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Refined Enumeration of Noncrossing Chains and Hook Formulas

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Abstract. In the combinatorics of finite Coxeter groups, there is a simple formula giving the number of maximal chains of noncrossing partitions. It is a reinterpretation of a result by Deligne which is due to Chapoton, and the goal of this article is to refine the formula. First, we prove a one-parameter generalization, by considering the enumeration of noncrossing chains where we put a weight on some relations. Second, we consider an equivalence relation on noncrossing chains coming from the natural action of the group on set partitions, and we show that each equivalence class has a simple generating function. Using this we recover Postnikov's hook length formula in type A and obtain a variant in type B.

Keywords: Euler numbers, finite Coxeter groups, set partitions

1. Introduction

Let W be a finite Coxeter group of rank n and h its Coxeter number. A formula due to Deligne [5] states that the number of factorizations of a Coxeter element as a product of n reflections is

$$\frac{n!}{|W|}h^n.$$

The value in the case of the symmetric group is $(n+1)^{n-1}$, and this number is also known to be the number of Cayley trees on n vertices. Chapoton [4] give another interpretation of Deligne's formula: this number counts the maximal chains in the lattice of noncrossing partitions [1].

Our first goal (in Section 3) is to prove a one-parameter generalization of this result. A noncrossing chain is a sequence $\hat{0} = \pi_0 \lessdot \pi_1 \lessdot \cdots \lessdot \pi_n = \hat{1}$ in the lattice of noncrossing partitions. By weighting some of the cover relations in these chains with a parameter q, the refined enumeration turns out to be

$$\frac{n!}{|W|}\prod_{i=1}^{n}(d_i + q(h-d_i)),$$

where d_i's are the degrees of the group. This is done by generalizing a recursion due to Reading [14], and using known results on Fuss-Catalan numbers [1].

Our second goal (in Section 4) is to study the equivalence classes of noncrossing chains defined as follows. The group W acts naturally on the set partition lattice, and there is an induced action on the set of maximal chains of set partitions. The number of orbits is an integer $K(W)$ that has been calculated in our previous work [10]. The subset of noncrossing chains is not stable under this action, but let us say that two noncrossing chains are equivalent if they are in the same orbit. We show that the generating function of each equivalence class has a simple form as a product.

Eventually (in Section 5), we show how our results lead to some hook-length formulas for trees in types A and B. More precisely, in type A we recover Postnikov's hook formula [6, 11] and in type B we obtain a variant.

2. Definitions

Let $S = \{s_1, \dots, s_n\}$ be the set of simple generators of W, and T the set of reflections. Let V be the standard geometric representation of W, i.e., an n-dimensional Euclidean space such that each $t \in T$ is an orthogonal reflection through the hyperplane $\mathrm{Fix}(t) = \{v \in V : t(v) = v\}$. These hyperplanes are called the *reflecting hyperplanes*. In particular, $H_i = \mathrm{Fix}(s_i)$ are called the *simple hyperplanes*.

Definition 2.1. *Let $\mathcal{P}(W)$ denote the set of (generalized) set partitions, i.e., linear subspaces of V that are an intersection of reflecting hyperplanes. It is partially ordered with reverse inclusion (i.e., $\pi \le \rho$ if $\rho \subseteq \pi$ as linear subspaces). Let $\mathcal{M}(W)$ denote the set of maximal chains of $\mathcal{P}(W)$.*

For each $\pi \in \mathcal{P}(W)$, we define the *stabilizer* and *pointwise stabilizer* as, respectively:

$$\mathrm{Stab}(\pi) = \big\{w \in W : w(L) = L\big\},$$

$$\mathrm{Stab}^*(\pi) = \big\{w \in W : \forall x \in L, w(x) = x\big\}.$$

In the classical case, an interval partition is a set partition where each block is a set of consecutive integers, for example, $123|4|56$. In the present context, there is a natural generalization (which might have been considered in previous work, with different terminology).

Definition 2.2. *An element $\pi \in \mathcal{P}(W)$ is an* interval partition *if it is an intersection of simple hyperplanes. Let $\mathcal{P}^I(W) \subseteq \mathcal{P}(W)$ denote the set of interval partitions, and $\mathcal{M}^I(W) \subset \mathcal{M}(W)$ denote the set of maximal chains in $\mathcal{P}^I(W)$.*

The set $\mathcal{P}^I(W)$ is a sublattice of $\mathcal{P}(W)$ and is isomorphic to a boolean lattice. It follows that $\mathcal{M}^I(W)$ has cardinality $n!$. The coatoms of $\mathcal{P}^I(W)$ are exactly the lines $L_1, \dots, L_n$ defined by:

$$L_i = \bigcap_{\substack{1 \le j \le n \\ j \ne i}} H_j. \tag{2.1}$$

Let $W_{(i)}$ denote the (standard maximal parabolic) subgroup of W generated by the s_j with $j \neq i$. Then $W_{(i)} = \mathrm{Stab}^*(L_i)$.

We will need the following fact (see [10, Proposition 3.3]) where w_0 denote the longest element of W (with respect to the simple generators s_i and the associated length function).

Proposition 2.3. *Each line $L \in \mathcal{P}(W)$ can be written as $w(L_i)$ for some $w \in W$ and $1 \leq i \leq n$. If $w \in W$ and $i \neq j$, then $w(L_i) = L_j$ implies $w_0(L_i) = L_j$.*

A consequence is the following:

Proposition 2.4. *Each orbit $O \in \mathcal{M}(W)/W$ contains an element of $\mathcal{M}^I(W)$.*

Proof. Let $C \in O$. By Proposition 2.3, there exists $w \in W$ such that the coatom L in the chain $w(C)$ is an interval partition, i.e., L is one of the L_i's previously defined. At this point we can make an induction on the rank.

Let us sketch how the induction work, using ideas in [10]. There is a natural bijection between $\mathcal{M}\left(W_{(i)}\right)$ and the chains in $\mathcal{M}(W)$ having L_i as a coatom. This bijection sends $\mathcal{M}^I\left(W_{(i)}\right)$ to the chains in $\mathcal{M}^I(W)$ having L_i as a coatom. By induction, there is $u \in W_{(i)}$ such that $uw(C) \in \mathcal{M}^I(W)$, whence the result. ∎

Let us motivate the next definition by some considerations in the "classical" case. Let π_1, π_2, π_3 be the noncrossing partitions represented in Figure 1 from left to right. Here, π_1, etc. are represented is represented by drawing an arc between two consecutive elements of each block. Both π_2 and π_3 are covered by π_1, and more precisely they are obtained from π_1 by splitting the block $\{1,2,5,7\}$. But we can make one distinction: π_2 is obtained by removing one arc from π_1, and its two blocks $\{1,2\}$ and $\{5,7\}$ form an interval partition of the block $\{1,2,5,7\}$ of π_1. This is not the case for π_3.

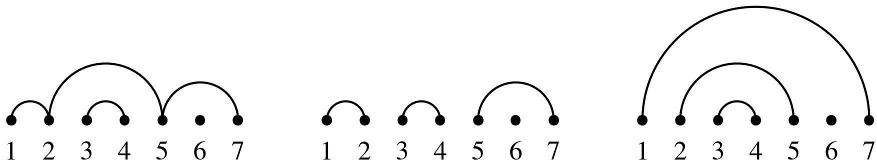

Figure 1: Noncrossing partitions.

To generalize this distinction, consider the group $\mathrm{Stab}^*(\pi_1) \subset \mathfrak{S}_7$. It has an irreducible factor $\mathfrak{S}_4$ acting on the block $\{1,2,5,7\}$. The simple roots of $\mathfrak{S}_7$ are $e_1 - e_2, \dots, e_6 - e_7$ where $(e_i)_{1 \leq i \leq 7}$ is the standard basis of $\mathbb{R}^7$. The ones of the irreducible factor $\mathfrak{S}_4$ of $\mathrm{Stab}^*(\pi_1)$ are $e_1 - e_2$, $e_2 - e_5$, $e_5 - e_7$. It can be seen that the simple roots of $\mathrm{Stab}^*(\pi_2)$ are included in the ones $\mathrm{Stab}^*(\pi_1)$, but it is not the case for π_3.

Let us turn to the general case. Let Φ be a root system of W (in the sense of Coxeter groups, see [9]), and let Φ^+ be a choice of positive roots. For each $\pi \in \mathcal{P}(W)$, the group $\mathrm{Stab}^*(\pi)$ is a reflection subgroup of W, and its set of roots is $\Phi \cap \pi^\perp$. We will always take $\Phi^+ \cap \pi^\perp$ as a natural choice of positive roots, and accordingly

$\mathrm{Stab}^*(\pi)$ has a natural set of simple roots and simple generators. In this setting, we have the following:

Definition 2.5. *Let $\pi_1, \pi_2 \in \mathcal{P}(W)$, we denote $\pi_2 \sqsubseteq \pi_1$ and say that π_2 is an* interval refinement *of π_1 if the simple roots of* $\mathrm{Stab}^*(\pi_2)$ *are included in the simple roots of* $\mathrm{Stab}^*(\pi_1)$.

Note that $\pi_2 \sqsubseteq \pi_1$ implies $\pi_1 \subseteq \pi_2$, i.e., $\pi_2 \le \pi_1$ in the lattice $\mathcal{P}(W)$. Also, interval partitions are exactly the interval refinements of the maximal partition.

Some preliminary definitions are needed before going to noncrossing partitions.

Definition 2.6. *Let $T \subset W$ be the set of reflections. A reduced T-word of w is a factorization $w = t_1 \cdots t_k$ where $t_1, \dots, t_k \in T$ and k is minimal. Let $u, v \in W$. The* absolute order *is defined by the condition that $u <_{abs} v$ if some reduced T-word of u is a subword of some reduced T-word of v.*

Definition 2.7. *If $\sigma \in \mathfrak{S}_n$, we call $c = s_{\sigma(1)} \cdots s_{\sigma(n)}$ a* standard Coxeter element *of W with respect to S. Any element conjugated in W to a standard Coxeter element is called a* Coxeter element.

This might differ from the terminology used in other references, but we need here some properties of the standard Coxeter elements that are not true in general. In what follows, we always assume that c is a standard Coxeter element.

Definition 2.8. *A set partition $\pi \in \mathcal{P}(W)$ is* noncrossing *with respect to c if $\pi =$* $\mathrm{Fix}(w)$ *for some $w \in W$ such that $w <_{abs} c$. This w is actually unique and will be denoted by $\underline{\pi}$ (see [3, Theorem 1]). Let $\mathcal{P}^{NC}(W,c) \subset \mathcal{P}(W)$ denote the subset of noncrossing partitions with respect to c, and let $\mathcal{M}^{NC}(W,c) \subset \mathcal{M}(W)$ denote the set of maximal chains of $\mathcal{P}^{NC}(W,c)$. If $\pi \in \mathcal{P}^{NC}(W,c)$, then $\underline{\pi}$ is the Coxeter element of a unique parabolic subgroup of W that we denote by $W_{(\underline{\pi})}$ or $W_{(\pi)}$ (although this interferes with the notation $W_{(s)}$ for maximal standard parabolic subgroup, there should be no confusion).*

Note in particular that $\mathrm{Fix}(\underline{\pi}) = \pi$. We refer to [1] for more on the subject of noncrossing partitions. In general, $\mathcal{P}^{NC}(W,c)$ is not stable under the action of W. But from the invariance of the absolute order under conjugation, we can see that $\mathcal{P}^{NC}(W,c)$ is stable under the action of c.

Remark 2.9. Noncrossing partitions are usually defined as a subset of W, but here it is natural to have the inclusion $\mathcal{P}^{NC}(W,c) \subset \mathcal{P}(W)$. These two points of view are equivalent under the correspondence $\underline{\pi} \leftrightarrow \pi$ and we will also allow to identify noncrossing partitions with a subset of W. For example, if $u, v \in W$ are noncrossing, the notion of interval refinement $u \sqsubseteq v$ is well defined, and $u \in W$ is called an interval partition if it is so as a noncrossing partition.

Proposition 2.10. *We have $\mathcal{P}^I(W) \subset \mathcal{P}^{NC}(W,c)$. Let $\pi_1 \in \mathcal{P}^{NC}(W,c)$ and $\pi_2 \in \mathcal{P}(W)$ with $\pi_2 \sqsubseteq \pi_1$, then $\pi_2 \in \mathcal{P}^{NC}(W,c)$.*

Proof. The maximal partition is noncrossing since $\{0\} = \mathrm{Fix}(c)$, so the first point follows from the second one.

To prove the second point, we need Proposition 3.4 from the next section. Let $r_1, \dots, r_k$ be the reflections associated with the simple roots of $\pi_1^\perp$, and we can assume there is $j \leq k$ such that $r_1, \dots, r_j$ are the reflections associated with the simple roots of $\pi_2^\perp$. Since π_1 is noncrossing, it means there is $u \in W$ with $u <_{abs} c$ and $\mathrm{Fix}(u) = \pi_1$. It is known that u is a Coxeter element of the subgroup $\mathrm{Stab}^*(\pi_1) \subset W$. But Proposition 3.4 shows more: it is a standard Coxeter element, so there is $\sigma \in \mathfrak{S}_k$ such that $u = r_{\sigma(1)} \cdots r_{\sigma(k)}$. Let v be obtained from this factorization by keeping only the factors $r_1, \dots, r_j$. Then, we have $v <_{abs} u <_{abs} c$ and $\mathrm{Fix}(v) = \pi_2$, so π_2 is noncrossing. ∎

Remark 2.11. It is interesting to note that similar results hold for *nonnesting partitions* in the sense of Postnikov (defined only in the crystallographic case). A set partition $\pi \in \mathcal{P}(W)$ is nonnesting when the simple roots of $\mathrm{Stab}^*(\pi)$ form an antichain in the poset of positive roots. A subset of an antichain being itself an antichain, if $\pi_2 \sqsubseteq \pi_1$ and π_1 is nonnesting, then π_2 is nonnesting. Any interval partition is nonnesting, since the simple roots form an antichain. Note also that the intuition from the "classical" case is clear: it is impossible to create a crossing or a nesting by removing arcs.

3. Chains of Noncrossing Partitions

Definition 3.1. *For any chain* $\Pi = (\pi_0, \dots, \pi_n) \in \mathcal{M}^{NC}(W, c)$, *let* $\mathrm{nir}(\Pi)$ *be the number of i such that π_i is not an interval refinement of π_{i+1}. Let*

$$M(W, q) = \sum_{\Pi \in \mathcal{M}^{NC}(W,c)} q^{\mathrm{nir}(\Pi)}.$$

It is not *a priori* obvious that $M(W, q)$ does not depend on the choice of the standard Coxeter element c. This will be proved below.

The coatoms of the lattice $\mathcal{P}^{NC}(W, c)$ are exactly the products ct for $t \in T$. Since T is stable by conjugation, the set cT of coatoms is stable by conjugation by c. An interesting property of standard Coxeter elements is that this action has good properties, (see Propositions 3.2 and 3.4) similar to those of a bipartite Coxeter element obtained by Steinberg [17].

In what follows, an orbit for the action of c will be called a *c-orbit*. Note that the action of c becomes conjugation when we see noncrossing partitions as elements of W, i.e., $\underline{c(\pi)} = c\underline{\pi}c^{-1}$ if $\pi \in \mathcal{P}^{NC}(W, c)$.

Proposition 3.2. *Let h be the Coxeter number of W (i.e., the order of c in W). For any $t \in T$, the c-orbit of ct satisfies one of the following two conditions*:

- *It contains h distinct elements, and exactly 2 interval partitions L_i and L_j, related by $L_i = w_0(L_j)$.*
- *Or it contains $\frac{h}{2}$ distinct elements, and exactly 1 interval partition L_i, satisfying $w_0(L_i) = L_i$. Moreover, $c^{h/2}$ restricted to L_i is -1 $\big($i.e., $c^{h/2} \notin W_{(i)}\big)$.*

The full proof is in Appendix A but let us give some comments. A standard Coxeter element $c = s_{\sigma(1)} \cdots s_{\sigma(n)}$ is called *bipartite* if there is j such that $s_{\sigma(1)}, \dots, s_{\sigma(j)}$

are pairwise commuting, and $s_{\sigma(j+1)}\cdots s_{\sigma(n)}$ too. Steinberg [17] proved that for a bipartite Coxeter element c, the c-orbit of a reflection contains either h elements and 2 simple reflections, or $\frac{h}{2}$ elements and 1 simple reflection. If h is even, another property of the bipartite Coxeter element is $c^{h/2}=w_0$. What we have is a variant that holds for any standard Coxeter element. It is natural to expect that our result can be seen as a consequence of Steinberg's but we have been unable to realize this in a uniform way.

Since the standard Coxeter element c is conjugated with a bipartite Coxeter element, and the bijection $t\mapsto ct$ from T to cT commutes with c-conjugation, we see that the c-orbit of ct contains either h or $\frac{h}{2}$ elements. In the case where w_0 is central, we can easily complete the proof of Proposition 3.2. It is known that in this case, h is even and $c^{h/2}=w_0=-1$, which acts trivially on $\mathcal{P}(W)$ (see [9, Section 3.19]). So every orbit has $\frac{h}{2}$ elements. Proposition 2.3 shows that there is at most one interval partition in each orbit, and the equality $\#T=\frac{nh}{2}$ shows that there is exactly one interval partition in each orbit. See Appendix A for other cases.

Remark 3.3. Suppose h is even and let L_i be such that $c^{h/2}(L_i)=L_i$. As mentioned above, we have $c^{h/2}=w_0$ when c is a bipartite Coxeter element. In the general case, since w_0 and $c^{h/2}$ are both in $\mathrm{Stab}(L_i)-\mathrm{Stab}^*(L_i)$, we have $w_0c^{h/2}\in W_{(i)}$. From the properties of $x\mapsto w_0xw_0$, one can deduce that the map $x\mapsto c^{h/2}xc^{h/2}$ permutes the irreducible factors of $W_{(i)}$ in the same way as $x\mapsto w_0xw_0$. This will be needed in the sequel.

It is known that parabolic Coxeter elements can be characterized with the absolute order, see [2, Lemma 1.4.3], so that ct is a Coxeter element of $W_{(ct)}$. The point of the next proposition is that it is actually a standard Coxeter element.

Proposition 3.4. *For any $t\in T$, ct is a standard Coxeter element of the parabolic subgroup $W_{(ct)}$ for the natural choice of simple generators.*

Proof. The elements ct ($t\in T$) are the coatoms of $\mathcal{P}^{NC}(W)$. By an immediate induction, the proposition implies (and therefore is equivalent to) the stronger fact that $\underline{\pi}$ is a standard Coxeter element of $\mathrm{Stab}^*(\pi)$ for each $\pi\in\mathcal{P}^{NC}(W)$. The proof of this has been provided by an anonymous referee, and relies on the results by Reading [13].

More specifically, the result follows from [13, Theorem 6.1]. A consequence of this theorem is that a noncrossing partition $\underline{\pi}$ is a product of its so-called *cover reflections*. Besides, [13, Lemma 1.3] states that these cover reflections are the simple generators of a parabolic subgroup. ∎

We are now ready to prove how $M(W,q)$ can be computed inductively, and in particular that it does not depend on the choice of a standard Coxeter element.

Proposition 3.5. *If W is irreducible, we have*

$$M(W,q)=\frac{2+q(h-2)}{2}\sum_{s\in S}M\big(W_{(s)},q\big). \tag{3.1}$$

Proof. For each $\Pi=(\pi_0,\ldots,\pi_n)\in\mathcal{M}^{NC}(W,c)$, let $\Pi'=(\pi_0,\ldots,\pi_{n-1})$. The coatom of Π is $\pi_{n-1}=ct$ for some $t\in T$, and the set of such Π with ct as a coatom is in bijection with $\mathcal{M}^{NC}\big(W_{(ct)}, ct\big)$ via the map $\Pi \mapsto \Pi'$. Moreover, $\mathrm{nir}(\Pi) = \mathrm{nir}(\Pi')$ if $ct \sqsubseteq c$ $\big($i.e., $ct \in \mathcal{P}^I(W)\big)$ and $\mathrm{nir}(\Pi) = \mathrm{nir}(\Pi') + 1$ otherwise. So, distinguishing the chains in $\mathcal{M}^{NC}(W, c)$ according to their coatoms gives

$$M(W, q) = \sum_{t\in T} q^{\chi[ct\notin\mathcal{P}^I(W)]} M\left(W_{(ct)}, q\right). \tag{3.2}$$

Note that to write this equation, we need to use Proposition 3.4. While it should be clear from the definition that the generating function of the chains $(\pi_0, \dots, \pi_{n-1}) \in \mathcal{M}^{NC}\big(W_{(ct)}, ct\big)$ with respect to the statistic nir is $M\big(W_{(ct)}, q\big)$, this quantity was only defined with respect to a standard Coxeter element. Since ct is indeed a standard Coxeter element of $W_{(ct)}$, we get the term $M\big(W_{(ct)}, q\big)$ which we assume we already know by induction.

Let $O \subset T$ be an orbit under conjugation by c. So if $t_1, t_2 \in O$, $W_{(ct_1)}$ and $W_{(ct_2)}$ are conjugated in W, so they are isomorphic and $M(W_{(ct_1)}, q) = M\big(W_{(ct_2)}, q\big)$. If $cO = \{co\colon o \in O\}$ contains $h/2$ elements and 1 interval partition L_i, we get

$$\sum_{t\in O} q^{\chi[\,ct\notin\mathcal{P}^I(W)\,]} M\left(W_{(ct)}, q\right) = \left(1 + q\left(\tfrac{h}{2} - 1\right)\right) M\left(W_{(i)}, q\right). \tag{3.3}$$

If it contains h elements and 2 interval partitions L_i and L_j, then

$$\sum_{t\in O} q^{\chi[\,ct\notin\mathcal{P}^I(W)\,]} M\left(W_{(ct)}, q\right) = (2 + q(h-2)) M\left(W_{(i)}, q\right),$$

and since the previous equation is true with i replaced with j, we also have

$$\sum_{t\in O} q^{\chi[\,ct\notin\mathcal{P}^I(W)\,]} M\left(W_{(ct)}, q\right) = \frac{2 + q(h-2)}{2} \left(M\left(W_{(i)}, q\right) + M\left(W_{(j)}, q\right)\right). \tag{3.4}$$

Now, we can split the sum in the right-hand side of (3.2) to group together the $t \in T$ that are in the same orbit, and using Equations (3.3) and (3.4), we get the desired formula for $M(W, q)$. ∎

Besides, in the reducible case it is straightforward to show that

$$M\left(W_1 \times W_2, q\right) = \binom{m+n}{m} M(W_1, q) \times M(W_2, q) \tag{3.5}$$

if the respective ranks of W_1 and W_2 are m and n.

Equations (3.1) and (3.5) can be used to compute $M(W, q)$ by induction for any W, with the initial value $M(A_1, q) = 1$.

This recursion permits to make a link with the Fuss-Catalan numbers $\mathrm{Cat}^{(m)}(W)$ (see [1, Chapter 5]). These numbers can be defined in terms of the degrees of the group $d_1, \dots, d_n$ and the Coxeter number $h = d_n$ by

$$\mathrm{Cat}^{(m)}(W) = \frac{1}{|W|} \prod_{i=1}^{n} (hm + d_i).$$

Chapoton [4] showed that $\mathrm{Cat}^{(m)}(W)$ is the number of multichains $\pi_1 \le \cdots \le \pi_m$ in $\mathcal{P}^{NC}(W,c)$, i.e., $\mathrm{Cat}^{(m)}(W) = Z(W,m+1)$ where $Z(W,m)$ is the zeta polynomial of $\mathcal{P}^{NC}(W,c)$. Fomin and Reading [8] introduced the so-called generalized cluster complex $\Delta^m(\Phi)$, and showed that its number of maximal simplices is $\mathrm{Cat}^{(m)}(W)$ (where Φ is the root system of W). Using this generalized cluster complex, they obtain in [8, Proposition 8.3] that

$$\mathrm{Cat}^{(m)}(W) = \frac{(m-1)h+2}{2n} \sum_{s\in S} \mathrm{Cat}^{(m)}\left(W_{(s)}\right) \tag{3.6}$$

in the irreducible case. Besides, there holds

$$\mathrm{Cat}^{(m)}(W_1 \times W_2) = \mathrm{Cat}^{(m)}(W_1) \times \mathrm{Cat}^{(m)}(W_2) \tag{3.7}$$

in the reducible case. Comparing the recursions (3.1), (3.5) and (3.6), (3.7) shows that

$$M(W,q) = n!(1-q)^n Z\left(W, \tfrac{1}{1-q}\right),$$

where we use the zeta polynomial rather than writing "$\mathrm{Cat}^{(\frac{q}{1-q})}(W)$" because it is generally assumed that $m \in \mathbb{N}$ when we write $\mathrm{Cat}^{(m)}(W)$. Then, the formula for $\mathrm{Cat}^{(m)}(W)$ in terms of the degrees proves the proposition below (note that the particular case $q=1$ is the result by Chapoton mentioned in the introduction):

Proposition 3.6.

$$M(W,q) = \frac{n!}{|W|} \prod_{i=1}^{n} (d_i + q(h-d_i)).$$

It is also possible to obtain this formula by solving the recursion (3.1) case by case. We will not give the details, since lengthy calculations are needed for the differential equations arising in the infinite families case. Let us just present the case of the group A_n, where we get that the series $A(z) = \sum_{n\ge 0} M(A_n,q)\frac{z^n}{n!}$ satisfies the differential equation

$$A' = A^2 + \tfrac{qz}{2}\left(A^2\right)'.$$

After multiplying the equation by A^{q-2}, it can be rewritten as

$$\left(\frac{A^{q-1}}{q-1}\right)' = (zA^q)'.$$

After checking the constant term, we arrive at the functional equation $A^{q-1} = 1 + (q-1)zA^q$. It would be possible to extract the coefficients of A with the Lagrange inversion formula. Another method is to use results about Fuss-Catalan numbers in type A. It is known that $\mathrm{Cat}^{(m-1)}(A_{n-1}) = \frac{1}{mn+1}\binom{mn+1}{n}$, which is the number of complete m-ary trees with n internal vertices, so that $F = 1 + \sum_{n\ge 1} \mathrm{Cat}^{(m-1)}(A_{n-1})z^n$ satisfies $F = 1 + zF^m$. The equation for A can be rewritten

$$A^{1-q} = 1 + z(1-q)A.$$

So, comparing the functional equations shows $F(z) = A\left(\frac{z}{1-q}\right)^{1-q}$ if $m = \frac{1}{1-q}$. This is also $F(z) = 1 + zA\left(\frac{z}{1-q}\right)$. Taking the coefficient of z^{n+1}, we obtain:

$$\frac{1}{\frac{n+1}{1-q}+1}\binom{\frac{n+1}{1-q}+1}{n+1} = \frac{1}{(1-q)^n n!}M(A_n, q),$$

hence,

$$M(A_n, q) = \frac{n!(1-q)^n}{\frac{n+1}{1-q}+1}\binom{\frac{n+1}{1-q}+1}{n+1} = \prod_{i=1}^{n-1}(i+1+q(n-i)).$$

4. Generating Functions of Equivalence Classes and Hook Formulas

Definition 4.1. *For any $\Pi \in \mathcal{M}^{NC}(W, c)$, let $[\Pi]$ denote its equivalence class for the W-action*:

$$[\Pi] = \{w(\Pi) \colon w \in W\} \cap \mathcal{M}^{NC}(W, c).$$

We also define the class generating function:

$$M([\Pi], q) = \sum_{\Omega \in [\Pi]} q^{\mathrm{nir}(\Omega)}.$$

These classes partition the set $\mathcal{M}^{NC}(W, c)$, so that we have

$$M(W, q) = \sum_{[\Pi]} M([\Pi], q), \tag{4.1}$$

where we sum over all distinct equivalence classes.

We need some definitions before giving the formula for $M([\Pi], q)$.

Let $\tau \lessdot \pi$ be a cover relation in $\mathcal{P}^{NC}(W, c)$. The group $W_{(\pi)}$ can be decomposed into irreducible factors (that can be thought of as "blocks" of the set partition π). There is only one of these factors where $\underline{\tau}$ and $\underline{\pi}$ differ, as can be seen from the factorization of the poset $\mathcal{P}\left(W_{(\pi)}\right)$ induced by the factorization of $W_{(\pi)}$.

Definition 4.2. *With τ and π as above, let $h(\tau, \pi)$ be the Coxeter number of the irreducible factor of $W_{(\pi)}$ where $\underline{\tau}$ and $\underline{\pi}$ differ.*

Definition 4.3. *Let $g(\tau, \pi)$ be minimal $g > 0$ such that $\underline{\pi}^g\, \underline{\tau}\, \underline{\pi}^{-g} = \underline{\tau}$ and the map $x \longrightarrow \underline{\pi}^g x \underline{\pi}^{-g}$ stabilizes each irreducible factor of $W_{(\tau)}$.*

Note that by examining the irreducible factors of $W_{(\pi)}$, we have $\underline{\pi}^{h(\tau,\pi)}\, \underline{\tau}\, \underline{\pi}^{-h(\tau,\pi)} = \underline{\tau}$. From $\underline{\pi}^g\, \underline{\tau}\, \underline{\pi}^{-g} = \underline{\tau}$ and Proposition 3.4, we have that either $g(\tau, \pi) = h(\tau, \pi)$ or $g(\tau, \pi) = \frac{1}{2}h(\tau, \pi)$. Note also that when $h(\tau, \pi)$ is even, as noted in Remark 3.3, we know that the map $x \rightarrow \underline{\pi}^{\frac{1}{2}h(\tau,\pi)} x \underline{\pi}^{-\frac{1}{2}h(\tau,\pi)}$ permutes the irreducible factors of $W_{(\tau)}$.

Proposition 4.4. *Let $\Pi = (\pi_0, \dots, \pi_n) \in \mathcal{M}^{NC}(W, c)$, let $h_i = h(\pi_{i-1}, \pi_i)$ and $g_i = g(\pi_{i-1}, \pi_i)$ for $2 \leq i \leq n$. Then we have*

$$M([\Pi], q) = \prod_{i=2}^{n}\left(\frac{2g_i}{h_i} + q\left(g_i - \frac{2g_i}{h_i}\right)\right).$$

The proof is rather similar to that of Proposition 3.5. We need a few lemmas.

Lemma 4.5. *If* $\Omega = (\omega_0, \dots, \omega_n) \in [\Pi]$, *there is* $k \geq 0$ *such that* $\omega_{n-1} = c^k(\pi_{n-1})$.

Proof. Let L_i (L_j, respectively) be an interval partition in the c-orbit of ω_{n-1} (π_{n-1}, respectively). The fact that these exist follows from Proposition 3.2. If $L_i = L_j$, the c-orbits are the same and this ends the proof.

Suppose now that $L_i \neq L_j$. Since $\Omega \in [\Pi]$, there is $w \in W$ such that $w(L_i) = L_j$, so Proposition 2.3 shows that $w_0(L_i) = L_j$. Then, Proposition 3.2 shows that L_i and L_j are in the same c-orbit. So ω_{n-1} and π_{n-1} are in the same c-orbit. ∎

Lemma 4.6. *Let* $\Omega = (\omega_0, \dots, \omega_n) \in [\Pi]$, *and assume inductively that Proposition 4.4 is true for the group* $W_{(\omega_{n-1})}$. *Let* $\langle \Omega \rangle$ *denote the class of* Ω *for the action of* $W_{(\omega_{n-1})}$, *i.e.,*

$$\langle \Omega \rangle = \left\{ w(\Omega) \colon w \in W_{(\omega_{n-1})} \right\} \cap \mathcal{M}^{NC}(W, c).$$

Then the generating function of $\langle \Omega \rangle$ *is*

$$M(\langle \Omega \rangle, q) = q^{\chi[\,\omega_{n-1} \notin \mathcal{P}^I(W)\,]} \prod_{i=2}^{n-1} \left(\frac{2g_i}{h_i} + q \left(g_i - \frac{2g_i}{h_i} \right) \right). \tag{4.2}$$

Proof. Let $\Omega' = (\omega_0, \dots, \omega_{n-1})$. Removing the last element of a chain gives a bijection between $\langle \Omega \rangle$ and

$$[\Omega'] = \left\{ w(\Omega') \colon w \in W_{(\omega_{n-1})} \right\} \cap \mathcal{M}^{NC}\left(W_{(\omega_{n-1})}, \underline{\omega_{n-1}} \right).$$

By induction, we can obtain $M([\Omega'], q)$. Since $\Omega \in [\Pi]$, it is straightforward to check that we have $g(\omega_{i-1}, \omega_i) = g(\pi_{i-1}, \pi_i)$ and $h(\omega_{i-1}, \omega_i) = h(\pi_{i-1}, \pi_i)$, although we see ω_{i-1}, ω_i as elements of $\mathcal{P}^{NC}\left(W_{(\omega_{n-1})}, \underline{\omega_{n-1}} \right)$ and π_{i-1}, π_i as elements of $\mathcal{P}(W, c)$. We have $M(\langle \Omega \rangle, q) = q^{\chi[\,\omega_{n-1} \notin \mathcal{P}^I(W)\,]} M([\Omega'], q)$, and this gives the formula for $M(\langle \Omega \rangle, q)$. ∎

Lemma 4.7. *The minimal integer* $g > 0$ *such that* $\langle \Pi \rangle = \langle c^g(\Pi) \rangle$ *is* g_n.

Proof. This g satisfies $c^g(\pi_{n-1}) = \pi_{n-1}$, so that either $g = h_n$ or $g = \frac{h_n}{2}$. If we are not in the case where $c^{h_n/2}(\pi_{n-1}) = \pi_{n-1}$, we have $g = h_n = g_n$. So, suppose $c^{h_n/2}(\pi_{n-1}) = \pi_{n-1}$.

Consider the factorization of the poset $\mathcal{P}\left(W_{(\pi_{n-1})} \right)$ induced by the factorization of $W_{(\pi_{n-1})}$ in irreducible factors. From the definition of g_n, the action of c^{g_n} stabilizes each factor of the poset, so it is the same action as some element $w \in W_{(\pi_{n-1})}$. So $\langle \Pi \rangle = \langle c^{g_n}(\Pi) \rangle$ and this proves $g \leq g_n$.

Reciprocally, suppose that $c^g(\Pi) = w(\Pi)$ for some $w \in W_{(\pi_{n-1})}$. It follows that c^g stabilizes the irreducible factors of $W_{(\pi_{n-1})}$. If the permutation on the factors is nontrivial, it would be possible to distinguish $c^g(\Pi)$ from $w(\Pi)$. So $g_n \geq g$, and eventually $g = g_n$. ∎

Lemma 4.8. *The classes* $\langle \Omega \rangle$ *form a partition of the set* $[\Pi]$. *A set of representatives is* $\left\{ \Pi, c(\Pi), \dots, c^{g_n - 1}(\Pi) \right\}$.

Proof. The first point is clear. From the previous lemma, the elements in the set $\{\Pi, c(\Pi), \ldots, c^{g_n-1}(\Pi)\}$ are in distinct classes. It remains to show that the list is exhaustive.

Knowing Lemma 4.5, it remains to prove that if $\Omega \in [\Pi]$ is such that $\omega_{n-1} = \pi_{n-1}$, then there is k such that $\langle\Omega\rangle = \langle c^k(\Pi)\rangle$. Let $w \in W$ such that $\Omega = w(\Pi)$. In particular, $w(\pi_{n-1}) = \pi_{n-1}$.

If $w \in W_{(\pi_{n-1})}$, we have $\langle\Omega\rangle = \langle\Pi\rangle$. Otherwise, $w \in \mathrm{Stab}(\pi_{n-1}) - \mathrm{Stab}^*(\pi_{n-1})$. Since the class $[\Pi]$ contains a chain of interval partitions, we might as well assume that π_{n-1} is an interval partition. It comes from Proposition 3.2 that $wc^{h/2} \in W_{(\pi_{n-1})}$. So we obtain $\langle\Omega\rangle = \langle c^{h/2}(\Pi)\rangle$. This completes the proof. ∎

We can now prove Proposition 4.4.

Proof of Proposition 4.4. Since the classes $\langle\Omega\rangle$ form a partition of $[\Pi]$, we have

$$M([\Pi], q) = \sum_{\langle\Omega\rangle} M(\langle\Omega\rangle, q),$$

and $M([\Pi], q)$ can be obtained by summing Equation (4.2).

From the previous lemma, the number of distinct classes $\langle\Omega\rangle$ is g_n. As we have seen above (just before Proposition 4.4), either $g_n = h_n$ or $g_n = \frac{1}{2}h_n$, so that $\frac{2g_n}{h_n}$ is an integer. From Proposition 3.2, $\frac{2g_n}{h_n}$ among the distinct classes $\langle\Omega\rangle$ are such that their coatom is an interval partition. So, we get

$$\sum_{\langle\Omega\rangle} q^{\chi[\,\omega_{n-1} \notin \mathcal{P}^I(W)\,]} = \left(\frac{2g_n}{h_n} + q\left(g_n - \frac{2g_n}{h_n}\right)\right).$$

So, we get the desired formula for $M([\Pi], q)$ by summing Equation (4.2) over the classes $\langle\Omega\rangle$. ∎

5. Hook Formulas for Types A and B

This section is devoted to explicit combinatorial description in types A and B, where Equation (4.1) can be interpreted as a hook-length formula for trees.

Definition 5.1. *Let $\mathcal{A}_n$ denote the set of* André trees *on n vertices, i.e., trees such that*:

- *each internal node has either one son or two unordered sons,*
- *the vertices are labeled with integers from* 1 *to n, and the labels are decreasing from the root to the leaves.*

The 5 elements of $\mathcal{A}_4$ are represented in Figure 2. These trees were introduced by Foata and Schützenberger [7, Chapter 5], who proved that $\#\mathcal{A}_n = T_n$ (in fact their definition requires increasing labels instead of decreasing here, but this is clearly equivalent). They were also used by Stanley [15] to prove $K(A_n) = T_n$.

Let us describe Stanley's bijection. We see it as a map $\mathcal{M}(A_{n-1}) \to \mathcal{A}_n$ that induces a bijection $\mathcal{M}(A_{n-1})/A_{n-1} \to \mathcal{A}_n$. We present an example in Figure 3 and

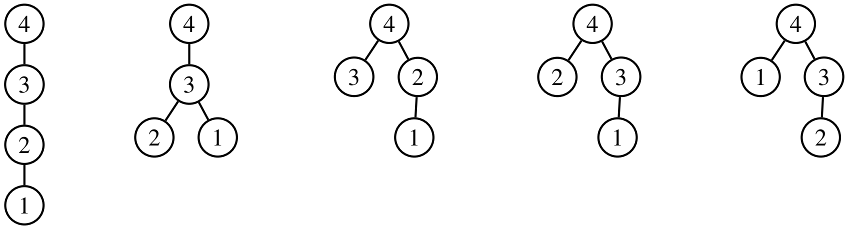

Figure 2: The André trees with 4 vertices.

refer to [15] for more details. Suppose that we start from the minimal partition 1|2|3|4|5|6|7 and at each step, two blocks merge into a larger block. We need 6 steps before arriving at the maximal partition 1234567. Each vertex v of the tree represents a subset b of $\{1,\dots,n\}$ of cardinality at least 2, which appears as a block of an element in the chain. This vertex v has label i if the block b appears after the ith merging. If v_1, v_2 are two vertices and b_1, b_2 are the corresponding subsets of $\{1,\dots,n\}$, then v_1 is below v_2 in the tree if $b_1 \subset b_2$. In the example of Figure 3, the correspondence between blocks and labels is: $46 \to 1$, $15 \to 2$, $37 \to 3$, $3467 \to 4$, $125 \to 5$, $1234567 \to 6$.

1234567
125|3467
15|2|3467
15|2|37|46
15|2|3|46|7
1|2|3|46|5|7
1|2|3|4|5|6|7

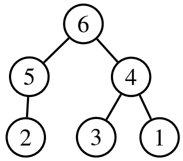

Figure 3: Stanley's bijection.

Proposition 5.2. *Let $\Pi \in \mathcal{M}^{NC}(A_{n-1})$, and $T \in \mathcal{A}_n$ its image under Stanley's bijection. Then we have*

$$M([\Pi],q) = \prod_{\substack{v\in T\\ h_v \neq 1}} (2+q(h_v-1)),$$

where h_v is the hook of the vertex v.

Proof. Let $2 \le i \le n$. There are $a>0$ and $b>0$ such that π_i is obtained from π_{i-1} by merging two blocks of sizes a and b into one block of size $a+b$. The integer h_i is the Coxeter number of $\mathfrak{S}_{a+b}$, i.e., $h_i = a+b$. If $a>1$ or $b>1$, i.e., one of the two blocks has cardinality at least 2, there is a nontrivial factor $\mathfrak{S}_a$ or $\mathfrak{S}_b$ that needs $a+b$ rotations through the cycle to go back to itself, so that $g_i = a+b$. But if $a=b=1$, we have $g_i = 1 = \frac{h_i}{2}$.

Let v be the vertex of T with label i. From the properties of the bijection, the two sons of v contain $a-1$ and $b-1$ vertices, and $h_v = a+b-1$. So, we obtain:

$$\frac{2g_i}{h_i} + q\left(g_i - \frac{2g_i}{h_i}\right) = \begin{cases} 2+q(h_v-1), & \text{if } h_v > 1,\\ 1, & \text{otherwise.}\end{cases}$$

So Proposition 4.4 specializes as stated above. ∎

As a consequence, Equation (4.1) gives the following:

Theorem 5.3.

$$\prod_{i=1}^{n-1}(i+1+q(n-i)) = \sum_{T\in\mathcal{A}_n}\prod_{\substack{v\in T\\ h_v\neq 1}}(2+q(h_v-1)). \tag{5.1}$$

For example, for $n=4$, and taking the 5 trees as in Figure 2, we get:

$$\begin{aligned}(2+3q)(3+2q)(4+q) &= (2+q)(2+2q)(2+3q)+(2+2q)(2+3q)\\ &\quad +(2+q)(2+3q)+(2+q)(2+3q)+(2+q)(2+3q).\end{aligned}$$

We have to make the connection with previously-known results. Let $\mathcal{T}_n$ denote the set of binary plane trees on n vertices, and $\mathcal{T}_n^\ell$ denote the set of pairs (T,L) where $T\in\mathcal{T}_n$ and L is a decreasing labeling of the vertices. It is well known that the number of such labelings L for a given T is

$$\frac{n!}{\prod_{v\in T}h_v}.$$

Moreover, there is a map $\mathcal{T}_n^\ell\to\mathcal{A}_n$ which consists in "forgetting" the notion of left and right among the sons of each internal vertex. It is such that each $T\in\mathcal{A}_n$ has $2^{\mathrm{in}(T)}$ preimages, where $\mathrm{in}(T)$ is the number of internal vertices of T (i.e., $v\in T$ such that $h_v>1$). Then, we can rewrite the right-hand side of (5.1):

$$\begin{aligned}\sum_{T\in\mathcal{A}_n}\prod_{\substack{v\in T\\ h_v\neq 1}}(2+q(h_v-1)) &= \frac{1}{2^n}\sum_{T\in\mathcal{A}_n}2^{\mathrm{in}(T)}\prod_{v\in T}(2+q(h_v-1))\\ &= \frac{1}{2^n}\sum_{T\in\mathcal{T}_n^\ell}\prod_{v\in T}(2+q(h_v-1))\\ &= \frac{n!}{2^n}\sum_{T\in\mathcal{T}_n}\prod_{v\in T}\left(\frac{2+q(h_v-1)}{h_v}\right).\end{aligned}$$

So we arrive at

$$\prod_{i=1}^{n-1}(i+1+q(n-i)) = \frac{n!}{2^n}\sum_{T\in\mathcal{T}_n}\prod_{v\in T}\left(q+\frac{2-q}{h_v}\right).$$

The particular case $q=1$ is Postnikov's hook-length formula [11, Corollary 17.3], proved by investigating the volume of generalized permutohedra. A one-parameter generalization was conjectured by Lascoux and proved by Du and Liu [6]. It is exactly the previous equation up to the change of variable $(q,2-q)\to(q,1)$.

Let us turn to the type B analogue, where we can adapt Stanley's bijection. (Note that a type B analogue of André trees or permutations has been considered by Purtill [12], in relation with type B Springer numbers.)

For brevity, the integers $-1, -2, \ldots, -n$ will be represented $\bar{1}, \bar{2}, \ldots, \bar{n}$. A set partition of type B is a set partition of $\{\bar{n}, \ldots, \bar{1}\} \cup \{1, \ldots, n\}$, unchanged under the map $x \to -x$, and such that there is at most one block b such that $b = -b$ (called the 0-block when it exists). For example, $1\bar{2}5|\bar{1}2\bar{5}|3\bar{3}6\bar{6}|4|\bar{4} \in \mathcal{P}(B_6)$.

Definition 5.4. *A* pointed André tree *is an André tree with a distinguished vertex* $v \in T$ *having* 0 *or* 1 *son. Let* $\mathcal{A}_n^*$ *denote the set of pointed André trees on n vertices.*

A tree $T \in \mathcal{A}_n^*$ is represented with the convention that the distinguished vertex has a starred label i^*. We can create a new tree as follows: increase the labels by 1, then add a new vertex with label 1 attached to the distinguished vertex. This is clearly a bijection between $\mathcal{A}_n^*$ and $\mathcal{A}_{n+1}$, showing that $\#\mathcal{A}_n^* = T_{n+1} = K(B_n)$. See Figure 4 for an example.

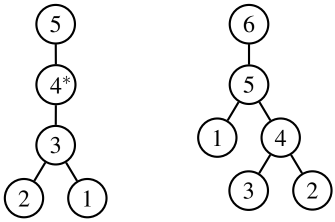

Figure 4: The bijection $\mathcal{A}_n^* \to \mathcal{A}_{n+1}$.

Let $\Pi = (\pi_0, \ldots, \pi_n) \in \mathcal{M}(B_n)$. We build a tree $T \in \mathcal{A}_n^*$ by adapting Stanley's map. A vertex in T represents either the 0-block in some π_i, or a pair of distinct opposite blocks in some π_i where the elements of the pair have cardinality at least 2. This vertex has label i if this 0-block, or pair of opposite blocks, appears in π_i but not in π_{i-1}. A vertex v_1 is below another vertex v_2 in the tree when the blocks represented by v_1 are included in the blocks represented by v_2. Eventually, we have the following rule: the distinguished vertex has label i if and only if π_i has a 0-block, and π_{i-1} has none. See Figure 5 for an example.

$$
\begin{array}{l}
1\bar{1}2\bar{2}3\bar{3}4\bar{4}5\bar{5}6\bar{6} \\
1\bar{1}3\bar{3}|2\bar{4}56|\bar{2}4\bar{5}\bar{6} \\
1\bar{1}3\bar{3}|25|\bar{2}\bar{5}|4\bar{6}|\bar{4}6 \\
13|\bar{1}\bar{3}|25|\bar{2}\bar{5}|4\bar{6}|\bar{4}6 \\
1|\bar{1}|3|\bar{3}|25|\bar{2}\bar{5}|4\bar{6}|\bar{4}6 \\
1|\bar{1}|2|\bar{2}|3|\bar{3}|5|\bar{5}|4\bar{6}|\bar{4}6 \\
1|\bar{1}|2|\bar{2}|3|\bar{3}|4|\bar{4}|5|\bar{5}|6|\bar{6}
\end{array}
$$

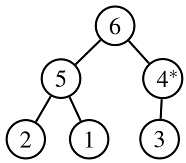

Figure 5: Stanley's bijection adapted to type B.

Proposition 5.5. *Let* $\Pi \in \mathcal{M}(B_n)$ *and* $T \in \mathcal{A}_n^*$ *its image under the bijection we have just defined. For any vertex v of the tree* $T \in \mathcal{A}_n^*$, *we define a factor* $\beta(v)$ *to be* $1 + q(h_v - 1)$ *if v belongs to the minimal path joining the root to the distinguished* *vertex,* $2+q(h_v-1)$ *otherwise. Then we have*

$$M([\Pi],q) = \prod_{\substack{v\in T\\ h_v\neq 1}} \beta(v).$$

Proof. Let $2 \leq i \leq n$, let v be the vertex with label i.

Suppose first that π_i is obtained from π_{i-1} by merging two pairs of distinct opposite blocks into a pair of distinct opposite blocks (such as $25|\bar{2}\bar{5}$ and $4\bar{6}|\bar{4}6$ in the example). This is the case when v is not in the minimal path from the root to the distinguished vertex. This means that $W_{(\pi_i)}$ is obtained from $W_{(\pi_{i-1})}$ by replacing a factor $\mathfrak{S}_a \times \mathfrak{S}_b$ with $\mathfrak{S}_{a+b}$. As in the type A case, we get $g_i = h_i = a+b+1$, and $a-1, b-1$ are the numbers of vertices in the subtrees of v. This gives $\frac{2g_i}{h_i} + q\big(g_i - \frac{2g_i}{h_i}\big) = \beta(v)$.

Suppose then that π_i is obtained from π_{i-1} by merging two pairs of distinct opposite blocks into a 0-block (such as 13 and $\bar{1}\bar{3}$ in the example). This is the case when v is the distinguished vertex. This means that $W_{(\pi_i)}$ is obtained from $W_{(\pi_{i-1})}$ by replacing a factor $\mathfrak{S}_j = A_{j-1}$ into B_j where j is the size of the 0-block, and also the hook-length of v. We obtain $h_i = 2j$, and $g_i = j$. Also in this case, this gives $\frac{2g_i}{h_i} + q\big(g_i - \frac{2g_i}{h_i}\big) = \beta(v)$.

Eventually, suppose that π_i is obtained from π_{i-1} by merging a pair of distinct opposite blocks to the 0-block (such as $2\bar{4}56|\bar{2}4\bar{5}\bar{6}$ in the example). This is the case when v is in the minimal path from the root to the distinguished vertex (but is not the distinguished vertex). This means that $W_{(\pi_i)}$ is obtained from $W_{(\pi_{i-1})}$ by replacing a factor $A_{j-1} \times B_k$ into B_{j+k}. Here, $k > 0$ is the number of vertices in the subtree of v containing the distinguished vertex, and $j-1 \geq 0$ is the number of vertices in the other subtree. We get $h_i = 2(j+k)$, $g_i = j+k = h_v$, and $\frac{2g_i}{h_i} + q\big(g_i - \frac{2g_i}{h_i}\big) = \beta(v)$.

So Proposition 4.4 specializes as stated above. ∎

So, in the type B case, Equation (4.1) gives:

Theorem 5.6.

$$\prod_{i=1}^{n}(i+q(n-i)) = \sum_{T\in\mathcal{A}_n^*} \prod_{\substack{v\in T\\ h_v\neq 1}} \beta(v).$$

For example, let $n=3$. We take the 5 elements of $\mathcal{A}_n^*$ as in Figure 2 after we apply the bijection $\mathcal{A}_{n+1} \to \mathcal{A}_n^*$, and we get:

$$\begin{aligned}3(2+q)(1+2q) = {} & (1+q)(1+2q) + (1+q)(1+2q) + (1+2q)\\ & + (1+2q) + (2+q)(1+2q).\end{aligned}$$

Strictly speaking, the identity in the previous theorem might not be considered as a hook-length formula since $\beta(v)$ does not depend only on the hook-length h_v. Still, it is on its own an interesting variant of the type A case.

Appendix A. Properties of the Standard Coxeter Elements

We sketch here a case-by-case proof of Propositions 3.2. As we have seen above, the result is proved in the case where the longest element is central. It remains only to prove the result for the infinite families A_{n-1}, D_n, and for the exceptional group E_6.

We shall use the notion of cyclic order and cyclic intervals. Recall that a sequence $i_1, \dots, i_n$ is *unimodal* if there is k such that $i_1 \le i_2 \le \dots \le i_k$ and $i_k \ge i_{k-1} \ge \dots \ge i_n$.

A.1. Case of A_{n-1}

Let $W = A_{n-1} = \mathfrak{S}_n$, $V = \{v \in \mathbb{R}^n : \sum v_i = 0\}$. Let $S = \{s_1, \dots, s_{n-1}\}$, where s_i acts by permuting the ith and $(i+1)$th coordinates. As a permutation, s_i is the simple transposition $(i, i+1)$. Let $c = s_{\sigma(1)} \cdots s_{\sigma(n-1)}$ be a standard Coxeter element. By exchanging pairs of commuting generators, we can write c as a product of s_{n-1} with a standard Coxeter element of A_{n-2} (where we do not specify the order of the product). By an easy induction, we see that we can write c as the cycle $(i_1, \dots, i_n)$ where $i_1, \dots, i_n$ is a unimodal sequence (and a permutation of $1, \dots, n$).

Any coatom of $\mathcal{P}^{NC}(A_{n-1}, c)$ is a pair of cyclic intervals of the sequence $i_1, \dots, i_n$, complementary to each other, and the action of c is the "rotation" along the cycle. Two such coatoms are in the same c-orbit if and only if they have the same block sizes. So, for each k with $1 \le k < \frac{n}{2}$, there is an orbit containing complementary cyclic intervals of sizes k and $n-k$. There are n such partitions, and the interval partitions among them are $1 \cdots k | k+1 \cdots n$ and $1 \cdots n-k | n-k+1 \cdots n$. Additionally, if n is even, there is an orbit containing two complementary cyclic intervals of size $\frac{n}{2}$. There are $\frac{n}{2}$ such partitions, and the only interval partition among them is $1 \cdots \frac{n}{2} | \frac{n}{2}+1 \cdots n$. Proposition 3.2 follows.

A.2. Case of B_n

Proposition 3.2 was already proved in this case, since the longest element is central. So the goal of this section is only to introduce some notation nedeed in the type D case (because we see D_n as a subgroup of B_n in the standard way). Let $W = B_n$ acting on $V = \mathbb{R}^n$. The group B_n is generated by $s_1, \dots, s_{n-1}$, i.e., generators of A_{n-1}, together with another generator s_0^B. The latter acts as $(v_1, \dots, v_n) \mapsto (-v_1, v_2, \dots, v_n)$. The simple roots are $-e_1$, together with $e_i - e_{i+1}$ for $1 \le i < n$. We identify B_n with the group of signed permutations, and s_0^B is the transposition $(1, -1)$. We use the notation $((a_1, \dots, a_n)) = (a_1, \dots, a_n)(-a_1, \dots, -a_n)$ and $[[a_1, \dots, a_n]] = (a_1, \dots, a_n, -a_1, \dots, -a_n)$ for the cycles of signed permutations.

A.3. Case of D_n

The group D_n is the subgroup of B_n generated by $s_1, \dots, s_{n-1}$ together with another generator s_0^D. The latter acts by the transformation

$$v = (v_1, \dots, v_n) \mapsto (-v_2, -v_1, v_3, \dots, v_n).$$

As a signed permutation, it is the transposition $((-1, 2))$. The simple roots are $-e_1 - e_2$, and $e_i - e_{i+1}$ for $1 \le i < n$. By exchanging pairs of commuting generators, we can see that a standard Coxeter element c of D_n is a product of s_0^D and a standard Coxeter element of A_{n-1}. So, either

$$c = (1, -1)[[i_1, \dots, i_{n-1}]],$$

where $i_1, \dots, i_{n-1}$ form a unimodal sequence, and a permutation of $2, \dots, n$, or

$$c = (2, -2)[[i_1, \dots, i_{n-1}]],$$

where $i_1, \dots, i_{n-1}$ form a unimodal sequence, and a permutation of $1, 3, \dots, n$. We only consider the first case, the other one being completely similar (it suffices to replace each 1 with a 2 in the text).

We have four kinds of products ct where t is a reflection:

$$\begin{aligned}
c((1, i_m)) &= ((1, i_{m+1}, \dots, i_{n-1}, -i_1, \dots, -i_m)),\\
c((-1, i_m)) &= ((1, -i_{m+1}, \dots, -i_{n-1}, i_1, \dots, i_m)),\\
c((i_\ell, i_m)) &= (1, -1)[[i_1, \dots, i_\ell, i_{m+1}, \dots, i_{n-1}]]((i_{\ell+1}, \dots, i_m)),\\
c((-i_\ell, i_m)) &= (1, -1)[[i_{\ell+1}, \dots, i_m]]((i_1, \dots, i_\ell, -i_{m+1}, \dots, -i_{n-1})).
\end{aligned}$$

Using the notation for type B set partitions, we obtain from the list above that the coatoms of $\mathcal{P}^{NC}(D_n, c)$ are:

- $1 i_{m+1} \cdots i_{n-1} \bar{i}_1 \cdots \bar{i}_m | \bar{1} \bar{i}_{m+1} \cdots \bar{i}_{n-1} i_1 \cdots i_m$,
- $\bar{1} i_{m+1} \cdots i_{n-1} \bar{i}_1 \cdots \bar{i}_m | 1 \bar{i}_{m+1} \cdots \bar{i}_{n-1} i_1 \cdots i_m$,
- $1 i_1 \cdots i_\ell i_{m+1} \cdots i_{n-1} \bar{1} \bar{i}_1 \cdots \bar{i}_\ell \bar{i}_{m+1} \cdots \bar{i}_{n-1} | i_{\ell+1} \cdots i_m | \bar{i}_{\ell+1} \cdots \bar{i}_m$,
- $1 i_{\ell+1} \cdots i_m \bar{1} \bar{i}_{\ell+1} \cdots \bar{i}_m | i_1 \cdots i_\ell \bar{i}_{m+1} \cdots \bar{i}_{n-1} | \bar{i}_1 \cdots \bar{i}_\ell i_{m+1} \cdots i_{n-1}$.

And the interval partitions among them are $1 \cdots n \,|\, \bar{1} \cdots \bar{n}$, $1\bar{2} \cdots \bar{n} | \bar{1} 2 \cdots n$, and

$$1 \cdots i \bar{1} \cdots \bar{i} \,|\, i+1 \cdots n \,|\, \overline{i+1} \cdots \bar{n},$$

where $2 \le i < n$. From these explicit description, we can check Proposition 3.2. We find that all orbits have size $\frac{h}{2}$ (here $h = 2n-2$), except that $1 \cdots n | \bar{1} \cdots \bar{n}$ and $1\bar{2} \cdots \bar{n} | \bar{1} 2 \cdots n$ are in a same orbit of size h if n is even.

A.4. Case of E_6

This can be done with the following Sage program [16] (tested with Sage 5.4).

```
W = WeylGroup(['E',6])
n = 6
h = 12

S = W.simple_reflections()
w0 = W.long_element()

def checkorbits(l):
  c = prod( S[i] for i in l )
  inte = []
  for i in range(1,n+1):
    inte.append( prod( S[j] for j in l if j!=i ) )
  for ct in inte:
    i=1; j=1; k= c * ct * c**(-1) ;
    while k != ct :
      i+=1
      if k in inte:
        j+=1
        ct2 = k
```

```
      k = c * k * c**(-1)
    if not (((j==2) and (i==h)) or ((mod(h,2)==0) and (i==h/2) and (j==1))):
      raise TypeError('ERROR!!!')
    if not (((j==2) and (ct2==w0*ct*w0)) or ((j==1) and (ct == w0*ct*w0))):
      raise TypeError('ERROR!!!')

for l in Permutations(n):
  checkorbits(l)
```

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