ASYMPTOTIC OF THE RESOLVENT SERIES OF A VOLTERRA OPERATOR AND ITS APPLICATION

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The asymptotic of the resolvent kernels of some Volterra integral operators for large λ is studied, and its application to the question of the expansion of finite perturbations of such operators in eigen- and associated functions is given.

Let M be the Volterra operator $\int_0^x M(x, t) f(t)\, dt$ $(0 \le x \le 1)$, whose kernel M(x, t) satisfies the conditions:

1) $\frac{\partial^{i+j}}{\partial x^i \partial t^j} M(x, t)$ $(i = 0, \ldots, n+2;\ j = 0, 1, 2)$ are continuous for $0 \le t \le x \le 1$, $n \ge 1$;

2) $\frac{\partial^i}{\partial x^i} M(x, t)_{|t=x} = \delta_{i, n-1}$ $(i = 0, \ldots, n)$, where $\delta_{i,k}$ is the Kronecker delta.

THEOREM 1. Let $M(x, t, \lambda)$ be the kernel of the operator $(E + \lambda M)^{-1}M$, where E is the unit operator. Then, the following asymptotic formulas are valid:

$$\frac{\partial^j}{\partial x^j} M(x, t, \lambda) = \sum_{k=1}^{p} \frac{d^j}{dx^j} y_k(x, \rho) z_k(t, \rho) + O(\rho^{j+1-n}) \quad (j = 0, \ldots, n-1), \tag{1}$$

where

$$\rho = \lambda^{1/n} \left(\arg \rho \in \left[-\frac{\pi}{2n}, \frac{3\pi}{2n}\right]\right),$$
$$\frac{d^j}{dx^j} y_k(x, \rho) = (\rho\omega_k)^j e^{\rho\omega_k x} [1 + O(\rho^{-1})],$$
$$z_k(t, \rho) = O(\rho^{1-n} e^{-\rho\omega_k t}),$$

ω_k are the n-th roots of -1, numbered so that $\mathrm{Re}\, \rho\omega_1 \ge \mathrm{Re}\, \rho\omega_2 \ge \ldots$,* and p is such that $\mathrm{Re}\, \rho\omega_p > 0 \ge \mathrm{Re}\, \rho\omega_{p+1}$. The estimates 0 are uniform in x, t and sufficiently large $|\rho|$.

Let A denote the operator

$$Af = Mf + \sum_{k=1}^{m} c_k(f) \psi_k(x), \tag{2}$$

where $c_k(f) = \int_0^1 f(t) \varphi_k(t)\, dt$ $(k = 1, \ldots, m)$, $\psi_k(x)$ $(k = 1, \ldots, m)$ is a linearly independent system of linear functionals in L[0, 1] and functions from C[0, 1].

*The numbering of ω_k depends on arg ρ.

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THEOREM 2. Let the kernel M(x, t) of the operator M in (2) satisfy the conditions 1) and 2), where we consider $j = 0, \ldots, n$ in 1). Let us moreover assume that $g_k(t)$ have continuous derivatives in some segment $[1-\delta_1, 1]$ $(0 < \delta_1 < 1)$, where $g_k^{(j)}(1) = \delta_{k,j+1}$ for $j < k$; $\psi_i(x) \equiv \varphi_k(x)$ in some segment $[0, \delta_2]$ $(0 < \delta_2 < 1)$, and $\{\varphi_k(x)\}$ is a linearly independent system of polynomials of degree $\leq n-1$; $n > n-2m > 2$. Then, if $f(x) \in L[0, 1]$, and

$$f(x) = \sum_{j=1}^{m} \sum_{k=0}^{\infty} a_{k,j} M^k \psi_j,$$

in the half-interval $[0, a)$ $(0 < a \leq 1)$ where $a_{k,j}$ are such that $\sum_{j=1}^{m} \sum_{k=0}^{\infty} |a_{k,j}| \frac{x^{kn}}{(kn)!} < +\infty$ for $x \in [0, a)$, then the Fourier series of the function $f(x)$ in eigen- and associated functions of the operator A will converge to $f(x)$ in $[0, a)$ in some subsequence of partial sums, where the convergence will be uniform in the whole segment $[0, a_1] \subset [0, a)$.*

As follows from [1] and [2], the demand that $f(x)$ be expanded in a series in the system $\{M^k\psi_j\}$ is essential for the validity of Theorem 2.

This paper is devoted to the proof of these theorems.

Let us use the following notation:

1) We shall denote the integral operators and their kernels as well as the determinants and cofactors of their elements by the same letter, where we shall write the arguments in the kernels, and the subscripts of the elements in the cofactors (thus, for example, A(x, t) is the kernel of the operator A, $\Delta_{j,k}$ is the cofactor of the element $a_{j,k}$ of the determinant $\Delta = \det \|a_{j,k}\|$);

2) $N(x, t) = (\partial^n/\partial x^n) M(x, t)$ and $\widetilde{N}(x, t)$ is the kernel of the operator $(E + N)^{-1} - E$;

3) $M(x, t)$, $M(x, t, \lambda)$, $N(x, t)$, $\widetilde{N}(x, t)$ will be considered to equal zero for $t > x$;

4) $\varepsilon(x, t) = 1$ for $t \leq x$, $\varepsilon(x, t) \equiv 0$ for $t > x$; $l[y] = (E + \widetilde{N})y^{(n)}$.

1. LEMMA 1. The integro-differential equation

$$y^{(n)} + \lambda [y + Ny] = 0 \tag{3}$$

has a system of solutions $\{y_k(x, \rho)\}_{k=1}^{n}$, for which the following asymptotic formulas are valid for large $|\rho|$:

$$\frac{d^j}{dx^j} y_k(x, \rho) = (\rho\omega_k)^j e^{\rho\omega_k x} [1 + O(\rho^{-1})], \text{ when } \operatorname{Re} \rho\omega_k \geqslant 0, \tag{4}$$

$$\frac{d^j}{dx^j} y_k(x, \rho) = (\rho\omega_k)^j [e^{\rho\omega_k x} + O(\rho^{-1})], \text{ when } \operatorname{Re} \rho\omega_k < 0 \tag{5}$$

$$(j = 0, \ldots, n-1; \; k = 1, \ldots, n).$$

Proof. Let us examine the integral equations

$$y_k = e^{\rho\omega_k x} - \lambda A_k N y_k \; (k = 1, \ldots, n), \tag{6}$$

where A_k is the operator $\int_0^1 A_k(x, t, \rho) f(t)\, dt$, whose kernel equals

$$A_k(x, t, \rho) = \frac{\varepsilon(t, x)}{n\rho^{n-1}} \sum_{j=1}^{n_k-1} \omega_j \exp \rho\omega_j (x-t) - \frac{\varepsilon(x, t)}{n\rho^{n-1}} \sum_{j=n_k}^{n} \omega_j \exp \rho\omega_j (x-t),$$

*It should be kept in mind that the problem of expanding the operator (2) in eigen- and associated functions is not always legitimate since it is easy to give an example when (2) will again be a Volterra operator. For instance

$$Af = \int_0^x f(t)\, dt - \int_0^1 f(t)\, dt = -\int_x^1 f(t)\, dt;$$

or when $g_k(t) \equiv 0$ for $t > \gamma$ and $\psi_k(x) \equiv 0$ for $x \leq \gamma$, where $0 < \gamma < 1$.

where $n_k = k$, for $\operatorname{Re} \rho\omega_k \geq 0$, and $n_k = p+1$, if $\operatorname{Re} \rho\omega_k < 0$. Since the solution (6) is also the solution of (3), then to prove the lemma it is sufficient to obtain the required asymptotic for the solution of (6). Let us convert (6). Let $\operatorname{Re} \rho\omega_k \geq 0$. Integrating by parts, we obtain the following formula for the kernel of the operator $\lambda A_k N$:

$$\lambda A_k N(x, t, \rho) = N(x, t) + B_0(x, t, \rho) + B_1(x, t, \rho), \tag{7}$$

where

$$B_0 = -\frac{1}{n} \sum_{j=1}^{k-1} N(1, t) \exp \rho\omega j(x-1),$$

$$B_1 = \int_0^1 N'_\tau(\tau, t) \left\{ \frac{\varepsilon(\tau, x)}{n} \sum_{j=1}^{k-1} \exp \rho\omega_j (x-\tau) - \frac{\varepsilon(x, \tau)}{n} \sum_{j=k}^{n} \exp \rho\omega_j (x-\tau) \right\} d\tau.$$

The obvious estimate

$$B_1(x, t, \rho) = O(\rho^{-1} \exp \rho\omega_k (x-t)). \tag{8}$$

is valid for B_1. By using (7) we convert (6) into

$$y_k = f_k + B_3 y_k,$$

where $f_k = (E + \widetilde{N}) e^{\rho\omega_k x} - B_0 (E + \widetilde{N}) e^{\rho\omega_k x}$, $B_3 = B_0^2 - B_0 B_2 + B_2$, $B_2 = -\widetilde{N} B_0 - (E + \widetilde{N}) B_1$. The estimate (8) is valid for $B_3(x, t, \rho)$. Let us put $z = y_k \exp(-\rho\omega_k x)$. Then

$$z = f_k \exp(-\rho\omega_k x) + B_4 z, \tag{9}$$

where $B_4(x, t, \rho) = B_3(x, t, \rho) \exp[-\rho\omega_k(t-x)]$. Since $B_4 = O(\rho^{-1})$, then we apply the principle of compressed mappings to (9). This means that (9) is solvable and $z = O(1)$ as $\rho \to \infty$. Hence, (4) follows for $j = 0$. For $j = 1, \ldots, n-1$, formulas (4) are obtained from (9) by differentiation and using the formula (4) already obtained for $j = 0$. Formulas (5) are obtained analogously (and even more simply since in this case, the exponentials in the estimates for B_1 and B_3 are omitted). The lemma is proved.

COROLLARY. For large $|\rho|$ the system $\{y_k(x, \rho)\}$ forms a fundamental system of solutions of (3).

Let $G(x, t, \lambda)$ denote the Green's function of the operator

$$l[y] + \lambda y = (E + \widetilde{N}) y^{(n)} + \lambda y \tag{10}$$

with normalized dissociating boundary conditions [3]

$$U_i(y) = 0 \qquad (i = 1, \ldots, n), \tag{11}$$

where, in contrast to (40) [3], we assume that the number m of boundary conditions taken at the point 1 satisfies the condition $2m \leq n$. It is known [1], that $\rho_k^0 = \lambda_k^{(0)1/n}$, where $\lambda_k^{(0)}$ are the eigenvalues of $y^{(n)} + \lambda y = 0$ with the conditions (11), will approach $\arg \rho = 0$ asymptotically if m is odd, and $\arg \rho = \pi/n$, if m is even. Let us remove ρ_k^0 together with circular neighborhoods of the same sufficiently small radius δ from $-\pi/2n \leq \arg \rho \leq 3\pi/2n$. Let S_δ denote the rest of the domain.

LEMMA 2. There exists a $G(x, t, \lambda)$ in the domain S_δ for large $|\rho|$, where the following asymptotic formulas are valid:

$$\begin{aligned} G_x^{(k)}(x, t, \lambda) = G_{0x}^{(k)}(x, t, \lambda) &+ \varepsilon(x, t) [O(\rho^{k-n}) + O(\rho^{k-n} \exp \rho\omega_{m+1}(x-t))] + \\ &+ \varepsilon(t, x) O(\rho^{k-n} \exp \rho\omega_m (x-t)) \quad (k = 0, 1, \ldots, n-1), \end{aligned} \tag{12}$$

where $G_0(x, t, \lambda)$ is the Green's function of the operator $y^{(n)} + \lambda y$ from (11).

Proof. The function G and G_0 are connected by the relationship

$$G = G_0 + GB, \tag{13}$$

where GB is an operator product, i.e., $GB = \int_0^1 G(x, \tau, \lambda) B(\tau, t, \lambda)\, d\tau$; $B = -\tilde{N} + \lambda \tilde{N} G_0$. Let us consider (13) as an integral equation in G. From the formulas presented in [4], it follows that

$$G_0(x, t, \lambda) = -n^{-1}\rho^{1-n}\Delta_0^{-1}(\rho)\sum_{j,k=1}^{n} \omega_j \exp([\rho\omega_k x - \rho\omega_j t])(\varepsilon(x, t) C_{j,k}(\rho) + \varepsilon(t, x) D_{j,k}(\rho)), \tag{14}$$

where $\Delta_0(\rho) = \det \| U_j(e^{\rho\omega_k x})\|_{j,k=1}^{n}$, $C_{j,k}(\rho)$, $D_{j,k}(\rho)$ are some determinants of order n, whose elements are independent of x and t. The expressions

$$C_{j,j} - D_{j,j} = \Delta_0(\rho),\ C_{j,k} = D_{j,k} \quad \text{for} \quad j \neq k \tag{15}$$

follow from the explicit expressions for these determinants, as do the estimates

$$\begin{aligned}
&C_{j,k}(\rho) = O\left(\rho^{\alpha} \exp\rho \left[\sum_1^{m+1} \omega_i - \omega_k\right]\right) \qquad (j, k \leqslant m),\\
&C_{j,k}(\rho) = O\left(\rho^{\alpha} \exp\rho \sum_1^{m} \omega_i\right) \qquad (j \leqslant m < k \text{ и } j = k > m),\\
&D_{j,k}(\rho) = O\left(\rho^{\alpha} \exp\rho \left[\sum_1^{m-1} \omega_i + \omega_j\right]\right) \qquad (k, j > m),\\
&D_{j,k}(\rho) = O\left(\rho^{\alpha} \exp\rho \left[\sum_1^{m} \omega_i + \omega_j - \omega_k\right]\right) \qquad (k \leqslant m < j \text{ и } k = j \leqslant m),
\end{aligned} \tag{16}$$

where $\alpha = \sum k_i + \sum k_i$ (see [3]). Let us still note the following lower bound for Δ_0 (see [4]):

$$|\Delta_0(\rho)| \geqslant C|\rho|^{\alpha}|\exp\rho\sum_1^{m}\omega_i|. \tag{17}$$

Furthermore, by integrating by parts, we have $B(x, t, \lambda) = B_0(x, t, \rho) + B_1(x, t, \rho)$, where

$$B_0 = \tilde{N}(x, 0)\, n^{-1}\Delta_0^{-1}(\rho)\sum_{j,k=1}^{n} \omega_j \omega_k^{-1} e^{-\rho\omega_j t} D_{j,k}(\rho)$$

and

$$B_1 = n^{-1}\Delta_0^{-1}(\rho)\sum_{j,k=1}^{n}\omega_j\omega_k^{-1}e^{-\rho\omega_j t}\int_0^1 \tilde{N}'_{\xi}(x, \xi)\{[\varepsilon(x, t)\varepsilon(t, \xi) + \varepsilon(t, x)] D_{j,k} + \varepsilon(x, t)\varepsilon(\xi, t) C_{j,k}\} e^{\rho\omega_k\xi} d\xi.$$

The estimate

$$B_1 = \varepsilon(x, t)[O(\rho^{-1}) + O(\rho^{-1}\exp\rho\omega_{m+1}(x - t))] + \varepsilon(t, x)\, O(\rho^{-1}\exp\rho\omega_m(x - t)). \tag{18}$$

follows from (16) and (17). Let us convert (13) into

$$G = f + GB_2, \tag{19}$$

where $f = G_0 + G_0B_0$ and $B_2 = B_2^0 + B_1B_0 + B_1$. It follows from the explicity expression for A_0 and (18) that the estimate (19) is true for B_2 also. Let $\operatorname{Re}\rho\omega_{m+1} > 0$. Let us assume $\Phi = G \exp\rho\omega_{m+1}(t-x)$. Then, we obtain from (19):

$$\Phi = f_1 + \Phi B_3, \tag{20}$$

where $f_1 = f\exp\rho\omega_{m+1}(t - x)$, $B_3(x, t, \rho) = B_2(x, t, \rho)\exp\rho\omega_{m+1}(t - x)$. Since $f_1 = O(\rho^{1-n})$ and $B_3 = O(\rho^{-1})$, then by the principle of compressed mappings the solution Φ of (20) exists in S_δ, where $\Phi = O(\rho^{1-n})$. This means that $G(x, t, \lambda)$ exists and $G = O(\rho^{1-n}\exp\rho\omega_{m+1}(x-t))$. Let us refine this estimate for $t \geq x$. Let us assume that $\Phi_1 = G\exp\rho\omega_m(t-x)$ for $t \geq x$. Then, taking account of the estimate obtained for G, we have from (19):

$$\Phi_1(x, t, \rho) = O(\rho^{1-n}) + \int_x^1 O(\rho^{-1})\Phi_1(x, \tau, \rho)\, d\tau, \qquad t \geqslant x.$$

Hence, $\Phi_1 = O(\rho^{1-n})$, and this means that $G = O(\rho^{1-n}\exp\rho\omega_n(x-t))$ for $t \geq x$. Substituting the estimate obtained into (19), we obtain (12) for k = 0 by virtue of (14)-(17). For k = 1, . . . , n−1 formulas (12) are obtained from (19) by differentiation and analogous reasoning. The case Re $\rho\omega_{m+1} \leq 0$ is investigated even more simply since in this case, the principle of compressed mappings is already applicable to (19). The lemma is proved.

Remark. It is shown analogously that Lemma 2 is even true when (11) are arbitrary regular boundary conditions [3]. In this case, the exponentials are missing from the right sides of (12).

Proof of Theorem 1. Let $y = (E + \lambda M)^{-1} Mf$. Then, y satisfies the equation $l[y] + \lambda y = f$ and the initial conditions $y^{(k)}(0) = 0$ (k = 0, ..., n−1). Let $G(x, t, \lambda)$ be the Green's function of the operator (10) with some regular boundary conditions. (If n is even, then, we take regular dissociating boundary conditions.) Then, according to the corollary to Lemma 1, the general solution of the equation $l[y] + \lambda y = f$ is $v = \sum_{k=1}^{n} \alpha_k y_k(x, \rho) + z(x)$, where $z(x) = \int_0^1 G(x, t, \lambda) f(t)\, dt$ and α_k are arbitrary constants. In order for v = y, it is necessary to determine α_k from the conditions $v^{(k)}(0) = 0$ (k = 0, ..., n−1). Therefore, by virtue of (4) and (5)

$$y = \sum_{k=1}^{n} \alpha_k y_k(x, \rho) + z(x), \tag{21}$$

where

$$\alpha_k = -\sum_{j=1}^{n} \Omega^{-1}\rho^{1-j}\Omega_{j,k}[1 + O(\rho^{-1})]\, z^{(j-1)}(0),$$

$$\Omega = \det \| \omega_k^{j-1} \|_{j,k=1}^{n}.$$

From (21), we obtain

$$M(x, t, \lambda) = G(x, t, \lambda) + \sum_{k=1}^{n} y_k(x, \rho)\, z_k(t, \rho),$$

where

$$z_k(t, \rho) = -\sum_{j=1}^{n} \Omega^{-1}\rho^{1-j}\Omega_{j,k}[1 + O(\rho^{-1})] \frac{\partial^{j-1}}{\partial x^{j-1}} G(x, t, \lambda)|_{x=0}.$$

From known estimates [3], and the remark to Lemma 2, follow the estimates $G_x^{(j)}(x, t, \lambda) = O(\rho^{j+1-n})$ (j = 0, ..., n−1). This means that according to Lemma 1

$$M_x^{(q)}(x, t, \lambda) = \sum_{k=1}^{p} y_k^{(q)}(x, \rho)\, z_k(t, \rho) + O(\rho^{q+1-n}). \tag{22}$$

In order to obtain the necessary estimates for $z_k(t, \rho)$, it is necessary to put x = t in (22) for q = 0, ..., p−1, and then by taking account of the equalities $M_x^{(q)}(x, x, \lambda) \equiv 0$ and (4), to solve the system obtained for $z_k(t, \rho)$. It should be kept in mind that the $z_k(t, \rho)$ constructed exist only in S_δ. However, if the regular boundary conditions are altered, then the domains S_δ will be altered, and we will consequently, achieve that (1) will be valid for all sufficiently large $|\lambda|$. The theorem is proved.

2. Let us turn to the proof of Theorem 2.

LEMMA 3. Let us put $Q_s = \det \| M(x_i, t_j, \lambda) \|_{i,j=1}^{s}$, where $0 \leq x_i, t_j \leq 1$. Then for $n > n-2m > 2$ the following estimates are valid:

1) $Q_1 = O(\exp \rho\omega_1 (x_1 - t_1))$,

2) $Q_s = O\left(\exp \rho \left[\sum_1^{m-1} \omega_j + \delta_{s,m}\omega_m \min\{x_1, 1 - t_1\}\right]\right)$ $(s = 2, \ldots, m)$,

3) $Q_{m+1} = O\left(\exp \rho \left[\sum_1^{m} \omega_j + \omega_m (x_1 - t_1)\right]\right)$, $t_1 > x_1$.

Proof. The estimate 1) is evident. In order to prove the remaining estimates, we have in conformity with (1)

$$M(x_i, t_j, \lambda) = \sum_{k=1}^{p} \varepsilon_{i,j} y_k(x, \rho)\, z_k(t, \rho) + O(\rho^{1-n}) \; (\varepsilon_{i,j} = \varepsilon(x_i, t_j)).$$

Then $Q_s = W_0 + W_1$, where $W_0 = \sum_{k_1, \ldots, k_s=1}^{p} A_{k_1, \ldots, k_s}$,

$$A_{k_1, \ldots, k_s} = \prod_{i=1}^{s} y_{k_i}(x_i, \rho) \det \| \varepsilon_{\alpha, \beta} z_{k_\alpha}(t^\beta, \rho) \|_{\alpha, \beta=1}^{s},$$

and W_1 is the sum of determinants for which some columns consist of $O(\rho^{1-n})$. The functions $A_{k_1}, \ldots, k_s$ are estimated in an elementary manner by using the estimates $y_k(x, \rho) = O(\exp \rho\omega_k x)$ and $z_k(t, \rho) = O(\exp \rho(-\omega_k t))$. In order to estimate W_1, each determinant in W_1 should be expanded in minors of the columns consisting of $O(\rho^{1-n})$, and then, the cofactors obtained should be estimated exactly as W_0.

LEMMA 4. If $R_\lambda(A) = (E + \lambda A)^{-1} A$, then for $f \in L[0, 1]$

$$R_\lambda(A) f = M_\lambda f + \Delta^{-1}(\lambda) \sum_{j, k=1}^{m} \psi_j(x, \lambda) c_k((E + \lambda M)^{-1} f) \Delta_{k, j}(\lambda), \tag{23}$$

where

$$M_\lambda = (E + \lambda M)^{-1} M, \ \psi_j(x, \lambda) = (E + \lambda M)^{-1} \psi_j, \ \Delta(\lambda) = \det \| \delta_{k,j} + \lambda c_k(\psi_j(x, \lambda)) \|_{k,j=1}^{m}.$$

Then, (23) is obtained by the same method as the formula for the resolvent of a degenerate integral equation.

Let us introduce two other operators

$$A_1 f = Mf + \sum_{k=1}^{m} c_k(f) \varphi_k(x), \ A_2 f = Mf + \sum_{k=1}^{m} d_k(f) \varphi_k(x),$$

where $d_k(f)$ are linear functionals, such that the domain of values of the operator A_2 satisfies some boundary conditions of the form (11).

LEMMA 5. If $R_\lambda(A_i) = (E + \lambda A_i)^{-1} A_i$ $(i = 1, 2)$, then

$$R_\lambda(A_1) f = R_\lambda(A_2) f + \sum_{k=1}^{m} V_k(Tf - (E + \lambda T) R_\lambda(A_2) f) \Phi_k(x, \rho), \tag{24}$$

where $V_k(y) = \sum_{s=0}^{\alpha_k} \xi_{k,s} y^{(s)}(0)$ $(k = 1, \ldots, n;\ 0 \leqslant \alpha_k \leqslant n-1)$ are linearly independent, $V_k(\varphi_j) = \delta_{k,j}$ and α_k ($k = m+1, \ldots, n$) are distinct:

$$T = A_1 - M;\ \Phi_k(x, \rho) = L^{-1}(\rho) \sum_{j=1}^{n} L_{k, j}(\rho) y_j(x, \rho);$$
$$L(\rho) = \det \| V_k((E + \lambda T) y_j) \|_{k, j=1}^{m}.$$

To prove (24), it is sufficient to note that $z = R_\lambda(A_1) f - R_\lambda(A_2) f$ satisfies the equation $l[z] + \lambda z = 0$ and hence, $z = \sum_{k=1}^{n} \alpha_k y_k(x, \rho)$. The constants α_k should be determined from the conditions $V_k((E + \lambda T) R_\lambda \cdot (A_1) f - Tf) = 0$ ($(k = 1, \ldots, n)$.

LEMMA 6. Let $G_i(x, t, \lambda)$ be the kernel of the operator $R_\lambda(A_i)$ $(i = 1, 2)$. Then, in some domain of the type S_δ

$$G_1(x, x, \lambda) = G_2(x, x, \lambda) + Q(\lambda^{-1}) + O(\rho^{1-n} \exp \rho(\omega_{m+1} - \omega_m)(1 - x)). \tag{25}$$

Proof. Let us put $\tilde{g}_k(t) = g_k(t) + \int_t^1 \tilde{N}(\tau, t) g_k(\tau) d\tau$. Then, by virtue of the conditions on $g_k(t)$, and (4), (5), we have

$$\lambda c_k(y_j) = -\int_0^1 \tilde{g}_k(t) y_j^{(n)}(t, \rho) dt = -\int_0^{1-\delta_1} - \int_{1-\delta_1}^{1} = \begin{cases} (-1)^{k-1}(\rho\omega_j)^{n-k} e^{\rho\omega_j}[1 + O(\rho^{-1})] + O(\rho^n \exp \rho\omega_j(1 - \delta_1)) & \text{when } \operatorname{Re} \rho\omega_j \geqslant 0 \\ O(\rho^n) & \text{when } \operatorname{Re} \rho\omega_j < 0. \end{cases}$$

Since $V_k((E+\lambda T)y_j) = V_k(y_j) + \lambda c_k(y_j)$ for $k = 1, \ldots, m$ and $V_k((E+\lambda T)y_j) = V_k(y_j)$ for $k = m+1, \ldots, n$, then, by the same method as in [1], we, hence, obtain the estimate

$$\Phi_k(x, \rho) = O(\rho^{k-n} \exp \rho\omega_m (x-1)). \tag{26}$$

Now, let $\Omega_k(t, \lambda)$ be determined from the equalities $V_k(Tf - (E+\lambda T)z) = \int_0^1 \Omega_k(t, \lambda) f(t)\, dt$, where $z = R_\lambda(A_2)f$. Let us give an estimate for $\Omega_k(t, \lambda)$. Let us put $g_k(t) = a_k(t) + b_k(t)$, where $a_k(t) \in C^n[0, 1]$ and $a_k(t) = g_k(t)$ for $t \in [1-\delta_1, 1]$, and $\tilde{a}_k(t) = a_k(t) + \int_t^1 \tilde{N}(\tau, t) a_k(\tau)\, d\tau$. Then, taking account of $f - \lambda z = l[z]$, we have

$$\int_0^1 \Omega_k(t, \lambda) f(t)\, dt = -V_k(z) + \int_0^1 \tilde{a}_k(t) z^{(n)}(t)\, dt + \int_0^1 b_k(t)[f(t) - \lambda z(t)]\, dt. \tag{27}$$

Applying n-fold integration by parts to $\int_0^1 \tilde{a}_k(t) z^{(n)}(t)\, dt$, we obtain such a formula for $\Omega_k(t, \lambda)$ from (27), from which the estimate

$$\Omega_k(t, \lambda) = O(\rho \exp \rho\omega_m (1-\delta_1-t)) + O(\rho^{1-k} \exp \rho\omega_{m+1}(1-t)). \tag{28}$$

follows by virtue of Lemma 2, and (14)-(17). Now (25) follows from Lemma 5, and the equalities (26) and (28).

LEMMA 7. Let $\Delta_i(\lambda)$ have the same meaning for A_i $(i = 1, 2)$, as does $\Delta(\lambda)$ for A. Then, in S_δ

$$\Delta_1^{-1}(\lambda)\Delta_1'(\lambda) = \Delta_2^{-1}(\lambda)\Delta_2^1(\lambda) + O(\lambda^{-1}) + O\left(|\rho|^{1-n} \int_0^1 |\exp \rho(\omega_{m+1} - \omega_m)x|\, dx\right). \tag{29}$$

The assertion in the lemma follows from (25), and the formulas $\int_0^1 G_i(x, x, \lambda)\, dx = \Delta_i^{-1}\Delta_i'$ $(i = 1, 2)$, whose validity follows from Lemma 4.

LEMMA 8. The estimate

$$|\Delta(\lambda)| \geqslant C(\varepsilon) \left|\exp \rho\left(\sum_1^m \omega_j - \varepsilon\right)\right|, \tag{30}$$

is valid in the domain S_δ, where $\varepsilon > 0$ is an arbitrarily small number.

Proof. It follows from (34)-(36) [3], that $\int_0^1 G_0(x, x, \lambda)\, dx = \tilde{\Delta}_0^{-1}\tilde{\Delta}_0'$, where $\tilde{\Delta}_0(\lambda) = n^{-n}\Omega(-1)^{\frac{n(n-1)}{2}} \rho^{\frac{-n(n-1)}{2}} \Delta_0(\rho)$. Hence, by virtue of (12), (29), and (17), it follows that (30) is valid for $\Delta_1(\lambda)$. Furthermore, by Lemma 3

$$\Delta(\lambda) - \Delta_1(\lambda) = O\left(\exp \rho\left[\sum_1^m \omega_j - \omega_m\delta_2 + \varepsilon\right]\right).$$

Since $\Delta = \Delta_1(1 + (\Delta - \Delta_1)\Delta_1^{-1})$, then the lemma is proved.

Now, the proof of Theorem 2 is completed exactly as in [2] by using Lemmas 3, 4, and 8.

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