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regarding the charged particle-surface interaction, which had to be postulated simply due to the lack of pertinent experimental evidence. It is hoped that, despite these shortcomings, some physical insight into the present problem may be obtained with the simplified analysis.

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Induced Oscillations in a Rarefied Plasma in a Magnetic Field*

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The excitation of collective plasma motion by a small charged object moving through a low-density unbounded plasma in an external uniform, static magnetic field is considered. The energy loss per unit path length due to the excitation of collective plasma motion is found as a function of magnetic field strength for an arbitrary angle between the velocity of the object and the field direction. It is assumed throughout this paper that the ion temperature is zero and the velocity of the charged object is small compared to the mean thermal electron speed.

A CHARGED object traveling through a plasma may lose energy to collective motion of the plasma particles (wave drag) in addition to losing energy by means of collisions with the individual particles. The energy loss due to excitation of collective motion of the plasma in the absence of an external magnetic field has been considered by Bohm and Pines[1] for the object velocity V_0 greater than the mean thermal electron speed $(\langle V_-^2\rangle)^{\frac{1}{2}}$, and by Krause and Watson[2] for V_0 less than $(\langle V_-^2\rangle)^{\frac{1}{2}}$ but greater than $(T_-/T_+\langle V_+^2\rangle)^{\frac{1}{2}}$, where $(\langle V_+^2\rangle)^{\frac{1}{2}}$ is the mean thermal ion speed, and T_-/T_+ is the ratio of the electron to the ion temperature.

This work will consider this particular form of energy loss (sometimes called electrohydrodynamic drag[2]) for a charged object traveling through a low-density unbounded, fully ionized plasma in an external, uniform, static magnetic field $\mathbf{B}_0$ for the case of $V_0^2 \ll \langle V_-^2\rangle$ and $T_+ \rightarrow 0$. In this work, as in the work of references 1 and 2, the object is assumed to be small in size compared to the electron Debye length λ_D of the plasma.

We will look for a steady-state solution for $\mathbf{E}(\mathbf{r} - \mathbf{V}_0 t)$, the electric field as a function of $\mathbf{r} - \mathbf{V}_0 t$, the radius vector measured from the position of the object, by solving Maxwell's equations together with the Boltzmann equations for the electrons and ions of the plasma. The energy loss per unit path length of the object can then be found from the electric field at the position of the object.

The techniques used in solving the equations are similar in many ways to those which appeared in a paper by I. B. Bernstein[3] as well as those used by Kraus and Watson.[2] The electromagnetic quantities and the electron and ion distribution functions are assumed to vary only slightly from specified zero-order quantities. The equations are then linearized and solved with the technique of a Fourier transformation. The zero-order distribution functions are assumed to be Maxwellian, and the plasma is assumed to be of sufficiently low density that the collision frequency can be taken to approach zero. Moreover, the charge to mass ratio Q/M of the object is assumed to be sufficiently small so that the curvature of the path of the object in the magnetic field is negligible. This implies that the

* Based on a paper presented at the Symposium on the Aerodynamics of the Upper Atmosphere, June 8–10, 1959.
[1] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).
[2] L. Kraus and K. M. Watson, Phys. Fluids **1**, 480 (1958).
[3] I. B. Bernstein, Phys. Rev. **109**, 10 (1958).

object is sufficiently massive so that its frequency of gyration in the magnetic field is small compared to the ion plasma frequency.

GENERAL EQUATIONS AND LINEARIZATION

Let $f(\mathbf{r}, \mathbf{v}, t)$ be the distribution function for a given kind of particle, ion, or electron. $f(\mathbf{r}, \mathbf{v}, t)$ satisfies the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f \pm \frac{e}{m}\left[\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right]\cdot\nabla_v f = \frac{\partial f}{\partial t_{\text{collisions}}}. \quad (1)$$

Here, $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, m the particle mass, and e the electronic charge. The upper sign is for positively charged, singly ionized ions; and, the lower sign is for the electrons. We will assume that the distribution function differs only slightly from a zero-order distribution function which we will take to be Maxwellian. We will also assume small perturbations in the electromagnetic field quantities, the zero-order magnetic field $\mathbf{B}_0$ being a constant, and the zero-order electric field zero. Therefore, we write

$$f = f_0(v) + f'(\mathbf{r}, \mathbf{v}, t) \quad (2)$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}'(\mathbf{r}, t)$$

$$\mathbf{E} = \mathbf{E}'(\mathbf{r}, t)$$

$$f_0(v) = n_0\left(\frac{m}{2\theta\pi}\right)^{\frac{3}{2}} e^{-mv^2/2\theta}$$

$$\theta = kT$$

$$n_{0+} = n_{0-} = \int f_{0+}\, d\mathbf{v} = \int f_{0-}\, d\mathbf{v}.$$

We will assume that the plasma is of sufficiently low density that we can write the collision term as[2]

$$(\partial f/\partial t)_{\text{collisions}} = \lim_{\nu\to 0}\,(-\nu f'), \quad (3)$$

where ν represents a collision frequency.

Finally, since we are considering the excitation of longitudinal oscillations, the electric field is derivable from a scalar potential ϕ. Therefore,

$$\frac{\partial f'}{\partial t} + \nu f' + \mathbf{v}\cdot\nabla f' \pm \frac{e}{mc}(\mathbf{v}\times\mathbf{B}_0)\cdot\nabla_v f' = \mp\frac{e}{\theta} f_0\mathbf{v}\cdot\nabla\phi. \quad (4)$$

In addition to Eq. (4), we have Poisson's equation

$$\nabla^2\phi = -4\pi\sigma, \quad (5)$$

where σ is the charge density, and is given by

$$\sigma = e\sum_{\pm} \pm \int f'\, d\mathbf{v} + \sigma_{\text{ext}}. \quad (6)$$

σ_{ext} is the charge density of the charged object moving through the plasma. We will assume the object to have a constant velocity $\mathbf{V}_0$, a charge Q, and a charge density $Qh(\mathbf{r} - \mathbf{V}_0 t)$.

Now perform a plane wave analysis of all the perturbed quantities as well as h:

$$\phi = \frac{1}{(2\pi)^3}\int d\mathbf{k}\, d\omega \phi^*(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad (7)$$

$$f' = \frac{1}{(2\pi)^3}\int d\mathbf{k}\, d\omega f^*(\mathbf{k}, \mathbf{v}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\sigma_{\text{ext}} = \frac{1}{(2\pi)^3}\int \sigma^*_{\text{ext}}(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\sigma^*_{\text{ext}}(\mathbf{k}, \omega) = Qh^*(\mathbf{k})\,\delta(\omega - \mathbf{k}\cdot\mathbf{V}_0).$$

For a "point" charge, $h^*(\mathbf{k}) = 1$.

In writing Eq. (4) in terms of the Fourier transforms, let us choose a rectangular coordinate system in velocity space,[3] in which the 1, 2, and 3 axes are parallel to $\mathbf{B}_0$, $\mathbf{k}_\perp$, and $\mathbf{B}_0\times\mathbf{k}_\perp$. Using cylindrical coordinates $v_\parallel$, $v_\perp$, and α for $\mathbf{v}$, we get

$$\left[\pm\frac{(\nu - i\omega + i\mathbf{k}\cdot\mathbf{v})}{\omega_c} - \frac{\partial}{\partial\alpha}\right] f^* = -\frac{ie}{\theta\omega_c}\phi^* f_0 \mathbf{k}\cdot\mathbf{v} \quad (8)$$

$$\omega_c = eB_0/mc.$$

Let us write f^* as

$$f^* = \mp(e\phi^*/\theta) f_0 + g^*. \quad (9)$$

The term $\mp(e\phi^*/\theta)f_0$ when added to f_0 represents the linearized equilibrium distribution in the scalar potential ϕ^*. In terms of g^*, Eq. (8) becomes

$$\left[\pm\frac{(\nu + i(k_\parallel v_\parallel - \omega) + ik_\perp v_\perp \cos\alpha)}{\omega_c} - \frac{\partial}{\partial\alpha}\right] g^* = -i\frac{\omega}{\omega_c}\frac{e}{\theta}\phi^* f_0. \quad (10)$$

The advantage of introducing g^* is that the right-hand side of the equation is now independent of α, which simplifies the integration. g^* is given by[3]

$$g^* = i\frac{\omega}{\omega_c}\frac{e}{\theta}\phi^* f_0 \lim_{\nu\to 0}\int_{\pm\infty}^{\alpha} G(\alpha, \alpha')\, d\alpha', \quad (11)$$

where

$$G(\alpha, \alpha') = \exp \pm\frac{1}{\omega_c}\left[(\nu + ik_\parallel v_\parallel - i\omega)(\alpha - \alpha') + ik_\perp v_\perp(\sin\alpha - \sin\alpha')\right].$$

Our problem can now be solved by combining Eq. (11) with Poissons equation

$$k^2\phi^* = 4\pi\sigma^*_{\text{ext}} + 4\pi e \sum_{+-} \pm \int f^* \, d\mathbf{v}$$

$$= 4\pi\sigma^*_{\text{ext}} - \sum_{+-} \frac{4\pi n_0 e^2}{\theta} \phi^* + 4\pi e \sum_{+-} \pm \int g^* \, d\mathbf{v} \quad (12)$$

$$\int g^* \, d\mathbf{v} = \frac{ie}{\theta} \phi^* \frac{n_0}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp -u^2 \, du$$

$$\cdot \int_0^\infty w \, dw \exp -w^2 \int_0^{2\pi} d\alpha \int_0^{\mp\infty} G(\alpha, y) \, dy,$$

where $y = (\omega/\omega_c)(\alpha - \alpha')$, $u = v_\parallel(m/2\theta)^{\frac{1}{2}}$, $w = v_\perp(m/2\theta)^{\frac{1}{2}}$.

In calculating $\int g^* \, d\mathbf{v}$, if we first integrate over α, then w and u, the three integrations are trivial; and we are reduced to a single integral

$$\int g^* \, d\mathbf{v} = \mp \frac{ie}{\theta} \phi^* n_0 N \quad (13)$$

$$N = \int_0^\infty \exp - \left[\frac{\mu_\parallel^2 y^2}{2} + \lambda_\perp^2 \left(1 - \cos \frac{\omega_c y}{\omega} \right) - iy \right] dy,$$

where $\mu^2 = k^2\theta/m\omega^2$, $\lambda^2 = k^2\theta/m\omega_c^2$, $|\mu|$ is a measure of the ratio of the mean plasma particle speed to the phase velocity of the disturbance, and $|\lambda|$ is a measure of the ratio of the Larmor radius of the plasma particles to the wavelength of the disturbance.

Poisson's equation now becomes

$$\left[k^2 + \sum_{\pm} \frac{4\pi n_0 e^2}{\theta} (1 + iN) \right] \phi^*$$

$$= 4\pi Q h^*(\mathbf{k}) \, \delta(\omega - \mathbf{k}\cdot\mathbf{V}_0). \quad (14)$$

Approximations to the integral N can be obtained for small and large values of the parameters μ^2 and λ^2. We are only going to consider the case where $\theta_+ \to 0$ so that μ_+^2 and $\lambda_+^2 \to 0$; and we are only considering $V_0^2 \ll \theta_-/m_-$, so that $\mu_-^2 \gg 1$. For $\theta = 0$, N can be calculated exactly

$$\frac{4\pi n_0 e^2}{\theta}(1 + iN) = -\frac{\omega_p^2}{\omega^2(\omega^2 - \omega_c^2)} (k^2\omega^2 - k_\parallel^2 \omega_c^2) \quad (15)$$

$$\omega_p^2 = 4\pi n_0 e^2/m.$$

For $\mu^2 \gg 1$, N can be easily approximated for λ^2 small and λ^2 large

$$N \sim \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\mu} + \frac{i}{\mu^2}, \qquad \mu^2, \lambda^2 \gg 1 \quad (16a)$$

$$N \sim \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\mu_\parallel} + \frac{i}{\mu_\parallel^2}, \qquad \mu^2 \gg 1, \qquad \lambda^2 \ll 1. \quad (16b)$$

In any case, for large μ^2 but arbitrary λ^2, we can write N as

$$1 + iN \sim 1 + i\epsilon, \quad (17)$$

where ϵ is small.

Therefore, for our case of $\theta_+ \to 0$, $\mu_-^2 \gg 1$, Eq. (14) reads

$$\left[K^2 + 1 + i\epsilon - \frac{\omega_{p+}^2}{\omega^2(\omega^2 - \omega_{c+}^2)} (K^2\omega^2 - K_\parallel^2 \omega_{c+}^2) \right] \phi^*$$

$$= 4\pi Q \lambda_D^2 h^*(\mathbf{k}) \, \delta(\omega - \mathbf{k}\cdot\mathbf{V}_0) \quad (18)$$

$$\mathbf{K} = \mathbf{k}\lambda_D, \qquad \lambda_D^2 = \theta_-/4\pi n_0 e^2.$$

DISPERSION RELATIONSHIP

The dispersion relationship for the low-frequency electrostatic oscillations can be found by equating the coefficient of ϕ^* in Eq. (18) to zero.

For the special case of $K = K_\parallel$, the dispersion relationship[4] is given by

$$\omega = \pm \frac{K\omega_{p+}}{(1 + K^2)^{\frac{1}{2}}} - \frac{iK\omega_{p+}}{(1 + K^2)^2} \left(\frac{m_-\pi}{m_+ 8} \right)^{\frac{1}{2}}. \quad (19)$$

This equation is independent of B_0 as expected, since charged particles oscillating parallel to the lines of force should not be affected by the magnetic field. We cannot consider the case of $K = K_\perp$, since our previous analysis breaks down in this limit.

For arbitrary direction of propagation, the dispersion relationship reads

$$\omega^4 - \omega^2 \left(\frac{K^2 \omega_{p+}^2}{1 + K^2 + i\epsilon} + \omega_{c+}^2 \right)$$

$$+ \frac{K_\parallel^2 \omega_{p+}^2 \omega_{c+}^2}{1 + K^2 + i\epsilon} = 0. \quad (20)$$

For $\omega_{c+} = 0$, this yields Eq. (19). At the other extreme, for $\omega_{c+} \to \infty$ and $\omega \ll \omega_{c+}$, we get

$$\omega = \pm \frac{K_\parallel \omega_{p+}}{(1 + K^2)^{\frac{1}{2}}} - \frac{iK_\parallel \omega_{p+}}{(1 + K^2)^2} \left(\frac{m_-\pi}{m_+ 8} \right)^{\frac{1}{2}}. \quad (21)$$

For this extreme, the plasma behaves as though the particles were tied to the lines of force.

For arbitrary direction of propagation and arbitrary magnetic field, Eq. (20) has two roots, ω_1 and ω_2, corresponding to two longitudinal disturbances. The two roots and the corresponding group velocities of the disturbances can be found approximately for $\omega_{c+}/\omega_{p+} \ll 1$, and for $\omega_{c+}/\omega_{p+} \gg 1$.

If we let β be the angle between the propagation vector and the magnetic field, we get approximately (neglecting the Landau damping)

[4] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), p. 61.

(a) $\alpha \equiv \frac{\omega_{c+}}{\omega_{p+}} \ll 1$ and $K^2 \gg \alpha^2$

$$\omega_1^2 \equiv \omega_{p+}^2\left(\frac{K^2}{1+K^2} + \alpha^2 \sin^2\beta\right)$$

$$V_{1g} = \left(\frac{\theta_-}{m_+}\right)^{\frac{1}{2}} \frac{1}{(1+K^2)^{\frac{3}{2}}} \cdot\left[1 - \frac{\alpha^2}{2}\sin^2\beta\,\frac{(1+K^2)}{K^2}\right] \tag{22}$$

$$\omega_2^2 = \omega_{c+}^2 \cos^2\beta\left[1 - \alpha^2\sin^2\beta\left(\frac{1+K^2}{K^2}\right)\right]$$

$$V_{2g} = \frac{\alpha^3}{K^3}\sin^2\beta\cos\beta\left(\frac{\theta_-}{m_+}\right)^{\frac{1}{2}}$$

(b) $\alpha \gg 1$

$$\omega_1^2 = \omega_{p+}^2\frac{K^2}{1+K^2}\cos^2\beta$$

$$V_{1g} = \left(\frac{\theta_-}{m_+}\right)^{\frac{1}{2}}\frac{1}{(1+K^2)^{\frac{3}{2}}}\cos\beta$$

$$\omega_2^2 = \omega_{c+}^2 + \omega_{p+}^2\frac{K^2}{1+K^2}\sin^2\beta$$

$$V_{2g} = \left(\frac{\theta_-}{m_+}\right)^{\frac{1}{2}}\frac{K}{(1+K^2)^2}\frac{\sin^2\beta}{\alpha},$$

where $V_g = (d\omega/dK)\lambda_D$ = the group velocity.

For the weak magnetic field, the fast wave corresponds to the expected ion electrostatic wave with a frequency slightly larger and a group velocity slightly smaller than the corresponding quantities in the absence of the magnetic field. In addition, there is a slow wave corresponding to oscillations at a frequency of the order of the ion cyclotron frequency. The group velocity of this wave goes to zero parallel to and perpendicular to the magnetic field, and has a maximum at $\sin^2\beta = \frac{2}{3}$. For large values of K (short wavelengths) both V_{1g} and V_{2g} are proportional to $1/K^3$ with $V_{2g} \sim \alpha^3 V_{1g}$.

For a very strong field, the fast wave is again the expected ion wave with a $\cos\beta$ factor in the frequency and group velocity which expresses the tendency of the strong magnetic field to inhibit oscillations perpendicular to the field. The slow wave again corresponds to oscillations at a frequency of order ω_{c+}. In the short wavelength limit $V_{2g} \to (\sin^2\beta/\alpha\cos\beta)V_{1g}$, both being proportional to $1/K^3$.

In all cases, we have considered only longitudinal waves ($\nabla \times \mathbf{E} = 0$), and we have not considered the effects of the possible coupling between longitudinal and transverse waves in the magnetic field.

ENERGY LOSS

The energy loss dW/ds per unit path length can be found from the force on the particle[1]

$$\frac{dW}{dS} = Q\frac{\mathbf{V}_0}{V_0}\cdot\mathbf{E}(\mathbf{r} - \mathbf{V}_0 t = 0). \tag{23}$$

The electric field is given by

$$\mathbf{E} = -\frac{4\pi i Q}{\lambda_D^2(2\pi)^3}\int\frac{\mathbf{K}h^*(\mathbf{K})\omega^2(\omega^2 - \omega_{c+}^2)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,\delta(\omega - \mathbf{k}\cdot\mathbf{V}_0)\,d\mathbf{k}\,d\omega}{(1+K^2+i\epsilon)\{\omega^4 - \omega^2[K^2\omega_{p+}^2/(1+K^2+i\epsilon)] + [K_\parallel^2\omega_{p+}^2\omega_{c+}^2/(1+K^2+i\epsilon)]\}}. \tag{24}$$

In performing the integrations over $\mathbf{K}$, it is convenient to use a coordinate system in which the 1-direction is parallel to $\mathbf{V}_0$ and the 2-direction is in the plane containing $\mathbf{V}_0$ and $\mathbf{B}_0$. By using cylindrical coordinates K_1, κ, and ζ for $\mathbf{K}$, the energy loss is given by

$$\frac{dW}{dS} = \tfrac{1}{2}\left(\lim_{R_1\to 0+} + \lim_{R_1\to 0-}\right)\left[\frac{-4\pi i Q^2}{\lambda_D^2(2\pi)^3}\right]\int_0^\infty \kappa\,d\kappa \cdot\int_0^{2\pi}d\zeta\int_{-\infty}^{\infty}\frac{h^*\,dK_1 e^{iK_1R_1}K_1^3(K_1^2 - \delta^2\alpha^2)}{D} \tag{25}$$

$$D = K_1^6 + K_1^4[1+\kappa^2+i\epsilon - \delta^2(1+\alpha^2)] - K_1^2\,\delta^2[\alpha^2(1+\kappa^2+i\epsilon-\delta^2\cos^2\psi)+\kappa^2] - 2K_1\kappa\cos\psi\sin\psi\cos\zeta\,\delta^4\alpha^2 + \delta^4\alpha^2\kappa^2\cos^2\zeta\sin^2\psi$$

$\delta^2 = \theta_-/m_+V_0^2 \ll 1$ ψ = angle between $\mathbf{V}_0$ and $\mathbf{B}_0$

$\alpha^2 = \omega_{c+}^2/\omega_{p+}^2$ $R_1 = (r_1 - V_0 t)/\lambda_D$.

For $\mathbf{V}_0$ parallel to $\mathbf{B}_0$, the roots of the denominator can be easily approximated for arbitrary α

$$D = K_1^2\left\{K_1^2 + 1 + \kappa^2 + i\epsilon - \delta^2\frac{[1-\delta^2(1-\alpha^2)]}{1+\kappa^2-\delta^2(1-\alpha^2)}\right\} \cdot\left\{K_1^2 - \delta^2\left[\alpha^2 + \frac{\kappa^2}{i\epsilon+1+\kappa^2-\delta^2(1-\alpha^2)}\right]\right\}. \tag{26}$$

For arbitrary values of ψ, the roots of the denominator can be found approximately for small and large values of α. Neglecting ϵ, we get

$$\alpha \ll 1$$

$$D \cong (K_1^2+1+\kappa^2)(K_1^2 - \delta^2\alpha^2\cos^2\zeta\sin^2\psi) \cdot\left(K_1^2 - \delta^2\left\{\frac{\kappa^2}{1+\kappa^2} + \alpha^2(1-\sin^2\psi\cos^2\zeta) + \frac{\delta^2\alpha^2}{1+\kappa^2}\left[\sin^2\psi - \frac{\kappa^2}{1+\kappa^2}\cdot(1-\cos^2\zeta\sin^2\psi)\right]\right\}\right); \tag{27}$$

$$\alpha \gg 1$$

$$D \cong (K_1^2 + 1 + \kappa^2)\left(K_1^2 - \frac{\delta^2\kappa^2\cos^2\zeta\sin^2\psi}{1+\kappa^2}\right) \cdot \left\{K_1^2 - \delta^2\left[\alpha^2 + \frac{\kappa^2(\sin^2\zeta + \cos^2\zeta\cos^2\psi) + \delta^2\alpha^2\sin^2\psi \pm 2\,\delta\alpha\kappa\cos\zeta\sin\psi\cos\psi}{1+\kappa^2+\delta^2\alpha^2}\right]\right\}.$$

If ϵ is neglected, in the integration over K_1 in Eq. (25), the contour is chosen so as to yield the retarded solution.

Finally, for the purpose of calculating the energy loss, D can be approximated for arbitrary ψ and α by the following expression

$$D \cong (1 + K_1^2 + \kappa^2)(K_1^4 + bK_1^2 + C)$$

$$b = -\delta^2 \cdot \left[\alpha^2 + \frac{\kappa^2 + \delta^2\alpha^2\sin^2\psi\left(1 + \frac{\kappa^2}{1+\kappa^2}\cos^2\zeta\right)}{1+\kappa^2+\delta^2\alpha^2}\right] \tag{28}$$

$$C = \frac{\delta^4\alpha^2\kappa^2\cos^2\zeta\sin^2\psi}{1+\kappa^2}.$$

The $\cos\zeta$ term has been neglected, since it averages out to zero in the energy loss calculation.

In calculating the energy loss, we will assume that the charged object has small but finite dimensions. The function h^* effectively imposes a cutoff[2] on the integral over κ at a value roughly of order κ_M, where

$$\kappa_M \sim \lambda_D/r_0 \equiv 1/R_0 \gg 1, \tag{29}$$

and r_0 is a characteristic size of the particle.

The results for the energy loss are given by

$$\delta^2\alpha^2 \ll 1/R_0^2 \tag{30a}$$

$$\frac{dW}{dS} = -\frac{2\pi n_0 Q^2 e^2}{m_+ V_0^2}\left[\ln\left(1 + \frac{1}{R_0^2}\right) - \frac{(1+\delta^2\alpha^2\cos^2\psi)}{1+\delta^2\alpha^2} - \frac{(1+\cos^2\psi)}{2}\ln(1+\delta^2\alpha^2)\right]$$

$$\delta^2\alpha^2 \ll 1$$

$$\frac{dW}{dS} = -\frac{2\pi n_0 Q^2 e^2}{m_+ V_0^2} \cdot \left[\ln\left(1 + \frac{1}{R_0^2}\right) - 1 + \delta^2\alpha^2\left(\tfrac{3}{2}\sin^2\psi - 1\right)\right]$$

$$\delta^2\alpha^2 \gg 1$$

$$\frac{dW}{dS} \cong -\frac{2\pi n_0 Q^2 e^2}{m_+ V_0^2}\left[\ln\left(1 + \frac{1}{R_0^2}\right) - \cos^2\psi - \frac{(1+\cos^2\psi)}{2}\ln(1+\delta^2\alpha^2)\right]$$

$$\delta^2\alpha^2 \gg 1/R_0^2 \tag{30b}$$

$$\frac{dW}{dS} \cong -\frac{\pi n_0 Q^2 e^2}{m_+ V_0^2} \cdot \left\{\sin^2\psi\left[\ln\left(1 + \frac{1}{R_0^2}\right) - 1\right] + \left(\frac{1}{R_0\,\delta\alpha}\right)^4\right\}.$$

We can see that the energy loss decreases with increasing magnetic field strength for $\sin^2\psi$ less than $\frac{2}{3}$. For $\sin^2\psi$ greater than $\frac{2}{3}$, the energy loss first increases with increasing field strength and then decreases. The maximum occurs at $\delta\alpha = 1$. This maximum energy loss is given by

$$\frac{(dW/dS)_{\delta\alpha=1} - (dW/dS)_{\delta\alpha=0}}{(dW/dS)_{\delta\alpha=0}} \cong \frac{\sin^2\psi - (1+\cos^2\psi)\ln 2}{2\ln(1+1/R_0^2)}, \quad \sin^2\psi > \tfrac{2}{3}. \tag{31}$$

For motion perpendicular to the magnetic field ($\sin^2\psi = 1$), this becomes

$$\frac{(dW/dS)_{\delta\alpha=1} - (dW/dS)_{\delta\alpha=0}}{(dW/dS)_{\delta\alpha=0}} \cong \frac{0.15}{\ln(1+1/R_0^2)}. \tag{32}$$

For arbitrary values of ψ, the energy loss approaches $(\sin^2\psi/2)(dW/dS)_{\alpha=0}$ as $\alpha \to \infty$. Thus, in the limit of infinite field strength, the energy loss goes to zero for motion parallel to the magnetic field, but approaches $\frac{1}{2}$ the energy loss for zero field for motion perpendicular to the magnetic field.