AIP | Journal of Mathematical Physics

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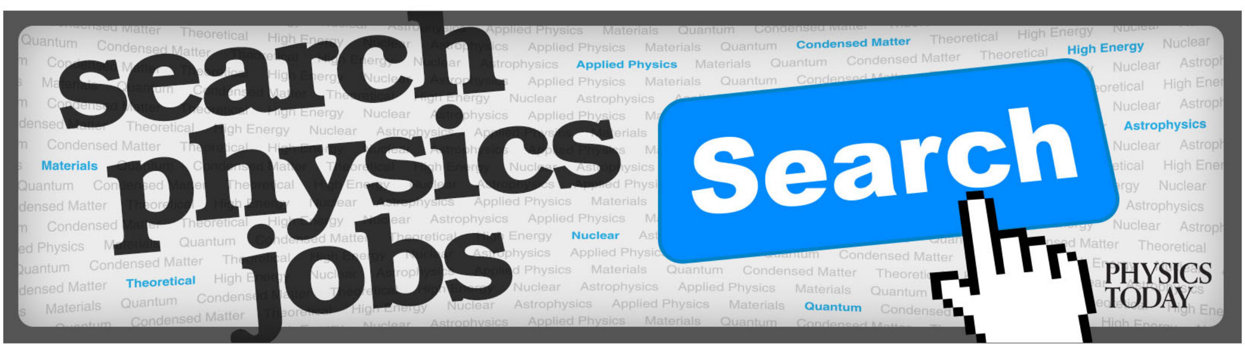
Citation: Journal of Mathematical Physics **26**, 1834 (1985); doi: 10.1063/1.526899
View online: http://dx.doi.org/10.1063/1.526899
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/26/7?ver=pdfcov
Published by the AIP Publishing





# Supertwistor fiber bundles as a formalism for supergravities

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(Received 13 April 1984; accepted for publication 15 March 1985)

A gauge theory of supergravity which allows one to obtain first-order Lagrangians that are locally gauge invariant by construction is presented. The formalism makes use of supertwistors as a representation space for the construction of a typical fiber of a vector bundle associated with a principal bundle, where the structural group is the super-Poincaré group. The approach proposed provides a means of resolving one of the central problems of gauge field theories of external symmetry groups, that is the satisfactory treatment of translations.

## I. INTRODUCTION

In earlier papers[1] we have discussed some of the essential difficulties which result from attempting to apply the Utimaya[2] procedure, appropriate for the gauging of internal symmetry groups, to external groups. In particular we analyzed the work of Kibble,[3] which enlarged Utimaya's formalism to include inhomogeneous Lorentz transformations, and found that some apparently well-established results could not be properly justified by means of such an approach.[4]

The same type of difficulties occur in several of the articles appearing in the current literature which utilize the above-mentioned procedures as a starting point to obtain gauge theories for the supersymmetric groups.[5]

The purpose of the present paper is to show how the twistor bundle formalism, which we developed previously[6] in order to construct gauge theories for noncompact groups correctly, can be extended to supersymmetric groups in a way which leads one to unambiguous supergravity theories.

We believe that through our formalism one of the central problems of gauge field theories that have the Poincaré group as a characteristic subgroup, which is the gauging of the translations in a satisfactory manner, has been adequately resolved.[7]

The essential features of our theory are the use of supertwistors[8] obtained by means of enlarging ordinary twistor space to a graded vector space given as an orthogonal direct sum of elements with even and odd Grassmann coefficients [one adjoins a one-dimensional ($N$-dimensional) complex vector space with odd Grassmann coefficients for simple (extended) supersymmetries]. Making use of these structures we can generate the required supertwistor bundles by a procedure based on an extension of our results in Ref. 6. Another important feature of our formalism is that the base space of our fiber bundles is a four-dimensional manifold which has no *a priori* additional structure imposed, but which acquires the metric structure of space-time as a consequence of the formalism. Thus, even though supertwistors are closely related to superspace in that the former may be regarded as the basic ingredients from which composite superspace is constructed,[9] the fibers in our theory are not based on superspace and superfields. Rather our approach relies on the formal point of view of considering supertwistor space as the fundamental complex five-dimensional [$(4+N)$-dimensional if the internal symmetry group $SU(N)$ is included] linear space representation of the graded group $SU(2,2/1)$ [or $SU(2,2/N)$ for extended supersymmetries]. This representation space serves to define a typical fiber from which our vector bundles are constructed.

Our procedure then leads to the calculation of a supertwistor curvature, in terms of which all possible Lagrangians permitted by the theory may be expressed as scalar functionals by means of appropriate contractions. These Lagrangians will be gauge invariant by construction; a feature of the theory which is made possible by the specific relations between the gravitino field and the Riemann and torsion tensors, imposed by the form of the supertwistor curvature.

It is interesting to note here that in the case of $N=0$ supertwistors, i.e., of ungraded standard twistor space, our formalism reduces to a gauge theory for ordinary gravitation (arrived at in a somewhat different, and perhaps simpler, manner than the one used in Ref. 6). This is the main reason for the particular choice of assignment of statistics in the construction of our supertwistors.

As a final remark, we point out that in this paper we have introduced the additional requirement that the null cone at infinity be part of the supertwistor structure. Thus we have broken superconformal invariance and retained only the group of super-Poincaré linear transformations which leave the infinity twistor invariant. Also, within this restriction we have only considered in detail the Lagrangian for simple supergravity, although this is only a particular case of the possible invariant scalars which are allowed by the theory.

The plan of the presentation is as follows: In Sec. II we give a very brief summary of the algebra of supertwistors which was developed in the immediately preceding paper (hereafter referred to as I). For details the reader is referred to that work. We also present in this section the essential arguments which lead, by incorporating these structures, to an extension of our formalism for a gauge theory of gravitation based on twistor bundles[6] to the construction of a gauge theory for the super-Poincaré group. The section also contains the basic aspects of supertwistor calculus which enable us to define supertwistor connections and arrive at a supertwistor curvature which is further related to the Riemann curvature, the torsion, and additional terms resulting from the gravitino field.

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Section III is dedicated to the construction of the Lagrangian density for simple supergravity, even though, as noted previously, this is only a particular case of the scalars that may be obtained from the more general functional form of the permissible invariant scalar Lagrangians.

We also outline in this section the procedure for obtaining field equations by a variational principle applied to the gauge fields, and show that in our formalism the fundamental quantities to be varied in an action principle are the supersymmetric connections. We further show how the variation of the supersymmetric connection may be split into three independent variations involving the gravitino field, the contorsion tensor, and one more variable, which is associated with the variation of the covariant gradient of the origin twist-tensor field. In Sec. IV we give concluding remarks and suggestions for further work.

At the end of the paper we include two appendices which contain five theorems where we prove several relations that are used throughout the text. In Appendix A, we derive a theorem on the product of skew-symmetric twistors and their duals, which considerably simplifies a number of algebraic manipulations that are performed with these types of twistors. In Appendix B we prove four theorems which serve to establish a natural map between the set of Dirac operators and the set of skew-symmetric twistors $J_i{}^{\alpha\beta} = D_i O^{\alpha\beta}$ (the covariant derivative of the origin twistor) at each point of the manifold. In addition, in the same appendix, we make use of this map to give a representation independent derivation of some properties of the Dirac gamma operators.

Finally some remarks about notation: Throughout the text we use Latin indices to denote space-time variables, while lowercase Greek indices will be reserved for twistors and capital Greek indices for supertwistors in order to follow, as close as possible, the notation conventionally utilized in twistor theory.

## II. SUPERTWISTOR SPACE AND BUNDLE STRUCTURES

### A. Supertwistors

In I we developed the concept of supertwistors by following a procedure somewhat different to that originally used by Ferber.[9] Here we summarize the essential aspects of the algebra of these structures which will be required throughout the paper.

We define a supertwistor as an element of the graded vector space given by the orthogonal direct sum

$$\mathscr{V}_{3,2} = \mathscr{V}_{2,2} \oplus \mathscr{V}_1 = (\mathscr{G}_e \otimes \mathscr{U}_{2,2}) \oplus (\mathscr{G}_0 \otimes \mathscr{W}_1), \qquad (2.1)$$

where $\mathscr{G}_e$ and $\mathscr{G}_0$ are the even and odd subsets, respectively, of a real Grassmann algebra $\mathscr{G}$ of dimension $2^d$ generated by a $d$-dimensional real vector space $\mathscr{K}_d$.

The subspace $\mathscr{V}_{2,2}$ is formally the same as ungraded twistor space, however supertwistors in $\mathscr{V}_{2,2}$ will be represented by the triple $(\omega^A, \pi_{A'}, 0)$, where the spinor components $\omega^A, \pi_{A'}$ are now complexified even elements of the Grassmann algebra $\mathscr{G}$. A supertwistor in $\mathscr{V}_{2,2}$ will be denoted by $u^\Sigma = (u^\sigma, 0) \leftrightarrow (\omega^A, \pi_{A'}, 0)$.

Supertwistors in the subspace $\mathscr{V}_1$ are of the form

$$\xi^\Sigma = \xi \tau^\Sigma, \qquad (2.2)$$

where $\xi$ is a complexified odd element of the Grassmann algebra $\mathscr{G}$, and $\tau^\Sigma$ is a normalized basis in the one-dimensional complex vector space $\mathscr{W}_1$ such that

$$\bar{\tau}_\Sigma \tau^\Sigma = i, \quad \bar{\tau}_\Sigma u^\Sigma = \bar{u}_\Sigma \tau^\Sigma = 0, \quad \forall\, u^\Sigma \in \mathscr{V}_{2,2}. \qquad (2.3)$$

Throughout the discussion, we will leave the dimensionality of the space $\mathscr{K}_d$ undefined in order to allow for all possible supersymmetric theories with increasing degree of complexity as $d \to \infty$.

A supertwistor in $\mathscr{V}_{3,2}$ is written as $Z^\Sigma = u^\Sigma + \xi^\Sigma$, where $u^\Sigma \in \mathscr{V}_{2,2}$ and $\xi^\Sigma \in \mathscr{V}_1$. Furthermore, since the subspaces $\mathscr{V}_{2,2}$ and $\mathscr{V}_1$ are assumed to be orthogonal, a Hermitian inner product in $\mathscr{V}_{3,2}$ may be defined as

$$\langle Z | W \rangle = \bar{Z}_\Sigma W^\Sigma = \bar{u}_\Sigma v^\Sigma + \bar{\xi}_\Sigma \theta^\Sigma, \qquad (2.4)$$

where $W^\Sigma = v^\Sigma + \theta^\Sigma$, and $\bar{Z}_\Sigma = \bar{u}_\Sigma + \bar{\xi}_\Sigma$ is the supertwistor conjugate to $Z^\Sigma$ that has values in the dual space $\mathscr{V}'_{3,2}$.

The anticommutativity of odd Grassmann numbers implies

$$\begin{aligned} \bar{Z}_\Sigma W^\Sigma &= \bar{u}_\Sigma v^\Sigma + \bar{\xi}_\Sigma \theta^\Sigma = v^\Sigma \bar{u}_\Sigma - \theta^\Sigma \bar{\xi}_\Sigma \\ &= \bar{u}_\sigma v^\sigma + i\xi^* \theta = v^\sigma \bar{u}_\sigma - i\theta\xi^*. \end{aligned} \qquad (2.5)$$

As pointed out in I, the graded group SU(2,2/1) is the set of linear transformations which leave (2.5) invariant.

We can now construct, in analogy to what is done with ordinary twistors, the tensor space $\mathscr{V}_{2,2}^{\wedge 2}$, $\mathscr{V}_{2,2}^{\wedge 4}$, $\mathscr{V}_{2,2}'^{\wedge 2}$, $\mathscr{V}_{2,2}'^{\wedge 4}$, $\mathscr{V}_{2,2} \otimes \mathscr{V}'_{2,2}$, etc. The only difference is that these spaces are now regarded as even-graded subspaces of the larger supertwistor spaces $\mathscr{V}_{3,2}^{\wedge 2}$, $\mathscr{V}_{3,2}^{\wedge 4}$, $\mathscr{V}_{3,2}'^{\wedge 2}$, $\mathscr{V}_{3,2}'^{\wedge 4}$, $\mathscr{V}_{3,2} \otimes \mathscr{V}'_{3,2}$, etc., respectively.

In particular we will also have here the special elements: (a) the totally antisymmetric supertwistor $\in \mathscr{V}_{2,2}^{\wedge 4}$,

$$\eta^{\Sigma\Gamma\Delta\Lambda} = \begin{cases} \eta^{\sigma\gamma\delta\lambda}, & \text{for } \Sigma,\Gamma,\Delta,\Lambda \text{ all different and values ranging from 0 to 3,} \\ 0, & \text{for any of the indices equal to 4;} \end{cases} \qquad (2.6)$$

(b) the vertex of the null cone at infinity, infinity supertwistor, or metric supertwistor $\in \mathscr{V}_{2,2}^{\wedge 2}$,

$$I^{\Sigma\Gamma} = \left[\begin{array}{c|c} I^{\sigma\gamma} & 0 \\ \hline 0 & 0 \end{array}\right]; \qquad (2.7)$$

and (c) the origin supertwistor $\in \mathscr{V}_{2,2}^{\wedge 2}$,

$$O^{\Sigma\Gamma} = \left[\begin{array}{c|c} O^{\sigma\gamma} & 0 \\ \hline 0 & 0 \end{array}\right]. \qquad (2.8)$$

All these special quantities are chosen from the corresponding homogeneous subspaces of degree zero.

The properties of these supertwistors are, up to a Grassmann factor, the same as those for the corresponding ordinary twistors, and so is the property of certain elements of $\mathscr{V}_{2,2}^{\wedge 2}$ being real elements.

### B. Gauge theory for the supergroup

The essential aspects of the philosophy that we will adopt for the construction of a gauge theory of the supergroup SU(2,2/1) [or SU(2,2/$N$) if internal symmetries are included] are fundamentally the same as those described in Ref. 6. For details we refer the reader to that paper. Here we will restrict ourselves to stressing the points of difference between the formalisms.

First, in place of $\mathscr{U}_{2,2}$ we start with $\mathscr{V}_{3,2}$ as the typical fiber of our twistor bundle.

If we also introduce $I^{\Sigma\Gamma}$ as part of the structure of $\mathscr{V}_{3,2}$, superconformal invariance will be broken and we will have a faithful representation of the super-Poincaré group.

Now, beginning with the four-dimensional base manifold $\mathscr{M}$, we can construct the bundles $\mathscr{V}_{3,2}(\mathscr{M})$, $\mathscr{V}^{\wedge 2}_{3,2}(\mathscr{M})$, $\mathscr{V}^{\wedge 4}_{3,2}(\mathscr{M})$ and $\mathscr{V}^{\wedge 5}_{3,2}(\mathscr{M})$. The cross section $I^{\Sigma\Gamma} \in\Gamma(\mathscr{M},\mathscr{V}^{\wedge 2}_{2,2}(\mathscr{M}))\subset\Gamma(\mathscr{M},\mathscr{V}^{\wedge 2}_{3,2}(\mathscr{M}))$ is taken as part of the structure of $\mathscr{V}_{3,2}(\mathscr{M})$ to give the bundle $(\mathscr{V}_{3,2},\ I^{\Sigma\Gamma})$ which from here on will be simply denoted by $\mathscr{V}_{3,2}(\mathscr{M})$.

At each point $q\in\mathscr{M}$, $(\mathscr{V}_{3,2})_q$ is the fiber above $q$ and $(\mathscr{V}_{2,2})_q$ is a subspace of $(\mathscr{V}_{3,2})_q$.

As in the case of the Poincaré group,[1,6] no metric structure is initially assumed for the manifold $\mathscr{M}$. It is by the selection of an origin twistor field $O^{\Sigma\Gamma}$ that it becomes possible to define a map which leads to a unique way of imposing a metric structure and connection on the tangent bundle $\mathscr{T}(\mathscr{M})$.

### C. Supertwistor connections

Let $D^S_X$ be a connection on the bundle $\mathscr{V}_{3,2}(\mathscr{M})$, in which $\mathscr{V}_{3,2}$ is a typical fiber. We have that $D^S_X$ satisfies the usual axioms of an arbitrary connection and is compatible with the inner product (2.5), i.e.,

$$X(\overline{Z}_\Sigma W^\Sigma)=(D^S_X\overline{Z}_\Sigma)\,W^\Sigma+\overline{Z}_\Sigma\,D^S_X W^\Sigma. \tag{2.9}$$

Moreover, $D^S_X$ preserves the structure of the typical fiber $(\mathscr{V}_{3,2},I^{\Sigma\Gamma})$ so we also have

$$D^S_X\,I^{\Sigma\Gamma}=0. \tag{2.10}$$

If we now recall that two connections can differ only by a transformation linear in the even Grassmann coefficients, we can express the supersymmetric connection as

$$D^S_X Z^\Sigma=D_X Z^\Sigma+B_x{}^\Sigma{}_\Gamma Z^\Gamma, \tag{2.11}$$

where $Z^\Sigma(q)\in(\mathscr{V}_{3,2})_q$, i.e., $Z^\Sigma$ is a cross section of the $\mathscr{V}_{3,2}(\mathscr{M})$ bundle, $D_X$ is the twistor connection which leaves invariant the subspace $\mathscr{V}_{2,2}$ on which it acts, and $B(q)$ is a tensor field with values in $\mathscr{T}'_q\otimes(\mathscr{V}_{3,2})_q\otimes(\mathscr{V}'_{3,2})_q$ whose components are $B_x{}^\Sigma{}_\Gamma$ and which represents the action of the connection due to the supertranslations.

The connection $D_x$ has the properties of being compatible with the inner product in $\mathscr{V}_{2,2}(\mathscr{M})$ and satisfies the conditions

$$D_X\eta^{\Sigma\Gamma\Delta\Lambda}=0, \tag{2.12}$$

$$D_X I^{\Sigma\Gamma}=0. \tag{2.13}$$

As an additional property, we shall require that

$$D_X\tau^\Sigma=0. \tag{2.14}$$

Note that Eqs. (2.11)–(2.14) serve to determine completely the action of $D_X$ on the bundle $\mathscr{V}_{3,2}(\mathscr{M})$. In particular, it can be readily shown that $D_X$ is also compatible with the inner product in $\mathscr{V}_{3,2}(\mathscr{M})$.

Now let us consider in detail the action of the supertranslations in (2.11). These can be given explicitly as the most general linear combination of the generators of supertranslations as derived in I [see Eq. (3.9)]. Thus we have

$$B_x{}^\Sigma{}_\Gamma=\psi_x{}^\Sigma\bar{\tau}_\Gamma-\tau^\Sigma\bar{\psi}_{x\Gamma}, \tag{2.15}$$

where the supertwistor $\psi_x{}^\Sigma$ is represented by the triple

$$\psi_x{}^\Sigma=(\psi_x{}^\sigma,0)\leftrightarrow(\rho_x{}^A+i\sigma_x{}^A,0,0), \tag{2.16}$$

and $\rho_x{}^A$, $\sigma_x{}^A$ are two-component real spinors which are odd in the Grassmann algebra $\mathscr{G}$.

### D. Lie and exterior derivative connections

In addition to the connections defined so far we will also need the concepts of Lie and exterior derivative connections. This construction will turn out to be most convenient in our later discussion of supertwistor curvatures.

To this end, let us first define a torsionless connection $D^0_X$ on the bundle $\mathscr{V}_{2,2}(\mathscr{M})$ by

$$D^0_X u^\Sigma=D_X u^\Sigma+(D_X O^{\Sigma\Gamma})I_{\Gamma\Lambda}u^\Lambda,\quad u^\Lambda(q)\in(\mathscr{V}_{2,2})_q. \tag{2.17}$$

Note that this connection has the following properties [which follow directly from (2.16)]: (a) for $u^\Sigma(q)\in\mathscr{G}_e\otimes(\mathscr{S}_I)_q$,

$$D^0_X u^\Sigma=D_X u^\Sigma, \tag{2.18}$$

(b) for $u^\Sigma(q)\in\mathscr{G}_e\otimes(\mathscr{S}_O)_q$,

$$\begin{aligned}D^0_X u^\Sigma&=D_X\left[(S_O)^\Sigma{}_\Gamma u^\Gamma\right]+(D_X O^{\Sigma\Gamma})I_{\Gamma\Lambda}u^\Lambda\\&=D_X(O^{\Sigma\Gamma}I_{\Lambda\Gamma}u^\Lambda)+(D_X O^{\Sigma\Gamma})I_{\Gamma\Lambda}u^\Lambda\\&=O^{\Sigma\Gamma}I_{\Lambda\Gamma}D_X u^\Lambda=(S_O)^\Sigma{}_\Lambda D_X u^\Lambda,\end{aligned} \tag{2.19}$$

where

$$(S_O)^\Sigma{}_\Gamma=O^{\Sigma\Lambda}I_{\Gamma\Lambda} \tag{2.20}$$

is the extension of the idempotent defined by (2.13) in I, i.e., the projection of $D_X u^\Sigma$ onto $\mathscr{G}_e\otimes(\mathscr{S}_O)_q$ gives $D^0_X u^\Sigma$. Also

$$\text{(c)}\quad D^0_X\,I^{\Sigma\Gamma}=0, \tag{2.21}$$

$$\text{(d)}\quad D^0_X\,O^{\Sigma\Gamma}=0, \tag{2.22}$$

and

$$\text{(e)}\quad[(D^0_X u)^c]^\Sigma=(D^0_X u^c)^\Sigma, \tag{2.23}$$

where $(u^c)^\Sigma$ is the charge conjugation of $u^\Sigma(q)$, i.e.,

$$(u^c)^\Sigma=\bar{u}_\Gamma(I^{\Gamma\Sigma}-O^{\Gamma\Sigma}). \tag{2.24}$$

[Compare to the definition given by Eq. (B56) in Appendix B.]

Recall that in Ref. 6, we defined the torsion tensor on the fiber as the action of the curvature tensor on the origin twistor [see Eq. (3.42) there]. Consequently, if we use the connection $D^0_X$ in particular, we have, by virtue of (2.22),

$$(\mathbf{T}^0_{\mathscr{T}})^{\Sigma\Gamma}(\mathbf{x},\mathbf{y})=(D^0_X\,D^0_Y-D^0_Y\,D^0_X-D^0_{[X,Y]})O^{\Sigma\Gamma}=0, \tag{2.25}$$

i.e., $D^0_X$ is indeed torsionless as asserted previously.

We can now introduce an operator $\mathscr{L}_X$, the Lie derivative connection, as a combined Lie derivative and twistor connection acting on fields of differential $p$ forms ($p=0,1,2,\ldots$),

$$u^\Sigma \in (\mathscr{V}_{2,2})_q\,, \quad \psi^\Sigma \equiv \psi_i{}^\Sigma\, dx^i \in \mathscr{T}'_q \otimes (\mathscr{V}_{2,2})_q\,,$$
$$\Phi^\Sigma \equiv \tfrac{1}{2}\, \Phi_{[i,j]}{}^\Sigma\, dx^i \wedge dx^j \in \mathscr{T}'^{\wedge 2}_q \otimes (\mathscr{V}_{2,2})_q\,,$$

etc. The action of $\mathscr{L}_X$ can be defined as

$$\mathscr{L}_X u^\Sigma = D^0_X u^\Sigma\,, \tag{2.26}$$

$$\begin{aligned}(\mathscr{L}_X \psi^\Sigma)(\mathbf{y}) &= \mathscr{L}_X[\psi^\Sigma(\mathbf{y})] - \psi^\Sigma(\mathscr{L}_X\, \mathbf{y}) \\ &= D^0_X[\psi^\Sigma(\mathbf{y})] - \psi^\Sigma([\mathbf{x},\mathbf{y}])\,,\end{aligned} \tag{2.27}$$

$$\begin{aligned}(\mathscr{L}_X \Phi^\Sigma)(\mathbf{y},\mathbf{z}) &= \mathscr{L}_X[\Phi^\Sigma(\mathbf{y},\mathbf{z})] - \Phi^\Sigma(\mathscr{L}_X\, \mathbf{y},\mathbf{z}) \\ &\quad - \Phi^\Sigma(\mathbf{y},\mathscr{L}_X\, \mathbf{z})\,, \text{ etc.,}\end{aligned} \tag{2.28}$$

where $\psi^\Sigma(\mathbf{y}) \equiv y^i\psi_i{}^\Sigma$, $\Phi^\Sigma(\mathbf{y},\mathbf{z}) = y^i z^j \Phi_{[ij]}{}^\Sigma$, and $\mathscr{L}_X u^\Sigma$, $\mathscr{L}_X \psi^\Sigma$, $\mathscr{L}_X \Phi^\Sigma$, at $q$ have values in $(\mathscr{V}_{2,2})_q, \mathscr{T}'_q \otimes (\mathscr{V}_{2,2})_q$, $\mathscr{T}'^{\wedge 2}_q \otimes (\mathscr{V}_{2,2})_q$, respectively.

An exterior derivative-connection $\mathscr{D}$, is a combined exterior derivative and twistor connection, defined on fields of $p$ forms $u^\Sigma, \psi^\Sigma, \Phi^\Sigma$, etc., where $\mathscr{D} u^\Sigma, \mathscr{D} \wedge \psi^\Sigma, \mathscr{D} \wedge \Phi^\Sigma$, etc., at $q$ are in $\mathscr{T}'_q \otimes (\mathscr{V}_{2,2})_q$, $\mathscr{T}'^{\wedge 2}_q \otimes (\mathscr{V}_{2,2})_q$, $\mathscr{T}'^{\wedge 3}_q \otimes (\mathscr{V}_{2,2})_q$, etc., respectively.

The action of $\mathscr{D}$ can also be defined by induction as

$$(\mathscr{D} u^\Sigma)(x) = D^0_X u^\Sigma\,, \tag{2.29}$$

$$(\mathscr{D} \wedge \psi^\Sigma)(\mathbf{x},\mathbf{y}) = (\mathscr{L}_X \psi^\Sigma)(\mathbf{y}) - (\mathscr{D}[\psi^\Sigma(\mathbf{x})])(\mathbf{y})\,, \tag{2.30}$$

$$\begin{aligned}(\mathscr{D} \wedge \Phi^\Sigma)(\mathbf{x},\mathbf{y},\mathbf{z}) &= (\mathscr{L}_X \Phi^\Sigma)(\mathbf{y},\mathbf{z}) \\ &\quad - (\mathscr{D} \wedge [\Phi^\Sigma(\mathbf{x})])(\mathbf{y},\mathbf{z}), \text{ etc.,}\end{aligned} \tag{2.31}$$

where $\Phi^\Sigma(\mathbf{x}) = x^i \Phi_{[ij]}{}^\Sigma\, dx^j$.

From (2.27), (2.29), and (2.30) we get, in addition, the useful result

$$D^0_X[\psi^\Sigma(\mathbf{y})] - D^0_Y[\psi^\Sigma(\mathbf{x})] - \psi^\Sigma([\mathbf{x},\mathbf{y}]) = (\mathscr{D} \wedge \psi^\Sigma)(\mathbf{x},\mathbf{y})\,. \tag{2.32}$$

Note that although the above definitions are given for supertwistors constructed from the subspace $\mathscr{V}_{2,2}$, they can be readily generalized to supertwistors and their tensors derived from $\mathscr{V}_{3,2}$ by making use of (2.14) and (2.17).

### E. Supertwistor curvature

We define the supertwistor curvature tensor $\mathfrak{S}^S$, with value at $q$ in $\mathscr{T}'_q \otimes \mathscr{T}'_q \otimes (\mathscr{V}_{3,2})_q \otimes (\mathscr{V}'_{3,2})_q$, by means of the expression

$$(\mathfrak{S}^S)^\Sigma{}_\Gamma(\mathbf{x},\mathbf{y}) Z^\Gamma = (D^S_X D^S_Y - D^S_Y D^S_X - D^S_{[X,Y]}) Z^\Sigma\,, \tag{2.33}$$

where $Z^\Sigma$ is a supertwistor field with $Z^\Sigma(q) \in (\mathscr{V}_{3,2})_q$.

Moreover, making use of (2.1)–(2.3) and of (2.14) and (2.15), one readily obtains

$$\begin{aligned}&(D^S_X D^S_Y - D^S_Y D^S_X - D^S_{[X,Y]}) Z^\Sigma \\ &\quad = (D_X D_Y - D_Y D_X - D_{[X,Y]})\, Z^\Sigma \\ &\qquad - \tau^\Sigma \{D_X[\bar\psi_\Gamma(\mathbf{y})] - D_Y[\bar\psi_\Gamma(\mathbf{x})] - \bar\psi_\Gamma([\mathbf{x},\mathbf{y}])\}\, Z^\Gamma \\ &\qquad + \{D_X[\psi^\Sigma(\mathbf{y})] - D_Y[\psi^\Sigma(\mathbf{x})] - \psi^\Sigma([\mathbf{x},\mathbf{y}])\}\, \bar\tau_\Gamma\, Z^\Gamma \\ &\qquad - i[\psi^\Sigma(\mathbf{x})\bar\psi_\Gamma(\mathbf{y}) - \psi^\Sigma(\mathbf{y})\bar\psi_\Gamma(\mathbf{x})]\, Z^\Gamma.\end{aligned} \tag{2.34}$$

Now substituting (2.32) into (2.34), and using the definition in (2.33) (and a similar one for the twistor curvature in $\mathscr{T}' \otimes \mathscr{T}' \otimes \mathscr{V}_{2,2} \otimes \mathscr{V}'_{2,2}$), yields

$$\begin{aligned}&(\mathfrak{S}^S)^\Sigma{}_\Gamma(\mathbf{x},\mathbf{y}) \\ &\quad = (\mathfrak{S})^\Sigma{}_\Gamma(\mathbf{x},\mathbf{y}) - \tau^\Sigma(\mathscr{D} \wedge \bar\psi_\Gamma)(\mathbf{x},\mathbf{y}) + (\mathscr{D} \wedge \psi^\Sigma)(\mathbf{x},\mathbf{y})\, \bar\tau_\Gamma \\ &\qquad - i\,[\psi^\Sigma(\mathbf{x})\bar\psi_\Gamma(\mathbf{y}) - \psi^\Sigma(\mathbf{y})\bar\psi_\Gamma(\mathbf{x})]\,.\end{aligned} \tag{2.35}$$

In order to relate the twistor curvature tensor $\mathfrak{S}$ appearing on the right-hand side of Eq. (2.35) to the Riemann curvature tensor we recall that [cf. Eq. (3.36) in Ref. 6]

$$\begin{aligned}&\mathbf{R}_{\mathscr{G}}(\mathbf{x},\mathbf{y})(u^\Sigma v^\Gamma - v^\Sigma u^\Gamma) \\ &\quad = (D_X D_Y - D_Y D_X - D_{[X,Y]})(u^\Sigma v^\Gamma - v^\Sigma u^\Gamma)\,,\end{aligned} \tag{2.36}$$

where $u^\Sigma, v^\Gamma$ are taken here to have values in $(\mathscr{V}_{2,2})_q$ in general and $\mathbf{R}_{\mathscr{G}}$ has values in $\mathscr{T}'_q \otimes \mathscr{T}'_q \otimes (\mathscr{V}^{\wedge 2}_{2,2})_q \otimes (\mathscr{V}'^{\wedge 2}_{2,2})_q$.

Observe now that

$$\begin{aligned}&(D_X D_Y - D_Y D_X - D_{[X,Y]})(u^\Sigma v^\Gamma - v^\Sigma u^\Gamma) \\ &\quad = [(D_X D_Y - D_Y D_X - D_{[X,Y]}) u^\Sigma]\, v^\Gamma \\ &\qquad + u^\Sigma\, [(D_X D_Y - D_Y D_X - D_{[X,Y]}) v^\Gamma] \\ &\qquad - [(D_X D_Y - D_Y D_X - D_{[X,Y]}) v^\Sigma]\, u^\Gamma \\ &\qquad - v^\Sigma\, [(D_X D_Y - D_Y D_X - D_{[X,Y]})\, u^\Gamma]\,,\end{aligned}$$

i.e.,

$$\begin{aligned}&\mathbf{R}_{\mathscr{G}}(\mathbf{x},\mathbf{y})(u^\Sigma v^\Gamma - v^\Sigma u^\Gamma) \\ &\quad = [(\mathfrak{S})^\Sigma{}_\Lambda(\mathbf{x},\mathbf{y}) u^\Lambda]\, v^\Gamma + u^\Sigma\, [(\mathfrak{S})^\Gamma{}_\Lambda(\mathbf{x},\mathbf{y}) v^\Lambda] \\ &\qquad - [(\mathfrak{S})^\Sigma{}_\Lambda(\mathbf{x},\mathbf{y}) v^\Lambda]\, u^\Gamma - v^\Sigma\, [(\mathfrak{S})^\Gamma{}_\Lambda(\mathbf{x},\mathbf{y}) u^\Lambda]\,.\end{aligned} \tag{2.37}$$

Moreover, if we use the symbol

$$E^\Gamma{}_\Pi \equiv (S_I)^\Gamma{}_\Pi + (S_O)^\Gamma{}_\Pi \in \mathscr{V}_{2,2} \otimes \mathscr{V}'_{2,2} \tag{2.38}$$

to represent the identity twistor acting on this subspace, we then have

$$\begin{aligned}&\mathbf{R}_{\mathscr{G}}(\mathbf{x},\mathbf{y})(u^\Sigma v^\Gamma - v^\Sigma u^\Gamma) \\ &\quad = [(\mathfrak{S})^\Sigma{}_\Lambda(\mathbf{x},\mathbf{y}) E^\Gamma{}_\Pi + E^\Sigma{}_\Lambda (\mathfrak{S})^\Gamma{}_\Pi(\mathbf{x},\mathbf{y})](u^\Lambda v^\Pi - v^\Lambda u^\Pi)\end{aligned} \tag{2.39a}$$

and

$$\begin{aligned}&(\mathbf{R}_{\mathscr{G}})^{\Sigma\Gamma\Omega\Phi}(\mathbf{x},\mathbf{y})\eta_{\Omega\Phi\Lambda\Pi} \\ &\quad = (\mathfrak{S})^\Sigma{}_{[\Lambda}(\mathbf{x},\mathbf{y}) E^\Gamma{}_{\Pi]} + E^\Sigma{}_{[\Lambda} (\mathfrak{S})^\Gamma{}_{\Pi]}(\mathbf{x},\mathbf{y})\,.\end{aligned} \tag{2.39b}$$

Putting the indices $\Lambda = \Gamma$ in (2.39b) leads to

$$(\mathbf{R}_{\mathscr{G}})^{\Sigma\Lambda\Omega\Phi}(\mathbf{x},\mathbf{y})\eta_{\Omega\Phi\Lambda\Pi} = -2\,\mathfrak{S}^\Sigma{}_\Pi(\mathbf{x},\mathbf{y}) - E^\Sigma{}_\Pi\, \mathfrak{S}^\Lambda{}_\Lambda(\mathbf{x},\mathbf{y})\,. \tag{2.40}$$

Equation (2.40) is our desired relation between the Riemann curvature and twistor curvature tensors. This expression, however, can be put in more familiar terms by explicitly accounting for torsion. We thus have [cf. Eq. (3.47) in Ref. 6]

$$\begin{aligned}&(\mathbf{R}_{\mathscr{G}})^{\Sigma\Lambda\Omega\Phi}(\mathbf{x},\mathbf{y}) \\ &\quad = (\mathbf{R}_{\mathscr{F}})^{\Sigma\Lambda\Omega\Phi}(\mathbf{x},\mathbf{y}) + \tfrac{1}{2}[(\mathbf{T}_{\mathscr{F}})^{\Sigma\Lambda}(\mathbf{x},\mathbf{y}) I^{\Omega\Phi} \\ &\qquad - I^{\Sigma\Lambda}(\mathbf{T}_{\mathscr{F}})^{\Omega\Phi}(\mathbf{x},\mathbf{y})]\,,\end{aligned} \tag{2.41}$$

where $\mathbf{R}_{\mathscr{F}}$ is the curvature tensor which has its values in $\mathscr{G}_e \otimes \mathscr{T}'_q \otimes \mathscr{T}'_q \otimes \mathscr{F}_q \otimes \mathscr{F}_q$; $\mathscr{F}_q$ is the subspace of real twistors orthogonal both to $I^{\Sigma\Gamma}$ and $O^{\Sigma\Gamma}$, and having a Minkowski inner product with signature $(+\,-\,-\,-)$, and $\mathbf{T}_{\mathscr{F}}$ is the torsion tensor with values in $\mathscr{G}_e \otimes \mathscr{T}'_q \otimes \mathscr{T}'_q \otimes \mathscr{F}_q$.

The curvature $\mathbf{R}_{\mathscr{F}}$ can be projected in turn onto the tangent bundle to yield the tensor $\mathbf{R}_{\mathscr{T}}$, with values in $\mathscr{G}_e \otimes \mathscr{T}'_q \otimes \mathscr{T}'_q \otimes \mathscr{T}_q \otimes \mathscr{T}_q$, by making use of the field

$$J_i^{\Sigma\Gamma} \equiv D_i O^{\Sigma\Gamma}, \tag{2.42}$$

which acts as a bijective map[6] of $\mathscr{T}_q$ on the subspace $\mathscr{G}_e \otimes \mathscr{F}_q \subset (\mathscr{V}^{\wedge 2}_{2,2})_q$. Thus, introducing an holonomic basis $\{\mathbf{e}_i\}$ in the tangent bundle and the dual basis $\{\mathbf{e}^i\}$ in the cotangent bundle, we get

$$(\mathbf{R}_{\mathscr{F}})^{\Sigma\Lambda\Omega\Phi}(\mathbf{e}_i,\mathbf{e}_j) = R_{ij}{}^{kl} J_k{}^{\Sigma\Lambda} J_l{}^{\Lambda\Phi}. \tag{2.43}$$

Similarly $\mathbf{T}_{\mathscr{F}}$ is related to the torsion tensor $\mathbf{T}_{\mathscr{T}}$ in the tangent bundle, with values in $\mathscr{G}_e \otimes \mathscr{T}'_q \otimes \mathscr{T}'_q \otimes \mathscr{T}_q$, by

$$(\mathbf{T}_{\mathscr{F}})^{\Sigma\Lambda}(\mathbf{e}_i,\mathbf{e}_j) = T_{ij}{}^k J_k{}^{\Sigma\Lambda}. \tag{2.44}$$

Substituting (2.43) and (2.44) into (2.41), results in

$$(\mathbf{R}_{\mathscr{E}})^{\Sigma\Lambda\Omega\Phi}(\mathbf{e}_i,\mathbf{e}_j) = R_{ij}{}^{kl} J_k{}^{\Sigma\Lambda} J_l{}^{\Omega\Phi} + \tfrac{1}{2} T_{ij}{}^k (J_k{}^{\Sigma\Lambda} I^{\Omega\Phi} - I^{\Sigma\Lambda} J_k{}^{\Omega\Phi}). \tag{2.45}$$

Replacing now the left side of (2.41) by (2.45), yields

$$2R_{ij}{}^{kl} J_k{}^{\Sigma\Lambda} J_{l\Lambda\Pi} + T_{ij}{}^k (J_k{}^{\Sigma\Lambda} I_{\Lambda\Pi} - I^{\Sigma\Lambda} J_{k\Lambda\Pi}) = -2\,\mathfrak{S}^{\Sigma}{}_{\Pi}(\mathbf{e}_i,\mathbf{e}_j) - E^{\Sigma}{}_{\Pi}\,\mathfrak{S}^{\Lambda}{}_{\Lambda}(\mathbf{e}_i,\mathbf{e}_j). \tag{2.46}$$

The value of $\mathfrak{S}^{\Lambda}{}_{\Lambda}(\mathbf{e}_i,\mathbf{e}_j)$ in the last term of (2.46) can be obtained by means of one additional contraction. We find

$$2R_{ij}{}^{kl} g_{kl} + T_{ij}{}^k (J_k{}^{\Sigma\Lambda} I_{\Sigma\Lambda} - I^{\Sigma\Lambda} J_{k\Sigma\Lambda}) = 6\,\mathfrak{S}^{\Lambda}{}_{\Lambda}(\mathbf{e}_i,\mathbf{e}_j), \tag{2.47}$$

where we have made use of the fact that

$$J_k{}^{\Sigma\Lambda} J_{l\Sigma\Lambda} = g_{kl}$$

[see (3.5a) below].

Moreover, since $J_k{}^{\Sigma\Lambda}$ and $I^{\Sigma\Lambda}$ are real twistors we also have

$$J_k{}^{\Sigma\Lambda} I_{\Sigma\Lambda} = J_{k\Sigma\Lambda}\, I^{\Sigma\Lambda} = 0,$$

since $J_k{}^{\Sigma\Lambda} \in \mathscr{F}$.

It follows readily that

$$\mathfrak{S}^{\Lambda}{}_{\Lambda}(\mathbf{e}_i,\mathbf{e}_j) = 0, \tag{2.48}$$

and (2.46) reduces to

$$\mathfrak{S}^{\Sigma}{}_{\Pi}(\mathbf{e}_i,\mathbf{e}_j) = -R_{ij}{}^{kl} J^{\Sigma\Lambda}_k J_{l\Lambda\Pi} + T_{ij}{}^k I^{\Sigma\Lambda} J_{k\Lambda\Pi}. \tag{2.49}$$

In arriving at (2.49) we have made use of

$$I^{\Sigma\Lambda} J_{k\Lambda\Pi} = -J_k{}^{\Sigma\Lambda} I_{\Lambda\Pi}, \tag{2.50}$$

which results directly from the identity (A1), derived in Appendix A.

Finally, as a consequence of (2.49) the expression of the supertwistor curvature tensor given in (2.35) takes the form

$$(\mathfrak{S}^S)_{ij}{}^{\Sigma}{}_{\Gamma} \equiv (\mathfrak{S}^S)^{\Sigma}{}_{\Pi}(\mathbf{e}_i,\mathbf{e}_j) = -R_{ij}{}^{kl} J_k{}^{\Sigma\Lambda} J_{l\Lambda\Pi} + T_{ij}{}^k I^{\Sigma\Lambda} J_{k\Lambda\Pi} - \tau^{\Sigma} \mathscr{D}_{[i} \bar{\psi}_{j]\Pi} + \mathscr{D}_{[i} \psi^{\Sigma}_{j]} \bar{\tau}_{\Pi} - i \psi_{[i}{}^{\Sigma} \bar{\psi}_{j]\Pi}, \tag{2.51}$$

where

$$(\mathscr{D} \wedge \boldsymbol{\psi}^{\Sigma})_{ij} = (\mathscr{D} \wedge \boldsymbol{\psi}^{\Sigma})(\mathbf{e}_i,\mathbf{e}_j) = \mathscr{D}_{[i} \psi^{\Sigma}_{j]} \tag{2.52}$$

and

$$\psi_i{}^{\Sigma} = \boldsymbol{\psi}^{\Sigma}(\mathbf{e}_i), \quad \bar{\psi}_{j\Sigma} = \bar{\boldsymbol{\psi}}_{\Sigma}(\mathbf{e}_j).$$

## III. SUPERGRAVITY LAGRANGIANS AND ACTION PRINCIPLES

Making use of the results in the preceding section we can now set up Lagrangian densities $\mathscr{L}[\mathfrak{S}^S(\mathscr{M})]$ for the gauge fields (i.e., the connection $D^S_X$) as scalar functionals of $\mathfrak{S}^S(\mathscr{M})$, which by construction will be locally super-Poincaré-invariant. Note that by virtue of Eq. (2.51) these Lagrangian densities will determine the allowed functional form of these Lagrangians in terms of the quantities $R_{ij}{}^{kl}$, $T_{ij}{}^k$, $\mathscr{D}_{[i}\psi_{j]}{}^{\Sigma}$, $\mathscr{D}_{[i}\bar{\psi}^{\Sigma}_{j]}$, $\psi_i{}^{\Sigma}$, and $\bar{\psi}_{i\Sigma}$. Note also that the supertwistor curvature has two files in the contangent bundle and the third and fourth files in $\mathscr{V}_{3,2}(\mathscr{M})$ and $\mathscr{V}'_{3,2}(\mathscr{M})$, respectively, i.e., in the fiber space above the base manifold. Thus, in order to be able to construct scalars[10] from this quantity we must first project (2.51) so that the resulting tensor has all its files in the same space or its dual. We can do this most directly by following a procedure analogous to that designed for such a purpose in Ref. 6 and which involves using the action of the connection on the origin twistor to construct a unique map from the tangent bundle to the supertwistor bundle.

We digress briefly in order to review the essential properties of this $J^S$ map which is explicitly defined by

$$x^i \to x^i (J^S)_i{}^{\Sigma\Gamma} \equiv D^S_X O^{\Sigma\Gamma}. \tag{3.1}$$

Moreover, making use of (2.14) and (2.15) it can be shown that

$$(J^S)_i{}^{\Sigma\Gamma} = D_i O^{\Sigma\Gamma} + B_i{}^{\Sigma}{}_{\Lambda} O^{\Lambda\Gamma} - B_i{}^{\Gamma}{}_{\Lambda} O^{\Lambda\Sigma} = J_i{}^{\Sigma\Gamma} - \tau^{\Sigma} \bar{\psi}_{i\Lambda} O^{\Lambda\Gamma} + \tau^{\Gamma} \bar{\psi}_{i\Lambda} O^{\Lambda\Sigma}. \tag{3.2}$$

Observe that $(\bar{J}^S)_{i\Sigma\Gamma}$ and $(J^S)_i{}^{\Sigma\Gamma}$ can also be used to map fields $Y^{\Sigma\Gamma}$ and $\bar{Z}_{\Sigma\Gamma}$ with values at $q$ in $(\mathscr{V}^{\wedge 2}_{3,2})_q$ and $(\mathscr{V}'^{\wedge 2}_{3,2})_q$, respectively, into fields $y_i$ and $z_i$ with values at $q$ in $\mathscr{G}_e \otimes \mathscr{T}'_q$ according to

$$y_i = (\bar{J}^S)_{i\Sigma\Gamma} Y^{\Sigma\Gamma}, \tag{3.3a}$$

$$z_i = (J^S)_i{}^{\Sigma\Gamma} \bar{Z}_{\Sigma\Gamma}. \tag{3.3b}$$

A similar procedure, based on induction from Eqs. (3.3), can be used for the mapping of tensors constructed from the spaces $\mathscr{V}^{\wedge 2}_{3,2}$ and $\mathscr{V}'^{\wedge 2}_{3,2}$ into cotangent bundle tensors.

Consider, in particular, the tensor $\frac{1}{2}\bar{\eta}_{\Sigma\Lambda\Phi\Pi}$ which acts as a metric tensor on $(\mathscr{V}^{\wedge 2}_{2,2})_q$ to give the inner product

$$L_{\Sigma\Lambda}\, M^{\Sigma\Lambda} = L^{\Sigma\Lambda} M_{\Sigma\Lambda}, \quad L^{\Sigma\Lambda}, \quad M^{\Sigma\Lambda} \in \mathscr{V}^{\wedge 2}_{2,2} \tag{3.4}$$

[cf. Eq. (2.6) in I].

Mapping $\frac{1}{2}\bar{\eta}_{\Sigma\Lambda\Phi\Pi}$ with $(J^S)_i{}^{\Sigma\Gamma}$, and making use of (3.2), yields

$$G_{ij} = \tfrac{1}{2} (J^S)_i{}^{\Sigma\Lambda} \bar{\eta}_{\Sigma\Lambda\Phi\Pi} (J^S)_j{}^{\Phi\Pi} = \tfrac{1}{2} J_i{}^{\Sigma\Lambda} \bar{\eta}_{\Sigma\Lambda\Phi\Pi} J_j{}^{\Phi\Pi} = J_i{}^{\Sigma\Lambda} J_{j\Sigma\Lambda} = J_{i\Sigma\Lambda} J_j{}^{\Sigma\Lambda}, \tag{3.5}$$

where $\mathbf{G}$ is valued in $\mathscr{G}_e \otimes \mathscr{T}'_q \otimes \mathscr{T}'_q$.

The tensor $\mathbf{G}$ serves to impose the metric structure given in the fibers on the tangent space $\mathscr{T}_q$, i.e., we can define an inner product in the tangent bundle by

$$\mathbf{x} \cdot \mathbf{y} = x^i G_{ij}\, y^j. \tag{3.6}$$

In terms of components in an holonomic basis, (3.5) and (3.6) read

$$g_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j = G_{ij} = J_{i\Sigma\Lambda} J_j{}^{\Sigma\Lambda}, \tag{3.5'}$$

$$\mathbf{x}\cdot\mathbf{y}=x^iy^jg_{ij}\,. \tag{3.6'}$$

We return now to the main objective of this section, which is the construction of scalar invariants from the supertwistor curvature tensor. In what follows we shall concentrate on contractions of (2.51) with (3.2) which lead to the Lagrangian for simple supergravity. The extension of the procedure to other permissible Lagrangians, and to Lagrangians for extended supergravities, is suggested by the approach here adopted and is rather straightforward.

First we transvect (2.51) with $(J^S)_m{}^{\Pi\Gamma}$, to get

$$\begin{aligned}(\mathfrak{S}^S)_{ij}{}^{\Sigma}{}_{\Pi}(J^S)_m{}^{\Pi\Gamma}&=-R_{ij}{}^{kl}J_k{}^{\Sigma\Lambda}J_{l\Lambda\Pi}\,J_m{}^{\Pi\Gamma}+R_{ij}{}^{kl}J_k{}^{\Sigma\Phi}J_{l\Phi\Pi}O^{\Pi\Lambda}\bar{\psi}_{m\Lambda}\tau^{\Gamma}\\&\quad+T_{ij}{}^kI^{\Sigma\Lambda}J_{k\Lambda\Pi}\,J_m{}^{\Pi\Gamma}-T_{ij}{}^kI^{\Sigma\Phi}J_{k\Phi\Pi}O^{\Pi\Lambda}\bar{\psi}_{m\Lambda}\tau^{\Gamma}\\&\quad-\tau^{\Sigma}(\mathscr{D}_{[i}\bar{\psi}_{j]\Pi})\,J_m{}^{\Pi\Gamma}+\tau^{\Sigma}\tau^{\Gamma}(\mathscr{D}_{[i}\bar{\psi}_{j]\Pi})O^{\Pi\Lambda}\bar{\psi}_{m\Lambda}\\&\quad+i\,(\mathscr{D}_{[i}\psi_{j]}{}^{\Sigma})O^{\Gamma\Pi}\bar{\psi}_{m\Pi}-i\psi_{[i}{}^{\Sigma}\bar{\psi}_{j]\Pi}\,J_m{}^{\Pi\Gamma}\,.\end{aligned}\tag{3.7}$$

The above equation still has two supertwistor indices and three indices in the contangent bundle free. It would seem natural that the next operation should then be a double contraction with $(\bar{J}^S)_{n\Sigma\Gamma}$. However, if we do this, all the important dynamical information contained in (3.7) relating to the gravitational and the $\psi_i{}^{\Sigma}$ fields will be lost.

To circumvent this problem we transvect (3.7) first on the right with $(\tilde{\gamma}_5)_{\Gamma}{}^{\Xi}$, where $(\tilde{\gamma}_5)_{\Gamma}{}^{\Xi}=(\gamma_5)^{\Xi}{}_{\Gamma}$ is the transpose of the Dirac gamma operator defined by [see Eq. (B51a) in Appendix B]

$$(\gamma_5)^{\Xi}{}_{\Gamma}=-i[(S_I)^{\Xi}{}_{\Gamma}-(S_O)^{\Xi}{}_{\Gamma}]\,, \tag{3.8}$$

and

$$(\gamma_5)^{\Sigma}{}_{\Gamma}(\gamma_5)^{\Gamma}{}_{\Lambda}=-E^{\Sigma}{}_{\Lambda}\,, \tag{3.9}$$

and contract the result from the left with $(\bar{J}^S)_{n\Sigma\Xi}$.

We therefore get

$$\begin{aligned}(\bar{J}^S)_{n\Sigma\Xi}(\mathfrak{S}^S)_{ij}{}^{\Sigma}{}_{\Pi}(J^S)_m{}^{\Pi\Gamma}(\tilde{\gamma}_5)_{\Gamma}{}^{\Xi}&=-R_{ij}{}^{kl}J_k{}^{\sigma\lambda}J_{l\lambda\alpha}\,J_m{}^{\alpha\beta}(\tilde{\gamma}_5)_{\beta}{}^{\xi}J_{n\sigma\xi}\\&\quad+T_{ij}{}^kI^{\sigma\alpha}J_{k\alpha\beta}\,J_m{}^{\beta\lambda}(\tilde{\gamma}_5)_{\lambda}{}^{\xi}J_{n\sigma\xi}\\&\quad+i(\mathscr{D}_{[i}\bar{\psi}_{j]\sigma})\,J_m{}^{\sigma\alpha}(\tilde{\gamma}_5)_{\alpha}{}^{\beta}O_{\beta\lambda}\psi_n{}^{\lambda}\\&\quad+i(\mathscr{D}_{[i}\psi_{j]}{}^{\sigma})\,J_{n\sigma\alpha}(\gamma_5)^{\alpha}{}_{\beta}O^{\beta\lambda}\bar{\psi}_{m\lambda}\\&\quad+i\psi_{[i}{}^{\sigma}J_{|n\sigma\lambda}(\gamma_5)^{\lambda}{}_{\alpha}\,J_{m|}{}^{\alpha\beta}\bar{\psi}_{j]\beta}\,.\end{aligned}\tag{3.10}$$

Note that in writing the right side of (3.10), we have made explicit use of the fact that this expression has no free supertwistor indices left and that the components in $\mathscr{V}_1$ and $\mathscr{V}'_1$ have been contracted out completely. Thus all supertwistors in the right of (3.10) are valued in $\mathscr{V}_{2,2}$, $\mathscr{V}'_{2,2}$ or are tensor constructed from these spaces. Consequently, by virtue of our previous definition (cf. Sec. II), we can use lowercase Greek indices (running from 0 to 3) to denote these supertwistors.

Also note that because of the spaces in which $\psi_i{}^{\sigma}$, $J_j{}^{\sigma\alpha}$, and $(\gamma_5)^{\alpha}{}_{\beta}$ are situated, the last term in (3.10) drops out. It is important to point out here that this cancellation has nothing to do with the fact that the coefficients of the fields $\psi_i{}^{\sigma}$ are Grassmann variables.

Furthermore, making use of the relation between the $J$'s and the Dirac gammas [see Appendix B, Eqs. (B61) and (B62)], we get

$$J_i{}^{\alpha\beta}=\tfrac{1}{2}(\gamma_i)^{\alpha}{}_{\lambda}(\gamma_5)^{\lambda}{}_{\kappa}C^{\kappa\beta}\,, \tag{3.11a}$$

$$J_{i\alpha\beta}=-\tfrac{1}{2}C_{\alpha\lambda}(\gamma_i)^{\lambda}{}_{\kappa}(\gamma_5)^{\kappa}{}_{\beta}\,, \tag{3.11b}$$

where

$$C^{\alpha\beta}=O^{\alpha\beta}-I^{\alpha\beta}\,,\quad C^{\alpha\beta}C_{\beta\kappa}=E^{\beta}{}_{\kappa}\,, \tag{3.12}$$

we have

$$\begin{aligned}-J_k{}^{\sigma\lambda}J_{l\lambda\alpha}J_m{}^{\alpha\beta}(\tilde{\gamma}_5)_{\beta}{}^{\xi}J_{n\sigma\xi}&=-\tfrac{1}{16}(\gamma_k)^{\sigma}{}_{\lambda}(\gamma_l)^{\lambda}{}_{\alpha}(\gamma_m)^{\alpha}{}_{\beta}(\gamma_n)^{\beta}{}_{\xi}(\gamma_5)^{\xi}{}_{\sigma}\\&=-\tfrac{1}{16}\,\mathrm{tr}(\gamma_k\gamma_l\gamma_m\gamma_n\gamma_5)\\&=-[1/16(4!)]\,\mathrm{tr}(\gamma_{[k}\gamma_l\gamma_m\gamma_{n]}\gamma_5)\\&=-[1/16(4!)^2]\epsilon_{klmn}\epsilon^{qrst}\,\mathrm{tr}(\gamma_{[q}\gamma_r\gamma_s\gamma_{t]}\gamma_5)\,.\end{aligned}\tag{3.13}$$

[In (3.13) we have deleted the tensor indices in the operations with the Dirac gammas, since multiplication of the mixed tensors $(\gamma_i)^{\sigma}{}_{\lambda}$ is just ordinary matrix algebra.]

Now, by virtue of (B43), (B46), and (B47) in Appendix B, we have

$$\begin{aligned}\frac{1}{(4!)^2}\epsilon^{qrst}\,\mathrm{tr}(\gamma_{[q}\gamma_r\gamma_s\gamma_{t]}\gamma_5)&=\sqrt{-g}\,[\epsilon^{qrst}/(4!)^2\sqrt{-g}\,]\,\mathrm{tr}(\gamma_{[q}\gamma_r\gamma_s\gamma_{t]}\gamma_5)\\&=\sqrt{-g}\,\mathrm{tr}(\gamma_5\gamma_5)=-4\sqrt{-g}\,.\end{aligned}\tag{3.14}$$

Hence,

$$-J_k{}^{\sigma\lambda}\,J_{l\lambda\alpha}\,J_m{}^{\alpha\beta}(\tilde{\gamma}_5)_{\beta}{}^{\xi}J_{n\sigma\xi}=\tfrac{1}{4}\sqrt{-g}\,\epsilon_{klmn}\,. \tag{3.15}$$

By analogous arguments, we find that

$$\begin{aligned}-I^{\sigma\alpha}J_{k\alpha\beta}+J_m{}^{\beta\lambda}(\tilde{\gamma}_5)_{\lambda}{}^{\xi}J_{n\sigma\xi}&=-\tfrac{1}{8}[\,I^{\sigma\alpha}O_{\alpha\beta}(\gamma_k)^{\beta}{}_{\lambda}(\gamma_m)^{\lambda}{}_{\kappa}(\gamma_n)^{\kappa}{}_{\sigma}\,]=0\,.\end{aligned}\tag{3.16}$$

Consequently, the second term in (3.10), which contains the torsion explicitly, also vanishes.

Taking into account (3.8), (3.9), and (3.11)–(3.16), Eq. (3.10) reduces to

$$\begin{aligned}(\bar{J}^S)_{n\Sigma\Xi}(\mathfrak{S}^S)_{ij}{}^{\Sigma}{}_{\Pi}(J^S)_m{}^{\Pi\Gamma}(\tilde{\gamma}_5)_{\Gamma}{}^{\Xi}&=+\tfrac{1}{4}\sqrt{-g}\,\epsilon_{klmn}R_{ij}{}^{kl}\\&\quad-(i/2)(\mathscr{D}_{[i}\bar{\psi}_{j]\sigma})(\gamma_m)^{\sigma}{}_{\lambda}\psi_n{}^{\lambda}\\&\quad-(i/2)(\mathscr{D}_{[i}\psi_{j]}^{\sigma})(\tilde{\gamma}_n)_{\sigma}{}^{\lambda}\bar{\psi}_{m\lambda}\,,\end{aligned}\tag{3.17}$$

where we have also used the relations [see Eqs. (B28) and (B29) in Appendix B]

$$(\tilde{\gamma}_i)_{\sigma}{}^{\lambda}=-C_{\sigma\alpha}(\gamma_i)^{\alpha}{}_{\beta}C^{\beta\lambda}\,, \tag{3.18}$$

$$(\tilde{\gamma}_5)_{\sigma}{}^{\lambda}=C_{\sigma\alpha}(\gamma_5)^{\alpha}{}_{\beta}C^{\beta\lambda}\,. \tag{3.19}$$

Note that in (3.17) all twistor indices are already contracted. Thus, in order to get a scalar density Lagrangian, we have only to contract on the vector indices in the contangent bundle. This can be readily accomplished by multiplication with the tensor element of volume $d\Omega^{ijmn}=d^4x\,\epsilon^{ijmn}$. The result is

$$\begin{aligned}\mathscr{L}\,d^4x&=d\Omega^{ijmn}(\bar{J}^S)_{n\Sigma\Xi}(\mathfrak{S}^S)_{ij}{}^{\Sigma}{}_{\Pi}(J^S)_m{}^{\Pi\Gamma}(\tilde{\gamma}_5)_{\Gamma}{}^{\Xi}\\&=\{\sqrt{-g}\,R_s+(i/2)\epsilon^{ijmn}[(\mathscr{D}(_{[i}\bar{\psi}_{j]\sigma})(\gamma_m)^{\sigma}{}_{\lambda}\psi_n{}^{\lambda}\\&\quad+(\mathscr{D}_{[i}\psi_{j]}^{\sigma})(\tilde{\gamma}_n)_{\sigma}{}^{\lambda}\bar{\psi}_{m\lambda}\,]\}\,d^4x\,,\end{aligned}\tag{3.20}$$

where $R_s$ is the Ricci scalar $R_s \equiv R_{ij}{}^{ij}$.

Using the fact that the ordinary complex conjugate of a scalar formed from supertwistors is obtained by taking the conjugate of each of the constituent twistors [cf. Eq. (2.39), and (2.40) in I], and also using the properties of the Dirac gammas, it can be easily checked that the Lagrangian density given by (3.20) is real, as required.

We can compare (3.20) with the usual form given in the literature[11] in terms of Majorana spinors, by noting that

$$(\psi^c)_i{}^\sigma = C^{\sigma\alpha}\bar{\psi}_{i\alpha} = (O^{\sigma\alpha} - I^{\sigma\alpha})\bar{\psi}_{i\alpha}\ , \tag{3.21a}$$

$$\psi_i{}^\sigma = (\psi^{cc})_i{}^\sigma\ , \tag{3.21b}$$

and that

$$(\psi^M)_i{}^\sigma \equiv \psi_i{}^\sigma + (\psi^c)_i{}^\sigma \tag{3.22}$$

is a Majorana spinor.

Also note that from (3.21a)

$$\psi_i{}^\sigma \in \mathscr{G}_o \otimes \mathscr{S}_I \rightarrow (\psi^c)_i{}^\sigma \in \mathscr{G}_0 \otimes \mathscr{S}_O\ ,$$

and

$$\psi_i{}^\sigma \in \mathscr{G}_o \otimes \mathscr{S}_O \rightarrow (\psi^c)_i{}^\sigma \in \mathscr{G}_o \otimes \mathscr{S}_I\ .$$

Therefore,

$$\begin{aligned}\mathscr{L}_g &\equiv \epsilon^{ijmn}[(\mathscr{D}_{[i}\bar{\psi}_{j]\sigma})(\gamma_m)^\sigma{}_\lambda \psi_n{}^\lambda + (\mathscr{D}_{[i}\psi_{j]}^\sigma)(\tilde{\gamma}_n)_\sigma{}^\lambda \bar{\psi}_{m\lambda}] \\ &= \epsilon^{ijmn}\{[(\mathscr{D}_{[i}(\psi^c)_{j]}^\sigma](\tilde{\gamma}_m)_\sigma{}^\beta(\bar{\psi}^c)_{n\beta} \\ &\quad - [\mathscr{D}_{[i}\psi_{j]}^\sigma](\tilde{\gamma}_m)_\sigma{}^\lambda \bar{\psi}_{n\lambda}\}\end{aligned} \tag{3.23}$$

after making use of (3.18) and taking into account that $\mathscr{D}_i C^{\alpha\beta} = 0$, because of Eqs. (2.21), (2.22), and (2.29).

But, since $\mathscr{D}_{[i}(\psi^c)_{j]}^\sigma \in \mathscr{G}_o \otimes \mathscr{S}_O$, and $(S_O)^\alpha{}_\beta = O^{\alpha\gamma}I_{\beta\gamma}$ is the projection operator onto $\mathscr{S}_O$, and $(S_I)^\alpha{}_\beta = I^{\alpha\gamma}O_{\beta\gamma}$ is the projection operator onto $\mathscr{S}_I$, we can write

$$\mathscr{D}_{[i}(\psi^c)_{j]}^\sigma = O^{\sigma\alpha}I_{\beta\alpha}\mathscr{D}_{[i}(\psi^c)_{j]}^\beta = [\mathscr{D}_{[i}(\psi^c)_{j]}^\beta]\, I_{\beta\alpha}\, O^{\sigma\alpha}\ , \tag{3.24}$$

$$\begin{aligned}&(S_I)^\sigma{}_\beta[\mathscr{D}_{[i}(\psi^c)_{j]}^\beta] \\ &\quad = 0 = I^{\sigma\alpha}O_{\beta\alpha}[\mathscr{D}_{[i}(\psi^c)_{j]}^\beta] = [\mathscr{D}_{[i}(\psi^c)_{j]}^\beta]O_{\beta\alpha}\, I^{\sigma\alpha}\ .\end{aligned} \tag{3.25}$$

Substracting (3.25) from (3.24) and utilizing (3.8), results in

$$\mathscr{D}_{[i}(\psi^c)_{j]}^\sigma = -i\,[\mathscr{D}_{[i}(\psi^c)_{j]}^\beta](\tilde{\gamma}_5)_\beta{}^\sigma\ . \tag{3.26}$$

By similar arguments we get

$$\mathscr{D}_{[i}\psi_{j]}^\sigma = i\,[\mathscr{D}_{[i}\psi_{j]}^\beta](\tilde{\gamma}_5)_\beta{}^\sigma\ . \tag{3.27}$$

Consequently, substituting (3.26) and (3.27) in (3.23) gives

$$\begin{aligned}\mathscr{L}_g = &-i\epsilon^{ijmn}\{[\mathscr{D}_{[i}(\psi^c)_{j]}^\sigma](\tilde{\gamma}_5)_\sigma{}^\lambda(\tilde{\gamma}_m)_\lambda{}^\beta(\bar{\psi}^c)_{n\beta} \\ &+ [\mathscr{D}_{[i}\psi_{j]}^\sigma](\tilde{\gamma}_5)_\sigma{}^\lambda(\tilde{\gamma}_m)_\lambda{}^\beta\bar{\psi}_{n\beta}\}\ .\end{aligned} \tag{3.28}$$

Furthermore, since

$$[\mathscr{D}_{[i}(\psi^c)_{j]}^\sigma](\tilde{\gamma}_5)_\sigma{}^\lambda(\tilde{\gamma}_m)_\lambda{}^\beta\bar{\psi}_{n\beta} = 0 \tag{3.29}$$

and

$$[\mathscr{D}_{[i}\psi_{j]}^\sigma](\tilde{\gamma}_5)_\sigma{}^\lambda(\tilde{\gamma}_m)_\lambda{}^\beta(\bar{\psi}^c)_{n\beta} = 0\ , \tag{3.30}$$

we can add these two noncontributing terms to (3.28) to get

$$\begin{aligned}\mathscr{L}_g &= -i\epsilon^{ijmn}\{[\mathscr{D}_{[i}(\psi^M)_{j]}^\sigma](\tilde{\gamma}_5)_\sigma{}^\lambda(\tilde{\gamma}_m)_\lambda{}^\beta(\bar{\psi}^M)_{n\beta}\} \\ &= i\epsilon^{ijmn}\{(\bar{\psi}^M)_{n\beta}(\gamma_m)^\beta{}_\lambda(\gamma_5)^\lambda{}_\sigma\mathscr{D}_{[i}(\psi^M)_{j]}^\sigma\} \\ &= -i\epsilon^{ijmn}\{(\bar{\psi}^M)_{n\beta}(\gamma_5)^\beta{}_\lambda(\gamma_m)^\lambda{}_\sigma\mathscr{D}_{[i}(\psi^M)_{j]}^\sigma\}\ .\end{aligned} \tag{3.31}$$

Finally, replacing this expression in (3.20), yields

$$\begin{aligned}\mathscr{L}\, d^4x = \{&\sqrt{-g}\, R_s - \tfrac{1}{2}\epsilon^{ijmn} \\ &\times[(\bar{\psi}^M)_{n\beta}(\gamma_5)^\beta{}_\lambda(\gamma_m)^\lambda{}_\sigma\mathscr{D}_{[i}(\psi^M)_{j]}^\sigma]\}\, d^4x\ .\end{aligned} \tag{3.32}$$

Converted to ordinary Dirac bispinors and matrices, Eq. (3.32) reads

$$\mathscr{L}\, d^4x = \{\sqrt{-g}\, R_s + (i/2)\,\epsilon^{ijmn}\bar{\psi}_n^M\gamma_5\gamma_m\mathscr{D}_{[i}\psi_{j]}^M\}\, d^4x\ , \tag{3.32$'$}$$

where $\psi_j{}^M$ is a one column Majorana bispinor and $\bar{\psi}_n{}^M$ is the adjoint Majorana bispinor defined, as usual, by $\bar{\psi}_n{}^M = (\psi_n^M)^\dagger\gamma^0$.

Aside from dimensionality factors, which can be readily made explicit, Eq. (3.32) is in agreement with the first order formulation of simple supergravity as it is usually presented in the literature.[5,11]

Now we will discuss variational principles. A few remarks concerning the derivation of field equations from our first-order Lagrangians seem to be in order. These equations of motion are obtained in our formalism by noting that the fundamental (gauge) quantities to be varied in an action principle are the connections $D_X^S$.

On the other hand, by virtue of (2.14) and (2.15), variation of $D_X^S$ is equivalent to independent variations of $D_X$ and the gravitino field $\psi_x{}^\sigma$.

Furthermore, since any two linear connections may differ only by a linear transformation, we have

$$(\delta D_X)z^\sigma = \delta M_x{}^\sigma{}_\gamma z^\gamma, \quad \forall z^\sigma(q) \in (\mathscr{V}_{2,2})_q\ , \tag{3.33}$$

where $\delta\mathbf{M}(q) \in \mathscr{T}'_q \otimes (\mathscr{V}_{2,2})_q \otimes (\mathscr{V}'_{2,2})_q$. In order to obtain additional insight into the structure of $\delta\mathbf{M}_x$, we note that

$$\begin{aligned}\delta[X(\bar{z}^\sigma w_\sigma)] &= 0 = [(\delta D_X)\bar{z}_\sigma]w^\sigma + \bar{z}_\sigma[(\delta D_X)w^\sigma] \\ &= \bar{z}_\sigma[\delta M_x{}^\sigma{}_\gamma + (\delta M_x^\dagger)^\sigma{}_\gamma]w^\gamma\ ,\end{aligned} \tag{3.34}$$

i.e., $\delta\mathbf{M}_x$ has to fulfill the same requirements as the infinitesimal generators of Poincaré transformations. Hence, the most general form of $\delta\mathbf{M}_x$ must be given as a linear combination of the generators of translations and Lorentz transformations.

Recalling the expressions for the generators of translations and Lorentz transformation given in Eqs. (3.12) and (3.16) of I, we can write in general

$$\delta M_x{}^\sigma{}_\gamma = \delta_1 M_x{}^\sigma{}_\gamma + \delta_2 M_x{}^\sigma{}_\gamma\ , \tag{3.35}$$

where

$$\delta_1\mathbf{M}(q) \in \mathscr{G}_e \otimes \mathscr{T}'_q \otimes [(\mathscr{S}_I)_q \otimes_{\rm aH} (\mathscr{S}_{\bar{I}})_q]\ , \tag{3.36}$$

$$\delta_2\mathbf{M}(q) \in \mathscr{G}_e \otimes \mathscr{T}'_q \otimes [(\mathscr{S}_I \otimes \mathscr{S}_{\bar{O}}) \otimes_{\rm aH} (\mathscr{S}_O \otimes \mathscr{S}_{\bar{I}})]_q\ , \tag{3.37}$$

and $\mathscr{S}_{\bar{I}}$ and $\mathscr{S}_{\bar{O}}$ are subspaces of $\mathscr{U}'_{2,2}$, conjugate to the Weyl spinor spaces $\mathscr{S}_I$ and $\mathscr{S}_O$, respectively. That is, the variation of $D_X$ acting on a twistor is made up of the two independent variations $\delta_1 M_x{}^\sigma{}_\gamma$ and $\delta_2 M_x{}^\sigma{}_\gamma$.

In addition, when $D_X$ acts on a real twistor $V^{\alpha\beta} \in \mathscr{G}_e \otimes \mathscr{E}$ we have

$$(\delta D_X)V^{\alpha\beta} = \delta B_x{}^{\alpha\beta}{}_{\gamma\delta} V^{\gamma\delta}, \tag{3.38}$$

where $\delta\mathbf{B}(q) \in \mathscr{G}_e \otimes \mathscr{T}'_q \otimes \mathscr{C}_q \otimes \mathscr{C}_q$.

Also, from the compatibility with the inner product we get

$$\delta[X(V^{\alpha\beta}W_{\alpha\beta})] = 0 = V^{\alpha\beta}[\delta B_{x\alpha\beta\gamma\delta} + \delta B_{x\gamma\delta\alpha\beta}]\, W^{\gamma\delta},$$

i.e,

$$\delta B_{x\alpha\beta\gamma\delta} = -\delta B_{x\gamma\delta\alpha\beta}. \tag{3.39}$$

It follows then from (3.39) that $\delta\mathbf{B}(q) \in \mathscr{G}_e \otimes \mathscr{T}'_q \otimes (\mathscr{C}_q \wedge \mathscr{C}_q)$. Moreover, from

$$(\delta D_X)I^{\alpha\beta} = 0 = \delta B_x{}^{\alpha\beta}{}_{\gamma\delta} I^{\gamma\delta}, \tag{3.40a}$$

$$\delta[X(I^{\alpha\beta}V_{\alpha\beta})] = 0 = I^{\alpha\beta}\delta B_{x\alpha\beta\gamma\delta} V^{\gamma\delta} \Rightarrow I^{\alpha\beta}\delta B_{x\alpha\beta\gamma\delta} = 0, \tag{3.40b}$$

$$2(D_X O_{\alpha\beta})O^{\alpha\beta} = \delta[X(O^{\alpha\beta}O_{\alpha\beta})] = 0$$
$$= 2O^{\alpha\beta}\delta B_{x\alpha\beta\gamma\delta} O^{\gamma\delta} = 2O^{\alpha\beta}\delta J_{x\alpha\beta}, \tag{3.40c}$$

it is easy to conclude that $\delta B_{i\alpha\beta\gamma\delta}$ has to be further restricted to the form

$$\delta B_{i\alpha\beta\gamma\delta} = \tfrac{1}{2}(\delta N)_i{}^j (J_{j\alpha\beta} I_{\gamma\delta} - I_{\alpha\beta} J_{j\gamma\delta}) + (\delta K)_i{}^{kl}(J_{k\alpha\beta} J_{l\gamma\delta}), \tag{3.41}$$

where

$$(\delta K)_i{}^{kl} = -(\delta K)_i{}^{lk}. \tag{3.42}$$

Equation (3.41), which was derived under the assumption that $D_X$ acted on real twistors, applies equally well to the variation of the connection acting on an arbitrary twistors valued in $\mathscr{V}^{\wedge 2}_{2,2}$, since the latter can always be expressed as a sum of elements in $\mathbb{C}\,\mathscr{C}$.

We can now relate the $\delta\mathbf{B}$ and $\delta\mathbf{M}$ variations by noting that

$$(\delta D_X)(u^\alpha v^\beta - v^\alpha u^\beta)$$
$$= \delta B_x{}^{\alpha\beta}{}_{\gamma\delta}(u^\gamma v^\delta - v^\gamma u^\delta)$$
$$= (\delta M_x{}^\alpha{}_\gamma E^\beta{}_\delta + E^\alpha{}_\gamma \delta M_x{}^\beta{}_\delta)(u^\gamma v^\delta - v^\gamma u^\delta). \tag{3.43}$$

Taking into account the antisymmetry of each of the tensor files in $\delta\mathbf{B}_x$, we can then write

$$\delta B_x{}^{\alpha\beta}{}_{\gamma\delta} = \tfrac{1}{4}(\delta M_{x[\gamma}^{[\alpha} E^{\beta]}_{\delta]} + E^{[\alpha}_{[\gamma}\delta M_{x\delta]}^{\beta]}). \tag{3.44}$$

Contracting (3.44) on the second and fourth tensor indices yields

$$\delta B_x{}^{\alpha\beta}{}_{\gamma\beta} = \delta M_x{}^\alpha{}_\gamma + \tfrac{1}{2}\delta M_x{}^\beta{}_\beta E^\alpha{}_\gamma, \tag{3.45}$$

where $\delta M_x{}^\beta{}_\beta$ is obtained by one additional contraction in (3.45), and is given by

$$\delta M_x{}^\alpha{}_\alpha = \tfrac{1}{3}\delta B_x{}^{\alpha\beta}{}_{\alpha\beta}. \tag{3.46}$$

Substituting the first term in (3.41) into (3.45) and (3.46) and using a holonomic basis results in

$$\delta_1 M_i{}^\alpha{}_\gamma = -\tfrac{1}{2}(\delta N)_i{}^j(J_j{}^{\alpha\beta} I_{\beta\gamma} - I^{\alpha\beta} J_{j\beta\gamma}). \tag{3.47}$$

Similarly, from the second term in (3.41) we get

$$\delta_2 M_i{}^\alpha{}_\gamma = -(\delta K)_i{}^{kl}(J_k{}^{\alpha\beta} J_{l\beta\gamma}). \tag{3.48}$$

Let us now consider the different quantities in (3.32) which are affected by these two independent variations.

First we have

$$\delta_1 J_i{}^{\alpha\beta} = (\delta_1 D_i)O^{\alpha\beta} = \delta_1 M_i{}^\alpha{}_\gamma O^{\gamma\beta} + \delta_1 M_i{}^\beta{}_\gamma O^{\alpha\gamma}$$
$$= -(\delta N)_i{}^j[O^{\alpha\gamma} I_{\gamma\lambda} J_j{}^{\lambda\beta} + I^{\alpha\gamma} O_{\gamma\lambda} J_j{}^{\lambda\beta}]$$
$$= (\delta N)_i{}^j[(S_O)^\alpha{}_\lambda + (S_I)^\alpha{}_\lambda]\, J_j{}^{\lambda\beta}$$
$$= (\delta N)_i{}^j J_j{}^{\alpha\beta}. \tag{3.49}$$

In analogy,

$$\delta_2 J_i{}^{\alpha\beta} = \delta_2 M_i{}^\alpha{}_\gamma O^{\gamma\beta} + \delta_2 M_i{}^\beta{}_\gamma O^{\alpha\gamma}$$
$$= -(\delta K)_i{}^{kl}(J_k{}^{\alpha\lambda} J_{l\lambda\gamma} O^{\gamma\beta} - J_k{}^{\beta\lambda} J_{l\lambda\gamma} O^{\gamma\alpha})$$
$$= -(\delta K)_i{}^{kl}(O^{\alpha\lambda} J_{k\lambda\gamma} J_l{}^{\gamma\beta} + O^{\alpha\gamma} J_{l\gamma\lambda} J_k{}^{\lambda\beta}) = 0 \tag{3.50}$$

[due to (3.42)].

In arriving at (3.49) and (3.50) we have used repeatedly the identity (A1) given in Appendix A.

Note that because of (3.11), the variation of the Dirac $\gamma_i$'s is determined by the variation of $J_i{}^{\alpha\beta}$. However, since $\gamma_5$ is defined according to (3.8), it will not vary.

Other quantities that need to be varied in (3.32) are the exterior derivative connections $\mathscr{D}$. To obtian an expression for $(\delta\mathscr{D}_{[i})\psi^\sigma_{j]}$, we first need to evaluate $\delta D^0_i$. This follows from (2.17)

$$(\delta D^0_X)u^\sigma = (\delta D_X)u^\sigma + \delta J_x{}^{\sigma\gamma} I_{\gamma\lambda} u^\lambda,$$

i.e.,

$$\delta\mathring{M}_x{}^\alpha{}_\gamma = \delta M_x{}^\alpha{}_\gamma + \delta J_x{}^{\alpha\lambda} I_{\gamma\lambda}. \tag{3.51}$$

From (3.47)–(3.50) we have, in particular,

$$\delta_1\mathring{M}_x{}^\alpha{}_\gamma = 0, \tag{3.52a}$$

and

$$\delta_2\mathring{M}_x{}^\alpha{}_\gamma = \delta_2 M_x{}^\alpha{}_\gamma. \tag{3.52b}$$

Consequently, making use of (2.32) we get

$$(\delta_1\mathscr{D}_{[i})\psi^\sigma_{j]} = (\delta_1\mathring{M}_{[i|}{}^\sigma{}_{\gamma|})\psi^\gamma_{j]} = 0, \tag{3.53}$$

$$(\delta_2\mathscr{D}_{[i})\psi^\sigma_{j]} = (\delta_2\mathring{M}_{[i|}{}^\sigma{}_{\gamma|})\psi^\gamma_{j]} = -(\delta K)_{[i}{}^{kl} J_{|k}{}^{\sigma\beta} J_{l|\beta\gamma}\psi^\gamma_{j]}. \tag{3.54}$$

Next consider

$$\delta g = g g^{ij}\delta g_{ij} = g g^{ij}\delta(J_{i\alpha\beta} J_j{}^{\alpha\beta}).$$

From (3.49) and (3.50), it follows immediately that

$$\delta_2 g = 0, \quad \delta_1 g = 2g g^{ij}(\delta N)_{ij}. \tag{3.55}$$

The last quantity that it is necessary to vary in (3.32) is the Ricci scalar $R_s$. Most of the work needed to obtain $\delta R_s$ has already been carried out in Ref. 6. Thus, we have only to substitute (3.41), (3.49), and (3.50) into the expression for $\delta R_s$ given by Eq. (4.30) in that paper. After some fairly straightforward calculations we get

$$\delta_1 R_s = -2(\delta_1 J_l{}^{\alpha\beta}) J_{m\alpha\beta} g^{mj}\delta^i_k R_{ij}{}^{kl}$$
$$= -2(\delta N)_l{}^n g_{nm} g^{mn} R_{ij}{}^{il}$$
$$= -2(\delta N)^{lj} R_{jl}, \tag{3.56}$$

$$\delta_2 R_s = [T_{kl}{}^i + 2T_{lm}{}^m\delta^i_k](\delta K)_i{}^{kl}, \tag{3.57}$$

where $R_{jl}$ is the nonsymmetric Ricci tensor and $T_{kl}{}^i$ is the torsion tensor in the tangent bundle.

Making use of (3.49), (3.50), and (3.53)–(3.57) in an action principle derived from (3.32) leads to the usual expressions for the field equations of simple supergravity.

As a final remark note that from the definition of the torsion

$$T_{ijk} = [(D_iD_j - D_jD_i)O^{\alpha\beta}]\, J_{k\alpha\beta}$$
$$= [D_i\, J_j{}^{\alpha\beta} - D_j\, J_i{}^{\alpha\beta}]\, J_{k\alpha\beta}\,, \tag{3.58}$$

we get

$$\delta_2 T_{ijk} = [\delta_2 B_i{}^{\alpha\beta}{}_{\gamma\delta}\, J_j{}^{\gamma\delta} - \delta_2 B_j{}^{\alpha\beta}{}_{\gamma\delta}\, J_i{}^{\gamma\delta}]\, J_{k\alpha\beta}$$
$$= [(\delta K)_{ikj} - (\delta K)_{jki}]\,. \tag{3.59}$$

Recalling (3.42) and the antisymmetry of $T_{ijk}$ in the first two indices, we can invert (3.59) to get

$$(\delta K)_{ijk} = \tfrac{1}{2}[\delta_2 T_{ikj} - \delta_2 T_{kji} - \delta_2 T_{ijk}]\,. \tag{3.60}$$

The term in the right of (3.60) may be recognized as the variation of the contorsion tensor. Thus, we see that the variation $\delta_2\mathbf{M}$ is equivalent to the variation of the contorsion in the tangent bundle.

## IV. SUMMARY AND CONCLUSIONS

We have proposed a formalism for the gauging of the super-Poincaré algebra which allows one to obtain first-order Lagrangians for supergravity. Our theory, which uses supertwistors as a representation space for the construction of a typical fiber, does not suffer from the conceptual problems which characterize some of the approaches followed in the literature and which are based on a direct attempt to extend to noncompact groups the Utimaya[2] procedure for gauging internal groups.

The above is made possible by means of the basic idea of treating the super-Poincaré group as an internal group. However, the theory differs from internal group theories in some significant aspects.

The first difference is that no metric structure or connection on the tangent bundle is assumed. In typical gauge theories the metric structure of the tangent bundle is given *a priori* together with a connection compatible with the metric. Nevertheless, a natural isomorphism can be achieved by selecting a given origin twistor field and introducing its covariant derivative $(J_i^S)^{\Sigma\Gamma} = D_i^S O^{\Sigma\Gamma}$, as a means to map structures originating in the fibers onto the tangent bundle, inducing in it a metric and connection [cf. Eqs. (3.5) and (3.6) in the text]. Furthermore, it may be shown that the selection of an origin twistor field imposes no special restriction on the theory.

Another way in which our procedure differs from the usual approach to internal gauge theories, is that there one starts with a principal bundle on which connections are defined. Gauge covariant differentiation is then specified on the associated vector bundle of this principal bundle. In our development we start with fiber bundles which may be regarded as vector bundles associated to a principal $G$ bundle, where the structural group $G$ is the super-Poincaré group. Note, however, that we can pass from our vector bundles to their associated frame bundles and vice versa by changing the space on which the transition functions act from the vector space to the group manifold and back.[12]

We contend also that our approach provides a means of resolving one of the central problems of gauge field theories of external symmetry groups, that is the satisfactory treatment of translations.

Another important feature of our theory is that the Lagrangian density is obtained as a functional of the supertwistor curvature. Such a curvature is constructed basically from spinorial entities, as opposed to the Riemann curvature tensor which is built up from four-vectors.

However, the supertwistor curvature may be related to the Riemann tensor, the torsion, and the gravitino field by means of Eq. (2.51) which we derived in Sec. II. The important consequence of this is that (2.51) determines completely the allowed functional form of the Lagrangian in terms of the gravitino field and the Riemann and torsion tensors in the tangent bundle. Thus our theory is gauge invariant by construction and provides a solid structure from which one can investigate all possible specific Lagrangians relating these fields in a gauge invariant fashion.

In addition, note that the covariant derivative $\mathscr{D}_i$, which is required to occur in supergravity defined according to its spin content only, is introduced in our theory in a straightforward manner through the exterior derivative connections defined in Eqs. (2.29)–(2.32) of Sec. II, and appears naturally in our supertwistor curvature tensor.

Equally natural is the incorporation of the concept of Majorana spinors in our formalism. Note, however, that both the Majorana spinors for the gravitino field and the Dirac gammas that we used are completely general, i.e., we do not introduce any specific choice of representation for them.

We remark, finally, that by enlarging the dimensionality of our supertwistor spaces extended supergravity theories can be readily accommodated within our formalism.


## ACKNOWLEDGMENTS

This work was supported in part by International Scientific Program Grants, National Science Foundation No. OIP75-09783A01 and Consejo Nacional de Ciencia y Tecnología No. PCAIEUA-790316.


## APPENDIX A: A TWISTOR IDENTITY

A useful tensor identity involving contractions of skew symmetric twistors is proved here. This identity is applied repeatedly throughout the text.

*Theorem:* For $A^{\mu\nu}, B^{\mu\nu} \in \mathscr{U}^{\wedge 2}_{2,2}$,

$$A^{\mu\lambda}B_{\lambda\nu} + B^{\mu\lambda}A_{\lambda\nu} = -\tfrac{1}{2}A^{\alpha\beta}B_{\alpha\beta}\delta^{\mu}_{\nu}\,. \tag{A1}$$

*Proof:* Let

$$M^{\mu}{}_{\nu\alpha\beta\gamma\lambda} = \delta^{\mu}_{\nu}\bar{\eta}_{\alpha\beta\gamma\lambda} - \delta^{\mu}_{\alpha}\bar{\eta}_{\nu\beta\gamma\lambda} - \delta^{\mu}_{\beta}\bar{\eta}_{\alpha\nu\gamma\lambda}$$
$$- \delta^{\mu}_{\gamma}\bar{\eta}_{\alpha\beta\nu\lambda} - \delta^{\mu}_{\lambda}\bar{\eta}_{\alpha\beta\gamma\nu}\,, \tag{A2}$$

where $\eta^{\alpha\beta\gamma\lambda}$ is the antisymmetric tensor in $\mathscr{U}^{\wedge 4}_{2,2}$ normalized as

$$\eta^{\alpha\beta\gamma\lambda}\bar{\eta}_{\alpha\beta\gamma\lambda} = 4!$$

and is used for constructing duals. Since $\mathscr{U}_{2,2}$ is four dimensional and $M^{\mu}{}_{\nu\alpha\beta\gamma\lambda}$ is antisymmetric in its five subscripts, it follows that

$$M^{\mu}{}_{\nu\alpha\beta\gamma\lambda} = 0 . \tag{A3}$$

By (A3) we have

$$0 = A^{\alpha\beta} B^{\gamma\lambda} M^{\mu}{}_{\nu\alpha\beta\gamma\lambda}$$
$$= 2A^{\alpha\beta} B_{\alpha\beta} \delta^{\mu}_{\nu} + 4A^{\mu\lambda} B_{\lambda\nu} + 4B^{\mu\lambda} A_{\lambda\nu} ,$$

and (A1) follows immediately from this.

## APPENDIX B: DIRAC GAMMA OPERATORS VIA TWISTOR THEORY

The space of Dirac $\gamma$ operators is related by a one-to-one correspondence to Minkowski space. We shall give here an explicit construction of this relationship by means of a map between two twistor spaces. The first space, the domain of the map, is isomorphic to Minkowski space, and it is the twistor space $\mathscr{F} \equiv \mathscr{W}_O$ defined in the preceding paper. The second space, the range of the map, is isomorphic to the space of Dirac $\gamma$ operators, and is a real subspace of the space of linear transformations $\mathscr{U}_{2,2} \otimes \mathscr{U}'_{2,2}$ on the space $\mathscr{U} \equiv \mathscr{U}_{2,2} \equiv (\mathscr{U}_{2,2}, \eta^{\alpha\beta\gamma\delta}, I^{\alpha\beta}, O^{\alpha\beta})$ defined in Ref. 6. Note that the space $\mathscr{U}_{2,2}$ is the twistor space, but the privileged elements $\eta^{\alpha\beta\gamma\delta}$, $I^{\alpha\beta}$, $O^{\alpha\beta}$ forming part of its structure make it isomorphic to the Dirac bispinor space.

The map which establishes the above-mentioned correspondence is the bijection $T^{\alpha\beta} \in \mathscr{U}^{\otimes 2} \rightarrow V^{\alpha}{}_{\beta} \in \mathscr{U} \otimes \mathscr{U}'$ given as

$$V^{\alpha}{}_{\beta} = L(T^{\alpha\beta}) = 2iT^{\alpha\gamma}(I_{\gamma\beta} + O_{\gamma\beta}) . \tag{B1}$$

The inverse map $V^{\alpha}{}_{\beta} \in \mathscr{U} \otimes \mathscr{U}' \rightarrow T^{\alpha\beta} \in \mathscr{U}^{\otimes 2}$ is

$$T^{\alpha\beta} = L^{-1}(V^{\alpha}{}_{\beta}) = \tfrac{1}{2} iV^{\alpha}{}_{\gamma}(I^{\gamma\beta} + O^{\gamma\beta}) . \tag{B2}$$

We now investigate some properties of this map. We shall define subspaces $\mathscr{J}$, $\mathscr{J}_5$, $\mathscr{J}_4$, $\mathscr{E}_{1,4}$, $\mathscr{E}_{1,3}$ of $\mathscr{U}^{\otimes 2}$, and subspaces $\mathscr{H}$, $\mathscr{H}_5$, $\mathscr{H}_4$, $\mathscr{D}_{1,4}$, $\mathscr{D}_{1,3}$ of $\mathscr{U} \otimes \mathscr{U}'$, which are the respective images of the above $\mathscr{U}^{\otimes 2}$ subspaces under the map $L$.

First define the subspaces $\mathscr{J} \subset \mathscr{U}^{\otimes 2}$ and $\mathscr{H} \subset \mathscr{U} \otimes \mathscr{U}'$ by

$$\mathscr{J} = \mathscr{U}^{\wedge 2} = \{ T^{\alpha\beta} | T^{\alpha\beta} \in \mathscr{U}^{\otimes 2} , \ T^{\beta\alpha} = -T^{\alpha\beta} \} , \tag{B3}$$

$$\mathscr{H} = \{ V^{\alpha}{}_{\beta} | V^{\alpha}{}_{\beta} \in \mathscr{U} \otimes \mathscr{U}' ,$$
$$(\tilde{V})_{\beta}{}^{\alpha} = -(I_{\beta\gamma} + O_{\beta\gamma}) V^{\gamma}{}_{\lambda} (I^{\lambda\alpha} + O^{\lambda\alpha}) \} . \tag{B4}$$

It is easily shown that under the map $L$

$$\mathscr{J} \rightarrow \mathscr{H} = L(\mathscr{J}) , \tag{B5}$$

where $L(\mathscr{J})$ denotes the image of $\mathscr{J}$ under the map $L$, i.e., $L(\mathscr{J}) \equiv \{ L(T^{\alpha\beta}) | T^{\alpha\beta} \in \mathscr{J} \}$.

To define the other subspaces, we shall make use of the following elements:

$$(E_6)^{\alpha\beta} = \tfrac{1}{2}(I^{\alpha\beta} + O^{\alpha\beta}) \in \mathscr{J} , \tag{B6a}$$

$$(E_5)^{\alpha\beta} = \tfrac{1}{2}(I^{\alpha\beta} - O^{\alpha\beta}) \in \mathscr{J} , \tag{B6b}$$

$$(U_6)^{\alpha}{}_{\beta} = -iE^{\alpha}{}_{\beta} = -i\,[(S_I)^{\alpha}{}_{\beta} + (S_O)^{\alpha}{}_{\beta}] \in \mathscr{H} , \tag{B6c}$$

$$(U_5)^{\alpha}{}_{\beta} = -i\,[(S_I)^{\alpha}{}_{\beta} - (S_O)^{\alpha}{}_{\beta}] \in \mathscr{H} , \tag{B6d}$$

where $E^{\alpha}{}_{\beta} = (S_I)^{\alpha}{}_{\beta} + (S_O)^{\alpha}{}_{\beta} \in \mathscr{U} \otimes \mathscr{U}'$ is the identity transformation on $\mathscr{U}$ [defined by (2.12) and (2.13) in I]. Under the map $L$, it follows that

$$(E_6)^{\alpha\beta} \rightarrow (U_6)^{\alpha}{}_{\beta} = L\,[(E_6)^{\alpha\beta}] , \tag{B7a}$$

$$(E_5)^{\alpha\beta} \rightarrow (U_5)^{\alpha}{}_{\beta} = L\,[(E_5)^{\alpha\beta}] . \tag{B7b}$$

Note also that

$$(E_6)^{\alpha\beta}(E_5)_{\alpha\beta} = 0 , \tag{B8a}$$

$$(E_6)^{\alpha\beta}(E_6)_{\alpha\beta} = 1 , \tag{B8b}$$

$$(E_5)^{\alpha\beta}(E_5)_{\alpha\beta} = -1 . \tag{B8c}$$

Now we define the subspace $\mathscr{J}_4 \subset \mathscr{J}_5 \subset \mathscr{J}$ and $\mathscr{H}_4 \subset \mathscr{H}_5 \subset \mathscr{H}$ as

$$\mathscr{J}_5 = \{ T^{\alpha\beta} | T^{\alpha\beta} \in \mathscr{J} , \ T^{\alpha\beta}(E_6)_{\alpha\beta} = 0 \} , \tag{B9}$$

$$\mathscr{J}_4 = \{ T^{\alpha\beta} | T^{\alpha\beta} \in \mathscr{J} , \ T^{\alpha\beta}(E_6)_{\alpha\beta} = 0 , \ T^{\alpha\beta}(E_5)_{\alpha\beta} = 0 \} , \tag{B10}$$

$$\mathscr{H}_5 = \{ V^{\alpha}{}_{\beta} | V^{\alpha}{}_{\beta} \in \mathscr{H} , \ V^{\alpha}{}_{\beta}(U_6)^{\beta}{}_{\alpha} = 0 \}$$
$$= \{ V^{\alpha}{}_{\beta} | V^{\alpha}{}_{\beta} \in \mathscr{H} , \ V^{\alpha}{}_{\alpha} = 0 \} , \tag{B11}$$

$$\mathscr{H}_4 = \{ V^{\alpha}{}_{\beta} | V^{\alpha}{}_{\beta} \in \mathscr{H} , \ V^{\alpha}{}_{\beta}(U_6)^{\beta}{}_{\alpha} = 0 , \ V^{\alpha}{}_{\beta}(U_5)^{\beta}{}_{\alpha} = 0 \}$$
$$= \{ V^{\alpha}{}_{\beta} | V^{\alpha}{}_{\beta} \in \mathscr{H} , \ V^{\alpha}{}_{\alpha} = 0 , \ V^{\alpha}{}_{\beta}(U_5)^{\beta}{}_{\alpha} = 0 \} . \tag{B12}$$

Note that for $T^{\alpha\beta} \in \mathscr{J}$ and $V^{\alpha}{}_{\beta} = L(T^{\alpha\beta}) \in \mathscr{H}$,

$$(E_6)_{\alpha\beta} T^{\alpha\beta} = (i/4)(I_{\alpha\beta} + O_{\alpha\beta})[V^{\alpha}{}_{\gamma}(I^{\gamma\beta} + O^{\gamma\beta})]$$
$$= -(i/4)(I_{\beta\alpha} + O_{\beta\alpha}) V^{\alpha}{}_{\gamma}(I^{\gamma\beta} + O^{\gamma\beta})$$
$$= (i/4)(\tilde{V})_{\beta}{}^{\beta} = (i/4) V^{\beta}{}_{\beta} . \tag{B13}$$

Consequently, under the map $L$ we have

$$\mathscr{J}_5 \rightarrow \mathscr{H}_5 = L(\mathscr{J}_5) . \tag{B14}$$

In addition, we shall need the following theorem [which can be readily proven by writing $A^{\gamma\delta} = -A^{\gamma\sigma}(I_{\sigma\tau} + O_{\sigma\tau})(I^{\tau\delta} + O^{\tau\delta}) = A^{\gamma\sigma}(\tilde{E})_{\sigma}{}^{\delta}$, and similarly for $B^{\gamma\beta}$ in the left-hand side of the twistor identity (A1) derived in Appendix A, using once more this identity to change the order of the products and finally resorting to (B13)].

*Theorem B.1:* For $(T_1)^{\alpha\beta}$, $(T_2)^{\alpha\beta} \in \mathscr{J}$, and $(V_1)^{\alpha}{}_{\beta} = L\,[(T_1)^{\alpha\beta}]$, $(V_2)^{\alpha}{}_{\beta} = L\,[(T_2)^{\alpha\beta}] \in \mathscr{H}$,

$$\tfrac{1}{2}[(V_1)^{\alpha}{}_{\beta}(V_2)^{\beta}{}_{\gamma} + (V_2)^{\alpha}{}_{\beta}(V_1)^{\beta}{}_{\gamma}]$$
$$- \tfrac{1}{4}[(V_1)^{\alpha}{}_{\gamma}(V_2)^{\beta}{}_{\beta} + (V_2)^{\alpha}{}_{\gamma}(V_1)^{\beta}{}_{\beta}]$$
$$= (T_1)_{\mu\nu}(T_2)^{\mu\nu} E^{\alpha}{}_{\gamma} . \tag{B15}$$

Observe that for $(V_1)^{\alpha}{}_{\beta}$, $(V_2)^{\alpha}{}_{\beta} \in \mathscr{H}_5$, (B15) leads to

$$\tfrac{1}{2}[(V_1)^{\alpha}{}_{\gamma}(V_2)^{\gamma}{}_{\beta} + (V_2)^{\alpha}{}_{\gamma}(V_1)^{\gamma}{}_{\beta}] = (T_1)_{\mu\nu}(T_2)^{\mu\nu} E^{\alpha}{}_{\beta} . \tag{B16}$$

Moreover, taking the trace of (B16) results in

$$(T_1)_{\mu\nu}(T_2)^{\mu\nu} = \tfrac{1}{4}(V_1)^{\alpha}{}_{\gamma}(V_2)^{\gamma}{}_{\alpha} . \tag{B17}$$

Thus for $(V_1)^{\alpha}{}_{\beta}$, $(V_2)^{\alpha}{}_{\beta} \in \mathscr{H}_5$, (B16) can be restated as

$$(V_1)^{\alpha}{}_{\gamma}(V_2)^{\gamma}{}_{\beta} + (V_2)^{\alpha}{}_{\gamma}(V_1)^{\gamma}{}_{\beta} = \tfrac{1}{2}[(V_1)^{\delta}{}_{\gamma}(V_2)^{\gamma}{}_{\delta}]\, E^{\alpha}{}_{\beta} . \tag{B18}$$

Note also that using $(T_2)^{\alpha\beta} = (E_5)^{\alpha\beta}$ and $(V_2)^{\alpha}{}_{\beta} = (U_5)^{\alpha}{}_{\beta}$ in (B17), leads to the result that under the map $L$ we have

$$\mathscr{J}_4 \rightarrow \mathscr{H}_4 = L(\mathscr{J}_4) . \tag{B19}$$

Next define the real subspaces $\mathscr{E}_{2,4} \subset \mathscr{J}$, $\mathscr{E}_{1,4} \subset \mathscr{J}_5$,

$\mathscr{E}_{1,3} \subset \mathscr{J}_4$, $\mathscr{D}_{1,4} \subset \mathscr{H}_5$, and $\mathscr{D}_{1,3} \subset \mathscr{H}_4$ as

$$\mathscr{E}_{2,4} = \{ T^{\alpha\beta} | T^{\alpha\beta} \in \mathscr{J}, \; \overline{T}^{\alpha\beta} = T^{\alpha\beta} \}, \tag{B20}$$

$$\mathscr{E}_{1,4} = \{ T^{\alpha\beta} | T^{\alpha\beta} \in \mathscr{J}_5, \; \overline{T}^{\alpha\beta} = T^{\alpha\beta} \}, \tag{B21}$$

$$\mathscr{E}_{1,3} \equiv \mathscr{F} = \{ T^{\alpha\beta} | T^{\alpha\beta} \in \mathscr{J}_4, \; \overline{T}^{\alpha\beta} = T^{\alpha\beta} \}, \tag{B22}$$

$$\mathscr{D}_{1,4} = \{ V^\alpha{}_\beta | V^\alpha{}_\beta \in \mathscr{H}_5, \; \tilde{\tilde{V}}^\alpha{}_\beta = V^\alpha{}_\beta \}, \tag{B23}$$

$$\mathscr{D}_{1,3} = \{ V^\alpha{}_\beta | V^\alpha{}_\beta \in \mathscr{H}_4, \; \tilde{\tilde{V}}^\alpha{}_\beta = V^\alpha{}_\beta \}. \tag{B24}$$

The space $\mathscr{E}_{1,3}$ is Minkowski space, and $\mathscr{D}_{1,3}$ is the space of Dirac gamma operators. In order to relate these two spaces under the map $L$ we need the following theorems.

*Theorem B.2:* For $T^{\alpha\beta} \in \mathscr{J}$,

$$T^{\alpha\beta} \in \mathscr{J}_5 \Leftrightarrow \overline{T}^{\alpha\beta} \in \mathscr{J}_5, \tag{B25}$$

and for $V^\alpha{}_\beta \in \mathscr{H}$,

$$V^\alpha{}_\beta \in \mathscr{H}_5 \Leftrightarrow \tilde{\tilde{V}}^\alpha{}_\beta \in \mathscr{H}_5. \tag{B26}$$

Relation (B25) follows directly from

$$\overline{T}^{\alpha\beta}(E_6)_{\alpha\beta} = \overline{T}_{\alpha\beta}(E_6)^{\alpha\beta} = [T^{\alpha\beta}(E_6)_{\alpha\beta}]^*,$$

where we have taken into account that $(E_6)^{\alpha\beta}$ is a real twistor.

To arrive at (B26) note that

$$\tilde{\tilde{V}}^\alpha{}_\alpha = (V^\alpha{}_\alpha)^*.$$

*Theorem B.3:* Under the map $L$ restricted to the domain $\mathscr{J}_5$, i.e., $T^{\alpha\beta} \in \mathscr{J}_5 \to V^\alpha{}_\beta = L(T^{\alpha\beta}) \in \mathscr{H}_5$, we have

$$\overline{T}^{\alpha\beta} \in \mathscr{J}_5 \to \tilde{\tilde{V}}^\alpha{}_\beta = L(\overline{T}^{\alpha\beta}) \in \mathscr{H}_5, \tag{B27}$$

and, consequently, for $T^{\alpha\beta} \in \mathscr{J}_5$ we have

$$\overline{T}^{\alpha\beta} = T^{\alpha\beta} \Leftrightarrow \tilde{\tilde{V}}^\alpha{}_\beta = V^\alpha{}_\beta. \tag{B28}$$

To prove this theorem we make use of the identity (A1) in order to show that

$$2i\overline{T}^{\alpha\beta}(I_{\beta\gamma} + O_{\beta\gamma}) = 2i(I_{\gamma\beta} + O_{\gamma\beta})\overline{T}^{\beta\alpha} = -2i\overline{T}_{\gamma\beta}(I^{\beta\alpha} + O^{\beta\alpha}) = \tilde{\tilde{V}}^\alpha{}_\gamma. \tag{B29}$$

Using Theorems B.2 and B.3, one can easily show that under the map $L$ we have

$$\mathscr{E}_{1,4} \to \mathscr{D}_{1,4} = L(\mathscr{E}_{1,4}), \tag{B30}$$

$$\mathscr{E}_{1,3} \to \mathscr{D}_{1,3} = L(\mathscr{E}_{1,3}). \tag{B31}$$

Equation (B31) expresses the result that the map $L$ establishes a one-to-one correspondence between Minkowski space $\mathscr{E}_{1,3}$ and the space $\mathscr{D}_{1,3}$ of Dirac gamma operators.

Combining the preceding results we can finally arrive at the following theorem.

*Theorem B.4:* Suppose $(W_5)^\beta{}_\alpha \in \mathscr{U} \otimes \mathscr{U}'$ has the following properties:

$$(\tilde{W}_5)_\alpha{}^\beta = -(I_{\alpha\gamma} + O_{\alpha\gamma})(W_5)^\gamma{}_\lambda(I^{\lambda\beta} + O^{\lambda\beta}), \tag{B32a}$$

$$(W_5)^\alpha{}_\alpha = 0, \tag{B32b}$$

$$V^\alpha{}_\gamma(W_5)^\gamma{}_\alpha = 0, \quad \text{for all } V^\alpha{}_\gamma \in \mathscr{D}_{1,3}, \tag{B32c}$$

$$(\tilde{\tilde{W}}_5)^\alpha{}_\beta = (W_5)^\alpha{}_\beta, \tag{B32d}$$

$$(W_5)^\alpha{}_\gamma(W_5)^\gamma{}_\beta = -E^\alpha{}_\beta. \tag{B32e}$$

Then

$$(W_5)^\alpha{}_\gamma = \pm (U_5)^\alpha{}_\gamma. \tag{B33}$$

To prove this theorem, first let $(F_5)^{\alpha\beta} = (i/2)(W_5)^\alpha{}_\gamma(I^{\gamma\beta} + O^{\gamma\beta})$. Then by the map $L$ we have $(W_5)^\alpha{}_\beta = L[(F_5)^{\alpha\beta}] = 2i(F_5)^{\alpha\gamma}(I_{\gamma\beta} + O_{\gamma\beta})$. Furthermore, by (B32a), (B32b), and (B32d) we find that $(W_5)^\alpha{}_\beta \in \mathscr{D}_{1,4} \subset \mathscr{H}_5 \subset \mathscr{H}$, and using (B13) and (B28) leads to $(F_5)^{\alpha\beta} \in \mathscr{E}_{1,4} \subset \mathscr{J}_5$. In addition, (B32c) and (B17) imply that $T_{\alpha\beta}(F_5)^{\alpha\beta} = 0$ for all $T^{\alpha\beta} \in \mathscr{E}_{1,3}$. But, by virtue of (B22) and (B10), $T^{\alpha\beta}(E_5)_{\alpha\beta} = 0$ for all $T^{\alpha\beta} \in \mathscr{E}_{1,3}$. Hence $(F_5)^{\alpha\beta}$ is in the same one-dimensional subspace in which $(E_5)^{\alpha\beta} \in \mathscr{E}_{1,4}$ lies, so $(F_5)^{\mu\nu} = \alpha(E_5)^{\mu\nu}$ for some real number $\alpha$. If we now make use of (B16), we get $(W_5)^\gamma{}_\delta(W_5)^\delta{}_\lambda = (F_5)_{\mu\nu}(F_5)^{\mu\nu}E^\gamma{}_\lambda$. Substituting (B32e) on the left-hand side of this equation yields $(F_5)_{\mu\nu}(F_5)^{\mu\nu} = -1$. Also since $(E_5)_{\mu\nu}(E_5)^{\mu\nu} = -1$, we arrive at $\alpha = \pm 1$ and $(F_5)^{\alpha\beta} = \pm (E_5)^{\alpha\beta}$. Thus, $(W_5)^\alpha{}_\beta = \pm 2i(E_5)^{\alpha\gamma}(I_{\gamma\beta} + O_{\gamma\beta}) = \pm (U_5)^\alpha{}_\beta$. Q.E.D.

Having established Theorem B.4, we are now in a position to elaborate further on the relation between Minkowski space and the space of Dirac gamma operators. For this purpose let $(E_i)^{\alpha\beta} \in \mathscr{E}_{1,3}$ for $i = 0, 1, 2, 3$ be any basis for $\mathscr{E}_{1,3}$, and $(E^i)^{\alpha\beta} \in \mathscr{E}_{1,3}$ for $i = 0, 1, 2, 3$ be the corresponding reciprocal basis. Thus

$$(E^i)^{\alpha\beta}(E_j)_{\alpha\beta} = \delta^i_j. \tag{B34}$$

The components $g_{ij}$ of the metric tensor for $\mathscr{E}_{1,3}$ are

$$g_{ij} = (E_i)^{\alpha\beta}(E_j)_{\alpha\beta}. \tag{B35}$$

Including $(E_5)^{\alpha\beta}$ with the $(E_i)^{\alpha\beta}$ given above, we have the elements $(E_A)^{\alpha\beta} \in \mathscr{E}_{1,4}$ for $A = 0, 1, 2, 3, 5$, which form a basis for $\mathscr{E}_{1,4}$. The elements $(E^A)^{\alpha\beta} \in \mathscr{E}_{1,4}$ for $A = 0, 1, 2, 3, 5$, where $(E^i)^{\alpha\beta}$ for $i = 0, 1, 2, 3$, is given above and $(E^5)^{\alpha\beta} = -(E_5)^{\alpha\beta}$ form the corresponding reciprocal basis for $\mathscr{E}_{1,4}$ since

$$(E^A)^{\alpha\beta}(E_B)_{\alpha\beta} = \delta^A_B. \tag{B36}$$

With respect to this basis the components of the metric tensor for $\mathscr{E}_{1,4}$ are

$$g_{AB} = (E_A)^{\alpha\beta}(E_B)_{\alpha\beta}, \tag{B37}$$

and we have $g_{ij}$ the same as given above for $i, j = 0, 1, 2, 3$, and $g_{i5} = g_{5i} = 0$, $g_{55} = -1$.

The elements $(U_i)^\alpha{}_\beta = L[(E_i)^{\alpha\beta}] = 2i(E_i)^{\alpha\gamma}(I_{\gamma\beta} + O_{\gamma\beta}) \in \mathscr{D}_{1,3}$, for $i = 0, 1, 2, 3$, form a basis for $\mathscr{D}_{1,3}$. Using $\frac{1}{4}(A^\alpha{}_\gamma B^\gamma{}_\alpha)$ as the inner product for $A^\alpha{}_\beta, B^\alpha{}_\beta \in \mathscr{D}_{1,3}$, the elements $(U^i)^\alpha{}_\beta = L[(E^i)^{\alpha\beta}] = 2i(E^i)^{\alpha\gamma}(I_{\gamma\beta} + O_{\gamma\beta})$ form the corresponding reciprocal basis for $\mathscr{D}_{1,3}$ since

$$\tfrac{1}{4}(U^i)^\alpha{}_\gamma(U_j)^\gamma{}_\alpha = (E^i)^{\alpha\beta}(E_j)_{\alpha\beta} = \delta^i_j. \tag{B38}$$

Also we have

$$\tfrac{1}{4}(U_i)^\alpha{}_\beta(U_j)^\beta{}_\alpha = (E_i)^{\alpha\beta}(E_j)_{\alpha\beta} = g_{ij}. \tag{B39}$$

Similarly, the elements $(U_A)^\alpha{}_\beta = L[(E_A)^{\alpha\beta}] = 2i(E_A)^{\alpha\gamma}(I_{\gamma\beta} + O_{\gamma\beta}) \in \mathscr{D}_{1,4}$ for $A = 0, 1, 2, 3, 5$ form a basis for $\mathscr{D}_{1,4}$. Using $\frac{1}{4}(A^\alpha{}_\beta B^\beta{}_\alpha)$ as the inner product for $A^\alpha{}_\beta$, $B^\alpha{}_\beta \in \mathscr{D}_{1,4}$ the elements $(U^A)^\alpha{}_\beta = L[(E^A)^{\alpha\beta}] = 2i(E^A)^{\alpha\gamma}(I_{\gamma\beta} + O_{\gamma\beta}) \in \mathscr{D}_{1,4}$ form the corresponding reciprocal basis for $\mathscr{D}_{1,4}$, since

$$\tfrac{1}{4}(U^A)^\alpha{}_\beta(U_B)^\beta{}_\alpha = (E^A)^{\alpha\beta}(E_B)_{\alpha\beta} = \delta^A_B. \tag{B40}$$

Also we have

$$\tfrac{1}{4}(U_A)^\alpha{}_\beta(U_B)^\beta{}_\alpha = (E_A)^{\alpha\beta}(E_B)_{\alpha\beta} = g_{AB}. \tag{B41}$$

Note that $(U^5)^\alpha{}_\beta = -(U_5)^\alpha{}_\beta$.

Introducing the customary notation for the Dirac gamma operators, we have

$$(\gamma_i)^\alpha{}_\beta = (U_i)^\alpha{}_\beta\,, \quad \text{for } i=0,1,2,3, \tag{B42a}$$

$$(\gamma_5)^\alpha{}_\beta = (U_5)^\alpha{}_\beta\,, \tag{B42b}$$

$$(\gamma^i)^\alpha{}_\beta = (U^i)^\alpha{}_\beta\,, \quad \text{for } i=0,1,2,3, \tag{B42c}$$

$$(\gamma^5)^\alpha{}_\beta = (U^5)^\alpha{}_\beta\,. \tag{B42d}$$

Note that

$$(\gamma^5)^\alpha{}_\beta = -(\gamma_5)^\alpha{}_\beta\,. \tag{B42e}$$

Now let

$$(W_5)^\alpha{}_\beta = (4!)^{-1}N^{ijkl}(U_i)^\alpha{}_\kappa(U_j)^\kappa{}_\lambda(U_k)^\lambda{}_\mu(U_l)^\mu{}_\beta$$
$$= (4!)^{-1}N^{ijkl}(\gamma_i)^\alpha{}_\kappa(\gamma_j)^\kappa{}_\lambda(\gamma_k)^\lambda{}_\mu(\gamma_l)^\mu{}_\beta\,, \tag{B43}$$

where $N^{ijkl}$ are components of an antisymmetric tensor $\mathbf{N}$ in $\mathscr{E}_{1,3}^{\wedge 4}$ normalized such that

$$N^{ijkl}N_{ijkl} = -4!\,. \tag{B44}$$

It can be shown that $(W_5)^\alpha{}_\beta$ satisfies the properties given in Theorem B.4, therefore

$$(W_5)^\alpha{}_\beta = \pm(U_5)^\alpha{}_\beta = \pm(\gamma_5)^\alpha{}_\beta\,. \tag{B45}$$

The choice of $\mathbf{N}$ is unique up to a sign. Assume a choice of $\mathbf{N}$ such that Eq. (B45) has a plus sign, i.e.,

$$(W_5)^\alpha{}_\beta = (U_5)^\alpha{}_\beta = (\gamma_5)^\alpha{}_\beta\,. \tag{B46}$$

For this $\mathbf{N}$ we say that a basis $(E_0)^{\alpha\beta}$, $(E_1)^{\alpha\beta}$, $(E_2)^{\alpha\beta}$, $(E_3)^{\alpha\beta}$ for $\mathscr{E}_{1,3}$ is right handed if the components $N^{ijkl}$ with respect to this basis satisfy $N^{0123}>0$.

In this case, we have

$$N^{ijkl} = \epsilon^{ijkl}/\sqrt{-g}\,, \tag{B47}$$

where $\epsilon^{ijkl}$ is the Levi–Civita symbol ($\epsilon^{0123}=\epsilon_{0123}=1$).

Suppose in addition that we have a basis $(E_i)^{\alpha\beta}$, for $\mathscr{E}_{1,3}$ which is orthonormal, i.e., such that

$$g_{ij}=0\,, \quad \text{for } i\neq j, \tag{B48}$$
$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1\,;$$

and suppose this basis is also right handed. Then it follows that

$$(\gamma_5)^\alpha{}_\beta = (\gamma_0)^\alpha{}_\kappa(\gamma_1)^\kappa{}_\lambda(\gamma_2)^\lambda{}_\mu(\gamma_3)^\mu{}_\beta\,. \tag{B49}$$

For convenience we summarize here some of the properties of the Dirac gamma operators which can be readily obtained from the results so far derived:

$$(\gamma_i)^\alpha{}_\beta = L\,[(E_i)^{\alpha\beta}] = 2i(E_i)^{\alpha\gamma}(I_{\gamma\beta}+O_{\gamma\beta})\in\mathscr{D}_{1,3}\,, \tag{B50a}$$

$$(\tilde{\gamma}_i)_\beta{}^\alpha = -(I_{\beta\lambda}+O_{\beta\lambda})(\gamma_i)^\lambda{}_\kappa(I^{\kappa\alpha}+O^{\kappa\alpha})\,, \tag{B50b}$$

$$(\gamma_i)^\alpha{}_\alpha = 0\,, \tag{B50c}$$

$$(\gamma_i)^\alpha{}_\lambda(\gamma_5)^\lambda{}_\alpha = 0\,, \tag{B50d}$$

$$(\tilde{\tilde{\gamma}}_i)^\alpha{}_\beta = (\gamma_i)^\alpha{}_\beta\,, \tag{B50e}$$

$$(\gamma_i)^\alpha{}_\kappa(\gamma_j)^\kappa{}_\beta + (\gamma_j)^\alpha{}_\kappa(\gamma_i)^\kappa{}_\beta = \tfrac{1}{2}(\gamma_i)^\mu{}_\nu(\gamma_j)^\nu{}_\mu E^\alpha{}_\beta = 2g_{ij}E^\alpha{}_\beta\,, \tag{B50f}$$

$$(\gamma_i)^\alpha{}_\kappa(\gamma_5)^\kappa{}_\beta + (\gamma_5)^\alpha{}_\kappa(\gamma_i)^\kappa{}_\beta = 0\,, \tag{B50g}$$

$$(\gamma_5)^\alpha{}_\beta = L\,[(E_5)^{\alpha\beta}] = 2i(E_5)^{\alpha\gamma}(I_{\gamma\beta}+O_{\gamma\beta}) - i\,[(S_I)^\alpha{}_\beta - (S_O)^\alpha{}_\beta]\in\mathscr{D}_{1,4}\,, \tag{B51a}$$

$$(\gamma_5)^\alpha{}_\beta = (\gamma_0)^\alpha{}_\kappa(\gamma_1)^\kappa{}_\lambda(\gamma_2)^\lambda{}_\mu(\gamma_3)^\mu{}_\beta \ [\text{if } (E_i)^{\alpha\beta} \text{ is a right-oriented orthonormal basis for } \mathscr{E}_{1,3}] \tag{B51b}$$

$$(\tilde{\gamma}_5)_\alpha{}^\beta = -(I_{\alpha\gamma}+O_{\alpha\gamma})(\gamma_5)^\gamma{}_\lambda(I^{\lambda\beta}+O^{\lambda\beta})\,, \tag{B51c}$$

$$(\gamma_5)^\alpha{}_\alpha = 0\,, \tag{B51d}$$

$$(\gamma_i)^\alpha{}_\beta(\gamma_5)^\beta{}_\alpha = 0 \quad (\text{for } i=0,1,2,3)\,, \tag{B51e}$$

$$(\tilde{\tilde{\gamma}}_5)^\alpha{}_\beta = (\gamma_5)^\alpha{}_\beta\,, \tag{B51f}$$

$$(\gamma_5)^\alpha{}_\lambda(\gamma_5)^\lambda{}_\beta = -E^\alpha{}_\beta\,. \tag{B51g}$$

In the main text we have used the space $\mathscr{F}\equiv\mathscr{E}_{1,3}$ as a typical fiber of a bundle $\mathscr{F}(\mathscr{M})\equiv\mathscr{E}_{1,3}(\mathscr{M})$. The $(E_i)^{\alpha\beta}$ (denoted there by $J_i{}^{\alpha\beta}$), $I^{\alpha\beta}+O^{\alpha\beta}$, and $(\gamma_i)^\alpha{}_\beta$ are cross sections of the bundles $\mathscr{E}_{1,3}(\mathscr{M})$, $\mathscr{E}_{2,4}(\mathscr{M})$, and $\mathscr{D}_{1,3}(\mathscr{M})$, respectively.

Taking note of this notational correspondence, (B50a) can be written as

$$(\gamma_i)^\alpha{}_\beta = 2iJ_i{}^{\alpha\gamma}(I_{\gamma\beta}+O_{\gamma\beta})\,, \tag{B52}$$

and its inverse, defined by (B2), is

$$J_i{}^{\alpha\beta} = (i/2)(\gamma_i)^\alpha{}_\gamma(I^{\gamma\beta}+O^{\gamma\beta})\,. \tag{B53}$$

Also, since $J_i{}^{\beta\alpha} = -J_i{}^{\alpha\beta}$, we have

$$J_i{}^{\alpha\beta} = (i/2)(I^{\alpha\gamma}+O^{\alpha\gamma})(\tilde{\gamma}_i)_\gamma{}^\beta\,. \tag{B54}$$

We now consider another map which we use extensively in the text in conjunction with the $\gamma$ operators. This is the charge conjugation operator.

We can define explicitly the charge conjugation operator $C^{\alpha\beta}\in\mathscr{E}_{2,4}$ by

$$C^{\alpha\beta} = O^{\alpha\beta} - I^{\alpha\beta}\,, \tag{B55}$$

and for $\psi^\alpha\in\mathscr{U}$, its charge conjugate $(\psi^c)^\alpha\in\mathscr{U}$ is

$$(\psi^c)^\alpha = (O^{\alpha\beta}-I^{\alpha\beta})\bar{\psi}_\beta\,. \tag{B56}$$

From Eqs. (2.12) and (2.13) in I, it is trivial to show that

$$C^{\alpha\gamma}C_{\gamma\beta} = E^\alpha{}_\beta\,, \tag{B57a}$$

$$C_{\alpha\gamma}C^{\gamma\beta} = (\tilde{E})_\alpha{}^\beta\,. \tag{B57b}$$

We can now express some of the relations derived above in terms of this charge conjugation operator.

First, multiplication of (B50b) by $C^{\lambda\beta}$ on the left and by $C_{\alpha\mu}$ on the right, and the use of (B50g) and (B51a), yields

$$(\gamma_i)^\alpha{}_\beta = -C^{\alpha\lambda}(\tilde{\gamma}_i)_\lambda{}^\kappa C_{\kappa\beta}\,. \tag{B58a}$$

The inverse of (B58a) is

$$(\tilde{\gamma}_i)_\beta{}^\alpha = -C_{\beta\kappa}(\gamma_i)^\kappa{}_\lambda C^{\lambda\alpha}\,, \tag{B58b}$$

which is readily obtained from (B58a) by making use of (B57b).

By means of a similar operation on (B51c), we also get

$$(\tilde{\gamma}_5)_\alpha{}^\beta = C_{\alpha\lambda}(\gamma_5)^\lambda{}_\kappa C^{\kappa\beta}\,. \tag{B59}$$

Finally, note that by virtue of (B51a) and (B51f)

$$I^{\alpha\beta}+O^{\alpha\beta} = -i(\gamma_5)^\alpha{}_\lambda C^{\lambda\beta} = -iC^{\alpha\lambda}(\tilde{\gamma}_5)_\lambda{}^\beta\,, \tag{B60a}$$

$$I_{\alpha\beta}+O_{\alpha\beta} = i(\tilde{\gamma}_5)_\alpha{}^\lambda C_{\lambda\beta} = iC_{\alpha\lambda}(\gamma_5)^\lambda{}_\beta\,. \tag{B60b}$$

It follows then, from (B53) and (B60a), that

$$J_i^{\alpha\beta}=\tfrac{1}{2}(\gamma_i)^{\alpha}{}_{\lambda}(\gamma_5)^{\lambda}{}_{\kappa}C^{\kappa\beta}. \tag{B61}$$

Similarly, taking the conjugate of (B54) and making use of (B50e) and (B60b) results in

$$J_{i\alpha\beta}=\tfrac{1}{2}C_{\alpha\lambda}(\gamma_5)^{\lambda}{}_{\kappa}(\gamma_i)^{\kappa}{}_{\beta}=-\tfrac{1}{2}C_{\alpha\lambda}(\gamma_i)^{\lambda}{}_{\kappa}(\gamma_5)^{\kappa}{}_{\beta}. \tag{B62}$$


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