respect to μ is equivalent to the same type of integrability with respect to Ψ thanks to Propositions 3.12. But Ψ is countably additive by Proposition 2.7, so Theorem 3.8 ensures that F is (simple)-Birkhoff integrable with respect to Ψ (i.e. μ) if and only if it is Gould integrable with respect to Ψ (i.e. μ). $\qquad\square$

3.2. The Case of σ-Finite Measure μ

In this section, a comparison will be obtained between simple-Birkhoff and Birkhoff integrability, in a more general setting, i.e. assuming that the measure μ is σ-finite, and also without requiring boundedness of F. Let us begin with a single-valued result:

Theorem 3.14. *Let $(T, \mathcal{A}, \mu)$ be any σ-finite measure space, with μ countably additive and non-negative. If $f : T \to X$ is simple-Birkhoff integrable then f is Pettis integrable.*

Proof. Assume that f is simple-Birkhoff integrable, with integral x. Then, for each $\varepsilon > 0$ there exists a partition $P_\varepsilon := \{E_1, \ldots, E_n, \ldots\}$ of T such that for every finer partition $P := \{B_1, \ldots, B_k, \ldots\}$ it holds:

$$\limsup_{k} \left\| \sum_{i=1}^{k} f(t_i)\mu(B_i) - x \right\| \le \varepsilon, \tag{11}$$

for every choice of the points $t_k \in B_k$, $k \in \mathbb{N}$. Then clearly, for each element $x^* \in X^*$, the mapping $\langle x^*, f \rangle$ is simple-Birkhoff integrable, with integral $\langle x^*, x \rangle$.

Now, the Birkhoff integrability of $\langle x^*, f \rangle$ will be proved for every x^*: to this aim, it will be sufficient to prove that, for each fixed x^*, all the series

$$\sum_k \langle x^*, f(t_k) \rangle \mu(B_k)$$

are absolutely convergent, as soon as $P := \{B_1, \ldots, B_k, \ldots\}$ is any finer partition of P_ε and for every choice $t_k \in B_k$, $k \in \mathbb{N}$. By contradiction, assume that one of these real-valued series is not absolutely convergent: then it is possible to rearrange its terms in such a way that the series diverge, and this would contradict the condition (11). So, the real-valued mapping $\langle x^*, f \rangle$ is Birkhoff integrable and, therefore, in L^1, for every element x^* of the dual space X^*. Clearly, it is also

$$\int_T \langle x^*, f \rangle \mathrm{d}\mu = \langle x^*, x \rangle$$

for all $x^* \in X^*$, which proves Pettis integrability of f. $\qquad \square$

Of course, this Theorem has an application to the multivalued case:

Corollary 3.15. *If $F : T \to cwk(X)$ is simple-Birkhoff integrable, then F and $j \circ F$ are Pettis integrable.*

Proof. By Theorem 3.14 and thanks to the Rådström embedding Theorem (Lemma 2.10), if F is simple Birkhoff integrable so is $j(F)$ in view of Corollary 3.4, where j denotes the Rådström embedding of $cwk(X)$ and, therefore, $j(F)$ is Pettis integrable. From this, and using the existence of the integral in $cwk(X)$, it follows also the Pettis integrability of F, proceeding like in the proof of [11, Proposition 3.5] without separability assumption. $\qquad \square$

Now, assuming that $(T, \mathcal{A}, \mu)$ is a σ-finite quasi-Radon space, we shall prove that any bounded simple-Birkhoff integrable function is also Birkhoff integrable.

Theorem 3.16. *Let $(T, \mathcal{A}, \mu)$ be a σ-finite measure space, If $F : T \to cwk(X)$ is a bounded simple-Birkhoff integrable mapping, then F is also Birkhoff integrable.*

Proof. Assume that T is a σ-finite measure space. Then T is the union of a sequence $(A_n)_n$ of sets of finite measure, on which F is bounded. Now, if F is simple-Birkhoff integrable in each A_n, it is also Birkhoff integrable there, thanks to Theorems 3.8 and 3.10, namely $j \circ F$ is Birkhoff integrable on each A_n. Moreover, thanks to Corollary 3.15, one can deduce that $j \circ F$ is Pettis integrable in T. So, the condition (ii) of [12, Lemma 3.2] is satisfied and, therefore, one can conclude that $j \circ F$ is Birkhoff integrable by virtue of that lemma: observe only that, though that lemma is stated in the setting of a finite measure space, the proof of the crucial implication (ii) $\Rightarrow$ (i) does not make use of this condition, so the conclusion holds also in this case. $\qquad \square$

To establish a more general result of equivalence, a lemma is needed, similar to [40, Lemma 5].

Lemma 3.17. *Let $(T, \mathcal{A}, \mu)$ be a σ-finite measure space, with μ non-negative and countably additive, and let $f : T \to X$ be any mapping. Assume that there exists a countable partition $P := \{E_1, E_2, \ldots, E_n, \ldots\}$ and a positive real number M such that*

$$\limsup_n \left\| \sum_{i=1}^{n} (f(t_i) - f(t_i'))\mu(E_i) \right\| \leq M, \tag{12}$$

for every choice of t_i and t_i' in E_i, $i \in \mathbb{N}$. Then the mapping f is bounded in every set E_n such that $\mu(E_n) \neq 0$.

Proof. Fix n such that $\mu(E_n) \neq 0$ and fix any two points t_n and t_n' in E_n. Choose also arbitrarily a point τ_j in every set E_j, $j \neq n$. Then, for N sufficiently large, $N > n$, it is

$$\left\| \sum_{1 \leq j \leq N, \ j \neq n} (f(\tau_j) - f(\tau_j))\mu(E_i) + (f(t_n) - f(t_n'))\mu(E_n) \right\| \leq M$$

thanks to (12), from which it follows clearly

$$\|f(t_n) - f(t_n')\|\mu(E_n) \leq M,$$

i.e.

$$\|f(t_n) - f(t_n')\| \leq \frac{M}{\mu(E_n)},$$

and this concludes the proof, by arbitrariness of t_n and t_n'. $\qquad\square$

Now, the final result of equivalence can be stated.

Theorem 3.18. *Let $(T, \mathcal{A}, \mu)$ be a σ-finite measure space, with μ non-negative and countably additive, and let $f : T \to X$ be any mapping. If f is simple-Birkhoff integrable, then it is Birkhoff integrable.*

Proof. Since f is simple-Birkhoff integrable, there exists a partition

$$P_1 := \{E_1, E_2, \ldots, E_n \ldots\}$$

such that for every choice of points $t_n \in E_n$ one has

$$\limsup_n \left\| \sum_{i=1}^{n} f(t_i)\mu(E_i) \right\| \leq M$$

for a suitable positive real number M (it suffices to take $\varepsilon = 1$ in the definition of simple-Birkhoff integrability). Then obviously also the condition (12) is satisfied. So, by Lemma 3.17, f is bounded in each set E_n such that $\mu(E_n) \neq 0$, and, since $\mu(E_n) < +\infty$ for all n, f turns out to be Birkhoff integrable in each set E_n of the partition P_1. Moreover, f is Pettis integrable by Theorem 3.14, so the result of [12, Lemma 3.2] can be applied also in this case, and f is Birkhoff integrable. $\qquad\square$

Of course, the result of the last Theorem holds true in the following case:

Corollary 3.19. *Let $F : T \to cwk(X)(ck(X))$. If F is simple-Birkhoff integrable, then it is Birkhoff integrable.*

Proof. It follows directly from Rådström embedding Theorem (Lemma 2.10) and Theorem 3.18. □

4. A Comparison with the Mc Shane Multivalued Integral

So far, it has been proved that the definition of simple-Birkhoff integrability is an extension of the usual Birkhoff integrability as introduced in [11] for $cwk(X)$-valued mappings. Another interesting property of simple-Birkhoff integrability is that it implies a corresponding notion of Mc Shane integrability (see, for example, [5]) for $b(X)$-valued functions.

Assume that $\mathscr{T}$ is a topology on T making $(T, \mathscr{T}, \mathcal{A}, \mu)$ a σ-finite quasi-Radon measure space which is *outer regular*, namely such that

$$\mu(B) = \inf\{\mu(G) : B \subseteq G \in \mathscr{T}\} \quad \text{for} \quad \text{all} \quad B \in \mathcal{A}.$$

A *generalized Mc Shane partition* Π of T ([29, Definitions 1A]) is a disjoint sequence $(E_i, t_i)_{i \in \mathbb{N}}$ of measurable sets of finite measure, with $t_i \in T$ for every $i \in \mathbb{N}$ and $\mu(T \setminus \bigcup_i E_i) = 0$.

A generalized Mc Shane partition $(E_i, t_i)_{i \in \mathbb{N}}$ is said to be *univocally tagged* (in short, u.t.) if each tag t_i corresponds to just one of the sets E_i.

If, moreover, $t_i \in E_i$ for all i, then the partition is said to be of the *Henstock* type. A *gauge* on T is a function $\Delta : T \to \mathscr{T}$ such that $s \in \Delta(s)$ for every $s \in T$. A generalized Mc Shane partition $(E_i, t_i)_i$ is Δ-*fine* if $E_i \subset \Delta(t_i)$ for every $i \in \mathbb{N}$.

From now on, let $\mathcal{P}_s$ be the class of all generalized Mc Shane partitions of T and by $\mathcal{P}_\Delta$ the set of all Δ-fine elements of $\mathcal{P}_s$.

Definition 4.1. Let $F : T \to b(X)$ be any multivalued mapping. F is said to be *Mc Shane integrable* if there exists an element $I \in bf(X)$ such that, for every $\varepsilon > 0$ there exists a gauge Δ on T with the property that, for every Δ-fine Mc Shane partition $P := \{(t_i, A_i), i \in \mathbb{N}\}$ the following holds:

$$\limsup_n h\left(\sum_{i=1}^n F(t_i)\mu(A_i), I\right) \leq \varepsilon.$$

In this case, I is said to be the *Mc Shane integral* of F. A formally more general notion is obtained, called *u.t. Mc Shane integrability*, if in the above requirements only u.t. partitions are involved.

Observe that this notion of Mc Shane integral includes the one given in [6] for $cwk(X)$- or $ck(X)$-valued mappings.

Lemma 4.2. *Assume that every point in T is contained in an open set of finite measure, and let $F : T \to b(X)$ be any bounded function. If F is u.t. Mc Shane integrable, with integral I, then it is also McShane integrable.*

Proof. Let M denote any positive constant, dominating $|F|$, and fix arbitrarily $\varepsilon > 0$. Then there exists a gauge Δ fitting the condition of u.t. Mc Shane integrability with respect to $\varepsilon 3^{-1}$, and, without loss of generality, one can assume that $\mu(\Delta(t)) < +\infty$ for all $t \in \Omega$.

Now, let $P \equiv \{(B_j, w_j) : j \in \mathbb{N}\}$ be any Δ-fine Mc Shane partition. For each tag w_k, let us denote by H_k the union of those sets B_j that are associated to the same tag w_k. Since μ is countably additive and $\mu(H_k) \leq \mu(\Delta(w_k)) < +\infty$, it is possible to find a finite number of these sets B_j, denoted by

$$B_1^k, \ldots, B_{j(k)}^k,$$

such that

$$\mu\left(H_k \setminus \left(\bigcup_{i=1}^{j(k)} B_i^k\right)\right) \leq \frac{\varepsilon}{4^k M}.$$

Hence, setting $Q := \bigcup_k (H_k \setminus (\bigcup_{i=1}^{j(k)} B_i^k))$, one has $\mu(Q) \leq \varepsilon(3M)^{-1}$. Now, observe that $\{H_k, w_k\} : k \in \mathbb{N}\}$ is a Δ-fine u.t. Mc Shane partition, so it satisfies

$$\limsup_n h\left(\sum_{k=1}^n F(w_k)\mu(H_k), I\right) \leq \frac{\varepsilon}{3}.$$

Hence, there exists an integer N such that

$$h\left(\sum_{k=1}^{N'} F(w_k)\mu(H_k), I\right) \leq \frac{\varepsilon}{3} \tag{13}$$

for all $N' \geq N$. In correspondence with N, there exists a (larger) integer N_1 such that the family $\{B_j : j \leq N_1\}$ includes all the sets of the type

$$B_i^k, i = 1, \ldots, j(k), \quad k = 1, \ldots, N.$$

Remark that, if $j > N$, then $B_j \subset Q$. So, for each integer $N_1' \geq N_1$ one has

$$h\left(\sum_{j=1}^{N_1'} F(w_j)\mu(B_j), \sum_{k=1}^N \sum_{i=1}^{j(k)} F(w_k)\mu(B_i^k)\right) \leq \frac{\varepsilon}{3} \tag{14}$$

since each set of the first summand which is not in the second one is included in Q, and $\mu(Q) \leq \varepsilon(3M)^{-1}$. For the same reason, one sees that

$$h\left(\sum_{k=1}^N F(w_k)\mu(H_k), \sum_{k=1}^N \sum_{i=1}^{j(k)} F(w_k)\mu(B_i^k)\right) \leq \frac{\varepsilon}{3}. \tag{15}$$

Now, taking into account of (13)–(15), it follows

$$h\left(\sum_{j=1}^{N_1'} F(w_j)\mu(B_j), I\right) \leq h\left(\sum_{j=1}^{N_1'} F(w_j)\mu(B_j), \sum_{k=1}^N \sum_{i=1}^{j(k)} F(w_k)\mu(B_i^k)\right)$$

$$+ h\left(\sum_{k=1}^N F(w_k)\mu(H_k), \sum_{k=1}^N \sum_{i=1}^{j(k)} F(w_k)\mu(B_i^k)\right) + h\left(\sum_{k=1}^N F(w_k)\mu(H_k), I\right) \leq \varepsilon.$$

Since $N_1' \geq N_1$ was arbitrary, this show that

$$\limsup_n h\left(\sum_{j=1}^n F(w_j)\mu(B_j), I\right) \leq \varepsilon$$

and, by arbitrariness of P, Mc Shane integrability of F follows. $\qquad\square$

Remark 4.3. If $F : T \to cwk(X)$ is Mc Shane integrable, then its integral coincides with the following set:

$$\Phi(F,T) = \Big\{ x \in X : \forall\, \varepsilon > 0,\ \exists \text{ a gauge } \Delta :\ \text{for every generalized } \mathcal{P}_\Delta$$

$$\text{Mc Shane partition}(E_i, t_i)_{i \in \mathbb{N}} \text{ there holds:}$$

$$\limsup_n d\Big(x, \sum_{i=1}^n F(t_i)\mu(E_i)\Big) \leq \varepsilon \Big\},$$

(see [6, Proposition 1]). The set $\Phi(F,T)$ is called the $(\star)$ integral of F. No measurability is required a priori; moreover, if F is single valued, then $\Phi(F,T)$ coincides with the classical Mc Shane integral, if it exists.

The simple Birkhoff integrability implies the Mc Shane one under suitable conditions: a first formulation holds in the countably additive case, for bounded functions F; a second theorem requires just finite additivity, but F must be bounded and with compact support. The pointwise non-atomicity of μ ensures that all the partitions in the Henstock sense will turn out to be equivalent to the same concepts in the Mc Shane sense, i.e. without requiring that the tags t_i are contained in the corresponding sets of the involved partitions (see also, for example, [4, Proposition 2.3]).

Theorem 4.4. *Suppose that μ is a pointwise non-atomic σ-finite measure. Let $F : T \to b(X)$ be a bounded simple Birkhoff integrable multifunction. Then F is also Mc Shane integrable and the two integrals agree.*

Proof. Thanks to the Lemma 4.2, it will be sufficient to prove that F is u.t. Mc Shane integrable. Let M be an upper bound for $|F|$. By simple Birkhoff integrability of F there exists $I_w \in b(X)$ such that for every $\varepsilon > 0$, there exists a countable partition $P_\varepsilon := \{E_n\}_{n \in \mathbb{N}}$ of T so that for every countable partition $P = \{A_n\}_{n \in \mathbb{N}}$ of T, with $P \geq P_\varepsilon$ and every $t_n \in A_n, n \in \mathbb{N}$, one has

$$\limsup_n h\left(\sum_{i=1}^n F(t_i)\mu(A_i), I_w\right) \leq \varepsilon/3. \tag{16}$$

Since μ is outer regular consider the sequence of open sets $(G_n)_n$ associated to P_ε such that for every $n \in \mathbb{N}$, $\mu(G_n) < +\infty$, $E_n \subset G_n \in \mathscr{T}$ and $\mu(G_n \setminus E_n) \leq \varepsilon(M4^n)^{-1}$.

Let now $\Delta^* : T \to \mathscr{T}$ be defined by:

$$\Delta^*(t) := \sum_{n \in \mathbb{N}} G_n 1_{E_n}.$$

An Extension of the Birkhoff Integrability

Let now $\Pi \in \mathcal{P}_{\Delta^*}, \Pi := \{(B_n, w_n), n \in \mathbb{N}\}$ be any u.t. Δ^*-fine Mc Shane partition. It will be proven that

$$\limsup_n h\left(\sum_{i=1}^n F(w_i)\mu(B_i), I_w\right) \leq \varepsilon.$$

Since μ is non-atomic, without loss of generality Π can be assumed to be of the Henstock type: otherwise it will suffice to replace every set B_n with $(B_n \cup \{w_n\})\backslash C$, where C is the set of all tags w_n. For every $n \in \mathbb{N}$ let $j(n)$ be the only index such that $w_n \in E_{j(n)}$, and so $B_n \subset G_{j(n)}$. Suitably enumerating the sets E_k , one can assume that the mapping $n \mapsto j(n)$ is increasing. Split each B_n into two parts: $B_n \cap E_{j(n)}$ and $B_n \backslash E_{j(n)}$ in such a way that:

$$\mu(B_n) = \mu(B_n \cap E_{j(n)}) + \mu(B_n \backslash E_{j(n)}) \leq \mu(B_n \cap E_{j(n)}) + \mu(G_{j(n)} \backslash E_{j(n)})$$
$$\leq \mu(B_n \cap E_{j(n)}) + \varepsilon(M4^{j(n)})^{-1} \qquad (17)$$

Moreover, thanks to countable additivity of μ, one has

$$\mu\left(\bigcup_n (B_n \backslash E_{j(n)})\right) \leq \varepsilon M^{-1} \sum_{j=1}^{+\infty} 4^{-j} = \varepsilon(3M)^{-1}.$$

For further convenience, the set $\bigcup_n (B_n \backslash E_{j(n)})$ will be denoted as Q. Since $\mu(Q) \leq \varepsilon(3M)^{-1}$, it is easy to see that

$$h\left(\sum_{i=1}^n F(w_i)\mu(B_i), \sum_{i=1}^n F(w_i)\mu(B_i \cap E_{j(i)})\right) \leq M\mu(Q) \leq \frac{\varepsilon}{3}, \qquad (18)$$

for all integers n. Now, to every set E_j of P_ε associate a tag t_j as follows: $t_j = w_{j(n)}$ if $E_j = E_{j(n)}$ for some n, otherwise let t_j be any arbitrary point of E_j. In this way P_ε becomes a Henstock-type partition, and certainly satisfies

$$\limsup_n h\left(\sum_{j=1}^n F(t_j)\mu(E_j), I_w\right) \leq \varepsilon/3. \qquad (19)$$

So, there exists an integer N' large enough that

$$h\left(\sum_{r=1}^{N''} F(t_r)\mu(E_r), I_w\right) \leq \varepsilon/3 \qquad (20)$$

holds, for every integer $N'' \geq j(N')$. In particular, if N is any integer larger that N', then $j(N) \geq j(N')$.

Except for a null set, $\Omega = Q \cup [\bigcup_n (B_n \cap E_{j(n)})]$, hence, for every integer k, one has

$$E_k = (E_k \cap Q) \cup [E_k \cap B_1 \cap E_{j(1)}] \cup [E_k \cap B_2 \cap E_{j(2)}] \cup \cdots$$

and, since the sets E_j are pairwise disjoint, all the intersections $E_k \cap B_i \cap E_{j(i)}$ are null sets, unless $E_k = E_{j(i)}$ for one i (and only one). So, if E_k is not of the type $E_{j(i)}$, then (except for a null set), $E_k = E_k \cap Q$; while, otherwise,

$E_k = (E_k \cap Q) \cup (B_i \cap E_{j(i)})$, (where $k = j(i)$). So, chosen arbitrarily $N \geq N'$, this leads to

$$\sum_{r=1}^{j(N)} F(t_r)\mu(E_r) = \sum_{r=1}^{j(N)} F(t_r)\mu(E_r \cap Q) + \sum_{i=1}^{N} F(t_i)\mu(B_i \cap E_{j(i)}).$$

This implies that

$$h\left(\sum_{i=1}^{N} F(t_i)\mu(B_i \cap E_{j(i)}), \sum_{r=1}^{j(N)} F(t_r)\mu(E_r)\right) \leq M\mu(Q) \leq \frac{\varepsilon}{3}. \qquad (21)$$

So, from (18), (20) and (21), one deduces that

$$h\left(\sum_{i=1}^{N} F(t_i)\mu(B_i), I_w\right) \leq h\left(\sum_{i=1}^{N} F(w_i)\mu(B_i), \sum_{i=1}^{N} F(w_i)\mu(B_i \cap E_{j(i)})\right)$$

$$+ h\left(\sum_{i=1}^{N} F(t_i)\mu(B_i \cap E_{j(i)}), \sum_{r=1}^{j(N)} F(t_r)\mu(E_r)\right)$$

$$+ h\left(\sum_{r=1}^{j(N)} F(t_r)\mu(E_r), I_w\right) \leq \varepsilon.$$

Since N is arbitrarily chosen larger than N', one gets

$$\limsup_{n} h\left(\sum_{i=1}^{n} F(t_i)\mu(B_i), I_w\right) \leq \varepsilon,$$

and, therefore, u.t. Mc Shane integrability for F, by arbitrariness of the Δ^*-fine partition Π. $\qquad \square$

So, it has been proved that, for $b(X)$-valued mappings, the simple-Birkhoff integrability implies the Mc Shane integrability, at least for bounded functions, likewise for the $cwk(X)$-valued case Birkhoff integrability implies Mc Shane integrability. Hence, simple-Birkhoff integrability is a natural extension of the Birkhoff one to the case of $b(X)$-valued functions.

The next Theorem 4.6 does not involve directly countable additivity. It is advisable to simplify notations and proofs when μ is a finite (countably additive) measure. Indeed, the following lemma holds true. To simplify notations, the symbol $s(P, F)$ will be adopted, to denote the summation $\sum_i F(t_i)\mu(A_i)$, whenever $P := \{(A_i, t_i) : i \in \mathbb{N}\}$ is a tagged partition.

Lemma 4.5. *Assume that $(T, \mathcal{A}, \mu)$ is a finite measure space, and $P := \{(E_i, t_i) : i \in \mathbb{N}\}$ is any Mc Shane partition, and assume that P is Δ-fine, for some gauge Δ on T. Then, for every bounded function $F : T \to cwk(X)$ and every $\varepsilon > 0$ there exists a Δ-fine finite partition $P' := \{(E'_j, t'_j)\}$ of T, in such a way that*

$$\limsup_{n} h\left(\sum_{j=1}^{n} F(t_i)\mu(E_i), \sum_{j=1}^{N} F(t'_i)\mu(E'_i)\right) \leq 2\varepsilon.$$

Proof. Let M denote any positive domination for $|F|$, and choose any integer N such that $\sum_{i=N+1}^{+\infty} \mu(E_i) \leq \varepsilon M^{-1}$. Then define $P' := \{(E_j, t_j), j = 1 \ldots, N, (B_j, w_j), j = N+1, \ldots, k\}$ in such a way that $P'' := \{(B_j, w_j), j = N+1, \ldots, k\}$ is a Δ-fine Mc Shane partition of the set $\bigcup_{j=N+1}^{+\infty} E_j$ (its existence is due to the well-known Cousin lemma, see [41, Proposition 1.7]). Then, for every $N' \geq N$ one has

$$h\left(\sum_{i=1}^{N'} F(t_i)\mu(E_i), s(P', F)\right) \leq h\left(\sum_{i=N+1}^{N'} F(t_i)\mu(E_i), s(P'', F)\right)$$

$$\leq h\left(\sum_{i=N+1}^{N'} F(t_i)\mu(E_i), \{0\}\right) + h(s(P'', F), \{0\}) \leq 2\varepsilon.$$

Since $N' \geq N$ is arbitrary, the conclusion is obvious. □

In some particular situations, the implication stated in the Theorem 4.4 holds also if the measure μ is just finitely additive, and with a much simpler proof.

Theorem 4.6. *Suppose that μ is a pointwise non-atomic finitely additive regular measure. Let $F : T \to cwk(X)$ be a bounded simple Birkhoff integrable multifunction in $cwk(X)$ with compact support. Then F is also Mc Shane integrable and the two integrals agree.*

Proof. Denote by S the compact support of F. Of course, it is possible to replace T with S in the rest of the proof. Then μ is bounded in the σ-algebra $\mathcal{A} \cap S$, and also countably additive there, as it is well known. Then, by Theorem 3.8, F is Gould integrable with the same integral. Let I denote the integral of F, and fix arbitrarily $\varepsilon > 0$. Then one can find a finite partition $P_\varepsilon := \{E_i : i = 1, \ldots, n\}$ such that

$$h(s(F, P'), I) = h\left(\sum_{j=1}^{k} F(t_j)\mu(A_j)\right) \leq \varepsilon$$

holds true, for every finite partition $P' := \{(A_j, w_j) : j = 1, \ldots, k\}$ finer than P and every choice of points $w_j \in A_j$, $j \in \{1, \ldots, k\}$. Thanks to the regularity of μ, there exist open sets G_i, $i = 1, \ldots, n$, such that $E_i \subset G_i$ and $\mu(G_i \setminus E_i) \leq \varepsilon(nM)^{-1}$ for all indexes i. Then define a gauge Δ on S, by setting, for all $t \in S$:

$$\Delta(t) = \sum_i G_i \, 1_{E_i}(t).$$

Now, choose any Δ-fine Mc Shane finite partition $\Pi := \{(B_r, v_r) : r = 1, \ldots, l\}$. Without loss of generality, one can assume that Π is u.t. (otherwise simply glue together the sets B_r that are associated to the same tag), and, by non-atomicity, that $v_r \in B_r$ for all r. Next, for all r the set $G(v_r)$ contains $E_{i(r)}$ for a suitable index $i(r)$ and $\mu(G(v_r) \setminus E_{i(r)}) \leq \varepsilon(nM)^{-1}$, hence $\mu(B_r \setminus E_{i(r)}) \leq \varepsilon(nM)^{-1}$. Take now the partition Π' consisting of all the sets $B_r \cap E_{i(r)}$ and all the intersections $(B_r \setminus E_{i(r)}) \cap E_l$, with $l \neq i(r)$. For all sets

in Π' choose tags as follows: to the sets of the type $B_r \cap E_{i(r)}$ associate the tag v_r, while, for all the other sets, the tag is any arbitrary point of the set. Since Π' is finer than P_ε, one has $h(s(\Pi', F), I) \leq \varepsilon$; but also

$$h(s(\Pi, F), s(\Pi', F)) \leq M \sum_r \mu(B_r \backslash E_{i(r)})$$

$$+M \sum_{r=1}^{n} \sum_{l \neq i(r)} \mu((B_r \backslash E_{i(r)}) \cap E_l) \leq 2\varepsilon;$$

hence, $h(s(\Pi, F), I) \leq 3\varepsilon$. Since Π is arbitrary, this concludes the proof. $\square$

As to the converse implication, namely if Mc Shane integrability implies the Birkhoff simple one, it does not hold in general, as the next example shows:

Example 4.7 In [44, Example 2.1], an example found by Phillips has been recalled, to show a mapping $f : [0,1] \rightarrow l^\infty([0,1])$ which is bounded, Mc Shane integrable and not Birkhoff integrable. Thanks to Lemma 4.5 (and subsequent sentence), f is not even simple Birkhoff integrable. Clearly, this function can be used to define a multivalued mapping F, simply by setting $F(t) = \{f(t)\}$ for all $t \in [0,1]$. Of course, integrability (in any sense) of F is equivalent to integrability (in the corresponding sense) of f. Thus, the mapping F gives an example of a bounded multivalued function, defined in a finite countably additive non-atomic measure space, which is Mc Shane integrable but not simple Birkhoff integrable.

A partial result could be the following:

Remark 4.8. Let $(T, \mathscr{T}, \mathcal{A}, \mu)$ be a σ-finite quasi-Radon outer regular measure space and let X a Banach space such that B_{X^*} is weak* separable. Then a bounded multifunction $F : T \rightarrow cwk(X)$ with compact support is simple Birkhoff integrable if and only if it is Mc Shane integrable and the two integrals agree. This is an easy consequence of Theorem 4.4 and of the remark of [7, Corollary 4.2].

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