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Real and p-adic Picard–Vessiot fields

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Abstract We consider differential modules over real and p-adic differential fields
K such that its field of constants k is real closed (resp., p-adically closed). Using
P. Deligne's work on Tannakian categories and a result of J.-P. Serre on Galois coho-
mology, a purely algebraic proof of the existence and unicity of real (resp., p-adic)
Picard–Vessiot fields is obtained. The inverse problem for real forms of a semi-simple
group is treated. Some examples illustrate the relations between differential modules,
Picard–Vessiot fields and real forms of a linear algebraic group.

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1 Introduction

We are grateful to J.-P. Serre for providing us with a proof of the following statement on Galois cohomology.

Theorem 1 *Let* $k \subset K$ *denote fields of characteristic zero such that*:
(i) *For every smooth variety* V *of finite type over* k, *one has that* $V(K) \neq \emptyset$ *implies* $V(k) \neq \emptyset$. (ii) *The natural map* $Gal(\overline{K}/K) \to Gal(\overline{k}/k)$ *is bijective.*
    Let G *be any linear algebraic group over* k, *then the natural map between the cohomology sets* $H^1(k, G(\overline{k})) \to H^1(K, G(\overline{K}))$ *is bijective.*

We will apply this result for the case that k, K are both real closed fields or both p-adically closed fields. For notational convenience a field F is called (*formally*) *p-adic* if there is a valuation ring $O \subset F$ (with field of fractions F) such that pO is the maximal ideal of O and $O/pO = \mathbb{F}_p$. The field F is called *p-adically closed* if moreover no proper algebraic extension of F is again a p-adic field. What we call "p-adic" is called "p-adic of rank one" in [8]. The case of higher rank p-adic fields can be treated in the same way.

We note that this theorem has also as a consequence that the classification of semi-simple Lie algebras over a real closed field (or a p-adically closed field) does not depend on the choice of that field.

In the sequel we will often write statements and proofs only for the real case and mention that the p-adic case is similar.

K denotes a real differential field with field of constants k. We will always suppose that $k \neq K$ and that k is a real closed field. Let M denote a differential module over K of dimension d, represented by a matrix differential equation $y' = Ay$ where A is a $d \times d$-matrix with entries in K. A *Picard–Vessiot field* L for M/K is a field extension of K such that:

(a) L is equipped with a differentiation extending the one of K,
(b) M has a full space of solutions over L, i.e., there exists an invertible $d \times d$-matrix F (called a fundamental matrix) with entries in L satisfying $F' = AF$,
(c) L is (as a field) generated over K by the entries of F,
(d) the field of constants of L is again k.

A *real Picard–Vessiot field* L for M/K is a Picard–Vessiot field which is also a real field. For a p-adic differential field K we always suppose that its field of constants is p-adically closed. The definition of a *p-adic Picard–Vessiot field* is similar. The main result of this paper is:

Theorem 2 $K \supset k$ *as above. Let* M/K *be a differential module.*

1. Existence. *There exists a real* (*resp., p-adic*) *Picard–Vessiot extension for* M/K.
2. Unicity for the real case. *Let* L_1, L_2 *denote two real Picard–Vessiot extensions for* M/K. *Suppose that* L_1 *and* L_2 *have total orderings which induce the same total ordering on* K. *Then there exists a* K-*linear isomorphism* $\phi : L_1 \to L_2$ *of differential fields.*
3. Unicity for the p-adic case. *Let* L_1, L_2 *denote two p-adic Picard–Vessiot extensions for* M/K. *Suppose that* L_1 *and* L_2 *have p-adic closures* L_1^+ *and* L_2^+ *such that the*

p-adic valuations of L_1^+ and L_2^+ induce the same p-adic valuation on K and such that $K \cap (L_1^+)^n = K \cap (L_2^+)^n$ for every integer $n \geq 2$ (where $F^n := \{f^n | f \in F\}$). Then there exists a K-linear isomorphism $\phi : L_1 \to L_2$ of differential fields.

Remark 1   1. It seems to be folklore that in the case the field of constants is $\mathbb{R}$, a Picard–Vessiot field (real or not) need not exist (compare [2, Remark 2.2]). This is due to a mistaken interpretation of an example of Seidenberg [11].

2. Consider part 2 of Theorem 2. Suppose that an isomorphism $\phi : L_1 \to L_2$ exists. Choose a total ordering of L_1 and define the total ordering of L_2 to be induced by ϕ. Then L_1 and L_2 induce the same total ordering on K. Therefore the condition in part 2 of Theorem 2 is necessary.

    If K happens to be real closed, then the assumption in part 2 of Theorem 2 is superfluous since K has a unique total ordering. On the other hand, consider the example $K = k(z)$ with differentiation $' = \frac{d}{dz}$ and the equation $y' = \frac{1}{2z}y$. Let $L_1 = K(t_1)$ with $t_1^2 = z$ and $L_2 = K(t_2)$ with $t_2^2 = -z$. Both fields are real Picard–Vessiot fields for this equation. They are not isomorphic as (differential) field extensions of K. We observe that z is positive for any total ordering of L_1 and z is negative for any total ordering of L_2.

3. Consider part 3 of Theorem 2. As in part 2, the condition is necessary. It is superfluous if K is p-adically closed. Consider the equation $y' = \frac{1}{2z}y$ over the differential field $\mathbb{Q}_p(z)$ with $z' = 1$. Then $L_j = K(t_j)$, $j = 1, 2, 3, 4$ and $t_1^2 = z$, $t_2^2 = pz$, $t_3^2 = az$, $t_4^2 = paz$, where the image of $a \in \mathbb{Z}_p^*$ in $\mathbb{F}_p^*$ is not a square, are non isomorphic p-adic Picard–Vessiot fields. Let L_j^+ denote a p-adic closure of L_j. It is possible to choose the L_j^+ such that the p-adic valuations induce the same p-adic valuation on K. However the sets $K \cap (L_j^+)^2$ are clearly distinct. $\square$

The proof of Theorem 2 uses Tannakian categories as presented in [4] and P. Deligne's fundamental paper [3]. We adopt much of the notation of [3]. Let $\langle M \rangle_\otimes$ denote the Tannakian category generated by the differential module M. The forgetful functor $\rho : \langle M \rangle_\otimes \to vect(K)$ associates to a differential module $N \in \langle M \rangle_\otimes$ the finite dimensional K-vector space N.

Lemma 1 *There exists a fibre functor $\omega : \langle M \rangle_\otimes \to vect(k)$.*

Proof We follow the reasoning of [3]. Let $\rho : \langle M \rangle_\otimes \to vect(K)$ denote the forgetful functor. Put $G = \underline{Aut}_k^\otimes(\rho)$ and let $Repr_K(G)$ denote the category of the representations of G on finite dimensional K-vector spaces. According to Théorème 1.12, ρ induces an equivalence of categories between $\langle M \rangle_\otimes$ and $Repr_K(G)$. It suffices to produce a fibre functor $Repr_K(G) \to vect(k)$.

According to the proof of [3, Corollaire 6.20], the field K contains a finitely generated k-subalgebra R such that there is a linear algebraic group G_0 over R and an isomorphism $K \times_R G_0 \to G$. Moreover, this isomorphism induces an equivalence between $Repr_R(G_0)$, i.e., the category of the representations of G_0 on finitely generated projective R-modules, and the category $Repr_K(G)$. The category $Repr_R(G_0)$ has a fibre functor $\tau : Repr_R(G_0) \to vect(R/\underline{m})$ where $\underline{m}$ is any maximal ideal of R.

Indeed, one composes the forgetful functor from $Repr_R(G_0)$ to the category $Proj_R$ of the finitely generated projective R-modules with the functor $Proj_R \to vect(R/\underline{m})$ given by $N \to R/\underline{m} \otimes N$, where $\underline{m}$ is a maximal ideal of R.

In our special case K is a real field (resp., p-adic field) and therefore R is a real (resp., p-adic) algebra, finitely generated over a real closed (resp., p-adically closed) field k. There exists $\underline{m}$ with $R/\underline{m} = k$ (see [6,8]). □

Now we recall some essential key results of [3, §9]. The functor $\underline{Aut}_k^\otimes(\omega)$ is represented by a linear algebraic group G over k. By Proposition 9.3, the functor $\underline{Isom}_K^\otimes(K \otimes \omega, \rho)$ is represented by a torsor P over $G_K := K \times_k G$. This torsor is affine, irreducible and its coordinate ring $O(P)$ has a natural differentiation extending the differentiation of K. Moreover, the field of fractions $K(P)$ of $O(P)$ is a Picard–Vessiot field for M/K and G identifies with the group of the K-linear differential automorphisms of $K(P)$.

On the other hand, let L be a Picard–Vessiot field for M/K. Define the fibre functor $\omega_L : \langle M \rangle_\otimes \to vect(k)$ by $\omega_L(N) = \ker(\partial : L \otimes_K N \to L \otimes_K N)$. Then ω_L produces a Picard–Vessiot field L' which is isomorphic to L as differential field extension of K. The conclusion is:

Proposition 1 [3, §9] *The above constructions yield a bijection between the (isomorphy classes of) fibre functors $\omega : \langle M \rangle_\otimes \to vect(k)$ and the (isomorphy classes of) Picard–Vessiot fields L for M/K.*

The following result will also be useful.

Proposition 2 [4, Theorem 3.2] *Let $\omega : \langle M \rangle_\otimes \to vect(k)$ be a fibre functor and $G = \underline{Aut}_k^\otimes(\omega)$.*

(a) *For any field $F \supset k$ and any fibre functor $\eta : \langle M \rangle_\otimes \to vect(F)$, the functor $\underline{Isom}_F^\otimes(F \otimes \omega, \eta)$ is representable by a torsor over $G_F = F \times_k G$.*

(b) *The map $\eta \mapsto \underline{Isom}_F^\otimes(F \otimes \omega, \eta)$ is a bijection between the (isomorphy classes of) fibre functors $\eta : \langle M \rangle_\otimes \to vect(F)$ and the (isomorphy classes of) G_F-torsors.*

The final ingredient in the proof of Theorem 2 is:

Theorem 3 *Suppose that K is real closed* (*resp., p-adically closed*). *Let L be a Picard–Vessiot field for M/K. Then L is a real field* (*resp., a p-adic field*) *if and only if the torsor $\underline{Isom}_K^\otimes(K \otimes \omega_L, \rho)$ is trivial.*

2 The proof of Theorem 2

2.1 Reduction to K is real (resp., p-adically) closed

For notational convenience, a differential module M/K is represented by a linear differential equation $\mathcal{L}(y) := y^{(d)} + a_{d-1}y^{(d-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0$. A Picard–Vessiot field L for $\mathcal{L}$ has the properties: k is the field of constants of L, the *solution space* $V = \{v \in L | \mathcal{L}(v) = 0\}$ is a k-linear space of dimension d and L is generated over the field K by V and all the derivatives of the elements in V. One writes $L = K\langle V \rangle$ for this last property.

Lemma 2 *Let $\tilde{K} \supset K$ be an extension of real (resp., p-adic) differential fields such that the field of constants of $\tilde{K}$ is k. Suppose that $\tilde{K} \otimes M$ has a real (resp., p-adic) Picard–Vessiot field $\tilde{L}$, then M has a real (resp., p-adic) Picard–Vessiot field.*

Proof Let $V \subset \tilde{L}$ denote the solution space of $\tilde{K} \otimes M$. Clearly, the field $L = K\langle V \rangle \subset \tilde{L}$ is a real (resp., p-adic) Picard–Vessiot field for M. □

Lemma 3 *Let L_1, L_2 be two real Picard–Vessiot fields for M over the real differential field K. Suppose that L_1 and L_2 have total orderings extending a total ordering τ on K. Let $K^r \supset K$ be the real closure of K inducing the total ordering τ. Then:*

The fields L_1, L_2 induce Picard–Vessiot fields $\tilde{L}_1$, $\tilde{L}_2$ for $K^r \otimes M$ over K^r. The fields $\tilde{L}_1$, $\tilde{L}_2$ are isomorphic as differential field extensions of K^r if and only if L_1 and L_2 are isomorphic as differential field extensions of K.

Proof Let, for $j = 1, 2$, τ_j be a total ordering on L_j inducing τ on K and let L_j^r be the real closure of L_j which induces the ordering τ_j. The algebraic closure K_j^r of K in L_j^r is real closed. Since τ_j induces τ, there exists a K-linear isomorphism $\phi_j : K^r \to K_j$. This isomorphism is unique since the only K-linear automorphism of K^r is the identity. We will identify K_j with K^r.

Let $V_j \subset L_j$ denote the solution space of M. Then, for $j = 1, 2$, the field $\tilde{L}_j := K^r \langle V_j \rangle \subset L_j^r$ is a real Picard–Vessiot field for $K^r \otimes M$.

Assume the existence of a K^r-linear differential isomorphism $\psi : K^r \langle V_1 \rangle \to K^r \langle V_2 \rangle$. Clearly $\psi(V_1) = V_2$ and ψ induces therefore a K-linear differential isomorphism $L_1 = K\langle V_1 \rangle \to L_2 = K\langle V_2 \rangle$.

On the other hand, an isomorphism $\phi : L_1 \to L_2$ (of differential field extensions of K) extends to an isomorphism $\tilde{\phi} : L_1^r \to L_2^r$. Clearly $\tilde{\phi}$ maps $\tilde{L}_1$ to $\tilde{L}_2$. □

The p-adic version of Lemma 3 is treated as follows.

Let L_1, L_2 be two p-adic Picard–Vessiot fields for the differential module M over the p-adic differential field K. Let L_1^+ and L_2^+ denote p-adic closures of L_1 and L_2 satisfying the condition of part 3 of Theorem 2.

For $j = 1, 2$, the algebraic closure K_j of K in L_j^+ is a p-adic closure of K, according to [8, Theorem 3.4]. Further $K \cap K_1^n = K \cap K_2^n$ for every integer $n \geq 2$ since $K \cap (L_1^+)^n = K \cap (L_2^+)^n$ holds. By [8, Corollary 3.11], there is a K-linear isomorphism of the p-adic fields $K_1 \to K_2$. Now we identify K_1 and K_2 and denote this field by $\tilde{K}$. Then:

The fields L_1, L_2 induce Picard–Vessiot fields $\tilde{L}_1$, $\tilde{L}_2$ for $\tilde{K} \otimes M$ over $\tilde{K}$. As in the proof of Lemma 3, one shows that $\tilde{L}_1$ and $\tilde{L}_2$ are isomorphic as differential field extensions of $\tilde{K}$ if and only if L_1 and L_2 are isomorphic as differential field extensions of K.

We conclude that it suffices to prove Theorem 2 for the case that K is real (resp., p-adically) closed.

2.2 Proof of the unicity

Theorem 4 (=Theorem 3) *Suppose that K is real (resp., p-adically) closed. Let L be a Picard–Vessiot field for a differential module M/K. Then L is a real (resp., p-adic) field if and only if the torsor $\underline{Isom}_K^\otimes(K \otimes \omega_L, \rho)$ is trivial.*

Proof We write the proof for the real case. The p-adic case is similar.

The coordinate ring of the affine torsor $\underline{Isom}_K^\otimes(K \otimes \omega_L, \rho)$ is denoted by R. We recall that L is the field of fractions of R.

If L is a real Picard–Vessiot field, then $R \subset L$ is a finitely generated real K-algebra. From the real Nullstellensatz and the assumption that K is real closed it follows that there exists a K-linear homomorphism $\phi : R \to K$ with $\phi(1) = 1$. The torsor $Spec(R)$ has a K-valued point and is therefore trivial.

If the torsor $Spec(R)$ is trivial, then the affine variety $Spec(R)$ has a K-valued point. It follows that the Picard–Vessiot field L, which is the function field of this variety, is real (see for instance [8, §1, Theorem 2]). □

In proving the unicity, we restrict to the real case. According to Sect. 2.1 we may assume that K is real closed. Let L_1, L_2 denote two real Picard–Vessiot fields for a differential module M/K. We will prove that L_1 and L_2 are isomorphic as differential extension fields of K.

Proof Write $\omega_j = \omega_{L_j} : \langle M \rangle_\otimes \to vect(k)$ for the corresponding fibre functors. Put $G = \underline{Aut}_k^\otimes(\omega_1)$. Then $\underline{Isom}_k^\otimes(\omega_1, \omega_2)$ is a G-torsor over k corresponding to an element $\xi \in H^1(k, G(\bar{k}))$.

The G_K-torsor $\underline{Isom}_K^\otimes(K \otimes \omega_1, K \otimes \omega_2)$ corresponds to an element $\eta \in H^1(K, G(\overline{K}))$. This element is the image of ξ under the natural map from $H^1(k, G(\bar{k}))$ to $H^1(K, G(\overline{K}))$, induced by the inclusion $G(\bar{k}) \subset G(\overline{K})$ and the observation that $Gal(\overline{K}/K) = Gal(\bar{k}/k)$). Since L_j is real, the torsor $\underline{Isom}_K^\otimes(K \otimes \omega_j, \rho)$ is trivial for $j = 1, 2$, by Theorem 3. Thus there exists isomorphisms $\alpha_j : K \otimes \omega_j \to \rho$ for $j = 1, 2$. The isomorphism $\alpha_2^{-1} \circ \alpha_1 : K \otimes \omega_1 \to K \otimes \omega_2$ implies that η is trivial.

Let ξ be represented by the 1-cocycle c with values in $G(\bar{k})$. Since its image in $H^1(K, G(\overline{K}))$ is trivial, there is an element $h \in G(\overline{K})$ such that $c(\alpha) = h^{-1}\alpha(h)$ for all $\alpha \in Gal(\overline{K}/K) = Gal(\bar{k}/k)$.

There exists a finitely generated k-algebra $B \subset K$ with $h \in G(\bar{k}B)$. Since B is real and k is real closed, there exists a k-linear homomorphism $\phi : B \to k$ with $\phi(1) = 1$ [6,8]. Further ϕ extends to a $\bar{k}$-linear homomorphism $\bar{k}B \to \bar{k}$, commuting with the actions of $Gal(\overline{K}/K) = Gal(\bar{k}/k)$. Applying ϕ to the identity $c(\alpha) = h^{-1}\alpha(h)$ one obtains $c(\alpha) = \phi(h)^{-1}\alpha(\phi(h))$. Thus c is a trivial 1-cocycle and there is an isomorphism $\omega_1 \to \omega_2$. Hence L_1 and L_2 are isomorphic as differential field extensions of K. □

2.3 Proof of Theorem 1

Let $k \subset K$ satisfy (i) and (ii). The final part of Sect. 2.2 proves the injectivity of $H^1(k, G(\bar{k})) \to H^1(K, G(\overline{K}))$. For notational convenience we write this as

$H^1(k, G) \rightarrow H^1(K, G)$. *Here we reproduce J.-P. Serre's proof of the surjectivity of this map, which he communicated to us by an email message on 04-07-2013.*

Proof 1. Let U be the unipotent radical of G. The map $H^1(k, G) \rightarrow H^1(k, G/U)$ is bijective. One can find the easy proof of this in lemma 7.20 of [5]. Since $H^1(K, G) \rightarrow H^1(K, G/U)$ is also bijective, we may divide by U, i.e. we may assume that the neutral component G^o of G is reductive.

2. Consider a commutative group C over k. It is given that the natural map $Gal(\overline{K}/K) \rightarrow Gal(\overline{k}/k)$ is a bijection. Hence there are natural maps $H^n(k, C) \rightarrow H^n(K, C)$ for all n. For every $n > 0$ (we only need $n = 1, 2$) these maps are bijective. Indeed, the commutative group $C(\overline{K})/C(\overline{k})$ is torsion free and divisible and so it has trivial Galois cohomology.

3. Let T be a maximal torus of G, and let N be its normalizer. By a result of T.A. Springer (lemma 6 of III.4.3 [10]), the map $H^1(K, N) \rightarrow H^1(K, G)$ is surjective. Hence it will be enough to prove surjectivity for N.

4. After replacing G by N, we have an exact sequence $1 \rightarrow C \rightarrow G \rightarrow F \rightarrow 1$, where C is a torus and F a finite group. This gives us a diagram for the H^1:

$$
\begin{array}{ccccc}
1 \rightarrow & H^1(k, C) & \rightarrow & H^1(k, G) & \rightarrow & H^1(k, F) \\
& \downarrow & & \downarrow & & \downarrow \\
1 \rightarrow & H^1(K, C) & \rightarrow & H^1(K, G) & \rightarrow & H^1(K, F)
\end{array}
$$

The map $H^1(k, F) \rightarrow H^1(K, F)$ is bijective since F is finite and $Gal(\overline{K}/K) = Gal(\overline{k}/k)$ and we identify the two sets. Let x be an element of $H^1(K, G)$ and let y be its image in $H^1(K, F)$. Thus we view y as an element of $H^1(k, F)$.

Claim *The element y belongs to the image of $H^1(k, G) \rightarrow H^1(k, F)$.*

Proof of the Claim We use prop. 41 of I.5.6 [10]. It tells us that y belongs to that image if and only if its coboundary $\Delta(k, y)$ is 0; this coboundary belongs to $H^2(k, C_y)$, where C_y is the Galois twist of C defined by a cocycle of the class y. This argument also applies over K, so that we have a class $\Delta(K, y)$ in $H^2(K, C_y)$, and it is clear that $\Delta(K, y)$ is the image of $\Delta(k, y)$ under the map $H^2(k, C_y) \rightarrow H^2(K, C_y)$. But, since y comes from x, we have $\Delta(K, y) = 0$, hence $\Delta(k, y) = 0$, by 2 above, applied to the commutative group C_y.

5. End of proof. By a little diagram chasing, the Claim and 2, one shows that x belongs to the image of $H^1(k, G)$.

□

2.4 Proof of the existence

We present the proof for the real case. The p-adic case is similar. One considers a differential module M over a real closed differential field K. We fix a fibre functor $\omega_0 : \langle M \rangle_\otimes \rightarrow vect(k)$ and write $G_0 := Aut^\otimes(\omega_0)$. Further $G_\rho := Aut^\otimes(\rho)$, where $\rho : \langle M \rangle_\otimes \rightarrow vect(K)$ is the forgetful functor.

We recall from Proposition 1, that $H^1(k, G_0(\overline{k}))$ can be identified with the set of the fibre functors $\omega : \langle M \rangle_\otimes \rightarrow vect(k)$ and that $H^1(K, G_\rho(\overline{K}))$ can be identified with the set of the right G_ρ-torsors.

Thus the map $\omega \mapsto \underline{Isom}(K \otimes \omega, \rho)$ from fibre functors to right G_ρ-torsors can be seen as a map $H^1(k, G_0(\overline{k})) \to H^1(K, G_\rho(\overline{K}))$. We want to show that the trivial element of $H^1(K, G_\rho(\overline{K}))$ is in the image, because this means that some fibre functor ω yields a trivial torsor, or translated, L_ω is a real Picard–Vessiot field. The above map factors as

$$H^1(k, G_0(\overline{k})) \overset{natG_0}{\to} H^1(K, G_0(\overline{K})) \overset{composition}{\to} H^1(K, G_\rho(\overline{K})).$$

Here $natG_0$ denotes the natural map and the map "*composition*" is defined as follows. An element in $H^1(K, G_0(\overline{K}))$ is a right $K \times_k G_0$-torsor. One can compose with $\underline{Isom}^\otimes(K \otimes \omega_0, \rho)$ which is a left $K \otimes_k G_0$-torsor and a right G_ρ-torsor. The result is a right G_ρ-torsor and thus an element in $H^1(K, G_\rho(\overline{K}))$. The map "*composition*" is bijective since $\underline{Isom}^\otimes(K \otimes \omega_0, \rho)$ has "inverse" $\underline{Isom}^\otimes(\rho, K \otimes \omega_0)$. According to Theorem 1, the map $natG_0$ is bijective. This finishes the proof of the existence.

3 Comments and examples

The proof of the unicity and existence of real (resp., p-adic) Picard–Vessiot fields uses almost exclusively properties of Tannakian categories and Galois cohomology of linear algebraic groups. This implies that the proof remains valid for other types of equations, such as:

(a) linear partial differential equations, like $\frac{\partial}{\partial z_j} Y = A_j Y$ for $j = 1, \ldots, n$,
(b) linear ordinary difference equations, like $Y(z + 1) = AY(z)$,
(c) linear q-difference equations, like $Y(qz) = AY(z)$, with $q \in \mathbb{R}^*$ (resp., $q \in \mathbb{Q}_p^*$).

Observations 1 *From Picard–Vessiot fields to differential Galois groups.*

Let K be a real closed differential field with field of constants k, and M/K a differential module. Let $\omega : \langle M \rangle_\otimes \to vect(k)$ be the unique fibre functor corresponding to the unique real Picard–Vessiot field L. Further G denotes the group of the differential automorphisms of L/K. Let H be the differential Galois group of $K(i) \otimes M$ over $K(i)$. We recall that G is a form of H over the field $k(i)$. Using the identification $k(i) \times_k G = H$, one obtains on H and on $Aut(H)$ a structure of algebraic group over k. Let $\{1, \sigma\}$ be the Galois group of $k(i)/k$. Then $H^1(\{1, \sigma\}, Aut(H))$ has a natural bijection to the set of forms of H over k.

Let $\eta : \langle M \rangle_\otimes \to vect(k)$ be another fibre functor. Then η is mapped, according to Proposition 2, to an element $\xi(\eta) \in H^1(\{1, \sigma\}, G(k(i)))$ (and this induces a bijection between the set of the classes of η's and this cohomology set). A 1-cocycle c for the group $\{1, \sigma\}$ has the form $c(1) = 1$, $c(\sigma) = a$ and a should satisfy $a \cdot \sigma(a) = 1$ (and is thus determined by a).

A 1-cocycle for $\xi(\eta)$ can be made as follows. The fibre functor η corresponds to a Picard–Vessiot field L_η. Both $L(i)$ and $L_\eta(i)$ are Picard–Vessiot fields for $K(i) \otimes M$ over $K(i)$. Thus there exists a $K(i)$-linear differential isomorphism $\phi : L(i) \to L_\eta(i)$. On the field $L(i)$ we write τ for the conjugation given by $\tau(i) = -i$ and τ is the identity on L. The similar conjugation on $L_\eta(i)$ is denoted by τ_η. Now $\tau_\eta \circ \phi \circ \tau : L(i) \to$

$L_\eta(i)$ is another $K(i)$-linear differential isomorphism. A 1-cocycle c for $\xi(\eta)$ is now $c(\sigma) = \phi^{-1} \circ \tau_\eta \circ \phi \circ \tau$.

Let G_η denote the group of the K-linear differential automorphisms of L_η. The group G_η is a form of G and produces an element in $H^1(\{1, \sigma\}, Aut(H))$ with $H = k(i) \times G$. We want to compute a 1-cocycle C for this element. Define the isomorphism $\psi : k(i) \times G \to k(i) \times G_\eta$ of algebraic groups over $k(i)$, by $\psi(g) = \phi \circ g \circ \phi^{-1}$. Define τ_G, the 'conjugation' on $k(i) \times G$, by the formula $\tau_G(g) = \tau \circ g \circ \tau$ for the elements $g \in G(k(i))$. Let τ_{G_η} be the similar conjugation on $k(i) \times G_\eta$. Now $\tau_{G_\eta} \circ \psi \circ \tau_G : k(i) \times G \to k(i) \times G_\eta$ is another isomorphism between the algebraic groups over $k(i)$. The 1-cocycle C is given by $C(\sigma) = \psi^{-1} \circ \tau_{G_\eta} \circ \psi \circ \tau_G$. One observes that $C(\sigma)(g) = c(\sigma)gc(\sigma)^{-1}$.

The map, which associates to $h \in G(k(i))$, the automorphism $g \mapsto hgh^{-1}$ of G, induces a map

$$H^1(\{1, \sigma\}, G(k(i))) \to H^1(\{1, \sigma\}, G/Z(G)(k(i))) \to H^1(\{1, \sigma\}, Aut(H)),$$

denoted by $\xi(\eta) \mapsto \tilde{\xi}(\eta)$. The forms corresponding to elements in the image of $H^1(\{1, \sigma\}, G/Z(G)(k(i))) \to H^1(\{1, \sigma\}, Aut(H))$ are called 'inner forms of G'. By §1, η induces a Picard–Vessiot field and a form $G(\eta)$ of H. Above we have verified (see [1] for a similar computation) that $G(\eta)$ is the inner form of G corresponding to the element $\tilde{\xi}(\eta)$. For the delicate theory of forms we refer to the informal manuscript [1] and the standard text [12]. In Examples 1 and in Proposition 3 we will investigate the subtle relation between Picard–Vessiot fields and (real) forms of the (complex) differential Galois group.

Examples 1 Picard–Vessiot fields and their groups of differential automorphisms. We continue with the notation and assumptions of Observations 1.

1. Let $M/K, \omega, L, G$ be such that $G = SL(n)_k$. Since $H^1(\{1, \sigma\}, SL(n)(k(i)))$ is trivial, L is the unique Picard–Vessiot field and is a real field (because a real Picard–Vessiot field exists).

   The group $SL(n)$ has non trivial forms. For instance, $SU(2)$ is an inner form of $SL(2)_\mathbb{R}$. There are examples, according to Proposition 3 below, of differential modules M/K having a real Picard–Vessiot field L with group of differential automorphisms of L/K equal to $SU(2)$.

   From [12], 12.3.7 and 12.3.9 one concludes that $H^1(\{1, \sigma\}, SU(2)(\mathbb{C}))$ is trivial. Again L is the only Picard–Vessiot field.

2. If G is the symplectic group $Sp(2n)_k$, then $H^1(\{1, \sigma\}, G(k(i)))$ is trivial. Therefore there is only one Picard–Vessiot field L and this is a real field.

3. Consider a k-form G of $SO(n)_k$ with odd $n \geq 3$. The center Z of G consists of the scalar matrices of order n, thus Z is the group $\mu_{n,k}$ of the nth roots of unity. Since n is odd, one has $Z(k) = \{1\}$. Further, again since n is odd, the automorphisms of $H = G_{k(i)}$ are inner and $Aut(H)(k(i)) = G/Z(k(i))$. We claim the following.

The natural map $H^1(\{1, \sigma\}, G(k(i))) \to H^1(\{1, \sigma\}, G/Z(k(i)))$ is a bijection.

Proof A 1-cocycle c for $G/Z(k(i))$ is given by $c(1) = 1$ and $c(\sigma) = a \in G/Z(k(i))$ with $a\sigma(a) = 1$. Choose an $A \in G(k(i))$ which maps to a. Thus $A\sigma(A) \in Z(k(i))$

and A commutes with σA. Further $\sigma(A\sigma(A)) = \sigma(A)A = A\sigma(A)$ and thus $A\sigma(A) \in \mu_n(k) = \{1\}$. Therefore C defined by $C(1) = 1$, $C(\sigma) = A$ is a 1-cocycle for $G(k(i))$ and maps to c. Hence the map is surjective.

Consider for $j = 1, 2$, the 1-cocycle C_j for G given by $C_j(\sigma) = A_j$. Suppose that the images of C_j as 1-cocycles for $G/Z(k(i))$ are equivalent. Then there exists $B \in G(k(i))$ such that $B^{-1}A_1\sigma(B) = xA_2$ for some element $x \in Z(k(i))$. We may replace B by yB with $y \in Z(k(i))$. Then x is changed into $xy^{-1}\sigma(y)$. And the latter is equal to 1 for a suitable y. This proves the injectivity of the map. $\qquad\square$

We conclude from the above result that there exists a (unique up to isomorphism) fibre functor $\eta : \langle M \rangle_\otimes \to vect(k)$ (or, equivalently, a Picard–Vessiot field) for every form of $H = SO(n)_{k(i)}$ over k. Moreover, only one of these fibre functors corresponds to a *real* Picard–Vessiot field.

Let $\omega : \langle M \rangle_\otimes \to vect(k)$ denote the fibre functor corresponding to a *real* Picard–Vessiot field L_ω and G_ω the group of the differential automorphisms of L_ω/K. *We want to identify this form G_ω of $H := \mathrm{SO}(n)_{k(i)}$.*

Since the differential Galois group of $K(i) \otimes M$ is $\mathrm{SO}(n)_{k(i)}$, there exists an element $F \in sym^2(K(i) \otimes M^*)$ with $\partial F = 0$. Further F is unique up to multiplication by a scalar and F is a non degenerate bilinear symmetric form. The non trivial automorphism σ of $K(i)/K$ and of $k(i)/k$ acts in an obvious way on $K(i) \otimes M$ and on constructions by linear algebra of $K(i) \otimes M$. Now $\sigma(F)$ has the same properties as F and thus $\sigma(F) = cF$ for some $c \in K(i)^*$. After changing F into aF for a suitable $a \in K(i)^*$, we may suppose that $\sigma(F) = F$. Then F belongs to $sym^2(M^*)$ and is a non degenerate form of degree n over the field K. Further F is determined by its signature because K is real closed. Moreover KF is the unique 1-dimensional submodule of $sym^2(M^*)$. We claim the following:

G_ω is the special orthogonal group over k corresponding to a form f over k which has the same signature as F.

Proof Let $V = \omega(M)$. The group G_ω is the special orthogonal group of some non degenerate bilinear symmetric form $f \in sym^2(V^*)$. Since L_ω is real, there exists an isomorphism $m : K \otimes_k \omega \to \rho$ of functors. Applying m to the modules M and $sym^2(M^*)$ one finds an isomorphism $m_1 : K \otimes_k V \to M$ of K-vector spaces which induces an isomorphism of K-vector spaces $m_2 : K \otimes_k sym^2(V^*) \to sym^2(M^*)$. The latter maps the subobject $K \otimes_k f$ to KF by the uniqueness of KF. One concludes that the forms f and F have the same signature. $\qquad\square$

Proposition 3 *Suppose that K is real closed. Given is a connected semi-simple group H over $k(i)$ and a form G of H over k. Then there exists a differential module M over K and a real Picard–Vessiot field for M/K such that the group of the differential automorphisms of L/K is G.*

Proof Let G be given as a subgroup of some $\mathrm{GL}_{n,k}$, defined by a radical ideal I. Then $k[G] = k[\{X_{k,l}\}_{k,l=1}^n, \frac{1}{\det}]/I$. The tangent space of G at $1 \in G$ can be identified with the k-linear derivations D of this algebra, commuting with the action of G. These derivations D have the form $(DX_{k,l}) = B \cdot (X_{k,l})$ for some matrix $B \in Lie(G)(k)$ (where $Lie(G) \subset \mathrm{Matr}(n, k)$ is the Lie algebra of G).

The same holds for $K[G] = K \otimes_k k[\{X_{k,l}\}_{k,l=1}^n, \frac{1}{\det}]/I$. Any K-linear derivation D on the algebra, commuting with the action of G, has the form $(DX_{k,l}) = A \cdot (X_{k,l})$ with $A \in Lie(G)(K)$. We choose A as general as possible.

The differential module M/K is defined by the matrix equation $y' = Ay$. It follows from [9, Proposition 1.31] that the differential Galois group of $K(i) \otimes M$ is contained in $H = G_{k(i)}$. Now one has to choose A such that the differential Galois group (which is connected because $K(i)$ is algebraically closed) is not a proper subgroup of H. Since H is semi-simple, there exists a Chevalley module for H. Using this Chevalley module one can produce a general choice of A such that the differential Galois group of $y' = Ay$ over $K(i)$ is in fact $G_{k(i)}$ (compare [9, §11.7] for the details which remain valid in the present situation).

The usual way to produce a Picard–Vessiot ring for the equation $y' = Ay$ is to consider the differential algebra $R_0 := K[\{X_{k,l}\}_{k,l=1}^n, \frac{1}{\det}]$, with differentiation defined by $(X_{k,l}') = A \cdot (X_{k,l})$, and to produce a maximal differential ideal in R_0. Since $A \in Lie(G)(K) \subset Lie(H)(K(i))$, the ideal $J \subset R_0[i]$, generated by I is a differential ideal. It is in fact a maximal differential ideal of $R_0[i]$, since the differential Galois group is precisely H. Then $J \cap R_0 = IR_0$ is a maximal differential ideal of R_0 and $K[G] = R = R_0/IR_0$ is a Picard–Vessiot ring for M over K. The field of fractions L of R is real because the G-torsor $Spec(K[G])$ is trivial. □

It seems that, imitating the proofs in [7], one can show that Proposition 3 remains valid under the weaker conditions: K is a real differential field and H is connected.

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