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Integrability and integrodifferential substitutions

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Chen, Lee, and Liu presented in 1979 an algorithm for establishing integrability of two-dimensional partial differential systems. It is proved here that this algorithm is invariant under the point transformations, differential substitutions, and some integrodifferential substitutions. It is also proved that canonical conserved densities of linearizable systems arising in the frameworks of the method are almost all trivial. The integrability of the non-Newtonian liquid equations is investigated and it is proved that there exist two integrable systems only. A preliminary classification of the third-order integrable evolution systems is presented. © *1997 American Institute of Physics.* [S0022-2488(97)02611-X]

I. INTRODUCTION

There are a few different approaches for the establishing of an integrability (or nonintegrability) of nonlinear partial differential systems with two independent variables. Most of them are presented in the book *What Is The Integrability?*.[1] The first systematic approach was the symmetry one (Kumei,[2] Olver,[3] and Fokas[4]). It deals with the Lie–Backlund vector fields. Then the ideas of the symmetry method were unified with the inverse scattering transform ideas and methods by Shabat and Ibragimov.[5,6] Considering the evolution equations $u_t=K(u)$, they introduced the notion of the formal symmetry as a formal pseudodifferential operator L commuting with the operator $D_t-K_iD_x^i$ of the linearized equation. It was proved that the residues of the fractional powers of L are the conserved densities. This unified approach is often called now the symmetry one. The most of the results of this approach and additional references one can find in the review article by Mikhailov, Shabat, and Sokolov[7] and in the papers cited therein.

We must also mention the Painleve test method (see Weiss, Tabor, and Carnevale[8] and the review article by Flashka, Newell, and Tabor[9]) and the prolongation structures approach by Estabrook and Wahlquist.[10] The Painleve test deals with an expansion of the solution of a nonlinear system near a singular manifold. The prolongation structures approach is as a matter of fact the direct method for construction the zero curvature representation of a nonlinear system. We are not going to consider these two methods here, as they are not relevant.

Another method was discovered by Chen, Lee, and C. S. Liu.[11] It is closely related with the symmetry approach and also gives the conserved densities (CD) of an evolution system under consideration. But in the frameworks of this method CD arise from the eigenfunction of a linear problem associated with the nonlinear one. The conserved densities, obtained by the both ''Russian'' and ''Chinese'' method, coincide (up to total derivatives) for all known examples. These densities are called the canonical ones. Flashka[12] proved the equivalence of these two algorithms in general form for a single evolution equation and many mathematicians are sure that it is true for the systems too, but the proof does not exist. That is why we investigated the Chinese algorithm, explained why it works, and generalized it for nonevolution systems in the previous article.[13] We shall call this method the CD method for brevity (canonical densities method).

Now we give the notation and a brief introduction in the CD method. Let $x^0=t$, and $x^1=x$ be the independent variables and u^α, $\alpha=1,...,m$ be the functions of x^i. We consider the space $\mathscr{C}^\infty(\mathscr{R}^2,\mathscr{R}^m)$ of infinitely differentiable functions u^α as the Frechet one. We shall write the partial differential system under consideration in the general form

$$F(u)=0, \tag{1}$$

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where the differential operator $F:\mathscr{C}^\infty(\mathscr{R}^2,\mathscr{R}^m)\to\mathscr{C}^\infty(\mathscr{R}^2,\mathscr{R}^m)$, is differentiable in the Hadamard sense (see the textbook by Yamamuro,[14] for instance). We suppose that solutions of the system (1) satisfy boundary conditions those define a manifold $M\subset\mathscr{C}^\infty(\mathscr{R}^2,\mathscr{R}^m)$ and the system (1) defines a submanifold $\mathscr{M}\subset M$. Let $n=(i,j)$ be the non-negative integer bi-index and $|n|=i+j$, then $u_n^\alpha=\partial^n u^\alpha=\partial_t^i\partial_x^j u^\alpha$ denotes the $|n|$-th-order partial derivative of u^α. We also use the usual notation $u_t=\partial u/\partial t$, $u_x=\partial u/\partial x$, $u_{tt}=\partial^2 u/\partial t^2$, etc., and the ordinary summation rule over the repeated indices.

Definition 1: Let a function Φ *depend on* t, x, *and* $\partial^n u(tx)$, *where* $0\leqslant|n|<\infty$. *Then* Φ *is said to be a local function of* u.

Sometimes the following nonlocal variables appear:

$$w^i=\int\varphi^i(t,x,u)dx, \tag{2}$$

where φ^i are local functions.

Definition 2: Let $D_t=d/dt$ *be the total differentiation operator on the manifold* $\mathscr{M}$. *If the nonlocal variables (2) satisfy the following equations:*

$$D_t w^i=f^i(t,x,u)+a_j^i(t)w^j,\quad i,j=1,...,N,$$

where f^i are local functions, then w^i are called the weakly nonlocal variables (see Sokolov and Svinolupov[15]).

Let $\mathscr{D}[a,b]\subset\mathscr{C}^\infty(\mathscr{R},\mathscr{R}^m)$ be the space of the basic functions with the support $[a,b]$ [that is, $\partial_x^n v(x)=0$, $\forall n$ and $\forall x\notin(a,b)$, if $v\in\mathscr{D}[a,b]$]. The symbol $\langle,\rangle$ denotes a linear functional on $\mathscr{D}[a,b]$:

$$\langle g,v\rangle=\int_a^b g_\alpha(x)v^\alpha(x)dx,\quad v\in\mathscr{D}[a,b],\quad g\in\mathscr{C}^\infty(\mathscr{R},\mathscr{R}^m).$$

Let $H=h_i\partial_x^i$ be a linear matrix operator on $\mathscr{C}^\infty(\mathscr{R},\mathscr{R}^m)$. Introducing the adjoint operator H^+ with the help of the equation $\langle f,Hv\rangle=\langle H^+f,v\rangle$, we obtain $H^+=(-\partial_x)^i h_i^T$, where T is the transposition symbol.

Admitting liberty of a language they call the system (1) the integrable if it is equivalent to one of the following operator equations:

$$U_t-V_x+[U,V]=0, \tag{3}$$

or

$$dL/dt=BL-LA. \tag{4}$$

Here the matrices U and V are local functions of u^α and a complex parameter λ; L, A, and B are the linear scalar differential operators depending on ∂_x only (but not ∂_t). Coefficients of the L, A, and B are local functions of u^α and λ (see Zakharov *et al.*,[16] for instance). Equation (3) is called the zero curvature representation of the system (1). Equation (4) is called the triadic representation of the system (1) or (L,A,B) representation. Let us consider the triadic representation for definiteness. It is obvious that an operator equation $H=0$ is equivalent to its adjoint $H^+=0$, therefore the equation (4) implies

$$dL^+/dt=L^+B^+-A^+L^+. \tag{5}$$

Equations (4) and (5) provide the compatibility for the following linear systems:

$$L\varphi=0, \quad \varphi_t=A\varphi, \tag{6}$$

or

$$L^+\chi=0, \quad \chi_t=-B^+\chi \tag{7}$$

accordingly. One of these systems are usually exploited for deriving conserved densities of the associated nonlinear system (see Zakharov *et al.*,[16] Zakharov, and Shabat[17]).

Definition 3: Suppose that the following conditions are valid: (i) the system (1) has a triadic representation (4); (ii) the operator $L(\lambda)$ has a pole at the point $\lambda=0$; (iii) the systems (6) and (7) have formal solutions in the following form:

$$\varphi=\exp\left\{\int \omega\, dx+\sigma\, dt\right\}, \quad \chi=\exp\left\{-\int \xi\, dx+\eta\, dt\right\}, \quad \omega=\sum_{i=0}^{\infty} \omega_i z^{i-n},$$

$$\sigma=\sum_{i=0}^{\infty} \sigma_i z^{i-k}, \quad \xi=\sum_{i=0}^{\infty} \xi_i z^{i-n}, \quad \eta=\sum_{i=0}^{\infty} \sigma_i z^{i-k}, \quad \lambda=z^p,$$

where $D_t\omega=D\sigma$, $D_t\xi=D\eta$, $(D=d/dx)$, and the numbers k, n, and p are natural; (iv) the functions ω_i, σ_i, ξ_i and η_i are local or weakly nonlocal. If the conditions (i)–(iv) are held we say the system (1) to be formally integrable.

Let us denote $G'(u)[v]$ the Hadamard derivative of an operator G at a point u in a direction v:

$$G'(u)[v]=\left(\frac{\partial}{\partial\epsilon} G(u+\epsilon v)\right)_{\epsilon=0}.$$

And G'^+ denotes the adjoint for G' operator with respect to the standard bilinear form:

$$\langle\varphi,\psi\rangle_1=\int \varphi_\alpha\psi^\alpha\, d^2x.$$

We proved in the previous paper[13] that for a wide set of two-dimensional formally integrable systems (1) their canonical conserved densities arise from the following equation:

$$\Phi(D_t+\theta,D+\rho)b=0, \tag{8}$$

where $\Phi(D_t,D)\equiv F'^+$, the 1-form $\Omega=\theta\, dt+\rho\, dx$ is closed on $\mathscr{M}$ and the column b satisfies the normalization condition $(c,b)=1$ with a constant vector c.

Let us consider the normal differential system:

$$F^\alpha(u)=\partial_t^{n(\alpha)}u^\alpha-H^\alpha(u_j^i)=0. \tag{9}$$

Here $u_j^{\alpha i}=\partial_t^i\partial_x^j u^\alpha$, $n(\alpha)>0$ and the functions H^γ do not depend on the variables $\partial_t^{n(\alpha)+i}u_j^\alpha$ for all $i\geqslant 0$, $\alpha=1,\ldots,m$. Note that for $n(\alpha)=1$ or 2 we have the evolution system or the Klein–Gordon-type system accordingly.

Theorem 1: *Let the system (1) take the form (9) or can be reduced into this form with the help of the linear nondegenerate transformation:*

$$x'^i=C_j^i x^j, \quad u'=u, \quad C_j^i=\text{const.}$$

If the system (1) is formally integrable, then the system (8) has a solution in the form of the formal Laurent expansions,

$$\rho=\sum_{i=0}^{\infty}\rho_i z^{i-n},\quad \theta=\sum_{i=p}^{\infty}\theta_i z^{i-n},\quad b=\sum_{i=0}^{\infty}b_i z^i, \tag{10}$$

where z is a parameter, $n>0$, or $n-p>0$; the coefficients ρ_i, θ_i, and b_i are local or weakly nonlocal functions of u^α.

The closedness condition of the 1-form Ω yields an infinite set of conservation laws for the system (1):

$$D_t\rho_i=D\theta_i \tag{11}$$

(some of the currents $\{\theta_i,\rho_i\}$ may be trivial). We remind that any trivial current takes the form $\{\theta,\rho\}=\{D_t\xi,D\xi\}$, where ξ is a local function.

The definition 3 and the theorem 1 were presented in our paper.[13] We would like to stress that the equation (8) generates some special conserved densities ρ_i that are called the canonical conserved densities (CCD). As we mentioned above, the CCD arising from the symmetry method and the CD method are equivalent probably. But the equation (8) is simpler for the computations of the CD.

In Sec. II we prove that the equations (8), (10), and (11) are invariant under the point transformations, differential substitutions, and some integrodifferential substitutions.

In Sec. III the notion of the c integrability by Calogero is considered. It is proved that canonical densities of linearizable systems are almost all trivial. The theorems presented in Sec. II and Sec. III have the same sense as the analogic theorems proved in the frameworks of the symmetry method. But we hope, nevertheless, that our results are useful as the equivalence of the two methods is not proved.

In Sec. IV we investigate the integrability of the non-Newtonian liquid equations and prove that only two systems satisfy the integrability conditions.

In Sec. V we present a preliminary classification of the third-order integrable evolution systems.

II. MAIN THEOREM

Let us consider a transformation $(x,u)\to(y,v)$:

$$y^i=f^i(x^k,u),\quad v^\alpha=g^\alpha(x^i,u), \tag{12}$$

where f and g are smooth differential or integrodifferential operators. If f and g are the zeroth-order differential operators we have a point transformation. If f and g are differential operators of a nonzero order, the transformation (12) was called by Sokolov[18] the differential substitution. Suppose that the transformation (12) maps the manifold M into a manifold N, and the operator F is transformed according to the formula

$$F(x,u)=P(y,v)\tilde{F}(y,v), \tag{13}$$

where $P(y,v)$ is a linear operator that is invertible and differentiable on the manifold N. Then the system (1) is equivalent to $\tilde{F}(y,v)=0$, $v\in N$. Choosing the differentials $dx^i=0$, and $du=\varphi(x)$, we obtain $dy=f'_u\varphi$, and $dv=g'_u\varphi$. Then the differentiation of the equation (13) in the direction $\{dy,dv\}$ gives

$$F'_u\varphi=(P'_y[dy]+P'_v[dv])\tilde{F}+P(\tilde{F}'_y f'_u+\tilde{F}'_v g'_u)\varphi. \tag{14}$$

Since $\tilde{F}'_y=D_y\tilde{F}-\tilde{F}'_v v'_y$ and $\tilde{F}=0$ on the manifold $\mathscr{M}$ then the equation (14) takes the following form on $\mathscr{M}$:

$$F'_u\varphi = P\widetilde{F}'_v S\varphi, \tag{15}$$

where

$$S = g'_u - v'_y f'_u \equiv g'_u - (\partial v/\partial y^i)(f^i)'_u. \tag{16}$$

Operator (16) plays a fundamental role in the Lie–Bäcklund theory. It maps the tangent space T_uM into the space T_vN. In particular, it maps a set of the Lie–Bäcklund fields of the system (1) into a corresponding set for the transformed system. If the transformation (12) is invertible in a region $H\subset\mathscr{M}$, then the operator (16) is also invertible in a subspace $L\subset T_uM$, where $u\in H$.

Setting $D_0 = D_t$, $D_1 = D$, $\widetilde{D}_i = d/dy^i$ we denote the inverse Jacobi matrix as $a = (Df)^{-1}$. Then we have $\widetilde{D}_i = a_i^j D_j$, obviously. It can be easily verified that the system (3) is invariant under the transformation (12)

$$\frac{\partial\widetilde{U}_i}{\partial y^j} - \frac{\partial\widetilde{U}_j}{\partial y^i} + [\widetilde{U}_i, \widetilde{U}_j] = 0,$$

where $\widetilde{U}_i = a_i^k U_k$, $U_0 = V$, $U_1 = U$. So, the transformed system (1) has a zero curvature representation. Therefore it has a triadic representation too. If the system (1) is formally integrable, then the equations (8), (10), and (11) are valid. But it is unclear *a priori* whether these equations are valid for the transformed system?

Let us introduce the operator $\widetilde{\Phi}(\widetilde{D}_i) = \widetilde{F}'^*_v$, where the asterisk denotes the conjugation with respect to the new bilinear form:

$$\langle\varphi,\psi\rangle_2 = \int \varphi_\alpha\psi^\alpha\, d^2y, \quad \langle\varphi,\psi\rangle_1 = \langle|a|\varphi,\psi\rangle_2.$$

Then we can formulate the following theorem.

Theorem 2: *Let us consider two differential operators F and $\widetilde{F}$ on the manifolds M and N accordingly, where N is the image of M under the map (12). If the operators F and $\widetilde{F}$ are connected by the formula (13), where*

$$P(y,v) = \sum_{i,j=-k}^{r} f_{ij}(y,v) D_t^i D^j \tag{17}$$

is an invertible on N operator with smooth coefficients and the set $S(T_uM)$ is dense in the set T_vN; then (i) the system (8) is equivalent to the following one:

$$\widetilde{\Phi}(\widetilde{D}_0 + \widetilde{\theta},\ \widetilde{D}_1 + \widetilde{\rho})\widetilde{b} = 0; \tag{18}$$

(ii) the functions $\widetilde{\theta}$, and $\widetilde{\rho}$ satisfy the equation $\widetilde{D}_0\widetilde{\rho} = \widetilde{D}_1\widetilde{\theta}$ and take the following form:

$$\widetilde{\rho} = a_1^0\theta + a_1^1\rho, \quad \widetilde{\theta} = a_0^0\theta + a_0^1\rho, \tag{19}$$

where θ, and ρ satisfy the equation (8);
(iii) $\widetilde{b}$ is a linear function of b satisfying the condition $(\widetilde{c},\widetilde{b}) = 1$ with a constant vector $\widetilde{c}$;
(iv) the functions $\widetilde{\rho}$, $\widetilde{\theta}$ and $\widetilde{b}$ take the form of the series expansions (10).

Proof: Let us perform the transformation (12) for the closed 1-form Ω: $\rho\, dx + \theta\, dt \equiv \omega_i\, dx^i = \omega_i a_j^i dy^j \equiv \widetilde{\omega}_j dy^j$. Denoting $\widetilde{\omega}_i = \{\widetilde{\theta},\widetilde{\rho}\}$, we obtain the formulas (19) and the series expansions (10) for $\widetilde{\rho}$ and $\widetilde{\theta}$. Then, the equation $d\Omega = 0$ is equivalent to the infinite sequence of the conservation laws $\widetilde{D}_0\widetilde{\rho}_i=\widetilde{D}_1\widetilde{\theta}_i$. To transform the equation (8) we consider the formula (15) that is valid according to the invertibility of the operator P. Multiplying the equation (15) by be^{ω} from the left-hand side and setting

$$\varphi=\psi\,\exp(-\omega),\quad \omega=\int\Omega,$$

we obtain

$$\langle be^{\omega},F'_u\psi e^{-\omega}\rangle_1=\langle e^{-\omega}F'^{+}_u e^{\omega}b,\psi\rangle_1=\langle\Phi(D_i+\omega_i)b,\psi\rangle_1=0=\langle be^{\omega},P\widetilde{F}'_v S\varphi\rangle_1$$
$$=\langle|a|be^{\omega},P\widetilde{F}'_v S\varphi\rangle_2=\langle e^{-\omega}\widetilde{F}'^{*}_v P^{*}e^{\omega}|a|b,e^{\omega}S\varphi\rangle_2=\langle\widetilde{\Phi}(\widetilde{D}_i+\widetilde{\omega}_i)\widetilde{b},\widetilde{\psi}\rangle_2.$$

Here we used the obvious identity $e^{-\omega}\widetilde{\Phi}(\widetilde{D}_i)e^{\omega}=\widetilde{\Phi}(\widetilde{D}_i+\widetilde{\omega}_i)$, and denoted $\widetilde{\psi}=e^{\omega}S_{\varphi},\widetilde{b}=e^{-\omega}P^{*}e^{\omega}|a|b\equiv Ab$. According to a denseness of the set $S(T_uM)$ we obtain the equation (18) with a broken normalization condition $(c,\widetilde{b})=(c,Ab)\neq 1$. But we proved[13] that one can change a normalization condition performing the gauge transformation $\Omega\to\Omega+d\xi$ that does not break the conservation laws (11). This transformation gives $(\widetilde{c},\widetilde{b})=1$, where $\widetilde{c}$ is a constant vector. So, validity of the equations (18) and (19) is proved.

To conclude the proof we must find a series expansion for $\widetilde{b}$. Let us consider the following expression:

$$\widetilde{h}=e^{-\omega}D_1^k e^{\omega}h,\quad \text{where } h=\sum_{i=0}^{\infty}h_i z^i,\quad k\in Z. \tag{20}$$

If $k=1$, then substituting from (10) ρ into the formula $\widetilde{h}=(D+\rho)h$, we easily obtain the following series:

$$\widetilde{h}=\sum_{i=0}^{\infty}z^{i-n}\left(D_1h_{i-n}+\sum_{j=0}^{i}\rho_j h_{i-j}\right),$$

for any $n\in Z$. If $k=-1$, then an elementary calculation gives

$$\widetilde{h}=\sum_{i=0}^{\infty}\widetilde{h}_i z^{i+n},\quad \widetilde{h}_i=\rho_0^{-1}\left(h_i-D_1\widetilde{h}_{i-n}-\sum_{j=1}^{i}\rho_j\widetilde{h}_{i-j}\right),\quad \text{for } n>0,$$

and

$$\widetilde{h}=\sum_{i=0}^{\infty}\widetilde{h}_i z^i,\quad \widetilde{h}_i=\begin{cases} D_1^{-1}\left(h_i-\sum_{j=0}^{i-|n|}\rho_j\widetilde{h}_{i-j-|n|}\right), & \text{for } n<0\\ (D_1+\rho_0)^{-1}\left(h_i-\sum_{j=1}^{i}\rho_j\widetilde{h}_{i-j}\right), & \text{for } n=0.\end{cases}$$

So, we have the power series for $\widetilde{h}$ when $k=\pm 1$. It is clear by induction that it is true for any $k\in Z$. Repeating the deduction for the expression $\widetilde{h}=e^{-\omega}D_0^k e^{\omega}h$, we conclude that $\widetilde{h}$ is the power series in this case too. According to the formula (17) the column $\widetilde{b}(z)$ is the linear combination of the terms $e^{-\omega}D_0^iD_1^je^{\omega}|a|b=(e^{-\omega}D_0^ie^{\omega})(e^{-\omega}D_1^je^{\omega})|a|b(z)$, where $b(z)$ is the Taylor series. Therefore we can consider each term in the $\widetilde{b}$ as the superposition of two transformations in the form (20) obviously. Hence the column $\widetilde{b}(z)$ is the power series for any case. Multiplying $\widetilde{b}$ by an appropriate positive power of z, we can obtain the Taylor series. The theorem is proved.

It is clear from this theorem that the functions $\tilde{b}_i$ may be both local and nonlocal. But when the both expansions (10) for $\rho(z)$ and $\theta(z)$ have poles at the point $z=0$, then the functions $\tilde{b}_i$ must be local. Thus, Theorem 2 allow us simplify a nonlinear system with the help of a suitable transformation (12) before applying the CD method.

Let us consider the most popular transformations. For the point transformations, the operators (16) and (17) are invertible matrices. Hence, the conditions of Theorem 2 are held. Besides, local functions $\{\theta_i,\rho_i,b_i\}$ are transformed into local $\{\tilde{\theta}_i,\tilde{\rho}_i,\tilde{b}_i\}$. For the Legendre transformation ($m=1$) $y^i=u_i$, $v=x^iu_i-u$ the operator (17) is a function and $S=-1$. Hence, the conditions of Theorem 2 are held again. Local functions $\{\theta_i,\rho_i,b_i\}$ are transformed into local $\{\tilde{\theta}_i,\tilde{\rho}_i,\tilde{b}_i\}$. For differential substitutions in the form $x^i=f^i(y)$, $u=g(y,v)$, the operators (16) and (17) are differential. Therefore, the conditions of Theorem 2 must be checked for such cases. A local current $\{\theta(u),\rho(u)\}$ is transformed into a local one $\{\tilde{\theta}(v),\tilde{\rho}(v)\}$. But the inverse transformation $\{\tilde{\theta},\tilde{\rho}\}\rightarrow\{\theta,\rho\}$ can produce nonlocal currents. Let us consider the following nonlocal transformation:

$$y^0=t,\quad y^1=\int\rho(u)dx+j(u)dt,\quad v^\alpha=u^\alpha, \tag{21}$$

where $D_t\rho=Dj$ on the manifold $\mathscr{M}$. (It may be the transformation between the Lagrange and Euler variables in the fluid dynamics, for example.) For this case we have $P=1$, but the operator (16) is integrodifferential. Nevertheless, S^{-1} exists for an invertible transformation (21) and the conditions of Theorem 2 are satisfied. A simple calculation gives the following expressions:

$$D_x=\rho\tilde{D}_1,\quad D_t=\tilde{D}_0+j\tilde{D}_1,\quad v_1=u_x\rho^{-1},\quad v_0=u_t-ju_x\rho^{-1}. \tag{22}$$

If the third and fourth equations (22) are algebraically solvable with respect to u_x and u_t, then the independent on x local currents are transformed into local currents. [The formulas (22) are exactly invertible if the order of $\rho(u)$ is zero.] For other cases nonlocal currents may appear.

III. *C* INTEGRABILITY

The notion of the c integrability was introduced by Calogero and Xiaoda.[19] Any c-integrable nonlinear system may be reduced into a linear one under a transformation (12). That is why we start from a linear system with the constant coefficients:

$$F(x,u)\equiv F(D_i)u=0.$$

For a linear system we have $F'_u=F(D_i)$, $\Phi(D_i)=F^+(D_i)$; therefore the system (8) admits a constant solution [$\rho=k_1$, $\theta=k_0$, $b=b(k_i)$]:

$$\Phi(k_i)b=0,\quad \det\,\Phi(k_i)=0. \tag{23}$$

The second equation (23) is called the dispersion relation. It defines a flat algebraic curve γ. The first equation defines the column b on the curve γ. To obtain the expansions (10) we prove a previous lemma.

Lemma: A complex irreducible flat algebraic curve $h(x,y)=0$ possesses the following parametrization:

$$x=z^{-n},\quad y=\sum_{i=0}^{\infty}c_iz^{i+p}, \tag{24}$$

where z is a parameter in a neighborhood of the zero, $n\in Z_+$, $p\in Z$.

Proof: To investigate the curve $h(x,y)=0$, when x runs to infinity, we perform the substitution $x=t^{-1}$. Then the equation of our curve can be written in the following form:

$$h(t,y)=y^m+\sum_{i\in N} y^{m-i}t^{k_i}\varphi_i(t)=0, \tag{25}$$

where $k_i\in Z$, $N\subseteq\{1,...,m\}$, $\varphi_i(t)=c_ip_i(t)/q(t)\not\equiv 0$, $c_i=\text{const}$, p_i and q are polynomials, $p_i(0)=q(0)=1$. If $N=\varnothing$, then $y=0$ and we have the equation (24), where $n=1$. If there are nonzero functions φ_i we were forced to consider some different cases.

(I) Let $m=1$; then $y=-t^k\varphi(t)$. Expanding the fraction $\varphi(t)$ into the Taylor series, we obtain the formulas (24), where $n=1$, $p=k$. We consider $m>1$ below.

(II) Let $k_i=0$ for $i\in I\neq\varnothing$, $k_i>0$ for $i\in J$, and $N=I\oplus J$, then we have

$$h(t,y)=y^m+\sum_{i\in I} y^{m-i}\varphi_i(t)+\sum_{i\in J} y^{m-i}t^{k_i}\varphi_i(t)=0. \tag{26}$$

When $t=0$, this equation implies

$$g(y)\equiv y^m+\sum_{i\in I} c_iy^{m-i}=0\Rightarrow y=y_0.$$

Setting $g(y)=(y-y_0)^n\lambda(y)$, where $\lambda(y_0)\neq 0$, we can write

$$h(t,y)=(y-y_0)^n\lambda(y)+t\psi(t,y)=0, \tag{27}$$

where ψ is a regular function,

$$\psi(t,y)=\sum_{i\in I} y^{m-i}\tilde{\varphi}_i(t)+\sum_{i\in J} y^{m-i}t^{k_i-1}\varphi_i(t), \quad \tilde{\varphi}_i(t)=c_i\,\frac{p_i-q}{tq}.$$

Substituting $t=z^n$ into (27), we obtain

$$\left(\frac{y-y_0}{z}\right)^n\lambda(y)+\psi(z^n,y)=0.$$

When z runs to zero we obtain $(y'(0))^n\lambda(y_0)+\psi(0,y_0)=0$. Since $\lambda(y_0)\neq 0$, then $y'(0)$ exists and the function $y(z)$ is holomorphic at zero. This implies the expressions (24), where $p=0$ when $y_0\neq 0$, and $p>0$ when $y_0=0$.

(III) $k_i>0$, for all $i\in N$. Let us find the following simplified fraction:

$$\min_{i\in N}\frac{k_i}{i}=\frac{p}{n},$$

and perform the substitution $t=z^n$, $y=uz^p$. Then there exist such sets $I\neq\varnothing$ and J that $nk_i-ip=0$ for $i\in I$, $nk_i-ip=n_i>0$ for $i\in J$ and $N=I\oplus J$. Therefore the equation (25) is reduced into the following form:

$$h(z,u)=u^m+\sum_{i\in I} u^{m-i}\varphi_i(z^n)+\sum_{i\in J} u^{m-i}z^{n_i}\varphi_i(z^n)=0, \tag{28}$$

analogous to the equation (26). Hence, repeating the deduction from the point (II), we obtain the expressions (24), where $n>0$, $p>0$.

(IV) $k_i = -r_i < 0$ for $i \in A \neq 0$, $k_i \geqslant 0$ for $i \in B$, and $N = A \oplus B$. In that case we have

$$h(t,y) = y^m + \sum_{i \in A} y^{m-i}\, t^{-z_i} \varphi_i(t) + \sum_{i \in B} y^{m-i} t^{k_i} \varphi_i(t) = 0. \tag{29}$$

Let us find the following simplified fraction:

$$\max_{i \in A} \frac{r_i}{i} = \frac{p}{n},$$

and perform the substitution $t = z^n$, $y = uz^{-p}$. There exists such subset $I \neq \varnothing$ in A that $ip - nr_i = 0$ for $i \in I$ and $ip - nr_i = n_i > 0$ for $i \in I' = A \setminus I$. It can be now verified that the equation (29) is reduced into the form (28), where $J = I' \oplus B$, $n_i > 0$. Therefore, repeating the deduction from the point (II), we obtain the expressions (24), where $n > 0$, $p < 0$. The lemma is proved.

Theorem 3: *If the system (1) can be reduced into a linear system with constant coefficients by means of an invertible transformation (12), then it possesses an infinite set of the trivial canonical conserved densities.*

Proof: To investigate the canonical conserved densities we may reduce the system (1) into the linear form $\widetilde{F}(\widetilde{D})v = 0$ according to Theorem 2. Let us consider now the constant solution $\widetilde{\theta} = k_0$, $\widetilde{\rho} = k_1$, $\widetilde{b} = b(k_i)$ of the system (23). Choosing an irreducible component of the dispersion curve, we can represent its equation in the form (24)

$$k_1 = z^{-n}, \quad k_0 = \sum_{i=-p}^{\infty} c_i z^{i-n}, \quad n > 0,$$

according to the lemma. Then for the original system (1) we have

$$\Omega = k_0\, dy^0 + k_1\, dy^1 = d\left[z^{-n} f^1(x,u) + f^0(x,u) \sum_{i=-p}^{\infty} c_i z^{i-n} \right] \equiv \rho\, dx + \theta\, dt.$$

Hence

$$\rho_i = D_x(c_i f^0 + \delta_{i0} f^1), \quad \theta_i = D_t(c_i f^0 + \delta_{i0} f^1).$$

If the functions f^i are local, these currents are trivial, obviously. If we deal with a transformation of the type (21), then one or two currents can be nontrivial. The theorem is proved.

If a linear system possesses the CCD dependent on the variables t,x, then the previous statement is also true. Really, the 1-form $\Omega = \rho(t,x)dx + \theta(t,x)dt$ is closed according to the definition. Hence $\Omega = d\psi(t,x)$, and we obtain the trivial CCD under the transformation (12). The nontrivial CCD may arise for a linearizable system if the CCD of the corresponding linear system depends on the variables u^α. The special case of this situation is considered in the next theorem.

Theorem 4: *If the system (1) can be reduced into a linear evolution system of the order m* $\geqslant 2$ by means of an invertible transformation (12), then its CCDs are almost all trivial.

Proof: For the linear evolution system $u_t = K(D)u$, where $K = K_m D^m + \cdots + K_0$ the equation (8) takes the following form:

$$[D_t + \theta + K^+(D + \rho)]b = 0. \tag{30}$$

If we set $\rho = \rho_0 z^{-n} + \rho_1 z^{1-n} + \cdots$, then $\theta = az^{-mn} + bz^{1-mn} + \cdots$ and the function $(-1)^{m+1} a \rho_0^{-mn}$ is an eigenvalue of the matrix K_m. Since $m > 1$ then $mn > n$ and $Da = 0$; hence $a = a(t)$ and $\rho_0 = \rho_0(t,x)$, as K_m does not depend on u^α. Other functions ρ_n, b_n are defined from recursion relations that follow from the equation (30):

$$\rho_n = R_n(\rho_0,\dots,\rho_{n-1},b_0,\dots,b_{n-1},\theta_0,\dots,\theta_{n-1}),$$

$$b_n = Q_n(\rho_0,\dots,\rho_{n-1},b_0,\dots,b_{n-1},\theta_0,\dots,\theta_{n-1}),$$

where R_n and Q_n are some differential polynomials. The functions θ_n are defined from the equations (11). Keeping in mind that the coefficients of the equation (30) do not depend on u^α, we conclude that the functions ρ_n, θ_n, and b_n do not depend on u^α. As it was mentioned above, $\rho\ dx + \theta\ dt = d\psi(t,x)$ for this case. Hence the CCDs are almost all trivial for the original system (1). One or two nontrivial densities may arise for a nonlocal linearizing transformation only. The theorem is proved.

If the system (1) is reducible to the normal linear system, Theorem 4 is true for this case too.

IV. NON-NEWTONIAN LIQUID

Let us consider the equations of the one-dimensional motion of a liquid without external forces

$$\frac{dv}{dt} = v u_x, \quad \frac{du}{dt} = v P_x, \quad \left(\frac{d}{dt} = \frac{\partial}{\partial t} + u\ \frac{\partial}{\partial x}\right), \tag{31}$$

$$\frac{de}{dt} = v(P u_x - q_x), \quad T\ \frac{ds}{dt} + v q_x = v\,\delta \geqslant 0. \tag{32}$$

Here $v = \rho^{-1}$ is the specific volume, u is the velocity of the flow, P is a component of the stress tensor, e is the density of the internal energy, s is the density of the entropy, q is a density of the heat flow, δ is a dissipation function, and T is the absolute temperature. Let us adopt the following assumptions:

$$P = P(v,T,u_x), \quad \partial P/\partial u_x \neq 0; \quad e = e(v,T), \quad \partial e/\partial T \neq 0; \quad q = -v\kappa(v,T)T_x,$$

then the first equation (32) takes the form

$$\frac{dT}{dt} = \frac{P - e_v}{e_T}\ v u_x + \frac{v}{e_T}\ (v\kappa T_x)_x. \tag{33}$$

We shall consider for simplicity that the second equation (32) defines the function δ. Then the equations (31), (33) are the closed system. A direct investigation of the integrability of this system is cumbersome. But introducing the Lagrange variables $(t,x) \to (t',y)$,

$$t' = t, \quad dy = v^{-1}(dx - u\ dt),$$

we obtain the essential simplification,

$$v_t = u_y, \quad u_t = P_y(v,T,u_y), \quad T_t = \gamma(v,T)[\kappa(v,T)T_y]_y + \Phi(v,T,u_y). \tag{34}$$

It is denoted here that $\gamma = e_T^{-1}$, $\Phi = (P - e_v)u_y e_T^{-1}$. The main theorem allows us to investigate the integrability of the system (34) instead of the system (31), (33). Applying the described method, we obtained two series of the CCD. A few first densities take the following form:

$$\rho_0 = P_1^{-1/2}, \quad 2\rho_1 = D\ \ln|\rho_0| - \rho_0\theta_0,$$

$$2\rho_2 = \rho_0 P_3 \Phi_1(\kappa\gamma - P_1)^{-1} - \rho_0^3 P_2 - \rho_0^{-1}\rho_1^2 - \rho_0\theta_1 + \rho_0^{-2}\rho_1 D\rho_0,$$

$$\widetilde{\rho}_0=(\kappa\gamma)^{-1/2},\quad 2\widetilde{\rho}_1=D\ \ln|\widetilde{\rho}_0\gamma^{-1}|-\widetilde{\theta}_0\widetilde{\rho}_0-2u_y^3(\ln|\widetilde{\rho}_0|)_3,$$

$$2\widetilde{\rho}_2=P_3\Phi_1\widetilde{\rho}_0(P_1-\kappa\gamma)^{-1}-\widetilde{\rho}_0(\widetilde{\theta}_1+\Phi_3)-\widetilde{\rho}_0D\kappa D\gamma-(\widetilde{\rho}_0\gamma)^{-1}\widetilde{\rho}_1D\gamma+\widetilde{\rho}_0u_y^3(\kappa D\gamma_3+\kappa_3D\gamma)$$
$$-2\widetilde{\rho}_0^{\,-2}\widetilde{\rho}_1u_y^3\widetilde{\rho}_{0,3}-\widetilde{\rho}_0^{\,-1}\widetilde{\rho}_1^{\,2}+\widetilde{\rho}_0^{\,-2}\widetilde{\rho}_1D\widetilde{\rho}_0.$$

Here we use the following notation for the derivatives $P_1=\partial P/\partial u_y$, $P_2=\partial P/\partial v$, $P_3=\partial P/\partial T$, etc. Besides, $P_1\neq\kappa\gamma$ here. The case $P_1=\kappa\gamma$ was investigated separately. Assuming the weak nonlocality of the conserved currents, we checked ten integrability conditions (11) with the help of the computer and found that two systems satisfy all conditions only:

$$u_t=cu_{yy}+bv_y,\quad v_t=u_y,\quad T_t=\gamma\kappa T_{yy}+\gamma cu_y^2+\varphi(v)u_y,\tag{35}$$

$$u_t=\frac{cu_{yy}}{(u_y+b)^2},\quad v_t=u_y,\quad T_t=\gamma\kappa T_{yy}+\gamma u_y\left(\varphi(v)-\frac{c}{(u_y+b)}\right).\tag{36}$$

Here c, b, γ, and κ are constants, and φ is an arbitrary function. The system (35) has the linear subsystem and can be solved directly. The first equation of the system (36) is reduced to the linear form $u'_\tau=cu'_{\xi\xi}$ with the help of the following transformation: $\tau=t$, $\xi=u+by$, $u'=(u_y+b)^{-1}$. If we find the function u, then the functions v and T can be easily obtained from the system (36). In the Euler variables the systems (35) and (36) take the following form:

$$u_t+uu_x=v(cvu_x+bv)_x,\quad v_t+uv_x=vu_x,$$
$$T_t+uT_x=\gamma\kappa v(vT_x)_x+c\gamma v^2u_x^2+v\varphi(v)u_x,\tag{37}$$

$$u_t+uu_x=cv(vu_x)_x(vu_x+b)^{-2},\quad v_t+uv_x=vu_x,$$
$$T_t+uT_x=\gamma\kappa v(vT_x)_x+\gamma vu_x[\varphi(v)-c(vu_x+b)^{-1}].\tag{38}$$

It can be easily verified that $\rho_0=v^{-1}$ is the unique nontrivial CCD and $\rho_i=\theta_i=\widetilde{\rho}_i=\widetilde{\theta}_i=0$ for $i>0$ for both the system (37) and the system (38). This fact is the consequence of the existence of the linear subsystems in the corresponding reduced systems.

V. QUASILINEAR EVOLUTION SYSTEMS OF THE THIRD ORDER

Here we shall consider the following system:

$$u_t^i=A_j^i(u_0)u_3^j+F^i(u_0,u_1,u_2),\quad i,j=1,2,\tag{39}$$

where $u_i^k=\partial^i u^k/\partial x^i$ and the matrix A possesses a nonzero eigenvalue λ. Assuming validity of the first integrability condition, we shall try to simplify the system (39). It is well known[7,13] that the first canonical density takes the form $\rho=\lambda^{-1/3}$. Let θ be a flux corresponding to the density ρ. Performing the transformation

$$\tau=t,\quad dy=\rho\ dx+\theta\ dt,\quad v^i=u^i,\tag{40}$$

we reduce the system (41) to the following form:

$$v_\tau^i+\theta v_y^i=A_j^i(v)(\rho D_y)^3v^j+F^i=\lambda^{-1}A_j^iv_3^j+\cdots.\tag{41}$$

It is obvious that one eigenvalue of the matrix $A'=\lambda^{-1}A$ is the unit. Let us denote $A'^1_1=a$, $A'^1_2=b$, $A'^2_1=c$, $A'^2_2=d$, then the characteristic equation for the matrix A takes the form $\lambda^2-\lambda(a+d)+ad-bc=0$. Since the eigenvalue $\lambda=1$ exists, then $bc=(a-1)(d-1)$. If $bc=0$, then A' is the triangular matrix. It is obvious that the cases $b=0$ and $c=0$ are equivalent with respect to the permutation $v^1 \leftrightarrow v^2$. Therefore we may consider two matrices only:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}.$$

If $a \neq 0$ then $a^{-1/3}=\rho$ is the conserved density for the corresponding system (39). Therefore, performing the transformation (40), we reduce the first matrix into the second one, where $d=a^{-1}$. Thus we have three different triangular matrices,

$$A_1=\begin{pmatrix} 1 & g \\ 0 & f \end{pmatrix}, \quad A_2=\begin{pmatrix} 0 & g \\ 0 & 1 \end{pmatrix}, \quad A_3=\begin{pmatrix} 1 & g \\ 0 & 0 \end{pmatrix}, \tag{42}$$

where $f \neq 0$.

Let us consider now the case $bc \neq 0$. Performing a point transformation $w^i=f^i(v)$, we can reduce the system (41) into the triangular form

$$w_t^i=\widetilde{A}{}^i_j w_3^j+\widetilde{F}{}^i, \quad \widetilde{A}{}^i_j=\frac{\partial w^i}{\partial v^s}\,\frac{\partial v^k}{\partial w^j}\,A'^s_k,$$

where the matrix $\widetilde{A}$ takes the above form A_1. To check this statement we rewrite the transformation formula for $\widetilde{A}$ in another form,

$$w^j_{,k}\widetilde{A}{}^i_j=w^i_{,s}A'^s_k,$$

where $w^i_{,s}=\partial w^i/\partial v^s$. Setting here $\widetilde{A}=A_1$, we obtain

$$(a-1)w^1_{,1}+cw^1_{,2}=gw^2_{,1}, \quad bw^1_{,1}+(d-1)w^1_{,2}=gw^2_{,2},$$

$$bw^2_{,1}+(1-a)w^2_{,2}=0, \quad (1-d)w^2_{,1}+cw^2_{,2}=0. \tag{43}$$

We used here the equation $f=a+d-1$ that follows from the fact that the characteristic polynomials coincide for the matrices A' and $\widetilde{A}$. There are two independent equations only in the system (43) because of the identity $bc=(a-1)(d-1)$. Besides $w^2_{,1}\neq 0$, $w^2_{,2}\neq 0$ as $bc \neq 0$. Therefore we shall consider the first and the third equations for definiteness:

$$(a-1)w^1_{,1}+cw^1_{,2}=gw^2_{,1}, \quad bw^2_{,1}+(1-a)w^2_{,2}=0, \tag{44}$$

The second of these equations defines the function w^2, the first defines the function g, and w^1 is an arbitrary function. Our statement is proved. Let us investigate now the first equation (44) more carefully.

If the vectors $\{a-1,c\}$ and $\{b,1-a\}$ are linearly independent we can choose w^1 as the solution of the equation $(a-1)w^1_{,1}+cw^1_{,2}=0$. Then the functions w^1 and w^2 will be independent and $g=0$. And vice versa, if $\{a-1,c\}=\gamma\{b,1-a\}$, then we may require $bw^1_{,1}+(1-a)w^1_{,2}=w^2_{,1}/\gamma$. For this case we obtain $g=f=1$. Thus we obtain two nonequivalent matrices:

$$\widetilde{A}_1=\begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}, \quad \widetilde{A}_2=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This result means that the matrix A_1 from (42) can be reduced to one of these forms. The systems (39) corresponding to the matrices A_2 and A_3 in the form (42) can be also simplified with the help of the transformation $w^1=h(v^1,v^2)$, $w^2=v^2$. Choosing $gh_{,1}+h_{,2}=0$ for A_2 and $gh_{,1}-h_{,2}=0$ for

A_3, we obtain $g=0$. Thus, both A_2 and A_3 give the same system that follows from the matrix $\widetilde{A}_1$ when $f=0$. Unifying the previous results, we have the following theorem.

Theorem 5: *If the matrix A of the system (39) possesses a nonzero eigenvalue $\lambda=\rho^{-3}$ that satisfies the equation $D_t\rho=D\theta$ with a local function θ, then the system (39) can be reduced into one of the following forms ($w^1=u$, $w^2=v$):*

$$u_t=u_3+F(u_0,v_0,u_1,v_1,u_2,v_2),\quad v_t=f(u,v)v_3+G(u_0,v_0,u_1,v_1,u_2,v_2), \tag{45}$$

$$u_t=u_3+v_3+F(u_0,v_0,u_1,v_1,u_2,v_2),\quad v_t=v_3+G(u_0,v_0,u_1,v_1,u_2,v_2), \tag{46}$$

by means of a superposition of the transformation (40) and a point transformation.

Remark 1: Setting $f=0$ in (45) we obtain the mentioned system corresponding to the matrices A_2 and A_3 (42).

Remark 2: If the matrix A has the zero eigenvalues only, then the system (39) can be reduced to the form

$$u_t=v_3+F(u_0,v_0,u_1,v_1,u_2,v_2),\quad v_t=G(u_0,v_0,u_1,v_1,u_2,v_2), \tag{47}$$

with the help of a point transformation.

For the classification the systems (45)–(47) the groups of the equivalence transformation are useful.

Theorem 6: *The systems (45)–(47) possess the following equivalence groups:*
(a) $u'=\varphi(u)$, $v'=\psi(v)$ *for (45), where* $f\neq 1$;
(b) $u'=\varphi(u,v)$, $v'=\psi(u,v)$ *for (45), where* $f=1$;
(c) $u'=u\varphi'(v)+\psi(v)$, $v'=\varphi(v)$, *for the systems (46) and (47).*

Both the transformation (40) and a point transformation are invertible and preserve a localness of canonical currents. Therefore one may investigate the integrability of the systems (45)–(47) instead of (39).

A. Classification of the systems (45)

The calculations are very cumbersome for the general case. Therefore we present here a list of the integrable systems (45) for the case $\mathrm{ord}(F,G)\leqslant 1$ only. Besides, we assume that $f(f-1)\neq 0$. Choosing $b^T=(1,a)$ or $b^T=(\widetilde{a},1)$ in the equation (8) we obtain two sequences of the CCD ρ_i and $\widetilde{\rho}_i$. A few first densities take the following form:

$$\rho_{-1}=1,\quad \rho_0=0,\quad \rho_1=-\tfrac{1}{3}F_{,u_1},\quad \rho_2=\tfrac{1}{3}F_{,u_0},$$

$$\rho_3=\frac{1}{3}\theta_1+\frac{G_{,u_1}F_{,v_1}}{3(f-1)},\quad \widetilde{\rho}_{-1}=f^{-1/3},\quad \rho_0=-\frac{2}{3}D\ln(f),$$

$$\widetilde{\rho}_1=\tfrac{1}{3}\widetilde{\rho}_{-1}\widetilde{\theta}_{-1}-\tfrac{1}{3}\widetilde{\rho}_{-1}^{2}G_{,v_1}+\tfrac{1}{3}f(D\widetilde{\rho}_{-1})^2.$$

Checking 14 integrability conditions (11) with the help of the computer, we found that there exist three integrable systems only:

$$u_t=u_3+6uu_1-12vv_1,\quad v_t=-2v_3-6uv_1, \tag{48}$$

$$u_t=u_3+uu_1+v_1,\quad v_t=-2v_3-uv_1, \tag{49}$$

$$u_t=u_3+3(u+v)u_1+\tfrac{3}{2}[(a+3)u-(11+5a)v]v_1,$$

$$v_t=bv_3-3[(a-2)u+v]u_1-\tfrac{3}{2}(a+3)(u+v)v_1, \tag{50}$$

where $a^2=5$, $b=-1/2(3a+7)$. Besides, there are many triangular systems satisfying the integrability conditions. These systems are not interesting and we omit them. Hirota and Satsuma[20] found the N-soliton solutions of the system (48) and it is called the Hirota–Satsuma system. The zero curvature representation for the system (48) was found by Drinfeld and Sokolov[21] and Dodd and Fordy[22] independently. The zero curvature representations for the systems (49) and (50) were found by Drinfeld and Sokolov.[23] But the system (50) was presented in another form therein:

$$u_t=4u_3-3v_3+(6u-3v/2)u_1+3uv_1,\quad v_t=-3u_3+v_3-6vu_1+3(v-u)v_1.$$

So, there are three and only three nonequivalent integrable systems (45) when ord$(F,G)<2$. For all of these systems $f=$const. After some preliminary calculations we suppose that there do not exist the integrable systems (45) when $f\neq$const and ord$(F,G)=2$ too. The classification of the integrable systems (45) for the case ord$(F,G)=2$ is being prepared now and will be presented in a separate publication.

B. Transformations of the integrable systems

The system (48) possesses the three zeroth-order conserved densities ρ_i $(D_t\rho_i=D\theta_i)$:

$$\rho_1=u,\quad \rho_2=u^2-2v^2,\quad \rho_3=3tu^2-6tv^2+xu,$$

$$\theta_1=u_2+3u^2-6v^2,\quad \theta_2=2uu_2+8vv_2-u_1^2-4v_1^2+4u^3,$$

$$\theta_3=6tuu_2+xu_2+24tvv_2-3u_1^2t-u_1-12v_1^2t+12u^3t+3u^2x-6v^2x.$$

Performing the transformation (21) in the form $t'=t$, $dy=\rho_1\,dx+\theta_1\,dt$, $U(t',y)=u(t,x)$, $V(t',y)=v(t,x)$ we obtain the new integrable system,

$$U_t=U^3U_3+3U^2U_1U_2-12VUV_1+3(U^2+2V^2)U_1,$$

$$V_t=-2U^3V_3-3U^2V_1U_2-6U^2U_1V_2-3UV_1U_1^2+3(2V^2-3U^2)V_1,$$

where $U_n=\partial U/\partial y^n$. The transformations (21) corresponding to the ρ_2 and ρ_3 give too cumbersome systems and we omit them. The system (48) also admits the following differential substitution (analogous to the Miura transformation for the KdV equation):

$$u=-U_1-U^2-V^2,\quad v=V_1+2UV.$$

The system (48) is reduced under this substitution into the following one:

$$U_t=U_3+6VV_2+6V_1^2+6(V^2-U^2)U_1+12UVV_1+\exp(-2U_{-1})\times(c_1(t)\exp(-2V_{-1})-c_2(t)\exp(2V_{-1})),$$

$$V_t=-2V_3-6VU_2-6U_1V_1+6(U^2-V^2)V_1+12UVU_1+\exp(-2U_{-1})\times(c_1(t)\exp(-2V_{-1})+c_2(t)\exp(2V_{-1})),$$

where $U_{-1}=\int U\,dx$, $V_{-1}=\int V\,dx$, and $c_i(t)$ are arbitrary functions. Besides, the system (48) admits the substitution $u\to u_1$.

The system (49) possesses the three zeroth-order conserved densities:

$$\rho_1=u, \quad \rho_2=u^2+2v, \quad \rho_3=tu^2+2xu+2tv,$$

$$\theta_1=u_2+1/2u^2+v, \quad \theta_2=2uu_2-4v_2-u_1^2+2/3u^3,$$

$$\theta_3=2tuu_2+2xu_2-4tv_2-u_1^2t-2u_1+2/3u^3t+u^2x+2xv.$$

The transformations $t'=t,\ dy=\rho_k\,dx+\theta_k dt,\ U(t',y)=u(t,x),\ V(t',y)=v(t,x),\ k=1,2,$ give the following integrable evolution systems:

$$U_t=U^3U_3+3U^2U_1U_2+1/2U^2U_1+UV_1-U_1V,$$

$$V_t=-2U^3V_3-3U^2V_1U_2-6U^2U_1V_2-3UV_1U_1^2-3/2U^2V_1-VV_1,$$

$$\begin{aligned}U_t={}&U_3\rho^3+6U_2\rho^2(V_1+U_1U)+6U_1V_2\rho^2+3U_1^3\rho^2+12UU_1^2V_1\rho+12U_1V_1^2\rho\\&+U_1U(2V+1/3U^2)+V_1\rho,\end{aligned}$$

$$\begin{aligned}V_t={}&-2V_3\rho^3-6UV_1U_2\rho^2-12V_2\rho^2(V_1+U_1U)\\&-3\rho U_1^2V_1(2V+5U^2)-12UV_1^2U_1\rho-1/3UV_1(5U^2+6V),\end{aligned}$$

accordingly $(\rho=U^2+2V)$. The analogous transformation with the ρ_3 yields the following mixed system:

$$\begin{aligned}U_t={}&U_3/X_1^3+6(tUU_1+UX_1+U_1X+tV_1)U_2/X_1^2+6tU_1V_2/X_1^2+2UU_1X_2/X_1^2+2(6tXV_1/X_1\\&+2tU^2+2XU+6t^2UV_1/X_1+3/X_1)U_1^2+12t^2U_1V_1^2/X_1+3tU_1^3/X_1^2+V_1/X_1\\&+(U/X_1-2/3U^3t-X/t/X_1+2X^2U/t+4U^2X_1+16V_1tU)U_1,\end{aligned}$$

$$\begin{aligned}V_t={}&-2V_3/X_1^3-6U_2(X+tU)V_1/X_1^2-12V_2(U_1X+UX_1+tUU_1+tV_1)/X_1^2-4UV_1X_2/X_1^2\\&-3(t/X_1+4X^2+4t^2U^2+8tXU)U_1^2V_1/X_1-4(2tUX_1+3tXU_1+3t^2UU_1)V_1^2/X_1\\&-2V_1U_1(3/X_1+10tU^2+10XU)+V_1(2X^2U/t-8U^2X_1-2/3U^3t-U/X_1-X/t/X_1),\end{aligned}$$

$$\begin{aligned}X_t={}&-2/X_1U_2(X+tU)+4/X_1tV_2+(t/X_1-4X^2-8tXU-4t^2U^2)U_1^2+8t^2V_1^2-4X_1U_1U(X+tU)\\&+8tUV_1X_1+4t(X+tU)V_1U_1+2U_1+2X_1(X^2U/t-1/3U^3t)-X/t,\end{aligned}$$

$$X_y=1/(tU^2+2tV+2XU),$$

where $X=x(t,y)$. The system (49) admits the substitutions $u\to u_1$ and $v_1\to v$. The previous transformation $dy=\rho_1\,dx+\theta_1\,dt$ will be the contact one $t'=t,\ x'=u,\ u'=u_1,\ v'=v$ under the first of the substitutions. If we differentiate the second equation (49) and perform the substitution $v_1\to v$, then the new conserved density $\rho=v$ and the new transformation (21) arise. For the original system this transformation takes the following form $t'=t,\ x'=v,\ u'=u,\ v'=v_1$ and yields the system $(u'=U,\ v'=V)$:

$$U_t=V^3U_3+3V^2V_1U_2+3VU_1(VV_2+V_1^2)+2UVU_1+V,$$

$$V_t=-2V^3V_3-6V^2V_1V_2-V^2U_1.$$

The system (50) possesses the three zeroth-order conserved densities:

$$\rho_1 = u + v, \quad \rho_2 = u^2 - 4v^2 a - 9v^2, \quad \rho_3 = -u^2 at + 3u^2 t - 7v^2 t - 3v^2 at + 2/3xu + 2/3vx,$$

$$\theta_1 = u_2 + bv_2 + 9/2u^2 - 3/2u^2 a - 21/2v^2 - 9/2v^2 a,$$

$$\theta_2 = 2uu_2 - 8vabv_2 - 18vbv_2 - u_1^2 + 4v_1^2 ab + 9v_1^2 b + 2u^3 + 9u^2 v + 3u^2 va + 24v^2 au + 54v^2 u + 47v^3 + 21v^3 a,$$

$$\theta_3 = -2uatu_2 + 6utu_2 + 2/3xu_2 + 2/3xbv_2 - 14vtbv_2 - 6vatbv_2 + u_1^2 at - 3u_1^2 t - 2/3u_1 - 2/3bv_1 + 7v_1^2 tb + 3v_1^2 atb + 3u^2 x + 12u^2 vt - u^2 xa + 6u^3 t + 42v^2 tu - 2u^3 at + 18v^2 atu - 7v^2 x + 16v^3 at + 36v^3 t - 3v^2 xa.$$

The transformation $t' = t$, $dy = \rho_1\, dx + \theta_1\, dt$, $U(t',y) = u(t,x)$, $V(t',y) = v(t,x)$ yields the following system:

$$U_t = U_3 \rho^3 + 3\rho^2 U_2 (U_1 + V_1) + 3/2\rho^2 V_2 U_1 (a+3) + 3/2\rho U_1^2 V_1 (a+3) + 3/2\rho U_1 V_1^2 (a+3) + 3/2U_1(-U^2 + 4UV + 9V^2 + U^2 a + 3V^2 a) - 3/2\rho V_1(-3U + 11V + 5\rho a - 6Ua),$$

$$V_t = bV_3 \rho^3 - 3/2\rho^2 U_2 V_1 (a+3) + 3\rho^2 bV_2 (U_1 + V_1) - 3/2\rho U_1^2 V_1 (a+3) - 3/2\rho U_1 V_1^2 (a+3) - 3\rho U_1(-2\rho + 3V + \rho a - Va) - 3V_1(3U^2 + 3UV - 2V^2 + VaU - V^2 a),$$

where $\rho = U + V$. Other transformations (21) corresponding to the ρ_2 and ρ_3 give too cumbersome systems and we omit them.

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