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Scaling behavior in string theory

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Abstract

In Calabi–Yau compactifications of the heterotic string there exist quantities which are universal in the sense that they are present in every Calabi–Yau string vacuum. It is shown that such universal characteristics provide numerical information, in the form of scaling exponents, about the space of ground states in string theory. The focus is on two physical quantities. The first is the Yukawa coupling of a particular antigeneration, induced in four dimensions by virtue of supersymmetry. The second is the partition function of the topological sector of the theory, evaluated on the genus-one worldsheet, a quantity relevant for quantum mirror symmetry and threshold corrections. It is shown that both these quantities exhibit scaling behavior with respect to a new scaling variable and that a scaling relation exists between them as well.

One of the outstanding questions in string theory concerns the structure of the space of ground-states. Much insight has been gained over the past years into some of its salient properties for (2,2)-supersymmetric vacua. Most of these results, such as the discovery of mirror symmetry [1], and the computation of the moduli dependence of Yukawa couplings [2], focus on the geometry associated to the topological characteristics of the internal manifold². Relevant in this context is the generation–antigeneration structure of the heterotic string, encoded in the two nontrivial Hodge numbers ($h^{(1,1)}$, $h^{(2,1)}$) of the internal Calabi–Yau manifold. These topological numbers provide a first parametrization of the space of string vacua and mirror symmetry, an operation that exchanges these two numbers, provides a profound

technique via which it is possible to extract physical information about individual ground states.

What is still missing is a framework which leads to insight into the more detailed structure of the space of string vacua and which will allow, eventually, to address problems like the vacuum degeneracy. In the absence of a physical “theory of moduli space” what is needed is some concrete information about this space, which we might hope to be encoded in some numerical characteristics. Such numerical information should be useful in providing a guideline as to what exactly it is that a future theory of moduli space has to explain. It is the purpose of this Letter to provide such characteristics, in the form of scaling exponents, thereby taking the first steps toward a quantitative understanding of the moduli space of the supersymmetric closed string.

The quantities which I will focus on in the following are based on the existence of a universal structure which exists on every Calabi–Yau manifold by virtue of the fact that they are projective, i.e. they are described by equations in a particular type of compact,

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² Or their field theoretic counterparts in the framework of Landau–Ginzburg theories or conformal field theories.

complex, Kähler manifold, so-called projective spaces. On such spaces there exists a natural structure, the so-called hyperplane bundle, denoted by L , which can be restricted to the Calabi–Yau manifold embedded in the ambient projective space. This bundle is of interest for physics because its first Chern class $c_1(L)$ leads to a universal massless mode, present in every Calabi–Yau manifold, which parametrizes one of the antigenerations observed in four dimensions. Moreover, on a three-complex dimensional Calabi–Yau manifold, the degree of this line bundle, defined as

$$L^3 \equiv \int_M c_1^3(L), \quad (1)$$

affords an interpretation as a Yukawa coupling κ_L of the antigereneration associated to L . Thus κ_L determines the strength of the Yukawa coupling of the corresponding four-dimensional mode. A second characteristic of any line bundle is the number h_L of its global functions. For every Calabi–Yau manifold the line bundle L thus leads to a pair of quantities (κ_L, h_L) which can be viewed as coordinates on the space of string vacua. The question arises whether these new coordinates allow us to draw some type of phase diagram for the spaces of interest, i.e. whether they provide a new sort of “structure diagram” for string vacua.

A final quantity of interest which is induced by the line bundle L is

$$L \cdot c_2 \equiv \int_M c_1(L) \wedge c_2(M), \quad (2)$$

where $c_2(M)$ is the second Chern class of the Calabi–Yau 3-fold. The physical interpretation of this quantity has been uncovered in Ref. [3], where it was shown that the generalized index

$$\mathcal{F} = \frac{1}{2} \int \frac{d^2\tau}{\text{Im } \tau} \text{Tr} [(-1)^F F_L F_R q^{H_L} \bar{q}^{H_R}], \quad (3)$$

introduced in [4], contains information about the threshold correction of the gauge couplings in string theory and that it is the key for the understanding of quantum mirror symmetry at one loop. Here the integral is over the fundamental domain of the moduli space of the torus, $F_{L,R}$ denote the left and right fermion numbers and the trace is over the Ramond sec-

tor for both the left- and right-movers³. In lowest order this index essentially reduces to $\frac{1}{24} \int_M K \wedge c_2(M)$, where K is the Kähler form of the manifold. Hence the second Chern class evaluated on L , (2), defines the universal contribution $\mathcal{F}_L^\dagger \equiv L \cdot c_2/24$ to the large volume limit of this partition function.

A class of Calabi–Yau spaces that is particularly amenable to an analysis in terms of the variables κ_L , \mathcal{F}_L^\dagger , and h_L is furnished by hypersurfaces embedded in weighted projective space $\mathbb{P}_{(k_1, \dots, k_5)}$. The complete class of such manifolds, consisting of 7555 configurations, has been constructed in [5,6]. The natural candidate for a line bundle on such spaces is the pull-back of the weighted form of the hyperplane bundle $L \equiv \mathcal{O}_{\mathbb{P}_{(k_1, \dots, k_5)}}^{(k)}$ defined on the weighted ambient space, where [7,8]

$$k = \text{lcm} \{ \{ \gcd(k_i, k_j) \mid i, j = 1, \dots, 5; i \neq j \} \cup \{ k_i \mid k_i \text{ does not divide } \sum_{i=1}^5 k_i \} \}. \quad (4)$$

The restriction of L from the weighted ambient space to the embedded Calabi–Yau manifold M induces an antigereneration which will also be denoted by L . The Yukawa coupling κ_L of this antigereneration leads, in the large volume limit, to the expression

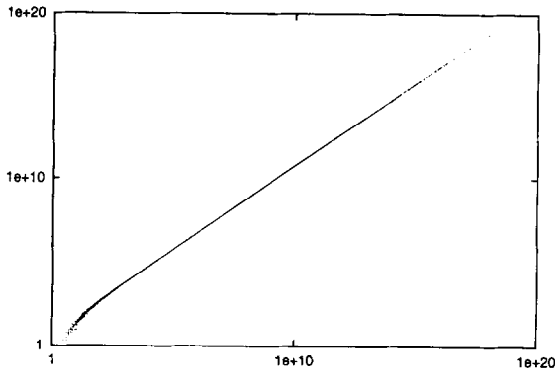
$$\kappa_L = \int_M (j^*(c_1(\mathcal{O}_{\mathbb{P}_{(k_1, \dots, k_5)}}^{(k)})))^3 = \left(\frac{\sum_{i=1}^5 k_i}{\prod_{i=1}^5 k_i} \right) k^3. \quad (5)$$

The number of global functions of the bundles $L = \mathcal{O}_{\mathbb{P}_{(k_1, \dots, k_5)}}^{(k)}$ can, finally, be obtained as

$$h_L = \frac{1}{k!} \frac{\partial^k}{\partial t^k} \left(\frac{(1 - t^{\sum_{i=1}^5 k_i})}{\prod_{i=1}^5 (1 - t^{k_i})} \right) \Big|_{t=0}. \quad (6)$$

In the following these variables will be used to explore the structure of the moduli space of Calabi–Yau manifolds. In general the two numbers κ_L and h_L are independent, completely uncorrelated, quantities. Hence in a diagram of these numbers we might expect a random distribution of data points, with no obvious pattern. The actual computation, the results of which

³ The contribution of the ground states of the supersymmetric Ramond sector to \mathcal{F} has to be deleted in order for the integral to converge.

Fig. 1. The Yukawa couplings κ_L versus h_L .

are shown in Fig. 1, however uncovers an unexpected simplicity.

The first thing to notice is the emergence of a well defined boundary which is described by a simple relation: all varieties in the class of Calabi–Yau 3-folds embedded in weighted $\mathbb{P}_{(k_1, \dots, k_5)}$ satisfy the inequality⁴

$$\kappa_L \leq 2(3h_L - 8). \quad (7)$$

This is in contrast to mirror symmetry where the boundary of the mirror diagram (see the first reference in [1]) does not yield to such an easy description, even though it is also well defined.

What is rather unexpected is that as the dimensions h_L increase for the individual Calabi–Yau hypersurfaces the Yukawa couplings approach the upper limit

$$\kappa_L^{\text{as}} \equiv 2(3h_L - 8) \quad (8)$$

very quickly, with very little scattering. Furthermore they do so according to a power law. This can be seen most clearly from Fig. 2, which contains the distances

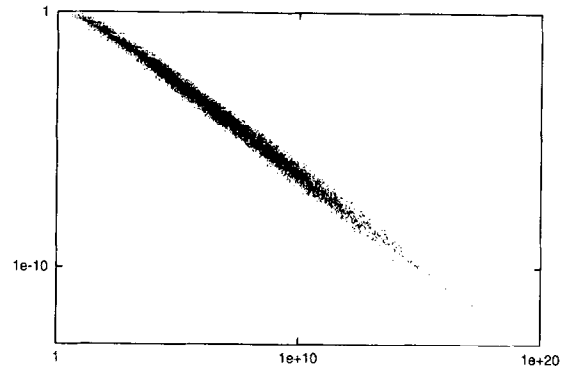
$$\mathcal{D} \equiv \frac{\kappa_L^{\text{as}}}{\kappa_L} - 1 \quad (9)$$

of the Yukawa couplings from κ_L^{as} as a function of h_L .

The result shows that the distance variable \mathcal{D} exhibits the scaling behavior

$$\mathcal{D} \sim A(h_L)^{-\alpha}, \quad (10)$$

⁴ The weaker limit $\kappa_L < 6h_L$ follows from the relation of Riemann–Roch–Hirzebruch and the positivity of the second Chern class.

Fig. 2. \mathcal{D} versus h_L .

where, approximately,

$$A = 5 \quad \text{and} \quad \alpha = 0.7, \quad (11)$$

via a least square fit. This leads to a behavior of the Yukawa couplings of the form⁵

$$\kappa_L \sim \frac{\kappa_L^{\text{as}}}{1 + A(h_L)^{-\alpha}} \quad (12)$$

and therefore the Yukawa coupling κ_L behaves like an order parameter with the dimension h_L assuming the rôle of the scaling variable.

It should be noted that the Yukawa couplings κ_L and the dimensions h_L take values over an enormously wide range of values, unprecedented in Calabi–Yau physics, spanning 18 orders of magnitude. Thus they lead to an enormous variation for the threshold corrections of these vacua. Furthermore the power law (12) for the coupling provides some measure of simplicity of Calabi–Yau manifolds: the larger the value of the

⁵ A rough idea of why such a scaling behavior might hold can be obtained by deriving an estimate for the Yukawa couplings in terms of the weights. It can be shown that

$$\begin{aligned} \kappa_L \leq 6h_L - \frac{1}{2 \prod_i k_i} \left[3k^2 \sum_{i < j} k_i k_j + k \sum_i k_i k_{i+1}^2 \right. \\ \left. + 3k \sum_i k_i k_{i+1} k_{i+2} \right]. \end{aligned}$$

Because the variable k grows faster than the weights themselves as the complexity of the singularity structure of the Calabi–Yau manifolds increases, this expression suggests that a scaling like behavior of the type (12) might occur. The fact that the deviations from this scaling equation are extremely small however can only be seen from an explicit computation.

scaling variable h_L , the more accurate the prediction of the Yukawa coupling via (12). Since, very roughly, the magnitude of the scaling variable is determined by the degree of complexity of the singular sets of the varieties that need to be resolved, spaces with more complicated resolution geometry are in fact simpler when it comes to scaling. This phenomenon is rather different from the conventional point of view which considers smooth projective hypersurfaces as the simplest type of manifolds. Finally, it is easy to see that the scaling behavior (10), (12) is a characteristic of Calabi–Yau manifolds, not of general projective 3-folds, not to mention general 3-folds: consider the hypersurfaces $\mathbb{P}_4[d]$ of degree d embedded in ordinary projective 4-space. For this (infinite) class of manifolds one finds $\kappa_L = d$, $h_L = 5$, $c_2 \cdot L = d(d^2 - 5d + 10)$.

Over the last years a confluence of ideas has occurred in physics and mathematics toward a further physically important quantity. Wilson [9], on the one hand, has reemphasized the importance of the linear form on the Picard group, defined by the second Chern class c_2 , for purposes of classification, a fact first recognized by Wall. Bershadsky, Cecotti, Ooguri, and Vafa, on the other, have found that the generalized index of [4] reduces in the large volume limit essentially to $\mathcal{F}_L^\dagger = c_2 \cdot L/24$, and have shown that this quantity arises in the 1-loop quantum mirror expansion of the Ray–Singer torsion, as well as in the threshold corrections of the gauge couplings. Thus the pair $(\mathcal{F}_L^\dagger, \kappa_L)$ provides two coordinates on the moduli space of Calabi–Yau manifolds with immediate physical meaning. To investigate a possible relation between these quantities is of particular interest since they arise in two different perturbation expansions of the string – the σ -model expansion at string tree level and the string loop expansion.

Wilson [9] has observed that for the Yukawa coupling κ_L and $L \cdot c_2$ an inequality $L \cdot c_2 \leq 10\kappa_L$ is obtained for models whose Fujita index Δ is greater than two. The analysis of the present class of weighted Calabi–Yau hypersurfaces shows that for all but 16 spaces the Fujita index is always larger than two⁶. Hence Wilson’s observation shows that except for those sixteen spaces the above inequality does hold in this class, i.e.

⁶ None of these spaces is of particular interest from a physics perspective.

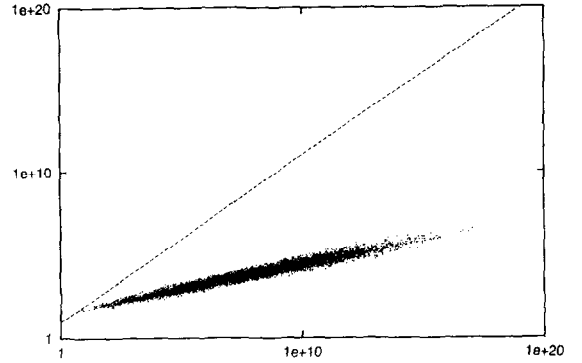


Fig. 3. $24\mathcal{F}_L^\dagger$ versus κ_L . The dotted line represents the upper limit $y = \frac{5}{12}x$.

$$\frac{\mathcal{F}_L^\dagger}{\kappa_L} \leq \frac{5}{12}. \quad (13)$$

No further information about the population below this upper limit is known.

The unexpected consequence of the analysis above of all Calabi–Yau hypersurfaces in weighted projective 4-space is that the vast majority of these vacua does not come even close to the upper limit (13) but instead takes values in a narrow region far below that line. For large values of \mathcal{F}_L^\dagger the Yukawa couplings lie more than ten orders of magnitude below the upper limit provided by Wilson’s observation. Furthermore the results, as shown in Fig. 3, uncover a definite correlation between the Yukawa coupling and \mathcal{F}_L^\dagger .

More precisely, a least square fit leads to the conclusion that \mathcal{F}_L^\dagger scales like

$$\mathcal{F}_L^\dagger \sim B \kappa_L^\beta, \quad (14)$$

with approximate values

$$B = 1.5, \quad \beta = 0.29. \quad (15)$$

To summarize, the results presented above suggest that the Yukawa coupling κ_L and the universal part \mathcal{F}_L^\dagger of the free energy should be interpreted as order parameters for which the number h_L of global functions behaves like a scaling variable. The scaling phenomena described above provide specific power laws with a particular set of exponents that any future physical theory of moduli space of the heterotic string will have to explain. Thus our scaling laws furnish the first quantitative insight into the structure of the string configuration space.

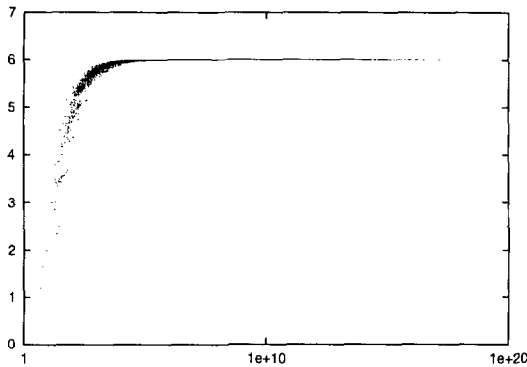


Fig. 4. $(h_L)^{-1}\kappa_L$ versus h_L .

The present situation is not without historical precedent: it is reminiscent of Bjorken's results many years ago when he argued that a certain scaling behavior should manifest itself in deep inelastic electron–nucleon scattering. Our scaling can in fact be presented in a form which is very similar to the original data of these historical results. Define a function $(h_L)^{-1}\kappa_L$ and consider its dependence on the scaling variable h_L . The result, which is displayed in Fig. 4, looks very much like the experimental results of the SLAC group for $\nu W_2(\nu, q^2)$. What is needed, then, is the analog of Feynman's quark-parton picture, or perhaps rather the analog of the QCD explanation of, say, the momentum fraction of the proton carried by the quarks.

Tantalizing questions remain: An immediate one is whether the exponents α, β are universal or specific to our class of Calabi–Yau hypersurfaces in weighted projective 4-space. There are two natural avenues for further exploration. The simpler of the two is provided by the toric framework of Ref. [10] which has been shown [11] to include all Calabi–Yau hypersurfaces in weighted projective 4-space. The second direction involves a natural generalization of the framework of Calabi–Yau compactification, motivated by mirror symmetry. In [12] a construction has been introduced which provides an embedding of Calabi–Yau spaces into a larger class of a special type of Fano manifolds, varieties whose positive first Chern class satisfies a particular quantization condition. It was shown in [12] that even though these special Fano manifolds are not consistent ground states themselves they do contain information about (2,2) supersymmetric string vacua. It was furthermore demonstrated in Ref.

[13] that such manifolds also lead to the correct behavior of the Yukawa couplings and in Ref. [14] that the framework of [12] lends itself to a toric analysis, generalizing the toric mirror construction of [10]. A natural question is whether our scaling behavior persists in these more general settings as well. Finally, the computations presented here are of lowest order in perturbation theory and an interesting problem is to explore the consequences of including higher order corrections.

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