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James P. Crawford

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The Dirac oscillator and local automorphism invariance

James P. Crawford
Department of Physics, Penn State-Fayette, Uniontown, Pennsylvania 15401

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The Dirac oscillator is a relativistic generalization of the quantum harmonic oscillator. In particular, the square of the Hamiltonian for the Dirac oscillator yields the Klein–Gordon equation with a potential of the form $(ar^2+b\mathbf{L}\cdot\mathbf{S})$, where a and b are constants. To obtain the Dirac oscillator, a "minimal substitution" is made in the Dirac equation, where the ordinary derivative is replaced with a covariant derivative. However, a very unusual feature of the covariant derivative in this case is that the potential is a nontrivial element of the Clifford algebra. A theory which naturally gives rise to gauge potentials which are nontrivial elements of the Clifford algebra is that based on local automorphism invariance. An exact solution of the pure automorphism gauge field equations which reproduces both the potential term and the mass term of the Dirac oscillator is presented herein.

I. INTRODUCTION

The Dirac oscillator exhibits many interesting features. It is the relativistic generalization of the classic nonrelativistic harmonic oscillator Schrödinger equation to the Dirac equation in the sense that the square of the Dirac Hamiltonian yields the relativistic Klein–Gordon equation for the spinor fields with a potential of the form $(ar^2+b\mathbf{L}\cdot\mathbf{S})$.[1–3] Here a and b are constants and we recognize this potential as that of the harmonic oscillator with an additional spin–orbit term. The Dirac oscillator is exactly solvable as in the nonrelativistic case,[4] and exhibits a hidden supersymmetry.[4–7] In addition, this particular form of potential has been used to model the interquark interactions in the hope of obtaining a realistic model of the hadrons.[8,9] Finally, an interesting version involving a different ("scalar") coupling has been investigated.[10]

A highly unusual feature of the Dirac oscillator is that the potential which is introduced as a "minimal substitution" is a nontrivial element of the Clifford algebra. This is to be contrasted with all "usual" gauge theories where the potentials are Clifford scalars (that is, the potentials multiply the unit element of the algebra). A theory which naturally incorporates gauge potentials which are general elements of the Clifford algebra is that based upon local automorphism invariance.[11,12] The basic idea underlying automorphism gauge theory is found in the observation that the particular matrix representation chosen for the Clifford algebra generators should not effect the physical predictions of the theory. If we then demand that this freedom of choice be allowed locally we obtain automorphism gauge theory.

In this article I present a set of exact "chiral" solutions of the pure automorphism gauge field equations which reproduces both the potential term and the mass term of the Dirac oscillator as a special case. In addition I show that the Dirac oscillator case is essentially unique in that the chiral solutions for the gauge fields can always be brought into the Dirac oscillator form by an appropriate gauge transformation.

II. THE DIRAC OSCILLATOR

The equation for the Dirac oscillator is most easily obtained by performing a "minimal substitution" of a covariant derivative for the ordinary derivative in the Dirac equation.[3,4] Written in terms of the momentum this substitution has the form

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$$\mathbf{p} \to \mathbf{p} - im\omega\beta\mathbf{r}, \tag{1}$$

where m is the mass of the particle and ω characterizes the strength of the interaction. Notice that the gauge potential is dependent upon a Clifford algebra generator ($\beta = \gamma_0$ is the Dirac matrix). This suggests therefore that the theory might be derived from the automorphism gauge theory, since in this case the gauge potentials naturally occur as general elements of the Clifford algebra (see the next section). This is the essence of my point of view, but I will first complete this review with the construction of the covariant form of the Dirac oscillator equation.

To obtain the covariant form of the Dirac oscillator equation we introduce a unit timelike four-vector u_μ and an antisymmetric tensor $r_{\mu\nu}$ formed from the timelike unit vector and the space-time coordinate vector

$$u^\mu u_\mu = 1, \quad r_{\mu\nu} \equiv (u_\mu x_\nu - u_\nu x_\mu). \tag{2}$$

In the "rest frame" these take the form

$$u_\mu = (1,0,0,0), \quad r_{0i} = x_i, \quad r_{ij} = 0. \tag{3}$$

Now the covariant Dirac equation may be written as

$$(\gamma^\mu p_\mu - m + \tfrac{1}{2} m\omega r_{\mu\nu}\gamma^{\mu\nu})\Psi = 0, \tag{4}$$

where the matrices $\gamma_{\mu\nu}$ are the bivector elements of the Clifford algebra basis.[13]

This equation has an electromagnetic interpretation as a particle with **zero charge** interacting via a **magnetic dipole moment** with a **radial electric field**. In this case the vector u_μ may be considered the four-velocity of the center of the electric field. Notice that **only the product of the magnetic moment and the electromagnetic field may be determined from this construction**, and in particular we find

$$\frac{\kappa e}{2m} F_{\mu\nu} = m\omega r_{\mu\nu}. \tag{5}$$

We may solve this equation for the classical angular frequency ω once the mass, magnetic moment, and electric field strength have been specified

$$(\mathbf{E})_i = -F_{0i}, \quad \mathbf{E} \equiv Mm^2\mathbf{r} \Rightarrow \omega = \tfrac{1}{2}\kappa e M, \tag{6}$$

where M is a parameter with dimensions of mass which characterizes the electric field. Note that I disagree with Benìtez *et al.*[4] on this point. In fact, their definitions of both the field strength tensor [Eq. (6) of their article] and the anomalous magnetic moment [just below Eq. (6) in their article] are dimensionally unsound.

Finally note that the electromagnetic interpretation is valid as long as we take Eq. (4) as our starting point. However, if we wish to view this equation as arising from a minimal substitution of a covariant derivative for an ordinary derivative as in Eq. (1), then the electromagnetic interpretation is untenable.

III. LOCAL AUTOMORPHISM INVARIANCE

I now approach the problem from the point of view of local automorphism invariance.[11,12,14] Although the theory of local automorphism invariance may be developed in spaces of arbitrary dimension and signature, we will restrict our attention to the case of four-dimensional space–time. If we assume that the particular matrix representation of the Clifford algebra generators may be chosen arbitrarily **at each point in space**, then we obtain a gauge theory based on the automorphism group U(2,2) (in four dimensions the automorphism group is isomorphic to the conformal group[14]). An automorphism transformation of the Clifford algebra generators may also be conceptualized as a transformation of the basis of the spinor space. Hence the transformation of the spinor fields is given by

$$\Psi' = \Theta\Psi \Rightarrow \bar{\Psi}' = \bar{\Psi}\bar{\Theta}, \tag{7}$$

where the "bar" indicates usual Dirac conjugation, and the transformation matrix is given by

$$\Theta = e^{i\theta^a \Gamma_a}, \quad \bar{\Theta} = e^{-i\theta^a \Gamma_a} \Rightarrow \Theta\bar{\Theta} = \bar{\Theta}\Theta = \mathbf{1}. \tag{8}$$

The matrices $\{\Gamma_a\}$ form a complete basis for the Clifford algebra[14] and the quantities θ^a are the real group parameters.

To incorporate this local invariance into the theory, the ordinary derivative must be replaced with the covariant derivative

$$\partial_\mu \to D_\mu = \partial_\mu + igA_\mu, \tag{9}$$

where the "spin connection = gauge potential" is a general element of the Clifford algebra[11–14]

$$A_\mu = a_\mu \mathbf{1} + a^\rho{}_\mu \gamma_\rho + \tfrac{1}{2} a^{\rho\sigma}{}_\mu \gamma_{\rho\sigma} - b^\rho{}_\mu \tilde{\gamma}_\rho - b_\mu \tilde{\gamma} \tag{10}$$

and the gauge potential transforms in the usual way

$$A'_\mu = \Theta A_\mu \bar{\Theta} + \frac{i}{g} (\partial_\mu \Theta)\bar{\Theta}. \tag{11}$$

This transformation law guarantees that the covariant derivative of a spinor transforms as a spinor under the automorphism transformations.

For the "spin curvature = field strength tensor" we find

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] = f_{\mu\nu}\mathbf{1} + f^\rho{}_{\mu\nu}\gamma_\rho + \tfrac{1}{2} f^{\rho\sigma}{}_{\mu\nu}\gamma_{\rho\sigma} - h^\rho{}_{\mu\nu}\tilde{\gamma}_\rho - h_{\mu\nu}\tilde{\gamma}. \tag{12}$$

The explicit form for the field strength tensor in terms of the gauge potentials can be found by the substitution of Eq. (10) into Eq. (12).

Since the gauge potential is a nontrivial element of the Clifford algebra and therefore does not commute with the Clifford algebra generators, care must be taken when performing the minimal substitution of the covariant derivative for the ordinary derivative.[11] We take the "symmetric" form of the free Dirac Lagrangian density as our starting point since this choice generates a manifestly real interacting Lagrangian density

$$L_\Psi = \frac{i}{2} \bar{\Psi}(\gamma^\mu \vec{\partial}_\mu - \overleftarrow{\partial}_\mu \gamma^\mu)\Psi \to \frac{i}{2} \bar{\Psi}(\gamma^\mu \vec{D}_\mu - \overleftarrow{D}_\mu \gamma^\mu)\Psi = \left(\frac{i}{2} \bar{\Psi}\gamma^\mu D_\mu \Psi\right) + \overline{\left(\frac{i}{2} \bar{\Psi}\gamma^\mu D_\mu \Psi\right)}, \tag{13}$$

where the second equality indicates that the interacting Lagrangian density so constructed is indeed real as desired.

The explicit form of the interacting Dirac Lagrangian density is now found to be

$$L_\Psi = \frac{i}{2} \bar{\Psi}(\gamma^\mu \vec{\partial}_\mu - \overleftarrow{\partial}_\mu \gamma^\mu)\Psi - g\bar{\Psi}(\Phi\mathbf{1} + a^\mu \gamma_\mu + 3\tilde{a}^\mu \tilde{\gamma}_\mu - \tilde{b}^{\mu\nu}\gamma_{\mu\nu})\Psi, \tag{14}$$

where we have made the definitions

$$\Phi \equiv a^{\rho}{}_{\rho}, \quad \tilde{a}_{\mu} \equiv \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} a^{\nu\rho\sigma}, \quad \tilde{b}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} b^{\rho\sigma} \tag{15}$$

and we see that the automorphism gauge fields couple to the fermion field through scalar, vector, pseudovector, and bivector (spin) interactions. Notice that we have not included the mass term in the basic Lagrangian since the scalar coupling (Yukawa interaction) will give rise to mass. Also note that there is no pseudoscalar interaction nor does the vector gauge potential b_{μ} associated with pseudoscalar transformations couple to the fermion. It is the symmetric form of the Lagrangian density which selects this particular subset of the gauge potentials for interaction with the spinor, and this is one of the interesting ways in which automorphism gauge theory differs from usual gauge theory. In general, the specific structure of this interaction depends on the dimension and signature of space–time.

We now make the observation that the interacting Lagrangian density (14) will yield the equation for the Dirac oscillator (4) if the following conditions are met:

$$g\Phi = m, \quad g\tilde{b}_{\mu\nu} = \tfrac{1}{2} m\omega r_{\mu\nu}, \quad a_{\mu} = 0, \quad \tilde{a}_{\mu} = 0. \tag{16}$$

It is remarkable that these particular expressions for the potentials form a subset of exact "chiral" solutions to the pure gauge field equations.

To obtain the field equations for the gauge potentials we proceed in the usual manner. Starting with the pure gauge field Lagrangian density

$$L_A = -\tfrac{1}{16} \operatorname{tr}(F^{\mu\nu} F_{\mu\nu}) \tag{17}$$

and demanding that the action be stationary with respect to arbitrary variations of the gauge fields results in the field equations

$$D_{\mu} F^{\mu\nu} = \partial_{\mu} F^{\mu\nu} + ig[A_{\mu}, F^{\mu\nu}] = 0. \tag{18}$$

Notice that we have not included the fermion source term in this equation. This is in concordance with the original approach to the Dirac oscillator in that the potential is introduced as an **external field**. Of course, even if we treat the gauge potentials as external fields, we must demand that they satisfy Eq. (18).

IV. CHIRAL SOLUTIONS

We now consider a special subset of solutions to Eq. (18). Consider the "chiral ansatz" defined as

$$a_{\mu} = 0, \quad a^{\rho\sigma}{}_{\mu} = 0, \quad b_{\mu} = 0, \quad b^{\rho}{}_{\mu} = \pm a^{\rho}{}_{\mu}. \tag{19}$$

We define the right-handed and left-handed chiral vectors of the Clifford algebra as

$$\rho_{\mu} \equiv \tfrac{1}{2}(\gamma_{\mu} - \tilde{\gamma}_{\mu}), \quad \lambda_{\mu} \equiv \tfrac{1}{2}(\gamma_{\mu} + \tilde{\gamma}_{\mu}) \tag{20}$$

and the gauge potential becomes

$$A_{\mu} = 2a^{\nu}{}_{\mu} \rho_{\nu}, \tag{21}$$

where we have chosen the plus sign in Eq. (19) for convenience. Another set of solutions may be obtained in the same fashion by choosing the minus sign.

Now observe that the chiral vectors form an Abelian subalgebra of the full Clifford algebra (viewed as a Lie algebra)

$$[\rho_\mu, \rho_\nu]=0, \quad [\lambda_\mu, \lambda_\nu]=0. \tag{22}$$

Thus we obtain from Eq. (12)

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu=2(\partial_\mu a^\kappa{}_\nu-\partial_\nu a^\kappa{}_\mu)\rho_\kappa=2f^\kappa{}_{\mu\nu}\rho_\kappa \tag{23}$$

and for the field equations (18) we find

$$D_\mu F^{\mu\nu}=\partial_\mu F^{\mu\nu}=2(\partial_\mu f_\kappa{}^{\mu\nu})\rho^\kappa=0. \tag{24}$$

Summarizing these results we have

$$f^\kappa{}_{\mu\nu}=\partial_\mu a^\kappa{}_\nu-\partial_\nu a^\kappa{}_\mu=h^\kappa{}_{\mu\nu}, \tag{25a}$$

$$\partial_\mu f_\kappa{}^{\mu\nu}=0. \tag{25b}$$

As expected, for the Abelian subgroup the field equations are linear.

There are, of course, many solutions to Eqs. (25), but consider the special case in which the field strength tensor is constant and uniform. In this case Eq. (25b) is clearly satisfied. To satisfy Eq. (25a) an obvious choice is to assume that the potential is linear in the space–time coordinate. Therefore we write

$$a^\kappa{}_\mu=c_1 m(1+d_1 m(u\cdot x))g^\kappa{}_\mu+c_2 m(1+d_2 m(u\cdot x))u^\kappa u_\mu+c_3 m^2 r^\kappa{}_\mu+c_4 m^2 \tilde{r}^\kappa{}_\mu+c_5 m^2 s^\kappa{}_\mu, \tag{26}$$

where the coefficients c_i and d_i are arbitrary dimensionless constants, and we have defined

$$\tilde{r}_{\mu\nu}\equiv\tfrac{1}{2}\epsilon_{\mu\nu\kappa\tau}r^{\kappa\tau}, \quad s_{\mu\nu}\equiv(u_\mu x_\nu+u_\nu x_\mu). \tag{27}$$

The parameter m is a quantity with the dimension of mass and it may be conveniently chosen to be the mass appearing in the Dirac oscillator. Finally note that Eq. (26) is the most general form which is both linear in the space–time coordinate and which involves only one arbitrary constant four-vector.

The field strength tensor may be calculated directly from Eq. (25a)

$$f^\kappa{}_{\mu\nu}=m^2(c_1 d_1+c_3-c_5)(u_\mu g^\kappa{}_\nu-u_\nu g^\kappa{}_\mu)+2m^2 c_4\epsilon^\kappa{}_{\mu\nu\tau}u^\tau. \tag{28}$$

As expected, the field strength tensor is constant and uniform, and therefore trivially satisfies Eq. (25b). For the part of the gauge potential which interacts directly with the fermion [see Eqs. (14) and (15)] we find

$$\Phi=(4c_1+c_2)m+(4c_1 d_1+c_2 d_2+2c_5)m^2(u\cdot x), \tag{29a}$$

$$\tilde{b}_{\mu\nu}=-c_3 m^2\tilde{r}_{\mu\nu}+c_4 m^2 r_{\mu\nu}, \quad a_\mu=0, \quad \tilde{a}_\mu=0. \tag{29b}$$

Notice that only the symmetric part of the gauge potential contributes to the scalar interaction [Eq. (29a)], and only the antisymmetric part of the gauge potential contributes to the spin interaction [Eq. (29b)]. This statement is generally true as may be seen by the inspection of Eq. (15).

We may now recover the Dirac oscillator interactions as a special case if we make the following choices [compare Eqs. (16) with Eqs. (29)]:

$$g(4c_1+c_2)=1, \quad 2gmc_4=\omega, \tag{30a}$$

$$(4c_1d_1+c_2d_2+2c_5)=0, \quad c_3=0. \tag{30b}$$

Actually, Eqs. (30a) may be viewed as simply a choice of scale, and Eqs. (30b) then guarantee the Dirac oscillator form of the potential. It is interesting to consider the case in which Eqs. (30b) are not satisfied. In particular, we would then also have a linearly time dependent scalar interaction, and a radial "magnetic" field coupling to the magnetic dipole moment (if we resort to the electromagnetic analogy). However, **Eqs. (30) can always be satisfied by an appropriate choice of gauge**. In other words, the potentials appearing in the Dirac oscillator (where we are including the mass as a constant scalar potential) are essentially unique chiral solutions of the automorphism gauge field equations, since the potentials may always be brought into this form [Eqs. (16)] by a chiral gauge transformation. To see this we must consider the chiral gauge transformations.

V. CHIRAL GAUGE TRANSFORMATIONS

Under arbitrary automorphism transformations (11) the gauge potentials do not generally maintain their chiral form. Therefore, we will restrict our attention to the chiral transformations[15] defined as follows:

$$\Xi=e^{i\xi^\nu\rho_\nu}, \quad \bar{\Xi}=e^{-i\xi^\nu\rho_\nu}, \tag{31}$$

where the quantities ξ^ν are the real parameters of the chiral subgroup, and we have chosen the right-handed transformation in agreement with our choice for the chiral potential [Eq. (21)]. These transformations form an Abelian subgroup of the full automorphism group as may be seen by the inspection of Eq. (22).

For these chiral transformations, the transformation law for the gauge potential becomes

$$A'_\mu=\Xi A_\mu\bar{\Xi}+\frac{i}{g}(\partial_\mu\Xi)\bar{\Xi}=A_\mu-\frac{1}{g}(\partial_\mu\xi^\nu)\rho_\nu \tag{32a}$$

$$\Rightarrow a^{\nu\prime}{}_\mu=a^\nu{}_\mu-\frac{1}{2g}\partial_\mu\xi^\nu, \tag{32b}$$

which is the expected form for an Abelian group.

Now we wish the gauge potential to remain a linear function of the space–time coordinates, so we consider the chiral transformation parameters ξ^ν to be quadratic functions of the space-time coordinates. The most general form (ignoring constant terms) involving only one constant four-vector may be written as follows:

$$\xi^\nu=2gm\{e_1x^\nu+e_2(u\cdot x)u^\nu+e_3m(u\cdot x)x^\nu+e_4m(u\cdot x)^2u^\nu\}, \tag{33}$$

where the e_i are arbitrary constants. Substitution of Eqs. (26) and (33) into Eq. (32b) now yields

$$a'_{\nu\mu}=m[(c_1-e_1)+(c_1d_1-e_3)m(u\cdot x)]g_{\nu\mu}+m[(c_2-e_2)+(c_2d_2-2e_4)m(u\cdot x)]u_\nu u_\mu$$
$$+m^2[(c_3+\tfrac{1}{2}e_3)r_{\nu\mu}+c_4\tilde{r}_{\nu\mu}+(c_5-\tfrac{1}{2}e_3)s_{\nu\mu}]. \tag{34}$$

As a check on this expression we may calculate the field strength tensor by substitution of Eq. (34) into Eq. (25b), yielding a result that is identical to Eq. (28).

If we now calculate the potentials which interact with the fermion field we find

$$\Phi' = [4(c_1 d_1 - e_3) + (c_2 d_2 - 2e_4) + (2c_5 - e_3)] m^2 (u \cdot x) + [4(c_1 - e_1) + (c_2 - e_2)] m, \tag{35a}$$

$$\widetilde{b}_{\mu\nu}{}' = -(c_3 + \tfrac{1}{2} e_3) m^2 \widetilde{r}_{\mu\nu} + c_4 m^2 r_{\mu\nu}. \tag{35b}$$

Now observe that if we choose the chiral transformation parameters as follows:

$$e_3 = -2c_3, \quad 4e_1 + e_2 = 4c_1 + c_2 - \frac{1}{g}, \tag{36a}$$

$$e_4 = 2c_1 d_1 + 5c_3 + \tfrac{1}{2} c_2 d_2 + c_5 \tag{36b}$$

we obtain

$$g\Phi' = m, \quad g\widetilde{b}_{\mu\nu}{}' = g c_4 m^2 r_{\mu\nu}, \tag{37}$$

which generates the Dirac oscillator (16). Therefore, the class of chiral potentials given in Eq. (26) may always be brought into a form which reproduces the Dirac oscillator potentials (16) by the chiral transformations given in Eqs. (31), (33), and (36).

There is a another very interesting and unusual aspect of these gauge transformations. Instead of the choice of transformation parameters given in Eq. (36a) we could just as well consider the following:

$$4e_1 + e_2 = 4c_1 + c_2. \tag{38}$$

In this case there would be no mass term appearing in the Dirac equation! This is a very peculiar result—that a change of gauge can make the mass vanish. Note that even if we had included a "bare" mass term in the original Lagrangian density, we could still make the total mass (bare mass plus that arising from the constant scalar potential) vanish by an appropriate choice of chiral gauge transformations. However, these chiral gauge transformations may also be interpreted as translations since the right and left chiral vectors of the Clifford algebra behave as the translation and special conformal transformation generators of the conformal group.[14] Therefore, these transformations can be thought of as shifting some of the energy momentum of the spinor field into the gauge field (the total energy momentum is, of course, conserved).

VI. SUMMARY AND CONCLUSIONS

I have shown that both the mass and the potential introduced into the Dirac equation to produce the Dirac oscillator may be viewed as a special case of a class of chiral solutions to the automorphism gauge field equations. In addition, this chiral solution is essentially unique in that a gauge transformation can always be found which puts the potential in the form displayed in the Dirac oscillator.

To gain insight into the physical interpretation of this system consider as an analogy the more familiar situation of an electron interacting with a constant magnetic field. In this case, since the field strength is constant and uniform, the electromagnetic potential will be a linear function of the space–time coordinate. As is well-known,[16] this system exhibits harmonic oscillator behavior in the two spatial directions perpendicular to the magnetic field. Therefore, the point of view considered in this article is actually very much like this situation in that there is a constant and uniform field strength and a corresponding linear potential yielding harmonic oscillator behavior. This should be contrasted with the direct electromagnetic interpretation of a particle with zero charge and nonzero magnetic moment interacting with a linear electric field (with a corresponding quadratic potential). Notice, however, that the case of a constant automorphism field strength does not lend itself easily to the construction of hadrons as advocated by Moshinsky *et al.*[8,9] since we do not view each particle as giving rise to the automorphism field (though they certainly must contribute to the automorphism field, as does the electron to the magnetic field in the preceding analogy, but this is here taken to be a "higher order effect"). In other words, to build the mesons (for example) we may consider a linearly rising potential between the quark–antiquark pair, but the situation with a constant automorphism field is more akin to putting several electrons in a constant magnetic field and neglecting the interactions between them. Each electron undergoes cyclotron motion but the centers and the phases of the individual cyclotron orbits are uncorrelated. These two pictures are clearly at odds, and we therefore do not necessarily expect the approach discussed in this article to yield a model of the hadrons. In fact, the fundamental motivation for considering local automorphism invariance is that it may be considered to arise from a generalization of the principle of equivalence, and an important anticipated goal is a truly unified approach to the electroweak and gravitational interactions, but these issues will be discussed elsewhere.[17]

The approach to the Dirac oscillator discussed in this article naturally generalizes to space–times of arbitrary dimension and signature. In particular, the specific cases of two, three, five, and six dimensions are likely to generate interesting results. In addition, the question of the existence of exact solutions to the full field equations (that is, including the fermion source term in the gauge field equations) is an issue that should be addressed.

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[1] D. Itô, K. Mori, and E. Carrieri, Nuovo Cimento A **51**, 1119 (1967).
[2] P. A. Cook, Lett. Nuovo Cimento **1**, 419 (1971).
[3] M. Moshinsky and A. Szczepaniak, J. Phys. A **22**, L817 (1989).
[4] J. Benítez, R. P. Martínez y Romero, H. N. Núñez-Yépez, and A. L. Salas-Brito, Phys. Rev. Lett. **64**, 1643 (1990); **65**, 2085E (1990).
[5] J. Beckers and N. Debergh, Phys. Rev. D **42**, 1255 (1990).
[6] R. P. Martínez Y Romero, M. Moreno, and A. Zentella, J. Phys. A **22**, L821 (1989); Phys. Rev. D **43**, 2036 (1991).
[7] O. Castaños, A. Frank, R. López, and L. F. Urrutia, Phys. Rev. D **43**, 544 (1991).
[8] M. Moshinsky, G. Loyola, A. Szczepaniak, C. Villegas, and N. Aquino, in *Relativistic Aspects of Nuclear Physics*, edited by T. Kodama *et al.* (World Scientific, Singapore, 1990), pp. 271–307.
[9] M. Moshinsky, G. Loyola, and C. Villegas, *Relativistic Mass Formula for Baryons,* in Notas de Fisica 13, Proceedings of the Oaxtepec Conference of Nuclear Physics 1990, pp. 187–196.
[10] V. V. Dixit, T. S. Santhanam, and W. D. Thacker, J. Math. Phys. **33**, 1114 (1992).
[11] J. P. Crawford, "Local automorphism invariance: Gauge boson mass without a Higgs particle," submitted to J. Math. Phys.
[12] J. P. Crawford, J. Math. Phys. **31**, 1991 (1990). This reference contains some motivating comments concerning local automorphism invariance.
[13] References 12 and 14 contain my definitions and particular notation for the Clifford algebra basis elements. These definitions and notational conventions are useful when dealing with space–times of arbitrary dimension and signature. The relationship between my notation and that of Bjorken and Drell [J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964)] for the case of four-dimensional space–time is given as follows: $\gamma_{\mu\nu}=\sigma_{\mu\nu}$, $\tilde{\gamma}_\mu=-\gamma_5\gamma_\mu$, $\tilde{\gamma}=-i\gamma_5$.
[14] J. P. Crawford, J. Math. Phys. **32**, 576 (1991). This reference contains a discussion of the automorphism group of the Clifford algebra.
[15] These transformations have been considered in a different context by A. O. Barut and J. McEwan, Phys. Lett. B **135**, 172 (1984); **139**, 464 (1984); Lett. Math. Phys. **11**, 67 (1986).
[16] L. D. Huff, Phys. Rev. **38**, 501 (1931); M. H. Johnson and B. A. Lippman, *ibid.* **77**, 702 (1950).
[17] J. P. Crawford, "An extension of general relativity via local spinor bases," in preparation.