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Reese T. Prosser

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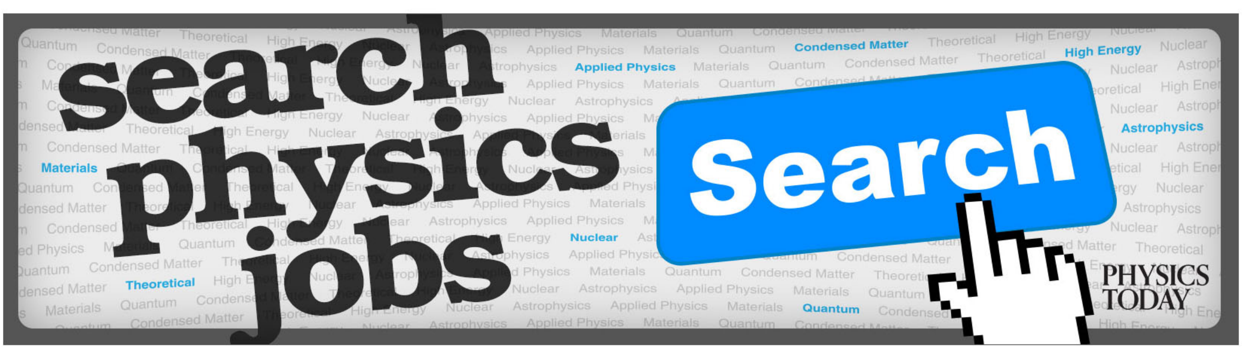

Formal solutions of inverse scattering problems. V

Reese T. Prosser
Dartmouth College, Hanover, New Hampshire 03755

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In part III of this series [J. Math. Phys. **21**, 2648 (1980)], it was shown that the backscatter function for the three-dimensional potential scattering problem can be used to recover the potential function, provided that either function is sufficiently small in a suitable norm. Here a new formula is given that relates the two functions directly, displays their invariance properties explicitly, and enables the inversion process to proceed under a more natural smallness condition.

I. INTRODUCTION

In part III of this series,[1] it was shown that the backscatter function for the three-dimensional potential scattering problem can be used to recover the potential function, provided that either function is sufficiently small in a suitable norm. Here a new formula is given that relates the two functions directly, displays their invariance properties explicitly, and enables the inversion process to proceed under a more natural smallness condition.

II. THE POTENTIAL FUNCTION

The scattering of a quantum-mechanical wave function $\phi(\mathbf{x},\mathbf{k})$ from a fixed potential function $V(\mathbf{x})$ is governed by the time-independent Schrödinger equation:

$$(\nabla^2+\mathbf{k}^2)\phi(\mathbf{x},\mathbf{k})=V(\mathbf{x})\phi(\mathbf{x},\mathbf{k}), \quad \mathbf{x},\mathbf{k}\in\mathbf{R}^3. \tag{1}$$

The solution, which is to consist of an incoming plane wave plus an outgoing scattered wave, may be expressed as

$$\phi(\mathbf{x},\mathbf{k})=\exp(i\mathbf{k}\cdot\mathbf{x}) - \int \frac{\exp(i|\mathbf{k}|\,|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|} V(\mathbf{y})\phi(\mathbf{y},\mathbf{k})d\mathbf{y}. \tag{2}$$

As $|x|\to\infty$, the behavior of $\phi(\mathbf{x},\mathbf{k})$ is given by

$$\phi(\mathbf{x},\mathbf{k})=\exp(i\mathbf{k}\cdot\mathbf{x})-\frac{\exp(i|\mathbf{k}|\,|\mathbf{x}|)}{4\pi|\mathbf{x}|}T(\mathbf{k}',\mathbf{k}) + o(1/|\mathbf{x}|). \tag{3}$$

Here $\mathbf{k}'=(|\mathbf{k}|/|\mathbf{x}|)\mathbf{x}$, and $T(\mathbf{k}',\mathbf{k})$ is given by

$$T(\mathbf{k}',\mathbf{k})=\int \exp(-i\mathbf{k}'\cdot\mathbf{y})V(\mathbf{y})\phi(\mathbf{y},\mathbf{k})d\mathbf{y}. \tag{4}$$

In this sense the "on-shell" T matrix $T(\mathbf{k}',\mathbf{k})$ with $|\mathbf{k}'|=|\mathbf{k}|$ contains all the scattering data.

Substituting Eq. (2) into Eq. (4), taking Fourier transforms, and rearranging, we find that $T(\mathbf{k}',\mathbf{k})$ satisfies the integral equation[2]

$$T(\mathbf{k}',\mathbf{k})=V(\mathbf{k}'-\mathbf{k})-\int \frac{V(\mathbf{k}'-\mathbf{k}'')T(\mathbf{k}'',\mathbf{k})}{\mathbf{k}''^2-\mathbf{k}^2+i0}d\mathbf{k}'', \tag{5}$$

or, more formally,

$$T=V-V\Gamma T, \tag{6}$$

where $\Gamma T(\mathbf{k}'',\mathbf{k})$ is the matrix

$$\Gamma T(\mathbf{k}'',\mathbf{k})=\frac{T(\mathbf{k}'',\mathbf{k})}{\mathbf{k}''^2-\mathbf{k}^2+i0}.$$

Equation (5) may be solved for $T(\mathbf{k}',\mathbf{k})$ by iteration, yielding the Born series for T:

$$T(\mathbf{k}',\mathbf{k})=V(\mathbf{k}'-\mathbf{k})-\int \frac{V(\mathbf{k}'-\mathbf{k}'')V(\mathbf{k}''-\mathbf{k})}{\mathbf{k}''^2-\mathbf{k}^2+i0}d\mathbf{k}'' + \iint \frac{V(\mathbf{k}'-\mathbf{k}'')V(\mathbf{k}''-\mathbf{k}''')V(\mathbf{k}'''-\mathbf{k})}{(\mathbf{k}''^2-\mathbf{k}^2+i0)(\mathbf{k}'''^2-\mathbf{k}^2+i0)} \times d\mathbf{k}''\,d\mathbf{k}'''-\cdots, \tag{7}$$

or, more formally,

$$T=V-V\Gamma V V\Gamma(V\Gamma V)-\cdots. \tag{8}$$

In part IV it is shown that we can choose an appropriate class of integral kernels $K(\mathbf{k}',\mathbf{k})$ (a *Friedrichs class*), and we can define a norm for this class (a *Friedrichs norm*), such that[2]

$$\|K(\Gamma M)\|\leqslant\|K\|\|M\|. \tag{9}$$

If the kernel $V(\mathbf{k}'-\mathbf{k})$ belongs to this class, and if

$$\|V\|=a<1, \tag{10}$$

then the Born series in Eq. (8) converges to $T(\mathbf{k}',\mathbf{k})$ in norm. Thus the kernel $T(\mathbf{k}',\mathbf{k})$ belongs to this class, and

$$\|T\| < a(1-a)^{-1}. \tag{11}$$

Moreover, if V and V' are two different potential functions, and if T and T' are the corresponding kernels, then we have

$$\|T'-T\| < (1-a)^{-2}\|V'-V\|. \tag{12}$$

III. THE BACKSCATTER FUNCTION

The scattering data are obtained from the T matrix $T(\mathbf{k}',\mathbf{k})$, in which $\mathbf{k}=|\mathbf{k}|\omega$ denotes the incoming vector of energy $|\mathbf{k}|$ and aspect direction ω, and $\mathbf{k}'=|\mathbf{k}'|\omega'$ denotes the outgoing vector of energy $|\mathbf{k}'|=|\mathbf{k}|$ and scattered direction ω'. In particular, the backscattering data are obtained by setting $\omega'=-\omega$, i.e., by setting $\mathbf{k}'=-\mathbf{k}$ in $T(\mathbf{k}',\mathbf{k})$.

Here, as in part III, it is convenient to introduce the *backscatter function B* by

$$B(-2\mathbf{k}) = T(-\mathbf{k},\mathbf{k}). \tag{13}$$

In the next section we shall show again, as in part III, that under suitable conditions the backscatter function B determines completely the potential function V. This depends on the fact that in Eq. (1) the potential function $V(\mathbf{x})$ serves as a multiplication operator, so that in Eq. (7) the corresponding integral kernel $V(\mathbf{k}'-\mathbf{k})$ serves as a convolution operator, and hence is translation invariant. Thus, if we subject Eq. (7) to the translation $\mathbf{k}\to\mathbf{k}-\mathbf{m}$, then all terms of the form $V(\mathbf{k}'-\mathbf{k}'')$ are invariant. The T matrix $T(\mathbf{k}',\mathbf{k})$ is not invariant, but if we choose for $\mathbf{m}$ the midpoint between $\mathbf{k}$ and $\mathbf{k}'$:

$$\mathbf{m} = \frac{\mathbf{k}'+\mathbf{k}}{2}, \tag{14}$$

then we get

$$T(\mathbf{k}',\mathbf{k}) \to T(\mathbf{k}'-\mathbf{m},\mathbf{k}-\mathbf{m}) = T\left(\frac{\mathbf{k}'-\mathbf{k}}{2},\frac{\mathbf{k}-\mathbf{k}'}{2}\right) = B(\mathbf{k}'-\mathbf{k}). \tag{15}$$

Moreover,

$$\begin{aligned}(\mathbf{k}''^2-\mathbf{k}^2) &= (\mathbf{k}''+\mathbf{k})\cdot(\mathbf{k}''-\mathbf{k}) \\ &\to (\mathbf{k}''+\mathbf{k}-2\mathbf{m})\cdot(\mathbf{k}''-\mathbf{k}) \\ &= -(\mathbf{k}'-\mathbf{k}'')\cdot(\mathbf{k}''-\mathbf{k}).\end{aligned} \tag{16}$$

Thus, after a translation by $\mathbf{m}$, Eq. (7) becomes

$$\begin{aligned}B(\mathbf{k}'-\mathbf{k}) &= V(\mathbf{k}'-\mathbf{k}) + \int \frac{V(\mathbf{k}'-\mathbf{k}'')V(\mathbf{k}''-\mathbf{k})}{(\mathbf{k}'-\mathbf{k}'')\cdot(\mathbf{k}''-\mathbf{k})+i0}\,d\mathbf{k}'' \\ &+ \iint \frac{V(\mathbf{k}'-\mathbf{k}'')V(\mathbf{k}''-\mathbf{k}''')V(\mathbf{k}'''-\mathbf{k})}{((\mathbf{k}'-\mathbf{k}'')\cdot(\mathbf{k}''-\mathbf{k})+i0)((\mathbf{k}'-\mathbf{k}''')\cdot(\mathbf{k}'''-\mathbf{k})+i0)}\,d\mathbf{k}'''\,d\mathbf{k}''+\cdots.\end{aligned} \tag{17}$$

Here we see that Eq. (17) gives the backscatter function directly in terms of the potential function, and that every term in this equation is translation invariant.

We can express this equation more formally in terms of the projection operator Θ introduced in part III. This operator acts on kernels $K(\mathbf{k}',\mathbf{k})$ according to the prescription

$$\Theta K(\mathbf{k}'-\mathbf{k}) = K\left(\frac{\mathbf{k}'-\mathbf{k}}{2},\frac{\mathbf{k}-\mathbf{k}'}{2}\right),$$

with the properties

$$\Theta V(\mathbf{k}'-\mathbf{k}) = V(\mathbf{k}'-\mathbf{k}),$$

$$\Theta T(\mathbf{k}',\mathbf{k}) = B(\mathbf{k}'-\mathbf{k}),$$

$$\Theta\Theta K(\mathbf{k}'-\mathbf{k}) = \Theta K(\mathbf{k}'-\mathbf{k}),$$

$$\|\Theta K\| \leqslant \|K\|.$$

In terms of this projection operator, the process of translating the kernel $K(\mathbf{k}',\mathbf{k})$ by the midpoint $\mathbf{m}=(\mathbf{k}'+\mathbf{k})/2$ becomes

$$K(\mathbf{k}',\mathbf{k}) \to K(\mathbf{k}'-\mathbf{m},\mathbf{k}-\mathbf{m}) = K\left(\frac{\mathbf{k}'-\mathbf{k}}{2},\frac{\mathbf{k}-\mathbf{k}'}{2}\right) = \Theta K(\mathbf{k}'-\mathbf{k}), \tag{18}$$

and Eq. (17) becomes

$$B = V - \Theta(V\Gamma V) + \Theta(V\Gamma(V\Gamma V)) - \cdots. \tag{19}$$

If the kernel $V(\mathbf{k}'-\mathbf{k})$ lies in some Friedrichs class with Friedrichs norm $\|V\|<a$, then the sum in Eq. (19) converges to B in this norm, so that the kernel $B(\mathbf{k}'-\mathbf{k})$ lies in the same class, with norm

$$\|B\| < (1-a)^{-1}\|V\|. \tag{20}$$

Moreover, if V and V' are two different potential functions with corresponding backscatter functions B and B', then we have

$$\|B-B'\| < (1-a)^{-2}\|V-V'\|, \tag{21}$$

as in part III.

IV. INVERSION

To show that the backscatter function B determines the potential function V, we have only to invert the series in Eq. (19). Here we describe two different inversion procedures for this purpose.

Our first procedure is suggested by the standard procedure for inverting an analytic function from its power series (see part II). We replace B by $B_1 z$, where $0\leqslant z\leqslant 1$, and V by

$$V(z)=\sum_{m=1}^{\infty} V_m z^m, \tag{22}$$

in Eq. (19), and then determine the coefficients V_m by equating powers of z:

$$\begin{aligned} &V_1=B_1,\\ &V_2=-\Theta(V_1\Gamma V_1),\\ &V_3=-\Theta(V_2\Gamma V_1)-\Theta(V_1\Gamma V_2)-\Theta(V_1\Gamma(V_1\Gamma V_1)),\\ &\vdots\\ &V_m=-\sum_{i=2}^{m}\sum_{r_1+\cdots+r_i=m}\Theta(V_{r_1}\Gamma(V_{r_2}\cdots\Gamma V_{r_i})). \end{aligned} \tag{23}$$

To determine the convergence of the resulting series in Eq. (22), we first note that if $\|B_1\|=b$, then

$$\begin{aligned} &\|V_1\|=b,\\ &\|V_2\|=\|\Theta(V_1\Gamma V_1)\|\leqslant\|V_1\|^2,\\ &\|V_3\|\leqslant\|V_2\|\|V_1\|+\|V_1\|\|V_2\|+\|V_1\|^3,\\ &\vdots\\ &\|V_m\|\leqslant\sum_{i=2}^{m}\sum_{r_1+\cdots+r_i=m}\|V_{r_1}\|\|V_{r_2}\|\cdots\|V_{r_i}\|. \end{aligned} \tag{24}$$

Hence if we put

$$\begin{aligned} &b_1=b,\\ &b_2=b_1^2,\\ &b_3=b_2b_1+b_1b_2+b_1^3,\\ &\vdots\\ &b_m=\sum_{i=2}^{m}\sum_{r_1+\cdots+r_i=m} b_{r_1}b_{r_2}\cdots b_{r_i}, \end{aligned} \tag{25}$$

then, from Eq. (24),

$$\|V_m\|\leqslant b_m. \tag{26}$$

Now we set

$$w=w(z)=\sum_{i=1}^{\infty} b_m z^m, \tag{27}$$

and observe that, because of Eq. (25),

$$w-bz=\sum_{i=2}^{\infty} b_m z^m=(1-w)^{-1}-1-w=w^2(1-w)^{-1}. \tag{28}$$

If we solve Eq. (28) for w, we get

$$w=\frac{(1+bz)\pm((1+bz)^2-8bz)^{1/2}}{4}. \tag{29}$$

As a function of z, w is analytic at $z=0$ and out to the nearest singularity, where

$$(1+bz)^2-8bz=0, \tag{30}$$

or where

$$bz=3-\sqrt{8}=0.172.... \tag{31}$$

Thus if

$$b<0.172..., \tag{32}$$

then the series defining w in Eq. (27) converges for $|z|\leqslant 1$, and it follows that the series for $V(z)$ in Eq. (22) converges for $z=1$. The sum V of this series then satisfies Eq. (18), and so determines a potential function whose corresponding backscatter function is B.

This inversion procedure gives us one solution for the inversion problem, but says nothing about the uniqueness of this solution. Our second inversion procedure, which is suggested by the fixed-point procedure used in part III, also shows that this solution is unique. For this purpose we define, for a given backscatter function B with $\|B\|=b$ and an arbitrary potential function V with $\|V\|=a<1$, a mapping F of potential functions by

$$F(V)=B+\Theta(V\Gamma V)-\Theta(V\Gamma(V\Gamma V))+\cdots. \quad (33)$$

If $\|V\|=a<1$, then this series converges to another potential function $F(V)$. If B happens to be the backscatter function corresponding to V, then, by Eq. (19), V is a fixed point of the mapping F:

$$F(V)=V. \quad (34)$$

Moreover, for any V,

$$\|F(V)\|\leqslant b+\sum_{m=2}^{\infty} a^m=b+a^2(1-a)^{-1}. \quad (35)$$

In particular, if $b<3-\sqrt{8}$ and $a=(1+\sqrt{1/2})b < 1-\sqrt{1/2}=0.293...$, then

$$\|F(V)\|<b+a^2(1-a)^{-1}<(1+\sqrt{1/2})b=a, \quad (36)$$

so that F maps the set of potential functions V with $\|V\| < a$ into itself. Moreover, if V and V' are any two potential functions in this set, then

$$F(V')-F(V)=\Theta((V'-V)\Gamma V')+\Theta(V\Gamma(V'-V)) -\cdots, \quad (37)$$

and so

$$\|F(V')-F(V)\|<c\|V'-V\|, \quad (38)$$

where

$$c=(2a+3a^2+\cdots)=(1-a)^{-2}-1<1. \quad (39)$$

It follows that the mapping F is a contraction on the set of potential functions V with $\|V\| \leqslant a < 1-\sqrt{1/2}$. The contraction mapping theorem now assures that there is a unique solution to Eq. (34), which may be obtained by iteration of F, starting from any convenient trial function $V^{(0)}$. It is easy to see that the convergence of the iterates is geometric:

$$\|V-V^{(m)}\|\leqslant c^m(1-c)^{-1}, \quad (40)$$

with c given by Eq. (39).

This second inversion procedure gives us a solution of the inverse problem that is unique, at least within the class of potential functions V satisfying the condition $\|V\| \leqslant a < 1-\sqrt{1/2}$. Now we have established a one-to-one correspondence between potential functions V and backscatter functions B:

$$V\leftrightarrow B,$$

such that if $\|V\|=a<1$, then

$$\|B\|\leqslant(1-a)^{-1}\|V\|, \quad (41)$$

and if $\|B\| = b < 3-\sqrt{8} = 0.172$, then

$$\|V\|\leqslant(1+\sqrt{1/2})\|B\|. \quad (42)$$

Moreover, if $V\leftrightarrow B$ and $V'\leftrightarrow B'$, then

$$\|B'-B\|\leqslant(1-a)^{-2}\|V'-V\|, \quad (43)$$

$$\|V'-V\|<\|B'-B\|. \quad (44)$$

These results provide a modest improvement over those presented in part III, where we assumed that $b < 1/8=0.125$(!) It is curious that this smallness condition should be so restrictive. Under this condition there are no apparent obstacles to the inversion procedure, such as bound states, in sight. For this reason we might expect that the inversion series in Eq. (21) might actually converge for larger values of b, i.e., we might expect that in the inversion series a certain amount of cancellation must take place. At this point, however, we see no way to demonstrate it.

[1] R. T. Prosser, (I), J. Math. Phys. 10, 1819 (1969); (II), 17, 1775 (1976); (III), 21, 2648 (1980); (IV), 23, 2127 (1982).

[2] *Added in proof:* The referee points out quite correctly that in using the Fourier transform the author has been careless with factors of 2π and ± 1 throughout this series. The proper treatment of these factors depends, of course, on the particular form of the Fourier transform used. The author has tacitly assumed throughout the series the symmetric forms $F(\mathbf{k}) = \int_{\mathbf{R}^3} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x})d\mathbf{x}$ and $f(\mathbf{x}) = \int_{\mathbf{R}^3} e^{i\mathbf{x}\cdot\mathbf{k}} F(\mathbf{k})d\mathbf{k}$, where $d\mathbf{k}=(2\pi)^{-3/2}\,dk_1\,dk_2\,dk_3$ and $d\mathbf{x}=(2\pi)^{-3/2}\,dx_1\,dx_2\,dx_3$. With these normalizations, the factors of 2π are absorbed into the integrals throughout. The sign of the term $i0$ in the denominator of Eqs. (5), (7), and in all similar equations should then be $+$, as it is here, but not in the earlier parts of this series.