OPTIMAL CONTROL APPLICATIONS & METHODS, VOL. 13, 95–104 (1992)

RATIONAL ADDICTIVE CYCLES ('BINGES') UNDER A BUDGET CONSTRAINT

GUSTAV FEICHTINGER
Institute for Econometrics, Operations Research and Systems Theory, University of Technology, Argentinierstrasse 8, A-1040 Vienna, Austria

SUMMARY

We consider an individual whose utility at any moment depends not only on the current consumption of a good but also on the habit accumulated by past consumption. The rational decision maker maximizes the discounted utility stream over an infinite time horizon subject to a budget which can be replenished by habit-dependent earnings. Addiction to the good requires that past consumption increases the marginal utility of current consumption. It is shown that strong complementarity might imply persistent oscillations in the optimal consumption pattern. Using Hopf bifurcation theory, we prove the existence of stable limit cycles in a numerical example.

KEY WORDS Optimal limit cycles Oscillatory consumption paths Maximum principle Rational addiction Life cycle model Replenishable budget

1. INTRODUCTION

Addictions play a role of increasing importance in our time. People become addicted to alcohol, nicotine, drugs, eating, television, work and many other activities. At a first look, addictive behaviour seems to be the result of irrationality. In an interesting paper, however, Becker and Murphy[1] showed recently that a wide variety of such consumer behaviour might be consistent with the maximization of a utility function. More specifically, rational consumers maximize their utility from stable preferences as they try to anticipate the future consequences of their choices. What Becker and Murphy have done is to derive conditions under which a drug user or an alcoholic weighs or even maximizes the future utility.

Becker and Murphy's analysis of rational addiction builds on work by Stigler and Becker,[2] Pollak,[3] Boyer[4,5] and Iannaccone.[6,7] In the literature on optimal growth theory it is sometimes assumed that instantaneous satisfaction depends both on instantaneous consumption and on the customary (past) consumption level, i.e. on the habit (see e.g. Ryder and Heal[8]).

The connection with addictive behaviour is obvious: addiction to a good requires an effect of past consumption of the good (habit) on current consumption. Generally, according to Iannaccone, a good may be called *addictive* if its current consumption increases as the habits derived from its previous consumption accumulate. (A commodity is denoted as *satiating* if the opposite occurs.) Thus the key assumption of addiction is that *past* consumption of the good affects its marginal utility of *current* consumption positively. We may think of a process of 'learning by doing'.

0143-2087/92/020095-10$10.00

Received 28 June 1990
Revised 3 February 1992

Let us denote the current consumption by c and the habit (expected level of consumption) at any moment by x. Then the utility of an individual depends both on c and x, i.e. $u = u(c, x)$. We assume that u is twice continuously differentiable and jointly strictly concave in (c, x).

According to Becker and Murphy, a necessary (but not sufficient) condition for addiction is

$$u_{cx}(c, x) > 0 \tag{1}$$

Becker and Murphy[1] define a '*binge*' as a cycle over time in the consumption of a good. Empirical evidence suggests that binges are common in alcoholism and overeating and therefore related to addictive behaviour. Although this may seem to be the prototype of irrational behaviour, Becker and Murphy show that it is under certain conditions consistent with rationality. In particular, they stress that for binges addiction is a necessary (but not sufficient) condition. In particular, they prove that under certain conditions addiction might result in oscillations of the consumption rate expressed as damped or explosive waves. Dockner and Feichtinger[9] extended their analysis to allow for persistent cycles. Note that the possibility of stable oscillations in consumption was mentioned already by Boyer.[4]

The purpose of the present paper is to analyse a variant of Becker and Murphy's primary model. Our main result will be to identify an intertemporal substitution effect between past and current consumption, implying a persistently oscillating optimal consumption pattern.

The paper is organized as follows. In Section 2 the model is presented. Section 3 discusses the first-order optimality conditions resulting in a modified Hamiltonian system. In Section 4 we present a numerical example that demonstrates the existence of a *stable* limit cycle as optimal solution.

2. THE MODEL

Following Becker and Murphy, we consider the *current* consumption $c(t)$ of a single good* at time t and the *expected* consumption (habit level, 'consumption capital') $x(t)$ whose change over time t is given by

$$\dot{x}(t) = c(t) - \delta x(t) \tag{2}$$

where δ denotes a constant depreciation rate. The initial level $x(0) = x_0 \geqslant 0$ is given.

The expenditures for consumption are paid out of a limited budget B. The remaining balance $y(t)$ in this budget at time t yields interest with a constant rate r and is replenished by the earnings at time t. These earnings are measured by the wage rate $w(x)$ depending on the accumulated consumption x. Let us assume that the consumed good is harmful in the sense that it has adverse effects on future earnings, i.e. $w' < 0$. The explanation of this assumption is as follows: the higher the accumulated consumption x, the worse is the physical and mental condition of the working individual. (One might think of alcohol as the consumed good.) Thus the wage rate per unit time depends negatively on the habit level. Moreover, we assume concavity of the wage function, i.e.

$$w(x) \geqslant 0 \quad w'(x) < 0 \quad \text{and} \quad w''(x) \leqslant 0 \quad \text{for } 0 \leqslant x \leqslant \bar{x} \tag{3}$$

* In the addiction literature[1,6,7] models typically assume at least *two* goods. Here we concentrate on what might be called the 'semi-reduced' form model, which involves only the allocation of a single consumption good over time. Dockner and Feichtinger[9] show that this model can be derived from a more general one that includes the intertemporal choice of leisure time and goods allocation subject to a wealth constraint. Becker and Murphy[1] consider two goods which are distinguished by assuming that current utility also depends on a measure of past consumption of the one but not of the other. Assuming perfect capital markets, the second 'normal' good can be maximized out with its first-order condition. This justifies the restriction to only one good c whose past consumption affects current utility through a process of learning by doing.

For perfect capital markets the budget constraint can now be formulated as follows:

$$\dot{y}(t) = ry(t) - pc(t) + w(x(t)), \qquad y(0) = B \tag{4}$$

Here p denotes the price of one unit of the consumed good, assumed to be constant. For the sake of simplicity we do not impose $y(t) \geqslant 0$ for all t, i.e. debt is allowed. It will turn out, however, that along the optimal solution the capital stock $y(t)$ is positive.

The objective of the decision maker is to maximize the present value of a stream of utilities $u(c(t), x(t), y(t))$ derived by the representative consumer at any instant t from the present consumption $c(t)$, the past consumption $x(t)$ and the remaining finance capital $y(t)$:

$$\int_0^\infty e^{-\rho t} u(c(t), x(t), y(t)) \, dt \to \max \tag{5}$$

Here we assume an infinite lifespan for the consumer and a constant rate of time preference, ρ.

Whereas the joint dependence of the utility function on c and x is standard in the literature on habit formation (see e.g. References 1, 7 and 8), the additional dependence on the remaining budget needs clarification. This assumption says that the utility of an individual is also influenced by the available finance capital stock (assets, etc.). Let us try to give two explanations for the occurrence of the capital stock in the utility function.* First, consider objects of art such as paintings, valuable books, etc. On the one hand they are an asset; on the other hand they delight their owner and create current utility by being in his/her possession. Secondly, and more importantly, large debts, i.e. a negative budget, usually cause disutility. Owning a high bank account provides a higher felicity. In other words, a given actual consumption rate c results in a higher utility if one has a higher bank account in the background rather than debts. An addictional justification for this assumption is given in Section 3 when referring to a reduced form of Boyer.[5]

For the utility function we assume the following:

$$u_c > 0, \qquad u_x < 0, \qquad u_y > 0 \tag{6a}$$

$$u_{cc} < 0, \qquad u_{xx} < 0, \qquad u_{yy} < 0 \tag{6b}$$

$$u_{cx} > 0, \qquad u_{cy} = 0, \qquad u_{xy} = 0 \tag{6c}$$

Moreover, $u(c, x, y)$ is assumed to be *strictly concave jointly* in (c, x, y). Since assumptions (6a) and (6b) are standard, only (6c) needs explanation. The increase in the marginal utility of c with consumption capital x is necessary for addiction.[1,7] The remaining assumptions in (6c) are made for mathematical simplicity.

With this notation, relations (2)–(6) describe an intertemporal optimal allocation problem of the single consumption good over time. The infinite-time optimal control problem has two state variables, x and y, and one control, c. Note that the time argument is mostly suppressed.

3. OPTIMALITY CONDITIONS

Applying standard optimal control theory in current value terms to this problem (see e.g. Feichtinger and Hart[10]), we obtain the *current value* Hamiltonian

$$H = u(c, x, y) + \lambda(c - \delta x) + \mu[ry - pc + w(x)] \tag{7}$$

* I acknowledge gratefully the help of William A. Brock, Engelbert J. Dockner and Franz Wirl in preparing these arguments.

The (current value) shadow prices λ and μ of consumption capital and budget, respectively, measure the marginal value of an additional unit of the state variables.

They have to satisfy the adjoint equations

$$\dot{\lambda} = \lambda(\rho + \delta) - u_x(c, x, y) - \mu w'(x) \tag{8}$$

$$\dot{\mu} = \mu(\rho - r) - u_y(c, x, y) \tag{9}$$

The Hamiltonian-maximizing condition yields

$$H_c = u_c(c, x, y) + \lambda - \mu p = 0 \tag{10}$$

Furthermore, to guarantee sufficiency, the following limiting transversality condition must be satisfied:

$$\lim \mathrm{e}^{-\rho t}\{\lambda(t)[x'(t) - x(t)] + \mu(t)[y'(t) - y(t)]\} = 0 \tag{11}$$

for all feasible states $x'(.)$, $y'(.)$.

By substituting the function $c = c(x, y)$ given implicitly by (10) into the state equations (2) and (4) as well as into the adjoint equations (8) and (9), we obtain the canonical system of the optimal control problem. Together with the initial conditions of the state variables ($x(0) = x_0$, $y(0) = B$) and the transversality condition (11) we obtain a two-point boundary value problem of dimension four.

To get insight to the dynamics of the optimal solutions of our model, a stability analysis has to be carried out. By linearizing the non-linear canonical system around the steady state (x^∞, y^∞, λ^∞, μ^∞, c^∞), we obtain the Jacobi matrix evaluated at the equilibrium.

An optimal steady state is a solution of

$$\begin{aligned} c - \delta x &= 0 \\ ry - pc + w(x) &= 0 \\ \lambda(\rho + \delta) - u_x(c, x, y) - \mu w'(x) &= 0 \\ \mu(\rho - r) - u_y(c, x, y) &= 0 \\ \mu_c(c, x, y) + \lambda - \mu p &= 0 \end{aligned}$$

In what follows we assume that there exists a solution of this system, i.e. an equilibrium (x^∞, y^∞, λ^∞, μ^∞, c^∞).

To simplify the cumbersome calculations, the following simplifying assumption is made:

$$w''(x) = 0 \tag{12}$$

Note that this assumption is no restriction since the main results remain valid for $w''(x) < 0$.

Then the Jacobian of the canonical system evaluated at the equilibrium has the following form:

$$J = \begin{bmatrix} -u_{cx}/u_{cc} - \delta & 0 & -1/u_{cc} & p/u_{cc} \\ p\,u_{cx}/u_{cc} + w' & r & p/u_{cc} & -p^2/u_{cc} \\ -u_{xx} + u_{cx}^2/u_{cc} & 0 & \rho + \delta + u_{cx}/u_{cc} & -w' - p\,u_{cx}/u_{cc} \\ 0 & -u_{yy} & 0 & \rho - r \end{bmatrix} \tag{13}$$

The determinant of the Jacobian evaluated at the steady state is given by the expression

$$\begin{aligned} \det J = {} & \delta(\rho + \delta)r(r - \rho) + \delta(\rho + \delta)p^2 \frac{u_{yy}}{u_{cc}} \\ & + \frac{r(r - \rho)}{u_{cc}}[u_{cx}(\rho + 2\delta) + u_{xx}] - (\rho + 2\delta)pw' \frac{u_{yy}}{u_{cc}} + w'^2 \frac{u_{yy}}{u_{cc}} \end{aligned} \tag{14}$$

To carry out a stability analysis of the four-dimensional canonical system, we have to calculate the eigenvalues of J, i.e. the roots of the characteristic equation $\det(\xi I - J) = 0$.

The eigenvalues are given by

$$\xi_{1,2,3,4} = \frac{\rho}{2} \pm \sqrt{\left\{\left(\frac{\rho}{2}\right)^2 - \frac{K}{2} \pm \sqrt{\left[\left(\frac{K}{2}\right)^2 - \det J\right]}\right\}} \tag{15}$$

(for details see e.g. Reference 9). For our model it turns out that K has the following form:

$$K = r(\rho - r) - \delta(\rho + \delta) + A_1 + A_2 \tag{16}$$

where

$$A_1 = -\frac{1}{u_{cc}}\,[u_{cx}(\rho + 2\delta) + u_{xx}] \tag{17}$$

$$A_2 = -p^2\,\frac{u_{yy}}{u_{cc}} \tag{18}$$

Note that while the sign of A_2 is always negative, the sign of A_1 is ambiguous according to the assumptions made in (6) and (12).

To identify economic mechanisms responsible for particular types of dynamic behaviour, e.g. stable limit cycles, the concepts of *adjacent complementarity* (AC) and *distant complementarity* (DC) are valuable. These concepts have been introduced by Ryder and Heal.[8] Iannaccone[6,7] recognized that addiction and satiation are the consequence of AC and DC, respectively. Becker and Murphy[1] pointed out that the concepts of AC and DC are virtually equivalent to the traditional concepts in economics of complementarity (for adjacent behaviour) and substitution (for distant behaviour). The first author to explicitly recognize the basic relationship between AC and addiction was Boyer.[5] He developed a model which provides in a sense a theoretical foundation to the assumption that the discretionary budget directly generates utility. Boyer's model leads to such a *reduced model* form.

Adjacent complementarity (AC) exists if increasing the consumption at some date t raises the marginal utility from the commodity at nearby dates (t_1) relative to distant ones (t_2). Under distant complementarity (DC) the situation is reversed.

By evaluating the condition that the marginal rate of substitution between t_1 and t_2 increases with a small increment in consumption at date t (see Reference 8, pp. 4 and 5), it can be shown that AC can be characterized by the *positivity* of expression A_1 given in (17).

Given the concavity of the utility function, it follows that $u_{cx} > 0$ is a necessary but by no means sufficient condition for $A_1 > 0$; see (17).

Thus the intertemporal substitution effect between consumption utilities today and tomorrow described above gives rise to the following result.

Proposition

There exists a pair of pure imaginary eigenvalues of the Jacobian (13) if and only if the following conditions are satisfied:

$$\det J - (K/2)^2 > 0 \tag{19}$$

$$\det J - (K/2)^2 - \rho^2 K/2 = 0 \tag{20}$$

Proof. First, from (15) we see that (19) is necessary for the existence of two imaginary characteristic roots.

Secondly, the real parts of the eigenvalues (15) are given as

$$\operatorname{Re} \xi_{1,2,3,4} = \frac{\rho}{2} \pm \sqrt{\left[\frac{1}{2}\left\{\sqrt{\left[\left(\frac{\rho}{2}\right)^4 - \left(\frac{\rho}{2}\right)^2 K + \det J\right]} + \left(\frac{\rho}{2}\right)^2 - \frac{K}{2}\right\}\right]}$$

In order that $\operatorname{Re} \xi_{3,4} = 0$, we need

$$\left(\frac{\rho}{2}\right)^2 + \frac{K}{2} \pm \sqrt{\left[\left(\frac{\rho}{2}\right)^4 - \left(\frac{\rho}{2}\right)^2 K + \det J\right]} \tag{21}$$

This together with $\det J > 0$ (which follows from (19)) implies that (21) cannot be satisfied for $K < 0$. Thus it holds that

$$K > 0 \tag{22}$$

From (21) we get

$$\left(\frac{\rho}{2}\right)^4 + \left(\frac{\rho}{2}\right)^2 K + \left(\frac{K}{2}\right)^2 = \left(\frac{\rho}{2}\right)^4 - \left(\frac{\rho}{2}\right)^2 K + \det J$$

or, equivalently, (20).

This completes the proof.

From (16) and the negative sign of A_2 we conclude that A_1 must be positive and sufficiently large to satisfy (22). On the other hand, from equation (20) we see that K must not be 'too large'. In Section 4 we will provide an example showing that both conditions (19) and (20) can be satisfied simultaneously.

In order to prove the existence of a limit cycle, we use Hopf's bifurcation theorem. The discount rate ρ has been selected as the bifurcation parameter. The bifurcation curve is given by (20). The Hopf theorem requires that the Jacobian of the canonical system has a pair of complex conjugate eigenvalues $\xi(\rho)$, $\bar{\xi}(\rho)$ with $\operatorname{Re} \xi(\rho) = 0$ at the bifurcation point $\rho = \rho_{crit}$. The technical details are sketched in the Appendix. There we discuss also the stability of the persistent cycle.

The economic content of this section may be summarized as follows: AC of the utility function $u(c, x, y)$ with respect to c and x of 'sufficient degree' is a necessary condition for the existence of a pair of pure imaginary eigenvalues of the Jacobian (13) in the optimal control problem (2)–(6).

In the following we present a numerical example which illustrates the application of the Hopf bifurcation theory.

4. A NUMERICAL EXAMPLE

The following numerical example establishes a stable limit cycle as optimal solution of our model. For this example we specify the wage rate as

$$w(x) = w_1 - w_2 x \tag{23}$$

Furthermore, the utility function is assumed to be a linear combination of a Cobb–Douglas function and a quadratic function as follows:

$$u(c, x, y) = a_1 c^{\alpha} x^{\beta} + a_2 y^{\gamma} - \frac{a_3}{2} c^2 - \frac{a_4}{2} x^2 + a_5 cx \tag{24}$$

Note that the utility function (21) is strictly concave jointly in (c, x, y) as long as

$$\alpha > 0, \quad \beta > 0, \quad \alpha + \beta < 1, \quad 0 < \gamma < 1$$

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_4 > 0, \quad a_3 a_4 - a_5^2 > 0$$

hold. Since the inequalities (6a) are not globally satisfied for the utility function (21), we restrict our calculations to that region of the (c, x, y)-space where (6a) holds.

For the numerical example we specify the model parameters as follows:

$$\delta = 0\cdot 12, \quad r = 3\cdot 75, \quad p = 4\cdot 74, \quad w_1 = 80, \quad w_2 = 1$$

$$\alpha = 0\cdot 05, \quad \beta = 0\cdot 90, \quad \gamma = 0\cdot 47$$

$$a_1 = 1, \quad a_2 = 309\cdot 305\,47, \quad a_3 = 21\cdot 8, \quad a_4 = 4\cdot 25, \quad a_5 = 8\cdot 7$$

Note that this example is not motivated by any specific consumption model. Since our purpose is to prove the existence (logical possibility) of limit cycles, no real-world data have been used.

With the discount rate as the bifurcation parameter the Jacobian evaluated at the unique steady state possesses two pure imaginary roots for the critical value

$$\rho_{\text{crit}} = 4\cdot 300\,311\,48$$

With these parameter values the unique steady state is given by

$$x^{\infty} = 73\cdot 869\,98, \quad y^{\infty} = 9\cdot 569\,93, \quad \lambda^{\infty} = -71\cdot 482\,04,$$

$$\mu^{\infty} = 79\cdot 798\,55, \quad c^{\infty} = 8\cdot 864\,40$$

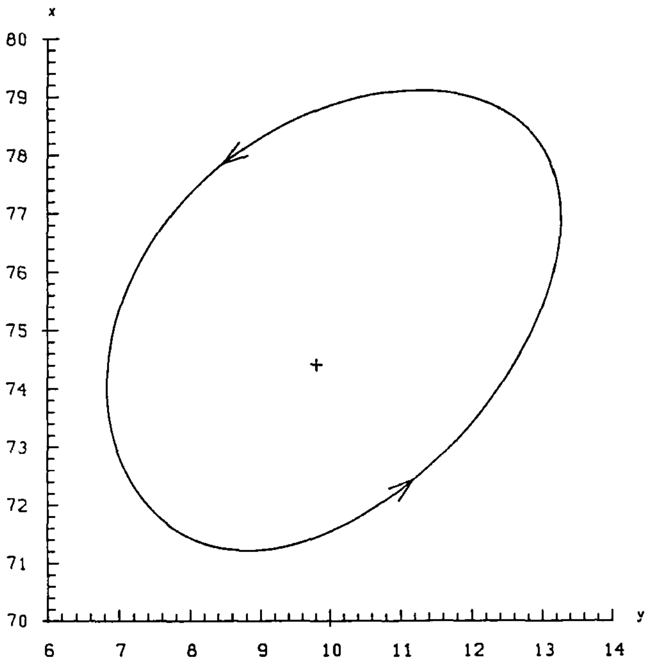

Figure 1. Projection of limit cycle in (y, x) phase portrait

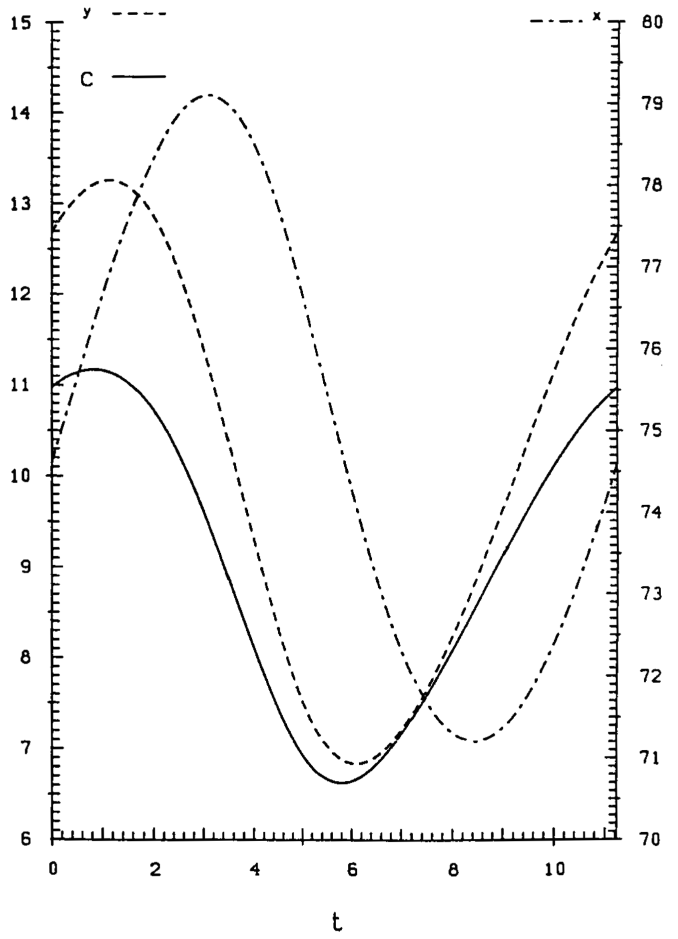

Figure 2. Time trajectories of consumption rate c, capital stock y and budget x

Evaluating A_1 and A_2 at this steady state, we get

$$A_1 = 1 \cdot 615\,3314, \qquad A_2 = -2 \cdot 502\,7992$$

This illustrates that the utility function shows AC with respect to x, but DC with respect to y. The pair of imaginary roots implies the existence of limit cycles for values of ρ close to the critical value ρ_{crit}. The stability of the cycles and the direction of the bifurcation are determined along with the sign of the coefficients A and A/D in the so-called normal form (see Appendix). Using the code 'BIFDD',[11] we calculate

$$A - 0 \cdot 005\,83, \qquad D = -0 \cdot 684\,47$$

This shows that the cycles are stable and occur for values of ρ slightly *smaller* than ρ_{crit}. Accordingly, a supercritical local Hopf bifurcation occurs when the pair of complex conjugate eigenvalues crosses the imaginary axis from the positive to the negative half-plane with increasing bifurcation parameter ρ.

In order to illustrate the cyclical behaviour of optimal consumption and habit for the value $\rho = 4 \cdot 29$, the TPBV problem solver 'COLSYS' has been applied.[12] It turns out that a closed orbit exists (see Figures 1 and 2). With $\rho = 4{\cdot}29$ instead of ρ_{crit} the steady state is shifted to (74·404 14, 9·804 08, −71·502 71, 80·308 01, 8·936 62).

5. CONCLUSIONS

In this paper we asked whether an addictive individual subject to a budget constraint can have persistent oscillatory consumption rates as an optimal pattern. Assuming (i) that the replenishing rate of the budget is inversely related to the habit level, (ii) a positive but decreasing marginal utility of the capital stock and (iii) addictive behaviour in the sense of Becker and Murphy of the utility function with respect to (c, x), we are able to find parameter constellations for which a limit cycle turns out to be optimal. Thus strict concavity and stable persistent oscillations may be consistent. The economic mechanism which is responsible for these persistent optimal fluctuations is identified as an *intertemporal substitution effect* between consumption today and tomorrow.

That alternating periods of heavy consumption and dieting may constitute an optimal behavioural pattern is at least intuitively plausible. In the first phase the individual consumes while the budget is exploited; then during the 'dry regime' the budget is replenished and the habit level recovers. It is addiction (adjacent complementarity) of sufficiently high degree which makes a long-run steady state suboptimal compared to a periodic consumption pattern.

It would be interesting to study also other *life cycle* models with respect to optimal cyclical solution paths. Human capital models of the Blinder–Weiss type provide an interesting framework in which periodic alternations between schooling and working (and possibly leisuring) could be optimal under special conditions. Blinder and Weiss[13] derive conditions by which the occurrence of cycles can be excluded.

Although the model was kept at a simple level, the existence of complex dynamics cannot be ruled out. This raises the question of whether more complex optimal solutions than limit cycles might occur in more complicated models. Periodic behaviour in alcoholism, smoking and overweighting–dieting are often observed in practice. More than that, the main result of Becker and Murphy,[1] Dockner and Feichtinger[9] and the present paper is that binges can be optimal provided that strong addiction prevails. Chaotic behaviour (in an intuitive sense) of addictive people can also be observed quite often. Thus it would be an interesting task to study deterministic, continuous-time models whose optimal consumption paths of addictive good are *chaotic*. This task should be seen in the light of the indeterminacy theorem of Boldrin and Montrucchio[14] (see also References 15 and 16). However, an answer to this question seems not to be easy.

APPENDIX

To prove the existence of a persistent periodic solution, we formulate the following result which is of central importance in the theory of limit cycles. Good references are Guckenheimer and Holmes[17] (Chap. 3) and Hassard *et al.*[11]

Hopf bifurcation theorem

Suppose that a system $\dot{z} = f_\rho(z)$, $z = (z_1, z_2)^{\mathrm{T}} \varepsilon \mathbb{R}^2, \rho \varepsilon \mathbb{R}$ has an equilibrium $(z_{\text{crit}}, \rho_{\text{crit}})$ at which the following property is satisfied:

> the Jacobian $D_z f_{\rho_{\text{crit}}}(z_{\text{crit}})$ has a simple pair of pure imaginary eignevalues and no other eigenvalues with zero real parts (25)

Then (25) implies that there is a smooth curve of equilibria $(\hat{z}(\rho), \rho)$ with $\hat{z}(\rho_{crit}) = z_{crit}$. The eigenvalues $\xi(\rho)$, $\bar{\xi}(\rho)$ of $D_z f_{\rho_{crit}}(\hat{z}(\rho))$ which are imaginary at $\rho = \rho_{crit}$, i.e. $\pm i\omega$, vary smoothly with ρ. If, moreover,

$$\frac{\mathrm{d}}{\mathrm{d}\rho}(\mathrm{Re}\ \xi(\rho))\,|_{\rho=\rho_{crit}} = D \neq 0 \tag{26}$$

then there is a unique three-dimensional centre manifold passing through (z_{crit}, ρ_{crit}) in $\mathbb{R}^n z \mathbb{R}$ and a smooth system of co-ordinates for which the Taylor expansion of degree three on the centre manifold is given by the following normal form:

$$\begin{aligned} z_1 &= [D(\rho - \rho_{crit}) + A(z_1^2 + z_2^2)]z_1 - [\omega + C(\rho - \rho_{crit}) + B(z_1^2 + z_2^2)]z_2 \\ z_2 &= [\omega + C(\rho - \rho_{crit}) + B(z_1^2 + z_2^2)]z_1 + [D(\rho - \rho_{crit}) + A(z_1^2 + z_2^2)]z_2 \end{aligned} \tag{27}$$

If $A \neq 0$, there is a surface of periodic solutions in the centre manifold which has quadratic tangency with the eignespace of $\xi(\rho_{crit})$, $\bar{\xi}(\rho_{crit})$, agreeing to second order with the paraboloid $\rho = \rho_{crit} - (A/D)(z_1^2 + z_2^2)$. Moreover, if $A < 0$, then the periodic solutions are *stable* limit cycles, while in the case of $A > 0$ the periodic solutions are repelling.

In our case the system $\dot{z} = f_\rho(z)$ is the non-linear canonical system of differential equations, i.e. $n = 4$. As bifurcation parameter we select the discount rate ρ.

REFERENCES

1. Becker, G. S., and K. M. Murphy, 'A theory of rational addiction', *J. Polit. Econ.*, **96**, 675–700 (1988).
2. Stigler, G. J., and G. S. Becker, 'De gustibus non est disputandum', *Am. Econ. Rev.*, **67**, 76–90 (March 1977).
3. Pollak, R. A., 'Habit formation and dynamic demand functions', *J. Polit. Econ.*, **78**, 745–763 (1970).
4. Boyer, M., 'A habit forming optimal growth model', *Int. Econ. Rev.*, **19**, 585–609 (1978).
5. Boyer, M., 'Rational demand and expenditures patterns under habit formation', *J. Econ. Theory*, **31**, 27–53 (October 1983).
6. Iannaccone, L. R., 'Consumption capital and habit formation with an application to religious participation', *Ph.D. Dissertation*, University of Chicago, 1984.
7. Iannaccone, L. R., 'Addiction and satiation', *Econ. Lett.*, **21**, 95–99 (1986).
8. Ryder Jr., H. E., and G. M. Heal, 'Optimal growth with intertemporally dependent preferences', *Rev. Econ. Stud.*, **40**, 1–31 (January 1973).
9. Dockner, E. J., and G. Feichtinger, 'Cyclical consumption patterns and rational addiction', *Forschungsbericht Nr. 121*, Instituts für Ökonometrie, Operations Research und Systemtheorie, Technische Univität Wien, May 1989; *Am. Econ. Rev.*, forthcoming in 1992 or 1993.
10. Feichtinger, G., and R. F. Hartl, *Optimale Kontrolle Ökonomischer Prozesse: Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften*, de Gruyter, Berlin, 1986.
11. Hassard, B. D., N. D. Kazarinoff, and Y.-H. Wan, *Theory and Application of Hopf Bifurcation*, *Lond. Math. Soc. Lecture Notes 41*, 1981.
12. Ascher, U., J. Christiansen, and R. D. Russell, 'A collocation solver for mixed order systems of boundary value problems', *Math. Comput.*, **33**, 659–679 (1978).
13. Blinder, A. S., and Y. Weiss, 'Human capital and labor supply: a synthesis', *J. Polit. Econ.*, **84**, 449–472 (1976).
14. Boldrin, M., and L. Montrucchio, 'On the indeterminacy of capital accumulation paths', *J. Econ. Theory*, **40**, 26–39 (1986).
15. Montrucchio, L., 'Dynamical systems that solve continuous-time concave optimization problems: anything goes', in Benhabib, J. (ed.), *Cycles and Chaos in Economic Equilibrium*, Princton University Press, Princeton, NJ, 1992, pp. 277–288.
16. Sorger, G., 'On the optimality of given feedback controls', *J. Optim. Theory Appl.*, **65**, 321–329 (1990).
17. Guckenheimer, J., and F. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York, 1983.