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Wavelet-Galerkin method for one-dimensional elastoplasticity and damage problems: Constitutive modeling and computational aspects

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Abstract

This work presents an analysis of the wavelet-Galerkin method for one-dimensional elastoplastic-damage problems. Time-stepping algorithm for non-linear dynamics is presented. Numerical treatment of the constitutive models is developed by the use of return-mapping algorithm. For spacial discretization we can use wavelet-Galerkin method instead of standard finite element method. This approach allows to locate singularities. The discrete formulation developed can be applied to the simulation of one-dimensional problems for elastic–plastic-damage models.

Keywords: Elastoplasticity; Damage mechanics; Wavelets; Galerkin method

1. Formulation of the problem

The governing equations obtained from the study of dynamic response of an elastoplastic-damage problem are represented by the mathematical model that we describe below.

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Let Ω a body with boundary $\partial\Omega$ in a one-dimensional space $\mathbb{R}$. The space variable is denoted by $x \in \Omega$, the time variable by $t \in I =]0, T[$ and $u(x,t) \in \mathbb{R}$ represent the displacement of a point x in the body. The linearized strain is represented by $\epsilon = u_{,x}(x,t)$ and the Cauchy stress σ is related to ϵ through a constitutive relation (specified in Section 6). The body force is written as $f : \Omega \times I \to \mathbb{R}$ and considering that it has axial motion, the problem is governed by the equation

$$\rho u_{,tt} = \sigma_{,x} + f, \quad \text{on } \Omega \times I, \tag{1}$$

where $\Omega =]0, L[$. The boundary and initial conditions are specified as

$$u(x,t) = g(x,t), \quad \text{on } x \in \partial\Omega_g,\ t \in I, \tag{2}$$

$$\sigma_n(x,t) = z(x,t), \quad \text{on } x \in \partial\Omega_z,\ t \in I, \tag{3}$$

$$u(x,0) = u_0(x), \quad \text{on } x \in \Omega, \tag{4}$$

$$\dot{u}(x,0) = \dot{u}_0(x), \quad \text{on } x \in \Omega, \tag{5}$$

where σ_n is the normal stress on the body. The sets $\partial\Omega_g$ and $\partial\Omega_h$ are Dirichlet and Neumann types. We have $\partial\Omega_g \cap \partial\Omega_z = \emptyset$ and $\overline{\partial\Omega_g \cup \partial\Omega_z} = \overline{\partial\Omega}$.

2. Variational formulation

To define the variational formulation of the dynamic problem (Eqs. (1)–(5)), we need to characterize two classes of functions.

For each t, let S_t denote the space of trial solutions,

$$S_t = \{u(.,t) : u(x,t) = g(x,t), x \in \partial\Omega_g, u(.,t) \in H^1(\Omega)\},$$

where $H^1(\Omega)$ is a Hilbert space and let V, the usual space of weighting functions, or test functions, satisfying zero-displacement Dirichlet boundary conditions, namely

$$V = \{w : w(x) = 0, x \in \partial\Omega_g, w \in H^1(\Omega)\}.$$

The weak formulation for the problem (Eqs. (1)–(5)) consists in find the solution $u(x,t) \in S_t$, $t \in [0,T]$ satisfying, for all $w \in V$,

$$\int_\Omega \rho u_{,tt} w \,\mathrm{d}V + G(\sigma, w) = 0 \tag{6}$$

and

$$\int_\Omega \rho u(x,0) w \,\mathrm{d}V = \int_\Omega \rho u_0(x) w \,\mathrm{d}V, \tag{7}$$

$$\int_\Omega \rho \dot{u}(x,0) w \,\mathrm{d}V = \int_\Omega \rho \dot{u}_0(x) w \,\mathrm{d}V, \tag{8}$$

where

$$G(\sigma, w) = \int_\Omega \sigma w_{,x} \,\mathrm{d}V - \int_{\partial\Omega_z} z w \,\mathrm{d}S - \int_\Omega f w \,\mathrm{d}V. \tag{9}$$

Here we suppose f, g, z, u_0 and $\dot{u}_0$ given functions.

3. Galerkin formulation

Galerkin methods are projections methods. In the Galerkin method we seek the solution which minimizes the error in an integral sense over the entire domain. In this section we describe the formulation of the (Bubnov-) Galerkin method for the governing equations. For our problem, it is defined via finite-dimensional subspaces approximations S_t^h and V^h to S_t and V, respectively.

We assume all members of V^h vanish, or vanish approximately, on boundary $\partial\Omega_g$ and that each member of S_t^h admits the representation $u^h = v^h + g^h$, where $v^h \in V^h$ and $g^h \in S_t^h$, results in satisfaction of boundary

condition $u = g$ on $\partial\Omega_g$. We can write the Galerkin formulation as follows: given $f, g, h, u_0, \dot{u}_0$, find $u^h(x,t) \in S_t^h, t \in [0,T]$, such that for all $w^h \in V^h$,

$$\langle \rho u^h_{,xx}, w^h \rangle + G(\sigma^h, w^h) = 0 \tag{10}$$

and

$$\langle \rho u^h(x,0), w^h \rangle = \langle \rho u_0(x), w^h \rangle, \tag{11}$$

$$\langle \rho \dot{u}^h(x,0), w^h \rangle = \langle \rho \dot{u}_0(x), w^h \rangle, \tag{12}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product

$$\langle u, w \rangle = \int_\Omega u\omega \, \mathrm{d}V.$$

Fixing bases in S_t^h and V^h, Eqs. (10)–(12) reduce to a system of ordinary differential equations, that are better described in Section 7.

4. Wavelet analysis

Wavelet transform and wavelet bases were originally used as a powerful tool for signal and image processing. It was developed as an extension of short time Fourier transform in order to decompose the frequency content of a signal in both time and frequency domain. The fundamental purpose of transformation techniques like Fourier transform is to give the frequency information of the time dependent signal. However, they do not provide any information about the frequencies in the time domain, i.e., what frequencies exist at what time. The short time Fourier transform analyzes the signal using fixed time windows to extract the frequency information in the time domain. Wavelet transform takes it one step further by using variable time windows for different frequencies, which enables the analysis of signals at different resolutions. That is, the Fourier transform is global and provides a description of the overall regularity of signals, but it is not well adapted for finding the location and the spatial distribution of singularities. This is a major motivation for studying the wavelet transform in mathematics and applied sciences.

Wavelets are functions generated from one single function called the mother wavelet by the operations of dilation and translation. A set of shifted and dilated self-similar functions

$$\psi_b^a = a^{-1/2}\psi\left(\frac{x-b}{a}\right)$$

form a basis, where b is the translation parameter and a is the dilation or scale parameter. Through translation by a factor of $j2^{-k}$ and dilation by a factor of 2^k, a set corresponding to resolution level k is generated

$$\psi_j^k = 2^{k/2}\psi(2^k x - j).$$

Therefore, by varying j the function is shifted on the x-axis and by varying k the amplitude of the function is varied.

Daubechies [1] has given a procedure to construct compactly supported wavelets with arbitrary regularity. However, the price for such nice property is lack of symmetry and wide support. This drawback disappears in the context of biorthogonal wavelets, a concept introduced by Cohen et al. [2]. In this context two non-orthogonal basis functions ψ_j^k and ψ_j^{*k} also called wavelets, are constructed from the scaled translates of two dual basic functions ψ and ψ^*. Instead of orthogonality, biorthogonality is required between the two families. In a biorthogonal setting one has more flexibility than in an orthogonal one. This additional degree of freedom can be used to adapt the bases to the specific problem at hand.

4.1. Multiresolution analysis of $L^2(\mathbb{R})$

Multiresolution analysis is a method, which allows us to communicate the information contained a lower resolution level to a higher resolution level and vice-versa.

We define a multiresolution analysis of $L^2(\mathbb{R})$ – $\{V^k,\phi\}$ – as a sequence of closed subspaces V^k of $L^2(\mathbb{R})$ with associated scaling function ϕ satisfying

1. $\cdots \subset V^{-2} \subset V^{-1} \subset V^0 \subset V^1 \subset V^2 \subset \cdots$,
   $\cap_{k\in\mathbb{Z}} V^k = \{0\}$,
   $L^2(\mathbb{R}) = \overline{\cup_{k\in\mathbb{Z}} V^k}$,
2. $f(x) \in V^k \Longleftrightarrow f(2x) \in V^{k+1}$;
3. $f(x) \in V^0 \Longleftrightarrow f(x-j) \in V^0$, $j \in \mathbb{Z}$;
4. $\phi(x-j)_{j\in\mathbb{Z}}$, form a Riesz basis of V^0.

Due to these properties, there is a sequence $h \in \ell^2$ so that ϕ satisfies the scale relation

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} h(k)\phi(2x-k)$$

and for each k the family

$$\phi_j^k(x) = 2^{k/2}\phi(2^k x - j), \quad j \in \mathbb{Z},$$

form a Riesz basis for V^k space.

For every $k \in \mathbb{Z}$, define W^k to be the complement of V^k in V^{k+1}. We have,

$$V^{k+1} = V^k + W^k.$$

It means that W^k contains the difference information between the resolution level k and the finest resolution level $k+1$. There are many multiresolution analysis, next we describe the biorthogonal multiresolution analysis.

Biorthogonal multiresolution analysis consists of a pair $\{V^k,\phi\}$ and $\{V^{*k},\phi^*\}$ of multiresolution analysis such that the scaling functions ϕ e ϕ^* satisfies the biorthogonality relation

$$\langle \phi(.-j), \phi^*(.-\ell)\rangle = \delta_{j,\ell},$$

where δ is the delta of Kronecker.

In the same manner, for fixed k, the families $\{\phi_j^k\}$ e $\{\phi_j^{*k}\}$ are biorthogonal

$$\langle \phi_j^k, \phi_\ell^{*k}\rangle = \delta_{j,\ell}.$$

The wavelet functions ψ and ψ^* are defined by

$$\psi(x) = 2\sum_{j\in\mathbb{Z}} g(j)\phi(2x-j),$$
$$\psi^*(x) = 2\sum_{j\in\mathbb{Z}} g^*(j)\phi^*(2x-j),$$

where

$$g(j) = (-1)^{j+1}h^*(-j+1),$$
$$g^*(j) = (-1)^{j+1}h(-j+1)$$

and the families of functions

$$\psi_j^k(x) = 2^{k/2}\psi(2^k x - j),$$
$$\psi_j^{*k}(x) = 2^{k/2}\psi^*(2^k x - j),$$

form a Riesz basis for W^k and W^{*k}.

The families of functions considered satisfy the biorthogonality conditions

$$\langle \phi_j^k, \psi_\ell^{*k} \rangle = 0,$$
$$\langle \phi_j^{*k}, \psi_\ell^k \rangle = 0,$$
$$\langle \psi_j^{*k}, \psi_\ell^m \rangle = \delta_{k,m}\delta_{j,\ell}.$$

In this context, if $f \in L^2(\mathbb{R})$ and $\mathscr{P}^k$ is the projection of f onto V^k then

$$\mathscr{P}^k f(x) = \sum_j \langle f, \phi_j^{*k} \rangle \phi_j^k(x).$$

The projection onto W^k defined by

$$\mathscr{Q}^k f(x) = \sum_j \langle f, \psi_j^{*k} \rangle \psi_j^k(x),$$

contains the difference of information between the levels k and $k+1$, i.e.,

$$\mathscr{Q}^k f(x) = [\mathscr{P}^{k+1} - \mathscr{P}^k] f(x).$$

So,

$$\mathscr{P}^{k+1} f(x) = [\mathscr{P}^k + \mathscr{Q}^k] f(x), \tag{13}$$

corresponds to the decomposition,

$$V^{k+1} = V^k + W^k.$$

For multiscale decomposition

$$\mathscr{P}^{k+1} f(x) = [\mathscr{P}^{k_0} + \mathscr{Q}^{k_0} + \cdots + \mathscr{Q}^k] f(x), \tag{14}$$

where $k_0 < k$. This corresponds to the sum

$$V^{k+1} = V^{k_0} + W^{k_0} + \cdots + W^k.$$

Defining

$$c_j^k = \langle f, \phi_j^{*k} \rangle, \quad d_j^k = \langle f, \psi_j^{*k} \rangle,$$

Eq. (13) can be write as

$$\sum_j c_j^{k+1} \phi_j^{k+1}(x) = \sum_j c_j^k \phi_j^k(x) + \sum_j d_j^k \psi_j^k(x) \tag{15}$$

and Eq. (14) as

$$\sum_j c_j^{k+1} \phi_j^{k+1}(x) = \sum_j c_j^{k_0} \phi_j^{k_0}(x) + \sum_{m=k_0}^{k} \sum_j d_j^m \psi_j^m(x). \tag{16}$$

From above we conclude that there are two forms to represent functions in wavelet analysis. The first (15) expands the function using the scale functions and the second (16) expands the solution in terms of wavelet functions. Besides, there exists many examples of biorthogonal wavelet, to solve our problem (1)–(5) we consider biorthogonal spline wavelet in which ϕ^* are splines functions.

Now, we analyse some properties of wavelets. We use Daubechies wavelets constructed in [1]. In this case, $\phi = \phi^*$ and $\psi = \psi^*$. Fig. 1 represents a wavelet and respective scaling function.

Wavelet has the property of double localization, that is localization in time and in frequency. We illustrate this in Fig. 2. There, a function is defined on the interval [0...1000]. It is composed by two types of oscillating wave. On the first half of the interval the function has a wave of low frequency and on the second part another wave of frequency which is ten times greater than the first. We note that wavelet coefficients indicates the region of transition between these oscillating wave. The Fourier transform can not recognize this region. It only detects the presence of the frequencies without information about the spatial localization of them.

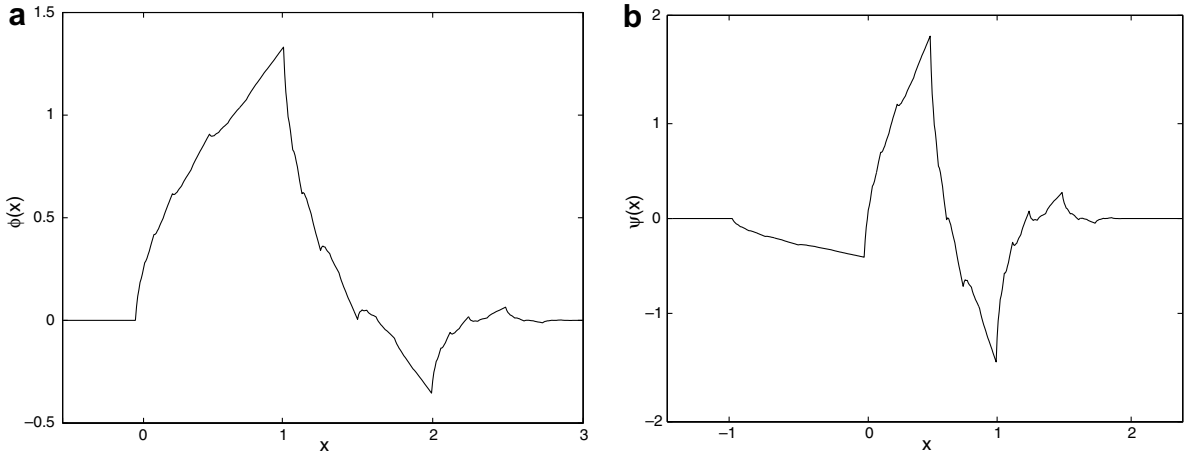

Fig. 1. Scaling and wavelet functions of Daubechies: (a) $\phi_2(t)$ and (b) $\psi_2(t)$.

Another interesting property is associated with the local regularity of analysed function. That is, the order of wavelet coefficient $d_j^k = \langle f, \psi_j^{*k} \rangle$ is associated with the smoothness of function f. So they can be used as indicators of smoothness of the function. They are small on smoothness regions and big on regions where the function has one discontinuity. As a consequence, expansion of the function using wavelets are more economic. We illustrate this in Fig. 3.

5. Wavelet-Galerkin method

When solving partial differential equations by the Galerkin method, the approximating spaces requires good approximation properties and they must also allow easy and fast computations. Besides, if the goal is the development of a multilevel method to detect and follow local singularities then hierarchial bases are necessary.

With the advent of wavelet analysis, a new approach has been successfully applied: the wavelet-Galerkin method.

Good approximation properties can be achieved using wavelet series. Furthermore, the multiresolution structure of bases provides a simple way to adapt computational refinements to the local regularity of the solution: high resolution computations are performed only in regions where singularities occur.

In a wavelet-Galerkin method the approximating spaces are spanned by the multiscale wavelet bases $\{\psi_j^k(x), j \in \mathbb{Z}\}$.

Eq. (1) can be written in compact form as

$$\mathscr{L}u(x,t) - f(x,t) = 0,$$

where $\mathscr{L}$ is the differential operator. In the wavelet-Galerkin method, the unknown solution $u(x,t)$ and weighting functions $w(x)$ are approximated in terms of the linear combination of the wavelets. The approximate solution can be expressed as

$$\begin{aligned} u^h &= \sum_j c_j^{k_0}(t)\phi_j^{k_0}(x) + \sum_{m=k_0}^{k} \sum_j d_j^m(t)\psi_j^m(x) \\ &= \sum_{m=k_0-1}^{k} \sum_j b_j^m(t)\psi_j^m(x). \end{aligned}$$

The application of the Galerkin method with weighing functions results in the following equation

$$\int_\Omega [\mathscr{L}u^h - f]w^h \, \mathrm{d}V = \int_\Omega R^k(x,t)w^h \, \mathrm{d}V = 0, \tag{17}$$

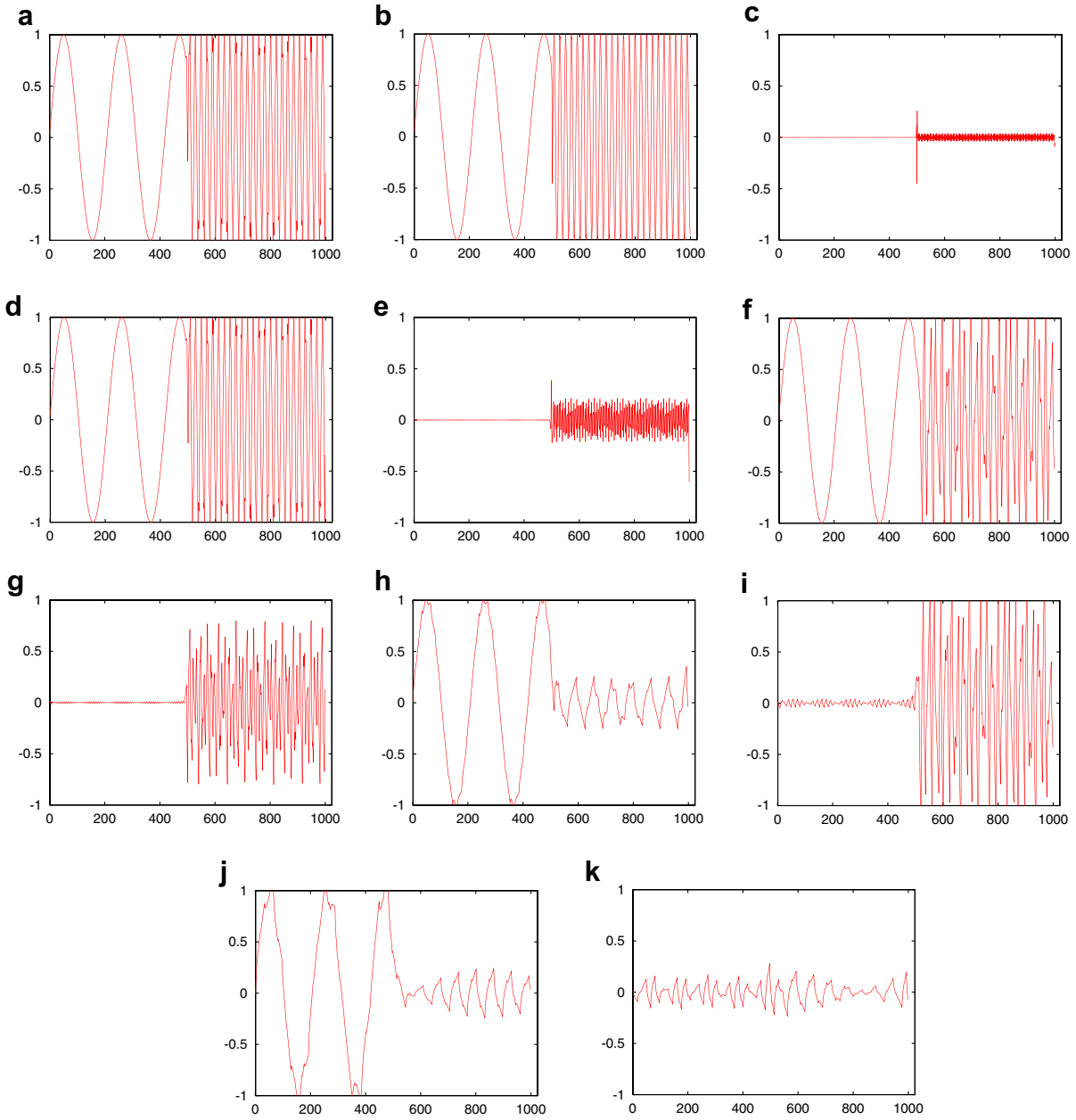

Fig. 2. Example of oscillating function with low and high frequencies and its discrete wavelet transform: (a) c^{k+1}; (b) c^{k}; (c) d^{k}; (d) c^{k-1}; (e) d^{k-1}; (f) c^{k-2}; (g) d^{k-2}; (h) c^{k-3}; (i) d^{k-3}; (j) c^{k-4}; and (k) d^{k-4}.

where $R^k(x,t)$ is the residual of the operator. The introduction of initial conditions result in additional equations for the wavelets unknown coefficients $b_j^m(t)$.

6. One-dimensional elastoplastic-damage model

The material models have a linear dependence for elasticity problems and non-linear for elastoplasticity problems. For the linear range, the elastic constitutive equation relates linearly stress with strain as

$$\sigma = Eu_{,x}, \tag{18}$$

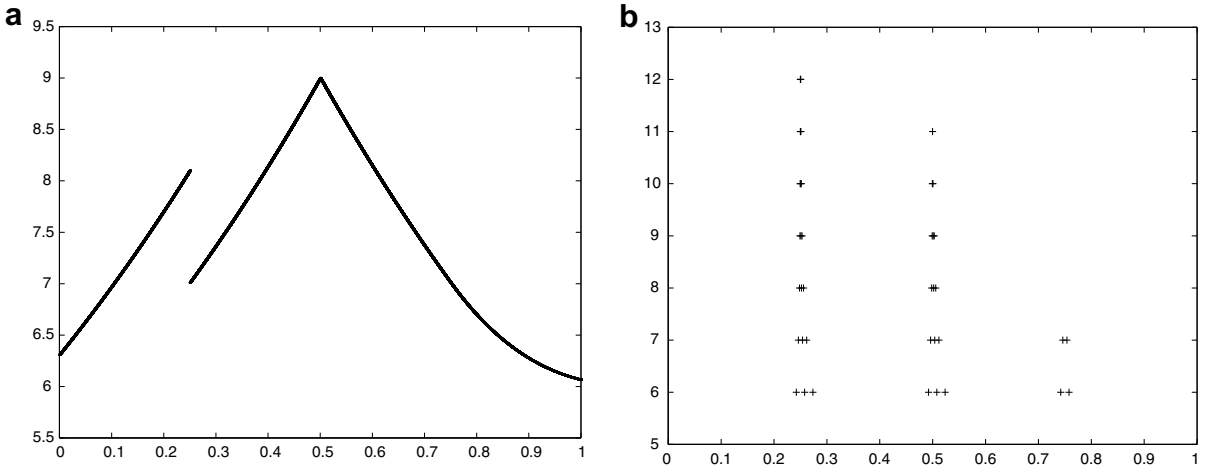

Fig. 3. Example of significant wavelet coefficient: (a) function; (b) position of $|d_j^k| > 10^{-2}$.

where the material parameter E is the Young's Modulus which is assumed to be constant over domain $\overline{\Omega}$. For materials with non-linear characteristics (plasticity), the stress is a non-linear function (possibly history-dependent) of ϵ. The total strain ϵ splits into two parts $\epsilon = \epsilon^{\mathrm{e}} + \epsilon^{\mathrm{p}}$, where the superscript e referred to the elastic part and p the plastic part. The stress is written as

$$\sigma = E(\epsilon - \epsilon^{\mathrm{p}}). \tag{19}$$

In many metals subjected to cyclic loading, it is experimentally observed that the center of the yield surface experiences a motion in the direction of the plastic flow. A phenomenological model that captures this kinematic effect is constructed by introducing an additional internal variable, denotes by q and called back stress, which defines the location of the center of the yield surface. The one-dimensional model of rate-independent plasticity with isotropic and kinematic hardening is examined. The hardening is isotropic in the sense that at any state of loading, the admissible relative stresses are constrained to lie in the closed interval $[-(\sigma_Y + K\alpha), \sigma_Y + K\alpha] \subset \mathbb{R}$. This leads to a yield criterion of the form

$$F(\sigma, q, \alpha) = |\sigma - q| - [\sigma_Y + K\alpha] \leqslant 0, \quad \alpha \geqslant 0, \tag{20}$$

where $\sigma_Y > 0$ (flow stress) and $K \geqslant 0$ (plastic modulus) are given constants. The variable $\alpha : [0, T] \to \mathbb{R}$ is a non-negative function of the amount of plastic flow, called an internal hardening variable. If $K < 0$, one speaks of a strain-softening response. The hardening is linear in the amount of plastic flow ($\dot{\alpha} = |\dot{\epsilon}^{\mathrm{p}}|$). The irreversible mechanism that governs the evolution of plastic flow, which is defined by the flow rule, remains unchanged. Then, we assume that

$$\dot{\epsilon}^{\mathrm{p}} = \gamma \operatorname{sign}(\sigma - q), \tag{21}$$

where $\gamma \geqslant 0$ is the rate at which slip takes place and q is the back stress, which defines the location of the center of the yield surface. The evolution of the back stress q is defined by Zieglers rule

$$\dot{q} = H\dot{\epsilon}^{\mathrm{p}}, \tag{22}$$

where H is the kinematic hardening modulus. The irreversible nature of plastic flow is again capture by means of the Karush–Kuhn–Tucker loading/unloading conditions [3], which in the present context read

$$\gamma \geqslant 0, \quad F(\sigma, q, \alpha) \leqslant 0, \quad \gamma F(\sigma, q, \alpha) = 0, \tag{23}$$

where $\gamma \geqslant 0$ is determined by the consistency condition

$$\gamma \dot{F}(\sigma, q, \alpha) = 0, \quad (\text{if } F(\sigma, q, \alpha) = 0). \tag{24}$$

The model is now completed with a damage dissipation mechanism. The stress σ is related with an effective stress $\tilde{\sigma}$ associated with the surface region that effectively resists the load

$$\tilde{\sigma} = \frac{\sigma}{1-D}, \tag{25}$$

where D is a variable bounded between 0 and 1, where $D = 0$ means undamaged material and $D = 1$ a fully broken one. In damage phenomenology has a strain equivalence principle, which states that any constitutive equation for a damaged material may be derived in the same way as for a virgin material, except that the stress is replaced by the effective stress.

The total potential energy function of the virgin material can be defined as

$$\omega_t = \Psi^0(\epsilon^e, \alpha, \epsilon^p) = \frac{1}{2}E(\epsilon^e)^2 + \frac{1}{2}K\alpha^2 + \frac{1}{2}H(\epsilon^p)^2. \tag{26}$$

The stress can be calculated as

$$\sigma = (1-D)\frac{\partial \Psi^0}{\partial \epsilon^e}.$$

The state of damage in the material at current time t is characterized by means of a damage criterion

$$g(\omega_t, r_t) = \omega_t - r_t \leqslant 0, \tag{27}$$

where r_t is the current damage threshold defined by $r_t = \max\{r_0, \omega_s\}$, $s \in [0,t]$ with r_0 denoting the initial damage threshold before any loading is applied. The equations of evolution for D_t and r_t are expressed as

$$\dot{D}_t = \dot{\mu}, \dot{r}_t = \dot{\mu}, \tag{28}$$

where $\dot{\mu} \geqslant 0$ is a damage consistency parameter which defines damage Kuhn–Tucker conditions

$$\dot{\mu} \geqslant 0, g(\omega_t, r_t) \leqslant 0, \dot{\mu} g(\omega_t, r_t) = 0. \tag{29}$$

The value of $\dot{\mu}$ is determined by the damage consistency condition that reads $\dot{\mu} = \dot{\omega}_t$ [3]. Venini and Morana [4] describe an adaptive wavelet-Galerkin method for an elastic–plastic-damage constitutive model.

7. Numerical issues

In this section, we present the numerical treatment for elastoplastic-damage problem. We describe the discretization in space and time, and we also detail the elastic-predictor, plastic-damage-corrector method.

7.1. Space discretization

At each time t we approximate the solution, $u^h(x,t)$, and weighting function $w^h(x)$ in different ways, such as using finite element method, wavelets, and other techniques. Following we describe some of them.

These approximations lead to the non-linear differential equation problem: given $\mathbf{F}^{\mathbf{ext}} :]0,T[\to \mathbb{R}^{n_{\mathrm{eq}}}$, $t \in [0,T]$, such that

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{F}^{\mathbf{int}}(\sigma^h) - \mathbf{F}^{\mathbf{ext}}(t) = 0, \tag{30}$$

$$\mathbf{d}(0) = \mathbf{d}_0, \tag{31}$$

$$\dot{\mathbf{d}}(0) = \dot{\mathbf{d}}_0, \tag{32}$$

where n_{eq} is the number of equations associated with the numbers points in $\overline{\Omega} - \partial\Omega_g$.

7.1.1. Finite element method

Considering shape functions $N_e(x)$, we can write

$$u_e^h(x,t) = \sum_{a=1}^{n} d(t)_e^a N_e^a(x),$$

$$w_e^h(x) = \sum_{a=1}^{n} c_e^a N_e^a(x).$$

where the subindex e represent the element, d is the displacement and c is an arbitrary function, with

$$\mathbf{d} = \{d_j\},$$
$$\mathbf{M} = [M_{jk}] = \mathbf{A}_{\mathbf{e=1}}^{\mathbf{n}}[\langle \rho N_e^a, N_e^b \rangle],$$
$$\mathbf{F}^{\mathbf{int}}(\sigma^h) = \{F^{\mathrm{int}}(\sigma^h)_j\} = \mathbf{A}_{\mathbf{e=1}}^{\mathbf{n}}\{\langle N_e^{a'}, \sigma^h(x,t) \rangle\},$$
$$\mathbf{F}^{\mathbf{ext}}(t) = \{F^{\mathrm{ext}}(t)_\ell\} = \mathbf{A}_{\mathbf{e=1}}^{\mathbf{n}}\{\langle f, N_e^a \rangle + \int_{\partial\Omega_z} N_e^a z \, \mathrm{d}S\},$$

where $j, k = 1 \ldots n_{\mathrm{eq}}$ $a, b = 1 \ldots n$, and $\mathbf{A}$ is the assemble operator.

7.1.2. Wavelets

At each time t we approximate the solution and weighting function with wavelet expansion

$$u^h(x,t) = \sum_{m=k_0-1}^{k} \sum_j b_j^m(t) \psi_j^m(x),$$
$$w^h(x) = \sum_{m=k_0-1}^{k} \sum_\ell b_\ell^{m,w} \psi_\ell^m(x)$$

and we obtain

$$\mathbf{d} = \{d_j\} = \{b_j^m(t)\},$$
$$\mathbf{M} = [M_{j\ell}] = [\langle \rho \psi_j, \psi_\ell \rangle],$$
$$\mathbf{F}^{\mathbf{int}}(\sigma^h) = \{F^{\mathrm{int}}(\sigma^h)_j\} = \{\langle \psi'_j, \sigma^h(x,t) \rangle\},$$
$$\mathbf{F}^{\mathbf{ext}}(t) = \{F^{\mathrm{ext}}(t)_\ell\} = \{\langle f, \psi_\ell \rangle + \int_{\partial\Omega_z} \psi_\ell z \, \mathrm{d}S\}.$$

7.2. Discretization in time

The computational procedure for direct integration of Eqs. (30)–(32) is carried over by the Newmark scheme having as unknown the coefficients $d_j(t)$ [5–7]. The crucial step in this procedure concerns the computation of the stress field $\sigma^h(x,t)$ for $x \in \Omega$ and time $t \in [0,T]$ [8] describe in Section 7.3. The Newmark method takes

$$\mathbf{M}\mathbf{a}_{\mathbf{n+1}} + \mathbf{F}_{\mathbf{n+1}}^{\mathbf{int}}(\sigma^h) - \mathbf{F}_{\mathbf{n+1}}^{\mathbf{ext}}(t) = 0,$$
$$\mathbf{d}_{\mathbf{n+1}} = \mathbf{d}_{\mathbf{n}} + \Delta t \mathbf{v}_{\mathbf{n}} + \frac{\Delta t^2}{2}[(1-2\beta)\mathbf{a}_{\mathbf{n}} + 2\beta \mathbf{a}_{\mathbf{n+1}}],$$
$$\mathbf{v}_{\mathbf{n+1}} = \mathbf{v}_{\mathbf{n}} + \Delta t[(1-\theta)\mathbf{a}_{\mathbf{n}} + \theta \mathbf{a}_{\mathbf{n+1}}],$$

where $\mathbf{d_n}$, $\mathbf{v_n} = \dot{\mathbf{d}}_{\mathbf{n}}$ and $\mathbf{a_n} = \ddot{\mathbf{d}}_{\mathbf{n}}$ are the approximations to the displacement, velocity and acceleration vectors respectively at time $t = n\Delta t$, Δt is the time step and β and θ are the Newmark parameters for stability conditions.

7.3. Elastic prediction and plastic-damage correction

Now we shows a return-mapping algorithm for one-dimensional rate independent plasticity with damage combining isotropic and kinematic hardening. After the usual elastic-predictor and plastic-corrector it is developed a damage corrector. In this procedure, the auxiliary variable ξ is the relative stress. Plastic and damage correctors are uncoupled from an algorithm point of view [3]. The following procedure allows to update the set of variables σ, ϵ^p, α, q, D at the time t_{n+1}:

1. Database at $x \in \Omega : \{\sigma_n, \epsilon_n^{\mathrm{p}}, \alpha_n, q_n, D_n, r_n\}$.
2. Given the displacement field u_{n+1} compute the total deformation $\epsilon_{n+1} = u_{n+1,x}$.
3. Compute elastic trial stress (elastic-predictor)

$$\begin{aligned}
\epsilon_{n+1}^{\mathrm{p,trial}} &= \epsilon_n^{\mathrm{p}},\\
\alpha_{n+1}^{\mathrm{trial}} &= \alpha_n,\\
q_{n+1}^{\mathrm{trial}} &= q_n,\\
D_{n+1}^{\mathrm{trial}} &= D_n,\\
\sigma_{n+1}^{\mathrm{trial}} &= (1 - D_{n+1}^{\mathrm{trial}})E(\epsilon_{n+1} - \epsilon_{n+1}^{\mathrm{p,trial}}),\\
\tilde{\sigma}_{n+1}^{\mathrm{trial}} &= \frac{\sigma_{n+1}^{\mathrm{trial}}}{(1 - D_{n+1}^{\mathrm{trial}})}.
\end{aligned}$$

4. Test for plastic loading (plastic-corrector)

$$\begin{aligned}
\xi_{n+1}^{\mathrm{trial}} &= \tilde{\sigma}_{n+1}^{\mathrm{trial}} - q_{n+1}^{\mathrm{trial}},\\
F_{n+1}^{\mathrm{trial}} &= |\xi_{n+1}^{\mathrm{trial}}| - [\sigma_Y + K\alpha_{n+1}^{\mathrm{trial}}].
\end{aligned}$$

If $F_{n+1}^{\mathrm{trial}} \leqslant 0$ then
Elastic step: $-\epsilon_{n+1}^{\mathrm{p}} = \epsilon_{n+1}^{\mathrm{p,trial}}$,
$-\alpha_{n+1} = \alpha_{n+1}^{\mathrm{trial}}$,
$-q_{n+1} = q_{n+1}^{\mathrm{trial}}$,
$-\tilde{\sigma}_{n+1} = \tilde{\sigma}_{n+1}^{\mathrm{trial}}$
–Proceed to step 6
else
Plastic step: Proceed to step 5

5. Plastic return-mapping corrector

$$\begin{aligned}
\Delta\gamma &= \frac{F_{n+1}^{\mathrm{trial}}}{E + [K + H]} > 0,\\
\tilde{\sigma}_{n+1} &= \tilde{\sigma}_{n+1}^{\mathrm{trial}} - \Delta\gamma E\,\mathrm{sign}(\xi_{n+1}^{\mathrm{trial}}),\\
\epsilon_{n+1}^{\mathrm{p}} &= \epsilon_n^{\mathrm{p}} + \Delta\gamma\,\mathrm{sign}(\xi_{n+1}^{\mathrm{trial}}),\\
q_{n+1} &= q_n + \Delta\gamma H\,\mathrm{sign}(\xi_{n+1}^{\mathrm{trial}}),\\
\alpha_{n+1} &= \alpha_n + \Delta\gamma.
\end{aligned}$$

6. State of variables $\{\tilde{\sigma}_{n+1}, \epsilon_{n+1}^{\mathrm{p}}, \alpha_{n+1}, q_{n+1}, D_n, r_n\}$.
7. Damage corrector

$$\begin{aligned}
\omega_{n+1} &= \Psi^0(\epsilon_{n+1} - \epsilon_{n+1}^{\mathrm{p}}, \alpha_{n+1}, \epsilon_{n+1}^{\mathrm{p}}),\\
D_{n+1} &= \begin{cases} D_n & \text{if } \omega_{n+1} - r_n \leqslant 0,\\ D_n + (\omega_{n+1} - \omega_n) & \text{otherwise} \end{cases}\\
r_{n+1} &= \max\{r_n; \omega_{n+1}\},\\
\sigma_{n+1} &= (1 - D_{n+1})\tilde{\sigma}_{n+1}.
\end{aligned}$$

8. Set of variables at time $t = t_{n+1}$,$\{\sigma_{n+1}, \epsilon_{n+1}^{\mathrm{p}}, \alpha_{n+1}, q_{n+1}, D_{n+1}, r_{n+1}\}$.

8. Final considerations

We present a description of the wavelet-Galerkin method for the resolution of elastoplastic-damage problem arising from damage mechanics. In the spatial discretization multiscale wavelet bases can be used as trial

and test functions. From this point of view we can realize experiments with several wavelet functions. Besides, this approach allows to locate discontinuities and to capture strain localization phenomena. The procedure proposed in this work can be extended to multidimensional problems, analysis of strain localization and for others material models.

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