LETTER

Solutions of All One-Dimensional Wave Equations with Time Independent Potential and Separable Variables

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Solutions, in terms of special functions, of all wave equations $u_{xx} - u_{tt} = V(x)u(t, x)$, characterised by eight inequivalent time independent potentials and by variables separation, have been found. The real valueness and the properties of the solutions produced by computer algebra programs are not always manifest and in this work we provide ready to use solutions. We discuss especially the potential $(m_1 + m_2 \sinh x) \cosh^{-2} x$. Such potential approximates the Schwarzschild black hole potential and its use for determining black holes quasi-normal modes is hinted to.

1. INTRODUCTION

In the frame of our research work on analytic solutions of black holes differential equations [1], we have determined the analytic solutions of all wave equations with time independent potential $V(x)$:

$$mx \tag{1}$$

$$mx^{-2} \tag{2}$$

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$$m \sin^{-2} x \tag{3}$$

$$m \sinh^{-2} x \tag{4}$$

$$m \cosh^{-2} x \tag{5}$$

$$m \exp x \tag{6}$$

$$(m_1 + m_2 \sin x) \cos^{-2} x \tag{7}$$

$$(m_1 + m_2 \sinh x) \cosh^{-2} x \tag{8}$$

$$(m_1 + m_2 \cosh x) \sinh^{-2} x \tag{9}$$

$$m_1 e^x + m_2 e^{2x} \tag{10}$$

$$m_1 + m_2 x^{-2} \tag{11}$$

$$m \tag{12}$$

for the wave equation:

$$u_{xx} - u_{tt} = V(x)u(t, x) \tag{13}$$

These potentials characterize the wave equation by variable separation [2]. They may be reduced to eight irreducible forms. The potential (6) is equivalent to potential (12) with the change of variables:

$$x' = \exp\left(\frac{x}{2}\right) \cosh \frac{t}{2} \qquad t' = \exp\left(\frac{x}{2}\right) \sinh \frac{t}{2}$$

while potentials (3,4,5) are equivalent to potential (2) with the following change of variables respectively:

$$x' = \tan \xi + \tan \eta \qquad t' = \tan \xi - \tan \eta \tag{14}$$

$$x' = \tanh \xi + \tanh \eta \qquad t' = \tanh \xi - \tanh \eta \tag{15}$$

$$x' = \coth \xi + \tanh \eta \qquad t' = \coth \xi - \tanh \eta \tag{16}$$

where

$$\xi = \frac{1}{2}(x + t) \qquad\qquad \eta = \frac{1}{2}(x - t)$$

The general form of the solution with separated variables of eq. (13) is:

$$u(t, x) = \phi_1(\omega_1)\phi_2(\omega_2) \tag{17}$$

where $\omega_1 = \omega_1(t, x)$, $\omega_2 = \omega_2(t, x)$, and $\phi_1(\omega_1)$ and $\phi_2(\omega_2)$ are arbitrary solutions of the separated ordinary differential equation:

$$\frac{d^2\phi_i}{d\omega_i^2} = [(c_i + g_i(\omega_i)]\phi_i \tag{18}$$

c_i being the separation constant and $i = t, x$.

The wave equations separate in several coordinate systems, among which the commonest and simplest poses $\omega_1 = t$, $\omega_2 = x$, $g_t = 0$ and $g_x = V(x)$. With the support of Maple 7 software, we have obtained the general solutions of all differential equations, analysed their properties and proved their real valueness. The solution for the time dependent function ϕ_t equation:

$$\frac{d^2\phi_t}{dt^2} - c_t\phi_t = 0 \tag{19}$$

is:

$$\phi_t = k_1 e^{i\sqrt{c_t}t} + k_2 e^{-i\sqrt{c_t}t} \tag{20}$$

The space dependent solutions in the appendix have been obtained by simplifying the results returned by the *odesolve* function of Maple 7. The solutions corresponding to potentials (7,8,9) require an additional effort to obtain a readable form[4]. It is the latter we directly show in the appendix. Instead, the applicability, the properties, and the real valueness of the solution corresponding to potential (8) have been considered in detail in the next section.

2. THE REGGE–WHEELER–ZERILLI EQUATION

Regge and Wheeler [4] proved the stability of the Schwarzschild black hole in vacuum for axial perturbations, while Zerilli [5] found the equation for polar perturbations. The latter is written in terms of the wave function Ψ_l, for each l–pole component :

$$\frac{d^2\Psi_l(r, t)}{dr^{*2}} - \frac{d^2\Psi_l(r, t)}{dt^2} - V_l(r)\Psi_l(r, t) = 0 \tag{21}$$

where

$$r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{22}$$

is the tortoise coordinate and the potential $V_l(r)$ is:

$$V_l(r) = \left(1 - \frac{2M}{r}\right)\frac{2\lambda^2(\lambda + 1)r^3 + 6\lambda^2 Mr^2 + 18\lambda M^2 r + 18M^3}{r^3(\lambda r + 3M)^2} \tag{23}$$

while $\lambda = \frac{1}{2}(l - 1)(l + 2)$. The Zerilli potential (23) may be approximated in a selected radial coordinate domain, including the maximum, by the potential (8). Blome and Mashhoon[6] have used the Eckart potential [7]:

$$V = V_0 e^{2\mu} - V_0\{\tanh[\alpha(x - x_0) + \mu] - \tanh\mu\}^2 \cosh^2\mu \tag{24}$$

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[4] Ref. [3] has been used for special function properties.
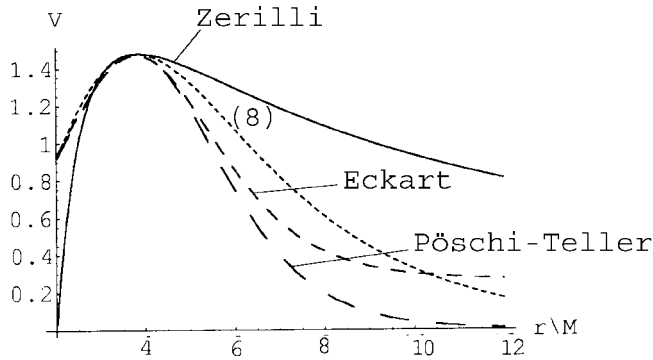
Figure 1. The Zerilli potential, the potential (8), the Eckart and the Pöschl–Teller potentials for $l = 2$ (quadrupole), starting from above. On the y axis the potential in $(1/M^2)$ units; on the x axis the radial coordinate in r/M units.

while Ferrari and Mashhoon[8], Beyer[9] have used the Pöschl–Teller potential[10]:

$$V = \frac{V_0}{\cosh^2 \alpha(x - x_0)} \tag{25}$$

for derivation of the QNM (quasi–normal modes) of a black hole. The ground state plus the first few excited states can be approximated by the bound states of the inverted potential. We note that (8) well reproduces the Zerilli potential and investigations on quasi–normal modes could be pursued. The Zerilli potential has not allowed any analytic determination of the QNM[5]. The polar potential is thus substituted by:

$$V_l'(x) = A[m_1 + m_2 \sinh(kr^*)] \cosh^{-2}(kr^*) \tag{26}$$

where A, m_1, m_2, k are parameters depending on l and M, for curve proper fitting. From fig. 1, we evince that the original black hole Zerilli potential is best replaced by potential (8) for $r/M < 10$ and after by the Eckart potential.

The general form of the solution is:

$$\Psi_l(r^*, t) = \psi_{lr^*}[\omega_{r^*}(r^*, t)]\psi_{lt}[\omega_t(r^*, t)] \tag{27}$$

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[5] Black holes perturbations equations do not admit exact solutions, apart of approximate solutions for portions of the frequency domain, e.g. [11]–[17], or post–Newtonian expansions in the weak field and slow motion, e.g. [18].

where ψ_{li}, $i = r^*, t$ are solutions of the o.d.e.:

$$\frac{d^2\psi_{li}(\omega_i)}{d\omega_i^2} = [c_i + g_i(\omega_i)]\psi_{li}(\omega_i) \tag{28}$$

The d'Alembert equation $[\partial_t^2 - \partial_{r^*}^2 - V'(r^*)]\Psi_l = 0$ separates into four coordinate systems, among which the following is the only with an explicit relation for ω_i:

$$\omega_{r^*} = r^* \quad \omega_t = t \quad f_{r^*} = A[m_1 + m_2 \sinh(kr^*)]\cosh^{-2}(kr^*) \quad f_t = 0 \tag{29}$$

In order to write the solution corresponding to the potential (8) in a more suitable way, we assume $c_x < 0$ and we introduce new real parameters k, x, μ_1, μ_2 and σ such that:

$$m_1 = 1/4 - \mu_1^2 + \mu_2^2, \quad m_2 = 2\mu_1\mu_2, \quad A = k^2, \quad c_x = -\sigma^2 k^2, \quad \sinh(kr^*) = x$$

Consequently:

$$\alpha_3 = 2(\mu_1 + i\mu_2)$$
$$a = 1/2 + \mu_1 - i\sigma$$
$$b = 1/2 + \mu_1 + i\sigma$$
$$c = 1 + \mu_1 + i\mu_2$$
$$z = (1 - ix)/2$$

Accordingly, the potential (23) can be rewritten as follows:

$$\psi_{lr^*} = (x - i)^{(\mu_1+1/2-i\mu_2)/2}(x + i)^{(\mu_1+1/2+i\mu_2)/2}w(z)$$

$$= \sqrt{x^2 + 1}^{(\mu_1+1/2)}e^{-\mu_2 \arctan(1/x)}w(z) \tag{30}$$

where the function $w(z)$ has to be a solution of the hypergeometric equation:

$$z(z - 1)\frac{d^2w(z)}{dz^2} + [z(1 + b + a) - c)]\frac{dw(z)}{dz} + abw(z) = 0 \tag{31}$$

and the general solution of which is:

$$w(z) = k_1 F(a, b; c; z) + k_2 z^{1-c} F(a - c + 1, b - c + 1; 2 - c; z) \tag{32}$$

Eq. (28) has, therefore, the following expression:

$$\psi_{lr^*} = [\cosh(kr^*)]^{(\mu_1+1/2)}e^{-\mu_2 \arctan[1/\sinh(kr^*)]}w\left(\frac{1 - i\sinh kr^*}{2}\right) \tag{33}$$

The solutions (32) must be real valued functions. This is possible since:

$$\bar{a} = b, \quad \bar{b} = a, \quad \bar{c} = a + b + 1 - c, \quad \bar{z} = 1 - z \tag{34}$$

so that:

$$\overline{F(a, b; c; z)} = F(\bar{a}, \bar{b}; \bar{c}; \bar{z}) = F(b, a; a + b + 1 - c; 1 - z) \tag{35}$$

and, moreover, for one of the identities between hypergeometric functions we know that:

$$F(b, a; a + b + 1 - c; 1 - z) = \sigma_1 F(a, b; c; z) \tag{36}$$

$$+ \sigma_2 z^{1-c} F(a - c + 1, b - c + 1; 2 - c; z)$$

where we set:

$$\sigma_1 = \frac{\Gamma(\bar{c})\Gamma(1 - c)}{\Gamma(a - c + 1)\Gamma(b - c + 1)}, \quad \sigma_2 = \frac{\Gamma(\bar{c})\Gamma(1 - c)}{\Gamma(a)\Gamma(b)} \tag{37}$$

Finally, for the potential (8) the space dependent solution is:

$$\phi_x = z^{(c/2 - 1/4)}(1 - z)^{[(a+b-c)/2+1/4]}$$

$$\left[k_1 F(a, b; c; z) + k_2 z^{(1-c)} F(1 + a - c, 1 + b - c; 2 - c; z) \right] \tag{38}$$

where

$$a = (\alpha_3 + \alpha_4)/4 + \beta + 1/2$$

$$b = (\alpha_3 + \alpha_4)/4 - \beta + 1/2$$

$$c = 1 + \alpha_3/2$$

$$z = (1 \pm i \sinh x)/2$$

with $\alpha_3 = \pm\sqrt{1 - 4m_1 \pm 4im_2}$, $\alpha_4 = \pm\sqrt{1 - 4m_1 \mp 4im_2}$ and $\beta = \pm\sqrt{c_x}$.

3. CONCLUSIONS

Solutions, in terms of special functions, of all wave equations $u_{xx} - u_{tt} = V(x)u$, characterized by eight inequivalent time independent potentials and by variable separation, have been found. One, among the potentials, $(m_1 + m_2 \sinh x) \cosh^{-2} x$, has a shape similar to a Schwarzschild black hole potential for polar perturbations.

APPENDIX

In this section we write the space dependent solutions for all potentials, obtained after some computation. For the potential (1), the solution can be written in terms of the Airy functions:

$$\phi_x = k_1 \text{Ai}\left(\frac{c_x + mx}{m^{2/3}}\right) + k_2 \text{Bi}\left(\frac{c_x + mx}{m^{2/3}}\right) \tag{39}$$

or in terms of Bessel K and I functions:

$$\phi_x = \sqrt{c_x + mx} \left\{ k_1 K \left(\frac{1}{3}, \frac{2(c_x + mx)^{3/2}}{3|m|} \right) \right.$$

$$\left. + k_2 \left[I \left(-\frac{1}{3}, \frac{2(c_x + mx)^{3/2}}{3|m|} \right) + I \left(\frac{1}{3}, \frac{2(c_x + mx)^{3/2}}{3|m|} \right) \right] \right\} \quad (40)$$

For the potential (2):

$$\phi_x = \sqrt{x} \left[k_1 J \left(\frac{\sqrt{1+4m}}{2}, x\sqrt{-c_x} \right) + k_2 Y \left(\frac{\sqrt{1+4m}}{2}, x\sqrt{-c_x} \right) \right] \quad (41)$$

where J and Y are Bessel functions.

For the potential (3):

$$\phi_x = \sqrt{\sin x} \left[k_1 P \left(i\sqrt{c_x} - \frac{1}{2}, \frac{1}{2}\sqrt{1+4m}, \cos x \right) \right.$$

$$\left. + k_2 Q \left(i\sqrt{c_x} - \frac{1}{2}, \frac{1}{2}\sqrt{1+4m}, \cos x \right) \right] \quad (42)$$

where $P(\nu, \mu, z) = P_\nu^\mu(z)$ and $Q(\nu, \mu, z) = Q_\nu^\mu(z)$ are associated Legendre functions of the first and second kind, respectively.

For the potential (4):

$$\phi_x = \sqrt{\sinh x} \left[k_1 P \left(\sqrt{c_x} - \frac{1}{2}, \frac{1}{2}\sqrt{1+4m}, \cosh x \right) \right.$$

$$\left. + k_2 Q \left(\sqrt{c_x} - \frac{1}{2}, \frac{1}{2}\sqrt{1+4m}, \cosh x \right) \right] \quad (43)$$

For the potential (5):

$$\phi_x = \sqrt{\cosh x} \left[k_1 P \left(\sqrt{c_x} - \frac{1}{2}, \frac{1}{2}\sqrt{1-4m}, i\,|\sinh x| \right) \right.$$

$$\left. + k_2 Q \left(\sqrt{c_x} - \frac{1}{2}, \frac{1}{2}\sqrt{1-4m}, i\,|\sinh x| \right) \right] \quad (44)$$

For the potential (6):

$$\phi_x = k_1 J \left(2\sqrt{c_x}, 2\sqrt{-m}\,e^{x/2} \right) + k_2 Y \left(2\sqrt{c_x}, 2\sqrt{-m}\,e^{x/2} \right) \quad (45)$$

For the potential (7):

$$\phi_x = z^{(c/2 - 1/4)}(1-z)^{[(a+b-c)/2 + 1/4]}$$

$$\left[k_1 F(a, b; c; z) + k_2 z^{(1-c)} F(1 + a - c, 1 + b - c; 2 - c; z) \right] \quad (46)$$

where F indicates Gauss hypergeometric function $_2F_1$ and the parameters a, b, c and z are defined by:

$$a = (\alpha_1 + \alpha_2)/4 + \beta + 1/2$$
$$b = (\alpha_1 + \alpha_2)/4 - \beta + 1/2$$
$$c = 1 + \alpha_2/2$$
$$z = (1 + \sin x)/2$$

with $\alpha_1 = \pm\sqrt{1 + 4m_1 + 4m_2}$, $\alpha_2 = \pm\sqrt{1 + 4m_1 - 4m_2}$ and $\beta = \pm\sqrt{-c_x}$.

For the potential (9):

$$\phi_x = z^{(c/2-1/4)}(z-1)^{[(a+b-c)/2+1/4]}$$
$$\times \left[k_1 F(a, b; c; z) + k_2 z^{(1-c)} F(1 + b - c, 1 + a - c; 2 - c; z)\right] \quad (47)$$

where

$$a = (\alpha_5 + \alpha_6)/4 + \beta + 1/2$$
$$b = (\alpha_5 + \alpha_6)/4 - \beta + 1/2$$
$$c = 1 + \alpha_6/2$$
$$z = (1 + \cosh x)/2$$

with $\alpha_5 = \pm\sqrt{1 + m_1 + m_2}$, $\alpha_6 = \pm\sqrt{1 + m_1 - m_2}$ and $\beta = \pm\sqrt{c_x}$.

For the potential (10):

$$\phi_x = e^{-x/2}\left[k_1 M\left(-\frac{m_1}{2\sqrt{m_2}}\sqrt{c_x}, -2\sqrt{m_2}\,e^x\right)\right.$$
$$\left. + k_2 W\left(-\frac{m_1}{2\sqrt{m_2}}, \sqrt{c_x}, -2\sqrt{m_2}\,e^x\right)\right] \quad (48)$$

where W and M are Whittaker's functions.

For the potential (11), the space dependent solution is equal to the solution for the potential (2), while for the potential (12), the solution is trivial:

$$\phi_x = K_1 e^{\sqrt{c_x}x} + K_2 e^{-\sqrt{c_x}x} \quad (49)$$

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