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Scroll codes over curves of higher genus

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Abstract We construct linear codes from scrolls over curves of high genus and study the higher support weights d_i of these codes. We embed the scroll into projective space $\mathbb{P}^{k-1}$ and calculate bounds for the d_i by considering the maximal number of $\mathbb{F}_q$-rational points that are contained in a codimension h subspace of $\mathbb{P}^{k-1}$. We find lower bounds of the d_i and for the cases of large i calculate the exact values of the d_i. This work follows the natural generalisation of Goppa codes to higher-dimensional varieties as studied by S.H. Hansen, C. Lomont and T. Nakashima.

Keywords Projective bundles on curves · Error-correcting codes

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1 Introduction

One way to produce linear q-ary codes with word length n and dimension k is to pick a geometric object T in the projective space $\mathbb{P}^{k-1}$, and let each of the, say n, points of T be represented by an element of $\mathbb{F}_q^k$. Using these k-tuples as the columns of a generator matrix, one defines the code via this generator matrix. The choice of representative for each point, and the ordering of the points, does not change the

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equivalence class of the code, and hence not the word length and dimension either. For a linear code C, the ith higher weight d_i is defined as the minimum support weight among all subcodes of C of dimension i. In particular, d_1 is equal to the minimum distance.

Moreover, it is well-known that for $i = 1, \dots, k$,

$$d_i = n - J_i,$$

where J_i is the the maximal number of $\mathbb{F}_q$- rational points from T on a codimension i linear subspace of $\mathbb{P}^{k-1}$. It is clear that also the d_i are independent of the choice of representative for each point of T.

The aim with this article is to investigate properties of linear error-correcting codes over a finite field $\mathbb{F}_q$, obtained from scrolls that are embeddings of projective bundles of higher rank over curves of higher genus. In [5], the authors studied properties of linear codes produced from rational normal scrolls, which are naturally embedded projective bundles of type $\mathbb{P}(\mathscr{E})$, where $\mathscr{E} = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_\Delta)$ is a bundle on $\mathbb{P}^1$. In the present work, we will study codes from the projectivised bundles over curves of higher genus in a similar way.

In the present paper, we will let X be a curve of genus g and $\mathscr{E}$ be a semi-stable vector bundle on X, both defined over $\mathbb{F}_q$ and therefore simultaneously over its algebraic closure, and we will embed $T' = \mathbb{P}(\mathscr{E})$ into some projective space $\mathbb{P}^{k-1}$ (over $\mathbb{F}_q$ and over its closure) by the natural line bundle $\mathcal{L}' = \mathcal{O}_{T'}(1)$ such that $k = h^0(X, \mathscr{E}) = h^0(T', \mathcal{L}')$. In this manner, the fibers of the projective bundle are embedded as linear, sub-projective spaces of $\mathbb{P}^{k-1}$.

In other papers, like [4,6] and [8], one also studies projective bundles $T' = \mathbb{P}(\mathscr{E})$ like this for the purpose of producing codes, and one even varies the complete linear system line bundle $a\mathcal{L}' + f_1 + \cdots + f_b$ by which one embeds $\mathbb{P}(\mathscr{E})$ into projective space, where $\mathcal{L}'$ is as described, and the f_j are fibres of $\mathbb{P}(\mathscr{E})$ over points $P_1, \dots, P_b$ on X. There one gives estimates for the minimum distance d_1 for the codes thus defined, in other words for (the number of points minus) the maximal number of $\mathbb{F}_q$-rational points in a codimension one space in the embedding space. In the present paper, it is not our main purpose to improve the estimates for d_1, but rather to say as much as possible about the d_i for higher $i \leq k$ for our particular linear system $\mathcal{L}'$. We will combine the insight of the mentioned articles about projective bundles in positive characteristic and the techniques of [5] for rational normal scrolls. To determine the d_i for large i (close to k) an important tool will be Riemann–Roch's theorem for vector bundles on curves, both defined over a finite field.

For somewhat smaller i a main tool to give lower bounds for the weights d_i will be Brill–Noether theory for vector bundles of higher ranks. Especially the non-existence results as in [2,3,9] and [7] will be useful. We believe that the demonstration of how this kind of mathematics can be applied in a code-theoretic setting is a main point of the article.

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2 Constructions and presentation of the problem

A linear code C is a linear subspace of $(\mathbb{F}_q)^n$ for some $n \in \mathbb{N}$. We usually denote the dimension of the code by k, and it is defined as $k = \log_q(\#(C))$. For $h = 1, 2, \ldots, k$, let D_h be the set of all linear subspaces of the code C generated by h linearly independent elements in C, and let

$$d_h = \min\{\#(\mathrm{Supp}(E)) \mid E \in D_h\}.$$

We call d_1 the *minimum distance* of the code C. One aim in coding theory is given q, n and k, to maximise d_1. In processes of trellis decoding, or in cryptology, using the generator matrix of C instead as a starting point in connection with the so-called wire-tap channel of type II, it can in some cases be interesting to maximise d_h for higher values of h.

Let X be a non-singular, projective curve of genus g defined over $\mathbb{F}_q$ (see [10, Chapter 5] for definitions), and let $\mathscr{E}$ be a locally free sheaf of rank r on X, where r is some positive integer. Let $\mathscr{E}$ *be defined over* $\mathbb{F}_q$ if there exists an open covering with transition functions consisting of elements of the function field over $\mathbb{F}_q$.

The following proposition is the Riemann–Roch theorem for vector bundles on curves defined over finite fields, and is used repeatedly by other authors, like in [4] and [8].

Proposition 1 *Over any field k, if X is a curve defined over k and $\mathscr{E}$ is a locally free sheaf of rank r on X, r any positive integer, then*

$$\chi(\mathscr{E}) = \deg(\mathscr{E}) + r(1-g).$$

We will from now on suppose the following: X will denote a non-singular, projective curve of genus $g \geq 0$ defined over the finite field $\mathbb{F}_q$, and $\mathscr{E}$ will denote a locally free, semistable sheaf of rank $r \geq 2$ (and some high degree) defined over $\mathbb{F}_q$ and where $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is very ample.

Let $T' = \mathbb{P}(\mathscr{E})$, and denote $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ by $\mathcal{L}'$. Use $\mathcal{L}'$ to embed T' into projective space $\mathbb{P}^{k-1}$, where $k = h^0(T', \mathcal{L}')$, and denote the isomorphic image by T. Let $\mathcal{L}$ be the line bundle on T corresponding to $\mathcal{L}'$ on T'. Then T will be a scroll in the sense that the fibres of T' over the points of X will be mapped into $\mathbb{P}^{k-1}$ as linear projective (sub)spaces. For each $\mathbb{F}_q$-rational point P on T, choose a set of coordinates $(x_1, \ldots, x_k)$ such that $x_1, \ldots, x_k \in \mathbb{F}_q$. We then define a matrix G where each column is of the form $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k$ are the chosen coordinates of a point P on T. We define C to be the linear code with generator matrix G. The choice of generators of $H^0(\mathcal{L})$ and the ordering of the columns will not affect the equivalence class of the code, and thus not the parameters $n, k, d_1, \ldots, d_k$ either. It is for example clear that

$$n = m\left(q^{r-1} + \cdots + q + 1\right),$$

where n simultaneously denotes the word length of the code and the number of $\mathbb{F}_q$-rational points on T, and m denotes the number of $\mathbb{F}_q$-rational points on X. We define:

$\mu(\mathscr{E}) := \deg(\mathscr{E})/r$. If $m > \mu(\mathscr{E})$, then the dimension of C is

$$k = h^0(T, \mathcal{L}) = h^0(X, \mathscr{E}).$$

This is true since $m > \mu(\mathscr{E})$ implies that $\deg(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_m)) = \deg(\mathscr{E}) - rm = r(\mu(\mathscr{E}) - m) < 0$, and hence, $h^0(T, \mathcal{L} \otimes \mathscr{O}(-f_1 - \cdots - f_m)) = h^0(X, \pi_*(\mathcal{L} - \mathscr{O}(f_1 + \cdots + f_m))) = h^0(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_m)) = 0$ since $\mathscr{E}$ and therefore also $\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_m)$ is semi-stable. Here f_i denotes the fibre of T over P_i, for $i = 1, \ldots, n$. Hence, the $\mathbb{F}_q$-rational points of T span all of $\mathbb{P}^{k-1}$.

We see that $\mathcal{L}' = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is very ample on T' if it embeds each fibre of T' as a projective $(r-1)$-subspace of $\mathbb{P}^{k-1}$, and if each pair of two such fibres are mapped onto disjoint $(r-1)$-subspaces, which together impose $2r$ conditions on the hyperplanes in $\mathbb{P}^{k-1}$. A sufficient condition for this to happen, if $\mathscr{E}$ is semistable, is $\deg(\mathscr{E}) > 2gr$, since then

$$\begin{aligned} &h^0(T', \mathcal{L}') - h^0(T', \mathcal{L}' - \mathscr{O}(f_1 + f_2)) \\ &\quad = h^0(X, \pi_*\mathcal{L}) - h^0(X, \pi_*(\mathcal{L} - \mathscr{O}(f_1 + f_2))) \\ &\quad = h^0(X, \mathscr{E}) - h^0(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - P_2)) \\ &\quad = (\deg(\mathscr{E}) + r(1-g)) - (\deg(\mathscr{E}) - 2r + r(1-g)) = 2r, \end{aligned}$$

since $\deg(\mathscr{E} \otimes \mathscr{O}(-P_1 - P_2)) > r(2g-2)$, and in both cases there is no h^1−term in Riemann–Roch's formula (we have for example: $H^1(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - P_2) = H^0(X, K_X \otimes \mathscr{O}(P_1 + P_2) \otimes \mathscr{E}^\vee) = 0$, since the bundle in the last parenthesis has negative degree and is semi-stable since $\mathscr{E}$ is).

Summing up, we obtain:

Remark 2 If $\mathscr{E}$ is semi-stable with $m > \mu(\mathscr{E}) > 2g$, where m is the number of $\mathbb{F}_q$-rational points on X, then $\mathcal{L}'$ is very ample. It follows that T is the isomorphic image of T' and C is an $[n, k]$-code, where $n = m(q^{r-1} + \cdots + q + 1)$ and $k = h^0(T, \mathcal{L}) = h^0(X, \mathscr{E}) = \deg(\mathscr{E}) + r(1-g)$.

Basic Assumption 3 In the rest of the paper (except in Example 35) we will assume that C is a code produced from a scroll T as in Remark 2, including the assumptions that $\mathscr{E}$ is semi-stable and $m > \mu(\mathscr{E}) > 2g$.

Our aim is to find a lower bound for $d_1, \ldots, d_k$. The number d_k is easily seen to be n, since otherwise there would be a point on T with all coordinates equal to zero, which is impossible.

Notation 4 We denote the maximal number of $\mathbb{F}_q$-rational points on T contained in a codimension h subspace by J_h.

It is well-known that

$$d_h = n - J_h. \tag{1}$$

In the rest of the article we will determine the J_h for as many h as possible and give good upper bounds for the J_h (lower bounds for the corresponding d_h) for the remaining h.

The following definition makes sense and will be useful:

Definition 5 Let $S_{h,0}$ be the maximal number of fibres of (T over X) contained in a codimension h subspace.

We then have the following obvious bound:

Remark 6 $J_h \leq (q^{r-1} + \cdots + q + 1) \cdot S_{h,0} + (q^{r-2} + \cdots + q + 1) \cdot (m - S_{h,0})$.

Using (1), we obtain

Proposition 7 $d_h \geq q^{r-1}(m - S_{h,0})$.

It is desirable to get a better upper bound by determining how a codimension h subspace L containing $S_{h,0}$ fibres intersects other fibres. The fact that the fibres of T over X are linear spaces reduces this to an issue of which dimension $f \cap L$ has for the other fibres f. It is also a priori possible that a codimension h subspace L containing less than $S_{h,0}$ fibres contains a maximal number of $\mathbb{F}_q$-rational points.

The following fact is obvious, but will be used so much throughout that we include it here anyway.

Observation 8 Let $f_1, \ldots, f_S$ be S fibres for some integer S. The fibres are contained in a codimension h subspace L if and only if

$$h^0(T, \mathcal{L} \otimes \mathcal{O}(-f_1 - \cdots - f_S)) \geq h.$$

We have the following preliminary result:

Proposition 9 *Let* $g \geq 0$ *and* $h \in \{1, \ldots, k\}$. *Then*

$$S_{h,0} \leq \mu(\mathcal{E}) - \lfloor (h-1)/r \rfloor .$$

Proof For $h = 1$, we observe that $h^0(X, \mathcal{E} \otimes \mathcal{O}(-P_1 - \cdots - P_{S_{1,0}})) \geq 1$, where $P_1, \ldots, P_{S_{1,0}}$ are points on X corresponding to $S_{1,0}$ fibres in a hyperplane. This implies that $\deg(\mathcal{E} \otimes \mathcal{O}(-P_1 - \cdots - P_{S_{1,0}})) \geq 0$, and hence $S_{1,0} \leq \deg(\mathcal{E})/r = \mu(\mathcal{E})$.

For $h \geq 1$, we show that $S_{h+r,0} \leq S_{h,0} - 1$, i.e., that a codimension $h + r$ subspace L' contains at most $S_{h,0} - 1$ fibres. For arbitrary j, and where C_0 denotes a hyperplane section, the Riemann–Roch theorem gives us

$$\begin{aligned} h^0(T, C_0 - (f_1 + \cdots + f_{j-1})) &= \deg(\mathcal{E}) - rj + (1-g)r \\ &\quad + h^1(T, C_0 - (f_1 + \cdots + f_{j-1})) + r \\ &\leq \deg(\mathcal{E}) - rj + (1-g)r \\ &\quad + h^1(T, C_0 - (f_1 + \cdots + f_{j-1} + f_j)) + r \\ &= h^0(T, C_0 - (f_1 + \cdots + f_{j-1} + f_j)) + r. \end{aligned}$$

Set $j = S_{h,0} + 1$. We get $h^0(T, C_0 - (f_1 + \cdots + f_{S_{h,0}})) \leq h^0(T, C_0 - (f_1 + \cdots + f_{S_{h,0}} + f_{S_{h,0}+1})) + r < h + r$, since $S_{h,0}$ is the greatest integer satisfying $h^0(T, C_0 - (f_1 + \cdots + f_{S_{h,0}})) \geq h$. So L' cannot contain $S_{h,0}$ fibres.

Hence, $S_{h+r,0} \leq S_{h,0} - 1$, and

$$S_{h,0} \leq \frac{\deg(\mathscr{E})}{r} - \lfloor (h-1)/r \rfloor. \qquad \square$$

The following result follows immediately from Remark 6 and Proposition 9 and is similar to results in [4,6], and [8].

Corollary 10 *$d(C) \geq q^{r-1}(m - \mu(\mathscr{E}))$. In general, $d_h \geq q^{r-1}(m - \mu(\mathscr{E})) + \lfloor (h-r)/r \rfloor)$.*

In Corollary 14, Lemma 17 and Proposition 33, we will improve the preliminary bound in Proposition 9 for h in certain (broad) ranges.

The following definitions will be instrumental for many h:

Definition 11 For each non-negative integer d, let $f(d)$ be the maximal value of $h^0(\mathscr{E})$ for all semi-stable vector bundles $\mathscr{E}$ of degree at most d on X, defined over the closure of $\mathbb{F}_q$.

Moreover, for each positive integer h, let $\phi(h) = \min\{d \,|\, h^0(\mathscr{E}) \geq h$ for some semi-stable vector bundle $\mathscr{E}$ of degree at most $d\} = \min\{d \mid f(d) \geq h\}$.

Let h be a positive integer. We now have:

Proposition 12

$$S_{h,0} \leq \frac{d}{r} - \frac{\phi(h)}{r} = \mu(\mathscr{E}) - \frac{\phi(h)}{r}.$$

Proof Assume a codimension h subspace contains S fibres, corresponding to the points $P_1, \ldots, P_S$ on C. Then $h^0(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_S)) \geq h$. This immediately implies $\deg(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_S)) \geq \phi(h)$. So we have $\deg(\mathscr{E}) - rS \geq \phi(h)$, which gives us

$$S \leq \frac{d}{r} - \frac{\phi(h)}{r} = \mu(\mathscr{E}) - \frac{\phi(h)}{r}. \qquad \square$$

We now for simplicity assume $g \geq 2$. We then have:

Proposition 13 – *For $0 \leq d \leq r(2g-2)$, we have $f(d) \leq r + \frac{d}{2}$.*
- *For $r(2g-2) \leq d \leq r(2g-1)$, we have $f(d) \leq rg$.*
- *For $d \geq r(2g-1)$, we have $f(d) \leq d + r(1-g)$, and $f(d) = d + r(1-g)$ if semistable bundles of degree d exist.*

Proof The first statement is the Clifford bound given in Theorem 1.1 of [2]. The second and third statements follow from the first statement and Riemann–Roch, which gives $h^0(\mathscr{E}) = d + r(1-g) + h^0(K_X \otimes \mathscr{E}^\vee) = d + r(1-g)$, since $K_X \otimes \mathscr{E}^\vee$ has negative degree and is semi-stable since $\mathscr{E}$ is. $\square$

Corollary 14 – *For $r \leq h \leq gr$, we have $\phi(h) \geq 2(h-r)$ and $S_{h,0} \leq \mu(\mathscr{E}) - \frac{2h}{r} + 2$.*
– *For $h \geq gr + 1$, we have $\phi(h) \geq h + r(g-1)$ and $S_{h,0} \leq \mu(\mathscr{E}) - \frac{h}{r} + (1-g)$.*

Proof The lower bounds for $\phi(h)$ follow immediately from Proposition 13. The upper bounds for $S_{h,0}$ follow immediately from Proposition 12 and the lower bounds for $\phi(h)$. □

To obtain better upper bounds on J_h than the ones we get using Remark 6 and the upper bounds on $S_{h,0}$, we have the following helpful result:

Proposition 15 *Let $0 \leq i \leq r-1$, and let L be a codimension h subspace that intersects $\geq s_j$ fibres in a $\mathbb{P}^{r-j-1}$ for $j = 0, \ldots, i$. Then*

$$s_0 + s_1 + \cdots + s_i \leq \mu(\mathscr{E}) - \frac{\phi(h - s_1 - 2s_2 - \cdots i s_i)}{r}.$$

Proof For $i = 0$, this is only Proposition 12. Let $i \geq 1$. We have $\geq s_0$ fibres contained in L, and in addition, L intersects $\geq s_i$ fibres in a $\mathbb{P}^{r-i-1}$ for each $1 \leq i \leq r-1$. For each of these i, choose s_i fibres that intersect L in a $\mathbb{P}^{r-1-i}$ and denote the set of these fibres by F_i. For each fibre in F_i, choose i points such that these points and the intersection of L with the fibre together span the fibre. Let L' be the linear span of L and the $s_1 + 2s_2 + \cdots + i s_i$ points we just chose. The codimension of L' is then at least $h - s_1 - 2s_2 - \cdots - i s_i$, and L' contains $\geq s_0 + s_1 + \cdots + s_i$ fibres. The proof of Proposition 12 then gives the conclusion. □

To improve the (effective) bounds for $f(d)$ and $\phi(h)$ in the range $0 < d < r(2g-2)$ and corresponding range $r < h < gr$, at least in some special cases under further assumptions on X and the bundle $\mathscr{E}$, is a matter of great interest and is essentially the socalled "non-existence" problem in Brill–Noether theory for bundles of higher rank, as addressed in [2,3,9] and [7]. We will return to this issue in Sect. 4. For $h \geq gr + 1$, there is not much room for improvement, as we will see in the beginning of the next section.

3 Particular bounds in the range $h \geq gr + 1$

We start this section by fixing the following notation.

Notation 16 As before, let $S_{h,0}$ be the maximal number of fibres contained in a codimension h subspace L, where the maximum is taken over all codimension h subspaces L in $\mathbb{P}^{k-1}$. Denote the set of all codimension h subspaces that contain $S_{h,0}$ fibres by $A_{h,0}$. For $1 \leq i \leq r$, denote by $S_{h,i}$ the maximal number of fibres that intersect a codimension h subspace $L \in A_{h,0}$ in a $\mathbb{P}^{r-i-1}$.

In this section we will now give some bounds for the $S_{h,i}$ for h large enough. In particular, we have the following lower bound for $S_{h,0}$:

Lemma 17 *For $h \geq rg + 1$, we have*

$$S_{h,0} \geq \mu(\mathscr{E}) - \frac{h}{r} - g + 1 - \frac{r-1}{r}.$$

It follows that

$$S_{h,0} = \left\lfloor \frac{\deg(\mathscr{E}) - h}{r} \right\rfloor - g + 1 = \frac{\deg(\mathscr{E}) - h'}{r} - g + 1,$$

where $h' = h^0(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}}))$ and $P_1, \ldots, P_{S_{h,0}}$ is any collection of points corresponding to fibres contained in a codimension h subspace that contains $S_{h,0}$ fibres of T.

Proof Let $P_1, \ldots, P_{S_{h,0}}$ be points corresponding to fibres as described, and let $h^0(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}})) = h'$. Then $h' \geq h \geq rg + 1$, and the Riemann–Roch theorem gives

$$S_{h,0} = \mu(\mathscr{E}) - \frac{h'}{r} + 1 - g,$$

since we have from Proposition 14 that $\deg(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}})) = \deg(\mathscr{E}) - rS_{h,0} \geq \phi(h') \geq (2g-1)r + 1 > (2g-2)r$. By the assumption that $\mathscr{E}$ is semi-stable, there is no h^1-term, and hence this equality follows.

Since $S_{h,0}$ is the largest integer such that there exist points P_i such that $h^0(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}})) \geq h$, we have $h' - h \leq r - 1$ because of the following argument: We just showed that $\deg(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}})) = d - rS_{h,0} \geq (2g-1)r + 1$. It follows that also $h^1(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}} - P_{S_{h,0}+1})) = 0$, and so $h^0(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}} - P_{S_{h,0}+1})) = h' - r$, which must be $< h$ because of the definition of $S_{h,0}$. It follows that $h' - h \leq r - 1$, and that the first inequality of the lemma holds.

The equalities at the end of the lemma now follow from Proposition 14, stating that $S_{h,0} \leq \mu(\mathscr{E}) - h/r - g + 1$, and from the fact that there is exactly one integer in the interval $[\mu(\mathscr{E}) - h/r - g + 1 - (r-1)/r,\ \mu(\mathscr{E}) - h/r - g + 1]$. □

Corollary 18 *For $h \geq rg + 1$, we have $d_h \geq q^{r-1}(m - \left\lfloor \frac{\deg(\mathscr{E})-h}{r} \right\rfloor + g - 1)$.*

We have the following result for $S_{h,i}$ with $i \geq 1$:

Corollary 19 *Let $0 \leq i \leq r - 1$, and let L be a codimension h subspace that intersects $\geq s_j$ fibres in a $\mathbb{P}^{r-j-1}$ for $j = 0, \ldots, i$, where $h \geq rg + s_1 + 2s_2 + \cdots + is_i + 1$. Then*

$$\left(s_0 - \mu(\mathscr{E}) + \frac{h}{r} + g - 1\right) + \left(\frac{r-1}{r}\right) s_1 + \left(\frac{r-2}{r}\right) s_2 + \cdots + \left(\frac{r-i}{r}\right) s_i \leq 0.$$

In particular, if $L \in A_{h,0}$*, we have:*

$$\left(\frac{r-1}{r}\right)s_1+\left(\frac{r-2}{r}\right)s_2+\cdots+\left(\frac{r-i}{r}\right)s_i \le \frac{h'-h}{r} \le \frac{r-1}{r},$$

where $h' = h^0(X, \mathscr{E}\otimes\mathscr{O}(-P_1-\cdots-P_{S_{h,0}}))$ *and the points* $P_1,\ldots,P_{S_{h,0}}$ *correspond to fibres contained in a codimension* h *subspace contained in* $A_{h,0}$ *(see Notation 16).*

Proof Since $h \ge rg+s_1+2s_2+\cdots+is_i+1$, we have $h-s_1-2s_2-\cdots-is_i \ge rg+1$, so $\phi(h-s_1-2s_2-\cdots-is_i) \ge h-s_1-2s_2-\cdots-is_i+r(g-1)$ by Corollary 14. Hence, Proposition 15 gives

$$s_0+s_1+\cdots+s_i \le \mu(\mathscr{E}) - \frac{h-s_1-2s_2-\cdots-is_i+r(g-1)}{r}.$$

Rearranging terms, we obtain the first part of the corollary.

The second part of the corollary follows since $h' = \deg(\mathscr{E})+r(1-g)-rS_0$, and $h'-h \le r-1$, as demonstrated in the proof of Lemma 17. □

Definition 20 Let $t = h'-h$, where h' was described in Corollary 19. Note also that Lemma 17 and its proof give the explicit formula: $t = h'-h = \deg(\mathscr{E}) - r\left\lfloor\frac{\deg(\mathscr{E})-h}{r}\right\rfloor - h$.

Remark 21 One might think of $k-h = k-h'+t = S_{h,0}r+t$ as the dimension of the affine cone in $(\mathbb{F}_q)^k$ of a linear space L in $\mathbb{P}^{k-1}$ containing $S_{h,0}$ fibres and t independent points in another fibre. This only makes sense if the fibres and points impose independent conditions on hyperplanes. We will show that if h and q are big enough, this is indeed the case.

We now make a few essential observations: The last part of Corollary 19 reads:

$$(r-1)s_1+(r-2)s_2+\cdots+(r-i)s_i \le t$$

if $h \ge rg+s_1+2s_2+\cdots+is_i+1$ and $s_0 = S_{h,0}$.

Assume $t=0$ and $h \ge rg+r$, and that L is a codimension h subspace that contains $S_{h,0}$ fibres and intersects $s_i = 1$ other fibre in a $\mathbb{P}^{r-i-1}$ for some $i \le r-1$. Then $h \ge rg+(r-1)s_i+1 \ge rg+is_i+1$. But then we obtain (with $s_j=0$ for all $j \ne i$) that $(r-i)s_i \le 0$, that is, $s_i = 0$. Hence, $S_{h,i}=0$ for $i \ge 1$ if $h \ge r(g+1)$ and $t=0$.

Assume t is any integer satisfying $1 \le t \le r-2$ and $h \ge rg+r-t$, and that L is a codimension h subspace that contains $S_{h,0}$ fibres and intersects $s_i=1$ other fibre in a $\mathbb{P}^{r-i-1}$ for some $1 \le i \le r-(t+1)$. Then $h \ge rg+(r-t-1)s_i+1 \ge rg+is_i+1$. But then we obtain (with $s_j=0$ for all $j\ne i$) that $(r-i)s_i \le t$, that is, $s_i=0$, since $r-i \ge t+1$. Hence, $S_{h,i}=0$ for $1 \le i \le r-(t+1)$ if $h \ge rg+r-t$.

If t is any integer satisfying $1 \le t \le r-1$ and h also satisfies $h \ge rg+2(r-t)+1$, then we conclude in an analogous way that $s_{r-t} \le 1$. Moreover, it is clear that if $S_{h,0}$ fibres span a codimension $h' = h+t$ plane, then we may just add t independent points in another fibre and thereby span a codimension h plane containing $S_{h,0}$ fibres and intersecting another one in a $\mathbb{P}^{t-1} = \mathbb{P}^{(r-1)-(r-t)}$. Hence, $S_{h,r-t} = 1$.

Moreover, it is then clear that if $h \geq rg+(r-t)+(r-1)+1 = rg+2r-t$ and that L is a codimension h subspace that contains $S_{h,0}$ fibres and intersects another one in a $\mathbb{P}^{t-1} = \mathbb{P}^{(r-1)-(r-t)}$, and $r-1 \geq i \geq 1$, $i \neq r-t$, then the equation $ts_{r-t} + is_i \leq t$ obtained from setting $s_j = 0$ for $j \neq i, r-t$, gives $s_i = 0$.

We sum this up as:

Proposition 22 *We have the following:*

(a) *If $h \geq rg + r - t$ and $0 \leq t \leq r-2$, then $S_{h,i} = 0$, for $i = 1, 2, \ldots, r-t-1$.*
(b) *If $h \geq rg + 2(r-t) + 1$ and $1 \leq t \leq r-1$, then $S_{h,r-t} = 1$. If moreover the stronger condition $h \geq rg + 2r - t$ holds, then any element in $A_{h,0}$ intersecting an additional fibre in a $\mathbb{P}^{t-1}$ intersects all other fibers empty.*

We then obtain:

Corollary 23 *We have the following:*

(a) *If $h \geq r(g+1)$ and $t = 0$, then the maximum number of intersection points between an element in $A_{h,0}$ and T is $S_{h,0}\frac{q^r-1}{q-1}$.*
(b) *If $h \geq rg + (t+1)(r-1) + 1$ and $1 \leq t \leq r-1$, and q is big enough, e.g. $q \geq (r-1)(r-2)$, then the maximum number of intersection points between an element in $A_{h,0}$ and T is $S_{h,0}\frac{q^r-1}{q-1} + \frac{q^t-1}{q-1}$.*

Proof Part (a) follows directly from the case $t = 0$ in part (a) of Proposition 22.

Because of Proposition 22 (b), part (b) of our corollary follows if we can prove that the number of points in a $\mathbb{P}^{t-1}$ is at least as large as the number of "additional" intersection points of any element in $A_{h,0}$ and T (meaning in addition to the points of the $S_{h,0}$ fibres that are contained in this intersection by definition). By Proposition 22 (a), we may restrict ourselves to looking at codimension h spaces that intersect the "additional" fibres of T only in m-spaces where $m < t$.

So assume L is such a codimension h space in $A_{h,0}$, and assume L intersects s_j fibres in a $\mathbb{P}^{r-j-1}$, where $r-1-j \leq t-2$. Then the first part of Corollary 19, with $s_0 = S_{h,0}$ and $s_l = 0$ for $l \neq 0, j$, gives $s_j \leq \frac{t}{r-j} \leq t$ if $h \geq rg + js_j + 1$. It will then be enough to assume $h \geq rg + (t+1)(r-1) + 1$ ($\geq rg + (t+1)j + 1$) to conclude $s_j \leq \frac{t}{r-j} \leq t$. (Pick a fibres such that the codimension h subspace contains $S_{h,0}$ fibres and intersects these a fibres in codimension h, where a is an integer with $\frac{t}{r-j} < a \leq t+1$. Then $h \geq rg + (t+1)(r-1) + 1 \geq rg + aj + 1$, and we conclude $a \leq \frac{t}{r-j}$ from Corollary 19, a contradiction that falsifies the possibility $\frac{t}{r-j} < a$, and we conclude $s_j \leq \frac{t}{r-j}$.) Then it will suffice to find conditions on q such that:

$$\frac{q^t-1}{q-1} \geq \sum_{i=1}^{t-1} \frac{t}{i} \cdot \frac{q^i-1}{q-1}. \tag{2}$$

By expanding both sides as polynomials in q, one sees that it suffices (but is far from necessary) that $q \geq \frac{t}{t-1} + \frac{t}{t-2} + \cdots + \frac{t}{2} + \frac{t}{1}$. It clearly suffices that $q \geq (r-1)(r-2) \geq t(t-1)$. □

3.1 A comparison between elements of $A_{h,0}$ and other codimension h planes

We observe from Corollary 23 above, using the identity $d_h = n - J_h$, that as long as J_h is computed by elements of $A_{h,0}$, then d_h is easy to compute as long as $h \geq rg + (t+1)(r-1) + 1$ and $q \geq (r-1)(r-2)$. To make sure that d_h and J_h really are computed by elements of $A_{h,0}$, we will have to impose further restrictions on h and q. Here is an analysis:

First we discuss how many fibres s_i that can intersect L in a $\mathbb{P}^{r-i-1}, i = 1, 2, \ldots, r-1$, when L contains $s_0 < S_{h,0}$ fibres.

Lemma 24 *Let L be a codimension h subspace that contains $S_{h,0} - i$ fibres and intersects s_1 fibres in a $\mathbb{P}^{r-2}$, where $h \geq rg + ir/(r-1) + 3$ and $i \in \{0, 1, \ldots, S_{h,0}\}$. Then*

$$s_1 \leq \left\lfloor \frac{ir}{r-1} \right\rfloor + 1.$$

Proof We use the first part of Proposition 19, with $s_0 = S_{h,0} - i$, and where we use the expression from the proof of Lemma 17 for $S_{h,0}$. We then get

$$\frac{h-h'}{r} - i + \left(\frac{r-1}{r}\right) s_1 \leq 0,$$

where $h' = h^0(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}}))$, where the P_i correspond to $S_{h,0}$ fibres contained in a codimension h-space L' which computes $S_{h,0}$. Using that $h' - h \leq r-1$, we get $s_1 \leq ir/(r-1) + 1$, if $h \geq rg + s_1 + 1$. But since we assume $h \geq rg + ir/(r-1) + 3$, the assumption $h \geq rg + s_1 + 1$ holds as long as $s_1 \leq ir/(r-1) + 2$. So in order for s_1 to be $> ir/(r-1) + 1$, we must have $s_1 > ir/(r-1) + 2$. But then we can't choose a subset of s_1' fibres for $ir/(r-1) + 1 < s_1' \leq ir/(r-1) + 2$, which is absurd, since this interval contains an integer. □

Proposition 25 *Assume q big enough, for example $q \geq 2g + 4$, and $h \geq (r-2)g + \mu(\mathscr{E}) + 1 + \frac{3(r-1)}{r}$ and $h \geq rg + 1$. Then there exists a codimension h space $L \in A_{h,0}$ that contains a maximal number of $\mathbb{F}_q$-rational points from T.*

Proof If $h \geq rg + \frac{rS_{h,0}}{r-1} + 3$, then Lemma 24 is applicable for all $i = (0), 1, \ldots, S_{h,0}$, and we conclude that L intersects at most $\frac{ir}{r-1} + 1$ fibres of T in a $\mathbb{P}^{r-2}$ if it contains $S_{h,0} - i$ fibres.

Since $h \geq gr + 1$, we have $S_{h,0} \leq \mu(\mathscr{E}) - \frac{h}{r} - g + 1$ by Lemma 17. If we insert the bigger value $\mu(\mathscr{E}) - \frac{h}{r} - g + 1$ for $S_{h,0}$ in the inequality $h \geq rg + \frac{rS_{h,0}}{r-1} + 3$, and the inequality thus obtained holds, then the original inequality also holds. But the condition $h \geq (r-2)g + \mu(\mathscr{E}) + 1 + \frac{3(r-1)}{r}$ in the proposition is precisely the inequality we obtain by inserting this value for $S_{h,0}$.

If $L \in A_{h,0}$, then L contains at least $S_{h,0}\frac{q^r-1}{q-1}$ points on T. If L' is not in $A_{h,0}$, then L' contains $S_{h,0} - i$ fibres, where $i \geq 1$. Then L' contains at most the following number of points:

$$(S_{h,0}-i)\frac{q^r-1}{q-1}+\left(\frac{ir}{r-1}+1\right)\frac{q^{r-1}-1}{q-1}+\left(m-S_{h,0}+i-\frac{ir}{r-1}-1\right)\frac{q^{r-2}-1}{q-1}.$$

It is then enough to prove that

$$i\frac{q^r-1}{q-1}\geq\left(\frac{ir}{r-1}+1\right)\frac{q^{r-1}-1}{q-1}+\left(m-S_{h,0}+i-\frac{ir}{r-1}-1\right)\frac{q^{r-2}-1}{q-1},$$

for $i=1,\dots,S_{h,0}$. Writing everything as polynomials in q, we see that it is enough to prove

$$iq^2\geq\left(\frac{ir}{r-1}+1\right)q+\left(m-S_{h,0}+i-\frac{ir}{r-1}-1\right).$$

This holds for all i if and only if it holds for $i=1$, and reduces to

$$q^2\geq\frac{2r-1}{r-1}q+\left(m-S_{h,0}-\frac{r}{r-1}\right). \tag{3}$$

Using the Hasse–Weil bound, we see that $m\leq q+1+2g\sqrt{q}\leq(2g+1)q+1$. Hence, the inequality holds if

$$q^2-\left(2g+\frac{3r-2}{r-1}\right)q+\left(S_{h,0}-1+\frac{r}{r-1}\right)\geq 0.$$

In particular, it holds if $q\geq 2g+4$. □

We observe that it is possible to modify the proof above to give alternative statements, possibly with harder restrictions on q and milder ones on h, for example like this:

Proposition 26 *Assume q big enough, for example $q\geq\max\{2g+4,\ \frac{4g^2}{i^2}+\frac{2}{i}\}$, and $h\geq rg+\frac{ir}{r-1}+3$, for some $i\in\{1,\dots S_{h,0}\}$. Then there exists a codimension h space $L\in A_{h,0}$ that contains a maximal number of $\mathbb{F}_q$-rational points from T.*

Proof The assumptions on h enable us to apply Lemma 24 in the cases where a codimension h plane contains $S_{h,0}-j$ fibres for $j\leq i$. The assumptions on q and the proof of Proposition 25 then give that elements of $A_{h,0}$ intersect T in more points than codimension h planes that contain $S_{h,0}-j$ fibres, for $j\leq i$. To prove that elements of $A_{h,0}$ intersect T in more points than codimension h planes that contain $S_{h,0}-j$ fibres for $j\geq i+1$, it suffices to prove that

$$(i+1)\frac{q^r-1}{q-1}\geq m\left(\frac{q^{r-1}-1}{q-1}\right).$$

Using the Hasse–Weil bound, we see that this holds if $iq\geq 2gq^{\frac{1}{2}}+1$, and in particular if $q\geq\frac{4g^2}{i^2}+\frac{2}{i}$. □

3.2 The main result for large h and q

Recall that $t = h' - h = \deg(\mathscr{E}) - r\left\lfloor \frac{\deg(\mathscr{E})-h}{r} \right\rfloor - h$.

Corollary 27 (a) *If* (2) *and* (3) *hold, in particular if* $q \geq \max\{(r-1)(r-2),\ 2g+4\}$, *and if in addition* $h \geq \max\{(r-2)g+\mu(\mathscr{E})+1+\frac{3(r-1)}{r},\ rg+(t+1)(r-1)+1\}$, *then*

$$J_h = S_{h,0}\left(\frac{q^r-1}{q-1}\right) + \frac{q^t-1}{q-1}.$$

(b) *If* (2) *and* (3) *hold, in particular if* $q \geq \max\{(r-1)(r-2),\ 2g+4\}$, *and if in addition* $q \geq \frac{4g^2}{i^2} + \frac{2}{i}$ *and* $h \geq rg + \frac{ir}{r-1} + 3$, *for some* $i \in \{1, 2, \ldots, S_{h,0}\}$, *and if* $h \geq rg + (t+1)(r-1) + 1$, *then*

$$J_h = S_{h,0}\left(\frac{q^r-1}{q-1}\right) + \frac{q^t-1}{q-1}.$$

Proof This follows directly from Corollary 23 and Propositions 25 and 26. □

Theorem 28 *Under the assumptions of Corollary 27, we have:*

$$\begin{aligned} d_h &= (m - S_{h,0})\left(q^{r-1}+\cdots+q+1\right) - \left(q^{t-1}+\cdots+q+1\right) \\ &= \left(m - \left\lfloor \frac{\deg(\mathscr{E})-h}{r} \right\rfloor + g - 1\right)\left(q^{r-1}+\cdots+q+1\right) - \left(q^{t-1}+\cdots+q+1\right). \end{aligned}$$

Proof This follows from Lemma 17 and Corollary 27. □

Remark 29 From Corollary 27 and the text preceeding Proposition 22, it follows that to find a codimension h space that computes J_h for the h in question, you may take the linear space in $\mathbb{P}^{k-1}$ spanned by any choice of $S_{h,0}$ fibres and any choice of $t = h' - h$ linearly independent points in any additional single fibre.

The appearance of the term $\mu(\mathscr{E})$ in the condition on h in part (a) of Corollary 27 implies that it holds for at most $(\frac{r-1}{r}) \cdot k$ of the numbers h between 0 and k (the biggest ones). In reality, since r and g also "count", we can only use (a) of Corollary 27 for a somewhat smaller fraction of the hs.

4 Bounds for low h

4.1 Bound for $h = 1$

The integer $S_{1,0}$ is the maximal number of fibres contained in a hyperplane, and is thus equal to the maximal number of points on the curve X that are zero in a global section of $\mathscr{E}$. If we let m be the number of $\mathbb{F}_q$-rational points on X, it is then clear that

$$J_1 = S_{1,0}(q^{r-1} + \cdots + q + 1) + (m - S_{1,0})(q^{r-2} + \cdots + q + 1),$$

since all fibres not contained in a hyperplane H must intersect H in a $\mathbb{P}^{r-2}$. Hence,

$$d_1 = n - J_1 = q^{r-1}(m - S_{1,0}).$$

Remark 30 Proposition 9 states that $S_{h,0} \leq \mu(\mathscr{E})$ for $h \leq r$. This is in a certain sense a sharp bound: We may construct curves with semi-stable bundles $\mathscr{E}$ of any rank with $S_{h,0} = \mu(\mathscr{E})$ for the corresponding scroll in the following way:

Let X be a curve in projective space such that there exists a hyperplane H that is zero in $\deg(X) > 2g$ distinct $\mathbb{F}_q$-rational points, and let $\mathscr{E} = \mathscr{O}_X(1) \oplus \cdots \oplus \mathscr{O}_X(1)$. Then $\mathscr{E}$ is obviously semistable and has $\mu(\mathscr{E}) = \deg(X)$. (It is easy to check that the tautological line bundle is very ample.) Since $\mathscr{O}_X(1)$ by assumption has a global section s which is zero in $\deg(X)$ distinct $\mathbb{F}_q$-rational points, then so do the global sections $(0, \ldots, 0, s, 0, \ldots, 0)$ of $\mathscr{E}$, and so $S_{h,0} = \mu(\mathscr{E})$ for $h \leq r$, as desired.

4.2 Bound for $h = 2$

For codimensions h, with $2 \leq h \leq r - 1$, it is difficult to say much about the J_h and the $S_{h,i}$. We do, however, have the following small result:

Proposition 31 *Suppose $S_{1,0} = S_{2,0}$. Then*

$$J_2 = S_{2,0}\left(\frac{q^r - 1}{q - 1}\right) + S_{2,1}\left(\frac{q^{r-1} - 1}{q - 1}\right) + (m - S_{2,0} - S_{2,1})\left(\frac{q^{r-2} - 1}{q - 1}\right),$$

where m is the number of $\mathbb{F}_q$-rational points on the curve X.

Proof We show that a codimension 2 plane intersecting a maximal number of points must contain a maximal number of fibres. The rest of the statement then follows naturally.

Suppose we have a codimension 2 plane L' containing $S_{2,0}$ fibres, let the plane be defined by two hyperplane sections z_1' and z_2', and let each z_i' contain s_i' fibres. Then L' intersects T in

$$\begin{aligned} J_2' &= S_{2,0}(q^{r-1} + \cdots + q + 1) + (s_1' + s_2' - 2S_{2,0})(q^{r-2} + \cdots + q + 1) \\ &\quad + (m - s_1' - s_2' + S_{h,0})(q^{r-3} + \cdots + q + 1) \\ &= S_{2,0}(q^{r-1} - q^{r-2}) + (s_1' + s_2')q^{r-2} + m(q^{r-3} + \cdots + q + 1) \end{aligned}$$

$\mathbb{F}_q$-rational points, where m is the number of $\mathbb{F}_q$-rational points on X.

Now suppose there is a codimension 2 plane L'' defined by hyperplane sections z_1'' and z_2'', each z_i'' containing s_i'' fibres, and such that L'' contains $S_{2,0} - j$ fibres for some $j \geq 1$. Then L'' intersects T in J_2'' points such that

$$J_2'' - J_2' = -j\left(q^{r-1} - q^{r-2}\right) + \left(s_1'' + s_2'' - s_1' - s_2'\right)q^{r-2}. \tag{4}$$

Now, we assumed that $S_{1,0} = S_{2,0}$, which means that since $z_1' \cap z_2'$ is zero in $S_{2,0}$ fibres, then z_1' and z_2' must each be zero along a maximal number of fibres, and so (4) must be negative, and J_2' must be maximal.

4.3 Bounds for $r + 1 \leq h \leq gr$

We now study the range $r + 1 \leq h \leq gr$ and look for possible improvements of Corollary 14, corresponding to possible improvements of the Clifford bound in Proposition 13. Recall the function $f(d)$ introduced in Definition 11. The most ambitious conjecture relating to the Clifford bound seems to be the following one, given in [7]. Mercat only states the conjecture for the two first intervals. The last one follows by duality from the first one.

Conjecture 32 Let X be a smooth curve of genus at least 4 and Clifford index γ. Let $\mathscr{E}$ be a semi-stable bundle of rank r, degree d and slope $\frac{d}{r}$. Then:

- If $1 \leq \frac{d}{r} \leq \gamma + 2$, then $h^0(\mathscr{E}) \leq f(d) \leq \frac{1}{\gamma+1}(d - r) + r$.
- If $\gamma + 2 \leq \frac{d}{r} \leq 2g - 4 - \gamma$, then $h^0(\mathscr{E}) \leq f(d) \leq \frac{d - r\gamma}{2} + r$.
- If $2g - 4 - \gamma \leq \frac{d}{r} \leq 2g - 3$, then $h^0(\mathscr{E}) \leq f(d) \leq r\left(2 - g + \frac{2g-3}{\gamma+1}\right) + d\left(\frac{\gamma}{\gamma+1}\right)$.

We will investigate the consequences of this conjecture:

First, we observe that $d = r$ implies $h^0(\mathscr{E}) \leq r$, and $d = (\gamma + 2)r$ implies $h^0(\mathscr{E}) \leq 2r$, and $d = (2g-4-\gamma)r$ implies $h^0(\mathscr{E}) \leq (g-1-\gamma)r$, and $d = (2g-3)r$ implies $h^0(\mathscr{E}) \leq (g - 1)r$, and that the bound for $h^0(\mathscr{E})$ is piecewise linear beteen these values of d. If in addition we set $h^0(\mathscr{E}) \leq r$ also for $0 \leq d \leq r - 1$, and $h^0(\mathscr{E}) \leq d + (2 - g)r$ for $(2g - 3)r \leq d \leq (2g - 2)r$ (the "outer ends" of the Clifford bound), we have an increasing piecewise linear continuous upper bound $L(d)$ for $h^0(\mathscr{E})$ as a function of $d = \deg(\mathscr{E})$ for $0 \leq d \leq (2g - 2)r$ (strictly increasing for $r \leq d \leq (2g - 2)r$, and which can be slightly improved for the two outer pieces, see Fig. 1 of [7]). We then obtain that $f(d)$ is dominated by this upper bound $L(d)$, and that $\phi(h)$ is at least the inverse of L. We then apply Proposition 12:

$$S_{h,0} \leq \mu(\mathscr{E}) - \frac{\phi(h)}{r}.$$

For the interval "in the middle", i.e., $2r \leq h \leq (g - 1 - \gamma)r$, corresponding to $(\gamma + 2)r \leq d \leq (2g - 4 - \gamma)r$ and $\phi(h) \geq 2h + r(\gamma - 2)$, this gives:

$$S_{h,0} \leq \mu(\mathscr{E}) - \frac{2h}{r} + 2 - \gamma. \tag{5}$$

This gives an improvement of γ for the upper bound on $S_{h,0}$ compared with the bound in Corollary 14.

For the interval with the smallest h's, i.e., $r \leq h \leq 2r$, corresponding to $r \leq d \leq r(\gamma + 2)$ and $\phi(h) \geq (\gamma + 1)h - r\gamma$, this gives

$$S_{h,0} \leq \mu(\mathscr{E}) - \frac{(\gamma + 1)h}{r} + \gamma.$$

As an example, if $h = 2r$, this gives an improvement of γ for the upper bound on $S_{h,0}$ compared with the bound in Corollary 14.

Luckily, there are other, although weaker, results that are theorems, not merely conjectures. In [7], the following theorem is also stated. Mercat only presents the result for the first two intervals, but the last one follows by duality from the first one.

Proposition 33 *If $\mathscr{E}$ is a semi-stable rank r bundle of degree d and X is a smooth curve with Clifford index at least 2, then the following holds:*

– *If $1 \leq \mu(\mathscr{E}) \leq 2 + \frac{2}{g-4}$, then $h^0(\mathscr{E}) \leq f(d) \leq \frac{d-r}{g-2} + r$.*
– *If $2 + \frac{2}{g-4} \leq \mu(\mathscr{E}) \leq 2g - 4 - \frac{2}{g-4}$, then $h^0(\mathscr{E}) \leq f(d) \leq \frac{d}{2}$.*
– *If $2g - 4 - \frac{2}{g-4} \leq \mu(\mathscr{E}) \leq 2g - 3$, then $h^0(\mathscr{E}) \leq f(d) \leq d + (3-g)r - \frac{d+r}{g-2}$.*

For the interval "in the middle", i.e., $\left(1 + \frac{1}{g-4}\right) r \leq h \leq \left(g - 2 - \frac{1}{g-4}\right) r$, corresponding to $\left(2 + \frac{2}{g-4}\right) r \leq d \leq \left(2g - 4 - \frac{2}{g-4}\right) r$ and $\phi(h) \geq 2h$, this gives

$$S_{h,0} \leq \mu(\mathscr{E}) - \frac{2h}{r}$$

and is an improvement of 2 for the upper bound for $S_{h,0}$, compared with the bound in Corollary 14.

5 Examples

Example 34 We consider the Hermitian curve $x^{j+1} + y^{j+1} + z^{j+1} = 0$ over $\mathbb{F}_q$, where $q = j^2$. We have $g = \frac{j^2-j}{2}$ and $m = j^3 + 1$, and we see that (3) holds if $j \geq 3$, i.e., $q \geq 9$, or $g \geq 3$. It also holds for $j = 2,\ q = 4,\ g = 1$ for h with $S_{h,0}$ big enough.

(a) In the case $g = 1,\ j = 2,\ q = 4$, we have an elliptic curve, and according to an unpublished PhD thesis by Agnes Tillmann, whose proof is recalled in [1] (see also the proof of Corollary 3.1 of [8]), there exists a canonical semistable vector bundle $\mathscr{E}_{d,r}$ defined over $\mathbb{F}_q$ of degree d and rank r for all integers $d \in \mathbb{Z}$ and $r \geq 1$, and hence in particular for all integers d and r that we are interested in.

We observe that (2) holds for all t in question for a lot of r, e.g. $r \leq 7$. Putting $r = 7$, and $\mu(\mathscr{E}) = 8$, we have $m > \mu(\mathscr{E}) > 2g$ as in Remark 2, and we obtain a code C with $k = h^0(\mathscr{E}) = r\mu(\mathscr{E}) = 56$ in addition to $n = 9 \cdot (4^6 + \cdots + 4 + 1) = 49149$. We may then use Corollary 27 (a) and conclude as in the corollary for $h \geq \max\{(r-2)g + \mu(\mathscr{E}) + 1 + \frac{3(r-1)}{r}, rg + (t+1)(r-1) + 1\} = \max\{17, rg + (t+1)(r-1) + 1\}$, provided that (3) holds. We can then determine d_h, using Theorem 28, for all $h \geq 44$, and for $h = 20, 21, 26, 27, 28, 32, 33, 34, 35, 38, 39, 40, 41, 42$. One can even determine d_{14} in the same manner, using Corollary 27 (b) for $i = 3$. For $h \geq 49$, the conclusion of Corollary 27 obviously holds (for all q), because then there are codimension h planes entirely contained in fibres of T and therefore in T.

We only need $S_{h,0} \geq 1$ to make (3) hold for these h, and this corresponds to $h \leq k - r = 49$. Hence the conclusion of Theorem 28 holds for all h listed, including all $h \geq 44$.

(b) We consider the case $j = 4,\ g = 6,\ q = 16,\ m = j^3 + 1 = 65$, a plane quintic curve. We assume that we have a semi-stable bundle $\mathscr{E}'$ of rank 5 and degree -10 (see Example 35 below for a candidate). We now tensor the bundle with $\mathcal{O}_{\mathbb{P}^2}(s)$, to obtain a bundle $\mathscr{E}$ of rank 5 and degree $25s - 10$ and slope $\mu = 5s - 2$. To satisfy $\mu(\mathscr{E}) < m = 65$ (See Observation 8), we must have $s \le 13$. So, for simplicity, we set $s = 13$. We observe that $q = 16 \ge (r-1)(r-2) = 12$, and $q \ge 2g+4 = 16$, so part (a) of Corollary 27 can be applied for $h \ge \max\{85, 35+4t\} = 85$ (since $t \le r-1 = 4$). In this case, the dimension of the code is $k = \deg(\mathscr{E}) + r(1-g) = 315 - 25 = 290$. The code length is much bigger: $n = 4543825$. But we may also use part (b) of Corollary 27 in the case $i = 4$, since we observe that $q = 16 \ge \frac{4g^2}{i^2} + \frac{2}{i} = \frac{19}{2}$, for $i = 4$. This part of the corollary can be applied for $h \ge \max\{38, 51\} = 51$ (putting $t = r-1$ in the condition).

To use $i = 4$, we must have $S_{h,0} \ge 4$, and this happens if $h \le k - 4r = 270$. Hence, the J_h and therefore the d_h are all determined by part (b) of Corollary 27 for $51 \le h \le 270$ in this case. But since part (a) covers the cases $271 \le h \le 290$, then all values of d_h for $h \ge 51$ are determined. Considering individual values of t in part (b) of the corollary furthermore gives us d_h for $h = 39, 40, 43, 44, 45, 47, 48, 49, 50$, hence for all $h \ge 47$.

For $h = 39$, we then get $d_h = 1048574$.

For $h \ge 31$, Corollary 18 gives $d \ge 16^4 \cdot \left(70 - \left\lfloor \frac{315-h}{5} \right\rfloor \cdot 5\right)$.

For $h = 39$, this is $16^4 \cdot 15 = 983040$.

For the range $6 \le h \le 30$, we combine Proposition 7 and Corollary 14 and obtain $d_h \ge 65536 \cdot \frac{2h}{5}$. For $h = 15$, this gives $d_h \ge 393216$. In this case, the Clifford index γ of the Hermitian curve X is 1, and if we can trust Conjecture 32 and Eq. (5), we may improve this by 65536 in the range $10 \le h \le 20$, for example to $d_{15} \ge 458752$. Unfortunately, Proposition 33 cannot be applied here because of the assumption $\gamma \ge 2$. Corollary 10 also gives $d(C) \ge 131072$.

For $1 \le h \le 5$, we have $d_h \ge 16^4 \cdot 2 = 131072$, by Proposition 9 and Remark 30.

(c) In the case $g = 21,\ j = 7,\ q = 49,\ m = j^3 + 1 = 344$, we have a plane octic curve, and we assume that there exists a semistable bundle $\mathscr{E}$ on X of degree -24 and rank 9. A candidate could be the kernel of the surjective bundle map $H^0(\mathcal{O}_X(3)) \otimes \mathcal{O}_X \to \mathcal{O}_X(3)$. We tensor this bundle with $\mathcal{O}_{\mathbb{P}^2}(s)$, to obtain a bundle $\mathscr{E}$ of rank 9 and degree $72s - 24$ and slope $8s - \frac{24}{9}$. To satisfy $\mu(\mathscr{E}) < m = 344$ (See Observation 8), we must have $s \le 43$. We choose $s = 40$. So $\deg(\mathscr{E}) = 2856$, and $\mu(\mathscr{E}) = 317.33 > 2g = 42$, so $\mathcal{L}'$ is very ample. We observe that (3) holds, and that $q = 49 \ge \frac{t}{t-1} + \frac{t}{t-2} + \cdots + \frac{t}{2} + t$, for $t = 1, 2, \ldots, r-1 = 8$, so (2) also holds. Hence, part (a) of Corollary 27 can be applied for $h \ge \max\{468, 198 + 8t\} = 468$ ($t \le r-1$). In this case, the dimension of the code is $k = \deg(\mathscr{E}) + r(1-g) = 2676$. The code length is much bigger: $n =$ approximately $1.17 \cdot 10^{16}$.

But we may also use part (b) of Corollary 27 in the case $i = 7$, since we observe that $q = 49 \ge \frac{4g^2}{i^2} + \frac{2}{i} = 36.29$, for $i = 7$. This part of the corollary can be applied for $h \ge \max\{200, 198 + 8t\}$, which gives us d_h for all $h \ge 257$, and several other values for h between 201 and 256. To use $i = 7$, we must have $S_{h,0} \ge 7$, and this happens if $h \le k - 7r = 2613$. Hence, the J_h and therefore the d_h are all determined by part (b) of Corollary 27 for $257 \leq h \leq 2613$ in this case. But since part (a) covers the cases $2614 \leq h \leq 2676$, then all values of d_h for $h \geq 257$ are determined.

For $190 \leq h \leq 256$, Corollary 18 can be applied to give a lower bound for d_h.

For the range $10 \leq h \leq 189$, we combine Proposition 7 and Corollary 14 and obtain $d_h \geq 49^8(\frac{80}{3} + \frac{2h}{9})$. In this case, the Clifford index γ of the Hermitian curve X is 4, and if we can trust Conjecture 32 and (5), we may improve this lower bound for d_h by 4×49^8 compared to this bound from Proposition 14 in the range $18 \leq h \leq 144$. Proposition 33 gives an improvement of 2×49^8 for $10 \leq h \leq 170$ compared to the bound from Proposition 14.

Example 35 Consider (as in Example 34 (b)) the (plane) Hermitian curve X given by $x^5 + y^5 + z^5 = 0$ over $\mathbb{F}_{16}$, and look at the vector bundle $\mathscr{E}$ given as follows: We let $\mathscr{E}'$ be the kernel of the surjective bundle map $H^0(X, \mathscr{O}(2)) \otimes \mathscr{O}_X \to \mathscr{O}(2)$, so that $\operatorname{rank}(\mathscr{E}') = 5$ and $\deg(\mathscr{E}') = -10$. We then let $\mathscr{E} = \mathscr{E}' \otimes \mathscr{O}(2)$, so that $\deg(\mathscr{E}) = 40$.

The vector bundle $\mathscr{E}$ has the following generators:

$$\begin{array}{llllll} x^2e_1, & x^2e_2, & x^2e_3, & x^2e_4, & x^2e_5, & \text{for } \; x \neq 0, \\ x^2e_3, & xye_3 - y^2e_1, & xze_3 - y^2e_2, & yze_3 - y^2e_4, & z^2e_3 - y^2e_5, & \text{for } \; y \neq 0. \end{array}$$

We have $h^0(X, \mathscr{E}) = 21$ (this has to be shown using the definition of $\mathscr{E}$, since we don't have $\mu(\mathscr{E}) > 2g-2$), and so the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ has 21 global sections. We can't see from the degree that this line bundle is very ample, since we don't have $\mu(\mathscr{E}) > 2g$, but this can be shown directly by regarding the global sections. We can choose 21 generators for the global sections of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$, and the polynomials corresponding to the zero sets of these can then act as the coordinates of $\mathbb{P}^{20}$, which we embed $\mathbb{P}(\mathscr{E})$ into. The global sections of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ correspond to the zero sets of the following polynomials on $\mathbb{P}(\mathscr{E})$:

$$\begin{array}{lll} t_0 = x^2e_1 & t_3 = x^2e_2 & t_6 = x(xe_3 - ye_1) \\ t_1 = xye_1 & t_4 = xye_2 & t_7 = y(xe_3 - ye_1) \\ t_2 = xze_1 & t_5 = xze_2 & t_8 = z(xe_3 - ye_1) \\ t_9 = x(ze_1 - xe_4) & t_{12} = x(xe_5 - ze_2) & \\ t_{10} = y(ze_1 - xe_4) & t_{13} = y(xe_5 - ze_2) & \\ t_{11} = z(ze_1 - xe_4) & t_{14} = z(xe_5 - ze_2) & \\ t_{15} = y(xe_4 - ye_2) & t_{17} = y(ze_4 - ye_5) & t_{19} = y(ze_3 - ye_4) \\ t_{16} = z(xe_4 - ye_2) & t_{18} = z(ze_4 - ye_5) & t_{20} = z(ze_3 - ye_4) \end{array}$$

Since the curve X is given by $x^5 + y^5 + z^5 = 0$, the embedded scroll then has the equations

$$\begin{aligned} t_0^5 + t_1^5 + t_2^5 &= 0 \\ t_3^5 + t_4^5 + t_5^5 &= 0 \\ t_6^5 + t_7^5 + t_8^5 &= 0 \end{aligned}$$

$$t_9^5 + t_{10}^5 + t_{11}^5 = 0$$
$$t_{12}^5 + t_{13}^5 + t_{14}^5 = 0,$$

in addition to the zero set of the 2×2 minors of the matrices

$$\begin{pmatrix} t_0 & t_3 & t_6 & t_9 & t_{12} \\ t_1 & t_4 & t_7 & t_{10} & t_{13} \\ t_2 & t_5 & t_8 & t_{11} & t_{14} \end{pmatrix}$$

and

$$\begin{pmatrix} t_1 & t_4 & t_7 & t_{10} & t_{13} & t_{15} & t_{17} & t_{19} \\ t_2 & t_5 & t_8 & t_{11} & t_{14} & t_{16} & t_{18} & t_{20} \end{pmatrix}.$$

We now try to say something about the minimum distance: Consider the hyperplane $t_{10} = 0$. This contains all fibres over the points on X where $y = 0$. It can be checked that $y = 0$ for 5 distinct $\mathbb{F}_q$-rational points on X. We see that $S_{h,0} \geq 5$ for $h = 1, \ldots, 8$, since there are eight linearly independent hyperplanes that contain the fibres over the points corresponding to $y = 0$ on X, namely $t_1, t_4, t_7, t_{10}, t_{13}, t_{15}, t_{17}, t_{19}$. So there exists a $\mathbb{P}^{12}$ that contains 5 fibres. If we add a sixth fibre, which is a $\mathbb{P}^4$, then these altogether 6 fibres must be contained in a $\mathbb{P}^{17}$. It follows that $S_{3,0} \geq 6$, and therefore also $S_{1,0} \geq 6$, and $d_1 \leq 16^4 \cdot (65-6) = 3866624$. We also have $S_{1,0} \leq \mu(\mathscr{E})$. Hence, $6 \leq S_{1,0} \leq 8 = \mu(\mathscr{E})$, and $3735552 \leq d_1 \leq 3866624$. The length of the scroll code is 4543825.

Since a fibre of T is a $\mathbb{P}^4$, it is clear that $S_{h,0} = 0$ for $h \geq 17$, and $S_{16,0} = 1$. The bounds using Conjecture 32 give $S_{h,0} \leq 9 - \frac{2h}{5}$ for $5 \leq h \leq 20$. This gives $S_{h,0} \leq 2, 2, 1, 1, 1$ for $h = 16, 17, 18, 19, 20$, respectively, so it is not always a sharp bound. It does however give $S_{8,0} \leq 5$, so if Conjecture 32 holds, we may conclude that $S_{8,0} = 5$.

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