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PARTICLE DIFFUSION IN ELECTROSTATIC PRECIPITATORS

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Abstract—A summary is presented of the main work in the transient diffusion of (mainly) passive scalars in tubes and channels with the viewpoint of seeking to apply methods of analysis in these areas to the consideration of the diffusion of particles in turbulent flow in a parallel plate electrostatic precipitator. Our theoretical studies indicate that while there is diffusion of dust particles away from the walls of the precipitator the effect of the applied electric field is such to drive the particles towards the walls. The theory appears to be applicable to two types of precipitator: (1) a uniform electric field applied between the plates and (2) a non-uniform electric field generated from wires placed in the flow field between the plates.

1. INTRODUCTION

Much engineering effort is currently being devoted to an understanding of the operation of electrostatic precipitators. One reason for this is the demand of the Environmental Protection Agency on the utilities in the U.S. to contain or control particulate emissions into the lower atmosphere. Experimental efforts to improve the efficiency of operation of these precipitators are of basic interest in connection with two-phase flow problems, in particular one would wish to significantly increase the collection efficiency of the precipitator in order to have as many particles as possible travel to the collecting plates from which they can then be removed by a variety of means.

In connection with this entrainment problem the objective of the present paper is to provide insight into a transient diffusion process which appears to offer promise of engineering benefits.

2. PROBLEM STATEMENT

In view of our conviction that diffusion plays a key role in improving the efficiency of the deposition of particulate matter in electrostatic precipitator problems, the following mathematical model is postulated as being (at this time) worthy of consideration.

The flow is assumed to be constrained between two parallel flat plates a distance a^* apart. Spherical particles of solid particulate matter are transported by a gas flow, which is taken to be a fully developed turbulent flow, and the continuous as well as dispersed phases are subject to the continuous application of an electric field applied between the plates. Many forces act on a particle as it is being transported by the moving stream of gas and, in our case, since the particles are heavy (their density divided by the density of the gas is a ratio much less than unity) two effects are considered to be important. These are now described. Consider the following equation for continuity of dust concentration in the flow:

$$\frac{\partial C}{\partial t} + \mathbf{v}\,\text{grad}\,C = \text{div}\,K\,\text{grad}\,C + \text{div}\left[k_1\left(\frac{\partial u}{\partial z}\right)^{\alpha} C\hat{\mathbf{k}} + k_2 EqC\hat{\mathbf{k}}\right]. \quad (1)$$

Equation (1) with $k_1 = k_2 = 0$ has been used in the analysis of a number of transient diffusion problems, e.g. Taylor[1, 2], Aris[3], Gill and Sankarasubramanian[4] which has many references, Fischer[5], Fife and Nicholes[6] and Swan[7]. The additional flux terms contained within the square brackets in (1) are described more fully below. Equation (1) can be derived from first principles as presented in Swan[8]. The quantity C is identified with the concentration of particles in the dispersed phase, $\mathbf{v} = u(z)\hat{\mathbf{i}}$ is the mass average velocity vector of the binary mixture, and K is the eddy diffusivity, which is considered to be a function of the distance coordinate z between the plates. A right-handed axis system is chosen with origin of coordinates fixed in one of the plates and is such that x is in the positive direction of the flow, the plates are parallel to the x, y plane and the z-axis is perpendicular to them; $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ denote the three unit vectors. The term $(\partial u/\partial z)^{\alpha}C$, $0 < \alpha < 1$ is assumed to be representative of the flux of concentration due to lift forces acting on the particle. This form for the lift force with $\alpha = 1/2$ is due to Saffman[9] and has been employed by various authors; see, e.g. DiGiovanni and Lee[10]. The effect of the electric field on the particles is assumed to be represented by the flux term k_2EqC with q denoting the charge on the particle. In (1), k_1 and k_2 are assumed to be constants. In the above model the effect of gravity on the particles has been neglected. (If one wished to include this effect then an appropriate flux term could be added. For heavy particles the flux term has been used by LoDato[11, p. 196] to describe Brownian motion in a gravitational force field.)

Our mathematical model is taken to be representative of an electrostatic precipitator consisting of either (1) a uniform electric field perpendicular to the plates with charged particles in the flow field, and consistent with experimental evidence that suggests that the effect of electric wind can be neglected, or (2) with a spatially varying electric field generated by passing current through wires, parallel to the plates, and located at fixed intervals between the plates.

The various techniques which were employed in the references cited above have provided insight into a number of basic problems in transient diffusion in pipes and channels. Accordingly it was felt that utilization of one or more of these approaches would provide insight into the diffusion problem governed by (1). Since the mathematical development of Gill *et al.*[12–17, 4] produced considerable insight, our choice of analytical approach lay in trying to apply the Gill approach to (1). However it was found that this did not work, which at first seemed rather surprising. The solution to (1) which *did* work is presented in Section 4 together with the physical insight gained for the diffusion process.

Our efforts in this paper are concentrated on making certain that the analytical and physical insights obtained are placed on a firm foundation. Subsequently it is intended to report on the experimental results and then on detailed numerical computations. Further details on these matters are discussed in Section 5.

Construction of the two types of electrostatic precipitator described in the present section is presently under way at Washington State University. It is hoped to have results from experiments on these precipitators in the not too distant future.

3. NONDIMENSIONAL PROBLEM

In the following portion of the paper, quantities with an asterisk are assumed to have physical dimension. Let (x^*, y^*, z^*) be a rectangular coordinate system and let t^* denote the time. Assume that the flow is in the x^* direction. Variations in the y^* direction will not be considered. Further, suppose that the origin of the coordinate system lies in the face of one of the collecting plates such that the positive z^* axis is directed towards the second plate. The distance between the plates is denoted by a^* as above. Then, eqn (1), including the flux terms mentioned above, which governs the diffusion of the dispersed ionized substance with concentration $C^* = C^*(x^*, z^*, t^*)$ in the host fluid becomes

$$\frac{\partial C^*}{\partial t^*} + u^*(z^*)\frac{\partial C^*}{\partial x^*} = K^*(z^*)\left(\frac{\partial^2 C^*}{\partial x^{*2}} + \frac{\partial^2 C^*}{\partial z^{*2}}\right) + \frac{dK^*(z^*)}{dz^*}\frac{\partial C^*}{\partial z^*} + k_1^*\left\{\alpha\left[\frac{du^*(z^*)}{dz^*}\right]^{\alpha-1}\frac{d^2u^*(z^*)}{dz^{*2}}C^* + \left[\frac{du^*(z^*)}{dz^*}\right]^{\alpha}\frac{\partial C^*}{\partial z^*}\right\} + k_2^* q^*\left[\frac{dE^*(z^*)}{dz^*}C^* + E^*(z^*)\frac{\partial C^*}{\partial z^*}\right] \tag{2}$$

where $0 < z^* < a^*$, $0 < x^* < b^*$, and $t^* > 0$. In addition, $u^*(z^*)$ is the mass average velocity profile of the mixture. It is assumed, in the precipitator itself, that there are no sources or sinks which could affect the concentration apart from the electric current sources. The initial and boundary conditions are given by

$$C^*(x^*, z^*, 0) = 0 \text{ for } 0 < x^* < b^* \text{ and } 0 < z^* < a^*; \tag{3}$$

and

$$C^*(0, z^*, t^*) = C_0^* \text{ for } 0 < z^* < a^* \text{ and } t^* > 0. \tag{4}$$

Here the concentration on the plane $x^* = 0$ is assumed (in a first approximation) to be a constant. Also, it seems reasonable to apply a restriction on either the size of the concentration or the magnitude of the flux at the downstream end of the precipitator, in this case $x^* = b^*$.

Now, introduce the Galilean transformation given by

$$\xi^* = x^* - V^* t^* \quad \text{and} \quad \eta^* = t^* \tag{5}$$

where the mean speed of the flow, denoted by V^*, is defined by

$$V^* = \frac{1}{a^*}\int_0^{a^*} u^*(z^*)\, dz^*. \tag{6}$$

In terms of the new coordinate system (ξ^*, z^*, η^*) which now moves at the velocity V^*, eqn (2) becomes

$$\frac{\partial \tilde{C}^*}{\partial \eta^*} + [u^*(z^*) - V^*]\frac{\partial \tilde{C}^*}{\partial \xi^*} = K^*(z^*)\left(\frac{\partial^2 \tilde{C}^*}{\partial \xi^{*2}} + \frac{\partial^2 \tilde{C}^*}{\partial z^{*2}}\right) + \frac{dK^*(z^*)}{dz^*}\frac{\partial \tilde{C}^*}{\partial z^*} + k_1^*\left\{\alpha\left[\frac{du^*(z^*)}{dz^*}\right]^{\alpha-1}\frac{d^2u^*(z^*)}{dz^{*2}}\tilde{C}^* + \left[\frac{du^*(z^*)}{dz^*}\right]^{\alpha}\frac{\partial \tilde{C}^*}{\partial z^*}\right\} + k_2^* q^*\left[\frac{dE^*(z^*)}{dz^*}\tilde{C}^* + E^*(z^*)\frac{\partial \tilde{C}^*}{\partial z^*}\right] \tag{7}$$

where $\tilde{C}(\xi^*, z^*, \eta^*) = C^*(x^*, z^*, t^*)$.

In order to rewrite eqn (7) in a non-dimensional form, it is necessary to assume the existence of appropriate velocity, length, and concentration scales in the ordinary sense as discussed by Lin and Segel[18]. Here, the length scale is assumed to be a^* and the concentration scale is taken to be C_0^* rather than an exact scale. Let u_0^* denote the velocity scale which is not explicitly specified, but its existence is assumed. In terms of these ordinary scales, the following non-dimensional quantities (no asterisk) are defined:

$$\xi = \frac{\xi^* K_{0d}^*}{a^{*2} u_0^*},\quad z = \frac{z^*}{a^*},\quad \eta = \frac{\eta^* K_{0d}^*}{a^{*2}},\quad C(\xi, z, \eta) = \frac{\tilde{C}^*(\xi^*, z^*, \eta^*)}{C_0^*}$$

$$u(z) = \frac{u^*(z^*)}{u_0^*},\quad V = \frac{V^*}{u_0^*},\quad K(z) = \frac{K^*(z^*)}{K_{0d}^*},\quad E(z) = \frac{E^*(z^*)}{E_0^*},$$

$$Pe = \frac{a^*u_0^*}{K_{0d}^*},\quad k_1 = \frac{k_1^*(u_0^*)^{\alpha-1}}{a^{*\alpha}},\quad \text{and}\quad k_2 = \frac{k_2^* q^* E_0^*}{u_0^*}. \tag{8}$$

In (8) Pe is the Peclet number. The number k_1 comes from the Saffman lift term in (1). At the present time it is not clear what effect significant changes in this quantity can have on the dispersed particles. However, k_1 can be physically interpreted as the $(1-\alpha)$th power of the ratio of a velocity due to lift and the mass average velocity. Similarly, k_2 is a ratio of the velocity due to the electric field and the mass average velocity.

One of the goals of any experimental work based on the present paper would be to establish ranges for the parameters k_1 and k_2 *in addition to* looking at the behavior of the flow by varying the Peclet number. Indeed this is one of the main complexities of arriving at information concerning actual flow situations—a flow with three defining parameters is much more difficult to describe than a flow with one parameter.

The mathematical statement of the non-dimensional problem is obtained by transforming eqn (7) and the conditions (2)–(3) by the coordinate transformations (8). This yields

$$\frac{\partial C}{\partial \eta} + [u(z) - V]\frac{\partial C}{\partial \xi} = \frac{K(z)}{Pe^2}\frac{\partial^2 C}{\partial \xi^2} + K(z)\frac{\partial^2 C}{\partial z^2} + F_1(z)C + F_2(z)\frac{\partial C}{\partial z} \tag{9}$$

where

$$F_1(z) = \alpha k_1 Pe\left(\frac{\mathrm{d}u(z)}{\mathrm{d}z}\right)^{\alpha-1}\frac{\mathrm{d}^2u(z)}{\mathrm{d}z^2} + k_2 Pe\frac{\mathrm{d}E(z)}{\mathrm{d}z} \tag{10}$$

and

$$F_2(z) = k_1 Pe\left(\frac{\mathrm{d}u(z)}{\mathrm{d}z}\right)^{\alpha} + k_2 Pe E(z) + \frac{\mathrm{d}K(z)}{\mathrm{d}z}. \tag{11}$$

The boundary and initial conditions are now given by

$$C(\xi, z, 0) = 0 \tag{12}$$

and

$$C(-V\eta, z, \eta) = 1. \tag{13}$$

The domain of the problem is now given by $0 < z < 1$, $-V\eta < \xi < b - V\eta$, and $\eta > 0$.

4. METHOD OF SOLUTION

The Gill solution technique can now be employed in constructing a solution to eqn (9). However, a direct application results in an inconsistency and, therefore, fails to generate a solution. This can be demonstrated by assuming a separation of variables expansion for the concentration in the form

$$C(\xi, z, \eta) = \phi(\xi, \eta) + \sum_{n=1}^{\infty}\psi_n(z, n)\frac{\partial^n \phi}{\partial \xi^n}(\xi, \eta), \tag{14}$$

where the longitudinal diffusion is assumed to dominate the transverse diffusion. Here, $\phi(\xi, \eta)$ represents the main contribution to this longitudinal diffusion. Gill and Sankarasubramanian have pointed out that eqn (14) is only valid when the initial distribution is uniform across the channel. We assume that this is the situation here so that (14) is analogous to the Gill expansion. Substitution of eqn (14) into eqn (9) yields

$$\frac{\partial \phi}{\partial \eta} + [u(z) - V]\frac{\partial \phi}{\partial \xi} - \frac{K(z)}{Pe^2}\frac{\partial^2 \phi}{\partial \xi^2} - F_1(z)\phi + \sum_{n=1}^{\infty}\left\{\frac{\partial \psi_n}{\partial \eta}\frac{\partial^n \phi}{\partial \xi^n} + \psi_n\frac{\partial^{n+1}\phi}{\partial \eta \partial \xi^n} + [u(z) - V]\psi_n\frac{\partial^{n+1}\phi}{\partial \xi^{n+1}} - \frac{K(z)}{Pe^2}\psi_n\frac{\partial^{n+2}\phi}{\partial \xi^{n+2}} - K(z)\frac{\partial^2\psi_n}{\partial z^2}\frac{\partial^n \phi}{\partial \xi^n} - F_1(z)\psi_n\frac{\partial^n \phi}{\partial \xi^n} - F_2(z)\frac{\partial \psi_n}{\partial z}\frac{\partial^n \phi}{\partial \xi^n}\right\} = 0. \tag{15}$$

Next, a generalized dispersion model is assumed for ϕ in the form

$$\frac{\partial \phi}{\partial \eta} = \sum_{n=0}^{\infty} K_n\frac{\partial^n \phi}{\partial \xi^n} \tag{16}$$

where the K_n are yet to be determined. Utilizing eqn (16) in eqn (15) gives

$$[K_0 - F_1(z)]\phi + \left\{K_1 + [u(z) - V] + [K_0 - F_1(z)]\psi_1 + \frac{\partial \psi_1}{\partial \eta} - F_2(z)\frac{\partial \psi_1}{\partial z} - K(z)\frac{\partial^2 \psi_1}{\partial z^2}\right\}\frac{\partial \phi}{\partial \xi} + \left\{K_2 + [K_0 - F_1(z)]\psi_2 + [K_1 + u(z) - V]\psi_1 - \frac{K(z)}{Pe^2} + \frac{\partial \psi_2}{\partial \eta} - F_2(z)\frac{\partial \psi_2}{\partial z} - K(z)\frac{\partial^2 \psi_2}{\partial z^2}\right\}\frac{\partial^2 \phi}{\partial \xi^2} + \sum_{n=3}^{\infty}\left\{K_n + \sum_{m=0}^{n}\psi_{n-m}K_m + [u(z) - V]\psi_{n-1} - \frac{K(z)}{Pe^2}\psi_{n-2} - F_1(z)\psi_n + \frac{\partial \psi_n}{\partial \eta} - F_2(z)\frac{\partial \psi_n}{\partial z} - K(z)\frac{\partial^2 \psi_n}{\partial z^2}\right\}\frac{\partial^n \phi}{\partial \xi^n} = 0. \tag{17}$$

This equation is satisfied if the coefficients of the $\partial^n\phi/\partial\xi^n$ vanish for each $n = 0, 1, 2, \ldots$. In the case of $n = 0$, this implies that $K_0 = F_1(z)$ which defines K_0 as a function of z. Similarly, K_n is defined as a function of z for each n. Therefore, the left-hand side of eqn (16) is a function of ξ and η only while the right-hand side is a function of ξ, z and η. It follows that the two assumptions given by eqns (14) and (16) are incompatible, and the procedure fails.

In order to circumvent this difficulty, the main contribution to the diffusion process is assumed to be in the transverse direction rather than the longitudinal direction. However, this transverse diffusion must decay with increasing x^* as the flow moves through the precipitator. The introduction of a new factor $A(\xi)$ reflects this decaying property and insures that the contribution of the ψ_n is indeed small. In this case the separation of variables expansion for the concentration assumes the form

$$C(\xi, z, \eta) = A(\xi)\left[\phi(z,\eta) + \sum_{n=1}^{\infty} \psi_n(\xi, z, \eta)\frac{\partial^n\phi(z,\eta)}{\partial z^n}\right]. \tag{18}$$

Substitution of eqn (18) into eqn (9) now yields

$$\begin{aligned}
&A(\xi)\left(\frac{\partial\phi}{\partial\eta} - F_1(z)\phi - F_2(z)\frac{\partial\phi}{\partial z} - K(z)\frac{\partial^2\phi}{\partial z^2}\right.\\
&+\sum_{n=1}^{\infty}\left\{\frac{\partial\psi_n}{\partial\eta}\frac{\partial^n\phi}{\partial z^n} + \psi_n\frac{\partial^{n+1}\phi}{\partial\eta\partial z^n} + [u(z)-V]\frac{\partial\psi_n}{\partial\xi}\frac{\partial^n\phi}{\partial z^n}\right.\\
&-\frac{K(z)}{Pe^2}\frac{\partial^2\psi_n}{\partial\xi^2}\frac{\partial^n\phi}{\partial z^n} - K(z)\left[\psi_n\frac{\partial^{n+2}\phi}{\partial z^{n+2}} + \frac{\partial\psi_n}{\partial z}\frac{\partial^{n+1}\phi}{\partial z^{n+1}}\right.\\
&\left.+\frac{\partial^2\psi_n}{\partial z^2}\frac{\partial^n\phi}{\partial z^n}\right] - F_1(z)\psi_n\frac{\partial^n\phi}{\partial z^n} - F_2(z)\left[\psi_n\frac{\partial^{n+1}\phi}{\partial z^{n+1}}\right.\\
&\left.\left.\left.+\frac{\partial\psi_n}{\partial z}\frac{\partial^n\phi}{\partial z^n}\right]\right\}\right) + \frac{dA(\xi)}{d\xi}\left\{[u(z)-V]\left(\phi + \sum_{n=1}^{\infty}\psi_n\frac{\partial^n\phi}{\partial z^n}\right)\right.\\
&\left.-2\sum_{n=1}^{\infty}\frac{\partial\psi_n}{\partial\xi}\frac{\partial^n\phi}{\partial z^n}\right\} - \frac{K(z)}{Pe^2}\frac{d^2A(\xi)}{d\xi^2}\left(\phi + \sum_{n=1}^{\infty}\psi_n\frac{\partial^n\phi}{\partial z^n}\right) = 0.
\end{aligned} \tag{19}$$

Again, a generalized dispersion model of the form (16) with ξ replaced by z is assumed. If the K_n are considered to be functions of η alone as Gill has hypothesized, eqns (18) and (16) are inconsistent in that $K_0(\eta) = F_1(z)$ as above. So, it is necessary to assume that $K_n = K_n(z, \eta)$ for all n where these K_n are yet to be determined. For clarity, eqn (16) is then rewritten as

$$\frac{\partial\phi}{\partial\eta} = \sum_{n=0}^{\infty} K_n(z,\eta)\frac{\partial^n\phi}{\partial z^n}. \tag{20}$$

Substitution of (20) into eqn (19) then yields after a little regrouping

$$\begin{aligned}
&\left\{A(\xi)\left[K_0(z,\eta) - F_1(z) + \sum_{n=1}^{\infty}\psi_n\frac{\partial^n K_0(z,\eta)}{\partial z^n}\right]\right.\\
&\left.+[u(z)-V]\frac{dA(\xi)}{d\xi} - \frac{K(z)}{Pe^2}\frac{d^2A(\xi)}{d\xi^2}\right\}\phi\\
&+\left(A(\xi)\left\{K_1(z,\eta)\right.\right.\\
&-F_2(z) - F_1(z)\psi_1 + \frac{\partial\psi_1}{\partial\eta} + [u(z)-V]\frac{\partial\psi_1}{\partial\xi}\\
&-\frac{K(z)}{Pe^2}\frac{\partial^2\psi_1}{\partial\xi^2} - F_2(z)\frac{\partial\psi_1}{\partial z} - K(z)\frac{\partial^2\psi_1}{\partial z^2}\\
&\left.+\sum_{n=1}^{\infty}\left[n\psi_n\frac{\partial^{n-1}K_0(z,\eta)}{\partial z^{n-1}} + \psi_n\frac{\partial^n K_1(z,\eta)}{\partial z^n}\right]\right\}\\
&\left.+\frac{dA(\xi)}{d\xi}\left\{\psi_1[u(z)-V] - 2\frac{\partial\psi_1}{\partial\xi}\right\} - \psi_1\frac{K(z)}{Pe^2}\frac{d^2A(\xi)}{d\xi^2}\right)\frac{\partial\phi}{\partial z}\\
&+\left(A(\xi)\left\{K_2(z,\eta) - K(z) - F_2(z)\psi_1 - F_1(z)\psi_2 + \frac{\partial\psi_2}{\partial\eta}\right.\right.\\
&+[u(z)-V]\frac{\partial\psi_2}{\partial\xi} - \frac{K(z)}{Pe^2}\frac{\partial^2\psi_2}{\partial\xi^2} - F_2(z)\frac{\partial\psi_2}{\partial z}\\
&-K(z)\frac{\partial^2\psi_2}{\partial z^2} - K(z)\frac{\partial\psi_1}{\partial z}\\
&+\sum_{n=1}^{\infty}\left[\frac{n(n+1)}{2}\psi_{n+1}\frac{\partial^{n-1}K_0(z,\eta)}{\partial z^{n-1}}\right.\\
&\left.\left.+n\psi_n\frac{\partial^{n-1}K_1(z,\eta)}{\partial z^{n-1}} + \psi_n\frac{\partial^n K_2(z,\eta)}{\partial z^n}\right]\right\}\\
&+\frac{dA(\xi)}{d\xi}\left\{\psi_2[u(z)-V]\right.\\
&\left.\left.-2\frac{\partial\psi_2}{\partial\xi}\right\} - \psi_2\frac{K(z)}{Pe^2}\frac{d^2A(\xi)}{d\xi^2}\right)\frac{\partial^2\phi}{\partial z^2}\\
&+\sum_{n=1}^{\infty}\left(A(\xi)\left\{K_{n+2}(z,\eta)\right.\right.\\
&-F_2(z)\frac{\partial\psi_{n+2}}{\partial z} - K(z)\frac{\partial^2\psi_{n+2}}{\partial z^2} - K(z)\frac{\partial\psi_{n+1}}{\partial z}\\
&-K(z)\psi_n - F_2(z)\psi_{n+1} - F_1(z)\psi_{n+2} + \frac{\partial\psi_{n+2}}{\partial\eta}\\
&+[u(z)-V]\frac{\partial\psi_{n+2}}{\partial\xi} - \frac{K(z)}{Pe^2}\frac{\partial^2\psi_{n+2}}{\partial\xi^2} + \sum_{m=1}^{\infty}\psi_m\frac{\partial^m K_{n+2}(z,\eta)}{\partial z^m}\\
&\left.+\sum_{m=1}^{n+2}\sum_{p=m}^{\infty}\binom{p}{m}\psi_p\frac{\partial^{p-m}K_{n+2-m}(z,\eta)}{\partial z^{p-m}}\right\}\\
&+\frac{dA(\xi)}{d\xi}\left\{\psi_{n+2}[u(z)-V] - 2\frac{\partial\psi_{n+2}}{\partial\xi}\right\}\\
&\left.-\psi_{n+2}\frac{K(z)}{Pe^2}\frac{d^2A(\xi)}{d\xi^2}\right)\frac{\partial^{n+2}\phi}{\partial z^{n+2}} = 0
\end{aligned} \tag{21}$$

where the usual definition for the binomial coefficients has been employed.

Equation (21) is certainly satisfied if the coefficients of the $\partial^n\phi/\partial z^n$ vanish for each n. Proceeding formally, this is assumed to be the case. Setting the coefficient of ϕ equal to zero in eqn (21) gives

$$\begin{aligned}
&A(\xi)\left[K_0(z,\eta) + \sum_{n=1}^{\infty}\psi_n\frac{\partial^n K_0(z,\eta)}{\partial z^n} - F_1(z)\right]\\
&+[u(z)-V]\frac{dA(\xi)}{d\xi} - \frac{K(z)}{Pe^2}\frac{d^2A(\xi)}{d\xi^2} = 0.
\end{aligned} \tag{22}$$

In order to proceed, it is necessary to obtain an explicit expression for K_0. To accomplish this, the choice for ϕ must be made more precise. Let ϕ be defined by

$$\phi(z,\eta) = \frac{1}{b}\int_{-V\eta}^{b-V\eta} C(\xi, z, \eta)A^{-1}(\xi)\,d\xi. \tag{23}$$

Here $A(\xi)$ is assumed to be a non-vanishing function of ξ on the interval $[-V\eta, b - V\eta]$ in analogy with an exponential decay although no specific form for $A(\xi)$ is yet assumed. Multiplying eqn (18) by $A^{-1}(\xi)$ and integrating with respect to ξ yields

$$\begin{aligned}
&\int_{-V\eta}^{b-V\eta} A^{-1}(\xi)C(\xi,z,\eta)\,d\xi\\
&= b\phi(z,\eta) + \sum_{n=1}^{\infty}\frac{\partial^n\phi(z,\eta)}{\partial z^n}\int_{-V\eta}^{b-V\eta}\psi_n(\xi,z,\eta)\,d\xi.
\end{aligned} \tag{24}$$

Dividing eqn (24) by b and using eqn (23) gives

$$\phi(z,\eta)=\phi(z,\eta)+\frac{1}{b}\sum_{n=1}^{\infty}\frac{\partial^n\phi(z,\eta)}{\partial z^n}\int_{-V\eta}^{b-V\eta}\psi_n(\xi,z,\eta)\,d\xi. \tag{25}$$

Equation (25) is satisfied provided that the auxiliary conditions,

$$\int_{-V\eta}^{b-V\eta}\psi_n(\xi,z,\eta)\,d\xi=0, \tag{26}$$

are obeyed for each $n=1,2,\ldots$. So, the ψ_n must be derived subject to this restriction. Now, multiplication of eqn (22) by $A^{-1}(\xi)$, integration with respect to ξ, division by b, and use of eqns (26) give an equation for K_0 as

$$K_0(z,\eta)-F_1(z)+\frac{1}{b}\left\{[u(z)-V]\ln\left[\frac{A(b-V\eta)}{A(-V\eta)}\right]-\frac{K(z)}{Pe^2}B(\eta)\right\}=0. \tag{27}$$

where

$$B(\eta)=\int_{-V\eta}^{b-V\eta}A^{-1}(\xi)\frac{d^2A(\xi)}{d\xi^2}\,d\xi. \tag{28}$$

Now, K_1 can be evaluated in a fashion similar to the above procedure for K_0. Equating the coefficient of $\partial\phi/\partial z$ in eqn (21) to zero and multiplying by $A^{-1}(\xi)$ gives

$$\begin{aligned}&K_1(z,\eta)-F_2(z)\frac{\partial\psi_1}{\partial z}-K(z)\frac{\partial^2\psi_1}{\partial z^2}-F_2(z)-F_1(z)\psi_1\\&+\frac{\partial\psi_1}{\partial\eta}+[u(z)-V]\frac{\partial\psi_1}{\partial\xi}-\frac{K(z)}{Pe^2}\frac{\partial^2\psi_1}{\partial\xi^2}\\&+\sum_{n=1}^{\infty}\left[n\psi_n\frac{\partial^{n-1}K_0(z,\eta)}{\partial z^{n-1}}+\psi_n\frac{\partial^nK_1(z,\eta)}{\partial z^n}\right]\\&+A^{-1}(\xi)\frac{dA(\xi)}{d\xi}\left\{\psi_1[u(z)-V]-2\frac{\partial\psi_1}{\partial\xi}\right\}\\&-\psi_1\frac{K(z)}{Pe^2}\frac{d^2A(\xi)}{d\xi^2}A^{-1}(\xi)=0.\end{aligned} \tag{29}$$

Integration of eqn (29), dividing by b, and using the conditions (26) yield

$$\begin{aligned}&K_1(z,\eta)-F_2(z)+\frac{1}{b}\left\{[u(z)-V][\psi_1(b-V\eta,z,\eta)\right.\\&-\psi_1(V\eta,z,\eta)]-\frac{K(z)}{Pe^2}\left[\frac{\partial\psi_1}{\partial\xi}(b-V\eta,z,\eta)\right.\\&\left.\left.-\frac{\partial\psi_1}{\partial\xi}(-V\eta,z,\eta)\right]\right\}+\frac{1}{b}[u(z)-V]\\&\times\int_{-V\eta}^{b-V\eta}\psi_1(\xi,z,\eta)A^{-1}(\xi)\frac{dA(\xi)}{d\xi}d\xi\\&-\frac{2}{b}\int_{-V\eta}^{b-V\eta}\frac{\partial\psi_1(\xi,z,\eta)}{\partial\xi}A^{-1}(\xi)\frac{dA(\xi)}{d\xi}d\xi\\&-\frac{K(z)}{bPe^2}\int_{-V\eta}^{b-V\eta}\psi_1(\xi,z,\eta)A^{-1}(\xi)\frac{d^2A}{d\xi^2}d\xi=0.\end{aligned} \tag{30}$$

A similar procedure applied to the coefficients of $\partial^2\phi/\partial z^2$ and $\partial^{n+2}\phi/\partial z^{n+2}$ in eqn (21) gives the following results for K_2 and K_{n+2}:

$$\begin{aligned}&K_2(z,\eta)-K(z)+\frac{1}{b}\left\{[u(z)-V]\right.\\&\times[\psi_2(b-V\eta,z,\eta)-\psi_2(-V\eta,z,\eta)]\\&\left.-\frac{K(z)}{Pe^2}\left[\frac{\partial\psi_2}{\partial\xi}(b-V\eta,z,\eta)-\frac{\partial\psi_2}{\partial\xi}(-V\eta,z,\eta)\right]\right\}\\&+\frac{1}{b}[u(z)-V]\int_{-V\eta}^{b-V\eta}\psi_2(\xi,z,\eta)A^{-1}(\xi)\frac{dA(\xi)}{d\xi}d\xi\\&-\frac{2}{b}\int_{-V\eta}^{b-V\eta}\frac{\partial\psi_2}{\partial\xi}(\xi,z,\eta)A^{-1}(\xi)\frac{dA(\xi)}{d\xi}d\xi\\&-\frac{K(z)}{bPe^2}\int_{-V\eta}^{b-V\eta}\psi_2(\xi,z,\eta)A^{-1}(\xi)\frac{d^2A(\xi)}{d\xi^2}d\xi=0\end{aligned} \tag{31}$$

and

$$\begin{aligned}&K_{n+2}(z,\eta)+\frac{1}{b}\left\{[u(z)-V]\right.\\&\times[\psi_{n+2}(b-V\eta,z,\eta)-\psi_{n+2}(-V\eta,z,\eta)]\\&\left.-\frac{K(z)}{Pe^2}\left[\frac{\partial\psi_{n+2}}{\partial\xi}(b-V\eta,z,\eta)-\frac{\partial\psi_{n+2}}{\partial\xi}(-V\eta,z,\eta)\right]\right\}\\&+\frac{1}{b}[u(z)-V]\int_{-V\eta}^{b-V\eta}\psi_{n+2}(\xi,z,\eta)A^{-1}(\xi)\frac{dA(\xi)}{d\xi}d\xi\\&-\frac{2}{b}\int_{-V\eta}^{b-V\eta}\frac{\partial\psi_{n+2}(\xi,z,\eta)}{\partial\xi}A^{-1}(\xi)\frac{dA(\xi)}{d\xi}d\xi\\&-\frac{K(z)}{bPe^2}\int_{-V\eta}^{b-V\eta}\psi_{n+2}(\xi,z,\eta)A^{-1}(\xi)\frac{d^2A(\xi)}{d\xi^2}d\xi=0.\end{aligned} \tag{32}$$

Combining eqns (20), (27), (30)–(32) the generalized dispersion model becomes

$$\begin{aligned}\frac{\partial\phi}{\partial\eta}=&\left(F_1(z)-\frac{1}{b}\left\{[u(z)-V]\ln\left[\frac{A(b-V\eta)}{A(-V\eta)}\right]-\frac{K(z)}{Pe^2}B(\eta)\right\}\right)\phi\\&+F_2(z)\frac{\partial\phi}{\partial z}+K(z)\frac{\partial^2\phi}{\partial z^2}-\sum_{n=1}^{\infty}\frac{\partial^n\phi}{\partial z^n}\left(\frac{1}{b}\left\{[u(z)-V]\right.\right.\\&\times\left[\psi_n(b-V\eta,z,\eta)-\psi_n(-V\eta,z,\eta)\right.\\&+\int_{-V\eta}^{b-V\eta}\psi_n(\xi,z,\eta)\\&\left.\times A^{-1}(\xi)\frac{dA(\xi)}{d\xi}d\xi\right]-\frac{K(z)}{Pe^2}\left[\frac{\partial\psi_n}{\partial\xi}(b-V\eta,z,\eta)\right.\\&\left.-\frac{\partial\psi_n}{\partial\xi}(-V\eta,z,\eta)+\int_{-V\eta}^{b-V\eta}\psi_n(\xi,z,\eta)A^{-1}(\xi)\frac{d^2A(\xi)}{d\xi^2}d\xi\right]\\&-2\int_{-V\eta}^{b-V\eta}A^{-1}(\xi)\frac{\partial\psi_2}{\partial\xi}(\xi,z,\eta)\frac{dA(\xi)}{d\xi}d\xi.\end{aligned} \tag{33}$$

Now $F_1(z)$, $F_2(z)$, $K(z)$ and $u(z)$ are assumed to be known and determinable from experimental data. So, eqn (33) is a partial differential equation for the determination of ϕ once the ψ_n and A are known. Equations for these quantities can be written from the assumption that the coefficients of eqn (21) vanish. In this case, the coefficient of $\partial^n\phi/\partial z^n$ provides an equation for the determination of ψ_n, and the coefficient of ϕ provides an equation for A. However, this equation for A is a particularly difficult integrodifferential equation. Since ϕ, ψ_n, and A are still to be determined such that the expression (18) satisfies the given problem, it is most convenient to choose a suitable form for A, and then to proceed to find ϕ and the ψ_n such that eqn (18) is a solution with this particular choice of A. An exponential function seems to be a natural choice for a diffusion process so let

$$A(\xi)=e^{-\lambda\xi} \tag{34}$$

where λ is yet to be determined from the boundary conditions.

With this choice of A the equations for the K_n become with the help of eqn (26)

$$K_0(z,\eta)=F_1(z)-\lambda[u(z)-V]+\lambda^2\frac{K(z)}{Pe^2}, \tag{35}$$

$$\begin{aligned}K_1(z,\eta)=F_2(z)-\frac{1}{b}\Big\{&[u(z)-V][\psi_1(b-V\eta,z,\eta)\\&-\psi_1(-V\eta,z,\eta)]-\frac{K(z)}{Pe^2}\Big[\frac{\partial\psi_1}{\partial\xi}(b-V\eta,z,\eta)\\&-\frac{\partial\psi_1}{\partial\xi}(-V\eta,z,\eta)\Big]\Big\},\end{aligned} \tag{36}$$

$$\begin{aligned}K_2(z,\eta)=K(z)-\frac{1}{b}\Big\{&[u(z)-V]\\&\times[\psi_2(b-V\eta,z,\eta)-\psi_2(-V\eta,z,\eta)]\\&-\frac{K(z)}{Pe^2}\Big[\frac{\partial\psi_2}{\partial\xi}(b-V\eta,z,\eta)-\frac{\partial\psi_2}{\partial\xi}(-V\eta,z,\eta)\Big]\Big\},\end{aligned} \tag{37}$$

$$\begin{aligned}K_{n+2}(z,\eta)=-\frac{1}{b}\Big\{&[u(z)-V]\\&\times[\psi_{n+2}(b-V\eta,z,\eta)-\psi_{n+2}(-V\eta,z,\eta)]\\&-\frac{K(z)}{Pe^2}\Big[\frac{\partial\psi_{n+2}}{\partial\xi}(b-V\eta,z,\eta)\\&-\frac{\partial\psi_{n+2}}{\partial\xi}(-V\eta,z,\eta)\Big].\end{aligned} \tag{38}$$

In this case, the generalized dispersion model becomes

$$\begin{aligned}\frac{\partial\phi}{\partial\eta}=&\Big\{F_1(z)-\lambda[u(z)-V]+\lambda^2\frac{K(z)}{Pe^2}\Big\}\phi+F_2(z)\frac{\partial\phi}{\partial z}\\&+K(z)\frac{\partial^2\phi}{\partial z^2}-\frac{1}{b}\sum_{n=1}^{\infty}\Big\{[u(z)-V][\psi_n(b-V\eta,z,\eta)\\&-\psi_n(-V\eta,z,\eta)]-\frac{K(z)}{Pe^2}\Big[\frac{\partial\psi_n}{\partial\xi}(b-V\eta,z,\eta)\\&-\frac{\partial\psi_n}{\partial\xi}(-V\eta,z,\eta)\Big]\Big\}\frac{\partial^n\phi}{\partial z^n}.\end{aligned} \tag{39}$$

Now, the equations for the ψ_n can be written from eqn (21). In order to shorten the notation, a partial differential operator L is defined by

$$\begin{aligned}L=&\frac{\partial}{\partial\eta}+[u(z)-V]\frac{\partial}{\partial\xi}-\frac{K(z)}{Pe^2}\frac{\partial^2}{\partial\xi^2}\\&-F_2(z)\frac{\partial}{\partial z}-K(z)\frac{\partial^2}{\partial z^2}.\end{aligned} \tag{40}$$

Then, using eqns (21), (34)–(38) and (40) yields the following system of equations for the ψ_n:

$$\begin{aligned}L\psi_1=&2\lambda\frac{\partial\psi_1}{\partial\xi}+\frac{1}{b}\Big\{[u(z)-V]\\&\times[\psi_1(b-V\eta,z,\eta)-\psi_1(-V\eta,z,\eta)]\\&-\frac{K(z)}{Pe^2}\Big[\frac{\partial\psi_1}{\partial\xi}(b-V\eta,z,\eta)-\frac{\partial\psi_1}{\partial\xi}(-V\eta,z,\eta)\Big]\Big\}\\&-\sum_{n=2}^{\infty}\Big[n\psi_n\frac{d^{n-1}K_0(z)}{dz^{n-1}}+\psi_{n-1}\frac{\partial^{n-1}K_1(z,\eta)}{\partial z^{n-1}}\Big],\end{aligned} \tag{41}$$

$$\begin{aligned}L\psi_2=&2\lambda\frac{\partial\psi_2}{\partial\xi}K(z)\frac{\partial\psi_1}{\partial z}+\frac{1}{b}\Big([u(z)-V]\Big\{[\psi_2(b-V\eta,z,\eta)\\&-\psi_2(-V\eta,z,\eta)]+\psi_1(\xi,\eta)\\&\times[\psi_1(b-V\eta,z,\eta)-\psi_1(-V\eta,z,\eta)]\\&-\frac{K(z)}{Pe^2}\Big\{\Big[\frac{\partial\psi_2}{\partial\xi}(b-V\eta,z,\eta)-\frac{\partial\psi_2}{\partial\xi}(-V\eta,z,\eta)\Big]\\&-\psi_1(\xi,\eta)\Big[\frac{\partial\psi_1}{\partial\xi}(b-V\eta,z,\eta)-\frac{\partial\psi_1}{\partial\xi}(-V\eta,z,\eta)\Big]\Big\}\Big)\\&-\sum_{n=2}^{\infty}\Big[\frac{n(n+1)}{2}\psi_{n+1}\frac{d^{n-1}K_0}{dz^{n-1}}+n\psi_n\frac{\partial^{n-1}K_1}{\partial z^{n-1}}\\&+\psi_{n-1}\frac{\partial^{n-1}K_2}{\partial z^{n-1}}\Big],\end{aligned} \tag{42}$$

and

$$\begin{aligned}L\psi_{n+2}=&K(z)\psi_n+F_2(z)\psi_{n+1}+F_1(z)\psi_{n+2}+K(z)\frac{\partial\psi_{n+1}}{\partial z}\\&+\frac{1}{b}\Big\{[u(z)-V]\\&\times[\psi_{n+2}(b-V\eta,z,\eta)-\psi_{n+2}(-V\eta,z,\eta)]\\&-\frac{K(z)}{Pe^2}\Big[\frac{\partial\psi_{n+2}}{\partial\xi}(b-V\eta,z,\eta)-\frac{\partial\psi_{n+2}}{\partial\xi}(-V\eta,z,\eta)\Big]\Big\}\\&-\lambda\Big[\psi_{n+2}[u(z)-V]-2\frac{\partial\psi_{n+2}}{\partial\xi}\Big]+\lambda^2\psi_{n+2}\frac{K(z)}{Pe^2}\\&-\sum_{m=1}^{\infty}\psi_m\frac{\partial^m K_{n+2}(z,\eta)}{\partial z^m}\\&-\sum_{m=1}^{n+2}\sum_{p=m}^{\infty}\binom{p}{m}\psi_p\frac{\partial^{p-m}K_{n+2-m}}{\partial z^{p-m}}.\end{aligned} \tag{43}$$

5. CONCLUSIONS

Equation (1) is to be regarded as an approximation to the real particle diffusion problem. One difficulty with (1) is that it is not easy to obtain information on the eddy diffusivity. This matter is currently under investigation and our initial findings suggest that information concerning it can be produced by using some adaptive system identification method. Briefly, from a knowledge of the experimental results it is reasonable to construct an approximate numerical method which can be optimized in order to produce estimates for the eddy diffusivity. The particular method is based on work reported by Angel and Bellman[19] and depends on applying quasilinearization; alternative methods are described by Phillipson[20].

This paper presents a theoretically consistent solution of the equation for concentration profiles in an electrostatic precipitator of rectangular cross-section. The solution (and flow) is characterized by three non-dimensional parameters and future work will be concerned with the examination of the changes in the particle concentrations when variations in these parameters are taken into consideration. The identification of these three parameters would appear to be of some importance in experimental situations.

Also, for the case of a variable electric field and a precipitator of finite size, more information is required on the effects of edge contributions, boundary layers, and the electric field itself, since the description by White[21, p. 97] is no longer appropriate.

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