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Evolution Problems with Nonlinear Nonlocal Boundary Conditions

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Abstract We provide a new approach to obtain solutions of evolution equations with nonlinear and nonlocal in time boundary conditions. Both, compact and noncompact semigroups are considered. As an example we show a "principle of huge growth": every control of a reaction-diffusion system necessarily leads to a profile preserving nonlinear huge growth for an appropriate initial value condition. As another example we apply the approach with noncompact semigroups also to a class of age-population models, based on a hyperbolic conservation law.

Keywords Nonlinear boundary condition · Nonlocal boundary condition · Function triple degree · Nonlinear Fredholm map · Semilinear partial differential equation · Nonuniqueness · Profile-preserving growth · Age-population model

1 Introduction

Consider a nonlinear evolution problem

$$u'(t) = Au(t) + f(t, u(t)) \qquad (0 < t \leq T) \tag{1}$$

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with nonlinear nonlocal “boundary” conditions occurring in time

$$F\big(u(0)\big) = \varphi(u). \tag{2}$$

Here, A denotes the generator of a linear C_0-semigroup U in a Banach space E, $f\colon [0,T]\times E \to E$ is a Carathéodory function (i.e. $f(t,\cdot)\colon E\to E$ is continuous for almost all $t\in[0,T]$, and $f(\cdot,u)\colon [0,T]\to E$ is Bochner measurable for every $u\in E$), and for some nonempty open bounded set $\Omega\subseteq E$ the function $F\colon \overline{\Omega}\to E$ is continuous and such that $F|_\Omega$ is a nonlinear Fredholm map of index zero (i.e., $F|_\Omega$ is C^1 with the derivative being a linear Fredholm map of index zero at every point of Ω). Finally, $\varphi\colon C([0,T],E)\to E$.

Problems with nonlinear nonlocal boundary condition of type (2) occur in many fields like pharmacokinetics, neural networks, epidemiology models etc, see e.g. [21, Sect. 10.2]. We will study a particular age-population model in Sect. 5. Moreover, in Sect. 3, we show that every controlled reaction-diffusion equation is subject to a shape-preserving huge growth: the precise assertion considered in Sect. 3 can also be formulated in the form (1) and (2).

For the particular case that (2) is a *local* (nonlinear) boundary condition

$$F\big(u(0)\big) = G\big(u(T)\big),$$

i.e.

$$\varphi(u) := G\big(u(T)\big), \tag{3}$$

it was sketched in [26] that this problem is accessible by the degree theory for function triples with a Fredholm function developed in that monograph. The considerations and results given there hold also for the more general case (2) without any change. In particular, this degree theory can be used to obtain e.g. existence and bifurcation results for the nonlocal nonlinear boundary value problem (1), (2) with general $\varphi\colon C([0,T],E)\to E$.

It is the purpose of this note to make these abstract considerations more precise. For simplicity, we will concentrate on existence results which can be obtained by applying the homotopy invariance of the mentioned degree (continuation principle) with the homotopy occurring naturally when φ is replaced by $\lambda\varphi$ $(0\le\lambda\le 1)$. We do not strive to give the most general formulation of that existence results but instead study the two probably most interesting cases in more detail which involve only classical hypotheses of an immediately compact semigroup U or, alternatively, some Lipschitz and compactness type condition on the nonlinearity. In particular, we try to avoid the perhaps less known concept of measures of noncompactness as far as possible, at least in the formulation of the results.

One of the features of our approach is that we do not suppose uniqueness of the solution of (1) with initial value condition

$$u(0) = u_0. \tag{4}$$

More precisely, by a solution of (1) on $[a,b]\subseteq[0,T]$, we understand a mild solution, i.e. a continuous function $u\colon [a,b]\to E$ satisfying the associated variations-of-constants formula

$$u(t) = U(t-a)u(a) + \int_a^t U(t-s)f\big(s,u(s)\big)\,\mathrm{d}s \quad \text{for all } t\in[a,b]. \tag{5}$$

Recall that the connection with strong solutions of (1) (i.e. absolutely continuous functions $u\colon [0,T]\to E$ satisfying (1) for almost all $t\in[0,T]$) is the following: every strong solution u for which $t\mapsto f(t,u(t))$ is Bochner integrable is a mild solution, and conversely, if E is reflexive and $f(t,u(t))$ belongs to the domain of A for almost every $t\in[a,b]$ with

$t \mapsto Af(t, u(t))$ being integrable then also the converse holds, see e.g. [15, Theorems 4.1.2 and 4.1.3]. It is well-known that without a Lipschitz condition on $f(t, \cdot)$, the problem (1) can have a continuum of strong (and thus mild) solutions $u \in C([0, T], E)$ satisfying (4).

Now we put

$$D := \Big\{u \in C([0, T], E) : u \text{ is a mild solution of (1) with } u(0) \in \overline{\Omega}\Big\}. \tag{6}$$

Finally, we are in a position to explain our approach: Let $\Phi \colon \overline{\Omega} \multimap D$ denote the (in general multivalued) map which associates to every $u_0 \in \overline{\Omega}$ the set of mild solutions of the initial value problem (1), (4). Then the boundary value problem (1), (2) can equivalently be rewritten as the abstract inclusion

$$F(u_0) \in \varphi\big(\Phi(u_0)\big). \tag{7}$$

Under some hypotheses about Φ, this is exactly the type of problems for which the degree theory of the function triple (F, Φ, φ) developed in [26] applies. As remarked above, we will only consider existence results which can be obtained by considering the homotopy invariance of that degree for the homotopy which occurs when φ is replaced by $\lambda\varphi$ $(0 \le \lambda \le 1)$. By this homotopy, the following continuation principle can be obtained. We use the convenient notation

$$\Phi(M) := \bigcup_{x \in M} \Phi(x) \quad \text{for } M \subseteq \overline{\Omega}.$$

The hypotheses and conclusions of the following result will be explained in a moment. However, let us already point out that, although at a first glance it may appear artificially general to formulate the compactness requirements only on *countable* subsets, this countability will play a crucial role for us (cf. Remark 14). By "countable" we mean throughout "finite or countably infinite".

Theorem 1 (continuation principle I) *Suppose that Φ is upper semicontinuous with R_δ values and $\varphi|_D$ is continuous. Put*

$$S := \Big\{u_0 \in \overline{\Omega} : \text{There is } \lambda \in [0, 1] \text{ with } F(u_0) \in \lambda\varphi\big(\Phi(u_0)\big)\Big\}.$$

Then S is closed. Suppose in addition that S satisfies the following properties:

1. *$\overline{S \cap \Omega}$ is compact and disjoint from $S \cap \partial\Omega$.*
2. *There is an open $\Omega_0 \subseteq \Omega$ with $S \cap \Omega \subseteq \Omega_0$ such that for each countable $C \subseteq \Omega_0$ and each countable $D_0 \subseteq \Phi(C) \subseteq D$ satisfying $F(C) \subseteq \overline{\mathrm{conv}}(\{0\} \cup \varphi(D_0))$ at least one of the sets $\overline{C}$ or $\overline{\varphi(D_0)}$ is compact.*

Then $S \cap \Omega$ is compact, and the degree for the function triple (F, Φ, φ) and the Benevieri-Furi degree $\deg(F, \Omega, 0)$ *are both defined and equal. If this degree is nonzero then (7) has a solution in Ω.*

Proof Since

$$\big\{(\lambda, u) \in [0, 1] \times \overline{\Omega} : F(u_0) \in \lambda\varphi(\Phi(u_0))\big\}$$

is closed (see e.g. [26, Corollary 2.115]), the compactness of $[0, 1]$ implies that S is closed, see e.g. [26, Corollary 2.112]. Now the result is a special case of [26, Theorem 14.49] with

$A = S \cap \Omega$. Indeed, since S is closed in $\overline{\Omega}$, the first hypothesis implies that A is closed in Ω and thus compact. Moreover, for any countable $C \subseteq \Omega_0$ the inclusion

$$F(C) \subseteq \overline{\operatorname{conv}}\big(\{0\} \cup \varphi(\Phi(C))\big)$$

implies that $N := \overline{\varphi(\Phi(C))}$ is compact. Indeed, since $\Phi(C)$ is a countable union of separable sets and thus separable, there is a countable dense $D_0 \subseteq \Phi(C)$, hence

$$\overline{\operatorname{conv}}\big(\{0\} \cup \varphi(\Phi(C))\big) = \overline{\operatorname{conv}}\big(\{0\} \cup \varphi(D_0)\big).$$

The hypothesis thus implies that either $\overline{C}$ or $\overline{\varphi(D_0)} = N$ is compact. The set N is compact in both cases, since if $\overline{C}$ is compact then also $\varphi(\Phi(\overline{C}))$ is compact, because $\varphi \circ \Phi$ is upper semicontinuous on $\overline{C}$ with closed and compact values, see e.g. [26, Proposition 2.100].

Note that if we consider the auxiliary boundary conditions

$$F\big(u(0)\big) = \lambda\varphi(u), \tag{8}$$

then S in Theorem 1 is just the set

$$S = \Big\{u(0) : u \text{ is a mild solution of (1), (8) for some } \lambda \in [0, 1]\Big\}.$$

For the first hypothesis of Theorem 1, we recall that $\Phi\colon E \multimap C([0, T], E)$ is upper semicontinuous if for each $u_0 \in E$ and each open $M \subseteq C([0, T], E)$ satisfying $\Phi(u_0) \subseteq M$ there is a neighborhood $N \subseteq E$ of u_0 satisfying $\Phi(N) \subseteq M$. Moreover, the condition that $\Phi(u_0)$ be an R_δ set means that $\Phi(u_0)$ is homeomorphic to the intersection of a decreasing sequence of nonempty compact contractible metric spaces, see e.g. [26] for more details.

The Benevieri-Furi degree was introduced in [7] for the case of a Fredholm map F equipped with an orientation. If F is non-oriented, an analogous degree can be considered with values in $\mathbb{Z}_2 = \{0, 1\}$ which we also call Benevieri-Furi degree, see e.g. [26]. For the case that F is oriented, the degrees coincide modulo 2.

In [26], instead, a degree for a triple (F, Φ, φ) is developed, both for oriented and non-oriented maps F.

The main assertion of Theorem 1 is that the calculation of the degree involving all three functions (F, Φ, φ) (where Φ represent the differential equation (1)) can be reduced to the calculation of the degree of a *single* function F which is not related with (1).

Moreover, the functions F occurring in boundary conditions (2) are often very simple (diffeomorphisms or odd), and for these functions, the degree is easy to calculate (and nonzero). Thus, the following result contains the probably most interesting special cases of Theorem 1. Recall that a map is called *proper* if preimages of compact sets are compact.

Theorem 2 (continuation principle II) *Suppose under the hypotheses of Theorem 1 that F is proper on closed in E subsets of Ω and $0 \notin F(\partial\Omega)$, that for some continuous map $F_0\colon \overline{\Omega} \to E$ with $F_0|_{\partial\Omega} = F|_{\partial\Omega}$ such that $F_0|_{\Omega}$ is of class C^1 and $F - F_0$ is compact, at least one of the following two hypotheses holds:*

1. *F_0 is a diffeomorphism onto $F_0(\Omega)$, and $0 \in F_0(\Omega)$.*
2. *$0 \in \Omega = -\Omega$, and $F_0(-x) = -F_0(x)$ for all $x \in \Omega$.*

Then the Benevieri-Furi degree $\deg(F, \Omega, 0)$ *is* ± 1 *or odd, respectively, and in particular,* (7) *has a solution in Ω.*

Proof Since $K := F - F_0$ is compact and C^1 on Ω, it follows that the derivative $dK(x)$ is compact for every $x \in \Omega$, see e.g. [11, Proposition 8.2]. This implies that $H(\lambda, x) := F(x) - \lambda K(x)$ $(0 \leq \lambda \leq 1)$ is a Fredholm homotopy of index 0. From $H(\lambda, \cdot)|_{\partial\Omega} = F|_{\partial\Omega}$ for any $\lambda \in [0, 1]$ and $0 \notin F(\partial\Omega)$, we obtain $H^{-1}(0) \subseteq [0, 1] \times \Omega$. Since $H^{-1}(0)$ is closed in $[0, 1] \times E$ and $[0, 1]$ is compact, it follows that $M := \bigcup_{\lambda\in[0,1]} \{x \in \Omega : H(\lambda, x) = 0\}$ is closed (see e.g. [26, Corollary 2.112]). Hence, putting $m(\lambda, y) := \lambda y$ and recalling that $(\lambda, x) \in H^{-1}(0)$ implies $F(x) = \lambda K(x)$ we obtain that $H^{-1}(0)$ is contained in the set

$$[0, 1] \times \big(M \cap F^{-1}\big(m([0, 1] \times \overline{K(\Omega)})\big)\big). \tag{9}$$

Since K is compact and F is proper, the set (9) is compact and thus $H^{-1}(0)$ is compact. Since $H(\lambda, \cdot)|_{\partial\Omega} = F|_{\partial\Omega}$, we also have $H^{-1}(0) \subseteq [0, 1] \times \Omega$. By the homotopy invariance of the Benevieri-Furi degree, we thus obtain

$$\deg(F, \Omega, 0) = \deg(F_0, \Omega, 0). \tag{10}$$

The latter degree is ± 1 by the normalization property of the degree for diffeomorphisms, or odd by the Borsuk theorem for the Benevieri-Furi degree, respectively: For orientable F, the Borsuk theorem has been shown in [6]. The proof for non-orientable F is analogous (if one calculates modulo 2 in the degrees throughout that proof); a different approach leading to a more general result (covering the non-orientable case) will be given in [27].

Remark 1 In the important case $F = F_0$ the equality (10) is trivial so that in this case we can drop in Theorem 2 the hypotheses about the properness of F and that $0 \notin F(\partial\Omega)$.

Remark 2 Although we avoid using measures of noncompactness as far as possible, we need them in one occasion in the second part of the paper, and it may be enlightening to the reader if we recall some definitions and properties here: By a *set function* γ on a metric space X, we understand a function which associates to every $M \subseteq X$ a (not necessarily finite) number $\gamma(M) \in [0, \infty]$. A set function γ is called *monotone* if $M \subseteq N$ implies $\gamma(M) \leq \gamma(N)$. If X is a Banach space, then a set function γ is called a *measure of noncompactness* if $\gamma(\overline{\operatorname{conv}} M) = \gamma(M)$ holds for all $M \subseteq X$.

An example of a monotone measure of noncompactness is the Hausdorff measure χ of noncompactness, i.e. $\chi(M)$ is defined as the infimum of all $\varepsilon \in (0, \infty]$ such that M can be covered in X by finitely many balls of radius ε.

In the second part of the paper, we will frequently use that if $g : X \to X$ is Lipschitz continuous with constant L then

$$\chi(g(M) + K) \leq L\chi(M) \quad \text{for all} M \subseteq X \text{and compact } \overline{K}. \tag{11}$$

Indeed, the proof of (11) is straightforward in case $K = \{0\}$, and it is easy to see that $\chi(N + K) = \chi(N)$ if $\overline{K}$ is compact and nonempty.

2 Immediately Compact Semigroups

We call a map compact if it maps bounded sets into relatively compact sets. In this section, we will assume that the semigroup U is immediately compact, i.e.

$$U(t) \;\text{ is compact for every } t > 0. \tag{12}$$

In this case, to apply Theorem 1, we need only rather mild hypotheses about f and φ.

One crucial hypothesis will be the boundedness of the set (6). Actually, later on, we will need the slightly more restrictive a priori boundedness hypothesis

$$B := \sup_{t\in[0,T]} \left\{\|u(t)\| : u \text{ is a mild solution of (1) on } [0,t] \text{ with } u(0) \in \overline{\Omega}\right\} < \infty. \quad (13)$$

Our first aim now is to prove that Φ is upper semicontinuous with R_δ values. To this end, it is natural to require (13). Indeed, related assertions can be found in [15, Corollary 5.2.1 and Theorem 5.3.1] which have as one hypothesis that every mild solution of (1) on $[0,t]$ can be extended to a mild solution on $[0,T]$, and that the set (6) of all global mild solutions on $[0,T]$ is bounded. This both together implies (13).

The hypothesis (13) holds e.g. if f has at most linear growth with respect to the second argument in the following sense.

Proposition 1 *Suppose that there is $a \in L_1([0,T])$ with*

$$\|f(t,v)\| \le a(t)(1+\|v\|) \qquad \textit{for all } v \in E \quad (14)$$

for almost all $t \in [0,T]$. Then (13) holds, and thus (6) is bounded in $C([0,T],E)$.

Proof Recall that the uniform boundedness principle implies

$$\sup_{t\in[0,T]} \|U(t)\| < \infty. \quad (15)$$

Hence, there is a constant $C \ge \|a\|_{L_1}$ which is a bound for $\overline{\Omega}$ and satisfies $\|U(t)\| \le C$ for all $t \in [0,T]$. Then we have for every $u \in D$ the estimate

$$\|u(t)\| \le \|U(t)u_0\| + \int_0^t \|U(t-s)\|a(s)(1+\|u(s)\|)\,\mathrm{d}s \le 2C^2 + C\int_0^t a(s)\|u(s)\|\,\mathrm{d}s,$$

and so the assertion follows from a variant of Gronwall's classical lemma.

Remark 3 The estimate obtained by Gronwall's lemma in the previous proof shows that there are even finite constants C_1, C_2, depending only on T, a, and the supremum in (15), such that (13) holds with

$$B \le C_1 + C_2\|u(0)\|.$$

Of course, instead of the linear growth condition (14), we could also suppose any other condition guaranteeing (13). One such example is the existence of a locally mildly invariant bounded set $M \subseteq E$ containing $\overline{\Omega}$, see Corollary 1.

We consider now sufficient conditions on the function Φ to be upper semicontinuous and to have R_δ values.

Lemma 1 *Suppose*

$$\sup_{u\in D}\|f(t,u(t))\| \le c(t) \quad \textit{for almost all } t \in [0,T] \textit{ and some } c \in L_1([0,T]). \quad (16)$$

Then (6) is bounded, and if additionally (12) holds then (6) is also closed in $C([0,T],E)$, and $\Phi : \overline{\Omega} \multimap D$ is upper semicontinuous and assumes compact values.

Remark 4 Note that (16) holds if (13) is satisfied and

$$\sup_{\|u\|\le B}\|f(t,u)\|\le c(t)\quad \text{for almost all } t\in[0,T] \text{ and some } c\in L_1([0,T]). \tag{17}$$

The latter holds in particular, if f maps bounded sets to bounded sets.

Proof of Lemma 1 The boundedness of D follows straightforwardly from

$$u(t)\in U(t)\overline{\Omega}+\int_0^t U(t-s)f(s,u(s))\,\mathrm{d}s \quad \text{for all } u\in D,\, t\in[0,T],$$

since Ω is bounded, and since we have the uniform estimates (15) and (16).

Now we show first that the Volterra operator $V\colon D\to C([0,T],E)$, defined by

$$V(u)(t):=\int_0^t U(t-s)f(s,u(s))\,\mathrm{d}s,$$

is continuous and compact. To see this, we show first that $V(D)$ is equicontinuous. Indeed, let $B_1:=\{u\in E:\|u\|\le 1\}$. Note that (12) implies that $W_\varepsilon:=U(\varepsilon)B_1$ is relatively compact for every $\varepsilon>0$. In particular, $(t,u)\mapsto U(t)u$ is uniformly continuous on the compact set $[0,T]\times\overline{W}_\varepsilon$, and so there is $\delta>0$ such that

$$\|U(\tau-\varepsilon)w-U(t-\varepsilon)w\|\le\varepsilon \quad \text{for all } w\in W_\varepsilon \text{ and } \tau,t\in[0,T] \text{ with } |t-\tau|\le\delta.$$

For $s\in[0,T]$, $\tau,t\in[s+\varepsilon,T]$, and $v\in B_1$, we have $U(\tau-s)v=U(\tau-s-\varepsilon)w$ and $U(t-s)v=U(t-s-\varepsilon)w$ with $w:=U(\varepsilon)v\in W_\varepsilon$, and so

$$\|U(\tau-s)v-U(t-s)v\|\le\varepsilon \quad \text{for all } v\in B_1,\ 0\le s<t-\varepsilon,\ t\le\tau\le T,\ \tau\le t+\delta.$$

Putting $M:=\sup_{t\in[0,T]}\|U(t)\|$, we obtain for $0\le t\le\tau\le T$ and $\tau\le t+\delta$, by splitting the integral, that

$$\|V(u)(\tau)-V(u)(t)\|\le\varepsilon\int_0^{\max\{t-\varepsilon,0\}}c(s)\,\mathrm{d}s+2M\int_{\max\{t-\varepsilon,0\}}^{t}c(s)\,\mathrm{d}s+M\int_t^\tau c(s)\,\mathrm{d}s.$$

Hence, the equicontinuity of $V(D)$ follows from the continuity of the integral function of c. Noting that Lebesgue's dominated convergence theorem implies by (16) and the uniform boundedness of U that for each $t\in[0,T]$

$$\lim_{[0,t]\supseteq I_n\downarrow\emptyset}\ \sup_{u\in D}\int_{I_n}\|U(t-s)f(s,u(s))\|\,\mathrm{d}s=0,$$

we obtain the precompactness of $V(D)$ and continuity of V on D e.g. from [24, Theorem 12.8] with $r(t,s)=0$ for almost all $s\ne t$, since $\{U(t-s)f(s,u(s)):u\in D\}$ is precompact for almost all $s<t$ by (16) and (12).

To see that D is closed, suppose that $u_n\in D$ converges to $u\in C([0,T],E)$. Since $f(s,u_n(s))\to f(s,u(s))$ for almost all s, we conclude from Vitali's convergence theorem (see e.g. [23, Theorem 3.3.3]) in view of (15) that we can pass to the limit $n\to\infty$ in the equation

$$u_n(t)=U(t)u_n(0)+\int_0^t U(t-s)f(s,u_n(s))\,\mathrm{d}s$$

and thus find that u satisfies (5) with $[a, b] = [0, T]$. Since $u_n(0) \in \overline{\Omega}$, we have also $u(0) \in \overline{\Omega}$ and thus $u \in D$.

We also define $W : \overline{\Omega} \to C([0, T], E)$ by $W(u_0)(t) := U(t)u_0$ and note that (15) implies that W is continuous.

If Φ would fail to be upper semicontinuous at $u_0 \in \overline{\Omega}$, there would be an open set $N \subseteq C([0, T], E)$ containing $\Phi(u_0)$ and a sequence $v_n \in \overline{\Omega}$ with $v_n \to u_0$ such that there are $u_n \in \Phi(v_n) \setminus N$. In particular, we have $u_n \in D$ and $u_n = W(v_n) + V(u_n)$. Since $W(v_n) \to W(u_0)$ and $V(D)$ is relatively compact in $C([0, T], E)$, we obtain that $\{u_1, u_2, \ldots\}$ is relatively compact in $C([0, T], E)$. Passing to some subsequence if necessary, we can thus assume that $u_n \to u$, and since D is closed, we have $u \in D$. The continuity of V at u now implies $u = W(u_0) + V(u)$, i.e. $u \in \Phi(u_0)$ which is a contradiction to $u_n \notin N$, since $u_n \to u \in N$. Repeating the same argument with $v_n = u_0$, we find that any sequence $u_n \in \Phi(u_0)$ has a subsequence converging to some $u \in \Phi(u_0)$, i.e. $\Phi(u_0)$ is compact.

Note that we did not assume (13) in Lemma 1. So we obviously cannot claim in Lemma 1 that the values of Φ or at least the set (6) are nonempty.

To prove that Φ assumes actually R_δ sets as values (which in particular implies $\Phi(u_0) \neq \emptyset$ for every $u_0 \in \overline{\Omega}$ and thus also $D \neq \emptyset$), we will assume the more restrictive a-priori boundedness condition (13).

Lemma 2 *Let* (12) *and* (13) *hold, and let* $f : [0, T] \times E \to E$ *be continuous and map bounded sets to bounded sets. Then* $\Phi : \overline{\Omega} \to D$ *is upper semicontinuous and assumes only* R_δ *values on* $\overline{\Omega}$.

Proof In view of Remark 4, the assertion about the upper semicontinuity is contained in Lemma 1.

Under assumption (12), in [4] the assertion about R_δ values has been proved for the case that f can be approximated uniformly by a uniformly bounded sequence of Lipschitz functions $f_n : [0, T] \times E \to E$. In fact, it suffices for that result that f_n satisfies a local Lipschitz condition, since the only place where the local Lipschitz condition is used in the proof of [4] is for the proof of the uniqueness of the solution of

$$u(t) = z(t) + \int_0^t U(t-s) f_n(s, u(s))\, \mathrm{d}s$$

where $z : [0, T] \to E$ is a given continuous function, see [4, (3)]. However, if f_n satisfies a local Lipschitz condition, this uniqueness assertion is true as well, for if $u, v : [0, T] \to E$ are two different solutions of that equation, then u and v are continuous with $u(0) = v(0) = z(0)$, and so there is a largest $t_0 \in (0, T)$ with $u|_{[0,t_0]} = v|_{[0,t_0]}$. Since f is locally Lipschitz in a neighborhood of $(t_0, u(t_0)) = (t_0, v(t_0))$ and (15) holds, there is $L < \infty$ such that

$$\|u(t) - v(t)\| = \left\| \int_{t_0}^t U(t-s)\big(f_n(s, u(s)) - f_n(s, v(s))\big)\, \mathrm{d}s \right\| \leq L \int_{t_0}^t \|u(s) - v(s)\|\, \mathrm{d}s$$

holds for all $t > t_0$ sufficiently close to t_0, and so $u|_{[t_0,t_0+\varepsilon]} = v|_{[t_0,t_0+\varepsilon]}$, contradicting the maximality of t_0.

Since every bounded continuous function $f : [0, T] \times E \to E$ can be approximated uniformly by a sequence f_n of (automatically bounded) locally Lipschitz functions, see e.g. [10, Lemma 1.1], we have shown the assertion about R_δ values for the case that in addition f is globally bounded.

Now in the general case, if B denotes the constant of (13), we apply this special case with f replaced by

$$\widetilde{f}(t,u) := \begin{cases} f(t,u) & \text{if } \|u\| \le B_0, \\ f\big(t,(B_0/\|u\|)u\big) & \text{if } \|u\| \ge B_0 \end{cases} \tag{18}$$

for some $B_0 > B$. Note that $\widetilde{f}$ is well-defined and thus continuous by the glueing lemma. The choice of B_0 implies by (13) that for $u_0 \in \Omega$ the value $\Phi(u_0)$ is the same for f as for $\widetilde{f}$, and so the assertion follows from the boundedness of $\widetilde{f}$ on $[0,T] \times E$.

A slightly different approach to Lemma 2 can be found in [2], see also the corresponding generalizations in [3,17] which would allow to slightly relax the continuity requirement.

Remark 5 In Lemma 2, the hypothesis that f maps bounded sets onto bounded sets can be slightly relaxed. In fact, our proof shows that it suffices that

$$\sup_{\substack{t\in[0,T]\\ \|u\|\le B_0}} \|f(t,u)\| < \infty \tag{19}$$

for some B_0 strictly larger than the left-hand side of (13).

Concerning the compactness conditions of Theorem 1, we note that the set (6) fails to be compact, in general. In view of our proof of Lemma 1, this failure of compactness is essentially only due to two reasons:

1. $\{u(0) : u \in D\} = \overline{\Omega}$ is noncompact.
2. D fails to be equicontinuous in a neighborhood of $t = 0$ (unless A is bounded).

However, for natural boundary conditions occurring in (2) the map $\varphi\colon C([0,T],E) \to E$ is compact on D, since φ should not take into account the value $u(0)$ (which should be treated by F instead in (2)) and thus e.g. depend only on $u|_{[\varepsilon,T]}$. For instance, if

$$\varphi(u) = \varphi_0\big(u(t_1),\dots,u(t_n)\big) \tag{20}$$

with fixed numbers $t_1,\dots,t_n \in (0,T]$, we obtain automatically a compact map, as we will show in a moment. Actually, φ might even depend on $u|_{(0,\varepsilon]}$, provided that this dependency is in some "integral" sense. To formulate this in a reasonably general way, allowing nonlinearities in the integrals, we consider the integral map

$$K(u) := \int_0^T k(t,u(t))\,\mathrm{d}t,$$

where $k\colon [0,T] \times E \to E_0$ is a Carathéodory function and E_0 is some Banach space.

Proposition 2 *Suppose that there exists $b \in L_1([0,T])$ such that*

$$\|k(t,u(t))\| \le b(t) \quad \text{a.e. on } [0,T] \text{ for every } u \in D. \tag{21}$$

Suppose in addition (12) and (16). Then for every $\varepsilon \in (0,T]$, every Banach space E_1, every continuous compact map $\varphi_1\colon D \subseteq C([0,T],E) \to E_1$, and every continuous (not necessarily compact) map

$$\varphi_0\colon C([\varepsilon,T],E) \times E_0 \times E_1 \to E \tag{22}$$

the composed map

$$\varphi(u) = \varphi_0\bigl(u|_{[\varepsilon,T]}, K(u), \varphi_1(u)\bigr) \tag{23}$$

is continuous and compact on D.

As in Remark 4, we can replace the hypotheses (16) by (13) and (17).

Example 1 Assume that (6) is bounded, $k_0\colon [0,T]\times E \to E_0$ is a bounded Carathéodory function, $g\colon E_0 \to E$ is continuous (no compactness requirements!), and

$$\varphi(u) = g\left(\int_0^T k_0(t,u(t))\,\mathrm{d}\mu(t)\right) \tag{24}$$

with μ being a measure on $[0,T]$ of bounded variation whose restriction to some $[0,\varepsilon)$ ($\varepsilon > 0$) is absolutely continuous with respect to the Lebesgue measure. Then φ has the form (23), and so $\varphi\colon C([0,T],E) \to E$ is continuous and compact on D if (13) holds.

Note that it is in particular admissible that the measure μ contains atoms, as long as some interval $[0,\varepsilon)$ remains free of atoms.

Indeed, denoting the Radon-Nikodym derivative of μ on $[0,\varepsilon)$ by α, we can put (with the Lebesgue measure in the integral)

$$K(u) := \int_0^\varepsilon k_0(t,u(t))\alpha(t)\,\mathrm{d}t,$$

$\varphi_1 \equiv 0$, and

$$\varphi_0(u,v,0) := g\Bigl(v + \int_\varepsilon^T k_0(t,u(t))\,\mathrm{d}\mu(t)\Bigr)$$

in (23).

Proof of Proposition 2 Defining V and W as in the proof of Lemma 1, we note that the definition of D implies $D \subseteq W(\overline{\Omega}) + V(D)$. Since the sets $V(D)$ and $\{W(u)(t) : u \in D\}$ are relatively compact for every $t \in (0,T]$, we find that $\overline{\{u(t) : u \in D\}}$ is compact for every $t \in (0,T]$. Hence, similarly to the proof of the continuity and compactness of V, we obtain from [24, Corollary 11.21] that K is actually continuous and compact on D.

Note also that $M := \bigl\{v|_{[\varepsilon,T]} : v \in W(\overline{\Omega})\bigr\}$ is equicontinuous with $\{v(t) : v \in M\}$ being relatively compact for every $t \in [\varepsilon,T]$. Hence, from the vector-valued Arzéla-Ascoli theorem, see e.g. [24, Theorem 12.5], we find that M is relatively compact in $C([\varepsilon,T],E)$. Since M and $N = \bigl\{v|_{[\varepsilon,T]} : v \in V(D)\bigr\}$ are relatively compact in $C([\varepsilon,T],E)$, it follows that $C_\varepsilon := \overline{M} + \overline{N}$ is compact. Note that $D \subseteq W(\overline{\Omega}) + V(D)$ implies that C_ε contains all restrictions of functions from D. In particular, putting $D_0 := C_\varepsilon \times \overline{K(D)} \times \overline{\varphi_1(D)}$, we have $\varphi(D) \subseteq \varphi_0(D_0)$. Since D_0 is compact, it follows that $\varphi(D)$ is relatively compact. Since we have already shown that K is continuous on D, the continuity of φ is trivial.

Remark 6 The proof of Proposition 2 shows that it suffices that φ_0 is defined (and continuous) on the set D_0 from the proof. It also suffices that k is defined (and Carathéodory) only on the unions of the graphs of the functions from D, and (21) can be replaced by the less restrictive condition

$$\lim_{[0,T]\supseteq I_n\downarrow\emptyset}\ \sup_{u\in D}\int_{I_n}\|k(t,u(t))\|\,\mathrm{d}t=0.$$

Indeed, $t\mapsto k(t,u(t))$ is Bochner measurable for every $u\in D$ by [24, Proposition 8.1], and so $K(u)$ is defined by [23, Lemma 3.3.7].

Now we can formulate our main abstract existence theorem for the case of an immediately compact semigroup.

Theorem 3 (abstract existence I) *Let* (12) *hold. Suppose that* $f\colon[0,T]\times E\to E$ *is continuous with* (13) *and* (19)*, and that* φ *has the form* (23) *as in Proposition* 2*. Suppose in addition that* F *is proper on closed in* E *subsets of* Ω *and that the following admissibility condition holds:*

$$\text{For every } u\in D \text{ with } u(0)\in\partial\Omega \text{ and every } \lambda\in[0,1) \text{ there holds } F\big(u(0)\big)\neq\lambda\varphi(u). \tag{25}$$

Then $\deg(F,\Omega,0)$ *is defined, and if it is nonzero, the boundary value problem* (1)*,* (2) *has a solution* u *with* $u(0)\in\overline{\Omega}$*.*

Recall that (13) holds by Proposition 1 if f satisfies the linear growth condition (14), and that every φ of the form (24) can be written in the form (23).

Remark 7 If F satisfies the additional hypotheses of Theorem 2 (or of Remark 1) then $\deg(F,\Omega,0)$ is nonzero.

Remark 8 In Theorem 3, we obtain an additional conclusion if there is no solution u of (1), (2) with $u(0)\in\partial\Omega$: Our subsequent proof shows that in this case the degree $\deg(F,\Phi,\varphi,\Omega)$ is defined and equal to $\deg(F,\Omega,0)$.

Remark 9 For the case $F=I$, a special case of Theorem 3 reduces essentially to the main existence result of [18]: In [18] some norm estimates are assumed which immediately imply (25) in case $F=I$; also the hypotheses in [18] guaranteeing the compactness of $\varphi|_D$ are rather similar. In fact, the classical Schauder fixed point theorem suffices to prove the main result in [18]. One cannot expect such a straightforward proof to work under the more general admissibility condition (25), even in case $F=I$. In case $F\neq I$, it is probably not even possible at all to apply any classical fixed point results to obtain a result similar to Theorem 3.

Proof of Theorem 3 Without loss of generality we can assume that (25) holds with $\lambda=1$, otherwise the existence of a solution is proved. We now prove that all hypotheses of Theorem 1 are satisfied. By our previous results, $\Phi\colon\overline{\Omega}\multimap D$ is upper semicontinuous with R_δ values, and $\varphi|_D$ is continuous and compact. In particular, $\overline{\varphi(\Phi(C))}$ is compact for every $C\subseteq\overline{\Omega}$. This implies the second hypothesis of Theorem 1 and, putting $m(\lambda,u):=\lambda u$, we obtain also that $M:=m\big([0,1]\times\overline{\varphi(D)}\big)$ is compact. The set S of Theorem 1 satisfies $S\subseteq F^{-1}(M)$, and the admissibility hypothesis (25) implies $S\cap\partial\Omega=\emptyset$, and so $S\subseteq\Omega$. Since S is closed in E, we have that $F|_S$ is proper. Hence, the compactness of M implies that also S is compact.

We call Theorem 3 an abstract existence theorem, because it is not immediately clear how to verify the admissibility hypothesis (25). In fact, a thorough study of this condition for a particular problem requires a study of the flow, simultaneously with considering the functions F and φ. However, in some cases the functions F and φ may have such a behavior that it is not necessary to know anything about the flow at all:

Example 2 Suppose that $F(\partial\Omega) \cap \lambda\varphi(D) = \emptyset$ for every $\lambda \in [0, 1)$. Then (25) is satisfied.

At a first glance, Example 2 may appear to be a rather trivial observation, but actually this hypothesis is satisfied in several cases, as we shall discuss now. Recall first that D is actually a bounded set by our hypothesis so that it is natural to expect that also φ is bounded on D. If this bound is just sufficiently small relative to F then we are in the situation of Example 2:

Example 3 Suppose that

$$0 \notin F(\partial\Omega), \qquad \inf_{u_0\in\partial\Omega} \|F(u_0)\| \geq \sup_{u\in D} \|\varphi(u)\|. \tag{26}$$

Then (25) is satisfied. Indeed, the condition of Example 2 holds.

Example 3 may appear even more artificial than Example 2 if one considers a fixed set $\partial\Omega$ and a corresponding set D and, e.g., F the identity function, since, depending on A and f, the norm of the elements in D might be much larger than those of $\partial\Omega$. However, if one is only interested in the existence of some solution of (1) and (2) and not in an a-priori estimate for it, this hypothesis only has to be satisfied for *some* set $\Omega \subseteq E$:

Example 4 Suppose that $\varphi\colon C([0, T], E) \to E$ has a bounded image and that $F\colon E \to E$ is weakly coercive, i.e. $\|F(u_0)\| \to \infty$ as $\|u_0\| \to \infty$. Then (26) and thus (25) holds if Ω is a sufficiently large open ball with center zero. In fact, it suffices even that

$$\liminf_{\|u_0\|\to\infty} \|F(u_0)\| \geq C > 0$$

for some upper bound $C > 0$ of the image of φ.

For the case that F is truly nonlinear and blows up near a boundary of a bounded set, we even only need that φ is bounded in the usual sense:

Example 5 Suppose that $\varphi\colon C([0, T], E) \to E$ maps bounded sets to bounded sets and that there is an open bounded set $\Omega_0 \subseteq E$ such that

$$\lim_{u_0\to\partial\Omega_0} \|F(u_0)\| = \infty. \tag{27}$$

Suppose also that the set D_{Ω_0} of all mild solutions $u\colon [0, T] \to E$ of (1) satisfying $u(0) \in \Omega_0$ is bounded. Then (26) and thus (25) holds if $\Omega \subseteq \Omega_0$ is chosen sufficiently large. Actually, in (27) one can replace ∞ by the upper bound for $\varphi(D_{\Omega_0})$.

If we assume the linear growth condition (14), we can say more. The abstract reason for this is that in this case we can use some knowledge about the flow.

Indeed, in this case, we can actually allow that φ and F both grow to ∞ on the whole space, provided that just φ growth slightly slower than F in the following sense:

Example 6 For $\varphi\colon C([0, T], E) \to E,\ F\colon E \to E$, and $r \geq 0$, put

$$\varphi_0(r) := \sup_{\|u\|\leq r} \|\varphi(u)\| \quad \text{and} \quad F_0(r) := \inf_{\|u\|=r} \|F(u)\|.$$

Suppose that (14) holds and

$$\limsup_{r\to\infty} \frac{\varphi_0(cr)}{F_0(r)} < 1 \tag{28}$$

for every sufficiently large $c > 1$. Then (26) and thus (25) holds if Ω is a sufficiently large open ball with center 0. In fact, it suffices that (28) holds with *some* $c > C_2$ where C_2 denotes the constant occurring in Remark 3.

To see this, it suffices to note that if Ω is a ball with radius r and center 0 then Remark 3 implies that, if r is sufficiently large, the corresponding set (6) is bounded by cr.

If one takes more properties of the flow induced by A and f into account, some of the above examples can be dramatically improved. As a particular case to illustrate this, we treat the case of invariant sets.

Since we are speaking about mild solutions, let us remark that for mild solutions, we have analogous "concatenation" properties as for strong solutions: If u and v are mild solutions on $[a, b]$ and $[b, c]$ respectively with $u(b) = v(b)$ then the "concatenated" function on $[a, c]$ is a mild solution on $[a, c]$; conversely, if u is a mild solution on $[a, c]$ then u is also a mild solution on $[a, b]$ and on $[b, c]$. This can be seen straightforwardly from the definition using the semigroup property of U.

Now it makes sense to call a set $M \subseteq E$ *locally mildly invariant* for (1) if for each $t_0 \in [0, T)$ and each mild solution u of (1) on some interval $[t_0,\ t_0+\delta]$ with $u(t_0) \in M$ there holds $u([t_0,\ t_0+\varepsilon]) \subseteq M$ for sufficiently small $\varepsilon > 0$.

Proposition 3 *If M is locally mildly invariant for* (1) *and closed then for each mild solution u of* (1) *on some interval* $[0, t] \subseteq [0, T]$ *with* $u(0) \in M$ *there holds* $u([0, t]) \subseteq M$.

Proof Since M is closed, there is a maximal value $t_0 \in [0, t]$ with $u([0, t_0]) \subseteq M$ which by the above properties would in case $t_0 < t$ contradict the maximality of t_0.

Corollary 1 *If $\overline{\Omega}$ is contained in a bounded set $M \subseteq E$ which is locally mildly invariant for* (1) *then* (13) *holds and D is bounded.*

Example 7 Let Ω be an open ball with radius $r > 0$ and center zero. Suppose that $\overline{\Omega}$ is locally mildly invariant, and that there is some $c > 0$ such that $\|F(u_0)\| \geq c\|u_0\|$ whenever $\|u_0\| = r$ and $\|\varphi(u)\| \leq c\|u\|$ whenever $\|u\| = r$. Then (13) and (25) holds.

Be aware, however, that in contrast to strong solutions, it seems not possible for mild solution to use techniques like standard Ljapunov or guiding functions to verify that $\overline{\Omega}$ is locally mildly invariant. The only achievement in this direction we are aware of is the following.

Proposition 4 *Assume that* (12) *holds, and that* $f\colon [0, T] \times E \to E$ *is continuous. Let $M \subseteq E$ be closed and such that for each $t_0 \in [0, T)$, $\varepsilon > 0$, $u_0 \in M$, there is at most one mild solution of* (1) *on* $[t_0,\ t_0+\varepsilon]$ *satisfying* $u(t_0) = u_0$. *Then M is locally mildly invariant if and only if*

$$\liminf_{h\to 0^+} \frac{1}{h} \operatorname{dist}\big(U(h)v + hf(t, v), M\big) = 0 \quad \text{for each } t \in [0, T),\ \ v \in M. \tag{29}$$

Proof Without the uniqueness hypothesis, it has been shown in [22] (and also in [8], see the comments in [2]) that (29) is equivalent to the existence of a mild solution u on $[t_0,\ t_0+\varepsilon]$ (for some $\varepsilon > 0$) with $u(t_0) = u_0$ and $u([t_0,\ t_0+\varepsilon]) \subseteq M$. Now the claim follows by the supposed uniqueness property.

Remark 10 It is well-known that the uniqueness hypothesis of Proposition 4 is satisfied e.g. if each $(t_0,\ u_0) \in [0, T] \times E$ has a neighborhood in which the Lipschitz estimate

$$\|f(t,u) - f(t,v)\| \le k(t)\|u - v\|$$

holds with an integrable function k. (In fact, a slight modification of our argument used in our proof Lemma 2 proves this uniqueness.)

If M is invariant under the semigroup U, one can formulate a convenient sufficient condition for (29) which is independent of U:

Proposition 5 *For closed $M \subseteq E$ the condition*

$$\liminf_{h\to 0^+} \frac{1}{h} \operatorname{dist}\big(v + hf(t,v), M\big) = 0 \quad \textit{for each } t \in [0,T),\ \ v \in M. \tag{30}$$

implies (29) *if $U((0,T])M \subseteq M$. Moreover, if M is closed and convex then* (30) *is equivalent to*

$$\begin{gathered}\textit{for every } v \in M \textit{ every bounded linear functional } \ell \textit{ on } E \textit{ with}\\ \|\ell\| = 1 \textit{ and } \operatorname{Re}\ell(v) = \sup_{u\in M} \operatorname{Re}\ell(u) \textit{ satisfies } \sup_{t\in[0,T)} \operatorname{Re}\ell\big(f(t,v)\big) \le 0.\end{gathered} \tag{31}$$

Proof The assertions are shown in the proof of [2, Theorem 2] and e.g. [12, p. 32], respectively.

3 Example: Every Control Implies a Profile-Preserving Nonlinear Huge Growth

As an application of the results in Sect. 2, we consider the reaction-diffusion system

$$\frac{\partial u(t,x)}{\partial t} = D\Delta u(t,x) + f_0(t,x,u(t,x)), \quad t > 0, \quad x \in G \tag{32}$$

in a bounded domain $G \subseteq \mathbb{R}^N$ with Lipschitz boundary together with the natural Neumann boundary conditions

$$\frac{\partial u(t,\,\cdot\,)}{\partial n} = 0 \quad \text{on } \partial G \ \text{ for all } t > 0. \tag{33}$$

Here, $D \subseteq \mathbb{R}^{M\times M}$ is a positive definite matrix, the unknown $u\colon [0,\infty) \times \overline{G} \to \mathbb{R}^M$ is a vector function, and the Laplacian Δ is understood with respect to the space-variable x only.

Physically, the M components of $u(t,x)$ can mean the difference of the concentration of M chemical substances to some equilibrium at time t and space x, and f_0 can be the sum of a reaction term (corresponding to these M chemicals) and of some control term (that is, a source or sink which may depend on $(t, x,\ u(t,x))$); in this case, D is typically a diagonal matrix, the diagonal entries corresponding to the diffusion speed of the M chemical substances. Note that the reaction term will vanish in the equilibrium $u = 0$, but we will assume that there really is a control, that is, that the control term does not vanish at $u = 0$: This excludes the case that the equilibrium $u(t,x) \equiv 0$ is a stationary solution of (32) (our subsequent observation is mathematically valid for this case, but trivial).

In fact, we will show a "principle of huge growth": *Any* attempt of such a (nontrivial) control, if applied to an appropriate initial condition, will automatically increase this initial value without changing its profile in arbitrarily small time by an arbitrary large (even in nonlinear sense) factor (if we would have no control term, i.e. if the constant equilibrium $u = 0$ would be a stationary solution of (32), the assertion would be correct, but trivial, because a large factor of 0 has no meaning). Moreover, in some sense, not only the factor can be described, but this factor can also be negative (of large absolute value) and even matrix-valued; it can contain also a given permutation of the M components.

In order to describe only the simplest semigroup setting (without fractional power spaces), we suppose that f_0 is measurable with respect to x, that $f_0(\cdot, x, \cdot)$ is continuous for almost all $x \in G$, and that we have the sublinear growth condition

$$\|f_0(t, x, u)\| \le a(t, x) + b(t)\|u\|,$$

where

$$\int_0^\tau \left(\int_G |a(t, x)|^2 \, \mathrm{d}x \right)^{1/2} \mathrm{d}t < \infty \quad \text{and} \quad \int_0^\tau |b(t)| \, \mathrm{d}t < \infty \quad \text{for all } \tau > 0.$$

It is well-known that the abstract form of (32), and (33) in the space $E = L_2(G, \mathbb{R}^M)$ then can be written as (1) with A being the operator $D\Delta$ in the sense of distributional derivatives defined on the set of all $u \in W^{2,2}(\Omega, \mathbb{R}^M) \subseteq L_2(G, \mathbb{R}^M)$ satisfying (33) in the sense of traces, and $f : [0, T] \times E \to E$ being defined as

$$f(t, v)(x) := f_0(t, x, v(x)).$$

Indeed, the interpretation of u' in (32) corresponds to the partial derivative in (32) by means of [23, Theorem 4.4.4]: The relation between the abstract problem and (32), (33) is such that if u is a strong solution of the abstract problem then (32) and (33) holds for almost every $x \in G$.

It is well-known that A is the generator of a C_0-semigroup on E, and since A is symmetric and has compact resolvent, it follows that this semigroup is immediately compact, see e.g. [13, XIX, Corollary 6.3]. Moreover, by our continuity and growth requirements, it follows that f is actually continuous, see e.g. [25].

The announced principle of huge growth means mathematically that we consider (for possibly small $T > 0$) the boundary condition

$$\eta(\|u(0)\|^2)u(0) = \mu(\|u(T)\|)u(T), \tag{34}$$

where $\mu : [0, \infty) \to \mathbb{R}^{M \times M}$ is continuous, $\eta : [0, \infty) \to \mathbb{R}^{M \times M}$ is of class C^1, and $\eta(r)$ is invertible for every $r \ge 0$. The idea is that e.g. $\eta(r)$ and $\mu(r)$ are positive multiples of two given permutation matrices. If the absolute value of this multiple of $\mu(r)$ is small compared to that of $\eta(r)$, the condition (34) describes the announced huge growth.

The surprising observation is that, if just the quotient of these absolute values is large enough (e.g. if the factor for $\eta(r)$ growth faster than for $\mu(r)$ as $r \to \infty$), then there is always a corresponding initial value u_0 such that there is a solution of the problem (32) and (33) for this initial value $u(0) = u_0$ which is subject to the increase (34). More precisely, the following holds:

Theorem 4 *In the above example, if $r > 0$ satisfies*

$$\|\eta(r^2)^{-1}\|^{-1} r \ge \max_{0 < s \le C_1 + C_2 r} \|\mu(s)\| s \tag{35}$$

where $C_1 \ge 0$, $C_2 > 0$, are constants depending only on a, b, and T, then (32) and (33) has a mild solution u on $[0, T]$ with $\|u(0)\| \le r$ such that (34) holds.

In fact, C_1, C_2 can be calculated explicitly by means of the Gronwall lemma: They are the constants of Remark 3.

Proof We put $F(u_0) := \eta(\|u_0\|^2)u_0$ for $u_0 \in \overline{\Omega}$ where $\Omega \subseteq E$ denotes the open ball with center 0 and radius r. By various straightforward applications of the chain rule, we obtain that $F: E \to E$ is of class C^1 with derivative

$$DF(u_0)(h) = 2\langle u_0, h\rangle \eta'\big(\|u_0\|^2\big)u_0 + \eta\big(\|u_0\|^2\big)h.$$

Hence, $DF(u_0)$ is a finite rank (even one-dimensional) perturbation of the isomorphism $h \mapsto \eta(\|u_0\|^2)h$ of E and thus a Fredholm map of index zero.

Next, we note that (13) holds by Proposition 1. Since $u \mapsto u(T)$ is compact on (6) by Proposition 2 and $u \mapsto \mu(\|u(T)\|)$ is trivially compact, it follows that $\varphi(u) := \mu(\|u(T)\|)u(T)$ is compact on D. Since Remark 3 implies for $\|u(0)\| = r$ that $\|u(T)\| \le C_1 + C_2 r$, the admissibility condition (25) follows from (35) similarly as in Example 6. Hence, Theorem 3 applies (with the additional hypotheses of Remark 1, since F is odd).

To clarify Theorem 4, we apply it to the following simple special case (with constant $\eta \equiv cI$ and $\mu \equiv I$)

$$\begin{aligned} u_t &= \Delta u + f_0(t, u) \\ u(T) &= cu(0). \end{aligned} \tag{36}$$

Then, for every sublinear f_0 and every $T > 0$ there is a constant $c_0 > 0$ such that for every $c \ge c_0$ there is a solution u of (36), i.e. a solution of the equation which increases by an arbitrarily large factor $c > 0$ (and thereby keeping its "shape") in an arbitrary small time interval $[0, T]$.

In a similar manner, one could replace (34) by the more general condition

$$\lambda(\|u(0)\|^2)u(0,x) = \mu\left(\int_0^T \|u(t)\|^2\,\mathrm{d}\alpha_1(t),\ \ldots,\ \int_0^T \|u(t)\|^2\,\mathrm{d}\alpha_n(t)\right)\int_0^T u(t,x)\,\mathrm{d}\alpha(t) \tag{37}$$

for almost all $x \in G$ where the integrals are understood in the Stieltjes sense and α is of bounded variation and in a neighborhood of zero of class C^1 (so that Example 1 applies to $u \mapsto \int_0^T u(t,x)\,\mathrm{d}\alpha(t)$). Note that no such requirement is needed for $\alpha_1, \ldots, \alpha_n$: It suffices that these functions are of bounded variation, and they may even jump at $t = 0$.

Note that condition (37) means that the profile of $u(0, \cdot)$ is preserved (and increased for small μ and large λ) in an averaged (measured with weight α) sense.

All results in this section hold also for the heat equation, if we replace the Neumann condition (33) by the Dirichlet condition $u|_{\partial G} = 0$. The only difference is that the semigroup U differs, and that the domain of A is the set of all $u \in W^{2,2}(G)$ satisfying $u|_{\partial G} = 0$ in the sense of traces.

4 C_0-Semigroups

If we drop the hypothesis (12), we need much stronger hypotheses about f, and we do not obtain "almost automatically" that φ is compact (Proposition 2), but we have to require it. Apart from these restrictions, the results of the previous section have analogous results for the richer class of C_0-semigroups which we discuss now.

More precisely, we consider the problem (1) and (2) with a nonlinearity satisfying the following hypothesis.

$$\begin{aligned} & f = f_1 + f_2 \ \text{ such that for some } q \in L_1([0,T]) \text{ and almost all } t \in [0,T]: \\ & \qquad \|f_1(t,u) - f_1(t,v)\| \le q(t)\|u - v\| \quad \text{and} \quad f_2(t,\,\cdot\,) \ \text{is compact.} \end{aligned} \tag{38}$$

Under this assumption, it is possible to prove a similar regularity result for the map $\Phi\colon \overline{\Omega} \multimap D$ as in Sect. 2:

Lemma 3 *Suppose that f is such that (38), (13) and (17) hold. Then $\Phi\colon \overline{\Omega} \multimap D$ is upper semicontinuous and assumes only R_δ values.*

In particular, Φ assumes only nonempty values, and $D \neq \emptyset$.

Proof Note that (38) implies in view of (11) that, for almost all $t \in [0, T]$,

$$\chi\big(f(\{t\} \times M)\big) \le q(t)\chi(M) \quad \text{for every bounded } M \subseteq E, \tag{39}$$

where χ denotes the Hausdorff measure of noncompactness. Hence, for the case that $f(t,\,\cdot\,)$ is globally bounded by $c(t)$, the assertion is a special case of [15, Corollary 5.2.2 and Theorem 5.3.1]; for a slightly different approach (for bounded f), see [2].

To reduce the general case to this setting, we use the same argument as in Lemma 2, replacing f by (18) with B being an upper bound for (13): By this change neither any of the hypotheses nor the map Φ is changed so that the assertion follows from the shown special case.

Using Lemma 3, we can obtain also a weaker conclusion under weaker hypotheses which is in the spirit of Lemma 1.

Corollary 2 *Suppose that f is such that (38) and (16) hold. Then $\Phi(u_0)$ is compact for every $u_0 \in \overline{\Omega}$.*

We can of course not conclude under the weaker hypotheses of Corollary 2 that $\Phi(u_0) \neq \emptyset$.

Proof Let $u_0 \in \overline{\Omega}$ be fixed. Since $\Phi(u_0) \subseteq D$ is bounded by Lemma 1, we can use similar arguments as in the last part of the proof of Lemma 2, replacing f by

$$\widetilde{f}(t,u) := \begin{cases} f(t,u) & \text{if } \|f(t,u)\| \le c(t), \\ \frac{c(t)}{\|f(t,u)\|} f(t,u) & \text{if } \|f(t,u)\| \ge c(t). \end{cases}$$

Note that $\widetilde{f}$ is a Carathéodory function, because $\widetilde{f}(t,\,\cdot\,)$ is continuous by the glueing lemma. Note that f and $\widetilde{f}$ coincide on the set

$$M := \Big\{(t, u(t)) : t \in [0,T],\ u \in \overline{\Phi(u_0)}\Big\},$$

and the modified function $\widetilde{f}$ has the property that $\widetilde{f}(t,\,\cdot\,)$ is globally bounded by $c(t)$. Defining $\widetilde{\Phi}$ in the same way as Φ but with $\widetilde{f}$ in place of f, we thus find that $\widetilde{\Phi}$ assumes only R_δ values. Moreover, since $\widetilde{f}$ and f coincide on M, we have $\Phi(u_0) \subseteq \widetilde{\Phi}(u_0)$ and $\overline{\Phi(u_0)} \cap \widetilde{\Phi}(u_0) = \Phi(u_0)$. Hence, $\Phi(u_0)$ is a closed subset of the compact set $\widetilde{\Phi}(u_0)$.

Now if $\varphi\colon C([0,T],E) \to E$ is compact on D, we obtain an analogous abstract result as in Sect. 2. However, we cannot verify this compactness condition by an analogue of Proposition 2, and therefore we keep for a moment a more general compactness hypotheses:

Theorem 5 (abstract existence II) *Suppose that f is such that (38), (13) and (17) hold. Suppose that $\varphi|_D$ is continuous, and that for each countable sets $C \subseteq \Omega$ and countable $D_0 \subseteq \varphi(\Phi(C)) \subseteq D$ satisfying $F(C) \subseteq \overline{\mathrm{conv}}(\{0\} \cup \varphi(D_0))$ at least one of the sets $\overline{C}$ or $\overline{\varphi(D_0)}$ is compact. Finally, suppose that the admissibility hypothesis (25) holds.*

Then $\deg(F, \Omega, 0)$ is defined, and if it is nonzero, the boundary value problem (1) and (2) has a solution u with $u(0) \in \overline{\Omega}$.

Proof The proof is analogous to the proof of Theorem 3. The only difference is that we use Lemma 3 to verify the hypotheses of Theorem 1.

Remark 11 In Theorem 5, we obtain the additional conclusion that $\deg(F, \Phi, \varphi, \Omega)$ is defined and equal to $\deg(F, \Omega, 0)$, if there is no solution u of (1) and (2) with $u(0) \in \partial\Omega$.

Remark 12 As mentioned above, the compactness condition of Theorem 5 is trivially satisfied if $\overline{\varphi(D)}$ is compact. Unfortunately, this is "typically" only the case in two somewhat degenerate situations:

1. If E is a function space and, if we denote the corresponding function argument by x, φ is an integral operator with respect to (t, x).
2. If the semigroup is eventually compact, i.e. if there is $t_0 > 0$ such that $U(t)$ is compact for some (and thus all) $t \geq t_0$, and if in this case $\varphi(u)$ depends only on $u|_{[t_0,t]}$ (the argument for the compactness is then similar to our proof of Proposition 2).

Therefore, it is important to us that the compactness condition of Theorem 5 can also be satisfied if $\varphi|_D$ fails to be compact. We come to this hypothesis later on.

The same arguments as in Examples 2–7 can be used to verify the admissibility hypothesis (25) of Theorem 5. Note that Proposition 3 and Corollary 1 do not require the compactness hypothesis (12) anyway. However, we need an analogous result to Proposition 4 for locally mildly invariant sets in the setting of Theorem 5.

Proposition 6 *Assume that (38) holds and that $f\colon [0, T] \times E \to E$ is continuous with (17). Let $M \subseteq E$ be closed and such that for each $t_0 \in [0, T)$, $\varepsilon > 0$, $u_0 \in M$, there is at most one mild solution of (1) on $[t_0, t_0 + \varepsilon]$ satisfying $u(t_0) = u_0$. Then M is locally mildly invariant if and only if (29) holds for each $u_0 \in M$. Moreover, if $U((0, T])M \subseteq M$ (and M is convex) then (30) (or the equivalent sign condition (31), respectively) is sufficient for M being locally mildly invariant.*

Proof The last sentence of the assertion follows with Proposition 5. As stated above, condition (38) implies (39). Hence, the proof is analogous to the proof of Proposition 4 with the exception that the sufficiency part is not covered by [22] but only by [8] (or, more directly, also by [19]).

As already observed in Remark 12, for most problems, we need a different mean to verify the compactness hypothesis of Theorem 5. This hypothesis can sometimes be verified with the aid of measures of noncompactness mentioned in Remark 2: Roughly speaking, this condition holds if "F is quantitatively more proper than $\varphi \circ \Phi$ is noncompact on countable sets" in the following sense:

Remark 13 Let γ be a monotone measure of noncompactness on E which satisfies $\gamma(\{0\} \cup M) = \gamma(M)$ for all $M \subseteq E$, and let $\beta\colon 2^{\Omega} \to [0, \infty)$. Suppose also that at least one of the set functions γ and β has the properties that it vanishes at most on relatively compact subsets of E. If there are constants $c_1 > c_0 \geq 0$ such that

$$\gamma(F(C)) \geq c_1\beta(C) \quad \text{and} \quad \gamma(\varphi(D_0)) \leq c_0\beta(C) \\ \text{for all countable } C \subseteq \Omega \text{ and countable } D_0 \subseteq \Phi(C) \subseteq D. \tag{40}$$

then the compactness hypothesis of Theorem 5 is satisfied.

Indeed, if C and D_0 are as in Theorem 5 then $F(C) \subseteq \overline{\operatorname{conv}}(\{0\} \cup \varphi(D_0))$ implies that $\gamma(F(C)) \leq \gamma(\varphi(D_0))$ which together with (40), $c_1 > c_0 \geq 0$, and $\beta(C) < \infty$ implies $\beta(C) = \gamma(\varphi(D_0)) = 0$.

Estimates for the constant c_1 in (40) from below are in many cases easy to obtain. For instance, if $\gamma = \beta = \chi$ and F is a (nonlinear) compact perturbation of a linear isomorphism J, we can choose $c_1 = \|J^{-1}\|^{-1}$ (and this estimate is best possible if E has infinite dimension). Estimates for c_0 from above have been studied in literature, if φ is the evaluation functional $\varphi(u) = u(t)$ for some given point $t \in [0, T]$, see e.g. [1,3]. We show in a moment how more general cases can be reduced to that. But first, we refine the results from [1,3], covering also certain non-separable spaces E under milder hypotheses.

More precisely, our estimate applies if the space E has the so-called retraction property from [24], that is, if for each separable subspace $E_0 \subseteq E$ there exists a mapping $R\colon E \to E$ with separable $R(E)$ such that $R(u) = u$ for all $u \in E_0$ and $\|R(u) - R(v)\| \leq \|u - v\|$ for all $u, v \in E_0$. Of course, all separable Banach spaces have this property. However, as discussed in [24], also all weakly compactly generated Banach spaces and thus in particular all separable Banach spaces have this property (in these cases, R can even be chosen to be a linear projection onto E_0).

Proposition 7 *Let E have the retraction property. Let $M > 0$ and $\omega \in \mathbb{R}$ be such that $\|U(t)\| \leq Me^{\omega t}$ for all $t \in (0, T]$. Suppose that f is such that (38) and (16) hold. Then (6) is bounded, and with $\varphi_t(u) := u(t)$, we have*

$$\chi\left(\varphi_t(\Phi(C))\right) \leq M \exp\Big(\omega t + \int_0^t q(s)\,\mathrm{d}s\Big)\chi(C) \quad \textit{for all countable } C \subseteq \overline{\Omega} \textit{ and all} \\ t \in [0, T]. \tag{41}$$

Proof The boundedness of (6) is contained in Lemma 1. Let $C \subseteq \overline{\Omega}$ be countable. Since $\Phi(C)$ is a countable union of separable sets by Corollary 2, there is a countable dense $D_0 \subseteq \Phi(C) \subseteq D$. By definition of Φ, we have

$$u(t) \in U(t)C + \int_0^t U(t-s)f(s, u(s))\,\mathrm{d}s \quad \text{for all } u \in D_0,\ t \in [0, T]. \tag{42}$$

For $t \in [0, T]$, let M_t denote the set of all functions $s \mapsto U(t-s)f(s, u(s))$ on $[0, t]$ with $u \in D_0$. Put now $m(s) := \chi(\{u(s) : u \in D_0\})$ and $m_t(s) := \chi(\{v(s) : v \in M_t\})$. Since E has the retraction property, D_0 is countable and the functions from D_0 and M_t are measurable, we obtain from [24, Proposition 11.12] that m and m_t are measurable on $[0, T]$ or $[0, t]$, respectively. Moreover, since the functions from M_t are uniformly dominated by the integrable function $s \mapsto Me^{\omega(t-s)}c(s)$, the same result [24, Proposition 11.12] also implies that

$$\chi\left(\left\{\int_0^t v(s)\,\mathrm{d}s : v \in M_t\right\}\right) \le \int_0^t m_t(s)\,\mathrm{d}s. \tag{43}$$

Combining this with (42), we obtain

$$m(t) \le \chi(U(t)C) + \int_0^t \chi\big(M_t(s)\big)\,\mathrm{d}s.$$

Using (11), we obtain the estimate

$$m(t) \le M\mathrm{e}^{\omega t}\chi(C) + \int_0^t M\mathrm{e}^{\omega(t-s)}q(s)m(s)\,\mathrm{d}s.$$

Note that the boundedness of $D_0 \subseteq D$ implies that we actually have $m \in L_\infty([0,T])$. Putting $m_0(t) := e^{-\omega t}m(t)$, we obtain

$$m_0(t) \le M\left(\chi(C) + \int_0^t q(s)m_0(s)\,\mathrm{d}s\right)$$

which in view of $m_0 \in L_\infty([0,T])$ implies by a variant of Gronwall's lemma that

$$m_0(t) \le M\chi(C)\exp\left(\int_0^t q(s)\,\mathrm{d}s\right). \tag{44}$$

Since the continuity of φ_t implies $\overline{\{u(t) : u \in D_0\}} = \overline{\varphi_t(\Phi(C))}$, we have

$$\chi(\varphi_t(\Phi(C))) = m(t) = \mathrm{e}^{\omega t}m_0(t),$$

and so the assertion follows from (44).

Remark 14 Our above proof of Proposition 7 shows why it is crucial that we have to verify the compactness conditions only with countable sets: For uncountable D_0, the function m_t in the proof fails to be measurable, in general. Moreover, even if one replaces m_t by a measurable majorant, the estimate (43) holds in this case only if an additional factor 2 is inserted on the right-hand side, see [24, Example 11.5 and Theorem 11.17]; this factor would occur as a factor of the integral in the formula (41).

However, this calculation is not even valid at all if m_t fails to be measurable. In [1], the latter problem could only be solved by requiring norm-continuity of U on $(0,\infty)$ which is too strong a requirement for most problems. By considering only countable sets, we have no such difficulties. A similar observation (using countable sets) was made in [3], but Proposition 7 is even stronger: We do not have the factor two in front of the integral.

Proposition 7 can be used directly only if φ is the evaluation functional. However, also for many other choices of φ simple additional observations are sufficient to obtain upper estimates for the quantity c_0 from (40). As a simple example let us consider the perhaps most important case of the boundary condition (3) where $\varphi(u) := G(u(T))$ with some function $G\colon E \to E$.

Example 8 Let E have the retraction property. If $G = G_1 + G_2$ where G_1 is Lipschitz with constant L and G_2 is compact, $\|U(t)\| \le Me^{\omega t}$ for all $t \in (0, T]$, and f satisfies (38) and (16), then the second estimate of (40) holds with $\gamma = \beta = \chi$ and

$$c_0 := LM \max_{t\in[0,T]} \exp\left(\omega t + \int_0^t q(s)\,\mathrm{d}s\right).$$

In particular, if F and $c_1 > c_0$ are such that also the first estimate of (40) holds with $\gamma = \beta = \chi$, then the compactness hypothesis of Theorem 5 is satisfied.

Actually, instead of the form $G = G_1 + G_2$, it suffices to require that G satisfies

$$\chi(G(C_0)) \le L\chi(C_0) \quad \text{for each countable bounded } C_0 \subseteq E.$$

Indeed, if $C \subseteq \Omega$ and $D_0 \subseteq \Phi(C)$ are countable then Proposition 7 implies with $C_0 := \varphi_T(D_0)$ that

$$\chi(\varphi(D_0)) = \chi(G(C_0)) \le L\chi(C_0) \le L\chi\big(\varphi_T(\Phi(C))\big) \le c_0\chi(C),$$

as required.

Remark 15 As our proofs show, all results in Sect. 4 remain valid if we replace (38) by the milder condition

$$\chi\big(f(\{t\}\times C)\big) \le q(t)\chi(C) \quad \text{for every countable bounded } C \subseteq E, \tag{45}$$

for almost all $t \in [0, T]$ (note that this implies (39) with q replaced by $2q$, see e.g. [26, (3.20)]). In Proposition 7 and Example 8, we can additionally relax the hypothesis $\|U(t)\| \le Me^{\omega t}$ to the hypothesis

$$\chi(U(t)C) \le M\mathrm{e}^{\omega t}\chi(C) \quad \text{for all countable } C \subseteq E$$

for all $t \in [0, T]$.

Moreover, by using the full power of the cited results [15, Corollary 5.2.2 and Theorem 5.3.1], we obtain analogous assertions for the results in this section even for the inclusion problem

$$u'(t) \in Au(t) + f(t, u(t)) \qquad (0 < t \le T)$$

when f is multivalued with nonempty closed convex values and "upper Carathéodory", that is, $f(\,\cdot\,, u)$ is strongly measurable for every $u \in E$ and $f(t,\,\cdot\,)$ is upper semicontinuous on E for almost all $t \in [0, T]$. The main difference is that for Corollary 2 and Proposition 7 one must replace D by $\overline{D}$ in the hypothesis (16). Since the rest needs no essentially new ideas for this multivalued case but just requires a more clumsy notation, we leave the further formulation of these generalizations to the reader.

5 Example: Age-Population Model

To demonstrate the results of Sect. 4, we apply them to a nonlinear hyperbolic conservation law, describing the time evolution of the age structure of a population. The most widely used versions of this application are the well known McKendrick-von Forster linear model [20,28] and the Gurtin-MacCamy nonlinear model [14], for which many modifications have been suggested. For a different approach to this model, see [5]. Similarly as in [9], we replace the averages (integrals) by pointwise evaluations of the function, but, as in [16], we consider a more general nonlinear boundary condition. Moreover, we add an integral term for covering an averaged death rate. More precisely, the problem we consider is the following:

$$\begin{cases} \frac{\partial u(t,x)}{\partial x} + \frac{\partial u(t,x)}{\partial t} + f_0(x,t,u(t,x))u(t,x) \\ \qquad + \int_0^\alpha \hat{f}_0(x,t,s,u(s,x))\,\mathrm{d}s = 0, \\ & x \in [0,T], t \in [0,\alpha], \\ u(0,x) = u_0(x), & x \in [0,T], \\ u(t,0) = g\Big(\int_0^T b(x)u(t,x)\,\mathrm{d}x\Big), & t \in [0,\alpha]. \end{cases} \tag{46}$$

The independent variables t and x denote time and age, respectively, and $u(t,x)$ represents the density of individuals of age x at time t. f_0 describes the death rate (and $\hat{f}_0$ can be used to model some part of this death rate in a more discrete way by replacing $f_0(t,x,u)u$ by a simpler, e.g. piecewise constant (or piecewise linear) function in t, with values e.g. depending on corresponding time intervals over $f_0(t,x,u)u$), and the boundary condition accounts for the birth in the population, depending on the fertility rate b and the reproduction rate g. After exchanging the roles of x and t, and after an obvious substitution to transform the original initial condition with respect to t (now a boundary condition with respect to x) to the case $u_0 \equiv 0$, assuming that $u_0 \in W^{1,p}([0,T],\mathbb{R})$ (with $1 \le p < \infty$ fixed) and $b \in L_1([0,T])$, we are led to the problem

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} = \dfrac{-\partial v(t,x)}{\partial x} - f_0(t,x,v(t,x)+u_0(t))(v(t,x)+u_0(t)) \\ \qquad + u_0'(t) - \displaystyle\int_0^\alpha \hat{f}_0(t,x,\xi,v(t,\xi)+u_0(t))\,\mathrm{d}\xi, \quad t \in [0,T],\ x \in [0,\alpha], \\ v(t,0) = 0, \quad t \in [0,T], \\ v(0,x) = g\Big(\displaystyle\int_0^T b(t)\big(v(t,x)+u_0(t)\big)\,\mathrm{d}t\Big) - u_0(0), \quad x \in [0,\alpha]. \end{cases} \tag{47}$$

Note that, due to our substitution, also negative solutions of this model have a meaningful interpretation. We assume that $f_0(\,\cdot\,,\,\cdot\,,u)$ and $\hat{f}_0(\,\cdot\,,\,\cdot\,,\,\cdot\,,u)$ are measurable for all $u \in \mathbb{R}$ and that $\hat{f}_0(t,x,\xi,\,\cdot\,)$ is continuous for almost all $(t,x,\xi) \in [0,T] \times [0,\alpha]^2$ and satisfies an at most linear growth condition

$$\left|\hat{f}_0(t,x,\xi,u)\right| \le a_0(t)(1+|u|)$$

with some $a_0 \in L_1([0,T])$. With the space $E := L_p([0,\alpha],\mathbb{R})$ and $f_2\colon [0,T] \times E \to E$,

$$f_2(t,u)(x) = \int_0^\alpha \hat{f}_0(t,x,\xi,u(\xi))\,\mathrm{d}\xi,$$

this implies that the Urysohn integral operator $f_2(t,\,\cdot\,)\colon E \to E$ is compact and continuous (see e.g. [24, Corollary 9.12]), and satisfies a growth estimate of the form (14) with some $a \in L_1([0,T])$. It also follows from [23, Theorem 4.4.2] that $f_2(\,\cdot\,,u)\colon [0,T] \to E$ is measurable for every $u \in E$. We define $\hat{f}_1\colon [0,T] \times E \to E$ by

$$\hat{f}_1(t,u)(x) := -f_0(t,x,u(x))u(x).$$

As in Sect. 3, we can now rewrite problem (47) in the abstract form in the space E, using that by [23, Theorem 4.4.4] we can identify the derivative in (1) with the partial derivative in (47). The problem then assumes the form (1) with $Ay = -y'$ in the sense of distributional derivatives defined on $\{u \in W^{1,p}([0,\alpha],\mathbb{R}) : u(0) = 0\}$, and $f : [0,T] \times E \to E$ being defined as $f = f_1 + f_2$ with

$$f_1(t,u) := \hat{f}_1(t, u + u_0(t)) + u_0'(t).$$

(Here, $u_0(t), u_0'(t) \in E$ are understood as constant functions). A is the generator of the translation C_0-semigroup on E, with $\|U(t)\| \leq 1$ for every $t \geq 0$, see e.g. [13, p. 420] (for $E = L_p([0,\infty))$, but our case is similar). The nonlinear boundary condition of (47) is rewritten in the abstract form as

$$v(0) = \varphi(v) := G(J(v + u_0)) - u_0(0),$$

where $G : E \to E$ is the superposition operator

$$G(w)(x) := g(w(x))$$

and $J : C([0,T],E) \to E$ is the integral operator

$$J(v)(x) := \Big(\int_0^T b(t)v(t)\,\mathrm{d}t\Big)(x).$$

Here, we use tacitly that every $v \in C([0,T],E)$ has a natural interpretation as a function $\hat{v}(t,x) = v(t)(x)$ such that $\hat{v}$ is measurable on $[0,T] \times [0,\alpha]$ and such that

$$J(v)(x) = \int_0^T b(t)\hat{v}(t,x)\,\mathrm{d}t \quad \text{for almost all } x \in [0,\alpha],$$

see [23, Theorems 4.4.2 and 4.4.3].

We will assume that f_0 satisfies a Lipschitz condition with respect to v such that the Lipschitz constant decreases on large balls, i.e.

$$|f_0(t,x,u) - f_0(t,x,v)| \leq a_1(t)\,|u - v|\,/\max\big\{1, \min\{|u|, |v|\}\big\}.$$

In this context, it is natural to suppose that f_0 is globally bounded, i.e.

$$|f_0(t,x,u)| \leq a_2(t). \tag{48}$$

Here, we suppose that $a_1,\ a_2 \in L_1([0,T],\mathbb{R})$.

Note that this hypothesis implies for all x with $|u(x)| \leq |v(x)|$ that

$$\begin{aligned}\Big|\hat{f}_1(t,u)(x) - \hat{f}_1(t,v)(x)\Big| &= |(f_0(t,x,u(x)) - f_0(t,x,v(x)))\,u(x) + f_0(t,x,v(x))\,(u(x) - v(x))| \\ &\leq (a_1(t) + a_2(t))\,|u(x) - v(x)|\,,\end{aligned}$$

and by symmetry reasons, an analogous estimate holds also for those x with $|u(x)| \geq |v(x)|$. Hence, $\hat{f}_1(t,\cdot) : E \to E$ and thus also $f_1(t,\cdot) : E \to E$ are Lipschitz with constant $q(t) := a_1(t) + a_2(t)$. It follows that f satisfies (38) and also (14). In particular, if we put $\Omega = \{u \in E : \|u\| \leq R\}$ (where $R \in (0,\infty)$ is arbitrary but fixed), we have (13) by Proposition 1, and also (17) holds.

Lemma 4 *Under the above hypotheses, we have for every countable $C \subseteq E$ that*

$$\chi\big(J(\Phi(C))\big) \leq \int_0^T |b(t)| \exp\Big(\int_0^t q(s)\,\mathrm{d}s\Big)\,\mathrm{d}t \cdot \chi(C).$$

Proof We can assume $\chi(C) < \infty$ and thus (increasing R if necessary) that $C \subseteq \Omega$. By Proposition 7, we have for all $t \in [0, T]$

$$\chi(\{b(t)v(t) : v \in \Phi(C)\}) = |b(t)|\, \chi(\{v(t) : v \in \Phi(C)\}) \\ \le r(t) := |b(t)| \exp\Big(\int_0^t q(s)\,\mathrm{d}s\Big)\chi(C).$$

Since the function which associates to every $t \in [0, T]$ the linear multiplication operator in E by the factor $b(t)$ is measurable as a function from $[0, T]$ into the space of bounded linear operators on E with the operator norm, we obtain from [24, Theorem 11.2] that

$$\chi\left(\left\{\int_0^T b(t)v(t) : v \in \Phi(C)\right\}\right) \le \int_0^T r(t)\,\mathrm{d}t$$

which is the assertion.

It follows similar to Example 8 in view of (11) that, if g (and thus G) is Lipschitz with constant L, we have (40) with $F = I$, $c_1 = 1$, and

$$c_0 := L \int_0^T |b(t)| \exp\Big(\int_0^t q(s)\,\mathrm{d}s\Big)\,\mathrm{d}t.$$

Hence, the compactness hypothesis of Theorem 5 holds if $c_0 < 1$. Theorem 5 thus implies:

Theorem 6 *If under the above hypotheses we have g Lipschitz with constant L and $c_0 < 1$, and if $R > 0$ is such that the admissibility condition*

$$\begin{aligned} &\textit{every mild solution } v \textit{ of } v'(t) = Av(t) + f(t, v(t)) \textit{ on } [0, T] \textit{ with } \|v(0)\| = R \\ &\quad \textit{satisfies } v(0) \ne \lambda\varphi(v) \textit{ for all } \lambda \in [0, 1) \end{aligned} \tag{49}$$

holds then (47) has a mild solution in $E = L_p([0, \alpha])$ satisfying $\|v(0, \cdot)\| \le R$.

The condition $c_0 < 1$ may appear rather restrictive, and indeed it is so if the function f_0 is bounded away from zero for large t. However, if one models essentially by using averages over fixed time intervals, that is, using $\hat{f}_0$ instead of f_0, this condition becomes much milder.

After all, the surprising assertion is that we are able to treat at all such a general problem for a hyperbolic conservation law, and moreover, that we can do this with topological methods. Roughly speaking, from a mathematical point of view, topological methods require some compactness which is not produced by $U(t)$. Heuristically, this noncompactness reflects possible shock waves in hyperbolic conservation laws. Note that for fixed f_0 the condition $c_0 < 1$ becomes a hypothesis *only* on the boundary condition, that is, our result can apply in certain situations even if shock waves occur in the equation.

We are now going to study two cases in which the admissibility condition (49) holds. The first follows from Example 6:

Corollary 3 *If under the above hypotheses we have g Lipschitz with constant L and $c_0 < 1$ and*

$$\lim_{y \to \pm\infty} g(y)/y = 0,$$

then (47) and thus also (46) has a mild solution.

Proof The hypothesis implies

$$\lim_{R\to\infty} \sup_{\|u\|\le R} \|G(u)\|/R = 0. \tag{50}$$

Indeed, by hypothesis we find for any $\varepsilon > 0$ some $r > 0$ such that $|g(y)| \le \varepsilon\,|y|$ if $|y| > r$. Since $|g(y)| \le C_r := |g(0)| + Lr$ if $|y| \le r$, we obtain $|g(y)| \le \varepsilon\,|y| + C_r$ for all y. Denoting by C_r also the constant function from E, we thus find for all $R \ge \|C_r\|/\varepsilon$ and all $u \in E$ with $\|u\| \le R$ that

$$\|G(u)\| \le \varepsilon\|u\| + \|C_r\| \le 2\varepsilon R,$$

and so (50) is established.

Since J is linear, a straightforward calculation shows that (50) implies (28) (with $F_0(r) = r$), and so we obtain from Example 6 that (49) holds with some $R > 0$.

As another example, we make use of mildly invariant sets. We observe first that by our hypothesis and according to [23, Theorems 4.4.2 and 4.4.3]

$$\|\varphi(u) - \varphi(v)\| \le L\|J\|\|u - v\| \le L \int_0^T |b(t)|\, \mathrm{d}t \cdot \|u - v\| \le c_0\|u - v\|,$$

and so in particular

$$\|\varphi(u)\| \le \|\varphi(0)\| + c_0\|u\| = \|G(J(u_0)) - u_0(0)\| + c_0\|u\|.$$

Since $c_0 < 1$, we thus find from Example 7 with $F_0(r) = r$ that, if

$$R \ge R_0 := \|G(J(u_0)) - u_0(0)\|/(1 - c_0)$$

and $\overline{\Omega}$ is locally mildly invariant, the admissibility Hypothesis (49) is satisfied.

In order to guarantee that $\overline{\Omega}$ is locally mildly invariant for all large $R > 0$, we need further assumptions:

In the following, we require in addition to the previous assumptions that

$$\begin{aligned} &u_0 \in C^1([0,T],\mathbb{R}),\ f_0(\,\cdot\,, x,\,\cdot\,) \text{ is continuous for almost all } x \in [0,\alpha],\\ &\quad \hat{f}_0 = 0 \text{ and } f_0(t,x,u) \ge c > 0 \text{ for all } (t,u) \in [0,T]\times\mathbb{R}, \quad \text{for almost all } x \in [0,\alpha]. \end{aligned} \tag{51}$$

The continuity hypothesis in (51) implies together with the growth hypothesis (48) that $\hat{f}_1$ and thus also f_1 are continuous, see e.g. [25]. Note that $\hat{f}_0 = 0$ implies $f_2 = 0$, and so f is continuous, and $f(t,\,\cdot\,) = f_1(t,\,\cdot\,)$ satisfies a Lipschitz condition with constant $q(t)$. In particular, the uniqueness hypothesis of Proposition 6 is satisfied by Remark 10. It remains to verify (29) with $M := \overline{\Omega}$ for sufficiently large $R > 0$. Since M is convex, and $\|U(t)\| \le 1$ implies $U((0,T])M \subseteq M$, it suffices to verify that (31) holds with sufficiently large $R > 0$.

To this end, we observe first that if ℓ is a linear functional on E with $\|\ell\| = 1$ then

$$\sup_{u\in M} |\ell(u)| = R,$$

so that a functional as in (31) can exist only if $\ell(v) = R$, hence if $\|v\| = R$. We can assume that ℓ in (31) has the form

$$\ell(u) = \int_0^\alpha u(x)w(x)\,\mathrm{d}x$$

with some $w \in L_{p/(p-1)}([0,\alpha])$ and, since we have equality in Hölder's inequality, it follows that

$$w(x) = R^{1-p}\,|v(x)|^{p-2}\,v(x) \quad \text{for almost all } x \in [0,\alpha] \text{ with } v(x) \neq 0.$$

With the shortcut $h(t,x) := f_0(t,x,v(x)+u_0(t))$, we calculate

$$\ell(f(t,v)) = -R^{1-p}\int_0^{\alpha} h(t,x)\,|v(x)|^p\,\mathrm{d}x + \ell\big(-h(t,\,\cdot\,)u_0(t) + u_0'(t)\big).$$

Since $h(t,x) \geq c$ by (51), $\|v\|^p = R^p$, and $|h(t,x)| \leq a_2(t)$, we obtain

$$\ell(f(t,v)) \leq -cR + \|a_2(t)\,|u_0(t)| + \left|u_0'(t)\right|\|.$$

Hence, if

$$R \geq R_1 := \sup_{t\in[0,T]} \big(a_2(t)\,|u_0(t)| + \left|u_0'(t)\right|\big)\alpha^{1/p}/c,$$

we obtain $\ell(f(t,v)) \leq 0$, and so $M = \overline{\Omega}$ is locally mildly invariant by Proposition 6. Summarizing, we conclude with Theorem 6:

Corollary 4 *If g Lipschitz with constant L and $c_0 < 1$, and if the additional hypotheses* (51) *and $R_1 < \infty$ are satisfied, then the problem* (47) *has a mild solution in $E = L_p([0,\alpha])$ satisfying*

$$\|v(t,\,\cdot\,)\| \leq \max\{R_0, R_1\} \quad \textit{for all } t \in [0,T].$$

In particular, (46) *has a mild solution.*

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