Ramanujan J
DOI 10.1007/s11139-015-9713-9

A positive proportion of some quadratic number fields with infinite Hilbert 2-class field tower

A. Mouhib[1]

Received: 14 December 2014 / Accepted: 8 June 2015

Abstract We show that a positive proportion of real and imaginary quadratic number fields with 2-class rank equal to 2 have 4-rank equal to 1 or 2 and infinite Hilbert 2-class field tower.

Keywords Class group · Class field tower · Quadratic number fields

Mathematics Subject Classification 11R29 · 11R32 · 11R37

1 Introduction

Let k be a number field, and let C_k be the class group of k. Let k^1 be the Hilbert 2-class field of k, i.e., the maximal unramified (including the infinite primes) abelian field extension of k whose degree over k is a power of 2. Let k^n for n a nonnegative integer, be defined inductively as $k^0 = k$ and $k^{n+1} = (k^n)^1$; then

$$k \subset k^1 \subset k^2 \subset \cdots \subset k^n \subset \cdots$$

is called the Hilbert 2-class field tower of k. If n is the minimal integer such that $k^n = k^{n+1}$, then n is called the length of the tower. If no such n exists, then the tower is said to be of infinite length.

✉ A. Mouhib
mouhibali@yahoo.fr

1 Université Mohammed Ben Abdellah, Fes LSI, Sciences and Engineering Laboratory, Polydisciplinary Faculty of Taza, B.P 1223 Taza-Gare, Morocco

Published online: 08 September 2015

We define the 2-rank of C_k, denoted by $r_2(k)$, as the dimension of the elementary abelian 2-group C_k/C_k^2 viewed as a vector space over $\mathbb{F}_2$:

$$r_2(k) = \dim_{\mathbb{F}_2}(C_k/C_k^2),$$

where $\mathbb{F}_2$ is the finite field with two elements. We define the 4-rank of C_k, denoted $r_4(k)$ by

$$r_4(k) = \dim_{\mathbb{F}_2}(C_k^2/C_k^4).$$

Assume that k is an imaginary quadratic number field. It is known that the Hilbert 2-class field tower of k is infinite if $r_2(k) \geq 5$ [1,5]. Also it is known that if $r_4(k) \geq 3$, then k has infinite Hilbert 2-class field tower [6,7]. In the case where $r_2(k) = 2$ or 3, the Hilbert 2-class field tower of k may be finite [9,13]. In the case where $r_2(k) = 4$, it has been conjectured that k has infinite Hilbert 2-class field tower [13]. This conjecture has been proved in the case where the discriminant of k is divisible by at most one prime $\equiv -1 \mod 4$ [14]. Note that for each $i \in \{0, 1, 2, 3\}$, there are infinitely many fields k such that

$$r_2(k) = 3, \;\; r_4(k) = i$$

and infinite Hilbert 2-class field tower [2,3]. On other hand, it is known by group theory that if $r_2(k) = 2$ and $r_4(k) = 0$, then k has finite Hilbert 2-class field tower of length at most 2 [9]. Moreover, B. Schmithals showed that the quadratic number field $k = \mathbf{Q}(\sqrt{-25355})$ with $r_2(k) = 2$ has infinite Hilbert 2-class field tower [17] and it is shown that there exist infinitely many imaginary quadratic number fields k such that

$$r_2(k) = 2, \;\; r_4(k) = 1$$

and infinite Hilbert 2-class field tower [14]. However, the case $r_2(k) = 2$ and $r_4(k) = 2$ is not treated any where.

Assume that k is a real quadratic number field. It is well known that if $r_2(k) \geq 6$, then the Hilbert 2-class field tower of k is infinite [5]. If $r_4(k) \geq 4$, then k has infinite Hilbert 2-class field tower [12]. Also a positive proportion of the fields k with $r_2(k) = 5$ for $0 \leq r_4(k) \leq 5$, $r_2(k) = 4$, for $0 \leq r_4(k) \leq 4$ and $r_4 = 3$ for $0 \leq r_4(k) \leq 3$ have infinite Hilbert 2-class field tower [2,4]. Moreover, there are infinitely many real quadratic number fields k such that

$$r_2(k) = 3, \;\; r_4(k) = 1$$

and finite Hilbert 2-class field tower [15]. However, it is not known that there is a positive proportion of real quadratic number fields k such that $r_2(k) = 2$ and $r_4(k) \in \{1, 2\}$ with infinite Hilbert 2-class field tower.

The aim of this article is to study the cases which are not treated in the literature, and then we prove the following theorem.

Theorem 1.1 *For each $i \in \{1, 2\}$, there exist infinitely many imaginary (resp. real) quadratic number fields k such that $r_2(k) = 2$, $r_4(k) = i$ and infinite Hilbert 2-class field tower.*

2 Preliminary results

2.1 Genus theory

Let p be a prime number and K/k be a Galois extension of number fields with degree p and Galois group G. Denote by E_k the unit group of k and $ram(K/k)$ the number of primes ramified in K/k. Denote by $B(K/k)$ the elementary abelian p-group $E_k/E_k \cap N_{K/k}(K^*)$. We define the p-rank of C_K, denoted $r_p(K)$ as the dimension of the elementary abelian p-group C_K/C_K^p viewed as a vector space over $\mathbb{F}_p$. We note that $B(K/k)$ is a vector space over $\mathbb{F}_p$, let $d_p(B(K/k))$ be its dimension.

By classical results of genus theory [8], we have

$$r_p(K) \geq ram(K/k) - dim_{\mathbb{F}_p}(B(K/k)) - 1;$$

where $ram(K/k)$ is the number of primes that ramify in the extension K/k, and $N_{K/k}$ is the norm map in the extension K/k. In the case where $p = 2$ and the class number of k is odd, the preceding inequality becomes an equality. Indeed, one can verify that the kernel of the homomorphism

$$\begin{aligned} f : C_K &\longrightarrow C_K^2 \\ C &\longmapsto C^2. \end{aligned}$$

is exactly the invariant group $(C_K)^G$. Hence, we have

$$r_2(K) = dim_{\mathbb{F}_2}(C_K/C_K^2) = dim_{\mathbb{F}_2}((C_K)^G) = ram(K/k) - dim_{\mathbb{F}_2}(B(K/k)) - 1.$$

We note that

$$dim_{\mathbb{F}_2}(B(K/k)) \leq \begin{cases} [k : \mathbf{Q}] & \text{if } k \text{ totally real,} \\ \frac{1}{2}[k : \mathbf{Q}] & \text{if not,} \end{cases}$$

2.2 4-Rank of the class group of quadratic number fields

Let k be a quadratic number field of discriminant d. We shall need the following results of Scholz, Rédei and Reichardt on the 2-class group of real quadratic number fields (see [16,18] and for more informations, see [11]).

A factorization of the discriminant d into relatively prime discriminants d_1 and d_2 : $d = d_1 \cdot d_2$ is called a C_4-factorization if

(1) d is not a sum of two squares and $\left(\frac{d_1}{p_2}\right) = \left(\frac{d_2}{p_1}\right) = 1$ for all primes $p_i | d_i$, or

(2) d is a sum of two squares, $\left(\frac{d_1}{p_2}\right) = \left(\frac{d_2}{p_1}\right) = 1$ for all primes $p_i|d_i$ and $\left(\frac{d_1}{d_2}\right)_4 = \left(\frac{d_2}{d_1}\right)_4$.

Suppose we have a C_4-factorization of d, and then there exists a cyclic extension over k of order 4 which is unramified outside ∞. We have the following proposition:

Proposition 2.1 *The* 4-rank $r_4(k)$ *of* k *equals the number of independant* C_4-*factorizations of* d.

2.3 Golod and Shafarevich inequality and Cebotarev density

Let k be a number field, C_k be the class group of k and E_k be the group of units of k. Then, from [1], we know that the Hilbert 2-class field tower of k is infinite if

$$r_2(k) \geq 2 + 2\sqrt{r_2(E_k) + 1}, \qquad (*)$$

where $r_2(E_k)$ is exactly the number of infinite primes of k.

We mention the following form of the Cebotarev density [10].

Proposition 2.2 *Let* K/k *be a Galois extension with group* G. *Let* $\sigma \in G$. *Let* $[K : k] = N$, *and let* c *be the number of elements in the conjugacy class of* σ *in* G. *Then those primes* $\mathcal{P}$ *of* k *which are unramified in* K *and for which there exists* $\mathcal{B}|\mathcal{P}$ *such that the Frobenius automorphism of* $\mathcal{B}$ *in the extension* K/k *verifies*

$$(\mathcal{B}, K/k) = \sigma$$

have a density, and this density is equal to $\frac{c}{N}$.

3 Proof of Theorem 1.1

To give the proof of Theorem 1.1, we distinguish between two cases:

Imaginary case: the case where $r_2(k) = 2$ and $r_4(k) = 1$ has been studied in [14]. Next, we construct infinitely many imaginary quadratic number fields k such that $r_2(k) = 2$ and $r_4(k) = 2$. Let p and p' be distinct prime numbers such that $p \equiv p' \equiv 1 \pmod 4$ and $F = \mathbf{Q}(\sqrt{pp'})$. Denote by F^1 the Hilbert 2-class field of F and $h = [F^1 : F]$. Suppose that h is divisible by 8. Introduce the number field $L = F^1(i)$, the compositum of F^1 and the imaginary quadratic number field $\mathbf{Q}(i)$. It is clear that $L/\mathbf{Q}$ is a Galois extension of degree divisible by $4h$. Let σ be the generator of the Galois group $Gal(L/F^1)$ of the extension L/F^1, so σ fixes F^1 and transforms i to $-i$. Let S be the set of primes ℓ of $\mathbf{Q}$ which are unramified in L and for which there exists $\mathcal{B}|\ell$ such that $(\mathcal{B}, L/\mathbf{Q}) = \sigma$. Hence by Proposition 2.2, S has a density equal to $\frac{c}{4h}$, where c is the number of elements in the conjugacy class of σ in $Gal(L/\mathbf{Q})$. Moreover, let $\mathcal{B}$ be a prime ideal of L over a prime ℓ of S such that $(\mathcal{B}, L/\mathbf{Q}) = \sigma$, then for each prime $\mathcal{L}$ of F^1 under $\mathcal{B}$, we have

$$\left(\mathcal{L}, F^1/\mathbf{Q}\right) = (\mathcal{B}, L/\mathbf{Q})\,|F^1 = 1 \text{ (restriction of } \sigma \text{ to } F^1)$$

since σ acts trivially on F^1. Hence all primes of S are totally decomposed in F^1 and inert in the extension M/F^1. It is clear that for each prime ℓ of S, we have $\ell \equiv 3 \pmod 4$, since ℓ is inert in $\mathbf{Q}(i)$. Also ℓ is decomposed in F, then $\left(\frac{pp'}{\ell}\right) = 1$ and we must have

$$\left(\frac{\ell}{p}\right) = \left(\frac{\ell}{p'}\right) = 1,$$

otherwise, we find $\left(\frac{\ell}{p}\right) = \left(\frac{\ell}{p'}\right) = -1$ and then the prime ideals of F above ℓ will be inert in $F(\sqrt{p})$ which is unramified over F, this is contrary to the fact that all primes of S are totally decomposed in F^1. So, let $k = \mathbf{Q}(\sqrt{-pp'\ell})$, where $\ell \equiv 3 \pmod 4)$ and $\left(\frac{\ell}{p}\right) = \left(\frac{\ell}{p'}\right) = 1$. It is clear by Genus theory that, $r_2(k) = 2$. Moreover, since h is divisible by 4, then by Proposition 2.1, $r_4(F) = 1$, and thus, $\left(\frac{p}{p'}\right) = 1$. It follows that we have two independant C_4-factorizations of the discriminant of k; hence by Proposition 2.1, the 4-rank of the class group of k is equal to 2. It remains to prove that the Hilbert 2-class field tower of k is infinite. We have the extension $F^1(\sqrt{-\ell})/F^1$ is ramified at the archimedean and the ℓ-adic primes of F^1. The number of infinite primes of F^1 is equal to $[F^1 : \mathbf{Q}] = 2h$, and also since ℓ is totally decomposed in F^1, the number of ℓ-adic primes of F^1 is equal to $2h$, so $ram(F^1(\sqrt{-\ell})/F^1) = 4h$. On other hand, we have $dim(B(F^1(\sqrt{-\ell})/F^1)) \leq [F^1 : \mathbf{Q}] = 2h$ and $r_2(E_{F^1(\sqrt{-\ell})}) = [F^1 : \mathbf{Q}] = 2h$. Hence, since h is divisible by 8, one can readily verify that

$$ram\big(F^1(\sqrt{-\ell})/F^1\big) - dim_{\mathbb{F}_2}\big(B\big(F^1(\sqrt{-\ell})/F^1\big)\big)\big) - 1 \geq 2+2\sqrt{r_2(E_{F^1(\sqrt{-\ell})})+1}.$$

By Sect. 2.1 , we have

$$r_2\big(F^1(\sqrt{-\ell})\big) \geq ram\big(F^1(\sqrt{-\ell})/F^1\big) - dim_{\mathbb{F}_2}\big(B\big(F^1(\sqrt{-\ell})/F^1\big)\big)\big) - 1,$$

so $F^1(\sqrt{-\ell})$ satisfies the inequality (*) of Sect. 2.3, and consequently $F^1(\sqrt{-\ell})$ has infinite Hilbert 2-class field tower. Therefore, since $F^1(\sqrt{-\ell})/k$ is an unramified 2-extension, k has infinite Hilbert 2-class field tower.

Real case: First we prove the Theorem for real quadratic number fields k with 2-rank equal to 2 and 4-rank equal to 2. Let $F = \mathbf{Q}(\sqrt{pp'})$ be the real quadratic number field introduced in the imaginary case and suppose that $h = [F^1 : F]$ is divisible by 8. In [19], the author proves that there exists an infinite set S of primes ℓ of $\mathbf{Q}$ which are totally decomposed in $F^1(\sqrt{E_{F^1}})$, where E_{F^1} is the group of units of F^1 and the Hilbert 2-class field tower of the field $k = \mathbf{Q}(\sqrt{pp'\ell})$ is infinite. It remains to prove that the infinite family of real quadratic number fields k verifies

$$r_2(k) = r_4(k) = 2.$$

Since each prime $\ell \in S$ is totally decomposed in $F^1(\sqrt{E_{F^1}})/\mathbf{Q}$, in particular is decomposed in $\mathbf{Q}(i)$, $\ell \equiv 1 \pmod 4$. Therefore, all primes of S are $\equiv 1 \pmod 4$, and by genus theory, we have $r_2(k) = 2$. Next we are going to prove that $r_4(k) = 2$, which is reduced by Proposition 2.1 to prove that we have 2 independant C_4-factorizations of the discriminant $d = \ell pp'$ of k. Since h is divisible by 4 then by Proposition 2.1, the discriminant pp' of the biquadratic number field F has a C_4 factorization, and this means that

$$\left(\frac{p'}{p}\right) = 1 \text{ and } \left(\frac{p}{p'}\right)_4 = \left(\frac{p'}{p}\right)_4. \tag{1}$$

On other hand, since $L = \mathbf{Q}(\sqrt{p}, \sqrt{p'})$ is an abelian unramified extension over F, each prime $\ell \in S$ is totally decomposed in L; therefore, we must have

$$\left(\frac{\ell}{p}\right) = \left(\frac{\ell}{p'}\right) = 1. \tag{2}$$

The real quadratic number fields $\mathbf{Q}(\sqrt{p})$ and $\mathbf{Q}(\sqrt{p'})$ are contained in F^1. Denote ε_p (resp. $\varepsilon_{p'}$) the fundamental unit of $\mathbf{Q}(\sqrt{p})$ (resp. $\mathbf{Q}(\sqrt{p'})$). Since ℓ is totally decomposed in $F^1(\sqrt{E_{F^1}})$, in particular is decomposed in $\mathbf{Q}(\sqrt{p})(\sqrt{\varepsilon_p})$ and in $\mathbf{Q}(\sqrt{p'})(\sqrt{\varepsilon_{p'}})$, for each prime ideal $\mathcal{L}$ (resp. $\mathcal{L}'$) of $\mathbf{Q}(\sqrt{p})$ (resp. $\mathbf{Q}(\sqrt{p'})$) above $\ell \in S$, $\mathcal{L}$ (resp. $\mathcal{L}'$) is totally decomposed in $\mathbf{Q}(\sqrt{p})(\sqrt{\varepsilon_p})$ (resp. $\mathbf{Q}(\sqrt{p'})(\sqrt{\varepsilon_{p'}})$). Hence, the value of the norm residue symbol,

$$\left(\frac{\varepsilon_p,\ \ell}{\mathcal{L}}\right) = \left(\mathcal{L}, \mathbf{Q}(\sqrt{p})(\sqrt{\varepsilon_p})/\mathbf{Q}(\sqrt{p})\right) = 1,$$

where the symbol in the middle is the Artin map applied to the prime ideal $\mathcal{L}$ in the extension $\mathbf{Q}(\sqrt{p})(\sqrt{\varepsilon_p})/\mathbf{Q}(\sqrt{p})$. The same occurs for

$$\left(\frac{\varepsilon_{p'},\ \ell}{\mathcal{L}'}\right) = \left(\mathcal{L}', \mathbf{Q}(\sqrt{p'})(\sqrt{\varepsilon_{p'}})/\mathbf{Q}(\sqrt{p'})\right) = 1.$$

Therefore, we obtain that all units of $\mathbf{Q}(\sqrt{p})$ (resp. $\mathbf{Q}(\sqrt{p'})$) are norms in $M = \mathbf{Q}(\sqrt{p}, \sqrt{\ell})$ (resp. $M' = \mathbf{Q}(\sqrt{p'}, \sqrt{\ell})$). Then, we have

$$dim_{\mathbb{F}_2}\left(B\left(M/\mathbf{Q}\left(\sqrt{p}\right)\right)\right) = dim_{\mathbb{F}_2}\left(B\left(M'/\mathbf{Q}\left(\sqrt{p'}\right)\right)\right) = 0.$$

On other hand, $ram(M/\mathbf{Q}(\sqrt{p})) = ram(M'/\mathbf{Q}(\sqrt{p'})) = 2$ and since the class number of $\mathbf{Q}(\sqrt{p})$ (resp. $\mathbf{Q}(\sqrt{p'})$) is odd, it follows from the discussion of Sect. 2.1 that

$$r_2(M) = r_2(M') = 1.$$

Consequently, the 2-class number of $\mathbf{Q}(\sqrt{p\ell})$ (resp. $\mathbf{Q}(\sqrt{p'\ell})$) is divisible by 4; thus, the 4-rank of the class group of $\mathbf{Q}(\sqrt{p\ell})$ (resp. $\mathbf{Q}(\sqrt{p'\ell})$) is equal to 1. Hence, using Proposition 2.1, we have the following equalities:

$$\left(\frac{p}{\ell}\right)_4 = \left(\frac{\ell}{p}\right)_4 \text{ and } \left(\frac{p'}{\ell}\right)_4 = \left(\frac{\ell}{p'}\right)_4. \tag{3}$$

Combining (1), (2) and (3), the discriminant $pp'\ell$ of k has 2 independant C_4-factorizations:

$$p \cdot \ell\ell' \text{ and } p' \cdot \ell\ell',$$

finishing the proof in the case where $r_2(k) = r_4(k) = 2$.

Now, we give infinitely many real quadratic number fields k such that $r_2(k) = 2$, $r_4(k) = 4$ with infinite Hilbert 2-class field tower.

Let $F = \mathbf{Q}(\sqrt{pp'})$ be the quadratic number field introduced above and suppose that h is divisible by 32, so we must have

$$\left(\frac{p}{p'}\right) = 1. \tag{4}$$

As in the imaginary case, there exist infinitely many prime numbers $\ell \equiv 3 \pmod 4$ such that ℓ is totally decomposed in the Hilbert 2-class field F^1 of F, and then we have

$$\left(\frac{\ell}{p}\right) = \left(\frac{\ell}{p'}\right) = 1. \tag{5}$$

Also, by the distribution of prime numbers in an arithmetic progression, there exist infinitely many prime numbers $\ell' \equiv 3 \pmod 4$ such that

$$\left(\frac{\ell'}{p}\right) = -\left(\frac{\ell'}{p'}\right) = 1. \tag{6}$$

Let $k = \mathbf{Q}(\sqrt{\ell\ell' pp'})$; it is clear by Genus theory that $r_2(k) = 2$. From the equalities (4), (5) and (6), we deduce that we have exactly one C_4-factorization of the discriminant $\ell\ell' pp'$ of k:

$$p \cdot p'\ell\ell'$$

Therefore, we have $r_4(k) = 1$. On other hand, ℓ is totally decomposed in F^1, and then the number of ℓ-adic primes of F^1 is equal to $2h$, where $h = [F^1 : F]$. Moreover, by equalities (6), ℓ' is inert in F; thus, the ℓ'-adic prime of F is principal. So by the reciprocity law applied in the extension F^1/F, the ℓ-adic prime of F is totally decomposed in F^1, and then the number of ℓ'-adic primes of F^1 is equal to h. Hence the number of ramified primes in the extension $F^1\big(\sqrt{\ell\ell'}\big)/F^1$ is equal to

$$ram\big(F^1\big(\sqrt{\ell\ell'}\big)/F^1\big) = 2h + h = 3h.$$

On other hand, we have $dim_{\mathbb{F}_2}(B(F^1(\sqrt{\ell\ell'})/F^1)) \leq [F^1 : \mathbf{Q}] = 2h$ and $r_2(E_{F^1(\sqrt{\ell\ell'})}) = 4h$. Hence, since $h \geq 32$, one can readily verify that

$$ram\big(F^1\big(\sqrt{\ell\ell'}\big)/F^1\big) - dim_{\mathbb{F}_2}\big(B\big(F^1\big(\sqrt{\ell\ell'}\big)/F^1\big)\big) - 1 \geq 2 + 2\sqrt{r_2(E_{F^1(\sqrt{\ell\ell'})}) + 1}.$$

By Sect. 2.1, we have

$$r_2\big(F^1\big(\sqrt{\ell\ell'}\big)\big) \geq ram\big(F^1\big(\sqrt{\ell\ell'}\big)/F^1\big) - dim_{\mathbb{F}_2}\big(B\big(F^1\big(\sqrt{\ell\ell'}\big)/F^1\big)\big) - 1,$$

so $F^1(\sqrt{\ell\ell'})$ satisfies the inequality (*) of Sect. 2.3, and consequently $F^1(\sqrt{\ell\ell'})$ has infinite Hilbert 2-class field tower. Therefore, since $F^1(\sqrt{\ell\ell'})/k$ is an unramified 2-extension, k has infinite Hilbert 2-class field tower. Hence, the proof of the theorem is now complete.

References

1. Cassels, J., Fröhlich, A.: Algebraic Number Theory. Academic Press, London (1986)
2. Gerth, F.: Quadratic fields with infinite Hilbert 2-class field towers. Acta Arith. **106**(2), 151–158 (2003)
3. Gerth, F.: A density result for some imaginary quadratic fields with infinite Hilbert 2-class field tower. Arch. Math. (Basel) **82**(1), 23–27 (2004)
4. Gerth, F.: Densities for some real quadratic fields with infinite Hilbert 2-class field towers. J. Number Theory **118**, 90–97 (2006)
5. Golod, E.S., Shafarevich, I.R.: On the class field tower. Izv. Akad. auk SSSR Ser. Math. **28**, 261–272 (1964) (in Russian); English translation in AMS Transl. 48
6. Hajir, F.: On a theorem of Koch. Pac. J. Math. **176**(1), 15–18 (1996)
7. Hajir, F.: Correction to 'On a theorem of Koch'. Pac. J. Math. **196**(2), 507–508 (2000)
8. Jehne, W.: On knots in algebraic number theory. J. Reine Angew. Math. **311**(312), 215–254 (1979)
9. Kisilevsky, H.: Number fields with class number congruent to 4 mod 8 and Hilbert's theorem 94. J. Number Theory **8**, 271–279 (1976)
10. Lang, S.: Algebraic Number Theory. Addison-Wesley, Reading (1970)
11. Lemmermeyer, F.: Higher Descent on Pell Conics. I. From Legendre to Selmer, arXiv: math/0311309v1
12. Maire, C.: Un raffinement du théorème de Golod-Safarevic. Nagoya Math. J. **150**, 1–11 (1998)
13. Martinet, J.: Tours de corps de classes et estimations de discriminants. Invent. Math. **44**, 65–73 (1978)
14. Mouhib, A.: Infinite Hilbert 2-class field tower of quadratic number fields. Acta Arith. **145**, 267–272 (2010)
15. Mouhib, A.: On 2-class field towers of some real quadratic number fields with 2-class groups of rank 3. Ill. J. Math. **57**, 1009–1018 (2013)
16. Rédei, L., Reichardt, H.: Die anzahl der durch 4 teilbaren invarienten der klasen-gruppe eines beliebigen quadratischen Zahlkorpers. J. Reine Angew. Math. **170**, 69–74 (1933)
17. Schmithals, B.: Konstruktion imaginärquadratischer Körper mit undendlichem Klassenkörperturm. Arch. Math. **34**, 307–312 (1980)
18. Scholz, A.: Über die Lösbarkeit der Gleichung $t^2 - Du^2 = -4$. Math. Z. **39**, 95–111 (1934)
19. Schoof, R.: Infinite class field towers of quadratic fields. J. Reine Angew. Math. **372**, 209–220 (1986)