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# Fully Transitive Homogeneously Separable Abelian Groups

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KEY WORDS: Abelian group, torsion-free group, fully transitive group, characteristic, homogeneously separable group, transitive group, $\chi$-group, reduced group.

A torsion-free Abelian group $G$ is said to be *homogeneously separable* if there exists a family $\mathbf{C}$ of homogeneous direct summands of this group such that each of the finite sets of elements of the group $G$ can be embedded in a direct summand of $G$ which is the direct sum of some groups of the family $\mathbf{C}$. In this case we say that the family $\mathbf{C}$ *defines* the *homogeneous separability* of the group $G$.

Note that, in particular, fully decomposable torsion-free Abelian groups, separable torsion-free Abelian groups, and homogeneously decomposable Abelian groups are homogeneously separable.

In the study of fully characteristic subgroups of torsion-free Abelian groups $G$, an interesting subclass consists of the groups each of whose fully characteristic subgroups has the form $G[v] = \{g \in G \mid \chi(g) \geq v\}$, where $v$ is a characteristic (i.e., a sequence of nonnegative integers and symbols $\infty$). In [1] such reduced torsion-free groups are called *$\chi$-groups*. (Note that in the study of fully characteristic subgroups of torsion-free Abelian groups we can restrict ourselves to reduced groups, because a nonzero subgroup $S$ of a torsion-free Abelian group $A = R \oplus V$, where $R$ and $V$ are the reduced part and the divisible part of the group $A$, respectively, is fully characteristic in $A$ if and only if $S = R_1 \oplus V$, where $R_1$ is a fully characteristic subgroup of the group $R$ [1, Lemma 3.3].) Each $\chi$-group is *fully transitive*, i.e., for any two elements $x$ and $y$ of the group such that $\chi(x) \leq \chi(y)$ ($\chi(x)$, $\chi(y)$ are the characteristics of the elements $x$ and $y$, respectively), there exists an endomorphism $\varphi$ such that $\varphi(x) = y$ [1, p. 63] (in [1], these groups are said to be *transitive*). In general, the converse assertion fails.

In the present note a characterization of fully transitive homogeneously separable groups is obtained; in particular, it is established that any fully transitive homogeneously separable group is a $\chi$-group.

Let $a$ be an element of a torsion-free Abelian group $G$ and let $p$ be a prime. Denote by $h_p(a)$ the $p$-height of an element $a$ and by $\chi(a)$ and $t(a)$ the characteristic and the type of this element, respectively. Further, let $\Pi$ be the set of primes, and let $\pi(G) = \{p \in \Pi \mid pG \neq G\}$. For a homogeneous group $G$, denote by $t(G)$ the *type of the group* $G$, i.e., the type of all its nonzero elements. Let $G$ be a homogeneously decomposable torsion-free Abelian group, i.e., let $G = \bigoplus_{i \in I} A_i$, where $A_i$ are homogeneous groups. On collecting the components $A_i$ of the same type and taking their direct sum, we obtain a canonical decomposition (the smallest homogeneous decomposition, see [2, p. 212 of the Russian translation]) $G = \bigoplus_{t \in T} G_t$, where $G_t$ are homogeneous groups of different types $t$.

**Theorem.** *Let $G$ be a homogeneously separable reduced group, let $\mathbf{C} = \{A_i\}_{i \in I}$ be a family of homogeneous torsion-free Abelian groups defining the homogeneous separability of the group $G$, and let $T$ be the set of types of the groups $A_i$ ($i \in I$). The following conditions are equivalent:*

1) *$G$ is a fully transitive group;*
2) *if $G_1$ is a homogeneously decomposable direct summand of the group $G$ and if $G_1 = \bigoplus_{t \in T_1} G_t$ is its canonical decomposition, then all groups $G_t$ are fully transitive and $\pi(G_{t_1}) \cap \pi(G_{t_2}) = \varnothing$ for $t_1 \neq t_2$ ($t_1, t_2 \in T_1$);*

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3) *each group* $A_i$ $(i \in I)$ *is fully transitive, and* $\pi(A_{i_1}) \cap \pi(A_{i_2}) = \varnothing$ *for* $t(A_{i_1}) \neq t(A_{i_2})$;
4) *the group* $G$ *is homogeneously decomposable, and* $G_t = \{g \in G \mid t(g) = t\} \cup \{0\}$ *is a subgroup of* $G$ *for any* $t \in T$; *moreover,* $G = \bigoplus_{t \in T} G_t$ *is a canonical decomposition of* $G$ *in which each of the groups* $G_t$ *is fully transitive and* $\pi(G_{t_1}) \cap \pi(G_{t_2}) = \varnothing$ *for* $t_1 \neq t_2$ $(t_1, t_2 \in T)$;
5) $G$ *is a* $\chi$*-group.*

**Proof.** The equivalence of 1) and 2) is proved in [1, Corollary 2.15]. Let us show that 1) $\Longrightarrow$ 3). Each of the groups $A_i$ is fully transitive as a direct summand of a fully transitive group [1, Corollary 2.4]. Assume that $t(A_{i_1}) \neq t(A_{i_2})$ $(i_1, i_2 \in I)$. To be definite, we assume that $t(A_{i_1}) \not\geq t(A_{i_2})$. We have $G = A_{i_1} \oplus B = A_{i_2} \oplus C$. Let $a \in A_{i_2}$, $a \neq 0$, and $a = a_1 + b$, where $a_1 \in A_{i_1}$ and $b \in B$. In this case, if $a_1 \neq 0$, then $\chi(a) \leq \chi(a_1)$, which is impossible because $t(A_{i_1}) \not\geq t(A_{i_2})$. Hence $a_1 = 0$ and $A_{i_2} \subset B$. We have $B = A_{i_2} \oplus (C \cap B)$, and therefore $G = A_{i_1} \oplus A_{i_2} \oplus (C \cap B)$; note that $A_{i_1} \oplus A_{i_2}$ is a homogeneously decomposable fully transitive group, and hence $\pi(A_{i_1}) \cap \pi(A_{i_2}) = \varnothing$ [1, Proposition 2.12].

Let us prove that 3) $\Longrightarrow$ 4). We first show that $G_t = \{g \in G \mid t(g) = t\} \cup \{0\}$ is a subgroup of the group $G$ $(t \in T)$. Let $a_1, a_2 \in G_t$, $a_1 \neq 0$, and $a_2 \neq 0$. Since $G$ is a homogeneously separable group and the family of groups $\{A_i\}_{i \in I}$ defines the homogeneous separability of this group, it follows that $a_1$ and $a_2$ can be embedded in a direct summand $A_{k_1} \oplus A_{k_2} \oplus \cdots \oplus A_{k_s}$ of the group $G$ $(k_j \in I$ and $j = 1, \dots, s)$, and we may assume that, for each of the groups $A_{k_j}$, at least one of the elements $a_1$ and $a_2$ has nonzero coordinate with respect to this group. At least one of the summands $A_{k_j}$ $(j = 1, \dots, s)$ has the type $t$, because otherwise it follows from the condition

$$\pi(A_{i_1}) \cap \pi(A_{i_2}) = \varnothing \qquad \text{for} \quad t(A_{i_1}) \neq t(A_{i_2}) \quad (i_1, i_2 \in I)$$

that $t(a_1) \neq t$ and $t(a_2) \neq t$. To be definite, we assume that $t(A_{k_1}) = t$. If for any prime $p$ we have $h_p(a_1) < \infty$, then, since the group $G$ is reduced, it follows that $t(A_{k_j}) = t(A_{k_1})$, $j = 2, \dots, s$.

Let $\mathcal{P}(A_{k_1}) = \{p \mid p$ is a prime and $pA_{k_1} = A_{k_1}\}$. Consider the case in which $\mathcal{P}(A_{k_1}) \neq 0$. Let us show that in this case we again have $t(A_{k_j}) = t(A_{k_1})$, $j = 2, \dots, s$. Assume the contrary: let there exist an index $k_l$ $(l = 2, \dots, s)$ such that $t(A_{k_l}) \neq t(A_{k_1})$. Considering an element $a_i$ $(i = 1, 2)$ that has a nonzero coordinate with respect to the group $A_{k_l}$ we see that there exists a prime $p \in \mathcal{P}(A_{k_1})$ for which $h_p(a_i) < \infty$, which contradicts the fact that $t(a_i) = t(A_{k_1}) = t$.

Thus $A_{k_1} \oplus A_{k_2} \oplus \cdots \oplus A_{k_s}$ is a homogeneous group of the type $t$. Hence, since we have

$$a_1 - a_2 \in A_{k_1} \oplus A_{k_2} \oplus \cdots \oplus A_{k_s},$$

it follows that $t(a_1 - a_2) = t$ and $a_1 - a_2 \in G_t$. Therefore, $G_t = \{g \in G \mid t(g) = t\} \cup \{0\}$ is a subgroup of the group $G$ for each $t \in T$. It is clear that, for each type $t \in T$, we have $G_t \cap \sum_{t' \in T} G_{t'} = 0$. Hence

$$\sum_{t \in T} G_t = \bigoplus_{t \in T} G_t.$$

It remains to show that $G = \bigoplus_{t \in T} G_t$.

Let $g \in G$. The element $g$ can be embedded in a direct summand (of $G$) that is decomposable into a direct sum of some groups from the family $\mathbf{C}$ $(\mathbf{C} = \{A_i\}_{i \in I})$. Each group of the family $\mathbf{C}$ is contained in a group of the form $G_t$, and hence $g \in \bigoplus_{t \in T} G_t$.

The validity of the implications 4) $\Longrightarrow$ 5) and 5) $\Longrightarrow$ 1) follows directly from [1, Corollary 3.16 and Proposition 2.1]. □

Since each torsion-free Abelian group of rank one is a fully transitive group, we obtain the following result.

**Corollary 1.** *Let $G$ be a reduced separable torsion-free Abelian group and let $T$ be the set of types of the direct summands of $G$ of rank one. The following conditions are equivalent:*

1) *$G$ is a fully transitive group;*
2) *if $G_1$ is a fully transitive direct summand of $G$ of finite rank and if $G_1 = \bigoplus_{i=1}^{n} B_i$ is a decomposition of $G_1$ into a direct sum of groups of rank one, then we have $\pi(B_{i_1}) \cap \pi(B_{i_2}) = \varnothing$ for $B_{i_1} \not\cong B_{i_2}$ $(i_1, i_2 = 1, \dots, n)$;*
3) *for any nonisomorphic direct summands $A$ and $B$ of $G$ of rank one we have $\pi(A) \cap \pi(B) = \varnothing$;*
4) *the group $G$ is homogeneously decomposable, the set $G_t = \{g \in G \mid t(g) = t\} \cup \{0\}$ is a subgroup of the group $G$ for each $t \in T$, and $G = \bigoplus_{t \in T} G_t$ is a canonical decomposition of the group $G$ in which each of the groups $G_t$ is fully transitive and $\pi(G_{t_1}) \cap \pi(G_{t_2}) = \varnothing$ for $t_1 \neq t_2$ $(t_1, t_2 \in T)$;*
5) *$G$ is a $\chi$-group.*

Not that the equivalence of conditions 1) and 2) is shown in [1, Corollary 2.16].

Following [3], we say that a torsion-free Abelian group $G$ is a *group of type* $\mathcal{P}^+$ if, for some prime $p$, the group $G$ is isomorphic to the additive group $\mathbb{Z}_p$ of all $p$-adic integers. A torsion-free Abelian group $G$ is said to be *separable of type* $\mathcal{P}^+$ if any finite subset of $G$ is contained in a direct summand of the group $G$ that is a direct sum of groups of the type $\mathcal{P}^+$ [3].

In particular, each reduced torsion-free Abelian group that can be endowed with the structure of a unitary $\mathbb{Q}_p^*$-module, where $\mathbb{Q}_p^*$ is the ring of $p$-adic integers, is a separable group of type $\mathcal{P}^+$. This follows from the fact that each cyclic submodule of a reduced torsion-free $\mathbb{Q}_p^*$-module $G$ that is generated by an element of zero $p$-height, is a direct summand of $G$ [4, p. 294; 5, p. 52].

Taking account of the fact that each group $\mathbb{Z}_p$ is fully transitive [1, p. 64], we obtain the following result.

**Corollary 2.** *Let $G$ be a separable group of the type $\mathcal{P}^+$. Then the following assertions hold:*

1) *$G$ is a fully transitive group;*
2) *$G$ is a homogeneously decomposable group that has a canonical decomposition $G = \bigoplus_{p \in \Pi_1} G_{(p)}$ in which $\Pi_1 \subset \Pi$, $pG_{(p)} \neq G_{(p)}$ for each prime $p \in \Pi_1$, and $qG_{(p)} = G_{(p)}$ for any $q \in \Pi$ such that $q \neq p$;*
3) *$G$ is a $\chi$-group.*

An Abelian group $A$ is said to be f.i.-*correct* if, for each Abelian group $B$, the relations $A \cong B'$ and $B \cong A'$, where $B'$ and $A'$ are fully characteristic subgroups of the groups $B$ and $A$, respectively, imply the relation $A \cong B$ [6]. Since each homogeneously decomposable fully transitive torsion-free Abelian group is f.i.-correct [6, Corollary 10], we obtain the following result.

**Corollary 3.** *Each reduced fully transitive homogeneously separable group (in particular, any separable group of type $\mathcal{P}^+$ and any fully transitive separable torsion-free group) is* f.i.*-correct.*

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