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Higher-dimensional unification by isometry

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A unified theory based on a homogeneous fiber bundle $Q(M,G/H)$ is discussed in detail. In spite of the fact that the theory retains the full G-gauge invariance, the physical gauge group K is shown to be $K = H^*/(H \cap H^*)$, where H^* is the centralizer of H in G. A principal fiber bundle $P(M,G,H)$ is also constructed by introducing an additional left action H on $P(M,G)$ that commutes with the right action G, and a unified theory based on $P(M,G,H)$ is discussed. It is shown that the theory based on $Q(M,G/H)$ is, in fact, the H projection of the Einstein–Hilbert action from $P(M,G,H)$, with the identification $Q(M,G/H) = P(M,G,H)/H$.

I. INTRODUCTION

It has long been recognized that[1,2] the gauge theory and gravitation could be unified into an Einstein–Hilbert action in a higher-dimensional space which unifies the four-dimensional space-time with an n-dimensional internal space. In this unification the gauge symmetry emerges from the isometry of the unified metric. In a prototype unification[2,3] where the Killing vector fields of the isometry G are linearly independent, the isometry makes the unified space a principal fiber bundle[4] $P(M,G)$ with the space-time M as the base manifold and G as the structure group. In this case the isometry becomes the physical gauge symmetry of the unified theory.

In the general case when the Killing vector fields of the isometry G are *not* linearly independent, the unified space has the structure of a homogeneous fiber bundle $Q(M,G/H)$.[5] In this case questions arise of how the isometry restricts the metric and the curvature on Q, how the Einstein–Hilbert action on Q can be reduced to a unified action on M, and what is the resulting gauge symmetry of the theory. These questions have been investigated recently.[5,6] In this paper we discuss them in more detail and compare with other attempts in the literature.[7,8] We show that, even though the isometry makes the theory gauge invariant under G, it restricts the physical gauge group (the holonomy group) K to be $K = H^*/(H \cap H^*)$, where H^* is the commutant subgroup (the centralizer) of H in G.

Other purposes of the paper are to construct a fiber bundle, which we denote by $P(M,G,H)$, by introducing a left action[5,9] H on $P(M,G)$ that commutes with the right action G, and to discuss the unified theory based on $P(M,G,H)$. The theory is interesting in its own right. But perhaps more importantly the left isometry provides us with a better understanding of the geometry of $P(M,G)$ and $Q(M,G/H)$. First, it gives a natural homomorphism between $P(M,G)$ and $Q(M,G/H)$, because $P(M,G,H)$ can also be identified as a principal fiber bundle $P(Q,H)$ with $Q(M,G/H) = P/H$. So it provides an alternative method to obtain the unified theory based on $Q(M,G/H)$. In fact, we show that theory based on $Q(M,G/H)$ is nothing but the H projection of the one obtained from $P(M,G,H)$. Another motivation behind the left isometry is that it gives us a natural tool to study the non-Abelian monopoles[9] of $P(M,G)$. The left isometry demands the holonomy group of the connection on $P(M,G)$ to be H^*. But when the second homotopy $\pi_2(G/H)$ defined by the left isometry becomes nonzero, the gauge potential becomes dual, capable of describing both electric and magnetic charges of H^*. In fact, choosing H to be Cartan's subgroup of G (in which case H^* coincides with H), one can describe all possible magnetic charges of $P(M,G)$. Furthermore this observation, together with the fact that the connection space forms an affine space, allows us to express the most general gauge potential on $P(M,G)$ as the sum of a dual potential of $H^* = H$ and a gauge-covariant vector field which has no neutral component (the valence potential). With this gauge-independent decomposition of the potential into the dual part and the valence part, one can construct the most general nontrivial non-Abelian gauge theory.[10,11]

An attractive aspect of our unification is that it provides a simple and consistent method of dimensional reduction. A central issue in any (supersymmetric or not) higher-dimensional unification is how to reduce the theory to a four-dimensional effective theory. So far a popular method of dimensional reduction has been the zero-mode approximation[12] of the harmonic expansion, obtained after a spontaneous compactification[13] of the internal space. Unfortunately this approximation bears many undesirable features: a logical ambiguity on the definition of the zero-modes due to the possibility of a spontaneous symmetry breaking among them,[14] the consistency problem,[15,16] the problem of quantum stability,[17] and others. Our approach provides an alternative method of dimensional reduction free of these undesirable features. In our case the dimensional reduction is obtained by the isometry or, in general, by the right invariance[18] when the matter fields are present, which automatically reduces the higher-dimensional fields to a *finite* number of physical modes whose internal space dependence is completely fixed. So there is no need of a spontaneous compactification[14] or a harmonic expansion. More importantly, as long as the isometry remains rigid against quantum fluctuations, the geometry will not only exclude any higher modes but also precludes any intrinsic inconsistency.

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The paper is organized as follows. In Sec. II we give a brief review of the higher-dimensional unification based on the principal fiber bundle $P(M,G)$ for later convenience. In Sec. III the unified theory based on $Q(M,G/H)$ is discussed in detail. In Sec. IV we discuss the consistency problem of a dimensional reduction and compare our method with others. In Sec. V we construct a principal fiber bundle $P(M,G,H)$ by introducing a left isometry H on $P(M,G)$. In Sec. VI we discuss a unified theory based on $P(M,G,H)$, and show that the theory based on $Q(M,G/H)$ is the H projection of the one obtained from $P(M,G,H)$. Finally in the last section we compare our unification with others and discuss physical implications of our results.

II. UNIFIED THEORY ON $P(M,G)$: A BRIEF REVIEW

We start from a unified space P that has the following properties.[2,3]

(i) P is a $(4+n)$-dimensional metric manifold with the metric g_{AB}.

(ii) There exist n linearly independent Killing vector fields ξ_a $(a=1,2,...,n)$ which form an isometry group G with the following commutation relations:

$$\mathcal{L}_{\xi_a} g_{AB}=0, \quad [\xi_a,\xi_b]=(1/\kappa) f_{ab}{}^c \xi_c\,, \tag{1}$$

where $\mathcal{L}_{\xi_a}$ is the Lie derivative along ξ_a, and κ is a scale parameter. We further require G to be unimodular for the reason that will become clear in the following.

(iii) The integral manifold of the Killing vectors is a metric submanifold, i.e., the metric

$$\phi_{ab}=g_{AB}\xi^A_a\xi^B_b \tag{2}$$

is invertible.[19] Notice that since the Killing vectors define an n-dimensional involutive distribution[4] on P, they admit a unique maximal integral manifold by virtue of the Frobenius theorem.

Now, let M be P/G and π be the canonical projection from P to M. Then one may view the unified space P as a principal fiber bundle $P(M,G)$ and the Killing vector fields as the fundamental vector fields that generate the right action[4] of G on P. We will identify M as the physical space-time and the vertical fiber (the integral manifold of the Killing fields) as the internal space.

The existence of the metric g_{AB} allows us to define the horizontal subspace H_p of the tangent space $T_p(P)$ at each $p\in P$ as the horizontal complement of the Killing vectors with respect to the metric. By virtue of the Killing symmetry H_p will be invariant under the right action G. Now, let U be an open neighborhood of $x\in M$, ∂_μ $(\mu=1,2,3,4)$ a local coordinate basis on U, and D_μ the horizontal lift of ∂_μ on $\pi^{-1}(U)$. Clearly $D_\mu\otimes\xi_a$ serves a basis on $\pi^{-1}(U)$. In this horizontal lift basis[2] the metric g_{AB} should become block diagonal:

$$g_{AB}=\left(\begin{array}{c|c} g_{\mu\nu} & 0 \\ \hline 0 & \phi_{ab}\end{array}\right). \tag{3}$$

In the following we will identify $g_{\mu\nu}$ as the physical metric on M up to a conformal transformation.

Let σ be a local cross section[4] in $\pi^{-1}(U)$ [i.e., a smooth mapping from $x\in U$ to $\sigma(x)\in\pi^{-1}(U)$ such that $\pi(\sigma(x))=x$], and let $\partial_\mu\otimes\xi_a$ be the local direct product basis[2] on $\pi^{-1}(U)\simeq U\times G$ defined by this local trivialization. Clearly this basis has the following commutation relations:

$$\begin{aligned} &[\partial_\mu,\partial_\nu]=0\,,\\ &[\partial_\mu,\xi_a]=0\,,\\ &[\xi_a,\xi_b]=(1/\kappa)f_{ab}{}^c\xi_c\,. \end{aligned} \tag{4}$$

Now, from the concept of the horizontality the definition of the gauge potential follows. Since the $\mathscr{G}$-valued ($\mathscr{G}$ is the Lie algebra of G) connection one-form on P defined by H_p is nothing but the dual one-form ω^a of ξ_a, one can define the gauge potential $A_\mu{}^a$ by[2]

$$A^a_\mu=(1/e\kappa)\omega^a_A\,\partial^A_\mu\,, \tag{5}$$

where e is a coupling constant. So the choice of a local cross section amounts to the choice of a local gauge. From (5) one has

$$D_\mu=\partial_\mu-e\kappa A_\mu{}^a\xi_a\,, \tag{6}$$

so that g_{AB} has the following expression in the local direct product basis[2]:

$$g_{AB}=\left(\begin{array}{c|c} g_{\mu\nu}+e^2\kappa^2\phi_{ab}A_\mu{}^aA_\nu{}^b & e\kappa A_\mu{}^a\phi_{ab} \\ e\kappa\phi_{ab}A_\nu{}^b & \phi_{ab}\end{array}\right). \tag{7}$$

From (4) one obtains the following commutation relations for the horizontal lift basis:

$$\begin{aligned} &[D_\mu,D_\nu]=-e\kappa F_{\mu\nu}{}^a\xi_a\,,\\ &[D_\mu,\xi_a]=0\,,\\ &[\xi_a,\xi_b]=(1/\kappa)f_{ab}{}^c\xi_c\,, \end{aligned} \tag{8}$$

where $F_{\mu\nu}{}^a$ is the field strength of the potential $A_\mu{}^a$,

$$F_{\mu\nu}{}^c=\partial_\mu A_\nu{}^c-\partial_\nu A_\mu{}^c+ef_{ab}{}^cA_\mu{}^aA_\nu{}^b\,.$$

This implies that the horizontal subspace H_p can be integrable if and only if the field strength vanishes.

Notice that the isometry (1) requires the metric g_{AB} to be *right invariant*,[3,18] and determines its internal space dependence completely:

$$\begin{aligned} &\partial_a g_{\mu\nu}=0\,,\\ &\partial_a A_\mu{}^c=-(1/\kappa)f_{ab}{}^cA_\mu{}^b\,,\\ &\partial_a\phi_{bc}=(1/\kappa)f_{ab}{}^d\phi_{dc}+(1/\kappa)f_{ac}{}^d\phi_{bd}\,, \end{aligned} \tag{9}$$

where we have put $\partial_a=\xi_a$. Now one can calculate the scalar curvature R of P. Assuming no torsion one finds[2]

$$\begin{aligned} R_P=&R_M+R_G+(e^2\kappa^2/4)\phi_{ab}F_{\mu\nu}{}^aF_{\mu\nu}{}^b\\ &+\tfrac14\phi^{ab}\phi^{cd}[(D_\mu\phi_{ac})(D_\nu\phi_{bd})+(D_\mu\phi_{ab})(D_\mu\phi_{cd})]\\ &+\nabla_\mu(\phi^{ab}D_\mu\phi_{ab})\,, \end{aligned} \tag{10}$$

where R_M and R_G are the scalar curvature of M and G, and ∇_μ is the gauge and generally covariant derivative. Notice that R_p has no fiber dependence, which is a direct consequence of the isometry (1).

To obtain the unified interaction, we start from the Einstein–Hilbert action on P,

$$I_P=-\frac{1}{16\pi G_0}\int\sqrt{g}\sqrt{\phi}\,(R_P+\Lambda)\,d^4x\,d^nG\,, \tag{11}$$

where G_0 and Λ are the $(4+n)$-dimensional gravitational and cosmological constants, $d^n G$ is the right-invariant Harr measure on G, and

$$g = |\det g_{\mu\nu}|, \quad \phi = |\det \phi_{ab}| .$$

Now, when G is unimodular one has

$$\partial_a \phi = \phi \times \phi^{bc} \partial_a \phi_{bc} = (2/\kappa)\phi f_{ab}{}^b = 0$$

so that ϕ becomes explicitly independent of the internal fiber. In this case the dimensional reduction amounts to performing the trivial integration over the fiber, after which one obtains the four-dimensional Lagrangian

$$L_P = -(\mu/16\pi G_0)\sqrt{g}\sqrt{\phi}(R_P + \Lambda) , \qquad (12)$$

where μ is the right-invariant volume of G. Then, identifying G_0/μ as the four-dimensional gravitational constant G and requiring

$$e^2\kappa^2/16\pi G = 1 , \qquad (13)$$

one obtains the desired unification.[2,3]

One can simplify the Lagrangian (12) further by putting $\phi_{ab} = \phi^{1/n}\rho_{ab}$ ($|\det \rho_{ab}| = 1$). Removing a total divergence one finds

$$L = -\frac{1}{16\pi G}\sqrt{g}\sqrt{\phi}\Big[R_M + R_G + 4\pi G \phi^{1/n}\rho_{ab} F_{\mu\nu}{}^a F_{\mu\nu}{}^b - \frac{n-1}{4n}\frac{(\partial_\mu \phi)^2}{\phi^2} + \frac{1}{4}\rho^{ab}\rho^{cd}(D_\mu \rho_{ac})(D_\mu \rho_{bd}) + \lambda(|\det \rho_{ab}| - 1)\Big] ,$$

where λ is introduced as a Lagrange multiplier. But now the Lagrangian appears *unstable* due to the negative kinetic term of the ϕ field. This defect, however, is superficial and can easily be removed by reparametrizing the fields. To see this we make the conformal transformation

$$g_{\mu\nu} \to \sqrt{\phi}\, g_{\mu\nu}$$

and find L is given, in terms of the *new* metric, by[18] (up to a total divergence)

$$L = -\frac{1}{16\pi G}\sqrt{g}\Big[R_M - \exp\Big(-\sqrt{\frac{n+2}{n}}\sigma\Big)\hat{R}_G + \exp\Big(-\sqrt{\frac{n}{n+2}}\sigma\Big)\Lambda + 4\pi G \exp\Big(\sqrt{\frac{n+2}{n}}\sigma\Big)\rho_{ab} F_{\mu\nu}{}^a F_{\mu\nu}{}^b + \frac{1}{2}(\partial_\mu \sigma)^2 + \tfrac{1}{4}(D_\mu \rho^{ab})(D_\mu \rho_{ab}) + \lambda(|\det \rho_{ab}| - 1)\Big] , \qquad (14)$$

where $\hat{R}_G = R_G(\rho_{ab})$ and we have introduced the dilaton field σ by

$$\sigma = \tfrac{1}{2}\sqrt{(n+2)/n}\, \log \phi .$$

This suggests that one should treat the new metric, but not the old one, as the physical space-time metric. There are three important aspects of the unified Lagrangian worth mentioning. First the gravitational coupling (i.e., the Newton's coupling) G_N of the Kaluza–Klein gauge field is given by

$$G_N = G e^{-c\sigma} ,$$

where $c = -\sqrt{(n+2)/n}$. In general, in the presence of matter fields one can show that[18] the value of the constant c depends on what kind of matter field one has, but the modification of the gravitational coupling by the dilaton is a generic feature of the higher-dimensional unification. Second, the dilaton acquires a nontrivial potential determined by $\hat{R}_G$ and Λ. Finally the dynamics of the internal metric ρ_{ab} is described by a generalized nonlinear sigma model, with the minimal gauge coupling to $A_\mu{}^a$ and the self-interaction potential specified by $\hat{R}_G$.

III. $Q(M,G/H)$

Notice that on $P(M,G)$ the isometry G acts freely on P, which restricts the internal space to be isomorphic to G itself. In general, however, one would like the internal space to be a homogeneous space G/H on which the isometry G acts effectively but not necessarily freely. Assuming that the isometry acts transitively on the internal space, this would be the most general type of isometry one can impose on the unified space. Under this circumstance the unified space becomes a homogeneous fiber bundle $Q(M,G/H)$ rather than a principal one. The problem then is to find how the isometry reduces the metric on Q down to four-dimensional fields, and what is the resulting unified theory. In this section we resolve this problem.[5,17]

Let Q be the $(4+m)$-dimensional unified space ($m = \dim G/H$) which admits an n-dimensional isometry G with the Killing fields h_a $(a = 1,2,\ldots,n)$:

$$\mathcal{L}_{h_a} g_{AB} = 0 , \quad [h_a, h_b] = (1/\kappa) f_{ab}{}^c h_c . \qquad (15)$$

Also let π be the projection of Q to $M = Q/G$, U a neighborhood of $x \in M$, and σ a local cross section on $\pi^{-1}(U)$. With this local trivialization we introduce local coordinates $(x^\mu, y^{\underline{a}})$ $(\underline{a} = 1,2,\ldots,m)$, which are the direct product of the space-time coordinates x^μ of M and the internal coordinates $y^{\underline{a}}$ of G/H. Then in the basis $\partial_\mu \otimes \partial_{\underline{a}}$ the metric can always be put into the following form:

$$g_{AB} = \left(\begin{array}{c|c} g_{\mu\nu} + e^2\kappa^2 g_{\underline{ab}} B_\mu{}^{\underline{a}} B_\nu{}^{\underline{b}} & e\kappa B_\mu{}^{\underline{a}} g_{\underline{ab}} \\ \hline e\kappa g_{\underline{ab}} B_\nu{}^{\underline{b}} & g_{\underline{ab}} \end{array}\right). \qquad (16)$$

Now with

$$h_a = \partial_a = h_a{}^{\underline{b}} \partial_{\underline{b}} ,$$
$$[h_a, \partial_{\underline{b}}] = F_{a\underline{b}}{}^{\underline{c}} h_{\underline{c}} = -(\partial_{\underline{b}} h_a{}^{\underline{b}})\partial_{\underline{c}} , \qquad (17)$$

one finds the following expression for the Killing condition (15):

$$\partial_a \phi_{\underline{bc}} = F_{a\underline{b}}{}^{\underline{d}} \phi_{\underline{dc}} + F_{a\underline{c}}{}^{\underline{d}} \phi_{\underline{bd}} ,$$
$$\partial_a B_\mu{}^{\underline{c}} = -F_{a\underline{b}}{}^{\underline{c}} B_\mu{}^{\underline{b}} , \quad \partial_a g_{\mu\nu} = 0 . \qquad (18)$$

This is the generalization of the Killing condition (9) to $Q(M,G/H)$.

To keep the analogy between $P(M,G)$ and $Q(M,G/H)$ whenever possible, it is very useful to introduce the "dual one-form" $\phi^a = \phi_{\underline{b}}{}^a dy^{\underline{b}}$ of the Killing fields h_a by

$$\phi_{\underline{a}}{}^c h_c{}^{\underline{b}} = \delta_{\underline{a}}{}^{\underline{b}} , \quad \partial_a \phi_{\underline{b}}{}^c = -(1/\kappa) f_{ab}{}^c \phi_{\underline{b}}{}^b + F_{a\underline{b}}{}^{\underline{c}} \phi_{\underline{c}}{}^c . \qquad (19)$$

The existence and uniqueness of such a dual one-form will be proved in Sec. V. Now with $F_{\underline{ab}}{}^{\underline{c}} = \phi_{\underline{a}}{}^{a} F_{a\underline{b}}{}^{\underline{c}}$, (18) can be written as

$$\partial_{\underline{a}} \phi_{\underline{bc}} = F_{\underline{ab}}{}^{\underline{d}} \phi_{\underline{dc}} + F_{\underline{ac}}{}^{\underline{d}} \phi_{\underline{bd}} ,$$
$$\partial_{\underline{a}} B_{\mu}{}^{\underline{c}} = - F_{\underline{ab}}{}^{\underline{c}} B_{\mu}{}^{\underline{b}}, \quad \partial_{\underline{a}} g_{\mu\nu} = 0 .$$

Notice that the first equality tells us that $F_{\underline{ab}}{}^{\underline{c}}$ is a metric connection on G/H. However, it is *not* Riemannian because it has a nonvanishing torsion $t_{\underline{ab}}{}^{\underline{c}}$,

$$t_{\underline{ab}}{}^{\underline{c}} = F_{\underline{ab}}{}^{\underline{c}} - F_{\underline{ba}}{}^{\underline{c}} = (1/\kappa) f_{ab}{}^{c} \phi_{\underline{a}}{}^{a} \phi_{\underline{b}}{}^{b} h_{c}{}^{\underline{c}} . \tag{20}$$

The torsion-free Riemannian connection $\Gamma_{\underline{ab}}{}^{\underline{c}}$ is given by

$$\Gamma_{\underline{ab}}{}^{\underline{c}} = F_{\underline{ab}}{}^{\underline{c}} - C_{\underline{ab}}{}^{\underline{c}} ,$$

where

$$C_{\underline{ab}}{}^{\underline{c}} = \tfrac{1}{2} (t_{\underline{ab}}{}^{\underline{c}} + t^{\underline{c}}{}_{\underline{ab}} + t^{\underline{c}}{}_{\underline{ba}})$$

is the contortion.

To find the curvature of the potential $B_{\mu}{}^{\underline{a}}$ let us define a horizontal basis D_{μ} by

$$D_{\mu} = \partial_{\mu} - e\kappa B_{\mu}{}^{\underline{a}} \partial_{\underline{a}} . \tag{21}$$

Clearly the metric (16) becomes block diagonal in the basis $(D_{\mu} \otimes \partial_{\underline{a}})$,

$$g_{AB} = \left(\begin{array}{c|c} g_{\mu\nu} & 0 \\ \hline 0 & g_{\underline{ab}} \end{array} \right) . \tag{22}$$

Now, in analogy with (8) we obtain

$$\begin{aligned} &[D_{\mu}, D_{\nu}] = - e\kappa G_{\mu\nu}{}^{\underline{a}} \partial_{\underline{a}} , \\ &[\partial_{\underline{a}}, D_{\mu}] = F_{\underline{ab}}{}^{\underline{c}} B_{\mu}{}^{\underline{b}}, \quad [\partial_{\underline{a}}, \partial_{\underline{b}}] = 0 , \end{aligned} \tag{23}$$

where the field strength $G_{\mu\nu}{}^{\underline{a}}$ of the potential $B_{\mu}{}^{\underline{a}}$ is given by

$$G_{\mu\nu}{}^{\underline{c}} = \partial_{\mu} B_{\nu}{}^{\underline{c}} - \partial_{\nu} B_{\mu}{}^{\underline{c}} + e\kappa t_{\underline{ab}}{}^{\underline{c}} B_{\mu}{}^{\underline{a}} B_{\nu}{}^{\underline{b}} .$$

Notice that the torsion determines the self-interaction of the potential.

To make the above geometry of $Q(M,G/H)$ more transparent let us define the "covariant" potential $B_{\mu}{}^{a}$ by

$$B_{\mu}{}^{a} = B_{\mu}{}^{\underline{a}} \phi_{\underline{a}}{}^{a} . \tag{24}$$

From the definition it follows that

$$G_{\mu\nu}{}^{\underline{a}} = G_{\mu\nu}{}^{a} h_{a}{}^{\underline{a}} ,$$

where $G_{\mu\nu}{}^{a}$ is the canonical field strength of $B_{\mu}{}^{a}$:

$$G_{\mu\nu}{}^{c} = \partial_{\mu} B_{\nu}{}^{c} - \partial_{\nu} B_{\mu}{}^{c} + e f_{ab}{}^{c} B_{\mu}{}^{a} B_{\nu}{}^{b} .$$

Now, in terms of the covariant potential the Killing condition (18) has the following familiar form:

$$\partial_{a} B_{\mu}{}^{c} = - (1/\kappa) f_{ab}{}^{c} B_{\mu}{}^{b} , \quad \partial_{a} G_{\mu\nu}{}^{c} = - (1/\kappa) f_{ab}{}^{c} G_{\mu\nu}{}^{b} . \tag{25}$$

Similarly one may introduce the "covariant" metric h_{ab} of G/H by

$$h_{ab} = g_{\underline{ab}} h_{a}{}^{\underline{a}} h_{b}{}^{\underline{b}}, \quad h^{ab} = g^{\underline{ab}} \phi_{\underline{a}}{}^{a} \phi_{\underline{b}}{}^{b} \tag{26}$$

and find the following Killing condition:

$$\begin{aligned} &\partial_{a} h_{bc} = (1/\kappa) f_{ab}{}^{d} h_{dc} + (1/\kappa) f_{ac}{}^{d} h_{bd} , \\ &\partial_{a} h^{bc} = - (1/\kappa) f_{ad}{}^{b} h^{dc} - (1/\kappa) f_{ad}{}^{c} h^{bd} . \end{aligned} \tag{27}$$

Thus, in terms of the covariant fields, the Killing condition on $Q(M,G/H)$ has exactly the same expression as the condition (9) on $P(M,G)$. However, this appearance is misleading because these "covariant" fields do *not* always represent the physical degrees of freedom. This must be obvious because, first of all, h_{ab} is singular as an $(n \times n)$ matrix. In fact the matrix $h_{a}{}^{b}$, defined by

$$h_{a}{}^{b} = h_{ac} h^{bc} = h_{a}{}^{\underline{a}} \phi_{\underline{a}}{}^{b} , \tag{28}$$

forms the projection operator from $\mathscr{G}$ to $\mathscr{G}/\mathscr{H}$ ($\mathscr{H}$ is the Lie algebra of H). Moreover one has

$$h_{ab} = h_{a}{}^{c} h_{b}{}^{d} h_{cd} , \quad B_{\mu}{}^{a} = h_{b}{}^{a} B_{\mu}{}^{b} , \tag{29}$$

so that both h_{ab} and $B_{\mu}{}^{a}$ can have only G/H degrees of freedom. This point will become important soon.

To construct the unified action one must calculate the scalar curvature R on Q. Assuming no torsion we find

$$\begin{aligned} R_{Q} = R_{M} &+ R_{G/H} + (e^2\kappa^2/4) h_{ab} G_{\mu\nu}{}^{a} G_{\mu\nu}{}^{b} \\ &+ \tfrac{1}{4} h^{ab} h^{cd} [(D_{\mu} h_{ac}) (D_{\mu} h_{bd}) \\ &+ (D_{\mu} h_{ab}) (D_{\mu} h_{cd})] \\ &+ \nabla_{\mu} (h^{ab} D_{\mu} h_{ab}) , \end{aligned} \tag{30}$$

where $D_{\mu} = \partial_{\mu} - e\kappa B_{\mu}{}^{a} \partial_{a}$ is the gauge-covariant derivative defined by (21), and ∇_{μ} is the gauge and generally covariant derivative. Notice that R_Q is *explicitly* G invariant. The fact that R_Q should have a G-invariant expression is obvious from the isometry (15). To obtain the above result, however, one has to do the calculation in the basis (23) first, and then express R_Q in the G-invariant form. To calculate $R_{G/H}$ one can make use of the identity

$$R_{\underline{abc}}{}^{\underline{d}} \phi_{\underline{d}}{}^{d} = - (\nabla_{\underline{a}} \nabla_{\underline{b}} - \nabla_{\underline{b}} \nabla_{\underline{a}}) \phi_{\underline{c}}{}^{d}$$

and find

$$\begin{aligned} R_{\underline{ab}} = &\tfrac{1}{2} \phi_{\underline{a}}{}^{a} \phi_{\underline{b}}{}^{b} f_{ac}{}^{d} f_{bd}{}^{c} + \tfrac{1}{2} t_{\underline{ac}}{}^{\underline{d}} t_{\underline{bc}}{}^{\underline{d}} \\ &- \tfrac{1}{4} t_{\underline{cda}} t_{\underline{cdb}} + \tfrac{1}{2} (t_{\underline{cab}} + t_{\underline{cba}}) t_{\underline{cd}}{}^{\underline{d}} . \end{aligned}$$

So, when G is unimodular, one obtains

$$R_{G/H} = (1/2\kappa^2) f_{ab}{}^{d} f_{cd}{}^{b} h^{ac} + (1/4\kappa^2) f_{ab}{}^{e} f_{cd}{}^{f} h^{ac} h^{bd} h_{ef} . \tag{31}$$

The result (30) looks exactly the same as (10), except here R_G is replaced by $R_{G/H}$. However, one has to keep in mind that the "covariant" fields in R_Q do not represent the physical degrees of freedom.

To determine the *physical* content of R_Q, notice first that the internal metric h_{ab} must be ad(H) invariant. This is so because any G-invariant metric on G/H has to be ad(H) invariant.[4] One can make this ad(H) invariance explicit by choosing a cross section σ_0 in $\pi^{-1}(U)$ on which the isotropy subgroup of G becomes exactly H. Indeed on σ_0 one has $h_a = 0$ when h_a belongs to $\mathscr{H}$, so that one finds

$$\partial_{a} h_{bc} = (1/\kappa) f_{ab}{}^{d} h_{dc} + (1/\kappa) f_{ac}{}^{d} h_{bd} = 0$$

when ∂_a belongs to $\mathscr{H}$. This proves the ad(H) invariance of h_{ab}. By the same token (25) tells us that $B_{\mu}{}^{a}$ (and $G_{\mu\nu}{}^{a}$) must also be ad(H) invariant. This together with (29) means that the physical gauge group K must be $K = H^*/(H \cap H^*)$, where H^* is the centralizer (the commutant) of H in G. In other words, in terms of the covariant potential $B_{\mu}{}^{a}$ the holonomy group has to be K, but not G. Notice that K can also be expressed as $K = N/H$, where N is the normalizer of H in G.

The dimensional reduction from $Q(M,G/H)$ can be obtained exactly the same way as before. One starts from the Einstein–Hilbert action on Q,

$$I_Q = -\frac{1}{16\pi G_0}\int \sqrt{g_M}\,\sqrt{g_{G/H}}\,(R_Q+\Lambda)d^4x\,d^my\,, \quad (32)$$

and notices that

$$\sqrt{g_{G/H}(x,y)}\;d^my = \sqrt{g_{G/H}(\sigma_0(x),0)}\,d^m\mu_{G/H} = \sqrt{h(x)}\,d^m\mu_{G/H}\,,$$

where $d^m\mu_{G/H}$ is the G-invariant measure on G/H and

$$h(x) = |\det g_{\underline{ab}}(\sigma_0(x))| = |\det h_{\underline{ab}}(\sigma_0(x))|\,.$$

So after the fiber integration one obtains the following four-dimensional Lagrangian (up to a total divergence):

$$L = \frac{\mu_{G/H}}{16\pi G_0}\sqrt{g_M}\,\sqrt{h}\Big[R_M + R_{G/H} + \frac{1}{4}\frac{e^2\kappa^2}{4}h_{\underline{ab}}G_{\mu\nu}{}^{\underline{a}}G_{\mu\nu}{}^{\underline{b}} + \frac{1}{4}h^{\underline{ab}}h^{\underline{cd}}[(D_\mu h_{\underline{ac}})(D_\mu h_{\underline{bd}}) - (D_\mu h_{\underline{ab}})(D_\mu h_{\underline{cd}})] + \Lambda\Big]\,, \quad (33)$$

where now the gauge group is restricted to K. There are two things to be noticed here. First, when H becomes the identity subgroup, the Lagrangian becomes exactly identical to (12), the one we obtained from $P(M,G)$. Second, the Lagrangian is explicitly invariant under K, because the choice of the cross section σ_0 still leaves the K-gauge degrees of freedom completely arbitrary. However, at a first glance the above dimensional reduction appears to depend on the choice of a cross section. Now we prove the σ independence of the dimensional reduction. To do this notice that under an infinitesimal change of the cross section from $\sigma_0(x)$ to $\sigma(x)$ generated by $\delta y^a(x)$, one has

$$\delta h(x) = \delta y^a\,\partial_a h(x)|_{\sigma_0} = (2/\kappa)h(x)\delta y^a f_{\underline{ab}}{}^{\underline{b}} = (2/\kappa)h(x)\delta y^a f_{ab}{}^{b}\,,$$

where the last equality follows from (19). Clearly this (together with the fiber independence of R_Q) tells us that, when G is unimodular, the reduction procedure is independent of the choice of a cross section.

IV. CONSISTENCY OF DIMENSIONAL REDUCTION

A central issue in any (supersymmetric or not) higher-dimensional unification is how to perceive the extra dimension. On this issue there are two different points of view. So far the popular view has been to treat the full $(4+n)$-dimensional space as physical, as is done in supersymmetric Kaluza–Klein unification.[12] Here the dimensional reduction is regarded as a low-energy approximation of the full theory, which one obtains by keeping only the "zero-modes" of the harmonic expansion after a spontaneous compactification[13] of the internal space. The alternative view is to treat only the four-dimensional space as physical, which one can do by imposing an exact isometry[3,6] as we did in this paper. In this view the dimensional reduction is not an approximation but an inevitable consequence of the isometry. No matter how one regards the dimensional reduction, however, a logical consistency requires that the resulting four-dimensional theory should remain compatible with the higher-dimensional theory. A minimum requirement of the consistency[15,16] is that the solutions of the four-dimensional effective theory must remain solutions of the higher-dimensional equations of motion obtained before the dimensional reduction. We will call a dimensional reduction procedure consistent if it satisfies this criterion.

One can easily show that the dimensional reduction by isometry described in the previous sections is consistent. In fact the four-dimensional equations of motion obtained from (14) or (33) become exactly identical to the higher-dimensional Einstein equations on $P(M,G)$ or $Q(M,G/H)$. The equivalence follows from the fact that when G is unimodular the action integrals before and after the dimensional reduction become equivalent to each other, as far as the variation of the action integral is concerned. This is so because when G is unimodular, the integral over the fiber does not involve averaging out the fiber dependence of the fields. So one can obtain the Euler–Lagrange equations either before the fiber integration or after, with the same result. Thus in our case the consistency is built in by the geometry. However, it must be emphasized that the consistency is guaranteed *only if G is unimodular*. In fact, one can easily show by constructing explicit examples[20] that the isometry alone is not sufficient to guarantee the consistency of the dimensional reduction.

Now we wish to make a few comments. First, when the matter fields are present the isometry of the metric should be generalized to the right invariance[18] (or the invariance under the right action of G) of *all* fields, including the matter fields. The right invariance will then determine the fiber dependence of fields uniquely and give us a consistent dimensional reduction. Another point is that, to apply our dimensional reduction method, we need to specify not just the internal space G/H, but both G and H. This is so because a given homogeneous space may admit more than one transitively acting group. For instance, S^7 topology has four transitively acting groups,[16] and can be identified as one of the following: SO(8)/SO(7), SO(7)/G_2, SU(4)/SU(3), or SO(5)/SO(3). So for the 11-dimensional supergravity the physical gauge group K resulting from our dimensional reduction method could be either identity, U(1), or SU(2), depending on which G/H one chooses. In general, for a given internal space one should choose the smallest isometry G to obtain the largest K. Finally, we emphasize that our dimensional reduction does *not* require the internal space to be compact, because the reduction is not based on a harmonic expansion. In fact in our approach one can easily construct a well-defined unified theory with a noncompact internal space.[14] In this respect we remark that our dimensional reduction is more general than the one proposed by some authors[8] recently. In their reduction the compactness of G/H has been a prerequisite for a consistent dimensional reduction. We do not require this. In contrast we require the unimodularity of the isometry as a necessary condition for a consistent dimensional reduction. This requirement has been absent in their reduction.

At this point it is perhaps instructive to compare our dimensional reduction method in more detail with the popular one widely accepted in supersymmetric Kaluza–Klein unification.[12] In this approach the dimensional reduction is regarded as a "zero-modes" approximation of the full theory which one obtains after a spontaneous compactification[13] of the internal space. The justification for this approximation is that when the internal space is compactified by a Planck scale, all the massive modes can safely be neglected in the low-energy limit. Unfortunately the matter is more complicated[17] and the approximation faces serious problems. First of all, it is not a simple matter to determine what are the "zero-modes" of the theory exactly. The zero-modes of the harmonic expansion do not necessarily become the massless modes because the physical (the four-dimensional) mass of the zero-modes can be determined only after one studies the possibility of a spontaneous symmetry breaking among them. On this problem the popular zero-modes ansatz does not help either. In fact the zero-modes ansatz, which identifies the isometry of the vacuum internal metric to be the physical gauge group of the four-dimensional effective theory, has a critical defect of its own.[14] This can be seen from our analysis of the previous section which tells us that, when the dimension of the isometry G is larger than that of the internal space, the gauge potential defined by the zero-modes ansatz should become *linearly dependent*. As a result not all the G-gauge degrees of freedom can become physical. In fact, one can argue that this is the origin of the consistency problem[15] of the zero-modes ansatz. To avoid this difficulty recently some authors have proposed the so-called "K invariance" of the zero-modes,[16] which, when applied to the 11-dimensional supergravity, apparently gives the same physical gauge group as our reduction method. In spite of this apparent similarity, however, it is impossible to miss the fundamental difference between the two approaches. To illustrate this point let us consider the case when the gauge group becomes SU(2). In this case they start from SO(8) as the vacuum isometry, but require the zero-modes to be singlets under the SO(5) subgroup (the K invariance) to obtain SU(2) as the physical gauge group.[16] Evidently this SU(2) is the subgroup of SO(8) that commutes with SO(5). In contrast, in our case SU(2) is obtained by identifying the internal space as SO(5)/SO(3), but here as the subgroup of SO(5) that commutes with SO(3). Furthermore in the scalar sector they seem to identify the scalars [the most general SO(5)-invariant metric on S^7] as SO(5) singlets. But in our case the scalars that describe the most general SO(5)-invariant metric on SO(5)/SO(3) are certainly *not* singlets of SO(5). They become singlets only under the SO(3) subgroup [the ad(H) invariance] of SO(5). What is more, in our case all the physical degrees of freedom are determined without ever mentioning SO(8). Of course, the difference between the two methods goes far beyond this. In their case the reduction is possible only after a spontaneous compactification of the internal space, which makes some of their "zero-modes" extremely heavy. Although this does not cause a problem for the consistency of the dimensional reduction it certainly makes the physical validity of the zero-modes approximation questionable.[14] In our case this problem of validity does not arise because our dimensional reduction does not involve any approximation.

But perhaps a more serious problem with the zero-modes approximation lies in its quantum instability.[17] This problem arises because, no matter what zero-modes one starts with, there is no way to keep them from interacting with the "higher-modes" in the high-energy limit. The question then is how does one know whether the nature of the four-dimensional effective theory (the zero-modes and their interaction) will remain unchanged when the quantum fluctuation turns on the interaction with the higher-modes. This is really the consistency problem *at the quantum level*, which could be potentially more serious than the consistency problem at the classical level that we have discussed above.

V. LEFT ISOMETRY

Now we go back to $P(M,G)$ of Sec. II and introduce another isometry[5,9] H on P with the Killing vector fields m_i ($i = 1, 2, ..., k$; $k = \dim H$),

$$\mathcal{L}_{m_i} g_{AB} = 0\,, \tag{34}$$

which has the following properties.

(i) They are linearly independent, and internal:

$$m_i = m_i^a \xi_a\,.$$

(ii) They commute with the right isometry G but formally form a subgroup H of G,

$$[m_i,\xi_a] = 0, \quad [m_i,m_j] = -(1/\kappa) f_{ij}^{(H)k} m_k\,. \tag{35}$$

To make sure that H is independent of G we further require that $\mathscr{H}$ does not contain the center of $\mathscr{G}$.

(iii) The integral manifold of the Killing fields is a metric submanifold, or the metric

$$g_{ij} = g_{AB} m_i^A m_j^B \tag{36}$$

is invertible. We will call this manifold the H fiber.

The above isometry makes each fiber $\pi^{-1}(x)$ of $P(M,G)$ a principle fiber bundle $G(G/H,H)$ of its own. To see this notice that (35) implies that each m_i^a forms an adjoint representation of G so that locally one may always find a cross section σ_H in $\pi^{-1}(U)$ on which m_i becomes exactly identical to ξ_i. In this local trivialization m_i may be regarded as the right translation of $\xi_i(\sigma_H(x))$ on the fiber. More precisely with the local parametrization of $p\in\pi^{-1}(x)$ by $p = (x,a)$ ($x\in M$, $a\in G$) one has

$$m_i(x,a) = R_{a*}\xi_i(x,e)\,,$$

where R_a is the right multiplication of a, and e is the identity element of G. So m_i generates a *left* action H on $\pi^{-1}(x)$. From this one may view $\pi^{-1}(x)$ as a principal fiber bundle $G(G/H,H)$, but this time with the structure group H acting on the left. We will denote the $P(M,G)$ that has the additional H structure by $P(M,G,H)$.

Notice that not all the subgroups H may be qualified to describe a left isometry. First H must admit a bi-invariant metric. But, more importantly, G/H must be *reductive* since the metric ϕ_{ab} on $\pi^{-1}(x)$ should be able to define a G-invariant connection on $G(G/H,H)$. In other words $\mathscr{G}$ must have the following reductive decomposition[4]:

$$\mathscr{G} = \mathscr{H} + \mathscr{M} \text{ (direct sum)}, \quad \text{ad}(H)\mathscr{M} = \mathscr{M}\,.$$

This is the necessary and sufficient condition for $G(G/H,H)$ to admit a G-invariant H connection.

With $Q = P(M,G,H)/H$, P may be regarded as a principal fiber bundle $P(Q,H)$, with the structure group acting *on the left*. Moreover the quotient space Q may be regarded as a homogeneous fiber bundle $Q(M,G/H)$ on which G acts effectively on the right. But here it is important to make a distinction between $Q(M,G/H)$ and the associated bundle[4] $E(M,G/H,G,P)$ of $P(M,G)$ that can be obtained by projecting out the right-isometry subgroup H_R of G from P. Although E is diffiomorphic to Q, notice that for E the group G acts on the left on the standard fiber G/H, but on Q it still acts on the right. More significantly $E = P/H_R$ can always be obtained from $P(M,G)$ without the introduction of the left isometry. The fact that this difference is not a matter of semantics will become obvious in the following.

The left isometry allows us to introduce a horizontal subspace $\widetilde{H}_P$ at each $p \in P(Q,H)$ which is horizontal with respect to the H fiber. The corresponding G-invariant connection one-form θ^i is given by the dual one-form of m_i:

$$\theta^{\,i}_{A} = g^{ij} g_{AB} m_j^B = g^{ij} \phi_{ab} \omega_A^a m_j^b = \theta^{\,i}_{\,a} \omega_A^a , \tag{37}$$

where ω^a is the connection one-form of $P(M,G)$. Now the projection operator of the tangent space $T(P)$ that projects out the horizontal component is given by

$$\hat{h}_A{}^B = \delta_A{}^B - \theta_A{}^i m_i{}^B = \delta_A{}^B - \hat{k}_A{}^B ,$$

where $\hat{k}_A{}^B$ is the projection operator for the vertical H component. Since the projection operates within the basis ξ_a of $\pi^{-1}(x)$, one may also express the projection operator by

$$\hat{h}_a{}^b = \delta_a{}^b - \theta_a{}^i m_i{}^b = \delta_a{}^b - \hat{k}_a{}^b . \tag{38}$$

From this one has the following decomposition of ξ_a and ω^a:

$$\begin{aligned} \xi_a &= \hat{h}_a{}^b \xi_b + \hat{k}_a{}^b \xi_b = \hat{h}_a + \theta^{\,i}_{\,a} m_i , \\ \omega^a &= \hat{h}_b{}^a \omega^b + \hat{k}_b{}^a \omega^b = \hat{\phi}^a + m_i^a \theta^{\,i} , \end{aligned} \tag{39}$$

where $\hat{h}_a$ and $\hat{\phi}^a$ are the horizontal components of ξ_a and ω^a. Notice that they (as well as ξ_a and ω^a) are left invariant,

$$\mathcal{L}_{m_i} \hat{h}_a = 0 , \quad \mathcal{L}_{m_i} \hat{\phi}^a = 0 , \tag{40}$$

which follows directly from (34).

Any tensor field on $P(Q,H)$ that is invariant under H (and horizontal with respect to H) may be projected down to a tensor field on Q. Conversely any tensor field on Q has a unique horizontal lift on $P(Q,H)$ with the above H invariance. To make this one-to-one correspondence more explicit let $\partial_\mu \otimes \partial_{\underline{a}}$ be the coordinate basis introduced in Sec. III, $\hat{h}_{\underline{a}}$ the horizontal lift of $\partial_{\underline{a}}$ to $P(Q,H)$, and $\hat{\phi}^{\underline{a}}$ the dual one-form of $\hat{h}_{\underline{a}}$. Then instead of ξ_a one may use $h_{\underline{a}} \otimes m_i$ as a basis on $\pi^{-1}(x)$. Now one can put

$$\begin{aligned} &\hat{h}_a = \hat{h}_a{}^{\underline{b}} \hat{h}_{\underline{b}} , \quad \hat{\phi}^a = \hat{\phi}_{\underline{b}}{}^a \hat{\phi}^{\underline{b}} , \\ &\hat{\phi}_{\underline{a}}{}^c \hat{h}_c{}^{\underline{b}} = \delta_{\underline{a}}{}^{\underline{b}} , \quad \hat{h}_a{}^{\underline{c}} \hat{\phi}_{\underline{c}}{}^b = \hat{h}_a{}^b , \end{aligned} \tag{41}$$

and obtain the following commutation relations:

$$\begin{aligned} &[\hat{h}_{\underline{a}}, \hat{h}_{\underline{b}}] = -(1/\kappa) f_{ab}{}^c \hat{\phi}_{\underline{a}}{}^a \hat{\phi}_{\underline{b}}{}^b \theta_c{}^i m_i , \\ &[\hat{h}_{\underline{a}}, m_j] = 0, \quad [m_i, m_j] = -(1/\kappa) f_{ij}^{(H)k} m_k . \end{aligned} \tag{42}$$

In this basis the metric ϕ_{ab} becomes block diagonal,

$$\phi_{ab} = \left(\begin{array}{c|c} \hat{g}_{\underline{ab}} & 0 \\ \hline 0 & g_{ij} \end{array} \right),$$

and the right invariance of ϕ_{ab} can be expressed by

$$\partial_a \hat{g}_{\underline{bc}} = \widehat{F}_{a\underline{b}}{}^{\underline{d}} \hat{g}_{\underline{dc}} + \widehat{F}_{a\underline{c}}{}^{\underline{d}} \hat{g}_{\underline{bd}} , \quad \partial_a g_{ij} = 0, \tag{43}$$

where $\widehat{F}_{a\underline{b}}{}^{\underline{c}}$ is defined by

$$[\xi_a, \hat{h}_{\underline{b}}] = \widehat{F}_{a\underline{b}}{}^{\underline{c}} \hat{h}_{\underline{c}} . \tag{44}$$

Actually this equality also follows from the right invariance of ϕ_{ab}. Now, notice that since $\hat{h}_a{}^{\underline{b}}$ and $\hat{\phi}_{\underline{a}}{}^b$ have no H dependence they may also be regarded as functions on $Q(M,G/H)$. In fact, after the projection on Q, $\hat{h}_a$ should be identified as the generators h_a of G on Q so that $\hat{h}_a{}^b$ becomes $h_a{}^b$. More significantly one can now recognize the dual one-form ϕ^a of h_a defined by (19) as the H projection of $\hat{\phi}^a$. Indeed (19) follows from (44). This proves the existence and uniqueness of ϕ^a on $Q(M,G/H)$, when G/H is reductive. At this point the parallel between $P(M,G,H)$ and $Q(M,G/H)$ becomes unmistakable. For instance, the covariant fields $B_\mu{}^a$ and h_{ab} on Q are nothing more than the horizontal components of $A_\mu{}^a$ and ϕ_{ab} on $P(Q,H)$. We close this section by pointing out that this kind of natural homomorphism does not exist between $P(M,G)$ and its associated bundle $E(M,G/H,G,P)$.

VI. UNIFICATION FROM *P(M,G,H)*

The unified theory based on $P(M,G,H)$ can easily be obtained from the action integral (11) of $P(M,G)$ by imposing the left isometry H on it. The left isometry requires the metric ϕ_{ab} to be ad(H) invariant. So the scalar curvature R_G on $\pi^{-1}(x)$ can be expressed as the following sum of two intrinsic curvatures of H and G/H, and an extrinsic part that comes from the nontrivial embedding of G/H into G:

$$R'_G = R_H + R_{G/H} + R_E , \tag{45}$$

where

$$\begin{aligned} R_H &= \tfrac{1}{4} g^{ik} f_{ij}{}^l f_{kl}{}^j , \\ R_{G/H} &= \tfrac{1}{2} h^{ab} f_{ac}{}^d f_{bd}{}^c + \tfrac{1}{4} h^{ab} h^{cd} h_{ef} f_{ac}{}^e f_{bd}{}^f , \\ R_E &= \tfrac{1}{4} g_{ij} h^{ab} h^{cd} \theta^{\,i}_{\,e} \theta^{\,j}_{\,f} f_{ac}{}^e f_{bd}{}^f . \end{aligned}$$

Notice that here h_{ab} represents the horizontal component of ϕ_{ab}. As for the gauge potential the left isometry requires that[5,9]

$$D_\mu m_i^a = 0. \tag{46}$$

This, together with (8), means that *the only nonvanishing components of $F^a_{\mu\nu}$ must be those of the little group H^* which leaves m_i^a invariant*. So H^* must be the commutant subgroup of H in G. Mathematically this means that[4] the holonomy group of the potential $A_\mu{}^a$ of $P(M,G)$ must become H^*, even though it has the apparent G-gauge degrees of freedom. In other words, the left isometry requires the connection on $P(M,G)$ to be reducible to a connection on a principal bundle $P^*(M,H^*)$.

With the above remarks the Einstein–Hilbert action on $P(M,G,H)$ can be written as

$$I = -\frac{1}{16\pi G_0} \int \sqrt{g_M} \sqrt{gh} \, (R'_P + \Lambda) d^4x \, d^nG, \tag{47}$$

where g_M and h are the same as before, $g = |\det g_{ij}|$, and R'_P is given by

$$R'_P = R_M + R'_G + \frac{e^2\kappa^2}{4}(h_{ab} + g_{ij}\,\theta^i_a\,\theta^i_b)F_{\mu\nu}{}^aF_{\mu\nu}{}^b$$
$$+\frac{1}{4}h^{ab}h^{cd}[(D_\mu h_{ac})(D_\mu h_{bd})$$
$$-(D_\mu h_{ab})(D_\mu h_{cd})]$$
$$+\frac{1}{2}g_{ij}h^{ab}(D_\mu\,\theta^i_a)(D_\mu\,\theta^j_b)$$
$$-\frac{k-1}{4k}\frac{(\partial_\mu g)^2}{g^2} - \frac{1}{2}\frac{\partial_\mu g}{g}\frac{\partial_\mu h}{h}.$$

Notice that although the action integral is expressed in an explicitly G-invariant form, one has to keep in mind that the *physical* gauge symmetry is restricted to the holonomy group H^*. Now, the dimensional reduction can easily be performed as before, and the unimodularity of G guarantees the consistency of the dimensional reduction.

It must become clear that, upon the H projection, the unified action (47) is reduced to the action integral (32) on $Q(M,G/H)$. This is so because under the projection g_{ij} and θ^i_a vanish, and H^* is reduced to $K = H^*/(H\cap H^*)$. In this sense the left isometry provides an alternative way to obtain the unified theory based on $Q(M,G/H)$. However, the above unified theory on $P(M,G,H)$ is interesting in its own right. First, it has a remarkably suggestive form in that the field θ^i_a could play an important role in a spontaneous symmetry breaking as a Higgs field. But, more importantly, the left isometry can be implemented in such a way that it adds a nontrivial topological structure to the theory.[9] To understand this notice that the Killing vector fields $m^a_i(x)$, regarded as a mapping from an S^2 of M to the homogeneous space G/H, determines the second homotopy $\pi_2(G/H)$ of G/H. So when the left isometry has an isolated singularity inside S^2, $\pi_2(G/H)$ becomes nonzero. In this case the gauge potential defined by (46) must necessarily contain a magnetic flux of a non-Abelian magnetic charge of H^*. This means that when the potential is reduced to $P^*(M,H^*)$, it will develop a string singularity in M and thus make $P^*(M,H^*)$ nontrivial. So the theory will effectively describe *a nontrivial non-Abelian gauge theory* of H^*. A particularly interesting case is obtained when H becomes Cartan's subgroup of G, in which case H^* coincides with H. In this case the topology of the theory can be chosen in such a way that the gauge field contains both electric and magnetic components of H^*, and becomes capable of describing all possible non-Abelian monopoles of $P(M,G)$. The resulting theory becomes *a dual gauge theory* of $H^* = H$. For this reason the left isometry is sometimes called the magnetic symmetry.[5,9]

VII. DISCUSSION

In this paper we have discussed a set of unified theories that can be obtained imposing an isometry G to the unified metric. The isometry provides a natural method of dimensional reduction guaranteed to be consistent when G is unimodular. When matter fields are present the isometry can easily be generalized to include the matter fields. The discussion in Sec. III was based on the existence of the one-form ϕ^a defined by (19). In this paper we were able to prove the existence and uniqueness of the one-form only when G/H becomes reductive. But we suppose that this will not pose a serious restriction from the physical point of view.

Recently some authors[7,8] have proposed a method to construct a unified theory based on $E(M,G/H)$ by identifying it as an associated bundle of $P(M,N/H)$, where N is the normalizer of H in G. Aside from the obvious discrepancy between their action integral on $E(M,G/H)$ and ours on $Q(M,G/H)$, which originates from their use of an incorrect volume element on E, their result is very similar to our result of Sec. III. Indeed as a manifold our $Q(M,G/H)$ can be viewed as identical to their $E(M,G/H)$. In spite of this similarity, however, we wish to emphasize that the difference, especially in the way we obtain the unified action, is also evident. To see the difference notice that on E they introduce *two actions* separately[7]: the right action (the "global" symmetry) G and the gauge action (the "local" symmetry) $K = N/H$ which commutes with (and thus is independent of) G. But in our case there is *only one right action* G on Q, and the gauge symmetry K is obtained as a subgroup of G. In fact, our K is obtained as the holonomy group of the isometry G. Besides, the natural homomorphism between $P(M,G)$ and $Q(M,G/H)$ that we have established in Sec. V does not come easily between $P(M,G)$ and $E(M,G/H)$. This discrepancy is obvious when H becomes the identity subgroup. In this limit Q becomes $P(M,G)$ itself with only one right action G, but E becomes an associated bundle $E(M,G,G,P)$ of P on which they still have two actions, the "global" G and the "local" G, which act independently from the right and from the left. The difference goes beyond this. In their case they require the existence of a compactifying ground state solution as a prerequisite[8] for a consistent dimensional reduction. In our reduction we find no reason why one must make such a requirement. In fact, one can easily show that[14] a perfectly consistent dimensional reduction is possible with a noncompact G/H. We emphasize, however, that one must require G to be a unimodular as a necessary (and sufficient) condition for the consistency of the dimensional reduction.

The introduction of $P(M,G,H)$ in Sec. V provides a general method of prolongation and reduction of a gauge symmetry. One can obtain the reduction of the physical gauge symmetry from $P(M,G)$ by making the left isometry H larger and larger starting from the identity subgroup. Conversely one may obtain the prolongation starting from $H = G$ and making H smaller and smaller. A potentially interesting case of the prolongation is *the extended gauge theory*. One can obtain this as a most general nontrivial non-Abelian gauge theory[10,11] by adding a valence potential[10] (i.e., a gauge-covariant vector field that has no neutral component) to the dual gauge theory of $H^* = H$. Since the connection space (the space of gauge potentials) forms an affine space, a most general gauge potential of $P(M,G)$ can be expressed as the sum of a gauge-covariant vector field that has no H^* component (the valence potential) and the dual potential of $H^* = H$. The result is the extended gauge theory in which the gauge potential of G is decomposed into the valence part and the dual part in a gauge-independent way. This prolongation allows us to have an alternative, completely unconventional and nevertheless physically very interesting, interpretation of a non-Abelian gauge symmetry.[10,11]

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[1]T. Kaluza, Sitzungsber Preuss. Akad. Wiss. **1921**, 966; O. Klein, Z. Phys. **37**, 895 (1926); B. deWitt, in *Relativity, Groups, and Topology* (Gordon and Breach, New York, 1965); R. Kerner, Ann. Inst. H. Poincaré **9**, 143 (1968); A. Trautmann, Rep. Math. Phys. **1**, 29 (1970).

[2]Y. M. Cho, J. Math. Phys. **16**, 2029 (1975); Y. M. Cho and P. G. O. Freund, Phys. Rev. D **12**, 1711 (1975); L. N. Chang, K. Macrae, and F. Mansouri, *ibid.* **13**, 235 (1976).

[3]Y. M. Cho and P. S. Jang, Phys. Rev. D **12**, 3189 (1975); F. Mansouri and L. Witten, Phys. Lett. B **127**, 341 (1983).

[4]See, e.g., S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vols. I & II (Interscience, New York, 1969).

[5]Y. M. Cho, CERN report TH-3414, 1982; see also Y. M. Cho, in *Proceedings of XIIIth International Colloquium on Group Theoretical Methods in Physics*, edited by W. Zachary (World Scientific, Singapore, 1984).

[6]Y. M. Cho, Phys. Lett. B **186**, 38 (1987).

[7]R. Coquereaux and A. Jadczyk, Commun. Math. Phys. **80**, 79 (1983).

[8]R. Coquereaux and A. Jadczyk, Nucl. Phys. B **276**, 617 (1986).

[9]Y. M. Cho, Phys. Rev. Lett. **44**, 1115 (1980); Phys. Rev. D **21**, 1080 (1980).

[10]Y. M. Cho, Phys. Rev. Lett. **46**, 302 (1981); Phys. Rev. D **23**, 2415 (1981).

[11]Y. M. Cho, Phys. Lett. B **167**, 432 (1986).

[12]E. Witten, Nucl. Phys. B **186**, 412 (1981); A. Salam and J. Strathdee, Ann. Phys. (NY) **141**, 316 (1982); M. J. Duff, Nucl. Phys. B **219**, 389 (1983).

[13]P. G. O. Freund and M. Rubin, Phys. Lett. B **97**, 233 (1980); F. Englert, *ibid.* **119**, 339 (1982).

[14]Y. M. Cho, Phys. Rev. Lett. **55**, 2932 (1985).

[15]M. J. Duff, B. Nilsson, C. Pope, and N. Warner, Phys. Lett. B **149**, 90 (1984).

[16]M. J. Duff and C. Pope, Nucl. Phys. B **255**, 355 (1985).

[17]Y. M. Cho, in *Proceedings of XIVth International Colloquium on Group Theoretical Methods in Physics*, edited by Y. M. Cho (World Scientific, Singapore, 1986); B. deWitt and H. Nicolai, Nucl. Phys. B **281**, 211 (1987).

[18]Y. M. Cho, Phys. Rev. D **35**, 2628 (1987).

[19]Here we use the index notation: The capital indices A and B indicate the type of tensors, and the lower case indices μ,ν,a,b are used to label them.

[20]M. A. H. MacCallum and A. H. Taub, Commun. Math. Phys. **25**, 173 (1972).