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A theorem on A-proper mappings and its application in scattering theory

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We propose a projectionally complete scheme yielding an approximate solution of the functional equation $Bf = g$ in a Hilbert space. We prove that B is an A-proper mapping. The result is applied to an integral equation with a kernel appearing in multichannel scattering theory.

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I. INTRODUCTION

Multichannel scattering integral equations of the structure

$$f = g + Af, \tag{1.1}$$

defined in a Hilbert space $\mathscr{H}$, with $g \in \mathscr{H}$ and A being a linear mapping from $\mathscr{H}$ into $\mathscr{H}$, have been solved successfully using the moments method[1] and the Padé method.[2,3] The moments method solves the projector equation

$$P^{(n)} f^{(n)} = P^{(n)} g + P^{(n)} A P^{(n)} f^{(n)}, \tag{1.2}$$

where $P^{(n)}$ is the orthogonal projection from $\mathscr{H}$ on the subspace

$$T_{A,g}^{(n)} = \text{linear span of } \{A^k g \mid k = 0,1,...,n\}. \tag{1.3}$$

The solution $f^{(n)}$ converges strongly to f (Ref. 4) if A is compact, $1 \in \rho(A)$, and $f \in T_{A,g}$, where

$$T_{A,g} = \overline{\bigcup_{n=0,1,...} T_{A,g}^{(n)}}. \tag{1.4}$$

The Padé method calculates (h,f) with $h \in \mathscr{H}$ by solving a projector equation like Eq. (1.2), but with $P^{(n)}$ now given by

$$T_{A^+,h}^{(n)} = \text{linear span of } \{A^{+k} h \mid k = 0,1,...,n\}, \tag{1.5}$$

$$P^{(n)}|_{T_{A,g}^{(n)}} = 1, \tag{1.6}$$

$$P^{(n)\prime}|_{T_{A^+,h}^{(n)}} = 1. \tag{1.7}$$

Recently, integral equations for multichannel scattering theory have been given by Chandler and Gibson[5] and Kröger and Perne[6] with kernels of a simple structure in contrast to the Faddeev-type equations. But these kernels are not connected and hence not compact. Thus it is not guaranteed that they can be approximated separably. Nevertheless one can try to solve the equations approximately using projector schemes. There are methods, introduced by Petryshyn,[7] which solve functional equations with so called A-proper mappings, which are more general than compact mappings.

In Sec. II we propose a projectionally complete scheme to solve $Bf = g$. We take the linear space spanned by g, Bg, $B^2 g, \cdots$ as projectional subspace. We prove that the projectional solution converges strongly to the solution f and that B is A-proper.

In Sec. III we investigate an integral equation with a nonconnected kernel of the type appearing in multichannel scattering equations and show that the previous results can be applied.

II. THEOREM ON A-PROPER MAPPINGS

Let us start with the definition of A-proper mappings and projectionally complete schemes given in Ref. 7. Let X, Y be Banach spaces, D a given subset of X, $T: D \subseteq X \to Y$ a possibly nonlinear mapping and $\Gamma = \{X_n, P_n; Y_n, Q_n\}$ a suitable approximation scheme for the equation

$$Tx = y, \ (x \in D, y \in Y). \tag{2.1}$$

The scheme Γ is called projectionally complete for (X,Y) provided that $\{X_n\} \subset X$ and $\{Y_n\} \subset Y$ are sequences of monotonically increasing finite-dimensional subspaces with $\dim X_n = \dim Y_n$ for each n and $P_n: X \to X_n$ and $Q_n: Y \to Y_n$ are linear projections such that $P_n x \to x$ and $Q_n y \to y$ for $x \in X$ and $y \in Y$. Here and in the following $\to$ denotes strong convergence. Let $D_n = D \cap X_n$, $T_n: D_n \to Y_n$, $T_n = Q_n T|_{D_n}$ and x_n be the solution of the approximate equation

$$T_n x_n = Q_n y. \tag{2.2}$$

The mapping T is said to be A-proper with respect to the projectionally complete scheme Γ if T_n is continuous for each n and if $\{u_n \mid u_n \in D_n\}$ is a bounded sequence such that $T_n u_n \to v$ for some v in Y, then there exists a subsequence $\{u_{n_\nu}\}$ and $u \in D$ such that $u_{n_\nu} \to u$ and $Tu = v$. Now we can prove the following result.

Theorem: Let $\mathscr{H}$ be a Hilbert space, let g be an element of $\mathscr{H}$, let $B: \mathscr{H} \to \mathscr{H}$ be a linear, bounded mapping, and assume the origin 0 to lie in the resolvent set $\rho(B)$. We define $T_{B,g}^{(n)}, T_{B,Bg}^{(n)}, T_{B,g}, T_{B,Bg}$ in analogy to Eqs. (1.3) and (1.4). Let $P^{(n)}$ be the orthogonal projection onto $T_{B,g}^{(n)}$ and let $Q^{(n)}$ be the orthogonal projection onto $T_{B,Bg}^{(n)}$. Let f be defined as the solution of

$$Bf = g, \tag{2.3}$$

and assume f to be an element of $T_{B,g}$. Let $f^{(n)}$ be defined as the solution of

$$Q^{(n)} B P^{(n)} f^{(n)} = Q^{(n)} g. \tag{2.4}$$

Then we claim that $P^{(n)} f^{(n)}$ converges strongly to f, the approximation scheme is projectionally complete, and B is an A-proper mapping.

Proof: Without loss of generality let $f^{(n)} \in T_{B,g}^{(n)}$, thus $Bf^{(n)} \in T_{B,Bg}^{(n)}$ and $f^{(n)}$ fulfills

$$Bf^{(n)} = Q^{(n)}g. \tag{2.5}$$

The definition of $Q^{(n)}$ implies the existence of $f^{(n)}$. The assumption $f \in T_{B,g}$ implies the existence of a sequence $\tilde{f}^{(n)} \in T^{(n)}_{B,g}$, such that

$$\tilde{f}^{(n)} \underset{n}{\longrightarrow} f. \tag{2.6}$$

The boundedness of B and Eq. (2.3) imply

$$B\tilde{f}^{(n)} \underset{n}{\longrightarrow} g. \tag{2.7}$$

From $B\tilde{f}^{(n)} \in T^{(n)}_{B,Bg}$ and $\|Q^{(n)}\| = 1$ it follows that

$$B\tilde{f}^{(n)} - Q^{(n)}g \underset{n}{\longrightarrow} 0. \tag{2.8}$$

Thus Eqs. (2.7) and (2.8) yield

$$Q^{(n)}g \underset{n}{\longrightarrow} g, \tag{2.9}$$

while Eqs. (2.3) and (2.5) yield

$$B\,(f - f^{(n)}) = g - Q^{(n)}g. \tag{2.10}$$

The property $0 \in \rho(B)$ implies the existence and boundedness of B^{-1}, thus follows from Eqs. (2.9) and (2.10)

$$f^{(n)} \underset{n}{\longrightarrow} f. \tag{2.11}$$

It remains to show the projectional completeness of the approximation scheme and the A-properness of the mapping. First, we prove the following relation:

$$T_{B,g} = T_{B,Bg}. \tag{2.12}$$

The implication $\supseteq$ is trivial; now consider $\subseteq$: Eq. (2.7) implies the existence of a sequence $B\tilde{f}^{(n)} \in T^{(n)}_{B,Bg} \subseteq T_{B,Bg}$, which strongly converges to g.
$T_{B,Bg}$ is closed, hence $g \in T_{B,Bg}$. For each $u \in T_{B,g}$ there exists a sequence $u^{(n)} \in T^{(n)}_{B,g}$

$$u^{(n)} \rightarrow u, \tag{2.13}$$

which can be written as

$$u^{(n)} = u_0^{(n)} g + \sum_{i=1}^{n} u_i^{(n)} B^i g. \tag{2.14}$$

One has

$u_0^{(n)} g \in T_{B,Bg}$, $\sum_{i=1}^{n} u_i^{(n)} B^i_{\ g} \in T^{(n-1)}_{B,Bg} \subseteq T_{B,Bg}$, hence $u^{(n)} \in T_{B,Bg}$. Because of Eq. (2.13) and $T_{B,Bg}$ being closed, one has $u \in T_{B,Bg}$, which proves Eq. (2.12). $T_{B,g}$ is a Hilbert space.

In order to establish the correspondence to the general definition of a projectionally complete approximation scheme and an A-proper mapping we define

$$\begin{aligned} &X = Y = D = T_{B,g}, \quad T = B\,|_{T_{B,g}}, \\ &X_n = T^{(n)}_{B,g}, \quad P_n = P^{(n)}|_{T_{B,g}}, \\ &Y_n = T^{(n)}_{B,Bg}, \quad Q_n = Q^{(n)}|_{T_{B,g}}, \\ &D_n = T^{(n)}_{B,g} \quad T_n = Q^{(n)}|_{T_{B,g}} B\,|_{T^{(n)}_{B,g}}. \end{aligned} \tag{2.15}$$

Equation (2.3) reads

$$B\,|_{T_{B,g}} f = g, \tag{2.16}$$

and Eq. (2.4) reads

$$Q^{(n)}|_{T_{B,g}} B\,|_{T^{(n)}_{B,g}} f^{(n)} = Q^{(n)}|_{T_{B,g}} g. \tag{2.17}$$

$T^{(n)}_{B,g}$ and $T^{(n)}_{B,Bg}$ are finite-dimensional, monotonically increasing subspaces of the same dimension. Because of Eq. (2.12) one has for each $h \in T_{B,g}$

$$P^{(n)}h \underset{n}{\longrightarrow} h, \; Q^{(n)}h \underset{n}{\longrightarrow} h, \tag{2.18}$$

$$Q^{(n)}|_{T_{B,g}} B\,|_{T^{(n)}_{B,g}}$$

is continuous for each n, because $Q^{(n)}$ and B are bounded mappings.

Let $\{u^{(n)} | u^{(n)} \in T^{(n)}_{B,g}\} \in T^{(n)}_{B,g}$ be a bounded sequence such that

$$Q^{(n)}|_{T_{B,g}} B_{T^{(n)}_{B,g}} u^{(n)} \underset{n}{\longrightarrow} v \tag{2.19}$$

for some $v \in T_{B,g}$. Equation (2.19) can be written as

$$Bu^{(n)} \underset{n}{\longrightarrow} v. \tag{2.20}$$

Because $0 \in \rho(B)$, B^{-1} exists and is bounded,

$$u^{(n)} \underset{n}{\longrightarrow} B^{-1} v = w, \tag{2.21}$$

with $w \in T_{B,g}$, hence

$$B\,|_{T_{B,g}} w = v, \tag{2.22}$$

which completes the proof of the theorem. This yields the following corollary.

Corollary 1: Let $\mathscr{H}$ be a Hilbert space, $g \in \mathscr{H}$, let A: $\mathscr{H} \rightarrow \mathscr{H}$ be a linear, bounded mapping with $z_0 \in \rho(A)$. We define

$$T^{(n)}_{A,g} = T^{(n)}_{z_0 - A,g}, \; T_{A,g} = T_{z_0 - A,g},$$
$$T^{(n)}_{z_0 - A,(z_0 - A)g}, T_{z_0 - A,(z_0 - A)g}$$

in analogy to Eqs. (1.3) and (1.4). Let $P^{(n)}, Q^{(n)}$ be the orthogonal projections onto $T^{(n)}_{A,g}$, $T^{(n)}_{z_0 - A,(z_0 - A)g}$, respectively. Let f, $f^{(n)}$ be defined as the solutions of

$$(z_0 - A)\,f = g, \tag{2.23}$$

and

$$Q^{(n)}(z_0 - A)P^{(n)}f^{(n)} = Q^{(n)}g, \tag{2.24}$$

respectively. We assume $f \in T_{A,g}$. Then $f^{(n)}$ converges to f in $\mathscr{H}$. The proof follows from $B = z_0 - A$ and the above theorem.

Corollary 2: We assume the conditions of Corollary 1, except $f \in T_{A,g}$. Instead we assume that the resolvent set $\rho(A)$ has a specific form, namely such that there is a tongue-shaped extension from the region outside the spectral radius $r(A)$ to the point z_0 (Fig. 1). Then the result of Corollary 1 holds.

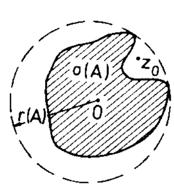

FIG. 1. Schematic plot of the spectrum $\sigma(A)$ with a tongue shaped extension from the region with $|z| > r(A)$ to the point z_0.

Proof: We will show $f\in T_{A,g}$, which completes the conditions of Corollary 1. The function $F(z)=(z-A)^{-1}g$ is an analytical function for $z\in\rho(A)$, i.e., in particular for $|z|>r(A)$. For $|z|>r(A)$ the Neumann series converges,

$$\sum_{k=0,1}^{n}\frac{1}{z}\left(\frac{A}{z}\right)^{k}g\underset{n}{\rightarrow}(z-A)^{-1}g. \tag{2.25}$$

Thus the property $F(z)\in T_{A,g}$ holds for $|z|>r(A)$, which implies $F(z)/T_{A,g}=0$ for $|z|>r(A)$. $F(z)$ is analytical for $z\in\rho(A)$ and so is $F(z)/T_{A,g}$. Thus the analytical continuation of $F(z)/T_{A,g}$ from the region $|z|>r(A)$ to $z=z_0$ yields $F(z_0)/T_{A,g}=0$ and thus $f=F(z_0)\in T_{A,g}$.

III. APPLICATION TO SCATTERING EQUATIONS

Transition operators $T^{(N)}$, which describe the amplitudes for N-nucleon scattering processes, obey Faddeev-type coupled integral equations of the form

$$T^{(N)}=I^{(N)}+K^{(N)}(T^{(N-1)})T^{(N)}. \tag{3.1}$$

The kernel $K^{(N)}$ depends on $(N-1)$-nucleon transition operators $T^{(N-1)}$. The kernels $K^{(N)}$ are disconnected; that means, the integration is not performed in all variables. An example for an integral equation with a disconnected kernel is

$$f(x,y)=g(x,y)+\int_a^b dx'\,k(x,y,x')f(x',y). \tag{3.2}$$

This kernel leads to a noncompact operator. One aims to set up integral equations for $T^{(N)}$ with connected kernels, which is usually possible by iteration of Eq. (3.1). Connectedness of the kernels and adequate treatment of their inherent singularities leads to compact kernels, which allows an approximate solution of the integral equation by some standard techniques like separable approximation of the kernels, for example.

In Refs. 5 and 6 scattering integral equations have been given, which have the advantage of simply structured kernels, compared with the kernels of Faddeev-type equations. But these kernels are disconnected and cannot become connected by iteration.

We want to show now for the three-nucleon case that a disconnected kernel, which is a typical part of the kernel of integral equations of Ref. 6, is A-proper. That means that an integral equation with only that kernel is approximately solvable by projector methods. We introduce some notation. The center of mass motion is dropped. The index α denotes a particle and the two-body subsystem, which does not contain the particle α. $\mathbf{p}_\alpha$ is the relative momentum between particle α and the c.m. of subsystem α, $\mathbf{q}_\alpha$ is the relative momentum between the particles in the subsystem α, m_α is the reduced mass of particle α and the subsystem α, and μ_α is the reduced mass of the particles of subsystem α. $|\mathbf{p},\mathbf{q}\rangle_\alpha$ denotes a plane wave state.

$$H_0=p_\alpha^2/2m_\alpha+q_\alpha^2/2\mu_\alpha \tag{3.3}$$

is the Hamiltonian of the free motion in momentum representation, $G_0(z)=(z-H_0)^{-1}$ is the corresponding Green's function and V_α is the two-body potential in subsytem α given by

$${}_\alpha\langle\mathbf{p}',\mathbf{q}'|V_\alpha|\mathbf{p},\mathbf{q}\rangle_\alpha=\delta(\mathbf{p}'-\mathbf{p})\langle\mathbf{q}'|V^{(2)}|\mathbf{q}\rangle, \tag{3.4}$$

where $V^{(2)}$ is a Hermitian potential in the two-body space, which we assume to be a rotational invariant.

The kernel we want to consider is

$$K=V_\alpha G_0(E+i0), \tag{3.5}$$

for a positive energy E. K, applied to a state f, reads in momentum-angular momentum representation

$$\begin{aligned}
{}_\alpha\langle pLM,qlm|K|f\rangle_\alpha
&=\lim_{\epsilon\to+0}\int_0^\infty dp'\,p'^2\int_0^\infty dq'\,q'^2\sum_{L'M'}\sum_{l'm'}\frac{\delta(p-p')}{p^2}\,\delta_{LM,L'M'}\,\delta_{lm,l'm'}\frac{\langle qlm|V^{(2)}|q'l'm'\rangle}{E+i\epsilon-\dfrac{p'^2}{2m_\alpha}-\dfrac{q'^2}{2\mu_\alpha}}\,{}_\alpha\langle p'L'M',q'l'm'|f\rangle_\alpha\\
&=\lim_{\epsilon\to+0}\int_0^\infty dq'q'^2\frac{V_l(q,q')}{E+i\epsilon-p^2/2m_\alpha-q'^2/2\mu_\alpha}f(pLM,q'lm),
\end{aligned} \tag{3.6}$$

where

$$V_l(q,q')=\langle qlm|V^{(2)}|q'lm\rangle \tag{3.7}$$

is independent of the quantum number m because of rotational invariance. In the following we omit the angular momentum quantum numbers and put $2m_\alpha=2\mu_\alpha=1$, without loss of generality and assume $0<E<1$. The singularity of the kernel can occur only for $0\leqslant p,q'\leqslant 1$. For sake of technical simplicity we modify K by restricting the variables to $0\leqslant p$, q, and $q'\leqslant 1$, but the same kind of result also holds without this restriction. Thus we finally consider the kernel

$$(Kf)(p,q)=\lim_{\epsilon\to+0}\int_0^1 dq'\,q'^2\,\frac{V(q,q')}{E+i\epsilon-p^2-q'^2}f(p,q'). \tag{3.8}$$

We want to show now that there is a Hilbert space $\mathscr{H}$, such that under some smoothness conditions on V, $K:\mathscr{H}\to\mathscr{H}$ becomes a linear, bounded operator, with the spectrum $\sigma(K)$ having the form required in the assumption of Corollary 2. Let $e(p)=E-p^2$, $s(p)=|e(p)|^{1/2}$ and define $\mathscr{H}$ by

$$L_2=\left\{f\,\Big|\int_0^1 dp\,p^2\int_0^1 dq\,q^2\,|f(p,q)|^2\text{exists}\right\},\quad \mathscr{H}=\left\{f\,\Big|\,f(p,q)\in L_2,\phi^f(p,q)=\frac{f(p,q)-f(p,s(p))}{q^2-e(p)}\in L_2\right\}. \tag{3.9}$$

A scalar product is introduced on $\mathscr{H}$.

$$(f,g)_{\mathscr{H}} = (f,g)_{L_2} + (\phi^f,\phi^g)_{L_2}. \quad (3.10)$$

We claim that $\mathscr{H},(.,.)_{\mathscr{H}}$ is a Hilbert space.

The completeness remains to be shown. Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{H}$, then $\{f_n\}$ is a Cauchy sequence in L_2 and there is a limit element $f\in L_2$,

$$f_n \underset{n}{\rightarrow} f. \quad (3.11)$$

Also ϕ^{f_n} is a Cauchy sequence in L_2 and there is a limit element $g\in L_2$,

$$\phi^{f_n} \underset{n}{\rightarrow} g. \quad (3.12)$$

$\mathscr{H}$ is complete if $\phi^f = g$.

(i) First, we claim that $\{f_n(p,s(p))\}$ is a Cauchy sequence in

$$L_2(p) = \left\{f \,\Big|\, \int_0^1 dp\, p^2\, |f(p)|^2 \text{ exists}\right\}, \quad (3.13)$$

because

$$\begin{aligned}\int_0^1 & dp\, p^2 |f_n[p,s(p)] - f_m[p,s(p)]|^2 \\ &= \frac{1}{3}\int_0^1 dp\, p^2 \int_0^1 dq\, q^2 |f_n[p,s(p)] - f_m[p,s(p)]|^2 \\ &= \frac{1}{3}\int_0^1 dp\, p^2 \int_0^1 dq\, q^2 |[\phi^{f_n}(p,q) - \phi^{f_m}(p,q)] \\ &\quad [e(p) - q^2] + f_n(p,q) - f_m(p,q)|^2\end{aligned} \quad (3.14)$$

tends to zero with $m,n\rightarrow\infty$ as $e(p) - q^2$ is bounded, and because of Eqs. (3.11) and (3.12). $L_2(p)$ is complete, hence there exists a limit element $F\in L_2(p)$,

$$f_n(p,s(p)) \underset{n}{\overset{L_2(p)}{\longrightarrow}} F(p). \quad (3.15)$$

(ii) The set $\{(p,q)|q = s(p)\}$ is a subset of measure 0 of the integration domain $\{(p,q)|0\leqslant p,q\leqslant 1\}$. We modify f on this set of measure 0, putting

$$f[p,s(p)] = F(p), \quad (3.16)$$

which does not change $f\in L_2$.

(iii) Now we can prove the relation

$$f(p,q) - f[p,s(p)] - g(p,q)[q^2 - e(p)] = 0 \quad (3.17)$$

almost everywhere, which follows from

$$\begin{aligned}\int_0^1 & dp\, p^2 \int_0^1 dq\, q^2\, |f(p,q) - f[p,s(p)] - g(p,q)[q^2 - e(p)]|^2 \\ &= \int_0^1 dpp^2 \int_0^1 dqq^2 |f(p,q) - f_n(p,q) - \{f[p,s(p)] \\ &\quad - f_n[p,s(p)]\} + [\phi^{f_n}(p,q) - g(p,q)][q^2 - e(p)]|^2\end{aligned} \quad (3.18)$$

which can be estimated by the triangle inequality. The first term tends to zero because of Eq. (3.11), the second term tends to zero because of Eqs (3.15) and (3.16), and the last term tends to zero because of Eq. (3.12). Equation (3.18) implies

$$\frac{f(p,q) - f[p,s(p)]}{q^2 - e(p)} = \phi^f(p,q) = g(p,q)\in L_2 \quad (3.19)$$

almost everywhere, thus proving the completeness of $\mathscr{H}$.

Now we want to show that K, defined by Eq. (3.8), becomes a linear, bounded operator mapping $\mathscr{H}$ into $\mathscr{H}$, provided that $V(q,q')$ is a sufficiently smooth function. Let $V(q,q')$ be a once continuously differentiable function in the domain $0\leqslant q,q'\leqslant 1$, and let $f\in\mathscr{H}$. Thus one can write the right hand side of Eq. (3.8)

$$\begin{aligned}\lim_{\epsilon\rightarrow +0} & \int_0^1 dq'\, q'^2 \frac{V(q,q')}{e(p) + i\epsilon - q'^2} f(p,q') \\ &= \int_0^1 dq'\, q'^2 \frac{V(q,q') - V[q,s(p)]}{e(p) - q'^2} f(p,q') \\ &\quad + V[q,s(p)] \int_0^1 dq'\, q'^2 \frac{f(p,q') - f[p,s(p)]}{e(p) - q'^2} \\ &\quad + V[q,s(p)] f[p,s(p)] \lim_{\epsilon\rightarrow +0} \int_0^1 dq'\, q'^2 \frac{1}{e(p) + i\epsilon - q'^2}.\end{aligned} \quad (3.20)$$

We abbreviate

$$\phi^V(p,q,q') = \frac{V(q,q') - V[q,s(p)]}{q'^2 - e(p)}, \quad (3.21)$$

$$\psi^V(p,q,q') = \frac{V(q,q') - V[q,s(p)] - V[s(p),q'] + V[s(p),s(p)]}{[q^2 - e(p)][q'^2 - e(p)]}. \quad (3.22)$$

From the smoothness property of $V(q,q')$ there follows the existence of an upper bound M, such that for $0\leqslant p,q$ and $q'\leqslant 1$,

$$|V(q,q')| \leqslant M, \quad (3.23)$$

$$|\phi^V(p,q,q')| \leqslant M, \quad (3.24)$$

$$|\psi^V(p,q,q')| \leqslant M. \quad (3.25)$$

Now the first term on the right-hand side of Eq. (3.20),

$$(K_1 f)(p,q) = -\int_0^1 dq'\, q'^2 \phi^V(p,q,q') f(p,q'), \quad (3.26)$$

can be estimated as

$$\begin{aligned}\|K_1 f\|^2_{L_2} &= \int_0^1 dp\, p^2 \int_0^1 dq\, q^2\, \left|\int dq'\, q'^2 \phi^V(p,q,q') f(p,q')\right|^2 \\ &\leqslant \int_0^1 dp\, p^2 \int_0^1 dqq^2 \left(\int_0^1 dq' q'^2 |\phi^V(p,q,q')|^2\right) \\ &\quad \times \left(\int_0^1 dq' q'^2 |f(p,q')|^2\right) \\ &\leqslant \frac{M^2}{9} \int_0^1 dp\, p^2 \int_0^1 dq\, q^2 |f(p,q)|^2 = \frac{M^2}{9} \|f\|^2_{L_2}.\end{aligned} \quad (3.27)$$

Similarly one has

$$\begin{aligned}\|\phi^{K_1 f}\|^2_{L_2} &= \int_0^1 dp\, p^2 \int_0^1 dq\, q^2 \left|\int_0^1 dq' q'^2 \psi^V(p,q,q') f(p,q')\right|^2 \\ &\leqslant \frac{1}{9} M^2 \|f\|^2_{L_2}\end{aligned} \quad (3.28)$$

and thus

$$\|K_1 f\|^2_{\mathscr{H}} = \|K_1 f\|^2_{L_2} + \|\phi^{K_1 f}\|^2_{L_2} \leqslant \frac{2M^2}{9} \|f\|^2_{L_2} \leqslant \frac{2M^2}{9} \|f\|^2_{\mathscr{H}}. \quad (3.29)$$

The second term on the right-hand side of Eq. (3.20) reads

$$(K_2 f)(p,q) = -V[q,s(p)] \int_0^1 dq'\, q'^2 \phi^f(p,q'). \tag{3.30}$$

The following estimates hold

$$\|K_2 f\|^2_{L_2} = \int_0^1 dp\, p^2 \int_0^1 dq\, q^2 |V[q,s(p)] \int_0^1 dq'\, q'^2 \phi^f(p,q')|^2 \leqslant \tfrac{1}{3} M^2 \|\phi_f\|^2_{L_2}, \tag{3.31}$$

$$\|\phi^{K_2 f}\|^2_{L_2} = \int_0^1 dp\, p^2 \int_0^1 dq\, q^2 | -\phi^V[p,s(p),q]^* \int_0^1 dq'\, q'^2 \phi^f(p,q')|^2 \leqslant \tfrac{1}{3} M^2 \|\phi^f\|^2_{L_2}, \tag{3.32}$$

and hence

$$\|K_2 f\|^2_{\mathscr{H}} \leqslant \tfrac{2}{3} M^2 \|f\|^2_{\mathscr{H}}. \tag{3.33}$$

The last term on the right-hand side of Eq. (3.20) reads

$$(K_3 f)(p,q) = V[q,s(p)] f[p,s(p)] c(p), \tag{3.34}$$

where

$$c(p) = \lim_{\epsilon \to +0} \int_0^1 dq\, q^2 \frac{1}{e(p) + i\epsilon - q^2} \tag{3.35}$$

exists for all p in $0 \leqslant p \leqslant 1$ and $c(p)$ is bounded,

$$|c(p)| \leqslant N. \tag{3.36}$$

$K_3 f$ can be estimated as

$$\begin{aligned}\|K_3 f\|^2_{L_2} &= \int_0^1 dp\, p^2 \int_0^1 dq\, q^2 |V[q,s(p)] f[p,s(p)] c(p)|^2 \\ &\leqslant M^2 N^2 \int_0^1 dp\, p^2 \int_0^1 dq\, q^2 |f[p,s(p)]|^2 \\ &= M^2 N^2 \int_0^1 dp\, p^2 \int_0^1 dq\, q^2 |\phi^f(p,q)[e(p) - q^2] + f(p,q)|^2 \\ &\leqslant M^2 N^2 (2\|\phi^f\|_{L_2} + \|f\|_{L_2})^2 \leqslant 9 M^2 N^2 \|f\|^2_{\mathscr{H}},\end{aligned} \tag{3.37}$$

where $|e(p) - q^2| \leqslant 2$ has been used. Analogously we have

$$\|\phi^{K_3 f}\|^2_{L_2} = \int_0^1 dp\, p^2 \int_0^1 dq\, q^2 |\phi^V[p,s(p),q]^* f[p,s(p)] c(p)|^2 \leqslant 9 M^2 N^2 \|f\|^2_{\mathscr{H}}, \tag{3.38}$$

hence

$$\|K_3 f\|^2_{\mathscr{H}} \leqslant 18 M^2 N^2 \|f\|^2_{\mathscr{H}}. \tag{3.39}$$

From Eqs. (3.29), (3.33), and (3.39) one concludes that range $(K) \subset \mathscr{H}$ and K is bounded.

Finally we investigate the spectrum $\sigma(K)$. We define for a fixed p, $0 \leqslant p \leqslant 1$,

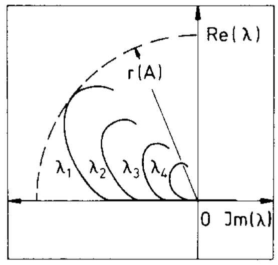

FIG. 2. Schematic plot of the eigenvalues $\lambda_\nu(p)$ [Weinberg trajectories given by Eq. (3.42) for an attractive potential].

$$\begin{aligned}\mathscr{H}_p = \Big\{ f \,|\, &\int_0^1 dq\, q^2 |f(q)|^2 \Big\} \text{exists}, \\ &\int_0^1 dq\, q^2 \left| \frac{f(q) - f[s(p)]}{q^2 - e(p)} \right|^2 \text{exists},\end{aligned} \tag{3.40}$$

and K_p is defined on $\mathscr{H}_p$

$$(K_p f)(q) = \lim_{\epsilon \to +0} \int_0^1 dq' q'^2 \frac{V(q,q')}{e(p) + i\epsilon - q'^2} f(q'). \tag{3.41}$$

It has been shown in Ref. 8 that $\mathscr{H}_p$ is a Hilbert space and K_p a compact mapping from $\mathscr{H}_p$ into $\mathscr{H}_p$. The spectrum of a compact operator consists only of the point spectrum and possibly the origin. Let λ_ν be the eigenvalues and φ_ν be the eigenvectors of K_p,

$$K_p \varphi_\nu = \lambda_\nu \varphi_\nu. \tag{3.42}$$

Of course λ_ν and φ_ν are dependent of p. Under variation of p the eigenvalues run along the so-called Weinberg trajectories[9] (Fig. 2). Thus the spectrum $\sigma(K)$ is given by

$$\sigma(K) = \{\lambda_\nu [e(p)] \,|\, \nu = 1,2,\ldots;\, 0 \leqslant p \leqslant 1\} \cup [0]. \tag{3.43}$$

Thus the spectrum $\sigma(K)$ has a shape, which allows a tongue-shaped extension from the region $|z| \geqslant r(K)$ into the region inside the spectral radius and fulfills for those values z_0 the assumption of Corallary 2, and hence K is A-proper.

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