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Optimal control of the viscous generalized Camassa–Holm equation

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ABSTRACT

In this paper, we study the optimal control problem for the viscous generalized Camassa–Holm equation. We deduce the existence and uniqueness of weak solution to the viscous generalized Camassa–Holm equation in a short interval by using Galerkin method. Then, by using optimal control theories and distributed parameter system control theories, the optimal control of the viscous generalized Camassa–Holm equation under boundary condition is given and the existence of optimal solution to the viscous generalized Camassa–Holm equation is proved.

1. Introduction

Recently, Holm and Staley [1] introduced the b-family PDEs that described the balance between convection and stretching for small viscosity in the dynamics of one dimensional nonlinear wave in fluids

$$m_t + \underbrace{um_x}_{convection} + \underbrace{bu_x m}_{stretching} = \underbrace{\varepsilon m_{xx}}_{viscosity}, \tag{1.1}$$

where $u = g * m$ denotes $u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)\mathrm{d}y$. The convolution relates velocity u to momentum density m by integration against the kernel $g(x)$.

When Eq. (1.1) is restricted to the peakon case $g(x) = \mathrm{e}^{-|x|/\alpha}$ with length scale α and $m = u - \alpha^2 u_{xx}$, it may be expressed solely in terms of the velocity $u(x, t)$ as (see [1])

$$u_t - \alpha^2 u_{xxt} - \varepsilon(u_{xx} - u_{xxxx}) + (b+1)uu_x = \alpha^2(bu_x u_{xx} + uu_{xxx}), \tag{1.2}$$

where b, α and ε are arbitrary real constant. Holm and Staley studied the effects of the balance parameter b and kernel $g(x)$ of the solitary wave structures and investigated their interactions analytically for $\varepsilon = 0$ and numerically for small viscosity $\varepsilon \neq 0$, of [1].

With $\varepsilon = 0$ in Eq. (1.2), it becomes the usual b-equation

$$u_t - \alpha^2 u_{xxt} + (b+1)uu_x = \alpha^2(bu_x u_{xx} + uu_{xxx}). \tag{1.3}$$

The b-equation (1.3) can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy for any $b \neq -1$ by an appropriate Kodama transformation, of [2,3]. For the case $b = -1$, the corresponding Kodama transformation is singular and the asymptotic ordering is violated, of [2,3]. The solutions of the b-equation (1.3) were studied numerically for various values of b in [1,4], where b was taken as a bifurcation parameter. The

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KdV equation, the Camassa–Holm equation, and the Degasperis–Procesi equation are the only three integrable equations in the b-equation (1.3), which was shown in [5,6] by using Painleve analysis. The b-equation (1.3) admits peakon solutions for any $b \in \mathbb{R}$, of [1,4,6]. The Cauchy problem for Eq. (1.3) with $\alpha \neq 0$ on the line has been discussed recently in [7]. The local well-posedness for the b-equation, a precise blowup scenario, several blowup results and global existence result of strong solutions, and the uniqueness and the existence of global weak solution to the b-equation on the line have been proved in [7].

If $\alpha = 0$, $b = 2$ and $\varepsilon = 0$, then Eq. (1.2) becomes the well-known Korteweg–de Vries equation which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity, of [8]. In this model $u(t, x)$ represents the wave height above a flat bottom, where x is proportional to the distance in the direction of propagation and t is proportional to the elapsed time. The KdV equation is completely integrable and its solitary waves are solitons [9,10]. The Cauchy problem of the KdV equation has been the subject of a number of studies, and a satisfactory local or global (in time) existence theory is now in hand (for example, see [11,12]). It is shown that the KdV equation is globally well-posed for $u_0 \in H^{-1}$ [12]. It is observed that the KdV equation does not accommodate wave breaking (by wave breaking we understand that the wave remains bounded but its slope becomes unbounded in finite time [13]).

For $\alpha = 1$, $b = 2$ and $\varepsilon = 0$, Eq. (1.2) becomes the Camassa–Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom, where $u(t, x)$ stands for the fluid velocity at time t in the spatial x direction [8,14–17]. The Camassa–Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [18,19]. It has a bi-Hamiltonian structure [20,21] and is completely integrable [14,22–25,40]. Its solitary waves are peaked [26]. The peaked solitons are orbital stable [27]. The explicit interaction of the peaked solitons is given in [28]. The peakons capture a characteristic of the traveling waves of greatest height—exact traveling solutions of the governing equations for water waves with a peak at their crest, of [29–31]. Simpler approximate shallow water models (like KdV) do not present traveling wave solutions with this feature (see [13]).

The Cauchy problems of the Camassa–Holm equation have been studied extensively. It has been shown that this equation is locally well-posed [32–36] for the initial data $u_0 \in H^s(I)$ with $s > 3/2$, where $I = \mathbb{R}$ or $I = \mathbb{R}/\mathbb{Z}$. More interestingly, it has global strong solutions [32,34,37,38] and also blow-up solutions in finite time [32,34,37–41]. On the other hand, it has global weak solutions in H^1 of [34,42–47]. It is observed that if u is the solution of the Camassa–Holm equation with the initial data u_0 in $H^1(\mathbb{R})$, we have for all $t > 0$,

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{2}\, \|u(t, \cdot)\|_{H^1(\mathbb{R})} \leqslant \sqrt{2}\, \|u_0(\cdot)\|_{H^1(\mathbb{R})} .$$

In comparison with the KdV equation, the Camassa–Holm equation has two advantages. First, the Camassa–Holm equation represents the next order in the asymptotic expansion for shallow water waves beyond the KdV equation [2,17]. Second, the Camassa–Holm equation admits peaked traveling waves, replicating a feature that is characteristic for waves of great height—waves of largest amplitude that are exact solutions of governing equations for water waves [29–31]. Moreover, these solutions are orbital stable—that is, their shape is stable under small perturbations and therefore these waves are recognizable physically [27,48,49]. The stability holds for the Camassa–Holm equation but not for the solutions of the governing equations, as water waves are due to the small-amplitude shallow water regime in which this is the valid model of [50]. It is worth pointing out that the equation models breaking waves [26,39]. The smooth solutions of the Camassa–Holm equation have an infinite propagation speed of [51].

For $\alpha = 1$, $b = 2$ and $\varepsilon \neq 0$ in Eq. (1.2), it becomes

$$u_t - u_{xxt} - \varepsilon(u - u_{xx})_{xx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.4}$$

which is the one-dimensional version of the three dimensional Navier–Stokes-alpha model for turbulence [52,53], we call Eq. (1.4) the viscous Camassa–Holm equation.

In addition, the field of optimal control was born in the 1950s with the discovery of the maximum principle, as a result of a competition in military affairs in the early days of the cold war. Concerning the maximum principle, there was a deuce at those times. The principle was firstly discovered by Hestenes in the US, however firstly proved by Pontryagin and his collaborators Boltyanskii and Gamkrelidze in the USSR. No doubt, the early successes of the theory in application in aerospace engineering paved the road for both theory and numerical methods of optimal control. In the meantime a wide spectrum of problems in applications can be solved by methods of optimal control, for example in robotics, chemical engineering, vehicle dynamics, and last but not least in economics. Today's challenges are also addressed: Optimal online control and PDE constrained optimization. Modern optimal control theories and applied models are both represented by ODE, which have developed perfectly. With the development and application of technology, it is necessary to solve the problem of optimal control theories for PDE. The optimal control theories for PDE are much more difficult to resolve. In particular, there are no unified theories and methods of nonlinear control theories for PDE. Two methods are introduced to study the control theories for PDE: one is using low model method, and then changing into ODE model [57]; the other is using quasi-optimal control method [58]. No matter which one we choose, it's necessary to prove the existence of an optimal solution according to the basic theory [59]. In order to realize optimal solutions of optimal control problems in praxis one must be able to recompute the optimal solutions in the presence of disturbances in real time unless one will give up optimality. We will show which mathematical theory and which related numerical methods can help to solve that problem. Optimal control problems with partial differential algebraic equations are one of today's really big challenges. At first glimpse is the optimal control of a fuel cell the dynamics of which is described by 28 quasi-linear partial differential algebraic equations of parabolic-hyperbolic type with non-standard boundary conditions.

In past decades, the optimal control of distributed parameter system had become much more active in academic field. In particular, the optimal control of nonlinear solitary wave equation lies in development for the intersection of mathematics, engineering and computer science. In the recent past considerable attention has been given to the problem of active control of fluids or combustion, where nonlinear effects actually can improve mixing. There are a lot of papers concerned with the study of asymptotic or steady state properties of solutions of nonlinear distributed parameter systems, such as Navier–Stokes equations, which contains both diffusion terms and nonlinear convection terms. A one dimensional simple model for convection-diffusion phenomena is the Burgers equation, such as shock wave, supersonic flow about airfoils, traffic flows, acoustic transmission etc. So, there is a great deal of literature devoted to studying the optimal control problem for the Burgers equation. Kunisch and Volkwein utilized proper orthogonal decomposition (POD) to solve open-loop and closed-loop optimal control problems for the Burgers equation [60] and discussed the instantaneous control of the Burgers equation [61]. Volkwein [62] analyzed the optimal control problems for the stationary Burgers equation using the Augmented Lagrangian-Sequential quadratic programming (SQP) method. In [63], Volkwein studied the distributed optimal control problem for time-dependent Burgers equation using the Augmented Lagrangian-SQP technique. Vedantham [64] developed a technique to utilize the Cole-Hopf transformation to solve an optimal control problem for the Burgers equation. Park et al. suggested an efficient method of solving optimal boundary control problems for the Burgers equation, which is practical as well as mathematically rigorous. Their eventual purpose is to extend this technique to the control problems of viscous fluid flows [65].

Apart for the case of the Burgers equation, there is plenty of research concerned with other optimal control problem. For example, Oksendal proved a sufficient maximum principle for the optimal control systems described by a quasi-linear stochastic heat equation [66]. Lagnese and Leugering [67] considered the problem of boundary optimal control of a wave equation with boundary dissipation in the way of a time-domain decomposition of the corresponding optimality system. Ghattas and Bark studied the optimal control of two and three dimensional compressible Navier–Stokes flows [68]. The existence and uniqueness of solutions for optimal control problems for the 2D Navier–Stokes equation in a 2D domain were established in [69]. Gengsheng Wang was concerned about necessary conditions for optimal control problems governed by some semi-linear parabolic differential equations, which may be non-well posed [70]. In [71], a unified existence theory of optimal controls for general semi-linear evolutionary distributed parameter systems is established under the framework of mild (or weak) solutions for evolution equations. Their theory can apply to optimal control problems with state equation being parabolic, hyperbolic partial differential equations and ordinary retarded differential equations. Their approach also applies to problems governed by elliptic partial differential equations as well as variational inequalities. Ahmed and Teo presented a closure theorem for the attainable trajectories of a class of control systems governed by a large class of nonlinear evolution equations in reflexive Banach space [72]. Since the Camassa–Holm equation satisfies the least Action principle [54–56] and the viscous Camassa–Holm equation (Eq. (1.4)) can be viewed as a one dimensional version of the three dimensional Navier–Stokes-alpha model for turbulence, L. Tian and C. Shen studied the optimal control problem for the viscous Camassa–Holm equation [73]. Recently, L. Tian and C. Shen studied the optimal control problem for the viscous Degasperis–Procesi equation and b-family equation of [74,75]. The Optimal control of the viscous Dullin–Gottwalld–Holm equation was discussed in [76].

In this Letter, we are interested in optimal control problem for the following equation

$$u_t - u_{xxt} - \varepsilon(u_{xx} - u_{xxxx}) + 3uu_x - 2u_x u_{xx} - uu_{xxx} + k(u - u_{xx})_x = 0, \tag{1.5}$$

where $\varepsilon(u_{xx} - u_{xxxx})$ is viscous item, k is a real constant, ku_x denotes the dissipative term and ku_{xxx} denotes the dispersive effect. We name Eq. (1.5) the viscous generalized Camassa–Holm equation.

With $y = u - u_{xx}$, the optimal control problem for Eq. (1.5) we intend to investigate is

$$(P)\begin{cases} \min J(y,\varpi) = \dfrac{1}{2}\,\|Cy - z\|_S^2 + \dfrac{\delta}{2}\,\|\varpi\|_{L^2(Q_o)}^2 \\ \text{s.t.}\quad y_t - \varepsilon y_{xx} + 2u_x y + uy_x + ky_x = B^*\varpi \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \qquad t \in (0,T) \\ y(0) = \phi(x), \qquad x \in (0,1). \end{cases} \tag{1.6}$$

Here, the control target is to match the given desired state z in L^2-sense by adjusting the body force, ϖ in a control volume $Q_o \subseteq Q = (0,T) \times \Omega$ in the L^2-sense, i.e. with minimal energy and work. The first term in the cost functional measures the physical objective, the second one is the size of the control, where the parameter $\delta > 0$ plays the role of a weight.

The remainder of the paper is organized as follows. In Section 2, we give some notations and definition of some space used in this paper. In Section 3, we prove the existence of the weak solution to the viscous generalized Camassa–Holm equation in a special space. At the same time, we discuss the relation among the norm of weak solution, initial value and control item. Section 4 is devoted to the study of problem (P). We discuss the optimal control of the viscous generalized Camassa–Holm equation and prove the existence of an optimal solution. Finally in Section 5, conclusions are obtained.

2. Notations

It is appropriate to introduce some notations that will be used throughout the paper.

For fixed $T > 0$, we set $\Omega = (0, 1)$ and $Q = (0, T) \times \Omega$. Let $Q_o \subseteq Q$ be an open set with positive measure.

Let $V = H^1_{0,1}(0, 1)$, $H = L^2(0, 1)$. $V^* = H^{-1}(0, 1)$ and $H^* = L^2(0, 1)$ are dual space respectively. It is supposed that V is dense in H so that, by identifying V^* and H^*, we have

$$V \subset_{\rightarrow} H = H^* \subset_{\rightarrow} V^*,$$

each embedding being dense.

We supply V with the inner product $\langle \varphi, \psi \rangle_V = \langle \varphi_x, \psi_x \rangle_H$, $\forall \varphi, \psi \in V$.

For $T > 0$ the space $L^2(0, T; V)$ and $C(0, T; V)$ denote the space of square integrable and continuous functions, respectively, in the sense of Bochner from $[0, T]$ to V. The space $W(0, T; V)$ is defined by

$$W(0, T; V) = \{\varphi : \varphi \in L^2(0, T; V), \varphi_t \in L^2(0, T; V)\},$$

which is a Hilbert space endowed with common inner product, see [77]. For brevity we write $L^2(V)$, $C(H)$ and $W(V)$ in place of $L^2(0, T; V)$, $C(0, T; V)$ and $W(0, T; V)$ respectively. Since $W(V)$ is continuously embedded into $C(H)$ [77], there exists an embedding constant $c > 0$ such that

$$\|\varphi\|_{C(H)} \leqslant c \, \|\varphi\|_{W(V)}, \quad \text{for all} \varphi \in W(V).$$

Further, the extension operator $B^* \in L(L^2(Q_o), L^2(V^*))$ is given by

$$B^* q = \begin{cases} q, & \text{in } Q_0 \\ 0, & \text{in } Q \setminus Q_o. \end{cases}$$

We denote $u(t)$ and $f(t)$ the functions $u(t, \cdot)$ and $f(t, \cdot)$ respectively, considered as functions of x only when t is fixed.

Define $\|u\|_{H^m(\Omega)} = \|D^m u\|_H$, where $D^m = \frac{\partial^m}{\partial x^m}$, $m = 0, 1, 2, \ldots$.

Because the following inequalities are used frequently in this work, we will give formation of them here and only refer to their names whenever necessary.

Young's inequality: Let $a, b \in [0, \infty)$ and $\varepsilon > 0$, then we have

$$ab \leqslant \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{-q/p} b^q}{q} \leqslant \varepsilon a^p + \varepsilon^{-q/p} b^q, \quad \text{where } 1 < p < \infty \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

Especially, when $p = q = 2$, it becomes

$$ab \leqslant \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2 \leqslant \varepsilon a^2 + \frac{1}{\varepsilon} b^2,$$

which is used frequently in this paper.

Holder's inequality: Let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, then we get

$$f \cdot g \in L^1(\Omega) \text{ and } \|f \cdot g\|_{L^1(\Omega)} \leqslant \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)},$$

where $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

3. The existence of a weak solution to the viscous generalized Camassa–Holm equation

Consider the following the viscous generalized Camassa–Holm equation

$$u_t - u_{xxt} - \varepsilon(u_{xx} - u_{xxxx}) + 3uu_x - 2u_x u_{xx} - uu_{xxx} + k(u - u_{xx})_x = B^* \varpi, \tag{3.1}$$

under the initial value

$$u(x, 0) = u_0(x),$$

and boundary condition

$$u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0,$$

where $x \in (0, 1)$, $t \in [0, T]$, $u_0(x) \in H^2$, $B^* \varpi \in L^2(V^*)$ and a control $\varpi \in L^2(Q_o)$.

Remark. According to theories of [14] and choosing $B^* \varpi = 0$ in $L^2(V^*)$, Eq. (3.1) leads to peaked solitary wave solutions as follows

$$u\,(x, t) = c\mathrm{e}^{-|x - ct|}.$$

With $y = u - u_{xx}$, Eq. (3.1) takes the form of a quasi-linear evolution equation of hyperbolic type:

$$y_t - \varepsilon y_{xx} + 2u_x y + u y_x + k y_x = B^* \varpi, \tag{3.2}$$

under the initial value

$$y(x, 0) = u_0(x) - u_{0,xx}(x) = \phi(x),$$

and boundary condition

$$u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0,$$

where $x \in (0, 1)$, $t \in [0, T]$, $\phi(x) \in H$, $B^*\varpi \in L^2(V^*)$ and a control $\varpi \in L^2(Q_o)$.

Since we will prove the existence of weak solution to the viscous generalized Camassa–Holm equation, we shall give the definition of the weak solution in the space $W(V)$ in order to purse our goal.

Definition 3.1. A function $y(x, t) \in W(V)$ is called a weak solution to Eq. (3.2), if

$$\frac{\mathrm{d}}{\mathrm{d}t}(y, \varphi)_H - \varepsilon(y_{xx}, \varphi)_H + (2u_x y, \varphi)_H + (u y_x, \varphi)_H + (k y_x, \varphi)_H = \langle B^*\varpi, \varphi\rangle_{V^*,V},$$

for all $\varphi \in V$ and a.e. $t \in [0, T]$ and $y(0) = \phi$ in H are valid.

By using the standard Galerkin method and a series of mathematical estimates, one can get the following theorem, which ensures the existence of a unique weak solution to the viscous generalized Camassa–Holm equation.

Theorem 3.1. *With $\phi \in H$ and $B^*\varpi \in L^2(V^*)$ holding, the Eq. (3.2) admits a weak solution $y(x, t) \in W(0, T; V)$ in the interval $[0, T]$.*

Proof. The Galerkin method is applied to the proof.

We denote $A = -\partial_x^2$ as a second differential operator. Clearly, the operator A is a linear unbounded self-adjoint operator in H with $D(A)$ dense in H, where H is a Hilbert space with a scalar product $(\cdot, \cdot)$ and a norm $\|\cdot\|_{L^2(\Omega)}$. We can then define the powers A^s of A for $s \in \mathbb{R}$. The space $D(A^s)$ is Hilbert space when endowed with norm $\|A^s \cdot\|$.

There exists orthogonal basis $\{\psi_i\}$ of H. Let $\{\psi_i\}_{i=1}^{\infty}$ be the eigenfunctions of the operator $A = -\partial_x^2$ with

$$A\omega_j = \lambda_j \omega_j, \qquad 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \lambda_j \to \infty, \quad \text{as } j \to \infty.$$

For $m \in \mathbb{N}$, define the discrete ansatz space by $V_m = span\{\psi_1, \psi_2 \ldots \psi_m\} \subset V$.

Set $y_m(t) = y_m(x, t) = \sum_{i=1}^{m} y_i^m(t)\psi_i(x)$ require $y_m(0, \cdot) \mapsto \phi$ in H holds true.

We will prove the existence of a unique weak solution to Eq. (3.2) by analyzing the limiting behavior of sequences of smooth functions $\{u_m\}$ and $\{y_m\}$.

Performing the Galerkin procedure for Eq. (3.2), we have

$$\begin{cases} y_{m,t} - \varepsilon y_{m,xx} + 2u_{m,x} y_m + u_m y_{m,x} + k y_{m,x} = B^*\varpi \\ u_m(0, t) = u_m(1, t) = u_{m,x}(0, t) = u_{m,x}(1, t) = u_{m,xx}(0, t) = u_{m,xx}(1, t) = 0, \\ y_m(x, 0) = \phi_m(x) \in H \end{cases} \tag{3.3}$$

where $y_m = u_m - u_{m,xx}$.

Eq. (3.3) are ordinary differential equations and according to ODE theory, there is a unique solution to Eq. (3.3) in the interval $[0, t_m)$. We should show that the solution is uniformly bounded when $t_m \to T$.

Multiplying Eq. (3.3) by u_m and integrating with respect to x on Ω, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_m\|_H^2 + \|u_m\|_V^2\right) + \varepsilon\left(\|u_m\|_V^2 + \|u_m\|_{H^2}^2\right) = \langle B^*\varpi, u_m\rangle_{V^*,V}. \tag{3.4}$$

Since $B^*\varpi \in L^2(V^*)$ is a control item, we can assume

$$\left\|B^*\varpi\right\|_{V^*} \leqslant M_1,$$

where M_1 is positive constant.

It follows from Young's inequality that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_m\|_H^2 + \|u_m\|_V^2\right) + \varepsilon\left(\|u_m\|_V^2 + \|u_m\|_{H^2}^2\right) \leqslant \varepsilon\|u_m\|_V^2 + \frac{M_1^2}{\varepsilon}, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_m\|_H^2 + \|u_m\|_V^2\right) \leqslant \frac{2M_1^2}{\varepsilon}. \end{aligned} \tag{3.5}$$

It thus transpires that

$$\|u_m\|_H^2 + \|u_m\|_V^2 \leqslant \frac{2M_1^2 t}{\varepsilon} + \left(\|u_0\|_H^2 + \|u_0\|_V^2\right) \underline{\underline{\Delta}} M_2^2 \quad , \tag{3.6}$$

where $\forall t \in [0, T]$.

From the above analysis, we know $\|u_m\|_H \leqslant M_2$, $\|u_m\|_V \leqslant M_2$, where M_2 is positive constant.

Next we prove a uniform $L^2(0, T; V)$ bound on a sequence $\{y_m\}$.

Multiplying Eq. (3.3) by y_m and integrating with respect to x on Ω, then we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_m\|_H^2 + \varepsilon\|y_m\|_V^2 - 3\int_0^1 u_m y_m y_{m,x}\mathrm{d}x = \langle B^*\varpi, y_m\rangle_{V^*,V}. \tag{3.7}$$

By Poincare inequality and Sobolev embedding theorem, we have

$$\begin{aligned}\left|3\int_0^1 u_m y_m y_{m,x}\mathrm{d}x\right| &\leqslant 3\|u_m\|_{L^\infty(\Omega)}\|y_m\|_H\left\|y_{m,x}\right\|_H \\ &\leqslant \frac{3}{2}K_1\|u_m\|_V\left(\|y_m\|_H^2 + \left\|y_{m,x}\right\|_H^2\right) \\ &\leqslant \frac{3}{2}K_1\|u_m\|_V\left(\lambda_1\left\|y_{m,x}\right\|_H^2 + \left\|y_{m,x}\right\|_H^2\right), \\ &\leqslant M_3\|y_m\|_V^2,\end{aligned} \tag{3.8}$$

where $M_3 = \frac{3}{2}K_1M_2(\lambda_1 + 1)$, K_1 is embedding constant and λ_1 is the Poincare coefficient.

It then follows from (3.7) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_m\|_H^2 + \varepsilon\|y_m\|_V^2 \leqslant M_3\|y_m\|_V^2 + \left\|B^*\varpi\right\|_{V^*}\|y_m\|_V. \tag{3.9}$$

Thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_m\|_H^2 + (\varepsilon - M_3)\|y_m\|_V^2 \leqslant M_1\|y_m\|_V, \tag{3.10}$$

where $\varepsilon > M_3$.

Using young's inequality, (3.10) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_m\|_H^2 + \frac{1}{2}(\varepsilon - M_3)\|y_m\|_V^2 \leqslant \frac{M_1^2}{2(\varepsilon - M_3)}, \tag{3.11}$$

where $\varepsilon > M_3$.

Integrating the above inequality with respect to t on $[0, T]$, we obtain

$$\|y_m\|_{L^2(0,T;V)}^2 \leqslant \frac{1}{(\varepsilon - M_3)}\left[\frac{M_1^2T}{(\varepsilon - M_3)} + \|\phi\|_H^2\right]\underline{\underline{\Delta}}r_1,$$

where $\varepsilon > M_3$.

From (3.11), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_m\|_H^2 \leqslant \frac{M_1^2}{2(\varepsilon - M_3)},$$

where $\varepsilon > M_3$.

Integrating the above inequality with respect to $t < T$ on $[0, t]$, we thus derive that

$$\|y_m\|_H^2 \leqslant \frac{M_1^2t}{(\varepsilon - M_3)} + \|\phi\|_H^2\underline{\underline{\Delta}}r_2, \quad \forall t \in [0, T],$$

where $\varepsilon > M_3$.

Next we prove a uniform $L^2(0, T; V^*)$ bound on a sequence $\{y_{m,t}\}$.

By Eq. (3.3) and Sobolev embedding theorem, we have

$$\begin{aligned}\left\|y_{m,t}\right\|_{V^*} &\leqslant \left\|B^*\varpi\right\|_{V^*} + \varepsilon\|y_m\|_V + 2\|u_m\|_H\|y_m\|_V + \|u_m\|_V\|y_m\|_H + k\|y_m\|_H \\ &\leqslant M_1 + \varepsilon\|y_m\|_V + 2M_2\|y_m\|_V + (M_2 + k)\|y_m\|_H.\end{aligned} \tag{3.12}$$

It follows from (3.12) that

$$\left\|y_{m,t}\right\|_{V^*}^2 \leqslant 3M_1^2 + 3(\varepsilon + 2M_2)^2\|y_m\|_V^2 + 3(M_2 + k)^2\|y_m\|_H^2. \tag{3.13}$$

Integrating (3.13) with respect to t on $[0, T]$, we deduce that

$$\left\|y_{m,t}\right\|_{L^2(0,T;V^*)}^2 \leqslant [3M_1^2 + 3(M_2 + k)^2r_2]T + 3(\varepsilon + 3M_2)^2r_1\underline{\underline{\Delta}}r_3.$$

Collecting the previous one has

(a) For every $t \in [0, T]$, r_1 and r_2 are two constants, the sequence $\{y_m\}_{m\in\mathbb{N}}$ is bounded in $L^2(0, T; H)$ as well as in $L^2(0, T; V)$, which is independent of the dimension of ansatz space m.

(b) For every $t \in [0, T]$ and r_3 is constant, the sequence $\{(y_m)_t\}_{m\in\mathbb{N}}$ is bounded in $L^2(0, T; V^*)$, which is independent of the dimension of ansatz space m.

Now (a) and (b) from above are equivalent to $\{y_m\}_{m\in\mathbb{N}} \subset W(0, T; V)$ bounded and since $W(0, T; V)$ is compact embedded into $C(0, T; H)$. One concludes convergence of a subsequence, again denoted by $\{y_m\}_{m\in\mathbb{N}}$ weak into $W(0, T; V)$, weak-star in $L^\infty(0, T; H)$ and strong in $L^2(0, T; H)$ to a function $y(x, t) \in W(0, T; V)$. Uniqueness of the solution is an immediate consequence of inequality (3.11) (see Ref. [61]).

This completes the proof of Theorem 3.1. □

In the following, we shall discuss the relation among the norm of weak solution in the space $W(V)$ and initial value and control item. The next theorem ensures that the norm of weak solution in the space $W(V)$ can be controlled by initial value and control item.

Theorem 3.2. *If $B^*\varpi \in L^2(V^*)$ and $\phi \in H$, then there exists a constant $C_1 > 0$, such that*

$$\|y\|^2_{W(V)} \leqslant C_1 \left[\|\phi\|^2_H + \|\varpi\|^2_{L^2(Q_0)} \right].$$

Proof. Multiplying Eq. (3.2) by y yields

$$yy_t - \varepsilon yy_{xx} + 2u_x y^2 + uyy_x + kyy_x = (B^*\varpi)y.$$

Integrating the above equation with respect to x on Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2_H + \varepsilon\int_0^1 y_x^2 \mathrm{d}x - 3\int_0^1 uyy_x \mathrm{d}x = \langle B^*\varpi, y\rangle_{V^*,V}. \tag{3.14}$$

By the Poincare inequality and Sobolev embedding theorem, we get

$$\begin{aligned}\left|3\int_0^1 uyy_x \mathrm{d}x\right| &\leqslant 3\|u\|_{L^\infty(\Omega)}\|y\|_H\|y_x\|_H \\ &\leqslant \frac{3}{2}\|u\|_{L^\infty(\Omega)}\left(\|y\|^2_H + \|y_x\|^2_H\right) \\ &\leqslant \frac{3}{2}K_1\|u\|_V\left(\lambda_1\|y_x\|^2_H + \|y_x\|^2_H\right),\end{aligned} \tag{3.15}$$

where K_1 is embedding constant and λ_1 is a Poincare coefficient.

Multiplying Eq. (3.1) by u and integrating the resulting equation on Ω. By using the same argument as in the proof of Theorem 3.1, we have

$$\|u\|_H \leqslant M_2, \qquad \|u\|_V \leqslant M_2,$$

where M_2 is positive constant.

It then follows from (3.15) and the above estimate that

$$\begin{aligned}\left|3\int_0^1 uyy_x \mathrm{d}x\right| &\leqslant \frac{3}{2}K_1M_2\left(\lambda_1\|y_x\|^2_H + \|y_x\|^2_H\right) \\ &\leqslant M_3\|y\|^2_V,\end{aligned} \tag{3.16}$$

where $M_3 = \frac{3}{2}K_1M_2(\lambda_1 + 1)$.

Combining (3.15) with (3.16), we can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2_H + \varepsilon\int_0^1 y_x^2 \mathrm{d}x \leqslant M_3\|y\|^2_V + \langle B^*\varpi,\ y\rangle_{V^*,V}. \tag{3.17}$$

Integrating (3.17) with respect to t on $[0, T]$, we derive that

$$\frac{1}{2}\|y(T)\|^2_H - \frac{1}{2}\|\phi\|^2_H + \varepsilon\|y\|^2_{L^2(V)} \leqslant \int_0^T \langle B^*\varpi, y\rangle_{V^*,V}\mathrm{d}t + M_3\|y\|^2_{L^2(V)}.$$

Thus

$$\frac{1}{2}\|y(T)\|^2_H - \frac{1}{2}\|\phi\|^2_H + (\varepsilon - M_3)\|y\|_{L^2(V)} \leqslant \int_0^T \langle B^*\varpi, y\rangle_{V^*,V}\mathrm{d}t, \tag{3.18}$$

where $\varepsilon > M_3$.

From Holder's inequality, we obtain

$$\int_0^T \langle B^*\varpi, y\rangle_{V^*,V} \mathrm{d}t \leqslant \int_0^T \left\| B^*\varpi \right\|_{V^*} \|y\|_V \, dt \leqslant \left\| B^*\varpi \right\|_{L^2(V^*)} \|y\|_{L^2(V)} . \tag{3.19}$$

Substituting (3.19) into (3.18) yields

$$\|y(T)\|_H^2 - \|\phi\|_H^2 + 2(\varepsilon - M_3) \|y\|_{L^2(V)}^2 \leqslant 2 \left\| B^*\varpi \right\|_{L^2(V^*)} \|y\|_{L^2(V)} . \tag{3.20}$$

From Young's inequality, we get

$$\left\| B^*\varpi \right\|_{L^2(V^*)} \|y\|_{L^2(V)} \leqslant \frac{(\varepsilon - M_3)}{2} \|y\|_{L^2(V)}^2 + \frac{1}{2(\varepsilon - M_3)} \left\| B^*\varpi \right\|_{L^2(V^*)}^2 . \tag{3.21}$$

Substituting (3.21) into (3.20) yields

$$(\varepsilon - M_3) \|y\|_{L^2(V)}^2 \leqslant \|\phi\|_H^2 + \frac{1}{\varepsilon - M_3} \left\| B^*\varpi \right\|_{L^2(V^*)}^2 .$$

It thus transpires that

$$\begin{aligned} \|y\|_{L^2(V)}^2 &\leqslant \frac{1}{\varepsilon - M_3} \|\phi\|_H^2 + \frac{1}{(\varepsilon - M_3)^2} \left\| B^*\varpi \right\|_{L^2(V^*)}^2 \\ &\leqslant \max\left\{ \frac{1}{\varepsilon - M_3}, \frac{1}{(\varepsilon - M_3)^2} \right\} \left(\|\phi\|_H + \left\| B^*\varpi \right\|_{L^2(V^*)} \right)^2 \\ &\leqslant C_0 \left(\|\phi\|_H + \left\| B^*\varpi \right\|_{L^2(V^*)} \right)^2 , \end{aligned} \tag{3.22}$$

where $C_0 = \max\{\frac{1}{\varepsilon - M_3}, \frac{1}{(\varepsilon - M_3)^2}\}$ and $\varepsilon > M_3$.

In view of (3.17), we can get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|y\|_H^2 \leqslant M_3 \|y\|_V^2 + \langle B^*\varpi, \quad y\rangle_{V^*,V} .$$

Integrating the above inequality with respect to t yields

$$\begin{aligned} \|y\|_H^2 &\leqslant \|\phi\|_H^2 + 2M_3 \|y\|_{L^2(V)}^2 + 2 \left\| B^*\varpi \right\|_{L^2(V^*)} \|y\|_{L^2(V)} \\ &\leqslant \|\phi\|_H^2 + 2C_0M_3 \left(\|\phi\|_H + \left\| B^*\varpi \right\|_{L^2(V^*)} \right)^2 + 2 \left\| B^*\varpi \right\|_{L^2(V^*)} \sqrt{C_0} \left(\|\phi\|_H + \left\| B^*\varpi \right\|_{L^2(V^*)} \right) \\ &\leqslant 2\max(1, 2\sqrt{C_0}, 2C_0M_3) \left(\|\phi\|_H + \left\| B^*\varpi \right\|_{L^2(V^*)} \right)^2 \quad . \end{aligned} \tag{3.23}$$

By (3.2), we deduce that

$$\|y_t\|_{V^*} \leqslant \left\| B^*\varpi \right\|_{V^*} + \varepsilon \|y\|_V + 2 \|u\|_H \|y\|_V + \|u\|_V \|y\|_H + k \|y\|_H .$$

In view of $\|u\|_H \leqslant M_2$ and $\|u\|_V \leqslant M_2$, we thus have

$$\begin{aligned} \|y_t\|_{V^*} &\leqslant \left\| B^*\varpi \right\|_{V^*} + \varepsilon \|y\|_V + 2M_2 \|y\|_V + (M_2 + k) \|y\|_H \\ &\leqslant \left\| B^*\varpi \right\|_{V^*} + (\varepsilon + 2M_2) \|y\|_V + (M_2 + k) \|y\|_H . \end{aligned}$$

Then,

$$\|y_t\|_{V^*}^2 \leqslant 3 \left\| B^*\varpi \right\|_{V^*}^2 + 3(\varepsilon + 2M_2)^2 \|y\|_V^2 + 3(M_2 + k)^2 \|y\|_H^2 .$$

Integrating the above inequality with respect to t on $[0, T]$, we derive that

$$\begin{aligned} \|y_t\|_{L^2(V^*)}^2 &\leqslant 3 \left\| B^*\varpi \right\|_{L^2(V^*)}^2 + 3(\varepsilon + 2M_2)^2 \|y\|_{L^2(V)}^2 + 3(M_2 + k)^2 \int_0^T \|y\|_H^2 \, \mathrm{d}t \\ &\leqslant 3 \left\| B^*\varpi \right\|_{L^2(V^*)}^2 + 3(\varepsilon + 2M_2)^2 C_0 \left(\|\phi\|_H + \left\| B^*\varpi \right\|_{L^2(V^*)} \right)^2 \\ &\quad + 6(M_2 + k)^2 T \cdot \max(1, 2\sqrt{C_0}, 2C_0M_3) \left(\|\phi\|_H + \left\| B^*\varpi \right\|_{L^2(V^*)} \right)^2 \\ &\leqslant \left[3 + 3(\varepsilon + 2M_2)^2 C_0 + 6(M_2 + k)^2 T \cdot \max(1, 2\sqrt{C_0}, 2C_0M_3) \right] (\|\phi\|_H + \left\| B^*\varpi \right\|_{L^2(V^*)})^2. \end{aligned} \tag{3.24}$$

Taking into account (3.22) and (3.24) yields

$$\begin{aligned}\|y\|^2_{W(V)} &= \|y\|^2_{L^2(V)} + \|y_t\|^2_{L^2(V^*)}\\ &\leqslant [C_0 + 3 + 3(\varepsilon + 2M_2)^2 C_0 + 6(M_2 + k)^2 T \cdot \max(1, 2\sqrt{C_0}, 2C_0 M_3)] \left(\|\phi\|_H + \left\|B^*\varpi\right\|_{L^2(V^*)}\right)^2\\ &\leqslant C_1 \left[\|\phi\|^2_H + \left\|B^*\varpi\right\|^2_{L^2(V^*)}\right],\\ &\leqslant C_1 \left[\|\phi\|^2_H + \|\varpi\|^2_{L^2(Q_0)}\right] \end{aligned} \tag{3.25}$$

where $C_1 = 2[C_0 + 3 + 3(\varepsilon + 2M_2)^2 C_0 + 6(M_2 + k)^2 T \cdot \max(1, 2\sqrt{C_0}, 2C_0 M_3)]$.

So we give the claim. □

4. The distributed optimal control of the viscous generalized Camassa–Holm equation

In this section, we discuss the distributed optimal control of the viscous generalized Camassa–Holm equation and prove the existence of optimal solution basing on Lions' theory.

Allowing a control $\varpi \in L^2(Q_o)$ we study the following Problem

$$(P)\begin{cases}\min J(y, \varpi) = \dfrac{1}{2}\|Cy - z\|^2_S + \dfrac{\delta}{2}\|\varpi\|^2_{L^2(Q_o)}\\ \text{s.t.}\quad y_t - \varepsilon y_{xx} + 2u_x y + u y_x + k y_x = B^*\varpi\\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \qquad t \in (0,T)\\ y(0) = \phi(x), \qquad x \in (0,1),\quad \phi(x) \in H,\end{cases}$$

where $y = u - u_{xx}$.

We know that there exists a weak solution y to Eq. (3.2) from Theorem 3.1.

Due to $u = (1 - \partial_x^2)^{-1} y$, then we can infer that there exists a weak solution u to Eq. (3.1).

Given an observation operator $C \in L(W(V), S)$, in which S is a real Hilbert space and C is continuous.

We choose performance index of tracking type

$$J(y, \varpi) = \frac{1}{2}\|Cy - z\|^2_S + \frac{\delta}{2}\|\varpi\|^2_{L^2(Q_o)}, \tag{4.1}$$

where $z \in S$ is a desired state and $\delta > 0$ is fixed.

Optimal control problem about the viscous generalized Camassa–Holm equation is

$$\min J(y, \varpi), \tag{4.2}$$

where (y, ϖ) satisfies Eq. (3.2) and initial value and boundary condition.

We set $X = W(V) \times L^2(Q_o)$ and $Y = L^2(V) \times H$.

We define an operator $e = e(e_1, e_2) : X \to Y$ by

$$e(y, \varpi) = \begin{bmatrix} G \\ y(x,0) - \phi(x)\end{bmatrix},$$

where $G = (-\Delta)^{-1}(y_t - \varepsilon y_{xx} + 2u_x y + u y_x + k y_x - B^*\varpi)$ and Δ is an operator from $H^1_{0,1}(\Omega)$ to $H^{-1}(\Omega)$.

Then we write (4.2) in following form

$$\min J(y, \varpi) \text{ subject to } e(y, \varpi) = 0. \tag{4.3}$$

The following theorem will give the existence of optimal solution to the viscous generalized Camassa–Holm equation in theory.

Theorem 4.1. *There exists an optimal control solution to the problem (P).*

Proof. Let $(y, \varpi) \in X$ satisfy the equation $e(y, \varpi) = 0$.

In view of (4.1), we have

$$J(y, \varpi) \geqslant \frac{\delta}{2}\|\varpi\|^2_{L^2(Q_o)}.$$

From Theorem 3.2, we then deduce that

$$\|y\|_{W(V)} \to \infty \text{ yields } \|\varpi\|_{L^2(Q_o)} \to \infty.$$

Hence,

$$J(y, \varpi) \to +\infty, \quad \text{when } \|(y, \varpi)\|_X \to \infty. \tag{4.4}$$

As the norm is weakly lowered semi-continuous [78], we achieve that J is weakly lowered semi-continuous.
Since $J(y,\varpi)\geqslant 0$, for all $(y,\varpi)\in X$ holds, there exist $\zeta\geqslant 0$ with

$$\zeta=\inf\{J(y,\varpi)|(y,\varpi)\in X,\quad \text{with } e(y,\varpi)=0\}.$$

This implies the existence of a minimizing sequence $\{(y^n,\varpi^n)\}_{n\in N}$ in X such that
$\zeta=\lim_{n\to\infty}J(y^n,\varpi^n)$and $e(y^n,\varpi^n)=0$ for all $n\in\mathbb{N}$.
Due to (4.4), there exists an element $(y^*,\varpi^*)\in X$ with

$$y^n \xrightarrow{weak} y^*, n\to\infty, y\in W(V), \tag{4.5}$$

$$\varpi^n \xrightarrow{weak} \varpi^*, n\to\infty, \varpi\in L^2(Q_o). \tag{4.6}$$

We can infer from (4.5) that

$$\lim_{n\to\infty}\int_0^T (y_t^n(t)-y_t^*,\varphi(t))_{V^*,V}\mathrm{d}t=0,\quad \forall\varphi\in L^2(V).$$

Since $W(V)$ is compactly embedded into $L^2(L^\infty)$ [79], we derive that $y^n\to y^*$ strongly in $L^2(L^\infty)$, as $n\to\infty$. Since $W(V)$ is continuously embedded into $C(H)$ [77], we can also derive that $y^n\to y^*$ strongly in $C(H)$, as $n\to\infty$. Then, we can infer $u^n\to u^*$ strongly in $C(H)$ also.

As the sequence $\{y^n\}_{n\in N}$ converges weakly, $\|y^n\|_{W(V)}$ is bounded [80]. From the embedding theorem, we deduce that $\|y^n\|_{L^2(L^\infty)}$ is also bounded.

Since $y^n\to y^*$ strongly in $L^2(L^\infty)$, then we can infer that $\|y^*\|_{L^2(L^\infty)}$ is bounded.
Thus, it follows from Holder's inequality that

$$\begin{aligned}
&\left|\int_0^T\int_0^1 (u_x^n y^n-u_x^*y^*)\varphi\mathrm{d}x\mathrm{d}t\right|\leqslant\left|\int_0^T\int_0^1 u_x^n(y^n-y^*)\varphi\mathrm{d}x\mathrm{d}t\right|+\left|\int_0^T\int_0^1(u_x^n-u_x^*)y^*\varphi\mathrm{d}x\mathrm{d}t\right|\\
&\quad\leqslant\int_0^T \left\|y^n-y^*\right\|_{L^\infty}\left\|u^n\right\|_H\|\varphi\|_V\,\mathrm{d}t+\int_0^T\left\|y^*\right\|_{L^\infty}\left\|u^n-u^*\right\|_H\|\varphi\|_V\,\mathrm{d}t\\
&\quad\leqslant\left\|y^n-y^*\right\|_{L^2(L^\infty)}\left\|u^n\right\|_{C(H)}\|\varphi\|_{L^2(V)}+\left\|u^n-u^*\right\|_{C(H)}\left\|y^*\right\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)}\\
&\quad\xrightarrow{n\to\infty}0,\quad \text{for }\forall\varphi\in L^2(V).\\
&\left|\int_0^T\int_0^1(u^ny_x^n-u^*y_x^*)\varphi\mathrm{d}x\mathrm{d}t\right|=\left|\int_0^T\int_0^1(u_x^*y^*\varphi+u^*y^*\varphi_x-u_x^ny^n\varphi-u^ny^n\varphi_x)\mathrm{d}x\mathrm{d}t\right|\\
&\quad\leqslant\left|\int_0^T\int_0^1(u_x^*y^*-u_x^ny^n)\varphi\mathrm{d}x\mathrm{d}t\right|+\left|\int_0^T\int_0^1(u^*y^*-u^ny^n)\varphi_x\mathrm{d}x\mathrm{d}t\right|\\
&\quad\leqslant\left|\int_0^T\int_0^1(u_x^*y^*-u_x^ny^n)\varphi\mathrm{d}x\mathrm{d}t\right|+\left\|u^*-u^n\right\|_{C(H)}\left\|y^*\right\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)}+\left\|u^n\right\|_{C(H)}\left\|y^*-y^n\right\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)}\\
&\quad\xrightarrow{n\to\infty}0,\quad \text{for }\forall\varphi\in L^2(V).
\end{aligned}$$

From (4.5), we can obtain

$$\left|\int_0^T\int_0^1(ky_x^n-ky_x^*)\varphi\mathrm{d}x\mathrm{d}t\right|\xrightarrow{n\to\infty}0,\quad\text{for all }\varphi\in L^2(V).$$

From (4.6), we can infer

$$\left|\int_0^T\int_0^1(B^*\varpi^n-B^*\varpi^*)\varphi\mathrm{d}x\mathrm{d}t\right|\xrightarrow{n\to\infty}0,\quad\text{for all }\varphi\in L^2(V).$$

In view of the above discussion, we can conclude that

$$e_1(y^*,\varpi^*)=0\quad\forall n\in N.$$

From $y^*\in W(V)$, we derive that $y^*(0)\in H$.
Since $y^n\xrightarrow{weak}y^*$ in $W(V)$, we can infer that $y^n(0)\xrightarrow{weak}y^*(0)$, when $n\to\infty$.
Thus we obtain

$$\langle y^n(0)-y^*(0),\psi\rangle_H\xrightarrow{n\to\infty}0,\quad\text{for }\forall\psi\in H,\quad\text{which gives } e_2(y^*,\varpi^*)=0.$$

Consequently, we can derive that

$$e(y^*,\varpi^*)=0\quad\text{in } Y.$$

In conclusion, there exists an optimal solution (y^*, ϖ^*) to the problem (P). In the meantime, we can infer that there exists an optimal solution (u^*, ϖ^*) to the viscous generalized Camassa–Holm equation due to $u = (1 - \partial_x^2)^{-1}y$. $\square$

5. Conclusion

Solving optimal control problems for nonlinear partial differential equations represents a significant numerical challenge due to the tremendous size and possible model difficulties (e.g. nonlinearities). The viscous generalized Camassa–Holm equation is an important mathematics physics equation that has many practical meanings. Optimal control problems for the viscous generalized Camassa–Holm equation have not been investigated so far because of the complexity of nonlinear items. In this paper, we study the distributed optimal control problem for the viscous generalized Camassa–Holm equation, using a series of mathematical estimates. Our research is motivated by the study of the optimal control problem for the Burgers equation and the existence theory of optimal control of distributed parameters systems. The aim of this paper is to find a general approach to investigate these problems. We also prove the existence of an optimal solution to the viscous generalized Camassa–Holm equation in theory. In order to realize optimal solutions of optimal control problems in praxis, one must be able to recompute the optimal solutions in the presence of disturbances in real time unless one will give up optimality. We will use mathematical theory and related numerical methods to solve that problem numerically, which is our purpose in the future. However, to the best of the author's knowledge, this is the first attempt to apply optimal control to the viscous generalized Camassa–Holm equation, which provides a theoretical basis for further study and application in engineering field.

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