EQUILIBRIUM AND STABILITY OF PLASMA IN AXIALLY SYMMETRIC TOROIDAL SYSTEMS

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UDC 533.9

The present article contains a study of the equilibrium and stability of a plasma confined by a magnetic field in axially symmetric systems of Tokamak type. The discussion is carried out within the framework of the magnetohydrodynamic approximation for a perfectly conducting plasma, when the equilibrium magnetic configuration takes the form of a system of enclosed toroidal magnetic surfaces surrounding a circular magnetic axis r = R. The equilibrium configurations investigated are magnetic configurations with a prescribed external surface Σ. Three cases will be considered, namely, when the surface Σ is a toroidal surface of elliptical, semi-elliptical, or rectangular cross section. In such systems the hydrodynamic sta bility of the plasma turns out to depend rather strongly on the ellipticity ε of the magnetic surface cross sections in the neighborhood of the magnetic axis and on the relative variation in the axial magnetic field $\delta B/B_0$ over the plasma cross section.

The equilibrium plasma configurations are investigated with the aid of the method of flux functions [1], the existence of which is ensured by the axial symmetry of the problem. In the stability investigations use is made of the criterion necessary to ensure localized stability [2-5] for an arbitrary plasma geometry. In the event of a quasiuniform magnetic field this criterion has been shown [6] to be a necessary and sufficient condition of stability. In the present article allowance is made for the effect on the equilibrium and stability of the plasma of its toroidal geometry and cross-sectional shape. The case is considered when the "meridian" cross section of the plasma torus are symmetrical relative to the "equatorial" plane z = 0, so that the normal cross sections of the magnetic surfaces in the neighborhood of the magnetic axis constitute in general ellipses orientated along or across the symmetry axis z.

General Relationships

Equilibrium. The equilibrium plasma configurations are determined by the equations

$$\nabla p = [\mathbf{jB}], \ \mathbf{j} = \operatorname{rot} \mathbf{B}, \ \operatorname{div} \mathbf{B} = 0 \tag{1}$$

where p is the plasma pressure, j is the peak density, and B is the magnetic field intensity. For axially symmetric problems these equations are most conveniently handled in cylindrical coordinates r, φ, z with the aid of the functions $\psi = rA_\varphi$ (A_φ is the corresponding component of the vector potential), $I_A(\psi) = rB_\varphi$ and $p(\psi)$. The components of the vectors B and j equal [1]:

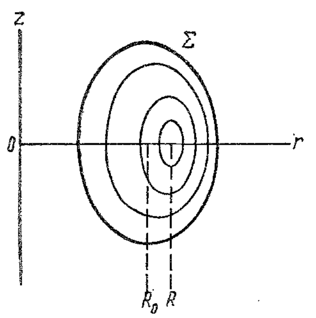

Fig. 1. Magnetic surfaces of an equilibrium plasma in a torus of elliptical cross section.

$$B_z = \frac{1}{r}\cdot\frac{\partial\psi}{\partial r}, \ B_r = -\frac{1}{r}\cdot\frac{\partial\psi}{\partial z}, \ B_\varphi = \frac{I_A}{r}, \tag{2}$$

$$j_z = \frac{I'_A}{r}\cdot\frac{\partial\psi}{\partial r}, \ j_r = -\frac{I'_A}{r}\cdot\frac{\partial\psi}{\partial z}, \ j_\varphi = rp' + \frac{I_A I'_A}{r}, \tag{3}$$

where the prime denotes differentiation with respect to ψ. The flux function $\psi(r, z)$ satisfies the equation

$$r\frac{\partial}{\partial r}\cdot\frac{1}{r}\cdot\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = -r^2 p' - I_A I'_A. \tag{4}$$

For given functions $p' \equiv -a(\psi)$ and $I_A I'_A \equiv -R^2 b(\psi)$ the solution of Eq. (4) determines a system of magnetic surfaces $\psi(r, z) = \text{const}$ corresponding to an equilibrium configuration.

Translated from Atomnaya Énergiya, Vol. 24, No. 5, pp. 453-459, May, 1968. Original article submitted August 16, 1967.

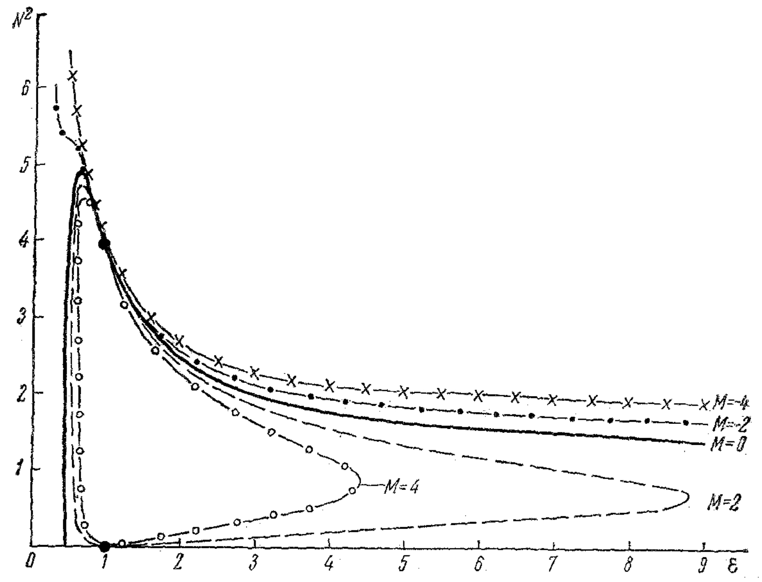

Fig. 2. Regions of stability of a plasma for various values of the parameter $M = (4\,\delta B^2/B_0^2)\cdot(R^2/\rho_2)$ (the regions of stability are located inside the closed curves or underneath the curves which are not closed).

We represent $p(\psi)$ and $I_A^2(\psi)$ in the form of expansions in powers of ψ in the neighborhood of the magnetic axis $r = R$, $z = 0$:

$$p = p_0 + p'(\psi - \psi_0) + \ldots;$$
$$I_A^2 = I_{A0}^2 + 2I_A I_A'(\psi - \psi_0) + \ldots, \tag{5}$$

where $\psi_0 = \psi(R, 0)$. Suppose the pressure $p(\psi)$ reduces to zero on some magnetic surface ψ_0. We then have in accordance with expressions (2) and (5)

$$\frac{b}{a} = -\frac{I_{A\sigma}^2 - I_{A0}^2}{2p_0 R^2} = -\frac{B_\sigma^2 - B_0^2}{2p_0}. \tag{6}$$

The ratio b/a is thus seen to characterize the diamagnetism of the plasma. We introduce the notation

$$\beta_0 = \frac{2p_0}{B_0^2};\ \delta B^2 = B_\sigma^2 - B_0^2;\ N = \frac{j_0 R}{B_0} \tag{7}$$

(the index zero denotes here that the corresponding quantity is taken on the magnetic axis) and obtain from Eqs. (3), (6)

$$\frac{b}{a}\beta_0 = -\frac{\delta B^2}{B_0^2}; \tag{8}$$

$$\frac{b}{a}\left(\frac{\psi_\sigma - \psi_0}{aR^4}\right)N^2 + \left(1 + \frac{b}{a}\right)^2 \frac{\delta B^2}{2B_0^2} = 0. \tag{9}$$

Eliminating the ratio b/a from these equations, we obtain for a known $\psi(r, z)$ an equilibrium equation connecting the quantities β_0, N^2, and $\delta B^2/B_0^2$.

Correct up to terms cubic in z and $r - R$, the solution of Eq. (4) does not depend on the derivatives of $a(\psi)$ and $b(\psi)$ and is given in fact by the expression

$$\psi = \frac{a+b}{1+\varepsilon^2}\Big[\frac{R^2}{2}\Big(1 + c\,\frac{r^2 - R^2}{R^2}\Big)z^2 + \frac{\varepsilon^2}{8}(r^2 - R^2)^2 - \frac{(1+\varepsilon^2)\,b - (1-c)(a+b)}{24R^2(a+b)}(r^2 - R^2)^3\Big]. \tag{10}$$

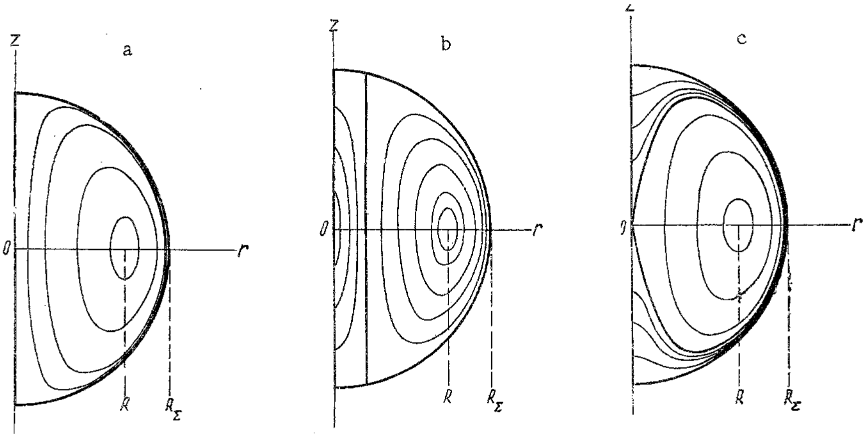

Fig. 3. Magnetic surface of equilibrium plasma in torus of semi-elliptical cross section: a) b = 0; b) R^2b/a = 0.008; c) R^2b/a = 0.008.

Cross sections of the magnetic surfaces in the neighborhood of the magnetic axis r = R, z = 0 have the form of an ellipse with a semi-axis ratio $l_z/l_r = \varepsilon$. In order to find the parameters ε and c it is necessary to solve Eq. (4) with a boundary magnetic surface ψ_Σ of the prescribed shape, after which ε and c can be determined in terms of the derivatives of ψ on the magnetic axis r = R:

$$\varepsilon^2 = \frac{\partial^2\psi/\partial r^2}{\partial^2\psi/\partial z^2}; \quad c = \frac{R}{2}\cdot\frac{\partial^3\psi/\partial z\,\partial r^2}{\partial^2\psi/\partial z^2}. \tag{11}$$

The position of the magnetic axis is determined by the equalities $\partial\psi/\partial r = 0$, $\partial\psi/\partial z = 0$.

Stability. The stability will be investigated with the aid of the localized stability criterion [2-4], which may be written in the following form

$$\left\langle \frac{S}{2} + \frac{\mathbf{jB}}{|\nabla V|^2} \right\rangle^2 - \left\langle \frac{\mathbf{B}^2}{|\nabla V|^2} \right\rangle \left\langle \Omega + \frac{\mathbf{j}^2}{|\nabla V|^2} \right\rangle \geqslant 0. \tag{12}$$

As shown in [6], criterion (12) constitutes a necessary and sufficient condition for the hydromagnetic stability of plasma configurations confined by a quasiuniform magnetic field.* The quantities Ω and S entering into expression (12), as also p, depend only on the current volume V of the system of enclosed magnetic surfaces, and are determined by the formulas

$$\dot{p} = \dot{I}\ddot{\varphi} - \dot{J}\ddot{\chi}, \quad \Omega = \dot{I}\dddot{\varphi} - \dot{J}\dddot{\chi}, \quad S = \dot{\chi}\dddot{\varphi} - \dot{\varphi}\dddot{\chi}, \tag{13}$$

where φ and χ are respectively the axial and transverse magnetic fluxes; J and I are respectively the axial and transverse currents within the magnetic surface bounding the volume V; the dots denote differentiation with respect to V; the angled brackets denote an average taken over a closed line of force $\langle f\rangle = \oint f\,\frac{dl}{B} \Big/ \oint\frac{dl}{B}$. For axially symmetric systems this average is equivalent to an average taken over the volume of the sheet between adjacent magnetic surfaces.

In the neighborhood of the magnetic axis r = R of axially symmetric plasma configurations, stability condition (12) acquires the form [6]

$$B_0^2 p' \frac{V''}{V'} - p'^2 \left\{1 + \frac{16\pi^2 B_0^2}{\chi'^2} \left\langle \frac{(r-R)^2}{|\nabla\varphi|^2} \right\rangle\right\} \geqslant 0, \tag{14}$$

where the primes denote differentiation with respect to φ. Utilizing expansion (10) and working out the various integrals, we can convert inequality (14) to the form [5]

$$N^2 \leqslant \left(1+\frac{1}{\varepsilon^2}\right) \times \left[\frac{2}{1+\varepsilon} + \frac{1-\varepsilon}{1+\varepsilon}\cdot\frac{1-\frac{b}{a}}{1+\frac{b}{a}} - \frac{1-\varepsilon^2}{1+\varepsilon^2}(1+2c)\right]. \tag{15}$$

* If the term $S^2/4$ is discarded, criterion (12) constitutes a sufficient condition for the stability of a low-pressure plasma.

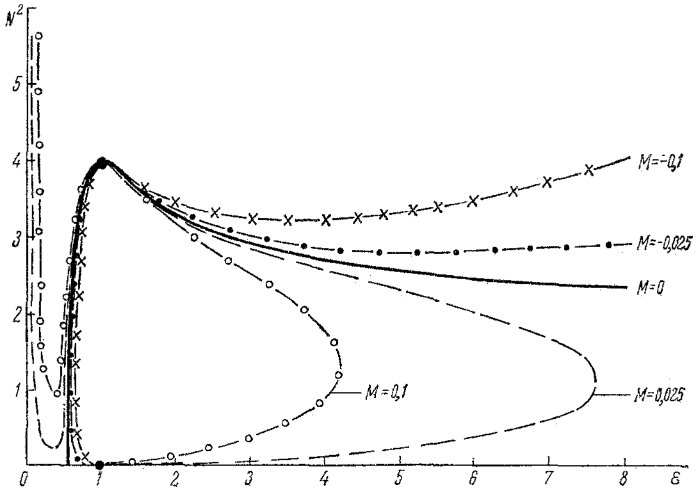

Fig. 4. Regions of stability for various values of the parameter $M = 2\delta B^2/B_0^2(2R^2/R^2\text{-}R^2)^2$.

We note that in the case of magnetic surfaces of circular cross section ($\varepsilon = 1$) near the axis, this stability condition does not depend on the parameters a, b, and c and has the universal form $N = Rj_0/B_0 < 2$.

If the plasma pressure reduces to zero on the magnetic surface ψ_0, the minimum distance of which from the z axis is denoted by R_1, then expression (15) implies the following restriction on β_0:

$$\beta_0 \leqslant \frac{1}{4} \cdot \frac{\varepsilon^2}{1+\varepsilon^2} \times \frac{1-R_1^2/R^2}{1+b/a}\left[1+\frac{1-R_1^2/R^2}{3\varepsilon^2}\left(c-\frac{1-\varepsilon^2 b/a}{1+b/a}\right)\right] N^2. \tag{16}$$

We note that the surface functions φ, χ, J, and I are connected with ψ and $I_A(\psi)$ by the relationships [5]:

$$\chi = -2\pi\psi; \quad I = -2\pi I_A; \tag{17}$$

$$2\pi\, d\varphi = I_A\, dU; \quad 2\pi\, dJ = p'\, dV + I_A I_A'\, dU, \tag{18}$$

where V and U are determined by integrals over the cross section of the magnetic surface:

$$V = 2\pi \int r\, dr\, dz; \quad U = 2\pi \int r^{-1}\, dr\, dz. \tag{19}$$

Let us now consider the equilibrium and stability of a plasma for an external magnetic surface ψ_Σ of various prescribed shapes.

Plasma in Torus of Elliptical Cross Section

Let the external magnetic surface ψ_Σ have an elliptical cross section $z^2 + \varepsilon_0^2(r-R_0)^2 = \varepsilon_0^2\rho^2$ with a center at $r = R_0$ and with a semi-axis ratio ε_0 (Fig. 1). We shall seek the solution of equilibrium equation

$$r\frac{\partial}{\partial r} \cdot \frac{1}{r} \cdot \frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = ar^2 + bR^2 + (a'r^2 + b'R^2)\,\psi + \ldots \tag{20}$$

in the form of a series in powers of the small parameters z/R_0 and $(r-R_0)/R_0$. We write $x = r-R_0$ and denote by ρ the half width of the elliptical cross section of the bounding magnetic surface $\psi = 0$. With this notation we have

$$\psi = (A + Bx + Cx^2 + Dz^2 + E\rho^2 + \ldots) \times (z^2 + \varepsilon_0^2 x^2 - \varepsilon_0^2\rho^2). \tag{21}$$

We introduce this expression into Eq. (20) and obtain the expansion coefficients A, B, ... (21) by equating like powers of x, z, and ρ^2:

$$A = \frac{1}{2} \cdot \frac{aR_0^2 + bR^2}{1+\varepsilon_0^2};$$

$$B=\frac{1}{2R_0}\cdot\frac{(2+3\varepsilon_0^2)\,aR_0^2+\varepsilon_0^2 bR^2}{(1+\varepsilon_0^2)(1+3\varepsilon_0^2)};$$

$$C=\frac{B}{2R_0}-\frac{A\varepsilon_0^2}{4R_0^2}\cdot\frac{6+\varepsilon_0^2-(5+\varepsilon_0^2)(a'R_0^2+b'R^2)R_0^2/3}{1+6\varepsilon_0^2+\varepsilon_0^4};\tag{22}$$

$$D=\frac{A}{4R_0^2}\cdot\frac{\varepsilon_0^2+(1+5\varepsilon_0^2)(a'R_0^2+b'R^2)R_0^2/3}{1+6\varepsilon_0^2+\varepsilon_0^4};$$

$$E=-\frac{5+\varepsilon_0^2}{1+\varepsilon_0^2}\varepsilon_0^2 D.$$

If the toroidal nature of the problem is not too strongly expressed, so that we may restrict ourselves in ψ to terms $\sim\rho^3/R_0^3$, then

$$\psi=(A+Bx)(z^2+\varepsilon_0^2x^2-\varepsilon_0^2\rho^2),\tag{25}$$

where

$$A=\frac{R_0^2}{2}\cdot\frac{a+b}{1+\varepsilon_0^2};\quad B=\frac{R_0}{2}\cdot\frac{(2+3\varepsilon_0^2)\,a+\varepsilon_0^2 b}{(1+\varepsilon_0^2)(1+3\varepsilon_0^2)}.\tag{26}$$

On equating to zero the derivatives of ψ with respect to x and z, we find the displacement of the magnetic axis $x_0 = R-R_0$ from the cross-sectional center $r = R_0$:

$$\frac{x_0}{R_0}=\frac{1}{2}\cdot\frac{2+3\varepsilon_0^2+\varepsilon_0^2 b/a}{(1+b/a)(1+3\varepsilon_0^2)}\cdot\frac{\rho^2}{R_0^2}.\tag{27}$$

Calculations to the same accuracy of the semi-axis ratio of elliptical magnetic surfaces in the neighborhood of the magnetic axis $r = R$ lead to

$$\varepsilon=\varepsilon_0\left(1+2\frac{x_0^2}{\rho^2}\right).\tag{28}$$

It is clear that the relative difference between R and R_0 and ε and ε_0 is small ($\sim\rho^2/R^2$).

We note that when $b/a < 0$ a second magnetic axis may appear with the formation within the toroid ψ_Σ of two families of enclosed magnetic surfaces separated by a "partition" $r-R_0 = -A/B$.

On neglecting terms $\sim\rho^2/R^2$, we obtain the quantity c appearing in stability condition (15):

$$c=\frac{1}{2}\cdot\frac{2+3\varepsilon^2+\varepsilon^2 b/a}{(1+b/a)(1+3\varepsilon^2)}.\tag{29}$$

The ratio b/a can be determined from Eq. (9) by putting $\psi_\sigma = 0$:

$$\frac{b}{a}=-\frac{\dfrac{\delta B^2}{B_0^2}}{\dfrac{\delta B^2}{B_0^2}+\dfrac{\varepsilon^2}{1+\varepsilon^2}\cdot\dfrac{\rho^2}{R^2}N^2}.\tag{30}$$

We substitute (30) into expression (8) to obtain the equilibrium equation:

$$\beta_0=\frac{\delta B^2}{B_0^2}+\frac{\varepsilon^2}{1+\varepsilon^2}\cdot\frac{\rho^2}{R^2}N^2.\tag{31}$$

Stability condition (15) may now be written in the form

$$N^4-\frac{(1+\varepsilon^2)(3\varepsilon^4+15\varepsilon^3-7\varepsilon^2+9\varepsilon-4)}{\varepsilon^3(1+\varepsilon)(1+3\varepsilon^2)}N^2+\frac{4(1-\varepsilon)^2(1+\varepsilon^2)^3}{\varepsilon^5(1+\varepsilon)(1+3\varepsilon^2)}\cdot\frac{\delta B^2}{B_0^2}\cdot\frac{R^2}{\rho^2}\leqslant 0.\tag{32}$$

The corresponding regions of stability are shown in Fig. 2.

In the case when the outside magnetic surface ψ_Σ is of circular cross section $\varepsilon_0 = 1$, the quantity ε is given approximately by

$$\varepsilon=1+\frac{1}{32}\left(\frac{5+b/a}{1+b/a}\right)^2\frac{\rho^2}{R^2}=1+\left(\frac{\frac{5}{4}N^2+\frac{2\delta B^2}{B_0^2}\cdot\frac{R^2}{\rho^2}}{N^2}\right)^2\frac{\rho^2}{2R^2}\tag{33}$$

and stability condition (32) takes the form

$$N^4-4N^2+\left(\frac{\frac{5}{4}N^2+\frac{2\delta B^2}{B_0^2}\cdot\frac{R^2}{\rho^2}}{N^2}\right)^4\frac{\delta B^2}{B_0^2}\cdot\frac{\rho^2}{R^2}\leqslant 0. \tag{34}$$

It can be seen from expression (34) that when $\delta B^2 > 0$ the plasma becomes unstable at small currents.

Plasma in Torus of Semi-Elliptical Cross Section

For constant a and b there exists a simple precise solution to the equilibrium equation (4) which satisfies the boundary condition $\psi = 0$ on the surface of an ellipsoid of revolution, the semi-axis ratio of which we denote by ε_Σ:

$$\psi=\frac{1}{2}\left(\frac{ar^2}{1+4\varepsilon_\Sigma^2}+bR^2\right)(z^2+\varepsilon_\Sigma^2 r^2-\varepsilon_\Sigma^2 R_\Sigma^2), \tag{35}$$

where R_Σ is the maximum radius of the boundary surface ψ_Σ. The magnetic surfaces of the equilibrium configuration (35) are shown in Fig. 3. The radius R of the magnetic axis and the semi-axis ratio ε of elliptical cross section magnetic surfaces in the neighborhood of the magnetic axis are connected with R_Σ and ε_Σ by the relationships

$$R^2=\frac{R_\Sigma^2}{2+(1+4\varepsilon_\Sigma^2)\,b/a}; \tag{36}$$

$$\varepsilon^2=\frac{4\varepsilon_\Sigma^2}{1+(1+4\varepsilon_\Sigma^2)\,b/a}. \tag{37}$$

On the magnetic axis r = R the quantity ψ equals

$$\psi_0=-\frac{aR^4}{8}\cdot\frac{\varepsilon^2}{1+\varepsilon^2}\cdot\frac{(1+b/a)^3}{(1-\varepsilon^2 b/a)^2}. \tag{38}$$

Let us denote the maximum distance between the plasma surface and the z axis by R_2. In accordance with expression (9) we have

$$\frac{b}{a}=-\frac{\delta B^2/B_0^2}{\frac{\delta B^2}{B_0^2}+\frac{\varepsilon^2 N^2}{1+\varepsilon^2}\left(\frac{R_2^2-R^2}{2R^2}\right)^2}. \tag{39}$$

On inserting this expression into (8), we arrive at the equilibrium equation

$$\beta_0=\frac{\delta B^2}{B_0^2}+\frac{\varepsilon^2 N^2}{1+\varepsilon^2}\left(\frac{R_2^2-R^2}{2R^2}\right)^2. \tag{40}$$

For the equilibrium configuration at present being considered the quantity c entering into stability criterion (15) is given by the expression

$$c=\frac{1-\varepsilon^2\frac{b}{a}}{1+\frac{b}{a}}. \tag{41}$$

The above results can be employed to convert stability criterion (15) to a form similar to (32):

$$N^4-2\frac{1+\varepsilon^2}{\varepsilon^3}\,\frac{\varepsilon^2+3\varepsilon-2}{1+\varepsilon}N^2+\frac{2\delta B^2}{B_0^2}\left(\frac{2R^2}{R_2^2-R^2}\right)^2\frac{1-\varepsilon}{\varepsilon^5}\,\frac{2+\varepsilon}{1+\varepsilon}(1+\varepsilon^2)^3\leqslant 0. \tag{42}$$

The corresponding regions of stability are shown in Fig. 4.

Let us consider in more detail the case $\delta B^2/B_0^2 = 0$, when b = 0 and the axial magnetic field $B_\varphi = I_A/r$ falls off as 1/r. From expressions (17), (18) we have

$$p'=\frac{2\pi J}{V};\quad N=\frac{4\pi^2 JR^3}{IV}, \tag{43}$$

where J is the axial current in the plasma; V is the plasma volume; I is the total current flowing through the solenoid producing the axial magnetic field.

If the toroidal nature of the problem is only weakly expressed ($V \approx 2\pi^2\rho^2 R\varepsilon$), the equilibrium value of β_0 coincides with Eq. (31):

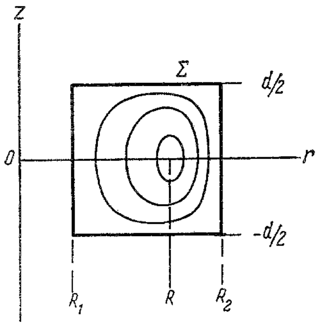

Fig. 5

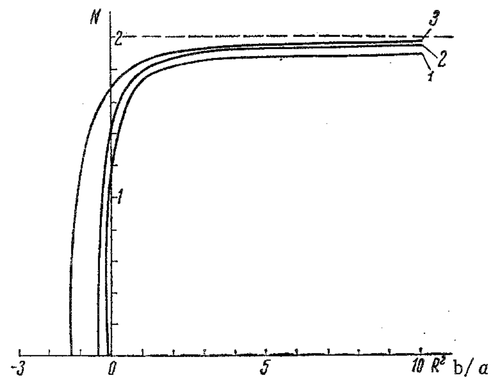

Fig. 6

Fig. 5. Magnetic surfaces of equilibrium plasma in a torus of rectangular cross section.

Fig. 6. Regions of stability of plasma in torus of square cross section for various values of R_1/d: 1) $R_1/d = 0.1$; 2) $R_1/d = 0.5$; 3) $R_1/d = 1$.

$$\beta_0 = \frac{\varepsilon^2}{1+\varepsilon^2} \cdot \frac{\rho^2}{R^2} N^2 = \frac{4}{1+\varepsilon^2} \cdot \frac{R^2}{\rho^2} \cdot \frac{J^2}{I^2}, \tag{44}$$

while the stability criterion reduces to a restriction on the current ratio J/I:

$$\frac{J^2}{I^2} \leqslant \frac{\rho^4}{R^4} \cdot \frac{1+\varepsilon^2}{2\varepsilon} \cdot \frac{\varepsilon^2+3\varepsilon-2}{1+\varepsilon}. \tag{45}$$

If the toroidal aspects of the problem are very strongly expressed, when the plasma torus occupies almost the entire volume ($V_\Sigma = 4\pi\varepsilon_\Sigma R^3{}_\Sigma/3$) within the bounding ellipsoid of revolution $z^2 + \varepsilon^2{}_\Sigma r^2 = \varepsilon^2{}_\Sigma R^2{}_\Sigma$, then

$$\beta_0 = \frac{\varepsilon^2}{1+\varepsilon^2} \cdot \frac{N^2}{4} = \frac{1}{8} \cdot \frac{9\pi^2}{1+\varepsilon^2} \cdot \frac{J^2}{I^2}, \tag{46}$$

while stability condition (43) acquires the form

$$\frac{J^2}{I^2} \leqslant \frac{8}{9\pi^2} \cdot \frac{1+\varepsilon^2}{2\varepsilon} \cdot \frac{\varepsilon^2+3\varepsilon-2}{1+\varepsilon}. \tag{47}$$

Here the maximum value of β_0 corresponds to $\varepsilon = 1 + \sqrt{2}$, and for $\varepsilon \approx 9/16$ the right side of inequality (47) reduces to zero.

Thus, in the strongly toroidal case and on the assumption that a quasiuniform axial magnetic field is realized ($b = 0$), it can be seen that the hydromagnetic stability condition enables quite large currents to flow in the plasma. For example, for $\varepsilon \approx 1$ the stability criterion permits $\beta_0 \approx 1/2$, $J/I \approx 1/3$.

Plasma in Torus of Rectangular Cross Section

We will restrict ourselves to the case $p' = -a = \text{const}$, $R^2 b = -I_A I_A{}' = \text{constant}$ and will seek a solution to Eq. (4) which satisfies the condition $\psi = 0$ on the surface Σ of a torus of rectangular cross section (Fig. 5). We take as a particular solution to the inhomogeneous equation (4) the function

$$\psi_1 = \frac{1}{2}(ar^2 + bR^2)(z^2 - d^2/4), \tag{48}$$

when the general solution of Eq. (4) may be written in the form

$$\psi = r \sum_{n=1}^{\infty} \left[\alpha_n I_1\left(\frac{n\pi z}{d}\right) + \beta_n K_1\left(\frac{n\pi z}{d}\right)\right] \cos\frac{n\pi z}{d} + \psi_1. \tag{49}$$

On expanding the function

$$f(z) = \begin{cases} z^2 - \frac{d^2}{4} & \text{for } 0 < z < \frac{d}{2}; \\ -\left[(z-d)^2 - \frac{d^2}{4}\right] & \text{for } \frac{d}{2} < z < d \end{cases} \tag{50}$$

as a Fourier series in the double interval $(-d, d)$, we find

$$f(z) = -\frac{8d^2}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin n\frac{\pi}{2}}{n^3} \cos\frac{n\pi z}{d}. \tag{51}$$

The coefficients α_n and β_n can be determined from the boundary conditions $\psi(R_1, z) = 0$, $\psi(R_2, z) = 0$. We finally obtain

$$\psi = \frac{4d^2}{\pi^3} r \sum_{n=1}^{\infty} \frac{\sin n\frac{\pi}{2}}{n^3 D_n(R_1, R_2)} \times \left[\left(aR_2 + \frac{bR^2}{R_2}\right) D_n(R_1, r) - \left(aR_1 + \frac{bR^2}{R_1}\right) D_n(R_2, r)\right] \cos\frac{n\pi z}{d} + \frac{r}{2}\left(ar + \frac{bR^2}{r}\right)\left(z^2 - \frac{d^2}{4}\right), \tag{52}$$

where $D_n(x, y)$ is expressed in terms of the modified Bessel functions I_1 and K_1:

$$D_n(x, y) = K_1\left(\frac{n\pi x}{d}\right) I_1\left(\frac{n\pi y}{d}\right) - I_2\left(\frac{n\pi x}{d}\right) K_1\left(\frac{n\pi y}{d}\right). \tag{53}$$

If the toroidal aspects of the problem are sufficiently weakly expressed, when $\pi R_1/d \gg 1$, we obtain on introducing the asymptotic form of the Bessel functions

$$\psi = \frac{4d^2}{\pi^3} r \sum_{n=1}^{\infty} \frac{\sin n\frac{\pi}{2}}{n^3 \operatorname{sh}\frac{n\pi}{d}(R_2 - R_1)} \times \left[\left(aR_2 + \frac{bR^2}{R^2}\right)\sqrt{\frac{R^2}{r}} \operatorname{sh}\frac{n\pi}{d}(r - R_1) + \left(aR_1 + \frac{bR^2}{R_1}\right)\sqrt{\frac{R_1}{r}} \operatorname{sh}\frac{n\pi}{d}(R_2 - r)\right] \cos\frac{n\pi z}{d} + \frac{r}{2}\left(ar + \frac{bR^2}{r}\right)\left(z^2 - \frac{d^2}{4}\right). \tag{54}$$

In the opposite limiting case of a strongly toroidal problem (as $R_1 \to 0$), the flux function ψ takes the form

$$\psi = \frac{4d^2}{\pi^3} r \sum_{n=1}^{\infty} \frac{\sin n\frac{\pi}{2}}{n^3} \times \left\{\left(aR_2 + \frac{bR^2}{R_2}\left[1 - \frac{n\pi R_2}{d} K_1\left(\frac{n\pi R_2}{d}\right)\right]\right) \times \frac{I_1\left(\frac{n\pi r}{d}\right)}{I_1\left(\frac{n\pi R_2}{d}\right)} - \frac{bR^2 n\pi}{d} K_1\left(\frac{n\pi r}{d}\right)\right\} \cos\frac{n\pi z}{d} + \frac{r}{2}\left(ar + \frac{bR^2}{r}\right)\left(z^2 - \frac{d^2}{4}\right). \tag{55}$$

On carrying out the necessary numerical calculations, it was found that in this case too, just as in the previous case of a torus of elliptical cross section, a second magnetic axis could appear when $b/a < 0$.

The region of stability was calculated utilizing the general formula (15), the parameters ε and c appearing therein being determined from expressions (11). The results obtained are cited in Fig. 6.

CONCLUSIONS

In the case when the magnetic surfaces have circular normal cross sections in the neighborhood of a magnetic axis ($\varepsilon = 1$), the stability criterion reduces to a limitation on the ratio of the current density to the magnetic field on the magnetic axis $j_0 R/B_0 < 2$. On the assumption of a uniform axial current, this criterion is equivalent to the Shafranov-Kruskal stability condition $RB_\theta(a)/aB_\varphi(a) < 1/m$ for the first mode of oscillation $m = 1$. If the current density falls off on going away from the magnetic axis, the stability criterion when written in the form of the Shafranov-Kruskal condition will correspond to $m > 1$. For example, in the case of the parabolic current density distribution $j_\varphi = j_0(1 - \rho^2/a^2)$, it turns out that $m = 2$.

To investigate the effect of the ellipticity of the magnetic surface cross sections, a study was made of the two cases when the external magnetic surface Σ had an elliptical or semielliptical cross section. As can be seen from Figs. 2 and 4, the limiting value of $j_0 R/B_0$ decreases rapidly when the semi-axis ratio is decreased below unity and decreases more smoothly when ε is increased above unity: ($\varepsilon = l_z/l_r$).

In the case when toroidal aspects of the problem are very strongly expressed, when the external boundary of the plasma has a semi-elliptical cross section with a semi-axis ratio $\varepsilon_\Sigma = 1/2$ (which corresponds to $\varepsilon = 1$), the limiting ratio of the axial current in the plasma to the current in the windings of the solenoid producing the axial magnetic field amounts to $J/I \approx 1/3$, which corresponds to a ratio of plasma pressure to magnetic pressure $\beta \approx 1/2$.

To investigate the effect of the toroidal geometry on the stability, we also investigated the case of a plasma torus in which the external magnetic surface was of rectangular cross section. As can be seen from Fig. 6, in the case when the axial magnetic field decreases on moving away from the magnetic axis ($b/a > 0$) the limiting value of the ratio $N = j_0 R/B_0$ is reduced somewhat with increasing toroidality, this reduction in N being associated with a corresponding increase in ε.

In summary, as far as the conditions of hydromagnetic stability for a prescribed magnetic field are concerned, an increase in toroidality should lead to an increase in the limiting plasma current and pressure.

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