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Mathematical analysis of scattering problems for partially coated obstacles and cracks

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We consider the scattering problem of time-harmonic electromagnetic waves from an infinite cylinder having an open arc and a bounded domain in R^2 as cross section. To this end, we solve a mixed scattering problem for the Helmholtz equation in R^2 where the scattering object is a combination of an open arc Γ and a bounded obstacle D. The well-posedness of the solution to the direct scattering problem is established by using the boundary integral equation method and the Fredholm theory.
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I. INTRODUCTION

In this paper, we consider the problem of scattering of electromagnetic waves from an infinite cylinder having an open arc Γ and a bounded domain D in R^2 as cross section. We assume that the obstacle is thin dielectric cylinder (bottom on the plane) whose properties do not change along the z axis and the incident time harmonic electric field is polarized parallel to the cylinder z axis. After factoring out the term $e^{-i\omega t}$, where ω is the fixed frequency, the only non-zero component u of the total electric field satisfies the Helmholtz equation in the domain $R^2 \setminus (\bar{D} \cup \bar{\Gamma})$, that is,

$$\Delta u + k^2 u = 0, \quad \text{in} \quad R^2 \setminus (\bar{D} \cup \bar{\Gamma}), \tag{1}$$

where k is the wavenumber, and we assume that $k > 0$ for simplicity.

We extend the arc Γ to an arbitrary piecewise smooth, simply connected, closed curve ∂G enclosing a bounded domain G such that the normal vector ν on Γ coincides with the normal vector on ∂G, and we assume that the domain D is completely contained in G, i.e., $D \subset G$ and $\partial D \cap \partial G = \emptyset$ (see Fig. 1).

Remark: The ν is the unit normal vector defined almost everywhere on ∂G and ∂D, respectively (except a finite number of points), and direct into the exterior of G and D, respectively.

More specific, the following boundary conditions on the open arc Γ are obtained for the component u of the total field (see Ref. 22)

$$[\frac{\partial u}{\partial \nu}] = 0 \quad \text{and} \quad [u] - i\lambda \frac{\partial u_+}{\partial \nu} = 0 \quad \text{on} \quad \Gamma, \tag{2}$$

where $u_\pm(x) = \lim_{h\to 0^+} u(x \pm h\nu)$ and $\frac{\partial u_\pm}{\partial \nu} = \lim_{h\to 0^+} \nu \cdot \nabla u(x \pm h\nu)$ for $x \in \Gamma$, $[u] = u_+ - u_-$ and $[\frac{\partial u}{\partial \nu}] = \frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu}$ are the respective jumps across Γ and the impedance coefficient λ is a positive constant.

The boundary ∂D of the domain D has a Lipshitz dissection $\partial D = \Gamma_D \cup \Pi \cup \Gamma_I$, where Γ_D and Γ_I are disjoint, relatively open subsets of ∂D, having Π as their common boundary in ∂D (see, e.g.,

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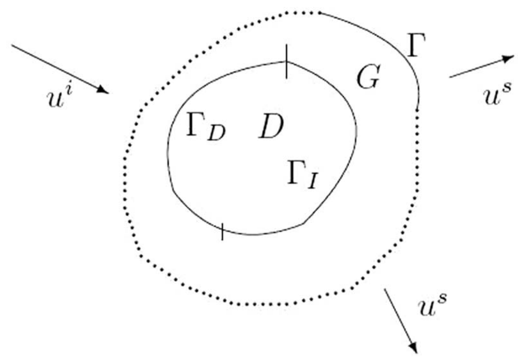

FIG. 1. The mixed scattering problem.

Ref. 4). Furthermore, boundary conditions of Dirichlet and impedance type are specified on Γ_D and Γ_I, respectively, i.e.,

$$u = 0 \quad \text{on} \quad \Gamma_D \tag{3}$$

and

$$\frac{\partial u}{\partial \nu} + ik\mu u = 0 \quad \text{on} \quad \Gamma_I \tag{4}$$

with positive constant μ.

In the region $R^2 \setminus (\bar{D} \cup \bar{\Gamma})$, the total wave u is the superposition of the given incident plane wave $u^i = e^{ikx \cdot d}$ (or point source $u^i = \Phi(x, x_0)$), and the scattered wave u^s which is required to satisfy the Sommerfeld radiation condition

$$\lim_{\nu \to \infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{5}$$

uniformly in all directions $x/|x|$, where $r = |x|$.

The direct and inverse scattering problems for cracks were initiated by Kress in 1995.[12] In the paper, Kress considered the direct and inverse scattering problem for a perfectly conducting crack, and used integral equation method to solve both the direct and inverse problems for a sound-soft crack. Mönch[18] extended this approach to a Neumann crack in 1997. In 2000, Kress's work was continued by Kirsch and Ritter in Ref. 11, and in the same year these results were generalized to the scattering problem with cracks for Maxwell equations in Ref. 1 by Ammari *et al.* Later in 2003, Cakoni and Colton in Ref. 2 discussed the direct and inverse scattering problems for cracks (possibly) coated on one side by a material with surface impedance λ. Extending to the impedance problem, Lee[14] considered the direct and inverse scattering problem for an impedance crack in 2008. In 2009, Yan and Yao considered a kind of scattering problem from a mixture of a crack and a bounded obstacle in Ref. 21. In the same year, Krutitskii in Ref. 13 discussed a crack scattering problem with different impedance boundary conditions on opposite sides of the crack. In 2010, Liu and Sini in Ref. 16 provided a method to reconstruct cracks of different types from far field measurements.

However, studying an inverse problem always requires a solid knowledge of the corresponding direct problem. So, in this paper we focus on the direct scattering problem, that is, we try to look for a function u satisfying (1)–(5) in some Sobolev spaces. The main challenge is to derive a suitable boundary integral system and show that the corresponding boundary integral operators are Fredholm of index zero. Then, the well-posedness of the solution to the direct problem can be obtained by employing the Fredholm theory.

To the author's knowledge, there are few results for the mixed scattering problem, that is, the obstacle is a mixture of the arc Γ and the domain D. In the case when the arc Γ disappears, the above problem is simply to solve the obstacle scattering problem, we refer the reader to Refs. 3–10, 15, 17, 19, and 20. Conversely, if there is not the domain D, we then just need to solve the scattering problem from cracks, we refer the reader to Refs. 2, 11, 14, and 16.

The plan of the paper is as follows. In Sec. II, we will introduce our problem and derive the system of boundary integral equations by employing the potential theory. In Sec. III, we establish well-posedness of the solution to the problem (1)–(5) by using the boundary integral equation method and the Fredholm theory.

II. THE DIRECT SCATTERING PROBLEM

In two-dimensional space, we suppose the arc Γ can be parameterized as

$$\Gamma = \{z(s) : s \in [s_0, s_1]\},$$

where $z : [s_0, s_1] \to R^2$ is an injective piecewise C^1 function.

In the following discussion, $H^1_{loc}(R^2\backslash(\bar{D} \cup \bar{\Gamma}))$ and $H^1(D)$ are usual Sobolev spaces. $H^{\frac{1}{2}}(\Gamma_D)$, $H^{-\frac{1}{2}}(\Gamma_I)$, and $H^{\pm\frac{1}{2}}(\Gamma)$ are the trace spaces. We define (see Ref. 17)

$$H^{\frac{1}{2}}(\Gamma) = \{u|_\Gamma : \ u \in H^{\frac{1}{2}}(\partial\Omega)\}, \tag{6}$$

$$\tilde{H}^{\frac{1}{2}}(\Gamma) = \{u \in H^{\frac{1}{2}}(\Gamma) : \quad \mathrm{supp} u \subseteq \bar{\Gamma}\}, \tag{7}$$

$$H^{-\frac{1}{2}}(\Gamma) = \left(\tilde{H}^{\frac{1}{2}}(\Gamma)\right)', \quad \text{the dual space of} \quad \tilde{H}^{\frac{1}{2}}(\Gamma), \tag{8}$$

$$\tilde{H}^{-\frac{1}{2}}(\Gamma) = \left(H^{\frac{1}{2}}(\Gamma)\right)', \quad \text{the dual space of} \quad H^{\frac{1}{2}}(\Gamma), \tag{9}$$

and we have the chain

$$\tilde{H}^{\frac{1}{2}}(\Gamma) \subset H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma) \subset \tilde{H}^{-\frac{1}{2}}(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma).$$

Consider the scattered field u^s, then our problem can be reformulated as the following.

Given $f \in H^{1/2}(\Gamma_D)$, $g \in H^{-1/2}(\Gamma_I)$, and $q \in H^{-1/2}(\Gamma)$, we try to find a function $u^s \in H^1_{loc}(R^2\backslash(\bar{D} \cup \bar{\Gamma}))$ such that

$$\begin{cases} \Delta u^s + k^2 u^s & = & 0 & \text{in} & R^2\backslash(\bar{D} \cup \bar{\Gamma}) \\ u^s & = & f & \text{on} & \Gamma_D \\ \frac{\partial u^s}{\partial \nu} + ik\mu u^s & = & g & \text{on} & \Gamma_I \\ [\frac{\partial u^s}{\partial \nu}] & = & 0 & \text{on} & \Gamma \\ [u^s] - i\lambda \frac{\partial u^s_+}{\partial \nu} & = & q & \text{on} & \Gamma \end{cases} \tag{10}$$

and u^s is required to satisfy the Sommerfeld radiation condition (5).

Remark: To the original problem (1)–(5), $f = -u^i$, $g = -\frac{\partial u^i}{\partial \nu} - ik\mu u^i$, and $q = i\lambda \frac{\partial u^i}{\partial \nu}$.

Theorem 2.1: *The problem (10) has at most one solution.*

Proof: Denote by B_R a sufficiently large ball with radius R containing $\bar{G}$ and by ∂B_R its boundary. Let u^s be a solution to the homogeneous problem (10), i.e., u^s satisfies the problem (10) with $f = g = q = 0$.

It is easy to check that this solution $u^s \in H^1(B_R\backslash\bar{G}) \cup H^1(G\backslash\bar{D})$ satisfies following transmission conditions on the complementary part $\partial G\backslash\bar{\Gamma}$ of ∂G,

$$\begin{cases} u^s_+ = u^s_- \\ \frac{\partial u^s_+}{\partial \nu} = \frac{\partial u^s_-}{\partial \nu}, \end{cases} \tag{11}$$

where "$\pm$" denote the limit approaching ∂G from outside and inside G, respectively.

Applying Green's formula for u^s and $\bar{u}^s$ in $G\backslash\bar{D}$ and $B_R\backslash\bar{G}$, we have

$$\int_{G\backslash\bar{D}} (u^s \Delta\bar{u}^s + \nabla\bar{u}^s \cdot \nabla u^s)dx = \int_{\partial G} u^s \frac{\partial \bar{u}^s}{\partial \nu} ds + \int_{\partial D} u^s \frac{\partial \bar{u}^s}{\partial \nu} ds \tag{12}$$

and

$$\int_{B_R\setminus\bar{G}}(u^s\triangle\bar{u}^s+\nabla\bar{u}^s\cdot\nabla u^s)dx=\int_{\partial G}u^s\frac{\partial\bar{u}^s}{\partial\nu}ds+\int_{\partial D}u^s\frac{\partial\bar{u}^s}{\partial\nu}ds. \tag{13}$$

Using the homogeneous boundary conditions and the above transmission boundary condition (11),

$$\int_{\partial B_R}u^s\frac{\partial\bar{u}^s}{\partial\nu}ds=(\int_{G\setminus\bar{D}}+\int_{B_R\setminus\bar{G}})(-k^2|u^s|^2+|\nabla u^s|^2)dx$$
$$+\int_{\Gamma_I}ik\mu|u^s|^2ds+\int_{\Gamma}i\lambda|\frac{\partial u^s}{\partial\nu}|^2ds.$$

Since k, μ, and λ are positive, then

$$Im(\int_{\partial B_R}u\frac{\partial\bar{u}^s}{\partial\nu}ds)\geq 0. \tag{14}$$

So, from Theorem 2.12 in Ref. 6 a unique continuation argument we obtain that $u^s=0$ in $R^2\setminus(\bar{D}\cup\bar{\Gamma})$.

III. EXISTENCE OF THE SOLUTION

Theorem 3.1: *Assume that B_R is a large ball with radius R containing $\bar{G}$, and $\Gamma_I\neq\emptyset$, then the problem (10) has a unique solution u^s which satisfies*

$$\| u^s \|_{H^1(B_R\setminus(\bar{D}\cup\bar{\Gamma}))}\leq C(\| f \|_{H^{\frac{1}{2}}(\Gamma_D)}+\| g \|_{H^{-\frac{1}{2}}(\Gamma_I)}+\| q \|_{H^{-\frac{1}{2}}(\Gamma)}), \tag{15}$$

where the positive constant C depends on R but not on f, g, and q.

Proof: By Green representation formula (see Refs. 6 and 5), we have

$$u^s(x)=\int_{\partial G}[\frac{\partial u^s}{\partial\nu}\Phi(x,y)-u^s\frac{\partial\Phi(x,y)}{\partial\nu}]ds_y$$
$$+\int_{\partial D}[\frac{\partial u^s}{\partial\nu}\Phi(x,y)-u^s\frac{\partial\Phi(x,y)}{\partial\nu}]ds_y \tag{16}$$

for $x\in G\setminus\bar{D}$, and

$$u^s(x)=\int_{\partial G}[u^s\frac{\partial\Phi(x,y)}{\partial\nu}-\frac{\partial u^s}{\partial\nu}\Phi(x,y))]ds_y \tag{17}$$

for $x\in R^2\setminus\bar{G}$, where

$$\Phi(x,y)=\frac{i}{4}H_0^{(1)}(k|x-y|) \tag{18}$$

is the fundamental solution to the Helmholtz equation in R^2 and $H_0^{(1)}$ is a Hankel function of the first kind of order zero.

We now denote by $\tilde{f}\in H^{\frac{1}{2}}(\partial D)$ and $\tilde{g}\in H^{-\frac{1}{2}}(\partial D)$ bounded extensions to the whole boundary ∂D of the boundary data f and g, respectively, and write

$$(\frac{\partial u^s}{\partial\nu}+ik\mu u^s)|_{\partial D}=a+\tilde{g},\quad u^s|_{\partial D}=b+\tilde{f}. \tag{19}$$

Obviously $a\in\tilde{H}^{-1/2}(\Gamma_D)$, $b\in\tilde{H}^{1/2}(\Gamma_I)$ since $a=0$ on Γ_I and $b=0$ on Γ_D.

If we denote by $\tilde{a}$ and $\tilde{b}$ zero extensions to the whole boundary ∂D of the data a and b, respectively. Then by calculation, we have

$$u^s|_{\partial D}=\tilde{b}+\tilde{f} \tag{20}$$

and

$$\frac{\partial u^s}{\partial\nu}|_{\partial D}=\tilde{g}-ik\mu\tilde{b}+\tilde{a}-ik\mu\tilde{f}. \tag{21}$$

We consider (16) and let x approach the boundary ∂D. Then making use of the known jump relationships of the single- and double-layer potentials across the boundary ∂D, interpreted in the sense of the trace theorem, we obtain the following expression for the Cauchy data of the solution on the boundary ∂D:

$$u^s|_{\partial D} = [K_D u^s - S_D \frac{\partial u^s}{\partial \nu} + S_G \frac{\partial u^s}{\partial \nu} - K_G u^s]|_{\partial D}, \tag{22}$$

$$\frac{\partial u^s}{\partial \nu}|_{\partial D} = [T_D u^s - K'_D \frac{\partial u^s}{\partial \nu} + K'_G \frac{\partial u^s}{\partial \nu} - T_G u^s]|_{\partial D}, \tag{23}$$

where S_D, K_D, S_G, K_G, K'_D, K'_G, T_D, and T_G denote boundary integral operators defined by

$$S_D\varphi(x) = 2\int_{\partial D} \varphi(y)\Phi(x,y)ds_y, \quad S_G\varphi(x) = 2\int_{\partial G} \varphi(y)\Phi(x,y)ds_y,$$

$$K_D\varphi(x) = 2\int_{\partial D} \varphi(y)\frac{\Phi(x,y)}{\partial \nu_y}ds_y, \quad K_G\varphi(x) = 2\int_{\partial G} \varphi(y)\frac{\Phi(x,y)}{\partial \nu_y}ds_y,$$

$$K'_D\varphi(x) = 2\int_{\partial D} \varphi(y)\frac{\partial\Phi(x,y)}{\partial \nu_x}ds_y, \quad K'_G\varphi(x) = 2\int_{\partial G} \varphi(y)\frac{\partial\Phi(x,y)}{\partial \nu_x}ds_y,$$

$$T_D\varphi(x) = 2\frac{\partial}{\partial \nu_x}\int_{\partial D} \varphi(y)\frac{\partial\Phi(x,y)}{\partial \nu_y}ds_y, \quad T_G\varphi(x) = 2\frac{\partial}{\partial \nu_x}\int_{\partial G} \varphi(y)\frac{\partial\Phi(x,y)}{\partial \nu_y}ds_y.$$

Now restricting (22) to Γ_D and using (19)–(21), we obtain

$$\begin{aligned} f(x) &= u^s|_{\Gamma_D} \\ &= [K_D u^s - S_D\frac{\partial u^s}{\partial \nu} + 2\int_{\partial G}(\frac{\partial u^s}{\partial \nu}\Phi(x,y) - \frac{\partial \Phi(x,y)}{\partial \nu}u^s)ds_y]|_{\Gamma_D} \\ &= [K_D(\tilde{b}+\tilde{f}) - S_D(\tilde{g} - ik\mu\tilde{b} + \tilde{a} - ik\mu\tilde{f})]|_{\Gamma_D} \\ &+ [2\int_{\partial G}(\frac{\partial u^s}{\partial \nu}\Phi(x,y) - \frac{\partial \Phi(x,y)}{\partial \nu}u^s)ds_y]|_{\Gamma_D} \\ &= [-S_D\tilde{a} + K_D\tilde{b} + ik\mu S_D\tilde{b}]|_{\Gamma_D} + [K_D\tilde{f} + ik\mu S_D\tilde{f} - S_D\tilde{g}]|_{\Gamma_D} \\ &+ 2\int_{\partial G}(\frac{\partial u^s_-}{\partial \nu}\Phi(x,y) - \frac{\partial \Phi(x,y)}{\partial \nu}u^s_-)ds_y|_{\Gamma_D}. \end{aligned} \tag{24}$$

Lemma 3.1: When $x \in \partial D$, then by using Green formula and the Sommerfeld radiation condition (5), we obtain

$$\int_{\partial G}[\frac{\partial u^s_+(y)}{\partial \nu}\Phi(x,y) - u^s_+(y)\frac{\partial \Phi(x,y)}{\partial \nu}u(y)]ds_y = 0. \tag{25}$$

Proof: Denote by B_R a sufficiently large ball with radius R containing $\bar{G}$ and use Green formula inside $B_R\backslash\bar{G}$. Furthermore, notice that $x \in \partial D$, $y \in \partial G$ and the Sommerfeld radiation condition (5), we can prove this lemma.

If we define

$$c = [u^s]|_\Gamma = (u^s_+ - u^s_-)|_\Gamma, \tag{26}$$

and use $\tilde{c}$ to denote zero extension of c to the whole boundary ∂G, then we have the following lemma.

Lemma 3.2: The last term in (24) can be rewritten as

$$2\int_{\partial G}(\frac{\partial u^s_-}{\partial \nu}\Phi(x,y) - \frac{\partial \Phi(x,y)}{\partial \nu}u^s_-)ds_y = 2\int_{\partial G}\tilde{c}\frac{\partial \Phi(x,y)}{\partial \nu}ds_y. \tag{27}$$

Proof: From Lemma 3.1, we have

$$
\begin{aligned}
&2\int_{\partial G}(\frac{\partial u^s}{\partial \nu}\Phi(x,y)-\frac{\partial \Phi(x,y)}{\partial \nu}u^s)ds_y\\
&=2\int_{\partial G}[(u_+^s-u_-^s)\frac{\partial \Phi(x,y)}{\partial \nu}-(\frac{\partial u_+^s}{\partial \nu}-\frac{\partial u_-^s}{\partial \nu})\Phi(x,y)]ds_y\\
&=2\int_{\partial G}(u_+^s-u_-^s)\frac{\partial \Phi(x,y)}{\partial \nu}ds_y\\
&=2\int_{\partial G}\tilde{c}\frac{\partial \Phi(x,y)}{\partial \nu}ds_y. \qquad (28)
\end{aligned}
$$

This ends the proof of the lemma.

Furthermore, we have

Lemma 3.3: On the boundary ∂G, we have

$$
2\frac{\partial}{\partial \nu(x)}\int_{\partial G}(\frac{\partial u_-^s}{\partial \nu}\Phi(x,y)-\frac{\partial \Phi(x,y)}{\partial \nu}u_-^s)ds_y=2\frac{\partial}{\partial \nu(x)}\int_{\partial G}\tilde{c}\frac{\partial \Phi(x,y)}{\partial \nu}ds_y. \qquad (29)
$$

By using boundary conditions on ∂G (see (10) and (11)), and Lemma 3.2, we continue to write (24) as

$$
\begin{aligned}
f(x)&=[-S_D\tilde{a}+K_D\tilde{b}+ik\mu S_D\tilde{b}]|_{\Gamma_D}+[K_D\tilde{f}+ik\mu S_D\tilde{f}-S_D\tilde{g}]|_{\Gamma_D}\\
&\quad+2\int_{\partial G}\tilde{c}\frac{\partial \Phi(x,y)}{\partial \nu}ds_y|_{\Gamma_D}\\
&=-S_{\Gamma_D\Gamma_D}a+K_{\Gamma_I\Gamma_D}b+ik\mu S_{\Gamma_I\Gamma_D}b+K_{\Gamma\Gamma_D}c\\
&\quad+[K_D\tilde{f}-S_D\tilde{g}+ik\mu S_D\tilde{f}]|_{\Gamma_D}, \qquad (30)
\end{aligned}
$$

where the operator $K_{\Gamma_I\Gamma_D}$ is the operator applied to a function with $supp\subseteq\bar{\Gamma}_I$ and evaluated on Γ_D, with analogous definitions for $S_{\Gamma_D\Gamma_D}$, $S_{\Gamma_I\Gamma_D}$, and $K_{\Gamma\Gamma_D}$. We have mapping properties (see Refs. 4 and 2)

$$
\begin{aligned}
&S_{\Gamma_D\Gamma_D}:\ \tilde{H}^{-1/2}(\Gamma_D)\ \to H^{1/2}(\Gamma_D),\quad S_{\Gamma_I\Gamma_D}:\ \tilde{H}^{-1/2}(\Gamma_I)\ \to H^{1/2}(\Gamma_D),\\
&K_{\Gamma_I\Gamma_D}:\ \tilde{H}^{1/2}(\Gamma_I)\ \to H^{1/2}(\Gamma_D),\quad K_{\Gamma\Gamma_D}:\ \tilde{H}^{1/2}(\Gamma)\ \to H^{1/2}(\Gamma_D).
\end{aligned}
$$

Formula (30) can be rewritten as

$$
S_{\Gamma_D\Gamma_D}a-K_{\Gamma_I\Gamma_D}b-ik\mu S_{\Gamma_I\Gamma_D}b-K_{\Gamma\Gamma_D}c=r_1(x), \qquad (31)
$$

where

$$
r_1(x)=[K_D\tilde{f}-S_D\tilde{g}+ik\mu S_D\tilde{f}]|_{\Gamma_D}-f(x). \qquad (32)
$$

Similarly, on Γ_I, by using (19)–(21), Lemmas 3.2 and 3.3, we obtain

$$
\begin{aligned}
g(x) &= (\frac{\partial u_+^s}{\partial \nu} + ik\mu u_+^s)|_{\Gamma_I} \\
&= (T_D u_+^s - K_D' \frac{\partial u_+^s}{\partial \nu})|_{\Gamma_I} + 2\frac{\partial}{\partial \nu(x)} \int_{\partial G} (\frac{\partial u_-^s}{\partial \nu}\Phi(x,y) - \frac{\partial \Phi(x,y)}{\partial \nu} u_-^s) ds_y|_{\Gamma_I} \\
&+ ik\mu(K_D u_+^s - S_D \frac{\partial u_+^s}{\partial \nu})|_{\Gamma_I} \\
&+ 2ik\mu \int_{\partial G} (\frac{\partial u_-^s}{\partial \nu}\Phi(x,y) - \frac{\partial \Phi(x,y)}{\partial \nu} u_-^s) ds_y|_{\Gamma_I} \\
&= (-K_D'\tilde{a} - ik\mu S_D \tilde{a} - k^2\mu^2 S_D \tilde{b})|_{\Gamma_I} + ik\lambda\mu(K_D\tilde{b} + K_D'\tilde{b})|_{\Gamma_I} + T_D\tilde{b}|_{\Gamma_I} \\
&+ 2\frac{\partial}{\partial \nu(x)} \int_{\partial G} \tilde{c} \frac{\partial \Phi(x,y)}{\partial \nu} ds_y|_{\Gamma_I} + 2\int_{\partial G} \tilde{c}\frac{\partial \Phi(x,y)}{\partial \nu} ds_y|_{\Gamma_I} \\
&+ [T_D \tilde{f} - K_D'\tilde{g} + ik\mu K_D'\tilde{f} + ik\mu K_D \tilde{f}]|_{\Gamma_I} \\
&+ [-ik\mu S_D \tilde{g} - k^2\mu^2 S_D \tilde{f}]|_{\Gamma_I}.
\end{aligned} \tag{33}
$$

Or we can rewrite (33) as

$$
\begin{aligned}
&(K'_{\Gamma_D\Gamma_I} + ik\mu S_{\Gamma_D\Gamma_I})a + [k^2\mu^2 S_{\Gamma_I\Gamma_I} - ik\mu(K_{\Gamma_I\Gamma_I} + K'_{\Gamma_I\Gamma_I}) - T_{\Gamma_I\Gamma_I}]b \\
&-(T_{\Gamma\Gamma_I} + K_{\Gamma\Gamma_I})c = r_2(x),
\end{aligned} \tag{34}
$$

where

$$
r_2(x) = [T_D\tilde{f} - K_D'\tilde{g} + ik\mu K_D'\tilde{f} + ik\mu K_D\tilde{f} - ik\mu S_D\tilde{g} - k^2\mu^2 S_D\tilde{f}]|_{\Gamma_I}.
$$

The new operators $K'_{\Gamma_D\Gamma_I}$, $K'_{\Gamma_I\Gamma_I}$, $T_{\Gamma_I\Gamma_I}$, and $T_{\Gamma\Gamma_I}$ have mapping properties

$$
\begin{aligned}
K'_{\Gamma_D\Gamma_I} &: \tilde{H}^{-1/2}(\Gamma_D) \to H^{-1/2}(\Gamma_I), \quad K'_{\Gamma_I\Gamma_I} : \tilde{H}^{-1/2}(\Gamma_I) \to H^{-1/2}(\Gamma_I), \\
T_{\Gamma_I\Gamma_I} &: \tilde{H}^{1/2}(\Gamma_I) \to H^{-1/2}(\Gamma_I), \quad T_{\Gamma\Gamma_I} : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma_I).
\end{aligned}
$$

Now, pay attention to the boundary ∂G, especially to the partial boundary Γ. When $x \in G\backslash\bar{D}$, and consider x approaches the boundary ∂G, we have

$$
\frac{\partial u_-^s}{\partial \nu} = K_G'\frac{\partial u_-^s}{\partial \nu} - T_G u_-^s + 2\frac{\partial}{\partial \nu(x)}\int_{\partial D} [\frac{\partial u^s}{\partial \nu}\Phi(x,y) - \frac{\partial \Phi(x,y)}{\partial \nu}u^s] ds_y. \tag{35}
$$

When $x \in R^2\backslash\bar{G}$, and consider x approaches the boundary ∂G, we have

$$
\frac{\partial u_+^s}{\partial \nu} = -K_G'\frac{\partial u_+^s}{\partial \nu} + T_G u_+^s. \tag{36}
$$

From (20) and (21), we restrict the last term in (35) on Γ, then

$$
\begin{aligned}
&2\frac{\partial}{\partial \nu(x)}\int_{\partial D}[\frac{\partial u^s}{\partial \nu}\Phi(x,y) - \frac{\partial \Phi(x,y)}{\partial \nu}u^s]ds_y|_\Gamma \\
&= (K_D'\frac{\partial u^s}{\partial \nu} - T_D u^s)|_\Gamma \\
&= [K_D'(\tilde{g} - ik\mu\tilde{b} + \tilde{a} - ik\mu\tilde{f}) - T_D(\tilde{b} + \tilde{f})]|_\Gamma \\
&= K'_{\Gamma_D\Gamma}a - ik\mu K'_{\Gamma_I\Gamma}b - T_{\Gamma_I\Gamma}b + (K_D'\tilde{g} - ik\mu K_D'\tilde{f} - T_D\tilde{f})|_\Gamma.
\end{aligned} \tag{37}
$$

Combine (35) and (36), and use (37), we have

$$\begin{aligned}
&(\frac{\partial u_+^s}{\partial\nu}+\frac{\partial u_-^s}{\partial\nu})|_\Gamma\\
&=[K'_G\frac{\partial u_+^s}{\partial\nu}+T_Gu_+^s+K'_G\frac{\partial u_-^s}{\partial\nu}-T_Gu_-^s]|_\Gamma\\
&+2\frac{\partial}{\partial\nu(x)}\int_{\partial D}[\frac{\partial u^s}{\partial\nu}\Phi(x,y)-\frac{\partial\Phi(x,y)}{\partial\nu}u^s]ds_y|_\Gamma\\
&=[-K'_G(\frac{\partial u_+^s}{\partial\nu}-\frac{\partial u_-^s}{\partial\nu})+T_G(u_+^s-u_-^s)]|_\Gamma\\
&+2\frac{\partial}{\partial\nu(x)}\int_{\partial D}[\frac{\partial u^s}{\partial\nu}\Phi(x,y)-\frac{\partial\Phi(x,y)}{\partial\nu}u^s]ds_y|_\Gamma\\
&=T_{\Gamma\Gamma}c+K'_{\Gamma_D\Gamma}a-ik\mu K'_{\Gamma_I\Gamma}b-T_{\Gamma_I\Gamma}b\\
&+(K'_D\tilde g-ik\mu K'_D\tilde f-T_D\tilde f)|_\Gamma,
\end{aligned}\tag{38}$$

where we used the condition $(\frac{\partial u_+^s}{\partial\nu}-\frac{\partial u_-^s}{\partial\nu})|_{\partial G}=0$.

On the other hand, boundary conditions in (10) imply that

$$\begin{aligned}
(\frac{\partial u_+^s}{\partial\nu}+\frac{\partial u_-^s}{\partial\nu})|_\Gamma&=[2\frac{\partial u_+^s}{\partial\nu}-(\frac{\partial u_+^s}{\partial\nu}-\frac{\partial u_-^s}{\partial\nu})]|_\Gamma\\
&=2\frac{\partial u_+^s}{\partial\nu}|_\Gamma\\
&=\frac{2}{i\lambda}[u^s]|_\Gamma-\frac{2}{i\lambda}q\\
&=\frac{2}{i\lambda}c-\frac{2}{i\lambda}q.
\end{aligned}\tag{39}$$

Combine (38) and (39), we obtain

$$-K'_{\Gamma_D\Gamma}a-(ik\mu K'_{\Gamma_I\Gamma}+T_{\Gamma_I\Gamma})b+\frac{2i}{\lambda}c+T_{\Gamma\Gamma}c=r_3(x),\tag{40}$$

where

$$r_3(x)=-\frac{1}{i\lambda}q(x)+(K'_D\tilde g-ik\mu K'_D\tilde f-T_D\tilde f)|_\Gamma$$

and the new operators $K'_{\Gamma_D\Gamma}$, $K'_{\Gamma_I\Gamma}$, $T_{\Gamma_I\Gamma}$, and $T_{\Gamma\Gamma}$ have mapping properties

$$K'_{\Gamma_D\Gamma}:\ \tilde H^{-1/2}(\Gamma_D)\ \to H^{-1/2}(\Gamma),\quad K'_{\Gamma_I\Gamma}:\ \tilde H^{-1/2}(\Gamma_I)\ \to H^{-1/2}(\Gamma),$$
$$T_{\Gamma_I\Gamma}:\ \tilde H^{1/2}(\Gamma_I)\ \to H^{-1/2}(\Gamma),\quad T_{\Gamma\Gamma}:\ \tilde H^{1/2}(\Gamma)\ \to H^{-1/2}(\Gamma).$$

Combine (31), (34), and (40), we obtain a system of boundary integral equations for the unknown data (a, b, c),

$$A\begin{pmatrix}a\\b\\c\end{pmatrix}=\vec R,\tag{41}$$

where the operator A is given by

$$A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\tag{42}$$

and

$$a_{11} = S_{\Gamma_D\Gamma_D}, \quad a_{12} = -(K_{\Gamma_I\Gamma_D} + ik\mu S_{\Gamma_I\Gamma_D}),$$
$$a_{21} = K'_{\Gamma_D\Gamma_I} + ik\mu S_{\Gamma_D\Gamma_I}, \quad a_{22} = k^2\mu^2 S_{\Gamma_I\Gamma_I} - ik\mu(K_{\Gamma_I\Gamma_I} + K'_{\Gamma_I\Gamma_I}) - T_{\Gamma_I\Gamma_I},$$
$$a_{31} = K'_{\Gamma_D\Gamma}, \quad a_{32} = -(ik\mu K'_{\Gamma_I\Gamma} + T_{\Gamma_I\Gamma}),$$
$$a_{13} = -K_{\Gamma\Gamma_D}, \quad a_{23} = -(T_{\Gamma\Gamma_I} + K_{\Gamma\Gamma_I}), \quad a_{33} = -\frac{2i}{\lambda}I - T_{\Gamma\Gamma}.$$

The right-hand side is given by

$$\vec{R} = \begin{pmatrix} r_1(x) \\ r_2(x) \\ r_3(x) \end{pmatrix}. \tag{43}$$

Therefore, we have that the operator A maps continuously

$$\tilde{H}^{-\frac{1}{2}}(\Gamma_D) \times \tilde{H}^{\frac{1}{2}}(\Gamma_I) \times \tilde{H}^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma_D) \times H^{-\frac{1}{2}}(\Gamma_I) \times H^{-\frac{1}{2}}(\Gamma). \tag{44}$$

For simplicity, we define $H = \tilde{H}^{-\frac{1}{2}}(\Gamma_D) \times \tilde{H}^{\frac{1}{2}}(\Gamma_I) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$, and its dual space $H^* = H^{\frac{1}{2}}(\Gamma_D) \times H^{-\frac{1}{2}}(\Gamma_I) \times H^{-\frac{1}{2}}(\Gamma)$.

From the existence and uniqueness of the solution to the system of integral equations (41), we can establish the existence of the solution to the problem (10). Once the unknown Cauchy data are determined from (20), (21), and (26) then the representation formula (16) and (17) determine the weak solution.

In the next two lemmas (Lemmas 3.4 and 3.5), we will prove that the operator A is Fredholm with index zero and $kernA = \{0\}$. This implies the solvability of the integral equation (41) and, therefore, of the original exterior mixed boundary value problem (10).

Once we prove the operator A: $H \to H^*$ is bijective, we will be able to assert that the inverse operator A^{-1} is bounded and furthermore the operators S, K, K', T involved in right-hand side of (41) are bounded as well. Hence, we may write

$$\begin{aligned} &\|a\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_D)} + \|b\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_I)} + \|c\|_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \\ &\leq C_1(\| f \|_{H^{\frac{1}{2}}(\Gamma_D)} + \| g \|_{H^{-\frac{1}{2}}(\Gamma_I)} + \| q \|_{H^{-\frac{1}{2}}(\Gamma)}). \end{aligned} \tag{45}$$

The representation formula (see (16) and (17)) now yields the following estimate for the weak solution u^s to (10),

$$\| u^s \|_{H^1(B_R\setminus(\bar{D}\cup\bar{\Gamma}))} \leq C(\| f \|_{H^{\frac{1}{2}}(\Gamma_D)} + \| g \|_{H^{-\frac{1}{2}}(\Gamma_I)} + \| q \|_{H^{-\frac{1}{2}}(\Gamma)}), \tag{46}$$

where the positive constant C depends on R but not on f, g, and q. This ends the proof of the theorem.

Lemma 3.4: If $\Gamma_I \neq \emptyset$, the operator A is Fredholm with index zero.

Proof: Notice that the operators $S_{\Gamma_D\Gamma_D}$ (or $S_{\Gamma_I\Gamma_I}$), $T_{\Gamma_I\Gamma_I}$, and $T_{\Gamma\Gamma}$ in the boundary integral system (41) are related to the operators S_D, T_D, and T_G, respectively (see (22) and (23)). It is known from Ref. 17, that the operators S_D and $-T_D$ are positive and bounded below up to a compact perturbation. In other words, there exist compact operators

$$L_S^0: \ H^{-\frac{1}{2}}(\partial D) \ \to H^{\frac{1}{2}}(\partial D), \quad L_T^0: \ H^{\frac{1}{2}}(\partial D) \ \to H^{-\frac{1}{2}}(\partial D)$$

such that

$$Re(< (S_D + L_S^0)\psi, \bar{\psi} >) \geq C||\psi||^2_{H^{-\frac{1}{2}}(\partial D)}, \quad \text{for } \psi \in H^{-1/2}(\partial D), \tag{47}$$

$$Re(- < (T_D + L_T^0)\psi, \ \bar{\psi} >) \geq C||\psi||^2_{H^{\frac{1}{2}}(\partial D)}, \quad \text{for } \psi \in H^{1/2}(\partial D), \tag{48}$$

where the brackets $< \cdot, \ \cdot >$ denote the duality between $H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$.

Similarly, the operator $-T_G$ is positive and bounded below up to a compact perturbation. That is, there exists a compact operator

$$L_T^1: \ H^{\frac{1}{2}}(\partial G) \ \to H^{-\frac{1}{2}}(\partial G)$$

such that

$$Re(- < (T_D + L_T^1)\psi, \ \bar{\psi} >) \geq C||\psi||^2_{H^{\frac{1}{2}}(\partial G)}, \quad \text{for } \psi \in H^{1/2}(\partial G), \tag{49}$$

where the brackets $< \cdot, \ \cdot >$ denote the duality between $H^{-\frac{1}{2}}(\partial G)$ and $H^{\frac{1}{2}}(\partial G)$.

Let us define $S_0 = S_D + L_S^0$, $T_0 = -(T_D + L_T^0)$, and $T_1 = -(T_G + L_T^1)$. Then S_0, T_0, and T_1 are bounded below up and are positive. Then if $\vec{\phi} = (a, b, c)^T \in H$ and $\tilde{a} \in H^{-\frac{1}{2}}(\partial D)$, $\tilde{b} \in H^{\frac{1}{2}}(\partial D)$ are the extension by zero on ∂D of a and b, respectively, and $\tilde{c} \in H^{\frac{1}{2}}(\partial G)$ is the extension by zero on ∂G of c, we define

$$A_0\vec{\phi} = \begin{pmatrix} (S_0\tilde{a} - ik\mu S_0\tilde{b})|_{\Gamma_D} - K_{\Gamma_I\Gamma_D} b \\ @ \\ (T_1\tilde{c} - \frac{2i}{\lambda}\tilde{c})|_{\Gamma} \end{pmatrix}, \tag{50}$$

where

$$@ = K'_{\Gamma_D\Gamma_I} S_1 a + (ik\mu S_0\tilde{a} + k^2\mu^2 S_0\tilde{b} + T_0\tilde{b})|_{\Gamma_I} - ik\mu(K_{\Gamma_I\Gamma_I} + K'_{\Gamma_I\Gamma_I})b$$

and

$$A_C\vec{\phi} = \begin{pmatrix} -(K_{\Gamma\Gamma_D} c - (L_S^0\tilde{a} - ik\mu L_S^0\tilde{b})|_{\Gamma D} \\ -(T_{\Gamma\Gamma_I} + K_{\Gamma\Gamma_I})c - (ik\mu L_S^0\tilde{a} + k^2\mu^2 L_S^0\tilde{b} + L_T^0\tilde{b})|_{\Gamma I} \\ K'_{\Gamma_D\Gamma_I} a - (ik\mu K'_{\Gamma_I\Gamma} + T_{\Gamma_I\Gamma})b + L_T^1\tilde{c}|_{\Gamma} \end{pmatrix} \tag{51}$$

such that $A = A_0 + A_C$. In this way the operator A_C: $H \to H^*$ is compact and A_0: $H \to H^*$ defines the sesquilinear form

$$\begin{aligned} &< A_0\vec{\phi}, \bar{\vec{\phi}} >_{H,H^*} \\ &= (S_0\tilde{a}, \tilde{a}) - ik\mu(S_0\tilde{b}, \tilde{a}) - (K_{\Gamma_I\Gamma_D} b, a)|_{\Gamma_D} + (K'_{\Gamma_D\Gamma_I} a, b)|_{\Gamma_I} \\ &+ik\mu(S_0\tilde{a}, \tilde{b}) + k^2\mu^2(S_0\tilde{b}, \tilde{b}) + (T_0\tilde{c}, \tilde{c}) \\ &-ik\mu((K_{\Gamma_I\Gamma_I} + K'_{\Gamma_I\Gamma_D})b, b)|_{\Gamma_I} + (T_1\tilde{c}, \tilde{c}) - \frac{2i}{\lambda}(\tilde{c}, \tilde{c}). \end{aligned} \tag{52}$$

Note that $(u, v)|_{\Gamma_0}$, for $\Gamma_0 \subseteq \partial D$ (or $\Gamma_0 \subseteq \partial G$) is the scalar product on $L^2(\Gamma_0)$ defined by $\int_{\Gamma_0} u\bar{v}ds$. From (47) and the fact that $supp\tilde{a} \subseteq \Gamma_D$, $supp\tilde{b} \subseteq \Gamma_I$ and $supp\tilde{c} \subseteq \Gamma$, we take the real part of the terms related to S_0 in (52) and obtain

$$\begin{aligned} &Re[(S_0\tilde{a}, \tilde{a}) - ik\mu(S_0\tilde{b}, \tilde{a}) + ik\mu(S_0\tilde{a}, \tilde{b}) + k^2\mu^2(S_0\tilde{b}, \tilde{b})] \\ &= Re[(S_0(\tilde{a} - ik\mu\tilde{b}), (\tilde{a} - ik\mu\tilde{b}))] \\ &\geq C \parallel \tilde{a} - ik\mu\tilde{b} \parallel^2_{H^{-\frac{1}{2}}(\partial D)} \\ &= C(\parallel a \parallel^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_D)} + k^2\mu^2 \parallel b \parallel^2_{\tilde{H}^{\frac{1}{2}}(\Gamma_I)}). \end{aligned} \tag{53}$$

Furthermore, since K and K' are adjoint we have

$$\begin{aligned} &Re[-(K_{\Gamma_I\Gamma_D} b, a)|_{\Gamma_D} + (K_{\Gamma_D\Gamma_I} a, b)|_{\Gamma_I}] \\ &= Re[-(K_{\Gamma_I\Gamma_D} b, a)|_{\Gamma_D} + (a, K_{\Gamma_I\Gamma_D} b)|_{\Gamma_D}] \\ &= Re[-(K_{\Gamma_I\Gamma_D} b, a)|_{\Gamma_D} + \overline{(K_{\Gamma_I\Gamma_D} b, a)}|_{\Gamma_D}] \\ &= 0 \end{aligned} \tag{54}$$

and

$$\begin{aligned} &Re[-ik\mu((K_{\Gamma_I\Gamma_I}+K'_{\Gamma_I\Gamma_D})b,b)] \\ &= k\mu Im[(K_{\Gamma_I\Gamma_I}b,b)+\overline{(K_{\Gamma_I\Gamma_I}b,b)}] \\ &= 0. \end{aligned} \tag{55}$$

Finally from (48) and (49) we have

$$Re[(T_0\tilde{b},\tilde{b})] \geq C \parallel \tilde{b} \parallel^2_{H^{\frac{1}{2}}(\partial D)} \geq C \parallel b \parallel^2_{\tilde{H}^{\frac{1}{2}}(\Gamma_I)} \tag{56}$$

and

$$Re[(T_1\tilde{c},\tilde{c})] \geq C \parallel \tilde{c} \parallel^2_{H^{\frac{1}{2}}(\partial G)} \geq C \parallel c \parallel^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)}\,. \tag{57}$$

Combining (53)–(57), we conclude that

$$Re(< (A-A_C)\vec{\phi},\ \bar{\vec{\phi}} >_{H,H^*}) \ \geq\ C||\vec{\phi}||^2_H, \quad \text{for} \quad \vec{\phi} \in H, \tag{58}$$

whence the operator A is Fredholm with index zero. This ends the proof of the lemma.

Lemma 3.5: The operator A has trivial kernel, that is, $kernA = \{0\}$.

Proof: Let $\vec{\psi} = (a,b,c)^T \in H$ be a solution of the homogeneous equation $A\vec{\psi} = \vec{0}$, and $\tilde{a} \in H^{-\frac{1}{2}}(\partial D)$, $\tilde{b} \in H^{\frac{1}{2}}(\partial D)$, $\tilde{c} \in H^{\frac{1}{2}}(\partial G)$ be their respective extensions by zero. We want to show that $\vec{\psi} \equiv \vec{0}$.

Define the potential

$$v(x) = S_D\tilde{a} - ik\mu S_D\tilde{b} - K_D\tilde{b} - K_G\tilde{c}, \tag{59}$$

which is a weak solution of the Helmholtz equation in D and $R^2\backslash(\bar{D}\cup\bar{\Gamma})$.

Consider the potential $v(x)$ and its normal derivative and approaches the boundary ∂D from inside the domain $R^2\backslash(\bar{D}\cup\bar{\Gamma})$, we obtain

$$2v(x)|_{\partial D} = [S_D\tilde{a} - ik\mu S_D\tilde{b} - K_D\tilde{b} - \tilde{b} - K_G\tilde{c}]|_{\partial D}, \tag{60}$$

$$\begin{aligned} 2(\frac{\partial v}{\partial \nu} + ik\mu v)|_{\partial D} &= [K'_D\tilde{a} - \tilde{a} - ik\mu K'_D\tilde{b} + ik\mu\tilde{b} - T_D\tilde{b} - T_G\tilde{c}]|_{\partial D} \\ &\quad + ik\mu[S_D\tilde{a} - ik\mu S_D\tilde{b} - K_D\tilde{b} - \tilde{b} - K_D\tilde{c}]|_{\partial D}. \end{aligned} \tag{61}$$

Restricting (60) and (61) on the partial boundaries Γ_D and Γ_I, respectively, and using the homogeneous equation $A\vec{\psi} = \vec{0}$, we have

$$2v(x)|_{\Gamma_D} = S_{\Gamma_D\Gamma_D}a - ik\mu S_{\Gamma_I\Gamma_D}b - K_{\Gamma_I\Gamma_D}b - \tilde{b} - K_{\Gamma\Gamma_D}c = 0 \tag{62}$$

and

$$\begin{aligned} 2(\frac{\partial v}{\partial \nu} + ik\mu v)|_{\Gamma_I} &= (K'_{\Gamma_D\Gamma_I} + ik\mu S_{\Gamma_D\Gamma_I})a + k^2\mu^2 S_{\Gamma_I\Gamma_I}b \\ &\quad - ik\mu(K'_{\Gamma_I\Gamma_I} + K_{\Gamma_I\Gamma_I})b - T_{\Gamma_I\Gamma_I}b - (T_{\Gamma\Gamma_I} + K_{\Gamma\Gamma_I})c \\ &= 0. \end{aligned} \tag{63}$$

Similarly, on the boundary ∂G, especially on the partial boundary Γ, we obtain

$$[v(x)]|_\Gamma = -c \tag{64}$$

and

$$[\frac{\partial v(x)}{\partial \nu}]|_\Gamma = 0. \tag{65}$$

By using (64) and the homogeneous equation $A\vec{\psi} = \vec{0}$, we obtain

$$\begin{aligned}([v] - i\lambda\frac{\partial v_+}{\partial \nu})|_\Gamma &= -c - \frac{i\lambda}{2}[K'_{\Gamma_D\Gamma}a - ik\mu K'_{\Gamma_I\Gamma}b - T_{\Gamma_I\Gamma}b - T_{\Gamma\Gamma}c] \\ &= -\frac{i\lambda}{2}[\frac{2c}{\lambda i} + K'_{\Gamma_D\Gamma}a - ik\mu K'_{\Gamma_I\Gamma}b - T_{\Gamma_I\Gamma}b - T_{\Gamma\Gamma}c] \\ &= -\frac{i\lambda}{2}[K'_{\Gamma_D\Gamma}a - ik\mu K'_{\Gamma_I\Gamma}b - T_{\Gamma_I\Gamma}b - T_{\Gamma\Gamma}c - \frac{2i}{\lambda}c] \\ &= 0. \end{aligned} \tag{66}$$

Therefore, (62)–(66) means that (59) is a weak solution of the homogeneous exterior mixed boundary problem (10) (with $f = g = q = 0$), and hence, from Theorem 2.1 in Sec. II, $v = 0$ in $R^2\backslash(\bar{D} \cup \bar{\Gamma})$. Thus,

$$c = -[v] = 0. \tag{67}$$

Approaching the boundary ∂D from the inside domain D, one can show that (59) is also a weak solution of the homogeneous interior mixed boundary problem, i.e.,

$$\begin{cases} \Delta v + k^2 v = 0 & \text{in} \quad D \\ v(x) = 0 & \text{on} \quad \Gamma_D \\ \frac{\partial v}{\partial \nu} + ik\mu v = 0 & \text{on} \quad \Gamma_I. \end{cases} \tag{68}$$

From Theorem 2.1 in Ref. 4, we have $v = 0$ in D. Thus

$$\tilde{b} = -[v]|_{\partial D} = 0, \tag{69}$$

$$\tilde{a} - ik\mu\tilde{b} = -[\frac{\partial v}{\partial \nu}]|_{\partial D} = 0. \tag{70}$$

Therefore,

$$a = 0, \qquad b = 0. \tag{71}$$

From (67) and (71), we obtain

$$\vec{\psi} = (a, b, c)^T = \vec{0}. \tag{72}$$

In Conclusion, we complete the proof of the lemma.

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