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On the Arf Invariant

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We proffer a swift and elementary treatment of the Arf invariant that simultaneously establishes that it *is* an invariant under equivalence for nondegenerate quadratic forms over fields of characteristic 2, and that is classifies these forms when the fields are perfect.

Let $Q(\mathbf{x})$ be a nondegenerate (nondefective) quadratic form on a vector space V of dimension $2n$ over a field K of characteristic 2. Then the associated polar form

$$B(\mathbf{x}, \mathbf{y}) = Q(\mathbf{x} + \mathbf{y}) + Q(\mathbf{x}) + Q(\mathbf{y}) \tag{1}$$

is a nonsingular alternating bilinear form. Let $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{2n}$ be a symplectic base of V, i.e., one with respect to which $B(\mathbf{x}, \mathbf{y})$ has the canonical coordinate form

$$B(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} (x_i y_{n+i} - y_i x_{n+i}). \tag{2}$$

The Arf invariant of $Q(\mathbf{x})$ with respect to this base is

$$\Delta(Q) = \sum_{i=1}^{n} Q(\mathbf{e}_i)\, Q(\mathbf{e}_{n+i}). \tag{3}$$

Let $L = \{\lambda^2 + \lambda : \lambda \in K\}$: L is the image of the homomorphism $\lambda \mapsto \lambda^2 + \lambda$ of the additive group K^+ whose kernel is $\{0, 1\} = \mathrm{GF}(2)$, and so L is a subgroup of K^+ isomorphic to $K^+/\mathrm{GF}(2)^+$. Write $\bar{\Delta}(Q)$ for the class of $\Delta(Q)$ modulo L. In precise terms we prove the following

THEOREM. (i) *$\bar{\Delta}(Q)$ is independent of the choice of symplectic base.*

(ii) *If $Q_1(\mathbf{x})$ and $Q_2(\mathbf{x})$ are equivalent nondegenerate quadratic forms on V then $\bar{\Delta}(Q_1) = \bar{\Delta}(Q_2)$.*

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(iii) *If K is perfect then two nondegenerate quadratic forms $Q_1(\mathbf{x})$, $Q_2(\mathbf{x})$ on* V *are equivalent if and only if $\bar{\Delta}(Q_1) = \bar{\Delta}(Q_2)$.*

Parts (i) and (ii) were first proved by Arf whose argument proceeds by induction on n and is rather lengthy [1, pp. 152–157]. Dieudonné [4] and Kneser [8] have given shorter proofs based on the properties of the Clifford algebra of $Q(\mathbf{x})$, and yet other approaches have been given by Witt and Klingenberg [7, 9]. Our method, which requires essentially nothing apart from the action of a transvection and a simple calculation for quadratic forms, yields (i), (ii), and (iii) at one fell stroke. Part (iii) is well known for finite K. It is then a consequence of the fact that $| K^+/L | = 2$ (see above) and Dickson's result [3, p. 197; 5, p. 34] that there are two types of nondegenerate quadratic forms on V. Bourbaki [2, pp. 112, 113] gives exercises for the reader leading to canonical forms for quadratic forms on V when K is perfect, and a field extension criterion for two such canonical forms to be equivalent. It is not difficult to deduce (iii) from this criterion. However, we shall adopt the reverse procedure and deduce the canonical form for $Q(\mathbf{x})$ as an immediate consequence of our approach to the Theorem.

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We now present the proof. If $\mathbf{f}_1$, $\mathbf{f}_2$,..., $\mathbf{f}_{2n}$ is a symplectic base for $B(\mathbf{x}, \mathbf{y})$ then, because of the form (2), the linear map $\mathbf{T}$ given by $\mathbf{Te}_u = \mathbf{f}_u$ $(1\ u \leqslant \leqslant 2n)$ is a member of the symplectic group of $B(\mathbf{x}, \mathbf{y})$ and so [6, p. 10] is the product of symplectic transvections. Hence, since each transvection takes one symplectic base to another, to establish (i) we need only prove that the difference of the Arf invariant of $Q(\mathbf{x})$ with respect to $\mathbf{Te}_1$,..., $\mathbf{Te}_{2n}$ and $\Delta(Q)$ is in L for each symplectic transvection $\mathbf{T}$. So suppose, now, that $\mathbf{T}$ is the transvection [6, pp. 9, 10],

$$\mathbf{T} : \mathbf{x} \to \mathbf{x} + \lambda B(\mathbf{x}, \mathbf{q})\mathbf{q}, \tag{4}$$

where $\mathbf{q} \neq \mathbf{0}$ in V and $\lambda \neq 0$ in K. Then, from (1),

$$Q(\mathbf{Tx}) = Q(\mathbf{x}) + [\lambda^2 Q(\mathbf{q}) + \lambda]\, [B(\mathbf{x}, \mathbf{q})]^2. \tag{5}$$

Writing $\mathbf{q} = \sum_{u=1}^{2n} q_u \mathbf{e}_u$ we see that if $\mu \in K$ then the Arf invariant with respect to the base $\mathbf{e}_1$,..., $\mathbf{e}_{2n}$ of

$$\hat{\mathrm{Q}}(\mathbf{x}) = Q(\mathbf{x}) + \mu[B(\mathbf{x}, \mathbf{q})]^2 \tag{6}$$

is

$$\Delta(\hat{Q}) = \sum_{i=1}^{n} [Q(\mathbf{e}_i) + \mu q_{n+i}^2][Q(\mathbf{e}_{n+i}) + \mu q_i^2]$$

$$= \Delta(Q) + \mu \sum_{i=1}^{n} [q_i^2 Q(\mathbf{e}_i) + q_{n+i}^2 Q(\mathbf{e}_{n+i})] + \mu^2 \sum_{i=1}^{n} q_i^2 q_{n+i}^2$$

$$= \Delta(Q) + \mu \left[Q(\mathbf{q}) + \sum_{i=1}^{n} q_i q_{n+i}\right] + \left[\mu \sum_{i=1}^{n} q_i q_{n+i}\right]^2$$

$$= \Delta(Q) + \mu Q(\mathbf{q}) + \mu \sum_{i=1}^{n} q_i q_{n+i} + \left[\mu \sum_{i=1}^{n} q_i q_{n+i}\right]^2, \tag{7}$$

by successive use of (2), (3), and (2) with (1). Thus $\Delta(\hat{Q}) - \Delta(\hat{Q}) \in L$ if and only if $\mu Q(\mathbf{q}) \in L$.

When $\hat{Q}(\mathbf{x}) = Q(\mathbf{Tx})$ then $\Delta(\hat{Q})$ is the Arf invariant of $Q(\mathbf{x})$ with respect to $\mathbf{Te}_1, \ldots, \mathbf{Te}_{2n}$, and, by (5), $\mu Q(\mathbf{q}) = [\lambda Q(\mathbf{q})]^2 + \lambda Q(\mathbf{q}) \in L$: (i) follows.

If $Q_2(\mathbf{x})$ is equivalent to $Q(\mathbf{x})$ then there is an $\mathbf{S}$ in $\mathrm{GL}_{2n}(K)$ such that $Q_2(\mathbf{x}) = Q(\mathbf{Sx})$. Then, from (1), the associated polar form of $Q_2(\mathbf{x})$ is $B(\mathbf{Sx}, \mathbf{Sy})$ and so $\mathbf{S}^{-1}\mathbf{e}_1, \ldots, \mathbf{S}^{-1}\mathbf{e}_{2n}$ form a symplectic base for $Q_2(\mathbf{x})$. Its Arf invariant with respect to this base is

$$\sum_{i=1}^{n} Q_2(\mathbf{S}^{-1}\mathbf{e}_i)\, Q_2(\mathbf{S}^{-1}\mathbf{e}_{n+i}) = \sum_{i=1}^{n} Q(\mathbf{SS}^{-1}\mathbf{e}_i)\, Q(\mathbf{SS}^{-1}\mathbf{e}_{n+i}) = \Delta(Q):$$

(ii) follows immediately.

Now suppose that K is perfect and that $\bar{\Delta}(Q_1) = \bar{\Delta}(Q_2)$. Since (2) is the canonical form for any nonsingular alternating bilinear form we need, in view of (ii), only prove that $Q_1(\mathbf{x})$ and $Q_2(\mathbf{x})$ are equivalent when $Q_1(\mathbf{x}) = Q(\mathbf{x})$ and $Q_2(\mathbf{x})$ has $B(\mathbf{x}, \mathbf{y})$ for its associated polar form in order to establish (iii). Then, from (1), the associated polar form of $Q_2(\mathbf{x}) - Q(\mathbf{x})$ is the null form and so $Q_2(\mathbf{x}) + Q(\mathbf{x}) = \sum_{u=1}^{2n} a_u x_u^2$ for some a_u $(1 \leqslant u \leqslant 2n)$ in K. Since K is perfect a_u is a square: write $a_i = q_{n+i}^2$, $a_{n+i} = q_i^2$ $(1 \leqslant i \leqslant n)$. Then, by (2), $Q_2(\mathbf{x})$ is the $\hat{Q}(\mathbf{x})$ of (6) with $\mu = 1$. Hence, by the remark after (7), $Q(\mathbf{q}) \in L$, and so there is a $\nu \in K$ such that $Q(\mathbf{q}) = \nu^2 + \nu$. As $\nu + 1$ will also serve we may assume that $\nu \neq 0$. Let $\lambda = \nu^{-1}$. Then

$$\lambda^2 Q(\mathbf{q}) + \lambda = \nu^{-2}[Q(\mathbf{q}) + \nu] = \nu^{-2}\nu^2 = 1.$$

If $\mathbf{T}$ is the transvection of (4) for this λ then, by (5) and (6),

$$Q_2(\mathbf{x}) = \hat{Q}(\mathbf{x}) = Q(\mathbf{x}) + [B(\mathbf{x}, \mathbf{q})]^2 = Q(\mathbf{Tx}),$$

and (iii) is proved.

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We have the

Corollary. *If $Q(\mathbf{x})$ is a nondegenerate quadratic form on a vector space of dimension $2n$ over a perfect field of characteristic 2 then coordinates may be chosen so that*

$$Q(\mathbf{x}) = \sum_{i=1}^{n} x_i x_{n+i} + \nu({x_n}^2 + x_{2n}^2).$$

Proof. This form has the $B(\mathbf{x}, \mathbf{y})$ of (2) for its polar form and with respect to $\mathbf{e}_1, \ldots, \mathbf{e}_{2n}$ its Arf invariant is ν^2. Since ν^2 takes all values in K as ν varies the result follows from (iii).

An immediate consequence is that $Q(\mathbf{x})$ has index n or $n - 1$.

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