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Three-dimensional natural convection motion in a confined porous medium

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Weakly nonlinear analysis is used to calculate the possible two- and three-dimensional convection patterns in a rectangular parallelepiped of saturated porous media when the horizontal dimensions are integral multiples of the vertical dimension. A two-term expansion for the Nusselt number is found for values of the Rayleigh number close to the critical values. It is shown that the two-dimensional roll configurations transfer heat more effectively than does the three-dimensional pattern of motion when the Rayleigh number is just above the critical value.

I. INTRODUCTION

Most studies of natural convection in porous media have been concerned with two-dimensional rolls.[1-3] For certain configurations of a rectangular parallelepiped, linear analysis[4,5] predicts that such motions are the preferred cellular mode. However, Fig. 2 of Beck[4] implies that many types of three-dimensional modes are possible if the aspect ratios of the container have appropriate values. This is particularly true for boxes with horizontal dimensions large compared with the vertical dimension.

In this paper we consider a situation where the preferred motion at the critical Rayleigh number is either one of a pair of different two-dimensional roll configurations or a combination of both. We use the method of weakly nonlinear analysis[3,6] to develop a two-term expansion for the temperature and velocity fields and the Nusselt number for values of the Rayleigh number (R) slightly above the critical value (R_c). The analysis is simplified considerably when the geometry of the container is such that $R_c = 4\pi^2$.

Our results show that for values of $R \to R_c$ the Nusselt numbers associated with either of the two-dimensional motions are equal and larger than the Nusselt number of the three-dimensional motion resulting from the nonlinear interaction of the same two-dimensional motions.

II. MATHEMATICAL SYSTEM

The physical system is shown in Fig. 1. The horizontal surfaces are kept at constant temperatures T_0^* (at $z^* = 0$) and $T_1^* > T_0^*$ (at $z^* = -L^*$). Here, the asterisk (*) refers to dimensional quantities. The vertical boundaries are insulated and all boundaries are impermeable. The nondimensional, constant properties, Boussinesq equations are[5]

$$u_x + v_y + w_z = 0\ , \tag{1a}$$

$$u_z = w_x - \theta_x\ , \tag{1b}$$

$$v_z = w_y - \theta_y\ , \tag{1c}$$

$$\theta_t + u\theta_x + v\theta_y + w\theta_z - w = R^{-1}\nabla^2\theta\ , \tag{1d}$$

where u, v, and w are the velocity components, and θ is the temperature perturbation from the conduction profile. The nondimensional quantities and the Rayleigh number,

$$R = (g^* k^* \alpha^* \Delta T^* L^* / \nu^{*2})(C^* \mu^* / \lambda_m^*)$$

are defined in Ref. 5.

The boundary conditions associated with (1) are

$$\begin{aligned} &u = \theta_x = 0\ , \quad x = 0, a\ , \quad a = l_x^*/L^*\ , \\ &v = \theta_y = 0\ , \quad y = 0, b\ , \quad b = l_y^*/L^*\ , \\ &w = \theta = 0\ , \quad z = 0, -1\ . \end{aligned} \tag{2}$$

III. LINEAR THEORY

The linearized steady state version of (1) can be written as

$$\nabla^2 w = \nabla_1^2\theta\ , \quad \nabla_1^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}\ , \quad \nabla^2\theta + Rw = 0\ . \tag{3}$$

Separation of variables leads to

$$\begin{aligned} &\theta = \sin\pi z \cos\alpha_n x \cos\beta_m y\ , \\ &\alpha_n = n\pi/a\ , \quad \beta_m = m\pi/b\ , \quad n, m = 0, 1, 2, \ldots\ , \end{aligned} \tag{4}$$

and

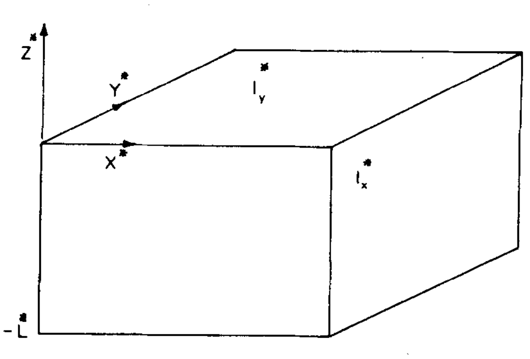

FIG. 1. The rectangular parallelepiped with horizontal dimensions l_x^*, l_y^*, and vertical dimensions L^*.

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$$R=(\alpha_n^2+\beta_m^2+\pi^2)^2/(\alpha_n^2+\beta_m^2) .$$

Two-dimensional motion in the x direction corresponds to $m=0$, while that in the y direction corresponds to $n=0$. We consider the two-dimensional motions given by

$$\theta^{(1)}=\sin\pi z\cos\alpha_n x , \qquad R_n^{(1)}=[(\alpha_n^2+\pi^2)/\alpha_n]^2 , \tag{5}$$

and

$$\theta^{(2)}=\sin\pi z\cos\beta_m y , \qquad R_m^{(2)}=[(\beta_m^2+\pi^2)/\beta_m]^2 . \tag{6}$$

If the two motions occur at the same value of R, then

$$R^{(1)}=R^{(2)}=R ,$$

and

$$\alpha_n=\beta_m .$$

It should be noted that this is the case for all points on the lines separating regions of two-dimensional motions in Fig. 2 of Beck's[4] paper. The condition $\alpha_n=\beta_m$ implies that each of the two-dimensional configurations is composed of an integer number of identical rolls (n rolls in the x direction and m rolls in the y direction).

The critical value of R is given by

$$R_c=\min_n\left(\frac{\alpha_n^2+\pi^2}{\alpha_n}\right)^2 . \tag{7}$$

In order to consider the most elementary nonlinear analysis one must examine the case for $R_c=4\pi^2$; then, $\alpha_n=\pi$ and both a and b are integers. This leads to considerable simplification in the necessary algebraic manipulations.

IV. WEAKLY NONLINEAR THEORY

For values of R slightly larger than $R_c(=4\pi^2)$, perturbation methods can be used to find the amplitude of the motion. Equations (1) can be written as

$$\nabla^2 w=\nabla_1^2\theta , \tag{8a}$$

$$\nabla^2\theta+R_c w=R\theta_t-(R-R_c)w+R(u\theta_x+v\theta_y+w\theta_z) . \tag{8b}$$

The terms on the right-hand side of (8b) are regarded as small compared with the linear terms on the left-hand side. The basic solution found in Sec. III and from (1b, c) and (2), is

$$\begin{aligned}\theta&=\theta_1=[A(t)\cos\pi y+B(t)\cos\pi x]\sin\pi z ,\\ w&=w_1=\tfrac{1}{2}\theta_1 ,\\ u&=u_1=-\tfrac{1}{2}B(t)\sin\pi x\cos\pi z ,\\ v&=v_1=-\tfrac{1}{2}A(t)\sin\pi y\cos\pi z ,\end{aligned} \tag{9}$$

where $A(t)$ and $B(t)$ are small amplitudes to be found. The first correction is found by letting

$$\theta=\theta_1+\theta_2 , \qquad w=w_1+w_2 ,$$

in (8). This leads to

$$\begin{aligned}&\nabla^2 w_2=\nabla_1^2\theta_2 , \qquad \nabla^2\theta_2+R_c w_2=G_1 ,\\ &G_1=R\theta_{1_t}-(R-R_c)w_1\\ &\qquad+(R\pi/4)(B^2+A^2+2AB\cos\pi y\cos\pi x)\sin 2\pi z .\end{aligned} \tag{10}$$

The compatibility conditions for solving (10) and (2) are found to be

$$\int_{-1}^0 dz\int_0^a dx\int_0^b dy\, G_1\sin\pi z\begin{pmatrix}\cos\pi x\\ \cos\pi y\end{pmatrix}=0 . \tag{11}$$

It follows that

$$\begin{aligned}\theta_2&=\frac{-R(A^2+B^2)}{16\pi}\sin 2\pi z-\frac{3RAB}{28\pi}\cos\pi y\cos\pi x\sin 2\pi z ,\\ w_2&=-\frac{RAB}{28\pi}\cos\pi y\cos\pi x\sin 2\pi z ,\end{aligned} \tag{12}$$

while $R\theta_{1t}-(R-R_c)w_1$ is $O(A^3)$ or $O(B^3)$. The corrections to u and v are then found from (1b, c) and (2) to be

$$\begin{aligned}u_2&=(RAB/28\pi)\cos\pi y\sin\pi x\cos 2\pi z ,\\ v_2&=(RAB/28\pi)\sin\pi y\cos\pi x\cos 2\pi z .\end{aligned} \tag{13}$$

The second correction is found by letting

$$\theta=\theta_1+\theta_2+\theta_3 ,$$

and similar expressions for w, u and v. This leads to

$$\nabla^2 w_3=\nabla_1^2\theta_3 , \qquad \nabla^2\theta_3+R_c w_3=G_2 , \tag{14}$$

where

$$\begin{aligned}G_2&=R(\dot A\cos\pi y+\dot B\cos\pi x)\sin\pi z\\ &\quad-\frac{(R-R_c)}{2}(A\cos\pi y+B\cos\pi x)\sin\pi z\\ &\quad+R(u_1\theta_{2x}+u_2\theta_{1x}+v_1\theta_{2y}+v_2\theta_{1y}+w_1\theta_{2z}+w_2\theta_{1z}) .\end{aligned}$$

The compatibility conditions (11) require that

$$R\dot B-\frac{(R-R_c)}{2}B+\frac{R^2B(A^2+B^2)}{32}+\frac{3}{224}R^2A^2B=0 \tag{15a}$$

and

$$R\dot A-\frac{(R-R_c)}{2}A+\frac{R^2A(A^2+B^2)}{32}+\frac{3}{224}R^2AB^2=0 . \tag{15b}$$

If the initial perturbation is such that $A(0)=0$, it follows from (15b) that $A(t)=0$ and from (15a) in the steady state ($\dot B=0$) that

$$B^2=16(R-R_c)/R^2 ,$$

and hence the Nusselt number

$$\begin{aligned}\mathrm{Nu}&=1+\frac{1}{ab}\int_0^a\int_0^b\left(\frac{\partial\theta}{\partial z}\right)_{z=0}dx\,dy\\ &=1+\frac{RB^2}{8}=1+2\,\frac{(R-R_c)}{R} .\end{aligned} \tag{16}$$

The motion is two-dimensional rolls in the x direction.

Similarly, if $B(0)=0$, then $B(t)=0$ and the motion is two-dimensional rolls in the y direction with the steady state given by

$$\begin{aligned}A^2&=16(R-R_c)/R^2 ,\\ \mathrm{Nu}&=1+2(R-R_c)/R .\end{aligned} \tag{17}$$

Comparison of the Nusselt number of the x rolls in (16) with that for the y rolls in (12) shows that the two are equal at the same value of R. This occurs because the Nusselt number is a measure of the nondimensional heat transfer per unit area for a collection of identical two-dimensional nonlinear rolls in either coordinate direction. The equivalence of the mode shapes for either of the nonlinear two-dimensional motions may be observed from the two-term representation of θ and w.

These two-dimensional motions are seen to result from specialized initial perturbations. However, a general perturbation to the system will be such that $A(0) \neq 0$ and $B(0) \neq 0$. In this case, the steady state solution ($\dot{B}=0$ and $\dot{A}=0$) is found from (15)

$$\frac{R-R_c}{2}=\frac{R^2}{32}(A^2+B^2)+\frac{3}{224}R^2A^2 ,$$
$$\frac{R-R_c}{2}=\frac{R^2}{32}(A^2+B^2)+\frac{3}{224}R^2B^2 , \tag{18}$$

hence,

$$A^2=B^2=\frac{112}{17}\frac{(R-R_c)}{R^2} ,$$
$$\mathrm{Nu}=1+\frac{RA^2}{4}=1+\left(\frac{28}{17}\right)\frac{(R-R_c)}{R} . \tag{19}$$

It should be noted that the first correction to the velocity field in (13) is identically zero for two dimensional rolls because either A or B is zero. When the motion is three-dimensional, the correction is finite.

V. CONCLUSIONS

By considering a simple model for three-dimensional natural convection in a confined porous media, arising from the nonlinear interaction of a pair of two-dimensional rolls which are excited at the same value of the critical Rayleigh number, we derived a two-term expansion for the velocity and temperature fields, and for the Nusselt number. If either of the two-dimensional motions prevails (depending on the form of the initial perturbation to the system), we find that the steady state Nusselt number is given by

$$\mathrm{Nu}_{2d}=1+2(R-R_c)/R .$$

However, if the initial disturbance to the system contains both modes, we find that

$$\mathrm{Nu}_{3d}=1+(28/17)[(R-R_c)/R] ,$$

which is clearly less than Nu_{2d}. This result which is valid for $R \to R_c^+$ is in qualitative agreement with the conclusions of Holst and Aziz.[7] These authors used numerical computations to show that for a cube, when the Rayleigh number is large, three-dimensional convection "... exhibits considerably more heat transfer" than the two-dimensional solutions, while at lower Rayleigh numbers "... the situation is reversed." If one accepts the Platzman[8] criterion that the solution exhibiting maximum heat transfer is physically preferred, then the two-dimensional motion is most likely to appear.

The results presented here represent a subset of all possible three-dimensional motions in a rectangular parallelepiped. More general results can be obtained in an analogous fashion although the algebraic complexity is significant. These considerations have applications to convection patterns in liquid-dominated geothermal basins. A variety of geophysical measurements suggest that the horizontal extent of these systems is of the magnitude of or greater than the vertical depth. Hence, it is likely that in the "large," when small-scale geological structure can be ignored, three-dimensional convection patterns will appear. In this situation the frequently computed two-dimensional rolls[6] may give misleading information about the heat and mass transfer in the system.

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[1]J. M. Straus, J. Fluid Mech. **64**, 51 (1974).
[2]J. P. Caltigirone, J. Fluid Mech. **72**, 269 (1975).
[3]E. Palm, J. E. Weber, and O. Kvernvold, J. Fluid Mech. **54**, 153 (1972).
[4]J. L. Beck, Phys. Fluids **15**, 1377 (1972).
[5]A. Zebib and D. R. Kassoy, Phys. Fluids **20**, 4 (1977).
[6]A. Zebib and D. R. Kassoy (to be published).
[7]P. H. Holst and K. Aziz, Int. J. Heat Mass Transfer **15**, 73 (1972).
[8]G. W. Platzman, J. Fluid Mech. **23**, 481 (1965).