Robust Stability Analysis Inspired by Classical Statistical Principles

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An analysis was made of robust stability by estimating the robust confidence region for the roots of the characteristic Equation of a process, which uses a typical PID controller with auto-tuning. From standard statistical concepts, considerations on the statistical distance associated with the chi-square distribution were taken into account when establishing the methodology, which enabled the region of robust stability to be found. The results show the robust stability region for all eigenvalues of characteristic equation, thus demonstrating the ability of the procedure. The region of stability can be updated online, thus enabling the stability of the system to be continuously evaluated, with little computational effort.

Keywords: controller, stability, robust confidence region, characteristic equation, root-locus analysis

INTRODUCTION

In the modern age, stability analysis has received particular attention from engineers and practitioners involved with process control, due basically to the effects of uncertainties present in systems, which can play a pivotal role for control systems.

Despite the high volume of research in the area, uncertainty remains a challenging problem for control engineers, who try to maintain the three basic foundations that determine the performance of a control system, namely: observability, controllability and stability.[1] Since the robustness of control systems can be viewed as the ability of the control structure to deal with all uncertainties present in the process, so that its performance remains satisfactory, it is evident that a statistical description of the approach is necessary. However, unsatisfactory attention has been given to the robust methods associated with a statistical approach which seek to limit the uncertainties in a region rather than represent them in the form of a probability measure, yet it is recognised that the results obtained with such an approach have been rather complex and therefore, in practice, they still leave much to be desired.[2,3] Most of them have received a deterministic treatment or they are concerned with probability measurements associated with well-known optimisation methods,[4] which involve strenuous mathematical efforts, making the analysis more complex and less attractive to practitioners.

There is a wide range of classical and modern methods that tackle the uncertainties of a model, and therefore, robustness, such as root locus and singular value analysis. Such methods are dealt deterministically without considering the boundaries of the process parameters, which results in a weak relationship of the metric to uncertainties. It should be emphasised that deterministic metrics can be substantially conservative. Structured singular value analysis[5] can, moreover, reduce conservatism to some degree, although such treatments continue to be deterministic. Furthermore, in most cases, these methods are difficult to understand and can be tedious to implement.

The method presented in this paper for analysing robust stability differs from others mainly due to its philosophy, which makes use of the universe of possible results of the process, describing its behaviour, including uncertainties and disturbances. Evidently, such a universe is open but the statistical methodology enables it to be rebuilt and closed with the degree of confidence required. Thus, the joint approach of such a universe applied to the roots of the characteristic equation together with the classical region of stability of the complex plane can provide the robustness of stability analysis. It should also be pointed out that the methodology has a simple form based on classical statistical concepts which are easy to understand and it focuses on establishing the confidence region for each root of the characteristic equation of the process under consideration. The technique takes into account the relationship between the Euclidian metric and the so-called statistical distance by virtue of using sample variances and covariances. What is meaningful is the relationship of these considerations with the chi-squared distribution. Hence, when applied to the real and imaginary parts of complex numbers such an approach makes it possible to determine the quadratic forms and compute the contours of the resulting ellipse, thus revealing the desired robustness.

THEORETICAL FORMULATION

Since the theoretical formulation is inspired by statistical concepts, it is very important to state briefly the statistical principles or philosophy that lies behind the methodology, that is: establishing the universe of all the possible results for the system considered. Although it is impractical to collect all the values of such a universe, the statistical procedures enable it to be rebuilt, and this is highly significant. It is also evident that it is impossible to work with an open universe. Therefore it is necessary to close it, which results in the so-called level of significance (α). Having set out this preliminary understanding, other essential considerations follow.

The index of central tendency of order '$\boldsymbol{n}$' can be defined as the value C_n that minimises the following equation:

$$D^n = \sum_i |x_i - C_n|^n f_i \tag{1}$$

for discrete variables, where x_i is a random variable.

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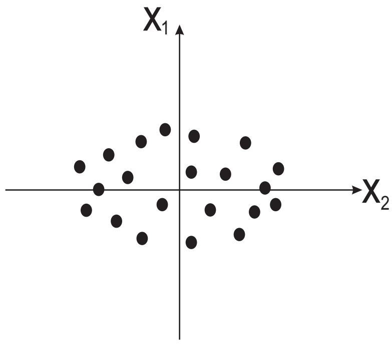

Figure 1. The generic scatter plot of variability.

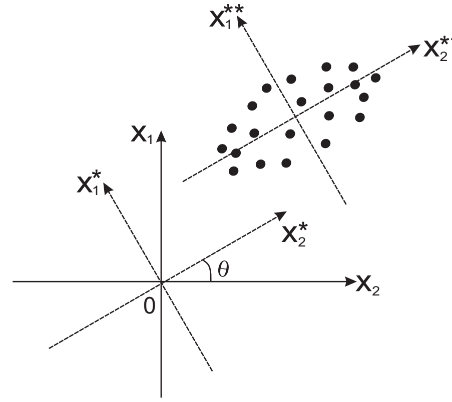

Figure 2. A cluster of points representing an ellipse.

It is straightforward to show that for $n = 2$, C_2 is the arithmetic average that minimises D^2, thus revealing how meaningful the choice of such a mean is for the performance of the second central moment. It should also be emphasised that the arithmetic average is an unbiased estimator of the expectation of a random variable.

Consider any point, P, of a plane or, in particular, of a complex plane with its real (x_1) and imaginary (x_2) coordinates. The Euclidian distance from P to origin O, given by:

$$d(O,P) = \sqrt{(x_1)^2 + (x_2)^2} \tag{2}$$

is seen to be inadequate for most statistical purposes, because, in most cases, the dispersions of the axial coordinates, representing variability, are not equal, as shown in Figure 1.[6]

Taking such dispersion into account, it is easy to derive, by using standardised coordinates, the so-called statistical distance as follows:

$$d(O,P) = \sqrt{\left(\frac{x_1}{\sigma_{x_1}}\right)^2 + \left(\frac{x_2}{\sigma_{x_2}}\right)^2} \tag{3}$$

or more generally in vector notation, thus

$$d^2(O,P) = x' \sum{}^{-1} x \tag{4}$$

where σ_{x_i} denotes the variance of components of the random vector x, $x' = [x_1, x_2]$ and

$$\sum = \begin{bmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{bmatrix} \tag{5}$$

Clearly, Equation (3) is a quadratic form, which represents an ellipse centred at the origin. It should also be observed that the Euclidian distance is a particular case of statistical distance when $\sigma_x = \sigma_y$.

In the case of an Ellipse, it is centred at the expected value of the random variable X, a point different from the origin, and also considering Σ as the variance–covariance matrix, Equation (4) can be generalised to yield:

$$d^2(O,P) = (X - \boldsymbol{\mu}_X)' \sum{}^{-1} (X - \mu_X) \tag{6}$$

For a two-dimensional variable X, Equation (6) describes the contours of a ellipse centred at $\mu(\mu_{x_1}, \mu_{x_2})$, the axes of which are not, necessarily, in the same direction as the coordinate plane.

The graph of Equation (6) for the plane can be illustrated in Figure 2.

If X_1 and X_2 are correlated which corresponds to a full variance–covariance Σ then by using combined movements of axial translations and rotations resulting from a linear transformation, it is possible to suppress the cross-covariance term involved in Equation (6). The linear transformation to be used in the translation can be expressed by:

$$(\boldsymbol{x} - \bar{\boldsymbol{x}}) = \boldsymbol{P}(\boldsymbol{x}^* - \bar{\boldsymbol{x}}^*) \tag{7}$$

where $\boldsymbol{P}$ is an orthogonal matrix given by:

$$\boldsymbol{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{8}$$

that is, the axes (x_1^*, x_2^*) are rotated by some angle θ from (x_1, x_2). It must be observed that the value of determinant $\boldsymbol{P}$ (det $\boldsymbol{P}$) is equal to 1, which ensures the movement of rotation.

Substituting Equation (7) into Equation (6), results in:

$$d^2(\mu, P) = [\boldsymbol{P}(\boldsymbol{x}^* - \boldsymbol{\mu}_{x^*})' \sum{}^{-1} [\boldsymbol{P}(\boldsymbol{x}^* - \boldsymbol{\mu}_{x^*})]$$

or

$$d^2(\mu, P) = (\boldsymbol{x}^* - \boldsymbol{\mu}_{x^*})' \boldsymbol{P}' \sum{}^{-1} \boldsymbol{P}(\boldsymbol{x}^* - \boldsymbol{\mu}_{x^*}) \tag{9}$$

It is easy to show that all (2×2) symmetric matrices can be diagonalised. Then, once the matrix Σ^{-1} is symmetrical and $\boldsymbol{P}$ is an orthogonal matrix, such a matrix can diagonalise Σ^{-1}, resulting in a diagonal matrix **D**:

$$D^{-1} = \boldsymbol{P}' \sum{}^{-1} \boldsymbol{P} \tag{10}$$

By replacing Equation (10) into Equation (9), the following equation can be found:

$$d^2(\mu_{X*}, P) = (\boldsymbol{x}^* - \boldsymbol{\mu}_{x^*})' \boldsymbol{D}^{-1} (\boldsymbol{x}^* - \boldsymbol{\mu}_{x^*}) \tag{11}$$

It should be emphasised that Equation (11) is the quadratic form of an ellipse centred at the expected value of the random variable X^* without the cross-covariance term.

Explicitly, Equation (11) can be rewritten for two dimensions as:

$$d^2(\mu_{X*}, P) = \frac{(x_1^* - \mu_{x_1^*})^2}{\sigma_{x_1^*}^2} + \frac{(x_2^* - \mu_{x_2^*})^2}{\sigma_{x_2^*}^2} \tag{12}$$

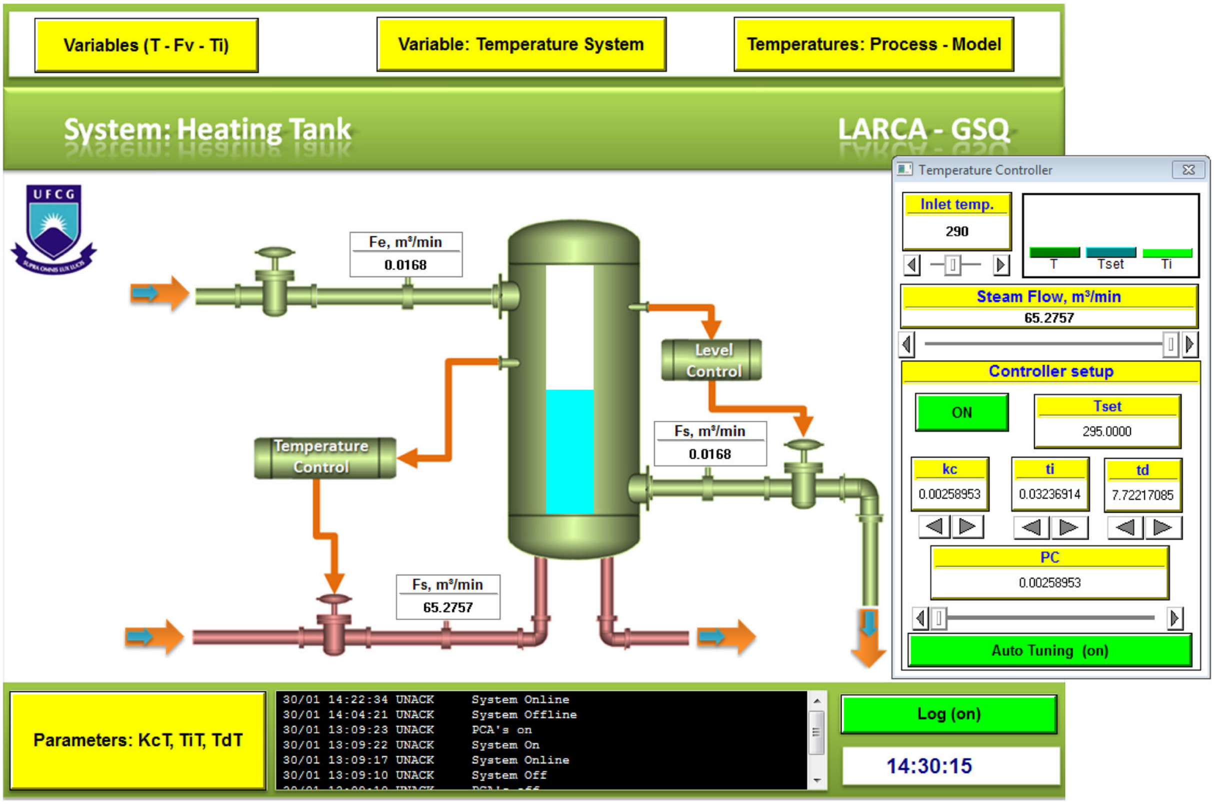

Figure 3. System under study.

Considering the definition of the random variable χ^2 (chi-square) given below:

$$\chi^2 = Z_1^2 + Z_2^2 + \cdots + Z_v^2 = \sum_{i=1}^{v} Z_i^1 = \sum_{i=1}^{v} \left(\frac{x_i - \mu_i}{\sigma_i} \right)^2 \tag{13}$$

where v denotes the degrees of freedom, and, therefore, all that needs to be noted here is that Equation (12) has a χ^2 distribution, that is it is equal to $\chi^2$with v degrees of freedom. It is important to observe the difference $(x_i - \mu_i)$ related to the mean as per the statement at the beginning of this section. Hence:

$$d^2(\mu_{X^*}, P) = \frac{(x_1^* - \mu_{x_1^*})^2}{\sigma_{x_1^*}^2} + \frac{(x_2^* - \mu_{x_2^*})^2}{\sigma_{x_2^*}^2} \cong \chi_2^2(\alpha) \tag{14}$$

that is the indicative of $d^2(\mu_{X^*}, P) = \chi^2$ enables the contour of an ellipse to be determined that contains $(1-\alpha)100\%$ of the probability, for which, α corresponds to the level of significance and $v = 2$.

Since x_1^* and x_2^* can describe the rectangular coordinates of the points representing the roots of the characteristic equation of a second-order system, which give us the first statement about the stability of the system, then the depicted ellipse is related to the robustness of the stability.

THE SYSTEM

As presented in Figure 3, a jacketed vessel heater connected to a control structure including a supervisory system was considered for analysis.

Figure 4 illustrates how information flows, and also gives the block diagram that represents the identification procedure and at what point the tuning parameters are set. There is a list of symbols at the end of this paper in which the input and output signals and intermediaries, as well as the control parameters, are all clearly defined.

In order to obtain the best set of conditions for monitoring and control to be used in industrial plants, a supervisory system was built in the language of object-oriented programming, as shown previously in Figure 3.

This enabled on-line information on the process variables and parameters to be extracted. Such information is needed to set up self-tuning control and is based on the work of Aström and Hagglund.[7] To simulate the operating conditions of an industrial plant, disturbances were introduced into the temperature of the inlet stream and into the stream itself.

THE CONTROL STRUCTURE

Building the Model

In order to provide a more realistic scenario, yet one in which the model differs in some way from the real process, this section describes the model and the process used for simulation purposes.

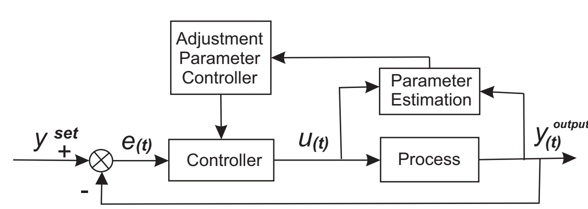

Figure 4. Diagram of a generalised closed-loop structure with identification block.

The dynamic modelling for the heating tank was derived from the conventional mass and energy balances, which results in the dynamic behaviour of the liquid level and the temperature of the system under study. From this point on, such a strategy will represent, in fact, the process.

The discrete-time model to be used in the following procedures corresponds to a combination of the convolution model and the autoregressive model with exogenous inputs, which can be expressed by:

$$y_{(t)} + a_1 y_{(t-1)} + a_2 y_{(t-2)} + \cdots + a_n y_{(t-n)} = b_1 u_{(t-1)} + b_2 u_{(t-2)} + \cdots + b_m u_{(t-m)} + v_{(t)} \tag{15}$$

where a_i and b_i are the coefficients obtained by regression, and $v_{(t)}$ denotes the combined effects of noise measurements, unmeasured disturbances and modelling errors. Since the order of the regressive model is described by n and m, the system may achieve better adjustment, relative to the mismatch between modelling and process, if large values are assumed for such parameters. However, higher-order models can present some difficulties, namely: the distinction between the poles which correspond to structural modes and spurious poles, the computational efforts and memory requirements. Therefore, models of a lower order are always desired. How to estimate the order of such a model can be found in Moore et al.[8]

Taking the backward shift operator into account and expressing this in a vectorial structure, the result is:

$$y_{(t)} = \boldsymbol{\theta}^{\mathrm{T}} \boldsymbol{\varphi}_{(t)} + v_{(t)} \tag{16}$$

where $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$ are the matrices of the coefficients and of the variables, respectively.

Given that the classical recursive least squares method (RLSM)[9] was used, the model parameters can be obtained by:

$$\hat{\boldsymbol{\theta}}_{(N)} = \left[\sum_{t=1}^{N} \varphi_{(t)} \varphi_{(t)}^{\mathrm{T}} \right]^{-1} \sum_{t=1}^{N} \varphi_{(t)} y_{(t)} \tag{17}$$

Since the difference between the process and model was minimised, then, the parameters obtained result in the best predictions for the output variable in the sense of minimum variance.

The Stability Analysis

A variant of the diagram shown in Figure 4 is representative of the control strategy to be adopted in this study, as shown in Figure 5, that is:

The simplified closed loop response for the process considered can be given by Equation (18):

$$\bar{y}(s) = \frac{G_P G_C}{1 + G_P G_C} \bar{y}_{sp}(s) + \frac{G_d}{1 + G_P G_C} \bar{d}(s) \tag{18}$$

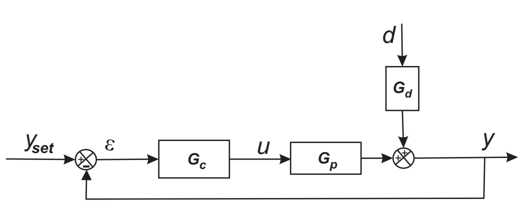

Figure 5. Basic block diagram for a generic closed loop system.

The classical stability criterion for a closed-loop system states that the system is considered stable if all the roots of its characteristic equation are located on the left of the imaginary axis. Therefore, the roots, which are the solutions of:

$$1 + G_P G_C = 0 \tag{19}$$

should obey such requirements.

Without losing generality, the equation above can be given by:

$$\tau^2 s^2 + 2\xi\tau s + 1 = 0 \tag{20}$$

Hence, the roots are performed by:

$$s = -\frac{\xi}{\tau} \pm j\frac{\sqrt{1-\xi^2}}{\tau} \tag{21}$$

Since the parameters ξ and τ can be expressed as a function of K_c, τ_I and τ_D, the tuning parameters of a PID controller, which are, by nature, random variables, then, ξ and τ are also stochastic variables, a basic characteristic of which is that a probability distribution is associated with each of them. Note that both real and imaginary parts of each complex root are, by consequence, stochastic, which if depicted in the complex plan, result in points dispersed around a mean. Thus, by considering the statistical distance previously approached as a metric, the contours of the solid ellipsoid that limit the region of the robustness of stability with a particular level of significance can be mapped. Although the probabilistic model is interesting, it should be observed that the robust control method looks much more for the limits of uncertainty than it does for its probability distribution.

Tuning Procedure

Note that classical auto-tuning is always implemented with the relay feedback connected to the process, which can give rise to some disadvantages because the process is uncontrolled during the tuning time. To overcome this difficulty, a slight modification was made to the strategy, as shown in Figure 6.

Two basic differences between classical auto-tuning and the strategy now developed can be noted, namely: classical auto-tuning is based on frequency response methods, while the approach proposed operates in the time domain besides making effective use of a model for the process. In this methodology, the convolution model presented in Building the Model Section was identified and used as the basic component in auto-tuning the relay. It can also be verified that the auto-tuning strategy may be run with the ongoing process and control system.

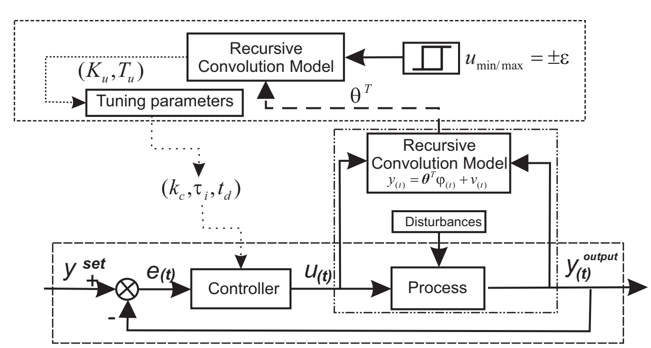

Figure 6. Block diagram of the generalised control structure.

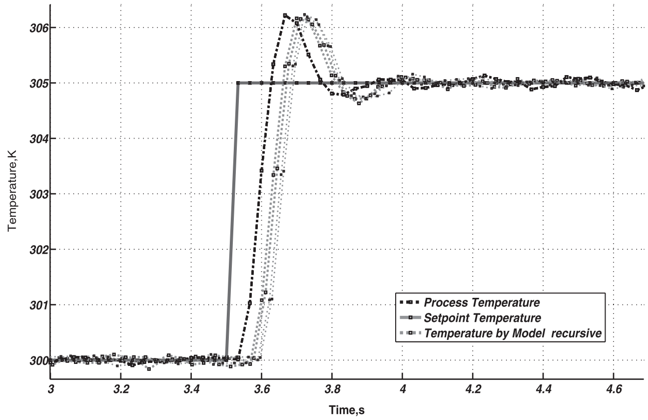

Figure 7. The behaviour of the model for n and m higher than 5.

The implementation of the auto-tuning procedure was carried out by means of generating a stimulus introduced in the relay $(\bar{u})$, which fluctuates between $\pm\varepsilon$, chosen suitably, for generating a controlled oscillation in the output variable of the model, $y^m_{(t)}$, with constant amplitude. By doing so, the ultimate gain and ultimate period can be determined in each sampling instance and the parameters of the controller are estimated using the classical tuning procedure.

By using the conventional rules reported by Aström and Hagglund,[7] the tuning parameters for the PID controller can be established. However, strategies more sophisticated for controller tuning can be implemented without conceptual changes in the methodology for establishing the robustness region.

RESULTS AND DISCUSSION

Despite the method of fitting being very good, the model to be used as expressed by Equation (15) depends on the values of n and m in order to choose the appropriate functional form, taking into account the one that best describes the process. After conducting a few simulations, such values can be found with minimum computational effort, as illustrated in Figure 7.

It can be verified that for high values, the functional form representing the model becomes sufficiently close to the process. In the case of $n = m = 100$, the result proves to be satisfactory. For convenience, it is assumed n and m have the same value. For the ongoing analysis and since the aim of the study is not to develop an auto-tuning method, it should be stressed that over-specifying the model order can be reasonable and of interest, given that the mismatch between the model and the process can be thus minimised, and thereby the additional interference of the model in such a procedure can be reduced. In practice, the values obtained for the analysis of interest are provided by the system, there being a need for an auto-tuning system like the one introduced in this paper.

Since the model parameters were calculated according to Equation (17), then the system can be run in order to verify the dynamic behaviour of the output variable, and compared to that of the process when it is submitted to a disturbance of 10% in the value of set point.

With the aim of adjusting the parameters of the PID controller on-line, a configuration for the control system was established, consisting of a relay, a convolution model and an algorithm for tuning parameters, as per Figure 6. The stimulus generated by the relay is based on adequate amplitude and is stabilised, in the form of a square wave of $\pm\varepsilon$, resulting in an output sinusoidal wave with constant amplitude. The relationship between the input waves and output present in Figure 8 enables the final period and ultimate gain to be determined, as per the following Equation (22):

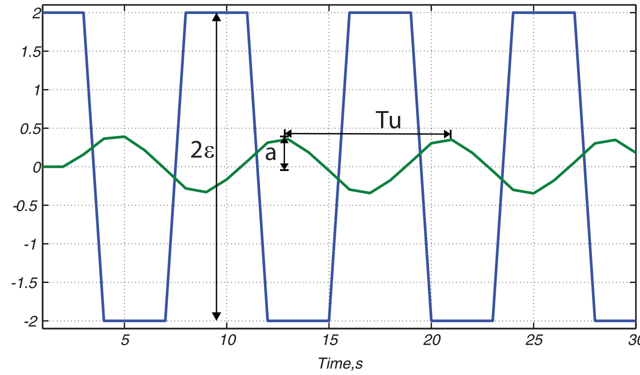

Figure 8. Stimulus generated by the relay and response of the output variable.

$$\mathrm{Ku} = \frac{4\varepsilon}{\pi a} \tag{22}$$

Since the system was running, the PID controller was automatically tuned, the tuning rules being used in accordance with Aström and Hagglund,[7] based on Ziegler–Nichols frequency response method gains:

Controller	$\rightarrow$	K_c	T_i	T_d
PID	$\rightarrow$	0.6 Ku	0.5 Tu	0.125 Tu

where Ku is given by Equation (22) and Tu the period of oscillation of the cycle, as shown in Figure 8.

Thus, a set of parameters in each sampling period was found which resulted in the closed loop response with the auto-tuning as shown in Figure 9.2. Figure 9.1 shows the corresponding close loop response without the auto-tuning procedure. The results indicate a very good agreement between the process and the model for both devices in spite of the presence of large disturbances. Note also that the system operating with the auto-tuning yields a superior performance.

Besides the well-known results that the performance can be improved with the auto-tuning scheme as shown above, it should be emphasised that implementing such a scheme has enabled the tuning parameters to be considered and dealt with as random variables. It is significant also to observe that the parameters of the model have also been considered as random variables.

With the auto-tuning device implemented on-line and having in mind the region of the robustness of stability to be drawn while taking into account the tuning parameters, the values of such parameters were recorded and distributed in accordance with a probability function, as shown in Figure 10.

It can be also verified that the tuning parameters follow a probability distribution which can be considered at least approximately normal $N(\mu, \sigma^2)$ (Figure 11).

Having in mind the use of classical PID controller in the process in study, the parameters ξ and τ from Equation (20) can

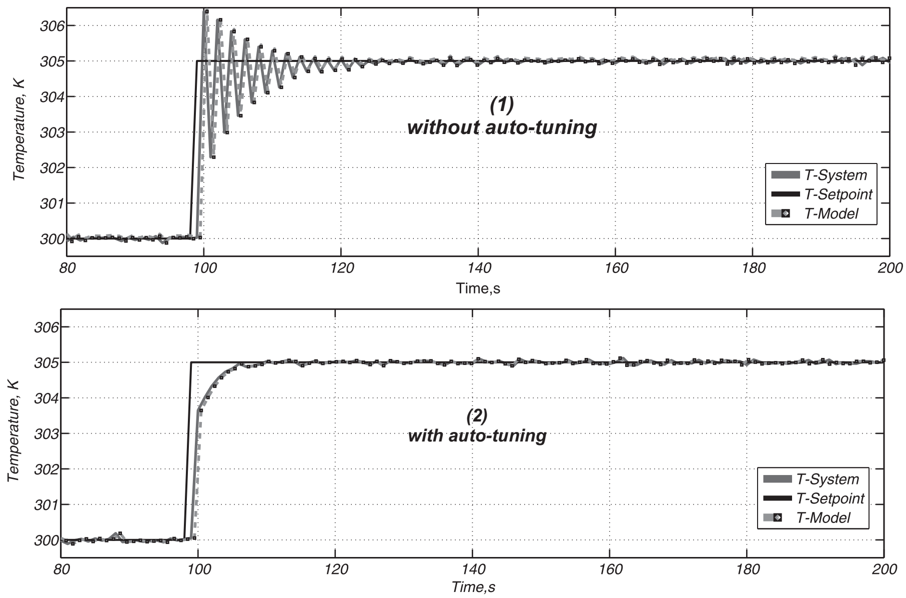

Figure 9. The behaviour of the output temperature for the model and process.

be given by:

$$\xi = \frac{1}{2}\left(\sqrt{\left(\frac{\tau_i}{k_p k_c} + \tau_i \tau_d\right)}\right)^{-1}\left(\frac{\tau_i}{k_p k_c} + \tau_i\right) \tag{23}$$

and

$$\tau = \sqrt{\left(\frac{\tau_i}{k_p k_c} + \tau_i \tau_d\right)} \tag{24}$$

On introducing all values for k_c, τ_L and τ_D in the equations previously mentioned, then the set of values for the real and imaginary parts of the complex number **s** (Equation 21) given by ξ/τ and $(\sqrt{1-\xi^2})/\tau$ can be expressed by the following probability distributions which can also be considered as being, at least, approximately normal.

Hence, drawing all the pairs of points that consist of the real and imaginary parts on the complex plane for both roots results in the diagram shown in Figure 12.

From Figure 12 and taking into account the axes, major and minor, of the mentioned ellipses, which are centred at the mean, it is easy to verify that the variability at each direction is not equal.

Thus, the statistical distance of the points P from mean, $d(\mu_{X^*}, P)$ or $d(\mu_{X^*}, P)^2$ representing the contours of the ellipse can be computed by using Equation (14), for the level of significance α to be established, as shown in Figure 13.

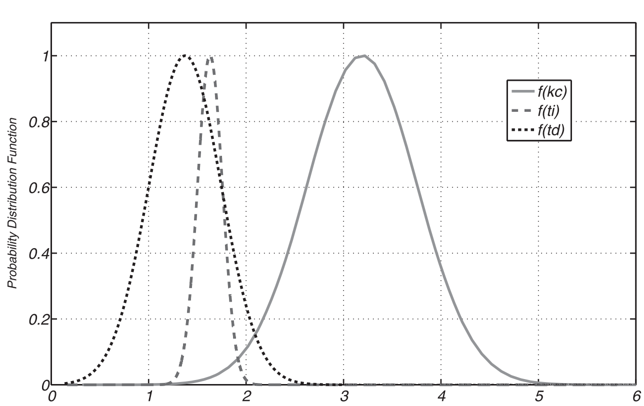

Figure 10. The probability density function for the tuning parameters of the PID controller.

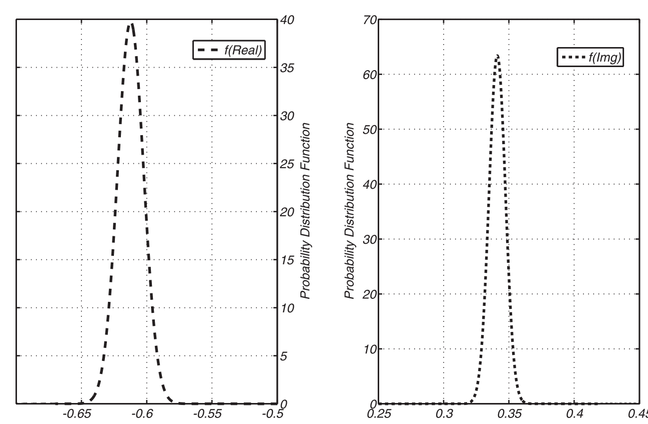

Figure 11. The probability distribution for the real and imaginary parts of the *s*.

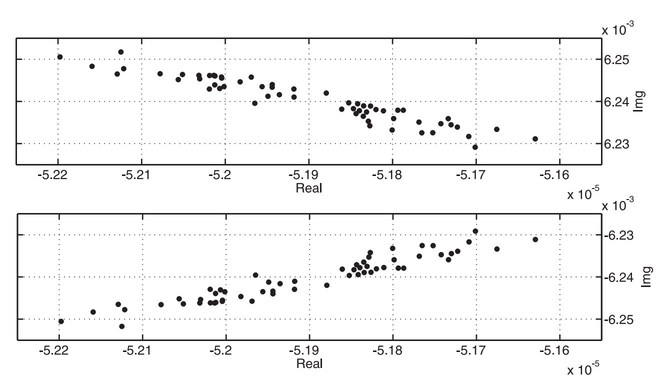

Figure 12. Representation of the roots of **s** in the complex plane.

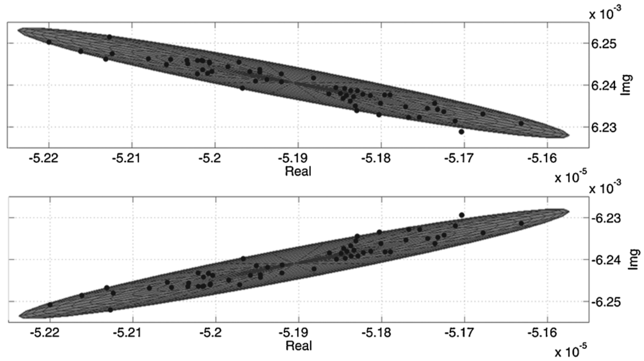

Figure 13. The robustness region for the other root of s also based on $\alpha = 5\%$ while the system is submitted to disturbances present in the inlet variables and set point.

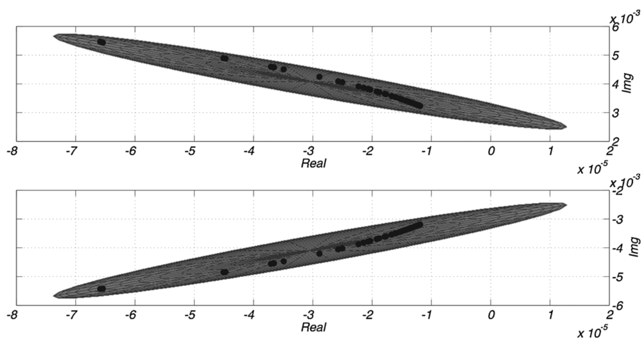

Figure 14. The robustness region for the roots of s based on $\alpha = 5\%$, when the control system is inoperative.

The figures above show the contours of the ellipses which correspond to the robustness confidence regions of stability for the system considered, assuming a level of significance α equal to 5%, when the system is submitted to disturbances in the inlet variables and in the set point. One has to consider also that such a process shows itself by nature, as being dynamic and thus the contours of the ellipses can change when new values of the random variable in the system are captured and incorporated into it. That is, the contour can be continuously evaluated and plotted on-line.

It should also be emphasised that all data obtained have been captured when the control system is operating with the auto-tuning. On the other hand, whenever for some motive, the control system for the process considered is inoperative, then, the process can be directed toward the unstable region, as shown in Figure 14.

It can be easily observed from Figure 14 that the robust confidence region is partially inside the unstable region for both roots, thus indicating a potentially unsustainable operating condition. Therefore, immediate action must be taken with a view to restoring the conditions of stability of the system. Furthermore, Figure 14 shows that the procedure developed is able to collect information and record it, thus allowing a very fast, on-line stability analysis of the operating conditions.

Due to the computational calculus used in the procedure for identification, auto-tuning and to perform and draw the regions of robustness, it should be pointed out that the processing time or CPU time spent on processing all the instructions of the program for identification system, plus auto-tuning and the program for establishing the regions of robustness corresponds to 0.61 s, short, when considered the process engineering time scale, consequently demand little computational effort. Such a time was obtained using a 2.2 GHz processor. If a high performance processor is used, this time can be substantially reduced.

CONCLUSIONS

Due to the need to capture the data from simulations, a structure consisting of a block for estimating recursively the parameters and automatic tuning connected to the process was developed, in which the parameters of the process model are updated on-line. This enabled the tuning parameters and those of the model to be considered as random variables with a view to statistical treatment, besides generating information on-line that will be used in determining the robust confidence region.

With the aim of providing a more realistic scenario for the purposes of simulation, a convolution model was used and fitted while the process corresponding to a stirred tank heater was modelled based on the first principle.

Although not the primary goal, a strategy for auto-tuning was also proposed in a way that allows a continuous operation of the process always to be governed by the control system. The results of the simulation, shown in Figure 9, indicate that the performance was satisfactory for the purposes of this paper.

Since robust stability theory can be viewed as a method that seeks to establish boundaries on the variables of interest rather than to express them in a form of probability distribution, then based on statistical principles that take the concept of statistical distance together with the chi-square distribution into account, a methodology was developed, which aimed to establish the robust confidence region for the stability, which corresponds to mapping the contours of the resulting ellipse associated with the roots of the characteristic equation. Such a procedure takes into account the level of significance α, to be appropriately chosen.

As to the system under study considering the feedback control configuration in which the auto tune PID controller is used, the results shown in Figures 13 and 14 illustrate the application of the procedure for $\alpha = 5\%$, a typical value which is acceptable in practice, thus presenting the region for the robust stability, as well as showing how dynamic such an analysis can be, besides the dynamic response presented by the procedure due to the control system being inoperative.

By examining the set of results the conclusion may be drawn that the procedure can be easily updated and implemented online, thus enabling the stability conditions to be analysed in real time, and therefore decisions on how to restore the stability of the system can be made quickly. A procedure like this can play a pivotal role in those processes in which the variables can change in space, as well in a short period of time, such as those for aircraft control systems.

Finally, based on basic statistical concepts, the procedure corresponds to the original purpose of the methodology which was that of being simple and not requiring a hard mathematical treatment, yet innovative from the standpoint of its theoretical application and having a practical sense. Such a procedure shows itself to be one that practitioners find easy to understand and deal with and one which can be implemented at low computational cost.

In future, a joint analysis of the performance and stability robustness should be considered, inspired by statistical principles.

NOMENCLATURE

$\boldsymbol{a}_n$, $\boldsymbol{b}_m$	coefficients of the dynamics recursive model
$\boldsymbol{b}_i$	coefficients' estimate of the regression coefficient of the model

$\boldsymbol{e}_{(t)}$	deviation variable
χ^2	chi-square distribution
ξ, τ	characteristic parameters
$\boldsymbol{G}_p$, $\boldsymbol{G}_c$	process and controller transfer functions
k_c, τ_I, τ_D	tuning parameters
$\sigma^2_{x_i}$	variance i variable
$\boldsymbol{\varepsilon}$	relay range stimulus
$\boldsymbol{u}_{(t)}$	process input variable
$\boldsymbol{u}^{max}_{(t)}$, $\boldsymbol{u}^{min}_{(t)}$	maximum and minimum value of variable input delay
$\boldsymbol{x}_i$, μ_{x_i}	deviation variable and average deviation of the variable
$\boldsymbol{y}_{(t)}$, $\boldsymbol{y}^m_{(t)}$	output variable of the process and of the model
$\hat{y}_i$	estimated process output with parameters $\hat{\beta}_j$ of the model
$\boldsymbol{\tau}_i$, $\boldsymbol{\tau}_d$	integral and derivative time parameter
PID	proportional integral and derivative

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