Comput Econ (2015) 46:275–285
DOI 10.1007/s10614-014-9463-6

An Improved RBF Method for Solving Variational Problems Arising from Dynamic Economic Models

A. Golbabai · A. Saeedi

Accepted: 11 August 2014 / Published online: 5 September 2014

Abstract This paper developes a direct method for solving variational problems via a set of Radial Basis Functions (RBFs) . Operational matrices of differentiation, the product of two RBF vectors and some other formulas are derived and are utilized to propose a method which essentially reduces a variational problem to the linear system of algebraic equations. National saving problem is considered and solved by proposed method which experimentally illustrates effectiveness and applicability of the method. Some experiments are conducted in order to compare the accuracy and stability of several shape parameter strategies in these type of problems. Finally a novel shape parameter strategy is proposed which promotes accuracy and stability of the method.

Keywords Radial basis functions · Direct methods · Operational matrix · Dynamic economic model · Variable shape parameter

1 Introduction

The calculus of variations investigates methods that permit maximal or minimal values of functionals (Elsgolts 1977). A direct method converts the variational problem into a mathematical programming problem. The idea of direct methods for solving variational problems consists in replacing the problem of searching for the extremum

A. Golbabai (✉) · A. Saeedi
School of Mathematics, Iran University of Science & Technology,
Narmak, Tehran 16844-13114, Iran
e-mail: golbabai@iust.ac.ir

A. Saeedi
e-mail: saeidi@iust.ac.ir

(usually, for the point of stationary) of a functional in the function space by a problem of searching for a solution in a finite set of parameters. The Ritz method (Gelfand and Fomin 1963) usually based on the subspaces of kinematically admissible complete functions, is the most commonly used approach in direct methods for solving variational problems.

Recently the operational matrices of integration and differentiation have been widely used for solving problems such as calculus of variations, differential equations and integral equations . Operational matrices of Bernstain polynomial by Yousefi and Behroozifar (2010), Chebyshev wavelets by Tavvasoli et al. (2009) are some examples.

In the last decades , Radial Basis Functios (RBFs) have been widely applied in different fields such as multivariate function interpolation and approximation, neural networks , solution of differential and integral equations.

Some advantages of the RBF-based methods are their ease of implementation , accuracy and efficiency , which are the reasons that this technique is getting popular. RBF methods have been actively investigated to numerically solve PDEs since the last decade. Indeed, they yield exponential convergence for smooth problems and they are meshless methods (Cheney 2000). Franke and Shaback (1998) used RBFs for solving PDEs , Golbabai and Seifollahi (2006) used RBFs for solving the second kind Integral Equations (IEs).

Recently, compact support RBFs have become more and more popular due to their computational advantages (Lin and Yuan 2006). Some detailes about error bounds of some RBFs are discussed by Schaback (1995, 2005) and Madych (1992); Madych and Nelson (1992). Convergence analysis of RBF interpolation has been carried out by several researchers (Madych 1992; Wendland 2001).

In this paper we use RBFs for solving variational problems arising from dynamic economic models . For this end, we introduce the RBFs opertaional matrices of differentiation and the product of two RBF vectors. The RBFs operational matrices can be used to solve problems in fields analysis ,calculus of variations and optimal control. The method consists of reducing the variational problem into a linear system of algebraic equations by first expanding the candidate functions as a linear combination of RBFs with unknown coefficients. Gauss–Lobatto quadrature is used to approximate integration. Finally we evaluate the coefficients of RBF in such a way that the necessary conditions for extermization of a multivariate function are imposed.

The paper is organized in the following way. In Sect. 2 some commonly used RBFs and their properties are presented. In Sect. 3 the operational matrix for RBF approximation, its derivatives and some other formulas are calculated. We proposed a novel varaiable Shape Parameter strategy. In Sect. 4 the RBF direct method for solving variational problems is described. In Sect. 5 it is shown that the RBF direct method provides a good approximate solution for the national saving model , then efficency of various shape parameter strategies and several grid choosing techniques are compared with numerical experiments. Experiments show the superiority of proposed strategy to other shape parameter strategies.

Table 1 Some commonly used RBFs

Name	$\phi(r, c)$
Gaussian (GA)	$\phi(r) = exp(-cr^2)$
Multiquadric (MQ)	$\phi(r) = \sqrt{c^2 + r^2}$
Inverse multiquadric (IMQ)	$\phi(r) = (c^2 + r^2)^{-1/2}$

2 Properties of RBFs

Let $x_1, \ldots, x_N$ be a given set of distinct points in R^d. The main idea behind the use of RBF interpolation by translation of a single function i.e. the interpolating RBFs approximation is considered as

$$y(x) \cong y^N(x) = \sum_{i=0}^{N} a_i \varphi_i(x) \tag{1}$$

where $\phi(x) = \Phi(\|x - x_i\|)$ and a_i's are unknown scalars for $i = 1, \ldots N$. Common choices for RBFs are listed in Table 1 where $r = \|x - x_i\|$ and $\|.\|$ denotes the euclidean norm, x_i's are centers of RBF and c is a positive shape parameter which controls the flatness (width) of the basis function.

Assume that we want to interpolate the given values $f_i = f(x_i)$, $i = 1, \ldots, N$. The unknown scalars a_i are chosen so that $y(x_j) = f_j$ for $j = 1, \ldots, N$, which results in the linear system $AX = f$, where $A_{ij} = \phi_i(x_j)$, $X = [a_1, \ldots, a_N]$, $f = [f_1, \ldots, f_N]$.

Since all applicable ϕ have global support, this method produces a dense matrix A, which is positive definite for distinct interpolation points for GA, IMQ, IQ, TPs by schoenberg theorem (Fasshuar 2007).

It has been shown that interpolants by RBF in R^d with finite smoothness of even order converge to a polynomial spline interploant as the scalar parameter of the RBFs goes to zero, i.e., the radial basis functions becomes increasingly flat (Song et al. 2012).

3 Some New Properties of RBF

The following properties of RBFs will be used:

Theorem 1 *Let* $y^N(x) = a^T\phi(x)$ *where* $a^T = [a_1, a_2, \ldots a_N]$ *and* $\phi(x) = [\varphi_1(x), \ldots, \varphi_N(x)]^T$, $\phi_i(x) = exp(-c\|x - x_i\|^2)$, *for* $i = 1, \ldots, N$. *Then there exist symmetric matrices* A, B, D *and square matrix* S *such that:*

$$(a)(y^N(x))^2 = a^T Ba, \tag{2}$$

where $B_{ij} = \varphi_i(x)\varphi_j(x)$.

$$(b)\frac{d}{dx} y^N(x) = a^T A\phi(x), \tag{3}$$

and $A = diag\{-2c(x-x_0), -2c(x-x_1), \ldots, -2c(x-x_N)\}$.

$$(c)\left(\frac{d}{dx}y^N(x)\right)^2 = a^T Sa, \tag{4}$$

where $S = A\phi\phi^T A^T$, $S_{ij} = 4c^2(x-x_i)(x-x_j)\varphi_i(x)\varphi_j(x)$.

Theorem 2 *Let* $y^N(x) = a^T\phi(x)$, , $\phi_i(x) = \sqrt{c^2 + \|x-x_i\|^2}$ *for* $i = 1, \ldots, N$. *Then there exist symmetric matrices* H, L *and vector* $\overline{\phi}(x)$, *square matrix* M *such that:*

$$(a)(y^N(x))^2 = a^T Ha, \tag{5}$$

where $H_{ij} = \varphi_i(x)\varphi_j(x)$.

$$(b)\frac{d}{dx}y^N(x) = a^T\overline{\phi}(x), \tag{6}$$

where $\overline{\phi}(x) = [\overline{\varphi_1}(x), \ldots, \overline{\varphi_N}(x)]^T$ *and* $\overline{\varphi_i}(x) = (x-x_i)(c^2 + \|x-x_i\|^2)^{\frac{-1}{2}}$.

$$(c)\left(\frac{d}{dx}y^N(x)\right)^2 = a^T La, \tag{7}$$

where $L_{ij} = (x-x_i)(x-x_j)\varphi_i^{-1}(x)\varphi_j^{-1}(x)$.

3.1 Shape Parameter Strategies

It is observed (Golbabai et al. 2014) that the accuracy of the RBF method depends heavily on the choice of shape parameter c.

There are some methods for choosing the shape parameter. The most typical is calculating the errors with different shape parameters and choosing the best one (trial and error procedure). Hardy (1971) and Franke (1979) define the relations (8) , (9) shape parameters respectively:

$$c = 0.815d \tag{8}$$

$$c = \frac{1.25D}{\sqrt{N}} \tag{9}$$

where $d = \frac{1}{N}\sum_{i=1}^{N} d_i$ and d_i is the distance from ith center to nearest neighbor , D is the Diameter of the smallest circle encompassing all the center locations and N is the number of centers.

A variable shape parameter (VSP) strategy refers to use a distinct value of the shape parameter at each center , using a VSP facilitate obtainig a different entries in the RBF matrices which yields in lower condition numbers. A drawback of using a VSP is that a related system is no longer symmetric.

The formula $\epsilon_j = \left[\epsilon_{min}^2 (\frac{\epsilon_{max}^2}{\epsilon_{min}^2})^{\frac{j-1}{N-1}}\right]^{\frac{1}{2}}$, $j = 1, \ldots, N$ gives a exponentially variying shape parameter, was suggested in order to have a distinct value shape parameter (Sarra and Sturgill 2009).

Linearly varying shape parameter is another strategy:

$$\epsilon_j = \epsilon_{min} + \left(\frac{\epsilon_{max} - \epsilon_{min}}{N-1}\right) j, \quad j = 0, \ldots, N-1,$$

also the random shape strategy $\epsilon_j = \epsilon_{min} + (\epsilon_{max} - \epsilon_{min}) rand(1, N)$.

The formula *rand(1,N)* is the MATLAB function that returns N uniformely distributed pseudo random numbers on the unit interval (Sarra and Sturgill 2009). Golbabai and Rabiei (2012) proposed sinusoidal shape parameter (SSP) which produces N shape parameters in the interval $[\epsilon_{min}, \epsilon_{max}]$

$$\epsilon_j = \epsilon_{min} + (\epsilon_{max} - \epsilon_{min}) sin\left((j-1)\frac{\pi}{2(N-1)}\right), \quad j = 1, \ldots, N. \tag{10}$$

In this paper we propose shape parameter strategy as:

$$\epsilon_j = \left(\epsilon_{min}^2 \epsilon_{max} \left(\frac{j-1}{N-1}\right)\right)^{\frac{1}{2}}, \quad j = 1, \ldots, N \tag{11}$$

that gives a *Square Root VSP*. Since the square root function is increasing and concave, aids generating different entries in the RBF matrices together whit a decrease in the condition number of matrices.

4 RBF Direct Method

Consider the problem of finding the extremum of the functional

$$min \int_{x_a}^{x_b} L(x, y, \dot{y}) dx, \tag{12}$$

where $y : [x_a, x_b] \to R^N$ is a sufficitly smooth function, $y(x_a) = A$, $y(x_b) = B$ which two points A and B in R^N are given.

Tonelli's theorem (Clarke 2013) says under assumptions :

(1) Coercivity of rank $r > 1$ for certain constants $\alpha > 0$ and β we have

$$L(t, x, v) \geq \alpha \|v\|^r + \beta. \, for \; every \; (t, x, v) \epsilon [a, b] \times R^N \times R^N$$

(2) Convexity: L_{vv} is everywhere positive semidefinite ($L_{vv} \geq 0$),

there exists a solution of the basic problem (12) relative to the class of absolutly continuous (A.C.) functions.

Under the hypothesis of Tonelli's theorem, if in addition the Lagrangian is autonomous, then any solution y^*, of the problem (12) is Lipschitz on $[x_a, x_b]$. The problem or its Lagrangian is said to be autonomous when L has no dependence on the t variable.

The problem leading to a necessary condition for $y(x)$ on extremizing $J(y)$ in the form of the well known Euler-Lagrange equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0, \tag{13}$$

which should accompany with appropriate boundary conditions.

The solution of (13) can be easily found only in a few classes, thus it is more practical to use numerical and direct methods such as the well known Ritz and Galerkin method for variational problem (12).

Comparing the various integration rules (Newton-cotes formulas, extrapolating methods, Gaussian integration), computational efforts being equal, Gaussian integration yields the most accurate results (Stoer and Bulirsch 2002). The Gauss–Lobatto may be more efficient than others at higher accuracies with smooth integrands.

Applying Theorems 1 and 2 combined with Gauss–Lobatto quadrature with m nodes yields

$$\int_{x_a}^{x_b} L(x, y, \dot{y})dx \cong \frac{x_b - x_a}{2} \sum_{k=1}^{m} w_k L(x_k, a^T \phi(\|x_k - x_i\|), a^T \frac{d\phi}{dx}(\|x_k - x_i\|)) \tag{14}$$

where $x_k = \frac{x_b+x_a}{2} + \frac{x_b-x_a}{2} z_k$, $z_k s$ and $w_k s$ are nodes and coefficients of Gauss–Lobatto quadrature and RBF centers x_i, $i = 1, \ldots, N$ are nodes which selected in a specific way.

Finally the variational problem (12) will be reduced to a constrained optimization problem of finding

$$\min J(a) = \frac{x_b - x_a}{2} \sum_{k=1}^{m} w_k L(x_k, a^T \phi(\|x_k - x_i)\|, a^T \frac{d\phi}{dx}(\|x_k - x_i\|)), \tag{15}$$

subject to

$$a^T \phi(\|x_a - x_i\|) = A, a^T \phi(\|x_b - x_i\|) = B. \tag{16}$$

In a constrained optimization problem if the objective function is quasiconvex and the constraint functions are all quasiconvex, then any local optimal solution to the problem is also globally optimal (Hoy et al. 2001).

One can easily investigate if Lagrangian in problem 12 is convex then objective function 15 will be convex function of unknown parameter $a = [a_1, \ldots, a_N]^T$.

To solve the optimization problem (15)-(16) our choice is to use Lagrange multipliers method , so we establish Lagrangian

$$J^*(a) = J(a) + \lambda_1(a^T \phi(\|x_a - x_i\|) - A) + \lambda_2(a^T \phi(\|x_b - x_i\|) - B), \quad (17)$$

then we should solve the following algebraic system in order to calculate coefficients $a^T = [a_0, a_1, \dots, a_N]$

$$\frac{\partial J^*}{\partial a} = 0, \quad (18)$$

$$\frac{\partial J^*}{\partial \lambda_i} = 0 \quad i = 1, 2. \quad (19)$$

5 National Saving Model

Suppose an economy developing over time where $C = C(t)$ denotes consumption, $K = K(t)$ the capital stock, and $Y = Y(t)$ net national product at time t. Assume that

$$Y = f(K),\ where\ f'(K) > 0,\ f''(K) \le 0, \quad (20)$$

so that the net national product is a strictly increasing , concave function of the capital stock alone. The relation

$$C(t) = f(K(t)) - K'(t). \quad (21)$$

is confirmed for each t (Seierstad and Sydseter 1987).

One must somehow find a way to reconsile the conflict between providing for the present and taking care of the future. To this aim, let us assume that the society has a utility function U, where $U(C)$ is the utility (flow) the country enjoys when the total consumption is C , and let us require that

$$U'(C) > 0, U''(C) < 0, \quad (22)$$

so that U is strictly increasing and strictly concave (Seierstad and Sydseter 1987).

If we suppose $f(K) = K$ and $U(C) = \sqrt{C}$ another way of formulating the problem is as follows:

Find the capital function $K = K(t)$, with $K(0) = K_0$ that maximizes

$$max \int_0^T e^{-\rho t} \sqrt{K - K'} dt \quad (23)$$

In order for the problem to have a solution some “terminal condition” on $K(t)$ is necessary, for example $K(T) = K_T$ is given .

If we assume $K(0) = 1$ and $K(T) = 3$ and let the period to be $T = 2.5$ years, using RBF direct method with $c = 1.20$ and $N = 20$, the solution will be obtained as in Fig. (1).

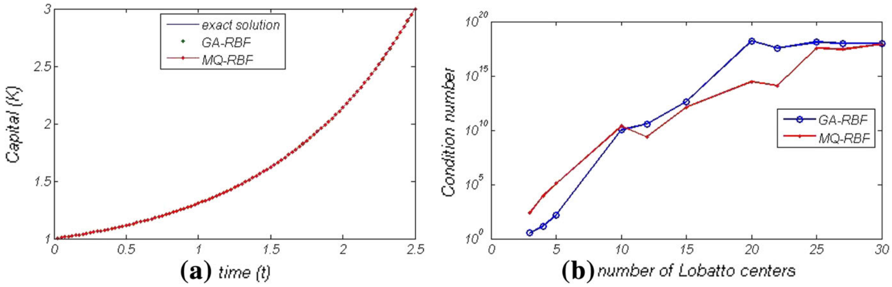

Fig. 1 **a** Capital Stock in time t, **b** Condition number versus Number of Lobatto centers

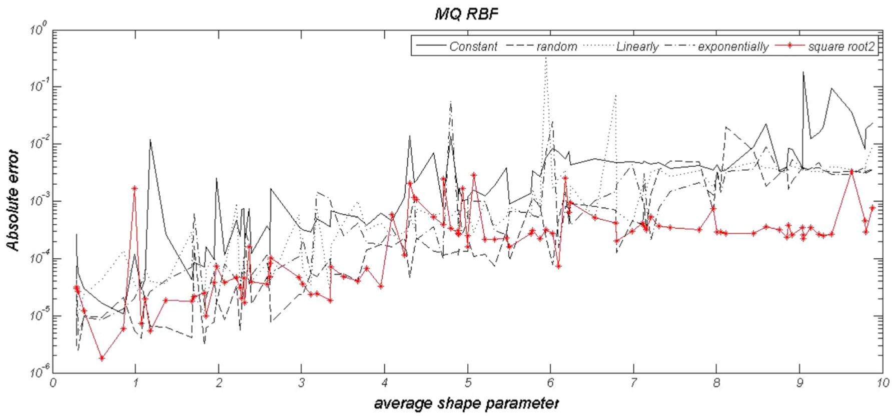

Fig. 2 Absolute error versus average shape parameter for several shape parameter strategies in MQ-RBF

The results are compared over a range of average shape parameter

$$\epsilon_{avg} = \frac{1}{2}(\epsilon_{max} + \epsilon_{min})$$

Figure 2 shows the applicability and efficiency of the RBF method with MQ basis for various Shape parameter strategies. It also shows that choosing Square Root VSP and random VSP result in better approxmiation (Figs. 3, 4).

Figure 3 shows that numerical stability of the Square Root VSP is comparable to other existing VSPs in MQ case. Figure 4 asserts that Gauss-Chebyshev centers act more efficient than other center points in small VSPs, for large VSPs random centers are better.

In order to inquire effiecincy of the proposed method in GA case and to consider the power of proposed VSP compared with other VSPs, the authors conducted some experiments. Results of experiments illustrated good performance of the RBF method (Fig. 5). Decrease in absoulte error of Square root VSP compared with other VSPs (Fig. 5), beside significant decrease in condition number (Fig. 6), which makes sense of more reliable computation, asserts the superiority of the proposed VSP.

Authors designed another numerical experiment to investigate effect of center point choosing on the accuracy of the solution. They considered uniform center points,

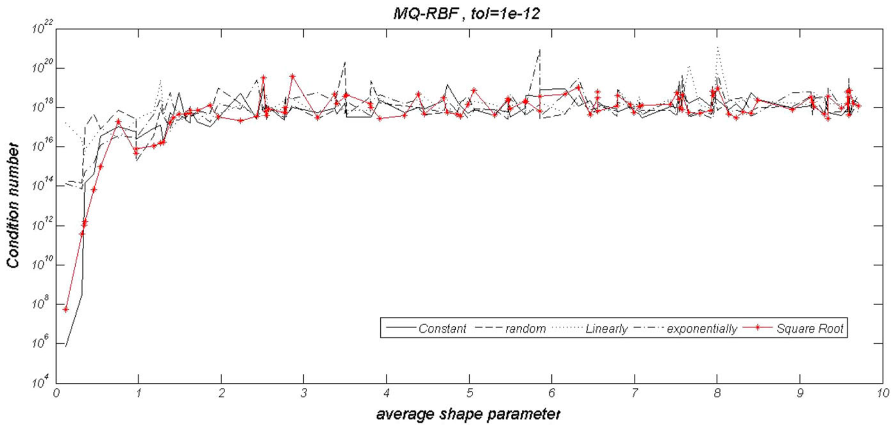

Fig. 3 Condition number versus average shape parameter for several shape parameter strategies in MQ-RBF

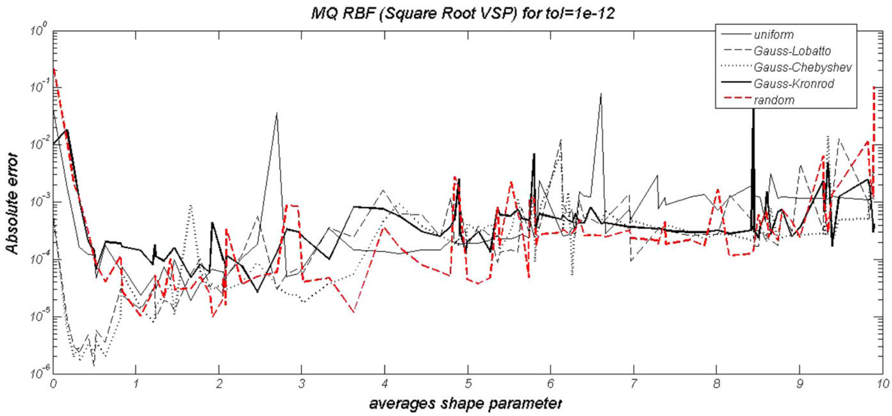

Fig. 4 Absolute error versus average shape parameter for various center point choosing in MQ case

random center points, Gauss–Lobatto , Gauss–Chebyshev and Gauss–Kronrod center points for RBF. The latter are the zeros of the Stieltjes polynomials. We used command *quadl* with *tol=1e-12* in MATLAB to decrease error of integration more and more. The *quadl* function may be more efficient than *quad* at higher accuracies with smooth integrands. In GA case it is observed that Gauss–Chebyshev, Gauss–Kronrod and random center points yilelds in to some extent similar results (Fig. 7).

6 Conclusion

In this paper some formulas for the RBFs operational matrices has been derived. These matrices can be used to solve problems in identification, analysis , optimal control and variational problems. The product of two RBF vectors have quadratic form, hence making RBF method computationally attractive.

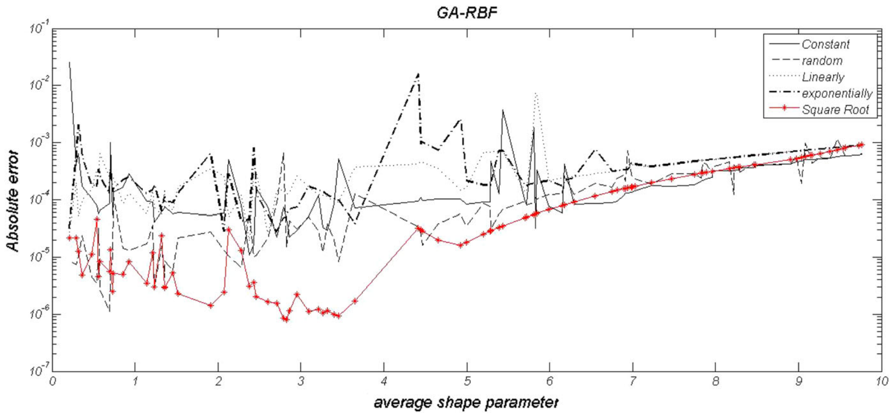

Fig. 5 Absolute error of several shape parameter strategies in GA case

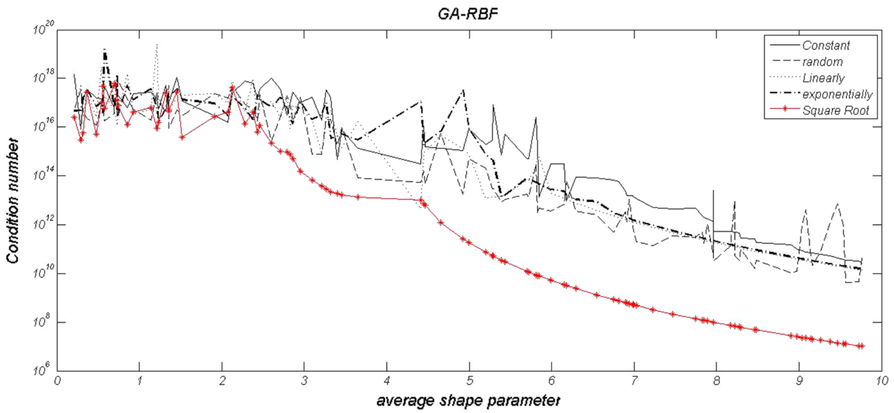

Fig. 6 Condition number vs Average Shape parameter for several Shape parameter strategies in GA case

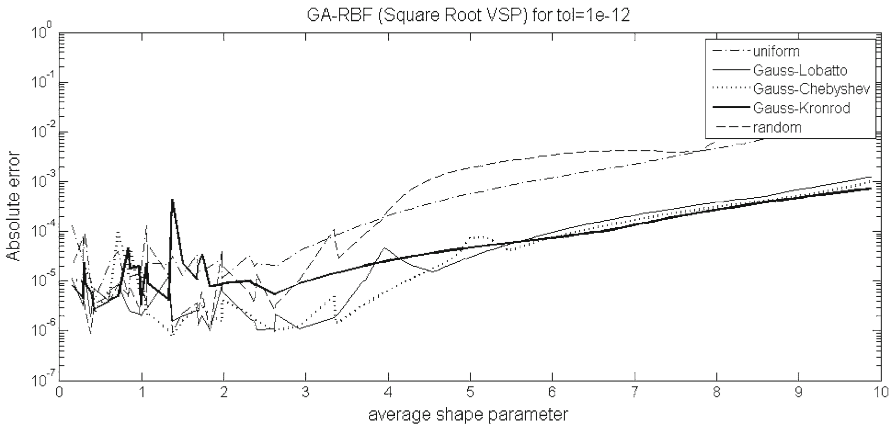

Fig. 7 Absolute error vs. Average shape parameter for various center point choosing in GA case

The RBF operational matrix combined with Gauss–Lobatto quadrature reduces a variational problem into a set of algebraic equations. In this paper RBF method was employed for finding the extremum of a variational problem arising from Dynamic Economic model. The high accuracy of the estimation and validity and applicability of the method was demonstrated by some experiments. Finally Authors proposed a Variable Shape Parameter strategy which successfully improved results of RBF direct method specially in GA case.

References

Cheney, W. (2000). *An introduction to approximation theory* (2nd ed.). New York: AMS Cheslea Publishing: American Mathematical Society.

Clarke, F. (2013). *Functional analysis, calculus of variations and optimal control*. New York: Springer.

Elsgolts, L. (1977). Differentiarl equations and calculus of variations. Moscow: Mir (translated from the Russian by G. Yankovsky).

Fasshuar, G. (2007). *Meshfree approximation methods with MATLAB.* World Scientific publishing.

Franke, R. (1979). A critical comparison of some methods for interpolation of scattered data, Ph.D. Thesis, Naval Posgraduate School Monterey, California.

Franke, C., & Shaback, R. (1998). Solving partial differential equations by collocation using radial basis functions. *Applied Mathematics and Computation*, *93*(1), 73–82.

Gelfand, I. M., & Fomin, S. V. (1963). *Calculus of variations*. Englewood cliffs, NJ: Prentice Hall.

Golbabai, A., Mohebian far, E., & Rabiei, H. (2014). On the role of shape parameter in approximating the eigenvalues of Fredholm integral equations: An RBF-Simpson approach. *Applied Mathematics and Information Sciences*, 1–14.

Golbabai, A., & Rabiei, H. (2012). A meshfree method based on radial basis functions for the eigenvalues of transient Stokes equations. *Engineering Analysis with Boundary Elements*, *36*(11), 1555–1559.

Golbabai, A., & Seifollahi, S. (2006). Numerical solution of the second kind integral equations using radial basis functions networks. *Applied Mathematics and Computation*, *174*, 877–883.

Hardy, R. L. (1971). Multiquadric equations of topology and other irregular surfaces. *Journal of Geophysical Research*, *76*(8), 1905–1915.

Hoy, M., Livernoise, J., Rees, R., & Stengos, T. (2001). *Mathematics for Economics* (2nd ed.). Cambridge, MA: MIT press.

Lin, Y., & Yuan, M. (2006). Convergence rate on compactly supported RBF regularization. *Statistica Sinica*, *16*, 425–439.

Madych, W. M. (1992). Miscellaneous error bounds for multiquadric and related interpolators. *Computers & Mathematics with Applications*, *24*(12), 121–138.

Madych, W. R., & Nelson, N. A. (1992). Bounds on multivariate polynomials and exponential error estimates for multiquadric interpolation. *Journal of Approximation Theory*, *70*(1), 94–114.

Sarra, S. A., & Sturgill, D. (2009). A random variable shape parameter strategy for radial basis function approximation. *Engineering Analysis with Boundary Elements*, *33*, 1239–1245.

Schaback, R. (1995). Error estimates and condition numbers for radial basis function interpolation. *Advances in Computational Mathematics*, *3*(3), 251–264.

Schaback, R. (2005). Multivariate interpolation by polynomials and radial basis functions. *Constructive Approximation*, *21*(3), 293–317.

Seierstad, A., & Sydseter, K. (1987). *Optimal control theory with economic applications*. Amsterdam: North-Holland.

Song, G., Riddle, J., Fasshauer, G., & Fred, J. (2012). Multivariate interpolation with increasingly at radial basis functions of finite smoothness. *Advances in Computational Mathematics, 36*(3), 485–501.

Stoer, J., & Bulirsch, R. (2002). *Introduction to numerical analysis* (3rd ed.). New York: Springer.

Tavvasoli Kajani, M., Hadi Venech, A., & Ghasemi, M. (2009). The Chebyshev wavelets operational matrix of integration and product operation matrix. *International Journal of systems sciences*, *86*, 1118–1125.

Wendland, H. (2001). *Trends in approximate theory*. Nashville: Vanderbilt University Press.

Yousefi, S. A., & Behroozifar, M. (2010). Operational matrices of Bernstein polynomials and their applications. *International Journal of systems sciences*, *41*(6), 709–716.