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Radiant-energy penetration effect in the thermal-diffusivity flash technique for layered and porous polymers

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Thermal insulation characterization of polymers and other materials is an important requirement for present and future aerospace missions, as well as for the home construction industry. Some dispersed composites and layered samples have been successfully characterized by the thermal-diffusivity flash technique, while the interpretation of these and other experiments for other systems remains problematical. One refinement for layered-sample data reduction is investigated here. An exponentially decaying spatial penetration of radiant energy into the sample is accounted for. Under the conditions chosen for numerical evaluation, an 18% reduction in the back-face rise time $t_{1/2}$ is predicted for a case in which only 5% as much radiant energy is deposited on the midplane as on the front surface. It is suggested how the use of the entire $V(t)$ response function might lead to experimental methods of estimating the penetration depth of the radiant energy for particular samples.

I. INTRODUCTION

Thermal insulation characterization of polymers is an important requirement for present and future aerospace systems. Examples include a Neoprene/Kevlar composite used in rocket motors, as well as insulating blankets of fused silica or other materials. Furthermore, the construction of new homes requires the use of lightweight fibrous insulation whose thermal properties need to be known.

The significance of thermal radiation in fibrous insulations has been demonstrated[1,2] by conducting guarded hot-plate experiments under vacuum. Tong and Tien[3,4] categorized the analyses of such experiments as those which experimentally determine effective conductivities caused by radiation and those which consider the detailed equations governing the intensity of radiation in an absorbing and scattering medium.

The thermal diffusivity α is the parameter of interest in transient-heat conduction problems, as opposed to steady-state problems in which only the thermal conductivity K plays a role. For a homogeneous body, α may be calculated from K, the density ρ, and the specific heat c. If α, ρ, and c are measured in a given temperature range, K may be calculated over that range. This is especially advantageous when steady-state calorimetry for K is difficult, e.g., at high temperatures.

One method of measuring the thermal diffusivity is the flash technique.[5–12] A flash of radiant energy is deposited over one face of a disk-shaped sample that is surrounded by a heating element and enclosed in a vacuum. For a homogeneous sample, α may be calculated from the sample thickness L and the time $t_{1/2}$ (the time for the rear face to achieve one half its ultimate temperature). To within a numerical factor, α is essentially $L^2/t_{1/2}$.

It has been observed[12] that problems involving large temperature differences and short heat-pulse propagation times may lead to situations in which the basic equations, boundary conditions, and assumptions of the flash technique data-reduction scheme are inapplicable. For these problems there may be no way to convert an observed oscilloscope trace into a meaningful value for some physical parameter. Conditions occurring during space vehicle reentry, laser irradiation problems, measurements at cryogenic temperatures, and materials with anisotropic conductivity may lead to situations in which the flash diffusivity method may not be viable.

On the other hand, some nonhomogeneous materials have been successfully characterized by the flash technique. The flash technique has been applied to some heterogeneously dispersed composites and layered samples, including some carbon fibers, fiber-reinforced materials, individual layers of layered composites, and dispersed composites.[12]

The continuing challenge of composite-sample characterization by means of the flash technique is illustrated by recent case histories and journal publications. Flash-technique analysis[13] of a Neoprene-covered Kevlar cloth composite revealed that the diffusivity value could not be calculated for the Kevlar layer solely on the basis of the measured rise times and the properties of the virgin Neoprene. The rise times were too short, indicating that the conductivity values for the virgin Neoprene were too small for the Neoprene in the composite. In a second example, fibrous insulation, it has been shown that steady-state and transient methods of determining thermal conductivity often give different results, especially for such porous insulating materials as rockwool and kapok.[2] This discepancy arises from the contribution of internal radiation to the conduction of heat and should be even larger for an intrinsically transparent material. Although recent publications[3,4] have addressed this issue from other viewpoints, internal radiation has not, to the present author's knowledge, previously been allowed for in the development of the flash technique.

Therefore, we see that some currently interesting sample types are in some way too complicated for available flash-technique analysis. The instrumentation is producing experimental outputs that are apparently in need of more refined interpretation. In any particular case, any one of a number of different adjustments may be necessary. These adjustments are necessitated by four categories of physical effects:

(1) Radiant-energy penetration during flash heating of the diffusivity sample (time = zero).

(2) The contribution of internal radiation and absorption to heat transfer (time greater than zero).

(3) Heterogeneous sample media.

(4) Anisotropic sample media.

An adjustment accounting for the first effect and applying to both the layered-composite and porous-insulation types is discussed in this paper. This mathematical modification of the current flash-technique data-reduction method explicitly allows for an exponentially decaying spatial penetration of incident radiant energy into a layered sample. A method of experimentally evaluating the characteristic length associated with the exponential penetration function will be proposed.

II. THEORETICAL DEFINITION OF THE PROBLEM

A. Homogeneous slab

For a homogeneous slab between $x=0$ and $x=L$, the governing equations (as in Ref. 14) are

$$\mathbf{J} = -K\nabla T, \tag{1}$$

$$\nabla\cdot\mathbf{J} + \rho c\frac{\partial T}{\partial t} = 0, \tag{2}$$

where $T = T(x,t)$ is the temperature as a function of position x and time t, K, ρ, and c are the thermal conductivity, density, and specific heat, respectively, and $\mathbf{J}$ is the flux (energy per area and time). Homogeneity implies constant K, ρ, and c, so combining the equations gives the partial differential equation (PDE)

$$\alpha\nabla^2 T = \frac{\partial T}{\partial t}, \tag{3}$$

where

$$\alpha = \frac{K}{\rho c}. \tag{4}$$

The boundary conditions (BC) for the homogeneous slab are $\nabla T = 0$ at $x=0$ and L (no heat loss). The initial conditions (IC) are

$$T(x,0) = \frac{Q}{\rho c}\delta(x), \tag{5}$$

where Q is the energy per area deposited on the slab surface ($x=0$ at $t=0$), and $\delta(x)$ is the Dirac delta function. Separation of variables $[T(x,t)\equiv W(x)\tau(t)]$ leads to the ordinary differential equations (ODE) $W'' + (\lambda/\alpha)W = 0$ and $\tau' + \lambda\tau = 0$, where λ is the separation constant and primes denote differentiation. In general for this Sturm–Liouville problem, $W_n = A_n \sin\beta_n x + B_n\cos\beta_n x$ and $\tau = e^{-\lambda t} = e^{-\beta_n^2\alpha t}$. The above BC imply that all $A_n = 0$, and the eigenvalue equation $\sin\beta_n L = 0$ leads to

$$\beta_n = \frac{n\pi}{L},\ n = 0,1,2\ldots \tag{6}$$

(including negative n only changes the B_n definition). The B_n are specified by the IC with the result

$$T(x,t) = \frac{Q}{\rho c L}\left[1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L}\exp\left(\frac{-n^2\pi^2}{L^2}\alpha t\right)\right]. \tag{7}$$

The ultimate $x=L$ temperature is $Q/\rho c L$, so the dimensionless temperature versus time profile at $x=L$ is

$$V = 1 + 2\sum_{n=1}^{\infty}(-1)^n\exp\left(\frac{-n^2\pi^2\alpha t}{L^2}\right). \tag{8}$$

Approximating the V result by the first two terms (valid as $t\rightarrow\infty$), one observes that $V = 1/2$ when

$$\alpha = \frac{L^2\ln 4}{\pi^2 t_{1/2}} = 0.1404\frac{L^2}{t_{1/2}}. \tag{9}$$

Using the full series expression,[12]

$$\alpha = 0.1388\frac{L^2}{t_{1/2}}. \tag{10}$$

Thus, the thermal diffusivity of a homogeneous slab can be measured by measuring the thickness L and the rise time $t_{1/2}$. This can be done to within 0.5% precision.[12] On the other hand, the IC and BC assumptions, such as delta function temperature pulse across the entire sample and complete thermal isolation, limit the accuracy of the measurement. Models have been developed that generate corrections up to 21% in magnitude in the measured values.[12]

B. Composite slab

Surprisingly enough, although a unique solution can also be shown to exist for a composite slab consisting of individual layers, that solution cannot be constructed by an extension of the Sturm–Liouville approach (p. 414 of Ref. 15). Instead, the Laplace transform technique (Chap. XII in Ref. 14) is used. The standard analysis for the simplest composite (two layers and one-dimensional heat flow) will first be summarized, using mostly the notation of Ref. 10. Then, in the following sections, recommendations for modifying the mathematics and the measurement algorithm will be stated, relying on the standard analysis as a starting point.

The two-layer problem is defined by the following equations.

Partial differential equations (PDE):

$$K_1\nabla^2 T_1 = \rho_1 c_1\frac{\partial T_1}{\partial t},\quad -l_1 < x < 0, \tag{11}$$

$$K_2\nabla^2 T_2 = \rho_2 c_2\frac{\partial T_2}{\partial t},\quad 0 < x < l_2. \tag{12}$$

Initial conditions (IC):

$$T_1(x,0) = 0,\quad -l_1 < x < 0, \tag{13}$$

$$T_2(x,0) = 0,\quad 0 < x < l_2. \tag{14}$$

Boundary conditions (BC):

$$-K_1\nabla T_1(-l_1,t) = Q\delta(t)\ \text{(flux pulse)}, \tag{15}$$

$$\nabla T_2(l_2,t) = 0\ \text{(no back-side heat loss)}, \tag{16}$$

$$T_1(0,t) = T_2(0,t)\ \text{(no interfacial resistance)}, \tag{17}$$

$$K_1\nabla T_1(0,t) = K_2\nabla T_2(0,t)\ \text{(zero divergence at interface)}. \tag{18}$$

Notice particularly that the temperature pulse in the IC of the Sturm–Loiuville problem has been replaced by a flux pulse in the BC of this Laplace problem. Combining the above equations, we have the following results:

PDE and IC:

$$K_1 \frac{d^2\overline{T}_1}{dx^2} - \rho_1 c_1 (s\overline{T}_1 - 0) = 0, \tag{19}$$

$$K_2 \frac{d^2\overline{T}_2}{dx^2} - \rho_2 c_2 (s\overline{T}_2 - 0) = 0. \tag{20}$$

ODE:

$$\frac{d^2\overline{T}_1}{dx^2} - \frac{s}{\alpha_1}\overline{T}_1 = 0, \tag{21}$$

$$\frac{d^2\overline{T}_2}{dx^2} - \frac{s}{\alpha_2}\overline{T}_2 = 0. \tag{22}$$

BC:

$$-K_1\left(\frac{d\overline{T}_1}{dx}\right)_{x=-l_1} = Q, \quad s>0, \tag{23}$$

$$\left(\frac{d\overline{T}_2}{dx}\right)_{x=l_2} = 0, \quad s>0, \tag{24}$$

$$\overline{T}_1(0,s) = \overline{T}_2(0,s), \quad s>0, \tag{25}$$

$$K_1\frac{d\overline{T}_1}{dx} = K_2\frac{d\overline{T}_2}{dx}, \quad x=0, \quad s>0. \tag{26}$$

Letting $\omega_i \equiv \sqrt{s/\alpha_i}$, the ODEs in Eqs. (21) and (22) have the solutions $\overline{T}_i = A_i \cosh \omega_i x + B_i \sinh \omega_i x$. Using the BCs in Eqs. (23)–(26), one obtains

$$\mathsf{a}\begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ p \end{pmatrix}, \tag{27}$$

where the matrix a has the elements a_{ij} $(i,j = 1,2,3,4)$, and $a_{13} = -\sinh \omega_2 l_2$, $a_{14} = -\cosh \omega_2 l_2$, $a_{21} = 1$, $a_{23} = -1$, $a_{32} = K_1\omega_1$, $a_{34} = -K_2\omega_2$, $a_{41} = \sinh \omega_1 l_1$, and $a_{42} = -\cosh \omega_1 l_1$, all other a_{ij} being zero. Furthermore, $p = Q/K_1\omega_1$.

Solving by determinants,

$$\det(\mathsf{a}) = K_2\omega_2 \cosh \omega_1 l_1 \sinh \omega_2 l_2 + K_1\omega_1 \sinh \omega_1 l_1 \cosh \omega_2 l_2, \tag{28}$$

$$A_1 = pK_1\omega_1 \cosh \omega_2 l_2/\det(\mathsf{a}), \tag{29}$$

$$B_1 = -pK_2\omega_2 \sinh \omega_2 l_2/\det(\mathsf{a}), \tag{30}$$

$$A_2 = pK_1\omega_1 \cosh \omega_2 l_2/\det(\mathsf{a}), \tag{31}$$

$$B_2 = -pK_1\omega_1 \sinh \omega_2 l_2/\det(\mathsf{a}). \tag{32}$$

Introducing

$$\sigma \equiv \frac{K_2\omega_2}{K_1\omega_1} = \sqrt{\frac{K_2\rho_2 c_2}{K_1\rho_1 c_1}} \text{ and } \sqrt{U_i s} \equiv l_i\omega_i,$$

and noting that $\omega_i K_i = \sqrt{s}\sqrt{K_i\rho_i c_i}$, Eqs. (28)–(32) become

$$\det(\mathsf{a}) = \sqrt{s}\sqrt{K_1\rho_1 c_1}(\sinh\sqrt{U_1 s}\cosh\sqrt{U_2 s} + \sigma\cosh\sqrt{U_1 s}\sinh\sqrt{U_2 s}), \tag{33}$$

$$A_1 = Q\cosh \omega_2 l_2/\det\ (\mathsf{a}), \tag{34}$$

$$B_1 = Q(-\sigma \sinh \omega_2 l_2)/\det\ (\mathsf{a}), \tag{35}$$

$$A_2 = Q\cosh \omega_2 l_2/\det\ (\mathsf{a}), \tag{36}$$

$$B_2 = Q(-\sinh \omega_2 l_2)/\det\ (\mathsf{a}). \tag{37}$$

Hence,

$$\overline{T}_1 = \frac{Q}{\det\ (\mathsf{a})}(\cosh \omega_2 l_2 \cosh \omega_1 x - \sigma \sinh \omega_2 l_2 \sinh \omega_1 x), \tag{38}$$

$$\overline{T}_2 = \frac{Q}{\det\ (\mathsf{a})}(\cosh \omega_2 l_2 \cosh \omega_2 x - \sinh \omega_2 l_2 \sinh \omega_2 x), \tag{39a}$$

$$\overline{T}_2 = \frac{Q}{\det\ (\mathsf{a})}\cosh[\sqrt{U_2 s}(1 - x/l_2)]. \tag{39b}$$

Combining Eqs. (33) and (39b) and introducing $T_0 \equiv Q/\sqrt{K_1\rho_1 c_1}$, we have

$$\frac{\overline{T}_2}{T_0} = \frac{1}{s^{1/2}}\frac{\cosh[\sqrt{U_2 s}(1 - x/l_2)]}{\sinh\sqrt{U_1 s}\cosh\sqrt{U_2 s} + \sigma\cosh\sqrt{U_1 s}\sinh\sqrt{U_2 s}}. \tag{40}$$

On the entire complex plane the series expansions of the factors in the denominator vary as $\sqrt{s}$ [$\sqrt{s}$ (1 + even powers)] (1 + even powers). Therefore, the denominator behaves as s near $s = 0$ (simple pole at $s = 0$). The nature of the remaining roots of the denominator has been investigated (pp. 324–326 of Ref. 14, pp. 409–410 of Ref. 15, and Appendix A of Ref. 16). There are no complex roots. The function $\overline{T}_2/T_0$ is meromorphic, with poles at $s = 0$ and $s = s_n$, $n = 1,2,3,\ldots$.

The Mittag–Leffler theorem states that any meromorphic function can be expressed as a sum of an entire function and a series of rational functions. In our case, letting $f(s) = \overline{T}_2/T_0$,

$$f(s) = \sum_{n=0}^{\infty}\frac{r_n}{s - s_n} + \frac{1}{2\pi i}\int_c \frac{f(s')ds'}{s' - s}, \tag{41a}$$

where r_n is the residue of the nth/pole. The contour integral in Eq. (41a) can be shown to be zero as in standard texts on complex analysis. Thus,

$$f(s) = \frac{r_0}{s} + \sum_{n=1}^{\infty}\frac{r_n}{s - s_n}, \tag{41b}$$

$$f(s) = \frac{1}{s}\lim_{s\to 0} sf(s) + \sum_{n=1}^{\infty}\frac{\mathrm{X}(s_n)}{\mathrm{Y}'(s_n)(s - s_n)}, \tag{41c}$$

where

$$\mathrm{X}(s) = \frac{\cosh[\sqrt{U_2 s}(1 - x/l_2)]}{\sqrt{s}}, \tag{42}$$

$$\mathrm{Y}(s) = \sinh\sqrt{U_1 s}\cosh\sqrt{U_2 s} + \sigma\cosh\sqrt{U_1 s}\sinh\sqrt{U_2 s}. \tag{43}$$

Thus, letting $X = \sqrt{U_1/U_2}$, we obtain

$$\frac{\overline{T}_2}{T_0} = \frac{1}{s\sqrt{U_2}(X + \sigma)} + \sum_{n=1}^{\infty}\frac{\mathrm{X}(s_n)}{\mathrm{Y}'(s_n)(s - s_n)}. \tag{44}$$

Using the inverse Laplace operator $\mathscr{L}^{-1}$, one obtains

$$\frac{T_2}{T_0} = \frac{1}{\sqrt{U_2}(X + \sigma)} + \sum_{n=1}^{\infty}\frac{\mathrm{X}(s_n)}{\mathrm{Y}'(s_n)}e^{s_n t}. \tag{45}$$

Consider the following definitions:

$$\beta_n = \sqrt{-U_2 s_n}, \quad n = 1,2,3\ldots, \tag{46a}$$

$$\Omega = \frac{X\sigma + 1}{X + 1}, \tag{46b}$$

$$\frac{1}{\sigma} = \frac{H}{X}, \tag{46c}$$

$$U_i = \frac{l_i^2}{\alpha_i}, \quad i = 1,2. \tag{46d}$$

After considerable algebra, it results that

$$T_2 = \frac{Q}{l_1 \rho_1 c_1 + l_2 \rho_2 c_2} \times \left(1 + 2 \sum_{n=1}^{\infty} \frac{e^{-\beta_n^2 t / U_2} \cos[\beta_n (1 - x/l_2)]}{\cos X\beta_n \cos \beta_n - \Omega \sin X\beta_n \sin \beta_n} \right), \tag{47a}$$

$$V(t) = \frac{T_2(l_2,t)}{T_{2,\max}}, \tag{47b}$$

where $T_{2,\max} = Q / [l_1 \rho_1 c_1 + l_2 \rho_2 c_2]$. The equation that gives the roots of the denominator in Eq. (40), when recast in terms of circular functions, determines the real, positive eigenvalues $\beta = \beta_n = \sqrt{-U_2 s_n}$ according to

$$\sin X\beta \cos \beta + \sigma \cos X\beta \sin \beta = 0. \tag{47c}$$

Furthermore, when the slabs have identical parameters,

$$T_2 = \frac{Q}{2 l_1 \rho_1 c_1} \times \left(1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{\frac{-n^2 \pi^2 \alpha_1 t}{4 l_1^2}} \cos\left[\frac{n\pi}{2} (1 - x/l_2) \right] \right), \tag{47d}$$

which corresponds to the V result given in the homogeneous sample analysis. This shows that the flux-pulse, dual-slab, Laplace analysis can reduce to the temperature-pulse, homogeneous-slab, Sturm–Liouville analysis for an appropriate choice of parameters. Finally, if $V(t)$ or $t_{1/2}$ is measured and compared to Eqs. (47a) or (47b), one unknown material parameter can be obtained by standard computer techniques.

III. ANALYSIS OF THE RADIANT-ENERGY PENETRATION EFFECT

Suppose that the incident electromagnetic energy is converted to heat throughout the depth of the first layer of a two-layer composite according to

$$Q(x) = \frac{Qke^{-k(x+l_1)}}{(1 - e^{-kl_1})}, \quad -l_1 < x < 0. \tag{48}$$

The normalization was chosen to ensure that Q energy units per area are incident. This is the same Q as in the standard solution above, in which all the electromagnetic energy is converted to heat at the front interface. We will first analyze what effect this penetration will have on the observable quantity $t_{1/2}$.

Equations (11)–(18) are the same as before, with the following exceptions:

$$T_1(x,0) = \frac{Qke^{-k(x+l_1)}}{\rho_1 c_1 (1 - e^{-kl_1})}, \quad -l_1 < x < 0, \tag{13'}$$

and

$$-K_1 \nabla T_1(-l_1,t) = 0. \tag{15'}$$

Now Eqs. (19)–(26) are the same as before, with the following exceptions:

$$K_1 \frac{d^2 \overline{T}_1}{dx^2} - \rho_1 c_1 \left(s\overline{T}_1 - \frac{Qke^{-k(x+l_1)}}{\rho_1 c_1 (1 - e^{-kl_1})} \right) = 0, \tag{19'}$$

and

$$-\left(K_1 \frac{d\overline{T}_1}{dx} \right)_{x=-l_1} = 0. \tag{23'}$$

The solution to the ODE, Eq. (19'), is somewhat different than before:

$$\overline{T}_1(x,s) = A_1 \cosh \omega_1 x + B_1 \sinh \omega_1 x + \frac{Qke^{-k(x+l_1)}}{K_1 (1 - e^{-kl_1})(\omega_1^2 - k^2)}. \tag{19''}$$

Now the matrix equation, Eq. (27) becomes

$$\mathbf{a} \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ p'' \\ p' \\ p \end{pmatrix}. \tag{27'}$$

All the entries of $\mathbf{a}$ are the same, but now

$$p'' = \frac{-Qke^{-kl_1}}{K_1 (1 - e^{-kl_1})(\omega_1^2 - k^2)}, \tag{49a}$$

$$p' = \frac{+Qke^{-kl_1}}{(1 - e^{-kl_1})(\omega_1^2 - k^2)}, \tag{49b}$$

$$p = \frac{-Qk^2}{\omega_1 K_1 (1 - e^{-kl_1})(\omega_1^2 - k^2)}. \tag{49c}$$

We will assume only a limited penetration consistent with $e^{-kl_1} \lesssim 0.05$, or $kl_1 \gtrsim 3$.

To a reasonable approximation, therefore, we set e^{-kl_1}, p'', and p' equal to zero, and

$$p \approx \frac{Q}{K_1 \omega_1 \left(1 - \frac{s}{k^2 \alpha_1} \right)}. \tag{50}$$

In the limit that $k \to \infty$ (delta function, or surface, energy deposition), $p \to Q / K_1 \omega_1$ as in the standard solution above. Therefore, we can follow the changing solution as surface deposition is continuously changed to deposition in depth. We can do this rigorously if nonzero p'' and p' are considered. Here we consider only the case of small penetration, $kl_1 \gtrsim 3$.

Since the problem has only changed to the extent of a modification of the factor Q in the quantity p, it is seen immediately that

$$\frac{\overline{T}_2}{T_0} = \frac{1}{s^{1/2}(1 - s/k^2 \alpha_1)} \times \frac{\cosh[\sqrt{U_2 s}(1 - x/l_2)]}{[\sinh\sqrt{U_1 s} \cosh\sqrt{U_2 s} + \sigma \cosh\sqrt{U_1 s} \sinh\sqrt{U_2 s}]}. \tag{40'}$$

Hence, the change in the problem which is due to finite radiant-energy penetration is seen in a change in the pole structure, as shown in Fig. 1. Thus, the previous solution, Eq. (41b), is modified to become

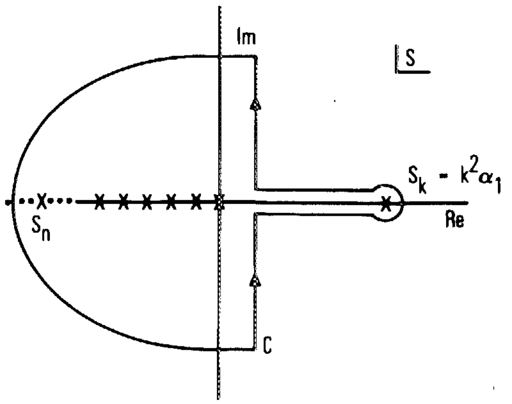

FIG. 1. The pole structure for the two-layer composite with a finite penetration depth of electromagnetic energy. The difference in this solution, compared to the case of electromagnetic energy conversion to heat at the surface only, is the appearance of a new pole at $s_k = k^2\alpha_1$. C is the contour for the integration that inverts $\overline{T}_2(x_1 s)/T_0$.

$$f(s) = \frac{r_0}{s} + \sum_{n=1}^{\infty} \frac{r_n}{s - s_n} + \frac{r_k}{s - s_k}. \tag{41b$'$}$$

Since $kl_1 \gtrsim 3$, k is positive, and $s_k > 0$. Now, considering Eqs. (42), (43), and (40′),

$$r_0 = \frac{1}{\sqrt{U_2}(X + \sigma)}, \tag{51a}$$

$$r_n = \frac{-s_k \mathrm{X}(s_n)}{(s_n - s_k)\mathrm{Y}'(s_n)}, \tag{51b}$$

$$r_k = -s_k \frac{\mathrm{X}(s_k)}{\mathrm{Y}(s_k)}. \tag{51c}$$

We note that r_k is proportional to $e^{-kl_1} \to 0$, since we have taken $e^{-kl_1} = 0$ as an approximation. Thus, the last term in Eq. (41b′) must be ignored. Only if $p'' \neq 0$, $p' \neq 0$, and $e^{-kl_1} \neq 0$ in Eq. (27′) would it be consistent to keep such terms. In our approximation, then, Eq. (41b′) differs from Eq. (41b) only through the $-s_k/(s_n - s_k)$ multiplicative factor in Eq. (51b).

Using $s_k = k^2\alpha_1$ and $s_n = -\beta_n^2/U_2$ Eq. (47a) becomes

$$\frac{T_2}{T_{2,\max}} = 1 + 2\sum_{n=1}^{\infty} \frac{1}{1 + \beta_n^2/U_2 k^2\alpha_1} \times \frac{e^{-\beta_n^2 t/U_2}\cos[\beta_n(1 - x/l_2)]}{\cos X\beta_n \cos\beta_n - \Omega \sin X\beta_n \sin\beta_n}. \tag{52}$$

Equation (52) is the master equation for the $p'' = p' = 0$ approximation to the finite-energy penetration problem. Two tasks now remain: First it will be demonstrated that this energy penetration can significantly affect experimental results. Then it will be indicated how the computer data-reduction algorithm might be modified to include codetermination of k along with the usual parameter evaluation.

The significance of Eq. (52) can be judged by evaluating it for the case of identical-layer parameters and in the first-term ($n = 1$) approximation analogous to Eq. (9):

$$\frac{1}{2} = 1 - \frac{2e^{-\pi^2\alpha_1 t_{1/2}/4l_1^2}}{1 + \pi^2/4k^2 l_1^2}, \tag{53a}$$

$$\frac{4l_1^2}{\pi^2\alpha_1}\left[\ln 4 - \ln\left(1 + \frac{\pi^2}{4k_1^2 l^2}\right)\right] = t_{1/2}. \tag{53b}$$

Letting $t_{1/2}^{(0)}$ be the rise time for the spatial delta-function energy deposition [Eq. (9)], we have

$$t_{1/2} = t_{1/2}^{(0)} - \frac{4l_1^2}{\pi^2\alpha_1}\ln\left(1 + \frac{\pi^2}{4k^2 l_1^2}\right). \tag{53c}$$

Letting SF be the fractional deviation (shortfall) of the rise time in the finite-deposition case, we obtain

$$\mathrm{SF} = \frac{t_{1/2} - t_{1/2}^{(0)}}{t_{1/2}^{(0)}} = -\frac{\ln(1 + \pi^2/4k^2 l_1^2)}{\ln 4} \tag{53d}$$

for the approximation $kl_1 \gtrsim 3$. Figure 2 shows the result for the "shortfall" of the rise time versus kl_1 or the characteristic penetration depth $1/k$.

The approximations used in the construction of Fig. 2 could be removed by the rigorous ($p'' \neq 0$, $p' \neq 0$, $e^{-kl_1} \neq 0$) solution of Eq. (27′), the subsequent inclusion of the r_k residue in Eq. (41b′), and the use of the entire series in Eq. (52). However, even with those refinements the $-$SF ($kl_1 = 3$) is still expected to be in the 10–20% range, even though the energy reaching the midplane of the sample is only $e^{-3} = 5\%$ of that converted to heat at the front interface.

We see that the finite penetration of radiation may cause a significant rise-time shortfall. How can this effect be allowed for in data reduction?

In one standard form (p. 493 of Ref. 10) of the flash-technique data reduction, the eigenvalue equation [Eq. (47c)] is solved with the measured auxiliary parameters and a first guess at α_1, which is chosen as the unknown. The first guess is $\alpha_1 = 0.1388\ l_1^2/t_{1/2}$, as in Eq. (10). {Presumably $\alpha_1 = 0.1404\, l_1^2/t_{1/2}$ [Eq. (9)] would also lead to a convergent series.} The deviation from $V(t = t_{1/2})$ is calculated, and an iteration algorithm is devised which terminates when $\alpha_1^{i+1} = \frac{1}{2}(\alpha_1^i + \alpha_1^{i-1})$ converges to a reasonable limit. Bulmer and Taylor use only the $t_{1/2}$ experimental output, because they

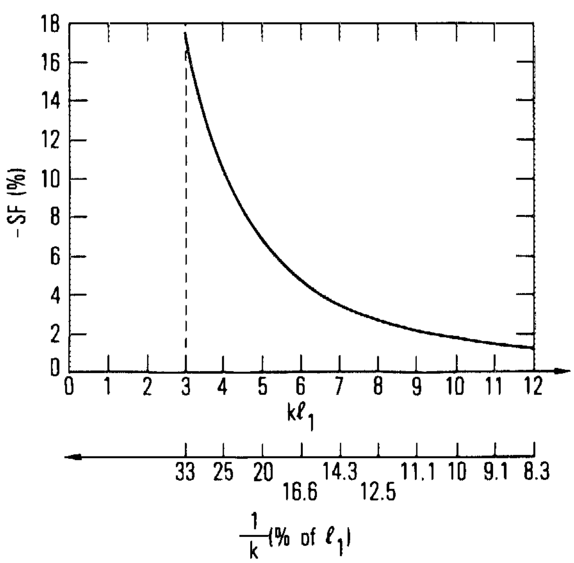

FIG. 2. The magnitude of the percent shortfall ($-$SF) of the measured rise time in the thermal diffusivity measurement of a homogeneous layer of width $2l_1$, graphed as a function of kl_1 and of the characteristic penetration depth $1/k$. In our approximation ($p'' = p' = e^{-kl_1} = 0$), $kl_1 \gtrsim 3$, and only the $n = 1$ rise times are compared.

feel that curve fitting to more points requires using extremes of the experimental curve which are inherently less accurate.

We would argue that the entire experimental $V(t)$ curve contains *some* relevant information, and that at the very least one additional parameter, k, should be derivable from this information. We propose that a figure of merit, RMS^2 $(k\rightarrow\infty, \alpha_1^{i+1})$, be formed for the first determination, α_1^{i+1}, in the following manner:

$$\mathrm{RMS}^2(k\rightarrow\infty,\ \alpha_1^{i+1}) = \frac{1}{N}\sum_{n=1}^{N}|V_{\exp}(t_n) - V_k(t_n)|^2, \qquad (54)$$

where N points t_n on the time scale are chosen. $V_{\exp}(t)$ is the experimental trace, and $V_k(t)$ is given in Eq. (52), in this case with $k\rightarrow\infty$. Now a different initial guess can be made for α_1 using a finite value of k:

$$\alpha_1 = \frac{4l_1^2}{\pi^2 t_{1/2}}\left[\ln 4 - \ln\left(1 + \frac{\pi^2}{4k^2 l_1^2}\right)\right]. \qquad (55)$$

After convergence to a new $\alpha_1^{i'+1}$, the quantity $\mathrm{RMS}^2(k, \alpha_1^{i'+1})$ can be calculated. When $\mathrm{RMS}^2(k, \alpha_1^{i'+1})$ reaches a minimum as a function of k, that is the best k for the measurement.

IV. SUMMARY AND CONCLUSIONS

Current and future aerospace and construction industry materials require efficient thermal analysis techniques, including the flash technique for thermal diffusivity. This is not possible unless the data can be adequately reduced for polymer composites and porous blankets. This paper has been concerned with radiant-energy penetration into a layered sample subjected to flash-technique conditions. A mathematical modification of the standard flash-technique solution for layered structures was presented which allows for an exponentially decaying penetration of radiant energy into the first of two layers. The following conclusions were drawn:

(1) A new (simple) pole in the Laplace-transformed solution appears on the complex s or β plane. Thus, the inverse Laplace transform, which gives the response function (back-face temperature versus time), contains an additional term. Furthermore, there is a small modification $(1+\beta_n^2/U_2k^2\alpha_1)^{-1}$ to the previous terms.

(2) The appearance of a new pole signifies in general the emergence of a new kind of physical behavior, because the residue of the new pole may in general be significantly larger than those of many of the original poles. However, to facilitate the evaluation of the importance of this effect, a small-penetration approximation was made in which e^{-kl_1} was set equal to zero. This is approximately true for $kl_1 \gtrsim 3$ $(e^{-kl_1} \lesssim 0.05)$. In this approximation, the new term is suppressed and only the $(1+\beta_n^2/U_2k^2\alpha_1)^{-1}$ modification retained. However, it must be kept in mind that for more extensive radiant energy penetration, this approximation would not apply, in which case the new term in the solution could specify a radical departure of the response function from its previous behavior. Furthermore, extensive penetration might also require the exponential function $Q(x)$ to extend over more than one layer.

(3) A numerical example was evaluated for the $e^{-kl_1} = 0$ approximation in the case of identical layer parameters and the first-term $(n = 1)$ approximation to the back-face rise time. It was found that the rise time shortfall $(1/k = 33\%$ of $l_1) \approx 18\%$ of the rise time that would obtain if all the radiant energy were deposited on the surface. Here, $1/k$ is the characteristic penetration depth of the exponential radiant-energy deposition function. Thus, even when the radiant energy deposited at the sample midplane is only 5% of that deposited on the surface, an 18% correction is necessary.

(4) The standard flash-technique data-reduction algorithm considers iterations, $\alpha_1^{(i+1)} = \frac{1}{2}[\alpha_1^{(i)} + \alpha_1^{(i-1)}]$, which are required to converge to a reasonable limit. Only the $t_{1/2}$ experimental data are used. If the entire $V(t)$ response function were used, however, a figure of merit might be defined that allows for the experimental determination of k.

(5) Modifications such as the ones reported here are expected to open up the flash technique to the analysis of advanced polymer composite and insulating blanket material.

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