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Two-Grid hp-Version DGFEMs for Strongly Monotone Second-Order Quasilinear Elliptic PDEs

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In this article we develop the *a priori* error analysis of so–called two-grid hp–version discontinuous Galerkin finite element methods for the numerical approximation of strongly monotone second–order quasilinear partial differential equations. In this setting, the fully nonlinear problem is first approximated on a coarse finite element space $V(\mathcal{T}_H, \boldsymbol{P})$. The resulting 'coarse' numerical solution is then exploited to provide the necessary data needed to linearize the underlying discretization on the finer space $V(\mathcal{T}_h, \boldsymbol{p})$; thereby, only a linear system of equations is solved on the richer space $V(\mathcal{T}_h, \boldsymbol{p})$. Numerical experiments confirming the theoretical results are presented.

This article is devoted to the *a priori* error analysis of hp–version symmetric interior penalty (SIP) discontinuous Galerkin finite element methods (DGFEMs) for the numerical approximation of strongly monotone second–order quasilinear partial differential equations. In particular, we shall consider the analysis of the so–called two-grid version of the underlying scheme. We point out that two-grid methods were originally introduced by Xu [1–3]; see, also, [4–11] for related work. The construction of a two-grid method to compute the numerical approximation of a nonlinear partial differential equation may be summarised as follows. Let X and Y be two Hilbert spaces. Further, we write $\mathcal{N}(\cdot;\cdot,\cdot) : X \times X \times Y \to \mathbb{R}$ to denote a semilinear form, with the convention that $\mathcal{N}(\cdot;\cdot,\cdot)$ is linear with respect to the arguments to the right of the semi-colon. We suppose that u is the unique solution to the variational problem: find u in X such that

$$\mathcal{N}(u;u,v) = 0 \qquad \forall v \in Y. \tag{1}$$

Problem (1) can be thought of as the weak formulation of a nonlinear partial differential equation on X whose unique solution is $u \in X$. In practice (1) cannot be solved in closed form but needs to be approximated numerically. For the purposes of this paper, we shall consider general hp–version finite element approximations to (1). In order to construct a Galerkin approximation to this problem, we consider a sequence of finite–dimensional spaces $\{X_{h,p}\}$, parameterized by the positive discretization parameters h and p. Simultaneously, consider a sequence of finite–dimensional spaces $\{Y_{h,p}\}$. For the purposes of this paper, $X_{h,p}$ and $Y_{h,p}$ can be thought of as finite element spaces consisting of piecewise polynomial functions of degree p on a partition $\mathcal{T}_h$, of granularity h, of the computational domain. The (standard) Galerkin approximation $u_{h,p}$ of u is then sought in $X_{h,p}$ as the solution of the finite–dimensional problem

$$\mathcal{N}_{h,p}(u_{h,p};u_{h,p},v_{h,p}) = 0 \qquad \forall v_{h,p} \in Y_{h,p}, \tag{2}$$

where $\mathcal{N}_{h,p}(\cdot;\cdot,\cdot) : X_{h,p} \times X_{h,p} \times Y_{h,p} \to \mathbb{R}$. The computation of $u_{h,p}$ defined in (2) involves the numerical solution of a potentially very large number of coupled nonlinear equations, which can be extremely computationally expensive. The key idea of the two-grid approach is as follows: given 'coarser' finite element spaces $X_{H,P} \subseteq X_{h,p}$ and $Y_{H,P} \subseteq Y_{h,p}$, first solve the nonlinear problem: find $u_{H,P} \in X_{H,P}$ such that

$$\mathcal{N}_{H,P}(u_{H,P};u_{H,P},v_{H,P}) = 0 \qquad \forall v_{H,P} \in Y_{H,P}. \tag{3}$$

Finally, using $u_{H,P}$ as appropriate data, compute the two grid approximation of (1) by solving the *linear* problem: find $u_{2G} \in X_{h,p}$ such that

$$\mathcal{N}_{h,p}(u_{H,P};u_{2G},v_{h,p}) = 0 \qquad \forall v_{h,p} \in Y_{h,p}. \tag{4}$$

In this article we consider the two-grid SIP DGFEM numerical approximation of the following quasi-linear elliptic boundary-value problem:

$$-\nabla\cdot(\mu(\boldsymbol{x},|\nabla u|)\nabla u) = f \qquad \text{in } \Omega, \tag{5}$$

$$u = 0 \qquad \text{on } \Gamma, \tag{6}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$, with boundary Γ and $f \in \mathrm{L}_2(\Omega)$.

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Assumption 1 We assume that the nonlinearity μ satisfies the following monotonicity conditions:

1. $\mu \in \mathrm{C}^0(\overline{\Omega} \times [0,\infty))$ and
2. there exist constants $0 < m_\mu \leq M_\mu$ such that

$$m_\mu(t-s) \leq \mu(\boldsymbol{x},t)t - \mu(\boldsymbol{x},s)s \leq M_\mu(t-s), \qquad t \geq s \geq 0, \quad \boldsymbol{x} \in \overline{\Omega}. \tag{7}$$

For ease of notation we shall suppress the dependence of μ on $\boldsymbol{x}$ and write $\mu(t)$ instead of $\mu(\boldsymbol{x},t)$.

The outline of this article is as follows. Section 1 introduces the two-grid SIP DGFEM for the numerical approximation of (5)–(6). In Section 2 we state an *a priori* error bound for the proposed numerical scheme, cf. [11]. Finally, in Section 3 we present some numerical experiments to validate the theoretical results.

1 Two-Grid hp–Version DGFEM

In this section we discuss the numerical approximation of the problem (5)–(6) based on employing both the hp–version of the (standard) SIP DGFEM, together with its so-called two-grid variant. To this end, we first introduce the necessary notation.

We consider shape-regular meshes $\mathcal{T}_h$ that partition $\Omega \subset \mathbb{R}^2$ into open disjoint triangles and/or parallelograms κ such that $\overline{\Omega} = \bigcup_{\kappa \in \mathcal{T}_h} \overline{\kappa}$. By h_κ we denote the element diameter of $\kappa \in \mathcal{T}_h$, $h = \max_{\kappa \in \mathcal{T}_h} h_\kappa$, and $\boldsymbol{n}_\kappa$ signifies the unit outward normal vector to κ. We allow the meshes $\mathcal{T}_h$ to be *1-irregular*, i.e., each edge of any one element $\kappa \in \mathcal{T}_h$ contains at most one hanging node (which, for simplicity, we assume to be the midpoint of the corresponding edge). Here, we suppose that $\mathcal{T}_h$ is of *bounded local variation*, i.e., there exists a constant $\rho_1 \geq 1$, independent of the element sizes, such that $\rho_1^{-1} \leq h_\kappa / h_{\kappa'} \leq \rho_1$, for any pair of elements $\kappa, \kappa' \in \mathcal{T}_h$ which share a common edge $e = \partial\kappa \cap \partial\kappa'$.

To each $\kappa \in \mathcal{T}_h$ we assign a polynomial degree $p_\kappa \geq 1$ (local approximation order) and define the degree vector $\boldsymbol{p} = \{p_\kappa : \kappa \in \mathcal{T}_h\}$. We suppose that $\boldsymbol{p}$ is also of bounded local variation, i.e., there exists a constant $\rho_2 \geq 1$, independent of the element sizes and $\boldsymbol{p}$, such that, for any pair of neighbouring elements $\kappa, \kappa' \in \mathcal{T}_h$, $\rho_2^{-1} \leq p_\kappa / p_{\kappa'} \leq \rho_2$. With this notation, we introduce the finite element space

$$V(\mathcal{T}_h, \boldsymbol{p}) = \{v \in \mathrm{L}_2(\Omega) : v|_\kappa \in \mathcal{S}_{p_\kappa}(\kappa) \quad \forall \kappa \in \mathcal{T}_h\},$$

where $\mathcal{S}_{p_\kappa}(\kappa) = \mathcal{P}_{p_\kappa}(\kappa)$ if κ is a triangle and $\mathcal{S}_{p_\kappa}(\kappa) = \mathcal{Q}_{p_\kappa}(\kappa)$ if κ is a parallelogram. Here, given $p \geq 0$, $\mathcal{P}_p(\kappa)$ denotes the space of all polynomials of degree at most p on κ, while $\mathcal{Q}_p(\kappa)$ is the space of all polynomials of degree at most p in each variable on κ.

We shall now define some suitable edge operators that are required for the definition of the proceeding DGFEM. To this end, associated with the mesh $\mathcal{T}_h$, we denote by $\mathcal{E}_h^{\mathcal{I}}$ the set of all interior edges of the partition $\mathcal{T}_h$ of Ω, and by $\mathcal{E}_h^{\mathcal{B}}$ the set of all boundary edges of $\mathcal{T}_h$. In addition, $\mathcal{E}_h = \mathcal{E}_h^{\mathcal{B}} \cup \mathcal{E}_h^{\mathcal{I}}$ denotes the set of all edges in the mesh $\mathcal{T}_h$.

Let v and $\boldsymbol{q}$ be scalar- and vector-valued functions, respectively, which are sufficiently smooth inside each element $\kappa \in \mathcal{T}_h$. Given two adjacent elements, $\kappa^+, \kappa^- \in \mathcal{T}_h$ which share a common edge $e \in \mathcal{E}_h^{\mathcal{I}}$, i.e., $e = \partial\kappa^+ \cap \partial\kappa^-$, we write $v^\pm$ and $\boldsymbol{q}^\pm$ to denote the traces of the functions v and $\boldsymbol{q}$, respectively, on the edge e, taken from the interior of $\kappa^\pm$, respectively. With this notation, the averages of v and $\boldsymbol{q}$ at $\boldsymbol{x} \in e$ are given by $\{\!\{v\}\!\} = {}^1\!/_2(v^+ + v^-)$ and $\{\!\{\boldsymbol{q}\}\!\} = {}^1\!/_2(\boldsymbol{q}^+ + \boldsymbol{q}^-)$, respectively. Similarly, the jumps of v and $\boldsymbol{q}$ at $\boldsymbol{x} \in e$ are given by $[\![v]\!] = v^+\boldsymbol{n}_{\kappa^+} + v^-_{\kappa^-}\boldsymbol{n}_{\kappa^-}$ and $[\![\boldsymbol{q}]\!] = \boldsymbol{q}^+ \cdot \boldsymbol{n}_{\kappa^+} + \boldsymbol{q}^- \cdot \boldsymbol{n}_{\kappa^-}$, respectively, where $\boldsymbol{n}_{\kappa^\pm}$ denotes the unit outward normal vector on $\partial\kappa^\pm$, respectively. On a boundary edge $e \in \mathcal{E}_h^{\mathcal{B}}$, we set $\{\!\{v\}\!\} = v$, $\{\!\{\boldsymbol{q}\}\!\} = \boldsymbol{q}$, $[\![v]\!] = v\boldsymbol{n}$ and $[\![\boldsymbol{q}]\!] = \boldsymbol{q} \cdot \boldsymbol{n}$, with $\boldsymbol{n}$ denoting the unit outward normal vector on the boundary Γ.

For an edge $e \in \mathcal{E}_h$, we define h_e to be the length of the edge; moreover, the edge polynomial degree p_e is defined by $p_e = \max(p_\kappa, p_{\kappa'})$, if $e = \partial\kappa \cap \partial\kappa' \in \mathcal{E}_h^{\mathcal{I}}$, and $p_e = p_\kappa$, if $e = \partial\kappa \cap \Gamma \in \mathcal{E}_h^{\mathcal{B}}$.

1.1 Standard interior penalty DGFEM discretization

In this section we first introduce the so-called *standard* SIP DGFEM for the numerical approximation of the problem (5)–(6). To this end, given a (fine) mesh partition $\mathcal{T}_h$ of Ω, together with a corresponding polynomial degree vector $\boldsymbol{p}$, the standard SIP DGFEM is defined as follows: find $u_{h,p} \in V(\mathcal{T}_h, \boldsymbol{p})$ such that

$$A_{h,p}(u_{h,p}; u_{h,p}, v_{h,p}) = F_{h,p}(v_{h,p}) \tag{8}$$

for all $v_{h,p} \in V(\mathcal{T}_h, \boldsymbol{p})$, where

$$\begin{aligned} A_{h,p}(\psi; u, v) = &\sum_{\kappa \in \mathcal{T}_h} \int_\kappa \mu(|\nabla_h \psi|) \nabla_h u \cdot \nabla_h v \,\mathrm{d}\boldsymbol{x} + \sum_{e \in \mathcal{E}_h} \int_e \sigma_{h,p} [\![u]\!] \cdot [\![v]\!] \,\mathrm{d}s \\ &- \sum_{e \in \mathcal{E}_h} \int_e \left(\{\!\{\mu(|\nabla_h \psi|) \nabla_h u\}\!\} \cdot [\![v]\!] + \{\!\{\mu(|\nabla_h \psi|) \nabla_h v\}\!\} \cdot [\![u]\!] \right) \mathrm{d}s, \\ F_{h,p}(v) = &\sum_{\kappa \in \mathcal{T}_h} \int_\kappa f v \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Here, $\sigma_{h,p} = \gamma p_e^2/h_e$, where $\gamma > 0$ is a sufficiently large constant (independent of the local element sizes and polynomial degrees), is the *interior penalty parameter*; cf. [12, 13], for example.

Remark 1.1 The SIP DGFEM scheme defined in (8) is identical to the method studied in [12], and represents a slight alternative to the parameterized DGFEMs considered in [13].

1.2 Two-grid interior penalty discretization

In this section, we now proceed to introduce the so–called two-grid SIP DGFEM approximation to (5)–(6). To this end, we consider two partitions $\mathcal{T}_h$ and $\mathcal{T}_H$ of the computational domain Ω, of granularity h and H, respectively. Here, we refer to $\mathcal{T}_h$ and $\mathcal{T}_H$ as the fine and coarse mesh partitions of Ω, respectively. In particular, we assume that $\mathcal{T}_h$ and $\mathcal{T}_H$ are nested in the sense that, for any $\kappa_h \in \mathcal{T}_h$ there exists an element $\kappa_H \in \mathcal{T}_H$ such that $\bar{\kappa}_h \subseteq \bar{\kappa}_H$. Moreover, to each mesh $\mathcal{T}_h$ and $\mathcal{T}_H$, we associate a corresponding polynomial degree distribution $\boldsymbol{p} = \{p_\kappa : \kappa \in \mathcal{T}_h\}$ and $\boldsymbol{P} = \{P_\kappa : \kappa \in \mathcal{T}_H\}$, respectively, with the property that, given $\kappa_h \in \mathcal{T}_h$ and the associated $\kappa_H \in \mathcal{T}_H$, such that $\bar{\kappa}_h \subseteq \bar{\kappa}_H$, the corresponding polynomial degrees satisfy the following condition: $p_{\kappa_h} \geq P_{\kappa_H}$. Given $\mathcal{T}_h$, $\boldsymbol{p}$ and $\mathcal{T}_H$, $\boldsymbol{P}$, we may construct the corresponding fine and coarse finite element spaces $V(\mathcal{T}_h, \boldsymbol{p})$ and $V(\mathcal{T}_H, \boldsymbol{P})$, respectively, which satisfy the following condition: $V(\mathcal{T}_H, \boldsymbol{P}) \subseteq V(\mathcal{T}_h, \boldsymbol{p})$.

With this notation, we now introduce the hp–version of the two-grid algorithm [10, Algorithm 1] for the SIP DGFEM discretization of (5)-(6):

1. Compute the coarse grid approximation $u_{H,P} \in V(\mathcal{T}_H, \boldsymbol{P})$ such that

$$A_{H,P}(u_{H,P}; u_{H,P}, v_{H,P}) = F_{H,P}(v_{H,P}) \qquad \forall v_{H,P} \in V(\mathcal{T}_H, \boldsymbol{P}). \tag{9}$$

2. Determine the fine grid solution $u_{2G} \in V(\mathcal{T}_h, \boldsymbol{p})$ such that

$$A_{h,p}(u_{H,P}; u_{2G}, v_{h,p}) = F_{h,p}(v_{h,p}) \qquad \forall v_{h,p} \in V(\mathcal{T}_h, \boldsymbol{p}). \tag{10}$$

Existence and uniqueness of the solution $u_{H,P}$ for this formulation is demonstrated in [12]. The formulation (10) is a symmetric interior penalty discretization of a linear elliptic PDE, where the coefficient $\mu(|\nabla_h u_{H,P}|)$ is a known function; thereby, provided that the constant γ is chosen sufficiently large, the existence and uniqueness of the solution u_{2G} to this problem follows immediately, cf., for example, [14].

2 Error Analysis

In this section, we develop the *a priori* error analysis of the two-grid SIP DGFEM defined by (9)–(10). To this end, we equip the finite element space $V(\mathcal{T}_h, \boldsymbol{p})$ with the following *energy norm*:

$$\|v\|_{h,p}^2 = \|\nabla_h v\|_{\mathrm{L}_2(\Omega)}^2 + \sum_{e \in \mathcal{E}_h} \int_e \sigma_{h,p} |[\![v]\!]|^2 \, \mathrm{d}s.$$

We first recall the following *a priori* error bound for the standard SIP DGFEM approximation (8) of the quasi-linear problem (5)–(6).

Lemma 2.1 *Assuming that $u \in \mathrm{C}^1(\Omega)$ and $u|_\kappa \in \mathrm{H}^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, for $\kappa \in \mathcal{T}_h$ then the solution $u_{h,p} \in V(\mathcal{T}_h, \boldsymbol{p})$ of (8) satisfies the error bound*

$$\|u - u_{h,p}\|_{h,p}^2 \leq C_1 \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{H^{k_\kappa}(\kappa)}^2, \tag{11}$$

with $1 \leq s_k \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}_h$, and C_1 is a positive constant independent of u, h and $\boldsymbol{p}$.

Proof. See [12] or [13]; we note, however, that the latter article employs a slightly different DGFEM formulation. □

Employing Lemma 2.1, we now deduce the following error bound for the two-grid approximation defined in (10).

Theorem 2.2 *Assuming that $u \in \mathrm{C}^1(\Omega)$, $u|_\kappa \in \mathrm{H}^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, for $\kappa \in \mathcal{T}_h$ and $u|_\kappa \in \mathrm{H}^{K_\kappa}(\kappa)$, $K_\kappa \geq 2$, for $\kappa \in \mathcal{T}_H$, then the solution $u_{2G} \in V(\mathcal{T}_h, \boldsymbol{p})$ of (10) satisfies the error bounds*

$$\|u_{h,p} - u_{2G}\|_{h,p}^2 \leq C_2 \sum_{\kappa \in \mathcal{T}_H} \frac{H_\kappa^{2S_\kappa - 2}}{P_\kappa^{2K_\kappa - 3}} \|u\|_{H^{K_\kappa}(\kappa)}^2, \tag{12}$$

$$\|u - u_{2G}\|_{h,p}^2 \leq C_1 \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 + C_2 \sum_{\kappa \in \mathcal{T}_H} \frac{H_\kappa^{2S_\kappa - 2}}{P_\kappa^{2K_\kappa - 3}} \|u\|_{H^{K_\kappa}(\kappa)}^2, \tag{13}$$

with $1 \leq s_k \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}_h$, $1 \leq S_k \leq \min\{P_\kappa + 1, K_\kappa\}$, $P_\kappa \geq 1$, for $\kappa \in \mathcal{T}_H$, and C_1 and C_2 are positive constants independent of u, $h, H, \boldsymbol{p}$ and $\boldsymbol{P}$.

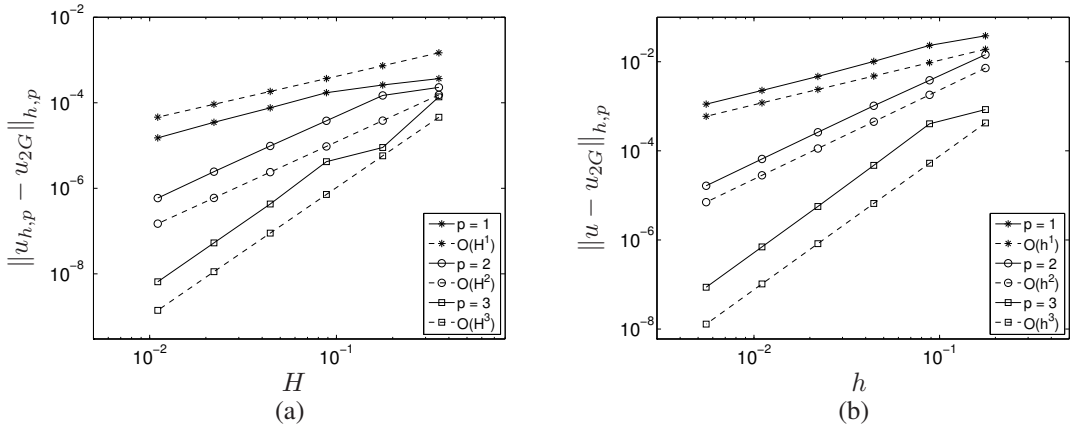

Fig. 1 (a) Plot of $\|u_{h,p} - u_{2G}\|_{h,p}$ against H, for a fixed fine mesh; (b) Plot of $\|u - u_{2G}\|_{h,p}$ against h as both the fine and coarse meshes are uniformly refined, with $H = h/2$.

Proof. See [11] for details. □

Remark 2.3 We note that due to the dependence of the nonlinear coefficient μ on $|\nabla u|$, the error bound derived in Theorem 2.2 indicates that the mesh and polynomial distribution of both the fine and coarse finite element spaces $V(\mathcal{T}_h, \boldsymbol{p})$ and $V(\mathcal{T}_H, \boldsymbol{P})$, respectively, should grow at roughly the same rate. This is in contrast to the h–version *a priori* error analysis undertaken in [10] in the case when $\mu = \mu(u)$. Indeed, in this setting, it is shown that for convergence, the coarse and fine mesh sizes H and h, respectively, should satisfy $H = \mathcal{O}(\sqrt{h})$, when the polynomial degree is (uniformly) set equal to one.

3 Numerical Experiment

In this section we present numerical experiments which confirm the theoretical results outlined in Theorem 2.2. To this end, we let $\Omega = (0,1)^2 \subset \mathbb{R}^2$ be the unit square, and define the nonlinear coefficient as $\mu(|\nabla u|) = 2 + \frac{1}{1+|\nabla u|}$. Furthermore, we select the right-hand forcing function f so that the analytical solution to (5)–(6) is given by $u(x,y) = x(1-x)y(1-y)(1-2y)\mathrm{e}^{-20(2x-1)^2}$. Firstly, we consider the case when the fine mesh $\mathcal{T}_h$ is fixed (256×256 uniform square mesh) and the coarse grid is uniformly refined. In Figure 1(a) we plot $\|u_{h,p} - u_{2G}\|_{h,p}$ against H in the case when the coarse and fine polynomial degrees $\boldsymbol{P}$ and $\boldsymbol{p}$, respectively, are both uniform and equal, i.e., $P_\kappa = p$ for all $\kappa \in \mathcal{T}_H$ and $p_\kappa = p$ for all $\kappa \in \mathcal{T}_h$; here, we consider the case when $p = 1, 2, 3$. We clearly observe that the error $\|u_{h,p} - u_{2G}\|_{h,p}$ converges to zero at the rate $\mathcal{O}(H^p)$, as H tends to zero, for each fixed polynomial degree, which is in full agreement with (12). Secondly, we now consider the case when the fine and coarse meshes are both simultaneously refined together. To this end, we again consider the case when $\boldsymbol{P}$ and $\boldsymbol{p}$ are both uniform and equal, and consider a sequence of uniformly refined meshes, such that $H = h/2$. From Figure 1(b), we observe that $\|u - u_{2G}\|_{h,p}$ convergences to zero at the rate $\mathcal{O}(h^p)$, as h tends to zero, for each fixed polynomial degree; this confirms (13).

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References

[1] J. Xu, SIAM J. Numer. Anal. **29**, 303–319 (1992).
[2] J. Xu, SIAM J. Sci. Comp. **15**, 231–237 (1994).
[3] J. Xu, SIAM J. Numer. Anal. **33**, 1759–1777 (1996).
[4] O. Axelsson and W. Layton, SIAM J. Numer. Anal. **33**(6), 2359–2374 (1996).
[5] C. N. Dawson, M. F. Wheeler, and C. S. Woodward, SIAM J. Numer. Anal. **35**, 435–452 (1998).
[6] T. Utnes, Comm. Numer. Meth. Engng. **13**(8), 675–684 (1997).
[7] M. Marion and J. Xu, SIAM J. Numer. Anal. **32**(4), 1170–1184 (1995).
[8] L. Wu and M. Allen, Numerical Methods for Partial Differential Equations **1999**, 589–604 (1999).
[9] C. Bi and V. Ginting, Numer. Math. **108**, 177–198 (2007).
[10] C. Bi and V. Ginting, J. Sci. Comput. (2011), In press.
[11] S. Congreve, P. Houston, and T. Wihler, Submitted for publication (2011).
[12] T. Gudi, N. Nataraj, and A. K. Pani, Numer. Math. **109**, 233–268 (2008).
[13] P. Houston, J. Robson, and E. Süli, IMA J. Numer. Anal. **25**, 726–749 (2005).
[14] T. P. Wihler, P. Frauenfelder, and C. Schwab, Comput. Math. Appl. **26**, 183–205 (2003).

