Thermoconvective Stability of Ferrofluids

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$$\frac{f(v,t)-f_2(v)}{f_1(v)-f_2(v)}=\frac{\exp(-t/\tau)}{1+\beta[1-\exp(-t/\tau)]}, \tag{9}$$

where

$$\tau=\frac{I_2-I_1}{I_{12}+I_{21}}, \tag{10}$$

$$\beta=\frac{I_{11}}{I_{12}+I_{21}}-1. \tag{11}$$

While the relaxation time found numerically by MacDonald, Rosenbluth, and Chuck[4] depends on v, the relaxation time τ in Eq. (9) is independent of v. This may be a restriction upon the Mott-Smith solution.

In order to evaluate the relaxation time τ the function $\Phi(v)$ and the initial distribution $f_1(v)$ must be specified. In this note Φ is chosen to be $\Phi = v^{2K}$ with $K = 2, 3, 4, \cdots$, and f_1 is specified to be a delta function

$$f_1(v)=\frac{n}{4\pi v^2}\,\delta\left[v-\left(\frac{3kT}{m}\right)^{1/2}\right]. \tag{12}$$

The equilibrium distribution f_2 is

$$f_2(v)=\frac{n}{(2\pi kT/m)^{3/2}}\exp\left(-\frac{mv^2}{2kT}\right). \tag{13}$$

Then the relaxation time τ is

$$\tau=\frac{m^{1/2}(3kT)^{3/2}}{8\pi ne^4(\ln\Lambda)\alpha}, \tag{14}$$

where

$$\alpha=\frac{K}{[(2K+1)!!/3^K-1]}\left[\left(\frac{2}{3\pi}\right)^{1/2}e^{-3/2}\cdot\left(-\sum_{L=1}^{K}\frac{(2K+1)!!}{3^{L-1}(2K-2L+1)!!}+5-2K\right)+\operatorname{erf}\left[\left(\tfrac{3}{2}\right)^{1/2}\right]\left(\frac{(2K+1)!!}{3^K}+\frac{2K-5}{3}\right)\right], \tag{15}$$

and erf (X) is the error function. Evaluating the relaxation constant α gives $\alpha = 0.892$ for $K = 2$ and $\alpha = 0.675$ for $K = 3$. Spitzer[2] and Montgomery and Tidman[3] evaluated the relaxation time by the test-particle formulation and obtained $\alpha = 0.714$ (Spitzer self-collision time). This indicates that the Mott-Smith v^4 and v^6 methods yield a relaxation time in agreement with the Spitzer self-collision time, which was obtained by quite different methods. However, the present results demonstrate that the relaxation constant α becomes too small for $K > 3$ (Fig. 1) and the relaxation time is extremely sensitive to the choice of moment equations. It is of interest to note the fact found by Rode and Tanenbaum[9] that the Mott-Smith shock thickness depends markedly on the v_x^p moment equation and becomes unacceptable for $p > 6$.

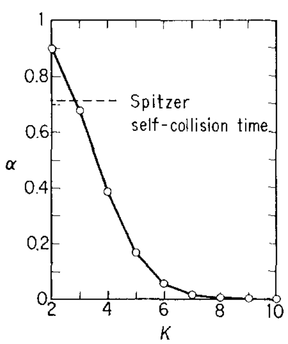

FIG. 1. Relaxation constant by the Mott-Smith v^{2K} method.

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The energy stability limit is obtained for a Boussinesq ferrofluid in the presence of gravity and of thermal and magnetic field gradients. For a motionless fluid heated from below the limit is identical to the linear theory result.

Since Rosensweig[1] succeeded in developing ferrofluids, numerous applications of their use have been proposed (see Resler and Rosensweig[2]). In view of this interest, in this note we will develop some interesting results regarding the nonlinear stability of ferrofluidic flows.

Ferrofluids, at present, are colloidal suspensions of a large number of magnetic particles in an otherwise nonconducting fluid (typical concentrations are of the order of 10^8 particles per cc). The particles, properly coated to prevent clustering, are usually dispersed in fluids like kerosene. Ferrofluids respond so rapidly to a magnetic torque that one can assume the following condition to hold:

$$\mathbf{M}\times\mathbf{H}=0, \tag{1}$$

where $\mathbf{M}$ is the magnetization and $\mathbf{H}$ is the magnetic field intensity. In addition to (1), the ferrofluids will also satisfy Maxwell's equations. In writing Maxwell's equations, one has to keep in mind that the conductivity of ferrofluids is very small and, therefore, in most present applications, we can assume Ampere's law to be of the form

$$\nabla \times \mathbf{H} = 0. \tag{2}$$

Furthermore, there is adequate coupling between the particles and the fluid so that the mixture can be described as a single continuum.

The equations governing the motion of ferrofluids when (1) and (2) hold are[3,4]

$$\nabla \cdot \mathbf{v} = 0, \tag{3}$$

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla p' = M \nabla H + \rho \mathbf{g} + \eta \nabla^2 \mathbf{v}, \tag{4}$$

$$\rho c_{v,B} \frac{DT}{Dt} + T \frac{\partial M}{\partial T}\bigg|_{v,H} \frac{DH}{Dt} = \nabla \cdot (k \nabla T) + \Phi, \tag{5}$$

where $\mathbf{v}$, ρ, v, T, $\mathbf{g}$, and $\mathbf{B}$ are the velocity, density, specific volume, temperature, gravity acceleration, and magnetic induction, respectively. In the above $H = |\mathbf{H}|$, $M = |\mathbf{M}|$, $B = |\mathbf{B}|$, and k, η, Φ are the heat conductivity, viscosity coefficient, and dissipation function, respectively. Furthermore, $c_{v,B}$ is the specific heat at constant v, B and

$$p' = p(\rho, T) + \mu_0 \int_0^H \left(\frac{\partial M v}{\partial v}\right)_{H,T} dH, \tag{6}$$

is the modified pressure. In (6), p is the regular hydrodynamic pressure and μ_0 is the permeability of free space. An equation of state which, in general, will specify M in terms of two thermodynamic variables only (say, T and H), is necessary to complete the system. In this note, we will consider the case where the magnetization is independent of the magnetic field intensity, so that $M = M(T)$ only. As a first approximation we assume

$$M = M_0[1 - \gamma(T - T_0)], \tag{7}$$

where M_0 is the magnetization at $T = T_0$, with T_0 being the reference temperature (usually equal to some average value), and $\gamma = (1/M_0)/(\partial M/\partial T)|_H$. More general equations are discussed by Curtis[5] and Zelazo and Melcher.[6] On the boundary $\partial\vartheta$, of the domain of interest ϑ, the velocity $\mathbf{v}$ and the temperature T are specified. The normal component of the magnetic induction and the tangential component of the magnetic field also have to match on $\partial\vartheta$. These conditions can determine the magnetic field $\mathbf{H}$, if induced fields are neglected. Furthermore, for free surfaces, one has to include the ferromagnetic stress in the total stress balance on $\partial\vartheta$ (for more details, see Rosensweig[3]).

The fluid is essentially incompressible although the temperature effect on density will be included in the buoyancy term (Boussinesq approximation) by taking

$$\rho \mathbf{g} = \rho_0[1 - \alpha(T - T_0)]\mathbf{g}, \tag{8}$$

where ρ_0 is the density at $T = T_0$ and α is the coefficient of thermal expansion.

Curtis[5] has calculated the approximate order of magnitude of the terms in the energy equation and found that, for a constant magnetic field gradient $\nabla H > 10^2$ G/cm, the dissipation function and the adiabatic magnetization terms can be neglected when compared with the conduction term. The equations in this approximation become

$$\nabla \cdot \mathbf{v} = 0, \tag{9}$$

$$\rho_0 \frac{D\mathbf{v}}{Dt} + \nabla p' = M_0 \nabla H[1 - \gamma(T - T_0)] + \rho_0 \mathbf{g}[1 - \alpha(T - T_0)] + \eta \nabla^2 \mathbf{v}, \tag{10}$$

$$\rho_0 c_{v,B} \frac{DT}{Dt} = \nabla \cdot (k \nabla T). \tag{11}$$

We shall now proceed to discuss the stability of ferrofluids that satisfy Eqs. (9)–(11) and the appropriate boundary conditions.

Investigations into the stability of flows usually follow either the linear theory[7] which yields criteria for certain instability, or the energy theory[8,9] which yields criteria for certain stability.

Recently, Curtis[5] and Finlayson[10] discussed the convective instability of ferrofluids by the linear method. Joseph,[9] on the other hand, used the energy method to investigate the stability of a Boussinesq fluid. The equations governing the motion of such a fluid, in the presence of a gravity field and a temperature gradient, are

$$\nabla \cdot \mathbf{v} = 0, \tag{12}$$

$$\rho_0 \frac{D\mathbf{v}}{Dt} + \nabla p = \rho_0[1 - \alpha(T - T_0)]\mathbf{g} + \eta \nabla^2 \mathbf{v}, \tag{13}$$

$$\rho_0 c \frac{DT}{Dt} = k \nabla^2 T, \tag{14}$$

where c is the specific heat, and the same boundary conditions exist on $\mathbf{v}$ and T. We observe that Eqs. (9)–(11) are identical to the system (12)–(14), if we take $k = \text{const}$, redefine p as the effective pressure p', replace c by $c_{v,B}$, and add the new buoyancy term due to the ferrofluid effect. Results similar to Joseph's[9] can, therefore, be obtained for our case.

Specifically, the energy stability limit of a ferrofluid motion with effective pressure p', reference density ρ_0, with μ_0, γ, α, η, k, $c_{v,B}$ as defined previously, in the presence of a gravity field $\mathbf{g}$, a temperature gradient ∇T and a constant magnetic field gradient ∇H, is identical to the limit obtained by Joseph[9] for an ordinary Boussinesq fluid with the Rayleigh number redefined as

$$\mathrm{Ra}_f = \left|\alpha \mathbf{g} + M_0\gamma \frac{\nabla H}{\rho_0}\right| \frac{\rho_0 c_{v,B} \mathbf{g}\beta\, d^4}{k\eta}, \tag{15}$$

where $\beta = \max|\nabla T|$. This implies that convection can start in a ferrofluid at a lower thermal Rayleigh number Ra. The above result holds when ∇H, ∇T, and $\mathbf{g}$ are taken to be parallel vector fields.

In the following we will illustrate this analogy by applying the above to a specific problem.

Let us consider the stability problem of a motionless ferrofluid, between two infinite plates a distance d apart, heated from below in the presence of gravity field $\mathbf{g}$ and a constant magnetic field gradient. Curtis,[5] using the linear method, found that instability sets in when

$$\mathrm{Ra}_f > 1708. \tag{16}$$

Since for typical ferrofluids $\nabla H > 10^2$ G/cm, $|\gamma M_0 \nabla H/\rho| > \alpha \mathbf{g}$, the flow can be unstable only when the magnetic term is in the same direction as gravity.

Joseph's[9] energy result, which includes finite amplitude disturbances insures the flow stability if

$$\mathrm{Ra} = \frac{\rho c \mathbf{g}\alpha\beta\, d^4}{k\eta} < 1708. \tag{17}$$

If we now follow the analogy and replace $\mathbf{g}$ in (17) by

$$\mathbf{g}' = \left|\frac{M_0\gamma\nabla H}{\rho_0\alpha} + \mathbf{g}\right|,$$

and c by $c_{v,B}$, we have certain stability for

$$\mathrm{Ra}_f < 1708. \tag{18}$$

This proves that the linear and energy theories give identical results for stationary ferromagnetic flow. Therefore, the derived criteria gives a necessary and sufficient condition for stability, thus eliminating the possibility that subcritical instabilities occur.[9]

In view of the many possible applications of ferrofluids, further studies of more complex cases should be considered and possible experimental investigations undertaken.

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Two-Stream Instability in the Two Maxwellian, Resonance, and Two Pole Models for Arbitrary Stream Temperatures and Densities

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The problem of stability of two hot streams of charged particles with Maxwellian distributions is solved, in principle, for arbitrary parameters. Stability limits are also found for two models often used for the velocity distributions and comparisons are made.

The stability limit for the two-stream instability has been the theme of many papers.[1–4] However, the problem of stability has never been completely solved for all stream parameters if the distribution functions are Maxwellians.

Here, the problem will be tackled by using the Nyquist diagram technique. For this purpose, one writes the dispersion relation in the form

$$K^2 = F(z, w, r, \gamma) = Z'(z) + \frac{\gamma}{r^2} Z'\left(\frac{z+w}{r}\right), \tag{1}$$

$$K = \left(\frac{\omega_{p_1}}{u_1}\right)^{-1} k, \quad z = \frac{(\omega/k - v_1)}{u_1},$$

$$w = \frac{(v_1 - v_2)}{u_1}, \quad r = \frac{u_2}{u_1}, \quad \gamma = \frac{\omega_{p_2}}{\omega_{p_1}}.$$

Here, ω_{p_i}, u_i, and v_i are the plasma frequency, thermal velocity, and mean velocity of the ith stream, respectively, and Z' is the derivative of the Hilbert-like transform of the normalized one-stream velocity distribution. Without loss of generality $v_1 \geq |v_2|$. We will only consider $\gamma \leq 1$, since stability for (w, r, γ) is equivalent to stability for $(wr^{-1}, r^{-1}, \gamma^{-1})$.

If we are on the boundary of the stability region, in the language of the Nyquist diagram, the $F(z)$, z real, curve either goes through $F = 0$ (*I*), or touches the real F axis at a point where

$$\mathrm{Im}\, F = 0, \quad \frac{\partial\, \mathrm{Im}\, F}{\partial z} = 0, \quad \mathrm{Re}\, F \geq 0,$$

and the first nonzero derivative of Im F is of even order (II). If II is satisfied, we must be on the line of marginal stability. (This can be seen from the general character of the Nyquist curves. They begin and end at the origin and have only one loop.