Acta Mathematica Sinica, English Series
Apr., 2015, Vol. 31, No. 4, pp. 715–730
Published online: March 15, 2015
DOI: 10.1007/s10114-015-4356-8
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Isomorphisms of Finite Semi-Cayley Graphs

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Abstract Let G be a finite group. A Cayley graph over G is a simple graph whose automorphism group has a regular subgroup isomorphic to G. A Cayley graph is called a CI-graph (Cayley isomorphism) if its isomorphic images are induced by automorphisms of G. A well-known result of Babai states that a Cayley graph Γ of G is a CI-graph if and only if all regular subgroups of $\mathrm{Aut}(\Gamma)$ isomorphic to G are conjugate in $\mathrm{Aut}(\Gamma)$. A semi-Cayley graph (also called bi-Cayley graph by some authors) over G is a simple graph whose automorphism group has a semiregular subgroup isomorphic to G with two orbits (of equal size). In this paper, we introduce the concept of SCI-*graph* (semi-Cayley isomorphism) and prove a Babai type theorem for semi-Cayley graphs. We prove that every semi-Cayley graph of a finite group G is an SCI-graph if and only if G is cyclic of order 3. Also, we study the isomorphism problem of a special class of semi-Cayley graphs.

Keywords Semi-Cayley graph, Cayley graph, CI-graph, semiregular subgroup

MR(2010) Subject Classification 20B25, 05C25, 05C60

1 Introduction and Results

In this paper, a graph means a finite, undirected and simple graph unless specified otherwise. For a graph Γ, we use $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set and the full automorphism group, respectively. Γ is called a vertex-transitive graph if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. For a group G and $g \in G$, the map $\tau_g : G \to G$ with the rule $x^{\tau_g} = g^{-1}xg$ is the inner automorphism of G induced by g. For the most part, our notation and terminology is standard and mainly taken from [4] (for permutation group theory) and [5] (for graph theory). We refer the reader to [4, 5] for the concepts not defined here.

Let S be a subset of a group G not containing the identity element of G. Recall that the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ of G with respect to S is the graph with vertex set G, where (x, y) is a directed edge if and only if $yx^{-1} \in S$. Also Γ is undirected if and only if $S = S^{-1}$. Each Cayley graph Γ of G admits $R(G)$, the right regular representation of G, as a subgroup of $\mathrm{Aut}(\Gamma)$. By a well-known result of Sabidussi (see for example [11, Proposition 1.1]), a graph Γ is a Cayley graph of a group G if and only if $\mathrm{Aut}(\Gamma)$ contains a regular subgroup (on $V(\Gamma)$) which is isomorphic to G. Resmini and Jungnickel [14], in analogous to Sabidussi's result, introduced the concept of *semi-Cayley graphs* in 1992, later called *bi-Cayley graphs* in [10] by Kovács and his co-authors. In this paper, we follow [14] to use the term semi-Cayley. A graph Γ is said to be a *semi-Cayley graph* over a group G if $\mathrm{Aut}(\Gamma)$ has a semiregular subgroup isomorphic

Received June 6, 2014, accepted October 8, 2014

to G with two orbits (of equal size). Resmini and Jungnickel [14] determined the structure representation of semi-Cayley graphs. Let Γ be a semi-Cayley graph over a group G. Then there exist subsets R, L and S of G such that $R = R^{-1}$, $L = L^{-1}$ and $R \cup L$ does not contain the identity element of G such that $\Gamma \cong \mathrm{SC}(G; R, L, S)$, where $\mathrm{SC}(G; R, L, S)$ is an undirected graph with vertex set $G \times \{1, 2\}$ and its edge set consists of three sets:

$$\begin{aligned} &\{\{(x,1),(y,1)\} \mid yx^{-1} \in R\} \quad \text{(right edges)},\\ &\{\{(x,2),(y,2)\} \mid yx^{-1} \in L\} \quad \text{(left edges)},\\ &\{\{(x,1),(y,2)\} \mid yx^{-1} \in S\} \quad \text{(spoke edges)}. \end{aligned}$$

Furthermore, $R_G := \{\rho_g \mid g \in G\}$, where $\rho_g : G \times \{1,2\} \to G \times \{1,2\}$ and $(x,i)^{\rho_g} = (xg, i)$, $i = 1, 2$, is a semiregular subgroup of $\mathrm{Aut}(\mathrm{SC}(G; R, L, S))$ isomorphic to G with two orbits $G \times \{1\}$ and $G \times \{2\}$. Also R_G acts regularly on $G \times \{1\}$ and $G \times \{2\}$.

If $|R| = |L| = s$, s is a non-negative integer, then the semi-Cayley graph $\mathrm{SC}(G; R, L, S)$ is said to be an *s-type semi-Cayley graph*, and if G is abelian, then $\mathrm{SC}(G; R, L, S)$ is simply called an *abelian semi-Cayley graph*. If $R = L = \emptyset$, then $\mathrm{BCay}(G, S)$ will be written for $\mathrm{SC}(G; \emptyset, \emptyset, S)$, by the notation of [15]. Note that an s-type semi-Cayley graph $\mathrm{SC}(G; R, L, S)$ is an $(s + |S|)$-regular graph and every 0-type semi-Cayley graph $\mathrm{BCay}(G, S)$ is a bipartite graph with partitions $G \times \{1\}$ and $G \times \{2\}$. We follow [15] and call it the bi-Cayley graph of G with respect to S. Note that in [10] bi-Cayley graph means semi-Cayley graph.

Recall that a Cayley graph $\mathrm{Cay}(G, S)$ is called a CI-graph if for any subset T of G whenever $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$, there exists an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $T = S^\alpha$. For a positive integer m, the group G is said to be an m-CI-group if all Cayley graphs $\mathrm{Cay}(G, S)$ of G where $|S| \le m$, are CI-graphs.

Recently, some authors studied the isomorphisms of bi-Cayley graphs (0-type semi-Cayley graphs), see [6–9] and [15], and they did not consider the general case. In this paper, we consider the general semi-Cayley graphs and study their isomorphisms. Given an arbitrary semi-Cayley graph $\Gamma = \mathrm{SC}(G; R, L, S)$, we define three natural isomorphisms arising from automorphisms of G, elements of G and swapping R and L with replacing S by S^{-1}. In fact, we can easily prove that (with the convention that for any automorphism σ of G and every two elements $x, y \in G$, $T = \emptyset$ if and only if $xT^\sigma y = \emptyset$):

Lemma 1.1 *Let $\Gamma = \mathrm{SC}(G; R, L, S)$. Then for each $\alpha \in \mathrm{Aut}(G)$ and $g, h \in G$, we have*

$$\begin{aligned} &\mathrm{SC}(G; R, L, S) \overset{\varphi_\alpha}{\cong} \mathrm{SC}(G; R^\alpha, L^\alpha, S^\alpha),\\ &\mathrm{SC}(G; R, L, S) \overset{\varphi_{g,h}}{\cong} \mathrm{SC}(G; g^{-1}Rg, h^{-1}Lh, h^{-1}Sg),\\ &\mathrm{SC}(G; R, L, S) \overset{\xi}{\cong} \mathrm{SC}(G; L, R, S^{-1}), \end{aligned}$$

where

$$\begin{aligned} &\varphi_\alpha : (x,1) \mapsto (x^\alpha, 1), \quad (x,2) \mapsto (x^\alpha, 2),\\ &\varphi_{g,h} : (x,1) \mapsto (g^{-1}x, 1), \quad (x,2) \mapsto (h^{-1}x, 2),\\ &\xi : (x,1) \mapsto (x,2), \quad (x,2) \mapsto (x,1). \end{aligned}$$

Furthermore, for each $a \in G$, $\varphi_\alpha^{-1}\rho_a\varphi_\alpha = \rho_{a^\alpha}$, $\xi^{-1}\rho_a\xi = \rho_a$ *and* $\varphi_{g,h}^{-1}\rho_a\varphi_{g,h} = \rho_a$. $\square$

We call the three isomorphisms given in Lemma 1.1 and their compositions *semi-Cayley isomorphisms.* It is easy to see that there exists a semi-Cayley isomorphism between $\mathrm{SC}(G; R, L, S)$ and $\mathrm{SC}(G; R_1, L_1, S_1)$ if and only if there exist $\alpha \in \mathrm{Aut}(G)$ and $g, h \in G$ such that one of the following holds:

$$R_1 = g^{-1}R^\alpha g, \quad L_1 = h^{-1}L^\alpha h, \quad S_1 = h^{-1}S^\alpha g, \tag{1.1}$$

$$R_1 = g^{-1}L^\alpha g, \quad L_1 = h^{-1}R^\alpha h, \quad S_1 = h^{-1}(S^{-1})^\alpha g. \tag{1.2}$$

Equivalently, by applying the inner automorphism τ_g to the above equalities, we can see that, there exists a semi-Cayley isomorphism between $\mathrm{SC}(G; R, L, S)$ and $\mathrm{SC}(G; R_1, L_1, S_1)$ if and only if there exist $\sigma \in \mathrm{Aut}(G)$ and $x \in G$ such that one of the following holds:

$$R_1 = R^\sigma, \quad L_1 = xL^\sigma x^{-1}, \quad S_1 = xS^\sigma, \tag{1.3}$$

$$R_1 = L^\sigma, \quad L_1 = xR^\sigma x^{-1}, \quad S_1 = x(S^{-1})^\sigma. \tag{1.4}$$

Now similar to the concepts of CI-graph and CI-group, we define SCI-graph and SCI-group:

Definition 1.2 *Let* $\Gamma = \mathrm{SC}(G; R, L, S)$ *be a semi-Cayley graph over a group* G.

(i) Γ *is called an* SCI-*graph* (SCI *stands for semi-Cayley isomorphism*), *if for any semi-Cayley graph* $\Sigma = \mathrm{SC}(G; R_1, L_1, S_1)$, *whenever* $\mathrm{SC}(G, R, L, S) \cong \mathrm{SC}(G, R_1, L_1, S_1)$, *there exists a semi-Cayley isomorphism between* Γ *and* Σ.

(ii) G *is called an* SCI-*group, if all semi-Cayley graphs of* G *are* SCI-*graphs.*

By the convention proceeding Lemma 1.1, the empty graph $\mathrm{SC}(G; \emptyset, \emptyset, \emptyset) \cong 2|G|K_1$ is an SCI-graph. Also it is clear that its complement, the complete graph $\mathrm{SC}(G; G\setminus\{1\}, G\setminus\{1\}, G) \cong K_{2|G|}$ is also an SCI-graph.

Babai [2, Lemma 3.1] proved that $\Gamma = \mathrm{Cay}(G, S)$ is a CI-graph if and only if every regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to $R(G)$ is conjugate to $R(G)$ in $\mathrm{Aut}(\Gamma)$. We prove a similar result for semi-Cayley graphs.

Theorem A *Let* Γ *be a semi-Cayley graph over a group* G. *Then the following are equivalent*:

(1) Γ *is an* SCI-*graph.*

(2) *Given a permutation* $\varphi \in \mathrm{Sym}(V(\Gamma))$ *with* $\varphi^{-1}R_G\varphi \le \mathrm{Aut}(\Gamma)$, R_G *and* $\varphi^{-1}R_G\varphi$ *are conjugate in* $\mathrm{Aut}(\Gamma)$.

(3) *Every semiregular subgroup of* $\mathrm{Aut}(\Gamma)$ *with two orbits and isomorphic to* R_G *is conjugate to* R_G *in* $\mathrm{Aut}(\Gamma)$.

In the following proposition, we establish a relation between SCI-groups and CI-groups.

Proposition 1.3 *Every* SCI-*group is a* CI-*group.*

Proof Let G be an SCI-group, $S = S^{-1}$ and $T = T^{-1}$ be two arbitrary subsets of $G \setminus \{1\}$ such that $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$. Then $\mathrm{SC}(G; S, S, \emptyset) \cong 2\mathrm{Cay}(G, S) \cong 2\mathrm{Cay}(G, T) \cong \mathrm{SC}(G; T, T, \emptyset)$. Since G is an SCI-group, there exist $\sigma \in \mathrm{Aut}(G)$ and $x \in G$ such that $T = S^\sigma = xS^\sigma x^{-1}$, which means that G is a CI-group. $\square$

Remark 1.4 In general, the converse of Proposition 1.3 is not true. For example, $\mathbb{Z}_2$ is a CI-group but it is not an SCI-group. To see this, consider the 4-cycle C_4. Then $\mathrm{SC}(\mathbb{Z}_2, R_1, L_1, S_1) \cong$

$\mathrm{SC}(\mathbb{Z}_2, R_2, L_2, S_2) \cong C_4$, where $\mathbb{Z}_2 = \langle a\rangle$, $R_1 = L_1 = \{a\}$, $S_1 = \{1\}$, $R_2 = L_2 = \emptyset$ and $S_2 = \mathbb{Z}_2$. Now it is clear that there is not any semi-Cayley isomorphism between these two graphs which means that $\mathbb{Z}_2$ is not an SCI-group.

Next we consider SCI-groups and obtain a characterization of $\mathbb{Z}_3$:

Theorem B *Let $G \neq 1$ be a finite group. Then G is an* SCI-*group if and only if $G \cong \mathbb{Z}_3$.*

Then we intend to study the isomorphism problem of a special class of semi-Cayley graphs, namely bi-Cayley graphs. Let G be a group and S be a non-empty subset of G. A bi-Cayley graph $\mathrm{BCay}(G, S)$ is called a BCI-graph if for any bi-Cayley graph $\mathrm{BCay}(G, T)$, whenever $\mathrm{BCay}(G, S) \cong \mathrm{BCay}(G, T)$ we have $T = gS^\alpha$ for some $g \in G$ and $\alpha \in \mathrm{Aut}(G)$. A group G is called a (connected) BCI-group, if all (connected) bi-Cayley graphs of G are BCI-graphs. Also G is called an m-BCI-group, if all bi-Cayley graphs of G of valency at most m are BCI-graphs (see [7, Definition 1]). In [7], the authors defined the concept of BCI-graph for bi-Cayley graphs and proved a necessary and sufficient condition for a finite group being a 2-BCI-group. Recently some authors studied the isomorphisms of bi-Cayley graphs. For example in [7], it is proved that only finite simple non-abelian 3-BCI group is A_5, the alternating group on five symbols. Also in [9] isomorphisms of connected bi-Cayley graphs of cyclic groups, with valency 4 are discussed. In [9, Corollary 2.7], a Babai type theorem for bi-Cayley graphs of finite cyclic groups is proved: a connected bi-Cayley graph $\Gamma = \mathrm{BCay}(\mathbb{Z}_n, S)$ is a BCI-graph if and only if every semiregular subgroup of A with the same orbits to that of R_G is conjugate to R_G in $\mathrm{Aut}(\Gamma)$, where $\mathbb{Z}_n$ is the cyclic group of order n. We extend this result to arbitrary groups:

Theorem C *Let $\Gamma = \mathrm{BCay}(G, S)$. Then the following are equivalent.*

(1) Γ *is a* BCI-*graph.*

(2) *For each $\varphi \in \mathrm{Sym}(V(\Gamma))$ where $\{G \times \{1\}, G \times \{2\}\}$ is φ-invariant and $\varphi^{-1}R_G\varphi \leq \mathrm{Aut}(\Gamma)$, R_G and $\varphi^{-1}R_G\varphi$ are conjugate in $\mathrm{Aut}(\Gamma)$ and $S^{-1} = gS^\alpha$ for some $g \in G$ and $\alpha \in \mathrm{Aut}(G)$.*

(3) *Every semiregular subgroup of $\mathrm{Aut}(\Gamma)$ with the same orbits of R_G which is isomorphic to R_G is conjugate to R_G in $\mathrm{Aut}(\Gamma)$ and $S^{-1} = gS^\alpha$ for some $g \in G$ and $\alpha \in \mathrm{Aut}(G)$.*

Finally, in the last section, we construct some BCI and non-BCI groups. In particular, we show that $\mathbb{Z}_p$, p a prime, and $\mathbb{Z}_9$ are BCI-groups. Also we show that each Sylow p-subgroup, $p > 3$ a prime, of a BCI-group is elementary abelian.

2 Proof of Theorem A

A graph Γ is said to be an n-Cayley graph over a group G, if its automorphism group has a semiregular subgroup isomorphic to G with n orbits (of equal size). Equivalently, there exist n^2 subsets T_{ij}, $1 \leq i, j \leq n$, of G such that vertex set of Γ is $G \times \{1, \ldots, n\}$ and $((x, i), (y, j)) \in E(\Gamma)$ if and only if $yx^{-1} \in T_{ij}$. Furthermore, Γ is undirected if and only if $T_{ij} = T_{ji}^{-1}$ for all $i, j \in \{1, \ldots, n\}$. Γ has no loop if and only if $T_{ii} \subseteq G\backslash\{1\}$, for all $i \in \{1, \ldots, n\}$. Also $R_G = \{\rho_g \mid g \in G\}$, where $(x, i)^{\rho_g} = (xg, i)$ for all $x \in G$ and $i = 1, \ldots, n$ is a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G with n orbits $G \times \{i\}$, $i = 1, \ldots, n$ (for more details see [1]). Clearly, every semi-Cayley graph is an undirected 2-Cayley graph.

In the following lemma, we characterize the elements of the normalizer of R_G in $\mathrm{Sym}(V(\Gamma))$.

Lemma 2.1 *Let Γ be an n-Cayley graph of a group G, $n \geq 2$. Then every element of $N_{\mathrm{Sym}(V(\Gamma))}(R_G)$ is of the form $\psi : V(\Gamma) \to V(\Gamma)$, where $(x, j)^\psi = (g_j x^\sigma g, j^\theta)$ for some $g_1 = 1, g_2, \ldots, g_n \in G$, $g \in G$, $\sigma \in \mathrm{Aut}(G)$ and $\theta \in S_n$.*

Proof Let $V = V(\Gamma)$ and $M = N_{\mathrm{Sym}(V)}(R_G)$. First we find the elements of $M_{(1,1)}$, the stabilizer of $(1, 1)$ in M. We claim that every element of $M_{(1,1)}$ is of the form

$$\begin{aligned}\alpha : V &\to V,\\ (x, j) &\mapsto (g_j x^\sigma, j^\pi)\end{aligned}$$

for some $g_1 = 1, g_2, \ldots, g_n \in G$, $\sigma \in \mathrm{Aut}(G)$ and $\pi \in S_n$ which fixes 1.

Let $\alpha \in M$ and $(1, 1)^\alpha = (1, 1)$. Since $\alpha^{-1} R_G \alpha = R_G$, for every $g \in G$, there exists a unique $g' \in G$ such that $\alpha^{-1}\rho_g\alpha = \rho_{g'}$. So there exists $\sigma \in \mathrm{Sym}(G)$ such that for every $g \in G$, $\alpha^{-1}\rho_g\alpha = \rho_{g^\sigma}$. Thus, for all $g \in G$, $(g, 1)^\alpha = (1, 1)^{\alpha^{-1}\rho_g\alpha} = (1, 1)^{\rho_{g^\sigma}} = (g^\sigma, 1)$, and $((g_1g_2)^\sigma, 1) = (g_1g_2, 1)^\alpha = (1, 1)^{\alpha^{-1}\rho_{g_1}\rho_{g_2}\alpha} = (1, 1)^{\rho_{g_1^\sigma}\rho_{g_2^\sigma}} = (g_1^\sigma g_2^\sigma, 1)$ for all $g_1, g_2 \in G$, and so $(g_1g_2)^\sigma = g_1^\sigma g_2^\sigma$. Consequently, $\sigma \in \mathrm{Aut}(G)$. Now for any $1 \leq j \leq n$, there exist $g_j \in G$ and $1 \leq l_j \leq n$ such that $(1, j)^\alpha = (g_j, l_j)$. Clearly, $g_1 = 1$ and $l_1 = 1$. Therefore, there exists $\pi \in S_n$ which fixes 1 and $(1, j)^\alpha = (g_j, j^\pi)$. So for each $(x, j) \in V$, $(x, j)^\alpha = (1, j)^{\rho_x\alpha} = (1, j)^{\alpha\rho_{x^\sigma}} = (g_jx^\sigma, j^\pi)$.

Conversely, let $\varphi : V \to V$ be a map and suppose that there exist $g_j \in G$, $j = 1, \ldots, n$, with $g_1 = 1$, $\sigma \in \mathrm{Aut}(G)$ and $\pi \in S_n$ fixing 1 and $(x, j)^\varphi = (g_jx^\sigma, j^\pi)$ for all $(x, j) \in V$. We show that $\varphi \in M_{(1,1)}$. Clearly, $(1, 1)^\varphi = (1, 1)$. Let $x, y \in G$, $1 \leq i_1, i_2 \leq n$ and $(x, i_1)^\varphi = (y, i_2)^\varphi$. Then $i_1^\pi = i_2^\pi$ and $g_{i_1}x^\sigma = g_{i_2}y^\sigma$. So $i_1 = i_2$ and $x = y$. Thus, φ is injective. Let $(x, j) \in V$. Then $((g_{j^{\pi^{-1}}}^{-1}x)^{\sigma^{-1}}, j^{\pi^{-1}})^\varphi = (x, j)$. So φ is surjective and $\varphi \in \mathrm{Sym}(V)$. Finally, let $g \in G$ and $(x, j) \in V$. Then $(x, j)^{\varphi^{-1}\rho_g\varphi} = (xg^\sigma, j) = (x, j)^{\rho_{g^\sigma}}$. Thus for all $g \in G$, $\varphi^{-1}\rho_g\varphi = \rho_{g^\sigma}$. This means that $\varphi \in N_{\mathrm{Sym}(V)}(R_G)$ and thus $\varphi \in M_{(1,1)}$, as desired.

Now since $R_G \cong G$ is a semiregular subgroup of $\mathrm{Sym}(V)$, by [4, Exercise 4.2.7], $C_{\mathrm{Sym}(V)}(R_G)$, the centralizer of R_G in $\mathrm{Sym}(V)$, is a transitive subgroup of M. Therefore, M is also transitive. For a permutation $\tau \in S_n$, we define $\overline{\tau} : V \to V$ by $(x, j)^{\overline{\tau}} = (x, j^\tau)$, $j = 1, \ldots, n$. Clearly, $\overline{\tau} \in M$. Put $H = \{\overline{\tau} \mid \tau \in S_n\}$. Then $H \leq C_{\mathrm{Sym}(V)}(R_G)$. For $(x, i_1), (y, i_2) \in V$, if $i_1 \neq i_2$, then $(x, i_1)^{\rho_{x^{-1}y}\overline{(i_1 i_2)}} = (y, i_2)$ and if $i_1 = i_2$, then $(x, i_1)^{\rho_{x^{-1}y}} = (y, i_2)$. Hence, $R_GH = HR_G$ is transitive on V. Now by [4, Exercise 1.4.1] we have $M = M_{(1,1)}R_GH$. This completes the proof. $\square$

Lemma 2.2 *Let Γ be a semi-Cayley graph over G, $\Gamma \overset{\lambda}{\cong} \Sigma$ and $\lambda \in N_{\mathrm{Sym}(V(\Gamma))}(R_G)$. Then λ is a semi-Cayley isomorphism.*

Proof Let $\Gamma = \mathrm{SC}(G; R, L, S)$ and $\Sigma = \mathrm{SC}(G; R_1, L_1, S_1)$. Since $\lambda \in N_{\mathrm{Sym}(V(\Gamma))}(R_G)$, by Lemma 2.1, there exist $\alpha \in \mathrm{Aut}(G)$, and $g, h \in G$ such that $(x, 1)^\lambda = (x^\alpha h, 1)$ and $(x, 2)^\lambda = (gx^\alpha h, 2)$ or $(x, 1)^\lambda = (x^\alpha h, 2)$ and $(x, 2)^\lambda = (gx^\alpha h, 1)$ for all $x \in G$. In the first case, $\Gamma \overset{\lambda}{\cong} \Sigma$ implies that $R_1 = R^\alpha$, $L_1 = gL^\alpha g^{-1}$ and $S_1 = gS^\alpha$ and in the latter case, $R_1 = gL^\alpha g^{-1}$, $L_1 = R^\alpha$ and $S_1 = (S^{-1})^\alpha g^{-1}$. This completes the proof. $\square$

Corollary 2.3 *Let Γ and Σ be two semi-Cayley graphs, $\Gamma \overset{\psi}{\cong} \Sigma$ and $\psi R_G \psi^{-1} = \beta^{-1} R_G \beta$ for some $\beta \in \mathrm{Aut}(\Gamma)$. Then $\beta\psi$ is a semi-Cayley isomorphism.*

Proof Since $\Gamma \overset{\beta\psi}{\cong} \Sigma$ and $\beta\psi \in N_{\mathrm{Sym}(V(\Gamma))}(R_G)$, $\beta\psi$ is a bi-Cayley isomorphism by Lem -ma 2.2. $\square$

Lemma 2.4 *Let Γ be a semi-Cayley graph over G and $\Gamma \overset{\lambda}{\cong} \Sigma$. Then Σ is also a semi-Cayley graph over G. In particular, if Γ is an s-type semi-Cayley graph over G and $\{G\times\{1\}, G\times\{2\}\}$ is λ-invariant, then Σ is also an s-type semi-Cayley graph over G.*

Proof For the first part, consider the map $\psi : \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\Sigma)$, where $\sigma^\psi = \lambda^{-1}\sigma\lambda$ for all $\sigma \in \mathrm{Aut}(\Gamma)$. Then ψ is a group isomorphism and the image of the semiregular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G with two orbits is a semiregular subgroup of $\mathrm{Aut}(\Sigma)$ isomorphic to G with two orbits. This means that Σ is also a semi-Cayley graph over G.

Now let $\Gamma = \mathrm{SC}(G; R, L, S)$ be an s-type semi-Cayley graph and $\{(G\times\{1\})^\lambda, (G\times\{2\})^\lambda\} = \{G\times\{1\}, G\times\{2\}\}$. Since $|R| = |L| = s$ and λ is a graph isomorphism, Σ is also an s-type semi-Cayley graph over G. $\square$

Now we are in the position that prove Theorem A.

Proof of Theorem A Let us start with the part (1) $\Rightarrow$ (2). Let Γ be an SCI-graph, $\varphi \in \mathrm{Sym}(V(\Gamma))$ with $\varphi^{-1}R_G\varphi \le \mathrm{Aut}(\Gamma)$. Let Σ be a graph with $V(\Sigma) = V(\Gamma)$ and $E(\Sigma) = \{\{\alpha,\beta\} \mid \{\alpha^\varphi,\beta^\varphi\} \in E(\Gamma)\}$. Then $\Sigma \overset{\varphi}{\cong} \Gamma$. So Σ is also a semi-Cayley graph over G, by Lemma 2.4. Since Γ is an SCI-graph, there exists a semi-Cayley isomorphism ψ between Γ and Σ. Now $\Gamma \overset{\psi}{\cong} \Sigma \overset{\varphi}{\cong} \Gamma$. So $\Gamma \overset{\psi\varphi}{\cong} \Gamma$ and $\theta = \psi\varphi \in \mathrm{Aut}(\Gamma)$. On the other hand, by Lemma 1.1, $\psi^{-1}R_G\psi = R_G$. This implies that $\theta^{-1}R_G\theta = \varphi^{-1}\psi^{-1}R_G\psi\varphi = \varphi^{-1}R_G\varphi$, which means that R_G and $\varphi^{-1}R_G\varphi$ are conjugate in $\mathrm{Aut}(\Gamma)$.

Now we prove the part (2) $\Rightarrow$ (1). Suppose that for each $\varphi \in \mathrm{Sym}(V(\Gamma))$, R_G and $\varphi^{-1}R_G\varphi \le \mathrm{Aut}(\Gamma)$ are conjugate in $\mathrm{Aut}(\Gamma)$. Let $\Gamma \overset{\varphi}{\cong} \Sigma$, where Σ is a semi-Cayley graph over G. Since $V(\Gamma) = V(\Sigma) = G\times\{1,2\}$, $\varphi^{-1} \in \mathrm{Sym}(V(\Gamma))$ and $\varphi R_G\varphi^{-1} \le \mathrm{Aut}(\Gamma)$. Hence, by our assumption there exists some $\beta \in \mathrm{Aut}(\Gamma)$ such that $\varphi R_G\varphi^{-1} = \beta^{-1}R_G\beta$. Now by Corollary 2.3, $\Gamma \overset{\beta\varphi}{\cong} \Sigma$ is a semi-Cayley isomorphism, which means that Γ is an SCI-graph.

We proved that (1) and (2) are equivalent. Now we prove the part (1) $\Rightarrow$ (3). Let Γ be an SCI-graph and H be a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ with two orbits β_1^H and β_2^H and $R_G \overset{\varphi}{\cong} H$. Let us denote ρ_g^φ with φ_g. Define $\lambda : V(\Gamma) \to V(\Gamma)$ such that $(g,i)^\lambda = \beta_i^{\varphi_g}$, $i = 1, 2$ and $g \in G$. Then λ is well defined and clearly onto. Also the semiregularity of H implies that λ is 1-1. Hence, $\lambda \in \mathrm{Sym}(V(\Gamma))$. On the other hand, $\rho_g\lambda = \lambda\varphi_g$ for all $g \in G$, which implies that $\lambda^{-1}R_G\lambda = H \le \mathrm{Aut}(\Gamma)$. Since (1) implies (2), R_G and $\lambda^{-1}R_G\lambda$ are conjugate in $\mathrm{Aut}(\Gamma)$. Hence, H and R_G are conjugate in $\mathrm{Aut}(\Gamma)$.

Finally, we turn to the part (3) $\Rightarrow$ (1). Suppose that each semiregular subgroup of $\mathrm{Aut}(\Gamma)$ with two orbits and isomorphic to R_G is conjugate to R_G in $\mathrm{Aut}(\Gamma)$. Let $\Gamma \overset{\lambda}{\cong} \Sigma$, where Σ is a semi-Cayley graph over G. Since $V(\Gamma) = V(\Sigma)$, $\varphi : \mathrm{Aut}(\Sigma) \to \mathrm{Aut}(\Gamma)$ with $a^\varphi = \lambda a\lambda^{-1}$, $a \in \mathrm{Aut}(\Sigma)$, is a group isomorphism. Furthermore, R_G is a subgroup of $\mathrm{Aut}(\Sigma)$ and so $R_G \cong R_G^\varphi = \lambda R_G\lambda^{-1} \le \mathrm{Aut}(\Gamma)$. Since R_G acts on $V(\Gamma)$ semiregularly, $\lambda R_G\lambda^{-1}$ also is semiregular on $V(\Gamma)$. Since R_G has two orbits on $V(\Gamma)$ and for each $g \in G$, $|\mathrm{Fix}(\lambda\rho_g\lambda^{-1})| = |\mathrm{Fix}(\rho_g)|$, $\lambda R_G\lambda^{-1}$ has two orbits on $V(\Gamma)$, by the Cauchy–Frobenius lemma. Thus, by hypothesis, there exists $\beta \in \mathrm{Aut}(\Gamma)$ such that $\beta^{-1}R_G\beta = \lambda R_G \lambda^{-1}$. Now Corollary 2.3 implies that there exists a semi-Cayley isomorphism between Γ and Σ, which means that Γ is a SCI-graph. This completes the proof. $\square$

Note that since the automorphism group of a graph and its complement are same, complement of any semi-Cayley graph is again a semi-Cayley graph. Furthermore, by Theorem A, a graph Γ is an SCI-graph if and only if its complement Γ^c is an SCI-graph.

3 Proof of Theorem B

In order to prove Theorem B, we need some lemmas.

Lemma 3.1 *Let G be a finite abelian group. If one of the following holds, then G is not an* SCI-*group.*

(1) *There exist two distinct non-involution same order elements a and b in G such that $a \neq b, b^{-1}$.*

(2) *There exist two distinct involutions $a, b \in G$.*

Proof First we prove (1). Let $R = L = R_1 = \{a, a^{-1}\}$, $L_1 = \{b, b^{-1}\}$ and $S = S_1 = \emptyset$. Set $\Gamma := \mathrm{SC}(G; R, L, S)$ and $\Sigma = \mathrm{SC}(G; R_1, L_1, S_1)$. Then $\Gamma \cong 2\mathrm{Cay}(G, R) \cong 2|G : \langle R\rangle|\mathrm{Cay}(\langle R\rangle, R) \cong 2|G : \langle R\rangle|C_n$, where $n = o(a) = o(b)$. Also $\Sigma \cong \mathrm{Cay}(G, R_1) + \mathrm{Cay}(G, L_1) \cong |G : \langle R\rangle|\mathrm{Cay}(\langle R\rangle, R) + |G : \langle L_1\rangle|\mathrm{Cay}(\langle L_1\rangle, L_1)$. Since $\langle R\rangle \cong \langle L_1\rangle \cong \mathbb{Z}_n$, $\Sigma \cong 2|G : \langle R\rangle|C_n$, which implies that $\Gamma \cong \Sigma$. Suppose, contrary to our claim, that G is an SCI-group. Then there exist $\sigma \in \mathrm{Aut}(G)$ and $x \in G$ such that $R_1 = R^\sigma$ and $L_1 = xL^\sigma x^{-1} = L^\sigma$, since G is abelian. Hence, $R_1 = L_1$, a contradiction.

Now let (2) hold. It is enough to consider semi-Cayley graphs $\Gamma = \mathrm{SC}(G, \{a\}, \{a\}, \emptyset)$ and $\mathrm{SC}(G, \{a\}, \{b\}, \emptyset)$ and follow the proof of (1). $\square$

In the following lemma, we prove that the property of being SCI-groups is inherited by characteristic subgroups and quotient groups by characteristic subgroups. Note that a similar result is proved by Babai and Frankl for CI-groups, see for example [11, Lemma 8.2]. We denote the lexicographic product of Γ_1 and Γ_2 by $\Gamma_1[\Gamma_2]$, see [5] for the definition.

Lemma 3.2 *Let G be a finite* SCI-*group and H be a characteristic subgroup of G. Then H and G/H are* SCI-*groups.*

Proof First we prove that H is an SCI-group. Let $\mathrm{SC}(H; R_1, L_1, S_1) \cong \mathrm{SC}(H; R_2, L_2, S_2)$ for some subsets R_i, L_i, S_i, $i = 1, 2$ of H. Since the complement of a semi-Cayley graph is again a semi-Cayley graph and it is an SCI-graph if and only if the original graph is an SCI-graph, we may assume that $\mathrm{SC}(H; R_1, L_1, S_1)$ and $\mathrm{SC}(H; R_2, L_2, S_2)$ are connected and therefore $S_1, S_2 \neq \emptyset$. It is easy to see that $\mathrm{SC}(G; R_1, L_1, S_1) \cong |G : H|\mathrm{SC}(H; R_1, L_1, S_1)$ and $\mathrm{SC}(G; R_2, L_2, S_2) \cong |G : H|\mathrm{SC}(H; R_2, L_2, S_2)$. This implies that $\mathrm{SC}(G; R_1, L_1, S_1) \cong \mathrm{SC}(G; R_2, L_2, S_2)$. Since G is an SCI-group, there exist $\sigma \in \mathrm{Aut}(G)$ and $x \in G$ such that one of the following holds:

(a) $R_2 = R_1^\sigma$, $L_2 = xL_1^\sigma x^{-1}$, $S_2 = xS_1^\sigma$,

(b) $R_2 = L_1^\sigma$, $L_2 = xR_1^\sigma x^{-1}$, $S_2 = x(S_1^{-1})^\sigma$.

Let α be the restriction of σ to H. Then $\alpha \in \mathrm{Aut}(H)$. Since S_1, S_2 are non-empty subsets of H, $x \in H$. This means that H is an SCI-group.

Now we prove that G/H is an SCI-group. Let $\Gamma = \mathrm{SC}(G/H; R, L, S)$, $\pi : G \to G/H$ be the natural projection homomorphism and $\Gamma^{\pi^{-1}}$ be the semi-Cayley graph $\mathrm{SC}(G; R^{\pi^{-1}}, L^{\pi^{-1}}, S^{\pi^{-1}})$, where $X^{\pi^{-1}} = \{g \in G \mid g^\pi \in X\}$. We claim that $\Gamma^{\pi^{-1}} \cong \Gamma[mK_1]$, where $m = |H|$. Let R be a right transversal of H in G. Then for any $g \in G$, there exists a unique $r \in R$ such that $Hg = Hr$. Define

$$\varphi : G \times \{1,2\} \to (G/H \times \{1,2\}) \times H$$
$$(g, i) \mapsto ((g^\pi, i), gr^{-1}),$$

where $Hg = Hr$, $(r \in R)$. If $(g_1, i_1)^\varphi = (g_2, i_2)^\varphi$, then $((g_1^\pi, i_1), g_1r_1^{-1}) = ((g_2^\pi, i_2), g_2r_2^{-1})$, where $Hg_1 = Hr_1$ and $Hg_2 = Hr_2$ for some $r_1, r_2 \in R$. So $i_1 = i_2$, $g_1^\pi = g_2^\pi$ and $g_1r_1^{-1} = g_2r_2^{-1}$, which implies that $r_1 = r_2$. Hence, $g_1 = g_2$ and $i_1 = i_2$, i.e., φ is 1-1. Now let $((Hx, i), h) \in (G/H \times \{0, 1\}) \times H$. Then there exists $r \in R$ such that $Hx = Hr$. So $(hr, i)^\varphi = ((Hhr, i), hrr^{-1}) = ((Hx, i), h)$. Hence, φ is onto.

It is easy to see that φ preservers the adjacency of right, left and spoke edges. Thus, φ is a graph isomorphism and the claim is proved.

Now let $\Gamma \cong \Sigma$, where $\Sigma = \mathrm{SC}(G/H; R_1, L_1, S_1)$. Then

$$\Gamma^{\pi^{-1}} \cong \Gamma[mK_1] \cong \Sigma[mK_1] \cong \Sigma^{\pi^{-1}}.$$

Since G is an SCI-group, there exist $\sigma \in \mathrm{Aut}(G)$ and $x \in G$ such that one of the following holds:

$$R_1^{\pi^{-1}} = (R^{\pi^{-1}})^\sigma, \quad L_1^{\pi^{-1}} = x(L^{\pi^{-1}})^\sigma x^{-1}, \quad S_1^{\pi^{-1}} = x(S^{\pi^{-1}})^\sigma,$$
$$R_1^{\pi^{-1}} = (L^{\pi^{-1}})^\sigma, \quad L_1^{\pi^{-1}} = x(R^{\pi^{-1}})^\sigma x^{-1}, \quad S_1^{\pi^{-1}} = x((S^{-1})^{\pi^{-1}})^\sigma.$$

Since H is a characteristic subgroup of G, $\psi : G/H \to G/H$ with the rule $(Hx)^\psi = Hx^\sigma$ is an automorphism of G/H. Since for every subset X of G/H, $((X^{\pi^{-1}})^\sigma)^\pi = X^\psi$, and π is a group homomorphism, we conclude that one of the following holds:

$$R_1 = R^\psi, \quad L_1 = yL^\psi y^{-1}, \quad S_1 = yS^\psi,$$
$$R_1 = L^\psi, \quad L_1 = yR^\psi y^{-1}, \quad S_1 = y(S^{-1})^\psi,$$

where $y = x^\pi \in G/H$. This shows that G/H is an SCI-graph. $\square$

Let us denote the incidence graph of the projective space $PG(n, q)$ and the Hadamard design $H(11)$ on 11 points with $B(PG(n, q))$ and $B(H(11))$ and their non-incidence graphs with $C(PG(n, q))$ and $C(H(11))$, respectively. The symmetry structure of semi-Cayley graphs over a group of prime order is fully given in [13]. For the convenience of the reader, we recall the results of this paper in the following theorem.

Theorem 3.3 ([13, Theorems 2.1 and 2.2]) *Let Γ be a semi-Cayley graph over a group $G = \langle x \rangle$ of prime order p. Then one of the following occurs.*

(1) Γ *or* $\Gamma^c \cong \Gamma_1 + \Gamma_2$, *where* Γ_i *are two non-isomorphic Cayley graphs of order* p, $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2)$ *and* Γ *is not transitive.*

(2) Γ *or* $\Gamma^c = \mathrm{SC}(G; G \setminus \{1\}, \emptyset, T)$ *and* $\mathrm{BCay}(G, T) \cong pK_2$, *in which case* $\mathrm{Aut}(\Gamma) \cong S_p$.

(3) Γ *or* $\Gamma^c = \mathrm{SC}(G; G \setminus \{1\}, \emptyset, T)$ *and* $\mathrm{BCay}(G, T) \cong B(PG(n, q))$ *where* $p = \frac{q^n - 1}{q - 1}$, *in which* $\mathrm{Aut}(\Gamma) = P\Sigma L(n, q)$.

(4) Γ *or* $\Gamma^c = \mathrm{SC}(G; G \setminus \{1\}, \emptyset, T)$ *and* $\mathrm{BCay}(G, T) \cong B(H(11))$, *in which case* $\mathrm{Aut}(\Gamma) \cong PSL(2, 11)$, $T = \{1, 3, 4, 5, 9\}$ *and* $p = 11$.

(5) *There exists* $\sigma \in \mathrm{Aut}(\Gamma)$ *such that* $\mathrm{Aut}(\Gamma) = R_G \rtimes \langle\sigma\rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_d$, *where* d *divides* $p-1$ (*for more details about the map* σ *and the structure of* Γ, *see* [13, Theorem 2.1(iii)]).

(6) Γ *or* $\Gamma^c \cong 2pK_1$, pK_2 *or* $2X$, *where* X *is connected Cayley graph of order* p *and* Γ *is transitive.*

(7) Γ *or* $\Gamma^c \cong P$, *where* P *is the Petersen graph,* $\mathrm{Aut}(\Gamma) \cong S_5$ *and* $p = 5$.

(8) Γ *or* $\Gamma^c \cong Y[2K_1]$, *where* Y *is a Cayley graph and* Γ *is imprimitive with only 2-blocks.*

(9) Γ *or* $\Gamma^c \cong B(PG(n, q))$ *or* $C(PG(n, q))$ *where* $p = \frac{q^n-1}{q-1}$, *in which* $\mathrm{Aut}(\Gamma) = P\Gamma L(n, q)$.

(10) Γ *or* $\Gamma^c \cong B(H(11))$ *or* $C(H(11))$, *in which* $\mathrm{Aut}(\Gamma) = PGL(2, 11)$ *and* $p = 11$.

(11) *There exist* $\alpha, \sigma \in \mathrm{Aut}(\Gamma)$ *such that* $\mathrm{Aut}(\Gamma) = \langle\alpha\rangle \rtimes \langle\sigma\rangle \cong \mathbb{Z}_{2p} \rtimes \mathbb{Z}_d$, *where* d *is a divisor of* $p-1$ *and* $\rho_x = \alpha^{p-1}$ (*for more details about the maps* α *and* σ *and the structure of* Γ, *see* [13, Theorem 2.2(v)]).

(12) *There exists* $\omega \in \mathrm{Aut}(\Gamma)$ *such that* $\mathrm{Aut}(\Gamma) = R_G \rtimes \langle\omega\rangle$, *where* $\langle\omega\rangle \cong \mathbb{Z}_{2d}$ *for some* d *dividing* $p-1$ (*for more details about the map* ω *and the structure of* Γ, *see* [13, Theorem 2.2(vi)]).

Now we are ready to prove Theorem B:

Proof of Theorem B First we prove that $G \cong \mathbb{Z}_3$ is an SCI-graph. Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be a semi-Cayley graph over G. Then Γ is one of the twelve type graphs given in Theorem 3.3. Since $|G| = 3$ and the only Cayley graph of G is the complete graph K_3, Γ cannot be of types (1), (4), (7) and (10). Now we examine other possibilities for Γ. In Cases (2), (3), (5) and (9), $|\mathrm{Aut}(\Gamma)| = 6$ and in Cases (11) and (12), $|\mathrm{Aut}(\Gamma)| = 12$. So in all of these cases, R_G is a Sylow 3-subgroup of $\mathrm{Aut}(\Gamma)$. Now by the Sylow Theorem and Theorem A, Γ is an SCI-graph. In Case (8), Γ or $\Gamma^c \cong K_3[2K_1]$ and so $\mathrm{Aut}(\Gamma) \cong S_2 \wr S_3$ is of order 48. This shows that again R_G is a Sylow 3-subgroup of $\mathrm{Aut}(\Gamma)$ and Γ is an SCI-graph. Now we consider the remaining Case (6). If Γ or $\Gamma^c \cong 6K_1$, then as mentioned after Definition 1.2, Γ is an SCI-graph. If Γ or $\Gamma^c \cong 3K_2$, then $\mathrm{Aut}(\Gamma) \cong S_2 \wr S_3$ and so again Γ is an SCI-graph. Finally, let Γ or $\Gamma^c \cong 2K_3$. Since Γ is an SCI-graph if and only if Γ^c is an SCI-graph, we may assume that $\Gamma \cong 2K_3$. If $R = L = \emptyset$, then $|S| = 2$ and so $\Gamma \cong C_6$, a contradiction. Hence, $|R| = |L| = s \geq 1$ and $s + |S| = 2$. Since $R = R^{-1}$ and $S = S^{-1}$, $R = S = \{x, x^{-1}\}$ and $S = \emptyset$. This shows that if $\Gamma \cong \mathrm{SC}(G; R_1, L_1, S_1)$, then $R_1 = L_1 = R = L = \{x, x^{-1}\}$ and $S_1 = S = \emptyset$, which means that Γ is an SCI-graph. Hence, we have showed that every semi-Cayley graph over G is an SCI-graph which implies that G is an SCI-group.

Now let G be an SCI-group. We claim that $G \cong \mathbb{Z}_3$. First suppose that G is abelian. If there exists a prime divisor p of $|G|$ greater than 3, then there exist two distinct non-involution elements a, b of order p in G such that $a \neq b, b^{-1}$, which contradicts Lemma 3.1. Hence, G is a $\{2, 3\}$-group. Let $1 \neq P$ be a Sylow 3-subgroup of G. Since P is a characteristic subgroup of G, Lemma 3.2 implies that P is an SCI-group. By Lemma 3.1, P has a unique subgroup of order p, and so it is cyclic, see [3, Lemma 4]. Now by Lemma 3.1 and Remark 1.4, $P \cong \mathbb{Z}_3$. If 2 divides the order of G then there exists a Sylow 2-subgroup of $1 \neq Q$ of G. Since, by Remark 1.4, $\mathbb{Z}_2$ is not an SCI-group, Lemma 4 of [3] implies that Q has at least two distinct subgroup of order 2, contradicting Lemma 3.1. Hence, we have proved that every abelian SCI-group is isomorphic to $\mathbb{Z}_3$.

Now we show that every finite SCI-group is abelian. Let G be a finite SCI-group. By Proposition 1.3, G is a CI-group and so is solvable, see [11, Theorem 8.6]. Let $G = G^{(0)} > G^{(1)} > \cdots > G^{(n)} = 1$, be the derived series of G. Since each $G^{(i)}$ is a characteristic subgroup of G and $G^{(i)}/G^{(i+1)}$ is abelian, Lemma 3.2 and the above discussion imply that G is a 3-group.

Now let H and K be two distinct subgroups of G of order 3. Let $R = H\backslash\{1\}$ and $L = K\backslash\{1\}$. Then $\mathrm{SC}(G; R, R, \emptyset) \cong 2|G : H|\mathrm{Cay}(H, R) \cong 2|G : H|K_3$ and similarly $\mathrm{SC}(G; L, L, \emptyset) \cong 2|G : K|K_3$. Since $|G : H| = |G : K|$ and G is an SCI-group, there exists $\sigma \in \mathrm{Aut}(G)$ such that $L = R^\sigma$. This means that all subgroups of G of order 3, are conjugate in $\mathrm{Aut}(G)$. Now by a result of Wilkens, see [11, Theorem 9.1], G is a homocyclic group and therefore is abelian. This completes the proof. □

4 Proof of Theorem C

Recently, some authors studied isomorphism problem of special classes of semi-Cayley graphs. The study of isomorphism problem for 0-type semi-Cayley graphs was started in [15]. Let G be a finite group and S be a non-empty subset G. Recall that a bi-Cayley graph $\mathrm{BCay}(G, S)$ is a bipartite 0-type semi-Cayley graph over G. Also $\mathrm{BCay}(G, S)$ is called a BCI-graph if for any bi-Cayley graph $\mathrm{BCay}(G, T)$, whenever $\mathrm{BCay}(G, S) \cong \mathrm{BCay}(G, T)$ we have $T = gS^\alpha$ for some $g \in G$ and $\alpha \in \mathrm{Aut}(G)$. A group G is called a (connected) BCI-group, if all (connected) bi-Cayley graphs of G are BCI-graphs. Also G is called an m-BCI-group, if all bi-Cayley graphs of G of valency at most m are BCI-graphs (see [7, Definition 1]).

We can restrict the isomorphism problem of semi-Cayley graphs to s-type semi-Cayley graphs:

Definition 4.1 *Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be an s-type semi-Cayley graph over a group G.*

(i) *Γ is called an s-type* SCI-*graph, if for any s-type semi-Cayley graph $\Sigma = \mathrm{SC}(G; R_1, L_1, S_1)$, whenever $\mathrm{SC}(G, R, L, S) \cong \mathrm{SC}(G, R_1, L_1, S_1)$ there exists a bi-Cayley isomorphism between Γ and Σ, or equivalently at least one of the relations* (1.1) *or* (1.2) *holds for some $\sigma \in \mathrm{Aut}(G)$ and $x \in G$.*

(ii) *G is called an s-type* SCI-*group, if all s-type bi-Cayley graphs of G are s-type* SCI-*graphs.*

Remark 4.2 Let $\Gamma = \mathrm{BCay}(G, S)$. Then Γ is a 0-type semi-Cayley graph. By our definition, Γ is a 0-type SCI-graph, if for any 0-type semi-Cayley graph $\mathrm{BCay}(G, T)$, whenever $\mathrm{BCay}(G, S) \cong \mathrm{BCay}(G, T)$ there exist $\sigma \in \mathrm{Aut}(G)$ and $x \in G$ such that at least one of the relations $T = xS^\sigma$ or $T = x(S^{-1})^\sigma$ holds. If always the first case holds, then Γ is called a BCI-*graph*, as defined in [7].

Note that if there exist $g \in G$ and $\alpha \in \mathrm{Aut}(G)$ such that $S^{-1} = gS^\alpha$, then the first equality holds if and only if the latter holds. Furthermore, since $\mathrm{BCay}(G, S) \cong \mathrm{BCay}(G, S^{-1})$ by Lemma 1.1, Γ is a BCI-graph if and only if it is a 0-type SCI-graph and there exist $g \in G$ and $\alpha \in \mathrm{Aut}(G)$ such that $S^{-1} = gS^\alpha$. In particular, every abelian 0-type semi-Cayley graph is a BCI-graph if and only if it is a 0-type SCI-graph.

In order to drive Theorem C, we need some preliminary lemmas.

Lemma 4.3 *Let* Γ *be an* s*-type* SCI*-graph. Then*

(1) *given a permutation* $\varphi \in \mathrm{Sym}(V(\Gamma))$*, where* $\varphi^{-1}R_G\varphi \leq \mathrm{Aut}(\Gamma)$ *and* $\{G\times\{1\}, G\times\{2\}\}$ *is* φ*-invariant,* R_G *and* $\varphi^{-1}R_G\varphi$ *are conjugate in* $\mathrm{Aut}(\Gamma)$*,*

(2) *every semiregular subgroup of* $\mathrm{Aut}(\Gamma)$ *with two orbits* $G\times\{1\}$ *and* $G\times\{2\}$ *and isomorphic to* R_G *is conjugate to* R_G *in* $\mathrm{Aut}(\Gamma)$*.*

Proof To prove (1), it is enough to follow the proof of (1) $\Rightarrow$ (2) in Theorem A and note that the graph Σ is an s-type bi-Cayley graph by Lemma 2.4.

Similarly, to prove (2), it is enough to follow the proof of (1) $\Rightarrow$ (3) in Theorem A and replace β_1^H and β_2^H with $G\times\{1\}$ and $G\times\{2\}$, respectively. □

Lemma 4.4 *Let* Γ *and* Σ *be isomorphic bi-Cayley graphs. Then there exists an isomorphism* $\Gamma \overset{\psi}{\cong} \Sigma$ *such that* $R_G \cong \psi R_G\psi^{-1} \leq \mathrm{Aut}(\Gamma)$ *and* $\{G\times\{1\}, G\times\{2\}\}$ *is* ψ*-invariant.*

Proof Let $\Sigma = \mathrm{BCay}(G,S)$, $\Gamma = \mathrm{BCay}(G,T)$ and $\Gamma \overset{\varphi}{\cong} \Sigma$. Let $H = \langle TT^{-1}\rangle$, $K = \langle SS^{-1}\rangle$, $m = |G:H|$ and $n = |G:K|$. Then $m = n$. Let $X = \{x_1 = 1, x_2, \ldots, x_m\}$ and $Y = \{y_1 = 1, y_2, \ldots, y_n\}$ be the right transversal to H and to K in G, respectively. Then it is easy to check that the connected components of Γ are Γ_i, $i = 1,\ldots,m$, where $V(\Gamma_i) = Hx_i\times\{1,2\}$ and $E(\Gamma_i) = \{\{(h_1t_i,1),(h_2t_i,2)\} \mid h_2h_1^{-1} \in T\}$. Also the connected components of Σ are Σ_i, $i = 1,\ldots,n$, where $V(\Sigma_i) = Ky_i\times\{1,2\}$ and $E(\Sigma_i) = \{\{(k_1t_i,1),(k_2t_i,2)\} \mid k_2k_1^{-1} \in S\}$. Furthermore, for each $i = 1,\ldots,n$, $\Gamma_i \cong \Gamma_1 \cong \mathrm{BCay}(H,Tt^{-1})$, for some $t\in T$ and $\Sigma_i \cong \Sigma_1 \cong \mathrm{BCay}(K,Ss^{-1})$ for some $s\in S$. So we may assume that the restriction of φ to Γ_1 is an isomorphism from Γ_1 to Σ_1. Let us denote this restriction with φ_1. Since Γ_1 and Σ_1 are connected and bipartite, $\{(H\times\{1\})^{\varphi_1}, (H\times\{2\})^{\varphi_1}\} = \{K\times\{1\}, K\times\{2\}\}$. Now for each $i = 2,\ldots,m$, we define $\varphi_i : V(\Gamma_i)\to V(\Sigma_i)$, where (hx_i,j) maps to (ky_i,l), $l,j\in\{1,2\}$, where $(h,j)^{\varphi_1} = (k,l)$. Then φ_i is a graph isomorphism. Finally, take ψ to be the isomorphism whose restriction to each component Γ_i is φ_i. Then $\psi\in\mathrm{Sym}(V(\Gamma))$, and since $R_G \leq \mathrm{Aut}(\Sigma)$, we have $R_G \cong \psi R_G\psi^{-1} \leq \mathrm{Aut}(\Gamma)$. Also

$$\{(G\times\{1\})^{\psi}, (G\times\{2\})^{\psi}\} = \{G\times\{1\}, G\times\{2\}\},$$

which completes the proof. □

Now we are ready to prove Theorem C.

Proof of Theorem C First recall that by Remark 4.2, Γ is a BCI-graph if and only if it is a 0-type SCI-graph and $S^{-1} = gS^{\alpha}$ for some $g\in G$ and $\alpha\in\mathrm{Aut}(G)$. Now (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are direct consequences of Lemma 4.3 and Remark 4.2.

Now let $\Gamma\cong\Sigma$, where $\Sigma = \mathrm{BCay}(G,T)$ is a bi-Cayley graph. By Lemma 4.4, there exists $\psi\in\mathrm{Sym}(V(\Gamma))$ such that $\Gamma \overset{\psi}{\cong} \Sigma$, $\psi R_G\psi^{-1} \leq \mathrm{Aut}(\Gamma)$ and $\{G\times\{1\}, G\times\{2\}\}$ is ψ-invariant. Hence, if (2) or (3) holds, then there exists $\beta\in\mathrm{Aut}(\Gamma)$ such that $\psi R_G\psi^{-1} = \beta^{-1}R_G\beta$. Hence by Corollary 2.3, there exists a semi-Cayley isomorphism between Γ and Σ, which means that Γ is a 0-type SCI-graph. Now by Remark 4.2, Γ is a BCI-graph. This proves (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1), which completes the proof. □

It is well known that every group of order $2p$, p an odd prime, is a CI-group, see [2]. In the following example, as an application of Theorem C, we prove that every 0-type bi-Cayley graph Γ over $\mathbb{Z}_{2p}$, p an odd prime, with $p^2 \nmid |\mathrm{Aut}(\Gamma)|$ is a BCI-graph. We leave the general case as an open question.

Example 4.5 Let Γ be a bi-Cayley graph of $G \cong \mathbb{Z}_{2p}$, p an odd prime, and $p^2 \nmid |\mathrm{Aut}(\Gamma)|$. Then Γ is a BCI-graph. To see this, let $\Gamma = \mathrm{BCay}(G,S)$ and $\varphi \in \mathrm{Sym}(V(\Gamma))$ such that $\varphi R_G \varphi^{-1} \leq \mathrm{Aut}(\Gamma)$ and $\{G \times \{1\}, G \times \{2\}\}$ is ψ-invariant. Put $A = \mathrm{Aut}(\Gamma)$ and $V = V(\Gamma)$. Since G is abelian there exists $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = S^{-1}$, so by Theorem C, it is enough to show that R_G and $\varphi R_G \varphi^{-1}$ are conjugate in A. Let P be a Sylow p-subgroup of G and $R_P = \{\rho_h \mid h \in P\}$. Since $R_P < R_G \leq A$ and $p^2 \nmid |A|$, R_P is a Sylow p-subgroup of A. Also $\varphi R_P \varphi^{-1}$ is a Sylow p-subgroup of A. Hence, there exists $\beta \in A$ such that $\varphi R_P \varphi^{-1} = \beta^{-1} R_P \beta$. So $\beta\varphi \in N_{\mathrm{Sym}(V(\Gamma))}(R_P)$.

Since R_P is a semiregular subgroup of A with 4 orbits, Γ is a 4-Cayley graph of P. Let $G = \langle a, t \mid a^p = t^2 = 1, at = ta \rangle \cong \mathbb{Z}_{2p}$, and $P = \langle a \rangle$. Then $G = P \cup tP$. Also orbits of P are $\Omega_1 = \{(a^i, 1) \mid i = 0, \ldots, p-1\}$, $\Omega_2 = \{(ta^i, 1) \mid i = 0, \ldots, p-1\}$, $\Omega_3 = \{(a^i, 2) \mid i = 0, \ldots, p-1\}$ and $\Omega_4 = \{(ta^i, 2) \mid i = 0, \ldots, p-1\}$. If we identify $(a^i,1)$, $(ta^i,1)$, $(a^i,2)$ and $(ta^i,2)$ with $(a^i,1)$, $(a^i,2)$, $(a^i,3)$ and $(a^i,4)$, respectively, Lemma 2.1 implies that there exist $g_1 = 1, g_2, g_3, g_4 \in P$, $g_0 \in P$, $\sigma \in \mathrm{Aut}(P)$ and $\theta \in S_4$ such that $(x,j)^{\beta\varphi} = (g_j x^\sigma g_0, j^\theta)$ for all $x \in P$ and $j = 1,2,3,4$. Note that all automorphisms of P extend to automorphisms of G and so we can assume that $\sigma \in \mathrm{Aut}(G)$.

Define $\alpha : V(\Gamma) \to V(\Gamma)$, where $(x,j)^\alpha = (x^\sigma g_0, j^\theta)$ for all $x \in P$ and $j = 1,2,3,4$. Clearly, $\alpha \in \mathrm{Sym}(V(\Gamma))$. Also for each $(x,j) \in V(\Gamma)$, we have

$$\begin{aligned}
(x,j)^{\beta\varphi\alpha^{-1}\rho_a} &= (g_j x^\sigma g_0, j^\theta)^{\alpha^{-1}\rho_a} \\
&= (g_j^{\sigma^{-1}} x, j)^{\rho_a} \\
&= (g_j^{\sigma^{-1}} xa, j) \\
&= (g_j x^\sigma a^\sigma g_0, j^\theta)^{\alpha^{-1}} \\
&= (xa, j)^{\beta\varphi\alpha^{-1}} \\
&= (x,j)^{\rho_a \beta\varphi\alpha^{-1}},
\end{aligned}$$

which implies that $\gamma := \beta\varphi\alpha^{-1} \in C_{\mathrm{Sym}(V(\Gamma))}(R_P)$. Also

$$\begin{aligned}
(x,j)^{\alpha\rho_a\alpha^{-1}} &= (x^\sigma g_0, j^\theta)^{\rho_a\alpha^{-1}} \\
&= (x^\sigma g_0 a, j^\theta)^{\alpha^{-1}} \\
&= (x^\sigma a g_0, j^\theta)^{\alpha^{-1}} \\
&= (x a^{\sigma^{-1}}, j) \\
&= (x,j)^{\rho_{a^{\sigma^{-1}}}},
\end{aligned}$$

and

$$\begin{aligned}
(x,j)^{\alpha\rho_t\alpha^{-1}} &= (x^\sigma g_0, j^\theta)^{\rho_t\alpha^{-1}} \\
&= (x^\sigma g_0 t, j^\theta)^{\alpha^{-1}} \\
&= (x^\sigma t g_0, j^\theta)^{\alpha^{-1}} \\
&= (x t^{\sigma^{-1}}, j)
\end{aligned}$$

$$= (x,j)^{\rho_{t^{\sigma^{-1}}}}$$

imply that $\alpha \in N_{\mathrm{Sym}(V(\Gamma))}(R_G)$. So there exists $\gamma \in C_{\mathrm{Sym}(V(\Gamma))}(R_P)$ such that $\varphi = \beta^{-1}\gamma\alpha$. Also $(x,j)^\gamma = (x,j)^{\beta\varphi\alpha^{-1}} = (g_j x^\sigma g_0, j^\theta)^{\alpha^{-1}} = (g_j^{\sigma^{-1}} x, j)$ for all $x \in P$ and $j = 1,2,3,4$. So for all $x \in P$, and $j = 1,2,3,4$, $(x,j)^{\gamma\rho_t\gamma^{-1}\rho_t} = (g_j^{\sigma^{-1}} xt, j)^{\gamma^{-1}\rho_t} = (xt,j)^{\rho_t} = (xt^2, j) = (x,j)$. Hence, $\gamma\rho_t = \rho_t\gamma$. Since $\gamma\rho_a = \rho_a\gamma$, this shows that $\gamma \in C_{\mathrm{Sym}(V(\Gamma))}(R_G)$. Hence,

$$\varphi R_G \varphi^{-1} = \beta^{-1}\gamma\alpha R_G \alpha^{-1}\gamma^{-1}\beta = \beta^{-1}\gamma R_G \gamma^{-1}\beta = \beta^{-1}R_G\beta,$$

which means that Γ is a BCI-graph.

In the rest of this section, we shall give some preliminary results which will be used in constructing non-BCI-graphs. Also we are going to find an infinite family of BCI-graphs. We prove that every group of prime order is a BCI-group. To prove this result, we have to prove some elementary results which most of them are a generalization of previous works.

Babai and Frankl proved that if $\mathrm{Cay}(G,S)$ is a CI-graph and $S \subseteq H \le G$, then $\mathrm{Cay}(H,S)$ is also a CI-graph, see for example [11, Lemma 8.2]. In the following lemma, we prove a similar result for BCI-graphs.

Lemma 4.6 *Let* $\mathrm{BCay}(G,S)$ *be a* BCI-*graph,* $H \le G$*,* $S \subseteq H$*. If* $\mathrm{BCay}(H,S)$ *is connected, then* $\mathrm{BCay}(H,S)$ *is a* BCI-*graph.*

Proof Let $T \subseteq H$ and $\mathrm{BCay}(H,S) \cong \mathrm{BCay}(H,T)$. Since $\mathrm{BCay}(H,S)$ is connected, by [7, p. 1259], $H = \langle SS^{-1}\rangle = \langle TT^{-1}\rangle$. So $\mathrm{BCay}(\langle SS^{-1}\rangle, S) \cong \mathrm{BCay}(\langle TT^{-1}\rangle, T)$ which implies that $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G,T)$, see [7, Lemma 2.8]. Since $\mathrm{BCay}(G,S)$ is a BCI-graph, there exist $\alpha \in \mathrm{Aut}(G)$ and $g \in G$ such that $T = gS^\alpha$. On the other hand,

$$H^\alpha = \langle SS^{-1}\rangle^\alpha = \langle S^\alpha (S^\alpha)^{-1}\rangle = \langle g^{-1}TT^{-1}g\rangle = g^{-1}\langle TT^{-1}\rangle g = g^{-1}Hg = H^{\tau_g}.$$

So $H^{\alpha\tau_{g^{-1}}} = H$ and $\alpha\tau_{g^{-1}} \in \mathrm{Aut}(H)$. Also $Tg^{-1} = S^{\alpha\tau_{g^{-1}}} \subseteq H^{\alpha\tau_{g^{-1}}} = H$ and $T \subseteq H$ imply that $g \in H$. Since $T = S^{\alpha\tau_{g^{-1}}}g$ there exist $\beta \in \mathrm{Aut}(H)$ and $h \in H$ such that $T = hS^\beta$. This shows that $\mathrm{BCay}(H,S)$ is a BCI-graph. $\square$

Let G be a finite group. Let S and T be two subsets of G both of which contain the identity. If $\mathrm{Cay}(G, S\setminus\{1\}) \cong \mathrm{Cay}(G, T\setminus\{1\})$, then $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G,T)$, see [7, Lemma 2.9]. In the following lemma we extend this result.

Lemma 4.7 *Let* G *and* H *be two groups,* $S \subseteq G$*,* $T \subseteq H$*,* $1_G \in S$ *and* $1_H \in T$*. If* $\mathrm{Cay}(G, S\setminus\{1_G\}) \cong \mathrm{Cay}(H, T\setminus\{1_H\})$*, then* $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(H,T)$*.*

Proof Let φ be an isomorphism from $\mathrm{Cay}(G, S\setminus\{1_G\})$ to $\mathrm{Cay}(H, T\setminus\{1_H\})$. By a similar argument to [7, Lemma 2.9], one can easily see that

$$\begin{aligned}\psi : G\times\{1,2\} &\to H\times\{1,2\}\\ (g,i) &\mapsto (g^\varphi, i), \quad i = 1,2.\end{aligned}$$

is a graph isomorphism from $\mathrm{BCay}(G,S)$ to $\mathrm{BCay}(H,T)$. $\square$

In [9, p. 218], it is shown that every bi-Cayley graph of a finite cyclic group is a Cayley graph and so is vertex-transitive. In the following lemma, we extend this result to abelian groups.

Lemma 4.8 *Let G be a finite abelian group and $\Gamma = \mathrm{BCay}(G, S)$ be a bi-Cayley graph of G. Then $\Gamma \cong \mathrm{Cay}(\langle\psi\rangle R_G, \psi R_S)$, where $R_S = \{\rho_s \mid s \in S\}$ and $\psi : V(\Gamma) \to V(\Gamma)$ is a function defined by $(x, 1)^\psi = (x^{-1}, 2)$ and $(x, 2)^\psi = (x^{-1}, 1)$.*

Proof It is an easy task to see that $\langle\psi\rangle R_G$ is a transitive subgroup of $\mathrm{Aut}(\Gamma)$. Since $\langle\psi\rangle \cong \mathbb{Z}_2$ and $\psi \in \mathrm{Aut}(\Gamma) \setminus R_G$, $R_G \cap \langle\psi\rangle = 1$ and so $|R_G\langle\psi\rangle| = 2|G| = |V(\Gamma)|$. This shows that $R_G\langle\psi\rangle$ is a regular subgroup of $\mathrm{Aut}(\Gamma)$.

Note that $(\psi R_G)^{-1} = \psi R_G$ and so $\mathrm{Cay}(\langle\psi\rangle R_G, \psi R_S)$ is an undirected Cayley graph. Now $\theta : V(\Gamma) \to \langle\psi\rangle R_G$, with $(g, 1)^\theta = \rho_g$ and $(g, 2)^\theta = \psi\rho_g$, is a graph isomorphism from Γ to $\mathrm{Cay}(D, \psi R_S)$. □

Let $D = \langle\psi\rangle R_G$ for bi-Cayley graphs of abelian groups G as defined in the proof of Lemma 4.8. Then the following corollary extends Corollary 2.2 of [9].

Corollary 4.9 *Let G be an abelian group with an element x such that $x^2 \neq 1$, and Γ be a bi-Cayley graph of G. If any regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to D is conjugate to D in $\mathrm{Aut}(\Gamma)$, then Γ is a BCI-graph. Furthermore, if D is a CI-group, then G is a BCI-group. In particular $\mathbb{Z}_p$, p a prime, and $\mathbb{Z}_9$ are BCI-groups.*

Proof Let $\Gamma = \mathrm{BCay}(G, S)$ and $\Gamma \cong \mathrm{BCay}(G, T)$. Then $\mathrm{BCay}(R_G, R_S) \cong \Gamma \cong \mathrm{BCay}(G, T) \cong \mathrm{BCay}(R_G, R_T)$, where $R_X = \{\rho_g | g \in X\}$. Also by Lemma 4.8, $\mathrm{Cay}(D, \psi R_S) \cong \Gamma \cong \mathrm{BCay}(G, T) \cong \mathrm{Cay}(D, \psi R_T)$. Now by Babai's theorem [2, Lemma 3.1], there exists $\varphi \in \mathrm{Aut}(D)$ such that $\psi R_T = (\psi R_S)^\varphi$. Let σ be the restriction of φ to R_G. Then $\sigma \in \mathrm{Aut}(R_G)$ and $\psi^\varphi \in D \setminus R_G$. So there exists $g \in G$ such that $\psi^\varphi = \psi\rho_g$, which implies that $R_T = \rho_g R_S^\varphi = \rho_g R_S^\sigma$. Therefore $\mathrm{BCay}(R_G, R_S)$ is a BCI-graph. Hence, Γ is a BCI-graph. The second part follows from the Babai theorem for Cayley graphs and the first part. It is easy to check that $\mathbb{Z}_2$ is a BCI-graph. Also D_{2p}, p odd prime, and D_{18} are CI-groups, [11, Theorem 8.9]. Since if G is a cyclic group of order n, then $D \cong D_{2n}$, and the result is clear by the second part. □

It is easy to check that $\psi : G\times\{1, 2\} \to G\times\{1, 2\}$, which maps $(x, 1)$ to $(x, 2)$ and maps $(x, 2)$ to $(x, 1)$ is an isomorphism from $\mathrm{BCay}(G, S)$ to $\mathrm{BCay}(G, S^{-1})$. So every BCI-graph is vertex-transitive by [12, Lemma 2.1 (5)]. Now the natural question is that is any vertex-transitive bi-Cayley graph a BCI-graph?

By Corollary 4.9, $\mathbb{Z}_9$ is a BCI-group and also it is easy to see that $\mathbb{Z}_4$ is a BCI-group. Now we consider the problem for $\mathbb{Z}_{p^2}$, $p > 3$ a prime in the following lemma which shows that the above question has negative answer.

Proposition 4.10 *Let $p > 3$ be a prime and $G = \langle a\rangle \cong \mathbb{Z}_{p^2}$. There exists a connected vertex-transitive bi-Cayley graph of G of valency $p+2$ which is not a BCI-graph. In particular, in a finite BCI-group G, for each prime $p > 3$ dividing order of G, each Sylow p-subgroup is elementary abelian.*

Proof By Lemma 4.8, every bi-Cayley graph of G is vertex-transitive. Also by Lemma 4.6, the second part follows immediately from the first part. Put $S = a\langle a^p\rangle \cup \{a^p\}$, $T = a\langle a^p\rangle \cup \{a^{2p}\}$, $S' = S \cup \{1\}$ and $T' = T \cup \{1\}$. Then $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$, see [11, p. 311]. So by Lemma 4.7, $\mathrm{BCay}(G, S') \cong \mathrm{BCay}(G, T')$. Suppose that there exist $g \in G$ and $\alpha \in \mathrm{Aut}(G)$ such that $gS'^\alpha = T'$. Since $1 \in S'$, $g \in T'$, we distinguish three cases:

Case I $g = 1$. Then $S'^{\alpha} = T'$ and so $S^{\alpha} = T$, which is a contradiction, see [11, p. 311].

Case II $g = a^{2p}$. Then

$$S'^{\alpha} = a^{-2p}T' = \{a^{(-2+r)p+1} \mid 0 \leq r \leq p-1\} \cup \{1\} \cup \{a^{-2p}\}.$$

Since a^{α} has order p^2, $a^{\alpha} = a^{(-2+k)p+1}$ for some $0 \leq k \leq p-1$. So $(a^p)^{\alpha} = (a^{\alpha})^p = a^{(-2+k)p^2+p} = a^p \in S'^{\alpha}$. Clearly $a^p \neq 1$. If $a^p = a^{(-2+r)p+1}$ for some $0 \leq r \leq p-1$, then $1 = (a^p)^p = (a^{(-2+r)p+1})^p = a^p$, a contradiction. Also if $a^p = a^{-2p}$, then $a^{3p} = 1$ and so $p^2 \mid 3p$ which means that $p = 3$, a contradiction.

Case III $g = a^{kp+1}$ for some $0 \leq k \leq p-1$. Then

$$S'^{\alpha} = g^{-1}T' = \{a^{(r-k)p} \mid 0 \leq r \leq p-1\} \cup \{a^{(2-k)p-1}\} \cup \{a^{-kp-1}\}.$$

Note that since $p \neq 2$, $a^{(2-k)p-1} \neq a^{-kp-1}$. Again since order of a^{α} is p^2, $a^{\alpha} = a^{(2-k)p-1}$ or a^{-kp-1}. In both cases we have $(a^p)^{\alpha} = (a^{\alpha})^p = a^{-p}$. If $a^{\alpha} = a^{(2-k)p-1}$, then $(a^{p+1})^{\alpha} = (a^p.a)^{\alpha} = (a^p)^{\alpha}a^{\alpha} = a^{(1-k)p-1} \in S'^{\alpha}$. Hence, $a^{(1-k)p-1} = a^{(2-k)p-1}$ or a^{-kp-1}. In both cases we can see that $a^p = 1$, which is a contradiction. Finally, if $a^{\alpha} = a^{-kp-1}$, then $(a^{p+1})^{\alpha} = a^{(-1-k)p-1} \in S'^{\alpha}$. Hence, $a^{(-1-k)p-1} = a^{(2-k)p-1}$ or a^{-kp-1}. The first case implies that $p = 3$ and the second case implies that $a^p = 1$, a contradiction.

Hence, in all cases we obtain a contradiction. This completes the proof. □

In Proposition 1.3, we proved that every SCI-group is a CI-group. In most cases, we find that also every BCI-group is a CI-group and we conjecture that this is true in general for BCI-groups.

Conjecture Every BCI-group is a CI-group.

Note that in general the converse of the above statement is not true. For example, $\mathbb{Z}_8$ is a CI-group, see [11, Theorem 8.9], but is not a 4-BCI group by [9, Remark 1.2].

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