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A characterization of the Σ_1-definable functions of $KP\omega$ + (*uniform AC*)

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Abstract. The subject of this paper is a characterization of the Σ_1-definable set functions of Kripke-Platek set theory with infinity and a uniform version of axiom of choice: $KP\omega$ + (*uniform AC*). This class of functions is shown to coincide with the collection of set functionals of type 1 primitive recursive in a given choice functional and $x \mapsto \omega$. This goal is achieved by a Gödel Dialectica-style functional interpretation of $KP\omega$ + (*uniform AC*) and a computability proof for the involved functionals.

1 Introduction

$KP\omega$, Kripke-Platek set theory with infinity, has been the subject of several recent proof-theoretic examinations. It contains axioms postulating extensionality, existence of pairs and unions, foundation of arbitrary classes and, of course, an infinity axiom. Separation and collection are allowed only with regard to Δ_0-formulas. A standard source for $KP\omega$ is J.Barwise, "Admissible Sets and Structures" [1].

The starting point of this work was the paper by Michael Rathjen, "A proof-theoretic characterization of the primitive recursive set functions" [11]. Rathjen was able to prove an interpretation theorem for a subsystem of $KP\omega$ with foundation restricted to Σ_1-formulas. This theorem yields that the Σ_1-definable set-functions of this subsystem are exactly those functions which are primitive recursive in $x \mapsto \omega$ (cf. the definition of primitive recursive set functions in [11]). Rathjen achieved his goal by partial cut-elimination. His paper led to the question whether an analogous result for all of $KP\omega$ could be established by means of a Gödel-style functional interpretation (see [4]), replacing set functions by set functionals of finite type. As P. Päppinghaus has published results aiming at a similar

Mathematics Subject Classification (1991):03F10, 03F25 conclusion (though by a completely different argument; see [7–9]), this at first seemed feasible. But it turned out that a Gödel-style functional interpretation of a theory axiomatized in the language of set theory seems to require necessarily the use of a kind of choice functional (cf. Sect. 5). Of course such a choice functional is not Σ_1-definable in the language of set theory. $KP\omega + (\mathit{uniform\ AC})$ turned out to be the theory which achieves a balance between set functionals of type 1 primitive recursive in a fixed choice functional and $x \mapsto \omega$ and Σ_1-definable functions. Our main theorem can be stated as follows (let F_C be a fixed choice functional of type 1, cf. Definition 2.2)[1]:

Theorem 1.1 *The set functionals of type 1 primitive recursive in F_C and $x \mapsto \omega$ are exactly the Σ_1-definable functions of $KP\omega + (\mathit{uniform\ AC})$.*

We now give a brief outline of this paper:
The paper is divided into three sections. First, set functionals are introduced and discussed. They are basically a generalization of Feferman's ordinal functionals used for a functional interpretation of ID_1 [3]. The second part is devoted to the presentation of the system $KP\omega$+$(\mathit{uniform\ AC})$ and the interpretation theorem. As $KP\omega$ is a classical system, the Shoenfield $\forall\exists$-translation (see [12]) is appropriate for this purpose, in contrast to most intuitionistic treatments (including Gödel's) using an $\exists\forall$-translation (see also Diller and Nahm [2]). As we employ a Hilbert-style calculus based on Shoenfield's system P' for arithmetic ([12], pp. 214, 215), we are able to refer to Shoenfield's proof to a considerable extent. In the third section we prove our functionals to be computable with unique output (therefore we needed a *uniform* axiom of choice). This is achieved by proving a Church-Rosser theorem. As this proof can be formalized within $KP\omega + (\mathit{uniform\ AC})$, this yields a complete characterization.

The analogous correspondence holds if we substitute the theory $KP\omega$ + $(\mathit{uniform\ AC})$ by $KP + (\mathit{uniform\ AC}) + \forall x \exists! y \varphi^G(x, y)$ for a Δ_0-definition φ^G of an arbitrary function G. For this purpose, we would add an appropriate functional f^G in place of the ω-functional (which would not even be necessary in the special case $G \equiv \lambda x.x$ yielding a theorem on $KP + (\mathit{uniform\ AC})$).

2 Set functionals

Definition 2.1 *(Finite types, the type structure) The collection $\mathscr{T}$ of finite linear type symbols is defined inductively by the following clauses:*

(i) $o \in \mathscr{T}$
(ii) $\sigma, \tau \in \mathscr{T} \Rightarrow \sigma \to \tau \in \mathscr{T}$.

[1] Added in proof: In the meantime the first author showed, that by means of a Diller-Nahm-style interpretation (cf. [2]) it is possible to eliminate the choice functional and give an interpretation with set functionals primitive recursive in $x \mapsto \omega$. This yields the characterization: The class of Σ-definable set functions of $KP\omega$ coincides with the collection of set functionals of type 1 primitive recursive in $x \mapsto \omega$.

We abbreviate $1 :\equiv o \to o, \sigma \to \tau \to \rho :\equiv \sigma \to (\tau \to \rho)$.
(See Shoenfield [12] page 215 for further details). We define inductively

(i) $\mathscr{V}_o := V$ *(the universe of sets)*
(ii) $\mathscr{V}_{\sigma\to\tau} := \{ f \mid f : \mathscr{V}_\sigma \to \mathscr{V}_\tau ; f \text{ function}\}$,

and call $\mathscr{V} := \bigcup\{\mathscr{V}_\sigma \mid \sigma \in \mathscr{T}\}$ *the full type structure.*

Definition 2.2 *($\mathscr{L}$-terms) We define $\mathscr{L}$-terms by the following clauses (and indicate by $t \in \sigma$ that t is an $\mathscr{L}$-Term of type σ) :*

(i) *(variables) For each $\sigma \in \mathscr{T}$ there are countably many variables* $x^\sigma, y^\sigma, \ldots$.
(ii) *(constants)* 0 *and* ω *are $\mathscr{L}$-Terms of type o.*
$M \in o \to 1$, $U \in 1 \to o \to o$ *and* $F_C \in 1$ *are $\mathscr{L}$-terms.*
For each $\sigma \in \mathscr{T}, C_\sigma \in \sigma \to \sigma \to o \to o \to \sigma$ *and*
$R_\sigma \in ((o \to \sigma) \to o \to \sigma) \to o \to \sigma$ *are $\mathscr{L}$-terms.*
(iii) *(application) For each pair of $\mathscr{L}$-terms $s \in \tau \to \sigma$ and $t \in \tau$, the application st is an $\mathscr{L}$-term of type σ.*
(iv) *(abstraction) If s is an $\mathscr{L}$-term of type σ, then $\lambda x^\tau.s$ is an $\mathscr{L}$-term of type $\tau \to \sigma$.*

We use boldface letters to denote sequences of variables or terms and introduce the following abbreviations:

$a\boldsymbol{b}$	$:\equiv ab_1..b_n$	for $a \in \tau_1 \to .. \to \tau_m \to \sigma, b_i \in \tau_i$ $(i = 1, ..., n)$
$\boldsymbol{ab}$	$:\equiv a_1\boldsymbol{b}, .., a_n\boldsymbol{b}$	for $a_i \in \tau_1 \to .. \to \tau_m \to \sigma_i, b_i \in \tau_i$ $(i = 1, .., n)$

When denoting variables of type o, we often omit the superscript, i.e. $x, y, z, \ldots$ stand for "set variables". We write $t[\boldsymbol{x}^{\boldsymbol{\sigma}}]$ to indicate that $\boldsymbol{x}^{\boldsymbol{\sigma}}$ is a complete list of the variables occuring free in t ($\boldsymbol{x}^{\boldsymbol{\sigma}}$ denoting a tuple $x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$).
We don't give a formal system for functionals. Instead, we will argue informally within $\mathscr{V}$. Application and abstraction work as usual. For example, we define zero-functionals by $Z_o :\equiv 0, Z_{\sigma\to\tau} :\equiv \lambda x^\sigma.Z_\tau$. Thus, the semantics of our terms should be obvious exept for the constants. We give a brief description of their intended meaning:
0 and ω denote the corresponding sets. The following constants respond to arguments of appropriate types as follows:

$$Mxy = x \cup \{y\}$$

$$UXy = \bigcup\{Xz \mid z \in y\}$$

$$C_\sigma XYab = \begin{cases} X, & \text{if } a \in b \\ Y, & \text{if } a \notin b. \end{cases}$$

$$R_\sigma Gx = G((R_\sigma G) \restriction x)x, \text{ where } t \restriction xy = C_\sigma(ty)Z_\sigma yx.$$

It is intended, that F_C is interpreted by a universal choice function, i.e. $F_C a \in a$ for all sets $\emptyset \neq a \in V$ and $F_C\emptyset = \emptyset$.

Definition 2.3 *Every $\mathscr{L}$-term $t[\boldsymbol{x}^{\boldsymbol{\sigma}}] \in \tau$ naturally corresponds to a (not necessarily unique) functional $f_t \in \mathscr{V}_{\sigma_1\to\ldots\to\sigma_n\to\tau}$. We call these set-functionals primitive recursive in F_C and $x \mapsto \omega$ of type $\sigma_1 \to \ldots \to \sigma_n \to \tau$.*

Our presentation of primitive recursive set functionals is an adaptation of Feferman's system PR_Ω of ordinal functionals (see [3]) with the following two exeptions: We consider in addition the choice functional F_C. Furthermore the functional C_σ is a generalisation of the primitive recursive set *function* C used by Rathjen [11] (it can be obtained by naturally combining two of Feferman's functionals: a comparison functional L and a restriction functional Res). Rathjen's set functions were introduced by Jensen and Karp [5].

Definition 2.4 *($\mathscr{L}$-formulas, generalized $\mathscr{L}$-formulas) The $\mathscr{L}$-formulas are defined inductively as follows:*

(i) *If s, t are $\mathscr{L}$-terms of type o, then $s \in t$ is an (atomic) formula of $\mathscr{L}$.*
(ii) *If A and B are $\mathscr{L}$-formulas and t is an $\mathscr{L}$-term of type o, then*

$$A \vee B, \neg A, \forall x \in tA$$

are $\mathscr{L}$-formulas.

We introduce the usual abbreviations $A \to B, A \wedge B$ and $\exists x \in tA$. An expression $\forall \boldsymbol{x} \exists \boldsymbol{y} A$ is called a generalized $\mathscr{L}$-formula, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are (possibly empty) tuples of distinct variables (possibly of higher type) and A is an $\mathscr{L}$-formula.

The semantics for $\mathscr{L}$ is standard, we write $\models A$ to express, that a (generalized) $\mathscr{L}$-formula is valid in $\mathscr{V}$. Equations of type o are abbreviations:

$$s = t :\equiv \forall x \in s(x \in t) \wedge \forall x \in t(x \in s).$$

For the functional interpretation we now proceed along the lines of Shoenfield's consistency proof for number theory [12] (Chap. 8.3). Thus, we should study the expressive power of $\mathscr{L}$-terms. First, we want to make sure that all the sets defined by Δ_0-Separation have their counterparts as $\mathscr{L}$-terms. The next step is to show that we may also introduce terms that depend on the validity of an $\mathscr{L}$-formula; we call this definition by cases. And finally, in dealing with the recursor R_σ, we have to generalize the introduction of terms $ta = s(t \restriction a, a)$ to finite sequences of terms (simultaneous recursion).

Lemma 2.5 *The following sets can be described by $\mathscr{L}$-terms and hence can be thought of as $\mathscr{L}$-terms:*

$$\{a\}, \bigcup a, a \cup b, a - b, a \cap b, \{t(x) \mid x \in a \wedge A(x)\}.$$

where $t \in o$ denotes an $\mathscr{L}$-term and A any $\mathscr{L}$-formula. As a consequence, for every set defined by separation with regard to an $\mathscr{L}$-formula we find a corresponding $\mathscr{L}$-term.

Proof. $\{a\} = M0a, \bigcup\{t(x) \mid x \in a\} = U(\lambda x.t(x))a$,
$\bigcup a = \bigcup\{x \mid x \in a\}, \{t(x) \mid x \in a\} = \bigcup\{\{t(x)\} \mid x \in a\}$,
$\{a, b\} = M\{a\}b, a \cup b = \bigcup\{a, b\}$,
$a - b = \bigcup\{C_0 0\{x\}xb \mid x \in a\}, a \cap b = a - (a - b)$;

Using these results, the assertion for $\{t(x) \mid x \in a \wedge A(x)\}$ is proved by an easy induction on the complexity of A. For instance

$$\{t(x) \mid x \in a \wedge s \in r\} = \bigcup\{C_0\{t(x)\}0sr \mid x \in a\}.$$

For $A \equiv \neg A_0$ we use

$$\{t(x) \mid x \in a \wedge A(x)\} = \{t(x) \mid x \in a\} - \{t(x) \mid x \in a \wedge A_0(x)\}. \quad \square$$

Lemma 2.6 *(Characteristic terms) For each $\mathscr{L}$-formula A there exists a characteristic term $t_A \in o$, which contains at most the free variables of A, such that (with $1 :\equiv \{0\} :\equiv M00$)*

$$\models A \leftrightarrow t_A = 0 \;\&\; \models \neg A \leftrightarrow t_A = 1.$$

Proof. Let $x \notin FV(A)$. We define $t_A :\equiv \bigcup\{\{0\} \mid x \in \{0\} \wedge A\}$. $\square$

Lemma 2.7 *(Definition by cases) For each pair of $\mathscr{L}$-terms $t_1[\boldsymbol{x}^{\boldsymbol{\sigma}}], t_2[\boldsymbol{x}^{\boldsymbol{\sigma}}]$ of type τ and any $\mathscr{L}$-formula $A[\boldsymbol{x}^{\boldsymbol{\sigma}}]$ we can find a third term $t[\boldsymbol{x}^{\boldsymbol{\sigma}}]$ containing at most the free variables of t_1, t_2 and A and fulfilling the following scheme of definition by cases:*

$$\models A \rightarrow t = t_1 \;\&\; \models \neg A \rightarrow t = t_2.$$

Proof. Let t_A be a characteristic term for A and define $t :\equiv C_\tau t_1 t_2 t_A 1$. $\square$

Lemma 2.8 *(Simultaneous recursion) Let $n \geq 1$. For each tuple $s_0(x_0, \ldots, x_n, a) \in \sigma_0, \ldots, s_n(x_0, \ldots, x_n, a) \in \sigma_n$ of $\mathscr{L}$-terms ($x_i \in o \rightarrow \sigma_i$) there are corresponding $\mathscr{L}$-terms $t_0 \in o \rightarrow \sigma_0, \ldots, t_n \in o \rightarrow \sigma_n$ such that the following scheme of simultaneous recursion holds:*

$$t_i a = s_i(t_0 \restriction a, \ldots, t_n \restriction a, a) \qquad i = 1, \ldots, n, a \in o.$$

Proof. Straightforward by means of pairing and decoding. Ordered n-tuples can be easily introduced by reducing the problem to type o. For example, let (for terms $s_0 \in o \rightarrow \sigma \rightarrow o, s_1 \in o \rightarrow \tau \rightarrow o, s \in 0 \rightarrow \sigma \rightarrow \tau \rightarrow o$ and fresh variables x^σ, y^τ, z)

$$< s_0, s_1 >:\equiv \lambda zxy. < s_0zx, s_1zy > \text{resp.}$$

$$(s)_0 :\equiv \lambda zx.(szxZ_\tau)_0 \text{ and } (s)_1 :\equiv \lambda zy.(szZ_\sigma y)_1.$$

We use a joint variable z to make sure that $(s \restriction a)_j = (s)_j \restriction a$ holds for j=0,1.

3 A functional interpretation of $KP\omega$ + (*uniform AC*)

We work in a Hilbert-style calculus T for $KP\omega + (\textit{uniform AC})$, based on the system P' given by Shoenfield [12] pp. 214, 215 (the calculus is not intended to be minimal). The language $\mathscr{L}_{\in}^{+}$ of T consists of free variables $a, b, c, \ldots$, bounded variables $x, y, z, \ldots$, the relation symbols $\in, =$, the unary function symbol F_C and the logical symbols $\neg, \vee, \forall$. $\mathscr{L}_{\in}^{+}$-terms are defined inductively as usual (free variables are terms, if t_0 is a term, then $F_C t_0$ is also a term). $\mathscr{L}_{\in}^{+}$-formulas are defined inductively:

(i) $t = s, t \in s$ are $\mathscr{L}_{\in}^{+}$-formulas (t, s $\mathscr{L}_{\in}^{+}$-terms).
(ii) If φ, ψ are $\mathscr{L}_{\in}^{+}$-formulas, then $\neg\varphi, \varphi \vee \psi$ are $\mathscr{L}_{\in}^{+}$-formulas.
(iii) If $\varphi(a)$ is an $\mathscr{L}_{\in}^{+}$-formula in which x does not occur, then $\forall x \varphi(x)$, $(\forall x \in b)\varphi(x)$ are $\mathscr{L}_{\in}^{+}$-formulas.

We employ the usual abbreviations $\varphi \to \psi, \varphi \wedge \psi, \varphi \leftrightarrow \psi, \exists y \varphi(x)$ and $(\exists x \in b)\varphi(x)$. The quantifiers $\forall x$ are called unbounded, one of the form $(\forall x \in b)$ is called bounded. The semantics for the bounded quantifier is obviously $\forall x(x \in b \to \varphi(x))$, but we emphasize that we do not treat them as abbreviations. The class Δ_0 consists of all formulas not containing unbounded quantifiers (see Barwise [1] for details).

Definition 3.1 *(A Hilbert-style calculus for $KP\omega + (\textit{uniform AC})$)*

(i) *Axioms* :

Logical axioms	$\neg\varphi \vee \varphi$	
Substitution	$\forall x \varphi(x) \to \varphi(t)$	(t *a term*)
	$(\forall x \in a)\varphi(x) \to (t \in a \to \varphi(t))$	(t *a term*)
Equality	$a = b \to (\varphi(a) \leftrightarrow \varphi(b))$	$(\varphi \in \Delta_0)$
Extensionality	$(\forall x \in a(x \in b) \wedge \forall x \in b(x \in a)) \to a = b$	
Pair	$\exists z(a \in z \wedge b \in z)$	
Union	$\exists z\ \forall y \in a \forall x \in y(x \in z)$	
Δ_0-Separation	$\exists z\ z = \{x \in a \mid \varphi(x)\}$	$(\varphi \in \Delta_0)$
Uniform AC	$\forall x(x \neq \emptyset \to F_C x \in x) \wedge F_C \emptyset = \emptyset$	
Δ_0-Collection	$\forall x \in a \exists y \varphi(x, y) \to$	
	$\exists z \forall x \in a \exists y(y \in z \wedge \varphi(x, y))$	$(\varphi \in \Delta_0)$
Infinity	$\exists \xi\, Lim(\xi)$	($Lim(\xi)$ *a* Δ_0 *formula expressing that* ξ *is a limit ordinal.*)

(ii) *Rules:*

Expansion	$\varphi \vdash \varphi \vee \psi$	
Contraction	$\varphi \vee \varphi \vdash \varphi$	
Associativity	$\varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi$	
$(\forall)$-*Introduction*	$\varphi(a) \vee \psi \vdash \forall x \varphi(x) \vee \psi$	$(a \notin \psi)$
$(b\forall)$-*Introduction*	$(b \in a \to \varphi(b)) \vee \psi \vdash (\forall x \in a)\varphi(x) \vee \psi$	$(b \notin \psi)$
Cut	$\varphi \vee \psi, \neg\varphi \vee \chi \vdash \psi \vee \chi$	
Foundation rule	$\forall z \in a \varphi(z) \to \varphi(a) \vdash \varphi(b)$	$(a \notin \varphi)$

Note that the Δ_0-Separation is an abbreviation for

$$\exists z\ [\forall u \in z(u \in a \wedge \varphi(a)) \wedge \forall u \in a(\varphi(u) \to u \in a)].$$

The use of foundation axioms would force us to handle longer formulas in the final proof, making it less transparent. On the other hand, it is easy to derive foundation axioms by applying the foundation rule (analogous to the case of the induction rule for number theory, see [12], p. 215). Note that in the collection axiom the subformula $\forall x \in a \exists y(y \in z \wedge \varphi(x,y))$ is not in Δ_0 (cf. proof of Theorem 3.3).

We will now give a variant of the Shoenfield translation which inductively assigns a generalized $\mathscr{L}$-formula to every $\mathscr{L}_{\in}^{+}$-formula φ.

Definition 3.2 *(*-translation, interpreting $\mathscr{L}$-terms)*

$\varphi^* \quad :\equiv \varphi, \qquad \varphi \in \Delta_0$

Now let $\varphi^* \equiv \forall \boldsymbol{v}\exists \boldsymbol{w}A(\boldsymbol{v},\boldsymbol{w})$; $\psi^* \equiv \forall \boldsymbol{x}\exists \boldsymbol{y}B(\boldsymbol{x},\boldsymbol{y})$ (A, B *$\mathscr{L}$-formulas*)

$(\varphi \vee \psi)^* \quad :\equiv \forall \boldsymbol{v}\forall \boldsymbol{x}\exists \boldsymbol{w}\exists \boldsymbol{y}(A(\boldsymbol{v},\boldsymbol{w}) \vee B(\boldsymbol{x},\boldsymbol{y}))$

$(\forall x\varphi)^* \quad :\equiv \forall \boldsymbol{v}\forall x\exists \boldsymbol{w}A(x,\boldsymbol{v},\boldsymbol{w})$

$(\forall x \in a\varphi)^* \quad :\equiv \forall \boldsymbol{v}\forall x\exists \boldsymbol{w}(x \in a \to A(x,\boldsymbol{v},\boldsymbol{w})), \qquad \varphi \notin \Delta_0$

$(\neg\varphi)^* \quad :\equiv \forall \boldsymbol{w}\exists \boldsymbol{v}\neg A(\boldsymbol{v},\boldsymbol{w}\boldsymbol{v})$

Let $A \equiv \forall \boldsymbol{v}\exists \boldsymbol{w}A'[\boldsymbol{v},\boldsymbol{w},\boldsymbol{a}]$ *be any generalized $\mathscr{L}$-formula. We call $\mathscr{L}$-terms* $\boldsymbol{t}[\boldsymbol{v},\boldsymbol{a}]$ *interpreting $\mathscr{L}$-terms for A, if the following holds:*

$$\mathscr{V} \models \forall \boldsymbol{v}A'[\boldsymbol{v},\boldsymbol{t}[\boldsymbol{v},\boldsymbol{a}],\boldsymbol{a}].$$

The idea which leads to the translation of negated formulas (clause iv) is that of a choice-process within $\mathscr{V}$:

$$\begin{aligned} & \neg\forall v\exists w A(v,w) \\ \leftrightarrow\ & \neg\exists W\forall v A(v,Wv) \\ \leftrightarrow\ & \forall W\exists v\neg A(v,Wv). \end{aligned}$$

We can use this to show by an easy induction on the complexity of $\varphi(\boldsymbol{a}) \in \mathscr{L}_{\in}^{+}$ that for all $\boldsymbol{a} \in V$

$$V \models \varphi(\boldsymbol{a}) \Leftrightarrow \mathscr{V} \models \varphi(\boldsymbol{a})^*$$

holds. For $\exists$-formulas the translation yields

$$(\exists u\varphi(u))^* \equiv \forall \boldsymbol{v}\exists u\exists \boldsymbol{w}\neg\neg A(u,\boldsymbol{v}u\boldsymbol{w},\boldsymbol{w}(\boldsymbol{v}u\boldsymbol{w})).$$

We mention that in this case it is sufficient to interpret the (modulo double negation equivalent) formula:

$$\forall \boldsymbol{v}\exists u\exists \boldsymbol{w}A(u,\boldsymbol{v}u\boldsymbol{w},\boldsymbol{w}(\boldsymbol{v}u\boldsymbol{w})).$$

Note that for $\boldsymbol{v} \equiv v_1,\ldots,v_n, \boldsymbol{w} \equiv w_1,\ldots,w_m, u$ of appropriate types we have

$$\boldsymbol{v}u\boldsymbol{w} \equiv v_1uw_1\ldots w_m,\ldots,v_nuw_1\ldots w_m.$$

This is our main theorem:

Theorem 3.3 *(Interpretation theorem) Let* $\varphi[\boldsymbol{a}]$ *be any* $\mathscr{L}_{\in}^{+}$ *formula with translation* $\varphi^* \equiv \forall \boldsymbol{v}\exists \boldsymbol{w}A[\boldsymbol{v},\boldsymbol{w},\boldsymbol{a}]$.
If $KP\omega$ + (*uniform* AC) $\vdash \varphi[\boldsymbol{a}]$, *then there exist interpreting $\mathscr{L}$-terms* $\boldsymbol{t}\,[\boldsymbol{a},\boldsymbol{v}]$ *such that* $A[\boldsymbol{v},\boldsymbol{t},\boldsymbol{a}]$ *is valid.*

Proof. We proceed by induction on the theorems of $KP\omega + (\textit{uniform AC})$. First of all, let's have a look at the cases Shoenfield has already dealt with. These are the propositional and substitutional axioms and the logical rules. Except for the $(b\forall)$-Introduction, the interpreting terms can be found in $\mathscr{L}$ in the same way as in Shoenfields system (e.g. for the contraction rule Lemma 2.7 (definition by cases) must be applied).

(•) The last inference is a $(b\forall)$-Introduction:

$$(b \in a_0 \to \varphi(b)) \vee \psi \vdash \forall x \in a_0 \varphi(x) \vee \psi, \ \ b \text{ not free in } \psi,$$

with translation $\psi[\boldsymbol{a}]^* :\equiv \forall \boldsymbol{v} \exists \boldsymbol{w} A[\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{a}]$ and $\varphi \in \Delta_0$ (the case $\varphi \notin \Delta_0$ is analogous to the $(\forall)$-Introduction). By the induction hypothesis we get terms $\boldsymbol{t}_0[\boldsymbol{v}, \boldsymbol{a}, a_0, b]$ with

$$(*) \ \ \models b \in a_0 \to \varphi[b, \boldsymbol{a}] \vee A[\boldsymbol{v}, \boldsymbol{t}_0[\boldsymbol{v}, \boldsymbol{a}, a_0, b], \boldsymbol{a}].$$

We have to find terms $\boldsymbol{t}\,[\boldsymbol{v}, \boldsymbol{a}, a_0]$ such that the following holds:

$$\models \forall x \in a_0 \varphi[x, \boldsymbol{a}] \vee A[\boldsymbol{v}, \boldsymbol{t}\,[\boldsymbol{v}, \boldsymbol{a}, a_0], \boldsymbol{a}].$$

Note that $\boldsymbol{t}$ must not contain the free variable b. We define

$$s[\boldsymbol{a}, a_0] := \{b \mid b \in a_0 \wedge \neg\varphi[b, \boldsymbol{a}]\} \ \ \text{and}$$

$$\boldsymbol{t}\,[\boldsymbol{v}, \boldsymbol{a}, a_0] := \boldsymbol{t}_0[\boldsymbol{v}, \boldsymbol{a}, a_0, F_C s].$$

This choice together with $(*)$ yields

$$\models \forall x \in a_0 \varphi[x, \boldsymbol{a}] \vee A[\boldsymbol{v}, \boldsymbol{t}\,[\boldsymbol{v}, \boldsymbol{a}, a_0], \boldsymbol{a}].$$

(•) Equality, $(\textit{uniform AC})$ and extensionality are unchanged by the translation and valid principles in $\mathscr{V}$. In the cases of pair, union and Δ_0-Separation axioms, Lemma 2.5 provides the appropriate interpreting terms. For the infinity axiom, naturally, we need the functional ω.

(•) We now consider Δ_0-Collection. The translation yields (modulo double negation):

$(\forall x \in a \exists y \varphi(x, y) \to \exists z \forall v \in a \exists w (w \in z \wedge \varphi(v, w)))^*$
$\Leftrightarrow \forall y \forall v \exists x \exists z \exists w [(x \in a \to \varphi(x, yx)) \to$
$(vzw \in a \to w(vzw) \in z \wedge \varphi(vzw, w(vzw)))]$.

(Note that the usual form of the collection axiom with a bounded quantifier would yield a different translation). We have to find terms $X(y, v, a)$, $Z(y, v, a) \in o$ and $W(y, v, a) \in 1$ such that the following holds:

$$(X \in a \to \varphi(X, yX)) \to$$

$$(vZW \in a \to W(vZW) \in Z \wedge \varphi(vZW, W(vZW))).$$

Remember that Z is intended to be a collection of examples b satisfying $\varphi(u, b)$ for $u \in a$. Therefore, the choice $W :\equiv y, Z :\equiv \{yu \mid u \in a\}$ and $X :\equiv vZW$ yields the assertion:

$$\models (X \in a \to \varphi(X, yX)) \to (X \in a \to yX \in Z \wedge \varphi(X, yX)).$$

($\bullet$) It remains to treat the foundation rule: $\forall u \in a \varphi(u) \to \varphi(a) \vdash \varphi(b)$. Let $\varphi(a)^* \equiv \forall \boldsymbol{v} \exists \boldsymbol{w} A(\boldsymbol{v}, \boldsymbol{w}, a)$ (omitting the other free variables $\boldsymbol{a}$). We have (modulo double negation)

$$(\forall u \in a \varphi(u) \to \varphi(a))^* \Leftrightarrow$$

$$\forall \boldsymbol{w} \forall \boldsymbol{x} \exists u \exists \boldsymbol{v} \exists \boldsymbol{y} [(u \in a \wedge \neg A(\boldsymbol{v}, \boldsymbol{w} u \boldsymbol{v}, u)) \vee A(\boldsymbol{x}, \boldsymbol{y}, a)].$$

The induction hypothesis provides terms $U(\boldsymbol{w}, \boldsymbol{x}, a), \boldsymbol{V}(\boldsymbol{w}, \boldsymbol{x}, a)$, $\boldsymbol{Y'}(\boldsymbol{w}, \boldsymbol{x}, a)$ such that

$$\models (U \in a \wedge \neg A(\boldsymbol{V}, \boldsymbol{w} U \boldsymbol{V}, U)) \vee A(\boldsymbol{x}, \boldsymbol{Y'}, a).$$

Setting $Y_i(\boldsymbol{w}, a) :\equiv \lambda \boldsymbol{x}. Y_i'$ for $i = 1, \ldots, n$ (with $\boldsymbol{Y'} = Y_1', \ldots, Y_n'$) yields:

$$(1) \ \models (U \in a \wedge \neg A(\boldsymbol{V}, \boldsymbol{w} U \boldsymbol{V}, U)) \vee A(\boldsymbol{x}, \boldsymbol{Y} \boldsymbol{v}, a).$$

Our aim is to find terms $\boldsymbol{W}(b, \boldsymbol{v})$ such that $\models A(\boldsymbol{v}, \boldsymbol{W}(b, \boldsymbol{v}), b)$.
We mention, that it suffices to find terms $\boldsymbol{W_0}$ of appropriate type such that $\models A(\boldsymbol{v}, \boldsymbol{W_0} b \boldsymbol{v}, b)$ holds: defining $\boldsymbol{W}(b, \boldsymbol{v}) :\equiv \boldsymbol{W_0} b \boldsymbol{v}$ (application) we get the interpreting terms for $\varphi(b)^*$.
By simultaneous recursion we define $\boldsymbol{W_0} z = \boldsymbol{Y}(\boldsymbol{W_0} \restriction z, z)$. After substituting $\boldsymbol{x} :\equiv \boldsymbol{v}, a :\equiv b$ and $\boldsymbol{w} :\equiv \boldsymbol{W_0} \restriction b$ in (1), this leads to

$$(2) \models (U(\boldsymbol{W_0} \restriction b, \boldsymbol{v}, b) \in b \wedge \neg A(\boldsymbol{V}(\boldsymbol{W_0} \restriction b, \boldsymbol{v}, b), (\boldsymbol{W_0} \restriction b) U(..) \boldsymbol{V}(..), U(..)))$$

$$\vee A(\boldsymbol{v}, \boldsymbol{Y}(\boldsymbol{W_0} \restriction b, b) \boldsymbol{v}, b).$$

Using this, we can now verify

$$(3) \qquad \models A(\boldsymbol{v}, \boldsymbol{W_0} b \boldsymbol{v}, b)$$

by $\in$-induction on b:
By the induction hypothesis we have for all $a \in b, \boldsymbol{v'} \in \mathscr{V}$:

$$(IV) \qquad \models A(\boldsymbol{v'}, \boldsymbol{W_0} a \boldsymbol{v}, a).$$

We choose $\boldsymbol{v} \in \mathscr{V}$ and show (3):
Case 1. $U(\boldsymbol{W_0} \restriction b, \boldsymbol{v}, b) \notin b$. (2) and $\boldsymbol{W_0} b \boldsymbol{v} = \boldsymbol{Y}(\boldsymbol{W_0} \restriction b, b) \boldsymbol{v}$ yield (3).
Case 2. $U(\boldsymbol{W_0} \restriction b, \boldsymbol{v}, b) \in b$. Substituting $\boldsymbol{v'} :\equiv \boldsymbol{V}(\boldsymbol{W_0} \restriction b, \boldsymbol{v}, b)$ and $a :\equiv U(\boldsymbol{W_0} \restriction b, \boldsymbol{v}, b)$ in (IV) we get

$$(4) \models A(\boldsymbol{V}(\boldsymbol{W_0} \restriction b, \boldsymbol{v}, b), \boldsymbol{W_0} U(\boldsymbol{W_0} \restriction b, \boldsymbol{v}, b) \boldsymbol{V}(\boldsymbol{W_0} \restriction b, \boldsymbol{v}, b), U(\boldsymbol{W_0} \restriction b, \boldsymbol{v}, b)).$$

Hence (3) follows from (2) und (4) together with

$$(\boldsymbol{W_0} \restriction b) U(..) \boldsymbol{V}(..) = \boldsymbol{W_0} U(..) \boldsymbol{y} \boldsymbol{V}(..).$$

This completes the proof of the theorem. $\square$

Corollary 3.4 *For every closed Π_2-theorem $\forall x \exists y \varphi(x,y) \in \mathscr{L}_{\in}^{+}$ of $KP\omega + (uniform\ AC)$ there are set functionals F and G primitive recursive in F_C of type 1, such that:*

(i) $\mathscr{V} \models \varphi(x, Fx)$
(ii) $\mathscr{V} \models (\forall x \in a)(\exists y \in Ga)\varphi(x,y)$.

Proof. Let F be the functional given by the interpreting $\mathscr{L}$-term f we find due to the interpretation theorem. Furthermore, let G be the functional corresponding to the set $\bigcup\{fx \mid x \in a\}$. □

4 Church-Rosser theorem and a computability theorem

In number theory, all closed functionals of type o are numerals and hence have primitive recursive names. With set functionals, this is different. Not every set has a primitive recursive “name”. So, when dealing with computability, it is not sufficient to restrict oneself to reductions of $\mathscr{L}$-terms. Therefore, we will introduce new terms including constant symbols for every set. As we are only interested in closed terms, we will in the process replace λ-abstraction with the combinators K and S.

Definition 4.1 *($\mathscr{L}_V$-terms): (By $s \in \sigma$, we now express that s is an $\mathscr{L}_V$-term of type σ.)*

i) (set constants)
For any set $x \in V$, let $\mathfrak{k}(x)$ be an $\mathscr{L}_V$-term of type o.

Let ρ, σ, τ be arbitrary type symbols.

ii) (constants of higher type)
M, U, C_σ and R_σ are $\mathscr{L}_V$-terms of the same types as in $\mathscr{L}$.

iii) (combinators)
$K_{\sigma\tau} \in \sigma \to \tau \to \sigma$ and $S_{\rho\sigma\tau} \in (\tau \to \rho \to \sigma) \to (\tau \to \rho) \to \tau \to \sigma$ are also $\mathscr{L}_V$-terms.

iv) (application)
If $s \in \sigma \to \tau$ and $t \in \sigma$ are $\mathscr{L}_V$-terms, so is $st \in \tau$.

The semantics of the combinators (for all suitable types) is given by

$$Kst = s, Srst = rt(st).$$

They allow a definition of combinatory λ-abstraction (see e.g. Diller and Nahm [2]). In particular, we define zero-functionals by

$$Z_o \equiv \mathfrak{k}(0),\ Z_{\sigma\to\tau} \equiv K_{\tau\sigma}Z_\tau.$$

Restriction now looks as follows (omitting subscripts):

$$t \upharpoonright s \equiv S(S(CtZ)(Ks))I$$

with the identity operator $I \equiv SKK$.

We now introduce the notion of reduction:

Definition 4.2 *(conversions, contractions, and reductions of $\mathscr{L}_V$-terms with reduction order $\alpha \in On$):*
(Let r, s, t etc. be arbitrary $\mathscr{L}_V$-terms of appropriate types.)

i) (conversions)

$(refl)$	$s \rightarrow_\alpha s$	$(\forall \alpha \geq 0)$
(M)	$M\mathfrak{k}(x)\mathfrak{k}(y) \rightarrow_\alpha \mathfrak{k}(x \cup \{y\})$	$(\forall \alpha \geq 0)$
(U)	$[G\mathfrak{k}(y) \twoheadrightarrow_{\alpha_y} \mathfrak{k}(x_y)\ \&\ \alpha_y < \alpha]$ *for all* $y \in x \Rightarrow$ $UG\mathfrak{k}(x) \rightarrow_\alpha \mathfrak{k}(\bigcup\{x_y \mid y \in x\})$	
(F_C)	$F_C\mathfrak{k}(x) \rightarrow_\alpha \mathfrak{k}(F_C(x))$	$(\forall \alpha \geq 0)$
(C_1)	$Cst\mathfrak{k}(x)\mathfrak{k}(y) \rightarrow_\alpha s$, *if* $x \in y$	$(\forall \alpha \geq 0)$
(C_2)	$Cst\mathfrak{k}(x)\mathfrak{k}(y) \rightarrow_\alpha t$, *if* $x \notin y$	$(\forall \alpha \geq 0)$
(R)	$RG\mathfrak{k}(x) \rightarrow_\alpha G((RG) \restriction \mathfrak{k}(x))\mathfrak{k}(x)$	$(\forall \alpha \geq 0)$
(K)	$Kst \rightarrow_\alpha s$	$(\forall \alpha \geq 0)$
(S)	$Srst \rightarrow_\alpha rt(st)$	$(\forall \alpha \geq 0)$

ii) (contractions)
A contraction consists of parallel conversions of subterms:

$$s_1 \rightarrow_\alpha t_1 \ \&\ s_2 \rightarrow_\alpha t_2 \Rightarrow s_1 s_2 \rightarrow_\alpha t_1 t_2.$$

iii) (reductions)
We write $s \twoheadrightarrow^n_\alpha t$, if there are terms $s_0, \ldots, s_n$ such that

$$s \equiv s_0 \rightarrow_\alpha s_1 \ldots \rightarrow_\alpha s_n \equiv t.$$

$s \twoheadrightarrow_\alpha t$ and $s \twoheadrightarrow t$ denote $\exists n s \twoheadrightarrow^n_\alpha t$ and $\exists \alpha \exists n s \twoheadrightarrow^n_\alpha t$ respectively.

Obviously, we have

- $s \rightarrow_\alpha t \Leftrightarrow s \twoheadrightarrow^1_\alpha t$.
- $s \twoheadrightarrow^n_\alpha t \ \&\ \beta > \alpha \Rightarrow s \twoheadrightarrow^n_\beta t$.

The notions of redex, normal form etc. are basically the same as in number theory (see, e.g., Troelstra [13]). There is, however, an important difference. Our U-conversion in general has an infinite number of premises depending on reduction chains of arbitrary length.
We mention that prima facie the reduct of a U-conversion is not unique! So the definition should be read as:

$$UG\mathfrak{k}(x) \rightarrow \mathfrak{k}(\bigcup z)$$

for any z meeting the following conditions:

i) $\forall u \in z \exists y \in x G\mathfrak{k}(y) \twoheadrightarrow \mathfrak{k}(u)$,
ii) $\forall y \in x \exists u \in z G\mathfrak{k}(y) \twoheadrightarrow \mathfrak{k}(u)$.

But this distinction will be obsolete by the time we have proved the forthcoming Church-Rosser theorem.

Our first attempt to prove this theorem was based on a paper by W. Maaß on a “Church-Rosser theorem for λ-calculi with infinitely long terms” [6]. The applied method was successful, but the step of retranslating the result from infinite terms back to our $\mathscr{L}_V$-terms seemed rather awkward. Finally, a less complicated assignment of reduction orders emerged, leading to a straightforward proof. The now-assigned orders roughly measure the $\in$-tree of the involved sets (their "$\in$-rank"), ignoring the actual number of reduction steps necessary.

Theorem 4.3 *(Church-Rosser theorem) For any $\mathscr{L}_V$-terms r, s, t ordinals α, β and $n, m \in \omega$ meeting*

$$r \twoheadrightarrow^n_\alpha s \;\&\; r \twoheadrightarrow^m_\beta t,$$

we can find an $\mathscr{L}_V$-term u such that

$$s \twoheadrightarrow^m_\beta u \;\&\; t \twoheadrightarrow^n_\alpha u.$$

Proof. By transfinite induction on $\alpha\#\beta$ and side-induction on $n + m$. If $n = 0$ or $m = 0$ we are done, as well as in case $n > 1$ or $m > 1$ because we may then directly apply the side-induction hypothesis. This leaves us with the case $n = m = 1$, that is

$$r \to_\alpha s \;\&\; r \to_\beta t.$$

We cannot avoid another induction, this time on the generation of r. There are three possibilities for reduction:

a) Two contractions.
   A trivial application of the induction hypothesis (on r).
b) A contraction $r \to_\alpha s$ and a conversion $r \to_\beta t$.
   This is again trivial except for the case that $r \equiv UG\mathfrak{k}(x)$. Suppose we have

$$(1) \qquad G \to_\alpha G'$$

   and therefore

$$r \to_\alpha s \equiv UG'\mathfrak{k}(x)$$

   and further for all $y \in x$

$$G\mathfrak{k}(y) \twoheadrightarrow_{\beta_y} \mathfrak{k}(x_y)$$

   and hence

$$r \to_\beta t \equiv \mathfrak{k}(\bigcup\{x_y \mid y \in x\}).$$

   (1) yields for all $y \in x$

$$G\mathfrak{k}(y) \to_\alpha G'\mathfrak{k}(y).$$

   The induction hypothesis (on the ordinals) gives for all $y \in x$ (since $\beta_y\#\alpha < \beta\#\alpha$)

$$G'\mathfrak{k}(y) \twoheadrightarrow_{\beta_y} \mathfrak{k}(x_y).$$

   Hence

$$UG'\mathfrak{k}(x) \twoheadrightarrow_\beta \mathfrak{k}(\bigcup\{x_y \mid y \in x\}) \equiv: u$$

   and u is the term we have been looking for.

c) Two conversions.
By the first symbol of r, the type of conversion (modulo (refl)) is fixed. Again, only the case of U-conversion is non-trivial. By isolating two premises

$$G\mathfrak{k}(y) \twoheadrightarrow_{\alpha_y} \mathfrak{k}(x_1) \;\&\; G\mathfrak{k}(y) \twoheadrightarrow_{\beta_y} \mathfrak{k}(x_2)$$

with $\alpha_y < \alpha, \beta_y < \beta, y \in x$, we find $x_1 = x_2$, due to the induction hypothesis on the ordinals and $\alpha_y \# \beta_y < \alpha \# \beta$. Thus, the reduct must be unique, and we actually have the trivial case $u \equiv s \equiv t$. $\square$

Corollary 4.4 *Any $\mathscr{L}_V$-term has at most one normal form. In particular, the reduct $\mathfrak{k}(\bigcup z)$ of a U-conversion*

$$UG\mathfrak{k}(x) \rightarrow_\alpha \mathfrak{k}(\bigcup z)$$

is unique.

Thus, our first definition of U-conversion has been vindicated. There is, however, one complication left. As we have no a priori knowledge of the (possibly) infinitely many premises required for a U-conversion, a term $UG\mathfrak{k}(x)$ could well be in normal form. In other words, normal form is no longer a mere syntactical notion. This leads to the following definition of computability:

Definition 4.5 *We inductively define the classes $Comp_\sigma$ of $\mathscr{L}_V$-terms of type σ:*

i) $s \in Comp_o :\Leftrightarrow s \in o$ *& there is a set x such that* $s \twoheadrightarrow \mathfrak{k}(x)$.
ii) $s \in Comp_{\sigma\to\tau} :\Leftrightarrow s \in \sigma \to \tau$ *& for all* $t \in Comp_\sigma, st \in Comp_\tau$.

Any $s \in Comp := \bigcup\{Comp_\sigma \,|\sigma \in \mathscr{T}\}$ *is called computable.*

It is now easy to verify the following statements:

Lemma 4.6 *i)* $s \in Comp_{\sigma\to\tau}$ & $t \in Comp_\sigma \Rightarrow st \in Comp_\tau$.
ii) $s \twoheadrightarrow t$ & $t \in Comp_\sigma \Rightarrow s \in Comp_\sigma$.
iii) $s \in Comp_{o\to\sigma} \Leftrightarrow$ *for all sets* $x, s\mathfrak{k}(x) \in Comp_\sigma$.
iv) Let $\sigma = \sigma_0, \dots, \sigma_n,\ s \in \sigma \to \tau$. *If* $st_0 \dots t_n \in Comp_\tau$ *for all $\mathscr{L}_V$-terms* $t_0, \dots, t_n$ *meeting*
– $t_j \equiv \mathfrak{k}(x)$ *with arbitrary x, if* $\sigma_j = o$;
– $t_j \in Comp_{\sigma_j}$, *if* $\sigma_j \neq o$;
then $s \in Comp_{\sigma\to\tau}$.

Before we prove the main result of this section we mention that by means of standard arithmetization it is possible to formalize the proof of the Church-Rosser theorem as well as the computability proof for *each separate term* within $KP\omega$ + (*uniform AC*). (Of course, the statement “All terms are computable” cannot be expressed in $KP\omega$ + (*uniform AC*).)

Theorem 4.7 *(Theorem on computability) Every $\mathscr{L}_V$-term s is computable.*

Proof. By induction on the generation of s. Using the results of the previous lemma, this is straightforward. Let’s have a quick glance at the critical cases:

– $s \equiv U$. We want to show

$$UG\mathfrak{k}(x) \in Comp_o \text{ for an arbitrary set } x \text{ and } G \in Comp.$$

$G \in Comp$ yields

$$\forall y \in x \exists x_y G\mathfrak{k}(y) \twoheadrightarrow \mathfrak{k}(x_y)$$

which provides the premises of U-conversion:

$$UG\mathfrak{k}(x) \rightarrow \mathfrak{k}(\bigcup\{x_y | y \in x\}).$$

Formalizing this in $KP\omega + (\textit{uniform AC})$, the process of collecting suitable x_y's is in fact an instance of Σ-Collection (see Barwise [1], chapter 1).

– $s \equiv R_\sigma$. We have to show

$$R_\sigma G\mathfrak{k}(a) \in \underset{\sigma}{Comp}$$

for arbitrary sets a, $G \in Comp$. We proceed by $\in$-induction on a. As induction hypothesis, we have

$$R_\sigma G\mathfrak{k}(z) \in \underset{\sigma}{Comp}$$

for every $z \in a$. By R-conversion we get

$$R_\sigma G\mathfrak{k}(a) \rightarrow G((R_\sigma G) \restriction \mathfrak{k}(a))\mathfrak{k}(a).$$

It is sufficient to examine the subterm $(R_\sigma G) \restriction \mathfrak{k}(a)$. Introducing an arbitrary set b, we get, with a C-conversion as last step:

$$(R_\sigma G) \restriction \mathfrak{k}(a)\mathfrak{k}(b) \twoheadrightarrow \begin{cases} R_\sigma G\mathfrak{k}(b) & \text{if } b \in a \\ Z_\sigma & \text{if } b \notin a. \end{cases}$$

Now, $R_\sigma G\mathfrak{k}(b)$ is computable for every $b \in a$, due to our induction hypothesis. Z_σ is obviously computable, so our lemma yields the claim. □

5 Remarks

In the introduction we have remarked, that it seems to be impossible to give a Gödel Dialectica-style functional interpretation of a theory axiomatized in the language of set theory without any kind of choice functional. Examining the proof of the interpretation-theorem, we can see that we used the choice functional to treat $(b\forall)$-Introduction. The question arises, whether this is a mere technical difficulty or a substantial point. The following example might illustrate the underlying problem: Obviously we have

$$(1) \quad KP\omega \vdash \forall x(x \neq \emptyset \rightarrow \exists y(y \in x)).$$

The translation yields

$$(2) \quad \forall v \exists w(v \neq \emptyset \rightarrow w \in v).$$

The “logical, internal” choice (1) made by $KP\omega$ becomes an external choice via the functional interpretation: any term interpreting (2) makes the choice explicit and corresponds to a kind of choice functional. This forces us to consider the additional functional F_C. In order to prove the Σ_1-definability of this additional functional, we obviously have to consider a stronger theory, namely $KP\omega$ + (*uniform AC*). The reason, that we add F_C as a new function symbol also to the language of $KP\omega$ is that we have to prove a Church-Rosser theorem within the theory, i.e. the theory needs a name to denote the unique element chosen by the choice functional F_C.

There is a further possibility to avoid the difficulties described in the example above: If we consider the full type-structure (cf. definition 2.1) relativized to the constructible hierarchy (i.e. starting with L instead V) and add a functional $\chi_{<_L}$ (denoting the characteristic function of the well-ordering $<_L$ of the constructible universe) to the class of primitive recursive set functionals, this additional functional also allows us to treat the $(b\forall)$-Introduction. To prove the computability of this new functional within the theory, we add the axiom of constructibility and obtain the theory $KP\omega + (V = L)$. In order to keep the balance between the class of primitive recursive set functionals and the theory we consider, it is again necessary to add a further functional. From [11] we know, that the function $\alpha \mapsto L_\alpha$ is a primitive recursive set function. But to interpret the new axiom $(V = L)$ i.e. $\forall x \exists \alpha (x \in L_\alpha)$ we need a rank-functional rk_L such that in the relativized type structure

$$x \in L_{rk_L(x)}$$

holds. This yields the interpretability of $V = L$.

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