DOI 10.1007/s10958-015-2422-5
Journal of Mathematical Sciences, Vol. 208, No. 1, July, 2015

EFFICIENT ALGORITHM FOR RECOGNIZING THE NIELSEN–THURSTON TYPE OF A THREE–STRAND BRAID

O. N. Biryukov
Moscow State Region Institute of Social Studies and Humanities
30, Zelenaya St., Kolomna 140410, Russia
oleg_biryukov@mail.ru

UDC 517.9

We propose an efficient algorithm for recognizing the Nielsen–Thurston type of a braid in the braid group B_3*. The algorithm has linear complexity on the input word length.* *Bibliography*: 11 *titles. Illustrations*: 5 *figures.*

Introduction

We consider a compact oriented two-dimensional surface M of zero genus with $n+1$ boundary components (a two-dimensional disk with removed n open disks from the interior). The group of homeotopies of M that leave fixed one of the boundary components (for example, the exterior boundary of the disk) is isomorphic to the classical Artin group B_n of n-strand braids admitting the representation

$$B_n = \langle\, \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i,\ 1 < i+1 < j < n,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \leqslant i \leqslant n-2 \,\rangle.$$

The surface M is hyperbolic if $n \geqslant 2$. According to the Nielsen–Thurston classification, homeotopies of hyperbolic surfaces are divided in groups of three types: periodic, pseudo-Anosov, or reducible. Respectively, similar types are introduced for braids and the problem of recognizing the type of a given braid arises.

There are several approaches (cf. [1]–[5]) to solution of this problem, based on either train-tracks or some modifications of summit sets. However, the efficiency (polynomial complexity) on the length of input word in the classical Artin generators σ_i or on the number of strands has not been established for these algorithms.

Efficient (on the input word length) algorithms for recognizing the type of a given braid are described in [6, 7], but these algorithms are also based on the use of summit sets (more exactly, supersummit sets). For these algorithms the operation time is estimated by $O(l^2)$ in the case of four strands and $O(l^3)$ in the case of an arbitrary number of strands, where l denotes the length of input word in the classical Artin generators σ_i.

In this paper, we propose an efficient (on the input word length) algorithm for recognizing

Translated from *Problemy Matematicheskogo Analiza* **79**, March 2015, pp. 45-53.

the type of a given three-strand braid. This algorithm is based on neither train-tracks nor summit sets. The main idea of the algorithm is to represent a three-strand braid as a motion of one strand about the other two and to encode the motion in a special way such that to provide an efficient recognition of the type of a given braid.

The central result of the paper is presented by Theorem 2.2 below.

1 The Braid Group and the Nielsen–Thurston Classification

In this paper, we consider the classical group of Artin braids (cf. [8, 9]). We recall some definitions. The *group B_n of n-strand braids* is defined by

$$B_n = \langle\, \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i,\ 1 < i+1 < j < n,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \leqslant i \leqslant n-2 \,\rangle.$$

The braid group is connected with groups of classes of maps (homeotopies) of two-dimensional surfaces. Namely, let M be a compact oriented two-dimensional surface of zero genus with $n+1$ boundary components $\gamma_0, \gamma_1, \dots, \gamma_n$ (a two-dimensional disk with n open disks removed from the interior). There is a known homomorphism from the braid group B_n to the group of homeotopies of the surface M

$$\varphi : B_n \to \mathrm{Homeot}\,(M) \tag{1.1}$$

which sends the generator σ_i, $i = 1, 2, \dots, n-1$, to the half Dehn twist along a curve enclosing the boundary components γ_i and γ_{i+1}.

For the visual representation of braids one uses a schematic drawing in the form of a collection of ascending curves, called *strands* of a braid. The strands of a braid β show the motion of the boundary components $\gamma_1, \dots, \gamma_n$ under the isotopy from the identity homeomorphism to a homeomorphism of a two-dimensional disk determined by the class of maps $\varphi(\beta)$ of the surface M.

Each braid induces a rearrangement of the set of boundary components $\gamma_1, \dots, \gamma_n$, which yields a homomorphism from the braid group to the symmetric group

$$B_n \to S_n. \tag{1.2}$$

The kernel of this homomorphism is called the *group of colored n-strand braids*.

A particular role in the braid group is played by the *Garside braid* Δ which is a positive half-twist of all braid strands:

$$\Delta = (\sigma_1\sigma_2 \dots \sigma_{n-1})(\sigma_1\sigma_2 \dots \sigma_{n-2}) \dots (\sigma_1\sigma_2)(\sigma_1).$$

The group B_1 is trivial. Therefore, we consider the case $n \geqslant 2$. Then the surface M is hyperbolic since its Euler characteristic is negative: $\chi(M) = 1 - n < 0$.

There exists the Nielsen–Thurston classification of elements of the group of classes of maps of a compact oriented hyperbolic surface (cf. [10, 11]). According to the Nielsen–Thurston classification, in each class there exists a representative of only one of the following three types: periodic, pseudo-Anosov, and reducible. According to the representative type, the classes are divided into periodic, pseudo-Anosov, and reducible ones.

Similarly, a braid $\beta \in B_n$ is said to be *periodic, pseudo-Anosov*, or *reducible* if its image under the homomorphism (1.1) belongs to the corresponding class.

Since Δ^2 generates the kernel of the homomorphism (1.1), the multiplication from the left of a braid by Δ^{2k}, $k \in \mathbb{Z}$, does not change the Nielsen–Thurston type of the braid.

We note that the braid group contains periodic braids in spite of the fact that it is a torsion-free group. Periodic braids are exactly those braids β for which there exists a natural number k such that β^k belongs to the kernel of the homomorphism φ. Thus, for example, the braid $\sigma_1\sigma_2 \in B_3$ is periodic since $(\sigma_1\sigma_2)^3 = \Delta^2 \in \mathrm{Ker}\ \varphi$.

There is an efficient algorithm for recognizing periodic braids (cf. [2]). Regarding pseudo-Anosov and reducible braids, there are several recognition methods [1]–[7] using either train-tracks or modifications of summit sets. In this paper. we propose a principally different method for recognizing the type of a three-strand braid.

2 Efficient Recognition of Pseudo–Anosov and Reducible Three–Strand Braids

Let β be a reducible three-strand braid. The reduction system of circles for such a braid consists of a single component surrounding two of three boundary components. The third component of the boundary is left invariant by the braid. Hence the following assertion holds.

Theorem 2.1. *A nonperiodic three-strand braid generating a transitive permutation on the set of strands* (*i.e., the permutation of a single cycle of length* 3) *is pseudo-Anosov.*

We note that the transitivity of permutation is sufficient, but not necessary for a braid to be pseudo-Anosov.

For the sake of definiteness we assume that the reducing circle of a reducible braid β surrounds the boundary components γ_1 and γ_2. In this case, we show that the braid β can be represented as a motion of the strand corresponding to the invariant component γ_3 around the other two strands which can be regarded as fixed ones.

We represent the surface M as a two-dimensional disk in the Euclidean plane and assume that the boundary components γ_1, γ_2, and γ_3 are horizontally arranged in a series (cf. Figure 1).

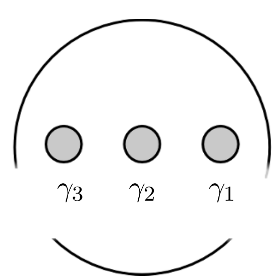

FIGURE 1.

We say that a strand is located at the left (centrally or at the right respectively) if it corresponds to the component γ_1 (γ_2 or γ_3 respectively). Thus, we start by tracing the motion of the strand located at the right.

If the traced strand is located at the right, then we set $u := \sigma_2\sigma_1$ and $r := \sigma_2\sigma_2$. The motion of the traced stand is presented in Figure 2. The strand is displaced leftwards after application of the braid u or remains at the right after application of the braid r.

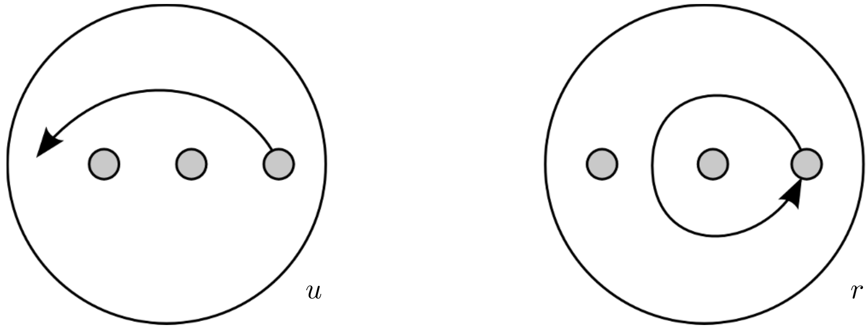

FIGURE 2.

If the traced strand is located at the left, then we set $d := \sigma_1\sigma_2$ and $l := \sigma_1\sigma_1$. The motion of the traced strand is shown in Figure 3. The strand is displaced rightwards under the action of the braid d or remains at the left if the braid l is applied.

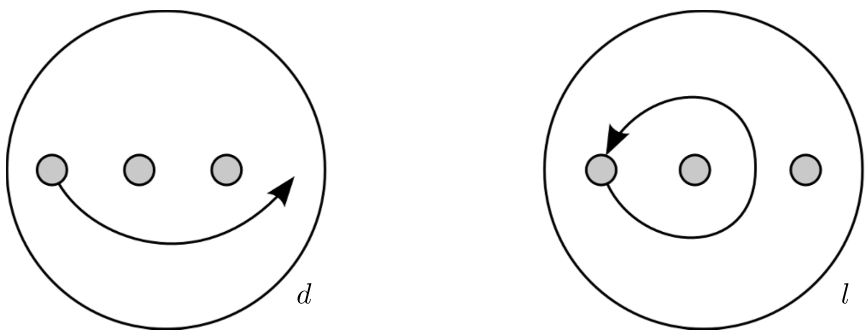

FIGURE 3.

If the traced strand is located at the right and the braid σ_1 is applied, then this means the motion of the first two strands located inside the reducing circle. We encode this situation as follows: $\sigma_1 = d^{-1}\Delta$. The goal of this encoding is that the strand located outside the reducing circle also moves, but additional factors of Δ arise. In a similar situation, but with the traced strand located at the left we set $\sigma_2 = u^{-1}\Delta$. We also set $a := \sigma_2\sigma_1^{-1}$ and $b := \sigma_1^{-1}\sigma_2$. The motion of the traced strand is shown in Figure 4.

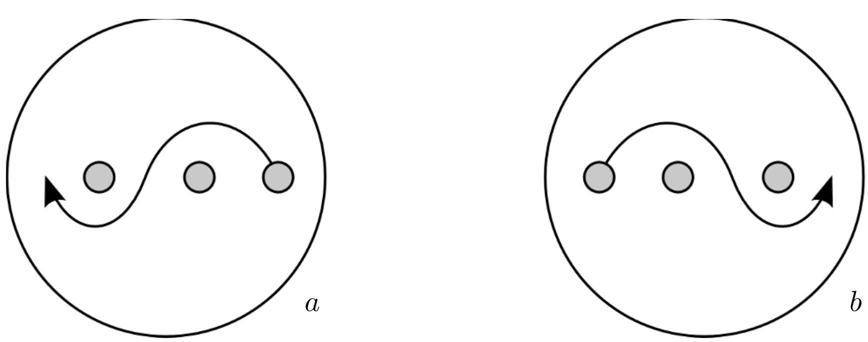

FIGURE 4.

Encoding the braid, we also trace the position of an invariant strand by assigning the subscript "l" if the invariant strand is located at the left or "r" if the invariant strand is located at the right. Initially, the subscript "l" or "r" is placed before the first letter of a word. For example if the invariant strand is initially located at the right, then

$$\beta = {}_{\mathrm{r}}\sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_k}^{\varepsilon_k}.$$

Travelling along the generators, we write a braid as a word in the alphabet

$$\mathscr{F} = \{u^{\pm 1}, d^{\pm 1}, r^{\pm 1}, l^{\pm 1}, a^{\pm 1}, b^{\pm 1}, \Delta^{\pm 1}\}$$

with the help of the following transformations:

$$\begin{gathered} {}_{\mathrm{r}}\sigma_1 = d^{-1}\Delta_{\mathrm{r}}, \quad {}_{\mathrm{r}}\sigma_1^{-1} = u\Delta^{-1}{}_{\mathrm{r}}, \quad {}_{\mathrm{r}}\sigma_2\sigma_1 = u_{\mathrm{l}}, \quad {}_{\mathrm{r}}\sigma_2\sigma_1^{-1} = a_{\mathrm{l}}, \quad {}_{\mathrm{r}}\sigma_2\sigma_2 = r_{\mathrm{r}}, \quad {}_{\mathrm{r}}\sigma_2^{-1}\sigma_1 = b^{-1}{}_{\mathrm{l}}, \\ {}_{\mathrm{r}}\sigma_2^{-1}\sigma_1^{-1} = d^{-1}{}_{\mathrm{l}}, \quad {}_{\mathrm{r}}\sigma_2^{-1}\sigma_2^{-1} = r^{-1}{}_{\mathrm{r}}, \quad {}_{\mathrm{l}}\sigma_2 = u^{-1}\Delta_{\mathrm{l}}, \quad {}_{\mathrm{l}}\sigma_2^{-1} = d\Delta^{-1}{}_{\mathrm{l}}, \quad {}_{\mathrm{l}}\sigma_1\sigma_2 = d_{\mathrm{r}}, \\ {}_{\mathrm{l}}\sigma_1\sigma_2^{-1} = a^{-1}{}_{\mathrm{r}}, \quad {}_{\mathrm{l}}\sigma_1\sigma_1 = l_{\mathrm{l}}, \quad {}_{\mathrm{l}}\sigma_1^{-1}\sigma_2 = b_{\mathrm{r}}, \quad {}_{\mathrm{l}}\sigma_1^{-1}\sigma_2^{-1} = u^{-1}{}_{\mathrm{r}}, \quad {}_{\mathrm{l}}\sigma_1^{-1}\sigma_1^{-1} = l^{-1}{}_{\mathrm{l}}. \end{gathered}$$

It is easy to see that these transformations exhaust all possible locations of the letters $\sigma_1^{\pm 1}$, $\sigma_2^{\pm 1}$ and the invariant strand. Indeed, if a current position of this invariant strand is "at the right," then the situation where ony one letter $\sigma_2^{\pm 1}$ is left in the braid is impossible. Similarly, if this strand is in the position "at the left," the letter $\sigma_1^{\pm 1}$ cannot be left single.

Example 2.1. We represent the braid $\sigma_2\sigma_1^3\sigma_2\sigma_1^{-1}\sigma_2$ as a word in the alphabet $\mathscr{F}$. This braid interchanges the left and central strands and leaves invariant the right strand. Therefore, we write the braid in the alphabet $\mathscr{F}$ by tracing the motion of the right strand. Initially, we have ${}_{\mathrm{r}}\sigma_2\sigma_1^3\sigma_2\sigma_1^{-1}\sigma_2$. By the above-listed transformations, ${}_{\mathrm{r}}\sigma_2\sigma_1$ is replaced with u_{l}: $u_{\mathrm{l}}\sigma_1^2\sigma_2\sigma_1^{-1}\sigma_2$. Then ${}_{\mathrm{l}}\sigma_1^2$ is replaced with l_{l} and so on,

$$ul_{\mathrm{l}}\sigma_2\sigma_1^{-1}\sigma_2 = ulu^{-1}\Delta_{\mathrm{l}}\sigma_1^{-1}\sigma_2 = ulu^{-1}\Delta b_{\mathrm{r}}.$$

Proposition 2.1. *Any three-strand braid can be written as a word in the alphabet* $\mathscr{F} = \{u^{\pm 1}, d^{\pm 1}, r^{\pm 1}, l^{\pm 1}, a^{\pm 1}, b^{\pm 1}, \Delta^{\pm 1}\}$ *in the time* $O(l)$*, where* l *denotes the length of an input word in the classical Artin generators* σ_i*.*

Proof. We assume that a braid β leaves invariant the left or right strand. Using the above transformations, we can write the braid in the alphabet $\mathscr{F}$ by tracing the motion of the invariant strand. Moreover, it is obvious that to transform an input word, it suffices to go entirely through this word one time. Therefore, the complexity of such a transformation is $O(l)$. Furthermore, a three-strand braid can be represented by a word in the alphabet $\mathscr{F}$ if, under the action of this braid, the left (right) strand goes to the right (left) strand. Since any three-strand braid possesses this property, we conclude that any three-strand braid can be represented as a word in the alphabet $\mathscr{F}$. $\square$

Remark 2.1. If a braid leaves invariant the central strand and either leaves invariant the left and right strands or transforms them into each other, then the braid can be represented as a word in the alphabet $\mathscr{F}$ by tracing the motion of the left or right strand.

For words in the alphabet $\mathscr{F}$ we will transfer all the letters $\Delta^{\pm 1}$ to the end of the word. For this purpose we consider how the inner automorphism τ (conjugation by the Garside braid) acts on elements of the $\mathscr{F}$:

$$u \overset{\tau}{\longleftrightarrow} d, \quad r \overset{\tau}{\longleftrightarrow} l, \quad a \overset{\tau}{\longleftrightarrow} a^{-1}, \quad b \overset{\tau}{\longleftrightarrow} b^{-1}.$$

Now, for transferring the letters $\Delta^{\pm 1}$ to the word end, we can use the equality

$$\Delta^{\pm 1}\beta = \tau(\beta)\Delta^{\pm 1}.$$

Proposition 2.2. *In the representation of braids in the form of words in the alphabet $\mathscr{F}$ the following relations arise*:

$$\begin{aligned} &ul^{-1} = a, \quad ub = r, \quad dr^{-1} = a^{-1}, \quad db^{-1} = l, \quad rd^{-1} = a, \quad rb^{-1} = u, \\ &lu^{-1} = a^{-1}, \quad lb = d, \quad ad = r, \quad al = u, \quad br^{-1} = u^{-1}, \quad bd^{-1} = l^{-1}, \end{aligned} \tag{2.1}$$

which allow us to diminish the word length to the minimal one in the time $O(l)$, where l denotes the length of input word in the alphabet $\mathscr{F}$.

Proof. The relations (2.1) are proved by a direct computation, for example,

$$ul^{-1} = \sigma_2\sigma_1 \cdot (\sigma_1\sigma_1)^{-1} = \sigma_2\sigma_1^{-1} = a.$$

It is easy to verify that only the relations (2.1) allow us to diminish the word length. Say, any two-letter expression started with the letter u, except for ul^{-1} and ub, cannot be reduced to a single letter. Indeed, $u = \sigma_2\sigma_1$ can be cancelled only with the letters started with σ_1^{-1}, i.e., $l^{-1} = \sigma_1^{-1}\sigma_1^{-1}$ and $b = \sigma_1^{-1}\sigma_2$. Further, from the relations (2.1) we can easily obtain two-letter expressions started with the letters u^{-1}, d^{-1}, r^{-1}, l^{-1}, a^{-1}, and b^{-1}.

It is obvious that to reduce the input word, it suffices to go through this word one time by transferring $\Delta^{\pm 1}$ to the end and applying relations of the form $uu^{-1} = 1$ and (2.1). After each application of such a relation, the word length is diminished by 1 or 2 letters. Therefore, the complexity of reducing a word is $O(l)$. □

Example 2.2. We consider the word $ulu^{-1}\Delta b$. Transferring the letter Δ to the word end (then b is replaced with b^{-1}), we get $ulu^{-1}b^{-1}\Delta$. Using the relation $lu^{-1} = a^{-1}$, we reduce the obtained word to the form $ua^{-1}b^{-1}\Delta$.

We say that a word w in the alphabet $\mathscr{F}$ defining a braid β is written in the *canonical form* if all the letters Δ are placed at the end of the word and the length of w is minimal, i.e., relations of the form $uu^{-1} = 1$ or (2.1) applyed to the word cannot diminish the length.

A word in the alphabet $\mathscr{F}$ is called a *ud-word* if it consists of alternating letters u and d or alternating letters u^{-1} and d^{-1} and, possibly, powers of the Garside braid Δ at the end.

Example 2.3. The words $dudud$, $udu\Delta$, $d^{-1}u^{-1}\Delta^4$, $u\Delta$, Δ are ud-words, whereas the words $dduu$, $dud^{-1}\Delta$, $du\Delta du$ ar not ud-words.

It is easy to see that ud-words have the canonical form.

Proposition 2.3. *A nonperiodic braid $\beta \in B_3$ is reducible if and only if there exists a conjugate braid represented by a ud-word in the alhpabet $\mathscr{F}$.*

Proof. *Necessity* Let a braid β be reducible. Passing to the conjugate braid, we can always reach the situation where the reducing circle is located in the canonical way (cf. Figure 5).

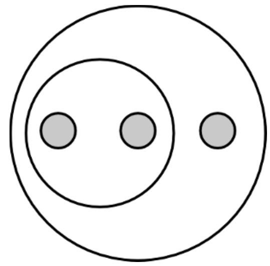

FIGURE 5.

Such a braid β can be regarded as the composition of the twisting of the first two strands, which is written as σ_1^k for some integer k, and the winding of the third strand around the other two, which is written as $(\sigma_2\sigma_1^2\sigma_2)^m$ for some integer m:

$$\beta = \sigma_1^k(\sigma_2\sigma_1^2\sigma_2)^m.$$

Moreover, $k \neq 2m$; otherwise, the braid β is periodic. The invariant strand is initially located at the right. Therefore, it can be written as $\beta = (d^{-1}\Delta)^k(ud)^m$ in the alphabet $\mathscr{F}$. Reducing to the canonical form, we obtain the ud-word

$$\beta = \begin{cases} \underbrace{udud\ldots}_{2m-k}\Delta^k, & 2m > k, \\ \underbrace{d^{-1}u^{-1}d^{-1}u^{-1}\ldots}_{k-2m}\Delta^k, & k > 2m. \end{cases}$$

Suffiency. A braid representable as a ud-word in the alphabet $\mathscr{F}$ is reducible with the reducing circle located in the canonical way. Then any conjugate brain is also reducible. $\square$

Lemma 2.1. *Let two brains be given by words in the canonical form in the alphabet $\mathscr{F}$:*

$$w = x_1x_2\ldots x_p\Delta^k, \quad z = y_1y_2\ldots y_q\Delta^s, \quad x_i, y_j \in \mathscr{F}\backslash\{\Delta\}, \quad k, s \in \mathbb{Z},$$

and let $x_p^{-1} \neq \tau^k(y_1)$. Then, in the process of reducing the word wz to the canonical form only one cancellation can occur, namely, two letters x_p and $\tau^k(y_1)$ can be reduced to a single one.

Proof. In the word wz, we transfer all the letters Δ to the word end

$$wz = x_1x_2\ldots x_p\tau^k(y_1)\tau^k(y_2)\ldots\tau^k(y_q)\Delta^{k+s}$$

and represent the obtained word in the alphabet $\{\sigma_1^{\pm1}, \sigma_2^{\pm1}\}$. Since the words w and z are written in the canonical form, cancellation can occur only at the place of joining the letters x_p and $\tau^k(y_1)$ by transforming two letters to one. It is obvious that further cancellations are impossible, which proves the required assertion. $\square$

Proposition 2.4. *A nonperiodic braid $\beta \in B_3$ given by a word w in the canonical form in the alphabet $\mathscr{F}$ is reducible if and only if the word w can be divided into two subwords $w = w_1w_2$ in such a way that the word w_2w_1 reduced to the canonical form is a ud-word.*

Proof. *Necessity.* By Proposition 2.3, a reducible braid β can be represented as $\beta = \gamma^{-1}\beta'\gamma$, where the braid β' is written as a ud-word in the alphabet $\mathscr{F}$. Consequently, the braid β is written as the word $z^{-1}wz$ in the alphabet $\mathscr{F}$.

Without loss of generality we assume that the words z^{-1}, w, and z are written in the canonical form. We transfer all the letters Δ in the word $z^{-1}wz$ to the end. By Lemma 2.1, cancellations in the word $z^{-1}wz$ can occur only at the place of joining the subwords z^{-1} and w or at the place of joining the subwords w and z. Moreover, one can show that simultaneous cancellations at both places are impossible.

In the first case, we denote by w_1 the word $z^{-1}w$ reduced to the canonical form and set $w_2 = z$. In the second case, we set $w_1 = z^{-1}$ and denote by w_2 the subword of wz reduced to the canonical form. It is easy to show that the word w_2w_1 reduced to the canonical form is a ud-word.

Sufficiency. The braid defined by the ud-word w_2w_1 is obtained from the braid β corresponding to the word $w = w_1w_2$ by conjugation of the braid corresponding to the word w_1. Taking into account Proposition 2.3, we arrive at the required assertion. □

We formulate an assertion necessary for the fast recognition of periodic braids.

Proposition 2.5. *A braid $\beta \in B_3$ is an integer power of the Garside braid if and only if the word w in the alphabet $\mathscr{F}$ corresponding to the braid β after reducing to the canonical form, has the form Δ^k for some $k \in \mathbb{Z}$; moreover, this fact is independent of which* (*left or right*) *strand was traced for composing the word w.*

Proposition 2.5 is proved by a direct computation with induction on k.

Below, we describe the algorithm for recognizing the Nielsen–Thurston type of a braid β.

Step 1. Compute the permutation s generated by the braid β on the set of strands.

Step 2. If the permutation s is transitive, then write the braid β^3 in the alphabet $\mathscr{F}$ by tracing the motion of the right or left strand and reduce the obtained word to the canonical form. If a result is the word Δ^{2k} for some $k \in \mathbb{Z}$, then the braid β is periodic; otherwise, β is pseudo-Anosov.

Step 3. If the permutation s contains a cycle of length 2, then write the braid β^2 in the alphabet $\mathscr{F}$ by tracing the motion of the right or left strand and reduce the obtained word to the canonical form. If a result is the word Δ^{2k} for some $k \in \mathbb{Z}$, then the braid β is periodic; otherwise, pass to Step 5.

Step 4. If the permutation s is trivial, then write the braid β in the alphabet $\mathscr{F}$ by tracing the motion of the right or left strand and reduce the obtained word to the canonical form. If a result is the word Δ^{2k} for some $k \in \mathbb{Z}$, then the braid β is periodic, otherwise, pass to Step 5.

Step 5. Find an invariant element of the permutation s. Tracing the motion of the strand corresponding to this element, perform Steps 6 and 7 for the braid β. If the permutation is trivial, then perform Steps 6 and 7 for each strand of the braid β. If the invariant strand is located centrally, then perform Steps 6 and 7 for the braid $\sigma_1^{-1}\beta\sigma_1$ the invariant strand of which is located at the left.

Step 6. Write the braid in the alphabet $\mathscr{F}$ and reduce the obtained word to the canonical form.

Step 7. Transfer the first letters to the word end and thereby pass to the conjugate brains. Reduce the obtained word to the canonical form. If a result is a *ud*-word, then the braid β is reducible. If the word is scrolled entirely, but we do not obtain a *ud*-word, then the braid is pseudo-Anosov.

Remark 2.2. If the permutation at Step 5 is trivial, then Steps 6 and 7 should be performed for each of the three strands. If at Step 7, we obtain a *ud*-word, then the braid β is reducible; otherwise, the braid is pseudo-Anosov.

Theorem 2.2 (the main result)**.** *For the braid group B_3 there exists an algorithm for recognizing the Nielsen–Thurston type of a braid with the operation time $O(l)$, where l is the length of the input word in the classical Artin generators σ_i.*

Proof. An algorithm for recognizing the type of a braid was described above. The well-posedness and the linear complexity on the input word length follow from Theorem 2.1 and Propositions 2.1–2.5. □

Example 2.4. We consider the braid $\beta = (\sigma_1\sigma_2^{-1})^{-1}\sigma_1^3(\sigma_1\sigma_2^{-1}) = \sigma_2\sigma_1^3\sigma_2^{-1}$. It is obvious that β is reducible since it is obtained from the reducible braid σ_1 by conjugation of the pseudo-Anosov braid $\sigma_1\sigma_2^{-1}$. We apply the above algorithm to the braid β.

The braid β generates a permutation containing the cycle (13). Therefore, we write the braid β^2 in the alphabet $\mathscr{F}$ by tracing the motion of the right strand:

$$ {}_{\mathrm{r}}\sigma_2\sigma_1^6\sigma_2^{-1} = u_{\mathrm{l}}\sigma_1^5\sigma_2^{-1} = ul_{\mathrm{l}}\sigma_1^3\sigma_2^{-1} = ull_{\mathrm{l}}\sigma_1\sigma_2^{-1} = ulla^{-1}. $$

It is seen that the braid β is not periodic.

The braid β interchanges the first and third strands and leaves invariant the central strand. Therefore, we pass to conjugation by the braid σ_1:

$$ \sigma_1^{-1}\beta\sigma_1 = \sigma_1^{-1}\sigma_2\sigma_1^3\sigma_2^{-1}\sigma_1. $$

The invariant strand is placed at the left. We write the braid β in the alphabet $\mathscr{F}$ by tracing the motion of the left strand:

$$ \begin{aligned} {}_{\mathrm{l}}\sigma_1^{-1}\sigma_2\sigma_1^3\sigma_2^{-1}\sigma_1 &= b_{\mathrm{r}}\sigma_1^3\sigma_2^{-1}\sigma_1 = bd^{-1}\Delta_{\mathrm{r}}\sigma_1^2\sigma_2^{-1}\sigma_1 = bd^{-1}\Delta d^{-1}\Delta_{\mathrm{r}}\sigma_1\sigma_2^{-1}\sigma_1 \\ &= bd^{-1}\Delta d^{-1}\Delta d^{-1}\Delta_{\mathrm{r}}\sigma_2^{-1}\sigma_1 = bd^{-1}\Delta d^{-1}\Delta d^{-1}\Delta b^{-1}. \end{aligned} $$

We reduce the word to the canonical form

$$ bd^{-1}\Delta d^{-1}\Delta d^{-1}\Delta b^{-1} = bd^{-1}u^{-1}d^{-1}b\Delta^3 = l^{-1}u^{-1}d^{-1}b\Delta^3. $$

We transfer the first letter to the end and reduce to the canonical form

$$ u^{-1}d^{-1}b\Delta^3 l^{-1} = u^{-1}d^{-1}br^{-1}\Delta^3 = u^{-1}d^{-1}u^{-1}\Delta^3. $$

As a result, we obtain a *ud*-word. Therefore, according to the algorithm, the braid is reducible.

References

1. M. Bestvina and M. Handel, "Train-tracks for surface homeomorphisms" *Topology*, **34**, No. 1, 109–140 (1995)
2. D. Bernardete, Z. Nitecki, and M. Gutierrez, "Braids and the Nielsen–Thurston classification," *J. Knot Theory Ramificat.* **4**, 549–618 (1995).
3. J. González-Meneses and B. Wiest, "Reducible braids and Garside theory," *Algebr. Geom. Topol.* **11**, No. 5, 2971–3010 (2011).
4. E.-K. Lee and S.-J. Lee, "A Garside-theoretic approach to the reducibility problem in braid groups," *J. Algebra* **320**, No. 2, 783–820 (2008).
5. J. E. Los, "Pseudo-Anosov maps and invariant train tracks in the disc: A finite algorithm," *Proc. Lond. Math. Soc., III* **66**, No. 2, 400–430 (1993).
6. M. Calvez, *Fast Nielsen–Thurston Classification of Braids*, *Algebr. Geom. Topol.* **14**, No. 3, 1745–1758 (2014).
7. M. Calvez and B. Wiest, "Fast algorithmic Nielsen–Thurston classification of four-strand braids," *J. Knot Theory Ramifications* **21**, No. 5, 1250043 (2012).
8. J. S. Birman, *Braids, Links, and Mapping Class Groups*, Princeton Univ. Press, Princeton, N.J. (1974).
9. J. S. Birman, T. E. Brendle, "Braids: a survey," In: *Handbook of Knot Theory*, pp. 19–103, Elsevier, Amsterdam (2005).
10. W. Thurston, "On the geometry and dynamics of diffeomorphisms of surfaces," *Bull. Am. Math. Soc., New Ser.* **19**, No. 2, 417–431 (1988).
11. A. J. Casson and S. A. Bleiler, *Automorpohisms of Surfaces after Nielsen and Thurston*, Cambridge Univ. Press, Cambridge (1988).

Submitted on August 29, 2014