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Linear Random Operator Equations in Mathematical Physics. II

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This paper is a continuation of Paper I [J. Math. Phys. 11, 1069 (1970)]. A general expression is determined for the stochastic Green's function (SGF) for two-point correlation functions, and various useful relationships are determined between the stochastic Green's functions for various statistical measures and between the stochastic Green's functions, random Green's functions, and ordinary Green's functions. In the author's dissertation and earlier papers, SGF was given as an ensemble average of the product of random Green's functions. This random Green's function is now specified in terms of an ordinary Green's function for a deterministic operator and a resolvent kernel which can be calculated for the random part of the stochastic operator. Hence that SGF is determinable which yields the desired statistical measure of the solution process directly. Second, the two-point correlation function of the solution process is found for the perturbation case. It is also demonstrated that, in the event that perturbation theory is adequate to deal with the randomness involved, the correct two-point correlation of the solution process is easily specialized from the general expression, i.e., the results of perturbation theory are obtained from the SGF of Adomian when perturbation theory is applicable.

1. GENERAL STOCHASTIC GREEN'S FUNCTION FOR CORRELATIONS

The present author[1] has defined the "stochastic Green's function" (SGF) for correlations as the kernel G_H relating the output (or solution process) correlation R_y to the input correlation function R_x in the expression

$$R_y(t_1, t_2) = \iint G_H(t_1, t_2, \tau_1, \tau_2) R_x(\tau_1, \tau_2) d\tau_1 d\tau_2. \tag{1.1}$$

A very detailed example, for the special case where stationarity could be assumed throughout[2] gave the spectral density for a randomly sampled random process and the well-known results of filter theory in the limiting case of a deterministic filter. Our purpose now is to look at the kernel G_H in a more comprehensive manner. Using terminology and symbols of Paper I, we can write the stochastic differential equation $\mathfrak{L}y = x$, where $\mathfrak{L}$ is a stochastic (differential) operator involving stochastic process coefficients $a_\nu(t,\omega)$, $t \in T$, $\omega \in (\Omega, \mathfrak{F}, \mu)$ as

$$y(t) = F(t) - \int K(t,\tau) y(\tau) d\tau, \tag{1.2}$$

where

$$F(t) = \int l(t,\tau) x(\tau) d\tau + \Sigma c_\nu \Phi_\nu.$$

Then

$$\begin{aligned}
R_y(t_1, t_2) &= \langle y(t_1) \overset{*}{y}(t_2) \rangle \\
&= \langle (F(t_1) - \int K(t_1,\tau_1) y(\tau_1) d\tau_1) \\
&\times (\overset{*}{F}(t_2) - \int \overset{*}{K}(t_2,\tau_2) \overset{*}{y}(\tau_2) d\tau_2) \rangle \\
&= \langle F(t_1) \overset{*}{F}(t_2) \rangle - \int \langle K(t_1,\tau_1) \overset{*}{F}(t_2) y(\tau_1) \rangle d\tau_1 \\
&- \int \langle \overset{*}{K}(t_2,\tau_2) F(t_1) \overset{*}{y}(\tau_2) \rangle d\tau_2 \\
&+ \iint \langle K(t_1,\tau_1) \overset{*}{K}(t_2,\tau_2) y(\tau_1) \overset{*}{y}(\tau_2) \rangle d\tau_1 d\tau_2
\end{aligned} \tag{1.3}$$

Using Adomian's iteration procedure in Paper I, we can write Eq. (2) in terms of iterated kernels as used by Sibul[3] thus:

$$y(t) = F(t) - \sum_{m=0}^{\infty} \int (-1)^{m+1} K_{m+1}(t, \tau) F(\tau) d\tau, \tag{1.4}$$

where $K_m(t,\tau)$ is defined by the recurrence formula

$$K_m(t,\tau) = \int K(t,\tau_1) K_{m-1}(\tau_1,\tau) d\tau_1, \tag{1.5}$$

with $K_1 = K$.

Thus

$$\begin{aligned}
y(t) &= F(t) - \int K(t,\tau) y(\tau) d\tau \\
&= F(t) - \int K(t,\tau)[y_0 - y_1 + y_2 + \cdots] d\tau \\
&= F(t) - \int K(t,\tau) y_0(\tau) d\tau + \int K(t,\tau) y_1(\tau) d\tau - \cdots \\
&= y_0 - y_1 + y_2 - \cdots \\
&= F(t) - \int K(t,\tau) F(\tau) d\tau \\
&+ \int K(t,\tau_1) K(\tau_1,\tau) F(\tau) d\tau \cdots \\
&= F(t) + \sum_{m=0}^{\infty} \int (-1)^{m+1} K_{m+1}(t,\tau) F(\tau) d\tau.
\end{aligned}$$

If the sum is uniformly convergent, then summation and integration can be interchanged[3] in (1.4).[4] Defining the resolvent kernel $\Gamma(t,\tau) = \sum_{m=0}^{\infty} (-1)^m K_{m+1}(t,\tau)$ allows us to write very conveniently

$$y(t) = F(t) + \int \Gamma(t,\tau) F(\tau) d\tau. \tag{1.6}$$

Writing $R_y(t_1, t_2)$, we now have

$$\begin{aligned}
R_y(t_1, t_2) &= \langle F(t_1) \overset{*}{F}(t_2) \rangle - \int \langle \Gamma(t_1,\tau_1) \overset{*}{F}(t_2) F(\tau_1) \rangle d\tau_1 - \int \langle \overset{*}{\Gamma}(t_2,\tau_2) F(t_1) \overset{*}{F}(\tau_2) \rangle d\tau_2 \\
&+ \iint \langle \Gamma(t_1,\tau_1) \overset{*}{\Gamma}(t_2,\tau_2) F(\tau_1) \overset{*}{F}(\tau_2) \rangle d\tau_1 d\tau_2.
\end{aligned} \tag{1.7}$$

In order to separate out $\langle F(\tau_1) \overset{*}{F}(\tau_2) \rangle$, we can rewrite (1.7) as

$$R_y(t_1,t_2) = \iint \langle F(\tau_1)\overset{*}{F}(\tau_2)\rangle \delta(\tau_1 - t_1)\delta(\tau_2 - t_2)d\tau_1 d\tau_2 - \iint \langle \Gamma(t_1,\tau_1)\rangle\langle F(\tau_1)\overset{*}{F}(\tau_2)\rangle \delta(\tau_2 - t_2)d\tau_1 d\tau_2$$
$$- \iint \langle \overset{*}{\Gamma}(t_2,\tau_2)\rangle\langle F(\tau_1)\overset{*}{F}(\tau_2)\rangle\delta(\tau_1 - t_1)d\tau_1 d\tau_2 + \iint\langle\Gamma(t_1,\tau_1)\overset{*}{\Gamma}(t_2,\tau_2)\rangle\langle F(\tau_1)\overset{*}{F}(\tau_2)\rangle d\tau_1 d\tau_2, \tag{1.8}$$

where

$$\langle F(\tau_1)\overset{*}{F}(\tau_2)\rangle = R_F(\tau_1,\tau_2) = \iint G(\tau_1,\sigma_1)\overset{*}{G}(\tau_2,\sigma_2)R_x(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2. \tag{1.9}$$

Substituting (9) into (8), we get

$$R_y(t_1,t_2) = \iiiint G(\tau_1,\sigma_1)\overset{*}{G}(\tau_2,\sigma_2)R_x(\sigma_1,\sigma_2)$$
$$\times\ \delta(\tau_1 - t_1)\delta(\tau_2 - t_2)\, d\tau_1 d\tau_2 d\sigma_1 d\sigma_2 - \iiiint\langle\Gamma(t_1,\tau_1)\rangle G(\tau_1,\sigma_1)\overset{*}{G}(\tau_2,\sigma_2)R_x(\sigma_1,\sigma_2)\delta(\tau_2 - t_2)$$
$$\times\ d\tau_1 d\tau_2 d\sigma_1 d\sigma_2 - \iiiint\langle\overset{*}{\Gamma}(t_2,\tau_2)\rangle G(\tau_1,\sigma_1)\overset{*}{G}(\tau_2,\sigma_2)R_x(\sigma_1,\sigma_2)\delta(\tau_1 - t_1)d\tau_1 d\tau_2 d\sigma_1 d\sigma_2$$
$$+ \iiiint\langle\Gamma(t_1,\tau_1)\overset{*}{\Gamma}(t_2,\tau_2)\rangle G(\tau_1,\sigma_1)\overset{*}{G}(\tau_2,\sigma_2)R_x(\sigma_1,\sigma_2)d\tau_1 d\tau_2 d\sigma_1 d\sigma_2,$$

which is the same as the general expression

$$R_y(t_1,t_2) = \iint G_H(t_1,t_2,\sigma_1,\sigma_2)R_x(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2,$$

where

$$G_H = \iint G(\tau_1,\sigma_1)\overset{*}{G}(\tau_2,\sigma_2)\delta(\tau_1 - t_1)\delta(\tau_2 - t_2)d\tau_1 d\tau_2 - \iint\langle\Gamma(t_1,\tau_1)\rangle G(\tau_1,\sigma_1)\overset{*}{G}(\tau_2,\sigma_2)\delta(\tau_2 - t_2)$$
$$\times\ d\tau_1 d\tau_2 - \iint\langle\overset{*}{\Gamma}(t_2,\tau_2)\rangle G(\tau_1,\sigma_1)\overset{*}{G}(\tau_2,\sigma_2)\delta(\tau_1 - t_1)d\tau_1 d\tau_2 + \iint\langle\Gamma(t_1,\tau_1)\overset{*}{\Gamma}(t_2,\tau_2)\rangle$$
$$\times\ G(\tau_1,\sigma_1)\overset{*}{G}(\tau_2,\sigma_2)d\tau_1 d\tau_2 \tag{1.10}$$

is the stochastic Green's function[5] if two-point correlations are used for the statistical measures of the input (forcing function) or output (solution process or dependent variable). [We have, as before, used statistical independence of the random coefficients and the forcing function to separate ensemble averages involving Γ and $F(t)$.] When there is no random term ($R = 0$) the last three terms of G_H are zero, so

$$R_y(t_1,t_2) = \iint G(t_1,\sigma_1)\overset{*}{G}(t_2,\sigma_2)R_x(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2,$$

where G's are, of course, ordinary (deterministic) Green's functions.

2. RELATIONSHIPS BETWEEN KERNEL FUNCTIONS (OR SGF'S FOR VARIOUS STATISTICAL MEASURES)

In I, the SGF's were given for various statistical measures of input and output [see Eqs. (I. 2. 1) for $R_y(t_1,t_2)$ or (I. 2. 3) for $R_y(\beta)$ (assuming stationarity) or (I. 2. 4) for the spectral density (also assuming stationarity holds)]. Equation (1.10) in this paper seems to show a more complicated kernel. Thus

$$R_y(t_1,t_2) = \iint G_H(t_1,t_2,\sigma_1,\sigma_2)R_x(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2,$$

where the SGF G_H is given by

$$G_H = G(t_1,\sigma_1)\overset{*}{G}(t_2,\sigma_2) - \int\langle\Gamma(t_1,\tau_1)\rangle G(\tau_1,\sigma_1)$$
$$\times\ \overset{*}{G}(t_2,\sigma_2)d\tau_1 - \int\langle\overset{*}{\Gamma}(t_2,\tau_2)\rangle G(t_1,\sigma_1)$$
$$\times\ \overset{*}{G}(\tau_2,\sigma_2)d\tau_2 + \iint\langle\Gamma(t_1,\tau_1)\overset{*}{\Gamma}(t_2,\tau_2)\rangle$$
$$\times\ G(\tau_1,\sigma_1)\overset{*}{G}(\tau_2,\sigma_2)d\tau_1 d\tau_2. \tag{2.1}$$

This kernel is of course identical to the corresponding kernel in Paper I. There, the H is a stochastic operator and $h(t,\sigma)$ is a random Green's function, not the Green's function G corresponding to the deterministic operator L^{-1}.[6] From (2.1) in Paper I we had

$$G_H(t_1,t_2,\sigma_1,\sigma_2) = \langle h(t_1,\sigma_1)h^*(t_2,\sigma_2)\rangle,$$

which is the same as (1.1) above. Hence we see immediately that

$$h(t,\sigma) = G(t,\sigma) - \int\Gamma(t,\tau)G(\tau,\sigma)d\tau, \tag{2.2}$$

a rather simple relation between the various Green's functions: the random Green's function h, a random quantity from which the SGF is easily found, the deterministic Green's function G (or sometimes l), and the resolvent kernel. Thus the results generalize and make more useful some of the results of the author.[2]

This expression clearly shows we can indeed compute the SGF for any statistical measure, e.g., for $R_y(t_1, t_2)$, as soon as h can be determined (by the indicated average of a product of h functions); and h can be determined since it depends on the known G and the resolvent kernel Γ, which can be found for reasonable operators R. Thus, for a given stochastic differential equation, after deciding we want a particular statistical measure (s.m.) of the solution process, we calculate the appropriate SGF which yields the desired s.m. in terms of the s.m. of the given random input.

3. PERTURBATION CASE FOR CORRELATIONS

In the event that perturbation theory is adequate to deal with the randomness involved (see I), we can let $R = \epsilon\mathfrak{L}_1$, where $\langle\mathfrak{L}_1\rangle = 0$ is the general expression for the stochastic Green's function. First, however, let us see the results to be expected by extending the conventional approach as used for the expectation $\langle y\rangle$ to the case of correlation measures $R_y(t_1, t_2) = \langle y(t_1)y(t_2)\rangle$:

$$\begin{aligned}
y &= L^{-1}x - \epsilon L^{-1}\mathfrak{L}_1 y \\
&= L^{-1}x - \epsilon L^{-1}\mathfrak{L}_1[y_0 + \epsilon y_1 + \epsilon^2 y_2 + O(\epsilon^3)] \\
&= L^{-1}x - \epsilon L^{-1}\mathfrak{L}_1 L^{-1}x + \epsilon^2 L^{-1}\mathfrak{L}_1 L^{-1}\mathfrak{L}_1 L^{-1}x \\
&\quad - O(\epsilon^3).
\end{aligned}$$

If $x = g_0 + \epsilon g_1 + \epsilon^2 g_2$, with $\langle g_1\rangle = \langle g_2\rangle = 0$, absorbing the means into $\langle g_0\rangle$, if they exist, leads immediately to[1]

$$\begin{aligned}
\langle y\rangle &= L^{-1}\langle x\rangle + \epsilon^2 L^{-1}\langle\mathfrak{L}_1 L^{-1}\mathfrak{L}_1\rangle L^{-1}\langle x\rangle, \\
\langle y\rangle &= [1 + \epsilon^2 L^{-1}\langle\mathfrak{L}_1 L^{-1}\mathfrak{L}_1\rangle]L^{-1}\langle g_0\rangle,
\end{aligned}$$

assuming R and x are uncorrelated, as in I. In the same manner we can extend to correlations $R_y(t_1, t_2)$:

$$y(t) = L^{-1}(t)x(t) - \epsilon L^{-1}(t)\mathfrak{L}_1(t)L^{-1}(t)x(t) + \epsilon^2 L^{-1}(t)\mathfrak{L}_1(t)L^{-1}(t)\mathfrak{L}_1(t)L^{-1}(t)x(t), \tag{3.1}$$

$$\begin{aligned}
y(t) &= L^{-1}g_0 + \epsilon L^{-1}g_1 + \epsilon^2 L^{-1}g_2 - \epsilon L^{-1}\mathfrak{L}_1 L^{-1}g_0 - \epsilon^2 L^{-1}\mathfrak{L}_1 L^{-1}g_1 + \epsilon^2 L^{-1}\mathfrak{L}_1 L^{-1}\mathfrak{L}_1 L^{-1}g_0, \\
R_y(t_1, t_2) &= \langle L^{-1}(t_1)g_0(t_1)L^{-1}(t_2)g_0(t_2)\rangle + \epsilon\langle L^{-1}(t_1)g_0(t_1)L^{-1}(t_2)g_1(t_2)\rangle + \epsilon^2\langle L^{-1}(t_1)g_0(t_1)L^{-1}(t_2)g_2(t_2)\rangle \\
&\quad - \epsilon\langle L^{-1}(t_1)g_0(t_1)L^{-1}(t_2)\mathfrak{L}_1(t_2)L^{-1}(t_2)g_0(t_2)\rangle - \epsilon^2\langle L^{-1}(t_1)g_0(t_1)L^{-1}(t_2)\mathfrak{L}_1(t_2)L^{-1}(t_2)g_1(t_2)\rangle \\
&\quad + \epsilon^2\langle L^{-1}(t_1)g_0(t_1)L^{-1}(t_2)\mathfrak{L}_1(t_2)L^{-1}(t_2)\mathfrak{L}_1(t_2)L^{-1}(t_2)g_0(t_2)\rangle + \epsilon\langle L^{-1}(t_1)g_1(t_1)L^{-1}(t_2)g_0(t_2)\rangle \\
&\quad + \epsilon^2\langle L^{-1}(t_1)g_1(t_1)L^{-1}(t_2)g_1(t_2)\rangle - \epsilon^2\langle L^{-1}(t_1)g_1(t_1)L^{-1}(t_2)\mathfrak{L}_1(t_2)L^{-1}(t_2)g_0(t_2)\rangle \\
&\quad + \epsilon^2\langle L^{-1}(t_1)g_2(t_1)L^{-1}(t_2)g_0(t_2)\rangle - \epsilon\langle L^{-1}(t_1)\mathfrak{L}_1(t_1)L^{-1}(t_1)g_0(t_1)L^{-1}(t_2)g_0(t_2)\rangle \\
&\quad - \epsilon^2\langle L^{-1}(t_1)\mathfrak{L}_1(t_1)L^{-1}(t_1)g_0(t_1)g_0(t_1)L^{-1}(t_2)g_1(t_2)\rangle - \epsilon^2\langle L^{-1}(t_1)\mathfrak{L}_1(t_1)L^{-1}(t_1)g_1(t_1)L^{-1}(t_2)g_0(t_2)\rangle \\
&\quad + \epsilon^2\langle L^{-1}(t_1)\mathfrak{L}_1(t_1)L^{-1}(t_1)\mathfrak{L}_1(t_1)L^{-1}(t_1)g_0(t_1)L^{-1}(t_2)g_0(t_2)\rangle.
\end{aligned}\tag{3.2}$$

The fourth, fifth, ninth, twelth, and thirteenth terms vanish because $\langle\mathfrak{L}_1\rangle = 0$. Rewriting in terms of the Green's function $l(t, \tau)$ rather than the operator L^{-1} and rearranging in powers of ϵ, we obtain

$$\begin{aligned}
R_y(t_1, t_2) &= \iint l(t_1, \tau_1)l(t_2, \tau_2)[\langle g_0(\tau_1)g_0(\tau_2)\rangle + \epsilon\langle g_0(\tau_1)g_1(\tau_2)\rangle + \epsilon\langle g_1(\tau_1)g_0(\tau_2)\rangle + \epsilon^2\langle g_0(\tau_1)g_2(\tau_2)\rangle \\
&\quad + \epsilon^2\langle g_1(\tau_1)g_1(\tau_2)\rangle + \epsilon^2\langle g_2(\tau_1)g_0(\tau_2)\rangle]d\tau_1 d\tau_2 + \epsilon^2\iiiint l(t_1, \tau_1)l(t_2, \tau_2)l(\tau_2, \tau_3)l(\tau_3, \tau_4) \\
&\quad \times \langle\mathfrak{L}_1(\tau_2)\mathfrak{L}_1(\tau_3)\rangle\langle g_0(\tau_1)g_0(\tau_4)\rangle d\tau_1\cdots d\tau_4 + \epsilon^2\iiiint l(t_1, \tau_1)l(\tau_1, \tau_2)l(\tau_2, \tau_3)l(t_2, \tau_4) \\
&\quad \times \langle\mathfrak{L}_1(\tau_1)\mathfrak{L}_1(\tau_2)\rangle\langle g_0(\tau_3)g_0(\tau_4)\rangle d\tau_1\cdots d\tau_4.
\end{aligned}\tag{3.3}$$

Rearranging the order of the last two terms and of the repeated τ's to see the symmetry better, we find

$$\begin{aligned}
R_y(t_1, t_2) &= \iint l(t_1, \tau_1)l(t_2, \tau_2)[\langle g_0(\tau_1)g_0(\tau_2)\rangle + \epsilon\langle g_0(\tau_1)g_1(\tau_2)\rangle + \epsilon\langle g_1(\tau_1)g_0(\tau_2)\rangle + \epsilon^2\langle g_0(\tau_1)g_2(\tau_2)\rangle \\
&\quad + \epsilon^2\langle g_1(\tau_1)g_1(\tau_2)\rangle + \epsilon^2\langle g_2(\tau_1)g_0(\tau_2)\rangle]d\tau_1 d\tau_2 + \epsilon^2\iiiint l(t_1, \tau_1)l(\tau_1, \tau_1')l(\tau_1', \tau_1'')l(t_2, \tau_2) \\
&\quad \times \langle\mathfrak{L}_1(\tau_1)\mathfrak{L}_1(\tau_1')\rangle\langle g_0(\tau_1'')g_0(\tau_2)\rangle d\tau_1 d\tau_1' d\tau_1'' d\tau_2 + \epsilon^2\iiiint l(t_2, \tau_2)l(\tau_2, \tau_2')l(\tau_2', \tau_2'')l(t_1, \tau_1)\rangle \\
&\quad \times \langle\mathfrak{L}_1(\tau_2)\mathfrak{L}_1(\tau_2')\rangle\langle g_0(\tau_1)g_0(\tau_2'')\rangle d\tau_2 d\tau_2' d\tau_2'' d\tau_1
\end{aligned}\tag{3.4}$$

4. PERTURBATION RESULTS FROM STOCHASTIC GREEN'S FUNCTION

We now use Adomian's stochastic Green's function (SGF) [for correlation measures of input and output from Eq. (1.10)] to verify the perturbation case, i.e., the case where randomness is relatively insignificant, and hence could be handled by conventional perturbation theory. To calculate the SGF, we need the resolvent kernel Γ, i.e.,

$$\Gamma(t_1,\tau_1)=\sum_m(-1)^m K_{m+1}(t_1,\tau_1).$$

Since $K_1(t_1,\tau_1)=G(t_1,\tau_1)R(\tau_1)$ and R is zero mean, clearly

$$\langle K_1\rangle = 0,$$

and

$$-K_2(t_1,\tau_1)=-\int K(t_1,\tau_1')K(\tau_1',\tau_1)d\tau_1'$$
$$=-\int G(t_1,\tau_1')\epsilon\mathfrak{L}_1(\tau_1')G(\tau_1',\tau_1)\epsilon\mathfrak{L}_1(\tau_1)d\tau_1',$$

which we get by replacing R with $\epsilon\mathfrak{L}_1$ (see Paper I). Hence

$$\langle\Gamma(t_1,\tau_1)\rangle = 0+\langle -K_2(t_1,\tau_1)\rangle+\mathcal{O}(\epsilon^3).$$

Thus, to order ϵ^2,

$$\langle\Gamma(t_1,\tau_1)\rangle=-\epsilon^2\int G(t_1,\tau_1')G(\tau_1',\tau_1)\langle\mathfrak{L}_1(\tau_1')\times\mathfrak{L}_1(\tau_1)\rangle d\tau_1'.$$

The four terms of the SGF G_H are now easily determined. The first term is simply $G(t_1,\sigma_1)G(t_2,\sigma_2)$. Since $x(t)=g_0(t)+\epsilon g_1(t)+\epsilon^2 g_2(t)$, we have

$$R_x(\sigma_1,\sigma_2)=\langle x(\sigma_1)x(\sigma_2)\rangle=\langle g_0(\sigma_1)g_0(\sigma_2)\rangle + \epsilon\langle g_0(\sigma_1)g_1(\sigma_2)\rangle+\epsilon^2\langle g_0(\sigma_1)g_2(\sigma_2)\rangle + \epsilon\langle g_1(\sigma_1)g_0(\sigma_2) + \epsilon^2\langle g_1(\sigma_1)g_1(\sigma_2)\rangle + \epsilon^2\langle g_2(\sigma_1)g_0(\sigma_2)\rangle.$$

We see[7] that

$$R_y(t_1,t_2)=\iint G(t_1,\sigma_1)G(t_2,\sigma_2)R_x(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2$$

gives exactly the double integral terms of Eq. (3.4). The remaining fourfold integral terms come from the terms of G_H which involve the resolvent kernel Γ, specifically, the second and third since the fourth is higher in ϵ than we are interested.

The second term of G_H is

$$-\int\langle\Gamma(t_1,\tau_1)\rangle G(\tau_1,\sigma_1)G(t_2,\sigma_2)d\tau_1,$$

into which we substitute the calculated $\langle\Gamma\rangle$ from above. Hence the second term for G_H becomes

$$\epsilon^2\iint G(t_1,\tau_1')G(\tau_1',\tau_1)G(\tau_1,\sigma_1)G(t_2,\sigma_2)\times\langle\mathfrak{L}_1(\tau_1')\mathfrak{L}_1(\tau_1)\rangle d\tau_1' d\tau_1.$$

The third term is

$$-\int\Gamma(t_2,\tau_2)G(t_1,\sigma_1)G(\tau_2,\sigma_2)d\tau_2$$

or

$$\epsilon^2\iint G(t_2,\tau_2')G(\tau_2',\tau_2)G(\tau_2,\sigma_2)G(t_1,\sigma_1)\times\langle\mathfrak{L}_1(\tau_2')\mathfrak{L}_1(\tau_2)\rangle d\tau_2' d\tau_2,$$

which when used in the expression for $R_y(t_1,t_2)$ obviously is identical to the fourfold integral terms of Eq. (3.4). Hence we have verified that the results of perturbation theory can be obtained from the SGF of Adomian when perturbation theory is applicable.

5. PERTURBATION RESULTS FROM THE RANDOM GREEN'S FUNCTION $h(t,\tau)$

The random Green's function $h(t,\tau)$ was given by

$$h(t,\tau)=G(t,\tau)-\int\Gamma(t,\sigma)G(\sigma,\tau)d\sigma; \tag{5.1}$$

hence $y(t)=\int h(t,\tau)x(\tau)d\tau$. Consequently,

$$y(t)=\int G(t,\tau)x(\tau)d\tau-\iint\Gamma(t,\sigma)G(\sigma,\tau)x(\tau)d\sigma\, d\tau;$$

averaging, we have

$$\langle y(t)\rangle=\int G(t,\tau)\langle x(\tau)\rangle d\tau-\iint\langle\Gamma(t,\sigma)G(\sigma,\tau)x(\tau)\rangle d\sigma\, d\tau.$$

Expanding the resolvent kernel $\Gamma(t,\sigma)$ in terms of iterated kernels, we have

$$\langle y(t)\rangle=\int G(t,\tau)\langle x(\tau)\rangle d\tau-\iint G(t,\tau_1)\langle R(\tau_1)\rangle G(\tau_1,t)\langle x(\tau)\rangle d\tau_1 d\tau+\iiint G(t,\tau_1)G(\tau_1,\tau_2)\langle R(\tau_1)R(\tau_2)\rangle \times G(\tau_2,\tau)\langle x(\tau)\rangle d\tau_2 d\tau_1 d\tau - \iiiint G(t_1,\tau_1)G(\tau_1,\tau_2)G(\tau_2,\tau_3)\langle R(\tau_1)R(\tau_2)R(\tau_3)\rangle G(\tau_3,\tau)\langle x(\tau)\rangle \times d\tau_3 d\tau_2 d\tau_1 d\tau+\cdots.$$

If $R=\epsilon\mathfrak{L}_1+\epsilon^2\mathfrak{L}_2$ and $x=g_0+\epsilon g_1+\epsilon^2 g_2=g$, as before, we have

$$\langle y(t)\rangle=\int G(t,\tau)\langle(g_0+\epsilon g_1+\epsilon^2 g_2)\rangle d\tau-\iint G(t,\tau_1)\langle\epsilon\mathfrak{L}_1(\tau_1)\rangle G(\tau_1,\tau)\langle g_0+\epsilon g_1+\epsilon^2 g_2\rangle d\tau_1 d\tau - \iint G(t,\tau_1)\langle\epsilon^2\mathfrak{L}_2(\tau_1)\rangle G(\tau_1,\tau)g_0 d\tau_1 d\tau+\iiint G(t,\tau_1)G(\tau_1,\tau_2)\langle\epsilon^2\mathfrak{L}_1(\tau_1)\mathfrak{L}_1(\tau_2)\rangle G(\tau_2,\tau) \times\langle g_0+\epsilon g_1+\epsilon^2 g_2\rangle d\tau_2 d\tau_1 d\tau+O(\epsilon^3)$$

or, equivalently,

$$\langle y(t)\rangle = L^{-1}\langle g_0 + \epsilon g_1 + \epsilon^2 g_2\rangle - \epsilon L^{-1}\langle \mathfrak{L}_1\rangle L^{-1}\langle g_0 + \epsilon g_1\rangle - \epsilon^2 L^{-1}\langle \mathfrak{L}_2\rangle L^{-1} g_0 + \epsilon^2 L^{-1}\langle \mathfrak{L}_1 L^{-1}\mathfrak{L}_1\rangle \times L^{-1} g_0 + O(\epsilon^3).$$

If $\langle \mathfrak{L}_1\rangle = \langle \mathfrak{L}_2\rangle = 0$,

$$\langle y(t)\rangle = L^{-1}\langle g\rangle + \epsilon^2 L^{-1}\langle \mathfrak{L}_1 L^{-1}\mathfrak{L}_1\rangle L^{-1} g_0$$

or, if $g_1 = g_2 = 0$ so that $g_0 = g$,

$$\langle y(t)\rangle = (1 + \epsilon^2 L^{-1}\langle \mathfrak{L}_1 L^{-1}\mathfrak{L}_1\rangle) L^{-1} g$$

as derived in Paper I. We have already demonstrated that the correlation result, for the case where perturbation theory is valid, can be derived from the SGF. It clearly could be found by writing $h(t,\tau)$.

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1 G. Adomian, J. Math. Phys. 11, 1069 (1970).
2 G. Adomian, Ph. D. thesis (UCLA, 1961) (unpublished).
3 L. H. Sibul, Ph. D. thesis (Pennsylvania State University, 1968) (unpublished).
4 A separate paper will deal with rates of convergence and comparison with hierarchy methods.
5 Obviously some integrations can be carried out, but we leave the expression this way.
6 l has also been used in the author's previous work.
7 $G(t,\tau) = l(t,\tau)$.

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Linear Random Operator Equations in Mathematical Physics. III

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The problem of wave motion in a stochastic medium is treated as an application of stochastic operator theory and as a generalization of Papers I and II (and previous work by the author) to the case of partial differential equations and random fields without monochromaticity assumptions and closure approximations. Connections to the theory of partial coherence are considered. The stochastic Green's function for the two-point correlation of the solution process can be determined so the correlation can be obtained. Spectral spreading in a "hot" medium is easily demonstrable and can be calculated.

INTRODUCTION

The problem of wave motion in a stochastic or randomly inhomogeneous or fluctuating medium arises in a number of interesting contexts in physics and as a natural generalization of Paper I (Ref. 1) to the case of partial differential equations. The refractive index is assumed to be a random point function or stochastic process (SP) rather than a random variable. Use of stochastic distributions is, of course, suggested; however, it is desirable to keep the formulation initially simple since there already exist a number of complexities not present in usual treatments. (For the same reason the nonlinear stochastic problem is deferred.) Nearly all studies of wave propagation in a random continuum utilize the monochromatic or quasi-monochromatic assumption of harmonic time dependence and essentially no spectral spreading, and therefore proceed immediately to a reduced wave equation or Helmholtz equation. Thus the field quantity becomes a functional of a random process $k(\overline{x})$ rather than $k(\overline{x}, t)$ or, more completely, $k(\overline{x}, t, \omega)$, where $\omega \in (\mathbf{\Omega}, \mathfrak{F}, \mu)$, a probability space. (See, e.g., Beran and Parrent[2] or Beran.[3]) Keller,[4] e.g., considered the scalar (reduced) wave equation $(\nabla^2 + k^2 n^2(\overline{x})]u = g(\overline{x})$, where $g(\overline{x})$ is the source distribution and there exists an appropriate condition on u and ∇u (radiation condition). k is the propagation constant for the medium, and n is the refractive index. Let $n^2(\overline{x}) = 1 + \mu(\overline{x})$, where $\mu(\overline{x})$ is a zero-mean SP. Then $(\nabla^2 + k^2)u + k^2\mu(\overline{x})u = g(\overline{x})$ or $\mathfrak{L} u = g$, where $\mathfrak{L}$ is a stochastic operator separable into the sum of a deterministic operator $L = \nabla^2 + k^2$ and a random operator R. The deterministic operator $L = \nabla^2 + k_0^2$. The Green's function $G(\overline{x} - \overline{x}') = G(r) = \exp(ik_0 r)/4\pi r = \exp(ik_0|x - x'|)/4\pi|x - x'|$. In a uniform or nonrandom medium, $n = 1$ and $(\nabla^2 + k^2)u = g$. The stochastic medium may, of course, be considered an ensemble of possible media with a probability distribution giving probabilities for various members of the ensemble.

The medium may not be known precisely, and the objective is to determine what is likely; it may be too complex to specify, or it may be randomly fluctuating. Most results involving stochastic equations in physics and engineering have been obtained by averaging procedures, closure, or truncation approximations (hierarchy methods, perturbation theories, etc.), self-consistent field approximations, and restriction to very special processes. Many of these have correlated well