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Optimal analytic extrapolation for the scattering amplitude from cuts to interior points

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Given a data function together with an error corridor for the scattering amplitude along some finite part of the cuts, one can construct effectively the whole set of analytic functions ("admissible amplitudes"), compatible with these conditions and bounded by a certain number M on the remaining part of the cuts. Depending on the actual value of an important constant ϵ_0 computed from the data function and the bound M, this set may be void. If not, in every point of the cut plane the set of values of the admissible amplitudes fills densely a circle; explicit formulas are given for its radius $\hat{\eta}(z)$ and center $\hat{f}(z)$, the latter being the best possible estimate for the whole set. In contrast to the linear extrapolation obtained by Poisson weighted dispersion relations, here nonlinear functional methods were used. This paper contains an appendix written by Professor C. Foias, on some functional analytical methods used in connection with the computation of the numerical value of the constant ϵ_0.

I. INTRODUCTION

In realistic particle-physics problems, information is available only along some limited parts of the cuts of the energy (or momentum transfer) complex plane of the scattering amplitude, and the problem one is usually faced with is to extract from this limited, error-affected knowledge, information on the behavior of the amplitude of other reactions or at energies outside the initial range.[1-5]

This is in general an ill-posed mathematical problem, in the sense that small changes (errors) in the input data could provide incontrollable responses in the output. Nevertheless, following an idea first emphasized[6] at the 1969 Lund Conference, Carleman weight functions can be used to write down[7,8] those dispersion relations (sum rules) which exploit this limited, error affected information in the most economical way, in the sense that any other weighted dispersion relation would lead to greater error-bounds in the results. As already emphasized in Ref. 7, the dispersion relations do not exhaust the optimization problem of the extrapolation procedures: The aim of the present paper is precisely to find by nonlinear methods this absolute optimum, as well as to construct *all possible* analytic functions $f(z)$ compatible with some given, error-affected, histogram $h(z)$, on some limited part of the energy (or momentum) cut complex plane. [z is here the relevant (energy, momentum, cosine, and so on) variable and $f(z)$ is the amplitude itself, or one of its combinations with some given complex functions].

We have perhaps all experienced the trying situation of being asked by some experimentalist friend to find a close form, say for the transverse momentum dependence of some cross-section, in terms of "usual" functions—cosines, logarithms, and so on. If possible, even exhibiting a Regge behavior. To our question "what for?", his answer would probably be "in order to have some easy-to-remember formula instead of these long intricate tables"; but this answer usually hides also the secret hope that our formula could apply to a much wider range of momenta than that where the measurements were made, representing, so to say, an "objective" physical reality.

This standpoint might seem naive, but under a more attentive consideration one sees that it prevails over the whole of theoretical physics. Indeed, we ought to remember that each successful theory is, in a metaphorical sense, a curve which lies inside the error corridor along the whole range of the present physical informations. (This goes for succesful theories only! For, usually, we are content with theories which only partially pass through the present error corridor!) Such a theory makes definite predictions also outside the range of the actual physical information, but, of course, one can have no confidence in these "predictions" for, in general, there are many possible theories passing through the same error corridor whose predictions can differ considerably outside the range of the present information. This is a serious problem, encountered, of course, not only in physics but in every branch of science.

A sensible solution to this problem would be to work simultaneously with the *whole set* of theories passing through the error corridor. The drawback of such an

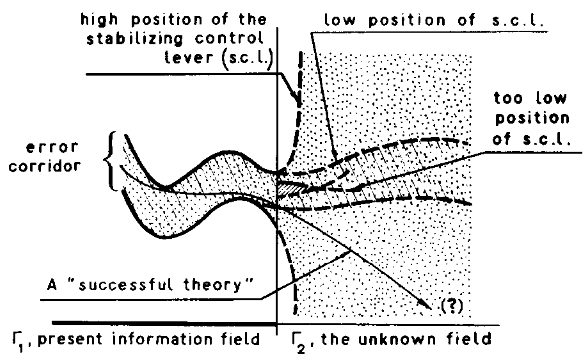

FIG. 1.

approach is, of course, evident, since these theories can in principle make arbitrary predictions outside the range of the present informations (see Fig. 1). In the good old days when the theorist felt the results before the actual calculations were done, there was an easier choice between the possible outcomes. Nowadays the situation has changed drastically, first of all because of the considerable broadening of the range of choice owing to the informational boom but also due to the ever increasing degree of abstraction. Research has become *more and more indirect*, with the consequence that the leading principles are now far outside the reach of the physical measurements. Even the most common concepts of theoretical physics, such as particles, resonances, exchanges and so on, are beyond the actual experimental range, and Landau, for instance, raised the question whether it would not be wiser to leave some of these principles out. (He included even the concept of "interaction" among the other presumable ill concepts.) Indeed, Calucci, Fonda and Ghiraldi[9] showed that a suitably chosen nonresonant background can simulate as well as one would like a Breit—Wigner resonance curve, so that those who are fitting cross-section bumps with two parameter resonance formulas get exactly what they had expected, from the beginning, to get!

In principle the new scientific approach would have to cope with these two, interwoven problems: (i) working with the whole set of theories passing through the error corridor of the present knowledge; (ii) finding new theoretical concepts to replace the dated ones and *controlling the behavior* of the set of possible theories *outside* the range of our present knowledge. Of great importance among these concepts are those which control "the opening" of the set of possible theories (see Fig. 1): We shall call them "regularizers" or "stabilizing" control levers. (Although the former term is already used in the theory of ill-posed problems of mathematical physics,[10] we shall give preference to the latter because of its more specific content.)

To make things more palpable, consider the problem of solving in some function space the equation $Af=h$, with f the input unknown function and h the experimental data, from which we try to deduce f. We assume that A is a continuous operator, with a unique inverse A^{-1}. The uniqueness of the inverse does not yet imply, in practice, the unique determination of f from h, since, if A^{-1} is discontinuous, arbitrarily small variation in h will cause uncontrollably large changes in f. This is a very frequently met situation, e.g., A is the operator of taking the restriction of an analytic function to an open curve or to a part of the boundary (note that A^{-1} is then the operator of analytic continuation of the function to all interior points, and hence, is unique); or consider the Fredholm operator of the first kind encountered in the theory of diffusion (solving backwards the heat equation), or in geophysics (solving in homogeneous Laplace equations with data given on an open boundary), etc.

The problem is: what complementary conditions should be imposed in order to stabilize the problem? Obviously a stabilizing lever of this problem would be every condition which restricts the set in which one searches for the solution (the admissible f's) to a compact set, since that is a sufficient condition to make A^{-1} continuous too.

The problem of devising a general stable approach for particle physics, as a whole, might seem a formidable task; nevertheless, special problems can be treated rather easily. An instructive example is provided by the problem of finding *all possible* amplitudes $f(z)$ passing for $z\in\Gamma_1$ (the actual range of measurements) through the error corridor

$$|f(z)-h(z)|_{z\in\Gamma_1}<\epsilon, \tag{1.1}$$

where the complex function $h(z)$—the experimentally measured histogram—is given along the part Γ_1 of the cuts of the analyticity domain (D) of $f(z)$. The analyticity of $f(z)$ in (D) represents itself a stabilizing concept, but alone it is unsufficient, as there are many analytic functions satisfying (1.1) which, however, differ arbitrarily much outside Γ_1. To turn the set of the admissible functions into a compact, we shall introduce also the stabilizing parameter M, adding to (1.1) the boundness condition

$$|f(z)|<M,\quad z\in\Gamma_2, \tag{1.2}$$

where $\Gamma=\Gamma_1+\Gamma_2$ represents the whole boundary (cuts) of the complex cut plane (D). As it was stated above, the aim of this paper is to construct effectively every possible analytic function $f(z)$ in D, satisfying the inequality (1.1) on Γ_1—$h(z)$ and ϵ being given. It will be shown that these functions $f(z)$ are labelled, not only by the value of the stabilizing parameter M, but also by some "running index" $\psi(\zeta)$, more precisely by a general unimodular function in the unit disk, to be defined later. The difference between the role of the "control level" M and the "running index" $\psi(\zeta)$ will appear clearly throughout this paper.

As a by-product of this theory, we shall find the value of the center $\hat{f}(z)$ of the set of the values of all possible $f(z)$ in every given point z, this center $\hat{f}(z)$ being the best estimate ever found for the extrapolation of the scattering amplitudes satisfying (1.1). A comparison with the optimal dispersion relation yield $\hat{h}(z)$ (see Refs. 7 and 8) is then performed.

II. DESCRIPTION OF THE PROBLEM

Our problem amounts to the construction of *analytic* functions to be used in the analytic continuation of the physical data in the holomorphy domain of the scattering amplitude. For what follows, it is convenient to transform this holomorphy domain[11] into the unit circle of a suitably chosen conformal variable $\zeta(z)$ (for technical details see Sec. 2 of Ref. 7), the physical region Γ_1, where the measurements were performed [where the function $h(\zeta)$ is given] being depicted on the right semicircle $\zeta=e^{i\theta}$, $-\pi/2<\theta<\pi/2$ (see Fig. 2). Following Ref. 6, in order to express the conditions (1.1) and (1.2) as a single one, we multiply both the amplitude $f(\zeta)$ $(\zeta\in D)$ and the histogram $h(\zeta)$ $(\zeta\in\Gamma_1)$ with a suitable chosen function of Carleman type

$$C_0(M/\epsilon;\zeta)=\exp\{-\ln(M/\epsilon)[\omega(\zeta)+i\tilde{\omega}(\zeta)]\}, \tag{2.1}$$

$$\tilde{f}(\zeta)=C_0(M/\epsilon;\zeta)f(\zeta), \tag{2.2}$$

$$\tilde{h}(\zeta)=C_0(M/\epsilon;\zeta)h(\zeta), \tag{2.3}$$

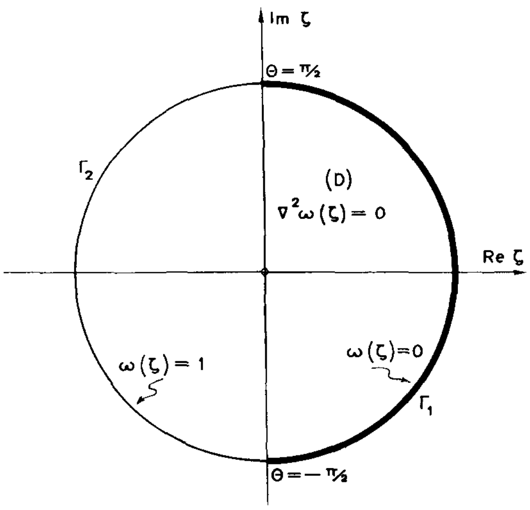

FIG. 2. The unit $\zeta(z)$ circle in which the z-cut plane was mapped. If the z cuts are $(-\infty, z_1)$ and (z_2, ∞) with data given along $\Gamma_1=(z_2, z_3)$, then $\zeta(z)=\{1-(1-u^2)^{1/2}\}/u$, where $u=[(z_1+z_2-2z_3)z+z_3(z_1+z_2)-2z_1z_2]/[(z-z_3)(z_2-z_1)]$.

where $\omega(\zeta)$ is a potential (a harmonic measure) defined to be *zero* on Γ_1 and *one* on the remainder part, Γ_2, of the cuts $\Gamma=\Gamma_1+\Gamma_2$,

$$\begin{aligned}\nabla^2\omega(\zeta)&=0 \quad \text{for } \zeta\in D,\\ \omega(\zeta)&=0 \quad \text{for } \zeta\in\Gamma_1,\\ \omega(\zeta)&=1 \quad \text{for } \zeta\in\Gamma_2,\end{aligned} \tag{2.4}$$

and where $\tilde{\omega}(\zeta)$ is its harmonic conjugate (the stream line function). For the case depicted in Fig. 2,

$$\begin{aligned}\omega(\zeta)+i\tilde{\omega}(\zeta)&=\tfrac{1}{2}-(2/\pi)\arctan\zeta\\ &\equiv\tfrac{1}{2}+(i/\pi)\ln[(1+i\zeta)/(1-i\zeta)].\end{aligned} \tag{2.5}$$

As, owing to (2.1) and (2.4), the modulus of $C_0(M/\epsilon;\zeta)$ is equal to 1 on Γ_1 and to ϵ/M on Γ_2, taking *by definition* $h(\zeta)$ equal to zero on Γ_2,

$$\tilde{h}(\zeta)=h(\zeta)\overset{\text{DEF}}{=}0 \quad \text{for } \zeta\in\Gamma_2, \tag{2.6}$$

[initially the histogram was defined only on the "known cut", so that we are free to complement this definition with Eq. (2.6)] the conditions (1.1) and (1.2) are, equivalent with the unique one,

$$|\tilde{f}(\zeta)-\tilde{h}(\zeta)|_{\zeta\in\Gamma_1+\Gamma_2}<\epsilon \tag{2.7}$$

for the weighted amplitude [see Eqs. (2.2) and (2.3)] $\tilde{f}(\zeta)$.

We are thus left with the well-stated problem that, given a function $\tilde{h}(\zeta)$ on the unit circle $\Gamma=\Gamma_1+\Gamma_2$,

$$\begin{aligned}\tilde{h}(e^{i\theta})&=h(e^{i\theta})\exp[-(i/\pi)\ln(M/\epsilon)\ln\tan(\pi/4-\theta/2)]\\ &\quad\text{for } -\pi/2<\theta<\pi/2,\\ \tilde{h}(e^{i\theta})&=0 \text{ for } \pi/2<\theta<3\pi/2,\end{aligned} \tag{2.8}$$

to find *all* functions $\tilde{f}(\zeta)$ *analytic* inside the unit circle D which approximate $\tilde{h}(\zeta)$ on Γ according to (2.7).

The stabilizing role of the parameter M is now transparent; indeed, we shall show in Sec. 4 that for every two "admissible" functions $\tilde{f}_1(\zeta)$ and $\tilde{f}_2(\zeta)$ we have

$$|\tilde{f}_1(\zeta)-\tilde{f}_2(\zeta)|\le 2\eta(\zeta), \tag{2.9}$$

so that, owing to (2.2), the difference between two admissible amplitudes cannot exceed

$$|f_1(\zeta)-f_2(\zeta)|\le 2\eta(\zeta)/|C_0(M/\epsilon;\zeta)|. \tag{2.10}$$

From (2.7) it is obvious that we have at least $\eta(\zeta)\le\epsilon$ and in Sec. 4 an algorithm will be given for the actual form of $\eta(\zeta)$.

In all previous extrapolation procedures we have, so far, assumed the existence of at least one analytic function (the amplitude itself) satisfying conditions (1.1) and (1.2), or the equivalent condition (2.7). Nevertheless, for some histograms $\tilde{h}(\zeta)$ and for some ϵ there *may be no analytic function* $\tilde{f}(\zeta)$ at all satisfying the condition (2.7). For instance, let us suppose that the Carleman weighted histogram $\tilde{h}(\zeta)$ has the special form

$$\tilde{h}(e^{i\theta})=\tilde{h}_1(e^{i\theta})+e^{-i\theta}, \tag{2.11}$$

where

$$\tilde{h}_1(e^{i\theta})=\sum_{n=0}^{\infty}c_ne^{in\theta} \tag{2.12}$$

is the limit of a function $\tilde{h}_1(\zeta)$ holomorphic inside (D). It can readily be shown that there are no holomorphic functions inside (D) which can approximate $\tilde{h}$ given by (2.11) with an error ϵ smaller than one. Indeed, putting

$$\tilde{f}(\zeta)=\tilde{h}_1(\zeta)-\chi_1(\zeta), \tag{2.13}$$

condition (2.7) reads

$$|-\chi_1(\zeta)-1/\zeta|_{\zeta\in\Gamma}\le\epsilon. \tag{2.14}$$

But on the unit circle we have

$$|\chi_1(\zeta)+1/\zeta|\equiv|1+\zeta\cdot\tilde{\chi}_1(\zeta)| \quad \text{for } |\zeta|=1,$$

and, as $1+\zeta\cdot\chi_1(\zeta)$ (in contradistinction to χ_1+1/ζ) is holomorphic inside (D) and equal to 1 at the origin, from the principle of the maximum of the modulus it follows that the value of ϵ of (2.14) cannot be smaller than 1!

Coming back to the general problem, we note that under very general conditions (Fourier expandibility of $\tilde{h}$ on the unit circle) the histogram can be cast into the form

$$\tilde{h}(\zeta)=\tilde{h}_1(\zeta)+\tilde{h}_2(\zeta), \tag{2.15}$$

where $\tilde{h}_1$ and $\tilde{h}_2$ are limits of functions holomorphic respectively inside and outside the unit circle:

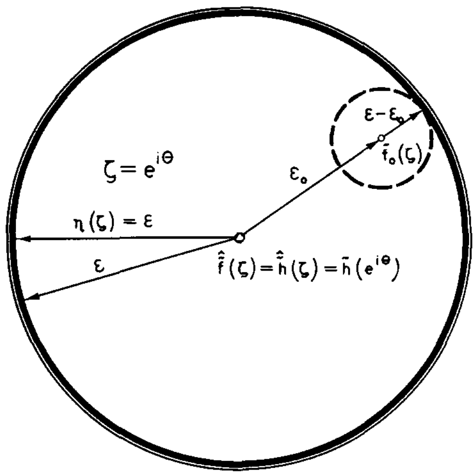

FIG. 3. In a *boundary point* $|\zeta|=1$ the circle filled by the admissible amplitudes coincides with the circle of radius ϵ, the "Nevanlinna bound" for the error encountered in Poisson weighted dispersion relations (Ref. 7). The point ζ being on the boundary, the harmonic function $\hat{h}(\zeta)$ coincides also with the w weighted histogram $\tilde{h}(\zeta)$. In every boundary point, the distance between $\hat{h}(\zeta)$ and the minimal function $\tilde{f}_0$ is ϵ_0. Admissible amplitudes which are nearer to the center at a particular point $\zeta \in \Gamma_1$, have to go away from it at other boundary points $|\zeta|=1$.

$$\tilde{h}_1(\zeta)=\sum_{n=0}^{\infty} c_n \zeta^n, \tag{2.16a}$$

$$\tilde{h}_2(\zeta)=\sum_{m=1}^{\infty} c_{-m} \zeta^{-m}, \tag{2.16b}$$

$$\tilde{h}(e^{i\theta})=\sum_{-\infty}^{\infty} c_n e^{in\theta}=\tilde{h}_1(e^{i\theta})+\tilde{h}_2(e^{i\theta}), \tag{2.16c}$$

where

$$c_{-n}=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} h(e^{i\theta})\exp\left\{\frac{-i}{\pi}\ln\left(\frac{M}{\epsilon}\right)\ln\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]\right\}e^{in\theta}\,d\theta. \tag{2.16d}$$

It is clear that in our problem, the trouble comes from the nonanalytic component $\tilde{h}_2(\zeta)$. Indeed, again writing $\tilde{f}(\zeta)$ in the form (2.13), we are left to find those holomorphic functions $\chi_1(\zeta)$ which according to (2.7) satisfy

$$|\chi_1(\zeta)+\tilde{h}_2(\zeta)|_\Gamma \leq \epsilon. \tag{2.17}$$

From the previous example [in which $\tilde{h}_2$ was set equal to $1/\zeta$, see (2.11)] it is apparent that we cannot approximate arbitrarily well the nonanalytic component $\tilde{h}_2(\zeta)$ of the histogram by analytic functions $-\chi_1(\zeta)$; in other words, for a given $\tilde{h}_2(\zeta)$ there exists a number ϵ_0 so that for every holomorphic function $\chi_1(\zeta)$ in D we have

$$\max_{\zeta\in\Gamma}|\chi_1(\zeta)+\tilde{h}_2(\zeta)| \geq \epsilon_0[\tilde{h}_2]. \tag{2.18}$$

That holomorphic function which reaches in (2.18) the lower bound ϵ_0 will be called "the minimalizing analytic function" and will be denoted by $\chi_1^0(\zeta)$ [in the previous example we had $\epsilon_0=1$ and $\chi_1^0(\zeta)\equiv 0$]. We should also like to stress that unlike ϵ, the quantity ϵ_0 does not depend only on the accuracy of the experiment, but on the experiment itself, ϵ_0 being a functional of $\tilde{h}_2(\zeta)$. Of course, in order to have at least one analytic function satisfying (2.7), ϵ has to be greater than ϵ_0,

$$\epsilon \geq \epsilon_0[\tilde{h}_2]. \tag{2.19}$$

Unless we are in the exceptional case $\epsilon=\epsilon_0$, the "minimal amplitude", constructed with the minimalizing function χ_1^0,

$$\tilde{f}_0=\tilde{h}_1-\chi_1^0 \tag{2.20}$$

is not the best approximation ["$\hat{\tilde{f}}(\zeta)$"] of all analytic functions in (D) satisfying (2.7): Indeed, if ϵ is strictly greater than ϵ_0, the set of the admissible function $\tilde{f}(\zeta)$ for every $\zeta\in\bar{D}$ fills densely a disk [the boldface circle in Figs. 3 and 4, corresponding, respectively, to the cases when ζ is a boundary and an interior point of (D)]. Obviously the center $\hat{\tilde{f}}(\zeta)$ of the circle represents for each ζ the best approximation, but as it will be proved in Sec. 4, this center *does not* coincide with the $f_0(\zeta)$.

The reader, making the intuitive assumption that the histogram is the limit of an analytic function (the amplitude itself), might get the feeling that all this trouble and soul searching is due only to the inaccuracy of the experiment which would produce a (small) nonanalytic term $\tilde{h}_2(\zeta)$. Contrary to the common belief, $\tilde{h}_2(\zeta)$ is by no means small (regardless of the experimental accuracy), being the direct product of a well-known theorem of Fourier decomposition applied to functions

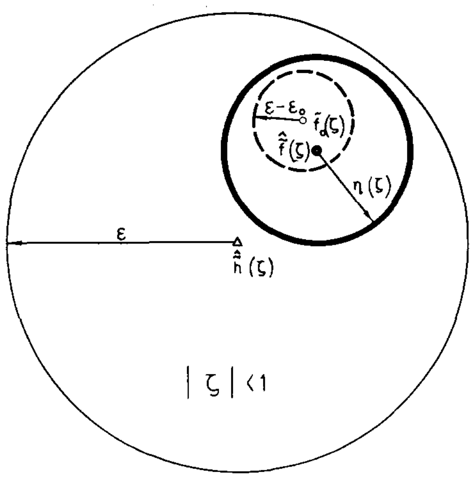

FIG. 4. Typical situation in an interior point $|\zeta|<1$. The values of all weighted holomorphic functions $\tilde{f}(\zeta)$ compatible with conditions (1.1) and (1.2) fill the boldface circle of radius $\eta(\zeta)$ around $\hat{f}(\zeta)$. There are no admissible amplitudes outside it, so that $\hat{f}(\zeta)$ is the best estimate for a random-taken amplitude. The dashed circle of radius $\epsilon-\epsilon_0$ around $\tilde{f}_0(\zeta)$ is still contained in the latter one, so that $\epsilon-\epsilon_0<\eta(\zeta)<\epsilon$. Especially for ϵ_0 close to ϵ, $\hat{f}(\zeta)$ may differ considerably from $\hat{h}(\zeta)$.

which are identically zero over a segment (Γ_2) of the boundary. (If a Fourier expandable function is identically zero over some segment of its period, the negative and positive frequencies are simultaneously present.) In other words $\tilde{h}_2(\zeta)$ is the consequence of our incomplete knowledge, reflecting our complete lack of knowledge on Γ_2 rather than the lack of accuracy on Γ_1!

Previous methods of extrapolation variously took into account this nonholomorphic component. For instance, the Carleman weighted dispersion relations,[6] written with a conventional Cauchy kernel,

$$\check{\tilde{h}}(\zeta)=\frac{1}{2\pi i}\int_\Gamma \frac{\tilde{h}(\zeta')}{\zeta'-\zeta}\,d\zeta', \tag{2.21a}$$

are completely insensitive to the presence of $\tilde{h}_2$, as

$$\check{\tilde{h}}(\zeta)\equiv\tilde{h}_1(\zeta),\quad \check{\tilde{h}}_2(\zeta)\equiv 0. \tag{2.21b}$$

On the other hand, the Poisson kernel used in Refs. 7 and 8 transforms $\tilde{h}_2(\zeta)$ into the complex valued harmonic (but not holomorphic!) function

$$\hat{\tilde{h}}_2(\zeta)=\sum_{p=1}^{\infty} c_{-p}\zeta^{*p}\quad (\zeta^* \text{ being the complex conjugate of } \zeta), \tag{2.22}$$

so that the extrapolated function[7,8]

$$\hat{\tilde{h}}(\zeta)=\frac{1}{2\pi i}\int \tilde{h}(\zeta')d(G(\zeta,\zeta')+iH(\zeta,\zeta'))\equiv\tilde{h}_1(\zeta)+\hat{\tilde{h}}_2(\zeta) \tag{2.23}$$

provides a "hundred per cent approximation" of the histogram on Γ. Indeed, owing to the limiting properties of harmonic functions on the boundary, we have on Γ

$$|\hat{\tilde{h}}(\zeta)-\tilde{h}(\zeta)|_{\zeta\in\Gamma}=0, \tag{2.24}$$

but inside the unit circle the harmonic function $\hat{\tilde{h}}(\zeta)$ could differ considerably (of course, less than ϵ!) from the values of the admissible holomorphic functions $f(\zeta)$. This is especially apparent from the previous example: If in Eq. (2.14) one sets ϵ equal to 1, the unique analytic function satisfying (2.14) is the minimalizing function $\chi_1^0(\zeta)\equiv 0$, i.e., the amplitude coincides both with the unique admissible function $\tilde{f}_0(\zeta)$ (the minimal amplitude) and with the Cauchy weighted integral $\check{\tilde{h}}_1(\zeta)$, while the corresponding Poisson extrapolation $\hat{\tilde{h}}=\tilde{h}_1(\zeta)+\zeta^*$ represents, with the exception of the origin $\zeta=\zeta^*=0$, a worse approximation. The coincidence between the minimal amplitude $\tilde{f}_0(\zeta)$ and the weighted Cauchy dispersion integral $\tilde{h}_1(\zeta)$ which occured in this example is purely incidental, as, in general, $\tilde{f}_0(\zeta)$, $\hat{\tilde{h}}(\zeta)$ and, especially, $\tilde{h}_1(\zeta)$ could differ considerably among them. This is especially apparent from the well-known example of the step function

$$\tilde{h}(e^{i\theta})=\begin{matrix}0 & \text{for } -\pi<\theta<0\\ 1 & \text{for } 0<\theta<\pi\end{matrix}$$
$$=1+\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{\sin(2p-1)\theta}{2p-1}, \tag{2.25}$$

which also emphasizes the importance of the $\tilde{h}_2$ term, as the corresponding

$$\tilde{h}_1(\zeta)=1-i\sum_{p=1}^{\infty}\frac{\zeta^{2p-1}}{(2p-1)} \tag{2.26}$$

becomes extremely great in the neighborhoods of the points $\zeta=1$ and $\zeta=-1$, while the sum $\tilde{h}_1+\hat{\tilde{h}}_2$—and, hence, also the weighted Poisson integral $\hat{\tilde{h}}(\zeta)=\tilde{h}_1(\zeta)+\hat{\tilde{h}}_2(\zeta)$, as well as the minimal amplitude $f_0(\zeta)$ and all the other admissible amplitudes—remain finite.

If, as in the step-function example, the Carleman weighted Cauchy dispersion integral could differ in an incontrollable manner from the true amplitude, the Carleman weighted Poisson dispersion relation always secures the error bound ϵ prescribed by the maximum of the modulus principle—or, if one comes back to the unweighted (real) amplitude $f(\zeta)$, (2.2), the Poisson extrapolation

$$\hat{h}(\zeta)\overset{\text{DEF}}{=}\hat{\tilde{h}}(\zeta)/C_0(M/\epsilon;\zeta) \tag{2.27}$$

secures the "Nevanlina bound"

$$|\hat{h}(\zeta)-f(\zeta)|_{\zeta\in\bar{D}}<\epsilon/C_0(\zeta)\equiv\epsilon\exp[\ln(M/\epsilon)\omega(\zeta)]$$
$$=\epsilon^{1-\omega(\zeta)}M^{\omega(\zeta)}. \tag{2.28}$$

As we have already stated above and as it will be proved in Sec. 4, all admissible weighted amplitudes $\tilde{f}(\zeta)$ fill densely, for every ζ, a disk of radius $\eta(\zeta)$, contained, of course, in the circle of radius ϵ centered around $\hat{\tilde{h}}(\zeta)$ (see Figs. 3 and 4).

Without going into the details of Sec. 4, we can immediately show that the circle of the admissible weighted amplitudes contains a circle of (constant) radius $\epsilon-\epsilon_0$, centered around the minimal amplitude $\tilde{f}_0(\zeta)$; for each function

$$\tilde{f}_{\alpha,\varphi}(\zeta)=\tilde{f}_0(\zeta)+\alpha(\epsilon-\epsilon_0)e^{i\varphi}\quad(0\le\alpha<1,\ 0\le\varphi<2\pi), \tag{2.29}$$

where α, φ are constants, is holomorfic inside D and satisfies condition (2.7) (the dashed line circles of Figs. 3 and 4). This provides us with a simple criterion to decide which method to use in some precise practical application. For that we need the numerical value only of ϵ_0 which will be computed in Sec. 3 [see Eq. (3.13)]:

(1) If $\epsilon_0\ll\epsilon$ so that the dashed circle of radius $\epsilon-\epsilon_0$ [and thus, *a fortiori*, the bold face circle of radius $\eta(\zeta)$, $\epsilon-\epsilon_0<\eta<\epsilon$] fills most of the circle of radius ϵ, the function $\hat{\tilde{h}}(\zeta)$ represents fairly well the middle point $\bar{\tilde{f}}(\zeta)$ of all possible weighted amplitudes $\tilde{f}(\zeta)$ (the center of the bold face circle), and thus the Carleman weighted dispersion relation [see Eqs. (2.23), (2.27) and Ref. 7] would give very satisfactory results.

(2) If the accuracy of the data along the initial physical region Γ_1 is high enough so that ϵ is small and only slightly greater than ϵ_0, it could happen that all admissible amplitudes would pile up in a very small region containing the minimal weighted amplitude as in Eq. (2.14). Therefore, if the radius $\epsilon-\epsilon_0$ of the dashed circle is small in comparison to ϵ, one has to compute the radius $\eta(\zeta)$ [Eq. (4.20)] and, if it differs also considerably from ϵ, find the center $\bar{\tilde{f}}(\zeta)$ of the circle of all the weighted admissible amplitudes. Of course, in both

cases the optimal approximation for the entire set of amplitudes is given by the nonanalytic function:

$$\hat{f}(\zeta)=\hat{\tilde{f}}(\zeta)/C_0(M/\epsilon;\zeta). \tag{2.30}$$

In Sec. 3 we shall deal with the problem of the minimal weighted amplitude $\tilde{f}_0(\zeta)$ and the computation of the constant ϵ_0. In Sec. 4 we shall then write down all possible amplitudes $f(\zeta)$ compatible with a given ϵ, as well as the explicit form of the (nonanalytic) function $\hat{f}(\zeta)$ giving the value of the center of the set of all the admissible amplitudes, as well as the value of the radius $\eta(\zeta)/C_0(M/\epsilon;\zeta)$ of this set. So far, the width of the error channel ϵ was kept constant: The physically important variable-error case is discussed in the concluding Sec. 5, where an out line of the L^2-norm problems is also given, together with a discussion about the two minimal amplitudes, $f_{\epsilon_0}(\zeta)$ (which approximates the best the histogram on Γ_1 *at given* M) and $f_{M_0}(\zeta)$ (the amplitude with least modulus M_0 on Γ_2, *at given* ϵ). Both extremal amplitudes, $f_{\epsilon_0}(\zeta)$ and $f_{M_0}(\zeta)$, as well as $f_0(\zeta)$, are contained in the circle of radius $\eta(\zeta)/C_0(M/\epsilon;\zeta)$ around the optimal function $\hat{f}(\zeta)$.

III. COMPUTATION OF $\epsilon_0[h;M/\epsilon]$

One of the problems described in Sec. 2 we shall now have to deal with is the construction of the holomorphic function $-\chi_1^0(\zeta)$, which approximates on the unit circle Γ, in the best way, the nonholomorphic part $\tilde{h}_2(\zeta)$ of the weighted histogram, i.e.,

$$\max_{\substack{\zeta\in\Gamma \\ \chi_1\to\chi_1^0}} |\chi_1(\zeta)+\tilde{h}_2(\zeta)|\to\min=\epsilon_0. \tag{3.1}$$

This is an important problem which has been under scrutiny for a long while by mathematicians, although it is virtually solved by the Schwartz lemma in the form of Pick and Schur,[12] which is nothing but a special case of the Lindelöf principle. Nevertheless, the last-time powerful function-analytical methods (Nehari,[13] and Krein[14]) have had a great impact on this problem, and in the present section we shall show how one can compute the constant ϵ_0 from Eq. (3.1) using the Foiaş–Nagy lifting theorem.[15] More precisely, we shall give here an outline of the proofs which can be found in their full extent in Appendix A. As the construction of the function $\chi_1^0(\zeta)$, once the numerical value of ϵ_0 is known, is quite similar to that of all other $\chi_1(\zeta)$'s satisfying (2.17)–(2.19), this second problem will be postponed until the next section.

Let the $\chi(e^{i\theta})$ be different functions, defined on the unit circle Γ, whose *negative-frequency* Fourier coefficients coincide with the *negative-frequency* coefficients of the weighted histogram $\tilde{h}(e^{i\theta})$:

$$Q_-\chi(e^{i\theta})=Q_-h(e^{i\theta}), \tag{3.2a}$$

where Q_- is the projection operator on the space $L^2\ominus H^2$ of functions with negative frequencies only. We shall write similarly to (2.16),

$$\chi(e^{i\theta})=\chi_1(e^{i\theta})+\chi_2(e^{i\theta}), \tag{3.3a}$$

with

$$\chi_1(e^{i\theta})=(1-Q_-)\chi(e^{i\theta}) \quad\text{and}\quad \chi_2(e^{i\theta})=Q_-\chi(e^{i\theta}), \tag{3.3b}$$

so that (3.2a) is nothing but

$$\chi_2(e^{i\theta})\equiv\tilde{h}_2(e^{i\theta}). \tag{3.2b}$$

Let us now remark that, as the L^∞ norm of a function,

$$\|\chi(\zeta)\|_{L^\infty}\equiv\operatorname*{ess.\,sup}_{\zeta\in\Gamma}|\chi(\zeta)|, \tag{3.4}$$

is always greater or equal to its L^2 norm, we have

$$\begin{aligned}\epsilon_0=\|\chi^0(\zeta)\|_{L^\infty}&\geq\|\chi^0(\zeta)\|_{L^2}\equiv\|\chi_1^0(\zeta)\|_{L^2}+\|\chi_2^0(\zeta)\|_{L^2}\\ &\geq\|\chi_2^0(\zeta)\|_{L^2}.\end{aligned} \tag{3.5}$$

Owing to (3.2), we get

$$\epsilon_0\geq\|\tilde{h}_2(\zeta)\|_{L^2}\equiv\Big(\sum_{-1}^{-\infty}|c_n|^2\Big)^{-1/2}, \tag{3.6}$$

which, by the way, proves the existence of ϵ_0 as a positive, not vanishing constant.

Now, as it is less easy to handle the L^∞ norm than the L^2 one, following C. Foiaş, one can transpose the whole problem into an L^2-norm problem for the suitable chosen *operators* Y_χ closely connected to the functions $\chi(e^{i\theta})$. Namely, if $\varphi(e^{i\theta})$ is some general L^2 function defined on Γ, we define

$$Y_\chi\varphi(e^{i\theta})\equiv\check{\chi}(e^{i\theta})^*U^+\check{\varphi}(e^{i\theta}), \tag{3.7a}$$

where we have denoted by $\check{\varphi}$ the "θ-reflected" function

$$\check{\varphi}(e^{i\theta})\equiv\varphi(e^{-i\theta}), \tag{3.7b}$$

and by U the multiplication operator with the factor $e^{i\theta}$:

$$\begin{aligned}U\varphi'(e^{i\theta})&=e^{i\theta}\varphi'(e^{i\theta}),\\ U^+\varphi'(e^{i\theta})&=e^{-i\theta}\varphi'(e^{i\theta}).\end{aligned} \tag{3.7c}$$

As neither of the two operations (3.7b) or (3.7c) change the norms, it is clear that the L^2 norm of *the operator* Y_χ coincides with the maximum of the function $\chi(e^{i\theta})$ on Γ, i.e.,

$$\|Y_\chi\|_{L^2}\equiv\|\chi(e^{i\theta})\|_{L^\infty}; \tag{3.8}$$

The additional unit-norm operators contained in Y_χ apart from the function $\chi(e^{i\theta})$ were taken precisely to secure the commutation relation

$$Y_\chi U^+=UY_\chi, \tag{3.9}$$

which is essential for the use of the Foiaş–Nagy lifting theorem (see Appendix A).

The crucial points of the proof (see Appendix A) are now the following two:

Firstly, the special form (3.7a) of Y_χ enables us to show that the restriction X of the operator[16] $Q_-Y_\chi^+$ on the subspace $L^2\ominus H^2$ of the negative frequency functions $\varphi_2=Q_-\varphi$,

$$X\varphi_2=Q_-Y_\chi^+\varphi_2=Q_-\chi(e^{i\theta})U^+\check{\varphi}_2, \tag{3.10}$$

depends solely [see further Eq. (3.15)] *on the negative frequency part* $\chi_2(e^{i\theta})$ of the function $\chi(e^{i\theta})$; its L^2 norm—which, as we shall show, coincides with ϵ_0 and which, as X itself, is also completely determined by the Fourier coefficients $c_{-1},c_{-2},\ldots$ of $\tilde{h}_2(e^{i\theta})$—is obviously[17] smaller than the L^2 norms of all the operators Y_χ corresponding to functions $\chi(e^{i\theta})$ with $\chi_2\equiv\tilde{h}_2$:

$$\|X\|_{L^2}\leq\|Y_\chi\|_{L^2}. \tag{3.11}$$

The second important point is that, using Foiaş–Nagy lifting theorem,[15,18,19] we can show *that there exists a* *function* $\chi = \chi^0(e^{i\theta})$ *satisfying* (3.2), *for which equality is reached in the inequality* (3.11); this enables us to define ϵ_0 as the norm of X:

$$\epsilon_0 \equiv \|X\|_{L^2} = \|Y_{\chi^0}\|_{L^2} \equiv \|\chi^0(e^{i\theta})\|_{L^\infty}. \tag{3.12a}$$

Indeed, combining (3.11) with (3.8), we get

$$\epsilon_0 < \text{all other } \|\chi(e^{i\theta})\|_{L^\infty} \text{ with } \chi_2(e^{i\theta}) \equiv \tilde{h}_2(e^{i\theta}), \tag{3.12b}$$

which corresponds to the previous definition (3.1) of ϵ_0 as being the maximum of the error produced by the best holomorphic approximant $-\chi_1^0$ of $\tilde{h}_2$ on Γ. The Foiaş—Nagy lifting theorem which made this point possible asserts, indeed, that if we have two isometric operators T and T' operating in the spaces K and K', respectively, and if S and S' are the *restrictions of their adjoints operators* T^+ and T'^+ on the *invariant* subspaces $H \subseteq K$ and $H' \subseteq K'$, and if the operator X transforms the subspace H into H' and satisfies the commutation relation $XS = S'X$, then there exists an extension Y_0 of X transforming K into K' and satisfying $Y_0T^+ = T'^+Y_0$, *whose norm coincides with that of* X. Now, to prove (3.12a), one has to take $T = U$ and $T' = U^+$ [see Eq. (3.9)] and apply the Foiaş—Nagy theorem twice (see Appendix A), once with $K = L^2$ and $H = H' = K' = L^2 \ominus H^2$ and once with $H = L^2 \ominus H^2$ and $K = H' = K' = L^2$. Further, one defines $\check{\chi}^0(e^{i\theta})^*$ to be that function one would get if one applied the minimal-norm operator Y_0 to the constant function 1 (and multiplied it by $e^{+i\theta}$):

$$\check{\chi}^0(e^{i\theta})^* = e^{i\theta} Y_0 \mathbf{1} \equiv U Y_0 \mathbf{1}.$$

Reciprocally, one can express then the minimal operator Y_0 in terms of χ^0 in the way of Eq. (3.7a). Indeed, if $\varphi(e^{i\theta})$ is any L^2 function defined on Γ,

$$\varphi(e^{i\theta}) = \sum_{-\infty}^{+\infty} a_n e^{in\theta} = \sum_{-\infty}^{+\infty} a_{-n}(U^+)^n \mathbf{1},$$

owing to (3.9):

$$Y_0\varphi = \sum_{-\infty}^{+\infty} a_{-n} Y_0 (U^+)^n \mathbf{1} = \sum_{-\infty}^{+\infty} a_{-n} U^n Y_0 \mathbf{1} = \sum_{-\infty}^{+\infty} a_{-n} U^n U^{-1} \check{\chi}^{0*} \equiv \check{\chi}^{0*} U^+ \check{\varphi}.$$

Hence,

$$\|\chi^0\|_{L^\infty} = \|Y_0\|_{L^2} = \|X\|_{L^2} \le \|Y_\chi\|_{L^2} \equiv \|\chi\|_{L^\infty}$$

and the proof of relations (3.12) is now complete.

Once the identity between the L^∞ norm of the optimal function χ^0 and the L^2 norm of the operator X has been established, one can evaluate ϵ_0 numerically as the square root of the greatest eigenvalue of the Hermitian operator XX^+, that is,

$$\epsilon_0 = \lim_{N' \to \infty} (\mathrm{Tr}[(XX^+)^{N'}])^{1/2N'}. \tag{3.13}$$

A matrix representation for X to be used in (3.13) can readily be found in the basis spanned by the eigenvectors $e^{-i\theta}, e^{-2i\theta}, e^{-3i\theta}, \cdots$ of the $L^2 \ominus H^2$ subspace. Indeed, by taking $\varphi_2 = e^{-ik\theta}$ from (3.10), it follows that

$$Xe^{-ik\theta} = Q_- \chi(e^{i\theta}) U^+ e^{+ik\theta} = Q_- \sum_{n=-\infty}^{n=\infty} c_n \exp[i(n+k-1)\theta], \tag{3.14}$$

so that, putting $|k\rangle \equiv \exp(-ik\theta)$, $k = 1, 2, 3, \cdots$, we have

$$\langle j|X|k\rangle = c_{-(k+j-1)}. \tag{3.15}$$

Thus,

$$\mathbf{X} = \begin{Bmatrix} c_{-1} & c_{-2} & c_{-3} & \cdot \\ c_{-2} & c_{-3} & \cdot & \cdot \\ c_{-3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{Bmatrix}, \tag{3.16a}$$

where $c_{-1}, c_{-2}, \cdots$, are the negative frequency Fourier coefficients (2.16d) of the weighted histogram. (Such a matrix is usually called a *Hankel-matrix*). In practical calculations one could set to zero in (3.16) all c_{-n} with n greater than a certain N, sufficiently so that

$$\tilde{h}_2^{(N)}(e^{i\theta}) = \sum_{n=1}^{N} c_{-n} \exp(-in\theta) \tag{3.17a}$$

would approximate sufficiently well $\tilde{h}_2(e^{i\theta})$; usually a fairly good approximation of ϵ_0 is reached with (3.12) in few N'-steps at a computer. However, one could spare computer loading [by smaller (3.16) matrices], using in (3.16) a quicker converging trigonometric series (for instance, the Fejer series $c'_{-k} = (1 - 1/k)c_{-k}$ or, better yet, the Chebysheff approximation), instead of the Fourier coefficients c_{-n}. Indeed, if

$$\tilde{h}_2'^{(N)}(e^{i\theta}) = \sum_{n=1}^{N} c'_{-n} \exp(-in\theta), \tag{3.17b}$$

with

$$\sup_{0 \le \theta < 2\pi} |\tilde{h}_2(e^{i\theta}) - \tilde{h}_2'^{(N)}(e^{i\theta})| = \eta^{(N)}, \tag{3.18}$$

[for a given N, $\eta^{(N)}$ is the smallest when (3.17b) is the Chebysheff approximation of order N to $\tilde{h}_2$], we have quick estimate of the accuracy with which $\epsilon_0^{(N)}$, computed with the help of the truncated matrix

$$X^{(N)} = \begin{Bmatrix} c'_{-1} & c'_{-2} & \cdots & c'_{-N} \\ c'_{-2} & c'_{-3} & \cdots & 0 \\ \vdots & & & \\ c'_{-N} & 0 & \cdots & 0 \end{Bmatrix}, \tag{3.16b}$$

represents the true ϵ_0. Indeed, from the previous theory it follows that if there exists a holomorphic function $-\chi_1^{0(N)}$ which approximates in the best way on Γ the negative frequency function $\tilde{h}_2'^{(N)}$, we have

$$\epsilon_0^{(N)} \equiv |\chi_1^{0(N)} + \tilde{h}_2'^{(N)}| \le \sup|\chi_1^0 + \tilde{h}_2 + \tilde{h}_2'^{(N)} - \tilde{h}_2| \le \epsilon_0 + \eta^{(N)}, \tag{3.19}$$

as well as

$$\epsilon_0 \equiv |\chi_1^0 + \tilde{h}_2| < \sup|\chi_1^{0(N)} + \tilde{h}_2'^{(N)} + \tilde{h}_2 - \tilde{h}_2'^{(N)}| \le \epsilon_0^{(N)} + \eta^{(N)}.$$

Hence,

$$\epsilon_0^{(N)} - \eta^{(N)} < \epsilon_0 < \epsilon_0^{(N)} + \eta^{(N)}, \tag{3.20}$$

so that, if the series (3.17b) approximates fairly well $\tilde{h}_2$, the constant $\epsilon_0^{(N)}$ computed via (3.13) and (3.16b) represents well the actual value of ϵ_0.

From (2.16d) and from (3.16) it is clear that ϵ_0 is a functional of the histogram $h(\zeta)$ and a function of the ratio M/ϵ:

$$\epsilon_0[\tilde{h}] \equiv \epsilon_0[h; M/\epsilon]. \tag{3.21}$$

In general, ϵ_0 *is not* the smallest ϵ for which there still exist holomorphic functions bounded by M on Γ_2 (1.2) and satisfying (1.1). Nevertheless, $\epsilon_0[h;M/\epsilon]$ being the smallest possible deviation *on* $\Gamma_1+\Gamma_2$ of an analytic function $\tilde{f}_0(\zeta)$ *from the* M/ϵ Carleman *weighted histogram* $\tilde{h}(\zeta)$, we obviously have

$$\epsilon_{k+1} \overset{\mathrm{DEF}}{=} \epsilon_0[h;M/\epsilon_k] < \epsilon_k, \tag{3.22}$$

so that the decreasing series $\epsilon_1>\epsilon_2>\epsilon_3>\cdots$ defines a minimal ϵ,

$$\epsilon_{00}=\lim_{n\to\infty}\epsilon_n \quad (>0), \tag{3.23}$$

satisfying the transcedent equation

$$\epsilon_0[h;M/\epsilon_{00}]=\epsilon_{00}. \tag{3.24}$$

The corresponding $\tilde{f}_0(\zeta)$ amplitude

$$|\tilde{f}_{00}(\zeta)-h(\zeta)C_0(M/\epsilon_{00};\zeta)|_\Gamma=\epsilon_{00} \tag{3.25a}$$

defines the minimal function

$$\begin{aligned} &f_{\epsilon_0}(\zeta)=\tilde{f}_{00}(\zeta)/C_0(M/\epsilon_{00};\zeta),\\ &|f_{\epsilon_0}(\zeta)|_{\Gamma_2}<M,\\ &|f_{\epsilon_0}(\zeta)-h(\zeta)|_{\Gamma_1}\le\epsilon_{00}, \end{aligned} \tag{3.25b}$$

the smallest value of ϵ for which such an analytic function still exists. The value ϵ_{00} can be found either as the limit (3.23), computing step-by-step ϵ_k (3.22), or directly, solving the trancedent equation on a computer, combining (3.24) with (2.16d), (3.13), and (3.16).

The second external problem consists in finding the amplitude $f_{M_0}(\zeta)$ of least module $M=M_0$ on Γ_2, ϵ being now fixed. As in (3.24), one would first have to determine M_0 from the equation [for a proof see (5.34)]

$$\epsilon_0[h;M_0/\epsilon]=\epsilon \quad (\epsilon \text{ being given}) \tag{3.26}$$

and then, using the methods of the next section, build the function $f_0(\zeta)$ corresponding to M_0 and ϵ:

$$f_{M_0}(\zeta)=\tilde{f}_0(\zeta)/C_0(M_0/\epsilon;\zeta). \tag{3.27}$$

We shall come back to this problem in Sec. 5.

IV. CONSTRUCTION OF THE SET OF ALL ADMISSIBLE AMPLITUDES

Once we have computed the constant ϵ_0, we turn back to the effective construction of all analytic functions $\tilde{f}(\zeta)$ satisfying (2.7):

$$|\tilde{f}(\zeta)-\tilde{h}(\zeta)|_{\Gamma_1+\Gamma_2}<\epsilon. \tag{4.1}$$

Of course, (2.19), ϵ has to be greater than ϵ_0, which is the smallest value of ϵ for which the set of admissible functions is not void; all the extremal functions $\tilde{f}_0(\zeta$ $\tilde{f}_0(\zeta)$, $\tilde{f}_{\epsilon_0}(\zeta)$, and $\tilde{f}_{M_0}(\zeta)$ are then constructed in a similar way to all other $\tilde{f}(\zeta)$ [there is still a difference, as will be seen, namely, that the function $\psi_N(\ )$ for $f_0(\zeta)$ reduces to zero, but this happens in an automatic way if ϵ is set equal to ϵ_0], by simply replacing ϵ by the corresponding ϵ_0 [by ϵ_{00} if the Carleman weight was $C_0(M/\epsilon_{00},\zeta)$, or by ϵ if one has used $C_0(M_0/\epsilon,\zeta)$].

According to (2.13) each admissible function $\tilde{f}(\zeta)$ contains besides the holomorphic part $\tilde{h}_1(\zeta)$ of the weighted amplitude,

$$\tilde{h}_1(\zeta)=\frac{1}{2\pi i}\int_{\Gamma_1+\Gamma_2}\frac{\tilde{h}(\zeta')}{\zeta'-\zeta}\,d\zeta' \quad (\zeta\in D), \tag{4.2}$$

a supplementary holomorphic part, also, $-\chi_1(\zeta)$, which (2.17) approximates the negative frequency part $\tilde{h}_2(\zeta)$ of the weighted histogram

$$\begin{aligned} &\tilde{f}(\zeta)=\tilde{h}_1(\zeta)-\chi_1(\zeta),\\ &|\chi_1(\zeta)+\tilde{h}_2(\zeta)|_{\zeta\in\Gamma}\le\epsilon. \end{aligned} \tag{4.3}$$

[In contradistinction with the L^2 norm problem (see Sec. 5), where, owing to the orthogonality of the positive and negative Fourier components, one *cannot* approximate $\tilde{h}_2$ by analytic functions in D.] In what follows we shall suppose for the sake of simplicity that the nonholomorphic part $\tilde{h}_2(e^{i\theta})$ of the weighted histogram contains only a finite number, N, of negative frequency Fourier coefficients ($\tilde{h}_2=\tilde{h}_2^{(N)}$ so that the function

$$\begin{aligned} \psi_0(\zeta)&=\frac{\zeta^N}{\epsilon}(\chi_1(\zeta)+\tilde{h}_2^{(N)}(\zeta))\equiv\frac{\zeta^N}{\epsilon}\chi(\zeta)\\ &=\frac{c_{-N}}{\epsilon}+\frac{c_{-N+1}}{\epsilon}\zeta+\cdots+\frac{c_{-1}}{\epsilon}\zeta^{N-1}\\ &\quad+\frac{\zeta^N}{\epsilon}\chi_1(\zeta) \end{aligned} \tag{4.4}$$

is *holomorphic* inside D. Moreover, according to (4.3) we have

$$|\psi_0(\zeta)|\le 1 \tag{4.5}$$

for every $|\zeta|\le 1$. Finding all amplitudes satisfying (4.1) reduces thus to finding all holomorphic functions satifying (4.5), and having N preassigned Taylor coefficients:

$$\psi_{0,0}=c_{-N}/\epsilon,\quad \psi_{0,1}=c_{-(N-1)}/\epsilon,\quad \cdots\ \psi_{0,N-1}=c_{-1}/\epsilon. \tag{4.6}$$

This problem can be solved in a simple recurrent way. Indeed, if the unity-bounded function $\psi_{k-1}(\zeta)$,

$$|\psi_{k-1}(\zeta)|\le 1,\quad |\zeta|\le 1, \tag{4.7}$$

has $N-(k-1)$ preassigned Taylor coefficients

$$\frac{1}{k!}\left.\frac{d^j\psi_{k-1}(\zeta)}{d\zeta^j}\right|_{\zeta=0}=\psi_{k-1,j}\quad (\text{given for all } 0\le j\le N-k), \tag{4.8}$$

where

$$|\psi_{k-1,0}|\lneq 1, \tag{4.9}$$

the function

$$\psi_k(\zeta)=\frac{1}{\zeta}\,\frac{\psi_{k-1}(\zeta)-\psi_{k-1,0}}{1-\psi_{k-1}(\zeta)\psi^*_{k-1,0}} \tag{4.10}$$

is also unity-bounded,

$$|\psi_k(\zeta)|\le 1,\quad |\zeta|\le 1, \tag{4.11}$$

analytic in D, and its $N-k$ first Taylor coefficients are completely determined by the first $N-k+1$ coefficients (4.7) of $\psi_{k-1}(\zeta)$. (The reverse statement being also valid):

$$\psi_{k,n}=\sum\frac{(\sum_{i=1}^{n+1}k_i)!\,\psi^{*(\Sigma_1^{n+1}k_i-1)}_{k-1,0}}{k_1!\ldots k_{n+1}!\,(1-|\psi_{k-1,0}|^2)}\psi^{k_1}_{k-1,1}\cdots\psi^{k_{n+1}}_{k-1,n+1}, \tag{4.12}$$

where the sum $\sum$ is extended to all combinations of nonnegative integer k_i with

$$k_1+2k_2+\cdots+(n+1)k_{n+1}=n+1. \tag{4.13}$$

The inequality (4.11) does not assure that $\psi_{k,0}$ satisfies the inequality (4.9) too, but this really happens for $\epsilon>\epsilon_0$.

Going further, one finally gets a holomorphic function $\psi_N(\zeta)$, which, beside the inequality

$$|\psi_N(\zeta)|<1, \quad |\zeta|\leq 1, \tag{4.14}$$

is completely free of supplementary conditions.

Hence, using the inverse of (4.10),

$$\psi_{k-1}(\zeta)=\frac{\zeta\psi_k(\zeta)+\psi_{k-1,0}}{1+\psi^*_{k-1,0}\zeta\psi_k(\zeta)}, \tag{4.15}$$

one finally gets the whole family of functions $\psi_0(\zeta)$ satisfying (4.4) and (4.5), in terms of a general, unity-bounded function $\psi_N(\zeta)$, and, of course, depending on the negative Fourier coefficients $c_{-1},\ldots, c_{-N}$, (2.16d), of the weighted histogram [via Eqs. (4.6), (4.12), and (4.15)]. Once one knows the functions $\psi_{0[\psi_N]}(\zeta)$, the most general holomorphic function satisfying the conditions (1.1) and (1.2) is given in the approximation

$$\tilde{h}_2(\zeta)\approx\tilde{h}_2^{(N)}(\zeta)=\sum_{k=1}^{N}c_{-k}\zeta^{-k} \tag{4.16}$$

by

$$f_{[\psi_N]}(\zeta)=[\tilde{h}_1(\zeta)+\tilde{h}_2^{(N)}(\zeta)-(\epsilon/\zeta^N)\psi_{0[\psi_N]}(\zeta)]/C_0(M/\epsilon,\zeta), \tag{4.17}$$

where $\tilde{h}_1$ is given by (4.2) [see also (2.1), (2.5), and (2.16d)]. Of course, the negative powers of ζ cancel out identically, owing to Eq. (4.4).

If in Eq. (4.4) one puts $\epsilon=\epsilon_0[h;M/\epsilon]$, at least the last $(k=N-1)$ of the constants $\psi_{k,0}$ has to be unimodular. If this were not the case, one would find a condition-free N unity-bounded iterate $\psi_N(\zeta)$, and this would be possible by continuity also for an ϵ slightly smaller than ϵ_0, when, by definition there are no more solutions. Thus, if $\psi_{k,0}$ infringes inequality (4.9),

$$|\psi_{k,0}|=1 \tag{4.18a}$$

from (4.11) and from the principle of the maximum of the modulus $[\psi_{k,0}=\psi_k(0)!]$, one gets

$$\psi_k(\zeta)\equiv\psi_{k,0}\quad(\epsilon=\epsilon_0), \tag{4.18b}$$

and the extremal weighted function $\tilde{f}_0(\zeta)$ can then be built again in a recursive (4.15) way; but, starting from the last not vanishing function $\psi_k(\zeta)$, given by (4.18b), rather than from $\psi_N(\zeta)$,

$$f_0(\zeta)=\tilde{h}_1(\zeta)+\tilde{h}_2^N(\zeta)-(\epsilon_0/\zeta^N)\psi_0(\zeta)C_0(M/\epsilon_0,\zeta), \tag{4.19}$$

where in, contradistinction to (4.17), $\psi_0(\zeta)$ is completely determined in terms of the Fourier coefficients c_{-n} $(1\leq n\leq N)$ of $\tilde{h}_2(\zeta)$. Unless $\epsilon=\epsilon_{00}$ [and hence $f_0(\zeta)=f_{\epsilon_0}(\zeta)$] or $M=M_0$ [see (3.26): $f_0(\zeta)=f_{M_0}(\zeta)$], the amplitudes $f_0(\zeta)$ have no special physical meaning; nevertheless, one should notice that for every weighted function $\tilde{f}_0(\zeta)$ one has the interesting property

$$|\tilde{f}_0(\zeta)-\tilde{h}(\zeta)|_{\Gamma_1+\Gamma_2}=\epsilon_0[h,M/\epsilon]\,(=\text{const}), \tag{4.20a}$$

and hence

$$\begin{aligned}&|f_0(\zeta)-h(\zeta)|_{\Gamma_1}=\epsilon_0,\\&|f_0(\zeta)|_{\Gamma_2}=M\epsilon_0/\epsilon.\end{aligned} \tag{4.20b}$$

Equation (4.20a) is a consequence of the fact that if $\psi_k(\zeta)=\psi_{k,0}$, the modules of all the functions $\psi_j(\zeta)$, $j\leq k$, are equal to one for the boundary points $|\zeta|=1$ and thus, owing to Eq. (4.4), with ϵ replaced by ϵ_0,

$$|\tilde{f}_0(\zeta)-\tilde{h}(\zeta)|_{\zeta\in\Gamma}=\epsilon_0|\psi_0(\zeta)/\zeta^N|_{\zeta\in\Gamma}=\epsilon_0,$$

which proves (4.20a).

In practical extrapolation problems it is perhaps more important to know the value $\hat{f}(\zeta)$ and $\hat{\eta}(\zeta)=\eta(\zeta)/C_0(\zeta)$ of the center and of the radius of the set of values of all the admissible functions $f_{[\psi_M]}(\zeta)$ in a given point ζ, rather than the admissible functions themselves. One can readily prove that, in every point ζ, this set of values fills densely a circle. Indeed, if in the recurrence formula (4.15) with fixed ζ and $\psi_{k-1,0}$ the possible values of $\psi_k(\zeta)$ fill a circle of center γ_k and radius η_k

$$\psi_k(\zeta)=\gamma_k+\alpha\eta_k e^{i\beta},\quad 0\leq\alpha\leq 1,\ 0\leq\beta\leq 2\pi, \tag{4.21}$$

then the values of $\psi_{k-1}(\zeta)$ will fill a circle too, whose center and radius are given by

$$\gamma_{k-1}=\frac{1}{\psi^*_{k-1,0}}\left(1-\frac{(1-|\psi_{k-1,0}|^2)(1+\psi_{k-1,0}\zeta^*\gamma_k^*)}{|1+\psi_{k-1,0}\zeta^*\gamma_k^*|^2-|\eta_k\psi_{k-1,0}\zeta|^2}\right), \tag{4.22a}$$

$$\hat{\eta}(\zeta)=\eta_{k-1}=\frac{\eta_k|\zeta|(1-|\psi_{k-1,0}|^2)}{|1+\psi_{k-1,0}\zeta^*\gamma_k^*|^2-|\eta_k\psi_{k-1,0}\zeta|^2}. \tag{4.22b}$$

Since the values of the arbitrary functions $\psi_N(\zeta)$ just fill—at fixed ζ—a circle of radius 1 and with the center in the origin, we have

$$\gamma_N\equiv 0,\quad \eta_N\equiv 1. \tag{4.22c}$$

Using (4.22a) and (4.22b) in a recurrent way, one gets finally $\gamma_0(\zeta)$ and $\eta_0(\zeta)$, as well as the center and radius of the (unweighted) admissible functions:

$$\begin{aligned}\hat{f}(\zeta)\equiv\hat{\tilde{f}}(\zeta)/C_0(M/\epsilon,\zeta)&=[\tilde{h}_1(\zeta)+\tilde{h}_2^{(N)}(\zeta)-(\epsilon/\zeta^N)\gamma_0(\zeta)]/\\&\quad C_0(M/\epsilon,\zeta),\end{aligned} \tag{4.22d}$$

$$\hat{\eta}(\zeta)=\eta(\zeta)/|C_0(M/\epsilon,\zeta)|=\epsilon\eta_0(\zeta)/|\zeta^N C_0(M/\epsilon,\zeta)|. \tag{4.22e}$$

Again, no poles appear in (4.20d), (4.20e) at $\zeta=0$, as one can see from (4.4) and from the denominator of (4.22b). One should also notice that for $|\zeta|=1$

$$\eta(\zeta)=\epsilon,\quad \zeta\in\Gamma \tag{4.23a}$$

$$\hat{\tilde{f}}(\zeta)=\tilde{h}(\zeta),\quad \zeta\in\Gamma, \tag{4.23b}$$

(see Fig. 3), but for $|\zeta|<1$, $\eta(\zeta)$ is usually much smaller than ϵ (see Fig. 4). This phenomenon is especially apparent when ϵ is only slightly different from ϵ_0, when some of the $|\psi_{k-1,0}|$ are close to 1 and, thus, the numerator of the formulas (4.22b) is small.

If $\epsilon=\epsilon_0$, η_k vanishes identically for $\zeta\in D$ [(4.22)], while for the boundary points we get the equations (4.20), i.e.,

$$|\tilde{f}_0(\zeta)-\tilde{h}(\zeta)|_\Gamma=\epsilon_0=\text{const}. \tag{4.24}$$

V. CONCLUDING REMARKS AND THE L^2 PROBLEM

A. Review of results

It has been shown in the previous sections that, given a data function $h(z)$ ("the histogram") along some finite parts Γ_1 of the cuts Γ of the amplitude, together with an error channel of width ϵ (1.1) and with a bound M (1.2) for the amplitude on the remaining parts $\Gamma_2=\Gamma\setminus\Gamma_1$ of the cuts, one can effectively construct the set of all analytic functions compatible with this input information. As it was shown in Sec. 4, the set of the values of all these possible ("admissible") amplitudes fill at each z

a disk (see Fig. 4) whose center $\hat{f}(\zeta)=\hat{\hat{f}}(\zeta)/C_0(\zeta)$ and radius $\hat{\eta}(\zeta)=\eta(\zeta)/|C_0(\zeta)|$ can be computed [see Eqs. (4.22d) and (4.22e) at each point $\zeta=\zeta(z)$ (see caption of Fig. 2) in terms of ϵ and M and of the negative Fourier coefficients $c_{-1}, c_{-2}, \cdots$ in the $\zeta(z)$ complex plane of the weighted [see Eqs. (2.1)–(2.3)] histogram $\tilde{h}(\zeta)$. There is no holomorphic amplitude at all, compatible with the initial conditions (1.1) and (1.2), if ϵ is smaller than the important constant ϵ_0—the norm of the matrix X (3.16), defined in terms of the negative Fourier coefficients $c_{-1}, c_{-2}, \cdots$. In the limit $\epsilon=\epsilon_0$, there remains a unique admissible amplitude $f_0(\zeta)=\tilde{f}_0(\zeta)/C_0(\zeta)$, as in this case the radius $\eta(\zeta)\to 0$ and $\hat{\tilde{f}}(\zeta)\to\tilde{f}_0(\zeta)$. Generally, $\tilde{f}_0(\zeta)$ is the "minimal" (weighted) amplitude, i.e., that holomorphic function which approximates best (4.24) the weighted data function $\tilde{h}(\zeta)$ on the boundary Γ.

As was emphasized in Sec. 2, in the case when $\epsilon_0 \ll \epsilon$, the center $\hat{f}(\zeta)$ of the set of values in ζ of all admissible amplitudes $f(\zeta)$ differs little from the best weighted dispersion relation extrapolated function $\hat{h}(\zeta)$.[7] Conversely, when ϵ_0 is close to ϵ, most of the circle of the Nevanlinna error bound $\epsilon/|C(\zeta)|$ (corresponding to the circle of radius ϵ of Fig. 4) is empty, the admissible amplitudes being clustered in a small circle around $\hat{f}(\zeta)$ which, in general, may differ considerably from $\hat{h}(\zeta)$; in this latter case, the techniques developed in Secs. 3 and 4 represent a net improvement over the Poisson weighted dispersion relations of Refs. 7 and 8. Nevertheless, in both cases the function $\hat{f}(\zeta)$ [although nonanalytic, in contrast to the function $f_0(\zeta)$ which, being itself an admissible amplitude, is holomorphic] is for each value of z the *most unbiased* estimate one can find for an admissible amplitude taken at random.

B. Variable error channel

So far the error-channel width ϵ was regarded to be a constant. The physically important variable error case can be readily reduced to the former one using the techniques of Sec. 4 of Ref. 7, namely by introducing a supplementary weight function.[20]

$$C_1(\zeta)=\exp\{-[w(\zeta)+i\tilde{w}(\zeta)]\}, \tag{5.1}$$

where

$$w(\zeta)+i\tilde{w}(\zeta)=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{e^{i\theta'}+\zeta}{e^{i\theta'}-\zeta}\ln\frac{\epsilon(\theta')}{\epsilon(\pi/2)}\,d\theta'. \tag{5.2}$$

Thus, the variable error channel conditions ($\zeta=e^{i\theta}$),

$$\begin{aligned} &|f(\zeta)-h(\zeta)|\le\epsilon(\theta) \quad \text{for } -\pi/2<\theta<\pi/2,\\ &|f(\zeta)|\le M \qquad\qquad \text{for } \pi/2<\theta<3\pi/2, \end{aligned} \tag{5.3}$$

reduce to the constant error ones for the weighted functions $C_1(\zeta)f(\zeta)$ and $C_1(\zeta)h(\zeta)$:

$$|C_1(\zeta)f(\zeta)-C_1(\zeta)h(\zeta)|\le\epsilon(\pi/2) \quad \text{for } \zeta\in\Gamma_1, \tag{5.4}$$

$$|C_1(\zeta)f(\zeta)|\le M \qquad \text{for } \zeta\in\Gamma_2 \tag{5.5}$$

and one then proceeds as in Sec. 4.

C. A probabilistic approach: L^∞ versus L^2 problems

We should like now to outline how the similar, but much simpler, L^2-problem can be solved. This problem can be connected in a natural way[3] to the χ^2 test if one makes the assumption that the data have a normal (Gaussian) distribution around the true amplitude. [It is well known that if ξ_i are independent random variables *of class* $N(0,\sigma)$ (centered Gaussian distributions of dispersion σ), the random variable $\eta=\Sigma_{i=1}^{s}\xi_i^2$ follows the usual χ^2 distribution, being of class $H(S,\sigma)$.] However, we should like to stress that the L^2 norm *does not* exhaust the possible connections with statistics; moreover, one can relax the normality assumption by using *parameter free* test of the Kolmogoroff type,[21] which lead to L^∞ norm problems, but which are much less known among physicists than the χ^2 one. For instance, if one takes a sample of volume n from an ordered population $\{\xi\}$ subjected to a certain repartition law[22] $F(\xi)$ yet unspecified) and if $\xi^{(1)}$ and $\xi^{(n)}$ are the minimal and the maximal value $\xi^{(j)}$ of the sample so that

$$\xi^{(n)}-\xi^{(1)}=\sup|\xi^{(i)}-\xi^{(j)}|, \tag{5.6}$$

then the probability P of finding a value ξ outside the range $(\xi^{(1)},\xi^{(n)})$ with a probability greater than α/n, equals asymptotically a universal function of the parameter α. Indeed, it can be proved[23] that

$$P=\int_0^\alpha h(x)\,dx,$$

where, asymptotically, $h(x)$ is the convolution of two $\gamma(1)$ (pure exponential) distributions

$$g(x)=e^{-x} \quad \text{(asymptotically)},$$

$$h(x)=\begin{cases}\int_0^x g(x-y)g(y)dy=xe^{-x} & \text{for } x\ge 0\\ 0 & \text{for } x<0.\end{cases}$$

Hence,

$$\begin{aligned}&\lim_{n\to\infty}P\{1-[F(\xi^{(n)})-F(\xi^{(1)})]<\alpha/n\}\\ &=\int_0^\alpha h(x)\,dx=1-e^{-\alpha}(1+\alpha).\end{aligned} \tag{5.7}$$

The fact that Eq. (5.7) is irrespective of the actual form of the repartition law, $F(\xi)$ has a great theoretical importance, especially when the values for the random variable ξ are not obtained by measurements of the characteristics of some palpable object, but, rather, are inferred themselves from some empirical repartition laws, as in the case of the scattering amplitude. Moreover, one could also write down the exact (nonasymptotic) form of Eq. (5.7), so that one could find the exact value for the volume n of the sample, in order to have all ξ inside the range $\xi^{(1)}, \xi^{(n)}$ with a probability greater than a given number, to a specified confidence level. For example, if the confidence level is 0.05 ($P=0.95$) and if the probability of ξ lying between $\xi^{(1)}$ and $\xi^{(n)}$ is 99%, we find $n=473$. It is apparent that Eq. (5.6) leads to the L^∞ problems of the previous sections, but this question needs more elaborations and will be treated elsewhere.

One of the first questions one could ask in connection with the L^2 problem might be that of finding those "amplitudes" which minimalize the L^2 norm (*over the* Γ_1 *cut*) of their difference to the histogram $h(\zeta)$, i.e., those functions which minimalize the lhs of the inequality

$$\|f-h\|^2_{L^2(\Gamma_1)}\equiv\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\rho(\theta)|h(e^{i\theta})-f(e^{i\theta})|^2d\theta<\epsilon, \tag{5.8}$$

where $\rho(\theta)$ is a suitable weight function. Obviously, this hardly could be the correctly posed physical problem, since, for instance, if one had to solve this problem for a finite sum (the discret points case) instead of the integral of the lhs of (5.8), one could reduce the sum to zero taking high enough polynomials; but, of course, the higher the degree of the polynomial, the stronger it would blow up outside the range Γ_1 of energies in which the data points were given! Therefore, as it was discussed in the Introduction, one would have to add to (5.8) supplementary stabilizing physical information, which, in the context of the L^2 norm problems could be of the form

$$\|f\|^2_{L^2(\Gamma_2)} = \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} \rho_2(\theta)|f(e^{i\theta})|^2 d\theta < \mathscr{M}^2. \tag{5.9}$$

D. Simplified L^2 problem

An incomplete but very simple way of treating this problem is the following: Divide (5.9) by $\mathscr{M}^2/\epsilon^2$ and add it to (5.8) and try to find the Carleman weighted function $\tilde{f} = C_0 f$ which minimalizes the L^2 norm integral over the whole unit circle $\Gamma_1 + \Gamma_2$:

$$\frac{1}{2\pi}\int_{\Gamma_1+\Gamma_2} |h-f|^2 |C_0|^2 \rho(\theta) d\theta = \left\|\tilde{f} - \tilde{h}\right\|^2_{L^2(\Gamma_1+\Gamma_2)}, \tag{5.10}$$

where, as in Sec. 2, $\tilde{h}(e^{i\theta})$ was settled equal to zero on Γ_2. Obviously this new formulation of the problem departs from the previous one as the minimum of (5.10) *by no means* implies the minimum of the lhs of (5.8) under condition (5.9); furthermore, we shall show how[24] the initial problem can be answered in a correct way.

In contrast to the L^∞ problem studied in Secs. 2 and 3, the solution of this simplified L^2 problem is immediate: First, define a new external ("Carleman") function $C_\rho(\zeta)$ satisfying on the unit circle the condition

$$|C_\rho(\zeta)|_\Gamma = \sqrt{\rho(\theta)}, \tag{5.11}$$

so that [see (5.1) and (5.2)]

$$C_\rho(\zeta) = \exp\left(\frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\log[\rho(\theta)]d\theta\right). \tag{5.12}$$

By introducing the weighted functions

$$\bar{h}(\zeta) = C_0(\zeta)C_\rho(\zeta)h(\zeta) \quad (\zeta \in \Gamma), \tag{5.13}$$

$$\bar{f}(\zeta) = C_0(\zeta)C_\rho(\zeta)f(\zeta) \quad (\zeta \in D), \tag{5.14}$$

our problem reduces to finding that analytic function $\bar{f}$ which minimalizes the unweighted L^2 norm on $\Gamma = \Gamma_1 + \Gamma_2$;

$$\|\bar{f} - \bar{h}\|_{L^2(\Gamma)} = \left[\frac{1}{2\pi}\int_\Gamma |\bar{f}(e^{i\theta}) - \bar{h}(e^{i\theta})|^2 d\theta\right]^{1/2} < \epsilon, \tag{5.15}$$

with the obvious solution

$$\bar{f}_{\min}(\zeta) = \bar{h}_1(\zeta) \equiv \frac{1}{2\pi i}\int_\Gamma \frac{h(\zeta')C_0(\zeta')C_\rho(\zeta')}{\zeta'-\zeta}\, d\zeta', \tag{5.16}$$

which is the direct consequence of the orthogonality of the positive and negative parts of $\bar{f} - \bar{h}$ on Γ:

$$\|\bar{f} - \bar{h}\|^2_{L^2} = \|\bar{f} - \bar{h}_1\|^2_{L^2} + \|\bar{h}_2\|^2_{L^2}. \tag{5.17}$$

In contrast to the L^∞ norm problem where the minimal function $\tilde{f}_0$ differs in general from the optimal approximation $\tilde{f}$, the center of the whole set of functions satisfying (5.15) coincides with $\bar{f}_{\min}(\zeta)$ defined by (5.16); indeed, owing again the orthogonality of the positive and negative frequencies on the unit circle Γ, a general L^2 admissible function can always be written in the form

$$\bar{f}(\zeta) = \bar{f}_{\min}(\zeta) + \bar{l}_1(\zeta) \quad [\bar{f}_{\min}(\zeta) \equiv \bar{h}_1(\zeta)], \tag{5.18}$$

where $\bar{l}_1(\zeta)$ is an arbitrary holomorphic function whose L^2 norm is smaller than

$$\|\bar{l}_1\|_{L^2(\Gamma)} < (\epsilon^2 - \epsilon^2_{\bar{h}_2})^{1/2}, \tag{5.19}$$

where

$$\epsilon_{\bar{h}_2} = \|\bar{h}_2\|_{L^2(\Gamma)}. \tag{5.20}$$

Indeed,

$$\|\bar{f} - \bar{h}\|^2_{L^2(\Gamma)} = \|\bar{f} - \bar{h}_1\|^2_{L^2} + \|\bar{h}_2\|^2_{L^2} = \|\bar{l}_1\|^2_{L^2} + \epsilon^2_{\bar{h}_2}. \tag{5.21}$$

of course, using L^2 conditions on Γ, one loses information about the behavior of $l(\zeta)$ in the special points ζ, so that the radius.

$$R(\zeta) = \sup|\bar{l}_1(\zeta)|. \tag{5.22}$$

Of the set of values of the possible admissible L^2 weighted function around $\bar{f}_{\min}(\zeta)$ exceeds considerably the L^∞ one, being equal to

$$R(\zeta) = (\epsilon^2 - \epsilon^2_{\bar{h}_2})^{1/2}/(1 - |\zeta|^2)^{1/2} \tag{5.23}$$

(see Appendix B), and blows up when ζ goes to the boundary Γ.

E. Complete L^2 problem

The logical drawback of this simplified approach is connected to the fact that the error channel condition (5.8) and the stabilizing condition (5.9) "mix" in (5.10) in an uncontrollable way. This mixing can be changed by changing the weight function $\rho(\theta)$ on the unknown cut by some given multiplicative factor—this amounts to the introduction in (5.10) of a supplementary Carleman function $C_0(\zeta)$—but, nevertheless, the mixing will subsist. The proper way to handle this problem is to look to all holomorphic functions satisfying (5.8) and (5.9) separately. This question was solved by Sabba Stefănescu[24] orthogonalizing the first N (N= sufficiently large) powers of ζ on both Γ_1 and Γ_2. For instance, one could first find, in a progressive way, the first N polynomials $P_n^1(\zeta)$ *of degree* n, which are *normal* and orthogonal on Γ_1 to all other polynomials of degree less than n. $P_n^{(1)}(\zeta)$ are nothing but the Legendre polynomials of the curve Γ_1 and, of course, are *not* orthogonal also on Γ_2, so that we can write

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} P_{n_1}^{(1)}(e^{i\theta})P_{n_2}^{(1)*}(e^{i\theta})\, d\theta = \delta_{n_1 n_2}, \tag{5.24}$$

$$\frac{1}{\pi}\int_{\pi/2}^{3\pi/2} P_{n_1}^{(1)}(e^{i\theta})P_{n_2}^{(1)*}(e^{i\theta})\, d\theta = B_{n_1 n_2}, \tag{5.25}$$

where $B_{n_1n_2}$ is an $N\times N$ Hermitian matrix. Sabba-Stefănescu then diagonalizes this matrix through a suitable basis change,

$$\mathcal{P}_m(\zeta) = \sum_1^N U_{mn}P_n^{(1)}(\zeta), \tag{5.26}$$

so that,

$$\frac{1}{\pi}\int_{\pi/2}^{3\pi/2} \mathcal{P}_{m_1}(\zeta)\mathcal{P}^*_{m_2}(\zeta)\, d\theta = \sum_{n_1n_2} U_{m_1n_1}B_{n_1n_2}U^+_{n_2m_2} = \lambda_{m_1}\delta_{m_1m_2}$$

$$(\text{all } \lambda_m > 0). \tag{5.26}$$

In contradistinction to the Legendre polynomials $P_n^{(1)}$ on Γ_1, *all* $\mathcal{P}_n(\zeta)$ are of *degree N* and their coefficients change when N is changed (to simplify the notations, we have dropped the index N on which they depend). Specific convergence problems arise in the infinite dimensional Hilbert, but they are carefully discussed in Ref. 24. If the $h_n^{(N)}$ are the expansion coefficients of the histogram in terms of $\mathcal{P}_n(\zeta)$ on Γ_1,

$$h_n^{(N)} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} h(e^{i\theta})\mathcal{P}_n^*(e^{i\theta})\,d\theta, \tag{5.27}$$

the conditions (5.8) and (5.9), for an admissible polynomial $f^{(N)}(\zeta)$ of degree N, are

$$\sum_i |h_i^{(N)} - f_i^{(N)}|^2 < \epsilon_N^2 \quad \text{where } \epsilon_N^2 = \epsilon^2 - \sum_{n=N+1}^{\infty}\left|\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} P_n^{(1)}(e^{i\theta}) \times h(e^{i\theta})\,d\theta\right|^2 \tag{5.28}$$

and

$$\sum_i \lambda_i |f_i^{(N)}|^2 < \mathfrak{M}^2, \tag{5.29}$$

so that the set of the admissible functions $f^{(N)}(\zeta)$ is the N dimensional Hilbert space region common to the hypersphere (5.28) centered around $h^{(N)}(\zeta)$ and the hyperelipsoid (5.29) with the center in the origin of the Hilbert space. If, as usually, $\|h\|_{L^2(\Gamma_1)} > \epsilon$, i.e., the origin is not contained in the sphere (5.28), there exists a smallest $\mathfrak{M} = \mathfrak{M}_0$ below which the ellipsoids no longer intersect the spheres:

$$\mathfrak{M}_0 = \lim_{N\to\infty}\mathfrak{M}_N^0, \tag{5.30}$$

where the $\mathfrak{M}_N^0$ are that $\mathfrak{M}$ for which the N-dimensional spheres (5.28) and ellipsoids (5.29) are tangent. It is a simple geometrical matter to find also the components of the tangent point vector $f_{0,n}^{(N)}$, the set $f_0^{(N)}(\zeta) = \sum f_{0,n}^{(N)}\mathcal{P}_n^N(\zeta)$ converging to the minimal function $f_{\mathfrak{M}_0}(\zeta)$. It is quite obvious (if two $N+1$ dimensional bodies are tangent in a N dimensional subspace, they have certainly at least one common point also in the $N+1$ dimensional space) that

$$\mathfrak{M}_{N+1}^0 < \mathfrak{M}_N^0, \tag{5.31}$$

so that the $\mathfrak{M}_N^0$ represent upper valued estimates for $\mathfrak{M}_0$; a lower bound for $\mathfrak{M}_0$ is yielded by the minimal $\mathfrak{M}$ of the simplified L^2 problem approach (5.10), namely lowering the parameter of the Carleman function C_0 until there remains only a single admissible amplitude (5.18), i.e., until

$$\|\tilde{h}\|_{L^2(\Gamma_1)} = \epsilon, \tag{5.32}$$

such that $\|l_1(\zeta)\|_{L^2(\Gamma_1)} \equiv 0$ see (5.19).

F. The analogous M_0 problem for the L^∞ case

As we have already shown at the end of the Sec. 3, there exists a minimal value M_0 also in the L^∞ norm problem, under which there are no more analytic functions satisfying (1.1) and (1.2). Let $M_2 < M_1$ and let ϵ_{0i} $(i = 1, 2)$ be the smallest (L^∞) deviations of a function holomorphic in D from the $(i = 1, 2)$ weighted histograms $\tilde{h}^{(i)}(\zeta) = h(\zeta)C_0(M_i/\epsilon, \zeta)$. Since[25]

$$\left|\frac{C_0(M_1/\epsilon,\zeta)}{C_0(M_2/\epsilon,\zeta)}\right|_\Gamma = \begin{cases}1, & \text{on } \Gamma_1\\ M_2/M_1 < 1, & \text{on } \Gamma_2,\end{cases}$$

we get

$$\left|\tilde{f}^{(2)}\frac{C_0(M_1/\epsilon,\zeta)}{C_0(M_2/\epsilon,\zeta)} - \tilde{h}^{(2)}\frac{C_0(M_1/\epsilon,\zeta)}{C_0(M_2/\epsilon,\zeta)}\right|_\Gamma = \begin{cases}\epsilon_{02}, & \text{on } \Gamma_1\\ <\epsilon_{02}, & \text{on } \Gamma_2.\end{cases}$$

Since $\tilde{h}^{(2)}(\zeta)C_0(M_1/\epsilon,\zeta)/C_0(M_2/\epsilon,\zeta) \equiv \tilde{h}^{(1)}(\zeta)$, from the definition of ϵ_{01} we have to have $\epsilon_{01} < \epsilon_{02}$; moreover, as it was shown in Ref. 25 this inequality is strict. Hence we get the important monotony property

$$\epsilon_0[h; M_1/\epsilon] < \epsilon_0[h; M_2/\epsilon] \quad \text{if } M_1 > M_2. \tag{5.33}$$

Thus, decreasing M, we see that ϵ_0 increases until it reaches, at $M = M_0$, the actual value of ϵ:

$$\epsilon_0[h; M_0/\epsilon] = \epsilon \quad [= (3.26)]. \tag{5.34}$$

There are no admissible amplitudes for M less than M_0, since, owing to (5.33), $\epsilon_0(M)$ would have to be greater than the channel error width ϵ. The numerical value of M_0 can be determined from Eq. (5.34), computer programming being much facilitated by the monotony property (5.33) (computer programmes are available at request, both for M_0 and for $\mathfrak{M}_0$). The corresponding minimal amplitude,

$$f_{M_0}(\zeta) = f_0(\zeta)\big|_{\epsilon_0 = \epsilon}, \tag{5.35}$$

is of course unique *and does not depend on* M (the stabilizing lever of this L^∞ problem), being a characteristic function of all the (M-dependent) sets of admissible amplitudes (4.17). If no information exists about the possible range of the true value of M, the minimal function $f_{M_0}(\zeta)$ could be used as a first reference for the amplitude. Owing to the fact that for every

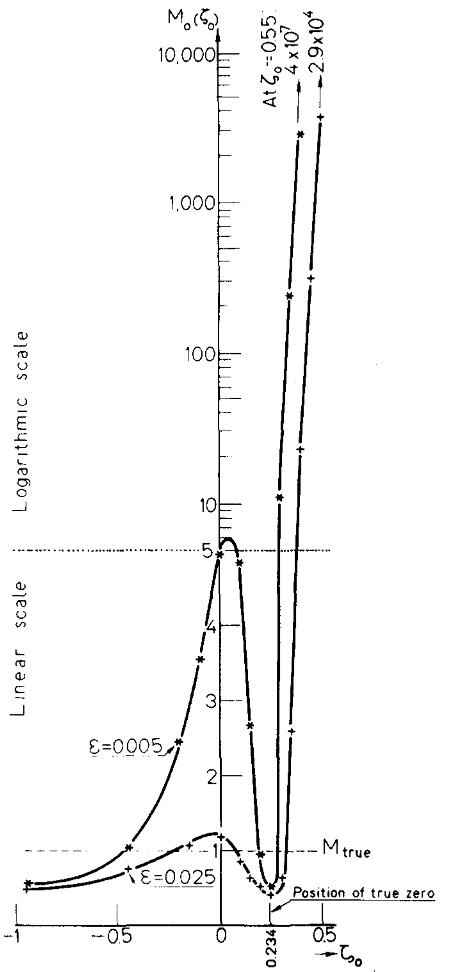

FIG. 5. Typical dependence (see Ref. 26) of M_0 versus the location of the artificial pole $\zeta_0 = \zeta(z_0)$ {here, $\zeta(z) = [30 - z + i\sqrt{195z(z-4)}]/(14z_0 - 30)$} of the function $F_1(z)/[(\zeta - \zeta_0)/(1 - \zeta_0\zeta)]$, introduced in order to locate the zero of the *model* amplitude $F_1(z) = (1 - \sqrt{4-z})/(1 + \sqrt{4-z})$. The dip corresponds to $z_0 = 3$, where the artificial pole disappears identically. Upper curve corresponds to 1% errors, while lower curve to 5% ones.

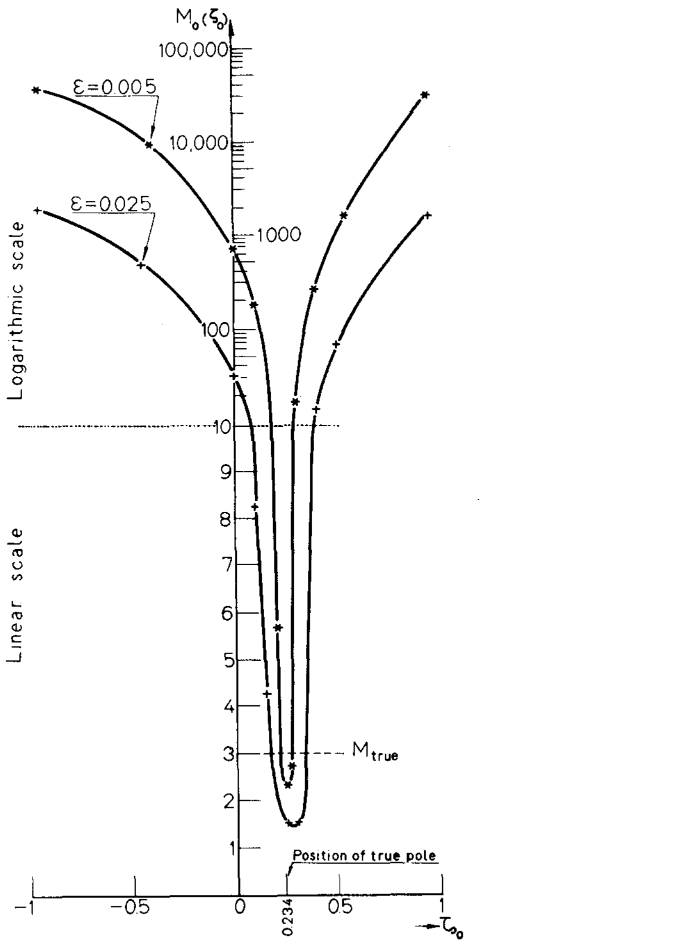

FIG. 6. Typical dependence of M_0 versus the location of the artificial zero $\zeta_0=\zeta(z_0)$ of the function $F_2(z)(\zeta-\zeta_0)/(1-\zeta_0\zeta)$, where $F_2(z)=1/F_1(z)$ of Fig. 5.

L^∞-minimal function, the module $|\tilde{h}-\tilde{f}_0|$ is constant along $\Gamma_1+\Gamma_2$ and equal to ϵ_0 [see (4.20)], $|f_{M_0}(\zeta)|=M_0$ on Γ_2 and, hence, its L^2 norm on Γ_2 coincides with M_0. Thus we get the inequality

$$\mathfrak{M}_0 \leq M_0. \tag{5.36}$$

G. Detection of singularities

On the other hand, M_0, as well as $\mathfrak{M}_0$, could be useful tools in a great variety of problems. For instance, they provide a sensitive device in the location of the zeros or poles of the amplitude.[26] For instance, if the amplitude $A(\zeta)$ has some zero in D, the function

$$f(\zeta)=\frac{A(\zeta)}{(\zeta-\zeta_0)/(1-\zeta_0^*\zeta)} \tag{5.37}$$

whose "histogram" $h(\zeta)$ can be constructed simply on Γ_1 from the data function for the amplitude, would be nonholomorphic in D unless the parameter ζ_0 has exactly the value of the zero of A. The curve $M_0(\zeta_0)$ (see Figs. 5 and 6), and Ref. 26) is very sensitive to that, especially when the error corridor is not too large; indeed, if ϵ is small, it would be very hard, i.e., M_0 would be very high—to find holomorphic function approximating the histogram of a nonanalytic function! If some theoretical information is available (unitarity, Froissart bound, etc.) limiting the upper value of M (the dashed lines $M=M_{\text{true}}$ of Figs. 5 and 6), then the only possible values of ζ_0 are those for which $M_0(\zeta_0)<M_{\text{true}}$, the distance between the M_0 curve and the line $M=M_{\text{true}}$ defining a sort of probability distribution for the location of the zero (poles) of the amplitude. One should notice in Fig. 5 the extremely steep wall on the right of the dip corresponding to the true position [for $\zeta_0=0.55$, the two M_0 values are 2.9×10^4 and 4×10^7 respectively!] of the zero, which is due to the artificial pole moving towards the physical cut Γ_1. In Fig. 6 the two branches of the curve are more symmetric, although also high, since here the pole is fixed, the artificial zero, here, being moving.

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APPENDIX A BY C. FOIAS

The aim of this short note is to give a comprehensive general foundation in the frame of the nowadays abstract operator theory for some of the mathematical questions considered in the present paper; it is hoped that this treatment might be useful also in other researches in the analytic theory of strong interactions.

Let K and K' be two Hilbert complex spaces. Let T and T' be two isometric operators in K, resp. K', i.e., linear operators such that

$$\|T\varphi\|=\|\varphi\|, \quad \|T'\varphi'\|=\|\varphi'\|$$

for all $\varphi\in K$, $\varphi'\in K'$, where $\|\cdot\|$ denote the norms in K and K'. We recall that a (closed linear) subspace $H\subset K$ (resp. $H'\subset K'$) is said to be invariant to T^* (resp. T'^*), the adjoint operator of T (resp. T') if $T^*H\subset H$ (resp. $T'^*H'\subset H'$). Denote now by S (resp. S') the restriction of T^* (resp. T'^*) to an invariant subspace H (resp. H'). Let Y be an operator from K in K' intertwining T^* and T'^*, i.e.,

$$YT^*=T'^*Y, \tag{A1}$$

and verifying

$$YH\subset H'. \tag{A2}$$

Let X denote the restriction $Y|_H$ of Y to H. This is obviously an operator from H to H' verifying

$$XS=S'X \quad \text{and} \quad \|X\|\leq\|Y\|. \tag{A3}$$

[where, let us recall, the norm of an operator, say X, is defined by

$$\|X\|=\sup\|X\varphi\|,$$

where the supremum is taken over all φ in the domain of X (i.e., $\varphi\in H$), verifying $\|\varphi\|=1$.]

Conversely, the following is valid (see Ref. 15):

(I) Let X be any operator from H in H' such that

$S'X = XS$.

Then there exists an operator Y_0 from K in K' verifying

$$Y_0 T^+ = T'^+ Y_0, \quad Y_0|_H = X, \quad \|Y_0\| = \|X\|. \tag{A4}$$

Remark: Let us remark that $\|Y_0\|$ is the infimum of the norms of all operators Y such that $Y|_H = X$.

We introduce now some notation of functional spaces. Let L^2 be the Hilbert space of function defined on $\theta \in [0, 2\pi]$, $\varphi(e^{i\theta})$, with the following scalar product:

$$(\varphi, \psi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta})\psi^*(e^{i\theta})\, d\theta,$$

representable as $\varphi(e^{i\theta}) = \sum_{-\infty}^{\infty} a_n e^{in\theta}$ with $\sum_{-\infty}^{\infty} |a_n|^2 < \infty$.

Let H^2 be the subspace of functions representable as

$$\varphi(e^{i\theta}) = \sum_0^{\infty} a_n e^{in\theta},$$

with positive frequencies only.

It is clear that every function $\varphi(e^{i\theta}) = \sum_0^\infty a_n e^{i\theta} \in H^2$ may be extended in a natural way to an analytic function $\varphi(z) = \sum_0^\infty a_n z^n$, whose boundary values $\lim_{r\to 1}\varphi(Re^{i\theta})$ coincides a.e. with $\varphi(e^{i\theta})$ (see Ref. 16, Chap. III).

We shall denote by U the unitary operator in L^2 defined by $U\varphi(e^{i\theta}) = e^{i\theta}\varphi(e^{i\theta})$. Obviously $UH^2 \subset H^2$ (i.e., H^2 is invariant to U) and shall denote by $U|_{H^2}$ the restriction to H^2 of U. This is an isometric operator in H^2. The space of all functions $\chi_1(z)$ bounded and analytic for $|z| \leq 1$ is denoted by $\mathcal{H}^\infty$ and it is included in a natural way in H^2, but it is endowed with the L^∞ norm:

$$\|\chi_1(z)\|_{L^\infty} = \underset{0\leq\theta<2\pi}{\text{ess. sup}} |\chi_1(e^{i\theta})|. \tag{A5}$$

(Essential superior means the superior on a given set, modulo a set of measure zero.)

Obviously $U^+ = U^{-1}$ is given by $U^+\varphi(e^{i\theta}) = e^{-i\theta}\varphi(e^{i\theta})$ and $U^{-1}(L^2\Theta H^2) \subset L^2\Theta H^2$ (where $L^2\Theta H^2$ denotes the orthogonal supplement of H^2 in L^2). Let us denote by Q_+ and Q_- the orthogonal projections of L^2 into H^2, resp. $L^2\Theta H^2$, i.e., for $\varphi = \sum_{-\infty}^{+\infty} a_n e^{i\theta}$,

$$Q_+ \sum_{-\infty}^{\infty} a_n e^{in\theta} = \sum_0^\infty a_n e^{in\theta}, \quad Q_- \sum_{-\infty}^{\infty} a_n e^{i\theta} = \sum_{-\infty}^{-1} a_n e^{in\theta}. \tag{A6}$$

Moreover, let us put $U_- = U^+|_{L^2\Theta H^2}$. (As $L^2\Theta H^2$ is not invariant to U, we *cannot* write $U_-^+ = U|_{L^2\Theta H^2}$, but $U_-^+ = Q_- U|_{L^2\Theta H^2}$.) Now let

$$\chi(e^{i\theta}) = \sum_{-\infty}^{\infty} c_n e^{in\theta} \equiv \tilde{h}(e^{i\theta}) \tag{A7}$$

be bounded, i.e.,

$$\underset{0\leq\theta<2\pi}{\text{ess. sup}} |\tilde{h}(e^{i\theta})| = \|\tilde{h}\|_{L^\infty} < \infty$$

be given; denoting by "$\smile$" the inversion $\check{\varphi}(e^{i\theta}) = \varphi(e^{-i\theta})$, we define X for every $\varphi_2 \in L^2\Theta H^2$ by

$$X\varphi_2 = Q_-(\chi U^+ \check{\varphi}_2). \tag{A8}$$

It is clear that X depends only on the negative index coefficients, c_{-k}, $k = 1, 2, \ldots,$ of $\chi(e^{i\theta})$. Then

$$XU_-\varphi_2 = Q_-(\chi U^+ \overset{\smile}{\overline{U^{-1}\varphi_2}}) = Q_-(\chi U^+ U\check{\varphi}_2)$$

$$= Q_-(\chi\check{\varphi}_2) = Q_- U(1 - Q_-)U^+\chi\check{\varphi}_2$$
$$+ Q_- U Q_- U^+ \chi\check{\varphi}_2 = Q_- U Q_-(\chi U^+ \cdot \check{\varphi}_2)$$
$$= Q_- U X\varphi_2 = U_-^+ X\varphi_2,$$

where we used, in order,

$$Q_- U Q_+ = 0, \quad U^+(\chi\varphi) = \chi U^+\varphi$$

(where $\varphi \in L^2$ and χ is bounded) and

$$Q_- U|_{L^2\Theta H^2} = U_-^+,$$

relations which can be easily verified. In this manner, if X is given by (A8), then

$$XU_- = U_-^+ X. \tag{A9}$$

It is obvious that

$$\|X\| \leq \|\chi\|_{L^\infty}, \tag{A10}$$

and if X is defined in the same manner as X but with $\bar{\chi}$ instead of χ, then

$$Q_-(\chi - \bar{\chi}) = 0 \Rightarrow X = \bar{X}. \tag{A11}$$

Moreover, if we consider in $L^2\Theta H^2$ the orthogonal basis $\{e^{-in\theta}\}_{n=1}^{n=\infty}$, then X corresponds to the matrix

$$\mathbf{X} = \begin{Bmatrix} c_{-1} & c_{-2} & c_{-3} & c_{-4} & \cdot & \cdot & \cdot \\ c_{-2} & c_{-3} & c_{-4} & \cdot & \cdot & & \cdot \\ c_{-3} & c_{-4} & \cdot & \cdot & \cdot & & \cdot \\ c_{-4} & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \end{Bmatrix}. \tag{A12}$$

It is an easy matter to see that if such a matrix is given (this is a Hankel matrix) and if the operator X defined by it in $L^2\Theta H^2$ by the intermediate of the basis $\{e^{-in\theta}\}_{n=1}^{n=\infty}$, then X verifies (A9). Apply now Theorem I in Sec. 1 with $K = L^2$, $T = U$,

$$H = L^2\Theta H^2, \; S = U_- \text{ and } K' = H' = L^2\Theta H^2, T' = U_-, \; S' = U_-^+;$$

it results that there exists an operator Z_0 from L^2 in $L^2\Theta H^2$ such that

$$Z_0 U^+ = U_-^+ Z_0, \tag{A13}$$

$$Z_0|_{L^2\Theta H^2} = X, \tag{A14}$$

$$\|Z_0\| = \|X\|. \tag{A15}$$

Now apply again Theorem I with $K = L^2$, $T = U$, $H = L^2\Theta H^2$, $S = U_-$, $K' = H' = L^2$, $T' = U^+$, $S' = U$ and with X replaced by Z_0^+: $Z_0^+ U_- = U Z_0^+$. It results that there exists an operator Y_0 from L^2 in L^2 such that

$$Y_0 U^+ = U Y_0, \tag{A16}$$

where

$$Y_0|_{L^2\Theta H^2} = Z_0^+, \tag{A17}$$

$$\|Y_0\| = \|Z_0^+\|. \tag{A18}$$

Then (A15) and (A18) give

$$\|Y_0\| = \|X\|. \tag{A19}$$

(A16) reads also

$$\overline{UY_0^+} = Y_0^+ U^+ \tag{A20}$$

and with (A14) and (A15) imply (for φ_2, $\psi_2 \in L^2\Theta H^2$)

$$(X\varphi_2, \psi_2) = (Z_0\varphi_2, \psi_2) = (\varphi_2, Q_- Z_0^+\psi_2) = (\varphi_2, Y_0\psi_2)$$
$$= (Y_0^+\varphi_2, \psi_2) = (Q_- Y_0^+\varphi_2, \psi_2). \qquad \text{(A21)}$$

The projection operator Q_- is redundant in the last equality of (A21), but we need it in order to have an operator under which the space $L^2 \ominus H^2$ is invariant. Hence,

$$X = Q_- Y_0^+|_{L^2\ominus H^2}, \quad \|X\| = \|Y_0\|. \qquad \text{(A22)}$$

Now denoting by $\mathbf{1}$ the constant function e^0, and putting $\chi^0(e^{i\theta}) \equiv e^{i\theta} Y_0^+ \mathbf{1}$, we have for any trigonometric polynomial

$$\varphi = \sum_{|n|\le N} a_n e^{in\theta} = \sum_{|n|\le N} (U^+)^{-n}\mathbf{1};$$

hence by (A20)

$$Y_0^+\varphi = \sum a_n U^{-n} Y_0^+\mathbf{1} = \chi^0 U^+ \check{\varphi}. \qquad \text{(A23)}$$

By using (A23), it is now an easy matter to verify that χ^0 is essentially bounded and that

$$\|Y_0\|_{L^2} = \|\chi^0\|_{L^\infty}. \qquad \text{(A24)}$$

Finally, from (A23) and (A22) we obtain (A8). In this manner we have obtained the following extrapolation theorem due to Z. Nehari[13] (see also Ref. 19):

(II) Let

$$\tilde{h}_2 = \sum_{k=1}^{\infty} c_{-k} e^{-ik\theta} \qquad \text{(A25)}$$

be given and suppose that the matrix corresponding to $\mathbf{X}$ *given by* (A12) *defines in* $L^2\ominus H^2$ *an operator* X *of norm* $\|X\| < \infty$; *then, if* ϵ_0 *denotes the infimum of the* $\|\chi\|_{L^\infty}$ *of all essentially bounded functions* χ *such that* $\chi_2 \equiv \tilde{h}_2$ (the negative frequency part of the histogram), *where*

$$Q_-\chi = \chi_2, \qquad \text{(A26)}$$

we have

$$\epsilon_0 = \|X\|; \qquad \text{(A27)}$$

moreover, this infimum is reached (namely such an optimal function is that constructed by the above successive application of Theorem I).

Remark: If $c_{-1} = 0$ for $k > N$, then the norm of the operator corresponding to the matrix $\mathbf{X}$ given by (A12) is identical to the norm of the linear transofrmation given by the matrix

$$\mathbf{X}_N = \begin{Bmatrix} c_{-1} & c_{-2} & c_{-1} & \cdots & c_{-N+1} & c_{-N} \\ c_{-2} & c_{-3} & c_{-4} & \cdots & c_{-N} & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ c_{-N+1} & c_{-N} & 0 & \cdots & 0 & 0 \\ c_{-N} & 0 & 0 & \cdots & 0 & 0 \end{Bmatrix} \qquad \text{(A28)}$$

in the N-dimensional complex euclidian space. Since this norm can be computed easily as the greatest eigenvalues of $\sqrt{\mathbf{X}_N^+\mathbf{X}_N}$, it is important to know if we may neglect the remainder of the terms expansion (A24).

We shall give some properties related to the proceding theorem.

First, let us remark that if $\tilde{h}_2$ is a polynomial in $e^{-in\theta}$, say $\sum_1^N c_{-k}e^{-ik\theta}$, then, since

$$\mathbf{X} = \begin{Bmatrix} \mathbf{X}_N & 0 \\ 0 & 0 \end{Bmatrix},$$

it is easy to see that the operator X is with finite-dimensional range. Suppose now that χ is continuous, and let

$$\chi_2^{(N)} = \sum_{k=1}^{N} \left(1 - \frac{k-1}{N}\right) c_{-k} e^{-ik\theta} \qquad \text{(A29)}$$

be the Fejer—Cesaro sequence for its Fourier expansion. Then

$$\|\chi_2^{(N)} - \chi\|_{L^\infty} \to 0.$$

Therefore, if $X^{(N)}$ denotes the operator corresponding to $\chi_2^{(N)}$, then instead of χ_2 we have

$$\|X^{(N)} - X\| \to 0 \quad \text{for } N \to \infty. \qquad \text{(A30)}$$

In particular, (A30) implies

$$\|X^{(N)}\| \to \|X\|. \qquad \text{(A30}')$$

On the other hand, by the remark made above, $X^{(N)}$ is with finite-dimensional range, so that (A30) implies that X is completely continuous. But then there exists a $\varphi_2^0 \in L^2\ominus H^2$, $\|\varphi_2^0\| = 1$, such that $\|X\varphi_2^0\| = \|X\|$ (take an eigenvector for the greatest eigenvalues of X^+X). But then, using (A8), (A20), and (A24), we have

$$\|X\| = \|X\varphi_2^0\| = \|Q_-\chi^0 U^+\check{\varphi}_2^0\| \le \|\chi^0 U^+\check{\varphi}_2^0\|$$
$$= \|\chi^0\check{\varphi}_2^0\| \le \|\chi^0\|_{L^\infty} = \|Y_0\| = \|X\|,$$

so that

$$\|\chi^0\check{\varphi}_2^0\|_{L^2} = \|\chi^0\|_{L^\infty}.$$

Hence, since $(\|\varphi_2^0\|_{L^2} = 1)$

$$\|\chi^0\check{\varphi}_2^0\|_{L^2} \le \|\chi^0\|_{L^2}\|\varphi_2^0\|_{L^2} = \|\chi^0\|_{L^2} \le \|\chi^0\|_{L^\infty},$$

we have $\|\chi^0\|_{L^2} = \|\chi^0\|_{L^\infty}$, i.e., we have

$$\frac{1}{2\pi}\int_0^{2\pi} (\|\chi^0\|^2_{L^\infty 1} - |\chi^0(e^{i\theta})|^2)\, d\theta = 0, \quad \text{i.e.,} \quad |\chi^0|$$
$$= \|\chi^0\|_{L^\infty} \text{ a.e.}$$

Thus, we obtain the following supplementary properties to the Theorem II:

(III) *Suppose that* $\tilde{h}_2$ *is continuous*; *then the norm of the matrix* [see (A29)],

$$\mathbf{X}_N' = \begin{Bmatrix} c_{-1} & (1-1/N)c_{-2} & \cdots & [1-(N-1)/N]c_{-N} \\ (1-1/N)c_{-2} & (1-2/N)c_{-3} & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ [1-(N-1)/N]c_{-N} & \cdot & \cdots & 0 \end{Bmatrix} \qquad \text{(A31)}$$

in E^N *tends for* $N \to \infty$ *to* ϵ_0. *Moreover, there exists a unique* χ^0 *such that*

$$Q_-\chi^0 = \tilde{h}_2 \quad \text{and} \quad \|\chi^0\|_{L^\infty} = \epsilon_0, \qquad \text{(A32)}$$

and this verifies

$$\|\chi^0\|_{L^\infty} = |\chi^0(e^{i\theta})| \quad \text{a.e. on } [0, 2\pi]. \qquad \text{(A33)}$$

We have only to prove the uniticity, knowing that any optimal χ [i.e., verifying (A32)] satisfies (A33). To this purpose, let χ^1 to be another optimal function. Then,

$$\chi' = \tfrac{1}{2}(\chi^1 + \chi^0)$$

is still an optimal function, thus

$$|\chi'(e^{i\theta})| = \epsilon_0 \text{ a.e.},$$

i.e.,

$$\tfrac{1}{2}|\chi^0(e^{i\theta}) + \chi^1(e^{i\theta})| = \epsilon_0 \text{ a.e.}$$

But this obviously implies

$$\chi^0(e^{i\theta}) = \chi^1(e^{i\theta}) \text{ a.e.},$$

by the strict convexity of the modulus.

Remarks:

(1) In the hypotheses of Theorem III we have for the optimal function χ^0,

$$\|\chi^0\|_{L^\infty} = \|\chi^0\|_{L^2} \geq \|\tilde{h}_2\|_{L^2} = \sum_1^\infty |c_{-k}|^2. \tag{A34}$$

(2) Let us denote by $\epsilon_0[\tilde{h}_2]$ the ϵ defined in Theorem II. The relations (A32) and (A34) show

$$\epsilon_0[\tilde{h}_2] \geq \|\tilde{h}_2\|_{L^2}. \tag{A35}$$

Let us remark that we have quite different behavior with respect to the L^∞ norm, namely,

$$\inf \epsilon_0[\tilde{h}_2]/\|\tilde{h}_2\|_{L^\infty} = 0, \tag{A36}$$

the infimum being taken for the continuous $\tilde{h}_2$.

To see this, let us suppose the contrary, that is,

$$\epsilon_0[\tilde{h}_2] \geq \delta . \|\tilde{h}_2\|_{L^\infty} \tag{A37}$$

for all continuous $\tilde{h}_2$ and a fixed δ. Take $\tilde{h}'$ essentially bounded, i.e., $\tilde{h}' \in L^\infty$, and put $\tilde{h}_2' = Q_-\tilde{h}'$. Also let σ_n' denote the Fejer—Cesaro sequence for the Fourier expansion of $\tilde{h}'$.

Then since the $Q_-\sigma_n'$ are continuous, we have

$$\|\tilde{h}'\|_{L^\infty} \geq \|\sigma_n\|_{L^\infty} \geq \epsilon_0[Q_-\sigma_n'] \geq \delta \|Q_-\sigma_n\|_{L^\infty}, \tag{A38}$$

where the first inequality follows from the well-known properties of the Fejer kernel. It is obvious that

$$Q_-\sigma_n' \to Q_-\tilde{h}' = \tilde{h}_2' \text{ in } L^2.$$

Therefore, (A38) implies easily that

$$\|Q_-\tilde{h}'\|_{L^\infty} \leq (1/\delta)\|\tilde{h}'\|_{L^\infty}, \tag{A39}$$

for all $\tilde{h}' \in L$. Or this is impossible, since, for instance, for $\tilde{h}' = \sum_1^\infty (1/n)\sin(n\theta)$ we have $Q_-\tilde{h}' = (1/2i)\sum_1^\infty (e^{-in\theta}/n)$, which does not belong to L^∞!

(3) Theorem III is a particular (though sufficient!) case of the results V.M. Adamjan, D.Z. Arov, and M.G. Krein have published in their papers (see, for instance, Sec. 3 in Ref. 14a and Secs. 2 and 4 in Ref. 14b). Moreover, we recommend the paper 14b for its explicit formula concerning the optimal function and its complete and definite study of such extrapolation questions.

APPENDIX B

We shall remind the reader here of a simple theorem about the supremum of the module of a function $\bar{l}(\zeta)$ holomorphic in the unit disk D, whose L^2 norm on the boundary $\Gamma(|\zeta|=1)$ equals $\epsilon_l = (\epsilon^2 - \epsilon_{h_2}^2)^{1/2}$. As

$$\|\bar{l}\|_{L^2}^2 = \sum_0^\infty |a_n|^2 < \epsilon_l^2, \tag{B1}$$

where a_n are the Fourier coefficients of $\bar{l}(e^{i\theta})$, [i.e., $\bar{l}(\zeta) = \sum_0^\infty a_n \zeta^n$], one readily finds that the value of the module of $\bar{l}(\zeta)$ in the origin

$$|\bar{l}(\zeta=0)| \equiv |a_0| \tag{B2}$$

cannot exceed ϵ_l.

A similar inequality can be derived for each interior point $\zeta_0 \in D$. Indeed, performing the usual transformation which leaves the unit circle invariant and brings the point $\zeta = \zeta_0$ into the origin,

$$\zeta' = (\zeta - \zeta_0)/(1 - \zeta_0^* \zeta), \tag{B3}$$

we get $d\theta = |1 - \zeta_0^*\zeta|^2/(1 - |\zeta_0|^2)\, d\theta'$, so that the unweighted problem on the $|\zeta| = 1$ circle becomes a weighted problem in the $|\zeta'| = 1$ circle. The weight

$$\rho = |1 - \zeta_0\zeta|^2/(1 - |\zeta_0|^2) \tag{B4}$$

can be absorbed by the exterior function

$$C_\rho = (1 - \zeta_0\zeta)/(1 - |\zeta_0|^2)^{1/2} \tag{B5}$$

[the function $C_\rho(\zeta')$ defined by (B5) has no singularities inside D!], so that one gets for the weighted function

$$\tilde{l}(\zeta) = C_\rho(\zeta)\bar{l}(\zeta), \tag{B6}$$

$$\frac{1}{2\pi}\int_0^{2\pi} |\bar{l}(\zeta')|^2 \rho(\zeta')\, d\theta' = \frac{1}{2\pi}\int_0^{2\pi} |\bar{l}(\zeta')C_\rho(\zeta')|^2 d\theta' = \frac{1}{2\pi}\int_0^{2\pi} |\tilde{\bar{l}}(\zeta')|^2 d\theta' < \epsilon_l^2. \tag{B7}$$

so, one gets in analogy to (B2),

$$|\tilde{l}(\zeta'=0)| \equiv |\bar{l}(\zeta_0)C_\rho(\zeta_0)| < \epsilon_l. \tag{B8}$$

Hence, as $C_\rho(\zeta_0) = (1 - |\zeta_0|^2)^{1/2}$, one finally gets

$$|\bar{l}(\zeta_0)| < \epsilon_l/(1 - |\zeta_0|^2)^{1/2}. \tag{B9}$$

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[16]To find Y_χ^+ [see Eq. (3.10)] from Eq. (3.7a) one first notices that the θ-reflection operator $T\varphi \equiv \check{\varphi}$ is self-adjoint. Indeed, if u_n are the basis vectors exp (in θ), we have $(T^+u_m u_n) \equiv (u_m, Tu_n) \equiv (u_m, \check{u}_n) = (u_m, u_{-n}) = \delta_{m,-n}$ and hence $T^+u_m = u_{-m}$, i.e., $T^+ = T$. Further, observing that $TU = U^+T$, starting from (3.7a) one gets $Y_\chi^+ = (\check{\chi}^* U^+T)^+ = T^+ U\check{\chi} = TU\check{\chi} = U^+\chi T = \chi U^+ T$, where the last equality holds because both χ and U^+ are multiplication operators.

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[20]All these weight functions (C_0, C_1, and so on) defined by the values of their moduli on the boundary and having no zeroes inside D are commonly known to mathematicians under the name of "exterior functions." The name "truncated Carleman kernels" or "Carleman functions" was introduced by us at the seminar at the 1969 Lund Conference [6] in analogy with Lavrentiev's "Carleman kernels", but this appears to have produced much confusion in the literature, since references to Carleman's book on quasianalytic functions are now floating around although there is no trace in it of "Carleman functions" at all! Similar weight functions were used, for instance, in the weighted L^2 norms [see Eq. (5.11)] in the early papers of Szegö and one can certainly say that the weight functions are as old as analysis itself. Therefore, in order to avoid further confusions, it might be better to call them simply "exterior weight functions."

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