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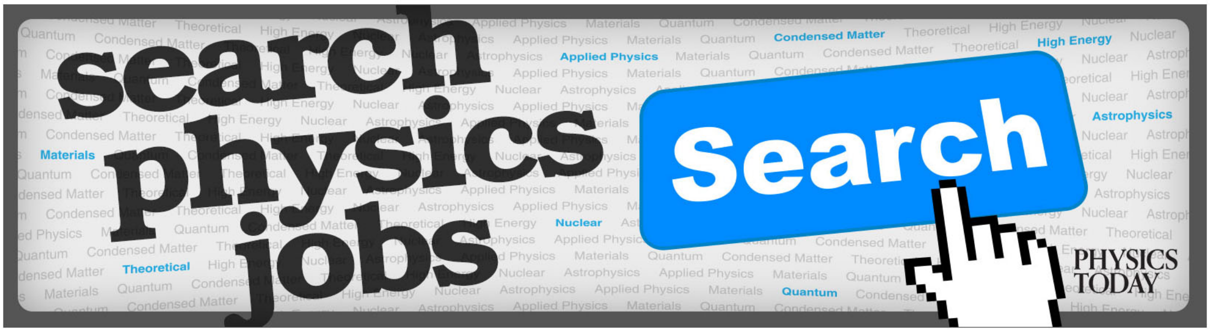

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# Exact Solution of a Family of Integral Equations of Anisotropic Scattering*

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It is shown that the solution of a certain Cauchy system provides the solution of a family of integral equations occurring in the theory of anisotropic scattering in a finite slab. Numerical experiments show that the Cauchy system is readily solved numerically, even in the case of very strong forward scattering.

## I. INTRODUCTION

A key role in the theory of anisotropic scattering in a finite slab is played by the integral equation

$$\begin{aligned} J(t, v, x, u) &= \tfrac{1}{2}c(v, -u)e^{-(x-t)/u} \\ &\quad + \int_t^x \int_0^1 c(v, -v')e^{-(y-t)/v'} J(y, -v', x, u)\frac{dv'}{v'}\,dy \\ &\quad + \int_0^t \int_0^1 c(v, +v')e^{-(t-y)/v'} J(y, v', x, u)\frac{dv'}{v'}\,dy, \\ &\qquad 0 \le t \le x,\ 0 \le u \le 1,\ -1 \le v \le +1. \qquad (1) \end{aligned}$$

It is assumed that the interval length $x$ is sufficiently small that Eq. (1) possesses a unique solution. For the physical background and analytical and computational approaches, see Refs. 1–3. In this paper it is shown that the family of integral equations (1) is equivalent to a Cauchy system which can readily be handled computationally. First, the theory is given,[4] and then results of numerical experiments are described.

## II. STATEMENT OF CAUCHY SYSTEM

We consider the Cauchy system for the auxiliary function $S$,

$$S(v, u, 0) = 0, \qquad (2)$$

where

$$\begin{aligned} S_x(v, u, x) &= -(u^{-1} + v^{-1})S(v, u, x) + 2c(v, -u) \\ &\quad + \int_0^1 c(v, v')S(v', u, x)\frac{dv'}{v'} \\ &\quad + 2\int_0^1 S(v, v', x)\frac{dv'}{v'}\Big(\tfrac{1}{2}c(-v', -u) \\ &\quad + \tfrac{1}{4}\int_0^1 \frac{dv''}{v''} c(-v', v'')S(v'', u, x)\Big), \\ &\qquad x \ge 0,\ 0 \le v, u \le 1. \qquad (3) \end{aligned}$$

We also consider the Cauchy system for the function $J$,

$$\begin{aligned} J_x(t, v, x, u) &= -u^{-1}J(t, v, x, u) + \int_0^1 \Big(c(-v', -u) \\ &\quad + \tfrac{1}{2}\int_0^1 \frac{dv''}{v''} c(-v', v'')S(v'', u, x)\Big)J(t, v, x, v')\frac{dv'}{v'}, \\ &\qquad x \ge t,\ 0 \le u \le 1,\ -1 \le v \le +1. \qquad (4) \end{aligned}$$

The initial condition on the function $J$ at $x = t$ is

$$J(t, v, t, u) = \tfrac{1}{2}c(v, -u) + \tfrac{1}{4}\int_0^1 \frac{dv'}{v'} c(v, v')S(v', u, t),$$
$$0 \le u \le 1,\ -1 \le v \le +1. \qquad (5)$$

It is assumed that this Cauchy system possesses a unique solution, at least for $x$ sufficiently small.

The aim is to show that the Cauchy system above and the integral equation (1) are equivalent.[4] These equations have been derived previously on physical grounds.[3]

## III. DERIVATION OF CAUCHY SYSTEM

We begin by differentiating both sides of Eq. (1) with respect to $x$, which yields

$$\begin{aligned} J_x(t, v, x, u) &= -\frac{1}{2u}c(v, -u)e^{-(x-t)/u} \\ &\quad + \int_0^1 \frac{dv'}{v'} c(v, -v')e^{-(x-t)/v'} J(x, -v', x, u) \\ &\quad + \int_t^x \int_0^1 \frac{dv'}{v'} c(v, -v')e^{-(y-t)/v'} J_x(y, -v', x, u)\,dy \\ &\quad + \int_0^t \int_0^1 \frac{dv'}{v'} c(v, +v')e^{-(t-y)/v'} J_x(y, v', x, u)\,dy. \qquad (6) \end{aligned}$$

By regarding Eq. (6) as an integral equation for the function $J_x$, keeping Eq. (1) in mind, and using the superposition principle for linear systems, we see that the solution of Eq. (6) for the function $J_x$ is

$$J_x(t, v, x, u) = -\frac{1}{u} J(t, v, x, u) + 2\int_0^1 J(x, -v', x, u) J(t, v, x, v') \frac{dv'}{v'}. \tag{7}$$

For $J(x, v, x, u)$ we may write, according to Eq. (1),

$$J(x, v, x, u) = \tfrac{1}{2} c(v, -u) + \int_0^x \int_0^1 \frac{dv'}{v'} c(v, v') e^{-(x-y)/v'} J(y, v', x, u)\, dy. \tag{8}$$

We introduce the function $S$ by means of the definition

$$S(v, u, x) = 4\int_0^x e^{-(x-y)/v} J(y, v, x, u)\, dy, \quad 0 \le v, u \le 1,\ x \ge 0. \tag{9}$$

Equation (8) may now be rewritten

$$J(x, v, x, u) = \tfrac{1}{2} c(v, -u) + \tfrac{1}{4}\int_0^1 \frac{dv'}{v'} c(v, v') S(v', u, x). \tag{10}$$

Next, we differentiate both sides of Eq. (9) with respect to $x$. This provides the equation

$$\begin{aligned} S_x(v, u, x) &= -\frac{1}{v} S(v, u, x) + 4J(x, v, x, u) \\ &\quad + 4\int_0^1 e^{-(x-y)/v}\left(-\frac{1}{u} J(y, v, x, u) \right. \\ &\quad \left. + 2\int_0^1 J(x, -v', x, u) J(y, v, x, v') \frac{dv'}{v'}\right) dy, \end{aligned} \tag{11}$$

where use has been made of Eq. (7) for the function $J_x$.

Equation (2) follows directly from the definition in Eq. (9), where the function $S$ is defined in terms of the function $J$. The differential equation for the function $S$, Eq. (3), comes directly from Eq. (11) by using Eqs. (8)–(10). The initial condition for the function $J$ at $x = t$ in Eq. (5) follows directly from Eq. (8) with the variable $x$ set equal to $t$. The differential equation (4) for the function $J$, valid for $x \ge t$, follows immediately from Eqs. (7) and (10).

## IV. VALIDATION OF CAUCHY SYSTEM

Now, we shall show that the solution of the Cauchy system (2)–(5) for the functions $S$ and $J$ provides a solution of the integral equation (1). Our first task is to demonstrate that

$$S(v, u, x) = 4\int_0^x e^{-(x-v)/v} J(y, v, x, u)\, dy, \quad x \ge 0,\ 0 \le v, u < 1. \tag{12}$$

Let the function $Q$ be defined by the equation

$$Q(v, u, x) = 4\int_0^x e^{-(x-y)/v} J(y, v, x, u)\, dy, \quad x \ge 0,\ 0 \le v, u \le 1. \tag{13}$$

It clearly satisfies the initial condition

$$Q(v, u, 0) = 0. \tag{14}$$

Furthermore, differentiation of both sides of Eq. (13) with respect to $x$ shows that

$$\begin{aligned} Q_x(v, u, x) &= -\frac{1}{v} Q(v, u, x) + 4J(x, v, x, u) \\ &\quad + 4\int_0^x e^{-(x-y)/v}\left[-\frac{1}{u} J(y, v, x, u) + \int_0^1 \left(c(-v', -u) \right.\right. \\ &\quad \left. + \tfrac{1}{2}\int_0^1 \frac{dv''}{v''} c(-v', v'') S(v'', u, x)\right) \\ &\quad \left. \times J(y, v, x, v') \frac{dv'}{v'}\right] dy. \end{aligned} \tag{15}$$

The last equation may be rewritten as

$$\begin{aligned} Q_x(v, u, x) &= -\left(\frac{1}{v} + \frac{1}{u}\right) Q(v, u, x) + 4J(x, v, x, u) \\ &\quad + 4\int_0^1 c(-v', -u) \frac{dv'}{v'} \int_0^x e^{-(x-y)/v} J(y, v, x, v')\, dy \\ &\quad + 2\int_0^1 \int_0^1 \frac{dv''}{v''} \frac{dv'}{v'} c(-v', v'') S(v'', u, x) \\ &\quad \times \int_0^1 e^{-(x-y)/v} J(y, v, x, v')\, dy \end{aligned} \tag{16}$$

or

$$\begin{aligned} Q_x(v, u, x) &= -\left(\frac{1}{u} + \frac{1}{u}\right) Q(v, u, x) + 2c(v, -u) \\ &\quad + \int_0^1 \frac{dv'}{v'} c(v, v') S(v', u, x) \\ &\quad + \int_0^1 c(-v', -u) \frac{dv'}{v'} Q(v, v', x) \\ &\quad + \tfrac{1}{2}\int_0^1 \int_0^1 \frac{dv''}{v''} \frac{dv'}{v'} c(-v', v'') S(v'', u, x) Q(v, v', x). \end{aligned} \tag{17}$$

Assuming that the linear Cauchy system for the function $Q$ in Eqs. (17) and (14) has a unique solution and keeping in mind the Cauchy system for the function $S$, we see that

$$Q = S \tag{18}$$

or

$$S(v, u, x) = 4\int_0^x e^{-(x-y)/v} J(y, v, x, u)\, dy, \quad x \geq 0,\ 0 \leq v,\ u \leq 1. \tag{19}$$

Next, we introduce the function $M$ by the equation

$$\begin{aligned} M(t, v, x, u) &= \tfrac{1}{2}c(v, -u)e^{-(x-t)/u} \\ &+ \int_t^x \int_0^1 \frac{dv'}{v'}\, c(v, -v')e^{-(y-t)/v'} J(y, -v', x, u)\, dy \\ &+ \int_0^t \int_0^1 \frac{dv'}{v'}\, c(v, v')e^{-(t-y)/v'} J(y, +v', x, u)\, dy, \\ &0 \leq t \leq x,\ 0 \leq u \leq 1,\ -1 \leq v \leq +1. \end{aligned} \tag{20}$$

At $x = t$ we find that

$$\begin{aligned} &M(t, v, t, u) \\ &= \tfrac{1}{2}c(v, -u) + \int_0^t \int_0^1 \frac{dv'}{v'}\, c(v, v')e^{-(t-y)/v'} J(y, v', t, u)\, dy \\ &= \tfrac{1}{2}c(v, -u) + \int_0^1 \frac{dv'}{v'}\, c(v, v') \int_0^t e^{-(t-y)/v'} J(y, v', t, u)\, dy \\ &= \tfrac{1}{2}c(v, -u) + \tfrac{1}{4}\int_0^1 \frac{dv'}{v'}\, c(v, v')Q(v', u, t) \\ &= \tfrac{1}{2}c(v, -u) + \tfrac{1}{4}\int_0^1 \frac{dv'}{v'}\, c(v, v')S(v', u, t) \\ &= J(t, v, t, u). \end{aligned} \tag{21}$$

In addition, differentiation of both sides of Eq. (20) with respect to $x$ shows that

$$\begin{aligned} &M_x(t, v, x, u) \\ &= -\frac{1}{2u}\, c(v, -u)e^{-(x-t)/u} \\ &+ \int_0^1 \frac{dv'}{v'}\, c(v, -v')e^{-(x-t)/v'} J(x, -v', x, u)\, dy \\ &+ \int_t^x \int_0^1 \frac{dv'}{v'}\, c(v, -v')e^{-(y-t)/v'}\, dy \\ &\times \left(-\frac{1}{u} J(y, -v', x, u) \right. \\ &\left. + 2\int_0^1 J(x, -v'', x, u)J(y, -v', x, v'')\frac{dv''}{v''}\right) \\ &+ \int_0^t \int_0^1 \frac{dv'}{v'}\, c(v, v')e^{-(t-y)/v'}\, dy \\ &\times \left(-\frac{1}{u} J(y, v', x, u) \right. \\ &\left. + 2\int_0^1 J(x, -v'', x, u)J(y, v', x, v'')\frac{dv''}{v''}\right). \end{aligned} \tag{22}$$

By collecting terms, this equation becomes

$$\begin{aligned} &M_x(t, v, x, u) \\ &\quad = -\frac{1}{u} M(t, v, x, u) \\ &\quad + 2\int_0^1 \frac{dv'}{v'} J(x, -v', x, u)M(t, v, x, v'), \quad x \geq t. \end{aligned} \tag{23}$$

From our uniqueness assumption, it follows that

$$M(t, v, x, u) \equiv J(t, v, x, u), \quad x \geq t, \tag{24}$$

which is precisely the family of integral equations (1).

## V. NUMERICAL RESULTS

The general scheme for solving the Cauchy system for the functions $S$ and $J$ numerically is to approximate the integrals occurring by the use of Gaussian quadrature formulas, which transforms the differential–integral equations into a system of ordinary differential equations.[5,6]

In the event that

$$c(v, u) = \text{const} = \tfrac{1}{2}\lambda, \tag{25}$$

the integral equation (1) becomes

$$J(t, x, u) = \tfrac{1}{4}\lambda e^{-(x-t)/u} + \tfrac{1}{2}\lambda \int_0^x E_1(|t-y|)J(y, x, u)\, dy, \quad 0 \leq t \leq x,\ 0 \leq u \leq 1, \tag{26}$$

where, as usual,

$$E_1(|t-y|) = \int_0^1 e^{-|t-y|/z}\frac{dz}{z}. \tag{27}$$

In addition, the Cauchy system for the function $S$ becomes

$$\begin{aligned} S_x(v, u, x) = &-\left(\frac{1}{u} + \frac{1}{v}\right)S(v, u, x) \\ &+ \tfrac{1}{2}\lambda \int_0^1 S(v', u, x)\frac{dv'}{v'} \\ &+ \tfrac{1}{2}\lambda \int_0^1 S(v, v', x)\frac{dv'}{v'} \\ &+ \tfrac{1}{4}\lambda \int_0^1\int_0^1 S(v, v', x)S(v'', u, x)\frac{dv''}{v''}\frac{dv'}{v'}. \end{aligned} \tag{28}$$

This is a well-known result for the case of isotropic scattering. Numerical results for $S$ and $J$, based on the numerical solution of the Cauchy system, using the method of lines, are available in Refs. 5 and 7.

To test the effects of stronger and stronger forward scattering, we selected a phase function of the form[3]

$$p(\cos\theta) = k(b - \cos\theta)^{-1}, \quad b > 1, \tag{29}$$

which corresponds to

$$c(v, u) = (\lambda k)2[(b - uv)^2 - (1 - u^2)(1 - v^2)]^{-\frac{1}{2}}. \tag{30}$$

The constant $k$ fulfills the normalization condition

$$k = 2[\log(b + 1)/(b - 1)]^{-1}. \tag{31}$$

The closer $b$ is to unity, the stronger the forward scattering is.

We found that, with $b = 1.1$ and $\lambda = 1.0$, use of a Gaussian quadrature formula of order 7 and an integration step size of 0.01 in an Adams–Moulton fourth-order integration routine resulted in about five accurate digits in the evaluation of the function $S$ for $0 \le x \le 1$, $0 \le v$, $u \le 1$. An IBM 7044 computer was used. When $b = 1.01$, about three accurate figures were obtained. When $b = 1.001$, we used a Gaussian quadrature formula of order 15 and a step size of 0.001 and obtained about one accurate figure. The front-to-back ratio passes from about 20 to 2000 in these cases. This shows the complete feasibility of the method from the numerical viewpoint, even for strongly peaked phase functions.

## VI. DISCUSSION

The method presented, when augmented by quasilinearization,[8] can be used to solve inverse problems. In this class of problems, we measure diffusely reflected radiation and wish to infer local scattering properties of the medium.

Extensions to cases involving internal sources of radiation and reflecting surfaces are readily made.

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# Groups of Curvature Collineations in Riemannian Space–Times Which Admit Fields of Parallel Vectors

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By definition, a Riemannian space $V_n$ admits a symmetry called a curvature collineation (CC) if the Lie derivative with respect to some vector $\xi^i$ of the Riemann curvature tensor vanishes. It is shown that if a $V_n$ admits a parallel vector field, then it will admit groups of CC's. It follows that every space–time with an expansion-free, shear-free, rotation-free, geodesic congruence admits groups of CC's, and hence gravitational pp waves admit such groups of symmetries.

## 1. INTRODUCTION

A Riemannian space $V_n$ with curvature tensor[1] $R^i_{jkm}$ is said to admit a symmetry called a curvature collineation[2] (CC) if $\pounds_\xi R^i_{jkm} = 0$ for some vector field[3] $\xi^i$. In this paper, we show that any $V_n$ which admits a field of parallel vectors also admits groups of CC's defined by $\xi^i$ related to the parallel field vectors in a simple manner. It follows that every space–time which admits a shear-free, expansion-free, rotation-free, geodesic congruence admits groups of CC's. As a consequence, all plane-fronted pure gravitational waves ($R_{ij} = 0$) with parallel ray vectors (pp waves) admit groups of CC's.

It can be shown (see Ref. 2) that a necessary and