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A risk model driven by Lévy processes

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SUMMARY

We present a general risk model where the aggregate claims, as well as the premium function, evolve by jumps. This is achieved by incorporating a Lévy process into the model. This seeks to account for the discrete nature of claims and asset prices. We give several explicit examples of Lévy processes that can be used to drive a risk model. This allows us to incorporate aggregate claims and premium fluctuations in the same process. We discuss important features of such processes and their relevance to risk modeling. We also extend classical results on ruin probabilities to this model. Copyright © 2003 John Wiley & Sons, Ltd.

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1. INTRODUCTION

A general insurance portfolio consists of several independent contracts issued for a limited time period (usually one year). During this period the company faces claims from policyholders, multiple claims from the same portfolio are possible. The classical risk reserve process for such a portfolio is

$$U(t) = u + ct - S(t) = u + ct - \sum_{n=1}^{N(t)} \zeta_n, \qquad t \geqslant 0 \tag{1}$$

where u is the initial surplus, $c > 0$ is a constant premium rate, N is a Poisson process with intensity λ modeling the total number of claims, while ζ_n, $n \geqslant 1$ are the claim sizes, typically assumed to be i.i.d. random variables independent of the Poisson process. $S(t) = \sum_{n=1}^{N(t)} \zeta_n$ is called the aggregate claims process. The quantity $\theta = (c - \lambda\mathbb{E}[\zeta_1])/(\lambda\mathbb{E}[\zeta_1])$ is called the security

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or safety loading factor. For an account on the theory we refer to Grandell [1], Asmussen [2] or Kaas *et al*. [3].

Lévy processes are processes with stationary and independent increments and are thus, in a way, generalizations of a Brownian motion. Unlike the latter, their increments are not normally distributed, the distribution of their increments belong to the wide class of infinitely divisible distributions. Lévy processes can be decomposed as the sum of three independent processes. One component is linear deterministic, the second a Brownian motion and the third a pure jump process.

Lévy processes have recently become an object of interest in finance modeling because they have diffusion-like and jump properties at the same time. In finance, as well as in insurance, this has been achieved by adding extra components into the model. In finance, large fluctuations are incorporated via a jump process and in insurance small fluctuations are incorporated via a diffusion. Lévy processes account for both types of structures.

These processes have been traditionally applied in risk theory as models for the aggregate claims process S. Compound Poisson process, Brownian motion and α-stable Lévy motion are examples of Lévy process commonly used in the actuarial literature. This approach can be extended to embrace more general Lévy processes. We believe that other Lévy processes have features that make them an object of interest in risk theory. General reference works on Lévy processes are by Bertoin [4] and Sato [5], for applications see Barndorff–Nielsen *et al*. [6] and Schoutens [7].

Lévy processes have already found their way in financial applications in the late 1980s and 1990s. Madan and Seneta [46] introduced the class of variance gamma (VG) distributions as a model for stock returns. The normal inverse Gaussian (NIG) distribution was proposed by Barndorff–Nielsen [8] (see also Barndorff–Nielsen [9] and Rydberg [10]). Eberlein and co-workers enlarged these distributions to define the Generalized Hyperbolic Lévy motion ([11–14]. See also Geman [15]). The CGMY model was introduced in Carr *et al*. [16] and the Meixner model was used in Schoutens [7]. For an overview of the use of Lévy processes in finance in connection with option pricing we refer to Schoutens [7]. Unifying approaches using stochastic differential equations and semimartingales to model risk processes have just been initiated recently (Sørensen [18] for instance). Our goal is to extend the use of Lévy processes in risk theory.

In the above mentioned stock models, a case has been made for the use of pure jump Lévy processes, the purely discontinuous feature of such processes accounts for the discrete nature of the real world. Diffusion with jumps had been long favored when it came to asset price modeling, however such an approach is being abandoned in favor of pure-jump Lévy processes (see LeBlanc and Yor [19] and Carr *et al*. [16]). In such processes the diffusion component is not present and an infinite number of small jumps drives its evolution.

Besides the pure jump features, other motivation for the use of some of the above mentioned Lévy processes in finance is the semi-heaviness of the tails of their marginal distributions. Semi-heaviness should be understood as linear decay of the tails of the logarithm of the density function. Semi-heavy tails fall somewhere between the tails of the normal distribution (its log density shows a quadratic decay) and the tails of α-stable distributions (power law decay). Grandell [20] and Furrer *et al*. [21] work out risk models with a Brownian motion and an α-stable process respectively.

Another interesting feature is that Lévy processes are semi-martingales. A remarkable result due to Monroe [22] is the fundamental characterization of all semimartingales. He shows that every semi-martingale can be written as a time-changed Brownian motion (possibly defined on a adequately extended probability space). For many of the popular Lévy processes, this time-change is explicitly available and independent of the Brownian motion involved.

Lévy processes can thus be seen as a Brownian motion running in business time instead of calendar time. Such a business time can be understood as an alternative time unit, as reckoned by a different clock, which evolves according to a random process. In finance, such a clock can be the traded volume or the number of trades of a particular asset. Moreover, this technique of a random time change has led to very attractive and adequate stochastic volatility models to describe the stock price behaviour (Carr *et al.* [23]). In the insurance context the time change could be the aggregate claims process of our portfolio. Empirical studies (see Chaubey *et al.* [24]) indicate that the inverse Gaussian distribution provides a good fit for aggregate claims. It turns out that an inverse Gaussian business time leads naturally to an NIG risk process, which we shall discuss in detail in Section 4.

We work with a general risk model along the lines of Sørensen [18],

$$U(t) = u + ct + Z(t) - S(t), \quad t \geqslant 0 \tag{2}$$

where u is the initial reserve, c is the constant loaded premium, Z is a process representing fluctuations in the risk premium, and S is the aggregate claims process. We focus on models where $X = Z - S = \{X(t) = Z(t) - S(t), t \geqslant 0\}$ is a (pure jump) Lévy process. The fact that X evolves by jumps captures the fact that changes in the premium rate and in the aggregate claim process are of discrete nature. As for the loaded premium c, it is natural to define it as $c = (1+\theta)\mathbb{E}(X_1)$, since it has to be greater than all random fluctuations to meet the net profit condition.

Very convenient choices for the process X are Meixner, VG, NIG and GH Lévy process. These choices yield tractable expressions for the ruin probabilities just as in the diffusion model of Grandell [20].

Section 2 recalls some properties of Lévy processes and give some explicit examples. Section 3 represents the main body of our discussion where we incorporate appealing features of Lévy processes into a general risk model. In this section, we also discuss our model as a time-changed diffusion. Section 4 discusses some specific examples like the NIG risk model, the VG model, the GH model and the Meixner model.

2. LÉVY PROCESSES

2.1. General theory

Suppose $\phi(u)$ is the characteristic function of a distribution. If for every positive integer n, $\phi(u)$ is also the nth power of a characteristic function, then we say that the distribution is infinitely divisible.

One can define for every such an infinitely divisible distribution a stochastic process, $X = \{X(t), t \geqslant 0\}$, called Lévy process, which starts at zero, has independent and stationary increments and such that the distribution of an increment over $[s, s+t]$, $s, t \geqslant 0$, i.e. $X(t+s) - X(s)$, has $(\phi(z))^{\mathrm{t}}$ as characteristic function: $\mathbb{E}[\exp(\mathrm{i}zX(t))] = (\phi(z))^{\mathrm{t}}$. The class of Lévy processes is in one-to-one correspondence with the class of infinitely divisible distributions.

Every infinitely divisible distribution generates a Lévy process and the increments of every Lévy process are infinitely divisible distributed.

Every Lévy process has a càdlàg modification which is itself a Lévy process. We always work with this càdlàg version of the process so that sample paths of a Lévy process are a.e. continuous from the right and have limits from the left.

The Lévy–Khintchine formula expresses the *characteristic exponent*, i.e. $\log\phi(z)$ of such an infinitely divisible distribution in the form

$$\log \phi(z) = \mathrm{i}\gamma z - \frac{\sigma^2}{2}z^2 + \int_{-\infty}^{+\infty} [\exp(\mathrm{i}zx) - 1 - \mathrm{i}zx\mathbb{1}_{\{|x|<1\}}(x)]\nu(\mathrm{d}x) \tag{3}$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geqslant 0$ and ν is a measure on $\mathbb{R}\backslash\{0\}$ with

$$\int_{-\infty}^{+\infty} \inf\{1, x^2\}\nu(\mathrm{d}x) = \int_{-\infty}^{+\infty} (1 \wedge x^2)\nu(\mathrm{d}x) < \infty$$

We say that our infinitely divisible distribution has a triplet of Lévy characteristics (or Lévy triplet for short) $[\gamma, \sigma^2, \nu(\mathrm{d}x)]$. The measure ν is called the *Lévy measure* of X. If the Lévy measure is of the form $\nu(\mathrm{d}x) = \rho(x)\mathrm{d}x$ we call $\rho(x)$ the Lévy density. The Lévy density has the same mathematical requirements as a probability density, except that it does not need to be integrable and must have zero mass at the origin.

If the Lévy process X is such that $\mathbb{E}(X_1) < \infty$ then (3) can be alternatively written as

$$\log \phi(z) = \mathrm{i}\gamma^* z - \frac{\sigma^2}{2}z^2 + \int_{-\infty}^{+\infty} (\exp(\mathrm{i}zx) - 1 - \mathrm{i}zx)\nu(\mathrm{d}x) \tag{4}$$

where the drift γ^* is the mean of the process (see Sato [5]). This is, if the process has a finite mean the Lévy–Khintchine characterization takes on a simpler form where the mean of the process appears as the drift term. This form (4) might seem more natural for a risk model. The new Lévy triplet is $[\gamma^*, \sigma^2, \nu(\mathrm{d}x)]$ with $\gamma^* = \gamma + \int_{\{|x|>1\}} x\nu(\mathrm{d}x) = \mathbb{E}(X_1)$.

From the Lévy–Khintchine formula, one sees that, in general, a Lévy process consists of three independent parts: a linear deterministic part, a Brownian part (if $\sigma^2 \neq 0$), and a pure jump part.

The Lévy measure $\nu(\mathrm{d}x)$ dictates how the jumps occur. The jumps of size in $\Lambda \in \mathbb{R}$ (provided that the closure $\bar{\Lambda}$ does not contain zero) form a compound Poisson process with rate $\int_\Lambda \nu(\mathrm{d}x)$ and jump density $\nu(\mathrm{d}x)/\int_\Lambda \nu(\mathrm{d}x)$.

The log-density of a Normal distribution has a quadratic decay. Time-series in finance show often an empirical log-density that seems to have a much more linear decay. This feature is typical for financial data and is often referred to as the semi-heaviness of the tails. We say that a distribution or its density function $f(x)$ has semi-heavy tails, if the tails of the density function behave as

$$\begin{aligned} f(x) &\sim C_-|x|^{\rho_-}\exp(-\eta_-|x|) \quad \text{as } x \to -\infty \\ f(x) &\sim C_+|x|^{\rho_+}\exp(-\eta_+|x|) \quad \text{as } x \to +\infty \end{aligned} \tag{5}$$

for some $\rho_-, \rho_+ \in \mathbb{R}$ and $C_-, C_+, \eta_-, \eta_+ \geqslant 0$. The Lévy processes that are discussed in this paper have semi-heavy tails.

2.2. *Examples*

2.2.1. *The Gamma process*

The density function of the Gamma distribution Gamma(a, b) with parameters $a > 0$ and $b > 0$ is given by

$$f_{\mathrm{Gamma}}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-xb), \quad x > 0$$

The density function clearly has a semi-heavy (right) tail. The characteristic function is given by

$$\phi_{\mathrm{Gamma}}(u; a, b) = (1 - \mathrm{i}u/b)^{-a}$$

Clearly, this characteristic function is infinitely divisible. The Gamma process $X^{(\mathrm{Gamma})} = \{X^{(\mathrm{Gamma})}(t), t \geqslant 0\}$ with parameters $a, b > 0$ is defined as the stochastic process which starts at zero and has stationary, independent Gamma distributed increments. The Lévy triplet of the Gamma process is given by

$$[a(1 - \exp(-b))/b, 0, a \exp(-bx)x^{-1} 1_{(x>0)}\, \mathrm{d}x]$$

2.2.2. *The VG process*

The characteristic function of the VG(σ, v, θ) law is given by

$$\phi_{VG}(u; \sigma, v, \theta) = (1 - \mathrm{i}u\theta v + \sigma^2 v u^2/2)^{-1/v}$$

The distribution is infinitely divisible and leads to the VG process $X^{(VG)}$. In Madan *et al.* [25], it was shown that the VG process may also be expressed as the difference of two independent Gamma processes.

This characterization allows the Lévy measure to be determined:

$$v_{VG}(\mathrm{d}x) = \begin{cases} C \exp(Gx)|x|^{-1}\, \mathrm{d}x & x < 0 \\ C \exp(-Mx)x^{-1}\, \mathrm{d}x & x > 0 \end{cases}$$

where

$$C = 1/v > 0$$

$$G = \left(\sqrt{\frac{\theta^2 v^2}{4} + \frac{\sigma^2 v}{2}} - \frac{\theta v}{2} \right)^{-1} > 0$$

$$M = \left(\sqrt{\frac{\theta^2 v^2}{4} + \frac{\sigma^2 v}{2}} + \frac{\theta v}{2} \right)^{-1} > 0$$

With this parameterization [we use the notation $VG(C, G, M)$], it is clear that $X^{(VG)}(t) = G \times (t)^{(1)} - G(t)^{(2)}$, where $G^{(1)} = \{G(t)^{(1)}, t \geqslant 0\}$ is a Gamma process with parameters $a = C$ and $b = M$, whereas $G^{(2)} = \{G(t)^{(2)}, t \geqslant 0\}$ is an independent Gamma process with parameters $a = C$ and $b = G$.

The Lévy measure has infinite mass, and hence a VG process has infinitely many jumps in any finite time interval. Since $\int_{-1}^{1} |x| v_{VG}(\mathrm{d}x) < \infty$, a VG process has paths of finite variation. A VG process has no Brownian component and its Lévy triplet is given by $[\gamma, 0, \nu_{VG}(\mathrm{d}x)]$, where

$$\gamma = \frac{-C(G(\exp(-M) - 1) - M(\exp(-G) - 1))}{MG}$$

Another way to define a variance gamma (VG) process is by seeing it as (gamma) time-changed Brownian motion with drift. More precisely, let $G = \{G(t), t \geqslant 0\}$ be a Gamma process with parameters $a = 1/\nu > 0$ and $b = 1/\nu > 0$. Let $W = \{W(t), t \geqslant 0\}$ denote a standard Brownian Motion, $\sigma > 0$ and let $\theta \in \mathbb{R}$, then the VG-gamma process $X^{(VG)} = \{X^{(VG)}(t), t \geqslant 0\}$ with parameters $\sigma > 0$, $\nu > 0$ and θ can alternatively be defined as

$$X^{(VG)}(t) = \theta G(t) + \sigma W(G(t))$$

When $\theta = 0$ then $G = M$ and the distribution is symmetric. Negative values of θ lead to the case where $G > M$ resulting in negatively skewness. The mean is given by $\theta = C(G - M)/(MG)$ and the variance is $\sigma^2 + \nu\theta^2 = C(G^2 + M^2)/(MG)^2$.

2.2.3. The NIG process

The NIG distribution with parameters $\alpha > 0$, $-\alpha < \beta < \alpha$ and $\delta > 0$, $\mathrm{NIG}(\alpha, \beta, \delta)$, has a characteristic function (see Barndorff–Nielsen [8]) given by

$$\phi_{\mathrm{NIG}}(u; \alpha, \beta, \delta) = \exp(-\delta(\sqrt{\alpha^2 - (\beta + \mathrm{i}u)^2} - \sqrt{\alpha^2 - \beta^2}))$$

One can clearly see that this is an infinitely divisible characteristic function. Hence we can define the NIG process $X^{(\mathrm{NIG})} = \{X^{(\mathrm{NIG})}(t), t \geqslant 0\}$: $X^{(\mathrm{NIG})}(t)$ has a $\mathrm{NIG}(\alpha, \beta, t\delta)$ law.

The Lévy measure for the NIG process is given by

$$\nu_{\mathrm{NIG}}(\mathrm{d}x) = \frac{\delta\alpha}{\pi} \frac{\exp(\beta x) \mathrm{K}_1(\alpha|x|)}{|x|} \mathrm{d}x \tag{6}$$

where $\mathrm{K}_\lambda(x)$ denotes the modified Bessel function of the third kind given by

$$K_\lambda(x) = \int_0^\infty u^{\lambda - 1} \mathrm{e}^{-(1/2)x(u^{-1} + u)} \, \mathrm{d}u, \qquad x > 0$$

A NIG process has no Brownian component and its Lévy triplet is given by $[\gamma_{\mathrm{NIG}}, 0, \nu_{\mathrm{NIG}}(\mathrm{d}x)]$, where

$$\gamma_{\mathrm{NIG}} = (2\delta\alpha/\pi) \int_0^1 \sinh(\beta x) K_1(\alpha x) \, \mathrm{d}x$$

Since there is no Gaussian constant, the NIG process is a pure jump process. The Lévy density $\rho(x) = (\delta/\pi)x^{-2} + o(x^{-2})$ as $x \downarrow 0$.

Since the NIG process has finite mean, we have the alternative Lévy triplet $[\gamma^*_{\mathrm{NIG}}, 0, \nu_{\mathrm{NIG}}(\mathrm{d}x)]$ where γ^*_{NIG} is the mean of the process given by

$$\gamma^*_{\mathrm{NIG}} = (\delta\beta)(\alpha^2 - \beta^2)^{-1/2}$$

In Figure 1 we can see different paths of NIG processes. Despite the apparent continuity, these paths are composed by an infinite number of small jumps.

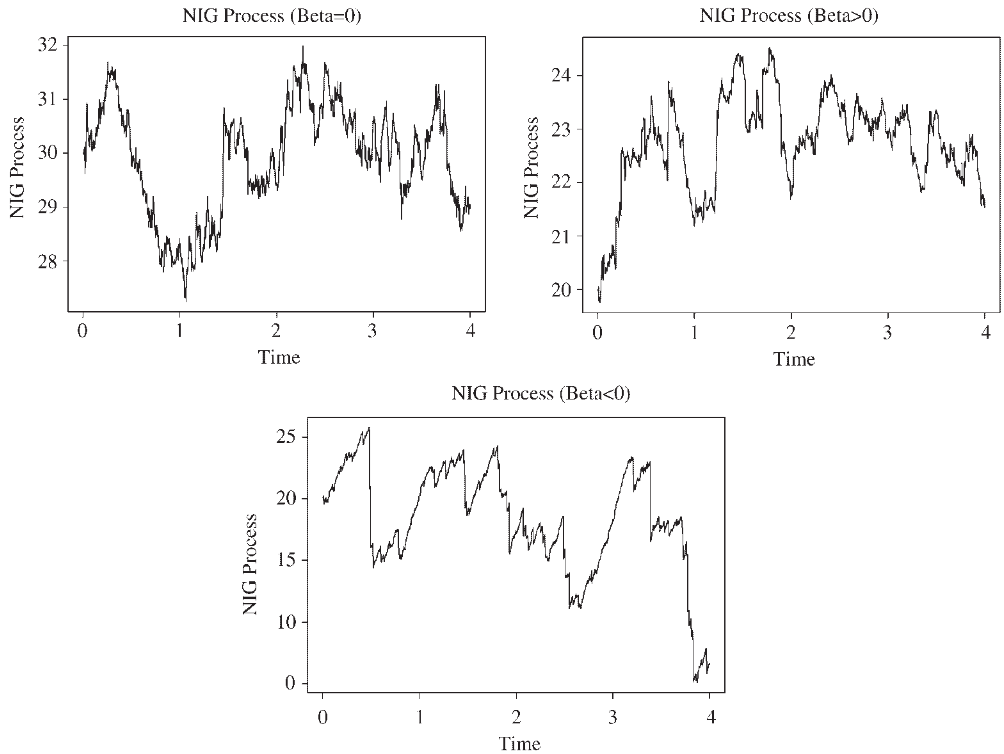

Figure 1. Simulated paths of a NIG Lévy processes for different values of β.

The density of the NIG(α, β, δ) distribution is given by

$$f_{\mathrm{NIG}}(x;\alpha,\beta,\delta)=\frac{\alpha\delta}{\pi}\exp(\delta\sqrt{\alpha^2-\beta^2}+\beta x)\frac{K_1(\alpha\sqrt{(\delta^2+x)^2})}{\sqrt{\delta^2+x^2}}$$

The mean and the variance are given respectively by $(\delta\beta)(\alpha^2-\beta^2)^{-1/2}$ and $(\delta\alpha^2)(\alpha^2-\beta^2)^{-3/2}$.

The NIG distribution was originally constructed in Barndorff–Nielsen [26] as a normal variance–mean mixture where the mixing distribution is an inverse Gaussian. This is, if X is a NIG distributed random variable then, the conditional distribution given $W = w$ is $N(\mu + \beta w, w)$ where W is inverse Gaussian distributed $\mathrm{IG}(\delta, \sqrt{\alpha^2-\beta^2})$, where an $\mathrm{IG}(a,b)$ law has a characteristic function

$$\phi_{\mathrm{IG}}(u;a,b)=\exp(-a(\sqrt{-2\mathrm{i}u+b^2}-b))$$

The IG distribution is infinitely divisible and leads to a IG process $X^{(\mathrm{IG})}=\{X^{(\mathrm{IG})}(t), t\geqslant 0\}$, with parameters $a, b > 0$. The density function of the $\mathrm{IG}(a,b)$ law is explicitly known:

$$f_{\mathrm{IG}}(x;a,b)=\frac{a}{\sqrt{2\pi}}\exp(ab)x^{-3/2}\exp(-(a^2x^{-1}+b^2x)/2), \quad x>0$$

See Jørgensen [27] for a reference on Inverse Gaussian distributions. Note that this gives a simple way of simulating NIG r.v., since there exist now standard techniques to simulate Normal and Inverse Gaussian variates (see for instance Devroye [28]).

Moreover, the variance–mean mixture representation leads to the construction of the NIG process via a time change (subordination). Let $W = \{W(t), t \geqslant 0\}$ be a standard Brownian motion and $I = \{I(t), t \geqslant 0\}$ a IG process with parameters $a = \delta$ and $b = \sqrt{\alpha^2 - \beta^2}$ then one can show that the stochastic process

$$X(t) = \beta I(t) + W(I(t))$$

is a NIG process with parameters α, β and δ.

2.2.4. The Meixner process

The density of the Meixner distribution [MEIXNER(α, β, δ)] is given by

$$f_{\text{Meixner}}(x; \alpha, \beta, \delta) = \frac{(2\cos(\beta/2))^{2\delta}}{2\alpha\pi\Gamma(2d)} \exp\left(\frac{bx}{a}\right) \left|\Gamma\left(\delta + \frac{\mathrm{i}x}{\alpha}\right)\right|^2$$

where $\alpha > 0, -\pi < \beta < \pi, \delta > 0$. It was introduced in Schoutens and Teugels [29] (see also Schoutens [30]).

The characteristic function of the MEIXNER(α, β, δ) distribution is given by

$$\phi_{\text{Meixner}}(u; \alpha, \beta, \delta) = \left(\frac{\cos(\beta/2)}{\cosh\frac{(\alpha u - \mathrm{i}\beta)}{2}}\right)^{2\delta}$$

The MEIXNER(α, β, δ) distribution is infinitely divisible and leads to the Meixner process $X^{(\text{Meixner})} = \{X^{(\text{Meixner})}(t), t \geqslant 0\}$. The distribution of $X^{(\text{Meixner})}(t)$ is given by the Meixner distribution MEIXNER$(\alpha, \beta, \delta t)$.

One can show (see Grigelionis [31]) that the process $X^{(\text{Meixner})} = \{X^{(\text{Meixner})}(t), t \geqslant 0\}$ has no Brownian part and a pure jump part governed by the Lévy measure

$$\nu(\mathrm{d}x) = \delta \frac{\exp(\beta x/\alpha)}{x \sinh(\pi x/\alpha)} \mathrm{d}x$$

The first parameter in the Lévy triplet equals

$$\gamma = \alpha\delta \tan(\beta/2) - 2\delta \int_1^\infty \sinh(\beta x/\alpha)/\sinh(\pi x/\alpha)\, \mathrm{d}x$$

Because $\int_{-1}^{+1} |x|\nu(\mathrm{d}x) = \infty$ the process is of infinite variation. Moments of all order of this distribution exist. The mean equals $\alpha\delta \tan(\beta/2)$ and the variance equals $\alpha^2\delta/2(\cos^{-2}(\beta/2))$.

The MEIXNER(α, β, δ) distribution has semiheavy tails (see Grigelionis [32]). We have in (5)

$$\rho_- = \rho_+ = 2\delta - 1, \quad \eta_- = (\pi - \beta)/\alpha, \quad \eta_+ = (\pi + \beta)/\alpha$$

The Meixner process is related to the process studied by Biane *et al.* [33] (see also Pitman and Yor [34])

$$\chi(t) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\Gamma_n(t)}{(n - \frac{1}{2})^2} \tag{7}$$

for a sequence of independent Gamma Processes (with $a = 1$ and $b = 1$) $\Gamma_n(t)$, i.e. a Lévy process with $\mathbb{E}[\exp(\mathrm{i}\theta\Gamma_n(t))] = (1 - \mathrm{i}\theta)^{-t}$.

In Biane *et al.* [33] one finds that $\chi(t)$ has characteristic function

$$\mathbb{E}[\exp(\mathrm{i}u\chi(t))] = \left(\frac{1}{\cosh\sqrt{-2u\mathrm{i}}}\right)^{\mathrm{t}}$$

Let $W = \{W(t), t \geqslant 0\}$ a standard Brownian motion, then the Brownian time change $W(\chi(t))$ has characteristic function

$$\mathbb{E}[\exp(\mathrm{i}uW(\chi(t)))] = \left(\frac{1}{\cosh u}\right)^{\mathrm{t}}$$

or equivalently $W(\chi(t))$ follows a MEIXNER$(2, 0, t)$ distribution.

2.2.5. The GH process

The generalized hyperbolic (GH) distribution GH$(\alpha, \beta, \delta, v)$ is defined in Barndorff–Nielsen [26] through its characteristic function:

$$\phi_{\mathrm{GH}}(u; \alpha, \beta, \delta, v) = \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + \mathrm{i}u)^2}\right)^{v/2} \frac{\mathrm{K}_v(\delta\sqrt{\alpha^2 - (\beta + \mathrm{i}u)^2})}{\mathrm{K}_v(\delta\sqrt{\alpha^2 - \beta^2})}$$

where K_v is the modified Bessel function. The GH distribution is a generalization of the NIG case: $\mathrm{GH}(\alpha, \beta, \delta, -1/2) = \mathrm{NIG}(\alpha, \beta, \delta)$.

The density of the GH$(\alpha, \beta, \delta, v)$ distribution is given by

$$f_{\mathrm{GH}}(x; \alpha, \beta, \delta, v) = a(\alpha, \beta, \delta, v)(\delta^2 + x^2)^{(v-(1/2))/2}\mathrm{K}_{v-(1/2)}(\alpha\sqrt{\delta^2 + x^2}) \exp(\beta x)$$

$$a(\alpha, \beta, \delta, v) = \frac{(\alpha^2 - \beta^2)^{v/2}}{\sqrt{2\pi}\alpha^{v-(1/2)}\delta^v \mathrm{K}_v(\delta\sqrt{\alpha^2 - \beta^2})}$$

where

$$\delta \geqslant 0, \quad |\beta| < \alpha \quad \text{if } v > 0$$

$$\delta > 0, \quad |\beta| < \alpha \quad \text{if } v = 0$$

$$\delta > 0, \quad |\beta| \leqslant \alpha \quad \text{if } v < 0$$

The GH distribution turns out to be infinitely divisible (see Barndorff–Nielsen and Halgreen [35]) and one can define a GH Lévy process $X^{(\mathrm{GH})} = \{X^{(\mathrm{GH})}(t), t \geqslant 0\}$ as the stationary process which starts at zero and has independent increments and where the distribution of $X^{(\mathrm{GH})}(t)$ has characteristic function

$$\mathbb{E}[\exp(\mathrm{i}uX^{(\mathrm{GH})}(t))] = (\phi_{\mathrm{GH}}(u; \alpha, \beta, \delta, v))^{\mathrm{t}}$$

Note that the GH process is in general not closed under convolution, only in the case $v = -1/2$—the NIG case-processes is closed under convolution. The Lévy measure $\nu(\mathrm{d}x)$ for the GH Process is rather involved and we refer to e.g. Schoutens [7] for the explicit expression.

The GH distributions have semiheavy tails, in particular

$$f_{\mathrm{GH}}(x;\alpha,\beta,\delta,v) \sim |x|^{v-1}\exp((\mp\alpha+\beta)x) \quad \text{as } x\to\pm\infty$$

up to a multiplicative constant. Note that, being a special case of the GH, the NIG distribution also has semi-heavy tails.

The GH distribution has mean $\beta\delta(\alpha^2-\beta^2)^{-1}\mathrm{K}_{v+1}(\zeta)\mathrm{K}_v^{-1}(\zeta)$ and variance

$$\delta^2\left(\frac{\mathrm{K}_{v+1}(\zeta)}{\zeta\mathrm{K}_v(\zeta)}+\frac{\beta^2}{\alpha^2-\beta^2}\left(\frac{\mathrm{K}_{v+2}(\zeta)}{\mathrm{K}_v(\zeta)}-\frac{\mathrm{K}_{v+1}^2(\zeta)}{\mathrm{K}_v^2(\zeta)}\right)\right) \quad \text{where } \zeta=\delta\sqrt{\alpha^2-\beta^2}$$

The GH distributions can also be represented as a normal variance–mean mixture:

$$f_{\mathrm{GH}}(x;\alpha,\beta,\delta,v)=\int_0^\infty f_{\mathrm{Normal}}(x;\mu+\beta w,w)f_{\mathrm{GIG}}(w;v,\delta,\sqrt{\alpha^2-\beta^2})\,\mathrm{d}w$$

were the Generalized Inverse Gaussian distribution $\mathrm{GIG}(\lambda,a,b)$ is given in terms of its density function:

$$f_{\mathrm{GIG}}(x;\lambda,a,b)=\frac{(b/a)^\lambda}{2\mathrm{K}_\lambda(ab)}x^{\lambda-1}\exp(-(a^2x^{-1}+b^2x)/2), \quad x>0$$

The parameters λ, a and b are such that $\lambda\in\mathbb{R}$ while a and b are both non-negative and not simultaneously 0.

The characteristic function is given by

$$\phi_{\mathrm{GIG}}(u;\lambda,a,b)=\frac{1}{\mathrm{K}_\lambda(ab)}(1-2\mathrm{i}u/b^2)^{\lambda/2}\mathrm{K}_\lambda(ab\sqrt{1-2\mathrm{i}ub^{-2}})$$

where $\mathrm{K}_\lambda(x)$ denotes the modified Bessel function of the third kind with index λ. For $\lambda=-1/2$, the GIG reduce to the IG distribution: $\mathrm{GIG}(-1/2,a,b)=\mathrm{IG}(a,b)$. It was shown by Barndorff–Nielsen and Halgreen [35] that this distribution is also infinitely divisible.

Like for the NIG distribution, the variance–mean mixture representation leads to the construction of the GH process via a time change (subordination). Let $W=\{W(t),t\geqslant 0\}$ be a standard Brownian motion and $I=\{I(t),t\geqslant 0\}$ a GIG process with parameters $v=\lambda$, $a=\delta$ and $b=\sqrt{\alpha^2-\beta^2}$ then one can show that the stochastic process

$$X(t)=\beta I(t)+W(I(t))$$

is a GH process with parameters α,β, δ and λ.

2.3. Introducing a drift parameter

In the above VG, NIG, Meixner and GH cases an additional 'drift' or location parameter $m\in\mathbb{R}$ can be introduced. This parameter will be linked with the premium rate c. Essentially, the transformation is completely of the same manner as the one which transforms a $\mathrm{Normal}(0,\sigma^2)$ random variable into a $\mathrm{Normal}(m,\sigma^2)$ random variable. Moreover, this extension does not influence the infinite divisibility property. Only in this section we will denote the original process with $\bar{X}$ and the newly obtained one with X. The same notation will be used for the characteristic function, and the ingredients of the Lévy triplet.

The (extended) distribution in the Meixner case, is denoted by $\mathrm{MEIXNER}(\alpha,\beta,\delta,m)$. For the others the new parameter will give rise to distributions which we denote by $\mathrm{VG}(\sigma,v,\theta,m)$ [or $\mathrm{VG}(C,G,M,m)$], $\mathrm{NIG}(\alpha,\beta,\delta,m)$, $\mathrm{GH}(\alpha,\beta,\delta,v,m)$.

The new distribution has a characteristic function ϕ in terms of the original characteristic function $\bar{\phi}$:

$$\phi(u) = \bar{\phi}(u)\exp(\mathrm{i}um)$$

This new parameter is just a shifting with the value $m \in \mathbb{R}$ of the distribution. In terms of the process this means a term mt is added to the process $\bar{X}$, i.e.

$$X(t) = \bar{X}(t) + mt \tag{8}$$

This is reflected only in the first parameter of the Lévy triplet which now equals

$$\gamma = \bar{\gamma} + m$$

$$\sigma^2 = \bar{\sigma}^2$$

$$\nu(\mathrm{d}x) = \bar{\nu}(\mathrm{d}x)$$

In terms of density functions this yields

$$f(x) = \bar{f}(x - m)$$

3. LÉVY RISK MODELS

3.1. The construction of the Lévy risk process as a transformed diffusion approximation

Grandell [20] constructed a sequence of risk reserve processes $\{U_n\}_{n=1,2,\ldots}$ of the form

$$U_n(t) = u_n + \kappa_n t - \sigma_n S_n(t), \quad t \geqslant 0$$

where u_n, $\kappa_n = \beta_n \rho_n$ are the corresponding sequences of initial reserves and premium rates such that $u_n \to u$, $\sigma_n^2 \to \sigma^2$ and $\kappa_n \to \kappa = \beta\rho$. The aggregate claim processes S_n is compound Poisson with mean β_n and variance σ_n^2. He showed that for claim sizes in the domain of attraction of the normal distribution, the sequence $\{U_n\}_{n=1,2,\ldots}$ converges weakly in the Skorohod topology to the diffusion process

$$U_D(t) = u + \kappa t - \sigma W(t), \quad t > 0 \tag{9}$$

as $n \to \infty$, where W is a standard Brownian motion.

Suppose we have a Lévy process X that is a time-changed Brownian motion with drift. More precisely, assume

$$X(t) = \tilde{\kappa}\tau(t) - \tilde{\sigma} W(\tau(t))$$

where $W(t)$ is standard Brownian motion and $\tau(t)$ is an independent subordinator (a nondecreasing Lévy process). Recall that in Section 2, we encounter different examples of this situation.

We can consider the following generalization of (9) via the subordinator τ:

$$\tilde{U}_D(t) = U_D(\tau(t)) = u + \kappa\tau(t) - \sigma W(\tau(t)), \quad t > 0 \tag{10}$$

Then the process $\tilde{U}_D(t) - u$ in (10) is the Lévy process $X(t)$ with $\tilde{\kappa} = \kappa$ and $\tilde{\sigma} = \sigma$.

If we wish to incorporate a drift parameter c we go a step further in the generalization yielding

$$\begin{aligned} U(t) &= ct + \tilde{U}_D(t) \\ &= u + ct + X(t) \\ &= u + ct + \kappa\tau(t) - \sigma W(\tau(t)), \quad t \geqslant 0 \end{aligned} \tag{11}$$

Note the complete similarity with (8). This last process (11) is the risk model that we will focus on. Note that the process in (11) is a transformation of the diffusion approximation of Grandell [20], it is a still a diffusion but operating in *business time*. The subordinator τ is as a random time transformation that accounts for different speeds at which the market evolves. In a way, the business does not flow continuously, but by an (often infinite) number of jumps of (possibly) different lengths which are represented by the subordinator τ.

Moreover, (11) is a generalization of (9). If we set $\tau(t) = t$, the business time flows just like regular time and we recover the diffusion risk process. We will see in Section 4 that if τ is an inverse Gaussian subordinator we obtain the so-called NIG model. Other choices of τ lead to the VG, the Meixner and the GH model (see Section 4). Note also that the diffusion model is also of the form (11) and that we can recuperate the α-stable model, if τ is an $\alpha/2$-stable subordinator and the drift term $\beta = 0$, then (11) is the α-stable model of Furrer *et al.* [21] (see Sato [5] and Cherny and Shiyaev [36] for a reference on semimartingales as a time-changed diffusions].

In finance, the time change τ is often referred to as the clock of the process (see Ané and Geman [37]). For instance, if $\tau(t) = t$ then the process runs on a calendar clock. Other subordinators τ are used to model different clocks, for instance τ might be the traded volume of a particular assets which reflects the business activity. In insurance, when talking about a clock for the process (11), the natural analogy for traded volume will be the aggregate claims process. This is the measure of insurance business activity, it represents the total claims filed up to time t.

3.2. *A purely discontinuous risk model*

Consider a general risk model as in (2)

$$U(t) = u + ct + Z(t) - S(t), \quad t \geqslant 0 \tag{12}$$

we now model $Z - S$ by one single pure jump Lévy process X with Lévy triplet $[\gamma_X, 0, \nu_X]$. This allows for great flexibility, moreover, it allows us to have the same features than the model of Dufresne and Gerber [38] in one single object instead of considering two different processes.

This reasoning leads to the following characteristic exponent for the process $U - u$:

$$\begin{aligned} \log \mathbb{E}[\exp(\mathrm{i}z(U-u))] &= \psi(z) \\ &= (c + \gamma_X)\mathrm{i}z + \int_{-\infty}^{\infty} \left[\mathrm{e}^{\mathrm{i}zx} - 1 - \mathrm{i}zx\mathbb{1}_{\{(-1,1)\}}(x)\right]\nu_X(\mathrm{d}x) \end{aligned}$$

The moment generating function $\vartheta(z)$ of $U - u$ is then given by

$$\vartheta(z) = \exp(\psi(-\mathrm{i}z))$$

which we assume to be finite for $z \in (-b_1, b2)$ ($b_1, b_2 > 0$ possibly infinity). Denote by $\Psi(z) = \log \vartheta(z)$, the corresponding Laplace exponent.

By choosing a Lévy process X with semi-heavy tails, we can model the aggregate claims with a compound Poisson process S having a medium-tailed claim size distribution of the form $C|x|^{-\rho}\mathrm{e}^{-\lambda x}$. As for the small positive and negative discrete fluctuations we incorporate them in a pure jump process Z.

Indeed, we can decompose X into Z and S in the following way. $-S$ is a compound Poisson process with Lévy triplet $[0, 0, v_{-S}(\mathrm{d}x)]$, where the Lévy measure $v_{-S}(\mathrm{d}x) = \mathbb{1}_{\{(-d,-\infty)\}}(x)v_X(\mathrm{d}x)$. The splitting point $d > 0$ can be seen as a level of retention and can be conveniently set to an arbitrary unit. The jumps (claims) follow a medium-tailed distribution of the form $C|x|^{-\rho}\mathrm{e}^{-\lambda x}$. Consequently, the Lévy triplet of Z is $[\gamma_X, 0, v_{-Z}(\mathrm{d}x)]$, where $v_Z(\mathrm{d}x) = \mathbb{1}_{\{(-d,\infty)\}}(x)v_X(\mathrm{d}x)$. The process Z would represent positive and small negative discrete fluctuations. The fact that we only allow Z to have small negative or positive jumps can be justified as follows: Since U is a risk reserve process we assume that the company has some seriously conservative policies that regulate any source of premium fluctuations. Investments and other source of fluctuations in the premiums are constrained to be minimum as to assure enough cash to face claims. Therefore there should not be large negative fluctuations coming from the process Z. This reasoning leads to the following characteristic exponent for the process $U - u$:

$$\Psi(z) = (c + \gamma_X)z + \int_{-d}^{\infty} [\mathrm{e}^{zx} - 1 - zx\mathbb{1}_{\{(-1,1)\}}(x)]v_Z(\mathrm{d}x) + \int_{-\infty}^{-d} [\mathrm{e}^{zx} - 1 - zx\mathbb{1}_{\{(-1,1)\}}(x)]v_S(\mathrm{d}x) \tag{13}$$

The loaded premium c has to satisfy $c > -\mathbb{E}(X_1)$ to meet the net profit condition.

Alternatively, if the process $U - u$ has finite mean we can rewrite (13) as

$$\Psi(z) = (c + \gamma_X^*)z + \int_{-d}^{\infty} [\mathrm{e}^{zx} - 1 - zx]v_Z(\mathrm{d}x) + \int_{-\infty}^{-d} [\mathrm{e}^{zx} - 1 - zx]v_S(\mathrm{d}x)$$

where $c + \gamma_X^*$ is the mean of the process $U - u$.

3.3. *A discontinuous risk model of finite variation*

If we start with a Lévy process X with path of finite variation, then we can write X as the difference of two increasing processes: $X = Z - S$. Negative jumps in X correspond to positive jumps in the aggregate claims process S and are interpreted as claims to be paid. The positive jumps of X are brought into Z and model the positive fluctuations in the premium process. An example of this situation is the VG process. Recall that a VG process can be decomposed into a difference of two Gamma processes. Note also that the Gamma process has also been proposed as a model for the aggregate claims (see Dufresne *et al.* [39]). Our model is then a generalization of this classical model that incorporates positive (gamma distributed) fluctuations in the premium process. Another example could be the CGMY process described in Carr *et al.* [16]. This four parameter model is of finite variation if the Y parameter is smaller than 1 and reduces to the VG case for $Y = 0$.

3.4. Claim rate

A Lévy process carries along its path some information from the distribution. Assume that our Lévy process has a finite second moment and a Lévy density $\rho(x)$ with asymptotic behavior $\rho(x) = ax^{-2} + o(x^{-2})$ as $x \downarrow 0$. For a fixed time $t > 0$, let

$$N_k = \frac{1}{kt}\#\{s \leqslant t, \Delta X_s > 1/k\}, \quad k = 1, 2, \ldots$$

Raible [40] proved that the sequence $(N_k, k = 1, 2, \ldots)$ converges to the value a. We can draw some analogies with the classical risk model: in a way a plays the role of the claim rate λ, it is the limit of the normalized number of jumps larger than $1/k$.

3.5. The martingale approach and ruin probabilities

The martingale approach of Schmidli [41] and Sørensen [18] can be applied to the risk model as in (12) to work out expressions for the ruin probability. We refer to Grandell [1] for a review of martingale methods in risk theory.

From the general theory of processes with independent increments (see Jacod and Shiryaev [42]) we have that, if X is a Lévy process with Laplace exponent Ψ, then the process

$$M^r(t) = \frac{e^{-rX(t)}}{e^{t\Psi(r)}}, \quad r \in \mathbb{R} \tag{14}$$

is a local martingale for r within the right domain. For a link of this martingale relation for Lévy process and orthogonal polynomials see Schoutens [30]. The finite dimensional distributions of M^r are the Esscher transforms of the finite dimensional distributions of X. We can define the new measure $\mathbb{Q}^r$ in the following way:

Definition 3.1

Let $X = \{X(t)\}_{t \geqslant 0}$ be a Lévy process on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, \mathbb{P})$. We define an Esscher transform as any change of $\mathbb{P}$ to a locally equivalent measure $\mathbb{Q}$ with a density process $M(t) = d\mathbb{Q}^r/d\mathbb{P}|_{\mathscr{F}_t}$ of the form (14) for some r in the domain of definition of the Laplace exponent Ψ.

In other words, this means that, for any $A \in \mathscr{F}_t$, the new measure $\mathbb{Q}^r$ is given by

$$\mathbb{Q}^r_t[A] = \mathbb{E}_{\mathbb{P}}[M^r(t); A]$$

Now, recall that the ruin probability is defined as

$$\psi(u) = \mathbb{P}\{\tau < \infty\}, \quad u \geqslant 0 \tag{15}$$

where $\tau = \inf[t > 0 \colon U(t) < 0]$ is the first time the process falls below zero. Since $U - u$ is a Lévy process we can write its associated ruin probability ψ in terms of the new measure as follows:

$$\psi(u) = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\tau<\infty\}}] = \mathbb{E}_{\mathbb{Q}^r}\left[\frac{1}{M^r(\tau)}\mathbb{1}_{\{\tau<\infty\}}\right] = \mathbb{E}_{\mathbb{Q}^r}[e^{\{r(U(\tau)-u)+\tau\Psi(r)\}}\mathbb{1}_{\{\tau<\infty\}}]$$

If we can find a value $r = R$ in the domain of definition of the Ψ such that $\Psi(R) = 0$ and $\tau < \infty$ a.s (under $\mathbb{Q}^R$), we could simply write

$$\psi(u) = \mathbb{E}_{\mathbb{Q}^R}[e^{RU(\tau)}]e^{-Ru}, \quad u \geqslant 0 \tag{16}$$

A first concern is how an Esscher transformation affects the Lévy process. It turns out that if ϕ is the characteristic function and $[\gamma, \sigma^2, \nu(dx)]$ is the Lévy triplet of X, then the characteristic function $\phi^{(r)}$ of the Esscher transformed measure is given by

$$\log \phi^{(r)}(u) = \log \phi(u + ir) - \log \phi(ir)$$

Moreover this law remains infinitely divisible and its Lévy triplet $[\gamma^{(r)}, (\sigma^{(r)})^2, \nu^{(r)}(dx)]$ is given by

$$\gamma^{(r)} = \gamma - \sigma^2 r + \int_{-1}^{1} (\exp(-rx) - 1)\nu(dx)$$

$$\sigma^{(r)} = \sigma$$

$$\nu^{(r)}(dx) = \exp(-rx)\nu(dx)$$

The Meixner process: If we have under $\mathbb{P}$ a MEIXNER$(\alpha, \beta, \delta, m)$ process X, then we have under the Esscher-based $\mathbb{Q}^r$ X will be a MEIXNER$(\alpha, \beta - \alpha r, \delta, m)$ process (see Grigelionis [31] and Schoutens [17]).

The NIG process: If we have under $\mathbb{P}$ our process is a NIG$(\alpha, \beta, \delta, m)$ process, then we have under $\mathbb{Q}^r$ a NIG$(\alpha, \beta - r, \delta, m)$ process.

The VG process: If we have under $\mathbb{P}$ our process is a VG(C, G, M, m) process, then we have under $\mathbb{Q}^r$ a VG$(C, G - r, M + r, m)$ process.

The GH process: This case is rather complicated. However the Esscher transform can be obtained numerically by applying Fourier inversion techniques. We refer to Prause [43].

This martingale approach defines a straight forward simulation scheme to evaluate ruin probabilities in the case R exists. This imposes a certain condition on the loaded premium. Although it might seem restrictive at first, such condition depends on other parameters that can be adjusted to allow for greater flexibility.

If the adjustment coefficient R exists then we can simulate the risk process U under the Esscher-induced change of measure. Under this measure the stopping rule for the simulated paths $\tau = \inf\{t > 0 | U(t) \leqslant 0\}$ is well defined since τ is finite $\mathbb{Q}^R$-almost surely. Then, for each path we evaluate the expression $e^{R[U(\tau) - u]}$ and we average over all simulated paths.

In order to simulate a Lévy process, we can exploit the well-known compound Poisson approximation of this process. The procedure has been suggested on intuitive grounds for some particular cases in Rydberg [10]. Further insight can be found in Asmussen and Rosinski [44]. For very specific processes, other techniques are available. This is for example the case if the Lévy process can be represented explicitly as a time-changed Brownian motion and techniques are already available to simulate the simpler subordinator (the time change). Sampling a path from the Lévy process, can be done by sampling first a path from the subordinator and the Brownian motion. The Lévy process is then obtained via a time transformation.

Altogether, we have shown that a Lévy process can be the basis of a risk reserve process for which classical ruin theory results still hold. Its counterintuitive but appealing features make it a model that accounts for the discrete (infinite) activity of the real world. By discrete *infinite* activity we refer to the fact that the evolution of the process seems to be governed by discrete jumps occurring at an infinite rate.

4. EXAMPLES

At this point we make some remarks about the Lévy measure that are important to our risk model. First, notice that if ν has infinite mass around the origin, as in the NIG, Meixner, GH and VG case. The process is composed of an infinite number of small jumps. In the scope of Section 3.2, this pure discontinuity will account for discrete changes in the constant premium. Moreover, we have in these cases that, the jump size distribution has medium or exponentially decaying tails, i.e. the jump size density behaves as $C|x|^{\rho}e^{-\lambda|x|}$, for $x \to \pm\infty$ and some real constants $C > 0$, ρ and λ. This makes a risk process driven by a such a Lévy motion a good model for medium tailed claims since the jump size distribution will account for claims over a certain threshold.

Next, we will consider some explicit examples of Lévy processes that can be used as risk models. We work out the calculation for the ruin probabilities for the NIG-case. Similar calculations along the same lines can be made for the other cases.

4.1. The NIG risk model

If we set in (11) $\kappa = \beta$ and $\sigma = 1$ and take for the subordinator $\tau(t)$ an IG process with parameters $a = \delta$ and $b = \gamma = \sqrt{\alpha^2 - \beta^2}$, we have that $U - u$ follows an NIG$(\alpha, \beta, \delta, c)$ process.

In terms of our model this means that the NIG risk model is the diffusion model of Grandell [20] distorted by a randomly changing business time. The inverse Gaussian (IG) distribution is traditionally used to model aggregate claims (see Chaubey *et al.* [24]). This thus leads to the so-called NIG risk model which is of the form (11). The NIG risk model models the risk reserves with a Brownian motion. However, this Brownian motion does not run in calendar time but in an IG business clock.

Figure 1 also illustrates the role played by the parameter β in the Lévy measure (6). By changing the sign of β we control the weight of positive versus negative jumps, this is because the only element in (6) affected by its sign is the exponential term. A parameter $\beta = 0$ implies a symmetric NIG distribution and hence an equilibrium between positive and negative jumps in the corresponding process. A value of $\beta > 0$ would induce larger and more frequent positive jumps. And not only that, the disparity between positive and negative jumps grows exponentially, i.e. positive jumps would be exponentially larger and more frequent than negative ones, which would be exponentially smaller and less frequent. The opposite applies for $\beta < 0$.

Thanks to the role of β we can incorporate aggregate claims and premium perturbations in the same process. By setting $\beta < 0$ for instance, we can obtain larger jumps downwards that represent the claims along with small negative and positive jumps accounting for premium fluctuations.

Let $\mathbb{Q}^r$ be an equivalent probability measure defined through the Esscher transform as in Definition 3.1. Looking at the Lévy density, we see that a NIG process remains a NIG process under an Esscher transform. Then, a NIG process X under $\mathbb{Q}^r$ is again a NIG Lévy process with parameter $\tilde{\beta} = \beta - r$.

Now, since for NIG processes stays a NIG under an Esscher change of measure with the parameter β acting exactly as the Esscher parameter r we can write the ruin probability ψ as in (16) as stated in the following result:

Proposition 4.1

Let U be a risk reserve process driven by a NIG Lévy process as in (12). Let $\gamma^2 = \alpha^2 - \beta^2$. Its associated ultimate ruin probability ψ satisfies:

(i) If $-\beta\delta/\gamma \leqslant c \leqslant \delta\gamma/(\alpha+\beta)$ then

$$\psi(u) = \mathbb{E}_{\mathbb{Q}^R}[\mathrm{e}^{RU(\tau)}]\mathrm{e}^{-Ru}, \quad u \geqslant 0$$

for $R = 2(\beta + \gamma(c/\delta))/1 + (c/\delta)^2$ and where $\mathbb{Q}^R$ is an equivalent measure induced by an Esscher transform with parameter R.

(ii) If $c > \delta\gamma/(\alpha+\beta)$ then

$$\psi(u) \leqslant \frac{\mathrm{e}^{-r^* u}}{\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{-\tau\Psi(r^*)}|\tau<\infty]}, \quad u \geqslant 0$$

for some r^* in such that $\beta - \alpha < r^* < \alpha + \beta$, where $\mathbb{P}$ is the original measure.

Proof

(i) We had already stated that, for $|\beta - r| < \alpha$,

$$\psi(u) = \mathbb{E}_{\mathbb{Q}^r}[\mathrm{e}^{\{r(U(\tau)-u)+\tau\Psi(r)\}}\mathbb{I}_{\{\tau<\infty\}}], \quad u \geqslant 0 \tag{17}$$

Now, if there exist a value $r = R$ in the domain of definition of Ψ such that $\Psi(R) = 0$ we would have half of the proof. Notice that, for a NIG distribution, the Laplace exponent Ψ is finite for all r in the domain of definition, including the endpoints. This implies a restriction in the possible values for c to insure that such a number R exists. This can be seen if we solve the equation $\Psi(r) = 0$, which implies the following relation:

$$cr = \delta(\gamma - \gamma_r), \quad -\alpha + \beta \leqslant r \leqslant \alpha + \beta \tag{18}$$

where $\gamma_r^2 = \alpha^2 - (\beta - r)^2$

The function in the right-hand side takes the value $\delta\gamma$ at both endpoints. If a positive solution R is to exist, the line cr should intersect the curve $\delta(\gamma - \gamma_r)$ at a lower point than $\delta\gamma$. This is, $c(\alpha+\beta) \leqslant \delta\gamma$ which implies the upper bound for c in the proposition.

Now, if we take derivatives we can see that the function Ψ is a convex function and that $\Psi'(0) = -(c + (\beta\delta)/\gamma)$. Since we assume that $c + (\beta\delta)/\gamma > 0$ to meet the net profit condition, we have that a positive solution R would exist as long as $-(\beta\delta)/\gamma \leqslant c \leqslant \delta\gamma/(\alpha+\beta)$. Solving (18) yields $R = 2(\beta + \gamma(c/\delta))/1 + (c/\delta)^2$ which is only positive and well defined in the specified range.

In order to complete the proof we have to show that, under $\mathbb{Q}^R$, $\tau < \infty$ almost surely. Since the process $U - u$ is still a NIG Lévy process, if we compute $\mathbb{E}_{\mathbb{Q}^R}[U(1) - u]$ we find

$$\mathbb{E}_{\mathbb{Q}^R}[U(1) - u] = c + \frac{(\beta - R)\delta}{\sqrt{\alpha^2 - (\beta - R)^2}} = -\Psi'(R)$$

Now, since Ψ is convex and strictly increasing on the positive axis, we have that $-\Psi'(R) < 0$ which implies that, under the measure $\mathbb{Q}^R$, the process $U - u$ drifts away to $-\infty$ and $\tau < \infty$ $\mathbb{Q}$-almost surely and $\mathbb{I}_{\{\tau<\infty\}} = 1$.

(ii) If $c > \delta\gamma/(\alpha+\beta)$ we have that there is no solution R in the domain of definition of the Laplace exponent such that $\Psi(R) = 0$. However, if we use the fact that M^r is a martingale, and therefore supermartingale, we have for $t > 0$,

$$1 \geqslant \mathbb{E}_{\mathbb{P}}[M^r(t \wedge \tau)] \geqslant \mathbb{E}_{\mathbb{P}}[M^r(t)|\tau < t]\mathbb{P}(\tau < t), \quad |\beta - r| < \alpha$$

This last equation follows from the optional stopping theorem and the fact that $M^r(0) = 1$. If we substitute the expression for M^r and we get

$$\mathbb{P}(\tau < t) \leqslant \frac{\mathrm{e}^{-ru}}{\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{-rU(\tau)-\tau\Psi(r)}|\tau < t]} \leqslant \frac{\mathrm{e}^{-ru}}{\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{-\tau\Psi(r)}|\tau < t]}, \quad u \geqslant 0$$

This last inequality comes from the fact that $U(\tau) < 0$ conditioned on the set $\tau < \infty$. Now, via Jensen's inequality and the previously established fact that $\Psi'(0) < 0$ because of the net profit condition, we have that there exists a point r^* in the domain of the function Ψ such that $\mathrm{e}^{-ru}/\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{-\Psi(r)}|\tau < t]$ attains its minimum. If we let $t \to \infty$, this leads to the form that has become standard in the actuarial literature (Grandell [1]):

$$\psi(u) \leqslant \frac{\mathrm{e}^{-r^*u}}{\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{-\tau\Psi(r^*)}|\tau < \infty]}, \quad u \geqslant 0 \qquad \square$$

Notice that from Proposition 4.1 we can recover Lundberg's inequality for the NIG risk process

$$\psi(u) \leqslant \mathrm{e}^{-\mathscr{R}u}$$

where $\mathscr{R} = \sup\{r|\Psi(r) \leqslant 0\}$. In the first case of Proposition 4.1 this yields the exact adjustment coefficient R for which $\Psi(R) = 0$. In the second case this yields r^*.

4.2. *Other examples*

4.2.1. *The Meixner risk model*

If we set in (11) $\kappa = 0$ and $\sigma = 1$ and take for the subordinator $\tau(t)$ the process χ_t as in (7), we have that $U - u$ follows an Meixner$(2, 0, 1, c)$ process. Introducing skewness (via β) and making the other parameters general leads to a MEIXNER$(\alpha, \beta, \delta, c)$. Next, we look for the $r = R \in ((\beta - \pi)/\alpha, (\beta + \pi)/\alpha)$ leading to $\Psi(R) = 0$. Note that Ψ is convex and goes to infinity in the boundary points $(\beta \pm \pi)/\alpha$. Note that $\Psi(0) = 0$. After some calculations one obtains that Ψ reaches its minimum value

$$2\delta \log\left(\cos(\beta/2)\sqrt{\frac{c^2}{\delta^2\alpha^2} + 1}\right) - \frac{c(\beta + 2\arctan(c/(\delta\alpha))}{\alpha}$$

in

$$\frac{\beta + 2\arctan(c/(\delta\alpha))}{\alpha}$$

If this value is positive, this would imply positive solution $R \in (0, (\beta + \pi)/\alpha)$. If the value is negative there is solution $R \in ((\beta - \pi)/\alpha, 0)$. One then can proceed as in the first part of Proposition 4.1. Having a positive R, one does not have to impose in this case the upper bound on c. The exact value of R can easily be computed numerically.

4.2.2. The GH risk model

If we set in (11) $\kappa = \beta$ and $\sigma = 1$ and take for the subordinator $\tau(t)$ a GIG-process with parameters λ, $a = \delta$ and $b = \gamma = \sqrt{\alpha^2 - \beta^2}$, we have that $U - u$ follows an GH$(\lambda, \alpha, \beta, \delta, c)$ process.

4.2.3. The VG risk model

If we set in (11) $\kappa = \theta$ and $\sigma = \tilde{\sigma}$ and take for the subordinator $\tau(t)$ a Gamma process with parameters $a = 1/v$ and $b = 1/v$, we have that $U - u$ follows a VG$(\tilde{\sigma}, v, \theta, c)$ process. Next, we look for the $r = R$ leading to $\Psi(R) = 0$. We do this in the CGM parameterization. We look for a $r = R \in (-M, G)$. Note that Ψ is convex and $\Psi(0) = 0$, moreover Ψ goes to infinity in the boundary points $-M$ and G. After some calculations one obtains that Ψ reaches its minimum value in

$$\frac{G}{2} - \frac{M}{2} - \frac{C}{c} - \frac{\sqrt{c^2(G+M)^2 + 4C^2}}{2c}$$

As in the Meixner case, if this value is positive, this would imply always another solution $R \in (0, G)$, one then can proceed as in the first part of Proposition 4.1. Having such a solution $R \in (0, G)$, one has not to impose an upper bound on c as in the NIG case. The exact value of R can easily be computed numerically.

Finally, we mention that along the same lines many other Lévy processes can be proposed. We think especially at Tempered Stable and Modified stable subordination described in Barndorff–Nielsen and Shephard [45].

5. CONCLUSIONS

We present a risk model based on a Lévy process. We show how the infinite activity feature of such family of processes can be used to account for discrete premium fluctuations as well as for semi-heavy tailed claims. Despite its counterintuitive properties, this risk process can be still incorporated into standard risk theory results.

The subordination construction of the Lévy processes implies that our risk model is a generalization of the diffusion risk model of Grandell [20]. The fact that the process is still a diffusion but operating in business time allows for larger fluctuations making it a better and more flexible model to fit risk reserves with exponentially decaying claims. The concept of business time for a transformed risk reserve process is used to generalize the diffusion model. In such a generalized process, time evolves by an infinite number of small jumps with occasional larger time jumps. The random time increments can be seen as a randomly varying market activity.

Classical simulation techniques for Lévy processes make it possible to evaluate its associated ruin probabilities using results such as described in Proposition 4.1.

Our discussion is mainly introductory. We present Lévy processes as well as some of its features within a risk theory context. Further research is needed to asses the performance of such processes compared to other risk models. We conclude by saying that risk processes driven by Lévy processes have merits to be considered an object of further research in risk theory.

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