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Inequalities for experimental tests of the Kochen-Specker theorem

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We derive inequalities for n-partite states under the assumption that the hidden-variable theoretical joint probability distribution for any pair of commuting observables is equal to the quantum mechanical one. Fine showed that this assumption is connected to the no-hidden-variables theorem of Kochen and Specker (KS theorem). These inequalities give a way to experimentally test the KS theorem. The fidelity to the Bell states which is larger than $1/2$ is sufficient for the experimental confirmation of the KS theorem. Hence, the Werner state is enough to test experimentally the KS theorem. Furthermore, it is possible to test the KS theorem experimentally using uncorrelated states. An n-partite uncorrelated state violates the n-partite inequality derived here by an amount that grows exponentially with n. *© 2005 American Institute of Physics.* [DOI: 10.1063/1.2081115]

I. INTRODUCTION

From the incompleteness argument of the EPR paper,[1] hidden-variable interpretation of quantum mechanics (QM) has been an attractive topic of research.[2,3] There are two main approaches to study this conceptual foundation of QM. One is the Bell-EPR theorem.[4] This theorem says that the statistical prediction of QM violates the inequality following from the EPR-locality principle. The EPR-locality principle tells that a result of measurement pertaining to one system is independent of any measurement performed simultaneously at a distance on another system.

The other is the no-hidden-variables theorem of Kochen and Specker (KS theorem).[5] The original KS theorem says the nonexistence of a real-valued function which is multiplicative and linear on commuting operators so that QM cannot be imbedded into the KS type of hidden-variable theory. The proof of the KS theorem relies on intricate geometric argument. Fine connected[6,7] the KS theorem to the assumption, to be called Fine's assumption, that the hidden-variable theoretical joint probability distribution for any commuting pair of observables is equal to a quantum mechanical one. Greenberger, Horne, and Zeilinger discovered[8] the so-called GHZ theorem for four-partite GHZ states and the KS theorem has taken very simple form since then (see also Refs. 9–12).

In 1990, Mermin considered the Bell-EPR theorem of multipartite systems and derived multipartite Bell's inequality.[13] It has shown that the n-partite GHZ state violates the Bell-Mermin inequality by an amount that grows exponentially with n. After this work, several multipartite Bell's inequalities have been derived.[14,15] They also exhibit that QM violates local hidden-variable theory by a similar way.

As for the KS theorem, most researches are related to "all versus nothing" demolition of the KS type of hidden-variable theory.[16] (Of course, Bell's inequalities are available for a test of the KS theorem.) Recently, it has begun to research the KS theorem using inequalities (see Refs. 17). To find such inequalities to test the KS theorem is particularly useful for experimental investigation.[18] Since the KS theorem was purely related to the algebraic structure of quantum operators and was independent of states, it might be possible to find an inequality that is violated by QM when the system is in an uncorrelated state.[19] If so, we ask what an amount of a violation is like.

In this paper, we shall derive two inequalities following from the assumption pointed out[6,7] by Fine as a test for the KS theorem for n spin-$1/2$ states. A violation of Fine's assumption implies that there exists a pair of commuting observables such that the hidden-variable theoretical joint distribution of them does not agree with QM, or hidden-variables cannot exist in the sense that the KS theorem holds.

One of the inequalities says that the fidelity to the Bell states, which is larger than $1/2$, allows a proof of the KS theorem. This says the Werner state[19] which admits local hidden-variable theory is enough to test experimentally the KS theorem. And we reveal possible modification of the Bell-Mermin inequality on combining Mermin's geometric idea[13] and a commutative operator group presented by Nagata *et al.*[20] We show that when n exceeds 2, not only n-partite GHZ states but also n-partite uncorrelated states violate the modified inequality derived here. The amount of a violation grows exponentially with n, which is a factor of $O(2^{n/2})$ at the macroscopic level.

Our result says that QM exhibits an exponentially stronger refutation of the KS type of hidden-variable theory, as the number of parties constituting the state increases linearly. The feature is independent of the requirement that the system be prepared in an entangled state. In other words, we can say that the KS theorem is more serious in high-dimensional settings than in low-dimensional ones. Further, we can see the local hidden-variable theory violates the KS type of hidden-variable theory.

This paper is organized as follows. In Sec. II, we fix several notations and prepare for arguments of this paper. In Sec. III, we review the statistical KS theorem and mention that its inequality version is necessary for an experimental test. In Sec. IV, we present an inequality which follows from Fine's assumption for two-spin-$1/2$ states and derive a sufficient condition to allow a proof of the KS theorem, which states that the fidelity to the Bell states is larger than $1/2$. Since the fidelity to the Bell states is $5/8$, the two-spin-$1/2$ Werner state violates the inequality. In Sec. V, we modify the Bell-Mermin inequality. It follows from Fine's assumption. And we show that not only n-partite GHZ states but also n-partite uncorrelated states violate the inequality by an amount that grows exponentially with n. Section VI summarizes this paper.

II. NOTATION AND PREPARATIONS

Throughout this paper, we assume von Neumann's projective measurements and we confine ourselves to the finite-dimensional and the discrete spectrum case. Let $\mathbf{R}$ denote the reals where $\pm\infty \notin \mathbf{R}$. We assume every eigenvalue in this paper lies in $\mathbf{R}$. Further, we assume that every Hermitian operator is associated with a unique observable because we do not need to distinguish between them in this paper.

We assume the validity of QM and we would like to investigate if the KS type of hidden-variable interpretation of QM is possible. Let $\mathcal{O}$ be the space of Hermitian operators described in a finite-dimensional Hilbert space, and $\mathcal{T}$ be the space of density operators described in the Hilbert space. Namely, $\mathcal{T}=\{\psi|\psi\in\mathcal{O}\wedge\psi\geqslant 0\wedge\mathrm{Tr}[\psi]=1\}$. Now we define the notation θ which represents one result of quantum measurement. Suppose that the measurement of a Hermitian operator A for a system in the state ψ yields a value $\theta(A)\in\mathbf{R}$. We assume that the following two propositions (BSF and QDJ) hold. Here, $\chi_\Delta(x),(x\in\mathbf{R})$ represents the characteristic function. Δ is any subset of the reals $\mathbf{R}$.

Proposition: BSF (*the Born statistical formula*),

$$\mathrm{Prob}(\Delta)^{\psi}_{\theta(A)}=\mathrm{Tr}[\psi\chi_\Delta(A)]. \tag{2.1}$$

The whole symbol $(\Delta)^{\psi}_{\theta(A)}$ is used to denote the proposition that $\theta(A)$ lies in Δ if the system is in the state ψ. And Prob denotes the probability that the proposition holds.

Proposition: QDJ (*the quantum-mechanical joint probability distribution for commuting observables*),

$$\mathrm{Prob}(\Delta,\Delta')^{\psi}_{\theta(A),\theta(B)} = \mathrm{Tr}[\psi\chi_{\Delta}(A)\chi_{\Delta'}(B)] \tag{2.2}$$

for every commuting pair A,B in $\mathcal{O}$. The notation on the LHS of (2.2) is a generalization of the symbol $(\Delta)^{\psi}_{\theta(A)}$ to express the proposition that $\theta(A)$ and $\theta(B)$ lie in Δ and in Δ', respectively, if the system is in the state ψ.

Let us consider a classical probability space $(\Omega,\Sigma,\mu_{\psi})$, where Ω is a nonempty sample space, Σ is a σ-algebra of subsets of Ω, and μ_{ψ} is a σ-additive normalized measure on Σ such that $\mu_{\psi}(\Omega)=1$. The subscript ψ expresses that the probability measure is determined uniquely when the state ψ is specified.

Let us introduce measurable functions (classical random variables) onto $\Omega(f:\Omega\mapsto\mathbf{R})$, which is written as $f_A(\omega)$ for an operator $A\in\mathcal{O}$. Here $\omega\in\Omega$ is a hidden variable. We introduce appropriate notation. $P(\omega)\simeq Q(\omega)$ means $P(\omega)=Q(\omega)$ holds almost everywhere with respect to μ_{ψ} in Ω. One may assume the probability measure μ_{ψ} is chosen such that the following relation is valid:

$$\mathrm{Tr}[\psi A] = \int_{\omega\in\Omega}\mu_{\psi}(\mathrm{d}\omega)f_A(\omega) \tag{2.3}$$

for every Hermitian operator A in $\mathcal{O}$. Please notice the assumption for the probability measure μ_{ψ} does not disturb the KS theorem. See the lemma (B1) in Appendix B.

Proposition: HV (*the deterministic hidden-variable interpretation of QM*).

Measurable function $f_A(\omega)$ exists for every Hermitian operator A in $\mathcal{O}$.

Proposition: D (*the probability distribution rule*),

$$\mu_{\psi}(f_A^{-1}(\Delta)) = \mathrm{Prob}(\Delta)^{\psi}_{\theta(A)}. \tag{2.4}$$

Proposition: JD (*the joint probability distribution rule*),

$$\mu_{\psi}(f_A^{-1}(\Delta)\cap f_B^{-1}(\Delta')) = \mathrm{Prob}(\Delta,\Delta')^{\psi}_{\theta(A),\theta(B)} \tag{2.5}$$

for every commuting pair A,B in $\mathcal{O}$.

Proposition: FUNC A.E. (*the functional rule holding almost everywhere*),

$$f_{g(A)}(\omega) \simeq g(f_A(\omega)) \tag{2.6}$$

for every function $g:\mathbf{R}\mapsto\mathbf{R}$.

Proposition: PROD A.E. (*the product rule holding almost everywhere*).

If Hermitian operators A and B commute, then

$$f_{AB}(\omega) \simeq f_A(\omega)\cdot f_B(\omega). \tag{2.7}$$

Theorem:[6]

$$\mathrm{HV}\wedge\mathrm{JD} \Rightarrow \mathrm{HV}\wedge\mathrm{D}\wedge\mathrm{FUNC\ A.E.} \tag{2.8}$$

Proof: See (B6) in Appendix B.

Theorem:[6]

$$\mathrm{HV}\wedge\mathrm{FUNC\ A.E.} \Rightarrow \mathrm{HV}\wedge\mathrm{PROD\ A.E.} \tag{2.9}$$

Proof: See (B14) in Appendix B.

III. THE STATISTICAL KOCHEN-SPECKER THEOREM

In this section, we want to review the statistical KS theorem (see also Ref. 17). In what follows, we assume HV and JD hold. This implies that we can use D, FUNC A.E., and PROD A.E.. We follow the statistical version of the KS theorem proposed by Peres[11] and refined by Mermin[12] for two-spin-1/2 states. One then can see that

$$X(\omega) := f_{\sigma_x^1\sigma_x^2}(\omega) f_{\sigma_y^1\sigma_y^2}(\omega) f_{\sigma_z^1\sigma_z^2}(\omega) \simeq f_{\sigma_x^1\sigma_x^2\sigma_y^1\sigma_y^2\sigma_z^1\sigma_z^2}(\omega) = f_{-I}(\omega)$$

$$\Rightarrow \int_{\omega\in\Omega} \mu_\psi(\mathrm{d}\omega) X(\omega) = \mathrm{Tr}[\psi(-I)] = -1, \tag{3.1}$$

where I represents the identity operator for the four-dimensional space. By the way we can factorize two of the terms as $f_{\sigma_x^1\sigma_x^2} \simeq f_{\sigma_x^1} f_{\sigma_x^2}$ and $f_{\sigma_y^1\sigma_y^2} \simeq f_{\sigma_y^1} f_{\sigma_y^2}$. Further, we have $f_{\sigma_x^1\sigma_y^2} \simeq f_{\sigma_x^1} f_{\sigma_y^2}$ and $f_{\sigma_y^1\sigma_x^2} \simeq f_{\sigma_y^1} f_{\sigma_x^2}$. Hence we get $f_{\sigma_x^1\sigma_x^2} f_{\sigma_y^1\sigma_y^2} \simeq f_{\sigma_x^1\sigma_y^2} f_{\sigma_y^1\sigma_x^2}$ and

$$X(\omega) \simeq f_{\sigma_x^1\sigma_y^2}(\omega) f_{\sigma_y^1\sigma_x^2}(\omega) f_{\sigma_z^1\sigma_z^2}(\omega) \simeq f_{\sigma_x^1\sigma_y^2\sigma_y^1\sigma_x^2\sigma_z^1\sigma_z^2}(\omega) = f_I(\omega) \Rightarrow \int_{\omega\in\Omega} \mu_\psi(\mathrm{d}\omega) X(\omega) = \mathrm{Tr}[\psi I] = 1 \tag{3.2}$$

in contradiction to (3.1). Thereby, we see that HV does not hold if we accept JD.

We follow the statistical version of the KS theorem proposed in Refs. 9 and 12 for three-spin-1/2 states. Then, one can see that

$$Y(\omega) := f_{\sigma_x^1\sigma_y^2\sigma_y^3}(\omega) f_{\sigma_y^1\sigma_x^2\sigma_y^3}(\omega) f_{\sigma_y^1\sigma_y^2\sigma_x^3}(\omega) f_{\sigma_x^1\sigma_x^2\sigma_x^3}(\omega) \simeq f_{\sigma_x^1\sigma_y^2\sigma_y^3\sigma_y^1\sigma_x^2\sigma_y^3\sigma_y^1\sigma_y^2\sigma_x^3\sigma_x^1\sigma_x^2\sigma_x^3}(\omega)$$

$$= f_{-I}(\omega) \Rightarrow \int_{\omega\in\Omega} \mu_\psi(\mathrm{d}\omega) Y(\omega) = \mathrm{Tr}[\psi(-I)] = -1, \tag{3.3}$$

where I represents the identity operator for the eight-dimensional space. By the way, we can factorize each of the four terms as

$$f_{\sigma_x^1\sigma_y^2\sigma_y^3}(\omega) \simeq f_{\sigma_x^1}(\omega) f_{\sigma_y^2}(\omega) f_{\sigma_y^3}(\omega) \tag{3.4}$$

and so on to get

$$Y(\omega) \simeq (f_{\sigma_x^1}(\omega))^2 (f_{\sigma_y^1}(\omega))^2 (f_{\sigma_x^2}(\omega))^2 (f_{\sigma_y^2}(\omega))^2 (f_{\sigma_x^3}(\omega))^2 (f_{\sigma_y^3}(\omega))^2 \simeq f_I(\omega) f_I(\omega) f_I(\omega) f_I(\omega) f_I(\omega) f_I(\omega)$$

$$\simeq f_I(\omega) \Rightarrow \int_{\omega\in\Omega} \mu_\psi(\mathrm{d}\omega) Y(\omega) = \mathrm{Tr}[\psi I] = 1 \tag{3.5}$$

in contradiction to (3.3).

These two examples provide the statistical KS theorem, which says demolition of HV or of JD. We have the following result.

Theorem: (*The statistical Kochen-Specker theorem*.)

For every quantum state described in a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ or $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, $(\mathrm{Dim}(\mathcal{H}_j) = 2, (j = 1, 2, 3))$,

$$\mathrm{HV} \wedge \mathrm{JD} \Rightarrow \perp. \tag{3.6}$$

That is, these two propositions do not hold at the same time.

These examples are sufficient to show that HV cannot be possible in any state if we accept JD. However, they are not of suitable form to test experimentally the KS theorem. Because, in a real experiment, we cannot claim a sharp value as an expectation with arbitrary precision. Therefore, its inequality version is necessary for an experimental test of the KS theorem.

IV. INEQUALITY FOR TWO-PARTITE STATES

In this section, we shall derive the inequality version statistical KS theorem for two-spin-1/2 states. Then, we show that the two-spin-1/2 Werner state[19] violates the inequality. Since the Werner state satisfies all Bell's inequalities, the inequality derived in this section does not belong to the category of Bell's inequalities. The inequality is just the inequality concerned with the KS theorem. In the following, we assume that HV and JD hold. Let x,y be real numbers with $x,y \in \{-1,+1\}$, then we have

$$(1+x+y-xy) = \pm 2. \tag{4.1}$$

Theorem:[21] For every state ψ described in a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, $(\mathrm{Dim}(\mathcal{H}_j)=2,(j=1,2))$,

$$\mathrm{HV} \wedge \mathrm{JD}(\wedge \sigma_x^1\sigma_y^2\sigma_y^1\sigma_x^2 = \sigma_z^1\sigma_z^2) \Rightarrow 1 + \mathrm{Tr}[\psi\sigma_x^1\sigma_x^2] + \mathrm{Tr}[\psi\sigma_y^1\sigma_y^2] - \mathrm{Tr}[\psi\sigma_z^1\sigma_z^2] \leqslant 2. \tag{4.2}$$

Proof: From PROD A.E., we have

$$(f_{\sigma_k^1\sigma_k^2}(\omega))^2 \simeq f_I(\omega) \simeq +1 \Leftrightarrow f_{\sigma_k^1\sigma_k^2}(\omega) \simeq \pm 1,(k=x,y). \tag{4.3}$$

Hence, the (4.1) says

$$U(\omega) \coloneqq 1 + f_{\sigma_x^1\sigma_x^2}(\omega) + f_{\sigma_y^1\sigma_y^2}(\omega) - f_{\sigma_x^1\sigma_x^2}(\omega)f_{\sigma_y^1\sigma_y^2}(\omega) \Rightarrow U(\omega) \simeq \pm 2 \tag{4.4}$$

and

$$\int_{\omega\in\Omega} \mu_\psi(\mathrm{d}\omega)U(\omega) \leqslant 2. \tag{4.5}$$

On using $f_{\sigma_x^1\sigma_x^2}f_{\sigma_y^1\sigma_y^2} \simeq f_{\sigma_x^1\sigma_y^2}f_{\sigma_y^1\sigma_x^2} \simeq f_{\sigma_z^1\sigma_z^2}$ we get

$$\int_{\omega\in\Omega} \mu_\psi(\mathrm{d}\omega)f_{\sigma_x^1\sigma_x^2}(\omega)f_{\sigma_y^1\sigma_y^2}(\omega) = \int_{\omega\in\Omega} \mu_\psi(\mathrm{d}\omega)f_{\sigma_x^1\sigma_y^2}(\omega)f_{\sigma_y^1\sigma_x^2}(\omega) = \int_{\omega\in\Omega} \mu_\psi(\mathrm{d}\omega)f_{\sigma_z^1\sigma_z^2}(\omega), \tag{4.6}$$

where we have used the quantum mechanical rule $\sigma_x^1\sigma_y^2\sigma_y^1\sigma_x^2 = \sigma_z^1\sigma_z^2$. Hence we conclude

$$\int_{\omega\in\Omega} \mu_\psi(\mathrm{d}\omega)U(\omega) \leqslant 2 \Leftrightarrow 1 + \mathrm{Tr}[\psi\sigma_x^1\sigma_x^2] + \mathrm{Tr}[\psi\sigma_y^1\sigma_y^2] - \mathrm{Tr}[\psi\sigma_z^1\sigma_z^2] \leqslant 2. \tag{4.7}$$

QED

A violation of the inequality (4.7) implies demolition of HV or of JD in the state ψ. Note the following quantum mechanical relation:

$$1 + \mathrm{Tr}[\psi\sigma_x^1\sigma_x^2] + \mathrm{Tr}[\psi\sigma_y^1\sigma_y^2] - \mathrm{Tr}[\psi\sigma_z^1\sigma_z^2] \leqslant 2 \Leftrightarrow \mathrm{Tr}[\psi|\pi\rangle\langle\pi|] \leqslant 1/2, \tag{4.8}$$

where

$$|\pi\rangle \coloneqq \frac{|+_1;-_2\rangle + |-_1;+_2\rangle}{\sqrt{2}}. \tag{4.9}$$

Therefore the statistical KS theorem holds if the fidelity to the Bell state $|\pi\rangle$ is larger than $1/2$. Note the fidelity to the Bell states of the two-spin-$1/2$ Werner state[19] is $5/8(>1/2)$. The Werner state W is

$$W = (1/2)|\pi\rangle\langle\pi| + (1/8)I, \tag{4.10}$$

where I is the identity operator on the four-dimensional space. Hence, this quantum state which admits local hidden-variable theory allows a proof of the KS theorem.

V. INEQUALITY FOR MULTIPARTITE STATES

In what follows, we shall modify the Bell-Mermin inequality.[13] We derive an n-partite inequality which is satisfied if both HV and JD hold. We show n-partite uncorrelated states violate the inequality when $n \geqslant 3$, by an amount that grows exponentially with n. Please note that any uncorrelated states satisfy all Bell's inequalities.[19] Hence, the modified inequality does not belong to the category of Bell's inequalities. In this section, we assume $n \geqslant 2$. Let us denote $\{1,2,\ldots,n\}$ by $\mathbf{N}_n$.

Definition: [*commutative group* (Λ_n) *of Hermitian operators*]

$O_p^n (p \in \{0,1,\ldots,2^n-1\}, n \geqslant 2)$ are Hermitian operators defined by

$$O_p^n := \prod_{j=1}^{n} (\sigma_z^j)^{b_j} (\sigma_x^j)^{b_0} = (\sigma_z^1)^{b_1} (\sigma_z^2)^{b_2} (\sigma_z^3)^{b_3} (\sigma_z^4)^{b_4} \cdots (\sigma_z^{n-1})^{b_{n-1}} (\sigma_z^n)^{b_n} \times (\sigma_x^1)^{b_0} (\sigma_x^2)^{b_0} (\sigma_x^3)^{b_0} \cdots (\sigma_x^{n-1})^{b_0} (\sigma_x^n)^{b_0}, \tag{5.1}$$

where the superscript j of the Pauli operators denotes the party j and the n-bit sequence $b_0 b_1 \cdots b_{n-1}$ is the binary representation of p, and $b_n \in \{0,1\} \wedge b_n \equiv \Sigma_{j=1}^{n-1} b_j (\mathrm{mod}\ 2)$. Thus, the parity of $b_1 b_2 \cdots b_n$ is even. (Here, σ_k^1 means $\sigma_k^1 \otimes_{j=2}^n I^j$ and so on. Omitting the identity operator, we abbreviate those as above.)

The operator O_0^n is the identity operator on the 2^n-dimensional space, and the other operators $O_1^n, \ldots, O_{2^n-1}^n$ have two eigenvalues, ± 1. In the following, there are the cases where we abbreviate O_0^n as I.

Example: If $p \in \{0,1,\ldots,2^{n-1}-1\}$, then $b_0 = 0$ and $(\sigma_x^j)^{b_0} = \otimes_{k=1}^n I^k = O_0^n$ for all j. That is, the binary representation of p takes, for example, the following form:

$$p_1 := 0\overbrace{1001 \cdots 01}^{B_1} (= b_0 b_1 \cdots b_{n-1}), \tag{5.2}$$

where B_1 represents the sum of the number of 1. Then, $b_n \in \{0,1\} \wedge b_n \equiv B_1 (\mathrm{mod}\ 2)$ holds. Suppose $b_n = 1$ holds, then $(\sigma_z^n)^{b_n} = \sigma_z^n$. Then, the corresponding Hermitian operator $O_{p_1}^n$ is as follows:

$$\begin{aligned} O_{p_1}^n &= \sigma_z^1 I^2 I^3 \sigma_z^4 \cdots I^{n-2} \sigma_z^{n-1} \sigma_z^n \times I^1 I^2 I^3 I^4 \cdots I^{n-1} I^n = (\sigma_z^1 I^1) I^2 I^3 (\sigma_z^4 I^4) \cdots I^{n-2} (\sigma_z^{n-1} I^{n-1}) (\sigma_z^n I^n) \\ &= \sigma_z^1 I^2 I^3 \sigma_z^4 \cdots I^{n-2} \sigma_z^{n-1} \sigma_z^n, \end{aligned} \tag{5.3}$$

where the number of $(\sigma_z I) = \sigma_z$ is even because of the definition of b_n.

Example: If $p \in \{2^{n-1}, 2^{n-1}+1, \ldots, 2^n-1\}$, then $b_0 = 1$ and $(\sigma_x^j)^{b_0} = \sigma_x^j$ for all j. That is, the binary representation of p takes, for example, the following form:

$$p_2 := 1\overbrace{1001 \cdots 01}^{B_2}, \tag{5.4}$$

where B_2 represents the sum of the number of 1. Then, $b_n \in \{0,1\} \wedge b_n \equiv B_2 (\mathrm{mod}\ 2)$ holds. Suppose $b_n = 0$ holds, then $(\sigma_z^n)^{b_n} = O_0^n$. Then the corresponding Hermitian operator $O_{p_2}^n$ is as follows:

$$\begin{aligned} O_{p_2}^n &= \sigma_z^1 I^2 I^3 \sigma_z^4 \cdots I^{n-2} \sigma_z^{n-1} I^n \times \sigma_x^1 \sigma_x^2 \sigma_x^3 \cdots \sigma_x^{n-1} \sigma_x^n = (\sigma_z^1 \sigma_x^1) \sigma_x^2 \sigma_x^3 (\sigma_z^4 \sigma_x^4) \cdots \sigma_x^{n-2} (\sigma_z^{n-1} \sigma_x^{n-1}) \sigma_x^n \\ &= (i\sigma_y^1) \sigma_x^2 \sigma_x^3 (i\sigma_y^4) \cdots \sigma_x^{n-2} (i\sigma_y^{n-1}) \sigma_x^n, \end{aligned} \tag{5.5}$$

where the number of $(\sigma_z \sigma_x) = i\sigma_y$ is even.

Example: The binary representation of 2^{n-1} takes the following form:

$$2^{n-1} = 1\overbrace{0000 \cdots 00}^{n-1}. \tag{5.6}$$

Then the corresponding Hermitian operator $O_{2^{n-1}}^n$ is as follows:

$$O_{2^{n-1}}^n = I^1 I^2 I^3 I^4 \cdots I^{n-1} I^n \times \sigma_x^1 \sigma_x^2 \sigma_x^3 \cdots \sigma_x^{n-1} \sigma_x^n = \sigma_x^1 \sigma_x^2 \sigma_x^3 \cdots \sigma_x^{n-1} \sigma_x^n. \tag{5.7}$$

Lemma: If $O_p^n, O_q^n \in \Lambda_n$, then

$$O_p^n O_q^n = O_{p\oplus q}^n (\in \Lambda_n), \tag{5.8}$$

where $p \oplus q$ is the bitwise XOR (exclusive OR) of p and q.

Proof: See (A1) in Appendix A.

From the lemma (5.8), the set of 2^n operators $\{O_p^n\}$ forms a commutative group isomorphic to $(Z_2)^n$. We have denoted this commutative group as Λ_n. Let us define another set of operators.

Definition: $R_p^n (p \in \{0,1,\ldots,2^n-1\}, n \geqslant 2)$ are operators defined by

$$R_p^n := \prod_{j=1}^{n} (\sigma_z^j)^{e_j} (\sigma_x^j)^{e_0}, \tag{5.9}$$

where the superscript j of the Pauli operators denotes the party j and the n-bit sequence $e_0 e_1 \cdots e_{n-1}$ is the binary representation of p, and $e_n \in \{0,1\} \wedge e_n \equiv \Sigma_{j=1}^{n-1} e_j + 1 (\mathrm{mod}\, 2)$. Thus, unlike O_p^n, the parity of $e_1 e_2 \cdots e_n$ is odd.

Example: If $p \in \{0,1,\ldots,2^{n-1}-1\}$, then $e_0 = 0$ and $(\sigma_x^j)^{e_0} = \otimes_{k=1}^n I^k = O_0^n$ for all j. That is, the binary representation of p takes, for example, the following form:

$$p_3 := 0\overbrace{1001 \cdots 01}^{B_3} (= e_0 e_1 \cdots e_{n-1}), \tag{5.10}$$

where B_3 represents the sum of the number of 1. Then, $e_n \in \{0,1\} \wedge e_n \equiv B_3 + 1 (\mathrm{mod}\, 2)$ holds. Suppose $e_n = 1$ holds, then $(\sigma_z^n)^{e_n} = \sigma_z^n$. Then, the corresponding Hermitian operator $R_{p_3}^n$ is as follows:

$$\begin{aligned} R_{p_3}^n &= \sigma_z^1 I^2 I^3 \sigma_z^4 \cdots I^{n-2} \sigma_z^{n-1} \sigma_z^n \times I^1 I^2 I^3 I^4 \cdots I^{n-1} I^n = (\sigma_z^1 I^1) I^2 I^3 (\sigma_z^4 I^4) \cdots I^{n-2} (\sigma_z^{n-1} I^{n-1})(\sigma_z^n I^n) \\ &= \sigma_z^1 I^2 I^3 \sigma_z^4 \cdots I^{n-2} \sigma_z^{n-1} \sigma_z^n, \end{aligned} \tag{5.11}$$

where the number of $(\sigma_z I) = \sigma_z$ is odd because of the definition of e_n.

Example: If $p \in \{2^{n-1}, 2^{n-1}+1, \ldots, 2^n-1\}$, then $e_0 = 1$ and $(\sigma_x^j)^{e_0} = \sigma_x^j$ for all j. That is, the binary representation of p takes, for example, the following form:

$$p_4 := 1\overbrace{1001 \cdots 01}^{B_4}, \tag{5.12}$$

where B_4 represents the sum of the number of 1. Then, $e_n \in \{0,1\} \wedge e_n \equiv B_4 + 1 (\mathrm{mod}\, 2)$ holds. Suppose $e_n = 0$ holds, then $(\sigma_z^n)^{e_n} = O_0^n$. Then the corresponding non-Hermitian operator $R_{p_4}^n$ is as follows:

$$\begin{aligned} R_{p_4}^n &= \sigma_z^1 I^2 I^3 \sigma_z^4 \cdots I^{n-2} \sigma_z^{n-2} I^n \times \sigma_x^1 \sigma_x^2 \sigma_x^3 \cdots \sigma_x^{n-1} \sigma_x^n = (\sigma_z^1 \sigma_x^1) \sigma_x^2 \sigma_x^3 (\sigma_z^4 \sigma_x^4) \cdots \sigma_x^{n-2} (\sigma_z^{n-1} \sigma_x^{n-1}) \sigma_x^n \\ &= (i\sigma_y^1) \sigma_x^2 \sigma_x^3 (i\sigma_y^4) \cdots \sigma_x^{n-2} (i\sigma_y^{n-1}) \sigma_x^n, \end{aligned} \tag{5.13}$$

where the number of $(\sigma_z \sigma_x) = i\sigma_y$ is odd. R_p^n / i and iR_p^n are Hermitian operators if $p \in \{2^{n-1}, 2^{n-1}+1, \ldots, 2^n-1\}$.

Lemma:

$$\frac{1}{2}\left(\prod_{j=1}^{n} (I^j + \sigma_z^j) + \prod_{j=1}^{n} (I^j - \sigma_z^j) \right) = \sum_{p=0}^{2^{n-1}-1} O_p^n,$$

$$\frac{1}{2}\left(\prod_{j=1}^{n} (I^j + \sigma_z^j) - \prod_{j=1}^{n} (I^j - \sigma_z^j) \right) = \sum_{p=0}^{2^{n-1}-1} R_p^n,$$

$$\frac{1}{2}\left(\prod_{j=1}^{n}(\sigma_x^j + i\sigma_y^j) + \prod_{j=1}^{n}(\sigma_x^j - i\sigma_y^j)\right) = \sum_{p=2^{n-1}}^{2^n-1} O_p^n,$$

$$\frac{1}{2}\left(\prod_{j=1}^{n}(\sigma_x^j + i\sigma_y^j) - \prod_{j=1}^{n}(\sigma_x^j - i\sigma_y^j)\right) = \sum_{p=2^{n-1}}^{2^n-1} R_p^n. \tag{5.14}$$

Proof: See (A7) and (A14) in Appendix A.
Lemma:

$$\mathrm{HV} \wedge \mathrm{FUNC\ A.E.} \Rightarrow \mathrm{Re}\left(\prod_{j=1}^{n}(f_{\sigma_x^j}(\omega) + i f_{\sigma_y^j}(\omega))\right) \simeq \sum_{p=2^{n-1}}^{2^n-1} f_{O_p^n}(\omega),$$

$$\mathrm{Im}\left(\prod_{j=1}^{n}(f_{\sigma_x^j}(\omega) + i f_{\sigma_y^j}(\omega))\right) \simeq \sum_{p=2^{n-1}}^{2^n-1} f_{R_p^n/i}(\omega). \tag{5.15}$$

Proof: See (A21) in Appendix A.
Theorem:[21] For every state ψ described in a Hilbert space $\otimes_{j=1}^{n}\mathcal{H}_j$, $(\mathrm{Dim}(\mathcal{H}_j)=2,(j\in\mathbf{N}_n, n \geqslant 2))$,

$$\mathrm{HV} \wedge \mathrm{JD}(\wedge(i\sigma_y^i)(i\sigma_y^j)\sigma_x^i\sigma_x^j = \sigma_z^i\sigma_z^j) \Rightarrow \sum_{p=0}^{2^{n-1}-1} \mathrm{Tr}[\psi O_p^n] \leqslant \begin{cases} 2^{n/2} & n = \text{even}, \\ 2^{(n-1)/2} & n = \text{odd}. \end{cases} \tag{5.16}$$

Proof: From PROD A.E., we have

$$(f_{\sigma_k^j}(\omega))^2 \simeq f_{O_0^n}(\omega) \simeq +1 \Leftrightarrow f_{\sigma_k^j}(\omega) \simeq \pm 1, \quad (j \in \mathbf{N}_n, k = x, y). \tag{5.17}$$

Now, we define F^{ψ} by

$$F^{\psi} := \int_{\omega\in\Omega} \mu_{\psi}(\mathrm{d}\omega) G(\omega), \tag{5.18}$$

where $G(\omega)$ is defined by

$$G(\omega) := \mathrm{Re}\left(\prod_{j=1}^{n}(f_{\sigma_x^j}(\omega) + i f_{\sigma_y^j}(\omega))\right)\prod_{j=1}^{n} f_{\sigma_x^j}(\omega). \tag{5.19}$$

From the geometric argument by Mermin in Ref. 13 and (5.17), we have

$$G(\omega) \leqslant \begin{cases} 2^{n/2} & n = \text{even} \\ 2^{(n-1)/2} & n = \text{odd} \end{cases} \qquad (\mu_{\psi} - \text{a.e.}). \tag{5.20}$$

In more detail, almost everywhere with respect to μ_{ψ} in Ω, the maximum of $G(\omega)$ is equal to the real part of a product of complex numbers each of which has magnitude of $\sqrt{2}$ and a phase of $\pm\pi/4$ or $\pm 3\pi/4$ since absolute value of $\Pi_{j=1}^{n} f_{\sigma_x^j}(\omega)$ is unity almost everywhere with respect to μ_{ψ}. When n is even the product can lie along the real axis and can attain a maximum value of $2^{n/2}$, when n is odd the product must lie along an axis at 45° to the real axis and its real part can only attain the maximum value $2^{(n-1)/2}$. Therefore, the value $G(\omega)$ is bounded as (5.20) almost everywhere in Ω, and hence F^{ψ} is bounded as

$$F^{\psi} \leqslant \begin{cases} 2^{n/2} & n = \text{even}, \\ 2^{(n-1)/2} & n = \text{odd}. \end{cases} \tag{5.21}$$

From (5.7), it is easy to see that

$$\prod_{j=1}^{n} f_{\sigma_x^j}(\omega) \simeq f_{O^n_{2^{n-1}}}(\omega). \tag{5.22}$$

Therefore, from (5.19) and the lemma (5.15), we have

$$G(\omega) \simeq \left(\sum_{p=2^{n-1}}^{2^n-1} f_{O_p^n}(\omega) \right) f_{O^n_{2^{n-1}}}(\omega). \tag{5.23}$$

Noting $[O_p^n, O_q^n] = \mathbf{0}, \forall O_p^n, O_q^n \in \Lambda_n$ [see the lemma (5.8)], PROD A.E. tells the following relations:

$$f_{O_p^n}(\omega) f_{O_q^n}(\omega) \simeq f_{O^n_{p\oplus q}}(\omega), \quad (\forall O_p^n, O_q^n \in \Lambda_n). \tag{5.24}$$

It is easy to see that

$$\{O_p^n O^n_{2^{n-1}} | p \in \{2^{n-1}, 2^{n-1}+1, \ldots, 2^n - 1\}\} = \{O_p^n | p \in \{0, 1, \ldots, 2^{n-1} - 1\}\}. \tag{5.25}$$

Here, we have used the quantum mechanical rule $(i\sigma_y^i)(i\sigma_y^j)\sigma_x^i\sigma_x^j = \sigma_z^i\sigma_z^j (i, j \in \mathbf{N}_n, i \neq j)$. [Equation (5.25) is also obvious from the expression (5.1) and (5.7)]. Therefore, we get

$$G(\omega) \simeq \sum_{p=0}^{2^{n-1}-1} f_{O_p^n}(\omega). \tag{5.26}$$

Thus from (5.18) we conclude

$$F^{\psi} = \int_{\omega \in \Omega} \mu_{\psi}(\mathrm{d}\omega) \left(\sum_{p=0}^{2^{n-1}-1} f_{O_p^n}(\omega) \right) = \sum_{p=0}^{2^{n-1}-1} \mathrm{Tr}[\psi O_p^n]. \tag{5.27}$$

QED

Now, it follows from the lemma (5.14) that

$$|+_1; +_2; \cdots; +_n\rangle\langle +_1; +_2; \cdots; +_n| + |-_1; -_2; \cdots; -_n\rangle\langle -_1; -_2; \cdots; -_n| = \frac{1}{2^n}\left(\prod_{j=1}^{n} (I^j + \sigma_z^j) + \prod_{j=1}^{n} (I^j - \sigma_z^j) \right)$$

$$= \frac{1}{2^{n-1}} \left(\sum_{p=0}^{2^{n-1}-1} O_p^n \right), \tag{5.28}$$

where $O_p^n = \Pi_{j=1}^n (\sigma_x^j)^{b_0} (\sigma_z^j)^{b_j}$ and $\sigma_z^j |\pm_j\rangle = \pm |\pm_j\rangle$. Hence we have

$$F^{\psi} = \sum_{p=0}^{2^{n-1}-1} \mathrm{Tr}[\psi O_p^n] = \mathrm{Tr}[\psi H_n], \tag{5.29}$$

where [see (5.28)]

$$H_n \coloneqq 2^{n-1}(|+_1; +_2; \cdots; +_n\rangle\langle +_1; +_2; \cdots; +_n| + |-_1; -_2; \cdots; -_n\rangle\langle -_1; -_2; \cdots; -_n|). \tag{5.30}$$

Now, let ψ be $|\Psi\rangle\langle\Psi|$ where

$$|\Psi\rangle \coloneqq \alpha|+_1;+_2;\cdots;+_n\rangle + \beta|-_1;-_2;\cdots;-_n\rangle, (|\alpha|^2 + |\beta|^2 = 1). \tag{5.31}$$

This state $|\Psi\rangle$ is an n-partite uncorrelated state if α or β is zero and $|\Psi\rangle$ is an n-partite GHZ state if $\alpha=\beta=1/\sqrt{2}$.

The quantum theoretical prediction says the expectation value $\mathrm{Tr}[|\Psi\rangle\langle\Psi|H_n]$ should take a value of 2^{n-1} for the state $|\Psi\rangle$ in spite of any value of α and of β, and we get

$$F^{|\Psi\rangle} = 2^{n-1}. \tag{5.32}$$

When n exceeds 2, this value $F^{|\Psi\rangle}$ is larger than the bound (5.21), which exceeds (5.21) by the exponentially larger factor of $2^{(n-2)/2}$ (for n even) or $2^{(n-1)/2}$ (for n odd). This implies demolition of HV or of JD in the state $|\Psi\rangle$. Thus, we have derived the exponentially stronger violation of HV$\wedge$JD, irrespective of quantum entanglement effects.

VI. SUMMARY

In summary, we showed that the fidelity to the Bell states which is larger than $1/2$ is sufficient to allow a proof of the KS theorem. Thus, the Werner state is enough to test experimentally the KS theorem. We also have derived an n-partite inequality following from HV$\wedge$JD. We have shown that an n-partite uncorrelated state violates the inequality by a factor of $O(2^{n/2})$ at the macroscopic level. Hence, it turns out that QM exhibits an exponentially stronger violation of HV$\wedge$JD, as the number of parties constituting the state increases, irrespective of entanglement effects.

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APPENDIX A

Lemma: If $O_p^n, O_q^n \in \Lambda_n$, then

$$O_p^n O_q^n = O_{p\oplus q}^n (\in \Lambda_n), \tag{A1}$$

where $p \oplus q$ is the bitwise XOR (exclusive OR) of p and q.

Proof: Suppose that the binary representations of p and q are $b_0 b_1 \cdots b_{n-1}$ and $c_0 c_1 \cdots c_{n-1}$, respectively. Suppose that $b_n \in \{0,1\} \wedge b_n \equiv \Sigma_{j=1}^{n-1} b_j (\mathrm{mod}\, 2)$ and $c_n \in \{0,1\} \wedge c_n \equiv \Sigma_{j=1}^{n-1} c_j (\mathrm{mod}\, 2)$ hold. This means that

$$c_j \in \{0,1\}\ \forall\, j \wedge \sum_{j=1}^{n} c_j \equiv 0 (\mathrm{mod}\, 2). \tag{A2}$$

This yields ($b_0 \in \{1,0\}$)

$$\left[\prod_{j=1}^{n} (\sigma_x^j)^{b_0}, \prod_{j=1}^{n} (\sigma_z^j)^{c_j}\right] = \mathbf{0}. \tag{A3}$$

Then from (5.1) we have

$$O_p^n = \prod_{j=1}^{n} (\sigma_z^j)^{b_j}(\sigma_x^j)^{b_0}, \quad O_q^n = \prod_{j=1}^{n} (\sigma_z^j)^{c_j}(\sigma_x^j)^{c_0} \Rightarrow O_p^n O_q^n = \prod_{j=1}^{n} (\sigma_z^j)^{b_j}(\sigma_x^j)^{b_0}(\sigma_z^j)^{c_j}(\sigma_x^j)^{c_0}$$

$$= \prod_{j=1}^{n} (\sigma_z^j)^{b_j}(\sigma_z^j)^{c_j}(\sigma_x^j)^{b_0}(\sigma_x^j)^{c_0} = \prod_{j=1}^{n} (\sigma_z^j)^{d_j}(\sigma_x^j)^{d_0} = O_{p\oplus q}^n, \tag{A4}$$

where

$$d_j \in \{0,1\} \wedge d_j \equiv b_j + c_j (\mathrm{mod}\, 2). \tag{A5}$$

Here,

$$d_n \equiv \sum_{j=1}^{n-1} b_j + \sum_{j=1}^{n-1} c_j (\mathrm{mod}\, 2) \equiv \sum_{j=1}^{n-1} (b_j + c_j)(\mathrm{mod}\, 2) \equiv \sum_{j=1}^{n-1} d_j (\mathrm{mod}\, 2). \tag{A6}$$

Hence, d_n can be assumed such that $d_n \in \{0,1\}$ and $d_n \equiv \Sigma_{j=1}^{n-1} d_j (\mathrm{mod}\, 2)$ hold. QED

Lemma:

$$\frac{1}{2}\left(\prod_{j=1}^{n} (I^j + \sigma_z^j) + \prod_{j=1}^{n} (I^j - \sigma_z^j)\right) = \sum_{p=0}^{2^{n-1}-1} O_p^n,$$

$$\frac{1}{2}\left(\prod_{j=1}^{n} (I^j + \sigma_z^j) - \prod_{j=1}^{n} (I^j - \sigma_z^j)\right) = \sum_{p=0}^{2^{n-1}-1} R_p^n. \tag{A7}$$

Proof: If the following relations hold for all $m, (2 \leqslant m \leqslant n)$,

$$\frac{1}{2}\left(\prod_{j=1}^{m} (I^j + \sigma_z^j) + \prod_{j=1}^{m} (I^j - \sigma_z^j)\right) = \sum_{p=0}^{2^{m-1}-1} O_p^m, \tag{A8}$$

$$\frac{1}{2}\left(\prod_{j=1}^{m} (I^j + \sigma_z^j) - \prod_{j=1}^{m} (I^j - \sigma_z^j)\right) = \sum_{p=0}^{2^{m-1}-1} R_p^m, \tag{A9}$$

then the theorem holds. Here, O_p^m means $O_p^m \otimes_{j=m+1}^n I^j$ and so on. Omitting the identity operator, we abbreviate those as above. Remember, σ_k^1 means $\sigma_k^1 \otimes_{j=2}^n I^j$ and so on.

In the case where $m=2$: The left-hand side (LHS) of (A8) is $(I^1 I^2 + \sigma_z^1 \sigma_z^2) \otimes_{j=3}^n I^j$ and right-hand side (RHS) of (A8) is also $(I^1 I^2 + \sigma_z^1 \sigma_z^2) \otimes_{j=3}^n I^j$. LHS of (A9) is $(I^1 \sigma_z^2 + \sigma_z^1 I^2) \otimes_{j=3}^n I^j$ and RHS of (A9) is also $(I^1 \sigma_z^2 + \sigma_z^1 I^2) \otimes_{j=3}^n I^j$. Therefore (A8) and (A9) hold when $m=2$. In the following, if possible, we omit the identity operator.

Suppose that (A8) and (A9) hold for $m=k-1$. Then we have

$$\prod_{j=1}^{k-1} (I^j + \sigma_z^j) = \sum_{p=0}^{2^{k-2}-1} O_p^{k-1} + \sum_{p=0}^{2^{k-2}-1} R_p^{k-1},$$

$$\prod_{j=1}^{k-1} (I^j - \sigma_z^j) = \sum_{p=0}^{2^{k-2}-1} O_p^{k-1} - \sum_{p=0}^{2^{k-2}-1} R_p^{k-1}, \tag{A10}$$

On the other hand, we have

$$
\begin{aligned}
(I^k+\sigma_z^k)&\left(\sum_{p=0}^{2^{k-2}-1} O_p^{k-1}+\sum_{p=0}^{2^{k-2}-1} R_p^{k-1}\right)\\
&=\left(\sum_{p=0}^{2^{k-2}-1} O_p^{k-1}I^k+\sum_{p=0}^{2^{k-2}-1} R_p^{k-1}\sigma_z^k\right)+\left(\sum_{p=0}^{2^{k-2}-1} O_p^{k-1}\sigma_z^k+\sum_{p=0}^{2^{k-2}-1} R_p^{k-1}I^k\right)\\
&=\sum_{p=0}^{2^{k-1}-1} O_p^k+\sum_{p=0}^{2^{k-1}-1} R_p^k
\end{aligned}
\tag{A11}
$$

and

$$
\begin{aligned}
(I^k-\sigma_z^k)&\left(\sum_{p=0}^{2^{k-2}-1} O_p^{k-1}-\sum_{p=0}^{2^{k-2}-1} R_p^{k-1}\right)\\
&=\left(\sum_{p=0}^{2^{k-2}-1} O_p^{k-1}I^k+\sum_{p=0}^{2^{k-2}-1} R_p^{k-1}\sigma_z^k\right)-\left(\sum_{p=0}^{2^{k-2}-1} O_p^{k-1}\sigma_z^k+\sum_{p=0}^{2^{k-2}-1} R_p^{k-1}I^k\right)\\
&=\sum_{p=0}^{2^{k-1}-1} O_p^k-\sum_{p=0}^{2^{k-1}-1} R_p^k.
\end{aligned}
\tag{A12}
$$

Therefore we have

$$
\begin{aligned}
\prod_{j=1}^{k}(I^j+\sigma_z^j)&=\sum_{p=0}^{2^{k-1}-1} O_p^k+\sum_{p=0}^{2^{k-1}-1} R_p^k,\\
\prod_{j=1}^{k}(I^j-\sigma_z^j)&=\sum_{p=0}^{2^{k-1}-1} O_p^k-\sum_{p=0}^{2^{k-1}-1} R_p^k.
\end{aligned}
\tag{A13}
$$

This implies that (A8) and (A9) hold for $m=k$. QED

Lemma:

$$
\begin{aligned}
\frac{1}{2}\left(\prod_{j=1}^{n}(\sigma_x^j+i\sigma_y^j)+\prod_{j=1}^{n}(\sigma_x^j-i\sigma_y^j)\right)&=\sum_{p=2^{n-1}}^{2^n-1} O_p^n,\\
\frac{1}{2}\left(\prod_{j=1}^{n}(\sigma_x^j+i\sigma_y^j)-\prod_{j=1}^{n}(\sigma_x^j-i\sigma_y^j)\right)&=\sum_{p=2^{n-1}}^{2^n-1} R_p^n.
\end{aligned}
\tag{A14}
$$

Proof: If the following relations hold for all $m,(2\leqslant m\leqslant n)$,

$$
\frac{1}{2}\left(\prod_{j=1}^{m}(\sigma_x^j+i\sigma_y^j)+\prod_{j=1}^{m}(\sigma_x^j-i\sigma_y^j)\right)=\sum_{p=2^{m-1}}^{2^m-1} O_p^m,
\tag{A15}
$$

$$
\frac{1}{2}\left(\prod_{j=1}^{m}(\sigma_x^j+i\sigma_y^j)-\prod_{j=1}^{m}(\sigma_x^j-i\sigma_y^j)\right)=\sum_{p=2^{m-1}}^{2^m-1} R_p^m,
\tag{A16}
$$

then the theorem holds. Here, O_p^m means $O_p^m \otimes_{j=m+1}^n I^j$ and so on.

In the case where $m=2$: LHS of (A15) is $(\sigma_x^1\sigma_x^2 + i\sigma_y^1 i\sigma_y^2)\otimes_{j=3}^n I^j$ and RHS of (A15) is also $(\sigma_x^1\sigma_x^2 + i\sigma_y^1 i\sigma_y^2)\otimes_{j=3}^n I^j$. LHS of (A16) is $(\sigma_x^1 i\sigma_y^2 + i\sigma_y^1\sigma_x^2)\otimes_{j=3}^n I^j$ and RHS of (A16) is also $(\sigma_x^1 i\sigma_y^2 + i\sigma_y^1\sigma_x^2)\otimes_{j=3}^n I^j$. Therefore (A15) and (A16) hold when $m=2$. In the following, if possible, we omit the identity operator.

Suppose that (A15) and (A16) hold for $m=k-1$. Then we have

$$\begin{aligned}\prod_{j=1}^{k-1}(\sigma_x^j + i\sigma_y^j) &= \sum_{p=2^{k-2}}^{2^{k-1}-1} O_p^{k-1} + \sum_{p=2^{k-2}}^{2^{k-1}-1} R_p^{k-1},\\ \prod_{j=1}^{k-1}(\sigma_x^j - i\sigma_y^j) &= \sum_{p=2^{k-2}}^{2^{k-1}-1} O_p^{k-1} - \sum_{p=2^{k-2}}^{2^{k-1}-1} R_p^{k-1}.\end{aligned} \tag{A17}$$

On the other hand, we have

$$\begin{aligned}&(\sigma_x^k + i\sigma_y^k)\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} O_p^{k-1} + \sum_{p=2^{k-2}}^{2^{k-1}-1} R_p^{k-1}\right)\\ &= \left(\sum_{p=2^{k-2}}^{2^{k-1}-1} O_p^{k-1}\sigma_x^k + \sum_{p=2^{k-2}}^{2^{k-1}-1} R_p^{k-1} i\sigma_y^k\right) + \left(\sum_{p=2^{k-2}}^{2^{k-1}-1} O_p^{k-1} i\sigma_y^k + \sum_{p=2^{k-2}}^{2^{k-1}-1} R_p^{k-1}\sigma_x^k\right)\\ &= \sum_{p=2^{k-1}}^{2^k-1} O_p^k + \sum_{p=2^{k-1}}^{2^k-1} R_p^k\end{aligned} \tag{A18}$$

and

$$\begin{aligned}&(\sigma_x^k - i\sigma_y^k)\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} O_p^{k-1} - \sum_{p=2^{k-2}}^{2^{k-1}-1} R_p^{k-1}\right)\\ &= \left(\sum_{p=2^{k-2}}^{2^{k-1}-1} O_p^{k-1}\sigma_x^k + \sum_{p=2^{k-2}}^{2^{k-1}-1} R_p^{k-1} i\sigma_y^k\right) - \left(\sum_{p=2^{k-2}}^{2^{k-1}-1} O_p^{k-1} i\sigma_y^k + \sum_{p=2^{k-2}}^{2^{k-1}-1} R_p^{k-1}\sigma_x^k\right)\\ &= \sum_{p=2^{k-1}}^{2^k-1} O_p^k - \sum_{p=2^{k-1}}^{2^k-1} R_p^k.\end{aligned} \tag{A19}$$

Therefore we have

$$\begin{aligned}\prod_{j=1}^{k}(\sigma_x^j + i\sigma_y^j) &= \sum_{p=2^{k-1}}^{2^k-1} O_p^k + \sum_{p=2^{k-1}}^{2^k-1} R_p^k,\\ \prod_{j=1}^{k}(\sigma_x^j - i\sigma_y^j) &= \sum_{p=2^{k-1}}^{2^k-1} O_p^k - \sum_{p=2^{k-1}}^{2^k-1} R_p^k.\end{aligned} \tag{A20}$$

This implies that (A15) and (A16) hold for $m=k$. QED

Lemma:

$$\mathrm{HV} \wedge \mathrm{FUNC\ A.E.} \Rightarrow \mathrm{Re}\left(\prod_{j=1}^{n} (f_{\sigma_x^j}(\omega) + i f_{\sigma_y^j}(\omega))\right) \simeq \sum_{p=2^{n-1}}^{2^n-1} f_{O_p^n}(\omega),$$
$$\mathrm{Im}\left(\prod_{j=1}^{n} (f_{\sigma_x^j}(\omega) + i f_{\sigma_y^j}(\omega))\right) \simeq \sum_{p=2^{n-1}}^{2^n-1} f_{R_p^n/i}(\omega). \tag{A21}$$

Proof: If the following relations hold for all $m, (2 \leqslant m \leqslant n)$,

$$\mathrm{Re}\left(\prod_{j=1}^{m} (f_{\sigma_x^j}(\omega) + i f_{\sigma_y^j}(\omega))\right) \simeq \sum_{p=2^{m-1}}^{2^m-1} f_{O_p^m}(\omega), \tag{A22}$$

$$\mathrm{Im}\left(\prod_{j=1}^{m} (f_{\sigma_x^j}(\omega) + i f_{\sigma_y^j}(\omega))\right) \simeq \sum_{p=2^{m-1}}^{2^m-1} f_{R_p^m/i}(\omega), \tag{A23}$$

then the theorem holds. Here, $f_{O_p^m}$ means $f_{O_p^m \otimes_{j=m+1}^n I^j}$ and so on.

In the case where $m=2$: LHS of (A22) is $f_{(\sigma_x^1 \sigma_x^2)\otimes_{j=3}^n I^j} + f_{(i\sigma_y^1 i\sigma_y^2)\otimes_{j=3}^n I^j}$ almost everywhere and RHS of (A22) is $f_{(\sigma_x^1 \sigma_x^2)\otimes_{j=3}^n I^j} + f_{(i\sigma_y^1 i\sigma_y^2)\otimes_{j=3}^n I^j}$. LHS of (A23) is $f_{(\sigma_x^1 \sigma_y^2)\otimes_{j=3}^n I^j} + f_{(\sigma_y^1 \sigma_x^2)\otimes_{j=3}^n I^j}$ almost everywhere and RHS of (A23) is $f_{(\sigma_x^1 \sigma_y^2)\otimes_{j=3}^n I^j} + f_{(\sigma_y^1 \sigma_x^2)\otimes_{j=3}^n I^j}$. Therefore (A22) and (A23) hold when $m=2$. Here, we have used PROD A.E.. In the following, if possible, we omit the identity operator.

Suppose that (A22) and (A23) hold for $m=k-1$. Then we have

$$\prod_{j=1}^{k-1} (f_{\sigma_x^j} + i f_{\sigma_y^j}) \simeq \sum_{p=2^{k-2}}^{2^{k-1}-1} f_{O_p^{k-1}} + \sum_{p=2^{k-2}}^{2^{k-1}-1} i f_{R_p^{k-1}/i},$$
$$\prod_{j=1}^{k-1} (f_{\sigma_x^j} - i f_{\sigma_y^j}) \simeq \sum_{p=2^{k-2}}^{2^{k-1}-1} f_{O_p^{k-1}} - \sum_{p=2^{k-2}}^{2^{k-1}-1} i f_{R_p^{k-1}/i}. \tag{A24}$$

FUNC A.E. says

$$(-1) f_A(\omega) \simeq f_{(-1)A}(\omega). \tag{A25}$$

PROD A.E. says

$$f_{O_p^{k-1}} f_{\sigma_x^k} \simeq f_{O_p^{k-1} \sigma_x^k} \tag{A26}$$

and so on. Hence, we have

$$\begin{aligned}
&(f_{\sigma_x^k}+if_{\sigma_y^k})\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} f_{O_p^{k-1}}+\sum_{p=2^{k-2}}^{2^{k-1}-1} if_{R_p^{k-1}/i}\right)\\
&\quad=\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} f_{O_p^{k-1}}f_{\sigma_x^k}+\sum_{p=2^{k-2}}^{2^{k-1}-1} if_{R_p^{k-1}/i}if_{\sigma_y^k}\right)+\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} f_{O_p^{k-1}}if_{\sigma_y^k}+\sum_{p=2^{k-2}}^{2^{k-1}-1} if_{R_p^{k-1}/i}f_{\sigma_x^k}\right)\\
&\quad\simeq\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} f_{O_p^{k-1}\sigma_x^k}+\sum_{p=2^{k-2}}^{2^{k-1}-1} f_{R_p^{k-1}i\sigma_y^k}\right)+\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} if_{O_p^{k-1}i\sigma_y^k/i}+\sum_{p=2^{k-2}}^{2^{k-1}-1} if_{R_p^{k-1}\sigma_x^k/i}\right)\\
&\quad=\sum_{p=2^{k-1}}^{2^{k}-1} f_{O_p^k}+\sum_{p=2^{k-1}}^{2^{k}-1} if_{R_p^k/i}
\end{aligned}\tag{A27}$$

and

$$\begin{aligned}
&(f_{\sigma_x^k}-if_{\sigma_y^k})\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} f_{O_p^{k-1}}-\sum_{p=2^{k-2}}^{2^{k-1}-1} if_{R_p^{k-1}/i}\right)\\
&\quad=\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} f_{O_p^{k-1}}f_{\sigma_x^k}+\sum_{p=2^{k-2}}^{2^{k-1}-1} if_{R_p^{k-1}/i}if_{\sigma_y^k}\right)-\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} f_{O_p^{k-1}}if_{\sigma_y^k}+\sum_{p=2^{k-2}}^{2^{k-1}-1} if_{R_p^{k-1}/i}f_{\sigma_x^k}\right)\\
&\quad\simeq\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} f_{O_p^{k-1}\sigma_x^k}+\sum_{p=2^{k-2}}^{2^{k-1}-1} f_{R_p^{k-1}i\sigma_y^k}\right)-\left(\sum_{p=2^{k-2}}^{2^{k-1}-1} if_{O_p^{k-1}i\sigma_y^k/i}+\sum_{p=2^{k-2}}^{2^{k-1}-1} if_{R_p^{k-1}\sigma_x^k/i}\right)\\
&\quad=\sum_{p=2^{k-1}}^{2^{k}-1} f_{O_p^k}-\sum_{p=2^{k-1}}^{2^{k}-1} if_{R_p^k/i}.
\end{aligned}\tag{A28}$$

Therefore we have

$$\begin{aligned}
\prod_{j=1}^{k}(f_{\sigma_x^j}+if_{\sigma_y^j})&\simeq\sum_{p=2^{k-1}}^{2^{k}-1} f_{O_p^k}+\sum_{p=2^{k-1}}^{2^{k}-1} if_{R_p^k/i},\\
\prod_{j=1}^{k}(f_{\sigma_x^j}-if_{\sigma_y^j})&\simeq\sum_{p=2^{k-1}}^{2^{k}-1} f_{O_p^k}-\sum_{p=2^{k-1}}^{2^{k}-1} if_{R_p^k/i}.
\end{aligned}\tag{A29}$$

This implies that (A22) and (A23) hold for $m=k$. QED

APPENDIX B

Lemma: Let S_A stand for the spectrum of the Hermitian operator A. If

$$\mathrm{Tr}[\psi A]=\sum_{y\in S_A}\mathrm{Prob}(\{y\})^{\psi}_{\theta(A)}y,$$

$$E_{\psi}(A):=\int_{\omega\in\Omega}\mu_{\psi}(\mathrm{d}\omega)f_A(\omega),$$

then

$$\mathrm{HV} \wedge \mathrm{D} \Rightarrow \mathrm{Tr}[\psi A] = E_{\psi}(A). \tag{B1}$$

Proof: Note

$$\omega \in f_A^{-1}(\{y\}) \Leftrightarrow f_A(\omega) \in \{y\} \Leftrightarrow y = f_A(\omega),$$

$$\int_{\omega \in f_A^{-1}(\{y\})} \frac{\mu_{\psi}(\mathrm{d}\omega)}{\mu_{\psi}(f_A^{-1}(\{y\}))} = 1,$$

$$y \neq y' \Rightarrow f_A^{-1}(\{y\}) \cap f_A^{-1}(\{y'\}) = \phi. \tag{B2}$$

Hence we have

$$\begin{aligned}\mathrm{Tr}[\psi A] &= \sum_{y \in S_A} \mathrm{Prob}(\{y\})^{\psi}_{\theta(A)} y = \sum_{y \in \mathbf{R}} \mathrm{Prob}(\{y\})^{\psi}_{\theta(A)} y = \sum_{y \in \mathbf{R}} \mu_{\psi}(f_A^{-1}(\{y\}))y \\ &= \sum_{y \in \mathbf{R}} \mu_{\psi}(f_A^{-1}(\{y\}))y \int_{\omega \in f_A^{-1}(\{y\})} \frac{\mu_{\psi}(\mathrm{d}\omega)}{\mu_{\psi}(f_A^{-1}(\{y\}))} \\ &= \sum_{y \in \mathbf{R}} \int_{\omega \in f_A^{-1}(\{y\})} \mu_{\psi}(f_A^{-1}(\{y\})) \frac{\mu_{\psi}(\mathrm{d}\omega)}{\mu_{\psi}(f_A^{-1}(\{y\}))} f_A(\omega) = \int_{\omega \in \Omega} \mu_{\psi}(\mathrm{d}\omega) f_A(\omega) = E_{\psi}(A).\end{aligned} \tag{B3}$$

QED

Lemma:

$$\chi_{\Delta}(g(x)) = \chi_{g^{-1}(\Delta)}(x), (x \in \mathbf{R})$$

and

$$\mathrm{Prob}(\Delta)^{\psi}_{\theta(g(A))} = \mathrm{Tr}[\psi \chi_{\Delta}(g(A))] = \mathrm{Tr}[\psi \chi_{g^{-1}(\Delta)}(A)] = \mathrm{Prob}(g^{-1}(\Delta))^{\psi}_{\theta(A)}. \tag{B4}$$

Proof: Obvious.
Lemma:

$$\mathrm{QJD} \Rightarrow \mathrm{BSF},$$

$$\mathrm{HV} \wedge \mathrm{JD} \Rightarrow \mathrm{HV} \wedge \mathrm{D} \tag{B5}$$

Proof: Obvious.
Theorem:[6]

$$\mathrm{HV} \wedge \mathrm{JD} \Rightarrow \mathrm{HV} \wedge \mathrm{D} \wedge \mathrm{FUNC\ A.E.} \tag{B6}$$

Proof: Suppose JD holds. Let y be any real number, and let $S := \{\omega | f_{g(A)}(\omega) = y\}$ and $T := \{\omega | g(f_A(\omega)) = y\}$. We want $\mu_{\psi}(\bar{S} \cap T) = \mu_{\psi}(S \cap \bar{T}) = 0$. This is valid if we have $\mu_{\psi}(S) = \mu_{\psi}(T) = \mu_{\psi}(S \cap T)$ since

$$\begin{aligned}\mu_{\psi}(S \cap \bar{T}) + \mu_{\psi}(S \cap T) = \mu_{\psi}(S), \\ \mu_{\psi}(\bar{S} \cap T) + \mu_{\psi}(S \cap T) = \mu_{\psi}(T).\end{aligned} \tag{B7}$$

Note

$$\omega \in f_{g(A)}^{-1}(\{y\}) \Leftrightarrow f_{g(A)}(\omega) \in \{y\} \Leftrightarrow y = f_{g(A)}(\omega) \tag{B8}$$

and

$$\omega \in f_A^{-1}(g^{-1}(\{y\})) \Leftrightarrow f_A(\omega) \in g^{-1}(\{y\}) \Leftrightarrow g(f_A(\omega)) \in \{y\} \Leftrightarrow y = g(f_A(\omega)). \tag{B9}$$

The lemma (B5) says that JD yields D. Then, from the lemma (B4), we have

$$\begin{aligned}\mu_\psi(T) &= \mu_\psi(\{\omega|\omega \in f_A^{-1}(g^{-1}(\{y\}))\}) \\ &= \mathrm{Prob}(g^{-1}(\{y\}))_{\theta(A)}^{\psi} = \mathrm{Prob}(\{y\})_{\theta(g(A))}^{\psi} = \mu_\psi(\{\omega|\omega \in f_{g(A)}^{-1}(\{y\})\}) = \mu_\omega(S).\end{aligned} \tag{B10}$$

Using the spectral representation of A, it follows that $\chi_\Delta(A)\chi_{g(\Delta)}(g(A)) = \chi_\Delta(A)$ for any set Δ, where $g(\Delta) = \{g(x) | x \in \Delta\}$. Because, $\chi_\Delta(z) = 1 \Leftrightarrow z \in \Delta \Rightarrow g(z) \in g(\Delta) \Leftrightarrow \chi_{g(\Delta)}(g(z)) = 1$ holds ($z \in \mathbf{R}$). Hence,

$$\mathrm{Prob}(\Delta, g(\Delta))_{\theta(A),\theta(g(A))}^{\psi} = \mathrm{Tr}[\psi\chi_\Delta(A)\chi_{g(\Delta)}(g(A))] = \mathrm{Tr}[\psi\chi_\Delta(A)] = \mathrm{Prob}(\Delta)_{\theta(A)}^{\psi}. \tag{B11}$$

On the other hand, we have $g(g^{-1}(\Delta)) = \Delta$ because $g(g^{-1}(\Delta)) = \{g(x) | x \in g^{-1}(\Delta)\} = \{g(x) | g(x) \in \Delta\} = \Delta$. Therefore, on substituting $g^{-1}(\{y\})$ into Δ, we have

$$\mathrm{Prob}(g^{-1}(\{y\}), \{y\})_{\theta(A),\theta(g(A))}^{\psi} = \mathrm{Prob}(g^{-1}(\{y\}))_{\theta(A)}^{\psi} = \mu_\psi(T). \tag{B12}$$

But, from JD we have

$$\mathrm{Prob}(g^{-1}(\{y\}), \{y\})_{\theta(A),\theta(g(A))}^{\psi} = \mu_\psi(f_A^{-1}(g^{-1}(\{y\})) \cap f_{g(A)}^{-1}(\{y\})) = \mu_\psi(T \cap S). \tag{B13}$$

QED

Theorem:[6]

$$\mathrm{HV} \wedge \mathrm{FUNC\ A.E.} \Rightarrow \mathrm{HV} \wedge \mathrm{PROD\ A.E.} \tag{B14}$$

Proof: Suppose that A and B are two commuting Hermitian operators. This means that there exists a basis $\{P_i\}$ by which we can expand $A = \Sigma_i a_i P_i$, and such that B can also be expanded in the form $B = \Sigma_i b_i P_i$. Now construct a Hermitian operator $O := \Sigma_i o_i P_i$ with real numbers o_i. None of them is equal. Namely, O is assumed to be nondegenerate by construction. Let us define functions j and k by $j(o_i) := a_i$ and $k(o_i) := b_i$, respectively. Then we can see that if A and B commute, there exists a nondegenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $AB = h(O)$ where $h := j \cdot k$. So we have the following:

$$f_{AB}(\omega) = f_{h(O)}(\omega) \simeq h(f_O(\omega)) = j(f_O(\omega)) \cdot k(f_O(\omega)) \simeq f_{j(O)}(\omega) \cdot f_{k(O)}(\omega) = f_A(\omega) \cdot f_B(\omega), \tag{B15}$$

where FUNC A.E. has been used. QED

Lemma:[7] If

$$\mu_\psi(\bar{S} \cap S') = \mu_\psi(\overline{S'} \cap S) = \mu_\psi(\bar{T} \cap T') = \mu_\psi(\overline{T'} \cap T) = 0,$$

then

$$\mu_\psi(S \cap T) = \mu_\psi(S' \cap T'). \tag{B16}$$

Proof: Note

$$\mu_\psi(\bar{S} \cap S' \cap T) + \mu_\psi(S \cap S' \cap T) = \mu_\psi(S' \cap T), \tag{B17}$$

$$\mu_\psi(\overline{S'} \cap S \cap T) + \mu_\psi(S \cap S' \cap T) = \mu_\psi(S \cap T).$$

If the following relation holds

$$\mu_\psi(\bar{S} \cap S') = \mu_\psi(\overline{S'} \cap S) = 0, \tag{B18}$$

then

$$\mu_\psi(\bar{S} \cap S' \cap T) = \mu_\psi(\overline{S'} \cap S \cap T) = 0. \tag{B19}$$

Therefore, from (B17), we have

$$\mu_\psi(S \cap S' \cap T) = \mu_\psi(S' \cap T) = \mu_\psi(S \cap T). \tag{B20}$$

Similar to the argument by changing S to T, S' to T', and T to S', we get

$$\mu_\psi(T \cap T' \cap S') = \mu_\psi(T' \cap S') = \mu_\psi(T \cap S'). \tag{B21}$$

From the relations (B20) and (B21), we conclude

$$\mu_\psi(T \cap S) = \mu_\psi(T' \cap S'). \tag{B22}$$

QED

Lemma:

$$\mathrm{HV} \wedge \mathrm{PROD\ A.E.} \Rightarrow f_{\chi_\Delta(A)}(\omega) \in \{0,1\}, \quad (\mu_\psi - \mathrm{a.e.}). \tag{B23}$$

Proof: Obvious.

Theorem:[7]

$$\mathrm{HV} \wedge \mathrm{D} \wedge \mathrm{PROD\ A.E.} \Rightarrow \mathrm{HV} \wedge \mathrm{JD} \tag{B24}$$

Proof: Suppose $[A,B]=\mathbf{0}$ holds. It follows from QJD, BSF, and D that

$$\begin{aligned}\mathrm{Prob}(\Delta,\Delta')^{\psi}_{\theta(A),\theta(B)} &= \mathrm{Tr}[\psi\chi_\Delta(A)\chi_{\Delta'}(B)] = \mathrm{Tr}[\psi\chi_{\{1\}}(\chi_\Delta(A)\chi_{\Delta'}(B))] = \mathrm{Prob}(\{1\})^{\psi}_{\theta(\chi_\Delta(A)\chi_{\Delta'}(B))} \\ &= \mu_\psi(f^{-1}_{\chi_\Delta(A)\chi_{\Delta'}(B)}(\{1\})).\end{aligned} \tag{B25}$$

PROD A.E. and the lemma (B23) say that

$$\begin{aligned}(\mathrm{B25}) &= \mu_\psi(\{\omega|\omega \in f^{-1}_{\chi_\Delta(A)\chi_{\Delta'}(B)}(\{1\})\}) = \mu_\psi(\{\omega|f_{\chi_\Delta(A)\chi_{\Delta'}(B)}(\omega) = 1\}) = \mu_\psi(\{\omega|f_{\chi_\Delta(A)}(\omega)\cdot f_{\chi_{\Delta'}(B)}(\omega) \\ &= 1\}) = \mu_\psi(\{\omega|f_{\chi_\Delta(A)}(\omega) = f_{\chi_{\Delta'}(B)}(\omega) = 1\}) = \mu_\psi(f^{-1}_{\chi_\Delta(A)}(\{1\}) \cap f^{-1}_{\chi_{\Delta'}(B)}(\{1\})).\end{aligned} \tag{B26}$$

On the other hand, we have

$$\begin{aligned}\mu_\psi(f^{-1}_{\chi_\Delta(A)}(\{1\}) \cap f_A^{-1}(\Delta)) &= \mu_\psi(\{\omega|f_{\chi_\Delta(A)}(\omega) = 1 \wedge f_A(\omega) \in \Delta\}) = \mu_\psi(\{\omega|f_{\chi_\Delta(A)}(\omega)\cdot f_A(\omega) \in \Delta\}) \\ &= \mu_\psi(\{\omega|f_{\chi_\Delta(A)\cdot A}(\omega) \in \Delta\}) = \mu_\psi(f^{-1}_{\chi_\Delta(A)\cdot A}(\Delta)) = \mathrm{Prob}(\Delta)^{\psi}_{\theta(\chi_\Delta(A)\cdot A)} \\ &= \mathrm{Tr}[\psi\chi_\Delta(\chi_\Delta(A)\cdot A)] = \mathrm{Tr}[\psi\chi_\Delta(A)].\end{aligned} \tag{B27}$$

We also obtain

$$\mu_\psi(f^{-1}_{\chi_\Delta(A)}(\{1\})) = \mathrm{Tr}[\psi\chi_{\{1\}}(\chi_\Delta(A))] = \mathrm{Tr}[\psi\chi_\Delta(A)] = \mu_\psi(f_A^{-1}(\Delta)). \tag{B28}$$

Note [see (B7)]

$$\mu_\psi(S \cap T) = \mu_\psi(S) = \mu_\psi(T) \Leftrightarrow \mu_\psi(S \cap \bar{T}) = \mu_\psi(\bar{S} \cap T) = 0. \tag{B29}$$

Therefore, from Eq. (B27) and Eq. (B28), we have

$$\mu_\psi(f^{-1}_{\chi_\Delta(A)}(\{1\}) \cap \overline{f_A^{-1}(\Delta)}) = \mu_\psi(\overline{f^{-1}_{\chi_\Delta(A)}(\{1\})} \cap f_A^{-1}(\Delta)) = 0. \tag{B30}$$

Similarly we can get

$$\mu_\psi(f^{-1}_{\chi_{\Delta'}(B)}(\{1\}) \cap f_B^{-1}(\Delta')) = \mathrm{Tr}[\psi\chi_{\Delta'}(B)], \quad \mu_\psi(f^{-1}_{\chi_{\Delta'}(B)}(\{1\})) = \mu_\psi(f_B^{-1}(\Delta')) = \mathrm{Tr}[\psi\chi_{\Delta'}(B)], \tag{B31}$$

and we have

$$\mu_\psi(f^{-1}_{\chi_{\Delta'}(B)}(\{1\}) \cap \overline{f_B^{-1}(\Delta')}) = \mu_\psi(\overline{f^{-1}_{\chi_{\Delta'}(B)}(\{1\})} \cap f_B^{-1}(\Delta')) = 0. \tag{B32}$$

Hence, from the lemma (B16), we have

$$\mu_\psi(f^{-1}_{\chi_\Delta(A)}(\{1\}) \cap f^{-1}_{\chi_{\Delta'}(B)}(\{1\})) = \mu_\psi(f_A^{-1}(\Delta) \cap f_B^{-1}(\Delta')). \tag{B33}$$

Therefore, from (B26), we conclude

$$\mathrm{Prob}(\Delta,\Delta')^\psi_{\theta(A),\theta(B)} = \mu_\psi(f_A^{-1}(\Delta) \cap f_B^{-1}(\Delta')), \tag{B34}$$

which is JD. QED

Theorem:[6]

$$\mathrm{HV} \wedge \mathrm{D} \wedge \mathrm{FUNC\ A.E.} \Rightarrow \mathrm{HV} \wedge \mathrm{JD} \tag{B35}$$

Proof: Suppose $[A,B]=\mathbf{0}$ holds. It follows from BSF, QJD, D, FUNC A.E., and PROD A.E. that

$$\begin{aligned}
\mathrm{Prob}(\Delta,\Delta')^\psi_{\theta(A),\theta(B)} &= \mathrm{Tr}[\psi\chi_\Delta(A)\chi_{\Delta'}(B)] = \mathrm{Tr}[\psi\chi_{\{1\}}(\chi_\Delta(A)\chi_{\Delta'}(B))] = \mathrm{Prob}(\{1\})^\psi_{\theta(\chi_\Delta(A)\chi_{\Delta'}(B))} \\
&= \mu_\psi(f^{-1}_{\chi_\Delta(A)\chi_{\Delta'}(B)}(\{1\})) = \mu_\psi(\{\omega|\omega \in f^{-1}_{\chi_\Delta(A)\chi_{\Delta'}}(\{1\})\}) = \mu_\psi(\{\omega|f_{\chi_\Delta(A)\chi_{\Delta'}(B)}(\omega) \\
&= 1\}) = \mu_\psi(\{\omega|f_{\chi_\Delta(A)}(\omega)\cdot f_{\chi_{\Delta'}(B)}(\omega) = 1\}) = \mu_\psi(\{\omega|\chi_\Delta(f_A(\omega))\cdot\chi_{\Delta'}(f_B(\omega)) = 1\}) \\
&= \mu_\psi(\{\omega|\chi_\Delta(f_A(\omega)) = \chi_{\Delta'}(f_B(\omega)) = 1\}) = \mu_\psi(\{\omega|f_A(\omega) \in \Delta \wedge f_B(\omega) \in \Delta'\}) \\
&= \mu_\psi(f_A^{-1}(\Delta) \cap f_B^{-1}(\Delta')).
\end{aligned} \tag{B36}$$

QED

Now we summarize the inclusion relation as follows:

$$\mathrm{HV} \wedge \mathrm{JD} \Leftrightarrow \mathrm{HV} \wedge \mathrm{D} \wedge \mathrm{FUNC\ A.E.} \Leftrightarrow \mathrm{HV} \wedge \mathrm{D} \wedge \mathrm{PROD\ A.E.} \tag{B37}$$

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] M. Redhead, *Incompleteness, Nonlocality, and Realism*, 2nd ed. (Clarendon, Oxford, 1989).
[3] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, The Netherlands, 1993).
[4] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
[5] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).
[6] A. Fine, J. Math. Phys. **23**, 1306 (1982).
[7] A. Fine, Synthese **29**, 257 (1974).
[8] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, The Netherlands, 1989), pp. 69–72; D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[9] C. Pagonis, M. L. G. Redhead, and R. K. Clifton, Phys. Lett. A **155**, 441 (1991).
[10] N. D. Mermin, Phys. Today **43**, 9 (1990); Am. J. Phys. **58**, 731 (1990).
[11] A. Peres, Phys. Lett. A **151**, 107 (1990).
[12] N. D. Mermin, Phys. Rev. Lett. **65**, 3373 (1990).
[13] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
[14] S. M. Roy and V. Singh, Phys. Rev. Lett. **67**, 2761 (1991); M. Ardehali, Phys. Rev. A **46**, 5375 (1992); A. V. Belinskii and D. N. Klyshko, Phys. Usp. **36**, 653 (1993); R. F. Werner and M. M. Wolf, Phys. Rev. A **61**, 062102 (2000).
[15] M. Żukowski, Phys. Lett. A **177**, 290 (1993); M. Żukowski and D. Kaszlikowski, Phys. Rev. A **56**, R1682 (1997); M.

Żukowski and Č. Brukner, Phys. Rev. Lett. **88**, 210401 (2002); R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001); R. F. Werner and M. M. Wolf, Quantum Inf. Comput. **1**, 1 (2001).

[16] See, for example, A. Cabello, Phys. Rev. Lett. **90**, 190401 (2003).

[17] C. Simon, Č. Brukner, and A. Zeilinger, Phys. Rev. Lett. **86**, 4427 (2001); J.-Å. Larsson, Europhys. Lett. **58**, 799 (2002); A. Cabello, Phys. Rev. A **65**, 052101 (2002).

[18] For a recent experimental report of tests for all versus nothing type KS theorem, see Y.-F. Huang, C.-F. Li, Y.-S. Zhang, J.-W. Pan, and G.-C. Guo, Phys. Rev. Lett. **90**, 250401 (2003).

[19] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).

[20] K. Nagata, M. Koashi, and N. Imoto, Phys. Rev. A **65**, 042314 (2002).

[21] We know that every proposition is true if the presupposition is false [see Eq. (3.6)]. Therefore, one might think that theorem (4.2) and theorem (5.16) are trivial. However, this is not the matter of our argument. We have used a quantum mechanical rule $\sigma_x^1\sigma_x^2\sigma_y^1\sigma_y^2\sigma_z^1\sigma_z^2=-I$ in the proof of the theorem (3.6). But, we have not used the quantum mechanical rule in the proof of the theorem (4.2). Likewise, a quantum mechanical rule $\sigma_x^i\sigma_y^j\sigma_y^i\sigma_x^j\sigma_z^i\sigma_z^j=I$, $(i,j\in\mathbf{N}_n, i\neq j)$ is needless to prove the theorem (5.16), while we have used the quantum mechanical rule in the proof of the theorem (3.6). Obviously, $\sigma_x^i\sigma_y^j\sigma_y^k\sigma_y^i\sigma_x^j\sigma_y^k\sigma_y^i\sigma_y^j\sigma_x^k\sigma_x^i\sigma_x^j\sigma_x^k=-I$, $(i,j,k\in\mathbf{N}_n, i\neq j\neq k\neq i)$ is needless to prove the theorem (5.16). Of course, Gleason's theorem is needless. Therefore, we can derive these inequalities (4.2) and (5.16) from more precise and weaker presupposition which should not be necessarily false.