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Perturbation calculation of magnetic field dependence of fluxon dynamics in long inline and overlap Josephson junctions

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The motion of a single fluxon in long Josephson-junctions of overlap and inline geometries is investigated in the presence of an applied external magnetic field. The form of the first zero-field step for various parameters is given in closed analytic forms in both cases, and the differences and similarities between the two geometries are emphasized.

PACS numbers: 74.50. + r, 84.40.Mk, 85.25. + k

I. INTRODUCTION

Fluxon excitation in Josephson transmission lines has been considered for applications, for example, as microwave generators in high-frequency integrated circuits[1] and in data processing systems.[2] For such applications, a general knowledge about the dependence of basic dynamic properties of single fluxons on external parameters is important. Such investigations have generally been carried out by numerical simulations because of the complexity of the nonlinear partial differential equation to be studied. Analytical solutions are, in general, rare. In the last few years, however, the method of a perturbation calculation on a modified sine-Gordon equation has had considerable success on a number of problems. Recently,[3] for example, it proved possible to calculate the shape of the zero-field steps in an inline junction (where the bias current is fed in the direction of propagating fluxons). Here we report on an analytic calculation of magnetic field dependence of zero-field steps in inline and overlap (where the bias current is fed at right angles to fluxon velocity) junctions.

II. THE OVERLAP JUNCTION

The overlap geometry is shown in Fig. 1. It is well known that the equation describing fluxon motion for this geometry is the perturbed sine-Gordon equation[4]

$$\phi_{xx} - \phi_{tt} - \sin\phi = \alpha\phi_t + \eta,$$

where $\phi(x,t)$ is the space- and time-dependent phase difference between the two superconducting films. The spatial variable x is measured in units of the Josephson penetration depth $\lambda_J = (\hbar/2\mu_0 edJ)^{1/2}$, and time t in units of the reciprocal plasma frequency ω_0^{-1}, where $\omega_0 = (2eJ/\hbar C)^{1/2}$. Here J is the maximum pair current density, d is the magnetic thickness of the barrier ($d = 2\lambda_L + t_0$), and C is the capacitance per unit area. The parameter α describes dissipative effects $\alpha = G(\hbar/2eJC)^{1/2}$, where G is the shunt conductance per unit area. ($\alpha = 1/\sqrt{\beta}$, where β is the usual McCumber parameter) η represents the uniformly distributed bias current; thus $\eta = I_{dc}/I_c^{ov}$, where I_{dc} is the dc bias current and $I_c^{ov} = JWL$ is the maximum supercurrent, and L and W are the length and width of the junction. It is assumed that $L \gg \lambda_J \gg W$. With α and η equal to zero, Eq. (1) is the pure sine-Gordon equation with the fluxon solution

$$\phi_0(x,t) = 4\tan^{-1}\{\exp[\gamma(u)(x - ut - x_0)]\}. \tag{2}$$

Here x_0 is the inital position of the fluxon with velocity u, and $\gamma(u)$ is the Lorentz factor

$$\gamma(u) = (1 - u^2)^{-1/2}. \tag{3}$$

The energy[4] of a single fluxon is

$$H_f = 8\gamma(u) \tag{4}$$

and the momentum[4] is

$$P_f = 8u\gamma(u). \tag{5}$$

With the terms η and $\alpha\phi_t$ treated as small perturbations, it is a well-known result[4] that a steady-state velocity u_∞ is obtained for the fluxon. u_∞ [normalized to $\bar{c} = c(t_0/\epsilon_r d)^{1/2}$] is given by

$$u_\infty = \left[1 + \left(\frac{4\alpha}{\pi\eta}\right)^2\right]^{-1/2}. \tag{6}$$

With a fluxon (antifluxon) reflected as an antifluxon (fluxon) at the junction ends, this give rise to a zero-field step at a voltage (normalized to $\hbar\omega_0/2e$) $V_{dc} = 2\pi u_\infty/l$. Here $l = L/\lambda_J$.

With a magnetic field H_{ext}, the boundary conditions are

$$\phi_x(0,t) = \phi_x(l,t) = \frac{H_{ext}}{\lambda_J J} \triangleq \kappa_{ext}. \tag{7}$$

Since the surface current flowing in the x direction is given by ϕ_x and the voltage is given by ϕ_t, and since there is a phase shift of $4\pi(-4\pi)$ during the reflexion, there is an energy input to the fluxon of the $4\pi\kappa_{ext}$ $(-4\pi\kappa_{ext})$ in each end due to the magnetic field. This is illustrated in Fig. 2, which shows schematically fluxon dynamic behavior with an ener-

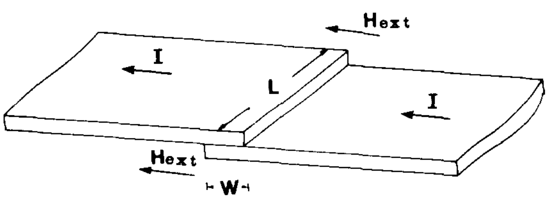

FIG. 1. The geometry of the overlap Josephson junction. Note that bias current is perpendicular to the length of the junction, and that the external magnetic field is applied in the direction of the current.

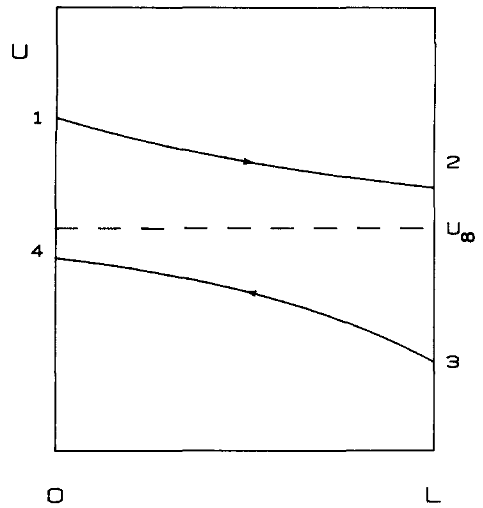

FIG. 2. A schematic picture of the periodic fluxon velocity vs position x in the overlap Josephson junction ($0<x<L$). From 1 to 2, the fluxon decelerates toward the power balance velocity u_∞. At $x=L$, the fluxon is reflected as an antifluxon with less energy. From 3 to 4, the antifluxon accelerates towards the velocity u_∞. At $x=0$, the antifluxon is reflected as a fluxon with larger energy.

gy input at $x=0$, and a decay in velocity towards u_∞ as it approaches $x=l$, where there is a velocity (energy) decrease. In order to calculate those trajectories, we introduce the following notation for convenience:

$$u=\tanh a\ ,\qquad z=u\gamma(u)=P_f/8, \tag{8}$$

which implies that

$$\gamma(u)=\cosh a\ ,\quad z=\sinh a\ ,\quad z_\infty=\pi\eta/4\alpha\ ,\quad u_\infty=\tanh a_\infty . \tag{9}$$

The equation of motion for a single fluxon in the perturbed sine-Gordon model was derived in Ref. 4. It is

$$\frac{dP_f}{dt}=-\alpha P_f+2\pi\eta. \tag{10}$$

Introducing the quantity from Eq. (9) and integrating, we obtain[5]

$$z=z_\infty+(z_0-z_\infty)e^{-\alpha t}, \tag{11}$$

where $z_0=u_0\gamma(u_0)$, u_0 being the initial velocity of the fluxon. Finding $u(t)$ from Eqs. (8) and (11), and integrating yields the trajectory $x(t)$[5]

$$x(t)=x_0+u_\infty t-\frac{1}{\alpha}\ln\frac{z+(z^2+1)^{1/2}}{z_0+(z_0^2+1)^{1/2}}-\frac{u_\infty}{\alpha}\times\ln\frac{1+z_\infty z_0+(z_\infty^2+1)^{1/2}(z_0^2+1)^{1/2}}{1+z\cdot z_0+(z^2+1)^{1/2}(z_0^2+1)^{1/2}}. \tag{12}$$

With reference to Fig. 2 and Eq. (4), the energy input at $x=l$ and $x=0$ may be expressed

$$8\gamma(u_1)-8\gamma(u_4)=8\gamma(u_2)-8\gamma(u_3)=4\pi\kappa_{\rm ext} \tag{13}$$

or

$$\frac{\pi\kappa_{\rm ext}}{2}=\cosh a_i-\cosh a_j;\quad i,j=\begin{cases}2,3\\1,4.\end{cases} \tag{14}$$

The times of flight t_{12} and t_{34} are determined from Eq. (11)

$$\alpha t_{ij}=\ln\frac{z_i-z_\infty}{z_j-z_\infty}=\ln\frac{\sinh a_i-\sinh a_\infty}{\sinh a_j-\sinh a_\infty}\quad i,j=\begin{cases}1,2\\3,4.\end{cases} \tag{15}$$

The third equation which introduces the length of the junction comes from using Eq. (12). It is

$$\frac{\alpha l}{u_\infty}=\alpha t_{ij}-\frac{1}{u_\infty}\ln\frac{z_j+(z_j^2+1)^{1/2}}{z_i+(z_i^2+1)^{1/2}}-\ln\frac{1+z_\infty z_i+(z_\infty^2+1)^{1/2}(z_i^2+1)^{1/2}}{1+z_\infty z_j+(z_\infty^2+1)^{1/2}(z_j^2+1)^{1/2}}, \tag{16}$$

$$\frac{\alpha l}{u_\infty}=\alpha t_{ij}-\frac{a_j-a_i}{u_\infty}-2\ln\frac{\cosh[(a_i+a_\infty)/2]}{\cosh[(a_j+a_\infty)/2]}\quad i,j=\begin{cases}1,2\\3,4.\end{cases} \tag{17}$$

The six equations Eqs. (14), (15), and (17) determine a_1, a_2, a_3, a_4, t_{12}, and t_{34}. The voltage of the zero-field step is given by the average velocity $u_{\rm av}$ defined by $u_{\rm av}=2l/(t_{12}+t_{34})$. We have not been able to find a closed expression for this quantity based on Eqs. (14), (15), and (17): however, an expansion of the equations to second order in the magnetic field, $\kappa_{\rm ext}$, yields:

$$\alpha(t_{12}+t_{34})=\frac{2\alpha l}{u_{\rm av}}=\frac{2\alpha l}{u_\infty}+\tfrac{1}{2}\tanh\left(\frac{\alpha l}{2u_\infty}\right)\times\frac{2+3\sinh^2 a_\infty}{(1+\sinh^2 a_\infty)\sinh^2 a_\infty}\left(\frac{\pi\kappa_{\rm ext}}{\sinh a_\infty}\right)^2 \tag{18}$$

and accordingly, the voltage of the first zero-field step is

$$V_{\rm dc}(H)=2\pi\frac{u_{\rm av}}{l}=V_{\rm dc}(H=0)\times\left\{1-\frac{\pi^2}{8}\left(\frac{\tanh(\alpha l/2u_\infty)}{\alpha l/2u_\infty}\right)\times\frac{2+3(\pi\eta/4\alpha)^2}{[1+(\pi\eta/4\alpha)^2](\pi\eta/4\alpha)^4}\kappa_{\rm ext}^2\right\}, \tag{19}$$

where $V_{\rm dc}(H=0)=2\pi u_\infty/l$ is the zero-field step in the absence of a magnetic field [Eq. (6) or Eq. (9)]. We note from Eq. (19) that the voltage of the zero-field step can only decrease, and that the effect is largest at the bottom of a step, where u_∞ is small. Figure 3 shows a calculation of the first zero-field step for various values of the magnetic field, and Fig. 4 shows the voltage as a function of magnetic field for various values of the bias current.

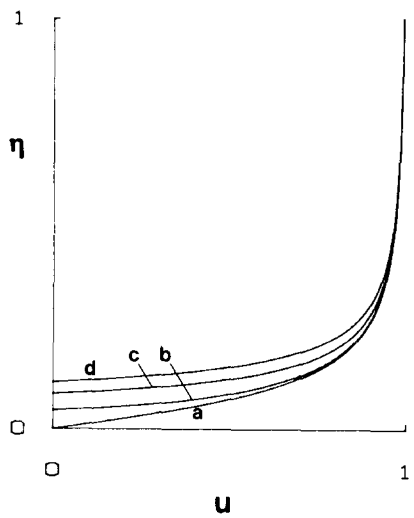

FIG. 3. The bias parameter η (current) vs average velocity u (voltage) for the first zero-field step in the overlap junction for various external magnetic fields. $\alpha = 0.1$ and $l = 10$. $\kappa_{\rm ext}$ = (a) 0, (b) 0.1, (c) 0.3, and (d) 0.5.

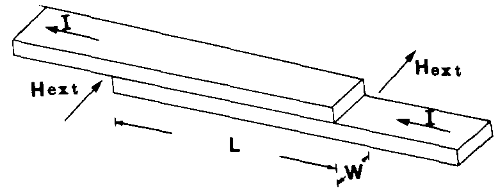

FIG. 5. The geometry of the inline Josephson junction. Note that the bias current is parallel to the length of the junction, and that the external magnetic field is applied perpendicular to the current.

III. THE INLINE JUNCTION

The inline geometry is shown in Fig. 5. Here the equation describing fluxon motion is

$$\phi_{xx} - \phi_{tt} - \sin\phi = \alpha\phi_t. \tag{20}$$

Note that when comparing to Eq. (1), the bias term η is absent. In the inline geometry, the dc bias current enters only through the boundary conditions. For the inline junction, the maximum supercurrent[3] is $I_c^{\rm in} = 4\lambda_J WJ$. It is known[6] that for $I_{\rm dc} > I_c^{\rm in}$ and $\alpha l \gg 1$, there exists a diplaced linear branch due to continuous creation of fluxons in one end and antifluxons in the other, with subsequent annihilation in the center. For $I_{\rm dc} < I_c^{\rm in}$ and $\alpha l \ll 1$, zero-field steps exist, and it was shown recently[3] that they have the same current voltage characteristic as those of the overlap junction, although the soliton dynamic picture is quite different. This is demonstrated in Fig. 6. In zero magnetic field, the soliton gets an energy input in each end of the junction and decays towards zero (since u_∞ from Eq. (6) is zero) during passage of the junction. In an external magnetic field, the fluxon gets a (further) increase in one end and a smaller increase in the other.

This is a consequence of the boundary conditions, which are

$$\begin{aligned} \phi_x(l) &= \kappa + \kappa_{\rm ext}, \\ \phi_x(0) &= -\kappa + \kappa_{\rm ext}, \end{aligned} \tag{21}$$

where $\kappa = I_{\rm dc}/JW2\lambda_J = 2I_{\rm dc}/I_c^{\rm in}$ represent the bias current. Hence $\kappa = 2$ corresponds to the critical current.

Following the arguments and notation used in connection with the overlap configuration, and with reference to Fig. 6, we obtain for the energy inputs at $x = 0$ and $x = l$:

$$\begin{aligned} \gamma(u_1) - \gamma(u_4) &= \frac{\pi}{2}(\kappa + \kappa_{\rm ext}) = \cosh a_1 - \cosh a_4, \\ \gamma(u_3) - \gamma(u_2) &= \frac{\pi}{2}(\kappa - \kappa_{\rm ext}) = \cosh a_3 - \cosh a_2. \end{aligned} \tag{22}$$

For the times of flight t_{12} and t_{34}, we obtain, using Eq. (11) (with $z_\infty = 0$)

$$\exp \alpha t_{ij} = \frac{\sinh a_i}{\sinh a_j} \quad i,j = \begin{cases} 1,2 \\ 3,4 \end{cases}. \tag{23}$$

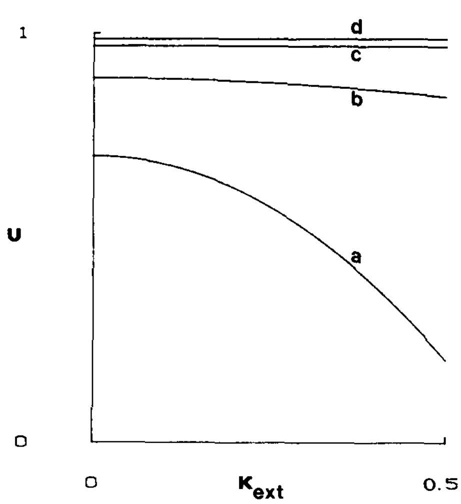

FIG. 4. Magnetic tuning. The average velocity (voltage) vs the applied magnetic field for various bias currents. $\alpha = 0.1$ and $l = 10$. η = (a) 0.125, (b) 0.25, (c) 0.5, and (d) 0.75.

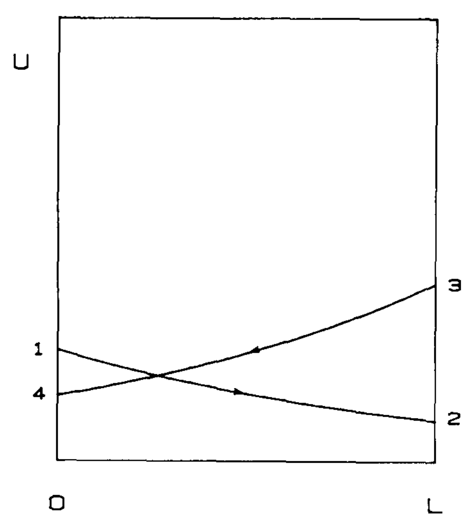

FIG. 6. A schematic picture of the periodic fluxon velocity vs position x in the inline Josephson junction ($0<x<L$). From 1 to 2, the fluxon velocity relaxes toward 0 ($u_\infty = 0$). At $x = L$, the fluxon is reflected as an antifluxon with larger energy. From 3 to 4, the antifluxon velocity relaxes toward 0. At $x = 0$, the antifluxon is reflected as a fluxon with larger energy.

Equation (23) enables us to find the average velocity u_{av} through

$$\exp \alpha(t_{12}+t_{34}) = \exp \alpha(2l/u_{\text{av}}) = \frac{\sinh a_1 \cdot \sinh a_3}{\sinh a_2 \cdot \sinh a_4}. \tag{24}$$

Finally, using Eq. (16) (with $u_\infty = 0$ and $z_\infty = 0$)

$$\alpha l = \ln \frac{\sinh a_i + \cosh a_i}{\sinh a_j + \cosh a_j} \quad i,j = \begin{cases} 1,2 \\ 3,4 \end{cases} \tag{25}$$

which implies

$$a_1 - a_2 = a_3 - a_4. \tag{26}$$

Again combining the six equations, Eqs. (22), (23), and (25), we may determine the six quantities a_1, a_2, a_3, a_4, t_{12}, and t_{34}, and, in particular, the average velocity $u_{\text{av}} = 2l/(t_{12}+t_{34})$ and the normalized voltage of the zero-field step, $V_{\text{dc}} = 2\pi u_{\text{av}}/l$. For the inline case, it is possible to obtain a closed analytic expression. If we, for convenience, introduce

$$A = \left(\frac{\pi\kappa}{4\sinh \alpha l/2}\right)^2, \quad B = \left(\frac{\pi\kappa_{\text{ext}}}{4\cosh \alpha l/2}\right)^2, \tag{27}$$

we find, after some rather lengthy calculations,

$$\exp(2\alpha l/u_{\text{av}}) = \frac{[(A-B)^{1/2}\cosh \alpha l/2 + (1+A-B)^{1/2}\sinh \alpha l/2]^2 - [B/(A-B)]}{[(A-B)^{1/2}\cosh \alpha l/2 - (1+A-B)^{1/2}\sinh \alpha l/2]^2 - [B/(A-B)]}, \tag{28}$$

from which the magnetic field dependence of the zero-field step may be calculated directly. In the limit of $\kappa_{\text{ext}} = 0$, Eq. (28) gives

$$\kappa = \frac{4}{\pi} \frac{\sinh \dfrac{\alpha l}{2}}{\left[\left(\dfrac{\tanh(\alpha l/2u_{\text{av}})}{\tanh(\alpha l/2)}\right)^2 - 1\right]^{1/2}}, \tag{29}$$

which gives the zero-field step when αl is not small. When $\alpha l \ll 1$, we obtain from Eq. (29)

$$u_{\text{av}}\gamma(u_{\text{av}}) = \frac{\pi I_{\text{dc}}}{4\alpha JWL}, \tag{30}$$

which gives the previously derived[3] expression for the zero-field step, valid for both inline and overlap junctions. Figure 7 shows a calculation of the zero-field step at $\kappa_{\text{ext}} = 0$ for various values of αl. Note the cutoff at finite current; for the

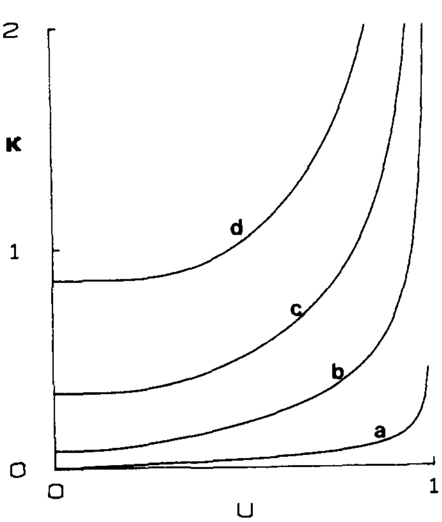

FIG. 7. The bias parameter κ (current) vs average velocity u (voltage) for the first zero-field step in the inline junction for $\alpha l =$ (a) 0.1, (b) 0.5, (c) 1, and (d) 1.5.

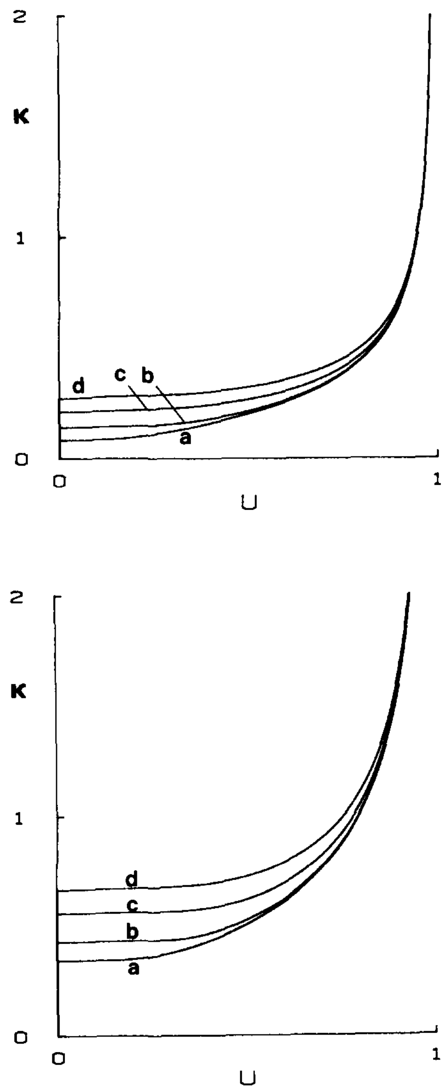

FIG. 8. The bias parameter κ (current) vs average velocity u (voltage) for the first zero-field step in the inline Josephson junction for various external magnetic fields. In 8(a) $\alpha l = 0.5$, and in 8(b) $\alpha l = 1$. In both 8(a) and 8(b), $\kappa_{\text{ext}} =$ (a) 0, (b) 0.1, (c) 0.3, and (d) 0.5.

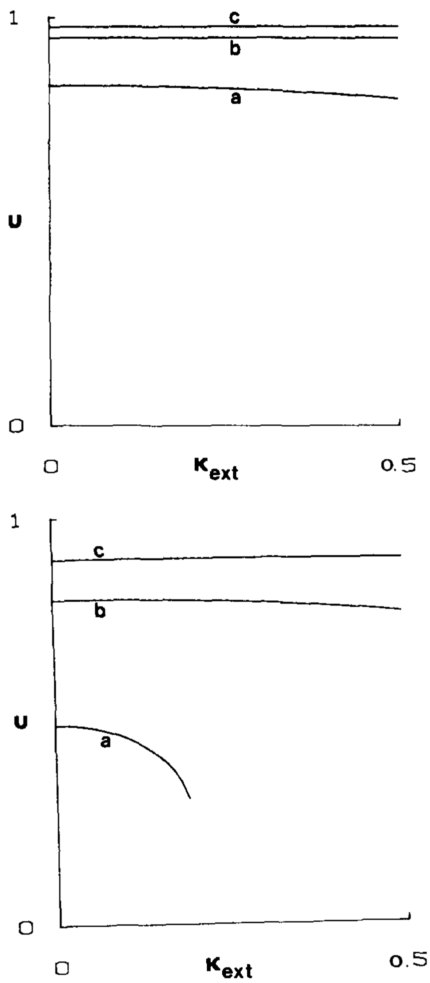

FIG. 9. Magnetic tuning. The average velocity (voltage) vs the applied magnetic field for various bias currents. In 9(a) $\alpha l = 0.5$, and in 9(b), $\alpha l = 1$. In both 9(a) and (b), $\kappa =$ (a) 0.5, (b) 1.0, and (c) 1.5.

corresponding overlap case, the zero-field step can be calculated directly from Eq. (30), and no cutoff takes place. Figure 8 shows for $\alpha l = 0.5$ and 1, the magnetic field dependence of the zero-field step, and finally Fig. 9 shows the tuning of the voltage with magnetic field for various values of the bias current. Note that also in the inline case, we find that the voltage can only decrease with magnetic field, and that the effect is largest for small values of the bias current.

IV. DISCUSSION OF THE PERTURBED SINE-GORDON EQUATION APPROACH

The results that we have obtained here should be taken with some reservations in mind. One assumption we have used implicitly is that a fluxon is reflected as an antifluxon. In some recent numerical calculations,[7] the range of validity of this assumption was discussed. Also, at the reflection of a soliton, a phase shift that is not taken into account in our model may be encountered. Hence our calculation is only valid in very long junctions where this phase shift is negligible.

Further, our calculation requires $\alpha \ll 1$ for the perturbation scheme to be valid. In some recent numerical calculations[8] on inline and overlap junctions where $l = 5$ and $\alpha l \simeq 1$, the magnetic tuning was found comparable to our result (even though that range of parameters is not directly usable in our scheme); however, in some cases, the voltage of the zero-field step increased with applied magnetic field, in contradiction to our results. Nevertheless, our results should give a reasonable approximation if the conditions for the perturbational approach are properly satisfied.

V. CONCLUSION

We have calculated the dependence of the IV curve of zero-field steps on the external magnetic field in long inline and overlap junctions. We used the scheme of the perturbed sine-Gordon model, which has been successful in many other contexts. The scheme requires $\alpha \ll 1$ and $l \gg 1$; however, this is a parameter range often encountered in experimental situations. We found that the voltage always decreases with applied field, the effect being largest when the bias current is smallest.

[1] S. N. Erné and R. D. Parmentier, J. Appl. Phys. **51**, 5025 (1980).
[2] T. V. Rajeevakumar, IEEE Trans. Magn. **MAG-17**, 591 (1981).
[3] O. A. Levring, N. F. Pedersen, and M. R. Samuelsen, Appl. Phys. Lett. **40**, 846 (1982).
[4] D. W. McLaughlin and A. C. Scott, Phys. Rev. A **18**, 1652 (1978).
[5] O. H. Olsen and M. R. Samuelsen, Phys. Rev. B **25**, 3181 (1982).
[6] A. C. Scott and W. J. Johnson, Appl. Phys. Lett. **14**, 316 (1969).
[7] O. H. Olsen and M. R. Samuelsen, J. Appl. Phys. **52**, 6247 (1981).
[8] S. N. Erné, A. Ferrigno, S. Di Genova, and R. D. Parmentier, Lett. Nuovo Cimento **34**, 121 (1982).