BIT
2001, Vol. 41, No. 1, pp. 127–148

0006-3835/01/4101-0127 $16.00

ON SHAPE PRESERVING C^2 HERMITE INTERPOLATION *

CARLA MANNI †

Dipartimento di Matematica, Universitá di Torino, Via Carlo Alberto 10
IT-10123 Torino, Italy. email: manni@dm.unito.it

Abstract.

We propose a general parametric local approach for functional C^2 Hermite shape preserving interpolation. The constructed interpolant is a parametric curve which interpolates values, first and second derivatives of a given function and reproduces the behavior of the data. The method is detailed for parametric curves with piecewise cubic components. For the selected space necessary and sufficient conditions are derived to ensure the convexity of the constructed interpolant. Monotonicity is also studied. The approximation order is investigated for both cases. The use of a parametric curve to interpolate data from a function can be considered a disadvantage of the scheme. However, the simple structure of the used curves greatly reduces such a disadvantage.

AMS subject classification: 41A05, 41A25, 41A29, 65D05.

Key words: Interpolation, shape preserving, parametric curves.

1 Introduction.

It is often necessary, in various applications, to generate a smooth function that interpolates a prescribed set of data. In many practical situations an interpolant of class C^1 or C^2 is required. At the same time, it is often desirable, or even strictly required for physical or aesthetical reasons, that the interpolating function preserves certain geometric shape properties of the data such as monotonicity or convexity.

There are many schemes available for the construction of a C^1 shape preserving interpolant (see for example [5, 6, 14, 15, 16, 24] and references quoted therein). Dealing with C^2 shape preserving interpolation is a more difficult task on the other hand. There are many papers related to this problem (see for example [2, 7, 9, 18, 20, 21, 22, 23] and references quoted therein) which propose different techniques to produce the required interpolant that, basically, can be summarised as follows. Assume the data (x_i, f_i), $i = 0, \ldots, n$, are given, distributed in a monotone and/or a convex position. At first a suitable space is selected where it is possible to find a solution (in any case, nonlinear spaces should be considered if success is generally desired for convexity preservation,

*Received February 2000. Communicated by Tom Lyche.
†This work was partially supported by MURST.

[19]). Suitable values for first and/or second derivatives at the data sites are then computed in order to obtain smoothness and shape preservation.

In this paper we face the problem from a different point of view since we are considering an Hermite interpolation problem up to second derivatives (see also [6]). In fact, in addition to the data positions, we assume that the first and second derivatives, f_i', f_i'', consistent with the data behavior, are also prescribed. The mentioned derivatives can be given as input or computed from (x_i, f_i), $i = 0, \ldots, n$. We look for a function, $s \in C^2[x_0, x_n]$, interpolating f and its first and second derivatives at the data points, having the same behavior of the data in $[x_i, x_{i+1}]$.

Classical spaces for constrained interpolation are polynomials and polynomial splines on refined grids. Regarding C^1 Hermite interpolation, it is well-known that comonotonicity and/or coconvexity cannot be achieved assuming s to be a polynomial with a preassigned degree in each subinterval $[x_i, x_{i+1}]$, [10].

Similar negative results hold if f_i, f_i', f_i'' must be interpolated at x_i. In [9], the set of *admissible* derivatives at the endpoints of $[x_i,\ x_{i+1}]$ for C^2 comonotone interpolation is carefully investigated in case the interpolant is a quintic in each interval; convexity preservation is also discussed.

Looking at splines on refined grids, it turns out that the space of cubic splines with two interior knots in $[x_i,\ x_{i+1}]$ is pliable enough to ensure the solution for C^1 Hermite comonotone and coconvex interpolation (see [10, 5]). But this is no longer sufficient for C^2 Hermite coconvex interpolation; see Example 4.1.

In this paper, to overcome these difficulties, we situate the problem in the parametric setting, and we consider the graph of the function $x \to s(x)$ as the image of a particular bidimensional parametric curve $(x(t), y(t))$ where $x(t)$ is strictly increasing (see [16, 17]). This different point of view provides the main tool of the approach but, as we will discuss later, also constitutes a disadvantage of the proposed scheme. By using this parametric approach the shape of the interpolant is easily controlled via the amplitude of the tangent vectors at the two endpoints of each segment which basically act as *tension parameters*. According to [19], the considered functions are well suited for constrained interpolation because they have a *nonlinear* dependence on their tension parameters.

After a general presentation of the parametric approach to shape preserving C^2 Hermite interpolation, we detail a case where the components of the curve belong to the space of piecewise cubics with two interior fixed knots in each subinterval. We show that it is always possible to select the above mentioned tension parameters in order to solve the C^2 Hermite comonotone and/or coconvex interpolation problem. In this regard, we provide conditions which are sufficient for monotonicity and necessary and sufficient for convexity, in order to ensure that the constructed interpolant is shape preserving. A simple automatic algorithm, which is based on the mentioned conditions, has been proposed for selecting values of the tension parameters that provide visually pleasing shape preserving interpolants.

The method is completely local: a change in the data at x_i only reflects in $[x_{i-1}, x_{i+1}]$.

We also investigate the approximation order of the presented scheme. It is well-known [8, 3] that monotonicity and/or convexity preserving approximation does not imply a reduction of the classical approximation power of polynomials or polynomial splines. However, for a large part of the schemes proposed in constrained interpolation, the approximation power is not investigated. Concerning C^2 shape preserving Lagrange interpolation, results on this topic can be found in [18] and [2], where the proposed schemes are shown to be third order accurate for C^3 functions with nonvanishing first (second) derivatives. Similarly strict monotonicity and/or convexity is required in [21] to ensure the existence of the interpolant, at least for small values of the grid spacing, and to investigate its accuracy (see also [7]).

The approach we propose here provides at least a third order approximation for C^3 monotone and/or convex functions and is fourth order accurate for data taken from a C^4 convex function without any restriction on its derivatives. For the sake of simplicity, throughout the paper we assume the first and second derivatives are exactly known. However we can immediately verify that the estimates for the approximation order still remain valid if the derivatives are approximated with a suitable accuracy. As an example the scheme is third order accurate if we work with second (first) order accurate estimates of the first (second) derivatives which are consistent with the behavior of the data.

Summarising the proposed scheme is a parametric tension method which allows an automatic local choice of the tension parameters ensuring shape preservation and accuracy.

It is worthwhile to mention that the method we are proposing differs from those in [2] and [18] non only for the kind of the interpolation problem and for the results on the accuracy, but also for the use of the space of piecewise cubics with additional knots. In [2] and [18] the interior knots are variable: their position is used to control the shape of the interpolant. Here the approach is completely different, variable knots are not necessary: the control of the shape is fully achieved via the amplitude of the tangent vectors at the ends of each subinterval.

We emphasize that here the parametric setting is a technical tool to introduce a space of interpolating functions whose shape can be easily controlled via tension parameters (the amplitude of the tangent vectors at the endpoints of each subinterval) having a nice geometric meaning. However, the method is tailored for *functional* interpolation so it is not pertinent to compare the presented scheme with other local methods for interpolation of parametric curves (see for example [4, 13], and references quoted therein) where no error estimates are provided since the authors are basically interested in controlling the shape of the curve.

It turns out that the parametric setting is, at the same time, the main tool and the main drawback of the proposed scheme. In fact, since s is obtained as a parametric curve with piecewise cubic components, its computational cost is comparable with that of classic functional schemes if only a plot or a tabulation of the interpolant is required but a third order equation must be solved to compute the value of s at a given fixed value $\hat{x}$. However, due to the very simple structure of the x component; see Section 4.1 (this can be efficiently done by classical methods). As an example, since the x component is a C^2 piecewise cubic increasing function, a few Newton–Raphson iterations suffice to compute the solution of the above mentioned equation to machine precision.

The remainder of this paper is divided into 4 sections. In the next section we state the problem while in Section 3 we present the general parametric approach to C^2 Hermite shape preserving interpolation. Section 4 with its subsections is devoted to presenting the details of the scheme and its performances if the components of the curve belong to the space of piecewise cubics with two interior, fixed a priori knots in each subinterval. Then we end in Section 5 with some numerical examples.

2 The problem.

Let (x_i, f_i, f_i', f_i''), $i = 0, \dots, n$, be given, where

$$f_i = f(x_i), \qquad f_i' = f'(x_i), \qquad f_i'' = f''(x_i), \quad i = 0, \dots, n. \tag{2.1}$$

We shall adopt in the sequel the following notation:

$$h_i = x_{i+1} - x_i, \qquad \Delta_i = f_{i+1} - f_i, \quad i = 0, \dots, n-1,$$

and we assume that $h_i > 0$. Concerning the shape of the data, according to the Taylor expansion we state the following definitions which are satisfied for a smooth function f.

Definition 2.1. *The data (x_l, f_l, f_l', f_l''), $l = 0, \dots, n$, are monotone increasing in $[x_i, x_{i+1}]$ if*

$$\Delta_i > 0, \quad f_{i+j}' \geq 0 \ \text{ and } \ \exists\, \epsilon_i^{(j)} > 0 \ : \ f_{i+j}' + (-1)^j \epsilon_i^{(j)} f_{i+j}'' \geq 0, \quad j = 0, 1,$$

or

$$\Delta_i = 0, \qquad f_i' = f_{i+1}' = f_i'' = f_{i+1}'' = 0.$$

Definition 2.2. *The data (x_l, f_l, f_l', f_l''), $l = 0, \dots, n$, are convex in $[x_i, x_{i+1}]$ if*

$$h_i f_i' < \Delta_i < h_i f_{i+1}', \qquad f_i'',\ f_{i+1}'' \geq 0,$$

or

$$h_i f_i' = \Delta_i = h_i f_{i+1}', \qquad f_i'' = f_{i+1}'' = 0.$$

Similar definitions can be stated for decreasing and/or concave data.

Our goal is to construct a function, $s \in C^2[x_0, x_n]$, interpolating the data:

$$s(x_i) = f_i, \qquad s'(x_i) = f_i', \qquad s''(x_i) = f_i'', \quad i = 0, \dots, n, \tag{2.2}$$

which is *comonotone* and/or *coconvex* with the data, i.e. $s'(x) \geq 0$, (≤ 0) and/or $s''(x) \geq 0$, (≤ 0) in $[x_i, x_{i+1}]$, $i = 0, \dots, n-1$, if the data are monotone increasing (decreasing) and/or convex (concave) in the interval. Here and throughout the paper dashes will denote derivatives with respect to the x variable.

3 The general construction of the interpolant.

Following [16], for each subinterval $[x_i, x_{i+1}]$ we want to construct a solution, s to the problem stated in the previous section by considering the graph of the function $x \to s(x)$ as the image of a bidimensional parametric curve with components belonging to a given linear space, V.

Let P_r be the linear space of polynomials of a degree less than or equal to r, and let V be a real linear space of functions such that

$$dim\ V = 6, \qquad P_0 \subset V \subset C^k[0,1], \quad k \geq 2.$$

To ensure the interpolation problem has only one solution, for each sequence $(d_0, d_1, D_0, D_1) \in R^4$ we assume there exists a unique $v(t; d_0, d_1, D_0, D_1) \in V$ so that

$$\begin{cases} v(j; d_0, d_1, D_0, D_1) = j, \\ \dfrac{d}{dt} v(t; d_0, d_1, D_0, D_1)_{|t=j} = d_j, \\ \dfrac{d^2}{dt^2} v(t; d_0, d_1, D_0, D_1)_{|t=j} = D_j, \end{cases} \qquad j = 0, 1. \tag{3.1}$$

For each pair $(h_i^{(0)},\ h_i^{(1)}) \in R^2$ it is thus possible to construct a parametric curve

$$\begin{cases} x = X_i(t; h_i^{(0)}, h_i^{(1)}) \\ y = Y_i(t; h_i^{(0)}, h_i^{(1)}) \end{cases}, \qquad i = 0, \dots, n-1, \quad 0 \leq t \leq 1, \tag{3.2}$$

such that

$$X_i(t; h_i^{(0)}, h_i^{(1)}),\ Y_i(t; h_i^{(0)}, h_i^{(1)}) \in V,$$

and

(3.3)

$$\begin{array}{ll}
X_i(0; h_i^{(0)}, h_i^{(1)}) = x_i, & X_i(1; h_i^{(0)}, h_i^{(1)}) = x_{i+1}, \\
\frac{d}{dt} X_i(t; h_i^{(0)}, h_i^{(1)})|_{t=0} = h_i^{(0)}, & \frac{d}{dt} X_i(t; h_i^{(0)}, h_i^{(1)})|_{t=1} = h_i^{(1)}, \\
\frac{d^2}{dt^2} X_i(t; h_i^{(0)}, h_i^{(1)})|_{t=0} = 0, & \frac{d^2}{dt^2} X_i(t; h_i^{(0)}, h_i^{(1)})|_{t=1} = 0, \\
Y_i(0; h_i^{(0)}, h_i^{(1)}) = f_i, & Y_i(1; h_i^{(0)}, h_i^{(1)}) = f_{i+1}, \\
\frac{d}{dt} Y_i(t; h_i^{(0)}, h_i^{(1)})|_{t=0} = h_i^{(0)} f_i', & \frac{d}{dt} Y_i(t; h_i^{(0)}, h_i^{(1)})|_{t=1} = h_i^{(1)} f_{i+1}', \\
\frac{d^2}{dt^2} Y_i(t; h_i^{(0)}, h_i^{(1)})|_{t=0} = (h_i^{(0)})^2 f_i'', & \frac{d^2}{dt^2} Y_i(t; h_i^{(0)}, h_i^{(1)})|_{t=1} = (h_i^{(1)})^2 f_{i+1}''.
\end{array}$$

If it is possible to choose positive parameters $h_i^{(0)},\ h_i^{(1)},\ i = 0, \dots, n-1$, such that

$$\frac{d}{dt} X_i(t; h_i^{(0)}, h_i^{(1)}) > 0, \tag{3.4}$$

the first component in (3.2) implicitly defines a function $t = t(x)$ so that

$$\frac{dt}{dx} = \left(\frac{dX_i}{dt}\right)^{-1}, \quad x_i \le x \le x_{i+1}. \tag{3.5}$$

Thus $Y_i(t; h_i^{(0)}, h_i^{(1)})$ can be expressed as a function of x. In this case, defining

$$s(x) = Y_i(t(x); h_i^{(0)}, h_i^{(1)}), \quad x_i \le x \le x_{i+1}, \quad i = 0, \dots, n-1, \tag{3.6}$$

from (3.2), (3.3) and (3.5) we have

$$\begin{aligned}
&s(x_i) = f_i, \\
&s'(x_i^+) = h_i^{(0)} f_i' \frac{1}{h_i^{(0)}} = f_i', && s'(x_i^-) = h_{i-1}^{(1)} f_i' \frac{1}{h_{i-1}^{(1)}} = f_i', \\
&s''(x_i^+) = (h_i^{(0)})^3 f_i'' \frac{1}{(h_i^{(0)})^3} = f_i'', && s''(x_i^-) = (h_{i-1}^{(1)})^3 f_i'' \frac{1}{(h_{i-1}^{(1)})^3} = f_i''.
\end{aligned}$$

Hence, for each pair of values $h_i^{(0)}$, $h_i^{(1)}$ satisfying (3.4), (3.6) defines a function of class $C^2[x_0, x_n]$ which satisfies the interpolation conditions (2.2).

The values $h_i^{(0)}$, $h_i^{(1)}$ determine the magnitude of the tangent vectors to the curve (3.2) at the endpoints and they control the shape of each component. To be more specific, let $v(t; 0, 0, 0, 0)$ be the unique element of V defined by (3.1) and by the sequence $(0, 0, 0, 0)$. Then, from linearity, and since V contains the constants, we have

$$\begin{aligned}
X_i(t; 0, 0) &= x_i + h_i v(t; 0, 0, 0, 0), \\
Y_i(t; 0, 0) &= f_i + \Delta_i v(t; 0, 0, 0, 0).
\end{aligned}$$

Thus, in the limit case $h_i^{(0)} = h_i^{(1)} = 0$, the curve (3.2) reduces to the straight line segment interpolating (x_i, f_i), (x_{i+1}, f_{i+1}); then $h_i^{(0)}$ and $h_i^{(1)}$ act as tension parameters.

Specifying suitable spaces of functions for V we can obtain several schemes to interpolate the data (2.1) with different performances both for the approximation order and for the graphical results. In the next section we study in detail one of the simplest choices for the mentioned space.

4 Piecewise cubic parametric interpolation.

In this section, we study the construction and the shape preserving and approximation properties of the scheme obtained considering V as the space of piecewise cubic splines with two interior knots.

4.1 Construction of the interpolant.

Let us consider in the interval $[0, 1]$ the linear space of C^2 cubic splines with two interior knots at $0 < \xi < \eta < 1$, i.e.

$$S_3^2[0, \xi, \eta, 1] = \{c \in C^2[0, 1] : \ c(t)|_{[0,\xi]} \in P_3, \ \ c(t)|_{[\xi,\eta]} \in P_3, \ c(t)|_{[\eta,1]} \in P_3\}.$$

For the sake of simplicity we choose $\xi = 1-\eta = \frac{1}{3}$, but the results we state hold true for general choices of the knots. Thus we will consider

$$V = S_3^2\left[0, \tfrac{1}{3}, \tfrac{2}{3},\ 1\ \right].$$

In this case it is not difficult to verify (see [18]) that the unique curve (3.2) which satisfies (3.3) can be represented in each subinterval in Bézier–Bernstein form as follows:

$$\begin{pmatrix} X_i\left(t; h_i^{(0)}, h_i^{(1)}\right) \\ Y_i\left(t; h_i^{(0)}, h_i^{(1)}\right) \end{pmatrix} = \sum_{j=0}^{3} \binom{3}{j} (1-u)^{3-j} u^j \mathbf{P}_i^{(3k+j)}, \tag{4.1}$$

$$t \in [k/3, (k+1)/3], \qquad u = 3(t-k/3), \quad k = 0,1,2,$$

where the Bézier–Bernstein coefficients (or *control points*) of the curve, $\mathbf{P}_i^{(l)}$, $l = 0, \dots, 9$, are

$$\begin{aligned}
&\mathbf{P}_i^{(0)} = \begin{pmatrix} x_i \\ f_i \end{pmatrix}, && \mathbf{P}_i^{(9)} = \begin{pmatrix} x_{i+1} \\ f_{i+1} \end{pmatrix}, \\
&\mathbf{P}_i^{(1)} = \begin{pmatrix} x_i + \frac{1}{9}h_i^{(0)} \\ f_i + \frac{1}{9}f_i' h_i^{(0)} \end{pmatrix}, && \mathbf{P}_i^{(8)} = \begin{pmatrix} x_{i+1} - \frac{1}{9}h_i^{(1)} \\ f_{i+1} - \frac{1}{9}h_i^{(1)} f_{i+1}' \end{pmatrix}, \\
&\mathbf{P}_i^{(2)} = \mathbf{B}_i - \tfrac{1}{9}\mathbf{b}_i, && \mathbf{P}_i^{(7)} = \mathbf{C}_i + \tfrac{1}{9}\mathbf{c}_i, \\
&\mathbf{P}_i^{(3)} = \mathbf{B}_i, && \mathbf{P}_i^{(6)} = \mathbf{C}_i, \\
&\mathbf{P}_i^{(4)} = \mathbf{B}_i + \tfrac{1}{9}\mathbf{b}_i, && \mathbf{P}_i^{(5)} = \mathbf{C}_i - \tfrac{1}{9}\mathbf{c}_i,
\end{aligned} \tag{4.2}$$

and

(4.3)

$$\begin{aligned}
\mathbf{B}_i &= \begin{pmatrix} x_i \\ f_i \end{pmatrix} + \frac{1}{9}\begin{pmatrix} h_i^{(0)} \\ h_i^{(0)} f_i' \end{pmatrix} + \frac{1}{9}\mathbf{a}_i + \frac{1}{9}\mathbf{b}_i, \\
\mathbf{C}_i &= \begin{pmatrix} x_{i+1} \\ f_{i+1} \end{pmatrix} - \frac{1}{9}\begin{pmatrix} h_i^{(1)} \\ h_i^{(1)} f_{i+1}' \end{pmatrix} - \frac{1}{9}\mathbf{c}_i - \frac{1}{9}\mathbf{d}_i, \\
\mathbf{a}_i &= \begin{pmatrix} h_i^{(0)}, \\ f_i' h_i^{(0)} + \frac{1}{6}(h_i^{(0)})^2 f_i'' \end{pmatrix}, \qquad \mathbf{d}_i = \begin{pmatrix} h_i^{(1)}, \\ f_{i+1}' h_i^{(1)} - \frac{1}{6}(h_i^{(1)})^2 f_{i+1}'' \end{pmatrix}, \\
\mathbf{b}_i &= \frac{1}{2}(\mathbf{a}_i + \mathbf{r}_i), \\
\mathbf{c}_i &= \frac{1}{2}(\mathbf{d}_i + \mathbf{r}_i), \\
\mathbf{r}_i &= \begin{pmatrix} 3h_i - h_i^{(0)} - h_i^{(1)}, \\ 3\Delta_i - h_i^{(0)} f_i' - h_i^{(1)} f_{i+1}' + \frac{1}{9}[(h_i^{(1)})^2 f_{i+1}'' - (h_i^{(0)})^2 f_i''] \end{pmatrix}.
\end{aligned}$$

Similarly, it is useful for the following to express the first derivative of the components of the curve with respect to the Bézier–Bernstein basis. We have

$$\begin{pmatrix} \frac{d}{dt}X_i\left(t;h_i^{(0)},h_i^{(1)}\right) \\ \frac{d}{dt}Y_i\left(t;h_i^{(0)},h_i^{(1)}\right) \end{pmatrix} = \begin{pmatrix} h_i^{(0)} \\ h_i^{(0)}f_i' \end{pmatrix}(1-u)^2 + 2\mathbf{a}_i u(1-u) + \mathbf{b}_i u^2 \quad \text{if } t \in [0,1/3]\,, \quad u = 3t, \tag{4.4}$$

$$\begin{pmatrix} \frac{d}{dt}X_i\left(t;h_i^{(0)},h_i^{(1)}\right) \\ \frac{d}{dt}Y_i\left(t;h_i^{(0)},h_i^{(1)}\right) \end{pmatrix} = \mathbf{b}_i(1-u)^2 + 2\mathbf{r}_i u(1-u) + \mathbf{c}_i u^2 \quad \text{if } t \in [1/3,2/3]\,, \quad u = 3(t-1/3), \tag{4.5}$$

$$\begin{pmatrix} \frac{d}{dt}X_i\left(t;h_i^{(0)},h_i^{(1)}\right) \\ \frac{d}{dt}Y_i\left(t;h_i^{(0)},h_i^{(1)}\right) \end{pmatrix} = \mathbf{c}_i(1-u)^2 + 2\mathbf{d}_i u(1-u) + \begin{pmatrix} h_i^{(1)} \\ h_i^{(1)}f_{i+1}' \end{pmatrix} u^2 \quad \text{if } t \in [2/3,1]\,, \quad u = 3(t-2/3). \tag{4.6}$$

Thanks to the properties of Bézier–Bernstein basis, (3.4) holds for $t \in [0,1]$ if the Bézier coefficients of the first derivative of X_i are positive. Then, from (4.3)–(4.6), the following bounds are sufficient to satisfy (3.4):

$$0 < h_i^{(0)}, h_i^{(1)} \le h_i. \tag{4.7}$$

Thus, if $h_i^{(0)}, h_i^{(1)}$ are positive parameters not exceeding h_i, then the curve defined in (4.1)–(4.3) implicitly defines (see (3.6)) a function $x \to s(x)$ belonging to $C^2[x_0,x_n]$, which interpolates the data (2.1).

We observe that, in the particular case $h_i^{(0)} = h_i^{(1)} = h_i$, we have

$$X_i(t;h_i,h_i) = x_i + th_i,$$

so that in $[x_i,x_{i+1}]$ s reduces to the classical piecewise cubic with interior knots at $x_i+\frac{h_i}{3}$, $x_{i+1}-\frac{h_i}{3}$ interpolating the data. As noticed in the previous section in the limit case $h_i^{(0)} = h_i^{(1)} = 0$, in $[x_i,x_{i+1}]$, s reduces to the straight line interpolating (x_i,f_i), (x_{i+1},f_{i+1}). Then $h_i^{(0)}$, $h_i^{(1)}$ act as *tension* parameters stretching s from the classical piecewise cubic to the piecewise linear interpolating the data. It turns out that they can be used to locally control the shape of s as discussed in the next subsections.

In this regard it is useful to observe that, in the classical functional setting, it is always possible to select the position of two interior knots in $[x_i,x_{i+1}]$ in order to obtain a C^2 piecewise cubic function interpolating (x_{i+j},f_{i+j}), $j = 0,1$, which is convex, [23]. However, this is no longer true if we must interpolate prescribed first and second derivatives at the data sites as well. This is shown in the next example.

EXAMPLE 4.1. Let the following convex data be given:

$$x_0 = 0,\ x_1 = 1,\ f_0 = f_0' = f_0'' = 0,\quad f_1 = 1,\ 1 < f_1' = p,\ 0 \le f_1'' = 2P.$$

From [23] we find that a C^2 piecewise cubic with knots at $0,\ \xi,\ \eta,\ 1$, interpolating the previous data is convex for $x \in [0,1]$ if and only if

$$\begin{aligned} 3 - (2-\xi-\eta)p + (1-\xi)(1-\eta)P &\ge 0, \\ -3 + (3-\eta-\xi)p - (1-\eta)(2-\xi)P &\ge 0. \end{aligned}$$

Since $0 < \xi < \eta < 1$, we have

$$0 \le -3 + (3-\eta-\xi)p - (1-\eta)(2-\xi)P \le -3 + 2p - (1-\eta)(P-p),$$

which cannot be satisfied if $p < \frac{3}{2}$ and $p < P$.

4.2 Preserving monotonicity.

In this subsection we provide sufficient conditions on the tension parameters $h_i^{(0)},\ h_i^{(1)}$ to ensure s is comonotone with the data in a given interval $[x_i, x_{i+1}]$. For the sake of simplicity we consider only the increasing case.

Since (3.4) holds, from

$$\frac{d}{dx}s(x) = \frac{dY_i}{dt}\frac{dt}{dx}, \quad x_i \le x \le x_{i+1}, \tag{4.8}$$

we find that the monotonicity of s in $[x_i, x_{i+1}]$ is determined by the monotonicity of $Y_i(t; h_i^{(0)}, h_i^{(1)})$ in $[0,1]$. Once again, thanks to the properties of the Bézier–Bernstein basis, $Y_i(t; h_i^{(0)}, h_i^{(1)})$ is increasing if the Bézier coefficients of its first derivative are nonnegative, then, from (4.3)–(4.6), if

$$\begin{aligned} & f_i' \ge 0, \qquad f_{i+1}' \ge 0, \\ & f_i' + \frac{1}{6}h_i^{(0)} f_i'' \ge 0, \\ & 3\Delta_i - h_i^{(0)} f_i' - h_i^{(1)} f_{i+1}' + \frac{1}{9}\big[(h_i^{(1)})^2 f_{i+1}'' - (h_i^{(0)})^2 f_i''\big] \ge 0, \\ & f_{i+1}' - \frac{1}{6}h_i^{(1)} f_{i+1}'' \ge 0. \end{aligned} \tag{4.9}$$

Then, if the data are increasing in $[x_i, x_{i+1}]$; see Definition 2.1, it is always possible to find $h_i^{(0)}$, $h_i^{(1)}$ which satisfy (4.7) and (4.9). In other words it is always possible to construct $s(x)$ which interpolates the data and is comonotone.

REMARK 4.1. Less restrictive sufficient conditions can be obtained by applying a subdivision to the Bézier net of Y_i; see [18].

Let us put

$$\mathcal{M}_i := \{(h_i^{(0)},\ h_i^{(1)}) \in R^2 : 0 < h_i^{(0)},\ h_i^{(1)} \le h_i \text{ satisfying } (4.9)\}.$$

In order to obtain a comonotone interpolant of the form (4.1) it suffices to select $(h_i^{(0)}, h_i^{(1)}) \in \mathcal{M}_i$. To choose the mentioned values of the tension parameters it is reasonable to project the pair (h_i, h_i) onto $\mathcal{M}_i$. However, the region $\mathcal{M}_i$ can be nonconvex (see Figure 4.1), so that the projection of (h_i, h_i) onto $\mathcal{M}_i$ can be nonunique. Moreover, projections onto convex sets are more easy to be treated from the computational point of view. Then, in order to simplify the practical choice of the tension parameters, we prefer to consider a convex subset of $\mathcal{M}_i$. We see immediately that (4.9) holds if

(4.10)

$$\begin{aligned}
&f_i' \geq 0, \qquad f_{i+1}' \geq 0,\\
&f_i' + \frac{1}{6} h_i^{(0)} f_i'' \geq 0, \qquad f_{i+1}' - \frac{1}{6} h_i^{(1)} f_{i+1}'' \geq 0.\\
&3\Delta_i - h_i^{(0)} f_i' - h_i^{(1)} f_{i+1}' + \frac{1}{9}\Big[(h_i^{(1)})^2 \min(0, f_{i+1}'') - (h_i^{(0)})^2 \max(0, f_i'')\Big] \geq 0,
\end{aligned}$$

and that the region

$$\mathcal{SM}_i := \{(h_i^{(0)},\ h_i^{(1)}) \in R^2 : 0 < h_i^{(0)},\ h_i^{(1)} \leq h_i \text{ satisfying (4.10)}\},$$

is convex (see Figure 4.1). Then, in order to obtain a comonotone interpolant of the form (4.1) we choose the tension parameters $h_i^{(0)}$, $h_i^{(1)}$ according to the following algorithm.

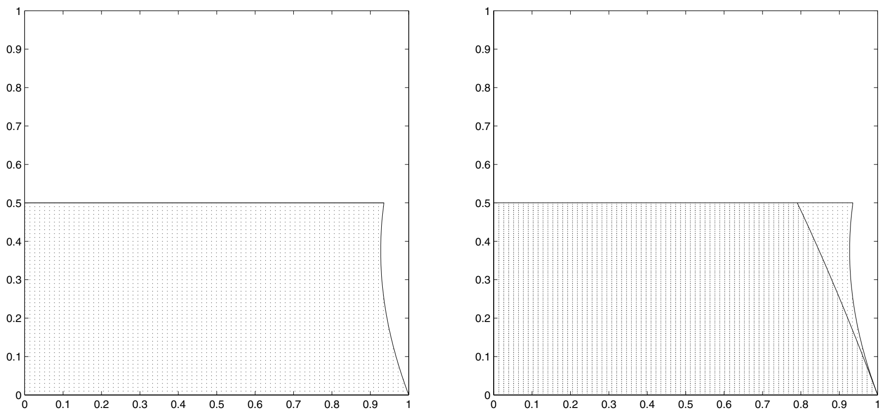


Figure 4.1: Left $\mathcal{M}_i$; right $\mathcal{M}_i$ and $\mathcal{SM}_i$; for $\Delta_i = \frac{1}{3}$, $h_i = 1$, $f_i' = 0$, $f_{i+1}' = \frac{3}{4}$, $f_i'' = f_{i+1}'' = 9$.

Algorithm 4.1 (Comonotonicity Algorithm).

1) *Let* $(x_j, f_j, f_j', f_j'',\ j = i, i+1)$ *be given.*

2) *Put* $h_i^{(0)} = h_i,\ h_i^{(1)} = h_i$.

3) *If the data are increasing in* $[x_i, x_{i+1}]$ *and* $(h_i, h_i) \notin \mathcal{M}_i$,
compute $(h_i^{(0)}, h_i^{(1)})$ *projecting* (h_i, h_i) *onto* $\mathcal{SM}_i$.

REMARK 4.2. From (4.8) and from the variation diminishing properties of the Bézier–Bernstein basis, the number of local extrema of s in (x_i, x_{i+1}) is bounded by the number of sign changes of the sequence

$$h_i^{(0)} f_i', \ (\mathbf{a}_i)_y, \ (\mathbf{b}_i)_y, \ (\mathbf{r}_i)_y, \ (\mathbf{c}_i)_y, \ (\mathbf{d}_i)_y, h_i^{(1)} f_{i+1}', \tag{4.11}$$

where $(\mathbf{p})_y$ denote the y component of $\mathbf{p}$. Thus, from the C^2 continuity conditions at the interior break points (see (4.2)–(4.3)) it turns out that s can have at maximum four local extrema in (x_i, x_{i+1}). Moreover, even when the monotonicity conditions are not satisfied in $[x_i, x_{i+1}]$, it is possible to find values $0 < h_i^{(0)}, h_i^{(1)} \le h_i$ such that the corresponding s has the minimum number of local extrema consistent with the data. Assume, as an example, that the data are increasing at x_i and decreasing at x_{i+1}, with the following configuration (see Definition 2.1)

$$f_i' \ge 0, f_{i+1}' \le 0 \text{ and exist } \epsilon_i^{(0)}, \epsilon_i^{(1)} > 0 \ : \ f_i' + \epsilon_i^{(0)} f_i'' > 0, \ f_{i+1}' - \epsilon_i^{(1)} f_{i+1}'' < 0.$$

From inequalities similar to (4.9) we can compute values $0 < h_i^{(0)}, \ h_i^{(1)} \le h_i$ ensuring that the sequence (4.11) has only a sign change. This implies that s has single a extremum in (x_i, x_{i+1}) which turns out to be a maximum.

4.3 Preserving convexity.

In this subsection we provide necessary and sufficient conditions on the tension parameters $h_i^{(0)}$ and $h_i^{(1)}$ to ensure that s is coconvex with the data in a given interval $[x_i, x_{i+1}]$. For the sake of simplicity, we consider the convex case only. As for the monotonicity conditions, the Bézier coefficients of the curve (4.1)–(4.3) play a fundamental role. The following theorem provides necessary and sufficient conditions for the convexity of our interpolant in $[x_i, x_{i+1}]$ imposing the control points of the curve to form a convex polygon.

THEOREM 4.1. *If* $0 < h_i^{(0)}, \ h_i^{(1)}$ *satisfy* (3.4) *the curve defined by* (4.1)–(4.3) *is convex if and only if the following inequalities hold:*

$$\begin{aligned}
&f_i'' \ge 0, \qquad f_{i+1}'' \ge 0,\\
&3(\Delta_i - h_i f_i') - \frac{1}{2} h_i h_i^{(0)} f_i'' - h_i^{(1)} (f_{i+1}' - f_i')\\
&\qquad + \frac{1}{18}(h_i^{(0)})^2 f_i'' + \frac{1}{6} h_i^{(0)} h_i^{(1)} f_i'' + \frac{1}{9}(h_i^{(1)})^2 f_{i+1}'' \ge 0,\\
&3(h_i f_{i+1}' - \Delta_i) - \frac{1}{2} h_i h_i^{(1)} f_{i+1}'' - h_i^{(0)} (f_{i+1}' - f_i')\\
&\qquad + \frac{1}{9}(h_i^{(0)})^2 f_i'' + \frac{1}{6} h_i^{(0)} h_i^{(1)} f_{i+1}'' + \frac{1}{18}(h_i^{(1)})^2 f_{i+1}'' \ge 0.
\end{aligned} \tag{4.12}$$

PROOF. Let us put $t_0 = 0$, $t_1 = \xi$, $t_2 = \eta$, $t_3 = 1$. In each subinterval $[t_k, t_{k+1}]$, $k = 0, 1, 2$, the curve (X_i, Y_i) is a Bézier cubic curve. From (3.4) it follows that the x components of the control point $\mathbf{P}_i^{(l)}$, $l = 0, \dots, 9$, are strictly increasing. Thus, from [12], the number of inflections of (X_i, Y_i) in each subinterval $[t_k, t_{k+1}]$ is bounded by the number of inflections of its control polygon. Hence (X_i, Y_i) is convex in $[t_k, t_{k+1}]$ if

$$\begin{aligned} (\mathbf{P}_i^{(3k+1)} - \mathbf{P}_i^{(3k)}) \times (\mathbf{P}_i^{(3k+2)} - \mathbf{P}_i^{(3k+1)}) > 0, \\ (\mathbf{P}_i^{(3k+2)} - \mathbf{P}_i^{(3k+1)}) \times (\mathbf{P}_i^{(3k+3)} - \mathbf{P}_i^{(3k+2)}) > 0, \end{aligned} \qquad k = 0, 1, 2, \tag{4.13}$$

where

$$\begin{pmatrix} x \\ y \end{pmatrix} \times \begin{pmatrix} v \\ w \end{pmatrix} = xw - yv.$$

Since, from the C^2 continuity at the interior break points (see (4.2)–(4.3)),

$$\begin{aligned} (\mathbf{P}_i^{(4)} - \mathbf{P}_i^{(3)}) \times (\mathbf{P}_i^{(5)} - \mathbf{P}_i^{(4)}) = (\mathbf{P}_i^{(2)} - \mathbf{P}_i^{(1)}) \times (\mathbf{P}_i^{(3)} - \mathbf{P}_i^{(2)}), \\ (\mathbf{P}_i^{(5)} - \mathbf{P}_i^{(4)}) \times (\mathbf{P}_i^{(6)} - \mathbf{P}_i^{(5)}) = (\mathbf{P}_i^{(7)} - \mathbf{P}_i^{(6)}) \times (\mathbf{P}_i^{(8)} - \mathbf{P}_i^{(7)}), \end{aligned} \tag{4.14}$$

after a simple manipulation we obtain (4.13) reducing to (4.12) provided that $h_i^{(0)}, h_i^{(1)} > 0$. Then (4.12) are sufficient to ensure the convexity of (X_i, Y_i). To show their necessity we notice ([11]) that the sign of the curvature of (X_i, Y_i) at the two ends of the subinterval $[t_k, t_{k+1}]$ is given, respectively, by the sign of

$$\begin{aligned} (\mathbf{P}_i^{(3k+1)} - \mathbf{P}_i^{(3k)}) \times (\mathbf{P}_i^{(3k+2)} - \mathbf{P}_i^{(3k+1)}), \\ (\mathbf{P}_i^{(3k+2)} - \mathbf{P}_i^{(3k+1)}) \times (\mathbf{P}_i^{(3k+3)} - \mathbf{P}_i^{(3k+2)}), \end{aligned} \qquad k = 0, 1, 2.$$

□

As for comonotonicity, from (4.12), it is always possible to find $h_i^{(0)}$, $h_i^{(1)}$ satisfying (4.7) and (4.12) whenever the data are convex in $[x_i, x_{i+1}]$ (see Definition 2.2), that is it is always possible to construct $s(x)$ which interpolates the data and is coconvex. Let us put

$$\mathcal{C}_i := \{(h_i^{(0)}, h_i^{(1)}) \in R^2 : 0 < h_i^{(0)},\ h_i^{(1)} \le h_i \text{ satisfying } (4.12)\}.$$

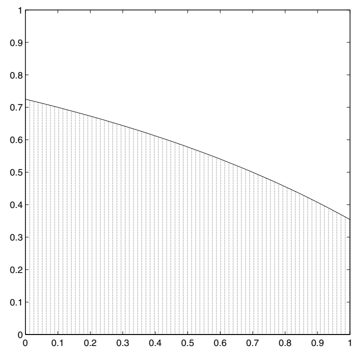
Once again in order to simplify the practical choice of the tension parameters we can consider a suitable subset of $\mathcal{C}_i$. In this case we can limit ourselves to considering linear constraints for $h_i^{(0)}$ and $h_i^{(1)}$. In fact, if the data are convex (see Definition 2.2) (4.12) are satisfied if the following inequalities hold:

$$\begin{aligned} & f_i'' \ge 0, \qquad f_{i+1}'' \ge 0, \\ & 3(\Delta_i - h_i f_i') - \frac{1}{2} h_i h_i^{(0)} f_i'' - h_i^{(1)} (f_{i+1}' - f_i') \ge 0, \\ & 3(h_i f_{i+1}' - \Delta_i) - \frac{1}{2} h_i h_i^{(1)} f_{i+1}'' - h_i^{(0)} (f_{i+1}' - f_i') \ge 0. \end{aligned} \tag{4.15}$$

Then, putting

$$\mathcal{SC}_i := \{(h_i^{(0)},\ h_i^{(1)}) \in R^2 : 0 < h_i^{(0)}, h_i^{(1)} \leq h_i \text{ satisfying (4.15)}\},$$

to obtain a coconvex interpolant, we can choose the tension parameters $h_i^{(0)}$, $h_i^{(1)}$ according to the following algorithm (see Figure 4.2).



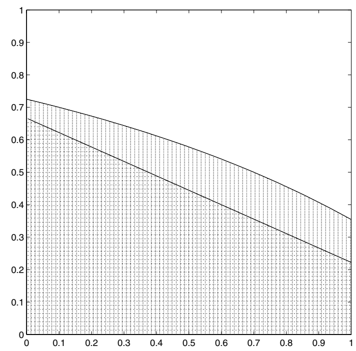


Figure 4.2: Left $\mathcal{C}_i$; right $\mathcal{C}_i$ and $\mathcal{SC}_i$, for $\Delta_i = 1$, $h_i = 1$, $f_i' = f_i'' = 0$, $f_{i+1}' = 2$, $f_{i+1}'' = 9$.

Algorithm 4.2 (Coconvexity Algorithm).

1) *Let* $(x_j, f_j, f_j', f_j'',\ j = i, i+1)$ *be given.*

2) *Put* $h_i^{(0)} = h_i,\ h_i^{(1)} = h_i$.

3) *If the data are convex in* $[x_i, x_{i+1}]$ *and* $(h_i, h_i) \notin \mathcal{C}_i$ *compute* $(h_i^{(0)}, h_i^{(1)})$ *projecting* (h_i, h_i) *onto* $\mathcal{SC}_i$.

Remark 4.3. The number of inflections of s in $[x_i, x_{i+1}]$ is bounded by the number of sign changes of the sequence (4.13) (see the proof of Theorem 4.1). Thus, from (4.14), it turns out that s can have at maximum three inflection points in $[x_i, x_{i+1}]$. Moreover, even when the convexity conditions are not satisfied in $[x_i, x_{i+1}]$, it is possible to find values $0 < h_i^{(0)},\ h_i^{(1)} \leq h_i$ such that the corresponding s has the minimum number of inflections consistent with the data. Assume, as an example, that the data are convex at x_i and concave at x_{i+1} with the following configuration (see Definition 2.2)

$$f_i'' \geq 0, \quad h_i f_i' < \Delta_i, \quad f_{i+1}'' \leq 0, \quad h_i f_{i+1}' < \Delta_i.$$

From inequalities similar to (4.12) we can compute values $0 < h_i^{(0)},\ h_i^{(1)} \leq h_i$ ensuring that the sequence (4.13) has only a sign change. This implies that s has only one inflection point in $[x_i, x_{i+1}]$.

4.4 Approximation order of the constrained interpolant.

In this section we study the approximation order of the proposed constrained interpolant. The main fact here is that, as noticed in Section 4.1, if $h_i^{(0)}, h_i^{(1)} = h_i$, then s reduces to the piecewise cubic with two interior knots in each interval $[x_i, x_{i+1}]$ interpolating the data, which provides a fourth order approximation to a C^4 function. If the tension parameters $h_i^{(0)}, h_i^{(1)}$ are not too far from h_i, it turns out that the approximation order will not reduce. This is established in the following two lemmas. In the following, D_l and K_l denote constants depending neither on h_i nor on $h_i^{(0)},\ h_i^{(1)}$.

LEMMA 4.2. *Assume that $f \in C^l[x_i, x_{i+1}]$, for some $l \in \{2,3,4\}$ and $0 < h_i^{(0)},\ h_i^{(1)} \le h_i$. If $l \in \{3,4\}$ further assume that*

$$\max_{x\in[x_i,x_{i+1}]} |h_i(h_i - h_i^{(j)})f''(x)| \le D_l h_i^l, \quad j = 0,1. \tag{4.16}$$

Then

$$\max_{x\in[x_i,x_{i+1}]} |f(x) - s(x)| \le K_l h_i^l, \quad l \in \{2,3,4\}.$$

PROOF. Let $\hat{x} \in [x_i, x_{i+1}]$ be fixed. Then, from (3.4) and (4.7) a unique $\hat{t} \in [0,1]$ exists so that $X_i(\hat{t}; h_i^{(0)},\ h_i^{(1)}) = \hat{x}$. To be specific let us assume that $\hat{t} \in [0, \frac{1}{3}]$.

Let us first consider the case $l = 4$. From the interpolation conditions at $t = 0$, we obtain

$$\begin{aligned}|f(\hat{x}) - s(\hat{x})| &= \left| f(X_i(\hat{t}; h_i^{(0)},\ h_i^{(1)})) - Y_i(\hat{t}; h_i^{(0)},\ h_i^{(1)}) \right| \\ &\le \left| \frac{d^3}{dt^3}[f(X_i(t; h_i^{(0)},\ h_i^{(1)})) - Y_i(t; h_i^{(0)},\ h_i^{(1)})]_{|t=\theta} \right|, \ \theta \in [0, 1/3]\,.\end{aligned}$$

Let us bound

$$\begin{aligned}&\left| \frac{d^3}{dt^3}[f(X_i(t; h_i^{(0)},\ h_i^{(1)})) - Y_i(t; h_i^{(0)},\ h_i^{(1)})] \right| \\ &= \left| \frac{d^3 f}{dx^3}\left(\frac{dX_i}{dt}\right)^3 + 3\frac{d^2 f}{dx^2}\left(\frac{dX_i}{dt}\right)\left(\frac{d^2 X_i}{dt^2}\right) + \frac{df}{dx}\left(\frac{d^3 X_i}{dt^3}\right) - \left(\frac{d^3 Y_i}{dt^3}\right) \right| =: A.\end{aligned}$$

From (4.3)–(4.4) we have

$$\begin{aligned}\frac{dX_i}{dt} &= h_i^{(0)} + \left[2(h_i - h_i^{(0)}) + (h_i - h_i^{(1)})\right]\frac{1}{2}u^2, \\ \frac{d^2 X_i}{dt^2} &= 3u\left[2(h_i - h_i^{(0)}) + (h_i - h_i^{(1)})\right], \\ \frac{d^2 Y_i}{dt^2} &= 3u\Big\{3\Delta_i - 2h_i^{(0)} f_i' - h_i^{(1)} f_{i+1}' + \frac{1}{9}\left[(h_i^{(1)})^2 f_{i+1}'' - (h_i^{(0)})^2 f_i''\right] \\ &\qquad - \frac{1}{6}(h_i^{(0)})^2 f_i''\Big\} + (1-u)(h_i^{(0)})^2 f_i'',\end{aligned} \tag{4.17}$$

$$\frac{d^3X_i}{dt^3} = 9\big[2(h_i - h_i^{(0)}) + (h_i - h_i^{(1)})\big],$$
$$\frac{d^3Y_i}{dt^3} = 9\Big\{3\Delta_i - 2h_i^{(0)}f_i' - h_i^{(1)}f_{i+1}' + \frac{1}{9}\big[(h_i^{(1)})^2 f_{i+1}'' - (h_i^{(0)})^2 f_i''\big] - \frac{1}{2}(h_i^{(0)})^2 f_i''\Big\}.$$

In addition, from the Taylor expansion

$$f'''(x)h_i^2(h_i - h_i^{(j)}) = f'''(x_i)h_i^2(h_i - h_i^{(j)}) + (x - x_i)f^{(4)}(\eta_i)h_i^2(h_i - h_i^{(j)}),$$

$\eta_i \in (x_i, x)$, $|x - x_i| \le h_i$, and

$$f''(x_{i+1}) - f''(x_i) - \frac{1}{2}f^{(4)}(\theta_i)h_i^2 = f'''(x_i)h_i, \quad \theta_i \in (x_i, x_{i+1}),$$

so that from (4.16) we have

$$|f'''(x)h_i^2(h_i - h_i^{(j)})| \le O(h_i^4), \quad j = 0, 1, \quad x \in [x_i, x_{i+1}]. \tag{4.18}$$

Then, from the Taylor expansion and from (4.16)–(4.18)

$$\begin{aligned} A &\le \left| f_i'''(h_i^{(0)})^3 + f_i'\left(\frac{d^3X_i}{dt^3}\right) - \left(\frac{d^3Y_i}{dt^3}\right)\right| + O(h_i^4) \\ &\le 9\bigg|\frac{1}{9}f_i'''(h_i^{(0)})^3 + f_i'\big[2(h_i - h_i^{(0)}) + (h_i - h_i^{(1)})\big] \\ &\quad - f_i'\big[2(h_i - h_i^{(0)}) + (h_i - h_i^{(1)})\big] - \bigg[\frac{3}{2}h_i^2 f_i'' + \frac{1}{2}h_i^3 f_i''' - h_i h_i^{(1)} f_i'' \\ &\quad - \frac{1}{2}h_i^2 h_i^{(1)} f_i''' - \frac{1}{2}(h_i^{(0)})^2 f_i'' + \frac{1}{9}h_i(h_i^{(1)})^2 f_i'''\bigg]\bigg| + O(h_i^4) \\ &\le \bigg| - 9\bigg[\frac{1}{2}f_i''(h_i^2 - (h_i^{(0)})^2) + f_i'' h_i(h_i - h_i^{(1)}) + \frac{1}{2}f_i''' h_i^2(h_i - h_i^{(1)})\bigg] \\ &\quad + f_i'''\big[(h_i^{(0)})^3 - h_i(h_i^{(1)})^2\big]\bigg| + O(h_i^4) \le K_4 h_i^4. \end{aligned}$$

Similarly we can conclude for $l = 2, 3$. Then the assertion holds for $\hat{t} \in [0, \frac{1}{3}]$.

Similar conclusions hold for $\hat{t} \in [\frac{2}{3}, 1]$.

Concerning the central subinterval, from the previous bounds we have

$$\left|\frac{d^q}{dt^q}[f(X_i(t; h_i^{(0)}, h_i^{(1)})) - Y_i(t; h_i^{(0)}, h_i^{(1)})]_{|t=\frac{1}{3}}\right| \le O(h_i^l), \quad l = 2, 3, 4, \ q = 0, 1, 2,$$

so that analogous arguments can be applied also to $\hat{t} \in [\frac{1}{3}, \frac{2}{3}]$. □

From the previous lemma, the shape preserving interpolant constructed in Sections 4.2 and 4.3 provides at least a second order approximation to a C^2 function. For a more thorough investigation of the approximation order of s it suffices to show that the tension parameters $h_i^{(0)}$, $h_i^{(1)}$ selected in the Comonotonicity and/or Coconvexity Algorithms satisfy (4.16). We have done so in the following lemmas beginning with the comonotone case.

LEMMA 4.3. *Let* $f \in C^3[x_i, x_{i+1}]$ *be monotone increasing. If the tension parameters* $h_i^{(0)}$, $h_i^{(1)}$ *are obtained from the Comonotonicity Algorithm then*

$$0 < h_i^{(0)}, \quad h_i^{(1)} \le h_i,$$

and

$$\max_{x\in[x_i,x_{i+1}]} |h_i(h_i - h_i^{(j)}) f''(x)| \le D_3 h_i^3, \quad j = 0, 1.$$

PROOF. From the structure of $\mathcal{M}_i$ and of $\mathcal{SM}_i$ we immediately find that $0 < h_i^{(0)},\ h_i^{(1)} \le h_i$. If $h_i^{(0)},\ h_i^{(1)} = h_i$, the assertion holds. If $h_i^{(0)} < h_i$ we find that the third and/or the fourth constraint in (4.10) holds as equality. If $f_i' + \frac{1}{6} h_i^{(0)} f_i'' = 0$, then, since f is increasing, $f_i'' \le 0$, and,

$$f_i' + \frac{1}{6} h_i f_i'' < 0, \tag{4.19}$$

then, from the Taylor expansion,

$$0 \le \Delta_i - h_i \left(f_i' + \frac{1}{6} h_i f_i'' \right) = \frac{1}{3} h_i^2 f_i'' + O(h_i^3),$$

hence

$$O(h_i^3) \le h_i^2 f_i'' \le h_i(h_i - h_i^{(0)}) f_i'' \le 0. \tag{4.20}$$

On the other hand, if

$$3\Delta_i - h_i^{(0)} f_i' - h_i^{(1)} f_{i+1}' + \frac{1}{9}\big[(h_i^{(1)})^2 \min(0, f_{i+1}'') - (h_i^{(0)})^2 \max(0, f_i'')\big] = 0,$$

recalling that $2\Delta_i - h_i f_i' - h_i f_{i+1}' = O(h_i^3)$, taking into account the various possibilities for the sign of f_i'' and f_{i+1}'', from the Taylor expansion we obtain

$$\begin{aligned} 0 =& 2\Delta_i - h_i f_i' - h_i f_{i+1}' + (h_i - h_i^{(0)}) f_i' + (h_i - h_i^{(1)}) f_{i+1}' \\ & - \frac{1}{9}\big(h_i^2 - (h_i^{(1)})^2\big) \min(0, f_{i+1}'') + \frac{1}{9}\big(h_i^2 - (h_i^{(0)})^2\big) \max(0, f_i'') \\ & + \Delta_i + \frac{1}{9} h_i^2 \min(0, f_{i+1}'') - \frac{1}{9} h_i^2 \max(0, f_i'') \\ \ge& (h_i - h_i^{(0)}) f_i' + (h_i - h_i^{(1)}) f_{i+1}' + \frac{1}{9}\big(h_i^2 - (h_i^{(0)})^2\big) \max(f_i'', 0) \\ & - \frac{1}{9}\big(h_i^2 - (h_i^{(1)})^2\big) \min(f_{i+1}'', 0) + O(h_i^3). \end{aligned}$$

Since f is increasing, the previous inequality provides

$$\begin{aligned} \big|(h_i - h_i^{(0)}) f_i'\big| &\le O(h_i^3), \\ \big|h_i(h_i - h_i^{(0)}) f_i''\big| &\le O(h_i^3), \quad \text{if } \ f_i'' \ge 0. \end{aligned} \tag{4.21}$$

If, on the contrary $f_i'' < 0$ there are two cases. If $f_i' + \frac{1}{6}h_i f_i'' \geq 0$, then, from (4.21)

$$0 \leq -(h_i - h_i^{(0)})h_i f_i'' \leq (h_i - h_i^{(0)})6f_i' \leq O(h_i^3).$$

Otherwise (4.19) holds so that (4.20) applies. Summarizing,

$$\left|h_i(h_i - h_i^{(0)})f_i''\right| \leq O(h_i^3).$$

and the assertion follows from the Taylor expansion.

We can make a similar conclusion when $h_i^{(1)} < h_i$. □

Concerning the convexity preservation we have the following lemma:

LEMMA 4.4. *Let $f \in C^l[x_i, x_{i+1}]$, for some $l \in \{3, 4\}$ be convex. If the tension parameters $h_i^{(0)}$, $h_i^{(1)}$ are obtained from the Coconvexity Algorithm then $0 < h_i^{(0)}$, $h_i^{(1)} \leq h_i$, and*

$$\max_{x\in[x_i,x_{i+1}]} |h_i(h_i - h_i^{(j)})f''(x)| \leq D_l h_i^l, \quad j = 0, 1, \quad l = 3, 4.$$

PROOF. Let us consider as an example the case $l = 4$.

If $h_i^{(0)}$ and/or $h_i^{(1)} < h_i$, then some of the constraints in (4.12) are not satisfied for $h_i^{(0)}$, $h_i^{(1)} = h_i$ that is

$$3(\Delta_i - h_i f_i') - \frac{1}{2}h_i^2 f_i'' - h_i(f_{i+1}' - f_i') + \frac{1}{18}h_i^2 f_i'' + \frac{1}{6}h_i^2 f_i'' + \frac{1}{9}h_i^2 f_{i+1}'' < 0, \tag{4.22}$$

or

$$3(h_i f_{i+1}' - \Delta_i) - \frac{1}{2}h_i^2 f_{i+1}'' - h_i(f_{i+1}' - f_i') + \frac{1}{9}h_i^2 f_i'' + \frac{1}{6}h_i^2 f_{i+1}'' + \frac{1}{18}h_i^2 f_{i+1}'' < 0.$$

Let us assume, for an example that (4.22) holds; then from the Taylor expansion we have

$$0 > \frac{1}{3}h_i^2 f_i'' + \frac{1}{9}h_i^3 f_i''' + O(h_i^4) = \frac{1}{3}h_i^2 f_{i+1}'' - \frac{2}{9}h_i^3 f_{i+1}''' + O(h_i^4). \tag{4.23}$$

Since f is convex, from (4.23) we obtain

$$\begin{aligned} 0 \leq\ & f_i''' h_i^3 \leq O(h_i^4), \quad \text{if } f_i''' \geq 0, \\ 0 \leq\ & -f_{i+1}''' h_i^3 \leq O(h_i^4), \quad \text{if } f_{i+1}''' \leq 0. \end{aligned}$$

Finally if $f_i''' < 0$ and $f_{i+1}''' > 0$, f''' vanishes somewhere in (x_i, x_{i+1}) so that, in any case,

$$|f'''(x)h_i^3| \leq O(h_i^4), \quad x \in [x_i, x_{i+1}]. \tag{4.24}$$

In addition, from the Coconvexity Algorithm,

$$3(\Delta_i - h_i f_i') - \frac{1}{2}h_i h_i^{(0)} f_i'' - h_i^{(1)}(f_{i+1}' - f_i') = 0, \tag{4.25}$$

or

$$3(h_i f'_{i+1} - \Delta_i) - \frac{1}{2} h_i h_i^{(1)} f''_{i+1} - h_i^{(0)}(f'_{i+1} - f'_i) = 0.$$

If (4.25) holds, from the Taylor expansion we obtain

$$\frac{1}{2} h_i f''_i (h_i - h_i^{(0)}) + h_i f''_i (h_i - h_i^{(1)}) + \frac{1}{2} h_i^2 (h_i - h_i^{(1)}) f'''_i + O(h_i^4) = 0.$$

Then, since f is convex and from (4.24)

$$0 \le h_i(h_i - h_i^{(j)}) f''_i \le O(h_i^4), \quad j = 0, 1,$$

and the assertion comes from (4.24) and from the Taylor expansion. We can make a similar conclusion in the remaining cases. □

Remark 4.4. It is not difficult to see that Lemma 4.4 still holds if, in the case where $(h_i, h_i) \notin \mathcal{C}_i$, (h_i, h_i) is projected onto any subset of $\mathcal{C}_i$ containing $\mathcal{SC}_i$. A similar remark holds for Lemma 4.3.

Finally, from Lemmas 4.2–4.4, we can deduce the approximation properties of s that we summarize in the following theorems.

Theorem 4.5. *Let $f \in C^l[x_i, x_{i+1}]$, $l = 2, 3$, be monotone increasing. Assuming that the tension parameters $h_i^{(0)}$, $h_i^{(1)}$ are determined by the Comonotonicity Algorithm, then s defined via* (3.6), (4.1)–(4.3), *interpolates the data* (2.1), *is monotone increasing in $[x_i, x_{i+1}]$ and*

$$\max_{x \in [x_i, x_{i+1}]} |f(x) - s(x)| \le K_l h_i^l, \quad l = 2, 3.$$

Theorem 4.6. *Let $f \in C^l[x_i, x_{i+1}]$, $l = 2, 3, 4$, be convex. Assuming that the tension parameters $h_i^{(0)}$, $h_i^{(1)}$ are determined by the Coconvexity Algorithm, then s defined via* (3.6), (4.1)–(4.3), *interpolates the data* (2.1) *is convex in $[x_i, x_{i+1}]$ and*

$$\max_{x \in [x_i, x_{i+1}]} |f(x) - s(x)| \le K_l h_i^l, \quad l = 2, 3, 4.$$

Remark 4.5. If $f' > 0$ in $[x_i, x_{i+1}]$ conditions (4.9) are satisfied if $h_i^{(0)} = h_i^{(1)} = h_i$ as $h_i \to 0$. Similarly (4.12) holds for $h_i^{(0)} = h_i^{(1)} = h_i$ when h_i is small enough if $f'' > 0$ in $[x_i, x_{i+1}]$. Then monotonicity (convexity) does not require any restriction on the tension parameters if the grid spacing is small enough and the first (second) derivative of f does not vanish. In this case the constrained interpolation problem stated in Section 2 is solvable in the space of piecewise cubics with knots at x_i, $x_i + \frac{h_i}{3}, x_{i+1} - \frac{h_i}{3}$, x_{i+1}, at least for small values of the grid spacing.

5 Numerical examples.

Before showing the performances of the proposed scheme on a few classical tests, we observe that additional degrees of freedom, which can be used to further

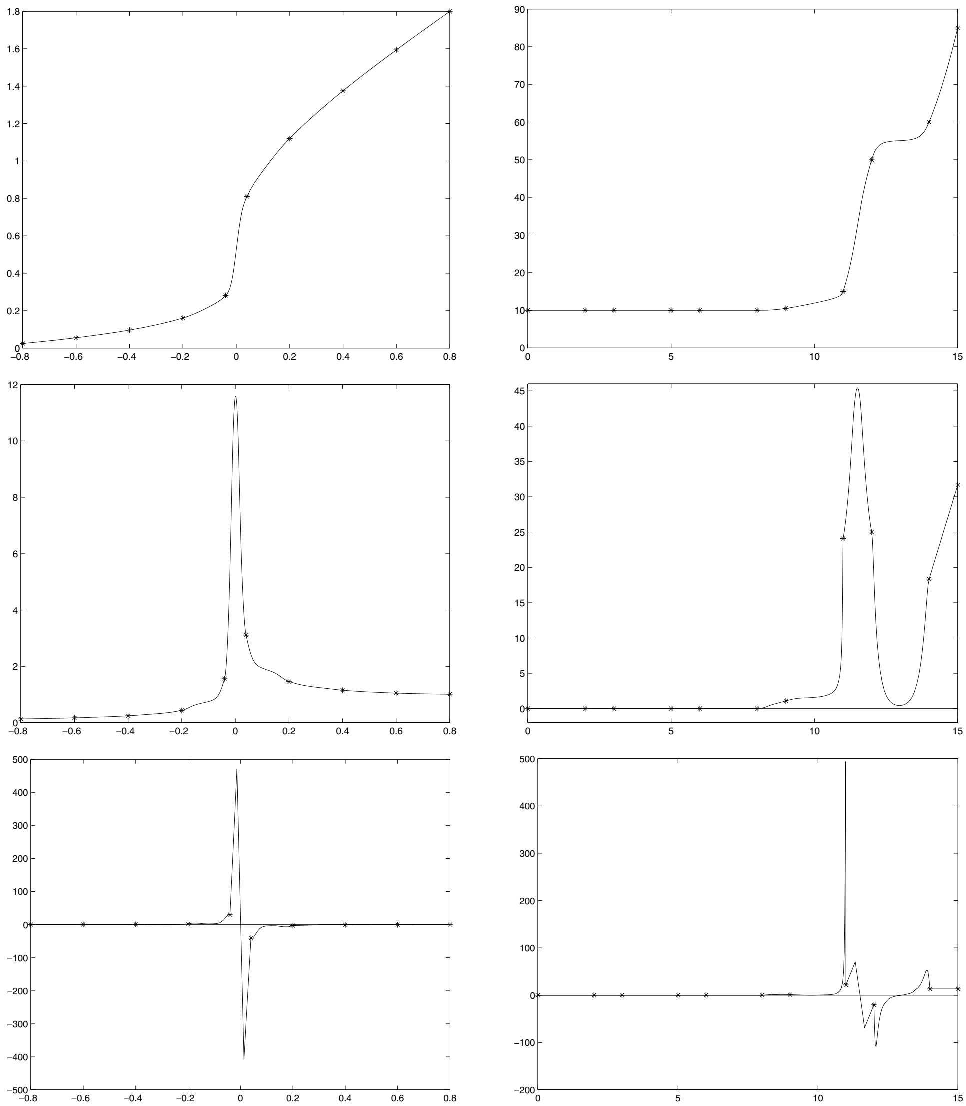


Figure 5.1: Top to bottom the interpolating function, the first derivative, and the second derivative. Left: Example 1. Right: Example 2.

control the shape of the interpolant, can be easily introduced in the Comonotonicity and/or Coconvexity Algorithms. In the Coconvexity Algorithm, for example, we can project (h_i, h_i) onto regions larger than $\mathcal{SC}_i$ while still contained in $\mathcal{C}_i$. This approach is similar to that used in [17], and, from Remark 4.4, it does not reduce the approximation order. Similar considerations can be made concerning the Comonotonicity Algorithm.

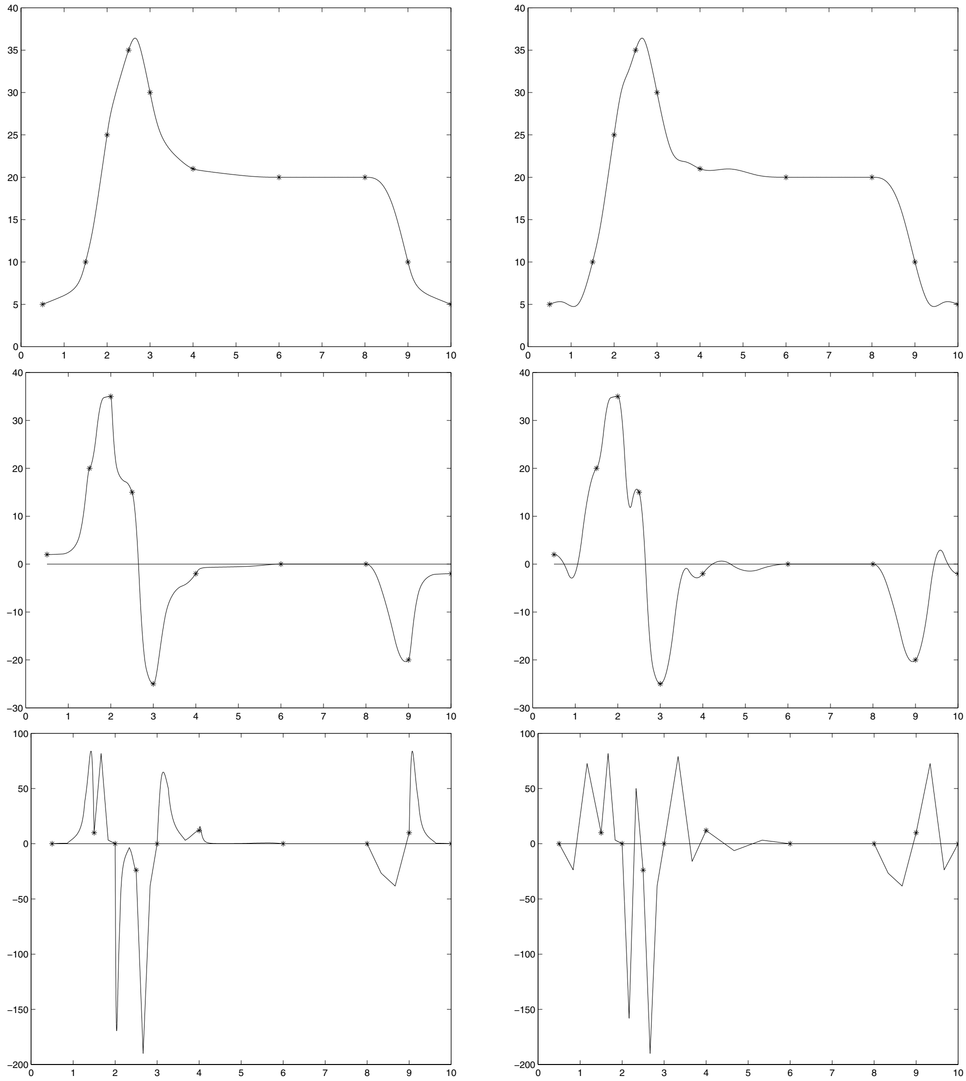


Figure 5.2: Example 3: left the constrained interpolant $s(x)$, right the piecewise cubic unconstrained interpolant. Top to bottom the interpolating function, the first derivative and the second derivative.

Fairness functionals can also be considered to improve the shape of s; see [20].

In the following examples the tension parameters have been computed according the Comonotonicity and/or Coconvexity Algorithm by using the MATLAB Optimization Toolbox.

In the first example we consider the data from the following function, [5]:

$$f(x) = \frac{1 + (x+\alpha)^{\frac{1}{3}}}{2 - (x+\alpha)^{\frac{1}{3}}}, \quad \alpha = 10^{-4},$$

at the abscissas:

$$x_i: \quad -0.8 \quad -0.6 \quad -0.4 \quad -0.2 \quad -0.04 \quad 0.04 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8\ .$$

Figure 5.1, left column, shows the interpolant produced by the proposed scheme and its first and second derivatives. The following table shows the ratios between the computed tension parameters and the grid steps:

$\frac{h_i^{(0)}}{h_i}$:	0.9875	0.9749	0.9313	0.8247	1	0.6531	0.9388	0.9736	0.9783
$\frac{h_i^{(1)}}{h_i}$:	0.9890	0.9792	0.9359	0.6270	1	0.8257	0.9345	0.9745	0.9849

In the second example we consider the classical Akima data, [1],

x_i :	0	2	3	5	6	8	9	11	12	14	15
f_i :	10	10	10	10	10	10	10.5	15	50	60	85

where we have computed the derivatives at each point following the Bessel scheme and have imposed zero derivatives at both ends of the intervals of constant data: Figure 5.1, right column, shows the obtained interpolant and its first and second derivatives.

Finally in the last example we deal with the following data similar to those considered in [22]:

x_i :	0.5	1.5	2	2.5	3	4	6	8	9	10
f_i :	5	10	25	35	30	21	20	20	10	5
f_i' :	2	20	35	15	-25	-2	0	0	-20	-2
f_i'' :	0	10	0	-24	0	12	0	0	10	0

The left column of Figure 5.2 shows the obtained interpolant and its first and second derivatives while in the right column we can see the behavior of the unconstrained piecewise cubic with knots at x_i, $x_i + \frac{h_i}{3}$, $x_{i+1} - \frac{h_i}{3}$, x_{i+1}, interpolating the data.

REFERENCES

1. H. Akima, *A new method of interpolation and smooth curve fitting based on local procedures*, J. Assoc. Comput. Mech., 17 (1970), pp. 589–602.
2. J. C. Archer and E. Le Gruyer, *Two shape preserving Lagrange C^2-interpolants*, Numer. Math., 64 (1993), pp. 1–11.
3. R. K. Beatson, *Convex approximation by splines*, SIAM J. Math. Anal., 12 (1981), pp. 549–559.
4. J. C. Clements, *A Convexity preserving C^2 parametric rational cubic interpolation*, Numer. Math., 63 (1992), pp. 165–171.

5. C. Conti and R. Morandi, *Piecewise C^1 shape preserving Hermite interpolation*, Computing 56 (1996), pp. 323–341.
6. P. Costantini, *An algorithm for computing shape-preserving interpolating splines of arbitrary degree*, J. Comput. Appl. Math., 22 (1988), pp. 89–136.
7. R. Delbourgo and J. A. Gregory, *C^2 rational quadratic spline interpolation to monotonic data*, IMA J. Numer. Anal., 3 (1983), pp. 141–152.
8. R. De Vore, *Monotone Approximation by splines*, SIAM J. Numer. Anal., 8 (1977), pp. 891–905.
9. R. L. Dougherty, A. Edelman, and J. M. Hyman, *Nonnegativity, monotonicity or convexity-preserving cubic and quintic Hermite interpolation*, Math. Comp., 52 (1989), pp. 471–494.
10. A. Edelman and C. A. Micchelli, *Admissible slopes for monotone and convex interpolation*, Numer. Math., 51 (1987), pp. 441–458.
11. F. G. Farin, *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, New York, 1993.
12. T. N. T. Goodman, *Total positivity and the shape of curves*, in Total Positivity and its Applications, M. Gasca and C. A. Micchelli, eds., Kluwer, Dordrecht, 1996, pp. 157–186.
13. T. N. T. Goodman and K. Unsworth, *Shape preserving interpolation by parametrically defined curves*, SIAM J. Numer. Anal., 25 (1988), pp. 1451–1465.
14. J. A. Gregory and R. Delbourgo, *Piecewise rational quadratic interpolation to monotonic data*, IMA J. Numer. Anal., 2 (1982), pp. 123–130.
15. A. Lahtinen, *Shape preserving interpolation by quadratic splines*, J. Comput. Appl. Math., 29 (1990), pp. 15–24.
16. C. Manni, *C^1 comonotone Hermite interpolation via parametric cubics*, J. Comput. Appl. Math., 69 (1996), pp. 143–157.
17. C. Manni, *Parametric shape preserving Hermite interpolation by piecewise quadratics*, in Advanced Topics in Multivariate Approximation, F. Fontanella, K. Jetter, and P. J. Laurent, eds., World Scientific, Singapore, 1996, pp. 211–226.
18. C. Manni and P. Sablonnière, *Monotone interpolation of order 3 by C^2 cubic splines*, IMA J. Numer. Anal., 17 (1997), pp. 305–320.
19. B. Mulansky and M. Neamtu, *Interpolation and approximation from convex sets*, Tech. report MATH-NM-02-1996, Technische Universität Dresden, Germany, 1996.
20. B. Mulansky and J. W. Schmidt, *Constructive methods in convex C^2 interpolation using quartic splines*, Numer. Algorithms, 12 (1996), pp. 111–124.
21. P. Oja, *Low degree rational spline interpolation*, BIT, 37 (1997), pp. 901–909.
22. S. Pruess, *Shape preserving C^2 cubic spline interpolation*, IMA J. Numer. Anal., 13 (1993), pp. 493–507.
23. J. W. Schmidt, W. Heß, *An always successful method in univariate convex C^2 interpolation*, Numer. Math., 71 (1995), pp. 237–252.
24. L. L. Schumaker, *On shape preserving quadratic spline interpolation*, SIAM J. Numer. Anal., 20 (1983), pp. 854–864.