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Generalized Tanaka–Webster and covariant derivatives on a real hypersurface in a complex projective space

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Abstract We consider real hypersurfaces M in complex projective space equipped with both the Levi-Civita and generalized Tanaka–Webster connections. For any non-null constant k and any vector field X tangent to M we can define an operator on M, $F_X^{(k)}$, related to both connections. We study commutativity problems of these operators and the shape operator of M.

Keywords g-Tanaka–Webster connection · Complex projective space · Real hypersurface · kth Cho operators

Mathematics Subject Classification 53C15 · 53B25

1 Introduction

Let $\mathbb{C}P^m$, $m \geq 2$, be a *complex projective space* endowed with the metric g of constant holomorphic sectional curvature 4. Let M be a *connected real hypersurface* of $\mathbb{C}P^m$ without boundary. Let ∇ be the Levi-Civita connection on M and J the complex structure of $\mathbb{C}P^m$. Take a locally defined unit normal vector field N on M and denote by $\xi = -JN$. This is a tangent vector field to M called the structure vector field

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on M. On M there exists an almost contact metric structure (ϕ, ξ, η, g) induced by the Kaehlerian structure of $\mathbb{C}P^m$, where ϕ is the tangent component of J and η is an one-form given by $\eta(X) = g(X, \xi)$ for any X tangent to M. The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [4,10–12]. His classification contains 6 types of real hypersurfaces. Among them we find type (A_1) real hypersurfaces that are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$ and type (A_2) real hypersurfaces that are tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m - 1$. We will call both types of real hypersurfaces type (A) real hypersurfaces.

Ruled real hypersurfaces can be described as follows: Take a regular curve γ in $\mathbb{C}P^m$ with tangent vector field X. At each point of γ there is a unique $\mathbb{C}P^{m-1}$ cutting γ so as to be orthogonal not only to X but also to JX. The union of these hyperplanes is called a ruled real hypersurface. It will be an embedded hypersurface locally, although globally it will in general have self-intersections and singularities. Equivalently, a ruled real hypersurface satisfies that the maximal holomorphic distribution on M, $\mathbb{D}$, given at any point by the vectors orthogonal to ξ, is integrable and its integral manifolds are $\mathbb{C}P^{m-1}$, or $g(A\mathbb{D}, \mathbb{D}) = 0$. For examples of ruled real hypersurfaces see [5] or [7].

The Tanaka–Webster connection, [13,15], is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR-manifold. As a generalization of this connection, Tanno, [14], defined the generalized Tanaka–Webster connection for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y. \tag{1.1}$$

Using the naturally extended affine connection of Tanno's generalized Tanaka–Webster connection, Cho defined the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ for a real hypersurface M in $\mathbb{C}P^m$ given, see [2,3], by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.2}$$

for any X,Y tangent to M where k is a non-zero real number. Then $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)}g = 0$, $\hat{\nabla}^{(k)}\phi = 0$. In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the g-Tanaka–Webster connection coincides with the Tanaka–Webster connection.

Here we can consider the tensor field of type (1, 2) given by the difference of both connections $F^{(k)}(X, Y) = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any X, Y tangent to M, see [6, Proposition 7.10, pages 234–235]. We will call this tensor the kth Cho tensor on M. Associated to it, for any X tangent to M and any nonnull real number k we can consider the tensor field of type (1, 1) $F^{(k)}_X$, given by $F^{(k)}_X Y = F^{(k)}(X, Y)$ for any $Y \in TM$. This operator will be called the kth Cho operator corresponding to X. The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $\hat{T}^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$ for any X, Y tangent to M.

The purpose of the present paper is to study real hypersurfaces M in $\mathbb{C}P^m$ such that the covariant and g-Tanaka–Webster derivatives of the shape operator coincide. $\nabla A = \bar{\nabla}^{(k)}A$ is equivalent to the fact that, for any X tangent to M, $AF^{(k)}_X = F^{(k)}_X A$.

The meaning of this condition is that every eigenspace of A is preserved by the kth Cho operator $F_X^{(k)}$ for any X tangent to M.

On the other hand $TM = Span\{\xi\} \oplus \mathbb{D}$. Thus we will obtain the following

Theorem 1 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $F_X^{(k)} A = AF_X^{(k)}$ for any $X \in \mathbb{D}$ and any nonnull constant k if and only if M is locally congruent to a ruled real hypersurface.*

Theorem 2 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $F_\xi^{(k)} A = AF_\xi^{(k)}$ for any nonnull constant k if and only if M is locally congruent to a type (A) real hypersurface.*

As a direct consequence of these Theorems we have

Corollary *There do not exist real hypersurfaces M in $\mathbb{C}P^m$, $m \geq 3$, such that for any nonnull constant k $F_X^{(k)} A = AF_X^{(k)}$ for any X tangent to M.*

2 Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi-Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M, see [1]. That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any tangent vectors X, Y to M. From (2.1) we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \phi AX \tag{2.4}$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & -2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \tag{2.6}$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M. We will call the maximal holomorphic distribution $\mathbb{D}$ on M to the following one: at any $p \in M$, $\mathbb{D}(p) = \{X \in T_pM | g(X, \xi) = 0\}$. We will say that M is Hopf if ξ is principal, that is, $A\xi = \alpha\xi$ for a certain function α on M.

In the sequel we need the following results:

Theorem 2.1 [9] *Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geq 2$. Then the following are equivalent:*

1. *M is locally congruent to either a geodesic hypersphere or a tube of radius r, $0 < r < \frac{\pi}{2}$ over a totally geodesic $\mathbb{C}P^n$, $0 < n < m - 1$.*
2. *$\phi A = A\phi$.*

Theorem 2.2 [8] *If ξ is a principal curvature vector with corresponding principal curvature α and $X \in \mathbb{D}$ is principal with principal curvature λ, then ϕX is principal with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

3 Proof of Theorem 1

If we suppose that $F_X^{(k)} A = A F_X^{(k)}$ for any $X \in \mathbb{D}$ we get

$$g(\phi AX, AY)\xi - \eta(AY)\phi AX = g(\phi AX, Y)A\xi - \eta(Y)A\phi AX \tag{3.1}$$

for any $Y \in TM$. Let us suppose that M is non Hopf. Thus locally we can write $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathbb{D}$, α and β are functions on M and $\beta \neq 0$.

If we take $X = Y = \xi$ in (3.1) and its scalar product with ξ we obtain $2g(\phi AX, A\xi) = 0 = 2\beta g(\phi AX, U)$ for any $X \in \mathbb{D}$. As we suppose $\beta \neq 0$ this means

$$A\phi U = 0. \tag{3.2}$$

Taking $Y = U$ in (3.1) we get $g(\phi AX, AU) = 0$ for any $X \in \mathbb{D}$. Thus $A\phi AU = g(A\phi AU, \xi) = g(\phi AU, \beta U) = -\beta g(A\phi U, U) = 0$, where we have applied (3.2). Therefore

$$A\phi AU = 0. \tag{3.3}$$

If we take the scalar product of (3.1) and U we obtain $0 = -\eta(AY)g(\phi AX, U) = \beta g(\phi AX, Y) - \eta(Y)g(A\phi AX, U) = \beta g(\phi AX, Y)$ for any $X \in \mathbb{D}$, $Y \in TM$. Taking $Y = \phi U$ we get $\beta g(AX, U) = 0$ for any $X \in \mathbb{D}$. Thus

$$AU = \beta\xi. \tag{3.4}$$

From (3.2) and (3.4) and the fact that $\phi AX = 0$ for any $X \in \mathbb{D}_U = Span\{\xi, U, \phi U\}^{\perp}$ we obtain $AX = 0$ for any $X \in \mathbb{D}_U$. Then M must be locally congruent to a ruled real hypersurface.

Suppose now that M is Hopf, that is, $A\xi = \alpha\xi$. Taking $Y = \xi$ in (3.1) we get

$$A\phi AX = \alpha\phi AX. \tag{3.5}$$

for any $X \in \mathbb{D}$. Suppose that $X \in \mathbb{D}$ satisfies $AX = \lambda X$. From (3.5) we obtain $\lambda A\phi X = \alpha\lambda\phi X$. Then we have either $\lambda = 0$ or $\lambda \neq 0$ and then $A\phi X = \alpha\phi X$.

Suppose $\lambda = 0$. Therefore $AX = 0$. The Codazzi equation gives $(\nabla_X A)\xi - (\nabla_\xi A)X = -\phi X$. Thus $X(\alpha)\xi + A\nabla_\xi X = -\phi X$. Taking its scalar product with ξ this yields $X(\alpha) = 0$. Then

$$g(\nabla_\xi X, A\phi X) = -1. \tag{3.6}$$

On the other hand $(\nabla_{\phi X} A)\xi - (\nabla_\xi A)\phi X = X$. Then $\nabla_{\phi X}\alpha\xi - A\phi A\phi X - \nabla_\xi A\phi X + A\nabla_\xi\phi X = X$. Taking its scalar product with X we get $g(\nabla_\xi X, A\phi X) = 1$ and bearing in mind (3.2) we arrive to a contradiction.

Therefore we must suppose that $\lambda \neq 0$ and $A\phi X = \alpha\phi X$. From Theorem 2.2 we obtain $\alpha^2 - \alpha\lambda + 2 = 0$. Thus $\alpha \neq 0$. Taking ϕX instead of X in (3.1) we get $-\alpha\lambda g(X, Y)\xi + \alpha^2\eta(Y)X = -\alpha^2 g(X, Y)\xi + \alpha\lambda\eta(Y)X$ for any $Y \in TM$. As $\alpha \neq 0$ this yields $-\lambda g(X, Y)\xi + \alpha\eta(Y)X = -\alpha g(X, Y)\xi + \lambda\eta(Y)X$. Taking $Y = \xi$ we get $\alpha = \lambda$ and M should be totally umbilical, which is impossible. Thus Hopf real hypersurfaces do not satisfy our condition.

Summing up our results, we have proved that M must be locally congruent to a ruled real hypersurface. As any ruled real hypersurface satisfies (3.1) we have finished the proof.

4 Proof of Theorem 2

Let us suppose that M is non Hopf. Thus we write $A\xi = \alpha\xi + \beta U$ for a unit $U \in \mathbb{D}$ and functions α and β on M, β being non-vanishing. If we suppose that $AF_\xi^{(k)} = F_\xi^{(k)}A$ we get

$$\begin{aligned}\alpha\beta g(\phi U, Y)\xi + \beta^2 g(\phi U, Y)U - \beta\eta(Y)A\phi U - kA\phi Y\\ = \beta g(A\phi U, Y)\xi - \beta\eta(AY)\phi U - k\phi AY\end{aligned} \tag{4.1}$$

for any $Y \in TM$.

If we take $Y = \xi$ in (4.1) we obtain

$$A\phi U = (\alpha + k)\phi U. \tag{4.2}$$

And if we take $Y = U$ in (4.1) we have $kA\phi U = \beta^2 \phi U + k\phi AU$. From (4.2) this yields $k\phi AU = -(\beta^2 - k(\alpha + k))\phi U$. Applying ϕ to this equality we get $kAU = k\beta\xi + (k(\alpha + k) - \beta^2)U$. Therefore

$$AU = \beta\xi + \left(\alpha + k - \frac{\beta^2}{k}\right) U. \tag{4.3}$$

This implies that $\mathbb{D}_U$ is A-invariant. Let $Y \in \mathbb{D}_U$ such that $AY = \lambda Y$. From (4.1) $kA\phi Y = k\phi AY$ and, as $k \neq 0$, we obtain $A\phi Y = \lambda\phi Y$. Thus any eigenspace in $\mathbb{D}_U$ is holomorphic (ϕ-invariant).

The Codazzi equation gives us $(\nabla_Y A)\phi Y - (\nabla_{\phi Y} A)Y = -2\xi$. That is $Y(\lambda)\phi Y + \lambda\nabla_Y \phi Y - A\nabla_Y \phi Y - (\phi Y)(\lambda)Y - \lambda\nabla_{\phi Y} Y + A\nabla_{\phi Y} Y = -2\xi$. If we take its scalar product with ϕY (respectively, with Y) we obtain

$$Y(\lambda) = (\phi Y)(\lambda) = 0. \tag{4.4}$$

The scalar product with ξ gives

$$\beta g([Y, \phi Y], U) = -2(\lambda^2 - \alpha\lambda - 1) \tag{4.5}$$

and the scalar product with U yields

$$\left(\lambda - \left(\alpha + k - \frac{\beta^2}{k}\right)\right) g([Y, \phi Y], U) = -2\lambda\beta. \tag{4.6}$$

From (4.5) and (4.6) we get

$$(\lambda^2 - \alpha\lambda - 1)\left(\lambda - \left(\alpha + k - \frac{\beta^2}{k}\right)\right) = \lambda\beta^2. \tag{4.7}$$

From (4.7) we easily see that $\lambda \neq \alpha + k$. We can also write (4.7) as

$$k(\lambda^2 - \alpha\lambda - 1)(\lambda - (\alpha + k)) = (-\lambda^2 + (\alpha + k)\lambda + 1)\beta^2. \tag{4.8}$$

If $-\lambda^2 + (\alpha + k)\lambda + 1 = 0$, either $\lambda = \alpha + k$, which is impossible, or $\lambda^2 - \alpha\lambda - 1 = 0$. This yields $k\lambda = 0$. Therefore $\lambda = 0$ which implies $1 = 0$, which is also impossible.

The Codazzi equation $(\nabla_{\phi U} A)Y - (\nabla_Y A)\phi U = 0$ yields $(\phi U)(\lambda)Y + \lambda\nabla_{\phi U} Y - A\nabla_{\phi U} Y - Y(\alpha)\phi U - (\alpha + k)\nabla_Y \phi U + A\nabla_Y \phi U = 0$. Taking its scalar product with ϕY we get $(\lambda - \alpha - k)g(\nabla_Y \phi U, \phi Y) = 0$. As $\lambda \neq \alpha + k$ we have $g(\nabla_Y \phi U, \phi Y) = 0$ and from (2.3) this gives

$$g(\nabla_Y U, Y) = 0. \tag{4.9}$$

We also have $0 = g((\nabla_\xi A)Y - (\nabla_Y A)\xi, Y) = \xi(\lambda) - \beta g(\nabla_Y U, Y)$. From (4.9) this yields

$$\xi(\lambda) = 0. \tag{4.10}$$

Now $(\nabla_\xi A)U - (\nabla_U A)\xi = \phi U$. This implies $\xi(\beta)\xi + \beta\phi A\xi + \xi(\alpha - \frac{\beta^2}{k})U + (\alpha + k - \frac{\beta^2}{k})\nabla_\xi U - A\nabla_\xi U - U(\alpha)\xi - \alpha\phi AU - U(\beta)U - \beta\nabla_U U + A\phi AU = \phi U$. Taking its scalar product with ξ we obtain

$$\xi(\beta) = U(\alpha). \tag{4.11}$$

Developing $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2\xi$ and taking its scalar product with ϕU we get

$$U(\alpha) + \frac{\beta^2}{k} g(\nabla_{\phi U} U, \phi U) = 0. \tag{4.12}$$

The same procedure applied to $g((\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi, \phi U) = 0$ yields

$$\xi(\alpha) - \beta g(\nabla_{\phi U} U, \phi U) = 0. \tag{4.13}$$

From (4.13) we have $\frac{\beta}{k}\xi(\alpha) - \frac{\beta^2}{k} g(\nabla_{\phi U} U, \phi U) = 0$. From (4.11) to (4.13) it follows

$$U(\alpha) = \xi(\beta) = -\frac{\beta}{k}\xi(\alpha). \tag{4.14}$$

From (4.7) and (4.10) we obtain $-k\lambda(\lambda - (\alpha + k))\xi(\alpha) - k(\lambda^2 - \alpha\lambda - 1)\xi(\alpha) = \lambda\beta^2\xi(\alpha) + 2(-\lambda^2 + (\alpha + k)\lambda + 1))\xi(\beta)$. As $-\lambda^2 + (\alpha + k)\lambda + 1 \neq 0$, we know that

$$\beta^2 = \frac{k(\lambda^2 - \alpha\lambda - 1)(\lambda - \alpha - k)}{-\lambda^2 + (\alpha + k)\lambda + 1}.$$

From this, (4.12) and (4.13), if we suppose $\xi(\alpha) \neq 0$ we obtain

$$\begin{aligned} &(-2k^2\lambda^2 + 2k^2\lambda\alpha + k^3\lambda + k^2)(-\lambda^2 + (\alpha + k)\lambda + 1) \\ &\quad + k^2\lambda(\lambda^2 - \lambda\alpha - 1)(\lambda - \alpha - k) - 2k(\lambda^2 - \alpha\lambda - 1)^2(\lambda - \alpha - k) = 0. \end{aligned} \tag{4.15}$$

By derivating several times this expression and bearing in mind the above relations we obtain $12k\lambda^2\xi(\alpha) = 0$. If $\xi(\alpha) \neq 0$ we have $\lambda = 0$. From (4.8) $k(\alpha + k) = \beta^2$. Thus $k\xi(\alpha) = 2\beta\xi(\beta) = -2\frac{\beta^2}{k}\xi(\alpha)$, where we have applied (4.14). Then $k^2 = -2\beta^2$, giving a contradiction. Thus we have proved

$$\xi(\alpha) = 0. \tag{4.16}$$

From (4.14) and (4.16) we also get

$$U(\alpha) = \xi(\beta) = 0. \tag{4.17}$$

From $g((\nabla_\xi A)Y - (\nabla_Y A)\xi, \xi) = 0$ we have

$$Y(\alpha) = -\beta g(\nabla_\xi Y, U) \tag{4.18}$$

and $g((\nabla_\xi A)Y - (\nabla_Y A)\xi, U) = 0$ yields

$$Y(\beta) = \left(\lambda - \alpha - k + \frac{\beta^2}{k}\right) g(\nabla_\xi Y, U). \tag{4.19}$$

From (4.18) and (4.19) we get

$$\beta Y(\beta) = \left(\alpha + k - \lambda - \frac{\beta^2}{k}\right) Y(\alpha). \tag{4.20}$$

Taking the derivative of (4.7) in the direction of Y we have

$$\begin{aligned}
&-\lambda\left(\lambda - \alpha - k + \frac{\beta^2}{k}\right) Y(\alpha) + (\lambda^2 - \alpha\lambda - 1)\left(-Y(\alpha) + \frac{2\beta}{k} Y(\beta)\right)\\
&\quad = 2\lambda\beta Y(\beta).
\end{aligned} \tag{4.21}$$

Introducing (4.20) in (4.21) and supposing that $Y(\alpha) \neq 0$ we get

$$\begin{aligned}
&-\lambda^2 + \lambda\alpha + \lambda k - \frac{\lambda}{k}\beta^2\\
&\quad +(\lambda^2 - \alpha\lambda - 1)\left(-1 + \frac{2}{k}\left(\alpha + k - \lambda - \frac{\beta^2}{k}\right)\right)\\
&\quad = 2\lambda\alpha + 2\lambda k - 2\lambda^2 - \frac{2\lambda}{k}\beta^2.
\end{aligned} \tag{4.22}$$

From (4.7), (4.22) yields $-\lambda k + 1 + \frac{2}{k}(-\beta^2\lambda) = -\frac{\lambda}{k}\beta^2$. Thus $-\lambda k + 1 = \frac{\lambda}{k}\beta^2$. This gives

$$Y(\beta) = 0 \tag{4.23}$$

If $\lambda = \alpha + k - \frac{\beta^2}{k}$, (4.23) yields $Y(\alpha) = 0$. If $\lambda \neq \alpha + k - \frac{\beta^2}{k}$ from (4.20) and (4.23) we arrive to the same result. Therefore

$$Y(\alpha) = 0. \tag{4.24}$$

By a linearity argument we also have

$$X(\alpha) = X(\beta) = 0 \tag{4.25}$$

for any $X \in \mathbb{D}_U$.

As $g((\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi, \xi) = 0$ we obtain

$$(\phi U)(\alpha) = -3\beta(\alpha + k) + \alpha\beta - \beta g(\nabla_\xi \phi U, U) \tag{4.26}$$

and from $g((\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi, U) = -1$ we get

$$(\phi U)(\beta) = \beta^2 \left(2 + \frac{\alpha}{k}\right) - \alpha k - k^2 + 1 - \frac{\beta^2}{k} g(\nabla_\xi U, \phi U). \tag{4.27}$$

From the Codazzi equation $g((\nabla_\xi A)U - (\nabla_U A)\xi, \phi U) = 1$. This yields

$$-\frac{\beta^2}{k} g(\nabla_\xi U, \phi U) - \beta g(\nabla_U U, \phi U) = -k\alpha + k^2 + 1. \tag{4.28}$$

Now $g((\nabla_U A)\phi U - (\nabla_{\phi U} A)U, U) = 0$. This implies

$$\frac{\beta^2}{k} g(\nabla_U \phi U, U) + 3\beta(\alpha + k) - \frac{\beta^3}{k} - (\phi U)(\alpha) + \frac{2\beta}{k}(\phi U)(\beta) = 0. \tag{4.29}$$

Introducing (4.26) and (4.27) into (4.29) we obtain

$$\begin{aligned} -\left(\beta + \frac{2\beta^3}{k^2}\right) g(\nabla_\xi U, \phi U) + \frac{\beta^2}{k} g(\nabla_U \phi U, U) \\ = -3\alpha\beta - 4k\beta - \frac{3\beta^3}{k} - \frac{2\beta^3\alpha}{k^2} - \frac{2\beta}{k}. \end{aligned} \tag{4.30}$$

Consider the system of linear equations given by (4.28) and (4.30). The matrix of coefficients has determinant $-\beta^2(1 + \frac{\beta^2}{k^2})$ that never vanishes. Thus the system has a unique solution. If we solve it we find

$$g(\nabla_\xi U, \phi U) = 3k\left(\frac{k^2 + \beta^2 + 1}{k^2 + \beta^2}\right) + 2\alpha. \tag{4.31}$$

Introducing this in (4.26) we get

$$(\phi U)(\alpha) = \frac{3k\beta}{k^2 + \beta^2}. \tag{4.32}$$

Now (4.16), (4.17), (4.25) and (4.32) yield

$$grad(\alpha) = \frac{3k\beta}{k^2 + \beta^2}\phi U. \tag{4.33}$$

As $g(\nabla_X grad(\alpha), Y) = g(\nabla_Y grad(\alpha), X)$ for any X, Y tangent to M we obtain $X(\omega)g(\phi U, Y) + \omega g(\nabla_X \phi U, Y) = Y(\omega)g(\phi U, Y) + \omega g(\nabla_Y \phi U, X)$ for any X, Y

tangent to M, where we have taken $\omega = \frac{3k\beta}{k^2+\beta^2}$. Taking $X = \xi$ we obtain $-\omega g(U, AX) = \omega g(\nabla_\xi \phi U, X)$. And as $\omega \neq 0$ we get

$$g(U, AX) = -g(\nabla_\xi \phi U, X) \tag{4.34}$$

for any X tangent to M. If we take $X = U$ and bear in mind (4.3) and (4.31) we have

$$\alpha + 2k + \frac{3k}{k^2+\beta^2} + \frac{\beta^2}{k} = 0. \tag{4.35}$$

As $(\nabla_U A)\xi - (\nabla_\xi A)U = -\phi U$, its scalar product with U gives

$$U(\beta) = 0. \tag{4.36}$$

From $g((\nabla_U A)\phi U - (\nabla_{\phi U} A)U, \xi) = -2$ and (4.28) we get

$$(\phi U)(\beta) = \frac{3\alpha\beta^2}{k} + k\left(\alpha + k - \frac{\beta^2}{k}\right) + 1 - 3\beta^2\left(\frac{k^2+\beta^2+1}{k^2+\beta^2}\right) = \mu. \tag{4.37}$$

As in the case of α we arrive to

$$grad(\beta) = \mu\phi U. \tag{4.38}$$

As above $g(\nabla_X grad(\beta), Y) = g(\nabla_Y grad(\beta), X)$ for any X, Y tangent to M. Thus we have either $\mu = 0$ or (4.35). If $\mu = 0$ from (4.38), β should be constant. Now from (4.35) α is also constant and $\omega = 0$ which is impossible. Therefore

$$\alpha = -2k - \frac{3k}{k^2+\beta^2} - \frac{\beta^2}{k}. \tag{4.39}$$

This yields

$$\beta^4 + (3k^2 + k\alpha)\beta^2 + k^3\alpha + 3k^2 + 2k^4 = 0. \tag{4.40}$$

Taking the derivative of (4.40) in the direction of ϕU bearing in mind (4.32) we obtain

$$3k^2(k^2+\beta^2) + 2((k^2+\beta^2)^2 - 3k^2)(\phi U)(\beta) = 0. \tag{4.41}$$

Introducing (4.37) into (4.41) and bearing in mind (4.39) we obtain an equation on k and β equal to 0. Therefore β is a solution of an equation with constant coefficients. Thus β is constant and we arrive to a contradiction.

This proves that M must be Hopf. Then the condition $F_\xi^{(k)} A = AF_\xi^{(k)}$ applied to $X \in \mathbb{D}$ yields $A\phi = \phi A$ on $\mathbb{D}$, because $k \neq 0$. As $A\phi\xi = \phi A\xi = 0$, M must be locally congruent to a real hypersurface of type (A) (Theorem 2.1). The converse is trivial and we have finished the proof.

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