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Linear and nonlinear stability analysis for two-dimensional ideal magnetohydrodynamics with incompressible flows

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The equilibrium and Lyapunov stability properties for two-dimensional ideal magnetohydrodynamic (MHD) plasmas with incompressible and homogeneous (i.e., constant density) flows are investigated. In the unperturbed steady state, both the velocity and magnetic field are nonzero and have three components in a Cartesian coordinate system with translational symmetry (i.e., one ignorable spatial coordinate). It is proved that (a) the solutions of the ideal MHD steady state equations with incompressible and homogeneous flows in the plane are also valid for equilibria with the axial velocity component being a free flux function and the axial magnetic field component being a constant, (b) the conditions of linearized Lyapunov stability for these MHD flows in the planar case (in which the fields have only two components) are also valid for symmetric equilibria that have a nonplanar velocity field component as well as a nonplanar magnetic field component. On using the method of convexity estimates, nonlinear stability conditions are established. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1828464]

I. INTRODUCTION

The equilibrium and stability properties of a magnetically confined plasmas are one of the basic objectives in fusion research and arises in a number of fields including astrophysics and solar physics. At present there are many difficulties surrounding the description of fully three-dimensional configurations and so it is necessary to consider configurations with additional symmetry. Symmetric configurations of plasmas with steady mass flow occur in both laboratory experiments (generally axial symmetric) and in a great variety of astrophysical situations. In many astrophysical situations such as stellar and extra-galactic winds or collimated outflows, axial symmetry is important while in solar physics translational symmetry is common in models of arcades and coronal loops.

Symmetric magnetohydrodynamic (MHD) flows were first considered by Chandrasekhar,[1] who treated the case of an axisymmetric incompressible plasma. A variational principle was formulated by Woltjer[2,3] for an axisymmetric configuration with constant entropy where the formulism of the magnetic flux function that had been introduced by Shafranov[4] for static equilibria was applied to the dynamic problem, deriving some integrals of the system. More general equations of state were treated by Hameiri.[5] A variational formulation based on minimizing the energy integral subject to constancy of the topological invariants of the ideal magnetostatic equations was given by Kruskal and Kulsrud.[6] The second variation of the energy integral yielded the known standard criterion for linear stability given by Bernstein *et al.*[7]

Explicit sufficient conditions for nonlinear stability of equilibrium solutions of a variety of fluid and plasma problems in one, two, and three dimensions were established by Holm *et al.*[8] In their analysis, they used the development of the Lyapunov technique for Hamiltonian systems due to Arnold.[9] The classical Lyapunov method finds criteria for stability of an equilibrium solution of a conservative dynamical system by seeking a constant of motion with a local extremum at the equilibrium. An important development for the applicability of the Lyapunov method to fluid dynamics due to Arnold[9,10] analysis of the stability of planar ideal incompressible fluid motion, providing nonlinear stability results that extend the classical linear theory of Lord Rayleigh.[11] Arnold's technique minimizes the Hamiltonian, subject to given values of constants of motion for the dynamical system. He shows that the minimizing state is a steady state of the system. Moreover, if the second variation of the Hamiltonian is strictly positive definite, and if some mild regularity conditions are satisfied, then the equilibrium is stable. Similar variational principles for the basic characterization of equilibrium states have been developed by Taylor[12,13] (see also the monograph of Biskamp,[14] and references therein, which gives details for these topics). Formal stability of fluids and plasmas has been established by several authors who employed some aspects of Arnold's method.[15–18] The Arnold method has been successfully applied to a wide range of the problems in fluid mechanics, astrophysics, and plasma physics.[8,19–22]

Equilibrium states with mass flow becoming increasingly interesting in magnetic fusion research. Theoretical studies of flowing plasmas were considered for the first time.[23,24] The problem of axisymmetric toroidal equilibrium with mass flows has been studied by a number of authors.[25–37] For incompressible toroidal equilibria with flow, the problem was formulated by Avinash *et al.*[25] An approach for finding exact solutions of axisymmetric stationary MHD equations describing the spatial structure of collimated flows in a magnetized plasma was introduced by Bacciotti and Chiuderi.[26] Several classes of exact analytic stationary equilibria with toroidal mass flow were obtained for isothermal as well as isentropic magnetic surfaces under various pressure and toroidal current density profiles.[27] An analytic equilibria with constant Mach number[28,29] M_A (defined in Sec. II) and with nonconstant Mach number[30] associated with monotonically increasing safety-factor profiles and differentially varying radial electric fields similar to those observed during the transition from the low- to the high-confinement mode in tokamaks were obtained.

For compressible equilibria with flow, the partial differential equation governing the axisymmetric equilibrium was derived by Zehrfeld and Green[31] assuming that the gas pressure proportional to the mass density as the equation of state, while Hameiri[32] used the adaibatic law as the equation of state. Exact solutions for the axisymmetric toroidal plasma with purely toroidal rotation were obtained.[33,34] A comparison of the properties of compressible and incompressible MHD oscillations in toroidal plasmas had been made by Wahlberg.[35] Variational principles based on the minimization of the energy subject to given values of the constants of motion of the dynamical system were formulated by Hameiri.[36] MHD equilibria with mass flow in an axisymmetric tokamak were constructed by Cheremnykh[37] assuming that the adiabatic law as the equation of state.

Two-dimensional equilibria with flow have been used in the context of the solar atmosphere to model coronal loops and arcades[38,39] and prominences.[40] Exact MHD equilibria of cylindrical plasma with arbitrary cross-sectional shape were obtained.[41–43] Lyapunov stability conditions for ideal MHD plasmas with mass flow in axisymmetric toroidal geometry[44] and in cylindrical geometry with arbitrary cross section[45] are determined in the Eulerian representation. For cylindrical and axisymmetric geometries, a work developed by Vladimirov *et al.*[46–48] for describing the stability properties of ideal MHD plasmas is applied to study the nonlinear stability of a wide class of incompressible MHD states.[49,50]

Previously, the stability of ideal incompressible MHD flows in the plane with constant density was investigated by Holm *et al.*[8] An approach to the study of the equilibrium and stability of ideal MHD flows was introduced by Vladimirov *et al.*[46,47] They dealt only with fields having two planar components,[47] and consequently obtained stability conditions equivalent to those obtained by Holm *et al.*[8]

Here we investigate the problem in the presence of nonplanar components of both the velocity and magnetic field. We use a principle of minimum constrained energy to derive the equilibrium equations for this class of MHD flows. The equilibrium solutions are associated to critical points of a nonlinearly conserved functional. We establish variational principles and obtain sufficient conditions for linear and nonlinear Lyapunov stability of the symmetric plasma equilibria with mass flow. The paper is organized as follows: In Sec. II, we introduce the governing equations of ideal incompressible MHD and derive the equilibrium equations in the steady state. In Sec. III, we discuss the constants of motion and associate the equilibrium states to critical points of a conserved Lyapunov functional. In Sec. IV, we establish sufficient conditions for linearized Lyapunov stability of the MHD equilibria given in Sec. III. In Sec. V, we establish sufficient conditions for nonlinear stability of these MHD equilibria. The paper is summarized in Sec. VI.

II. GOVERNING AND EQUILIBRIUM EQUATIONS

In some cases of plasma equilibria the compressible flow is impossible. For example, as shown by Throumoulopoulos and Pantis[41] when they considered the equilibrium of a cylindrical plasma with purely poloidal flow they remarked that only incompressible flows are possible. Clearly this case of equilibrium plasma flow is included in the considered stability problem. The assumption of incompressibility condition for the plasma flow results in a considerable simplification of the problem in which the equilibrium equations reduce to an elliptic partial differential equation which is analytically tractable, while the equation is hyperbolic in the compressible case. Moreover, the study of incompressible flows may be leads to some hidden properties for the equilibrium state. Indeed, Throumoulopoulos and Tasso[42] had showed that for the case of cylindrical isothermal plasma with incompressible flow the magnetic surfaces necessarily have a circular cross section. As we mentioned in the Introduction, a comparison of compressible and incompressible MHD oscillations in toroidal plasmas was made by Wahlberg.[35] It is shown that in an incompressible plasma, the Alfvén wave equation describing toroidal Alfvén eigenmodes in a low β tokamak has two distinguishing features as compared with the corresponding equation in the compressible case, viz., (i) the Alfvén continua are modified and (ii) a coupling between m (poloidal mode number) and $m\pm 2$ takes place.

In the following treatment we shall prove that the solutions of the ideal MHD steady state equations with incompressible and homogeneous flows in the plane are also valid for equilibria with the axial velocity component being a free flux function and the axial magnetic field component being a constant.

The ideal incompressible MHD plasma flows are governed by the following set of equations, written in standard notations and convenient units: The momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \mathbf{J}\wedge\mathbf{B}, \tag{1}$$

the incompressibility condition

$$\nabla\cdot\mathbf{v} = 0, \tag{2}$$

Faraday's law

$$\nabla\wedge\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

Ampère's law

$$\nabla\wedge\mathbf{B} = \mathbf{J}, \tag{4}$$

the divergence-free Gauss law

$$\nabla\cdot\mathbf{B} = 0, \tag{5}$$

and Ohm's law

$$\mathbf{E}+\mathbf{v}\wedge\mathbf{B}=0, \tag{6}$$

where t, $\mathbf{v}$, p, $\mathbf{B}$, $\mathbf{J}$, and $\mathbf{E}$ stand as usual for the time, fluid velocity, gas pressure, magnetic field (induction), electric current density, and electric field, respectively. Under assumption of translational symmetry, we consider an incompressible, homogeneous (with unit density), inviscid, and perfectly conducting fluid contained in a domain D with fixed boundary ∂D. For this configuration all the physical quantities are invariant under translations in fixed direction, which we may take to be the direction O_z of a Cartesian coordinate system O_{xyz} with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$. The domain D is then of cylindrical shape with arbitrary cross section. The boundary conditions are taken to be

$$\mathbf{n}\cdot\mathbf{v}=0, \quad \mathbf{n}\cdot\mathbf{B}=0 \text{ on } \partial D, \tag{7}$$

where $\mathbf{n}=(n_1,n_2,0)$ is the outward unit normal to ∂D.

From the symmetry assumption, it is possible to write the divergence-free fields $\mathbf{B}$ and $\mathbf{v}$ in the form

$$\mathbf{B}=\boldsymbol{\nabla}\psi\wedge\mathbf{e}_z+B_z\mathbf{e}_z, \tag{8}$$

$$\mathbf{v}=\boldsymbol{\nabla}\phi\wedge\mathbf{e}_z+v_z\mathbf{e}_z, \tag{9}$$

where $\psi(x,y,t)$, $\phi(x,y,t)$, $B_z(x,y,t)$, and $v_z(x,y,t)$ are the poloidal magnetic flux function, the poloidal stream function, the z component of the magnetic field and the z component of the velocity field, respectively.

From Eqs. (8) and (9), both the electric current density $\mathbf{J}$ and vorticity $\boldsymbol{\omega}$ can be expressed as

$$\mathbf{J}=\boldsymbol{\nabla}B_z\wedge\mathbf{e}_z-\nabla^2\psi\mathbf{e}_z, \tag{10}$$

$$\boldsymbol{\omega}=\boldsymbol{\nabla}v_z\wedge\mathbf{e}_z-\nabla^2\phi\mathbf{e}_z. \tag{11}$$

Since $\mathbf{B}\cdot\boldsymbol{\nabla}\psi=\mathbf{v}\cdot\boldsymbol{\nabla}\phi=0$, the magnetic and flow surfaces are characterized, respectively, by the conditions $\psi(x,y,t)=\text{const}$ and $\phi(x,y,t)=\text{const}$. Besides, since they are related to the flux of the corresponding vector field, they are generally referred to as flux functions.

Equations (8) and (10) yield the following representation of the magnetic force:

$$\mathbf{J}\wedge\mathbf{B}=\boldsymbol{\nabla}B_z\wedge\boldsymbol{\nabla}\psi-\boldsymbol{\nabla}\left(\frac{B_z^2}{2}\right)-\nabla^2\psi\nabla\psi. \tag{12}$$

Using the identity

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}=\boldsymbol{\nabla}\left(\frac{v^2}{2}\right)-\mathbf{v}\wedge(\boldsymbol{\nabla}\wedge\mathbf{v}), \tag{13}$$

we have

$$\begin{aligned}(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}-\mathbf{J}\wedge\mathbf{B}=&\tfrac{1}{2}\boldsymbol{\nabla}(v^2-v_z^2+B_z^2)-\nabla^2\phi\boldsymbol{\nabla}\phi\\&+\boldsymbol{\nabla}v_z\wedge\boldsymbol{\nabla}\phi+\nabla^2\psi\boldsymbol{\nabla}\psi\\&+\boldsymbol{\nabla}\psi\wedge\boldsymbol{\nabla}B_z.\end{aligned} \tag{14}$$

In the steady state $(\partial/\partial t=0)$ Faraday's law (3) becomes $\boldsymbol{\nabla}\wedge\mathbf{E}=0$, and hence the electric field is expressed as $\mathbf{E}=-\boldsymbol{\nabla}\Phi$, where Φ is the electric potential. Thus Ohm's law is projected along $\mathbf{e}_z$ and $\mathbf{B}$ yielding, respectively,

$$\mathbf{e}_z\cdot(\boldsymbol{\nabla}\phi\wedge\mathbf{e}_z)\wedge(\boldsymbol{\nabla}\psi\wedge\mathbf{e}_z)=0 \tag{15}$$

and

$$\mathbf{B}\cdot\boldsymbol{\nabla}\Phi=0. \tag{16}$$

Equations (15) and (16) imply that $\phi=\phi(\psi)$ and $\Phi=\Phi(\psi)$, respectively. This means that the stream function and the electric potential are flux functions. Two additional flux functions are found from the component of Eq. (6) perpendicular to a magnetic surface:

$$v_z-B_z\phi'=\Phi', \tag{17}$$

and from the component of the momentum Eq. (1) along $\mathbf{e}_z$:

$$\phi'v_z-B_z=F(\psi), \tag{18}$$

with an arbitrary function $F(\psi)$. The prime denotes differentiation with respect to ψ.

Solving the set of Eqs. (17) and (18) for B_z and v_z, one obtains

$$B_z=\frac{\phi'\Phi'-F}{1-(\phi')^2}, \tag{19}$$

$$v_z=\frac{\Phi'-\phi'F}{1-(\phi')^2}. \tag{20}$$

Equations (19) and (20) imply that B_z and v_z are flux functions $[B_z=B_z(\psi)$ and $v_z=v_z(\psi)]$. They have a singularity when $(\phi')^2=1$.

The Alfvénic Mach number M_A for the components of the velocity normal to the invariance direction $\mathbf{e}_z$ is

$$M_A\equiv\frac{|\mathbf{v}\wedge\mathbf{e}_z|}{|\mathbf{v}_A\wedge\mathbf{e}_z|}=\phi', \tag{21}$$

where $\mathbf{v}_A=\mathbf{B}$ is the Alfvén velocity. Then the Alfvénic Mach number is a flux function and the singularity of Eqs. (19) and (20) can be written as $M_A^2=1$.

From Eqs. (8), (9), and (17) we obtain the following relation between the velocity and magnetic field:

$$\mathbf{v}=\phi'\mathbf{B}+\Phi'\mathbf{e}_z, \tag{22}$$

from which it is clear that the two vectors $\mathbf{v}$ and $\mathbf{B}$ are parallel only if $\Phi'=0$ [i.e., if $\mathbf{E}=0$, according to Ohm's law (6)].

With the aid of Eq. (14), the momentum Eq. (1) in the steady state reads

$$\nabla\left[p+\tfrac{1}{2}(v^2-v_z^2+B_z^2)\right]-\nabla^2\phi\boldsymbol{\nabla}\phi+\nabla^2\psi\boldsymbol{\nabla}\psi=0, \tag{23}$$

where we used the functional dependence on ψ of the quantities ϕ, B_z, and v_z.

Using Eq. (9), Eq. (23) becomes

$$\boldsymbol{\nabla}\left[p+\frac{(\phi')^2}{2}|\nabla\psi|^2\right]+\left[\nabla^2\psi-\phi'\nabla^2\phi+\frac{(B_z^2)'}{2}\right]\boldsymbol{\nabla}\psi=0. \tag{24}$$

Scalar multiplication of Eq. (24) by $\mathbf{B}$ yields

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left[p+\frac{(\phi')^2}{2}|\nabla\psi|^2\right]=0. \quad (25)$$

Equation (25) can be integrated to yield an expression for the pressure

$$p=K(\psi)-\frac{(\phi')^2}{2}|\nabla\psi|^2, \quad (26)$$

where $K(\psi)$ is the static pressure.

Substitution of Eq. (26) into Eq. (24) yields

$$\nabla^2\psi-\phi'\nabla^2\phi+\left(K+\frac{B_z^2}{2}\right)'=0. \quad (27)$$

Equation (27) is commonly known as generalized Grad–Shafranov equation, as it represents the generalization to dynamical equilibria of the equation obtained for the static case. If the analytical form of the arbitrary functions of ψ is chosen, then Eq. (27) becomes a nonlinear second-order partial differential equation for the unknown function $\psi(x,y)$.

Substitution of $\nabla^2\phi=\phi'\nabla^2\psi+\phi''|\boldsymbol{\nabla}\psi|^2$ into Eq. (27) yields the elliptic partial differential equation

$$[1-(\phi')^2\ \nabla^2\psi-\phi'\phi''|\nabla\psi|^2+\left(K+\frac{B_z^2}{2}\right)'=0. \quad (28)$$

In the following we discuss special cases of Eq. (27).

(i) When $B_z=$const, Eq. (27) is then

$$\nabla^2\psi-\phi'\nabla^2\phi+K'=0, \quad (29)$$

which is the generalized Grad–Shafranov equation in the plane. Note that Eq. (27) does not contain the axial velocity v_z. Hence the solutions of the ideal incompressible MHD equations which describes the equilibrium states of a plasma in the plane are also valid for equilibria with the axial velocity component being a free flux function and the axial magnetic field component being a constant.

(ii) When $\phi=\alpha\psi$ (with $\alpha^2\neq 1$), Eq. (27) is reduced to

$$\nabla^2\psi+\frac{1}{1-\alpha^2}\left(K+\frac{B_z^2}{2}\right)'=0. \quad (30)$$

This equation is similar in the form to the equation governing static equilibria; the only explicit difference is the presence of α. Equation (30) can be linearized for several choices of $K+B_z^2/2$ and a variety of analytic solutions of the linearized equation can be derived. Moreover, it can still posses several classes of exact solutions for the nonlinear case (Ref. 51 and references there in).

III. CONSTANTS OF MOTION AND VARIATIONAL PRINCIPLE

In this section we formulate a variational principle for MHD equilibria given in Sec. II.

Using Eq. (6), Faraday's law (3) can be written in the form

$$\frac{\partial\mathbf{B}}{\partial t}=\boldsymbol{\nabla}\wedge(\mathbf{v}\wedge\mathbf{B}). \quad (31)$$

The (x,y) components of Eq. (31) yield

$$\frac{\partial\psi}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\psi=0, \quad (32)$$

which means that the flux of $\mathbf{B}$ through any closed material circuit being conserved.

The ideal MHD equations conserve the total energy

$$E=\int_D\frac{1}{2}(v^2+B^2)d\tau, \quad (33)$$

where $d\tau=dxdy$. The other invariants for ideal incompressible MHD in the cylindrical case are the flux weighted

$$C_1=\int_D F_1(\psi)d\tau, \quad (34)$$

the momentum

$$C_2=\int_D v_zF_2(\psi)d\tau, \quad (35)$$

the magnetic helicity

$$C_3=\int_D B_zF_3(\psi)d\tau, \quad (36)$$

and the rotation

$$C_4=\int_D \omega_zF_4(\psi)d\tau, \quad (37)$$

where F_1, F_2, F_3, and F_4 are arbitrary functions. The first of these, the flux-weighted C_1, is conserved with the aid of Eqs. (2) and (32), which readily imply

$$\frac{\partial}{\partial t}[F_1(\psi)]=-\boldsymbol{\nabla}\cdot[\mathbf{v}F_1(\psi)]. \quad (38)$$

Hence C_1 will be conserved for perfectly conducting plasmas by using the boundary conditions (7). The second invariant, the momentum C_2, yields from Eqs. (1), (2), (5), (8), and (32), which readily imply

$$\frac{\partial}{\partial t}[v_zF_2(\psi)]=\boldsymbol{\nabla}\cdot[F_2(\psi)(B_z\mathbf{B}-v_z\mathbf{v})]. \quad (39)$$

The invariance of the magnetic helicity C_3 follows from Eqs. (2), (5), (8), (31), and (32) which yield

$$\frac{\partial}{\partial t}[B_zF_3(\psi)]=\boldsymbol{\nabla}\cdot[F_3(\psi)(v_z\mathbf{B}-B_z\mathbf{v})]. \quad (40)$$

Finally, the invariant of the rotation C_4 arises by taking the curl of Eq. (1) which yields

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\boldsymbol{\nabla}\wedge(\boldsymbol{\omega}\wedge\mathbf{v})=\boldsymbol{\nabla}\wedge(\mathbf{J}\wedge\mathbf{B}). \quad (41)$$

Then with the aid of the identity

$$\boldsymbol{\nabla}\wedge(\mathbf{a}\wedge\mathbf{b})=\mathbf{a}\boldsymbol{\nabla}\cdot\mathbf{b}+(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a}-\mathbf{b}\boldsymbol{\nabla}\cdot\mathbf{a}-(\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b}, \quad (42)$$

the z component of Eq. (41) reads

$$\frac{\partial \omega_z}{\partial t} + \mathbf{v} \cdot \nabla \omega_z - \boldsymbol{\omega} \cdot \nabla v_z = \mathbf{B} \cdot \nabla J_z - \mathbf{J} \cdot \nabla B_z. \tag{43}$$

Using Eqs. (10) and (11), we find that the third term in the left-hand side and the second term in the right-hand side vanish. Thus Eq. (43) becomes

$$\frac{\partial \omega_z}{\partial t} + \mathbf{v} \cdot \nabla \omega_z = \mathbf{B} \cdot \nabla J_z. \tag{44}$$

Hence using Eqs. (2), (5), (8), and (32), we have

$$\frac{\partial}{\partial t}[\omega_z F_4(\psi)] = \nabla \cdot [F_4(\psi)(J_z \mathbf{B} - \omega_z \mathbf{v})]. \tag{45}$$

The identity (45) proves the invariance of C_4 for the boundary conditions (7). Thus, for cylindrical ideal incompressible MHD plasmas the four integrals C_1, C_2, C_3, and C_4 are conserved. The constants of motion E, C_1, C_2, C_3, and C_4 in Eqs. (33)–(37) will be used to characterize the equilibrium states for ideal incompressible MHD.

Consider now the functional

$$\mathfrak{R} = E + C_1 + C_2 + C_3 + C_4. \tag{46}$$

The equilibria for MHD flows may be sought as critical points of the functional $\mathfrak{R}$ in Eq. (46). The first variation of $\mathfrak{R}$ is

$$\begin{aligned} \delta\mathfrak{R} = \int_D [&\nabla \phi \cdot \nabla \delta\phi + \nabla \psi \cdot \nabla \delta\psi + (v_z + F_2)\delta v_z \\ &+ (B_z + F_3)\delta B_z + (F_1' + v_z F_2' + B_z F_3' + \omega_z F_4')\delta\psi \\ &+ F_4 \delta\omega_z] d\tau. \end{aligned} \tag{47}$$

Applying Gauss divergence theorem to the first and second terms of Eq. (47), we get

$$\begin{aligned} \delta\mathfrak{R} = \int_D [&(\phi + F_4)\delta\omega_z + (v_z + F_2)\delta v_z + (B_z + F_3)\delta B_z \\ &+ (F_1' + v_z F_2' + B_z F_3' + \omega_z F_4' + J_z)\delta\psi d\tau \\ &+ \int_{\partial D} \mathbf{n} \cdot (\phi \nabla \delta\phi + \delta\psi \nabla \psi) ds. \end{aligned} \tag{48}$$

The surface integral in Eq. (48) vanishes with the boundary conditions

$$\phi = 0, \quad \delta\psi = 0 \text{ on } \partial D. \tag{49}$$

The boundary condition $\delta\psi = 0$ is consistent with the second condition in Eq. (7), which implies that $\psi = \text{const}$ on ∂D. Hence Eq. (48) becomes

$$\begin{aligned} \delta\mathfrak{R} = \int_D [&(\phi + F_4)\delta\omega_z + (v_z + F_2)\delta v_z + (B_z + F_3)\delta B_z \\ &+ (F_1' + v_z F_2' + B_z F_3' + \omega_z F_4' + J_z)\delta\psi] d\tau. \end{aligned} \tag{50}$$

From Eq. (50), the sufficient conditions for $\delta\mathfrak{R} = 0$ are

$$\phi + F_4 = 0, \tag{51a}$$

$$V_z + F_2 = 0, \tag{51b}$$

$$B_z + F_3 = 0, \tag{51c}$$

$$F_1' + v_z F_2' + B_z F_3' + \omega_z F_4' + J_z = 0. \tag{51d}$$

Equations (51a)–(51c) imply that the quantities ϕ, v_z and B_z are flux functions as we proved in Sec. II. Also we find that the velocity and magnetic field are related by

$$\mathbf{v} = -F_4' \mathbf{B} - (F_2 + F_3 F_4')\mathbf{e}_z, \tag{52}$$

which coincide with the relation (22).

Equation (51d) is the generalized Grad–Shafranov equation. If we choose $F_2 = F_4 = 0$, then by Eq. (52) we find that $\mathbf{v} = 0$, i.e., we have a static equilibrium and hence Eq. (51d) is reduced to the magnetostatic Grad–Shafranov equation.

Now we show that the critical point conditions for $\mathfrak{R}$, Eqs. (51a)–(51d), imply the equilibrium relations for ideal incompressible MHD equations [Eqs. (1)–(6) with vanishing time derivatives]. Relation (52) gives immediately

$$\nabla \cdot \mathbf{v} = -\mathbf{B} \cdot \nabla F_4' = 0, \tag{53}$$

$$\mathbf{v} \wedge \mathbf{B} = -(F_2 + F_3 F_4') \nabla \psi = \nabla \Phi, \tag{54}$$

which satisfy the incompressibility condition (2) and Ohm's law (6) at the steady state. Besides, the z component of Eq. (52) coincides with Eq. (17). Formula (52) also implies the relation

$$\mathbf{v} \cdot \nabla \psi = 0, \tag{55}$$

which satisfies Eq. (32) at the steady state.

Substitution of Eqs. (51a)–(51c) into Eq. (51d) yields

$$J_z - \phi' \omega_z + \left(F_1 - \frac{v_z^2 + B_z^2}{2}\right)' = 0. \tag{56}$$

If we choose the function F_1 on the form

$$F_1 = \frac{v_z^2}{2} - K, \tag{57}$$

and substitute $J_z = -\nabla^2 \psi$ and $\omega_z = -\nabla^2 \phi$ [from Eqs. (10) and (11)] into Eq. (56), then Eq. (56) becomes identical with Eq. (27). This shows that Eq. (56) represents the component of the momentum Eq. (1) perpendicular to a magnetic surface at the steady state. Therefore the critical point conditions (51a)–(51d) of the functional $\mathfrak{R}$ in Eq. (46) imply the equilibrium relations for ideal incompressible MHD equations.

IV. LINEARIZED LYAPUNOV STABILITY

In this section we establish sufficient conditions of linearized Lyapunov stability for the MHD equilibria given in Sec. III.

According to the general theory of Arnold,[9,10] if the functional $\mathfrak{R}$ is extremal (maximum or minimum) for all admissible variations $\delta\psi$, $\delta\phi$, δv_z, and δB_z then the system considered is linearly stable. Thus the conditions on the equilibrium flow for $\delta^2\mathfrak{R}$ to be definite in sign are sufficient conditions for linearized Lyapunov stability (see Holm *et al.*[8] and also Spies,[52] and references therein for additional discussions of Lyapunov methods[8] and the energy principle methods[52] in plasma physics).

From Eq. (47), the second variation of $\mathfrak{R}$ in Eq. (46) gives

$$\delta^2\mathfrak{R} = \int_D \{(\nabla \delta\phi)^2 + (\nabla \delta\psi)^2 + (\delta v_z)^2 + (\delta B_z)^2 + 2(F_2'\delta v_z \delta\psi + F_3'\delta B_z \delta\psi) + (F_1'' + v_z F_2'' + B_z F_3'' + \omega_z F_4'')(\delta\psi)^2 + 2F_4'\delta\omega_z \delta\psi\} d\tau. \quad (58)$$

The last term in Eq. (58) can be integrated by parts to yields

$$\int_D 2F_4'\delta\psi\delta\omega_z d\tau = \int_D 2\,\nabla(F_4'\delta\psi)\cdot\nabla\,\delta\phi d\tau - \int_{\partial D} 2F_4'\delta\psi \mathbf{n}\cdot\nabla\,\delta\phi ds. \quad (59)$$

The surface integral vanishes by using the boundary conditions (49) and then Eq. (59) becomes

$$\int_D 2F_4'\delta\psi\delta\omega_z d\tau = \int_D 2\,\nabla(F_4'\delta\psi)\cdot\nabla\,\delta\phi d\tau = \int_D 2(F_4'\nabla\,\delta\psi\cdot\nabla\,\delta\phi + \delta\psi\nabla F_4'\cdot\nabla\,\delta\phi)d\tau. \quad (60)$$

Substituting Eq. (60) into Eq. (58), we get

$$\delta^2\mathfrak{R} = \int_D [(\nabla \delta\phi)^2 + (\nabla \delta\psi)^2 + (\delta v_z)^2 + (\delta B_z)^2 + 2(F_2'\delta v_z \delta\psi + F_3'\delta B_z \delta\psi + F_4'\nabla\,\delta\psi\cdot\nabla\,\delta\phi + \delta\psi\nabla F_4'\cdot\nabla\,\delta\phi) + (F_1'' + v_z F_2'' + B_z F_3'' + \omega_z F_4'') \times(\delta\psi)^2] d\tau. \quad (61)$$

Let $(\partial_x\delta\phi, \partial_y\delta\phi)$ and $(\partial_x\delta\psi, \partial_y\delta\psi)$ are the (x,y) components of $\nabla\delta\phi$ and $\nabla\delta\psi$, respectively, then the second variation Eq. (61) can be rearranged into matrix quadratic form as follows:

$$\delta^2\mathfrak{R} = \int_D (\partial_x\delta\phi\ \partial_y\delta\phi\ \delta v_z\ \partial_x\delta\psi\ \partial_y\delta\psi\ \delta B_z\ \delta\psi) \times \begin{pmatrix} 1 & 0 & 0 & F_4' & 0 & 0 & \partial_x F_4' \\ 0 & 1 & 0 & 0 & F_4' & 0 & \partial_y \delta\psi \\ 0 & 0 & 1 & 0 & 0 & 0 & F_2' \\ F_4' & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & F_4' & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & F_3' \\ \partial_x F_4' & \partial_y \delta\psi & F_2' & 0 & 0 & F_3' & f \end{pmatrix} \begin{pmatrix} \delta v_1 \\ \partial_x\delta\phi \\ \partial_y\delta\phi \\ \delta v_z \\ \partial_x\delta\psi \\ \partial_y\delta\psi \\ \delta B_z \\ \delta\psi \end{pmatrix} d\tau, \quad (62)$$

where $f = F_1'' + v_z F_2'' + B_z F_3'' + \omega_z F_4''$.

The purely algebraic quadratic form is positive definite if and only if each of its subdeterminants along the principal diagonal (principal minors) is positive definite. The seven principal minors of the symmetric 7×7 matrix in Eq. (62) are

$$\mu_1 = \mu_2 = \mu_3 = 1, \quad \mu_4 = 1 - (F_4')^2, \quad \mu_5 = \mu_6 = (1 - (F_4')^2)^2,$$

$$\mu_7 = (1 - (F_4')^2)\{[1 - (F_4')^2][f - (F_2')^2 - (F_3')^2] - (\partial_x F_4')^2 - (\partial_y F_4')^2\}.$$

Therefore the second variation $\delta^2\mathfrak{R}$ is positive definite provided

$$(F_4')^2 \leqslant 1, \quad (63a)$$

$$(1 - (F_4')^2)(f - (F_2')^2 - (F_3')^2) \geqslant |\nabla F_4'|^2. \quad (63b)$$

Differentiating Eq. (51d) with respect to ψ and using Eqs. (51a)–(51c), one gets

$$f - (F_2')^2 - (F_3')^2 = \phi'\omega_z' - J_z' = \frac{\phi'\nabla\psi\cdot\nabla\omega_z - \nabla\psi\cdot\nabla J_z}{|\nabla\psi|^2}. \quad (64)$$

Substituting Eqs. (64) and (51a) into Eqs. (63a) and (63b), we obtain the following stability criterion.

A. Criterion A

For velocity and magnetic fields tangent to the boundary in a cylindrical domain with arbitrary cross section, the steady state solutions of ideal MHD plasmas with incompressible flows are linearly stable provided

$$(\phi')^2 \leqslant 1, \quad (65a)$$

$$[1 - (\phi')^2]\left(\frac{\phi'\nabla\psi\cdot\nabla\omega_z - \nabla\psi\cdot\nabla J_z}{|\nabla\psi|^2}\right) \geqslant |\nabla\phi'|^2, \quad (65b)$$

throughout D.

Based on the definition (21) for the Alfvénic Mach number, condition (65a) can be written as $M_A^2 \leqslant 1$, which means that the flow must be sub-Alfvénic. The algebraic condition (65b) places a constraint on the fields **v** and **B**. Conditions (65a) and (65b) require in fact the cylindrical equilibrium to be in the elliptic regime. Now we rearrange the second variation $\delta^2\mathfrak{R}$ into another form which yields other stability conditions. The second variation (61) can be rewritten as

$$\delta^2\mathfrak{R} = \int_D [[\nabla\delta\phi + \nabla(F_4'\delta\psi)]^2 + (\delta v_z + F_2'\delta\psi)^2 + (\delta B_z + F_3'\delta\psi)^2 + (\nabla\delta\psi)^2 + [f - (F_2')^2 - (F_3')^2] \times(\delta\psi)^2 - [\nabla(F_4'\delta\psi)]^2]d\tau. \quad (66)$$

The last term in Eq. (66) can be integrated as follows:

$$\int_D [\nabla(F_4'\delta\psi)]^2 d\tau = \int_D [(F_4')^2(\nabla\delta\psi)^2 + \nabla\cdot[(\delta\psi)^2F_4'\nabla F_4'] - (\delta\psi)^2F_4'\nabla^2F_4']d\tau = \int_D [(F_4')^2(\nabla\delta\psi)^2 - (\delta\psi)^2F_4'\nabla^2F_4']d\tau. \quad (67)$$

Substituting Eq. (67) into Eq. (66), we obtain

$$\delta^2\mathfrak{R} = \int_D [[\nabla\delta\phi + \nabla(F_4'\delta\psi)]^2 + (\delta v_z + F_2'\delta\psi)^2 + (\delta B_z + F_3'\delta\psi)^2 + [1-(F_4')^2](\nabla\delta\psi)^2 + [f - (F_2')^2 - (F_3')^2 + F_4'\nabla^2F_4'](\delta\psi)^2]d\tau. \quad (68)$$

Using Eqs. (64) and (51a), the following stability criterion can be obtained.

B. Criterion B

The ideal MHD steady state mentioned in Criterion A is linearly stable provided

$$(\phi')^2 \leqslant 1, \quad (69a)$$

$$\frac{\phi'\nabla\psi\cdot\nabla\omega_z - \nabla\psi\cdot\nabla J_z}{|\nabla\psi|^2} + \phi'\nabla^2\phi' \geqslant 0, \quad (69b)$$

throughout D.

Criterion B confirms the requirement that the equilibrium flow should be sub-Alfvénic in a rotating frame determined on each flux surface. This does not mean that every configuration with sub-Alfvénic flow velocity is stable, but it should satisfy additional condition (rather than the sub-Afvénic condition) on the fields **v** and **B** in order to the equilibrium state being stable. This is shown in condition (69b). In the special case, when the magnetic field lines are perpendicular to a magnetic surface while the flow velocity is tangent to it; the equilibrium state is stable if the current density decreases with respect to the magnetic flux function.

Conditions (69a) and (63b) are equivalent to those obtained in Refs. 8 and 47. It should be pointed out that these authors investigated the equilibrium and stability of incompressible MHD plasmas with planar flow and magnetic field ($v_z = B_z = 0$), while we have considered the problem in the presence of the axial velocity and magnetic field ($v_z \neq 0, B_z \neq 0$). We note that conditions (65a), (65b), (69a), and (69b) do not contain any terms of v_z or B_z. So the stability conditions for planar incompressible MHD are also valid for symmetric equilibria that has a nonplanar velocity field component and a nonplanar magnetic field component.

C. Examples

(i) The Grad–Shafranov solutions ($\mathbf{v}=0$): For these equilibria, $\phi'=0$. Therefore the steady state is linearly stable provided that

$$J_z' \leqslant 0. \quad (70)$$

(ii) The Alfvén solutions ($\mathbf{v}=\mathbf{B}$): In this case $\phi'=1$, so $\omega_z = J_z = -\nabla^2\psi$ and hence the conditions (65a) and (65b) of Criterion A and Eqs. (69a) and (69b) of Criterion B are immediately satisfied. Then the Alfvén solutions are linearly stable.

(iii) Flow and field are perpendicular to the plane: Suppose that

$$\mathbf{v} = v_z\mathbf{e}_z, \quad \mathbf{B} = B_z\mathbf{e}_z. \quad (71)$$

In this case $\phi = \psi = J_z = 0$, which satisfies the conditions (65a), (65b), (69a), and (69b). Then the state (71) is linearly stable.

V. NONLINEAR STABILITY

In this section we establish nonlinear stability conditions for the MHD equilibria given in Sec. III. We use the stability algorithm introduced in Holm *et al.*[8] in the sense of the Lyapunov definition of nonlinear stability which states that, in terms of a norm $\| \ \|$, an equilibrium point u_e of a dynamical system is said to be nonlinearly stable if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|u(0) - u_e\| < \delta$, then $\|u(t) - u_e\| < \varepsilon$ for $t > 0$ (t is the time).

Briefly, we list the nonlinear stability algorithm of Holm *et al.*[8]

A. Stability algorithm

(a) Choose a Banach space **U** of fields u and write the equations of motion on **U** as

$$\frac{\partial u}{\partial t} = X(u), \quad (72)$$

for a nonlinear operator X mapping a domain in **U** to **U**.

(b) Find a conserved functional H for (72), usually representing the total energy; that is find a map $H : \mathbf{U} \to \mathbf{R}$ (the real numbers) such that $dH(u)/dt = 0$ for continuously differentiable solution u of Eq. (72).

(c) Find a family of constants of motion for Eq. (72). That is, find a collection of functionals C on **U** such that $dC(u)/dt = 0$ for any continuously differentiable solution u of Eq. (72).

(d) Relate an equilibrium solution u_e of Eq. (72) to the constant of motion C by requiring that $\mathfrak{R} := H + C$ have a critical point at u_e.

(e) Find quadratic forms (convexity estimates) Q_1 and Q_2 on $\mathbf{U}$ such that

$$Q_1(\Delta u) \leqslant H(u_e + \Delta u) - H(u_e) - DH(u_e) \cdot \Delta u, \tag{73a}$$

$$Q_2(\Delta u) \leqslant C(u_e + \Delta u) - C(u_e) - DC(u_e) \cdot \Delta u, \tag{73b}$$

for all $\Delta u = u - u_e$ in $\mathbf{U}$. Then require that

$$Q_1(\Delta u) + Q_2(\Delta u) > 0, \text{ for all } \Delta u \neq 0 \text{ in } \mathbf{U}. \tag{74}$$

(f) If steps from (a) to (e) have been carried out, then for any solution u of Eq. (72) we have the following *a priori* estimate on Δu:

$$Q_1(\Delta u) + Q_2(\Delta u) \leqslant \mathfrak{R}[u(0)] - \mathfrak{R}(u_e). \tag{75}$$

(g) Set $\|\Delta u(t)\|^2 = Q_1(\Delta u) + Q_2(\Delta u)$, so $\|\Delta u(t)\|$ defines a norm on $\mathbf{U}$ and the functional $\mathfrak{R}$ is continuous in this norm at u_e, then u_e is nonlinearly stable.

Now we apply the above procedure to the considered problem in Sec. III. For simplicity we denote u for the equilibrium point instead of u_e.

All the steps (a)–(d) have been carried out in Secs. II and III, where the conserved functional H in step (b) has taken to be the total energy, i.e., $H = E$ and the collection of a family of constants of motion in step (c) is $C = C_1 + C_2 + C_3 + C_4$. What remains is to apply steps (e)–(g).

Since E is quadratic, we choose $Q_1 = E$. Next consider

$$\begin{aligned}\hat{C} &= C(\psi+\Delta\psi, v_z+\Delta v_z, B_z+\Delta B_z, \omega_z+\Delta\omega_z) - C(\psi, v_z, B_z, \omega_z) - DC(\psi, v_z, B_z, \omega_z)C(\Delta\psi, \Delta v_z, \Delta B_z, \Delta\omega_z)\\ &= \int_D [F_1(\psi+\Delta\psi) + (v_z+\Delta v_z)F_2(\psi+\Delta\psi) + (B_z+\Delta B_z)F_3(\psi+\Delta\psi) - F_1(\psi) - (v_z+\Delta v_z)F_2(\psi)\\ &\quad - (B_z+\Delta B_z)F_3(\psi) - (\omega_z+\Delta\omega_z)F_4(\psi) - (F_1'(\psi) + v_zF_2'(\psi) + B_zF_3'(\psi) + \omega_zF_4'(\psi))\Delta\psi]d\tau,\end{aligned} \tag{76}$$

where $\Delta(\cdot)$ refers to perturbations.

Using the following notation:

$$\hat{F}_i = F_i(\psi+\Delta\psi) - F_i(\psi) - F_i'(\psi)\Delta\psi, \tag{77a}$$

$$F_i^* = F_i(\psi+\Delta\psi) - F_i(\psi), \tag{77b}$$

with $i = 1, \ldots, 4$; thus Eq. (76) becomes

$$\begin{aligned}\hat{C} = \int_D &[\hat{F}_1(\Delta\psi) + v_z\hat{F}_2(\Delta\psi) + B_z\hat{F}_3(\Delta\psi) + \omega_z\hat{F}_4(\Delta\psi)\\ &+ (\Delta v_z)F_2^*(\Delta\psi) + (\Delta B_z)F_3^*(\Delta\psi) + (\Delta\omega_z)F_4^*(\Delta\psi)]d\tau.\end{aligned} \tag{78}$$

Using Taylor's expansion with remainder in Lagrange's form, we obtain

$$\begin{aligned}\hat{C} = \frac{1}{2}\int_D &\{[F_1''(\psi_1) + v_zF_2''(\psi_1) + B_zF_3''(\psi_1) + \omega_zF_4''(\psi_1)]\\ &\times(\Delta\psi)^2 + 2[F_2'(\psi_2)\Delta v_z + F_3'(\psi_2)\Delta B_z\\ &+ F_4'(\psi_2)\Delta\omega_z]\Delta\psi\}d\tau,\end{aligned} \tag{79}$$

where

$$\psi_1 = \psi + \theta_1\Delta\psi, \quad \psi_2 = \psi + \theta_2\Delta\psi, \quad 0 < \theta_1, \ \theta_2 < 1. \tag{80}$$

For finite-amplitude perturbations $\psi_{min} \leqslant \psi_1, \psi_2 \leqslant \psi_{max}$, we use the function f defined after Eq. (62) to write $\hat{C}$ as the following form:

$$\begin{aligned}\hat{C} = \frac{1}{2}\int_D &\{[f(\psi_1) - [F_2'(\psi_2)]^2 - [F_3'(\psi_2)]^2](\Delta\psi)^2\\ &+ [\Delta v_z + F_2'(\psi_2)\Delta\psi]^2 + [\Delta B_z + F_3'(\psi_2)\Delta\psi]^2\\ &- (\Delta v_z)^2 - (\Delta B_z)^2 + 2F_4'(\psi_2)\Delta\omega_z\Delta\psi\}d\tau.\end{aligned} \tag{81}$$

Consequently, for $(F_2')_{min} \leqslant F_2'(\psi_2) \leqslant (F_2')_{max}$, $(F_3')_{min} \leqslant F_3'(\psi_2) \leqslant (F_3')_{max}$, and $f_{min} \leqslant f(\psi_1) \leqslant f_{max}$, we take the quadratic functional Q_2 in Eq. (73b) to be

$$\begin{aligned}Q_2 = \frac{1}{2}\int_D &\{[f_{min} - (F_2'^2)_{max} - (F_3'^2)_{max}](\Delta\psi)^2 + [\Delta v_z\\ &+ F_2'(\psi_2)\Delta\psi]^2 + [\Delta B_z + F_3'(\psi_2)\Delta\psi]^2 - (\Delta v_z)^2\\ &- (\Delta B_z)^2 + 2F_4'(\psi_2)\Delta\omega_z\Delta\psi\}d\tau,\end{aligned} \tag{82}$$

and hence

$$\begin{aligned}&(Q_1 + Q_2)(\Delta\psi, \Delta v_z, \Delta B_z, \Delta\omega_z)\\ &\quad= \frac{1}{2}\int_D \{[f_{min} - (F_2'^2)_{max} - (F_3'^2)_{max}](\Delta\psi)^2\\ &\qquad+ [\Delta v_z + F_2'(\psi_2)\Delta\psi]^2 + [\Delta B_z + F_3'(\psi_2)\Delta\psi]^2\\ &\qquad+ [\boldsymbol{\nabla}(\Delta\phi)]^2 + [\boldsymbol{\nabla}(\Delta\psi)]^2 + 2F_4'(\psi_2)\Delta\omega_z\Delta\psi\}d\tau.\end{aligned} \tag{83}$$

As in Eq. (59), the last term in Eq. (83) integrated by parts and then

$$Q_1 + Q_2 = \frac{1}{2}\int_D \{[f_{min} - (F_2'^2)_{max} - (F_3'^2)_{max}](\Delta\psi)^2 + [\Delta v_z + F_2'(\psi_2)\Delta\psi]^2 + [\Delta B_z + F_3'(\psi_2)\Delta\psi]^2 + [\nabla(\Delta\phi) + \nabla(F_4'(\psi_2)\Delta\psi)]^2 + [\nabla(\Delta\psi)]^2 - [\nabla(F_4'(\psi_2)\Delta\psi)]^2\} d\tau. \quad (84)$$

The last term in Eq. (84) can be integrated like as in Eq. (67) and hence Eq. (84) becomes

$$Q_1 + Q_2 = \frac{1}{2}\int_D \{[1 - F_4'^2(\psi_2)][\nabla(\Delta\psi)]^2 + [f_{min} - (F_2'^2)_{max} - (F_3'^2)_{max} + F_4'(\psi_2)\nabla^2 F_4'(\psi_2)](\Delta\psi)^2 + [\Delta v_z + F_2'(\psi_2)\Delta\psi]^2 + [\Delta B_z + F_3'(\psi_2)\Delta\psi]^2 + [\nabla(\Delta\phi) + \nabla(F_4'(\psi_2)\Delta\psi)]^2\} d\tau. \quad (85)$$

Condition (74) holds when Q_1+Q_2 is positive; this holds if

$$F_4'^2(\psi_2) < 1, \quad (86a)$$

$$f_{min} - (F_2'^2)_{max} - (F_3'^2)_{max} + F_4'(\psi_2)\nabla^2 F_4'(\psi_2) > 0. \quad (86b)$$

Thus, in the norm

$$\|(\Delta\psi, \Delta v_z, \Delta B_z, \Delta\omega_z)\|^2 = (Q_1 + Q_2)(\Delta\psi, \Delta v_z, \Delta B_z, \Delta\omega_z), \quad (87)$$

where Q_1+Q_2 is given by Eq. (85), the *a priori* estimate (75) holds. Put

$$A(x,y) = \frac{\phi'(\psi)\nabla\psi\cdot\nabla\omega_z - \nabla\psi\cdot\nabla J_z}{|\nabla\psi|^2}, \quad \text{for all } x,y \in D, \quad (88)$$

and return to Eqs. (51) and (64), we can summarize our findings in the following nonlinear stability criterion.

B. Nonlinear stability criterion

Let $(\psi, v_z, B_z, \omega_z)$ be an equilibrium solution of the system (1)–(6). Suppose the following.

(i) Equations (51a)–(51d) are satisfied for some twice continuously differentiable functions $F_1(\psi)$, $F_2(\psi)$, $F_3(\psi)$, and $F_4(\psi)$.

(ii) For $-\infty < \psi_{min} \leqslant \psi(x,y) \leqslant \psi_{max} < \infty$, the functions $F_2(\psi)$, $F_3(\psi)$, and $f(\psi) = F_1'' + v_z F_2'' + B_z F_3'' + \omega_z F_4''$ satisfy

$$-\infty < (F_2')_{min} \leqslant F_2'(\psi) \leqslant (F_2')_{max} < \infty, \quad (89a)$$

$$-\infty < (F_3')_{min} \leqslant F_3'(\psi) \leqslant (F_3')_{max} < \infty, \quad (89b)$$

$$-\infty < f_{min} \leqslant f(\psi) \leqslant f_{max} < \infty. \quad (89c)$$

(iii) For $-\infty < \psi_{min} \leqslant \psi(x,y) \leqslant \psi_{max} < \infty$,

$$[\phi'(\psi)]^2 < 1, \quad (90a)$$

$$A_{min} + \phi'(\psi)\nabla^2\phi'(\psi) > 0; \quad (90b)$$

then $(\psi, v_z, B_z, \omega_z)$ is nonlinearly stable relative to the norm

$$\|(\Delta\psi, \Delta v_z, \Delta B_z, \Delta\omega_z)\|^2 = \frac{1}{2}\int_D ([1 - \phi'(\psi)^2][\nabla(\Delta\psi)]^2 + [A_{min} + \phi'(\psi)\nabla^2\phi'(\psi)](\Delta\psi)^2 + [\Delta v_z + F_2'(\psi)\Delta\psi]^2 + [\Delta B_z + F_3'(\psi)\Delta\psi]^2 + \{\nabla(\Delta\phi) - \nabla[\phi'(\psi)\Delta\psi]\}^2) d\tau. \quad (91)$$

Note that the nonlinear stability criterion admits comparison with the linear stability Criteria A and B. That is the sub-Alfvénic condition on the flow remains necessary. Condition (90b) is more stringent than condition (69b) of Criterion B.

VI. SUMMARY

Under assumption of translational symmetry (in which the fields have three components with an ignorable coordinate), the equilibrium and stability properties of incompressible flows have been investigated within the framework of nonlinear ideal MHD theory. It has been proved that the solutions of the ideal, incompressible, and homogeneous MHD flows in the plane ($v_z = B_z = 0$) are valid for equilibria with the axial velocity component being a free flux function and the axial magnetic field component being a constant [$v_z = v_z(\psi)$ and B_z=const]. We have established nonlinear Lyapunov stability conditions for the MHD flows considered by using the method of convexity estimates, as explained in Holm *et al.*[8]

Comparing our results with other approaches used in the literature are discussed in the following.

In recent works, the equilibrium and stability properties of ideal incompressible flows in the plane with constant density were investigated by Holm *et al.*, Vladimirov and Moffatt (Part I),[46] and Vladimirov *et al.* (Part II).[47] In Part I, variational principles for incompressible MHD flows were established and a frozen-in field (generalized vorticity) was constructed. The existence of this frozen-in field has consequences for the construction of Casimirs, the integral invariants that play an essential role in the derivation of sufficient conditions for stability (or stability criteria) for steady solutions of the governing equations. In Part II, the approach of Part I is applied to two-dimensional MHD flows where stability criteria were established. By comparing our Criteria with the results of Part II,[47] we find that conditions (69a) and (69b) of criterion B coincide with those obtained there although they are derived there by following a different approach, where the authors[47] described the constant of motion in a non explicit way involving Lagrangian variables. Further, their procedure requires that (as they mentioned in their conclusions) the velocity and magnetic fields have components only in the x and y direction, while we have considered the problem in the presence of the axial velocity and magnetic field components. Criterion A gives another estimate for linear stability of the considered problem. Conditions (69a) and (69b) are also identical with those obtained by Holm *et al.* (Sec. 5.1 there). This proves that the stability conditions for ideal MHD steady states with incompressible flows and constant density in a planar domain are also valid for symmetric equilibria that have a nonplanar velocity field component and a nonplanar magnetic field component ($v_z \neq 0$ and $B_z \neq 0$).

Previously, a sufficient condition for linear stability of force-free fields ($\mathbf{J}=\lambda\mathbf{B}$ with constant λ) was obtained by Tasso[53] (Sec. 3.1 there). This condition is $\int_D[(\boldsymbol{\nabla}\wedge\mathbf{A})^2 - \lambda\mathbf{A}_{perp}\cdot\boldsymbol{\nabla}\wedge\mathbf{A}]d\tau \geqslant 0$, where $\mathbf{A}$ is the vector potential of the magnetic field and $\mathbf{A}_{perp}$ is the part of $\mathbf{A}$ perpendicular to $\mathbf{B}$. Note that this condition is satisfied if $\lambda\mathbf{A}_{perp}\cdot\boldsymbol{\nabla}\wedge\mathbf{A}\leqslant 0$. By comparing this to Criteria A and B, we find that the corresponding condition is $\lambda\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}B_z \leqslant 0$. In the nonlinear case his criterion (Sec. 4.1 in Ref. 53) was $\int_D[(\nabla\wedge\mathbf{A})^2 - \lambda\mathbf{A}\cdot\nabla\wedge\mathbf{A}]d\tau \geqslant 0$, which is satisfied when $\lambda\mathbf{A}\cdot\boldsymbol{\nabla}\wedge\mathbf{A}\leqslant 0$. The corresponding condition by our nonlinear criterion is $(\lambda\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}B_z)_{max}\leqslant 0$. Further, as an agreement with the nonlinear stability of a straight z pinch or tokamak surrounded by perfectly conducting walls with homogeneous current density and no flow in the unperturbed fluid in which he proved several statements (Sec. 4.2 in Ref. 53); we find that this MHD state is nonlinearly stable by our nonlinear stability criterion.

As shown by Tasso,[54] the two-dimensional flows parallel to a vacuum magnetic field are nonlinearly stable. An agreement with this by our criteria, we find that this MHD state is linearly stable by Criteria A and B [example (iii)] as well as it is nonlinearly stable by conditions (90a) and (90b).

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