Lithuanian Mathematical Journal, Vol. 35, No. 3, 1995

ON QUANTALOIDS AND QUANTAL CATEGORIES

R. P. Gylys

0. INTRODUCTION

The notion of quantale, a complete lattice provided with an additional binary operation subject to certain laws, was introduced by C. J. Mulvey [5] as a possible setting for constructive foundations for quantum mechanics (see also [2, 7, 12]). A. M. Pitts [6] and K. I. Rosenthal [8, 9] extended quantales to quantaloids in a way similar to the generalization of groups to groupoids. So a quantaloid can be viewed as a "quantale with many objects" or a quantale (with unit) can be a quantaloid with only one object. It seems that quantaloids already play a new role of great promise as in the theory of Grothendieck toposes [6] as well as in several areas of theoretical computer science [10, 11].

In [3] we proposed a non-commutative version of some notions and constructions of [4] and considered a class of quantales which might possibly assist in understanding the representation theory of non-commutative C^*-algebras. In this paper we extend these quantales to quantaloids.

We begin in Section 1 with a definition and a discussion of quantaloids with which we shall work. Section 2, devoted to the generalization (in our setting) of Girard quantaloids, is not absolutely necessary for what follows. In Section 3 we introduce the notion of quantal category and indicate its relationship with that of K. I. Rosenthal's Q-enriched category. In Section 4 we show that there is a monad on the category of quantal categories, which assigns to every quantal category its associated quantal category of singletons. Finally, in Section 5 we establish that Eilenberg–Moore algebras for this monad include "sheaf structures."

1. QUANTALOIDS

We recall some basic definitions concerning quantaloids [6, 8–11].

Definition 1.1. A *quantaloid* Q (under the heading "*Sl-category*" in [6]) is a locally small category whose hom-sets are complete lattices and whose composition preserves sups in each variable separately:

(i) $(\bigvee_j b_j) \circ a = \bigvee_j (b_j \circ a)$ and $b \circ (\bigvee_j a_j) = \bigvee_j (b \circ a_j)$

for all morphisms $a\colon u \to v$, $b\colon v \to w$ and all families of morphisms $\{a_j\colon u \to v\}$, $\{b_j\colon v \to w\}$ in Q.

On Q there exist two further binary operations $()\setminus()$ and $()/()$ *left* and *right* (*Kan*) *extensions*, respectively: for any morphisms $a\colon u \to v$, $b\colon v \to w$, $c\colon u \to w$ in Q, the maps $b \circ ()\colon Q(u, v) \to Q(u, w)$ and $() \circ a\colon Q(v, w) \to Q(u, w)$ preserve suprema and hence have right adjoints: $b \setminus ()\colon Q(u, w) \to Q(u, v)$ and $()/a\colon Q(u, w) \to Q(v, w)$. Thus two different "*implications*," the left extension $b \setminus c$ of c along b and the right extension c/a of c along a, are given by

(ii) $b \setminus c = \bigvee\{d\colon u \to v \mid b \circ d \leqslant c\}$ and $c/a = \bigvee\{e\colon v \to w \mid e \circ a \leqslant c\}$

and satisfy

(iii) $d \leqslant b \setminus c \iff b \circ d \leqslant c$ and $e \leqslant c/a \iff e \circ a \leqslant c$,

respectively.

In this paper the term of quantaloid has one of its slightly wider meanings: Q is with possible lack of two-sided units (morphisms id_u, id_v such that $id_v \circ a = a = a \circ id_u$ for all $a\colon u \to v$ in Mor(Q)), i.e., two-sided

---

Institute of Mathematics and Informatics, Akademijos 4, 2600 Vilnius, Lithuania. Published in Lietuvos Matematikos Rinkinys, Vol. 35, No. 3, pp. 266–296, July–September, 1995. Original article submitted October 4, 1994.

0363-1672/95/3503-0210$12.50

unitalness of hom-sets is not at all necessary for us because we shall never use it. But we shall work with a quantaloid Q which, in addition, will satisfy

Axioms 1.2. (1) Relations

(i) $T(v,v)\circ a \leqslant a \leqslant a\circ T(u,u)$,

(ii) $(b\circ T(v,u))\wedge(a''\circ T(v',u)) = ((b\circ T(v,v'))\wedge a'')\circ T(v',u)$

and

$(T(w,v)\circ a)\wedge(T(w,u')\circ c'') = T(w,v)\circ(a\wedge(T(v,u')\circ c''))$

for $a\colon u\to v$, $a''\colon v'\to w$, $b\colon v\to w$, $c''\colon u\to u'$, $u,u',v,v',w\in \mathrm{Obj}(Q)$, always hold, where $T(v'',u'')$ denotes the top morphism in a hom-set $Q(u'',v'')$. (The reader may find the fact that the reversal of order of objects in $T(\ ,\ )$ is rather confusing.)

(2) Given two morphisms $b\colon v\to w$, $c\colon u\to w$ in Q, there exist morphisms $b'\colon w\to w$, $c'\colon u\to v$ such that

(iii) $(b\circ T(v,u))\wedge c\leqslant b\circ c'\leqslant c$ and $(b\circ T(v,u))\wedge c\leqslant b'\circ c\leqslant b\circ T(v,u)$.

Recall that a quantaloid with one object is just a quantale which in our setting is not necessarily unital (that is quite unacceptable in [6, 8–11]). In this case the additional Axioms 1.2 coincide with Axioms 1.2 [3] in turn generalizing those for quantales considered in [1, 2, 4]. Let us give an example of a situation in the more general "many objects" case which will be of concern in the motivation of Axioms 1.2.

Let Q be a modular quantaloid, i.e., it is with two-sided units and there is an involution $(\ )^{\#}\colon Q^{\mathrm{op}}\to Q$ satisfying Freyd's law of modularity: given morphisms $a\colon u\to v$, $b\colon v\to w$ and $c\colon u\to w$ in Q, then

$$(b\circ a)\wedge c\leqslant b\circ(a\wedge(b^{\#}\circ c))$$

(see [6, 9], where the latter relation is written in a dual form). Let us also suppose that units of Q coincide with top morphisms. Then, obviously, relations 1.2(i) hold. It is easy to check that the remainder of Axioms 1.2 is also satisfied. To see this, put $a=T(v,u)$, $c=a''\circ T(v',u)$ in Freyd's law of modularity. Then, the following holds:

$$\begin{aligned} b\circ T(v,u)\wedge a''\circ T(v',u) &\leqslant b\circ(T(v,u)\wedge b^{\#}\circ a''\circ T(v,'u)) = b\circ b^{\#}\circ a''\circ T(v',u)\\ &\leqslant (b\circ T(v,v')\wedge a'')\circ T(v',u) \qquad \text{(by 1.2(i))}\\ &(\leqslant b\circ T(v,u)\wedge a''\circ T(v',u) \qquad \text{(obviously)}). \end{aligned}$$

We thus obtain the first equality of 1.2(ii). To prove the second one, we let $b=T(w,v)$, $c=T(w,u')\circ c''$ in Freyd's law of modularity and calculate as follows:

$$\begin{aligned} T(w,v)\circ a\wedge T(w,u')\circ c'' &\leqslant T(w,v)\circ(a\wedge T(w,v)^{\#}\circ T(w,u')\circ c'')\\ &\leqslant T(w,v)\circ(a\wedge T(v,u')\circ c'')(\leqslant T(w,v)\circ a\wedge T(w,u')\circ c'' \qquad \text{(obviously)}). \end{aligned}$$

In order to prove 1.2(iii), we put $a=T(v,u)$ in Freyd's law of modularity. Then we have that

$$\begin{aligned} b\circ T(v,u)\wedge c &\leqslant b\circ b^{\#}\circ c\ (=b\circ c')\\ &\leqslant T(w,w)\circ c \qquad \text{(by 1.3(ii) below)}\\ &\leqslant c \qquad \text{(by 1.2(i))} \end{aligned}$$

and

$$b\circ T(v,u)\wedge c\leqslant b\circ b^{\#}\circ c\ (=b'\circ c)\leqslant b\circ T(v,u) \qquad \text{(by 1.3(ii))},$$

i.e., relations 1.2(iii) hold. Thus, a modular quantaloid, units of which are top morphisms, provides a useful example of a quantaloid satisfying Axioms 1.2. □

Let us now list some properties of quantaloids that will be of interest to us.

LEMMA 1.3. *Let Q be a quantaloid satisfying Axioms 1.2. Then for all morphisms a, $a'\colon u\to v$, $b\colon v\to w$, $c\colon u\to w$, $c'\colon u\to u'$, $d\colon w\to v$, $d'\colon w\to u$, $e\colon v\to u$, $e'\colon u'\to u$ and for all families $\{a_j\}\subseteq Q(u,v)$, $\{b_j\}\subseteq Q(v,w)$, $\{c_j\}\subseteq Q(u,w)$ of morphisms in Q, the following relations hold:*

(i) $b\circ(b\setminus c)\leqslant c$, $(c/a)\circ a\leqslant c$;

(ii) $a \leqslant a'$ *implies* $a \circ e' \leqslant a' \circ e'$, $b \circ a \leqslant b \circ a'$;

(iii) $T(v, v) \circ a = a$, $\perp(w, v) \circ a = \perp(w, u)$, $a \circ \perp(u, w) = \perp(v, w)$ (*here* $\perp(v', u') = \bigvee \varnothing$ *is the bottom morphism of* $Q(u', v')$);

(iv) $(a \circ e') \setminus d = e' \setminus (a \setminus d)$, $c'/(b \circ a) = (c'/a)/b$, $e' \setminus (e/b) = (e' \setminus e)/b$;

(v) $(\bigvee_j b_j) \setminus c = \bigwedge_j (b_j \setminus c)$, $c/\bigvee_j a_j = \bigwedge_j (c/a_j)$;

(vi) $b \setminus \bigwedge_j c_j = \bigwedge_j (b \setminus c_j)$, $(\bigwedge_j c_j)/a = \bigwedge_j (c_j/a)$;

(vii) $\bigvee_j (b \setminus c_j) \leqslant b \setminus \bigvee_j c_j$, $\bigvee_j (c_j/a) \leqslant (\bigvee_j c_j)/a$;

(viii) $\bigvee_j (b_j \setminus c) \leqslant (\bigwedge_j b_j) \setminus c$, $\bigvee_j (c/a_j) \leqslant c/\bigwedge_j a_j$;

(ix) $(d \setminus a) \circ e' \leqslant d \setminus (a \circ e')$, $b \circ (a/c') \leqslant (b \circ a)/c'$, $(b/e) \circ T(u, u) = b/e$;

(x) $(e' \setminus e) \circ (e \setminus d') \leqslant e' \setminus d'$, $(c/a) \circ (a/c') \leqslant c/c'$;

(xi) $d' \setminus e' \leqslant (a \circ d') \setminus (a \circ e')$, $c/a \leqslant (c \circ e')/(a \circ e')$;

(xii) $(b \circ T(v, u)) \wedge c = b \circ (b \setminus c) = ((b \circ T(v, u))/c) \circ c$;

(xiii) $(b \circ T(v, u)) \wedge (T(w, v) \circ a) = b \circ (b \setminus T(w, v) \circ a) = ((b \circ T(v, u))/a) \circ a$;

(xiv) $[(\bigvee_j b_j) \circ T(v, u)] \wedge c = \bigvee_j [(b_j \circ T(v, u)) \wedge c]$, $[b \circ T(v, u)] \wedge \bigvee_j c_j = \bigvee_j [(b \circ T(v, u)) \wedge c_j]$.

Proof. Relations (i), (vii), (ix)–(xi) follow immediately from 1.1(iii) and 1.2(i). To prove (ii) note that $a \leqslant a'$ is equivalent to $a \vee a' = a'$ which by 1.1(i) yields $a' \circ e' = (a \vee a') \circ e' = (a \circ e') \vee (a' \circ e')$ and $b \circ a' = b \circ (a \vee a') = (b \circ a) \vee (b \circ a')$, i.e., implication (ii) holds. Next, from (ii) and 1.2(i), (iii) we can deduce easily that $a = T(v, u) \wedge a \leqslant T(v, u) \circ T(u, u) \wedge a \leqslant T' \circ a \leqslant T(v, v) \circ a \leqslant a$, i.e., $T(v, v)$ is the left-sided unit in Q, while the following two equalities of (iii) are just the case "$\{j\}$ is an empty set" in Axiom 1(i). Obviously (ii) and 1.1(iii) imply (iv), (vi), (viii). Further, we verify the left relation of (v) by the following series of implications:

$$
\begin{aligned}
&\Big(\bigvee_j b_j\Big) \circ \bigg(\Big(\bigvee_j b_j\Big) \setminus c\bigg) \leqslant c \quad \text{(by (i))} \implies \bigvee_j \bigg[b_j \circ \bigg(\Big(\bigvee_i b_i\Big) \setminus c\bigg)\bigg] \leqslant c \\
&\implies (\forall j) \quad b_j \circ \bigg(\Big(\bigvee_i b_i\Big) \setminus c\bigg) \leqslant c \implies (\forall j) \quad \Big(\bigvee_i b_i\Big) \setminus c \leqslant b_j \setminus c \qquad \text{(by 1.1(iii))} \\
&\implies \Big(\bigvee_i b_i\Big) \setminus c \leqslant \bigwedge_i (b_i \setminus c), \qquad (\forall j) \quad \bigwedge_i (b_i \setminus c) \leqslant b_j \setminus c \\
&\implies (\forall j) \quad b_j \circ \bigwedge_i (b_i \setminus c) \leqslant c \implies \bigvee_j \Big(b_j \circ \bigwedge_i (b_i \setminus c)\Big) \leqslant c \\
&\implies \Big(\bigvee_j b_j\Big) \circ \bigwedge_i (b_i \setminus c) \leqslant c \implies \bigwedge_i (b_i \setminus c) \leqslant \Big(\bigvee_i b_i\Big) \setminus c,
\end{aligned}
$$

while the verification of the other equality in (v) is similar. Now we prove the left side of (xii). We have

$$
\begin{aligned}
b \circ (b \setminus c) &= b \circ \Big(\bigvee \{a \in Q(u, v) \mid b \circ a \leqslant c\}\Big) = \bigvee \{b \circ a \in Q(u, w) \mid b \circ a \leqslant c\} \qquad \text{(by 1.1(i))} \\
&\geqslant (b \circ T(v, u)) \wedge c \qquad \text{(by 1.2(iii))}.
\end{aligned}
$$

Conversely, $b \circ (b \setminus c) \leqslant b \circ T(v, u)$ and $b \circ (b \setminus c) \leqslant c$, so that

$$b \circ (b \setminus c) \leqslant b \circ T(v, u) \wedge c.$$

The argument for the right side of (xii) and (xiii) is similar. Finally, the first relation in (xiv) holds, since

$$\left[\left(\bigvee_i b_i\right) \circ T(v,u)\right] \wedge c = \left(\bigvee_i b_i\right) \circ \left(\left(\bigvee_j b_j\right) \setminus c\right) \qquad \text{(by (xii))}$$

$$= \bigvee_i \left[b_i \circ \left(\left(\bigvee_j b_j\right) \setminus c\right)\right] \qquad \text{(by 1.1 (i))}$$

$$= \bigvee_i \left(b_i \circ \bigwedge_i (b_j \setminus c)\right) \qquad \text{(by (v))}$$

$$\leqslant \bigvee_i b_i \circ (b_i \setminus c) = \bigvee_i (b_i \circ T(v,u) \wedge c) \qquad \text{(by (xii))}$$

and

$$(\forall j) \quad b_j \circ T(v,u) \wedge c \leqslant \left[\left(\bigvee_i b_i\right) \circ T(v,u)\right] \wedge c \Longrightarrow \quad \bigvee_i (b_i \circ T(v,u) \wedge c) \leqslant \left[\left(\bigvee_i b_i\right) \circ T(v,u)\right] \wedge c,$$

while the last relation is verified in a similar way. □

Note that by 1.3(xiv) each hom-lattice $Q(u,v)$ is actually a quantale, with the product of a, a': $u \to v$ being $(a \circ T(u,u)) \wedge a'$ and the left-sided unit being $T(v,u)$. (The associativity of this multiplication follows from 1.2(ii). Since $T(v,u) = T(v,u) \circ T(u,u)$ (by 1.2(i)), it follows that $(T(v,u) \circ T(u,u)) \wedge a = a$.)

2. GENERALIZED GIRARD QUANTALOIDS

In this section we shall deal with a class of quantaloids together with "dualizing families" which coincide with the so-called Girard quantaloids (introduced by K. I. Rosenthal in [9], and in the case of quantales – by D. Yetter in [12]) when these dualizing families are "cyclic" and $T(u,u) = id_u$ ($\forall u \in \mathrm{Obj}(Q)$).

Definition 2.1 (cf. Definition 2.6 [9]). Let Q be a quantaloid. $O = \{O_u\colon u \to v \mid u \in Q\}$, a family of morphisms of Q such that $O_u \circ T(u,u) = O_u$ is a *dualizing family* iff given a: $u \to v$ in $\mathrm{Mor}(Q)$, then

$$(O_u/a) \setminus O_u = O_v/(a \setminus O_v) = a \circ T(u,u).$$

Q is a *generalized Girard quantaloid* iff it has a dualizing family O of morphisms.

If Q is a generalized Girard quantaloid and a: $u \to v$ is a morphism of Q, then we shall use a° and ${}^\circ a$ to denote two different (in general) "negations" $a \setminus O_v$ and O_u/a, respectively. Note that $({}^\circ a)^\circ = {}^\circ(a^\circ)(=: {}^\circ a^\circ) = a \circ T(u,u)$, $a \leqslant b^\circ$ iff $b \leqslant {}^\circ a$ (by 1.1(iii)), $a^\circ \circ T(v,v) = a^\circ$ (since $O_v \circ T(v,v) = O_v$ implies $(a \setminus O_v) \circ T(v,v) \leqslant a \setminus (O_v \circ T(v,v))$ (by 1.3(ix))$= a \setminus O_v$), ${}^\circ a \circ T(v,v) = {}^\circ a$ (since $(O_u/a) \circ a = (O_u/a) \circ T(v,v) \circ a$ (by 1.3(iii))$\leqslant O_u$ imply $(O_u/a) \circ T(v,v) \leqslant O_u/a$ (by 1.1(iii))), ${}^\circ(a^\circ)^\circ = a^\circ$, ${}^\circ({}^\circ a)^\circ = {}^\circ a$, $({}^\circ a^\circ)^\circ = a^\circ$ (since $a \circ ((a \circ T(u,u)) \setminus O_v) = a \circ T(u,u) \circ (a \circ T(u,u) \setminus O_v) \leqslant O_v$ imply $(a \circ T(u,u)) \setminus O_v \leqslant a \setminus O_v$ and since $a \circ T(u,u) \circ (a \setminus O_v) = a \circ (a \setminus O_v) \leqslant O_v$ imply $a \setminus O_v \leqslant (a \circ T(u,u)) \setminus O_v$) and ${}^\circ({}^\circ a^\circ) = {}^\circ a$ (since $(O_u/(a \circ T(u,u))) \circ a \leqslant (O_u/(a \circ T(u,u))) \circ a \circ T(u,u) \leqslant O_u$ imply $O_u/(a \circ T(u,u)) \leqslant O_u/a$ and since $(O_u/a) \circ a \circ T(u,u) \leqslant O_u \circ T(u,u) = O_u$ imply $O_u/a \leqslant O_u/(a \circ T(u,u))$).

The following lemmas list some basic facts about generalized Girard quantaloids.

LEMMA 2.2 (cf. [9, Proposition 2.3] and [4, Lemma 1.4]). *Let Q be a generalized Girard quantaloid with a dualizing family $O = \{O_u\colon u \to u \mid u \in \mathrm{Obj}(Q)\}$ and let a, a': $u \to v$, b: $v \to w$, c: $u \to w$, e: $v \to u$, d': $w \to u$, $\{a_i\colon u \to v\}$, $\{b_i\colon u \to v\}$ be, respectively, morphisms and families of morphisms in Q. Then:*

(i) $(\bigvee_j a_j)^\circ = \bigwedge_j a_j^\circ$, ${}^\circ(\bigvee_j a_j) = \bigwedge_j {}^\circ a_j$, $(\bigwedge_j {}^\circ a_j^\circ)^\circ = \bigvee_j a_j^\circ$, ${}^\circ(\bigwedge_j {}^\circ a_j^\circ) = \bigvee_j {}^\circ a_j$;

(ii) $a \setminus b^\circ = (b \circ a)^\circ$, ${}^\circ a/b = {}^\circ(b \circ a)$, ${}^\circ b \setminus {}^\circ a^\circ = {}^\circ b^\circ \setminus a^\circ$ $(= ({}^\circ a \circ {}^\circ b)^\circ = {}^\circ(a^\circ \circ b^\circ))$;

(iii) $c^\circ/b^\circ = c \setminus {}^\circ b^\circ$, ${}^\circ c \setminus {}^\circ a = {}^\circ c^\circ/a$;

(iv) ${}^\circ a^\circ \vee {}^\circ a'^\circ = ({}^\circ a^\circ/a') \setminus {}^\circ a^\circ = {}^\circ a^\circ/(a' \setminus {}^\circ a^\circ)$;

(v) ${}^\circ a^\circ \vee \bigwedge_j {}^\circ a_j^\circ = \bigwedge_j({}^\circ a^\circ \vee {}^\circ a_j^\circ)$;

(vi) $b \circ \bigwedge_j {}^\circ a_j^\circ = \bigwedge_j b \circ {}^\circ a_j^\circ$, $(\bigwedge_j {}^\circ b_j^\circ) \circ {}^\circ a^\circ = \bigwedge_j (b_j \circ {}^\circ a^\circ)$.

Proof. (i) The first two equalities are none other than 1.3(v). Since $\bigwedge_j {}^\circ a_j^\circ = {}^\circ(\bigvee_j a_j^\circ) = (\bigvee_j {}^\circ a_j)^\circ$, it follows that $(\bigwedge_j {}^\circ a_j^\circ)^\circ = {}^\circ(\bigvee_j a_j^\circ)^\circ = (\bigvee_j a_j^\circ) \circ T(v,v) = \bigvee_j (a_j^\circ \circ T(v,v)) = \bigvee_j a_j^\circ$ and ${}^\circ(\bigwedge_j {}^\circ a_j^\circ) = {}^\circ(\bigvee_j {}^\circ a_j)^\circ = (\bigvee_j {}^\circ a_j) \circ T(v,v) = \bigvee_j ({}^\circ a_j \circ T(v,v)) = \bigvee_j {}^\circ a_j$.
(ii) The first two equalities follow immediately from 1.3(iv). Next, we have that $({}^\circ b \setminus {}^\circ a^\circ) \circ a^\circ = ({}^\circ b \setminus {}^\circ a^\circ) \circ ({}^\circ a^\circ)^\circ = ({}^\circ b \setminus {}^\circ a^\circ) \circ ({}^\circ a^\circ \setminus O_v) \leqslant {}^\circ b \setminus O_v$ (by 1.3(x))$= {}^\circ b^\circ$ imply ${}^\circ b \setminus {}^\circ a^\circ \leqslant {}^\circ b^\circ / a^\circ$ (by 1.1(iii)) and ${}^\circ b \circ ({}^\circ b^\circ / a^\circ) = {}^\circ({}^\circ b^\circ) \circ ({}^\circ b^\circ / a^\circ) = (O_v / {}^\circ b^\circ) \circ ({}^\circ b^\circ / a^\circ) \leqslant O_v / a^\circ = {}^\circ a^\circ$ imply ${}^\circ b^\circ / a^\circ \leqslant {}^\circ b \setminus {}^\circ a^\circ$.
(iii) In view of 1.3(x) and 1.1(iii) we obtain

$$\begin{aligned}
&c \circ (c^\circ / b^\circ) = c \circ T(u,u) \circ (c^\circ / b^\circ) = (O_w / c^\circ) \circ (c^\circ / b^\circ) \leqslant {}^\circ b^\circ \implies c^\circ / b^\circ \leqslant c \setminus {}^\circ b^\circ, \\
&(c \setminus {}^\circ b^\circ) \circ b^\circ = (c \setminus {}^\circ b^\circ) \circ ({}^\circ b^\circ)^\circ \leqslant c^\circ \implies c \setminus {}^\circ b^\circ \leqslant c^\circ / b^\circ, \\
&({}^\circ c \setminus {}^\circ a) \circ a \leqslant ({}^\circ c \setminus {}^\circ a) \circ ({}^\circ a \setminus O_u) \leqslant {}^\circ c^\circ \implies {}^\circ c \setminus {}^\circ a \leqslant {}^\circ c^\circ / a, \\
&{}^\circ c \circ ({}^\circ c^\circ / a) = {}^\circ({}^\circ c^\circ) \circ ({}^\circ c^\circ / a) \leqslant {}^\circ a \implies {}^\circ c^\circ / a \leqslant {}^\circ c \setminus {}^\circ a.
\end{aligned}$$

(iv) We have

$$\begin{aligned}
a^\circ \vee a'^\circ &= ({}^\circ a^\circ \wedge {}^\circ a'^\circ)^\circ \qquad \text{(by (i))} \\
&= (a \circ T(u,u) \wedge {}^\circ a'^\circ)^\circ = (a \circ (a \setminus {}^\circ a'^\circ))^\circ \qquad \text{(by 1.3(xi))} \\
&= (a \setminus {}^\circ a'^\circ) \setminus a^\circ \qquad \text{(by (ii))} \\
&= (a^\circ / a'^\circ) \setminus a^\circ \qquad \text{(by (iii))},
\end{aligned}$$

while the remainder follows by a similar calculation.
(v) Because of (i) and 1.3(xiv) we obtain that

$$\begin{aligned}
{}^\circ a^\circ \vee \bigwedge_j {}^\circ a_j^\circ &= ({}^\circ a)^\circ \vee \bigwedge_j ({}^\circ a_j)^\circ = ({}^\circ a)^\circ \vee \Big(\bigvee_j {}^\circ a_j\Big)^\circ \\
&= \Big({}^\circ({}^\circ a)^\circ \wedge {}^\circ\Big(\bigvee_j {}^\circ a_j\Big)^\circ\Big)^\circ = \Big({}^\circ({}^\circ a)^\circ \wedge \bigvee_j {}^\circ({}^\circ a_j)^\circ\Big)^\circ \\
&= \Big(\bigvee_j ({}^\circ({}^\circ a)^\circ \wedge {}^\circ({}^\circ a_j)^\circ)\Big)^\circ = \bigwedge_j ({}^\circ({}^\circ a)^\circ \wedge {}^\circ({}^\circ a_j)^\circ)^\circ = \bigwedge_j ({}^\circ a^\circ \vee {}^\circ a_j^\circ).
\end{aligned}$$

(vi) In view of

$$\begin{aligned}
b \setminus \bigwedge_j b \circ {}^\circ a_j^\circ \Bigg[&= \bigwedge_j (b \setminus b \circ {}^\circ a_j^\circ) \qquad \text{(by 1.3(vi))} \\
&= \bigwedge_j (b \setminus {}^\circ(b \circ a_j)^\circ) = \bigwedge_j (b^\circ / (b \circ a_j)^\circ) = \bigwedge_j (b^\circ / (a_j \setminus b^\circ)) \\
&= \bigwedge_j ({}^\circ(b^\circ)^\circ / (a_j \setminus {}^\circ(b^\circ)^\circ)) = \bigwedge_j ({}^\circ(b^\circ)^\circ \vee {}^\circ a_j^\circ) \\
&= {}^\circ(b^\circ)^\circ \vee \bigwedge_j {}^\circ a_j^\circ \qquad \text{(by (v))} = {}^\circ(b^\circ)^\circ \vee {}^\circ\Big(\bigwedge_j {}^\circ b_j^\circ\Big)^\circ \\
&= {}^\circ(b^\circ)^\circ / \Big(\bigwedge_j {}^\circ a_j^\circ \setminus {}^\circ(b^\circ)^\circ\Big) = b^\circ / \Big(\bigwedge_j {}^\circ a_j^\circ \setminus b^\circ\Big) \\
&= b^\circ / \Big(b \circ \bigwedge_j {}^\circ a_j^\circ\Big)^\circ = b / {}^\circ\Big(b \circ \bigwedge_j {}^\circ a_j^\circ\Big)^\circ = b \setminus b \circ {}^\circ\Big(\bigwedge_j {}^\circ a_j^\circ\Big)^\circ \Bigg] \\
&= b \setminus \Big(b \circ \bigwedge_j {}^\circ a_j^\circ\Big)
\end{aligned}$$

we obtain that

$$\bigwedge_j b \circ {}^\circ a_j^\circ = b \circ T(v,v) \wedge \bigwedge_j b \circ {}^\circ a_j^\circ = b \circ \left(b \setminus \bigwedge_j b \circ {}^\circ a_j^\circ\right) = b \circ \left(b \setminus b \circ \bigwedge_j {}^\circ a_j^\circ\right)$$
$$= b \circ T(v,v) \wedge \left(b \circ \bigwedge_j {}^\circ a_j^\circ\right) = b \circ \bigwedge_j {}^\circ a_j^\circ.$$

Now interchanging the role of pairs (b, a_j) and (b_j, a) we can verify the last equality. □

LEMMA 2.3. *Let $(Q, \{O_v\})$ be a generalized Girard quantaloid. Then it has a second composition, $()+()$, defined by*

$$b + a = ({}^\circ a \circ {}^\circ b)^\circ \big[= {}^\circ(a^\circ \circ b^\circ) = {}^\circ b \setminus {}^\circ a^\circ = {}^\circ b^\circ / a^\circ \quad (by\ 2.2(ii))\big]$$

and satisfying

(i) $(b+a)+e' = b + (a+e')$;
(ii) $O_v + a = a + O_u = a \circ T(u,u)$;
(iii) $({}^\circ a + {}^\circ b)^\circ = {}^\circ(a^\circ + b^\circ) = b \circ a \circ T(u,u)$;
(iv) $b + \bigvee_j a_j = \bigvee_j (b + a_j)$, $(\bigvee_j b_j) + a = \bigvee_j (b_j + a)$;
(v) $b + \bigwedge_j {}^\circ a_j^\circ = \bigwedge_j (b + a_j)$, $(\bigwedge_j {}^\circ b_j^\circ) + a = \bigwedge_j (b_j + a)$;
(vi) ${}^\circ a^\circ \vee {}^\circ a'^\circ = (a' \circ a^\circ) + a = a + ({}^\circ a \circ a')$;
(vii) $b \circ (a + e') \leqslant (b \circ a) + e'$, $(b+a) \circ e' \leqslant b + (a \circ e')$ ($\circ$+ *associativity*);
(viii) ${}^\circ e'^\circ \setminus (e+d) \leqslant (e' \setminus {}^\circ e^\circ) + d$,

where a, a': $u \to v$, b: $v \to w$, e': $u' \to u$, d: $w \to v$, $\{a_j\} \subseteq Q(u,v)$, $\{b_j\} \subseteq Q(v,w)$ are, respectively, morphisms and families of morphisms in Q.

Proof. To prove (i) note that

$$(b+a)+e' = {}^\circ(b+a) \setminus {}^\circ e'^\circ = ({}^\circ a \circ {}^\circ b) \setminus {}^\circ e'^\circ = {}^\circ b \setminus ({}^\circ a \setminus {}^\circ e'^\circ) \qquad \text{(by 1.3(iv))}$$
$$= {}^\circ b \setminus (a+e') = {}^\circ b \setminus {}^\circ(a+e')^\circ = b + (a+e'),$$

while relations (ii), (iii) follow immediately from definition of +. Next, we have that

$$b + \bigvee_j a_j = {}^\circ\!\left(\left(\bigvee_j a_i\right)^\circ \circ b^\circ\right) = {}^\circ\!\left(\left(\bigwedge_j a_j^\circ\right) \circ b^\circ\right)$$
$$= {}^\circ\!\left(\left(\bigwedge_j {}^\circ(a_j^\circ)^\circ\right) \circ {}^\circ(b^\circ)^\circ\right) = {}^\circ\!\left(\bigwedge_j a_j^\circ \circ {}^\circ(b^\circ)^\circ\right) \qquad \text{(by 2.2(vi))}$$
$$= \bigvee_j {}^\circ(a_j^\circ \circ b^\circ) = \bigvee_j (b + a_j)$$

and

$$\left(\bigvee_j b_j\right) + a = {}^\circ\!\left(a^\circ \circ \bigwedge_j b_j^\circ\right) = {}^\circ\!\left(a^\circ \circ \bigwedge_j {}^\circ(b_j^\circ)^\circ\right) = {}^\circ\!\left(\bigwedge_j a^\circ \circ {}^\circ(b_j^\circ)^\circ\right) \qquad \text{(by 2.2(vi))}$$
$$= \bigvee_j {}^\circ(a^\circ \circ b_j^\circ) = \bigvee_j (b_j + a).$$

To verify (v) we proceed as follows:

$$b+\bigwedge_j{}^{\circ}a_j^{\circ}={}^{\circ}\!\left(\left(\bigwedge_j{}^{\circ}a_j^{\circ}\right)^{\circ}\circ b^{\circ}\right)={}^{\circ}\!\left(\left(\bigvee_j a_j^{\circ}\right)\circ b^{\circ}\right)\qquad\text{(by 2.2(i))}$$
$$={}^{\circ}\!\left(\bigvee_j(a_j^{\circ}\circ b^{\circ})\right)\qquad\text{(by 1.1(i))}$$
$$=\bigwedge_j{}^{\circ}(a_j^{\circ}\circ b^{\circ})=\bigwedge_j(b+a_j),$$

$$\left(\bigwedge_j{}^{\circ}b_j^{\circ}\right)+a={}^{\circ}\!\left(a^{\circ}\circ\left(\bigwedge_j{}^{\circ}b_j^{\circ}\right)^{\circ}\right)={}^{\circ}\!\left(a^{\circ}\circ\bigvee_j b_j^{\circ}\right)={}^{\circ}\!\left(\bigvee_j(a^{\circ}\circ b_j^{\circ})\right)=\bigwedge_j{}^{\circ}(a^{\circ}\circ b_j^{\circ})=\bigwedge_j(b_j+a).$$

Finally, (vi) follows from 2.2(iv), while (vii) and (viii) follow from 1.1(iii). □

LEMMA 2.4. *Let Q be a generalized Girard quantaloid. Then Q has two "coimplications," $()\rightharpoondown()$ and $()\leftharpoondown()$, such that ${}^{\circ}(b\rightharpoondown c)^{\circ}=b\rightharpoondown c$, ${}^{\circ}(c\leftharpoondown a)^{\circ}=c\leftharpoondown a$ with*
(i) $b+a\geqslant c\iff {}^{\circ}a^{\circ}\geqslant b\rightharpoondown c$ *($+\rightharpoondown$residuation),*
(ii) $b+a\geqslant c\iff {}^{\circ}b^{\circ}\geqslant c\leftharpoondown a$ *($+\leftharpoondown$residuation),*
and it holds true that
(iii) $b\rightharpoondown c=\bigwedge\{{}^{\circ}a^{\circ}\mid b+a\geqslant c\}$,
(iv) $c\leftharpoondown a=\bigwedge\{{}^{\circ}b^{\circ}\mid b+a\geqslant c\}$,
where $a\colon u\to v$, $b\colon v\to w$, $c\colon u\to w$ are morphisms in Q.

Proof. Assume $b\rightharpoondown c$ to be given. By (i) ${}^{\circ}(b\rightharpoondown c)^{\circ}=b\rightharpoondown c$ implies $b+(b\rightharpoondown c)\geqslant c$. So it is clear that the infimum in (iii) does not exceed ${}^{\circ}(b\rightharpoondown c)^{\circ}(=b\rightharpoondown c)$. But if $\bigwedge\{{}^{\circ}a^{\circ}\mid b+a\geqslant c\}<b\rightharpoondown c$, then there exists some $d\in\mathrm{Mor}(Q)$ with ${}^{\circ}d^{\circ}<b\rightharpoondown c$ and $b+d\geqslant c$. For such d we get a contradiction by ${}^{\circ}d^{\circ}\geqslant b\rightharpoondown c>{}^{\circ}d^{\circ}$.
Let the coimplication $\rightharpoondown$ be defined by (iii). Then by 2.2(i) and 2.3(v) we have

$${}^{\circ}\!\left(\bigwedge\{{}^{\circ}a^{\circ}\mid b+a\geqslant c\}\right)^{\circ}={}^{\circ}\!\left(\bigvee\{a^{\circ}\mid b+a\geqslant c\}\right)=\bigwedge\{{}^{\circ}a^{\circ}\mid b+a\geqslant c\},$$

$$b+a\geqslant c\implies {}^{\circ}a^{\circ}\geqslant\bigwedge\{{}^{\circ}a'^{\circ}\mid b+a'\geqslant c\}=b\rightharpoondown c,$$

$${}^{\circ}a^{\circ}\geqslant b\rightharpoondown c\implies b+a=b+{}^{\circ}a^{\circ}\geqslant b+(b\rightharpoondown c)=b+\bigwedge\{{}^{\circ}a'^{\circ}\mid b+a'\geqslant c\}$$
$$=\bigwedge\{b+a'\mid b+a'\geqslant c\}\geqslant c,$$

i.e., $+\rightharpoondown$ residuation holds, while the argument for the other coimplication is similar. □

PROPOSITION 2.5. *Let Q be a generalized Girard quantaloid. For all a, $a'\colon u\to v$, $b\colon v\to w$, $c\colon u\to w, c'\colon u\to u'$, $d\colon w\to v$, $d'\colon w\to u$, $e\colon v\to u$, $e'\colon u'\to u$, $l,m,n\colon u\to u$, $\{a_j\}\subseteq Q(u,v)$, $\{b_j\}\subseteq Q(v,w)$, $\{c_j\}\subseteq Q(u,w)$, the following relations hold:*
(i) $b\leftharpoondown e=b\circ e^{\circ}$, $c\rightharpoondown b={}^{\circ}c\circ b\circ T(v,v)$;
(ii) $b+(b\rightharpoondown c)\geqslant c$, $(c\leftharpoondown a)+a\geqslant c$;
(iii) $a\geqslant a'$ *implies* $a+e'\geqslant a'+e'$, $b+a\geqslant b+a'$;
(iv) $c'\leftharpoondown(b+a)=(c'\leftharpoondown a)\leftharpoondown b$, $(a+e')\rightharpoondown c=e'\rightharpoondown(a\rightharpoondown c)$, $e'\rightharpoondown(e\leftharpoondown b)=(e'\rightharpoondown e)\leftharpoondown b$;
(v) $c\leftharpoondown\bigwedge_j{}^{\circ}a_j^{\circ}=\bigvee_j(c\leftharpoondown a_j)$, $(\bigwedge_j{}^{\circ}b_j^{\circ})\rightharpoondown c=\bigvee_j(b_j\rightharpoondown c)$;
(vi) $(\bigvee_j c_j)\leftharpoondown a=\bigvee_j(c_j\leftharpoondown a)$, $b\rightharpoondown\bigvee_j c_j=\bigvee_j(b\rightharpoondown c_j)$;
(vii) $\bigwedge_j(c_j\leftharpoondown a)\geqslant(\bigwedge_j c_j)\leftharpoondown a$, $\bigwedge_j(b\rightharpoondown c_j)\geqslant b\rightharpoondown\bigwedge_j c_j$;
(viii) $\bigwedge_j(c\leftharpoondown a_j)\geqslant c\leftharpoondown(\bigvee_j a_j)$, $\bigwedge_j(b_j\rightharpoondown c)\geqslant(\bigvee_j b_j)\rightharpoondown c$;
(ix) $b+(a\leftharpoondown c)\geqslant(b+a)\leftharpoondown c$, $(d\rightharpoondown b)+e'\geqslant d\rightharpoondown(b+e')$;
(x) $(c\leftharpoondown a)+(a\leftharpoondown c')\geqslant c\leftharpoondown c'$, $(e'\rightharpoondown e)+(e\rightharpoondown d')\geqslant e'\rightharpoondown d'$;

(xi) $c \leftharpoondown a \geqslant (c + e') \leftharpoondown (a + e')$, $d' \rightharpoondown e' \geqslant (a + d') \rightharpoondown (a + e')$;
(xii) ${}^\circ a^\circ \vee {}^\circ a'^\circ = a + (a \rightharpoondown a') = (a' \leftharpoondown a) + a$;
(xiii) $a \leftharpoondown m \leqslant {}^\circ a^\circ$, $m \rightharpoondown e \leqslant {}^\circ e^\circ$;
(xiv) $b \geqslant b'$ *implies* $b \leftharpoondown e \geqslant b' \leftharpoondown e$ *and* $c \rightharpoondown b \geqslant c \rightharpoondown b'$, $a \geqslant a'$ *implies* $c \leftharpoondown a \leqslant c \leftharpoondown a'$ *and* $a \rightharpoondown d \leqslant a' \rightharpoondown d$;
(xv) ${}^\circ(b \leftharpoondown e) = {}^\circ e \setminus {}^\circ b$, $(c \rightharpoondown b)^\circ = b^\circ / c^\circ$;
(xvi) $(m \leftharpoondown n) \rightharpoondown m = m \leftharpoondown (n \rightharpoondown m) = {}^\circ m^\circ \wedge {}^\circ n^\circ$;
(xvii) $(m \leftharpoondown n) \rightharpoondown (m \leftharpoondown c) = ({}^\circ m^\circ \wedge n) \leftharpoondown c$,
$(e \rightharpoondown m) \leftharpoondown (n \rightharpoondown m) = e \rightharpoondown ({}^\circ m^\circ \wedge n)$;
(xviii) $(m \leftharpoondown l) \leftharpoondown (({}^\circ m^\circ \wedge n) \leftharpoondown l) = (m \leftharpoondown l) \wedge (m \leftharpoondown n)$,
$(l \rightharpoondown ({}^\circ m^\circ \wedge n)) \rightharpoondown (l \rightharpoondown m) = (l \rightharpoondown m) \wedge (n \rightharpoondown m)$;
(xix) $m \leftharpoondown \perp(u, u) = \perp(u, u) \rightharpoondown m = {}^\circ m^\circ$ *(recall that* $\perp(u, u)$ *is the bottom morphism of* $Q(u, u)$*)*;
(xx) $m \leftharpoondown m = m \rightharpoondown m = \perp(u, u)$;
(xxi) *If* $m \leftharpoondown n = \perp(u, u)$ *or* $n \rightharpoondown m = \perp(u, u)$, *then* ${}^\circ m^\circ \leqslant {}^\circ n^\circ$;
(xxii) $m \leftharpoondown n = {}^\circ m^\circ = n \rightharpoondown m \iff {}^\circ m^\circ \wedge {}^\circ n^\circ = \perp(u, u)$.

Proof. (i) Since ${}^\circ(b \circ e^\circ)^\circ / e^\circ \geqslant b$ always, it follows that $b \circ e^\circ + e \geqslant b$, i.e., $b \circ e^\circ \geqslant b \leftharpoondown e$. Further, by 2.4(iv) we have

$$b \leftharpoondown e = \bigwedge \{{}^\circ c^\circ \colon u \to w \mid c + e \geqslant b\}.$$

But $c + e \geqslant b$ iff $b \circ e^\circ \leqslant {}^\circ c^\circ$. So

$$b \circ e^\circ \leqslant \bigwedge \{{}^\circ c^\circ \mid c + e \geqslant b\} = b \leftharpoondown e.$$

The next equality in (i) can be checked similarly.

Relations (ii)–(xv) follow immediately from (i). (They may be considered as a dual case of 1.3.)
(xvi) Using (i), 2.2(ii), 1.3(xii), and 1.2(ii), we get

$$\begin{aligned}(m \leftharpoondown n) \rightharpoondown m &= {}^\circ(m \circ n^\circ) \circ m \circ T(u, u) = (n \circ T(u, u)/m) \circ m \circ T(u, u)\\ &= (n \circ T(u, u) \wedge m) \circ T(u, u) = n \circ T(u, u) \wedge m \circ T(u, u)\\ &= m \circ T(u, u) \wedge {}^\circ n^\circ = m \circ (m \setminus {}^\circ n^\circ) = m \circ ({}^\circ n \circ m)^\circ = m \leftharpoondown (n \rightharpoondown m).\end{aligned}$$

(xvii) In view of (iii), (xvi), 1.2(ii), and 1.3(iii) we obtain

$$\begin{aligned}(m \leftharpoondown n) \rightharpoondown (m \leftharpoondown c) &= ((m \leftharpoondown n) \rightharpoondown m) \leftharpoondown c = (m \circ T(u, u) \wedge n \circ T(u, u)) \leftharpoondown c\\ &= (m \circ T(u, u) \wedge n) \circ T(u, u) \circ c^\circ = (m \circ T(u, u) \wedge n) \circ c^\circ\\ &= (m \circ T(u, u) \wedge n) \leftharpoondown c,\\ (e \rightharpoondown m) \leftharpoondown (n \rightharpoondown m) &= e \rightharpoondown (m \leftharpoondown (n \rightharpoondown m)) = e \rightharpoondown (m \circ T(u, u) \wedge n \circ T(u, u))\\ &= {}^\circ e \circ (m \circ T(u, u) \wedge n) \circ T(u, u) \circ T(u, u) = e \rightharpoondown (m \circ T(u, u) \wedge n).\end{aligned}$$

(xviii) Using (xvii), (xvi) we get

$$\begin{aligned}(m \leftharpoondown l) \leftharpoondown (({}^\circ m^\circ \wedge n) \leftharpoondown l) &= (m \leftharpoondown l) \leftharpoondown ((m \leftharpoondown n) \rightharpoondown (m \leftharpoondown l))\\ &= (m \leftharpoondown l) \wedge (m \leftharpoondown n),\\ (l \rightharpoondown ({}^\circ m^\circ \wedge n)) \rightharpoondown (l \rightharpoondown m) &= ((l \rightharpoondown m) \leftharpoondown (n \rightharpoondown m)) \rightharpoondown (l \rightharpoondown m)\\ &= (l \rightharpoondown m) \wedge (n \rightharpoondown m).\end{aligned}$$

(xix) From (xvi), (xiv), (xiii) we have

$$m \circ T(u, u) = (m \leftharpoondown m) \rightharpoondown m \leqslant \perp(u, u) \rightharpoondown m \leqslant m \circ T(u, u),$$

$$m \circ T(u, u) = m \leftharpoondown (m \rightharpoondown m) \leqslant m \leftharpoondown \perp(u, u) \leqslant m \circ T(u, u),$$

which imply the assertion.

(xx) From (i), (xvi), (xix) we get

$$\begin{aligned} m \leftharpoondown m = m \circ m^{\circ} = m \circ (m \circ T(u,u))^{\circ} = m \leftharpoondown m \circ T(u,u) = m \leftharpoondown (\perp(u,u) \rightharpoondown m) \\ = m \circ T(u,u) \wedge \perp(u,u) \circ T(u,u) = \perp(u,u) \end{aligned}$$

and

$$m \rightharpoondown m = {}^{\circ}m \circ m = {}^{\circ}(m \circ T(u,u)) \circ m = m \circ T(u,u) \rightharpoondown m = (m \leftharpoondown \perp(u,u)) \rightharpoondown m = \perp(u,u).$$

(xxi) From (xix), (xvi) we get

$$\begin{aligned} &{}^{\circ}m^{\circ} = m \leftharpoondown \perp(u,u) = m \leftharpoondown (n \rightharpoondown m) = {}^{\circ}m^{\circ} \wedge {}^{\circ}n^{\circ} \quad \text{or} \\ &{}^{\circ}m^{\circ} = \perp(u,u) \rightharpoondown m = (m \leftharpoondown n) \rightharpoondown m = {}^{\circ}m^{\circ} \wedge {}^{\circ}n^{\circ}, \qquad \text{i.e.,}\ {}^{\circ}m^{\circ} \leqslant {}^{\circ}n^{\circ}. \end{aligned}$$

(xxii) Let $m \leftharpoondown n = {}^{\circ}m^{\circ}$. Then ${}^{\circ}m^{\circ} \wedge {}^{\circ}n^{\circ} = (m \leftharpoondown n) \rightharpoondown m = {}^{\circ}m^{\circ} \rightharpoondown m = m \rightharpoondown m = \perp(u,u)$. Let ${}^{\circ}m^{\circ} \wedge {}^{\circ}n^{\circ} = \perp(u,u)$. Then $(m \leftharpoondown n) \rightharpoondown m = m \leftharpoondown (n \rightharpoondown m) = \perp(u,u)$, and by (xxi) we have ${}^{\circ}m^{\circ} \leqslant m \leftharpoondown n$, ${}^{\circ}m^{\circ} = n \rightharpoondown m$, which imply, by (xiii), $m \leftharpoondown n = {}^{\circ}m^{\circ} = n \rightharpoondown m$. □

We finish this section with an "example" of a generalized Girard quantaloid. Recall that every hom-lattice $Q(u,v)$ of a quantaloid Q obeying 1.2 is a quantale with the product $(a,b) \mapsto (b \circ T(u,u)) \wedge a$ and with the left-sided unit $T(v,u)$. Let us consider the case of quantaloids where, in addition, every $Q(u,v)$ is such that for all a in $Q(u,v)$,

$$ {}^{\ulcorner}\!(a^{\urcorner}) = ({}^{\ulcorner}a)^{\urcorner} = a \circ T(u,u), \tag{1}$$

$$a \setminus \perp(v,u) \leqslant T(u,v) \circ a^{\urcorner}, \quad \perp(v,u)/a \leqslant {}^{\ulcorner}a \circ T(u,v), \tag{2}$$

where

$$a^{\urcorner} := \bigvee \{ b \in Q(u,v) \mid (a \circ T(u,u)) \wedge b = \perp(v,u) \}$$

and

$${}^{\ulcorner}a := \bigvee \{ b \in Q(u,v) \mid (b \circ T(u,u)) \wedge a = \perp(v,u) \}$$

are "complements" of a in $Q(u,v)$. (We still keep the notation $\perp(v,u)$ for the bottom morphism of $Q(u,v)$.) Moreover, let the equality

$$T(u,v) \circ T(v,u) = T(u,u) \tag{3}$$

always hold.

Notice some analogy between (1) and the Boolean law of "double negation": $a^{\urcorner\urcorner} (:= \neg\neg a) = a$. By 1.3(xiv), $Q(u,v)$ is a Heyting algebra with the operation of "pseudocomplementation" $\overline{(\,)}^{\urcorner}$ $(= {}^{\ulcorner}\overline{(\,)})$, assuming $T(u,u) = id_u$. Thanks to (1), this operation is precisely what can be kept as an "authentic complementation," making $Q(u,v)$ into a Boolean algebra. Observe also that conditions (2) are satisfied if, for example, for all $a, b \in Q(u,v)$, $b \circ T(u,v) \circ a = \perp(v,u)$ implies $b \circ T(u,u) \wedge a = \perp(v,u)$.

PROPOSITION 2.6 (cf. [9, Theorem 2.3]). *Let Q be a quantaloid satisfying additional Axioms* (1)–(3). *Then Q is a generalized Girard quantaloid with the dualizing family $\perp = \{\perp(u,u) \mid u \in Q\}$.*

Proof. We first need to verify the following inequalities:

$$a \setminus \perp(v,v) \leqslant T(u,v) \circ a^{\urcorner} \circ T(u,v), \qquad \perp(u,u)/a \leqslant T(u,v) \circ {}^{\ulcorner}a \circ T(u,v). \tag{i}$$

From (2) we have

$$(a \setminus \perp(v,u))/T(v,u) \leqslant (T(u,v) \circ a^{\urcorner})/T(v,u).$$

But

$$\begin{aligned}\perp(v,u)/T(v,u) &\leqslant (\perp(v,u)\circ T(u,v))/(T(v,u)\circ T(u,v)) \qquad \text{(by 1.3(xi))}\\ &= \perp(v,v)/T(v,v) \qquad \text{(by (3))}\\ &\leqslant (\perp(v,v)/T(v,v))\circ T(v,v) \qquad \text{(by 1.2(i))}\\ &\leqslant \perp(v,v),\end{aligned}$$

so that, by 1.3(iv), 1.3(xi), 1.2(i),

$$\begin{aligned}a\setminus\perp(v,v) = (a\setminus\perp(v,u))/T(v,u) &\leqslant (T(u,v)\circ a^{\urcorner})/T(v,u)\\ &\leqslant (T(u,v)\circ a^{\urcorner}\circ T(u,v))/(T(v,u)\circ T(u,v))\\ &= (T(u,v)\circ a^{\urcorner}\circ T(u,v))/T(v,v) \qquad \text{(by (3))}\\ &\leqslant (T(u,v)\circ a^{\urcorner}\circ T(u,v)/T(v,v))\circ T(v,v) \leqslant T(u,v)\circ a^{\urcorner}\circ T(u,v),\end{aligned}$$

while the argument for the other inequality of (i) is similar. Next, $a\circ T(u,v)\circ a^{\urcorner}\leqslant a^{\urcorner}$, ${}^{\ulcorner}a\circ T(u,v)\circ a\leqslant a$ imply, by 1.1(iii), $T(u,v)\circ a^{\urcorner}\leqslant a\setminus a^{\urcorner}$ and $T(u,v)\circ a\leqslant {}^{\ulcorner}a\setminus a$. Using the latter pair of inequalities, we proceed as follows:

$$\begin{aligned}&a\circ T(u,v)\circ a^{\urcorner}\leqslant a\circ(a\setminus a^{\urcorner}) = a\circ T(u,u)\wedge a^{\urcorner} = \perp(v,u) \qquad \text{(by 1.3(xii))},\\ &{}^{\ulcorner}a\circ T(u,v)\circ a\leqslant {}^{\ulcorner}a\circ({}^{\ulcorner}a\setminus a) = ({}^{\ulcorner}a\circ T(u,u))\wedge a = \perp(v,u),\\ &a\circ T(u,v)\circ a^{\urcorner}\circ T(u,v)\leqslant \perp(v,u)\circ T(u,v) = \perp(v,v),\\ &T(u,v)\circ {}^{\ulcorner}a\circ T(u,v)\circ a\leqslant T(u,v)\circ\perp(v,u) = \perp(u,u).\end{aligned}$$

So in view of 1.1(iii), (2), and (i) we obtain that for every $a\in Q(u,v)$,

$$\begin{aligned}&a\setminus\perp(v,u) = T(u,v)\circ a^{\urcorner}, \qquad \perp(v,u)/a = {}^{\ulcorner}a\circ T(u,v),\\ &a\setminus\perp(v,v) = T(u,v)\circ a^{\urcorner}\circ T(u,v), \qquad \perp(u,u)/a = T(u,v)\circ {}^{\ulcorner}a\circ T(u,v).\end{aligned} \tag{ii}$$

Now we are ready to verify that $\perp$ is a dualizing family. We have

$$\begin{aligned}\perp(v,v)/(a\setminus\perp(v,v)) &= \perp(v,v)/(T(u,v)\circ a^{\urcorner}\circ T(u,v)) \qquad \text{(by (ii))}\\ &= \perp(v,v)/T(u,v))/(T(u,v)\circ a^{\urcorner}) \qquad \text{(by 1.3(iv))}.\end{aligned}$$

But

$$\begin{aligned}\perp(v,v)/T(u,v) &\leqslant (\perp(v,v)\circ T(v,u))/(T(u,v)\circ T(v,u)) \qquad \text{(by 1.3(xi))}\\ &= \perp(v,u)/T(u,u) \qquad \text{(by (3))}\\ &\leqslant (\perp(v,u)/T(u,u))\circ T(u,u)\leqslant \perp(v,u).\end{aligned}$$

Hence

$$\begin{aligned}\perp(v,v)/(a\setminus\perp(v,v)) &= \perp(v,u)/(T(u,v)\circ a^{\urcorner}) = (\perp(v,u)/a^{\urcorner})/T(u,v) = ({}^{\ulcorner}a^{\urcorner}\circ T(u,v))/T(u,v) \qquad \text{(by (ii))}\\ &= (a\circ T(u,u)\circ T(u,v))/T(u,v) \qquad \text{(by (1))}\\ &= (a\circ T(u,v))/T(u,v).\end{aligned}$$

But

$$\begin{aligned}a\circ T(u,u) &\leqslant (a\circ T(u,v))/T(u,v)\leqslant (a\circ T(u,v)\circ T(v,u))/(T(u,v)\circ T(v,u))\\ &= (a\circ T(u,u))/T(u,u) \qquad \text{(by (3))}\\ &\leqslant (a\circ T(u,u)/T(u,u))\circ T(u,u))\leqslant a\circ T(u,u).\end{aligned}$$

Therefore

$$\perp(v,v)/(a\setminus\perp(v,v)) = a\circ T(u,u).$$

Further,

$$
\begin{aligned}
(\perp(u,u)/a) \setminus \perp(u,u) &= (T(u,v) \circ \overline{a} \circ T(u,v)) \setminus \perp(u,u) && \text{(by (ii))} \\
&= (\overline{a} \circ T(u,v)) \setminus (T(u,v) \setminus \perp(u,u)) && \text{(by 1.3(iv))} \\
&= (\overline{a} \circ T(u,v)) \setminus \perp(v,u) = T(u,v) \setminus (\overline{a} \setminus \perp(v,u)) \\
&= T(u,v) \setminus (T(u,v) \circ \overline{\overline{a}}) && \text{(by (ii))} \\
&= T(u,v) \setminus (T(u,v) \circ a \circ T(u,u)) && \text{(by (1)).}
\end{aligned}
$$

Since

$$
\begin{aligned}
a \circ T(u,u) \leqslant T(u,v) \setminus (T(u,v) \circ a \circ T(u,u)) &\leqslant (T(v,u) \circ T(u,v)) \setminus (T(v,u) \circ T(u,v) \circ a \circ T(u,u)) \\
&= T(v,v) \setminus (T(v,v) \circ a \circ T(u,u)) \qquad \text{(by (3))} \\
&= T(v,v) \circ (T(v,v) \setminus a \circ T(u,u)) \qquad \text{(by 1.3(iii))} \\
&\leqslant a \circ T(u,u),
\end{aligned}
$$

we conclude that

$$
(\perp(u,u)/a) \setminus \perp(u,u) = a \circ T(u,u),
$$

as claimed. □

3. QUANTAL CATEGORIES

A quantaloid Q can also be viewed as a locally ordered base bicategory in the sense of R. H. Street, so it is natural to consider categories with hom-sets enriched in Q (as K. I. Rosenthal [9, 10] does). However, we shall go a little bit outside the usual context of enriched category theory, since our quantaloid is not very ordinary and convenient in the categorical sense.

We thus turn to a "non-commutative many-object" version of U. Höhle's [4] concept of set structured with an "existence predicate."

Definition 3.1. Let Q be a quantaloid defined in 1.2. A *Q-category* (or *quantal category*) is a triple $(X, \widetilde{()}, E)$ (below we shall suppress writing $\widetilde{()}$ explicitly and these data will take the form (X, E)), where X is a (small) set, $\widetilde{()}$: $X \to \mathrm{Obj}(Q)$ is a map assigning to $x \in X$ an object $\tilde{x} \in Q$, E: $X \times X \to \mathrm{Mor}(Q)$ is a map ("*enrichment*"), which assigns to every pair $x, x' \in X$ a morphism $E(x, x')$: $\tilde{x'} \to \tilde{x}$ in Q such that

(i) $E(x,x') \leqslant (E(x,x) \circ T(\tilde{x}, \tilde{x'})) \wedge (T(\tilde{x}, \tilde{x'}) \circ E(x', x'))$ (strictness),

(ii) $E(x,x') \circ (E(x',x') \setminus E(x',x'')) \leqslant E(x',x'')$ (transitivity)

for all x, x' x'' in X. On any Q-category (X, E) we define *extents of existence* and (*ordinary*) *equivalence* by

(iii) $Ex = E(x,x)$, and

(iv) $x \approx x' \iff \tilde{x} = \tilde{x'}$, $Ex \vee Ex' \leqslant E(x,x') \wedge E(x',x)$.

A Q-category (X, E) is *separated* iff (X, E) satisfies the additional axiom:

(v) $x \approx x'$ implies $x = x'$ (separation).

We can associate with every Q-category (X, E) a *separated Q-category* $(\hat{X}, \hat{E})$ in usual way:

(vi) $\hat{X} = X/\approx$, $\hat{E}(\hat{x}, \hat{x'}) = E(y, y')$, where $y \in \hat{x}$, $y' \in \hat{x'}$.

Axioms 3.1 generalize to the "many-object" case those for Q-sets (cf. [3, 4]). Here are some easy consequences of the definitions from 1.3(iii), (xiii), 1.2(ii).

PROPOSITION 3.2. *Let Q be a quantaloid and (X, E) be a Q-category. For all $x, x', x'' \in X$,*

(i) $E(x,x') \circ (Ex' \setminus E(x',x'')) = (E(x,x') \circ T(\tilde{x'}, \tilde{x'})/Ex') \circ E(x',x'')$,

(ii) $(E(x,x') \circ T(\tilde{x'}, \tilde{x'})) \wedge (T(\tilde{x}, \tilde{x'}) \circ Ex') = E(x,x')$,

(iii) $E(x,x') = Ex \circ (Ex \setminus E(x,x')) = (E(x,x') \circ T(\tilde{x'}, \tilde{x'})/Ex') \circ Ex'$.

Let us compare quantal categories over Q obeying Axioms 3.1 with quantal categories in the sense of [9].

Definition 3.3 ([9, Definition 3.1]). Let Q be a quantaloid containing two-sided units but in general unsatisfying Axioms 1.2. A set X is a *Q-category* iff it comes equipped with the following data:

(1) a map $\widetilde{()}$ assigning to $x \in X$ an object $\tilde{x} \in Q$;

(2) an enrichment which assigns to every pair $x, x' \in X$ a morphism $E(x, x')$: $\tilde{x}' \to \tilde{x}$ in Q such that

(i) $id_{\tilde{x}} \leqslant E(x, x)$ for all $x \in X$,

(ii) $E(x, x') \circ E(x', x'') \leqslant E(x, x'')$ for all x, x', $x'' \in X$.

PROPOSITION 3.4. *Let Q be a quantaloid having two-sided units and satisfying Axioms* 1.2. *Then every Q-category in the sense of* 3.3 *is also a Q-category in the sense of* 3.1.

Proof. Let (X, E) be a Q-category in the sense of 3.3. Then 3.3(i) implies $E(x, x) = T(\tilde{x}, \tilde{x})$, since, by 1.3(iii), $id_{\tilde{x}} = T(\tilde{x}, \tilde{x})$. Consequently, in view of 1.3(iii), (xii) we have

$$\begin{aligned} E(x', x') \setminus E(x', x'') &= T(\tilde{x}', \tilde{x}') \setminus E(x', x'') = T(\tilde{x}', \tilde{x}') \circ (T(\tilde{x}', \tilde{x}') \setminus E(x', x'')) \\ &= T(\tilde{x}', \tilde{x}') \circ T(\tilde{x}', \tilde{x}'') \wedge E(x', x'') = T(\tilde{x}', \tilde{x}'') \wedge E(x', x'') = E(x', x''). \end{aligned}$$

Hence (X, E) is a Q-category in the sense of 3.1. □

It is interesting to see that in a certain sense these definitions lead to the same things.

PROPOSITION 3.5. *Let (X, E) be a Q-category in the sense of* 3.1, *in the setting of the preceding proposition. Then (X, E') is a Q-category in the sense of* 3.3, *where E' is defined by $E'(x, x') = Ex \setminus E(x, x')$.*

Proof. Because of 3.2(iii) and 1.1(iii), we have that inequality 3.1(ii) takes the form

$$(Ex \setminus E(x, x')) \circ (Ex' \setminus E(x', x'')) \leqslant Ex \setminus E(x, x''),$$

i.e., E' obeys 3.3(ii), while 3.3(i) follows immediately from 1.1(iii):

$$Ex \circ id_{\tilde{x}} \leqslant Ex \implies id_{\tilde{x}} \leqslant Ex \setminus Ex = E'(x, x). \qquad \square$$

Note that below we shall not consider Q-categories in the sense of 3.3, rather Q-category will always refer to 3.1.

Definition 3.6. If (X, E), (Y, F) are separated Q-categories (in the sense of 3.1), a *Q-functor* f: $(X, E) \to (Y, F)$ of Q-categories is a map f: $X \to Y$ satisfying

(i) $\tilde{x} = \widetilde{f(x)}$, $Ex = Ff(x)$ (strictness),

(ii) $E(x, x') \leqslant F(f(x), f(x'))$ (preservation of structure)

for all x, x' in X.

Obviously separated Q-categories and Q-functors together with the usual composition of maps constitute some category Q-CAT.

4. A MONAD ON Q-CAT

In this section we construct a monad on the category Q-CAT. This can be done by extending Höhle's [4] notion of singletons.

Definition 4.1. For a Q-category (X, E), a *singleton* is a pair s of maps $\langle\,, s\rangle$, $\langle s,\,\rangle$: $X \to \mathrm{Mor}(Q)$ assigning to every $x \in X$ morphisms $\langle x, s\rangle$: $\tilde{s} \to \tilde{x}$ and $\langle s, x\rangle$: $\tilde{x} \to \tilde{s}$ (for some object $\tilde{s} \in Q$) and satisfying the following conditions:

(i) $[\![s]\!] \circ T(\tilde{s}, \tilde{s}) = \bigvee_{x \in X} \langle s, x\rangle \circ T(\tilde{x}, \tilde{s})$,

(ii) $E(x, x') \circ (Ex' \setminus \langle x', s\rangle) \leqslant \langle x, s\rangle$, $\langle s, x\rangle \circ (Ex \setminus E(x, x')) \leqslant \langle s, x'\rangle$ (extensionality),

(iii) $\langle x, s\rangle \circ ([\![s]\!] \setminus \langle s, x'\rangle) \leqslant E(x, x')$ (singleton condition),

(iv) $\langle x, s\rangle \leqslant T(\tilde{x}, \tilde{s}) \circ [\![s]\!]$, $\langle s, x\rangle \leqslant [\![s]\!] \circ T(\tilde{s}, \tilde{x})$, $[\![s]\!] \setminus [\![s]\!] \leqslant \langle x, s\rangle \setminus \langle x, s\rangle$

for all x, x' in X, where $[\![s]\!]$: $= \bigvee_{x \in X} T(\tilde{s}, \tilde{x}) \circ \langle x, s\rangle$.

Here are some consequences of the definition.

PROPOSITION 4.2. *Let s be a singleton of (X, E). For all $x, x', x'' \in X$,*
(i) $\langle x, s\rangle \circ ([\![s]\!] \setminus \langle s, x'\rangle) = (\langle x, s\rangle \circ T(\tilde{s}, \tilde{s})/[\![s]\!]) \circ \langle s, x'\rangle$,
(ii) $\langle x, s\rangle \leqslant Ex \circ T(\tilde{x}, \tilde{s})$, $\langle s, x\rangle \leqslant T(\tilde{s}, \tilde{x}) \circ Ex$ (*strictness*),
(iii) $E(x, x') \circ (Ex' \setminus \langle x', s\rangle) = (E(x, x') \circ T(\tilde{x'}, \tilde{x'})/Ex') \circ \langle x', s\rangle$,
(iv) $\langle s, x\rangle \circ (Ex \setminus E(x, x')) = (\langle s, x\rangle \circ T(\tilde{x}, \tilde{x})/Ex) \circ E(x, x')$,
(v) $(\langle s, x\rangle \circ T(\tilde{x}, \tilde{x})) \wedge (T(\tilde{s}, \tilde{x}) \circ Ex) = \langle s, x\rangle$,
(vi) $(\langle x, s\rangle \circ T(\tilde{s}, \tilde{s})) \wedge (T(\tilde{x}, \tilde{s}) \circ [\![s]\!]) = \langle x, s\rangle$.

Proof. (vi) We have

$$\begin{aligned}\langle x, s\rangle &\leqslant \langle x, s\rangle \circ T(\tilde{s}, \tilde{s}) \wedge T(\tilde{x}, \tilde{s}) \circ [\![s]\!] \qquad \text{(by 4.1(iv), 1.2(i))}\\ &= (\langle x, s\rangle \circ T(\tilde{s}, \tilde{s})/[\![s]\!]) \circ [\![s]\!] \qquad \text{(by 1.3(xiii))}\\ &\leqslant (\langle x, s\rangle \circ T(\tilde{s}, \tilde{s})/[\![s]\!]) \circ [\![s]\!] \circ (\langle x, s\rangle \setminus \langle x, s\rangle) \qquad \text{(by 4.1(iv))}\\ &\leqslant \langle x, s\rangle \circ T(\tilde{s}, \tilde{s}) \circ (\langle x, s\rangle \setminus \langle x, s\rangle)\\ &= \langle x, s\rangle \circ (\langle x, s\rangle \setminus \langle x, s\rangle) = \langle x, s\rangle \circ T(\tilde{s}, \tilde{s}) \wedge \langle x, s\rangle = \langle x, s\rangle,\end{aligned}$$

which implies (vi).
(i) The following holds:

$$\begin{aligned}\langle x, s\rangle \circ ([\![s]\!] \setminus \langle s, x'\rangle) &= (\langle x, s\rangle \circ T(\tilde{s}, \tilde{s}) \wedge T(\tilde{x}, \tilde{s}) \circ [\![s]\!]) \circ ([\![s]\!] \setminus \langle s, x'\rangle) \qquad \text{(by (vi))}\\ &= (\langle x, s\rangle \circ T(\tilde{s}, \tilde{s})/[\![s]\!]) \circ [\![s]\!] \circ ([\![s]\!] \setminus \langle s, x'\rangle)\\ &= (\langle x, s\rangle \circ T(\tilde{s}, \tilde{s})/[\![s]\!]) \circ ([\![s]\!] \circ T(\tilde{s}, \tilde{x'}) \wedge \langle s, x'\rangle) \qquad \text{(by 1.3(xiii))}\\ &= (\langle x, s\rangle \circ T(\tilde{s}, \tilde{s})/[\![s]\!]) \circ \langle s, x'\rangle \qquad \text{(by 4.1(iv))}.\end{aligned}$$

(ii) In view of (vi), 1.2(i), 1.3(xiii), 4.1(i), (iii), (iv) and 3.1(i) we obtain

$$\begin{aligned}\langle x, s\rangle &= \langle x, s\rangle \circ T(\tilde{s}, \tilde{s}) \wedge T(\tilde{x}, \tilde{s}) \circ [\![s]\!] \leqslant (\langle x, s\rangle \circ T(\tilde{s}, \tilde{s})/[\![s]\!]) \circ [\![s]\!] \circ T(\tilde{s}, \tilde{s})\\ &= (\langle x, s\rangle \circ T(\tilde{s}, \tilde{s})/[\![s]\!]) \circ \bigvee_{x' \in X} \langle s, x'\rangle \circ T(\tilde{x'}, \tilde{s})\\ &\leqslant \bigvee_{x' \in X} E(x, x') \circ T(\tilde{x'}, \tilde{s}) \leqslant \bigvee_{x' \in X} Ex \circ T(\tilde{x}, \tilde{x'}) \circ T(\tilde{x'}, \tilde{s}) = Ex \circ T(\tilde{x}, \tilde{s})\end{aligned}$$

and

$$\begin{aligned}\langle s, x\rangle &= [\![s]\!] \circ T(\tilde{s}, \tilde{x}) \wedge \langle s, x\rangle = [\![s]\!] \circ ([\![s]\!] \setminus \langle s, x\rangle)\\ &= \bigvee_{x' \in X} T(\tilde{s}, \tilde{x'}) \circ \langle x', s\rangle \circ ([\![s]\!] \setminus \langle s, x\rangle) \leqslant \bigvee_{x' \in X} T(\tilde{s}, \tilde{x'}) \circ E(x', x)\\ &\leqslant \bigvee_{x' \in X} T(\tilde{s}, \tilde{x'}) \circ T(\tilde{x'}, \tilde{x}) \circ Ex = T(\tilde{s}, \tilde{x}) \circ Ex.\end{aligned}$$

(iii) The proof is based on (ii), 3.2(iii) and 1.3(xiii):

$$\begin{aligned}E(x, x') \circ (Ex' \setminus \langle x', s\rangle) &= (E(x, x') \circ T(\tilde{x'}, \tilde{x'})/Ex') \circ Ex' \circ (Ex' \setminus \langle x', s\rangle)\\ &= (E(x, x') \circ T(\tilde{x'}, \tilde{x'})/Ex') \circ (Ex' \circ T(\tilde{x'}, \tilde{s}) \wedge \langle x', s\rangle)\\ &= (E(x, x') \circ T(\tilde{x'}, \tilde{x'})/Ex') \circ \langle x', s\rangle).\end{aligned}$$

(iv) By virtue of (ii), 1.2(ii), 1.3(xiii) and 3.2(iii) we have that

$$\begin{aligned}\langle s, x\rangle \circ (Ex \setminus E(x, x')) &= \langle s, x\rangle \circ T(\tilde{x}, \tilde{x}) \circ (Ex \setminus E(x, x'))\\ &= (\langle s, x\rangle \circ T(\tilde{x}, \tilde{x}) \wedge T(\tilde{s}, \tilde{x}) \circ Ex \circ T(\tilde{x}, \tilde{x})) \circ (Ex \setminus E(x, x'))\\ &= (\langle s, x\rangle \circ T(\tilde{x}, \tilde{x}) \wedge T(\tilde{s}, \tilde{x}) \circ Ex) \circ T(\tilde{x}, \tilde{x}) \circ (Ex \setminus E(x, x'))\\ &= (\langle s, x\rangle \circ T(\tilde{x}, \tilde{x})/Ex) \circ Ex \circ (Ex \setminus E(x, x'))\\ &= (\langle s, x\rangle \circ T(\tilde{x}, \tilde{x})/Ex) \circ E(x, x').\end{aligned}$$

(v) The equality follows from

$$\begin{aligned}\langle s,x\rangle &= \langle s,x\rangle \wedge T(\tilde{s},\tilde{x})\circ Ex \qquad \text{(by (ii))}\\ &\leqslant \langle s,x\rangle \circ T(\tilde{x},\tilde{x}) \wedge T(\tilde{s},\tilde{x})\circ Ex\\ &= (\langle s,x\rangle \circ T(\tilde{x},\tilde{x})/Ex)\circ Ex \qquad \text{(by 1.3(xiii))}\\ &\leqslant \langle s,x\rangle \qquad \text{(by (iv), 4.1(ii)).}\end{aligned}$$

□

PROPOSITION 4.3. *Given a quantaloid Q, a Q-category (X, E), an element $x \in X$, an object $u \in \mathrm{Obj}(Q)$ with $T(\tilde{x},u)\circ T(u,\tilde{x}) = T(\tilde{x},\tilde{x})$, and a morphism $a\colon u \to \tilde{x}$ in $Q(u,\tilde{x})$, we produce singletons E^x and $E^{(x,a)}$ (of (X,E)) by putting*

(i) $\langle x', E^x\rangle = E(x',x)$, $\langle E^x, x'\rangle = E(x,x')$ (*with* $[\![E^x]\!] = Ex$);

(ii) $\langle x', E^{(x,a)}\rangle = E(x',x)\circ(Ex \setminus a)$ $(= (E(x',x)\circ T(\tilde{x},\tilde{x})/Ex)\circ((Ex\circ T(\tilde{x},u))\wedge a)$ (*by* 3.2(iii), 1.3(xii)) $= (E(x',x)\circ T(\tilde{x},\tilde{x})/Ex)\circ T(\tilde{x},u)\circ [\![E^{(x,a)}]\!])$,

$$\begin{aligned}\langle E^{(x,a)}, x'\rangle &= T(u,\tilde{x})\circ(a\circ T(u,\tilde{x}))/Ex)\circ E(x,x')\\ &(= T(u,\tilde{x})\circ((a\circ T(u,\tilde{x}))\wedge Ex)\circ(Ex\setminus E(x,x')) \qquad (\textit{by } 3.2(\text{iii}),\ 1.3(\text{xiii}))\\ &= T(u,\tilde{x})\circ((Ex\circ T(\tilde{x},u))\wedge a)\circ T(u,\tilde{x})\circ(Ex\setminus E(x,x')) \qquad (\textit{by } 1.2(\text{ii}))\\ &= [\![E^{(x,a)}]\!]\circ T(u,\tilde{x})\circ(Ex\setminus E(x,x')))\end{aligned}$$

(*with* $[\![E^{(x,a)}]\!] = T(u,\tilde{x})\circ((Ex\circ T(\tilde{x},u))\wedge a)$.

PROPOSITION 4.4. *On the set $S(X,E)$ of all singletons of (X,E), there is a Q-category structure given by the map $\widetilde{(\,)}$: $S(X,E)\to \mathrm{Obj}(Q)$ assigning to every $s\in S(X,E)$ the object $\tilde{s}\in Q$ (defined in 4.1) and by the enrichment $[\![\,,\,]\!]$: $S(X,E)\times S(X,E)\to \mathrm{Mor}(Q)$:*

$$[\![s,s']\!] := \Big([\![s]\!]\circ \bigwedge_{x\in X}\langle x,s\rangle\setminus\langle x,s'\rangle\Big)\wedge\Big(\Big(\bigwedge_{x\in X}\langle s,x\rangle/\langle s',x\rangle\Big)\circ[\![s']\!]\Big).$$

Moreover, $[\![s,s]\!] = [\![s]\!]$.

Proof. The last equality follows immediately from 4.1(iv), 1.3(iii) and the fact that $\langle s,x\rangle/\langle s,x\rangle = T(\tilde{s},\tilde{s})$. Now the strictness 3.1(i) of $[\![\,,\,]\!]$ is evident. Let us prove the transitivity of $[\![\,,\,]\!]$. We have that

$$\begin{aligned}[\![s,s']\!]\circ([\![s',s']\!]\setminus[\![s',s'']\!]) &= \Big(\Big([\![s]\!]\circ\bigwedge_{x\in X}\langle x,s\rangle\setminus\langle x,s'\rangle\Big)\\ &\quad\wedge\Big(\bigwedge_{x\in X}\langle s,x\rangle/\langle s',x\rangle\Big)\circ[\![s']\!]\Big)\circ\Big([\![s']\!]\setminus\Big([\![s']\!]\circ\bigwedge_{x\in X}\langle x,s'\rangle\ \langle x,s''\rangle\Big)\wedge\Big(\bigwedge_{x\in X}\langle s',x\rangle/\langle s'',x\rangle\Big)\circ[\![s'']\!]\Big)\\ &\leqslant\Big(\Big([\![s]\!]\circ\bigwedge_{x\in X}\langle x,s\rangle\setminus\langle x,s'\rangle\Big)\circ T(\tilde{s},\tilde{s}')\wedge T(\tilde{s},\tilde{s}')\circ[\![s']\!]\Big)\circ\Big([\![s']\!]\setminus[\![s']\!]\circ\bigwedge_{x\in X}\langle x,s'\rangle\setminus\langle x,s''\rangle\Big)\\ &=\Big(\Big([\![s]\!]\circ\bigwedge_{x\in X}\langle x,s\rangle\setminus\langle x,s'\rangle\Big)\circ T(\tilde{s},\tilde{s}')/[\![s']\!]\Big)\circ[\![s']\!]\circ\Big([\![s']\!]\setminus[\![s']\!]\circ\bigwedge_{x\in X}\langle x,s'\rangle\setminus\langle x,s''\rangle\Big) \qquad \text{(by 1.3(xiii))}\\ &\leqslant[\![s]\!]\circ\Big(\bigwedge_{x\in X}\langle x,s\rangle\setminus\langle x,s'\rangle\Big)\circ\bigwedge_{x\in X}\langle x,s'\rangle\setminus\langle x,s''\rangle \qquad \text{(1.3(i) twice)}\\ &\leqslant[\![s]\!]\circ\bigwedge_{x\in X}((\langle x,s\rangle\setminus\langle x,s'\rangle)\circ(\langle x,s'\rangle\setminus\langle x,s''\rangle))\leqslant[\![s]\!]\circ\bigwedge_{x\in X}\langle x,s\rangle\setminus\langle x,s''\rangle \qquad \text{(by 1.3(x))}\end{aligned}$$

and that

$$\begin{aligned}
[\![s,s']\!]\circ([\![s',s']\!]\setminus[\![s',s'']\!]) &\leqslant \Big(\bigwedge_{x\in X}\langle s,x\rangle/\langle s',x\rangle\Big)\circ[\![s']\!]\circ\Big([\![s']\!]\setminus\Big(\bigwedge_{x\in X}\langle s',x\rangle/\langle s'',x\rangle\Big)\circ[\![s'']\!]\Big)\\
&\leqslant \Big(\bigwedge_{x\in X}\langle s,x\rangle/\langle s',x\rangle\Big)\circ\Big(\bigwedge_{x\in X}\langle s',x\rangle/\langle s'',x\rangle\Big)\circ[\![s'']\!] \qquad \text{(by 1.3(i))}\\
&\leqslant \Big(\bigwedge_{x\in X}(\langle s,x\rangle/\langle s',x\rangle)\circ(\langle s',x\rangle/\langle s'',x\rangle)\Big)\circ[\![s'']\!]\\
&\leqslant \Big(\bigwedge_{x\in X}\langle s,x\rangle/\langle s'',x\rangle\Big)\circ[\![s'']\!] \qquad \text{(by 1.3(x))}.
\end{aligned}$$

These imply the transitivity of $[\![\;,\;]\!]$. □

PROPOSITION 4.5.
(i) $\Sigma(X,E) := (S(X,E),[\![\;,\;]\!])$ *is a separated* Q*-category;*
(ii) $[\![E^x,s]\!]=\langle x,s\rangle$ *and* $[\![s,E^x]\!]=\langle s,x\rangle$ *for each* (x,s) *in* $X\times S(X,E)$.

Proof. (i) We need to verify the following implication:

$$\tilde{s}=\tilde{s}',\qquad [\![s]\!]\vee[\![s']\!]\leqslant[\![s,s']\!]\wedge[\![s',s]\!]\implies s=s'.$$

Thus, to establish the separation it suffices to show that if

$$\begin{aligned}
\tilde{s}=\tilde{s}',\qquad [\![s]\!]&\leqslant[\![s]\!]\circ\bigwedge_{x\in X}\langle x,s\rangle\setminus\langle x,s'\rangle,\\
[\![s']\!]&\leqslant\Big(\bigwedge_{x\in X}\langle s',x\rangle/\langle s,x\rangle\Big)\circ[\![s]\!],\\
[\![s']\!]&\leqslant[\![s']\!]\circ\bigwedge_{x\in X}\langle x,s'\rangle\setminus\langle x,s\rangle,\\
[\![s]\!]&\leqslant\Big(\bigwedge_{x\in X}\langle s,x\rangle/\langle s',x\rangle\Big)\circ[\![s']\!],
\end{aligned}$$

then $s=s'$, and this is so since these datum lead to the following relations:

$$\begin{aligned}
\langle x,s\rangle &= (\langle x,s\rangle\circ T(\tilde{s},\tilde{s})/[\![s]\!])\circ[\![s]\!] \qquad \text{(by 4.2(vi), 1.3(xiii))}\\
&\leqslant(\langle x,s\rangle\circ T(\tilde{s},\tilde{s})/[\![s]\!])\circ[\![s]\!]\circ\bigwedge_{x\in X}\langle x,s\rangle\setminus\langle x,s'\rangle\\
&\leqslant\langle x,s\rangle\circ T(\tilde{s},\tilde{s})\circ\bigwedge_{x\in X}\langle x,s\rangle\setminus\langle x,s'\rangle\\
&\leqslant\langle x,s\rangle\circ(\langle x,s\rangle\setminus\langle x,s'\rangle)\leqslant\langle x,s'\rangle,
\end{aligned}$$

$$\begin{aligned}
\langle s,x\rangle &= [\![s]\!]\circ T(\tilde{s},\tilde{x})\wedge\langle s,x\rangle \qquad \text{(by 4.1(iv))}\\
&=[\![s]\!]\circ([\![s]\!]\setminus\langle s,x\rangle) \qquad \text{(by 1.3(xiii))}\\
&\leqslant\Big(\bigwedge_{x\in X}\langle s',x\rangle/\langle s,x\rangle\Big)\circ[\![s]\!]\circ([\![s]\!]\setminus\langle s,x\rangle)\\
&\leqslant(\langle s',x\rangle/\langle s,x\rangle)\circ\langle s,x\rangle\leqslant\langle s',x\rangle \qquad \text{(by 1.3(i))},
\end{aligned}$$

$\langle x,s'\rangle\leqslant\langle x,s\rangle$ and $\langle s',x\rangle\leqslant\langle s,x\rangle$ (interchanging the role of s and s').

(ii) Let s be in $\Sigma(X, E)$ and let $x \in X$. Then

$$[\![E^x, s]\!] = \Big(Ex \circ \bigwedge_{x' \in X} E(x', x) \setminus \langle x', s\rangle\Big) \wedge \Big(\bigwedge_{x' \in X} E(x, x')/\langle s, x'\rangle\Big) \circ [\![s]\!].$$

Observe that

$$\begin{aligned}\langle x, s\rangle &\leqslant Ex \circ (Ex \setminus \langle x, s\rangle) \leqslant Ex \circ \bigwedge_{x' \in X} E(x', x) \setminus \langle x', s\rangle \qquad \text{(by 4.1(iii), 1.1(iii))}\\ &\leqslant Ex \circ (Ex \setminus \langle x, s\rangle) \leqslant \langle x, s\rangle,\end{aligned}$$

which imply

$$Ex \circ \bigwedge_{x' \in X} E(x', x) \setminus \langle x', s\rangle = \langle x, s\rangle.$$

Noting still more that

$$\begin{aligned}\Big(\bigwedge_{x' \in X} E(x, x')/\langle s, x'\rangle\Big) \circ [\![s]\!] &\geqslant ((\langle x, s\rangle \circ T(\bar{s}, \bar{s})/[\![s]\!]) \circ [\![s]\!] \qquad \text{(by 4.2(i), 4.1(iii), 1.1(iii))}\\ &\geqslant \langle x, s\rangle \circ T(\bar{s}, \bar{s}) \geqslant)\langle x, s\rangle,\end{aligned}$$

we prove the left equality of (ii).
Similarly,

$$[\![s, E^x]\!] = \Big([\![s]\!] \circ \bigwedge_{x' \in X} \langle x', s\rangle \setminus E(x', x)\Big) \wedge \Big(\bigwedge_{x' \in X} \langle s, x'\rangle/E(x, x')\Big) \circ Ex = \langle s, x\rangle,$$

since 4.1(iii), 1.1(iii), 1.3(i) imply

$$\langle s, x\rangle \leqslant [\![s]\!] \circ \bigwedge_{x' \in X} \langle x', s\rangle \setminus E(x', x)$$

and since the following relations hold:

$$\begin{aligned}\langle s, x\rangle &\leqslant ((\langle s, x\rangle \circ T(\tilde{x}, \tilde{x})/Ex) \circ Ex \leqslant \Big(\bigwedge_{x' \in X} \langle s, x'\rangle/E(x, x')\Big) \circ Ex \qquad \text{(by 4.2(iv), 4.1(ii), 1.1(iii))}\\ &\leqslant ((\langle s, x\rangle/Ex) \circ Ex \leqslant \langle s, x\rangle.\end{aligned}$$

□

Now we are going to define a monad on the category Q-CAT. We begin with a series of lemmas, the straightforward proofs of which will be omitted for reasons of space. (Those proofs parallel the respective proofs given in [3].)

LEMMA 4.6. *Let f: $(X, E) \to (Y, F)$ be a morphism in Q-CAT. Then the map $\Sigma(f)$: $\Sigma(X, E) \to \Sigma(Y, F)$ assigning to every $s \in S(X, E)$ the singleton $(\Sigma(f))(s)$ in $S(Y, F)$ defined by*

$$\begin{aligned}\langle y, (\Sigma(f))(s)\rangle :&= \bigvee_{x \in X} F(y, f(x)) \circ (Ex \setminus \langle x, s\rangle)\Big(= \bigvee_{x \in X} (F(y, f(x)) \circ T(\tilde{x}, \tilde{x})/Ex) \circ \langle x, s\rangle\\ &\qquad (by\ 3.2(iii), 3.6(i), 1.3(xiii), 4.2(ii))\Big),\end{aligned}$$

$$\begin{aligned}\langle(\Sigma(f))(s), y\rangle :&= \bigvee_{x \in X} \langle s, x\rangle \circ (Ex \setminus F(f(x), y))\Big(= \bigvee_{x \in X} ((\langle s, x\rangle \circ T(\tilde{x}, \tilde{x})/Ex) \circ F(f(x), y)\\ &\qquad (by\ 4.2(v), 1.3(xiii), 3.6(i), 3.2(iii))\Big)\end{aligned}$$

is a morphism in Q-CAT.

LEMMA 4.7. *Let Σ be an object map which associates with every separate Q-category (X, E) the singleton space $\Sigma(X, E)$ and a morphism map which associates with every morphism $f\colon (X, E) \to (Y, F)$ in Q-*CAT *the morphism $\Sigma(f)\colon \Sigma(X, E) \to \Sigma(Y, F)$ defined in* 4.6. *Then Σ is a (covariant) functor in Q-*CAT.

LEMMA 4.8. *If $\sigma(s) = (\langle s, \sigma\rangle\colon \tilde{\sigma} \to \tilde{s},\ \langle\sigma, s\rangle\colon \tilde{s} \to \tilde{\sigma})$, $s \in S(X, E)$, is a singleton of $\Sigma(X, E)$, then $\sigma(E^x) = (\langle E^x, \sigma\rangle\colon \tilde{\sigma} \to \tilde{x},\ \langle\sigma, E^x\rangle\colon \tilde{x} \to \tilde{\sigma})$, $x \in X$, is a singleton of (X, E) such that*

$$[\![\sigma(E^{()})]\!]_{S(X,E)} = [\![\sigma]\!]_{S\left(\Sigma(X,E)\right)}.$$

(Recall that $E^x = (E(\ ,x), E(x,\ ))$.)

The following lemma completes the list of preparatory results.

LEMMA 4.9. *There are natural transforms $\eta\colon \mathrm{Id} \to \Sigma$ and $\mu\colon \Sigma^2 \to \Sigma$ given by*

$$\eta_{(X,E)}(x) = E^x, \qquad \mu_{(X,E)}(\sigma) = \sigma(E^{()})$$

*for all objects (X, E) of Q-*CAT, *where* Id *is the identily functor on Q-*CAT, *and Σ is the functor produced in* 4.6.

Proof. First, we must be sure that the maps $\eta_{(X,E)}$ and $\mu_{(X,E)}$ are morphisms of Q-CAT for any separated Q-category (X, E). Axioms 3.6(i), (ii) for $\eta_{(X,E)}$ follow immediately from 4.5(ii), while the strictness of $\mu_{(X,E)}$ is already met in 4.8(i). In order to verify 3.6(ii) for $\mu_{(X,E)}$ we proceed as follows:

$$\begin{aligned}
[\![\sigma, \sigma']\!]_{S\left(\Sigma(X,E)\right)} &= \Bigl([\![\sigma]\!] \circ \bigwedge_{s\in S(X,E)} \langle s, \sigma\rangle \setminus \langle s, \sigma'\rangle\Bigr) \wedge \Bigl(\Bigl(\bigwedge_{s\in S(X,E)} \langle\sigma, s\rangle / \langle\sigma', s\rangle\Bigr) \circ [\![\sigma']\!]\Bigr)\\
&\leqslant \Bigl(\bigl[\sigma(E^{()})\bigr] \circ \bigwedge_{x\in X} \langle E^x, \sigma\rangle \setminus \langle E^x, \sigma'\rangle\Bigr) \wedge \Bigl(\Bigl(\bigwedge_{x\in X} \langle\sigma, E^x\rangle / \langle\sigma', E^x\rangle\Bigr) \circ \bigl[\sigma'(E^{()})\bigr]\Bigr)\\
&= \bigl[\sigma(E^{()}, \sigma'(E^{()})\bigr]_{S(X,E)}.
\end{aligned}$$

Next, the diagrams

$$\begin{tikzcd}
(X,E) \arrow[r, "\eta_{(X,E)}"] \arrow[d, "f"] & \Sigma(X,E) \arrow[d, "\Sigma(f)"] \\
(Y,F) \arrow[r, "\eta_{(Y,F)}"] & \Sigma(Y,F)
\end{tikzcd},$$

$$\begin{tikzcd}
\Sigma^2(X,E) \arrow[r, "\mu_{(X,E)}"] \arrow[d, "\Sigma^2(f)"] & \Sigma(X,E) \arrow[d, "\Sigma(f)"] \\
\Sigma^2(Y,F) \arrow[r, "\mu_{(Y,F)}"] & \Sigma(Y,F)
\end{tikzcd}$$

always commute: we verify the commutativity of the first diagram as follows:

$$
\begin{aligned}
\left\langle y, (\Sigma(f))(\eta_{(X,E)}(x))\right\rangle &= \bigvee_{x'\in X} (F(y, f(x')) \circ T(\tilde{x}', \tilde{x}')/Ex') \circ E(x', x) \\
&\leqslant \bigvee_{x'\in X} (F(y, f(x')) \circ T(\tilde{x}', \tilde{x}')/Ff(x')) \circ F(f(x'), f(x)) \qquad \text{(by 3.6)} \\
&\leqslant F(y, f(x)) \qquad \text{(by 3.2(i), 3.1(ii))} \\
&\quad \left(= \langle y, \eta_{(X,F)}(f(x))\rangle\right) \\
&= (F(y, f(x)) \circ T(\widetilde{f(x)}, \widetilde{f(x)})/Ff(x)) \circ Ff(x) \qquad \text{(by 3.2(iii))} \\
&= (F(y, f(x)) \circ T(\tilde{x}, \tilde{x})/Ex) \circ Ex \qquad \text{(3.6(i))} \\
&\leqslant \bigvee_{x'\in X} (F(y, f(x')) \circ T(\tilde{x}', \tilde{x}')/Ex') \circ E(x', x) \\
&= \left\langle y, (\Sigma(f))(\eta_{(X,E)}(x))\right\rangle,
\end{aligned}
$$

$$
\left\langle (\Sigma(f))(\eta_{(X,E)}(x)), y\right\rangle = \left\langle \eta_{(Y,F)}(f(x)), y\right\rangle \qquad \text{(similarly).}
$$

Now we establish the commutativity of the second one: if σ is in $S(\Sigma(X, E))$ and y in Y, then

$$
\begin{aligned}
\left\langle y, \mu_{(Y,F)}\Big(\big(\Sigma(\Sigma(f))\big)(\sigma)\Big)\right\rangle &= \left\langle F^{Y}, \big(\Sigma(\Sigma(f))\big)(\sigma)\right\rangle \qquad \text{(by definition of } \mu) \\
&= \bigvee_{s\in S(X,E)} \Big[F^{y}, (\Sigma(f))(s)\Big]_{S(Y,F)} \circ ([\![s]\!] \setminus \langle s, \sigma\rangle) \qquad \text{(by 4.6)} \\
&= \bigvee_{s\in S(X,E)} \left\langle y, (\Sigma(f))(s)\right\rangle \circ ([\![s]\!] \setminus \langle s, \sigma\rangle) \qquad \text{(by 4.5(ii))} \\
&= \bigvee_{s\in S(X,E)} \left(\bigvee_{x\in X} (F(y, f(x)) \circ T(\tilde{x}, \tilde{x})/Ex) \circ \langle x, s\rangle\right) \circ ([\![s]\!] \setminus \langle s, \sigma\rangle) \\
&= \bigvee_{x\in X} (F(y, f(x)) \circ T(\tilde{x}, \tilde{x})/Ex) \circ \bigvee_{s\in S(X,E)} [\![E^{x}, s]\!] \circ ([\![s]\!] \setminus \langle s, \sigma\rangle) \\
&= \bigvee_{x\in X} (F(y, f(x)) \circ T(\tilde{x}, \tilde{x})/Ex) \circ \langle E^{x}, \sigma\rangle \qquad \text{(by 4.1(ii), 1.3(xii), 4.2(ii))} \\
&= \left\langle y, (\Sigma(f))(\sigma(E^{()}))\right\rangle,
\end{aligned}
$$

$$
\left\langle \mu_{(Y,F)}\Big(\big(\Sigma(\Sigma(f))\big)(\sigma)\Big), y\right\rangle = \left\langle (\Sigma(f))(\sigma(E^{()})), y\right\rangle \qquad \text{(analogously).} \qquad \square
$$

Finally, we arrive at

THEOREM 4.10. *The triple* (Σ, η, μ) *consisting of the functor* Σ: Q-CAT $\to$ Q-CAT *and the two natural transforms* η: Id $\to$ Σ, μ: Σ^2 $\to$ Σ (*described in* 4.6, 4.9) *is a monad on the category* Q-CAT, *i.e., the* *following diagrams always commute*:

$$\begin{tikzcd} \Sigma^3 \arrow[r, "\Sigma\mu"] \arrow[d, "\mu\Sigma"'] & \Sigma^2 \arrow[d, "\mu"] \\ \Sigma^2 \arrow[r, "\mu"] & \Sigma \end{tikzcd}$$

(*associativity*),

$$\begin{tikzcd} \mathrm{Id}\,\Sigma \arrow[r, "\eta\Sigma"] \arrow[d, equal] & \Sigma^2 \arrow[d, "\mu"] & \Sigma\,\mathrm{Id}_{\Sigma\eta} \arrow[l] \arrow[d, equal] \\ \Sigma \arrow[r, equal] & \Sigma \arrow[r, equal] & \Sigma \end{tikzcd}$$

(*unit laws*).

Proof. A verification of the unit laws for η is easy. Let us prove the associativity of μ. If τ is in $S(\Sigma(\Sigma(X, E)))$ and x is in X, then we have

$$\begin{aligned}
\Big\langle x, \mu_{(X,E)}\Big(\big(\Sigma(\mu_{(X,E)})\big)(\tau)\Big)\Big\rangle &= \Big\langle E^x, \big(\Sigma(\mu_{(X,E)})\big)(\tau)\Big\rangle \qquad \text{(by definition of } \mu) \\
&= \bigvee_{\sigma\in S(\Sigma(X,E))} \Big[E^x, \mu_{(X,E)}(\sigma)\Big]_{S(X,E)} \circ ([\![\sigma]\!] \setminus \langle\sigma, \tau\rangle) \qquad \text{(by 4.6)} \\
&= \bigvee_{\sigma\in S(\Sigma(X,E))} \Big\langle x, \mu_{(X,E)}\Big\rangle \circ ([\![\sigma]\!] \setminus \langle\sigma, \tau\rangle) \qquad \text{(by 4.5(ii))} \\
&= \bigvee_{\sigma\in S(\Sigma(X,E))} \big\langle E^x, \sigma\big\rangle \circ ([\![\sigma]\!] \setminus \langle\sigma, \tau\rangle) \\
&= \bigvee_{\sigma\in S(\Sigma(X,E))} [([\ , E^x], [E^x, \ ]), \sigma]_{S(\Sigma(X,E))} \circ ([\![\sigma]\!] \setminus \langle\sigma, \tau\rangle) \qquad \text{(by 4.5(ii))} \\
&= \big\langle([\ , E^x], [E^x, \ ]), \tau\big\rangle \qquad \text{(by 4.1(ii), 1.3(xii), 4.2(ii))} \\
&= \Big\langle E^x, \mu_{\Sigma(X,E)}(\tau)\Big\rangle \\
&= \Big\langle x, \mu_{(X,E)}\big(\mu_{\Sigma(X,E)}(\tau)\big)\Big\rangle \qquad \text{(by definition of } \mu),
\end{aligned}$$

$$\Big\langle \mu_{(X,E)}\Big(\big(\Sigma(\mu_{(X,E)})\big)(\tau)\Big), x\Big\rangle = \Big\langle \mu_{(X,E)}\big(\mu_{\Sigma(X,E)}(\tau)\big), x\Big\rangle \qquad \text{(analogously).}$$

□

5. SHEAVES OVER QUANTALOIDS

Having defined the monad (Σ, η, μ) on the category Q-CAT, it is now time to extend Höhle's concept of sheaves.

Definition 5.1. Let (X, E) be a separated Q-category and u be an object of Q.

(i) *A* subset B of X is called *compatible* when $\{(Ex \circ T(\bar{x}, u),\ T(u, \bar{x}) \circ Ex) \mid x \in B\}$ constitute a singleton of the Q-category $(B, E\,|_{B\times B})$, where $E\,|_{B\times B}$ stands for the restriction of $E(\ ,\ )$ to $B \times B$.

(ii) An element x_B of X is said to be a *join* of a compatible subset B of (X, E) when $\tilde{x}_B = u$,

$$E(x, x_B) = \bigvee_{x' \in B} E(x, x') \circ T(\tilde{x}', u) \quad \text{and} \quad E(x_B, x) = \bigvee_{x' \in B} T(u, \tilde{x}') \circ E(x', x).$$

PROPOSITION 5.2. *Let (X, E) be a separated Q-category, u an object of Q, B a subset of X, and s a pair of maps $\langle\,, s\rangle$, $\langle s,\,\rangle$: $B \to \mathrm{Mor}(Q)$ assigning to every $x \in B$ morphisms $\langle x, s\rangle$: $u \to \tilde{x}$ and $\langle s, x\rangle$: $\tilde{x} \to u$, respectively. Then the following assertions are equivalent*:

(i) *s is a singleton of $(B, E|_{B\times B})$*;

(ii) *The pair $\hat{s}$ of maps $\langle\,, \hat{s}\rangle$ and $\langle \hat{s},\,\rangle$ from X to* Mor(Q) *defined by*

$$\langle x, \hat{s}\rangle = \bigvee_{x' \in B} E(x, x') \circ (Ex' \setminus \langle x', s\rangle) \qquad \Big(= \bigvee_{x' \in B} (E(x, x') \circ T(\tilde{x}', \tilde{x}')/Ex') \circ \langle x', s\rangle\Big)$$

and

$$\langle \hat{s}, x\rangle = \bigvee_{x' \in B} ((\langle s, x'\rangle \circ T(\tilde{x}', \tilde{x}')/Ex') \circ E(x, x') \qquad \Big(= \bigvee_{x' \in B} \langle s, x'\rangle \circ (Ex' \setminus E(x', x))\Big)$$

is a singleton of (X, E) such that $\hat{s}(x) = s(x)$ for any x in B and

$$[\![\hat{s}]\!] = \bigvee_{x \in B} T(u, \tilde{x}) \circ \langle x, s\rangle.$$

Moreover, for two such extensions $\hat{s}$ and $\hat{s'}$ of $(B, E|_{B\times B})$-singletons s and s', respectively, the following equality:

$$[\![\hat{s}, \hat{s'}]\!]_{S(X,E)} = [\![s, s']\!]_{S(B,E|_{B\times B})}$$

always holds.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. In order to verify (i) $\Rightarrow$ (ii) we first observe that if $x \in B$, then

$$\begin{aligned}
\langle x, s\rangle &= Ex \circ (Ex \setminus \langle x, s\rangle) \leqslant \bigvee_{x' \in B} E(x, x') \circ (Ex' \setminus \langle x', s\rangle) \qquad (= \langle x, \hat{s}\rangle)\\
&\leqslant \langle x, s\rangle \qquad \text{(by 4.1(ii))}\\
\langle s, x\rangle &= (\langle s, x\rangle \circ T(\tilde{x}, \tilde{x})/Ex) \circ Ex \leqslant \bigvee_{x' \in B} ((\langle s, x'\rangle \circ T(\tilde{x}', \tilde{x}')/Ex') \circ E(x', x) \qquad (= \langle \hat{s}, x\rangle)\\
&\leqslant \langle s, x\rangle,
\end{aligned}$$

i.e., $\langle x, \hat{s}\rangle = \langle x, s\rangle$ and $\langle \hat{s}, x\rangle = \langle s, x\rangle$. Let us now prove that $\hat{s}$ is a singleton of (X, E). But this is an immediate consequence of 4.6. Moreover, by 4.6 the map $\hat{}$: $s \mapsto \hat{s}$ is a Q-CAT-morphism from $(B, E|_{B\times B})$ to (X, E), i.e., for every s in $S(B, E|_{B\times B})$, $\hat{s}$ is in $S(X, E)$, $[\![\hat{s}]\!] = [\![s]\!]$, and $[\![s, s']\!] \leqslant [\![\hat{s}, \hat{s'}]\!]$. Thus, it only remains to verify the opposite inequality: $[\![\hat{s}, \hat{s'}]\!] \leqslant [\![s, s']\!]$, but this is true, since

$$\begin{aligned}
[\![\hat{s}, \hat{s'}]\!] &= \Big([\![\hat{s}]\!] \circ \bigwedge_{x \in X} \langle x, \hat{s}\rangle \setminus \langle x, \hat{s'}\rangle\Big) \wedge \Big(\Big(\bigwedge_{x \in X} \langle \hat{s}, x\rangle / \langle \hat{s'}, x\rangle\Big) \circ [\![\hat{s'}]\!]\Big)\\
&\leqslant \Big([\![\hat{s}]\!] \circ \bigwedge_{x \in B} \langle x, \hat{s}\rangle \setminus \langle x, \hat{s'}\rangle\Big) \wedge \Big(\Big(\bigwedge_{x \in B} \langle \hat{s}, x\rangle / \langle \hat{s'}, x\rangle\Big) \circ [\![\hat{s'}]\!]\Big)\\
&= \Big([\![s]\!] \circ \bigwedge_{x \in B} \langle x, s\rangle \setminus \langle x, s'\rangle\Big) \wedge \Big(\Big(\bigwedge_{x \in B} \langle s, x\rangle / \langle s', x\rangle\Big) \circ [\![s']\!]\Big) = [\![s, s']\!].
\end{aligned}$$

□

COROLLARY 5.3 (cf. [4, Lemma 4.2]). *Let (X, E) be a separated Q-category, u an object of Q, and B a subset of X. Then the following assertions are equivalent:*

(i) *B is compatible;*

(ii) *The pair $\hat{s}$ of maps $\langle\ ,\hat{s}\rangle$, $\langle\hat{s},\ \rangle$ from X to* $\mathrm{Mor}(Q)$ *defined by*

$$\langle x, \hat{s}\rangle = \bigvee_{x' \in B} E(x, x') \circ T(\tilde{x'}, u) \quad \textit{and} \quad \langle \hat{s}, x\rangle = \bigvee_{x' \in B} T(u, \tilde{x'}) \circ E(x', x)$$

is a singleton of (X, E) with $[\![\hat{s}]\!] = \bigvee_{x \in B} T(u, \tilde{x}) \circ Ex \circ T(\tilde{x}, u)$.

Before going further, we recall the definition of (Eilenberg–Moore) algebras for a monad.

Definition 5.4. Let $M = (\Sigma, \eta, \mu)$ be the monad specified in 4.10. An *M-algebra* is a pair $((X, E), \xi)$ of a separated Q-category (X, E) and a Q-CAT-morphism ξ: $\Sigma(X, E) \to (X, E)$ which makes both diagrams

$$\begin{tikzcd}
\Sigma^2(X,E) \arrow[r, "\Sigma(\xi)"] \arrow[d, "\mu_{(X,E)}"] & \Sigma(X,E) \arrow[d, "\xi"] \\
\Sigma(X,E) \arrow[r, "\xi"] & (X,E)
\end{tikzcd},$$

$$\begin{tikzcd}
(X,E) \arrow[r, "\eta_{(X,E)}"] \arrow[dr, "id_{(X,E)}"'] & \Sigma(X,E) \arrow[d, "\xi"] \\
 & (X,E)
\end{tikzcd}$$

commute.

PROPOSITION 5.5 (cf. [4, Proposition 4.3]). *Let $((X, E), \xi)$ be an M-algebra, u an object of Q and B a compatible subset of (X, E). Then B has a (unique) join.*

Proof. Consider the maps $\langle\ , s\rangle$, $\langle s,\ \rangle$: $X \to \mathrm{Mor}(Q)$ defined by

$$\langle x, s\rangle = \bigvee_{x' \in B} E(x, x') \circ T(\tilde{x'}, u) \quad \text{and} \quad \langle s, x\rangle = \bigvee_{x' \in B} T(u, \tilde{x'}) \circ E(x', x).$$

Since $s = (\langle\ , s\rangle, \langle s,\ \rangle)$ is a singleton of (X, E) (by 5.3) and ξ is a Q-CAT-morphism, we obtain $\langle x, s\rangle = [\![E^x, s]\!] \leqslant E(x, \xi(s))$ and $\langle s, x\rangle = [\![s, E^x]\!] \leqslant E(\xi(s), x)$ for all x in X. To establish the opposite inequalities, we proceed as follows:

$$
\begin{aligned}
E(x, \xi(s)) &= (E(x, \xi(s)) \circ T(u, u)/E\xi(s)) \circ E\xi(s) \qquad \text{(by 3.2(iii))} \\
&= (E(x, \xi(s)) \circ T(u, u)/E\xi(s)) \circ [\![s]\!] \qquad \text{(by 3.6(i))} \\
&= (E(x, \xi(s)) \circ T(u, u)/E\xi(s)) \circ \bigvee_{x' \in B} T(u, \tilde{x'}) \circ Ex' \circ T(\tilde{x'}, u) \qquad \text{(by 5.3)} \\
&= \bigvee_{x' \in B} (E(x, \xi(s)) \circ T(u, u)/E\xi(s)) \circ \langle s, x' \rangle \circ T(\tilde{x'}, u) \\
&\quad (\text{since} \langle s, x' \rangle = T(u, \tilde{x'}) \circ Ex', \text{ when } x' \text{is in } B) \\
&= \bigvee_{x' \in B} (E(x, \xi(s)) \circ T(u, u)/E\xi(s)) \circ [\![s, E^{x'}]\!] \circ T(\tilde{x'}, u) \qquad \text{(by 4.5(ii))} \\
&\leqslant \bigvee_{x' \in B} (E(x, \xi(s)) \circ T(u, u)/E\xi(s)) \circ E(\xi(s), x') \circ T(\tilde{x'}, u) \qquad \text{(by 3.6(ii))} \\
&\leqslant \bigvee_{x' \in B} E(x, x') \circ T(\tilde{x'}, u) \qquad \text{(by 3.2(i), 3.1(ii))} \\
&= \langle s, x \rangle.
\end{aligned}
$$

$E(\xi(s), x) = \langle s, x \rangle$ (analogously). Hence $\xi(s)$ is the join of B. Since we consider only separated Q-categories, this join is unique. □

PROPOSITION 5.6 (cf. [4, Proposition 4.4] and [3, Proposition 4.6]). *Every M-algebra $((X, E), \xi)$ induces a "preasheaf structure" $(X, \urcorner, E)$ on X over Q by*

$$x \urcorner a = \xi\big(E^{(x,a)}\big), \tag{i}$$

with the following properties: for all x in X, u, v in $\mathrm{Obj}(Q)$ *with the relations:* $T(\tilde{x}, u) \circ T(u, \tilde{x}) = T(\tilde{x}, \tilde{x}) = T(\tilde{x}, v) \circ T(v, \tilde{x})$, $T(u, v) \circ T(v, u) = T(u, u)$, *and for all $a\colon u \to \tilde{x}$, $b\colon v \to u$,*

$$
\begin{aligned}
&x \urcorner Ex = x, \qquad E(x \urcorner a) = T(u, \tilde{x}) \circ ((Ex \circ T(\tilde{x}, u)) \wedge a) \qquad \Big(= \big[E^{(x,a)}\big]\Big), \\
&(x \urcorner a) \urcorner b = x \urcorner ((a \circ T(u, v)) \wedge (T(\tilde{x}, u) \circ b)),
\end{aligned} \tag{ii}
$$

where $E^{(x,a)}$ is determined by 4.3(ii), *and E by* 3.1(iii). *In particular, $(X, \urcorner, E)$ is compatible with the underlying Q-category structure by*

$$E(x \urcorner a, x \urcorner (T(\tilde{x}, u) \circ b)) = T(u, \tilde{x}) \circ ((Ex \circ T(\tilde{x}, v)) \wedge (a \circ T(u, v))) \wedge (T(\tilde{x}, u) \circ b)). \tag{iii}$$

Proof. The strictness 3.6(i) of ξ immediately implies the first two equalities of (ii). In order to prove the last equality of (ii), for given $x \in X$, u, $v \in \mathrm{Obj}(Q)$ (with $T(\tilde{x}, u) \circ T(u, \tilde{x}) = T(\tilde{x}, \tilde{x}) = T(\tilde{x}, v) \circ T(v, \tilde{x})$ and $T(u, v) \circ T(v, u) = T(u, u)$), $a\colon u \to \tilde{x}$ and $b\colon v \to u$, let us consider a singleton $\sigma = (\langle\ , \sigma \rangle, \langle \sigma,\ \rangle)$ of $\Sigma(X, E)$ defined by $\langle s, \sigma \rangle = [\![s, E^{(x,a)}]\!] \circ \big([\![E^{(x,a)}]\!] \setminus b\big)$ and $\langle \sigma, s \rangle = T(v, u) \circ \big(b \circ T(v, u)/\ [\![E^{(x,a)}]\!]\big) \circ [\![E^{(x,a)}, s]\!]$ (cf.

4.3(ii)). Put $s = E^{x'}$, then

$$\begin{aligned}\left\langle E^{x'},\sigma\right\rangle &= \left[E^{x'},E^{(x,a)}\right]\circ\left(\left[E^{(x,a)}\right]\setminus b\right) = \left\langle x',E^{(x,a)}\right\rangle\circ\left(\left[E^{(x,a)}\right]\setminus b\right) \qquad \text{(by 4.5(ii))}\\ &= (E(x',x)\circ T(\tilde{x},\tilde{x})/Ex)\circ T(\tilde{x},u)\circ\left[E^{(x,a)}\right]\circ\left(\left[E^{(x,a)}\right]\setminus b\right) \qquad \text{(by 4.3(ii))}\\ &= (E(x',x)\circ T(\tilde{x},\tilde{x})/Ex)\circ T(\tilde{x},u)\circ\left(\left[E^{(x,a)}\right]\circ T(u,v)\wedge b\right) \qquad \text{(by 1.3(xii))}\\ &= (E(x',x)\circ T(\tilde{x},\tilde{x})/Ex)\circ T(\tilde{x},u)\circ((T(u,\tilde{x})\circ(Ex\circ T(\tilde{x},u)\wedge a)\circ T(u,v))\wedge b)\\ &= (E(x',x)\circ T(\tilde{x},\tilde{x})/Ex)\circ(Ex\circ T(\tilde{x},v)\wedge(a\circ T(u,v)\wedge T(\tilde{x},u)\circ b)) \qquad \text{(by 1.2(ii))}\\ &= \left\langle x',E^{(x,a\circ T(u,v)\wedge T(x,u)\circ b)}\right\rangle,\end{aligned}$$

$\langle\sigma,E^{x'}\rangle = \langle E^{(x,a\circ T(u,v)\wedge T(\tilde{x},u)\circ b)},x'\rangle$ (similarly). Hence $\mu_{(X,E)}(\sigma)(=\sigma(E^{()}))$ (by 4.9) $= E^{(x,a\circ T(u,v)\wedge T(\tilde{x},u)\circ b)}$. We further have

$$\begin{aligned}\left\langle x',\left(\Sigma(\xi)\right)(\sigma)\right\rangle &= \bigvee_{s\in S(X,E)}(E(x',\xi(s))\circ T(\tilde{s},\tilde{s})/[\![s]\!])\circ\langle s,\sigma\rangle \qquad \text{(by 4.6)}\\ &= \bigvee_{s\in S(X,E)}(E(x',\xi(s))\circ T(\tilde{s},\tilde{s})/[\![s]\!])\circ\left[s,E^{(x,a)}\right]\circ\left(\left[E^{(x,a)}\right]\setminus b\right)\\ &= \left(E(x',\xi\left(E^{(x,a)}\right)\right)\circ\left(\left[E^{(x,a)}\right]\setminus b\right) \qquad \text{(by 3.6, 3.1(ii))}\\ &= E(x',x\urcorner a)\circ(E(x\urcorner a)\setminus b) = \left\langle x',E^{(x\urcorner a,b)}\right\rangle,\end{aligned}$$

$\langle(\Sigma(\xi))(\sigma),x'\rangle = \langle E^{(x\urcorner a,b)},x'\rangle$ (analogously). Hence

$$\begin{aligned}x\urcorner(a\circ T(u,v)\wedge T(\tilde{x},u)\circ b) &= \xi\left(E^{(x,a\circ T(u,v)\wedge T(\tilde{x},u)\circ b)}\right) = \xi\left(\mu_{(X,E)}(\sigma)\right)\\ &= \xi\left(\left(\Sigma(\xi)\right)(\sigma)\right) \qquad \text{(by 5.4)}\\ &= \xi\left(E^{(x\urcorner a,b)}\right) = (x\urcorner a)\urcorner b.\end{aligned}$$

To prove (iii) note that

$$\begin{aligned}\left[E^{(x,a)},E^{(x,T(\tilde{x},u)\circ b)}\right] &\leqslant \left[E^{(x,a)}\right]\circ\bigwedge_{x'\in X}(E(x',x)\circ(Ex\setminus a)\setminus E(x',x)\circ(Ex\setminus T(\tilde{x},u)\circ b)) \qquad \text{(by 4.4, 4.3(ii))}\\ &\leqslant \left[E^{(x,a)}\right]\circ(Ex\circ(Ex\setminus a)\setminus Ex\circ(Ex\setminus T(\tilde{x},u)\circ b))\\ &= T(u,\tilde{x})\circ((Ex\circ T(\tilde{x},u)\wedge a)\\ &\quad\circ T(u,v)\wedge(Ex\circ T(\tilde{x},v)\wedge T(\tilde{x},u)\circ b)) \qquad \text{(by 1.3(xii) twice)}\\ &= T(u,\tilde{x})\circ(Ex\circ T(\tilde{x},v)\wedge a\circ T(u,v)\wedge T(\tilde{x},u)\circ b) \qquad \text{(by 1.2(ii))}\end{aligned}$$

and

$$\begin{aligned}\left[E^{(x,a)},E^{(x,T(\tilde{x},u)\circ b)}\right] \geqslant &\Bigg[\left[E^{(x,a)}\right]\circ\bigwedge_{x'\in X}\Big(\left(E(x',x)\circ T(\tilde{x},\tilde{x})/Ex\right)\circ T(\tilde{x},u)\\ &\circ\left[E^{(x,a)}\right]\setminus(E(x',x)\circ T(\tilde{x},\tilde{x})/Ex)\circ T(\tilde{x},u)\circ T(u,v)\circ\left[E^{(x,T(\tilde{x},u)\circ b)}\right]\Big)\Bigg]\\ &\wedge\Bigg[\Big(\bigwedge_{x'\in X}\left[E^{(x,a)}\right]\circ T(u,v)\circ T(v,\tilde{x})\circ(Ex\setminus E(x,x'))/\left[E^{(x,T(\tilde{x},u)\circ b)}\right]\circ T(v,\tilde{x})\\ &\circ(Ex\setminus E(x,x'))\Big)\circ\left[E^{(x,T(\tilde{x},u)\circ b)}\right]\Bigg] \qquad \text{(by 4.4, 4.3(ii))}\end{aligned}$$

$$\geqslant \left(\left[E^{(x,a)}\right] \circ \left(\left[E^{(x,a)}\right] \setminus T(u,v) \circ \left[E^{(x,T(\tilde{x},u)\circ b)}\right]\right)\right)$$
$$\wedge \left(\left(\left[E^{(x,a)}\right] \circ T(u,v)/\left[E^{(x,T(\tilde{x},u)\circ b)}\right]\right) \circ \left[E^{(x,T(\tilde{x},u)\circ b)}\right]\right) \qquad \text{(by 1.3(xi))}$$
$$= \left(\left[E^{(x,a)}\right] \circ T(u,v) \wedge T(u,v) \circ \left[E^{(x,T(\tilde{x},u)\circ b)}\right]\right)$$
$$\wedge \left(\left[E^{(x,a)}\right] \circ T(u,v) \wedge T(u,v) \circ \left[E^{(x,T(\tilde{x},u)\circ b)}\right]\right) \qquad \text{(by 1.3(xiii))}$$
$$= (T(u,\tilde{x}) \circ (Ex \circ T(\tilde{x},u) \wedge a) \circ T(u,v)) \wedge (T(u,v) \circ T(v,\tilde{x})$$
$$\circ (Ex \circ T(\tilde{x},v) \wedge T(\tilde{x},u) \circ b))$$
$$= T(u,\tilde{x}) \circ Ex \circ T(\tilde{x},v) \wedge T(u,\tilde{x}) \circ a \circ T(u,v) \wedge T(u,\tilde{x})$$
$$\circ Ex \circ T(\tilde{x},v) \wedge T(u,u) \circ b \qquad \text{(by 1.2(ii))}$$
$$= T(u,\tilde{x}) \circ (Ex \circ T(\tilde{x},v) \wedge a \circ T(u,v) \wedge T(\tilde{x},u) \circ b),$$

i.e., $\left[E^{(x,a)}, E^{(x,T(\tilde{x},u)\circ b)}\right] = T(u,\tilde{x}) \circ (Ex \circ T(\tilde{x},v) \wedge a \circ T(u,v) \wedge T(\tilde{x},u) \circ b)$. Then

$$T(u,\tilde{x}) \circ (Ex \circ T(\tilde{x},v) \wedge a \circ T(u,v) \wedge T(\tilde{x},u) \circ b)$$
$$= \left[E^{(x,a)}, E^{(x,T(\tilde{x},u)\circ b)}\right] \leqslant \left[E^{(x,a)}\right] \circ T(u,v) \wedge T(u,v) \circ \left[E^{(x,T(\tilde{x},u)\circ b)}\right] \qquad \text{(by 3.1(i))}$$
$$= T(u,\tilde{x}) \circ (Ex \circ T(\tilde{x},u) \wedge a) \circ T(u,v) \wedge T(u,v) \circ T(v,\tilde{x}) \circ (Ex \circ T(\tilde{x},v) \wedge T(\tilde{x},u) \circ b)$$
$$= T(u,\tilde{x}) \circ Ex \circ T(\tilde{x},v) \wedge T(u,\tilde{x}) \circ a \circ T(u,v) \wedge T(u,\tilde{x}) \circ Ex \circ T(\tilde{x},v) \wedge T(u,u) \circ b \qquad \text{(by 1.2(ii))}$$
$$= T(u,\tilde{x}) \circ (Ex \circ T(\tilde{x},v) \wedge a \circ T(u,v) \wedge T(\tilde{x},u) \circ b) \qquad \text{(by 1.2(ii))},$$

which proves (iii). □

Note that in the case where Q is a commutative quantale U. Höhle [4] calls M-algebras also sheaves over Q.

REFERENCES

1. U. Berni-Canani, F. Borceux, and R. Succi Cruciani, A theory of quantal sets, *J. Pure Appl. Algebra*, **62**, 123–136 (1989).
2. F. Borceux and G. van den Bossche, Quantales and their sheaves, *Order*, **3**, 61–87 (1986).
3. R. P. Gylys, Quantal sets and sheaves over quantales, *Lith. Math. J.*, **34**, 8–29 (1994).
4. U. Höhle, M-valued sets and sheaves over integral commutative cl-monoids, in: *Applications of Category Theory to Fuzzy Subsets*, S. E. Rodabaug *et al.* (eds), Kluwer Academic Publishers, The Netherlands (1992), pp. 33–72.
5. C. J. Mulvey, &, *Rend. Circ. Mat. Palermo*, **12**, 99–104 (1986).
6. A. M. Pitts, Applications of sup-lattice enriched category theory to sheaf theory, *Proc. London Math. Soc.*, **57**, 433–480 (1988).
7. K. I. Rosenthal, *Quantales and Their Applications*, Pitman Research Notes in Math. 234, Longman, Scientific and Technical, Harlow (1990).
8. K. I. Rosenthal, Free quantaloids, *J. Pure Appl. Algebra*, **72**, 67–82 (1991).
9. K. I. Rosenthal, Girard quantaloids, *Math. Struct. Comp. Science*, **2**, 93–108 (1992).
10. K. I. Rosenthal, Quantaloidal nuclei, the syntactic congruence and tree automata, *J. Pure Appl. Algebra*, **77**, 189–205 (1992).
11. K. I. Rosenthal, A categorical look at tree automata and context-free languages, to appear in *Math. Struct. Comp. Science*.
12. D. Yetter, Quantales and (non-commutative) linear logic, *J. Symbolic Logic*, **55**, 41–64 (1990).