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Motivated by the fluctuating plume model of turbulent mixing downstream of a point source, a flapping model is formulated for application to other configurations. For the scalar mixing layer, simple expressions for single-point scalar fluctuation statistics are obtained that agree with measurements. For a spatially homogeneous scalar mixing field, the family of probability density functions previously derived using mapping closure is reproduced. It is inferred that single-point scalar statistics may depend primarily on large-scale flapping motions in many cases of interest, and thus that multipoint statistics may be the principal indicators of finer-scale mixing effects.

Gifford's fluctuating plume model[1,2] provides a simple characterization of scalar concentration fluctuations in the plume downstream of a continuous point source in turbulent flow. It is assumed that the instantaneous spatial distribution of concentration in a given transverse plane is of a specified functional form. Fluctuations are represented by random transverse displacements of this profile, characterized by a probability density function (pdf) of displacements. Assuming a Gaussian form both for the instantaneous spatial distribution and for the displacement pdf, Gifford obtained an expression for the single-point scalar pdf that was later shown to account for qualitative features of measured plume fluctuation statistics.[3] The experimental configuration did not allow a precise quantitative comparison. In any event, Gifford's model would appear to be no more than qualitative in view of the evident role of small eddies that generate complicated non-Gaussian instantaneous distributions. In fact, rigid displacements of the scalar concentration field have no effect on molecular mixing and thus cannot account for mixing enhancement by fluid motions.

Nevertheless, it is shown here that a simple flapping model motivated by Gifford's formulation can provide a good quantitative as well as qualitative characterization of single-point scalar fluctuation statistics for several configurations of interest. This does not imply that fluid motions in these configurations do not enhance mixing, but rather that single-point scalar fluctuation statistics may not be very sensitive to the mixing motions of principal interest.

Gifford considered an axisymmetric configuration. The configurations considered here are simpler in that they have planar symmetry. Denoting the streamwise coordinate as x and the transverse coordinates as y and z, the scalar mixing layer is the scalar field $c(x,y,z,t)$ that develops downstream of a transverse plane, denoted $x=0$, in which the scalar distribution $c(0,y,z,t)=H(y)$ is maintained (H denotes the Heaviside function). The scalar mixing layer has been studied experimentally in grid turbulence, with several different techniques employed to establish the scalar distribution in the plane $x=0$.[4,5] Measured transverse profiles of single-point scalar fluctuation statistics, depending only on y for given x, are found to relax to self-preserving functional forms, reflecting the relaxation of the scalar-to-velocity length-scale ratio to a constant value.[5,6] Here, measured self-preserving profiles of root-mean-square fluctuation c', skewness S, and kurtosis K are compared to profiles predicted using a flapping model formulated as follows.

Following Gifford, it is assumed that the single-point scalar fluctuation statistics reflect the flapping of a scalar profile of fixed functional form $\chi(y)$, specified shortly. The flapping is assumed to induce a Gaussian distribution of displacements $\hat{y}$ of this profile (with zero mean displacement), so that the instantaneous scalar profile is of the form $\chi(y-\hat{y})$. The cumulative distribution function (cdf) of the scalar at given y is denoted as $F(C)$ (suppressing the y dependence). Here, $F(C)$ is equal to $\mathrm{Prob}[c<C]=\mathrm{Prob}[\chi(y-\hat{y})<C]$. Taking χ to be an increasing function of y, denote as y_C the value of y for which $\chi(y)=C$. Then, $F(C)=\mathrm{Prob}[y-\hat{y}<y_C] = \mathrm{Prob}[\hat{y}>y-y_C] = 1-\mathrm{Prob}[\hat{y}<y-y_C]$. The pdf $f(C)$ at y is obtained by taking the C derivative of $F(C)$, giving

$$f(C)=-\frac{(\partial/\partial y_C)\mathrm{Prob}[\hat{y}<y-y_C]}{\partial C/\partial y_C}. \tag{1}$$

By assumption, the numerator is the Gaussian distribution of argument $y-y_C$ and (as yet undetermined) standard deviation m. The denominator is determined implicitly from the relation $\chi(y_C)=C$.

Consistent with conventional turbulent transport analysis,[7] measured far-field transverse profiles of the mean scalar $\bar{c}$ relax to an error-function form conveniently expressed as[5]

$$\bar{c}=\tfrac{1}{2}[1+\mathrm{erf}(0.477y/L)], \tag{2}$$

where L is the y location at which $\bar{c}=0.75$. It is convenient for analysis to express this as

$$\bar{c}=\tfrac{1}{2}[1+\mathrm{erf}(y/\sqrt{2}\sigma)], \tag{3}$$

where $\sigma=L/(0.477\sqrt{2})$ is the standard deviation of the $d\bar{c}/dy$ profile, which is Gaussian.

The function $\chi(y)$ is chosen so that the model yields a mean scalar profile of this form. This prescription can be implemented formally by evaluating the expectation value of C with respect to the pdf given by Eq. (1). The final result can be obtained more directly by taking the y derivative of the expression

$$\bar{c} = \frac{1}{\sqrt{2\pi} m} \int_{-\infty}^{\infty} d\hat{y} \exp\left(-\frac{\hat{y}^2}{2m^2}\right) \chi(y - \hat{y}) \tag{4}$$

for the mean profile given by the model. Since $d\bar{c}/dy$ is Gaussian and since the derivative of the right-hand side is equal to the convolution of $d\chi(y)/dy$ with a Gaussian profile, $d\chi(y)/dy$ is Gaussian in y with standard deviation $\sigma_0 = \sqrt{\sigma^2 - m^2}$. [This standard result is conveniently verified substituting this result for $d\chi(y)/dy$ into the y derivative of Eq. (4) and taking the Fourier transform.] Thus $\chi(y)$ has the same functional form as $\bar{c}(y)$, but with σ_0 in place of σ in Eq. (3).

Based on these results, the final expression for the pdf of C is

$$f(C) = \left(\sqrt{2\pi} m \frac{\partial C}{\partial y_C}\right)^{-1} \exp\left(-\frac{(y - y_C)^2}{2m^2}\right), \tag{5}$$

from which y_C can be eliminated using the relation

$$C = \tfrac{1}{2}[1 + \mathrm{erf}(y_C/\sqrt{2}\sigma_0)]. \tag{6}$$

The nth moment of C is obtained by integrating C^n times Eq. (5). A change of integration variable yields

$$\overline{c^n} = \frac{1}{2^n \sqrt{2\pi} m} \int_{-\infty}^{\infty} du \exp\left(-\frac{(u-y)^2}{2m^2}\right) \times \left[1 + \mathrm{erf}\left(\frac{u}{\sqrt{2}\sigma_0}\right)\right]^n. \tag{7}$$

Quantities such as c', S, and K readily evaluated using this result.

Before applying this result to the scalar mixing layer, implications with regard to stationary, homogeneous turbulent mixing fields are noted. Molecular diffusion causes a stirred fluid consisting initially of equal volumes of unmarked ($c = 0$) and marked ($c = 1$) components to develop error-function scalar profiles at local marked/unmarked interfaces. Convective distortion and the influence of neighboring interfaces cause the tails of the profiles to deviate from this form. The resulting mixing field may be idealized as a collection of clipped error-function profiles. Namely, each profile is of the form given by Eq. (6), but restricted to a finite interval $|y_C| \leqslant w$ in the direction locally normal to the interface, where w is a non-negative random variable whose cdf is denoted $H(w)$. This cdf may be viewed as representing the distribution of separations of neighboring interfaces. Under these assumptions, the relation

$$f(C) = \frac{\partial y_C}{\partial C} \frac{1 - H(|y_C|)}{2\int_0^1 w\, dH(w)} \tag{8}$$

is readily derived. As before, y_C is eliminated using Eq. (6). If the cdf of w is assumed to have the form $H(w) = 1 - \exp(-w^2/2m^2)$, then Eq. (8) becomes equivalent to Eq. (5) for the case $y = 0$.

Under these assumptions, Eqs. (5) and (6) with $y = 0$ may be interpreted as a family of concentration pdf's for mixing in stationary, homogeneous turbulence. The parameter m/σ_0 represents the ratio of interface wrinkling length scale to interface width. As this parameter decreases from infinity to zero, the pdf evolves from double-delta-function initial form to the final, fully mixed asymptote ($c = 1/2$ everywhere).

Equations (5) and (6) with $y = 0$ give the identical family of pdf's previously derived by applying mapping closure[8] to the problem of stationary, homogeneous mixing within a closed volume with equal amounts of two initially unmixed fluids.[9,10] It has been noted[9] that this family of pdf's agrees well with pdf's determined by direct numerical simulation of this configuration.[11] However, neither the present approach nor mapping closure provides a time parametrization of the predicted family of pdf's. Time development depends on dynamical aspects not treated by either approach.

The consistency of the flapping model with mapping closure in this instance merits further comment. First, the physical reasoning that motivates the application of Eqs. (5) and (6) to homogeneous mixing is not based on the flapping mechanism, although the mathematical implementation is the same as for flapping. In this and other applications of the flapping model, good agreement with measured properties does not necessarily identify the flapping mechanism as the unique interpretation of the observations. Second, the fact that elementary reasoning yields a result consistent with mapping closure should not be interpreted as providing a simple alternative derivation of mapping closure. The adoption of Gaussian statistics is *ad hoc* in both contexts, but is introduced in a different manner in each context. Third, from a purely formal viewpoint, Eqs. (5) and (6) constitute a natural generalization of the mapping-closure result to an inhomogeneous configuration. In view of empirical evidence supporting those equations, as outlined next, it would be interesting to investigate whether generalization of mapping closure to inhomogeneous configurations reproduces Eqs. (5) and (6) or gives a different result.

Equations (5) and (6), in conjunction with the earlier expression for σ_0, implicitly determine the scalar pdf at any location y for given m/σ. Here, the transverse profiles of the quantities c', S, and K are evaluated using Eq. (7) for comparison to measurements. The parameter m/σ is chosen so that $c'(0)$ matches the measured value 0.19 obtained in the self-preserving regime of scalar mixing layers.[5] Numerical solution based on Eq. (7) gives $m/\sigma = 0.48$. Using the earlier relation between σ and L, lengths are scaled by L and Eq. (7) is used to compute the quantities plotted in Figs. 1–3.

In those figures, dashed curves demarcate the range of values reported by Ma and Warhaft,[5] based on their own measurements and those of LaRue and Libby.[4] The solid curves are based on Eq. (7) with $m/\sigma = 0.48$. Model results fall within the experimental range in all instances. It is interesting to note that the measured kurtosis at $y = 0$ ranges from 2.0 to 2.7, indicating a non-Gaussian pdf. (For a Gaussian pdf, $K = 3$.) The model captures this effect, predicting $K = 2.3$ at $y = 0$.

The flapping model can also be applied to mixing in plumes downstream of continuous line sources. Warhaft's[12] measurements of the cross correlation of diffusive scalar species emitted from two line sources are of

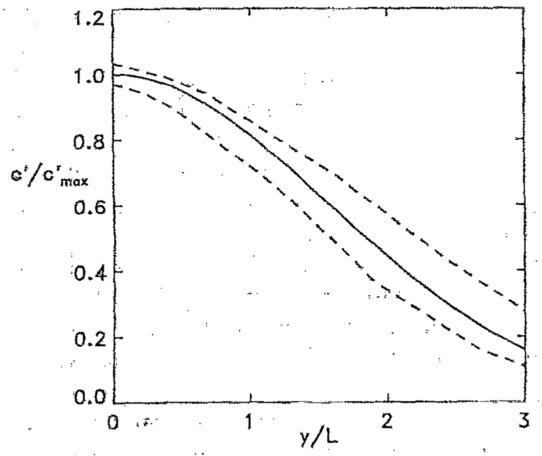

FIG. 1. Transverse profile of the root-mean-square scalar fluctuation c' in the scalar mixing layer, normalized by its maximum value. The transverse coordinate y is scaled by the half-width L of the mean profile. Dashed curves: range of reported measurements.[4,5] (Measured values can deviate from unity at $y=0$ because the measured maximum need not occur at the centerline.) Solid curve: flapping-model prediction.

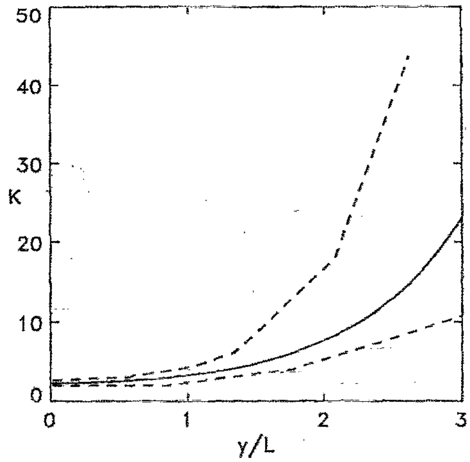

FIG. 3. Transverse profile of the scalar kurtosis K. Notation as in Fig. 1.

particular interest. This and other aspects of plume mixing are addressed elsewhere in the context of a comparison of the present model to a more complete mixing model.[13]

In summary, single-point scalar fluctuation statistics for several configurations of interest are captured by a model that involves no representation of mixing enhancement by convection. At least in these instances, such statistics appear to embody little information concerning the mixing process.

One is therefore led to consider what measured quantities are, in fact, sensitive to mixing. In this regard, the following considerations should be noted. First, the evolution of these statistics parametrized by time or an equivalent quantity has not been addressed. It is shown elsewhere[13] that quantitatively correct treatment of time evolution appears to require a mixing model. Second, inferences are limited to the statistics of conserved scalars. It is well recognized that chemical product evolution in a reacting flow reflects the dynamics of the mixing process.[14] Finally, the relevance of spatial structure (as represented by multipoint statistics or related quantities such as the joint pdf of the scalar and its gradient) to turbulent mixing is well recognized, but a satisfactory generalization of turbulent mixing models based on single-point closure to incorporate such information has not yet been achieved.[15–17]

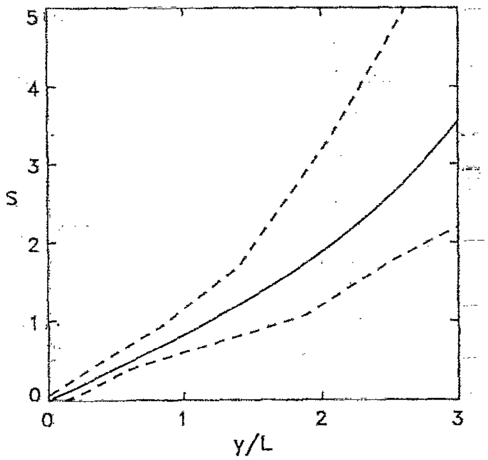

FIG. 2. Transverse profile of the absolute magnitude S of the scalar skewness. (The skewness is an odd function of y.) Notation as in Fig. 1.

The present work highlights the importance of incorporating some representation of multipoint fluctuation properties into turbulent mixing models. Multipoint statistics (or equivalent spatial structure statistics) do not merely supplement the information provided by single-point statistics. Multipoint statistics appear, in fact, to be the principal indicators of mixing effects.

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