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Global Space-time Symmetry Gauging and Kaluza-Klein Dimensional Reduction

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A relation between dimensional reduction and space-time symmetry gauging is outlined.

1. Foreword

One of the miracles of String theory is its higher dimensional nature. But since everyone can directly feel and observe only four-dimensional world, it is a great puzzle how to connect the String/M-theory living at a Planck energy scale in ten/eleven space-time dimensions with phenomenological models describing experimental data. A way to resolve this problem is to compactify additional dimensions.

In this volume in memory of the 75th anniversary of D. V. Volkov it is worth to emphasize that Dmitrij Vasilievich Volkov was one of the pioneers in the investigation of mechanisms of spontaneous compactifications of Kaluza-Klein (KK) theories [1]–[6]. In [5], [6] D. V. Volkov and V. I. Tkach proposed for the first time the mechanism of spontaneous compactification based on embedding the spin connection of the compactified manifold into a gauge field connection, which was then applied in [7]–[10] for studying KK supergravities.

To compactify extra dimensions one needs as a starting point an *ansatz* being the classical solution for the low energy effective superstring theory (supergravity) equations of motion [11], and the choice of every ansatz compatible with the equations of motion determines its own independent classical vacuum state. Small fluctuations over the vacuum form a spectrum of massless and massive modes with masses depending on internal properties of the compactified manifold. Because there is no additional rule of the selection of the ground state metric and antisymmetric tensor fields (for instance, classical stability of vacuum and compactified manifold can be broken by quantum corrections and non-perturbative effects) there exists a problem of the choice of the true classical vacuum state. However, taking into account the requirement of unbroken supersymmetry one can restrict the consideration to the ansätze which correspond to backgrounds which are maximally symmetric from the lower-dimensional observer point of view, i.e. which are invariant under the space-time group/supergroup of rotations and translations generated by a set of Killing vectors/spinors. The form of the maximally-symmetric ground-state ansatz is governed by the properties of Killing vectors and in general involves a "warp" factor, which is a function of internal coordinates in the Randall-Sundrum (RS1)/Rubakov-Shaposhnikov (RS2) scenarios [12], [13].

This restriction leads to the space-time manifolds of a constant curvature or Einstein spaces among which only Minkowski and anti-de Sitter spaces are selected as the spaces compatible with supersymmetry. In its turn the internal manifold may also possesses the invariance under the action of an isometry group, which becomes an

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internal group of the lower-dimensional theory after compactification and which is defined by another set of Killing vectors. The latter play an important rôle in the definition of the massless states in the *complete* Kaluza-Klein ansatz especially in the case of dimensional reduction when all massive modes are ignored and the massless sector of the compactified theory is considered independently [3]. The form of the KK *ansatz for dimensional reduction* is further restricted by the requirement of its consistency with higher-dimensional equations of motion. This poses an additional problem of finding a consistent ansatz in the case of nontrivial (and nonlinear) reduction.

At this point it is necessary to recall that the above arguments do not appeal to the existence of Killing vectors in the original *arbitrary* higher-dimensional background. However, things change if from the beginning a higher-dimensional background allows for the existence of an isometry group. This feature is important for the construction of T(arget-space)-dual theories [14] as well as for the investigation of properties of "non-standard" BPS objects such as KK-monopoles [15] and space-time-filling branes (see, for instance, [16] and Refs. therein). In this case the space-time isometry group is generated by a set of the space-time Killing vectors and the requirement of local invariance under isometry group transformations leads to the introduction of compensator fields akin to that of the KK theories. The goal of this note is to find the conditions for the application of the global space-time symmetry gauging technique to the description of dimensionally reduced theories and to establish a way to recover dimensionally reduced theories starting from geometrical properties of manifolds with isometries but without appealing to a *dimensional reduction ansatz.*

To this end we begin with the consideration of the dimension reduction in Kaluza-Klein theories, after that the philosophy of gauged σ-models will be discussed. Having these thesis and anti-thesis the syn-thesis is realized in establishing a bridge between these two techniques. We conclude by discussing the results and open questions.

[3]Since all massive modes arising in the standard KK compactification have an order of the Planck mass they are not important for low-energy phenomenology. Note however that this is not true for quantum KK theories and also for RS1/2 scenarios.

2. Dimensional reduction in Kaluza-Klein theories

Because the dimensional reduction in Kaluza-Klein theories is a sufficiently well-known subject, we only remind what kind of geometrical objects appears in this scheme [11].

In the Kaluza-Klein approach dealing with theories in space-time dimensions higher than four it is assumed that the higher-dimensional space-time manifold M^D is the direct product of an internal manifold M^k and a space-time manifold M^{D-k}. If a group G is the isometry group of the metric tensor in M^k, then G is generated by the set of the Killing vectors k_i^A $(A = 1, \dots, dim\ G)$ having the following properties:

$$\mathcal{L}_k g_{ij} = 0 \Longleftrightarrow \nabla_{(i}k^A_{j)} \equiv 2\partial_{(i}k^A_{j)} - 2\Gamma^l{}_{ij}k^A_l = 0,$$

$$[k^A, k^B] = f^{AB}{}_C k^C, \qquad k^A = k^{iA}\frac{\partial}{\partial Z^i},$$

where $\mathcal{L}_k$ is the Lie derivative in the Killing vector direction, g_{ij} is the metric tensor in the internal manifold M^k parameterized by the coordinates Z^i, ∇ is the covariant derivative with the standard definition of the Christoffel connection

$$\Gamma^l{}_{ij} = \frac{1}{2}g^{lk}(\partial_j g_{ki} + \partial_i g_{kj} - \partial_k g_{ij}),$$

and $f^{AB}{}_C$ are the structure constants of the isometry group G.

The D-dimensional metric tensor can be split on to M^k and M^{D-k} blocks as follows [4]

$$\hat{g}_{\underline{\hat{m}}\underline{\hat{n}}}(X, Z) = \begin{pmatrix} g_{\underline{mn}} & g_{\underline{m}i} \\ g_{i\underline{n}} & g_{ij} \end{pmatrix}.$$

To dimensionally reduce the theory it is necessary to choose an *ansatz*, which shall define the

[4]Throughout the paper all hatted quantities indicate an object in D-dimensional space-time with $\hat{X}^{\underline{\hat{m}}}$ coordinates, unhatted quantities belong to the $(D-k)$-dimensional space-time with $X^{\underline{m}}$ coordinates, and the small indices from the middle of the Latin alphabet are the indices of the internal space parameterized by Z^i.

form of the matrix blocks and will be consistent with equations of motion for the original D-dimensional theory. It is known that, in particular, the following choice of the off-diagonal matrix element

$$g_{\underline{m}i}(X,Z) = A^A_{\underline{m}}(X)k^A_i(Z) + \ massive \ \ modes$$

is self-consistent (at least in the case of toroidal compactification). Here a new vector field $A^A_{\underline{m}}$ as a massless mode appears. Because we deal with dimensional reduction, all massive modes are ignored. The complete form of the Kaluza-Klein ansatz can be extracted from the expression for the line element written as

$$d\hat{s}^2 = dX^{\underline{m}} \otimes dX^{\underline{n}} g_{\underline{mn}} - (dZ^i - A^A k^{Ai})$$
$$\otimes (dZ^j - A^A k^{Aj}) g_{ij}.$$

Under the D-dimensional general coordinate transformations $\hat{X}^{\underline{\hat{m}}} \hookrightarrow \hat{X}^{\underline{\hat{m}}} - \hat{\xi}^{\underline{\hat{m}}}(\hat{X})$ the metric tensor transforms as

$$\delta \hat{g}_{\underline{\hat{m}\hat{n}}} = 2\nabla_{(\underline{\hat{m}}}\hat{\xi}_{\underline{\hat{n}})}.$$

For a special choice of the local parameter

$$\hat{\xi}^{\underline{\hat{m}}}(X,Z) = (0, \ \epsilon^A(X)k^{Ai}),$$

which corresponds to local coordinate shifts in the internal manifold coordinate directions a transformation rule for the vector field $A^A_{\underline{m}}$ has the form of the gauge transformation for a Yang-Mills gauge field with the gauge group G

$$\delta A^A_{\underline{m}}(X) = \partial_{\underline{m}}\epsilon^A(X) - f^A_{\ BC}A^B_{\underline{m}}\epsilon^C.$$

This observation is one of the key points of the Kaluza-Klein philosophy: The gauge group in $D-k$ space-time dimensions is connected to the isometry group of the extra dimensions and is a subgroup of the D-dimensional general coordinate transformation (g.c.t.) group [17]–[21].

3. Global space-time symmetries and their gauging

Let us outline now a resemblance of the construction above to that of gauged sigma-model formulation. To this end consider a σ-model (p-brane) described by the action [22]–[24]

$$S = \int d^{p+1}\xi[\sqrt{-h}(\partial_m \hat{X}^{\underline{\hat{m}}}\hat{g}_{\underline{\hat{m}\hat{n}}}\partial_n\hat{X}^{\underline{\hat{n}}}h^{mn} + (p-1))$$
$$+\varepsilon^{m_1\ldots m_{p+1}}\hat{B}_{\underline{\hat{m}}_1\cdots\underline{\hat{m}}_{p+1}}\partial_{m_1}\hat{X}^{\underline{\hat{m}}_1}\ldots\partial_{m_{p+1}}\hat{X}^{\underline{\hat{m}}_{p+1}}]$$

with h_{mn} being the metric on the p-brane worldvolume.

For special backgrounds allowing for the existence of isometry directions this action is invariant under the following *global* space-time transformations [25]–[27]

$$\delta\hat{X}^{\underline{\hat{m}}} = \epsilon^A k^{\underline{\hat{m}}A}$$

with a constant parameter ϵ^A. Here $k^{\underline{\hat{m}}A}$ are the Killing vectors generating the isometry group of the background, i.e.

$$\mathcal{L}_k \ \hat{g}_{\underline{\hat{m}\hat{n}}} = 0,$$

and the target space antisymmetric gauge field $\hat{B}$ should satisfy the equation

$$\mathcal{L}_k \ (d\hat{B})_{\underline{\hat{m}}_1\cdots\underline{\hat{m}}_{p+1}} = 0.$$

For further consideration it is important that having the *global* target-space invariance we can gauge the isometry group [5] , i.e. to require the invariance under the *local* space-time transformations

$$\delta\hat{X}^{\underline{\hat{m}}} = \epsilon^A(\hat{X}) \ k^{\underline{\hat{m}}A}.$$

To keep the action invariance we have to introduce a gauge field A^A_m which is minimally coupled to the σ-model and transforming as

$$\delta A^A_m(\hat{X}) = \partial_m\epsilon^A - f^A_{\ BC}A^B_m\epsilon^C,$$

which enters the derivative covariant with respect to the local space-time transformations

$$\mathsf{D}_m\hat{X}^{\underline{\hat{m}}} = \partial_m\hat{X}^{\underline{\hat{m}}} - A^A_m k^{\underline{\hat{m}}A}.$$

[5]Other aspects of space-time symmetry gauging can be found in [26] (and Refs. therein). From the modern point of view the existence of a global symmetry in the original theory is a sufficient but not necessary (see, for instance, Ref. [28]) condition for a possibility of constructing its dual [29] .

Gauged σ-model action can be obtained after replacing the usual derivative with the covariant one. For keeping the minimal coupling to the A_m^A we should require [26], [27]

$$\mathcal{L}_k \hat{B}_{\underline{\hat{m}}_1 \cdots \underline{\hat{m}}_{p+1}} = 0.$$

Therefore

$$S_{g.} = \int d^{p+1}\xi[\sqrt{-h}(\mathsf{D}_m \hat{X}^{\underline{\hat{m}}} \hat{g}_{\underline{\hat{m}}\underline{\hat{n}}} \mathsf{D}_n \hat{X}^{\underline{\hat{n}}} h^{mn} + (p-1))$$
$$+\varepsilon^{m_1 \ldots m_{p+1}} \hat{B}_{\underline{\hat{m}}_1 \cdots \underline{\hat{m}}_{p+1}} \mathsf{D}_{m_1} \hat{X}^{\underline{\hat{m}}_1} \ldots \mathsf{D}_{m_{p+1}} \hat{X}^{\underline{\hat{m}}_{p+1}}].$$

4. The bridge

We have observed that one of the essential ingredients of the abovementioned approaches is the notion of the Killing vector. The properties of dimensional reduction and global space-time symmetry gauging based on the features of Killing vectors are summarized in the Table 1.

Therefore, we can fit the global symmetry gauging approach to the description of dimensionally reduced theories if we apply the gauged sigma-model-like technique to a subgroup of g.c.t. group and restrict fields and gauge parameters by the requirement that they do not depend on several spatial coordinates regarded as the coordinates of an effective "internal" manifold.

To realize this observation consider a space-time manifold with isometries. For the sake of simplicity we shall consider the manifold with only one isometry direction. Restrict now the field configuration of a theory by the requirement

$$\mathcal{L}_k\ \Phi = 0$$

for any field Φ. It is always possible to rotate the coordinate system in such a way that the Killing vector $k^{\underline{\hat{m}}}$ will be settled along one of the spatial directions, say Z. In this so-called adapted frame $k^{\underline{\hat{m}}} = \delta^{\hat{m}}_{\underline{Z}}$, and our restriction becomes

$$\mathcal{L}_k\ \Phi = \partial_{\underline{Z}}\Phi = 0,$$

which is the standard requirement for the dimensional reduction in $\underline{Z}$ direction.

To describe the geometry of an arbitrary Riemannian manifold we have to introduce a vielbein and connection one-forms which should be invariant under the local coordinate transformations along the Killing vector

$$\mathcal{L}_k \hat{E}^{\underline{\hat{a}}} = 0, \qquad \mathcal{L}_k \hat{\omega}^{\underline{\hat{a}}\underline{\hat{b}}} = 0.$$

If in addition we impose the zero torsion condition

$$\mathcal{D}\hat{E}^{\underline{\hat{a}}} = d\hat{E}^{\underline{\hat{a}}} - \hat{E}^{\underline{\hat{b}}} \wedge \hat{\omega}_{\underline{\hat{b}}}{}^{\underline{\hat{a}}} \equiv \hat{T}^{\underline{\hat{a}}} = 0$$

we can recover the general expression for the vielbeins [30]

$$\hat{E}^{\underline{a}} = e^{\alpha\phi} E^{\underline{a}}, \qquad \hat{E}^{\underline{z}} = e^{\beta\phi}(dZ - A)$$

in the Kaluza-Klein ansatz which form triangular matrix

$$\hat{E}_{\underline{\hat{m}}}{}^{\underline{\hat{a}}}(X,Z) = \begin{pmatrix} \hat{E}_{\underline{m}}{}^{\underline{a}}(X) & \hat{E}_{\underline{m}}{}^{\underline{z}}(X) \\ 0 & \hat{E}_{\underline{Z}}{}^{\underline{z}}(X) \end{pmatrix}.$$

It is very important to note that the form of the Kaluza-Klein ansatz is completely defined by the geometry of the space-time manifold.

We can arrive at the same result by fitting the gauged σ-model technique. To this end note that the requirement of the Kaluza-Klein scheme to consider the special choice of the general coordinate transformations – $\delta Z = \epsilon(X)$ in the case under consideration – is equivalent to the local space-time transformation

$$\delta\hat{X}^{\underline{\hat{m}}} = \epsilon(X) k^{\underline{\hat{m}}}$$

with the parameter ϵ which does not depend on Z. Then, to require the invariance of our theory under this transformation we can introduce the differential

$$\mathsf{D}\hat{X}^{\underline{\hat{m}}} = d\hat{X}^{\underline{\hat{m}}} - Ak^{\underline{\hat{m}}},$$

which is covariant with respect to the local shift in Z direction when

$$\delta A = d\epsilon(X),$$

and the covariant vielbein one-form

$$\hat{E}^{\underline{\hat{a}}}(\mathsf{D}) \equiv \mathsf{D}\hat{X}^{\underline{\hat{m}}} \hat{E}_{\underline{\hat{m}}}{}^{\underline{\hat{a}}}.$$

An important point is that to establish the connection with the KK approach the component structure of the vielbein one-form should

Table 1
Properties of different approaches.

Dimensional reduction in KK theories	Global space-time symmetry gauging
$M^D = M^{D-k} \otimes M^k$	M^D
$G^{(k)}$ – isometry group of M^k (a subgroup of g.c.t. group)	$G^{(D)}$ – isometry group of M^D
$\delta \hat{X}^{\underline{\hat{m}}} = \epsilon^A(X) k^{Ai}(Z) \delta_i^{\underline{\hat{m}}}$	$\delta \hat{X}^{\underline{\hat{m}}} = \epsilon^A(\hat{X}) k^{A\underline{\hat{m}}}(\hat{X})$
$\delta A^A_{\underline{m}}(X) = \partial_{\underline{m}} \epsilon^A - f^A_{BC} A^B_{\underline{m}} \epsilon^C$	$\delta A^A_{\underline{\hat{m}}}(\hat{X}) = \partial_{\underline{\hat{m}}} \epsilon^A - f^A_{BC} A^B_{\underline{\hat{m}}} \epsilon^C$

be restricted to the very special diagonal form of $\hat{E}_{\underline{\hat{m}}}{}^{\hat{\underline{a}}}(X, Z)$

$$\hat{E}_{\underline{\hat{m}}}{}^{\hat{\underline{a}}}(X, Z) = \begin{pmatrix} \hat{E}_{\underline{m}}{}^{\underline{a}}(X) & 0 \\ 0 & \hat{E}_{\underline{Z}}{}^{\underline{z}}(X) \end{pmatrix},$$

which, in particular, means that $g_{\underline{zm}} = 0$. This kind of representation is similar to that of Ref. [31] where the construction of a target-space geometry in the presence of isometries has been discussed. In its turn, the transition to the covariant differential deforms this structure to the triangular Kaluza-Klein form. Therefore, in this representation the line element can be simply written as

$$d\hat{s}^2 = \hat{E}^{\hat{\underline{a}}}(\mathsf{D}) \otimes \hat{E}^{\hat{\underline{b}}}(\mathsf{D}) \hat{\eta}_{\hat{\underline{a}}\hat{\underline{b}}}.$$

For antisymmetric tensor fields the form $\hat{C}^{(n)}(\mathsf{D})$ is apparently invariant under the local transformation in the Killing vector direction. Rearranging n-form in terms of usual differentials we arrive at

$$\hat{C}^{(n)}(\mathsf{D}) = C'^{(n)}(d) + C^{(n-1)}(d) \wedge dZ,$$

$$C'^{(n)} = C^{(n)} - C^{(n-1)} \wedge A,$$

which is nothing but the standard rule of reduction for the n-form gauge field in the Kaluza-Klein theories.

In particular, since the dynamics of a p-brane is described by the pullback of target-space fields, we conclude that under assumptions above *the gauged $(p+1)$-dimensional σ-model describes the dynamics of a directly dimensionally reduced p-brane.*

This observation is in agreement with a result of Ref. [32] adapted to the massless case.

5. Discussion and open questions

To summarize, we have outlined the connection between dimensional reduction and global space-time symmetries gauging. By presenting the line element in a more "economic" form by means of "covariant" vielbein one-forms and the reduction law for the antisymmetric target-space tensor fields, the investigation of the reduced theory symmetry structure is simplified especially in the case when the original theory has a non-trivial symmetry structure. An example of such a theory is the covariant formulation of the M5-brane [33], [34], [35]. The action for the bosonic M5-brane propagating in a D=11 background has the form

$$S_{M5} = -\int d^6\xi \, [\sqrt{-\det(\hat{g}_{mn} + i\hat{H}^*_{mn})}$$

$$+ \frac{\sqrt{-\det \hat{g}_{mn}}}{4} \hat{H}^{*mn} \hat{H}_{mn}] + \int_{\mathcal{M}^6} \mathcal{L}_{WZ}(d).$$

Here $\hat{g}_{mn} = \partial_m \hat{X}^{\underline{\hat{m}}} \hat{g}_{\underline{\hat{m}}\underline{\hat{n}}} \partial_n \hat{X}^{\underline{\hat{n}}}$ is the induced metric, $H^{(3)} = db^{(2)} - \hat{C}^{(3)}$ is the curl of the worldvolume second-rank chiral form and

$$\hat{H}^*_{mn} = \frac{1}{\sqrt{-\partial a \hat{g} \partial a}} \hat{H}^*_{mnr} \partial^r a \equiv (\hat{H}^* \cdot \hat{v})_{mn},$$

$$\hat{H}_{mn} = (H \cdot \hat{v})_{mn}, \hat{H}^{*mnl} = \frac{1}{3!\sqrt{-\hat{g}}}\epsilon^{mnlpqr}H_{pqr}.$$

The explicit form of the Wess-Zumino term is

$$\int_{\mathcal{M}^6} \mathcal{L}_{WZ}(d) = \int_{\mathcal{M}^6} (\hat{C}^{(6)} + \frac{1}{2}db^{(2)} \wedge \hat{C}^{(3)}).$$

The details of the symmetry structure of this action can be found in literature.

Let us now consider an eleven-dimensional background having isometry along the coordinate Z. Then after transition to the target space fields which contain covariant derivative or differential, i.e.

$$\hat{g}_{mn} \to \hat{g}_{mn}(\mathsf{D}) = \mathsf{D}_m\hat{X}^{\underline{\hat{m}}}\hat{E}_{\underline{\hat{m}}}^{\hat{a}}\hat{\eta}_{\underline{\hat{a}\hat{b}}}\mathsf{D}_n\hat{X}^{\underline{\hat{n}}}\hat{E}_{\underline{\hat{n}}}^{\hat{b}},$$

$$\hat{C}^{(n)}(d) \to \hat{C}^{(n)}(\mathsf{D})$$

and leaving the worldvolume fields ($b^{(2)}$ and a) the same we get the M5 gauged sigma-model action

$$S_{M5g.} = -\int d^6\xi\, [\sqrt{-\det(\hat{g}_{mn}(\mathsf{D}) + i\hat{H}^{*}_{mn}(\mathsf{D}))}$$
$$+\frac{\sqrt{-\det \hat{g}_{mn}(\mathsf{D})}}{4}\hat{H}^{*mn}(\mathsf{D})\hat{H}_{mn}(\mathsf{D})]$$
$$+\int_{\mathcal{M}^6} \mathcal{L}_{WZ}(\mathsf{D}).$$

In view of the statement of the previous section, this action is nothing but the action for an NSIIA five-brane [36]. Because the structure of the NSIIA five-brane action formally remains the same it is evident that the symmetry structure of the NSIIA5-brane is akin to that of the M5-brane. New gauge symmetries arising under reduction can be derived from the definition of the covariant derivative.

Consider now the application of the proposed scheme to the description of dimensionally reduced (super)gravity theories. To this end recall that the procedure of dimensional reduction, say, onto multidimensional torus can be reformulated in the following way. Imagine that our action functional is described by a p-form integrated over p-dimensional manifold $\mathcal{M}^p$ being the direct product of the space-time manifold $\mathcal{M}^{p-n}$ and n-dimensional torus T^n. The latter is parameterized by Z^i coordinates with $i = 1, \ldots, n$. Then we can write down the action as

$$S = \int_{\mathcal{M}^p} \mathcal{L}^{(p)}$$

$$= \int_{\mathcal{M}^{p-n}\times\mathcal{M}^n} [\alpha\, i_{i_1} \ldots i_{i_n} (\mathcal{L}^{(p)} \wedge dZ^{i_1} \wedge \ldots \wedge dZ^{i_n})$$
$$+\omega\, i_{i_1} \ldots i_{i_n} \mathcal{L}^{(p)} \wedge dZ^{i_1} \wedge \ldots \wedge dZ^{i_n}]$$

with numerical coefficients α and ω. The first integral is zero as it should be for a p-form on the p-dimensional manifold, which does not contain n differentials of the manifold coordinates. Upon integration over Z^i the second term takes the form

$$S_{Red.} = \Omega_{\mathcal{M}^n} \int_{\mathcal{M}^{p-n}} \epsilon^{i_1\ldots i_n} i_{i_1} \ldots i_{i_n} \mathcal{L}^{(p)},$$

$$\Omega^{i_1\ldots i_n}_{\mathcal{M}^n} = \int_{\mathcal{M}^n} \omega\, dZ^{i_1} \wedge \ldots dZ^{i_n},$$

where $\Omega_{\mathcal{M}^n}$ is the global volume of the internal space which we take to be of a unite value.

In the simplest case of $\mathcal{M}^p = \mathcal{M}^{p-1} \times \mathcal{M}^1$

$$\int_{\mathcal{M}^p} \mathcal{L}^{(p)}(d) = \int_{\mathcal{M}^p} i_Z\mathcal{L}^{(p)} \wedge dZ. \qquad (1)$$

On the other hand, by definition

$$\mathcal{L}^{(p)}(\mathsf{D}) = \frac{1}{p!}\mathsf{D}X^{\underline{\hat{m}_1}} \wedge \ldots \wedge \mathsf{D}X^{\underline{\hat{m}_p}}\mathcal{L}_{\underline{\hat{m}_p\ldots\hat{m}_1}}$$
$$= \mathcal{L}^{(p)}(d) - i_k\mathcal{L}^{(p)} \wedge A.$$

Therefore taking into account eq. (1) and choosing the adapted coordinate frame one arrives at

$$\mathcal{L}^{(p)}(\mathsf{D}) = i_Z\mathcal{L}^{(p)}(d) \wedge (dZ - A). \qquad (2)$$

Integrating (2) over Z we recover the result of dimensional reduction.

To illustrate how it works, consider the classical example of the reduction of a five-dimensional Kaluza-Klein gravity down to four dimensions in a way presented in [30]. The $D = 5$ gravity action

$$S = \int_{\mathcal{M}^5} \frac{1}{3!}\hat{R}^{\underline{\hat{a}_1\hat{a}_2}} \wedge \hat{E}^{\underline{\hat{a}_3}} \wedge \hat{E}^{\underline{\hat{a}_4}} \wedge \hat{E}^{\underline{\hat{a}_5}}\epsilon_{\underline{\hat{a}_1\ldots\hat{a}_5}}$$

$$\equiv \int d^5x\,\sqrt{-\hat{g}}\hat{R}.$$

Consider now the Lagrangian 5-form

$$\mathcal{L}^{(5)}(\mathsf{D}) = \frac{1}{3!}\hat{R}^{\underline{\hat{a}_1\hat{a}_2}}(\mathsf{D})\wedge\hat{E}^{\underline{\hat{a}_3}}(\mathsf{D})\ldots\hat{E}^{\underline{\hat{a}_5}}(\mathsf{D})\epsilon_{\underline{\hat{a}_1}\ldots\underline{\hat{a}_5}},$$

which by use of the representation for the curvature and vielbein forms

$$\hat{R}^{\underline{\hat{a}\hat{b}}}(\mathsf{D}) = \hat{R}^{\underline{\hat{a}\hat{b}}}(d) - dA(i_k\hat{\omega}^{\underline{\hat{a}\hat{b}}}) - A\mathcal{D}(i_k\hat{\omega}^{\underline{\hat{a}\hat{b}}}),$$
$$\hat{E}^{\underline{\hat{a}}}(\mathsf{D}) = \hat{E}^{\underline{\hat{a}}}(d) - A(i_k\hat{E}^{\underline{\hat{a}}})$$

is written as

$$\mathcal{L}^{(5)}(\mathsf{D}) = -\frac{1}{3!}[3(\hat{R}^{\underline{a_1a_2}} - dA(i_k\hat{\omega}^{\underline{a_1a_2}}))$$
$$\wedge\hat{E}^{\underline{a_3}}\wedge\hat{E}^{\underline{a_4}}\wedge(dZ - A)\epsilon_{\underline{a_1}\ldots\underline{a_4}} \quad (3)$$
$$-3\mathcal{D}(i_k\hat{\omega}^{\underline{a_1a_2}})\wedge\hat{E}^{\underline{a_3}}\wedge\hat{E}^{\underline{a_4}}\wedge A\wedge(dZ-A)\epsilon_{\underline{a_1}\ldots\underline{a_4}}],$$

where we have used that $i_k\hat{E}^{\underline{\hat{a}_5}} = \delta_{\underline{z}}^{\underline{\hat{a}_5}}$ and $\epsilon_{\underline{\hat{a}_1}\ldots\underline{\hat{a}_4}\underline{z}} = -\epsilon_{\underline{a_1}\ldots\underline{a_4}}$.

What we can say about the last term of eq. (3)? It is not so difficult to observe that the torsion two-form $\hat{T}^{\underline{a}}$ vanishes. The Bianchi identity $\mathcal{D}\hat{T}^{\underline{a}} = \hat{E}^{\underline{b}}\hat{R}_{\underline{b}}{}^{\underline{a}}$ implies in particular $i_k\hat{R}_{[\underline{abc}]} = 0$, or, in view of the requirement $\mathcal{L}_k\hat{\omega}^{\underline{\hat{a}\hat{b}}} = 0$,

$$\mathcal{D}_{[\underline{a}}i_k\hat{\omega}_{\underline{bc}]} = 0.$$

Hence, the term under consideration vanishes and

$$i_k\hat{\omega}_{\underline{ab}} = -\frac{1}{2}\partial_{[\underline{a}}A_{\underline{b}]} = -\frac{1}{4}F_{\underline{ab}}.$$

Integrating over Z and taking into account

$$*F_{\underline{cd}} = -\frac{1}{2}\epsilon^{\underline{ab}}{}_{\underline{cd}}F_{\underline{ab}}$$

one arrives at

$$S_{Red.} = -\int_{\mathcal{M}^4}[\frac{1}{2!}\hat{R}^{\underline{a_1a_2}}\wedge\hat{E}^{\underline{a_3}}\wedge\hat{E}^{\underline{a_4}}\epsilon_{\underline{a_1}\ldots\underline{a_4}}$$
$$-\frac{1}{2}F^{(2)}\wedge *F^{(2)}].$$

By rescaling the vielbeins we can rewrite the action in the Einstein frame

$$S_{Red.} = \int d^4x\sqrt{-g}[R - \frac{1}{2}(\partial\phi)^2 + \frac{1}{4}e^{-\sqrt{3}\phi}F^2],$$

which is a well known result (cf. [11]).

Thus, in the simplest case it does work. Moreover, it works in the case when antisymmetric tensor fields enter the action which describes a bosonic sector of supergravity. It is clear also, that the generalization to the reduction onto T^n is straightforward and in this case, because the reduction can be performed in a step-by-step manner, we have the embedding chain, which we call "matryoshka", of the lower-dimensional theories into the original one which should lead to the result similar to that of [37].

Several concluding remarks are in order. Firstly, we may expect that this scheme can be modified to describe more complicated cases of the reduction onto G/H coset manifolds and S^n, although, apparently, the hint with fixing an adapted coordinate frame will not work. Secondly, the approach we discuss is restricted to the pure bosonic consideration and the application of this scheme has allowed, for instance, to make a new test for T-duality in the bosonic string theory [38].

Therefore, the problems for further study are to extend this approach to the non-Abelian and to the supersymmetric cases. We hope to make a progress in these directions.

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