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A Method of Time-Domain Approximation Using the Transfer Function Whose Attenuation Poles are Restricted on the Imaginary Axis

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SUMMARY

In the transfer function with zeros restricted on the imaginary axis of S-plane the phase characteristic is essentially determined by that of the denominator polynomial. As long as the denominator polynomial is a Hurwitz polynomial, the tangent of its phase characteristic, i.e., the ratio of the imaginary part (odd part) to the real part (even part), is a reactance function. This paper proposes a waveform approximation method which approximates the phase tangent of the Fourier transform of the object waveform by a reactance function using an optimization technique. The Hurwitz polynomial, which is the sum of the numerator and denominator of the reactance function, is adopted as the denominator polynomial of the approximate transfer function. The numerator polynomial of the approximate transfer function is derived by adjusting the zeros on the imaginary axis by an optimization technique to approximate the amplitude of the Fourier transform of the object waveform. An evaluation measure in the proposed approximation method is presented, describing its problems and physical implications. It is shown that the proposed method can provide a satisfactory approximation for symmetrical waveforms. For asymmetrical waveforms it is shown that the approximation can be improved by adding a delay time. An appropriate weighting function is proposed for the evaluation measure. Lastly, the proposed method is compared with other methods of approximation.

1. Introduction

The problem of approximating a given waveform by the impulse response of a circuit whose transfer function is a rational function of ω is encountered in data transmission and radar practice and has been the subject of various investigations [1 ∿ 3]. It is required in such an approximation to ensure the stability of the circuit, making the denominator polynomial of the approximate transfer function a Hurwitz polynomial. It is not easy, however, to determine the denominator polynomial satisfying this condition and minimizing the approximation error in the time domain.

From the above considerations, a method is generally employed in which the denominator is detemined as a particular Hurwitz polynomial and the numerator polynomial is determined so that the time response is a good approximation to the object waveform. However, if the coefficients of the numerator polynomial are determined by the least-mean-square method or by Taylor series expansion [2, 4], the numerator polynomial usually possesses complex zeros. Then a complicated circuit such as a Darlington section is needed to realize such a transfer function by a resistor-terminated reactance two-port.

On the other hand, it is known that the necessary and sufficient condition for a polynomial to be a Hurwitz polynomial is that the ratio of its even and odd parts is a reactance function. It is also known that when the zeros of a transfer function are located on the imaginary axis, it can be realized by an LC ladder circuit with at most the addition of Brune sections. Then the phase of the transfer function is essentially determined by the phase of the denominator polynomial.

Based on the above properties, the paper derives the denominator polynomial of

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the approximate transfer function with ensured stability. A point in the derivation is that the tangent phase of the Fourier transform of the object waveform is approximated by a reactance function using an optimization technique and the Hurwitz polynomial is constructed from that reactance function. Then the amplitude of Fourier transform of the object waveform is approximated using only the attenuation poles on the imaginary axis. It is shown by examples that a satisfactory result is obtained by the proposed method and the proposed method is compared with other methods of approximation.

2. Preliminaries

Let f(t) be the object waveform and g(t) be the impulse response of the resistor-terminated two-port approximating that waveform. Their Laplace transforms are denoted by F(s) and G(s), respectively. For $s = j\omega$ it is written that

$$F(j\omega) = |F(j\omega)|\,e^{-j\varphi(\omega)} \tag{1}$$

$$G(j\omega) = |G(j\omega)|\,e^{-j\theta(\omega)} \tag{2}$$

For convenience, $|F(j\omega)|$ and $|G(j\omega)|$ are called the amplitudes of $F(j\omega)$ and $G(j\omega)$ and $\varphi(\omega)$ and $\theta(\omega)$ are called the phases of $F(j\omega)$ and $G(j\omega)$, respectively.

Obviously, G(s) is a rational function of s. N(s) and D(s) are numerator and denominator polynomials, respectively, of the approximate transfer function G(s). In other words,

$$G(s) = N(s)/D(s) \tag{3}$$

The mean-square error E of the object waveform and approximate waveform is defined by

$$E = \int_0^{\infty} |f(t) - g(t)|^2 dt \tag{4}$$

3. Determination of Denominator Polynomial from Phase Tangent

A. Outline of determination of denominator polynomial

Assume that zeros of the approximate transfer function (i.e., its attenuation poles) exist only on the imaginary axis. For real frequency ω, the numerator polynomial $N(j\omega)$ takes real value when M is even and pure imaginary value when M is odd, where M is the degree of $N(j\omega)$. Consequently, disregarding the discontinuity of π at the attenuation pole, the phase of G(s) is essentially determined by the denominator polynomial $D(j\omega)$. The following equations apply:

$$\theta(\omega) = \begin{cases} -\arg D(j\omega) & M:\ \text{even} \\ -\arg D(j\omega) - \pi/2 & M:\ \text{odd} \end{cases} \tag{5a}$$

$$e^{j2\theta(\omega)} = \begin{cases} D(-j\omega)/D(j\omega) & M:\ \text{even} \\ -D(-j\omega)/D(j\omega) & M:\ \text{odd} \end{cases} \tag{5b}$$

In this paper it is assumed that $F(0) \neq 0$ and consequently the degree of the numerator polynomial N(s) of G(s) is set as even in the following.

In order to make the denominator polynomial D(s) a Hurwitz polynomial, $\exp[j2\varphi(\omega)]$ is rewritten as follows:

$$e^{j2\varphi(\omega)} = \frac{\alpha(\omega) - j\beta(\omega)}{\alpha(\omega) + j\beta(\omega)} = \frac{1 - jH(\omega)}{1 + jH(\omega)} \tag{6}$$

where $\alpha(\omega)$ and $\beta(\omega)$ are the even and odd parts of $F(j\omega)$, respectively, and $jH(\omega) = j\beta(\omega)/\alpha(\omega)$.

Similarly, Eq. (5b) is rewritten as

$$\frac{D(-j\omega)}{D(j\omega)} = \frac{\gamma(\omega) - j\xi(\omega)}{\gamma(\omega) + j\xi(\omega)} = \frac{1 - jX(\omega)}{1 + jX(\omega)} \tag{7}$$

where $\gamma(\omega)$ and $\xi(\omega)$ are the even and odd parts of $D(j\omega)$, respectively, and $jX(\omega) = j\xi(\omega)/\gamma(\omega)$.

The necessary and sufficient condition for a polynomial to be a Hurwitz polynomial is that the ratio of its even and odd parts is a reactance function. Consequently, as is obvious by comparing Eqs. (6) and (7), D(s) is ensured to be a Hurwitz polynomial as long as $jH(\omega)$ is approximated by a reactance function $jX(\omega)$. Although the reactance function to be considered is $jX(\omega)$, $X(\omega)$ obtained by deleting j is also called a reactance function in the following.

In determination of the denominator polynomial D(s) within the range of Hurwitz polynomial, the following relation is used as the measure to represent the approximation of $X(\omega)$ to $H(\omega)$:

$$I = \int_0^{\omega_a} W(\omega)\,|H(\omega) - X(\omega)|^2 d\omega \tag{8}$$

where ω_a and $W(\omega)$ are the constant and the weighting function described later. The implication of approximating $H(\omega)$ by $X(\omega)$ from the viewpoint of waveform approximation is discussed in the next section with respect to the moment of the waveform.

From Eq. (6), $H(\omega)$ satisfies the following equation:

$$H(\omega) = \tan\varphi(\omega) \tag{9}$$

Since $\varphi(\omega)$ is an odd function, $\varphi(0) = 0$. Consequently, it is assumed that $X(0) = 0$. In other words, we consider the reactance function which takes the value 0 at the origin of S-phase. Denoting the numerator and the denominator polynomials of $X(\omega)$ by $N_X(j\omega)$ and $D_X(j\omega)$, respectively, and letting their degrees be M_X and N_X, respectively, M_X is odd and N_X is even. Obviously, $|M_X - N_X| = 1$. Let the zeros of $X(\omega)$ in $\omega \geqq 0$ be a_m ($m = 0 \sim (M_X - 1)/2$) and let the poles be b_n ($n = 1 \sim N_X/2$). Then $jX(\omega)$ can be represented as

$$jX(\omega) = A\frac{N_X(j\omega)}{D_X(j\omega)} = A\frac{j\omega \cdot \prod_{m=1}^{(M_X-1)/2}(-\omega^2 + a_m^2)}{\prod_{n=1}^{N_X/2}(-\omega^2 + b_n^2)} \tag{10}$$

where A is a suitable constant. The relation $0 = a_0 < b_1 < a_1 < \cdots < a_{(M_X-1)/2} < b_{N_X/2}$ holds among the poles and zeros and consequently $0 = a_0 < b_1 < a_1 < \cdots < b_{N_X/2} < a_{(M_X-1)/2}$.

Let the degree of the denominator polynomial D(s) be N. Letting $A = x_1^2$, $b_1 = x_2^2$, $a_1 = b_1 + x_3^2 = x_2^2 + x_3^2$, and $b_2 = a_1 + x_4^2 = x_2^2 + x_3^2 + x_4^2, \cdots$ $(x_k \neq 0,\ k = 1 \sim N)$ in Eq. (10), x_k ($k = 1 \sim N$) is used as the parameter for optimization. Using the method of [5], I in Eq. (8) is minimized. The method in [5] is an improvement of Powell's method, which is a conjugate gradient method without using the derivatives of the object function. For the details see [5].

B. Determination of parameters and discussions

Here we discuss the range of integration in Eq. (8), the weighting function and initial settings for the parameters in Eq. (10). Before proceeding, $H(\omega)$ should be examined. It is easily seen that when the object waveform is symmetrical, the phase $\varphi(\omega)$ of the Fourier transform of the object waveform satisfies $\varphi(\omega) = K\omega$ (K is a constant) and $H(\omega) = \tan(K\omega)$.

As an example of the case where f(t) is not symmetrical, consider the asymmetrical trapezoidal waveform shown in Fig. 1, which is used in radar [6, 11]. Several examples of $H(\omega)$ of trapezoidal waveforms are

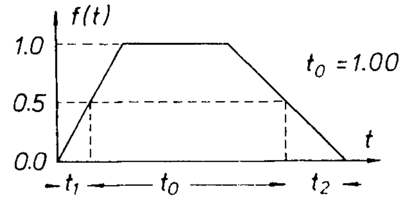

Fig. 1. The desired trapezoid waveform.

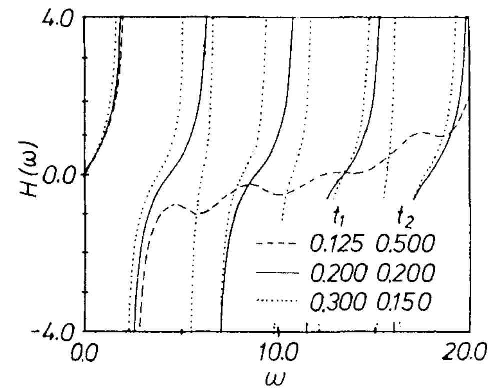

Fig. 2. Frequency characteristics of $H(\omega)$.

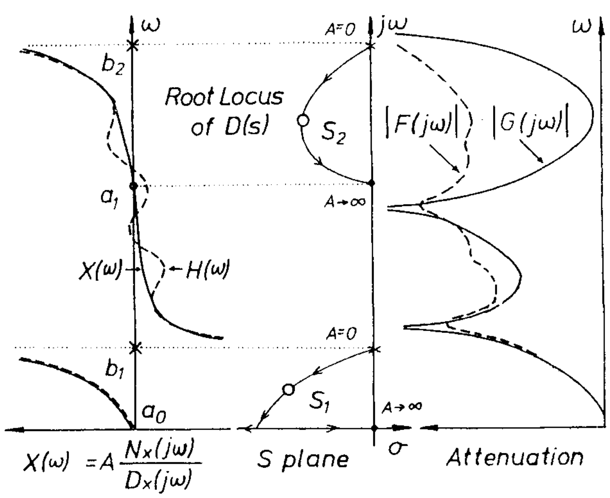

Fig. 3. An example of $X(\omega)$, root locus of D(s) and attenuation.

shown in Fig. 2. The dashed line in the figure corresponds to H( ) of $t_1 < t_2$ ($t_1 = 0.125$, $t_2 = 0.500$), the dotted line to $t_1 > t_2$ ($t_1 = 0.300$, $t_2 = 0.150$) and the solid line to $t_1 = t_2$ ($t_1 = t_2 = 0.200$). As is seen from the figure, the trapezoidal waveform of $t_1 < t_2$ includes a frequency range for which the derivative $dH(\omega)/d\omega$ of $H(\omega)$ is negative, which cannot be faithfully approximated by a reactance function, for which the derivative should always be positive.

One may intuitively conclude that if $X(\omega)$ can be set as shown by the solid line on the left of the figure, a good approximation to $H(\omega)$ of the dashed line can be obtained. If such a setting is used, however, the roots of the obtained Hurwitz polynomial

$$D(s) = A \cdot N_X(s) + D_X(s) \tag{11}$$

exhibit the root locus for values of A as shown for the roots S_1 and S_2 in the middle of Fig. 3. Consequently, it happens that some of the poles of the approximate transfer function exist in the frequency range which should be regarded as the stopband of the amplitude of $F(j\omega)$ (S_2 in the figure), as shown in the right of Fig. 3. Thus, if $X(\omega)$ is set as in Fig. 3, the approximate transfer function $G(j\omega)$ cannot achieve the required attenuation in the stopband, resulting in an unsatisfactory approximation. Considering the above problem, the initial values for the poles and zeros of $X(\omega)$ are based on the poles and the zeros of $H(\omega)$, but they are not set far from the passband of $F(j\omega)$.

Let us briefly consider the meaning of the approximation of $H(\omega)$ by $X(\omega)$ [7]. Differentiation in Eq. (9) with respect to ω, letting $\omega = 0$ and noting that $\varphi(\omega)$ is an odd function, we obtain $(dH(\omega)/d\omega)_{\omega=0} = \varphi^{(1)}(0)$. Letting $\omega = 0$ in the second-order derivative of $H(\omega)$, $(d^2H(\omega)/d\omega^2)_{\omega=0} = \varphi^{(2)}(0) = 0$. Letting $\omega = 0$ in the third-order derivative of $H(\omega)$, $(d^3H(\omega)/d\omega^3)_{\omega=0} = \varphi^{(3)}(0) + 2\{\varphi^{(1)}(0)\}^3$. On the other hand, the center of the object waveform $\mu = \int_{-\infty}^{\infty} t \cdot f(t)\,dt / \int_{-\infty}^{\infty} f(t)\,dt$ is given by $\mu = \theta^{(1)}(0)$. The third moment μ_3 with respect to μ is given by $\mu_3 = \int_{-\infty}^{\infty} (t - \mu)^3 f(t)\,dt = -|F(0)|\varphi^{(3)}(0)$ [8]. Consequently, letting $|F(0)| = 1$, the higher-order derivatives of $H(\omega)$ can be represented uniquely by the center of the object waveform and the third-order moment with respect to the center. As can easily be verified, the approximation of $H(\omega)$ by $X(\omega)$ near $\omega = 0$ is equivalent to approximating the center and the third-order moment of the object waveform by those of the approximate waveform.

Table 1 compares the first- and third-order derivatives of $H(\omega)$ and $X(\omega)$ for the object trapezoidal waveform. Some examples are considered in Tables 3 and 4 and the agreement is satisfactory up to sixth order of $X(\omega)$. Since the numerator of $X(\omega)$ is odd, the second-order derivative of $X(\omega)$ is obviously 0.

Thus, the approximation of $H(\omega)$ by $X(\omega)$ near $\omega = 0$ is important in the sense of the moment of the waveform. Consequently, ω_a, indicating the range of integration, is set a little less than b_1 (e.g., $\omega_a = 0.995 \times b_1$). Determining the range of integration as 0 to ω_a, we use the weighting function

Table 1. Comparison of $H(\omega)$ and $X(\omega)$

t_1	t_2	$H^{(1)}(0)$	$H^{(3)}(0)$	$X^{(1)}(0)$	$X^{(3)}(0)$	MX/NX
0.150	0.300	0.6583	0.5836	0.6520	0.5909	8/10
0.200	0.200	0.7000	0.6860	0.6973	0.6906	6/8
0.300	0.150	0.7888	0.9760	0.7861	0.9956	8/10

$$W(\omega) = \frac{1}{(1 + \omega^{M_1})(1 + |H(\omega)|^{M_2})} \tag{12}$$

where M_1 and M_2 are set so that a satisfactory approximation can be obtained. The values of ω_a, M_1 and M_2 to obtain a satisfactory approximation in most cases are discussed later.

Although Eq. (12) is not the best weighting function, it becomes smaller at higher frequencies and near the poles of $H(\omega)$ and can be considered as a natural choice when the waveform is to be approximated by the impulse response of a circuit of low-pass type. $W(\omega)$ in the case of high asymmetry, where it is difficult to approximate $H(\omega)$ by $X(\omega)$, is discussed in Sect. 6 with a weighting function different from Eq. (12). Other than the method described above, there can be a method in which $X(\omega)$ is determined to pass through specified points from which the denominator polynomial can be determined. The details of this method are described in [7].

4. Optimization of Numerator Polynomial

In this approximation, only the attenuation poles on the imaginary axis are used to approximate the amplitude of $F(j\omega)$:

$$N(j\omega) = \prod_{i=1}^{M/2} (-\omega^2 + \omega_i^2) \tag{13}$$

where M is the degree of the numerator polynomial and ω_i is the attenuation pole on the imaginary axis.

The square root of ω_i is used as the parameter for optimization and the square error

$$E_f = \int_0^{\omega_b} \left| F(j\omega) - \frac{N(j\omega)}{D(j\omega)} \right|^2 d\omega \tag{14}$$

is minimized in the same way as in Sect. 3 by the method of [5]; where $D(j\omega)$ is the denominator polynomial determined in Sect. 3. ω_b in this equation represents the bandwidth in which the error is considered. In this calculation it is set that $\omega_b = 20.0$. The initial value for the attenuation pole ω_i on the imaginary axis in the optimization is set by referring to the amplitude of $F(j\omega)$.

5. Approximation of Symmetrical Waveform

When the object waveform is a pulse waveform which is symmetrical around the center of the waveform, $H(\omega)$ exhibits a behavior following a tangential curve. Consequently, the denominator polynomial of the approximate transfer function in this method, which is represented by Eq. (11), is the same, as long as the waveform is symmetrical and the center of the object waveform is the same, independently of the waveform shape.

Table 2 shows the parameter values of the reactance function $X(\omega)$ for the denominator polynomials of degree 5 to 8 for the center of the waveform $\mu = 1.0$. Figure 4 compares $X(\omega)$ of 6th degree and $H(\omega)$. The dashed line in the figure is the object waveform and the solid line is the approximate waveform characteristic. The same convention is employed in the remaining figures. For reference, the dotted line shows the ratio of even and odd parts of the maximally, flat delay (Thomson) polynomial, which is often used in waveform approximation. It is intuitively seen from this figure that the denominator polynomial obtained by the proposed method is more useful in the sense of the linear phase.

Values of 2 ∿ 4 are adopted for the constants M_1 and M_2 in the weighting function $W(\omega)$ in Eq. (12). When M_1 and M_2 are larger the convergence to the optimum value is fast, but the degree of approximation is deteriorated. Consequently, the above range of values is desirable.

Using those denominator polynomials and the numerator polynomial determined as shown in the previous section, the impulse response of the transfer function is constructed for waveform approximation. As an example, a symmetrical trapezoidal waveform of $t_1 = 0.2$, $t_2 = 0.2$ ($\mu = 0.7$) is approximated by $G(j\omega)$ of 6th degree. Figure 5 shows the attenuation characteristics of $F(j\omega)$ and $G(j\omega)$ and Fig. 6 shows the time response of $f(t)$ and $g(t)$. The square error between $f(t)$ and $g(t)$ is 9.25×10^{-4} ($t = 0.0 \sim 10.0$).

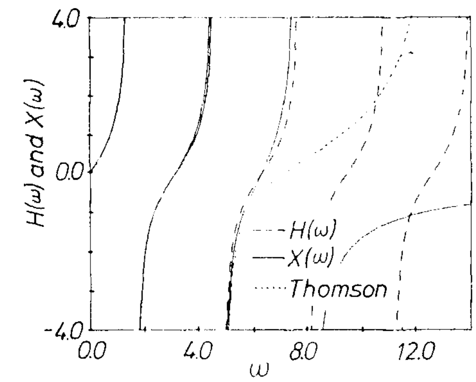

Fig. 4. Frequency characteristics of $H(\omega)$ and $X(\omega)$.

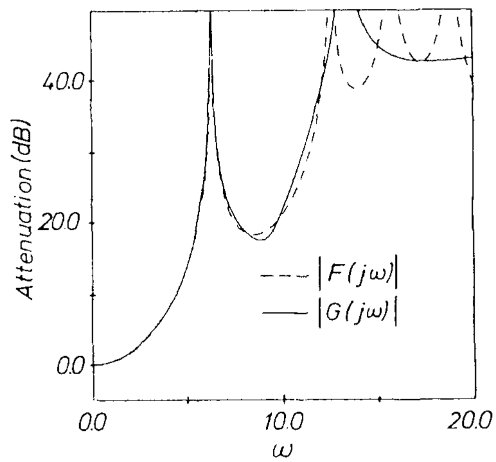

Fig. 5. Attenuation.

Table 2. Parameters of $X(\omega)$ for denominator polynomial approximating symmetrical waveform

N \ $X(\omega)$	A	b_1	a_1	b_2	a_2	b_3	a_3	b_4
5 th	7.0844	1.5708	3.1416	4.7073	6.2781			
6 th	8.6077	1.5708	3.1416	4.7124	6.2921	7.8629		
7 th	10.2201	1.5708	3.1416	4.7124	6.2832	7.8540	9.4248	
8 th	12.1544	1.5708	3.1416	4.7124	6.2825	7.8540	9.4308	11.1808

Table 3. Result of approximation for symmetrical waveforms

	w_1	w_2	w_3	$E(t=0-10)$	Thomson deg.	
Trapezoidal						
$(t_1=t_2=0.2)$	5.2042	10.4588	16.4614	4.9900E-3	1.0565E-2	6/8
$(t_1=t_2=0.3)$	5.7060	11.4392	18.5172	6.2941E-4	4.2762E-3	6/8
$(t_1=t_2=0.4)$	6.2870	13.2879	—	9.2509E-4	5.0028E-3	4/6
$(t_1=t_2=0.4)$	6.2788	12.6856	16.0173	5.2447E-4	9.1245E-4	6/8
Triangular	9.1610	9.1644	—	1.2919E-3	1.4801E-3	4/6
	8.9109	9.0776	18.2696	2.4384E-4	1.2366E-3	6/8
2 Squared cosine	9.0237	21.9255	—	4 6531E-4	2.1296E-3	4/6
	8.1562	8.1145	—	9.9981E-5	1.8704E-4	4/6
	Input: square wave, height 0.7, width 0.7.					

Table 3 summarizes the results of approximation for other symmetrical pulse waveforms. The numerical values for the Thomson type are for the square error where the maximally flat delay polynomial is used as the denominator and the numerator polynomial is optimized by the method in the previous section. In that case, the root of the denominator polynomial is normalized so that the center of the waveform of the impulse response of the all-pole transfer function, with the maximally flat delay polynomial as the denominator, is the same as that of the object function. As is seen from the table, the proposed method exhibits error which is 20 to 50% of the result by the maximally flat delay polynomial, which is very satisfactory.

In the case of the symmetrical pulse waveform considered in this section a denominator polynomial with the desired characteristics can be obtained and in most cases the amplitude of the object waveform is easy to approximate using only the attenuation poles on the imaginary axis. This is the reason for excellent approximation. Although the details are omitted, the approximation is also satisfactory if the input is not an impulse but a square pulse. For example, when the object waveform is a squared-cosine wave, the square pulse input results in approximately 20% of the error of the case of impulse input.

6. Approximation of Asymmetrical Waveform

As an example of the case where the object waveform is an asymmetrical waveform, the asymmetrical trapezoidal waveform of Fig. 1 is considered. In the case of a

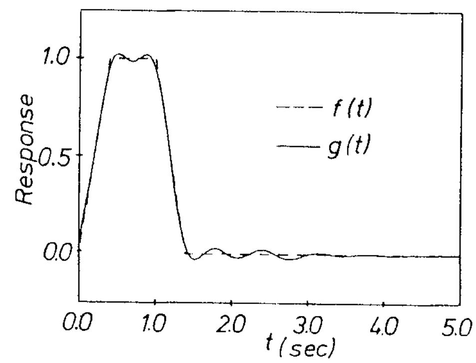

Fig. 6. Time response.

trapezoidal waveform with $t_1 < t_2$, $H(\omega)$ as shown in Fig. 2 has a frequency range in which the derivative is negative. Consequently, the initial value of the reactance function $X(\omega)$ is set so that $H(\omega)$ is well approximated outside of that range. Among the poles and zeros of $H(\omega)$, those which can be used as the poles and zeros of $X(\omega)$ are adopted and are fixed. If there are two fixed poles (zeros) on both sides of a zero (pole), then, letting those poles (zeros) be p_n and p_{n+2}, the unfixed zero (pole) is assumed as given by $p_{n+1} = (p_n + p_{n+2})/2 + (p_{n+2} - p_n) \cdot \tan^{-1}(x_k)/\pi$, where x_k is the optimization parameter of [5], which can be in the range of $-\infty$ to $+\infty$, and k is an integer in the range of $1 \sim N$. n is an appropriate natural number. Also in the case where a pole or zero lies between fixed poles or zeros, the above method is applied to a certain point set between unfixed poles or zeros. Thus, ω_a indicating the range of integration is extended to be set as a value larger than the pole a_1 of $H(\omega)$ ($\omega_a = 20.0$ in this case). As the weighting function we

Table 4. Result of approximation for asymmetrical trapezoidal waveform

	a_1 / b_1	a_2 / b_2	a_3 / b_3	a_4 / b_4	A / b_5	w_1 / w_2	w_3 / w_4	E (t=0.0-10.0)	τ deg.
t_1 = 0.150	5.0550	10.5560	16.5073	20.4287	25.5000	6.1779	19.7120	2.8212E-3	τ = 0.0
t_2 = 0.300	2.3936	6.7664	15.3025	19.2973	23.5307	15.7474	19.7275		8/10
〃	4.2562	7.6056	14.2000	18.7927	51.7154	6.2862	22.9256	2.3343E-3	τ = 0.1
	2.0750	6.3136	12.7500	16.3999	25.8947	15.0401	——		6/10
t_1 = 0.175	4.7525	9.3000	13.0080	17.2002	19.8488	6.2613	18.7474	1.1175E-3	τ = 0.0
t_2 = 0.233	2.3201	6.8500	11.5806	15.0500	19.2522	12.1050	15.2027		8/10
〃	4.0762	7.9237	12.5417	15.1759	23.1922	6.4001	15.8578	1.3717E-3	τ = 0.1
	2.0202	6.3164	10.5512	13.7817	18.0588	12.8463	——		6/10
t_1 = 0.300	3.8833	6.1701	8.2584	10.8405	13.5500	6.2627	12.7374	4.1799E-3	τ = 0.0
t_2 = 0.150	1.9819	5.4156	6.9008	9.6875	12.4123	6.2630	12.8430		8/10

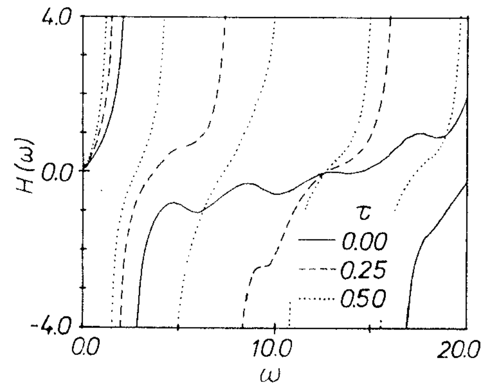

Fig. 7. Effect of delay time τ in H(ω).

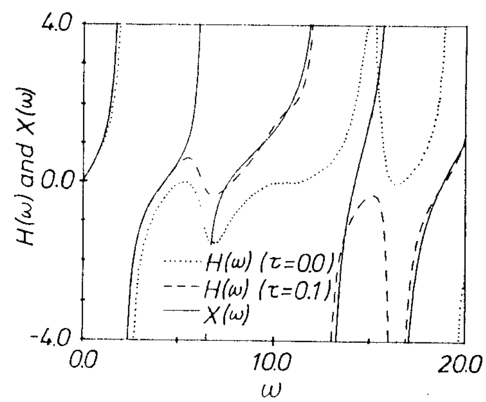

Fig. 8. Frequency characteristics of H(ω) and X(ω).

use

$$W(\omega)=\frac{\exp[M_3\cdot\varphi^{(1)}(\omega)]}{(1+\omega^{M_1})(1+|H(\omega)|^{M_2})\exp[M_3\cdot\varphi^{(1)}(0)]} \tag{15}$$

or

$$W(\omega)=\frac{|F(j\omega)|^2\cdot\exp[M_3\cdot\varphi^{(1)}(\omega)]}{(1+|H(\omega)|^{M_2})\cdot\exp[M_3\cdot\varphi^{(1)}(0)]} \tag{16}$$

$\exp[M_3\cdot\varphi^{(1)}(\omega)]/\exp[M_3\cdot\varphi^{(1)}(0)]$ in Eqs. (15) and (16) is set so that the weighting function is small in the range where the derivative of H(ω) is negative. The value of M_3 is set so that the approximation is satisfactory. Applying Eq. (15) to the symmetrical waveform, the result is the same as in Eq. (12). In Eq. (16) the weight of the power spectrum of the object waveform is used instead of $(1+\omega^{M_1})$. In this case, it is set that $M_1 = M_2 = 2 \sim 4$ and $M_3 = 14 \sim 16$.

With an increase in t_2/t_1, the frequency range in which the derivative of H(ω) is negative is increased, making it difficult to approximate it by X(ω). In this case H(ω) can be more easily approximated by adding a suitable delay to the object trapezoidal waveform. Figure 7 shows the change of H(ω) when delay τ is added. However, in order to modify H(ω) to the extreme, as is shown in the figure, the degrees of the denominator and numerator polynomials must be sufficiently increased, which is not practical. It is useful to add a certain delay, since it improves the approximation accuracy. As an example, a delay of τ = 0.1 is added to the trapezoidal wave of τ = 0.1. Figure 8 shows the frequency characteristics of the trapezoidal wave of τ = 0.1 and X(ω). The dotted line in the figure indicates H(ω) of the trapezoidal wave of τ = 0.0. Figure 9 shows the trapezoidal wave and the time response of the approximation function G(jω) of 10th degree.

In this case the square error of the time response is improved by some 20% compared with the case without time delay. However, since the time delay is realized by

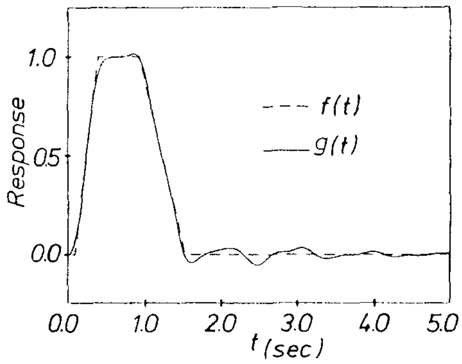

Fig. 9. Time response.

the difference in degrees of the denominator and the numerator of the transfer function, the addition of a time delay is not always effective when the degree of the transfer function is low or in the case of a symmetrical waveform, where $H(\omega)$ is improved little by adding the time delay.

In the case of the trapezoidal wave of $t_1 > t_2$, $H(\omega)$ has larger slope than $H(\omega)$ (tangential curve) of the symmetrical wave of Fig. 2. Since the number of poles and zeros is increased, the error seems to be decreased by increasing the degree of $X(\omega)$. However, since the distance between a pole and a zero of $H(\omega)$ is sometimes small, the root locus of Fig. 3 is also small, making the root of $D(s)$ approach the imaginary axis, decreasing the attenuation of $G(s)$. One may conceive a scheme in which the effect of the pole near the imaginary axis can be cancelled by the root of $N(s)$ (i.e., the zero on the imaginary axis). Since $N(s)$ is limited, however, the approximation error in the time response cannot be effectively improved.

The results of approximation for various kinds of trapezoidal waves are shown in Table 4 and discussed in Sect. 7. In the approximation method proposed in this paper, the phase tangent of the object function is approximated by a reactance function in order to determine the optimum function under the restriction of the reactance function. It is not always true, however, that the Hurwitz polynomial obtained from the reactance function is the denominator polynomial that minimizes the square error in the time domain.

From such a point of view, the square error E of Eq. (4) is modified using Percival's formula as

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{|D(j\omega)|^2}\, |D(j\omega)\cdot F(j\omega) - N(j\omega)|^2\, d\omega \tag{17}$$

Regarding $1/|D(j\omega)|^2$ as a new weighting function, the notation $1/|\tilde{D}(j\omega)|^2$ is used in order to distinguish it from $D(j\omega)$ multiplied by the Fourier transform $F(j\omega)$ of the object waveform. Then $D(j\omega)$ in the weighting function of Eq. (17) (i.e., $\tilde{D}(j\omega)$) is used as the denominator polynomial and the numerator polynomial is used as $N(j\omega)$ in Eq. (17) and $D(j\omega)$ is determined so that the square error E is minimized.

This newly determined polynomial may not be a strict Hurwitz polynomial. However, since $\tilde{D}(j\omega)$ is a Hurwitz polynomial and $N(j\omega)/\tilde{D}(j\omega) \cong F(j\omega)$, the stability will be maintained as long as the square error is small. Using this new denominator polynomial $D(j\omega)$, the numerator polynomial $N(j\omega)$ may be calculated again using the method of Sect. 4. It was verified by an example that if $t_1 < t_2$, this method can improve the square error E by approximately 15% compared with the square-error determined only by the method of Sects. 3 and 4. Thus, redetermining $D(j\omega)$ as $\tilde{D}(j\omega)$, further optimization can be achieved, which is omitted in this discussion.

7. Discussion

A. ω_a for range of integration

In principle, ω_a indicating the range of integration in the determination of $X(\omega)$ is set a little less than b_1. In this approximation method, ω_a should originally be set as $\omega_a = +\infty$. In that case, however, the numerical integration becomes difficult. Considering determination of the passband characteristics by the denominator polynomial of the transfer function, ω_a should have the magnitude of passband. From discussion of the derivative and the moment of $H(\omega)$, it is seen to be important to provide a good approximation of $H(\omega)$ by $X(\omega)$ near zero frequency.

From such a point of view, ω_a is set a little less than b_1 in this paper. However, the behavior of $X(\omega)$ depends on the object waveform and it may be permissible to increase ω_a if this does not degrade the approximation near zero frequency. In such a case, using the weighting functions of Eq. (15) or (16), the phase can well be approximated over a wide range, although the function is not always the optimum. The optimization, however, may fall into a local minimum and it is desirable to repeat the optimization by modifying the initial values for the optimizing parameters and the constants. If satisfactory initial values are obtained for poles and zeros, the approximation is repeated by again setting ω_a a little less than b_1 in order to minimize the error between $H(\omega)$ and $X(\omega)$ near zero frequency.

B. Optimization in determination of $X(\omega)$

In optimization by the method of [5] a one-dimensional search is made to determine the gradient of the object function. The step in the search is set as 0.01 and a golden mean search [9] is used. The convergence is determined by comparing the absolute value of the parameter change x_k, $k = 1 \sim N$ with 1×10^{-3}.

The convergence greatly depends on the setting of the initial values. In determination of the poles and zeros of the 6th-degree reactance function approximating the tangent function, for example, if the values of ω at the poles and zeros of the tangent function are adopted as the initial values, the computation converges after calculating the value of the object function at approximately 80 points. The convergence is worse when the trapezoidal waveform of $t_1 < t_2$ is to be approximated with Eq. (15) or (16) as the weighting function. If the values of ω at poles and zeros of $H(\omega)$ are used as the corresponding initial values for pole b_1 and zero a_1, they remain almost fixed during the optimization procedure, making it possible to reduce the number of parameters. For reference, fewer optimization parameters are required in determination of $N(j\omega)$ and some 20 points are sufficient to converge the process.

C. Comparison with the root locations by approximation of [10]

Zschunke [10] has shown the location of the root of the denominator polynomial resulting from optimization of the approximate transfer function. The roots obtained by the present method agree well with those. For the details, see [7]. Table 5 shows, as an example, the case of an 8th-degree polynomial.

D. Approximation of asymmetrical trapezoidal waveform

It is known that when the denominator polynomial is set as an appropriate Hurwitz polynomial and the numerator polynomial is determined by an approximation without any particular restriction, such as the least-mean-square or Padé approximation, the approximation can be made more efficient by employing an asymmetrical trapezoidal wave ($t_1 < t_2$) [11]. As a special case, it is reported that when the numerator polynomial is determined by the least-mean-square method, if a transitional Butterworth-Thomson polynomial is set as the denominator polynomial, which has a root on the segment connecting the roots of the maximally flat amplitude polynomial and the maximally flat delay polynomial, and if the location on the segment is adjusted, the approximation is quite satisfactory independently of the degree of asymmetry of the trapezoidal waveform when the root is close to the maximally flat delay polynomial [12].

Further, examining cases where a trapezoidal waveform with $t_1 < t_2$ is very accurately approximated, it is seen that the numerator polynomial of the approximate transfer function is a minimum-phase polynomial. This is a natural consequence of the relation between the time delay of the waveform and the phase delay of the transfer function. For this reason it is anticipated that the approximation error increases in the proposed method for the trapezoidal waveform of $t_1 < t_2$, where the zeros of the approximate transfer function are restricted to the imaginary axis. This is also seen from the fact that in Table 4 the approximation is worse when the degree of asymmetry is increased. In the case of a symmetrical waveform, on the other hand, the result in the proposed method is as good as the result in the literature. It is thus concluded that, if the zeros of the approximate transfer function are restricted to the imaginary axis, it is possible to specify the object waveform as a symmetrical or nearly symmetrical waveform.

8. Conclusion

This paper presented an answer to the problem of to what extent approximation can be made under the restriction that the waveform is to be approximated by the transfer function of a resistor-terminated reactance filter with attenuation poles only on the imaginary axis. In the proposed approximation method the denominator polynomial can easily be determined as a Hurwitz polynomial. Another feature is that although an optimization procedure is used, the degree of approximation can be verified at every stage of the process by an intuitive evaluation.

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