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Jürgen Tolksdorf

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Real $\mathbb{Z}_2$-bigradings, Majorana modules, and the standard model action

Jürgen Tolksdorf[a)]
Max-Planck Institute for Mathematics in the Sciences, D-04103 Leipzig, Germany

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The action functional of the standard model of particle physics is intimately related to a specific class of first order differential operators called Dirac operators of Pauli type ("Pauli–Dirac operators"). The aim of this article is to carefully analyze the geometrical structure of this class of Dirac operators on the basis of real Dirac operators of simple type. On the basis of simple type Dirac operators, it is shown how the standard model action (STM action) may be viewed as generalizing the Einstein–Hilbert action in a similar way that the Einstein–Hilbert action is generalized by a cosmological constant. Furthermore, we demonstrate how the geometrical scheme presented allows to naturally incorporate also Majorana mass terms within the standard model. For reasons of consistency, these Majorana mass terms are shown to dynamically contribute to the Einstein–Hilbert action by a "true" cosmological constant. Due to its specific form, this cosmological constant can be very small. Nonetheless, this cosmological constant may provide a significant contribution to dark matter/energy. In the geometrical description presented, this possibility arises from a subtle interplay between Dirac and Majorana masses.
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I. INTRODUCTION

The dynamical description of fermions and bosons is usually based on different geometrical schemes. The fermionic actions are always defined in terms of Dirac-type operators. In contrast, the gravitational and the Yang–Mills functionals are defined in terms of the respective curvatures associated with connections. Accordingly, the following "ambiguity" in the definition of the fermionic action is not taken into account. Let $\psi \in \mathrm{Sec}(M,\mathcal{E})$ be a section of the Hermitian Clifford module

$$(\mathcal{E},\gamma_{\mathcal{E}}) \twoheadrightarrow (M,g_{\mathrm{M}}) \tag{1}$$

over an oriented (semi)Riemannian manifold of even dimension and arbitrary signature. Also, let $\not{D}_{\mathcal{E}}$ be a Dirac (type) operator (see below) that acts on $\mathrm{Sec}(M,\mathcal{E})$. The fermionic action is defined in terms of the smooth function: $\langle \psi, \not{D}_{\mathcal{E}}\psi\rangle_{\mathcal{E}}$, with $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ being the Hermitian form on $\mathcal{E}$. Clearly, nothing changes when a (tricky) null is added, i.e.,

$$\langle \psi, \not{D}_{\mathcal{E}}\psi\rangle_{\mathcal{E}} \equiv \langle \psi, \not{D}_{\mathcal{E}}\psi\rangle_{\mathcal{E}} + \langle \psi, \Phi_{\mathcal{E}}\psi\rangle_{\mathcal{E}} - \langle \psi, \Phi_{\mathcal{E}}\psi\rangle_{\mathcal{E}} =: \left\langle \begin{pmatrix}\psi\\ \psi\end{pmatrix}, \begin{pmatrix}\not{D}_{\mathcal{E}} & -\Phi_{\mathcal{E}}\\ \Phi_{\mathcal{E}} & \not{D}_{\mathcal{E}}\end{pmatrix}\begin{pmatrix}\psi\\ \psi\end{pmatrix}\right\rangle_{{}^2\mathcal{E}}. \tag{2}$$

Here, $\Phi_{\mathcal{E}} \in \mathrm{Sec}(M,\mathrm{End}(\mathcal{E}))$ denotes an arbitrary zero order operator and ${}^2\mathcal{E} := \mathcal{E}\oplus\mathcal{E}$ denotes the "doubling" of $\mathcal{E}$ with an appropriately induced Hermitian form and Clifford structure.

This apparently trivial observation may become meaningful, actually, if the bosonic action is also defined in terms of Dirac operators. In fact, it has been shown that both the fermionic part and the bosonic part of the standard model (STM) action—the latter also including the Einstein–

[a)]Electronic mail: juergen.tolksdorf@mis.mpg.de.

Hilbert functional—can be geometrically described in terms of a single Dirac operator [cf., for instance, Refs. 1 and 34 with respect to the combined Einstein–Hilbert–Yang–Mills and the Einstein–Hilbert–Yang–Mills–Higgs (EHYMH) action in terms of the noncommutative residue],

$$\not{P}_{\mathrm{D}} = \begin{pmatrix} i\not{\partial}_{\mathrm{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}} & -\not{F}_{\mathrm{D}} \\ \not{F}_{\mathrm{D}} & i\not{\partial}_{\mathrm{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}} \end{pmatrix} \equiv i\not{\partial}_{\mathrm{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}} + \mathcal{I}_{\mathcal{E}} \circ \not{F}_{\mathrm{D}}. \tag{3}$$

Here, respectively, the Dirac operator

$$\not{D}_{\mathcal{E}} \equiv i\not{\partial}_{\mathrm{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}} \tag{4}$$

belongs to the distinguished class of Dirac operators of simple type on the Clifford module (1) and $\not{F}_{\mathrm{D}}$ is the "quantized" relative curvature of (4). The details of these and the following notions will be summarized in Sec. II.

The specific class of Dirac operators (4) will play a crucial role in what follows (see also, for example, Refs. 31 and 6 for the role of simple type Dirac operators in the case of the family index theorem and Ref. 12 of noncommutative geometry). When evaluated with respect to (3), the "total Dirac action" (see below)

$$\mathcal{I}_{\mathrm{D,tot}} := \int_M (\langle \Psi, \not{P}_{\mathrm{D}} \Psi \rangle_{^2\mathcal{E}} + \mathrm{tr}_\gamma(\mathrm{curv}(\not{P}_{\mathrm{D}}) - \varepsilon ev_g(\omega_{\mathrm{D}}^2))) d\mathrm{vol}_{\mathrm{M}} \tag{5}$$

decomposes into the various parts of the STM action, including gravity described in terms of the Einstein–Hilbert functional. In particular, the fermionic part reduces to the usual Dirac–Yukawa action,

$$\int_M \langle \Psi, \not{P}_{\mathrm{D}} \Psi \rangle_{^2\mathcal{E}} d\mathrm{vol}_{\mathrm{M}} = \int_M \langle \psi, (i\not{\partial}_{\mathrm{A}} + \phi_{\mathcal{E}})\psi \rangle_{\mathcal{E}} d\mathrm{vol}_{\mathrm{M}}, \tag{6}$$

provided the sections $\Psi \in \mathrm{Sec}(M, {}^2\mathcal{E})$ on the doubled Clifford module

$$({}^2\mathcal{E} \equiv \mathcal{E} \oplus \mathcal{E},\ \ \tau_{^2\mathcal{E}} \equiv \tau_{\mathcal{E}} \ominus \tau_{\mathcal{E}},\ \ \gamma_{^2\mathcal{E}} \equiv \gamma_{\mathcal{E}} \oplus \gamma_{\mathcal{E}}) \twoheadrightarrow (M, g_{\mathrm{M}}) \tag{7}$$

are restricted to "diagonal sections" $\Psi = (\psi, \psi)$ and the sections $\psi \in \mathrm{Sec}(M, \mathcal{E})$ are restricted, furthermore, to the "physical subbundle" $\mathcal{E}_{\mathrm{phys}} \hookrightarrow \mathcal{E} \twoheadrightarrow M$ of the underlying Clifford module (cf. Ref. 37 and the corresponding references therein).

The specific form of the Dirac operator (3), acting on the sections of the doubled Clifford module, parallels the first order differential operator

$$i\not{\partial}_{\mathrm{A}} - m - i\not{F}_{\mathrm{A}}, \tag{8}$$

with F_{A} being the electromagnetic field strength that was introduced to account for the anomalous magnetic moment of the proton at a time when it was not yet clear that the proton is a composite of quarks but considered as "elementary" (see, for example, Chap. 2-2-3 in Ref. 22). However, when the quarks entered the stage of particle physics, the *Pauli term* $i\not{F}_{\mathrm{A}}$ became superfluous. Moreover, the additional fermionic interaction caused by the Pauli term rendered the quantum field theory based on (8) nonrenormalizable.

It is a remarkable feature of "Dirac-type gauge theories" that the complete STM action (including gravity) can be geometrically described in terms of the "Pauli-type Dirac operator" (3). It has been shown that this description of the STM allows us to make a prediction for the value of the mass of the Higgs boson, which is consistent with all the otherwise known data from the STM. In other words, the geometrical description of the STM based on the geometry of $\not{P}_{\mathrm{D}}$ renders the STM even more predictive than it is the case with respect to its usual description [cf. Ref. 36, where one can also find a brief comparison to similar results presented in Ref. 9 (see also Ref. 15)]. For this matter, it seems worth investigating more closely the specific form of Pauli-type

Dirac operators and the restrictions made with respect to the fermionic sector that guarantee the Pauli-type term $\mathcal{I}_{\mathcal{E}}\circ\not{F}_{\mathrm{D}}$ to only contribute to the bosonic part of the total Dirac action (5).

In this paper, we carefully discuss the fact that in the bosonic part of (5), only curvature terms enter, whereas the fermionic part is determined by connections only. This subtle interplay between the fermionic and the bosonic part of the total Dirac action permits to geometrically regard the Yang–Mills action as a "covariant generalization" of the Einstein–Hilbert action and the STM action as a natural generalization of the Einstein–Hilbert action with cosmological constant. Moreover, the geometrical analysis of the operator (3) permits us to also naturally include the notion of Majorana masses within the scheme of Dirac-type gauge theories. It will be shown that the thus described Majorana masses dynamically contribute to the bosonic part of (5) in the form of Einstein's "biggest blunder."

Some of the features presented seem close to the geometrical description of the STM in terms of Connes' noncommutative geometry (cf., for example, Refs. 13, 11, 9, and 15). However, the geometrical setup presented is different in various respects. For example, the relation between Dirac operators and connections is based on the canonical first order decomposition of any Dirac (type) operator (cf. Sec. II). As a consequence, the Higgs boson is intimately tied to gravity in the setup presented. Indeed, the Higgs boson is shown to generalize the Yang–Mills connection via the metric. Furthermore, the bosonic part of the total Dirac action (5) is based on the canonical second order decomposition of any Dirac (type) operator. This, indeed, provides a canonical generalization of the Einstein–Hilbert action with cosmological constant (cf. Sec. III). These basic features of Dirac-type gauge theories will be the starting point of everything that follows.

This paper is organized as follows. Section II provides a summary of some of the basic notions already used in Sec. I. Also, some motivation for the ensuing constructions are presented. In Sec. III, we present the geometrical picture that underlies Dirac-type gauge theories. In particular, we discuss the Einstein–Hilbert action from the point of view of Dirac operators. In Sec. IV, we discuss Pauli-type Dirac operators in view of "real, $\mathbb{Z}_2$-bigraded Clifford modules" ("real Clifford modules," for short, see the work of Atiyah *et al.*,[2] which may serve as a kind of a standard reference). We present some examples of particular interest. In Sec. V, we discuss the geometrical description of Majorana masses within Dirac-type gauge theories. In particular, we discuss a generalization of the STM action when Majorana masses are taken into account. Section VI is devoted to a discussion of the STM action in terms of real Dirac operators of simple type. This will provide a new geometrical picture of the STM action and how the latter is related to the Einstein–Hilbert functional of general relativity. Finally, Sec. VII summarizes the main conclusions. Before we get started, however, it might be worth presenting a brief summary of the main results obtained.

The presented geometrical discussion of the operators (3), defining the bosonic part of the total Dirac action (5), is based on a careful analysis of the geometrical background of the Dirac equation and the Majorana equation,

$$i\not{\partial}\chi = m_{\mathrm{D}}\chi \Leftrightarrow \begin{cases} i\not{\partial}\chi_{\mathrm{R}} = m_{\mathrm{D}}\chi_{\mathrm{L}} \\ i\not{\partial}\chi_{\mathrm{L}} = m_{\mathrm{D}}\chi_{\mathrm{R}}, \end{cases} \tag{9}$$

$$i\not{\partial}\chi = m_{\mathrm{M}}\chi^{\mathrm{cc}} \Leftrightarrow \begin{cases} i\not{\partial}\chi_{\mathrm{R}} = m_{\mathrm{M}}\chi_{\mathrm{R}}^{\mathrm{cc}} \\ i\not{\partial}\chi_{\mathrm{L}} = m_{\mathrm{M}}\chi_{\mathrm{L}}^{\mathrm{cc}}. \end{cases} \tag{10}$$

Here, χ_{R} and χ_{L} are the "chiral" eigensections, respectively, m_{D} is the "Dirac mass," m_{M} is the "Majorana mass," and "cc" has the physical meaning of "charge conjugation." It will be shown here how a specific interplay between the two basic $\mathbb{Z}_2$-gradings, realized in nature by chirality and charge conjugation, allows us to overcome the issue of fermion doubling. The latter is usually encountered in the description of the fermionic action in terms of simple type Dirac operators. Furthermore, the interplay between parity and charge conjugation will also give the Pauli–Dirac operators their geometrical meaning. The geometrical background of Pauli–Dirac operators in terms of real Clifford modules has been partially discussed before in Ref. 35. There, however, only the reduced Dirac action,

$$\mathcal{I}_{\mathrm{D,red}} := \int_M \mathrm{tr}_\gamma \, \mathrm{curv}(\not{D}) d\mathrm{vol}_{\mathrm{M}} \tag{11}$$

was used. Also, the requirements imposed on "particle-antiparticle modules" (cf. Definition 3 in Ref. 35) turn out to be too restrictive and do not allow one to geometrically describe, for example, Dirac's first order differential operator $i\not{\partial} - m_{\mathrm{D}}$ in terms of simple type Dirac operators. Thus, it does not account for the issue of the fermion doubling already mentioned. This drawback is remedied in this work in terms of Dirac modules associated with Majorana modules (cf. Sec. IV). Also, we take the opportunity to generalize formulas (110) and (113) of Lemma 1 in Ref. 35, which hold true only in the special case of $\Phi \in \mathrm{Sec}(M, \mathrm{End}_\gamma(\mathcal{E}))$ [cf. the corresponding formulas (110) and (111) of Lemma 4.1].

In this work, emphasis is put on *real Dirac operators of simple type*, which turns out to yield an appropriate geometrical description of both the fermionic and the bosonic action of the STM. Indeed, on the basis of real Dirac operators of simple type, the STM action will be shown to be described by the Einstein–Hilbert action with a "cosmological constant" term (cf. Sec. VI),

$$\mathcal{I}_{\mathrm{EHYMH}} = \int_M \mathrm{tr}_\gamma [\mathrm{curv}(\not{\partial}_{\mathcal{A}}) + \Phi_{\mathrm{YMH}}^2] d\mathrm{vol}_{\mathrm{M}}. \tag{12}$$

This may be regarded as a generalization of Lovelock's theorem (cf. Ref. 28 and Sec. III). In contrast to this theorem, however,

$$\Lambda_{\mathrm{YMH}} := \mathrm{tr}_\gamma(\Phi_{\mathrm{YMH}}^2) \tag{13}$$

also depends on the metric and, in fact, is shown to coincide with the (Hodge dual of the) STM Lagrangian density $\mathcal{L}_{\mathrm{YMH}} \in \Omega^n(M)$ plus a "true" cosmological constant term that is determined by Dirac and Majorana masses of an otherwise noninteracting species of particles, collectively called "cosmological neutrinos" (cf. Sec. IV),

$$\Lambda_{\mathrm{YMH}} = * \mathcal{L}_{\mathrm{YMH}} - \Lambda_{\mathrm{DM},\nu},$$

$$\Lambda_{\mathrm{DM},\nu} \equiv a' \, \mathrm{tr}_{\mathcal{W}_\nu} m_{\mathrm{D},\nu}^4 - b' \, \mathrm{tr}_{\mathcal{W}_\nu} m_{\mathrm{D},\nu}^2 + a' \, \mathrm{tr}_{\mathcal{W}_\nu} m_{\mathrm{M},\nu}^4 - b' \, \mathrm{tr}_{\mathcal{W}_\nu} m_{\mathrm{M},\nu}^2 - 2a' \, \mathrm{tr}_{\mathcal{W}_\nu} (m_{\mathrm{D},\nu} \circ m_{\mathrm{M},\nu})^2. \tag{14}$$

Here, $a', b' > 0$ are numerical constants that are determined by the dimension of the underlying (space-time) manifold (cf. Secs. IV and V). Though not discussed in detail in this work, the point to be emphasized here is that, due to its specific form, the cosmological constant $\Lambda_{\mathrm{DM},\nu}$ can be arbitrarily small, though, for example, the contribution of the Majorana masses $m_{\mathrm{M},\nu}$ to the "dark matter/energy" of the universe can be very high.

II. PRELIMINARIES

For the sake of self-consistency and for the convenience of the reader, we summarize some facts about general Clifford modules, although later on we shall be mainly concerned with the case of twisted Grassmann bundles. Nonetheless, it seems worth starting with the general case to clarify the general scheme. Afterward, we shall introduce some facts concerning the case of twisted Grassmann bundles (subbundles thereof).

The bundle of Grassmann and Clifford algebras with respect to (M, g_{M}) is supposed to be generated by the cotangent bundle of M. In what follows, however, we shall be mainly concerned with their complexifications. In particular, all Clifford modules are considered as complex vector bundles.

A. General Clifford modules

To get started, let $(\mathcal{E},\gamma_{\mathcal{E}}) \twoheadrightarrow (M,g_{\mathrm{M}})$ be a general bundle of Clifford modules over a smooth, orientable (semi)Riemannian manifold of even dimension $n=p+q$ and signature $s=p-q$. Let $\mathrm{Cl}_{\mathrm{M}} \twoheadrightarrow M$ be the algebra bundle of (complexified) Clifford algebras with respect to the (semi)metric g_{M} that is generated by the cotangent bundle $T^*M \twoheadrightarrow M$. The mapping $T^*M \stackrel{\gamma_{\mathcal{E}}}{\longrightarrow} \mathrm{End}(\mathcal{E})$ denotes a Clifford mapping,

$$\gamma_{\mathcal{E}}(\alpha)^2 = \varepsilon g_{\mathrm{M}}(\alpha,\alpha)\mathrm{id}_{\mathcal{E}}, \tag{15}$$

for all $\alpha \in T^*M$. Here, the use of $\varepsilon \in \{\pm 1\}$ allows one to treat both signatures $\pm s$ simultaneously. Especially, it takes into account that both signatures are physically indistinguishable.

By abuse of notation, Clifford mappings also denote the induced representations of the Clifford bundle on the corresponding algebra bundles of endomorphisms $\mathrm{End}(\mathcal{E}) \twoheadrightarrow M$. Also, we do not distinguish between (semi)metrics of signature $s=p-q$ on M and sections of the "Einstein–Hilbert bundle"

$$\mathcal{E}_{\mathrm{EH}} \coloneqq \mathcal{F}_{\mathrm{M}} \times_{\mathrm{GL(n)}} \mathrm{GL(n)/SO(p,q)} \twoheadrightarrow M, \tag{16}$$

which is associated with the frame bundle $\mathcal{F}_{\mathrm{M}} \twoheadrightarrow M$ of M. Finally, the scalar products on the tangent and the cotangent bundle of M are also denoted by g_{M}.

On every Clifford module, there exists a canonical one-form $\Theta \in \Omega^1(M,\mathrm{End}(\mathcal{E}))$, which locally reads as

$$\Theta \stackrel{\text{loc.}}{=} \frac{\varepsilon}{n} e^k \otimes \gamma_{\mathcal{E}}(e_k^\flat). \tag{17}$$

Here $(e_1,\ldots,e_n)$ is a local (orthonormal) basis of $TM \twoheadrightarrow M$ and $(e^1,\ldots,e^n)$ its dual. The mappings, ${}^{\flat/\sharp}: TM \rightleftarrows T^*M$, are the "musical" isomorphisms induced by g_{M}.

The canonical one-form (17) is thus the (normalized) soldering form of the frame bundle of M lifted to $\mathcal{E} \twoheadrightarrow M$. It also plays a basic role in the definition of the twistor operator in conformal geometry (cf. p. 164, Lecture 6 in Ref. 7). (The author would like to thank M. Schneider for pointing him out this relation.) Indeed, the canonical one-form provides a right inverse of the restriction of the canonical mapping,

$$\delta_\gamma : \Omega^*(M,\mathrm{End}(\mathcal{E})) \to \mathrm{Sec}(M,\mathrm{End}(\mathcal{E})),$$

$$\omega \otimes \chi \mapsto \gamma_{\mathcal{E}}(\sigma_{\mathrm{Ch}}^{-1}(\omega)) \circ \chi \tag{18}$$

to one-forms via

$$\mathrm{ext}_\Theta : \Omega(M,\mathrm{End}(\mathcal{E})) \to \Omega^1(M,\mathrm{End}(\mathcal{E})),$$

$$\Phi \mapsto \Theta \wedge \Phi. \tag{19}$$

Here,

$$\sigma_{\mathrm{Ch}} : \mathrm{Cl}_{\mathrm{M}} \to \Lambda_{\mathrm{M}},$$

$$a \mapsto \gamma_{\mathrm{Ch}}(a) 1_\Lambda \tag{20}$$

denotes Chevalley's canonical linear isomorphism between the Clifford bundle and the Grassmann bundle $\Lambda_{\mathrm{M}} \twoheadrightarrow M$ of M. It is based on the Clifford mapping,

$$\gamma_{\mathrm{Ch}}: T^*M \to \mathrm{End}(\Lambda_{\mathrm{M}}),$$

$$\alpha \mapsto \begin{cases} \Lambda_{\mathrm{M}} \to \Lambda_{\mathrm{M}} \\ \omega \mapsto \varepsilon\ \mathrm{int}_g(\alpha)\omega + \mathrm{ext}(\alpha)\omega. \end{cases} \tag{21}$$

Here, respectively, $\mathrm{int}_g(\alpha)\omega$ is the contraction (inner derivative) of ω with respect to $\alpha^\sharp \in TM$ and $\mathrm{ext}(\alpha)\omega$ denotes the exterior multiplication of ω with respect to $\alpha \in T^*M$.

The isomorphism (20) is referred to as "symbol map" and its inverse as "quantization map." Although misleading from a physical point of view, we shall still use this common term and call the section $\not{\alpha} \equiv \delta_\gamma(\alpha) \in \mathrm{Sec}(M, \mathrm{End}(\mathcal{E}))$ the "quantization" of the "noncommutative superfield" $\alpha \in \Omega^*(M, \mathrm{End}(\mathcal{E})) = \oplus_{k=0}^n \mathrm{Sec}(M, \Lambda^k T^*M \otimes \mathrm{End}(\mathcal{E}))$.

On the affine set $\mathcal{A}(\mathcal{E})$ of all (linear) connections on $\mathcal{E} \twoheadrightarrow M$ there exists a distinguished affine subset, consisting of what is called *Clifford connections*. This subset may be characterized as follows:

$$\mathcal{A}_{\mathrm{Cl}}(\mathcal{E}) := \{\partial_{\mathrm{A}} \in \mathcal{A}(\mathcal{E}) | \partial_{\mathrm{A}}^{T^*M \otimes \mathrm{End}(\mathcal{E})}\Theta \equiv 0\}. \tag{22}$$

We call a first order differential operator $\not{D}$, acting on sections of $\mathcal{E} \twoheadrightarrow M$, of *Dirac type*, provided that it fulfills

$$[\not{D}, f] = \gamma_{\mathcal{E}}(df), \tag{23}$$

for all $f \in \mathcal{C}^\infty(M)$. The set of all such operators is denoted by $\mathcal{D}(\mathcal{E})$.

An odd Dirac-type operator $\not{D} \in \mathcal{D}(\mathcal{E})$ on a $\mathbb{Z}_2$-graded Clifford module bundle $(\mathcal{E}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}})$ is called a *Dirac operator*. Here, $\tau_{\mathcal{E}} \in \mathrm{End}(\mathcal{E})$ denotes the underlying grading involution such that $\not{D} \in \mathcal{D}(\mathcal{E})$ is a Dirac operator provided that it satisfies $\not{D} \circ \tau_{\mathcal{E}} = -\tau_{\mathcal{E}} \circ \not{D}$.

At this point, we have to warn the reader. Usually, every Dirac-type operator is assumed to be odd. For reasons that will become clear in Sec. II B, however, we want to distinguish between Dirac-type operators and Dirac operators on a Clifford module. Clearly, every Dirac operator is of Dirac type. Moreover, every Dirac-type operator may be written as

$$\not{D} = \not{\partial}_{\mathrm{A}} + \Phi_{\mathrm{A}}, \tag{24}$$

where $\not{\partial}_{\mathrm{A}} \equiv \delta_\gamma \circ \partial_{\mathrm{A}}$ and $\Phi_{\mathrm{A}} \in \mathrm{Sec}(M, \mathrm{End}(\mathcal{E}))$, in general, will depend on the choice of the Clifford connection $\partial_{\mathrm{A}} \in \mathcal{A}_{\mathrm{Cl}}(\mathcal{E})$.

Every Dirac-type operator $\not{D} \in \mathcal{D}(\mathcal{E})$ has a unique first order and second order decomposition,

$$\not{D} = \not{\partial}_{\mathrm{B}} + \Phi_{\mathrm{D}}, \tag{25}$$

$$\not{D}^2 = \Delta_{\mathrm{B}} + V_{\mathrm{D}}. \tag{26}$$

Here, $\not{\partial}_{\mathrm{B}} \equiv \delta_\gamma \circ \partial_{\mathrm{B}}$ is the quantization of the *Bochner connection* $\partial_{\mathrm{B}} \in \mathcal{A}(\mathcal{E})$ that is uniquely defined by $\not{D}$ via

$$2ev_g(df, \partial_{\mathrm{B}}\psi) := \varepsilon([\not{D}^2, f] - \delta_g df)\psi, \tag{27}$$

for all $f \in \mathcal{C}^\infty(M)$ and $\psi \in \mathrm{Sec}(M, \mathcal{E})$. The second order operator,

$$\triangle_{\mathrm{B}} := \varepsilon ev_g(\partial_{\mathrm{B}}^{T^*M \otimes \mathcal{E}} \circ \partial_{\mathrm{B}}), \tag{28}$$

is the induced *Bochner–Laplacian* (or "trace/connection Laplacian," see, for example, Refs. 8 and 17 as well as Chap. 2.1 of Ref. 5).

With every Dirac-type operator $\not{D} \in \mathcal{D}(\mathcal{E})$, there is naturally associated with a connection $\partial_{\mathrm{D}} \in \mathcal{A}(\mathcal{E})$ such that $\not{\partial}_{\mathrm{D}} \equiv \delta_\gamma \circ \partial_{\mathrm{D}} = \not{D}$. This *Dirac connection* is given by (cf. Ref. 37)

$$\partial_{\mathrm{D}} := \partial_{\mathrm{B}} + \mathrm{ext}_\Theta(\Phi_{\mathrm{D}}). \tag{29}$$

We call the one-form $\omega_{\mathrm{D}} := \mathrm{ext}_{\Theta}(\not{D} - \not{\partial}_{\mathrm{B}})$, uniquely associated with $\not{D} \in \mathcal{D}(\mathcal{E})$, the *Dirac form* and the tangent vector field: $\xi_{\mathrm{D}} := -\varepsilon(\mathrm{tr}_{\mathcal{E}}\, \omega_{\mathrm{D}})^{\sharp}$, the *Dirac vector field*.

It follows that

$$\mathrm{tr}_{\mathcal{E}}\, V_{\mathrm{D}} = \mathrm{tr}_{\gamma}(\mathrm{curv}(\not{D}) - \varepsilon ev_g(\omega_{\mathrm{D}}^2)) + \mathrm{div}\, \xi_{\mathrm{D}}, \tag{30}$$

where

$$\mathrm{curv}(\not{D}) := \partial_{\mathrm{D}} \wedge \partial_{\mathrm{D}} \in \Omega^2(M, \mathrm{End}(\mathcal{E})) \tag{31}$$

is the curvature of the Dirac-type operator $\not{D} \in \mathcal{D}(\mathcal{E})$ and $\mathrm{tr}_{\gamma} \equiv \mathrm{tr}_{\mathcal{E}}\, \delta_{\gamma}$ is the "quantized trace."

We call the Hodge dual of the smooth function (30) the *universal Dirac–Lagrangian*,

$$\mathcal{L}_{\mathrm{D}} := \, * \,\mathrm{tr}_{\mathcal{E}}\, V_{\mathrm{D}}. \tag{32}$$

Its cohomology class is generated by the top form: $\mathrm{tr}_{\gamma}(\mathrm{curv}(\not{D}) - \varepsilon ev_g(\omega_{\mathrm{D}}^2)) d\mathrm{vol}_{\mathrm{M}}$.

It follows that two Dirac-type operators $\not{D}, \not{D}' \in \mathcal{D}(\mathcal{E})$ define the same Bochner connection provided that $\not{D}' - \not{D}$ anticommutes with the Clifford connection (cf. Corollary 4.1). Therefore, on every $\mathbb{Z}_2$-graded Clifford module, there is a distinguished class of Dirac-type operators depending on whether $\Phi_{\mathrm{D}} = \not{D} - \not{\partial}_{\mathrm{B}}$ is even or odd. In particular, we call a Dirac operator of *simple type* if it reads as

$$\not{D} = \not{\partial}_{\mathrm{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}}, \tag{33}$$

where $\phi_{\mathcal{E}} \in \mathrm{Sec}(M, \mathrm{End}_{\gamma}^{-}(\mathcal{E}))$. Here, $\mathrm{End}_{\gamma}(\mathcal{E}) \hookrightarrow \mathrm{End}(\mathcal{E}) \twoheadrightarrow M$ is the algebra subbundle of all $\gamma_{\mathcal{E}}$-invariant endomorphisms and $\tau_{\mathcal{E}}$ is the underlying grading involution on $\mathcal{E} \twoheadrightarrow M$.

The specific form (33) of simple type Dirac operators is determined by the condition that the corresponding Bochner connections are given by Clifford connections. Of course, every $\not{\partial}_{\mathrm{A}} \in \mathcal{D}(\mathcal{E})$ is of simple type.

Likewise, one may consider Dirac-type operators of the form

$$\not{D} := \not{\partial}_{\mathrm{A}} + \Phi_{\mathrm{H}},$$

$$\Phi_{\mathrm{H}} \in \mathrm{Sec}(M, \mathrm{End}_{\gamma}(\mathcal{E})). \tag{34}$$

These Dirac-type operators have the property that their Dirac connections read as

$$\partial_{\mathrm{D}} = \partial_{\mathrm{B}} + \mathrm{ext}_{\Theta}(\Phi_{\mathrm{D}}) = \partial_{\mathrm{A}} + \mathrm{ext}_{\Theta}(n\varepsilon\Phi_{\mathrm{H}}) + \mathrm{ext}_{\Theta}[(1 - n\varepsilon)\Phi_{\mathrm{H}}] = \partial_{\mathrm{A}} + \Phi_{\mathrm{H}}\Theta \equiv \partial_{\mathrm{A}} + H. \tag{35}$$

We call the one-form,

$$H := \Phi_{\mathrm{H}}\Theta \overset{\mathrm{loc.}}{=} \frac{\varepsilon}{n} e^k \otimes \gamma_{\mathcal{E}}(e_k^{\flat}) \circ \Phi_{\mathrm{H}}, \tag{36}$$

the *Higgs gauge potential* and the connections: $\partial_{\mathrm{YMH}} := \partial_{\mathrm{A}} + H \overset{\mathrm{loc.}}{=} \partial + A + H$, *Yang–Mills–Higgs connections* on the Clifford module $\mathcal{E} \twoheadrightarrow M$.

We remark that for $\Phi_{\mathrm{H}} \in \mathrm{Sec}(M, \mathrm{End}_{\gamma}^{+}(\mathcal{E}))$, the Yang–Mills–Higgs connections are odd connections. They have the property that the (locally defined) Yang–Mills gauge potentials $A \in \Omega^1(M, \mathrm{End}_{\gamma}^{+}(\mathcal{E}))$ provide connections which respect the subbundles $\mathcal{E}^{\pm} \twoheadrightarrow M$. The Yang–Mills part of a Yang–Mills–Higgs connection is thus "chirality preserving." In contrast, the Higgs gauge potential $H \in \Omega^1(M, \mathrm{End}^{-}(\mathcal{E}))$ provides a connection between these subbundles of $\mathcal{E} \twoheadrightarrow M$ and thus constitutes the "chirality violating" part of the Yang–Mills–Higgs gauge potential. This is similar to the geometrical interpretation of the connections constructed within the original Connes–Lott description of the Yang–Mills–Higgs sector of the STM [cf. Refs. 14, 38, 12, 33, and 24 (see also Refs. 30 and 29 in the case of alternative approaches)]. However, in contrast to noncommutative geometry, where mainly the algebraic structure of connections is taken into account, Dirac connections are always related to the underlying geometry that is encoded within the canonical one-form (17). In particular, within the scheme presented, the Higgs gauge potential is intimately related to gravity.

Definition 2.1: A Clifford module bundle $(\mathcal{E},\gamma_{\mathcal{E}})\twoheadrightarrow(M,g_{\mathrm{M}})$ *is said to be "partially flat" provided that there is a Clifford connection* $\partial_{\mathrm{A}}\in\mathcal{A}_{\mathrm{Cl}}(\mathcal{E})$ *fulfilling*

$$\mathrm{curv}(\not\partial_{\mathrm{A}})=\not\mathrm{R}\mathrm{iem}(g_{\mathrm{M}}), \tag{37}$$

where $\not\mathrm{R}\mathrm{iem}(g_{\mathrm{M}})\in\Omega^2(M,\mathrm{End}(\mathcal{E}))$ *is the Riemann curvature with respect to* g_{M} *lifted to the Clifford module.*

We call a Clifford connection satisfying (37) a *partially flat Clifford connection* and denote it by $\partial\in\mathcal{A}_{\mathrm{Cl}}(\mathcal{E})$.

Definition 2.2: A ($\mathbb{Z}_2$*-graded, Hermitian*) *vector bundle* $\mathcal{E}\twoheadrightarrow M$ *is called a "Clifford bimodule" provided that it carries a representation* $\gamma_{\mathcal{E}}$ of $\mathrm{Cl}_{\mathrm{M}}\twoheadrightarrow M$ *and a representation* $\gamma_{\mathcal{E},\mathrm{op}}$ of $\mathrm{Cl}_{\mathrm{M}}^{\mathrm{op}}\twoheadrightarrow M$ *(the bundle of opposite Clifford algebras) such that both representations commute.*

Note that, in particular, $\gamma_{\mathcal{E},\mathrm{op}}(\mathfrak{a})\in\mathrm{End}_{\gamma}(\mathcal{E})$, for all $\mathfrak{a}\in\mathrm{Cl}_{\mathrm{M}}^{\mathrm{op}}$. Furthermore, on a Clifford bimodule there exists a distinguished class of connections.

Definition 2.3: Let $(\mathcal{E},\gamma_{\mathcal{E}},\gamma_{\mathcal{E},\mathrm{op}})\twoheadrightarrow M$ *be a Clifford bimodule. A connection* $\nabla^{\mathcal{E}}\in\mathcal{A}(\mathcal{E})$ *is called "S-reducible" provided that it is a right Clifford connection,*

$$\nabla_{\xi}^{\mathrm{End}(\mathcal{E})}\gamma_{\mathcal{E},\mathrm{op}}(\mathfrak{a}):=[\nabla_{\xi}^{\mathcal{E}},\gamma_{\mathcal{E},\mathrm{op}}(\mathfrak{a})]=\gamma_{\mathcal{E},\mathrm{op}}(\nabla_{\xi}^{\mathrm{Cl}^{\mathrm{op}}}\mathfrak{a}) \tag{38}$$

for all $\mathfrak{a}\in\mathrm{Sec}(M,\mathrm{Cl}_{\mathrm{M}}^{\mathrm{op}})$ *and* $\xi\in\mathrm{Sec}(M,TM)$. *The (affine) subspace of S-reducible connections is denoted by* $\mathcal{A}_{\mathrm{S}}(\mathcal{E})\subset\mathcal{A}(\mathcal{E})$.

We make use of the following (common) terminology: a "Clifford module" generically means a Clifford left module. Accordingly, "Clifford connections" always refer to the appropriate left action. Likewise, the notion of "Dirac (type) operators" also refers to this left action. However, on a Clifford bimodule every Dirac (type) operator $\not D\in\mathcal{D}(\mathcal{E})$ may be considered to act on $\mathrm{Sec}(M,\mathcal{E})$ either from the left or from the right. That is, one has to distinguish between "left-Dirac (type) operators" and "right-Dirac (type) operators." In the sequel, "Dirac (type) operators" always mean left-Dirac (type) operators. Note that for every right-Dirac (type) operator there is a unique Dirac (type) operator $\not D_{\mathrm{op}}\in\mathcal{D}(\mathcal{E})$, which acts from the left on $\mathrm{Sec}(M,\mathcal{E})$ via $\gamma_{\mathcal{E},\mathrm{op}}$. Clearly, every $\not D\in\mathcal{D}(\mathcal{E})$ uniquely defines an appropriate $\not D_{\mathrm{op}}\in\mathcal{D}(\mathcal{E})$ and vice versa. We thus call $\not D_{\mathrm{op}}$ the *opposite Dirac (type) operator* associated with $\not D\in\mathcal{D}(\mathcal{E})$.

Note that S-reducible connections on a Clifford bimodule may also be characterized by the requirement

$$\nabla^{T^*M\otimes\mathrm{End}(\mathcal{E})}\Theta_{\mathrm{op}}=0. \tag{39}$$

Here, $\Theta_{\mathrm{op}}\overset{\mathrm{loc.}}{=}(\varepsilon/n)e^k\otimes\gamma_{\mathcal{E},\mathrm{op}}(e_k^{\flat})$ is the canonical one-form represented on the Clifford bimodule via $\gamma_{\mathcal{E},\mathrm{op}}$. Hence, $\delta_{\gamma_{\mathrm{op}}}\circ\mathrm{ext}_{\Theta_{\mathrm{op}}}=(\varepsilon/n)\gamma_{\mathcal{E},\mathrm{op}}(e_k^{\flat}e^k)=\mathrm{id}_{\mathcal{E}}$.

Clearly, the Grassmann bundle provides the archetypical example of a Clifford bimodule.

Definition 2.4: A Dirac (type) operator $\not D\in\mathcal{D}(\mathcal{E})$ *on a Clifford bimodule is called "S-reducible" provided that its Dirac connection is S-reducible*: $\partial_{\mathrm{D}}\in\mathcal{A}_{\mathrm{S}}(\mathcal{E})$. *The set of all S-reducible Dirac (type) operators is denoted by* $\mathcal{D}_{\mathrm{S}}(\mathcal{E})$.

In this section, we summarized some basic notions with respect to general Clifford (bi)modules. In the sequel, we shall restrict our further discussion mainly to the more specific case of twisted Grassmann bundles. On the one hand, this case is broad enough to geometrically describe most of the cases encountered in physics. On the other hand, it is topologically less restrictive than the case of a twisted spinor bundle.

B. Twisted Grassmann bundles

Basically, the advantage in restricting to twisted Grassmann bundles is provided by the fact that each section of the Einstein–Hilbert bundle then yields a natural Clifford action and thus turns the twisted Grassmann bundle into a Clifford module bundle. This Clifford action, of course, is given by Chevalley's canonical isomorphism (20), which only takes the metric structure into account.

Therefore, let $E \twoheadrightarrow M$ be any Hermitian and (maybe trivially) $\mathbb{Z}_2$-graded complex vector bundle of finite rank. We then consider

$$\mathcal{S} := \Lambda_{\mathrm{M}} \otimes_{\mathbb{C}} E \twoheadrightarrow M. \tag{40}$$

Any section $g_{\mathrm{M}} \in \mathrm{Sec}(M, \mathcal{E}_{\mathrm{EH}})$ turns (40) into a bundle of Clifford left modules according to the action,

$$Cl_{\mathrm{M}} \times_M \mathcal{S} \twoheadrightarrow \mathcal{S},$$

$$(a, \omega \otimes \chi) \mapsto \sigma_{\mathrm{Ch}}(a \sigma_{\mathrm{Ch}}^{-1}(\omega)) \otimes \chi, \tag{41}$$

where the Clifford multiplication is denoted by juxtaposition. The underlying Clifford mapping is denoted, again, by γ_{Ch} and no distinction is made between the Clifford mapping and its induced homomorphism.

Note that

$$\mathrm{End}_{\gamma}(\mathcal{S}) = \mathrm{Cl}_{\mathrm{M}}^{\mathrm{op}} \otimes_M \mathrm{End}(E), \tag{42}$$

with $\mathrm{Cl}_{\mathrm{M}}^{\mathrm{op}} \twoheadrightarrow M$ acting from the right on $\mathcal{S} \twoheadrightarrow M$. Also note that any $g_{\mathrm{M}} \in \mathrm{Sec}(M, \mathcal{E}_{\mathrm{EH}})$ turns $\mathcal{S} \twoheadrightarrow M$ into a Hermitian vector bundle,

$$\langle \omega_1 \otimes \chi_1, \omega_2 \otimes \chi_2 \rangle_{\mathcal{S}} = g_{\Lambda\mathrm{M}}(\omega_1, \omega_2) \langle \chi_1, \chi_2 \rangle_{\mathrm{E}}. \tag{43}$$

Here $\langle \cdot, \cdot \rangle_{\mathrm{E}}$ is the Hermitian product on $E \twoheadrightarrow M$ and $g_{\Lambda\mathrm{M}}$ is the induced (semi)metric on the Grassmann bundle $\Lambda_{\mathrm{M}} \twoheadrightarrow M$.

A first order differential operator $\not{D}$, acting on sections of $\mathcal{S} \twoheadrightarrow M$, is called of Dirac type if there is a section $g_{\mathrm{M}} \in \mathrm{Sec}(M, \mathcal{E}_{\mathrm{EH}})$ such that

$$[\not{D}, f] = \gamma_{\mathrm{Ch}}(df) \tag{44}$$

for all $f \in \mathcal{C}^{\infty}(M)$. The set of all of these operators is denoted by $\mathcal{D}(\mathcal{S})$. Similar to the general case, a Dirac operator on a twisted Grassmann bundle is defined to be an odd Dirac-type operator with respect to the particular grading involution,

$$\tau_{\mathcal{S}} = \tau_{\mathrm{M}} \circ \iota_{\mathcal{S}},$$

$$\iota_{\mathcal{S}} := \mathrm{id}_{\Lambda} \otimes \tau_{\mathrm{E}}. \tag{45}$$

Here, τ_{E} is a grading involution on $E \twoheadrightarrow M$ that can also be trivial. The *chirality* involution $\tau_{\mathrm{M}} \in \mathrm{End}(\mathcal{E})$ is defined by

$$\tau_{\mathrm{M}} := \sqrt{(-1)^{n(n-1)/2+q}} \delta_{\gamma}(d\mathrm{vol}_{\mathrm{M}}), \tag{46}$$

with $d\mathrm{vol}_{\mathrm{M}} \in \Omega^n(M)$ being the metric induced volume form. We call $\iota_{\mathcal{S}} \in \mathrm{End}_{\gamma}(\mathcal{S})$ the "inner involution."

The *universal Dirac action* is (formally) defined by the functional

$$\mathcal{I}_{\mathrm{D}}:\mathcal{D}(\mathcal{S}) \to \mathbb{C},$$

$$\not{D} \mapsto \langle\langle [M],[\mathcal{L}_{\mathrm{D}}]\rangle\rangle \equiv \int_M \mathcal{L}_{\mathrm{D}}. \tag{47}$$

[In general, the domain of definition of the universal Dirac action is an appropriate subset of $\mathcal{D}(\mathcal{E})$, only, for M is not supposed to be compact.] The *total Dirac action* is given by the functional,

$$\mathcal{I}_{\mathrm{D,tot}}:\mathrm{Sec}(M,\mathcal{S}) \times \mathcal{D}(\mathcal{S}) \to \mathbb{C},$$

$$(\psi,\not{D}) \mapsto \langle\langle \psi,\not{D}\psi\rangle\rangle + \mathcal{I}_{\mathrm{D}}(\not{D}), \tag{48}$$

with

$$\langle\langle \psi,\not{D}\psi\rangle\rangle \coloneqq \int_M \langle \psi,\not{D}\psi\rangle_{\mathcal{S}} d\mathrm{vol}_{\mathrm{M}}. \tag{49}$$

For every (symmetric) $\not{D} \in \mathcal{D}(\mathcal{S})$, the functional (49) is considered as a (real-valued) quadratic form on $\mathrm{Sec}(M,\mathcal{S})$. It is called the *fermionic* part of the total Dirac action. Accordingly, the universal Dirac action (47) is referred to as the *bosonic* part of the total Dirac action.

It follows that the *gauge group of the total Dirac action* is given by the semidirect product,

$$\mathcal{G}_{\mathrm{D}} = \mathcal{D}\mathrm{iff}(M) \ltimes \mathcal{G}_{\mathcal{S}}, \tag{50}$$

with $\mathcal{G}_{\mathcal{S}}$ being the gauge group of $\mathcal{S} \twoheadrightarrow M$. For every section g_{M}, this gauge group explicitly reads as

$$\mathcal{G}_{\mathcal{S}} = \mathcal{G}_{\mathrm{EH}} \times \mathcal{G}_{\mathrm{YM}}. \tag{51}$$

Here, $\mathcal{G}_{\mathrm{YM}} \simeq \mathrm{Aut}_\gamma(\mathcal{S})$ is the subgroup of all bundle automorphisms on $\mathcal{S} \twoheadrightarrow M$, which are γ-invariant and $\mathcal{G}_{\mathrm{EH}}$ is the gauge group of the SO(p,q)-reduced frame bundle.

In contrast, the *gauge group of the universal Dirac action* is provided by the affine group as

$$\mathcal{P}_{\mathrm{D}} = \mathcal{G}_{\mathrm{D}} \ltimes \mathcal{T}_{\mathrm{D}}, \tag{52}$$

with the *translational group* being given by

$$\mathcal{T}_{\mathrm{D}} \simeq \Omega^1(M,\mathrm{End}_\gamma(\mathcal{S})). \tag{53}$$

Its action on $\mathcal{D}(\mathcal{S})$ reads as $\not{D} \mapsto \not{D} + \not{\alpha}$. We stress that the universal Dirac–Lagrangian is invariant with respect to this action.

We close this section with the following remarks concerning the case of *twisted spinor bundles*. (The author would like to thank V. Soucek for appropriate remarks.) For this, let M be an even-dimensional, orientable spin manifold. In this case, every Clifford module bundle $\mathcal{E} \twoheadrightarrow M$ is equivalent to a twisted spinor bundle $\mathcal{S} \otimes \mathcal{W} \twoheadrightarrow M$. Here, the (total space of the) vector bundle $\mathcal{W} \coloneqq \mathrm{Hom}_\gamma(\mathcal{S},\mathcal{E}) \twoheadrightarrow M$ is defined by the γ-equivariant homomorphisms (cf. Ref. 2 and Sec. 3.3 in Ref. 5). Basically, this follows from Wedderburn's structure theorems about invariant linear mappings (see, for example, Chap. 11 in Ref. 19). Accordingly, the abovementioned equivalence is provided by the evaluation map. Although the spinor module carries a canonical Clifford action according to the identification $Cl_{\mathrm{M}} \otimes \mathbb{C} \simeq \mathrm{End}(\mathcal{S})$, there are usually different (i.e., inequivalent) spin structures for given $g_{\mathrm{M}} \in \mathrm{Sec}(M,\mathcal{E}_{\mathrm{EH}})$ (see, for example, Chap. 3 in Ref. 5 and Sec. 1.8 in Ref. 23). This makes the actual domain of definition of the Dirac action geometrically more interesting, for one may ask how the Dirac action changes with a change of the spin structure (cf., for example, Refs. 4 and 16). One may also take into account what is called "generalized spin structures" or "canonically generalized spin structures" (cf., for example, Refs. 3, 20, and 21). In fact, in the case of real representations, the appropriate discussions presented in Ref. 21 seem to fit with the discussion presented in this work. Clearly, if the spin structure is basically unique, then the case of twisted spinor bundles can be treated similar to the case of twisted Grassmann bundles.

III. THE GEOMETRICAL PICTURE OF DIRAC-TYPE OPERATORS AND THE EINSTEIN–HILBERT ACTION

In this section, we briefly discuss the geometrical picture that underpins the Einstein–Hilbert action, $\mathcal{I}_{\mathrm{EH}}$, when the latter is expressed in terms of Dirac-type operators. From the usual Lichnerowicz/Schrödinger decomposition of $\not\partial_{\mathrm{A}}^2$ (cf. Ref. 25 and 32),

$$\not\partial_{\mathrm{A}}^2 - \varepsilon ev_g(\partial_{\mathrm{A}}^{\mathcal{T}^*M\otimes\mathcal{S}} \circ \partial_{\mathrm{A}}) = \delta_\gamma(\mathrm{curv}(\not\partial_{\mathrm{A}})) = -\frac{\varepsilon}{4}\mathrm{scal}(g_{\mathrm{M}}) + \delta_\gamma(F_{\mathrm{A}}), \tag{54}$$

it follows that

$$\mathcal{I}_{\mathrm{EH}}(g_{\mathrm{M}}) \sim \int_M * \mathrm{tr}_\gamma(\mathrm{curv}(\not\partial_{\mathrm{A}})). \tag{55}$$

Note that the "relative curvatures," $F_{\mathrm{A}} \coloneqq \mathrm{curv}(\not\partial_{\mathrm{A}}) - \not R\mathrm{iem}(g_{\mathrm{M}})$, of Clifford connections have the peculiar property that $F_{\mathrm{A}} \in \Omega^2(M, \mathrm{End}_\gamma^+(\mathcal{S}))$. (In the case of Clifford connections, the relative curvature of $\not\partial_{\mathrm{A}}$ is also called "twisting curvature.") Therefore, $\mathrm{tr}_\gamma(F_{\mathrm{A}}) \equiv 0$.

This description of the Einstein–Hilbert action allows one to point out a subtle difference between the fermionic and the bosonic actions, usually not taken into account. This difference provides the geometrical origin of the difference between the respective gauge groups of the fermionic and the bosonic part of the total Dirac action. The discussion presented in this section will eventually yield some motivation for the "Pauli map" that is introduced in Sec. VI, which permits one to interpret the Yang–Mills and the STM action as natural generalizations of the Einstein–Hilbert action with a "cosmological constant term."

Let $\mathcal{S}_{\mathrm{D}} \coloneqq \mathcal{E}_{\mathrm{EH}} \times_M \mathrm{End}(\mathcal{S}) \twoheadrightarrow M$. We consider the quotient

$$\Gamma_{\mathrm{D}} \coloneqq \mathrm{Sec}(M, \mathcal{S}_{\mathrm{D}})/\mathcal{T}_{\mathrm{D}}, \tag{56}$$

with the equivalence relation given by

$$(g_{\mathrm{M}}, \Phi) \sim (g'_{\mathrm{M}}, \Phi') :\Leftrightarrow \begin{cases} g'_{\mathrm{M}} = g_{\mathrm{M}} \\ \Phi' = \Phi + \not\alpha. \end{cases} \tag{57}$$

It follows that $\Gamma_{\mathrm{D}} \simeq \mathcal{D}(\mathcal{S})/\mathcal{T}_{\mathrm{D}}$. Therefore, the restriction to S-reducible Dirac connections yields a principal fibering

$$\mathcal{D}_{\mathrm{S}}(\mathcal{S}) \twoheadrightarrow \Gamma_{\mathrm{D}},$$

$$\not D = \not\partial_{\mathrm{A}} + \Phi_{\mathrm{A}} \mapsto [(g_{\mathrm{M}}, \Phi_{\mathrm{A}})], \tag{58}$$

with typical fiber given by the Abelian group $\Omega^1(M, \mathrm{End}(E))$.

The principal fibering (58) is clearly trivial but only in a noncanonical way unless the twisting part of $\mathcal{S} \twoheadrightarrow M$ is given by the trivial bundle $E = M \times \mathbb{C}^{\mathrm{N}} \twoheadrightarrow M$. This holds true also in the case where $\mathcal{S} \twoheadrightarrow M$ is supposed to be partially flat for every g_{M}. Indeed, every choice of a connection on $E \twoheadrightarrow M$ yields a trivializing section,

$$\sigma_{\mathrm{A}} : \Gamma_{\mathrm{D}} \to \mathcal{D}_{\mathrm{S}}(\mathcal{S}),$$

$$[(g_{\mathrm{M}}, \Phi)] \mapsto \not\partial_{\mathrm{A}} + \Phi. \tag{59}$$

It follows that $\sigma_{\rm A}^{*}\mathcal{I}_{\rm D}$ is independent of the choice of the trivializing section because of the translational invariance of the universal Dirac action. In particular, when restricted to the distinguished subset,

$$\Gamma_{\rm EH} := \{[(g_{\rm M},\Phi)] \in \Gamma_{\rm D} | \Phi \sim \not{\alpha}\} \simeq \mathrm{Sec}(M,\mathcal{E}_{\rm EH}), \tag{60}$$

every trivializing section (59) yields the Einstein–Hilbert functional,

$$\sigma_{\rm A}^{*}\mathcal{I}_{\rm D}:\mathrm{Sec}(M,\mathcal{E}_{\rm EH}) \to \mathbb{C},$$

$$g_{\rm M} \mapsto \mathcal{I}_{\rm EH}(g_{\rm M}). \tag{61}$$

The sections $g_{\rm M} \in \mathrm{Sec}(M,\mathcal{E}_{\rm EH})$ are thus geometrically represented on $\mathcal{D}_{\rm S}(\mathcal{S})$ by the trivializing sections (59),

$$\sigma_{\rm A}(g_{\rm M}) = \not{\partial}_{\rm A} = d_{\rm A} + \varepsilon\,\delta_{g,A}. \tag{62}$$

Here, $\delta_{g,A}$ denotes the formal adjoint of the exterior covariant derivative $d_{\rm A}$ that is defined with respect to some Clifford connection.

Accordingly, the geometrical meaning of these gauge sections is to make the metric on M "covariant" on $\mathcal{S}$. This geometrical view of the gauge sections becomes most apparent for flat modules (i.e., for flat $E \twoheadrightarrow M$). In this case, one gets

$$\sigma_{\rm A}(g_{\rm M}) = \not{\partial} + \not{A} = d + \varepsilon\,\delta_{g} + \not{A}, \tag{63}$$

with the Gauss–Bonnet–Hodge–de Rham operator, $d+\varepsilon\,\delta_g$, being determined by the (semi)metric $g_{\rm M}$.

We emphasize that this geometrical picture of the (semi)metric is provided by the translational invariance of the universal Dirac–Lagrangian. Finally, any Dirac (type) operator $\not{D} \in \mathcal{D}(\mathcal{S})$ may be locally regarded as a "generalized covariance" of its underlying (semi)Riemannian metric $g_{\rm M}$,

$$\sigma_{\Phi_A}(g_{\rm M}) \equiv \sigma_{\rm A}([(g_{\rm M},\Phi)]) \overset{\text{loc.}}{=} d + \varepsilon\,\delta_g + \Phi_{\rm A}, \tag{64}$$

where locally: $\Phi_{\rm A} := \Phi + \not{A}$.

So far, we discussed the geometrical picture of the Einstein–Hilbert action, when the latter is described in terms of the universal Dirac action. It is natural to ask for the appropriate substitute of the Einstein–Hilbert action with a cosmological constant $\Lambda \in \mathbb{R}$,

$$\mathcal{I}_{\rm EH,\Lambda}(g_{\rm M}) = \int_M *(\mathrm{scal}(g_{\rm M}) + \Lambda). \tag{65}$$

To answer this question, we take into account (55) that expresses the Einstein–Hilbert action in terms of the curvature of the quantized Yang–Mills connection $\partial_{\rm YM} \equiv \partial_{\rm A} \overset{\text{loc.}}{=} \partial + A$. It may thus not come as a big surprise that the functional (65) turns out to be expressible in terms of the curvature of the quantized Yang–Mills–Higgs connection $\partial_{\rm YMH} \overset{\text{loc.}}{=} \partial + A + H$,

$$\mathcal{I}_{\rm EH,\Lambda}(g_{\rm M}) \sim \int_M * \mathrm{tr}_\gamma(\mathrm{curv}(\not{\partial}_{\rm YMH}) - \varepsilon ev_g(\omega_{\rm D}^2)) = \int_M *(\mathrm{tr}_\gamma\,\mathrm{curv}(\not{\partial}_{\rm YM}) - \Lambda_{\rm H}), \tag{66}$$

whereby the cosmological constant reads as

$$\Lambda \equiv \Lambda_{\mathrm{H}} := \lambda \ \mathrm{tr}_g \, H^2 = \lambda' \ \mathrm{tr}_{\mathcal{S}} \, \Phi_{\mathrm{H}}^2. \tag{67}$$

Here, $\lambda, \lambda' \in \mathbb{R}$ are numerical constants determined by the dimension of M. Notice that the right-hand side of (67) is indeed independent of the metric although the Higgs gauge potential itself is metric dependent.

The point to be emphasized is that the Einstein equations do not demand the Higgs gauge potential $H = \mathrm{ext}_\Theta(\Phi_{\mathrm{H}})$ itself to be constant but only to take values on the sphere bundle of radius Λ/λ'. Consequently, if the Yang–Mills gauge group $\mathcal{G}_{\mathrm{YM}} \subset \mathcal{P}_{\mathrm{D}}$ is supposed to act transitively on the sphere bundle (like in the case of the ordinary Higgs potential), then the Higgs gauge potential becomes gauge equivalent to the one-form $im_{\mathrm{D}}\Theta$, with $m_{\mathrm{D}} \in \mathrm{Sec}(M, \mathrm{End}_\gamma(\mathcal{S}))$ being a constant section of length Λ/λ'. Clearly, such a section exists if and only if the Yang–Mills gauge group is reducible to the isotropy group of m_{D}. The rank of the reduced gauge group is determined by the codimension of the sphere bundle depending on the representation of the Yang–Mills gauge group on the Clifford module. This completely parallels the usual Higgs mechanism used in the STM description of particle physics and indicates how spontaneous symmetry breaking can be described when gravity is taken into account. In fact, we claim that the Higgs potential is only needed to provide the Higgs boson itself with mass but the symmetry reduction is triggered by gravity in the way indicated. As mentioned earlier, this geometrical interpretation of spontaneous symmetry breaking is based on the intimate relation between gravity and the Higgs that is formally provided by the geometrical construction of Dirac connections in terms of the canonical one-form. We also point to the fact that the Higgs mass term (67) is of the same physical dimension as the scalar curvature, as opposed to the fourth order term in the usual Higgs potential, which is dimensionless like the quadratic Yang–Mills–Lagrangian (in four dimensions). The same holds true for the "kinetic term" of the Higgs. Hence, from a geometrical point of view one may regard the Higgs sector of the STM as the sum of various terms having different geometrical origins. This is most clearly exhibited when the Einstein–Hilbert action with cosmological constant is expressed in terms of the Yang–Mills–Higgs connection ∂_{YMH} and when also the *Yang–Mills–Higgs curvature*,

$$F_{\mathrm{YMH}} := \mathrm{curv}(\not{\partial}_{\mathrm{YMH}}) - \not{R}\mathrm{iem}(g_{\mathrm{M}}) = F_{\mathrm{YM}} + d_{\mathrm{A}}H + H \wedge H, \tag{68}$$

is taken into account, where $F_{\mathrm{YM}} \equiv F_{\mathrm{A}} \in \Omega^2(M, \mathrm{End}_\gamma^+(\mathcal{E}))$ is the usual *Yang–Mills curvature* ("twisting curvature").

Note that

$$d_{\mathrm{A}}H + H \wedge H = (d_{\mathrm{A}}\Phi_{\mathrm{H}} + \Phi_{\mathrm{H}}^2\Theta) \wedge \Theta. \tag{69}$$

Therefore, the Yang–Mills–Higgs connection is not flat, in general, even if the Yang–Mills connection is supposed to be flat and $\Phi_{\mathrm{H}} = im_{\mathrm{D}}$ is a constant section, for in this case

$$F_{\mathrm{YMH}} = - m_{\mathrm{D}}^2 \Theta \wedge \Theta. \tag{70}$$

The (square of the) Dirac mass may be geometrically interpreted as curvature.

To summarize, we briefly discussed how the Einstein–Hilbert action and the Einstein–Hilbert action with a cosmological constant can be geometrically described in terms of, respectively, Yang–Mills and Yang–Mills–Higgs connections. Because of the translational invariance of the universal Dirac–Lagrangian, the latter does not depend on the chosen Yang–Mills connection. The Higgs part of the Yang–Mills–Higgs connection may serve to provide a symmetry reduction in the underlying Yang–Mills gauge group and thus contributes only by a constant section. Therefore, the bosonic part of the total Dirac action does not explicitly depend on the choice of the Yang–Mills part of the Yang–Mills–Higgs connection. It is (up to gauge) locally determined by

$$\not{\partial}_{\mathrm{YMH}} \overset{\mathrm{loc.}}{=} d + \varepsilon \delta_g + im_{\mathrm{D}}, \tag{71}$$

which is but the general relativistic analog of Dirac's original first order differential operator $i\not{\partial} - m$. We point out that the Dirac connection of (71) is basically identical to the notion of the "extended connection" in terms of a "frame field" as discussed, for example, in Ref. 10 and the corresponding references cited therein. In fact, the local term $-i\gamma_\mu M/4$ (cf. the beginning of p. 547 in Ref. 10) is but a special case of a (locally defined) Dirac form $\omega_{\mathrm{D}}=\mathrm{ext}_\Theta(\not{D}-\not{\partial}_{\mathrm{B}})$ (cf. Sec. II A).

When the fermionic part of the total Dirac action is taken into account, the translational symmetry of the bosonic part is broken. As a quadratic form that is determined by $\not{D}\in\mathcal{D}(\mathcal{S})$, this gauge reduction occurs since the fermionic part of the total action only depends on the choice of the Dirac operator. In contrast, the universal Dirac action is defined in terms of the corresponding curvature of the chosen Dirac operator. This subtle interplay between the fermionic and the bosonic part of the total Dirac action will be geometrically analyzed more carefully in the following section in terms of "real Clifford modules" and the Pauli map.

IV. REAL CLIFFORD MODULES AND THE PAULI MAP

Let $(\mathcal{E},\gamma_{\mathcal{E}})\twoheadrightarrow(M,g_{\mathrm{M}})$ be a Hermitian Clifford module. The Hermitian product is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{E}}$.

Definition 4.1: A Hermitian Clifford module is called a "real $\mathbb{Z}_2$-bigraded Hermitian Clifford module" ("real Clifford module" for short) if it is endowed, in addition, with a $\mathbb{C}$-linear involution $\tau_{\mathcal{E}}$, making $\mathcal{E}=\mathcal{E}^+\oplus\mathcal{E}^-\twoheadrightarrow M$ $\mathbb{Z}_2$-graded, and a $\mathbb{C}$-antilinear involution $J_{\mathcal{E}}$, making $\mathcal{E}=\mathcal{M}_{\mathcal{E}}\otimes\mathbb{C}\twoheadrightarrow M$ real, such that

$$\tau_{\mathcal{E}}\circ\gamma_{\mathcal{E}}(\alpha)=-\gamma_{\mathcal{E}}(\alpha)\circ\tau_{\mathcal{E}},$$

$$J_{\mathcal{E}}\circ\gamma_{\mathcal{E}}(\alpha)=\pm\gamma_{\mathcal{E}}(\alpha)\circ J_{\mathcal{E}},$$

$$J_{\mathcal{E}}\circ\tau_{\mathcal{E}}=\pm\tau_{\mathcal{E}}\circ J_{\mathcal{E}},$$

$$\langle J_{\mathcal{E}}(z),J_{\mathcal{E}}(w)\rangle_{\mathcal{E}}=\pm\langle w,z\rangle_{\mathcal{E}} \tag{72}$$

for all $\alpha\in T^*M$ and $z,w\in\mathcal{E}$.

A real Clifford module is called a "Majorana module" provided that

$$J_{\mathcal{E}}\circ\tau_{\mathrm{M}}=-\tau_{\mathrm{M}}\circ J_{\mathcal{E}}. \tag{73}$$

We make use of the following abbreviation: $B^{\mathrm{cc}}\equiv J_{\mathcal{E}}\circ B\circ J_{\mathcal{E}}$ for all $B\in\mathrm{End}(\mathcal{E})$. Similarly, $\not{D}_{\mathcal{E}}^{\mathrm{cc}}\equiv J_{\mathcal{E}}\circ\not{D}_{\mathcal{E}}\circ J_{\mathcal{E}}$ for all $\not{D}_{\mathcal{E}}\in\mathcal{D}(\mathcal{E})$. An operator $B\in\mathrm{End}(\mathcal{E})$ is called "real" ("imaginary") if $B^{\mathrm{cc}}=B$ ($B^{\mathrm{cc}}=-B$). We denote by $\mathcal{D}_{\mathrm{real}}(\mathcal{E})\subset\mathcal{D}(\mathcal{E})$ the subset of all real Dirac (type) operators: $\not{D}_{\mathcal{E}}^{\mathrm{cc}}=\not{D}_{\mathcal{E}}$.

Let

$$(\mathcal{E},\langle\cdot,\cdot\rangle_{\mathcal{E}},\tau_{\mathcal{E}},\gamma_{\mathcal{E}},J_{\mathcal{E}}) \tag{74}$$

be a real Clifford module bundle over (M,g_{M}) such that

$$\tau_{\mathcal{E}}^{\mathrm{cc}}=\pm\tau_{\mathcal{E}}, \tag{75}$$

$$\gamma_{\mathcal{E}}^{\mathrm{cc}}=+\gamma_{\mathcal{E}}. \tag{76}$$

We denote by

$$(\mathcal{P},\langle\cdot,\cdot\rangle_{\mathcal{P}},\tau_{\mathcal{P}},\gamma_{\mathcal{P}}) \tag{77}$$

the doubling of the Clifford module $(\mathcal{E},\langle\cdot,\cdot\rangle_{\mathcal{E}},\tau_{\mathcal{E}},\gamma_{\mathcal{E}})$. That is,

$$\mathcal{P} := {}^2\mathcal{E} \equiv \underset{\mathcal{E}}{\overset{\mathcal{E}}{\oplus}} = \mathcal{E} \otimes \mathbb{C}^2, \tag{78}$$

$$\langle \cdot , \cdot \rangle_{\mathcal{P}} := \tfrac{1}{2}(\langle \cdot , \cdot \rangle_{\mathcal{E}} + \langle \cdot , \cdot \rangle_{\mathcal{E}}), \tag{79}$$

$$\tau_{\mathcal{P}} := \tau_{\mathcal{E}} \otimes \tau_2, \tag{80}$$

$$\gamma_{\mathcal{P}} := \gamma_{\mathcal{E}} \otimes \mathbf{1}_2. \tag{81}$$

Here and in the sequel: $\mathbf{1}_2 \in \mathbb{C}(2)$ and $\tau_2, \varepsilon_2, \mathrm{I}_2 \in \mathbb{C}(2)$ denote, respectively, the two-by-two unit matrix and

$$\tau_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varepsilon_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{I}_2 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{82}$$

The real structure on $\mathcal{E}$ then allows one to also introduce a real structure on the doubled Clifford module $\mathcal{P} = {}^2\mathcal{E}$,

$$J_{\mathcal{P}} := J_{\mathcal{E}} \otimes \varepsilon_2, \tag{83}$$

such that

$$(\mathcal{P}, \langle \cdot , \cdot \rangle_{\mathcal{P}}, \tau_{\mathcal{P}}, \gamma_{\mathcal{P}}, J_{\mathcal{P}}) \tag{84}$$

becomes, again, a real Clifford module over (M, g_{M}). It follows that

$$\tau_{\mathcal{P}}^{\mathrm{cc}} = \pm \tau_{\mathcal{P}} \Leftrightarrow \tau_{\mathcal{E}}^{\mathrm{cc}} = \mp \tau_{\mathcal{E}}, \tag{85}$$

$$\gamma_{\mathcal{P}}^{\mathrm{cc}} = + \gamma_{\mathcal{P}}. \tag{86}$$

With respect to the real structure $J_{\mathcal{P}}$, the doubled Clifford module may be regarded as the complexification of the real vector bundle

$$\mathcal{M}_{\mathcal{P}} := \left\{ \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z}^{\mathrm{cc}} \end{pmatrix} \in \mathcal{P} \middle| \mathfrak{z} \in \mathcal{E} \right\} \twoheadrightarrow M. \tag{87}$$

This real vector bundle contains a distinguished real subvector bundle

$$\mathcal{V}_{\mathcal{P}} := \left\{ \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z} \end{pmatrix} \in \mathcal{P} \middle| \mathfrak{z} \in \mathcal{M}_{\mathcal{E}} \right\} \twoheadrightarrow M, \tag{88}$$

whoses complexification $\mathcal{V}_{\mathcal{P}}^{\mathbb{C}}$ of the total space may be identified with the diagonal embedding

$$\mathcal{E} \hookrightarrow {}^2\mathcal{E},$$

$$\mathfrak{z} \mapsto \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z} \end{pmatrix}. \tag{89}$$

Here, $\mathcal{M}_{\mathcal{E}} := \{ \mathfrak{z} \in \mathcal{E} | J_{\mathcal{E}}(\mathfrak{z}) = \mathfrak{z} \} \subset \mathcal{E}$ is the (total space) of the induced real subvector bundle, such that $\mathcal{E} = \mathcal{M}_{\mathcal{E}}^{\mathbb{C}}$.

A general real Dirac operator on the real Clifford module (84) reads (for a proof see, for example, Ref. 35, Theorem 1) as

$$\not{D}_{\mathcal{P}} = \begin{pmatrix} \not{D}_{\mathcal{E}} & \phi_{\mathcal{E}} - \mathcal{F}_{\mathcal{E}} \\ \phi_{\mathcal{E}} + \mathcal{F}_{\mathcal{E}} & \not{D}_{\mathcal{E}}^{\mathrm{cc}} \end{pmatrix}. \tag{90}$$

Here, respectively, $\not{D}_{\mathcal{E}} \in \mathcal{D}(\mathcal{E})$ is any Dirac operator on (74) and

$$\phi_{\mathcal{E}}^{\mathrm{cc}} = + \phi_{\mathcal{E}},$$

$$\mathcal{F}_{\mathcal{E}}^{\mathrm{cc}} = - \mathcal{F}_{\mathcal{E}} \tag{91}$$

are general sections of $\mathrm{End}^{+}(\mathcal{E}) \twoheadrightarrow M$.

The (affine) set of all real Dirac operators on the doubled Clifford module (84) contains a distinguished (affine) subset, consisting of those Dirac operators where in addition $\not{D}_{\mathcal{E}}^{\mathrm{cc}} = \not{D}_{\mathcal{E}}$ is a real Dirac operator on (74). In particular, the Dirac operators

$$\not{D}_{\mathcal{P}} = \begin{pmatrix} \not{D}_{\mathcal{E}} & \phi_{\mathcal{E}} \\ \phi_{\mathcal{E}} & \not{D}_{\mathcal{E}} \end{pmatrix} = \not{D}_{\mathcal{E}} \otimes \mathbf{1}_2 + \phi_{\mathcal{E}} \otimes \varepsilon_2 \tag{92}$$

also preserve $\mathrm{Sec}(M, \mathcal{V}_{\mathcal{P}})$.

In contrast, one may consider the distinguished class of Dirac operators on the doubled real Clifford module (84), which are determined already by the real Dirac operators on (74),

$$\not{P}_{\mathrm{D}} := \begin{pmatrix} \not{D}_{\mathcal{E}} & -\mathcal{F}_{\mathcal{E}} \\ \mathcal{F}_{\mathcal{E}} & \not{D}_{\mathcal{E}} \end{pmatrix} = \not{D}_{\mathcal{E}} \otimes \mathbf{1}_2 + \mathcal{F}_{\mathcal{E}} \otimes \mathrm{I}_2, \tag{93}$$

with $\mathcal{F}_{\mathcal{E}}$ being defined by the (relative) curvature of $\not{D}_{\mathcal{E}}$,

$$\mathcal{F}_{\mathcal{E}} := \mathcal{F}_{\mathrm{D}} \equiv i\,\delta_{\gamma}(\mathrm{curv}(\not{D}_{\mathcal{E}}) - \not{R}\mathrm{iem}(g_{\mathrm{M}})) = i\not{F}_{\mathrm{D}}. \tag{94}$$

Note that $\not{F}_{\mathrm{D}} \in \mathrm{Sec}(M, \mathrm{End}^{+}(\mathcal{E}))$ is even and real for real (or imaginary) Dirac operators $\not{D}_{\mathcal{E}} \in \mathcal{D}(\mathcal{E})$. Whence, $\mathcal{F}_{\mathrm{D}}^{\mathrm{cc}} = -\mathcal{F}_{\mathrm{D}}$.

By a slight abuse of notation, we rewrite the *Pauli-type* Dirac operators (93) as

$$\not{P}_{\mathrm{D}} := \not{D}_{\mathcal{E}} + \iota\mathcal{F}_{\mathrm{D}} \tag{95}$$

to bring them most closely to Dirac's first order operator including the Pauli term $i\not{F}$. Here,

$$\iota\mathcal{F}_{\mathrm{D}} \equiv \begin{pmatrix} 0 & -\mathrm{id}_{\mathcal{E}} \\ \mathrm{id}_{\mathcal{E}} & 0 \end{pmatrix} \circ \begin{pmatrix} \mathcal{F}_{\mathrm{D}} & 0 \\ 0 & \mathcal{F}_{\mathrm{D}} \end{pmatrix}. \tag{96}$$

In the sequel, we shall consider Pauli-type Dirac operators on the doubled Clifford module (84) as mappings,

$$\not{P}_{\mathrm{D}} : \mathrm{Sec}(M, \mathcal{V}_{\mathcal{P}}^{\mathbb{C}}) \to \mathrm{Sec}(M, \mathcal{P}),$$

$$^{2}\psi = \begin{pmatrix} \psi \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} \not{D}_{\mathcal{E}}\psi - \mathcal{F}_{\mathrm{D}}\psi \\ \not{D}_{\mathcal{E}}\psi + \mathcal{F}_{\mathrm{D}}\psi \end{pmatrix}. \tag{97}$$

Therefore, the restriction of our original Pauli-type Dirac operators to the diagonal embedding $\mathcal{E} \hookrightarrow {}^{2}\mathcal{E}$ may formally be interpreted as the restriction of the real Dirac operators (93) to the sections of the distinguished subbundle

$$\mathcal{V}_{\mathcal{P}}^{\mathbb{C}} \hookrightarrow \mathcal{P} \twoheadrightarrow M. \tag{98}$$

For this matter, we also call this bundle the *Pauli bundle* associated with the real Clifford bundle (74).

We put emphasize on the following fact: the Lagrangian density that is defined by the smooth function $\langle \Psi, \not{P}_{\rm D}\Psi\rangle_{\mathcal{P}}$ reduces to $\langle \psi, \not{D}_{\mathcal{E}}\psi\rangle_{\mathcal{E}}$ when the Pauli-type Dirac operators are restricted to the sections of the Pauli bundle. Therefore, the two fermionic functionals,

$$\mathcal{I}_{\rm D,ferm}:\mathrm{Sec}(M,\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{C},$$

$$(\psi, \not{D}_{\mathcal{E}}) \mapsto \int_M \langle \psi, \not{D}_{\mathcal{E}}\psi\rangle_{\mathcal{E}} d\mathrm{vol}_{\rm M}, \tag{99}$$

$$\mathcal{I}'_{\rm D,ferm}:\mathrm{Sec}(M,\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{C},$$

$$(\psi, \not{D}_{\mathcal{E}}) \mapsto \int_M \langle {}^2\psi, \not{P}_{\rm D}{}^2\psi\rangle_{\mathcal{P}} d\mathrm{vol}_{\rm M}, \tag{100}$$

contain the same information, actually.

The (generalized) Pauli term thus does not alter the fermionic action. In particular, the fermionic action is fully determined by the (Dirac) connections on the vector bundle $\mathcal{E} \twoheadrightarrow M$ and not, in addition, by the curvature of these (Dirac) connections. As mentioned earlier, this fact is known to play a fundamental role in quantizing the fermionic action. Of course, when the functional $\mathcal{I}'_{\rm D,ferm}$ is actually regarded as being a functional on $\mathrm{Sec}(M, \mathcal{V}_{\mathcal{P}}^{\mathbb{C}})$, then a stationary point of this functional has to satisfy the more restrictive condition:

$$\psi \in \ker(\not{D}_{\mathcal{E}}) \cap \ker(\mathcal{F}_{\rm D}). \tag{101}$$

The equivalence of the two fermionic actions $\mathcal{I}_{\rm D,ferm}$ and $\mathcal{I}'_{\rm D,ferm}$ (when both are regarded as being functionals on the same domain) is very basic for the structure of Dirac-type gauge theories. Indeed, these equivalent geometrical descriptions of the fermionic action seem to provide a deep relation between the fermionic part of the total Dirac action and its corresponding bosonic part.

To formalize the above discussed equivalence of the functionals $\mathcal{I}_{\rm D,ferm}$ and $\mathcal{I}'_{\rm D,ferm}$, we introduce the following.

Definition 4.2: Let $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}, J_{\mathcal{E}})$ *be a real Clifford module bundle over* $(M, g_{\rm M})$ *satisfying the requirements imposed on (74). Also, let* $\mathcal{D}_{\rm real}(\mathcal{E}) \subset \mathcal{D}(\mathcal{E})$ *be the (affine) set of real Dirac operators acting on* $\mathrm{Sec}(M, \mathcal{E})$.

We call the mapping

$$\mathcal{P}_{\rm D}:\mathcal{D}_{\rm real}(\mathcal{E}) \to \mathcal{D}_{\rm real}(\mathcal{P}),$$

$$\not{D}_{\mathcal{E}} \mapsto \not{P}_{\rm D}, \tag{102}$$

which associates with every real Dirac operator on $\mathcal{E} \twoheadrightarrow M$ *the appropriate Pauli-type Dirac operator on the doubled Clifford module* $\mathcal{P} = {}^2\mathcal{E} \twoheadrightarrow M$, *the Pauli map.*

Geometrically, one may regard the fermionic action as a mapping from $\mathcal{D}(\mathcal{E})$ into the quadratic forms on $\mathrm{Sec}(M, \mathcal{E})$,

$$\mathcal{I}_{\rm D,ferm}:\mathcal{D}(\mathcal{E}) \to \mathrm{Map}(\mathrm{Sec}(M,\mathcal{E}), \mathbb{C}),$$

$$\not{D}_{\mathcal{E}} \mapsto \begin{cases} \mathrm{Sec}(M,\mathcal{E}) \to \mathbb{C} \\ \psi \mapsto \mathcal{I}_{\rm D,ferm}(\psi, \not{D}_{\mathcal{E}}). \end{cases} \tag{103}$$

When restricted to $\mathcal{D}_{\rm real}(\mathcal{E})$, the Pauli map thus allows one to lift the quadratic form $\mathcal{I}_{\rm D,ferm}$ on $\mathrm{Sec}(M, \mathcal{E})$ to the quadratic form $\mathcal{I}'_{\rm D,ferm}$ on $\mathrm{Sec}(M, \mathcal{P})$,

$$\mathcal{I}'_{\mathrm{D,ferm}} = \mathcal{I}_{\mathrm{D,ferm}} \circ \mathcal{P}_{\mathrm{D}}, \tag{104}$$

such that $\mathcal{P}_{\mathrm{D}}$ acts like the identity when $\mathcal{I}'_{\mathrm{D,ferm}}$ is restricted to $\mathrm{Sec}(M, \mathcal{V}_{\mathcal{P}}^{\mathbb{C}}) \subset \mathrm{Sec}(M, \mathcal{P})$.

As mentioned earlier, on every Clifford module there exists a distinguished class of Dirac operators called of simple type. Explicitly, they read as

$$\not{D}_{\mathcal{E}} = \not{\partial}_{\mathrm{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathrm{D}}, \tag{105}$$

where $\phi_{\mathrm{D}} \in \mathrm{Sec}(M, \mathrm{End}_{\gamma}^{-}(\mathcal{E}))$. In general, however, these Dirac operators are not real. Therefore, our original Pauli-type Dirac operators (3) fail to be real and our geometrical understanding of this class of Dirac operators in terms of the Pauli map (102) is not yet complete.

Of course, this flaw may most straightforwardly be remedied by giving up the restriction of the Pauli map to real Dirac operators. However, this will then not yield any new insight concerning the structure of the original Pauli-type operator (3). Even worse, one loses significant information as will be shown in Sec. IV A. Indeed, there it will be shown that the Pauli map (102) allows one to naturally include the geometrical description of "Majorana masses" in terms of real Dirac operators of simple type.

A. Majorana masses and real Dirac operators of simple type

Let $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}}, \tau_{\mathcal{S}}, \gamma_{\mathcal{S}}, J_{\mathcal{S}})$ be a real Clifford module bundle over (M, g_{M}). We put

$$\mathcal{E} := {}^2\mathcal{S} = \mathcal{S} \otimes \mathbb{C}^2, \tag{106}$$

$$\langle\cdot,\cdot\rangle_{\mathcal{E}} := \tfrac{1}{2}(\langle\cdot,\cdot\rangle_{\mathcal{S}} + \langle\cdot,\cdot\rangle_{\mathcal{S}}), \tag{107}$$

$$\tau_{\mathcal{E}} := \begin{pmatrix} \tau_{\mathcal{S}} & 0 \\ 0 & -\tau_{\mathcal{S}} \end{pmatrix} = \tau_{\mathcal{S}} \otimes \tau_2, \tag{108}$$

$$\gamma_{\mathcal{E}} := \begin{pmatrix} \gamma_{\mathcal{S}} & 0 \\ 0 & \gamma_{\mathcal{S}}^{\mathrm{cc}} \end{pmatrix}, \tag{109}$$

$$J_{\mathcal{E}} := \begin{pmatrix} 0 & J_{\mathcal{S}} \\ J_{\mathcal{S}} & 0 \end{pmatrix} = J_{\mathcal{S}} \otimes \varepsilon_2. \tag{110}$$

It follows that

$$\tau_{\mathcal{E}}^{\mathrm{cc}} = \pm \tau_{\mathcal{E}} \Leftrightarrow \tau_{\mathcal{S}}^{\mathrm{cc}} = \mp \tau_{\mathcal{S}},$$

$$\gamma_{\mathcal{E}}^{\mathrm{cc}} = + \gamma_{\mathcal{E}}. \tag{111}$$

Definition 4.3: Let $\not{D}_{\mathcal{S}} \in \mathcal{D}(\mathcal{S})$ be a Dirac-type operator on the real Clifford module $\mathcal{S} \twoheadrightarrow M$. The real Dirac-type operator on the induced real Clifford module $\mathcal{E} \twoheadrightarrow M$,

$$\not{D}_{\mathcal{E}} := \begin{pmatrix} \not{D}_{\mathcal{S}} & 0 \\ 0 & \not{D}_{\mathcal{S}}^{\mathrm{cc}} \end{pmatrix},$$

$$\equiv \not{D}_{\mathcal{S}} \oplus \not{D}_{\mathcal{S}}^{\mathrm{cc}}, \tag{112}$$

is called the "real form" of $\not{D}_{\mathcal{S}}$.

Proposition 4.1: The most general real Dirac operator of simple type, acting on $\mathrm{Sec}(M, \mathcal{E})$, *explicitly reads as*

$$\not{D}_{\mathcal{E}} = \not{\partial}_{\mathcal{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}}, \tag{113}$$

whereby $\not{\phi}_{\mathcal{A}} := \not{\phi}_{\rm A} \oplus \not{\phi}_{\rm A}^{\rm cc}$ *is the real form of* $\not{\phi}_{\rm A}$ *and*

$$\phi_{\mathcal{E}} := \begin{pmatrix} \chi_{\mathcal{S}} & \pm \phi_{\mathcal{S}}^{\rm cc} \\ -\phi_{\mathcal{S}} & \mp \chi_{\mathcal{S}}^{\rm cc} \end{pmatrix}, \tag{114}$$

depending on whether $\tau_{\mathcal{S}}^{\rm cc} = \pm \tau_{\mathcal{S}}$. *Moreover,* $\phi_{\mathcal{S}} \in \mathrm{Sec}(M, \mathrm{End}_\gamma^+(\mathcal{S}))$ *is explicitly given by*

$$\phi_{\mathcal{S}} \equiv \begin{cases} \chi_{\mathcal{S}}' + \tau_{\mathcal{S}} \circ \delta_\gamma(\sigma_{\mathcal{S}}), & \textit{for} \quad \gamma_{\mathcal{S}}^{\rm cc} = +\gamma_{\mathcal{S}}, \\ \tau_{\mathcal{S}} \circ \mu_{\rm M} + \delta_\gamma(\sigma_{\mathcal{S}}), & \textit{for} \quad \gamma_{\mathcal{S}}^{\rm cc} = -\gamma_{\mathcal{S}}. \end{cases} \tag{115}$$

Here, $\mu_{\rm M}, \chi_{\mathcal{S}}' \in \Omega^0(M, \mathrm{End}_\gamma^+(\mathcal{S}))$, $\chi_{\mathcal{S}} \in \Omega^0(M, \mathrm{End}_\gamma^-(\mathcal{S}))$ and $\sigma_{\mathcal{S}} \in \Omega^1(M, \mathrm{End}_\gamma^-(\mathcal{S}))$.

The proof of the above statement is based on the following statement and a corollary thereof. Both of which are interesting in its own and will be useful also later on.

Lemma 4.1: Let $(\mathcal{E}, \gamma_{\mathcal{E}}) \twoheadrightarrow (M, g_{\rm M})$ *be a general Clifford module over a smooth (semi)Riemannian manifold. Also, let* $\not{D}_k + \Phi_k \in \mathcal{D}(\mathcal{E})(k=1,2)$ *be two Dirac-type operators, acting on* $\mathrm{Sec}(M, \mathcal{E})$. *The Laplace type operator*

$$H := (\not{D}_1 + \Phi_1) \circ (\not{D}_2 + \Phi_2) \tag{116}$$

has the explicit Lichnerowicz decomposition: $H = \Delta_{\rm H} + V_{\rm H}$, *where the second order part is defined in terms of the connection,*

$$\partial_{\rm H} := \partial_{\rm B} + \alpha_{\rm H},$$

$$\alpha_{\rm H}(v) := \frac{\varepsilon}{2}(\gamma_{\mathcal{E}}(v^\flat) \circ \Phi_2 + \Phi_1 \circ \gamma_{\mathcal{E}}(v^\flat) + (\not{D}_1 - \not{D}_2) \circ \gamma_{\mathcal{E}}(v^\flat)) \tag{117}$$

for all $v \in TM$. *The zero order part explicitly reads as*

$$V_{\rm H} := V_{\rm D} + \delta_\gamma(\partial_{\rm B}\Phi_2) - \varepsilon ev_g(\partial_{\rm H}\alpha_{\rm H}) - \varepsilon ev_g(\alpha_{\rm H}^2) + \Phi_{\rm D} \circ \Phi_2 + (\Phi_1 + (\not{D}_1 - \not{D}_2)) \circ (\Phi_2 + \Phi_{\rm D}). \tag{118}$$

Here, $\partial_{\rm B} \in \mathcal{A}(\mathcal{E})$ *denotes the Bochner connection that is defined by* $\not{D}_2 \equiv \not{D}$, *and*

$$V_{\rm D} := \not{D}^2 - \Delta_{\rm B},$$

$$\Phi_{\rm D} := \not{D} - \not{\partial}_{\rm B}. \tag{119}$$

Proof: First, we again put $\not{D} \equiv \not{D}_2$ and abbreviate $\Phi_{12} \equiv \not{D}_1 - \not{D}_2$ to rewrite H as

$$H = \not{D}^2 + [\not{D}, \Phi_2] + (\Phi_1 + \Phi_2 + \Phi_{12}) \circ \not{D} + (\Phi_1 + \Phi_{12}) \circ \Phi_2. \tag{120}$$

It follows that for all $f \in \mathcal{C}^\infty(M)$,

$$[[\not{D}, \Phi_2], f] = [\delta_\gamma(df), \Phi_2] \tag{121}$$

and thus

$$[H, f] = [\not{D}^2, f] + \delta_\gamma(df) \circ \Phi_2 + \Phi_1 \circ \delta_\gamma(df) + \Phi_{12} \circ \delta_\gamma(df). \tag{122}$$

This yields the explicit formula (117) for the connection $\partial_{\rm H}$.

The explicit formula (118) for the zero order term is then obtained from the identity $V_{\rm H} = H - \Delta_{\rm H}$ and

$$\Delta_{\rm H} \equiv \varepsilon ev_g(\partial_{\rm H} \circ \partial_{\rm H}) = \Delta_{\rm D} + \varepsilon ev_g(\partial_{\rm H}\alpha_{\rm H}) + \varepsilon ev_g(\alpha_{\rm H}^2) + 2\varepsilon ev_g(\alpha_{\rm H}, \partial_{\rm B}). \tag{123}$$

This proves the statement. □

For later convenience, we consider $V_{\mathrm{H}} \equiv V_{\mathrm{D}}$ in the case where $H = D\!\!\!\!/^2$ and $D\!\!\!\!/ = \partial\!\!\!/_{\mathrm{A}} + \Phi$. From Lemma 4.1, it follows for $D\!\!\!\!/_1 = D\!\!\!\!/_2 \equiv \partial\!\!\!/_{\mathrm{A}}$ and $\Phi_1 = \Phi_2 \equiv \Phi$ that

$$V_{\mathrm{D}} = \delta_\gamma(\mathrm{curv}(\partial\!\!\!/_{\mathrm{A}})) + \delta_\gamma(\partial_{\mathrm{A}}\Phi) + \Phi^2 - \varepsilon ev_g(\alpha_{\mathrm{D}}^2) - \varepsilon ev_g(\partial_{\mathrm{B}}\alpha_{\mathrm{D}}), \tag{124}$$

whereby $\partial_{\mathrm{B}} = \partial_{\mathrm{A}} + \alpha_{\mathrm{D}}$ and

$$\alpha_{\mathrm{D}}(v) := \frac{\varepsilon}{2}\{\gamma_{\mathcal{E}}(v^\flat), \Phi\}. \tag{125}$$

Clearly, Lemma 4.1 generalizes the well-known formula by Lichnerowicz/Schrödinger (54) with respect to $\partial\!\!\!/_{\mathrm{A}}^2$ to general Laplacians which can be factorized by arbitrary Dirac-type operators. The next statement yields an easy characterization of simply type Dirac operators.

Corollary 4.1: A Dirac operator $D\!\!\!\!/$ on a $\mathbb{Z}_2$-graded Clifford module $(\mathcal{E}, \gamma_{\mathcal{E}}) \twoheadrightarrow (M, g_{\mathrm{M}})$ is of simple type if and only if

$$\{D\!\!\!\!/ - \partial\!\!\!/_{\mathrm{B}}, \gamma_{\mathcal{E}}(\alpha)\} \equiv 0 \tag{126}$$

for all $\alpha \in T^*M$. Here, $\partial\!\!\!/_{\mathrm{B}} \equiv \delta_\gamma \circ \partial_{\mathrm{B}}$ is the quantized Bochner connection that is defined by $D\!\!\!\!/ \in \mathcal{D}(\mathcal{E})$.

Proof: It follows from Lemma 4.1 that two Dirac-type operators $D\!\!\!\!/', D\!\!\!\!/ \in \mathcal{D}(\mathcal{E})$ share the same Bochner connection if and only if the zero order operator $D\!\!\!\!/' - D\!\!\!\!/$ anticommutes with the Clifford action [cf. formula (125)]. Whence, $D\!\!\!\!/$ and $\partial\!\!\!/_{\mathrm{B}}$ have the same Bochner connection ∂_{B} if and only if $D\!\!\!\!/ - \partial\!\!\!/_{\mathrm{B}}$ anticommutes with the Clifford action. However, Clifford connections $\partial_{\mathrm{A}} \in \mathcal{A}_{\mathrm{Cl}}(\mathcal{E})$ are the only connections with the property that the three notions of Dirac connection, Clifford connection, and Bochner connection coincide, i.e.,

$$\partial_{\mathrm{D}} = \partial_{\mathrm{A}} = \partial_{\mathrm{B}}. \tag{127}$$

Whence, the Dirac-type operator $\partial\!\!\!/_{\mathrm{B}} \in \mathcal{D}(\mathcal{E})$ yields the Bochner connection ∂_{B} if and only if $\partial_{\mathrm{B}} \in \mathcal{A}_{\mathrm{Cl}}(\mathcal{E})$. This proves the statement. □

We now turn back to the proof of Proposition 4.1.

Proof of Proposition 4.1: The most general real Dirac operator, acting on $\mathrm{Sec}(M, \mathcal{E})$, reads as

$$D\!\!\!\!/'_{\mathcal{E}} = \begin{pmatrix} D\!\!\!\!/_{\mathcal{S}} & \Phi_{\mathcal{S}}^{\mathrm{cc}} \\ \Phi_{\mathcal{S}} & D\!\!\!\!/_{\mathcal{S}}^{\mathrm{cc}} \end{pmatrix}, \tag{128}$$

whereby $\Phi_{\mathcal{S}} \in \mathrm{Sec}(M, \mathrm{End}^+(\mathcal{S}))$. We may rewrite this real Dirac operator as

$$D\!\!\!\!/'_{\mathcal{E}} = D\!\!\!\!/_{\mathcal{E}} + \Phi'_{\mathcal{E}} \tag{129}$$

with $D\!\!\!\!/_{\mathcal{E}}$ being the real form of $D\!\!\!\!/_{\mathcal{S}}$ and

$$\Phi'_{\mathcal{E}} \equiv \begin{pmatrix} 0 & \Phi_{\mathcal{S}}^{\mathrm{cc}} \\ \Phi_{\mathcal{S}} & 0 \end{pmatrix}. \tag{130}$$

Let, respectively, $\partial_{\mathrm{B}'}$ and ∂_{B} be the Bochner connections of $D\!\!\!\!/'_{\mathcal{E}}$ and $D\!\!\!\!/_{\mathcal{E}}$. Then, Lemma 4.1 implies that

$$\partial_{\mathrm{B}'} = \partial_{\mathrm{B}} + \alpha_{\mathrm{D}'},$$

$$\alpha_{\mathrm{D}'}(v) = \frac{\varepsilon}{2}\{\gamma_{\mathcal{E}}(v^\flat), \Phi'_{\mathcal{E}}\}. \tag{131}$$

By assumption, $\partial_{\mathrm{B}'} \in \mathcal{A}_{\mathrm{Cl}}(\mathcal{E})$. We show that also ∂_{B} is a Clifford connection and thus $\alpha_{\mathrm{D}'}$ has to commute with the Clifford action. This condition will eventually give us the explicit form of the zero order operator $\Phi'_{\mathcal{E}}$.

Indeed, it follows that

$$\not{D}'_{\mathcal{E}} = \not{\partial}_{\mathrm{B}'} + \Phi_{\mathrm{D}'} = \not{\partial}_{\mathrm{B}} + \not{\alpha}_{\mathrm{D}'} + \Phi_{\mathrm{D}'} = \not{\partial}_{\mathrm{B}} + \Phi_{\mathrm{D}} + \Phi'_{\mathcal{E}}, \tag{132}$$

where $\Phi_{\mathrm{D}} = \not{D}_{\mathcal{E}} - \not{\partial}_{\mathrm{B}}$. Therefore,

$$\Phi_{\mathrm{D}'} = \tau_{\mathcal{E}} \circ \phi_{\mathrm{D}'} = \Phi_{\mathrm{D}} + \Phi'_{\mathcal{E}} - \not{\alpha}_{\mathrm{D}'}. \tag{133}$$

Whence,

$$\phi_{\mathrm{D}'} = \tau_{\mathcal{E}} \circ \Phi_{\mathrm{D}} + \tau_{\mathcal{E}} \circ (\Phi'_{\mathcal{E}} - \not{\alpha}_{\mathrm{D}'}) \tag{134}$$

and the condition $\phi_{\mathrm{D}'} \in \mathrm{Sec}(M, \mathrm{End}^{-}_{\gamma}(\mathcal{E}))$ yields the equivalence,

$$[\phi_{\mathrm{D}'}, \gamma_{\mathcal{E}}(\alpha)] = 0 \Leftrightarrow \begin{cases} \{\Phi_{\mathrm{D}}, \gamma_{\mathcal{E}}(\alpha)\} = 0 \\ \{(\Phi'_{\mathcal{E}} - \not{\alpha}_{\mathrm{D}'}), \gamma_{\mathcal{E}}(\alpha)\} = 0 \end{cases} \tag{135}$$

for all $\alpha \in T^*M$.

According to Corollary 4.1, the first relation of (135) implies that also $\partial_{\mathrm{B}} \in \mathcal{A}_{\mathrm{Cl}}(\mathcal{E})$. Whence, $\not{D}_{\mathcal{E}}$ is of simple type

$$\not{D}_{\mathcal{E}} = \begin{pmatrix} \not{\partial}_{\mathrm{A}} + \tau_{\mathcal{S}} \circ \chi_{\mathcal{S}} & 0 \\ 0 & (\not{\partial}_{\mathrm{A}} + \tau_{\mathcal{S}} \circ \chi_{\mathcal{S}})^{\mathrm{cc}} \end{pmatrix}, \tag{136}$$

with $\chi_{\mathcal{S}} \in \mathrm{Sec}(M, \mathrm{End}^{-}_{\gamma}(\mathcal{S}))$.

Moreover, being the difference of two Clifford connections, it follows that

$$[\alpha_{\mathrm{D}'}(v), \gamma_{\mathcal{E}}(\alpha)] \equiv 0 \tag{137}$$

for all $v \in TM$ and $\alpha \in T^*M$. Taking into account the explicit form of $\alpha_{\mathrm{D}'}$, the condition (137) is seen to be equivalent to

$$[[\Phi_{\mathcal{S}}, \gamma_{\mathcal{S}}(\alpha_1)]_{\pm}, \gamma_{\mathcal{S}}(\alpha_2)]_{\mp} \equiv 0 \tag{138}$$

for all $\alpha_1, \alpha_2 \in T^*M$. Here, $[x,y]_{\pm} \equiv xy \pm yx$, with the relative sign referring to $\gamma^{\mathrm{cc}}_{\mathcal{S}} = \pm \gamma_{\mathcal{S}}$.

It follows that

$$\Phi_{\mathcal{S}} = \begin{cases} \delta_{\gamma}(\sigma_{\mathcal{S}}) + \tau_{\mathcal{S}} \circ \chi'_{\mathcal{S}} & \text{for } \gamma^{\mathrm{cc}}_{\mathcal{S}} = +\gamma_{\mathcal{S}} \\ \mu_{\mathrm{M}} + \tau_{\mathcal{S}} \circ \delta_{\gamma}(\sigma_{\mathcal{S}}) & \text{for } \gamma^{\mathrm{cc}}_{\mathcal{S}} = -\gamma_{\mathcal{S}}, \end{cases} \tag{139}$$

with $\chi'_{\mathcal{S}}, \mu_{\mathrm{M}} \in \mathrm{Sec}(M, \mathrm{End}^{+}_{\gamma}(\mathcal{S}))$ and $\sigma_{\mathcal{S}} \in \Omega^1(M, \mathrm{End}^{-}_{\gamma}(\mathcal{S}))$.

For reasons of consistency, we still have to verify the second relation of (135) in order to complete the proof of Proposition 4.1. However, this is done straightforwardly taking the explicit solution (139) of (137) into account. □

The significance of Proposition 4.1 is given by generalizing the notion of simple type Dirac operators to those which are also real. These are certainly distinguished Dirac operators on the real Clifford module $\mathcal{E} = {}^2\mathcal{S} \twoheadrightarrow M$ on which one may then apply the Pauli map (102). Even more, these real simple type Dirac operators also allow one to incorporate Majorana masses within the scheme of Dirac-type gauge theories. For this, let $(\mathcal{S}, \partial)$ be a flat Majorana module with an imaginary Clifford action and grading involution. The stationary points of the fermionic action $\mathcal{I}_{\mathrm{D,ferm}}$, which is defined by the real Dirac operator of simple type

$$\not{D}_{\mathrm{M}} := \begin{pmatrix} \not{\partial} & i\mu_{\mathrm{M}} \\ -i\mu_{\mathrm{M}} & -\not{\partial} \end{pmatrix} \in \mathcal{D}_{\mathrm{real}}(\mathcal{E}) \tag{140}$$

with $\mu_{\mathrm{M}} \in \mathrm{Sec}(M, \mathrm{End}^{+}_{\gamma}(\mathcal{S}))$ being real, fulfill the Majorana equations,

$$i\partial\!\!\!/\chi = \mu_{\mathrm{M}}\chi^{\mathrm{cc}},$$

$$i\partial\!\!\!/\chi^{\mathrm{cc}} = \mu_{\mathrm{M}}\chi. \tag{141}$$

We note that the (total space of the) real subvector bundle $\mathcal{M}_{\mathcal{E}} \twoheadrightarrow M$, whose complexification equals $\mathcal{E} \twoheadrightarrow M$, reads as

$$\mathcal{M}_{\mathcal{E}} = \left\{ \begin{pmatrix} z \\ z^{\mathrm{cc}} \end{pmatrix} \in \mathcal{E} \middle| z \in \mathcal{S} \right\}. \tag{142}$$

Hence, $D\!\!\!\!/_{\mathrm{M}}$ leaves the real module $\mathrm{Sec}(M,\mathcal{M}_{\mathcal{E}})$ invariant.

Equation (141) is diagonal with respect to the grading involution $\tau_{\mathcal{S}}$. In particular, it is diagonal with respect to the chirality involution τ_{M},

$$D\!\!\!\!/_{\mathrm{M}}\psi = 0 \Leftrightarrow \begin{cases} i\partial\!\!\!/\chi_{\mathrm{R}} = \mu_{\mathrm{M}}\chi_{\mathrm{R}}^{\mathrm{cc}} \\ i\partial\!\!\!/\chi_{\mathrm{L}} = \mu_{\mathrm{M}}\chi_{\mathrm{L}}^{\mathrm{cc}}, \end{cases} \tag{143}$$

plus the corresponding conjugate equations. Here, we have put

$$\psi = \begin{pmatrix} \chi \\ \chi^{\mathrm{cc}} \end{pmatrix} \in \mathrm{Sec}(M,\mathcal{M}_{\mathcal{E}})$$

an the chiral eigensections of τ_{M} are again denoted by $\chi_{\mathrm{R}},\chi_{\mathrm{L}} \in \mathrm{Sec}(M,\mathcal{S})$ such that $\chi = \chi_{\mathrm{R}} + \chi_{\mathrm{L}}$.

In this section, we discussed how the Majorana equations can be described in terms of real Dirac operators of simple type on real Clifford modules. We turn now to the corresponding discussion of the *Dirac–Yukawa equation*:,

$$i\partial\!\!\!/_{\mathrm{A}}\chi = \varphi_{\mathrm{D}}\chi \Leftrightarrow \begin{cases} i\partial\!\!\!/_{\mathrm{A}}\chi_{\mathrm{R}} = \varphi_{\mathrm{D}}\chi_{\mathrm{L}} \\ i\partial\!\!\!/_{\mathrm{A}}\chi_{\mathrm{L}} = \varphi_{\mathrm{D}}\chi_{\mathrm{R}}. \end{cases} \tag{144}$$

The *Yukawa (coupling) term* φ_{D} generalizes in a gauge covariant manner the usual mass term m_{D} of the Dirac equation (9) with help of the Higgs field.

B. Dirac masses and real Dirac operators of simple type

In Sec. VII we have shown how Majorana masses can be geometrically described in terms of a real Clifford module if the latter is considered as being the doubling of a Majorana module. In order to also geometrically describe Dirac masses within Dirac-type gauge theories we have to consider special Majorana modules $\mathcal{S} \twoheadrightarrow M$, called *Dirac modules*. More precisely, we make the following.

Definition 4.4: A real Clifford module

$$(\mathcal{S},\langle\cdot,\cdot\rangle_{\mathcal{S}},\tau_{\mathcal{S}},\gamma_{\mathcal{S}},J_{\mathcal{S}}) \tag{145}$$

is called a "Dirac module" provided that there is a Majorana module $(\mathcal{W},\langle\cdot,\cdot\rangle_{\mathcal{W}},\tau_{\mathcal{W}},\gamma_{\mathcal{W}},J_{\mathcal{W}})$ *over* (M,g_{M}) *such that*

$$\mathcal{S} = {}^{2}\mathcal{W} = \mathcal{W} \otimes \mathbb{C}^{2} \tag{146}$$

and

$$\tau_{\mathcal{S}} = \begin{pmatrix} \mathrm{id}_{\mathcal{W}} & 0 \\ 0 & -\mathrm{id}_{\mathcal{W}} \end{pmatrix} = \mathrm{id}_{\mathcal{W}} \otimes \tau_{2}, \tag{147}$$

$$\gamma_{\mathcal{S}} = \begin{pmatrix} 0 & \gamma_{\mathcal{W}} \\ \gamma_{\mathcal{W}} & 0 \end{pmatrix} = \gamma_{\mathcal{W}} \otimes \varepsilon_{2}, \tag{148}$$

$$J_{\mathcal{S}} = \begin{pmatrix} 0 & J_{\mathcal{W}} \\ J_{\mathcal{W}} & 0 \end{pmatrix} = J_{\mathcal{W}} \otimes \varepsilon_2. \tag{149}$$

Finally,

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_{\mathcal{S}} = \langle u_1, v_2 \rangle_{\mathcal{W}} \pm \langle u_2, v_1 \rangle_{\mathcal{W}}, \tag{150}$$

depending on whether $\langle J_{\mathcal{W}}(u), J_{\mathcal{W}}(v)\rangle_{\mathcal{W}} = \pm \langle v, u\rangle_{\mathcal{W}}$ *for all* $u, v \in \mathcal{W}$.

It follows that

$$\tau_{\mathcal{S}}^{\mathrm{cc}} = -\tau_{\mathcal{S}},$$

$$\gamma_{\mathcal{S}}^{\mathrm{cc}} = \pm \gamma_{\mathcal{S}} \Leftrightarrow \gamma_{\mathcal{W}}^{\mathrm{cc}} = \pm \gamma_{\mathcal{W}}. \tag{151}$$

We remark that the relation between Majorana and Dirac modules parallels the relation between the "Dirac matrices" and the "Pauli matrices" (i.e., the construction of $\mathrm{Cl}_{1,3}$ from Cl_3). The corresponding change in the Hermitian product is taken into account in (150), which is not positive definite even if the Hermitian product on the underlying Majorana module is positive definite. To have a specific example in mind, one may choose for the Majorana module the Pauli spinors. A corresponding Dirac module is then provided by the Dirac spinors. (I would like to thank the referee for appropriate comments.) Note that the definition of the grading involution, the Clifford action and the real structure of a Dirac module is similar to the Majorana representation of (the complexification of) $\mathrm{Cl}_{1,3}$.

Let $\partial\!\!\!/_{\mathrm{A}} \in \mathcal{D}(\mathcal{W})$ and $\varphi_{\mathrm{D}} \in \mathrm{Sec}(M, \mathrm{End}_{\gamma}(\mathcal{W}))$. Furthermore, we assume that $\partial\!\!\!/_{\mathrm{A}} \pm i\varphi_{\mathrm{D}}$ are S-reducible if $\mathcal{W} \hookrightarrow \Lambda_{\mathrm{M}} \otimes E \twoheadrightarrow M$. We put

$$D\!\!\!\!/_{\mathrm{D}} := \begin{pmatrix} 0 & \partial\!\!\!/_{\mathrm{A}} - i\varphi_{\mathrm{D}} \\ \partial\!\!\!/_{\mathrm{A}} + i\varphi_{\mathrm{D}} & 0 \end{pmatrix} \in \mathcal{D}(\mathcal{S}), \tag{152}$$

which is easily seen to be of simple type. In fact, one may rewrite $D\!\!\!\!/_{\mathrm{D}}$ as

$$D\!\!\!\!/_{\mathrm{D}} = \partial\!\!\!/_{\mathrm{A}} + i\mu_{\mathrm{D}}, \tag{153}$$

with

$$\mu_{\mathrm{D}} \equiv -\tau_{\mathcal{S}} \circ \phi_{\mathrm{D}},$$

$$\phi_{\mathrm{D}} := \varphi_{\mathrm{D}} \otimes \varepsilon_2 \in \mathrm{Sec}(M, \mathrm{End}_{\gamma}^{-}(\mathcal{S})). \tag{154}$$

Here, by a slight abuse of notation, we identify $\partial_{\mathrm{A}} \in \mathcal{A}_{\mathrm{Cl}}(\mathcal{W})$ with

$$\partial_{\mathrm{A}} = \begin{pmatrix} \partial_{\mathrm{A}} & 0 \\ 0 & \partial_{\mathrm{A}} \end{pmatrix} \in \mathcal{A}_{\mathrm{Cl}}(\mathcal{S}), \tag{155}$$

such that always $\partial\!\!\!/_{\mathrm{A}} = \delta_{\gamma} \circ \partial_{\mathrm{A}}$ and, respectively, $\partial\!\!\!/_{\mathrm{A}} \in \mathcal{D}(\mathcal{W})$, or $\partial\!\!\!/_{\mathrm{A}} \in \mathcal{D}(\mathcal{S})$, depending on whether "$\gamma$" denotes either $\gamma_{\mathcal{W}}$ or $\gamma_{\mathcal{S}}$.

Note that the simple type Dirac operator $D\!\!\!\!/_{\mathrm{D}} \in \mathcal{D}(\mathcal{S})$ is not real. Also, the first order operators $\partial\!\!\!/_{\mathrm{A}} \pm i\varphi_{\mathrm{D}} \in \mathcal{D}(\mathcal{W})$ are not Dirac operators, in general. Clearly, for constant sections $\varphi_{\mathrm{D}} = m_{\mathrm{D}}$, these two operators are but the complex factors of the *Klein–Gordon operator*: $D\!\!\!\!/_{\mathrm{D}}^2 = \partial\!\!\!/_{\mathrm{A}}^2 + m_{\mathrm{D}}^2$.

Also note that the most general Dirac operators on a Dirac module read as

$$D\!\!\!\!/_{\mathcal{S}} := \begin{pmatrix} 0 & D\!\!\!\!/_{\mathcal{W},1} \\ D\!\!\!\!/_{\mathcal{W},2} & 0 \end{pmatrix} \in \mathcal{D}(\mathcal{S}), \tag{156}$$

where, respectively, $D\!\!\!\!/_{\mathcal{W},1}, D\!\!\!\!/_{\mathcal{W},2} \in \mathcal{D}(\mathcal{W})$ are of Dirac type. In particular, the most general real Dirac operators on a Dirac module are given by

$$D\!\!\!\!/_{\mathcal{S}} := \begin{pmatrix} 0 & D\!\!\!\!/_{\mathcal{W}}^{\mathrm{cc}} \\ D\!\!\!\!/_{\mathcal{W}} & 0 \end{pmatrix} \in \mathcal{D}_{\mathrm{real}}(\mathcal{S}), \tag{157}$$

In either case, the Dirac operators on a Dirac module are thus parametrized by general first order differential operators, acting on $\mathrm{Sec}(M,\mathcal{W})$, such that their principal symbols are determined by the Clifford action of the underlying Majorana module. Then, our Lemma 4.1 provides an explicit (global) formula for the corresponding Lichnerowicz/Schrödinger decomposition of any such Dirac operator $D\!\!\!\!/_{\mathcal{S}} \in \mathcal{D}(\mathcal{S})$ in terms of the underlying Dirac operators $D\!\!\!\!/_{\mathcal{W},1}, D\!\!\!\!/_{\mathcal{W},2} \in \mathcal{D}(\mathcal{W})$.

Finally, the most general Dirac operator of simple type, acting on sections of a Dirac module, takes the form

$$D\!\!\!\!/_{\mathcal{S}} := \begin{pmatrix} 0 & \partial\!\!\!/_{\mathrm{A}} - \Phi_{\mathcal{W}} \\ \partial\!\!\!/_{\mathrm{A}} + \Phi_{\mathcal{W}} & 0 \end{pmatrix} \in \mathcal{D}(\mathcal{S}), \tag{158}$$

with $\Phi_{\mathcal{W}} \in \mathrm{Sec}(M, \mathrm{End}_{\gamma}(\mathcal{W}))$ being a general section. Indeed, for

$$\mu_{\mathcal{S}} := \Phi_{\mathcal{W}} \otimes \mathrm{I}_2 \in \mathrm{Sec}(M, \mathrm{End}(\mathcal{S})) \tag{159}$$

one obtains that for all $\alpha \in T^*M$,

$$\{\gamma_{\mathcal{S}}(\alpha), \mu_{\mathcal{S}}\} = \gamma_{\mathcal{W}}(\alpha) \circ \Phi_{\mathcal{W}} \otimes \{\varepsilon_2, \mathrm{I}_2\} = 0. \tag{160}$$

Therefore, $D\!\!\!\!/_{\mathcal{S}} \equiv \partial\!\!\!/_{\mathrm{A}} + \mu_{\mathcal{S}}$ is of simple type, whereby

$$\mu_{\mathcal{S}} := -\,\tau_{\mathcal{S}} \circ \phi_{\mathcal{S}},$$

$$\phi_{\mathcal{S}} := \Phi_{\mathcal{W}} \otimes \varepsilon_2 \in \mathrm{Sec}(M, \mathrm{End}_{\gamma}^{-}(\mathcal{S})). \tag{161}$$

Note that real Dirac operators on a Dirac module cannot be of simple type and vice versa.

To clarify how the Dirac–Yukawa equation (144) may arise from the Dirac functional $\mathcal{I}'_{\mathrm{D,ferm}}$, one simply considers the real form of the (symmetric) simple type Dirac operator $D\!\!\!\!/_{\mathrm{D}} = \partial\!\!\!/_{\mathrm{A}} + i\mu_{\mathrm{D}}$ on the Dirac module $\mathcal{S} \twoheadrightarrow M$, thereby defining a real Dirac operator of simple type on the associated real Clifford module $\mathcal{E} \twoheadrightarrow M$. Clearly,

$$\mathcal{I}_{\mathrm{D,ferm}}(\mathcal{P}_{\mathrm{D}}(D\!\!\!\!/_{\mathrm{D}} \oplus D\!\!\!\!/_{\mathrm{D}}^{\mathrm{cc}}))({}^2\Psi) = \mathcal{I}_{\mathrm{D,ferm}}(\psi, D\!\!\!\!/_{\mathrm{D}}). \tag{162}$$

Here,

$$\psi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \in \mathrm{Sec}(M, \mathcal{S}), \quad \Psi = \begin{pmatrix} \psi \\ \psi^{\mathrm{cc}} \end{pmatrix} \in \mathrm{Sec}(M, \mathcal{E}), \tag{163}$$

where $\chi_1, \chi_2 \in \mathrm{Sec}(M, \mathcal{W})$ are arbitrary sections, which are in one-to-one correspondence with arbitrary eigensections of the involution $\tau_{\mathcal{S}}$ (not of $\tau_{\mathcal{W}}$!). Hence, to recover the Dirac–Yukawa equation (144) one may restrict to the eigensections of $\tau_{\mathcal{S}}$, corresponding to the eigenvalue equal to $+1$. That is,

$$i\partial\!\!\!/_{\mathrm{A}}\chi = \varphi_{\mathrm{D}}\chi \Leftrightarrow \begin{cases} D\!\!\!\!/_{\mathrm{D}}\psi = 0 \\ \tau_{\mathcal{S}}\psi = \psi. \end{cases} \tag{164}$$

This "solves" the issue of fermion doubling already mentioned in Sec. I [and carefully discussed, for example, Refs. 26, 27, 18, and 37 (see also Ref. 15)].

For $\varphi_{\rm D} \in \mathrm{Sec}(M, \mathrm{End}_\gamma(\mathcal{W}))$, the Dirac–Yukawa equation is odd with respect to the chirality involution (46),

$$i\partial\!\!\!/_{\rm A}\chi = \varphi_{\rm D}\chi \Leftrightarrow \begin{cases} i\partial\!\!\!/_{\rm A}\chi_{\rm R} = \varphi_{\rm D}\chi_{\rm L} \\ i\partial\!\!\!/_{\rm A}\chi_{\rm L} = \varphi_{\rm D}\chi_{\rm R}. \end{cases} \tag{165}$$

Again, $\chi_{\rm R}, \chi_{\rm L} \in \mathrm{Sec}(M, \mathcal{W})$ denote the chiral eigensections: $\tau_{\rm M}\chi_{\rm R/L} = \pm\chi_{\rm R/L}$ such that $\chi = \chi_{\rm R} + \chi_{\rm L}$.

For $\varphi_{\rm D} \in \mathrm{Sec}(M, \mathrm{End}^+_\gamma(\mathcal{W}))$ the Dirac–Yukawa equation is gauge covariant only if the chiral eigensections carry the same representation of the underlying gauge group (i.e., the fermions are considered "left-right gauge symmetric"). Otherwise, the Yukawa coupling term has to be odd: $\varphi_{\rm D} \in \mathrm{Sec}(M, \mathrm{End}^-_\gamma(\mathcal{W}))$.

The first order differential operators $\partial\!\!\!/_{\rm A} \pm i\varphi_{\rm D}$ on the Majorana module $\mathcal{W} \twoheadrightarrow M$ are of Dirac type, in general. In contrast, the induced first order operator $D\!\!\!\!/_{\rm D} \in \mathcal{D}(\mathcal{S})$ is always a Dirac operator (of simple type) on the corresponding Dirac module. We stress once more that in any case both Dirac-type operators $\partial\!\!\!/_{\rm A} \pm i\varphi_{\rm D}$ are needed, actually, to define a simple type Dirac operator on the Dirac module, thereby excluding the reality of $D\!\!\!\!/_{\rm D}$. Finally, the Dirac–Yukawa equation is clearly diagonal with respect to the action of "charge conjugation" $J_{\mathcal{W}}$.

In Sec. V, we discuss the combined Dirac–Yukawa–Majorana (DYM) equation and its implication for the Dirac action. We also briefly discuss the bundle structure that allows one to regard the Majorana masses as constant sections of the Dirac module bundle associated with a Majorana module.

V. THE PAULI MAP OF THE COMBINED DIRAC–YUKAWA AND DIRAC–MAJORANA OPERATORS

In the following, let $(\mathcal{W}, \partial)$ be a (partially) flat Majorana module over $(M, g_{\rm M})$ such that $\gamma^{\rm cc}_{\mathcal{W}} = -\gamma_{\mathcal{W}}$.

We may complement the Majorana operator $D\!\!\!\!/_{\rm M} \in \mathcal{D}(\mathcal{E})$ by the replacement of the real Dirac operator $\partial\!\!\!/ \ominus \partial\!\!\!/$ on the Dirac module $\mathcal{E} \twoheadrightarrow M$ by the real form of $D\!\!\!\!/_{\rm D} = \partial\!\!\!/_{\rm A} + i\mu_{\rm D} \in \mathcal{D}(\mathcal{S})$ to obtain the following real Dirac operator of simple type:

$$D\!\!\!\!/_{\rm YM} := \begin{pmatrix} \partial\!\!\!/_{\rm A} + i\mu_{\rm D} & i\mu_{\rm M} \\ -i\mu_{\rm M} & (\partial\!\!\!/_{\rm A} + i\mu_{\rm D})^{\rm cc} \end{pmatrix} \equiv \partial\!\!\!/_{\mathcal{A}} + i\mu_{\rm YM} \in \mathcal{D}_{\rm real}(\mathcal{E}). \tag{166}$$

Here, respectively,

$$\partial\!\!\!/_{\mathcal{A}} := \partial\!\!\!/_{\rm A} \oplus \partial\!\!\!/^{\rm cc}_{\rm A} \in \mathcal{D}_{\rm real}(\mathcal{E}) \tag{167}$$

is the real form of $\partial\!\!\!/_{\rm A} \in \mathcal{D}(\mathcal{S})$ and

$$\mu_{\rm YM} := \tau_{\mathcal{E}} \circ \phi_{\rm YM}, \tag{168}$$

$$\phi_{\rm YM} := \begin{pmatrix} \tau_{\mathcal{S}} \circ \mu_{\rm D} & \tau_{\mathcal{S}} \circ \mu_{\rm M} \\ \tau_{\mathcal{S}} \circ \mu_{\rm M} & -(\tau_{\mathcal{S}} \circ \mu_{\rm D})^{\rm cc} \end{pmatrix} \equiv \begin{pmatrix} -\phi_{\rm D} & \phi_{\rm M} \\ \phi_{\rm M} & \phi^{\rm cc}_{\rm D} \end{pmatrix} \in \mathrm{Sec}(M, \mathrm{End}^-_\gamma(\mathcal{E})). \tag{169}$$

We call in mind that the *Majorana mass operator* $\mu_{\rm M} \in \mathrm{Sec}(M, \mathrm{End}^+_\gamma(\mathcal{S}))$ is supposed to be real. In contrast, no such reality assumption is imposed on the *Dirac mass operator* $\mu_{\rm D} \in \mathrm{Sec}(M, \mathrm{End}^-(\mathcal{S}))$, which has to fulfill the requirement,

$$\{\mu_{\rm D}, \gamma_{\mathcal{S}}(\alpha)\} = 0, \tag{170}$$

for all $\alpha \in T^*M$.

We call $D\!\!\!\!/_{\rm YM} = \partial\!\!\!/_{\mathcal{A}} + i\mu_{\rm YM} \in \mathcal{D}_{\rm real}(\mathcal{E})$ the *DYM operator*.

Let $\chi \in \mathrm{Sec}(M, \mathcal{W})$ and $\psi = \binom{\chi}{0} \in \mathrm{Sec}(M, \mathcal{S})$ be the associated eigensection of $\tau_{\mathcal{S}}$ that corresponds to the eigenvalue equal to +1. Also let

$$\not{P}_{\mathrm{DYM}} := \mathcal{P}_{\mathrm{D}}(\not{D}_{\mathrm{YM}}) \in \mathcal{D}_{\mathrm{real}}(\mathcal{P}) \tag{171}$$

and

$$\Psi = \begin{pmatrix} \psi \\ \psi^{\mathrm{cc}} \end{pmatrix} \in \mathrm{Sec}(M,\mathcal{E})$$

Note that

$$\psi^{\mathrm{cc}} = \begin{pmatrix} 0 \\ \chi^{\mathrm{cc}} \end{pmatrix} \in \mathrm{Sec}(M,\mathcal{S}).$$

Clearly,

$$\langle {}^2\Psi, \not{P}_{\mathrm{DYM}}{}^2\Psi\rangle_{\mathcal{P}} = \langle \Psi, \not{D}_{\mathrm{YM}}\Psi\rangle_{\mathcal{E}} \tag{172}$$

and

$$\not{D}_{\mathrm{YM}}\Psi = \begin{pmatrix} (\not{\partial}_{\mathrm{A}} + i\mu_{\mathrm{D}})\psi + i\mu_{\mathrm{M}}\psi^{\mathrm{cc}} \\ (\not{\partial}_{\mathrm{A}} + i\mu_{\mathrm{D}})^{\mathrm{cc}}\psi^{\mathrm{cc}} - i\mu_{\mathrm{M}}\psi \end{pmatrix}. \tag{173}$$

Whence,

$$\begin{aligned}\langle {}^2\Psi, \not{P}_{\mathrm{DYM}}{}^2\Psi\rangle_{\mathcal{P}} &= \tfrac{1}{2}(\langle \psi, (\not{\partial}_{\mathrm{A}} + i\mu_{\mathrm{D}})\psi\rangle_{\mathcal{S}} + \langle \psi, i\mu_{\mathrm{M}}\psi^{\mathrm{cc}}\rangle_{\mathcal{S}} + \langle \psi^{\mathrm{cc}}, (\not{\partial}_{\mathrm{A}} + i\mu_{\mathrm{D}})^{\mathrm{cc}}\psi^{\mathrm{cc}}\rangle_{\mathcal{S}} - \langle \psi^{\mathrm{cc}}, i\mu_{\mathrm{M}}\psi\rangle_{\mathcal{S}}) \\ &= \tfrac{1}{2}(\langle \chi, (\not{\partial}_{\mathrm{A}} + i\varphi_{\mathrm{D}})\chi\rangle_{\mathcal{W}} + \langle \chi, im_{\mathrm{M}}\chi^{\mathrm{cc}}\rangle_{\mathcal{W}} + \langle \chi^{\mathrm{cc}}, (\not{\partial}_{\mathrm{A}} + i\varphi_{\mathrm{D}})^{\mathrm{cc}}\chi^{\mathrm{cc}}\rangle_{\mathcal{W}} - \langle \chi^{\mathrm{cc}}, im_{\mathrm{M}}\chi\rangle_{\mathcal{W}}),\end{aligned} \tag{174}$$

where we have put

$$\mu_{\mathrm{M}} \equiv \begin{pmatrix} m_{\mathrm{M}} & 0 \\ 0 & m_{\mathrm{M}} \end{pmatrix}, \quad m_{\mathrm{M}} \in \mathrm{Sec}(M, \mathrm{End}_{\gamma}(\mathcal{W})) \text{ real and constant} \tag{175}$$

according to the definition (and the physical interpretation) of the Majorana mass operator.

Thus, the quadratic form $\mathcal{I}_{\mathrm{D,ferm}}(\not{P}_{\mathrm{DYM}})$ on $\mathrm{Sec}(M,\mathcal{E})$ yields the Euler–Lagrange equations,

$$i\not{\partial}_{\mathrm{A}}\psi = \mu_{\mathrm{D}}\psi + \mu_{\mathrm{M}}\psi^{\mathrm{cc}}, \tag{176}$$

$$(i\not{\partial}_{\mathrm{A}}\psi)^{\mathrm{cc}} = \mu_{\mathrm{D}}^{\mathrm{cc}}\psi^{\mathrm{cc}} + \mu_{\mathrm{M}}\psi. \tag{177}$$

When restricted to $\tau_{\mathcal{S}}\psi = \psi$, these equations become equivalent to

$$i\not{\partial}_{\mathrm{A}}\chi = \varphi_{\mathrm{D}}\chi + m_{\mathrm{M}}\chi^{\mathrm{cc}}, \tag{178}$$

$$(i\not{\partial}_{\mathrm{A}}\chi)^{\mathrm{cc}} = \varphi_{\mathrm{D}}^{\mathrm{cc}}\chi^{\mathrm{cc}} + m_{\mathrm{M}}\chi. \tag{179}$$

In order to geometrically describe the Yukawa coupling term (and thus the "Dirac mass" after spontaneous symmetry breaking) in terms of (real) Dirac operators of simple type, one simply has to go from the underlying Majorana module to the corresponding Dirac module quite similar to how the Pauli matrices are lifted to the Dirac matrices (the latter being considered in the Majorana representation). The doubling of the Dirac module then allows to also geometrically describe the characteristic "particle-antiparticle" coupling that arises by the Majorana mass term, in terms of real, simple type Dirac operators. In fact, this is where the real structure necessarily enters the scheme, thereby turning the Clifford module into a real Clifford module. Finally, the Pauli map of the DYM, which describes both the "left-right" coupling and the "particle-antiparticle" coupling on the same geometrical footing, then allows one to geometrically describe the Pauli term in a way that does not alter the fermionic action. For reasons of renormalization, this is actually necessary.

Before we discuss the bosonic part of the full Dirac action with respect to the real Dirac operator $\not{P}_{\rm DYM}$, we still comment on the gauge invariance of Eqs. (178) and (179). Of course, this is related to the dynamical discrepancy between the fermionic left-right coupling, provided by the Dirac mass, and the particle-antiparticle coupling that is invoked on the fermions by the Majorana mass.

For the sake of gauge invariance, the underlying Majorana module $\mathcal{W} \twoheadrightarrow M$ has to be *partially flat* when Majorana masses are taken into account. In this case, $\partial_{\rm A} \neq \partial$ only for $\chi \in \ker(m_{\rm M})$. In geometrical terms this and the constant Majorana mass operator may be described by the assumption that the Majorana module splits,

$$\mathcal{W} = \begin{matrix} \mathcal{W}_\nu \\ \oplus \\ \mathcal{W}_e \end{matrix} \twoheadrightarrow M, \tag{180}$$

where the subbundle $\mathcal{W}_\nu \twoheadrightarrow M$ carries the trivial representation of the Yang–Mills gauge group $\mathcal{G}_{\rm YM} \subset \mathcal{G}_{\rm D}$ and

$$m_{\rm M} \equiv \begin{pmatrix} m_{{\rm M},\nu} & 0 \\ 0 & 0 \end{pmatrix}. \tag{181}$$

Accordingly, $\partial_{\rm A} \in \mathcal{A}_{\rm Cl}(\mathcal{W})$ and $\varphi_{\rm D} \in {\rm Sec}(M, {\rm End}_\gamma(\mathcal{W}))$ may be decomposed as

$$\partial_{\rm A} \equiv \begin{pmatrix} \partial & 0 \\ 0 & \partial_{\rm A} \end{pmatrix},$$

$$\varphi_{\rm D} \equiv \begin{pmatrix} m_{{\rm D},\nu} & 0 \\ 0 & \varphi_e \end{pmatrix}, \tag{182}$$

with $m_{{\rm M},\nu}, m_{{\rm D},\nu} \in {\rm Sec}(M, {\rm End}_\gamma(\mathcal{W}_\nu))$ being real with respect to $J_{\mathcal{W}}$ and constant. Furthermore, $\varphi_e \in {\rm Sec}(M, {\rm End}_\gamma(\mathcal{W}_e))$.

The combined Dirac–Majorana equations (178) and (179) become equivalent to

$$i\not{\partial}\nu = m_{{\rm D},\nu}\nu + m_{{\rm M},\nu}\nu^{\rm cc}, \tag{183}$$

$$i\not{\partial}_{\rm A} e = \varphi_e e, \tag{184}$$

together with the corresponding complex (or charge) conjugate equations. Here, we have put $\chi \equiv (\nu, e) \in {\rm Sec}(M, \mathcal{W}_\nu \oplus \mathcal{W}_e)$ for the "uncharged sections" and the "charged sections," respectively, of the Majorana module $\mathcal{W} \twoheadrightarrow M$. Generically, the uncharged sections $\nu \in {\rm Sec}(M, \mathcal{W}_\nu)$ are referred to as "cosmological neutrinos." They are carriers of Dirac and/or Majorana masses or are massless, depending on $\ker(m_{{\rm D},\nu})$ and $\ker(m_{{\rm M},\nu})$. Clearly, in the case of *Majorana neutrinos*: $\nu^{\rm cc} = \nu \in {\rm Sec}(M, \mathcal{M}_{\mathcal{W},\nu}) \subset {\rm Sec}(M, \mathcal{M}_{\mathcal{W}})$ (whereby $\mathcal{W} = \mathcal{M}_{\mathcal{W}}^{\mathbb{C}}$), the notions of Dirac and Majorana masses coincide and (183) reduce to

$$i\not{\partial}\nu = m_\nu \nu \quad (\nu^{\rm cc} = \nu). \tag{185}$$

Only the submodule

$$\ker(m_{\rm M}) = \mathcal{W}_e \hookrightarrow \mathcal{W} \twoheadrightarrow M \tag{186}$$

of the Majorana module carries a nontrivial representation of the Yang–Mills gauge subgroup of $\mathcal{G}_{\rm D}$.

In the case of the STM, the cosmological neutrinos should not be confounded with the electrically neutral (left-handed) component of $e \in {\rm Sec}(M, \mathcal{W}_e)$ after the mechanism of spontaneous symmetry break has been established. Indeed, the sections $\nu \in {\rm Sec}(M, \mathcal{W}_\nu)$ represent a kind of new species of particles which do not contribute to any yet known kind of interaction besides gravity. This "ghostlike species" of particles may thus serve as candidates for "dark matter" ("dark energy"). Of course, the masses of the cosmological neutrinos cannot be dynamically generated by the mechanism of spontaneous symmetry breaking since the cosmological neutrinos only carry the trivial representation of the Yang–Mills gauge group. This is certainly unsatisfying but may change with the upcoming experiments made at the Large Hadron Collider (LHC) at CERN/Swiss.

The Dirac mass matrix is known to only couple particles of different chirality but respects the particle-antiparticle grading. This is opposed to the Majorana mass matrix. Since the latter is "nondynamical," one may wonder as to what extent the Majorana masses may nonetheless dynamically contribute, for example, to the STM? A partial answer to this question within Dirac-type gauge theories will be discussed next.

A. The Dirac action concerning DYM

So far, we have carefully discussed the fermionic action of the total Dirac action. In this section we discuss the bosonic part of the latter with respect to the corresponding DYM. Since the DYM operator $\not{D}_{\rm YM}=\not{\partial}_{\mathcal{A}}+i\mu_{\rm YM}\in\mathcal{D}_{\rm real}(\mathcal{E})$ is of simple type it lifts to $\mathcal{D}_{\rm real}(\mathcal{P}={}^2\mathcal{E})$ via the Pauli map. It thereby generalizes the operator (3). The latter operator has been shown earlier to yield the STM action including gravity (cf. Refs. 34, 37, and 36). This time, however, also Majorana masses are taken into account. We therefore summarize the basic steps allowing us to express the Lagrangian density

$$\mathcal{L}_{\rm DYM}:= *\,{\rm tr}_\gamma({\rm curv}(\not{P}_{\rm DYM})-\varepsilon ev_g(\omega_{\rm D}^2)) \tag{187}$$

in terms of the sections given by the metric $g_{\rm M}$, the Yang–Mills gauge field A, the Higgs field $\varphi_{\rm D}$ (φ_e), and the Majorana (Dirac) masses $m_{\rm M}$ ($m_{\rm D}$) which altogether parametrize the DYM operator $\not{D}_{\rm YM}\in\mathcal{D}_{\rm S,real}(\mathcal{E})$.

Following the calculation, it will be shown that this density is automatically real and thus takes values in $\Omega^n(M)$. Furthermore, the calculation will also allow to reveal a subtle relation between simple type Dirac operators and the "kinetic term" of the Higgs field within the STM action.

To get started, we put $\not{P}_{\rm DYM}=\not{\partial}_{\mathcal{A}}+i\Phi_{\rm DYM}$, where

$$\Phi_{\rm DYM}\equiv\begin{pmatrix}\mu_{\rm YM} & -\not{F}_{\rm DYM}\\ \not{F}_{\rm DYM} & \mu_{\rm YM}\end{pmatrix},\quad \mu_{\rm YM}:=\begin{pmatrix}\mu_{\rm D} & \mu_{\rm M}\\ -\mu_{\rm M} & -\mu_{\rm D}^{\rm cc}\end{pmatrix} \tag{188}$$

and $\not{F}_{\rm DYM}\in{\rm Sec}(M,{\rm End}^+(\mathcal{E}))$ is the (quantized) relative curvature of $\not{D}_{\rm YM}$. Since the latter is of simple type, it follows that

$$F_{\rm DYM}=F_{\mathcal{A}}-(d_{\mathcal{A}}(i\mu_{\rm YM})+(i\mu_{\rm YM})^2\Theta)\wedge\Theta\in\Omega^2(M,{\rm End}^+(\mathcal{E})). \tag{189}$$

Here, $d_{\mathcal{A}}$ is the exterior covariant derivative with respect to the (real) Clifford connection $\partial_{\mathcal{A}}\in\mathcal{A}_{\rm Cl}(\mathcal{E})$ and $F_{\mathcal{A}}\in\Omega^2(M,{\rm End}_\gamma^+(\mathcal{E}))$ its twisting curvature. Hence,

$$\not{F}_{\rm DYM}=\not{F}_{\mathcal{A}}+\frac{n-1}{n}(\delta_\gamma(d_{\mathcal{A}}(i\mu_{\rm YM}))+(i\mu_{\rm YM})^2). \tag{190}$$

We may then take advantage of Lemma 4.1 to obtain

$${\rm tr}_{\mathcal{P}}\,V_{\rm D}={\rm tr}_\gamma({\rm curv}(\not{\partial}_{\mathcal{A}}))-{\rm tr}_{\mathcal{P}}\,\Phi_{\rm DYM}^2+\frac{\varepsilon}{4}g_{\rm M}(e_i,e_j){\rm tr}_{\mathcal{P}}(\{\gamma_{\mathcal{P}}(e^i),\Phi_{\rm DYM}\}\{\gamma_{\mathcal{P}}(e^j),\Phi_{\rm DYM}\}), \tag{191}$$

where we have neglected an appropriate boundary term and $e_1,\ldots,e_n\in TM$ is any local ($g_{\rm M}$-orthonormal) basis with dual basis $e^1,\ldots,e^n\in T^*M$.

It follows that

$$\mathrm{tr}_{\mathcal{P}}\,\Phi_{\mathrm{DYM}}^2 = 2\ \mathrm{tr}_{\mathcal{E}}(\mu_{\mathrm{YM}}^2 - \not{F}_{\mathrm{DYM}}^2),$$

$$\mathrm{tr}_{\mathcal{P}}(\{\gamma_{\mathcal{P}}(e^i),\Phi_{\mathrm{DYM}}\}\{\gamma_{\mathcal{P}}(e^j),\Phi_{\mathrm{DYM}}\}) = 2\ \mathrm{tr}_{\mathcal{E}}(\{\gamma_{\mathcal{E}}(e^i),\mu_{\mathrm{YM}}\}\{\gamma_{\mathcal{E}}(e^j),\mu_{\mathrm{YM}}\}) - 2\ \mathrm{tr}_{\mathcal{E}}(\{\gamma_{\mathcal{E}}(e^i),\not{F}_{\mathrm{DYM}}\} \times\{\gamma_{\mathcal{E}}(e^j),\not{F}_{\mathrm{DYM}}\}). \tag{192}$$

Furthermore,

$$\mathrm{tr}_{\mathcal{E}}\,\not{F}_{\mathrm{DYM}}^2 = -\frac{1}{2}\mathrm{tr}_g\,F_{\mathcal{A}}^2 + \varepsilon\left(\frac{n-1}{n}\right)^2\mathrm{tr}_g(\partial_{\mathcal{A}}\mu_{\mathrm{YM}})^2 + \left(\frac{n-1}{n}\right)^2\mathrm{tr}_{\mathcal{E}}\,\mu_{\mathrm{YM}}^4 \tag{193}$$

and

$$\mathrm{tr}_{\mathcal{E}}(\{\gamma_{\mathcal{E}}(e^i),\mu_{\mathrm{YM}}\}\{\gamma_{\mathcal{E}}(e^j),\mu_{\mathrm{YM}}\}) = 0, \tag{194}$$

$$\mathrm{tr}_{\mathcal{E}}(\{\gamma_{\mathcal{E}}(e^i),\not{F}_{\mathrm{DYM}}\}\{\gamma_{\mathcal{E}}(e^j),\not{F}_{\mathrm{DYM}}\}) = \mathrm{tr}_{\mathcal{E}}(\{\gamma_{\mathcal{E}}(e^i),\not{F}_{\mathcal{A}}\}\{\gamma_{\mathcal{E}}(e^j),\not{F}_{\mathcal{A}}\}) + \left(\frac{1-n}{n}\right)^2\mathrm{tr}_{\mathcal{E}}(\{\gamma_{\mathcal{E}}(e^i),\not{\alpha}_{\mathrm{YM}}\} \times\{\gamma_{\mathcal{E}}(e^j),\not{\alpha}_{\mathrm{YM}}\}) + 4\left(\frac{1-n}{n}\right)^2\mathrm{tr}_{\mathcal{E}}(\gamma_{\mathcal{E}}(e^i)\gamma_{\mathcal{E}}(e^j)\mu_{\mathrm{YM}}^4), \tag{195}$$

where we abbreviated $\not{\alpha}_{\mathrm{YM}} \equiv \delta_\gamma(d_{\mathcal{A}}(i\mu_{\mathrm{YM}}))$. Also,

$$\frac{\varepsilon}{4}g_{\mathrm{M}}(e_i,e_j)\mathrm{tr}_{\mathcal{E}}(\{\gamma_{\mathcal{E}}(e^i),\not{F}_{\mathcal{A}}\}\{\gamma_{\mathcal{E}}(e^j),\not{F}_{\mathcal{A}}\}) = \frac{2-n}{2}\mathrm{tr}_g(F_{\mathcal{A}}^2), \tag{196}$$

$$\left(\frac{1-n}{n}\right)^2\frac{\varepsilon}{4}g_{\mathrm{M}}(e_i,e_j)\mathrm{tr}_{\mathcal{E}}(\{\gamma_{\mathcal{E}}(e^i),\not{\alpha}_{\mathrm{YM}}\}\{\gamma_{\mathcal{E}}(e^j),\not{\alpha}_{\mathrm{YM}}\}) = -\varepsilon\frac{(1-n)^3}{n^2}\mathrm{tr}_g(\partial_{\mathcal{A}}\mu_{\mathrm{YM}})^2, \tag{197}$$

$$\varepsilon\left(\frac{1-n}{n}\right)^2 g_{\mathrm{M}}(e_i,e_j)\mathrm{tr}_{\mathcal{E}}(\gamma_{\mathcal{E}}(e^i)\gamma_{\mathcal{E}}(e^j)\mu_{\mathrm{YM}}^4) = n\left(\frac{1-n}{n}\right)^2\mathrm{tr}_{\mathcal{E}}(\mu_{\mathrm{YM}}^4). \tag{198}$$

Finally, one ends up with

$$\begin{aligned}
-\,\mathrm{tr}_{\mathcal{P}}\,\Phi_{\mathrm{DYM}}^2 + \frac{\varepsilon}{4}g_{\mathrm{M}}(e_i,e_j)\mathrm{tr}_{\mathcal{P}}(\{\gamma_{\mathcal{P}}(e^i),\Phi_{\mathrm{DYM}}\}\{\gamma_{\mathcal{P}}(e^j),\Phi_{\mathrm{DYM}}\}) &= -2\ \mathrm{tr}_{\mathcal{E}}\,\mu_{\mathrm{YM}}^2 + 2\ \mathrm{tr}_{\mathcal{E}}\,\not{F}_{\mathrm{DYM}}^2 \\
-\frac{\varepsilon}{2}g_{\mathrm{M}}(e_i,e_j)\mathrm{tr}_{\mathcal{E}}\{\gamma_{\mathcal{E}}(e^i),\not{F}_{\mathrm{DYM}}\}\{\gamma_{\mathcal{E}}(e^j),\not{F}_{\mathrm{DYM}}\} &= (n-3)\mathrm{tr}_g(F_{\mathcal{A}}^2) - 2\varepsilon(n-2)\left(\frac{n-1}{n}\right)^2\mathrm{tr}_g(\partial_{\mathcal{A}}\mu_{\mathrm{YM}})^2 \\
-2\frac{(n-1)^3}{n^2}\mathrm{tr}_{\mathcal{E}}(\mu_{\mathrm{YM}}^4) - 2\ \mathrm{tr}_{\mathcal{E}}(\mu_{\mathrm{YM}}^2),
\end{aligned} \tag{199}$$

which for $\varepsilon := +1$ and anti-Hermitian μ_{YM} has the form of the STM Lagrangian. We stress that the "kinetic term" of the Higgs, $\mathrm{tr}_g(\partial_{\mathcal{A}}\mu_{\mathrm{YM}})^2$, drops out if $\not{D}_{\mathrm{YM}}$ was not of simple type.

The explicit form of the combined Dirac–Majorana mass operator $\mu_{\mathrm{YM}} \in \mathrm{Sec}(M,\mathrm{End}(\mathcal{E}))$ yields

$$\mathrm{tr}_g(\partial_{\mathcal{A}}\mu_{\mathrm{YM}})^2 = -4\ \mathrm{Re}\ \mathrm{tr}_g(\partial_{\mathrm{A}}\varphi_e)^2, \tag{200}$$

$$a\ \mathrm{tr}_{\mathcal{E}}\,\mu_{\mathrm{YM}}^4 + \mathrm{tr}_{\mathcal{E}}\,\mu_{\mathrm{YM}}^2 = 4\ \mathrm{Re}(a\ \mathrm{tr}_{\mathcal{W}_e}\,\varphi_e^4 - \mathrm{tr}_{\mathcal{W}_e}\,\varphi_e^2 + \Lambda_{\mathrm{DM},\nu}), \tag{201}$$

whereby $a \equiv 2[(n-1)^3/n^2]$ and

$$\Lambda_{\mathrm{DM},\nu} \equiv a\,\mathrm{tr}_{\mathcal{W}_\nu}\, m_{\mathrm{D},\nu}^4 - \mathrm{tr}_{\mathcal{W}_\nu}\, m_{\mathrm{D},\nu}^2 + a\,\mathrm{tr}_{\mathcal{W}_\nu}\, m_{\mathrm{M},\nu}^4 - \mathrm{tr}_{\mathcal{W}_\nu}\, m_{\mathrm{M},\nu}^2 - 2a\,\mathrm{tr}_{\mathcal{W}_\nu}(m_{\mathrm{D},\nu} \circ m_{\mathrm{M},\nu})^2 \tag{202}$$

is the "true cosmological constant," which naturally occurs in the Einstein–Hilbert action when Majorana masses are taken into account within the geometrical frame of Dirac-type gauge theories. Its possible phenomenological consequences, for instance, with respect to the mass of the Higgs boson and the cosmological issue of dark matter, will be discussed separately in a forthcoming paper. However, because of the significance of the cosmological constant, we summarize the basic steps to obtain the result (202). This will also enlighten the subtle interplay between simple type Dirac operators and the peculiar form of $\Lambda_{\mathrm{DM},\nu}$ as the sum of two Higgs potentials and an "interaction term" for the Dirac and Majorana masses.

First, it follows that μ_{YM}^2 structurally reads as

$$\mu_{\mathrm{YM}}^2 = \begin{pmatrix} u & z \\ z^{\mathrm{cc}} & u^{\mathrm{cc}} \end{pmatrix},$$

$$u \equiv \mu_{\mathrm{D}}^2 - \mu_{\mathrm{M}}^2,$$

$$z \equiv \mu_{\mathrm{D}} \circ \mu_{\mathrm{M}} - \mu_{\mathrm{M}} \circ \mu_{\mathrm{D}}^{\mathrm{cc}}. \tag{203}$$

Because of the explicit form of the sections $\varphi_{\mathrm{D}} \in \mathrm{Sec}(M, \mathrm{End}_\gamma(\mathcal{W}_\nu \oplus \mathcal{W}_e))$ and $m_{\mathrm{M}} \in \mathrm{Sec}(M, \mathrm{End}_\gamma^+(\mathcal{W}_\nu \oplus \mathcal{W}_e))$, one gets

$$m_{\mathrm{M}} \circ \varphi_{\mathrm{D}}^{\mathrm{cc}} = m_{\mathrm{M}} \circ \varphi_{\mathrm{D}} \Rightarrow \mu_{\mathrm{M}} \circ \mu_{\mathrm{D}}^{\mathrm{cc}} = -\,\mu_{\mathrm{M}} \circ \mu_{\mathrm{D}}. \tag{204}$$

Therefore,

$$a\,\mathrm{tr}_{\mathcal{E}}\,\mu_{\mathrm{YM}}^4 + \mathrm{tr}_{\mathcal{E}}\,\mu_{\mathrm{YM}}^2 = 4\,\mathrm{Re}[a\,\mathrm{tr}_{\mathcal{E}}\,\varphi_{\mathrm{D}}^4 - \mathrm{tr}_{\mathcal{E}}\,\varphi_{\mathrm{D}}^2 + a\,\mathrm{tr}_{\mathcal{E}}\,m_{\mathrm{M}}^4 - \mathrm{tr}_{\mathcal{E}}\,m_{\mathrm{M}}^2 - 2a\,\mathrm{tr}_{\mathcal{E}}(\varphi_{\mathrm{D}} \circ m_{\mathrm{M}})^2], \tag{205}$$

where the occurrence of the Higgs potentials of φ_{D} and m_{M} is due to the fact that the DYM operator $\not{D}_{\mathrm{YM}} = \not{\partial}_{\mathcal{A}} + i\mu_{\mathrm{YM}} \in \mathcal{D}_{\mathrm{real}}(\mathcal{E})$ is of simple type.

Note that also $\mathrm{tr}_g(F_{\mathcal{A}}^2) = 4\,\mathrm{Re}\,\mathrm{tr}_g(F_{\mathrm{A}}^2)$. Hence, the total Dirac action with respect to the real Dirac operator $\not{P}_{\mathrm{DYM}} \in \mathcal{D}(\mathcal{P})$ is a real-valued functional, actually. This is independent of whether the section $\varphi_e \in \mathrm{Sec}(M, \mathrm{End}_\gamma(\mathcal{W}_e))$ and the simple type Dirac operator $\not{\partial}_{\mathrm{A}} \in \mathcal{D}(\mathcal{W})$ are supposed to be Hermitian or anti-Hermitian.

VI. REAL CLIFFORD BIMODULES AND THE "π_{D}-MAP"

The Pauli map (102) is defined for general real Clifford modules. In Sec. V we demonstrated how the Pauli map of the simple type Dirac operator defined in terms of a Yang–Mills–Higgs connection on a Majorana module encodes the full STM action functional.

In this section we discuss once again the STM action in view of the Dirac operators (3), this time, however, in the case where the underlying Majorana module is supposed to have the structure of a Clifford bimodule. The discussion of Sec. V exhibited the importance of simple type Dirac operators. However, the Pauli map does not preserve the structure of simple type Dirac operators. Although the Yukawa coupling term and the Pauli term are geometrically treated almost in the same manner, there is yet a basic asymmetry between these two terms. Basically, this is because $i\mu_{\mathrm{YM}} \in \mathrm{Sec}(M, \mathrm{End}^-(\mathcal{P}))$ anticommutes with the Clifford action in contrast to $\iota\mathcal{F}_{\mathrm{D}} \in \mathrm{Sec}(M, \mathrm{End}^-(\mathcal{P}))$. This apparent asymmetry, however, may be easily overcome in the case where the underlying Majorana module is (embedded into) a Clifford bimodule. This will yield a straightforward geometrical interpretation of the STM action in terms of the Einstein–Hilbert action including a cosmological constant term.

Definition 6.1: Let $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}, \gamma_{\mathcal{E},\mathrm{op}}, J_{\mathcal{E}})$ *be a real Clifford bimodule over* (M, g_{M}). *The mapping*

$$\pi_{\mathrm{D}}:\mathcal{D}_{\mathrm{real}}(\mathcal{E}) \to \mathcal{D}_{\mathrm{real}}(\mathcal{P}),$$

$$\not{D} \mapsto \not{P}_{\mathrm{D}} := \not{D} + i\mathcal{F}_{\mathrm{D}} \tag{206}$$

is called the "π_{D} -map." Here,

$$\mathcal{F}_{\mathrm{D}} := \begin{pmatrix} 0 & -\tau_{\mathcal{E}}\circ \not{\mathcal{F}}_{\mathrm{D}} \\ \tau_{\mathcal{E}}\circ \not{\mathcal{F}}_{\mathrm{D}} & 0 \end{pmatrix} \equiv -\tau_{\mathcal{P}}\circ \not{\mathcal{F}}_{\mathrm{D}} \in \mathrm{Sec}(M,\mathrm{End}^{-}(\mathcal{E})), \tag{207}$$

$$\not{\mathcal{F}}_{\mathrm{D}} := \not{F}_{\mathrm{D,op}} \otimes \varepsilon_2 \in \mathrm{Sec}(M,\mathrm{End}^{-}_{\gamma}(\mathcal{E})), \tag{208}$$

and

$$\not{F}_{\mathrm{D,op}} \in \mathrm{Sec}(M,\mathrm{End}^{+}_{\gamma}(\mathcal{E})) \tag{209}$$

is the relative curvature of $\not{D}_{\mathrm{op}}$ quantized with respect to $\gamma_{\mathcal{E},\mathrm{op}}$.

Apparently, the real Pauli-type Dirac operator that is defined by (206) is most analogous to the real, simple type Dirac operator (152). In particular, if $\not{D} \in \mathcal{D}(\mathcal{E})$ is of simple type, then so is its associated Pauli-type operator $\not{P}_{\mathrm{D}} \in \mathcal{D}(\mathcal{P} = {}^2\mathcal{E})$. In other words, in contrast to the Pauli map (102), the map (206) preserves the distinguished structure of simple type Dirac operators. According to its definition, however, the π_{D}-map does not preserve S-reducibility, as opposed to the Pauli map $\mathcal{P}_{\mathrm{D}}$.

Starting again with a Yang–Mills–Higgs connection $\partial_{\mathrm{YMH}} \in \mathcal{A}(\mathcal{W})$ on the Majorana module $\mathcal{W} \twoheadrightarrow M$, we may consider the real Dirac operator of simply type,

$$\not{P}_{\mathrm{DYM}} := \pi_{\mathrm{D}}(\not{\partial}_{\mathcal{A}} + i\mu_{\mathrm{YM}}) \equiv \not{\partial}_{\mathcal{A}} + i(\mu_{\mathrm{YM}} + \mathcal{F}_{\mathrm{DYM}}), \tag{210}$$

with $\mathcal{F}_{\mathrm{DYM}} := -\tau_{\mathcal{P}}\circ(\not{F}_{\mathrm{DYM,op}} \otimes \varepsilon_2)$ being defined by the ($\gamma_{\mathcal{E},\mathrm{op}}$-quantized) relative curvature of $\not{D}_{\mathrm{YM,op}} = \not{\partial}_{\mathcal{A},\mathrm{op}} + i\mu_{\mathrm{YM}}$. Notice that $\not{D}_{\mathrm{YM,op}} \notin \mathcal{D}_{\mathrm{S}}(\mathcal{E})$, in contrast to $\not{D}_{\mathrm{YM}}$. However, the $\not{D}_{\mathrm{YM}}$-induced Pauli-type curvature term $i\mathcal{F}_{\mathrm{DYM}}$ does not contribute to the fermionic part of the total Dirac action. Instead, the latter is fully determined by the Dirac connection

$$\partial_{\mathrm{DYM}} := \partial_{\mathcal{A}} + i\,\mathrm{ext}_{\Theta}(\mu_{\mathrm{YM}}) \equiv \partial_{\mathcal{YMH}} + i\,\mathrm{ext}_{\Theta}(\mu_{\mathrm{M}}) \tag{211}$$

of the simple type Dirac operator $\not{D}_{\mathrm{YM}} = \delta_{\gamma}\circ\partial_{\mathrm{DYM}} \in \mathcal{D}_{\mathrm{S,real}}(\mathcal{E})$. Here, $\partial_{\mathcal{YMH}} \in \mathcal{A}_{\mathrm{S}}(\mathcal{E})$ denotes the real form of the Dirac connection of the simple type Dirac operator (152) on the Dirac module $\mathcal{S} \twoheadrightarrow M$ that is induced by the Yang–Mills–Higgs connection $\partial_{\mathrm{YMH}} \in \mathcal{A}_{\mathrm{S}}(\mathcal{W})$ on the underlying Majorana module $\mathcal{W} \twoheadrightarrow M$.

Since the real Dirac operator $\not{P}_{\mathrm{DYM}}$ is of simple type, it becomes straightforward to express the Dirac action

$$\mathcal{I}_{\mathrm{DYM}} := \int_M *\mathrm{tr}_{\gamma}(\mathrm{curv}(\not{P}_{\mathrm{DYM}}) - \varepsilon ev_g(\omega_{\mathrm{D}}^2)) \tag{212}$$

in terms of the sections parametrizing $\not{D}_{\mathrm{YM}} \in \mathcal{D}_{\mathrm{S,real}}(\mathcal{E})$.

First, we mention that the Dirac vector field $\xi_{\mathrm{D}} \in \mathrm{Sec}(M,TM)$ of any simple type Dirac operator vanishes identically. This holds true for arbitrary Clifford modules $(\mathcal{E},\gamma_{\mathcal{E}}) \twoheadrightarrow (M,g_{\mathrm{M}})$. Therefore,

$$\mathrm{tr}_{\mathcal{E}}(\not{D}^2 - \Delta_{\mathrm{B}}) = \mathrm{tr}_{\gamma}(\mathrm{curv}(\not{D}) - \varepsilon ev_g(\omega_{\mathrm{D}}^2)) \tag{213}$$

does not hold true only up to boundary terms but is an identity for simple type Dirac operators $\not{D} \in \mathcal{D}(\mathcal{E})$.

Similar to the last section, we put $\Phi_{\rm DYM} := i(\mu_{\rm YM} + \mathcal{F}_{\rm DYM}) \in \mathrm{Sec}(M, \mathrm{End}^{-}(\mathcal{P}))$ and apply once more Lemma 4.1. This time, however, we may take advantage of $\{\gamma_{\mathcal{P}}(\alpha), \Phi_{\rm DYM}\} \equiv 0$, for all $\alpha \in T^{*}M$. Consequently, $\Theta \wedge \Phi_{\rm DYM} = -\Phi_{\rm DYM} \wedge \Theta$ and thus

$$ev_g(\omega_{\rm D}^2) = -g_{\rm M}(e^i, e^j)\Theta(e_i) \circ \Theta(e_j) \circ \Phi_{\rm DYM}^2 = -\frac{\varepsilon^2}{n^2} g_{\rm M}(e_i, e_j)\gamma_{\mathcal{P}}(e^i) \circ \gamma_{\mathcal{P}}(e^j) \circ \Phi_{\rm DYM}^2 = -\frac{\varepsilon}{n}\Phi_{\rm DYM}^2. \tag{214}$$

Furthermore,

$$\begin{aligned}\mathcal{I}_{\rm DYM} &= \int_M \left[\mathrm{tr}_{\gamma}(\mathrm{curv}(\partial\!\!\!/_{\mathcal{A}}) - (d_{\mathcal{A}}\Phi_{\rm DYM} + \Phi_{\rm DYM}^2\Theta) \wedge \Theta) + \frac{1}{n}\mathrm{tr}_{\mathcal{P}}\,\Phi_{\rm DYM}^2\right] d\mathrm{vol}_{\rm M} \\ &= \int_M [\mathrm{tr}_{\gamma}\,\mathrm{curv}(\partial\!\!\!/_{\mathcal{A}}) + \mathrm{tr}_{\mathcal{P}}\,\Phi_{\rm DYM}^2] d\mathrm{vol}_{\rm M}.\end{aligned} \tag{215}$$

The Dirac action with respect to $P\!\!\!/_{\rm DYM} = \pi_{\rm D}(\partial\!\!\!/_{\mathcal{A}} + i\mu_{\rm YM}) \in \mathcal{D}_{\rm real}(\mathcal{P})$ thus dynamically generalizes the Einstein–Hilbert action (66) and (67) with the cosmological constant induced by the Yang–Mills–Higgs connection, whose quantization (together with the Majorana masses) defines the fermionic action. (It makes no sense to take the Majorana masses into account directly on the Majorana module $\mathcal{W} \twoheadrightarrow M$. Indeed, for this one has to make use of the induced Dirac module.) This time, however, the cosmological constant does depend on the metric as opposed to (67). Indeed, according to the explicit form of $\Phi_{\rm DYM}$, it follows that

$$\begin{aligned}\mathrm{tr}_{\mathcal{P}}\,\Phi_{\rm DYM}^2 &= 2\,\mathrm{tr}_{\mathcal{E}}(F\!\!\!/_{\rm DYM,op}^2 - \mu_{\rm YM}^2) = -\,\mathrm{tr}_g\,F_{\mathcal{A}}^2 + 2\varepsilon\left(\frac{n-1}{n}\right)^2 \mathrm{tr}_g(\partial_{\mathcal{A}}\mu_{\rm YM})^2 + 2\left(\frac{n-1}{n}\right)^2 \mathrm{tr}_{\mathcal{E}}\,\mu_{\rm YM}^4 \\ &\quad - 2\,\mathrm{tr}_{\mathcal{E}}\,\mu_{\rm YM}^2.\end{aligned} \tag{216}$$

For $\varepsilon := -1$ and Hermitian $\mu_{\rm YM}$ the "cosmological constant term" has the form of the usual Lagrangian of the STM such that $\mathcal{I}_{\rm DYM}$, again, takes the form of the combined EHYMH action. (For $\varepsilon := +1$, one gets the corresponding Lagrangian with respect to the Euclidean signature of $g_{\rm M}$.) This functional basically coincides with what has been derived from the Pauli–Dirac operator $\mathcal{P}_{\rm D}(\partial\!\!\!/_{\mathcal{A}} + i\mu_{\rm YM}) \in \mathcal{D}_{\rm S,real}(\mathcal{P})$ in Sec. V. Note, however, that there is a difference concerning the conditions imposed on $(\varepsilon, \mu_{\rm YM})$. Also note that there is a significant difference between $\mathcal{P}_{\rm D}$ and $\pi_{\rm D}$ in dimension 2.

VII. CONCLUSION

In this article, we discussed the geometrical structure of Pauli-type Dirac operators which encode the STM action including gravity. This has been done by carefully analyzing the corresponding structure of the Dirac equation and the Majorana equation in terms of real Clifford (bi)modules and Dirac operators of simple type. It has been shown how the geometrical frame presented allows one to overcome the issue of "fermion doubling" and how the combined EHYMH action can be derived from the distinguished class of real Dirac operators of simple type. The latter description allows one to geometrically recast the EHYMH action into a form which formally looks identical to the Einstein–Hilbert action with a cosmological constant. On this basis, we have demonstrated how Majorana masses are naturally included within the geometrical frame of Dirac-type gauge theories and how they dynamically contribute to the combined EHYMH action in terms of a peculiar cosmological constant. This cosmological constant may have interesting phenomenological consequences with respect to dark matter/energy and the mass of the Higgs boson to be discussed in a forthcoming work.

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