Acta Mathematica Sinica, New Series
1990, Vol.6, No.4, pp. 349 — 353

Exponential Convergence for One Dimensional Contact Processes*)

Chen Jinwen ***)(陈金文)
Department of Mathematics, Beijing Normal University

Richard Durrett **)
Cornell University, USA

Liu Xiufang ***)(刘秀芳)
Department of Mathematics, Beijing Normal University

Received March 23, 1988 Revised February 30, 1989

Abstract. The complete convergence theorem implies that starting from any initial distribution the one dimensional contact process converges to a limit as $t\to\infty$. In this paper we give a necessary and sufficient condition on the initial distribution for the convergence to occur with exponential rapidity.

1. Introduction

Since Harris (1974) introduced the contact process, many papers have been written on this topic, especially for the basic contact process in one dimension (see [1], [4], [6], [7], and [8]). In this paper we will consider the one dimensional nearest neighbor case, that is, a Markov process $\xi_t\subset\mathbb{Z}$ with:

$$P(x\in\xi_{t+s}|\xi_t)=\beta_x(\xi_t)s+o(s)\quad\text{when } x\notin\xi_t,$$
$$P(x\notin\xi_{t+s}|\xi_t)=\delta_x(\xi_t)s+o(s)\quad\text{when } x\in\xi_t,$$

where the birth and death rates are given by

$$\beta_x(\xi)=\begin{cases}0 & \text{if}\,|\xi\cap\{x-1,x+1\}|=0,\\ \lambda & \text{if}\,|\xi\cap\{x-1,x+1\}|=1,\\ \theta\lambda & \text{if}\,|\xi\cap\{x-1,x+1\}|=2,\end{cases}$$

and $\delta_x(\xi)=1$. Here $\theta\geqslant 1$ is considered to be fixed while $\lambda>0$ is varied. When $\theta=2$ we get the *basic contact process*.

If $A\subset\mathbb{Z}$, let ξ_t^A denote the process with $\xi_0^A=A$. We assume that $\theta\geqslant1$ so the system is *attractive*: *if* $A\subset B$ then we can construct ξ_t^A and ξ_t^B on the same space with $\xi_t^A\subset\xi_t^B$ for all t. A consequence of attractiveness is that $\xi_t^{\mathbb{Z}}\Rightarrow\nu$ as $t\to\infty$, where $\Rightarrow$ denotes the weak convergence, which in this setting is just the convergence of finite dimensional distributions. ν is a stationary distribution for the contact process, but may be the trivial one: $\delta| = a$ pointmass on ϕ. Let $\lambda_c(\theta)=\inf\{\lambda:\nu\neq\delta\phi\}$. It is known that if $\theta\geqslant1$ then $\lambda_c(\theta)\leqslant\lambda_c(1)\leqslant4$, and that the following *complete convergence theorem* holds for $\lambda>\lambda_c(\theta)$:

*) This work was discussed while the authors were visiting the Nankai Mathematics Institute in Tianjin.

**) Partially supported by the National Science Foundation, the Army Research Office through the Mathematical Sciences Institute at Cornell University, and a Guggenheim fellowship.

***) Research supported by the National Science Foundation of China.

(*) $$\xi_t^\mu \Rightarrow P(\tau^\mu < \infty)\delta\phi + P(\tau^\mu = \infty)\nu.$$

Here ξ_t^μ denotes the contact process with initial distribution μ and $\tau^\mu = \inf\{t : \xi_t^\mu = \phi\}$. This result was proved in [1] for $1 \leqslant \theta \leqslant 2$. Using Theorem 4 in [5] it is easy to extend the proof to $\theta > 2$.

The purpose of this paper is to identify the initial distributions for which the convergence in (*) occurs with exponential rapidity, that is, for which the following conclusion holds:

(**) For any nonempty finite $B \subset \mathbb{Z}$, there are constants C, $\gamma \in (0,\infty)$ such that

$$|P(B \subset \xi_t^\mu) - P(\tau^\mu = \infty)\nu(\xi : \xi \supset B)| \leqslant Ce^{-\gamma t}.$$

Since any event involving finitely many coordinates can be written in terms of the events $\{\xi : B \subset \xi\}$, (**) gives the exponential convergence of finite dimensional distributions.

In [5] (see Theorem 2 on p. 383) it was shown that (**) holds when $\mu = \delta_{\mathbb{Z}}$. In [6] (see p. 172) it was shown that if μ is a product measure with density p, i.e.

$$\mu(\xi : B \subset \xi) = p^{|B|},$$

where $|B|$ = the number of points, then (**) holds when $\theta = 2$ and $\lambda \geqslant 14$.

Before this work, the first and third authors showed that (**) holds for one dimensional supercritical contact processes (i.e. $\theta \geqslant 1$, $\lambda > \lambda_c(\theta)$) when $\mu = \delta_A$, where A is a finite set or $A \supset k_0\mathbb{Z}_+$ for some integer k_0 with $|k_0| \geqslant 1$. The next result shows that exponential convergence always holds for product measures and includes the case mentioned above.

Theorem. *Let* $\lambda > \lambda_c(\theta)$. (**) *holds if and ony if there are constants* C, δ, $\gamma \in (0,\infty)$ *such that for all* $n \in \mathbb{Z}_+$,

$$\mu(\xi : |\xi \cap [-n, n]| \leqslant \delta n,\ \xi \cap [-n, n]^c \neq \phi) \leqslant Ce^{-\gamma n}.$$

Before explaining the intuition behind the theorem, we need to state some known results:

(1.1) Let $\tau^A = \inf\{t : \xi_t^A = \phi\}$. There are $C, \gamma \in (0, \infty)$ such that for all $t \geqslant 0$ and $A \subset \mathbb{Z}$, $P(t < \tau^A < \infty) \leqslant Ce^{-\gamma t}$.

(1.2) There are C, $\gamma \in (0,\infty)$ such that $P(\tau^A < \infty) \leqslant Ce^{-\gamma|A|}$ for all $A \subset \mathbb{Z}$.

(1.3) Let $r_t^A = \sup \xi_t^A$, $r_t^- = r_t^{(-\infty, 0]}$, and $\alpha(\lambda) = \lim Er_t^- / t$, which is > 0 for $\lambda > \lambda_c(\theta)$. Then for any $a < \alpha$ and $b > \alpha$ there are $C, \gamma \in (0,\infty)$ such that for all $t \geqslant 0$,

$$P(r_t^- \leqslant at) \leqslant Ce^{-\gamma t} \text{ and } P(r_t^- \geqslant bt) \leqslant Ce^{-\gamma t}.$$

These conclusions were proved first for $1 \leqslant \theta \leqslant 2$ in [4] and extended to $\theta > 2$ in [5]. Here and in what follows C, γ denote positive finite constants whose values are unimportant and will change from line to line.

To explain why the condition is necessary we begin by considering what happens when $\mu(\xi : |\xi| = \infty) = 1$ and hence $\xi_t^\mu \Rightarrow \nu$. In this case if $\mu(\xi : |\xi \cap [-n, n]| \leqslant \delta n) \geqslant e^{-\varepsilon n}$ then with probability at least $e^{-\varepsilon n}e^{-\Gamma\delta n}$ all the particles in $[-n, n]$ die by time 1 without giving birth and it follows from (1.3) that with probability at least $e^{-\varepsilon n}e^{-\Gamma\delta n} - 2Ce^{-\gamma n/2\alpha}$ there will be no particles in $[-n/3, n/3]$ at time

$t=n/2\alpha$.

The second part of the condition $\xi\cap[-n,n]^c\neq\phi$ is needed to take care of finite initial configurations. Our theorem implies that for a fixed finite initial configuration exponential convergence always occurs. If for example ξ_0 is a single particle at X_0, reasoning as in the last paragraph shows that exponential convergence occurs if and only if $P(|X_0|>n)\leqslant Ce^{-\gamma n}$.

The proof of sufficiency, given in Section 2, is more technical. The key to the proof is a coupling result given in (2.1), from which the conclusion follows in a straightforward manner by using (1.1)−(1.3). The proof of necessity is given in Section 3.

2. Proof of Sufficiency

We begin by constructing the process. Define independent Poisson processes $\{S_n^x: n\geqslant 1\}$, $\{T_n^x: n\geqslant 1\}$, and $\{U_n^x: n\geqslant 1\}$ for each $x\in\mathbb{Z}$ with rates 1, λ, and $(\theta-1)\lambda$ respectively. As the reader can probably guess from the rates, at times

S_n^x we kill a particle at x if one is present,
T_n^x a particle is born at x if $x-1$ or $x+1$ is occupied,
U_n^x a particle is born at x if $x-1$ and $x+1$ are both occupied.

It is easy to see that using this "graphical representation" we can construct for each μ and s the process starting from distribution μ at time s: $\{\xi_t^{\mu,s};\ t\geqslant s\}$. See [5] for more details. In what follows it will be useful to use also the coordinate notation for our processes: $\xi_t^{\mu,s}(x)=1$ if $x\in\xi_t^{\mu,s}$, $=0$ otherwise.

To prove that our condition is sufficient we will prove several lemmas. The first one is a coupling property that is a special property of the nearest neighbor case. Let $l_t^A=\inf\xi_t^A$.

(2.1) **Lemma.** *Let* $E_t=\{r_t^{(-\infty,-at]}\geqslant bt\}$, $F_t=\{l_t^{[at,\infty)}\leqslant -bt\}$, $G_t=\{\tau^{A\cap[-at,at]}>t\}$, *where* $a,b>0$. *On* $E_t\cap F_t\cap G_t$, $\xi_t^A(x)=\xi_t^{\mathbb{Z}}(x)$ *for* $x\in[-bt,bt]$.

Proof. This follows easily from the proof of Lemma 13 in [5].

Pick $\varepsilon<\alpha/4$, and let $a=(\alpha-2\varepsilon)$ and $b=\varepsilon$. It follows from (1.3), translation invariance, and symmetry that

$$P(l_t^{[at,\infty)}>-\varepsilon t)=P(r_t^{(-\infty,-at]}<\varepsilon t)\leqslant Ce^{-\gamma t}. \tag{2.2}$$

If μ satisfies the hypothesis of our theorem, then there are η, $C,\gamma\in(0,\infty)$ such that

$$\mu(\xi:|\xi\cap[-at,at]|\leqslant\eta t,\ \xi\cap[-at,at]^c\neq\phi)\leqslant Ce^{-\gamma t}. \tag{2.3}$$

With (2.1)−(2.3) being established the rest is straightforward. Let

$$M_t=\{\xi:|\xi\cap[-at,at]|>\eta t\},\quad N_t=\{\xi\subset[-at,at]\},$$

$$p(t)=\left|\int\{P(B\subset\xi_t^A)-P(\tau^A=\infty)\nu(\xi:B\subset\xi)\}\mu(dA)\right|,$$

and for $i=1,2,3$ let

$$p_i(t)=\int_{\Omega_i}|P(B\subset\xi_t^A)-P(\tau^A=\infty)\nu(\xi:B\subset\xi)|\mu(dA),$$

where $\Omega_1 = M_t^c \cap N_t^c$, $\Omega_2 = M_t$, and $\Omega_3 = N_t$. Clearly $p(t) \leqslant p_1(t) + p_2(t) + p_3(t)$. Since the integrand is a difference of two probabilities, it is $\leqslant 1$ and (2.3) implies

$$p_1(t) \leqslant Ce^{-\gamma t}.$$

For the second term we observe

$$p_2(t) \leqslant \int_{M_t} \{|P(B \subset \xi_t^A) - \nu(\xi : B \subset \xi)| + P(\tau^A < \infty)\}\mu(dA).$$

If $A \in M_t$ then $P(\tau^{A \cap [-at, at]} < \infty) \leqslant Ce^{-\gamma t}$ by (1.2); so it follows from (2.2) and (2.1) that $p_2(t) \leqslant Ce^{-\gamma t}$. To bound the third term we observe

$$p_3(t) \leqslant \int_{N_t} \{ |P(B \subset \xi_t^A) - P(t/2 < \tau^A) P(B \subset \xi_{t/2}^{\mathbb{Z}})| + P(t/2 < \tau^A < \infty)$$
$$+ |P(B \subset \xi_{t/2}^{\mathbb{Z}}) - \nu(\xi : B \subset \xi)| P(\tau^A = \infty)\} \mu(dA).$$

The last two terms in the integrand are $\leqslant Ce^{-\gamma t}$ by (1.1) and the fact that exponential convergence holds for $\mu = \delta_{\mathbb{Z}}$. To estimate the first term, we observe that

$$|P(B \subset \xi_t^A) - P(t/2 < \tau^A) P(B \subset \xi_{t/2}^{\mathbb{Z}})|$$
$$= |P(B \subset \xi_t^A,\ \tau^A > t/2) - P(B \subset \xi_t^{\mathbb{Z}, t/2},\ \tau^A > t/2)|$$

because $B \subset \xi_t^{\mathbb{Z}, t/2}$ and $\tau^A > t/2$ are independent. When $A \in N_t$, $A \subset [-at, at]$ so (2.1) implies the last difference is smaller than

$$P(t/2 < \tau^A \leqslant t) + P(E_t^c) + P(F_t^c).$$

The last quantity is $\leqslant Ce^{-\gamma t}$ by (1.1) and (2.2), and the proof is complete.

3. Proof of Necessity

The main step in proving necessity is to establish

(3.1) **Lemma.** *Suppose* $|A \cap [-n, n]| \leqslant \delta n$ *and* $A \cap [-n, n]^c \neq \phi$. *Let* $t = n/2\alpha$. *Then*

$$P(\tau^A > t/2)\, P(0 \in \xi_t^{\mathbb{Z}, t/2}) - P(0 \in \xi_t^A) \geqslant Ke^{-(\theta\lambda+1)\delta n} - 2\, P(r_t^- \geqslant 2\alpha t),$$

where $K = e^{-1} P(\tau^0 = \infty) \nu(\xi : 0 \in \xi)$.

Note. Of course, $P(r_t^- \geqslant 2\alpha t) \leqslant Ce^{-\gamma t}$ by (1.3). We have written the result in the above form to emphasize that the error term, $-2P(r_t^- \geqslant 2\alpha t)$, does not depend on δ.

Proof. Let $B_n = \{$All particles in $A \cap [-n, n]$ die by time 1 and do not give birth$\}$,

$$D_n = \{r_t^{(-\infty, -n]} < 0,\quad l_t^{[n, \infty)} > 0\}$$

$$x_n = \text{ the point in } A \cap [-n, n]^c \text{ closest to } 1/3.$$

Since $\tau^A > t/2$ and $0 \in \xi_t^{\mathbb{Z}, t/2}$ are independent, we have

$$P(\tau^A > t/2)\, P(0 \in \xi_t^{\mathbb{Z}, t/2}) - P(0 \in \xi_t^A) = p_1(t) + p_2(t) + p_3(t),$$

where

$$p_1(t) = P(\tau^A > t/2,\ 0 \in \xi_t^{\mathbb{Z}, t/2}) - P(\tau^A > t/2,\ 0 \in \xi_t^{\mathbb{Z}, t/2},\ B_n^c),$$

$$p_2(t)=P(\tau^A>t/2,\ 0\in\xi_t^{\mathbb{Z},t/2},\ B_n^c)-P(\tau^A>t/2,\ 0\in\xi_t^{\mathbb{Z},t/2},\ B_n^c\cup D_n^c),$$
$$p_3(t)=P(\tau^A>t/2,\ 0\in\xi_t^{\mathbb{Z},t/2},\ B_n^c\cup D_n^c)-P(\tau^A>t/2,\ 0\in\xi_t^A).$$

On $B_n\cap D_n$ or $\{0\notin\xi_t^{\mathbb{Z},t/2}\}$, $0\notin\xi_t^A$ so $p_3(t)\geqslant 0$. Clearly
$$p_2(t)\geqslant -P(D_n^c)\geqslant -2P(r_t^-\geqslant n),$$
by translation invariance. As for the remaining term,
$$\begin{aligned}p_1(t)&=P(\tau^A>t/2,\ 0\in\xi_t^{\mathbb{Z},t/2},\ B_n)\\&\geqslant P(\text{the particle at } x_n \text{ does not die by time } 1,\ \xi_{t/2}^{x_n,1}\neq\phi,\ 0\in\xi_t^{\mathbb{Z},t/2},\ B_n)\\&\geqslant e^{-1}P(\tau^0=\infty)\nu(\xi:0\in\xi)\{e^{-\theta\lambda}(1-e^{-1})\}^{|A\cap[-n,n]|},\end{aligned}$$

since the four events are independent. Replacing $1-e^{-1}$ by e^{-1} gives the desired bound.

to prove necessity now we write
$$P(\tau^\mu=\infty)\nu(\xi:0\in\xi)-P(0\in\xi_t^\mu)=q_1(t)+q_2(t)+q_3(t),$$
where
$$q_1(t)=P(\tau^\mu=\infty)\nu(\xi:0\in\xi)-P(\tau^\mu>t/2)\nu(\xi:0\in\xi),$$
$$q_2(t)=P(\tau^\mu>t/2)\nu(\xi:0\in\xi)-P(\tau^\mu>t/2)P(0\in\xi_t^{\mathbb{Z},t/2}),$$
$$q_3(t)=P(\tau^\mu>t/2)P(0\in\xi_t^{\mathbb{Z},t/2})-P(0\in\xi_t^\mu).$$

By (1.1), $q_1(t)\geqslant -Ce^{-\gamma t}$. Exponential convergence for the case $\mu=\delta_{\mathbb{Z}}$ implies that
$$q_2(t)\geqslant -Ce^{-\gamma t}.$$

Notice that in both cases the constants C,γ do not depend on μ. Using (3.1) on the third term we see that
$$q_3(t)\geqslant Ke^{-(\theta\lambda+1)\delta n}\mu(A:|A\cap[-n,n]|<\delta n,\ A\cap[-n,n]^c\neq\phi)-2P(r_t^-\geqslant 2\alpha t)$$
and the proof of necessity is complete.

References

[1] Durrett, R., On the growth of one dimensional contact processes, *Ann. Prob.*, **8** (1980), 890— 907.
[2] Durrett, R., Oriented percolation in two dimensions, *Ann. Prob.*, **12** (1984), 999— 1040.
[3] Durrett, R., Lecture Notes on Particle Systems and Percolation, Wadsworth Pub. Co., Pacific Grove CA, 1988.
[4] Durrett, R. and Griffeath, D., Supercritical contact processes on Z, *Ann. Prob.*, **11** (1983), 1— 15.
[5] Gray, L., Duality for general attractive spin systems with application in one dimension, *Ann. Prob.*, **14** (1986), 371— 396.
[6] Griffeath, D., The basic contact process, *Stoch. Processes Appl.*, **11** (1981), 151— 185.
[7] Harris, T. E., Contact interactions on a lattice, *Ann. Prob.*, **2** (1974), 969— 988.
[8] Liggett, T. M., Interacting Particle Systems, Springer Verlag, New York, 1985.