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Moduli of Continuity of Local Times of Strongly Symmetric Markov Processes via Gaussian Processes[1]

Michael B. Marcus and Jay Rosen

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Let X be a strongly symmetric standard Markov process on a locally compact metric space S with 1-potential density $u^1(x, y)$. Let $\{L_t^y, (t, y) \in R^+ \times S\}$ denote the local times of X and let $G = \{G(y), y \in S\}$ be a mean zero Gaussian process with covariance $u^1(x, y)$. In this paper results about the moduli of continuity of G are carried over to give similar moduli of continuity results about L_t^y considered as a function of y. Several examples are given with particular attention paid to symmetric Lévy processes.

KEY WORDS: Gaussian processes; Markov processes; local times; moduli of continuity.

1. INTRODUCTION

In this paper we show that the moduli of continuity of the local times of a large class of Markov processes can be easily obtained if one knows the moduli of continuity of the Gaussian processes that are associated with the Markov process. And, since the moduli of continuity results for the Gaussian processes are either well known, or in many cases easy to obtain, we get similar moduli of continuity results for the local times. This paper is an application of Ref. 16 in which we obtained necessary and sufficient

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[2] Department of Mathematics, The City College of CUNY, New York, New York 10031.
[3] Department of Mathematics, College of Staten Island, CUNY, Staten Island, New York 10301.

conditions for continuity and boundedness of the local times of strongly symmetric standard Markov processes by establishing the equivalence of these conditions with those of the associated Gaussian processes. The key ingredient in this work is an application of Lemma 4.3 of Ref. 16 which is itself a corollary of an isomorphism theorem of Dynkin,[5,6] which relates local times and Gaussian processes. (For another application of these methods see Ref. 17).

Let (S, ρ) be a locally compact metric space with a countable base and let $X = (\Omega, \mathscr{F}_t, X_t, P^x)$, $t \in R^+$, be a strongly symmetric standard Markov process with state space S. In saying that X is symmetric we mean that there is a σ-finite measure $m(\cdot)$ on S such that the Markov transition function P_t satisfies

$$(P_t f, g) = (f, P_t g) \qquad \forall t \in R^+$$

for all measureable functions f and g in $L^2(S)$ where $(f, g) \equiv \int fg\,dm$ is the usual inner product. In saying that X is strongly symmetric we mean that in addition to X being symmetric the measure $U^\alpha = U^\alpha(x, \cdot)$ given by

$$U^\alpha(x, \cdot) = \int_0^\infty e^{-\alpha t} P_t(x, \cdot)\, dt$$

is absolutely continuous with respect to m for some $\alpha > 0$, (and hence for all $\alpha > 0$). In this case there is a canonical symmetric α-excessive density $u^\alpha = u^\alpha(x, y)$ for U^α. Moreover a strongly symmetric standard Markov process X has a symmetric transition density function $p_t(x, y)$ and

$$u^\alpha(x, y) = \int_0^\infty e^{-\alpha t} p_t(x, y)\, dt$$

Let $L = \{L_t^y, (t, y) \in R^+ \times S\}$ denote the local times of X. It is known that a necessary and sufficient condition for the existence of a local time for a strongly symmetric standard Markov process is that

$$u^\alpha(x, y) < \infty \qquad \forall x, y \in S$$

We assume this throughout this paper and normalize the local time by setting

$$E^x\left(\int_0^\infty e^{-\alpha t}\, dL_t^y\right) = u^\alpha(x, y) \tag{1.1}$$

(If Eq. (1.1) holds for any $\alpha > 0$ it holds for all $\alpha > 0$).

It is well known, (see e.g. Ref. 16, Theorem 3.3), that the function $u^\alpha(x, y)$ is positive definite on $S \times S$ for each $\alpha > 0$. Therefore, for each $\alpha > 0$, we can define a mean zero Gaussian process $\{G_\alpha(y),\ y \in S\}$ with covarience

$$E(G_\alpha(x)\, G_\alpha(y)) = u^\alpha(x, y) \qquad \forall x, y \in S$$

The processes X and $\{G_\alpha(y),\ y \in S\}$, which we take to be independent, are related through the α-potential density $u^\alpha(x, y)$ and are referred to as associated processes. To simplify the statement of our results we will always consider X and the associated Gaussian process corresponding to $\alpha = 1$, that is $\{G_1(y),\ y \in S\}$. In what follows we denote this process by $G = \{G(y),\ y \in S\}$ and note that the natural L^2 metric induced by G is a function of the 1-potential density, i.e.

$$d(x, y) \equiv (E(G(x) - G(y))^2)^{1/2} = (u^1(x, x) + u^1(y, y) - 2u^1(x, y))^{1/2} \tag{1.2}$$

The fact that d is a metric and not a pseudo-metric is because $u^1(x, x) > 0$ for all $x \in S$, (see e.g. Ref. 16, Lemma 3.6). For simplicity we assume that d induces the same topology on S as the original metric ρ. (Although it is easy to handle the general case as we point out in Remark 3.4).

Let $K \subset S$ be compact. Under very general conditions, whenever a Gaussian process $\{G(y),\ y \in K\}$ has continuous sample paths it also has both an exact uniform and an exact local modulus of continuity. To be more precise we call ω: $R^+ \to R^+$ an exact uniform modulus of continuity for $\{G(y),\ y \in K\}$ if

$$\lim_{\delta \to 0} \sup_{\substack{d(x, y) \leq \delta \\ x, y \in K}} \frac{|G(x) - G(y)|}{\omega(d(x, y))} = 1 \qquad \text{a.s.} \tag{1.3}$$

We call ρ: $R^+ \to R^+$ an exact local modulus of continuity for $\{G(y),\ y \in S\}$ at some fixed $y_0 \in S$ if

$$\lim_{\delta \to 0} \sup_{\substack{d(y, y_0) \leq \delta \\ y \in S}} \frac{|G(y) - G(y_0)|}{\rho(d(y, y_0))} = 1 \qquad \text{a.s.} \tag{1.4}$$

We use the expressions uniform and local moduli of continuity for functions ω and ρ for which the equality signs in Eqs. (1.3) and (1.4) are replaced by "less than or equal" signs. We use the terms lower uniform and local moduli of continuity for functions ω and ρ for which the equality signs in Eqs. (1.3) and (1.4) are replaced by "greater than or equal" signs. We will always assume, in our discussions of moduli of continuity, that $\{G(y),\ y \in K\}$ is continuous on (S, d). Other ways of defining moduli of continuity, some of which are equivalent to the definitions given in Eqs. (1.3) and (1.4) and some which are not, are discussed in Section 4.

Our methods for studying moduli of continuity of local times enables us to only consider the Markov processes up to, but not including, their lifetimes. We shall denote the lifetime of the strongly symmetric standard Markov process X by ζ. We obtain the following results relating the moduli of continuity of the local times of strongly symmetric Markov processes with the moduli of continuity of their associated Gaussian processes.

Theorem 1.1. Let X be a strongly symmetric standard Markov process and let $G=\{G(y),\ y\in S\}$ be the associated Gaussian process. Let $\{L_t^y,\ (t,y)\in R^+\times S\}$ be the local time of X. Then if ρ is an exact local modulus of continuity for G at $y_0\in S$

$$\lim_{\delta\to 0}\sup_{\substack{d(y,y_0)\leqslant\delta\\ y\in S}}\frac{|L_t^y-L_t^{y_0}|}{\rho(d(y,y_0))}=\sqrt{2}\,(L_t^{y_0})^{1/2}\qquad\text{for almost all } t\in[0,\zeta)\text{ a.s.}\tag{1.5}$$

We note that if ρ is simply a local modulus of continuity for $\{G(y),\ y\in S\}$ then the expression in Eq. (1.5) holds with the equality sign replaced by a "less than or equal sign."

Theorem 1.2. Let X be a strongly symmetric standard Markov process and let $G=\{G(y),\ y\in S\}$ be the associated Gaussian process. Let $\{L_t^y,\ (t,y)\in R^+\times S\}$ be the local time of X and let $K\subset S$ be compact. Then if ω is a uniform modulus of continuity for $\{G(y)),\ y\in K\}$

$$\lim_{\delta\to 0}\sup_{\substack{d(x,y)\leqslant\delta\\ x,y\in K}}\frac{|L_t^x-L_t^y|}{\omega(d(x,y))}\leqslant\sqrt{2}\sup_{y\in K}(L_t^y)^{1/2}\qquad\text{for almost all } t\in[0,\zeta)\text{ a.s.}\tag{1.6}$$

The next theorem shows that if ω is an exact uniform modulus of continuity for G then it is "best possible" in Eq. (1.6).

Theorem 1.3. Let X be a strongly symmetric standard Markov process and let $G=\{G(y),\ y\in S\}$ be the associated Gaussian process. Let $\{L_t^y,\ (t,y)\in R^+\times S\}$ be the local time of X and let $K\subset S$ be compact. Then if ω is an exact uniform modulus of continuity for $\{G(y),\ y\in K\}$ there exists a $y_0\in K$ such that

$$\lim_{\delta\to 0}\sup_{\substack{d(x,y)\leqslant\delta\\ x,y\in K}}\frac{|L_t^x-L_t^y|}{\omega(d(x,y))}\geqslant\sqrt{2}\,(L_t^{y_0})^{1/2}\qquad\text{for almost all } t\in[0,\zeta)\text{ a.s.}\tag{1.7}$$

We can improve on Eq. (1.7) when the associated Gaussian process and the state space S have sufficient regularity.

Theorem 1.4. Let X be a strongly symmetric standard Markov process and let $G=\{G(y),\ y\in S\}$ be the associated Gaussian process. Let $\{L_t^y,\ (t,y)\in R^+\times S\}$ be the local time of X. Furthermore, let (S,d) be a locally homogeneous metric space; i.e., any two points in S have isometric neighborhoods in the metric d, and let $K\subset S$ be a compact set which is the closure of its interior. Then if ω is an exact uniform modulus of continuity for $\{G(y),\ y\in K\}$

$$\lim_{\delta\to 0}\sup_{\substack{d(x,y)\leqslant\delta\\ x,y\in K}}\frac{|L_t^x-L_t^y|}{\omega(d(x,y))}=\sqrt{2}\sup_{y\in K}(L_y^t)^{1/2}\qquad\text{for almost all } t\in[0,\zeta)\text{ a.s.}\tag{1.8}$$

The fact that we can only obtain these results for almost all t is a weakness of our method which is explained in Ref. 16.

Lemma 4.3, in Ref. 16, as well as the Dynkin Isomorphism Theorem, relate local times to the squares of Gaussian process. Thus we need to investigate properties of the uniform and local moduli of continuity for the squares of Gaussian processes. These questions have not been loked at earlier because they are not natural from the viewpoint of the theory of Gaussian processes. This will be done in Section 2.

In Section 3, we will give the proofs of Theorems 1.1–1.4. Since these Theorems allow one to lift results for Gaussian processes to those of the associated local times, in Section 4 we will survey known results about the moduli of continuity of Gaussian processes and also present some new results to tie up loose ends. Lastly, in Section 5, we will use our results to study the moduli of continuity of real valued Markov processes and in particular of Lévy processes. Some of our results complement interesting work of Barlow[1] on the uniform modulus of continuity of the local times of Lévy processes.

2. MODULI OF CONTINUITY FOR THE SQUARES OF GAUSSIAN PROCESSES

The first Lemma is simple and probably known. It establishes conditions for a Gaussian process to have an exact uniform modulus of continuity. We give it to make our presentation more complete. To avoid trivialities in what follows we will assume, whenever we are considering the local modulus of continuity of $G=\{G(u),\ u\in S\}$ at a point $u_0\in S$, that G and (S,d) are such that

$$\sup_{d(u,u_0)\leqslant\delta}|G(u)-G(u_0)|>0\qquad\forall\delta>0\tag{2.0}$$

and whenever we are considering the uniform modulus of continuity of G, that

$$\sup_{\substack{d(u,v)\leqslant\delta\\ u,v\in K}} G(u)-G(v)>0 \qquad \forall\delta>0 \tag{2.1}$$

where K is a compact subset of S.

Lemma 2.1. Let $\{G(u),\ u\in S\}$ be a mean zero Gaussian process with continuous metric d, as given in Eq. (1.2). Let τ: $R^+\to R^+$ and $K\subset S$ be a compact set. For the following three statements:

$$\lim_{\delta\to 0}\sup_{\substack{d(u,v)\leqslant\delta\\ u,v\in K}}\frac{G(u)-G(v)}{\tau(d(u,v))}\leqslant C \qquad \text{a.s. for some } 0\leqslant C<\infty \tag{2.2}$$

$$\lim_{\delta\to 0}\sup_{\substack{d(u,v)\leqslant\delta\\ u,v\in K}}\frac{d(u,v)}{\tau(d(u,v))}=0 \tag{2.3}$$

and

$$\lim_{\delta\to 0}\sup_{\substack{d(u,v)\leqslant\delta\\ u,v\in K}}\frac{G(u)-G(v)}{\tau(d(u,v))}=C' \qquad \text{a.s. for some} \qquad 0\leqslant C'\leqslant\infty \tag{2.4}$$

we have that Eq. (2.2) implies Eq. (2.3) implies Eq. (2.4). (Obviously, if Eq. (2.2) holds then $C'<\infty$ in Eq. (2.4) but Eq. (2.3) implies Eq. (2.4) even if the limit superior in Eq. (2.2) is not finite almost surely.)

Proof. To show that Eq. (2.2) implies Eq. (2.3) we show that if Eq. (2.3) does not hold then Eq. (2.2) does not hold. Suppose that Eq. (2.3) does not hold but that Eq. (2.2) does. Then there exists a sequence of pairs $\{(u_k, v_k)\}_{k=1}^{\infty}$ such that for all k, u_k, $v_k\in K$ and $d(u_k, v_k)\geqslant\varepsilon\tau(d(u_k, v_k))$ and $\lim_{k\to\infty} d(u_k, v_k)=0$. It follows from Eq.(2.2) that almost surely

$$C\geqslant\lim_{\delta\to 0}\sup_{\substack{d(u,v)=\delta\\ u,v\in K}}\frac{G(u)-G(v)}{\tau(d(u,v))}\geqslant\overline{\lim_{k\to\infty}}\frac{\varepsilon(G(u_k)-G(v_k))}{d(u_k, v_k)}$$

which gives

$$\overline{\lim_{k\to\infty}}\frac{G(u_k)-G(v_k)}{d(u_k, v_k)}\leqslant\frac{C}{\varepsilon} \qquad \text{a.s.} \tag{2.5}$$

But for each $k \geqslant 1$, $(G(u_k) - G(v_k))/d(u_k, v_k)$ is a normal random variable with mean zero and variance 1 and Eq. (2.5) can not be finite almost surely for such a sequence. (This is because $\bigcup_{k \geqslant n} \{\xi_k > C/\varepsilon\} \supset \{\xi_n > C/\varepsilon\}$ and the probability of this last set is greater than zero for any sequence of identically distributed unbounded random variables.) Thus we see that Eq. (2.2) implies Eq. (2.3).

To show that Eq. (2.3) implies Eq. (2.4) we express G in terms of it's Karhunen-Loeve expansion, (see e.g. Ref. 16, Theorem 2.6).

$$G(u) = \sum_{j=1}^{\infty} \xi_j \phi_j(u) \qquad u \in S$$

where $\{\phi_j\}_{j=1}^{\infty}$ are continuous functions on (S, d) and $\{\xi_j\}_{j=1}^{\infty}$ are independent indentically distributed normal random variables with mean zero and variance 1. Clearly, in this case

$$d(u, v) = \left(\sum_{j=1}^{\infty} (\phi_j(u) - \phi_j(v))^2 \right)^{1/2}$$

Set

$$G_N(u) = \sum_{j=1}^{N} \xi_j \phi_j(u) \qquad u \in S$$

and note that

$$\begin{aligned} |G_N(u) - G_N(v)| &\leqslant \left(\sum_{j=1}^{N} |\xi_j| \right) \sup_{1 \leqslant j \leqslant N} |\phi_j(u) - \phi_j(v)| \\ &\leqslant \left(\sum_{j=1}^{N} |\xi_j| \right) d(u, v) \end{aligned} \tag{2.6}$$

It follows from Eqs. (2.3) and (2.6) that

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u, v \in K}} \frac{G_N(u) - G_N(v)}{\tau(d(u, v))} = 0 \qquad \text{a.s.}$$

Therefore for $0 < C'' \leqslant \infty$ the event

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u, v \in K}} \frac{G(u) - G(v)}{\tau(d(u, v))} \geqslant C'' \qquad \text{a.s.}$$

is a tail event and hence occurs with probability 0 or 1. This implies Eq. (2.4). □

The next lemma shows that an exact uniform modulus of continuity for a Gaussian process on a compact set K is also an exact modulus of continuity for the process on some arbitrarily small compact subset of K. However, since we are also interested in one sided results we will state the next lemma and some of the subsequent results for the lower uniform modulus of continuity. To avoid unnecessary complications, from now on we will always assume that the Gaussian processes have continuous sample paths and that the moduli of continuity go to zero as δ goes to zero. In what follows we will say that a modulus, say ω, satisfies Eq. (2.3), if Eq. (2.3) holds with τ replaced by ω.

Lemma 2.2. Let $G = \{G(u),\ u \in S\}$ be a mean zero Gaussian process with continuous sample paths and let $K \subset S$ be a compact set. Assume that ω is a lower uniform modulus of continuity for $\{G(u),\ u \in K\}$ that satisfies Eq. (2.3). Then there exists a $u_0 \in K$ such that for all $\varepsilon > 0$

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u, v \in B(u_0, \varepsilon) \cap K}} \frac{G(u) - G(v)}{\omega(d(u, v))} \geqslant 1 \qquad \text{a.s.} \tag{2.7}$$

where $B(u_0, \varepsilon) = \{u \in S \colon d(u, u_0) \leqslant \varepsilon\}$.

Proof. We will first show that for all $\varepsilon_n > 0$ there exists a $u_0(\varepsilon_n) \in K$ such that

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u, v \in B(u_0(\varepsilon_n), \varepsilon_n) \cap K}} \frac{G(u) - G(v)}{\omega(d(u, v))} \geqslant 1 \qquad \text{a.s.} \tag{2.8}$$

Consider the sets $\{B(x, \varepsilon_n/2) \colon x \in K\}$. Obviously, $K \subset \bigcup_{x \in K} B(x, \varepsilon_n/2)$. Let $B(x_1, \varepsilon_n/2), \ldots, B(x_m, \varepsilon_n/2)$ be a finite cover of K. Note that if $u, v \in K$ such that $d(u, v) < \varepsilon_n/2$ then both u and v are contained in $B(x_j, \varepsilon_n)$ for some $1 \leqslant j \leqslant m$. Thus, for $0 < \delta < \varepsilon_n/2$

$$\sup_{1 \leqslant j \leqslant m} \sup_{\substack{d(u,v) \leqslant \delta \\ u, v \in B(x_j, \varepsilon_n) \cap K}} \frac{G(u) - G(v)}{\omega(d(u, v))} = \sup_{\substack{d(u,v) \leqslant \delta \\ u, v \in K}} \frac{G(u) - G(v)}{\omega(d(u, v))} \tag{2.9}$$

Suppose there exists an $\varepsilon' > 0$ and a $1 \leqslant j \leqslant m$ such that

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u, v \in B(x_j, \varepsilon_n) \cap K}} \frac{G(u) - G(v)}{\omega(d(u, v))} \leqslant 1 - \varepsilon' \tag{2.10}$$

on a set of positive measure. Then, since Eq. (2.3) holds on $B(x_j, \varepsilon_n) \cap K$, it follows from Lemma 2.1 that the event in Eq. (2.10) must hold almost surely. If the event in Eq. (2.10) holds almost surely for all $1 \leqslant j \leqslant m$, we can take the lim sup as δ goes to zero of the two sides in Eq. (2.9) to get

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u,v \in K}} \frac{G(u)-G(v)}{\omega(d(u,v))} \leqslant 1-\varepsilon' \qquad \text{a.s.} \tag{2.11}$$

which contradicts the fact that ω is a lower uniform modulus of continuity for G. Thus the limit superior in Eq. (2.10) is greater than or equal to 1 almost surely for some $1 \leqslant j \leqslant m$. We set $u_0(\varepsilon_n) = x_j$ for some j for which this occurs. Now choose a sequence $\varepsilon_n \to 0$ and consider the balls $B(u(\varepsilon_n), \varepsilon_n)$ for which Eq. (2.8) holds. Since K is compact there exists a sequence $\{u_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k \to \infty} u_{n_k} = u_0$ for some $u_0 \in K$. It is easy to see that for all $\varepsilon > 0$ there exists a $k_0(\varepsilon)$ such that for $k \geqslant k_0(\varepsilon)$, $B(u(\varepsilon_{n_k}), \varepsilon_{n_k}) \subset B(u_0, \varepsilon)$. This observation and Eq. (2.8) gives Eq. (2.7). □

The next lemma is in preparation for the study of the moduli of continuity for squares of Gaussian processes. To avoid confusion and since, because of Lemma 3.6, Ref. 16, we will only be concerned with Gaussian processes for which

$$EG^2(u) > 0 \qquad \forall u \in S \tag{2.12}$$

we will assume this condition in the remainder of this Section.

Lemma 2.3. Let $G = \{G(u), u \in S\}$ be a mean zero Gaussian process which is continuous on some compact set $K \subset S$ and which satisfies Eq. (2.12). The given any $\varepsilon' > 0$ and $u_0 \in K$ there exists an $\varepsilon > 0$ such that

$$\inf_{u,v \in B(u_0,\varepsilon) \cap K} \frac{|G(u)+G(v)|}{|G(u_0)|} \geqslant 2(1-\varepsilon') \tag{2.13}$$

on a set of measure greater than or equal to $(1-\varepsilon')$.

Proof. Since

$$|G(u)+G(v)| \geqslant (2\,|G(u_0)| - |G(u)-G(u_0)| - |G(v)-G(u_0)|)$$

we see that for all $u, v \in B(u_0, \varepsilon) \cap K$, for some $\varepsilon > 0$, we have

$$\frac{|G(u)+G(v)|}{|G(u_0)|} \geqslant 2\left(1 - \sup_{u,v \in B(u_0,\varepsilon) \cap K} \frac{|G(u)-G(v)|}{|G(u_0)|}\right)$$

Therefore

$$\inf_{u,v\in B(u_0,\varepsilon)\cap K}\frac{|G(u)+G(v)|}{|G(u_0)|}\geqslant 2\left(1-\sup_{u,v\in B(u_0,\,\varepsilon)\cap K}\frac{|G(u)-G(v)|}{|G(u_0)|}\right) \tag{2.14}$$

Since, by Eq. (2.12), $|G(u_0)|>0$ almost surely we see that

$$\lim_{\varepsilon\to 0}\sup_{u,v\in B(u_0,\varepsilon)\cap K}\frac{|G(u)-G(v)|}{|G(u_0)|}=0 \qquad \text{a.s.}$$

and so given $\varepsilon'>0$ we can find an ε such that

$$P\left(\sup_{u,v\in B(u_0,\varepsilon)\cap K}\frac{|G(u)-G(v)|}{|G(u_0)|}\leqslant\varepsilon'\right)\geqslant 1-\varepsilon' \tag{2.15}$$

Using Eq. (2.15) in Eq. (2.14) we get Eq. (2.13). □

We now obtain the inequality for Gaussian processes which gives the lower-bound in Theorem 1.1.

Theorem 2.4. Let $G=\{G(u),\ u\in S\}$ be a mean zero continuous Gaussian process which satisfies Eq. (2.12) and which has a lower uniform modulus of continuity ω on a compact set $K\subset S$ satisfying Eq. (2.3). Then there exists a $u_0\in K$ such that

$$\lim_{\delta\to 0}\sup_{\substack{d(u,v)\leqslant\delta\\ u,v\in K}}\frac{G^2(u)-G^2(v)}{2\omega(d(u,v))}\geqslant |G(u_0)| \qquad \text{a.s.} \tag{2.16}$$

Proof. Fix $\delta'>0$ and $\varepsilon'>0$ and note that for all $\varepsilon>0$

$$\begin{aligned}&\sup_{\substack{d(u,v)\leqslant\delta'\\ u,v\in B(u_0,\varepsilon)\cap K}}\frac{G^2(u)-G^2(v)}{2\omega(d(u,v))\,|G(u_0)|}\\ &\qquad\geqslant\sup_{\substack{d(u,v)\leqslant\delta'\\ u,v\in B(u_0,\varepsilon)\cap K}}\frac{|G(u)-G(v)|}{2\omega(d(u,v))}\inf_{\substack{d(u,v)\leqslant\delta'\\ u,v\in B(u_0,\varepsilon)\cap K}}\frac{|G(u)+G(v)|}{|G(u_0)|}\end{aligned} \tag{2.17}$$

Let u_0 be the element of K for which Lemma 2.2 holds. For this u_0 choose ε in Eq. (2.17) so that Lemma 2.3 holds for the fixed ε'. Now repeat Eq. (2.17) with the chosen values of u_0 and ε but with δ' replaced by $\delta\leqslant\delta'$. Since the last term in Eq. (2.17) is greater than the left-hand side of Eq. (2.13) we see that

$$\sup_{\substack{d(u,v)\leqslant\delta\\ u,v\in B(u_0,\varepsilon)\cap K}}\frac{G^2(u)-G^2(v)}{2\omega(d(u,v))\,|G(u_0)|}\geqslant(1-\varepsilon')\sup_{\substack{d(u,v)\leqslant\delta\\ u,v\in B(u_0,\varepsilon)\cap K}}\frac{|G(u)-G(v)|}{\omega(d(u,v))} \tag{2.18}$$

on a set of measure greater than or equal to $(1-\varepsilon')$. Taking the limit superior as $\delta \to 0$ in Eq. (2.18) and using Eq. (2.7) we see that

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u,v \in B(u_0,\varepsilon) \cap K}} \frac{G^2(u)-G^2(v)}{2\omega(d(u,v))\,|G(u_0)|} \geqslant (1-\varepsilon')$$

on a set of measure greater than or equal to $(1-\varepsilon')$ for all ε sufficiently small. Since ε' is arbitrary we get Eq. (2.16). □

Theorem 2.5. Let (S, d) be locally homogeneous, i.e. any two points in S have isometric neighborhoods in the metric d, and let $K \subset S$ be a compact set which is the closure of its interior. Let $G = \{G(u), u \in S\}$ be a mean zero continuous Gaussian process which satisfies Eq. (2.12) and which has a lower uniform modulus of continuity ω on a compact set $K \subset S$ satisfying Eq. (2.3). Then

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u,v \in K}} \frac{G^2(u)-G^2(v)}{2\omega(d(u,v))} \geqslant \sup_{u \in K} |G(u)| \qquad \text{a.s.} \tag{2.19}$$

Proof. Since K is the closure of it's interior there exists a $u_0 \in K$ and an $\varepsilon > 0$ such that $B(u_0, \varepsilon) \subset K$. It then follows from the homogeneity of (S, d) that Eq. (2.7) is satisfied for this u_0 and this ε. In fact, for every $u_0 \in \operatorname{int} K$, Eq. (2.7) is satisfied for some $\varepsilon > 0$. Using the homogeneity of S with respect to d, we can follow the proof of Theorem 2.4 to see that Eq. (2.16) holds for all $u_0 \in \operatorname{int} K$. In particular, if $\{x_j\}_{j=1}^n$ are points in the interior of K then

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u,v \in K}} \frac{G^2(u)-G^2(v)}{2\omega(d(u,v))} \geqslant \sup_{1 \leqslant j \leqslant n} |G(x_j)| \qquad \text{a.s.}$$

Since this inequality is independent of n, it also holds for $\{x_j\}_{j=1}^\infty$ contained in a countable dense subset of K. Therefore, since G is uniformly continuous on K, we get Eq. (2.19). □

Theorem 2.6. Let $G = \{G(u), u \in S\}$ be a mean zero continuous Gaussian process which satisfies Eq. (2.12) and which has a uniform modulus of continuity ω on a compact set $K \subset S$. Then

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u,v \in K}} \frac{G^2(u)-G^2(v)}{2\omega(d(u,v))} \leqslant \sup_{u \in K} |G(u)| \qquad \text{a.s.} \tag{2.20}$$

Proof. This is immediate since

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u,v \in K}} \frac{G^2(u) - G^2(v)}{2\omega(d(u,v))}$$

$$\leqslant \lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u,v \in K}} \frac{|G(u) - G(v)|}{\omega(d(u,v))} \lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u,v \in K}} \frac{|G(u) - G(v)|}{2} \tag{2.21}$$

The first term on the right side of the inequality in Eq. (2.21) is less than or equal to 1 because ω is a uniform modulus of continuity for G. The rest is obvious. □

Theorem 2.7. Let (S, d) be locally homogeneous, i.e. any two points in S have isometric neighborhoods in the metric d, and let $K \subset S$ be a compact set which is the closure of its interior. Let $G = \{G(u), u \in S\}$ be a mean zero continuous Gaussian process which satisfies Eq. (2.12) and which has an exact uniform modulus of continuity ω on a compact set $K \subset S$. Then

$$\lim_{\delta \to 0} \sup_{\substack{d(u,v) \leqslant \delta \\ u,v \in K}} \frac{G^2(u) - G^2(v)}{2\omega(d(u,v))} = \sup_{u \in K} |G(u)| \qquad \text{a.s.} \tag{2.22}$$

Proof. We need only note that because ω is an exact uniform modulus of continuity of Eq. (2.2) holds with $C = 1$. This implies Eq. (2.3) which enables us to use Theorem 2.5 for the lower bound in Eq. (2.22). The upper-bound follows immediately from Theorem 2.6. □

The results pertaining to the local modulus of continuity are much simpler.

Theorem 2.8. Let $G = \{G(y), y \in S\}$ be a mean zero continuous Gaussian process which satisfies Eq. (2.12) and which has an exact local modulus of continuity ρ at a point $y_0 \in S$. Then

$$\lim_{\delta \to 0} \sup_{\substack{d(y,y_0) \leqslant \delta \\ y \in S}} \frac{|G^2(y) - G^2(y_0)|}{2\rho(d(y, y_0))} = |G(y_0)| \qquad \text{a.s.} \tag{2.23}$$

Furthermore, if ρ is a local modulus of continuity for G, Eq. (2.23) holds with a less than or equal sign and if ρ is a lower local modulus of continuity for G, Eq. (2.23) holds with a greater than or equal sign.

Proof. This is immediate since

$$\begin{aligned}
&\sup_{d(y, y_0) \leqslant \delta} \frac{|G(y)-G(y_0)|}{\rho(d(y, y_0))}\left(|G(y_0)|-\frac{1}{2} \sup_{d(y, y_0) \leqslant \delta}|G(y)-G(y_0)|\right) \\
&\quad \leqslant \sup_{d(y, y_0) \leqslant \delta} \frac{|G^2(y)-G^2(y_0)|}{2\rho(d(y, y_0))} \\
&\quad \leqslant \sup_{d(y, y_0) \leqslant \delta} \frac{|G(y)-G(y_0)|}{\rho(d(y, y_0))}\left(|G(y_0)|+\frac{1}{2} \sup_{d(y, y_0) \leqslant \delta}|G(y)-G(y_0)|\right)
\end{aligned} \tag{2.24}$$

We see that Eq. (2.23) follows by continuity. It is also clear from Eq. (2.24) that the one-sided results also hold. □

3. PROOFS OF THEOREMS 1.1–1.4

We begin by presenting the material necessary to state part of Lemma 4.3 in Ref. 16. In the notation of Section 1, let X be a strongly symmetric standard Markov process and $G=\{G(u), u \in S\}$ be the associated Gaussian process. Let (Ω_G, P_G) denote the probability space of G and let $L=\{L_t^y, (t, y) \in R^+ \times S\}$ be the local time of S. Let $\{x_i\}_{i=1}^{\infty}$ be a countable dense subset of S and consider $\{L_t^{x_i}\}_{i=1}^{\infty}$ and $\{G(x_i)\}_{i=1}^{\infty}$ as R^{∞} valued random variables. Let $\mathscr{C}$ denote the σ-algebra generated by the cylinder sets of R^{∞} and τ denote Lebesgue measure on R^+. The following Lemma is taken from Ref. 16, Lemma 4.3.

Lemma 3.1. Let $B \in \mathscr{C}$ be such that $P_G(G^2(\cdot)/2 \in B)=1$. Then for almost all $\omega' \in \Omega_G$ with respect to P_G

$$P^x\left(L_t+\frac{G^2(\cdot, \omega')}{2} \in B \quad \text{for almost all } t \in [0, \zeta)\right)=1 \tag{3.1}$$

Remark 3.2. Suppose G has a version with continuous sample paths on (S, d). Then by Ref. 16, Theorem I, we can find a continuous version for L on (S, d). Clearly, in this case, Lemma 3.1 holds for the continuous versions. By hypotheses the Gaussian processes considered in Theorems 1.1–1.4 are continuous and the corresponding local times will be taken to be the continuous versions.

Proof. (Theorem 1.1). Let G be defined on the probability space (Ω_G, P_G). Since G satisfies Eq. (2.12), by Lemma 3.1, Remark 3.2 and Theorem 2.8, we have that for almost all $\omega' \in \Omega_G$ with respect to P_G

$$\lim_{\delta \to 0} \sup_{\substack{d(y, y_0) \leqslant \delta \\ y \in S}} \frac{|L_t^y - L_t^{y_0} + \frac{1}{2}(G^2(y, \omega') - G^2(y_0, \omega'))|}{\rho(d(y, y_0))}$$
$$= \sqrt{2} \left| L_t^{y_0} + \frac{G^2(y_0, \omega')}{2} \right|^{1/2} \tag{3.2}$$

for almost all $t \in [0, \zeta)$ almost surely with respect to P^x. (We obtain Eq. (3.2) by replacing $G^2(\cdot)/2$ by $L_t + G^2(\cdot)/2$ in Eq. (2.23).) It follows from Eqs. (3.2) and (2.23) that for almost all $\omega' \in \Omega_G$ with respect to P_G

$$\lim_{\delta \to 0} \sup_{\substack{d(y, y_0) \leqslant \delta \\ y \in S}} \frac{|L_t^y - L_t^{y_0}|}{\rho(d(y, y_0))} \leqslant \sqrt{2} \left| L_t^{y_0} + \frac{G^2(y_0, \omega')}{2} \right|^{1/2} + |G(y_0, \omega')| \tag{3.3}$$

for almost all $t \in [0, \zeta)$ almost surely with respect to P^x. Note that $G(y_0)$ is a mean zero normal random variable with finite variance. Hence, given $\varepsilon > 0$, we can choose ω' so that $|G(y_0, \omega')| < \varepsilon$. Thus we get Eq. (1.5) with a less than or equal sign. A similar argument gives the opposite inequality in Eq. (1.5). It is also clear from this proof that if ρ is a local modulus of continuity for G then we get Eq. (1.5) with a less than or equal sign. □

In order to prove Theorems 1.2 and 1.4 we need the following lemma about continuous Gaussian processes.

Lemma 3.3. Let $G = \{G(y), y \in K\}$, K a compact separable metric space, be a mean zero Gaussian process with continuous sample paths. Then for all $\varepsilon > 0$, we have

$$P(\sup_{y \in K} |G(y)| \leqslant \varepsilon) > 0 \tag{3.4}$$

Proof. Consider the Karhunen-Loeve expansion of G

$$G(y) = \sum_{j=1}^{\infty} \xi_j \phi_j(y) \qquad y \in K \tag{3.5}$$

where $\{\xi_j\}_{j=1}^{\infty}$ are independent normal random variables with mean 0 and variance 1 and $\{\phi_j(\cdot)\}_{j=1}^{\infty}$ are continuous functions on K, as described in Ref. 16, Theorem 2.6. Since G is continuous, it follows from this Theorem that the series in Eq. (3.5) converges uniformly on K. Hence given any $\varepsilon_1 > 0$ we can find an $N(\varepsilon_1)$ such that

$$E \sup_{y \in K} \left| \sum_{j = N(\varepsilon_1) + 1}^{\infty} \xi_j \phi_j(y) \right| < \varepsilon_1 \tag{3.6}$$

Let $\|\cdot\|$ denote the sup-norm on $C(K)$, then

$$P\left(\left\|\sum_{j=1}^{\infty}\xi_j\phi_j\right\|\leqslant\varepsilon\right)\geqslant P\left(\left\|\sum_{j=1}^{N(\varepsilon_1)}\xi_j\phi_j\right\|\leqslant\frac{\varepsilon}{2}\right)\left(1-P\left(\left\|\sum_{j=N(\varepsilon_1)+1}^{\infty}\xi_j\phi_j\right\|>\frac{\varepsilon}{2}\right)\right)$$

Thus if $\varepsilon_1=\varepsilon/4$, we get from Eq. (3.6) and Chebysev's Inequality, that

$$P\left(\left\|\sum_{j=1}^{\infty}\xi_j\phi_j\right\|\leqslant\varepsilon\right)\geqslant\frac{1}{2}P\left(\left\|\sum_{j=1}^{N(\varepsilon_1)}\xi_j\phi_j\right\|\leqslant\varepsilon/2\right)\tag{3.7}$$

It is easy to see that this last probability is strictly positive since the ϕ_j are bounded on K and the $\{\xi_j\}_{j=1}^{N(\varepsilon_1)}$ are simply a collection of independent normal random variables with mean zero and variance one. Clearly Eq. (3.4) follows from Eq. (3.7). □

Proof. *(Theorem 1.2).* The proof of this Theorem follows precisely the proof of Theorem 1.1 except that Lemma 3.1 and Remark 3.2 are applied to Theorem 2.6 and instead of Eq. (3.3) we get that for all $\omega'\in\Omega_G$ with respect to P_G

$$\lim_{\delta\to 0}\sup_{\substack{d(x,y)\leqslant\delta\\ x,y\in K}}\frac{|L_t^x-L_t^y|}{\omega(d(x,y))}\leqslant\sup_{y\in K}\sqrt{2}\left|L_t^y+\frac{G^2(y,\omega')}{2}\right|^{1/2}+\sup_{y\in K}|G(y,\omega')|\tag{3.8}$$

We see that Eq. (1.6) now follows by Lemma 3.3. □

Proof. *(Theorem 1.3).* The proof of this Theorem is precisely the same as the proof of Theorem 1.1 except that Lemma 3.1 and Remark 3.2 are applied to Theorem 2.4, which along with Theorem 2.6 gives us that for almost all $\omega'\in\Omega_G$ with respect to P_G

$$\lim_{\delta\to 0}\sup_{\substack{d(x,y)\leqslant\delta\\ x,y\in K}}\frac{|L_t^x-L_t^y|}{\omega(d(x,y))}\geqslant\sqrt{2}\left|L_t^{y_0}+\frac{G^2(y_0,\omega')}{2}\right|^{1/2}+\sup_{y\in K}|G(y,\omega')|\tag{3.9}$$

Using Lemma 3.3 we get Eq. (1.7). □

Proof. *(Theorem 1.4).* The upper-bound in Eq. (1.8) is given in Theorem 1.2. The lower-bound uses Lemma 3.1 and Remark 3.2 applied to Theorem 2.7 and Lemma 3.3, as in the proof of Theorem 1.2. □

Remark 3.4. In this paper we made the simplifying assumption that the given metric ρ and d induce the same topology on S. (This is the case in all examples that we know.) Because we are lifting results from the theory of Gaussian processes and applying them to the study of local times our results are stated in terms of d. The simplifying assumption guarantees that (S, d) is also locally compact and separable. Let us now consider the general case in which we just consider the given metric space (S, ρ). Since we are studying moduli of continuity we naturally assume that $\{L_t^y, y \in S\}$ is continuous for some t, and hence for all t by in Ref. 16, Theorem VI. With this assumption we have by in Ref. 16, Theorem 3.7, that d is continuous on (S, ρ). We now see that all our results on the uniform moduli of continuity hold without modification. This is because, by it's very nature, results on the uniform modulus of continuity relate to a compact subset of S, (e.g. K in Eq. (1.3)) and ρ and d induce the same topology on a compact subset of (S, ρ). In order to consider local moduli of continuity in the general case we make a minor adjustment in the statement of our results. In Eq. (1.4), and all similar expressions about local moduli, we replace $y \in S$ by $y \in K$, for K a compact subset of (S, ρ). This is really no restriction in considering local moduli since (S, ρ) is locally compact.

4. NECESSARY AND SUFFICIENT CONDITIONS FOR MODULI OF CONTINUITY OF GAUSSIAN PROCESSES

In this paper we show how results about the moduli of continuity of the local times of strongly symmetric standard Markov processes can be obtained from corresponding properties of Gaussian processes. The main point of all this is that, in most cases, necessary and sufficient coditions are known for these properties of the Gaussian processes and therefore we now have them for the local times. We will survey some of these conditions for Gaussian processes in this Section.

There are two aspects to the consideration of the moduli of continuity of Gaussian processes. First we have the question of their existence and then the problem of estimating them. Not all Gaussian processes have local or uniform moduli of continuity. The process $G(y) = gy$, $y \in [0, 1]$, where g is a normal random variable with mean zero and variance one has neither since $|G(x) - G(y)|/|x - y| = |g|$ for $x, y \in [0, 1]$. The next theorem gives a necessary and sufficient condition for the existence of an exact local modulus of continuity and an exact uniform modulus of continuity for a Gaussian process. The major portion of the theorem, Eq. (4.6) and (4.7), is due to Fernique.[8]

Let $\{X(z), z \in S\}$ be a separable Gaussian process where S is an arbitrary index set. The median of $\sup_{z \in S} |X(z)|$ is well defined. It is the real number m satisfying both

$$P(\sup_{z \in S} |X(z)| \leqslant m) \geqslant \tfrac{1}{2} \quad \text{and} \quad P(\sup_{z \in S} |X(z)| \geqslant m) \geqslant \tfrac{1}{2} \tag{4.1}$$

Moreover,

$$|E \sup_{z \in S} |X(z)| - m| \leqslant \sigma \sqrt{\frac{2}{\pi}} \tag{4.2}$$

where

$$\sigma = \sup_{z \in S} (E(X(z))^2)^{1/2} \tag{4.3}$$

(For details see Ref. 16, Remark 2.3, and material that follows it).

Theorem 4.1. Let $G = \{G(y),\ y \in K\}$, (K, d) a compact metric space, be a mean zero Gaussian process with continuous sample paths. For $\delta > 0$ let

$$m(\delta) = \text{median of } \sup_{\substack{d(y, y_0) \geqslant \delta \\ t \in K}} \frac{|G(y) - G(y_0)|}{d(y, y_0)} \tag{4.4}$$

for some fixed $y_0 \in K$ and

$$\hat{m}(\delta) = \text{median of } \sup_{\substack{d(x, y) \geqslant \delta \\ x, y \in K}} \frac{G(x) - G(y)}{d(x, y)} \tag{4.5}$$

Then

If $\lim_{\delta \to 0} m(\delta) = \infty$, $\rho(\delta) = \delta m(\delta)$ is an exact local modulus of continuity for G. (4.6)

If $\lim_{\delta \to 0} \hat{m}(\delta) = \infty$, $\omega(\delta) = \delta \hat{m}(\delta)$ is an exact uniform modulus of continuity for G. (4.7)

If $m(\delta)$ is bounded, G does not have an exact local modulus of continuity. (4.8)

If $\hat{m}(\delta)$ is bounded, G does not have an exact uniform modulus of continuity. (4.9)

Proof. The statements in Eqs. (4.6) and (4.7) follow immediately from Corollaries 3.24 and 3.23 respectively in Ref. 8. These Corollaries are expressed in a slightly different way but it is clear that they are equivalent to Eqs. (4.6) and (4.7).

To obtain Eq. (4.9) let us assume that $\hat{m}(\delta)$ is bounded but that G does have an exact uniform modulus of continuity ω, i.e. that Eq. (1.3) holds. The condition that $\hat{m}(\delta)$ is bounded implies that there exists a constant C such that for all $0 < \delta < \delta'$, for some $\delta' > 0$

$$E\left(\sup_{\substack{d(x,y) \geq \delta \\ x, y \in K}} \frac{G(x) - G(y)}{d(x,y)}\right) \leq C \tag{4.10}$$

(To see this we use Eq. (4.2) applied to the process $(G(x) - G(y))/d(x, y)$. We note that in this case the parameter σ, defined in Eq. (4.3), is equal to 1.) It follows from Eq. (4.10) and the Monotone Convergence Theorem that

$$E\left(\varlimsup_{\delta \to 0} \sup_{\substack{d(x,y) = \delta \\ x, y \in K}} \frac{G(x) - G(y)}{\delta}\right) \leq C$$

and hence that

$$\varlimsup_{\delta \to 0} \sup_{\substack{d(x,y) = \delta \\ x, y \in K}} \frac{G(x) - G(y)}{\delta} < \infty \qquad \text{a.s.} \tag{4.11}$$

But now, writing

$$\varlimsup_{\delta \to 0} \sup_{\substack{d(x,y) = \delta \\ x, y \in K}} \frac{G(x) - G(y)}{\omega(\delta)} = \varlimsup_{\delta \to 0} \sup_{\substack{d(x,y) = \delta \\ x, y \in K}} \frac{G(x) - G(y)}{\delta} \frac{\delta}{\omega(\delta)}$$

and using Eqs. (4.11) and (2.3), which follow by our assumption that ω is an exact uniform modulus of continuity, we get that

$$\varlimsup_{\delta \to 0} \sup_{\substack{d(x,y) = \delta \\ x, y \in K}} \frac{G(x) - G(y)}{\omega(\delta)} = 0 \qquad \text{a.s.}$$

which contradicts the assumption that ω is an exact uniform modulus of continuity. Thus we have established Eq. (4.9).

The proof of Eq. (4.8) is exactly the same as the proof of Eq. (4.9) except that we need the fact that if ρ is an exact local modulus of continuity for G then it satisfies the analogue of Eq. (2.3) for the local modulus. The proof of this fact is similar to the proof that Eq. (2.2) implies Eqs. (2.3) in Lemma 2.1 but with ω replaced by ρ. □

We next present another result of Fernique [Ref. 8, Theorem 3.25] which gives local and uniform moduli of continuity of Gaussian processes in terms of more familiar quantities than $m(\delta)$ and $\hat{m}(\delta)$.

Theorem 4.2. Let $G=\{G(y),\ y\in K\}$, (K,d) a compact metric space, be a mean zero Gaussian process with continuous sample paths. For $\delta>0$ set

$$f_{y_0}(\delta)=\text{median of} \sup_{\substack{d(y,y_0)\leqslant\delta\\ y\in K}} |G(y)-G(y_0)| \tag{4.12}$$

for some fixed $y_0\in K$ and

$$f(\delta)=\text{median of} \sup_{\substack{d(x,y)\leqslant\delta\\ x,y\in K}} G(x)-G(y) \tag{4.13}$$

Then, for all $y_0\in K$

$$\lim_{\delta\to 0} \sup_{\substack{d(y,y_0)\leqslant\delta\\ y\in K}} \frac{|G(y)-G(y_0)|}{f_{y_0}(d(y,y_0))+d(y,y_0)(2\log\log 1/f_{y_0}(d(y,y_0)))^{1/2}}\leqslant 1 \quad \text{a.s.} \tag{4.14}$$

and

$$\lim_{\delta\to 0} \sup_{\substack{d(x,y)\leqslant\delta\\ x,y\in K}} \frac{G(x)-G(y)}{f(d(x,y))+d(x,y)(2\log\log 1/f(d(x,y)))^{1/2}}\leqslant 1 \quad \text{a.s.} \tag{4.15}$$

Theorem 4.2 shows how to estimate uniform and local moduli of continuity for Gaussian processes. Note that if follows from Eq. (4.2) that

$$|E \sup_{\substack{d(y,y_0)\leqslant\delta\\ y\in K}} |G(y)-G(y_0)|-f_{y_0}(\delta)|=\delta\sqrt{\frac{2}{\pi}} \tag{4.16}$$

and

$$|E \sup_{\substack{d(x,y)\leqslant\delta\\ x,y\in K}} (G(x)-G(y))-f(\delta)|=\delta\sqrt{\frac{2}{\pi}} \tag{4.17}$$

and as we remarked earlier, $\delta=o(f(\delta))$ if $f(\delta)$ is a uniform modulus of continuity for G, and similarly for f_{y_0}. Thus, very often, we can work with expectations rather than with the median.

Generally, continuity of Gaussian processes is proved by giving an upper bound for the expectation in Eq. (4.17). By a result of Fernique [Ref. 9, Sec. 6], (see also Ref. 19, Theorem 17), we have that

$$E \sup_{\substack{d(x,y)\leqslant\delta\\ x,y\in K}} G(x)-G(y)\leqslant C\sup_{y\in K}\int_0^{\delta}\left(\log\frac{1}{m(B(y,\varepsilon))}\right)^{1/2} d\varepsilon \tag{4.18}$$

for all probability measures m on (K, d). In this inequality and in the one that follows C is an absolute constant, not necessarily the same in each case.

In terms of metric entropy we have the following extension of Dudley's[4] sufficient condition for the continuity of Gaussian processes, (see also [Ref. 15, II Theorem 3.1]).

$$E \sup_{\substack{d(x,y) \leqslant \delta \\ x, y \in K}} G(x) - G(y) \leqslant C \left(\int_0^\delta (\log N(K, \varepsilon))^{1/2} \, d\varepsilon + \hat{d}\phi(\delta/4\hat{d}) \right) \tag{4.19}$$

were $\phi(x) = x(\log \log 1/x)^{1/2}$, $\hat{d} \equiv \sup_{x, y \in K} d(x, y)$ and $N(K, \varepsilon)$ is the minimum number of D balls of radius ε that covers K.

The existence of uniform or local moduli of continuity do not imply the existence of an exact uniform or local modulus of continuity because the limits in Eqs. (1.3) and (1.4) could be zero. However, the functions $f_{y_0}(\delta)$ and $f(\delta)$, defined in Theorem 4.2, are often exact moduli continuity. The next result is an immediate corollary of Theorem 4.2.

Theorem 4.3. Let $G = \{G(y), y \in K\}$, (K, d) a compact metric space, be a mean zero Gaussian process with continuous sample paths. Let $f_{y_0}(\delta)$ and $f(\delta)$ be as defined in Eqs. (4.12) and (4.13). If

$$\delta(\log \log 1/\delta)^{1/2} = o(f_{y_0}(\delta)) \tag{4.20}$$

$f_{y_0}(\delta)$ is an exact local modulus of continuity for G at y_0. If

$$\delta(\log \log 1/\delta)^{1/2} = o(f(\delta)) \tag{4.21}$$

$f(\delta)$ is an exact uniform modulus of continuity for G.

Proof. It is obvious that $f_{y_0}(\delta) \geqslant c\delta$ for some constant c independent of δ and so Eqs. (4.20) and (4.14) imply that $f_{y_0}(\delta)$ is a local modulus of continuity for G. Also, by the definition of median

$$P\left(\sup_{\substack{d(y, y_0) \leqslant \delta \\ y \in S}} \frac{|G(y) - G(y_0)|}{f_{y_0}(\delta)} \geqslant 1 \right) \geqslant \frac{1}{2}$$

This implies that there exists a set Ω' with $P_G(\Omega') \geqslant 1/2$ such that on Ω'

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{d(y, y_0) \leqslant \delta \\ y \in S}} \frac{|G(y) - G(y_0)|}{f_{y_0}(\delta)} \geqslant 1 \tag{4.22}$$

However, by Eq. (4.20) and the analogue of Eq. (2.3) implies Eq. (2.4) for the local modulus, which is also valid, we see that Eq. (4.22) holds almost surely. It is easy to see, since $f_{y_0}(\delta)$ is monotone, that Eq. (4.22) implies that $f_{y_0}(\delta)$ is also a lower modulus of continuity for G and hence an exact modulus of continuity for G. A similar proof gives the result for the exact uniform modulus of continuity. This completes the proof of Theorem 4.3. □

By Theorem 4.2, since $f(\delta) \geqslant c\delta$, we see that

$$f(\delta) + \delta(2 \log\log 1/\delta)^{1/2}$$

is a uniform modulus of continuity for G. But exact uniform moduli of continuity are often of the order of $\delta(2 \log 1/\delta)^{1/2}$ as in [Ref. 18, Sec. 3] and [Ref. 1, Theorem 2] so Eq. (4.21) of Theorem 4.3 is often satisfied. (Although, it is possible for Gaussian processes to have uniform moduli on the order of $\delta(\log\log 1/\delta)^{1/2}$, see Ref. 13, Sec. 3). On the other hand exact local moduli are often on the order of $\delta(\log\log 1/\delta)^{1/2}$ and $f_{y_0}(\delta)$ is often not a local modulus. However, there are important examples when it is. We shall say more about this in Section 5.

The next Theorem gives a sufficient condition for a Gaussian process with stationary increments to have exact moduli of continuity.

Theorem 4.4. Let $G = \{G(y),\ y \in I\}$, I a closed interval of R, be a mean zero continuous Gaussian process with stationary increments. Assume that $d^2(x, y) = \sigma^2(|x-y|)$ where σ^2 is regularly varying at zero with index $0 \leqslant \alpha < 2$, i.e. $\sigma^2(|u|) = |u|^\alpha L(|u|)$ where L is slowly varying at zero and $0 \leqslant \alpha < 2$. Then G has an exact uniform modulus of continuity on $[0, 1]$ and an exact local modulus of continuity at all points of $[0, 1]$.

Proof. We consider the uniform modulus of continuity first. By Theorem 4.1 we need only show that $\lim_{\delta \to 0} \hat{m}(\delta) = \infty$ for $\hat{m}(\delta)$ as defined in Eq. (4.5). Let M, N, l be integers with $N > M > l > 1$. Consider

$$\xi_k = \frac{G\left(\dfrac{kl+1}{lN}\right) - G\left(\dfrac{k}{N}\right)}{\sigma\left(\dfrac{1}{lN}\right)} \qquad k = 0,\ldots, M-1 \tag{4.23}$$

For $j, k = 0,\ldots, M-1$; $j \neq k$ we have

$$E\xi_j\xi_k = \frac{\sigma^2\left(\dfrac{|(j-k)l+1|}{lN}\right) + \sigma^2\left(\dfrac{|(j-k)l-1|}{lN}\right) - 2\sigma^2\left(\dfrac{|j-k|}{lN}\right)}{2\sigma^2\left(\dfrac{1}{lN}\right)} \tag{4.24}$$

where $|j-k|<M$. Note that since L is slowly varying at zero we have

$$\sup_{\substack{j,k=0,\dots,M-1\\ a=-1,0,1}}\left|\frac{L\left(\dfrac{|(j-k)+a/l|}{N}\right)}{L\left(\dfrac{1}{lN}\right)}\right|\leqslant 1+\varepsilon'(N)$$

where $\lim_{N\to\infty}\varepsilon'(N)=0$. Using this in Eq. (4.24) we get

$$E\xi_j\xi_k\leqslant\frac{3\alpha|\alpha-1|}{(|j-k|\,l)^{2-\alpha}}+5\varepsilon'(N)M^{\alpha}\qquad j,k=0,\dots,M-1\qquad j\neq k \tag{4.25}$$

for l sufficiently large. Therefore, given $\varepsilon>0$ we then take N and l sufficiently large such that

$$E\xi_j\xi_k\leqslant\varepsilon\qquad j,k=0,\dots,M-1\qquad j\neq k \tag{4.26}$$

Also, clearly

$$E\xi_k^2=1 \tag{4.27}$$

Let $\{\eta_k\}_{k=0}^{M-1}$ and ρ be independent normal random variables with mean zero and with $E\eta_k^2=1-\varepsilon$, $k=0,\dots,M-1$ and $E\rho^2=\varepsilon$. Define $\rho_k=\rho+\eta_k$, $k=0,\dots,M-1$. Note that

$$E\xi_k^2=E\rho_k^2=1\qquad k=0,\dots,M-1$$

$$E\xi_k\xi_j\leqslant E\rho_k\rho_j\qquad j,k=0,\dots,M-1\qquad j\neq k$$

Therefore, by Slepian's Lemma (see e.g. [Ref. 10, II Lemma 4.3])

$$P(\sup_{0\leqslant k\leqslant M-1}\xi_k>\lambda)\geqslant P(\sup_{0\leqslant k\leqslant M-1}\rho_k>\lambda) \tag{4.28}$$

This last probability is easy to calculate and we see that for M sufficiently large

$$P(\sup_{0\leqslant k\leqslant M-1}\xi_k>(2(1-\varepsilon)\log M)^{1/2})\geqslant 3/4 \tag{4.29}$$

It is clear from the definition of ξ_k that

$$\hat{m}\left(\frac{1}{lN}\right)>\text{median of}\sup_{0\leqslant k\leqslant M-1}\xi_k$$

and so $\hat{m}(1/lN) \geqslant (\log M)^{1/2}$. But we can take M as large as we like as long as we allow N to increase. Thus we see that $\lim_{\delta \to 0} \hat{m}(\delta) = \infty$ which is what we wanted to prove. □

The proof that an exact local modulus of continuity exists is similar. Since G is stationary we need only show this at $y = 0$. We consider

$$\xi_k' = \frac{G(2^{-(N-k)l}) - G(0)}{\sigma(2^{-(N-k)l})} \qquad k = 0,..., M-1 \tag{4.30}$$

where, as before, M, N, l are integers with $N > M > l$. Because of the assumption of regular variation we can obtain Eqs. (4.26) and (4.27) for $\{\xi_k'\}_{k=0}^{M-1}$ and following the previous proof we can show that $\lim_{\delta \to 0} m(\delta) = \infty$, for m defined in Eq. (4.4).

Theorem 4.4 gives a condition for the exact uniform and local modulus of continuity of a Gaussian process to exist. Moreover, the moduli are given in Eqs. (4.6) and (4.7). However, it is generally not clear how to compute them. In the next Section we will obtain more concrete expressions for the exact moduli of the local times of certain strongly symmetric standard Markov processes.

There are other ways to describe moduli of continuity for Gaussian processes that also lead to moduli of continuity for the local times of the associated Markov processes. Let K be a compact subset of R^n. An increasing function $\hat{\sigma}$ for which

$$d(x, y) \leqslant \sigma(|x-y|) \qquad \forall x, y \in K \tag{4.31}$$

is called a monotone majorant for d. For a Gaussian process on R^1 the right-hand side of Eq. (4.19) can be bounded by an integral expression involving a monotone majorant for d since the metric entropy with respect to d can be bounded by a function of the monotone majorant. In fact this is how one can derive part of the following result which is given in [Ref. 10, IV Theorem 1.3], for $I = [0, 1]$.

Theorem 4.5. Let $G = \{G(y), y \in I\}$, I a closed interval of R, be a mean zero Gaussian process with continuous sample paths. Let $\hat{\sigma}$ satisfy Eq. (4.31). Define

$$g(\delta) = \int_0^{1/2} \frac{\hat{\sigma}(\delta u)}{u(\log 1/u)^{1/2}}\, du + \hat{\sigma}(\delta)(\log\log 1/\delta)^{1/2} \tag{4.32}$$

and

$$h(\delta) = \int_0^{\delta} \frac{\hat{\sigma}(u)}{u(\log 1/u)^{1/2}}\, du + \hat{\sigma}(\delta)(\log 1/\delta)^{1/2} \tag{4.33}$$

Then there exist finite constants C and C', independent of I, such that for each $y_0 \in I$

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{|u| \leqslant \delta \\ y_0 + u \in I}} \frac{|G(y_0+u) - G(y_0)|}{g(\delta)} \leqslant C \qquad \text{a.s.} \tag{4.34}$$

and

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{|x-y| \leqslant \delta \\ x, y \in I}} \frac{|G(x) - G(y)|}{h(\delta)} \leqslant C' \qquad \text{a.s.} \tag{4.35}$$

Proof. As we remarked earlier this result is given in Ref. 16, IV Theorem 1.3, for $I = [0, 1]$. However, it is clear, because of the homogeneity of the condition in Eq. (4.31), that the result is valid for all closed intervals $I \subset R$. □

The results in Eqs. (4.34) and (4.35) are for different types of moduli than those considered in Eqs. (1.3) and (1.4). The moduli defined in Eqs. (1.3) and (1.4) are given in terms of the L^2 metric $d = d(x, y)$, defined in Eq. (1.2), which determines a mean zero Gaussian process, (up to a fixed Gaussian variable), and relates it to a local time through the 1-potential density. One generally wants d to be continuous with respect to the metric on the state space but, even when this is the case, the relationship between the two metrics can be very irregular. However, in classical probability, in studying stochastic processes on R^n, it has been customary to consider moduli with respect to the Euclidean distance on R^n. Generally, the processes considered are those for which the metric d is nicely behaved in relation to the Euclidean metric. With respect to the processes considered in Theorem 4.5, if $d(x, y) = \hat{\sigma}(|x-y|)$ and $\hat{\sigma}$, g and h are strictly increasing; the relationships between the moduli considered in Eqs. (4.34) and (4.35), and Eqs. (1.4) and (1.3) are immediate and the two sets of expressions are equivalent. If these functions are not all strictly increasing but are fairly smooth with respect to each other, then one can pass between Eqs. (4.34) and (4.35), and Eqs. (1.4) and (1.3) on a case by case basis. However, there is a better way for us to handle the different ways in which the moduli are defined. It is to reprove Theorems 1.2–1.4 for the different types of moduli. Theorems 1.1–1.4 are simple consequences of Lemma 3.1 and Remark 3.2 applied to events for Gaussian processes that have measure one. The results in Section 2, which are used to establish the probability one events can also be obtained if the moduli of continuity are defined in different ways. In particular, using Lemma 3.1 and Remark 3.2, the almost sure events in Eqs. (4.34) and (4.25) and obvious analogues of Theorems 2.6 and 2.8, we obtain the following theorem.

Theorem 4.6. Let X be a strongly symmetric standard real valued Markov process with 1-potential density density $u^1(x, y)$. Let $\{L_t^y, (t, y) \in R^+ \times R\}$ be the local time of X and assume that for $x, y \in I$

$$(u^1(x, x) + u^1(y, y) - 2u^1(x, y))^{1/2} = d(x, y) \leqslant \hat{\sigma}(|x - y|) \tag{4.36}$$

Then for g, h, C and C' as given in Theorem 4.5 we have for each $y_0 \in I$, that

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{|u| \leqslant \delta \\ y_0 + u \in I}} \frac{|L_t^{y_0+u} - L_t^{y_0}|}{g(\delta)} \leqslant \sqrt{2}\, C(L_t^{y_0})^{1/2} \qquad \text{for almost all } t \in [0, \zeta) \quad \text{a.s.} \tag{4.37}$$

and

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{|x - y| \leqslant \delta \\ x, y \in I}} \frac{|L_t^x - L_t^y|}{h(\delta)} \leqslant \sqrt{2}\, C' \sup_{y \in I} (L_t^y)^{1/2} \qquad \text{for almost all } t \in [0, \zeta) \quad \text{a.s.} \tag{4.38}$$

Here is still another way to define moduli of continuity that is commonly used. Employing the notation of the paragraph containing Eqs. (1.3) and (1.4) we call $\omega_m(\delta)$ an exact uniform m-modulus of continuity for $\{G(y), y \in K\}$ if

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{d(x, y) \leqslant \delta \\ x, y \in K}} \frac{|G(x) - G(y)|}{\omega_m(\delta)} = 1 \qquad \text{a.s.} \tag{4.39}$$

We call $\rho_m(\delta)$ an exact local m-modulus of continuity for $\{G(y), y \in S\}$ at some fixed $y_0 \in S$ if

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{d(y, y_0) \leqslant \delta \\ y \in S}} \frac{|G(y) - G(y_0)|}{\rho_m(\delta)} = 1 \qquad \text{a.s.} \tag{4.40}$$

As in Section 1 we also define m-moduli and lower m-moduli. We use the expression m-modulus because $\omega_m(\delta)$ and $\rho_m(\delta)$ can essentially always be taken to be monotone. (To be more specific, let B denote the range of $d(x, y)$ for $x, y \in K$ and assume $0 \in B$. Assume that for $u \in B$, $\lim_{u \to 0} \omega_m(u) = 0$ and that there exists an $\varepsilon > 0$ contained in B such that for all $\delta \in B$ such that $\delta > 0$, $\inf_{u \in B_{\delta,\varepsilon}} \omega_m(u) > 0$, where $B_{\delta,\varepsilon} \equiv \{B \cap [\delta, \varepsilon]\}$. Then for $0 \leqslant \delta \leqslant \varepsilon$, $\tilde{\omega}_m(\delta) = \inf_{u \in B_{\delta,\varepsilon}} \omega_m(u)$ is monotone and an excact uniform m-modulus of continuity for $\{G(y), y \in K\}$. Clearly, we can define

$\tilde{\omega}_m(\delta)$ arbitrarily for $\delta > \varepsilon$. A similar argument also applies to the exact local m-modulus of continuity.)

The moduli ω and ρ, defined in Section 1, are not necessarily monotone. In fact ω and ρ are highly nonunique since, in many cases, we can take $\overline{\lim}_{\delta \to 0} \omega(\delta) = \infty$ and still get Eq. (1.3) and similarly for ρ. Of course one tries to find moduli which are monotone. An exact uniform modulus of continuity for $\{G(y), y \in K\}$ that is monotone is also an exact uniform m-modulus of continuity for $\{G(y), y \in K\}$ and similarly for the local moduli. Furthermore, everything in this paper in Sections 1–3 remain exactly the same if we use m-moduli throughout instead of moduli. That is, Theorems 1.1–1.4 are valid for m-moduli as is all of Section 2 which is used to prove these Theorems. We chose the definitions of uniform and local moduli given in Eqs. (1.3) and (1.4) because Theorem 4.1 gives us necessary and sufficient conditions for the existence of the moduli defined in Eqs. (1.3) and (1.4) but not for the m-moduli defined in Eqs. (4.39) and (4.40). However, in all the classical examples that we know of the moduli are monotone or equivalent to a monotone function so the distrinction between the two definitions have not been important up to now. (One might say that Paul Levy's uniform modulus of continuity for Brownian motion is generally written in the form of Eq. (1.3) rather than Eq. (4.39) and so this justifies our choice. However, since Levy's modulus is monotone, the two definitions are equivalent.)

Another often used expression for the exact uniform modulus of continuity of a stochastic process is given by writing the expression in Eq. (1.3) with $d(x, y) \leqslant \delta$ replaced by $d(x, y) = \delta$ and the limit replaced by the limit superior. It is easy to see that this is just an equivalent formulation of Eq. (1.3). The same argument applies to the exact modulus defined in Eq. (1.4).

5. REAL VALUED MARKOV PROCESSES

We now give some results on the local and uniform modulus of continuity for local times of real valued Markov processes. Although these results are not restricted to Lévy processes, many of them are new for Lévy processes.

Theorem 5.1. Let X be a strongly symmetric standard real valued Markov process with 1-potential density density $u^1(x, y)$. Let $\{L_t^y, (t, y) \neq R^+ \times R\}$ be the local time of X and let $d(x, y)$ be as given in Eq. (1.2). Suppose that for some $\delta' > 0$, there exists a nondecreasing function $\rho(\delta)$, $\delta \in [0, \delta']$ and constants $0 < C_0 \leqslant C_1 < \infty$ such that for all $x, y \in I$, a closed interval of R,

$$C_0\rho(|x-y|) \leqslant d(x, y) \leqslant C_1\rho(|x-y|) \qquad \forall\, |x-y| \leqslant \delta' \tag{5.1}$$

$$\rho^2(2^{-n}) - \rho^2(2^{-n-1}) \qquad \text{is nonincreasing as } n \to \infty \tag{5.2}$$

$$\rho^2(2^{-n}) \leqslant 2^\alpha \rho^2(2^{-n-1}) \qquad \text{for some } \alpha < 2 \quad \forall n \text{ sufficiently large} \tag{5.3}$$

and

$$I_\rho(\delta) \equiv \int_0^{1/2} \frac{\rho(\delta u)}{u(\log 1/u)^{1/2}}\, du < \infty$$

Let

$$\psi(\delta) = \max(I_\rho(\delta), \rho(\delta)(\log\log 1/\delta)^{1/2})$$

Then there exist constants $0 < C_2 \leqslant C_3 < \infty$ such that for each $y_0 \in I$

$$\varlimsup_{\delta \to 0} \sup_{\substack{|u| \leqslant \delta \\ y_0 + u \in I}} \frac{|L_t^{y_0+u} - L_t^{y_0}|}{\psi(u)} = C(y_0)(L_t^{y_0})^{1/2} \qquad \text{for almost all } t \in [0, \zeta) \qquad \text{a.s.} \tag{5.4}$$

where $C_2 \leqslant C(y_0) \leqslant C_3$.

Proof. By Ref. 13, Theorem 3.8, (p. 303), if $\{G(x),\ x \in I\}$ is a Gaussian process with metric $d(x, y)$ then there exist constants $0 < C_2 \leqslant C_3 < \infty$ such that for each $y_0 \in I$

$$C_2 \leqslant \varlimsup_{\delta \to 0} \sup_{\substack{|u| \leqslant \delta \\ y_0 + u \in I}} \frac{|G(y_0+u) - G(y_0)|}{\psi(u)} \leqslant C_3 \qquad \text{a.s.} \tag{5.5}$$

(Reference 13, Theorem 3.8, is written for $I = [0, 2\pi]$ but can be extended to any closed interval I by covering I with closed intervals of length 2π.) Since Eq. (2.3) implies Eq. (2.4) is also valid for the local modulus of continuity, for each y_0 the limit in Eq. (5.5) is a constant $C'(y_0)$ which, by Eq. (5.5) satisfies $C_2 \leqslant C'(y_0) \leqslant C_3$. By Theorem 1.1, which also holds for this kind of modulus, we get Eq. (5.4).

Corollary 5.2. Continuing the notation of Theorem 5.1, we have the following examples of functions ρ which satisfy Eqs. (5.1)–(5.3) and functions $\tilde{\psi}$ for which Eq. (5.4) holds with ψ replaced by $\tilde{\psi}$. The constants C_2 and C_3 depend on $\tilde{\psi}$.

ρ is regularly varying with index greater than 0 (5.6)

$\tilde{\psi}(u) = \rho(u)(\log\log 1/|u|)^{1/2}$

$\rho^2(u) = \exp(-(\log 1/|u|)^\alpha(\log\log 1/|u|)^\beta),\ 0<\alpha<1,\ -\infty<\beta<\infty$ (5.7)

$\tilde{\psi}(u) = \rho(|u|)((\log 1/|u|)^{1-\alpha}(\log\log 1/|u|)^{-\beta})^{1/2}$

$\rho(u) = (\log 1/|u|)^{-\alpha},\ \alpha > 1/2$ (5.8)

$\tilde{\psi}(u) = \rho(|u|)(\log 1/|u|)^{1/2}$

$\rho(u) = ((\log 1/|u|)(\log\log 1/|u|)^\beta)^{-1/2},\ \beta > 2$ (5.9)

$\tilde{\psi}(u) = \rho(|u|)(\log 1/|u|)^{1/2}\log\log 1/|u|$

Proof. In each of the four cases it is straightforward to verify that ρ satisfies Eqs. (5.1)–(5.3). We will show that $\psi(u) \approx \tilde{\psi}(u)$ where we use $\psi(u) \approx g(u)$ to mean that there exist constants $0 < c \leqslant c' < \infty$ such that $c \leqslant \psi(u)/g(u) \leqslant c'$, for all $u \in [0, u_0]$ for some $u_0 > 0$. This gives Eq. (5.5) with ψ replaced by $\tilde{\psi}$ and possibly different constants. Nevertheless, this still implies Eq. (5.4) with ψ replaced by $\tilde{\psi}$ just as in the proof of Theorem 5.1. It is easy to check the approximation for ψ in Eq. (5.6). The approximation for ψ given in Eq. (5.7) comes from Ref. 13, Lemma 3.9. The approximations for ψ given in Eqs. (5.8) and (5.9) can be easily calculated using the following lemma which expresses $I_\rho(\delta)$ in terms of an integral that is easier to compute. □

Lemma 5.3. Let ρ be a nondecreasing function which satisfies Eq. (5.3), let

$$\phi(\delta) = \int_0^\delta \frac{\rho(u)}{u(\log 1/u)^{1/2}}\,du \tag{5.10}$$

and assume that

$$\rho(\delta)(\log 1/\delta)^{1/2} = O(\phi(\delta)) \tag{5.11}$$

Then $\phi(\delta) \approx I_\rho(\delta)$.

Proof. By Eq. (5.3), $\rho(2u) \leqslant 2^\alpha \rho(u)$. Using this we see that

$$\phi(\delta) \leqslant 2^\alpha \psi(\delta)$$

The other direction follows from Ref. 13, Lemma 4.4. □

Of course these results can be applied to real valued Lévy processes. In the rest of this paper we will concentrate on these processes. Let $\{X(t), t \in R^+\}$ be a real valued symmetric Lévy process, i.e.

$$E \exp(i\lambda X(t)) = \exp(-t\psi(\lambda)) \tag{5.12}$$

where

$$\psi(\lambda) = 2 \int_0^\infty (1 - \cos \lambda u)\, \nu(du) \tag{5.13}$$

for ν a Levy measure. It is known that X has a local time if and only if $(1+\psi(\lambda))^{-1} \in L^1(\mathbb{R}^+)$, (see e.g. Refs. 3 and 11). It is easy to see, by considering the characteristic function of X, that when $(1+\psi(\lambda))^{-1} \in L^1(R^+)$

$$u^1(x, y) + u^1(y, y) - 2u^1(x, y) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos \lambda(x-y)}{1+\psi(\lambda)}\, d\lambda \equiv \sigma^2(|x-y|) \tag{5.14}$$

Thus by Eq. (1.2) a symmetric real valued Lévy process is associated with a stationary Gaussian process for which $d(x, y) = \sigma(|x-y|)$.

The following Lemma shows that for symmetric Lévy processes on the real line, the function σ^2, defined in Eq. (5.14), can not be regularly varying with index greater than 1.

Lemma 5.4. Let $\{X(t), t \in R^+\}$ be a real valued symmetric Levy process and let ψ and σ be as defined in Eqs. (5.13) and (5.14). Then

$$\psi(\lambda) = O(\lambda^2) \qquad \text{as} \quad \lambda \to \infty \tag{5.15}$$

and

$$x = O(\sigma^2(x)) \qquad \text{as} \quad x \to 0 \tag{5.16}$$

Proof. Equation (5.15) is well known but we include it because it is used for Eq. (5.16). We have

$$\begin{aligned}\psi(\lambda) &= 2 \int_0^\infty \sin^2 \frac{\lambda x}{2}\, \nu(x) \\ &\leqslant \frac{\lambda^2}{2} \int_0^1 x^2 \nu(dx) + 2 \int_1^\infty \nu(dx)\end{aligned}$$

which, using properties of the Levy measure ν, gives Eq. (5.15). By

Eq. (5.15) there exist constants C and λ_0 such that $\psi(\lambda) \leqslant C\lambda^2$ for all $\lambda \geqslant \lambda_0$. Let $N \geqslant \lambda_0$ be such that $1/N^2 \leqslant C$. Then for $x < 1/N$ we have

$$\begin{aligned}\sigma^2(x) &= C_1 \int_0^\infty \sin^2 \frac{\lambda x}{2} \frac{1}{1+\psi(\lambda)} \, d\lambda \\ &\geqslant C_2 x^2 \int_N^{1/x} \frac{1}{1/N^2 + C} \, d\lambda \\ &\geqslant \frac{C_2 x^2}{2C} \left(\frac{1}{x} - N \right)\end{aligned}$$

where C_1 and C_2 are strictly positive constants. Clearly, Eq. (5.17) implies Eq. (5.16). □

We now present examples in which we can obtain exact moduli of continuity for some symmetric Lévy processes. The next theorem gives an exact local modulus of continuity and complements a similar result of Barlow,[1] Theorem 2, which deals with the exact uniform modulus of continuity.

Theorem 5.5. Let X be a real valued symmetric Lévy process with local time $L = \{L_t^y, (t, y) \in R^+ \times R\}$. Let $\sigma(|u|)$ be as defined in Eq. (5.14), (so that for symmetric Lévy processes $\sigma(|x-y|) = d(x, y)$) and let I be a closed interval of R. Then, if σ^2 is regularly varying with index $0 < \alpha \leqslant 1$, for each $y_0 \in I$

$$\varlimsup_{\delta \to 0} \sup_{\substack{|u| \leqslant \delta \\ y_0 + u \in I}} \frac{|L_t^{y_0+u} - L_t^{y_0}|}{(\sigma^2(|u|) \log\log 1/|u|)^{1/2}} = 2(L_t^{y_0})^{1/2} \qquad \text{for almost all } t \text{ a.s.} \tag{5.18}$$

Proof. This result is a direct consequence of Theorem 1.1 in this paper and Ref. 12, Theorems 5 and 6; keeping in mind, as we have remarked earlier, that Theorem 1.1 is also valid for the modulus in Eq. (5.18) and also that $\zeta = \infty$ for Lévy processes. (Kôno's[12] Theorems 5 and 6 contains a version of Eq. (1.4) for stationary Gaussian processes with L^2 metric σ^2 satisfying the conditions of this Theorem. In fact Kôno's conditions on σ^2 is actually weaker that the one in the statement of Theorem 5.5. Kôno[12] adds the condition that σ^2 is concave but he only uses the regular variation of σ^2 in the parts of these Theorems that we are using here.) □

Results of the form of Eq. (5.18) can also be obtained in some cases when σ^2 is slowly varying at zero by using the Theorems of Kôno mentioned or Theorem 6 in Ref. 14. Moreover, in dealing with symmetric

Lévy processes, the constant $C(y_0)$ in Eq. (5.4) is independent of y_0, since by Eq. (5.14), the Gaussian processes associated with Lévy processes are stationary.

Theorem 5.5 is not completely analagous to Ref. 1, Theorem 2, because the latter result is valid for all $t \in R^+$ not just almost all t. Barlow's Theorem 2 gives a uniform modulus of continuity for the local time of Lévy processes when σ^2 is regularly varying with index $0 < \alpha \leqslant 1$. We can not get this result by our methods. Our approach is to infer results for local times from corresponding ones for Gaussian processes. But special properties of the Gaussian processes associated with Lévy processes seem to be involved in Ref. 1, Theorem 2. We say this because the analogue of Theorem 5.3 in Ref. 1 is false for Gaussian processes. Let us be more precise. The approach of this paper is to use results for Gaussian processes to obtain results for the local times of the associated Markov processes. The result for Gaussian processes that might give Theorem 5.3 in Ref. 1 would be that $\sigma(u)(2 \log 1/u)^{1/2}$ is a lower uniform modulus of continuity for Gaussian processes for which σ satisfies the hypotheses of [Ref. 1, Theorem 5.3]. But this is not true as can be seen by the example in Ref. 13. Clearly, there are special properties of the Gaussian processes which are associated with Lévy processes that we would need to use to obtain Theorem 5.3 in Ref. 1. So far we have been unable to isolate these properties.

Exact uniform moduli of continuity are known for a wide class of Gaussian processes with stationary increments. However, all the results we know of require that σ^2 be concave on $[0, \delta]$ for some $\delta > 0$. This is not so great a restriction in the study of Gaussian processes in general since for every such function σ^2 one can find a stationary Gaussian process, say $\{G(x),\ x \in R\}$, such that $E(G(x+h) - G(x))^2 = \sigma^2(h)$, for $h \in [0, \delta]$, (see e.g. Chapter XV.3, Example b, Ref. 7). However, in general, it is hard to see whether the functions σ^2 defined in Eq. (5.14) are concave on $[0, \delta]$ for some $\delta > 0$. In fact, they are for symmetric stable processes with index greater than 1. However, we will use a slightly different approach to obtain exact uniform moduli of continuity for the local times of these processes. Of course this is contained in Ref. 1, Theorem 2; but our proof is simple and illustrates a useful technique of manipulating Gaussian processes. See also Ref. 1, Theorem 3.

Theorem 5.6. Let X be a symmetric stable process of index $1 < p \leqslant 2$. In particular, let $\psi(\lambda) = \lambda^p$ in Eq. (5.12). Let $\{L_t^y,\ (t, y) \in R^+ \times R\}$ be the local time of X. Then, for I a closed interval of R

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{|x-y| \leqslant \delta \\ x, y \in I}} \frac{|L_t^x - L_t^y|}{(|x-y|^{p-1} \log 1/|x-y|)^{1/2}} = 2(C_p)^{1/2} \sup_{y \in I} (L_t^y)^{1/2} \quad \text{for almost all } t \text{ a.s.} \qquad (5.19)$$

where

$$C_p = \frac{2}{\pi} \int_0^\infty \frac{(1-\cos y)}{y^p} \, dy \tag{5.20}$$

Proof. The proof follows immediately from Theorem 1.4, which also holds for this kind of modulus, (see the remarks following the proof of Theorem 4.5), and the fact that

$$(2C_p |x-y|^{p-1} \log 1/|x-y|)^{1/2} \tag{5.21}$$

in an exact uniform modulus of continuity for the stationary Gaussian process $\{G_p(x), x \in I\}$ for which

$$\sigma_p^2(|x-y|) = E(G_p(x) - G_p(y))^2 = \frac{2}{\pi} \int_0^\infty \frac{1-\cos \lambda(x-y)}{1+\lambda^p} \, d\lambda$$

To see this we introduce two Gaussian processes with stationary increments, $G_{p,i}(x), x \in I\}$, $i = 1, 2$, defined by

$$G_{p,i}(x) = \frac{1}{\sqrt{\pi}} \left(\int_0^\infty (1-\cos \lambda x) f_i(\lambda) \, dB(\lambda) + \int_0^\infty \sin \lambda x f_i(\lambda) \, dB'(\lambda) \right) \tag{5.22}$$

where

$$f_1(\lambda) = \frac{1}{\lambda^{p/2}} \qquad \text{and} \qquad f_2(\lambda) = \frac{1}{(\lambda^p(1+\lambda^p))^{1/2}}$$

and B and B' are independent Brownian motions. Let G_p, $G_{p,1}$ and $G_{p,2}$ be independent and note that $G_{p,1}(x)$ and $G_p(x) - G_p(0) + G_{p,2}(x)$ are equivalent Gaussian processes. (They have the same covariance.) Furthermore, we see by a change of variables that

$$E(G_{p,1}(x+h) - G_{p,1}(x))^2 = \frac{2}{\pi} \int_0^\infty \frac{1-\cos \lambda h}{\lambda^p} \, d\lambda = C_p h^{p-1}$$

It follows from Ref. 14, Theorem 7, that Eq. (5.21) is an exact uniform modulus of continuity for $G_{p,1}$. (The Theorem just cited is written for $I = [0, 1]$. It is easy to see, by a change of scale that it is valid for any closed interval I.) Therefore, Eq. (5.21) is an exact uniform modulus of continuity for $G_p(x) - G_p(0) + G_{p,2}(x)$. However, by Theorem 4.5 the term in $G_{p,2}$ is "little o" of this modulus. Thus Eq. (5.21) is an exact uniform modulus of continuity for G_p, which is what we wanted to show. This completes the proof of Theorem 5.6 □

Using Theorem 5.1 we can obtain some information about the uniform modulus of continuity of the local time of symmetric Lévy processes that is not contained in Ref. 1.

Theorem 5.7. Let X be a symmetric Lévy process and let $\{L_t^y, (t, y) \in R^+ \times R\}$ be the local time of X. Assume that

$$C_0 \rho(|x-y|) \leqslant \sigma(|x-y|) \leqslant C_1 \rho(|x-y|) \qquad \forall\, |x-y| \leqslant \delta'$$

for some $\delta' > 0$ where ρ is as given in Theorem 5.1 and also satisfies Eqs. (5.2) and (5.3) and σ is given in Eq. (5.14). Let $\phi(\delta)$ be as defined in Eq. (5.10) and assume that Eq. (5.11) holds. Let I be a closed interval of R. Then there exists a constant $0 < C < \infty$, independent of I, such that

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{|x-y| \leqslant \delta \\ x, y \in I}} \frac{|L_t^x - L_t^y|}{\phi(|x-y|)} = C \sup_{y \in I} (L_t^y)^{1/2} \qquad \text{for almost all } t \text{ a.s.} \tag{5.23}$$

In particular if

$$\rho(\delta) = (\log 1/\delta)^{-\alpha} \qquad \alpha > 1/2$$

then

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{|x-y| \leqslant \delta \\ x, y \in I}} \frac{|L_t^x - L_t^y|}{\rho(|x-y|)(\log 1/|x-y|)^{1/2}} = C \sup_{y \in I} (L_t^y)^{1/2} \qquad \text{for almost all } t \text{ a.s.} \tag{5.24}$$

and if

$$\rho(\delta) = ((\log 1/|\delta|)(\log\log 1/|\delta|)^{\beta})^{-1/2} \qquad \beta > 2$$

then

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{|x-y| \leqslant \delta \\ x, y \in I}} \frac{|L_t^x - L_t^y|}{\rho(|x-y|)((\log 1/|x-y|)\log\log 1/|x-y|)^{1/2}} = C \sup_{y \in I} (L_t^y)^{1/2} \qquad \text{for almost all } t \text{ a.s.} \tag{5.25}$$

(The constants C are not necessarily the same.)

Proof. Fix an interval I and let $\{G(x), x \in I\}$ be a mean zero Gaussian process with $E(G(x) - G(y))^2 = \sigma^2(|x-y|)$. It follows from Theorem 4.5, Eq. (5.11) and the fact that $\phi(\delta)$ is monotone that

$$\overline{\lim_{\delta \to 0}} \sup_{\substack{|x-y| \leqslant \delta \\ x, y \in I}} \frac{|G(x) - G(y)|}{\phi(|x-y|)} \leqslant D_1 \qquad \text{a.s.} \tag{5.26}$$

for some constant $D_1 < \infty$. Also the limit superior in Eq. (5.26) is greater than or equal to a constant $D_2 > 0$, since by Lemma 5.3 we get Eq. (5.5) with ψ replaced by ϕ and, of course, the uniform modulus is greater than the local modulus. It now follows from Lemma 2.1 that the limit superior in Eq. (5.26) is equal to a constant and it is clear, because of the homogeneity of the condition on σ, that this constant is independent of I. We then get Eq. (5.23) by Theorem 1.4, which also holds for this kind of modulus, since the Gaussian processes associated with Lévy processes are stationary. Equations (5.24) and (5.25) follow from Eq. (5.23). □

In this discussion of moduli of continuity we have confined our attention to the real line because the 1-potential density of Lévy processes on R^2 is not finite. However, there are examples of Markov processes on R^2 with continuous local times, such as in the recent work of Barlow and Bass[2] on the Sierpinski carpet. All the results on moduli of continuity in this Section can be extended to R^n whenever to local time is continuous.

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