Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **55**, 113501 (2014); doi: 10.1063/1.4899082
View online: http://dx.doi.org/10.1063/1.4899082
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/55/11?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **55**, 113501 (2014)

Multi-indexed Jacobi polynomials and Maya diagrams

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(Received 27 November 2013; accepted 10 October 2014; published online 4 November 2014)

Multi-indexed Jacobi polynomials are defined by the Wronskian of four types of eigenfunctions of the Pöschl-Teller Hamiltonian. We give a correspondence between multi-indexed Jacobi polynomials and pairs of Maya diagrams, and we show that any multi-indexed Jacobi polynomial is essentially equal to some multi-indexed Jacobi polynomial of two types of eigenfunction. As an application, we show a Wronskian-type formula of some special eigenstates of the deformed Pöschl-Teller Hamiltonian. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4899082]

I. INTRODUCTION

Recently systems of orthogonal polynomials which are out of range of Bochner's theorem are studied actively (see Refs. 4–7, 9, 11, 13–20, and references therein). A typical example of them is the multi-indexed Jacobi polynomial, which is a generalization of the exceptional Jacobi polynomial. Here we recall the system of quantum mechanics which is related to the multi-indexed Jacobi polynomial. The Hamiltonian with the Pöschl-Teller (PT) potential is given by

$$\mathcal{H} = -\frac{d^2}{dx^2} + U(x;g,h), \quad U(x;g,h) = \frac{g(g-1)}{\sin^2 x} + \frac{h(h-1)}{\cos^2 x} - (g+h)^2. \tag{1.1}$$

The eigenstate with the energy $\mathcal{E}_n(g,h) = 4n(n+g+h)$ ($n = 0, 1, 2, \dots$) is given by

$$\phi_n(x;g,h) = (\sin x)^g (\cos x)^h P_n^{(g-1/2,h-1/2)}(\eta(x)), \quad \eta(x) = \cos(2x), \tag{1.2}$$

where $P_n^{(\alpha,\beta)}(\eta)$ is the Jacobi polynomial in the variable η defined by

$$P_n^{(\alpha,\beta)}(\eta) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{k!(\alpha+1)_k}\left(\frac{1-\eta}{2}\right)^k. \tag{1.3}$$

Then the eigenstate is square-integrable, i.e., $\int_0^{\pi/2}\phi_n(x;g,h)^2dx < +\infty$ in the case $g > -1/2$, $h > -1/2$. To define the multi-indexed Jacobi polynomials,[15,7] we introduce three types of seed polynomial solutions indexed by $\mathrm{v} \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
\tilde{\phi}^{\mathrm{I}}_{\mathrm{v}}(x;g,h) &= (\sin x)^g(\cos x)^{1-h}P_{\mathrm{v}}^{(g-1/2,1/2-h)}(\eta(x)),\\
\tilde{\phi}^{\mathrm{II}}_{\mathrm{v}}(x;g,h) &= (\sin x)^{1-g}(\cos x)^{h}P_{\mathrm{v}}^{(1/2-g,h-1/2)}(\eta(x)),\\
\tilde{\phi}^{\mathrm{III}}_{\mathrm{v}}(x;g,h) &= (\sin x)^{1-g}(\cos x)^{1-h}P_{\mathrm{v}}^{(1/2-g,1/2-h)}(\eta(x)).
\end{aligned} \tag{1.4}$$

They are solutions of the Schrödinger equation (1.1) with the eigenvalues $\tilde{\mathcal{E}}^{\mathrm{I}}_{\mathrm{v}}(g,h) = -4(g+\mathrm{v}+1/2)(h-\mathrm{v}-1/2)$, $\tilde{\mathcal{E}}^{\mathrm{II}}_{\mathrm{v}}(g,h) = \tilde{\mathcal{E}}^{\mathrm{I}}_{-(\mathrm{v}+1)}(g,h)$, $\tilde{\mathcal{E}}^{\mathrm{III}}_{\mathrm{v}}(g,h) = \mathcal{E}_{-(\mathrm{v}+1)}(g,h)$, respectively, which are not square-integrable in the case $g \geq 3/2$, $h \geq 3/2$. In this paper, we assume that $g \pm h \notin \mathbb{Z}$ and $g, h \notin \mathbb{Z}+1/2$ under which the distinct eigenstates and seed solutions are linearly independent. Let φ_j be a seed solution or an eigenstate for $j = 1, \dots, \mathcal{N}$ and assume that $\varphi_1, \dots, \varphi_{\mathcal{N}}$ are distinct. Let

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$W[\varphi_1, \dots, \varphi_{\mathcal{N}}](x)$ be the Wronskian with respect to the derivative of the variable x. Then it follows from the typical argument[2,1,10] that

$$\phi_n^{(\mathcal{N})}(x) = \frac{\mathrm{W}[\varphi_1, \dots, \varphi_{\mathcal{N}}, \phi_n](x)}{\mathrm{W}[\varphi_1, \dots, \varphi_{\mathcal{N}}](x)} \tag{1.5}$$

is an eigenfunction of the deformed Hamiltonian

$$\mathcal{H}^{(\mathcal{N})} = -\frac{d^2}{dx^2} + U(x; g, h) - 2\frac{d^2 \log \mathrm{W}[\varphi_1, \dots, \varphi_{\mathcal{N}}](x)}{dx^2} \tag{1.6}$$

with the same eigenvalue $\mathcal{E}_n = 4n(n + g + h)$, provided that the deformed potential is non-singular on the open interval $(0, \pi/2)$, $\varphi_1, \dots, \varphi_{\mathcal{N}}, \phi_n$ are distinct, and g, h are enough large (see also Ref. 15). The multi-indexed Jacobi polynomial is defined by the polynomial part of the denominator of $\phi_n^{(\mathcal{N})}$. In this paper, we extend the notion of the multi-indexed Jacobi polynomial such that the polynomial part of the Wronskian $\mathrm{W}[\varphi_1, \dots, \varphi_{\mathcal{N}}, \phi_n](x)$ or $\mathrm{W}[\varphi_1, \dots, \varphi_{\mathcal{N}}](x)$ in the variable η. For example, the Wronskian $\mathrm{W}[\tilde{\phi}_1^{\mathrm{I}}\tilde{\phi}_2^{\mathrm{II}}\tilde{\phi}_1^{\mathrm{III}}](x)$ is equal to $(2g - 1)(2h + 1)(\sin x)^{1-g}(\cos x)^{1-h}P(\eta(x))/16$, where $P(\eta)$ is a multi-indexed Jacobi polynomial of degree 5 such that the coefficient of η^5 is $(g - h + 2)(g - h - 1)(g - h - 3)(g - h - 4)(g + h - 3)$. We remark that the deformed potentials may coincide for a different choice of seed solutions $\varphi_1, \dots, \varphi_{\mathcal{N}}$ and $\varphi'_1, \dots, \varphi'_{\mathcal{N}'}$.

The relationship between Wronskians and Maya diagrams (or Young diarams) in the soliton theory is well known (see Ref. 8 and references therein). In this paper, we connect a tuple of seed solutions $\varphi_1, \dots, \varphi_{\mathcal{N}}$ to a pair of Maya diagram with a division, and we show that the Maya diagrams describe relations among the Wronskians. As a corollary, the polynomial part of the Wronskian is proportional to the polynomial part of some Wronskian which constitutes the type I seed solutions and the square-integrable eigenstates with shifted parameters. Let us explain our results by an example. The tuple $\tilde{\phi}_1^{\mathrm{I}}\tilde{\phi}_2^{\mathrm{II}}\tilde{\phi}_1^{\mathrm{III}}$ corresponds to

$$\text{(III)} \begin{array}{c} \cdots \bullet\bullet\circ\bullet | \circ\circ\circ\circ \dots \\ \dots 3\,2\,1\,0\;\;0\,1\,2\,3 \dots \end{array} \text{(eigenstates)},\quad \text{(II)} \begin{array}{c} \cdots \bullet\bullet\circ\bullet\bullet | \circ\bullet\circ\circ \dots \\ \dots 4\,3\,2\,1\,0\;\;0\,1\,2\,3 \dots \end{array} \text{(I)},$$

where the white (resp. black) beads in the left (resp. right) of the division of the first Maya diagram represent the type III seed solutions (resp. the eigenstates) and the white (resp. black) beads in the left (resp. right) of the division of the second Maya diagram represent the type II (resp. type I) seed solutions. We move the division of the second Maya diagram one step to the left. Then the resulting Maya diagrams with divisions are

$$\begin{array}{c} \cdots \bullet\bullet\circ\bullet | \circ\circ\circ\circ \dots \\ \dots 3\,2\,1\,0\;\;0\,1\,2\,3 \dots \end{array}, \quad \begin{array}{c} \cdots \bullet\bullet\circ\bullet | \bullet\circ\bullet\circ\circ \dots \\ \dots 3\,2\,1\,0\;\;0\,1\,2\,3\,4 \dots \end{array},$$

and the corresponding tuple of states is $\tilde{\phi}_0^{\mathrm{I}}\tilde{\phi}_2^{\mathrm{I}}\tilde{\phi}_1^{\mathrm{II}}\tilde{\phi}_1^{\mathrm{III}}$. On the two tuples, we have a relation between the Wronskians, namely $W[\tilde{\phi}_1^{\mathrm{I}}\tilde{\phi}_2^{\mathrm{II}}\tilde{\phi}_1^{\mathrm{III}}](x; g, h) \propto W[\tilde{\phi}_0^{\mathrm{I}}\tilde{\phi}_2^{\mathrm{I}}\tilde{\phi}_1^{\mathrm{II}}\tilde{\phi}_1^{\mathrm{III}}](x; g - 1, h + 1)(\sin x)^{1-g}(\cos x)^h$. Details are discussed in Sec. III. Note that similar formulas were obtained in Refs. 15, 16, 13, although the discussion on the movement of the division of the Maya diagram was not archieved. We move the division of the first and the second Maya diagrams repeatedly to the left. Then we have the Maya diagrams with divisions

$$\begin{array}{c} \cdots \bullet\bullet | \circ\bullet\circ\circ\circ\circ \dots \\ \dots 1\,0\;\;0\,1\,2\,3\,4\,5 \dots \end{array}, \quad \begin{array}{c} \cdots \bullet\bullet | \circ\bullet\bullet\circ\bullet\circ\circ \dots \\ \dots 1\,0\;\;0\,1\,2\,3\,4\,5\,6 \dots \end{array},$$

and the relation $W[\tilde{\phi}_1^{\mathrm{I}}\tilde{\phi}_2^{\mathrm{II}}\tilde{\phi}_1^{\mathrm{III}}](x; g, h) \propto W[\tilde{\phi}_1^{\mathrm{I}}\tilde{\phi}_2^{\mathrm{I}}\tilde{\phi}_4^{\mathrm{I}}\phi_1](x, g - 5, h + 1)(\sin x)^{15-5g}(\cos x)^h$, which is a special case of Theorem III.3. Thus the given Wronskian is proportional to the Wronskian which consists of the first seed solutions and the eigenstates. Note that Odake[13] established that the polynomial part of the Wronskian $\mathrm{W}[\varphi_1, \dots, \varphi_{\mathcal{N}}](x)$ where φ_i is a type I seed solution or a type II seed solution is proportional to the polynomial part of some Wronskian which constitutes the type I seed solutions with shifted parameters, where the systems include the discrete quantum mechanics.

We give an application to the eigenstates of the deformed PT system which are represented by deleting a seed solution in Sec. IV. In an example of the system governed by the Hamiltonian

$$\mathcal{H}^{(3)} = -\frac{d^2}{dx^2} + U(x; g, h) - 2\frac{d^2 \log \mathrm{W}[\tilde{\phi}^{\mathrm{I}}_1 \tilde{\phi}^{\mathrm{II}}_2 \tilde{\phi}^{\mathrm{III}}_m](x)}{dx^2}, \tag{1.7}$$

the function

$$\phi^{(3)}_{-m-1}(x) = \frac{\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_1 \tilde{\phi}^{\mathrm{II}}_2](x)}{\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_1 \tilde{\phi}^{\mathrm{II}}_2 \tilde{\phi}^{\mathrm{III}}_m](x)} \tag{1.8}$$

is an eigenfunction of the Hamiltonian with the eigenvalue $\mathcal{E}_{-m-1} = -4(m+1)(g+h-m-1)$. As a consequence, the positions of the white beads of the first Maya diagram describe the energies of the system. Note that the eigenfunctions represented by deleting a seed solution were considered in Refs. 17 and 4 in the different situations.

This article is organized as follows. In Sec. II, we obtain formulas which are used later. Section III is the main part of this paper. In Sec. IV, we give an application to the extra eigenstates of the deformed PT system. In Sec. V, we give concluding remarks.

II. DARBOUX TRANSFORMATION AND RELATIONS OF WRONSKIAN

We apply Darboux transformation with respect to the seed solution $\tilde{\phi}^{\mathrm{I}}_0(x; g, h) = (\sin x)^g (\cos x)^{1-h}$ in the PT system. Then

$$\phi^{(1)}_n(x) = \frac{\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_0, \phi_n](x; g, h)}{\tilde{\phi}^{\mathrm{I}}_0(x; g, h)}, \quad \tilde{\phi}^{(1),\mathrm{I}}_{\mathrm{v}}(x) = \frac{\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_0, \tilde{\phi}^{\mathrm{I}}_{\mathrm{v}}](x; g, h)}{\tilde{\phi}^{\mathrm{I}}_0(x; g, h)}, \tag{2.1}$$

$$\tilde{\phi}^{(1),\mathrm{II}}_{\mathrm{v}}(x) = \frac{\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_0, \tilde{\phi}^{\mathrm{II}}_{\mathrm{v}}](x; g, h)}{\tilde{\phi}^{\mathrm{I}}_0(x; g, h)}, \quad \tilde{\phi}^{(1),\mathrm{III}}_{\mathrm{v}}(x) = \frac{\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_0, \tilde{\phi}^{\mathrm{III}}_{\mathrm{v}}](x; g, h)}{\tilde{\phi}^{\mathrm{I}}_0(x; g, h)},$$

are eigenfunctions of the deformed Hamiltonian

$$\mathcal{H}^{(1)} = -\frac{d^2}{dx^2} + U(x; g, h) - 2\frac{d^2 \log((\sin x)^g (\cos x)^{1-h})}{dx^2} = -\frac{d^2}{dx^2} + U(x; g+1, h-1) \tag{2.2}$$

with the eigenvalues $\mathcal{E}_n(g, h) = 4n(n+g+h)$, $\tilde{\mathcal{E}}^{\mathrm{I}}_{\mathrm{v}}(g, h) = -4(g+\mathrm{v}+1/2)(h-\mathrm{v}-1/2)$, $\tilde{\mathcal{E}}^{\mathrm{II}}_{\mathrm{v}}(g, h) = -4(g-\mathrm{v}-1/2)(h+\mathrm{v}+1/2)$, $\tilde{\mathcal{E}}^{\mathrm{III}}_{\mathrm{v}}(g, h) = -4(\mathrm{v}+1)(g+h-\mathrm{v}-1)$, respectively. On the other hand, the functions $\phi_n(x; g+1, h-1)$, $\tilde{\phi}^{\mathrm{I}}_{\mathrm{v}-1}(x; g+1, h-1)$, $\tilde{\phi}^{\mathrm{II}}_{\mathrm{v}+1}(x; g+1, h-1)$, $\tilde{\phi}^{\mathrm{III}}_{\mathrm{v}}(x; g+1, h-1)$ are also an eigenfunction of $-\frac{d^2}{dx^2} + U(x; g+1, h-1)$ with the eigenvalues $4n(n+g+h)$, $-4(g+\mathrm{v}+1/2)(h-\mathrm{v}-1/2)$, $-4(g-\mathrm{v}-1/2)(h+\mathrm{v}+1/2)$, $-4(\mathrm{v}+1)(g+h-\mathrm{v}-1)$, respectively. By comparing eigenfunctions with the same eigenvalue, we have the following relations:

$$\begin{aligned}
&\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_0, \phi_n](x; g, h) \propto \phi_n(x; g+1, h-1)\tilde{\phi}^{\mathrm{I}}_0(x; g, h), \\
&\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_0, \tilde{\phi}^{\mathrm{I}}_n](x; g, h) \propto \tilde{\phi}^{\mathrm{I}}_{n-1}(x; g+1, h-1)\tilde{\phi}^{\mathrm{I}}_0(x; g, h), \\
&\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_0, \tilde{\phi}^{\mathrm{II}}_n](x; g, h) \propto \tilde{\phi}^{\mathrm{II}}_{n+1}(x; g+1, h-1)\tilde{\phi}^{\mathrm{I}}_0(x; g, h), \\
&\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_0, \tilde{\phi}^{\mathrm{III}}_n](x; g, h) \propto \tilde{\phi}^{\mathrm{III}}_n(x; g+1, h-1)\tilde{\phi}^{\mathrm{I}}_0(x; g, h).
\end{aligned} \tag{2.3}$$

Proof. The functions $\phi^{(1)}_n(x)$ and $\phi_n(x; g+1, h-1)$ satisfy the linear differential equation $\{-\frac{d^2}{dx^2} + U(x; g+1, h-1) - 4n(n+g+h)\} f(x) = 0$. Thus they belong to the two dimensional vector space of solutions of the differential equation. The differential equation has a regular singularity along $x = 0$ and the exponents are $g+1$ and $-g$, and any solutions of the differential equation is written as a liner combination of $x^{g+1} f_1(x^2)$ and $x^{-g} f_2(x^2)$, where $f_1(t)$ and $f_2(t)$ are convergent power series of t such that $f_1(1) = f_2(1) = 1$. The function $\phi_n(x; g+1, h-1)$ is expanded as $x^{g+1}(c_0 + c_2 x^2 + c_4 x^4 + \dots)$. On the other hand, the function $\phi^{(1)}_n(x) = \phi'_n(x; g, h) - \phi_n(x; g, h)\tilde{\phi}^{\mathrm{I}}_0(x; g, h)'/\tilde{\phi}^{\mathrm{I}}_0(x; g, h)$ are expanded as $x^{g+1}(c'_0 + c'_2 x^2 + c'_4 x^4 + \dots)$. Therefore, the function $\phi^{(1)}_n(x)$ is proportional to the function $\phi_n(x; g+1, h-1)$, i.e., $\mathrm{W}[\tilde{\phi}_0^{\mathrm{I}}, \phi_n](x; g, h) \propto \phi_n(x; g+1, h-1)\tilde{\phi}_0^{\mathrm{I}}(x; g, h)$. Other relations are shown similarly. □

We denote the square-integrable eigenstate by the type "null" eigenfunction, i.e., $\tilde{\phi}_n^{\mathrm{N}}(x; g, h) = \phi_n(x; g, h)$. Then the following proposition is proved by similar ways to obtain Eq. (2.3).

Proposition 2.1. Let $J \in \{I, II, III, N\}$. We have

$$\mathrm{W}[\tilde{\phi}_0^{\mathrm{I}}, \tilde{\phi}_n^J](x; g, h) \propto \tilde{\phi}^J_{n-(\mathrm{I},J)}(x; g+1, h-1)\tilde{\phi}_0^{\mathrm{I}}(x; g, h), \tag{2.4}$$

$$\mathrm{W}[\tilde{\phi}_0^{\mathrm{II}}, \tilde{\phi}_n^J](x; g, h) \propto \tilde{\phi}^J_{n-(\mathrm{II},J)}(x; g-1, h+1)\tilde{\phi}_0^{\mathrm{II}}(x; g, h),$$

$$\mathrm{W}[\tilde{\phi}_0^{\mathrm{III}}, \tilde{\phi}_n^J](x; g, h) \propto \tilde{\phi}^J_{n-(\mathrm{III},J)}(x; g-1, h-1)\tilde{\phi}_0^{\mathrm{III}}(x; g, h),$$

$$\mathrm{W}[\phi_0, \tilde{\phi}_n^J](x; g, h) \propto \tilde{\phi}^J_{n-(N,J)}(x; g+1, h+1)\phi_0(x; g, h),$$

where

$$(J, J') = \begin{cases} 1 & J = J', \\ -1 & \{J, J'\} = \{I, II\} \text{ or } \{III, N\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

By applying the Jacobi's formula of Wronskians

$$\mathrm{W}[\varphi_1, \dots, \varphi_M, f, g](x)\mathrm{W}[\varphi_1, \dots, \varphi_M](x) = \mathrm{W}[\mathrm{W}[\varphi_1, \dots, \varphi_M, f](x), \mathrm{W}[\varphi_1, \dots, \varphi_M, g](x)] \tag{2.6}$$

(see Ref. 2), we have the following relations:

Proposition 2.2. Let $t_j \in \{I, II, III, N\}$ and $n_j \in \{0, 1, 2, \dots\}$. We have

$$\mathrm{W}[\tilde{\phi}_0^{I}, \tilde{\phi}_{n_1}^{t_1}, \dots, \tilde{\phi}_{n_M}^{t_M}](x; g, h) \propto \mathrm{W}[\tilde{\phi}^{t_1}_{n_1-(I,t_1)}, \dots, \tilde{\phi}^{t_M}_{n_M-(I,t_M)}](x; g+1, h-1)\tilde{\phi}_0^{I}(x; g, h), \tag{2.7}$$

$$\mathrm{W}[\tilde{\phi}_0^{II}, \tilde{\phi}_{n_1}^{t_1}, \dots, \tilde{\phi}_{n_M}^{t_M}](x; g, h) \propto \mathrm{W}[\tilde{\phi}^{t_1}_{n_1-(II,t_1)}, \dots, \tilde{\phi}^{t_M}_{n_M-(II,t_M)}](x; g-1, h+1)\tilde{\phi}_0^{II}(x; g, h),$$

$$\mathrm{W}[\tilde{\phi}_0^{III}, \tilde{\phi}_{n_1}^{t_1}, \dots, \tilde{\phi}_{n_M}^{t_M}](x; g, h) \propto \mathrm{W}[\tilde{\phi}^{t_1}_{n_1-(III,t_1)}, \dots, \tilde{\phi}^{t_M}_{n_M-(III,t_M)}](x; g-1, h-1)\tilde{\phi}_0^{III}(x; g, h),$$

$$\mathrm{W}[\phi_0, \tilde{\phi}_{n_1}^{t_1}, \dots, \tilde{\phi}_{n_M}^{t_M}](x; g, h) \propto \mathrm{W}[\tilde{\phi}^{t_1}_{n_1-(\mathrm{N},t_1)}, \dots, \tilde{\phi}^{t_M}_{n_M-(N,t_M)}](x; g+1, h+1)\phi_0(x; g, h).$$

Proof. We show the first relation by the induction of M, because the others are shown similarly. It follows from Proposition 2.1 that the case $M = 1$ is true. We assume that the case $M \leq k$ is true. Then

$$\begin{aligned}
&\mathrm{W}[\tilde{\phi}_0^{\mathrm{I}}, \tilde{\phi}_{n_1}^{t_1}, \dots, \tilde{\phi}_{n_{k-1}}^{t_{k-1}}, \tilde{\phi}_{n_k}^{t_k}, \tilde{\phi}_{n_{k+1}}^{t_{k+1}}](x; g, h)\mathrm{W}[\tilde{\phi}_0^{\mathrm{I}}, \tilde{\phi}_{n_1}^{t_1}, \dots, \tilde{\phi}_{n_{k-1}}^{t_{k-1}}](x; g, h) \\
&= \mathrm{W}[\mathrm{W}[\tilde{\phi}_0^{\mathrm{I}}, \tilde{\phi}_{n_1}^{t_1}, \dots, \tilde{\phi}_{n_{k-1}}^{t_{k-1}}, \tilde{\phi}_{n_k}^{t_k}](x; g, h), \mathrm{W}[\tilde{\phi}_0^{\mathrm{I}}, \tilde{\phi}_{n_1}^{t_1}, \dots, \tilde{\phi}_{n_{k-1}}^{t_{k-1}}, \tilde{\phi}_{n_{k+1}}^{t_{k+1}}](x; g, h)] \\
&\propto \mathrm{W}[\mathrm{W}[\tilde{\phi}^{t_1}_{n_1-(\mathrm{I},t_1)}, \dots, \tilde{\phi}^{t_{k-1}}_{n_{k-1}-(\mathrm{I},t_{k-1})}, \tilde{\phi}^{t_k}_{n_k-(\mathrm{I},t_k)}](x; g+1, h-1)\tilde{\phi}_0^{\mathrm{I}}(x; g, h), \\
&\qquad \mathrm{W}[\tilde{\phi}^{t_1}_{n_1-(\mathrm{I},t_1)}, \dots, \tilde{\phi}^{t_{k-1}}_{n_{k-1}-(\mathrm{I},t_{k-1})}, \tilde{\phi}^{t_{k+1}}_{n_{k+1}-(\mathrm{I},t_{k+1})}](x; g+1, h-1)\tilde{\phi}_0^{\mathrm{I}}(x; g, h)] \\
&= \mathrm{W}[\mathrm{W}[\tilde{\phi}^{t_1}_{n_1-(\mathrm{I},t_1)}, \dots, \tilde{\phi}^{t_{k-1}}_{n_{k-1}-(\mathrm{I},t_{k-1})}, \tilde{\phi}^{t_k}_{n_k-(\mathrm{I},t_k)}](x; g+1, h-1), \\
&\qquad \mathrm{W}[\tilde{\phi}^{t_1}_{n_1-(\mathrm{I},t_1)}, \dots, \tilde{\phi}^{t_{k-1}}_{n_{k-1}-(\mathrm{I},t_{k-1})}, \tilde{\phi}^{t_{k+1}}_{n_{k+1}-(\mathrm{I},t_{k+1})}](x; g+1, h-1)]\tilde{\phi}_0^{\mathrm{I}}(x; g, h)^2 \\
&= \mathrm{W}[\tilde{\phi}^{t_1}_{n_1-(\mathrm{I},t_1)}, \dots, \tilde{\phi}^{t_{k-1}}_{n_{k-1}-(\mathrm{I},t_{k-1})}, \tilde{\phi}^{t_k}_{n_k-(\mathrm{I},t_k)}, \tilde{\phi}^{t_{k+1}}_{n_{k+1}-(\mathrm{I},t_{k+1})}](x; g+1, h-1) \\
&\qquad \mathrm{W}[\tilde{\phi}^{t_1}_{n_1-(\mathrm{I},t_1)}, \dots, \tilde{\phi}^{t_{k-1}}_{n_{k-1}-(\mathrm{I},t_{k-1})}](x; g+1, h-1)]\tilde{\phi}_0^{\mathrm{I}}(x; g, h)^2.
\end{aligned}$$

Combining with

$$\begin{aligned}&\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_0, \tilde{\phi}^{t_1}_{n_1}, \ldots, \tilde{\phi}^{t_{k-1}}_{n_{k-1}}](x; g, h)\\ &\propto \mathrm{W}[\tilde{\phi}^{t_1}_{n_1-(\mathrm{I},t_1)}, \ldots, \tilde{\phi}^{t_{k-1}}_{n_{k-1}-(\mathrm{I},t_{k-1})}](x; g+1, h-1)\tilde{\phi}^{\mathrm{I}}_0(x; g, h),\end{aligned}$$

we obtain the proposition for the case $M = k + 1$. □

III. MAYA DIAGRAMS AND WRONSKIAN

A Maya diagram is a placement of white beads (○) or black ones (•) in a line such that the beads at a sufficiently negative (left) position are black and the ones at a sufficiently positive (right) position are white. We set a division (|) on the Maya diagram and we number the beads from 0 for both the right side and the left side from the division. An example of the Maya diagram with a division which is numbered is

$$\begin{array}{c}\cdots \bullet\bullet\bullet\circ\bullet\bullet\bullet \,|\, \bullet\bullet\circ\circ\circ \ldots\\ \ldots 6\,5\,4\,3\,2\,1\,0\quad 0\,1\,2\,3\,4 \ldots\end{array}.$$

For the tuple of states

$$\tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_1}, \tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_2}, \ldots, \tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_{M_{\mathrm{I}}}}, \tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_1}, \ldots, \tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_{M_{\mathrm{II}}}}, \tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_1}, \ldots, \tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_{M_{\mathrm{III}}}}, \phi_{d^{\mathrm{N}}_1}, \ldots, \phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}}, \tag{3.1}$$

we associate a couple of Maya diagrams with a division as follows:

1. The beads of the number $d^{\mathrm{III}}_1, \ldots, d^{\mathrm{III}}_{M_{\mathrm{III}}}$ on the left of the division of the first Maya diagram is white (○) and the residual beads on them are black (•).
2. The bead of the number $d^{\mathrm{N}}_1, \ldots, d^{\mathrm{N}}_{M_{\mathrm{N}}}$ on the right of the division of the first Maya diagram is black (•) and the residual beads on them are white (○).
3. The bead of the number $d^{\mathrm{II}}_1, \ldots, d^{\mathrm{II}}_{M_{\mathrm{II}}}$ on the right of the division of the second Maya diagram is black (•) and the residual beads on them are white (○).
4. The bead of the number $d^{\mathrm{I}}_1, \ldots, d^{\mathrm{I}}_{M_{\mathrm{I}}}$ on the left of the division of the second Maya diagram is white (○) and the residual beads on them are black (•).

An example was given in the Introduction. We give another example here. The tuple of states $\tilde{\phi}^{\mathrm{I}}_2\tilde{\phi}^{\mathrm{I}}_3\tilde{\phi}^{\mathrm{II}}_0\tilde{\phi}^{\mathrm{II}}_2\tilde{\phi}^{\mathrm{III}}_3\phi_0\phi_1$ corresponds to the following Maya diagrams:

$$\cdots \bullet\bullet\bullet\circ\bullet\bullet\bullet \,|\, \bullet\bullet\circ\circ\circ \ldots, \qquad \cdots \bullet\bullet\bullet\circ\bullet\circ \,|\, \circ\circ\bullet\bullet\circ\circ\circ \ldots.$$

We go back to the general situation where the tuple of state is given by Eq. (3.1) and assume that $0 \le d^{\mathrm{J}}_1 < d^{\mathrm{J}}_2 < \cdots < d^{\mathrm{J}}_{M_{\mathrm{J}}}$ for $J \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{N}\}$. We move the division of the second Maya diagram one step to the left. Then the corresponding tuple of states is

$$\begin{aligned}&\tilde{\phi}^{\mathrm{I}}_0, \tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_1+1}, \ldots, \tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_{M_{\mathrm{I}}}+1}, \tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_1-1}, \ldots, \tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_{M_{\mathrm{II}}}-1}, \tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_1}, \ldots, \tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_{M_{\mathrm{III}}}}, \phi_{d^{\mathrm{N}}_1}, \ldots, \phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}}, \qquad d^{\mathrm{II}}_1 \neq 0,\\ &\tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_1+1}, \ldots, \tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_{M_{\mathrm{I}}}+1}, \tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_2-1}, \ldots, \tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_{M_{\mathrm{II}}}-1}, \tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_1}, \ldots, \tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_{M_{\mathrm{III}}}}, \phi_{d^{\mathrm{N}}_1}, \ldots, \phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}}, \qquad d^{\mathrm{II}}_1 = 0.\end{aligned} \tag{3.2}$$

In the example of $\tilde{\phi}^{\mathrm{I}}_2\tilde{\phi}^{\mathrm{I}}_3\tilde{\phi}^{\mathrm{II}}_0\tilde{\phi}^{\mathrm{II}}_2\tilde{\phi}^{\mathrm{III}}_3\phi_0\phi_1$, we have $d^{\mathrm{II}}_1 = 0$, the changed Maya diagrams are

$$\cdots \bullet\bullet\bullet\circ\bullet\bullet\bullet \,|\, \bullet\bullet\circ\circ\circ \ldots, \qquad \cdots \bullet\bullet\bullet\circ\bullet \,|\, \circ\circ\circ\bullet\bullet\circ\circ\circ \ldots.$$

and the tuple of state corresponding to the changed Maya diagrams is $\tilde{\phi}^{\mathrm{I}}_3\tilde{\phi}^{\mathrm{I}}_4\tilde{\phi}^{\mathrm{II}}_1\tilde{\phi}^{\mathrm{III}}_3\phi_0\phi_1$.

It follows from Proposition 2.2 that the Wronskian

$$\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_1}, \ldots, \tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_{M_{\mathrm{I}}}}, \tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_1}, \ldots, \tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_{M_{\mathrm{II}}}}, \tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_1}, \ldots, \tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_{M_{\mathrm{III}}}}, \phi_{d^{\mathrm{N}}_1}, \ldots, \phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}}](x; g, h) \tag{3.3}$$

is proportional to

$$\begin{aligned}&\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_0, \tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_1+1}, \ldots, \tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_{M_{\mathrm{I}}}+1}, \tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_1-1}, \ldots, \tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_{M_{\mathrm{II}}}-1}, \tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_1}, \ldots, \tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_{M_{\mathrm{III}}}}, \phi_{d^{\mathrm{N}}_1}, \ldots\\ &\quad \ldots, \phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}}](x, g-1, h+1)/\tilde{\phi}^{\mathrm{I}}_0(x; g-1, h+1)\end{aligned} \tag{3.4}$$

for $d_1^{\mathrm{II}} \neq 0$ and to

$$\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_{d_1^{\mathrm{I}}+1}, \dots, \tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_{M_{\mathrm{I}}}+1}, \tilde{\phi}^{\mathrm{II}}_{d_2^{\mathrm{II}}-1}, \dots, \tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_{M_{\mathrm{II}}}-1}, \tilde{\phi}^{\mathrm{III}}_{d_1^{\mathrm{III}}}, \dots, \tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_{M_{\mathrm{III}}}}, \phi_{d_1^{\mathrm{N}}}, \dots \\ \dots, \phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}}](x, g-1, h+1)\tilde{\phi}^{\mathrm{II}}_0(x; g, h) \tag{3.5}$$

for $d_1^{\mathrm{II}} = 0$. Note that in the case that the states in the Wronskian are types I and II, the relations were obtained by Odake and Sasaki,[15,16] although the proof seems to be different from ours. By combining with $1/\tilde{\phi}^{\mathrm{I}}_0(x; g-1, h+1) = \tilde{\phi}^{\mathrm{II}}_0(x; g, h) = (\sin x)^{1-g}(\cos x)^h$, we have the following proposition on the movement of the division of the second Maya diagram.

Proposition 3.1. If the division of the second Maya diagram is moved one step to the left, then the original Wronskian is proportional to the product of the function $(\sin x)^{1-g}(\cos x)^h$ *and the Wronskian corresponding to the moved division where the parameters are shifted to* $g - 1$ *and* $h + 1$. *Namely the Wronskian in Eq. (3.3) is proportional to Eq. (3.4) or Eq. (3.5).*

On the movement of the division of the first Maya diagram, we have the following.

Proposition 3.2. If the division of the first Maya diagram is moved one step to the left, then the original Wronskian is proportional to the product of the function $(\sin x)^{1-g}(\cos x)^{1-h}$ *and the Wronskian corresponding to the moved division where the parameters are shifted to* $g - 1$ *and* $h - 1$, i.e., *the Wronskian given as Eq. (3.3) is proportional to*

$$\mathrm{W}[\tilde{\phi}^{I}_{d_1^{I}}, \dots, \tilde{\phi}^{I}_{d^{I}_{M_I}}, \tilde{\phi}^{II}_{d_1^{II}}, \dots, \tilde{\phi}^{II}_{d^{II}_{M_{II}}}, \tilde{\phi}^{III}_{d_1^{III}-1}, \dots, \tilde{\phi}^{III}_{d^{III}_{M_{III}}-1}, \phi_0, \phi_{d_1^{\mathrm{N}}+1}, \dots \\ \dots, \phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}+1}](x, g-1, h-1)(\sin x)^{1-g}(\cos x)^{1-h} \tag{3.6}$$

for $d_1^{\mathrm{III}} \neq 0$ *and to*

$$\mathrm{W}[\tilde{\phi}^{I}_{d_1^{I}}, \dots, \tilde{\phi}^{I}_{d^{I}_{M_I}}, \tilde{\phi}^{II}_{d_2^{II}}, \dots, \tilde{\phi}^{II}_{d^{II}_{M_{II}}}, \tilde{\phi}^{III}_{d_2^{III}-1}, \dots, \tilde{\phi}^{III}_{d^{III}_{M_{III}}-1}, \phi_{d_1^{\mathrm{N}}+1}, \dots \\ \dots, \phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}+1}](x, g-1, h-1)(\sin x)^{1-g}(\cos x)^{1-h} \tag{3.7}$$

for $d_1^{\mathrm{III}} = 0$.

For a given tuple of states, we associate a pair of Maya diagrams with a division and we move the divisions to the left. By applying the propositions repeatedly, we find that the Wronskian of a given tuple of states is equal to some Wronskian which constitutes the type I states and the square-integrable states up to a scalar multiplication. Namely, we obtain

Theorem 3.3. *Let* $\bar{\mathcal{D}}_{\mathrm{J}} = \{0, 1, 2, \dots, d^{\mathrm{J}}_{M_{\mathrm{J}}}\} \setminus \{d^{\mathrm{J}}_{M_{\mathrm{J}}} - d^{\mathrm{J}}_1, d^{\mathrm{J}}_{M_{\mathrm{J}}} - d^{\mathrm{J}}_2, \dots, d^{\mathrm{J}}_{M_{\mathrm{J}}} - d^{\mathrm{J}}_{M_{\mathrm{J}}} (= 0)\}$ *and write* $\bar{\mathcal{D}}_{\mathrm{J}} = \{e^{\mathrm{J}}_1, e^{\mathrm{J}}_2, \dots, e^{\mathrm{J}}_{\bar{M}_{\mathrm{J}}}\}$ $(e^{\mathrm{J}}_1 < e^{\mathrm{J}}_2 < \cdots < e^{\mathrm{J}}_{\bar{M}_{\mathrm{J}}}, \bar{M}_{\mathrm{J}} = d^{\mathrm{J}}_{M_{\mathrm{J}}} + 1 - M_{\mathrm{J}})$ *for* $J \in \{I, II, III, N\}$. *We have*

$$\begin{aligned} &\mathrm{W}[\tilde{\phi}^{I}_{d_1^{I}}, \dots, \tilde{\phi}^{I}_{d^{I}_{M_I}}, \tilde{\phi}^{II}_{d_1^{II}}, \dots, \tilde{\phi}^{II}_{d^{II}_{M_{II}}}, \tilde{\phi}^{III}_{d_1^{III}}, \dots, \tilde{\phi}^{III}_{d^{III}_{M_{III}}}, \phi_{d_1^{\mathrm{N}}}, \dots, \phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}}](x; g, h) \\ &\propto \mathrm{W}[\tilde{\phi}^{I}_{e_1^{II}}, \dots, \tilde{\phi}^{I}_{e^{II}_{\bar{M}_{II}}}, \tilde{\phi}^{I}_{d_1^{I}+d^{II}_{M_{II}}+1}, \dots, \tilde{\phi}^{I}_{d^{I}_{M_I}+d^{II}_{M_{II}}+1}, \phi_{e_1^{III}}, \dots, \phi_{e^{III}_{\bar{M}_{III}}}, \phi_{d_1^{\mathrm{N}}+d^{III}_{M_{III}}+1}, \dots \\ &\quad , \phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}+d^{III}_{M_{III}}+1}](x, g - d^{II}_{M_{II}} - d^{III}_{M_{III}} - 2, h + d^{II}_{M_{II}} - d^{III}_{M_{III}})(\sin x)^{g_{I,\mathrm{N}}}(\cos x)^{h_{I,\mathrm{N}}}, \end{aligned} \tag{3.8}$$

where $g_{I,N} = (d^{II}_{M_{II}} + d^{III}_{M_{III}} + 2)\{-g + (d^{II}_{M_{II}} + d^{III}_{M_{III}} + 3)/2\}$ *and* $h_{I,N} = (d^{II}_{M_{II}} - d^{III}_{M_{III}})\{h + (d^{II}_{M_{II}} - d^{III}_{M_{III}} - 1)/2\}$.

Proof. We move the division of the second Maya diagram of the given states $d^{\mathrm{II}}_{M_{\mathrm{II}}} + 1$ times to the left. By noticing the placement of the black beads on the right of the moved division, we find that the left-hand side of Eq. (3.8) is proportional to

$$\begin{aligned}&\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_{e^{\mathrm{II}}_1},\dots,\tilde{\phi}^{\mathrm{I}}_{e^{\mathrm{II}}_{\bar{M}_{\mathrm{II}}}},\tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_1+d^{\mathrm{II}}_{M_{\mathrm{II}}}+1},\dots,\tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_{M_{\mathrm{I}}}+d^{\mathrm{II}}_{M_{\mathrm{II}}}+1},\tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_1},\tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_2},\dots,\tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_{M_{\mathrm{III}}}},\\ &\qquad\phi_{d^{\mathrm{N}}_1},\phi_{d^{\mathrm{N}}_2},\dots,\phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}}]](x,g-d^{\mathrm{II}}_{M_{\mathrm{II}}}-1,h+d^{\mathrm{II}}_{M_{\mathrm{II}}}+1)(\sin x)^{g'}(\cos x)^{h'},\end{aligned} \tag{3.9}$$

where $g'=(d^{\mathrm{II}}_{M_{\mathrm{II}}}+1)\{-g+(d^{\mathrm{II}}_{M_{\mathrm{II}}}+2)/2\}$ and $h'=(d^{\mathrm{II}}_{M_{\mathrm{II}}}+1)(h+d^{\mathrm{II}}_{M_{\mathrm{II}}}/2)$. We move further the division of the first Maya diagram $d^{\mathrm{III}}_{M_{\mathrm{III}}}+1$ times to the left. Then we obtain the theorem.

In the example of $\tilde{\phi}^{\mathrm{I}}_2\tilde{\phi}^{\mathrm{I}}_3\tilde{\phi}^{\mathrm{II}}_0\tilde{\phi}^{\mathrm{II}}_2\tilde{\phi}^{\mathrm{III}}_3\phi_0\phi_1$, we have $\bar{\mathcal{D}}_{\mathrm{II}}=\{0,1,2\}\setminus\{2,0\}=\{1\}$, $\bar{\mathcal{D}}_{\mathrm{III}}=\{1,2,3\}$ and

$$\begin{aligned}&\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_2\tilde{\phi}^{\mathrm{I}}_3\tilde{\phi}^{\mathrm{II}}_0\tilde{\phi}^{\mathrm{II}}_2\tilde{\phi}^{\mathrm{III}}_3\phi_0\phi_1](x;g,h)\\ &\propto \mathrm{W}[\tilde{\phi}^{\mathrm{I}}_1\tilde{\phi}^{\mathrm{I}}_5\tilde{\phi}^{\mathrm{I}}_6\phi_1\phi_2\phi_3\phi_4\phi_5](x,g-7,h-1)(\sin x)^{28-7g}(\cos x)^{1-h},\end{aligned} \tag{3.10}$$

and the corresponding Maya diagrams are

$$\cdots\bullet\bullet\bullet|\circ\bullet\bullet\bullet\bullet\bullet\bullet\circ\circ\circ\dots,\qquad \cdots\bullet\bullet\bullet|\circ\bullet\circ\circ\circ\circ\bullet\bullet\circ\circ\circ\dots.$$

Note that the left-hand side of Eq. (3.8) is also proportional to each Wronskian of the followings:

$$\begin{aligned}&\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_{e^{\mathrm{II}}_1},\dots,\tilde{\phi}^{\mathrm{I}}_{e^{\mathrm{II}}_{\bar{M}_{\mathrm{II}}}},\tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_1+d^{\mathrm{II}}_{M_{\mathrm{II}}}+1},\dots,\tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_{M_{\mathrm{I}}}+d^{\mathrm{II}}_{M_{\mathrm{II}}}+1},\tilde{\phi}^{\mathrm{III}}_{e^{\mathrm{N}}_1},\dots,\tilde{\phi}^{\mathrm{III}}_{e^{\mathrm{III}}_{\bar{M}_{\mathrm{N}}}},\tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_1+d^{\mathrm{N}}_{M_{\mathrm{N}}}+1},\dots\\ &\qquad\dots,\tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_{M_{\mathrm{III}}}+d^{\mathrm{N}}_{M_{\mathrm{N}}}+1}](x,g-d^{\mathrm{II}}_{M_{\mathrm{II}}}+d^{\mathrm{N}}_{M_{\mathrm{N}}},h+d^{\mathrm{II}}_{M_{\mathrm{II}}}+d^{\mathrm{N}}_{M_{\mathrm{N}}}+2)(\sin x)^{g_{\mathrm{I,III}}}(\cos x)^{h_{\mathrm{I,III}}},\\ &\mathrm{W}[\tilde{\phi}^{\mathrm{II}}_{e^{\mathrm{I}}_1},\dots,\tilde{\phi}^{\mathrm{II}}_{e^{\mathrm{I}}_{\bar{M}_{\mathrm{I}}}},\tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_1+d^{\mathrm{I}}_{M_{\mathrm{I}}}+1},\dots,\tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_{M_{\mathrm{II}}}+d^{\mathrm{I}}_{M_{\mathrm{I}}}+1},\phi_{e^{\mathrm{III}}_1},\dots,\phi_{e^{\mathrm{III}}_{\bar{M}_{\mathrm{III}}}},\phi_{d^{\mathrm{N}}_1+d^{\mathrm{III}}_{M_{\mathrm{III}}}+1},\dots\\ &\qquad\dots,\phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}+d^{\mathrm{III}}_{M_{\mathrm{III}}}+1}](x,g+d^{\mathrm{I}}_{M_{\mathrm{I}}}-d^{\mathrm{III}}_{M_{\mathrm{III}}},h-d^{\mathrm{I}}_{M_{\mathrm{I}}}-d^{\mathrm{III}}_{M_{\mathrm{III}}}-2)(\sin x)^{g_{\mathrm{II,N}}}(\cos x)^{h_{\mathrm{II,N}}},\\ &\mathrm{W}[\tilde{\phi}^{\mathrm{II}}_{e^{\mathrm{I}}_1},\dots,\tilde{\phi}^{\mathrm{II}}_{e^{\mathrm{I}}_{\bar{M}_{\mathrm{I}}}},\tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_1+d^{\mathrm{I}}_{M_{\mathrm{I}}}+1},\dots,\tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_{M_{\mathrm{II}}}+d^{\mathrm{I}}_{M_{\mathrm{I}}}+1},\tilde{\phi}^{\mathrm{III}}_{e^{\mathrm{N}}_1},\dots,\tilde{\phi}^{\mathrm{III}}_{e^{\mathrm{III}}_{\bar{M}_{\mathrm{N}}}},\tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_1+d^{\mathrm{N}}_{M_{\mathrm{N}}}+1},\dots\\ &\qquad\dots,\tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_{M_{\mathrm{III}}}+d^{\mathrm{N}}_{M_{\mathrm{N}}}+1}](x,g+d^{\mathrm{I}}_{M_{\mathrm{I}}}+d^{\mathrm{N}}_{M_{\mathrm{N}}}+2,h-d^{\mathrm{I}}_{M_{\mathrm{I}}}+d^{\mathrm{N}}_{M_{\mathrm{N}}})(\sin x)^{g_{\mathrm{II,III}}}(\cos x)^{h_{\mathrm{II,III}}},\end{aligned}$$

where $g_{\mathrm{I,III}}=(d^{\mathrm{II}}_{M_{\mathrm{II}}}-d^{\mathrm{N}}_{M_{\mathrm{N}}})\{g+(d^{\mathrm{II}}_{M_{\mathrm{II}}}-d^{\mathrm{N}}_{M_{\mathrm{N}}}-1)/2\}$, $h_{\mathrm{I,III}}=(d^{\mathrm{II}}_{M_{\mathrm{II}}}+d^{\mathrm{N}}_{M_{\mathrm{N}}}+2)\{h+(d^{\mathrm{II}}_{M_{\mathrm{II}}}+d^{\mathrm{N}}_{M_{\mathrm{N}}}+1)/2\}$, $g_{\mathrm{II,N}}=(d^{\mathrm{I}}_{M_{\mathrm{III}}}-d^{\mathrm{III}}_{M_{\mathrm{III}}})\{g+(d^{\mathrm{I}}_{M_{\mathrm{I}}}-d^{\mathrm{III}}_{M_{\mathrm{III}}}-1)/2\}$, $h_{\mathrm{II,N}}=(d^{\mathrm{I}}_{M_{\mathrm{I}}}+d^{\mathrm{III}}_{M_{\mathrm{III}}}+2)\{-h+(d^{\mathrm{I}}_{M_{\mathrm{I}}}+d^{\mathrm{III}}_{M_{\mathrm{III}}}+3)/2\}$, $g_{\mathrm{II,III}}=(d^{\mathrm{N}}_{M_{\mathrm{N}}}+d^{\mathrm{I}}_{M_{\mathrm{I}}}+2)\{g+(d^{\mathrm{N}}_{M_{\mathrm{N}}}+d^{\mathrm{I}}_{M_{\mathrm{I}}}+1)/2\}$ and $h_{\mathrm{II,III}}=(d^{\mathrm{N}}_{M_{\mathrm{N}}}-d^{\mathrm{I}}_{M_{\mathrm{I}}})\{h+(d^{\mathrm{N}}_{M_{\mathrm{N}}}-d^{\mathrm{I}}_{M_{\mathrm{I}}}-1)/2\}$.

IV. APPLICATION TO EXTRA EIGENSTATES

We give an application of the relations of the Wronskians in Sec. III to the extra eigenstates of the deformed PT system.

Proposition 4.1. Let $\varphi_1,\dots,\varphi_{\mathcal{N}}$ be distinct seed solutions (or eigenstates) and assume that $\varphi_\ell=\tilde{\phi}^{III}_m$. Then

$$\phi^{(\mathcal{N})}_{-m-1}(x)=\frac{\mathrm{W}[\varphi_1,\dots,\varphi_{\ell-1},\varphi_{\ell+1},\dots\varphi_{\mathcal{N}}](x;g,h)}{\mathrm{W}[\varphi_1,\dots,\varphi_{\mathcal{N}}](x;g,h)} \tag{4.1}$$

is an eigenfunction of the deformed PT Hamiltonian

$$\mathcal{H}^{(\mathcal{N})}=-\frac{d^2}{dx^2}+U(x;g,h)-2\frac{d^2\log \mathrm{W}[\varphi_1,\dots,\varphi_{\mathcal{N}}](x;g,h)}{dx^2} \tag{4.2}$$

with the eigenvalue $\mathcal{E}_{-m-1}=-4(m+1)(g+h-m-1)$, provided that the deformed potential is non-singular on the open interval $(0,\pi/2)$ and g,h are enough large.

Proof. We associate the tuple $\varphi_1,\dots,\varphi_{\mathcal{N}}$ to a pair of Maya diagrams with a division. Then the bead of the number m on the left of the division of the first Maya diagram is white. We move the division of the first Maya diagram $m+1$-times to the left, and denote the corresponding tuple by $\varphi'_1, \ldots, \varphi'_{\mathcal{N}'}$. Since the bead of the number 0 on the right of the division is white, no one of $\varphi'_1, \ldots, \varphi'_{\mathcal{N}'}$ is proportional to ϕ_0. Hence the function

$$\phi_0^{(\mathcal{N}')}(x) = \frac{\mathrm{W}[\varphi'_1, \ldots \varphi'_{\mathcal{N}'}, \phi_0](x; g-m-1, h-m-1)}{\mathrm{W}[\varphi'_1, \ldots, \varphi'_{\mathcal{N}'}](x; g-m-1, h-m-1)} \tag{4.3}$$

is an eigenfunction of the deformed Hamiltonian

$$\mathcal{H}^{(\mathcal{N}')} = -\frac{d^2}{dx^2} + V, \tag{4.4}$$
$$V = U(x; g-m-1, h-m-1) - 2\frac{d^2 \log \mathrm{W}[\varphi'_1, \ldots, \varphi'_{\mathcal{N}'}](x; g-m-1, h-m-1)}{dx^2}$$

with the eigenvalue $\mathcal{E}_0 = 0$. The Maya diagrams of the tuple $\varphi'_1, \ldots \varphi'_{\mathcal{N}'}, \phi_0$ are obtained from the ones of the tuple $\varphi'_1, \ldots, \varphi'_{\mathcal{N}'}$ by changing the bead of the number 0 on the right of the division of the first Maya diagram to black. We move the divisions of the first Maya diagrams of $\varphi'_1, \ldots, \varphi'_{\mathcal{N}'}$ and that of $\varphi'_1, \ldots \varphi'_{\mathcal{N}'}, \phi_0$ $m+1$-times to the right. Then we recover $\varphi_1, \ldots, \varphi_{\mathcal{N}}$ from $\varphi'_1, \ldots, \varphi'_{\mathcal{N}'}$ and we have $\varphi_1, \ldots, \varphi_{\ell-1}, \varphi_{\ell+1}, \ldots \varphi_{\mathcal{N}}$ from $\varphi'_1, \ldots \varphi'_{\mathcal{N}'}, \phi_0$. It follows from $\mathrm{W}[\varphi'_1, \ldots, \varphi'_{\mathcal{N}'}](x; g-m-1, h-m-1) \propto \mathrm{W}[\varphi_1, \ldots, \varphi_{\mathcal{N}}](x; g, h)(\sin x)^{(m+1)\{g-(m+2)/2\}}(\cos x)^{(m+1)\{h-(m+2)/2\}}$ that

$$V = U(x; g, h) - 2\frac{d^2 \log \mathrm{W}[\varphi_1, \ldots, \varphi_{\mathcal{N}'}](x; g, h)}{dx^2} + 4(m+1)(g+h-m-1). \tag{4.5}$$

Combining with $\mathrm{W}[\varphi'_1, \ldots, \varphi'_{\mathcal{N}'}, \phi_0](x; g-m-1, h-m-1) \propto \mathrm{W}[\varphi_1, \ldots, \varphi_{\ell-1}, \varphi_{\ell+1}, \ldots \varphi_{\mathcal{N}}](x; g, h)(\sin x)^{(m+1)\{g-(m+2)/2\}}(\cos x)^{(m+1)\{h-(m+2)/2\}}$, we have the proposition. □

We investigate the eigenvalues of the deformed PT Hamiltonian given by Eq. (4.2) where the states $\varphi_1, \ldots, \varphi_{\mathcal{N}}$ are described as Eq. (3.1), the deformed potential is non-singular on the open interval $(0, \pi/2)$ and g, h are enough large. It follows from Proposition IV.1 that the eigenvalue $\mathcal{E}_{-m-1} = -4(m+1)(g+h-m-1)$ $(m \in \{d_1^{\mathrm{III}}, \ldots, d_{M_{\mathrm{III}}}^{\mathrm{III}}\})$ is permitted. The eigenvalue $\mathcal{E}_n = 4n(g+h+n)$ $(n \in \mathbb{Z}_{\geq 0} \setminus \{d_1^{\mathrm{N}}, \ldots, d_{M_{\mathrm{N}}}^{\mathrm{N}}\})$ is also permitted and the eigenfunction is given by Eq. (1.5). Thus the labeling of the eigenvalues corresponds to the position of the white beads of the first Maya diagram.

In the example of the deformed PT Hamiltonian given by the states $\tilde{\phi}_3^{\mathrm{I}}\tilde{\phi}_2^{\mathrm{II}}\tilde{\phi}_1^{\mathrm{III}}\tilde{\phi}_4^{\mathrm{III}}\ \tilde{\phi}_5^{\mathrm{III}}\phi_1\phi_3$, the permitted eigenvalues are $\mathcal{E}_{-6}$, $\mathcal{E}_{-5}$, $\mathcal{E}_{-2}$, $\mathcal{E}_0$, $\mathcal{E}_2$, $\mathcal{E}_4$, $\mathcal{E}_5$, $\mathcal{E}_6, \ldots$ and the first Maya diagram with division is

$$\begin{array}{c} \cdots\bullet\ \bullet\bullet\ \circ\circ\ \bullet\bullet\ \circ\bullet \,|\, \circ\bullet\circ\bullet\circ\circ\circ\ldots \\ \ldots\text{-9-8-7-6-5-4-3-2-1}\ \ 0\,1\,2\,3\,4\,5\,6\ldots \end{array}.$$

V. CONCLUDING REMARKS

In this article, we gave a correspondence between tuples of states and pairs of Maya diagrams with a division. We have shown that a movement of the division corresponds to an equality of Wronskians of the states, and that any Wronskian of four types of states is essentially equal to a Wronskian of eigenstates and type I seed solutions. Here we propose a problem that the condition

$$\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_{d_1^{\mathrm{I}}}, \ldots, \tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_{M_{\mathrm{I}}}}, \phi_{d_1^{\mathrm{N}}}, \ldots, \phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}}](x, g, h) \propto \tag{5.1}$$
$$\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_{\tilde{d}_1^{\mathrm{I}}}, \ldots, \tilde{\phi}^{\mathrm{I}}_{\tilde{d}^{\mathrm{I}}_{\tilde{M}_{\mathrm{I}}}}, \phi_{\tilde{d}_1^{\mathrm{N}}}, \ldots, \phi_{\tilde{d}^{\mathrm{N}}_{\tilde{M}_{\mathrm{N}}}}](x, g+m_1, h+m_2)(\sin x)^{m_3}(\cos x)^{m_4},$$

$(0 < d_1^{\mathrm{J}} < \cdots < d^{\mathrm{J}}_{M_{\mathrm{J}}}, 0 < \tilde{d}_1^{\mathrm{J}} < \cdots < \tilde{d}^{\mathrm{J}}_{M_{\mathrm{J}}},\ (\mathrm{J} = \mathrm{I}, \mathrm{N}),\ m_1, m_2 \in \mathbb{Z},\ m_3, m_4 \in \mathbb{R})$ for any g and h leads to that the two tuples coincide and $m_1 = m_2 = m_3 = m_4 = 0$ or not? If g and h are special, then the above problem is negative because we have a counterexample which follows from the coincidence of the system $\tilde{\phi}_2^{\mathrm{I}}\tilde{\phi}_1^{\mathrm{III}}$ $(g = 3(h-3)/(4h-9))$ and the one $\tilde{\phi}_1^{\mathrm{I}}\tilde{\phi}_2^{\mathrm{III}}$ $(g = 3h/(4h-9)))$ obtained in Ref. 9.

Our results in this article hold essentially true for the multi-indexed Laguerre polynomials. We express the corresponding results for the multi-indexed Laguerre polynomials by using the notations in Ref. 19. We also associate tuples of eigenstates and three types of seed solutions Maya diagrams with couples of Maya diagrams with divisions. Then a movement of the division corresponds to an identity of Wronskians of the states. It is also shown that any Wronskian of four types of states is essentially equal to a Wronskian of eigenstates and type I seed solutions. Namely, we have

$$\begin{aligned}
&\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_1},\dots,\tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_{M_{\mathrm{I}}}},\tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_1},\dots,\tilde{\phi}^{\mathrm{II}}_{d^{\mathrm{II}}_{M_{\mathrm{II}}}},\tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_1},\dots,\tilde{\phi}^{\mathrm{III}}_{d^{\mathrm{III}}_{M_{\mathrm{III}}}},\phi_{d^{\mathrm{N}}_1},\dots,\phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}}](x;g) \qquad (5.2)\\
&\propto \mathrm{W}[\tilde{\phi}^{\mathrm{I}}_{e^{\mathrm{II}}_1},\dots,\tilde{\phi}^{\mathrm{I}}_{e^{\mathrm{II}}_{\tilde{M}_{\mathrm{II}}}},\tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_1+d^{\mathrm{II}}_{M_{\mathrm{II}}}+1},\dots,\tilde{\phi}^{\mathrm{I}}_{d^{\mathrm{I}}_{M_{\mathrm{I}}}+d^{\mathrm{II}}_{M_{\mathrm{II}}}+1},\phi_{e^{\mathrm{III}}_1},\dots,\phi_{e^{\mathrm{III}}_{\tilde{M}_{\mathrm{III}}}},\phi_{d^{\mathrm{N}}_1+d^{\mathrm{III}}_{M_{\mathrm{III}}}+1},\dots\\
&\qquad ,\phi_{d^{\mathrm{N}}_{M_{\mathrm{N}}}+d^{\mathrm{III}}_{M_{\mathrm{III}}}+1}](x,g-d^{\mathrm{II}}_{M_{\mathrm{II}}}-d^{\mathrm{III}}_{M_{\mathrm{III}}}-2)x^{g_{\mathrm{I,N}}}\exp((d^{\mathrm{III}}_{M_{\mathrm{III}}}-d^{\mathrm{II}}_{M_{\mathrm{II}}})x^2/2),
\end{aligned}$$

instead of Eq. (3.8), where the notations in Theorem III.3 are used. Key relations are

$$\begin{aligned}
\mathrm{W}[\tilde{\phi}^{\mathrm{I}}_0,\tilde{\phi}^{J}_n](x;g) &\propto \tilde{\phi}^{J}_{n-(\mathrm{I},J)}(x;g+1)\tilde{\phi}^{\mathrm{I}}_0(x;g), \qquad (5.3)\\
\mathrm{W}[\tilde{\phi}^{\mathrm{II}}_0,\tilde{\phi}^{J}_n](x;g) &\propto \tilde{\phi}^{J}_{n-(\mathrm{II},J)}(x;g-1)\tilde{\phi}^{\mathrm{II}}_0(x;g),\\
\mathrm{W}[\tilde{\phi}^{\mathrm{III}}_0,\tilde{\phi}^{J}_n](x;g) &\propto \tilde{\phi}^{J}_{n-(\mathrm{III},J)}(x;g-1)\tilde{\phi}^{\mathrm{III}}_0(x;g),\\
\mathrm{W}[\phi_0,\tilde{\phi}^{J}_n](x;g) &\propto \tilde{\phi}^{J}_{n-(N,J)}(x;g+1)\phi_0(x;g),
\end{aligned}$$

which corresponds to Eq. (2.4).

Odake and Sasaki[16] extended the multi-indexed orthogonal polynomials to multi-indexed Wilson and Askey-Wilson polynomials, which appear in the discrete quantum mechanics, and Odake[13] established that any multi-indexed Wilson (Askey-Wilson) polymonial which is expressed by type I seed solutions and the type II seed solutions is proportional to some multi-indexed Wilson (Askey-Wilson) polymonial which is expressed by only type I seed solutions. We believe that Odake's results are generalized and the results in this article are extended to the multi-indexed Wilson and Askey-Wilson polynomials.

Felder, Hemery, and Veselov observed relationships between the pattern of zeros of multi-indexed Hermite polynomials and the shape of Young diagrams.[3] A correspondence between Young diagrams and Maya diagrams is known well (e.g., see Ref. 12). The contents in Ref. 3 may lead to new development of multi-indexed polynomials.

ACKNOWLEDGMENTS

The author is grateful to Satoru Odake and Ryu Sasaki for valuable comments and fruitful discussions. He also thanks C.-L. Ho for hospitality during his visit in Taipei in August 2013 where this work was started. Thanks are due to the referee for valuable comments, especially for indicating the paper by Felder, Hemery, and Veselov.[3] He is supported in part by the grant-in-aid for Young Scientists from the Japan Society for the Promotion of Science (JSPS), No. 22740107 and by Chuo University Grant for Special Research.

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