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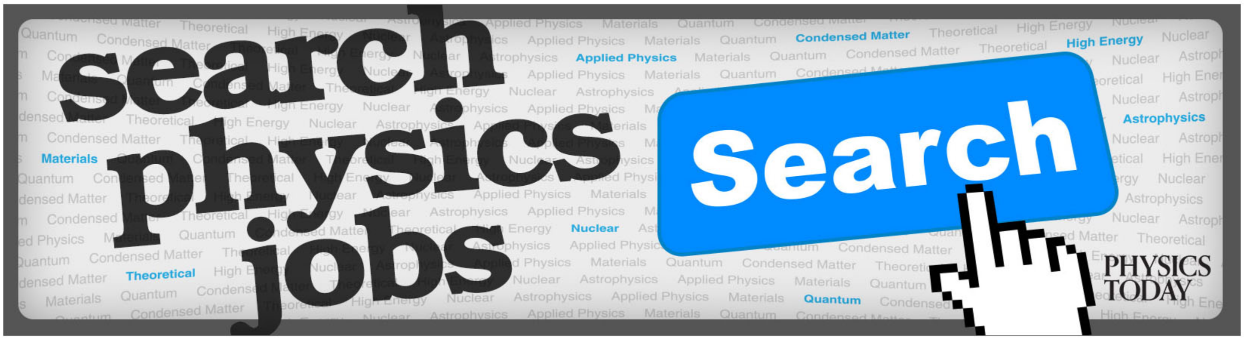

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Neutrino in the presence of gravitational fields: Separation of variables

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It is well known that the most complete information about single-particle states is contained in its wave function. For spin-$\frac{1}{2}$ particles this means that it is necessary to have exact solutions of the Dirac equation. In particular, in the case of neutrinos in the presence of gravity, it is necessary to solve the covariant Dirac equation. At present, the existence of neither massless neutrinos (electron neutrinos) nor massive neutrinos (muon and τ neutrinos) cannot be excluded. Since for massless neutrinos any solution of the Dirac equation is also a solution of the Weyl equation, there exists the possibility of studying, from a unified point of view, massive as well as massless neutrinos by means of the Dirac equation. In the search of exact solutions of systems of partial differential equations one can proceed as follows: (a) separation of variables and (b) solution of the corresponding ordinary differential equations. In the present paper, a complete analysis of the separation of variables in the Dirac equation for massive as well as for massless neutrinos is carried out by means of the algebraic method [J. Math. Phys. **30**, 2132 (1989)]. It is found that for the massless neutrinos, there are further possibilities of separation of variables, not valid for the massive case.

I. INTRODUCTION

Perhaps it is impossible to indicate another particle, which, like neutrino, has been so important in all the spheres of influence in the nature of matter. The neutrino was introduced in theoretical physics by Pauli in order to preserve the laws of conservation of energy, impulse, and angular momentum in the β-decay processes, i.e., at first, it was theoretically discovered, and, nowadays, the neutrino is present in all the areas of modern physics: from weak interaction to relativistic astrophysics. The neutrino has some null properties: zero mass—at least for electron neutrinos, zero electric charge, zero magnetic momentum; therefore neutrino interacts very weakly with matter. It was not coincidence that it was necessary about 30 years in order to identify the neutrinos experimentally, while its existence was out of doubt from the theoretical point of view.

In virtue of the fact that the neutrino is a spin-$\frac{1}{2}$ particle it must be described by the Dirac equation that, for massless particles, splits in two Weyl equations, corresponding to two opposite longitudinal polarization. In the case of free neutrinos, Dirac and Weyl equations can exactly be solved, obtaining in this way the well-known plane spinor waves.

For many years it was enough to have plane waves for computing scattering cross sections, energy spectra, angular distribution, and polarization characteristics in the processes of transmutation of elementary particles in the presence of neutrinos in weak interactions. We have to say that, using plane waves as exact solutions, the results obtained in the theory of elementary particles are, in fact, approximate and restricted by the limitations of perturbation theory, being the most developed of them, the S-matrix theory.

A new period in the study of the neutrino begins with the appearance of the neutrino in relativistic astrophysics. In fact, the contemporary relativistic astrophysics is formulated on the basis of a unified theory, including quantum theory and the theory of gravitation. The present quantum physics has reached a level such that it has become necessary to consider not only relativistic, but gravitational effects. On the other hand, a consistent theory of gravitation must take into account quantum processes. Here, we have to mention the well-known Hawking effect, theoretical discovery, that has changed, in a radical way, our understanding of black holes.

The development of the unified quantum-gravitational theory is impossible without an exact description of single-particle states, i.e., without exact solutions of the covariant wave equations. In particular, when we work with neutrinos we need exact solutions of the massive or massless Dirac equation and Weyl equations in the presence of external gravitational fields.

Since contemporary high-energy experimental physics, on one hand, does not exclude the existence of the massless neutrino (electron neutrino) and, on the other hand, it is almost sure that the muon neutrino and τ neutrino are massive, it would be of interest to study,

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from a unified point of view, the possibility of an exact description of massive and massless neutrinos in the presence of gravity, i.e., to analyze the most general possibilities of finding exact solutions of the Dirac equation for massive ($m\neq 0$) and massless ($m=0$) particles.

During the last few years, a great effort has been devoted to the search of exact solutions of the Dirac equation. We have to mention the pioneer works of Brill and Wheeler[1] and Chandrasekhar.[2,3] Brill and Wheeler[1] studied Dirac equation in a central symmetric gravitational field associated with a diagonal metric; they succeeded, using a normal diagonal tetrad, in separating variables in the Dirac equation, and constructing the generalized angular momentum operator. Chandrasekhar[2,3] considers the axially symmetric nondiagonal Kerr metric. Using an isotropic tetrad and the Newman–Penrose spin coefficients method,[4,5] he was able to separate variables in the Dirac equation in a complicated geometrical configuration.

Since the apparition of the Brill and Wheeler article, other authors began to analyze the possibilities of separation of variables in the Dirac in the presence of gravitational fields associated with a diagonal metric. In particular, the separation of variables in central symmetric fields, in some special tetrads, was carried out in Ref. 6. The necessary and sufficient conditions guaranteeing a complete or particular separation of variables in the Dirac equation in the presence of diagonal metrics, using for this purpose diagonal tetrads, in a general approach was studied in Ref. 7. Separation of variables in the Dirac equation in diagonal metrics, without restricting themselves to diagonal tetrads, has been considered by Bagrov and their co-workers.[8,9]

After Chandrasekhar's[2] article was published, a series of works,[10–14] devoted to the problem of separation of variables in the Dirac equation in axially symmetric metrics, in particular, with rotational symmetry. In those articles there is a further application of the method and results obtained in Chandrasekhar's work. The results presented in Ref. 2 generalize, for the massive Dirac case, the results obtained by Teukolsky[15] for the Weyl neutrino equation. Here we have to mention that Chandrasekhar succeeded in separating variables in the Dirac equation in the Kerr background because he performed the separation of variables prior decoupling. A general feature of the articles[2,10–14] is the use of noncommuting separating operators. Also, an article[16] is devoted to the separation of variables in the Dirac equation when the metric is nondiagonal, using, in contrast with other articles,[2,10–14] commuting separating operators.

In the article by Kalnins *et al.*[17] it is presented a rigorous proof of the theorem, showing that, for the Dirac equation in Minkowski space-time, second-order symmetry operators can be expressed in terms of first-order symmetry operators. In spite of the fact that, in Cartesian coordinates, it is trivial that any solution of the free Dirac equation also has to be a solution of the Klein–Fock equation, we have to recognize the importance of the theorem demonstrated by Kalnins *et al.*

During the last few years, many articles have been dedicated to the study of the exact description of massless neutrinos in external gravitational fields. Besides the works on the separation of variables in a self-coupled system of several field equations, we have to notice that the Brill and Wheeler article[1] is perhaps the first work on the separation of variables in the Weyl equation in central symmetric gravitational fields. Undoubtedly, the results of articles[2,8–13] can be extended to the massless case by taking the zero mass limit, however, it would be of interest to analyze the further possibilities of the separation of variables for the massless neutrino equation if, from the same beginning, we equate to zero the mass term in the Dirac equation.

Kamran and McLenaghan[11] have shown that there exists a coordinate system and null tetrad for space-times admitting a two-parameter Abelian orthogonally transitive isometry group and a pair of shear-free geodesic null congruences in which the neutrino equation is solvable by separation of variables when the metric is Petrov type D; they also discuss the conditions of separability for the massive Dirac equation. In Ref. 18, they also constructed a first-order differential operator whose commutator with the Weyl neutrino is proportional to it in the class $\mathscr{D}$ of Petrov type D.

Finally, we have to mention the result obtained by Fels and Kamran,[19] where high-order symmetry operators are related to the possibility of nonfactorizable, in the sense of Miller, separation of variables, in the Dirac equation in curved backgrounds. Let us recall that the Kalnins *et al.*[17] theorem was shown for the free case, and it may not be valid in the presence of external fields. The last one becomes evident if we remember that the quadratic Dirac equation in the presence of fields does not coincide with the corresponding Klein–Fock equation, because of the presence of additional derivatives of the fields.

In recent years there has been a growing interest in the study of exact solutions of the Dirac equation for massive and massless particles. Since the present paper is written devoting special care to massless neutrinos, we are going to present some results on exact solutions of the Dirac equation when $m\equiv 0$.

First of all, we have to mention that most of the exact solutions of the Dirac equation for massless neutrinos, reported in the literature, are related to metrics, depending only on one space-time variable.

The metric,

$$ds^2=e^{2u}(dx^2-dt^2)+e^{2v}(dy^2+dz^2),\qquad u=u(x),\quad v=v(x), \tag{1.1}$$

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was used in solving Dirac equation in the articles by Davis[20] and Pechenik.[21]

An exact solution of the Dirac equation in the Marder metric,

$$ds^2 = e^{2(\alpha-\beta)}(dr^2 - dt^2) + r^2 e^{-2\beta} d\varphi^2 + e^{2(\beta+\nu)} dz^2, \tag{1.2}$$

$$\alpha = \alpha(r), \quad \beta = \beta(r), \quad \nu = \nu(r),$$

was obtained by Krori[22,23] and Patra.[24,25]

The neutrino in the Kasner metric,

$$ds^2 = t^{2a} dx^2 + t^{2b} dy^2 + t^{2c} dz^2 - dt^2, \quad a+b+c = a^2+b^2+c^2 = 1, \tag{1.3}$$

was studied by Srivastava.[26]

Exact solutions of the Dirac equation for the massless neutrino in the spatially flat Robertson–Walker metric,

$$ds^2 = a^2(t)(dx^2+dy^2+dz^2) - dt^2, \tag{1.4}$$

were obtained by Barut[27] and Lotze;[28] also, exact solutions for spatially closed and open Robertson–Walker universes were obtained by Villalba and Percoco.[29]

Finally, we have to mention the exact solutions of the Dirac equation for massive and massless neutrinos obtained, for the Kerr–Newman metric, by Einstein and Finkelstein,[30] and for rotating universes by Krori *et al.*[31]

All the above mentioned solutions were obtained using a complete separation of variables scheme. Namely, the possibility of separating variables, almost in all the cases, determines the possibility of finding exact solutions in any partial differential equation, and, in particular, in the Dirac equation.

The problem of separating variables in the Dirac equation in the presence of gravitational fields can be studied as the problem of finding all the types of metric allowing a complete separation of variables. In fact, this is the purpose of the present paper. Considering the most general form of the metric tensor, we compute all gravitational field configurations, allowing separation of variables in the Dirac equation. In this paper we restrict ourselves to diagonal metrics. We analyze, in a parallel way, massive and massless neutrinos. Obviously the results obtained for $m \neq 0$ (massive Dirac particle, but not necessarily neutrino), are valid in the case $m=0$. However, the case $m \equiv 0$ gives place to new possibilities of the separation of variables not included when $m \neq 0$.

In the present paper, the separation of variables in the Dirac equation is carried out by means of the algebraic method, which we have applied in the presence of external vector fields.[32]

II. ALGEBRAIC METHOD OF SEPARATION OF VARIABLES

In this paper we analyze the separation of variables in the Dirac equation in the presence of gravitational fields associated with a diagonal metric of the form

$$ds^2 = a_i(dx^i)^2 + a_j(dx^j)^2 + a_m(dx^m)^2 + a_n(dx^n)^2, \tag{2.1}$$

where the metric functions $a_k(x^i,x^j,x^m,x^n)$ are definite positive and satisfy the necessary conditions of differentiability. The timelike variable (in general, we do not fix it beforehand) is imaginary, this one allows us to consider only Hermitian Dirac matrices. This avoids us to work with the squares of an anti-Hermitian matrix, which arises when we have to deal with real timelike variables. The advantage of this convention becomes clear under separation of variables in the Dirac equation.

The covariant generalization of the Dirac equation has the form

$$\{\gamma^\mu(\partial_\mu - \Gamma_\mu) + m\}\widetilde{\Psi} = 0, \tag{2.2}$$

where Γ_μ are the spinor connections, and γ^μ are the Dirac matrices associated with the line element (2.1). γ^μ are related to the γ^m-standard flat Dirac matrices as follows:

$$\gamma^\mu = h^\mu{}_m \gamma^m, \quad \gamma_\mu = h_\mu{}^m \gamma_m. \tag{2.3}$$

The tetrad coefficients are determined by the equations

$$g^{\mu\nu} = h^\mu{}_m h^\nu{}_n \eta^{mn}, \quad g_{\mu\nu} = h_\mu{}^m h_\nu{}^n \eta_{mn}, \tag{2.4}$$

where $g_{\mu\nu}$ and η_{mn} are the metric tensors in the space-time (2.1) and in the Minkowski space-time, respectively,

$$g_{\mu\nu} = \mathrm{diag}(a_i,a_j,a_m,a_n), \quad \eta_{mn} = \mathrm{diag}(1,1,1,1). \tag{2.5}$$

Matrices γ^μ and γ^m satisfy the following commutation relations:

$$[\gamma^\mu,\gamma^\nu]_+ = 2g^{\mu\nu}I, \quad [\gamma^m,\gamma^n]_+ = 2\eta^{mn}I. \tag{2.6}$$

The Greek indices are related to space-time (2.1), and the Latin indices are related to the Minkowski space, in both cases the indices take the values *i, j, m,* and *n* in correspondence with (2.1).

Choosing to work with a normal diagonal tetrad of the form

$$h^\mu{}_m = \mathrm{diag}(a_i{}^{1/2}, a_j{}^{1/2}, a_m{}^{1/2}, a_n{}^{1/2}), \tag{2.7}$$

after computing the spinor connections Γ_μ, Eq. (2.2) takes the form

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$$\left\{\frac{\gamma^i}{a_i^{1/2}}\left(\partial_i-\frac{\partial_i a_i}{4a_i}\right)+\frac{\gamma^j}{a_j^{1/2}}\left(\partial_j-\frac{\partial_j a_j}{4a_j}\right)+\frac{\gamma^m}{a_m^{1/2}}\left(\partial_m-\frac{\partial_m a_m}{4a_m}\right)\right.$$

$$\left.\times\frac{\gamma^n}{a_n^{1/2}}\left(\partial_n-\frac{\partial_n a_n}{4a_n}\right)+m\right\}\Psi=\{H\}\Psi=0, \tag{2.8}$$

$$\Psi=\{a_i a_j a_m a_n\}^{-1/4}\widetilde{\Psi}. \tag{2.9}$$

In Eq. (2.8) there is no summation over the repeated indices i,j,m,n, which only indicate different space-time variables in accordance with (2.1).

The separation of variables in Eq. (2.8) is carried out by means of the algebraic method,[32] according to which the initial equation is changed by a new equivalent equation using the following scheme:

$$\{H\}\Psi=0\Rightarrow\mathscr{F}\{H\}\Gamma\Gamma^{-1}\Psi=(\widehat{K}_\alpha+\widehat{K}_\beta)\Phi=0, \tag{2.10}$$

$$\Psi=\Gamma\Phi, \tag{2.11}$$

the symbol "$\Rightarrow$" indicates the mapping defined by the equal sign in (2.10), where α and β are group indices of the separated variables, $\widehat{K}_\alpha$ and $\widehat{K}_\beta$ are first-order commuting matrix differential operators,

$$\widehat{K}_\alpha\widehat{K}_\beta-\widehat{K}_\beta\widehat{K}_\alpha=0, \tag{2.12}$$

which we shall name separating operators; Γ is a constant nonsingular separating matrix and $\mathscr{F}$ is a scalar function that we have to find (if they exist) in order to fulfill the condition (2.12).

Here two different ways of a complete separation of variables emerge. (a) Consecutive separation of variables, when one variable is separated from the remaining three variables ($x^i,x^j,x^m,x^n\Rightarrow x^i/x^j,x^m,x^n$), then one variable is separated from two variables ($x^j,x^m,x^n\Rightarrow x^j/x^m,x^n$), and finally, the remaining two variables are separated ($x^m,x^n\Rightarrow x^m/x^n$). (b) Pairwise separation of variables ($x^i,x^j,x^m,x^n\Rightarrow x^i,x^j/x^m,x^n$), and then the separation is carried out in each pair ($x^i,x^j\Rightarrow x^i/x^j,x^m,x^n\Rightarrow x^m/x^n$). Because of the covariant formulation of the problem, indices do not have an absolute character: we can begin the separation from any of the four variables.

Here we have to mention that if it is possible to separate variables partially or completely in the Dirac equation with zero rest mass, such a separation is also possible in the metrics conformally equivalent to (2.1):

$$ds^2=f(x^i,x^j,x^m,x^n)\{a_i(dx^i)^2+a_j(dx^j)^2+a_m(dx^m)^2$$
$$+a_n(dx^n)^2\}. \tag{2.13}$$

The Dirac equation in this metric takes the form (2.8) after introducing a new unknown function as follows:

$$\Psi\to f^{1/4}\Psi \tag{2.14}$$

(Miller's R separation[33]).

Then, all the consequent results on the separation of variables in Eq. (2.8), for $m\equiv0$, are also valid for the metrics of the form (2.13).

It was shown, in Ref. 7, that in order to separate variables in Eq. (2.8) we require a multiplicative separation of the metric functions. We find convenient, for the subsequent study, to introduce the following notation: (a) for consecutive separation of variables,

$$\begin{aligned}a_k(x^i,x^j,x^m,x^n)&\Rightarrow\tilde{a}_k(x^i)b_k(x^j,x^m,x^n)=\tilde{a}_kb_k\\&\Rightarrow\tilde{a}_k\tilde{b}_k(x^j)c_k(x^m,x^n)=\tilde{a}_k\tilde{b}_kc_k\\&\Rightarrow\tilde{a}_k\tilde{b}_k\tilde{c}_k(x^m)\tilde{d}_k(x^n)=\tilde{a}_k\tilde{b}_k\tilde{c}_k\tilde{d}_k;\end{aligned} \tag{2.15}$$

(b) for pairwise separation of variables,

$$\begin{aligned}a_k(x^i,x^j,x^m,x^n)&\Rightarrow v_k(x^i,x^j)w(x^m,x^n)\\&\Rightarrow\tilde{a}_k(x^i)\tilde{b}_k(x^j)\tilde{c}_k(x^m)\tilde{d}_k(x^n)\\&=\tilde{a}_k\tilde{b}_k\tilde{c}_k\tilde{d}_k.\end{aligned} \tag{2.16}$$

The overtilde indicates that the corresponding function depends only on its own variable.

The following will be useful for understanding the notation:

$$\begin{array}{c|cccc} & i & j & m & n\\ \hline i & \tilde{a}_i & \tilde{b}_i & \tilde{c}_i & \tilde{d}_i\\ j & \tilde{a}_j & \tilde{b}_j & \tilde{c}_j & \tilde{d}_j\\ m & \tilde{a}_m & \tilde{b}_m & \tilde{c}_m & \tilde{d}_m\\ n & \tilde{a}_n & \tilde{b}_n & \tilde{c}_n & \tilde{d}_n\end{array}. \tag{2.17}$$

III. CONSECUTIVE SEPARATION OF VARIABLES

A. Separation $x^i,x^j,x^m,x^n\Rightarrow x^i/x^j,x^m,x^n$

Let us consider the separation of variables in Eq. (2.8) according to the scheme

$$\{H\}\Psi=0\Rightarrow\mathscr{F}\{H\}\Gamma\Gamma^{-1}\Psi=(\widehat{K}_i+\widehat{K}_{jmn})\Phi, \tag{3.1}$$

$$\Psi=\Gamma\Phi,$$

$$\widehat{K}_i\widehat{K}_{jmn}-\widehat{K}_{jmn}\widehat{K}_i=0, \tag{3.2}$$

where the "$\Rightarrow$" symbol, like in (2.10), indicates the mapping defined by the equal sign.

The most general form of the operators $\widehat{K}_i$ and $\widehat{K}_{jmn}$ corresponding to the explicit form of Eq. (2.8) can be written as follows:

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$$\hat{K}_i = \mathscr{F}\left\{\frac{\gamma^i}{a_i^{1/2}}\left(\partial_i - \frac{\partial_i \tilde{a}_i}{4\tilde{a}_i}\right) + \alpha m\right\}\Gamma, \tag{3.3}$$

$$\hat{K}_{jmn} = \mathscr{F}\left\{\frac{\gamma^j}{a_j^{1/2}}\left(\partial_j - \frac{\partial_j b_j}{4b_j}\right) + \frac{\gamma^m}{a_m^{1/2}}\left(\partial_m - \frac{\partial_m b_m}{4b_m}\right) + \frac{\gamma^n}{a_n^{1/2}}\left(\partial_n - \frac{\partial_n b_n}{4b_n}\right) + (1-\alpha)m\right\}\Gamma, \tag{3.4}$$

where α is a numeric parameter.

It is not difficult to see that the metric functions must be multiplicative in the separated variables as follows:

$$a_k(x^i,x^j,x^m,x^n) = \tilde{a}_k(x^i)b_k(x^j,x^m,x^n) = \tilde{a}_k b_k. \tag{3.5}$$

If the condition (3.5) is not fulfilled there will not be a function $\mathscr{F}$ guaranteeing the commutation relations (3.2).

Substituting (3.3) and (3.4) into (3.2) and taking into account (3.5), we obtain the following system of equations and identities:

$$\frac{\gamma^i}{a_i^{1/2}}\partial_i\Gamma\mathscr{F}\frac{\gamma^r}{a_r^{1/2}}\partial_r - \frac{\gamma^r}{a_r^{1/2}}\partial_r\Gamma\mathscr{F}\frac{\gamma^i}{a_i^{1/2}}\partial_i = 0,$$

$$\frac{\gamma^i}{a_i^{1/2}}\partial_i\Gamma\mathscr{F}\frac{\gamma^r}{a_r^{1/2}}\frac{\partial_r a_r}{4a_r} - \frac{\gamma^r}{a_r^{1/2}}\frac{\partial_r a_r}{4a_r}\Gamma\mathscr{F}\frac{\gamma^i}{a_i^{1/2}}\partial_i = 0,$$

$$(\gamma^i/a_i^{1/2})\partial_i\Gamma\mathscr{F}(1-\alpha)m - (1-\alpha)m\Gamma\mathscr{F}(\gamma^i/a_i^{1/2})\partial_i = 0,$$

$$\frac{\gamma^i}{a_i^{1/2}}\frac{\partial_i a_i}{4a_i}\Gamma\mathscr{F}\frac{\gamma^r}{a_r^{1/2}}\partial_r - \frac{\gamma^r}{a_r^{1/2}}\partial_r\Gamma\mathscr{F}\frac{\gamma^i}{a_i^{1/2}}\frac{\partial_i a_i}{4a_i} = 0,$$

$$\frac{\gamma^i}{a_i^{1/2}}\frac{\partial_i a_i}{4a_i}\Gamma\mathscr{F}\frac{\gamma^r}{a_r^{1/2}}\frac{\partial_r a_r}{4a_r} - \frac{\gamma^r}{a_r^{1/2}}\frac{\partial_r a_r}{4a_r}\Gamma\mathscr{F}\frac{\gamma^i}{a_i^{1/2}}\frac{\partial_i a_i}{4a_i} = 0,$$

$$\frac{\gamma^i}{a_i^{1/2}}\frac{\partial_i a_i}{4a_i}\Gamma\mathscr{F}(1-\alpha)m - (1-\alpha)m\Gamma\mathscr{F}\frac{\gamma^i}{a_i^{1/2}}\frac{\partial_i a_i}{4a_i} = 0,$$

$$\alpha m\Gamma\mathscr{F}(\gamma^r/a_r^{1/2})\partial_r - (\gamma^r/a_r^{1/2})\partial_r\Gamma\mathscr{F}\alpha m = 0,$$

$$\alpha m\Gamma\mathscr{F}\frac{\gamma^r}{a^{1/2}}\frac{\partial_r a_r}{4a_r} - \frac{\gamma^r}{a^{1/2}}\frac{\partial_r a_r}{4a_r}\Gamma\mathscr{F}\alpha m = 0,$$

$$\alpha m\Gamma\mathscr{F}(1-\alpha)m - (1-\alpha)m\Gamma\mathscr{F}\alpha m \equiv 0. \tag{3.6}$$

Here and thereafter $r=j,m,n$.

After some simplifications and taking into account only nonequivalent equations in (3.6), we get the following system:

$$\begin{aligned}
&\Gamma\gamma^i\gamma^r - \gamma^r\gamma^i\Gamma = 0,\\
&(\Gamma\gamma^i - \gamma^i\Gamma)(1-\alpha)m = 0,\\
&(\Gamma\gamma^r - \gamma^r\Gamma)\alpha m = 0,\\
&\partial_i(\mathscr{F}a_r^{-1/2}) = 0, \quad \partial_r(\mathscr{F}a_i^{-1/2}) = 0,\\
&\partial_i\mathscr{F}(1-\alpha)m = 0, \quad \partial_r\mathscr{F}\alpha m = 0.
\end{aligned} \tag{3.7}$$

We call Eqs. (3.7) a system of separating equations. In the above system of equations there is a subsystem of three equations that only contain matrices [the first file of Eqs. (3.7)] that do not depend on the remaining equations. Then this subsystem must be solved separately. In order to find the solution of this subsystem, we consider each of the 16 linearly independent (given in terms of the Dirac matrices) elements of the 4×4 matrices. After substituting the basis matrices into the subsystem, we obtain the following possibilities for Γ:

$$\Gamma = \gamma^i, \gamma^j\gamma^m\gamma^n, \quad c_1\gamma^i + c_2\gamma^j\gamma^m\gamma^n, \tag{3.8}$$

where c_1 and c_2 are nonzero arbitrary constants.

Substituting the solutions (3.8) into the remaining equations of system (3.7) we obtain the following variants of separation of variables x^i from x^j,x^m,x^n:

$$[1]\ \Gamma = \gamma^i, \quad \alpha = 0, \quad \mathscr{F} = b_i^{1/2}, \quad a_i = \tilde{a}_i b_i, \quad a_r = b_r; \tag{3.9}$$

$$[2]\ \Gamma = \gamma^j\gamma^m\gamma^n, \quad \alpha = 1, \quad \mathscr{F} = \tilde{a}^{1/2}, \quad a_i = \tilde{a}_i, \quad b_r = \tilde{a}b_r; \tag{3.10}$$

$$[3]\ \Gamma = c_1\gamma^i + c_2\gamma^j\gamma^m\gamma^n, \quad m\equiv 0, \quad \mathscr{F} = (\tilde{a}b_i)^{1/2}, \quad a_i = \tilde{a}_i b_i, \quad a_r = \tilde{a}b_r. \tag{3.11}$$

In cases [1] and [2], it is not necessary to impose $m=0$, although separation of variables for the massless neutrino is possible in the gravitational fields (3.9) and (3.10). In the case (3.11) the separation of variables is only possible for the massless neutrino.

Since the separating matrices of the variants [1] and [2] are particular cases of the separating matrix in [3], the variants of separation (3.9) and (3.10) are particular cases of (3.11).

It is well known that the states with a given helicity are fixed in the wave function of the massless neutrino by means of the projection operators $(1\mp\gamma^5)$, which are present in the separating matrix in the case (3) up to a nonsingular matrix factor. This situation takes place

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when $c_1=1$, $c_2=\mp 1$, i.e., according (3.1), those states are present in the solution $\Psi=\Gamma\Phi$.

B. Separation $x^j,x^m,x^n \Rightarrow x^j/x^m,x^n$

In order to separate variables x^j from x^m,x^n, we rewrite Eq. (3.1) as an eigenvalue and eigenfunction problem,

$$\hat{K}_j\Phi = -\hat{K}_{jmn}\Phi = \lambda^i\Phi, \tag{3.12}$$

where λ^i is a constant of the separation $x^i/x^j,x^m,x^n$.

According to (2.10), we accomplish the separation of variables using the scheme

$$(\hat{K}_{jmn}+\lambda^i)\Phi \Rightarrow \mathcal{G}(\hat{K}_{jmn}+\lambda^i)\Sigma\Sigma^{-1}\Phi \Rightarrow (\hat{K}_j+\hat{K}_{mn})\Omega,$$

$$\Phi=\Sigma\Omega, \tag{3.13}$$

$$\hat{K}_j\hat{K}_{mn}-\hat{K}_{mn}\hat{K}_j=0, \tag{3.14}$$

where Σ is the separating matrix and $\mathcal{G}$ is a scalar function guaranteeing the fulfillment of (3.14).

Now, let us analyze the separation of variables in the operator $\hat{K}_{jmn}$ according to the variants [1]–[3].

Substituting (3.9) into (3.4), we get

$$\hat{K}_{jmn}=\left\{\gamma^j\left(\frac{b_i}{b_j}\right)^{1/2}\left(\partial_j-\frac{\partial_j b_j}{4b_j}\right)+\gamma^m\left(\frac{b_i}{b_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m b_m}{4b_m}\right)\right.$$
$$\left.+\gamma^n\left(\frac{b_i}{b_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n b_n}{4b_n}\right)+b_i^{1/2}m\right\}\gamma^i. \tag{3.15}$$

Then we have that the most general form of the separating operators can be written as follows:

$$\hat{K}_j=\mathcal{G}\left\{-\gamma^i\gamma^j\left(\frac{b_i}{b_j}\right)^{1/2}\left(\partial_j-\frac{\partial_j\tilde{b}_j}{4b_i}\right)+\gamma^i b_i^{1/2}\alpha m\right.$$
$$\left.+\beta\lambda^i\right\}\Sigma, \tag{3.16}$$

$$\hat{K}_{mn}=\mathcal{G}\left\{-\gamma^i\gamma^m\left(\frac{b_i}{b_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m c_m}{4c_m}\right)\right.$$
$$-\gamma^i\gamma^n\left(\frac{b_i}{b_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n c_n}{4c_n}\right)+\gamma^i b_i^{1/2}(1$$
$$\left.-\alpha)m+\beta\lambda^i\right\}\Sigma. \tag{3.17}$$

Moreover,

$$b_k=\tilde{b}_k c_k,\quad \beta=\text{const.} \tag{3.18}$$

Then the corresponding system of separating equations is

$$\Sigma\gamma^j\gamma^s+\gamma^j\gamma^s\Sigma=0,$$
$$(\Sigma\gamma^j+\gamma^j\Sigma)(1-\alpha)m=0,$$
$$(\Sigma\gamma^i\gamma^j-\gamma^i\gamma^j\Sigma)(1-\beta)\lambda^i=0,$$
$$(\Sigma\gamma^s-\gamma^s\Sigma)\alpha m=0,$$
$$(\Sigma\gamma^i\gamma^s-\gamma^i\gamma^s\Sigma)\beta\lambda^i=0,$$
$$(\Sigma\gamma^i-\gamma^i\Sigma)\alpha m(1-\beta)\lambda^i=0,$$
$$(\Sigma\gamma^i-\gamma^i\Sigma)\beta\lambda^i(1-\alpha)m=0,$$
$$\partial_j\{\mathcal{G}(b_i/b_s)^{1/2}\}=0,\quad \partial_s\{\mathcal{G}(b_i/b_j)^{1/2}\}=0,$$
$$\partial_j\{\mathcal{G}b_i^{1/2}\}(1-\alpha)m=0,\quad \partial_j\{\mathcal{G}\}(1-\beta)\lambda^i=0,$$
$$\partial_s\{\mathcal{G}b_i^{1/2}\}\alpha m=0,\quad \partial_s\{\mathcal{G}\}\beta\lambda^i=0,\quad s=m,n. \tag{3.19}$$

The matrix subsystem in Eqs. (3.19) gives as a result

$$\Sigma=\gamma^j,\gamma^i\gamma^j,\gamma^m\gamma^n,\gamma^i\gamma^m\gamma^n,\quad c_1\gamma^j+c_2\gamma^i\gamma^m\gamma^n,$$
$$c_3\gamma^i\gamma^j+c_4\gamma^m\gamma^n, \tag{3.20}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants. Other linear combinations of these solutions requires that the constant of separation λ^i in Eq. (3.12) must be equated to zero.

Substituting (3.20) into (3.19) we obtain the following possibilities of separation $x^j/x^m,x^n$:

$$[1.1]\quad \Sigma=\gamma^i\gamma^j,\quad \alpha=0,\quad \beta=0,\quad \mathcal{G}=(c_j/c_i)^{1/2},\quad b_i=c_i,\quad b_j=\tilde{b}_jc_j,\quad b_s=c_s; \tag{3.21}$$

$$[1.2]\quad \Sigma=\gamma^j,\quad \alpha=1,\quad \beta=1,\quad \mathcal{G}=(\tilde{b}/\tilde{b}_i)^{1/2},\quad b_i=\tilde{b}_i,\quad b_j=\tilde{b}_j,\quad b_s=\tilde{b}c_s; \tag{3.22}$$

$$[1.3]\quad \Sigma=\gamma^m\gamma^n,\quad \alpha=1,\quad \beta=0,\quad \mathcal{G}=c_i^{-1/2},\quad b_i=\tilde{b}c_i,\quad b_j=\tilde{b}_j,\quad b_s=\tilde{b}c_s; \tag{3.23}$$

$$[1.4]\quad \Sigma=\gamma^i\gamma^m\gamma^n,\quad \alpha=0,\quad \beta=1,\quad \mathcal{G}=\tilde{b}_i^{-1/2},\quad b_i=\tilde{b}_ic,\quad b_j=\tilde{b}_jc,b_sc_s; \tag{3.24}$$

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$$[1.5]\ \Sigma=c_1\gamma^i\gamma^j+c_2\gamma^m\gamma^n,\quad m\equiv 0,\quad \beta=0,\quad \mathcal{G}=(c_j/c_i)^{1/2},\quad b_i=\widetilde{b}c_i,\quad b_j=\widetilde{b}_jc_j,\quad b_s=\widetilde{b}c_s; \tag{3.25}$$

$$[1.6]\ \Sigma=c_3\gamma^j+c_4\gamma^i\gamma^m\gamma^n,\quad m\equiv 0,\quad \beta=1,\quad \mathcal{G}=(\widetilde{b}_s/\widetilde{b}_i)^{1/2};\quad b_i=\widetilde{b}_ic,\quad b_j=\widetilde{b}_jc,\quad b_s=\widetilde{b}_sc_s. \tag{3.26}$$

The cases [1.1]–[1.4] are valid for massive and massless neutrinos. The cases [1.5] and [1.6] are valid only when $m\equiv 0$. Let us remark that [1.1] and [1.3] are particular cases of [1.5]. In a similar way, [1.2] and [1.4] are included in [1.6]. Notice that the variants [1.5] and [1.6] automatically allow the possibility of considering two different helicities for the neutrino.

Substituting (3.10) into (3.4) we have that the operator $\widehat{K}_{jmn}$ reads as

$$\widehat{K}_{jmn}=\left\{\frac{\gamma^j}{b_j^{1/2}}\left(\partial_j-\frac{\partial_jb_j}{4b_j}\right)+\frac{\gamma^m}{b_m^{1/2}}\left(\partial_m-\frac{\partial_mb_m}{4b_m}\right)+\frac{\gamma^n}{b_n^{1/2}}\left(\partial_n-\frac{\partial_nb_n}{4b_n}\right)\right\}\gamma^j\gamma^m\gamma^n. \tag{3.27}$$

According to the scheme of separation (3.13) we obtain that the operators $\widehat{K}_j$ and $\widehat{K}_{mn}$ take the form

$$\widehat{K}_j=\mathcal{G}\left\{\frac{\gamma^m\gamma^n}{b_j^{1/2}}\left(\partial_j-\frac{\partial_j\widetilde{b}_j}{4b_j}\right)+\beta\lambda^i\right\}\Sigma, \tag{3.28}$$

$$\widehat{K}_{mn}=\mathcal{G}\left\{-\frac{\gamma^j\gamma^n}{b_m^{1/2}}\left(\partial_m-\frac{\partial_mc_m}{4c_m}\right)+\frac{\gamma^j\gamma^m}{b_n^{1/2}}\left(\partial_n-\frac{\partial_nc_n}{4c_n}\right)+(1-\beta)\lambda^i\right\}\Sigma. \tag{3.29}$$

Substituting (3.28) and (3.29) into (3.14) we obtain the following system of separating equations:

$$\begin{gathered}\Sigma\gamma^i\gamma^s+\gamma^i\gamma^s\Sigma=0,\\ (\Sigma\gamma^m\gamma^n-\gamma^m\gamma^n\Sigma)(1-\beta)\lambda^i=0,\\ (\Sigma\gamma^j\gamma^s+\gamma^j\gamma^s\Sigma)\beta\lambda^i=0,\\ \partial_j(\mathcal{G}/b_s^{1/2})=0,\quad \partial_s(\mathcal{G}/b_j^{1/2})=0,\\ \partial_j(\mathcal{G})(1-\beta)\lambda^i=0,\quad \partial_s(\mathcal{G})\beta\lambda^i=0.\end{gathered} \tag{3.30}$$

Then, we obtain the following possibilities for the separating matrix Σ:

$$\Sigma=\gamma^i,\quad \gamma^i\gamma^j,\quad \gamma^m\gamma^n,\quad \gamma^j\gamma^m\gamma^n, \tag{3.31}$$

and also their linear combinations. After substituting (3.31) into the system (3.30), we obtain that for all the solutions in (3.31) there is only one variant of separation of x^j from x^m,x^n:

$$[2.1]\ \beta=0,\quad \mathcal{G}=c_j^{1/2},\quad b_j=\widetilde{b}_jc_j,\quad b_s=c_s. \tag{3.32}$$

Since in (3.10) the mass term does not appear, (3.32) is valid for massive and massless neutrinos.

Finally, substituting (3.11) into (3.4) we get

$$\widehat{K}_{jmn}=\left\{\gamma^j\left(\frac{b_i}{b_j}\right)^{1/2}\left(\partial_j-\frac{\partial_jb_j}{4b_j}\right)+\gamma^m\left(\frac{b_i}{b_m}\right)^{1/2}\left(\partial_m-\frac{\partial_mb_m}{4b_m}\right)+\gamma^n\left(\frac{b_i}{b_n}\right)^{1/2}\left(\partial_n-\frac{\partial_nb_n}{4b_n}\right)\right\}(c_1\gamma^i+c_2\gamma^j\gamma^m\gamma^n). \tag{3.33}$$

Then the separating operators $\widehat{K}_j$ and $\widehat{K}_{mn}$ take the form

$$\widehat{K}_j=\mathcal{G}\left\{(-c_1\gamma^i\gamma^j+c_2\gamma^m\gamma^n)\left(\frac{b_i}{b_j}\right)^{1/2}\left(\partial_j-\frac{\partial_j\widetilde{b}_j}{4b_j}\right)+\beta\lambda^i\right\}\Sigma, \tag{3.34}$$

$$\widehat{K}_{mn}=\mathcal{G}\left\{(-c_1\gamma^i\gamma^m-c_2\gamma^j\gamma^n)\left(\frac{b_i}{b_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m\widetilde{c}_m}{4\widetilde{c}_m}\right)+(-c_1\gamma^i\gamma^n+c_2\gamma^j\gamma^m)\left(\frac{b_i}{b_n}\right)^{1/2}\times\left(\partial_n-\frac{\partial_n\widetilde{c}_n}{4\widetilde{c}_n}\right)+(1-\beta)\lambda^i\right\}\Sigma, \tag{3.35}$$

from which, taking into account (3.14), we obtain the following system of separating equations:

$$\begin{gathered}\Sigma\gamma^i\gamma^s+\gamma^i\gamma^s\Sigma=0,\quad \Sigma\gamma^j\gamma^s+\gamma^j\gamma^s\Sigma=0,\\ (\Sigma\gamma^i\gamma^j+\gamma^i\gamma^j\Sigma)(1-\beta)\lambda^i=0,\\ (\Sigma\gamma^m\gamma^n+\gamma^m\gamma^n\Sigma)(1-\beta)\lambda^i=0,\\ \partial_j\{\mathcal{G}(b_i/b_s)^{1/2}\}=0,\quad \partial_s\{\mathcal{G}(b_i/b_j)^{1/2}\}=0,\\ \partial_j(\mathcal{G})(1-\beta)\lambda^i=0,\quad \partial_s(\mathcal{G})\beta\lambda^i=0.\end{gathered} \tag{3.36}$$

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From the system (3.36) we find only one variant of the separation of variables,

$$[3.1]\ \Sigma=\gamma^i\gamma^j,\quad \gamma^m\gamma^n,\quad \beta=0,\quad \mathscr{G}=(c_j/c_i)^{1/2},$$
$$b_i=\widetilde{b}c_i,\quad b_j=\widetilde{b}_jc_j,\quad b_s=\widetilde{b}c_s. \tag{3.37}$$

Notice that the cases [1.5] and [2.1] are included in the variant [3.1].

C. Separation $x^m,x^n \Rightarrow x^m/x^n$

The eigenvalues and eigenfunction problem for the operators $\widehat{K}_j$ and $\widehat{K}_{mn}$, according to (3.13), can be written as

$$\widehat{K}_j\Omega=-\widehat{K}_{mn}\Omega=\lambda^i\Omega, \tag{3.38}$$

where λ^i is a constant of separation.

The separation of variables in the operators $\widehat{K}_{mn}$ is accomplished as follows:

$$(\widehat{K}_{mn}+\lambda^j)\Omega\rightarrow\mathscr{H}(\widehat{K}_{mn}+\lambda^j)\Xi\Xi^{-1}\Omega\rightarrow(\widehat{K}_m+\widehat{K}_n)\Theta,$$
$$\Omega=\Xi\Theta, \tag{3.39}$$

$$\widehat{K}_m\widehat{K}_n-\widehat{K}_n\widehat{K}_m=0. \tag{3.40}$$

The separating matrix Ξ and the scalar function $\mathscr{H}$ are to be defined in order to satisfy the commutation relation (3.40).

Since all the further computations are analogous to the one presented, we are going to show only the most important equations and results. Now, we proceed to consider all the possible variants for the operator $\widehat{K}_{mn}$ arising from Sec. III B.

From (3.17) and (3.21) we get

$$\widehat{K}_{mn}=-\gamma^j\gamma^m\left(\frac{c_j}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m c_m}{4c_m}\right)-\gamma^j\gamma^n\left(\frac{c_j}{c_n}\right)^{1/2}\times\left(\partial_n-\frac{\partial_n c_n}{4c_n}\right)+\gamma^j\left(\frac{c_j}{c_i}\right)^{1/2}m+\gamma^i\gamma^j\left(\frac{c_j}{c_i}\right)^{1/2}\lambda^i, \tag{3.41}$$

$$\widehat{K}_m=\mathscr{H}\left\{-\gamma^j\gamma^m\left(\frac{c_j}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m\widetilde{c}_m}{4\widetilde{c}_m}\right)+\gamma^j\left(\frac{c_j}{c_i}\right)^{1/2}\alpha m+\gamma^i\gamma^j\left(\frac{c_j}{c_i}\right)^{1/2}\beta\lambda^i+\delta\lambda^j\right\}\Xi, \tag{3.42}$$

$$\widehat{K}_n=\mathscr{H}\left\{-\gamma^j\gamma^n\left(\frac{c_j}{c_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n\widetilde{d}_n}{4\widetilde{d}_n}\right)+\gamma^j\left(\frac{c_j}{c_i}\right)^{1/2}(1-\alpha)m+\gamma^i\gamma^j\left(\frac{c_j}{c_i}\right)^{1/2}(1-\beta)\lambda^i+(1-\delta)\lambda^j\right\}\Xi. \tag{3.43}$$

Then, we obtain that the different possibilities of separation of variables x^m from x^n are

$$[1.1.1]\ \Xi=\gamma^m,\quad \gamma^j\gamma^m,\quad \mathscr{H}=(\widetilde{c}/\widetilde{c}_j)^{1/2},\quad \delta=1,\ \alpha=0,\ \beta=1\quad\text{or}\quad\alpha=1,\ \beta=0, c_i=\widetilde{c}\widetilde{d},$$
$$c_j=\widetilde{c}_j\widetilde{d},\quad c_m=\widetilde{c}_m\widetilde{d},\quad c_n=\widetilde{c}\widetilde{d}_n; \tag{3.44}$$

$$[1.1.2]\ \Xi=\gamma^n,\quad \gamma^j\gamma^n,\quad \mathscr{H}=(\widetilde{d}/\widetilde{d}_j)^{1/2},\quad \delta=0, \alpha=0,\quad \beta=1\quad\text{or}\quad\alpha=1,$$
$$\beta=0,\quad c_i=\widetilde{c}\widetilde{d},\quad c_j=\widetilde{c}\widetilde{d}_j,\quad c_m=\widetilde{c}_m\widetilde{d},\quad c_n=\widetilde{c}\widetilde{d}_n; \tag{3.45}$$

$$[1.1.3]\ \Xi=\gamma^i\gamma^m,\quad \alpha=\beta=\delta=0,\quad \mathscr{H}=(\widetilde{d}_m/\widetilde{d}_j)^{1/2}, c_i=\widetilde{c}\widetilde{d}_i,\quad c_j=\widetilde{c}\widetilde{d}_j,\quad c_m=\widetilde{c}_m\widetilde{d}_m,\quad c_n=\widetilde{c}\widetilde{d}_n; \tag{3.46}$$

$$[1.1.4]\ \Xi=\gamma^i\gamma^m,\quad \alpha=\beta=\delta=1,\quad \mathscr{H}=(\widetilde{c}_n/\widetilde{c}_j)^{1/2}, c_i=\widetilde{c}_i\widetilde{d},\quad c_j=\widetilde{c}_j\widetilde{d},\quad c_m=\widetilde{c}_m\widetilde{d},\quad c_n=\widetilde{c}_n\widetilde{d}_n; \tag{3.47}$$

$$[1.1.5]\ \Xi=\gamma^i\gamma^j\gamma^n,\quad \alpha=\beta=0,\quad \delta=1,\quad \mathscr{H}=(\widetilde{c}/\widetilde{c}_j)^{1/2}, c_i=\widetilde{c}\widetilde{d}_i,\quad c_j=\widetilde{c}_j\widetilde{d},\quad c_m=\widetilde{c}_m\widetilde{d},\quad c_n=\widetilde{c}\widetilde{d}_n; \tag{3.48}$$

$$[1.1.6]\ \Xi=\gamma^i\gamma^j\gamma^m,\quad \alpha=\beta=1,\quad \delta=0,\quad \mathscr{H}=(\widetilde{d}/\widetilde{d}_j)^{1/2}, c_i=\widetilde{c}_i\widetilde{d},\quad c_j=\widetilde{c}\widetilde{d}_j,\quad c_m=\widetilde{c}_m\widetilde{d},\quad c_n=\widetilde{c}\widetilde{d}_n. \tag{3.49}$$

The case [1.1.1] is included in [1.1.5]; also, the case [1.1.2] is contained in [1.1.6]. All these variants take place for massive and massless particles. There are not further possibilities for $m\equiv 0$. Here, we have to mention the existence trivial symmetry respecting to the permutation of x^m and x^n.

From (3.17) and (3.22) we obtain

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$$\hat{K}_{mn}=\frac{\gamma^i\gamma^j\gamma^m}{c_m^{1/2}}\left(\partial_m-\frac{\partial_m c_m}{4c_m}\right)+\frac{\gamma^i\gamma^j\gamma^n}{c_n^{1/2}}\left(\partial_n-\frac{\partial_n c_n}{4c_n}\right), \tag{3.50}$$

$$\hat{K}_m=\mathscr{H}\left\{\frac{\gamma^i\gamma^j\gamma^m}{c_m^{1/2}}\left(\partial_m-\frac{\partial_m\tilde{c}_m}{4\tilde{c}_n}\right)+\delta\lambda^j\right\}\Xi, \tag{3.51}$$

$$\hat{K}_n=\mathscr{H}\left\{\frac{\gamma^i\gamma^j\gamma^n}{c_n^{1/2}}\left(\partial_n-\frac{\partial_n\tilde{d}_n}{4\tilde{d}_n}\right)+(1-\delta)\lambda^j\right\}\Xi. \tag{3.52}$$

Then for this case we have the following variants of separation:

$$[1.2.1]\ \ \Xi=\gamma^n,\gamma^i\gamma^n,\gamma^j\gamma^n,\gamma^i\gamma^j\gamma^n,\quad \delta=1,\quad \mathscr{H}=\tilde{c}_n^{1/2},\quad c_m=\tilde{c}_m,\quad c_n=\tilde{c}_n\tilde{d}_n; \tag{3.53}$$

$$[1.2.2]\ \ \Xi=\gamma^m,\gamma^i\gamma^m,\gamma^j\gamma^m,\gamma^i\gamma^j\gamma^m,\quad \delta=0,\quad \mathscr{H}=\tilde{d}_m^{1/2},\quad c_m=\tilde{c}_m\tilde{d}_m,\quad c_n=\tilde{d}_n. \tag{3.54}$$

Substituting (3.23) into (3.17), we have

$$\hat{K}_{mn}=-\frac{\gamma^i\gamma^n}{c_m^{1/2}}\left(\partial_m-\frac{\partial_m c_m}{4c_m}\right)+\frac{\gamma^i\gamma^m}{c_n^{1/2}}\left(\partial_n-\frac{\partial_n c_n}{4c_n}\right)+\frac{\gamma^m\gamma^n}{c_i^{1/2}}\lambda^i, \tag{3.55}$$

$$\hat{K}_m=\mathscr{H}\left\{-\frac{\gamma^i\gamma^n}{c_m^{1/2}}\left(\partial_m-\frac{\partial_m\tilde{c}_m}{4\tilde{c}_m}\right)+\frac{\gamma^m\gamma^n}{c_i^{1/2}}\beta\lambda^i+\delta\lambda^j\right\}\Xi, \tag{3.56}$$

$$\hat{K}_n=\mathscr{H}\left\{\frac{\gamma^i\gamma^m}{c_n^{1/2}}\left(\partial_n-\frac{\partial_n\tilde{d}_n}{4\tilde{d}_n}\right)+\frac{\gamma^m\gamma^n}{c_i^{1/2}}(1-\beta)\lambda^i+(1-\delta)\lambda^j\right\}\Xi. \tag{3.57}$$

Then, here we have the following variants of separation:

$$[1.3.1]\ \ \Xi=\gamma^m,\gamma^i\gamma^n,\gamma^j\gamma^m,\gamma^i\gamma^j\gamma^n,\quad \beta=\delta=0,\quad \mathscr{H}=\tilde{c}_n^{1/2},\quad c_i=\tilde{c}_i,\quad c_m=\tilde{c}_m,\quad c_n=\tilde{c}_n\tilde{d}_n; \tag{3.58}$$

$$[1.3.2]\ \ \Xi=\gamma^n,\gamma^i\gamma^m,\gamma^j\gamma^n,\gamma^i\gamma^j\gamma^m,\quad \beta=\delta=1,\quad \mathscr{H}=\tilde{d}_m^{1/2},\quad c_i=\tilde{d}_i,\quad c_m=\tilde{c}_m\tilde{d}_m,\quad c_n=\tilde{d}_n. \tag{3.59}$$

The results (3.58) and (3.59) are valid for massive and massless particles.
Using the expressions (3.17) and (3.24) we obtain

$$\hat{K}_{mn}=\gamma^n\left(\frac{c}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m c_m}{4c_m}\right)-\gamma^m\left(\frac{c}{c_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n c_n}{4c_n}\right)+\gamma^m\gamma^n c^{1/2}m, \tag{3.60}$$

$$\hat{K}_m=\mathscr{H}\left\{\gamma^n\left(\frac{c}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m\tilde{c}_m}{4\tilde{c}_m}\right)+\gamma^m\gamma^n c^{1/2}\alpha m+\delta\lambda^j\right\}\Xi, \tag{3.61}$$

$$\hat{K}_n=\mathscr{H}\left\{-\gamma^m\left(\frac{c}{c_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n\tilde{d}_n}{4\tilde{d}_n}\right)+\gamma^m\gamma^n c^{1/2}(1-\alpha)m+(1-\delta)\lambda^j\right\}\Xi. \tag{3.62}$$

Here arise the following possibilities of separation of variables x^m and x^n in Eq. (2.9) for massive and massless particles:

$$[1.4.1]\ \ \Xi=\gamma^m,\gamma^i\gamma^n,\gamma^j\gamma^n,\gamma^i\gamma^j\gamma^m,\quad \alpha=\delta=1,\quad \mathscr{H}=\tilde{c}_n^{-1/2},c=\tilde{c},\quad c_m=\tilde{c}_m,\quad c_n=\tilde{c}_n\tilde{d}_n; \tag{3.63}$$

$$[1.4.2]\ \ \Xi=\gamma^n,\gamma^i\gamma^m,\gamma^j\gamma^m,\gamma^i\gamma^j\gamma^n,\quad \alpha=\delta=0,\quad \mathscr{H}=\tilde{d}_m^{-1/2},c=\tilde{c},\quad c_m=\tilde{c}_m\tilde{d}_m,\quad c_n=\tilde{d}_n. \tag{3.64}$$

There are not additional possibilities for $m\equiv 0$.
Substituting (3.25) into (3.17), we obtain

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$$\hat{K}_{mn}=(-c_1\gamma^j\gamma^m-c_2\gamma^i\gamma^n)\left(\frac{c_j}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m c_m}{4c_m}\right)$$
$$+(-c_1\gamma^j\gamma^n+c_2\gamma^i\gamma^m)\left(\frac{c_j}{c_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n c_n}{4c_n}\right)+(c_1\gamma^i\gamma^j+c_2\gamma^m\gamma^n)\left(\frac{c_j}{c_i}\right)^{1/2}\lambda^i, \tag{3.65}$$

$$\hat{K}_m=\mathscr{H}\left\{(-c_1\gamma^j\gamma^m-c_2\gamma^i\gamma^n)\left(\frac{c_j}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m\tilde{c}_m}{4\tilde{c}_m}\right)+(c_1\gamma^i\gamma^j+c_2\gamma^m\gamma^n)\left(\frac{c_j}{c_i}\right)^{1/2}\beta\lambda^i+\delta\lambda^j\right\}\Xi, \tag{3.66}$$

$$\hat{K}_n=\mathscr{H}\left\{(-c_1\gamma^j\gamma^n+c_2\gamma^i\gamma^m)\left(\frac{c_j}{c_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n\tilde{d}_n}{4\tilde{d}_n}\right)+(c_1\gamma^i\gamma^j+c_2\gamma^m\gamma^n)\left(\frac{c_j}{c_i}\right)^{1/2}(1-\beta)\lambda^i+(1-\delta)\lambda^j\right\}\Xi. \tag{3.67}$$

Equations (3.66) and (3.67) give place to the following cases of separation:

$$[1.5.1]\quad \Xi=\gamma^i\gamma^m,\gamma^j\gamma^n,\quad \beta=\delta=1,\quad \mathscr{H}=(\tilde{c}_n/\tilde{c}_j)^{1/2},$$
$$c=\tilde{c}_i\tilde{d},\quad c_j=\tilde{c}_j\tilde{d},\quad c_m=\tilde{c}_m\tilde{d},\quad c_n=\tilde{c}_n\tilde{d}_n: \tag{3.68}$$

$$[1.5.2]\quad \Xi=\gamma^i\gamma^n,\gamma^j\gamma^m,\quad \beta=\delta=0,\quad \mathscr{H}=(\tilde{d}_m/\tilde{d}_j)^{1/2},\quad c=\tilde{c}\tilde{d}_i,\quad c_j=\tilde{c}\tilde{d}_j,\quad c_m=\tilde{c}_m\tilde{d}_m,\quad c_n=\tilde{c}\tilde{d}_n. \tag{3.69}$$

Gathering (3.17) and (3.25) we get

$$\hat{K}_{mn}=(c_3\gamma^i\gamma^j\gamma^m+c_4\gamma^n)\left(\frac{c}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m c_m}{4c_m}\right)+(c_3\gamma^i\gamma^j\gamma^n-c_4\gamma^m)\left(\frac{c}{c_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n c_n}{4c_n}\right), \tag{3.70}$$

$$\hat{K}_m=\mathscr{H}\left\{(c_3\gamma^i\gamma^j\gamma^m+c_4\gamma^n)\left(\frac{c}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m\tilde{c}_m}{4\tilde{c}_m}\right)+\delta\lambda^j\right\}\Xi, \tag{3.71}$$

$$\hat{K}_n=\mathscr{H}\left\{(c_3\gamma^i\gamma^j\gamma^n-c_4\gamma^m)\left(\frac{c}{c_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n\tilde{d}_n}{4\tilde{d}_n}\right)+(1-\delta)\lambda^j\right\}\Xi. \tag{3.72}$$

Then for the present case we have the following variants of separations:

$$[1.6.1]\quad \Xi=\gamma^n,\quad \delta=1,\quad \mathscr{H}=(\tilde{c}_n/\tilde{c})^{1/2},\quad c=\tilde{c}\tilde{d},\quad c_m=\tilde{c}_m\tilde{d},\quad c_n=\tilde{c}_n\tilde{d}_n; \tag{3.73}$$

$$[1.6.2]\quad \Xi=\gamma^m,\quad \delta=0,\quad \mathscr{H}=(\tilde{d}_m/\tilde{d})^{1/2},\quad c=\tilde{c}\tilde{d},\quad c_m=\tilde{c}_m\tilde{d}_m,\quad c_n=\tilde{c}\tilde{d}_n; \tag{3.74}$$

The cases [1.5.1], [1.5.2], [1.6.1], and [1.6.2] are only valid for massless neutrinos.

The conditions of separability of variables (3.32) were obtained for the general case when it is possible to consider $m\neq 0$ or $m=0$. Substituting (3.32) into (3.39), we find

$$\hat{K}_{mn}=-\gamma^i\gamma^j\gamma^n\left(\frac{c_j}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m c_m}{4c_m}\right)+\gamma^i\gamma^j\gamma^m\left(\frac{c_j}{c_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n c_n}{4c_n}\right)+\gamma^i c_j^{1/2}\lambda^i, \tag{3.75}$$

$$\hat{K}_m=\mathscr{H}\left\{-\gamma^i\gamma^j\gamma^n\left(\frac{c_j}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m\tilde{c}_m}{4\tilde{c}_m}\right)+\gamma^i c_j^{1/2}\beta\lambda^i+\delta\lambda^j\right\}\Xi, \tag{3.76}$$

$$\hat{K}_n=\mathscr{H}\left\{\gamma^i\gamma^j\gamma^m\left(\frac{c_j}{c_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n\tilde{d}_n}{4\tilde{d}_n}\right)+\gamma^i c_j^{1/2}(1-\beta)\lambda^i+(1-\delta)\lambda^j\right\}\Xi. \tag{3.77}$$

Substituting Eqs. (3.76) and (3.77) into (3.39) we obtain the following variants of separation:

$$[2.1.1]\quad \Xi=\gamma^m,\gamma^j\gamma^m,\quad \beta=\delta=1,\quad \mathscr{H}=(\tilde{c}_n/\tilde{c}_j)^{1/2},\quad c_j=\tilde{c}_j,\quad c_m=\tilde{c}_m,\quad c_n=\tilde{c}_n\tilde{d}_n; \tag{3.78}$$

$$[2.1.2]\quad \Xi=\gamma^n,\gamma^j\gamma^n,\quad \beta=\delta=0,\quad \mathscr{H}=(\tilde{d}_m/\tilde{d}_j)^{1/2},\quad c_j=\tilde{d}_j,\quad c_m=\tilde{c}_m\tilde{d}_m,\quad c_n=\tilde{d}_n; \tag{3.79}$$

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$$[2.1.3]\ \Xi=\gamma^i\gamma^m,\gamma^i\gamma^j\gamma^n,\quad \beta=1,\quad \delta=0,\quad \mathscr{H}=\widetilde{d}_j^{-1/2},\quad c_j=\widetilde{c}\widetilde{d}_j,\quad c_m=\widetilde{c}_m,\quad c_n=\widetilde{c}\widetilde{d}_n; \tag{3.80}$$

$$[2.1.4]\ \Xi=\gamma^i\gamma^n,\gamma^i\gamma^j\gamma^m,\quad \beta=0,\quad \delta=1,\quad \mathscr{H}=\widetilde{c}_j^{-1/2},\quad c_j=\widetilde{c}_j\widetilde{d},\quad c_m=\widetilde{c}_m\widetilde{d},\quad c_n=\widetilde{d}_n. \tag{3.81}$$

Here [2.1.1]–[2.1.4] are valid for massive as well as for massless neutrinos.

Finally, we consider the separation of variables associated with Eqs. (3.35) and (3.37), i.e., the cases valid only for the massless neutrino. Substituting (3.37) into (3.35), we obtain

$$\widehat{K}_{mn}=(-c_1\gamma^j\gamma^m+c_2\gamma^i\gamma^n)\left(\frac{c_j}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m c_m}{4c_m}\right)(-c_1\gamma^j\gamma^n+c_2\gamma^i\gamma^m)\left(\frac{c_j}{c_n}\right)^{1/2}\left(\partial_m-\frac{\partial_n c_n}{4c_n}\right)\gamma^i\gamma^j)\left(\frac{c_j}{c_i}\right)^{1/2}\lambda^i \tag{3.82}$$

$$\widehat{K}_m=\mathscr{H}\left\{(-c_1\gamma^j\gamma^m+c_2\gamma^i\gamma^n)\left(\frac{c_j}{c_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m\widetilde{c}_m}{4\widetilde{c}_m}\right)+\gamma^i\gamma^j\left(\frac{c_j}{c_i}\right)^{1/2}\beta\lambda^i+\delta\lambda^j\right\}\Xi, \tag{3.83}$$

$$\widehat{K}_m=\mathscr{H}\left\{(-c_1\gamma^j\gamma^n+c_2\gamma^i\gamma^m)\left(\frac{c_j}{c_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n\widetilde{d}_n}{4\widetilde{d}_n}\right)+\gamma^i\gamma^j\left(\frac{c_j}{c_i}\right)^{1/2}(1-\beta)\lambda^i+(1-\delta)\lambda^j\right\}\Xi. \tag{3.84}$$

Here two equivalent variants of separation emerge:

$$[3.1.1]\ \Xi=\gamma^i\gamma^m,\gamma^j\gamma^n,\quad \beta=\delta=1,\quad \mathscr{H}=(\widetilde{c}_n/\widetilde{c}_j)^{1/2},\quad c_i=\widetilde{c}_i\widetilde{d},\quad c_j=\widetilde{c}_j\widetilde{d},\quad c_m=\widetilde{c}_m\widetilde{d},\quad c_n=\widetilde{c}_n\widetilde{d}_n; \tag{3.85}$$

$$[3.1.2]\ \Xi=\gamma^i\gamma^n,\gamma^j\gamma^m,\quad \beta=\delta=0,\quad \mathscr{H}=(\widetilde{d}_m/\widetilde{d}_j)^{1/2},\quad c_i=\widetilde{c}\widetilde{d}_i,\quad c_j=\widetilde{c}\widetilde{d}_j,\quad c_m=\widetilde{c}_m\widetilde{d}_m,\quad c_n=\widetilde{c}\widetilde{d}_n. \tag{3.86}$$

We did not write the explicit expressions for the separating operators $\widehat{K}_i$, $\widehat{K}_j$, $\widehat{K}_m$, $\widehat{K}_n$ for the concrete possible separations, but they can be obtained in a straightforward way, by substituting the corresponding metric functions a_k into the general form for the operators exhibited in Sec. III.

The structure of the metric coefficients a_k, allowing consecutive separation of variables in the Dirac equation, are presented in Table I.

IV. PAIRWISE SEPARATION OF VARIABLES

A. Separation $x^i,x^j,x^m,x^n \Rightarrow x^i,x^j/x^m,x^n$

Let us analyze the separation of variables of x^i,x^j from x^m,x^n in Eq. (2.8). Using Eqs. (2.9) and (2.10) and taking into account the explicit form of Eq. (2.8), we can write the most general form of the separating operator:

$$\widehat{K}_{ij}=\mathscr{F}\left\{\frac{\gamma^i}{a_i^{1/2}}\left(\partial_i-\frac{\partial_i v_i}{4v_i}\right)+\frac{\gamma^j}{a_j^{1/2}}\left(\partial_j-\frac{\partial_j v_j}{4v_j}\right)+\alpha m\right\}\Gamma, \tag{4.1}$$

$$\widehat{K}_{mn}=\mathscr{F}\left\{\frac{\gamma^m}{a_m^{1/2}}\left(\partial_m-\frac{\partial_m w_m}{4w_m}\right)+\frac{\gamma^n}{a_n^{1/2}}\left(\partial_n-\frac{\partial_n w_n}{4w_n}\right)+(1-\alpha)m\right\}\Gamma, \tag{4.2}$$

where

$$a_k(x^i,x^j,x^m,x^n)=v_k(x^i,x^j)w_k(x^m,x^n),k=i,j,m,n. \tag{4.3}$$

Imposing that

$$\widehat{K}_{ij}\widehat{K}_{mn}-\widehat{K}_{mn}\widehat{K}_{ij}=0, \tag{4.4}$$

we find the following system of separating equations:

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TABLE I. Structure of the metric functions a_k for consecutive separation of variables in the Dirac equation. The variant [1.1.1] is included in [1.1.5] and the variant [1.2.2] is included in [1.1.6]. The variants indicated by ( ) are equivalent to the variants shown in this table, up to a change between x^m and x^n.

Variant	From	Condition	i	j	m	n
(initial)			a_i	a_j	a_m	a_n
[1]	initial	$m\neq 0,\ m=0$	$\tilde a_i\ b_i$	b_j	b_m	b_n
[1.1]	[1]	$m\neq 0,\ m=0$	$\tilde a_i\ c_i$	$\tilde b_j\ c_j$	c_m	c_n
[1.1.3] ([1.1.4])	[1.1]	$m\neq 0,\ m=0$	$\tilde a_i\ \tilde c_i\ \tilde d$	$\tilde b_j\ \tilde c_j\ \tilde d$	$\tilde c_m\ \tilde d$	$\tilde c_n\ \tilde d_n$
[1.1.5] ([1.1.6])	[1.1]	$m\neq 0,\ m=0$	$\tilde a_i\ \tilde c\ \tilde d_i$	$\tilde b_j\ \tilde c_j\ \tilde d$	$\tilde c_m\ \tilde d$	$\tilde c\ \tilde d_n$
[1.2]	[1]	$m\neq 0,\ m=0$	$\tilde a_i\ \tilde b_i$	$\tilde b_j$	$\tilde b\ c_m$	$b\ c_n$
[1.2.1] ([1.2.2])	[1.2]	$m\neq 0,\ m=0$	$\tilde a_i\ \tilde b_i$	$\tilde b_j$	$\tilde b\ \tilde c_m$	$\tilde b\ \tilde c_n\ \tilde d_n$
[1.3]	[1]	$m\neq 0,\ m=0$	$\tilde a_i\ \tilde b\ c_i$	$\tilde b_j$	$\tilde b\ c_m$	$\tilde b\ c_n$
[1.3.1] ([1.3.2])	[1.3]	$m\neq 0,\ m=0$	$\tilde a_i\ \tilde b\ \tilde c_i$	$\tilde b_j$	$\tilde b\ \tilde c_m$	$\tilde b\ \tilde c_n\ \tilde d_n$
[1.4]	[1]	$m\neq 0,\ m=0$	$\tilde a_i\ \tilde b_i\ c$	$\tilde b_j\ c$	c_m	c_n
[1.4.1] ([1.4.2])	[1.4]	$m\neq 0,\ m=0$	$\tilde a_i\ \tilde b_i\ \tilde c$	$\tilde b_j\ \tilde c$	$\tilde c_m$	$\tilde c_n\ \tilde d_n$
[1.5]	[1]	$m\equiv 0$	$\tilde a_i\ \tilde b\ c_i$	$\tilde b_j\ c_j$	$\tilde b\ c_m$	$\tilde b\ c_n$
[1.5.1] ([1.5.2])	[1.5]	$m\equiv 0$	$\tilde a_i\ \tilde b\ \tilde c_i\ \tilde d$	$\tilde b_j\ \tilde c_j\ \tilde d$	$\tilde b\ \tilde c_m\ \tilde d$	$\tilde b\ \tilde c_n\ \tilde d_n$
[1.6]	[1]	$m\equiv 0$	$\tilde a_i\ \tilde b_i\ c$	$\tilde b_j\ c$	$\tilde b_m\ c_m$	$\tilde b_n\ c_n$
[1.6.1] ([1.6.2])	[1.6]	$m\equiv 0$	$\tilde a_i\ \tilde b_i\ \tilde c\ \tilde d$	$\tilde b_j\ \tilde c\ \tilde d$	$\tilde b_m\ \tilde c_m\ \tilde d$	$\tilde b_n\ \tilde c_n\ \tilde d_n$
[2]	initial	$m\neq 0,\ m=0$	$\tilde a_i$	$\tilde a\ b_j$	$\tilde a\ b_m$	$\tilde a\ b_n$
[2.1]	[2]	$m\neq 0,\ m=0$	$\tilde a_i$	$\tilde a\ \tilde b_j\ c_j$	$\tilde a\ c_m$	$\tilde a\ c_n$
[2.1.1] ([2.1.2])	[2.1]	$m\neq 0,\ m=0$	$\tilde a_i$	$\tilde a\ \tilde b_j\ \tilde c_j$	$\tilde a\ \tilde c_m$	$\tilde a\ \tilde c_n\ \tilde d_n$
[2.1.3] ([2.1.4])	[2.1]	$m\neq 0,\ m=0$	$\tilde a_i$	$\tilde a\ \tilde b_j\ \tilde c\ \tilde d$	$\tilde a\ \tilde c_m$	$\tilde a\ \tilde c\ \tilde d_n$
[3]	initial	$m\equiv 0$	$\tilde a_i\ b_i$	$\tilde a\ b_j$	$\tilde a\ b_m$	$\tilde a\ b_n$
[3.1]	[3]	$m\equiv 0$	$\tilde a_i\ \tilde b\ c_i$	$\tilde a\ \tilde b_j\ c_j$	$\tilde a\ \tilde b\ c_m$	$\tilde a\ \tilde b\ c_n$
[3.1.1] ([3.1.2])	[3.1]	$m\equiv 0$	$\tilde a_i\ \tilde b\ \tilde c_i\ \tilde d$	$\tilde a\ \tilde b_j\ \tilde c_j\ \tilde d$	$\tilde a\ \tilde b\ \tilde c_m\ \tilde d$	$\tilde a\ \tilde b\ \tilde c_n\ \tilde d_n$

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$$\Gamma\gamma^r\gamma^s+\gamma^r\gamma^s\Gamma=0,\quad (\Gamma\gamma^r-\gamma^r\Gamma)(1-\alpha)m=0,$$
$$(\Gamma\gamma^s-\gamma^s\Gamma)\alpha m=0,\quad \partial_r(\mathscr{F}a_s^{-1/2})=0,\quad \partial_s(\mathscr{F}a_r^{-1/2})=0, \tag{4.5}$$
$$\partial_r(\mathscr{F})(1-\alpha)m=0,\quad \partial_s(\mathscr{F})\alpha m=0,\quad r=i,j,\quad s=m,n.$$

From the system (4.5) we get the following possibilities of separation of variables $x^i,x^j,x^m,x^n \Rightarrow x^i,x^j/x^m,x^n$:

$$[1]\ \Gamma=\gamma^i\gamma^j,\quad \alpha=1,\quad \mathscr{F}=\{v(x^i,x^j)\}^{1/2},\quad a_r=v_r,\quad a_s=vw_s; \tag{4.6}$$

$$[2]\ \Gamma=\gamma^m\gamma^n,\quad \alpha=0,\quad \mathscr{F}=\{w(x^m,x^n)\}^{1/2},\quad a_r=v_rw,\quad a_s=w_s; \tag{4.7}$$

$$[3]\ \Gamma=c_1\gamma^i\gamma^j+c_2\gamma^m\gamma^n,\quad m=0,\quad \mathscr{F}=(vw)^{1/2},\quad a_r=v_rw,\quad a_s=vw_s. \tag{4.8}$$

The variants (1) and (2) are valid for $m\neq 0$ and $m=0$. The variant [3] takes place only for $m=0$. Here [1] and [2] are particular cases of [3]. Finally, we have to mention the symmetry existing between x^i,x^j and x^mx^n.

B. Separation $x^i,x^j \Rightarrow x^i/x^j$ and $x^m,x^n \Rightarrow x^m/x^n$

Substituting (4.6) into (4.1), we obtain

$$\hat{K}_{ij}=\gamma^j\left(\frac{v}{v_i}\right)^{1/2}\left(\partial_i-\frac{\partial_iv_i}{4v_i}\right)-\gamma^i\left(\frac{v}{v_j}\right)^{1/2}\left(\partial_j-\frac{\partial_jv_j}{4v_j}\right)+\gamma^i\gamma^jv^{1/2}m, \tag{4.9}$$

$$\hat{K}_i=\mathscr{G}\left\{\gamma^j\left(\frac{v}{v_i}\right)^{1/2}\left(\partial_i-\frac{\partial_i\tilde{a}_i}{4\tilde{a}_i}\right)+\gamma^i\gamma^jv^{1/2}\alpha m+\beta\lambda\right\}\Sigma, \tag{4.10}$$

$$\hat{K}_j=\mathscr{G}\left\{-\gamma^i\left(\frac{v}{v_j}\right)^{1/2}\left(\partial_j-\frac{\partial_j\tilde{b}_j}{4\tilde{b}_j}\right)+\gamma^i\gamma^jv^{1/2}(1-\alpha)m+(1-\beta)\lambda\right\}\Sigma, \tag{4.11}$$

where λ is a constant, arising after pairwise separation of variables, and

$$v_i=\tilde{a}_i\tilde{b}_i,\quad v_j=\tilde{a}_j\tilde{b}_j,\quad v=\tilde{a}\tilde{b}. \tag{4.12}$$

Imposing that operators $\hat{K}_i$ and $\hat{K}_j$ commute, we obtain the system of separating equations:

$$\Sigma\gamma^i\gamma^j+\gamma^i\gamma^j\Sigma=0,\quad (\Sigma\gamma^i+\gamma^i\Sigma)(1-\alpha)m=0,\quad (\Sigma\gamma^j+\gamma^j\Sigma)\alpha m=0,$$
$$(\Sigma\gamma^j-\gamma^j\Sigma)(1-\beta)\lambda=0,\quad (\Sigma\gamma^i-\gamma^i\Sigma)\beta\lambda=0,$$
$$\partial_i\{\mathscr{G}(v/v_j)^{1/2}\}=0,\quad \partial_j\{\mathscr{G}(v/v_i)^{1/2}\}=0,\quad \partial_i(\mathscr{G}v^{1/2})(1-\alpha)m=0,\quad \partial_j(\mathscr{G}v^{1/2})\alpha m=0. \tag{4.13}$$

From (4.13), we obtain the following possibilities of separation of variables x^i from x^j:

$$[1.1]_{ij}\ \Gamma=\gamma^i,\gamma^j\gamma^m,\gamma^j\gamma^n,\gamma^i\gamma^m\gamma^n,\quad \alpha=\beta=1,\quad \mathscr{G}=(\tilde{a}_j/\tilde{a})^{1/2},\quad v_i=\tilde{a}_i,\quad v_j=\tilde{a}_j\tilde{b}_j,\quad v=\tilde{a}; \tag{4.14}$$

$$[1.2]_{ij}\ \Gamma=\gamma^j,\gamma^i\gamma^m,\gamma^i\gamma^n,\gamma^j\gamma^m\gamma^n,\quad \alpha=\beta=0,\quad \mathscr{G}=(\tilde{b}_i/\tilde{b})^{1/2},\quad v_i=\tilde{a}_i\tilde{b}_i,\quad v_j=\tilde{b}_j,\quad v=\tilde{b}. \tag{4.15}$$

Obviously, there is a natural symmetry between x^i and x^j.
From (4.2) and (4.6) we obtain

$$\hat{K}_{mn}=\frac{\gamma^i\gamma^j\gamma^m}{w_m^{1/2}}\left(\partial_m-\frac{\partial_mw_m}{4w_m}\right)+\frac{\gamma^i\gamma^j\gamma^n}{w_n^{1/2}}\left(\partial_n-\frac{\partial_nw_n}{4w_n}\right), \tag{4.16}$$

then, for the separating operators, we have

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$$\widehat{K}_m=\mathscr{H}\left\{\frac{\gamma^i\gamma^j\gamma^m}{w_m^{1/2}}\left(\partial_m-\frac{\partial_m\widetilde{a}_m}{4\widetilde{a}_m}\right)-\delta\lambda\right\}\Xi, \tag{4.17}$$

$$\widehat{K}_n=\mathscr{H}\left\{\frac{\gamma^i\gamma^j\gamma^n}{w^{1/2}}\left(\partial_n-\frac{\partial_n\widetilde{b}_n}{4\widetilde{b}_n}\right)-(1-\delta)\lambda\right\}\Xi. \tag{4.18}$$

The commutativity between $\widehat{K}_m$ and $\widehat{K}_n$ gives as a result the following system of separating equations:

$$\Xi\gamma^m\gamma^n+\gamma^m\gamma^n\Xi=0,\quad (\Xi\gamma^i\gamma^j\gamma^m-\gamma^i\gamma^j\gamma^m\Xi)(1-\delta)\lambda=0,\quad (\Xi\gamma^i\gamma^j\gamma^n-\gamma^i\gamma^j\gamma^n\Xi)\delta\lambda=0,$$

$$\partial_m(\mathscr{H}w_n^{1/2})=0,\quad \partial_n(\mathscr{H}w_m^{1/2})=0,\quad \partial_m(\mathscr{H})(1-\delta)\lambda=0,\quad \partial_n(\mathscr{H})\delta\lambda=0. \tag{4.19}$$

From the system (4.19) we have the following variants of separation of variables x^m from x^n:

$$[1.1]_{mn}\ \Xi=\gamma^n,\gamma^i\gamma^n,\gamma^j\gamma^n,\gamma^i\gamma^j\gamma^n,\quad \delta=1,\quad \mathscr{H}=\widetilde{c}_n^{1/2},\quad w_m=\widetilde{c}_m,\quad w_n=\widetilde{c}_n\widetilde{d}_n; \tag{4.20}$$

$$[1.2]_{mn}\quad \Xi=\gamma^m,\gamma^i\gamma^m,\gamma^j\gamma^m,\gamma^i\gamma^j\gamma^m,\quad \delta=0,\quad \mathscr{H}=\widetilde{d}_m^{1/2},\quad w_m=\widetilde{c}_m\widetilde{d}_m,\quad w_n=\widetilde{d}_n. \tag{4.21}$$

Here, there is present a symmetry under permutation of x^m and x^n.

In virtue of the symmetry under permutation between the pairs x^i,x^j and x^mx^n, the separation of variables x^i from x^j, and x^m from x^n, in the operators $\widehat{K}_{ij}$ and $\widehat{K}_{mn}$ in [2] is completely equivalent to Eqs. (4.20) and (4.21).

Since the variant [1] contains the mass term, the possibilities of separation [1.1] and [1.2] are valid for massive as well as for massless particles.

Now, we proceed to separate variables x^i from x^j, and x^m from x^n, in the operators $\widehat{K}_{ij}$ and $\widehat{K}_{mn}$, only for the massless neutrino.

From (4.1), (4.2), and (4.8), we have

$$\widehat{K}_{ij}=\gamma^j\left(\frac{v}{v_i}\right)^{1/2}\left(\partial_i-\frac{\partial_iv_i}{4v_i}\right)-\gamma^i\left(\frac{v}{v_j}\right)^{1/2}\left(\partial_j-\frac{\partial_jv_j}{4v_j}\right), \tag{4.22}$$

$$\widehat{K}_{mn}=\gamma^i\gamma^j\gamma^m\left(\frac{w}{w_m}\right)^{1/2}\left(\partial_m-\frac{\partial_mw_m}{4w_m}\right)+\gamma^i\gamma^j\gamma^n\left(\frac{w}{w_n}\right)^{1/2}\left(\partial_n-\frac{\partial_nw_n}{4w_n}\right). \tag{4.23}$$

For the corresponding separating operators, we have

$$\widehat{K}_i=\mathscr{G}\left\{\gamma^j\left(\frac{v}{v_i}\right)^{1/2}\left(\partial_i-\frac{\partial_i\widetilde{a}_i}{4\widetilde{a}_i}\right)+\beta\lambda\right\}\Sigma, \tag{4.24}$$

$$\widehat{K}_j=\mathscr{G}\left\{-\gamma^i\left(\frac{v}{v_j}\right)^{1/2}\left(\partial_j-\frac{\partial_j\widetilde{b}_j}{4\widetilde{b}_j}\right)+(1-\beta)\lambda\right\}\Sigma, \tag{4.25}$$

$$\widehat{K}_m=\mathscr{H}\left\{\gamma^i\gamma^j\gamma^m\left(\frac{w}{w_m}\right)^{1/2}\left(\partial_m-\frac{\partial_m\widetilde{c}_m}{4\widetilde{c}_m}\right)-\delta\lambda\right\}\Xi, \tag{4.26}$$

$$\widehat{K}_n=\mathscr{H}\left\{\gamma^i\gamma^j\gamma^n\left(\frac{w}{w_n}\right)^{1/2}\left(\partial_n-\frac{\partial_n\widetilde{d}_n}{4\widetilde{d}_n}\right)-(1-\delta)\lambda\right\}\Xi. \tag{4.27}$$

Now, in a complete analogous way to (4.9)–(4.13), we obtain the following variants of separation of x^i from x^j, and x^m from x^n:

$$[3.1]_{ij}\ \Sigma=\gamma^i,\gamma^j\gamma^m,\gamma^j\gamma^n,\gamma^i\gamma^m\gamma^n,\quad \beta=1,\quad \mathscr{G}=(\widetilde{a}_j/\widetilde{a})^{1/2},\quad v_i=\widetilde{a}_i\widetilde{b},\quad \widetilde{v}_j=\widetilde{a}_j\widetilde{b}_j,\quad v=\widetilde{a}\widetilde{b}; \tag{4.28}$$

$$[3.2]_{ij}\ \Sigma=\gamma^j,\gamma^i\gamma^m,\gamma^i\gamma^n,\gamma^j\gamma^m\gamma^n,\quad \beta=0,\quad \mathscr{G}=(\widetilde{b}_i/\widetilde{b})^{1/2},\quad v_i=\widetilde{a}_i\widetilde{b}_i,\quad \widetilde{v}_j=\widetilde{a}\widetilde{b}_j,\quad v=\widetilde{a}\widetilde{b}; \tag{4.29}$$

$$[3.1]_{mn}\ \Xi=\gamma^n,\gamma^i\gamma^n,\gamma^j\gamma^n,\gamma^i\gamma^j\gamma^n,\quad \delta=1,\quad \mathscr{H}=(\widetilde{c}_n/\widetilde{c})^{1/2},\quad w_m=\widetilde{c}_m\widetilde{d},\quad w_n=\widetilde{c}_n\widetilde{d}_n,\quad w=\widetilde{c}\widetilde{d}; \tag{4.30}$$

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$$[3.2]_{mn}\ \Xi=\gamma^m,\gamma^j\gamma^m,\gamma^j\gamma^m,\gamma^i\gamma^j\gamma^m,\quad \delta=0,\ \mathscr{H}=(\widetilde{d}_m/\widetilde{d})^{1/2},\quad w_m=\widetilde{c}_m\widetilde{d}_m,\quad w_n=\widetilde{c}\widetilde{d}_n,\quad w=\widetilde{c}\widetilde{d}. \tag{4.31}$$

The structure of the metric coefficients a_k, allowing pairwise separation of variables in the Dirac equation, are presented in Table II.

Taking into account the structure of the metric functions allowing a complete or partial separation of variables in the Dirac equation, based on the algebraic method, we arrive at the result that the metric functions must have a multiplicative structure for consecutive as well as for pairwise separation.

V. SOME SPECIAL CASES OF SEPARABILITY

In this section we are going to explore the possibilities of a complete separation of variables in the Dirac equation when the metric coefficients do not have a multiplicative form. In particular, we consider the line element

$$ds^2=a(x^i,x^j)b(x^m,x^n)\big((dx^i)^2+t(x^j)(dx^j)^2\big)+c(x^i,x^j)d(x^m,x^n)\big((dx^m)^2+r(x^m)dx^n)^2\big), \tag{5.1}$$

where $a(x^i,x^j)$, $b(x^m,x^n)$, $c(x^i,x^j)$, and $d(x^m,x^n)$ are arbitrary functions. The scheme of separation to be applied will determine the form that the functions a, b, c, and d have to be in order to allow a complete separation of variables in the Dirac equation.

After a trivial change of variables, we reduce the line element (4.1) to the form

$$ds^2=a(x^i,x^j)b(x^m,x^n)\big((dx^i)^2+(dx^j)^2\big)+c(x^i,x^j)d(x^m,x^n)\big((dx^m)^2+(dx^n)^2\big). \tag{5.2}$$

The massless Dirac equation in the metric (5.1) reads as

$$\Big\{(ab)^{-1/2}\Big(\gamma^i\Big(\partial_i-\frac{1}{4}\frac{\partial_i a}{a}\Big)+\gamma^j\Big(\partial_j-\frac{1}{4}\frac{\partial_j a}{a}\Big)\Big)+(cd)^{-1/2}\Big(\gamma^m\Big(\partial_m-\frac{1}{4}\frac{\partial_n d}{d}\Big)+\gamma^n\Big(\partial_n-\frac{1}{4}\frac{\partial_n d}{d}\Big)\Big)\Big\}\Psi=0, \tag{5.3}$$

TABLE II. Structure of the metric functions a_k for pairwise separation of variables in the Dirac equation. The variants of separation indicated by ( ) can be obtained from the variants shown in this table after changing x^i, x^j by x^m, x^n; x^i by x^j; and x^m by x^n.

				$[1.1]_{ij},[1.2]_{mn}$				
		v_i		$([1.2]_{ij},[1.2]_{mn};$	$\widetilde{a}_i$			
		v_j		$[2.1]_{ij},[2.1]_{mn};$	$\widetilde{a}_j$	$\widetilde{b}_j$		
	[1] ([2])			$[2.2]_{ij},[2.2]_{mn})$				
	$m\neq0,\ m=0$	v	w_m	$m\neq0,\ m=0$	$\widetilde{a}$		$\widetilde{c}_m$	
a_i		v	w_n		$\widetilde{a}$		$\widetilde{c}_n$	$\widetilde{d}_n$
a_j								
a_m								
a_n								
		v_i	w	$[3.1]_{ij},[3.1]_{mn}$	$\widetilde{a}_i$	$\widetilde{b}$	$\widetilde{c}$	$\widetilde{d}$
	[3]	v_j	w	$([3.2]_{ij},[3.2]_{mn})$	$\widetilde{a}_j$	$\widetilde{b}_j$	$\widetilde{c}$	$\widetilde{d}$
	$m\equiv0$	v	w_m	$m\equiv0$	$\widetilde{a}$	$\widetilde{b}$	$\widetilde{c}_m$	$\widetilde{d}$
		v	w_n		$\widetilde{a}$	$\widetilde{b}$	$\widetilde{c}_n$	$\widetilde{d}_n$

then Eq. (5.3) can be separated as follows:

$$\hat{K}_{ij}=(c/a)^{1/2}(\gamma^i\partial_i+\gamma^j\partial_j)\gamma^m\gamma^n\Phi, \tag{5.4}$$

$$\hat{K}_{mn}=(b/d)^{1/2}(\gamma^m\partial_m+\gamma^n\partial_n)\gamma^m\gamma^n\Phi, \tag{5.5}$$

with

$$\hat{K}_{ij}\Phi=-\hat{K}_{mn}\Phi=\lambda\Phi, \tag{5.6}$$

$$\hat{K}_{ij}\hat{K}_{mn}-\hat{K}_{mn}\hat{K}_{ij}=0, \tag{5.7}$$

$$\Phi=(ad)^{-1/4}\Psi. \tag{5.8}$$

In addition to the multiplicative case, where c/a and b/d must be functions of a single variable in order to allow a complete separation of variables following the scheme developed in Ref. 32, we can consider an additive dependence as follows:

$$(a/c)=s(x^i)+t(x^j), \tag{5.9}$$

$$(d/b)=q(x^m)+r(x^n); \tag{5.10}$$

then the metric (5.2) becomes

$$ds^2=c(x^i,x^j)b(x^m,x^n)\big(s(x^i)+t(x^j)\big)\big((dx^i)^2+(dx^j)^2\big)c(x^i,x^j)b(x^m,x^n)\big(q(x^m)+r(x^n)\big)\big((dx^m)^2+(dx^n)^2\big), \tag{5.11}$$

or, in a more compact form,

$$=c(x^i,x^j)b(x^m,x^n)\{ds^2_{ij}+ds^2_{mn}\}, \tag{5.12}$$

where ds_{ij} and ds_{mn} are given by

$$ds^2_{ij}=\big(s(x^i)+t(x^j)\big)\big((dx^i)^2+(dx^j)^2\big), \tag{5.13}$$

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$$ds^2_{mn} = (q(x^m) + r(x^n))((dx^m)^2 + (dx^n)^2), \quad (5.14)$$

then we have that, using in the standard Jauch and Rohrlich[34] representation,

$$\gamma^i = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix},$$
$$\gamma^m = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad \gamma^n = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.15)$$

where σ^1, σ^2, and σ^3 are the standard Pauli matrices; that equation $\hat{K}_{ij}\Phi = \lambda\Phi$, splits as follows:

$$(\sigma^2\partial_i - \sigma^1\partial_j)\Phi_1 = i\lambda(s(x^i) + t(x^j))^{1/2}\Phi_1, \quad (5.16)$$

$$(-\sigma^2\partial_i + \sigma^1\partial_j)\Phi_2 = -i\lambda(s(x^i) + t(x^j))^{1/2}\Phi_2, \quad (5.17)$$

with

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ d\sigma^3\,\Phi_1 \end{pmatrix}, \quad (5.18)$$

where $d = d(x^m,x^n)$.

Equations (5.5) and (5.6), for the variables x^m and x^n, can be reduced to the form (5.16) and (5.17) by means of the matrix transformation T:

$$T = \tfrac{1}{2}(1 + \gamma^n\gamma^j)(1 + \gamma^m\gamma^i); \quad (5.19)$$

then we have

$$(\sigma^2\partial_m - \sigma^1\partial_n)\tilde{\Phi}_1 = -i\lambda(q(x^m) + r(x^n))^{1/2}\tilde{\Phi}_1, \quad (5.20)$$

$$(-\sigma^2\partial_m + \sigma^1\partial_n)\tilde{\Phi}_2 = -i\lambda(q(x^m) + r(x^n))^{1/2}\tilde{\Phi}_2, \quad (5.21)$$

with $T\tilde{\Phi} = \Phi$,

$$\tilde{\Phi} = \begin{pmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{pmatrix} = \begin{pmatrix} \tilde{\Phi}_1 \\ c\sigma^3\,\tilde{\Phi}_1 \end{pmatrix}, \quad (5.22)$$

where $c = c(x^i,x^j)$.

In order to determine the explicit form of the functions s, t, q, and r for which a complete separation of variables in Eqs. (5.16) and (5.17) is possible, let us introduce the similarity transformation

$$\Phi_1 = \exp(\alpha(x^i,x^j))\exp(i\beta(x^i,x^j)\sigma^3)\Omega = \Delta\Omega, \quad (5.23)$$

applying the transformation (5.23) onto the spinor equation, and taking into account the subsidiary conditions

$$\partial_i\alpha = \partial_j\beta, \quad \partial_j\alpha = -\partial_i\beta, \quad (5.24)$$

$$\partial^2_{ii}\alpha + \partial^2_{jj}\alpha = 0, \quad \partial^2_{ii}\beta + \partial^2_{jj}\beta = 0, \quad (5.25)$$

we find

$$[\sigma^2\partial_i - \sigma^1\partial_j + i\lambda(\alpha(x^i) + \beta(x^j))^{1/2} \times \exp(2i\beta(x^i,x^j)\sigma^3)]\Omega = 0; \quad (5.26)$$

then, in order to separate variables in (5.26), we impose the relation

$$\exp(2i\beta) = [p(x^i) + iq(x^j)\sigma^3]/[(p(x^i)^2 + q(x^j)^2)^{1/2}], \quad (5.27)$$

with

$$\beta = \tfrac{1}{2}\arctan(q(x^j)/p(x^i)), \quad (5.28)$$

$$p(x^i)^2 + q(x^j)^2 = s(x^i) + t(x^j); \quad (5.29)$$

in this way, Eq. (5.26) can be written as follows:

$$\{\sigma^2\partial_i + i\lambda p(x^i) - i\sigma^3(\sigma^2\partial_j - \lambda q(x^j))\}\Omega = 0. \quad (5.30)$$

Here we are able to separate variables by introducing the auxiliary spinor W,

$$\Omega = [\sigma^3(\sigma^2\partial_i + i\lambda p(x^i)) + \sigma^2\partial_j - \lambda q(x^j)]W; \quad (5.31)$$

substituting (5.31) into (5.30) we obtain two systems of second-order differential equations:

$$(\partial^2_i + i\lambda\sigma^2 p_{,i} - \lambda^2 p^2)\Delta = k^2\Delta, \quad (5.32)$$

$$(\partial^2_j - \lambda\sigma^2 q_{,j} + \lambda^2 q^2)\Delta = k^2\Delta, \quad (5.33)$$

where k is a constant of separation.

Substituting (5.28) into (5.24) and (5.25), we find that a necessary condition for $p(x^i)$ and $q(x^j)$ is given by the relation

$$q^2(\partial^2_{ii}p/p) + p^2(\partial^2_{jj}q/q) = f(x^j) + g(x^j); \quad (5.34)$$

then, solving Eq. (5.34) and imposing (5.24), we obtain the following configurations:

(a) $p = x^i, \quad q = x^j$;

the expression (5.13) takes the form

$$ds^2_{ij} = ((x^i)^2 + (x^j)^2)((dx^i)^2 + (dx^j)^2); \quad (5.35)$$

(b) $p = 1/x^i, \quad q = 1/x^j$,

with

$$ds^2_{ij} = ([(x^i)^2 + (x^j)^2]/(x^i)^2(x^j)^2)((dx^i)^2 + (dx^j)^2); \quad (5.36)$$

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(c) $p=\exp(x^i)$, $q=\exp(ix^j)$,

and

$$ds_{ij}^2=(\exp(2x^i)+\exp(2ix^j))((dx^i)^2+(dx^j)^2), \quad (5.37)$$

where we have assumed that the coordinate x^j is imaginary (a timelike variable). An analogous expression is obtained if we consider x^i (instead of x^j) as the timelike variable;

(d) $p=\cos(x^i)$, $\sinh(x^j)$,

$$ds_{ij}^2=(\cos^2(x^i)+\sinh^2(x^j))((dx^i)^2+(dx^j)^2); \quad (5.38)$$

(e) $p=(\cos(x^i))^{-1}, q=(\sinh(x^j))^{-1}$,

$$ds_{ij}^2=\left(\frac{\cos^2(x^i)+\sinh^2(x^j)}{\cos^2(x^i)\sinh^2(x^j)}\right)((dx^i)^2+(dx^j)^2). \quad (5.39)$$

Changing x^i by x^j and vice versa, we find equivalent line elements, where, also, it is possible to separate variables.

Regretfully, the line elements (5.35), (5.37), and (5.38) are those obtained where we analyze the separation of variables in the Dirac equation in curvilinear coordinates and, therefore, they do not represent a particular interest for the problem we have stated because they can be reduced to the standard two-dimensional Minkowski line element by means of the coordinate transformations described in Refs. 35 and 36. Analogously, (5.36) and (5.39) can be reduced to a conformally flat metric in a similar fashion. As a simple illustration, let us take the metric

$$ds^2=\left(\frac{(x^i)^2+(x^j)^2}{(x^i)^2(x^j)^2}\right)((dx^i)^2+(dx^j)^2) + \left(\frac{\cos^2(x^m)+\sinh^2(x^n)}{\cos^2(x^m)\sinh^2(x^n)}\right)((dx^m)^2+(dx^n)^2), \quad (5.40)$$

after making the change of variables

$$y^i=[(x^i)^2-(x^j)^2]/2,$$

$$y^j=(x^i)(x^j),$$

$$y^m=\sin(x^m)\cosh(x^n),$$

$$y^n=\cos(x^m)\sinh(x^n), \quad (5.41)$$

the metric (5.40) becomes

$$ds^2=\frac{1}{(y^i)^2(y^n)^2}[(y^n)^2((dy^i)^2+(dy^j)^2) + (y^i)^2((dy^m)^2+(dy^n)^2)], \quad (5.42)$$

which admits a complete separation of variables according to the scheme presented in Secs. III and IV.

For the massive Dirac equation we have that the results presented in this section are valid if we put $s=1$ and $t=0$ in (5.9) or $q=1$ and $r=0$ in Eq. (5.10).

It is straightforward to show that only flat metrics, written in curvilinear coordinates, allow a complete separation of variables in the Dirac equation.

VI. DISCUSSION

Analyzing the results on a separation of variables obtained in this paper, first of all we would like to remark that, at least to us, all the known multiplicative diagonal metrics (see Ref. 37) admit a complete separation of variables in the Dirac equation, according to some of the schemes presented in this paper.

For example, consecutive separation of variables in the Dirac equation for the Schwarzschild metric can be achieved, according to the scheme [1.1.3], [1.1.5], [1.2.1], [1.4.1] ($m\neq 0$ or $m=0$). Further possibilities of separation corresponding to the massless neutrino can be considered following the scheme [1.5.1] and [1.6.1]. For the Schwarzschild metric, pairwise separation of variables is possible according to the schemes [1.1] ($m\neq 0$ or $m=0$), and [3.1] ($m\equiv 0$).

The metrics (1.1) and (1.4) allow pairwise, as well as consecutive separation of variables. The metric (1.2), for arbitrary functions $\alpha(r)$, $\beta(r)$, $\nu(r)$, and the metric (1.3), for arbitrary parameters a_1, a_2, a_3, allow only consecutive separation of variables. Particularization for massive or massless particles can be achieved according to Tables I and II.

As an illustration of the capabilities of the algebraic method of separation of variables, we have that the Dirac equation (2.8) is separable in the nonfactorizable (in the sense of the Miller) metric,[19]

$$ds^2=-dt^2+a^2(t)(dx^2+b^2(x)[dy^2+c^2(y)dz^2]). \quad (6.1)$$

Substituting (6.1) into (2.8), we obtain

$$\left\{\gamma^0\partial_t+\frac{1}{a}\gamma^1\partial_x+\frac{1}{ab}\gamma^2\partial_y+\frac{1}{abc}\gamma^3\partial_z+m\right\}\Psi=0, \quad (6.2)$$

Eq. (6.2) can be written as

$$(\hat{K}_1+\hat{K}_2)\Phi=0, \quad \Psi=\gamma^1\gamma^2\gamma^3\Phi, \quad (6.3)$$

$$\hat{K}_1=a(\gamma^0\partial_0+m)\gamma^1\gamma^2\gamma^3, \quad (6.4)$$

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$$\hat{K}_2=\left(\gamma^1\partial_x+\frac{1}{b}\gamma^2\partial_y+\frac{1}{bc}\gamma^3\partial_z\right)\gamma^1\gamma^2\gamma^3, \tag{6.5}$$

the operators $\hat{K}_1$ and $\hat{K}_2$ commute, and satisfy the eigenvalue equations:

$$\hat{K}_1\Phi=-\hat{K}_2\Phi=\lambda\Phi; \tag{6.6}$$

the next step is to separate variables in the equation $\hat{K}_2\Phi+\lambda\Phi=0$. Considering (6.5) and (6.6), we reduce this equation to the form

$$(\hat{K}_3+\gamma^1\gamma^2\hat{K}_4)\phi=0, \quad [\hat{K}_3,\hat{K}_4]_-=0, \tag{6.7}$$

with

$$\hat{K}_3=b(\gamma^2\gamma^3\partial_x+\lambda),$$

$$\hat{K}_4=(-\gamma^2\partial_y+1/c\gamma^3\partial_z)\gamma^3, \tag{6.8}$$

It is not difficult to see that the operator $\hat{K}_4$ commutes with ∂_z, and therefore the eigenvalue equation $\hat{K}_4\phi=\delta\phi$ can be rewritten as follows:

$$(\hat{K}_5+\hat{K}_6)\Xi=0, \quad [\hat{K}_5,\hat{K}_6]_-=0, \tag{6.9}$$

with

$$-\hat{K}_5\Xi=\hat{K}_6\Xi=\epsilon\Xi$$

and

$$\hat{K}_5=\partial_z, \quad \hat{K}_6=c(\gamma^2\gamma^3\partial_y-\delta), \quad \Xi=\Omega. \tag{6.10}$$

In this way, we have reduced Eq. (6.2) to four systems of ordinary differential equations corresponding to the operators $\hat{K}_1$, $\hat{K}_3$, $\hat{K}_5$, and $\hat{K}_6$. Then the Dirac equation is separable in the line element (6.1).

Finally, let us remark the following aspect of our results: In the literature we can find articles where Einstein equations are solved under the additional requirement of separability of the Dirac equation. For example, Jing[38] obtains exact solutions of Einstein equations for space-time with local rotational symmetry in which the Dirac equation separates. We have to say that the requirement of separability of the Dirac equation reduces the classes of solutions of Einstein equations. Solving the Einstein equations without imposing separability of variables in the Dirac equation we can obtain a wider class of solutions. Then, after obtaining the metric it would be possible to analyze whether or not the Dirac equation is separable. We hope that the results presented in this paper will be helpful in investigating such kinds of problems.

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