Girth and Fractional Chromatic Number of Planar Graphs

Amir Pirnazar and Daniel H. Ullman*

DEPARTMENT OF MATHEMATICS
THE GEORGE WASHINGTON UNIVERSITY
WASHINGTON, DC
E-mail: dullman@math.gwu.edu

Received October 25, 1999; revised October 19, 2001

Abstract: In 1959, even before the Four-Color Theorem was proved, Grötzsch showed that planar graphs with girth at least 4 have chromatic number at the most 3. We examine the fractional analogue of this theorem and its generalizations. For any fixed girth, we ask for the largest possible fractional chromatic number of a planar graph with that girth, and we provide upper and lower bounds for this quantity. © 2002 Wiley Periodicals, Inc. J Graph Theory 39: 201–217, 2002; DOI 10.1002/jgt.10024

Keywords: *fractional chromatic number; girth; planar graphs; four-color theorem*

1. INTRODUCTION

We write $\nu(G)$ for the number of vertices (the *order* of G), $e(G)$ for the number of edges (the *size* of G), $\alpha(G)$ for the size of the largest independent set of vertices (the *independence number* of G), $g(G)$ for the length of the shortest cycle (the *girth* of G), and $\chi(G)$ for the chromatic number of the graph G. We write $\chi_f(G)$ for the fractional chromatic number of G, the infimum of all ratios a/b such that

*Correspondence to: Daniel H. Ullman, Department of Mathematics, George Washington University, Washington, DC 20052. E-mail: dullman@math.gwu.edu

there is a coloring of G using a colors that assigns b colors to each vertex with adjacent vertices getting disjoint sets of colors. Such an assignment is called an a:b-coloring. For background on the fractional chromatic number, see Ref. [11].

One of the celebrated triumphs of the probabilistic method is Erdős's proof [2] that $g(G)$ and $\chi(G)$ of a graph can be simultaneously large. The proof proceeds by showing that the ratio between $\nu(G)$ and $\alpha(G)$ in a random graph is likely to be large and by using the inequality $\chi(G) \geq \nu(G)/\alpha(G)$. The same proof shows that $g(G)$ and $\chi_f(G)$ if a Graph G can be simultaneously large, since $\chi_f(G) \geq \nu(G)/\alpha(G)$ also.

If we restrict to planar graphs, however, the story is different. First of all, the chromatic number cannot exceed 4 in the first place [1], regardless of girth. Moreover, Grötzsch [5] showed that the chromatic number cannot exceed 3 when the girth is at least 4. One is tempted to ask if the chromatic number can be forced any lower by insisting that the girth be sufficiently large. Of course the answer is no; consider the case of the odd cycles. In the fractional world, however, the answer is yes. For any $\epsilon > 0$, there is an integer g such that, if the girth of a planar graph G is at least g, then the fractional chromatic number of G is at most $2 + \epsilon$. This fact has been noticed by several researchers [4]. We provide a proof here on our way to more detailed results.

For an integer $g \geq 3$, let

$$f(g) = \sup\{\chi_f(G)\colon G \text{ is a planar graph with girth } g\}.$$

This is our object of study in this article. Let us begin with a few simple observations. The function f is nonincreasing, because the addition of a cycle of length $g - 1$ to a graph with girth g can only increase the fractional chromatic number. Since the complete graph K_4 on 4 vertices is planar and has $\chi_f(K_4) = 4$, the four-color theorem implies that $f(3) = 4$. Since $\chi_f(G) \leq \chi(G)$, Grötzsch's Theorem [5] implies that $f(4) \leq 3$. It is also clear that $f(g) \geq 2$ for every g, since the fractional chromatic number of any tree, in fact any bipartite graph, is 2. Without much effort, we show that $\lim_{g\to\infty} f(g) = 2$. But we do not know any exact values for $f(g)$ with $g \geq 5$.

Let $K_{a:b}$ be the *Kneser* graph defined as follows. The vertices of $K_{a:b}$ are the b-element subsets of $\{1, 2, \ldots, a\}$, and two such vertices are connected by an edge if and only if they are disjoint. These graphs play a central role in fractional coloring, because a graph has an a:b-coloring just when there is a homomorphism from the graph to $K_{a:b}$. In this sense, $K_{a:b}$ plays the same role for a:b-coloring that the complete graph K_n plays for n-coloring.

In Section 2, we make a careful study of paths in Kneser graphs, in order to understand when partial fractional colorings can be extended. In Section 3, we find upper bounds for $f(g)$ for arbitrary g. To do this, we find a certain set of unavoidable configurations in any planar graph with girth g or larger. We then show that each configuration in our set is reducible in the sense that no smallest counterexample to our assertion about upper bounds can have such a configuration. In Section 4, we find lower bounds for $f(g)$. Here, the object is to construct graphs G with girth g but with $\chi_f(G)$ not too small. In particular, we show that $f(4) = 3$. In Section 5, we indicate briefly how these results can be extended to higher genus graphs, we consider connections to other work, and we mention open problems.

2. PATHS IN KNESER GRAPHS

Before embarking on our main theme, we study paths in Kneser graphs. Later, we appeal several times to results in this section when trying to extend partial fractional colorings.

We begin with a general result that characterizes precisely when paths of a certain length can be found between two vertices in a Kneser graph. Because the vertices of the Kneser graph $K_{a:b}$ are sets—subsets of $\{1, 2, \ldots, a\}$ of size b, in fact—we may speak of the intersection of two vertices.

Theorem 2.1. *Suppose that a and b are positive integers with $a > 2b$. Then the following two conditions on nonnegative integers k and l are equivalent.*

(1) *For any two (not necessarily distinct) vertices v and w in $K_{a:b}$ with $|v \cap w| = k$, there is a walk of length exactly l in $K_{a:b}$ beginning at v and ending at w.*
(2) *l is even and $k \geq b - (l/2)(a - 2b)$, or l is odd and $k \leq ((l-1)/2)\ (a - 2b)$.*

Proof. When $l = 0$, the result asserts the triviality that there is a path of length 0 from v to w if and only if $v = w$. When $l = 1$, the result asserts that there is an edge from v to w if and only if $|v \cap w| = 0$. This reproduces the definition of $K_{a:b}$. We proceed by induction on l.

Let $l \geq 2$ be even and suppose that $|v \cap w| \geq b - (l/2)(a - 2b)$. Then $|v - w\ | = |w - v| \leq (l/2)(a - 2b)$. Choose $x \in V(K_{a:b})$ disjoint from v so that $|x \cap w|\ \leq (l/2 - 1)(a - 2b)$. This is possible by choosing for x elements of $\{1, 2, \ldots, a\}\ - v - w$ (there are at least $a - b - (l/2)(a - 2b) = b - ((l/2) - 1)$ $(a - 2b)$ of them) and choosing the remaining elements for x if need be from $w - v$. There is an edge between v and x, and there is a walk of length $l - 1$ between x and w by the induction hypothesis. Hence there is a walk of length l from v to w.

Conversely, assume $l \geq 2$ is even and there is a path of length l from v to w. Suppose that x is the vertex on this path adjacent to v. Then $|v \cap x| = 0$ and $|x \cap w| \leq (l/2 - 1)(a - 2b)$ by the induction hypothesis. It follows that

$$|\{1, 2, \ldots, a\} - x - w| \leq a - (b + b - (l/2 - 1)(a - 2b)) = (l/2)(a - 2b),$$

and hence $|v \cap w| \geq b - (l/2)(a - 2b)$.

The case of odd $l \geq 3$ works similarly. Suppose that $|v \cap w| \leq ((l-1)/2)(a-2b)$. Choose a vertex x disjoint from v so that $|x \cap w| \geq b - ((l-1)/2)\ (a - 2b)$. This is guaranteed if one makes certain that $w - v \subset x$. Then the induction hypothesis gives a path of length $l - 1$ from x to w. Adding the edge from v to x extends this to the required path of length l.

Conversely, assume $l \geq 3$ is odd and there is a path of length l from v to w, and let x be the vertex on this path adjacent to v. Then $|x \cap w| \geq b - ((l-1)/2)\,(a-2b)$ by the induction hypothesis. It follows that $|v \cap w| \leq ((l-1)/2)\,(a-2b)$. ■

We use the following specializations of Theorem 2.1 in Sections 3 and 4.

Corollary 2.2. *If v and w are any two vertices in $K_{(2n+1):n}$, then there is a walk of length $2n$ beginning at v and ending at w.*

Proof. Put $a = 2n + 1$, $b = n$, and $l = 2n$ in Theorem 2.1. Then the condition on k is $k \geq n - n = 0$, which is vacuous. Hence the required walk exists no matter what k is. ■

Corollary 2.3. *If n is odd and v and w are any two vertices in $K_{(4n):(2n-1)}$ with $|v \cap w| = n - 1$, then there is a walk of length n beginning at v and ending at w.*

Proof. Put $a = 4n$, $b = 2n - 1$, and $l = n$ in Theorem 2.1. Then the condition on k is $k \leq n - 1$, which holds because $k = n - 1$. ■

3. UPPER BOUNDS

A path in a graph G is *singular* if all its internal vertices have degree 2 in G. A tree with exactly 3 leaves is a *tripod*. Every tripod has exactly one vertex of degree 3, which we call the *hub*.

First, we obtain a simple bound on the number of edges that a planar graph with fixed girth can have.

Proposition 3.1. *A planar graph G with girth at least g has at most $g(\nu - 2)/(g - 2)$ edges.*

Proof. Every face in such a graph is bounded by at least g edges, hence $gf \leq 2e$, where f is the number of faces. Euler's Formula then yields $2 = \nu - e + f \leq \nu - e + 2e/g$, which simplifies to the desired bound on e. ■

To illustrate the central ideas in this section, we begin with a proposition that is strictly weaker than our main theorem.

Proposition 3.2. *Let n be a positive integer. If G is planar and $g(G) \geq 10n - 4$, then $\chi_f(G) \leq 2 + (1/n)$.*

Proof. We show in fact that there is a $(2n + 1){:}n$-coloring of any such graph G. We proceed by induction on the order of G. The result is clear if G has only a few vertices, since then the hypothesis requires that G is a tree. Trees are bipartite, and bipartite graphs can even be $2n{:}n$-colored.

Assume then that the result holds for all graphs whose order is smaller than $\nu(G)$. If G has a vertex v of degree 0 or 1, we may remove it, apply the induction hypothesis to $G - v$, and then extend the $(2n+1)$:n-coloring of $G - v$ to all of G with no difficulty. So we may assume that every vertex of G has degree at least 2.

Draw G in the plane, let G^* be the planar dual of G, and let H be the simple graph underlying G^*. Since H is planar, it must have a vertex v of degree 5 or less. Since the minimum degree of G^* is the girth of G, the vertex v in G^* has at least $10n - 4$ incident edges, yet at most 5 adjacent vertices. Hence at least one neighbor w of v is connected to v by at least $\lceil(10n-4)/5\rceil = 2n$ parallel edges. These parallel edges in G^* correspond to a singular path P of length $2n$ in G.

Suppose that P has vertices $v_0, v_1, \ldots, v_{2n}$ labeled in order. Apply the induction hypothesis to $G - \{v_1, v_2, \ldots, v_{2n-1}\}$ to obtain a $(2n+1)$:n-coloring that assigns to each vertex v the color set $\phi(v)$. We are left only to extend this coloring to the internal vertices in P. This can be done by using Corollary 2.2 to identify a path of length $2n$ in $K_{(2n+1):n}$ starting at the vertex $\phi(v_0)$ and ending at $\phi(v_{2n})$. The vertices along this path give sets of colors appropriate for the vertices along P in order. Hence the $(2n+1)$:n-coloring can be extended to all of G. ■

Proposition 3.2 yields the bound $f(g) \leq 2 + 10/(g+4)$ for $g \equiv 6 \pmod{10}$. Already this implies that $\lim_{g\to\infty} f(g) = 2$. We improve this bound next, with the result stated as Corollary 3.6. Even then, the best upper bound that we provide seems too large. The lower bounds from Section 4 are much smaller and are probably closer to the truth.

The proof of Proposition 3.2 shows that the singular path P_{2n} is unavoidable in a planar graph with minimum degree 2 of girth at least $10n - 4$ and that P_{2n} is a reducible configuration. We now provide a somewhat improved upper bound for $f(g)$. Although in this setting P_{2n} is avoidable, we identify a set of unavoidable configurations each of which is reducible. Let $T_{a,b,c}$ be the tripod formed from $K_{1,3}$ by subdividing the three edges $a-1$ times, $b-1$ times, and $c-1$ times, respectively. In other words, take three paths of length $a+1$, $b+1$, and $c+1$, respectively. Select one end from each path and identify these three ends to form the hub of the tripod. Figure 1 shows $T_{2,3,6}$.

Given a tree T, we say that the configuration T is found in G to mean that G has a subgraph isomorphic to T and that the only edges in G with precisely one end in T are incident to a leaf of T.

Our main goal in this section is to prove the following result, which improves Theorem 3.2 by lowering 10 to 8.

Theorem 3.3. *Let n be a positive integer. If G is planar and $g(G) \geq 8n - 4$, then $\chi_f(G) \leq 2 + (1/n)$.*

We accomplish our goal by showing that

$$S_n = \{T_{z,z,2n-z} : n \leq z < 2n\} \cup \{P_{2n}\}$$

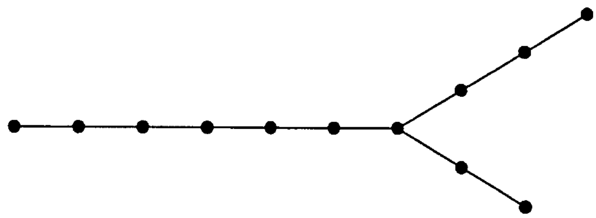

FIGURE 1. The tripod $T_{2,3,6}$.

is an unavoidable set of reducible configurations. We handle the two aspects of this proof as separate lemmas. We begin by showing that S_n is unavoidable. We employ an electrical discharging argument.

Lemma 3.4. *Let $n \geq 2$ be a positive integer. Every planar graph with minimum degree 2 and girth at least $8n-4$ has a configuration from the set S_n.*

Proof. Assume that G is a graph with minimum degree 2, girth $8n-4$, but no configuration from the set S_n. On each vertex in G, place an electrical charge according to the following scheme. First place on each vertex v a preliminary charge of magnitude $d(v)-(8n-4)/(4n-3)$, where $d(v)$ is the degree of v. Since

$$\sum_{v\in V(G)}\left(d(v)-\frac{8n-4}{4n-3}\right)=2e(G)-\frac{8n-4}{4n-3}\nu(G)$$
$$\leq 2\frac{8n-4}{8n-6}(\nu(G)-2)-\frac{8n-4}{4n-3}\nu(G)=-4\frac{8n-4}{8n-6}<0$$

by Proposition 3.1, the sum of the preliminary charges is negative. Now transfer a charge of magnitude $1/(4n-3)$ from each vertex v of degree greater than 2 to every vertex of degree 2 that lies on a singular path that ends at v. The hypothesis on girth and the absence of a singular P_{2n} imply that no component of G can be a cycle. Thus every vertex of degree 2 inherits a charge of magnitude $1/(4n-3)$ from two other vertices, and so the resulting charge on a vertex of degree 2 is

$$2-\frac{8n-4}{4n-3}+\frac{2}{4n-3}=0.$$

Since G has no singular paths of length $2n$, a vertex of degree d transfers some of its preliminary charge to at most $(2n-2)d$ vertices of degree 2. Hence the resulting charge on a vertex v of degree $d \geq 4$ is at least

$$d-\frac{8n-4}{4n-3}-\frac{(2n-2)d}{4n-3}=\frac{2n-1}{4n-3}d-\frac{8n-4}{4n-3}\geq 0.$$

We now focus on vertices of degree 3. We claim that each such vertex v is called upon to transfer charge to at most $4n-5$ vertices of degree 2. Assume that v is at the hub of a tripod configuration $T_{a,b,c}$ with $a \geq b \geq c$. Since G has no singular paths of length $2n$, we have $a \leq 2n-1$. Since G has no configuration isomorphic to a tripod in S_n, we must have $c < 2n-b$. It follows that $a+b+c \leq 2n-1+2n-1 = 4n-2$. The number of vertices to receive a transfer of charge from v is then $(a-1)+(b-1)+(c-1) = a+b+c-3 \leq 4n-5$, as claimed. The resulting charge on a vertex v of degree 3 is therefore at least

$$3 - \frac{8n-4}{4n-3} - \frac{4n-5}{4n-3} = 0.$$

The sum of the preliminary charges on the vertices of G was negative. After the transfer of charges, however, all the vertices carry a nonnegative charge. This violates the obvious conservation law, and we have our contradiction. ∎

We now move on to the notion of reducibility. We want to show that no configuration in S_n can exist in any smallest counterexample to Theorem 3.3. We examine the extendability of partial fractional colorings.

Lemma 3.5. *Let T be any tripod in S_n. If the three leaves of T are assigned n-element subsets of $\{1,2,\ldots,2n+1\}$, then the remaining vertices of T can also be assigned n-element subsets of $\{1,2,\ldots,2n+1\}$ in such a way as to produce a proper $(2n+1)$:n-coloring of T.*

Proof. Suppose that T is isomorphic to $T_{z,z,2n-z}$, where $n \leq z < 2n$. The idea of the proof is to first construct an appropriate n-element subset for the hub of T and then to appeal to Theorem 2.1 to extend the coloring to the remaining vertices in T.

Assume first that z is even. Let A, B, and C be the n-element subsets of $\{1,2,\ldots,2n-1\}$ assigned to the leaves of T, with A and B assigned to the leaves whose distance from the hub of T is z and with C assigned to the leaf whose distance from the hub is $2n-z$. Our goal is to construct an n-element subset D of $\{1,2,\ldots,2n-1\}$ satisfying

$$\left.\begin{aligned} &|A \cap D| \geq n - z/2, \\ &|B \cap D| \geq n - z/2, \\ \text{and}\quad &|C \cap D| \geq z/2. \end{aligned}\right\} \tag{1}$$

If we can construct such a set D, then appeal to Theorem 2.1 with $a = 2n+1$, $b = n$, and $k = z$ or $k = 2n-z$ allows us to extend the fractional coloring to all of T.

We now construct D. If $|(A \cup B) \cap C| \geq z/2$, then we select $z/2$ elements from $(A \cup B) \cap C$. Say that r of these elements are in $A \cap C$ and s of them are in $B \cap C$ with $r + s = z/2$. Then select any $n - r - z/2$ further elements from A and $n - s - z/2$ further elements from B. The selected elements form a set D that satisfies (1), and the number of selected elements is

$$\frac{z}{2} + \left(n - r - \frac{z}{2}\right) + \left(n - s - \frac{z}{2}\right) = 2n - z \leq n.$$

Hence we complete the construction by adding to D arbitrary elements until $|D| = n$.

If $|(A \cup B) \cap C| < z/2$, select all the elements of $(A \cup B) \cap C$. Now suppose that $|A \cap B \cap C| = r$ and $|((A \cup B) \cap C) - (A \cap B \cap C)| = s$. Next select precisely $(z/2) - (r + s)$ additional elements from C. Were $|(A \cap B) - C| < n - r - z/2$, then we would have

$$\begin{aligned}|(A \cap B) \cup (A \cap C) \cup (B \cap C)| &= |(A \cap B) - C| + |(A \cup B) \cap C| \\ &\leq \left(n - r - \frac{z}{2} - 1\right) + \left(\frac{z}{2} - 1\right) = n - r - 2,\end{aligned}$$

which would imply

$$\begin{aligned}2n + 1 &\geq |A \cup B \cup C| \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &\geq 3n - ((n - r - 2) + 2r) + r = 2n + 2,\end{aligned}$$

a contradiction. Hence $|(A \cap B) - C| \geq n - r - z/2$. We may therefore select $n - r - z/2$ elements from $(A \cap B) - C$. The selected elements form a set D that satisfies (1) and this time we have so far selected

$$(r + s) + (z/2 - (r + s)) + (n - r - z/2) = n - r \leq n$$

elements. We complete the construction by adding to D arbitrary elements until $|D| = n$.

We now consider the case when z is odd. This time we need to find $E \subset \{1, 2, \ldots, 2n + 1\}$ of size n with

$$\left.\begin{aligned}|A \cap E| &\leq (z - 1)/2, \\ |B \cap E| &\leq (z - 1)/2, \\ \text{and} \quad |C \cap E| &\leq n - (z + 1)/2.\end{aligned}\right\} \tag{2}$$

The set E can be assigned to the hub of the tripod, and Theorem 2.1 implies that this assignment can be extended to a fractional coloring of the entire tripod.

Since $z-1$ is even, we may construct a set D with $|D| = n$ that satisfies

$$\begin{aligned} |A \cap D| &\geq n-(z-1)/2, \\ |B \cap D| &\geq n-(z-1)/2, \\ \text{and} \quad |C \cap D| &\geq (z-1)/2. \end{aligned}$$

Then the complement $\tilde{D} = \{1, 2, \ldots, 2n+1\} - D$ satisfies

$$\begin{aligned} |A \cap \tilde{D}| &\leq (z-1)/2, \\ |B \cap \tilde{D}| &\leq (z-1)/2, \\ \text{and} \quad |C \cap \tilde{D}| &\leq n-(z-1)/2. \end{aligned}$$

That $|\tilde{D}| = n+1$ gives us a bit of wiggle room. Set $E = \tilde{D} - \{x\}$, where $x \in \tilde{D}$ is chosen arbitrarily unless $|C \cap \tilde{D}| = n-(z-1)/2$, in which case we require that $x \in C \cap \tilde{D}$. Then $|E| = n$ and E satisfies (2). ■

Now that we have disposed of Lemmas 3.4 and 3.5, there remain just a few words to say to complete the proof of the main theorem.

Proof of Theorem 3.3. When $n = 1$, the result follows from Grötzsch's Theorem [5]. Hence we may assume that $n \geq 2$. We proceed by induction on the order of G. The result is clear if the order is small enough. Let G be a planar graph with girth at least $8n-4$. If G has any vertex v of degree 0 or 1, we fractionally color $G - v$ using the induction hypothesis and then extend this fractional coloring without any difficulty to v. Hence we may assume that all vertices of G have degree 2 or more. Then, by Lemma 3.4, G contains one of the trees $T \in S_n$. Remove from G all the vertices of T except the leaves of T and apply the induction hypothesis to this graph to obtain a $(2n+1)$:n-coloring. If T is the path P_{2n}, then use Corollary 2.2 to extend this fractional coloring to all of G. If T is a tripod, then use Lemma 3.5 to extend the fractional coloring to all of G. ■

Corollary 3.6. *If $g \geq 4$, then $f(g) \leq 2 + 1/\lfloor (g+4)/8 \rfloor$.*

Proof. Let $h = 8\lfloor (g+4)/8 \rfloor - 4$. Then $g \geq h$, and so $f(g) \leq f(h)$. Now $f(h) \leq 2 + 1/\lfloor (g+4)/8 \rfloor$ by Theorem 3.3, and we have our result. ■

4. LOWER BOUNDS

Finding lower bounds for $f(g)$ requires constructing planar graphs with large girth and with fractional chromatic number not too small. The cycle C_{2n+1} has fractional chromatic number $2 + 1/n$. Substituting g for $2n + 1$, this yields $f(g) \geq 2 + 2/(g - 1)$ for odd g. We begin in Theorem 4.1 with a modest improvement of this bound. In Theorem 4.2 below, we further improve this bound for most values of g employing a more complicated construction.

Theorem 4.1. *If* $g \equiv 3 \pmod 4$, *then* $f(g) \geq 2 + 2/(g - 2)$. *If* $g \equiv 1 \pmod 4$, *then* $f(g) \geq 2 + 2(g + 1)\big/(g(g - 1))$.

Proof. Suppose that $g = 2n + 1$, and let G_n be the wheel with $2n + 1$ spokes and with every spoke subdivided $n - 1$ times. More precisely, G_n is formed as follows. Begin with the cycle C_{2n+1} (forming the *rim* of G_n) on the vertex set $\{v_1, v_2, \ldots, v_{2n+1}\}$ in that cyclic order and with $2n + 1$ paths $P_1, P_2, \ldots P_{2n+1}$ (forming the *spokes* of G_n), each of length n, with P_i on vertex set $\{w_{i,0}, w_{i,1}, \ldots, w_{i,n}\}$ in that order. Then identify v_i with $w_{i,n}$ for every $i \in \{1, 2, \ldots, 2n + 1\}$, and identify the $2n + 1$ vertices $w_{1,0}, w_{2,0}, \ldots, w_{2n+1,0}$ (the resulting vertex is the *hub* of G_n). Figure 2 shows G_3.

Clearly G_n has girth $2n + 1$. We assume first that n is odd, and we show that $\chi_f(G_n) = 2 + 2/(2n - 1) = 2 + 2/(g - 2)$. (Strictly speaking, we need only show the inequality $\chi_f(G_n) \geq 2 + 2/(2n - 1)$, but verifying that equality holds shows that another construction is needed if we hope to improve the lower bound in the theorem.)

To show that $\chi_f(G_n) \leq 2 + 2/(2n - 1)$, we provide a $(4n)$:$(2n - 1)$-coloring of G_n. Assign the set of colors $\{1, 2, \ldots 2n - 1\}$ to the hub of G_n. Next assign colors to the rim of G_n in two steps. First assign $n - 1$ colors chosen from $\{1, 2, \ldots, 2n - 1\}$ to each of the vertices around the rim; this can be done because there is a $(2n - 1)$:$(n - 1)$-coloring of C_{2n+1}. Then assign n colors chosen from $\{2n, 2n + 1, \ldots, 4n\}$ to each of the vertices around the rim; this can be done because there is a $(2n + 1)$:n-coloring of C_{2n+1}. In fact, $\chi_f(C_{2n+1}) = (2n + 1)/n$, so this last fractional coloring is optimal. We now have assigned $2n - 1$ colors to each vertex around the rim in such a way that every vertex around the rim shares exactly $n - 1$ colors with the hub. We are left

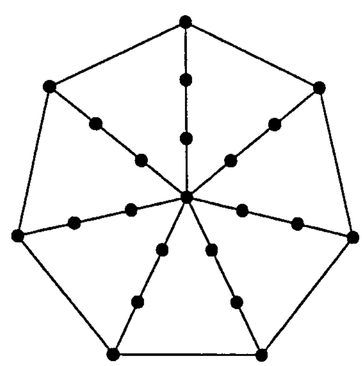

FIGURE 2. The graph G_3.

only to complete the $(4n)$:$(2n-1)$-coloring to the spokes of G_n. This is done by appeal to Corollary 2.3.

To show that $\chi_f(G_n) \geq 2 + 2/(2n-1)$, we switch perspectives to the dual problem. A *fractional clique* on a graph G is a function λ: $V(G) \to [0, 1]$ that satisfies $\sum_{v \in I} \lambda(v) \leq 1$ for every independent set $I \subseteq V(G)$. The value $\sum_{v \in V(G)} \lambda(v)$ is called the *weight* of the fractional clique. The problem of finding the maximum weight fractional clique is dual (in the sense of linear programming) to the problem of finding the optimal fractional coloring. Hence $\chi_f(G)$ is bounded below by the weight of any fractional clique.

We therefore need only to construct a fractional clique on G_n of weight $4n/(2n-1)$. Let $\lambda(h) = 2n/(2n^2 + n - 1)$ if h is the hub of G_n, and let $\lambda(v) = 2/(2n^2 + n - 1)$ for all other vertices v. If an independent set $I \subseteq V(G_n)$ contains the hub h, then it can contain at most $(n-1)/2$ other vertices on each spoke, and so

$$\sum_{v \in I} \lambda(v) \leq \frac{2n}{(2n^2 + n - 1)} + (2n+1)\frac{n-1}{2}\frac{2}{2n^2 + n - 1} = 1,$$

as required. If, on the other hand, I does not contain the hub h, then it contains at most $(n+1)/2$ other vertices on each spoke, but it can contain $(n+1)/2$ vertices on at most n spokes, because on each such spoke the vertex on the rim where that spoke attaches must be in I, but $\alpha(C_{2n+1}) = n$. Hence

$$\sum_{v \in I} \lambda(v) \leq n\frac{n+1}{2}\frac{2}{(2n^2 + n - 1)} + (n+1)\frac{n-1}{2}\frac{2}{2n^2 + n - 1} = 1,$$

and so λ is the desired fractional clique.

A similar argument shows that $\chi_f(G_n) = 2 + (2n+2)/(2n^2 + n) = 2 + 2(g+1)/(g(g-1))$ when n is even. ■

It does not seem worth completing the details of the proof of Theorem 4.1 when n is even, because we show next how to obtain a stronger result. In the previous result, we provided a lower bound L for $f(g)$ by constructing, for every odd g, a single planar graph with girth g and $\chi_f = L$. In the next result, we obtain a lower bound by constructing, for every odd g, a sequence $G_1, G_2, \ldots$ of planar graphs with girth g and with $\chi_f(G_m) \uparrow L$.

Theorem 4.2. *If $g = 7$ or if $g \geq 11$ is odd, then $f(g) \geq 2 + 2/(g-3)$.*

Proof. We describe slightly different constructions depending on the remainder when g is divided by 4. Let us first consider the case $g \equiv 3 (\text{mod } 4)$. Say that $g = 4n + 3$ with $n \geq 1$. Let $G_0 = C_g$, the cycle on g vertices, let a_0 be any vertex in G_0 and let b_0c_0 be the edge opposite a_0 in G_0. Construct the graph G_m with distinguished vertices a_m, b_m, c_m recursively as follows. Add to G_{m-1} a path P_m of length $4n+1$ with vertices $v_0, v_1, \ldots v_{4n}$ in that order. Then add edges

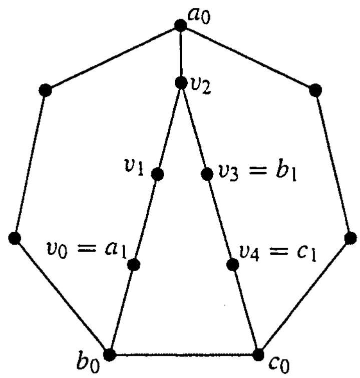

FIGURE 3. The graph G_1 with $g = 7$.

$b_{m-1}v_0$, $a_{m-1}v_{2n}$, and $c_{m-1}v_{4n}$. Finally, let $a_m = v_0$, $b_m = v_{2n+1}$, and $c_m = v_{2n+2}$. Figure 3 shows G_1 with $g = 7$.

The construction places a_m, b_m, c_m on a newly formed face cycle, with vertex a_m opposite edge $b_m c_m$. Thus the construction preserves planarity. The girth of G_m is $g = 4n + 3$ for all m. Also $\nu(G_m) = \nu(G_{m-1}) + 4n + 1$, and since $\nu(G_0) = 4n + 3$ we obtain $\nu(G_m) = 4n + 3 + m(4n + 1) = g + m(g - 2)$.

We claim that $\alpha(G_m) = 2n + 1 + 2nm$. This is clear when $m = 0$, so we proceed by induction. Let I be an independent set of vertices in G_m. If $|I \cap V(P_m)| \leq 2n$, then $|I| \leq |I \cap V(G_{m-1})| + 2n$, which is less than or equal to $2n + 1 + 2n(m - 1) + 2n = 2n + 1 + 2nm$ by the induction hypothesis. Because P_m has $4n + 1$ vertices, we have $\alpha(P_m) = 2n + 1$, so there is left only to consider the case $|I \cap V(P_m)| = 2n + 1$. In this case, I must consist precisely of the even-indexed vertices of P_m, including v_0, v_{2n}, and v_{4n}. Therefore, I may not contain any of the three distinguished vertices $a_{m-1}, b_{m-1}, c_{m-1}$. But then $|I \cap P_{m-1}|$ is at most $2n - 1$, since $P_{m-1} - \{a_{m-1}, b_{m-1}, c_{m-1}\}$ consists of a path of length $2n - 1$ and a path of length $2n - 3$, whose union has independence number only $2n - 1$. Hence $|I| \leq |I \cap V(G_{m-2})| + (2n - 1) + (2n + 1)$, which is less than or equal to $2n + 1 + 2n(m - 2) + 4n = 2n + 1 + 2nm$ by the induction hypothesis. This proves the claim.

Since $\nu(G_m) = 4n + 3 + (4n + 1)m$ and $\alpha(G_m) = 2n + 1 + 2nm$, we obtain

$$f(4n+3) \geq \sup_m \chi_f(G_m) \geq \sup_m \frac{4n + 3 + (4n + 1)m}{2n + 1 + 2nm} = 2 + \frac{1}{2n} = 2 + \frac{2}{g - 3},$$

as required.

When $g \equiv 1 (\text{mod } 4)$, the construction is slightly different. Let $g = 4n + 1$ with $n \geq 3$. (The construction fails when $n \leq 2$.) As before, let $G_0 = C_g$, the cycle on g vertices. Let a_0 be any vertex in G_0 and let b_0 and c_0 be the two vertices on G_0 that are at distance $2n - 1$ from a_0. This puts b_0 and c_0 at distance 3 from one another. Construct the graph G_m with distinguished vertices a_m, b_m, c_m

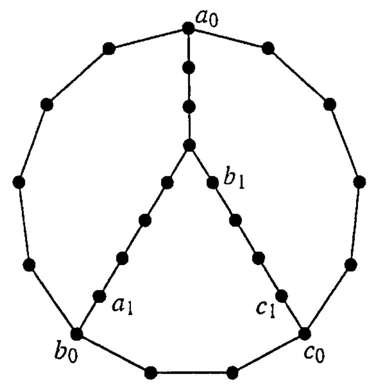

FIGURE 4. The graph G_1 with $g = 13$.

recursively as follows. Add to G_{m-1} a tripod $T_{3,2n-1,2n-1}$. Identify the three ends of the tripod with a_{m-1}, b_{m-1}, and c_{m-1}, respectively, with the end of the shortest leg matching with a_{m-1}. Finally, let a_m be a vertex on the tripod adjacent to the end of one of the long legs, and let b_m and c_m be chosen along the other long leg of the tripod so that they are each at distance $2n-1$ from a_m. Figure 4 shows G_1 with $g = 13$.

We compute that $\nu(G_m) = 4n + 1 + (4n-1)m$ and an induction argument like the one above shows that $\alpha(G_m) = 2n + (2n-1)m$. Hence

$$f(4n+1) \geq \sup_m \chi_f(G_m) \geq \sup_m \frac{4n+1+(4n-1)m}{2n+(2n-1)m} = 2 + \frac{1}{2n-1} = 2 + \frac{2}{g-3},$$

as required. ■

If one attempts the construction in this proof for girth 9, there does not seem to be a choice of a_m, b_m, c_m that gives $\alpha(G_m) = 2n + (2n-1)m$ as desired. Ironically, if one attempts the construction for girth 5, one does obtain $\alpha(G_m) = 2n + (2n-1)m$, but the girth of G_m fails to be 5 as desired. Unintended cycles of length 4 present themselves; find them in Figure 5. We learn in any case a fact about girth 4. Grötzsch's Theorem about planar graphs of girth at least 4 cannot be improved in the fractional case.

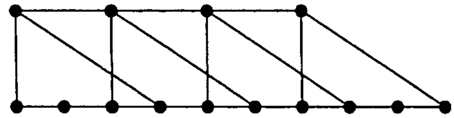

FIGURE 5. The graph G_3 with $g = 5$.

Theorem 4.3. $f(4) = 3$.

Proof. We noted from the outset, as a simple consequence of Grötzsch's Theorem, that $f(4) \leq 3$. Let G_m be the graphs from the proof of Theorem 4.2 with $g = 5$. The graph G_3 is pictured in Figure 5, drawn with crossing edges for convenience. (These graphs appear also in [3].) The graphs G_m are planar, have girth 4, and have $\chi_f(G_m) \geq \nu(G_m)/\alpha(G_m) = (5 + 3m)/(2 + m)$, so $\chi_f(G_m) \uparrow 3$. Hence $f(4) \geq 3$. ∎

We now explore the case of girth 5. The best lower bound for $f(5)$ that we know of is $11/4$.

Proof. Let G be the graph with 11 vertices pictured and labeled as in Figure 6. Build a graph G_m out of m copies of G, adding edges to connect, for each $k \in \{2, 3, \ldots, m\}$, (1) vertex a in copy k to vertex z in copy $k - 1$, (2) vertex b in copy k to vertex v in copy $k - 1$, and (3) vertex e in copy k to vertex x in copy $k - 1$. The graph G_2 is pictured in Figure 7.

The graph G_m is planar, has girth 5, and has $11m$ vertices. We now show that the independence number of G_m is $1 + 4m$. This is clearly so for $m = 1$ (and the only independent sets of size 5 contains $\{v, x, z\}$), so we proceed by induction with the following strengthened induction hypothesis. If I is an independent set in G_m, then $|I| \leq 4m + 1$, and if $|I| = 4m + 1$ then I contains the vertices labeled v, x, and z in copy m. Let I be an independent set in G_m. If $|I \cap V(G_{m-1})| \leq 4(m - 1)$, then

$$|I| \leq |I \cap V(G_{m-1})| + 5 \leq 4(m - 1) + 5 = 1 + 4m.$$

Moreover, if we have equality here, then the 5 vertices in I in copy m include those labeled x, v, and z. Otherwise, by the induction hypothesis, $|I \cap V\ (G_{m-1})| = 1 + 4(m - 1)$ and the vertices labeled v, x, and z in copy $m - 1$ are all in I. The vertices in copy m that are not adjacent to those labeled x, v, z

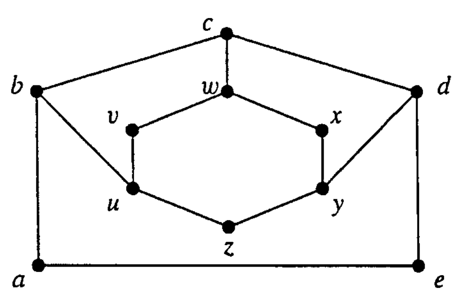

FIGURE 6. A hexagon in a pentagon, labeled.

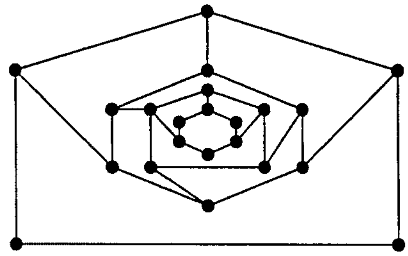

FIGURE 7. A hexagon in a pentagon in a hexagon in a pentagon.

in copy $m-1$ form a graph whose independence number is merely 4. Hence $|I \cap V\ (G_m)| \leq 4$, and

$$|I| \leq |I \cap V(G_{m-1})| + |I - V(G_{m-1})| \leq 1 + 4(m-1) + 4 = 1 + 4m.$$

Moreover, if we have equality here, then the 4 vertices in I in copy m are labeled either c, x, v, z or d, x, v, z. This completes the proof by induction.

We conclude that

$$f(5) \geq \sup_m \chi_f(G_m) \geq \sup_m \frac{\nu(G_m)}{\alpha(G_m)} = \sup_m \frac{11m}{1+4m} = \frac{11}{4}. \qquad \blacksquare$$

5. OTHER DIRECTIONS

The results of this paper leave many questions unanswered. One might hope to compute $f(n)$ exactly for all n or at least to improve the bounds provided in this paper. Further refinement of discharging and reducibility arguments can be expected to give stronger upper bounds. Stronger lower bounds require new constructions. A modest question is whether the result of Theorem 4.2 holds in the case $g = 9$. Also, since it is odd cycles, not even cycles that create obstacles in coloring, one might conjecture that $f(2n) = f(2n+1)$ for all $n \geq 2$ or at least for $n \geq 3$. This could be viewed as an extension of the obvious fact that $f(2) = f(3)$, which holds because the parallel edges permitted when girth is 2 impose no extra constraint on coloring.

The *star* (or *circular*) *chromatic number* $\chi^*(G)$ of a graph G is the smallest fraction a/b with the property that there is a coloring $c\colon V(G) \to \{1, 2, \ldots, a\}$ such that $b \leq |c(v) - c(w)| \leq a - b$, whenever $vw \in E(G)$. There has been much recent interest in the star chromatic number of planar graphs [4, 8, 12]. One can ask many questions about χ^* along the same lines as the questions addressed in this paper about χ_f. Let

$$f^*(n) = \sup\{\chi^*(G)\colon G \text{ is planar and } g(G) = n\}.$$

Since $\chi_f(G) \le \chi^*(G) \le \chi(G)$ for any graph G, any upper bound for $f^*(n)$ is an upper bound for $f(n)$ as well. It is not hard to see that Proposition 3.2 can be strengthened to yield the same result with χ^* replacing χ_f. In fact, this seems to be a folk theorem [6]. We do not know, however, if Theorem 3.3 holds with χ^* replacing χ_f; certainly a new idea would be needed in the proof. Yet it remains possible that $f^*(n) = f(n)$ for all n.

Many of the results in this paper can be extended to the class of graphs of fixed genus greater than 0. Let

$$f_j(n) = \sup\{\chi_f(G)\colon G \text{ has genus } j \text{ and girth } n\}.$$

The function f studied in this paper is f_0. Certainly $f_j(n)$ is nondecreasing in j and nonincreasing in n, and $\lim_{j\to\infty} f_j(n) = \infty$ for every n. When $j \ge 1$, the Heawood Map Coloring Theorem [10] yields $f_j(3) = \lfloor (7+\sqrt{1+48j})/2 \rfloor$. One can obtain upper bounds for $f_j(n)$ for general j and odd n along the lines of Proposition 3.2. Lower bounds that improve the results in Section 4 for small positive genus, however, seem hard to come by. For example, to find a genus j with $f_j(4) > 3$, one must construct a triangle-free graph G with fractional chromatic number greater than 3 and put j equal to the genus of G. One such graph is the triangle-free graph G with $\chi(G) = 5$ obtained by the construction of Mycielski [9]. This graph has fractional chromatic number equal to $941/290 \approx 3.245$ [7], and so $f_j(4) \ge 3.24$ when j is the genus of G. We leave to the diligent reader the task of computing this value of j.

REFERENCES

[1] K. Appel and W. Haken, Every planar map is four colorable, Bull Amer Math Soc 82 (1976), 711–712.

[2] P. Erdős, Graph theory and probability, Canad J Math 11 (1959), 34–38.

[3] K. Fraughnaugh, Minimum independence graphs with maximum degree four, Graphs and Applications, Wiley, New York, 1985, pp. 221–230.

[4] G. Gao, Y. Wang, and H. Zhou, Star chromatic numbers of some planar graphs, J Graph Theory 27 (1998), 33–42.

[5] H. Grötzsch, Ein Dreifarbensatz für dreikreisfrei Netze auf der Kugel, Wiss Z Martin-Luther-Univ, Halle-Wittenberg, Math-Nat Reihe 8 (1959), 109–120.

[6] W. Klostermeyer and C. Q. Zhang, $(2+\epsilon)$-coloring of planar graphs with large odd girth, J Graph Theory, 33 (2000), 109–119.

[7] M. Larsen, J. Propp, and D. Ullman, The fractional chromatic number of Mycielski's graphs, J Graph Theory 19 (1995), 411–416.

[8] D. Moser, The star-chromatic number of planar graphs, J Graph Theory 24 (1997), 33–43.

[9] J. Mycielski, Sur le coloriage des graphes, Colloq Math 3 (1955), 161–162.

[10] G. Ringel and J. W. T. Youngs, Solution of the Heawood map-coloring problem, Proc Natl Acad Sci 60 (1968), 438–445.

[11] E. R. Scheinerman and D. H. Ullman, Fractional Graph Theory, Wiley, New York, 1997.

[12] X. Zhu, Planar graphs with circular chromatic numbers between 3 and 4, J Comb Theory 76 (1999), 170–200.