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OPTIMALITY OF SINGULAR CURVES IN THE PROBLEM OF A CAR WITH n TRAILERS

N. B. Mel'nikov

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CONTENTS

Introduction

The mathematical model of a car with n trailers is one of the first nontrivial examples of a nonholonomic system in robot technology. Initially, the stabilization problem of this control system was mainly considered (see [6, 9, 14] and references in them). Later on, it was shown that the system is described by a Goursat 2-distribution on the manifold $\mathbb{R}^2 \times (\mathcal{S}^1)^{n+1}$ (see [7, 10, 12, 16]) and that the local classification of this 2-distribution and an arbitrary Goursat distribution of corank $(n+1)$ is the same problem [12]. In particular, every singularity of an arbitrary Goursat distribution of corank $n+1$ can be realized as a certain singular configuration of a car with n trailers (see [12]). The singular configurations of the car with n trailers are described by the following simple geometric condition (see [7]): at least one of the trailers (except for the last) is located at a right angle to the previous. The curves in $\mathbb{R}^2 \times (\mathcal{S}^1)^{n+1}$ defined by this condition are called *singular curves.*

M. I. Zelikin conjectured in [17] that singular curves can be described as singular trajectories yielding the minimum in the time-optimal control problem with constant linear and bounded angular velocities of the car. In [3, 4], by explicitly constructing the field of extremals in a neighborhood of a singular trajectory (see [18]), the optimality (strong minimality) of small parts of one of the $n-1$ possible types of singular curves was proved.

In the present paper, we prove that a sufficiently small part of *any* first-order singular trajectory yields a weak minimum (Theorem 2). Moreover, an *arbitrary* singular curve is a singular trajectory, and any (not necessarily small) part of this trajectory yields a weak minimum of the problem (Theorem 3). Note, that if a singular control is a boundary control, then the minimum is simultaneously strong. The formulations of the main results and schemes of proofs were published in [11].

The further presentation is organized as follows. In Sec. 1, we collect necessary assertions concerning the time-optimal control problem for an arbitrary system with scalar control linearly entering the system. In Sec. 2, we consider the kinematic model of a car with n trailers (Sec. 2.1), present the necessary definitions and facts concerning Goursat distributions (Sec. 2.3), and give proofs of the main results: Theorems 2 (Sec. 2.2) and Theorem 3 (Sec. 2.4). The proof of Theorems 2 and 3 uses the second-order optimality conditions for singular trajectories obtained in [1]. In Sec. 2.5, for illustration, we consider the case of a system with two trailers.

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1. Auxiliary Assertions

1.1. Statement of the problem and reduction. Let us consider the following time-optimal control problem for a control-linear system of the general form with scalar control:

$$T \longrightarrow \min \tag{1}$$

$$\dot{q} = f_0(q) + u f_1(q), \tag{2}$$

$$q(0) = a, \quad q(T) = b, \quad |u| \le \omega, \tag{3}$$

where $f_0(q)$ and $f_1(q)$ are arbitrary smooth functions in a certain domain of the n-dimensional Euclidean space.

We first reduce this time-optimal control problem to a problem on a fixed closed interval (see [2]). For this purpose, we make the change of time setting $t(\tau) = z\tau$, $z > 0$. Considering z as an additional variable, we pass from the initial system (2) to the extended system

$$\dot{z} = 0, \quad \dot{q} = z(f_0(q) + u f_1(q)), \quad \tau \in [0, t_1]. \tag{4}$$

Under such an extension of system (2), the time-optimal functional (1) passes to the terminal functional

$$\mathcal{J} = z(0) \longrightarrow \min \tag{5}$$

under the same constraints

$$q(0) = a, \quad q(t_1) = b, \quad |u| \le \omega. \tag{6}$$

We study this minimum problem among all absolutely continuous n-dimensional vector-valued functions $q(t)$ and measurable scalar-valued functions $u(t)$. The existence of the minimum is guaranteed by the Filippov lemma [5] under the following additional constraints on the growth of the right-hand side of the system:

$$(f_i(q), q) \le C(\|q\|^2 + 1), \quad i = 1, 2.$$

The weak minimum in the problem (4)–(6) is defined by the norm $|z| + \|x\|_C + \|u\|_\infty$; the Pontryagin (here, merely L_1) minimum is given by the norm $|z| + \|x\|_C + \|u\|_1$; the strong minimum is given by the seminorm $|z| + \|x\|_C$ for a free control. Here, $\|\cdot\|_p$ is the norm in the corresponding Banach space $L_p^n[0, t_1]$, $p \ge 1$; $\|\cdot\|_C$ is the norm of uniform convergence in $C^n[0, t_1]$. Since the control is scalar-valued, the quadratic necessary and sufficient condition for the Pontryagin and weak minima coincide. Therefore, the main results Theorems 2 and 3) formulated in terms of the weak minimum remain valid if we replace the weak minimum by the Pontryagin minimum. Moreover, if an optimal control is boundary: $u(t) = \omega$ for almost all $t \in [0, t_1]$, then the Pontryagin minimum is equivalent to the strong minimum (see [1]).

1.2. Maximum principle and second variation. Let us write the Pontryagin maximum principle in the form [1, 2] for the problem (4)–(6). Let $\hat{w}(t) = (\hat{z}(t), \hat{q}(t), \hat{u}(t))$, $t \in [0, t_1]$ be a trajectory yielding the Pontryagin (and, the more so, the weak) minimum. Then there exist Lipschitz functions $\psi_z(t)$, $\psi_q(t)$ of dimensions 1 and n, respectively, and numbers $\alpha_0 \ge 0$, β_0, and β_1 such that

(1) the tuple $\lambda = (\psi_z, \psi_q, \alpha_0, \beta_0, \beta_1)$ of Lagrange multipliers is nontrivial:

$$\max_t |\psi_z(t)| + \max_t |\psi_q(t)| + \alpha_0 + |\beta_0| + |\beta_1| \ne 0; \tag{7}$$

(2) the functions $\psi_z(t)$ and $\psi_q(t)$ are a solution of the adjoint system

$$\dot{\psi}_z = -\partial_z \hat{H}[\lambda] = -\psi_q(\hat{f}_0 + \hat{u}\hat{f}_1), \quad \dot{\psi}_q = -\partial_q \hat{H}[\lambda] = -\psi_q \hat{z}(\hat{f}_0' + \hat{u}\hat{f}_1'), \tag{8}$$

where $H[\lambda]$ is the Pontryagin function:

$$H[\lambda](z, q, u) = \psi_q z(f_0(q) + u f_1(q)); \tag{9}$$

(3) the following transversality conditions hold:

$$\begin{aligned} \psi_z(0) &= \partial_{z(0)}\hat{l}[\lambda] = \alpha_0, & \psi_z(t_1) &= -\partial_{z(t_1)}\hat{l}[\lambda] = 0, \\ \psi_q(0) &= \partial_{q(0)}\hat{l}[\lambda] = \beta_0, & \psi_q(t_1) &= -\partial_{q(t_1)}\hat{l}[\lambda] = -\beta_1, \end{aligned} \tag{10}$$

where $l[\lambda]$ is the endpoint Lagrange function:

$$l[\lambda](z(0), z(t_1), q(0), q(t_1)) = \alpha_0 z(0) + \beta_0 q(0) + \beta_1 q(t_1); \tag{11}$$

(4) the following maximum condition holds:

$$\max_{|u|\le\omega} H[\lambda](\hat{z}, \hat{q}, u) = H[\lambda](\hat{z}, \hat{q}, \hat{u}) = \psi_q \hat{z}(f_0(\hat{q}) + \hat{u} f_1(\hat{q})) = \text{const} \ge 0. \tag{12}$$

Without loss of generality, we set $\hat{z} = 1$. Then $t_1 = \hat{T}$ is the optimal time in the problem (1)–(3). It is easy to see from the equality of the Pontryagin function to a constant on the optimal trajectory that $\dot{\psi}_z = -\psi_q(f_0(\hat{q}) + \hat{u} f_1(\hat{q})) = -\,\text{const}$. Taking into account the transversality conditions, we find from this that $\psi_z = \alpha_0(\hat{T} - t)/\hat{T}$. Hence the maximum condition can be improved:

$$\max_{|u|\le\omega} H[\lambda](\hat{z}, \hat{q}, u) = H[\lambda](\hat{z}, \hat{q}, \hat{u}) = \psi_q(f_0(\hat{q}) + \hat{u} f_1(\hat{q})) = \frac{\alpha_0}{\hat{T}} \ge 0. \tag{13}$$

Note that the numbers α_0, β_0, and β_1 are uniquely found by the function $\psi_q(t)$ because of the latter relation and the transversality conditions. Therefore, instead of the complete tuple of the Lagrange multipliers λ, it suffices to only consider the function $\psi_q(t)$, which will be denoted by $\psi(t)$ for brevity in what follows (the subscripts will be used for denoting partial derivatives). Denote by Ψ the set of all function ψ corresponding to a given trajectory w. A pair (ψ, w) for which relations (7)–(12) of the maximum principle hold is called an extremal.

For every extremal, let us consider the Lagrange function

$$\Phi[\psi](w) = l[\psi](w(0), w(T)) + \int_0^T ((\psi, \dot{q}) - H[\psi](w))\, dt \tag{14}$$

and write its second variation (multiplied by $\frac{1}{2}$ for convenience) on the extremal $\hat{w}$:

$$\Omega[\psi](\bar{w}) = -\int_0^{\hat{T}} \left(\bar{z}\hat{H}_{zq}[\psi]\bar{q} + \frac{1}{2}\,(\hat{H}_{qq}[\psi]\bar{q}, \bar{q}) + \bar{z}\bar{u}\hat{H}_{zu}[\psi] + \bar{u}\hat{H}_{qu}[\psi]\bar{q} \right) dt, \tag{15}$$

where

$$\begin{aligned} \hat{H}_{zq}[\psi] &= \psi(\hat{f}_0' + \hat{u}\hat{f}_1'), & \hat{H}_{qq}[\psi] &= \psi(\hat{f}_0'' + \hat{u}\hat{f}_1''), \\ \hat{H}_{zu}[\psi] &= \psi\hat{f}_1, & \hat{H}_{qu}[\psi] &= \psi\hat{f}_1'. \end{aligned}$$

Here, the Legendre condition degenerates, $\hat{H}_{uu}[\psi] = 0$, because of the linearity of $H[\psi]$ in the control, and there are no terms outside the integral because the terminal Lagrange function l is linear. The cone of critical variations of the problem (4)–(6) without control constraint is given by the relations

$$\dot{\bar{z}} = 0, \quad \dot{\bar{q}} = \bar{z}\left(\hat{f}_0 + \hat{u}\hat{f}_1\right) + \left(\hat{f}_0' + \hat{u}\hat{f}_1'\right)\bar{q} + \bar{u}\hat{f}_1, \tag{16}$$

$$\bar{z} \le 0, \quad \bar{q}(0) = 0, \quad \bar{q}(T) = 0. \tag{17}$$

1.3. Singular trajectories.

Definition 1. A trajectory $\hat{w}$ of the problem (4)–(6) is said to be singular if on any extremal $(\psi, \hat{w})$, $\psi \in \Psi$ corresponding to it, the Pontryagin function (9) is independent of the control:

$$\hat{H}_u = H_u(\psi, \hat{w}) = 0.$$

Definition 2. A trajectory $\hat{w}$ of the problem (4)–(6) is said to be regular (normal) if $\alpha_0 > 0$.

Under the condition $\alpha_0 > 0$ the adjoint variable $\psi(t)$ is unique. In this case, for a scalar-valued control, the concept of the order of a singular trajectory is defined.

Definition 3. The number m is called the order of a singular trajectory $\hat{w}$ of the problem (4)–(6) if the following relations hold on the extremal $(\psi, \hat{w})$ corresponding to it:

$$\begin{aligned} &\frac{\partial}{\partial u}\left(\frac{d^k}{dt^k}\frac{\partial H}{\partial u}\right)\bigg|_{(\psi,\hat{w})} = 0, \qquad k = 1, \ldots, 2m-1; \\ &\frac{\partial}{\partial u}\left(\frac{d^{2m}}{dt^{2m}}\frac{\partial H}{\partial u}\right)\bigg|_{(\psi,\hat{w})} \neq 0. \end{aligned} \tag{18}$$

If such a number m does exist, then it is even, $m \in \mathbb{N}$ [8, 15]. Thus, the control u arises at an even step of differentiation. If relation (18) holds for all $k \in \mathbb{N}$, then we say that $\hat{w}$ is of infinite order.

Since $\hat{H}_u = \psi \hat{f}_1 = 0$ on the singular trajectory $\hat{w}$, the second variation (15) becomes

$$\Omega[\psi](\bar{w}) = -\int_0^T \left(\bar{z}\psi(\hat{f}_0' + \hat{f}_1')\bar{q} + \frac{1}{2}\,\psi((\hat{f}_0'' + \hat{f}_1'')\bar{q}, \bar{q}) + \bar{u}\psi(\hat{f}_1'\bar{q})\right) dt. \tag{19}$$

Assertion 1. *Let $\hat{w}$ be a regular singular trajectory. Then $\bar{z} = 0$.*

Proof. On the singular trajectory $\hat{w}(t)$, taking into account (8), (13), (16), and (17), we have

$$\begin{aligned} 0 &= (\psi(\hat{T}), \bar{q}(\hat{T})) - (\psi(0), \bar{q}(0)) = \int_0^{\hat{T}} \frac{d}{dt}(\psi, \bar{q})\, dt \\ &= \int_0^{\hat{T}} \Big\{-(\psi, (\hat{f}_0' + \hat{u}\hat{f}_1')\,\bar{q}) + (\psi, \bar{z}(\hat{f}_0 + \hat{u}\hat{f}_1) + (\hat{f}_0' + \hat{u}\hat{f}_1')\bar{q} + \bar{u}\hat{f}_1)\Big\}\, dt \\ &= \int_0^{\hat{T}} \Big\{\bar{z}(\psi, (\hat{f}_0 + \hat{u}\hat{f}_1)) + \bar{u}(\psi, \hat{f}_1))\Big\}\, dt = \int_0^{\hat{T}} \left\{\bar{z}\frac{\alpha_0}{\hat{T}} + \bar{u}(\psi, \hat{f}_1))\right\} dt = \bar{z}\alpha_0. \end{aligned}$$

Thus, $\bar{z} = 0$ on the regular singular trajectory $\hat{w}(t)$, and relations (16) and (17) define not a cone but a half-space Π of critical variations. ◻

To use the quadratic optimality conditions, we make the Goh transform $(\bar{z}, \bar{q}) \mapsto (\bar{z}, \bar{\xi}, \bar{u}, \bar{v})$, where

$$\bar{\xi} = \bar{q} - \bar{v}\hat{f}_1, \quad \dot{\bar{v}} = \bar{u}, \quad \bar{v}(0) = 0. \tag{20}$$

We accept the following definition of the commutator of vector fields in local coordinates: $[f(q), g(q)] = -(f'(q)g(q) - g'(q)f(q))$ (in some works, the expression for the commutator differs by the sign).

Assertion 2. *Let $\hat{w} = (\hat{z} = 1, \hat{q}, \hat{u} = 1)$ be a singular trajectory. Then in the Goh variables* (20), *the variation system* (16) *is written in the form*

$$\dot{\bar{z}} = 0, \quad \dot{\bar{\xi}} = \bar{z}\left(\hat{f}_0 + \hat{f}_1\right) + \left(\hat{f}_0' + \hat{f}_1'\right)\bar{\xi} - \bar{v}\widehat{[f_0, f_1]}, \tag{21}$$

and the second variation (19) *becomes*

$$\Omega[\psi](\bar w) = \frac{1}{2}\bar v^2\psi\hat f_1'\hat f_1\Big|_{t=T} + \int\limits_0^T \Big(-\bar z\psi(\hat f_0' + \hat f_1')\bar\xi$$
$$-\frac{1}{2}\psi((\hat f_0'' + \hat f_1'')\bar\xi,\bar\xi) + \bar v\psi\widehat{[f_0,f_1]}'\bar\xi + \frac{1}{2}\,\bar v^2\psi[f_1,\widehat{[f_0,f_1]}]\Big)\,dt. \quad (22)$$

Proof. The differentiation of the first of the relations (20) in accordance with the second and system (16), after collecting similar terms, yields (21).

To prove formula (22), we substitute $\bar q = \bar\xi + \bar v\hat f_1$ and $\bar u = \dot{\bar v}$ in expression (19) for the second variation in the initial variables. Collecting similar terms, we obtain (omotting ˆ for brevity)

$$\Omega[\psi](\bar w) = -\int\limits_0^T \Big(\bar z\psi(f_0' + f_1')\bar\xi + \bar z\bar v\psi(f_0' + f_1')f_1 + \frac{1}{2}\psi((f_0'' + f_1'')\bar\xi,\bar\xi)$$
$$+\bar v\psi((f_0'' + f_1'')\bar\xi, f_1) + \frac{1}{2}\bar v^2\psi((f_0'' + f_1'')f_1, f_1) + \dot{\bar v}\psi(f_1'\,\bar\xi) + \dot{\bar v}\bar v\psi(f_1'\,f_1)\Big)\,dt. \quad (23)$$

Exclude $\dot{\bar v}$ in the two last terms. Integrate by parts the next to the last term in (23):

$$-\int\limits_0^T \dot{\bar v}\psi(f_1'\,\bar\xi)\,dt = -\bar v\psi(f_1'\,\bar\xi)\Big|_{t=T} + \int\limits_0^T \Big\{ -\bar v\psi((f_0' + f_1')f_1', \bar\xi)$$
$$+\bar v\psi(f_1''(f_0 + f_1),\bar\xi) + \bar v\psi\left(\bar z f_1'(f_0 + f_1) + f_1'(f_0' + f_1')\bar\xi - \bar v f_1'[f_0, f_1]\right)\Big\}\,dt. \quad (24)$$

Now integrate by parts the last term of expression (23):

$$-\int\limits_0^T \dot{\bar v}\,\bar v\,\psi(f_1'\,f_1)\,dt = -\frac{1}{2}\int\limits_0^T (\bar v^2)^{\cdot}\,\psi(f_1'\,f_1)\,dt = -\,\frac{1}{2}\bar v^2\psi(f_1'\,f_1)\Big|_{t=T}$$
$$+\frac{1}{2}\int\limits_0^T \Big\{ -\bar v^2\psi(f_0' + f_1')f_1'\,f_1 + \bar v^2\psi(f_1''(f_0 + f_1), f_1) + \bar v^2\psi f_1' f_1'(f_0 + f_1)\Big\}\,dt. \quad (25)$$

Collect similar terms in (23)–(25). Collecting the terms with $\bar z\bar v$ in (23) and (24), we obtain

$$\int\limits_0^T \bar z\bar v\left\{-\psi(f_0' + f_1')f_1 + \psi f_1'(f_0 + f_1)\right\}\,dt = \int\limits_0^T \bar z\bar v\psi[f_0, f_1]\,dt = 0, \quad (26)$$

since $\psi[f_0, f_1] = \frac{d}{dt}(\psi f_1) = 0$ on a singular trajectory.

Analogously, collecting terms containing $\bar v\bar\xi$, we have

$$\bar v\int\limits_0^T \psi\Big(-(f_0''f_1',\bar\xi) + (f_1''f_0,\bar\xi\,) - f_0'f_1'\bar\xi - f_1'f_0'\bar\xi\Big)dt = \bar v\int\limits_0^T \psi[f_0, f_1]'\bar\xi\,dt. \quad (27)$$

From (23)–(25), we find the coefficient of $\bar v^2$:

$$-\,\frac{1}{2}\psi(f_1'\,f_1)\Big|_{t=T} + \frac{1}{2}\int\limits_0^T \psi\Big(-\,2f_1'[f_0, f_1] - (f_0''f_1', f_1') - f_0'f_1'f_1 + (f_1''f_0, f_1) + f_1'f_1'f_0\Big)\,dt. \quad (28)$$

It is easy to see that this expression coincides with

$$-\frac{1}{2}\psi(f_1' f_1)\Big|_{t=T} + \frac{1}{2}\int\limits_0^T \psi[f_1,[f_0,f_1]]\,dt. \tag{29}$$

The term outside the integral,

$$-\bar{v}\psi(f_1'\bar{\xi})\big|_{t=T} - \frac{1}{2}\,\bar{v}^2\psi\hat{f}_1'\hat{f}_1\Big|_{t=T},$$

is equal to

$$\frac{1}{2}\,\bar{v}^2\psi\hat{f}_1'\hat{f}_1\Big|_{t=T},$$

since $\bar{\xi}(T) = -f_1(q(T))\bar{v}(T)$. Thus we have proved (22). □

Corollary 1. *On a regular singular trajectory* $\hat{w} = (\hat{z} = 1, \hat{q}, \hat{u} = 1)$*, the variation system* (19) *and the second variation* (16) *can be written in the following form in the Goh variables*:

$$\dot{\bar{\xi}} = \left(\hat{f}_0' + \hat{f}_1'\right)\bar{\xi} - \bar{v}\widehat{[f_0,f_1]}, \tag{30}$$

$$\Omega(\bar{w}) = \frac{1}{2}\,\bar{v}^2\psi\hat{f}_1'\hat{f}_1\Big|_{t=T} + \int\limits_0^T \left(-\frac{1}{2}\,\psi((\hat{f}_0'' + \hat{f}_1'')\bar{\xi},\bar{\xi}) + \bar{v}\psi\widehat{[f_0,f_1]}'\bar{\xi} + \frac{1}{2}\,\bar{v}^2\psi\widehat{[f_1,[f_0,f_1]]}\right)dt. \tag{31}$$

Remark 1. Formula (22) can be obtained as a particular case of [2, (7.11)] for $k = 1$. For this purpose, in [2, (7.11)], it is necessary to replace r_0 by $r_0 + r_1$ (resp. f_0 and $f_0 + f_1$ in our notation), change the sign in the definition of commutator, and collect similar terms. We give here the deduction of formula (22), since for a scalar-valued control, there arises considerable simplifications in the proof as compared with the general case.

Let us formulate the following necessary and sufficient conditions for the weak minimum (and in the case of a scalar-valued control, for the Pontrygin minimum) of a regular singular trajectory $\hat{w}$ of the problem (4)–(6).

Theorem 1 (see [1])**.** (a) *Let a trajectory* $\hat{w}$ *yield a weak minimum. Then the Kelley condition holds*: $\psi\widehat{[f_1,[f_0,f_1]]} \ge 0$*, and, moreover,*

$$\Omega(\bar{w}) \ge 0 \qquad \forall \bar{w} \in \Pi.$$

(b) *On the trajectory* w*, let the strengthened Kelley condition hold*: $\psi\widehat{[f_1,[f_0,f_1]]} > 0$*, and let there exist a constant* $c > 0$ *such that*

$$\Omega(\bar{w}) > \bar{v}^2(\hat{T}) + c\int\limits_0^{\hat{T}} \bar{v}^2\,dt \quad \forall \bar{w} \in \Pi. \tag{32}$$

Then $\hat{w}$ *yields the weak minimum.*

Note that if the coefficient of $\bar{v}^2$ in (22) is strictly positive, $\psi\widehat{[f_1,[f_0,f_1]]} > 0$, then the quadratic form $\Omega(\bar{w})$ is Legendre. The absence of conjugate points on the closed interval, as in the problem of the classical calculus of variations, is a sufficient condition for positivity (nonnegativity) of a Legendre quadratic form.

2. Car with n Trailers and Goursat Distributions

2.1. Kinematic model. We use the same coordinate notation as in [6, 7, 16]. Let (x, y) be the Euclidean coordinate of the *last* trailer, θ_0 be the angle of direction of motion of the last trailer (with the abscissa axis), θ_i be the angle of the direction of motion of the $(n-i)$th trailer, and let θ_n be the angle of direction of motion of the car. Let the distances between the trailers be equal and be normalized by 1. Then denoting $q = (x, y, \theta_1, \ldots, \theta_n) \in \mathbb{R}^2 \times (\mathcal{S}^1)^{n+1}$, we can write the equations of motion on the form

$$\dot{q} = v f_0(q) + u f_1(q), \tag{33}$$

$$q(0) = a, \quad q(T) = b, \tag{34}$$

where v and u are linear and angular velocities of the car and

$$f_0(q) = \Big(v_0 \cos\theta_0,\ v_0 \sin\theta_0,\ v_1 \sin(\theta_1 - \theta_0),\ \ldots,\ v_i \sin(\theta_i - \theta_{i-1}),\ \ldots,\ v_n \sin(\theta_n - \theta_{n-1}),\ 0\Big),$$

$$v_i = v \prod_{j=i+1}^{n} \cos(\theta_j - \theta_{j-1}), \quad i = 0, \ldots, (n-1), \quad v_n = v, \tag{35}$$

$$f_1(q) = (0, \ldots, 0, 1).$$

Assertion 3 (see [7, 9])**.** *The system "car with n trailers" is completely controllable, i.e., any pair of points* (34) *can be connected by a trajectory of system* (33) *for a certain $T \geq 0$ and controls $u(t)$ and $v(t)$ from $C^1[0, T]$.*

2.2. Singular trajectories of time-optimal control problem. One of the most natural functional for kinematic models of mechanical systems is the length of the passed path. In the natural parametrization: $v \equiv 1$, the length of the path passed by the car is equal to the time of motion. Therefore, we arrive at the time-optimal control problem under the constraint imposed on the angular velocity in the form (4)–(6).

By definition 2, on the first-order singular trajectory of the problem (4)–(6), together with $\psi \hat{f}_1 = 0$, the following chain of relations holds:

$$\left.\frac{d(\psi f_1)}{dt}\right|_{w=\hat{w}} = \psi \widehat{[f_0, f_1]} = 0, \tag{36}$$

$$\left.\frac{d^2(\psi f_1)}{dt^2}\right|_{w=\hat{w}} = \psi \widehat{[f_0, [f_0, f_1]]} + \hat{u}\, \psi \widehat{[f_1, [f_0, f_1]]} = 0. \tag{37}$$

In the system describing the motion of a car with n trailer, the latter commutator can be explicitly calculated.

Lemma 1. *Let vector fields f_0 and f_1 be defined by formulas* (35)*. Then $[f_1, [f_0, f_1]] = f_0$.*

Proof. Starting from the explicit form (35) of the vector fields f_0 and f_1, we directly obtain

$$[f_1, [f_0, f_1]] = [f_0, f_1]' f_1 = -(f_0'' f_1, f_1) = -\frac{\partial^2 f_0}{\partial \theta_n^2} = f_0.$$

The lemma is proved. $\square$

On the other hand, it follows from $\psi \hat{f}_1 = 0$ that on a singular trajectory, the Hamiltonian (13) takes the form

$$\hat{H} = \psi \hat{f}_0 = \frac{\alpha_0}{\hat{T}} \geq 0. \tag{38}$$

Therefore, the following assertion holds.

Lemma 2. *A regular singular trajectory $\hat{w}$ of the problem* (4)–(6), (35) *is of order $m = 1$. Moreover, on $\hat{w}$, the strict Kelley condition holds*:

$$\frac{\partial}{\partial u}\left(\frac{d^2}{dt^2}\frac{\partial H}{\partial u}\right)\Bigg|_{w=\hat{w}} > 0. \tag{39}$$

We have from the explicit formula (35) for the vector field f_1 that $f_1' = 0$. Therefore, the variation system (16) and the adjoint system (8) are written in the form

$$\dot{\bar{q}} = \hat{f}_0'\bar{q} + \bar{u}\hat{f}_1, \quad \dot{\psi} = -\psi\hat{f}_0', \tag{40}$$

and the second variation (19) takes the following form on the first-order singular trajectory $\hat{w}$:

$$\Omega(\bar{w}) = -\int_0^{\hat{T}} \frac{1}{2}\psi(\hat{f}_0''\bar{q},\bar{q})\,dt. \tag{41}$$

Remark 2. Here, passage to the Goh variables (20) reduces to the last component $\bar{q}_n$ being taken as a new control. Indeed, the critical variations satisfy the system (30) with zero boundary conditions on the left endpoint:

$$\dot{\bar{\xi}} = \hat{f}_0'\bar{\xi} + \bar{v}\hat{f}_0'\hat{f}_1, \quad \bar{\xi}(0) = 0, \quad \bar{v}(0) = 0. \tag{42}$$

It is easy to see that $\dot{\bar{\xi}}_n = 0$. Hence $\bar{q}_n = \bar{v}$ and $\bar{q}_i = \bar{\xi}_i$, $i = 1,\ldots,n-1$. In particular, this implies $\bar{v}(\hat{T}) = 0$. Therefore, the comparison functional (32) contains no terms outside the integral, so that the following estimate is a sufficient weak minimum condition:

$$\Omega(\bar{w}) \geq c\int_0^{\hat{T}} \bar{v}^2(t)\,dt, \quad c = \text{const} > 0.$$

Taking into account Assertion 1, for the second variation, we have

$$\Omega[\psi](\bar{w}) = \int_0^{T}\left(-\frac{1}{2}\psi(\hat{f}_0''\bar{\xi},\bar{\xi}) - \bar{v}\psi(\hat{f}_0''\hat{f}_1,\bar{\xi}) + \frac{1}{2}\bar{v}^2\psi\hat{f}_0\right)dt. \tag{43}$$

Since the strict Kelley condition at the point $\tilde{t}$ (generalization of the Legendre condition of the classical calculus of variations) is sufficient for the positive definiteness of the quadratic form (43) on a small interval $(\tilde{t}-\varepsilon,\tilde{t}+\varepsilon)$ (see [1]), the previous arguments are summarized in the following result.

Theorem 2. *Small parts of any first-order singular trajectory yield the weak minimum in the problem* (4)–(6), (35).

2.3. Goursat distributions. Following [12, 13], we give necessary definitions. Let M be a smooth manifold, and let D be a distribution on M (a subbundle of constant rank in the tangent bundle TM).

Definition 4. A distribution D of corank $d \geq 2$ on the manifold M of dimension $m \geq 4$ is called a Goursat distribution if $D^{i+1} = D^i + [D^i, D^i]$, $D^1 = D$, is a distribution for every $i = 1,\ldots,d$, and the condition $\dim D^{i+1} = \dim D^i + 1$ holds.

The following two definitions refer to an arbitrary distribution, not necessary a Goursat distribution.

Definition 5. The big growth vector of a distribution D at a point $q \in M$ is the tuple

$$(\dim D^1(q), \dim D^2(q), \dim D^3(q), \ldots).$$

It is easy to see that Definition 4 is equivalent to the following: the big growth vector has a constant dimension at each point and is equal to $(r, r+1, r+2\ldots, m)$, where $r = m - d$.

Definition 6. The small growth vector of a distribution D at a point $q \in M$ is the tuple

$$(\dim D^1(q), \dim D^2(q), \dim D^3(q), \ldots),$$

where $D_{i+1} = D_i + [D, D_i]$ and $D_1 = D$.

Note that the big and small growth vectors are invariantly defined (with accuracy up to an arbitrary diffeomorphism of a neighborhood).

Definition 7. The dimension β of the small growth vector is called the nonholonomy degree of the distribution.

The inequality $\dim D^i(q) \geq \dim D_i(q)$ holds at an arbitrary point $q \in M$.

Definition 8. The points of a Goursat distribution at which the big and small growth vectors coincide are said to be regular.

Definition 9. The points of a Goursat distribution at which the small growth vector is not equal to $(r, r+1, r+2 \ldots, m)$, i.e., the nonholonomy degree exceeds $d+1$, are said to be singular. The set of all singular points is called the singularity domain.

The distribution generated by the pair of vector fields (35) is a Goursat distribution of corank $d = n+1$ and models all normal forms of an arbitrary Goursat distribution of corank $n+1$ (see [12]). The singularity domain of the problem on a car with n trailers consists of those configurations of the system in which a certain pair of trailers (except for the last) is at a right angle with respect to one another. More precisely, the nonholonomy degree at a singular point is finite and satisfies the inequality $n+2 < \beta \leq F_{n+3}$, where F_i is the ith Fibonacci number (see [7, 10, 16]).

Note that for $n \leq 5$, the small growth vector is a complete invariant of the distribution of the problem of a car with n trailers, and, therefore, it is that of an arbitrary Goursat distribution of corank $n+1$: the number of possible local normal forms of the distribution coincides with the number of all possible types of the small growth vector and is equal to F_{2n-1}. For $n > 5$, the number of normal forms of a 2-Goursat distribution becomes infinite (a normal form depends on an arbitrary function), and the number of possible types of small growth vector remains equal to F_{2n-1} (see [7, 10, 16]).

2.4. Optimality of singular curves. Let $\Gamma_k = \{q \in \mathbb{R}^2 \times (\mathcal{S}^1)^{n+1} \mid \cos(\theta_k - \theta_{k-1}) = 0\}$, $k = 1, \ldots, n$. The union Γ_k over all $k > 1$ is the singularity domain of the distribution (35).

Theorem 3. *Let $\hat{w}(t)$ be a trajectory of the system* (4), (6), (35), *and let $\gamma = \{\hat{q}(t) \mid t \in [0, \hat{T}]\} \subset \Gamma_k$, $k = 1, \ldots, n$. Let $\omega > 1$. Then the curve γ is a singular trajectory and yields a weak minimum in the problem* (4)–(6), (35).

Proof. (I) Let us explain the idea of the proof in the case $\gamma \subset \Gamma_n$. For definiteness, let $\hat{\theta}_n(t) - \hat{\theta}_{n-1}(t) = \pi/2$, $t \in [0, \hat{T}]$. Then it follows from the explicit form of system (48) that $\hat{x} = \text{const}$, $\hat{y} = \text{const}$, and $\hat{\theta}_i = \text{const}$, $i = 0, \ldots, n-2$. Moreover, $\dot{\theta}_{n-1} = 1$. This implies $\dot{\theta}_{n-1} = \hat{u} = 1$. If $\omega > 1$, then the control is inner. Hence the trajectory $\hat{w}$ is singular.

The singularity of the trajectory $\hat{w}(t)$ in this case means that the last component of the vector of adjoint variables is equal to zero: $\psi \hat{f}_1 = \psi_{n+3} = 0$. With account for (13) this implies that the next to the last component of the vector ψ is constant: $\hat{H} = \psi \hat{f}_0 = \psi_{n+2} = \text{const}$. By the relation $\cos(\hat{\theta}_n - \hat{\theta}_{n-1}) \equiv 0$, in the adjoint system (40), we have the relations $\dot{\psi}_i = 0$, $i = 1, \ldots, n+1$. Therefore, $\psi = (0, \ldots, 0, 1, 0) \in \hat{\Psi}$ is a (unique up to a proportionality, since $\alpha_0 > 0$) solution of the adjoint system (40) on the trajectory $\hat{w}$.

Let us consider the second variation on the extremal $(\psi, \hat{w})$. The first summand in formula (43) becomes

$$-\int_0^{\hat{T}} \frac{1}{2} \psi(\hat{f}_0'' \bar{\xi}, \bar{\xi})\, dt = -\int_0^{\hat{T}} \frac{1}{2} ((\widehat{f_0^{n-1}})'' \bar{\xi}, \bar{\xi})\, dt = \int_0^{\hat{T}} \frac{1}{2} (\bar{\theta}_n - \bar{\theta}_{n-1})^2\, dt.$$

Here, $(\widehat{f_0^{n-1}})''$ is the Hessian of the $(n-1)$th component $f_0^{n-1} = \sin(\theta_n - \theta_{n-1})$ of the vector field f_0 evaluated on the trajectory $\hat{w}$. The second summand in formula (43) on the extremal $(\psi, \hat{w})$ is equal to

$$-\int_0^{\hat{T}} \bar{v}\psi(\hat{f}_0''\hat{f}_1, \bar{\xi})\,dt = \int_0^{\hat{T}} \bar{v}(\bar{\theta}_n - \bar{\theta}_{n-1})\,dt.$$

It is directly verified that $\bar{\theta}_{n-1} = \bar{\theta}_n = 0$ in the system (42) on the trajectory $\hat{w}$. Hence the first two summands in the integrand of (43) vanish on the extremal $(\psi, \hat{w})$. Therefore,

$$\Omega(\bar{w}) = \frac{1}{2}\int_0^{\hat{T}} \bar{v}^2\,dt, \tag{44}$$

and the sufficient weak minimum condition for a singular trajectory holds on $\hat{w}$.

(II) Now let us consider $\gamma \subset \Gamma_k$ in the case of an arbitrary $k = 1, \ldots, n$. Introduce the following notation: $c_i = \cos(\theta_i - \theta_{i-1})$ and $s_i = \sin(\theta_i - \theta_{i-1})$. Then the equations for the variables θ_i, $i = 0, \ldots, n-1$, can be rewritten in the form (here, for brevity, we write f^i instead of f_0^i)

$$\begin{cases} \dot{\theta}_0 = f^3 = c_n c_{n-1}\cdots c_{i+1}c_i c_{i-1}\cdots c_2 s_1, \\ \dot{\theta}_1 = f^4 = c_n c_{n-1}\cdots c_{i+1}c_i c_{i-1}\cdots s_2, \\ \cdots \\ \dot{\theta}_{i-2} = f^{i+1} = c_n c_{n-1}\cdots c_{i+1}c_i s_{i-1}, \\ \dot{\theta}_{i-1} = f^{i+2} = c_n c_{n-1}\cdots c_{i+1}s_i, \\ \dot{\theta}_i = f^{i+3} = c_n c_{n-1}\cdots s_{i+1}, \\ \cdots \\ \dot{\theta}_{n-2} = f^{n+1} = c_n s_{n-1}, \\ \dot{\theta}_{n-1} = f^{n+2} = s_n. \end{cases} \tag{45}$$

Lemma 3. *The optimal control on the trajectory $\gamma \subset \Gamma_k$, $k = 1, \ldots, n$, is constant and equal to $\hat{u} = \pm(n-k+1)^{-1/2}$.*

Proof. Let $\dot{\hat{\theta}}_{k-1} = \cdots = \dot{\hat{\theta}}_n = s_{n-k+1}$. Then the following chain of recursive relations holds: $s_i = c_i s_{i-1}$, $i = k+1, \ldots, n$. After elementary transformations, we obtain $s_i^{-2} = s_{i-1}^{-2} + 1$, $s_1^2 = 1$. From this, we find that $s_i^2 = i^{-1}$. Hence $s_i = \pm 1/\sqrt{i}$ and $c_i = \pm\sqrt{i-1}/\sqrt{i}$. Then $\hat{u} = \dot{\hat{\theta}}_n = \pm(n-k+1)^{-1/2}$. □

Corollary 2. *Let $\gamma \subset \Gamma_k$, $k = 1, \ldots, n$, be an optimal trajectory. Then γ is a singular trajectory.*

Proof. It follows from Lemma 3 that the optimal control $\hat{u} = (n-k+1)^{-1/2}$ on the trajectory γ is inner: $\hat{u} \in (0, \omega)$. Hence γ is a singular trajectory. □

Lemma 4. *Let $\gamma \subset \Gamma_k$, $k = 1, \ldots, n$, be an optimal trajectory. Then a regular extremal ($\alpha_0 > 0$) corresponds to γ, and a (unique) solution of the adjoint system has the form $\psi = (0, \ldots, 0, 1, \ldots, 1, 0)$, where the unity start from the $(k+2)$th coordinate.*

Proof. It follows from the singularity condition $\psi\hat{f}_1 = \psi_n = 0$ that

$$\hat{H} = \psi\hat{f}_0 = \psi_0 + \cdots + \psi_{n-1} = \text{const} > 0.$$

Hence $\alpha_0 = \hat{T} > 0$.

In order not to complicate the calculations by additional indices, we consider the case $k = 1$. It is directly verified that the following relations hold on $\gamma \subset \Gamma_1$:

$$\begin{aligned}
\frac{\partial f^{i+3}}{\partial \theta_i} &= -c_n \cdots c_{i+3} c_{i+2} c_{i+1} = -\frac{\sqrt{n}}{n}\sqrt{i}, \qquad i = 0, \dots, n-1, \\
\frac{\partial f^{i+3}}{\partial \theta_{i+1}} &= c_n \cdots c_{i+3} s_{i+2} s_{i+1} + c_n \cdots c_{i+3} c_{i+2} c_{i+1} \\
&= \frac{\sqrt{n}}{n}\left(\sqrt{i} + \frac{\sqrt{i+1}}{i+1}\right), \qquad i = 0, \dots, n-2, \\
\frac{\partial f^{i+3}}{\partial \theta_j} &= -c_n \cdots c_{j+2} c_{j+1} s_j c_{j-1} \cdots c_{i+2} s_{i+1} + c_n \cdots c_{j+2} s_{j+1} c_j c_{j-1} \cdots c_{i+2} s_{i+1} \\
&= \frac{\sqrt{n}}{n}\left(-\frac{\sqrt{j-1}}{j-1} + \frac{\sqrt{j}}{j}\right), \qquad i = 0, \dots, n-1, \quad j = i+2, \dots, n-1,
\end{aligned}$$

and also

$$\frac{\partial f^{n+2}}{\partial \theta_n} = \frac{\sqrt{n}}{n}\sqrt{n-1}, \qquad \frac{\partial f^{i+3}}{\partial \theta_n} = -\frac{\sqrt{n}}{n}\frac{\sqrt{n-1}}{n-1}.$$

Therefore, the matrix $\dfrac{\sqrt{n}}{n} f_0'$ has the form

$$\begin{bmatrix}
0 & 0 & \cos\theta_0 & -\cos\theta_0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \sin\theta_0 & -\sin\theta_0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 + \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} & \dots & -\frac{\sqrt{n-3}}{n-3} + \frac{\sqrt{n-2}}{n-2} & -\frac{\sqrt{n-2}}{n-2} + \frac{\sqrt{n-1}}{n-1} & -\frac{\sqrt{n-1}}{n-1} \\
0 & 0 & 0 & -1 & 1 + \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} & \dots & -\frac{\sqrt{n-3}}{n-3} + \frac{\sqrt{n-2}}{n-2} & -\frac{\sqrt{n-2}}{n-2} + \frac{\sqrt{n-1}}{n-1} & -\frac{\sqrt{n-1}}{n-1} \\
0 & 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} + \frac{\sqrt{3}}{3} & \dots & -\frac{\sqrt{n-3}}{n-3} + \frac{\sqrt{n-2}}{n-2} & -\frac{\sqrt{n-2}}{n-2} + \frac{\sqrt{n-1}}{n-1} & -\frac{\sqrt{n-1}}{n-1} \\
0 & 0 & 0 & 0 & 0 & -\sqrt{3} & \dots & -\frac{\sqrt{n-3}}{n-3} + \frac{\sqrt{n-2}}{n-2} & -\frac{\sqrt{n-2}}{n-2} + \frac{\sqrt{n-1}}{n-1} & -\frac{\sqrt{n-1}}{n-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & -\sqrt{n-2} & \sqrt{n-2} + \frac{\sqrt{n-1}}{n-1} & -\frac{\sqrt{n-1}}{n-1} \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & -\sqrt{n-1} & \sqrt{n-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

It follows from the obtained formulas that the function $\psi(t) = (0, 0, 1, \dots, 1, 0) \in \hat{\Psi}$ is a solution of the adjoint system (40) on the trajectory $\hat{w}$. The case $\gamma \subset \Gamma_k$ for an arbitrary $k \leq n$ is verified analogously. $\square$

Corollary 3. *The trajectory γ is a regular first-order singular trajectory.*

Lemma 5. *The second variation $\Omega(\bar{w})$ on the extremal $(\psi, \hat{w})$ is a nonnegative quadratic form for any $\hat{T}$.*

Proof. As in Lemma 4, we consider the case $k = 1$: $\cos(\hat{\theta}_1 - \hat{\theta}_0) = 0$ on the trajectory $\hat{w}$ (the case of an arbitrary $k = 2, \dots, n$ is analogous). For definiteness, let $\hat{\theta}_1(t) - \hat{\theta}_0(t) = \pi/2$, $t \in [0, \hat{T}]$. It is easy to see that

$$\psi(f_0''\bar{\xi}, \bar{\xi}) = \Big(((f^3)'' + \dots + (f^{n+2})'')\bar{\xi}, \bar{\xi}\Big).$$

Denote $S = n^{-1/2}((f^3)'' + \dots + (f^{n+2})'')$ and $\Sigma = f^3 + \dots + f^{n+2}$. $\square$

Lemma 6. *The matrix $\hat{S}$ is tridiagonal and has the form*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -5 & 3 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -7 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -(2n-3) & (n-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & (n-1) & -(2n-1) & n \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & n & -n \end{bmatrix}. \tag{46}$$

Proof. Let us show that the following relations hold on $\gamma \subset \Gamma_1$:

$$\frac{\partial^2 \Sigma}{\partial \theta_i^2} = -\frac{2i+1}{\sqrt{n}}, \quad i = 0, \dots, n,$$
$$\frac{\partial^2 \Sigma}{\partial \theta_i \partial \theta_{i-1}} = \frac{1}{\sqrt{n}}, \quad i = 1, \dots, n,$$
$$\frac{\partial^2 \Sigma}{\partial \theta_i \partial \theta_j} = 0, \quad i = 0, \dots, n, \quad |i-j| > 1.$$

(1) Note that $f^i = n^{-1/2}$, $i = 3, \dots, n+2$, on Γ_1. It is directly verified that

$$\frac{\partial^2 \Sigma}{\partial \theta_0^2} = \frac{\partial^2 f^3}{\partial \theta_0^2} = \left(\prod_{j=2}^{n} c_j\right)(-s_1) = -f^3 = -\frac{1}{\sqrt{n}},$$

$$\frac{\partial^2 \Sigma}{\partial \theta_1^2} = \frac{\partial^2 (f^3 + f^4)}{\partial \theta_1^2} = \left(\prod_{j=3}^{n} c_j\right)\left(\frac{\partial^2 c_2}{\partial \theta_0^2} s_1 + 2\frac{\partial c_2}{\partial \theta_0}\frac{\partial s_1}{\partial \theta_0} + c_2 \frac{\partial^2 s_1}{\partial \theta_0^2}\right) + \left(\prod_{j=3}^{n} c_j\right)(-s_2)$$
$$= \left(\prod_{j=3}^{n} c_j\right)(-c_2 s_1 + 2 s_2 c_1 - c_2 s_1) + \left(\prod_{j=3}^{n} c_j\right)(-s_2) = -2f^3 - f^4 = -\frac{3}{\sqrt{n}}.$$

For $i = 2, \dots, n-2$, we have

$$\frac{\partial^2 \Sigma}{\partial \theta_i^2} = \frac{\partial^2}{\partial \theta_i^2}(f^3 + \dots + f^{i+3}).$$

Here,

$$\frac{\partial^2 f^{l+2}}{\partial \theta_i^2} = -2f^{l+2} - \left(\prod_{j=i+2}^{n} c_j\right) s_{i+1} s_i \left(\prod_{j=l+1}^{i-1} c_j\right) s_l, \quad l = 1, \dots, i-1,$$
$$\frac{\partial^2 f^{i+2}}{\partial \theta_i^2} = -2f^{i+2} + \left(\prod_{j=i+2}^{n} c_j\right) s_{i+1} c_i, \quad \frac{\partial^2 f^{i+3}}{\partial \theta_i^2} = -f^{i+3}.$$

Summing up, we find that

$$\frac{\partial^2\Sigma}{\partial\theta_i^2} = -2\sum_{j=3}^{i+2} f^j - f^{i+3} - \left(\prod_{j=i+2}^{n} c_j\right) s_{i+1}\Big(s_i c_{i-1}\dots c_2 s_1 + \cdots + s_i c_{i-1} c_{i-2} s_{i-3} + s_i c_{i-1} s_{i-2} + s_i s_{i-1} - c_i\Big) = -\frac{2n+1}{\sqrt{n}} - \left(\frac{i-1}{\sqrt{i}\sqrt{i-1}} - \frac{\sqrt{i-1}}{\sqrt{i}}\right) = -\frac{2n+1}{\sqrt{n}}.$$

Analogously, we obtain

$$\frac{\partial^2\Sigma}{\partial\theta_{n-1}^2} = -\frac{2n-1}{\sqrt{n}}, \quad \frac{\partial^2\Sigma}{\partial\theta_n^2} = -\frac{n}{\sqrt{n}}.$$

(2) For the mixed derivatives, we find that $\partial\theta_i\partial\theta_{i-1}$:

$$\frac{\partial^2\Sigma}{\partial\theta_0\partial\theta_1} = \frac{\partial^2 f^3}{\partial\theta_0\partial\theta_1} = -\left(\prod_{j=3}^{n} c_j\right)(s_2 c_1 - c_2 s_1) = f^3 = \frac{1}{\sqrt{n}},$$

$$\frac{\partial^2\Sigma}{\partial\theta_1\partial\theta_2} = \frac{\partial^2(f^3+f^4)}{\partial\theta_1\partial\theta_2} = f^3 + \left(\prod_{j=4}^{n} c_j\right)(s_3 c_2 c_1 - c_3 s_2 c_1 + s_3 s_2 s_1) + f^4 + \left(\prod_{j=4}^{n} c_j\right)(-s_3 c_2) = f^3 + f^4 = \frac{2}{\sqrt{n}}.$$

For $i = 3, \dots, n-1$, we have

$$\frac{\partial^2\Sigma}{\partial\theta_i\partial\theta_{i-1}} = \frac{\partial^2}{\partial\theta_i\partial\theta_{i-1}}(f^3 + \cdots + f^{i+2}).$$

Here, for $l = 1, \dots, i-2$,

$$\frac{\partial^2 f^{l+2}}{\partial\theta_i\partial\theta_{i-1}} = \left(\prod_{j=i+2}^{n} c_j\right)(s_{i+1}s_i c_{i-1} - s_{i+1}c_i s_{i-1} + c_{i+1}c_i c_{i-1} - c_{i+1}s_i s_{i-1})\left(\prod_{j=l+1}^{i-2} c_j\right) s_l$$
$$= f^{l+2} + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{i}\sqrt{i-1}} - \frac{1}{\sqrt{i-1}\sqrt{i-2}} - \frac{1}{\sqrt{i}\sqrt{i-2}}\right),$$

$$\frac{\partial^2 f^{i+1}}{\partial\theta_i\partial\theta_{i-1}} = \left(\prod_{j=i+2}^{n} c_j\right)(s_{i+1}s_i s_{i-1} + s_{i+1}c_i c_{i-1} + c_{i+1}c_i s_{i-1} + c_{i+1}s_i c_{i-1})$$
$$= f^{i+1} + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{i}\sqrt{i-1}} + \frac{\sqrt{i-2}}{\sqrt{i}} + \frac{\sqrt{i-2}}{\sqrt{i-1}}\right),$$

$$\frac{\partial^2 f^{i+2}}{\partial\theta_i\partial\theta_{i-1}} = \left(\prod_{j=i+2}^{n} c_j\right)(-s_{i+1}c_i + c_{i+1}s_i) = f^{i+2} - \frac{1}{\sqrt{n}}\frac{\sqrt{i-1}}{\sqrt{i}}.$$

Summing up, we find the required expression. The cases $i = 2$ and $i = n$ are verified analogously

(3) For the mixed derivatives $\partial\theta_i\partial\theta_j$, $i > j+1$, for $i = 3, \dots, n-1$, we find that

$$\frac{\partial^2\Sigma}{\partial\theta_i\partial\theta_j} = \frac{\partial^2}{\partial\theta_i\partial\theta_j}(f^3 + \cdots + f^{j+3}).$$

Here, for $l = 1, \dots, j-1$,

$$\frac{\partial^2 f^{l+2}}{\partial\theta_i \partial\theta_j} = \left(\prod_{p=i+2}^{n} c_p\right) (s_{i+1}c_i - c_{i+1}s_i) \left(\prod_{p=j+2}^{i-1} c_p\right) (s_{j+1}c_j - c_{j+1}s_j) \left(\prod_{p=l+1}^{l+1} c_p\right) s_l$$

$$= \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i-1}}\right) \left(\frac{1}{\sqrt{j}} - \frac{1}{\sqrt{j-1}}\right),$$

$$\frac{\partial^2 f^{j+2}}{\partial\theta_i \partial\theta_j} = \left(\prod_{p=i+2}^{n} c_p\right) (s_{i+1}c_i - c_{i+1}s_i) \left(\prod_{p=j+2}^{i-1} c_p\right) (s_{j+1}s_j + c_{j+1}c_j)$$

$$= \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i-1}}\right) \left(\frac{1}{\sqrt{j}} - \sqrt{j-1}\right),$$

$$\frac{\partial^2 f^{j+3}}{\partial\theta_i \partial\theta_j} = \left(\prod_{p=i+2}^{n} c_p\right) (s_{i+1}c_i - c_{i+1}s_i) \left(\prod_{p=j+2}^{i-1} c_p\right) (-c_{j+1})$$

$$= \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i-1}}\right) \sqrt{j}.$$

Summing up, we obtain the required expression. The case $i = n$ is verified analogously. □

Corollary 4. *The second variation on the extremal $(\psi, \hat{w})$ has the form*

$$\Omega(\bar{w}) = \frac{1}{2}(n-k+1)^{-1/2} \int_0^{\hat{T}} \sum_{j=k+3}^{n+3} (j-k-2)\,(\bar{q}_j - \bar{q}_{j-1})^2\, dt. \tag{47}$$

Proof. Note that $S = S_1 + \dots + S_{n-k+1}$, where

$$S_i = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & i & -i & & \\ & & -i & i & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

for $i = 1, \dots, n-k+1$. Therefore, $(S_i\bar{q}, \bar{q}) = i(\bar{q}_{i+k+2} - \bar{q}_{i+k+1})^2$. □

Corollary 5. *On the extremal $\gamma \subset \Gamma_k$, $k = 1, \dots, n$, the second variation $\Omega(\bar{w})$ is a positive quadratic form for any $\hat{T}$.*

Proof. Assume that the quadratic form (47) vanishes on a vector $(\bar{x}, \bar{y}, \bar{\theta}_0 \dots, \bar{\theta}_n)$. Then $\bar{\theta}_k - \bar{\theta}_{k-1} = 0$, …, $\bar{\theta}_n - \bar{\theta}_{n-1} = 0$. With account for the boundary conditions, this implies $\bar{\theta}_k = 0$, …, $\bar{\theta}_n = 0$. The variation equation for other variables on $\gamma \subset \Gamma_k$ have the following form:

$$\begin{cases} \dot{\bar{x}} = c_{k-1} \dots c_1 c_0 (\bar{\theta}_k - \bar{\theta}_{k-1}), \\ \dot{\bar{y}} = c_{k-1} \dots c_1 s_0 (\bar{\theta}_k - \bar{\theta}_{k-1}), \\ \dot{\bar{\theta}}_0 = c_{k-1} \dots c_2 s_1 (\bar{\theta}_k - \bar{\theta}_{k-1}), \\ \quad \dots \\ \dot{\bar{\theta}}_i = c_{k-1} \dots c_i s_{i+1} (\bar{\theta}_k - \bar{\theta}_{k-1}), \\ \quad \dots \\ \dot{\bar{\theta}}_{k-2} = s_{k-1} (\bar{\theta}_k - \bar{\theta}_{k-1}), \end{cases}$$

This implies $\dot{\bar{q}} = 0$. Taking into account the boundary conditions $\bar{q}(0) = \bar{q}(\hat{T}) = 0$, we obtain $\bar{q}(t) \equiv 0$. $\square$

Lemma 7. *The following estimate holds on the extremal* $(\psi, \hat{w})$:

$$\Omega(\bar{w}) \geq \frac{\sqrt{n}}{2} \int_0^{\hat{T}} \bar{v}^2(\tau)\, d\tau.$$

Proof. As in Lemmas 4 and 5, in order not to complicate the calculations, we consider the case $\gamma \subset \Gamma_1$. Note that

$$0 = \frac{1}{2} \int_0^{\hat{T}} (\bar{\theta}_{n-1}^2)^{\cdot}\, dt = \int_0^{\hat{T}} \dot{\bar{\theta}}_{n-1} \bar{\theta}_{n-1}\, dt = \frac{1}{\sqrt{n}} \int_0^{\hat{T}} (\bar{\theta}_{n-1} - \bar{v})\bar{\theta}_{n-1}\, dt.$$

Hence the expression for the second variation can be written in the form

$$\begin{aligned} 2\sqrt{n}\,\Omega(\bar{w}) &= \int_0^{\hat{T}} \left\{(\bar{\theta}_0 - \bar{\theta}_1)^2 + 2(\bar{\theta}_1 - \bar{\theta}_2)^2 + \cdots + (n-1)(\bar{\theta}_{n-2} - \bar{\theta}_{n-1})^2 + n\bar{\theta}_{n-1}^2 - 2n\bar{\theta}_{n-1}\bar{v} + n\bar{v}^2\right\} dt \\ &= \int_0^{\hat{T}} \left\{(\bar{\theta}_0 - \bar{\theta}_1)^2 + 2(\bar{\theta}_1 - \bar{\theta}_2)^2 + \cdots + (n-1)(\bar{\theta}_{n-2} - \bar{\theta}_{n-1})^2 - n\bar{\theta}_{n-1}^2 + n\bar{v}^2\right\} dt. \end{aligned}$$

Consider the following change of variables:

$$\begin{bmatrix} \bar{s} \\ \bar{d}_1 \\ \bar{d}_2 \\ \vdots \\ \bar{d}_{n-3} \\ \bar{d}_{n-2} \\ \bar{d}_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{\theta}_0 \\ \bar{\theta}_1 \\ \bar{\theta}_2 \\ \vdots \\ \bar{\theta}_{n-3} \\ \bar{\theta}_{n-2} \\ \bar{\theta}_{n-1} \end{bmatrix}.$$

The inverse change is given by the matrix

$$\frac{1}{n} \begin{bmatrix} 1 & (n-1) & (n-2) & (n-3) & \ldots & 2 & 1 \\ 1 & -1 & (n-2) & (n-3) & \ldots & 2 & 1 \\ 1 & -1 & -2 & (n-3) & \ldots & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & -2 & -3 & \ldots & 2 & 1 \\ 1 & -1 & -2 & -3 & \ldots & -(n-2) & 1 \\ 1 & -1 & -2 & -3 & \ldots & -(n-2) & -(n-1) \end{bmatrix},$$

so that $\theta_{n-1} = (\bar{s} + \bar{d}_1 + 2\bar{d}_2 + \cdots + (n-1)\bar{d}_{n-1})/n$. Since $\bar{s} = \bar{\theta}_0 + \bar{\theta}_1 + \cdots + \bar{\theta}_{n-1} = 0$, the expression for the second variation transforms as follows:

$$\begin{aligned} 2\sqrt{n}\,\Omega(\bar{w}) &= \int_0^{\hat{T}} \left\{(\bar{\theta}_0 - \bar{\theta}_1)^2 + 2(\bar{\theta}_1 - \bar{\theta}_2)^2 + \cdots + (n-1)(\bar{\theta}_{n-2} - \bar{\theta}_{n-1})^2 - n\bar{\theta}_{n-1}^2 + n\bar{v}^2\right\} dt \\ &= \int_0^{\hat{T}} \left\{\bar{d}_1^2 + 2\bar{d}_2^2 + \cdots + (n-1)\bar{d}_{n-1}^2 - \frac{1}{n}\left(\bar{d}_1 + 2\bar{d}_2 + \cdots + (n-1)\bar{d}_{n-1}\right)^2 + n\bar{v}^2\right\} dt. \end{aligned}$$

By the Jensen inequality,

$$\frac{f(a_1) + \cdots + f(a_{n-1})}{n} \geq f\left(\frac{a_1 + \cdots + a_{n-1}}{n}\right),$$

where f is an arbitrary convex function ($f(x) = x^2$ in our case), we have

$$\bar{d}_1^2 + 2\bar{d}_2^2 + \cdots + (n-1)\bar{d}_{n-1}^2 \geq \frac{1}{n}\left(\bar{d}_1 + 2\bar{d}_2 + \cdots + (n-1)\bar{d}_{n-1}\right)^2.$$

Thus, the lemma is proved. □

The weak minimum $\hat{w}$ is ensured by the second-order condition: item (b) of Theorem 1 (see also Remark 1). Thus, Theorem 3 is completely proved. □

2.5. Example. Let us illustrate the obtained result in the simplest case. Assume that we have two trailers ($n = 2$). Taking into account the explicit form (35) of the vector fields f_0 and f_1, we write system (2) in the coordinates $(x, y, \theta_0, \theta_1, \theta_2)$ of the configuration space $M^5 = \mathbb{R}^2 \times (\mathcal{S}^1)^3$:

$$\begin{cases} \dot{x} = \cos(\theta_2 - \theta_1)\cos(\theta_1 - \theta_0)\cos\theta_0, \\ \dot{y} = \cos(\theta_2 - \theta_1)\cos(\theta_1 - \theta_0)\sin\theta_0, \\ \dot{\theta}_0 = \cos(\theta_2 - \theta_1)\sin(\theta_1 - \theta_0), \\ \dot{\theta}_1 = \sin(\theta_2 - \theta_1), \\ \dot{\theta}_2 = u. \end{cases} \tag{48}$$

In this case, a unique singularity of the Goursat distribution is given by the relation

$$\cos(\theta_2 - \theta_1) = 0.$$

For definiteness, let $\hat{\theta}_2(t) - \hat{\theta}_1(t) \equiv \pi/2$, $t \in [0, T]$, $\hat{u} \equiv 1$. Without loss of generality, we assume that $\hat{q} = (0, 0, 0, t, t + \pi/2)$.

Introduce the following notation for the component of the vector of adjoint variables: $\psi = (\psi_x, \psi_y, \psi_0, \psi_1, \psi_2)$. In this notation, the adjoint system (40) takes the form

$$\begin{cases} \dot{\psi}_x = 0, \\ \dot{\psi}_y = 0, \\ \dot{\psi}_0 = 0, \\ \dot{\psi}_1 = -\cos(\hat{\theta}_1 - \hat{\theta}_0)(\psi_x\cos\hat{\theta}_0 + \psi_y\sin\hat{\theta}_0) - \sin(\hat{\theta}_1 - \hat{\theta}_0)\psi_0, \\ \dot{\psi}_2 = \cos(\hat{\theta}_1 - \hat{\theta}_0)(\psi_x\cos\hat{\theta}_0 + \psi_y\sin\hat{\theta}_0) + \sin(\hat{\theta}_1 - \hat{\theta}_0)\psi_0. \end{cases} \tag{49}$$

The following assertion coincides with [3, Theorem 2] with accuracy up to the coordinate system used.

Assertion 4. *The first-order singular extremals $(\psi, \hat{w})$ of the problem (4)–(6), (48) such that the trajectory $\hat{w}$ belongs to the singularity Γ^+ of the Goursat distribution compose a smooth submanifold of codimension 4 in the state space T^*M, and this submanifold is defined by the relations*

$$\begin{cases} \cos(\hat{\theta}_2 - \hat{\theta}_1) = 0, \\ \psi_2 = 0, \\ \psi_x\cos\hat{\theta}_0 + \psi_y\sin\hat{\theta}_0 = 0, \\ \psi_0 = 0. \end{cases} \tag{50}$$

It follows from the assumption that the order is equal to 1 that $\psi f_1 = \hat{\psi}_0 \neq 0$. From this, we obtain the strict Kelley condition $\psi_1 > 0$, in full accordance with Assertion 2.

In (50), only the two latter relations need to be proved. We have $\psi \hat{f}_1 = \psi_2 = 0$ on the singular trajectory. Since $\psi\widehat{[f_0, f_1]} = -\psi \hat{f}_0' \hat{f}_1$ is the last row of the adjoint system (49), relation (36) here is

$$\dot{\psi}_2 = \cos(\hat{\theta}_1 - \hat{\theta}_0)(\psi_x \cos\hat{\theta}_0 + \psi_y \sin\hat{\theta}_0) + \sin(\hat{\theta}_1 - \hat{\theta}_0)\psi_0 = 0.$$

With account for (48), relation (37) becomes

$$\ddot{\psi}_2 = -\sin(\hat{\theta}_1 - \hat{\theta}_0)(\psi_x \cos\hat{\theta}_0 + \psi_y \sin\hat{\theta}_0) + \cos(\hat{\theta}_1 - \hat{\theta}_0)\psi_0 = 0.$$

This implies the two latter relations of (50). The nonsingularity of the Jacobian is directly verified.

Therefore, on the first-order singular extremal, the vector of adjoint variables is constant and equal to $\psi = (0, 0, 0, \psi_1, 0)$, $\psi_1 > 0$ (accounting for the normalization, we may assume that $\psi_1 = 1$). The variation system becomes

$$\begin{cases} \dot{\xi}_1 = \bar{v}\cos(\hat{\theta}_1 - \hat{\theta}_0)\cos\hat{\theta}_0, \\ \dot{\xi}_2 = \bar{v}\cos(\hat{\theta}_1 - \hat{\theta}_0)\sin\hat{\theta}_0, \\ \dot{\xi}_3 = \bar{v}\sin(\hat{\theta}_1 - \hat{\theta}_0), \\ \dot{\xi}_4 = 0, \\ \dot{\xi}_5 = 0. \end{cases} \tag{51}$$

The second variation is positive-definite:

$$\Omega(\bar{w}) = \psi_1 \int_0^T \bar{v}^2\, dt, \tag{52}$$

and the trajectory $\hat{w}$ yields the weak minimum for an arbitrary T.

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N. B. Mel'nikov
Central Economico-Mathematical Institute, Russian Academy of Sciences
M. V. Lomonosov Moscow State University
E-mail: melnikov@cs.msu.su