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SOLUTIONS OF AN EVOLUTION EQUATION WHOSE INHOMOGENEOUS PART IS AN ENTIRE FUNCTION

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We study the evolution equation $y'(t) = Ay(t) + f(t)$, $y(0) = y_0$; $t \in [0, \infty)$, in a Banach space B, where the operator A is the infinitesimal generator of a bounded C_0-semigroup and $f(t)$ is an entire function. We establish sufficient conditions under which the solution can be represented in the form $y(t) = v(t) + g(t)$, where $g(t)$ is an entire function and $v(t)$ is the solution of the homogeneous equation with a certain condition. We also consider the inverse problem.

We consider the inhomogeneous evolution equation in a Banach space

$$\frac{dy(t)}{dt} = Ay(t) + f(t), \tag{1}$$

where A is the generator of a bounded semigroup of class C_0.

The authors [1] have studied Eq. (1) in the case when $f(t) = \sum_{k=0}^{n} a_k t^k$ is a polynomial. The asymptotics of the solution at infinity have been studied, and the inverse problem has been solved: *from a given solution, recover the right-hand side of the equation.* By analogy the problem arises of replacing the polynomial by an entire or analytic function. The following example shows that in general there is no analogous theorem. Consider Eq. (1) in a Hilbert space and assume that A is the generator of a semigroup $U(t)$ of class C_0, but $U(t)$ is not an analytic semigroup. Let $f(t) = \sum_{n=0}^{\infty} \frac{t^n e_n}{n!}$, where e_n is an orthonormal basis. The function $f(t)$ is an entire function. From the formula for solving the inhomogeneous equation (see [3]), the solution of the equation is not an analytic function in any nonzero sector.

We recall that the Cesàro limit of the function $\psi(t)$, $0 \le t \le \infty$, is defined as follows: $(C, 1)\text{-}\lim_{t\to\infty} \psi(t) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \psi(\xi)d\xi$ (see [4], p. 504).

It is known that if A is the generator of a bounded semigroup of class C_0, the Banach space decomposes into the direct sum of the closed image and the kernel of the operator A, that is, $B = \overline{R(A)} \oplus \operatorname{Ker} A$ (see [4], Theorem 18.6.2). The projection on $\operatorname{Ker} A$ will be denoted P.

Theorem 1. *Let $U(z)$ be a bounded analytic semigroup in a sector Θ that contains the real semi-axis, and let $f(t)$ be an entire function. Then the solution of Eq.* (1) *can be represented as a sum $v(t) + g(t)$, where $v(t)$ is the solution of the homogeneous equation corresponding to* (1) *with the condition $(C, 1)\text{-}\lim_{t\to\infty} v(t) = 0$, and $g(t)$ is an entire function.*

Proof. By Theorem 6.6 of [3, p. 159] the solution of Eq. (1) can be represented in the form $y(t) = U(t)y_0 + \int_0^t U(t-s)f(s)ds$. Taking account of the facts that the semigroup $U(z)$ is analytic in a sector Θ and the function $f(t)$ is entire, we see from the representation formula that the solution is an entire function.

As in [1], the inverse problem arises.

Theorem 2. *Suppose given an entire function $g(t)$ such that $Ag(t)$ is also an entire function and the linear operator A is the generator of a bounded semigroup of class C_0. Then there exists an entire function $f(t) = \sum_{k=0}^{\infty} a_k t^k$ such that any*

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solution of Eq. (1) *can be represented in the form* $y(t) = v(t) + g(t)$, *where* $v(t)$ *is the solution of the corresponding homogeneous equation with the condition* $(C, 1)\text{-}\lim_{t\to\infty} v(t) = 0$, *if and only if* $P(y_0 - b_0) = 0$.

Proof. Let $g(t) = \sum_{k=0}^{\infty} b_k t^k$. We find the solution $y(t)$ in the form $y(t) = v(t) + g(t)$, where $(C, 1)\text{-}\lim_{t\to\infty} v(t) = 0$. Because $g(t) \in D(A)$ and A is a closed operator, we can show that $b_k \in D(A)$, $k = 1, 2, \ldots$. Indeed, letting t tend to zero and using the fact that A is closed, we obtain $b_0 \in D(A)$. Dividing by t for $t \neq 0$, we have $b_1 + b_2 t + \ldots + b_n t^{n-1} + \ldots \in D(A)$. Continuing to use the fact that A is closed, we find $b_k \in D(A)$ $k = 1, 2, \ldots, n, \ldots$. From this we find that $Ag(t) = \sum_{k=0}^{\infty} Ab_k t^k$. We shall seek the function $f(t)$ in the form of a series $\sum a_k t^k$. Substituting into the equation, we obtain the equality $\sum_{k=1}^{\infty} kb_k t^{k-1} = A(\sum_{k=0}^{\infty} b_k t^k) + \sum_{k=0}^{\infty} a_k t^k$. Because A is closed, we have

$$\sum_{k=1}^{\infty} kb_k t^{k-1} = \sum_{k=0}^{\infty} (Ab_k + a_k) t^k.$$

Hence we obtain the following relations among the coefficients b_k and a_k:

$$\begin{aligned} a_0 &= b_1 - Ab_0, \\ a_1 &= 2b_2 - Ab_1, \\ &\ldots\ldots\ldots\ldots, \\ a_k &= (k+1)b_{k+1} - Ab_k, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

Since the function $\sum_{k=0}^{\infty} b_k t^k$ is entire, the function $\sum_{k=0}^{\infty} kb_k t^k$ is also. It then follows that $f(t)$ is an entire function, being the sum of entire functions. The second author [2] has established a necessary and sufficient condition for the existence of a solution of the following problem:

$$\frac{dy(t)}{dt} = Ay(t) + p, \qquad y(0) = y_0, \qquad (C, 1)\text{-}\lim_{t\to\infty} y(t) = y_\infty,$$

where p is a parameter and the operator A is the generator of a bounded semigroup of class C_0 (see [2]). The condition is that $Py_0 = Py_\infty$, where P is the projection on the kernel of the operator. Hence we find that the condition $P(y_0 - b_0) = 0$ necessarily holds, which completes the proof of Theorem 2.

Literature Cited

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