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Influence of the centrifugal force and parallel dynamics on the toroidal momentum transport due to small scale turbulence in a tokamak

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The paper derives the gyro-kinetic equation in the comoving frame of a toroidally rotating plasma, including both the Coriolis drift effect [A. G. Peeters *et al.*, Phys. Rev. Lett. **98**, 265003 (2007)] as well as the centrifugal force. The relation with the laboratory frame is discussed. A low field side gyro-fluid model is derived from the gyro-kinetic equation and applied to the description of parallel momentum transport. The model includes the effects of the Coriolis and centrifugal force as well as the parallel dynamics. The latter physics effect allows for a consistent description of both the Coriolis drift effect as well as the ExB shear effect [R. R. Dominguez and G. M. Staebler, Phys. Fluids B **5**, 3876 (1993)] on the momentum transport. Strong plasma rotation as well as parallel dynamics reduce the Coriolis (inward) pinch of momentum and can lead to a sign reversal generating an outward pinch velocity. Also, the ExB shear effect is, in a similar manner, reduced by the parallel dynamics and stronger rotation. © *2009 American Institute of Physics*.
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I. INTRODUCTION

In recent years much progress has been made in the description of the anomalous transport in tokamak plasmas using the gyro-kinetic formalism. Although this work can be considered far from complete, a basic understanding of the physics that governs ion heat,[1–6] electron heat,[7–11] and particle transport[12–15] has been developed. The understanding of anomalous impurity and toroidal momentum transport, however, is less developed and consequently receives much current interest.[16–27] While for the former neoclassical effects might be of importance,[28–30] the neoclassical transport coefficients for toroidal momentum are small,[31] and it is the anomalous transport that determines the rotation profiles.

Toroidal rotation is of interest since plasma flows have a stabilizing influence on plasma instabilities. The study of the underlying theory is further motivated by the large amount of recent experiments.[32–40] These experiments have revealed that the toroidal momentum transport is not purely diffusive as peaked rotation profiles are observed even in the absence of a torque on the plasma. The explanation of these observations requires a consistent development of the gyro-kinetic theory, including the effects of the plasma flow. Such a development is the subject of the present paper.

In this paper the formulation of the gyro-kinetic equations in the comoving frame of a toroidally rotating plasma will be presented. The paper expands upon earlier work[23] in which the Coriolis drift effect was identified to play a key role in the physics of parallel momentum transport. It gives more details on the derivation but also extends the formalism to include the centrifugal force as well as the finite beta effects of the equilibrium. From the gyro-kinetic equation a low field side gyro-fluid model is then derived, in which the effects of the Coriolis drift, centrifugal force, as well as the parallel dynamics are kept. This model allows for the consistent evaluation of both the Coriolis drift effect as well as the ExB shear effect[19] within the same model and, furthermore, allows for a discussion on the influence of strong rotation. Although the paper focuses on the transport of parallel momentum, the formalism also lays the basis for the discussion of the influence of toroidal rotation on various transport channels.[41]

This paper is structured as follows. In Sec. II the equations of motion are derived in the comoving system and the relation to the laboratory frame will be discussed. Section III derives the gyro-kinetic equation from the comoving frame and Sec. IV develops a gyro-fluid model by building moments of this equation. Section V discusses the implications of the model for toroidal momentum transport. Conclusions are drawn in Sec. VI.

II. THE EQUATIONS OF MOTION

The analysis presented here is carried out adopting the gyro-kinetic formalism. The electric field connected with the toroidal rotation, however, does not satisfy the standard ordering of gyro kinetics in which the ExB velocity is of the order of the diamagnetic velocity and, hence, much smaller than the thermal velocity. To deal with rotation velocities approaching the thermal velocity, a transformation of the Lagrangian to the comoving system will be made similar to (but not exactly equivalent with) the work presented in Ref. 42. The problem of a rotating plasma has been treated by several authors within the drift-kinetic theory.[31,43–46] Since the scale length of the electric potential connected with the toroidal rotation will be assumed much larger than the Larmor radius, there is a large overlap between the drift and gyro-kinetic formalism. However, the main interest of the literature on the drift kinetic equation has been the collision dominated transport and, consequently, the equations were manipulated in a different way when compared with the present work.

A. Lagrangian of a rotating system

The Lagrangian one form of a charged particle in an electromagnetic field is

$$\gamma = (Ze\mathbf{A} + m\mathbf{v}) \cdot d\mathbf{x} - \left(Ze\phi + \frac{1}{2}mv^2\right)dt, \tag{1}$$

where $Z(m)$ is the charge number (mass) of the particle, e is the elementary charge, $\mathbf{A}$ and ϕ are the magnetic and electric potentials, respectively, and $(\mathbf{x},\mathbf{v})$ are the position and velocity of the particle. The Lagrangian [Eq. (1)] can be transformed from the fixed to the rotating frame using the transformation

$$\mathbf{v} \rightarrow \mathbf{v} + \mathbf{u}_0,$$

$$\frac{d\mathbf{x}}{dt} \rightarrow \frac{d\mathbf{x}}{dt} + \mathbf{u}_0 \Leftrightarrow ,$$

$$d\mathbf{x} \rightarrow d\mathbf{x} + \mathbf{u}_0 dt. \tag{2}$$

Here, the background velocity $\mathbf{u}_0$ is assumed to be a constant rigid body toroidal rotation with angular frequency $\mathbf{\Omega}$,

$$\mathbf{u}_0 = \mathbf{\Omega} \times \mathbf{x} = R^2\Omega\,\nabla\varphi, \quad \mathbf{\Omega} = (\nabla R \times \nabla\varphi)R\Omega, \tag{3}$$

where φ is the toroidal angle. This is the equilibrium solution of a collisional plasma except for a small poloidal rotation of the order of the diamagnetic velocity,[47,48] which will be neglected here. It is stressed that in this paper the frame always rotates like a rigid body (Ω=constant). In general the angular rotation frequency measured in the laboratory frame is described by a radial profile $\omega_\phi = \omega_\phi(\psi)$ with ψ being the poloidal flux. We will transform to a coordinate system that rotates with the plasma at one particular radial location $\psi = \psi_r$. (To avoid confusion the rotation of the frame will be denoted with a constant $\Omega = \omega_\phi(\psi_r)$, whereas the plasma rotation in the laboratory frame is denoted by ω_ϕ, a function of ψ). The comoving system is suited for a local analysis since the rotation in this frame vanishes at ψ_r, but it is to be noted that in the comoving system there is still a finite gradient in the rotation. Transforming from the laboratory frame to a coordinate system in which the plasma rotation is zero everywhere, i.e., including a differential rotation in the coordinate transformation, however, has the undesirable property that the distance between two stationary points in the comoving system is increasing in time, i.e., the metric tensor is time dependent.

Besides the transformation of the coordinates it is essential to also transform the electric field through the Lorentz transform

$$\mathbf{E} = -\nabla\phi \rightarrow \mathbf{E} + \mathbf{u}_0 \times \mathbf{B}, \tag{4}$$

while the magnetic field, and therefore the poloidal flux, is assumed unchanged. The latter approximation is justified through the nonrelativistic nature of the rotation velocities. Since for a tokamak toroidal symmetry applies

$$\mathbf{A} = \nabla g(R,\psi) \times \nabla\varphi + \psi\nabla\varphi, \tag{5}$$

therefore,

$$\mathbf{A} \cdot \mathbf{u}_0 = \psi\Omega, \tag{6}$$

and the electric potential transformation is given by

$$\phi \rightarrow \phi + \mathbf{A} \cdot \mathbf{u}_0. \tag{7}$$

With both the transformation of the coordinates and the electric field the Lagrangian one form is

$$\gamma = [Ze\mathbf{A} + m(\mathbf{v} + \mathbf{u}_0)] \cdot d\mathbf{x} - \left(Ze\phi + \tfrac{1}{2}mv^2 - \tfrac{1}{2}m\mathbf{u}_0^2\right)dt. \tag{8}$$

Following the same steps as in Ref. 42 the gyro-kinetic Lagrangian one form can be derived with the help of the Lie transforms

$$\Gamma = [Ze\mathbf{A} + m(v_\parallel \mathbf{b} + \mathbf{u}_0)] \cdot d\mathbf{X} + \mu d\alpha - Hdt, \tag{9}$$

where the Hamiltonian H is

$$H = Ze\langle\phi\rangle + \tfrac{1}{2}mv_\parallel^2 + \mu B - \tfrac{1}{2}m\mathbf{u}_0^2. \tag{10}$$

Here, $\mu = mv_\perp^2/2B$ is the magnetic moment, α is the gyro angle, $\mathbf{b} = \mathbf{B}/B$ is the unit vector parallel to the magnetic field, and $v_\parallel = \mathbf{v} \cdot \mathbf{b}$ is the particle velocity parallel to the magnetic field. In this paper, unless stated explicitly otherwise, the gyro-kinetic coordinates $(\mathbf{X},\alpha,v_\parallel,\mu)$ are those of the comoving system.

B. Equations of motion

Again following the steps outlined in Ref. 42 the equations of motion follow directly from the Lagrangian Eq. (9),

$$\frac{d\mathbf{X}}{dt} = \{\mathbf{X},H\} = \frac{\mathbf{b}}{ZeB_\parallel^*} \times \nabla H + \frac{\mathbf{B}^*}{mB_\parallel^*}\frac{\partial H}{\partial v_\parallel},$$

$$\frac{dv_\parallel}{dt} = \{v_\parallel,H\} = -\frac{\mathbf{B}^*}{mB_\parallel^*} \cdot \nabla H \Leftrightarrow , \tag{11}$$

$$mv_\parallel \frac{dv_\parallel}{dt} = -\frac{d\mathbf{X}}{dt} \cdot \nabla H. \tag{12}$$

Here, $\mathbf{B}^*$ is the generalized magnetic field

$$\mathbf{B}^* = \mathbf{B} + \frac{m}{Ze}\nabla \times (v_\parallel \mathbf{b} + \mathbf{u}_0), \tag{13}$$

and

$$B_\parallel^* = \mathbf{b} \cdot \mathbf{B}^*. \tag{14}$$

The ratio of the two appearing in the equations of motion can be written as

$$\frac{\mathbf{B}^*}{B_\parallel^*} = \mathbf{b} - \frac{m}{ZeB_\parallel^*}\mathbf{b} \times \{\mathbf{b} \times [\nabla \times (v_\parallel \mathbf{b} + \mathbf{u}_0)]\}. \tag{15}$$

Furthermore substituting the expression for $\mathbf{u}_0$ gives

$$\nabla \times (v_\parallel \mathbf{b} + \mathbf{u}_0) = v_\parallel \nabla \times \mathbf{b} + 2\boldsymbol{\Omega}, \tag{16}$$

and Eq. (15) becomes

$$\frac{\mathbf{B}^*}{B_\parallel^*} = \mathbf{b} + \frac{mv_\parallel}{ZeB_\parallel^*}\mathbf{b} \times (\mathbf{b} \cdot \nabla)\mathbf{b} + \frac{2m}{ZeB_\parallel^*}\boldsymbol{\Omega}_\perp. \tag{17}$$

Finally, a rigid body rotation is assumed

$$\nabla \mathbf{u}_0^2 = 2R\Omega^2 \nabla R. \tag{18}$$

Substituting the relations derived above into Eq. (11) the velocity is obtained in the form

$$\frac{d\mathbf{X}}{dt} = v_\parallel \mathbf{b} + \frac{mv_\parallel^2}{ZeB_\parallel^*}\mathbf{b} \times (\mathbf{b} \cdot \nabla)\mathbf{b} + \frac{\mu}{Ze}\frac{\mathbf{b} \times \nabla B}{B_\parallel^*} + \frac{\mathbf{b} \times \nabla\langle\phi\rangle}{B_\parallel^*} + 2\frac{mv_\parallel}{ZeB_\parallel^*}\boldsymbol{\Omega}_\perp - \frac{m\Omega^2 R}{ZeB_\parallel^*}\mathbf{b} \times \nabla R. \tag{19}$$

The different terms in this equation have a straightforward interpretation. The first term is the motion along the magnetic field. The second and third are the familiar curvature and grad-B drifts

$$\mathbf{v}_{\mathrm{cu}} = \frac{mv_\parallel^2}{ZeB_\parallel^*}\mathbf{b} \times (\mathbf{b} \cdot \nabla)\mathbf{b}, \tag{20}$$

$$\mathbf{v}_{\nabla B} = \frac{\mu}{Ze}\frac{\mathbf{b} \times \nabla B}{B_\parallel^*}. \tag{21}$$

The fourth term is the ExB drift in the comoving frame

$$\mathbf{v}_E = \frac{\mathbf{b} \times \nabla\langle\phi\rangle}{B_\parallel^*}, \tag{22}$$

and the fifth and sixth terms are related with the plasma rotation. Using the formula to calculate the drift velocity due to a force perpendicular to the field (this equation has been modified compared with the textbook equation to account for the effect of $B_\parallel^*$)

$$\mathbf{v} = \frac{\mathbf{F} \times \mathbf{B}}{ZeBB_\parallel^*}, \tag{23}$$

and substituting for $\mathbf{F}$ the Coriolis (due to the parallel motion) and centrifugal force

$$\mathbf{F}_{\mathrm{co}} = 2mv_\parallel \mathbf{b} \times \boldsymbol{\Omega}, \tag{24}$$

$$\mathbf{F}_{\mathrm{cf}} = m\Omega^2 R \nabla R, \tag{25}$$

one directly obtains the drift velocities

$$\mathbf{v}_{\mathrm{co}} = 2\frac{mv_\parallel}{ZeB_\parallel^*}\boldsymbol{\Omega}_\perp, \tag{26}$$

$$\mathbf{v}_{\mathrm{cf}} = -\frac{m\Omega^2 R}{ZeB_\parallel^*}\mathbf{b} \times \nabla R. \tag{27}$$

The fifth and sixth terms of Eq. (19) can therefore be interpreted as Coriolis drift and centrifugal drift, respectively.

The equation for the change in the parallel velocity directly follows from Eq. (12),

$$mv_\parallel \frac{dv_\parallel}{dt} = -\frac{d\mathbf{X}}{dt} \cdot [Ze \nabla \langle\phi\rangle + \mu \nabla B - m\Omega^2 R \nabla R] \tag{28}$$

and can be put in an alternative form by observing that

$$-Ze\mathbf{v}_{\nabla B} \cdot \nabla\langle\phi\rangle = \mu \mathbf{v}_E \cdot \nabla B, \tag{29}$$

$$-Ze\mathbf{v}_{\mathrm{cf}} \cdot \nabla\langle\phi\rangle = -\mathbf{v}_E \cdot m\Omega^2 R \nabla R, \tag{30}$$

$$-\mathbf{v}_{\mathrm{cf}} \cdot \mu \nabla B = -\mathbf{v}_{\nabla B} \cdot m\Omega^2 R \nabla R, \tag{31}$$

which allows for the reformulation of Eq. (28),

$$mv_\parallel \frac{dv_\parallel}{dt} = -(v_\parallel \mathbf{b} + \mathbf{v}_{\mathrm{cu}} + \mathbf{v}_{\mathrm{co}}) \cdot (Ze \nabla \langle\phi\rangle + \mu \nabla B - m\Omega^2 R \nabla R). \tag{32}$$

Equation (19), together with either Eq. (32) or Eq. (28), describes the motion of the particle in the comoving frame. In Sec. III they will be used to construct the gyro-kinetic equation.

C. Conserved quantities

The equations derived above satisfy energy as well as toroidal angular momentum conservation, as will be shown in this section. The energy conservation follows from Noether's theorem and the gyro-kinetic Lagrangian [Eq. (9)]. For a more complete discussion on the subject see Ref. 49. From Noether's theorem the energy

$$E = \frac{\partial L}{\partial \dot{q}}\dot{q} - L = H.$$

$$E = Ze\langle\phi\rangle + \tfrac{1}{2}mv_\parallel^2 + \mu B - \tfrac{1}{2}m\mathbf{u}_0^2 \tag{33}$$

is conserved. Indeed this is rather obvious from Eq. (28). Compared with the nonrotating system an additional potential energy proportional to Ω^2 appears in the expression for the energy. The Coriolis force (proportional to Ω) is perpendicular to the particle motion and does not do any work. Consequently, there is no contribution to the energy, and the effect of the Coriolis force enters the equations of motion only through the Coriolis drift that appears completely symmetric compared with the drifts due to the magnetic field inhomogeneity. The centrifugal force generates both a drift velocity as well as a contribution to the energy of the particles. Particles moving from the outboard side of a magnetic surface toward the inboard side (toward smaller R) will lose parallel kinetic energy due to this contribution, and the centrifugal force will therefore enhance particle trapping in agreement with the outward directed centrifugal force.

Since axisymmetry is assumed here for all quantities except the turbulent fields, we expect the toroidal angular momentum to be a conserved quantity in the absence of turbulence. The Lagrangian [Eq. (9)] is independent of the toroidal angle φ if it is assumed that the electric potential ϕ is a "background" quantity, that is, independent of φ. The toroidal angular momentum

$$\begin{aligned} P_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} \\ &= [ZeA + m(v_\parallel \mathbf{b} + \mathbf{u}_0)] \cdot R^2 \nabla \varphi \\ &= Ze\psi + \frac{mv_\parallel RB_t}{B} + m\Omega R^2, \end{aligned} \tag{34}$$

where B_t is the toroidal component of the magnetic field, is therefore a conserved quantity.

A perturbed electric potential $\langle \phi \rangle(\varphi)$ connected with a turbulent field breaks the toroidal symmetry and, therefore, the conservation of P_φ. The Euler–Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0 \tag{35}$$

shows that the acceleration of the particles in the toroidal direction changes the value of P_φ,

$$\frac{dP_\varphi}{dt} = -e\frac{\partial \langle \phi \rangle}{\partial \varphi}. \tag{36}$$

D. Laboratory frame

The equations derived so far are formulated in the comoving system. This allows for an elegant and compact derivation, as well as a physical interpretation of the derived phenomena. It is instructive, however, to compare the results of the comoving frame with those of the laboratory frame. The Coriolis and centrifugal force are absent in the laboratory frame, and it is not directly obvious how the same physics effects are obtained in the two frames. It will be shown in this section that the effect of the Coriolis and centrifugal drift in the comoving frame is incorporated in the curvature drift in the laboratory frame. The curvature drifts in the two frames are different due to the different values of the parallel velocities that are obtained through a transformation from one frame into the other. Roughly speaking, one could think of the curvature drift in the laboratory frame to include the effects of the Coriolis and centrifugal drifts. The statement above, however, is not exact. The Coriolis and centrifugal drifts of the comoving frame contain not only the effect of the curvature due to the parallel motion but also curvature effects associated with the ExB motion. The latter are absent in the laboratory frame if one uses the ordering that the ExB velocity is of the order of the diamagnetic velocity. It will be shown that for the usual tokamak parameters the effects related with the curvature of the ExB velocity are small compared with the curvature of the parallel velocity and that the formulations in the two frames yield nearly identical results even if the ExB motion in the laboratory frame is ordered to be equal to the diamagnetic velocity. Of course, in the laboratory frame the plasma rotation must be incorporated into the background distribution to obtain the same physics effects. The discussion above makes clear how a recent formulation of the toroidal momentum pinch in the laboratory frame[25] is completely incorporated in the derivation based on the Coriolis drift[23] (for comparison see also Ref. 50).

In this section the relation between the two frames is briefly outlined. The strategy is as follows: we will start by formulating the equations of motion in the laboratory frame which will be denoted by the index L. A transformation to the comoving frame

$$v_{\parallel L} = u_\parallel + v_\parallel \quad \frac{d\mathbf{X}_L}{dt} = \mathbf{u}_0 + \frac{d\mathbf{X}}{dt}, \tag{37}$$

where $u_\parallel = RB_t\Omega/B$ is the parallel component of the frame rotation ($\mathbf{u}_0 = R^2\Omega\nabla\varphi$), will then be made. The equations thus obtained can be compared with the equations derived in the comoving frame. To simplify the discussion, the low beta approximation will be used here. In general, the curvature drift can be written as

$$\frac{mv_\parallel^2}{ZeB_\parallel^*}\mathbf{b} \times (\mathbf{b}\cdot\nabla)\mathbf{b} = \frac{mv_\parallel^2}{ZeB_\parallel^*}\left(\frac{\mathbf{B}\times\nabla B}{B^2} + \frac{\mathbf{b}\times\nabla p}{B^2/\mu_0}\right). \tag{38}$$

The last term in the brackets is of the order of the plasma beta compared with the first term and will be neglected below. Furthermore, RB_t will be assumed constant.

In the laboratory frame the equation for the evolution of the parallel velocity $v_{\parallel L}$ is

$$\begin{aligned} m\frac{dv_{\parallel L}}{dt} &= -Ze\left[\mathbf{b} + \frac{mv_{\parallel L}}{ZeB_\parallel^*}\frac{\mathbf{B}\times\nabla B}{B^2}\right]\cdot\nabla\langle\phi_L + \phi\rangle \\ &\quad - \mu\mathbf{b}\cdot\nabla B. \end{aligned} \tag{39}$$

Note that the electrostatic potential $\phi_L = \phi_L(\psi)$ connected with the ExB velocity ($\mathbf{v}_{EL}$) of the toroidal rotation must be kept in the laboratory frame [see also Eq. (4)]. This potential is such that the ExB velocity ($\mathbf{v}_{EL}$), together with a parallel flow ($u_\parallel = RB_t\Omega/B$), generates the toroidal rotation

$$\mathbf{v}_{EL} + u_\parallel\mathbf{b} = \mathbf{u}_0 = R^2\Omega\nabla\varphi. \tag{40}$$

Of course, the averaged parallel flow in the laboratory frame is determined by an averaged motion of the particles and, therefore, does not enter explicitly in Eq. (39). Because the combination of $u_\parallel\mathbf{b}$ and $\mathbf{v}_{EL}$ is in the symmetry direction,

$$\mathbf{v}_{EL}\cdot\nabla B = -u_\parallel\mathbf{b}\cdot\nabla B, \tag{41}$$

with a similar equation applying to all quantities, which are independent of the toroidal angle.

Now using the transformation of Eq. (37), rearranging the vector products connected with $\langle\nabla\phi_L\rangle$, and using the equation above yields

$$\begin{aligned} m\frac{dv_\parallel}{dt} &= -Ze\left[\mathbf{b} + \frac{m(v_\parallel + u_\parallel)}{ZeB_\parallel^*}\frac{\mathbf{B}\times\nabla B}{B^2}\right]\cdot\nabla\langle\phi\rangle \\ &\quad - \left[\mu + \frac{mu_\parallel(v_\parallel + u_\parallel)}{B}\right]\mathbf{b}\cdot\nabla B - m\frac{du_\parallel}{dt}. \end{aligned} \tag{42}$$

The time derivative of $u_\parallel$ follows from

$$\frac{du_\parallel}{dt} = \frac{d\mathbf{X}_L}{dt} \cdot \nabla\left(\frac{RB_t\Omega}{B}\right)$$
$$= -\frac{u_\parallel}{B}[(v_\parallel + u_\parallel)\mathbf{b} + \mathbf{v}_{EL} + \mathbf{v}_E] \cdot \nabla B. \quad (43)$$

Combining the two equations above and again rearranging the vector products yields

$$m\frac{dv_\parallel}{dt} = -Ze\left[\mathbf{b} + \frac{m(v_\parallel + 2u_\parallel)}{ZeB_\parallel^*}\frac{\mathbf{B}\times\nabla B}{B^2}\right] \cdot \nabla\langle\phi\rangle - \left(\mu\mathbf{b} + \frac{mu_\parallel^2}{B}\mathbf{b}\right)\cdot\nabla B. \quad (44)$$

The velocity of the particle can be more readily obtained from the transformation of Eq. (37),

$$\frac{d\mathbf{X}}{dt} = v_\parallel\mathbf{b} + \mathbf{v}_E + \frac{m(v_\parallel^2 + 2v_\parallel u_\parallel + u_\parallel^2) + \mu B}{ZeB_\parallel^*}\frac{\mathbf{B}\times\nabla B}{B^2}. \quad (45)$$

Comparing the equations above with the equations of the comoving system derived Sec. II B one finds agreement if the Coriolis and centrifugal drifts as well as the magnetic field strength can be approximated as

$$\mathbf{v}_{\rm co} = \frac{2mv_\parallel u_\parallel}{ZeB_\parallel^*}\frac{\mathbf{B}\times\nabla B}{B^2}, \quad (46)$$

$$\mathbf{v}_{\rm cf} = \frac{mu_\parallel^2}{ZeB_\parallel^*}\frac{\mathbf{B}\times\nabla B}{B^2}, \quad (47)$$

$$B \propto \frac{1}{R}. \quad (48)$$

The last of these three conditions makes that the terms due to ∇R in Eq. (32) can be approximated as $mu_\parallel^2\mathbf{b}\cdot\nabla B/B$. Assuming the poloidal magnetic field (B_p) of the order ϵ compared with the toroidal field (B_t), where $\epsilon = r/R$ is the inverse aspect ratio, one arrives at

$$B_t = B + \mathcal{O}(\epsilon^2) \propto \frac{1}{R} + \mathcal{O}(\epsilon^2), \quad (49)$$

$$u_\parallel = R\Omega + \mathcal{O}(\epsilon^2), \quad (50)$$

$$\frac{\mathbf{B}\times\nabla B}{B^2} = -\frac{1}{R}\mathbf{b}\times\nabla R + \mathcal{O}(\epsilon^2), \quad (51)$$

$$\mathbf{\Omega}_\perp = -\Omega\mathbf{b}\times\nabla R + \mathcal{O}(\epsilon^2). \quad (52)$$

The approximations of Eqs. (46)–(48) are, therefore, valid up to first order in the inverse aspect ratio.

The derivation above shows that starting from the laboratory frame and transforming the equations to a comoving frame, terms appear which are very similar to the Coriolis and centrifugal drifts directly derived from the Lagrangian in the comoving frame. It follows that the physics effects connected with the Coriolis and centrifugal drifts in the comoving frame will be largely recovered in the laboratory frame. The drift responsible for the physics effects in the laboratory frame is the curvature drift. A hand waving derivation can be obtained through the replacement of the parallel velocity ($v_{\parallel L} = v_\parallel + u_\parallel$) in the curvature drift

$$\frac{mv_{\parallel L}^2}{ZeB_\parallel^*}\frac{\mathbf{B}\times\nabla B}{B^2} \to \frac{mv_\parallel^2}{ZeB_\parallel^*}\frac{\mathbf{B}\times\nabla B}{B^2} + \frac{2mv_\parallel u_\parallel}{ZeB_\parallel^*}\frac{\mathbf{B}\times\nabla B}{B^2} + \frac{mu_\parallel^2}{ZeB_\parallel^*}\frac{\mathbf{B}\times\nabla B}{B^2}, \quad (53)$$

and identifying the first term with the curvature drift in the comoving frame, the second with the Coriolis drift in the comoving frame, and the third with the centrifugal force in the comoving frame. The small difference between the formulation in the laboratory and comoving frame can be traced to the curvature drift related with the background ExB velocity ($\mathbf{v}_{EL}$). The equations of the laboratory frame used here do not include this curvature drift, but the effect is included for the comoving frame through the inertial term connected with $\mathbf{u}_0$. The curvature drift connected with $\mathbf{v}_{EL}$ can be added to the equations of motion in the laboratory frame. It can be shown that

$$\mathbf{v}_{De} = \frac{mv_{\parallel L}}{ZeB_\parallel^*}\mathbf{b}\times[(\mathbf{v}_{EL}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{v}_{EL}]$$
$$= \frac{mv_{\parallel L}}{ZeB_\parallel^*}\left(2\mathbf{\Omega}_\perp - 2u_\parallel\frac{\mathbf{B}\times\nabla B}{B^2}\right), \quad (54)$$

i.e., the additional (linearized) contribution to the curvature drift due to background ExB velocity exactly makes up for the difference with the Coriolis drift obtained in comoving frame.

III. THE GYRO-KINETIC EQUATION

Using the phase space conservation

$$\frac{1}{B_\parallel^*}\left[\nabla\cdot\left(B_\parallel^*\frac{d\mathbf{X}}{dt}\right) + \frac{\partial}{\partial v_\parallel}\left(B_\parallel^*\frac{dv_\parallel}{dt}\right)\right] = 0, \quad (55)$$

one can write the gyro-kinetic equation in the form

$$\frac{\partial F}{\partial t} + \frac{d\mathbf{X}}{dt}\cdot\frac{\partial F}{\partial\mathbf{X}} + \frac{dv_\parallel}{dt}\frac{\partial F}{\partial v_\parallel} = 0. \quad (56)$$

As is custom we split the distribution function (F) in a perturbed distribution (f) and a Maxwellian background (F_M),

$$F_M = \frac{n_0}{(2\pi T/m)^{3/2}}\exp\left[-\frac{m(v_\parallel - u_\parallel)^2}{2T} - \frac{\mu B}{T}\right]. \quad (57)$$

In the case of a finite centrifugal force the equilibrium will require the existence of a background potential, which will be denoted as Φ, i.e., in the derived equations the potential will be replaced as $\phi \to \Phi + \phi$. The parallel flow occurring in the Maxwellian distribution is the parallel flow relative to the comoving frame

$$u_\parallel = \frac{RB_t}{B}[\omega_\phi(\psi) - \Omega]. \quad (58)$$

A local model will be constructed here with $\Omega = \omega_\phi$ satisfied at the local magnetic surface considered. Therefore, $u_\parallel = 0$ but does have a finite radial gradient. The temperature will be assumed a flux function $T=T(\psi)$.

The Maxwellian must satisfy the equilibrium equation which at the surface for which $\Omega=\omega_\phi$ and neglecting the drifts is given by

$$v_\parallel \mathbf{b}\cdot\nabla F_M - \frac{1}{m}\mathbf{b}\cdot[Ze\nabla\langle\Phi\rangle + \mu\nabla B - m\Omega^2 R\nabla R]\frac{\partial F_M}{\partial v_\parallel} = 0. \quad (59)$$

The derivatives of the Maxwellian in this equation can then be evaluated to be

$$v_\parallel \mathbf{b}\cdot\nabla F_M = v_\parallel \mathbf{b}\cdot\left(\frac{\nabla n_0}{n_0} - \frac{\mu\nabla B}{T}\right)F_M. \quad (60)$$

$$\frac{\partial F_M}{\partial v_\parallel} = -\frac{m v_\parallel}{T}F_M, \quad (61)$$

yielding

$$\nabla_\parallel \ln(n_0) = -\frac{Ze\nabla_\parallel\langle\Phi\rangle}{T} + \frac{m\Omega^2 R\nabla_\parallel R}{T} \quad (62)$$

with the solution

$$n_0 = n_{R_0}\exp\left[-\frac{Ze\langle\Phi\rangle}{T} + \frac{m\Omega^2(R^2-R_0^2)}{2T}\right], \quad (63)$$

where R_0 is the integration constant which will be chosen to be the radius of the magnetic axis. The integration constant of the equilibrium equation is more commonly chosen to be the flux surface average of R^2 such that, for not too large rotation, n_{R_0} represents the flux surface average of the density. Here this choice is not adopted because the flux surface average of R^2 is a function of the poloidal flux and its gradient would appear in the equations. The choice made above implies that n_{R_0} represents the density at $R=R_0$.

In the equations above the equilibrium potential Φ has been introduced. As mentioned earlier the centrifugal force, which has a different magnitude for ions and electrons, requires such a potential in order to satisfy quasineutrality. With Ω given, the quasineutrality constraint will completely determine Φ. For a plasma consisting of singly charged ions one can readily derive

$$e\langle\Phi\rangle = \frac{T_e T_i}{T_e+T_i}\left[\frac{m_i}{T_i} - \frac{m_e}{T_e}\right]\frac{1}{2}\Omega^2(R^2-R_0^2), \quad (64)$$

but for a plasma with more species the determination of the potential is not as straightforward.

To proceed, we define an energy (species dependent)

$$\mathcal{E} = Ze\langle\Phi\rangle - \tfrac{1}{2}m\Omega^2(R^2-R_0^2). \quad (65)$$

The Maxwellian background can then be written as

$$F = \frac{n_{R_0}}{(2\pi T/m)^{3/2}}\exp\left[-\frac{m(v_\parallel-u_\parallel)^2/2 + \mu B + \mathcal{E}}{T}\right], \quad (66)$$

and the ExB velocity due to the background potential (Φ) can be combined with the centrifugal drift into one term, i.e.,

$$\frac{d\mathbf{X}}{dt} = v_\parallel\mathbf{b} + \frac{m v_\parallel^2}{ZeB_\parallel^*}\mathbf{b}\times(\mathbf{b}\cdot\nabla)\mathbf{b} + \frac{\mu}{ZeB_\parallel^*}\mathbf{b}\times\nabla B + \frac{1}{ZeB_\parallel^*}\mathbf{b}\times\nabla\mathcal{E} + \frac{\mathbf{b}\times\nabla\langle\phi\rangle}{B_\parallel^*} + \frac{2mv_\parallel}{ZeB_\parallel^*}\mathbf{\Omega}_\perp. \quad (67)$$

In the equation above ϕ refers only to the perturbed potential.

Although the Maxwellian background has been derived here retaining only the parallel derivatives, it can be shown that all terms in the equilibrium equation containing ∇B or $\nabla\mathcal{E}$ cancel, and the given Maxwellian distribution is an exact solution in the absence of potential fluctuations if both the density as well as the temperature are uniform. Finite radial density and temperature gradients, however, lead to nonvanishing terms in the equilibrium equation, and the Maxwellian as given is no longer an exact solution. Such terms will be neglected here. This approximation is similar to the usual approximation made for the nonrotating system in which the Maxwellian is also not an exact equilibrium solution. The drifts in the gradients of density and temperature lead to small corrections that are connected with the neoclassical transport fluxes.

For the perturbed distribution we will furthermore make the δf approximation assuming

$$\frac{\partial f}{\partial v_\parallel} \ll \frac{\partial F_M}{\partial v_\parallel}, \quad (68)$$

i.e., neglecting the parallel velocity nonlinearity. This results in the gyro-kinetic equation

$$\frac{\partial f}{\partial t} + \frac{d\mathbf{X}}{dt}\cdot\nabla f - \frac{\mathbf{b}}{m}\cdot(\mu\nabla B + \nabla\mathcal{E})\frac{\partial f}{\partial v_\parallel} = -\mathbf{v}_E\cdot\nabla_p F_M - \frac{d\mathbf{X}}{dt}\cdot\frac{Ze\nabla\langle\phi\rangle}{T}F_M, \quad (69)$$

where

$$\nabla_p F = \left[\frac{\nabla n_{R_0}}{n_{R_0}} + \left(\frac{m v_\parallel^2/2 + \mu B + \mathcal{E}}{T} - \frac{3}{2}\right)\frac{\nabla T}{T} + \frac{m v_\parallel R B_t}{BT}\nabla\omega_\phi\right]F_M. \quad (70)$$

The equations above are the gyro-kinetic formulation of a toroidally rotating plasma in the local limit under the delta-f approximation. As noted in Ref. 23 the Coriolis force leads only to an additional drift. The influence of the centrifugal force enters the equations in several ways, all of which can be identified through the energy $\mathcal{E}$. The centrifugal effects lead to a drift, but also an enhanced trapping represented by the third term in Eq. (69), and a modification of the background as is clear from Eq. (70). The effect of the Coriolis drift, neglecting the centrifugal force, i.e., putting $\mathcal{E}=0$, has previously been implemented in the gyro-kinetic code GKW,[51] and has been shown to lead to an inward pinch of toroidal momentum[23] in linear simulations. Recent results[26] based on GYRO (Ref. 52) have shown this effect to persist in the nonlinear regime.

IV. FLUID MODEL

Much insight in the physics can be gained from a low field side gyro-fluid model. Such a model is motivated by the ballooning nature of the instabilities, leading to a localization of the mode on the low field side. In this model a magnetic field structure of the form

$$\mathbf{B} = B\mathbf{e}_y, \quad \nabla B = -\frac{B}{R}\mathbf{e}_x \tag{71}$$

is assumed, where R is the major radius of the considered surface at the low field side position. The low beta approximation is used and the rotation vector is

$$\boldsymbol{\Omega} = \Omega \mathbf{e}_z. \tag{72}$$

For simplicity only one ion species will be considered here, but the formalism can readily be extended to include more than one species.

To describe the effect of the rotation, three dimensionless quantities are introduced,

$$u \equiv \frac{R\Omega}{v_{\rm th}}, \quad u' \equiv -\frac{R^2}{v_{\rm th}}\frac{d\omega_\phi}{dx}, \quad E = \frac{\mathcal{E}}{T}, \tag{73}$$

where $v_{\rm th} = \sqrt{2T/m}$ is the thermal velocity. All gradients will be defined positive for the usual case of peaked profiles. The normalized density, temperature, and rotation energy gradients are

$$\frac{R}{L_N} \equiv -\frac{R}{n_{R_0}}\frac{dn_{R_0}}{dx}, \quad \frac{R}{L_T} \equiv -\frac{R}{T}\frac{dT}{dx}, \quad E' = -\frac{R}{T}\frac{d\mathcal{E}}{dx}. \tag{74}$$

With these definitions the gyro-kinetic equation can be written in the form

$$\begin{aligned}\omega f &+ \left[\frac{mv_\parallel^2 + mv_\perp^2/2 + 2mv_\parallel v_{\rm th}u}{ZT} + \frac{E'}{Z}\right]f - \frac{v_\parallel k_\parallel}{\omega_D}f \\ &= \left[\frac{R}{L_N} + \left(\frac{mv_\parallel^2 + mv_\perp^2}{2T} + E - \frac{3}{2}\right)\frac{R}{L_T} + \frac{mv_\parallel v_{\rm th}}{T}u' \right. \\ &\quad \left. - \frac{mv_\parallel^2 + mv_\perp^2/2 + 2mv_\parallel v_{\rm th}u}{T} - E' + Z\frac{v_\parallel k_\parallel}{\omega_D}\right]F\phi. \end{aligned} \tag{75}$$

In the equation above, the frequency has been normalized to the drift frequency

$$\omega_D = -\frac{k_z T}{eBR}, \tag{76}$$

with k_z representing the (poloidal) component of the wave vector, and $k_\parallel$ the component parallel to the magnetic field. The potential has been normalized to e/T, and finite Larmor radius (FLR) effects have been neglected.

The gyro-fluid equations for the density, parallel flow velocity, and temperature are obtained from the equation above by taking the appropriate moments. The integrals over the Maxwellian can readily be performed. To write the equations in a compact form, we define the average

$$\{G\}_F = \exp[E]\int d^3\mathbf{v}GF. \tag{77}$$

Note that the common factor $\exp[-E]$ that appears in all integrals over the Maxwellian is canceled by the exponent in front of the integrals. One then arrives at

$$\begin{aligned}&\{1\}_F = n_{R_0}, \quad \{mv_\parallel^2\}_F = n_{R_0}T, \\ &\{mv_\perp^2\}_F = 2n_{R_0}T, \quad \{m^2v_\parallel^4\}_F = 3n_{R_0}T^2, \\ &\{m^2v_\perp^4\}_F = 8n_{R_0}T^2, \quad \{m^2v_\parallel^2v_\perp^2\}_F = 2n_{R_0}T^2.\end{aligned} \tag{78}$$

All odd moments in $v_\parallel$ are zero. The moments of the perturbed distribution are evaluated in a similar way,

$$\{G\}_f = \exp[E]\int d^3\mathbf{v}Gf. \tag{79}$$

For these moments a Maxwellian closure will be assumed. The various quantities can then be directly obtained through a linearization of the results given in Eq. (78),

$$\begin{aligned}&\{1\}_f = n_{R_0}\tilde{n}, \\ &\{mv_\parallel^2\}_f = n_{R_0}T(\tilde{n} + \tilde{T}), \\ &\{mv_\perp^2\}_f = 2n_{R_0}T(\tilde{n} + \tilde{T}), \\ &\{m^2v_\parallel^4\}_f = 3n_{R_0}T^2(\tilde{n} + 2\tilde{T}), \\ &\{m^2v_\perp^4\}_f = 8n_{R_0}T^2(\tilde{n} + 2\tilde{T}), \\ &\{m^2v_\parallel^2v_\perp^2\}_f = 2n_{R_0}T^2(\tilde{n} + 2\tilde{T}).\end{aligned} \tag{80}$$

The perturbed density $(\tilde{n})$ is normalized to the local density $n_{R_0}\exp[-E]$, and the perturbed temperature $(\tilde{T})$ to the background temperature T. Finally the odd moments of the perturbed distribution are evaluated as

$$\begin{aligned}&\{v_\parallel\}_f = n_{R_0}v_{\rm th}\tilde{w}, \\ &\{mv_\parallel^3\}_f = 3n_{R_0}Tv_{\rm th}\tilde{w}, \\ &\{mv_\perp^2v_\parallel\}_f = 2n_{R_0}Tv_{\rm th}\tilde{w},\end{aligned} \tag{81}$$

which can be directly derived from a shifted Maxwellian with a perturbed velocity $\tilde{w}$. The latter velocity has been normalized to the thermal ion velocity $v_{\rm th}$. It is noted here that the equations above neglect the parallel heat fluxes, which is a clear simplification (see Refs. 53–59).

Using the relations above the gyro-fluid equations can be written in the form

$$\omega\tilde{n} + \frac{2}{Z}\tilde{n} + \frac{2}{Z}\tilde{T} + \frac{4}{Z}(u + k_{\parallel N})\tilde{w} + \frac{1}{Z}E'\tilde{n} = \left[\frac{R}{L_N} - 2 + E\frac{R}{L_T} - E'\right]\phi, \quad (82)$$

$$\omega\tilde{w} + \frac{4}{Z}\tilde{w} + \frac{2}{Z}(u + k_{\parallel N})\tilde{n} + \frac{2}{Z}(u + k_{\parallel N})\tilde{T} + \frac{1}{Z}E'\tilde{w} = [u' - 2(u + k_{\parallel N})]\phi, \quad (83)$$

$$\omega\tilde{T} + \frac{4}{3Z}\tilde{n} + \frac{14}{3Z}\tilde{T} + \frac{8}{3Z}(u + k_{\parallel N})\tilde{w} + \frac{1}{Z}E'\tilde{T} = \left[\frac{R}{L_T} - \frac{4}{3}\right]\phi, \quad (84)$$

where $k_{\parallel N}$ is the normalized parallel wave vector

$$k_{\parallel N} = \frac{k_\parallel R}{2k_z\rho_Z} \quad \text{with} \quad \rho_Z = \frac{mv_{\text{th}}}{ZeB}. \quad (85)$$

Finally the quasineutrality equation is

$$Z\tilde{n}\tau = \phi, \quad (86)$$

where $\tau = T_e/T$, with T_e being the electron temperature. The equations above are the gyro-fluid equations for the rotating system. The reader can readily verify that they go over in the fluid equation given in Ref. 23 when the centrifugal effects (essentially E and E') as well as the parallel wave vector are set to zero.

V. IMPLICATIONS OF THE MODEL

The model derived in Sec. IV contains several extensions when compared with the model derived in Ref. 23. The effect of the centrifugal force and the parallel dynamics is kept. Interestingly, it can be cast in the same form as the expressions given in Ref. 23. To do this it is assumed here that $Z=1$, $\tau=1$. Since the density is nonuniform on the flux surface R/L_N in the equations above does not represent the local density gradient on the low field side. The local density is $n_{R_0}\exp[-E]$ and its gradient length can be expressed as

$$\frac{R}{L_{N*}} = \frac{R}{L_N} + E\frac{R}{L_T} - E'. \quad (87)$$

Furthermore, the effect of the centrifugal drift on the left hand side of the equations can be absorbed into a new definition of the frequency

$$\omega_* = \omega + \frac{1}{Z}E', \quad (88)$$

and finally, we note that the Coriolis drift terms and the parallel wave vector appear in a symmetric way. Therefore, a new normalized velocity can be defined as

$$\hat{u} = u + k_{\parallel N}. \quad (89)$$

The definitions above allow the fluid equations to be written in the form

$$\omega_*\tilde{n} + 2\tilde{n} + 2\tilde{T} + 4\hat{u}\tilde{w} = \left[\frac{R}{L_{N*}} - 2\right]\phi, \quad (90)$$

$$\omega_*\tilde{w} + 4\tilde{w} + 2\hat{u}\tilde{n} + 2\hat{u}\tilde{T} = [u' - 2\hat{u}]\phi, \quad (91)$$

$$\omega_*\tilde{T} + \frac{4}{3}\tilde{n} + \frac{14}{3}\tilde{T} + \frac{8}{3}\hat{u}\tilde{w} = \left[\frac{R}{L_T} - \frac{4}{3}\right]\phi, \quad (92)$$

i.e., the equations have the same form as those in Ref. 23.

It follows that the centrifugal force only affects the real part of the frequency and does not influence the growth rate of the instability or the radial fluxes of momentum and energy. This conclusion can be understood as follows. The centrifugal drift is proportional to the square of the frame rotation and independent of the particle velocity. The drift is the same for all ions independent of their velocity and, therefore, does not lead to a coupling of density, temperature, and parallel velocity perturbations. It merely leads to a Doppler shift in the frequency represented by Eq. (88). The centrifugal force does lead to a redistribution of the density within the magnetic surface, but this effect can be included through the use of the local density gradient in Eq. (87). Of course, the result that the centrifugal force does not have any influence is related to the approximations made to obtain the fluid model and can be expected to be only approximately true for the gyro-kinetic simulations in full toroidal geometry. At any location, other than the low field side, the centrifugal drift has a component perpendicular to the flux surface and its effect can no longer be described as a simple Doppler shift. Furthermore the Doppler shift as given in Eq. (88) is dependent on the charge and mass of the species, such that for the case of kinetic electrons or impurity ions, one cannot transform the frequency with a single shift.

In Ref. 23 the set of equations above was used to derive the radial flux of parallel momentum neglecting all terms that are quadratic in the toroidal velocity (u). Below a similar derivation is made, keeping u to all orders and including the effect of the parallel dynamics. Multiplying the density equation with $\hat{u}$ and subtracting it from the momentum equation one can derive

$$[\omega_* + 4(1 - \hat{u}^2)]\tilde{w} = \left[u' + \left(\omega_* - \frac{R}{L_{N*}}\right)\hat{u}\right]\phi, \quad (93)$$

where an adiabatic electron response ($\tilde{n}=\phi$) has been assumed. The radial flux of parallel momentum (Γ_ϕ) is proportional to $\text{Im}(\phi^\dagger\tilde{w})$,[60,61] where Im refers to the imaginary part and the dagger denotes the complex conjugate. Then splitting the frequency in a real and imaginary part $\omega_* = \omega_{*R} + i\gamma$ one can derive

$$\Gamma_\phi \propto -\gamma|\phi|^2\left[u' - 4(1 - \hat{u}^2)\hat{u} - \frac{R}{L_N}\hat{u}\right]. \quad (94)$$

Here the flux will be split into three contributions: the diagonal part, the Coriolis pinch, and the effect due to a finite parallel wave vector

$$\Gamma_\phi = \chi_\phi u' + RV_\phi u + D_k\frac{k_\parallel R}{k_z\rho}. \quad (95)$$

The equation above yields the ratio of the transport coefficients

$$\frac{RV_\phi}{\chi_\phi} = -4(1-u^2) + 12k_{\parallel N}^2 - \frac{R}{L_N}, \tag{96}$$

$$\frac{D_k}{\chi_\phi} = -2(1-k_{\parallel N}^2) + 6u^2 - \frac{1}{2}\frac{R}{L_N}. \tag{97}$$

Since $k_{\parallel N}$ appears in the equation for V_ϕ and u appears in the equation for D_k, this splitting of the fluxes is somewhat arbitrary. The choice made above is such that no linear $k_{\parallel N}(u)$ terms appear in the expression for $V_\phi(D_k)$.

In a tokamak the parallel wave vector will have a finite value, but in most cases the modes with positive and negative $k_\parallel$ are equally unstable. This can be modeled by assuming $k_\parallel=0$, but $k_\parallel^2 \approx 1/q^2R^2 \neq 0$. In the presence of an ExB shearing, a finite parallel wave vector is generated,[19,20,24] which will be assumed much smaller than $1/qR$. With these approximations the equation above yields

$$\frac{RV_\phi}{\chi_\phi} = -4(1-u^2) + \frac{3}{q^2(k_z\rho)^2} - \frac{R}{L_N}, \tag{98}$$

$$\frac{D_k}{\chi_\phi} = -2 + \frac{1}{2q^2(k_z\rho)^2} + 6u^2 - \frac{1}{2}\frac{R}{L_N}. \tag{99}$$

To proceed, an expression for the parallel wave vector due to the ExB shear is required. Here, the equations derived in Ref. 24 appropriate for the ITG,

$$\frac{k_\parallel R}{k_z\rho} = \frac{1}{2}\left[1 + \frac{L_N}{L_T}\right]\frac{L_N}{L_s}\frac{R}{v_{\mathrm{th}}}\frac{dv_{Ez}}{dx}, \tag{100}$$

will be used. In the equation above $L_s=qR/\hat{s}$ is the length scale associated with the magnetic shear. Then using the radial force balance

$$\mathbf{E} = \frac{\nabla p}{en} - \mathbf{V}\times\mathbf{B}, \tag{101}$$

one can derive

$$\frac{R}{v_{\mathrm{th}}}\frac{dv_{Ez}}{dx} = \frac{\rho_*}{2}\left[\frac{R}{L_T}\frac{R}{L_N} - \left(\frac{R}{L_N}\right)^2 + \frac{R^2}{n}\frac{d^2n}{dr^2} + (1-k_{\mathrm{neo}})\frac{R^2}{T}\frac{d^2T}{dr^2}\right] - \frac{\epsilon}{q}u', \tag{102}$$

where only derivatives of the profile functions have been retained. The equation above is similar to that of Ref. 20, which discusses the spontaneous spin of the plasma through the second derivative of the temperature gradient.

VI. DISCUSSION/CONCLUSIONS

In this paper, starting from a Lagrangian description in the comoving frame, the gyro-kinetic formalism for the description of small scale instabilities in a toroidally rotating plasma has been further developed. The gyro-kinetic equation has been formulated retaining the effects of the Coriolis and centrifugal forces. A gyro-fluid model is derived by taking moments of the gyro-kinetic equation. This model allows for the investigation of the influence of the centrifugal force on toroidal momentum transport and also for the discussion of both the effects due to the Coriolis drift as well as the ExB shearing within one model.

The work presented in this paper allows for the following conclusions:

- The Coriolis force enters the equations only through the Coriolis drift, which appears symmetric with the drift due to the magnetic field inhomogeneity. If the centrifugal force is neglected there is no change in the energy of the particles.
- The centrifugal force appears through the centrifugal drift but also through an enhanced mirror force. The background distribution is modified through the latter effect and the particle energy contains an additional potential energy connected with the rotation of the frame.
- The influence of the centrifugal force on the toroidal momentum transport is expected to be weak with the main effect being due to the redistribution of the density within the magnetic surface.
- The parallel dynamics weakens the Coriolis pinch effect. From Eq. (98) the pinch is expected to be smaller (less negative) for smaller values of the safety factor q, as well as for larger poloidal wavelengths, and can change sign leading to an outward momentum pinch. It is noted that gyro-kinetic simulations of the Coriolis pinch presented in Ref. 23 indeed support these observations, although the change in the pinch velocity with q and k_z appears to be weaker in the simulations. The influence of the parallel dynamics as given in Eq. (98) by the second term is relatively large. For small wavelengths and a small safety factor the pinch would change sign and would be outward. Although a change in sign is possible it is expected that the term overestimates the effect since $k_\parallel = 1/qR$ is not a good approximation for long wavelength ($k_z\rho \ll 1$) modes. Furthermore, all quantities in the fluid model are evaluated on the low field side. For modes that are extended along the field this is obviously not a good approximation. In this respect we note that the original fluid model was shown to be of limited accuracy when compared with the gyro-kinetic simulations.[23] The fluid model, however, does capture the trends and is useful in developing the physics insight.
- Also stronger plasma rotation weakens the Coriolis pinch effect. A substantial deviation, however, requires toroidal Mach numbers close to unity, which is larger than most plasma experiments.
- In a symmetric way the ExB shearing effect is reduced by the parallel dynamics as well as the finite toroidal rotation. The latter effect again requires relatively large rotation velocities.
- The ExB shear is linked to the profiles through the radial force balance, but the contribution of the density and temperature profiles scales as $\rho_* = \rho/R$ in agreement with the scaling arguments presented in Ref. 62. Under many circumstances this contribution can be expected to be small, but it is to be noted that the profiles enter through the second derivative. In the core of the plasma $(R/L_T)^2 \approx 40$ can be easily reached, and in the edge even larger values

can be expected. Both for small tokamaks (larger ρ_*) as well as in the edge of larger tokamaks (steeper profiles) the effect can therefore be expected to be non-negligible.

- The ExB shear effect also introduces an additional contribution proportional to u'. The splitting of the equations in three contributions—diagonal, pinch, and ExB shear—is therefore somewhat arbitrary. The contribution to the diagonal element generated by the ExB shear enhances the diffusion coefficient as also noted in Ref. 20. It presents a small but not entirely negligible contribution.

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