Journal of Applied Mathematics and Physics (ZAMP)
Vol. 37, January 1986

0044-2275/86/001012-15 $ 4.50/0

Chaotic solutions of systems with almost periodic forcing

By Jürgen Scheurle, Mathematisches Institut A der Universität Stuttgart, Pfaffenwaldring 57, D-7000 Stuttgart 80, West Germany

1. Introduction

It is well known that the chaotic behaviour of certain dynamical systems can be explained by the presence of homoclinic or heteroclinic orbits. Consider, for instance, a diffeomorphism F on a two-dimensional manifold. If F has a hyperbolic saddle point x_0 and a corresponding transversal homoclinic point p, i.e. the stable and the unstable manifold of x_0 are intersecting each other transversally at p, then there is a set Λ near p which is invariant under a certain iterate F^m of F, and the flow on Λ generated by successively applying this iterate is topologically conjugate to the Bernoulli shift on a finite number of symbols. For two symbols this has been proved by Smale [23] using the horseshoe construction. Moser [19] generalized this result to any finite number of symbols. See also Guckenheimer and Holmes [9]. Examples of such diffeomorphisms F arise as period maps of periodic systems of differential equations.

Poincaré [21] was the first, who observed the importance of homoclinic and heteroclinic points. Further contributions were made by Birkhoff [3]. The so-called Melnikov function $M(t_0)$ provides a sufficient condition for the existence of a transversal homoclinic point for the period map of two-dimensional systems that result from adding a small periodic forcing term to an autonomous system which has a hyperbolic saddle point and a corresponding homoclinic orbit. See Melnikov [18] and also Arnold [1]. One considers the period map which advances solutions by one period starting at time t_0. Due to the robustness of hyperbolic saddle points, this map has a unique saddle point near the unperturbed one. So there are (local) stable and unstable manifolds corresponding to the perturbed saddle point, and $M(t_0)$ is nothing else but a first order approximation with respect to the amplitude of the forcing term for the distance of these manifolds in a certain one-dimensional cross section. If $M(t_0)$ has simple zeroes, then the period map has transversal homoclinic points. This method, nowadays known as Poincaré-Melnikov-Arnold method (see [17]), enables one to detect "chaos" in specific problems. Important contributions have recently been made by Chirikov [4], Holmes [11], Chow, Hale and Mallet-Paret [6], and Holmes and Marsden [12], [13]. A very nice presentation, in particular of the dynamical consequences, and an application to the periodically forced pendulum equation is given in Kirchgraber [16]. Finally, Palmer [20] generalized the whole theory to more than two dimensions based on the notion of exponential dichotomies. Palmer's theory also applies in certain situations, where the unforced equation has heteroclinic orbits rather than a homoclinic orbit.

Actually the present paper is a slight generalization of results contained in [20]. In fact, in two dimensions a "Melnikov function" $M(t_0)$ can even be defined for arbitrary bounded forcing terms, although then one does not have a period map anymore. One just considers the suspended system in the extended phase space $\mathbb{R}^3$ by adding the time variable to the original phase variables. Then the perturbed saddle point corresponds to a one-dimensional hyperbolic invariant set and one can measure the distance of the corresponding (local) stable and unstable manifolds in a certain one-dimensional cross section for each fixed value t_0 of the time variable. Again, $M(t_0)$ is just some approximation of this distance, and any simple zero of $M(t_0)$ corresponds to a transversal intersection of these manifolds along a solution curve in $\mathbb{R}^3$. But the question is, does this still imply the existence of a whole variety of solutions which behave more or less chaotically. Of course, one expects that in general the behaviour is even more irregular than in the periodic case. But is there still some order in this chaotic behaviour?

In this paper we shall give a positive answer to this question for almost periodic forcing terms, thereby generalizing the periodic case. Such forcing terms can oscillate with a large number of independent frequencies. In particular, different time scales can be involved. For example, this seems to be relevant for a deterministic description of turbulence in Hydrodynamics (cf. Kirchgässner [14], [15]). Also in bifurcation theory situations like this occur, when one considers unfoldings of singularities, the linear part of which has two or more pairs of purely imaginary eigenvalues (cf. Guckenheimer and Holmes [9], Chow and Hale [5], and Scheurle and Marsden [22]).

For a precise definition of what we mean by almost periodic, we refer to Definition 2.6. Our main result is stated in Theorem 2.11. It is stated and proved for the homoclinic case. But a straightforward generalization to certain situations involving heteroclinic orbits is possible (see Remark 2.12). As an example we consider the pendulum equation with a non-periodic forcing term (Example 2.13).

Acknowledgements. I would like to thank the referee for some helpful comments. Also I would like to thank Professor W. A. Coppel for his interest in this work and some suggestions for further research.

2. Results

In order to formulate and to prove our results we need to provide some knowledge on the notion of exponential dichotomies for ordinary differential equations. This is essentially taken from Coppel's book [7] and Palmer's paper [20].

Let $A(t)$ be a real $n \times n$ matrix function, continuous on an interval $J \subset \mathbb{R}$. Denote by $X(t)$ a fundamental matrix function for the system

$$\dot{x} = A(t)x. \tag{2.1}$$

This system is said to have an *exponential dichotomy on J* if there is a projection P in $\mathbb{R}^n$ and constants $K \geqq 1$, $\alpha > 0$ such that

$$\begin{aligned} |X(t) P X(s)^{-1}| &\leqq K e^{-\alpha(t-s)} \qquad (s \leqq t) \\ |X(t)(E-P) X^{-1}(s)| &\leqq K e^{-\alpha(s-t)} \qquad (s \geqq t) \end{aligned}. \tag{2.2}$$

Here E denotes the $n \times n$ unit matrix. For the projection matrix function $X(t) P X(t)^{-1}$ we use the notation $\mathscr{P}(t)$. When $J = [0, \infty)$ the range of $\mathscr{P}(0)$ is called *stable subspace*, and when $J = (-\infty, 0]$ the kernel of $\mathscr{P}(0)$ is called *unstable subspace*. These subspaces are uniquely determined in these cases, but not $\mathscr{P}(t)$. However, when $J = \mathbb{R}$, then $\mathscr{P}(t)$ is uniquely determined.

Proposition 2.1 (Coppel [7, p. 19]). *Let $A(t)$ be an $n \times n$ matrix function, defined and continuous on $\mathbb{R}$. Then system* (2.1) *has an exponential dichotomy on $\mathbb{R}$ if and only if it has an exponential dichotomy on both* $[0, \infty)$ *and* $(-\infty, 0]$, *and $\mathbb{R}^n$ is the direct sum of the stable and unstable subspaces.*

An important property of exponential dichotomies is their roughness with respect to small perturbations of the system. The following propositions are precise statements of this fact.

Proposition 2.2 (Coppel [7, p. 34]). *Let $A(t)$ and $B(t)$ be $n \times n$ matrix functions, bounded and continuous on $\mathbb{R}$, such that the Eq.* (2.1) *has an exponential dichotomy* (2.2) *on $\mathbb{R}$ with projection matrix function $\mathscr{P}(t)$. If*

$$\kappa = \sup_{t \in \mathbb{R}} |B(t)| < \alpha/4K^2$$

then the perturbed equation

$$\dot{x} = [A(t) + B(t)]\, x \tag{2.3}$$

also has an exponential dichotomy on $\mathbb{R}$, say with constants L and β, and if $\mathscr{Q}(t)$ is the corresponding projection matrix function, then

$$\sup_{t \in \mathbb{R}} |\mathscr{P}(t) - \mathscr{Q}(t)| = O(\kappa) \quad \text{as} \quad \kappa \to 0.$$

Moreover, L and β can be chosen as certain functions of K, α and κ such that

$$\begin{matrix} |K - L| = O(1) \\ |\alpha - \beta| = O(\kappa) \end{matrix} \quad \text{as} \quad \kappa \to 0.$$

Proposition 2.3 (Palmer [20]). *Let $A(t)$ and $B(t)$ be $n \times n$ matrix functions bounded and continuous on $[t_0, \infty)$ or $(-\infty, t_0]$ for some $t_0 \in \mathbb{R}$. Assume that the Eq. (2.1) has an exponential dichotomy on that interval with projection matrix function $\mathscr{P}(t)$, and that $B(t) \to 0$ as $t \to \pm\infty$, respectively. Then the perturbed Eq. (2.3) also has an exponential dichotomy on that interval, and if $\mathscr{Q}(t)$ is a corresponding projection matrix function then*

$$|\mathscr{P}(t) - \mathscr{Q}(t)| \to 0 \quad \text{as} \quad t \to \pm\infty,$$

respectively.

Next we prove a perturbation result which is based on the notion of an exponential dichotomy. A similar theorem can be found in Coppel [8, p. 137].

Lemma 2.4. *Let $g(t, x)$ and $h(t, x)$ be functions which are defined and continuous on $\mathbb{R} \times O$ with values in $\mathbb{R}^n$, where O is an open subset of $\mathbb{R}^n$. Furthermore, assume that the partial derivatives g_x and h_x exist and that g_x is uniformly continuous and h_x continuous in $\mathbb{R} \times O$. Finally assume that the equation $\dot{x} = g(t, x)$ has a solution $x = x_0(t)$ defined and contained in O for all $t \in \mathbb{R}$, and strictly bounded away from the boundary of O, such that the variational equation $\dot{x} = g_x(t, x_0(t))\, x$ has an exponential dichotomy on $\mathbb{R}$ with constants say K and α. Then there exist a positive constant η_0 and a function $\eta_1(\eta)$ depending only on g, K and α such that, if $0 < \eta \leqq \eta_0$,*

$$\sup_{(t,x)} |h(t, x)| < \eta_1(\eta) \quad \text{and} \quad \sup_{(t,x)} |h_x(t, x)| < \alpha/2K, \tag{2.4}$$

then the equation

$$\dot{x} = g(t, x) + h(t, x) \tag{2.5}$$

has a unique solution $x(t)$ satisfying

$$|x(t) - x_0(t)| \leqq \eta \qquad (t \in \mathbb{R}).$$

Proof: Let $\mathscr{B}$ denote the Banach space of continuous and bounded functions $f: \mathbb{R} \to \mathbb{R}^n$ equipped with the norm

$$\|f\| = \sup_{t \in \mathbb{R}} |f(t)|.$$

The ball of all $f \in \mathscr{B}$ with $\|f\| \leqq \eta$ will be denoted by $B(\eta)$.

Since the equation $\dot{x} = g_x(t, x_0(t))\,x$ has an exponential dichotomy on $\mathbb{R}$, there is a fundamental matrix function $X(t)$ and a projection P in $\mathbb{R}^n$ such that (2.2) holds. Consequently, the integral operator $\mathscr{K}: \mathscr{B} \to \mathscr{B}$ defined by

$$\mathscr{K} f(t) = \int_{-\infty}^{t} X(t)\, P\, X(\tau)^{-1} f(\tau)\, d\tau - \int_{t}^{\infty} X(t)\,(E - P)\, X(\tau)^{-1} f(\tau)\, d\tau$$

is bounded, and we have

$$\| \mathscr{K} f \| \leqq \frac{2K}{\alpha} \| f \| \qquad (f \in \mathscr{B}). \tag{2.6}$$

Set $x(t) = x_0(t) + \xi(t)$. Then, by the variation of constant formula, finding solutions of (2.5) which are defined and bounded on $\mathbb{R}$, is equivalent with solving the following fixed point problem in $\mathscr{B}$:

$$\xi = \mathscr{K}\, \Phi(\xi). \tag{2.7}$$

Here the operator $\Phi: \mathscr{B} \to \mathscr{B}$ is defined by

$$\Phi(\xi)(t) = g(t, x_0(t) + \xi(t)) - g(t, x_0(t)) - g_x(t, x_0(t))\, \xi(t) + h(t, x_0(t) + \xi(t))$$

in some ball $B(\eta)$. Moreover, if η is sufficiently small and if (2.4) holds with $\eta_1(\eta)$ sufficiently small, then Φ is a contraction in the ball $B(\eta)$. This follows by straight forward estimates using the assumptions. Thus the lemma follows by the contraction mapping principle. □

A second basic fact upon which the proof of our main result relies is the following shadow lemma for non-autonomous systems due to Palmer [20].

Proposition 2.5 (Shadow lemma). *Let $f(t, x)$ be a vector function, defined and continuous in $\mathbb{R} \times O$, where O is an open subset of $\mathbb{R}^n$. Suppose also that the partial derivative $f_x(t, x)$ exists, is bounded and uniformly continuous. Assume that for each integer k the system*

$$\dot{x} = f(t, x) \tag{2.8}$$

has a solution $w_k(t)$ defined on an interval $[t_{k-1}, t_k]$ such that

(i) *the variational equation $\dot{x} = f_x(t, w_k(t))\, x$ has an exponential dichotomy on $[t_{k-1}, t_k]$ with projection matrix function $\mathscr{P}_k(t)$ and constants K, α independent of k;*

(ii) $|w_{k-1}(t_{k-1}) - w_k(t_{k-1})| \leqq \delta$;

(iii) $|\mathscr{P}_{k-1}(t_{k-1}) - \mathscr{P}_k(t_{k-1})| \leqq \delta$;

(iv) $t_k - t_{k-1} \geqq \tau$.

Then there exist positive constants ε_0, $\bar{\tau}_0$ and a function $\delta_0(\varepsilon)$ all of which only depend on $f(t, x)$, K and α but not on particular solutions $w_k(t)$, such that if $\tau \geqq \bar{\tau}_0$, $0 < \varepsilon \leqq \varepsilon_0$ and $\delta \leqq \delta_0(\varepsilon)$, then Eq. (2.8) has a unique solution $w(t)$ satisfying

$$|w(t) - w_k(t)| \leqq \varepsilon \qquad (t_{k-1} \leqq t \leqq t_k,\ k \in \mathbb{Z}).$$

Now we introduce the notion of an almost periodic function which we shall use in the following discussion. Basically it is Bohr's definition [2, p. 38]. The extension to parameter dependent functions is given in Hale's book [10].

Definition 2.6. A vector or matrix valued function $f(t)$ defined and continuous for $-\infty < t < \infty$ is said to be *almost periodic in t* if for any $\varepsilon > 0$ there exists an $l = l(\varepsilon) > 0$ such that in any interval of length l there is a socalled *translation number* τ such that

$$|f(t + \tau) - f(t)| < \varepsilon \qquad (t \in \mathbb{R}).$$

A function $f(t, x)$ defined and continuous in $\mathbb{R} \times \Lambda$, where Λ is a compact subset of $\mathbb{R}^n$, is called *almost periodic in t uniformly with respect to x in Λ*, if the quantities l and τ in the previous definition can be chosen independently of $x \in \Lambda$.

We shall not use many properties of almost periodic functions. Let us just mention that almost periodic functions in t are bounded and uniformly continuous on $\mathbb{R}$. Also, if two functions $f(t)$ and $g(t)$ are almost periodic, then the function $t \mapsto (f(t), g(t))$ is almost periodic, too (cf. Bohr [2, p. 36]). Both properties remain valid for functions which are almost periodic in t uniformly with respect to any other variable. Periodic functions are special almost periodic functions. In this case integer multiples of the minimal period are possible translation numbers. Another interesting subclass of almost periodic functions is formed by the quasiperiodic functions which are of the form $f(t) = g(\omega_1 t, \ldots, \omega_m t)$ where $g(x_1, \ldots, x_m)$ is 2π-periodic in each argument and the frequencies ω_k are independent over the rationals.

Now we are ready to state and to prove our results. Consider the equation

$$\dot{x} = g(x) + \mu h(t, x, \mu), \tag{2.9}$$

where μ is supposed to be a small parameter. Furthermore, we make the following assumptions.

(H 1) Let g be a continuously differentiable vector field in $\mathbb{R}^n$ which has a hyperbolic saddle point x_0, i.e. $g(x_0) = 0$ and all eigenvalues of the Jacobian matrix $g_x(x_0)$ are lying off the imaginary axis.

(H 2) Assume that the function $h = h(t, x, \mu)$: $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous and that the partial derivative h_x exists. Let both h and h_x be almost periodic functions in t uniformly with respect to $(x, \mu) \in \bar{O} \times [-\bar{\mu}, \bar{\mu}]$, where O is a bounded open subset of $\mathbb{R}^n$ containing x_0, $\bar{O}$ denotes the closure of O, and $\bar{\mu}$ is a positive constant.

Remark 2.7. Here almost periodicity of h_x in t uniformly with respect to (x, μ) is equivalent with h_x being uniformly continuous on $\mathbb{R} \times \bar{O} \times [-\bar{\mu}, \bar{\mu}]$.

Lemma 2.8. *Under the assumptions* (H1) *and* (H2) *there exist a positive constant* η_0 *and a function* $\mu_0(\eta)$ *such that if* $0 < \eta \leqq \eta_0$ *and* $|\mu| \leqq \mu_0(\eta)$, *then Eq.* (2.9) *has a unique solution* $u^\mu(t)$ *satisfying*

$$|u^\mu(t) - x_0| \leqq \eta \qquad (t \in \mathbb{R}).$$

Moreover, $u^\mu(t)$ *is almost periodic, and the corresponding variational equation has an exponential dichotomy on* $\mathbb{R}$.

Proof: For each μ Eq. (2.9) is of type (2.5). Since x_0 is a hyperbolic saddle point of the equation $\dot{x} = g(x)$, the corresponding variational equation has an exponential dichotomy on $\mathbb{R}$. The partial derivative g_x is uniformly continuous on O, since O is bounded. Given any small $\eta > 0$, we can find a positive constant $\mu_0 = \mu_0(\eta)$ such that the remaining assumptions of Lemma 2.4 are satisfied, too, if $|\mu| \leqq \mu_0$. Here we use the boundedness of h and h_x. Thus Lemma 2.4 applies to prove the first part of Lemma 2.8. Furthermore, if μ_0 is sufficiently small then the roughness theorem Proposition 2.2 applies to prove that the variational equation corresponding to the solution $u^\mu(t)$ of (2.9) has an exponential dichotomy on $\mathbb{R}$.

To prove that $u^\mu(t)$ is almost periodic in t we argue as follows. Consider the equation

$$\dot{x} = g(x) + \mu h(t + \tau, x, \mu) \tag{2.10}$$

for any $\tau \in \mathbb{R}$. The previous argument shows that for $|\mu| \leqq \mu_0$ this equation has a unique solution $u_\tau^\mu(t)$ satisfying $|u_\tau^\mu(t) - x_0| \leqq \eta$ for all $t \in \mathbb{R}$. But if $u^\mu(t)$ is a solution of (2.9), then $u^\mu(t + \tau)$ is a solution of (2.10). Hence, by uniqueness

$$u_\tau^\mu(t) = u^\mu(t + \tau) \qquad (t, \tau \in \mathbb{R}).$$

On the other hand we can apply Lemma 2.4 to Eq. (2.10) by setting $g(t, x) = g(x) + \mu h(t, x, \mu)$, $h(t, x) = \mu[h(t + \tau, x, \mu) - h(t, x, \mu)]$, and $x_0(t) = u^\mu(t)$. Thus, for small μ, irrespective of τ, $|u^\mu(t) - u_\tau^\mu(t)|$ is arbitrarily small provided that

$$\sup_{(t, x) \in \mathbb{R} \times O} |h(t + \tau, x, \mu) - h(t, x, \mu)|$$

is sufficiently small. Hence, almost periodicity of h in t uniformly with respect to $(x, \mu) \in \bar{O} \times [-\bar{\mu}, \bar{\mu}]$ implies the almost periodicity of $u^\mu(t)$ in t. □

Now we are going to state another assumption which is crucial for the construction of chaotic solutions for Eq. (2.9).

(H 3) Suppose that for some μ in $0 < |\mu| \leqq \mu_0(\eta)$ Eq. (2.9) has a solution $v^\mu(t) \not\equiv u^\mu(t)$ contained in O, such that

(i) $|v^\mu(t) - u^\mu(t)| \to 0 \quad \text{as} \quad |t| \to \infty$;

(ii) the variational equation corresponding to $v^\mu(t)$ possesses an exponential dichotomy on $\mathbb{R}$ with constants L and β.

Remark 2.9. This condition generalizes the notion of a transversal homoclinic point for the period map of a periodic system (see Palmer [20]). In geometrical terms it has the following meaning. Since the variational equation corresponding to $u^\mu(t)$ has an exponential dichotomy on $\mathbb{R}$, $\{(t, u^\mu(t)) | t \in \mathbb{R}\} \subset \mathbb{R}^{n+1}$ is a hyperbolic invariant set for the suspended system

$$\begin{aligned} \dot{x} &= g(x) + \mu h(t, x, \mu) \\ \dot{t} &= 1 \end{aligned} \tag{2.11}$$

By (H 3) the corresponding stable and unstable manifolds are intersecting each other transversally along the orbit $\{(t, v^\mu(t)) | t \in \mathbb{R}\}$. A criterium for (H 3) to hold is the following (see Palmer [20]). Let $g(x)$ be a twice continuously differentiable vector function defined in an open subset $O \subset \mathbb{R}^n$ such that the system $\dot{x} = g(x)$ has a hyperbolic saddle point x_0 and a corresponding homoclinic solution $\zeta(t)$, the closure of the range of which is contained in O. Suppose also that $\dot{\zeta}(t)$ is the unique (up to a scalar multiple) bounded solution of the variational equation corresponding to $\zeta(t)$. Let $h(t, x, \mu)$ be a bounded continuous vector function in $\mathbb{R} \times O \times [-\bar{\mu}, \bar{\mu}]$ such that the partial derivatives $h_t, h_x, h_\mu, h_{xx}, h_{x\mu}, h_{\mu x}, h_{\mu\mu}$ exist, are bounded, continuous in t for each fixed (x, μ) and continuous in (x, μ) uniformly with respect to (t, x, μ). Then, if the function

$$\Delta(t_0) = \int_{-\infty}^{\infty} \Psi(t) \cdot h(t + t_0, \zeta(t), 0)\, dt \tag{2.12}$$

has a simple zero at some point t_0, condition (H 3) is satisfied for sufficiently small $|\mu| \neq 0$. Moreover, for the corresponding solutions $v^\mu(t)$

$$\sup_{t \in \mathbb{R}} |v^\mu(t) - \zeta(t - t_0)| = O(\mu) \quad \text{as} \quad \mu \to 0$$

holds. Here $\Psi(t)$ is the unique (up to a scalar multiple) bounded solution of the system adjoint to the variational equation of $\zeta(t)$ and "$\cdot$" denotes the inner product in $\mathbb{R}^n$. In the special case $n = 2$ the function $\Delta(t_0)$ is just given by

$$\Delta(t_0) = -\int_{-\infty}^{\infty} g(\zeta(t)) \wedge h(t + t_0, \zeta(t), 0)\, e^{-\int_0^{t+t_0} \operatorname{trace} g_x(\zeta(s))\, ds}\, dt. \tag{2.13}$$

Here the wedge product is defined by $x \wedge y = x_1 y_2 - x_2 y_1$ for two-vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$. This is the Melnikov function up to a certain multiplicative factor (cf. Sect. 1). When (2.9) is in Hamiltonian form with Hamiltonian $H_0(x, y) + \mu H_1(t, x, y, \mu)$, where x and y are m-vectors, then $\Delta(t_0)$ can be written as

$$\Delta(t_0) = \int_{-\infty}^{\infty} \{H_0, H_1\}(t + t_0, \zeta(t), 0)\,\mathrm{d}t. \tag{2.14}$$

Here $\{H_0, H_1\}$ denotes the Poisson bracket given by

$$H_{0,x} \cdot H_{1,y} - H_{0,y} \cdot H_{1,x}.$$

Note, that $\Delta(t_0)$ is almost periodic in t_0, when h is almost periodic in t uniformly in (x, μ). Hence, as it is easily seen, one simple zero implies infinitely many simple zeroes of $\Delta(t_0)$ and so infinitely many solutions $v^\mu(t)$ of Eq. (2.9) such that (H 3) is satisfied. During the proof of the following theorem we shall deduce this fact directly from (H 3).

Remark 2.10. In Lemma 2.8 we did not really use the assumption that h_x is almost periodic in t. But this assumption will be important in the following theorem. The reason is that in (H 3) we allow $L \to \infty$ and $\beta \to 0$ as $\mu \to 0$. As a matter of fact we fix μ from now on, and for simplicity, we shall even omit it in the notation occasionally.

Theorem 2.11. *Let* (H 1), (H 2) *and* (H 3) *hold. Then there exist a positive constant* ε_0 *and functions* $T_0 = T_0(\varepsilon)$, $T = T(\varepsilon)$, *and* $\sigma = \sigma(\varepsilon)$ *such that if* $0 < \varepsilon \leqq \varepsilon_0$ *and given any interval* $I_0 \subset \mathbb{R}$ *with length* σ *and any sequence of real numbers* $\tau_k \geqq 0$ $(k \in \mathbb{Z})$, *then there exists a sequence of real numbers* t_k *with* $t_0 \in I_0$ *and*

$$T + \tau_k \leqq t_k - t_{k-1} \leqq T + \tau_k + \sigma,$$

such that (2.9) *has a unique solution* $w^\mu(t)$ *satisfying*

$$|w^\mu(t) - v^\mu(T_0 + t - t_{k-1})| \leqq \varepsilon \tag{2.15}$$

for $t \in [t_{k-1}, t_k]$ *and all* k. *Moreover,*

$$|w^\mu(t) - u^\mu(t)| \leqq \varepsilon \tag{2.16}$$

for $t \in [t_{k-1} + T, t_k]$. *Furthermore, if* $\tau_l = \infty$ *either for some* $l > 0$ *or for some* $l \leqq 0$, *then we set* $t_l = \infty$ *and* $t_{l-1} = -\infty$, *respectively. In this case* (2.15) *makes sense only for* $k \leqq l$ *and* $k \geqq l + 1$; *but* (2.16) *holds for* $k \leqq l$ *and* $k \geqq l$ *(except at* $t = \pm\infty$, *of course), and* $|w^\mu(t) - u^\mu(t)| \to 0$ *as* $t \to \pm\infty$, *respectively.*

So for $t \in [t_{k-1} + T, t_k]$ the solution $w(t) = w^\mu(t)$ actually stays close to the unperturbed saddle point x_0, since $u(t) = u^\mu(t)$ does it. But inbetween these time intervals it leaves a certain neighborhood of x_0 for some time, provided that $v(t) = v^\mu(t)$ does it within the time interval $[T_0, T_0 + T]$. These "humps" seem to occur quite randomly, because the τ_k are arbitrary nonnegative real numbers. However, in general τ_k determines the difference $t_k - t_{k-1}$ and thus the length of the time interval between two consecutive humps only up to the constant σ which can be very large as the proof will show. In fact, this is the main difference between the general almost periodic case and the periodic case, where the t_k can be chosen as T_0 plus certain integer multiples of the period of the system. This then leads to the construction of subsets of $\mathbb{R}^n$ which are invariant and hyperbolic with respect to certain iterates of the period map, and on which these iterates are topologically conjugate to the Bernoulli shift on a finite number of symbols (see Palmer [20]). Thus in the periodic case we can find slightly more order in the chaotic behaviour of the solutions $w(t)$ than in the general almost periodic case. Of course, this is not very surprising. In fact, it should also be possible to give a refined description of the solutions $w(t)$, when the perturbation h is quasiperiodic. We will possibly do this in a forthcoming paper.

We also remark that a particular t_l does not depend on the whole sequence τ_k, but only on I_0 and the τ_k with $1 \leqq k \leqq l$ when $l \geqq 0$, and with $l+1 \leqq k \leqq 0$ when $l \leqq 0$. This will become clear by the proof of Theorem 2.11. As a matter of fact, if for fixed ε we take the solution $w(t)$ corresponding to some interval I_0 and some sequence τ_k and compare it to solutions $w_j(t)$ corresponding to the same interval I_0 and to sequences τ_k^j with $\tau_k^i = \tau_k$ for $|k| \leqq j$ ($j \in \mathbb{N}$), then we find

$$|w(t) - w_j(t)| \to 0 \qquad \text{as} \qquad j \to \infty \tag{2.17}$$

uniformly on compact t-intervals for arbitrary τ_k^j with $|k| > j$. Indeed, assume the contrary to be true. Then there are a subsequence of solutions $w_j(t)$, points $\bar{t}_j$ in some compact interval and a positive constant δ such that $|w(\bar{t}_j) - w_j(\bar{t}_j)| \geqq \delta$. Again by choosing subsequences if necessary, we can assume $\bar{t}_j \to \bar{t} \in \mathbb{R}$ and $w_j(\bar{t}_j) \to \bar{x} \in \mathbb{R}^n$ as $j \to \infty$. This then implies $|w(\bar{t}) - \bar{x}| \geqq \delta$. Now let $x(t)$ be the unique solution of (2.9) with $x(\bar{t}) = \bar{x}$. It follows by continuous dependence on initial values that $w_j(t) \to x(t)$ as $j \to \infty$ uniformly on compact t-intervals, where $w_j(t)$ is the subsequence of solutions selected above. Now we use the fact that $w(t)$ is uniquely determined by property (2.15). But by the above mentioned property of t_l, the limit $x(t)$ of the solutions $w_j(t)$ also fulfills (2.15) with the same numbers t_k as $w(t)$. Thus, by uniqueness we conclude $w(t) \equiv x(t)$. So (2.17) follows by contradiction. But this shows that the long-time behaviour of the solutions $w(t)$ is highly sensitive with respect to small perturbations of the initial values being another manifestation of their chaotic character.

Proof of Theorem 2.11: Recall that μ is fixed and will be omitted occasionally. The idea is to approximate $w(t)$ by different solutions $w_k(t)$ on the intervals $[t_{k-1}, t_k]$ and then to prove its existence by the shadow lemma Proposition 2.5. First we construct the approximating solutions $w_k(t)$ from $v(t)$. To this end let

$$\mu \sup_{(t,x) \in \mathbb{R} \times O} (|h(t+\tau, x, \mu) - h(t, x, \mu)|, |h_x(t+\tau, x, \mu) - h_x(t, x, \mu)| \leqq \kappa \tag{2.18}$$

hold for some real constants κ and τ. Then, by Lemma 2.4, given any $\delta > 0$ sufficiently small, say $\delta \leqq \delta_1$, there are unique solutions $u_\tau(t)$ and $v_\tau(t)$ of Eq. (2.10) such that for all $t \in \mathbb{R}$

$$\begin{aligned} &|u_\tau(t) - u(t)| \leqq \delta/6 \\ &|v_\tau(t) - v(t)| \leqq \delta/6 \end{aligned}, \tag{2.19}$$

provided that κ is sufficiently small. By the roughness theorem Proposition 2.2 the variational equations corresponding to $u_\tau(t)$ and $v_\tau(t)$ have exponential dichotomies on $\mathbb{R}$, since those corresponding to $u(t)$ and $v(t)$ haves ones. Moreover, the corresponding projection matrix functions satisfy

$$\begin{aligned} &|\mathscr{Q}_\tau(t) - \mathscr{Q}(t)| \leqq \delta/6 \\ &|\mathscr{P}_\tau(t) - \mathscr{P}(t)| \leqq \delta/6 \end{aligned} \qquad (t \in \mathbb{R}), \tag{2.20}$$

respectively, where $\mathscr{Q}(t)$ and $\mathscr{P}(t)$ denote those for $u(t)$ and $v(t)$. The corresponding constants can be chosen uniformly for any τ and small $|\kappa|$, say to be $\tilde{L}$ and $\tilde{\beta}$ in case of $v_\tau(t)$. Finally, by (H3) (i) and Proposition 2.3 there are functions $T_0 = T_0(\delta)$ and $T = T(\delta)$ such that

$$\begin{aligned} &|u(t) - v(t)| \leqq \delta/6 \\ &|\mathscr{Q}(t) - \mathscr{P}(t)| \leqq \delta/6 \end{aligned} \qquad (t \notin (T_0, T_0 + T)). \tag{2.21}$$

From these estimates we conclude

$$\begin{aligned} &|u_\tau(t) - v_\tau(t)| \leqq \delta/2 \\ &|\mathscr{Q}_\tau(t) - \mathscr{P}_\tau(t)| \leqq \delta/2 \end{aligned} \qquad (t \notin (T_0, T_0 + T)). \tag{2.22}$$

Now, note that $u_\tau(t - \tau)$ and $v_\tau(t - \tau)$ are solutions of (2.9). Obviously, the corresponding variational equations have exponential dichotomies on $\mathbb{R}$ with projection matrix functions $\mathscr{Q}_\tau(t - \tau)$ and $\mathscr{P}_\tau(t - \tau)$ and the same constants as $u_\tau(t)$ and $v_\tau(t)$, respectively. By uniqueness it follows $u_\tau(t - \tau) = u(t)$ and $\mathscr{Q}_\tau(t - \tau) = \mathscr{Q}(t)$, since by (2.19) and Lemma 2.8 $|u_\tau(t - \tau) - x_0| \leqq |u_\tau(t - \tau) - u(t - \tau)| + |u(t - \tau) - x_0| \leqq \frac{\delta}{6} + \eta$ for all t. Thus we conclude from (2.22)

$$\begin{aligned} &|u(t) - v_\tau(t - \tau)| \leqq \delta/2 \\ &|\mathscr{Q}(t) - \mathscr{P}_\tau(t - \tau)| \leqq \delta/2 \end{aligned} \qquad (t \notin (\tau + T_0, \tau + T_0 + T)). \tag{2.23}$$

Now we proceed by induction to construct the t_{k-1} and the solutions $w_k(t)$ for $k = 1, 2, \ldots$ and $k = 0, -1, -2, \ldots$. Let us just indicate how the argument works. Let ε_0, $\bar{\tau}_0$ and $\delta_0(\varepsilon/2)$ be chosen according to Proposition 2.5 as functions of the constants $\tilde{L}$ and $\tilde{\beta}$ introduced above; set $\delta = \min(\delta_0(\varepsilon/2), \varepsilon, \delta_1)$ for any given ε in $0 < \varepsilon \leqq \varepsilon_0$. Then choose κ and the quantities T_0 and T according to the discussion above, where we can assume $T \geqq \bar{\tau}_0$. Finally we make use of the assumption that the functions h and h_x both are almost periodic in t uniformly with respect to (x, μ). Therefore there is a constant $\sigma = \sigma(\delta) > 0$ such that each time interval of length σ contains a translation number τ such that (2.18) is fulfilled.

Given any interval I_0 with length σ, we now take t_0 to be $T_0 + \tau$, where τ is a translation number in the interval $I_0 - T_0 = \{t \in \mathbb{R} \mid t + T_0 \in I_0\}$. Hence, $t_0 \in I_0$. Then we take $w_1(t)$ to be the corresponding solution $v_\tau(t - \tau)$ of Eq. (2.9) constructed above. Next we consider the interval $I_1 = [t_0 + T + \tau_1, t_0 + T + \tau_1 + \sigma]$ and define t_1 to be $T_0 + \tau$, where τ is a translation number in $I_1 - T_0$. Hence $t_1 \in I_1$. Again, $w_2(t)$ is defined to be the corresponding solution $v_\tau(t - \tau)$; and so on for $k = 3, 4, \ldots$. In the negative t-direction we proceed as follows. Let $I_{-1} = [t_0 - \tau_0 - T - \sigma, t_0 - \tau_0 - T]$. Set $t_{-1} = T_0 + \tau$, where τ is a translation number in $I_{-1} - T_0$, i.e. $t_{-1} \in I_{-1}$, and take $w_0(t)$ to be the solution $v_\tau(t - \tau)$ of (2.9) corresponding to τ; and so on for $k = -1, -2, \ldots$ (see Fig. 2.1).

Thus, by induction it follows the existence of a sequence of solutions $w_k(t)$ of (2.9) ($k \in \mathbb{Z}$), such that

$$|w_k(t) - v(t - (t_{k-1} - T_0))| \leqq \delta/6 \qquad (t \in \mathbb{R}) \tag{2.24}$$

and, by (2.23),

$$|w_k(t) - u(t)| \leqq \delta/2 \qquad (t \notin (t_{k-1}, t_{k-1} + T)). \tag{2.25}$$

This last estimate shows that assumption (ii) of Proposition 2.5 is satisfied. Moreover, the variational equation corresponding to $w_k(t)$ has an exponential dichotomy on the interval $[t_{k-1}, t_k]$ with constants $\tilde{L}$ and $\tilde{\beta}$ independent of k. By (2.23), assumption (iii) of Proposition 2.5 is satisfied for the corresponding projection matix functions. Finally, by the choice of T we have $t_k - t_{k-1} \geqq T + \tau_k \geqq \bar{\tau}_0$. Hence, Proposition 2.5 applies to prove the existence of a unique solution $w(t)$ of Eq. (2.9) with the property

$$|w(t) - w_k(t)| \leqq \varepsilon/2 \qquad (t \in [t_{k-1}, t_k], k \in \mathbb{Z}). \tag{2.26}$$

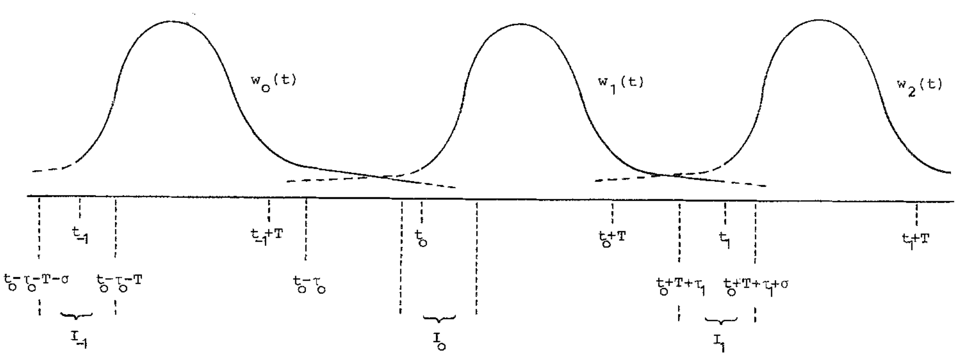

Figure 2.1

Since $\delta \leqq \varepsilon$, (2.24), (2.25) and (2.26) imply (2.15) and (2.16). In particular, if $\tau_l = \infty$ either for some $l > 0$ or for some $l \leqq 0$, then we use w_l to approximate w in the whole interval $[t_{l-1}, \infty)$ and w_{l+1} to approximate w in the whole interval $(-\infty, t_{l+1}]$, respectively. Thus $|w(t) - u(t)| \leqq \varepsilon$ for $t \geqq t_{l-1} + T$ and $t \leqq t_l$, respectively. But, if ε is sufficiently small this means that the orbit $(t, w(t))_{t \in \mathbb{R}}$ of the suspended system (2.11) belongs either to the stable or to the unstable manifold of the hyperbolic invariant set $(t, u(t))_{t \in \mathbb{R}}$ (cf. Remark 2.9). Hence, $|w(t) - u(t)| \to 0$ as $t \to \pm\infty$, respectively. Thus Theorem 2.11 is completely proved now. □

Remark 2.12. If Eq. (2.9) has more than one solution $v^\mu(t)$ which satisfy (H 3) with uniform constants L and β, then for each k one can use any of these solutions to construct $w_k^\mu(t)$. This then leads to even more chaotic solutions $w^\mu(t)$. We also mention that certain cases of "heteroclinic chaos" can be treated in this way. In fact, if (2.9) has a whole sequence of perturbed saddle points $u_k^\mu(t)$ $(k = 0, \pm 1, \pm 2, \ldots)$ and solutions $v_k^\mu(t)$ in O such that $|v_k^\mu(t) - u_{k-1}^\mu(t)| \to 0$ as $t \to -\infty$ and $|v_k^\mu(t) - u_k^\mu(t)| \to 0$ as $t \to +\infty$, and that for each k the variational equations corresponding to $u_k^\mu(t)$ and $v_k^\mu(t)$ have exponential dichotomies on $\mathbb{R}$ with constants independent of k, then the assertion of Theorem 2.11 remains true, when we replace $u^\mu(t)$ by $u_k^\mu(t)$, $v^\mu(t)$ by $v_k^\mu(t)$ and T, T_0 by constants T_k, $T_{0,k}$ for each k. In the last part of the assertion $u^\mu(t)$ has to be replaced by a certain $u_k^\mu(t)$.

Example 2.13. Consider the equation for a quasiperiodically forced pendulum

$$\ddot{\phi} + \sin\phi = \mu \sin\omega_1 t + \mu \sin\omega_2 t, \tag{2.27}$$

where $\omega_1, \omega_2 > 0$ are constant angular forcing frequencies which are independent over the rationals, and μ is a small real parameter. Let $x = (\phi, \dot{\phi})$, so (2.27) becomes

$$\frac{d}{dt}\begin{pmatrix}\phi \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix}\dot{\phi} \\ -\sin\phi\end{pmatrix} + \mu \begin{pmatrix}0 \\ \sin\omega_1 t + \sin\omega_2 t\end{pmatrix}, \tag{2.28}$$

that is an equation of type (2.9). For $\mu = 0$ (2.28) has hyperbolic saddle points $((2k+1)\pi, 0)$ $(k \in \mathbb{Z})$. But since the right-hand side is 2π-periodic in ϕ, we can identify these as one saddle point x_0. This means that we are looking at the unperturbed equation on $S^1 \times \mathbb{R}$, where S^1 stands for the circle, elements of which are regarded as the variable ϕ. In this sense x_0 has two homoclinic orbits given by

$$\zeta_\pm(t) = \begin{pmatrix}\pm 2 \arctan(\sinh t) \\ \pm 2 \operatorname{sech} t\end{pmatrix}.$$

According to (2.13) we compute the corresponding Melnikov function to be

$$\begin{aligned} \Delta(t_0) &= \mp \int_{-\infty}^{\infty} 2(\sin\omega_1(t+t_0) + \sin\omega_2(t+t_0)) \operatorname{sech} t \, dt \\ &= \mp 2\pi \left(\operatorname{sech} \frac{\pi\omega_1}{2} \sin\omega_1 t_0 + \operatorname{sech} \frac{\pi\omega_2}{2} \sin\omega_2 t_0 \right). \end{aligned}$$

The improper integral is evaluated by the method of residues. Obviously, $t_0 = 0$ is a simple zero of $\Delta(t_0)$. Hence, for small $|\mu| \neq 0$ there are transversal homoclinic solutions $v^{\mu}_{\pm}(t)$ corresponding to the perturbed saddle point, and Theorem 2.11 applies to prove existence of chaotic solutions $w^{\mu}(t)$. According to Remark 2.12, for each k we can either take $v^{\mu}_{+}(t)$ or $v^{\mu}_{-}(t)$ in (2.15). Physically this means that the pendulum rotates counterclockwise or clockwise during the time segments $[t_{k-1}, t_{k-1} + T]$. By (2.16), during the time segments $[t_{k-1} + T, t_k]$ it stays close to the vertically upward position described by x_0.

References

[1] V. Arnold, *Instability of dynamical systems with several degrees of freedom.* Dokl. Akad. Nauk. SSSR. *156*, 9–12 (1964).

[2] H. Bohr, *Almost periodic functions.* Chelsea, New York 1947.

[3] G. D. Birkhoff, *Dynamical systems.* Am. Math. Soc. Colloq. Publ. *9* (1927).

[4] B. V. Chirikov, *A universal instability of many-dimensional oscillator systems.* Phys. Rep. *52*, 265–379 (1979).

[5] S. N. Chow and J. K. Hale, *Methods of bifurcation theory.* Springer, New York 1982.

[6] S. N. Chow, J. K. Hale and J. Mallet-Paret, *An example of bifurcation to homoclinic orbits.* J. Diff. Equat. *37*, 351–373 (1980).

[7] W. A. Coppel, *Dichotomies in stability theory.* Lect. Notes in Maths. *629*, Springer, Berlin 1978.

[8] W. A. Coppel, *Stability and asymptotic behaviour of differential equations.* D. C. Heath, Boston 1965.

[9] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcation of vector fields.* Springer, New York 1983.

[10] J. K. Hale, *Ordinary differential equations.* Wiley and Sons, New York 1969.

[11] P. Holmes, *Averaging and chaotic motion in forced oscillations.* SIAM J. Appl. Math. *38*, 68–80 (1980); *Errata and addenda,* SIAM J. Appl. Math. *40*, 167–168 (1982).

[12] P. J. Holmes and J. E. Marsden, *A partial differential equation with infinitely many periodic orbits: Chaotic oscillations of a forced beam.* Arch. Rat. Mech. Anal. *76*, 135–166 (1981).

[13] P. J. Holmes and J. E. Marsden, *Horseshoes in perturbations of Hamiltonian systems with two degrees of freedom.* Comm. Math. Phys. *82*, 523–544 (1982).

[14] K. Kirchgässner, *Solitary waves under external forcing.* Lect. Notes in Phys. *195*, Ciarlet-Roseau edts., Springer, Berlin 1984, pp. 211–234.

[15] K. Kirchgässner, *Nonlinear waves and homoclinic bifurcation.* To appear in Proc. XVIth. Int. Congr. Theor. Appl. Mech. 1984.

[16] U. Kirchgraber, *Erratische Lösungen der periodisch gestörten Pendelgleichung.* Preprint, Universität Würzburg 1982.

[17] J. E. Marsden, *Chaos in dynamical systems by the Poincaré-Melnikov-Arnold method.* Proc. ARO Workshop on Dynamics, SIAM 1984.

[18] V. K. Melnikov, *On the stability of the center for time periodic perturbations.* Trans. Moscow Math. Soc. *12*, 1–57 (1963).

[19] J. Moser, *Stable and random motions in dynamical systems.* Annals of Math. Studies, Princeton University Press 1973.

[20] K. J. Palmer, *Exponential dichotomies and transversal homoclinic points*. J. Diff. Equat. *55* (2), 225–256 (1984).
[21] H. Poincaré, *Les méthods nouvelles de la mécanique céleste, Vol. 3*. Gauthier-Villars, Paris 1892.
[22] J. Scheurle and J. E. Marsden, *Bifurcation to quasiperiodic tori in the interaction of steady state and Hopf bifurcations*. SIAM J. Math. Anal. *15* (6), 1055–1074 (1984).
[23] S. Smale, *Differentiable dynamical systems*. Bull. Am. Math. Soc. *73*, 747–817 (1967).

Summary

In a recent paper [J. Diff. Equat. 55 (2) (1984), 225–256], J. Palmer proved Smale's theorem on the embedding of the Bernoulli shift in the context of a periodic differential system (*) $\dot{x} = f(t, x)$, $x \in \mathbb{R}^n$, using a nonautonomous shadow lemma. By means of this lemma, we show that one does get a similar kind of chaotic motion when f is almost periodic in t. Actually, we do not consider the equation (*). In order to show that the hypotheses can be satisfied, we rather consider a parameter-dependent equation of the form $x = g(x) + \mu h(t, x, \mu)$, where $\mu \in \mathbb{R}$ is the parameter.

Zusammenfassung

In einer kürzlich erschienenen Arbeit [J. Diff. Equat. 55 (2) (1984), 225–256] bewies J. Palmer den Satz von Smale über die Einbettung des Bernoullischifts für periodische Differentialgleichungssysteme der Form (*) $\dot{x} = f(t, x)$, $x \in \mathbb{R}^n$, unter Verwendung eines Schattenlemmas für nicht-autonome Systeme. Mit Hilfe dieses Lemmas zeigen wir, daß man eine ähnliche chaotische Bewegung erhält, wenn f fast-periodisch in t ist. Genau genommen betrachten wir nicht die Gleichung (*). Um zu zeigen, daß die Voraussetzungen erfüllt werden können, betrachten wir vielmehr eine parameterabhängige Gleichung der Form $\dot{x} = g(x) + \mu h(t, x, \mu)$, wobei $\mu \in \mathbb{R}$ der Parameter ist.

(Received: May 14, 1985)