AIP | Journal of Mathematical Physics

Bounds on the admittance for KMS states

J. Naudts and A. Verbeure

Citation: Journal of Mathematical Physics **17**, 419 (1976); doi: 10.1063/1.522909
View online: http://dx.doi.org/10.1063/1.522909
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/17/3?ver=pdfcov
Published by the AIP Publishing

Bounds on the admittance for KMS states

J. Naudts

University of Antwerpen, Belgium

A. Verbeure

*University of Leuven, Belgium**

(Received 18 August 1975)

Upper and lower bounds are proved for the static admittance of observables in a KMS state on a von Neumann algebra. As an application some exact results for the transverse Ising model are derived.

I. INTRODUCTION

Recently Roepstorff[1] derived a new upper and lower bound for the state admittance of observables in a Gibbs state. They are given in the case of a finite lattice system.

As is well-known the states of infinite systems (i.e., in the thermodynamic limit) are no longer of Gibbs' type, and it is now widely accepted that an equilibrium state of an infinite system should be described by a state satisfying the KMS condition.[2]

In this paper we give rigorous proofs of the upper and lower bounds given in Ref. 1, and derive new bounds for KMS states on a von Neumann algebra of observables. Hence the results are valid not only for infinite lattice systems but also for continuous systems.

In Sec. II we introduce the necessary mathematical material and prove some more properties of the Bogoliubov or Kubo–Mori scalar product, which was studied in Ref. 3. In Sec. III two upper bounds and essentially two lower bounds of the static admittance are derived. Finally in Sec. IV we apply the inequalities to prove some exact results for the transverse Ising model.

II. MATHEMATICAL FRAMEWORK

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Let $t \to U_t$ be a strongly continuous map from the real numbers $\mathbb{R}$ into the group of unitaries on $\mathcal{H}$, then there exists a self-adjoint operator H on $\mathcal{H}$ such that $U_t = \exp itH$, and let $x_t = U_t x U_t^*$. Furthermore let ω be any vector state on $\mathcal{M}$, i.e., $\omega(x) = (\Omega, x\Omega)$ for all $x \in \mathcal{M}$, with with Ω a cyclic element of $\mathcal{H}$. The state ω is called an equilibrium state if it satisfies the following definition.

Definition II.1: The state ω on $\mathcal{M}$ satisfies the KMS condition at inverse temperature $\beta = 1/kT$, if for any pair x, y of observables in $\mathcal{M}$, there exists a complex function $F_{x,y}(z)$, defined, bounded and continuous on the strip $-\beta \leq \operatorname{Im} z \leq 0$, and analytic inside, with boundary values $F_{x,y}(t) = \omega(x_t y)$, $F_{xy}(t - i\beta) = \omega(y x_t)$.

Without restriction of generality, let $\beta = 1$ in the sequel. Any state ω satisfying the KMS condition has the following properties[4]:

(i) the vector Ω is separating;

(ii) for all t, $U_t\Omega = \Omega$;

(iii) there exists an operator $\Delta = \exp(-H)$ on $\mathcal{H}$, such that $\Delta = FS$ where S is the closure of the map $x\Omega \to x^*\Omega$, $x \in \mathcal{M}$, F is the adjoint of S; furthermore, $S = J\Delta^{1/2}$ is the polar decomposition of S and $J\Delta^{it} = \Delta^{it}J$, $\Delta^{it} = U_t$, $J\Omega = \Omega$, $\Delta\Omega = \Omega$;

(iv) there exists a subset $\mathcal{B}$ of elements of $\mathcal{M}$ such that $\mathcal{B}$ is invariant under left multiplication with Δ^α, $\alpha \in C$ (complex numbers), and such that $\mathcal{B}$ is dense in the Hilbert space $\mathcal{D}(\Delta^\alpha)$ (domain of Δ^α). $\mathcal{B}$ is generally called the set of analytic elements.

In Ref. 3 we defined the Hilbert space $\tilde{\mathcal{H}}$, as the closure of $\mathcal{M}$ with respect to the scalar product $(x, y)_\sim = (Tx\Omega, Ty\Omega)$; $x, y \in \mathcal{M}$, $T = [(\Delta - 1)/\ln\Delta]^{1/2}$.

Furthermore the following results were proved:

(i) there exists a unitary operator U from $\tilde{\mathcal{H}}$ onto $\mathcal{H}$, defined by $Ux = Tx\Omega$, $x \in \mathcal{M}$;

(ii) let

$$\chi_{x,y}(0) = \lim_{z\to 0\pm} i\int_0^\infty dt \exp(\mp izt)\,\omega([x_t, y])$$

be the static admittance of the pair of observables $x, y \in \mathcal{M}$, then

$$(x^*, y)_\sim = \chi_{x,y}(0) + (\Omega, xE_0 y\Omega), \tag{1}$$

where E_0 is the orthogonal projection on the set of U_t-invariant vectors of $\mathcal{H}$.

If E_0 is one-dimensional then

$$(x^*, y)_\sim = \chi_{x,y}(0) + \omega(x)\,\omega(y).$$

In the following we derive bounds for the scalar product $(x, y)_\sim$; the implications for the static admittance are given by formula (1).

(iii) for each pair x, y of elements in $\mathcal{M}$,

$$\begin{aligned}(x, y)_\sim &= \int_{-1}^0 dt\, F_{x,y}(it),\\ &= \int_0^{1/2} dt\{(x\Omega, \Delta^t y\Omega) + (y^*\Omega, \Delta^t x^*\Omega)\}.\end{aligned} \tag{2}$$

Now we prove some more properties of this scalar product.

Proposition II.2: For all $x, y \in \mathcal{M}$, we have

$$(x, y)_\sim = \tfrac{1}{2}\int_{-1}^1 dt\ (\Delta^{(1+t)/4}x\Omega,\ \Delta^{(1+t)/4}y\Omega). \tag{3}$$

Proof: Starting from formula (2), after a few substitutions for the integration variable one gets

$$(x,y)_\sim = \tfrac{1}{2}\int_{-1}^{0} dt\ (\Delta^{(1+t)/2}x\Omega,\ y\Omega)\,dt + \tfrac{1}{2}\int_0^1 dt(\Delta^{(1-t)/2}y^*\Omega, x^*\Omega).$$

But

$$\begin{aligned}(\Delta^{(1/2)(1-t)}y^*\Omega,\ x^*\Omega) &= (\Delta^{(1-t)/4}J\Delta^{1/2}y\Omega,\ \Delta^{(1-t)/4}J\Delta^{1/2}x\Omega)\\ &= (J\Delta^{(t-1)/4}\Delta^{1/2}y\Omega,\ J\Delta^{(t-1)/4}\Delta^{1/2}x\Omega)\\ &= (J\Delta^{(1+t)/4}y\Omega,\ J\Delta^{(1+t)/4}x\Omega)\\ &= (\Delta^{(1+t)/4}x\Omega,\ \Delta^{(1+t)/4}y\Omega),\end{aligned}$$

and the result follows. QED

For any $x \in \mathcal{M}$, denote

$$f_x(t) = \log\|\Delta^{(1+t)/4}x\Omega\|^2. \tag{4}$$

As is easily checked:

$$f_x(-1) = \log\omega(x^*x), \tag{5}$$

$$f_x(1) = \log\omega(xx^*). \tag{6}$$

Furthermore

$$2\exp[f_x(t)]\frac{d}{dt}f_x(t) = -(\Delta^{(1+t)/4}x\Omega,\ H\Delta^{(1+t)/4}x\Omega) \quad \text{for } t\in(-1,1).$$

Hence, if $x^*\Omega \in \mathcal{D}(H)$

$$f_x'(1) = \omega(xHx^*)/2\omega(xx^*), \tag{7}$$

and if $x\Omega \in \mathcal{D}(H)$

$$f_x'(-1) = -\omega(x^*Hx)/2\omega(x^*x). \tag{8}$$

Proposition II. 3:

(i) for all $x\in\mathcal{M}$, the function $t \to f_x(t)$ is convex on the interval $[-1,+1]$;

(ii) for all $x = x^* \in \mathcal{M}$, the function satisfies $f_x(t) = f_x(-t)$.

Proof: For any $x \in \mathcal{B}$ (analytic elements), the function $t \to f_x(t)$ is analytic, hence it is sufficient to prove $f_x((t+s)/2) \le \frac{1}{2}[f_x(t)+f_x(s)]$. But this follows from Schwartz's inequality,

$$\begin{aligned}\|\Delta^{[1+(t+s)/2]/4}x\Omega\|^2 &= (\Delta^{(1+t)/4}x\Omega,\ \Delta^{(1+s)/4}x\Omega)\\ &\le \|\Delta^{(1/4)(1+t)}x\Omega\|\,\|\Delta^{(1/4)(1+s)}x\Omega\|.\end{aligned}$$

Let now x be any element of $\mathcal{M}$, from definition II. 1 (iv), there exists a sequence of elements $\{x_n\}$ in $\mathcal{B}$, such that $x_n\Omega$ tends to $x\Omega$ and $\Delta^{1/2}x_n\Omega$ tends to $\Delta^{1/2}x\Omega$.

Hence $f_{x_n}(t)$ tends to $f_x(t)$, and $f_x(t)$ is also a convex function as the limit of convex functions. This proves (i).

Now, if $x = x^*$, then

$$\begin{aligned}\|\Delta^{(1/4)(1+t)}x\Omega\|^2 &= \|J\Delta^{(1+t)/4}x\Omega\|^2\\ &= \|J\Delta^{(t-1)/4}\Delta^{1/2}x\Omega\|^2\\ &= \|\Delta^{(1/4)(1-t)}x^*\Omega\|^2 = \|\Delta^{(1/4)(1-t)}x\Omega\|^2,\end{aligned}$$

and (ii) follows. QED

III. BOUNDS

Theorem III. 1 (*Roepstorff*): For all $x\in\mathcal{M}$, we have

$$(x,x)_\sim \le \omega([x,x^*])/\log\frac{\omega(xx^*)}{\omega(x^*x)}.$$

Proof: From formula (4) for all $x\in\mathcal{M}$,

$$(x,x)_\sim = \tfrac{1}{2}\int_{-1}^{1} dt\ \exp f_x(t).$$

From Proposition II. 3 (i), $t\in[-1,1]$,

$$\frac{f_x(1)-f_x(-1)}{2}t + \frac{f(1)+f(-1)}{2} \ge f(t) \quad \text{[see Fig. 1, curve (1)]},$$

hence

$$(x,x)_\sim \le \frac{\exp f_x(1) - \exp f_x(-1)}{f_x(1)-f_x(-1)}$$

and by (5) and (6),

$$(x,x)_\sim \le \omega([x,x^*])/\log[\omega(xx^*)/\omega(x^*x)]. \qquad \text{QED}$$

Corollary III. 2. (*Bogoliubov–Roepstorff*): For all y of $\mathcal{M}$ and elements of x of $\mathcal{M}$ such that $x\Omega$ and $x^*\Omega$ belong to the domain $\mathcal{D}(H)$ of H we have,

$$|\omega([y^*,x])|^2 \le \tfrac{1}{2}\omega(\{y,y^*\})\omega([[x.H],\ x^*])g(r),$$

where

$$g(r) = \frac{2r}{\log(1+r/1-r)} \le 1,$$

$$r = \omega([y,y^*])/\omega(\{y,y^*\}).$$

Proof: Under the conditions of the corollary, as in Ref. 3, Theorem III. 8, one gets

$$|\omega([y^*,x])|^2 \le (y,y)_\sim\,\omega([[x,H],x^*]).$$

Using Theorem III. 1 to majorize $(y,y)_\sim$, the corollary follows. QED

Remark: $g(r=0)=1$, so that the inequality above is not stronger than the original Bogoliubov inequality in the case $\omega(yy^*)=\omega(y^*y)$, in particular if $y=y^*$ or if ω is

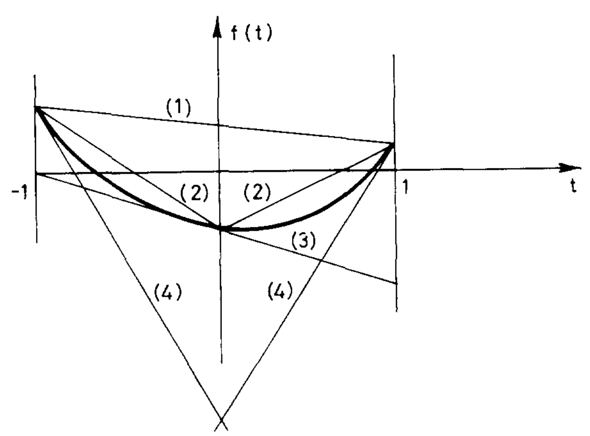

FIG. 1.

a central state. On the other hand $\lim_{r\to 1} g(r) = 0$. Hence the interest of the stronger version lies in the region $r \approx 1$. A stronger upper bound is obtained in the following theorem.

Theorem III.3: For all $x \in \mathcal{M}$

$$(x,x)_\sim \leq \frac{1}{2}\left(\frac{\omega(x^*[\exp(-H/2)-1]x)}{\log[\omega(x^*\exp(-H/2)x)/\omega(x^*x)]} + \frac{\omega(x[\exp(-H/2)-1]x^*)}{\log[\omega(x\exp(-H/2)x^*)/\omega(xx^*)]}\right).$$

Proof: From Proposition II.3 (i) [see Fig. 1, curve (2)],

$$(f_x(0) - f_x(-1))\,t + f_x(0) \geq f_x(t), \quad t \in [-1, 0],$$
$$(f_x(1) - f_x(0))t + f_x(0) \geq f_x(t), \qquad t \in [0, 1].$$

Hence

$$(x,x)_\sim = \tfrac{1}{2}\int_{-1}^{1} dt\; \exp f_x(t) \leq \frac{1}{2}\left(\frac{\exp[f_x(0)] - \exp[f_x(-1)]}{f_x(0) - f_x(-1)} + \frac{\exp[f_x(1)] - \exp[f_x(0)]}{f_x(1) - f_x(0)}\right).$$

Using formulas (5) and (6), and the fact that

$$f_x(0) = \log\omega(x^*\exp(-H/2)x) = \log\omega(x\exp(-H/2)x^*),$$

one gets the result. QED

Remark: In any case, the inequality of Theorem III.3 is a much stronger inequality than the one of Theorem III.1. The associated Bogoliubov inequality will also be much better. We do not elaborate on this point here.

Now we turn to the lower bounds.

Theorem III.4: For all $x \in \mathcal{M}$,

$$(x,x)_\sim \geq \sup_{-1\leq s\leq 1} \exp\{f_x(s) - s f_x'(s)\}\,\sinh[f_x'(s)]/f_x'(s).$$

In particular,

(i) if $x\Omega \in \mathcal{D}(H)$,

$$(x,x)_\sim \geq \frac{\omega(x^*x)^2}{2\omega(x^*Hx)}\left[1 - \exp\left(-2\,\frac{\omega(x^*Hx)}{\omega(x^*x)}\right)\right] \;;$$

(ii) if $x^x\Omega \in \mathcal{D}(H)$,

$$(x,x)_\sim \geq \frac{\omega(xx^*)^2}{2\omega(xHx^*)}\left[1 - \exp\left(-2\,\frac{\omega(xHx^*)}{\omega(xx^*)}\right)\right];$$

(iii) if $x = x^*$,

$$(x,x)_\sim \geq \omega(x\exp(-H/2)x).$$

Proof: From Proposition II.3 (i), for all $s \in (-1, 1)$,

$$f_x(t) \geq f_x(s) + (t-s)\,f_x'(s) \quad \text{[see Fig. 1, curve (3)]},$$

hence

$$(x,x)_\sim \geq \tfrac{1}{2}\int_{-1}^{1} dt\; \exp[f(s) + (t-s)f'(s)].$$

After integration one gets

$$(x,x)_\sim \geq \exp[f(s) - sf'(s)]\,\sinh[f'(s)]/f'(s),$$

yielding the first part of the theorem.

If $x\Omega \in \mathcal{D}(H)$, then using (5) and (8) one gets (i) by taking $s = -1$.

If $x^*\Omega \in \mathcal{D}(H)$, put $s = 1$ and use (7) and (8) to get (ii).

If $x = x^*$, then by Proposition II.3 (ii) the function $f_x(t)$ is symmetric around $t = 0$ and $f_x'(t=0) = 0$, hence

$$(x,x)_\sim \geq \exp f_x(0) = \omega(x\exp(-H/2)x). \qquad \text{QED}$$

Finally we derive another lower bound, which was first derived in Ref. 1 for finite lattice systems. We prove it only in the case of elements $x = x^* \in \mathcal{M}$.

Using the same proof it may be derived in the general case.

Theorem III.5: For all elements $x = x^*$ of $\mathcal{M}$ such that $x\Omega \in \mathcal{D}(H)$,

$$(x,x)_\sim \geq \omega(x^2)\left(\frac{1 - e^{-C}}{C}\right)$$

where

$$C = \tfrac{1}{2}\,\omega(xHx)/\omega(x^2).$$

Proof: From Proposition II.3 (i) and (ii) it follows that [see Fig. 1, curve (4)],

$$f_x(t) \geq f_x'(-1)(t+1) + f_x(-1) \quad \text{if } -1 \leq t \leq 0,$$
$$f_x(t) \geq f_x'(1)(t-1) + f_x(1) \qquad \text{if } 0 \leq t \leq 1.$$

Hence

$$(x,x)_\sim \geq \tfrac{1}{2}\int_{-1}^{0} dt\; \exp[f_x'(-1)t + f_x(-1) + f_x'(-1)] + \tfrac{1}{2}\int_{0}^{1} dt\; \exp[f_x'(1)t + f_x(1) - f_x'(1)]$$

Using (5), (6), (7), and (8) and the fact that $f_x(1) = f_x(-1)$; $f_x'(-1) = -f_x'(1)$, and one gets the proof. QED

Remark: There is no strict relation between the two lower bounds which we proved. However the last inequality is stronger than the particular cases (i) or (ii) of Theorem III.4. The essential difference consists in minorizing the convex function respectively by one or by two straight lines.

IV. APPLICATION

As an application we derive some exact results on the transverse Ising model,[5] sometimes called the Blinc model or Tunnel model,[6] described by the following Hamiltonian. Let Z^ν be a ν-dimensional lattice, Λ any finite subset of Z^ν, then the local Hamiltonian is given by

$$H_\Lambda = \Omega\sum_{k\in\Lambda}\sigma_k^x + \tfrac{1}{2}\sum_{k,1\in\Lambda} J(|k-1|)\,\sigma_k^z\sigma_1^z,$$

where Ω and $J(|k|)$ are real numbers and we take $J(0) = 0$. Furthermore, $\sigma_p^\alpha;\ \alpha = x, y, z;\ p \in Z^\nu$ are the Pauli matrices, satisfying $[\sigma_p^\alpha, \sigma_q^\beta] = 2i\epsilon_{\alpha\beta\gamma}\sigma_p^\gamma \delta_{p,q}$. The local algebra of observables for the volume Λ is the completed tensor product of the 2×2 matrices M_2,

$$\mathfrak{A}_\Lambda = \bigotimes_{k\in\Lambda} M_2.$$

The C^*-algebra of observables $\mathfrak{A}$ is then the norm closure of $U_{\Lambda\in Z^\nu}\mathfrak{A}_\Lambda$.

We suppose that $\sum_{k\in Z^\nu} |J(k)| < \infty$, such that the map

$$t\in \mathrm{R} \to \alpha_t(x) = \lim_{\Lambda\to\infty} \exp(itH_\Lambda)\, x \exp(-itH_\Lambda), \quad x\in\mathfrak{A}$$

exists and yields a strongly continuous group of $*$-automorphisms of $\mathfrak{A}$.[7]

As is easily checked, the following limits exist, and define a not necessarily bounded derivation H of the C^*-algebra,

$$H(\sigma_p^x) = \lim_{\Lambda\to\infty} [H_\Lambda, \sigma_p^x] = 2i\tau_p\sigma_p^y \tag{9}$$

where

$$\tau_p = \sum_{k\in Z^\nu} J(|k-p|)\sigma_k^z,$$

$$H(\sigma_p^y) = \lim_{\Lambda\to\infty} [H_\Lambda, \sigma_p^y] = 2i\Omega\sigma_p^z - 2i\tau_p\sigma_p^x, \tag{10}$$

$$H(\sigma_p^z) = \lim_{\Lambda\to\infty} [H_\Lambda, \sigma_p^z] = -2i\Omega\sigma_p^y. \tag{11}$$

Let ω be a time-invariant state on Ω, and let (π, Ω, U_t) be its GNS representation, i.e.,

$$\omega(x) = (\Omega, \pi(x)\Omega), \quad x\in\mathfrak{A},$$

where Ω is a cyclic vector of $\mathcal{H}$ for $\pi(\mathfrak{A})$,

$$\pi(\alpha_t(x)) = U_t\, \pi(x)\, U_t^*$$

$$M = \pi(\mathfrak{A})''.$$

Consider the extension of ω to M; we denote it by the same symbol ω. From now on we drop the notation π. Finally let us suppose that ω is a KMS state of M for the time evolution $x_t = U_t x U_t^*$. Now we are in a position to apply the inequalities.

Using (11) and Ref. 3, Theorem III. 2 we get

$$(\sigma_p^y, \sigma_p^y)_\sim = -(1/\Omega)\,\omega(\sigma_p^x) \geq 0. \tag{12}$$

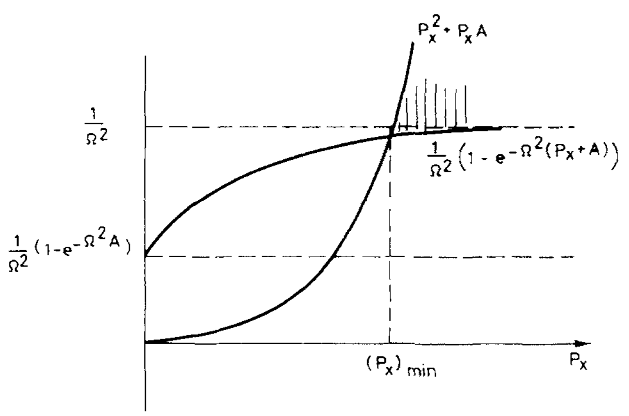

FIG. 2.

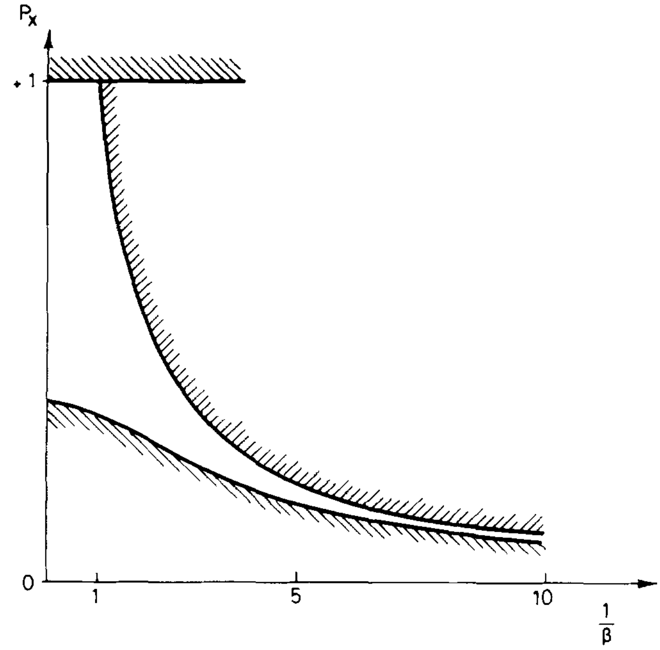

FIG. 3.

Let us first apply Corollary III. 2 with $y = \sigma_p^y$ and $x = \sigma_p^z$. Using Eqs. (9)–(12) one gets immediately

$$0 \leq -\omega(\sigma_p^x)/\Omega \leq 1. \tag{13}$$

This means that for any fixed temperature $\omega(\sigma_p^x)$ tends to zero, if the frequency Ω tends to zero. Or in other words the phase transition in the Ising models (i.e., $\Omega = 0$) is not due to a breaking of the symmetry along the x direction, which makes the model a quantum-mechanical one.

A lower bound for the spin polarization in the x direction at fixed temperature and fixed frequency Ω, can be found by applying the inequality of Theorem III. 5. Take $x = \sigma_p^y$, then again using (9)

$$-(1/\Omega)\omega(\sigma_p^x) \geq 1 - e^{-C}/C, \tag{14}$$

where $C = \Omega^2[-\omega(\sigma_p^x)/\Omega + A]$,

$$A = -\omega(\tau_p\sigma_p^z)/\Omega.$$

As now

$$(H\sigma_p^x, H\sigma_p^x) = -4\omega(\tau_p\sigma_p^z) \geq 0,$$

it follows that $A \geq 0$.

Denote

$$P_x = -\omega(\sigma_p^x)/\Omega,$$

then (14) becomes

$$P_x^2 + AP_x \geq \frac{1}{\Omega^2}\{1 - \exp[-\Omega^2(P_x + A)]\},$$

yielding a minimum value $(P_x)_{\min}$ for the polarization P_x as is shown in Fig. 2.

It is easy to reintroduce the inverse temperature β in the formulas and then Fig. 3 represents a numerical calculation of the bounds (13) and (14) as a function of the temperature. The quantity A in (14) has been majorized as follows:

$$A \leq \sum_{k \in Z^\nu} |J(k)|.$$

*Postal address: Institute for Theoretical Physics, K. U. L., Celestijnenlaan 200 D, B-3030 Heverlee, Belgium.

[1] G. Roepstorff, "Correlation inequalities in Quantum Statistical Mechanics and their Application in the Kondo Problem," preprint.
[2] R. Haag, N. M. Hugenholtz, and M. Winnink, Commun. Math. Phys. 5, 215 (1967).
[3] J. Naudts, A. Verbeure, and R. Weder, Commun. Math. Phys. **44**, 87 (1975).
[4] M. Takesaki, "Tomita's Theory of Modular Hilbert Algebras and its applications," (Lecture Notes in Mathematics, Springer, 1970), near p. 128.
[5] J. Ginibre, Commun. Math. Phys. 14, 205 (1969).
[6] R. Blinc, J. Phys. Chem. Solids 13, 204 (1960); P. G. de Gennes, Solid State Commun. 1, 132 (1963).
[7] D. W. Robinson, Commun. Math. Phys. **7**, 337 (1968).