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The remaining quantity needed is $\cos \alpha_2$ as a function of s_1 and θ_2, for this is the argument of one of the reflectivities. Thus, from Eq. (A6),

$$\begin{aligned}\cos \alpha_2 &= \cos [\tfrac{1}{2}\pi + \tfrac{1}{2}(\theta_2 - \theta_1)] \\ &= \sin [\tfrac{1}{2}(\theta_1 - \theta_2)] \\ &= -\sin [\tfrac{1}{2}(\theta_0 + \theta_2) - \beta_1] \\ &= -(1 + s_1^2)^{-\frac{1}{2}}\{\sin [\tfrac{1}{2}(\theta_0 + \theta_2)] \\ &\quad + s_1 \cos [\tfrac{1}{2}(\theta_0 + \theta_2)]\}. \qquad \text{(A9)}\end{aligned}$$

The reflectivities will be written here for convenience. If we distinguish them as $r_V = |R_V|^2$ and $r_H = |R_H|^2$ to denote vertical and horizontal polarization, respectively, then we have

$$R_V(\cos \alpha_i) = \frac{K \cos \alpha_i - (K - 1 + \cos^2 \alpha_i)^{\frac{1}{2}}}{K \cos \alpha_i + (K - 1 + \cos^2 \alpha_i)^{\frac{1}{2}}},$$

$$R_H(\cos \alpha_i) = \frac{\cos \alpha_i - (K - 1 + \cos^2 \alpha_i)^{\frac{1}{2}}}{\cos \alpha_i + (K - 1 + \cos^2 \alpha_i)^{\frac{1}{2}}},$$

$$K = \frac{\epsilon}{\mu}.$$

Here, ϵ is the relative complex permittivity, and μ is the relative permeability.

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Boundary-Value Problems of Linear-Transport Theory—Green's Function Approach

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Case's technique utilizing Green's functions for dealing with boundary-value problems of the neutron linear-transport theory is exploited. We show that the Fourier coefficients of the Green's function over the Case spectrum are precisely the normal modes. In particular, if we assume that the scattering kernel is rotationally invariant (which indeed we do assume) and approximate it by a degenerate kernel consisting of spherical harmonics, the set of modes is deficient for problems lacking azimuthal symmetry. We also show that the expansion of the scattering kernel, in terms of spherical harmonics (or any set of orthogonal functions for that matter), permits the linear factorization of the Fourier coefficients of the Green's function in terms of the lowest element, with the proportionality functions consisting of complete orthogonal polynomials. As a consequence of this attribute of Fourier coefficients, the eigenfunctions (continuum and discrete) also factorize, which then permits decoupling of the appropriate singular integral equations. To illustrate our idea, we solve half-space and slab problems. However, the basic procedure is kept sufficiently general so that the extension to problems involving other geometrics remains straightforward.

1. INTRODUCTION

The normal-mode (eigenfunction) expansion technique of Case,[1] in dealing with boundary-value problems, has achieved considerable success in the types of problems for which the normal modes (continuum plus discrete) form a complete orthogonal set. However, there are several problems of interest, for instance, in the theory of neutron diffusion and kinetic theory of gases,[2] where the sets of modes are either deficient or the appropriate integral equations are regular. In particular, in a recent paper by Case *et al.*,[3] it has been shown for spherical geometry that one cannot directly adapt the above-mentioned technique. In this paper, we consider the Green's function approach also due to Case.[4] We show that the Fourier coefficients of the Green's function for the appropriate neutron 1-speed transport equation over the Case spectrum are precisely the normal modes. In particular, if we assume that the scattering kernel is rotationally invariant (which indeed we do assume) and approximate it by a degenerate kernel consisting of the spherical harmonics, then the set of modes is deficient for problems lacking azimuthal symmetry. However, if the index of degeneracy is allowed to approach infinity, then the deficiency of that set vanishes. Furthermore, we also show that the expansion of the scattering kernel, in terms of spherical harmonics (or any set of orthogonal functions for that matter), permits the linear factorization of the Fourier coefficients of the Green's function in terms of the lowest element with the proportionality functions which consist of complete orthogonal polynomials. This attribute of Fourier coefficients then leads to the factorization of eigenfunctions (continuum and discrete) and the eventual decoupling of the singular integral equations. The main advantage of Green's function technique over the normal-mode expansion technique is that the normal modes appear "naturally" in the Green's function, with the additional terms (if any) which make the set complete also appearing as an integral part of it.

To illustrate our idea, we solve half-space and slab problems. The latter type of problems are treated in somewhat greater detail than the former. In particular, two limiting cases of thick and thin slabs are considered. We begin by first presenting the basic formulas[4] and relevant mathematical tools.

2. BASIC FORMULAS

In the 1-speed approximation,[4] the neutron-transport equation we consider is

$$(1 + \mathbf{\Omega} \cdot \nabla)\Psi(\mathbf{r}, \mathbf{\Omega}) = \int d\Omega' f(\mathbf{\Omega} \cdot \mathbf{\Omega}')\Psi(\mathbf{r}, \mathbf{\Omega}') + Q(\mathbf{r}, \mathbf{\Omega}), \quad (1)$$

where $\mathbf{\Omega}$ is the unit velocity vector, Ψ is the angular density, Q is some given source function, and $f(\mathbf{\Omega} \cdot \mathbf{\Omega}')$ is a rotationally invariant scattering kernel. The appropriate Green's function satisfies

$$(1 + \mathbf{\Omega} \cdot \nabla)G(\mathbf{r}, \mathbf{\Omega}; \mathbf{r}_0, \mathbf{\Omega}_0) = \int d\Omega' f(\mathbf{\Omega} \cdot \mathbf{\Omega}')G(\mathbf{r}, \mathbf{\Omega}'; \mathbf{r}_0, \mathbf{\Omega}_0) + \delta(\mathbf{r} - \mathbf{r}_0)\delta(\mathbf{\Omega} \cdot \mathbf{\Omega}_0). \quad (2)$$

The quadrature for the angular density is

$$\Psi(\mathbf{r}, \mathbf{\Omega}) = \int_V d\Omega' d^3r' G(\mathbf{r}, \mathbf{\Omega}; \mathbf{r}', \mathbf{\Omega}')Q(\mathbf{r}', \mathbf{\Omega}') + \int_S d\Omega' dS' G(\mathbf{r}, \mathbf{\Omega}; \mathbf{r}_s', \mathbf{\Omega}')\hat{n}_i(\mathbf{r}_s') \cdot \mathbf{\Omega}'\Psi(\mathbf{r}_s', \mathbf{\Omega}'), \quad (3)$$

where V is the volume in which the angular density is to be determined, S is the boundary of V, $\mathbf{r}_s'$ is a point on S, and $\hat{n}_i$ is a unit normal pointing into V. The integral equation for the surface distribution $\Psi(\mathbf{r}_s, \mathbf{\Omega})$ is

$$\Psi(\mathbf{r}_s, \mathbf{\Omega}) = \int_V d\Omega' d^3r' G(\mathbf{r}_s, \mathbf{\Omega}; \mathbf{r}', \mathbf{\Omega}')Q(\mathbf{r}', \mathbf{\Omega}') + \int_{S'} d\Omega' dS' G_{\pm}(\mathbf{r}_s, \mathbf{\Omega}; \mathbf{r}_s', \mathbf{\Omega}')\hat{n}_i(\mathbf{r}_s') \cdot \mathbf{\Omega}'\Psi(\mathbf{r}_s', \mathbf{\Omega}') = 0. \quad (4)$$

The object is to construct the Green's function from Eq. (2) and solve the integral Eq. (4) for the surface distribution[5] $\Psi(\mathbf{r}_s, \mathbf{\Omega})$. Having obtained $\Psi(\mathbf{r}_s, \mathbf{\Omega})$, we then determine the angular density $\Psi(\mathbf{r}, \mathbf{\Omega})$ by Eq. (3). The basic mathematical tools relevant to such a treatment are the elementary use of Fourier transforms and the theory of singular integral equations of the type

$$\tfrac{1}{2}B(\mu)\Gamma(\mu) + \frac{1}{2\pi i}\mathcal{P}\int_L \frac{d\nu}{\nu - \mu} A(\mu, \nu)\Gamma(\nu) = f(\mu). \quad (5)$$

Reduction of Eq. (4) to the integral equation (5) should become obvious soon.

3. GREEN'S FUNCTION FOR THE 1-SPEED TRANSPORT EQUATION AND EIGENFUNCTIONS

In this section, we take a cursory look at the relationship between the eigenfunctions of the 1-speed transport equation and the Fourier components of the corresponding Green's function. We express the scattering kernel $f(\mathbf{\Omega} \cdot \mathbf{\Omega}')$ in Eq. (1) in the degenerate form

$$f(\mathbf{\Omega} \cdot \mathbf{\Omega}') = \sum_{l=0}^{N} \frac{2l + 1}{4\pi} b_l P_l(\mathbf{\Omega} \cdot \mathbf{\Omega}'), \quad (6)$$

where N is arbitrary. Using the addition theorem for spherical harmonics, i.e.,

$$P_l(\mathbf{\Omega} \cdot \mathbf{\Omega}') = \sum_{m=-l}^{l} \frac{4\pi}{2l + 1} Y_{lm}^*(\mathbf{\Omega})Y_{lm}(\mathbf{\Omega}') \quad (7)$$

in Eq. (6), the 1-speed transport equation then may be written as

$$(1 + \mathbf{\Omega} \cdot \nabla)\Psi(\mathbf{r}, \mathbf{\Omega}) = \sum_{l=0}^{N}\sum_{m=-l}^{l} b_l Y_{lm}^*(\mathbf{\Omega})\langle \Psi Y_{lm}\rangle, \quad (8)$$

where the inner product is defined by

$$\langle fg\rangle = \int d\Omega f(\mathbf{\Omega})g(\mathbf{\Omega}). \quad (9)$$

Let us consider the Fourier transform of Eq. (8), i.e., set

$$\Psi(\mathbf{r}, \mathbf{\Omega}) = \frac{1}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{r}}\psi^k(\mathbf{\Omega}). \quad (10)$$

Then, Eq. (8) becomes

$$(1 + i\mathbf{k}\cdot\mathbf{\Omega})\psi^k(\mathbf{\Omega}) = \sum_{l=0}^{N}\sum_{m=-l}^{l} b_l Y_{lm}^*(\mathbf{\Omega})\langle\psi^k Y_{lm}\rangle. \quad (11)$$

The appropriate Green's function satisfies

$$(1 + \mathbf{\Omega}\cdot\nabla)G(\mathbf{r},\mathbf{\Omega};\mathbf{r}_0,\mathbf{\Omega}_0) = \sum_{l=0}^{N}\sum_{m=-l}^{l} b_l Y_{lm}^*(\mathbf{\Omega})\langle G Y_{lm}\rangle + \delta(\mathbf{r}-\mathbf{r}_0)\delta(\mathbf{\Omega}\cdot\mathbf{\Omega}_0). \quad (12)$$

To construct the Green's function, let us take the Fourier transform of Eq. (12); i.e., set

$$G(\mathbf{r},\mathbf{\Omega};\mathbf{r}_0,\mathbf{\Omega}_0) = \frac{1}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)} g(k,\mathbf{\Omega};\mathbf{\Omega}_0). \quad (13)$$

The result is

$$g(k,\mathbf{\Omega},\mathbf{\Omega}_0) = \sum_{l,m} b_l \frac{Y_{lm}^*(\mathbf{\Omega})}{1+i\mathbf{k}\cdot\mathbf{\Omega}}\langle g Y_{lm}\rangle + \frac{\delta(\mathbf{\Omega}\cdot\mathbf{\Omega}_0)}{1+i\mathbf{k}\cdot\mathbf{\Omega}}. \quad (14)$$

Now, every solution of Eq. (14) must be of the form

$$g = \sum_{l,m} b_l \frac{Y_{lm}^*(\mathbf{\Omega})}{1+i\mathbf{k}\cdot\mathbf{\Omega}}\xi_{lm}(k,\mathbf{\Omega}_0) + \frac{\delta(\mathbf{\Omega}\cdot\mathbf{\Omega}_0)}{1+i\mathbf{k}\cdot\mathbf{\Omega}}, \quad (15)$$

where $\xi_{lm} \equiv \langle g Y_{lm}\rangle$ are to be determined. If we multiply both sides of Eq. (15) with $Y_{l'm'}(\mathbf{\Omega})$ and integrate over $\mathbf{\Omega}$, we get a system of linear inhomogeneous equations for ξ_{lm}. They are

$$\sum_{l,m}\xi_{lm}(k,\mathbf{\Omega}_0)\left(\delta_{ll'}\delta_{mm'} - b_l\left\langle\frac{Y_{lm}^* Y_{l'm'}}{1+i\mathbf{k}\cdot\mathbf{\Omega}}\right\rangle\right) = \frac{Y_{l'm'}(\mathbf{\Omega}_0)}{1+i\mathbf{k}\cdot\mathbf{\Omega}_0}. \quad (16)$$

Simple calculations will show that

$$\left\langle\frac{Y_{lm}^* Y_{l'm'}}{1+i\mathbf{k}\cdot\mathbf{\Omega}}\right\rangle = 2\pi\delta_{mm'}\int_{-1}^{1}\frac{d\mu'}{1+ik\mu} Y_{lm}(\mu,0)Y_{l'm'}(\mu,0).$$

By using this simplification in Eq. (16), we get

$$\sum_{l=|m|}^{N}\xi_{lm}\mathcal{A}_{ll'}^m(k) = \frac{Y_{l'm}(\mathbf{\Omega}_0)}{1+i\mathbf{k}\cdot\mathbf{\Omega}_0}, \quad (17)$$

where

$$\mathcal{A}_{ll'}^m(k) = \delta_{ll'} - 2\pi b_l\int_{-1}^{1}\frac{d\mu}{1+ik\mu} Y_{lm}(\mu,0)Y_{l'm}(\mu,0). \quad (18)$$

When the determinant (the dispersion function)

$$\Lambda_m(k) = \det|\mathcal{A}_{ll'}^m| \quad (19)$$

of the system (17) is nonzero, for any fixed m, we have the unique solution

$$\xi_{lm} = \sum_{l'=|m|}^{N}\frac{d_m\binom{l}{l'}}{\Lambda_m}\frac{Y_{l'm}(\mathbf{\Omega}_0)}{1+i\mathbf{k}\cdot\mathbf{\Omega}_0}, \quad m \le l, \quad (20)$$

where $d_m\binom{l}{l'}$ denotes the signed minor of the matrix $(\mathcal{A}_{ll'}^m)$ associated with the lth row and the l'th column. In particular, the homogeneous equations

$$g - \sum_{l,m} b_l\frac{Y_{lm}^*(\mathbf{\Omega})}{1+i\mathbf{k}\cdot\mathbf{\Omega}}\xi_{lm}(k,\mathbf{\Omega}_0) = 0 \quad (21)$$

and

$$\sum_{l=|m|}^{N}\xi_{lm}\mathcal{A}_{ll'}^m = 0 \quad (22)$$

then have the unique solutions $g = 0$ and $\xi_{lm} = 0$. On the other hand, when $\Lambda_m = 0$, Eq. (22) and, consequently, Eq. (16) have nonzero solutions, and the number of linearly independent solutions is equal to the nullity of the matrix $(\mathcal{A}_{ll'}^m)$ (i.e., the difference between its order and its rank). In any event, the most general Fourier representation of G is of the form

$$G(\mathbf{r},\mathbf{\Omega};\mathbf{r}_0,\mathbf{\Omega}_0) = \frac{1}{(2\pi)^3}\sum_{l,m} b_l Y_{lm}^*(\mathbf{\Omega}) \times\int d^3k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)}\frac{\xi_{lm}(k,\mathbf{\Omega}_0)}{1+i\mathbf{k}\cdot\mathbf{\Omega}} + \delta(\mathbf{\Omega}\cdot\mathbf{\Omega}_0)\frac{1}{(2\pi)^3}\int d^3k\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)}}{1+i\mathbf{k}\cdot\mathbf{\Omega}}. \quad (23)$$

We note that the Fourier components ξ_{lm} of G, given by Eq. (20), are sectionally holomorphic functions in the complex k-vector space, with a branch cut for $k = -i\infty$ to $-i$ and i to $i\infty$, and they have poles at the zeros of the dispersion function Λ_m. In what follows, we look at ξ_{lm} in terms of their relation to the eigenfunctions of Eq. (11) over this spectrum (the Case spectrum),[6] and also examine a certain recurrence relation leading to the factorization of ξ_{lm} in terms of the lowest element ξ_{mm}.

Our first immediate observation is that, for a fixed direction of $\mathbf{k}$, the difference of boundary values of ξ_{lm} about its branch cut are precisely the continuum eigenfunctions[2] of Eq. (11); i.e., if we denote such functionals by $E_{lm}(k,\mathbf{\Omega})$, then

$$E_{lm}(k,\Omega_k,\phi) = \xi_{lm}^-(k,\Omega_k,\phi) - \xi_{lm}^+(k,\Omega_k,\phi) \quad (24)$$

or, explicitly,

$$E_{lm} = \sum_{l'=|m|}^{N} Y_{l'm}(\Omega_k,\phi) \times\left(\frac{d_m^-\binom{l}{l'}}{\Lambda_m^-(1+ik\Omega_k)_-} - \frac{d_m^+\binom{l}{l'}}{\Lambda_m^+(1+ik\Omega_k)_+}\right) \quad (25)$$

satisfy Eq. (11). Here, $\Omega_k = \hat{\mathbf{k}} \cdot \mathbf{\Omega}$, and $+$ ($-$) denotes the boundary value as k approaches the branch cut from the left (right) side. On the other hand, if k_j is a simple zero[7] of $\Lambda_m(k)$, then the discrete eigenfunction [of Eq. (11)] is given by

$$F_{lm}(k_j, \Omega_k, \phi) = \lim_{k \to k_j} (k - k_j)\xi_{lm}(k, \Omega_k, \phi), \quad (26)$$

i.e.,

$$F_{lm}(k_j, \Omega_k, \phi) = \frac{1}{\Lambda_m'(k_j)} \sum_{l'=|m|}^{N} d_m\begin{pmatrix} l \\ l' \end{pmatrix}, k_j\Big) \frac{Y_{l'm}(\Omega_k, \phi)}{1 + ik_j\Omega_k}, \quad (27)$$

where $\Lambda_m'(k_j)$ is the derivative of $\Lambda_m(k)$ evaluated at $k = k_j$.

It may seem peculiar at first sight that, for a fixed point in the Case spectrum, there are N number of eigenfunctions for l ranges from $|m|$ to N. However, we shall see presently that all such eigenfunctions are not distinct. In fact, they differ from the lowest eigenfunction ($l = |m|$) by a multiplicative factor which is a polynomial in (i/k). To see that, consider Eq. (16) rewritten in the form

$$\sum_{l,m} \xi_{lm}\left(\delta_{ll'}\delta_{mm'}(1 - b_l) + b_l\left\langle Y_{lm}^* \frac{i\mathbf{k}\cdot\mathbf{\Omega}}{1 + i\mathbf{k}\cdot\mathbf{\Omega}} Y_{l'm'}\right\rangle\right) = \frac{Y_{l'm'}(\mathbf{\Omega}_0)}{1 + i\mathbf{k}\cdot\mathbf{\Omega}_0}. \quad (28)$$

Using the recurrence relation for spherical harmonics,

$$\Omega_k Y_{lm}(\mathbf{\Omega}) = A_{lm}Y_{l+1\,m}(\mathbf{\Omega}) + A_{l-1\,m}Y_{l-1\,m}(\mathbf{\Omega}), \quad (29)$$

where

$$A_{lm} = \left(\frac{(l + 1 - m)(l + 1 + m)}{(2l + 1)(2l + 3)}\right)^{\frac{1}{2}}, \quad (30)$$

we obtain

$$z(b_l - 1)\xi_{lm} + A_{lm}\xi_{l+1\,m} + A_{l-1\,m}\xi_{l-1\,m} = -zY_{lm}(\mathbf{\Omega}_0), \quad (31)$$

where, for convenience, we have put $k = i/z$. From this equation, we conclude that

$$\xi_{lm} = h_{lm}(z)\xi_{mm} + W_{lm}(z, \mathbf{\Omega}_0), \quad (32)$$

where $h_{lm}(z)$ are complete orthogonal polynomials (in the Stieltjes sense) satisfying the following 3-term recurrence relation:

$$A_{lm}h_{l+1\,m}(z) + z(b_l - 1)h_{lm}(z) + A_{l-1\,m}h_{l-1\,m}(z) = 0, \quad (33)$$

and $W_{lm}(z, \mathbf{\Omega})$ are also polynomial in z. Equation (32) gives us the desired factorization of ξ_{lm} (mentioned above) in terms of the lowest element ξ_{mm}. Two immediate consequences of this equation are (1) the factorization of eigenfunctions and (2) a convenient representation of the dispersion function Λ_m. In other words, we have

$$E_{lm}(\nu, \mathbf{\Omega}) = h_{lm}(\nu)E_{mm}(\nu, \mathbf{\Omega}), \quad (34)$$

$$F_{lm}(\nu_i, \mathbf{\Omega}) = h_{lm}(\nu_i)F_{mm}(\nu_i, \mathbf{\Omega}), \quad (35)$$

and

$$\Lambda_m(z) = 1 - z\sum_{l=|m|} b_l h_{lm}(z)\left\langle\frac{Y_{lm}^* Y_{mm}}{z - \Omega_k}\right\rangle. \quad (36)$$

Equations (34) and (35), of course, follow by definitions (24) and (26), while Eq. (36) is obtained merely by substituting ξ_{lm} in Eq. (28) by means of Eq. (32). In particular, for the lowest element ξ_{mm}, we have

$$\Lambda_m(z)\xi_{mm}(z, \mathbf{\Omega}) = Y_{mm}(z)\frac{z}{z - \Omega_k} - \sum_{l=|m|}^{N} W_{lm}(z, \mathbf{\Omega}) \times \left(\delta_{lm} - b_l\left\langle Y_{lm}^* \frac{z}{z - \Omega_k} Y_{mm}\right\rangle\right), \quad (37)$$

from which we may readily construe the explicit forms of the lowest eigenfunctions.

The results of this section may be summarized as follows:

(i) The Green's function for a degenerate kernel of the form given by Eq. (1) was Fourier transformed. For the Fourier components (ξ_{lm}) of G, we obtained a set of inhomogeneous linear algebraic relations.

(ii) It was then shown that the difference of boundary values of ξ_{lm} about the Case-spectral line gave rise to the continuum eigenfunctions of Eq. (11), while the discrete ones consisted of the

$$\lim_{z \to \nu_j} (z - \nu_j)\xi_{lm};$$

ν_j is a simple zero of Λ_m.

(iii) Using the recurrence relation for spherical harmonics, we obtained a 3-term inhomogeneous recurrence relation for ξ_{lm} which permitted us to express all ξ_{lm} linearly in terms of the lowest coefficient ξ_{mm}. As a consequence of this factorization, all the eigenfunctions for fixed m and ν (or ν_j) become proportional to the corresponding lowest eigenfunction, with the factors being orthogonal polynomials in ν (or ν_j).

We may remark here that result (ii) is valid independently of the geometry, the type of functions used to express the scattering kernel, and the rank N of degeneracy. Result (iii), on the other hand, though valid for any geometry, is crucially dependent on the fact that we expanded the scattering kernel in terms of orthogonal functions. In other words, the Fourier coefficients of G satisfy a 3-*term* inhomogeneous recurrence relation of the type given by Eq. (31) if and only if the scattering kernel is expanded in terms of a set of orthogonal functions. The coefficients then factorize in the way given by Eq. (32) and the corresponding eigenfunctions as given by Eqs. (34) and (35). As a final remark, we wish to state that the above factorization of ξ_{lm}, in terms of a *single* lowest element, is not possible if the scattering kernel is a function of all velocity components, such as in the energy-dependent case.[2]

In what follows we shall restrict our treatment to 1-dimensional problems. In particular, for the purpose of illustrating the general formulation discussed above, we shall consider half-space and slab problems. For the latter, the angular density in two asymptotic limits of thick and thin slabs will be given.

4. ONE-DIMENSIONAL PROBLEMS (GENERAL FORMULATION)

The 1-dimensional version of the Fourier representation of the Green's function [Eq. (23)] is

$$G(x, \mathbf{\Omega}; x_0, \mathbf{\Omega}_0) = \frac{1}{2\pi}\sum_{l,m} b_l Y^*_{lm}(\mathbf{\Omega}) \times \int_{-\infty}^{\infty} dk \, \frac{e^{ik(x-x_0)}}{1+ik\mu}\, \xi_{lm}(k, \mathbf{\Omega}_0) + \delta(\mathbf{\Omega}\cdot\mathbf{\Omega}_0)\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, \frac{e^{ik(x-x_0)}}{1+ik\mu}, \tag{38}$$

where $\mu = \hat{x}\cdot\mathbf{\Omega}$. Let $G_>$ denote G for $x > x_0$, and $G_<$ for $x < x_0$; the point source is presumed to be at x_0. First, consider $x > x_0$. In order to express G in terms of eigenfunctions of Eq. (11), as discussed previously, consider the integral in Eq. (38) over the contour C shown in Fig. 1. Assuming that Λ_m has no zeros on the real k axis, the sum of the integrals from $-\infty$ to ∞ and that around the branch cut equals the residue arising from the zeros of the dispersion function $\Lambda_m(k)$ in the upper-half k plane[8]; since $x > x_0$, the integral along the semicircle at infinity gives zero contribution. Hence

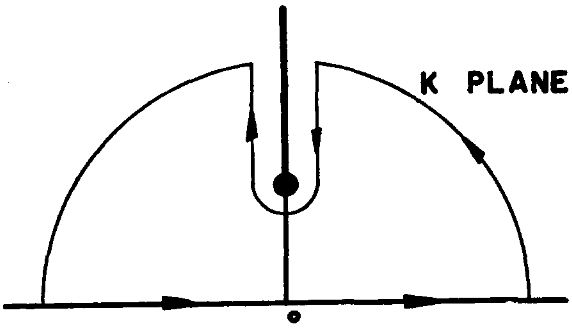

FIG. 1. Contour $x > x_0$.

$$G_> = \frac{1}{2\pi}\sum_{l,m} b_l Y^*_{lm}(\mathbf{\Omega}) \times \int_i^{i\infty} dk\, e^{ik(x-x_0)}\left(\frac{\xi^-_{lm}}{(1+ik\mu)_-} - \frac{\xi^+_{lm}}{(1+ik\mu)_+}\right) + i\sum_{l,m} b_l Y^*_{lm}(\mathbf{\Omega})\sum_{j=1}^{M} e^{-(x-x_0)/\nu_{m_j}} F_{lm}(\nu_{m_j}, \mathbf{\Omega}_0) + \delta(\mathbf{\Omega}\cdot\mathbf{\Omega}_0) e^{-(x-x_0)/\mu}\,\frac{\Theta(\mu)}{\mu}, \tag{39}$$

where M is the total number of zeros ν_{m_j} of Λ_m in the upper-half k plane, $\Theta(\mu)$ is the Heaviside step function, and $F_{lm}(\nu_{m_j}, \mathbf{\Omega}_0)$ are the discrete eigenfunctions of Eq. (11). The explicit form of F_{lm} is as given by Eq. (27), with k_j replaced by i/ν_{m_j}.

Putting $k = i/\nu$ in Eq. (30) and using the Plemelj formula

$$1/(\nu-\mu)_\pm = \mathcal{P}[1/(\nu-\mu)] \mp i\pi\delta(\nu-\mu), \tag{40}$$

we re-express $G_>$ in the form

$$G_> = \frac{1}{2\pi i}\sum_{m=-N}^{N}\sum_{l=|m|}^{N} b_l Y^*_{lm}(\mathbf{\Omega}) \times \left(\mathcal{P}\int_0^1 \frac{d\nu}{(\nu-\mu)\nu}\, e^{-(x-x_0)/\nu} E_{lm}(\nu, \mathbf{\Omega}_0) + \pi i[\xi^-_{lm}(\mu, \mathbf{\Omega}_0) + \xi^+_{lm}(\mu, \mathbf{\Omega}_0)] e^{-(x-x_0)/\mu}\,\frac{\Theta(\mu)}{\mu} + 2\pi i \sum_{j=1}^{M} e^{-(x-x_0)/\nu_{m_j}} F_{lm}(\nu_{m_j}, \mathbf{\Omega}_0)\right) + \delta(\mathbf{\Omega}\cdot\mathbf{\Omega}_0) e^{-(x-x_0)/\mu}\,\frac{\Theta(\mu)}{\mu}, \tag{41}$$

where we have now identified $\xi^-_{lm}(\nu, \mathbf{\Omega}_0) - \xi^+_{lm}(\nu, \mathbf{\Omega}_0)$ with the continuum eigenfunctions $E_{lm}(\nu, \mathbf{\Omega}_0)$ [Eqs. (24) and (25)] and have used the identity

$$\sum_{l=0}^{N}\sum_{m=-l}^{l} A_{lm} = \sum_{m=-N}^{N}\sum_{l=|m|}^{N} A_{lm}. \tag{42}$$

We note that the singular part of $G_>$ is appropriately expressed in terms of the continuum eigenfunctions and has a Cauchy-type kernel, but that the second term on the right-hand side contains the sum of the boundary values of ξ_{lm}, which are not eigenfunctions.

However, if we write Eq. (17) in the form

$$\sum_{l=|m|}^{N} \xi_{lm}(z, \boldsymbol{\Omega}_0)\mathcal{A}_{ll'}^{m}(z) = \frac{z}{z-\mu_0} Y_{l'm}(\boldsymbol{\Omega}_0) \quad (43)$$

and consider the difference of its boundary values as z approaches the cut $(-1 \leq \nu \leq 1)$ from the top and the bottom, then we obtain

$$\sum_{l=|m|}^{N} [\mathcal{A}_{ll'}^{m+}(\mu) - \mathcal{A}_{ll'}^{m-}(\mu)](\xi_{lm}^{+} + \xi_{lm}^{-}) = \sum_{l=|m|}^{N} (\mathcal{A}_{ll'}^{m+} + \mathcal{A}_{ll'}^{m-})(\xi_{lm}^{+} - \xi_{lm}^{-}) - 4\pi i\mu\delta(\mu-\mu_0)Y_{l'm}(\boldsymbol{\Omega}_0), \quad (44)$$

which relates $\xi_{lm}^{+} + \xi_{lm}^{-}$ to the eigenfunctions

$$\xi_{lm}^{+} - \xi_{lm}^{-} (\equiv E_{lm}).$$

By means of this equation, we may now replace the second term in Eq. (41) by the right-hand side of Eq. (44), if we note that [see Eq. (18)]

$$\mathcal{A}_{ll'}^{m+}(\mu) - \mathcal{A}_{ll'}^{m-}(\mu) = 4\pi^2 i b_l \mu Y_{lm}^{*}(\boldsymbol{\Omega})Y_{l'm}(\boldsymbol{\Omega}). \quad (45)$$

Thus, using Eqs. (44), (45), and the factorizations given by Eqs. (34) and (35) in Eq. (41), we get

$$\begin{aligned} G_{>} = \sum_{m=-N}^{N} \frac{e^{-im\phi}}{\mu Y_{mm}(\mu, 0)} &\times \Bigg(\frac{1}{2\pi i}\mathcal{P}\int_0^1 \frac{d\nu}{\nu-\mu}\, e^{-(x-x_0)/\nu} A_m(\mu,\nu)\frac{E_{mm}(\nu,\boldsymbol{\Omega}_0)}{4\pi^2 i\nu} \\ &+ \tfrac{1}{2}B_m(\mu)\frac{E_{mm}(\mu,\boldsymbol{\Omega}_0)}{4\pi^2 i\mu}\, e^{-(x-x_0)/\mu}\Theta(\mu) \\ &+ \sum_{j=1}^{M} e^{-(x-x_0)/\nu_{m_j}} A_m(\mu,\nu_{m_j})\frac{F_{mm}(\nu_{m_j},\boldsymbol{\Omega}_0)}{4\pi^2}\Bigg) \\ &+ \frac{1}{\mu}\,\delta(\mu-\mu_0)e^{-(x-x_0)/\mu}\Theta(\mu) \\ &\times \left(\delta(\phi-\phi_0) - \frac{1}{2\pi}\sum_{m=-N}^{N} e^{im(\phi_0-\phi)}\right), \end{aligned} \quad (46)$$

where

$$A_m(\mu,\nu) = \sum_{l=|m|}^{N} [\mathcal{A}_{lm}^{m+}(\mu) - \mathcal{A}_{lm}^{m-}(\mu)]h_{lm}(\nu), \quad (47)$$

$$B_m(\mu) = \sum_{l=|m|}^{N} [\mathcal{A}_{lm}^{m+}(\mu) + \mathcal{A}_{lm}^{m-}(\mu)]h_{lm}(\mu), \quad (48)$$

$$\mathcal{A}_{lm}^{m\pm}(\mu) = \delta_{lm} - 2\pi b_l \times \int_{-1}^{1} d\mu' \frac{\mu}{(\mu-\mu')_{\pm}}\, Y_{lm}(\mu',0)Y_{mm}(\mu',0), \quad (49)$$

and $h_{lm}(\nu)$ are polynomials given by the recurrence relation (33).

Similarly, for $x < x_0$, we have

$$\begin{aligned} G_{<} = -\sum_{m=-N}^{N} \frac{e^{-im\phi}}{\mu Y_{mm}(\mu, 0)} &\times \Bigg(\frac{1}{2\pi i}\mathcal{P}\int_{-1}^{0} \frac{d\nu}{\nu-\mu}\, e^{-(x-x_0)/\nu} A_m(\mu,\nu)\frac{E_{mm}(\nu,\boldsymbol{\Omega}_0)}{4\pi^2 i\nu} \\ &+ \tfrac{1}{2}B_m(\mu)\frac{E_{mm}(\mu,\boldsymbol{\Omega}_0)}{4\pi^2 i\mu}\, e^{-(x-x_0)/\mu}\Theta(-\mu) \\ &+ \sum_{j=1}^{M} e^{-(x-x_0)/\nu_{m_j}} A_m(\mu,-\nu_{m_j})\frac{F_{mm}(-\nu_{m_j},\boldsymbol{\Omega}_0)}{4\pi^2}\Bigg) \\ &- \mu\delta(\mu-\mu_0)e^{-(x-x_0)/\mu}\Theta(-\mu) \\ &\times \left(\delta(\phi-\phi_0) - \sum_{m=-N}^{N} e^{im(\phi_0-\phi)}\right). \end{aligned} \quad (50)$$

A few remarks are due here. In the expression (46), the last two terms cannot cancel so long as N, the rank of degeneracy of the scattering kernel, is finite. In other words, for the problems lacking azimuthal symmetry, the set of eigenfunctions (E_{mm}, F_{mm}) do not possess half-range completeness for degenerate kernels. This was to be expected, because any arbitrary function of ϕ cannot be expanded in terms of a finite set of $e^{im\phi}$. Consequently, the last two terms are there to substantiate the deficiency of the set (E_{mm}, F_{mm}), as may be seen by letting N approach infinity; the terms cancel, and hence the deficiency becomes zero. On the other hand, for azimuthally symmetric problems, the above set is complete over the half-range of ν; this is readily seen by integrating Eq. (46) with respect to ϕ from 0 to 2π. The same remarks apply to $G_{>}$.

5. APPLICATIONS

A. Half-Space Problems

As an application of the above formulation, let us first consider the half-space problems. Shifting the point source to the origin ($x_0 = 0$), we may write the integral for $\Psi(x, \boldsymbol{\Omega})$ [see Eq. (3)] in the form

$$\begin{aligned} \Psi(x,\boldsymbol{\Omega}) = f(x,\boldsymbol{\Omega}) + \sum_{m=-N}^{N} \frac{e^{-im\phi}}{\mu Y_{mm}(\mu,0)} &\times \Bigg(\frac{1}{2\pi i}\mathcal{P}\int_0^1 \frac{d\nu}{\nu-\mu}\, e^{-x/\nu} A_m(\mu,\nu)\Gamma_m(\nu) \\ &+ \tfrac{1}{2}B_m(\mu)e^{-x/\mu}\Theta(\mu)\Gamma_m(\mu) \\ &+ \sum_{j=1}^{M} e^{-x/\nu_{m_j}} A_m(\mu,\nu_{m_j})\Gamma_m^0(\nu_{m_j})\Bigg) \\ &+ e^{-x/\mu}\Theta(\mu)\Bigg(\Psi(0,\boldsymbol{\Omega}) - \frac{1}{2\pi}\sum_{m=-N}^{N} e^{-im\phi} \\ &\times \int_0^{2\pi} d\phi' e^{im\phi'}\Psi(0,\mu,\phi')\Bigg), \end{aligned} \quad (51)$$

where $f(x, \boldsymbol{\Omega})$ is the angular density due to source and where

$$\Gamma_m(\nu) = \int d\Omega' \mu' \Psi(0, \mathbf{\Omega}') \frac{E_{mm}(\nu, \mathbf{\Omega}')}{4\pi^2 i\nu} \quad (52)$$

and

$$\Gamma_m^0(\nu_{m_j}) = \int d\Omega' \mu' \Psi(0, \mathbf{\Omega}') \frac{F_{mm}(\nu_{m_j}, \mathbf{\Omega}')}{4\pi^2} \quad (53)$$

are the coefficients to be determined from the given boundary condition. An equation that determines them is

$$\begin{aligned}\Psi(0, \mathbf{\Omega}) = f(0, \mathbf{\Omega}) &+ \sum_{m=-N}^{N} \frac{e^{-im\phi}}{\mu Y_{mm}(\mu, 0)} \\ &\times \left(\frac{1}{2\pi i} \mathcal{P} \int_0^1 \frac{d\nu}{\nu - \mu} A_m(\mu, \nu)\Gamma_m(\nu) \right. \\ &+ \tfrac{1}{2} B_m(\mu)\Gamma_m(\mu)\Theta(\mu) \\ &\left. + \sum_{j=1}^{M} A_m(\mu, \nu_{m_j})\Gamma_m^0(\nu_{m_j})\right) \\ &+ \Theta(\mu)\left(\Psi(0, \mathbf{\Omega}) - \frac{1}{2\pi} \sum_{m=-N}^{N} e^{-im\phi} \right. \\ &\left. \times \int_0^{2\pi} d\phi' e^{im\phi'} \Psi(0, \mu, \phi')\right). \end{aligned} \quad (54)$$

In solving this integral equation for any specific problem, we assume that $\Psi(0, \mathbf{\Omega})$ for $\mu > 0$ is known, so that

$$\Psi_m(0, \mu) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{im\phi} \Psi(0, \mathbf{\Omega}), \quad \mu > 0, \quad (55)$$

is also known. This entails a considerable amount of simplification in the solution of the integral Eq. (54). If we multiply it by $e^{im'\phi}$ and integrate over ϕ from 0 to 2π, we obtain a set of $2N + 1$ decoupled integral equations of the form

$$\frac{1}{2\pi i} \mathcal{P} \int_0^1 \frac{d\nu}{\nu - \mu} A_m(\mu, \nu)\Gamma_m(\nu) + \tfrac{1}{2} B_m(\mu)\Gamma_m(\mu) = \Phi_m(\mu), \quad (56)$$

where

$$\begin{aligned}\Phi_m(\mu) &= \mu Y_{mm}(\mu, 0)[\Psi_m(0, \mu) - f_m(\mu)] \\ &- \sum_{j=1}^{M} A_m(\mu, \nu_{m_j})\Gamma_m^0(\nu_{m_j})\end{aligned} \quad (57)$$

and

$$f_m(\mu) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{im\phi} f(0, \mathbf{\Omega}). \quad (58)$$

The set of Eqs. (56) are singular integral equations, which may be solved by the standard procedure due to Muskhelishvili.[9] In fact, an elaborate solution for $m = 0$, but arbitrary N, has been given by Mika.[10] Since the procedure for $m \neq 0$ is the same as for $m = 0$, we merely state the pertinent results.

Let us assume that the zeros of $\Lambda_m(z)$ [see Eq. (36)] are nondegenerate and the polynomials $h_{lm}(z)$ [see Eq. (33)] are simple, i.e., of degree precisely $l - m$. Splitting the kernel in Eq. (56) into the singular and the regular parts, we rewrite it in the form

$$\frac{\Lambda_m^+(\mu) - \Lambda_m^-(\mu)}{2\pi i} \mathcal{P} \int_0^1 \frac{d\nu}{\nu - \mu} \Gamma_m(\nu) + \tfrac{1}{2}[\Lambda_m^+(\mu) + \Lambda_m^-(\mu)]\Gamma_m(\mu) = \tilde{\Phi}_m(\mu), \quad (59)$$

where

$$\tilde{\Phi}_m(\mu) = \Phi_m(\mu) - \frac{1}{2\pi i} \int_0^1 d\nu \Gamma_m(\nu) \frac{A_m(\mu, \nu) - A_m(\mu, \mu)}{\nu - \mu} \quad (60)$$

and where we have used the fact that

$$A_m(\mu, \mu) = \Lambda_m^+(\mu) - \Lambda_m^-(\mu) \quad (61)$$

and

$$B_m(\mu) = \Lambda_m^+(\mu) + \Lambda_m^-(\mu) \quad (62)$$

[compare Eqs. (15), (16), and (17) with Eq. (36)]. In Eq. (60), the integral may be written as a sum over the moments of $\Gamma_m(\nu)$ as follows:

$$\frac{A_m(\mu, \nu) - A_m(\mu, \mu)}{\nu - \mu} = 4\pi^2 i\mu Y_{mm}(\mu, 0) \sum_{l=|m|}^{N} b_l \frac{Y_{lm}(\mu, 0)}{\nu - \mu} [h_{lm}(\nu) - h_{lm}(\mu)]. \quad (63)$$

If we write $h_{lm}(\nu)$ as

$$h_{lm}(\nu) = \sum_{k=0}^{l-|m|} C_k(l, m)\nu^k, \quad (64)$$

then

$$\frac{h_{lm}(\nu) - h_{lm}(\mu)}{\nu - \mu} = \sum_{k=1}^{l-|m|} \sum_{j=0}^{k-1} C_k(l, m)\mu^{k-j-1}\nu^j. \quad (65)$$

Substituting the appropriate ratio in Eq. (63) by means of Eq. (65), we obtain

$$\begin{aligned}\frac{A_m(\mu, \nu) - A_m(\mu, \mu)}{\nu - \mu} &= 4\pi^2 i\mu Y_{mm}(\mu, 0) \sum_{l=|m|+1}^{N} b_l Y_{lm}(\mu, 0) \\ &\times \sum_{k=1}^{l-|m|} \sum_{j=0}^{k-1} C_k(l, m)\mu^{k-j-1}\nu^j. \end{aligned} \quad (66)$$

Denoting the moments of $\Gamma_m(\nu)$ by g_{jm}, i.e.,

$$g_{jm} = \int_0^1 d\nu \nu^j \Gamma_m(\nu), \quad (67)$$

we re-express $\tilde{\Phi}_m$ [Eq. (60)] in the form

$$\begin{aligned}\tilde{\Phi}_m(\mu) = \Phi_m(\mu) &- 2\pi i\mu Y_{mm}(\mu, 0) \sum_{l=|m|+1}^{N} b_l Y_{lm}(\mu, 0) \\ &\times \sum_{k=1}^{l-|m|} \sum_{j=0}^{k-1} C_k(l, m)\mu^{k-j-1} g_{jm}. \end{aligned} \quad (68)$$

The solution of Eq. (59) is

$$\Gamma_m(\nu) = \frac{\Lambda_m^+(\nu) + \Lambda_m^-(\nu)}{2\Lambda_m^+(\nu)x_m^-(\nu)}\tilde{\Phi}_m(\nu) - \frac{\Lambda_m^+(\nu) - \Lambda_m^-(\nu)}{\Lambda_m^+(\nu)\Lambda_m^-(\nu)}\mathcal{P}\frac{1}{2\pi i}\int_0^1 d\mu\,\frac{\tilde{\Phi}_m(\mu)}{\nu-\mu}\frac{x_m^-(\mu)}{\Lambda_m^-(\mu)}, \tag{69}$$

while the conditions that determine $\Gamma_m^0(\nu_{m_j})$ are

$$\int_0^1 d\mu\mu^j\,\frac{x_m^-(\mu)}{\Lambda_m^-(\mu)}\tilde{\Phi}_m(\mu) = 0, \quad j = 0, 1, \cdots, M-1, \tag{70}$$

where

$$x_m(z) = \frac{1}{(1-z)^M}\exp\left(\frac{1}{\pi}\int_0^1 \frac{d\mu}{\mu - z}\arg\Lambda_m^+(\mu)\right). \tag{71}$$

Equations (70) give just the sufficient number of conditions to determine the unknown discrete coefficients $\Gamma_m^0(\nu_{m_j})$. The moments g_{jm} may be evaluated by using Eq. (67).

B. Slab Problems

Let us take the volume V under consideration to be the slab between $x = -\frac{1}{2}L$ and $x = \frac{1}{2}L$. Assuming that there are no sources ($Q = 0$), we see that the integral representation of $\Psi(x, \mathbf{\Omega})$, by virtue of Eqs. (3), (46), and (50), is then

$$\begin{aligned}\Psi(x,\mathbf{\Omega}) = \sum_{m=-N}^{N}\frac{e^{-im\phi}}{\mu Y_{mm}(\mu,0)} &\times\Bigg(\mathcal{P}\frac{1}{2\pi i}\int_0^1\frac{d\nu}{\nu-\mu}\,e^{-(x+\frac{1}{2}L)/\nu}A_m(\mu,\nu)\Gamma_m^{(1)}(\nu)\\ &+\tfrac{1}{2}B_m(\mu)e^{-(x+\frac{1}{2}L)/\mu}\Theta(\mu)\Gamma_m^{(1)}(\mu)\\ &+\sum_{j=1}^{M}e^{-(x+\frac{1}{2}L)/\nu_{m_j}}A_m(\mu,\nu_{m_j})D_m^{(1)}(\nu_{m_j})\Bigg)\\ &+e^{-(x+\frac{1}{2}L)/\mu}\Theta(\mu)\Bigg(\Psi(-\tfrac{1}{2}L,\mathbf{\Omega})-\frac{1}{2\pi}\\ &\times\sum_{m=-N}^{N}e^{-im\phi}\int_0^{2\pi}d\phi' e^{im\phi'}\Psi(-\tfrac{1}{2}L,\mu,\phi')\Bigg)\\ &-\sum_{m=-N}^{N}\frac{e^{-im\phi}}{\mu Y_{mm}(\mu,0)}\\ &\times\Bigg(\mathcal{P}\frac{1}{2\pi i}\int_{-1}^0\frac{d\nu}{\nu-\mu}\,e^{-(x-\frac{1}{2}L)/\nu}\Gamma_m^{(2)}(\nu)A_m(\mu,\nu)\\ &+\tfrac{1}{2}B_m(\mu)e^{-(x-\frac{1}{2}L)/\mu}\Theta(-\mu)\Gamma_m^{(2)}(\mu)\\ &+\sum_{j=1}^{M}e^{-(x-\frac{1}{2}L)/\nu_{m_j}}A_m(\mu,-\nu_{m_j})D_m^{(2)}(\nu_{m_j})\Bigg)\\ &-e^{-(x-\frac{1}{2}L)/\mu}\Theta(-\mu)\Bigg(\Psi(\tfrac{1}{2}L,\mathbf{\Omega})-\frac{1}{2\pi}\\ &\times\sum_{m=-N}^{N}e^{-im\phi}\int_0^{2\pi}d\phi' e^{im\phi'}\Psi(\tfrac{1}{2}L,\mu,\phi')\Bigg).\end{aligned} \tag{72}$$

The coefficients $\Gamma_m^{(1),(2)}(\nu)$ and $D_m^{(1),(2)}(\nu_{m_j})$ in Eq. (72), which are to be determined, are defined as

$$\Gamma_m^{(1),(2)}(\nu) = \int d\Omega'\mu'\Psi(\mp\tfrac{1}{2}L,\mathbf{\Omega}')\,\frac{E_{mm}(\nu,\mathbf{\Omega}')}{4\pi^2 i\nu} \tag{73}$$

and

$$D_m^{(1),(2)}(\nu_{m_j}) = \int d\Omega'\mu'\Psi(\mp\tfrac{1}{2}L,\mathbf{\Omega}')\,\frac{F_{mm}(\pm\nu_{m_j},\mathbf{\Omega}')}{4\pi^2}, \tag{74}$$

where $\Psi(\mp\frac{1}{2}L, \mathbf{\Omega})$ is the surface distribution at $x = \mp\frac{1}{2}L$. The rest of the symbols have the same meaning as previously.

In dealing with any particular problem, we assume that $\Psi(\mp\frac{1}{2}L, \mathbf{\Omega})$ for $(^{\mu>0}_{\mu<0})$ are known. In that case, we may reduce Eq. (73) to two sets of decoupled singular integral equations by letting x approach $\mp\frac{1}{2}L$, multiplying both sides by $e^{im'\phi}$, and integrating over ϕ' from 0 to 2π. The result is

$$\begin{aligned}\tfrac{1}{2}B_m(\mu)\Gamma_m^{(1)}(\mu) &+ \mathcal{P}\frac{1}{2\pi i}\int_0^1\frac{d\nu}{\nu-\mu}A_m(\mu,\nu)\Gamma_m^{(1)}(\nu)\\ &+\sum_{j=1}^{M}A_m(\mu,\nu_{m_j})D_m^{(1)}(\nu_{m_j})\\ &-\sum_{j=1}^{M}e^{-L/\nu_{m_j}}A_m(\mu,-\nu_{m_j})D_m^{(2)}(\nu_{m_j})\\ &-\frac{1}{2\pi i}\int_{-1}^0\frac{d\nu}{\nu-\mu}\,e^{L/\nu}A_m(\mu,\nu)\Gamma_m^{(2)}(\nu)\\ &=\mu Y_{mm}(\mu,0)\Psi_m(-\tfrac{1}{2}L,\mu),\quad \mu>0,\end{aligned} \tag{75}$$

$$\begin{aligned}\tfrac{1}{2}B_m(\mu)\Gamma_m^{(2)}(\mu) &+ \mathcal{P}\frac{1}{2\pi i}\int_{-1}^0\frac{d\nu}{\nu-\mu}A_m(\mu,\nu)\Gamma_m^{(2)}(\nu)\\ &+\sum_{j=1}^{M}A_m(\mu,-\nu_{m_j})D_m^{(2)}(\nu_{m_j})\\ &-\sum_{j=1}^{M}e^{-L/\nu_{m_j}}A_m(\mu,\nu_{m_j})D_m^{(1)}(\nu_{m_j})\\ &-\frac{1}{2\pi i}\int_0^1\frac{d\nu}{\nu-\mu}\,e^{-L/\nu}A_m(\mu,\nu)\Gamma_m^{(1)}(\nu)\\ &=-\mu Y_{mm}(\mu,0)\Psi_m(\tfrac{1}{2}L,\mu),\quad \mu<0.\end{aligned} \tag{76}$$

Clearly, exact solutions of these integral equations are not feasible. However, they are well suited for approximations in the asymptotic limits.

1. Thick Slabs ($L \gg 1$)

For this limiting case we can solve Eqs. (75) and (76) for the coefficients by the iterative procedure discussed in Ref. 4. Thus, in the zeroth approximation, we ignore the terms involving the exponentials $e^{-L/\nu}$. Equations (75) and (76) then reduce to Eq. (56) for the half-space problems. Let us, therefore, assume for a moment that $\Gamma_m^{(2)}(\nu)$, $D_m^{(2)}(\nu_{m_j})$, and $\Gamma_m^{(1)}(\nu)$, $D_m^{(1)}(\nu_{m_j})$ are known in Eqs. (75) and (76), respectively. Then, formally, solutions of Eqs. (75) and (76) are

$$\Gamma_m^{(1)}(\nu) = \frac{\Lambda_m^+(\nu) + \Lambda_m^-(\nu)}{2\Lambda_m^+(\nu)x_m^-(\nu)}\Phi_m^{(1)}(\nu) - \frac{\Lambda_m^+(\nu) - \Lambda_m^-(\nu)}{\Lambda_m^+(\nu)\Lambda_m^-(\nu)} I_m^{(1)}(\nu), \quad (77)$$

where

$$\begin{aligned}\Phi_m^{(1)}(\nu) = {} & \nu Y_{mm}(\nu, 0)\Psi_m(-\tfrac{1}{2}L, \nu) \\ & - \sum_{j=1}^{M} A_m(\nu, \nu_{m_j}) D_m^{(1)}(\nu_{m_j}) \\ & + \sum_{j=1}^{M} e^{-L/\nu_{m_j}} A_m(\nu, -\nu_{m_j}) D_m^{(2)}(\nu_{m_j}) \\ & + \frac{1}{2\pi i}\int_{-1}^{0}\frac{d\nu'}{\nu' - \nu} e^{L/\nu'} A_m(\nu, \nu')\Gamma_m^{(2)}(\nu') \\ & - 2\pi i \nu Y_{mm}(\nu, 0)\sum_{l=|m|+1}^{N} b_l Y_{lm}(\nu, 0) \\ & \times \sum_{k=1}^{l-|m|}\sum_{j=0}^{k-1} C_k(l, m)\nu^{k-j-1} g_{jm}^{(1)}, \end{aligned} \quad (78)$$

$$g_{jm}^{(1)} = \int_0^1 d\nu \nu^j \Gamma_m^{(1)}(\nu), \quad (79)$$

and

$$\begin{aligned} I_m^{(1)}(\nu) = {} & \mathcal{P}\frac{1}{2\pi i}\int_0^1 \frac{d\mu'}{\nu - \mu'}\frac{x_m^-(\mu')}{\Lambda_m^-(\mu')} \\ & \times \mu' Y_{mm}(\mu', 0)\Psi_m(-\tfrac{1}{2}L, \mu') \\ & - \sum_{j=1}^{M}[D_m^{(1)}(\nu_{m_j}) R_m^{(1)}(\nu, \nu_{m_j}) \\ & - e^{-L/\nu_{m_j}} D_m^{(2)}(\nu_{m_j}) R_m^{(1)}(\nu, -\nu_{m_j})] \\ & + \frac{1}{2\pi i}\int_{-1}^{0}\frac{d\nu'}{\nu - \nu'} e^{L/\nu'}\Gamma_m^{(2)}(\nu') \\ & \times [R_m^{(1)}(\nu', \nu') - R_m^{(1)}(\nu, \nu')], \end{aligned} \quad (80)$$

with

$$R_m^{(1)}(\nu, \nu') = \mathcal{P}\frac{1}{2\pi i}\int_0^1 \frac{d\mu'}{\nu - \mu'}\frac{x_m^-(\mu')}{\Lambda_m^-(\mu')} A_m(\mu', \nu'). \quad (81)$$

The additional conditions that determine the discrete coefficients are

$$\int_0^1 d\mu \mu^j \frac{x_m^-(\mu)}{\Lambda_m^-(\mu)}\Phi_m^{(1)}(\mu) = 0, \quad j = 0, 1, \cdots, M-1. \quad (82)$$

Similarly, for Eq. (76), we have

$$\Gamma_m^{(2)}(\nu) = \frac{\Lambda_m^+(\nu) + \Lambda_m^-(\nu)}{2\Lambda_m^+(\nu)x_m^-(\nu)}\Phi_m^{(2)}(\nu) - \frac{\Lambda_m^+(\nu) - \Lambda_m^-(\nu)}{\Lambda_m^+(\nu)\Lambda_m^-(\nu)} I_m^{(2)}(\nu), \quad (83)$$

where

$$\begin{aligned}\Phi_m^{(2)}(\mu) = {} & -\Big(\mu Y_{mm}(\mu, 0)\Psi_m(\tfrac{1}{2}L, \mu) \\ & - \sum_{j=1}^{M} A_m(\mu, -\nu_{m_j}) D_m^{(2)}(\nu_{m_j}) \\ & + \sum_{j=1}^{M} e^{-L/\nu_{m_j}} A_m(\mu, \nu_{m_j}) D_m^{(1)}(\nu_{m_j}) \\ & + \int_0^1 \frac{d\nu}{\nu - \mu} e^{-L/\nu} A_m(\mu, \nu)\Gamma_m^{(1)}(\nu) \\ & - 2\pi i \mu Y_{mm}(\mu, 0)\sum_{l=|m|+1}^{N} b_l Y_{lm}(\mu, 0) \\ & \times \sum_{k=1}^{l-|m|}\sum_{j=0}^{k-1} C_k(l, m)\mu^{k-j-1} g_{jm}^{(2)}\Big), \end{aligned} \quad (84)$$

$$g_{jm}^{(2)} = \int_{-1}^{0} d\nu \nu^j \Gamma_m^{(2)}(\nu), \quad (85)$$

$$\begin{aligned} I_m^{(2)}(\nu) = {} & -\mathcal{P}\frac{1}{2\pi i}\int_{-1}^{0}\frac{d\mu'}{\nu - \mu'}\frac{x_m^-(\mu')}{\Lambda_m^-(\mu')} \\ & \times \mu' Y_{mm}(\mu', 0)\Psi_m(\tfrac{1}{2}L, \mu') \\ & + \sum_{j=1}^{M}[D_m^{(2)}(\nu_{m_j}) R_m^{(2)}(\nu, -\nu_{m_j}) \\ & - e^{-L/\nu_{m_j}} D_m^{(1)}(\nu_{m_j}) R_m^{(2)}(\nu, \nu_{m_j})] \\ & - \frac{1}{2\pi i}\int_0^1 \frac{d\nu'}{\nu - \nu'} e^{-L/\nu'}\Gamma_m^{(1)}(\nu') \\ & \times [R_m^{(2)}(\nu', \nu') - R_m^{(2)}(\nu, \nu')], \end{aligned} \quad (86)$$

$$R_m^{(2)}(\nu, \nu') = \mathcal{P}\frac{1}{2\pi i}\int_{-1}^{0}\frac{d\mu'}{\nu - \mu'}\frac{x_m^-(\mu')}{\Lambda_m^-(\mu')} A_m(\mu', \nu'). \quad (87)$$

The additional conditions are

$$\int_{-1}^{0} d\mu \mu^j \frac{x_m^-(\mu)}{\Lambda_m^-(\mu)}\Phi_m^{(2)}(\mu) = 0, \quad j = 0, 1, \cdots, M.$$

Consider Eq. (77) first. In the zeroth approximation, ignore all the terms involving the exponentials. The coefficients $\Gamma_m^{(1)}(\nu)$ {denoting the degree of approximation as $[\Gamma_m^{(1)}(\nu)]_n$} are then given by

$$[\Gamma_m^{(1)}(\nu)]_0 = \frac{\Lambda_m^+(\nu) + \Lambda_m^-(\nu)}{2\Lambda_m^+(\nu)x_m^-(\nu)}[\Phi_m^{(1)}(\nu)]_0 - \frac{\Lambda_m^+(\nu) - \Lambda_m^-(\nu)}{\Lambda_m^+(\nu)\Lambda_m^-(\nu)}[I_m^{(1)}(\nu)]_0, \quad (88)$$

where

$$[\Phi_m^{(1)}(\mu)]_0 = \mu Y_{mm}(\mu, 0)\Psi_m(-\tfrac{1}{2}L, \mu) - \sum_{j=1}^{M} A_m(\mu, \nu_{m_j}) D_m^{(1)}(\nu_{m_j}) - 2\pi i \mu Y_{mm}(\mu, 0) \sum_{l=|m|+1}^{N} b_l Y_{lm}(\mu, 0) \times \sum_{k=1}^{l-|m|} \sum_{j=0}^{k-1} C_k(l, m) \mu^{k-j-1} g_{jm}^{(2)} \tag{89}$$

and

$$[I_m^{(1)}(\nu)]_0 = \mathcal{P} \frac{1}{2\pi i} \int_0^1 \frac{d\mu'}{\nu - \mu'} \frac{x_m^-(\mu')}{\Lambda_m^-(\mu')} \times \mu' Y_{mm}(\mu', 0)\Psi_m(-\tfrac{1}{2}L, \mu') - \sum_{j=1}^{M} D_m^{(1)}(\nu_{m_j}) R_m^{(1)}(\nu, \nu_{m_j}). \tag{90}$$

Similar quantities for $\Gamma_m^{(2)}$ should be obvious. In the first approximation, the correction to Eq. (89) for $\Gamma_m^{(1)}(\nu)$ is obtained simply by retaining the exponential terms in Eqs. (78) and (80), with $\Gamma_m^{(2)}(\nu)$ and $D_m^{(2)}(\nu_{m_j})$ replaced by $[\Gamma_m^{(2)}(\nu)]_0$ and $[D_m^{(2)}(\nu_{m_j})]_0$, respectively. Thus,

$$[\Gamma_m^{(1)}(\nu)]_1 = \frac{\Lambda_m^+(\nu) + \Lambda_m^-(\nu)}{2\Lambda_m^+(\nu) x_m^-(\nu)} [\Phi_m^{(1)}(\nu)]_1 - \frac{\Lambda_m^+(\nu) - \Lambda_m^-(\nu)}{\Lambda_m^+(\nu)\Lambda_m^-(\nu)} [I_m^{(1)}(\nu)]_1, \tag{91}$$

where

$$[\Phi_m^{(1)}(\nu)]_1 = [\Phi_m^{(1)}(\nu)]_0 + \sum_{j=1}^{M} e^{-L/\nu_{m_j}} A_m(\nu, -\nu_{m_j}) [D_m^{(2)}(\nu_{m_j})]_0 + \frac{1}{2\pi i} \int_{-1}^{0} \frac{d\nu'}{\nu' - \nu} e^{L/\nu'} A_m(\nu, \nu') [\Gamma_m^{(2)}(\nu')]_0 \tag{92}$$

and

$$[I_m^{(1)}(\nu)]_1 = [I_m^{(1)}(\nu)]_0 + \sum_{j=1}^{M} e^{-L/\nu_{m_j}} [D_m^{(2)}(\nu_{m_j})]_0 R_m^{(1)}(\nu, -\nu_{m_j}) + \frac{1}{2\pi i} \int_{-1}^{0} \frac{d\nu'}{\nu - \nu'} e^{L/\nu'} [\Gamma_m^{(2)}(\nu')]_0 \times [R_m^{(1)}(\nu', \nu') - R_m^{(1)}(\nu, \nu')]. \tag{93}$$

The same iterative procedure may be followed to approximate the discrete coefficients $D_m^{(1)}(\nu_{m_j})$ which are determined by Eq. (82). The procedure for obtaining $\Gamma_m^{(2)}$ and $D_m^{(2)}$ is exactly the same. Here we omit the details.

2. Thin Slabs

Because this situation is physically much simpler than the limiting case ($L \gg 1$) considered previously, one can obtain the integral representation for $\Psi(x, \mathbf{\Omega})$ by dealing directly with Eq. (3). The approximation procedure for various other situations is discussed in Refs. 2 and 4. To avoid repetition, we merely state the pertinent results here. Thus, if we write

$$\Psi(-\tfrac{1}{2}L, \mathbf{\Omega}) = \Psi(\tfrac{1}{2}L, \mathbf{\Omega}) + \tilde{\Psi}(-\tfrac{1}{2}L, \mathbf{\Omega}), \quad \mu < 0, \tag{94}$$

$$\Psi(\tfrac{1}{2}L, \mathbf{\Omega}) = \Psi(-\tfrac{1}{2}L, \mathbf{\Omega}) + \tilde{\Psi}(\tfrac{1}{2}L, \mathbf{\Omega}), \quad \mu > 0, \tag{95}$$

where $\tilde{\Psi}(\mp\tfrac{1}{2}L, \mathbf{\Omega})$ are to be of order L, then one can show that[11]

$$\tilde{\Psi}(\tfrac{1}{2}L, \mathbf{\Omega}) = \int d\Omega' \mu' [\Psi(-\tfrac{1}{2}L, \mathbf{\Omega}')\Theta(\mu') + \Psi(\tfrac{1}{2}L, \mathbf{\Omega}')\Theta(-\mu')] \cdot \{G_>(\tfrac{1}{2}L, \mathbf{\Omega}; -\tfrac{1}{2}L, \mathbf{\Omega}') - G_+(-\tfrac{1}{2}L, \mathbf{\Omega}; -\tfrac{1}{2}L, \mathbf{\Omega}') + G_<(-\tfrac{1}{2}L, \mathbf{\Omega}; \tfrac{1}{2}L, \mathbf{\Omega}') - G_-(-\tfrac{1}{2}L, \mathbf{\Omega}; -\tfrac{1}{2}L, \mathbf{\Omega}')\}, \tag{96}$$

where

$$G_\pm(-\tfrac{1}{2}L, \mathbf{\Omega}; -\tfrac{1}{2}L, \mathbf{\Omega}') = \lim_{x \to -\frac{1}{2}L \left\{\substack{\text{From within } V \\ \text{From without } V}\right\}} G(x, \mathbf{\Omega}; -\tfrac{1}{2}L, \mathbf{\Omega}'). \tag{97}$$

For a homogeneous medium we have

$$\tilde{\Psi}(-\tfrac{1}{2}L, \mathbf{\Omega}) = -\tilde{\Psi}(\tfrac{1}{2}L, \mathbf{\Omega}). \tag{98}$$

The combination of Green's functions, occurring in the right-hand side of Eq. (96), may be calculated explicitly by means of Eqs. (46) and (50). It is given by

$$G_> - G_+ + G_< - G_- = \sum_{m=-N}^{N} \frac{e^{-im\phi}}{\mu Y_{mm}(\mu, 0)} \times \Bigg(\int_0^1 d\nu (e^{-L/\nu} - 1) E_{mm}(\nu, \mathbf{\Omega}') H_m(\nu, \mu) + \frac{1}{4\pi^2} \sum_{j=1}^{M} (e^{-L/\nu_{m_j}} - 1)(A_m(\mu, \nu_{m_j}) F_{mm}(\nu_{m_j}, \mathbf{\Omega}') - A_m(\mu, -\nu_{m_j}) F_{mm}(-\nu_{m_j}, \mathbf{\Omega}')) \Bigg) + \frac{1}{\mu} \delta(\mu - \mu') \left(\delta(\phi - \phi') - \frac{1}{2\pi} \sum_{m=-N}^{N} e^{im(\phi' - \phi)} \right) \times [(e^{-L/\mu} - 1)\Theta(\mu) - (e^{L/\mu} - 1)\Theta(-\mu)], \tag{99}$$

where for convenience we have defined

$$H_m(\nu,\mu)=\frac{1}{2\pi i}\mathcal{P}\frac{1}{\nu-\mu}\frac{A(\mu,\nu)}{4\pi^2 i\nu}+\frac{B_m(\mu)}{8\pi^2 i\mu}\delta(\nu-\mu). \tag{100}$$

In general, the contribution from the terms involving discrete eigenfunctions [in Eq. (99)] is small compared to the terms involving the continuum eigenfunctions. Let us therefore ignore that term and further approximate the terms involving exponentials as follows:

$$\begin{aligned}\int_0^1 d\nu(e^{-L/\nu}-1)F(\nu)&=\int_0^1 d\nu[F(\nu)-F(0)](e^{-L/\nu}-1)\\&\quad+F(0)\int_0^1 d\nu(e^{-L/\nu}-1)\\&=\sum_{n=1}^{\infty}\frac{(-1)^nL^n}{n!}\int_0^1 d\nu\nu^{-n}[F(\nu)-F(0)]\\&\quad+F(0)\int_0^1 d\nu(e^{-L/\nu}-1).\end{aligned}$$

Since

$$\int_0^1 d\nu e^{-L/\nu}=\sum_{\substack{n=0\\n\neq 1}}^{\infty}\frac{(-L)^n}{n!\,(1-n)}+L(\log L-1+\gamma),$$

where $\gamma=0.577216$ is the Euler's constant, we get

$$\begin{aligned}\int_0^1 d\nu(e^{-L/\nu}-1)F(\nu)&=\sum_{n=1}^{\infty}\frac{(-L)^n}{n!}\int_0^1 d\nu\nu^{-n}(F(\nu)-F(0))\\&\quad+F(0)\left(\sum_{n=2}^{\infty}\frac{(-L)^n}{n!\,(1-n)}+L(\log L-1+\gamma)\right).\end{aligned} \tag{101}$$

Retaining terms only up to quadratic in L, we see that Eq. (99), by means of Eq. (101), becomes

$$\begin{aligned}&G_>-G_++G_<-G_-\\&=\sum_{m=-N}^{N}\frac{e^{-im\phi}}{\mu Y_{mm}(\mu,0)}\Bigg[L\log LE_{mm}(0,\mathbf{\Omega}')H_m(0,\mu)\\&\quad+L\Bigg((\gamma-1)E_{mm}(0,\mathbf{\Omega}')H_m(0,\mu)\\&\quad-\int_0^1 d\nu\nu^{-1}(E_{mm}(\nu,\mathbf{\Omega}')\cdot H_m(\nu,\mu)\\&\quad-E_{mm}(0,\mathbf{\Omega}')H_m(0,\mu))\Bigg)\\&\quad+\tfrac{1}{2}L^2\Bigg(\int_0^1 d\nu\nu^{-2}(E_{mm}(\nu,\mathbf{\Omega}')H_m(\nu,\mu)\\&\quad-E_{mm}(0,\mathbf{\Omega}')H_m(0,\mu))-E_{mm}(0,\mathbf{\Omega}')H_m(0,\mu)\Bigg)\Bigg].\end{aligned} \tag{102}$$

The angular density Ψ may now be calculated simply by inserting the expression (102) for the given combination of Green's function in Eq. (97).

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[5] The two integral equations (4) for $\Psi(\mathbf{r}_s,\mathbf{\Omega})$ are actually identical by virtue of the jump condition.

$$\hat{n}_i(r_s')\cdot\mathbf{\Omega}'[G_+(\mathbf{r}_s,\mathbf{\Omega};r_s',\mathbf{\Omega}')-G_-(\mathbf{r}_s,\mathbf{\Omega};\mathbf{r}_s',\mathbf{\Omega}')]=\delta^{(s)}(\mathbf{r}_s-\mathbf{r}_s')\delta(\mathbf{\Omega}\cdot\mathbf{\Omega}'),$$

where $G_\pm$ is the boundary value of G as $\mathbf{r}$ approaches the surface from (within/without) V.

[6] We shall also consider the Case spectrum in the complex z plane with $z=i/k$. In this plane, all the appropriate singular parts of the Green's function have branch cuts on the real axis for $-1\leq\nu\leq 1$, while the zeros of Λ_m are also transformed accordingly.

[7] For real zeros of Λ_m one needs to modify the procedure as given in Refs. 2 and 3.

[8] We assume that $\Lambda_m^+(k)\Lambda_m^-(k)\neq 0$ for any value of m; i.e., no zeros of $\Lambda_m(k)$ are imbedded in its branch line $k=\pm i\infty$ to $\pm i$.

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[11] In arriving at Eq. (97), we assumed that $L/\mu \ll 1$.