Radio Science, Volume 4, Number 8, pages 703–708, August 1969

Approximate boundary conditions in the electrodynamics of stratified tensor media

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(Received December 12, 1968; revised April 21, 1969.)

Approximate boundary conditions in the first and second approximations are obtained for stratified tensor media. The approximate conditions make possible the solution of electrodynamic problems without examination of the fields inside the anisotropic layers. The layer thickness in all cases is assumed to be small compared with the wavelength in any layer. The accuracy of the approximate boundary conditions is estimated for one particular example. It is shown that boundary value problems involving ferrite layers near the waveguide walls reduce to solutions of eigenvalue problems with definite boundary conditions along the periphery.

STATEMENT OF PROBLEM: THE BOUNDARY CONDITIONS IN THE FIRST APPROXIMATION

In the electrodynamics of anisotropic media, it is of interest to give approximate boundary conditions for electromagnetic waves in the presence of thin anisotropic layers (for example, in plasmas or ferrites). The customary method of solution of the boundary value problems requires 'matching' of fields. This method, even in the case of simple regions, involves rather cumbersome calculations. The effect of anisotropic layers on the field may be taken into account by means of special boundary conditions that make possible the consideration of waves in the exterior without treating in detail the fields in the layers. In this case the boundary conditions for a transition layer are given analogously to the well-known boundary conditions formulated by *Leontovitch* [1948] for highly conducting bodies.

For simplicity we obtain first the boundary conditions for only one layer with thickness δ and tensor parameters $||\epsilon^{\mathrm{II}}||$ and $||\mu^{\mathrm{II}}||$ located on a conducting surface of finite conductivity σ, with ϵ^{I}, μ^{I} denoting the parameters of the upper isotropic medium (Figure 1). We can use the second Maxwell equation in integral form

$$\oint_{(\mathcal{L})} \mathbf{E} \cdot d\mathbf{l} = ik \iint_{(s)} \mathbf{B} \cdot d\mathbf{s} \qquad (1)$$

where k is the wavenumber, λ is the wavelength in vacuum, and $e^{-i\omega t}$ is the time dependence.

Applying (1) to the contour $\mathcal{L}_1$ (Figure 1), we take into account that $\delta \ll \lambda$ and that the normal component of $\mathbf{E}$ along the contour is a slowly varying function. The magnetic field $\mathbf{H}$, related to $\mathbf{B}$ via

$$\mathbf{B} = ||\mu^{\mathrm{II}}||\, \mathbf{H} \qquad (2)$$

is assumed constant inside the layer for $\delta \ll \lambda$.

Then for $\mathcal{L}_1$ we can write[1]

$$E_2^{\mathrm{I}} l - {}_{(2)}E_1^{\mathrm{II}} \delta + {}_{(1)}E_1^{\mathrm{II}} \delta - E_2^{\mathrm{II}} l \Big|_{x_1=-\delta} = ik\, ||\mu^{\mathrm{II}}||\, \mathbf{H}^{\mathrm{II}} \mathbf{i}_3\, \delta l + O(\delta^2) \qquad (3)$$

Suppose that in (3), l is going to zero as usual; utilizing the boundary conditions given by *Leontovitch* [1948]

$$\mathbf{E} = w(\mathbf{n} \times \mathbf{H}); \qquad w = \frac{i-1}{2} k\, d\mu_\sigma$$

where μ_σ is permeability of body ($x_1 < -\delta$), d is skin depth, and $\mathbf{n}$ is external normal to metal surface, we can find

$$E_2^{\mathrm{I}} - \delta \frac{\partial}{\partial x_2} E_1^{\mathrm{II}} + wH_3^{\mathrm{II}} = ik\, ||\mu^{\mathrm{II}}||\, \mathbf{H}^{\mathrm{II}} \mathbf{i}_3 \delta \qquad (4)$$

By analogy we can write for the contour $\mathcal{L}_2$:

$$E_3^{\mathrm{I}} - \delta \frac{\partial}{\partial x_3} E_1^{\mathrm{II}} - wH_2 = ik\, ||\mu^{\mathrm{II}}||\, \mathbf{H}^{\mathrm{II}} \mathbf{i}_2 \delta \qquad (5)$$

Now we can express the field components of the

[1] Dot product operations between vectors and matrices are implied. The operation $||\mu||\mathbf{H}\mathbf{i}_j \equiv \mathbf{i}_j||\mu||\mathbf{H}$ is defined in the equation following (6).

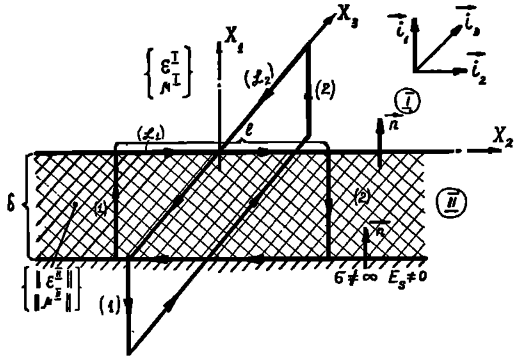

Fig. 1. Anisotropic layer of thickness δ located on a conducting surface.

layer in terms of the fields in the upper half-space. Using the continuity equation for $(\mathbf{n} \times \mathbf{H})$, we can replace H_3^{II} by H_3^I and H_2^{II} by H_2^I in (3) and (4), respectively. From the condition $\mathbf{D}^{II} \cdot \mathbf{i}_1 = \mathbf{D}^I \cdot \mathbf{i}_1$ for $x_1 = 0$, and taking into account continuity of tangential components $\mathbf{E}$ at the boundary, we get

$$E_1^{II} = \frac{1}{\epsilon_{11}} (\epsilon^I E_1^I - \epsilon_{12}^{II} E_2^I - \epsilon_{13}^{II} E_3^I) \tag{6}$$

The right hand parts in (4) and (5)

$$||\mu^{II}||\ \mathbf{H}^{II}\mathbf{i}_3 = \mu_{31}^{II} H_1^{II} + \mu_{32}^{II} H_2^{II} + \mu_{33}^{II} H_3^{II}$$

$$||\mu^{II}||\ \mathbf{H}^{II}\mathbf{i}_2 = \mu_{21}^{II} H_1^{II} + \mu_{22}^{II} H_2^{II} + \mu_{23}^{II} H_3^{II}$$

are equal to (in expanded form, taking into account the continuity of the tangential components of $\mathbf{H}$ and normal components of $\mathbf{B}$)

$$\left.\begin{matrix} ||\mu^{II}||\ \mathbf{H}^{II}\mathbf{i}_3 \\ ||\mu^{II}||\ \mathbf{H}^{II}\mathbf{i}_2 \end{matrix}\right\} = \frac{\mu^{II}_{\left\{\substack{31\\21}\right\}}}{\mu_{11}^{II}} (\mu^I H_1^I - \mu_{12}^{II} H_2^I - \mu_{13}^{II} H_3^I) + \mu^{II}_{\left\{\substack{32\\22}\right\}} H_2^I + \mu^{II}_{\left\{\substack{33\\23}\right\}} H_3^I$$

Finally, omitting the index in the expression for the fields in the upper half-space, the approximate boundary condition for an anisotropic layer on a metallic surface, in the first approximation, is of the form

$$E_{\left\{\substack{2\\3}\right\}} = \delta \frac{1}{\epsilon_{11}^{II}} \left(\epsilon^I \frac{\partial E_1}{\partial x_{\left\{\substack{2\\3}\right\}}} - \epsilon_{12}^{II} \frac{\partial E_2}{\partial x_{\left\{\substack{2\\3}\right\}}} - \epsilon_{13}^{II} \frac{\partial E_3}{\partial x_{\left\{\substack{2\\3}\right\}}} \right)$$

$$\pm ik\delta \left[\frac{\mu^{II}_{\left\{\substack{31\\21}\right\}}}{\mu_{11}^{II}} (\mu^I H_1 - \mu_{12}^{II} H_2 - \mu_{13}^{II} H_3) + \mu^{II}_{\left\{\substack{32\\22}\right\}} H_2 + \mu^{II}_{\left\{\substack{33\\23}\right\}} H_3 \right] \pm \frac{i-1}{2} \mu_\sigma k\, dH_{\left\{\substack{3\\2}\right\}} \tag{7}$$

It can be shown that if the medium constants of the upper half-space are tensors $||\epsilon^I||$ and $||\mu^I||$, expression 7 may be simply generalized to accommodate this case if we replace ϵ^I and μ^I in (7) by corresponding tensors $||\epsilon^I||$ and $||\mu^I||$, as well as E_1 and H_1 by $\mathbf{D} \cdot \mathbf{i}_1$ and $\mathbf{B} \cdot \mathbf{i}_1$.

Now we can formulate boundary conditions for the more general case when on the conducting surface there are N layers and each is characterized by its own tensor $||\epsilon^j||$ and $||\mu^j||$, where $j = 1, 2 \cdots N$. In this case we may write approximate boundary conditions that permit us to consider the process in general, not for each individual layer, provided that the total thickness of the layers is essentially less than the wavelength. From a physical point of view this means for sufficiently large N that the layers degenerate into films with anisotropic parameters. For

$$\sum_{i=1}^{N} \delta_i \ll \lambda$$

expression 4 in this case will have the form

$$E_2^I - \sum_{i=1}^{N} \delta_i \frac{\partial}{\partial x_2} E_1^{(i)} + wH_3^{(1)} = ik \sum_{i=1}^{N} ||\mu^{(i)}||\ \mathbf{H}^{(i)}\mathbf{i}_3$$

Utilizing this limitation and supposing that the tangential components of magnetic field do not change during transition from one layer to another, we get after simple manipulations

$$E_{\left\{\substack{2\\3}\right\}} = \sum_{i=1}^{N} \left\{ \delta_i \frac{1}{\epsilon_{11}^{(i)}} \frac{\partial}{\partial x_{\left\{\substack{2\\3}\right\}}} (||\epsilon||\ \mathbf{E}\mathbf{i}_1 - \epsilon_{12}^{(i)} E_2 - \epsilon_{13}^{(i)} E_3) \pm ik\delta_i \left[\frac{\mu^{(i)}_{\left\{\substack{31\\21}\right\}}}{\mu_{11}^{(i)}} (||\mu||\ \mathbf{H}\mathbf{i}_1 - \mu_{12}^{(i)} H_2 - \mu_{13}^{(i)} H_3) + \mu^{(i)}_{\left\{\substack{32\\22}\right\}} H_2 + \mu^{(i)}_{\left\{\substack{33\\23}\right\}} H_3 \right] \right\} \pm \frac{i-1}{2} \mu_\sigma k\, dH_{\left\{\substack{3\\2}\right\}} \tag{8}$$

We have taken into account here that the upper half-space is a tensor medium with parameters $||\epsilon||$ and $||\mu||$. The same method can be utilized to find the boundary conditions in the first approximation for the case where an anisotropic layer with thickness $\delta \ll \lambda$ does not have any metallic 'support.' (See Figure 2.) As can be shown, the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are discontinuous in the presence of the layer, and the difference of fields at opposite sides of the layer is equal to

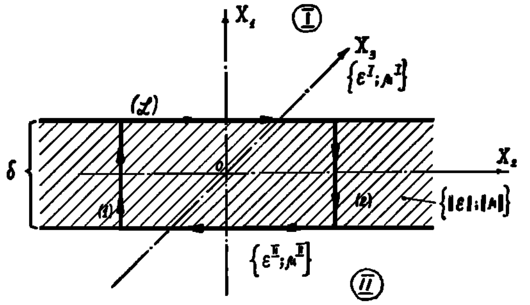

Fig. 2. Anisotropic layer of thickness δ without metallic 'support.'

ary conditions in the first approximation are insufficient, and it is necessary to use more exact relations. It is therefore of interest to develop conditions for successive approximations, i.e., to take into account terms of the order δ^2, δ^3, and higher. Naturally, in this case the difficulties of calculation are increased, but, knowing the conditions for the second approximation, we can, if necessary, estimate the error in application of the conditions of the first approximation. The following terms in the expansion involving the thickness of the layer can be obtained by assuming that the fields in the layer are variable and are ex-

$$
\begin{aligned}
&\left.E^{\mathrm{I}}_{\{\substack{2\\3}\}}\right|_{x_1=\delta/2} - \left.E^{\mathrm{II}}_{\{\substack{2\\3}\}}\right|_{x_1=-\delta/2} = \frac{\delta}{2}\frac{\partial}{\partial x_{\{\substack{2\\3}\}}}\frac{1}{\epsilon_{11}}\Big(\epsilon^{\mathrm{I}}\left.E_1^{\mathrm{I}}\right|_{x_1=\delta/2} + \epsilon^{\mathrm{II}}\left.E_1^{\mathrm{II}}\right|_{x_1=-\delta/2} - \epsilon_{12}\big(\left.E_2^{\mathrm{I}}\right|_{x_1=\delta/2} + \left.E_2^{\mathrm{II}}\right|_{x_1=-\delta/2}\big) \\
&\quad - \epsilon_{13}\big(\left.E_3^{\mathrm{I}}\right|_{x_1=\delta/2} + \left.E_3^{\mathrm{II}}\right|_{x_1=-\delta/2}\big) \pm ik\frac{\delta}{2}\left\{\frac{\mu_{\{\substack{31\\21}\}}}{\mu_{11}}\Big[\mu^{\mathrm{I}}\left.H_1^{\mathrm{I}}\right|_{x_1=\delta/2} + \mu^{\mathrm{II}}\left.H_1^{\mathrm{II}}\right|_{x_1=-\delta/2} - \mu_{12}\big(\left.H_2^{\mathrm{I}}\right|_{x_1=\delta/2} + \left.H_2^{\mathrm{II}}\right|_{x_1=-\delta/2}\big)\right. \\
&\quad \left. - \mu_{13}\big(\left.H_3^{\mathrm{I}}\right|_{x_1=\delta/2} + \left.H_3^{\mathrm{II}}\right|_{x_1=-\delta/2}\big)\Big] + \mu_{\{\substack{32\\22}\}}\big(\left.H_2^{\mathrm{I}}\right|_{x_1=\delta/2} + \left.H_2^{\mathrm{II}}\right|_{x_1=-\delta/2}\big) + \mu_{\{\substack{33\\23}\}}\big(\left.H_3^{\mathrm{I}}\right|_{x_1=\delta/2} + \left.H_3^{\mathrm{II}}\right|_{x_1=-\delta/2}\big)\right\}
\end{aligned}
\tag{9}
$$

For the very thin layer ($\delta \ll \lambda$) expression 9 can be simplified and the conditions for the fields transposed to the middle of the layer; that is, we suppose that all of the values of (9) relate to $x = +0$ and $x = -0$ (see also *Wait* [1960]).

Let us note that the boundary conditions obtained in (7), (8), and (9) can be used for problems with curvilinear boundaries, subject in general to the same limitations that occur during application of the Leontovitch boundary conditions. In particular the thickness of the layer must also be small compared with the radii of curvature of the body and with the distance to the sources of the field.

BOUNDARY CONDITIONS IN THE SECOND APPROXIMATION

In cases when the inequality $\delta \ll \bar{\lambda}$ (where $\bar{\lambda}$ = wavelength in the layer) is fulfilled weakly, the boundary conditions in the first approximation are insufficient, and it is necessary to use more exact relations.

pressed through slowly varying functions in their dependence on the coordinate along the normal to the layer. They can be obtained by expansion in Taylor series with respect to the small parameter δ. The expansion of the field components is performed conveniently in relation to a point on the external boundary of the layer. It can be written

$$
\begin{aligned}
E_1^{\mathrm{II}}(x_1) &= E_1^{\mathrm{II}}(0) + x_1\frac{\partial E_1^{\mathrm{II}}(0)}{\partial x_1} + \cdots \\
E_2^{\mathrm{II}}(x_1) &= E_2^{\mathrm{II}}(0) + x_1\frac{\partial E^{\mathrm{II}}(0)}{\partial x_1} + \cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
H_3^{\mathrm{II}}(x_1) &= H_3^{\mathrm{II}}(0) + x_1\frac{\partial H_3^{\mathrm{II}}(0)}{\partial x_1} + \cdots
\end{aligned}
\tag{10}
$$

Expression 1 for both contours $\mathfrak{L}_1$ and $\mathfrak{L}_2$ is of the form

$$
\begin{aligned}
&E^{\mathrm{I}}_{\{\substack{2\\3}\}} - \frac{\partial}{\partial x_{\{\substack{2\\3}\}}}\int_{-\delta}^{0}\left[E_1^{\mathrm{II}}(0) + \frac{\partial E_1^{\mathrm{II}}(0)}{\partial x_1}x_1\right]dx_1 \pm w\left.\left[H^{\mathrm{II}}_{\{\substack{3\\2}\}}(0) + x_1\frac{\partial H^{\mathrm{II}}_{\{\substack{3\\2}\}}}{\partial x_1}\right]\right|_{x_1=-\delta} \\
&\quad = ik\int_{-\delta}^{0}||\mu^{\mathrm{II}}||\left[\mathbf{H}^{\mathrm{II}}(0) + \frac{\partial \mathbf{H}^{\mathrm{II}}(0)}{\partial x_1}x_1\right]\mathbf{i}_{\{\substack{3\\2}\}}\,dx_1
\end{aligned}
\tag{11}
$$

Expression 11 forms the starting point for deriving the boundary conditions in the second approximation. In (11) results from previous expressions are introduced for the first derivatives of the field components in the layer. They can be represented by the basic Maxwell equations for the layer and the continuity conditions for the tangential components of E and H at the boundary. Omitting the cumbersome calculations, we can write the boundary conditions in the second approximation in the final form

$$
\begin{aligned}
E_{\{\substack{2\\3}\}} = {} & \delta \frac{\epsilon^{\mathrm{I}}}{\epsilon_{11}} \frac{\partial}{\partial x_{\{\substack{2\\3}\}}} E_1 - \delta \frac{1}{\epsilon_{11}} \left(||\epsilon|| \frac{\partial}{\partial x_{\{\substack{2\\3}\}}} \mathbf{E}\mathbf{i}_1 \right)_\perp \pm ik\delta \Bigg\{ \mu^{\mathrm{I}} \frac{\mu_{\{\substack{31\\21}\}}}{\mu_{11}} H_1 - \frac{\mu_{\{\substack{31\\21}\}}}{\mu_{11}} (||\mu||\, \mathbf{H}\mathbf{i}_1)_\perp \\
& + \mu_{\{\substack{32\\22}\}} H_2 + \mu_{\{\substack{33\\23}\}} H_3 \Bigg\} \pm ik \frac{\delta^2}{2} \Bigg\{ \mu_{\{\substack{31\\21}\}} \Bigg[-\frac{1}{\mu_{11}} \operatorname{div}_\perp ||\mu||\, \mathbf{H} - \mu^{\mathrm{I}} \frac{1}{\mu_{11}} \left(\mu_{12} \frac{\partial}{\partial x_2} + \mu_{13} \frac{\partial}{\partial x_3} \right) H_1 \\
& + \frac{1}{(\mu_{11})^2} \left(\mu_{12} \left\langle ||\mu|| \frac{\partial \mathbf{H}}{\partial x_2} \mathbf{i}_1 \right\rangle_\perp + \mu_{13} \left\langle ||\mu|| \frac{\partial \mathbf{H}}{\partial x_3} \mathbf{i}_1 \right\rangle_\perp \right) + ik\epsilon^{\mathrm{I}} \frac{\mu_{12}\epsilon_{31} + \mu_{13}\epsilon_{21}}{\epsilon_{11}\mu_{11}} E_1 \\
& - ik \frac{\mu_{12}\epsilon_{31} + \mu_{13}\epsilon_{21}}{\epsilon_{11}\mu_{11}} (||\epsilon||\, \mathbf{E}\mathbf{i}_1)_\perp + ik \frac{\mu_{12}(||\epsilon||\, \mathbf{E}\mathbf{i}_3)_\perp - \mu_{13}(||\epsilon||\, \mathbf{E}\mathbf{i}_2)_\perp}{\mu_{11}} \Bigg] + \mu_{\{\substack{32\\22}\}} \Bigg[\frac{\mu^{\mathrm{I}}}{\mu_{11}} \frac{\partial}{\partial x_2} H_1 \\
& - \frac{1}{\mu_{11}} \left(||\mu|| \frac{\partial \mathbf{H}}{\partial x_2} \mathbf{i}_1 \right)_\perp - ik\epsilon^{\mathrm{I}} \frac{\epsilon_{31}}{\epsilon_{11}} E_1 + ik \frac{\epsilon_{31}}{\epsilon_{11}} (||\epsilon||\, \mathbf{E}\mathbf{i}_1)_\perp + ik(||\epsilon||\, \mathbf{E}\mathbf{i}_3)_\perp \Bigg] \\
& + \mu_{\{\substack{33\\23}\}} \Bigg[\frac{\mu^{\mathrm{I}}}{\mu_{11}} \frac{\partial}{\partial x_3} H_1 - \frac{1}{\mu_{11}} \left(||\mu|| \frac{\partial \mathbf{H}}{\partial x_3} \mathbf{i}_1 \right)_\perp - ik\epsilon^{\mathrm{I}} \frac{\epsilon_{21}}{\epsilon_{11}} E_1 + ik \frac{\epsilon_{21}}{\epsilon_{11}} (||\epsilon||\, \mathbf{E}\mathbf{i}_1)_\perp - ik(||\epsilon||\, \mathbf{E}\mathbf{i}_2)_\perp \Bigg] \Bigg] \Bigg\} \\
& - \frac{\delta^2}{2} \frac{\partial}{\partial x_{\{\substack{2\\3}\}}} \Bigg\{ -\frac{1}{\epsilon_{11}} \operatorname{div}_\perp ||\epsilon||\, \mathbf{E} - \epsilon^{\mathrm{I}} \frac{1}{(\epsilon_{11})^2} \left(\epsilon_{12} \frac{\partial}{\partial x_2} + \epsilon_{13} \frac{\partial}{\partial x_3} \right) E_1 \\
& + \frac{\epsilon_{12}(||\epsilon||\, (\partial \mathbf{E}/\partial x_2)\mathbf{i}_1)_\perp + \epsilon_{13}(||\epsilon||\, (\partial \mathbf{E}/\partial x_3)\mathbf{i}_1)_\perp}{(\epsilon_{11})^2} - ik\mu^{\mathrm{I}} \frac{\epsilon_{12}\mu_{31} + \epsilon_{13}\mu_{21}}{\epsilon_{11}\mu_{11}} H_{11} \\
& + ik \frac{\epsilon_{12}\mu_{31} + \epsilon_{13}\mu_{21}}{\epsilon_{11}\mu_{11}} (||\mu||\, \mathbf{H}\mathbf{i}_1)_\perp - ik \frac{1}{\epsilon_{11}} [\epsilon_{12}(||\mu||\, \mathbf{H}\mathbf{i}_3)_\perp - \epsilon_{13}(||\mu||\, \mathbf{H}\mathbf{i}_2)_\perp] \Bigg\} \\
& \pm wH_{\{\substack{3\\2}\}} \pm w\delta \Bigg\{ \frac{\mu^{\mathrm{I}}}{\mu_{11}} \frac{\partial}{\partial x_{\{\substack{3\\2}\}}} H_1 - \frac{1}{\mu_{11}} \left(||\mu|| \frac{\partial}{\partial x_{\{\substack{3\\2}\}}} \mathbf{H}\mathbf{i}_1 \right)_\perp - ik\epsilon^{\mathrm{I}} \frac{\epsilon_{\{\substack{21\\21}\}}}{\epsilon_{11}} E_1 \\
& + ik \frac{\epsilon_{\{\substack{21\\31}\}}}{\epsilon_{11}} (||\epsilon||\, \mathbf{E}\mathbf{i}_1)_\perp - ik(||\epsilon||\, \mathbf{E}\mathbf{i}_{\{\substack{2\\3}\}})_\perp \Bigg\} + O(\delta^3)
\end{aligned}
\tag{12}
$$

where

$$(||T||\, \mathbf{A}\mathbf{i}_j)_\perp = T_{j2}A_2 + T_{j3}A_3$$

and

$$\operatorname{div}_\perp ||T||\, \mathbf{A} = T_{21}\frac{\partial A_1}{\partial x_2} + T_{22}\frac{\partial A_2}{\partial x_2} + T_{23}\frac{\partial A_3}{\partial x_2} + T_{31}\frac{\partial A_1}{\partial x_3} + T_{32}\frac{\partial A_2}{\partial x_3} + T_{33}\frac{\partial A_3}{\partial x_3}$$

$||T||$ is an arbitrary tensor and A is an arbitrary vector.

It is clear that for arbitrary ϵ and μ the boundary conditions in the second approximation are too cumbersome and practically not usable for actual problems. But for some particular cases, the expression can be simplified substantially. For example, for a ferrite layer on a conducting ground plane, expression 12 has the form

$$
\begin{aligned}
E_{\{\substack{2\\3}\}} = {} & \delta \frac{\partial}{\partial x_{\{\substack{2\\3}\}}} E_1 \pm ik\delta\{\mu_{\{\substack{31\\21}\}} H_1 + \mu_{\{\substack{32\\22}\}} H_2 + \mu_{\{\substack{33\\23}\}} H_3\} \\
& \pm ik \frac{\delta^2}{2} \left\{ \mu_{\{\substack{31\\21}\}} \frac{\partial H_1}{\partial x_1} + \mu_{\{\substack{32\\22}\}} \frac{\partial H_2}{\partial x_1} + \mu_{\{\substack{33\\23}\}} \frac{\partial H_3}{\partial x_1} \right\} - \frac{\delta^2}{2} \frac{\partial}{\partial x_{\{\substack{2\\3}\}}} \frac{\partial E_1}{\partial x_1} \pm wH_{\{\substack{3\\2}\}} \pm w\delta \frac{\partial}{\partial x_1} H_{\{\substack{3\\2}\}} + O(\delta^3)
\end{aligned}
$$

If we take into account the values of the various tensor components, the last expression can be simplified still further and can be utilized for practical cases.

EXAMPLES OF APPLICATION OF THE BOUNDARY CONDITIONS

The boundary conditions (7), (8), and (9) for the ferrite medium are simplified because ϵ will be a scalar, and the tensor $||\mu||$ is hermitean with many zero terms. When the constant magnetic field is directed along the x_2 axis, the tensor for the ferrite medium is (in the absence of losses)

$$||\mu|| = \begin{Vmatrix} \mu_1 & 0 & -i\mu_2 \\ 0 & \mu_3 & 0 \\ i\mu_2 & 0 & \mu_1 \end{Vmatrix}$$

Usually one introduces a definition of the effective magnetic permeability $\mu_\perp = (\mu_1^2 - \mu_2^2)/\mu_1^2$. In this case the boundary conditions are

$$\begin{aligned} E_2 &= \delta \frac{\epsilon^{\mathrm{I}}}{\epsilon^{\mathrm{II}}} \frac{\partial E_1}{\partial x_2} + ik\delta\left\{i\mu^{\mathrm{I}} \frac{\mu_2}{\mu_1} H_1 + \mu_\perp H_3\right\} \\ &\qquad - \frac{i-1}{2} \mu_\sigma k\, dH_3 \\ E_3 &= \delta \frac{\epsilon^{\mathrm{I}}}{\epsilon^{\mathrm{II}}} \frac{\partial E_1}{\partial x_3} - ik\delta\mu_3 H_2 + \frac{i-1}{2} \mu_\sigma k\, dH_2 \end{aligned} \tag{13}$$

In the same way, formula 9 is simplified to

$$\begin{aligned} E_2^{\mathrm{I}} - E_2^{\mathrm{II}} &= \frac{\delta}{2}\left(\frac{\epsilon^{\mathrm{I}}}{\epsilon} \frac{\partial E_1^{\mathrm{I}}}{\partial x_2} + \frac{\epsilon^{\mathrm{II}}}{\epsilon} \frac{\partial E_1^{\mathrm{II}}}{\partial x_2}\right) \\ &\quad + ik \frac{\delta}{2}\left[\frac{i\mu_2}{\mu_1} (\mu^{\mathrm{I}} H_1 + \mu^{\mathrm{II}} H_1) + \mu_\perp (H_3^{\mathrm{I}} + H_3^{\mathrm{II}})\right] \\ E_3^{\mathrm{I}} - E_3^{\mathrm{II}} &= \frac{\delta}{2}\left(\frac{\epsilon^{\mathrm{I}}}{\epsilon} \frac{\partial E_1^{\mathrm{I}}}{\partial x_3} + \frac{\epsilon^{\mathrm{II}}}{\epsilon} \frac{\partial E_1^{\mathrm{II}}}{\partial x_3}\right) \\ &\quad - ik \frac{\delta}{2} \mu_3 (H_2^{\mathrm{I}} + H_2^{\mathrm{II}}) \end{aligned} \tag{14}$$

Let us analyze the quality of the above boundary conditions. For example, consider the propagation of the H_{10} mode through a waveguide which has near one surface a ferrite plate with constant magnetization along the Y axis (Figure 3). Since the field of the H_{10} mode is independent of the coordinate Y, the conditions (13) are simplified and take the following form

$$E_y = k\delta\left(-\frac{\mu_2}{\mu_1} H_z + i\mu_\perp H_z\right) \tag{15}$$

The boundary conditions (15) when $x = 0$ may be satisfied in terms of a Hertz magnetic vector. The scalar function ψ is introduced to satisfy the Helmholtz equation

$$\Delta_\perp \psi + \kappa^2 \psi = 0 \tag{16}$$

where $\kappa = (k^2 - h^2)^{1/2}$ is the eigenvalue of the boundary value problem. The components of the fields are determined according to the well-known formula

$$E_y = ik \frac{\partial \psi}{\partial x}, \quad H_x = \frac{\partial^2 \psi}{\partial x\, \partial z}, \quad H_z = \left(k^2 + \frac{\partial^2}{\partial z^2}\right)\psi \tag{17}$$

Using (15) and (17) and changing $(\partial/\partial x)$ into $(\partial/\partial n)$, we obtain the following

$$\frac{\partial \psi}{\partial n} = -\delta \frac{\mu_\perp (k^2 - h^2)}{1 - \delta \frac{\mu_2}{\mu_1} h} \psi \tag{18}$$

where $\mathbf{n}$ is the internal normal to the surface. Thus the problem of the waveguide with a ferrite plate may be formulated in the following manner. Let us obtain the solutions of the scalar wave equation with the boundary conditions $(\partial\psi/\partial n) = 0$ on the contour L and conditions 18 when $x = 0$. This problem is equivalent to finding the free oscillations of a membrane, one side of which is sealed elastically. It is easy to see that condition 18 contains a dependence on the direction of propagation of the electromagnetic waves (through $\pm h$). It is obvious that equation 16 with the above conditions is satisfied when $\psi = \bar{A} e^{ihz} \cos \kappa(a - \delta - x)$, where κ are the eigenvalues

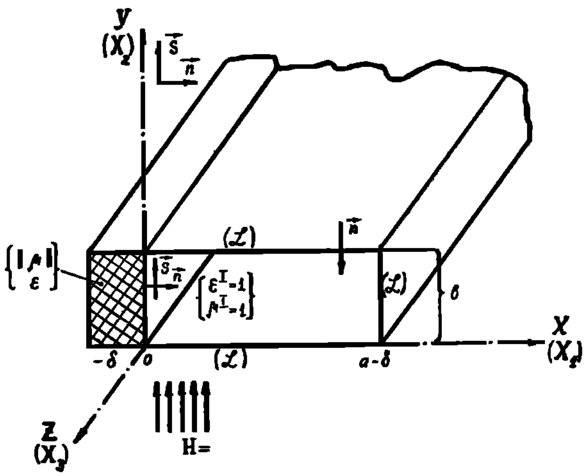

Fig. 3. Rectangular waveguide with a ferrite plate subject to constant magnetization along the Y axis.

of the characteristic equation of the boundary value problem

$$\pm\frac{\mu_2}{\mu_1} h\delta \tan [(a - \delta)(k^2 - h^2)^{1/2}] + \tan [(a - \delta) \cdot(k^2 - h^2)^{1/2}] + \mu_\perp(k^2 - h^2)^{1/2}\delta = 0 \qquad (19)$$

The exact characteristic equation of this problem is [*Mikaeljan and Pistolkors,* 1955]

$$\pm\frac{\mu_2}{\mu_1} h \tan [\delta(k^2\epsilon\mu_\perp - h^2)^{1/2}] \tan [(a - \delta)(k^2 - h^2)^{1/2}] + (k^2\epsilon\mu_\perp - h^2)^{1/2} \tan [(a - \delta)(k^2 - h^2)^{1/2}] + \mu_\perp(k^2 - h^2)^{1/2} \tan [\delta(k^2\epsilon\mu_\perp - h^2)^{1/2}] = 0 \qquad (20)$$

For a thin ferrite layer the following series is justified

$$\tan [\delta(k^2\epsilon\mu_\perp - h^2)^{1/2}] \approx \delta(k^2\epsilon\mu_\perp - h^2)^{1/2} + \tfrac{1}{3}\delta^3[(k^2\epsilon\mu_\perp - h^2)^{1/2}]^3 + \cdots \qquad (21)$$

If we neglect the second term in series 21, we obtain from (20) the characteristic equation 19, which has been derived also from the approximate boundary condition. It is possible to indicate the limit of applicability of this boundary condition. From (21) it follows

$$\delta(k^2\epsilon\mu_\perp - h^2)^{1/2} \gg \tfrac{1}{3}\delta^3[(k^2\epsilon\mu_\perp - h^2)^{1/2}]^3$$

and

$$\delta \ll [(3)^{1/2}/(k^2\epsilon\mu_\perp - h^2)^{1/2}] \qquad (22)$$

For example, for ferrite one usually has $1 < \epsilon\mu_\perp < 10$, $\epsilon\mu_\perp \approx 9$; therefore $\delta \leq 0.1\lambda$, where λ is the wavelength in the air. Thus, formula 22 for a ferrite medium can estimate the range of validity of the approximate boundary conditions.

Acknowledgment. The author expresses his sincere thanks to Professor L. B. Felsen for considerable assistance with the translation and the preparation of this manuscript for publication.

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