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Operator density current and relativistic localization problem

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The well-known difficulties with the relativistic localization in quantum theory of one particle are revisited. Among them the noncausal propagation is shown to be connected with the positivity of the time translations generator. A proposal for solving this problem is presented, namely searching for an operator probability density current being the 4-vector quantity. The operator of multiplication by the argument of the Dirac wavefunction is an example of a causal position operator.

INTRODUCTION

The problem of defining position operators and their eigenstates in the framework of relativistic quantum mechanics was generally solved by Newton and Wigner.[1] These authors showed that for elementary systems (i.e., irreducible representations of the Poincaré group) three commuting components of position operator exist together with the continuous spectrum of their eigenvalues.[2]

This Newton—Wigner solution has, however, some disadvantages:

(I) The three components of the position operator are not a part of any four-vector quantity. Therefore, it is difficult to imagine any simple Lorentz covariance.

(II) A state which is localized in the origin (that means that it is an eigenstate of the three position operators to the eigenvalue zero) in one coordinate system is not localized in a moving coordinate system even when the origins coincide at $t=0$.

(III) The localized states have noncausal propagation—this property may be also called instantaneous spreading out of the wavepacket. This means that the scalar product of two localized states (ψ_a, ψ_b), where a and b are points in the Minkowski space, is different from zero also for $(a-b)^2<0$, i.e., for spacelike separations. This implies that the probability of propagation faster than light is nonzero.

A solution of the problem I has been given by Fleming[3,4] who ascertained that a position operator *should* depend on a spacelike hyperplane on which the particle is to be localized. The Lorentz transformation changes the hyperplane of localization, therefore one should expect that it also changes the position operator.

Let us briefly sketch the Fleming hyperplane formalism. Each hyperplane Σ may be uniquely characterized by a four-dimensional vector η orthogonal to it and by its "distance" τ from the origin. These two quantities enter into the equation of the hyperplane,

$$\eta\cdot x=\tau.$$

The quantity τ is a scalar—it is the proper time in a frame of the hyperplane. Since the hyperplane is spacelike, $\eta^2>0$, we choose $\eta^2=1$, $\eta_0>0$. The dependence of the hyperplane on η, τ will be denoted $\Sigma=\Sigma(\eta,\tau)$.

In the Heisenberg picture we have a family of position operators $X_\mu(\eta,\tau)$ fulfilling the operator equation

$$\eta^\mu X_\mu(\eta,\tau)=\tau\cdot 1.$$

This formula says that the four components $X_\mu(\eta,\tau)$ are linearly dependent, so only three of them are independent. In particular, for purely space hyperplanes [i.e., for $\overset{0}{\eta}=(1,\mathbf{0})$] the above formula gives

$$X_0(\overset{0}{\eta},\tau)=\tau\cdot 1,$$

that is, the zeroth component of position is merely the time parameter.

Now the Lorentz covariance has the form

$$U(\Lambda)X_\mu(\eta,\tau)U(\Lambda)^{-1}=(\Lambda^{-1})_\mu{}^\nu X_\nu(\Lambda\eta,\tau).$$

In this way problem I is solved.

Problem II remains unsolved since an eigenstate of $X(\eta,0)$, localized in the origin goes into an eigenstate of $X(\Lambda\eta,0)$ under Lorentz transformation $U(\Lambda)$, but physically one expects that it should remain the eigenstate of $X(\eta,0)$. In Fleming's case one may generally have a family of eigenstates at the same space—time point, indexed by hyperplanes. It may, however, happen that the set of eigenstates of $X(\eta,0)$ localized in one space—time point is not one-dimensional and the Lorentz transformations do not lead out from this set. This situation is already acceptable from the physical point of view and it occurs in our example of Sec. 3.

Our note gives a proposal of seeking for a solution of the third problem. It is based on a useful tool which is an operator probability density distribution $\rho(\mathbf{x},t)$ and an operator probability density current $\mathbf{j}(\mathbf{x},t)$ which together form a four-vector probability density current $j_\mu(x)$. This notion has been used for instance in the paper of Barut and Malin[5] where it was assumed that the expectation values of these operators $(\psi,\rho(\mathbf{x},t)\psi)$ and $(\psi,\mathbf{j}(\mathbf{x},t)\psi)$ give the ordinary probability density distribution and probability density current of quantum mechanics, respectively. Petzold and coworkers[6,7] also investigated four-vector operator density currents in the specific case of scalar particles and showed that for positive energies it cannot have causal propagation.

1. SIGN OF THE TIME TRANSLATION GENERATOR

There are some indications that the positivity of energy causes difficulty III. Let us present a quite elegant argument concerning this question which is due to Jadczyk (private communication).

Suppose we have a localization on spacelike regions Δ of three-dimensional volumes. Suppose also that it is given by projection operators $E(\Delta)$ associated with all such regions. Assume that the localization propagates causally, which means that for two spacelike regions Δ_1 and Δ_2 the projections $E(\Delta_1)$ and $E(\Delta_2)$ are orthogonal, i.e., $E(\Delta_1)E(\Delta_2)=0$. We shall show that this cannot be reconciled with the positivity of the Hamiltonian. For this purpose we need a theorem by Borchers:

Theorem: There exists a one-parameter group $U(t) = \exp(-iHt)$ where the generator $H \geqslant c > -\infty$. Denote $F_t = U(t)FU(t)^{-1}$ for any operator F. If there exists a pair of projectors E, F such that for $|t| < \epsilon$, $EF_t = 0$ then for any $t \in R$, $EF_t = 0$.

The proof is based on analyticity properties of spectral measures and can be found in Ref. 8. We shall use this theorem for different one-parameter groups $U_\eta(\tau)$ of time translations in different frames, namely $U_\eta(\tau) = \exp[-i(P\circ\eta)\tau]$ where P^μ is the four-vector translation generator. The operator $(P\circ\eta)$ is the Hamiltonian in the proper frame of η. According we shall consider expressions $F_{\eta\tau} = U_\eta(\tau)FU_\eta(\tau)^{-1}$.

Now we proceed to the argument of Jadczyk. We start from two parallel regions Δ_1 and Δ_2 of the same size, spacelike separated as shown in Fig. 1. We have $E(\Delta_1)E(\Delta_2)=0$. We also have $E(\Delta_1)E(\Delta_2)_{\eta\tau}=0$ for τ small such that $\Delta_2+\eta\tau$ has no intersection with the causal shadow of Δ_1. Then by virtue of the Borchers theorem we have

$$E(\Delta_1)E(\Delta_2)_{\eta\tau}=0$$

for arbitrary τ. Hence we may write

$$E(\Delta_1)E(\Delta_3)=0,$$

where Δ_3 is as shown in Fig. 1. In the same way we can show that $E(\Delta_1)E(\Delta_2)_{\eta'\tau}=0$ for sufficiently small τ

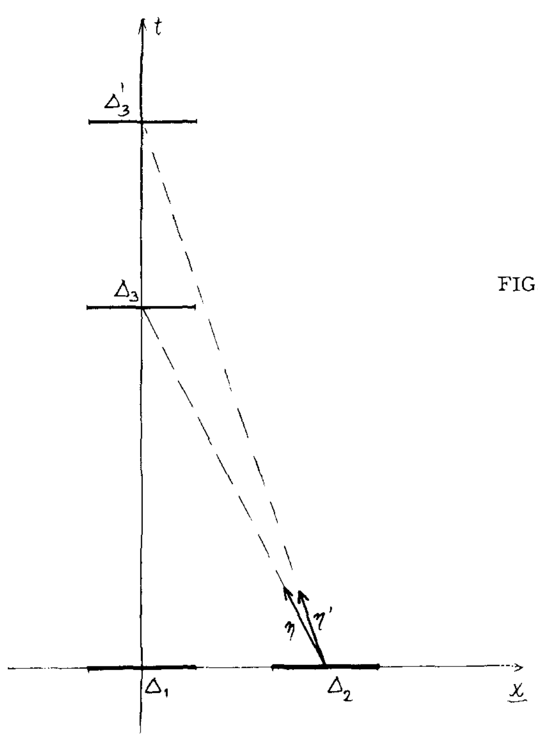

FIG. 1.

and then $E(\Delta_1)E(\Delta_3')=0$. Thus we are allowed to move Δ_3 in the pure time direction without destroying the orthogonality relation,

$$E(\Delta_1)E(\Delta_3)_t=0 \quad \text{for} \quad |t|<\epsilon.$$

Now, using the Borchers theorem we may write down

$$E(\Delta_1)E(\Delta_3)_t=0 \quad \text{for any} \quad t\in R.$$

If we translate Δ_3 back in time far enough to cover it with Δ_1 we obtain in this way

$$E(\Delta_1)^2=0,$$

which implies

$$E(\Delta_1)=0.$$

Thus our family of projectors contains only zero operators. Therefore, we have to agree that such projectors cannot describe any localization. In this way the positivity of time translation generator and the causality of localization cannot be reconciled.

There are also other arguments leading to the same conclusion, see Refs. 9 and 10.

Now we understand why the Newton–Wigner solution of the localization problem has to propagate noncausally: This is so since these authors work with irreducible representations of the Poincaré group which of course have the time translation generator bounded from below. If one wishes to secure the causal propagation one has to give up the positivity of this operator.

In our opinion the relativistic localization problem ought to be stated in the following form:

Find a general form of the Fleming position operators with the following spectral decomposition,

$$X^\mu(\eta,\tau)=\int_{\Sigma(\eta,\tau)} dE_\eta(x)x^\mu, \tag{1}$$

and of the Poincaré group representation under which they are covariant. The spectral measures should satisfy the condition of causal propagation,

$$E_\eta(\Delta)E_{\eta'}(\Delta')=0 \quad \text{for} \quad \Delta \times \Delta'. \tag{2}$$

The most interesting part of the problem is to find the Poincaré group representation and especially its Hamiltonian. Now we are convinced that these representations cannot be irreducible and should contain negative as well as positive Hamiltonian parts. Therefore, it is understandable why Petzold[6] obtained noncausal propagation for positive energy pions.

If one wants to retain the probabilistic interpretation for the states of one sign of energy only, one may use nonspectral measures,

$$\tilde{E}_\eta(\Delta)=PE_\eta(\Delta)P+(1-P)E_\eta(\Delta)(1-P),$$

called positive operator-valued measures[11,12] where P is a projection operator onto the one sign of Hamiltonian vectors, and then use new position operators

$$\tilde{X}^\mu(\eta,\tau)=\int_{\Sigma(\eta,\tau)} d\tilde{E}_\eta(x)x^\mu,$$

which do not lead out from the space of states of one sign of the energy. Jauch and Piron[13] proposed using positive operator valued-measures for a localization of the photons, see also Ref. 14.

However in this language it is difficult to consider the causality of propagation, since the condition $\tilde{E}(\Delta_1)\tilde{E}(\Delta_2)=0$ for $\Delta_1 \times \Delta_2$ makes no sense because even for equal times $\tilde{E}(\Delta_1)\tilde{E}(\Delta_2)\neq 0$ when $\tilde{E}$ is not a spectral measure. Obviously for such nonspectral measures the above argument of Jadczyk cannot be applied since they do not have the idempotence property. Therefore, we must face the following dilemma: what should be given up—positivity of the Hamiltonian or "projectivity" of the localization in finite regions—to ensure the causality of propagation of the localization? In this paper we choose the first possibility.

In this connection a very puzzling question arises: How is it possible that the energy of an elementary system has both positive and negative signs? In our opinion it is worthwhile to discuss the following answer: The time translation generator H is not merely the energy operator but a product of energy $E=(\mathbf{P}^2+m^2)^{1/2}$ and some other quantity Ξ,

$$H=E\cdot\Xi. \tag{3}$$

These two quantities commute, hence they can be simultaneously determined. The nature of quantity Ξ is not yet clear. Anyway

$$\Xi=\frac{P_0}{(\mathbf{P}^2+m^2)^{1/2}}$$

is Lorentz invariant, which can be easily checked by virtue of the Poincaré group Lie algebra. It has an obvious property $\Xi^2=1$ for the one-particle systems, i.e., for $P_0^2-\mathbf{P}^2=m^2$.

In order to maintain the Lorentz covariance of (3) one should also change the interpretation of the linear momentum, namely $\Xi\cdot\mathbf{P}=\mathbf{p}$ should be viewed as momentum operator instead of $\mathbf{P}$ which may be called only space translation generators. From the point of view of the conservation laws both $\mathbf{P}$ and $\mathbf{p}$ are equally physical since both commute with the Hamiltonian.

In the case of the massless Dirac particle, i.e., for the neutrino, Ξ is just the product of helicity and chirality (see Sec. 4). Thus the necessity of using both signs of the Hamiltonian may be expressed in the following way: In order to have causal relativistic covariant localization of the neutrino one has to work with both signs of helicity and/or chirality.

There is a paper by Bertrand[15] in which the hyperplane formalism of Fleming is applied to a special limiting case of null planes appropriate for massless particles and specifically to the photon. Bertrand has been led to use both signs of the Hamiltonian only because of the very existence of the position operators for the photon and without adducing the causal propagation. This is understandable since there are difficulties with the existence of any (causal or not) localization of the photon, see Refs. 13 and 14.

2. OPERATOR PROBABILITY DENSITY CURRENT

In the light of the previous section it is natural that the localization problem needs a new approach for solving it. Our goal is to present a useful tool for this aim. Namely, an operator-valued probability density distribution $\rho(x)$ together with an operator-valued probability density current $\mathbf{j}(x)$ which together form a probability density current 4-vector,

$$j^\mu(x)=(\rho(x),\mathbf{j}(x)).$$

Mathematically it is an operator-valued distribution of class $\mathcal{O}'_M$ in spacelike directions,[16] which means that after smearing out over spacelike hyperplanes with smooth functions growing polynomially[17] it yields an (unbounded) operator in a Hilbert space. Let us denote such a class of distributions $\mathcal{O}'_{M,s}$, where s stands for "spacelike." The expectation values of this operator quantity $(\psi,j^\mu(x)\psi)$ are the ordinary probability currents of quantum mechanics, see Ref. 5. We may summarize our proposal in the following assumption:

There exists an operator-valued 4-vector distribution $j^\mu(\cdot)\in\mathcal{O}'_{M,s}$ such that the Radon—Nikodym derivative of the spectral measure $E_\eta(\Delta)$ occurring in (1) with respect to the three-dimensional volume V in the proper frame of η can be expressed by the formula

$$\frac{dE_\eta(x)}{dV}=\eta\cdot j(x).$$

The current is assumed to be conserved,

$$\partial^\mu j_\mu(x)=0.$$

This property assures that the integral

$$\int_{\Sigma(\eta,\tau)} j^\mu(x)\,d\sigma_\mu(\eta,x),$$

where $d\sigma(\eta,x)=\eta_\mu\,dV(x)$, does not depend on η nor τ.[18] From the physical interpretation we demand

$$\int_{\Sigma(\eta,\tau)} j^\mu(x)\,d\sigma_\mu(\eta,x)=1 \tag{4}$$

as an operator identity. We assume also that the current is Poincaré covariant,

$$U(\Lambda,a)j^\mu(x)U(\Lambda,a)^{-1}=(\Lambda^{-1})^\mu{}_\nu j^\nu(\Lambda x+a).$$

Having this we define the position operator[19]

$$X^\mu(\eta,\tau)=\int_{\Sigma(\eta,\tau)} y^\mu j^\nu(y)\,d\sigma_\nu(\eta,y). \tag{5}$$

Let us check the Fleming condition $X_\mu(\eta,\tau)\eta^\mu=\tau\cdot 1$,

$$\eta^\mu X_\mu(\eta,\tau)=\int_{\Sigma(\eta,\tau)}(\eta\cdot y)j^\nu(y)\,d\sigma_\nu(\eta,y).$$

Since $\eta\cdot y=\tau$ for $y\in\Sigma(\eta,\tau)$ we have

$$\eta^\mu X_\mu(\eta,\tau)=\tau\int_{\Sigma(\eta,\tau)} j^\nu(y)\,d\sigma_\nu(\eta,y)=\tau 1$$

by virtue of (4). Let us also examine the Lorentz transformation law of $X(\eta,\tau)$. Toward this purpose we need the transformation properties of the integration element $d\sigma_\mu$,

$$\Lambda_\mu{}^\nu\,d\sigma_\nu(\eta,x)=d\sigma_\mu(\Lambda\eta,\Lambda x). \tag{6}$$

Checking, we have:

$$\begin{aligned}
&U(\Lambda)X_\mu(\eta,\tau)U(\Lambda)^{-1}\\
&\quad=\int_{\Sigma(\eta,\tau)} y_\mu U(\Lambda)j^\nu(y)U(\Lambda)^{-1}\,d\sigma_\nu(\eta,y)\\
&\quad=\int_{\Sigma(\eta,\tau)} y_\mu(\Lambda^{-1})^\nu{}_\lambda j^\lambda(\Lambda y)\,d\sigma_\nu(\eta,y)\\
&\quad=\int_{\Sigma(\eta,\tau)} y_\mu j^\lambda(\Lambda y)\Lambda_\lambda{}^\nu\,d\sigma_\nu(\eta,y)\\
&\quad=\int_{\Sigma(\eta,\tau)} y_\mu j^\lambda(\Lambda y)\,d\sigma_\lambda(\Lambda\eta,\Lambda y)
\end{aligned}$$

changing the variables $y' = \Lambda y$

$$= \int_{\Sigma(\Lambda\eta,\tau)} (\Lambda^{-1} y')_\mu j^\lambda(y')\, d\sigma_\lambda(\Lambda\eta, y')$$

$$= (\Lambda^{-1})_\mu{}^\rho X_\rho(\Lambda\eta, \tau). \qquad \text{QED}$$

The probability density current has to have one more property in order to have expression (5) as a spectral decomposition, namely the idempotence property. A spectral measure is given by the formula

$$E_\eta(\Delta) = \int_\Delta j^\nu(y)\, d\sigma_\nu(\eta, y) \tag{7}$$

for $\Delta \subset \Sigma(\eta, \tau)$. In order to have the idempotence $E_\eta(\Delta)^2 = E_\eta(\Delta)$ it is sufficient to postulate

$$\eta_\mu j^\mu(x) j^\nu(y) = \delta^3_\eta(x-y) j^\nu(y) \qquad \text{for} \quad (x-y)^2 < 0 \quad \text{or} \quad x = y, \tag{8}$$

where η is a unit vector orthogonal to $(x-y)$ and the distribution δ^3_η, introduced by Fleming in Ref. 20, fulfils

$$\delta(\eta x)\delta^3_\eta(x) = \delta^4(x).$$

This distribution can be also written as

$$\delta^3_\eta(x) = (2\pi)^{-3} \int d^4k\, \delta(\eta k) \exp(ikx)$$

or

$$\delta^3_\eta(x) = \delta^3(\mathbf{x}'), \tag{9}$$

where x' are the coordinates of x in the proper frame of η. The δ^3_η has its support on the axis $\{x = \lambda\eta,\ \lambda \in \mathbf{R}\}$ and has the property

$$\int_{\Sigma(\eta,\tau)} d\sigma_\mu(\eta, x)\delta^3_\eta(x-y) f(x) = f(y)\eta_\mu \quad \text{for} \quad y \in \Sigma(\eta, \tau). \tag{10}$$

The proof of sufficiency of (8) for the idempotence may be found in Ref. 21.

If one wants to work with one sign of Hamiltonian position operators $\widetilde{X}(\eta, \tau)$ one may assume a weaker condition

$$\eta_\mu \tilde{j}^\mu(x) > 0 \tag{11}$$

for each timelike $\eta, \eta_0 > 0$. This means that in the proper frame of η the zeroth component of this current

$$\tilde{\rho}'(x) = \tilde{j}^{0\prime}(x) = \eta^\mu \tilde{j}_\mu(x)$$

(which is just the probability density in that frame) has the ordinary positive probabilistic interpretation. It ensures that the current $\tilde{j}$ put in (7) gives the positive operator-valued measure. The inequality (11) also says that $\tilde{j}(x)$ is a timelike vector for each x. Of course a current satisfying (8) also satisfies (11).

In this way we are sure that the postulating of the existence of a covariant conserved 4-vector probability current yields the Fleming position operator by formula (5).

Now we complete our list of postulates by assuming the property

$$j_\mu(x) j_\mu(y) = 0 \quad \text{for} \quad (x-y)^2 < 0, \tag{12}$$

which quarantees that the projectors (7) satisfy condition (2) of causal propagation. It might seem that (12) follows from (8) but this is not the case since we have in (8) a timelike vector η which can not be arbitrary and should be absent in (12).

Thus we formulate the problem of relativistic localization translated into the language of the probability current as follows:

Find a general (not necessarily irreducible) representation of the Poincaré group and and operator valued distribution $j(\cdot) \in \mathcal{O}'_{M,s}$ such that

(i) $\partial_\mu j^\mu(x) = 0$;

(ii) $\int_{\Sigma(\eta,\tau)} j^\mu(x)\, d\sigma_\mu(\eta, x) = 1$;

(iii) $U(\Lambda, a) j^\mu(x) U(\Lambda, a)^{-1} = (\Lambda^{-1})^\mu{}_\nu j^\nu(\Lambda x + a)$;

(iv) $\eta_\mu j^\mu(x) j^\nu(y) = \delta^3_\eta(x-y) j^\nu(y)$ for $(x-y)^2 < 0$ or $x = y$;

(v) $j^\mu(x) j^\nu(y) = 0$ for $(x-y)^2 < 0$.

Postulates (i), (ii), and (iii) serve to fit the Fleming's solution of the problem I mentioned in the Introduction, postulate (iv) ensures that expression (7) gives a spectral measure, and postulate (v) solves problem III. Up to now we did not touch upon the solution of problem II since we do not see how to express it in purely algebraic language.

3. DIRAC PARTICLE—AN EXAMPLE SATISFYING THE POSTULATES

Now we proceed in presenting an example which fulfills postulates (i)—(v), namely the Dirac particle. The free relativistic spin-$\frac{1}{2}$ particle is described by a Hilbert space $\mathcal{H}$ of four-component functions Ψ satisfying the Dirac equation:

$$i\partial_0 \Psi + i\alpha^k \partial_k \Psi - \beta m \Psi = 0, \tag{13}$$

where α^1, α^2, α^3, and β are 4×4 Hermitian matrices which anticommute with each other and have their squares equal to one. It is useful for our purposes to introduce the following representation for them

$$\alpha^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}, \quad k \in \{1, 2, 3\}, \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where σ^k are 2×2 Pauli matrices, and I is a 2×2 unit matrix. The α matrices may be completed with the unit matrix $\mathbb{1}$ to form the 4-vectors

$$\alpha^\mu = (\mathbb{1}, \alpha), \quad \tilde{\alpha}^\mu = (\mathbb{1}, -\alpha).$$

With the help of 2×2 matrix 4-vectors $\sigma^\mu = (I, \sigma)$ and $\tilde{\sigma}^\mu = (I, -\boldsymbol{\sigma})$, we may write

$$\alpha^\mu = \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \tilde{\sigma}^\mu \end{pmatrix}, \quad \tilde{\alpha}^\mu = \begin{pmatrix} \tilde{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}.$$

These matrices have the property

$$\tilde{\alpha}^\mu \alpha^\nu + \tilde{\alpha}^\nu \alpha^\mu = 2g^{\mu\nu}. \tag{14}$$

Using the matrix 4-vector α^μ we may rewrite Eq. (13), (14)

$$i(\alpha \cdot \partial)\Psi - \beta m \Psi = 0, \tag{15}$$

where $(\alpha \cdot \partial) = \alpha^\mu \partial_\mu$.

The scalar product in $\mathcal{H}$ is given by

$$(\Phi, \Psi) = \int_{x^0 = \text{const}} \Phi^+(x) \Psi(x)\, d^3x. \tag{16}$$

It is positive definite and does not depend on x^0.

A matrix distribution $G(x)$ satisfying (15) and such that

$$G(\mathbf{x},0)=\delta^3(\mathbf{x})\mathbf{I} \tag{17}$$

will be called a propagator for (15). It can be found[22] to have the form

$$G(x)=-[(\tilde{\alpha}\cdot\partial)-im\beta]\Delta(x,m^2), \tag{18}$$

where Δ is the well-known Pauli—Jordan invariant function vanishing outside the light cone. Hence the distribution (18) also has its support in the closed light cone. It has, moreover, the properties:

$$G(x)^+=G(-x), \tag{19}$$

$$i\partial_\mu G\alpha^\mu - mG\beta=0. \tag{20}$$

If $A\in \mathrm{SL}(2,C)$ then $A\to\Lambda(A)$ is the well-known homomorphism of $\mathrm{SL}(2,C)$ onto $L_+^{\dagger}$—the proper orthochronous Lorentz group, given by the formulas

$$A^+\sigma^\mu A=\Lambda^\mu{}_\nu\sigma^\nu \quad\text{or}\quad \tilde{A}^+\tilde{\sigma}^\mu\tilde{A}=\Lambda^\mu{}_\nu\tilde{\sigma}^\nu, \tag{21}$$

where $\tilde{A}=(A^+)^{-1}$.[23] We introduce the four-dimensional representations of $\mathrm{SL}(2,C)$ by the formula

$$B(A)=\begin{pmatrix}A & 0\\ 0 & \tilde{A}\end{pmatrix},\quad \tilde{B}(A)=\begin{pmatrix}\tilde{A} & 0\\ 0 & A\end{pmatrix}.$$

Then we have, by virtue of (21),

$$B^+(A)\alpha^\mu B(A)=\Lambda^\mu{}_\nu\alpha^\nu \quad\text{and}\quad \tilde{B}^+(A)\tilde{\alpha}^\mu\tilde{B}(A)=\Lambda^\mu{}_\nu\tilde{\alpha}^\nu. \tag{22}$$

It follows from (22) that

$$B(A)\tilde{\alpha}^\mu B(A)^+=(\Lambda^{-1})^\mu{}_\nu\tilde{\alpha}^\nu$$

and

$$\tilde{B}(A)\alpha^\mu\tilde{B}(A)^+=(\Lambda^{-1})^\mu{}_\nu\alpha^\nu. \tag{23}$$

Moreover it is easy to verify that

$$B(A)^+\beta B(A)=\beta \quad\text{for each } A\in \mathrm{SL}(2,C). \tag{24}$$

Using the above formulas it is straightforward to see that G has the following covariance property:

$$B(A)G(x)B^+(A)=G(\Lambda x). \tag{25}$$

The term "propagator" has been used for G because it yields the relation

$$\Psi(x)=\int_{y^0=\mathrm{const}} d^3y\,G(x-y)\Psi(y) \tag{26}$$

for any solution of Eq. (15) and for arbitrary y^0.

In the Hilbert space $\mathcal{H}$ the following unitary representation of $\mathrm{SL}(2,C)$ is given, $\Psi\to U(A)\Psi$, where

$$(U(A)\Psi)(x)=B(A)\Psi(\Lambda^{-1}x).$$

Using the unitarity of these operators and the formulas (22), one may show that the scalar product (16) can be written

$$(\Psi,\Phi)=\int_{\Sigma(\eta,\tau)}\Psi^+(x)\alpha^\mu\Phi(x)\,d\sigma_\mu(\eta,x). \tag{27}$$

Similarly relation (26) after employing (25) may be generalized to the arbitrary spacelike hyperplane,

$$\Psi(x)=\int_{\Sigma(\eta,\tau)} d\sigma_\mu(\eta,y)G(x-y)\alpha^\mu\Psi(y). \tag{28}$$

In particular, since G is also the solution of (15), we have

$$G(x-z)=\int_{\Sigma(\eta,\tau)} d\sigma_n(\eta,y)G(x-y)\alpha^\mu G(y-z).$$

The boundary condition (17) may be also generalized onto a spacelike hyperplane passing through the origin. To this purpose one may use a Lorentz transformation Λ such that $\Lambda\eta=\overset{0}{\eta}$. The explicit form of it is

$$\Lambda_0{}^0=\eta_0,\quad \Lambda_0{}^k=\eta^k,\quad \Lambda_j{}^0=-\eta_j,$$

$$\Lambda_j{}^k=\frac{\eta_j\eta_k}{\eta_0+1}+\delta_j{}^k. \tag{29}$$

Denote $\Lambda x=x'$, of course $x'^0=0$ when $x\in\Sigma(\eta,0)$. Due to (25) we have

$$G(x)=G(\Lambda^{-1}x')=B(A^{-1})G(x')B(A^{-1})^+$$

for A corresponding to Λ given by (29). We take $x\in\Sigma(\eta,0)$, then $x'^0=0$, and by virtue of (17) we have

$$G(x)=B(A^{-1})\delta^3(\mathbf{x}')B(A^{-1})^+$$
$$=\delta^3(\mathbf{x}')B(A^{-1})\tilde{\alpha}^0B(A^{-1})^+.$$

We use (23) and (9),

$$G(x)=\delta^3_\eta(x)\Lambda^0{}_\nu\tilde{\alpha}^\nu.$$

Therefore, by virtue of (29) we have

$$G(x)=\delta^3_\eta(x)(\eta\cdot\tilde{\alpha}) \quad\text{for}\quad x\in\Sigma(\eta,0) \tag{30}$$

and this is the needed generalization of (17).

Now we are prepared to introduce our operator-valued density current. For any Ψ of class $\mathcal{S}$ as a function of $\mathbf{x}$ (the set D of such elements from $\mathcal{H}$ constitutes the domain of definition of j),

$$(j^\mu(x)\Psi)(y)=G(y-x)\alpha^\mu\Psi(x). \tag{31}$$

First of all we check the hermicity of this operator. We have

$$(\Psi,j^\mu(x)\Phi)=\int_{y^0=\mathrm{const}}\Psi^+(y)(j^\mu(x)\Phi)(y)\,d^3y$$
$$=\int\Psi^+(y)G(y-x)\alpha^\mu\Phi(x)\,d^3y$$
$$=\left[\int G(y-x)^+\Psi(y)\,d^3y\right]^+\alpha^\mu\Phi(x)$$

which using (19),

$$=\left[\int G(x-y)\Psi(y)\,d^3y\right]^+\alpha^\mu\Phi(x)$$

and using (26),

$$=\Psi^+(x)\alpha^\mu\Phi(x).$$

We also have

$$(j^\mu(x)\Psi,\Phi)=\int_{y^0=\mathrm{const}}(j^\mu(x)\Psi)^+(y)\Phi(y)\,d^3y$$
$$=\int[G(y-x)\alpha^\mu\Psi(x)]^+\Phi(y)\,d^3y$$
$$=\int\Psi^+(x)\alpha^\mu G(y-x)^+\Phi(y)\,d^3y$$
$$=\Psi^+(x)\alpha^\mu\int G(x-y)\Phi(y)\,d^3y$$
$$=\Psi(x)^+\alpha^\mu\Phi(x).$$

Hence we have

$$(\Psi,j^\mu(x)\Phi)=(j^\mu(x)\Psi,\Phi)=\Psi(x)^+\alpha^\mu\Phi(x)$$

and the hermicity is checked. We also see in this formula that the expectation value of the operator $j^\mu(x)$ is indeed an ordinary c-number probability current of quantum mechanics.

The wavefunctions belonging to the Hilbert space $\mathcal{H}$ are functions of four space—time coordinates. These functions are determined uniquely by their values on an arbitrary spacelike hyperplane[24] since they satisfy the wave equation (15) and the scalar product (27) is independent on the hyperplane. Therefore, when considering an (improper) element $j^\mu(x)\Psi$ of $\mathcal{H}$ we may confine ourselves for the function $(j^\mu(x)\Psi)(y)$, only to y belonging to some specific hyperplane $\Sigma(\eta,\tau)$. By virtue of (30), definition (31) then gives

$$(j^\mu(x)\Psi)(y)=\delta^3_\eta(y-x)(\eta\cdot\tilde{\alpha})\alpha^\mu\Psi(x) \quad \text{for} \quad x,y\in\Sigma(\eta,\tau). \tag{32}$$

Now we integrate this with a test function f over the hyperplane $\Sigma(\eta,\tau)$,

$$I=[\int_{\Sigma(\eta,\tau)} j^\mu(x)f(x)\,d\sigma_\mu(\eta,x)\Psi](y)$$
$$=\int_{\Sigma(\eta,\tau)}\delta^3_\eta(y-x)(\eta\cdot\tilde{\alpha})\alpha^\mu\,d\sigma_\mu(\eta,x)f(x)\Psi(x).$$

If we change the variables $x'=\Lambda x$ for Λ given in (29) and use (9) we obtain

$$I=\int_{x'^0=y'^0}\delta^3(\mathbf{y}'-\mathbf{x}')(\eta\cdot\tilde{\alpha})(\eta\cdot\alpha)f(\Lambda^{-1}x')\Psi(\Lambda^{-1}x')\,d^3x'.$$

If we employ the identity

$$(\eta\cdot\tilde{\alpha})(\eta\cdot\alpha)=(\eta\cdot\eta)=1 \tag{33}$$

following from (14), we obtain

$$I=\int_{x'^0=y'^0}\delta^3(\mathbf{y}'-\mathbf{x}')f(\Lambda^{-1}x')\Psi(\Lambda^{-1}x')\,d^3x'$$
$$=f(\Lambda^{-1}y')\Psi(\Lambda^{-1}y')$$
$$=f(y)\Psi(y).$$

Thus we have proven the identity

$$[\int_{\Sigma(\eta,\tau)}j^\mu(x)f(x)\,d\sigma_\mu(\eta,x)\Psi](y)=f(y)\Psi(y) \quad \text{for } y\in\Sigma(\eta,\tau). \tag{34}$$

We see from it that the smeared current operators $j(f)$ leave our domain D invariant when f is of class $\mathcal{O}_{M,s}$. In this way we have shown that $j^\mu(x)$ is the operator-valued distribution of class $\mathcal{O}'_{M,s}$ on the domain D of vectors which have Schwartz type $\mathcal{S}$ behavior on spacelike hyperplanes.

Now we verify the postulates of Sec. 2.

(i) The current conservation:

$$\left[\frac{\partial}{\partial x^\mu}j^\mu(x)\Psi\right](y)$$
$$=\frac{\partial}{\partial x^\mu}\left[G(y-x)\alpha^\mu\Psi(x)\right]$$
$$=-(\partial_\mu G)(y-x)\alpha^\mu\Psi(x)+G(y-x)\alpha^\mu\partial_\mu\Psi(x)$$
$$=[-\partial_\mu G(y-x)\alpha^\mu-imG(y-x)\beta]\Psi(x)$$
$$+G(y-x)(\alpha^\mu\partial_\mu+im\beta)\Psi(x).$$

Both terms vanish by virtue of (20) and (15), therefore we have

$$\partial_\mu j^\mu=0.$$

(ii) Normalization: It is sufficient to use (34) for $f(x)\equiv 1$,

$$\int_{\Sigma(\eta,\tau)}d\sigma_\mu(\eta,x)j^\mu(x)\Psi=\Psi.$$

Hence

$$\int_{\Sigma(\eta,\tau)}d\sigma_\mu(\eta,x)j^\mu(x)=1.$$

(iii) Lorentz covariance:

$$[U(A)j^\mu(x)U(A)^{-1}\Psi](y)$$
$$=B(A)[j^\mu(x)U(A)^{-1}\Psi](\Lambda^{-1}y)$$
$$=B(A)G(\Lambda^{-1}y-x)\alpha^\mu[U(A)^{-1}\Psi](x)$$
$$=B(A)G(\Lambda^{-1}(y-\Lambda x))B(A)^+B(A^{-1})^+\alpha^\mu B(A^{-1})\Psi(\Lambda x)$$

using (22) and (25),

$$=G(y-\Lambda x)(\Lambda^{-1})^\mu{}_\nu\alpha^\nu\Psi(\Lambda x)$$
$$=(\Lambda^{-1})^\mu{}_\nu[j^\nu(\Lambda x)\Psi](y).$$

Hence

$$U(A)j^\mu(x)U(A)^{-1}=(\Lambda^{-1})^\mu{}_\nu j^\nu(\Lambda x).$$

The translation covariance is easy to check.

(iv) The projection property: Let x,y be such that $(x-y)^2<0$. Then there exists a unit positive timelike vector η such that $\eta\cdot(x-y)=0$ and a hyperplane $\Sigma(\eta,\tau)$ such that $x,y\in\Sigma(\eta,\tau)$. Let us assume that $z\in\Sigma(\eta,\tau)$ and consider the following expression

$$K=[j^\mu(x)j^\nu(y)\Psi](z)=G(z-x)\alpha^\mu G(x-y)\alpha^\nu\Psi(y). \tag{35}$$

We have $z-x\in\Sigma(\eta,0)$ and $x-y\in\Sigma(\eta,0)$, therefore we can use (30),

$$K=\delta^3_\eta(z-x)(\eta\cdot\tilde{\alpha})\alpha^\mu\delta^3_\eta(x-y)(\eta\cdot\tilde{\alpha})\alpha^\nu\Psi(y)$$
$$=\delta^3_\eta(x-y)(\eta\cdot\tilde{\alpha})\alpha^\mu\delta^3_\eta(z-y)(\eta\cdot\tilde{\alpha})\alpha^\nu\Psi(y)$$
$$=\delta^3_\eta(x-y)(\eta\cdot\tilde{\alpha})\alpha^\mu G(z-y)\alpha^\nu\Psi(y)$$
$$=\delta^3_\eta(x-y)(\eta\cdot\tilde{\alpha})\alpha^\mu[j^\nu(y)\Psi](z).$$

If we multiply both sides by η_μ and use (33) we get

$$[\eta_\mu j^\mu(x)j^\nu(y)\Psi](z)=\delta^3_\eta(x-y)[j^\nu(y)\Psi](z)$$

and since Ψ is arbitrary from D,

$$\eta_\mu j^\mu(x)j^\nu(y)=\delta^3_\eta(x-y)j^\nu(y).$$

(v) Causality: We know from (18) that $G(x-y)=0$ for $(x-y)^2<0$. Therefore, the equality (35) gives

$$[j^\mu(x)j^\nu(y)\Psi](z)=0$$

for arbitrary $\Psi\in D$, hence

$$j^\mu(x)j^\nu(y)=0 \quad \text{for} \quad (x-y)^2<0. \qquad \text{QED}$$

Now we find the position operator corresponding to the current (31). We calculate the action of expression (5) on a function $\Psi\in D$,

$$[X^\mu(\eta,\tau)\Psi](z)=[\int_{\Sigma(\eta,\tau)}y^\mu j^\nu(y)\,d\sigma_\nu(\eta,y)\Psi](z).$$

By virtue of (34) we have

$$[X^\mu(\eta,\tau)\Psi](z)=z^\mu\Psi(z) \quad \text{for} \quad z\in\Sigma(\eta,\tau). \tag{36}$$

Thus $X^\mu(\eta,\tau)$ is the operator of multiplication by the argument of the wavefunction when this function is taken on the hyperplane $\Sigma(\eta,\tau)$. Therefore, for α^k we have the usual interpretation as the velocity operators together with the well-known difficulties of the Zitterbewegung.

This velocity seems to be disconnected with the translation generators, since the equality

$$\mathbf{P}=C\boldsymbol{\alpha},$$

where C might be some operator coefficient, is not satisfied. Instead of this we have a connection between velocity and spin,

$$\mathbf{S} = \Omega\alpha, \tag{37}$$

where in the representation chosen for this paper

$$\Omega = \frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

It is easy to check that $\Omega = \frac{1}{2}\gamma_5 = \frac{1}{2}i\alpha^1\alpha^2\alpha^3$. In this way the velocity becomes an internal variable of the particle since it does not affect the space—time variables x of its wavefunction. Note that the quantity Ω commutes with spin and velocity. We propose to call it *velocity helicity* in distinction to the ordinary helicity which relates spin with the momentum. The quantity γ_5 is commonly called chirality.

Let us now find the (generalized, since from the continuous spectrum) eigenvectors of the operator $X^\mu(\eta, \tau)$;

$$X^\mu(\eta, \tau)\Psi_{a,\xi} = a^\mu\Psi_{a,\xi}.$$

We use two indices, the first refers to the position eigenvalue, the second refers to some other quantities (for example spin and velocity helicity). After using (36) the above equation takes the form

$$z^\mu\Psi_{a,\xi}(z) = a^\mu\Psi_{a,\xi}(z) \quad \text{for} \quad z \in \Sigma(\eta, \tau).$$

A solution of this equation can only have the form

$$\Psi_{a,\xi}(z) = \delta^3_\eta(z - a)b'_\xi \quad \text{for} \quad z \in \Sigma(\eta, \tau),$$

where b'_ξ is an arbitrary column. In order to calculate the function $\Psi_{a,\xi}$ in other space—time points we use the formula (28),

$$\Psi_{a,\xi}(y) = \int_{\Sigma(\eta,\tau)} d\sigma_\mu(\eta, z)G(y - z)\alpha^\mu\delta^3_\eta(z - a)b'_\xi.$$

Due to (10),

$$\Psi_{a,\xi}(y) = G(y - a)(\eta\cdot\alpha)b'_\xi.$$

Since b'_ξ is arbitrary and $(\eta\cdot\alpha)$ is a reversible matrix we may argue that $b_\xi = (\eta\cdot\alpha)b'_\xi$ is arbitrary and write

$$\Psi_{a,\xi}(y) = G(y - a)b_\xi. \tag{38}$$

We see that the set of all vectors localized at one space—time point a is a four-dimensional manifold[25]—this is natural since the Dirac particle has internal degrees of freedom. The different states localized at the same point differ for instance by velocities.

Now we look at the Lorentz transform of the vector (37),

$$\begin{aligned}[U(A)\Psi_{a,\xi}](y) &= B(A)\,\Psi_{a,\xi}(\Lambda^{-1}y) \\ &= B(A)G(\Lambda^{-1}y - a)b_\xi \\ &= B(A)G(\Lambda^{-1}y - a)\,B(A)^+B(A^{-1})^+b_\xi\end{aligned}$$

using (25),

$$= G(y - \Lambda a)\tilde{B}(A)b_\xi.$$

If we denote $b''_\xi = \tilde{B}(A)b_\xi$ we obtain

$$[U(A)\Psi_{a,\xi}](y) = G(y - \Lambda a)b''_\xi,$$

i.e., again a vector of the form (38). We see from this that the eigenvector of $X_\mu(\eta, 0)$ localized at the origin (that is $a = 0$) after a Lorentz transformation is again an eigenvector of $X_\mu(\eta, 0)$ localized at the same point. It is the velocity (and also the spin) which is the transformed quantity in this case. Therefore, the Dirac particle is an example in which disadvantage II of the Introduction is absent.

4. CONCLUDING REMARKS

The notion of the operator-valued probability density current may be very useful in finding admissible relativistic Hamiltonians. Borowiec and Jadczyk[26] have shown, for instance, that for the current $\mathbf{j}$ and density ρ linked by the relation

$$\mathbf{j} = \tfrac{1}{2}(\mathbf{v}\rho + \rho\mathbf{v}),$$

where $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity, the Hamiltonian ensuring the covariance (iii) under time translations should be linear in the momentum operators. Therefore, for example, the scalar Klein—Gordon particle in the usual formulation is not such a solution since for it $H = (\pm\,\mathbf{P}^2 + m^2)^{1/2}$ and this function evidently is not linear in the P operators. The Duffin—Kemmer equation seems to be more appropriate for our purposes.

There is a hope to localize also the photons on the spacelike hyperplanes (not on null planes as in Ref. 15) since Dirac-like equations have been proposed for the photons.[27]

Barut and Malin maintained[5] that the operator probability density ρ for the scalar particle is not the zeroth component of any 4-vector quantity. We have seen in Sec. 3 that for the Dirac particle the density ρ and the current $\mathbf{j}$ form a covariant 4-vector.

From the example elaborated in Sec. 3 we also learned another lesson: The linear manifold of eigenvectors of $X(\eta, \tau)$ for fixed η and τ, localized in one space—time point should be Lorentz invariant in order to not have disadvantage II of the Introduction. On the other hand, this manifold cannot be one-dimensional since otherwise this would imply that it consists of eigenvectors of Lorentz transformations and thus the Lorentz transformations would commute with $X(\eta, \tau)$. Thus the particle should have internal degrees of freedom. In our example the velocity yields such internal degrees of freedom.

For the massless Dirac particle the Hamiltonian has the form $H = \boldsymbol{\alpha}\cdot\mathbf{P}$. Using the relation $\boldsymbol{\alpha} = 2\gamma_5\mathbf{S}$ following from (37), we get $H = 2\gamma_5\mathbf{S}\cdot\mathbf{P}$. Introducing the helicity $h = 2\mathbf{S}\cdot\mathbf{P}/|\mathbf{P}| = 2\mathbf{S}\cdot\mathbf{P}/E$, we obtain $H = \gamma_5 hE$. Hence the quantity introduced in (3) is $\Xi = \gamma_5 h$. Note that the three quantities appearing in this relation commute with each other.

There is a paper by Durand[28] in which a quantized field version of position operators is introduced for the spin-$\frac{1}{2}$ field. This "Dirac position operator" is expressible as a "weighted average of x over the charge density." The charge density obviously is not positive definite and therefore when restricted to the one-particle subspace it cannot be identified with our density $\rho = j^0$. The density of Durand's paper when integrated with $\mathbf{x}$ gives dipole moment rather than "position of the center of charge." Durand discusses a commutativity of the density at spacelike separated points and calls it a condition for a causal theory, but does not intend to show that the product itself vanishes at such points. Therefore she does not touch the question of the causal propagation in our sense.

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[1]T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).

[2]There are exceptions to this solution, namely that particles with zero mass and spin greater than $\frac{1}{2}$ for which such operators cannot exist.

[3]G. G. Fleming, Phys. Rev. B **137**, 188 (1965).

[4]G. N. Fleming, Phys. Rev. B **139**, 963 (1965).

[5]A. O. Barut and S. Malin, Rev. Mod. Phys. **40**, 632 (1968).

[6]J. Petzold, Ann. Phys. **31**, 361, 372 (1974).

[7]B. Gerlach, D. Gromes, J. Petzold, and P. Rosental, Z. Phys. **208**, 381 (1968).

[8]H. J. Borchers, Commun. Math. Phys. **4**, 315 (1967).

[9]G. C. Hegerfeldt, Phys. Rev. D **10**, 3320 (1974).

[10]B. S. Skagerstam, Int. J. Theor. Phys. **15**, 213 (1976).

[11]S. K. Berberian, *Notes on Spectral Theory* (Van Nostrand, Princeton, New Jersey, 1966).

[12]M. A. Naimark, Dokl. Acad. Sci. USSR **41**, 359 (1961).

[13]J. M. Jauch and C. Piron, Helv. Phys. Acta **40**, 559 (1967).

[14]A. Z. Jadczyk and B. Jancewicz, Bull. Acad. Pol., Sér. Math. Astr. Phys. **21**, 477 (1973).).

[15]J. Bertrand, Nuovo Cimento A **15**, 299 (1973).

[16]Similar c-number valued distributions of class $\mathcal{S}'$ on spacelike hypersurfaces were recently discussed by H. Wakita, Rep. Math. Phys. **10**, 311 (1976).

[17]L. Schwartz, *Théorie des distributions* (Hermann, Paris, 1966).

[18]In proving this fact one also needs the property that $j^\mu(x)$ decreases sufficiently quickly at spacelike infinity. This property is already formalized in the prior assumption $j(\cdot) \in \mathcal{O}'_{M,s}$.

[19]In fact we are able to define more general position operators $X(\Sigma)$ where Σ is not a hyperplane but an arbitrary spacelike hypersurface,

$$X^\mu(\Sigma) = \int_\Sigma y^\mu j^\nu(y) d\sigma_\nu(\eta(y), y)$$

where $\eta(y)$ is a unit vector orthogonal to Σ at $y \in \Sigma$; see Ref. 16.

[20]G. N. Fleming, J. Math. Phys. **9**, 193 (1968).

[21]W. Cegła and B. Jancewicz, Rep. Math. Phys. **11**, 53 (1977).

[22]S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, New York, 1961), Chap. 8.

[23]I. Białynicki-Birula and Z. Białynicka-Birula, *Quantum Electrodynamics* (Pergamon and PWN, Warsaw, 1975), Appendix I.

[24]This property of relativistic wave equations has been thoroughly discussed by Dirac in P. A. M. Dirac, Phys. Rev. **73**, 1092 (1948). That paper of Dirac can be viewed as a harbinger of the Fleming hyperplane formalism.

[25]This manifold, of course, is not a subspace of $\mathcal{H}$ because it consists of generalized eigenvectors, i.e., vectors belonging to a larger space dual to the domain D.

[26]A. Borowiec and A. Z. Jadczyk, "Relativistic systems with a conserved probability current," in preparation.

[27]R. Mignani, E. Recami, and M. Baldo, Lett. Nuovo Cimento **11**, 568 (1974).

[28]B. Durand, Phys. Rev. D **14**, 299 (1976).