Algebra and Logic, Vol. 50, No. 3, July, 2011
(Russian Original Vol. 50, No. 3, May-June, 2011)

TURING JUMPS IN THE ERSHOV HIERARCHY

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UDC 510.53

Keywords: *Turing jumps, Ershov hierarchy, constructive ordinals, superlow sets.*

We look at infinite levels of the Ershov hierarchy in the natural system of notation, which are proper for jumps of sets. It is proved that proper infinite levels for jumps are confined to Δ_a^{-1}-levels, where a stands for an ordinal $\omega^n > 1$.

A set A is said to be *n-computably enumerable* (n-c.e.), where $n \in \omega$, if there exists a computable function f in two variables such that for any x, we have $A(x) = \lim_s f(s,x)$, $f(0,x) = 0$, and

$$|\{s : f(s+1,x) \neq f(s,x)\}| \leqslant n.$$

Classes of n-c.e. sets, where $n \in \omega$, are initial levels of the Ershov hierarchy [1-3]. For ordinals greater than or equal to ω^2, appropriate classes depend on the notation for these ordinals. In the present paper, we opt for the following natural system of notation for ordinals less than ω^ω:

$$\nu_C(a) = \omega^m k_0 + \omega^{m-1} k_1 + \ldots + k_m,$$

where a is a number of a row $k_0, \ldots, k_m$ of length $m+1$.

In [1-3], it was shown that each level of the Ershov hierarchy contains sets not belonging to lower-lying levels. Here we describe levels containing Turing jumps missing at lower-lying levels of the Ershov hierarchy. It is not hard to show that not each such level properly contains the Turing jump of a set. Thus, for instance, there do not exist a set A and a number $n > 1$ such that A' would be properly n-c.e. In fact, if a jump A' of a set A is n-c.e. then A is also n-c.e. If A is computable then A' is c.e. Suppose A is not computable. Denote by B a set such that $\varnothing <_T B \leqslant_T A$ and B is c.e. By using the resolution method, we can construct a non n-c.e. set C computable with respect to B. In this event C' is 1-1 reducible to A', which is a contradiction

*Supported by RFBR (project Nos. 09-01-97010, 10-01-00399, MK-1784.2010.1), by the Federal Program “Scientific and Scientific-Pedagogical Cadres of Innovative Russia” (gov. contracts P267 and 14.740.11.1142), and by the Russian Ministry of Education through the Analytical Departmental Target Program (ADTP) “Development of Scientific Potential of the Higher School of Learning” (project No. 2.1.1/5367).

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Translated from *Algebra i Logika*, Vol. 50, No. 3, pp. 399-414, May-June, 2011. Original article submitted June 21, 2008; revised March 16, 2011.

with A' being n-c.e. Note that such an argument fails with A' belonging to some infinite level of the Ershov hierarchy.

All sets under consideration are subsets of the set of natural numbers $\omega = \{0, 1, 2, \ldots\}$. For a partial function $\psi : \omega^n \to \omega$, $\operatorname{dom}\psi$ denotes its domain. We write $\psi(x_1, \ldots, x_n)\downarrow$ if an n-tuple $(x_1, \ldots, x_n)$ is in the domain of ψ and write $\psi(x_1, \ldots, x_n)\uparrow$ otherwise. Denote by φ_e a function that is partial computable on a Turing machine with Gödel number e. Further, $\varphi_{e,s}$ is used to denote a part of a function φ_e defined after taking s steps by the Turing machine computing the function, in which case arguments on which $\varphi_{e,s}$ is defined may be only numbers less than s.

Denote by W_e^A a domain of a Turing functional Φ_e^A computable on a Turing machine with an oracle and Gödel number e. Assume $\Phi_{e,s}^A$ means the same as $\varphi_{e,s}$, but with Φ_e^A taken in place of φ_e. For a set A, by A' we denote its T-jump, i.e., the set $\{i : \Phi_i^A(i)\downarrow\}$. For a functional Φ_e^A, we write $\operatorname{use}(A; e, x, s)$ for a use-function computing $\Phi_e^A(x)$, i.e., the sum of 1 and the greatest number whose belonging to the set A is verified in computing $\Phi_e^A(x)$ if $\Phi_{e,s}^A(x)$ is defined; if $\Phi_{e,s}^A(x)$ is undefined, we assume $\operatorname{use}(A; e, x, s) = 0$.

For a set A and an element $x \in \omega$, we denote by $A{\upharpoonright}x$ the set $A \cap \{0, \ldots, x\}$, and by $\overline{A} = \{y : y \notin A\}$ the complement of A in ω. Below a set A and its characteristic function are identified. For a finite set $\{y_0, \ldots, y_n\}$, its canonical index is defined to be $z = 2^{y_0} + \ldots + 2^{y_n}$, and the set $\{y_0, \ldots, y_n\}$ is denoted D_n ($D_0 = \varnothing$ by definition). If $x \leqslant y$, then $[x, y]$ stands for a set of the form $\{z : x \leqslant z \leqslant y\}$. For a pair (x, y), $\langle x, y\rangle$ denotes its image with respect to a standard pairing function $(x^2 + 2xy + y^2 + 3x + y)/2$, and for an n-tuple $(x_1, \ldots, x_n)$, we define $\langle x_1, \ldots, x_n\rangle = \langle x_1, \ldots, \langle x_{n-1}, x_n\rangle \ldots\rangle$. By definition, a set A is 1-reducible to a set B (written $A \leqslant_1 B$) if there exists a computable injective function f such that $x \in A$ iff $f(x) \in B$. A set A is tt-reducible to a set B (written $A \leqslant_{tt} B$) if there exist computable functions f and g such that $x \in A$ iff $B{\upharpoonright}f(x) = D_y$ for some $y \in D_{g(x)}$.

If P is a unary predicate defined on some set of ordinals, then $\mu\beta(P(\beta))$ denotes a least ordinal β such that $P(\beta)$. We use lower-case Latin letters for natural numbers, and use lower-case Greek letters, for instance, α, β, γ (with or without indices), for ordinals.

Lastly, we give definitions of a natural notation system for constructive ordinals and of levels of the Ershov hierarchy in this system of notation.

Definition 1. A pair $\langle D_C, \nu_C\rangle$, where the set D_C and the function ν_C mapping D_C into a segment of ordinal numbers less than ω^ω are defined by setting

$$D_C = \{x : \exists m, k_0, \ldots, k_m (x = \langle m, k_0, \ldots, k_m\rangle \mathbin{\&} (m \neq 0 \to k_0 \neq 0))\},$$
$$\nu_C(\langle m, k_0, \ldots, k_m\rangle) = \omega^m k_0 + \omega^{m-1} k_1 + \ldots + k_m,$$

is called a *natural system of notation* for ordinals less than ω^ω.

Definition 2. Let $X \subset \omega$ and $a \in D_C$. The set X belongs to a Σ_a^{-1}-*level in the Ershov hierarchy* (written $X \in \Sigma_a^{-1}$) if there exists a binary partial computable function ψ such that for all x,

$$x \in X \Leftrightarrow \exists t \in D_C \forall t' \in D_C (\nu_C(t) < \nu_C(a) \,\&\, \psi(t,x)\downarrow = 1$$
$$\&\, (\psi(t',x)\downarrow \Rightarrow \nu_C(t) \leqslant \nu_C(t'))).$$

We say that a function ψ such as in Definition 2 defines X as a Σ_a^{-1}-set. Obviously, the natural system of notation is univalent and is order recursive; so ordinals less than ω^ω will be identified with their designations. For an ordinal $\alpha < \omega^\omega$, a Σ_α^{-1}-level is defined thus: $X \in \Sigma_\alpha^{-1}$ iff there exists a partial computable function ψ such that

$$x \in X \Leftrightarrow \exists \beta < \alpha(\psi(\beta,x)\downarrow) \,\&\, \psi(\mu\gamma(\psi(\gamma,x)\downarrow),x) = 1 \tag{1}$$

for all x.

Furthermore, Π_α^{-1}- and Δ_α^{-1}-*levels in the Ershov hierarchy* are defined by setting

$$\Pi_\alpha^{-1} = \{A \subset \omega : \overline{A} \in \Sigma_\alpha^{-1}\},\ \Delta_\alpha^{-1} = \Sigma_\alpha^{-1} \cap \Pi_\alpha^{-1}.$$

It is easy to see that $X \in \Delta_\alpha^{-1}$ iff there exists a partial computable function ψ in two variables such that for each x, there is $\beta < \alpha$ with which $\psi(\beta,x)\downarrow$ and X and ψ satisfy condition (1). In this event we say that ψ defines X as a Δ_α^{-1}-set or $X \in \Delta_\alpha^{-1}$ with a function ψ.

There are other ways to define levels of the Ershov hierarchy in the natural system of notation. For instance, $X \in \Delta_\omega^{-1}$ iff there exist computable functions f and g such that for any x, we have $X(x) = \lim\limits_s f(s,x)$ and

$$|\{s : f(s,x) \neq f(s+1,x)\}| \leqslant g(x).$$

Also we couch alternative definitions for $\Delta_{\omega 2}^{-1}$- and $\Delta_{\omega^2}^{-1}$-levels. A set X is $\Delta_{\omega 2}^{-1}$ iff there exist computable functions f and g and a partial computable function h such that for any x, we have $X(x) = \lim\limits_s f(s,x)$ and

$$|\{s : f(s,x) \neq f(s+1,x)\}| \leqslant g(x) \ \vee\ h(x)\downarrow \,\&\, |\{s : f(s,x) \neq f(s+1,x)\}| \leqslant h(x).$$

Finally, $X \in \Delta_{\omega^2}^{-1}$ iff there are computable functions f and g and a partial computable function h such that for all x, it is true that $X(x) = \lim\limits_s f(s,x)$, $h(g(x),x)\downarrow$, and

$$|\{s : f(s,x) \neq f(s+1,x)\}| \leqslant \max\{h(t,x) : t \leqslant g(x) \,\&\, h(t,x)\downarrow\}.$$

From the definitions above, we see that the Δ_ω^{-1}-level consists of sets that are limit computable, and the number of errors in their limit computation is computably bounded. In order to set bounds on the number of errors, we have to make two attempts in computing $\Delta_{\omega 2}^{-1}$-sets, and a computably bounded number of attempts in computing $\Delta_{\omega^2}^{-1}$-sets. For the general definitions of a notation system and of levels of the Ershov hierarchy, we ask the reader to consult [1-4].

Suppose $\alpha < \omega^\omega$ and $\beta < \omega^\omega$ are given. The ordinals α and β are decomposed with respect to ω as follows:

$$\alpha = \omega^m p_0 + \omega^{m-1} p_1 + \ldots + p_m,\ \ \beta = \omega^m q_0 + \omega^{m-1} q_1 + \ldots + q_m.$$

A *natural sum* of α and β is the ordinal

$$\alpha(+)\beta = \omega^m(p_0 + q_0) + \omega^{m-1}(p_1 + q_1) + \ldots + (p_m + q_m).$$

It is easy to see that the natural sum grows with increasing both the first and the second summands.

In view of [5], the condition $X \in \Delta^{-1}_\omega$ is equivalent to $X \leqslant_{tt} \varnothing'$. Therefore, sets A whose jumps are in Δ^{-1}_ω are characterized by the condition that $A' \leqslant_{tt} \varnothing'$. In the literature, such sets are referred to as *superlow* (see, e.g., [6, 7]). The results of the present paper enable us to develop a criterion verifying whether a set A is superlow: namely, A is superlow iff $A' \in \Sigma^{-1}_{\omega n}$ for some natural n.

THEOREM 1. Let a set A and an ordinal $\alpha < \omega^\omega$ be such that $A' \in \Pi^{-1}_\alpha$. Then $A' \in \Delta^{-1}_\alpha$.

Proof. Suppose that a binary function ψ defines $\overline{A'}$ as a Σ^{-1}_α-set. We construct a set F that is c.e. with respect to A, and using the recursion theorem, assume that at the beginning of the construction, the A-c.e. index of F and, consequently, a computable function f such that $x \in F \Leftrightarrow f(x) \in A'$ for all x are known. Put

$$B = \{x : \exists \beta < \alpha(\psi(\beta, x)\downarrow)\}.$$

Clearly, the set B is c.e. Let $\{B_s\}_{s\in\omega}$ be some computable enumeration of B. Choose $e \in \omega$ and a sequence $\{A_s\}_{s\in\omega}$ uniformly computable relative to s, for which $A = \lim\limits_s A_s$ and $A' = \lim\limits_s W^{A_s}_{e,s}$.

We describe how to construct the set F.

Step $s = 0$. Let $F_0 = \varnothing$.

Step $s + 1$. If $i \leqslant s$, $f(i) \in B_s$, and $\Phi^A_{e,s}\downarrow(i)$, then i is enumerated into F_{s+1}.

Put $F = \bigcup\limits_s F_s$.

We claim that $f(i) \in B$ for each i. In fact, if $f(i) \notin B$ for some i, then $i \notin F$, and hence $f(i) \in \overline{A'} \subset B$. The construction of F immediately implies that $i \in A'$ iff $i \in F$.

A partial computable function θ is defined thus: $\theta(t, x) = 1$ if $\psi(t, f(x))\downarrow \neq 1$; $\theta(t, x) = 0$ if $\psi(t, f(x)) = 1$; in all other cases $\theta(t, x)$ is undefined. Obviously, the function θ defines A' as a Δ^{-1}_α-set. $\square$

Actually Theorem 1, as distinct from subsequent results in this paper, can be proved by using an arbitrary system of notation. Our further reasoning is underpinned by the obvious fact that if a set A belongs to some level of the Ershov hierarchy, and $B \leqslant_1 A$, then B belongs to the same level as A.

THEOREM 2. Let $A' \in \Sigma^{-1}_{\omega^n}$, where $n > 0$. Then $A' \in \Delta^{-1}_{\omega^n}$.

Proof. Assume $n > 0$ and $A' \in \Sigma^{-1}_{\omega^n}$. Define

$$S = \{2^x(2y+1) : x \in A' \ \& \ y \in \omega\}.$$

Obviously, S and A' are recursively isomorphic. We choose a partial computable function ψ that defines S as a $\Sigma^{-1}_{\omega^n}$-set. A c.e. set B is defined thus:

$$B = \{x : \exists \beta < \alpha(\psi(\beta, x)\downarrow)\}.$$

Let $\{B_s\}_{s\in\omega}$ be a computable enumeration of B. Choose $e \in \omega$ and a sequence $\{A_s\}_{s\in\omega}$ uniformly computable relative to s, for which $A = \lim_s A_s$ and $S = \lim_s W^{A_s}_{e,s}$. For all x, i, and s, put

$$\begin{aligned} r(x,i) &= 2^x 3^i, \\ q(x,i,s) &= |\{t < s : \Phi^{A_t}_{e,t}(r(x,i)) \neq \Phi^{A_{t+1}}_{e,t+1}(r(x,i))\}|, \\ p(x,i,s) &= 3^x 5^{i+1} 7^{q(x,i,s)}. \end{aligned}$$

For each partial computable function φ_n, we define partial computable functions h^n_0 and h^n_1. Let $h^n_{0,0} = h^n_{1,0} = \varnothing$. Suppose $h^n_{0,s}$ and $h^n_{1,s}$ are already defined, letting $i = \min\{k : h^n_{0,s}(k)\uparrow\}$. We find a least number $x \leqslant s$ for which $\varphi_{n,s}(r(x,i))\downarrow \in B_s$, $\varphi_{n,s}(p(x,i,s))\downarrow \in B_s$, and $\Phi^{A_s}_{e,s}(r(x,i))\uparrow$. If such x exists, then we define $h^n_{0,s+1} = h^n_{0,s} \cup \{(i, r(x,i))\}$ and $h^n_{1,s+1} = h^n_{1,s} \cup \{(i, p(x,i,s))\}$. Otherwise, put $h^n_{0,s+1} = h^n_{0,s}$ and $h^n_{1,s+1} = h^n_{1,s}$. Assume $h^n_0 = \bigcup_s h^n_{0,s}$ and $h^n_1 = \bigcup_s h^n_{1,s}$. From the definitions of functions h^n_0 and h^n_1, we see that $h^n_{0,s}(i)\downarrow$ iff $h^n_{1,s}(i)\downarrow$. The functions h^n_0 and h^n_1 have disjoint ranges since the ranges of r and p are disjoint.

A step-by-step construction of a set F that is c.e. with respect to A is as follows. Using the recursion theorem, we may assume that the A-c.e. index of F is known at the beginning of the construction, and hence there exists a computable function f such that $x \in F \Leftrightarrow f(x) \in S$. Choose a number n for which $f = \varphi_n$. Put $h_k = h^n_k$ and $h_{k,s} = h^n_{k,s}$ for $k = 0, 1$.

Step $s = 0$. Let $F_0 = \varnothing$.

Step $s + 1$.

Step $(s+1, 1)$. For all $x \leqslant s$ and for a least integer $i \notin \operatorname{dom} h_{0,s}$, the following actions are executed:

(a) If $\Phi^{A_s}_{e,s}(r(x,i))\downarrow$, and

$$A_s \restriction \mathrm{use}\,(A_s; e, r(x,i), s) = A \restriction \mathrm{use}\,(A_s; e, r(x,i), s),$$

then $r(x,i)$ is enumerated into F_{s+1}.

(b) If

$$A_s \restriction \max\{\mathrm{use}\,(A_t; e, r(x,i), t) : t \leqslant s\} = A \restriction \max\{\mathrm{use}\,(A_t; e, r(x,i), t) : t \leqslant s\},$$

then $p(x,i,s)$ is enumerated into F_{s+1}.

Step $(s+1,2)$. For all $j \in \operatorname{dom} h_{0,s}$ such that $\Phi^{A}_{e,s+1}(j)\downarrow$, $h_{0,s}(j)$ and $h_{1,s}(j)$ are enumerated into F_{s+1}.

Put $F = \bigcup\limits_{s} F_s$.

There are two cases to consider.

Case 1. A function h_0 is not total. Let $i = \min\{k : h_0(k)\uparrow\}$. In this event it is true that if $f(r(x,i)) \in B$ then $r(x,i) \in S$ for all x. Indeed, suppose $f(r(x,i)) \in B$. Since $A = \lim\limits_{s} A_s$ and $S = \lim\limits_{s} W^{A_s}_{e,s}$, it follows that there exists a step s such that

$$A_s\!\upharpoonright\!\max\{\mathrm{use}\,(A_t; e, r(x,i), t) : t \leqslant s\} = A\!\upharpoonright\!\max\{\mathrm{use}\,(A_t; e, r(x,i), t) : t \leqslant s\},$$

which is verifiable at nonzero steps in (b). Moreover, s can be chosen so that the equality above is respected at all successive steps. Therefore, $p(x,i,t) \in F$ for all $t \geqslant s$, and hence $f(p(x,i,t)) \in B$. The value of $h_0(i)$ is undefined, and so $\Phi^{A}_{e}(r(x,i))$ is defined, i.e., $r(x,i) \in S$. On the other hand, item (a) for nonzero steps ensures that

$$\{r(x,i) : r(x,i) \in F \;\&\; x \in \omega\} = \{r(x,i) : r(x,i) \in S \,\&\, x \in \omega\}$$

in the case under consideration. Consequently, if $r(x,i) \in S$ then $f(r(x,i)) \in B$. (Recall that $F \leqslant_1 S$ is realized via f.) Thus $A' \leqslant_1 B$ with a function $g(x) = f(r(x,i))$. Therefore, A' will be c.e., and hence $A' \in \Delta^{-1}_{\omega^n}$.

Case 2. A function h_0 is total. In this event h_1 is also total. Let $i \in \omega$ be arbitrary and s be a least step such that $h_{0,s+1}(i)\downarrow$ (at this step, $\Phi^{A_s}_{e,s}(h_0(i))\uparrow$ by the definition of h_0), and

$$s_0 = \min\{t \leqslant s : \forall s' \in [t,s](\Phi^{A_{s'}}_{e,s'}(h_0(i))\uparrow)\}.$$

Obviously, $q(x,i,t_0) = q(x,i,t_1)$ for all $t_0, t_1 \in [s_0, s]$. Therefore, $h_1(i) = p(x,i,s_0)$. Clearly, $p(x,i,s_0) \notin F_{s_0}$.

We claim that either $h_0(i) \notin F_{s+1}$ or $h_1(i) \notin F_{s+1}$. In fact, assume that $h_0(i) \in F_{s+1}$. Then $\Phi^{A_t}_{e,t}(h_0(i))\downarrow$ at some step $t < s_0$, and

$$A_t\!\upharpoonright\!\mathrm{use}\,(A_t; e, h_0(i), t) = A\!\upharpoonright\!\mathrm{use}\,(A_t; e, h_0(i), t).$$

Since $\Phi^{A_u}_{e,u}(h_0(i))\uparrow$ for all $u \in [s_0, s]$, it follows that for all $u \in [s_0, s]$,

$$A_u\!\upharpoonright\!\max\{\mathrm{use}\,(A_v; e, h_0(i), v) : v \leqslant s_0\} \neq A\!\upharpoonright\!\max\{\mathrm{use}\,(A_v; e, h_0(i), v) : v \leqslant s_0\}.$$

Consequently, $p(x,i,s_0) \notin F_{s+1}$. Thus, with due regard for the actions at step $(s+1,2)$, we obtain

$$\forall i (i \in S \Leftrightarrow (f(h_0(i)) \in S \,\&\, f(h_1(i)) \in S)). \tag{2}$$

For given i, we choose a least number s such that $h_{0,s+1}(i)\downarrow$. As shown, either $h_0(i) \notin F_{s+1}$ or $h_1(i) \notin F_{s+1}$. The definition of F implies that $h_0(i) \in F_t$ and $h_1(i) \in F_t$ for some $t > s+1$ iff $i \in S$. Note that $F \leqslant_1 S$ via f, and so formula (2) is true.

At the moment, we construct a function θ which will define S as a $\Delta^{-1}_{\omega^n}$-set. Let ψ_s denote a part of a function ψ defined by the end of step s. For given i, we find ordinals $\beta_0, \beta_1 < \omega^n$ and a step $v \in \omega$ such that $\psi_v(\beta_0, f(h_0(i)))\downarrow$ and $\psi_v(\beta_1, f(h_1(i)))\downarrow$. (Note that these exist necessarily, since the construction of h_0 and h_1 immediately implies that $f(h_0(j)) \in B$ and $f(h_1(j)) \in B$ for all j.) Put

$$\theta_0(\beta_0(+)\beta_1, i) = \psi(\beta_0, f(h_0(i)))\psi(\beta_1, f(h_1(i))),$$
$$\beta_{0,0} = \beta_0,\ \beta_{1,0} = \beta_1.$$

Assume $\theta_s(\beta_{0,s}(+)\beta_{1,s}, i)$ is already defined. Let

$$\gamma_k = \mu\delta(\psi_{v+s}(\delta, f(h_k(i)))\downarrow), \quad k = 0, 1.$$

Define

$$\theta_{s+1}(\gamma_0(+)\gamma_1, i) = \psi(\gamma_0, f(h_0(i)))\psi(\gamma_1, f(h_1(i))),$$
$$\beta_{0,s+1} = \gamma_0,\ \beta_{1,s+1} = \gamma_1.$$

Put $\theta(t, i) = \bigcup\limits_s \theta_s(t, i)$.

We claim that θ does in fact define S as a $\Delta^{-1}_{\omega^n}$-set. Let x be arbitrary. Put $\beta_k = \mu\beta(\psi(\beta, f(h_k(x)))\downarrow)$, with $k = 0, 1$. Then $\beta_0(+)\beta_1 = \mu\gamma(\theta(\gamma, x)\downarrow)$. In view of $\beta_0 < \omega^n$ and $\beta_1 < \omega^n$, we have $\beta_0(+)\beta_1 < \omega^n$. Lastly, $S(x) = \psi(\beta_0, f(h_0(x)))\psi(\beta_1, f(h_1(x))) = \theta(\beta_0(+)\beta_1, x)$ by virtue of relation (2). As in the previous case, we see that $A' \in \Delta^{-1}_{\omega^n}$. $\square$

Theorems 1 and 2 give negative answers to the questions posed in [8] which ask if there exist sets whose jumps are in $\Sigma^{-1}_{\omega} - \Delta^{-1}_{\omega}$ and in $\Pi^{-1}_{\omega} - \Delta^{-1}_{\omega}$.

THEOREM 3. If a set A is such that $A' \in \Sigma^{-1}_{\omega^n m}$ for some n, $m > 0$, then $A' \in \Delta^{-1}_{\omega^n}$.

Proof. For $m = 1$ and for an arbitrary number $n > 0$, the statement of the theorem follows immediately from Theorem 2. By induction, we assume that it is true for some $m \geqslant 1$ and for all $n > 0$. Let $A' \in \Sigma^{-1}_{\omega^n(m+1)}$ and $n > 0$. Put

$$S = \{2^x(2y + 1) : x \in A' \ \&\ y \in \omega\}.$$

Choose a partial computable function ψ defining S as a $\Sigma^{-1}_{\omega^n(m+1)}$-set. Put

$$B = \{x : \exists\beta < \omega^n(\psi(\beta, x)\downarrow)\}.$$

Let $\{B_s\}_{s\in\omega}$ be a computable enumeration of the set B.

Now we choose $e \in \omega$ and a sequence $\{A_s\}_{s\in\omega}$ uniformly computable relative to s, for which $A = \lim\limits_s A_s$ and $S = \lim\limits_s W^{A_s}_{e,s}$. For all x, i, and s, put

$$r(x, i) = 2^x 3^i,$$

$$q(x,i,s) = |\{t < s : \Phi_{e,t}^{A_t}(r(x,i)) \neq \Phi_{e,t+1}^{A_{t+1}}(r(x,i))\}|,$$
$$p(x,y,i,s) = 2^y 3^x 5^{i+1} 7^{q(x,i,s)}.$$

For each partial computable function φ_n, partial computable functions h_0^n and h_1^n are defined as follows. Assume $h_{0,0}^n = h_{1,0}^n = \varnothing$. Suppose $h_{0,s}^n$ and $h_{1,s}^n$ are already defined, letting $i = \min\{k : h_{0,s}^n(k)\uparrow\}$. We find least $\langle x,y\rangle \leqslant s$ for which $\varphi_{n,s}(r(x,i))\downarrow \in B_s$, $\varphi_{n,s}(p(x,y,i,s))\downarrow \in B_s$, and $\Phi_{e,s}^{A_s}(r(x,i))\uparrow$. If such x and y exist, then we put $h_{0,s+1} = h_{0,s} \cup \{(i, r(x,i))\}$ and $h_{1,s+1} = h_{1,s} \cup \{(i, p(x,y,i,s))\}$; otherwise put $h_{0,s+1} = h_{0,s}$ and $h_{1,s+1} = h_{1,s}$.

At the moment, we construct a set F that is c.e. with respect to A. By the recursion theorem, we may assume that a function f, via which $F \leqslant_1 S$, is known. Choose a number n such that $f = \varphi_n$. Put $h_k = h_k^n$ and $h_{k,s} = h_{k,s}^n$ for $k = 0, 1$.

A step-by-step construction for F is as follows.

Step $s = 0$. Let $F_0 = \varnothing$.

Step $s+1$.

Step $(s+1, 1)$. For all $x, y \leqslant s$ and for a least integer $i \notin \operatorname{dom} h_{0,s}$, the following actions are executed:

(a) If $\Phi_{e,s}^{A_s}(r(x,i))\downarrow$, and

$$A_s \restriction \operatorname{use}(A_s; e, r(x,i), s) = A \restriction \operatorname{use}(A_s; e, r(x,i), s),$$

then $r(x,i)$ is enumerated into F_{s+1}.

(b) If $\Phi_{e,s}^{A_s}(p(x,y,i,s))\downarrow$, and

$$\begin{aligned} &A_s \restriction (\max\{\operatorname{use}(A_t; e, r(x,i), t) : t \leqslant s\} + \operatorname{use}(A_s; e, p(x,y,i,s), s)) \\ =&A \restriction (\max\{\operatorname{use}(A_t; e, r(x,i), t) : t \leqslant s\} + \operatorname{use}(A_s; e, p(x,y,i,s), s)), \end{aligned}$$

then $p(x,y,i,s)$ is enumerated into F_{s+1}.

Step $(s+1, 2)$. For all $j \in \operatorname{dom} h_{0,s}$ such that $\Phi_{e,s+1}^{A}(j)\downarrow$, $h_{0,s}(j)$ and $h_{1,s}(j)$ are enumerated into F_{s+1}.

Put $F = \bigcup_s F_s$.

As in the proof of Theorem 2, the following two cases are possible.

Case 1. A function h_0 is not total. Let $i = \min\{k : h_0(k)\uparrow\}$. Then the definitions of h_0 and h_1 imply that

$$\forall x \forall y \forall s (f(r(x,i)) \in B_s \,\&\, \Phi_{e,s}^{A_s}(r(x,i))\uparrow \,\&\, \langle x,y\rangle \leqslant s \Rightarrow f(p(x,y,i,s)) \notin B_s). \tag{3}$$

Assume, to the contrary, that (3) does not hold true. Let s, x, and y be chosen so that $\langle x,y\rangle \leqslant s$, $f(r(x,i)) \in B_s$, $\Phi_{e,s}^{A_s}(r(x,i))\uparrow$, and $f(p(x,y,i,s)) \in B_s$. Then $h_{0,t}(i)\downarrow$ for some $t \leqslant s+1$. Thus relation (3) is true.

First suppose that for all x,

$$f(r(x,i)) \in B \Rightarrow r(x,i) \in S.$$

Then the implication

$$f(r(x,i)) \in B \cup S \Rightarrow r(x,i) \in S$$

holds for all x. Indeed, by (a) at step $(s+1,1)$, the equality

$$\{r(x,i) : r(x,i) \in F \;\&\; x \in \omega\} = \{r(x,i) : r(x,i) \in S \;\&\; x \in \omega\} \tag{4}$$

holds for a given integer i, and $F \leqslant_1 S$ via f. On the other hand, (4) implies that if $r(x,i) \in S$ then $f(r(x,i)) \in S \cup B$. Consequently, $A' \leqslant_1 S \cup B$ via $g(x) = f(r(x,i))$.

We construct a function ξ which will define $S \cup B$ as a $\Sigma^{-1}_{\omega^n m}$-set by setting

$$\begin{aligned}\xi(0,x) &= 1 \text{if } x \in B,\\ \xi(0,x) &\text{ is undefined if } x \notin B,\\ \xi(\beta+1,x) &= \psi(\omega^n+\beta,x).\end{aligned}$$

Thus $A' \in \Sigma^{-1}_{\omega^n m}$, and by the induction hypothesis, $A' \in \Delta^{-1}_{\omega^n}$. Suppose now that there is x such that $f(r(x,i)) \in B$ and $r(x,i) \notin S$. Let $\widetilde{q} = \lim\limits_s q(x,i,s)$ (this limit exists since there exists $\lim\limits_s W^{A_s}_{e,s}$). Put $\widetilde{p}(y) = 2^y 3^x 5^{i+1} 7^{\widetilde{q}}$. Formula (3) implies that $f(\widetilde{p}(y)) \notin B$ for all y. By virtue of the fact that item (b) at step $(s+1,1)$ ensures that

$$\{\widetilde{p}(y) : \widetilde{p}(y) \in F\} = \{\widetilde{p}(y) : \widetilde{p}(y) \in S\},$$

we conclude that $y \in A'$ iff $f(\widetilde{p}(y)) \in S \cup B$. Hence $A' \in \Delta^{-1}_{\omega^n}$ by the induction hypothesis.

Case 2. A function h_0 is total. Therefore, h_1 is also total. By the definition of h_0 and h_1, for all i, we have $f(h_k(i)) \in B$, with $k = 0,1$. We claim that

$$\forall i (i \in S \Leftrightarrow (f(h_0(i)) \in S \cap B \,\&\, f(h_1(i)) \in S \cap B)). \tag{5}$$

Obviously, if formula (5) is true then the function θ constructed in proving Theorem 2 will define S as a $\Delta^{-1}_{\omega^n}$-set. Let i be arbitrary, s a least step such that $h_{0,s+1}(i){\downarrow}$ (at this step, $\Phi^{A_s}_{e,s}(h_0(i)){\uparrow}$), and

$$s_0 = \min\{t \leqslant s : \forall s' \in [t,s](\Phi^{A_{s'}}_{e,s'}(h_0(i)){\uparrow})\}.$$

Assume $h_1(i) = p(x,y,i,s_0)$. (Note that $p(x,y,i,s_0) = p(x,y,i,t)$ for $s_0 \leqslant t \leqslant s$.) We claim that either $h_0(i) \notin F_{s+1}$ or $h_1(i) \notin F_{s+1}$. Indeed, let $h_0(i) \in F_{s+1}$. Then $\Phi^{A_t}_{e,t}(h_0(i)){\downarrow}$ at some step $t < s_0$, and

$$A_t{\restriction}\operatorname{use}(A_t; e, h_0(i), t) = A{\restriction}\operatorname{use}(A_t; e, h_0(i), t).$$

Since $\Phi^{A_u}_{e,u}(h_0(i)){\uparrow}$ for all $u \in [s_0,s]$, it follows that for all $u \in [s_0,s]$,

$$\begin{aligned}&A_u{\restriction}(\max\{\operatorname{use}(A_v; e, h_0(i), v) : v \leqslant s_0\} + \operatorname{use}(A_u; e, p(x,y,i,u), u))\\ \neq{}&A{\restriction}(\max\{\operatorname{use}(A_v; e, h_0(i), v) : v \leqslant s_0\} + \operatorname{use}(A_u; e, p(x,y,i,u), u)).\end{aligned}$$

Thus $h_1(i) \notin F_{s+1}$. With the actions at step $(s+1, 2)$ in mind, we conclude that (5) is true. In view of $S \equiv_1 A'$, we obtain $A' \in \Delta^{-1}_{\omega^n}$. □

COROLLARY. If $n > 0$, $\omega^n \leqslant \alpha < \omega^{n+1}$, and $A' \in \Sigma^{-1}_{\alpha}$, then $A' \in \Delta^{-1}_{\omega^n}$.

Proof. Let $n > 0$ and $\omega^n \leqslant \alpha < \omega^{n+1}$. Choose m so that $\alpha < \omega^n m$. Using Theorem 3 and the fact that $\Sigma^{-1}_{\alpha} \subset \Sigma^{-1}_{\omega^n m}$, we see that for any set A,

$$A' \in \Sigma^{-1}_{\alpha} \Rightarrow A' \in \Sigma^{-1}_{\omega^n m} \Rightarrow A' \in \Delta^{-1}_{\omega^n}. \ \square$$

In the next theorem, we show that each level $\Delta^{-1}_{\omega^n}$, $n > 0$, properly contains the Turing jump of a set.

THEOREM 4. For every $n > 0$, there exists a set A such that $A' \in \Delta^{-1}_{\omega^{n+1}} - \Delta^{-1}_{\omega^n}$.

Proof. For a given number $n > 0$, we construct a converging sequence $\{A_s\}_{s\in\omega}$ of sets, which is uniformly computable relative to s, has A as its limit, and for all e, satisfies the following requirements:

$$N_e: \ \exists^\infty s(\Phi^{A_s}_{e,s}(e)\downarrow) \to \Phi^A_e(e)\downarrow,$$

$$R_e: \forall x \exists \alpha < \omega^n(\varphi_e(\alpha, x)\downarrow) \to \exists x(\varphi_e(\mu\alpha(\varphi_e(\alpha, x)\downarrow), x) \neq A(x)).$$

Clearly, satisfaction of all N_e and R_e ensures satisfaction of $A'(i) = \lim\limits_s \Phi^{A_s}_{i,s}(i)$ for all i and $A \notin \Delta^{-1}_{\omega^n}$; consequently, $A' \notin \Delta^{-1}_{\omega^n}$.

We construct A by using the finite injury priority method. Requirements are arranged in order of decreasing priority: $N_0, R_0, N_1, R_1, N_2, R_2, \ldots$. A parameter $r(e, s)$ definable in the construction will be called an *active witness for a requirement* R_e *at step* s. We say that a requirement R_e *requires attention at step* $s+1$ *with a witness* $y = r(e, s+1)$ if $A_s(y) = \varphi_{e,s}(\alpha, y)$, where α is least among all ordinals β (whose designations do not exceed s) such that $\varphi_{e,s}(\beta, y)\downarrow$.

We describe how to construct $\{A_s\}_{s\in\omega}$.

Step $s = 0$. Let $A_0 = \varnothing$ and $r(e, 0) = \langle 0, e\rangle$ for all e.

Step $s+1$. For all i, put

$$r(i, s+1) = \min\{\langle x, i\rangle : \langle x, i\rangle > \max\{\mathrm{use}\,(A_s; j, j, s) : j \leqslant i\}\}.$$

Find a least number $e \leqslant s$ such that R_e requires attention with a witness $r(e, s+1)$. If such e exists, then we define $A_{s+1}(r(e, s+1)) = 1 - A_s(r(e, s+1))$.

It is not hard to verify that all requirements R_e and N_e are satisfied.

Our next goal is to set a computable upper bound on the number of possible witnesses for R_e on which A is subject to change. With this in mind, we define a function g as follows:

$$g(0) = 2,$$
$$g(p+1) = 2(p+1)(g(0) + \ldots + g(p)).$$

We prove that every requirement R_p uses at most $g(p)$ witnesses. That at most $g(0)$ witnesses are needed to fulfill the requirement R_0 follows from the observation that until $\Phi^A_0(0)$ is computed,

R_0 works with the first witness, and if will $\Phi_0^A(0)$ be computed in several steps, then R_0 changes its witness, to keep the new witness forever. If at most $g(m)$ witnesses are needed to fulfill all requirements R_m, $m \leqslant p$, then the number of witnesses necessary for fulfilling R_{p+1} can be estimated thus: with every new witness for R_m, the computation of $\Phi_k^A(k)$, $m \leqslant k \leqslant p+1$, may be stopped twice (i.e., the computation of $\Phi_k^A(k)$ is subject to be assigned two values of the use-function, depending on whether a given witness is in A); thus at most $2(p+1)(g(0)+\ldots+g(p))$ witnesses are required for R_{p+1} to be fulfilled.

Now that the function g has been defined, using a simple but cumbersome argument, we can derive a partial function ψ defining A' as a $\Delta^{-1}_{\omega^{n+1}}$-set. We sketch the argument. Let $e \in \omega$ be fixed. Put $k = g(0)+\ldots+g(e)$. Wait till $\Phi_{e,s}^{A_s}(e)$ becomes defined at some step s. If $\Phi_{e,s}^{A_s}(e)$ remains undefined, then $\psi(\alpha, e)\uparrow$ for all α. Assume that such a step s exists. Then we look at how many active witnesses for requirements R_i, $i < e$, exist in the moment. Let x be one such witness. Then the number of changes of a value for $A(x)$ may be bounded by an ordinal less than ω^n. Thus an ordinal α can be chosen so that $\omega^n(k-1) \leqslant \alpha < \omega^n k$ and changes of a value for $A'(e)$ are bounded by α in an interval $[\omega^n(k-1), \omega^n k)$. As soon as a new active witness appears, we start working with an interval $[\omega^n(k-2), \omega^n(k-1))$ and so on, until every R_i, $i < e$, stops working. $\square$

Using Theorem 4 and the corollary plugged in Theorem 3, we will see that some infinite level of the Ershov hierarchy properly contains the Turing jump of a set iff that level is $\Delta^{-1}_{\omega^n}$ for some $n > 0$.

Question. How can levels of the Ershov hierarchy properly containing Turing jumps of sets be described in the Kleene system of notation?

REFERENCES

1. Yu. L. Ershov, "On a hierarchy of sets I," *Algebra Logika*, **7**, No. 1, 47-73 (1968).
2. Yu. L. Ershov, "On a hierarchy of sets II," *Algebra Logika*, **7**, No. 4, 15-47 (1968).
3. Yu. L. Ershov, "On a hierarchy of sets III," *Algebra Logika*, **9**, No. 1, 34-51 (1970).
4. M. M. Arslanov, *The Ershov Hierarchy*, A Special Course in Mathematics, Kazan State Univ., Kazan (2007).
5. H. G. Carstens, "Δ_2^0-Mengen," *Arch. Math. Logik Grundl.*, **18**, 55-65 (1976).
6. B. Schaeffer, "Dynamic notions of genericity and array noncomputability," *Ann. Pure Appl. Log.*, **95**, Nos. 1-3, 37-69 (1998).
7. S. Walk, "Towards a definition of the array computable degrees," Ph. D. Thesis, Univ. Notre Dame (1999).
8. I. Sh. Kalimullin, "Some notes on degree spectra of the structures," *Lect. Notes Comput. Sci.*, **4497**, Springer-Verlag, Berlin (2007), pp. 389-397.