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EXISTENCE OF GLOBAL ATTRACTORS FOR A NONLINEAR EVOLUTION EQUATION IN SOBOLEV SPACE H^k*

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Abstract In this paper we prove that the initial-boundary value problem for the nonlinear evolution equation $u_t = \Delta u + \lambda u - u^3$ possesses a global attractor in Sobolev space H^k for all $k \geq 0$, which attracts any bounded domain of $H^k(\Omega)$ in the H^k-norm. This result is established by using an iteration technique and regularity estimates for linear semigroup of operator, which extends the classical result from the case $k \in [0, 1]$ to the case $k \in [0, \infty)$.

Key words semigroup of operator; global attractor; evolution equation; regularity of attractor

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1 Introduction

The dynamic properties of the evolution equation, i.e., the global asymptotic behaviors of solutions and existence of global attractors, are important for the study of reaction diffusion equations, because they determine the stability of reaction diffusion phenomena and provide a mathematical foundation for the study of reaction diffusion phenomena. Extensive studies on the existence of global attractors for general nonlinear diffusion dynamic systems were carried out in the past thirty years. For the classical results we refer to the monographs [1, 6, 8, 11]. Recently, Zhong et al. developed a global attractor theory where they replaced the uniformly compact conditions in the existence theorems of global attractors by the so-called Condition (C), which is more suitable and convenient for the applications to partial differential equations, see [2, 3, 9, 10] for details. In this paper, we consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \lambda u - u^3, & \text{in } \ \Omega \times (0, \infty), \\ u = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = \varphi(x), & \text{in } \ \Omega \times (0, \infty), \end{cases} \tag{1.1}$$

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where $\Omega \subset R^3$ is a bounded domain with smooth boundary $\partial\Omega$. For this problem, the existence of global solutions and global attractors in $L^2(\Omega)$ and $H_0^1(\Omega)$ follows from the well-known theory of general reaction diffusion equations, see [8]. Specifically, the following assertions are known:

• Problem (1.1) has a unique global strong solution $u \in C^0([0,\infty), H_{\frac{1}{2}}) \cap L^2((0,\infty), H_1), u_t \in L^2([0,\infty), H)$ for any $\varphi \in H_{\frac{1}{2}}$.

• Problem (1.1) has a unique strong solution $u \in L^2((0,T), H_1) \cap H^1((0,T), H)$ for any $T > 0, \varphi \in H$.

• Problem (1.1) has a global attractor $\mathcal{A} \subset H$, and $\mathcal{A}$ attracts any bounded set of H in the H-norm.

where, the spaces $H, H_{\frac{1}{2}}$ and H_1 are defined, respectively, as follows:

$$H = L^2(\Omega), \quad H_{\frac{1}{2}} = \big\{u \in H^1(\Omega) \cap H; u|_{\partial\Omega} = 0\big\}, \quad H_1 = H^2(\Omega) \cap H_{\frac{1}{2}}. \tag{1.2}$$

Thanks to the above results, it can be shown that the problem (1.1) is equivalent to a semigroup $\{S(t)\}_{t\geq 0}$ defined by $S(t) : H_0^1(\Omega) \to H_0^1(\Omega)$ such that

$$S(t)u_0 = u(\cdot, t)$$

The existence of the global attractor in $H_0^1(\Omega)$ for the semigroup $\{S(t)\}_{t\geq 0}$ associated with the problem (1.1) was proved by Temam in [8] and Ma et.al. in [4].

The aim of this paper is to show the existence of global attractors for the problem (1.1) in the Sobolev space $H^k(\Omega)$ for all $k \geq 0$. Specifically, we prove that (1.1) possesses a global attractor which attracts any bounded set of $H^k(\Omega)$ in H^k-norm for every $k \geq 0$. To prove this result, we adopt an idea from references [4, 5], which combines an iteration technique and regularity estimates for linear semigroups of operator with a classical theorem of global attractors.

The remainder of the paper is organized as follows. In Section 2 we recall some properties of operator semigroups which will be used later. In Section 3 we present detailed proofs of the main existence theorem and its some further results.

2 Semigroups of Operator and Their Properties

Let X and X_1 be two Banach spaces, $X_1 \subset X$ be a compact and dense inclusion. Suppose $L : X_1 \to X$ is a linear operator, and $G : X_1 \to X$ is a nonlinear operator. Consider the following abstract nonlinear evolution equation defined on X: Find $u(t) : (0,\infty) \to X$ such that

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = Lu + G(u), \\ u(0) = \varphi. \end{cases} \tag{2.1}$$

A family of operators $S(t) : X \to X$ $(t \geq 0)$ is called as an semigroup of operator generated by (2.1), if $S(t)$ satisfies the following properties:

(1) $S(t) : X \to X$ is a continuous mapping for any $t \geq 0$,

(2) $S(0) = \mathrm{id} : X \to X$ is the identity operator,

(3) $S(t + s) = S(t) \cdot S(s)$ for all $t,\ s \geq 0$

and the solution of (2.1) can be expressed as

$$u(t, \varphi) = S(t)\varphi.$$

In the following, we introduce the concepts and definitions of invariant sets, global attractors, ω-limit sets for the semigroup of operator $S(t)$.

Definition 2.1 Let $S(t)$ be an semigroup of operator defined on X. A set $\Sigma \subset X$ is called an invariant set of $S(t)$ if $S(t)\Sigma = \Sigma, \forall t \geq 0$. An invariant set Σ is an attractor of $S(t)$ if Σ is compact, and there exists a neighborhood $U \subset X$ of Σ such that for any $\varphi \in U$ we have

$$\inf_{v \in \Sigma} \|S(t)\varphi - v\|_X \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty.$$

In this case, we say that Σ attracts U. Especially, if Σ attracts any bounded set of X, then Σ is called a global attractor of $S(t)$.

For a set $D \subset X$, we define the ω-limit set of D by

$$\omega(D) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)D},$$

where the closure is taken in the X-norm.

The following is a classical existence theorem of global attractors, which can be found in [8].

Theorem 2.1 Let $S(t) : X \to X$ be an semigroup of operator generated by (2.1). Suppose that

(i) $S(t)$ has a bounded absorbing set $B \subset X$, that is, for any bounded set $A \subset X$ there exists a time $t_A \geq 0$ such that $S(t)\varphi \in B, \forall \varphi \in A$ and $t > t_A$,

(ii) $S(t)$ is uniformly compact, i.e., for any bounded set $U \subset X$ and some $T > 0$ sufficiently large, the set $\overline{\bigcup_{t \geq T} S(t)U}$ is compact in X,

then, the ω-limit set $\mathcal{A} = \omega(B)$ of B is a global attractor of (2.1) and $\mathcal{A}$ is connected provided that B is connected.

Hereafter, we assume that the linear operator $L : X_1 \to X$ in (2.1) is a sectorial operator, which generates an analytic semigroup e^{tL}. It is well-known that for a sectorial operator L there exists a constant $\lambda > 0$ such that $L - \lambda I$ generates the fractional power operators $\mathcal{L}^\alpha$ and fractional order spaces X_α for $\alpha \in R^1$, where $\mathcal{L} = -(L - \lambda I)$. Without loss of generality, we assume that L generates the fractional power operator $\mathcal{L}^\alpha$ and fractional order space X_α as follows

$$\mathcal{L}^\alpha = (-L)^\alpha : \ X_\alpha \longrightarrow X, \quad \alpha \in R^1,$$

where $X_\alpha = D(\mathcal{L}^\alpha)$ is the domain of $\mathcal{L}^\alpha$, By the semigroup theory of linear operators (see [7]), for any $\beta > \alpha$,

$$X_\beta \subset X_\alpha \text{ is a compact inclusion.}$$

Thus, Theorem 2.1 can be equivalently stated as follows.

Theorem 2.2 Let $u(t, \varphi) = S(t)\varphi$ ($\varphi \in X, \ t \geq 0$) be a solution of (2.1), $S(t)$ be the semigroup of operator generated by (2.1). Let X_α be the fractional order space generated by L. If

(i) for some $\alpha \geq 0$ there is a bounded set $B \subset X_\alpha$, as $\varphi \in X_\alpha$ there exists $t_\varphi > 0$ such that

$$u(t, \varphi) \in B, \quad \forall \ t > t_\varphi,$$

(ii) there is a constant $\beta > \alpha$, for any bounded set $U \subset X_\beta$ there are $T > 0$ and $C > 0$ such that

$$\|u(t, \varphi)\|_{X_\beta} \leq C, \quad \forall\ t > T \text{ and } \varphi \in U,$$

then, (2.1) has a global attractor $\mathcal{A} \subset X_\alpha$ and $\mathcal{A}$ attracts any bounded set of X_α in the X_α-norm.

In addition, in Section 3 we will make use of the following properties of sectorial operators, which can be found in [7].

Theorem 2.3 Let $L : X_1 \to X$ be a sectorial operator, which generates an analytic semigroup $T(t) = \mathrm{e}^{tL}$. If all eigenvalues λ of L satisfy $\mathrm{Re}\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathcal{L}^\alpha$ ($\mathcal{L} = -L$) we have

(i) $T(t) : X \to X_\alpha$ is bounded for all $\alpha \in R^1$ and $t > 0$,

(ii) $T(t)\mathcal{L}^\alpha x = \mathcal{L}^\alpha T(t)x, \ \ \forall\ x \in X_\alpha$,

(iii) for each $t > 0$, $\mathcal{L}^\alpha T(t) : X \to X$ is bounded, and

$$\|\mathcal{L}^\alpha T(t)\| \leq C_\alpha t^{-\alpha} \mathrm{e}^{-\delta t},$$

for some $\delta > 0$, where $C_\alpha > 0$ is a constant only depending on α,

(iv) the X_α-norm can be computed by

$$\|x\|_{X_\alpha} = \|\mathcal{L}^\alpha x\|_X. \tag{2.2}$$

3 Existence of Global Attractors for System (1.1)

As mentioned in Section 1, many results on the existence of global attractors for problem (1.1) and for general reaction–diffusion equations in $L^2(\Omega)$ and $H_0^1(\Omega)$ were established in the literature (cf. [5, 8]). In this section, we shall study the existence of global attractors for problem (1.1) in Sobolev spaces $H^k(\Omega)$ for all $k \geq 0$. Our goal is to extend those classical results from the cases $k = [0, 1]$ to the general case $k \in [0, \infty)$.

Let H, H_1 be same as in (1.2). Define the operators $L : H_1 \to H$ and $G : H_1 \to H$ by

$$Lu = \Delta u, \quad Gu = g(u) = \lambda u - u^3. \tag{3.1}$$

Then the problem (1.1) can be rewritten in the abstract form (2.1).

It is well known that the above linear operator L is a sectorial operator and $L = \Delta$. Let $\{\lambda_j\}$ denote the eigenvalues of the Laplacian Δ in $H_0^1(\Omega)$, then (see [8])

$$-\infty < \cdots < \lambda_n < \lambda_{n-1} < \cdots < \lambda_2 < \lambda_1 < 0.$$

It's easy to check that all the requirements on the spaces and the operator L in Theorem 2.1 and Theorem 2.3 are satisfied. To infer the desired results from Theorem 2.2, we need the following two lemmas.

Lemma 3.1 For each $\sigma \geq 0$, the solution of problem (1.1) with any $\varphi \in U \subset H_\sigma$ is uniformly bounded for all $t \geq 0$. Here, U is any bounded subset of H_σ.

Proof From [8] (see Section 1 for the precise statement) we know that problem (1.1) has a unique strong solution if $\varphi \in H^1$. Hence, its solution $u(t, \varphi)$ can be written analytically

$$u(t, \varphi) = \mathrm{e}^{tL}\varphi + \int_0^t \mathrm{e}^{(t-\tau)L} G(u) \mathrm{d}\tau. \tag{3.2}$$

It follows from the existence of global attractors in $H_0^1(\Omega)$ for the semigroup $\{S(t)\}_{t\geq 0}$ associated with problem the (1.1) that for any bounded set $U\subset H_{\frac{1}{2}}$, there exists a constant $C>0$ such that

$$\|u(t,\varphi)\|_{H_{\frac{1}{2}}}\leq C\quad \forall\ t\geq 0,\quad \varphi\in U\subset H_{\frac{1}{2}}. \tag{3.3}$$

It is clear that $g:H_{\frac{1}{2}}\to H$ is bounded. In fact, $H_{\frac{1}{2}}=H^1\hookrightarrow L^6(\Omega)$ and

$$\begin{aligned}\|g(u)\|_H^2&=\int_\Omega |g(u)|^2\mathrm{d}x=\int_\Omega|\lambda u-u^3|^2\mathrm{d}x\\&\leq\int_\Omega(|\lambda|u^2+2|\lambda|u^4+u^6)\mathrm{d}x\\&\leq C\big(\|u\|^2_{H_{\frac{1}{2}}}+\|u\|^4_{H_{\frac{1}{2}}}+\|u\|^6_{H_{\frac{1}{2}}}\big)\leq C.\end{aligned}$$

We divide the remainder of proof into three steps.

Step 1 In this step we prove that for any bounded set $U\subset H_\sigma$ ($\frac{1}{2}<\sigma<1$), there exists $C>0$, which only depends on φ, such that

$$\|u(t,\varphi)\|_{H_\sigma}\leq C\qquad\forall\ t\geq 0,\ \varphi\in U. \tag{3.4}$$

By (3.2) we have

$$\begin{aligned}\|u(t,\varphi)\|_{H_\sigma}&\leq\|\mathrm{e}^{tL}\varphi\|_{H_\sigma}+\Big\|\int_0^t\mathrm{e}^{(t-\tau)L}g(u)\mathrm{d}\tau\Big\|_{H_\sigma}\\&\leq\|\mathrm{e}^{tL}\varphi\|_{H_\sigma}+\int_0^t\|(-L)^\sigma\mathrm{e}^{(t-\tau)L}\|\cdot\|g(u)\|_H\mathrm{d}\tau.\end{aligned} \tag{3.5}$$

Since $g:H_\sigma$ ($\frac{1}{2}<\sigma<1$) $\to H$ is bounded, it follows from (3.5) and Theorem 2.3 that

$$\|u(t,\varphi)\|_{H_\sigma}\leq\|\varphi\|_{H_\sigma}+C\int_0^t\tau^{-\beta}\mathrm{e}^{-\delta\tau}\mathrm{d}\tau\leq C\qquad\forall\ t\geq 0,\ \varphi\in U\subset H_\sigma,$$

where $\delta=-\lambda_1>0,\beta=\sigma$ ($0<\beta<1$). Hence, (3.4) holds with $C>0$ being a constant which only depends on φ.

Step 2 The task of this step is to show that for any bounded set $U\subset H_\sigma$ ($1\leq\sigma<\frac{3}{2}$), there exists $C>0$ such that

$$\|u(t,\varphi)\|_{H_\sigma}\leq C\qquad\forall\ t\geq 0,\ \varphi\in U,\ \sigma<\frac{3}{2}. \tag{3.6}$$

First, we note that $g:H_\sigma$ ($\frac{3}{4}<\sigma<1$) $\to H_{\frac{1}{2}}$ is bounded. In fact, by the embedding theorem for fractional order spaces (cf. [7]) there holds

$$H_\sigma=H^{2\sigma}\hookrightarrow C^0(\Omega)\cap H^1(\Omega)\ \text{ as }\ \sigma>\frac{3}{4}.$$

Hence,

$$\begin{aligned}\|g(u)\|^2_{H_{\frac{1}{2}}}&=\int_\Omega|\nabla g(u)|^2\mathrm{d}x=\int_\Omega|\nabla(\lambda u-u^3)|^2\mathrm{d}x\\&\leq\int_\Omega(|\lambda\nabla u-3u^2\nabla u|^2\mathrm{d}x\leq\int_\Omega(\lambda-3u^2)^2|\nabla u|^2\mathrm{d}x\\&\leq\sup_{\bar\Omega}\big(|\lambda|+3u^2\big)^2\int_\Omega|\nabla u|^2\mathrm{d}x\leq C\|u\|^2_{H_\sigma}\leq C,\end{aligned} \tag{3.7}$$

which implies $g : H_\sigma(\frac{3}{4} < \sigma < 1) \to H_{\frac{1}{2}}$ is bounded, so does $g : H_\sigma(1 \leq \sigma < \frac{3}{2}) \to H_{\frac{1}{2}}$.

Next, by (3.2) and (2.2) we have

$$\begin{aligned}\|u(t,\varphi)\|_{H_\sigma} &= \Big\|\mathrm{e}^{tL}\varphi + \int_0^t \mathrm{e}^{(t-\tau)L} g(u)\mathrm{d}\tau\Big\|_{H_\sigma} \\ &\leq \|\varphi\|_{H_\sigma} + \int_0^t \|(-L)^\sigma \mathrm{e}^{(t-\tau)L} g(u)\|_H \mathrm{d}\tau \\ &\leq \|\varphi\|_{H_\sigma} + \int_0^t \|(-L)^{\sigma-\frac{1}{2}} \mathrm{e}^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{\frac{1}{2}}} \mathrm{d}\tau. \end{aligned} \tag{3.8}$$

It then follows from (3.8) and Theorem 2.3 that

$$\|u(t,\varphi)\|_{H_\sigma} \leq \|\varphi\|_{H_\sigma} + C\int_0^t \tau^{-\beta}\mathrm{e}^{-\delta\tau}\mathrm{d}\tau \leq C \qquad \forall\ t \geq 0,\ \varphi \in U \subset H_\sigma,$$

where $\delta = -\lambda_1 > 0,\ \beta = \sigma - \frac{1}{2}\ (0 < \beta < 1)$. Hence, (3.6) is verified.

Step 3 Finally, we prove that for any bounded set $U \subset H_\sigma$ $(\sigma \geq 0)$ there is a constant $C > 0$ such that

$$\|u(t,\varphi)\|_{H_\sigma} \leq C \qquad \forall\ t \geq 0,\ \varphi \in U \subset H_\sigma,\ \sigma \geq 0. \tag{3.9}$$

Using the result of Step 2 it suffices to show (3.9) for the case $\sigma \geq \frac{3}{2}$. Again, by the embedding theorem of fractional order spaces (cf. [7]) there holds

$$H_1 = H^2 \hookrightarrow W^{1,6}, \quad H_\alpha \subset C^0(\Omega) \cap H^2(\Omega) \quad \forall\ 1 < \sigma < \frac{3}{2}. \tag{3.10}$$

Hence,

$$\begin{aligned}\|g(u)\|_{H_1}^2 &= \int_\Omega |\Delta g(u)|^2 \mathrm{d}x = \int_\Omega |\Delta(\lambda u - u^3)|^2 \mathrm{d}x \\ &= \int_\Omega |\lambda\Delta u| - 6u(\nabla u)^2 - 3u^2\Delta u|^2 \mathrm{d}x \\ &\leq C\int_\Omega ((\Delta u)^2 + u^2(\nabla u)^4 + u^4(\Delta u)^2)\mathrm{d}x \\ &\leq C\Big(\int_\Omega (\Delta u)^2\mathrm{d}x + \sup_{\bar\Omega}|u|^2\int_\Omega (\nabla u)^4\mathrm{d}x + \sup_{\bar\Omega}|u|^4\int_\Omega(\Delta u)^2\mathrm{d}x\Big) \\ &\leq C(\|u\|_{H_1}^2 + \|u\|_{H_{\frac{1}{2}}}^4 + \|u\|_{H_1}^2) \\ &\leq C(\|u\|_\sigma^2 + \|u\|_\sigma^4 + \|u\|_\sigma^2 \leq C, \end{aligned} \tag{3.11}$$

which implies that $g : H_\sigma \to H_1$ is bounded for $1 < \sigma < \frac{3}{2}$, so does $g : H_\sigma \to H_1$ for $\frac{3}{2} \leq \sigma < 2$. Therefore, by (3.2), Theorem 2.3 and (3.11) we get

$$\begin{aligned}\|u(t,\varphi)\|_{H_\sigma} &\leq \|\mathrm{e}^{tL}\varphi\|_{H_\sigma} + \int_0^t \|(-L)^{\sigma-1}\mathrm{e}^{(t-\tau)L}\| \cdot \|g(u)\|_{H_1}\mathrm{d}\tau \\ &\leq \|\mathrm{e}^{tL}\varphi\|_{H_\sigma} + C\int_0^t \tau^{-\beta}\mathrm{e}^{-\delta\tau}\mathrm{d}\tau \leq C, \end{aligned} \tag{3.12}$$

where $\delta = -\lambda_1 > 0,\ \ \beta = \sigma - 1\ (0 < \beta < 1)$. Hence, (3.9) holds for $\frac{3}{2} \leq \sigma < 2$. Similarly, it follows from (3.12) that

$$\|u(t,\varphi)\|_{H_\sigma} \leq C \qquad \forall\ t \geq 0,\ \varphi \in U \subset H_\sigma,\ \ \sigma \geq 2. \tag{3.13}$$

Repeating the above argument, we can prove that (3.9) holds for all $\sigma \geq 0$. The proof of Lemma 3.1 is now completed.

Lemma 3.2 For any $\sigma \geq 0$, problem (1.1) has a bounded absorbing set in H_σ.

Proof For the case $\sigma = \frac{1}{2}$, the proof of the assertion can be found in [8], also see Section 1 for the precise statement.

We now show that problem (1.1) has a bounded absorbing set in H_k for any $k \geq \frac{1}{2}$, that is, for any bounded set $U \subset H_k$ there exist constants $T > 0$ and $C > 0$ independent on φ such that

$$\|u(t,\varphi)\|_{H_k} \leq C \qquad \forall\ t \geq T,\ \varphi \in U \subset H_k. \tag{3.14}$$

From [8] we know that (3.14) holds for $k = \frac{1}{2}$. By (3.2) we have

$$u(t,\varphi) = \mathrm{e}^{(t-T)L}u(T,\varphi) + \int_T^t \mathrm{e}^{(t-\tau)L}g(u)\mathrm{d}\tau. \tag{3.15}$$

Let $B \subset H_{\frac{1}{2}}$ be the bounded absorbing set of (1.1), and $T_0 > 0$ be a number such that

$$u(t,\varphi) \in B \qquad \forall\ t > T_0,\ \varphi \in U \subset H_k,\ k \geq \frac{1}{2}. \tag{3.16}$$

Also, it is well-known that (cf.[7])

$$\|\mathrm{e}^{tL}\| \leq C\mathrm{e}^{-t\delta},$$

where $\delta = -\lambda_1 > 0$ is the smallest eigenvalue of the negative Laplacian $-\Delta$ in $H_0^1(\Omega)$. Hence, for any given $T > 0$ and $\varphi \in U \subset H_k$ $(k \geq \frac{1}{2})$ we have

$$\|\mathrm{e}^{(t-T)}u(T,\varphi)\|_{H_k} \longrightarrow 0 \ \text{ as } t \to \infty. \tag{3.17}$$

It follows from (3.15), (3.16), and Theorem 2.3 that for any $\frac{1}{2} \leq k < 1$

$$\begin{aligned}\|u(t,\varphi)\|_{H_k} &\leq \|\mathrm{e}^{(t-T_0)L}u(T_0,\varphi)\|_{H_k} + \int_{T_0}^t \|(-L)^k\mathrm{e}^{(t-\tau)L}g(u)\|_H\mathrm{d}\tau \\ &\leq \|\mathrm{e}^{(t-T_0)L}u(T_0,\varphi)\|_{H_k} + C\int_0^{t-T_0} \tau^{-k}\mathrm{e}^{-\delta\tau}\mathrm{d}\tau,\end{aligned} \tag{3.18}$$

where $C > 0$ is a constant independent on φ.

Then, it follows from (3.17) and (3.18) that (3.14) holds for all $\frac{1}{2} \leq k < 1$. By an iteration argument, we conclude that (3.14) holds for all $k \geq \frac{1}{2}$. The proof is completed.

Thanks to the above two lemmas, an application of Theorem 2.2 immediately yields the following main result of this paper, which says that problem (1.1) has a global attractor in the kth-order space H_k for all $k \geq 0$.

Theorem 3.1 For any $\sigma \geq 0$, the initial-boundary problem (1.1) has a global attractor $\mathcal{A}$ in H_σ, and $\mathcal{A}$ attracts any bounded set of H_σ in the H_σ-norm.

Remark 3.1 The attractors $\mathcal{A}_\alpha \subset H_\alpha$ in Theorem 3.1 are the same for all $\alpha \geq 0$, i.e., $\mathcal{A}_\alpha = \mathcal{A}$ for all $\alpha \geq 0$. Hence, $\mathcal{A} \subset C^\infty(\Omega)$. Theorem 3.1 implies that for any $\varphi \in H$, the solution $u(t,\varphi)$ of (1.1) satisfies that $\lim\limits_{t\to\infty}\inf\limits_{v\in\mathcal{A}} \|u(t,\varphi) - v\|_{C^k} = 0 \ \ \forall\ k \geq 1$.

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