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Khokhlov–Zabolotskaya–Kuznetsov type equation: nonlinear acoustics in heterogeneous media

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Abstract

The KZK type equation introduced in this Note differs from the traditional form of the KZK model known in acoustics by the assumptions on the nonlinear term. For this modified form, a global existence and uniqueness result is established for the case of non-constant coefficients. Afterwards the asymptotic behaviour of the solution of the KZK type equation with rapidly oscillating coefficients is studied. ***To cite this article: I. Kostin, G. Panasenko, C. R. Mecanique 334 (2006).***

Résumé

L'équation de type Khokhlov–Zabolotskaya–Kuznetsov : acoustique non linéaire dans un milieu hétérogène. L'équation de type KZK introduit dans cette Note est une version modifiée du modèle KZK connu en acoustique (ces modifications concernent les hypothèses sur le terme non linéaire). Pour cette forme modifiée, un résultat d'existence et unicité globales est établi dans le cas des coefficients variables. Ensuite le comportement asymptotique de la solution de l'équation de type KZK avec les coefficients rapidement oscillants est étudié. ***Pour citer cet article : I. Kostin, G. Panasenko, C. R. Mecanique 334 (2006).***

Keywords: Acoustics; Nonlinear acoustics; KZK equation; Homogenization

Mots-clés : Acoustique ; Acoustique non linéaire ; Équation KZK ; Homogéneisation

1. Introduction

The so called Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation [1,2] belongs to the set of non-linear acoustics models, such as the well-known Riemann wave equation (or the non-linear transfer equation), Burgers equation, Korteweg–de Vries (KdV) equation, Khokhlov–Zabolotskaya (KZ) equation (see [1,3–6]), Zakharov–Kuznetsov equation [7,8] and Rudenko–Sukhorukov equation [9–11]. These models are derived from the linear or non-linear

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wave equation for the acoustic pressure, usually, under the hypothesis of small variations of this pressure. More precisely the KZK equation has the form

$$\alpha u_{z\tau} = \big(f(u_\tau)\big)_\tau + \beta u_{\tau\tau\tau} + \gamma u_\tau + \Delta_x u \tag{1}$$

where $u_\tau = u_\tau(z, x, \tau)$ is the acoustic pressure, $(z, x) \in \mathbb{R}^d \times \mathbb{R}$, $d = 1, 2$, are space variables and τ is the retarded time. The nonlinear function f in the KZK equation is quadratic, i.e., $f(s) = \theta s^2$, although for the description of space-limited beams subject to the diffraction and self-action effects it can be taken as cubic: $f(s) = \theta s^3$ (see [2]).

However, in the real physical setting, f has a more complicated shape. Indeed, all these models are derived under the assumption of small oscillations of the pressure. If this assumption of small variations of the pressure is valid, one can always consider f as quadratic for $|s| \leqslant s^*$ and anything different for $|s| > s^*$. If $|u_\tau|$ is smaller than s^*, then the two models (with $f(s) = \theta s^2$, $\forall s$ and $f(s) = \theta s^2$ for $|s| \leqslant s^*$) coincide. The advantage of this modified shape of f is that (as it will be proved below) we can get the global existence theorem as soon as f has bounded derivative.

These arguments motivate us to consider the 'KZK type equation', that is, Eq. (1) with a nonlinearity f admitting a bounded derivative. Let us emphasize that this shape gives a more convenient physical description than the classical quadratic shape.

Another particularity of the model we consider in the present paper is that the coefficients are rapidly oscillating functions of z. This corresponds to the heterogeneous (stratified in the direction of the axis z) acoustic media. This feature complicates the problem, although it allows us to apply the homogenization method (see [12]) to obtain the homogenized model. Its solution is close to the one of the initial problem.

It seems that so far there was no any publication on the existence and uniqueness of the solution for the KZK (or KZK type) equation, although the authors discovered that independently and simultaneously these questions (as well as the derivation of the KZK equation form the Navier–Stokes model) in the case of constant coefficients were studied by Bardos and Rozanova [13,14]. They consider the KZK equation in the whole space ($x \in \mathbb{R}^d$) in the case of constant coefficients. In the present study we shall consider the varying (and even rapidly oscillating) coefficients of the KZK type equation set for $x \in \omega$, where $\omega \subset \mathbb{R}^d$ is a bounded domain. The boundedness of f' will ensure the global existence, while in [13,14] it is proved for small data only. In the case of stratified media we homogenize the KZK type equation and prove the closeness of the solutions of the homogenized and initial models.

2. Problem setting

Let ω be a bounded open domain in $\mathbb{R}^d$ with a smooth boundary $\partial\omega$. For a real function $u = u(z, x, \tau)$ of the variables $z \in \mathbb{R}_+$, $\tau \in \mathbb{R}$ and $x \in \omega$ consider the following PDE:

$$\alpha u_{z\tau} = \big(f(u_\tau)\big)_\tau + \beta u_{\tau\tau\tau} + \gamma u_\tau + \Delta u \tag{2}$$

where the Laplace operator Δ (as well as ∇ below) derives with respect to the variable $x \in \omega$. The positive coefficients α, β and γ are functions of z and x. The nonlinearity f may depend on z and x as well, but neither depends on τ.

In the mathematical setting, the variable z is considered as the evolutionary variable, while x and τ shall be called space variables. Let us impose the following boundary conditions on u:

$$u(z, x, \tau) = u(z, x, \tau + 2\pi) \tag{3}$$

$$\nu \cdot \nabla u|_{\partial\omega} = 0 \tag{4}$$

where ν denotes the unit normal vector of ω. It is clear that these conditions are not sufficient to ensure the uniqueness. Indeed, if the coefficients are constant, then any function u depending only on z solves the problem. This liberty is eliminated by the additional orthogonality condition

$$\int_0^{2\pi} u(z, x, \tau)\,\mathrm{d}\tau = 0, \quad \forall x,\ \forall z \tag{5}$$

Finally, the evolutionary problem (2)–(5) requires an initial condition

$$u(0, x, \tau) = u_0(x, \tau) \tag{6}$$

To avoid heavy notation, in the estimates below we shall denote by M any large positive constant depending only on ω, f, α, β, γ, and Z. By 'large' we mean that the estimate stays true with M replaced by any larger value.

Here is the list of assumptions under which the existence and uniqueness result for problem (2)–(6) will be established.

- (ω) $\omega \subset \mathbb{R}^d$ is an open bounded set. Its boundary $\partial\omega$ is twice continuously differentiable.
- ($f1$) $f(s) = f(s,z,x)$ is continuously differentiable with respect to s and its partial derivative $f' \equiv f_s(s,z,x)$ is uniformly bounded on $\mathbb{R} \times \mathbb{R}_+ \times \omega$.
- ($f2$) $f(0,z,x) = 0$ for all $z \in \mathbb{R}_+$ and $x \in \omega$. As is easy to see, this is not a constraint (otherwise one can always replace $f(s,z,x)$ by $f(s,z,x) - f(0,z,x)$ leaving the equation intact). Together with the previous assumption this one implies $|f(s,z,x)| \leqslant M|s|$.
- ($f3$) Denote by F the primitive of f with respect to the first argument such that $F(0,z,x) = 0$. Clearly, $|F(s,z,x)| \leqslant M|s|^2$. We shall assume that F is differentiable with respect to z and $|F_z(v,z,x)| \leqslant M|v|^2$.
- ($\alpha\beta\gamma$) $\alpha = \alpha(z,x)$, $\beta = \beta(z,x)$, $\gamma = \gamma(z,x)$ are bounded functions defined on $\mathbb{R}_+ \times \omega$. The functions α and β are positive and uniformly bounded away from zero. The derivatives α_z, α_x, β_z, β_x are assumed to be bounded functions.

This list has to be completed by an assumption concerning the regularity of u_0. It will be given along with the definition of a solution.

3. Existence and uniqueness

Let us introduce some functional spaces. First consider the spaces of functions of the variables x and τ. Set $\Omega = \omega \times (0,2\pi)$. The scalar product and the norm in $L_2(\Omega)$ are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let D_Ω be the set of infinitely smooth real functions $v = v(x,\tau)$ defined on $\bar\Omega$ and satisfying

$$\int_0^{2\pi} v(x,\tau)\,\mathrm{d}\tau = 0, \quad \forall x \in \bar\omega \tag{7}$$

$$\frac{\partial^k}{\partial\tau^k} v(x,0) = \frac{\partial^k}{\partial\tau^k} v(x,2\pi), \quad \forall x \in \bar\omega,\ \forall k = 0,1,\ldots \tag{8}$$

$$\nu(x) \cdot \nabla v(x,t) = 0, \quad \forall x \in \partial\omega,\ \forall\tau \in [0,1] \tag{9}$$

Denote by X_0 the closure of D_Ω in the $L_2(\Omega)$-norm. Given a function $v \in X_0$, we shall denote by $\tilde v$ its (unique in X_0) primitive with respect to τ, that is,

$$\tilde v_\tau = v, \quad \int_0^{2\pi} \tilde v\,\mathrm{d}\tau = 0$$

The closure of D_Ω with respect to the norm

$$\|v\|_2^2 = \|v_{\tau\tau}\|^2 + \|\Delta\tilde v\|^2$$

is denoted by X_2. Recall that, as well as the Laplace operator, the gradient is assumed to derive only with respect to $x \in \omega$.

Now let us consider spaces of functions of three variables. For a given positive Z, set $Q = (0,Z) \times \omega \times (0,2\pi)$. We shall use the triple bars to denote the $L_2(Q)$-norm:

$$|||u|||^2 = \int_0^Z \big\|u(z,\cdot,\cdot)\big\|^2\,\mathrm{d}z$$

Let D_Q be the set of infinitely smooth real functions $u = u(z,x,\tau)$ defined on $\bar Q$ such that $u(z,\cdot,\cdot) \in D_\Omega$ for all $z \in [0,Z]$. The closure of D_Q in $L_2(Q)$ is denoted U_0. The space U_2 is defined as the closure of D_Q in the norm

$$|||u|||_2^2 = |||u_{z\tau}|||^2 + |||u_{\tau\tau\tau}|||^2 + |||\Delta u|||^2$$

If $u \in U_2$, then each term in Eq. (2) is a function from U_0, so that the equation writes down as an identity in U_0. Note also that the functions of U_2 belong to $C([0, Z], X_0)$, so that the initial condition $u(0, \cdot, \cdot) = u_0$ can be understood as an identity in X_0. These two observations justify the following definition.

Definition. For given $u_0 \in X_2$ and $Z > 0$, a function $u \in U_2$ is called *strong solution* to problem (2)–(6) on $[0, Z]$ if Eq. (2) holds almost everywhere in Q and $u(0, \cdot, \cdot) = u_0$ almost everywhere in Ω.

Theorem 1. *For any $u_0 \in X_2$ and any $Z > 0$ problem* (2)–(6) *admits a unique strong solution u. It satisfies the following estimates*:

$$\sup_z\bigl[\|u_\tau\|^2 + \|\nabla u\|^2\bigr] + |||u_{\tau\tau}|||^2 + |||\nabla u_\tau|||^2 \leqslant M\bigl[\|u_{0\tau}\|^2 + \|\nabla u_0\|^2\bigr] \tag{10}$$

$$\sup_z \|u_{\tau\tau}\|^2 + |||u_{\tau\tau\tau}|||^2 \leqslant M\|u_{0\tau\tau}\|^2 \tag{11}$$

$$\sup_z \|u_z\|^2 + |||u_{z\tau}|||^2 + |||\Delta u^m|||^2 \leqslant M\|u_{0\tau\tau}\|^2 + M\|\Delta\tilde{u}_0\|^2 \tag{12}$$

If u^1 and u^2 are two strong solutions of problem (2)–(6) *corresponding to the initial conditions $u^1(0) = u_0^1$ and $u^2(0) = u_0^2$, respectively, then*

$$\sup_z\bigl\|u_\tau^1 - u_\tau^2\bigr\|^2 + \bigl|\bigl|\bigl|u_{\tau\tau}^1 - u_{\tau\tau}^2\bigr|\bigr|\bigr|^2 \leqslant M\bigl\|u_{0\tau}^1 - u_{0\tau}^2\bigr\|^2. \tag{13}$$

4. Stratified media

The present section is devoted to the study of the asymptotic behaviour of the solutions to problem (2)–(6) as its coefficients rapidly oscillate. This study will be undertaken under the following additional assumptions:

- $(\beta\gamma)$ The coefficients α, β and γ depend only on z. It is clear that in this case the replacement of z by the new variable $\tilde{z} = \int_0^z \alpha(s)\,\mathrm{d}s$ will bring the equation to the form with $\alpha = 1$. Thus we assume that $\alpha = 1$ throughout the section. The regularity of β and γ is the same as above, that is, β, β_z and γ are bounded while β is also positive and bounded away from zero.
- $(f4)$ The nonlinear function f is of the form $f(s, z, x) = \rho(z)\phi(s)$, where ρ is bounded with a bounded derivative, while ϕ satisfies the same assumptions as f in $(f1)$–$(f3)$, that is, ϕ admits a bounded continuous derivative and $\phi(0) = 0$. Denote by Φ the primitive of ϕ satisfying $\Phi(0) = 0$.

Thus the equation we study takes the form

$$u_{z\tau} = \rho\bigl(\phi(u_\tau)\bigr)_\tau + \beta u_{\tau\tau\tau} + \gamma u_\tau + \Delta u \tag{14}$$

In order to introduce the oscillating character of the coefficients, let us assume that they are of the form

$$\rho(z) = r(z, z/\varepsilon), \qquad \beta(z) = b(z, z/\varepsilon), \qquad \gamma(z) = g(z, z/\varepsilon)$$

where ε is a small parameter. The last assumption we introduce is the following.

- $(\rho\beta\gamma)$ The functions $r(z, \zeta)$, $b(z, \zeta)$ and $g(z, \zeta)$ are bounded as well as the derivatives r_z, r_ζ, b_z, b_ζ. The function b is positive and bounded away from zero. Finally, the functions $r(z, z/\varepsilon)$, $b(z, z/\varepsilon)$ and $g(z, z/\varepsilon)$ admit limits as $\varepsilon \to 0$ in the following weak sense: there exists $q > 1/2$ such that for any $Z > 0$ we have

$$\int_0^Z \bigl(r(z, z/\varepsilon) - \bar{\rho}(z)\bigr)\,\mathrm{d}z = \mathrm{O}(\varepsilon^q), \quad \int_0^Z \bigl(b(z, z/\varepsilon) - \bar{\beta}(z)\bigr)\,\mathrm{d}z = \mathrm{O}(\varepsilon^q), \quad \int_0^Z \bigl(g(z, z/\varepsilon) - \bar{\gamma}(z)\bigr)\,\mathrm{d}z = \mathrm{O}(\varepsilon^q),$$

  where the functions $\bar{\rho}(z)$, $\bar{\beta}(z)$ and $\bar{\gamma}(z)$, as well as the derivatives $\bar{\rho}_z(z)$, $\bar{\beta}_z(z)$ are bounded, while $\bar{\rho}(z)$ are $\bar{\beta}(z)$ are also positive and bounded away from zero.

  Note that if r, b and g are Lipschitzian with respect to the first argument and periodic with respect to the second one, then the above convergence conditions hold with $q = 1$ (see [12,9,10]).

The main result of this section is the following.

Theorem 2. *Let u be the strong solution of* (14) *corresponding to the initial condition $u(0)=u_0 \in X_2$. Denote by v the strong solution of the equation*

$$v_{z\tau} = \bar{\rho}\big(\phi(v_\tau)\big)_\tau + \bar{\beta} v_{\tau\tau\tau} + \bar{\gamma} v_\tau + \Delta v \tag{15}$$

satisfying the initial condition $v(0)=u_0$. Then

$$\sup_z \|u-v\|^2 \leqslant M\varepsilon^{q-1/2}\big[\|u_{0\tau\tau}\|^2 + \|\Delta\tilde{u}_0\|^2\big] \tag{16}$$

5. Concluding remarks

The obtained global existence and uniqueness results for Eq. (2) for the case of nonlinearity f with bounded derivative show that this formulation may have some mathematical advantages in comparison with the classical KZK equation with the existence problems (see [13,14]). Although for small variations of the acoustic pressure (which is one of the assumptions of the physical model used to derive the equation) these two models coincide.

The introduction of the varying coefficients allows us to model the acoustic beams propagation in a stratified heterogeneous media (for example, in the atmosphere), while the asymptotic analysis by means of the homogenization techniques simplifies the problem reducing it to the case of constant coefficients, when in some situations an analytical solution is possible.

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