PROJECTIONS OF TOPOLOGICAL GROUPS

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The set L(G) of all closed subgroups of a topological group G is a lattice with respect to the operations of intersection $A \cap B$ and taking the smallest closed subgroup $A \vee B = \overline{\{A, B\}}$, containing subgroups A and B in L(G). By a projection of the group G onto a group H we mean an isomorphism φ between the lattices L(G) and L(H).

Graev [1] studied projections of Abelian locally compact groups satisfying the second axiom of countability. It turned out that under certain conditions the projection φ is induced by an isomorphism of the Abelian groups G and H. The proof of this result required imposing restrictions of continuity type on the map φ: the set S(G) of all discrete infinite cyclic subgroups in G is provided with a definite topology (with S(H) having the corresponding topology), and it is required that the map $\varphi: S(G) \to S(H)$ be continuous in this topology. Mukhin [2] studied projections of pure nilpotent groups under a similar continuity condition. It is shown in [3] that the following properties are preserved under projection of a locally compact group G: connectedness, zero-dimensionality, connectedness and compactness, and the property of being a compact Lie group.

It is natural to study the properties of a projection of a topological group G onto another topological group which is continuous in the topology introduced into the whole lattice L(G) of closed subgroups of an arbitrary locally compact group G.

In this paper we suggest introducing into L(G) a topology in analogy with the topology previously considered in the space of closed subsets of an arbitrary topological space, called the exponential topology (see [4]).

We choose an open subset U in the topological group G and denote by $D_1(U)$ the set of all closed subgroups of G contained in U, and by $D_2(U)$ the set of all closed subgroups having nonempty intersection with U. As basis for the topology in the lattice L(G) we take all the sets of the form

$$D_1(U) \cap D_2(U_1) \cap D_2(U_2) \cap \ldots \cap D_2(U_n),$$

where U, U_1, U_2, . . ., U_n are finite collections of open sets in the space G. This topology will be called a topology of the exponential type, and the lattice itself with the above topology is denoted by $L_E(G)$.

Definition. An E-projection φ of a topological group G onto a topological group H is a continuous lattice isomorphism of $L_E(G)$ onto $L_E(H)$. The image H of the group G under an E-projection φ will be denoted by G^φ.

In the following it is assumed that E-projections are considered exclusively in the class of locally compact groups. Only closed subgroups will be called subgroups of a topological group.

The following simple proposition follows directly from the definition.

Proposition. Let A be a subgroup, B a normal subgroup of the topological group G. Then

1) the lattice $L_E(A)$ is E-projected onto the interval $[\{e\}, A]$ of the lattice $L_E(G)$ (where e is the group identity);

2) the lattice $L_E(G/B)$ is E-projected onto the interval [B, G] of the lattice $L_E(G)$.

THEOREM 1. The following properties of a topological group are preserved under E-projection: 1) being a Lie group; 2) discreteness; 3) being monothetic; 4) compactness; 5) finiteness of general and special ranks.

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Proof. 1) As is well known, the Lie groups are precisely the locally compact groups without arbitrarily small subgroups, and they therefore admit the following characterization in terms of the lattice $L_E(G)$: G is a Lie group if and only if $\{e\}$ is isolated in $L_E(G)$. Now let the Lie group G be E-projected onto G^φ. Since

$$\varphi\colon \{e\} \to \{e_1\} \in L_E(G^\varphi),$$

the identity subgroup of G^φ is isolated in $L_E(G^\varphi)$, and therefore G^φ is a Lie group.

2) If G is discrete then G^φ is a Lie group by 1). Moreover, G^φ is zero-dimensional, i.e., G^φ is discrete.

3) First assume that the monothetic group G is zero-dimensional. Since zero-dimensionality is preserved under projection, G^φ is also zero-dimensional. By a well-known lemma of Pontryagin, either G is isomorphic to the infinite discrete cyclic group, or else it is compact. In the first case G^φ is also isomorphic to the infinite discrete cyclic group (a proof of this simple fact is given in [5]). In the second case, $G \simeq \Pi_p G_p$ is the direct product of p-Sylow subgroups G_p, where G_p is either a p-primary cyclic subgroup, or else the additive group of the ring of p-adic integers. It is easy to show that the special rank of G is equal to 1 (this is a well-known fact which follows from [6, Theorem 1]). But a locally compact group has rank 1 if and only if L(G) is distributive [5]. Thus, $L(G^\varphi)$ is distributive, and G^φ is a group of rank 1, so that, in particular, G^φ is Abelian. Let U^φ be an open compact subgroup of G^φ. Since the identity subgroup of G^φ corresponds under projection to the identity subgroup of G, there exists a neighborhood $D_1(V)$ of the identity of the lattice $L_E(G)$ for the neighborhood $D_1(U^\varphi)$ such that $\varphi(D_1(V)) \subset D_1(U^\varphi)$, where V is an open subgroup of G. Consequently, $V^\varphi \subset U^\varphi$. But G/V is a finite cyclic group and therefore, G^φ/V^φ is finite cyclic. It follows that G^φ is compact and any neighborhood of it contains a subgroup such that the quotient group is cyclic, i.e., G^φ is monothetic.

We assume that G is a connected monothetic group (hence it is necessarily Abelian and compact). The proof that the group G^φ is monothetic in this case can be taken in its entirety from [3]. We give only a short sketch. First of all, G^φ is Abelian. It is well-known that a connected Abelian group G is monothetic if and only if the cardinality of its character group X(G) does not exceed the continuum. Thus, $|L(X(G))| \leqslant 2^c$. Since X(G) projects onto $X(G^\varphi)$, we have $|L(X(G^\varphi))| \leqslant 2^c$. Since $X(G^\varphi)$ is a discrete torsion-free Abelian group, necessarily $|X(G^\varphi)| \leqslant c$. Hence G^φ is monothetic.

Turning to the general case, let G_0 denote the connected component of the identity of the compact monothetic group G. Since by hypothesis G/G_0 E-projects onto G^φ/G_0^φ, G^φ is compact. Choose any Lie quotient group G^φ/N^φ in G^φ. Its inverse image under the E-projection is an Abelian Lie group $G/N \simeq T^n \times K$, where K is a finite cyclic group, T a toral group. We consider in $L_E(G/N)$ a sequence of cyclic subgroups S_m converging to the element G/N. By the continuity of the projection, S_m^φ converges to G^φ/N^φ. It follows directly from this that G^φ/N^φ is Abelian. Since N^φ is arbitrary, G^φ is Abelian. Thus, G_0^φ is monothetic, G^φ/G_0^φ is monothetic, and G^φ is compact Abelian, so that G^φ is monothetic (see [3, p. 68]).

4) First assume that G is zero-dimensional. We show that G^φ is compact. Consider a net $\{y_\alpha, \alpha \in J\}$ in G^φ. Since φ^{-1} is an E-projection of G^φ onto G and E-projection takes monothetic subgroups into monothetic subgroups, we have

$$\varphi^{-1}\{\overline{y_\alpha}\} = \{\overline{x_\alpha}\} \in L_E(G).$$

Since G is compact, the net $\{x_\alpha, \alpha \in J\}$ has an adherent point x. We show that $\overline{\{x\}}$ is a point adherent to the net $\{\overline{\{x_\alpha\}}, \alpha \in J\}$. Let $z_1, z_2, \ldots, z_n$ be points in $\overline{\{x\}}$, U an arbitrary neighborhood of the identity of G, V an open set containing $\overline{\{x\}}$. We show that the neighborhood

$$D(\overline{\{x\}}) = D_1(V) \cap D_2(Uz_1) \cap D_2(Uz_2) \cap \ldots \cap D_r(Uz_n)$$

of the subgroup $\overline{\{x\}}$ necessarily contains elements of the net $\{\overline{\{x_\alpha\}}, \alpha \in J\}$. By the choice of the points $z_1, z_2, \ldots, z_n$ there exists a neighborhood U(x) and integers $m_1, \ldots, m_n$ such that

$$U^{m_1}(x) \subset Uz_1, \quad U^{m_2}(x) \subset Uz_2, \ldots, U^{m_n}(x) \subset Uz_n.$$

If now $x_{\alpha_0} \in U(x)$, then

$$\overline{\{x_{\alpha_0}\}} \in D_2(Uz_1) \cap \ldots \cap D_2(Uz_n).$$

Furthermore, since G is a zero-dimensional compact group, there exists an open normal subgroup H for the open set V such that $\overline{\{x\}}H \subset V$. Thus, $\overline{\{z\}} \subset V$ for every $z \in xH$. Hence if now $x_{\beta_0} \in xH \cap U(x)$, then $\overline{\{x_{\beta_0}\}} \in D(\overline{\{x\}})$, i.e., {x} is an adherent point of the net $\{\overline{\{x_\alpha\}}, \alpha \in J\}$. By the continuity of the projection, $\varphi\overline{\{y\}} = \varphi\overline{\{x\}}$ is an adherent point of the net $\{\overline{\{y_\alpha\}}, \alpha \in J\}$. We remark that by 3) $\overline{\{y\}}$ is compact. Therefore, there exists an open set V in G^φ with compact closure such that $\overline{\{y\}} \subset V$. Consider the subnet $\{\overline{\{y_\alpha\}}, \alpha \in J\}$, consisting of all subgroups $\overline{\{y_\gamma\}}$, $\gamma \in J_0$, which are contained in V. This means that all the y_γ, $\gamma \in J_0$, belong to the compact set $\overline{V}$, and hence the net $\{y_\gamma, \gamma \in J_0\}$ has an adherent point, which proves that G^φ is compact.

Turning now to the general case, let G_0 denote the connected component of the identity of an arbitrary compact group G. Since connectedness and compactness are preserved even under ordinary projection, we have that $(G^\varphi)_0 = G_0^\varphi$ is compact. By hypothesis, G/G_0 E-projects onto G^φ/G_0^φ and by what was proved above, the zero-dimensional group G^φ/G_0^φ is compact, so that G^φ is also compact.

5) Let G be a group of rank n, $b_1, b_2, \ldots, b_s \in G^\varphi$. By 3) $\varphi^{-1}\overline{\{b_i\}} = \overline{\{a_i\}}$. Since $\overline{\{a_1, \ldots, a_s\}} = \overline{\{c_1, \ldots, c_n\}}$, we have $\overline{\{b_1, \ldots, b_s\}} = \varphi\overline{\{c_1, \ldots, c_n\}} = \overline{\{\varphi\overline{\{c_1\}}, \ldots, \varphi\overline{\{c_n\}}\}} = \overline{\{d_1, \ldots, d_n\}}$, i.e., G^φ has rank n. The proof that the general rank is preserved is similar.

As a direct application of Theorem 1, we can obtain an analog of a local theorem of L. E. Sadovskii for E-projections.

THEOREM 2. Let φ be an E-projection of the inductive limit G = lim ind G_λ of groups G_λ, $\lambda \in J$, onto the group G^φ and assume that the set A_λ of all isomorphisms of G_λ onto G_λ^φ, inducing the projection φ, is compact for every λ in the compact-open topology. Then $G^\varphi =$ lim ind G_λ^φ, and if all the A_λ are nonempty, then φ is induced by an isomorphism of G onto G^φ.

Let G be a compact zero-dimensional group. Then as is well known, there exists a projective system $\{G_\alpha, \varphi_{\beta\alpha}, \alpha \in J\}$ of finite groups such that $G = \lim \mathrm{pr}\, G_\alpha$. $\varphi_{\beta\alpha}$ defines a natural homomorphism $\overline{\varphi}_{\beta\alpha}$ of the lattice $L_E(G_\beta)$ onto $L_E(G_\alpha)$, which is continuous since $L_E(G_\beta)$ is finite. Since $\overline{\varphi}_{\beta\alpha} = \overline{\varphi}_{\beta\gamma}\overline{\varphi}_{\alpha\gamma}$ for $\alpha \leqslant \gamma \leqslant \beta$, we have that $\{L_E(G_\alpha), \overline{\varphi}_{\beta\alpha}, \alpha \in J\}$ is also a projective system of finite sets. By a lemma of Kurosh, the projective limit $L_E(G_\alpha)$ is nonempty. We now establish a relation between the lattice $L_E(G)$ and lim pr $L_E(G_\alpha)$.

THEOREM 3. The lattice $L_E(G)$ of a zero-dimensional compact group is a projective limit of lattices of finite groups.

Proof. By definition, lim pr $L_E(G_\alpha)$ is the closed subset of the topological direct product $\Pi_{\alpha\in J} E_E(G_\alpha)$, consisting of elements K such that $\overline{\varphi}_{\beta\alpha}(\mathrm{pr}_\beta K) = \mathrm{pr}_\alpha K$, where $\mathrm{pr}_\alpha K$ is the projection of K into $L_E(G_\alpha)$. Let H be a closed subgroup of G. Then H = lim pr $(\mathrm{pr}_\alpha H)$ and we have thereby established a natural correspondence f between $L_E(G)$ and lim pr $L_E(G_\alpha)$. It is easy to see by direct verification that f is a structure isomorphism. We show that f is a homeomorphism. Let $H \in L_E(G)$. If V is a neighborhood of f(H), then

$$V = \{K \in \lim \mathrm{pr}\, L_E(G_\alpha)\colon \mathrm{pr}_{\alpha_1} K = \mathrm{pr}_{\alpha_1} H,\ \mathrm{pr}_{\alpha_2} K = \mathrm{pr}_{\alpha_2} H, \ldots, \mathrm{pr}_{\alpha_n} K = \mathrm{pr}_{\alpha_n} H,\ \alpha_1, \alpha_2, \ldots, \alpha_n \in J\}.$$

Let $N_{\alpha_1}, \ldots, N_{\alpha_n}$ denote open normal subgroups of G such that $G/N_{\alpha_i} = G_{\alpha_i}$, $i = 1, 2, \ldots, n$. It is clear that $f(S) \in V$, if

$$S/(N_{\alpha_1} \cap \ldots \cap N_{\alpha_n}) = H/(N_{\alpha_1} \cap \ldots \cap N_{\alpha_n}).$$

However, the set of closed subgroups S in G satisfying the last equality is open in $L_E(G)$. Reversing these arguments, we obtain that f^{-1} is continuous. This proves the theorem.

Remark 1. It is asserted in Mukhin and Khomenko [3] that monotheticity is preserved under projection.

The following example shows that monotheticity, compactness, discreteness, and the property of being a Lie group are not preserved under projection. It is more convenient for us to construct an example of a nonmonothetic group which projects onto a monothetic group.

Let $G = \prod_{i=1}^\infty G_{p_i}$ be the algebraic direct product of cyclic groups G_{p_i} of prime orders for all distinct primes $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, . . . Let H denote the topological direct product (in the Tikhonov topology) of the groups G_{p_i}. The group G projects onto H. In fact, let A be a subgroup of G. It splits into the direct product of a collection of subgroups of the set G_{p_i} (i = 1, 2, . . .). The analogous assertion is true also for the compact group H. We define a map φ of the lattice L(G) onto the lattice L(H) in the natural way: if $A \in L(G)$, then we put $\varphi(A) = \bar{A}$, where $\bar{A}$ is the closure of A in H, if A is regarded as a subset of H. The subgroup $\varphi(A)$ splits into a topological direct product of the same set of subgroups G_{p_i} as A and G (see [6, Theorem 2]). The map φ is a lattice isomorphism of L(G) and L(H). The group G is not monothetic, whereas H is monothetic.

Remark 2. A proposition analogous to Theorem 2 was stated in [3] without an additional assumption of continuity type of the map φ. The above example shows that this proposition is not valid in this form. In [7], the same example is cited in connection with similar remarks.

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INTERSECTIONS OF CONVEX CONES

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We retain the definitions, conventions, and notation of [1]. As in [1], all the results can be stated in terms of spherically convex sets. We first present some auxiliary facts.

THEOREM A [2, p. 450]. The set $X \subset \mathbf{R}^n \, (0 \notin X)$ is s.p. (strongly positive) independent if and only if the space R^n can be written as a direct sum of subspaces

$$\mathbf{R}^n = \mathbf{R}^{d_0} \oplus \mathbf{R}^{d_1} \oplus \ldots \oplus \mathbf{R}^{d_m}$$

so that 1) $X \subset \bigcup_{i=0}^{m} \mathbf{R}^{d_i}$, 2) $X \cap \mathbf{R}^{d_0}$ is linearly independent, 3) $X \cap \mathbf{R}^{d_i}$ is a minimal positive basis of the space $\mathbf{R}^{d_i}$ $(i = 1, \ldots, m)$.

LEMMA 1. If some proper subset of the set X admits a positive decomposition, then X itself has a positive decomposition, which is moreover not unique.

The proof of Lemma 1 is obvious.

Let X be a set of nonzero vectors of R^n and put l = lin pos X. Then the subspace $\mathbf{R}^l =$ pos X $\cap$ (− pos X), contained in the cone pos X, is the boundary of this cone, and therefore, $\text{pos}\,(X \cap R^l) = R^l$. It follows from this and Steinitz' theorem [3] that a positive basis B of $\mathbf{R}^l$ can be chosen from the set $X \cap \mathbf{R}^l$. Moreover, $l + 1 \leqslant |B| \leqslant 2l$ if $l > 0$, and $|B| = 0$ if $l = 0$.

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