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REAL FINITE-GAP REGULAR SOLUTIONS OF THE KAUP–BOUSSINESQ EQUATION

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Smooth real finite-gap solutions of the Kaup–Boussinesq equation are found together with simple reductions of the general smooth real two-gap and three-gap solutions to one-dimensional theta functions. The Abelian integrals that occur in the solutions are reduced to elliptic integrals.

Introduction

In this paper, we continue study of the equation first derived by Boussinesq [1, 2] in the theory of wave propagation on shallow water:

$$\begin{cases} \pi_\tau=\Phi_{xx}+\beta^2\Phi_{xxxx}-\varepsilon(\Phi_x\pi)_x, \\ \pi=\Phi_\tau+{}^1/_2\varepsilon\Phi_x^2. \end{cases} \tag{I}$$

The complete integrability of this equation was first established in 1975 by Kaup [3, 4], and we shall therefore call (I) the Kaup–Boussinesq (KB) equation. In [5, 6], Lukomskii showed that this equation describes in the long-wave approximation the longitudinal motions of an electron plasma in a waveguide with magnetic field $H_0 \to \infty$ and $0<\beta\ll1$, $0<\varepsilon\ll1$. On the basis of the scattering problem posed by Kaup, he also found an infinite set of integrals of the motion. In [7], Matveev and Yavor obtained complex finite-gap multiphase solutions expressed in terms of Riemann theta functions, used the degeneracy procedure to find multisoliton solutions, and established their asymptotic behavior. Here, we continue the investigation of finite-gap solutions of the KB equation by finding smooth real solutions, and we also find simple reductions of the general smooth two- and three-gap solutions to one-dimensional Riemann theta functions by means of the technique developed by Babich, Bobenko, and Matveev [8, 9].

One of the important aspects of "finite-gap integration" is the finding of effective conditions or reality of the constructed finite-gap solutions. In this field, an important contribution was made by Dubrovin and Natanzon, who investigated the reality of the sine–Gordon equation. The method which they proposed in [10] will be used on the KB equation.

It should be noted that there exists a different technique for reduction from two- to one-dimensional theta functions that is not associated with Appel's theorem [11-14]. By means of it Belokolos and Enol'skii found a two-gap solution of the sine–Gordon equation in Jacobi theta functions that does not reduce to the "Lamb ansatz."

At the end, in order to make the obtained solutions truly effective, we reduce the Abelian integrals to elliptic integrals.

1. Almost Periodic Solutions of the Kaup–Boussinesq Equation

The system (I) can be obtained as the condition of compatibility of the two nonlinear equations

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$$\begin{cases}\psi_{xx}+(\lambda^2+{}^1/_4\beta^{-2}+i\lambda q-r)\psi=0,\\ \psi_\tau={}^1/_2\beta q_x\psi-(2i\beta\lambda+\beta q)\psi_x;\end{cases}\quad q=\varepsilon\beta^{-1}\Phi_x/2,\quad r={}^1/_4\varepsilon\beta^{-2}(\Phi_\tau+{}^3/_4\varepsilon\Phi_x^2). \tag{1.1}$$

We introduce the Riemann surface Γ of genus g of the function $w(\lambda)$:

$$w^2=\prod_{j=1}^{2g+2}(\lambda-E_j). \tag{1.2}$$

On this surface, we consider the canonical system of cycles $a_1,\ldots,a_g,b_1,\ldots,b_g$ with intersection index matrix $\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$. To it there corresponds a normalized basis of holomorphic differentials

$$dU_\nu=\frac{\sum_{k=1}^{g}c_{\nu k}\lambda^{g-k}}{w(\lambda)}d\lambda, \tag{1.3}$$

which are such that

$$\oint_{a_k}dU_\nu=\delta_{k\nu}. \tag{1.4}$$

We determine the matrix of b periods of the Riemann surface Γ:

$$B_{jk}=\oint_{b_k}dU_j. \tag{1.5}$$

It is well known (see, for example, [15]) that B is a symmetric matrix with positive-definite imaginary part.

Using the matrix B we construct the Riemann theta function with characteristics $\alpha,\beta\,(\alpha,\beta\in R^g)$:

$$\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\mathrm{x}\,|\,B)=\sum_{\mathrm{m}\in Z^g}\exp\{\pi i\langle B(\mathrm{m}+\alpha),\mathrm{m}+\alpha\rangle+2\pi i\langle\mathrm{x}+\beta,\mathrm{m}+\alpha\rangle\}, \tag{1.6}$$

where $\mathrm{x}\in C^g$, the summation is over the integral g-dimensional lattice, and $\langle,\rangle$ denotes the standard scalar product.

For brevity, we denote $\Theta(\mathrm{x}|B)=\Theta\begin{bmatrix}0\\ 0\end{bmatrix}(\mathrm{x}|B)$, and if it is clear what matrix B is being considered we shall simply write $\Theta(\mathrm{x})$.

The Riemann theta function of the Riemann surface Γ is defined by

$$\Theta=\Theta(\mathrm{U}(P)-\mathrm{e}), \tag{1.7}$$

$$U_j(P)=\int_{E_{2g+2}}^{P}dU_j. \tag{1.8}$$

On this surface Γ we take g points P_j forming a divisor of general position, and we choose the vector **e** in the form

$$\mathrm{e}=\mathrm{K}+\sum_{j=1}^{g}\mathrm{U}(P_j),\quad K_m=\frac{1}{2}\sum_{j=1}^{g}B_{mj}-\frac{m}{2}. \tag{1.9}$$

Then the theta function will have exactly g zeros of first order at the given points and no other zeros. If $\pi(P_j)=\pi(P_k)$ for certain j, k (π is the canonical projection of the point of Γ onto the complex plane C), the theta function is identically zero.

On Γ, we also define two Abelian integrals of the second kind with zero a periods and having the following asymptotic behaviors at the only poles, $\infty^\pm$, that they have:

$$\omega_1\sim\pm\left(2\lambda^2+\frac{K_0}{2}+\sum_{n=1}^{\infty}K_n\lambda^{-n}\right),\quad P\to\infty^\pm, \tag{1.10}$$

$$\omega_2 \sim \mp\left(\lambda - \frac{R_0}{2} + \sum_{n=1}^{\infty} R_n \lambda^{-n}\right), \quad P \to \infty^{\pm}, \quad \lambda = \pi(P). \tag{1.11}$$

Let the function Ψ satisfy the following two conditions (Akhiezer axioms):

1) $\Psi(x, t, P)$ is a single-valued function of the point P of the Riemann surface Γ, is meromorphic on Γ except at the points $\infty^{\pm}$, and has g simple poles at the points P_j, these forming a divisor of general position;

2) $\Psi(0, 0, P) = 1$;

$$\Psi \sim \exp\{i\lambda x - 2i\lambda^2 t\} \sum_{j=0}^{\infty} \frac{\varphi_j(x,t)}{\lambda^j}, \quad P \to \infty^-; \; \Psi \sim \exp\{-i\lambda x + 2i\lambda^2 t\} \sum_{k=0}^{\infty} \frac{\chi_k(x,t)}{\lambda^k}, \quad P \to \infty^+.$$

Then Ψ is uniquely determined by these conditions and can be constructed in accordance with the formula

$$\Psi(x,t,P) = \exp\{i\omega_1(P)t + i\omega_2(P)x\} \frac{\Theta(\mathbf{U}(P) - \mathbf{e}(x,t))}{\Theta(\mathbf{U}(P) - \mathbf{e}(0,0))} \times \left\{\frac{\Theta(\mathbf{e}(0,0) - \mathbf{r})\Theta(\mathbf{e}(x,t) + \mathbf{r})}{\Theta(\mathbf{e}(x,t) - \mathbf{r})\Theta(\mathbf{e}(0,0) + \mathbf{r})}\right\}^{1/2} \tag{1.12}$$

where $e(0, 0)$ is determined by formula (1.9),

$$e_j(x,t) = e_j(0,0) + 2ic_{j1}x - 4i(c_{j1}c_0/2 + c_{j2})t, \; c_0 = \sum_{k=1}^{2g+2} E_k, \quad r_j = U_j(\infty^+). \tag{1.13}$$

All finite-gap solutions of the KB equation can be constructed in terms of the function Ψ. They have the form (see [7])

$$\Phi(x,\tau) = \varepsilon^{-1}(1 - 2\beta^2 K_0)\tau + 2i\beta\varepsilon^{-1}R_0 x + 2\beta\varepsilon^{-1}\ln J_1(x, i\beta\tau) + c', \quad \forall c' \in C, \tag{1.14}$$

where

$$J_1(x,t) = \Theta(\mathbf{e}(x,t) - \mathbf{r}) / \Theta(\mathbf{e}(x,t) + \mathbf{r}).$$

2. Spectral Analysis of Quadratic Sheaf

We consider the second-order differential equation with periodic real potentials

$$\psi_{xx} + (\lambda^2 + i\lambda q + r)\psi = 0, \; \operatorname{Im} q = \operatorname{Im} r = 0, \quad q(x+T) = q(x), \; r(x+T) = r(x). \tag{2.1}$$

For it, a standard basis of solutions and monodromy matrix are determined:

$$\varphi(x,\lambda)|_{x=0} = 0, \quad \varphi_x(x,\lambda)|_{x=0} = 1; \; \theta(x,\lambda)|_{x=0} = 1, \quad \theta_x(x,\lambda)|_{x=0} = 0,$$

$$\begin{pmatrix} \varphi(x+T,\lambda) \\ \theta(x+T,\lambda) \end{pmatrix} = M(\lambda) \begin{pmatrix} \varphi(x,\lambda) \\ \theta(x,\lambda) \end{pmatrix}, \; M(\lambda) = \begin{pmatrix} \varphi_x(T,\lambda) & \varphi(T,\lambda) \\ \theta_x(T,\lambda) & \theta(T,\lambda) \end{pmatrix}$$

The eigenfunctions $\psi_{1,2}(x, \lambda)$ of the monodromy matrix are called Bloch functions, and the eigenvalues satisfy the relations

$$\psi_k(x+T,\lambda) = \varkappa_k(\lambda)\psi_k(x,\lambda), \; \varkappa_k = \exp\{\pm i\gamma(\lambda)\}, \quad \cos\gamma(\lambda) = {}^1/_2 \operatorname{Sp} M(\lambda).$$

By the spectrum of the problem we shall understand the points λ at which there exist two linearly independent solutions of Eq. (2.1) bounded on the complete axis $x \in R$. Since any two linearly independent solutions can be expressed in terms of Bloch functions, we must have $|\varkappa_1| = |\varkappa_2| = 1$. In other words, $\operatorname{Im} \operatorname{Sp} M(\lambda) = 0$, $[\operatorname{Sp} M(\lambda)]^2 \leqslant 4$.

Considering the analogous equation

$$\psi_{xx} + (\bar\lambda^2 - i\bar\lambda q + r)\psi = 0,$$

we arrive at the relations

$$\psi_k(x, -\bar\lambda) = \overline{\psi_k(x,\lambda)}, \quad \varkappa_k(-\bar\lambda) = \overline{\varkappa_k(\lambda)},$$

from which symmetry of the spectrum with respect to the axis $\operatorname{Re} \lambda = 0$ follows. Using the standard technique, we find that the spectrum consists of bands situated on curves:

1) $\operatorname{Re} \lambda = 0$ or

2) $\operatorname{Im}\lambda = -\frac{1}{2}\int_0^T q(x)\,|\psi_i(x,\lambda)|^2 dx \Big/ \int_0^T |\psi_i(x,\lambda)|^2 dx,$

$$(\operatorname{Re}\lambda)^2 = \int_0^T \{|\psi_{ix}|^2 - r(x)\,|\psi_i|^2\}dx \Big/ \int_0^T |\psi_i(x,\lambda)|^2 dx - (\operatorname{Im}\lambda)^2.$$

Further analysis of the spectrum leads to the following results:

a) the band edges satisfy the relations

$$\operatorname{Sp} M(E_j) = \pm 2, \quad \varkappa_1 = \varkappa_2 = \pm 1,$$

i.e., are eigenvalues of a periodic or an antiperiodic problem. In addition, $\operatorname{Sp} M(-\bar{E}_j) = \operatorname{Sp} M(E_j)$, and therefore they are situated symmetrically with respect to the axis $\operatorname{Re}\lambda = 0$ and symmetric eigenvalues correspond to the same problem;

b) in the limit $|\lambda| \to \infty$, the asymptotic behavior of the spectrum has the form

$$\operatorname{Im}\lambda = -\frac{1}{2T}\int_0^T q(x)\,dx\cdot[1 + O(|\lambda|^{-2})],$$

from which it also follows that there is an even number of band edges E_j;

c) the zeros of the function $\varphi(T, \lambda_k)$ are situated symmetrically with respect to the axis $\operatorname{Re}\lambda = 0$;

d) generally speaking, the spectrum can have a fairly complicated structure. But in the construction of the Riemann surface the cuts need not be taken through the spectrum; the important thing is that they join the band edges E_j.

Therefore, the Riemann surface Γ can have the types of cuts shown in Fig.1.

3. Reality of Finite-Gap Solutions of the KB Equation

It follows from (1.14) for a finite-gap solution of the KB equation that a sufficient condition of reality is the fulfillment of three conditions:

$$\operatorname{Im} K_0 = 0, \tag{3.1}$$

$$\operatorname{Re} R_0 = 0, \tag{3.2}$$

$$\ln J_1(x, i\beta\tau) + \frac{c'}{2\beta}\varepsilon = \overline{\ln J_1(x, i\beta\tau) + \frac{c'}{2\beta}\varepsilon}. \tag{3.3}$$

LEMMA 1. If on the surface Γ there exists an antiholomorphic automorphism $\tau: (w, \lambda) \to ((-1)^\varepsilon \bar{w}, -\bar{\lambda})$, then the conditions (3.1) and (3.2) are satisfied automatically for an appropriate choice of the basis of cycles.

Proof. We choose the cycle basis as shown in Fig.2. The antiholomorphic automorphism transforms it as follows:

$$\begin{aligned}\tau\mathbf{a} &= \mathbf{a},\\ \tau\mathbf{b} &= \Lambda\mathbf{a} - \mathbf{b},\end{aligned} \text{ where } \Lambda = \begin{pmatrix} -1 & -1 & 1 & 1\\ -1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0\end{pmatrix}. \tag{3.4}$$

We consider the Abelian integral of the second kind

$$\hat{\omega}_2(P) = \overline{\omega_2(\tau[P])}.$$

It has zero a periods and in the limits $P \to \infty^\pm$ asymptotic behaviors of the form (see (1.11))

$$\hat{\omega}_2(P) \sim \mp\left(\lambda + \frac{1}{2}\bar{R}_0 + \sum_{n=1}^{\infty}(-1)^{n+1}\bar{R}_n\lambda^{-n}\right), \quad P \to \infty^\pm. \tag{3.5}$$

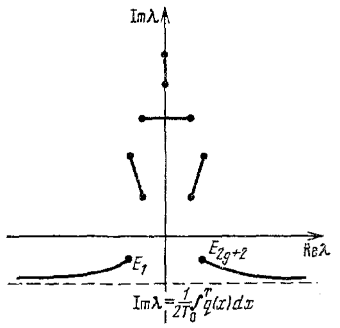

Fig. 1

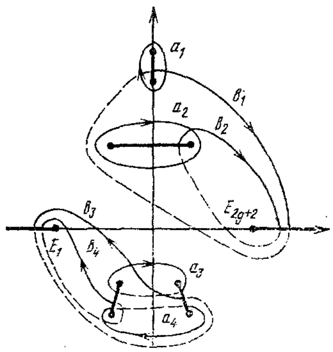

Fig. 2

From the uniqueness theorem for an Abelian integral of the second kind with given singularities and zero a periods we find that $\hat{\omega}_2(P)$ and $\omega_2(P)$ differ only by a constant term:

$$\int\limits_{E_{2g+2}}^{P} d\omega_2=\int\limits_{E_1}^{P} d\hat{\omega}_2+\text{const.}$$

It follows from this that $d\hat{\omega}_2=d\omega_2$ and, therefore, const $=0$ (since the integral along the path joining the points E_1 and E_{2g+2}, is equal to the half-sum of the integrals around the a cycles).

It follows from (1.11) and (3.5) that $\operatorname{Re} R_0=0$. Proceeding similarly with the Abelian integral $\omega_1(P)$, we obtain $\operatorname{Im} K_0=0$. Thus, Lemma 1 has been proved.

LEMMA 2. If the condition of Lemma 1 is satisfied,

$$4\operatorname{Re} U_k(\infty^+)\in Z.$$

Proof. Consider $\tau^* dU_k$, a holomorphic Abelian differential of the first kind. It follows from (3.4) that

$$\oint\limits_{a_k}\overline{\tau^* dU_j}=\oint\limits_{\tau a_k}\overline{dU_j}=\oint\limits_{a_k}\overline{dU_j}=\delta_{kj},$$

and therefore

$$\oint\limits_{a_k}(\overline{\tau^* dU_j}-dU_j)=0,\quad \forall k=1,\dots,g.$$

Therefore, by the uniqueness theorem for holomorphic Abelian differentials of the first kind $\overline{\tau^* dU_j}=dU_j$, $\forall j=1,\dots,g$ and

$$\bar{c}_{vk}=(-1)^{k+1}c_{vk}. \tag{3.6}$$

Suppose the cycle $A=\sum\limits_{k=1}^{g}\alpha_k a_k$, $\alpha_k\in Z$, surrounds all the finite cuts. Then $\oint\limits_A dU_j=\alpha_j$. On the other hand, it is easy to show that $\oint\limits_A dU_j=4\operatorname{Re} U_j^{(R)}(\infty^+)$. Finally, bearing in mind that (from (3.4)) $\operatorname{Re} B=\frac{1}{2}\Lambda$, we find that $4\operatorname{Re} U_j(\infty^+)\in Z$, where the integration is along an arbitrary path of the Riemann surface Γ connecting the points E_{2g+2} and ∞^+.

The proof of Lemma 2 is completed.

LEMMA 3. If $\bar{B}=\Lambda-B$, $\Lambda_{ik}\in Z$, then $\overline{\Theta}(\mathbf{z}|B)=\Theta(\bar{\mathbf{z}}+\lambda|B)$, where $\lambda_i={}^1/{}_2\Lambda_{ii}$.

The proof follows from the definition of the function $\Theta(\mathbf{z}|B)$.

THEOREM 1. Suppose the Riemann surface Γ has an antiholomorphic automorphism τ: $(w,\lambda)\to((-1)^g\bar{w},-\bar{\lambda})$. We choose a basis of cycles for which $\tau\mathbf{a}=\mathbf{a}$. Then if the divisor of the zeros P_j (see (1.9)) satisfies the condition

$$\frac{1}{2}\sum_{j\neq k}\Lambda_{jk}+2\sum_{j=1}^{g}\operatorname{Re} U_k(P_j)-2\operatorname{Re} U_k(\infty^+)\in Z, \tag{3.7}$$

the solution (1.14) will be unique.

Proof. Since the conditions of Lemma 1 hold, (3.1) and (3.2) will be satisfied. For Eq. (3.3) to hold, it is sufficient that $\overline{\Theta}(\mathbf{e}-\mathbf{r})=\Theta(\mathbf{e}-\mathbf{r})$, $\overline{\Theta}(\mathbf{e}+\mathbf{r})=\Theta(\mathbf{e}+\mathbf{r})$. It then follows from Lemma 3 that

$$\mathbf{e}-\mathbf{r}=\pm(\bar{\mathbf{e}}-\bar{\mathbf{r}}+\lambda)+B\mathbf{m}+\mathbf{n},\quad \mathbf{m},\ \mathbf{n}\in Z^g,\ \mathbf{e}+\mathbf{r}=\pm(\bar{\mathbf{e}}+\bar{\mathbf{r}}+\lambda)+B\mathbf{m}'+\mathbf{n}',\quad \mathbf{m}',\ \mathbf{n}'\in Z^g.$$

Bearing in mind that, by virtue of (1.13) and (3.6), only the imaginary component of the argument of the theta function depends on x and τ, we obtain

$$2\mathrm{Re}\,\mathbf{e}-2\mathrm{Re}\,\mathbf{r}+\lambda=\mathbf{n},\quad 2\mathrm{Re}\,\mathbf{e}+2\mathrm{Re}\,\mathbf{r}+\lambda=\mathbf{n}'.$$

From this two requirements follow (we use (1.9)):

$$\frac{1}{2}\sum_{j\neq k}\Lambda_{jk}+2\sum_{j=1}^{g}\mathrm{Re}\,U_k(P_j)-2\,\mathrm{Re}\,r_j\in Z,\quad 4\mathrm{Re}\,r_j\in Z.$$

The first is satisfied by hypothesis, the second follows from Lemma 2. This proves the theorem.

Remark 1. By means of the generalized Liouville theorem one could show that on the transition to a different cycle basis ($\tau'\mathbf{a}=L\mathbf{a}$, $\tau'\mathbf{b}=(L^t)^{-1}\mathbf{b}$) the following quantity is conserved:

$$\frac{\Theta(\mathbf{U}(P)-\mathbf{e})}{\Theta(\mathbf{U}(P)+\mathbf{e})}\exp\left\{-2\pi i\sum_{k=1}^{g}U_k(P)\right\}.$$

Therefore, if the condition (3.3) is satisfied in one of the cycle bases, it will hold in the other, except that the constant C' will be different.

Remark 2. The spectral analysis of the quadratic sheaf has shown that the projections of the zeros of the theta function must be placed symmetrically with respect to the axis Re λ = 0. We take two points (P_j, P_j') whose projections are symmetric. It is not difficult to show that if the points are on one sheet

$$\mathbf{U}(P_j')+\overline{\mathbf{U}(P_j)}=2\mathrm{Re}\,\mathbf{U}(\infty^+)+B\mathbf{m}+\mathbf{n};\quad \mathbf{m},\ \mathbf{n}\in Z^g.$$

LEMMA 4. If a solution of the KB equation is real, the divisor of the zeros of the theta function is invariant with respect to the antiholomorphic automorphism $\hat{\tau}$: $(w,\lambda)\to((-1)^{g+1}\bar{w}, -\bar{\lambda})$.

Proof. In the divisor of the zeros of the theta function we introduce a dependence on x and τ, representing the vector $\mathbf{e}(x, i\beta\tau)$ in the form

$$\mathbf{e}(x, i\beta\tau)=\mathbf{K}+\sum_{k=1}^{g}\mathbf{U}(P_k(x,\tau)).$$

Then for the projections $\mu_k(x,\tau)=\pi(P_k)$ of the zeros of the theta function we have the differential equation

$$\frac{\partial\mu_k}{\partial x}=\frac{2iw(\mu_k)}{\prod\limits_{j\neq k}(\mu_k-\mu_j)}$$

from which the assertion of the lemma follows, the projections having the symmetry property.

COROLLARY. Suppose the Riemann surface Γ possesses the antiholomorphic automorphism $\hat{\tau}$: $(w,\lambda)\to((-1)^{g+1}\bar{w}, -\bar{\lambda})$. We choose a cycle basis for which $\hat{\tau}\mathbf{a}$ = $-\mathbf{a}$. A necessary and sufficient condition for a solution of the KB equation to be real is that the divisor of the zeros of the theta function be invariant with respect to this automorphism.

The necessity follows from Lemma 4. To prove the sufficiency, we use Remark 1 and take a cycle basis satisfying these requirements: a) $\tau\mathbf{a}$ = $-\mathbf{a}$, b) within the cycle a_k ($k=1,\dots,g$) there are only two branch points $-E_{2k}$, E_{2k+1}; c) the direction of the a cycles is chosen in such a way that their sum forms a contour surrounding all finite cuts: $A=\sum_{k=1}^{g}a_k$. It is then easy to show that

$$4\mathrm{Re}U_k^{(\mathbb{R})}(\infty^+)=-1,\quad \sum_{j\neq k}\Lambda_{jk}=g-1,\quad k=1,\dots,g.$$

Finally, taking into account the invariance of the divisor of the zeros with respect to $\hat{\tau}$, which means that symmetric points are situated on one sheet, we obtain fulfillment of the requirement (3.7) and, therefore, reality of the solution of the KB equation. This is what we wanted to prove.

4. Smoothness of Real Finite-Gap Solutions of the KB Equation

The solution of the KB equation can have singularities only on account of the zeros of the Riemann theta function

$$\Theta(\mathbf{e}(x, i\beta\tau)\pm\mathbf{r}).$$

If the theta function is equal to zero, then there exists a point in the divisor of the zeros for which one of the following two equations is satisfied:

$$\Theta(\mathbf{U}(\infty^\pm)-\mathbf{e}(x, i\beta\tau))=\Theta(\mathbf{U}(P_k)-\mathbf{e}(0, 0)). \qquad (4.1)$$

PROPOSITION. For any Riemann surface Γ ($g \geq 3$) it is possible to choose the divisor of the zeros of the theta function in such a way that the corresponding solution of the KB equation is smooth.

Proof. The relation (4.1) is equivalent to one of the four following equations holding:

$$\mathbf{U}(P_k)=\mathbf{e}(0, 0)\pm\{\mathbf{U}(\infty^\pm)-\mathbf{e}(x, i\beta\tau)\},$$

where the signs $\pm$ are in no way correlated. We go over to a basis of unnormalized Abelian differentials of the first kind:

$$d\eta_k=\frac{\lambda^{g-k}}{w}d\lambda, \quad \eta(P_k)=\mathbf{e}_\eta(0,0)\pm\{\eta(\infty^\pm)-\mathbf{e}_\eta(x, i\beta\tau)\}.$$

Then, using the expressions (1.3) and (1.13), we see that in the system of g scalar equations only the first two depend on x and τ. Therefore, for $g \geq 3$ we can choose the position of the zeros in such a way that Eq. (4.1) will not be satisfied for any x and τ.

5. Holomorphic Automorphisms of Riemann Surfaces

The existence of holomorphic automorphisms on Γ leads to restrictions on the constants and on the matrix B. Here, we shall give only the final results, since the process of obtaining them is analogous to the case of an antiholomorphic automorphism.

A. Suppose Γ possesses an involution τ_1: $(w, \lambda)\to(w, -\lambda)$. Then

$$C_0=0, \quad R_0=0. \qquad (5.1)$$

If $A=\sum_{k=1}^{g}\alpha_k a_k$, $\alpha_k \in Z$, is a contour surrounding all finite cuts, and if the cycle basis is transformed by means of the matrix M: $\tau_1\mathbf{a}=M\mathbf{a}$, $\tau_1\mathbf{b}=M^t\mathbf{b}$, then

$$\tau_1{}^*dU_\nu=\sum_{k=1}^{g}M_{k\nu}dU_k, \qquad (5.2)$$

$$B=M^tBM, \qquad (5.3)$$

$$c_{\nu j}=(-1)^{g+1-j}\sum_{k=1}^{g}M_{k\nu}c_{kj}, \qquad (5.4)$$

$$\sum_{k=1}^{g}\{M_{k\nu}+\delta_{k\nu}\}U_k(\infty^+)=-\frac{\alpha_\nu}{2}. \qquad (5.5)$$

B. If on Γ there exists the involution

$$\tau_2: (w, \lambda)\to(-\lambda^{-g-1}w, \lambda^{-1}),$$

then obviously formulas (5.2) and (5.3) remain valid, but only with a different matrix, and instead of (5.4) we have (for $\tau_2\mathbf{a}=N\mathbf{a}$, $\tau_2\mathbf{b}=N^t\mathbf{b}$)

$$c_{vj}=\sum_{k=1}^{g} N_{kv}c_{k,g-j+1}. \tag{5.6}$$

6. Appel's Theorem

Because we shall reduce the multidimensional theta functions ($g = 2, 3$) by means of Appel's theorem, which is not very widely known, it will be appropriate to give its formulation here. The proof of Appel's theorem can be found in [16], and the special case that we shall use in [9].

Suppose the final column of the matrix B satisfies the relations

$$n_k B_{kg}=q_k, \quad k=1,\ldots,\nu; \; n_k B_{kg}=n_g B_{gg}+q_k, \quad k=\nu+1,\ldots,g-1. \tag{6.1}$$

Here n_k, $q_k \in Z$. One can always choose $n_g>0$, $n_k>0$, $k=1,\ldots,\nu$. Suppose now $n_k<0$, $k>\nu$. We now make the following transformation: we change the sign of z_k (the argument of the theta function $\Theta(\mathbf{z}|B)$), m_k (the summation sign in formula (1.6)), and B_{kj}, B_{jk}, $j=1,\ldots,g$, $j\neq k$. Then the theta function $\Theta(\mathbf{z}|B)$ is unchanged, and n_k in the expression (6.1) changes sign. Therefore, in what follows we shall assume that n_k are natural numbers. In addition, we shall take $n_k \in N$ to be the smallest of those possible for which Eq. (6.1) is still satisfied; in particular, if $B_{jg} = 0$, we shall assume that $n_j = 1$.

THEOREM 2 (Appel). If (6.1) holds, then

$$\Theta_g(\mathbf{z}\,|\,B)=\sum_{\mathbf{s}\in Z^g(\hat{\mathbf{n}})} e^{\alpha}\Theta_{g-1}(\mathbf{p}\,|\,A)\,\Theta_1(\alpha_g\,|\,B_{gg}n_g^2),$$

where the summation over $\mathbf{s} \in Z^g(\hat{\mathbf{n}})$ means a finite sum over $\mathbf{s}$: $0\leqslant s_j\leqslant n_j-1$, $j=1,\ldots,g$; Θ_g, Θ_{g-1}, and Θ_1 are, respectively, g-dimensional, $(g-1)$-dimensional, and one-dimensional theta functions, and the parameters are given by

$$\alpha=2\pi i\langle \mathbf{s},\mathbf{z}\rangle+\pi i\langle B\mathbf{s},\mathbf{s}\rangle,\; p=(\alpha_1,\alpha_2,\ldots,\alpha_\nu,\alpha_{\nu+1}-\alpha_g,\ldots,\alpha_{g-1}-\alpha_g),\; \alpha_k=n_k z_k+\frac{1}{2}n_k\frac{\partial}{\partial s_k}\langle B\mathbf{s},\mathbf{s}\rangle,\quad k=1,\ldots,g,$$
$$A_{ii}=n_i^2B_{ii},\quad i\leqslant\nu,\quad A_{ii}=n_i^2B_{ii}-n_g^2B_{gg},\quad i>\nu,\quad A_{ij}=n_in_jB_{ij},\quad \text{if } i \text{ or } j\leqslant\nu,\quad A_{ij}=n_in_jB_{ij}-n_g^2B_{gg},\quad \text{if } i,\,j>\nu. \tag{6.2}$$

THEOREM 2a. Suppose the last column of the matrix B satisfies the relations (6.1), and $q_k = 0$, $k = 1,\ldots,g$. Then

$$\Theta_g(\mathbf{z}\,|\,B)=\sum_{\mathbf{s}\in Z^g(\hat{\mathbf{n}})} \Theta_{g-1}\begin{bmatrix}\gamma\\0\end{bmatrix}(\mathbf{f}\,|\,A)\,\Theta_1\begin{bmatrix}\delta\\0\end{bmatrix}(n_gz_g\,|\,n_g^2B_{gg}), \tag{6.3}$$

where $\gamma=(\gamma_1,\ldots,\gamma_{g-1})$, $\gamma_k=\dfrac{s_k}{n_k}$, $k=1,\ldots,g-1$; $\delta=\displaystyle\sum_{k=\nu+1}^{g}\left(\frac{s_k}{n_k}\right)$, the matrix A is given by the expression (6.2), and

$$f_k=n_kz_k,\quad k=1,\ldots,\nu,\quad f_k=n_kz_k-n_gz_g,\quad k=\nu+1,\ldots,g-1. \tag{6.4}$$

COROLLARY. Under the conditions of Theorem 2a,

$$\Theta_g\begin{bmatrix}\alpha\\0\end{bmatrix}(\mathbf{z}\,|\,B)=\sum_{\mathbf{s}\in Z^g(\hat{\mathbf{n}})}\exp\{-2\pi i\langle B\alpha,\mathbf{s}\rangle\}\,\Theta_{g-1}\begin{bmatrix}\varphi\\0\end{bmatrix}(\mathbf{f}\,|\,A)\,\Theta_1\begin{bmatrix}\psi\\0\end{bmatrix}(n_gz_g\,|\,n_g^2B_{gg}),$$

where

$$\varphi_i=\frac{s_i+\alpha_i}{n_i},\quad \psi=\sum_{k=\nu+1}^{g}\frac{s_k+\alpha_k}{n_k},$$

and A and $\mathbf{f}$ are given by (6.2) and (6.4).

In what follows, we use the fact that the matrix of b periods of symmetric Riemann surfaces satisfies the condition of Theorem 2a.

7. Real Smooth Solutions of the KB Equation in One-Dimensional Theta Functions

A. Consider the curve $(g=2)$: $w^2=(\lambda^2-t^2)(\lambda^2+v^2)(\lambda^2+u^2)$. We choose a canonical basis of cycles as shown in Fig. 3. On this surface, we consider the anti-involution τ: $(w,\lambda)\to(\bar{w},-\bar{\lambda})$ and the involution τ_1: $(w,\lambda)\to(w,-\lambda)$.

The anti-involution transforms the cycles as follows:

$$\begin{aligned}\tau\mathbf{a}&=\mathbf{a},\\ \tau\mathbf{b}&=-\mathbf{b}+\Lambda\mathbf{a}\end{aligned},\quad \Lambda=\begin{pmatrix}1&0\\0&0\end{pmatrix}.$$

Hence, using the results of Sec. 3, we readily find that

$$\operatorname{Re}B=\frac{1}{2}\begin{pmatrix}1&0\\0&0\end{pmatrix},\tag{7.1}$$

$$\operatorname{Re}\int_t^{\infty^+}dU_1=-\frac{1}{4},\quad \operatorname{Re}\int_t^{\infty^+}dU_2=0.\tag{7.2}$$

Under the holomorphic automorphism, the cycle basis is transformed by means of the matrix $M=\begin{pmatrix}1&0\\1&-1\end{pmatrix}$: $\tau_1\mathbf{a}=M\mathbf{a}$, $\tau_1\mathbf{b}=M^t\mathbf{b}$. Therefore, using formula (5.3), we find that the matrix B satisfies the conditions of Appel's theorem,

$$B=\begin{pmatrix}\alpha&-\delta\\-\delta&2\delta\end{pmatrix},\tag{7.3}$$

and therefore

$$\Theta(\mathbf{z}|B)=\Theta(-2z_1-z_2|4\alpha-2\delta)\Theta(z_2|2\delta)+\Theta\begin{bmatrix}1/2\\0\end{bmatrix}(-2z_1-z_2|4\alpha-2\delta)\Theta\begin{bmatrix}1/2\\0\end{bmatrix}(z_2|2\delta).\tag{7.4}$$

In addition, the restrictions on the coefficients c_{ik} that follow from the relations (5.4) and (5.5) have the consequence that

$$e_1(x,i\beta\tau)=e_1(0,0)+2ic_{11}x+4\beta c_{12}\tau,\ e_2(x,i\beta\tau)=e_2(0,0)-8\beta c_{12}\tau,\ \int_t^{\infty^+}dU_1=-\frac{1}{4}-\frac{1}{2}\int_t^{\infty^+}dU_2.\tag{7.5}$$

Thus,

$$\Phi(x,\tau)=\varepsilon^{-1}(1-2\beta^2K_0)\tau+2\beta\varepsilon^{-1}\ln\frac{\Theta(\mathbf{z}^-|B)}{\Theta(\mathbf{z}^+|B)}+c',$$

where

$$z_2^{\pm}=e_2(0,0)-8\beta c_{12}\tau\pm\int_t^{\infty^+}dU_2,\quad -2z_1^{\pm}-z_2^{\pm}=-2e_1(0,0)-e_2(0,0)-4ic_{11}x\pm{}^1/_2.$$

B. As curve with $g=3$, we take the Riemann surface of the function $w^2=(\lambda^2-t^2)(\lambda^2-t^{-2})(\lambda^4-2\lambda^2\cos2\varphi+1)$.

On it we consider the anti-involution τ: $(w,\lambda)\to(-\bar{w},-\bar{\lambda})$ and two involutions τ_1: $(w,\lambda)\to(w,-\lambda)$, τ_2: $(w,\lambda)\to(-\lambda^{-4}w,\lambda^{-1})$. Under these automorphisms of the surface, the cycle basis shown in Fig. 4 transforms as follows:

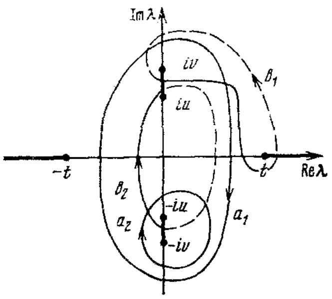

Fig. 3

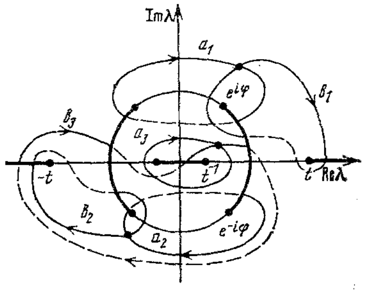

Fig. 4

$$\tau\tilde{\mathbf{a}}=\tilde{\mathbf{a}},\quad \tau\tilde{\mathbf{b}}=-\tilde{\mathbf{b}}+\tilde{\Lambda}\tilde{\mathbf{a}},\quad \tilde{\Lambda}=\begin{pmatrix}0&1&-1\\1&0&-1\\-1&-1&2\end{pmatrix};\quad \tau_1\tilde{\mathbf{a}}=\tilde{M}\tilde{\mathbf{a}},\quad \tau_1\tilde{\mathbf{b}}=\tilde{M}^t\tilde{\mathbf{b}},\quad \tilde{M}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix};\quad \tau_2\tilde{\mathbf{a}}=\tilde{N}\tilde{\mathbf{a}},\quad \tau_2\tilde{\mathbf{b}}=N^t\tilde{\mathbf{b}},\quad \tilde{N}=\begin{pmatrix}0&-1&0\\-1&0&0\\-1&-1&1\end{pmatrix}.$$

In order to use Appel's theorem, we go over to a different cycle basis:

$$\mathbf{a}=D\tilde{\mathbf{a}},\quad \mathbf{b}=(D^t)^{-1}\tilde{\mathbf{b}},\quad D=\begin{pmatrix}1&-1&0\\0&1&0\\1&1&-1\end{pmatrix}. \tag{7.6}$$

Under the automorphisms, it will transform by means of different matrices:

$$\Lambda=\begin{pmatrix}0&1&-1\\1&2&-2\\-1&-2&2\end{pmatrix},\quad M=\begin{pmatrix}-1&0&0\\1&1&0\\0&0&1\end{pmatrix},\quad N=\begin{pmatrix}1&0&0\\-1&-1&0\\-1&-2&1\end{pmatrix}. \tag{7.7}$$

Considering the Abelian differentials, normalized in the new basis, and using the expressions (5.3) and (5.5), we obtain

$$B=\begin{pmatrix}\alpha&\delta&-\gamma\\\delta&2\delta&-2\gamma\\-\gamma&-2\gamma&2\gamma\end{pmatrix},\quad \operatorname{Re}B=\frac{1}{2}\begin{pmatrix}0&1&-1\\1&2&-2\\-1&-2&2\end{pmatrix}, \tag{7.8}$$

$$\operatorname{Re}U_1(\infty^+)=0,\quad U_2(\infty^+)=0,\quad U_3=(\infty^+)=-{}^1/_4. \tag{7.9}$$

The relations (5.4) and (5.6) lead us to

$$e_1(x,i\beta\tau)=e_1(0,0)+2ic_{11}x+4\beta c_{12}\tau,\quad e_2(x,i\beta\tau)=e_2(0,0)+4ic_{11}x,\quad e_3(x,i\beta\tau)=e_3(0,0)-2i(c_{11}+c_{13})x. \tag{7.10}$$

Now, bearing in mind that $\Theta(\mathbf{z}|B)=\Theta(\hat{\mathbf{z}}|\hat{B})$, where

$$\hat{z}_1=-z_1,\quad \hat{z}_2=-z_2,\quad \hat{z}_3=z_3,\quad \hat{B}=\begin{pmatrix}\alpha&\delta&\gamma\\\delta&2\delta&2\gamma\\\gamma&2\gamma&2\gamma\end{pmatrix},$$

we can use Appel's theorem with $\nu=0$, $n_1=2$, $n_2=1$, $n_3=1$. We then obtain

$$\Theta_3(\hat{z}|\hat{B})=\Theta_2\begin{bmatrix}0&0\\0&0\end{bmatrix}\left(\begin{matrix}-2z_1-z_3\\-z_2-z_3\end{matrix}\,\middle|\,A\right)\Theta\begin{bmatrix}0\\0\end{bmatrix}(z_3|2\gamma)+\Theta_2\begin{bmatrix}{}^1/_2&0\\0&0\end{bmatrix}\left(\begin{matrix}-2z_1-z_3\\-z_2-z_3\end{matrix}\,\middle|\,A\right)\Theta\begin{bmatrix}{}^1/_2\\0\end{bmatrix}(z_3|2\gamma),$$

where

$$A=\begin{pmatrix}4\alpha-2\gamma&2\delta-2\gamma\\2\delta-2\gamma&2\delta-2\gamma\end{pmatrix}.$$

Noting that the matrix A has the necessary form, we apply to the first term Appel's theorem and to the second the corollary ($\nu=0$, $n_1=1$, $n_2=1$). As a result, the three-dimensional theta function is represented as a sum of two terms, each of which is a product of three one-dimensional theta functions:

$$\Theta(\mathbf{z}|B)=\Theta(-2z_1+z_2|4\alpha-2\delta)\Theta(-z_2-z_3|2\delta-2\gamma)\Theta(z_3|2\gamma)+$$
$$\Theta\begin{bmatrix}{}^1/_2\\0\end{bmatrix}(-2z_1+z_2|4\alpha-2\delta)\Theta\begin{bmatrix}{}^1/_2\\0\end{bmatrix}(-z_2-z_3|2\delta-2\gamma)\times\Theta\begin{bmatrix}{}^1/_2\\0\end{bmatrix}(z_3|2\gamma). \tag{7.11}$$

Thus, the solution of the KB equation constructed using the curve $w^2=(\lambda^2-t^2)(\lambda^2-t^{-2})(\lambda^4-2\lambda^2\cos 2\varphi+1)$, has the form

$$\Phi(x,\tau)=\varepsilon^{-1}(1-2\beta^2K_0)\tau+2\beta\varepsilon^{-1}\ln\frac{\Theta(\mathbf{z}^-|B)}{\Theta(\mathbf{z}^+|B)},$$

where the expression for $\Theta(\mathbf{z}^{\pm}|B)$ is given by (7.11) and

$$z_3^{\pm}=e_3(0,0)-2i(c_{11}+c_{13})x\mp{}^1/_4,\quad -z_2^{\pm}-z_3^{\pm}=-e_2(0,0)-e_3(0,0)-2i(c_{11}-c_{13})x\pm{}^1/_4,$$

$$-2z_1^{\pm}+z_2^{\pm}=e_2(0,0)-2e_1(0,0)-8\beta c_{12}\tau\mp\int_t^{\infty^+}dU_1.$$

8. Reduction of Abelian Integrals to Elliptic Integrals

A. The curve $w^2=(\lambda^2-t^2)(\lambda^2+u^2)(\lambda^2+v^2)$. Consider the basis of unnormalized Abelian differentials

$$dW_0=\frac{d\lambda}{w},\quad dW_1=\frac{\lambda d\lambda}{w}.$$

Making the change of variables $y=\pm\lambda^2$, we reduce them to elliptic differentials. Bearing in mind that $dU_1=c_{11}dW_1+c_{12}dW_0$, $dU_2=-2c_{12}dW_0$, we can express c_{11}, c_{12}, α, δ, $U_2(\infty^+)$ in terms of the elliptic integrals

$$Q_0=4(v\sqrt{t^2+u^2}+u\sqrt{t^2+v^2})^{-1}K\left[\frac{v\sqrt{t^2+u^2}-u\sqrt{t^2+v^2}}{v\sqrt{t^2+u^2}+u\sqrt{t^2+v^2}}\right],\ Q_\infty=4(\sqrt{v^2+t^2}+\sqrt{u^2+t^2})^{-1}K\left[\frac{\sqrt{v^2+t^2}-\sqrt{u^2+t^2}}{\sqrt{v^2+t^2}+\sqrt{u^2+t^2}}\right],$$

$$S_0=4(v\sqrt{t^2+u^2}+t\sqrt{v^2-u^2})^{-1}K\left[\frac{v\sqrt{t^2+u^2}-t\sqrt{v^2-u^2}}{v\sqrt{t^2+u^2}+t\sqrt{v^2-u^2}}\right],\ S_\infty=4(\sqrt{v^2+t^2}+\sqrt{v^2-u^2})^{-1}K\left[\frac{\sqrt{v^2+t^2}-\sqrt{v^2-u^2}}{\sqrt{v^2+t^2}+\sqrt{v^2-u^2}}\right]$$

$$S=\frac{1}{2}S_0+2(v\sqrt{t^2+u^2}+t\sqrt{v^2-u^2})^{-1}F\left(\varphi\left|\frac{v\sqrt{t^2+u^2}-t\sqrt{v^2-u^2}}{v\sqrt{t^2+u^2}+t\sqrt{v^2-u^2}}\right.\right),\ \varphi=\arcsin\left\{\frac{v\sqrt{v^2-u^2}-t\sqrt{t^2+u^2}}{v\sqrt{v^2-u^2}+t\sqrt{t^2+u^2}}\right\}.$$

In this notation,

$$c_{11}=-\frac{1}{2}Q_\infty^{-1},\quad c_{12}=\frac{i}{2}Q_0^{-1},\quad 2\delta=2i\frac{S_0}{Q_0},\ 4\alpha-2\delta=2+2i\frac{S_\infty}{Q_\infty},\quad U_2(\infty^+)=-\frac{i}{2}\frac{S}{Q_0}.$$

B. The curve $w^2=(\lambda^2-t^2)(\lambda^2-t^{-2})(\lambda^4-2\lambda^2\cos 2\varphi+1)$. We consider the basis of unnormalized Abelian differentials

$$dW=\frac{\lambda d\lambda}{w},\quad dW^+=\frac{\lambda^2+1}{w}d\lambda,\quad dW^-=\frac{\lambda^2-1}{w}d\lambda.$$

Then the basis of normalized Abelian differentials can be written in the form

$$dU_1=c_{12}dW+{}^1/_2(c_{11}+c_{13})dW^++{}^1/_2(c_{11}-c_{13})dW^-,\ dU_2=(c_{11}+c_{13})dW^++(c_{11}-c_{13})dW^-,\quad dU_3=-(c_{11}+c_{13})dW^+.$$

It is easy to show that

$$c_{12}=\left[\oint_{a_1}dW\right]^{-1},\quad c_{11}+c_{13}=-\left[\oint_{a_3}dW^+\right]^{-1},\quad c_{11}-c_{13}=\left[\oint_{a_3}dW^-\right]^{-1}.$$

The Abelian differentials dW, dW^+, dW^- can be reduced to elliptic ones by means of the following substitutions:

a) if the integration is along the real axis, then $\lambda=\pm\exp\{\pm{}^1/_2\operatorname{arch}y\}$;

b) if the integration is around the unit circle, then $\lambda=\pm\exp\{\pm{}^1/_2 i\arccos y\}$.

The signs are chosen in accordance with the limits of integration. As an example, we consider one of the integrals:

$$\oint_{a_1}dW=2\left\{\int_{-e^{-i\varphi}}^{e^{i\varphi}}dW+\int_{e^{-i\varphi}}^{-e^{i\varphi}}dW\right\}=4\int_{-e^{-i\varphi}}^{e^{i\varphi}}\frac{\lambda d\lambda}{w}=8\int_{-e^{-i\varphi}}^{-e^{-i\pi/2}}\frac{\lambda d\lambda}{w}.$$

Using the substitution $\lambda=-\exp\{-{}^1/_2 i\arccos y\}$ and bearing in mind that on the given path of integration

$$w=-\sqrt{(\lambda^2-t^2)(\lambda^2-t^{-2})(\lambda^4-2\lambda^2\cos 2\varphi+1)},$$

we obtain

$$\oint_{a_1}dW=2i\int_{-1}^{\cos 2\varphi}\left\{(1-y^2)\left(\frac{t^2+t^{-2}}{2}-y\right)(\cos 2\varphi-y)\right\}^{-1/2}dy.$$

We then reduce the elliptic integral to canonical form. On the basis of this, we write down a number of Abelian integrals:

$$\oint dW=\frac{8i}{u+(t+t^{-1})\sin\varphi}K\left[\frac{u-(t+t^{-1})\sin\varphi}{u+(t+t^{-1})\sin\varphi}\right],\quad \oint_{a_3}dW^+=\frac{-8}{u+t-t^{-1}}K\left[\frac{u-t+t^{-1}}{u+t-t^{-1}}\right],$$

$$\oint_{a_2} dW^- = \frac{-8}{t+t^{-1}+u} K\left[\frac{t+t^{-1}-u}{t+t^{-1}+u}\right], \quad \oint_{b_1} dW^- = \frac{-4i}{t+t^{-1}+2\cos\varphi} K\left[\frac{t+t^{-1}-2\cos\varphi}{t+t^{-1}+2\cos\varphi}\right],$$

$$\oint_{b_1} dW^+ = \frac{4}{u+t-t^{-1}} K\left[\frac{u-t+t^{-1}}{u+t-t^{-1}}\right] + \frac{4i}{u+2\sin\varphi} K\left[\frac{u-2\sin\varphi}{u+2\sin\varphi}\right],$$

$$\oint_{b_1} dW = \frac{-2}{u+(t-t^{-1})\cos\varphi} K\left[\frac{u-(t-t^{-1})\cos\varphi}{u+(t-t^{-1})\cos\varphi}\right] - \frac{2i}{u+(t+t^{-1})\sin\varphi} K\left[\frac{u-(t+t^{-1})\sin\varphi}{u+(t+t^{-1})\sin\varphi}\right],$$

$$\int_t^{\infty^+} dW = \frac{1/2}{u+(t-t^{-1})\cos\varphi}\left\{K\left[\frac{u-(t-t^{-1})\cos\varphi}{u+(t-t^{-1})\cos\varphi}\right] + F\left(\arcsin\frac{4\cos\varphi-(t-t^{-1})u}{4\cos\varphi+(t-t^{-1})u}\,\middle|\,\frac{u-(t-t^{-1})\cos\varphi}{u+(t-t^{-1})\cos\varphi}\right)\right\}$$

$$u = \{(t-t^{-1})^2 + 4\sin^2\varphi\}^{1/2}.$$

In terms of the unnormalized Abelian differentials, the b periods of the one-dimensional theta functions take a particularly simple form:

$$4\alpha-2\delta = 4\left[\oint_{a_1} dW\right]^{-1}\oint_{b_1} dW, \quad 2\delta-2\gamma = 2\left[\oint_{a_2} dW^-\right]^{-1}\oint_{b_1} dW^-, \quad 2\gamma = -2\left[\oint_{a_2} dW^+\right]^{-1}\oint_{b_1} dW^+,$$

and in addition

$$\int_{t'}^{\infty^+} dU_1 = \left[\oint_{a_1} dW\right]^{-1}\int_t^{\infty^+} dW.$$

9. Concluding Remarks

The interest in reductions of multidimensional theta functions is due to the fact that when the series (1.6) is approximated by a finite number of terms it is necessary to take into account many of them in the case of high dimensions. In connection with the fact that two-dimensional theta functions can already be approximated to high accuracy by a small number of terms of the series, one can also use a different reduction technique (see [17]), by means of which the three-dimensional theta function is "broken down" into two- and one-dimensional functions and the four-dimensional function into two-dimensional functions. This technique can be used if on a surface possessing weak symmetry there exists a cyclic automorphism (τ^p = I) or an automorphism (τ^2 = I) possessing n pairs of fixed points.

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INTEGRABILITY CONDITIONS FOR SYSTEMS OF TWO EQUATIONS OF THE FORM $u_t = A(u)u_{xx} + F(u, u_x)$. II

A. V. Mikhailov and A. B. Shabat

The conditions (integrability conditions) that the right-hand side of any system of equations of the form indicated in the title must satisfy if the system is to have a rich set of conservation laws are found. The integrability conditions form an overdetermined system of nonlinear partial differential equations. This over-determined system of equations can be integrated by quadrature and the form of the right-hand side completely determined.

1. Introduction

The aim of the present paper* is to describe a system of two nonlinear equations of the form

$$\mathbf{u}_t=\mathbf{A}(\mathbf{u})\mathbf{u}_{xx}+\mathbf{F}(\mathbf{u}, \mathbf{u}_x) \tag{1.1}$$

that possess a rich set of local conservation laws.

The system of equations (1.1) arises frequently in various problems of theoretical physics, and therefore an intensive search for integrable cases has been made during the last decade. We list the systems so far known that possess an infinite set of conservation laws.

The nonlinear Schrödinger equation ([2], 1971):

$$u_t=u_{xx}+u^2v, \quad v_t=-v_{xx}-v^2u. \tag{1.2}$$

The Boussinesq equation, whose integrability was proved in [3] (1973), can also be represented in the form (1.1):

$$u_t=u_{xx}+vv_x, \quad v_t=-v_{xx}+u_x. \tag{1.3}$$

The Heisenberg model $(J = 0)$ ([4, 5], 1977) and the Landau–Lifshitz equation $J = \operatorname{diag}(J_1, J_2, J_3)$ ([6, 7], 1978–1979):

$$\mathbf{S}_t=\mathbf{S}_{xx}\times\mathbf{S}+\mathbf{S}\times J\mathbf{S}, \quad S_1^2+S_2^2+S_3^2=1. \tag{1.4}$$

The nonlinear Schrödinger equation with derivative was integrated for $\alpha = \beta$ in ([8], 1978), for $\alpha = 0$ in ([9], 1979), and for $\beta = 0$ it can be found in ([10], 1981):

$$u_t=u_{xx}+2\beta uvu_x+\alpha u^2v_x-\alpha(\alpha-\beta)u^3v^2/2, \qquad v_t=-v_{xx}+2\beta uvv_x+\alpha v^2u_x+\alpha(\alpha-\beta)v^3u^2/2. \tag{1.5}$$

The integrable discretization of the Kadomtsev–Petviashvili equation ([11], 1979) (see also [12]) for $N = 3$ can be reduced to the form (1.1):

$$u_t=u_{xx}+2v_{xx}+u_x^2+2u_xv_x+\exp(-2u-2v)-\exp(2v), \quad v_t=-2u_{xx}-v_{xx}-v_x^2-2u_xv_x+\exp(2u)-\exp(-2u-2v). \tag{1.6}$$

In 1979, integrability of the following system ([12], 1981; [13], 1980) was proved:

$$u_t=u_{xx}+(v^2)_x, \quad v_t=-v_{xx}+(u^2)_x. \tag{1.7}$$

*The results of our previous paper [1] are used essentially in this paper.

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