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Quantization of massive vector fields in curved space–time

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We develop a cannonical quantization for massive vector fields on a globally hyperbolic Lorentzian manifold. © *1999 American Institute of Physics.* [S0022-2488(99)04506-5]

I. INTRODUCTION

Rigorous theories have been developed for scalar, Dirac, and electromagnetic quantum fields on globally hyperbolic manifolds.[1–6] However, apparently few rigorous results exist for massive vector fields on such manifolds.[7] A particle described by such a field is the ω-meson with a mass $m_\omega = 783$ MeV.[8] In this article, we develop a canonical quantization for massive vector fields on a globally hyperbolic Lorentzian manifold. The analysis is divided into two parts, the classical problem and the quantum problem.

For the classical problem, we start with the equations of motion $(\delta^{(4)}d^{(4)}+m^2)\mathcal{A}=0$ which are the curved space–time generalizations of Proca's equations. We reduce these to a hyperbolic system $(\Box+m^2)\mathcal{A}=0$, and a constraint $\delta^{(4)}\mathcal{A}=0$. The hyperbolic system has global fundamental solutions $E_m^{\pm(1)}$ and a propagator $E_m^{(1)}=E_m^{+(1)}-E_m^{-(1)}$.[9–12] We introduce a series of operators $\rho_{(0)}$, $\rho_{(d)}$, $\rho_{(\delta)}$, and $\rho_{(n)}$ that map a solution $\mathcal{A}$ to its Cauchy data $A_{(0)}$, $A_{(d)}$, $A_{(\delta)}$, and $A_{(n)}$, respectively. These operators have transposes $\rho'_{(0)}$, $\rho'_{(d)}$, $\rho'_{(\delta)}$, and $\rho'_{(n)}$. We construct a series of operators $E_m^{(1)}\rho'_{(0)}$, $E_m^{(1)}\rho'_{(d)}$, $E_m^{(1)}\rho'_{(\delta)}$, and $E_m^{(1)}\rho'_{(n)}$ and show that these operators collectively map Cauchy data to a unique field solution. We apply these operators to obtain a global solution for Proca's equations wherein we satisfy the constraint by restricting the Cauchy data. Our representation of the solution is apparently new, and especially useful for field quantization as we demonstrate. This method also applies to Maxwell's equations. Since these equations are also of interest, but not needed for our quantum problem, we treat them separately in an appendix.

For the quantum problem, we start with a representation $(\phi,\pi,\mathcal{H})$ of the CCRs on an arbitrary Cauchy surface. We construct a space–time field operator $\mathcal{A}$ in terms of the data (ϕ,π) in accordance with the classical initial value problem. We then pass to the Weyl form W of the CCRs and study the C^* algebra of observables $\mathfrak{A}$ generated by W. We show that $\mathfrak{A}$ is independent of the representation on a given Cauchy surface, and also independent of the Cauchy surface. This work generalizes Dimock's treatment of scalar fields.[3]

II. PRELIMINARY CONCEPTS

Let $(\mathcal{M},g)$ be a globally hyperbolic, orientable, time-orientable, space–time consisting of a smooth four-dimensional manifold $\mathcal{M}$ endowed with a smooth Lorentzian metric g with signature $(-1,1,1,1)$. As a consequence of global hyperbolicity, there is a (nonunique) smooth time coordinate t, and $(\mathcal{M},g)$ can be foliated by a one-parameter family of Cauchy surfaces $\{t\}\times\Sigma_t$ giving it the topology $\mathcal{M}\approx\mathbb{R}\times\Sigma$.[2,13] Each Cauchy surface Σ inherits a smooth, proper Riemannian metric γ. We label events in $\mathcal{M}$ using (t,x) where $x\in\Sigma$, and adopt the standard convention in which Greek subscripts apply to $(\mathcal{M},g)$ taking values from 0 to 4, and Latin subscripts apply to (Σ,γ) and range from 1 to 3.

Let $\mathcal{E}^{(p)}(\mathcal{M})$ denote the space of smooth, real-valued p-forms on $(\mathcal{M},g)$ and $\mathfrak{D}^{(p)}(\mathcal{M})$ specify such forms with compact support. These spaces have duals $\mathcal{E}'^{(p)}(\mathcal{M})$ and $\mathfrak{D}'^{(p)}(\mathcal{M})$ with topologies similar to those defined for the corresponding spaces on $\mathbb{R}^n$.[10–12] We use the standard notation $\langle \mathcal{T},\mathcal{F}\rangle$ to denote the action of a distribution $\mathcal{T}$ on a test function $\mathcal{F}\in\mathfrak{D}^{(p)}(\mathcal{M})$. Specializing to $\mathcal{T}\in\mathfrak{D}^{(p)}(\mathcal{M})\subset\mathfrak{D}'^{(p)}(\mathcal{M})$ we have $\langle \mathcal{T},\mathcal{F}\rangle=\langle \mathcal{T},\mathcal{F}\rangle_{\mathcal{M}}$, where

$$\langle \mathcal{T},\mathcal{F}\rangle_{\mathcal{M}}\equiv\int_{\mathcal{M}}\mathcal{T}\wedge *^{(4)}\mathcal{F}, \tag{1}$$

is the global inner product, and $*^{(4)}:\mathfrak{D}^{(p)}(\mathcal{M})\rightarrow\mathfrak{D}^{(4-p)}(\mathcal{M})$ is the Hodge star operator with respect to g ($(*^{(4)})^2=(-1)^{p+1}$). On $(\mathcal{M},g)$ we have an exterior derivative $d^{(4)}$ and codifferential $\delta^{(4)}=*^{(4)}d^4*^{(4)}$ which are the formal adjoints of one another $\langle d^{(4)}\mathcal{A},\mathcal{F}\rangle_{\mathcal{M}}=\langle\mathcal{A},\delta^{(4)}\mathcal{F}\rangle_{\mathcal{M}}$, where $\mathcal{A}\in\mathfrak{D}^{(p-1)}(\mathcal{M})$ and $\mathcal{F}\in\mathfrak{D}^{(p)}(\mathcal{M})$. There is also the D'Alembertian $\Box=(\delta^{(4)}d^{(4)}+d^{(4)}\delta^{(4)})$ which is formally self-adjoint (on Minkowski space $\Box=\partial_0^2-\nabla^2$).[14,15]

Let $\mathcal{E}^{(k)}(\Sigma)$, $\mathfrak{D}^{(k)}(\Sigma)$, $\mathcal{E}'^{(k)}(\Sigma)$ and $\mathfrak{D}'^{(k)}(\Sigma)$ denote the corresponding spaces on (Σ,γ). For $T\in\mathfrak{D}^{(k)}(\Sigma)\subset\mathfrak{D}'^{(k)}(\Sigma)$ we have $\langle T,F\rangle=\langle T,F\rangle_{\Sigma}$, where

$$\langle T,F\rangle_{\Sigma}\equiv\int_{\Sigma}T\wedge *^{(3)}F, \tag{2}$$

is the global inner product on (Σ,γ), and $*^{(3)}$: $\mathfrak{D}^{(k)}(\Sigma)\rightarrow\mathfrak{D}^{(3-k)}(\Sigma)$ is the Hodge star operator with respect to γ ($(*^{(3)})^2=1$). The exterior derivative $d^{(3)}$ and codifferential $\delta^{(3)}=(-1)^k*^{(3)}d^{(3)}*^{(3)}$ on (Σ,γ) are formal adjoints of one another $\langle d^{(3)}A,F\rangle_{\Sigma}=\langle A,\delta^{(3)}F\rangle_{\Sigma}$, with $A\in\mathfrak{D}^{(k-1)}(\Sigma)$ and $F\in\mathfrak{D}^{(k)}(\Sigma)$. The Laplace–Beltrami operator $\triangle=\delta^{(3)}d^{(3)}+d^{(3)}\delta^{(3)}$ is formally self-adjoint.[14,15] Finally, we work in units in which $c=1$.

III. PROCA'S EQUATIONS

We study the initial value problem for a vector field $\mathcal{A}$ of mass $m\in(0,\infty)$ satisfying the curved space–time generalization of Proca's equations

$$(\delta^{(4)}d^{(4)}+m^2)\mathcal{A}=0. \tag{3}$$

These equations reduce to the hyperbolic system

$$(\Box+m^2)\mathcal{A}=0, \tag{4}$$

and a constraint

$$\delta^{(4)}\mathcal{A}=0. \tag{5}$$

The system (4) has unique fundamental solutions $E_m^{\pm(1)}(p,q)\in\mathfrak{D}'^{(1)}(\mathcal{M}_p\times\mathcal{M}_q)$ where $E_m^{\pm(1)}(p,q):\mathfrak{D}^{(1)}(\mathcal{M}_q)\rightarrow\mathcal{E}^{(1)}(\mathcal{M}_p)$ ($p,q\in\mathcal{M}$ represent space–time events and $\mathcal{M}_p$, $\mathcal{M}_q$ identify the action of $E_m^{\pm(1)}(p,q)$).[9,10,16] These solutions satisfy

$$(\Box_p+m^2)E_m^{\pm(1)}(p,q)=\delta^{(1)}(p,q), \tag{6}$$

where $\delta^{(1)}(p,q)$ is the Dirac 1-tensor kernel, and supp $E_m^{\pm(1)}(\cdot,q)\subset J^{\pm}(q)$. Also $\mathcal{A}^{\pm}(p)=\langle E_m^{\pm(1)}(p,q),\mathcal{F}(q)\rangle_{\mathcal{M}_q}$ satisfies $(\Box+m^2)\mathcal{A}^{\pm}=\mathcal{F}$, with

$$\mathrm{supp}(\mathcal{A}^{\pm})\subset J^{\pm}(\mathrm{supp}(\mathcal{F})), \tag{7}$$

where $J^{\pm}(S)$ is the set of point in $(\mathcal{M},g)$ that can be reached from the set $S\subset\mathcal{M}$ by a future/past directed causal curve.[10]

The kernels $E_m^{\pm(1)}(p,q)$ are identified with operators $E_m^{\pm(1)}:\mathfrak{D}^{(1)}(\mathcal{M})\rightarrow\mathcal{E}^{(1)}(\mathcal{M})$ and from (6),

$$E_m^{\pm(1)}(\Box + m^2) = (\Box + m^2)E_m^{\pm(1)} = I. \tag{8}$$

The $E_m^{\pm(1)}$, which are linear and continuous, give rise to transpose operators $E_m^{\pm\prime(1)}:\mathcal{E}'^{(1)}(\mathcal{M}) \to \mathfrak{D}'^{(1)}(\mathcal{M})$ which are also continuous.[17] Since $\Box + m^2$ is formally self-adjoint $E_m^{\pm\prime(1)} = E_m^{\mp(1)}$ on $\mathfrak{D}^{(1)}(\mathcal{M})$. Moreover, since $E_m^{\pm\prime(1)}$ are linear and continuous on $\mathcal{E}'^{(1)}(\mathcal{M})$ we extend $E_m^{\pm(1)}$ from $\mathfrak{D}^{(1)}(\mathcal{M}) \subset \mathcal{E}'^{(1)}(\mathcal{M})$ to $\mathcal{E}'^{(1)}(\mathcal{M})$, i.e.,

$$E_m^{\mp(1)} = E_m^{\pm\prime(1)} : \mathcal{E}'^{(1)}(\mathcal{M}_p) \to \mathfrak{D}'^{(1)}(\mathcal{M}_q). \tag{9}$$

Lastly, we introduce the propagator

$$E_m^{(1)} = E_m^{+(1)} - E_m^{-(1)}, \ \text{(propagator)},$$

where $(\Box + m^2)E_m^{(1)} = 0$. This operator has a transpose $E_m'^{(1)} = E_m^{+\prime(1)} - E_m^{-\prime(1)}$ and from (9) we have

$$E_m'^{(1)} = -E_m^{(1)} : \mathcal{E}'^{(1)}(\mathcal{M}_p) \to \mathfrak{D}'^{(1)}(\mathcal{M}_q). \tag{10}$$

Next, we introduce a series of operators that collectively map a solution of (4) to its data. Choose any Cauchy surface say $\{0\}\times\Sigma$, and let $i:\Sigma \to \mathcal{M}$ be the inclusion operator with pullback i^*. We define the operators

$$\begin{aligned}
\rho_{(0)} &\equiv i^* \quad \text{(pullback)}\\
\rho_{(d)} &\equiv -*^{(3)} i^* *^{(4)} d^{(4)} \quad \text{(forward normal derivative)}\\
\rho_{(\delta)} &\equiv i^* \delta^{(4)} \quad \text{(pullback of divergence)}\\
\rho_{(n)} &\equiv -*^{(3)} i^* *^{(4)} \quad \text{(forward normal)},
\end{aligned} \tag{11}$$

where $\rho_{(0)}$, $\rho_{(d)} : \mathcal{E}^{(1)}(\mathcal{M}) \to \mathcal{E}^{(1)}(\Sigma)$, and $\rho_{(\delta)}$, $\rho_{(n)} : \mathcal{E}^{(1)}(\mathcal{M}) \to \mathcal{E}^{(0)}(\Sigma)$. These operators can be applied to any smooth p-form. The motivation for (11) comes from an analysis of Green's identity for $\Box + m^2$ (Appendix A).

Next, let $\mathcal{A} \in \mathcal{E}^{(1)}(\mathcal{M})$ and define

$$A_{(0)} \equiv \rho_{(0)}\mathcal{A}, \tag{12}$$

$$A_{(d)} \equiv \rho_{(d)}\mathcal{A}, \tag{13}$$

$$A_{(n)} \equiv \rho_{(n)}\mathcal{A}, \tag{14}$$

and

$$A_{(\delta)} \equiv \rho_{(\delta)}\mathcal{A}, \tag{15}$$

with $A_{(0)}$, $A_{(d)} \in \mathcal{E}^{(1)}(\Sigma)$ and $A_{(n)}$, $A_{(\delta)} \in \mathcal{E}^{(0)}(\Sigma)$. Specifying $A_{(0)}$, $A_{(d)}$, $A_{(n)}$, and $A_{(\delta)}$ is equivalent to specifying the Cauchy data for $\mathcal{A}$. To see this, let (n, e_i) be a surface normal reference frame for (Σ, γ), then (12) and (14) specify $\mathcal{A}_\mu(0,x)$ from which we obtain $\nabla_k \mathcal{A}_\mu(0,x)$. Given $\nabla_k \mathcal{A}_\mu(0,x)$, (13) specifies $\nabla_0 \mathcal{A}_k(0,x)$ and finally, (15) gives $\nabla_0 \mathcal{A}_0(0,x)$. Thus, the operators (11) collectively map a solution $\mathcal{A}$ to its Cauchy data $(\mathcal{A}_\mu, n^\alpha \nabla_\alpha \mathcal{A}_\mu)$. Alternatively, we view $A_{(0)}$, $A_{(d)}$, $A_{(\delta)}$ and $A_{(n)}$ as Cauchy data for (4).

The operators (11) which are continuous and linear, give rise to continuous transpose operators $\rho'_{(0)}$, $\rho'_{(d)} : \mathcal{E}'^{(1)}(\Sigma) \to \mathcal{E}'^{(1)}(\mathcal{M})$, and $\rho'_{(n)}$, $\rho'_{(\delta)} : \mathcal{E}'^{(0)}(\Sigma) \to \mathcal{E}'^{(1)}(\mathcal{M})$.[17] We construct the following continuous operators $E_m^{(1)}\,\rho'_{(d)}$, $E_m^{(1)}\,\rho'_{(n)}$, $E_m^{(1)}\,\rho'_{(\delta)}$, and $E_m^{(1)}\,\rho'_{(0)}$, where

$$E_m^{(1)}\rho'_{(0)}, E_m^{(1)}\rho'_{(d)} : \mathfrak{D}^{(1)}(\Sigma) \subset \mathcal{E}'^{(1)}(\Sigma) \to \mathfrak{D}'^{(1)}(\mathcal{M}), \tag{16}$$

and

$$E_m^{(1)}\rho'_{(\delta)}, E_m^{(1)}\rho'_{(n)} : \mathfrak{D}^{(0)}(\Sigma) \subset \mathcal{E}'^{(0)}(\Sigma) \to \mathfrak{D}'^{(1)}(\mathcal{M}). \tag{17}$$

These play a crucial role in the classical Cauchy problem as is now shown.

Theorem 1: *Let $A_{(0)}$, $A_{(d)} \in \mathfrak{D}^{(1)}(\Sigma)$, and $A_{(n)}$, $A_{(\delta)} \in \mathfrak{D}^{(0)}(\Sigma)$ specify Cauchy data on (Σ, γ), and let $m \in [0,\infty)$. Then,*

$$\mathcal{A}' = -E_m^{(1)}\rho'_{(d)}A_{(0)} - E_m^{(1)}\rho'_{(n)}A_{(\delta)} + E_m^{(1)}\rho'_{(\delta)}A_{(n)} + E_m^{(1)}\rho'_{(0)}A_{(d)} \tag{18}$$

is the unique smooth solution of $(\square + m^2)\mathcal{A}' = 0$ with these data. Moreover, $\mathcal{A}' \in \mathcal{E}^{(1)}(\mathcal{M})$ is continuously dependent on the data.

Proof: First, from (16) and (17) we know that (18) makes sense. Let $\mathcal{F} \in \mathfrak{D}^{(1)}(\mathcal{M})$ be any 1-form test function, and consider

$$\langle \mathcal{A}', \mathcal{F} \rangle = \langle A_{(0)}, \rho_{(d)}E_m^{(1)}\mathcal{F} \rangle_\Sigma + \langle A_{(\delta)}, \rho_{(n)}E_m^{(1)}\mathcal{F} \rangle_\Sigma - \langle A_{(n)}, \rho_{(\delta)}E_m^{(1)}\mathcal{F} \rangle_\Sigma - \langle A_{(d)}, \rho_{(0)}E_m^{(1)}\mathcal{F} \rangle_\Sigma .$$

From Theorem 5 we know that if $\mathcal{A}$ is a smooth solution with data (12)–(15), then $\langle \mathcal{A}', \mathcal{F} \rangle = \langle \mathcal{A}, \mathcal{F} \rangle_{\mathcal{M}}$ which implies that $\mathcal{A}' = \mathcal{A}$ in a distributional sense. Thus $\mathcal{A}'$ is identified with the unique smooth solution $\mathcal{A} \in \mathcal{E}^{(1)}(\mathcal{M})$.

It remains to show that $\mathcal{A}'$ is continuously dependent on the data. For this result, we assume that Σ is compact. From the first part of the proof we have $E_m^{(1)}\ \rho'_{(0)}$, $E_m^{(1)}\ \rho'_{(d)} : \mathfrak{D}^{(1)}(\Sigma) \to \mathcal{E}^{(1)}(\mathcal{M})$ and $E_m^{(1)}\rho'_{(\delta)}, E_m^{(1)}\rho'_{(n)} : \mathfrak{D}^{(0)}(\Sigma) \to \mathcal{E}^{(1)}(\Sigma)$. It suffices to show that these restrictions are continuous. The same analysis applies to both sets of operators so we need only consider the former. Recall that $E_m^{(1)}\ \rho'_{(0)}$, $E_m^{(1)}\ \rho'_{(d)} : \mathcal{E}'^{(1)}(\Sigma) \to \mathfrak{D}'^{(1)}(\mathcal{M})$ are continuous with respect to the weak topologies of dual spaces. Thus, the graphs of the restrictions $E_m^{(1)}\ \rho'_{(0)}$, $E_m^{(1)}\ \rho'_{(d)}\ \mathfrak{D}^{(1)}(\Sigma) \to \mathcal{E}^{(1)}(\mathcal{M})$ are closed with respect to these weak topologies and it follows that they are closed with respect to the topologies of $\mathfrak{D}^{(1)}(\Sigma)$ and $\mathcal{E}^{(1)}(\mathcal{M})$ as well. Finally, since $\mathfrak{D}^{(1)}(\Sigma)$ and $\mathcal{E}^{(1)}(\mathcal{M})$ are Frechet spaces (assuming Σ is compact) it follows from the closed graph theorem that the restrictions of $E_m^{(1)}\ \rho'_{(0)}$ and $E_m^{(1)}\rho'_{(d)}$ are continuous. As noted, a similar analysis applies to the restrictions of $E_m^{(1)}\rho'_{(\delta)}$ and $E_m^{(1)}\rho'_{(n)}$ and therefore $\mathcal{A}' \in \mathcal{E}^{(1)}(\mathcal{M})$ is continuously dependent on the data. This presumably holds for noncompact Σ as well. ■

Thus, we obtain a global solution to (4) as a mapping of Cauchy data. Notice that Theorem 1 applies for $m = 0$ which we use in our study of Maxwell's equations (Appendix B).

Corollary 1: On $\mathfrak{D}^{(1)}(\Sigma)$ we have

$$\rho_{(0)}E_m^{(1)}\rho'_{(d)} = -I, \quad \rho_{(d)}E_m^{(1)}\rho'_{(d)} = 0, \quad \rho_{(\delta)}E_m^{(1)}\rho'_{(d)} = 0, \quad \rho_{(n)}E_m^{(1)}\rho'_{(d)} = 0, \tag{19}$$

$$\rho_{(0)}E_m^{(1)}\rho'_{(0)} = 0, \quad \rho_{(d)}E_m^{(1)}\rho'_{(0)} = I, \quad \rho_{(\delta)}E_m^{(1)}\rho'_{(0)} = 0, \quad \rho_{(n)}E_m^{(1)}\rho'_{(0)} = 0, \tag{20}$$

and on $\mathfrak{D}^{(0)}(\Sigma)$ we have

$$\rho_{(0)}E_m^{(1)}\rho'_{(n)} = 0, \quad \rho_{(d)}E_m^{(1)}\rho'_{(n)} = 0, \quad \rho_{(\delta)}E_m^{(1)}\rho'_{(n)} = -I, \quad \rho_{(n)}E_m^{(1)}\rho'_{(n)} = 0, \tag{21}$$

$$\rho_{(0)}E_m^{(1)}\rho'_{(\delta)} = 0, \quad \rho_{(d)}E_m^{(1)}\rho'_{(\delta)} = 0, \quad \rho_{(\delta)}E_m^{(1)}\rho'_{(\delta)} = 0, \quad \rho_{(n)}E_m^{(1)}\rho'_{(\delta)} = I. \tag{22}$$

Proof: These identities follow from (18). For example, the first identity in (19) is obtained by applying $\rho_{(0)}$ to (18) with $A_{(\delta)}$, $A_{(n)}$, $A_{(d)} = 0$. The remaining indentities are obtained in a similar fashion. □

We are finally ready for the main result of this section. Notice that Theorem (1) gives a global solution to (4). We apply it to Proca's equations and obtain a global solution to the initial value problem by restricting the data.

Theorem 2: *Let $A_{(0)}$, $A_{(d)}\in\mathfrak{D}^{(1)}(\Sigma)$, and $A_{(n)}$, $A_{(\delta)}\in\mathfrak{D}^{(0)}(\Sigma)$ specify Cauchy data on Σ and let $m\in[0,\infty)$. Set*

$$A_{(\delta)}=0, \tag{23}$$

and

$$\delta^{(3)}A_{(d)}=m^2A_{(n)}. \tag{24}$$

Then,

$$\mathcal{A}=-E_m^{(1)}\rho'_{(d)}A_{(0)}+E_m^{(1)}\rho'_{(\delta)}A_{(n)}+E_m^{(1)}\rho'_{(0)}A_{(d)} \tag{25}$$

is the unique smooth solution of $(\delta^{(4)}d^{(4)}+m^2)\mathcal{A}=0$ with these data. Moreover, $\mathcal{A}\in\mathcal{E}^{(1)}(\mathcal{M})$ is continuously dependent on the data. When $m>0$ (Proca's equations) we satisfy (24) with arbitrary $A_{(d)}$ and $A_{(n)}=[\delta^{(3)}A_{(d)}/m^2]$, thus

$$\mathcal{A}=-E_m^{(1)}\rho'_{(d)}A_{(0)}+E_m^{(1)}\left(\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right)\rho'_{(0)}A_{(d)}\quad(m>0), \tag{26}$$

where we have used $\rho'_{(\delta)}\delta^{(3)}=d^{(4)}\delta^{(4)}\rho'_{(0)}$ on $\mathfrak{D}^{(1)}(\Sigma)$. When $m=0$ we satisfy (24) with arbitrary $A_{(n)}$, and with $A_{(d)}$ satisfying $\delta^{(3)}A_{(d)}=0$ (see Maxwell's equations in Appendix B).

Proof: We show that $\mathcal{A}$ is a smooth solution of $(\Box+m^2)\mathcal{A}=0$ with $\delta^{(4)}\mathcal{A}=0$. The former follows from Theorem 1. For the latter, it suffices to show that $\langle\delta^{(4)}\mathcal{A},f\rangle_M=0$, for all $f\in\mathfrak{D}^{(0)}(\mathcal{M})$. Consider,

$$\begin{aligned}\langle\delta^{(4)}\mathcal{A},f\rangle_{\mathcal{M}}&=\langle\mathcal{A},d^{(4)}f\rangle_{\mathcal{M}},\\&=\langle A_{(0)},\rho_{(d)}E_m^{(1)}d^{(4)}f\rangle_\Sigma-\langle A_{(n)},\rho_{(\delta)}E_m^{(1)}d^{(4)}f\rangle_\Sigma,\\&\quad-\langle A_{(d)},\rho_{(0)}E_m^{(1)}d^{(4)}f\rangle_\Sigma.\end{aligned} \tag{27}$$

Now, since $E_m^{(1)}d^{(4)}=d^{(4)}E_m^{(0)}$ on $\mathfrak{D}^{(0)}(\mathcal{M})$ we have $\rho_{(d)}E_m^{(1)}d^{(4)}f=0$ and the first term on the right-hand side of (27) is zero. Consider the second term on the right-hand side of (27),

$$\begin{aligned}\rho_{(\delta)}E_m^{(1)}d^{(4)}f&=i^*\delta^{(4)}E_m^{(1)}d^{(4)}f,\\&=i^*\delta^{(4)}d^{(4)}E_m^{(0)}f,\\&=i^*(\Box-d^{(4)}\delta^{(4)})E_m^{(0)}f,\\&=-\rho_{(0)}m^2E_m^{(0)}f,\end{aligned} \tag{28}$$

where we have used $(\Box+m^2)E_m^{(0)}f=0$, and $\delta^{(4)}E_m^{(0)}f=0$ ($E_m^{(0)}f$ is a function). Substituting (28) into (27) and making use of $\rho_{(0)}E_m^{(1)}d^{(4)}f=d^{(3)}\rho_{(0)}E_m^{(0)}f$ we have

$$\begin{aligned}\langle\delta^{(4)}\mathcal{A},f\rangle_{\mathcal{M}}&=\langle A_{(n)},\rho_{(0)}m^2E_m^{(0)}f\rangle_\Sigma-\langle A_{(d)},d^{(3)}\rho_{(0)}E_m^{(0)}f\rangle_\Sigma\\&=\langle m^2A_{(n)}-\delta^{(3)}A_{(d)},\rho_{(0)}E_m^{(0)}f\rangle_\Sigma,\\&=0,\end{aligned}$$

where, in the last step we have used (24). Thus $\delta^{(4)}\mathcal{A}=0$ which is compatible with (23).

Lastly, from Theorem 1 we know that $\mathcal{A}$ is unique. We also have that $\mathcal{A}$ is continuously dependent on the data when Σ is compact. ■

From Theorem 2 we have the following additional result which is need for the quantum problem.

Corollary 2: The operator identity

$$\left(\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right)E_m^{(1)}=-E_m^{(1)}\rho_{(d)}'\rho_{(0)}\left(\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right)E_m^{(1)}+E_m^{(1)}\left(\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right)\rho_{(0)}'\rho_{(d)}E_m^{(1)}, \quad (29)$$

holds on $\mathfrak{D}^{(1)}(\mathcal{M})$ *for* $m>0$.

Proof: Let $\mathcal{F}\in\mathfrak{D}^{(1)}(\mathcal{M})$. It is easy to check that $(d^{(4)}\delta^{(4)}/m^2+I)E_m^{(1)}\mathcal{F}$ is a smooth solution to Proca's equations. Thus, (29) follows from Theorem 2 with $\mathcal{A}=[(d^{(4)}\delta^{(4)}/m^2)+I]E_m^{(1)}\mathcal{F}$ and the fact that $\rho_{(d)}[d^{(4)}\delta^{(4)}/m^2]E_m^{(1)}\mathcal{F}=0$. ■

This last result completes the prerequisite classical work. We proceed to the quantum problem.

IV. THE QUANTUM PROBLEM

Our approach to field quantization closely follows the work of Dimock.[3] We start with a representation $(\phi,\Pi,\mathcal{H})$ of the CCRs on an arbitrary Cauchy surface Σ. Let $\mathfrak{h}$ denote the completion of smooth complex-valued 1-forms on Σ with respect to the norm $\|\cdot\|_{\mathfrak{h}}^2=\langle\cdot,\cdot\rangle_{\mathfrak{h}}$, where

$$\langle F,G\rangle_{\mathfrak{h}}=\int(\overline{F},G)_x\,d\tau, \quad (30)$$

and

$$(F,G)_x=F_n(x)G^n(x), \quad (n=1,2,3) \quad (31)$$

with $d\tau=\sqrt{\gamma}dx^1\wedge dx^2\wedge dx^3$, where $x=(x^1,x^2,x^2)$ are local coordinates for (Σ,γ). Next, construct the Bose–Fock space $\mathcal{H}$ over $\mathfrak{h}$,

$$\mathcal{H}=\mathbb{C}\oplus\left(\bigoplus_{n=1}^{\infty}\mathfrak{h}^{(n)}\right), \quad (32)$$

where $\mathfrak{h}^{(n)}=\otimes_s^n\mathfrak{h}$, and the subscript s denotes the symmetric tensor product. Let $\mathbf{a}(\cdot)$, $\mathbf{a}^*(\cdot)$ denote the usual creation and annihilation operators defined on finite particle vectors in $\mathcal{H}$ with $[\mathbf{a}(F),\mathbf{a}^*(G)]=\langle F,G\rangle_{\mathfrak{h}}$.[18] Let

$$\phi(F)\equiv\frac{1}{\sqrt{2}}[\mathbf{a}(F)+\mathbf{a}^*(F)], \quad (33)$$

and

$$\Pi(F)\equiv\frac{i}{\sqrt{2}}[\mathbf{a}^*(F)-\mathbf{a}(F)], \quad (34)$$

and then take the closure of (33) and (34) (keeping the same notation) to obtain self-adjoint ϕ and Π on $\mathcal{H}$ with $[\phi(F),\Pi(G)]=i\langle F,G\rangle_{\mathfrak{h}}$ for $F,G\in\mathfrak{D}^{(1)}(\Sigma)$.[18] This gives the representation $(\phi,\Pi,\mathcal{H})$.

Now, given a representation $(\phi,\Pi,\mathcal{H})$ on Σ, not necessarily as above, we define a space–time field operator

$$\mathcal{A} \equiv -E_m^{(1)} \rho'_{(d)} \phi + E_m^{(1)} \left(\frac{d^{(4)} \delta^{(4)}}{m^2} + I \right) \rho'_{(0)} \Pi , \tag{35}$$

in accordance with the classical initial value problem (Theorem 2 with $m > 0$). This holds in a distributional sense, therefore,

$$\mathcal{A}(\mathcal{F}) = \phi(\rho_{(d)} E_m^{(1)} \mathcal{F}) - \Pi \left(\rho_{(0)} \left[\frac{d^{(4)} \delta^{(4)}}{m^2} + I \right] E_m^{(1)} \mathcal{F} \right)$$

for $\mathcal{F} \in \mathfrak{D}^{(1)}(\mathcal{M})$. This makes sense because $E_m^{(1)} \mathcal{F} \in \mathcal{E}^{(1)}(\mathcal{M})$ and $J^{\pm}(\mathrm{supp}(\mathcal{F})) \cap \Sigma$ is compact.[3]

Theorem 3: *Let* $(\phi, \Pi, \mathcal{H})$ *be a representation of the CCRs over* $\mathfrak{D}^{(1)}(\Sigma)$, *and let* $\mathcal{A}$ *be the field operator (35) with test functions* $\mathcal{F} \in \mathfrak{D}^{(1)}(\mathcal{M})$. *Then* $\mathcal{A}$ *satisfies the Proca's equations in a distributional sense,*

$$(\delta^{(4)} d^{(4)} + m^2) \mathcal{A} = 0, \tag{36}$$

and

$$[\mathcal{A}(\mathcal{F}), \mathcal{A}(\mathcal{F}')] = -i \left\langle \mathcal{F}, \left(\frac{d^{(4)} \delta^{(4)}}{m^2} + I \right) E_m^{(1)} \mathcal{F}' \right\rangle_{\mathcal{M}} . \tag{37}$$

Proof: First we verify (36). Consider,

$$\begin{aligned}
(\delta^{(4)} d^{(4)} + m^2) \mathcal{A}(\mathcal{F}) &= \mathcal{A}((\delta^{(4)} d^{(4)} + m^2) \mathcal{F}), \\
&= \phi(\rho_{(d)} E_m^{(1)} (\delta^{(4)} d^{(4)} + m^2) \mathcal{F}) \\
&\quad - \Pi \left(\rho_{(0)} \left[\frac{d^{(4)} \delta^{(4)}}{m^2} + I \right] (\delta^{(4)} d^{(4)} + m^2) E_m^{(1)} \mathcal{F} \right), \\
&= -\phi(\rho_{(d)} d^{(4)} \delta^{(4)} E_m^{(1)} \mathcal{F}) - \Pi(\rho_{(0)} [\Box + m^2] E_m^{(1)} \mathcal{F}), \\
&= 0,
\end{aligned}$$

where we have used $\rho_{(d)} d^{(4)} = 0$ and $(\Box + m^2) E_m^{(1)} = 0$ on $\mathcal{E}^{(1)}(\mathcal{M})$.

Next, we verify (37). Consider,

$$\begin{aligned}
[\mathcal{A}(\mathcal{F}), \mathcal{A}(\mathcal{F}')] &= -i \left\langle \rho_{(d)} E_m^{(1)} \mathcal{F}, \rho_{(0)} \left(\frac{d^{(4)} \delta^{(4)}}{m^2} + I \right) E_m^{(1)} \mathcal{F}' \right\rangle_{\Sigma} \\
&\quad + i \left\langle \rho_{(0)} \left(\frac{d^{(4)} \delta^{(4)}}{m^2} + I \right) E_m^{(1)} \mathcal{F}, \rho_{(d)} E_m^{(1)} \mathcal{F}' \right\rangle_{\Sigma} , \\
&= i \left\langle \mathcal{F}, E_m^{(1)} \rho'_{(d)} \rho_{(0)} \left(\frac{d^{(4)} \delta^{(4)}}{m^2} + I \right) E_m^{(1)} \mathcal{F}' \right\rangle_{\mathcal{M}} \\
&\quad - i \left\langle \mathcal{F}, E_m^{(1)} \left(\frac{d^{(4)} \delta^{(4)}}{m^2} + I \right) \rho'_{(0)} \rho_{(d)} E_m^{(1)} \mathcal{F}' \right\rangle_{\mathcal{M}} , \\
&= -i \left\langle \mathcal{F}, \left(\frac{d^{(4)} \delta^{(4)}}{m^2} + I \right) E_m^{(1)} \mathcal{F}' \right\rangle_{\mathcal{M}} ,
\end{aligned}$$

where we have applied Corollary 2. ■

Notice from (7) that

$$\operatorname{supp}\left(\left[\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right]E_m^{(1)}\mathcal{F}'\right)\subset J^+(\operatorname{supp}(\mathcal{F}'))\cup J^-(\operatorname{supp}(\mathcal{F}')),$$

and therefore (37) implies causality. Also, (37) reduces to the usual relation on Minkowski space.[19]

We now show that a representation $(\phi,\Pi,\mathcal{H})$ on Σ induces a representation on any other Cauchy surface.

Corollary 3: Let $(\phi,\Pi,\mathcal{H})$ *be a representation of the CCRs over* $\mathfrak{D}^{(1)}(\Sigma)$, *and let* $\mathcal{A}$ *be the field operator (35). Let* $\hat{\Sigma}$ *be another Cauchy surface in* $\mathcal{M}$, *and let* $\hat{\phi}$ *and* $\hat{\Pi}$ *be data of* $\mathcal{A}$ *on* $\hat{\Sigma}$, *i.e.,*

$$\hat{\phi}\equiv\hat{\rho}_{(0)}\mathcal{A},$$

and,

$$\hat{\Pi}\equiv\hat{\rho}_{(d)}\mathcal{A}.$$

Then $(\hat{\phi},\hat{\Pi},\mathcal{H})$ *is a representation of the CCRs over* $\mathfrak{D}^{(1)}(\hat{\Sigma})$. *Furthermore, let*

$$\hat{\mathcal{A}}\equiv-E_m^{(1)}\hat{\rho}'_{(d)}\hat{\phi}+E_m^{(1)}\left(\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right)\hat{\rho}'_{(0)}\hat{\Pi}, \tag{38}$$

then

$$\hat{\mathcal{A}}(\mathcal{F})=\mathcal{A}(\mathcal{F}). \tag{39}$$

Proof: We first show that $(\hat{\phi},\hat{\Pi},\mathcal{H})$ is a representation. Let $F\in\mathfrak{D}^{(1)}(\hat{\Sigma})$, then

$$\begin{aligned}\hat{\phi}(F)&=\mathcal{A}(\hat{\rho}'_{(0)}F),\\&=\phi(\rho_{(d)}E_m^{(1)}\hat{\rho}'_{(0)}F)-\Pi\left(\rho_{(0)}\left[\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right]E_m^{(1)}\hat{\rho}'_{(0)}F\right),\end{aligned} \tag{40}$$

and

$$\begin{aligned}\hat{\Pi}(F)&=\mathcal{A}(\hat{\rho}'_{(d)}F),\\&=\phi(\rho_{(d)}E_m^{(1)}\hat{\rho}'_{(d)}F)-\Pi\left(\rho_{(0)}\left[\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right]E_m^{(1)}\hat{\rho}'_{(d)}F\right).\end{aligned} \tag{41}$$

These make sense because $E_m^{(1)}\hat{\rho}'_{(0)},E_m^{(1)}\hat{\rho}'_{(d)}:\mathfrak{D}^{(1)}(\hat{\Sigma})\rightarrow\ \mathcal{E}^{(1)}(\mathcal{M})$ (Theorem 1) and $J^{\pm}(\operatorname{supp}(F))\cap\Sigma$ is compact.[3] Consider,

$$[\hat{\phi}(F),\hat{\Pi}(F')]=-\left[\phi(\rho_{(d)}E_m^{(1)}\hat{\rho}'_{(0)}F),\Pi\left(\rho_{(0)}\left[\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right]E_m^{(1)}\hat{\rho}'_{(d)}F'\right)\right]$$
$$+\left[\phi(\rho_{(d)}E_m^{(1)}\hat{\rho}'_{(d)}F'),\Pi\left(\rho_{(0)}\left[\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right]E_m^{(1)}\hat{\rho}'_{(0)}F\right)\right]$$
$$=-i\langle\rho_{(d)}\mathcal{F},\rho_{(0)}\mathcal{F}'\rangle_\Sigma+i\langle\rho_{(0)}\mathcal{F},\rho_{(d)}\mathcal{F}'\rangle_\Sigma$$
$$\equiv i\Omega_\Sigma(\mathcal{F},\mathcal{F}') \tag{42}$$

where $\mathcal{F}\equiv\{[d^{(4)}\delta^{(4)}/m^2]+I\}E_m^{(1)}\hat{\rho}'_{(0)}F$ and $\mathcal{F}'\equiv\{[d^{(4)}\delta^{(4)}/m^2]+I\}E_m^{(1)}\hat{\rho}'_{(d)}F'$ are smooth solutions of Proca's equations. It follows from Green's identity that $\Omega_\Sigma(\cdot,\cdot)$ is independent of the Cauchy surface for such solutions (see the vector potential formulation of Maxwell's equations).[2,4] Thus, (42) is independent of the Cauchy surface and we have

$$[\hat{\phi}(F),\hat{\Pi}(F')]=i\Omega_{\hat{\Sigma}}(\mathcal{F},\mathcal{F}')$$
$$=-i\langle\hat{\rho}_{(d)}\mathcal{F},\hat{\rho}_{(0)}\mathcal{F}'\rangle_{\hat{\Sigma}} \tag{43}$$
$$+i\langle\hat{\rho}_{(0)}\mathcal{F},\hat{\rho}_{(d)}\mathcal{F}'\rangle_{\hat{\Sigma}}$$
$$=i\langle F,F'\rangle_{\hat{\Sigma}}, \tag{44}$$

where in the last step we have applied Corollary (1) to $\hat{\Sigma}$. Thus $(\hat{\phi},\hat{\Pi},\mathcal{H})$ is a representation.

We now verify (39). Substitute (40) and (41) into (38) and obtain

$$\hat{\mathcal{A}}(\mathcal{F})\equiv\phi(\rho_{(d)}\hat{\mathcal{F}})-\Pi\left(\rho_{(0)}\left(\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right)\hat{\mathcal{F}}\right),$$
$$=\phi(\rho_{(d)}\tilde{\mathcal{F}})-\Pi\left(\rho_{(0)}\left(\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right)\tilde{\mathcal{F}}\right)$$
$$=\phi(\rho_{(d)}E_m^{(1)}\mathcal{F})-\Pi\left(\rho_{(0)}\left(\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right)E_m^{(1)}\mathcal{F}\right)$$
$$=\mathcal{A}(\mathcal{F}),$$

where

$$\hat{\mathcal{F}}=\left[E_m^{(1)}\hat{\rho}'_{(0)}\hat{\rho}_{(d)}E_m^{(1)}-\hat{\rho}'_{(d)}\hat{\rho}_{(0)}\left(\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right)E_m^{(1)}\right]\mathcal{F}$$

and

$$\tilde{\mathcal{F}}=\left[\left(\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right)E_m^{(1)}-\frac{d^{(4)}\delta^{(4)}}{m^2}E_m^{(1)}\hat{\rho}'_{(0)}\hat{\rho}_{(d)}E_m^{(1)}\right]\mathcal{F}.$$

In the second step we have used Corollary 2 and in subsequent steps we have used $\rho_{(d)}d^{(4)}=0$ on $\mathcal{E}^{(1)}(\mathcal{M})$ and $\delta^{(4)}\mathcal{G}=0$ for any smooth solution $\mathcal{G}$ of Proca's equations. ■

At this point we have a field operator $\mathcal{A}$ defined in terms of a representation $(\phi,\Pi,\mathcal{H})$ on an arbitrary Cauchy surface Σ. Unlike Minkowski space–time, there is, in general, no preferred representation (i.e., no natural positive and negative frequency decomposition).[2] Thus, we must consider a class of representations, and show that the theory is independent of the representation, and independent of the Cauchy surface. To this end, we pass to the Weyl form of the CCRs, construct an algebra of observables $\mathfrak{A}$, and exploit known results on the equivalence of such algebras.

Given a representation $(\phi,\Pi,\mathcal{H})$ let

$$W(F,F')=\exp(i(\phi(F)-\Pi(F'))).$$

Then $W(\cdot,\cdot)$is a map from $\mathfrak{D}^{(1)}(\Sigma)\times\mathfrak{D}^{(1)}(\Sigma)$ to unitary operators on $\mathcal{H}$ satisfying

$$W(F,F')W(G,G')=W(F+G,F'+G')\exp(-\,i/2\,\sigma_\Sigma((F,F'),(G,G'))), \tag{45}$$

where

$$\sigma_\Sigma((F,F'),(G,G'))\equiv\langle F,G'\rangle_\Sigma-\langle F',G\rangle_\Sigma\,, \tag{46}$$

is symplectic on $\mathfrak{D}^{(1)}(\Sigma)\times\mathfrak{D}^{(1)}(\Sigma)$. Also, $t\to W(tF,tF')$ is strongly continuous. Thus we have the Weyl system $(W,\mathcal{H})$ with the Weyl form of the CCRs (45). As an aside, note that $\Omega_\Sigma(\mathcal{F},\mathcal{F}')=\sigma_\Sigma((\rho_{(0)}\mathcal{F},\rho_{(d)}\mathcal{F}),(\rho_{(0)}\mathcal{F}',\rho_{(d)}\mathcal{F}'))$ for smooth solutions $\mathcal{F}$ and $\mathcal{F}'$ of Proca's equations.

Alternatively, given a representation $(W,\mathcal{H})$ we recover self-adjoint ϕ and Π via Stone's Theorem, i.e.,

$$e^{i\phi(F)t}=W(tF,0), \tag{47}$$

and

$$e^{-i\mathbf{\Pi}(F)t}=W(0,tF). \tag{48}$$

We also obtain a self-adjoint field operator $\mathcal{A}(\mathcal{F})$ via Stone's Theorem,

$$e^{i\mathcal{A}(\mathcal{F})t}=W\left(t\rho_{(d)}E_m^{(1)}\mathcal{F},t\rho_{(0)}\left[\frac{d^{(4)}\delta^{(4)}}{m^2}+I\right]E_m^{(1)}\mathcal{F}\right), \tag{49}$$

where $\mathcal{F}\in\mathfrak{D}^{(1)}(\mathcal{M})$.

Next, we define an algebra of observables $\mathfrak{A}$. Given a Weyl system $(W,\mathcal{H})$ take the set of all finite sums of the form

$$\sum_\alpha c_\alpha W(F_\alpha,F'_\alpha),\quad c_\alpha\in\mathbb{C},$$

where $F_\alpha,F'_\alpha\in\mathfrak{D}^{(1)}(\Sigma)$ and define $\mathfrak{A}$ to be the norm closure of this set in the Banach space of all bounded operators on $\mathcal{H}$.

We now exploit known results on the equivalence of such algebras.

Theorem 4: *Let $(W,\sigma_\Sigma,\mathfrak{A},\mathcal{H})$ and $(\tilde{W},\sigma_{\hat{\Sigma}},\tilde{\mathfrak{A}},\tilde{\mathcal{H}})$ be representations on Cauchy surfaces Σ and $\hat{\Sigma}$, respectively. There is a unique *-isomorphism $\alpha:\mathfrak{A}\to\tilde{\mathfrak{A}}$ with $\alpha:e^{i\mathcal{A}(\mathcal{F})}\to e^{i\tilde{\mathcal{A}}(\mathcal{F})}$.*

Proof: First, consider the case $\Sigma=\hat{\Sigma}$. Given $(W,\sigma_\Sigma,\mathfrak{A},\mathcal{H})$ and $(\tilde{W},\sigma_\Sigma,\tilde{\mathfrak{A}},\tilde{\mathcal{H}})$ over $\mathfrak{D}^{(1)}(\Sigma)\times\mathfrak{D}^{(1)}(\Sigma)$ there is a unique $*-$isomorphism $\alpha:\mathfrak{A}\to\tilde{\mathfrak{A}}$ such that $\alpha(W(F,F'))=\tilde{W}(F,F')$ (see Theorem 5.2.8 in 18). It follows from (49) that $\alpha:e^{i\mathcal{A}(\mathcal{F})}\to e^{i\tilde{\mathcal{A}}(\mathcal{F})}$. Thus $\mathfrak{A}$ is independent of the representation on Σ in this sense.

Next, let Σ and $\hat{\Sigma}$ be different, and let $(\phi,\Pi,\mathcal{H})$ and $(\hat{\phi},\hat{\Pi},\mathcal{H})$ be respective representations as defined in Corollary 3. It follows that $e^{i\mathcal{A}(\mathcal{F})}=e^{i\hat{\mathcal{A}}(\mathcal{F})}$ and therefore $\mathfrak{A}=\hat{\mathfrak{A}}$. Moreover, from the first part of the proof we have $\alpha:\hat{\mathfrak{A}}\to\tilde{\mathfrak{A}}$ and therefore $\alpha:\mathfrak{A}\to\tilde{\mathfrak{A}}$ with $\alpha:e^{i\mathcal{A}(\mathcal{F})}\to e^{i\tilde{\mathcal{A}}(\mathcal{F})}$. Thus $\mathfrak{A}$ is independent of the Cauchy surface in this sense. ■

From this last result, we see that the Fock representation $(\phi,\Pi,\mathcal{H})$ defined by (33) and (34) gives rise to an algebra of observables $\mathfrak{A}$ that is unique up to $*-$isomorphism. This concludes the quantum problem.

V. CONCLUSION

We have obtained classical and quantum results for the propagation of massive vector fields on a globally hyperbolic Lorentzian manifold. Our classical results include solutions of the initial value problem for Proca's equations (3), the vector Klein Gordon equation (4), and Maxwell's equations (Appendix B). The form of these solutions is apparently new and useful for field quantization. Our quantum results include a causal field operator constructed from a representation of the CCRs on an arbitrary Cauchy surface. The algebra of observables generated by this field operator is independent of the representation and independent of the Cauchy surface.

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APPENDIX A: GREEN'S IDENTITY

In this section we develop Green's identity for $\Box+m^2$. Start with Stoke's Theorem,

$$\int_{\mathcal{O}} d^{(4)}\mathcal{G}=\int_{\partial\mathcal{O}} i^*\mathcal{G}, \tag{A1}$$

where $\mathcal{O}\subset\mathcal{M}$, $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$, $i:\partial\mathcal{O}\to\mathcal{O}$ is the natural inclusion, i^* is the pullback, and $\mathcal{G}\in\mathfrak{D}^{(3)}(\mathcal{M})$.[20–22] Let $\mathcal{G}^{(1)}=\delta^{(4)}\mathcal{F}\wedge*^{(4)}\mathcal{A}-\delta^{(4)}\mathcal{A}\wedge*^{(4)}\mathcal{F}$ and $\mathcal{G}^{(2)}=\mathcal{A}\wedge*^{(4)}d^{(4)}\mathcal{F}-\mathcal{F}\wedge*^{(4)}d^{(4)}\mathcal{A}$, with $\mathcal{A}$, $\mathcal{F}\in\mathfrak{D}^{(1)}(\mathcal{M})$. We apply (A1) with $\mathcal{G}=\mathcal{G}^{(1)}+\mathcal{G}^{(2)}$ and obtain

$$\begin{aligned}\int_{\mathcal{O}}\mathcal{A}\wedge*^{(4)}\Box\mathcal{F}-\mathcal{F}\wedge*^{(4)}\Box\mathcal{A}=&-\int_{\partial\mathcal{O}} i^*(\mathcal{A}\wedge*^{(4)}d^{(4)}\mathcal{F}+\delta^{(4)}\mathcal{A}\wedge*^{(4)}\mathcal{F})\\ &+\int_{\partial\mathcal{O}} i^*(\delta^{(4)}\mathcal{F}\wedge*^{(4)}\mathcal{A}+\mathcal{F}\wedge*^{(4)}d^{(4)}\mathcal{A}).\end{aligned} \tag{A2}$$

Finally, add $\mathcal{A}\wedge*^{(4)}m^2\mathcal{F}-\mathcal{F}\wedge*^{(4)}m^2\mathcal{A}$ to the left-hand side of (A2) and obtain Green's identity for $(\Box+m^2)$,

$$\begin{aligned}\int_{\mathcal{O}}\mathcal{A}\wedge*^{(4)}(\Box+m^2)\mathcal{F}-\mathcal{F}\wedge*^{(4)}(\Box+m^2)\mathcal{A}=&-\int_{\partial\mathcal{O}} i^*(\mathcal{A}\wedge*^{(4)}d^{(4)}\mathcal{F}+\delta^{(4)}\mathcal{A}\wedge*^{(4)}\mathcal{F})\\ &+\int_{\partial\mathcal{O}} i^*(\delta^{(4)}\mathcal{F}\wedge*^{(4)}\mathcal{A}+\mathcal{F}\wedge*^{(4)}d^{(4)}\mathcal{A}).\end{aligned} \tag{A3}$$

Next, apply (A3) to the regions $\mathcal{O}=\Sigma^{\pm}\equiv J^{\pm}(\Sigma)\setminus\Sigma$, with $\partial\mathcal{O}=\Sigma$ and obtain

$$\begin{aligned}\int_{\Sigma^{\pm}}\mathcal{A}\wedge*^{(4)}(\Box+m^2)\mathcal{F}-\mathcal{F}\wedge*^{(4)}(\Box+m^2)\mathcal{A}=&\mp\{\langle\rho_{(0)}\mathcal{A},\rho_{(d)}\mathcal{F}\rangle+\langle\rho_{(\delta)}\mathcal{A},\rho_{(n)}\mathcal{F}\rangle_{\Sigma}\\ &-\langle\rho_{(n)}\mathcal{A},\rho_{(\delta)}\mathcal{F}\rangle_{\Sigma}-\langle\rho_{(d)}\mathcal{A},\rho_{(0)}\mathcal{F}\rangle_{\Sigma},\end{aligned} \tag{A4}$$

where $\rho_{(0)}$, $\rho_{(n)}$, $\rho_{(d)}$, and $\rho_{(\delta)}$ are as defined in (11), and, for example, $\int_{\Sigma} i^*\delta^{(4)}\mathcal{F}\wedge*^{(3)}(-*^{(3)}i^**^{(4)})\mathcal{A}=\langle\rho_{(\delta)}\mathcal{F},\rho_{(n)}\mathcal{A}\rangle_{\Sigma}$ (the standard orientation is used for both regions $\Sigma^{\pm}$).

The action of $\rho_{(0)}$ and $\rho_{(\delta)}$ is obvious, however, $\rho_{(n)}$ and $\rho_{(d)}$ are more subtle. Specifically, $\rho_{(n)}$ and $\rho_{(d)}$ are the forward normal, and pullback of the forward normal derivative operators, respectively. To see this, let $x^{\mu}=(t,x^{i})$, then $e_{\mu}=\partial_{x^{\mu}}$ represents the standard basis for the tangent space of $(\mathcal{M},g)$. Let (Σ,γ) be a Cauchy surface with forward normal n. Following the presentation of Misner, Thorne and Wheeler we have $n=n^{\mu}e_{\mu}$,

$$n=N^{-1}(\partial_{t}-N^{i}\partial_{x^{i}})$$

where $N=(-g^{00})^{-(1/2)}$ is the lapse function, and $N^{i}=g^{ik}g_{ok}$ are the components of the shift vector.[14] Notice that $n=-N\ dt$ in covariant form, and recall that $\sqrt{|g|}=N\sqrt{\gamma}$. Now, let $\mathcal{A}=\mathcal{A}_{\mu}dx^{\mu}$ and consider,

$$\begin{aligned}
\rho_{(n)}\mathcal{A}&=-*^{(3)}i**^{(4)}\mathcal{A}_{\mu}dx^{\mu},\\
&=-*^{(3)}i*\frac{1}{3!}\sqrt{|g|}\mathcal{A}^{\mu}\epsilon_{\mu\alpha\beta\eta}dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\eta},\\
&=-*^{(3)}\frac{1}{3!}N\sqrt{\gamma}\mathcal{A}^{0}\epsilon_{0ijk}dx^{i}\wedge dx^{j}\wedge dx^{k},\\
&=-\frac{1}{3!}N\gamma\mathcal{A}^{0}\epsilon_{0lmn}\epsilon_{ijk}\gamma^{il}\gamma^{jm}\gamma^{kn},\\
&=-N\mathcal{A}^{0},\\
&=\mathcal{A}(n),
\end{aligned}$$

where $\epsilon_{\mu\alpha\beta\eta}$ is the Levi Cevita symbol.[14]

For the analysis of $\rho_{(d)}$ it is convenient to work with a basis $\hat{e}_{\mu}$ where $\hat{e}_{0}=n$ and $\hat{e}_{i}=\partial_{x^{i}}$. The dual for this basis is $\hat{\omega}^{\mu}$ where $\hat{\omega}^{0}=Ndt$ and $\hat{\omega}^{i}=dx^{i}+N^{i}dt$.[14] We also have $|\hat{g}|=\gamma$. Let $\mathcal{A}=\hat{\mathcal{A}}_{\mu}\hat{\omega}^{\mu}$, and $\mathcal{F}=d\mathcal{A}=(1/2!)\hat{\mathcal{F}}_{\mu\nu}\hat{\omega}^{\mu}\wedge\hat{\omega}^{\nu}$. It follows that $\hat{\mathcal{F}}_{0s}=\mathcal{F}(\hat{e}_{0},\hat{e}_{s})=\mathcal{F}(n^{\mu}e_{\mu},e_{s})=n^{\mu}\mathcal{F}_{\mu s}$ and therefore,

$$n^{\mu}\mathcal{F}_{\mu s}dx^{s}=\rho_{(0)}\mathcal{F}(n,\cdot). \tag{A5}$$

Consider,

$$\begin{aligned}
\rho_{(d)}\mathcal{A}&=-*^{(3)}i**^{(4)}d^{(4)}\hat{\mathcal{A}}_{\mu}\hat{\omega}^{\mu},\\
&=-*^{(3)}i**^{(4)}\frac{1}{2!}\hat{\mathcal{F}}_{\mu\nu}\hat{\omega}^{\mu}\wedge\hat{\omega}^{\nu},\\
&=-*^{(3)}i*\frac{1}{2!}\sqrt{|\hat{g}|}\frac{1}{2!}\hat{g}^{\mu\alpha}\hat{g}^{\nu\beta}\hat{\mathcal{F}}_{\alpha\beta}\epsilon_{\mu\nu\sigma\tau}\hat{\omega}^{\sigma}\wedge\hat{\omega}^{\tau},\\
&=-*^{(3)}\sqrt{\gamma}\hat{g}^{0\alpha}\hat{g}^{i\beta}\hat{\mathcal{F}}_{\alpha\beta}\epsilon_{0ijk}dx^{j}\wedge dx^{k},\\
&=\gamma\frac{1}{2!}\hat{\mathcal{F}}_{0s}\gamma^{is}\gamma^{jm}\gamma^{kn}\epsilon_{0imn}\epsilon_{jkp}dx^{p},\\
&=\frac{1}{2!}\hat{\mathcal{F}}_{0s}\epsilon_{sjk}\epsilon_{jkp}dx^{p},\\
&=\hat{\mathcal{F}}_{0s}dx^{s},\\
&=n^{\alpha}\mathcal{F}_{\alpha s}dx^{s}\\
&=\rho_{(0)}(d^{(4)}\mathcal{A})(n,\cdot),
\end{aligned}$$

where in the last step we have used (A5).

We are finally ready to prove

Theorem 5: *Let $\mathcal{A}$ be a smooth solution of $(\Box+m^2)\mathcal{A}=0$ with Cauchy data $A_{(0)}\equiv\rho_{(0)}\mathcal{A}$, $A_{(d)}\equiv\rho_{(d)}\mathcal{A}\in\mathfrak{D}^{(1)}(\Sigma)$, and $A_{(n)}\equiv\rho_{(n)}\mathcal{A}$, $A_{(\delta)}\equiv\rho_{(\delta)}\mathcal{A}\in\mathfrak{D}^{(0)}(\Sigma)$. Then,*

$$\langle\mathcal{A},\mathcal{F}\rangle_{\mathcal{M}}=\langle A_{(0)},\rho_{(d)}E_m^{(1)}\mathcal{F}\rangle_\Sigma+\langle A_{(\delta)},\rho_{(n)}E_m^{(1)}\mathcal{F}\rangle_\Sigma-\langle A_{(n)},\rho_{(\delta)}E_m^{(1)}\mathcal{F}\rangle_\Sigma-\langle A_{(d)},\rho_{(0)}E_m^{(1)}\mathcal{F}\rangle_\Sigma, \tag{A6}$$

for any test function $\mathcal{F}\in\mathfrak{D}^{(1)}(\mathcal{M})$.

Proof: Since $(\Box+m^2)\mathcal{A}=0$, the second term on the left-hand side of (A4) is zero. For the remaining integrals, substitute $\mathcal{F}=E_m^{\mp(1)}\mathcal{F}'$ for the regions $\Sigma^\pm$, respectively. All integrals are well-defined because they entail integrations of smooth functions over compact sets. Specifically, for the left-hand side of (A4) we have

$$\mathrm{supp}(E_m^{\mp(1)}\mathcal{F}')\subset J^{\mp}(\mathrm{supp}(\mathcal{F}')),$$

with $J^{\mp}(\mathrm{supp}(\mathcal{F}'))\cap J^{\pm}(\Sigma)$ compact ($\mathcal{M}$ is globally hyperbolic), and for the right-hand side, Σ is compact by assumption.[3] Next, sum the integrations over the $\Sigma^\pm$ regions substituting $(\Box+m^2)E_m^{\pm(1)}=I$ in $\Sigma^\pm$ integrals and $E_m^{(1)}=E_m^{+(1)}-E_m^{-(1)}$ in Σ integrals. The sum of the $\Sigma^\pm$ integrals gives an integral over $\mathcal{M}$ (Σ constitutes a set of measure zero relative to this integration). Finally, relabel $\mathcal{F}'\to\mathcal{F}$ and obtain (A6). This completes the proof. ■

APPENDIX B: MAXWELL'S EQUATIONS

In this section we study Maxwell's equations,

$$d^{(4)}\mathcal{F}=0 \tag{B1}$$

and

$$\delta^{(4)}\mathcal{F}=0, \tag{B2}$$

where $\mathcal{F}$ is the field strength 2-form (not a test function as above). We pose an initial value problem for these equations following the presentation in Wald.[13] Specifically, we specify the initial data for $\mathcal{F}$ in terms of the electric and magnetic fields $E\equiv\rho_{(n)}\mathcal{F}$ and $B\equiv\rho_{(0)}\mathcal{F}$, respectively, which are regarded as a 1-form and 2-form on Σ, respectively. This data satisfies additional constraints $\delta^{(3)}E=0$ and $d^{(3)}B=0$. Given these data, we obtain a field solution $\mathcal{F}$ that satisfies (B1) and (B2) on $\mathcal{M}$. Our approach is similar to Dimock's; the differences being that our emphasis is on the fields rather than the vector potential, and we give an explicit representation for $\mathcal{F}$.[4]

Before we proceed, we make a further restriction on $\mathcal{M}$. Specifically, we assume that Σ is compact and contractible. If Σ is contractible then any closed p-form on Σ is exact, that is, $K\in\mathfrak{D}^{(p)}(\Sigma)$ with $d^{(3)}K=0\Rightarrow K=d^{(3)}H$ for some $H\in\mathfrak{D}^{(p-1)}(\Sigma)$ $(p>0)$.[20,21]

Theorem 6: *Let $E\in\mathfrak{D}^{(1)}(\Sigma)$ and $B\in\mathfrak{D}^{(2)}(\Sigma)$ be data for the field strength $\mathcal{F}$, i.e.,*

$$\rho_{(n)}\mathcal{F}=E, \tag{B3}$$

and

$$\rho_{(0)}\mathcal{F}=B, \tag{B4}$$

where

$$\delta^{(3)}E=0, \tag{B5}$$

and

$$d^{(3)}B=0. \tag{B6}$$

Then, given these data, there is a smooth potential $\mathcal{A}$ *such that* $\mathcal{F}=d^{(4)}\mathcal{A}$ *satisfies Maxwell's equations* $d^{(4)}\mathcal{F}=0$ *and* $\delta^{(4)}\mathcal{F}=0$, *as well as (B3)–(B6). Moreover, any two such potentials are gauge equivalent i.e., they differ by the exterior derivative of a scalar. Here we assume that* Σ *is compact and contractible.*

Proof: Existence: We choose data for $\mathcal{A}$ as follows:

$$A_{(d)}=E, \tag{B7}$$

$$d^{(3)}A_{(0)}=B, \tag{B8}$$

with

$$\delta^{(3)}A_{(d)}=0, \tag{B9}$$

$A_{(\delta)}=0$, and $A_{(n)}$ arbitrary. These choices of data are compatible with the field constraints (B5), and (B6) in that (B7) and (B9) imply (B5), and (B8) implies (B6). Regarding (B8), we know that such an $A_{(0)}$ exists because Σ is contractible and B is exact. These choices of data satisfy (23) and (24) of Theorem 2 for the $m=0$ case. Consequently, there is a unique smooth $\mathcal{A}$ that satisfies

$$\delta^{(4)}d^{(4)}\mathcal{A}=0, \tag{B10}$$

and $\mathcal{F}=d^{(4)}\mathcal{A}$ is our desired field strength. Next we show that $\mathcal{F}$ renders the data (B3) and (B4). From Theorem 2 we have

$$\mathcal{A}=-E_0^{(1)}\rho'_{(d)}A_{(0)}+E_0^{(1)}\rho'_{(\delta)}A_{(n)}+E_0^{(1)}\rho'_{(0)}A_{(d)}. \tag{B11}$$

Consider,

$$\begin{aligned}\rho_{(n)}\mathcal{F}&=-\rho_{(n)}d^{(4)}E_0^{(1)}\rho'_{(d)}A_{(0)}+\rho_{(n)}d^{(4)}E_0^{(1)}\rho'_{(\delta)}A_{(n)}+\rho_{(n)}d^{(4)}E_0^{(1)}\rho'_{(0)}A_{(d)},\\ &=-\rho_{(d)}E_0^{(1)}\rho'_{(d)}A_{(0)}+\rho_{(d)}E_0^{(1)}\rho'_{(\delta)}A_{(n)}+\rho_{(d)}E_0^{(1)}\rho'_{(0)}A_{(d)},\\ &=A_{(d)},\\ &=E,\end{aligned}$$

and

$$\begin{aligned}\rho_{(0)}\mathcal{F}&=-\rho_{(0)}d^{(4)}E_0^{(1)}\rho'_{(d)}A_{(0)}+\rho_{(0)}d^{(4)}E_0^{(1)}\rho'_{(\delta)}A_{(n)}+\rho_{(0)}d^{(4)}E_0^{(1)}\rho'_{(0)}A_{(d)},\\ &=-d^{(3)}\rho_{(0)}E_0^{(1)}\rho'_{(d)}A_{(0)}+d^{(3)}\rho_{(0)}E_0^{(1)}\rho'_{(\delta)}A_{(n)}+d^{(3)}\rho_{(0)}E_0^{(1)}\rho'_{(0)}A_{(d)},\\ &=d^{(3)}A_{(0)},\\ &=B,\end{aligned}$$

where we have used the results of Corollary 1.

Uniqueness: We want to show that any two potentials $\mathcal{A}'$ and $\tilde{\mathcal{A}}$ that satisfy (B10) with data (B7)–(B9) are gauge equivalent, i.e.,

$$\mathcal{A}'=\tilde{\mathcal{A}}+d^{(4)}f,$$

where $f\in\mathfrak{D}^{(0)}(\mathcal{M})$.[4] It suffices to show that any solution is gauge equivalent to the unique solution $\mathcal{A}$ above. Consider a solution $\mathcal{A}'$, we want to show that f' exists such that

$$\mathcal{A}=\mathcal{A}'+d^{(4)}f'.$$

Since $\mathcal{A}$ is unique, it suffices to construct $f'\in\mathfrak{D}^{(0)}(\mathcal{M})$ such that $\mathcal{A}'+d^{(4)}f'$ satisfies

$$\Box(\mathcal{A}'+d^{(4)}f')=0 \tag{B12}$$

on $\mathcal{M}$ with the same data as $\mathcal{A}$ on Σ. First, notice that (B12) is equivalent to

$$\delta^{(4)}d^{(4)}\mathcal{A}'+d^{(4)}\delta^{(4)}\mathcal{A}'+d^{(4)}\delta^{(4)}d^{(4)}f'=0. \tag{B13}$$

The first term in (B13) is zero because $\mathcal{A}'$ satisfies (B10). Therefore (B12) reduces to

$$\Box f'=-\delta^{(4)}\mathcal{A}'. \tag{B14}$$

Next, we study the data. Recall from Theorem 2 that the unique solution $\mathcal{A}$ has data $A_{(\delta)}$, $A_{(n)}$, $A_{(0)}$, and $A_{(d)}$, where

$$A_{(\delta)}=0, \tag{B15}$$

$$\delta^{(3)}A_{(d)}=0, \tag{B16}$$

and $A_{(0)}$ and $A_{(n)}$ are arbitrary. We need to construct f' so that $\mathcal{A}'+d^{(4)}f'$ renders the same data. To this end, we specify

$$\rho_{(\delta)}(\mathcal{A}'+d^{(4)}f')=0, \tag{B17}$$

$$\rho_{(n)}\mathcal{A}=\rho_{(n)}(\mathcal{A}'+d^{(4)}f'), \tag{B18}$$

$$\rho_{(0)}\mathcal{A}=\rho_{(0)}(\mathcal{A}'+d^{(4)}f'), \tag{B19}$$

and

$$\rho_{(d)}\mathcal{A}=\rho_{(d)}(\mathcal{A}'+d^{(4)}f'). \tag{B20}$$

To satisfy (B17) we impose (B14). The conditions (B18) and (B19) are satisfied when

$$\rho_{(d)}f'=\rho_{(n)}(\mathcal{A}-\mathcal{A}'), \tag{B21}$$

and

$$d^{(3)}\rho_{(0)}f'=\rho_{(0)}(\mathcal{A}-\mathcal{A}'), \tag{B22}$$

respectively. Regarding (B22), we know that such a $\rho_{(0)}f'$ exists because Σ is contractible and, by assumption, $\rho_{(0)}(\mathcal{A}-\mathcal{A}')$ is exact, i.e., $d^{(3)}\rho_{(0)}(\mathcal{A}-\mathcal{A}')=B-B=0$. Finally, (B20) is satisfied because, by assumption, $\mathcal{A}$ and $\mathcal{A}'$ satisfy (B7), and we know that $\rho_{(d)}d^{(4)}f'=0$.

Now, by assumption, we are given $\mathcal{A}$ and $\mathcal{A}'$ so we view (B14), (B21) and (B22) as specifying a Cauchy problem for the scalar field f'. That is, (B21) and (B22) specify the Cauchy data $\rho_{(d)}f'$ and $\rho_{(0)}f'$ for the nonhomogeneous linear hyperbolic equation (B14). A unique solution to this problem is known to exist which gives us the desired f'.[13] This shows that any solution $\mathcal{A}'$ is gauge equivalent to the unique $\mathcal{A}$ and therefore, given two different solutions $\mathcal{A}'$ and $\tilde{\mathcal{A}}$ we have $\mathcal{A}=\mathcal{A}'+d^{(4)}f'$, and $\mathcal{A}=\tilde{\mathcal{A}}+d^{(4)}\tilde{f}$ which shows that $\mathcal{A}'=\tilde{\mathcal{A}}+d^{(4)}f$, where $f=\tilde{f}-f'$. Thus any two such solutions give rise to the same field strength $\mathcal{F}$. This completes the proof. ∎

We obtain an explicit expression for $\mathcal{F}$ as follows:

Corollary 4: Let $\mathcal{A}$ be a vector potential with data $A_{(0)}$, $A_{(d)}$, $A_{(n)}$, and $A_{(\delta)}$ satisfying the conditions of Theorem 6. The field strength is given by

$$\mathcal{F} = -d^{(4)} E_0^{(1)} \rho'_{(d)} A_{(0)} + d^{(4)} E_0^{(1)} \rho'_{(0)} A_{(d)}, \tag{B23}$$

where $\mathcal{F} \in \mathcal{E}^{(2)}(\mathcal{M})$ is continuously dependent on $A_{(0)}$, $A_{(d)} \in \mathfrak{D}^{(1)}(\Sigma)$.

Proof: From Theorem 2 we have

$$\mathcal{A} = -E_0^{(1)} \rho'_{(d)} A_{(0)} + E_0^{(1)} \rho'_{(\delta)} A_{(n)} + E_0^{(1)} \rho'_{(0)} A_{(d)} \tag{B24}$$

with

$$\delta^{(3)} A_{(d)} = 0. \tag{B25}$$

We show that (B23) equals $d^{(4)}\mathcal{A}$. Let $\mathcal{G} \in \mathfrak{D}^{(2)}(\mathcal{M})$ be a 2-form test function, and consider,

$$\begin{aligned}
\langle d^{(4)}\mathcal{A}, \mathcal{G}\rangle_{\mathcal{M}} &= \langle \mathcal{A}, \delta^{(4)}\mathcal{G}\rangle_{\mathcal{M}}, \\
&= \langle A_{(0)}, \rho_{(d)} E_0^{(1)} \delta^{(4)}\mathcal{G}\rangle_\Sigma - \langle A_{(n)}, \rho_{(\delta)} E_0^{(1)} \delta^{(4)}\mathcal{G}\rangle_\Sigma \\
&\quad - \langle A_{(d)}, \rho_{(0)} E_0^{(1)} \delta^{(4)}\mathcal{G}\rangle_\Sigma \\
&= \langle -d^{(4)} E_0^{(1)} \rho'_{(d)} A_{(0)} + d^{(4)} E_0^{(1)} \rho'_{(0)} A_{(d)}, \mathcal{G}\rangle_\Sigma,
\end{aligned}$$

where, in the last step we have used $\rho_{(\delta)} E_0^{(1)} \delta^{(4)}\mathcal{G} = 0$. Thus, (B23) is satisfied in a distributional sense. From Theorem 1 we know that $E_0^{(1)}\rho'_{(0)}$, $E_0^{(1)}\rho'_{(d)} : \mathfrak{D}^{(1)}(\Sigma) \to \mathcal{E}^{(1)}(\mathcal{M})$ are continuous, and we also know that $d^{(4)} : \mathcal{E}^{(1)}(\mathcal{M}) \to \mathcal{E}^{(2)}(\mathcal{M})$ is continuous, therefore $\mathcal{F} \in \mathcal{E}^{(2)}(\mathcal{M})$ is continuously dependent on $A_{(0)}$ and $A_{(d)}$. ■

From this final result, we see that $\mathcal{F}$ depends only on the data $A_{(0)}$ and $A_{(d)}$. Thus, we can set $A_{(n)} = 0$ in (B24). This choice of data is useful when quantizing the electromagnetic field.[4]

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