J. Math. Anal. Appl. 356 (2009) 237–241

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Codimension reduction and second fundamental form of CR submanifolds in complex space forms ☆

Mirjana Djorić

University of Belgrade, Faculty of Mathematics, Studentski trg 16, pb. 550, 11000 Belgrade, Serbia

ARTICLE INFO

Article history:
Received 25 August 2008
Available online 10 March 2009
Submitted by H.R. Parks

Keywords:
CR submanifold
Complex space form
Second fundamental form

ABSTRACT

Considering n-dimensional real submanifolds M of a complex space form $(\overline{M}^{n+p}, \bar{g}, J)$ which are CR submanifolds of CR dimension $\frac{n-m}{2}$, we study the condition $h(FX, Y) + h(X, FY) = 0$ on the structure tensor F naturally induced from the almost complex structure J of the ambient manifold and on the second fundamental form h of submanifolds M.
© 2009 Elsevier Inc. All rights reserved.

1. Introduction

The differential geometry of a submanifold M of an almost Hermitian manifold $(\overline{M}, J)$ depends on the behavior of the tangent bundle of M relative to the action of the almost complex structure J. According to this behavior, there exist three typical classes of submanifolds: holomorphic submanifolds (if the tangent bundle $T(M)$ of M is invariant by J, i.e., $J(T_x(M)) = T_x(M)$, for each $x \in M$), totally real submanifolds (if $J(T_x(M)) \subset T_x^{\perp}(M)$, for each $x \in M$) and CR submanifolds (if M is endowed with a pair of mutually orthogonal and complementary distributions $(\mathcal{D}, \mathcal{D}^{\perp})$ such that for any $x \in M$ we have $J\mathcal{D}_x = \mathcal{D}_x$ and $J\mathcal{D}_x^{\perp} \subset T_x^{\perp}(M)$), where $T_x^{\perp}(M)$ is the normal space to M at x.

Holomorphic submanifolds and totally real submanifolds are particular cases of CR submanifolds. Especially, each real hypersurface of $\overline{M}$ is a CR submanifold of CR dimension $\frac{n-1}{2}$, namely CR submanifold of maximal CR dimension, which is neither a holomorphic submanifold nor a totally real submanifold. Therefore, the generalization of some results which are valid for real hypersurfaces to CR submanifolds may be expected. It is known that real hypersurfaces of complex space forms have been a fertile field of research for years and many authors have described a lot of their geometric properties (see for example [10–12,14,17,13] for the fundamental definitions and results and for further references). Moreover, a real hypersurface M of a complex space form $\overline{M}$ has two geometric structures: namely, an almost contact structure ϕ induced from the complex structure J of $\overline{M}$, and a submanifold structure represented by the second fundamental tensor H of M in $\overline{M}$. In this sense, Okumura in [14] and Montiel and Romero in [12], studied and classified the real hypersurfaces M of a complex projective space and of a complex hyperbolic space, respectively, which satisfy the commutativity condition $\phi H = H\phi$. Above all, they gave a geometric meaning of the commutativity of the second fundamental tensor of the real hypersurface of a complex space form and its induced almost contact structure. In [4] and [6] Okumura and the author of this paper considered the same problem by studying CR submanifolds of CR dimension $\frac{n-1}{2}$ in complex space forms. Moreover, in [5] (resp. [7]) we studied and classified all CR submanifolds of CR dimension $\frac{n-1}{2}$ in complex Euclidean space (resp. complex projective space) which satisfy the condition $h(FX, Y) - h(X, FY) = g(FX, Y)\eta$, $\eta \in T^{\perp}(M)$, on the structure

☆ Research partially supported by the Ministry of Science, Serbia, project 144032D.
E-mail address: mdjoric@matf.bg.ac.yu.

0022-247X/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2009.03.020

tensor F naturally induced from the almost complex structure J of the ambient manifold and on the second fundamental form of submanifolds M.

In the present article we continue the above-mentioned study by considering complete n-dimensional real submanifolds of codimension p of complex space forms which are CR submanifolds of CR dimension $\frac{n-m}{2}$, $m > 1$, i.e. such that $\dim \mathcal{D}^{\perp} = m$ for a holomorphic distribution $\mathcal{D}$ of a tangent space, whose orthogonal complement is totally real. In Section 2 we collect some basic material concerning submanifolds, especially we discuss the notion of CR submanifolds of Kähler manifolds and we derive formulas for later use. Section 3 is devoted to the study of CR submanifolds of Kähler manifolds which satisfy the condition $h(FX,Y) + h(X,FY) = 0$ on the structure tensor F naturally induced from the almost complex structure J of the ambient manifold and on the second fundamental form h of a submanifold M. Finally, in Section 4, using codimension reduction theorems for complex space forms, we derive one characteristic property of CR submanifolds of complex space forms satisfying this condition.

2. Preliminaries

Let M be an n-dimensional real submanifold of codimension p of a Kähler manifold $\overline{M}$ with structure tensor field J and the Hermitian metric $\bar{g}$, where $n > 1$. Also, we denote by $\imath$ the immersion of M into $\overline{M}$ and the differential of the immersion. However, sometimes we omit to mention $\imath$ for brevity of notation. Then the tangent bundle $T(M)$ is identified with a subbundle of $T(\overline{M})$ and a Riemannian metric g of M is induced from the Riemannian metric $\bar{g}$ of $\overline{M}$ in such a way that $g(X,Y) = \bar{g}(\imath X, \imath Y)$ where $X, Y \in T(M)$. The normal bundle $T^{\perp}(M)$ is the subbundle of $T(\overline{M})$ consisting of all $\overline{X} \in T(\overline{M})$ which are orthogonal to $T(M)$ with respect to the Riemannian metric $\bar{g}$.

The purpose of this paper is to continue the study of CR submanifolds defined in [2] and we first recall that a submanifold M of a Kähler manifold $(\overline{M}, \bar{g}, J)$ is called a CR *submanifold* if there is a differentiable distribution $\mathcal{D} : x \mapsto \mathcal{D}_x \subset T_x(M)$ on M satisfying the following conditions:

(a) $\mathcal{D}$ is holomorphic, i.e. $J\mathcal{D}_x = \mathcal{D}_x$ for any $x \in M$, and
(b) the complementary orthogonal distribution $\mathcal{D}^{\perp} : x \mapsto \mathcal{D}_x^{\perp} \subset T_x(M)$ is totally real, i.e. $J\mathcal{D}_x^{\perp} \subset T_x^{\perp}(M)$ for each $x \in M$.

If $\dim \mathcal{D}_x^{\perp} = 0$ (respectively, $\dim \mathcal{D}_x = 0$), then the CR submanifold M is a *holomorphic submanifold* (respectively, *totally real submanifold*). We shall always denote by m the *real dimension* of $\mathcal{D}_x^{\perp}$, i.e. $m = \dim_{\mathbb{R}} \mathcal{D}_x^{\perp}$. Therefore, the *complex dimension* of $\mathcal{D}_x$ is $\dim_{\mathbb{C}} \mathcal{D}_x = \frac{n-m}{2}$ and it is called the CR dimension of a CR submanifold M.

Further, for a CR submanifold M of a Kähler manifold $\overline{M}$ with complex structure J, we denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^{\perp}$ in the normal bundle $T^{\perp}M$. Hence we have the following orthogonal direct sum decomposition $T^{\perp}(M) = J\mathcal{D}^{\perp} \oplus \nu(M)$. Using the Hermitian property of J and having in mind that J is a skew-symmetric endomorphism of $T(\overline{M})$, it follows that $\nu(M)$ is J-invariant (see [15]). Therefore, for any $X \in T(M)$, choosing a local orthonormal basis $\eta_1, \ldots, \eta_m, \xi_1, \ldots, \xi_{p-m}$ of vectors normal to M, such that $\eta_1, \ldots, \eta_m$ span $J\mathcal{D}_x^{\perp}$ and $\xi_1, \ldots, \xi_{p-m}$ span $\nu(M)$, we have the following decomposition into tangential and normal components:

$$J\imath X = \imath FX + \sum_{c=1}^{m} u^c(X)\eta_c, \tag{1}$$

$$J\eta_c = -\imath U_c, \quad c = 1, \ldots, m, \tag{2}$$

$$J\xi_a = \sum_{b=1}^{p-m} p_a^b \xi_b, \quad a = 1, \ldots, p-m. \tag{3}$$

Here F is a skew-symmetric endomorphism acting on $T(M)$, namely the structure tensor naturally induced from the almost complex structure of the ambient manifold $\overline{M}$, and U_c and u^c, $c = 1, \ldots, m$, are tangent vector fields and one-forms on M, respectively.

Furthermore, using (1), (2) and (3), the Hermitian property of J implies

$$F^2 X = -X + \sum_{c=1}^{m} u^c(X)U_c. \tag{4}$$

Next, let us denote by $\overline{\nabla}$ and ∇ the Riemannian connection of $\overline{M}$ and M, respectively, and by D the normal connection induced from $\overline{\nabla}$ in the normal bundle of M. Then the Gauss and Weingarten formulae for M are given respectively by

$$\overline{\nabla}_{\imath X}\imath Y = \imath \nabla_X Y + h(X,Y), \tag{5}$$

$$\overline{\nabla}_{\imath X}\xi = -\imath A_{\xi} X + D_X \xi, \tag{6}$$

for any vector fields X, Y tangent to M and any vector field ξ normal to M, where h denotes the second fundamental form and A_{ξ} denotes the shape operator (second fundamental tensor) corresponding to ξ.

Now, let us suppose that $\mu(M) \subset T^{\perp}(M)$ is J-invariant, that is $J\mu_x(M) = \mu_x(M)$, for all $x \in M$. From now on let us denote the orthonormal basis of $\mu(M)$ by $\xi_1, \ldots, \xi_{\frac{e}{2}}, \xi_{1^*}, \ldots, \xi_{\frac{e}{2}^*}$, where $\xi_{i^*} = J\xi_i$, $\dim_{\mathbb{R}} \mu(M) = e$. Further, let us suppose that $\mu(M)$ is also invariant with respect to the normal connection D, namely $D_X Y \in \mu(M)$ for all $Y \in \mu(M)$, $X \in T(M)$. Then the Weingarten formula (6) reads

$$\overline{\nabla}_{\imath X}\xi_b = -\imath A_b X + D_X \xi_b = -\imath A_b X + \sum_{d=1}^{\frac{e}{2}} \{ s_{bd}(X)\xi_d + s_{bd^*}(X)\xi_{d^*} \}, \tag{7}$$

$$\overline{\nabla}_{\imath X}\xi_{b^*} = -\imath A_{b^*} X + D_X \xi_{b^*} = -\imath A_{b^*} X + \sum_{d=1}^{\frac{e}{2}} \{ s_{b^*d}(X)\xi_d + s_{b^*d^*}(X)\xi_{d^*} \}, \tag{8}$$

for $b = 1, \ldots, \frac{e}{2}$, where s's are the coefficients of the normal connection D and A_b, A_{b^*}, are the shape operators corresponding to the normals ξ_b, ξ_{b^*}, respectively.

Since the ambient manifold is a Kähler manifold, using (1), (7) and (8), it follows

$$A_{b^*}X = FA_b X, \tag{9}$$

$$A_b X = -FA_{b^*}X, \tag{10}$$

$$u^c(A_b X) = 0, \qquad u^c(A_{b^*}X) = 0, \tag{11}$$

for all $X \in T(M)$, $b = 1, \ldots, \frac{e}{2}$, $c = 1, \ldots, m$. Moreover, since F is skew-symmetric, and A_b and A_{b^*} are symmetric, (9) and (10) imply

$$A_b F + FA_b = 0, \qquad A_{b^*}F + FA_{b^*} = 0, \quad \text{for all } b = 1, \ldots, \frac{e}{2}. \tag{12}$$

3. CR submanifolds of Kähler manifolds satisfying $h(FX, Y) + h(X, FY) = 0$

Let M^n be a complete CR submanifold of a Kähler manifold $\overline{M}^{n+p}$, with $\dim \mathcal{D}^{\perp} = m$. In this section we study such CR submanifolds M^n which satisfy the condition

$$h(FX, Y) + h(X, FY) = 0, \quad \text{for all } X, Y \in T(M). \tag{13}$$

Examples. In [4] and [6] the authors obtained the complete classification of CR submanifolds M^n of complex space forms $\overline{M}^{n+k}$ whose CR dimension is $\frac{n-1}{2}$ and which satisfy the condition (13). Namely, under these conditions, M is congruent to $\mathbb{E}^n$, $\mathbb{S}^n$ or $\mathbb{S}^{2p+1} \times \mathbb{E}^{n-2p-1}$ (when $\overline{M}^{n+k} = \mathbb{C}^{\frac{n+k}{2}}$), to $M^C_{p,q}$, for some p, q satisfying $2p + 2q = n - 1$ (when $\overline{M}^{n+k} = \mathbb{C}P^{\frac{n+k}{2}}$) or to M^*_n or $M^H_{p,q}(r)$, for some p, q satisfying $2p + 2q = n - 1$ (when $M^{n+k} = \mathbb{C}H^{\frac{n+k}{2}}$).

Moreover, after a straightforward computation it follows that CR submanifold $\underbrace{S^1 \times \cdots \times S^1}_{q} \times \mathbb{C}^{n-q}$ of $\mathbb{C}^{n+1}$ satisfies the condition (13), where S^1 is one-dimensional unit sphere. Similarly, since $M_{m_1,\ldots,m_k} = S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k)$ immersed in $\mathbb{C}^{\frac{n+k}{2}}$, where $S^m(r)$ is m-dimensional sphere of radius r and $m_1, \ldots, m_k$ are odd numbers, $n = \sum_{i=1}^k m_i$, $\sum_{i=1}^k (r_i)^2 = 1$, is a CR submanifold of $\mathbb{C}^m$ $(2m > n + k)$ with parallel mean curvature and flat normal connection, using [16], it follows that $\pi(M_{m_1,\ldots,m_k})$ is a CR submanifold of CP^m $(2m + 1 > n + k)$ which satisfies the condition (13). For more details and formulas we refer to [18, pp. 99, 106].

Since the second fundamental form h and the shape operators A_c, A_a, A_{a^*} corresponding to normals $\eta_c \in J\mathcal{D}^{\perp}$, $\xi_a, \xi_{a^*} \in \nu(M)$, $a = 1, \ldots, \frac{q}{2}$, $q = p - m$, respectively, are related by

$$h(X, Y) = \sum_{c=1}^{m} g(A_c X, Y)\eta_c + \sum_{a=1}^{\frac{q}{2}} \{ g(A_a X, Y)\xi_a + g(A_{a^*}X, Y)\xi_{a^*} \}, \tag{14}$$

it follows

$$\begin{aligned} h(FX, Y) + h(X, FY) = {} & \sum_{c=1}^{m} \{ g(A_c FX, Y) + g(A_c X, FY) \} \eta_c \\ & + \sum_{a=1}^{\frac{q}{2}} \{ \big(g(A_a FX, Y) + g(A_a X, FY) \big)\xi_a + \big(g(A_{a^*}FX, Y) + g(A_{a^*}X, FY) \big)\xi_{a^*} \}. \end{aligned} \tag{15}$$

Therefore, since F is skew-symmetric, relation (13) is equivalent to

$$A_c F = F A_c, \tag{16}$$

$$A_a F = F A_a, \tag{17}$$

$$A_{a^*} F = F A_{a^*}, \tag{18}$$

with $c = 1, \ldots, m$ and $a = 1, \ldots, \frac{q}{2}$. Consequently, making use of (12) and (11), we can easily prove the following

Lemma 1. *Let M be a complete n-dimensional CR submanifold of CR dimension $\frac{n-m}{2}$ of a complex space form $\overline{M}^{n+p}$. Let $\mu(M)$ be J-invariant subspace of $T^{\perp}M$ which is also invariant with respect to the normal connection D, with $\dim \mu(M) = e \leqslant p - m$. If the condition (13) is satisfied, then $A_a = 0 = A_{a^*}$, $a = 1, \ldots, \frac{e}{2}$, where A_a, A_{a^*} are the shape operators for the normals ξ_a, ξ_{a^*}, respectively.*

4. Codimension reduction of CR submanifolds in complex space forms

For a submanifold M of a Riemannian manifold $\overline{M}$, if there exists a totally geodesic submanifold M' of $\overline{M}$ such that $M \subset M'$, we say that we can reduce the codimension of the submanifold M. The codimension reduction problem was investigated by Allendoerfer [1] in the case when the ambient manifold is a Euclidean space and by Erbacher [8] in the case when the ambient manifold is a real space form. Then Cecil [3] proved a complex analogue for complex submanifold of complex projective space. Okumura proved in [15] the corresponding theorem for real submanifolds of complex projective space and Kawamoto in [9] proved it for real submanifolds of complex hyperbolic space.

More precisely, let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\overline{M}$. For $x \in M$, the first normal space $N_1(x)$ is the orthogonal complement in $T_x^{\perp}(M)$ of the set

$$N_0(x) = \{V \in T_x^{\perp}(M):\ A_V = 0\}.$$

If, for any vector field V with $V_x \in N_1(x)$, we have $D_X V \in N_1(x)$ for any vector field X of M at x, then the first normal space $N_1(x)$ is said to be parallel with respect to the normal connection.

We have the following reduction theorem of codimension in the case when the ambient manifold is a real space form:

Theorem 1. *(See [8].) Let M be an n-dimensional submanifold of an m-dimensional complete simply connected space form $\overline{M}^m(c)$. Suppose the first normal space $N_1(x)$ has constant dimension k, and is parallel with respect to the normal connection. Then there is a totally geodesic $(n+k)$-dimensional submanifold M^{n+k} of $\overline{M}^m(c)$ which contains M.*

Now, let M be an n-dimensional submanifold of a complex projective space $\overline{M}$ and let $H_0(x) = JN_0(x) \cap N_0(x)$. Then $H_0(x)$ is the maximal J-invariant subspace of $N_0(x)$ and since J is an isomorphism, it follows $JH_0(x) = H_0(x)$. The holomorphic first normal space $H_1(x)$ is the orthogonal complement of $H_0(x)$ in $T_x^{\perp}(M)$. By definition is $N_1(x) \subset H_1(x)$ in $T_x^{\perp}(M)$. Using Theorem 1, Okumura proved the following

Theorem 2. *(See [15].) Let M be an n-dimensional real submanifold of a real $(n+p)$-dimensional complex projective space $\mathbb{C}P^{(n+p)/2}$ and $H(x)$ be a J-invariant subspace of $H_0(x)$. If the orthogonal complement $H_2(x)$ of $H(x)$ in $T_x^{\perp}(M)$ is invariant under parallel translation with respect to the normal connection and q is the constant dimension of H_2, then there exists a real $(n+q)$-dimensional totally geodesic complex projective subspace $\mathbb{C}P^{(n+q)/2}$ such that $M \subset \mathbb{C}P^{(n+q)/2}$.*

Moreover, Kawamoto proved the following theorem for the complex hyperbolic space.

Theorem 3. *(See [9].) Let $\imath : M \to \mathbb{C}H^{(n+p)/2}$ be an isometric immersion of a connected n-dimensional real submanifold into a real $(n+p)$-dimensional complex hyperbolic space $\mathbb{C}H^{(n+p)/2}$ and let $H(x)$ be a J-invariant subspace of $H_0(x)$. If the orthogonal complement $H_2(x)$ of $H(x)$ in $T_x^{\perp}(M)$ is invariant under parallel translation with respect to the normal connection and if the dimension q of H_2 is constant, then there exists a real $(n+q)$-dimensional totally geodesic complex hyperbolic subspace $\mathbb{C}H^{(n+q)/2}$ in $\mathbb{C}H^{(n+p)/2}$ such that $M \subset \mathbb{C}H^{(n+q)/2}$.*

Now, let us suppose that M^n is a complete CR submanifold of a complex space form $\overline{M}^{n+p}$, whose CR dimension $\frac{n-m}{2}$, i.e. $\dim \mathcal{D}^{\perp} = m$ for a holomorphic distribution $\mathcal{D}$ whose orthogonal complement $\mathcal{D}^{\perp}$ is totally real. If the condition (13) is satisfied, let us consider the two cases:

- $J\mathcal{D}^{\perp}$ is invariant with respect to the normal connection D,
- $\nu_0(M)$ is J-invariant subspace of $\nu(M)$ which is also invariant with respect to the normal connection D.

Using the results of the previous section and codimension reduction theorems, we obtain the following results.

Theorem 4. *Let M^n be a complete CR submanifold of CR dimension $\frac{n-m}{2}$ of a complex space form $\overline{M}^{n+p}$. If $J\mathcal{D}^\perp$ is invariant with respect to the normal connection D and if the condition* (13) *is satisfied, then there exists a totally geodesic complex space form M', $\dim M' = n+m$ of $\overline{M}$ such that $M \subset M'$.*

Proof. First, let us define $N_0(x) = \{\xi \in T_x^\perp(M) \mid A_\xi = 0\}$ and let $H_0(x)$ be the maximal J-invariant subspace of $N_0(x)$, that is, $H_0(x) = JN_0(x) \cap N_0(x)$. Then, since $\nu_x(M)$ is J-invariant, using Lemma 1, it follows that $N_0(x) = \nu_x(M)$. Since $\nu(M)$ is J-invariant, it follows $H_0(x) = \operatorname{span}\{\xi_1(x), \ldots, \xi_q(x), \xi_{1^*}(x), \ldots, \xi_{q^*}(x)\}$, $q = \frac{p-m}{2}$. Hence the orthogonal complement $H_2(x)$ of $H_0(x)$ in $T^\perp(M)$ is spanned by $\eta_1, \ldots, \eta_m$, i.e. $H_2(x) = J\mathcal{D}^\perp$. Further, since $J\mathcal{D}^\perp$ is invariant with respect to the normal connection, we can apply the codimension reduction theorems for real submanifolds of complex projective subspace and complex hyperbolic space (Theorems 2 and 3) and conclude that there exist real $(n+m)$-dimensional totally geodesic complex projective space and complex hyperbolic subspace M' such that M is a subset of M'. In the case when $\overline{M}$ is a complex Euclidean space and the condition (13) is satisfied, applying the codimension reduction theorem for submanifolds of real space forms (Theorem 1), it follows that there exists an $(n+m)$-dimensional totally geodesic submanifold M' such that M is a subset of M'. It can be easily checked that $T(M')$ is J-invariant subspace of $T(\overline{M})$, which means that M' is also a complex Euclidean space. □

Theorem 5. *Let M^n be a complete CR submanifold of CR dimension $\frac{n-m}{2}$, of a complex space form $\overline{M}^{n+p}$. Let $\nu_0(M)$ be J-invariant subspace of $\nu(M)$ which is also invariant with respect to the normal connection D, with $\dim \nu_0(M) = l \leqslant p-m$. If the condition* (13) *is satisfied, then there exists a totally geodesic complex space form M', $\dim M' = n+p-l$ in $\overline{M}$ such that $M \subset M'$.*

Proof. Using Lemma 1, it follows that $N_0(x) = \{\xi \in T_x^\perp(M) \mid A_\xi = 0\} = \nu_0(x)$. Therefore, using the assumption that $\nu_0(M)$ is a J-invariant subspace of $\nu(M)$, it follows that $H_0(x)$, which is the maximal J-invariant subspace of $N_0(x)$, is given by $H_0(x) = JN_0(x) \cap N_0(x) = \nu_0(x)$ and consequently $H_0(x) = \operatorname{span}\{\xi_1(x), \ldots, \xi_{\frac{l}{2}}(x), \xi_{1^*}(x), \ldots, \xi_{\frac{l}{2}^*}(x)\}$. Hence the orthogonal complement $H_2(x)$ of $H_0(x)$ in $T^\perp(M)$ is spanned by $p-l$ vectors. Further, since $\nu_0(M)$ is invariant with respect to the normal connection D, using relation (3), it follows that $H_2(x)$ is invariant with respect to the normal connection D. Similarly as in the proof of Theorem 4, we can apply the codimension reduction theorems and conclude that there exist real $(n+p-l)$-dimensional totally geodesic complex space forms M' such that M is a subset of M'. □

Acknowledgments

The author would like to express her sincere gratitude to Masafumi Okumura, for inspiration and his valuable suggestions during the preparation of this paper.

References

[1] C.B. Allendoerfer, Rigidity for spaces of class greater than one, Amer. J. Math. 61 (1939) 633–644.
[2] A. Bejancu, CR-submanifolds of a Kähler manifold I, Proc. Amer. Math. Soc. 69 (1978) 135–142.
[3] T.E. Cecil, Geometric applications of critical point theory to submanifolds of complex projective space, Nagoya Math. J. 55 (1974) 5–31.
[4] M. Djorić, M. Okumura, CR submanifolds of maximal CR dimension in complex space forms and second fundamental form, in: Proceedings of the Workshop Contemporary Geometry and Related Topics, Belgrade, May 15–21, 2002, 2004, pp. 105–116.
[5] M. Djorić, M. Okumura, Certain condition on the second fundamental form of CR submanifolds of maximal CR dimension of complex Euclidean space, Ann. Global Anal. Geom. 30 (2006) 383–396.
[6] M. Djorić, M. Okumura, Certain CR submanifolds of maximal CR dimension of complex space forms, Differential Geom. Appl. 26 (2) (2008) 208–217.
[7] M. Djorić, M. Okumura, Certain condition on the second fundamental form of CR submanifolds of maximal CR dimension of complex projective space, Israel J. Math. 169 (2009) 47–59.
[8] J. Erbacher, Reduction of the codimension of an isometric immersion, J. Differential Geom. 5 (1971) 333–340.
[9] S. Kawamoto, Codimension reduction for real submanifolds of complex hyperbolic space, Rev. Mat. Complut. 7 (1994) 119–128.
[10] M. Kimura, Sectional curvatures of holomorphic planes on a real hypersurface in $P^n(C)$, Math. Ann. 276 (1987) 487–497.
[11] Y. Maeda, On real hypersurfaces of a complex projective space, J. Math. Soc. Japan 28 (1976) 529–540.
[12] S. Montiel, A. Romero, On some real hypersurfaces of a complex hyperbolic space, Geom. Dedicata 20 (1986) 245–261.
[13] R. Niebergall, P.J. Ryan, Real hypersurfaces in complex space forms, in: T.E. Cecil, S.-S. Chern (Eds.), Tight and Taut Submanifolds, in: Math. Sci. Res. Inst. Publ., vol. 32, Cambridge Univ. Press, Cambridge, 1997, pp. 233–305.
[14] M. Okumura, On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. 212 (1975) 355–364.
[15] M. Okumura, Codimension reduction problem for real submanifolds of complex projective space, Colloq. Math. Soc. János Bolyai 56 (1989) 574–585.
[16] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J. 13 (1966) 459–469.
[17] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math. 10 (1973) 495–506.
[18] K. Yano, M. Kon, CR Submanifolds of Kaehlerian and Sasakian Manifolds, Progr. Math., vol. 30, Birkhäuser, 1983.