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A connection between nonlinear evolution equations and ordinary differential equations of P-type. I

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We develop here two aspects of the connection between nonlinear partial differential equations solvable by inverse scattering transforms and nonlinear ordinary differential equations (ODE) of P-type (i.e., no movable critical points). The first is a proof that no solution of an ODE, obtained by solving a linear integral equation of a certain kind, can have any movable critical points. The second is an algorithm to test whether a given ODE satisfies necessary conditions to be of P-type. Often, the algorithm can be used to test whether or not a given nonlinear evolution equation may be completely integrable.

I. INTRODUCTION

The development of the inverse scattering transform (IST; e.g., see Ref. 1) has shown that certain nonlinear evolution equations possess a number of remarkable properties, including the existence of solitons, an infinite set of conservation laws, and an explicit set of action-angle variables. It has been noted[2] that there is a connection between these nonlinear partial differential equations (PDE's) solvable by IST and nonlinear ordinary differential equations (ODE's) without movable critical points. (Some definitions: a *critical point* is a branch point or an essential singularity in the solution of the ODE. It is *movable* if its location in the complex plane depends on the constants of integration of the ODE. A family of solutions of the ODE without movable critical points has the P-property: here P stands for Painlevé. The ODE is of P-type if all of its solutions have this property.) The reductions of some of the equations solvable by IST to the classical Painlevé transcendents have been discussed in the literature.[2–6] In Ref. 3 we announced a number of results which indicate that this connection to ODE's of P-type is yet another remarkable property of these special nonlinear PDE'S. The purpose of this paper is to develop some of those announced results in more detail.

In particular, it was conjectured[3] that:

> Every nonlinear ODE obtained by an exact reduction of a nonlinear PDE of IST class is of P-type.

Here a nonlinear PDE is of IST class if nontrivial solutions of it can be found by solving a linear integral equation of the Gelfand–Levitan–Marchenko form. No proof of this conjecture is available yet, but in Sec. II of this paper we prove a more restricted result in this direction. It is known that under scaling transformations certain nonlinear PDE's of IST class reduce to ODE's. Moreover, the solutions of these ODE's may be obtained by solving linear integral equations.[2] We show in Sec. II that every such family of solutions has the P-property.

The reader should note that this conjecture relates to ODE's obtained from equations solved *directly* by IST. There are many examples of equations solved only indirectly by IST; the sine–Gordon equation is perhaps the best known example. An ODE obtained from an equation solved indirectly by IST need not be of P-type, but it must be related through a simple transformation to an ODE that is. We discuss this point in more detail in our following paper.[7]

One consequence of this conjecture is an explicit test of whether or not a given PDE may be of IST class; namely, reduce it to an ODE, and determine whether the ODE is of P-type. To this end, we identify in Sec. III certain necessary conditions that an ODE must satisfy to be of P-type and describe an explicit algorithm to determine whether an ODE meets these necessary conditions. Examples are given to illustrate the main ideas. In many cases, this algorithm seems to be simpler than the α-method of Painleve and his coworkers,[8] which also determines whether an ODE satisfies necessary conditions to be of P-type. It is similar to the method of Kovalevskya,[9] who made major contributions to the theory of the motion of a rigid body about a fixed point after first determining the choices of parameters for which the equations of motion had no movable critical points. It is interesting to note that she found the complete solution whenever the equations were of P-type, and that no solutions are known to this day when they are not of P-type.

II. LINEAR INTEGRAL EQUATIONS AND ODE's OF P-TYPE

In this section we demonstrate exact reductions of nonlinear PDE's of IST class to ODE's such that solutions of the ODE may be obtained *via* a linear integral equation with a nonsingular kernel. Then we prove that any family of solutions obtained in this way necessarily has the P-property, by applying Fredholm's theory of linear integral equations.[10]

Consider a nonlinear PDE of IST class, with two independent variables, x and t. Solutions of the PDE may be

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obtained *via* a linear integral equation of the form

$$\mathscr{K}(x,y;t) = \mathscr{F}(x+y;t) + \int_x^\infty \mathscr{K}(x,z;t)\mathscr{N}(x;z,y;t)\,dz, \qquad (2.1)$$

where $\mathscr{N}$ is explicitly given in terms of $\mathscr{F}$ (possibly with one or more integrals from x to ∞). For instance,

$$\mathscr{N}(x;z,y;t) = \mathscr{F}(z+y;t), \qquad (2.2a)$$

$$\mathscr{N}(x;z,y;t) = \pm \int_x^\infty \mathscr{F}(z+u;t)\mathscr{F}(u+y;t)\,du, \qquad (2.2b)$$

$$\mathscr{N}(x;z,y;t) = \pm \int_x^\infty \bar{\mathscr{F}}(z+u;t)\mathscr{F}(u+y;t)\,du, \qquad (2.2c)$$

where $\bar{\mathscr{F}}(x;t) = [\mathscr{F}(x^*;t^*)]^*$ and ($*$) means complex conjugate,

$$\mathscr{N}(x;z,y;t) = \pm i \int_x^\infty \partial_z\bar{\mathscr{F}}(z+u;t)\mathscr{F}(u+y;t)\,du, \qquad (2.2d)$$

$$\mathscr{N}(x;z,y;t) = \int_x^\infty \int_x^\infty \mathscr{F}(z+u;t)\mathscr{F}(u+v;t) \times \mathscr{F}(v+y;t)\,du\,dv, \qquad (2.2e)$$

etc.

If $\mathscr{F}$ satisfies a suitable linear PDE and decays rapidly enough as $x \to +\infty$ for the integral equation to be meaningful and have a unique solution $\mathscr{K}$, then

$$\mu(x;t) = \mathscr{K}(x,x;t) \quad \text{or} \quad \mu = \frac{d}{dx}\mathscr{K}(x,x;t)$$

satisfies the original nonlinear PDE. This is the "direct" method,[11,12,7] first introduced by Zakharov and Shabat; it involves no reference to a scattering problem.

Among the PDE's of IST class, we restrict our attention to those that admit self-similar solutions of the scaling type. Thus, we assume that the integral equation is compatible with the ansatz

$$\mathscr{F}(x+y;t) = \phi(x+y;t)F[\psi(x,t)+\psi(y,t)], \qquad (2.3a)$$

$$\mathscr{K}(x,y;t) = \phi(x+y;t)K[\psi(x,t),\psi(y,t)], \qquad (2.3b)$$

where ϕ and ψ are known functions, and that this ansatz reduces the linear PDE for $\mathscr{F}$ to a linear ODE for F. If the integral equation in this form can be solved, its solution is the self-similar solution of the nonlinear PDE; i.e.,

$$u(\xi) = K(\xi,\xi) \quad \left[\text{or}\ \frac{d}{d\xi}K(\xi,\xi)\right]$$

satisfies a nonlinear ODE that is an exact reduction of the PDE. As an example, we note that a one-parameter family of solutions of Painlevé's second equation,

$$w'' = 2w^3 + zw, \qquad (2.4)$$

was constructed in this way.[2] More examples will be given in.[7] If it could be proved, the conjecture stated above would assert that any ODE obtained in this way must be of P-type, i.e., its general solution has no movable singularities other than poles.

Next, we prove that there is a family of solutions of the nonlinear ODE which have no movable singularities other than poles, provided that there are solutions of the corresponding linear ODE that decay rapidly enough.

First we rewrite the integral Eq. (2.1) as

$$K(\xi;\eta) = F(\xi+\eta) + \int_\xi^\infty K(\xi,\zeta)N(\xi;\zeta,\eta)\,d\zeta, \qquad (2.5)$$

where N is given in terms of F. Here F satisfies a linear homogeneous ODE, and its general solution has no movable singularities whatsoever; its only singularities are fixed and their positions can be determined by direct examination of the equation.

Among those solutions F, some may decay at $+\infty$ fast enough for the integral equation to make sense and be solvable. Since its ODE is linear and homogeneous, the set of such solutions is a vector subspace. From these solutions, a family of solutions of the nonlinear ODE can be obtained by solving the integral equation.

So far, the variable x has been considered to be real. Now, we allow x to be complex valued. To do this we must define the integral more carefully. The same applies to the integral(s) that define N in terms of F.

For each ξ, choose a contour $\mathscr{C}(\xi)$ with ξ as one endpoint, and going to $+\infty$ while staying in a strip of finite width along the positive real axis. If F and N have (fixed) singularities, it may be necessary to make cuts in the complex plane, starting at the singular points of F, and to demand that for any ξ the contour $\mathscr{C}(\xi)$ crosses no cuts. But these cuts are fixed and delimit a fixed domain $\mathscr{A}$ in which N depends on ξ only and is analytic.

Now fix ξ and a contour $\mathscr{C}(\xi)$. Consider the following Fredholm integral equation:

$$K(\lambda;\eta) = F(\xi+\eta) + \lambda \oint K(\lambda;\zeta)N(\xi;\zeta,\eta)\,d\zeta. \qquad (2.6)$$

Assume that the decay of F when its argument goes to infinity in the strip defined previously is fast enough for the following conditions to be satisfied.

$$|N(\xi;\zeta,\eta)| < M(\xi;\eta) \ \text{for all}\ \zeta, \ \text{such that} \qquad (2.7a)$$

$$\left|\oint M(\xi;\eta)d\eta\right| = \mathscr{M}(F) < \infty, \qquad (2.7b)$$

$$\left|\oint F(\xi+\eta)M(\xi;\eta)\,d\eta\right| < +\infty. \qquad (2.7c)$$

Then the powerful results of Fredholm theory apply (cf. Appendix). Namely, the integral equation has a unique solution

$$K(\lambda;\eta) = F(\xi+\eta) + \lambda \oint F(\xi+\eta) \times \mathscr{N}(\mathscr{C}(\xi);\lambda;\zeta,\eta)\,d\zeta, \qquad (2.8a)$$

with

$$\mathscr{N}(\mathscr{C}(\xi);\lambda;\zeta,\eta) = \mathscr{D}_2(\mathscr{C}(\xi);\lambda;\zeta,\eta)/\mathscr{D}_1(\mathscr{C}(\xi);\lambda). \qquad (2.8b)$$

$\mathscr{D}_1$ and $\mathscr{D}_2$ are given as series in powers of λ

$$\mathscr{D}_i = \sum_{n=0}^{\infty} \lambda^n \mathscr{D}_i^{(n)}, \quad i = 1,2,$$

where the $\mathscr{D}_i^{(n)}$ are multiple integrals over $\mathscr{C}$ of polynomials in N. Fredholm theory states that the radius of convergence of these series in λ is infinite. Since N is analytic in the fixed domain $\mathscr{A}$, the $\mathscr{D}_i^{(n)}(\mathscr{C}(\xi);\lambda;\zeta,\eta)$ are analytic in ζ and η in the domain $\mathscr{A}$. Moreover, as long as $(\mathscr{C})$ does not cross a cut, the $\mathscr{D}_i^{(n)}$ do not depend on $\mathscr{C}$ but only on its endpoint ξ and this dependence is analytic for ξ in $\mathscr{A}$.

Hence $\mathscr{D}_1(\xi;\lambda)$ and $\mathscr{D}_2(\xi;\lambda;\zeta,\eta)$ are analytic in all their arguments, in λ in the entire complex plane, and in ξ,ζ,η in a fixed domain $\mathscr{A}$. Since

$$\lim_{\xi\to\infty} \mathscr{D}_1(\xi,\lambda) = 1$$

in the strip for any λ, $\mathscr{D}_1(\xi,\lambda)$ is not identically 0 in ξ and its zeros in $\mathscr{A}$ are isolated.

There is therefore a family of solutions $u(\xi)$ of the nonlinear ODE given by

$$u(\xi) = K(\xi,\xi) = F(2\xi) + \frac{1}{\mathscr{D}_1(\xi;1)} \times \oint F(\xi+\zeta)\mathscr{D}_2(\xi;1;\zeta,\xi)\,d\xi, \quad (2.9)$$

where F belongs to some vector subspace. The only singularities of the solution u, apart from the fixed singularities of F, come from the zeros of $\mathscr{D}_1(\xi,1)$. Since $\mathscr{D}_1$ is analytic in $\mathscr{A}$, these movable singularities are indeed poles.

III. SINGULAR POINT ANALYSIS

For an ODE to be of P-type, it is necessary that it have no movable branch points, either algebraic or logarithmic. We now describe an algorithm to determine whether a nonlinear ODE (or system of ODE's) admits movable branch points. The presentation is simplified considerably if we make two assumptions.

(i) The nth order system of ODE's has the form

$$\frac{d}{dz} w_j = F_j(z;w_1,w_2,\ldots,w_n), \quad j = 1,\ldots,n, \quad (3.1)$$

where each F_j is analytic in z and rational in its other arguments. An important special case is a nth order ODE

$$\frac{d^n w}{dz^n} = F\left(z;w,w',\ldots,\frac{d^{n-1}w}{dz^{n-1}}\right), \quad (3.2)$$

where F is analytic in z and rational in its other arguments.

(ii) The dominant behavior of the function in a sufficiently small neighborhood of the (movable) singularity is algebraic, i.e.,

$$w_j \sim \alpha_j (z - z_o)^{p_j} \text{ as } z\to z_o. \quad (3.3)$$

This does not exclude logarithmic branch points; it does exclude branch points in which the dominant behavior is logarithmic (but see Example 6). Neither of these restrictions is essential to the method, but we make no attempt to remove them in this paper.

An ODE without movable branch points might still admit movable essential singularities. This method does not identify essential singularities, and therefore it provides only necessary conditions for an ODE to be of P-type. That these conditions are not also sufficient may be seen from the ODE,[8]

$$w'' = (w')^2[(2w-1)/(w^2+1)].$$

Its general solution is

$$w = \tan \ln(Az+B),$$

where A and B arbitrary. The solution has a movable essential singularity at $z = -B/A$, but the equation nevertheless satisfies the necessary conditions we will describe.

Let us consider first the case of a single ODE, (3.2) and assume that the function becomes infinite at the singularity. There are basically three steps to the algorithm.

A. Find the Dominant Behavior

Look for a solution of (3.2) in the form

$$w \sim \alpha (z - z_o)^p, \quad (3.4)$$

where $\mathrm{Re}(p) < 0$ and z_0 is arbitrary. Substituting (3.4) into (3.2) shows that for certain values of p, two or more terms in the equation may balance (depending on α), and the rest can be ignored as $z\to z_o$. For each such choice of p, the terms which can balance are called the *leading terms*. Requiring that the leading terms do balance (usually) determines α.

Example 1:

$$w''' + ww'' - 2w^3 + \lambda w^2 + \mu w = 0. \quad (3.5)$$

There are two possible choices:

(i) $p = -1$, $\alpha = 3$, the leading terms are w''' and ww''.

(ii) $p = -2$, $\alpha = 3$, the leading terms are ww'' and $-2w^3$. ■

Example 2:

$$w''' + aww'' + b(w')^2 + cw^4 + dww' + ew^3 + fw^2 + gw = 0. \quad (3.6)$$

The only possible choice is $p = -1$, and α may be any of the three roots of

$$-6 + (2a+b)\alpha + c\alpha^3 = 0.$$

The first four terms in (3.6) are all leading terms. ■

Example 3:

$$w'' = 2w(w')^2/(w^2-1). \quad (3.7)$$

The only possible choice is $p = -1$, but α is entirely unrestricted. [This particular equation is of P-type; its general solution is

$$w = \tanh(Az+B),$$

for arbitrary A and B. It has no movable critical points.] ■

A given equation may have several choices of p. If *any* of the possible p's is not an integer, and if (3.4) actually is asymptotic near z_o, then it represents the dominant behavior in the neighborhood of a movable algebraic branch point of order p. The existence of such a branch point means that the equation is not of P-type.

To prove that (3.4) is asymptotic, define a new variable

$$v = w^{1/p}, \quad (3.8)$$

and rewrite (3.2) in terms of v. By construction, v vanishes at z_o, and v' is finite. We must show from its ODE that $v(z)$ is analytic at z_o. Then it follows from (3.8) that w has a branch point of order p at z_o.

Expample 4:

$$w'' + 10w^4 = 0. \tag{3.9}$$

The only possible choice is $p = -2/3$, so we define

$$w = v^{-2/3}.$$

The equation for v is

$$\tfrac{3}{5}vv'' = (v')^2 - (\tfrac{3}{2})^2. \tag{3.10}$$

There is a solution of (3.10) that is regular at z_0 if

$$v(z_o) = 0, \quad v'(z_o) = \pm 3/2, \text{ and } v''(z_o) \text{ is finite.}$$

Then $v(z)$ is analytic at z_o, and (3.9) has a movable branch point of order $-2/3$. It is not of P-type. ■

Even if a given equation has a movable algebraic branch point, a simple transformation may turn it into one of P-type.

Example 5:

$$ww'' = \tfrac{5}{2}(w')^2. \tag{3.11}$$

The only possible choice is $p = -2/3$, so we define

$$w = v^{-2/3}.$$

The equation for v is

$$v'' = 0.$$

This equation is linear and therefore of P-type. ■

If all possible p's are integers, then for each p, (3.4) *may* represent the first term in the Laurent series, valid in a deleted neighborhood of a movable pole. In this case, a solution of (3.2) is

$$w(z) = (z - z_o)^p \sum_{j=0}^{\infty} a_j (z - z_o)^j, \quad 0 < |z - z_o| < R. \tag{3.12}$$

Here z_o is an arbitrary constant. If $(n-1)$ of the coefficients $\{a_j\}$ are also arbitrary, these are the n constants of integration of the ODE, and (3.12) is the general solution in the deleted neighborhood. The powers at which these arbitrary constants enter are called *resonances*.

B. Find the Resonances

For each (p,α) from step A, construct a simplified equation that retains only the leading terms of the original equation. Substitute

$$w = \alpha(z - z_o)^p + \beta(z - z_o)^{p+r} \tag{3.13}$$

into the simplified equation. To leading order in β, this equation reduces to

$$Q(r)\beta(z - z_o)^q = 0, \quad q \geqslant p + r - n. \tag{3.14}$$

If the highest derivative of the original equation is a leading term, $q = p + r - n$, and $Q(r)$ is a polynomial of order n. If not, $q > p + r - n$, and the order of the polynomial $Q(r)$ equals the order of highest derivative among the leading terms ($< n$).

The roots of $Q(r)$ determine the resonances. [$Q(r) = 0$ corresponds to the "indicial equation" in the method of Frobenius for finding solutions of a linear ODE near a regular singular point.]

(i) One root is always (-1). It represents the arbitrariness of x_o.

(ii) If α is arbitrary in step 1, another root is (0).

(iii) Ignore any roots with $\mathrm{Re}(r) < 0$, because they violate the hypothesis that $(z - z_o)^p$ is the dominant term in the expansion near z_o.

(iv) Any root with $\mathrm{Re}(r) > 0$, but r not a real integer, indicates a (movable) branch point at $z = z_o$. There is no need to continue the algorithm, but it remains to prove that the equation actually has such a branch point (cf. Example 4).

(v) If for every possible (p,α) from step A, all of the roots of $Q(r)$ (except -1 and possibly 0) are positive real integers, then there are no algebraic branch points. Proceed to step C to check for logarithmic branch points.

(vi) To represent the general solution of the nth order ODE in the neighborhood of a movable pole, $Q(r)$ must have $(n-1)$ nonnegative distinct roots, all real integers. If for *every* (p,α) from step A, $Q(r)$ has fewer than $(n-1)$ such roots, then none of the local solutions is general. This suggests that (3.4) misses an essential part of the solution.

Example 6:

$$w'' + 4ww' + 2w^3 = 0. \tag{3.15}$$

The only dominant algebraic behavior is

$$w \sim \alpha(z - z_o)^{-1}, \tag{3.16}$$

where α satisfies

$$2 - 4\alpha + 2\alpha^2 = 0. \tag{3.17}$$

Substituting (3.13) into (3.15) leads to

$$r^2 + r = 0.$$

The root (-1) corresponds to the arbitrariness of x_o. The root (0) corresponds not to the *arbitrariness* of α but to the fact that $\alpha = 1$ is a double root of (3.17). Thus, (3.16) cannot be the first term in a Laurent series of a *general* solution of (3.15). In fact, under the transformation

$$w = f(\ln(z - z_o))/(z - z_o),$$

(3.15) becomes

$$f'' + (4f - 3)f' + 2f(f - 1)^2 = 0. \tag{3.18}$$

Clearly, (3.18) has a variety of regular solutions, including those for which $f(0) = 1$. Every nonexponential solution of (3.18) corresponds to a movable logarithmic branch point of (3.15). ■

C. Find the Constants of Integration

For a given (p,α) from step A, let $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_s$ denote the positive integer roots of $Q(r)$; $(s \leqslant n - 1)$. Substitute

$$w = \alpha(z - z_o)^p + \sum_{j=1}^{r_s} a_j (z - z_o)^{p+j} \tag{3.19}$$

into the full equation, (3.2). [This portion of the analysis is similar to a certain portion of the method of Frobenius for linear problems. One might think of substituting (3.12), but little is gained in most nonlinear ODE's by continuing beyond the largest root of $Q(r)$.] The coefficient of $(z - z_o)^{p+j-n}$, which must vanish identically, is

$$Q(j)a_j - R_j(z_o, \alpha, a_1, \ldots, a_{j-1}) = 0. \tag{3.20}$$

(i) For $j < r_1$, (3.20) determines a_j.

(ii) For $j = r_1$, (3.20) becomes

$$0 \cdot a_{r_1} - R_{r_1}(z_o, \alpha, a_1, \ldots, a_{r_1 - 1}) = 0.$$

If

$$R_{r_1}(z_o, \alpha, a_1, \ldots, a_{r_1 - 1}) \neq 0, \tag{3.21}$$

then (3.20) cannot be satisfied. There is no solution of the form (3.19), and we must introduce logarithmic terms into the expansion. Replace (3.19) with

$$w = \alpha(z - z_o)^p + \sum_{j=1}^{r-1} a_j (z - z_o)^{p+j} + [a_{r_1} + b_{r_1} \ln(z - z_o)](z - z_o)^{p + r_1} + \cdots. \tag{3.22}$$

Now the coefficient of $\{(z - z_o)^{p + r_1 - n} \ln(z - z_o)\}$ is

$$Q(r_1) b_{r_1} = 0,$$

but b_{r_1} is determined by demanding that the coefficient of $(z - z_o)^{p + r_1 - n}$ vanish; a_{r_1} is arbitrary. Continuing the expansion (3.22) to higher orders introduces more and more logarithmic terms. Thus, (3.21) signals a (movable) logarithmic branch point. Using methods similar to those in Example 4, one may prove that this series is asymptotic. Then it follows that the equation is not of P-type.

(iii) If it happens that (3.21) is false (i.e., $R_{r_1} = 0$), then a_{r_1} is an arbitrary constant of integration. Proceed to the next coefficient.

(iv) Any resonance that is a multiple root of $Q(r)$ represents a (movable) logarithmic branch point with an arbitrary coefficient. If the assumed representation is asymptotic, the equation is not of P-type.

Example 7:

$$w'''' = 27 w'' w^2 + 21 w' w^3 - 9 w^5. \tag{3.23}$$

One approximate solution is

$$w \sim \alpha (z - z_o)^{-1},$$

with $\alpha = 1$. There are three other choices of α possible. Substituting

$$w \sim (z - z_o)^{-1} + \beta (z - z_o)^{r-1},$$

yields

$$Q(r) = (r+1)(r-1)^2(r-9) = 0. \tag{3.24}$$

At the first resonance, $r = 1$, the approximate solution is

$$w \sim (z - z_o)^{-1} + c_1 + c_2 \{22 \ln(z - z_o) + 27 [\ln(z - z_o)]^2\} + \cdots, \tag{3.25}$$

where (z_0, c_1, c_2) are all arbitrary constants. If (3.25) is asymptotic as $z \to z_o$, then (3.23) is not of P-type. ■

(v) At each nonresonant power, (3.20) determines a_j. At each resonance, either

$$R_{r_j} \neq 0,$$

logarithmic terms must be introduced into (3.19), and the equation is not of P type, or

$$R_{r_j} = 0,$$

and a_{r_j} is a arbitrary constant of integration.

(vi) If no logarithms are introduced at any of the resonances, one could in principle compute all of the terms in the series. However, because the recursion relations are nonlinear, it is usually not feasible to determine the region of convergence of the series, as one does in a linear problem. An alternative is to prove directly from the ODE that each arbitrary constant, (a_j), is the coefficient of an analytic function (cf. Ref. 8, §14.41).

(vii) If no logarithms are introduced at any of the resonances for all possible (p, α) from step A, then the equation has met the necessary conditions, under the assumption that $p < 0$ in (3.4). Cases for which $p > 0$ are treated below. This completes the algorithm.

To this point, we have assumed that the function becomes infinite at the singularity. Other possibilities, where the function remains finite while some derivative becomes singular, may be treated either directly (with $p > 0$) or by considering the ODE to be a system of first-order ODE's which we consider next.

The basic steps of the algorithm for a system of first-order ODE's, (3.1), are not essentially different from what we have already discussed.

1. Dominant behavior of the system

Substitute

$$w_j \sim \alpha_j (z - z_o)^{p_j}, \quad j = 1, \ldots, n, \tag{3.26}$$

into (3.1) and determine both the $\mathbf{p}(= \{p_j\}_{j=1}^n)$ for which there is a balance of leading terms, and what the leading terms are. Ordinarily, the $\{\alpha_j\}$ are not entirely determined, but must satisfy k relations, with $k \leqslant n$. The algorithm stops unless the only possible p_j's are integers.

2. Resonances of the system

For each p, construct a simplified equation from (3.1) that retains only the leading terms. Substitute into this simplified equation

$$w_j = \alpha_j (z - z_o)^{p_j} + \beta_j (z - z_o)^{p_j + r}, \quad j = 1, \ldots, n, \tag{3.27}$$

with the same r for every w_j. To leading order in $\boldsymbol{\beta}$, this becomes

$$[Q(r)] \boldsymbol{\beta} = 0,$$

where $[Q]$ is an $n \times n$ matrix, whose elements depend on r. The resonances are the nonnegative roots of

$$\det[Q(r)] = 0, \tag{3.28}$$

a polynomial of order $\leqslant n$. One root is always (-1); zero may also be a root, with multiplicity depending on how many α_j's were determined in step 1. The algorithm stops unless all of the resonances are integers.

3. The constants of integration

Substitute into (3.1)

$$w_j \sim \alpha_j (z - z_o)^{p_j} + \sum_{k=1}^{r_s} a_{j_k} (z - z_o)^{p_j + k}, \tag{3.29}$$

where r_s is the largest resonance. The coefficient of each power of $(z - z_o)$, which must vanish, has the form of a matrix generalization of (3.20). Its treatment is identical to the previous case, and we omit the details.

Because we are interested in the analytic properties of functions, special attention must be given to equations (or systems) where the complex conjugate of an unknown function appears explicitly. The algorithm still can be applied, with these changes.

(i) Write the ODE for w, and the ODE for w^*.

(ii) Treat $v = w^*$ as a new variable. Then apply the algorithm to this system of two ODE's, without assuming *any* relation between v and w. Because the equation for v is the formal complex conjugate of that for w, if the initial conditions at a point of the *real axis* are formally complex conjugate, then the solutions will satisfy

$$v(z) = [w(z^*)]^*. \tag{3.30}$$

The behavior of v at a singular point z_o is related to the behavior of w at z_o^*, but *not* at z_o unless z_o is real. If among the possible leading behaviors, there are some where v and w differ, it only means that if complex conjugate initial data is given, there can be no such singularities on the real axis.

Example 8. For real x,d,

$$w'' = xw + 2\,|w|^2 w + d. \tag{3.31}$$

For real z, this implies

$$w'' = zw + 2\,w^2 w^* + d,$$

$$(w^*)'' = zw^* + 2\,(w^*)^2 w + d.$$

Therefore, consider

$$\begin{aligned} w'' &= zw + 2\,w^2 v + d,\\ v'' &= zv + 2\,v^2 w + d. \end{aligned} \tag{3.32}$$

1. Substitute

$$w \sim \alpha_1 (z - z_o)^p, \quad v \sim \alpha_2 (z - z_o)^q.$$

The only possibility is

$$p = q = -1, \quad \alpha_1 \alpha_2 = 1. \tag{3.33}$$

2. The simplified equations are

$$w'' \sim 2w^2 v, \quad v'' \sim 2v^2 w.$$

Substitute

$$w \sim \alpha (z - z_o)^{-1} + \beta_1 (z - z_o)^{r-1},$$

$$v \sim (1/\alpha)(z - z_o)^{-1} + \beta_2 (z - z_o)^{r-1},$$

and find

$$Q(r) = [(r-1)(r-2) - 4\ \ ]^2 - 4 = 0, \tag{3.34}$$

with roots $r = -1,0,3,4$. The first two roots correspond to the arbitrary constants x_o and α.

3. Set $\xi = z - z_o$, and substitute into (3.32)

$$\begin{aligned} w &\sim \alpha \xi^{-1} + \sum_0^3 a_n \xi^n,\\ v &\sim \frac{1}{\alpha}\xi^{-1} + \sum_0^3 b_n \xi^n. \end{aligned} \tag{3.35}$$

Then find that

$$a_o = b_o = 0,$$

$$a_1 = -\alpha z_o/6, \quad b_1 = -z_o/6\alpha.$$

The first resonance occurs at the next order, where logarithmic terms arise unless

$$d = \alpha^2 d. \tag{3.36}$$

But α is arbitrary, so it follows from (3.36) that (3.32) is not of P-type unless

$$d = 0. \tag{3.37}$$

With (3.37), we find that b_2 is arbitrary, and

$$a_2 = \alpha/2 - \alpha^2 b_2.$$

The last resonance arises at the next order, where b_3 is arbitrary and

$$a_3 = \alpha^2 b_3.$$

It follows that (3.32) satisfies the necessary conditions to be of P-type if and only if d vanishes. With (3.37), we have four free constants, and (3.35) apparently represents the general solution of (3.32) in a deleted neighborhood of the movable pole at z_o. ■

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APPENDIX

Here we show that the conditions in (2.7) are sufficient to apply the results of Fredholm theory to the linear integral equation in (2.6). For fixed ξ and contour $\mathscr{C}(\xi)$, let $(s_1, s_2, \ldots, s_n)$ denote a finite ordered sequence of points on $\mathscr{C}$. We need suitable bounds on Fredholm determinants of the form:

$$\Delta_n = \begin{vmatrix} N(s_1,s_1) N(s_1,s_2) & \cdots & N(s_1,s_n)\\ N(s_2,s_1) N(s_2,s_2) & \cdots & N(s_2,s_n)\\ \vdots & & \\ N(s_n,s_1) N(s_n,s_2) & \cdots & N(s_n,s_n) \end{vmatrix}, \tag{A1}$$

where the ξ dependence of N was suppressed. The bound comes from a complex generalization of Hadamard's Lemma [10]

$$\begin{aligned} |\Delta_n|^2 &\leqslant \prod_{\nu=1}^n \sum_{\mu=1}^n |N(s_\mu, s_\nu)|^2\\ &\leqslant \prod_{\nu=1}^n n \sup_{s \text{ on } \mathscr{C}} |N(s,s_\nu)|^2\\ &\leqslant n^n \prod_{\nu=1}^n M^2(S_\nu), \end{aligned}$$

using (2.7a). Therefore,

$$|\Delta_n| \leqslant n^{n/2} \prod_{\nu=1}^n M(s_\nu). \tag{A2}$$

Fredholm's first series has the form

$$\mathscr{D}_1(\mathscr{C}(\xi);\lambda) = 1 + \sum_{n=1}^\infty \frac{(-\lambda)^n}{n!} \oint \cdots \oint \Delta_n \, ds_1 \cdots ds_n. \tag{A3}$$

The magnitude of the nth term in this series does not exceed

$$\frac{(-\lambda)^n}{n!} \left(n^{n/2} \prod_{\nu=1}^n \oint M(s_\nu)\, ds_\nu \right) \leqslant n^{\ \ n/2} (c_1 \lambda)^n c_2, \tag{A4}$$

where c_1, c_2 are constants. It follows that the series converges for all values of λ and for all (ξ). Proof of the convergence of

Fredholm's second series follows similar lines, and these two series define the resolvent kernel. Then (2.7c) is needed to solve (2.6).

[1]M.J. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur, Stud. Appl. Math. **53**, 249 (1974).
[2]M.J. Ablowitz and H. Segur, Phys. Rev. Lett. **38**, 1103 (1977).
[3]M.J. Ablowitz, A. Ramani, and H. Segur, Lett. Nuovo Cimento **23**, 333 (1978).
[4]G.L. Lamb, Rev. Mod. Phys. **43**, 99 (1977).
[5]R. Rosales, Proc. R. Soc. London A **361**, 265 (1978).
[6]R. Nakach, Tech. Report #2, School of Elect. Eng., Chalmers Univ. Techn., Göteborg, Sweden (1977).
[7]M.J. Ablowitz, A. Ramani, and H. Segur, "A Connection between nonlinear evolution equations and ordinary differential equations of P-type. II", to appear in J. Math. Phys.
[8]E.L. Ince, *Ordinary Differential Equations* (Dover, New York, 1944).
[9]V.V. Golubev, *Lectures on Integrals of Equations of Motion* (State Publ. House, Moscow, 1953).
[10]W. Pogorzelski, *Integral Equations and Their Applications* (Pergamon, New York, 1966), Vol. I.
[11]V.E. Zakharov and A.B. Shabat, Funct. Anal. Appl. **8**, 226 (1974).
[12]H. Cornille, J. Math. Phys. **18**, 1855 (1977).