AIP Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **33**, 3509 (1992); doi: 10.1063/1.529902
View online: http://dx.doi.org/10.1063/1.529902
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/33/10?ver=pdfcov
Published by the AIP Publishing

Gravitational equations in space-time with torsion

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(Received 24 July 1991; accepted for publication 5 March 1992)

Various representations of equations of gravitational theories in space-time with torsion are considered. Special attention is paid to the case of quadratic Poincaré gauge theory of gravity. Advantages of each representation are discussed. The complete algorithm for calculation of gravitational equations in the framework of the most efficient formalism of spinor-valued exterior forms is presented.

I. INTRODUCTION

It is well known that reformulation of a mathematical problem in more suitable terms usually gives new insight into it and can play a crucial role in its solution. A good example is supplied by Newman–Penrose formalism, which plays a very important role in search and classification of exact solutions of an Einstein equation.

The scope of this paper is the investigation of equations of gravity theories in space-time with torsion. These theories are far generalization of Einstein theory of gravity. For such models the role played by a formalism used in its investigation is far more significant than in general theory of relativity. It is connected with the complexity of the gravitational equations. Indeed, an Einstein equation is a rather trivial algebraic relation in terms of curvature, but equations of gravitation theories in space-time with torsion include complicated differential and nonlinear algebraic expressions with torsion and curvature. Additional complexity is concerned with a large number of unknown parameters in the theory.[1]

Two independent structures exist in space with torsion (or so-called Riemann–Cartan space): metric, $g_{\alpha\beta}$, and connection, $\Gamma^{\alpha}{}_{\beta\gamma}$. The only restriction imposed on these objects is metricity condition

$$\nabla_\alpha g_{\beta\gamma}=0. \tag{1}$$

Solving Eq. (1) one can obtain the following representation for connection:

$$\Gamma^{\alpha}{}_{\beta\gamma}=\{{}^{\alpha}_{\beta\gamma}\}+K^{\alpha}{}_{\beta\gamma}, \tag{2}$$

where $\{{}^{\alpha}_{\beta\gamma}\}$ are usual Christoffel symbols, $K^{\alpha}{}_{\beta\gamma}$ is a contorsion tensor,

$$K_{\alpha\beta\gamma}=\tfrac{1}{2}(Q_{\alpha\beta\gamma}-Q_{\gamma\alpha\beta}+Q_{\beta\gamma\alpha}),$$

and $Q^{\alpha}{}_{\beta\gamma}$ is a torsion tensor,

$$Q^{\alpha}{}_{\beta\gamma}=\Gamma^{\alpha}{}_{\beta\gamma}-\Gamma^{\alpha}{}_{\gamma\beta}. \tag{3}$$

Further, we use tetrad formalism in which all quantities are represented respectively to a certain ungolonomic basis. Metric tensor in the ungolonomic basis is a constant, symmetric, invertible matrix η_{ab}:

$$g_{\mu\nu}=\eta_{ab}h^a_\mu h^b_\nu, \tag{4}$$

$$\eta_{ab}=g_{\mu\nu}h^\mu_a h^\nu_b, \tag{5}$$

where h^a_μ is a tetrad and h^μ_a is an inverse tetrad ($h^\mu_a h^b_\mu=\delta^b_a$). Tetrad connection can be obtained from the identity

$$\nabla_\alpha h^a_\beta=0, \tag{6}$$

$$\omega^a{}_{b\mu}=h^a_\alpha\Gamma^\alpha{}_{\beta\mu}h^\beta_b-\partial_\mu h^a_\beta h^\beta_b. \tag{7}$$

The metricity condition (1) is equivalent to

$$\nabla_\mu\eta_{ab}\equiv-\omega_{ab\mu}-\omega_{ba\mu}=0. \tag{8}$$

Curvature and torsion in tetrad formalism take the form

$$R^a{}_{b\mu\nu}=\partial_\mu\omega^a{}_{b\nu}-\partial_\nu\omega^a{}_{b\mu}+\omega^a{}_{c\mu}\omega^c{}_{b\nu}-\omega^a{}_{c\nu}\omega^c{}_{b\mu}, \tag{9}$$

$$Q^a{}_{\mu\nu}=\partial_\nu h^a_\mu-\partial_\mu h^a_\nu+h^b_\mu\omega^a{}_{b\nu}-h^b_\nu\omega^a{}_{b\mu}. \tag{10}$$

Let us introduce some useful operators:

$$\tilde{\nabla}_\pi X^{\pi\cdots}\overset{\text{def}}{=}(\nabla_\pi-Q^\alpha{}_{\pi\alpha})X^{\pi\cdots},\quad\forall X^{\alpha\cdots}, \tag{11}$$

$$\mathscr{D}_\pi X^{\alpha\pi\cdots}\overset{\text{def}}{=}\tilde{\nabla}_\pi X^{\alpha\pi\cdots}-\tfrac{1}{2}Q^\alpha{}_{\pi\kappa}X^{\pi\kappa\cdots},$$
$$\forall X^{\alpha\beta\cdots}=X^{[\alpha\beta]\cdots}. \tag{12}$$

We use $\overset{\{\}}{\nabla}$ for covariant derivative without torsion.

Dualization operation for tensors

$$\overset{*}{X}_{\alpha_1\cdots\alpha_{4-p}}=\frac{1}{p!}\mathscr{E}_{\alpha_1\cdots\alpha_{4-p}}{}^{\beta_1\cdots\beta_p}X_{\beta_1\cdots\beta_p} \tag{13}$$

and for exterior forms

$$\chi=\frac{1}{p!}X_{\beta_1\cdots\beta_p}dx^{\beta_1}\wedge\cdots\wedge dx^{\beta_p},\tag{14}$$

$$*\chi=\frac{1}{(4-p)!}\overset{*}{X}_{\alpha_1\cdots\alpha_{4-p}}dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_{4-p}},$$

where $\mathscr{E}^{\alpha\beta\gamma\delta}$ is a totally antisymmetric tensor with

$$\mathscr{E}^{0123}=\frac{1}{\sqrt{-\det(g_{\mu\nu})}}.\tag{15}$$

We use overlain symbols above variables for complex conjugation.

II. GRAVITATIONAL THEORIES IN SPACE-TIME WITH TORSION

One of the most important of Einstein's ideas is the identification of space-time geometry and gravity. Presumably, it is the reason for the so-called "theoretical beauty" of general theory of relativity. The theories under consideration are based on direct extension of this Einstein's idea to more general types of geometry than a Riemannian one. Why do we concentrate especially on the spaces with torsion and not on more general affine-metric spaces? The theories in space-time with torsion have many attractive features. For example, the sources in its gravitational equations are both energy-momentum and spin tensors of matter fields.[1] Contrary to the case of affine-metric theories in spaces with torsion there are no troubles with physical interpretation of nonmetricity and corresponding current-hypermomentum.[2] Attempts to treat gravity as a gauge theory of a Poincaré group always result in theories in Riemann–Cartan space.[1,3]

Thus, in accordance with the principle of identification of geometry and gravity we assume that tetrad h^a_μ and antisymmetric tetrad connection $\omega^a{}_{b\mu}$ are independent gravitational potentials. Let gravitational action include only first derivatives of the potentials

$$S=S_g+S_m$$
$$=\int\left[\frac{1}{16\pi G}\mathscr{L}_g(h,\partial h,\omega,\partial\omega)+\mathscr{L}_m\right]h\,d^4x,$$
$$h=\det(h^a_\mu).\tag{16}$$

If action is invariant under general coordinate and local Lorenzian transformations, then using the Noether theorem one can easily conclude that (16) can be expressed in the terms of curvature and torsion only:[1,4]

$$\mathscr{L}_g=\mathscr{L}_g(h^a_\mu,Q^a{}_{\mu\nu},R^a{}_{b\mu\nu}).\tag{17}$$

Using (17) and the second Noether theorem one can obtain the following expressions for the left parts of gravitational equations:[1]

$$Z^\mu{}_a\overset{\text{def}}{=}16\pi G\,h^{-1}\frac{\delta S_g}{\delta h^a_\mu}$$
$$=-\mathscr{D}_\pi\widetilde{Q}_a{}^{\mu\pi}-\widetilde{Q}^{\alpha\beta\mu}Q_{\alpha\beta a}-\widetilde{R}^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma a}+h^\mu_a\mathscr{L}_g,\tag{18}$$

$$V^{ab\mu}\overset{\text{def}}{=}16\pi G\,h^{-1}\frac{\delta S_g}{\delta\omega_{ab\mu}}=\mathscr{D}_\pi\widetilde{R}^{ab\mu\pi}+\widetilde{Q}^{[ab]\mu},\tag{19}$$

where

$$\widetilde{R}^{ab\mu\nu}=2\frac{\partial\mathscr{L}_g}{\partial R_{ab\mu\nu}},\tag{20}$$

$$\widetilde{Q}_a{}^{\mu\nu}=2\frac{\partial\mathscr{L}_g}{\partial Q^a{}_{\mu\nu}}.\tag{21}$$

Variational derivatives (18) and (19) are not independent. They satisfy the identities for arbitrary values of h^a_μ and $\omega_{ab\mu}$:

$$-\widetilde{\nabla}_\pi Z^\pi{}_\mu+V^{ab\pi}R_{ab\pi\mu}+Z^\alpha{}_bQ^b{}_{\alpha\mu}=0,\tag{22}$$

$$Z_{[ab]}-\widetilde{\nabla}_\pi V_{ab}{}^\pi=0.\tag{23}$$

Certainly, gravitational sources must satisfy (22) and (23), too.

It is obvious from (23) that the antisymmetric part of the $\delta/\delta h$ equation can be eliminated without loss of information. Consequently, we have gravitational equations for the action (17) in the form

$$-\tfrac{1}{2}Z_{(ab)}=8\pi GT_{ab},\tag{24}$$

$$V_{ab\mu}=-16\pi G\mathscr{S}_{ab\mu},\tag{25}$$

where

$$T_{ab}=t_{(ab)}\equiv t_{ba}+\widetilde{\nabla}_\pi\mathscr{S}_{ab}{}^\pi,\tag{26}$$

$$t^\mu{}_a\overset{\text{def}}{=}h^{-1}\frac{\delta S_m}{\delta h^a_\mu},\tag{27}$$

$$\mathscr{S}^{ab\mu}\overset{\text{def}}{=}h^{-1}\frac{\delta S_m}{\delta\omega_{ab\mu}},\tag{28}$$

and S_m is the action of matter fields.

Equations for (17) [exactly in the form of (24) and (25)] can be obtained by another way if we take metric $g_{\mu\nu}$ and general affine connection $\Gamma^\mu{}_{\nu\lambda}$ as independent variables and take into account metricity condition with the help of Lagrange multipliers.

In order to proceed further we must choose some concrete form of gravitational action. Usually, both linear in curvature and various quadratic terms are included in (17). We shall not discuss the reasons for this choice in details.[1,3,4] We have to stress only that quadratic terms are natural from the point of view of gauge approach (direct analogy with Yang–Mills Lagrangian), and linear term provides correct Newtonian limit and good correspondence with experimental data.[1]

The most general quadratic in torsion and curvature Lagrangian reads[1,2]

$$\mathscr{L}_g=\lambda R-2\Lambda+R_{abcd}(l_1R^{abcd}+l_2R^{cdab}+l_3R^{acbd}) +R_{ab}(l_4R^{ab}+l_5R^{ba})+l_6R^2+Q_{abc}(a_1Q^{abc} +a_2Q^{cba})+a_3Q_aQ^a, \tag{29}$$

where $R_{ab}=R^n{}_{anb}$, $R=R^n{}_n$, and $Q_a=Q^n{}_{an}$.

We call theory with (29) quadratic Poincaré gauge theory of gravity, or simply $R+R^2+Q^2$ theory.

This model is very wide. Lagrangian (29) includes nine independent parameters excluding gravitational constant G and Λ-term [we have only nine parameters owing to existence of total divergence $h(R_{abcd}R^{cdab}-4R_{ab}R^{ba}+R^2)=\text{div}$]. In the rest of paper we restrict our attention to the case of $R+R^2+Q^2$ theory.

III. EQUATIONS OF $R+R^2+Q^2$ THEORY IN THE TERMS OF IRREDUCIBLE CURVATURE AND TORSION COMPONENTS

Direct substitution of (29) into (18)–(21) results in very complicated and inconvenient equations. For example, they include more than ten curvature squired terms with three or four pairs of contracted indexes in various positions. Such terms are undesired from a practical point of view. However, one can eliminate these terms and sufficiently reduce the complexity of the equations. For this purpose one must use decomposition of curvature and torsion on irreducible parts.[5,6]

Curvature tensor in space with torsion has only the following symmetries:

$$R_{abcd}=R_{[ab]cd}=R_{ab[cd]},$$

and in the general case it can be represented as the sum of six algebraically independent components:

$$R_{abcd}=C_{abcd}+T_{abcd}+S_{abcd}+A_{abcd}+B_{abcd}+D_{abcd},$$

$$T_{abcd}=2\eta_{[a[c}C_{b]d]},$$

$$S_{abcd}=(R/6)\eta_{[a[c}\eta_{b]d]},$$

TABLE I. Irreducible curvature components.

X_{abcd}	X_{cdab}	$2X_{[a[c\|b]d]}$	$^*X^*_{abcd}$	$X^m{}_{amb}$	$X_{a[bcd]}$
C_{abcd}	C_{abcd}	C_{abcd}	$-C_{abcd}$	0	0
T_{abcd}	T_{abcd}	T_{abcd}	T_{abcd}	C_{ab}	0
S_{abcd}	S_{abcd}	S_{abcd}	$-S_{abcd}$	$\frac{1}{4}\eta_{ab}R$	0
A_{abcd}	$-A_{abcd}$	0	$-A_{abcd}$	A_{ab}	$A_{a[bcd]}$
B_{abcd}	$-B_{abcd}$	0	B_{abcd}	0	$B_{a[bcd]}$
D_{abcd}	D_{abcd}	$-2D_{abcd}$	$-D_{abcd}$	0	D_{abcd}

$$A_{abcd}=2\eta_{[a[c}A_{b]d]}, \tag{30}$$

$$B_{abcd}=\mathscr{E}_{ab[c}{}^pB_{p|d]},$$

$$D_{abcd}=-(E/12)\mathscr{E}_{abcd},$$

where C_{abcd} is the Weyl tensor, $C_{ab}=R_{(ab)}-\frac{1}{4}\eta_{ab}R$, $A_{ab}=R_{[ab]}$, $B_{ab}=D_{(ab)}-\frac{1}{4}\eta_{ab}E$, $E=D^m{}_m$, $D_{ab}=R^{*p}{}_{apb}$, and $R^*_{abcd}=\frac{1}{2}\mathscr{E}_{cd}{}^{mn}R_{abmn}$.

Irreducible curvature components have important algebraic properties that are presented in Table I (where $^*X^*_{abcd}=\frac{1}{4}\mathscr{E}_{abmn}\mathscr{L}_{cdpq}X^{mnpq}$). Bianchi identity $\mathscr{D}_m{}^*R^*_{abc}{}^m=0$ can be rewritten in terms of (30):[1,7]

$$\mathscr{D}_m(C_{abc}{}^m-T_{abc}{}^m+S_{abc}{}^m+A_{abc}{}^m-B_{abc}{}^m+D_{abc}{}^m)=0. \tag{31}$$

Analogously, torsion tensor can be represented as the sum of three irreducible components:

$$Q_{abc}=C_{abc}+T_{abc}+P_{abc},$$

$$T_{abc}=\tfrac{2}{3}\eta_{a[c}Q_{b]}, \tag{32}$$

$$P_{abc}=\tfrac{1}{3}\mathscr{E}_{abcm}P^m,$$

where $Q_a=Q^n{}_{an}$, $P_a=Q^{*n}{}_{an}$, and $Q^*_{abc}=\frac{1}{2}\mathscr{E}_{bc}{}^{mn}Q_{amn}$.

Algebraic properties of (32) are presented in Table II.

In the case of $R+R^2+Q^2$ theory $\widetilde{R}_{abcd}$ and $\widetilde{Q}_{abc}$ tensors are linear combinations of irreducible components with arbitrary weights

TABLE II. Irreducible torsion components.

X_{abc}	$X_{[bc]a}$	$\frac{1}{2}(X_{abc}-X_{cab}+X_{bca})$	$X^n{}_{an}$	$X_{[abc]}$
C_{abc}	$-\frac{1}{2}C_{abc}$	$-C_{cab}$	0	0
T_{abc}	$-\frac{1}{2}T_{abc}$	$-T_{cab}$	Q_a	0
P_{abc}	P_{abc}	$\frac{1}{2}P_{cab}$	0	P_{abc}

$$\tilde{R}_{abcd}=2\lambda\eta_{[a[c}\eta_{b]d]}+L_1C_{abcd}+L_2T_{abcd}+L_3S_{abcd}+L_4A_{abcd}+L_5B_{abcd}+L_6D_{abcd}, \quad (33)$$

$$\tilde{Q}_{abc}=M_1C_{abc}+M_2T_{abc}+M_3P_{abc}. \quad (34)$$

Coefficients L_i and M_i in (33) and (34) can be expressed as linear combinations of l_i and a_i from (29):

$$\begin{aligned}
&\Lambda_0=4(l_1+l_2)+2l_3, \\
&\Lambda_1=4(l_1+l_2)+2l_3+l_4+l_5, && L_1=\Lambda_0, \\
&\Lambda_2=4l_1+l_3, && L_2=2\Lambda_1-\Lambda_0, \\
&\Lambda_3=4(l_1+l_2)+2l_3+4(l_4+l_5)+12l_6, && L_3=2\Lambda_3-2\Lambda_1+\Lambda_0, \\
&\Lambda_4=4l_1-2l_3, && L_4=2\Lambda_5-2\Lambda_1+\Lambda_0, \\
&\Lambda_5=4(l_1-l_2)+l_4-l_5, && L_5=2\Lambda_2-\Lambda_0, \\
&\Lambda_6=4l_1+l_3+2l_4, && L_6=2\Lambda_4-2\Lambda_2+\Lambda_0,
\end{aligned} \quad (35)$$

$$\begin{aligned}
&\mu_1=4(a_1-a_2)-\lambda, && M_1=-2(\mu_3+\lambda), \\
&\mu_2=-2a_1-a_2-3a_3+2\lambda, && M_2=-2(\mu_2-2\lambda), \\
&\mu_3=-2a_1-a_2-\lambda, && M_3=\mu_1+\lambda.
\end{aligned} \quad (36)$$

Substituting (33) and (34) into (24) and (25) and taking in account algebraic properties of irreducible components one can obtain equations of $R+R^2+Q^2$ theory in the form

$$\lambda C_{ab}-\frac{1}{4}\eta_{ab}(\lambda R-4\Lambda)+\Lambda_1C_{ambn}C^{mn}+\Lambda_2C^{*}_{ambn}B^{mn}+\frac{\Lambda_3}{6}RC_{ab}+\frac{\Lambda_4}{6}EB_{ab}+\Lambda_5B_{m(a}\overset{*}{A}{}^{m}{}_{b)}+\Lambda_6C_{m(a}A^{m}{}_{b)}$$
$$-(\mu_3+\lambda)\left[\overset{\{\}}{\nabla}_mC_{(ab)}{}^{m}+\frac{1}{2}C_a{}^{mn}C_{bmn}-\frac{1}{4}\eta_{ab}(C_{mnp})^2-\frac{1}{3}C_{(ab)m}Q^m-\frac{1}{6}C^{*}_{(ab)m}P^m\right]$$
$$-\frac{(\mu_2-2\lambda)}{3}\left[\overset{\{\}}{\nabla}_{(a}Q_{b)}-\eta_{ab}\overset{\{\}}{\nabla}_mQ^m-\frac{1}{3}Q_aQ_b-\frac{1}{6}\eta_{ab}(Q_m)^2-C_{(ab)m}Q^m\right]$$
$$+\frac{(\mu_1+\lambda)}{18}\left[P_aP_b+\frac{1}{2}\eta_{ab}(P_m)^2-6C^{*}_{(ab)m}P^m\right]=8\pi GT_{ab}, \quad (37)$$

$$2\Lambda_1\mathscr{D}_mT_{abc}{}^{m}+2(\Lambda_3-\Lambda_1)\mathscr{D}_mS_{abc}{}^{m}+2(\Lambda_5-\Lambda_2)\mathscr{D}_mA_{abc}{}^{m}+2\Lambda_2\mathscr{D}_mB_{abc}{}^{m}$$
$$+2(\Lambda_4-\Lambda_2)\mathscr{D}_mD_{abc}{}^{m}+\mu_3C_{cab}+\mu_2T_{cab}+\mu_1P_{cab}=-16\pi G\mathscr{S}_{abc}. \quad (38)$$

In the process of deducing (38) we have used Bianchi identity (31) in order to eliminate the $\mathscr{D}_mC_{abc}{}^{m}$ term. In the general case other terms can be excluded in (38).

Equations of $R+R^2+Q^2$ theory in the form of (37) and (38) provide radical simplification. For example, the above-mentioned curvature squared terms have been reduced to six comparatively simple terms in (37). Usually, in order to obtain the left parts of gravitational equations one must perform the following chain of calculations: metric, torsion $\xrightarrow{1}$ connection $\xrightarrow{2}$ curvature $\xrightarrow{3}$ equations. In the case of Eqs. (37) and (38) one additional step is needed: metric, torsion $\xrightarrow{1}$ connection $\xrightarrow{2}$ curvature $\xrightarrow{2'}$ irreducible parts of torsion and curvature $\xrightarrow{3}$ equations. Nevertheless, practice shows that expanses for this step are fully compensated by simplification of the Step 3.

Let us add some technical remarks. Working with Eqs. (37) and (38) one can use either coordinate or tetrad formalism. Formally, these methods are equivalent. However, in the case of $R+R^2+Q^2$ theory there is one important practical difference. Equations (37) and (38) contain torsion and curvature with numerous contractions and with various positions (lower or upper) of indexes. It is obvious that calculation of such terms for nondiagonal metrics can be very complicated. This trouble is absent in the framework of tetrad formalism in spite of possible complexity of metric (if we don't use "exotic" forms of η_{ab}).

IV. EQUATIONS OF $R+R^2+Q^2$ THEORY IN SPINORIAL FORMALISM

The role played by spinorial formalism in Einstein theory is well known.[8,9] The use of this method is highly productive, especially in combination with algebraic classification schemes. This approach can be successfully applied to the investigation of $R+R^2+Q^2$ gravity, too. The main advantage of spinorial formalism is concerned with considerable simplification of index algebra and reduction of complicated symmetries.

Let us illustrate it with the case of Weyl tensor. This irreducible tensor C_{abcd} has a comparatively complicated set of symmetries (see Table I). In four-dimensional space it has ten independent components. However, explicit splitting of these components causes some troubles. Spinorial analog of the Weyl tensor is very simple totally symmetric spinor C_{ABCD}. It has exactly five independent complex components

$$C_0 \overset{\text{def}}{=} C_{0000}, \quad C_1 \overset{\text{def}}{=} C_{1000}=C_{0100}=\cdots=C_{0001},\ldots,C_4 \overset{\text{def}}{=} C_{1111}.$$

As in this example all irreducible tensors have spinorial analogs that always are totally symmetric (irreducible spinors). Thus their independent components can be trivially slitted.

Let us remember basic notions of spinorial formalism.[9,10] Spinors can have two types of indexes. Undotted indexes take values 0 and 1 and are transformed with the help of $SL(2,C)$ matrices. Dotted indexes are transformed with the conjugated matrices.

Raising and lowering of the indexes can be performed with the help of antisymmetric metric spinor:

$$\varepsilon_{AB}=\varepsilon^{AB}=\varepsilon_{\dot A\dot B}=\varepsilon^{\dot A\dot B}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{39}$$

$$\varphi^A=\varphi_B\varepsilon^{BA}, \quad \varphi_A=\varepsilon_{AB}\varphi^B. \tag{40}$$

Correspondence between tensors and spinors is established with the help of σ-matrices:

$$X^a\sigma_a{}^{A\dot A} \overset{\text{def}}{=} X^{A\dot A}, \quad \forall X^a. \tag{41}$$

Further, we shall use sign $\rightarrow$ for this correspondence in the style of (41) conserving relation between a tetrad and pair of conjugated spinor indexes as $a\rightarrow A\dot A$, $b\rightarrow B\dot B$, etc.

Let us consider spinorial analogs of some tensors[6]

$$\eta_{ab}\rightarrow -\varepsilon_{AB}\varepsilon_{\dot A\dot B}, \tag{42}$$

$$\mathscr{E}_{abcd}\rightarrow i(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{\dot A\dot D}\varepsilon_{\dot B\dot C}-\varepsilon_{\dot A\dot C}\varepsilon_{\dot B\dot D}\varepsilon_{AD}\varepsilon_{BC}), \tag{43}$$

$$C_{abcd}\rightarrow C_{ABCD}\varepsilon_{\dot A\dot B}\varepsilon_{\dot C\dot D}+C_{\dot A\dot B\dot C\dot D}\varepsilon_{AB}\varepsilon_{CD}, \tag{44}$$

$$C_{ab}\rightarrow C_{AB\dot A\dot B}, \tag{45}$$

$$B_{ab}\rightarrow B_{AB\dot A\dot B}, \tag{46}$$

$$A_{ab}\rightarrow A_{AB}\varepsilon_{\dot A\dot B}+A_{\dot A\dot B}\varepsilon_{AB}, \tag{47}$$

$$C_{abc}\rightarrow C_{\dot A ABC}\varepsilon_{\dot B\dot C}+C_{A\dot A\dot B\dot C}\varepsilon_{BC}, \tag{48}$$

$$Q_a\rightarrow Q_{A\dot A}, \tag{49}$$

$$P_a\rightarrow P_{A\dot A}. \tag{50}$$

All spinors C_{ABCD}, $C_{AB\dot A\dot B}$, $B_{AB\dot A\dot B}$, A_{AB}, and $C_{\dot A ABC}$ are irreducible, i.e., totally symmetric in dotted and undotted indexes separately. The pairs of terms in (44), (47), and (48) are conjugated to each other and spinors (45), (46), (49), and (50) are Hermitian:

$$C_{\dot A\dot B\dot C\dot D}=\overline{C_{ABCD}}, \quad C_{A\dot B\dot C\dot D}=\overline{C_{\dot A BCD}}, \quad \overline{C_{AB\dot C\dot D}}=C_{CD\dot A\dot B},$$

$$A_{\dot A\dot B}=\overline{A_{AB}}, \quad Q_{A\dot B}=\overline{Q_{B\dot A}}, \quad P_{A\dot B}=\overline{P_{B\dot A}}. \tag{51}$$

Now, inserting (42)–(50) into (37) and (38) one finds

$$\begin{aligned}
&\lambda C_{AB\dot A\dot B}+\Lambda_1[C_{ABMN}C^{MN}{}_{\dot A\dot B}+\text{c.c.}]+\Lambda_2[-iC_{ABMN}B^{MN}{}_{\dot A\dot B}+\text{c.c.}]+\frac{\Lambda_3}{6}RC_{AB\dot A\dot B}+\frac{\Lambda_4}{6}EB_{AB\dot A\dot B}+\Lambda_5[-iB^M{}_{(A|\dot A\dot B}A_{B)M}+\text{c.c.}]\\
&\quad+\Lambda_6\Big[C^M{}_{(A|\dot A\dot B}A_{B)M}+\text{c.c.}]-(\mu_3+\lambda)\Big[-\overset{\{\}}{\nabla}{}^P{}_{(\dot A}C_{\dot B)ABP}+\text{c.c.}+C_{(\dot A(A|MN}C_{\dot B)B)}{}^{MN}+\text{c.c.}+\frac{1}{3}Q^P{}_{(\dot A}C_{\dot B)ABP}+\text{c.c.}\\
&\quad-\frac{i}{6}P^P{}_{(\dot A}C_{\dot B)ABP}+\text{c.c.}\Big]-\frac{(\mu_2-2\lambda)}{3}\Big[\overset{\{\}}{\nabla}{}_{(A(\dot A}Q_{B)\dot B)}-\frac{1}{3}Q_{(A(\dot A}Q_{B)\dot B)}+Q^P{}_{(\dot A}C_{\dot B)ABP}+\text{c.c.}\Big]\\
&\quad+\frac{(\mu_1+\lambda)}{18}[P_{(A(\dot A}P_{B)\dot B}-6iP^P{}_{(\dot A}C_{\dot B)ABP}+\text{c.c.}]=8\pi GT_{AB\dot A\dot B},
\end{aligned} \tag{52}$$

$$-\lambda R+4\Lambda-(\mu_3+\lambda)[C_{\dot ABCD}C^{\dot ABCD}+\text{c.c.}]-(\mu_2-2\lambda)\left[\overset{\{\}}{\nabla}{}_{M\dot M}Q^{M\dot M}+\frac{1}{3}Q_{M\dot M}Q^{M\dot M}\right]-\frac{(\mu_1+\lambda)}{6}P_{M\dot M}P^{M\dot M}=8\pi GT^m_m, \tag{53}$$

$$
\begin{aligned}
&2\Lambda_1\left[-\frac{1}{2}\overset{\{\}}{\nabla}_C{}^{\dot P}C_{AB\dot C\dot P}-K_{(A|}{}^{M}{}_{C}{}^{\dot N}C_{M|B)\dot C\dot N}\right]+2\Lambda_2\left[\frac{i}{2}\overset{\{\}}{\nabla}_C{}^{\dot P}B_{AB\dot C\dot P}+iK_{(A|}{}^{M}{}_{C}{}^{\dot N}B_{M|B)\dot C\dot N}\right]+2(\Lambda_5-\Lambda_2)\\
&\times\left[-\overset{\{\}}{\nabla}{}^{P}{}_{\dot C}A_{ABCP}-2K_{(A|}{}^{MN}{}_{\dot C}A_{B)MCN}\right]+2(\Lambda_3-\Lambda_1)\left[-\overset{\{\}}{\nabla}{}^{P}{}_{\dot C}S_{ABCP}-2K_{(A|}{}^{MN}{}_{\dot C}S_{B)MCN}\right]+2(\Lambda_4-\Lambda_2)\\
&\times\left[-\overset{\{\}}{\nabla}{}^{P}{}_{\dot C}D_{ABCP}-2K_{(A|}{}^{MN}{}_{\dot C}D_{B)MCN}\right]+\mu_3C_{\dot CCAB}+\frac{\mu_2}{3}Q_{(A|\dot C}\epsilon_{B)C}+\frac{i\mu_1}{3}P_{(A|\dot C}\epsilon_{B)C}=-16\pi G\mathscr{S}_{ABC\dot C},
\end{aligned}
\tag{54}
$$

where

$$K_{abc}\to K_{ABC\dot C}\epsilon_{\dot A\dot B}+\text{c.c.},\tag{55}$$

$$T_{abcd}\to\tfrac{1}{2}C_{AB\dot C\dot D}\varepsilon_{\dot A\dot B}\varepsilon_{CD}+\text{c.c.},\tag{56}$$

$$S_{abcd}\to S_{ABCD}\varepsilon_{\dot A\dot B}\varepsilon_{\dot C\dot D}+\text{c.c.},$$

$$S_{ABCD}=R/24(\varepsilon_{AC}\epsilon_{BD}+\varepsilon_{AD}\epsilon_{BC}),\tag{57}$$

$$A_{abcd}\to A_{ABCD}\varepsilon_{\dot A\dot B}\varepsilon_{\dot C\dot D}+\text{c.c.},\quad A_{ABCD}=-\varepsilon_{(A(C}A_{B)D)},\tag{58}$$

$$B_{abcd}\to-(i/2)B_{AB\dot C\dot D}\varepsilon_{\dot A\dot B}\varepsilon_{CD}+\text{c.c.},\tag{59}$$

$$D_{abcd}\to D_{ABCD}\varepsilon_{\dot A\dot B}\varepsilon_{\dot C\dot D}+\text{c.c.},$$

$$D_{ABCD}=iE/24(\varepsilon_{AC}\epsilon_{BD}+\varepsilon_{AD}\epsilon_{BC}).\tag{60}$$

Equation (53) is the trace of (37), and (52) is the traceless part of (37). For the right part of (52) we have by definition

$$T_{ab}-\tfrac{1}{4}\eta_{ab}T^m_m\to T_{AB\dot A\dot B}.\tag{61}$$

Equation (54) is the "half" of (38) as for right part (spin tensor)

$$\mathscr{S}_{abc}\to\mathscr{S}_{ABC\dot C}\epsilon_{\dot A\dot B}+\mathscr{S}_{\dot A\dot B\dot CC}\epsilon_{AB}.\tag{62}$$

V. EQUATION OF $R+R^2+Q^2$ THEORY IN THE FORMALISM OF EXTERIOR FORMS

There is one more efficient method in differential geometry and theory of gravitation: formalism of exterior forms.[8,11,12] Let us rewrite the equations of $R+R^2+Q^2$ theory in the framework of this formalism.

For arbitrary gravitational Lagrangian we have analogs of (18) and (19):[12,13]

$$
\begin{aligned}
Z_a&=*(Z^\mu{}_a\,dx_\mu)\\
&=-D\widetilde\Theta_a-(\partial_a\rfloor\Theta^b)\wedge\widetilde\Theta_b-(\partial_a\rfloor\Omega^{bc})\wedge\widetilde\Omega_{bc}\\
&\quad+\partial_a\rfloor\widetilde L_g,
\end{aligned}
\tag{63}
$$

$$V_{ab}=*(V_{ab\mu}\,dx^\mu)=D\widetilde\Omega_{ab}+\theta_{[a}\wedge\widetilde\Theta_{b]},\tag{64}$$

$$t_a=*(t^\mu{}_a\,dx_\mu),\tag{65}$$

$$\mathscr{S}_{ab}=*(\mathscr{S}_{ab\mu}\,dx^\mu),\tag{66}$$

where two-forms of torsion and curvature are

$$\Theta^a=\tfrac{1}{2}Q^a{}_{\mu\nu}\,dx^\mu\wedge dx^\nu,\tag{67}$$

$$\Omega^{ab}=\tfrac{1}{2}R^{ab}{}_{\mu\nu}\,dx^\mu\wedge dx^\nu;\tag{68}$$

analogs of $\widetilde Q_{abc}$, $\widetilde R_{abcd}$ are

$$\widetilde\Theta^a=-\tfrac{1}{2}\widetilde Q^a{}_{\mu\nu}*(dx^\mu\wedge dx^\nu),\tag{69}$$

$$\widetilde\Omega^{ab}=-\tfrac{1}{2}\widetilde R^{ab}{}_{\mu\nu}*(dx^\mu\wedge dx^\nu);\tag{70}$$

and θ^a, ∂_a-one-forms of tetrad and vectors of inverse tetrad are

$$\theta^a=h^a_\mu\,dx^\mu,\tag{71}$$

$$\partial_a=h^\mu_a\partial_\mu.\tag{72}$$

The Lagrangian is represented in (66) as four-form

$$\widetilde L_g=\mathscr{L}_g v,\tag{73}$$

$$v=\sqrt{-g}dx^0\wedge dx^1\wedge dx^2\wedge dx^3.\tag{74}$$

The symmetric part of $Z^\mu{}_a$ can be obtained with the help of

$$Z_{ab}=\partial_{(a}\rfloor*Z_{b)}=*(\theta_{(a}\wedge Z_{b)}).\tag{75}$$

For a particular case of $R+R^2+Q^2$ theory we have

$$
\begin{aligned}
\widetilde L_g=&-2\Lambda v-\lambda*S_{ab}\wedge\Omega^{ab}-\frac{1}{2}\sum_{i=1\ldots6}L_i*\Omega^{(i)}_{ab}\wedge\Omega^{ab}\\
&-\frac{1}{2}\sum_{i=1\ldots3}M_i*\Theta^{(i)}_a\wedge\Theta^a,
\end{aligned}
\tag{76}
$$

where

$$\Omega^{(1)}_{ab}=\tfrac{1}{2}C_{abcd}S^{cd},\quad \Omega^{(2)}_{ab}=\tfrac{1}{2}T_{abcd}S^{cd},\quad \Omega^{(3)}_{ab}=\tfrac{1}{2}S_{abcd}S^{cd},$$

$$\Omega^{(4)}_{ab}=\tfrac{1}{2}A_{abcd}S^{cd},\quad \Omega^{(5)}_{ab}=\tfrac{1}{2}B_{abcd}S^{cd},\quad \Omega^{(6)}_{ab}=\tfrac{1}{2}D_{abcd}S^{cd}, \tag{77}$$

$$\Theta^{a}_{(1)}=\tfrac{1}{2}C^{a}{}_{bc}S^{bc},\quad \Theta^{a}_{(2)}=\tfrac{1}{2}T^{a}{}_{bc}S^{bc},\quad \Theta^{a}_{(3)}=\tfrac{1}{2}P^{a}{}_{bc}S^{bc}, \tag{78}$$

$$S^{ab}=\theta^{a}\wedge\theta^{b},\quad *S^{ab}=\tfrac{1}{2}\mathscr{E}^{ab}{}_{cd}S^{cd}. \tag{79}$$

Here $\widetilde{\Omega}_{ab}$ and $\widetilde{\Theta}_{a}$ are as follows:

$$\widetilde{\Omega}_{ab}=-\sum_{i=1\dots6}L_{i}*\Omega^{(i)}_{ab}-\lambda*S_{ab}, \tag{80}$$

$$\widetilde{\Theta}_{a}=-\sum_{i=1\dots3}M_{i}*\Theta^{(i)}_{a}. \tag{81}$$

Finally, we have equations of $R+R^2+Q^2$ model,

$$Z_a=-16\pi Gt_a, \tag{82}$$

$$V_{ab}=-16\pi G\mathscr{S}_{ab}. \tag{83}$$

with (63)–(66), (76)–(81). The symmetric part of (82) can be extracted with the help of (75).

In some sense Eqs. (82) and (83) are more complicated than (37) and (38). For example, expression (63) contains curvature squared terms [like $R^{abcd}R_{abce}$ and $\delta^{d}_{e}(R_{klmn})^2$] that are explicitly canceled in (37).

Nevertheless, these equations can be successfully used in practice owing to efficiency of exterior forms.

VI. EQUATIONS OF $R+R^2+Q^2$ THEORY IN THE FORMALISM OF SPINOR-VALUED EXTERIOR FORMS

Finally, let us consider so-called formalism of spinor-valued exterior forms. This method was used in Einstein theory[14] but not widely enough. Unifying advantages of spinors and exterior forms provides the most powerful tool for practical calculations. Objective comparison with the help of computer confirms it.

Let us introduce spinorial analogs of required objects: S-forms:

$$S_{ab}=\theta_a\wedge\theta_b\to S_{AB}\epsilon_{\dot A\dot B}+S_{\dot A\dot B}\epsilon_{AB}; \tag{84}$$

one-forms of spinorial connection ω_{AB}, $\omega_{\dot A\dot B}$:

$$\omega_{ab}=\omega_{ab\mu}\,dx^{\mu}\to\omega_{AB}\epsilon_{\dot A\dot B}+\omega_{\dot A\dot B}\epsilon_{AB}; \tag{85}$$

two-forms of spinorial curvature Ω_{AB}, $\Omega_{\dot A\dot B}$:

$$\Omega_{ab}\to\Omega_{AB}\epsilon_{\dot A\dot B}+\Omega_{\dot A\dot B}\epsilon_{AB}. \tag{86}$$

Representation (86) has to be applied for all curvature-type forms $\widetilde{\Omega}_{ab}$, $\Omega^{(i)}_{ab}$.

All spinors S_{AB}, ω_{AB}, Ω_{AB} and their conjugated analogs $S_{\dot A\dot B}$, $\omega_{\dot A\dot B}$, $\Omega_{\dot A\dot B}$ are symmetric (irreducible).

The two-form of torsion can be expressed in spinorial basis:

$$\Theta_a=\tfrac{1}{2}Q_{abc}S^{bc}=\vartheta_a+\overline{\vartheta_a},$$

$$\vartheta_a=Q_{aBC}S^{BC},\quad \overline{\vartheta_a}=Q_{a\dot B\dot C}S^{\dot B\dot C}, \tag{87}$$

$$Q_{abc}\to Q_{aBC}\epsilon_{\dot B\dot C}+Q_{a\dot B\dot C}\epsilon_{BC}.$$

Formula (87) has to be applied for all torsion-type two-forms $\widetilde{\Theta}_a$, $\Theta^{(i)}_a$.

Spinorial analogs of S-forms (84) are eigenvectors of dualization operator

$$*S_{AB}=-iS_{AB}. \tag{88}$$

Thus, for $\widetilde{\Omega}_{AB}$ and $\widetilde{\Theta}_a$ we have

$$\widetilde{\Omega}_{AB}=i\Big(\lambda S_{AB}+\sum_{k=1,3,4,6}L_k\Omega^{(k)}_{AB}-\sum_{k=2,5}L_k\Omega^{(k)}_{AB}\Big), \tag{89}$$

$$\widetilde{\Theta}_a=\widetilde{\vartheta}_a+\text{c.c.},\quad \widetilde{\vartheta}_a=i\sum_{k=1\dots3}M_k\vartheta^{(k)}_a. \tag{90}$$

The Lagrangian takes the form

$$\widetilde{L}_g=-2\Lambda v+2i\lambda S_{AB}\wedge\Omega^{AB}+\text{c.c.}+i\Big(\sum_{k=1,3,4,6}L_k\Omega^{(k)}_{AB}-\sum_{k=2,5}L_k\Omega^{(k)}_{AB}\Big)\wedge\Omega^{AB}+\text{c.c.}+\frac{1}{2}\widetilde{\Theta}_a\wedge\Theta^a. \tag{91}$$

The left part of (82) is as follows:

$$Z_a=-D\widetilde{\Theta}_a-(\partial_a\lrcorner\,\Theta^b)\wedge\widetilde{\Theta}_b-2(\partial_a\lrcorner\,\Omega^{AB})\wedge\widetilde{\Omega}_{AB}+\text{c.c.}+\partial_a\lrcorner\,\widetilde{L}_g, \tag{92}$$

and the spinorial analog of V_{ab} is

$$V_{ab}\to V_{AB}\epsilon_{\dot A\dot B}+\text{c.c.}, \tag{93}$$

$$V_{AB}=D\widetilde{\Omega}_{AB}+\widetilde{\Sigma}_{AB}, \tag{94}$$

$$\theta_{[a}\wedge\widetilde{\Theta}_{b]}\to\widetilde{\Sigma}_{AB}\epsilon_{\dot A\dot B}+\text{c.c.} \tag{95}$$

It is well known that all equations of spinorial formalism can be rewritten in component form without contractions. This approach is used in Newman–Penrose formalism[9] and is the main reason of their success. We can rewrite in this style our equations (52)–(54). Instead of it we rewrite in components equations of spinor-valued forms method. A full set of formulas for calculations in the framework of this formalism is given in Appendix A.

It is worth mentioning that those formulas demonstrate remarkable simplicity (compare with original Newman–Penrose method!) in spite of a more general type of geometry than a Riemannian one.

The method of spinor-valued external forms is realized in computer algebra program GRG.[15,16]

There are many articles concerned with $R+R^2+Q^2$ theory of gravitation. They use various notations for geometrical objects and parameters of the theory. For convenience translation between our notation and notations of other authors are presented in Appendix B.

APPENDIX A: FORMULAS OF THE SPINOR-VALUED EXTERIOR FORMS FORMALISM

Working with spinors it is natural to use complex null tetrad

$$ds^2=-\theta^0\otimes\theta^1-\theta^1\otimes\theta^0+\theta^2\otimes\theta^3+\theta^3\otimes\theta^2, \tag{A1}$$

$$\eta_{ab}=\eta^{ab}=\begin{pmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{A2}$$

With complex conjugation rules

$$\theta^0=\overline{\theta^0},\quad \theta^1=\overline{\theta^1},\quad \theta^2=\overline{\theta^3}. \tag{A3}$$

All other objects with tensorial indexes have the same rules of conjugation.

In this basis σ-matrix can be chosen in the form (only nonzero components are presented)

$$\sigma_0{}^{1\dot 1}=\sigma_1{}^{0\dot 0}=\sigma_2{}^{1\dot 0}=\sigma_3{}^{0\dot 1}=1, \tag{A4}$$

$$\sigma^0{}_{1\dot 1}=\sigma^1{}_{0\dot 0}=\sigma^2{}_{1\dot 0}=\sigma^3{}_{0\dot 1}=-1. \tag{A5}$$

It is obvious from (A4) and (A5) that correspondence between spinorial and tensorial indexes are reduced to relabeling:

$$\forall X^a\to X^{A\dot A}\quad X^0=X^{1\dot 1},\quad X^1=X^{0\dot 0},$$
$$X^2=X^{1\dot 0},\quad X^3=X^{0\dot 1}. \tag{A6}$$

In accordance with this we use less abusive tensorial notation for θ^a and similar objects with one tensorial index (P^a, Q^a, Θ^a, etc).

For irreducible (symmetric) sets of spinorial indexes we use summed notation. For example, if $X_{AB}=X_{(AB)}$, then $X_0\overset{\text{def}}{=}X_{00}, X_1\overset{\text{def}}{=}X_{01}=X_{10}, X_2\overset{\text{def}}{=}X_{11}$.

We have the following auxiliary objects:

spinorial S-forms:

$$S_0=-\theta^0\wedge\theta^2, \tag{A7}$$

$$S_1=\tfrac{1}{2}(\theta^0\wedge\theta^1-\theta^2\wedge\theta^3), \tag{A8}$$

$$S_2=\theta^1\wedge\theta^3, \tag{A9}$$

$$S_{\dot 0}=-\theta^0\wedge\theta^3, \tag{A10}$$

$$S_{\dot 1}=\tfrac{1}{2}(\theta^0\wedge\theta^1+\theta^2\wedge\theta^3), \tag{A11}$$

$$S_{\dot 2}=\theta^1\wedge\theta^2. \tag{A12}$$

We should like to stress that all other antisymmetric in pair of indexes tensors (Ω_{ab}, ω_{ab}, $\mathscr{S}_{ab}$, A_{ab}, etc.) follow the same rule of tensor $\to$ spinor correspondence

$$\forall X_{ab}=X_{[ab]}\to X_{AB}\varepsilon_{\dot A\dot B}+\text{c.c.},$$
$$X_{13}=X_0,\quad \tfrac{1}{2}(X_{01}-X_{23})=-X_1,\quad X_{02}=-X_2\,. \tag{A13}$$

Spinorial S-forms have only the following nonzero exterior products

$$S_0\wedge S_2=-iv,\quad S_1\wedge S_1=(i/2)v. \tag{A14}$$

The volume element is

$$v=i\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3,\quad *v=1. \tag{A15}$$

The following formulas are presented in the form of algorithm for calculation the equations of $R+R^2+Q^2$ theory starting with tetrad and torsion. We present only required a minimum of formulas. All the components that do not carry additional information (complex conjugated versions for spinors and third components for tensors) are missed.

Step 1. Calculation of connection: Spinorial connection one-forms can be determined with help of first structural equations for S-forms:

$$dS_0-2\omega_1\wedge S_0+2\omega_0\wedge S_1+\Sigma_0=0, \tag{A16}$$

$$dS_1-\omega_2\wedge S_0+\omega_0\wedge S_2+\Sigma_1=0, \tag{A17}$$

$$dS_2-2\omega_2\wedge S_1+2\omega_1\wedge S_2+\Sigma_2=0, \tag{A18}$$

where

$$\Sigma_0=\Theta^2\wedge\theta^0-\Theta^0\wedge\theta^2, \tag{A19}$$

$$\Sigma_1=\tfrac{1}{2}(\Theta^0\wedge\theta^1-\Theta^1\wedge\theta^0-\Theta^2\wedge\theta^3+\Theta^3\wedge\theta^2), \tag{A20}$$

$$\Sigma_2=\Theta^1\wedge\theta^3-\Theta^3\wedge\Theta^1. \tag{A21}$$

The solution of (A16)–(A18) is

$$\omega_0 = i*[(dS_1+\Sigma_1)\wedge\theta^2]\theta^0 - i*[(dS_0+\Sigma_0)\wedge\theta^0]\theta^1$$
$$-i*[(dS_1+\Sigma_1)\wedge\theta^0]\theta^2$$
$$+i*[(dS_0+\Sigma_0)\wedge\theta^2]\theta^3, \qquad \text{(A22)}$$

$$\omega_1 = \frac{i}{2}*[(dS_2+\Sigma_2)\wedge\theta^2 - (dS_1+\Sigma_1)\wedge\theta^1]\theta^0$$
$$+\frac{i}{2}*[(dS_0+\Sigma_0)\wedge\theta^3 - (dS_1+\Sigma_1)\wedge\theta^0]\theta^1$$
$$+\frac{i}{2}*[(dS_1+\Sigma_1)\wedge\theta^3 - (dS_2+\Sigma_2)\wedge\theta^0]\theta^2$$
$$+\frac{i}{2}*[(dS_1+\Sigma_1)\wedge\theta^2 - (dS_0+\Sigma_0)\wedge\theta^1]\theta^3, \qquad \text{(A23)}$$

$$\omega_2 = -i*[(dS_2+\Sigma_2)\wedge\theta^1]\theta^0 + i*[(dS_1+\Sigma_1)\wedge\theta^3]\theta^1$$
$$+i*[(dS_2+\Sigma_2)\wedge\theta^3]\theta^2$$
$$-i*[(dS_1+\Sigma_1)\wedge\theta^1]\theta^3. \qquad \text{(A24)}$$

Note that dualization in (A22)–(A24) is a trivial operation owing to (A15).

Step 2. Calculation of curvature two-forms:

$$\Omega_0 = d\omega_0 + 2\omega_0\wedge\omega_1, \qquad \text{(A25)}$$

$$\Omega_1 = d\omega_1 + \omega_0\wedge\omega_2, \qquad \text{(A26)}$$

$$\Omega_2 = d\omega_2 + 2\omega_1\wedge\omega_2. \qquad \text{(A27)}$$

Step 3. Calculation of irreducible curvature spinors: At first one must calculate spinors $\mathscr{R}_{ABCD}$ and $\mathscr{R}_{AB\dot{C}\dot{D}}$, which are precise coefficients of Ω_{AB} in the basis of spinorial S-forms

$$\Omega_{AB} = \mathscr{R}_{ABCD}S^{CD} + \mathscr{R}_{AB\dot{C}\dot{D}}S^{\dot{C}\dot{D}}.$$

For calculation of this coefficient one can use

$$X_{AB}S^{AB} = X_0S_2 - 2X_1S_1 + X_2S_0, \quad \forall X_{AB} = X_{(AB)}, \qquad \text{(A28)}$$

or

$$\mathscr{R}_{00} = i*[\Omega_0\wedge S_0], \quad \mathscr{R}_{0\dot{0}} = -i*[\Omega_0\wedge S_{\dot{0}}],$$
$$\mathscr{R}_{01} = i*[\Omega_0\wedge S_1], \quad \mathscr{R}_{0\dot{1}} = -i*[\Omega_0\wedge S_{\dot{1}}], \qquad \text{(A29)}$$
$$\cdots \quad \cdots$$

Spinor $\mathscr{R}_{ABCD}$ is reducible (not totally symmetric) and has only symmetries $\mathscr{R}_{ABCD} = \mathscr{R}_{(AB)CD} = \mathscr{R}_{AB(CD)}$. Therefore, a pair of indexes is used in (A29) (each of them takes values $0=00$, $1=01=10$, $2=11$).

Now, we can calculate irreducible curvature spinors: Weyl spinor:

$$C_0 = \mathscr{R}_{00},$$
$$C_1 = \tfrac{1}{2}(\mathscr{R}_{10} + \mathscr{R}_{01}),$$
$$C_2 = \tfrac{1}{6}(\mathscr{R}_{20} + \mathscr{R}_{02}) + \tfrac{2}{3}\mathscr{R}_{11}, \qquad \text{(A30)}$$
$$C_3 = \tfrac{1}{2}(\mathscr{R}_{12} + \mathscr{R}_{21}),$$
$$C_4 = \mathscr{R}_{22};$$

traceless symmetric Ricci part:

$$C_{0\dot{0}} = \mathscr{R}_{0\dot{0}} + \overline{\mathscr{R}_{0\dot{0}}},$$
$$C_{0\dot{1}} = \mathscr{R}_{0\dot{1}} + \overline{\mathscr{R}_{1\dot{0}}}, \qquad \text{(A31)}$$
$$\cdots;$$

scalar curvature:

$$R = 2(\mathscr{R}_{20} + \mathscr{R}_{02} - 2\mathscr{R}_{11}) + \text{c.c.}; \qquad \text{(A32)}$$

antisymmetric Ricci part:

$$A_0 = \mathscr{R}_{10} - \mathscr{R}_{01},$$
$$A_1 = \tfrac{1}{2}(\mathscr{R}_{20} - \mathscr{R}_{02}), \qquad \text{(A33)}$$
$$A_2 = \mathscr{R}_{21} - \mathscr{R}_{12};$$

traceless symmetric deviation:

$$B_{0\dot{0}} = i(\mathscr{R}_{0\dot{0}} - \overline{\mathscr{R}_{0\dot{0}}}),$$
$$B_{0\dot{1}} = i(\mathscr{R}_{0\dot{1}} - \overline{\mathscr{R}_{1\dot{0}}}), \qquad \text{(A34)}$$
$$\cdots;$$

scalar deviation:

$$E = -2i(\mathscr{R}_{20} + \mathscr{R}_{02} - 2\mathscr{R}_{11}) + \text{c.c.} \qquad \text{(A35)}$$

If $Q_{abc}=0$, then the formulas (A30)–(A35) can be simplified: (A33)–(A35) are zero and $2C_{AB\dot{C}\dot{D}}$ coincides with $\mathscr{R}_{AB\dot{C}\dot{D}}$.

Note that, for Einstein theory, this step is final because (A31) and (A32) are precise left parts of Einstein equation.

Explicit form of tensor→spinor correspondence for Weyl tensor (44) is

$$C_{1313}=C_0,\quad C_{1323}=C_1,\quad C_{1302}=-C_2,\quad C_{0223}=-C_3,\quad C_{0202}=C_4, \tag{A36}$$

and for traceless symmetric tensors C_{ab} (45) and B_{ab} (46) is

$$\forall X_{ab}=X_{(ab)}\to X_{AB\dot A\dot B}$$

$$X_{11}=X_{0\dot 0},\quad X_{01}=X_{1\dot 1},\quad X_{00}=X_{2\dot 2},\quad X_{12}=X_{1\dot 0},\quad X_{22}=X_{2\dot 0},\quad X_{02}=X_{2\dot 1}. \tag{A37}$$

Step 4. Calculation of irreducible curvature two-forms: For calculation of irreducible curvature two-forms the following formulas can be used:

$$\Omega_0^{(1)}=C_2S_0-2C_1S_1+C_0S_2,\quad \Omega_1^{(1)}=C_3S_0-2C_2S_1+C_1S_2,\quad \Omega_2^{(1)}=C_4S_0-2C_3S_1+C_2S_2; \tag{A38}$$

$$\Omega_0^{(2)}=\tfrac12 C_{0\dot 2}S_{\dot 0}-C_{0\dot 1}S_{\dot 1}+\tfrac12 C_{0\dot 0}S_{\dot 2},\quad \Omega_1^{(2)}=\tfrac12 C_{1\dot 2}S_{\dot 0}-C_{1\dot 1}S_{\dot 1}+\tfrac12 C_{1\dot 0}S_{\dot 2},\quad \Omega_2^{(2)}=\tfrac12 C_{2\dot 2}S_{\dot 0}-C_{2\dot 1}S_{\dot 1}+\tfrac12 C_{2\dot 0}S_{\dot 2}; \tag{A39}$$

$$\Omega_0^{(3)}=\frac{R}{12}S_0,\quad \Omega_1^{(3)}=\frac{R}{12}S_1,\quad \Omega_2^{(3)}=\frac{R}{12}S_2; \tag{A40}$$

$$\Omega_0^{(4)}=-A_1S_0+A_0S_1,\quad \Omega_1^{(4)}=-\tfrac12 A_2S_0+\tfrac12 A_0S_2,\quad \Omega_2^{(4)}=-A_2S_1+A_1S_2; \tag{A41}$$

$$\Omega_0^{(5)}=-\frac{i}{2}B_{0\dot 2}S_{\dot 0}+iB_{0\dot 1}S_{\dot 1}-\frac{i}{2}B_{0\dot 0}S_{\dot 2},\quad \Omega_1^{(5)}=-\frac{i}{2}B_{1\dot 2}S_{\dot 0}+iB_{1\dot 1}S_{\dot 1}-\frac{i}{2}B_{1\dot 0}S_{\dot 2},\quad \Omega_2^{(5)}=-\frac{i}{2}B_{2\dot 2}S_{\dot 0}+iB_{2\dot 1}S_{\dot 1}-\frac{i}{2}B_{2\dot 0}S_{\dot 2}; \tag{A42}$$

$$\Omega_0^{(6)}=\frac{iE}{12}S_0,\quad \Omega_1^{(6)}=\frac{iE}{12}S_1,\quad \Omega_2^{(6)}=\frac{iE}{12}S_2. \tag{A43}$$

Step 5. Calculation of torsion irreducible spinors: At first one must calculate $Q^a{}_{BC}$, $Q^a{}_{\dot B\dot C}$—coefficients of decomposition of Θ^a in the basis of spinorial S-forms $\Theta^a=Q^a{}_{BC}S^{BC}+Q^a{}_{\dot B\dot C}S^{\dot B\dot C}$.

For this purpose one can use (A28) or

$$Q^0{}_0=i*[\Theta^0\wedge S_0],\quad Q^0{}_{\dot 0}=-i*[\Theta^0\wedge S_{\dot 0}],\quad Q^0{}_1=i*[\Theta^0\wedge S_1],\quad Q^0{}_{\dot 1}=-i*[\Theta^0\wedge S_{\dot 1}],\quad \ldots\ \ldots \tag{A44}$$

Now, one can calculate irreducible torsion components:

traceless torsion spinor:

$$C_{\dot 0 0}=Q^0{}_0,\quad C_{\dot 0 1}=\tfrac13(-Q^3{}_0+2Q^0{}_1),\quad C_{\dot 0 2}=\tfrac13(Q^0{}_2-2Q^3{}_1),\quad C_{\dot 0 3}=-Q^3{}_2,$$
$$C_{\dot 1 0}=-Q^2{}_0,\quad C_{\dot 1 1}=\tfrac13(Q^1{}_0-2Q^2{}_1),\quad C_{\dot 1 2}=\tfrac13(-Q^2{}_2+2Q^1{}_1),\quad C_{\dot 1 3}=Q^1{}_2; \tag{A45}$$

torsion trace:

$$Q^0=-Q^3{}_0-Q^0{}_1-Q^2{}_{\dot 0}-Q^0{}_{\dot 1},\quad Q^1=Q^1{}_1+Q^2{}_2+Q^1{}_{\dot 1}+Q^3{}_{\dot 2},\quad Q^2=-Q^1{}_0-Q^2{}_1+Q^2{}_{\dot 1}+Q^0{}_{\dot 2}; \tag{A46}$$

torsion pseudotrace:

$$P^0=i(Q^3{}_0+Q^0{}_1-Q^2{}_{\dot 0}-Q^0{}_{\dot 1}),\quad P^1=i(-Q^1{}_1-Q^2{}_2+Q^1{}_{\dot 1}+Q^3{}_{\dot 2}),\quad P^2=i(Q^1{}_0+Q^2{}_1+Q^2{}_{\dot 1}+Q^0{}_{\dot 2}). \tag{A47}$$

The explicit form of tensor→spinor correspondence for traceless torsion C_{abc} (48) is

$$C_{113}=-C_{\dot 0 0},\ C_{213}=-C_{\dot 0 1},\quad C_{102}=C_{\dot 0 2},\ C_{202}=C_{\dot 0 3},\quad C_{313}=-C_{\dot 1 0},\quad C_{013}=-C_{\dot 1 1},\quad C_{302}=C_{\dot 1 2},\quad C_{002}=C_{\dot 1 3}. \tag{A48}$$

Step 6. Calculation of torsion irreducible two-forms: Irreducible torsion two-forms can be calculated with help of following formulas:

$$\Theta^0_{(i)}=\vartheta^0_{(i)}+\overline{\vartheta^0_{(i)}},\quad \Theta^1_{(i)}=\vartheta^1_{(i)}+\overline{\vartheta^1_{(i)}},$$
$$\Theta^2_{(i)}=\vartheta^2_{(i)}+\overline{\vartheta^3_{(i)}},\quad i=1,2,3; \tag{A49}$$

$$\vartheta^0_{(1)}=C_{\dot 0 0}S_2-2C_{\dot 0 1}S_1+C_{\dot 0 2}S_0,$$
$$\vartheta^1_{(1)}=C_{\dot 1 1}S_2-2C_{\dot 1 2}S_1+C_{\dot 1 3}S_0,$$
$$\vartheta^2_{(1)}=-C_{\dot 1 0}S_2+2C_{\dot 1 1}S_1-C_{\dot 1 2}S_0, \tag{A50}$$
$$\vartheta^3_{(1)}=-C_{\dot 0 1}S_2+2C_{\dot 0 2}S_1-C_{\dot 0 3}S_0;$$

$$\vartheta^0_{(2)}=\tfrac{1}{3}Q^0S_1+\tfrac{1}{3}Q^3S_0,\quad \vartheta^1_{(2)}=-\tfrac{1}{3}Q^2S_2-\tfrac{1}{3}Q^1S_1,$$
$$\vartheta^2_{(2)}=\tfrac{1}{3}Q^2S_1+\tfrac{1}{3}Q^1S_0,\quad \vartheta^3_{(2)}=-\tfrac{1}{3}Q^0S_2-\tfrac{1}{3}Q^3S_1; \tag{A51}$$

$$\vartheta^0_{(3)}=\frac{i}{3}P^0S_1+\frac{i}{3}P^3S_0,\quad \vartheta^1_{(3)}=-\frac{i}{3}P^2S_2-\frac{i}{3}P^1S_1,$$
$$\vartheta^2_{(3)}=\frac{i}{3}P^2S_1+\frac{i}{3}P^1S_0,\quad \vartheta^3_{(3)}=-\frac{i}{3}P^0S_2-\frac{i}{3}P^3S_1. \tag{A52}$$

Step 7. Equations of $R+R^2+Q^2$ gravity: Using irreducible curvature and torsion two-forms one must calculate the following quantities:

$\widetilde{\Omega}_{AB}$ two-forms:

$$\widetilde{\Omega}_{AB}=i\lambda S_{AB}+iL_1\Omega^{(1)}_{AB}-iL_2\Omega^{(2)}_{AB}+iL_3\Omega^{(3)}_{AB}+iL_4\Omega^{(4)}_{AB}$$
$$-iL_5\Omega^{(5)}_{AB}+iL_6\Omega^{(6)}_{AB}; \tag{A53}$$

$\widetilde{\Theta}^a$ two-forms:

$$\widetilde{\Theta}^a=iM_1\vartheta^a_{(1)}+\text{c.c.}+iM_2\vartheta^a_{(2)}+\text{c.c.}+iM_3\vartheta^a_{(3)}+\text{c.c.}; \tag{A54}$$

$\widetilde{L}_g$ four-form:

$$\widetilde{L}_g=-2\Lambda v+(i\lambda S_0+\widetilde{\Omega}_0)\wedge\Omega_2+\text{c.c.}-2(i\lambda S_1+\widetilde{\Omega}_1)\wedge\Omega_1$$
$$+\text{c.c.}+(i\lambda S_2+\widetilde{\Omega}_2)\wedge\Omega_0+\text{c.c.}-\tfrac{1}{2}\widetilde{\Theta}^0\wedge\Theta^1$$
$$-\tfrac{1}{2}\widetilde{\Theta}^1\wedge\Theta^0+\tfrac{1}{2}\widetilde{\Theta}^2\wedge\Theta^3+\text{c.c.} \tag{A55}$$

And finally we have the left part of the first $R+R^2+Q^2$ equation:

$$Z_0=d\widetilde{\Theta}^1+\omega_1\wedge\widetilde{\Theta}^1+\text{c.c.}+\omega_2\wedge\widetilde{\Theta}^2+\text{c.c.}+\partial_0\lrcorner\Theta^0\wedge\widetilde{\Theta}^1$$
$$+\partial_0\lrcorner\Theta^1\wedge\widetilde{\Theta}^0-\partial_0\lrcorner\Theta^2\wedge\widetilde{\Theta}^3+\text{c.c.}$$
$$-2[\partial_0\lrcorner\Omega_0\wedge\widetilde{\Omega}_2-2\partial_0\lrcorner\Omega_1\wedge\widetilde{\Omega}_1+\partial_0\lrcorner\Omega_2\wedge\widetilde{\Omega}_0]$$
$$+\text{c.c.}+\partial_0\lrcorner\widetilde{L}_g; \tag{A56}$$

$$Z_1=d\widetilde{\Theta}^0-\omega_1\wedge\widetilde{\Theta}^0+\text{c.c.}-\omega_0\wedge\widetilde{\Theta}^3+\text{c.c.}+\partial_1\lrcorner\Theta^0\wedge\widetilde{\Theta}^1$$
$$+\partial_1\lrcorner\Theta^1\wedge\widetilde{\Theta}^0-\partial_1\lrcorner\Theta^2\wedge\widetilde{\Theta}^3+\text{c.c.}-2[\partial_1\lrcorner\Omega_0$$
$$\wedge\widetilde{\Omega}_2-2\partial_1\lrcorner\Omega_1\wedge\widetilde{\Omega}_1+\partial_1\lrcorner\Omega_2\wedge\widetilde{\Omega}_0]+\text{c.c.}$$
$$+\partial_1\lrcorner\widetilde{L}_g; \tag{A57}$$

$$Z_2=-d\widetilde{\Theta}^3-\omega_2\wedge\widetilde{\Theta}^0+\omega_{\dot 0}\wedge\widetilde{\Theta}^1+(\omega_{\dot 1}-\omega_1)\wedge\widetilde{\Theta}^3$$
$$+\partial_2\lrcorner\Theta^0\wedge\widetilde{\Theta}^1+\partial_2\lrcorner\Theta^1\wedge\widetilde{\Theta}^0-\partial_2\lrcorner\Theta^2\wedge\widetilde{\Theta}^3$$
$$-\partial_2\lrcorner\Theta^3\wedge\widetilde{\Theta}^2-2[\partial_2\lrcorner\Omega_0\wedge\widetilde{\Omega}_2-2\partial_2\lrcorner\Omega_1\wedge\widetilde{\Omega}_1$$
$$+\partial_2\lrcorner\Omega_2\wedge\widetilde{\Omega}_0]-2[\partial_2\lrcorner\Omega_{\dot 0}\wedge\widetilde{\Omega}_{\dot 2}-2\partial_2\lrcorner\Omega_{\dot 1}\wedge\widetilde{\Omega}_{\dot 1}$$
$$+\partial_2\lrcorner\Omega_{\dot 2}\wedge\widetilde{\Omega}_{\dot 0}]+\partial_2\lrcorner\widetilde{L}_g. \tag{A58}$$

The left part of the second $R+R^2+Q^2$ equation is

$$V_0=d\widetilde{\Omega}_0-2\omega_1\wedge\widetilde{\Omega}_0+2\omega_0\wedge\widetilde{\Omega}_1-\tfrac{1}{2}\widetilde{\Theta}^2\wedge\theta^0+\tfrac{1}{2}\widetilde{\Theta}^0\wedge\theta^2, \tag{A59}$$

$$V_1=d\widetilde{\Omega}_1-\omega_2\wedge\widetilde{\Omega}_0+\omega_0\wedge\widetilde{\Omega}_2-\tfrac{1}{4}(\widetilde{\Theta}^0\wedge\theta^1-\widetilde{\Theta}^1\wedge\theta^0$$
$$-\widetilde{\Theta}^2\wedge\theta^3+\widetilde{\Theta}^3\wedge\theta^2), \tag{A60}$$

$$V_2=d\widetilde{\Omega}_2-2\omega_2\wedge\widetilde{\Omega}_1+2\omega_1\wedge\widetilde{\Omega}_2-\tfrac{1}{2}\widetilde{\Theta}^1\wedge\theta^3+\tfrac{1}{2}\widetilde{\Theta}^3\wedge\theta^1. \tag{A61}$$

APPENDIX B: CORRESPONDENCE WITH NOTATIONS OF OTHER AUTHORS

In this appendix we present a translation between our notation and notations of other authors.

All articles cited below and our article use similar Lorentzian signature $(-,+,+,+)$.

Left parts of presented equalities contain quantities in our notation and right parts in notation of cited article.

The following equations refer to Ref. 5:

curvature:

$$R_{abcd}=F_{abcd},\quad R_{ab}=F_{ab},\quad R=F,$$

$$\tfrac{1}{2}(C_{abcd}-C_{adbc})=B_{abcd},$$

$$C_{ab}=I_{ab},$$

$$A_{ab}=E_{ab},$$

$$D_{abcd}=A_{abcd};$$

torsion:

$$Q_{abc}=T_{abc}\,,$$

$$-Q_a=v_a\,,$$

$$\tfrac{1}{3}P_a=a_a\,,$$

$$C_{(ab)c}=t_{abc}\,;$$

parameters of $R+R^2+Q^2$ theory (our $\lambda=1$):

$$\frac{1}{16\pi G}=a,$$

$$\frac{1}{16\pi G}\frac{L_3}{24}=a_6,\quad \frac{1}{16\pi G}\frac{L_2}{2}=a_5,\quad \frac{1}{16\pi G}\frac{L_4}{2}=a_4\,,$$

$$\frac{1}{16\pi G}\frac{L_6}{4}=a_1,\quad \frac{1}{16\pi G}\frac{L_5}{4}=a_3,\quad \frac{1}{16\pi G}\frac{L_1}{4}=\frac{3}{4}a_2\,,$$

$$\frac{1}{16\pi G}\frac{M_1}{4}=\frac{3}{4}\alpha,\quad \frac{1}{16\pi G}\frac{M_2}{6}=\beta,\quad -\frac{1}{16\pi G}\frac{M_3}{6}=\frac{\gamma}{9}.$$

These equations refer to Ref. 12:
curvature:

$$\Omega_{ab}=-R_{ab},$$

$$\Omega_{ab}^{(1)}=-{}^{(1)}R_{ab},\quad \Omega_{ab}^{(2)}=-{}^{(4)}R_{ab},\quad \Omega_{ab}^{(3)}=-{}^{(6)}R_{ab}\,,$$

$$\Omega_{ab}^{(4)}=-{}^{(5)}R_{ab},\quad \Omega_{ab}^{(5)}=-{}^{(2)}R_{ab},\quad \Omega_{ab}^{(6)}=-{}^{(3)}R_{ab}\,;$$

torsion:

$$\Theta^a=-T^a,$$

$$\Theta^a_{(i)}=-{}^{(i)}T^a,\quad i=1,2,3;$$

parameters of $R+R^2+Q^2$ theory:

$$\frac{1}{16\pi G}=\frac{1}{2l^2},\quad \lambda=-A_0,\quad \Lambda=-\Lambda,$$

$$\tfrac{1}{2}M_i=A_i,\quad i=1,2,3,$$

$$\frac{1}{2}L_1=\frac{l^2}{\kappa}B_1,\quad \frac{1}{2}L_2=\frac{l^2}{\kappa}B_4,\quad \frac{1}{2}L_3=\frac{l^2}{\kappa}B_6\,,$$

$$\frac{1}{2}L_4=\frac{l^2}{\kappa}B_5,\quad \frac{1}{2}L_5=\frac{l^2}{\kappa}B_2,\quad \frac{1}{2}L_6=\frac{l^2}{\kappa}B_3\,.$$

This final set of equations refers to Ref. 7:
curvature:

$$R_{ab\mu\nu}=F_{\mu\nu ba},\quad R_{ab}=F_{ba},\quad R=F;$$

torsion:

$$Q_{a\mu\nu}=F_{\nu\mu a}\,;$$

parameters of $R+R^2+Q^2$ theory:

$$\frac{1}{16\pi G}=\frac{1}{2\chi l^2},\quad \Lambda=-\frac{\chi}{\mu l^2},$$

$$\frac{1}{16\pi G}a_i=\frac{1}{4l^2}d_i,\quad i=1,2,3,$$

$$\frac{1}{16\pi G}l_1=-\frac{1}{4\kappa},\quad \frac{1}{16\pi G}l_2=-\frac{1}{4\kappa}f_2,$$

$$\frac{1}{16\pi G}l_3=-\frac{1}{4\kappa}f_1\,,$$

$$\frac{1}{16\pi G}l_4=-\frac{1}{4\kappa}f_3,\quad \frac{1}{16\pi G}l_5=-\frac{1}{4\kappa}f_4,$$

$$\frac{1}{16\pi G}l_6=-\frac{1}{4\kappa}f_5\,.$$

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