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Influence of finite particle temperature on a density distribution in beams

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Based on the gas dynamics description, the structural dynamics of a non-neutralized charged-particle beam is examined within limited spaces with reference to Lagrangian coordinates. The spatial behavior of density for beams with a finite particle temperature is analyzed. © *1999 American Institute of Physics.* [S1070-664X(99)02104-7]

Nonlinear phenomena in gas and plasma flows, as a rule, are described by relations having singular peculiarities.[1–6] The simplest possible example of such flow is focusing charged-particle beam.[7,8] The internal dynamics of cold sheet beams were studied in Lagrangian variables in Refs. 7 and 8, where it was shown that the beam density can become infinite at a certain point of the beam trajectory. However, this result holds true when the beam particles are assumed to be cold. In this Brief Communication we shall show that temperature effects will eliminate singularity.

We shall study the nonrelativistic, warm sheet charged-particle beam in vacuum propagating along the z axis. We consider the case when any spatial variation of the beam in the transverse direction y is neglected and the beam velocity in this direction is zero, so that the beam may be nonuniform only in the transverse direction x. The beam is assumed to be symmetric in the direction of transverse motion, x, with respect to the z axis at $x=0$. Thus it is sufficient to consider only the upper-half of the beam, where $x\geqslant 0$.

Since there is no self-consistent magnetic field of the beam, we can consider only the nonlinear electric field of the beam. The beam energy is assumed to be much higher than the space–charge potential energy. It is possible to suppose that all the beam particles have the same energy and, furthermore, transverse velocities, as a rule, are much smaller than axial particle velocities. Therefore all beam particles have the same constant axial velocity v_b along the beam trajectory. Also, the axial scale length is large compared to the traversed scale length, so that the gradient of the self-field for the beam in the z direction might be neglected. In this case we can use a time-like coordinate $t=z/v_b$ instead of z.

Then the dynamics of the two-dimensional sheet beam can be described by time-dependent, one-dimensional gas dynamic and Poisson's equations, i.e.,

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nU)=0, \tag{1}$$

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=-\frac{1}{Mn}\frac{\partial p}{\partial x}+\frac{q}{M}E, \tag{2}$$

$$\frac{\partial}{\partial t}\left[Mn\left(\frac{U^2}{2}+h\right)\right]+\frac{\partial}{\partial x}\left[MUn\left(\frac{U^2}{2}+w\right)\right]=qnUE, \tag{3}$$

$$\frac{\partial E}{\partial x}=4\pi qn, \tag{4}$$

where q and M are the charge and the mass of the beam particles, E is the electric field strength of the beam, and p is the particle pressure. We shall use a polytropic equation of state for the beam particles

$$p=p_0\left(\frac{n}{n_0}\right)^{\gamma}, \tag{5}$$

where γ is the polytropic exponent which lies in the interval $\gamma>1$. The internal energy h is described by the following relation:

$$h=\frac{pV}{\gamma-1}$$

and the specific enthalpy w is defined by

$$w=\frac{\gamma pV}{\gamma-1},$$

where $V=1/n$ is the specific volume.

We must supplement the above-given equations [(1)–(4)] with boundary conditions, following from the problem symmetry over $x=0$:

$$E(x=0,t)=0,\quad U(x=0,t)=0. \tag{6}$$

We shall consider the case when

$$N_b=\int_0^{\infty}n(x,t)\,dx<\infty, \tag{7}$$

where the half-width may be used for convenience, since we have assumed that density is an even function of coordinate.

Note that equivalent to Eq. (4) is Ampere's law in the electrostatic approximation

$$\frac{\partial E}{\partial t}=-4\pi qnU. \tag{8}$$

Using (8) we rewrite (3) in conservative form

$$\frac{\partial}{\partial t}\left[Mn\left(\frac{U^2}{2}+h\right)+\frac{E^2}{8\pi}\right]+\frac{\partial}{\partial x}\left[MUn\left(\frac{U^2}{2}+w\right)\right]=0. \tag{9}$$

Bearing in mind condition (7), we have that the quantity ϵ, defined by

$$\epsilon = \int_0^\infty \left[Mn\left(\frac{U^2}{2}+h\right)+\frac{E^2}{8\pi}\right]dx<\infty, \tag{10}$$

is constant in time.

In order investigate the structural dynamics of a warm, charged-particle beam we transform the system (1)–(4) from Eulerian to Lagrangian coordinates. First of all we reduce (1)–(4) to a simpler set in the Eulerian coordinates. It is convenient to use the total number of particles per unit length for $0\leqslant x'\leqslant x$,

$$N(x,t)=\int_0^x n(x',t)dx'. \tag{11}$$

Differentiating the relation (11) with respect to N, we get the following identity:

$$n\frac{\partial x}{\partial N}\equiv 1. \tag{12}$$

Taking into account the boundary condition for the electric field and using (12), Eqs. (1), (2), and (4) are reduced to

$$\frac{\partial N}{\partial t}+U\frac{\partial N}{\partial x}=0, \tag{13}$$

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=-\frac{1}{M}\frac{\partial p}{\partial N}+\frac{4\pi q^2}{M}N. \tag{14}$$

Applying the transformation[2]

$$\tau=t,\quad \xi=x-\int_0^t U(\xi,t')dt' \tag{15}$$

to Eqs. (13) and (14), we obtain

$$\frac{\partial}{\partial\tau}N(\xi,\tau)=0, \tag{16}$$

$$\frac{\partial U}{\partial\tau}=\frac{4\pi q^2}{M}N-\frac{1}{M}\frac{\partial p}{\partial N}. \tag{17}$$

In this case under the translation $V(\xi,\tau)=1/n(\xi,\tau)$ relation (12) becomes

$$\frac{\partial x}{\partial N}=V. \tag{18}$$

Then from (18), using

$$U(\xi,\tau)=\left(\frac{\partial x}{\partial\tau}\right)_\xi,$$

we find

$$\frac{\partial U}{\partial N}=\frac{\partial V}{\partial\tau}. \tag{19}$$

Differentiating the relation (17) with respect to N and using (16), we obtain

$$\frac{\partial^2 V}{\partial\tau^2}+a^2\frac{\partial^2}{\partial N^2}\left(\frac{1}{V}\right)^\gamma=\beta^2, \tag{20}$$

where $a=\sqrt{p_0/(Mn_0^\gamma)}$ and $\beta=\sqrt{4\pi q^2/M}$.

Let us consider an initial-value problem for Eq. (20). At $t=0$, where the beam is launched, we determine the initial beam density $n(x,t=0)=n_0(x)$ and the transverse velocity $U(x,t=0)=U_0(x)$. Here n_0 is an even, decreasing function and U_0 is an odd, decreasing function.

Using the functions n_0 and U_0, for Eq. (20) we can write the following conditions;

$$V(\tau=0,N)=V_0=1/n_0, \tag{21}$$

$$\left|\frac{\partial V}{\partial\tau}\right|_{\tau=0}=\left.\frac{\partial U}{\partial N}\right|_{\tau=0}=\left[\frac{\partial N}{\partial\xi}\right]^{-1}\left.\frac{\partial U}{\partial\xi}\right|_{\tau=0}=V_0\frac{\partial U_0}{\partial\xi}. \tag{22}$$

It should be noted that Eq. (20) belongs to a class of equations that mathematically admits singular solutions for both $a\neq 0$ and $\beta\neq 0$. In the general case such behavior is possible owing to the nonlinear character of the equations of continuity and momentum from which this equation has been derived.

As is known, equations of similar type have been used in order to investigate the nonlinear phenomena in the neutral gas[1] and the quasineutral plasma.[2–5] These phenomena correspond to a time-asymptotic state of a system which has been determined by a form of wave equation and initial conditions. In order to find a time-asymptotic state in the evolution of a system we can use a self-similar description[9] for the initial-value problem. However, the self-similar solutions of (20) with (21) and (22) have a physical bearing if and only if the solutions of (20) satisfy condition (10). Using the relation $dN=ndx$ we rewrite this condition in new coordinates (τ,N) as

$$\epsilon(\tau,N)=\int_0^\infty\left[M\left(\frac{U^2}{2}+h\right)+\frac{VE^2}{8\pi}\right]dN<\infty. \tag{23}$$

In particular, since all members in (23) are positive, we have the condition

$$H=\int_0^\infty\frac{pV}{\gamma-1}dN<\infty, \tag{24}$$

which will be useful in the analysis of Eq. (20). Thus, the form of the solutions having a physical bearing has been restricted by condition (24).

To illustrate some basic peculiarities of beam dynamics, we consider the initial-value problem for $a=0$. In this simplest case the general solution of (20) is

$$V(\tau,N)=\frac{4\pi q^2}{M}\frac{\tau^2}{2}+C_2\tau+C_1,$$

where C_1 and C_2 are an arbitrary constants. Then using (21), (22), we find

$$n(\xi,\tau)=n_0(\xi)\left[\frac{4\pi q^2 n_0}{M}\frac{\tau^2}{2}+\frac{\partial U_0}{\partial\xi}\tau+1\right]^{-1}. \tag{25}$$

As is seen from (25), for focusing beams the density may become infinite at certain values of ξ and τ. However, since $p_0\equiv 0$, condition (24) is fulfilled even if $n=\infty$.

We are now going to show that Eq. (20) for $a \neq 0$ has no solutions which acquire the zero value in a point $\tau = \tau_c$ ($\tau_c < \infty$) simultaneously satisfying the natural physical requirement (24).

In order to prove this we assume that there exists at least one particular point $\tau = \tau_c$, where $V \to 0$ for $\tau \to \tau_c$, and investigate the behavior of the solution for Eq. (20) near this point $\tau = \tau_c$ by self-similar theory.[9] Equation (20) is invariant under the transformation

$$\tau \to \alpha(\tau - \tau_c), \quad N \to \alpha^{-\gamma} N, \quad V \to \alpha^2 V, \tag{26}$$

where α is the parameter of transformation. The corresponding self-similar solution of Eq. (20) has the following form:

$$V(\tau,N) = (\tau - \tau_c)^2 \theta(\zeta), \quad \zeta = (\tau - \tau_c)^{\gamma} N. \tag{27}$$

Using (27) and (5), from (24) we find

$$H = Ma^2 (\tau - \tau_c)^{2-3\gamma} \int_0^\infty \frac{\theta^{1-\gamma}}{\gamma - 1} d\zeta < \infty. \tag{28}$$

Since

$$\int_0^\infty \frac{\theta^{1-\gamma}}{\gamma - 1} d\zeta \neq 0$$

Eq. (28) near $\tau = \tau_c$ is finite only for $\gamma \leqslant 2/3$. It is clear that the value $\epsilon(\tau,N)$ will be finite for the same γ. However, this case has no physical bearing since $\gamma \geqslant 1$. In case $\gamma > 2/3$, $\epsilon(\tau,N)$ becomes infinite. Consequently, this unphysical feature of the self-similar solution in the form (27) shows that the flow with $V \to 0$ for $\tau \to \tau_c$ cannot exist for beams with a finite temperature when $\gamma \geqslant 1$. So we see that our assumption about the existence of a particular point $\tau = \tau_c$ is wrong.

We would like to stress that such behavior is possible owing to the space charge force and the gradient of pressure, which over the symmetry of the problem [see Eq. (26)] defines the character of the solution having form (27). Thus, the warm, beam plasmas differ from conventional quasineutral plasma and neutral gas since some singular peculiarities can exist in such physical systems. To illustrate this, it will be useful to consider some of the simplest examples.

When $\beta = 0$, Eq. (20) describes an ordinary gas compression. In order to determine the self-similar solution for this case, one more requirement of Eq. (20) must be satisfied. For example, we can consider Eq. (20) with initial distribution

$$V(\tau = 0, N) = N^m, \quad m > 0. \tag{29}$$

The initial function describing the density distribution of the beam is a monotonic decreasing function with respect to coordinate ξ. The problem (20) and (29) is invariant under the transformation

$$\tau \to \alpha(\tau - \tau_c), \quad N \to \alpha^{\delta} N, \quad V \to \alpha^{m\delta} V, \tag{30}$$

where $\delta = 1/[1 + m(1+\gamma)/2] < 1$. The self-similar solution of (20) and (29) may be presented in the following form:

$$V(\tau,N) = (\tau - \tau_c)^{m\delta} \theta(\zeta), \quad \zeta = N/(\tau - \tau_c)^{\delta}. \tag{31}$$

In this case

$$H = Ma^2 (\tau - \tau_c)^{m\delta(1-\gamma)+\delta} \int_0^\infty \frac{\theta^{1-\gamma}}{\gamma - 1} d\zeta < \infty$$

is finite for

$$m \leqslant \frac{1}{\gamma - 1}, \tag{32}$$

i.e., the solutions in form (31) reflecting the dynamics of the focused neutral beams account for the possibility of singularity if the requirement (24) is satisfied.

Let us finally call attention to the existence of singular flows in a plasma.[4,5] In the isothermal case, $\gamma = 1$, the wave equation describing the behavior of nonlinear ion waves in warm plasma is[5]

$$\frac{\partial^2 V}{\partial \tau^2} + v_0^2 \frac{\partial^2 V}{\partial N^2} = -\lambda \frac{\partial^2}{\partial N^2}\left(\frac{1}{V}\right), \tag{33}$$

where v_0 is the unperturbed drift of the electrons, λ is the parameter representing the thermal effect. As is seen, the wave equation—Eq. (33)—is similar to Eq. (20). However, in contrast to Eq. (20), Eq. (33) admits a new kind of singular solution.[4,5]

The possible form of the self-similar solution for Eq. (20) is determined by the scale-invariance transformation (30) for $\delta = 1$ and $m = 0$. For the present limit, Eq. (31) reduces to

$$V(\tau,N) = (\tau - \tau_c)\theta(\zeta), \quad \zeta = N/(\tau - \tau_c).$$

In the present case it is possible to estimate the internal energy as $h = C_v T_i$ where T_i is the ion temperature and C_v is the heat capacity. Since T_i = constant we have $H = C_v T_i N_b$, i.e., $\epsilon(\tau,N)$ will be finite for any moment of time. This means that collapse for Eq. (20) may occur for all time.

Thus, the singular peculiarities can exist in the neutral flows and the warm plasma but are impossible in the charged-particle beams. In conclusion, we again would like to emphasize that the nonsingular flows in the charged-particle beams are brought about by the balance between the space charge force and the gradient of pressure.

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