J. math. fluid mech. 10 (2008) 89–105
1422-6928/08/010089-17

DOI 10.1007/s00021-006-0218-6

Journal of Mathematical Fluid Mechanics

On Spatial Decay Estimates for Derivatives of Vorticities with Application to Large Time Behavior of the Two-Dimensional Navier–Stokes Flow

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Communicated by Y. Giga

Abstract. In this paper we establish spatial decay estimates for derivatives of vorticities solving the two-dimensional vorticity equations equivalent to the Navier–Stokes equations. As an application we derive asymptotic behaviors of derivatives of vorticities at time infinity. It is well known by now that the vorticity behaves asymptotically as the Oseen vortex provided that the initial vorticity is integrable. We show that each derivative of the vorticity also behaves asymptotically as that of the Oseen vortex.

Mathematics Subject Classification (2000). 35Q30, 35Q35, 76D05.

Keywords. Two-dimensional Navier–Stokes equations, vorticity equations, large time behavior.

1. Introduction

We are interested in the two-dimensional flow of a viscous incompressible fluid. The velocity of the fluid is described by the Navier–Stokes equations:

$$\begin{cases} u_t - \Delta u + (u,\nabla)u + \nabla p = 0 & \text{for} \quad t>0, \quad x\in\mathbb{R}^2, \\ \nabla\cdot u = 0 & \text{for} \quad x\in\mathbb{R}^2, \end{cases} \tag{1}$$

where $u = u(x,t) \in \mathbb{R}^2$ is the fluid velocity, $p(x,t) \in \mathbb{R}$ is the pressure, $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, $\Delta = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$ and $u_t = \partial_t u = \partial u/\partial t$. The kinematic viscosity has been rescaled to be 1. We are concerned with the vorticity $\omega =$rot $u = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ when initial vorticity is integrable. For this purpose, instead of (1), we consider an equation for the vorticity which is obtained by taking the curl of (1):

$$\omega_t - \Delta\omega + (u,\nabla)\omega = 0, \qquad t>0, \quad x\in\mathbb{R}^2. \tag{2}$$

The velocity u is obtained in terms of ω via Biot–Savart law

$$u(x,t)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^{\perp}}{|x-y|^2}\omega(y,t)dy,\quad t>0,\quad x\in\mathbb{R}^2, \tag{3}$$

where $x^{\perp}=(-x_2,x_1)$. Equations (2)–(3) are formally equivalent to (1).

The global well-posedness of the two-dimensional vorticity equations in $L^1(\mathbb{R}^2)$ is first obtained by Y. Giga, T. Miyakawa and H. Osada [10]. In fact they constructed a global solution even when initial data is a finite measure. This result is extended by various authors for example by M. Ben-Artzi [1], H. Brezis [2], and T. Kato [13]. Although the uniqueness of solution was known by [10] when the point mass part of the initial data is small, it is quite recent that the uniqueness is proved for a general measure by I. Gallagher and Th. Gallay [5]. In the past papers, several estimates for vorticities have been established. For example, we already know L^p estimates of vorticities and velocities as follows.

Let $p\in[1,\infty]$ and $q\in(2,\infty]$. Let $|f|_p$ denotes the norm of f in L^p; if f is a vector (f_1,f_2), by $|f|_p$ we mean $|(|f_1|^2+|f_2|^2)^{\frac{1}{2}}|_p$. Then, we have

$$|\partial_t^b\partial_x^\beta\omega(\cdot,t)|_p\le\frac{W_1}{t^{b+\frac{|\beta|}{2}+1-\frac{1}{p}}}|\omega_0|_1, \tag{4}$$

$$|\partial_t^b\partial_x^\beta u(\cdot,t)|_q\le\frac{W_2}{t^{b+\frac{|\beta|}{2}+\frac{1}{2}-\frac{1}{q}}}|\omega_0|_1, \tag{5}$$

where $W_1=W_1(b,\beta,p,|\omega_0|_1)$ and $W_2=W_2(b,\beta,q,|\omega_0|_1)$. Here, $\partial_x^\beta=\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}$ for multi-index $\beta=(\beta_1,\beta_2)\in\mathbb{N}_0\times\mathbb{N}_0$, where $\partial_{x_i}=\partial/\partial x_i$ and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, the set of all nonnegative integers.

The above estimates (4), (5) were proved by T. Kato [13] for $p\in(1,\infty)$ by using an interpolation method, and by Y. Giga and M.-H. Giga [9] for $p\in[1,\infty]$ by a Gronwall-type argument (see also Y. Giga [11], or Y. Giga and O. Sawada [12]). In this paper we establish spatial decay estimates for derivatives of vorticities. Our main result is

Theorem 1.1. *Assume that $p\in[1,\infty]$, $q\in(2,\infty]$. Let ω be the solution of* (2)–(3) *with initial vorticity $\omega_0\in L^1(\mathbb{R}^2)$ and u be the velocity field associated with ω via Biot–Savart law. Then, there exists a positive constant $W_3=W_3(b,\beta,p,|\omega_0|_1)$ such that for all $R\ge1$ and $t>0$,*

$$|\partial_t^b\partial_x^\beta\omega(\cdot,t)|_{p,2R}\le\frac{W_3}{t^{1-\frac{1}{p}+b+\frac{|\beta|}{2}}}\left\{\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}}+|\omega_0|_{1,R}\right\}, \tag{6}$$

where $|\omega(\cdot,t)|_{p,R}:=\left(\int_{|x|>R}|\omega(x,t)|^pdx\right)^{\frac{1}{p}}$, $|\omega(\cdot,t)|_{\infty,R}:=\mathrm{ess.sup}_{|x|>R}|\omega(x,t)|$.

When $b=0$ and $|\beta|=0$, the spatial decay estimates similar to (6) are obtained by A. Carpio [4] and by Y. Giga and M.-H. Giga [9], but we give its proof for completeness in Section 2.1. In order to establish the estimate (6), we shall show three spatial decay estimates. First one is for ω itself, which is obtained by using pointwise estimate for the fundamental solution of the perturbed heat equation, $\omega_t - \Delta\omega + (u, \nabla)\omega = 0$. Second one is for the velocity u. Since u is represented by ω via Biot–Savart law (3), we shall estimate the well-known Riesz potential; see Section 2.2.1. The last one is for the solution of the heat equation; see Section 2.2.2. Collecting these estimates, one can derive the estimate (6); see Section 2.2.3.

As an application of Theorem 1.1, we study the large time behaviors of derivatives of vorticities. It is well-known that the vorticity itself behaves like a constant multiple of the Gauss kernel $g(x,t) = (4\pi t)^{-1}\exp(-\frac{|x|^2}{4t})$ at time infinity. Let us recall its precise form:

Theorem 1.2 ([8], [4], [6]). *Assume that* $p \in [1,\infty]$, $q \in (2,\infty]$. *Let* ω *be the solution of* (2)–(3) *with initial vorticity* $\omega_0 \in L^1(\mathbb{R}^2)$. *Let* $m = \int_{\mathbb{R}^2} \omega_0(x)dx$, *and* $g(x,t) = \frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}$. *Then*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} |\omega(\cdot,t) - mg(\cdot,t)|_p = 0, \tag{7}$$

$$\lim_{t\to\infty} t^{\frac{1}{2}-\frac{1}{q}} |u(\cdot,t) - mv^g(\cdot,t)|_q = 0. \tag{8}$$

Here v^g *is the velocity field associated with* g *via Biot–Savart law* (3).

The above theorem shows that the vorticity behaves asymptotically as mg which is called the Oseen vortex. Note that the Gauss kernel is a solution of (2)–(3) with a Dirac mass as the initial data. The quantity $m = \int_{\mathbb{R}^2}\omega_0(x)dx$ is called "total circulation" and it is preserved by the semi-flow defined by (2)–(3) in $L^1(\mathbb{R}^2)$;

$$\int_{\mathbb{R}^2} \omega(x,t)dx = \int_{\mathbb{R}^2} \omega_0(x)dx, \quad t \geq 0. \tag{9}$$

Y. Giga and T. Kambe [8] first proved Theorem 1.2 when the Reynolds number $\int_{\mathbb{R}^2} |\omega_0(x)|dx$ is sufficiently small by giving the delicate estimates of the bilinear form of the integral equation associated with (2). Later A. Carpio [4] proved Theorem 1.2 under the assumption that $|m|$ is small by rescaling solutions: $\omega_k(x,t) = k^2\omega(kx,k^2t)$, $u_k(x,t) = ku(kx,k^2t)$ for $k > 0$. Recently, Th. Gallay and C. E. Wayne [6] proved for a general initial vorticity in $L^1(\mathbb{R}^2)$ by introducing entropy-like Lyapunov function for a renormalized equation. After this work was completed, the author was informed of a recent work of I. Gallagher, Th. Gallay and P.-L. Lions [7] which give another proof for Theorem 1.2 using the rearrangement argument.

With spatial decay estimates for derivatives of vorticities, we shall prove that each derivative of vorticities behaves asymptotically as that of the Oseen vortex. That is, we have the following theorem.

Theorem 1.3. *Assume that $p \in [1,\infty]$, $q \in (2,\infty]$, $b \in \mathbb{N}_0$ and β is a multi-index. Let ω be the solution of (2)–(3) with initial vorticity $\omega_0 \in L^1(\mathbb{R}^2)$, $m = \int_{\mathbb{R}^2} \omega_0(x)dx$, and $g(x,t) = \frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}$. Then, we have*

$$\lim_{t\to\infty} t^{b+\frac{|\beta|}{2}+1-\frac{1}{p}} |\partial_t^b \partial_x^\beta \omega(\cdot,t) - \partial_t^b \partial_x^\beta m g(\cdot,t)|_p = 0. \tag{10}$$

$$\lim_{t\to\infty} t^{b+\frac{|\beta|}{2}+\frac{1}{2}-\frac{1}{q}} |\partial_t^b \partial_x^\beta u(\cdot,t) - \partial_t^b \partial_x^\beta m v^g(\cdot,t)|_q = 0. \tag{11}$$

Let us give the outline of the proof of Theorem 1.3 for $\alpha = 0$, $|\beta| = 1$. First we consider the same rescaling as was used in A. Carpio [4]. We shall see that the convergence of $\partial_x \omega(x,t)$ as time goes to infinity is equivalent to the convergence of the rescaled functions $\partial_x \omega_k(x,1)$ as k goes to infinity. Once we obtain Theorem 1.1, we can apply Ascoli–Arzelà type compactness theorem in L^p to the family of rescaled functions $\{\partial_x \omega_k(x,1)\}_{k\geq 1}$. So every subsequence of $\{\partial_x \omega_{k(l)}(x,1)\}_{l=1}^{\infty}$ ($k(l) \to \infty$ as l goes to infinity) has a convergent subsequence in L^p. Theorem 1.2 implies that the limit function is unique, so we obtain Theorem 1.3. By the induction we see that Theorem 1.3 also holds for higher derivatives of the solution. The details are given in Section 3.

In fact, to prove the convergence results on derivatives in Theorem 1.3, there is an alternative method by appealing interpolation together with the convergence results of the vorticity ω itself and global estimates on derivatives (4). In particular, spatial decay estimates in Theorem 1.1 are not involved. However, our method which uses compactness argument with the spatial decay estimates in Theorem 1.1 has an advantage if there is an inhomogeneous term in the vorticity equations of the form

$$\omega_t - \Delta\omega + (u,\nabla)\omega = f,$$

for the details, see Section 4.

Acknowledgments. The author is grateful to Professor Yoshikazu Giga, Professor Shin'ya Matsui for critical and useful advices. The author is also grateful to Professor Yoshinori Morimoto and an anonymous referee for some useful comments.

2. Estimates near space infinity

In this section we shall prove the main theorem 1.1.

2.1. Spatial decay estimates of vorticities

In this section we show the spatial decay estimates of the vorticity itself. We recall next pointwise estimate due to E. A. Carlen and M. Loss ([3, Theorem 3]).

For any $\theta \in (0,1)$, there exists $C_\theta > 0$ (depending only on θ and $|\omega_0|_1$) such that

$$|\omega(x,t)| \leq C_\theta \int_{\mathbb{R}^2} \frac{1}{t} e^{-\theta \frac{|x-y|^2}{4t}} |\omega_0(y)| dy, \quad x \in \mathbb{R}^2, \quad t > 0. \tag{12}$$

Remark that the constant C_θ does not depend on u.

Proof of Theorem 1.1 for $b = 0$, $|\beta| = 0$. Fix any $R > 0$. We decompose the right side of the estimate (12) as $|\omega(x,t)| \leq \omega_1(x,t) + \omega_2(x,t)$;

$$\omega_1(x,t) = C_\theta \int_{|y| \leq R} \frac{1}{t} e^{-\theta \frac{|x-y|^2}{4t}} |\omega_0(y)| dy,$$

$$\omega_2(x,t) = C_\theta \int_{|y| \geq R} \frac{1}{t} e^{-\theta \frac{|x-y|^2}{4t}} |\omega_0(y)| dy.$$

It is easy to check the next inequality:

$$\frac{1}{t^l} e^{-\frac{A^2}{t}} \leq \frac{C_l}{A^{2l}}, \quad {}^\forall t > 0, \tag{13}$$

where $C_l > 0$ is a positive constant depending only on l.

If $|x| \geq 2R$, $|y| \leq R$, then $|x - y| \geq |x| - |y| \geq \frac{|x|}{2}$, so we observe that

$$\begin{aligned} \omega_1(x,t) &\leq C_\theta |\omega_0|_1 \frac{1}{t} e^{-\theta \frac{|x|^2}{16t}} \\ &\leq \frac{C}{t} e^{-\theta \frac{R^2}{32t}} e^{-\theta \frac{|x|^2}{32t}}, \end{aligned}$$

where C depends only on θ, $|\omega_0|_1$. Hence, by (13) we have

$$\begin{aligned} |\omega_1(\cdot,t)|_{p,2R} &\leq C t^{-1+\frac{1}{p}} e^{-\theta \frac{R^2}{32t}} \\ &\leq C t^{-\frac{3}{4}+\frac{1}{p}} R^{-\frac{1}{2}} \end{aligned}$$

where C depends only on $p, |\omega_0|_1, \theta$. Next we estimate ω_2. By Young's inequality,

$$\begin{aligned} |\omega_2(\cdot,t)|_{p,2R} &\leq |\omega_2(\cdot,t)|_p \leq \frac{C_\theta}{t} \left(\int_{\mathbb{R}^2} e^{-p\theta \frac{|y|^2}{4t}} dy \right)^{\frac{1}{p}} |\omega_0|_{1,R} \\ &\leq \frac{C}{t^{1-\frac{1}{p}}} |\omega_0|_{1,R} \end{aligned}$$

where C depends only on $\theta, p, |\omega_0|_1$. Summing up, we get the estimate (6) for $b = |\beta| = 0$.

Remark 2.1. Fix any $\delta \in (0,1)$. Since $|x - y| \geq |x| - |y| \geq (1 - \delta)|x|$ whenever $|x| \geq 2R$ and $|y| \leq 2\delta R$, arguing as the same above, we observe that there exists $W = W(p, |\omega_0|_1, \delta)$ such that

$$|\omega(\cdot,t)|_{p,2R} \leq \frac{W}{t^{1-\frac{1}{p}}} \left(\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}} |\omega_0|_1 + |\omega_0|_{1,2\delta R} \right). \tag{14}$$

Remark 2.2. The spatial decay estimate of solutions ω like (14) has already obtained by A. Carpio [4] or by Y. Giga and M.-H. Giga [9, Chapter II]. Their methods are different from ours. They use a cut of function $\phi_R(x)$ whose support is outside of the ball with center at origin and radius R, and consider the equation which $\omega_R(x,t) := \omega(x,t)\phi_R(x)$ satisfies.

2.2. Spatial decay estimates for derivatives of solution

In this section, we prove the following estimate instead of Theorem 1.3 (6). Fix $\delta \in (0,1)$. Then, for any $R \geq 1$

$$|\partial_t^b \partial_x^\beta \omega(\cdot,t)|_{p,2R} \leq \frac{C}{t^{1-\frac{1}{p}+b+\frac{|\beta|}{2}}} \left\{ \frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,2\delta^{2+|\beta|+2b}R} \right\}, \tag{15}$$

where C depends only on b, β, p, $|\omega_0|_1$, and δ. Clearly, this implies that the estimate (6) holds. Note that by Remark 2.1.1, the estimate (15) hods for $b = |\beta| = 0$. So We first prove (15) for the case $b = 0$ by induction with respect to $|\beta|$, and next we shall prove for the case for arbitrary $b \in \mathbb{N}_0$. The solution ω is the solution of the associated integral equation

$$\begin{aligned} \omega(x,t) &= e^{t\Delta} f - \int_0^t e^{(t-s)\Delta}(u(s),\nabla)\omega(s)ds \\ &= e^{\frac{t}{2}\Delta}\omega(\tfrac{t}{2}) - \int_{\frac{t}{2}}^t \nabla \cdot e^{(t-s)\Delta} u(s)\omega(s)ds, \end{aligned} \tag{16}$$

where $e^{t\Delta}$ is the heat semigroup, whose representation is

$$e^{t\Delta} f = \frac{1}{4\pi t} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4t}} f(y)dy. \tag{17}$$

Differentiating both sides by x,

$$\partial_x^\beta \omega(x,t) = \partial_x^\beta e^{\frac{t}{2}\Delta}\omega(\tfrac{t}{2}) - \int_{\frac{t}{2}}^t \nabla \cdot e^{(t-s)\Delta}\partial_x^\beta (u(s)\omega(s))ds. \tag{18}$$

Using the representation (18), we shall show the decay estimate for the derivatives of the solution. In Section 2.2.1 we estimate the spatial decay of the velocity $u(x,t)$, which may be used in estimating nonlinear terms of (18). Moreover, in Section 2.2.2, we shall estimate the decay for the derivatives of functions convoluted with Gaussian. Summing up these results, we get (15) for $b = 0$.

2.2.1. Spatial decay estimate for the Riesz potential

In this section, we shall estimate the spatial decay of the velocity field $u(x,t)$. Since $u(x,t)$ is represented by Biot–Savart law (3), it is sufficient to show the decay estimate for the Riesz potential. Here we may assume that the spatial dimension is n. The Riesz potential for $\alpha \in (0, n)$ is given by

$$I_\alpha(f)(x) := \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y)dy, \quad f \in C_0(\mathbb{R}^n), \tag{19}$$

where $C_0(\mathbb{R}^n)$ denotes the space of all continuous functions with compact support in $\mathbb{R}^n$.

Lemma 2.1. (a) *Let $p, r \in (1, \infty)$ satisfy $\frac{1}{p} = \frac{1}{r} - \frac{\alpha}{n}$, $\quad n - \alpha - \frac{n}{p} > 0$. Let $\tilde{r} \in [1, \infty]$ be the number satisfying $\frac{1}{\tilde{r}} - \frac{1}{p} > \frac{\alpha}{n}$. Then there exist positive constants $M_1 = M_1(n, p, \tilde{r}, \alpha)$, $M_2 = M_2(n, p, \alpha)$ such that for all $R > 0$,*

$$|I_\alpha(f)|_{p,2R} \le \frac{M_1}{R^{n(\frac{1}{\tilde{r}} - \frac{1}{p}) - \alpha}} |f|_{\tilde{r}} + M_2 |f|_{r,R}. \tag{20}$$

(b) *Let $p \in \left(\frac{n}{\alpha}, \infty\right]$. Then, there exist positive constants $M_1' = M_1'(n, p, \alpha)$, $M_2' = M_2'(n, \alpha)$ such that for all $R > 0$,*

$$|I_\alpha(f)|_{\infty,2R} \le \frac{M_1'}{R^{\frac{n}{p} - \alpha}} |f|_p + M_2' |f|_{1,R}^{\frac{\alpha}{n}} |f|_{\infty,R}^{1-\frac{\alpha}{n}}. \tag{21}$$

Proof. The idea of the proof is the same as the previous section. That is, we decompose as

$$\begin{aligned} |I_\alpha(f)| &\le \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} |f(y)| dy = I_1 + I_2, \\ I_1 &= \int_{|y| \le R} \frac{1}{|x-y|^{n-\alpha}} |f(y)| dy, \\ I_2 &= \int_{|y| \ge R} \frac{1}{|x-y|^{n-\alpha}} |f(y)| dy. \end{aligned}$$

Since $|x - y| \ge R$ whenever $|x| \ge 2R$ and $|y| \le R$, so

$$I_1 = \int_{|y| \le R} \frac{1}{|x-y|^{n-\alpha}} \chi_R(x-y) |f(y)| dy,$$

where $\chi_R(x) = 1$ if $|x| \ge R$ and $\chi_R(x) = 0$ otherwise. Set $\tilde{q}$ as $\frac{1}{p} = \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} - 1$. Thus by Young's inequality, we have

$$|I_1|_{p,2R} \le |I_1|_p \le \left(\int_{|x| \ge R} \frac{1}{|x|^{(n-\alpha)\tilde{q}}} \, dx \right)^{\frac{1}{\tilde{q}}} |f|_{\tilde{r}} \le \frac{M_1}{R^{n(\frac{1}{\tilde{r}} - \frac{1}{p}) - \alpha}} |f|_{\tilde{r}},$$

where M_1 depends only on n, p, α. Clearly, this holds also for $p = \infty$. Recalling Hardy–Littlewood–Sobolev's inequality (see for instance [9, Chapter VI]), for $1 < p, r < \infty$ with $\frac{1}{p} = \frac{1}{r} - \frac{\alpha}{n}$, we have

$$|I_2|_{p,2R} \le |I_2|_p \le C(\alpha, p) |f|_{r,R},$$

this proves (a).

Let us estimate $|I_2|_{\infty,R}$. For $L>0$ we decompose $I_2 = I_{2,1} + I_{2,2}$:

$$I_{2,1} = \int_{|y|\geq R, |x-y|\leq L} \frac{1}{|x-y|^{n-\alpha}} |f(y)| dy,$$

$$I_{2,2} = \int_{|y|\geq R, |x-y|\geq L} \frac{1}{|x-y|^{n-\alpha}} |f(y)| dy.$$

Using Young's inequality, we have

$$|I_{2,1}|_{\infty,R} \leq \int_{|y|\leq L} \frac{1}{|y|^{n-\alpha}} dy |f|_{\infty,R} = C_{2,1} L^{\alpha} |f|_{\infty,R},$$

where $C_{2,1}$ depends only on $n,\ \alpha$. Using Young's inequality again, we have

$$|I_{2,2}|_{\infty,R} \leq \frac{1}{L^{n-\alpha}} |f|_{1,R}.$$

Taking $L = \left(\frac{|f|_{1,R}}{|f|_{\infty,R}}\right)^{\frac{1}{n}}$, we get

$$|I_2|_{\infty,R} \leq |I_{2,1}|_{\infty,R} + |I_{2,2}|_{\infty,R} \leq M_2' |f|_{1,R}^{\frac{\alpha}{n}} |f|_{\infty,R}^{1-\frac{\alpha}{n}},$$

where M_2' depends only on $n,\ \alpha$. This completes the proof. □

Remark 2.3. Fix $\delta \in (0,1)$. Arguing as in Remark 2.1.1, we have, instead of (20) and (21),

$$|I_\alpha(f)|_{p,2R} \leq \frac{M_1}{R^{n\left(\frac{1}{\tilde{r}}-\frac{1}{p}\right)-\alpha}} |f|_{\tilde{r}} + M_2 |f|_{r,2\delta R}, \tag{22}$$

where M_1 depends only on $n,\ p,\ \tilde{r},\ \alpha$, and δ, and, M_2 depends only on $n,\ p,\ \alpha$.

$$|I_\alpha(f)|_{\infty,2R} \leq \frac{M_1'}{R^{\frac{n}{p}-\alpha}} |f|_p + M_2' |f|_{1,2\delta R}^{\frac{\alpha}{n}} |f|_{\infty,2\delta R}^{1-\frac{\alpha}{n}}, \tag{23}$$

where M_1' depends only on $n,\ p,\ \alpha$ and δ, and, M_2' depends only on $n,\ \alpha$.

2.2.2. Decay estimate for the derivatives of the convolution with Gauss kernel

In this section we estimate the spatial decay property for the derivatives of the semigroup, or the derivatives of the convolution with the Gauss kernel

$$\partial_x^\beta e^{\Delta} f = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \partial_x^\beta e^{-\frac{|x-y|^2}{4t}} f(y) dy. \tag{24}$$

Lemma 2.2. *Let $p,\ r,\ \tilde{r} \in [1,\infty]$, $|\beta| = k$. Then, there exist positive constants M_3 (depending only on $n,\ p,\ r,\ k$) and M_4 (depending only on $n,\ \tilde{r},\ k$) such that for all $R>0$,*

$$|\partial_x^\beta e^{t\Delta} f|_{p,2R} \leq \frac{M_3}{t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{p})+\frac{k}{2}-\frac{1}{4}} R^{\frac{1}{2}}} |f|_r + \frac{M_4}{t^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})+\frac{k}{2}}} |f|_{\tilde{r},R} \quad \text{for all } f \in C_0(\mathbb{R}^n). \tag{25}$$

Proof. Set $q,\ \tilde{q} \in [1,\infty]$ as satisfying $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1,\ \frac{1}{p} = \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} - 1$, respectively. Note that

$$|\partial_x^\beta e^{\Delta} f(x,t)| \le \sum_{\substack{i,j\ge 0\\ 2j-i=k}} C_{i,j} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{|x-y|^i}{t^j} e^{-\frac{|x-y|^2}{4t}} |f(y)|dy,$$

where $C_{i,j}$ depends only on i and j. We decompose each term as

$$\begin{aligned}
&\frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{|x-y|^i}{t^j} e^{-\frac{|x-y|^2}{4t}} |f(y)|dy \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{|y|\le R} \frac{|x-y|^i}{t^j} e^{-\frac{|x-y|^2}{4t}} |f(y)|dy + \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{|y|\ge R} \frac{|x-y|^i}{t^j} e^{-\frac{|x-y|^2}{4t}} |f(y)|dy \\
&= J_{i,j,1} + J_{i,j,2}.
\end{aligned}$$

Arguing as in Lemma 2.1 (a), we see that if $|x| \ge 2R$,

$$J_{i,j,1}(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{|y|\le R} \frac{|x-y|^i}{t^j} e^{-\frac{|x-y|^2}{4t}} \chi_R(x-y)|f(y)|dy.$$

Thus we have

$$\begin{aligned}
|J_{i,j,1}|_{p,2R} &\le \frac{C}{t^{\frac{n}{2}+j}} \left|\chi_R(x)|x|^i e^{-\frac{|x|^2}{4t}}\right|_{L_x^q} |f|_r \\
&\le \frac{C}{t^{\frac{n}{2}+j-\frac{i}{2}-\frac{1}{q}}} e^{-\frac{R^2}{32t}} |f|_r \\
&\le \frac{C}{t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{p})+\frac{k}{2}-\frac{1}{4}} R^{\frac{1}{2}}} |f|_r.
\end{aligned}$$

Next we estimate $J_{i,j,2}$. Using Young's inequality, we have

$$\begin{aligned}
|J_{i,j,2}|_{p,2R} &\le |J_{i,j,2}|_p \le \frac{C}{t^{\frac{n}{2}+j}} \left||y|^i e^{-\frac{|y|^2}{4t}}\right|_{L_y^{\tilde{q}}} |f|_{\tilde{r},R} \\
&\le \frac{C}{t^{\frac{n}{2}-\frac{1}{\tilde{q}}+j-\frac{i}{2}}} |f|_{\tilde{r},R} \\
&= \frac{C}{t^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})+\frac{k}{2}}} |f|_{\tilde{r},R}.
\end{aligned}$$

Collecting these above, we have the desired estimates. □

Remark 2.4. Fix $\delta \in (0,1)$. Arguing as in Remark 2.1.1, we have, instead of (25),

$$|\partial_x^\beta e^{t\Delta} f|_{p,2R} \le \frac{M_3}{t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{p})+\frac{k}{2}-\frac{1}{4}} R^{\frac{1}{2}}} |f|_r + \frac{M_4}{t^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})+\frac{k}{2}}} |f|_{p,2\delta R}, \tag{26}$$

where M_3 depends only on $n,\ p,\ r,\ k$ and δ, and, M_4 depends only on n, $\tilde{r}$, k.

Remark 2.5. The spatial decay estimates of functions convolved with the heat kernel like (25) are already obtained by Y. Giga and M.-H. Giga [9, Chapter I] for the case $|\beta| = 0$.

2.2.3. Proof of the spatial decay estimates for derivatives

We are now in position to prove the estimate (15) for $b = 0$ by induction. The constants in this section may depend on δ, but we do not refer to it for simplicity of the notation.

Proof of the estimate (15) *for* $b = 0$. Assume that the estimate (15) holds for multi-index β with $|\beta| = k - 1 \geq 0$. We have to prove (15) for $|\beta| = k$. By the representation (18) we observe that

$$|\partial_x^\beta \omega(\cdot,t)|_{p,2R} \leq \left|\partial_x^\beta e^{\frac{t}{2}\Delta}\omega\Big(\frac{t}{2}\Big)\right|_{p,2R} + \int_{\frac{t}{2}}^{t} \big|\nabla\cdot e^{(t-s)\Delta}\partial_x^\beta h(s)\big|_{p,2R} ds =: K_1 + K_2, \tag{27}$$

where $h(s) := u(s)\omega(s)$. Let us estimate K_1. It follows from Remark 2.2.2 that

$$K_1 \leq \frac{C}{t^{\frac{3}{4}-\frac{1}{p}+\frac{k}{2}}R^{\frac{1}{2}}}\left|\omega\Big(\frac{t}{2}\Big)\right|_1 + \frac{C}{t^{1-\frac{1}{p}+\frac{k}{2}}}\left|\omega\Big(\frac{t}{2}\Big)\right|_{1,2\delta R},$$

and by (4) and (14) for $p = 1$, we have

$$K_1 \leq \frac{C}{t^{\frac{3}{4}-\frac{1}{p}+\frac{k}{2}}R^{\frac{1}{2}}}|\omega_0|_1 + \frac{C}{t^{1-\frac{1}{p}+\frac{k}{2}}}\left(\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}}|\omega_0|_1 + |\omega_0|_{1,2\delta^2 R}\right) \tag{28}$$

$$\leq \frac{c}{t^{1-\frac{1}{p}+\frac{k}{2}}}\left(\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,2\delta^2 R}\right), \tag{29}$$

where C depends only on $k,\ p,\ |\omega_0|_1$.

Next we estimate K_2. Again, from Remark 2.2.2 it follows that

$$\big|\nabla\cdot e^{(t-s)\Delta}\partial_x^\beta h(s)\big|_{p,2R} \leq \frac{C}{(t-s)^{\frac{1}{4}}R^{\frac{1}{2}}}\big|\partial_x^\beta h(s)\big|_p + \frac{C}{(t-s)^{\frac{1}{2}}}\big|\partial_x^\beta h(s)\big|_{p,2\delta R}$$
$$=: K_{2,1} + K_{2,2}.$$

By (4) and (5) for $q = \infty$ we have

$$\big|\partial_x^\beta h(s)\big|_p \leq \sum_{\alpha\leq\beta} C_\alpha \big|\partial_x^\alpha u(s)\big|_\infty \big|\partial_x^{\beta-\alpha}\omega(s)\big|_p \leq \frac{C}{s^{\frac{k+3}{2}-\frac{1}{p}}}, \tag{30}$$

where C depends only on k and $|\omega_0|_1$, so we have

$$K_{2,1} \leq \frac{C}{(t-s)^{\frac{1}{4}}s^{\frac{k+3}{2}-\frac{1}{p}}R^{\frac{1}{2}}}. \tag{31}$$

Now we estimate $|\partial_x^\beta h(s)|_{p,2\delta R}$. By Young's inequality,

$$|\partial_x^\beta h(s)|_{p,2\delta R} \leq |u(s)|_{\infty,2\delta R}|\partial_x^\beta \omega(s)|_p + \sum_{\alpha\neq 0}|\partial_x^\alpha u(s)|_\infty |\partial_x^{\beta-\alpha}\omega(s)|_{p,2\delta R}$$
$$=: K_{2,2,1} + K_{2,2,2}.$$

By Remark 2.2.1 (23) for $n = 2,\ p = \frac{4}{3},\ \alpha = 1$, and by (4), we observe that

$$K_{2,2,1} \le \left(\frac{C}{R^{\frac{1}{2}}} |\omega(s)|_{\frac{4}{3}} + C|\omega(s)|^{\frac{1}{2}}_{1,2\delta^2 R} |\omega(s)|^{\frac{1}{2}}_{\infty,2\delta^2 R} \right) \frac{W_1|\omega_0|_1}{s^{1+\frac{k}{2}-\frac{1}{p}}}.$$

Here, by (14), we have

$$\begin{aligned} |\omega(s)|_{1,2\delta^2 R}|\omega(s)|_{\infty,2\delta^2 R} &\le C\left(\frac{s^{\frac{1}{4}}}{R^{\frac{1}{2}}}|\omega_0|_1 + |\omega_0|_{1,2\delta^3 R}\right)\left(\frac{1}{s^{\frac{3}{4}}R^{\frac{1}{2}}} \frac{|\omega_0|_1}{s^{\frac{1}{2}}} + \frac{|\omega_0|_{1,2\delta^3 R}}{s}\right) \\ &= C\left(\frac{|\omega_0|_1}{s^{\frac{1}{4}}R^{\frac{1}{2}}} + \frac{|\omega_0|_{1,2\delta^3 R}}{s^{\frac{1}{2}}}\right)^2, \end{aligned}$$

so

$$|\omega(s)|^{\frac{1}{2}}_{1,2\delta^2 R}|\omega(s)|^{\frac{1}{2}}_{\infty,2\delta^2 R} \le C\left(\frac{|\omega_0|_1}{s^{\frac{1}{4}}R^{\frac{1}{2}}} + \frac{|\omega_0|_{1,2\delta^3 R}}{s^{\frac{1}{2}}}\right).$$

Thus since $|\omega(s)|_1 \le |\omega_0|_1$, we have

$$\begin{aligned} K_{2,2,1} &\le \left\{ \frac{C}{s^{\frac{1}{4}}R^{\frac{1}{2}}}|\omega_0|_1 + C\left(\frac{|\omega_0|_1}{s^{\frac{1}{4}}R^{\frac{1}{2}}} + \frac{|\omega_0|_{1,2\delta^3 R}}{s^{\frac{1}{2}}}\right)\right\} \frac{W_1|\omega_0|_1}{s^{1+\frac{k}{2}-\frac{1}{p}}} \\ &\le \frac{C}{s^{\frac{k+3}{2}-\frac{1}{p}}}\left(\frac{s^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,2\delta^3 R}\right), \end{aligned} \tag{32}$$

where C depends only on $k,\ p,\ |\omega_0|_1$.

Next we estimate $K_{2,2,2}$. By the assumption of the induction we have

$$|\partial_x^{\beta-\alpha}\omega(s)|_{p,2\delta R} \le \frac{B_{\alpha,\beta}}{s^{1-\frac{1}{p}+\frac{|\beta-\alpha|}{2}}}\left\{ \frac{s^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,2\delta^{3+|\beta-\alpha|}R}\right\},$$

where $B_{\alpha,\beta}$ depends only on $|\beta-\alpha|,\ p,\ |\omega_0|_1$. Note that $|\beta-\alpha| = |\beta| - |\alpha|$. Then,

$$\begin{aligned} K_{2,2,2} &\le \sum_{\alpha\neq 0} \frac{C_\alpha}{s^{\frac{|\alpha|}{2}+\frac{1}{2}}}|\omega_0|_1 \frac{B_{\alpha,\beta}}{s^{1-\frac{1}{p}+\frac{|\beta-\alpha|}{2}}}\left\{ \frac{s^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,2\delta^{3+|\beta-\alpha|}R}\right\} \\ &\le \frac{C}{s^{\frac{k+3}{2}-\frac{1}{p}}}\left(\frac{s^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,2\delta^{2+k}R}\right), \end{aligned} \tag{33}$$

where C depends only on $k,\ p,\ |\omega_0|_1$. Collecting estimates (31)–(33), we have

$$\begin{aligned} &|\nabla\cdot e^{(t-s)\Delta}\partial_x^\beta h(s)|_{p,2R} \\ &\le \frac{C}{(t-s)^{\frac{1}{4}}s^{\frac{k+3}{2}-\frac{1}{p}}R^{\frac{1}{2}}} + \frac{C}{(t-s)^{\frac{1}{2}}s^{\frac{k+3}{2}-\frac{1}{p}}}\left(\frac{s^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,2\delta^{2+k}R}\right), \end{aligned}$$

where C depends only on $k,\ p,\ |\omega_0|_1$. This implies that the estimate (15) holds for $b = 0$.

Proof of the estimate (15) *for arbitrary* $\beta,\ b$. Let us show that (15) holds for arbitrary multi-index β and $b \in \mathbb{N}_0$. First we consider the case $b = 1$. Since ω is a solution of (2), i.e.,

$$\partial_t\omega = \Delta\omega - (u,\nabla)\omega,$$

we have

$$\partial_t\partial_x^\beta\omega = \partial_x^\beta\Delta\omega - \partial_x^\beta(u,\nabla)\omega.$$

Thus

$$\begin{aligned}
|\partial_t\partial_x^\beta\omega(\cdot,t)|_{p,2R} &\le |\partial_x^\beta\Delta\omega(\cdot,t)|_{p,2R} + |\partial_x^\beta(u,\nabla)\omega(\cdot,t)|_{p,2R} \\
&\le |\partial_x^\beta\Delta\omega(\cdot,t)|_{p,2R} + \sum_{\alpha\le\beta}|\partial_x^\alpha u\partial_x^{\beta-\alpha}\nabla\omega(\cdot,t)|_{p,2R} \\
&=: A_1 + A_2.
\end{aligned}$$

The estimate (15) for $b=0$ yields

$$A_1 \le \frac{C}{t^{2-\frac{1}{p}+\frac{|\beta|}{2}}}\left(\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,2\delta^{4+|\beta|}R}\right). \tag{34}$$

Moreover, it follows from (5) and (6) for $b=0$,

$$\begin{aligned}
A_2 &\le \sum_{\alpha\le\beta}|\partial_x^\alpha u|_\infty|\partial_x^{\beta-\alpha}\nabla\omega(\cdot,t)|_{p,2R} \\
&\le C\sum_{\alpha\le\beta}\frac{1}{t^{\frac{|\alpha|}{2}+\frac{1}{2}}}|\omega_0|_1\frac{1}{t^{\frac{1-\frac{1}{p}+|\beta-\alpha|}{2}}}\left(\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,2\delta^{3+|\beta-\alpha|}R}\right) \\
&\le \frac{C}{t^{2-\frac{1}{p}+\frac{|\beta|}{2}}}\left(\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,2\delta^{3+|\beta|}R}\right),
\end{aligned} \tag{35}$$

where C depends only on $\beta,\ p,\ |\omega_0|_1$. By (34) and (35), the estimate (15) holds for $b=1,\ \beta$. The case $b\ge 2$ can be shown as the same above, we omit it. This completes the proof of Theorem 1.3.

3. Large time behavior of derivatives

3.1. Rescaling

We are now in position to prove the main theorem. We rescale the pair (ω,u) by

$$\omega_k(x,t) = k^2\omega(kx,k^2t), \quad k>0$$

$$u_k(x,t) = ku(kx,k^2t), \quad k>0$$

where (ω,u) is the solution of (2)–(3) with initial vorticity $\omega_0\in L^1(\mathbb{R}^2)$. Then the rescaled pair (ω_k,u_k) satisfies vorticity equation (2) with initial vorticity $\omega_k(0) = k^2\omega_0(kx)$, and u_k is the velocity field associated with ω_k via Biot–Savart law.

Remark that $|\omega_k(0)|_1 = |\omega_0|_1$ and the Gauss kernel g is invariant with respect to this rescaling, i.e., for all $k > 0,\ g_k(x,t) = g(x,t),\ {}^\forall x \in \mathbb{R}^2,\ t > 0$. We can easily check the following lemma.

Lemma 3.1. *Assume that $p \in [1,\infty]$, $b \in \mathbb{N}_0$ and β is a multi-index. Let ω be the solution of (2)–(3) with initial vorticity $\omega_0 \in L^1(\mathbb{R}^2)$, $m = \int_{\mathbb{R}^2} \omega_0(x)dx$, and $g(x,t) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}$. Then, the following two formulae are equivalent.*

$$\lim_{t\to\infty} t^{b+\frac{|\beta|}{2}+1-\frac{1}{p}} |\partial_t^b \partial_x^\beta \omega(\cdot,t) - \partial_t^b \partial_x^\beta mg(\cdot,t)|_p = 0, \tag{36}$$

$$\lim_{k\to\infty} |\partial_t^b \partial_x^\beta \omega_k(\cdot,1) - \partial_t^b \partial_x^\beta mg(\cdot,1)|_p = 0. \tag{37}$$

Proof. Note that

$$\partial_t^b \partial_x^\beta (\omega_k - mg)(x,t) = k^{2+2b+|\beta|} \partial_t^b \partial_x^\beta (\omega - mg)(kx, k^2 t),$$

so that

$$\begin{aligned} |\partial_t^b \partial_x^\beta \omega_k(\cdot,1) - \partial_t^b \partial_x^\beta mg(\cdot,1)|_p &= k^{2+2b+|\beta|} |\partial_t^b \partial_x^\beta \omega(k\cdot,k^2) - \partial_t^b \partial_x^\beta mg(k\cdot,k^2)|_p \\ &= k^{2+2b+|\beta|-\frac{2}{p}} |\partial_t^b \partial_x^\beta \omega(\cdot,k^2) - \partial_t^b \partial_x^\beta mg(\cdot,k^2)|_p. \end{aligned}$$

Taking $k^2 = t$, we observe that the statement holds. □

3.2. Compactness of rescaled functions

Lemma 3.1.1 implies that we should study the compactness of the family of the rescaled functions $\{\omega_k(x,1)\}_{k\geq 1}$. Theorem 1.1 yields compactness for rescaled functions.

Lemma 3.2. *Assume that $p \in [1,\infty]$, $b \in \mathbb{N}_0$ and β is a multi-index. Let $\{\omega_k(x,1)\}_{k\geq 1}$ be a family of the rescaled functions as above. Then, the following statements hold.*

(a) *There exists a positive constants $C = C(\beta,\ b,\ p,\ |\omega_0|_1)$ such that for all $k \geq 1$,*

$$|\partial_t^b \partial_x^\beta \omega_k(\cdot,1)|_p \leq C \tag{38}$$

holds.

(b) *For any $\epsilon > 0$, there exists $\delta > 0$ such that for all $k \geq 1$ and $|y| < \delta$,*

$$|\partial_t^b \partial_x^\beta \omega_k(\cdot - y,1) - \partial_t^b \partial_x^\beta \omega_k(\cdot,1)|_p \leq \epsilon \tag{39}$$

holds.

(c) *For any $\epsilon > 0$, there exists $R \geq 1$ such that for all $k \geq 1$,*

$$|\partial_t^b \partial_x^\beta \omega_k(\cdot,1)|_{p,R} \leq \epsilon \tag{40}$$

holds.

Proof. Note that $|\omega_k(0)|_1 = |\omega_0|_1$, $|\omega_k(0)|_{1,R} \le |\omega_0|_{1,R}$ for all $k \ge 1$. Hence (a), (b) clearly follows from by (4). The statement (c) follows from the estimate Theorem 1.1 (6). The details will be omitted. □

Applying the Riesz criterion ([14, Theorem XIII.66]) to the above lemma, we know that the family of the rescaled functions $\{\omega_k(x,1)\}_{k\ge1}$ is relatively compact with valued in Sobolev space $W^{|\beta|,p}(\mathbb{R}^2)$ for $p \in [1,\infty)$ and all multi-index β. We observe that $\{\omega_k(x,1)\}_{k\ge1}$ is also relatively compact with valued in Sobolev space $W^{|\beta|,\infty}(\mathbb{R}^2)$ by using Ascoli–Arzelà theorem. Let $\{\omega_{k(l)}(x,1)\}_{l=1}^{l=\infty}$($k(l)$ goes to ∞ as l goes to ∞) be a convergent subsequence of $\{\omega_k(x,1)\}_{k\ge1}$ in $W^{|\beta|,p}(\mathbb{R}^2)$. We have already known that $\omega_k(x,1) \to mg(x,1)$ in $L^p(\mathbb{R}^2)$ as k goes to ∞ by Theorem 1.2 and Lemma 3.1.1. So $\omega_{k(l)}(x,1) \to mg(x,1)$ in $W^{|\beta|,p}(\mathbb{R}^2)$. This implies $\omega_k(x,1) \to mg(x,1)$ in $W^{|\beta|,p}(\mathbb{R}^2)$ as k goes to ∞. So Lemma 3.1.1 (37) holds for $b=0$.

Next we consider the case $b=1$. Since $\partial_t\omega_k = \Delta\omega_k - (u_k,\nabla)\omega_k$, we have

$$\partial_t\partial_x^\beta\omega_k = \partial_x^\beta\Delta\omega_k - \partial_x^\beta(u_k,\nabla)\omega_k.$$

Thus if we note the fact $\partial_t g = \Delta g = \Delta g - (v^g,\nabla)g$, where v^g is a function associated with the Gauss kernel g by Biot–Savart law (3), we observe that

$$\begin{aligned}
&|\partial_t\partial_x^\beta\omega_k(\cdot,1) - \partial_t\partial_x^\beta mg(\cdot,1)|_p \\
&\le |\partial_x^\beta\Delta\omega_k(\cdot,t) - \partial_x^\beta\Delta mg(\cdot,1)|_p + |\partial_x^\beta(u_k,\nabla)\omega_k(\cdot,1) - \partial_x^\beta(v^g,\nabla)mg(\cdot,1)|_p \\
&\le |\partial_x^\beta\Delta\omega_k(\cdot,1) - \partial_x^\beta\Delta mg(\cdot,1)|_p + |\partial_x^\beta(u_k - mv^g,\nabla)\omega_k(\cdot,1)|_p \\
&+ |\partial_x^\beta(mv^g,\nabla)(\omega_k - mg)(\cdot,1)|_p \\
&=: G_1 + G_2 + G_3.
\end{aligned}$$

From the result of the case $b=0$, the first term G_1 goes to zero. By Biot–Savart law (3) we observe that

$$\partial_x^\alpha(u_k - v^g)(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\partial_x^\alpha(\omega_k - mg)(y,t)dy,$$

and by Hardy–Littlewood–Sobolev's inequality, if $1 < q, r < \infty$, $\frac{1}{q} = \frac{1}{r} - \frac{1}{2}$, then

$$|\partial_x^\alpha(u_k - v^g)(\cdot,1)|_q \le C(q)|\partial_x^\alpha(\omega_k - mg)(\cdot,1)|_r. \tag{41}$$

Moreover, by Gagliardo–Nirenberg's inequality (see for instance [9, Chapter VI]), for $2 < q < \infty$ we have

$$|\partial_x^\alpha(u_k - v^g)(\cdot,1)|_\infty \le C'|\partial_x^\alpha(u_k - v^g)(\cdot,1)|_q^{1-\frac{2}{q}}|\nabla\partial_x^\alpha(u_k - v^g)(\cdot,1)|_q^{\frac{2}{q}}. \tag{42}$$

Combining (41) and (42) with the estimate (4) and the results (10) for $b=0$, we observe that G_2 goes to zero as $k\to\infty$. Finally, since $|\partial_x^\alpha v^g(\cdot,1)|_\infty \le C(\alpha)$, G_3 goes to zero. So (10) also holds for $b=1$. By induction, arguing as the same above, we also know that (10) holds for any β, b. Once we obtain (10), then (11) is obvious if we use inequalities such as (41), (42). We omit the details. The proof of Theorem 1.3 is now complete.

4. Alternative method

In this section, we derive the convergence in Theorem 1.3 by an alternative way which uses interpolation inequalities instead of spatial decay estimates of vorticities. We shall show the proof only for the case $b = 0$ and $|\beta| = 1$.

First, note that we have the interpolation inequalities such as

$$|f|_{1,p} \le C|f|_p^{\frac{1}{2}}|f|_{2,p}^{\frac{1}{2}}, \text{ for all } f \in W^{2,p}(\mathbb{R}^n), \tag{43}$$

where C depends only on n and $p \in [1, \infty]$. So we see

$$\begin{aligned} |\omega_k - mg|_{1,p} &\le C|\omega_k - mg|_p^{\frac{1}{2}}|\omega_k - mg|_{2,p}^{\frac{1}{2}} \\ &\le C|\omega_k - mg|_p^{\frac{1}{2}}(|\omega_k|_{2,p} + |mg|_{2,p})^{\frac{1}{2}} \\ &\le C|\omega_k - mg|_p^{\frac{1}{2}}, \end{aligned}$$

where C depends only on p and $|\omega_0|_1$. Here, the last inequality follows from the global estimates (4). Since we already have $\lim_{k\to\infty} |\omega_k(\cdot, 1) - mg(\cdot, 1)|_p = 0$, the desired convergence follows.

This proof is very simple compared with the proof using compactness argument together with spatial decay estimates for derivatives of vorticities (6). However, the above interpolation method has a disadvantage if there is an inhomogeneous term in the vorticity equations of the form

$$\omega_t - \Delta\omega + (u, \nabla)\omega = f. \tag{44}$$

Under appropriate conditions on f, we can prove the global existence and uniqueness of the solution of (44). Moreover, we can also show the large time behaviors of solutions similar to those in Theorem 1.2. Let us state the typical results for the inhomogeneous case without proofs.

Theorem 4.1. *Assume that a function $f \in L^1(\mathbb{R}^2 \times (0, \infty))$ satisfies that $tf(\cdot, t) \in L^\infty(0, \infty; L^1(\mathbb{R}^2))$. Let $\omega_0 \in L^1(\mathbb{R}^2)$. Then, there exists a unique solution $\omega \in C([0, \infty); L^1(\mathbb{R}^2))$ of (44) with initial vorticity ω_0. The vorticity ω satisfies that for $p \in [1, \infty)$ and $q \in [1, 2)$,*

$$\sup_{t>0} t^{1-\frac{1}{p}}|\omega(\cdot, t)|_p \le C, \qquad \sup_{t>0} t^{\frac{3}{2}-\frac{1}{q}}|\partial_x\omega(\cdot, t)|_q \le C,$$

$$\begin{aligned} &|\omega(\cdot, t)|_{p,2R} \\ &\le \frac{C}{t^{1-\frac{1}{p}}}\left(\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,R}\right) + C\int_0^t \frac{1}{(t-s)^{1-\frac{1}{p}}}\left(\frac{(t-s)^{\frac{1}{4}}}{R^{\frac{1}{2}}}|f(\cdot, s)|_1 + |f(\cdot, s)|_{1,R}\right)ds, \end{aligned}$$

where C depends only on p, q, $|\omega_0|_1$, and

$$C_f := \int_0^\infty \int_{\mathbb{R}^2} |f(x,t)|dxdt + \text{ess.sup}_{t>0} t|f(\cdot,t)|_1.$$

Moreover, we have

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} |\omega(\cdot,t) - (m+m_f)g(\cdot,t)|_p = 0.$$

Here, $m = \int_{\mathbb{R}^2} \omega_0(x)dx$ and $m_f = \int_0^\infty \int_{\mathbb{R}^2} f(x,t)\, dxdt$.

If we use interpolation inequalities such as (43) in order to derive the large time behaviors of derivatives of vorticities solving (44), we are forced to assume unnecessary regularity conditions on the inhomogeneous term f. On the other hand, by arguing as in Section 3 with spatial decay estimates for derivatives of solutions of (44), we can derive the large time behavior of derivatives of solutions without irrelevant regularity assumptions on f. Precisely, we have

Theorem 4.2. *Assume that a function f satisfies the conditions in Theorem 4.1. Let $\omega_0 \in L^1(\mathbb{R}^2)$ and $q \in [1,2)$. Then, the solution ω satisfies that*

$$\begin{aligned} &|\partial_x \omega(\cdot,t)|_{q,2R} \\ &\leq \frac{C}{t^{\frac{3}{2}-\frac{1}{q}}} \left(\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,R} \right) + C\int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{q}}} \left(\frac{(t-s)^{\frac{1}{4}}}{R^{\frac{1}{2}}} |f(\cdot,s)|_1 + |f(\cdot,s)|_{1,R} \right) ds, \end{aligned}$$

where C depends only on q, $|\omega_0|_1$, and C_f. Moreover, we have

$$\lim_{t\to\infty} t^{\frac{3}{2}-\frac{1}{q}} |\partial_x \omega(\cdot,t) - \partial_x (m+m_f) g(\cdot,t)|_q = 0.$$

The proof of the above theorem is quite similar to that of the homogeneous case, we omit the details here.

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(accepted: February 10, 2006; published Online First: September 19, 2006)