Applied Mathematics and Computation 230 (2014) 164–173

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Hopf bifurcation of an epidemic model with a nonlinear birth in population and vertical transmission ☆

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ARTICLE INFO

Keywords:
Nonlinear birth
Vertical transmission
Hopf bifurcation
Normal theory
Center manifold theorem

ABSTRACT

In this paper, an epidemic model involving a nonlinear birth in population and vertical transmission was studied. When $\mathscr{R}_0 < 1$, the disease-free equilibrium was stable, while if $\mathscr{R}_0 > 1$, the disease-free equilibrium was unstable. We researched the existence of Hopf bifurcation and obtained the stability and direction of the Hopf bifurcation by using the normal theory and the center manifold theorem. Numerical simulations were carried out to illustrate the main theoretical results and a brief discussion was given to conclude this work.

1. Introduction

Mathematical epidemiology, i.e. the construction and analysis of mathematical models are one of the major areas of biology to describe the spread and control of infectious diseases. Since Kermack and Mckendrick constructed a system of ODE [1] to study epidemiology in 1927, the concept of "Compartment model" have been used until now. Most of the research literatures described the spread of a non-lethal disease in a large population by dividing the total population into three classes: the susceptible (S), the infectious (I), and the recovered (i.e. with a permanent or temporary acquired immunity) (R). We usually call these compartmental models SIR models or SIRS models, and each letter in SIR or SIRS denotes a compartment. And there must be an individual belong to one compartment.

In real life, some diseases may be passed from one individual to another via vertical transmission. That is to say, vertical transmission of diseases refer to diseases are infected to the offspring by their infected parentage. In recent years, a few studies of vertical transmission have been conducted to describe the effects of various and demographical factors [2–5]. For example, Busenberg and Cooke [5] discussed a variety of diseases which contained both of horizontally and vertically transmitting, a comprehensive and formulation survey. They also provided the mathematical analysis of compartmental models including vertical transmission. Some examples of such diseases are AIDS, Rubella, Hepatitis, etc.

2. The model

Classical epidemic models assume that the total population size is constant, and that concentrate on describing the spread of disease based on the population. In recent years more and more models pay attention to a variable population size, and then disease causing death for a longer time scale should be taken into account to reduce reproduction. For example, according to the paper [6], a nonlinear birth term $B(N)$ is considered and we can also find that the form $B(N)N$ is important

☆ This work is supported by Natural Science Foundation of Shanxi province (2013011002-2).
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0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.12.084

in determining the qualitative dynamics. In the absence of disease, the paper [7] assumes that the total population size N changes according to a population growth equation

$$\frac{dN(t)}{dt} = B(N)N - dN.$$

Here $d > 0$ is the death rate constant, and $B(N)N$ is a birth rate function with $B(N)$ satisfying with following basic assumptions for $N \in (0, \infty)$:

(A1) $B(N) > 0$;
(A2) $B(N)$ is continuously differentiable with $B'(N) < 0$;
(A3) $B(0^+) > d > B(\infty)$.

Based on the above description, we choose a $B(N) = \frac{A}{N} + B$. It is clear that $B(N)$ meets (A1) and (A2). Because of (A3), we can choose $B, \mu = \min\{\mu_1, \mu_2, \mu_3\}$ satisfying $B < \mu$. Under these assumptions, the following epidemic model is considered as one model with nonlinear birth in population and nonlinear incidence.

$$\begin{cases} \frac{dS(t)}{dt} = A - (\mu_1 - B)S(t) + BqI(t) + BR(t) - \frac{\beta S(t)I(t)}{1+\alpha I(t)} + \gamma e^{-\mu_3\tau} I(t-\tau), \\ \frac{dI(t)}{dt} = \frac{\beta S(t)I(t)}{1+\alpha I(t)} + BpI(t) - (\gamma + \mu_2)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - \mu_3 R(t) - \gamma e^{-\mu_3\tau} I(t-\tau), \end{cases} \tag{2.1}$$

where $A, B, \mu_i (i = 1, 2, 3), q, p, \beta, \alpha, \gamma$ are nonnegative. $S(t), I(t), R(t)$ denote the number of susceptible, infected and recover population stage at time t, respectively. A is constant immigrants, B is the birth rate and $\mu_i (i = 1, 2, 3)$ are natural death rate. q is the probability that a child who is born from infectious mother is susceptible; p is the probability that a child who is born from infectious mother is infected, then $p + q = 1$. β is contact rate between the susceptible and the infection. γ is the recovery rate. τ is the time delay.

The initial conditions for system (2.1) are

$$(\psi_1(\theta), \psi_2(\theta), \psi_2(\theta)) \in C_+ = C([-\tau, 0], \mathbb{R}^3_+), \quad \psi_i(0) > 0, \; i = 1, 2, 3, \tag{2.2}$$

where

$$\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \geqslant 0, i = 1, 2, 3\}.$$

Theorem 2.1. *For any solution $S(t), I(t), R(t)$ of system (2.1) with initial conditions (2.2), $S(t) < M, I(t) < M, R(t) < M$ for all large t, where $M = \frac{A}{\mu - B}$.* □

This paper is organized as following: in the next section, we obtain the basic reproduction number by the next generation method and the existence of equilibriums. We verify when $\mathscr{R}_0 < 1$, the disease-free equilibrium was stable, while if $\mathscr{R}_0 > 1$, the disease-free equilibrium was unstable. Then we focus on the local stability of the endemic equilibrium and the existence of the Hopf bifurcation. In Section 4, we obtain the stability and direction of the Hopf bifurcation by using the normal theory and the center manifold theorem. Numerical simulations are carried out in Section 5 to illustrate the main theoretical results and a brief discussion is given in last part to conclude this work.

3. The existence and stability of equilibria

3.1. The existence of equilibria and the stability of the disease-free equilibrium

It is easy to obtain the disease-free equilibrium $E_0 = (S_0, 0, 0) = \left(\frac{A}{\mu_1 - B}, 0, 0\right)$. Then, we define the basic reproduction number $\mathscr{R}_0$ of our model by directly using the next generation method presented in Diekmann et al. [8] and van den Driessche and Watmough [9].

Then

$$\mathscr{R}_0 = \frac{\beta A}{(\mu_1 - B)(\mu_2 + \gamma - Bp)}.$$

Theorem 3.1. *When $\mathscr{R}_0 < 1$, there exists the unique disease-free equilibrium E_0. When $\mathscr{R}_0 > 1$, there also exist a endemic equilibrium $E^* = (S^*(\tau), I^*(\tau), R^*(\tau))$, where $S^*(\tau) = \frac{(\mu_2+\gamma-Bp)(1+\alpha I^*)}{\beta}$, $R^*(\tau) = \frac{\gamma I^*}{b_3}(1 - e^{-\mu_3\tau})$ and $I^*(\tau) = \frac{[\beta A - (\mu_1 - B)(\mu_2+\gamma-Bp)]\mu_3}{\alpha\mu_3(\mu_1-B)(\mu_2+\gamma-Bp) + \beta\gamma B(1+e^{-\mu_3\tau}) + \beta\mu_3(\mu_2-B) + \beta\gamma\mu_3(1-e^{-\mu_3\tau})}$.*

When $\tau = 0, I^*(0) = \frac{[\beta A-(\mu_1-B)(\mu_2+\gamma-Bp)]\mu_3}{\alpha\mu_3(\mu_1-B)(\mu_2+\gamma-Bp)+2\beta\gamma B+\beta\mu_3(\mu_2-B)}$.

Then we find the characteristic equation of any equilibrium $\overline{E} = (\overline{S}, \overline{I}, \overline{R})$ as follows:

$$\det(\lambda I - J(\overline{E})) = \begin{vmatrix} \lambda + \mu_1 - B + \frac{\beta\bar{I}}{1+\alpha\bar{I}} & \frac{\beta\bar{S}}{(1+\alpha\bar{I})^2} - Bq - \gamma e^{-(\mu_3+\lambda)\tau} & -B \\ -\frac{\beta\bar{I}}{1+\alpha\bar{I}} & \lambda + \mu_2 + \gamma - Bp - \frac{\beta\bar{S}}{(1+\alpha\bar{I})^2} & 0 \\ 0 & \gamma e^{-(\mu_3+\lambda)\tau} - \gamma & \lambda + \mu_3 \end{vmatrix} = 0 \tag{3.1}$$

Theorem 3.2. *When $\mathscr{R}_0 < 1$, the disease-free equilibrium E_0 is locally asymptotically stable for all $\tau \geqslant 0$; When $\mathscr{R}_0 > 1, E_0$ is unstable for all $\tau \geqslant 0$.* □

3.2. The locally stability of the endemic equilibrium and the Hopf bifurcation

In this section we will analyze the local stability of the positive equilibrium E^*. It is not easy to find rigorously local stability condition of E^*. In the following, using stability switch criteria, we try to analyze local stability of E^*. The criteria can indicate that the stability of a given steady state is simply determined by the graph which is expressed as some functions of τ and thus can be easily depicted by Matlab. We will show the stability switch criteria for E^*.

The characteristic of the system (2.1) near the infected equilibrium E^* is given by

$$P(\lambda, \tau) + Q(\lambda, \tau)e^{-\lambda\tau} = 0, \tag{3.1}$$

where

$$\begin{aligned} &P(\lambda, \tau) = \lambda^3 + A_1(\tau)\lambda^2 + A_2(\tau)\lambda + A_3(\tau); \\ &Q(\lambda, \tau) = A_4(\tau)\lambda + A_5(\tau) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} A_1(\tau) &= \mu_1 + \mu_2 + \mu_3 + \gamma - Bp - B + \frac{\beta I^*(\tau)}{1 + \alpha I^*(\tau)} - \frac{\beta S^*(\tau)}{(1 + \alpha I^*(\tau))^2}, \\ A_2(\tau) &= (\mu_1 + \mu_3 - B)\left(\mu_2 + \gamma - Bp - \frac{\beta S^*(\tau)}{(1 + \alpha I^*(\tau))^2}\right) + \mu_3\left(\mu_1 - B + \frac{\beta I^*(\tau)}{1 + \alpha I^*(\tau)}\right) + (\mu_2 + \gamma - B)\frac{\beta I^*(\tau)}{1 + \alpha I^*(\tau)}, \\ A_3(\tau) &= \mu_3(\mu_1 - B)\left(\mu_2 + \gamma - Bp - \frac{\beta S^*(\tau)}{(1 + \alpha I^*(\tau))^2}\right) + \mu_3(\mu_2 - B)\frac{\beta I^*(\tau)}{1 + \alpha I^*(\tau)} + (\mu_3 - B)\frac{\gamma\beta I^*(\tau)}{1 + \alpha I^*(\tau)}, \\ A_4(\tau) &= -\frac{\gamma\beta I^*(\tau)}{1 + \alpha I^*(\tau)}e^{-\mu_3\tau}, \quad A_5(\tau) = -(\mu_3 - B)\frac{\gamma\beta I^*(\tau)}{1 + \alpha I^*(\tau)}e^{-\mu_3\tau}. \end{aligned}$$

Theorem 3.3. *When $\mathscr{R}_0 > 1, E^*$ is locally asymptotically stable for $\tau = 0$.*

Proof. When $\tau = 0$, the characteristic equation at E^* is of the form:

$$\lambda^3 + m_1(0)\lambda^2 + m_2(0)\lambda + m_3(0) = 0,$$

where

$$\begin{aligned} m_1(0) &= A_1(0); \\ m_2(0) &= A_2(0) + A_4(0) = (\mu_1 + \mu_3 - B)\left(\mu_2 + \gamma - Bp - \frac{\beta S^*(0)}{(1 + \alpha I^*(0))^2}\right) + \mu_3\left(\mu_1 - B + \frac{\beta I^*(0)}{1 + \alpha I^*(0)}\right) + (\mu_2 - B)\frac{\beta I^*(0)}{1 + \alpha I^*(0)}; \\ m_3(0) &= A_3(0) + A_5(0) = \mu_3(\mu_1 - B)\left(\mu_2 + \gamma - Bp - \frac{\beta S^*(0)}{(1 + \alpha I^*(0))^2}\right) + \mu_3(\mu_2 - B)\frac{\beta I^*(0)}{1 + \alpha I^*(0)}; \end{aligned}$$

For the fact that $\frac{\beta S^*(0)}{(1+\alpha I^*(0))^2} = \frac{\mu_2+\gamma-Bp}{1+\alpha I^*(0)} < \mu_2 + \gamma - Bp$, we can obtain $m_k(0) > 0 (k = 1, 2, 3)$. Then

$$\triangle_1 = m_1(0), \quad \triangle_2 = m_1(0)m_2(0) - m_3(0).$$

We obtain $\triangle_k > 0 (k = 1, 2)$ by directly computing. From the well-known Routh–Hurwitz criterion, we can obtain that when $\mathscr{R}_0 > 1$, the endemic equilibrium E^* is locally asymptotically stable for all $\tau = 0$. □

In the following, we investigate the existence of purely imaginary roots $\lambda = i\omega(\omega > 0)$ to Eq. (3.1). For simplify, we will use the abbreviations S^*, I^*, R^* instead of $S^*(\tau), I^*(\tau), R^*(\tau)$. Eq. (3.1) takes the form of a third-degree exponential polynomial in λ, with all the coefficients of P and Q depending on τ. Beretta and Kuang [10] established a geometrical criterion which gives the existence of purely imaginary root of a characteristic equation with delay dependent coefficients.

In order to apply the criterion due to Beretta and Kuang [10], we need to verify the following properties for all $\tau \in [0, \tau_{max})$, where τ_{max} is the maximum value which E^* exists.

(a) $P(0,\tau) + Q(0,\tau) \neq 0$.
(b) $P(i\omega,\tau) + Q(i\omega,\tau) \neq 0$.
(c) $\limsup\{|\frac{Q(\lambda,\tau)}{P(\lambda,\tau)}| : |\lambda| \to \infty, \mathrm{Re}\lambda \geqslant 0\} < 1$.
(d) $F(\omega,\tau) = |P(i\omega,\tau)|^2 - |Q(i\omega,\tau)|^2$ has a finite number of zeros.
(e) Each positive root $\omega(\tau)$ of $F(\omega,\tau) = 0$ is continuous and differentiable in τ whenever it exists.

Here, $P(\lambda,\tau)$ and $Q(\lambda,\tau)$ are defined as in (3.2).

Let $\tau \in [0, \tau_{max})$. It is easy to see that $P(i\omega,\tau) + Q(i\omega,\tau) = A_3(\tau) + A_5(\tau) \neq 0$. This implies that (a) is satisfied. And (b) is obviously true because

$$P(i\omega,\tau) + Q(i\omega,\tau) = \left[-A_1(\tau)\omega^2 + A_3(\tau) + A_5(\tau)\right] + i\omega\left[-\omega^2 + A_2(\tau) + A_4(\tau)\right] \neq 0.$$

From (3.2) we know that

$$\lim_{|\lambda|\to\infty}\left|\frac{Q(\lambda,\tau)}{P(\lambda,\tau)}\right| = \lim_{|\lambda|\to\infty}\left|\frac{A_4(\tau)\lambda + A_5(\tau)}{\lambda^3 + A_1(\tau)\lambda^2 + A_2(\tau)\lambda + A_3(\tau)}\right| = \lim_{|\lambda|\to\infty}\left|\frac{A_4(\tau)}{3\lambda^2 + 2A_1(\tau)\lambda + A_2(\tau)}\right| = 0$$

Therefore (c) follows.

Let F be defined as in (d). From

$$|P(i\omega,\tau)|^2 = \left(-\omega^2 + A_2(\tau)\omega\right)^2 + \left(-A_1(\tau)\omega^2 + A_3(\tau)\right)^2 = \omega^6 + [A_1^2(\tau) - 2A_2(\tau)]\omega^4 + [A_2^2(\tau) - 2A_1(\tau)A_3(\tau)]\omega^2 + A_3^2(\tau)$$

and

$$|Q(i\omega,\tau)|^2 = A_4^2(\tau)\omega^2 + A_5^2(\tau),$$

we have

$$F(\omega,\tau) = \omega^6 + p(\tau)\omega^4 + q(\tau)\omega^2 + r(\tau),$$

where $p(\tau) = A_1^2(\tau) - 2A_2(\tau), \quad q(\tau) = A_2^2(\tau) - 2A_1(\tau)A_3(\tau) - A_4^2(\tau), \quad r(\tau) = A_3^2(\tau) - A_5^2(\tau)$.

Let $\omega^2 = h$, then we have that

$$F(h) = h^3 + p(\tau)h^2 + q(\tau)h + r(\tau) = 0. \tag{3.3}$$

Theorem 3.4. *Suppose that (3.3) has no positive roots, then when $\mathscr{R}_0 > 1, E^*$ is locally asymptotically stable for all $\tau \geqslant 0$.* □

Suppose that (3.3) has positive roots, then (d) is satisfied. Let (ω_0, τ_0) be a point of its domain of definition such that $F(\omega_0, \tau_0) = 0$. We know the partial derivatives F_ω and F_τ exist and are continuous in a certain neighborhood of (ω_0, τ_0), and $F_\omega(\omega_0, \tau_0) \neq 0$. By Implicit Function Theorem, (e) is also satisfied.

Now let $\lambda = i\omega(\omega > 0)$ be a root of Eq. (3.1), and from which we have that

$$-i\omega^3 - A_1(\tau)\omega^2 + iA_2(\tau)\omega + A_3(\tau) + [iA_4(\tau)\omega + A_5(\tau)]e^{-i\omega\tau} = 0.$$

Hence, we have that

$$\begin{aligned} &\omega^3 - A_2(\tau)\omega = A_4(\tau)\omega\cos(\omega\tau) - A_5(\tau)\sin(\omega\tau); \\ &A_1\omega^2 - A_3(\tau)\omega = A_4(\tau)\omega\sin(\omega\tau) + A_5(\tau)\cos(\omega\tau). \end{aligned} \tag{3.4}$$

From (3.4) it follows that

$$\sin(\omega\tau) = \frac{[A_1(\tau)A_4(\tau) - A_5(\tau)]\omega^3 + [A_2(\tau)A_5(\tau) - A_3(\tau)A_4(\tau)]\omega}{A_4^2(\tau)\omega^2 + A_5^2(\tau)}, \tag{3.5a}$$

$$\cos(\omega\tau) = \frac{A_4(\tau)\omega^4 + [A_1(\tau)A_5(\tau) - A_2(\tau)A_4(\tau)]\omega^2 - A_3(\tau)A_5(\tau)}{A_4^2(\tau)\omega^2 + A_5^2(\tau)}. \tag{3.5b}$$

By the definitions of $P(\lambda,\tau), Q(\lambda,\tau)$ as in (3.2), and applying the property (a), (3.5a) and (3.5b) can be written as

$$\sin(\omega\tau) = \mathrm{Im}\frac{P(i\omega,\tau)}{Q(i\omega,\tau)} \tag{3.6a}$$

and

$$\cos(\omega\tau) = -\mathrm{Re}\frac{P(i\omega,\tau)}{Q(i\omega,\tau)}, \tag{3.6b}$$

which yields

$$|P(i\omega,\tau)|^2 = |Q(i\omega,\tau)|^2.$$

Assume that $I \in \mathbb{R}_{+0}$ is the set where $\omega(\tau)$ is a positive root of

$$F(\omega,\tau) = |P(i\omega,\tau)|^2 - |Q(i\omega,\tau)|^2$$

and for $\tau\neg \in I$, $\omega(\tau)$ is not defined. Then for all τ in I, $\omega(\tau)$ satisfied

$$F(\omega,\tau) = 0. \tag{3.7}$$

Assume that Eq. (3.3) has only one positive real root, we denote by h_+ this positive real root. Thus, Eq. (3.7) has only one positive real root $\omega = \sqrt{h_+}$. And the critical values of τ and $\omega(\tau)$ are impossible to solve explicitly, so we shall use the procedure described in Beretta and Kuang [10]. According to this procedure, we define $\theta(\tau) \in [0, 2\pi)$ such that $\sin\theta(\tau)$ and $\cos\theta(\tau)$ are given by the right-hand sides of (3.5a) and (3.5b), respectively, with $\theta(\tau)$ given by (3.6)

And the relation between the argument θ and $\omega\tau$ in (3.6) for $\tau > 0$ must be

$$\omega\tau = \theta + 2n\pi, \quad n = 0,1,2,\ldots \tag{3.8}$$

Hence we can define the maps $\tau_n : I \to \mathbb{R}_{+0}$ given by

$$\tau_n(\tau) := \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad \tau_n > 0,\ n = 0,1,2,\ldots,$$

where a positive root $\omega(\tau)$ of (3.3) exists in I.

Let us introduce the functions $S_n(\tau) : I \to \mathbb{R}$,

$$S_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad n = 0,1,2,\ldots,$$

that are continuous and differentiable in τ. Thus, we give the following theorem which is due to Beretta and Kuang [10].

Theorem 3.5. *Assume that $\omega(\tau)$ is a positive root of (3.1) defined for $\tau^* \in I$, $I \subseteq \mathbb{R}_{+0}$, and at some $\tau^* \in I, S_n(\tau^*) = 0$ for some $n \in N_0$. Then a pair of simple conjugate pure imaginary roots $\lambda = \pm i\omega$ exists at $\tau = \tau^*$ which crosses the imaginary axis from left to right if $\delta(\tau^*) > 0$ and crosses the imaginary axis from right to left if $\delta(\tau^*) < 0$, where*

$$\delta(\tau^*) = \mathrm{sign}\{F'_\omega(\omega\tau^*,\tau^*)\}\mathrm{sign}\left\{\frac{dS_n(\tau)}{d\tau}\bigg|_{\tau=\tau^*}\right\}.$$

Applying Theorems 3.3, 3.4 and 3.5 and the Hopf bifurcation theorem for functional differential equation [11], we can conjecture the existence of a Hopf bifurcation as stated in Theorem 3.6.

Theorem 3.6 (A conjecture). *For system (2.1), then there exists $\tau \in I$, such that the equilibrium E^* is asymptotically stable for $0 \leqslant \tau < \tau^*$, and becomes unstable for τ staying in some right neighborhood of τ^*, with a Hopf bifurcation occurring when $\tau = \tau^*$.*

4. Direction and stability of the Hopf bifurcation

In this section, we shall study the direction of the Hopf bifurcations and stability of bifurcating periodic solutions by applying the normal theory and the center manifold theorem introduced by Hassard et al. [12].

Let $u_1 = S - S^*, u_2 = I - I^*, u_3 = R - R^*, \widetilde{u}_i(t) = u_i(\tau t)(i = 1,2,3)$, $\tau = \nu + \tau^*$ and dropping the bars for simplification of notations, system (2.1) becomes an functional differential equation in $C = C([-1,0],\mathbb{R}^3)$ as

$$\begin{cases} \frac{du_1(t)}{dt} = (\tau^*+\nu)\Big[-\Big(\mu_1 - B + \frac{\beta I^*}{1+\alpha I^*}\Big)u_1(t) + \Big(Bq - \frac{\beta S^*}{(1+\alpha I^*)^2}\Big)u_2(t) + Bu_3(t) + \frac{\beta u_1u_2 + \alpha\beta I^* u_1u_2 - \alpha\beta S^* u_2^2}{(1+\alpha I^*)^2(1+\alpha I^*+\alpha u_2)} + \gamma e^{-\mu_3(\nu+\tau^*)}u_2(t-1)\Big], \\ \frac{du_2(t)}{dt} = (\tau^*+\nu)\Big[\frac{\beta I^*}{1+\alpha I^*}u_1(t) + \Big(\frac{\beta S^*}{(1+\alpha I^*)^2} + Bp - \mu_2 - \gamma\Big)u_2(t) + \frac{\alpha\beta S^* u_2^2 - \alpha\beta I^* u_1u_2 - \beta u_1u_2}{(1+\alpha I^*)^2(1+\alpha I^*+\alpha u_2)}\Big], \\ \frac{du_3(t)}{dt} = (\tau^*+\nu)\big[\gamma u_2(t) - \mu_3u_3(t) - \gamma e^{-\mu_3(\nu+\tau^*)}u_2(t-1)\big]. \end{cases}$$

Expanding the nonlinear part by Taylor expansion, we obtain

$$\begin{cases} \frac{\mathrm{d}u_1(t)}{\mathrm{d}t} = (\tau^* + \nu)\left[-\left(\mu_1 - B + \frac{\beta I^*}{1+\alpha I^*}\right)u_1(t) + \left(Bq - \frac{\beta S^*}{(1+\alpha I^*)^2}\right)u_2(t) + Bu_3(t) - \frac{\beta u_1 u_2}{(1+\alpha I^*)^2} + \frac{\alpha\beta S^* u_2^2(t)}{(1+\alpha I^*)^3} + \frac{\alpha\beta u_1(t)u_2^2(t)}{(1+\alpha I^*)^3} + \cdots + \gamma e^{-\mu_3(\nu+\tau^*)}u_2(t-1)\right], \\ \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} = (\tau^* + \nu)\left[\frac{\beta I^*}{1+\alpha I^*}u_1(t) + \left(\frac{\beta S^*}{(1+\alpha I^*)^2} + Bp - \mu_2 - \gamma\right)u_2(t) + \frac{\beta u_1(t)u_2(t)}{(1+\alpha I^*)^2} - \frac{\alpha\beta S^* u_2^2(t)}{(1+\alpha I^*)^3} - \frac{\alpha\beta u_1(t)u_2^2(t)}{(1+\alpha I^*)^3} + \cdots\right], \\ \frac{\mathrm{d}u_3(t)}{\mathrm{d}t} = (\tau^* + \nu)[\gamma u_2(t) - \mu_3 u_3(t) - \gamma e^{-\mu_3(\nu+\tau^*)}u_2(t-1)]. \end{cases}$$

We denote the above system by

$$\dot{u}(t) = L_\nu(u_t) + f(\nu, u_t), \tag{4.1}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^{\mathrm{T}} \in \mathbb{R}^3$, and $L_\nu : C \to \mathbb{R}^3, f : \mathbb{R} \times C \to \mathbb{R}^3$ are given, respectively, by

$$L_\nu(\phi) = (\tau^* + \nu)\begin{pmatrix} -(\mu_1 - B + \frac{\beta I^*}{1+\alpha I^*}) & Bq - \frac{\beta S^*}{(1+\alpha I^*)^2} & B \\ \frac{\beta I^*}{1+\alpha I^*} & \frac{\beta S^*}{(1+\alpha I^*)^2} + Bp - (\mu_2 + \gamma) & 0 \\ 0 & \gamma & -\mu_3 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + (\tau^* + \nu)\begin{pmatrix} 0 & \gamma e^{-\mu_3(\tau^*+\nu)} & 0 \\ 0 & 0 & 0 \\ 0 & -\gamma e^{-\mu_3(\tau^*+\nu)} & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix} \tag{4.2}$$

and

$$f(\nu, \phi) = (\tau^* + \nu)\left\{\begin{pmatrix} -\frac{\beta\phi_1(0)\phi_2(0)}{(1+\alpha I^*)^2} \\ \frac{\beta\phi_1(0)\phi_2(0)}{(1+\alpha I^*)^2} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\alpha\beta S^*\phi_2^2(0)}{(1+\alpha I^*)^3} \\ -\frac{\alpha\beta S^*\phi_2^2(0)}{(1+\alpha I^*)^3} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\alpha\beta\phi_1(0)\phi_2^2(0)}{(1+\alpha I^*)^3} \\ -\frac{\alpha\beta\phi_1(0)\phi_2^2(0)}{(1+\alpha I^*)^3} \\ 0 \end{pmatrix} + \cdots\right\}. \tag{4.3}$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \nu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\nu(\phi) = \int_{-1}^{0} \mathrm{d}\eta(\theta, \nu)\phi(\theta), \quad \text{for } \phi \in C. \tag{4.4}$$

In fact, we can choose

$$\eta(\theta, \nu) = (\tau^* + \nu)\begin{pmatrix} -\left(\mu_1 - B + \frac{\beta I^*}{1+\alpha I^*}\right) & Bq - \frac{\beta S^*}{(1+\alpha I^*)^2} & B \\ \frac{\beta I^*}{1+\alpha I^*} & \frac{\beta S^*}{(1+\alpha I^*)^2} + Bp - (\mu_2 + \gamma) & 0 \\ 0 & \gamma & \mu_3 \end{pmatrix}\delta(\theta) - (\tau^* + \nu)\begin{pmatrix} 0 & \gamma e^{-\mu_3(\tau^*+\nu)} & 0 \\ 0 & 0 & 0 \\ 0 & -\gamma e^{-\mu_3(\tau^*+\nu)} & 0 \end{pmatrix}\delta(\theta + 1), \tag{4.5}$$

where δ denote the Dirac delta function. For $\phi \in C([-1, 0], \mathbb{R}^3)$, define

$$A(\nu)\phi = \begin{cases} \frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} \mathrm{d}\eta(\theta, \nu)\phi(\theta), & \theta = 0 \end{cases}$$

and

$$R(\nu)(\phi) = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\nu, \phi), & \theta = 0. \end{cases}$$

Then, system (4.1) is equivalent to

$$\dot{u}(t) = A(\nu)u_t + R(\nu)u_t, \tag{4.6}$$

where $u_t = u(t + \theta)$ for $\theta \in [-1, 0]$.

For $\psi \in C([0, 1], (\mathbb{R}^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0, 1], \\ \int_{-1}^{0} \psi(-t)\mathrm{d}\eta(t, 0), & s = 0. \end{cases}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\sigma=0}^{\theta} \overline{\psi}(\sigma - \theta)\mathrm{d}\eta(\theta)\phi(\sigma)\mathrm{d}\sigma, \tag{4.7}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators. By the discussion in Section 3.2, we know that $\pm i\xi^*\tau^*$ are eigenvalues of $A(0)$. Hence, they are also eigenvalues of A^*. We first need to compute the eigenvectors of $A(0)$ and A^* corresponding to $i\xi^*\tau^*$ and $-i\xi^*\tau^*$, respectively.

Assume that $q(\theta) = (1, q_1, q_2)^{\mathrm{T}} e^{i\xi^*\tau^*\theta}$ is the eigenvector of $A(0)$ corresponding to $i\xi^*\tau^*$, then $A(0)q(\theta) = i\xi^*\tau^* q(\theta)$. Then from the definition of $A(0)$ and (4.2), (4.4) and (4.5), and for $q(-1) = q(0)e^{-i\xi^*\tau^*}$, we have

$$\begin{pmatrix} -\mu_2 - B + \frac{\beta I^*}{1+\alpha I^*} & Bq - \frac{\beta S^*}{(1+\alpha I^*)^2} + \gamma e^{-(\mu_3+i\xi^*)\tau^*} & B \\ \frac{\beta I^*}{1+\alpha I^*} & \frac{\beta S^*}{(1+\alpha I^*)^2} + Bp - (\mu_2+\gamma) & 0 \\ 0 & \gamma - \gamma e^{-(\mu_3+i\xi^*)\tau^*} & -\mu_3 \end{pmatrix} \begin{pmatrix} 1 \\ q_1(0) \\ q_2(0) \end{pmatrix} = i\xi^* \begin{pmatrix} 1 \\ q_1(0) \\ q_2(0) \end{pmatrix}.$$

Then we obtain

$$\begin{cases} q_1 = \frac{\beta I^*(1+\alpha I^*)}{(i\xi^*+\mu_2+\gamma-Bp)(1+\alpha I^*)^2-\beta S^*}, \\ q_2 = \frac{\beta\gamma I^*(1+\alpha I^*)(1-e^{-(\mu_3+i\xi^*)\tau^*})}{(i\xi^*+\mu_3)[(i\xi^*+\mu_2+\gamma-Bp)(1+\alpha I^*)^2-\beta S^*]} \end{cases}.$$

Similarly, we can calculate the eigenvector $q^*(s) = D(1, q_1^*, q_2^*)^{\mathrm{T}} e^{i\xi^*\tau^*}$ of A corresponding to $-i\xi^*\tau^*$. Where

$$\begin{cases} q_1^* = \frac{(-i\xi^*+\mu_1-B)(1+\alpha I^*)+\beta I^*}{\beta I^*}, \\ q_2^* = \frac{B}{-i\xi^*+\mu_3}. \end{cases}$$

We normalize q and q^* by the condition $\langle q^*(s), q(\theta)\rangle = 1$. Clearly, $\langle q^*(s), \overline{q}(\theta)\rangle = 0$. In order to assure $\langle q^*(s), q(\theta)\rangle = 1$, we need to determine the value of D. By (4.7), we have

$$\begin{aligned} \langle q^*(s), q(\theta)\rangle &= \overline{D}(1, \overline{q}_1^*, \overrightarrow{q}_2^*)(1, q_1, q_2)^{\mathrm{T}} - \int_{-1}^{0}\int_{\sigma=0}^{\theta} \overline{D}(1, \overline{q}_1^*, \overline{q}_2^*) e^{-i\xi^*\tau^*(\sigma-\theta)} \mathrm{d}\eta(\theta)(1, q_1, q_2)^{\mathrm{T}} e^{i\xi^*\tau^*\sigma} \mathrm{d}\sigma \\ &= \overline{D}\{1 + q_1\overline{q}_1^* + q_2\overline{q}_2^* - \int_{-1}^{0} (1, \overline{q}_1^*, \overline{q}_2^*)\theta e^{i\xi^*\tau^*\theta} \mathrm{d}\eta(\theta)(1, q_1, q_2)^{\mathrm{T}}\} \\ &= \overline{D}\{1 + q_1\overline{q}_1^* + q_2\overline{q}_2^* + \tau^* q_1\gamma e^{-\mu_3(\tau^*+\nu)}(1 - \overline{q}_2^*) e^{-i\xi^*\tau^*}\}. \end{aligned}$$

Therefore, we can choose D as

$$D = \frac{1}{1 + \overline{q}_1 q_1^* + \overline{q}_2 q_2^* + \tau^*\overline{q}_1\gamma e^{-\mu_3(\tau^*+\nu)}(1 - q_2^*) e^{i\xi^*\tau^*}}.$$

Following the algorithms given in [12] and using similar computation process in [13], we can get the coefficients which will be used to determine the important quantities

$$\begin{aligned} g_{20} &= \frac{2\beta\tau^*\overline{D}(\overline{q}_1^*-1)}{(1+\alpha I^*)^2} q_1 + \frac{2\alpha\beta\tau^* S^*\overline{D}}{(1+\alpha I^*)^3}(1-\overline{q}_1^*) q_1^2; \\ g_{11} &= \frac{2\beta\tau^*\overline{D}(\overline{q}_1^*-1)}{(1+\alpha I^*)^2} \mathrm{Re}\{q_1\} + \frac{2\alpha\beta\tau^* S^*\overline{D}}{(1+\alpha I^*)^3}(1-\overline{q}_1^*)|q_1|^2; \\ g_{02} &= \frac{2\beta\tau^*\overline{D}(\overline{q}_1^*-1)}{(1+\alpha I^*)^2} \overline{q}_1 + \frac{2\alpha\beta\tau^* S^*\overline{D}}{(1+\alpha I^*)^3}(1-\overline{q}_1^*)\overline{q}_1^2; \\ g_{21} &= \frac{\beta\tau^*\overline{D}(\overline{q}_1^*-1)}{(1+\alpha I^*)^2}\left[W_{20}^{(1)}(0)\overline{q}_1 + 2q_1 W_{11}^{(1)}(0) + W_{20}^{(2)}(0) + 2W_{11}^{(2)}(0)\right] \\ &\quad + \frac{2\alpha\beta\tau^* S^*\overline{D}}{(1+\alpha I^*)^3}(1-\overline{q}_1^*)[\overline{q}_1 W_{20}^{(2)}(0) + 2q_1 W_{11}^{(2)}(0)] + \frac{2\alpha\beta\tau^*\overline{D}}{(1+\alpha I^*)^3}(1-\overline{q}_1^*)[q_1^2 + 2|q_1|^2], \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\xi^*\tau^*} q(0) e^{i\xi^*\tau^*\theta} + \frac{i\overline{g}_{02}}{3\xi^*\tau^*}\overline{q}(0) e^{-i\xi^*\tau^*\theta} + \left[W_{20}(0) + \frac{g_{20}}{i\xi^*\tau^*} q(0) + \frac{\overline{g}_{02}}{3i\xi^*\tau^*}\overline{q}(0)\right] e^{2i\xi^*\tau^*\theta} \triangleq \frac{ig_{20}}{\xi^*\tau^*} q(0) e^{i\xi^*\tau^*\theta} \\ &\quad + \frac{i\overline{g}_{02}}{3\xi^*\tau^*}\overline{q}(0) e^{-i\xi^*\tau^*\theta} + E_1 e^{2i\xi^*\tau^*\theta} \end{aligned} \tag{4.9}$$

and

$$W_{11}(\theta) = \frac{ig_{11}}{\xi^*\tau^*}\overline{q}(0) e^{-i\xi^*\tau^*\theta} + E_2. \tag{4.10}$$

Moreover E_1, E_2 satisfy the following equations, respectively,

$$\begin{pmatrix} 2i\xi^* + \mu_1 - B + \frac{\beta I^*}{1+\alpha I^*} & -Bq + \frac{\beta S^*}{(1+\alpha I^*)^2} - \gamma e^{-(\mu_3+i\xi^*)\tau^*} & -B \\ -\frac{\beta I^*}{1+\alpha I^*} & 2i\xi^* + (\mu_2+\gamma) - Bp - \frac{\beta S^*}{(1+\alpha I^*)^2} & 0 \\ 0 & \gamma e^{-(\mu_3+i\xi^*)\tau^*} - \gamma & 2i\xi^* + \mu_3 \end{pmatrix} E_1 = \frac{2\beta\tau^* q_1}{(1+\alpha I^*)^2}(-1,1,0)^{\mathrm{T}} + \frac{2\alpha\beta\tau^* S^* q_1^2}{(1+\alpha I^*)^3}(1,-1,0)^{\mathrm{T}}$$

and

$$\begin{pmatrix} \mu_1 - B + \frac{\beta I^*}{1+\alpha I^*} & -Bq + \frac{\beta S^*}{(1+\alpha I^*)^2} - \gamma e^{-(\mu_3+i\xi^*)\tau^*} & -B \\ -\frac{\beta I^*}{1+\alpha I^*} & \mu_2 + \gamma - Bp - \frac{\beta S^*}{(1+\alpha I^*)^2} & 0 \\ 0 & \gamma e^{-(\mu_3+i\xi^*)\tau^*} - \gamma & \mu_3 \end{pmatrix} E_2 = \frac{2\beta\tau^*\mathrm{Re}\{q_1\}}{(1+\alpha I^*)^2}(-1,1,0)^{\mathrm{T}} + \frac{2\alpha\beta\tau^* S^*|q_1|^2}{(1+\alpha I^*)^3}(1,-1,0)^{\mathrm{T}}.$$

Because each g_{ij} is expressed by the parameters and display in (4.8), we can compute the following quantities:

$$\begin{aligned} &c_1(0) = \frac{i}{2\xi^*\tau^*}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|}{3}\right) + \frac{g_{21}}{2}, \quad v_2 = -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\frac{d\lambda(\tau^*)}{d\tau}\}}, \\ &\beta_2 = 2\mathrm{Re}\{c_1(0)\}, \quad T_2 = -\frac{\mathrm{Re}\{c_1(0)\} + v_2\mathrm{Re}\{\frac{d\lambda(\tau^*)}{d\tau}\}}{\xi^*\tau^*}. \end{aligned} \tag{4.11}$$

By the result of Hassard et al. [12], we have the following:

Theorem 4.1. *In (4.11), the following results hold:*

(i) *the sign of v_2 determines the directions of the Hopf bifurcation: if $v_2 > 0(v_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau^*(\tau < \tau^*)$;*
(ii) *the sign of β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0(\beta_2 > 0)$;*
(iii) *the sign of T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0(T_2 < 0)$.*

5. Numerical example

In this section, we present some numerical results of system (2.1). In Section 3, we have obtained that when $\mathscr{R}_0 < 1$, the disease-free equilibrium was stable, while if $\mathscr{R}_0 > 1$, the disease-free equilibrium was unstable. We have researched the existence of Hopf bifurcation and have obtained the stability and direction of the Hopf bifurcation according to the normal theory and the center manifold theorem. Our theoretical results show that the time delay τ must be responsible for the observed regular cycles of disease incidence.

We consider the following two set of the parameters:

(1) $A = 0.2; \mu_1 = 0.1; \mu_2 = 0.2; \mu_3 = 0.1; \beta = 4; \alpha = 1; \gamma = 10; B = 0.0001; q = 0.2; p = 0.8;\ \tau = 5$ and $S(0) = 3; I(0) = 1; R(0) = 1$. By directly computing, we obtain $R_0 = 0.7851 < 1$. According to Theorem 3, we know the "infection-free" periodic solution of system (2.1) is locally asymptotically stable for this case (see Fig. 1).
(2) $A = 0.2; \mu_1 = 0.1; \mu_2 = 0.2; \mu_3 = 0.1; \beta = 4; \alpha = 1; \gamma = 5; B = 0.0001; q = 0.2; p = 0.8;\ \tau = 5$ and $S(0) = 3; I(0) = 1; R(0) = 1$. By directly computing, we obtain $R_0 = 1.54 > 1$. For this case, the diseases will be permanent (see Figs. 2 and 3).

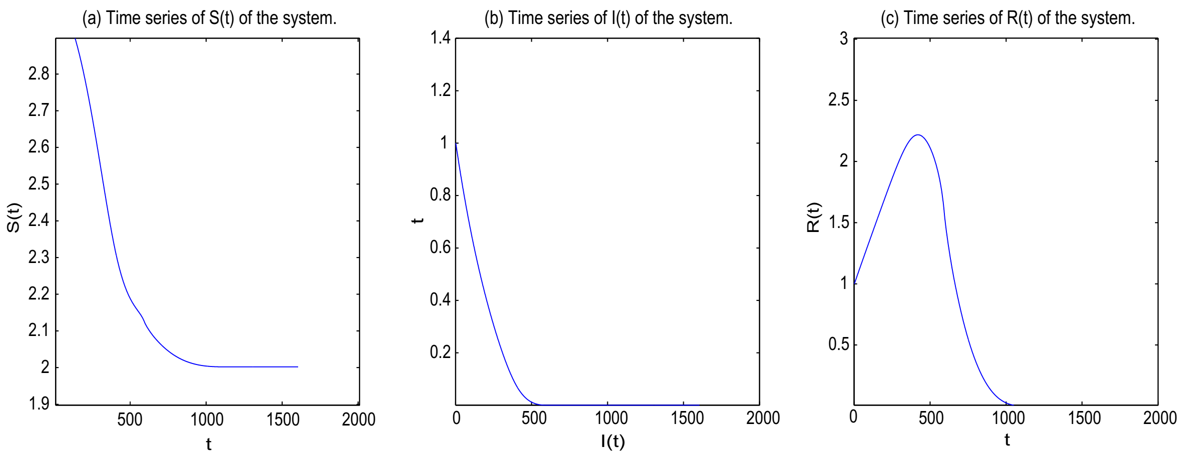

Fig. 1. Time series on the threshold values $R_0 = 0.7851 < 1$. The "infection-free" periodic solution of system (2.1) is locally asymptotically stable.

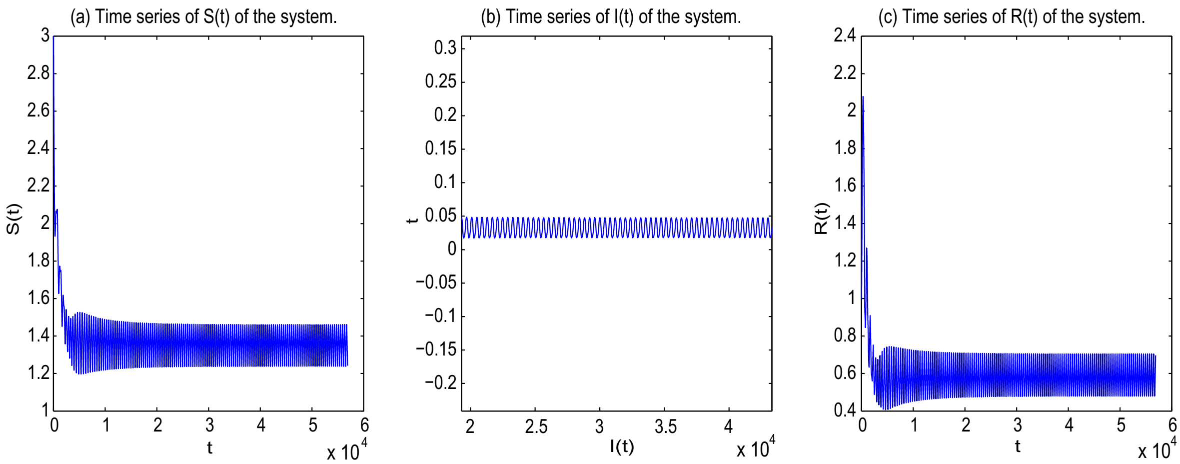

Fig. 2. Time series on the threshold values $R_0 = 1.54 > 1$. The diseases will be permanent.

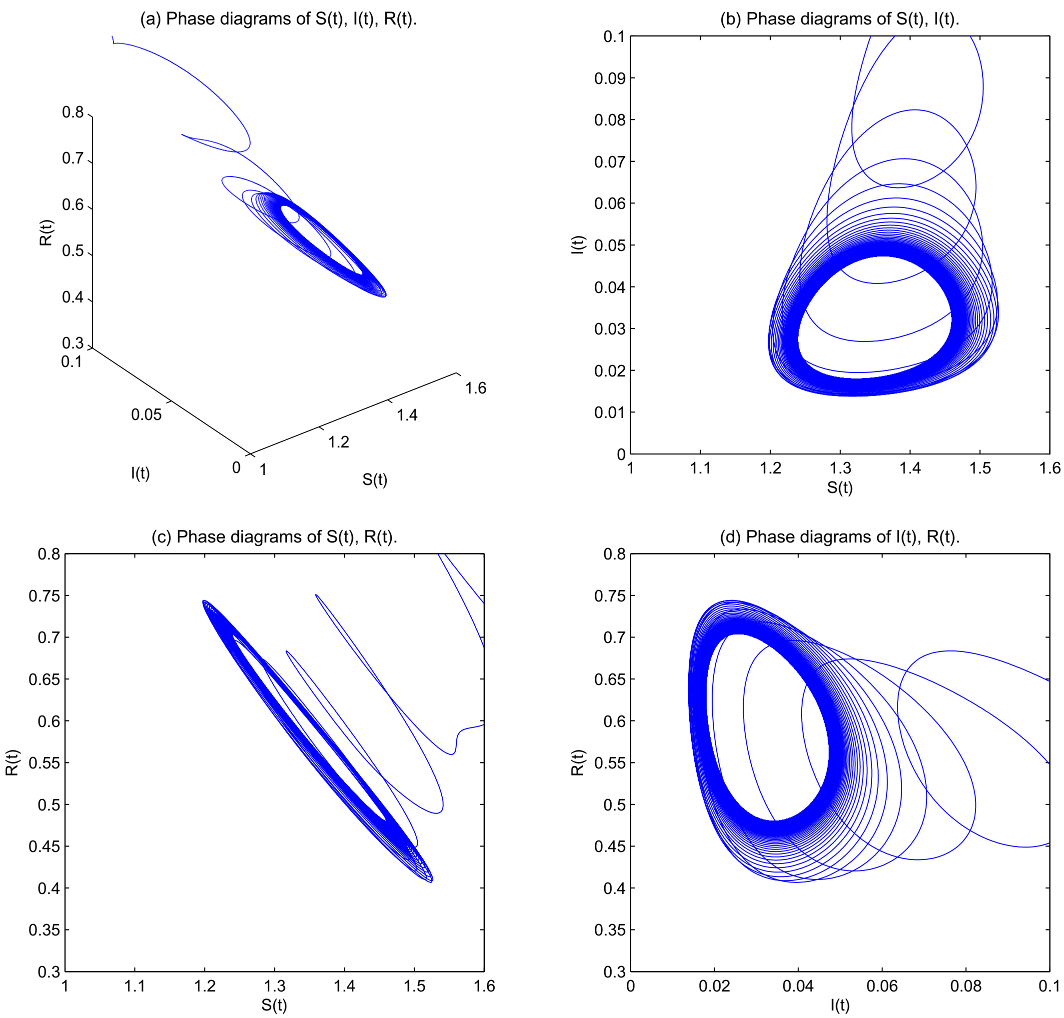

Fig. 3. Phase diagram on the threshold values $R_0 = 1.54 > 1$.

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