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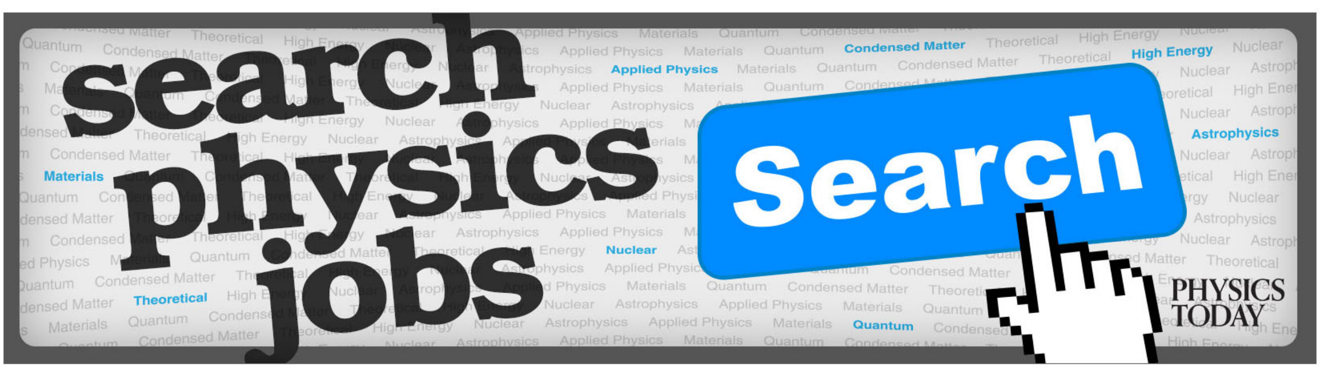
Citation: Physics of Plasmas **11**, 866 (2004); doi: 10.1063/1.1642655
View online: http://dx.doi.org/10.1063/1.1642655
View Table of Contents: http://scitation.aip.org/content/aip/journal/pop/11/3?ver=pdfcov
Published by the AIP Publishing



PHYSICS OF PLASMAS VOLUME 11, NUMBER 3 MARCH 2004

O-X mode conversion in an axisymmetric plasma at electron cyclotron frequencies

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(Received 23 April 2003; accepted 24 November 2003)

The cold plasma dielectric function is employed to represent the plasma response of an axisymmetric steady state to incident waves at frequencies on the order of the electron cyclotron frequency. A boundary layer analysis of Maxwell's equation valid near the mode conversion region is carried out to obtain an approximate system of wave equations. The independent variables of the wave equations are the flux surface label and poloidal angle. The case is analyzed in which the O mode cut-off surface and X mode cut-off surface intersect. An integral representation of solutions is given. It is shown that for a wide class of incident waves all such waves are transmitted with total mode conversion and no mode reflection. This work cannot assure that solutions with mode reflection do not occur, although a broad search for solutions, possibly with mode reflection, was unsuccessful. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1642655]

I. INTRODUCTION

Quite some time ago Preinhälter and Kopecký[1] studied mode conversion of O to X modes at electron cyclotron frequencies by means of WKB solutions for a plasma with a cold plasma dielectric tensor and with the assumption of perpendicular stratification. In a later paper with another author[2] we examined the same problem to identify more precisely where mode conversion might occur. We also applied the relatively standard boundary layer-type of stretched variables expansion to obtain a simpler wave equation valid in the mode conversion region. We constructed solutions which gave quantitative values for mode conversion coefficients and which could be matched to the geometrical optics solutions. The results, obtained numerically, agreed with those of Preinhälter and Kopecký and provided a modest extension in generality in terms of the assumption concerning the plasma state and incident wave. We found, as is typical of WKB tunneling solutions, that the mode conversion dropped off exponentially as one moved away from the ideal situation. More precisely, if ϵ is the small geometric optics expansion parameter, then for a mismatch of $O(\epsilon^{1/2})$ in a wave or steady state parameter, there is a reduction of order one in the mode conversion coefficient, see Eq. (18) of Ref. 2.

Recently, there has been renewed experimental and theoretical interest in the mode conversion problem, and Wendelstein 7-AS (W7-AS) has demonstrated considerable success with mode conversion.[3] While one can certainly apply geometrical optics away from the mode conversion region, its ability to predict mode conversion effects is open to question. The wave propagation properties of a plasma depend critically on the two functions: the number density n and the modulus of the magnetic field B. Since the surfaces $n=\text{constant}$ and $B=\text{constant}$ are not approximately parallel in a typical toroidally confined plasma, there is little justification in the use of results from a perpendicularly stratified medium. This paper addresses the mode conversion problem in plasma modeled by the cold plasma dispersion function in the simplest nontrivial toroidal plasma of axisymmetry. Our aim is to examine the possibility of mode conversion in such systems.

We follow the general procedure of our earlier paper. We apply boundary layer expansions to obtain appropriate wave equations near the mode conversion region. We expect solutions of this equation to be continued onto geometrical optics solutions outside the region. Wave cut-offs occur on surfaces, and we expect mode conversion to be significant when these surfaces intersect along a curve, or when they are close to each other along curves on each surface. It is clear that the latter possibility is easily changed by a modest variation of plasma or wave parameters into the case where the surfaces intersect. In any case, we concentrate here on the case of intersecting surfaces. The choice made here, in which the cut-off surfaces intersect at an angle which is not small (or close to π), has a consequence that the problem treated cannot be reduced by a variation of parameters to the case of a stratified medium studied in Refs. 1 and 2. If the cut-off surfaces were approximately parallel and close together along some curve, the work of Refs. 1 and 2 could apply. We consider the case treated here to be "typical" and the situation in which Refs. 1 and 2 apply to be exceptional. When we expand Maxwell's equations about the curve on which both the O and X modes are cut off, we obtain wave equations. This system consists of two complex valued equations with one component each of **E** and **B** as the dependent variables and the flux surface label and poloidal angle as independent variables. We also find the appropriate Poynting theorem for the system. Each equation has both first derivatives with respect to its arguments and coefficients which are linear functions of the arguments.

The wave equations appear to be of a type of elliptic equations not studied before, and so we must develop solutions and properties of the solutions. We must determine the expansion of the solutions far from the mode conversion curve, so that we may match the solutions to O and X waves, and we must also identify incoming and outgoing wave solutions. We show that a change of dependent variables converts the wave equations into a system for which it is possible to obtain integral representations of solutions. There is an arbitrary analytic function in the representation, which we choose in a manner to make further analysis possible and to ensure that any wave solutions will have finite nonzero energy flux. The final representation of a family of solutions involves an integral with respect to one complex variable and which depends on the two independent variables.

The analysis of the asymptotic expansions far from the mode conversion curve is rather intricate and involves the properties of a particular curve which spirals out of one point in the complex plane of integration and spirals into a second point in that plane. With the arbitrary function as chosen, it is easily seen that the integral representation is effectively even in the two independent variables. Thus, any solution which may be found and which might represent an incoming wave with a reflected and transmitted wave must of necessity have no reflected wave and indicate mode transmission with total conversion. We show that there are infinitely many solutions of the type representing waves incident from either side of the mode conversion region. Most of these solutions have small angular extent, whose direction depend on the centers of the spirals. For appropriate plasma parameters these solutions may cover relatively large angular regions. A consequence of the nontrivial two-dimensional variation of the solutions is that the precise restriction to parameters close to resonance is not so severe as in the perpendicularly stratified medium case. The parameters may vary by amounts of order $\epsilon^{1/2}$, and the same results on total mode conversion of the transmitted wave apply. As indicated, in the stratified medium such parameter variation would result in losses of order one in the mode conversion.

We cannot assert that we have all relevant solutions of this problem, or that there are not solutions in which mode reflection occurs. Our methods are constructive, and other solutions might exist. Other solutions which may exhibit mode reflections may be given by different choices of the arbitrary function in the integral representation. A broad examination of other possible functions which give solutions of relevance for wave propagation problems show no other solutions with or without reflections.

II. THE EXPANSION IN THE MODE CONVERSION REGION

The expansion of Maxwell's equations with the cold plasma dielectric tensor in the mode conversion region is complicated by the lack of parallel or perpendicular stratification. Nevertheless, we may carry over the approach and scaling assumptions used in the perpendicularly stratified case.[2] We indicate the similarities as they occur. It is clear, however, that we must expect to examine partial differential equations instead of ordinary differential equations. In this section we obtain the partial differential equations which characterize the electromagnetic fields in the region of interest.

We work in nondimensional variables in which frequencies are measured in units of the wave frequency ω and distances in units of the free space wavelength c/ω. We introduce the characteristic equilibrium scale length L, and the small parameter

$$\epsilon = c/(\omega L), \tag{1}$$

the reciprocal of the usual (large) geometrical optics expansion parameter. The equilibrium magnetic field has the standard form in an axisymmetric plasma equilibrium

$$\mathbf{b} = -Q(\psi)\frac{\nabla\psi}{\epsilon}\times\frac{\nabla\theta}{\epsilon} + \frac{\nabla\psi}{\epsilon}\times\frac{\nabla\phi}{\epsilon}, \tag{2}$$

where $\psi = \psi(\epsilon r, \epsilon z)$, $\theta = \theta(\epsilon r, \epsilon z)$, and r, ϕ, z are cylindrical coordinates, ψ is the flux function, θ the poloidal angle, and $Q(\psi)$ is the safety factor. The poloidal and toroidal components of $\mathbf{b}$ are

$$\mathbf{b}_p = \frac{\nabla\psi}{\epsilon}\times\frac{\nabla\phi}{\epsilon}, \quad \mathbf{b}_t = -Q(\psi)\frac{\nabla\psi}{\epsilon}\times\frac{\nabla\theta}{\epsilon}, \tag{3}$$

and we note the usual identities

$$\mathbf{b}\cdot\frac{\nabla\theta}{\epsilon} = b_t/(\epsilon r Q), \quad \mathbf{b}\cdot\frac{\nabla\phi}{\epsilon} = b_t/(\epsilon r), \tag{4a}$$

and

$$\frac{\nabla\psi}{\epsilon}\times\frac{\nabla\theta}{\epsilon}\cdot\frac{\nabla\phi}{\epsilon} = -b_t/(\epsilon r Q). \tag{4b}$$

With these assumptions Maxwell's equations become

$$\nabla\times\mathbf{E} = -i\mathbf{B}, \tag{5a}$$

$$\nabla\times\mathbf{B} = i\boldsymbol{\kappa}\cdot\mathbf{E}, \tag{5b}$$

where the dielectric tensor in a system with three orthonormal, right-handed basis vectors and with $\mathbf{b}$ in the third direction is

$$\boldsymbol{\kappa} = \begin{pmatrix} \kappa_\perp & -i\kappa_\wedge & 0 \\ i\kappa_\wedge & \kappa_\perp & 0 \\ 0 & 0 & \kappa_\parallel \end{pmatrix} \tag{6}$$

and

$$\kappa_\perp = 1 - \sum_{(\alpha)} \frac{\omega_{p\alpha}^2}{1-\Omega_{c\alpha}^2}, \tag{7a}$$

$$\kappa_\wedge = \sum_{(\alpha)} \frac{\Omega_{c\alpha}\omega_{p\alpha}^2}{1-\Omega_{c\alpha}^2}, \tag{7b}$$

$$\kappa_\parallel = 1 - \sum_{(\alpha)} \omega_{p\alpha}^2. \tag{7c}$$

In (7) α is the species index, $\omega_{p\alpha}$ is the species plasma frequency, and $\Omega_{c\alpha}$ is the signed species cyclotron frequency. We assume that ω_{p_e} and Ω_{c_e} are both of order one, while the ion contributions are typically small. We might ignore the ions, but they do not modify the analysis in any significant manner.

It is convenient to represent the electromagnetic wave fields in terms of two distinct basis vectors. For any vector $\mathbf{V}$ we introduce

$$\mathbf{V}=V_\psi\frac{\nabla\psi}{\epsilon}+V_\theta\frac{\nabla\theta}{\epsilon}+V_\phi\frac{\nabla\phi}{\epsilon} \tag{8}$$

and

$$\mathbf{V}=V_1\mathbf{e}_1+V_2\mathbf{e}_2+V_3\mathbf{e}_3, \tag{9}$$

where

$$\mathbf{e}_1=\frac{\nabla\psi}{\epsilon}\left|\frac{\nabla\psi}{\epsilon}\right|^{-1}, \tag{10a}$$

$$\mathbf{e}_2=\mathbf{b}\times\mathbf{e}_1/b, \tag{10b}$$

and

$$\mathbf{e}_3=\mathbf{e}_1\times\mathbf{e}_2=\mathbf{b}/b. \tag{10c}$$

The transformation from ψ, θ, ϕ components to 1,2,3 components is

$$V_1=\left[V_\psi\frac{\nabla\psi}{\epsilon}+V_\theta\frac{\nabla\theta}{\epsilon}\right]\cdot\mathbf{e}_1, \tag{11a}$$

$$V_2=[-b_t^2V_\theta+Qb_p^2V_\phi]\left[Qb\left|\frac{\nabla\psi}{\epsilon}\right|\right]^{-1}, \tag{11b}$$

$$V_3=(V_\theta+QV_\phi)b_t/(\epsilon rQb). \tag{11c}$$

We now have the apparatus with which to examine Maxwell's equations. We start with the Faraday law (5a), which we treat exactly. We use the representation (8) and we assume that the wave fields are functions of ψ, θ, and ϕ. More precisely since ψ and ϕ vary on the slow equilibrium scale, the quantities are functions of ψ/ϵ and θ/ϵ. Thus, we obtain

$$\begin{aligned}\epsilon\frac{\nabla\phi}{\epsilon}\times\frac{\nabla\psi}{\epsilon}(E_{\psi,\phi}-E_{\phi,\psi})+\epsilon\frac{\nabla\psi}{\epsilon}\times\frac{\nabla\theta}{\epsilon}(E_{\theta,\psi}-E_{\psi,\theta})\\+\epsilon\frac{\nabla\theta}{\epsilon}\times\frac{\nabla\phi}{\epsilon}(E_{\phi,\theta}-E_{\theta,\phi})=-i\mathbf{B}.\end{aligned} \tag{12}$$

Since the equilibrium is axisymmetric we may assume that the wave fields vary as $e^{iN\phi}$, where N is large, more precisely $\epsilon N=O(1)$. Thus, we may replace $\partial/\partial\phi$ by iN. As just noted, for the wave fields $\epsilon(\partial/\partial\psi)$ and $\epsilon(\partial/\partial\theta)$ are O(1). Hence, all the terms in (12) are O(1) in ϵ.

It is straightforward to obtain the three relations from (11) and (12)

$$\begin{aligned}iN\epsilon(E_1\pm iE_2)=(D_1\pm iD_2)E_\phi\mp(B_1\pm iB_2)(\epsilon rb_t/b)\\+iB_3(\epsilon rb_p/b),\end{aligned} \tag{13a}$$

$$\begin{aligned}iN\epsilon E_3=-iB_1(\epsilon rb_p/b)+(\epsilon E_{\phi,\theta}+iN\epsilon QE_\phi)\\\times[b_t/(\epsilon rQb)],\end{aligned} \tag{13b}$$

where the differential operators D_1 and D_2 are defined as

$$D_1=\mathbf{e}_1\cdot\epsilon\left(\frac{\nabla\psi}{\epsilon}\frac{\partial}{\partial\psi}+\frac{\nabla\theta}{\epsilon}\frac{\partial}{\partial\theta}\right), \tag{14a}$$

$$D_2=\left[Qb\left|\frac{\nabla\psi}{\epsilon}\right|\right]^{-1}\left(-b_t^2\epsilon\frac{\partial}{\partial\theta}+i\epsilon NQb_p^2\right). \tag{14b}$$

The three relations (13a) with the + and − signs and (13b) are not linearly independent, as the difference of the relations (13a) with + and with − yields (13b). Nonetheless, it is useful to use both forms. In order to satisfy fully the Faraday law, at the appropriate time we adjoin the relation $\nabla\cdot\mathbf{B}=0$.

The dielectric tensor $\boldsymbol{\kappa}$ in the Ampère law (with displacement current) (5b) has a simpler representation in terms of the 1,2,3 components, even though the curl operator becomes more complicated. We use the representation (9), and

$$\begin{aligned}i(\kappa_\perp E_1-i\kappa_\wedge E_2)=-(\epsilon B_{2,\theta}+iN\epsilon QB_2)b_t/(\epsilon rQb)\\+D_2B_3+O(\epsilon),\end{aligned} \tag{15a}$$

$$\begin{aligned}i(\kappa_\perp E_2+i\kappa_\wedge E_1)=(\epsilon B_{1,\theta}+iN\epsilon QB_1)b_t/(\epsilon rQb)\\-D_1B_3+O(\epsilon),\end{aligned} \tag{15b}$$

$$i\kappa_\parallel E_3=+D_2B_1-D_1B_2+O(\epsilon). \tag{15c}$$

The error terms in (15) all arise from derivatives of the equilibrium quantities $\mathbf{e}_1,\mathbf{e}_2$, or $\mathbf{e}_3$. We could have avoided these terms if we had used the representation (8), but then we would have had more complicated equations. We know from the analysis of the perpendicularly stratified medium that we must solve Maxwell's equations only correct through order $\epsilon^{1/2}$. Since same type of expansion is used here, the error terms in (15) do not affect any results. Finally, we must add the equation $\nabla\cdot\mathbf{B}=0$ to the system and

$$(\mathbf{e}_3\cdot\nabla)B_3+D_1B_1+D_2B_2=O(\epsilon). \tag{16}$$

We may combine (15a) and (15b), so that together with (13a) we obtain

$$\begin{aligned}(\kappa_\perp\mp\kappa_\wedge)[(D_1\pm iD_2)E_\phi\mp(B_1\pm iB_2)(\epsilon rb_t/b)\\+iB_3(\epsilon rb_p/b)]\\=\pm i\epsilon N[b_t/(\epsilon rbQ)]\left(\frac{\partial}{\partial\theta}+iN\epsilon Q\right)(B_1\pm iB_2).\end{aligned} \tag{17}$$

The relations (13b), (15c), (16), and (17) complete our system.

We are now prepared to develop an expansion of Maxwell's equations in the mode conversion region. We expect the solution outside this region to be in the form of a typical geometrical optics solution which has a smooth transition into the mode conversion region. In the mode conversion region we follow the general structure and parameter scaling in the solution found in the perpendicularly stratified medium case. Consistent with geometrical optics and the perpendicularly stratified medium case we expect **E** and **B** to have local parallel and perpendicular wave numbers, and we expect the wave amplitudes to vary more rapidly than the equilibrum scale length L but slower than the wave variation ψ/ϵ and θ/ϵ. Specifically, we look for solutions of the form

$$\begin{aligned}(\mathbf{E},\mathbf{B})=[\mathbf{E}'(\psi\epsilon^{-1/2},\theta\epsilon^{-1/2}),\mathbf{B}'(\psi\epsilon^{-1/2},\theta\epsilon^{-1/2})]\\\times\exp i[X(\psi,\theta)/\epsilon].\end{aligned} \tag{18}$$

In (18) the exponential factor is the typical geometric optics form, while the amplitudes are appropriate to the mode conversion region and should match to the geometrical optics region. The representation (18) is far from unique, even when **E** and **B** are given. We specify the representation more precisely shortly.

With the representation (18) the local wave number is

$$\mathbf{k}=\frac{\nabla\psi}{\epsilon}X_{,\psi}+\frac{\boldsymbol{\nabla}\theta}{\epsilon}X_{,\theta}+\frac{\epsilon N}{\epsilon r}\hat{\phi}. \tag{19}$$

The components of $\mathbf{k}$ perpendicular to $\mathbf{b}$ are k_1 and k_2, when

$$k_1=D_1X=\frac{\nabla X}{\epsilon}\cdot\mathbf{e}_1, \tag{20a}$$

$$k_2=D_2X=(-b_t^2X_{,\theta}+\epsilon NQb_p^2)\left(Qb\left|\frac{\boldsymbol{\nabla}\psi}{\epsilon}\right|\right)^{-1}, \tag{20b}$$

while the component parallel to $\mathbf{b}$ is k_3

$$k_3=(X_{,\theta}+N\epsilon Q)b_t/(\epsilon rQ). \tag{20c}$$

We expect that in the mode conversion region both k_1 and k_2 are small $\mathrm{O}(\sqrt{\epsilon})$. We then find from (20b) and (20c)

$$k_3\equiv k_\parallel=\epsilon Nb/(\epsilon rb_t) \tag{20c$'$}$$

and that both D_1 and D_2 are operators $\mathrm{O}(\sqrt{\epsilon})$. Hence (16) shows that $B_3=\mathrm{O}(\sqrt{\epsilon})$, so that (17) reduces to

$$(\kappa_\perp\mp\kappa_\wedge-k_\parallel^2)(B_1\pm iB_2)=\mathrm{O}(\sqrt{\epsilon}). \tag{21}$$

We are thus left with two possibilities

$$\kappa_\perp-\kappa_\wedge-k_\parallel^2=\mathrm{O}(\sqrt{\epsilon})\quad\text{and}\quad B_1-iB_2=\mathrm{O}(\sqrt{\epsilon}) \tag{22a}$$

or

$$\kappa_\perp+\kappa_\wedge-k_\parallel^2=\mathrm{O}(\sqrt{\epsilon})\quad\text{and}\quad B_1+iB_2=\mathrm{O}(\sqrt{\epsilon}). \tag{22b}$$

The two possibilities are formally equivalent and we choose (22a) corresponding to the usual X mode cut-off condition

$$k_\parallel^2=1-\frac{\omega_{pe}^2}{1+|\Omega_{ce}|}. \tag{23}$$

We must now complete the expansion through $\mathrm{O}(\sqrt{\epsilon})$. We can carry out the general form of the expansion fairly easily, but before we are able to make the equations explicit we must return to (18) and the properties of the phase $X(\psi,\theta)$. We give the simpler parts first. From (13b) and (20c) it follows that

$$k_3E_\phi=\epsilon NE_3+(\epsilon rb_p/b)B_1+\mathrm{O}(\sqrt{\epsilon}), \tag{24}$$

while (16) becomes

$$ik_3B_3=-D_1B_1-D_2B_2+\mathrm{O}(\epsilon). \tag{25}$$

When we combine (17), (22a), (24), and (25) we obtain

$$\begin{aligned}k_3[&(D_1+iD_2)\epsilon NE_3+2i(\epsilon rb_p/b)D_2B_1]\\&-2B_1(\kappa_\perp-\kappa_\wedge-k_3^2)(\epsilon rb_t/b)\\&=2\left\{iN\epsilon[b_t/(\epsilon rQb)]\left(\frac{\partial}{\partial\theta}+iN\epsilon Q\right)\right.\\&\qquad\left.+k_3^2(\epsilon rb_t/b)\right\}B_1+\mathrm{O}(\epsilon).\end{aligned} \tag{26}$$

An elementary calculation shows that

$$\begin{aligned}&iN\epsilon[b_t/(\epsilon rQb)]\left(\frac{\partial}{\partial\theta}+iN\epsilon Q\right)+k_3^2\epsilon rb_t/b\\&\quad=-ik_3\epsilon r(b_p/b)D_2,\end{aligned} \tag{27}$$

so that (27) reduces to

$$\begin{aligned}&(D_1+iD_2)E_3+[4i\epsilon rb_p/(\epsilon N)]D_2B_1\\&\quad-2B_1(\kappa_\perp-\kappa_\wedge-k_3^2)\epsilon rb_t/(\epsilon Nk_3)=\mathrm{O}(\epsilon).\end{aligned} \tag{28}$$

The O mode cut-off requires that $\kappa_\parallel=\kappa_\parallel(\psi)=\mathrm{O}(\sqrt{\epsilon})$, so that

$$+\kappa_\parallel E_3=(D_1-iD_2)B_1+\mathrm{O}(\epsilon). \tag{29}$$

We see from (29) that

$$4iD_2B_1=2(D_1+iD_2)B_1-2\kappa_\parallel E_3+\mathrm{O}(\epsilon) \tag{30}$$

so that (28) becomes

$$\begin{aligned}&(D_1+iD_2)\{E_3+[2\epsilon rb_p/(\epsilon N)]B_1\}-2\kappa_\parallel E_3\epsilon rb_p/(\epsilon N)\\&\quad-2B_1(\kappa_\perp-\kappa_\wedge-k_3^2)\epsilon rb_t/(\epsilon Nk_3)=\mathrm{O}(\epsilon).\end{aligned} \tag{31}$$

We see that it is convenient to introduce a new dependent variable

$$\tilde{E}_3=E_3+2B_1\epsilon rb_p/(\epsilon N), \tag{32}$$

and our system becomes

$$(D_1-iD_2)B_1=\kappa_\parallel\tilde{E}_3-2\kappa_\parallel(\psi)B_1\epsilon rb_p/(\epsilon N)+\mathrm{O}(\epsilon) \tag{33a}$$

and

$$\begin{aligned}(D_1+iD_2)\tilde{E}_3=&+2\kappa_\parallel(\psi)\tilde{E}_3\epsilon rb_p/(\epsilon N)+2B_1\\&\times\{-2\kappa_\parallel(\psi)[\epsilon rb_p/(\epsilon N)]^2\\&+(\kappa_\perp-\kappa_\wedge-k_3^2)(\epsilon rb_t/\epsilon Nk_3)\}+\mathrm{O}(\epsilon).\end{aligned} \tag{33b}$$

The system we finally expand is (33).

The mode conversion region is the neighborhood of a point, which we identify as $\psi=\theta=0$. The point is uniquely defined by the conditions

$$\kappa_\parallel(0)=\kappa_\perp(0,0)-\kappa_\wedge(0,0)-k_3^2(0,0)=0. \tag{34}$$

If we expand the phase in (18) we obtain

$$\begin{aligned}[X(\psi,\theta)-X(0,0)]/\epsilon=&X_{,\psi}\psi/\epsilon+X_{,\theta}\theta/\epsilon+\tfrac{1}{2}(X_{,\psi\psi}\psi^2\\&+2X_{,\psi\theta}\psi\theta+X_{,\theta\theta}\theta^2)/\epsilon+\cdots.\end{aligned} \tag{35}$$

We have assumed that both k_1 and k_2 are $O(\sqrt{\epsilon})$, hence, see (20),

$$X_{,\psi}\epsilon^{-1/2}=O(1) \tag{36a}$$

and

$$X_{,\theta}=(\epsilon NQb_p^2/b_t^2)|_{0,0}+O(\sqrt{\epsilon}). \tag{36b}$$

Thus, we may express $X(\psi,\theta)$ in the form

$$[X(\psi,\theta)-X(0,0)]/\epsilon=(\epsilon NQb_p^2/b_t^2)|_{0,0}\frac{\theta}{\epsilon} + f\left(\frac{\psi}{\sqrt{\epsilon}},\frac{\theta}{\sqrt{\epsilon}},\epsilon\right). \tag{37}$$

We now modify the phase and amplitude functions in (18) by setting

$$X(\psi,\theta)/\epsilon=X(0,0)/\epsilon+(\epsilon NQb_p^2/b_t^2)|_{0,0}\theta/\epsilon \tag{38}$$

and multiplying $\mathbf{E}'$ and $\mathbf{B}'$ by

$$\exp\left[i\ \ f\left(\frac{\psi}{\sqrt{\epsilon}},\frac{\theta}{\sqrt{\epsilon}},\epsilon\right)\right].$$

After this specification of the phase and amplitude functions we have arranged the system so that k_1 and k_2 are both $O(\epsilon)$. We also have the explicit form for k_3

$$k_3=\epsilon N[(Qb_p^2/b_t^2)|_{0,0}+Q(\psi)]b_t/(\epsilon rQb). \tag{39}$$

We note that the ability to make these modifications to the phase and amplitude depends critically on the fact that the solution and background plasma depend nontrivially on both ψ and θ. In a stratified plasma the θ dependence would have been fixed as $\exp(i\theta k_\theta)$, where k_θ is a constant. The more general θ dependence allows greater freedom in the structure of the solutions. It is also clear that the modifications to the phase and amplitude are also consistent with the transition of a solution from the mode conversion region into the geometrical optics region. We note in passing that we could have omitted the modification (38), but the analysis would have become more complicated.

It is now easy to obtain the equations for the wave fields in the mode conversion region. We may expand near $\psi=\theta=0$ so that

$$-2\kappa_\parallel(\psi)\epsilon rb_p/(\epsilon N)=\sqrt{\epsilon}c_1\psi/\sqrt{\epsilon}+\cdots, \tag{40a}$$

$$\begin{aligned}Y&\equiv-2\kappa_\parallel(\psi)[\epsilon rb_p/(\epsilon N)]^2+(\kappa_\perp-\kappa_\wedge-k_3^2)\\&\quad\times\epsilon rb_t/(\epsilon Nk_3)\\&=\sqrt{\epsilon}(c_2\psi/\sqrt{\epsilon}+c_3\theta/\sqrt{\epsilon}).\end{aligned} \tag{40b}$$

We note that if the two cut-off surfaces become parallel rather than touch we have $c_3=0$. We explore this possibility elsewhere. We also assume $c_1>0$ as is typical, but not essential. We assume $c_3\neq0$ and we introduce new independent variables

$$y'=[\theta+c_2\psi/c_3]\epsilon^{-1/2}, \tag{41a}$$

$$x'=\psi\epsilon^{-1/2}. \tag{41b}$$

In terms of the operators

$$D_1'=\mathbf{e}_1\cdot\left(\frac{\nabla\psi}{\epsilon}\frac{\partial}{\partial x'}+\left(\frac{1}{\epsilon}\frac{\partial Y}{\partial\theta}\right)^{-1}\frac{\nabla Y}{\epsilon}\frac{\partial}{\partial y'}\right) \tag{42a}$$

$$=d_1'\frac{\partial}{\partial x'}+d_2'\frac{\partial}{\partial y'}, \tag{42b}$$

$$D_2'=-\left[b_t^2\left(Qb\frac{|\nabla\psi|}{\epsilon}\right)^{-1}\right]_{0,0}\frac{\partial}{\partial y'}=d_3'\frac{\partial}{\partial y'} \tag{42c}$$

and the additional constant

$$\alpha=[\epsilon rb_p/(\epsilon N)]|_{0,0} \tag{42d}$$

the system becomes

$$(D_1'+iD_2')\tilde{E}_3=+\alpha c_1x'\tilde{E}_3+(2B_1)c_3y', \tag{43a}$$

$$(D_1'-iD_2')(2B_1)=+c_1x'\tilde{E}_3-(2B_1)\alpha c_1x'. \tag{43b}$$

Our last transformation puts the system in a form with fewer constants. We set

$$x'=e_1x,\quad y'=e_2y, \tag{44a}$$

$$2B_1=e_3B,\quad \tilde{E}_3'=E, \tag{44b}$$

and if

$$e_1^2=d_1'/(\alpha c_1), \tag{45a}$$

$$e_2e_1=\alpha d_1'/c_3, \tag{45b}$$

$$e_3=1/\alpha, \tag{45c}$$

and

$$D_1=\frac{\partial}{\partial x}+d_2\frac{\partial}{\partial y},\quad D_3=d_3\frac{\partial}{\partial y}, \tag{46}$$

with

$$d_2=d_2'c_3(d_1'\alpha^2c_1)^{-1}, \tag{47a}$$

$$d_3=d_3'c_3(d_1'\alpha^2c_1)^{-1}, \tag{47b}$$

we have

$$(D_1+iD_2)E=xE+yB, \tag{48a}$$

$$(D_1+iD_2)B=xE-xB. \tag{48b}$$

We may write the system in terms of the one complex constant

$$d=d_2+id_3 \tag{49}$$

as

$$\left(\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}\right)E=xE+yB, \tag{50a}$$

$$\left(\frac{\partial}{\partial x}+\bar{d}\frac{\partial}{\partial y}\right)B=x(E-B). \tag{50b}$$

This is our basic system. We have the identity

$$\left(\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}\right)(\bar{B}E)=yB\bar{B}+xE\bar{E}, \tag{51}$$

which is the Poynting relation for the system. In particular, the energy flux relation is

$$\mathrm{Im}\left[\left(\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}\right)\overline{B}E\right]=0, \tag{52}$$

or

$$\left(\frac{\partial}{\partial x}+d_r\frac{\partial}{\partial y}\right)\mathrm{Im}\,\overline{B}E+d_i\frac{\partial}{\partial y}\mathrm{Re}(\overline{B}E)=0. \tag{53}$$

Thus, the energy flow in the $\nabla\psi$ direction is $\mathrm{Im}(\overline{B}E)$. It is easy to verify that this energy flux is exactly the same as the standard form $\widehat{\nabla\psi}\cdot\,\mathrm{Im}(\overline{\mathbf{B}}\times\mathbf{E})$. We study the system in the form (50) in the next sections.

III. AN INTEGRAL REPRESENTATION OF SOLUTIONS

We would like to construct solutions of the system (50) in order to study the properties of the waves in the mode conversion region. To this end we look to develop integral representations of Laplace integral type. We cannot easily deal with the system as it stands, but we may reduce it to a simpler form, and one which makes explict the elliptic nature of the system. We introduce new dependent variables by

$$(E,B)=(\tilde{E},\tilde{B})\exp[\tfrac{1}{2}(Kx^2+2Lxy+My^2)], \tag{54}$$

so that $\tilde{E}$ and $\tilde{B}$ satisfy the equations

$$\tilde{E}_{,x}+d\tilde{E}_{,y}=\{x[(1-K)-dL]-y(L+dM)\}\tilde{E}+y\tilde{B}, \tag{55a}$$

$$\tilde{B}_{,x}+\overline{d}\tilde{B}_{,y}=xE-[x(1+K+\overline{d}L)+y(L+\overline{d}M)]\tilde{B}. \tag{55b}$$

Provided

$$L+dM=1+K+\overline{d}L \tag{55c}$$

and there is a constant μ such that

$$1/\mu=K+dL-1=1/(L+\overline{d}M), \tag{56}$$

$\tilde{E}$ and $\tilde{B}$ satisfy

$$\tilde{E}_{,x}+d\tilde{E}_{,y}=-x\tilde{E}/\mu+y\tilde{B}, \tag{57a}$$

and

$$\tilde{B}_{,x}+\overline{d}\tilde{B}_{,y}=x\tilde{E}-\mu y\tilde{B} \tag{57b}$$

so that there is a function $F(x,y)$ such that

$$\tilde{E}=F_{,x}+\overline{d}F_{,y}, \tag{58a}$$

$$\tilde{B}=-\mu(F_{,x}+dF_{,y}), \tag{58b}$$

while F satisfies a manifestly elliptic second order complex differential equation

$$\left(\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+\overline{d}\frac{\partial}{\partial y}\right)F+\frac{x}{\mu}\left(\frac{\partial}{\partial x}+\overline{d}\frac{\partial}{\partial y}\right)F$$
$$+\mu y\left(\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}\right)F=0. \tag{59}$$

We see that (59) is in a form for which integral transforms should be possible as (59) is linear in x and y. We note that the transformation (54) would yield results similar to those of (58)–(59) for general linear forms on the right-hand side of (50).

We must also show that we can determine K, L, M, and μ. We find easily

$$d\mu^2-2\mu-1=0, \tag{60}$$

$$M=\mu/(\overline{d}-d), \tag{61a}$$

$$L=-d\mu/(\overline{d}-d), \tag{61b}$$

$$K=-1+d\overline{d}\mu/(\overline{d}-d). \tag{61c}$$

We select a particular root of (60) as soon as we examine the transformation (54) further. We define

$$\chi(x,y)=\tfrac{1}{2}(Kx^2+2Lxy+My^2) \tag{62}$$

and with (60), (61) we find easily

$$\mathrm{Re}\,\chi(x,y)=\frac{1}{4d_i|\mu|^2}\left(\mathrm{Im}\frac{1}{\mu}\right)(x+|\mu|^2y)^2. \tag{63}$$

Since

$$\frac{1}{\mu}=-1\pm\sqrt{1+d} \tag{64}$$

and $d_i<0$, we may choose this root to ensure that $\mathrm{Re}\,\chi<0$. The need for this choice in order to construct acceptable solutions of a wave propagation problem will become clear shortly. The other choice generates solutions of the differential equations, but uninteresting ones.

We seek an integral representation of solutions of (59) in the form

$$F=\int\int s\,ds\ \ dt\,G(s,t)\exp[is(tx+y)]. \tag{65}$$

$G(s,t)$ is assumed analytic in some domain and the paths of integration in the s and t planes are such that the integrand vanishes rapidly at the ends of the contour—or at infinity. Thus $G(s,t)$ must satisfy

$$-s^3(t+d)(t+\overline{d})G=\frac{1}{\mu}\frac{\partial}{\partial t}[s(t+\overline{d})G]+\mu(t+d)$$
$$\times\frac{\partial(s^2G)}{\partial s}-\mu s\frac{\partial}{\partial t}[t(t+d)G]. \tag{66}$$

With (60) the equation simplifies to

$$\mu s(t+d)\frac{\partial\log G}{\partial s}-\left(\mu t^2+2t-\frac{\overline{d}}{\mu}\right)\frac{\partial\log G}{\partial t}$$
$$=-s^2(t+\overline{d})(t+d)-2\left(1+\frac{1}{\mu}\right), \tag{67}$$

a simple linear first order partial differential equation. With standard methods we find

$$G=\left(\frac{t+\beta_-}{t+\beta_+}\right)^{\gamma}\exp\left[\frac{-s^2(t^2/2+td+\overline{d}/2)}{2(1+1/\mu)}\right]$$
$$\cdot\tilde{G}[s(t+\beta_+)^{(1-\gamma)/2}(t+\beta_-)^{(1+\gamma)/2}], \tag{68}$$

where $\tilde{G}(z)$ is an arbitrary analytic function and

$$\beta_+=\overline{d}\overline{\mu}/\mu, \quad \beta_-=-(\mu\overline{\mu})^{-1}, \tag{69}$$

$$\gamma=(1+1/\mu)/(1+1/\overline{\mu})=(1+d)/|1+d|. \tag{70}$$

When we insert (68) into (65) we have an integral representation of solutions of (59). We are interested in a class of solutions which may represent wave fields, and we are particularly interested in the behavior of solutions for large $|x|$ and $|y|$, so that we may match the solutions to the geometrical optics solutions outside the mode conversion region. We choose a particular form for $\tilde{G}$ that will later be seen to give solutions of interest; we set

$$\tilde{G}(z)=z^{\nu}\exp(-\lambda z^2/2), \tag{71}$$

where we will determine the constant ν shortly. With the representation of the Weber function[4]

$$D_{\nu}(z)\exp(-z^2/4)=2^{\nu}(2/\pi)^{1/2}e^{-i\nu\pi/2}\int_{i\sigma-\infty}^{i\sigma+\infty}t^{\nu}$$
$$\times\exp(-2t^2+2itz)dt \tag{72}$$

for $\sigma>0$ and ν arbitrary, we have an expression for $F(x,y)$, where we have dropped constant factors in front of the integral in (72)

$$F(x,y)=\int dt[(t+\beta_-)/(t+\beta_+)]^{\gamma(1+\nu/2)}[(t+\beta_+)$$
$$\times(t+\beta_-)]^{\nu/2}[\Delta_0(t)]^{-2-\nu}\cdot D_{\nu+1}$$
$$\times[(tx+y)/\Delta_0(t)]\cdot\exp[-(tx+y)^2/4\Delta_0^2(t)], \tag{73}$$

where

$$(\Delta_0)^2=[t^2/2+dt+d\overline{d}/2+\lambda(t+\beta_+)^{1-\gamma}(t+\beta_-)^{1+\gamma}]$$
$$\times(1+1/\mu)^{-1}. \tag{74}$$

In the next section we examine the expansion of (74) in some detail for large x and y, y/x fixed, but we can obtain substantial information on the properties of the solution if we only assume that an expansion for E and B is possible in which the electromagnetic fields are either exponentially small or oscillating—as one would expect. We start from the expansions of D_{ν} for large argument[4]

$$e^{-z^2/4}D_{\nu}(z)=e^{-z^2/2}z^{\nu}[1+O(1/z^2)], \quad |\arg z|<3\pi/4, \tag{75a}$$

$$e^{-z^2/4}D_{\nu}(z)=\left[e^{-z^2/2}z^{\nu}-\frac{(2\pi)^{1/2}}{\Gamma(-\nu)}e^{\nu\pi i}z^{-\nu-1}\right]$$
$$\times\left[1+O\left(\frac{1}{z^2}\right)\right], \quad +\pi/4<\arg z<5\pi/4, \tag{75b}$$

$$e^{-z^2/4}D_{\nu}(z)=\left[e^{-z^2/2}z^{\nu}-\frac{(2\pi)^{1/2}}{\Gamma(-\nu)}z^{-\nu-1}e^{-\nu\pi i}\right]$$
$$\times\left[1+O\left(\frac{1}{z^2}\right)\right], \quad -\pi/4>\arg z>\frac{-5\pi}{4}. \tag{75c}$$

We see from (75) that the entire function $e^{-z^2/4}D_{\nu}(z)$ always has one term proportional to $z^{\nu}e^{-z^2/2}$, but may also have an additional decreasing power series in z. With the choice $\mathrm{Re}\,\chi<0$, see (54) and (62), we see that the additional decreasing power series that appear in (75b), (75c) are exponentially small in x and y and never form a propagating wave. Thus, we only need be concerned with the first term in (75a), (75b), (75c) whatever the argument of z. It is relatively straightforward, but tedious to consider the solutions constructed with the value of μ for which $\mathrm{Re}\,\chi>0$. These solutions are not significantly different from the ones presented here, although different arguments and contours of integration are needed.

Although the asymptotic expansion of the Weber function exhibits a Stokes phenomenon in Eq. (75), the added terms in the asymptotic expansion of the solution have no effect on the asymptotic wave properties. This situation is completely different in a stratified medium. It is easy to verify that the solution of Eq. (13) of Ref. 2 for the wave amplitude in a stratified medium is $D_{\nu}(\lambda y')$, where $\nu=i(\ell^2+a^2)/2$ and $\lambda^2=-i/2$. As y' changes from positive to negative the asymptotic expansion of the Weber function jumps, just as it does in (95) here. In the stratified medium case the jump produces a reflected wave for one sign of y. The jump here has no significant effect on the solution. Thus, in principle, mode reflection possibilities are included in the solution (73), they just do not occur. Loosely speaking, there is a reflected wave, it just happens to be damped. Other choices for $\tilde{G}(z)$ instead of (71), for instance Weber functions or other exponentials of quadratic functions times a Weber function give the same results as those to be presented in the sequel. It appears necessary that the arbitrary function must be the exponential of a function of order two in z in order that propagating waves appear at all. Thus, these choices appear fairly general.

We expect that the solutions will be oscillitory in the sector $c_1<y/x<c_2$, x large, and exponentially small elsewhere. We require that the total energy flux be neither zero nor infinite. We would then require according to (53) that both B and E be asymptotically of order $(x^2+y^2)^{-1/4}$. Provided that only one value of t contribute to the asymptotic expansion of (73) we find from (75)

$$F\sim(x^2+y^2)^{\nu/2}\exp\left\{-\frac{[xt(y/x)+y]^2}{2\Delta_0^2(t)}\right\}, \tag{76}$$

and thus from (58)

$$(E,B)\sim(x^2+y^2)^{(\nu+1)/2}e^{iA}, \tag{77}$$

where A is some real phase. We choose

$$\nu=-3/2 \tag{78}$$

in (74) to obtain finite, nonzero, energy flux.

Before we address the question of what determines an incoming or an outgoing wave, we make one change of variable of integration which is less than obvious but which simplifies some of the subsequent analysis, we introduce the new variable of integration τ defined as

$$\mu\tau=-(t+\bar{d})/(t+d), \tag{79}$$

so that (73) and (74) with (78) becomes

$$F(x,y)=\int d\tau\left(\frac{\tau+\delta_-}{\tau+\delta_+}\right)^{\gamma/4}\frac{(\tau^2-2\tau-\bar{d})^{-3/4}}{\Delta^{1/2}(\tau)}\times\exp\left\{-\frac{[\mu\tau(dx-y)+\bar{d}x-y]^2}{4\Delta^2}\right\}\cdot D_{-1/2}\times\left[-\left(\frac{\mu\tau(dx-y)+(\bar{d}x-y)}{\Delta}\right)\right], \tag{80}$$

where

$$\Delta^2=(\bar{d}-d)\left\{\frac{\tau^2}{2}d\mu^2+d\mu\tau+\frac{\bar{d}}{2}-\Lambda\left(\frac{\tau+\delta_-}{\tau+\delta_+}\right)^{\gamma}\times\left(\frac{\tau^2}{2}-\tau-\frac{\bar{d}}{2}\right)\right\}\left(1+\frac{1}{\mu}\right)^{-1} \tag{81}$$

and

$$\Lambda=-2\lambda[(d-\beta_-)/(d-\beta_+)]^{\gamma}, \tag{82a}$$

$$\delta_+=-\bar{d}\bar{\mu}, \tag{82b}$$

$$\delta_-=1/\bar{\mu}. \tag{82c}$$

We find this form useful in the analysis of asymptotic expansions.

We can now return to the Poynting theorem to distinguish incoming and outgoing waves. We note first that

$$\bar{B}E=-\bar{\mu}\left[\left(\frac{\partial}{\partial x}+\bar{d}\frac{\partial}{\partial y}\right)\bar{F}\right]\left[\left(\frac{\partial}{\partial x}+\bar{d}\frac{\partial}{\partial y}\right)F\right]\exp\left[\frac{K+\bar{K}}{2}x^2+(L+\bar{L})(xy)+\frac{M+\bar{M}}{2}y^2\right]. \tag{83}$$

When we complete an asymptotic expansion we expect it to be of the form

$$F\sim\exp\left\{-\frac{[\mu\tau(dx-y)+\bar{d}x-y]^2}{2\Delta^2(\tau)}\right\}\cdot P(x,y,\tau), \tag{84}$$

where $\tau=\tau(y/x)$ and $P(x,y,\tau)$ is sum of power of x and y multiplied by function of τ. When we carry out the differentiations in (83) the leading order terms involve the derivatives of the exponential only and we have

$$\bar{B}E=-\bar{\mu}\frac{[\overline{\mu\tau}(\bar{d}x-y)+dx-y][\overline{\mu\tau}\bar{d}+d-\bar{d}(\overline{\mu\tau}+1)]}{\bar{\Delta}(\tau)^2}\cdot\frac{[\mu\tau(dx-y)+\bar{d}x-y][\mu\tau d+\bar{d}-\bar{d}(\mu\tau+1)]}{\Delta(\tau)^2}P\bar{P} \tag{85}$$

or

$$\bar{B}E=\tau\left|\frac{[\mu\tau(dx-y)+\bar{d}x-y]\mu P(d-\bar{d})}{\Delta^2(\tau)}\right|^2. \tag{86}$$

To obtain (85) and (86) we have assumed that the real parts of the exponents in (83) cancel exactly and that both B and E are purely oscillating. Since the Poynting vector in the direction of $\nabla\psi$ is $\mathrm{Im}(\bar{B}E)$ the waves are incoming or outgoing as $\mathrm{Im}\,\tau<0$ or $\mathrm{Im}\,\tau>0$. The characterization of incoming or outgoing is a convention. This simplicity of this calculation and the clear result illustrate the utility of the variable of integration τ as opposed to the original variable t.

IV. THE ASYMPTOTIC EXPANSION OF THE INTEGRAL

In the previous two sections we have developed a system of equations to describe wave propagation in a mode conversion region, obtained formal integral representations of solutions and extracted a few properties of the solution under the assumption that the solution found is indeed an acceptable solution for a wave propagation problem. In this section we justify the form of the integral representation by identifying the contour of integration and we also obtain an asymptotic representation of the solution for x large and y/x fixed. We proceed by assuming that the solution is acceptable, deriving properties of the presumed solution, and then *a posteriori* justifing the entire procedure.

We start from the integral representation (80), and for x and y large we may use the asymptotic expansions (75). As indicated before the terms on the right-hand side of (75) without the exponential factor are irrelevant since F is multiplied by a term exponentially small to obtain E or B. Thus we consider

$$F\sim\int d\tau\left(\frac{\tau+\delta_-}{\tau+\delta_+}\right)^{\gamma/4}\frac{(\tau^2-2\tau-\bar{d})^{-3/4}}{[\mu\tau(dx-y)+\bar{d}x-y]^{1/2}}\times\exp\left[-\frac{[\mu\tau(dx-y)+\bar{d}x-y]^2}{2\Delta^2}\right]. \tag{87}$$

We assume that x and y are large, but that

$$\zeta=y/x \tag{88}$$

is fixed. We are then interested in the properties of the exponent that appears in E or B:

$$\Phi(\zeta,\tau)=\frac{K}{2}+L\zeta+\frac{M\zeta^2}{2}-\frac{[\mu\tau(d-\zeta)+\bar{d}-\zeta]^2}{2\Delta^2(\tau)}. \tag{89}$$

As usual we are interested in stationary points of the integrand of (87): values of τ such that

$$\frac{\partial\Phi}{\partial\tau}=0. \tag{90}$$

This relation defines a function $\tau(\zeta)$, or $\zeta(\tau)$. We would like $\mathrm{Re}\,\Phi(\zeta,\tau(\zeta))=0$ for some interval in ζ, $\zeta_1<\zeta<\zeta_2$, and thus we would like to satisfy

$$\mathrm{Re}\frac{\partial\Phi}{\partial\zeta}=\mathrm{Re}\frac{d}{d\zeta}\Phi(\zeta,\tau(\zeta))=0. \tag{91}$$

In a straightforward manner we find that (90) holds only if

$$\mu\tau(d-\zeta)+\bar{d}-\zeta=0 \tag{92}$$

or

$$\Lambda\left(\frac{\tau+\delta_-}{\tau+\delta_+}\right)^{\gamma}=\frac{\tau\mu d+\zeta}{\tau/\mu-\zeta}. \tag{93}$$

We do not consider the possibility (92), as it also implies that $\mathrm{Re}\,\Phi<0$, corresponding to nonpropagating solutions. When (93) holds

$$\Delta^2=(\bar{d}-d)\tau[\tau\mu(d-\zeta)+\bar{d}-\zeta]/(\tau/\mu-\zeta) \tag{94}$$

and we obtain after a short calculation that

$$\mathrm{Re}\,\Phi=(1+\zeta/|\tau|^2)(-\tau_i(\zeta-d_r)/d_i+\tau_r) \tag{95}$$

and

$$\mathrm{Re}\frac{\partial\Phi}{\partial\zeta}=-[\tau_i/(2d_i)](1+\zeta/|\tau|^2). \tag{96}$$

Although there are degenerate situations we examine in detail the possibility

$$\zeta=-\tau\bar{\tau} \tag{97}$$

for which both $\mathrm{Re}\,\Phi$ and $\mathrm{Re}\,\Phi_{,\zeta}$ vanish. When we insert (97) into (93) we obtain

$$-\Lambda=\left(\frac{\tau+\delta_+}{\tau+\delta_-}\right)^{\gamma}\left(\frac{\bar{\tau}+\bar{\delta}_+}{\bar{\tau}+\bar{\delta}_-}\right). \tag{98}$$

One would usually expect that (98) would determine only one value of τ. However, the distinguishing characteristic of γ is, see (70)

$$\gamma\bar{\gamma}=1, \tag{99}$$

so that if we set

$$z=\frac{\tau+\delta_+}{\tau+\delta_-}=|z|e^{i\theta}, \tag{100}$$

then (98) becomes

$$\log(-\Lambda)=[1+i\gamma_i/(1+\gamma_r)][(1+\gamma_r)\log|z|-\gamma_i\theta]. \tag{101}$$

Thus, on the spiral in the z plane

$$|z|=r_0\exp[\theta\gamma_i/(1+\gamma_r)] \tag{102}$$

and with

$$\log(-\Lambda)=[(1+\gamma_r)+i\gamma_i]\log r_0, \tag{103}$$

(93) and (97) are satisfied and hence on this spiral $\mathrm{Re}\,\Phi=\mathrm{Re}\,\Phi_{,\zeta}=0$. If we set

$$1+\frac{1}{\mu}=\left|1+\frac{1}{\mu}\right|e^{i\omega}, \tag{104}$$

then

$$\gamma=e^{2i\omega}. \tag{105}$$

Further, from (70) $\mathrm{Im}\,\gamma=\mathrm{Im}\,d<0$, thus (63), (64) and the condition $\mathrm{Re}\,\chi<0$ imply

$$\pi>\omega>\pi/2. \tag{106}$$

Thus, the spiral in the z plane is

$$|z|=r_0\exp(\theta\tan\omega), \tag{107}$$

so that the spiral in the z plane has $|z|$ decreasing as θ increases.

We must now restate the results in the τ plane. It follows from (100) that

$$\tau=\frac{\delta_+-z\delta_-}{z-1}. \tag{108}$$

The spiral in the z plane maps into a curve with a spiral centered on $-\delta_+$ connected to a spiral centered on $-\delta_-$. The spirals wrap around each of the points in a counterclockwise sense. Since neither δ_+ nor δ_- is real and $\delta_++\delta_-=2$, it follows that one center is in the lower half-plane and the other is in the upper half-plane. The spiral is determined by the background state, i.e., the parameter ω, and the particular incident wave, the parameter r_0. Given a spiral, there corresponds to each point τ_0 on the spiral a value of ζ_0 given by (97), $\zeta_0=-\tau_0\bar{\tau}_0$. For this value of ζ, the exponent $\Phi(\zeta_0,\tau)$ is stationary as function of τ at $\tau=\tau_0$ and $\Phi(\zeta_0,\tau_0)$ is pure imaginary, as one would expect for a propagating wave.

Our intention is to start with a given spiral, which corresponds to a given equilibrium and incident wave. We pick a point on the spiral and corresponding value $\zeta_0=-\tau_0\bar{\tau}_0$. We define the path of integration for this value of the parameter ζ_0 as the curve through τ_0 on which τ varies and $\mathrm{Im}\,\Phi(\zeta_0,\tau_0)=\mathrm{Im}\,\Phi(\zeta_0,\tau)$, while $\mathrm{Re}\,\Phi(\zeta_0,\tau)$ is monotone decreasing as τ moves away from τ_0. We say more about this path of integration later. We can vary ζ_1 away from the value ζ_0, and find a corresponding value of τ_1 on the spiral such that $\zeta_1=-\tau_1\bar{\tau}_1$. The contour of integration in the τ plane for this value of ζ_1 is then also moved away from the contour for the value ζ_0. We may think of ζ_1 as determined by τ_1 on the spiral, instead of τ_1 by ζ_1. We can continue to vary τ_1 along any segment of the spiral provided that (i) $|\tau|$ is monotone on the segment and (ii) $\mathrm{Im}\,\tau$ is of definite sign, so that the associated wave is either purely incoming or purely outgoing. Since we are obtaining an asymptotic expansion of an integral that is unchanged under $x\to-x$, $y\to-y$, it is clear that we can only construct solutions with no reflected waves. Since δ_+ and δ_- are in the upper and lower half-plane, respectively, and also the spirals wrap around each of those points, it is clear that there are sections of the spiral on which $|\tau|$ is monotone and Im τ is of one sign.

We have determined the paths of integration in a range $|\tau_m|<|\tau|<|\tau_M|$, where τ varies on a section of a spiral on which $|\tau|$ is monotone and varies between $|\tau_m|$ and $|\tau_M|$. Thus, we have the starting points of the paths of integration for values of ζ, $-|\tau_m|^2>\zeta>-|\tau_M|^2$. We must determine the paths of integration for all ζ. To that end we examine what happens at the limits of a segment of the spiral on which $|\tau|$ is monotone. We distinguish two cases, (a) $\tau_0=0$ so that the spiral passes through the origin and (b) $\tau_0\neq 0$. We start with the case $\tau_0\neq 0$. We introduce the function which characterizes the stationary points and spirals, see (93)

$$Z(\tau,\zeta)=\gamma[\log(\tau+\delta_-)-\log(\tau+\delta_+)]+\log(\tau\mu d+\zeta) -\log(\tau/\mu-\zeta), \tag{109}$$

$$Z_{,\tau}=2\left(1+\frac{1}{\mu}\right)[\tau^2(\zeta-d)+\zeta(\zeta-\overline{d})][(\tau^2-2\tau-\overline{d})(\tau^2 d -2\tau\zeta-\zeta^2)]^{-1}, \tag{110a}$$

and

$$Z_{,\zeta}=\tau/(\tau^2 d-2\tau\zeta-\zeta^2). \tag{110b}$$

We may use (109), (110) to determine the behavior of the roots $\tau(\zeta)$, where $\tau(\zeta)$ is a stationary point of the phase. Except for $\tau=0,\ -\delta_+,-\delta_-$, the only possible points on the spiral at which $Z_{,\tau}$ or $Z_{,\zeta}$ can vanish or become infinite, are points for which $0=\tau^2(\zeta-d)+\zeta(\zeta-\overline{d})$, and on the spiral this condition is equivalent to $\mathrm{Im}[\tau(\zeta-d)]=0$. In the neighborhood of such a point ζ_0,τ_0, the root $\zeta(\tau)$ has the expansion

$$(\zeta-\zeta_0)=A(\tau-\tau_0)^2+B(\tau-\tau_0)^3+\cdots. \tag{111}$$

When we now expand the phase function $\Phi(\tau,\ \zeta)$ about this point we obtain

$$\Phi(\tau,\zeta)=\Phi(\tau_0,\zeta_0)+\tfrac{1}{2}\Phi_{,\tau\tau}(\tau_0,\tau_0)(\tau-\tau_0)^2+\Phi_{,\zeta_0} \times(\zeta-\zeta_0)+\Phi_{,\tau_0\zeta_0}(\tau-\tau_0)(\zeta-\zeta_0)+\tfrac{1}{2}\Phi_{,\zeta_0\zeta_0} \times(\zeta-\zeta_0)^2+\cdots, \tag{112}$$

where we have used the fact that τ_0,ζ_0 is a stationary point of the phase, so that $\Phi_{,\tau}(\tau_0,\zeta_0)=0$. If we differentiate this latter relation

$$\frac{d\tau}{d\zeta}\Phi_{,\tau\tau}+\Phi_{,\tau\zeta}(\tau_0,\zeta_0)=0 \tag{113}$$

so that at τ_0,ζ_0, see (111) $d\zeta/d\tau=0$, so that $\Phi_{,\tau\tau}(\tau_0,\zeta_0)=0$. Further, $\Phi(\tau_0,\zeta_0)$ is pure imaginary on the spiral, as is $\Phi_{,\zeta}(\tau_0,\zeta_0)$. Thus,

$$\Phi(\tau,\zeta)=\Phi(\tau_0,\zeta_0)+\Phi_{,\zeta_0}(\zeta-\zeta_0)+\Phi_{,\tau_0\zeta_0}(\tau-\tau_0) \times(\zeta-\zeta_0)+\tfrac{1}{6}\Phi_{,\tau_0\tau_0\tau_0}(\tau-\tau_0)^3+\cdots. \tag{114}$$

Thus, the phase is stationary in τ at the point

$$(\zeta-\zeta_0)=-(1/2)[\Phi_{,\tau_0\tau_0\tau_0}/\Phi_{,\tau_0\zeta_0}](\tau-\tau_0)^2+\cdots \tag{115}$$

compare (111). We know that along the curve, $\zeta-\zeta_0$ must be real. Thus, if the curve is given as $\tau=\tau_0+e^{i\alpha}s$, s real then

$$(\zeta-\zeta_0)=-(1/2)e^{2i\alpha}(\Phi_{,\tau_0\tau_0\tau_0}/\Phi_{,\tau_0\zeta_0})s^2. \tag{116}$$

If ζ is a maximum (minimum) at $s=0$, then $e^{2i\alpha}\Phi_{,\tau_0\tau_0\tau_0}/\Phi_{,\tau_0\zeta_0}$ is real and positive (negative). In the orthogonal directions s, pure imaginary, ζ continues to increase (decrease). Thus, when $|\tau|$ is stationary on the spiral two points of stationary phase on the spiral meet and correspond to the same values of ζ. Further, two pairs $(\zeta,\ \tau)$ corresponding to points of stationary phase move off the curve in directions perpendicular to the curve. When we examine $\mathrm{Re}\,\Phi(\tau,\zeta)$, given that both $\Phi(\tau_0,\zeta_0)$ and $\Phi_{,\zeta_0}(\tau_0,\zeta_0)$ are pure imaginary, we see that $\mathrm{Re}\,\Phi(\tau,\zeta)$ is cubic in $(\tau-\tau_0)$. Hence of the two stationary points off the curve one has $\mathrm{Re}\,\Phi<0$ and the other $\mathrm{Re}\,\Phi>0$. Thus, we see that we may follow the stationary points $\tau(\zeta)$ along the spiral as long as $|\tau|$ is monotone. When $|\tau|$ becomes stationary we may follow a stationary point off the spiral for which $\mathrm{Re}\,\Phi<0$, so that the wave is damped. When $\tau_0=0$, the argument is not significantly different; it is still true that $d\zeta/d\tau=0$ at $\zeta=\tau>0$ so that $\Phi_{,\tau\tau}(0,0)=0$. With this condition the remainder of the argument is exactly unchanged.

To summarize, we start with a section of the spiral on which $|\tau|$ is monotone, and where $|\tau|$ is stationary at the end points. We may associate a stationary point of the phase which each value of τ on the spiral, corresponding to $\zeta=-|\tau|^2$. For all these values $\mathrm{Re}\,\Phi=0$. For ζ outside this range there is a value of $\tau(\zeta)$ corresponding to a stationary point of the phase and for which $\mathrm{Re}\,\Phi<0$. We have established this property only for ζ close to its largest and smallest values. We know we can continue to find stationary points of the phase $\tau(\zeta)$, but we can show that $\mathrm{Re}\,\Phi$ remain negative and monotone decreasing. We can think of the curve of stationary points $\tau(\zeta)$. Near the spiral and for a stationary point $\mathrm{Re}\,\Phi_{,\zeta}$ is nonzero and can change sign only if $\tau_i=0$ or τ lies on the spiral, see (96). Now τ cannot lie on the spiral, as $\mathrm{Re}\,\Phi=0$ there. Thus we must examine the possibility that $\tau_i=0$ at a stationary point which is not on the spiral.

We ask first what is the condition that τ_i vanish? We see

$$-\tau+1=\left|1+\frac{1}{\mu}\right|e^{-i\omega}[1+2/(z-1)], \tag{117}$$

so that $\tau_i=0$ if and only if $|z|$ and θ satisfy

$$2\cot\omega\sin\theta+|z|-1/|z|=0. \tag{118}$$

If we define

$$w=\frac{\tau\mu d+\zeta}{-\tau/\mu+\zeta}, \tag{119}$$

then at a stationary point, see (93)

$$w=r_0^{(1+\gamma)}z^{-\gamma}, \tag{120}$$

where r_0 and γ define the phase state and wave properties of the system. We may rewrite (120) as

$$\zeta/\tau+1=\left|1+\frac{1}{\mu}\right|e^{i\omega}\left(1+\frac{2}{w-1}\right), \tag{121}$$

compare (117). Thus $\tau_i=0$ only if w satisfies

$$-2\cot\omega\sin\phi+|w|-1/|w|=0, \tag{122}$$

where

$$w=|w|e^{i\phi}. \tag{123}$$

Thus, a stationary point crosses the real axis in the τ plane only if w and z satisfy (118), (122), where they are related by (120). If, in addition, $\overline{w}=z$, then the point in question lies on the spiral generated by r_0 and ω. Imre has carried an extensive numerical search for stationary points on the real axis and not on the spiral.[5] For wide ranges of values of ω and r_0, no such points have been found. Thus, we take as given that $\mathrm{Re}\,\Phi_{,\zeta}\neq 0$ at stationary points not on the given spiral. Thus, we can continue the curve $\tau(\zeta)$ off the spiral and $\mathrm{Re}\,\Phi(\zeta,\tau(\zeta))$ is monotone decreasing off the spiral.

With a few more observations we can lay out the full argument concerning the path of integration for the integral representation and justify the process as completely as we can. There are two branch points in the integrands of the integral representation (73) or (80), β_+ or δ_+, and β_- or δ_- and we need to know the wave phase at these points. The wave phase is a property of the point and not the chosen representation (73) or (80). It is easy to verify using (73) for β_+ and (80) for δ_- that

$$\mathrm{Re}\,\Phi(\zeta_{,-})\leqslant 0 \tag{124}$$

for either branch point. Another issue is that ζ is monotone on the section of the spiral, and when $|\tau(\zeta)|$ ceases to be monotone one may continue to determine values of stationary points with $\mathrm{Re}\,\Phi(\tau(\zeta),\zeta)<0$ for ζ continuing to increase or decrease, depending on whether $\tau(\zeta))$ is a minimum or maximum. Finally, it is easy to see from the condition for stationary points (93), that stationary points exist as $\zeta\rightarrow\pm\infty$. Thus, we expect stationary points to exist for $-\infty<\zeta<\infty$ and we expect that curve $\tau=\tau(\zeta)$, $-\infty<\zeta<\infty$ to be a section of the spiral on which $\mathrm{Re}\,\Phi=0$ and then curves on which $\mathrm{Re}\,\Phi$ is monotone decreasing as ζ is monotone.

We must finally return to the determination of the contour of integration of the integral representation of the wave field. We start with a section of a spiral on which $|\tau|$ is monotone and we define $\zeta=-\tau\overline{\tau}$, or equivalently $\tau=\tau(\zeta)$. For these values of ζ, $\mathrm{Re}\,\Phi=0$. We may extend $\tau(\zeta)$ for values of $\zeta\rightarrow\pm\infty$ such that $\tau(\zeta)$ is a stationary point of $\Phi(\tau,\,\zeta)$ and $\mathrm{Re}\,\Phi(\tau,\zeta)$ is monotone decreasing as ζ increases or decreases. We next turn to the form (80). We see that the integrand has branch points connecting infinitely many Riemann surfaces at $\tau=-\delta_\pm$ and a branch point connecting two surfaces at $\Delta^2=0$. The integrand is absolutely intgrable at the branch points $\tau=-\delta_\pm$ and at infinity. The various functions are all analytic in the neighborhood of the point at infinity; thus we may consider ∞ as an ordinary point of an analytic function. We now start with a point on the spiral $\tau(\zeta)$, ζ. We then follow the contour on which $\mathrm{Re}\,\Phi(\tau,\zeta)$ is monotone decreasing and $\mathrm{Im}\,\Phi(\tau,\zeta)=\mathrm{Im}\,\Phi(\tau(\zeta),\zeta)$. This contour can only end at a singular point of the function $\Phi(\tau,\,\zeta)$. It is easy to verify that $\mathrm{Im}\,\Phi(\tau(\zeta),\zeta)$ does not depend on ζ alone, but also on $\arg\tau$. Thus, except for isolated special values of ζ and τ—and we return to this situation shortly—the contour of integration can end only at $\mathrm{Re}\,\Phi=-\infty$, which occurs on any of the Riemann surfaces only at $\Delta^2=0$. In this process we may extend the contour of integration through the point at infinity at which all functions are analytic. As we vary ζ, the stationary point $\tau(\zeta)$ varies and the contour of integration as well. At the stationary points $\mathrm{Im}\,\Phi(\tau(\zeta),\zeta)$ is monotone decreasing as $\tau(\zeta)$ moves off the spiral, since $\mathrm{Re}\,(d\Phi/d\zeta)$ at the stationary point cannot change sign. We cannot rule out the possibility that for some value, say ζ_1, the contour of integration passes through a branch point $\tau=-\delta_\pm$. For $\zeta>\zeta_1$ the path of integration will be on one side of the branch point, and on the other side for $\zeta<\zeta_1$. These two forms are related by an integral looping around the branch point. At the branch point $\mathrm{Re}\,\Phi\leqslant 0$ and $\mathrm{Re}\,\Phi$ decreases away from that point, thus the integral is asymptotically small in $|x|$, if such loop integrals do occur. This loop integral can also be made to vary continually with ζ as ζ continues to vary away from ζ_1. The loop integrals cannot loop around the same branch point on other Riemann surfaces as $\mathrm{Re}\,\Phi$ at the branch points is independent of the particular surface. Thus, the path of integration, defined initially for one value of $\tau(\zeta)$, ζ only can be extended to all ζ. Further, on this contour the asymptotic expansion shows that E and B oscillate for $|x|$ large and ζ corresponding to a sector in which $\tau(\zeta)$ lies on a spiral, $\mathrm{Im}\,\tau(\zeta)$ is of one sign and $|\tau(\zeta)|$ is monotone. All of these solutions, of which there are infinitely many, both incoming and outgoing waves, mode convert with no reflection.

We see that many of the solutions found approximate a ray, as in ray optics, for which $\zeta=y/x=-|\delta_\pm|^2$. Depending on the plasma and wave parameters, solutions may exist for which a larger range of ζ is possible. Perhaps the most important parameter in assessing the possible solutions is ω, see (104) where $\pi/2<\omega<\pi$ and we have

$$\left|1+\frac{1}{\mu}\right|^2 e^{2i\omega}=1+d=1+d_r+id_i, \tag{125}$$

and if we interpret the result in terms of the physical parameters

$$\left|1+\frac{1}{\mu}\right|^2 e^{2i\omega}=[\epsilon r b_p/(\epsilon N)]^2\kappa_{\parallel,\psi}+Y_{,\psi}$$
$$+i\ \ b_t^2[Qb|\nabla\psi/\epsilon|]^{-1}, \tag{126}$$

where Y is given by (40b). The largest angular ranges of the asymptotic expansions would occur for

$$Y_{,\psi}+[\epsilon r b_p/(\epsilon N)]^2\kappa_{\parallel,\psi}<0. \tag{127}$$

For all cases there will be the ray-like solutions of fully mode-converted solutions. The solutions with largest angular extent occur when the spiral passes through the origin and there is no other crossing of the axis $\tau_i=0$. For the origin to be on a spiral $z=\delta_+/\delta_-$, or $z=-(2\overline{\mu}+1)$, where (100) and (107) must hold. If ω is such that (118) has only one root when (107) holds, then the segment of the spiral starts at the origin and goes towards δ_+ or δ_-, generating solutions of large angular extent.

V. CONCLUSIONS

We have examined mode conversion of O to X waves at electron cyclotron frequencies in an axisymmetric geometry modelling the plasma with a cold plasma dielectic function. Such a treatment avoids the assumption of a stratified plasma. We have assumed the wavelengths are much shorter than the equilibrium plasma gradient lengths. Thus, we expect geometrical optics to apply except possibly near cut-off layers, absorption layers, or possibly mode conversion layers. We have concentrated on the possible general case in which the O mode cut-off layer and the X mode cut-off layer cross at an angle not as small as the square root of the small geometrical optics expansion parameter. This assumption, necessary in the analysis, precludes the possibility of matching the solution found here to the solutions of Refs. 1 and 2. We expand Maxwell's equations in the region near the intersection of the two cut-off surfaces. We find approximate full wave equations appropriate for this region. We expect these wave solutions to match to the geometrical optics solution outside the mode conversion region. The resulting wave equation depends nontrivially on two coordinates, the flux surface label, x, and the poloidal angle y.

After suitable transformations we have found two equivalent integral representations of the wave fields. We show how it is possible to obtain asymptotic expansions of solutions for $|x|,|y|$ large. We identify incoming and outgoing waves and also show that our waves have finite, and nonzero energy flux in or out of the system. Our method yields infinitely many solutions with pure mode conversion and no mode reflection, with either incoming or outgoing waves. Many of the solutions approximate a ray-like wave solution with y/x close to either of two specific values. Other solutions of the form $0 \geqslant k_1 \geqslant y/x \geqslant k_2$ may be possible, depending on the plasma parameters.

The results on transmission without mode conversion may be easily interpreted and understood in terms of the results from the stratified medium. The results of Ref. 2 in a stratified medium depend on two parameters, see Ref. 2, Eqs. (13) and (17), a, which represents the separation of the cut-off surfaces, and ℓ, which represents the magnitude of $k_\perp$, which should be exactly zero for total mode transmission with no mode conversion. The value of $k_\parallel$ is also automatically fixed for the ideal case, see Ref. 2, Eq. (6). Here, whatever $k_\parallel$ may be, one may look along the cut-off surface $\omega_p^2 = 1$ to find the point at which the correct value of $k_\parallel$ is found. The variability of quantities on a flux surface allows such a point to be found. Thus, in comparison with Ref. 2, the parameter a is zero. Next, in a stratified medium the value of $k_\perp$ is fixed and may not change. Since here there is variation in two dimensions, $k_\perp$ may be space dependent. The solution in two dimensions allows variables in $k_\perp$, and while there is no obvious reason that it must be so, the solutions found here show that $k_\perp$ adjusts itself to come into the resonance point at the value zero, so that the wave, even in the stratified medium case, would be fully transmitted with no mode conversion. In the stratified medium no relaxation of $k_\perp$ is possible, here it appears that relaxation occurs. We note that the value of $k_\perp$ does not even appear as a parameter in the basic system (50), it disappears as a result of the transformation following (38). Far from the critical curve, the solution of this system matches to any solutions of the dispersion relation, provided only that $k_\perp = O(\sqrt{\epsilon})$. This restriction limits the range in which the present results apply, but it is far more general than in the case of a stratified medium. Thus, it would seem that the results are consistent with Ref. 2, although they might not be predicted by Ref. 2.

The results in the present case, in which the waves depend nontrivially on two independent variables, are less restrictive on the equilibrium or incident wave parameters. The parameters may vary from the exact parametes for the ideal mode conversion conditions $\kappa_\parallel = 0 = 1 - \kappa_\wedge - \kappa_\perp - \kappa_\parallel^2$ by terms of order of the square root of the geometrical optics parameter without any significant drop in the conversion of the modes. This property depends entirely on the essential two-dimensional character of the waves.

We cannot be assured that we have all solutions to the wave propagation. There may be other solutions with mode reflection and reduced mode conversion with transmission. Other choices of the arbitrary analytic functions may generate such solutions. However, we have explored a number of other possibilities without having found any results different from those given here. As the discussion following Eq. (75) notes, the Stokes phenomenon present in the expansions (75) in the stratified medium case yield mode conversion with wave reflection. Here the "reflected wave" is irrelevant as it is exponentially damped. We suspect that for the present case of essentially two dimensional problems we have found all relevant solutions.

ACKNOWLEDGMENTS

The author was stimulated by the interesting work of Dr. H. Laqua to attempt this problem. The author has had many valuable discussions with Dr. K. Imre concerning this problem. His contributions are gratefully acknowledged.

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