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ON ASYMPTOTICS OF SOLUTIONS OF PARABOLIC EQUATIONS WITH NONLOCAL HIGH-ORDER TERMS

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Abstract. In the paper, we study the Cauchy problem for second-order differential-difference parabolic equations containing translation operators acting to the high-order derivatives with respect to spatial variables. We construct the integral representation of the solution and investigate its long-term behavior. We prove theorems on asymptotic closeness of the constructed solution and the Cauchy problem solutions for classical parabolic equations; in particular, conditions of the stabilization of the solution are obtained.

Introduction

The theory of differential-difference equations (i.e., equations containing translation operators apart from differential ones) is a modern and actively developing research area (see, e.g., [1–3] and the references therein); it also has different applications. However, this mainly refers to *ordinary* differential-difference equations and to the differential-difference equations in the Banach and Hilbert spaces (see, e.g., [4–6] and the references therein). Partial differential-difference equations are investigated in a relatively fewer number of papers. Elliptic differential-difference equations were studied in [3] (see also the references therein). Parabolic equations containing translation operators with respect to the temporal variable were studied in [7–11]. Parabolic equations containing translation operators with respect to spatial variables were studied in [12–18] (see also the references therein).

The latter cited paper is devoted to the long-term behavior of the Cauchy problem solutions for the specified equations; theorems on their (weighted asymptotic) closeness to the Cauchy problem solutions for the classical parabolic equations are proved. However, the investigated equations contain only *low-order* nonlocal terms, more exactly, zero-order differential-difference terms. In this paper, we consider the Cauchy problem for parabolic equations with *high-order* differential-difference terms. We prove that the closeness (and, as a corollary, the stabilization) theorem is valid for the solutions of the specified problems.

1. The Definition of the Fundamental Solution

Let $a, h \in \mathbb{R}^m$. The following equation is considered in $\mathbb{R}^1 \times (0, +\infty)$:

$$\frac{\partial u}{\partial t} = Lu \overset{\text{def}}{=} \frac{\partial^2 u}{\partial x^2} + \sum_{k=1}^{m} a_k \frac{\partial^2 u}{\partial x^2}(x + h_k, t). \tag{1}$$

Equations of the above kind arise, e.g., in the models of nonclassical optics (see [19, 20]).

Consider the real part of the symbol of the operator L (or, which is the same, the symbol of the operator $L + L^*$):

$$\operatorname{Re} L(\xi) = -\xi^2 - \xi^2 \sum_{k=1}^{m} a_k \cos h_k \xi$$

(see [3, Sec. 8]). We say that $-L(\xi)$ is *positive definite* if there exists a positive C such that $-\operatorname{Re} L(\xi) \geq C\xi^2$ for any $\xi \in \mathbb{R}^1$. Similarly to the case of differential operators (see, e.g., [25, p. 66 and p. 78]), the operator

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$-L$ possessing the above property can be called a second-order operator *strongly elliptic* in the whole space.

Below we will assume that the operator $-L$ is strongly elliptic.

Note that the coefficients of the equation can be arbitrarily large under the assumption of strong ellipticity (see, e.g., [3, Ex. 8.1]).

The initial-value condition

$$u|_{t=0} = u_0(x), \tag{2}$$

where $u_0(x)$ is continuous and bounded in $\mathbb{R}^1$, is considered together with Eq. (1).

We define the following function on $\mathbb{R}^1 \times (0, +\infty)$:

$$\mathcal{E}(x,t) \overset{\text{def}}{=} \mathcal{E}_{a,h}(x,t) \overset{\text{def}}{=} \int_0^\infty e^{-t\xi^2\left(1+\sum\limits_{k=1}^m a_k \cos h_k\xi\right)} \cos\left(x\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k\xi\right) d\xi. \tag{3}$$

Since $-L$ is strongly elliptic, we have

$$1 + \sum_{k=1}^m a_k \cos h_k\xi \geq C$$

for any $\xi \neq 0$. Let us prove that the latter inequality also holds (maybe with another positive constant) for $\xi = 0$, which means that

$$1 + \sum_{k=1}^m a_k > 0.$$

Suppose, to the contrary, that

$$1 + \sum_{k=1}^m a_k \leq 0.$$

Then, for any $\xi \neq 0$, we have

$$\begin{aligned} C &\leq 1 + \sum_{k=1}^m a_k - \sum_{k=1}^m a_k + \sum_{k=1}^m a_k \cos h_k\xi = 1 + \sum_{k=1}^m a_k + \sum_{k=1}^m a_k(\cos h_k\xi - 1) \\ &= 1 + \sum_{k=1}^m a_k - 2\sum_{k=1}^m a_k \sin^2 \frac{h_k\xi}{2} = 1 + \sum_{k=1}^m a_k - \frac{1}{2}\sum_{k=1}^m a_k h_k^2 \xi^2 \left(\frac{\sin\frac{h_k\xi}{2}}{\frac{h_k\xi}{2}}\right)^2 \leq -\frac{\xi^2}{2}\sum_{k=1}^m a_k h_k^2 \left(\frac{\sin\frac{h_k\xi}{2}}{\frac{h_k\xi}{2}}\right)^2. \end{aligned}$$

Now, choosing a sufficiently small positive ξ, we obtain a contradiction to the positivity of the constant C. Therefore,

$$|\mathcal{E}(x,t)| \leq \int_0^\infty e^{-Ct\xi^2}\, d\xi = \sqrt{\frac{\pi}{4Ct}};$$

hence, for any $t_0, T \in (0, +\infty)$, integral (3) converges absolutely and uniformly with respect to $(x,t) \in \mathbb{R}^1 \times [t_0, T]$. Thus, $\mathcal{E}(x,t)$ is well defined on $\mathbb{R}^1 \times (0, +\infty)$.

Let us formally differentiate $\mathcal{E}$ with respect to t inside the integral; we have

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} = &-\int_0^\infty \xi^2\left(1 + \sum_{k=1}^m a_k \cos h_k\xi\right) e^{-t\xi^2\left(1+\sum\limits_{k=1}^m a_k \cos h_k\xi\right)} \cos\left(x\xi - t\xi^2\sum_{k=1}^m a_k \sin h_k\xi\right) d\xi \\ &+ \int_0^\infty e^{-t\xi^2\left(1+\sum\limits_{k=1}^m a_k \cos h_k\xi\right)} \sin\left(x\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k\xi\right) \sum_{k=1}^m a_k\xi^2 \sin h_k\xi\, d\,\xi \end{aligned}$$

$$= \sum_{k=1}^{m} a_k \int_0^\infty \xi^2 e^{-t\xi^2\left(1+\sum\limits_{k=1}^{m} a_k \cos h_k\xi\right)}$$

$$\times \left[\sin\left(x\xi - t\xi^2 \sum_{k=1}^{m} a_k \sin h_k\xi\right)\sin h_k\xi - \cos\left(x\xi - t\xi^2 \sum_{k=1}^{m} a_k \sin h_k\xi\right)\cos h_k\xi\right] d\xi$$

$$- \int_0^\infty \xi^2 e^{-t\xi^2\left(1+\sum\limits_{k=1}^{m} a_k \cos h_k\xi\right)} \cos\left(x\xi - t\xi^2 \sum_{k=1}^{m} a_k \sin h_k\xi\right) d\xi$$

$$= \int_0^\infty \xi^2 e^{-t\xi^2\left(1+\sum\limits_{k=1}^{m} a_k \cos h_k\xi\right)}$$

$$\times \left(\sum_{k=1}^{m} a_k \cos\left[(x+h_k)\xi - t\xi^2 \sum_{k=1}^{m} a_k \sin h_k\xi\right] + \cos\left(x\xi - t\xi^2 \sum_{k=1}^{m} a_k \sin h_k\xi\right)\right) d\xi.$$

Further, formally differentiating $\mathcal{E}$ with respect to x inside the integral, we have

$$\frac{\partial^2 \mathcal{E}}{\partial x^2} = -\int_0^\infty \xi^2 e^{-t\xi^2\left(1+\sum\limits_{k=1}^{m} a_k \cos h_k\xi\right)} \cos\left(x\xi - t\xi^2 \sum_{k=1}^{m} a_k \sin h_k\xi\right) d\xi.$$

The absolute values of the above integrals are bounded from above by a linear combination of integrals of the form

$$\int_0^\infty \xi^2 e^{-Ct\xi^2}\, d\xi = \frac{\sqrt{\pi}}{4Ct^{\frac{3}{2}}};$$

hence, they converge absolutely and uniformly with respect to $(x,t) \in \mathbb{R}^1 \times [t_0, T]$ for any $t_0, T \in (0, +\infty)$. Therefore, the above differentiation inside the integral is legitimate. This means that

$$\frac{\partial \mathcal{E}}{\partial t} - \frac{\partial^2 \mathcal{E}}{\partial x^2} = -\int_0^\infty \xi^2 e^{-t\xi^2\left(1+\sum\limits_{k=1}^{m} a_k \cos h_k\xi\right)} \sum_{k=1}^{m} a_k \cos\left[(x+h_k)\xi - t\xi^2 \sum_{k=1}^{m} a_k \sin h_k\xi\right] d\xi$$

$$= -\sum_{k=1}^{m} a_k \int_0^\infty \xi^2 e^{-t\xi^2\left(1+\sum\limits_{k=1}^{m} a_k \cos h_k\xi\right)} \cos\left[(x+h_k)\xi - t\xi^2 \sum_{k=1}^{m} a_k \sin h_k\xi\right] d\xi = \sum_{k=1}^{m} a_k \frac{\partial^2 \mathcal{E}}{\partial x^2}(x+h_k, t)$$

in $\mathbb{R}^1 \times (0, +\infty)$. Thus, $\mathcal{E}(x,t)$ satisfies (in the classical sense) Eq. (1) in $\mathbb{R}^1 \times (0, +\infty)$.

2. The Convolution of the Fundamental Solution with Bounded Functions

For a fixed positive t, we estimate the behavior of $\mathcal{E}(x,t)$ and its derivatives as $x \to \infty$. To do this, we decompose $\mathcal{E}$ into its even and odd with respect to x parts $\mathcal{E}_1(x,t)$ and $\mathcal{E}_2(x,t)$:

$$\mathcal{E}_1(x,t) = \int_0^\infty e^{-t\xi^2\left(1+\sum\limits_{k=1}^{m} a_k \cos h_k\xi\right)} \cos x\xi \cos\left(t\xi^2 \sum_{k=1}^{m} a_k \sin h_k\xi\right) d\xi,$$

$$\mathcal{E}_2(x,t) = \int_0^\infty e^{-t\xi^2\left(1+\sum\limits_{k=1}^{m} a_k \cos h_k\xi\right)} \sin x\xi \sin\left(t\xi^2 \sum_{k=1}^{m} a_k \sin h_k\xi\right) d\xi.$$

Let us prove the following assertion.

Lemma 1. *Let $t > 0$. Then $x^2\mathcal{E}(x,t)$ is bounded in $(-\infty, +\infty)$.*

Proof. Fix an arbitrary positive t and integrate

$$\int_0^\infty e^{-t\xi^2\left(1+\sum\limits_{k=1}^m a_k\cos h_k\xi\right)}\cos\left(t\xi^2\sum_{k=1}^m a_k\sin h_k\xi\right)\cos x\xi\,d\xi$$

by parts two times. This yields

$$\frac{1}{x^2}\int_0^\infty\left[e^{-t\xi^2\left(1+\sum\limits_{k=1}^m a_k\cos h_k\xi\right)}\cos\left(t\xi^2\sum_{k=1}^m a_k\sin h_k\xi\right)\right]''\cos x\xi\,d\xi$$

(it easy to check that all the terms outside the integral vanish).

The latter integral is a bounded function of the variable x; therefore, $x^2\mathcal{E}_1(x,t)$ is bounded. The boundedness of the function $x^2\mathcal{E}_2(x,t)$ is proved in the same way. $\square$

Thus, the following function is defined in $\mathbb{R}^1\times(0,+\infty)$:

$$u(x,t)\overset{\text{def}}{=}\frac{1}{\pi}\int_{-\infty}^{+\infty}\mathcal{E}(x-\xi,t)u_0(\xi)\,d\xi. \tag{4}$$

Further, the following assertion is valid.

Lemma 2. *Let $t>0$. Then $x^2\frac{\partial^2\mathcal{E}}{\partial x^2}(x,t)$ is bounded in $(-\infty,+\infty)$.*

To prove it we decompose $\frac{\partial^2\mathcal{E}}{\partial x^2}$ into parts even and odd with respect to x; then the even (for definiteness) part

$$-\int_0^\infty \xi^2 e^{-t\xi^2\left(1+\sum\limits_{k=1}^m a_k\cos h_k\xi\right)}\cos\left(t\xi^2\sum_{k=1}^m a_k\sin h_k\xi\right)\cos x\xi\,d\xi$$

is integrated by parts twice. The further proof is entirely similar to the proof of Lemma 1.

Obviously, Lemma 2 remains valid if $\frac{\partial^2\mathcal{E}}{\partial x^2}$ is taken at any point $(x+h_k,t)$, $k=\overline{1,m}$, instead of the point (x,t). Since $\mathcal{E}(x,t)$ satisfies Eq. (1) in $\mathbb{R}^1\times(0,+\infty)$, as we prove in the previous section, the following assertion follows from Lemmas 1 and 2.

Lemma 3. *Let $t>0$. Then $x^2\frac{\partial\mathcal{E}}{\partial t}(x,t)$ is bounded in $(-\infty,+\infty)$.*

Lemmas 1–3 and the fact that $\mathcal{E}(x,t)$ satisfies Eq. (1) in $\mathbb{R}^1\times(0,+\infty)$ evidently imply

Theorem 1. *Let the operator $-L$ be strongly elliptic in $\mathbb{R}^1$. Then function* (4) *satisfies* (*in the classical sense*) *Eq.* (1) *in* $\mathbb{R}^1\times(0,+\infty)$.

Remark 1. It is well known that function (4) satisfies problem (1), (2) *in the sense of distributions* (see, e.g., [21]). The only new result of Theorem 1 is the fact that it is a *classical solution* in $\mathbb{R}^1\times(0,+\infty)$.

To prove the uniqueness of this solution, we investigate, according to [21], the real part of the symbol of the elliptic operator L contained in Eq. (1). The specified symbol $\mathcal{P}(z)\overset{\text{def}}{=}\mathcal{P}(\sigma+i\tau)$ is equal to

$$\begin{aligned}
-z^2\left(1+\sum_{k=1}^m a_k e^{-ih_kz}\right)&=(\tau^2-\sigma^2-2i\sigma\tau)\left(1+\sum_{k=1}^m a_k e^{-ih_kz}\right)\\
&=(\tau^2-\sigma^2-2i\sigma\tau)\left(1+\sum_{k=1}^m a_k e^{h_k\tau-ih_k\sigma}\right)\\
&=(\tau^2-\sigma^2-2i\sigma\tau)\left(1+\sum_{k=1}^m a_k e^{h_k\tau}\cos h_k\sigma-i\sum_{k=1}^m a_k e^{h_k\tau}\sin h_k\sigma\right).
\end{aligned}$$

Thus,

$$\operatorname{Re}\mathcal{P}(z) = (\tau^2 - \sigma^2)\left(1 + \sum_{k=1}^{m} a_k e^{h_k\tau}\cos h_k\sigma\right) - 2\sigma\tau\sum_{k=1}^{m} a_k e^{h_k\tau}\sin h_k\sigma.$$

Now let us estimate the function $\mathcal{Q}(z, t_0, t) \stackrel{\text{def}}{=} e^{(t-t_0)\mathcal{P}(z)}$:

$$|\mathcal{Q}(z, t_0, t)| \le e^{(t-t_0)[C_1(1+\sigma^4)+C_2 e^{C_3\tau}]}.$$

It follows from the latter estimate (see [21, Chap. 2, Appendix 1]) that problem (1), (2) has no more than one solution in the sense of distributions.

Remark 2. In general, the uniqueness of the solution of problem (1), (2) (in the corresponding spaces of distributions) holds for much wider classes of initial-value functions than the class of continuous bounded functions. In particular, it holds for the Tikhonov classes and their generalizations (see [22,23]). However, we consider only continuous, bounded, initial-value functions because we study the closeness of the solution of the specified problem to the solutions of *classical* parabolic problems.

Using the proved uniqueness of the solution, we can compute the integral of the fundamental solution over the whole real axis.

Lemma 4. $\int\limits_{-\infty}^{\infty} \mathcal{E}(x, t)\, dx = \pi.$

Proof. Consider $u_0(x) \equiv 1$; it is continuous and bounded; hence, in $\mathbb{R}^1 \times (0, +\infty)$ the function

$$y(x, t) \stackrel{\text{def}}{=} \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} \mathcal{E}(x - \xi, t)\, d\xi$$

satisfies Eq. (1) and the initial-value condition $y(x, 0) \equiv 1$. However $y(x, t)$ does not depend on x:

$$\int\limits_{-\infty}^{+\infty} \mathcal{E}(x - \xi, t)\, d\xi = \int\limits_{-\infty}^{+\infty} \mathcal{E}(\xi, t) d\xi = \pi y(t).$$

Therefore, $y(t)$ satisfies the *ordinary* differential equation $y' = 0$ and the initial-value condition $y(0) = 1$. Hence, $y(t) \equiv 1$. □

3. Asymptotic Properties of the Solution

The long-term behavior of $u(x, t)$ is studied in this section. We consider the Cauchy problem for the heat equation with the same initial-value condition (2) and denote its classical bounded solution by $v(x, t)$. We also denote the positive constant $1 + \sum\limits_{k=1}^{m} a_k$ by p.

The following assertion is valid.

Theorem 2. $\lim\limits_{t\to\infty}[u(x, t) - v(x, pt)] = 0$ *for any real* x.

Proof. Fix an arbitrary real x_0 and consider $u(x_0, t)$. The change of variables $\eta = \frac{x_0-\xi}{2\sqrt{t}}$ yields

$$u(x_0, t) = \frac{2\sqrt{t}}{\pi}\int\limits_{-\infty}^{+\infty} \mathcal{E}(2\sqrt{t}\eta, t) u_0(x_0 - 2\sqrt{t}\eta)\, d\eta.$$

Further,

$$\sqrt{t}\,\mathcal{E}(2\sqrt{t}\eta,t)=\sqrt{t}\int\limits_0^\infty e^{-t\xi^2\left(1+\sum\limits_{k=1}^m a_k\cos h_k\xi\right)}\cos\Big(2\sqrt{t}\eta\xi-t\xi^2\sum_{k=1}^m a_k\sin h_k\xi\Big)d\xi$$

$$=\int\limits_0^\infty e^{-z^2\left(1+\sum\limits_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}\right)}\cos\Big(2z\eta-z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\Big)dz.$$

This implies that

$$u(x_0,t)=\frac{2}{\pi}\int\limits_{-\infty}^{+\infty}u_0(x_0-2\sqrt{t}\eta)\int\limits_0^\infty e^{-z^2\left(1+\sum\limits_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}\right)}\cos\Big(2z\eta-z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\Big)dz\,d\eta.$$

Then

$$u(x_0,t)-v(x_0,pt)$$

$$=\frac{2}{\pi}\int\limits_{-\infty}^{+\infty}u_0(x_0-2\sqrt{t}\eta)\int\limits_0^\infty\Big[e^{-z^2\left(1+\sum\limits_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}\right)}\cos\Big(2z\eta-z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\Big)-e^{-pz^2}\cos 2z\eta\Big]\,dz\,d\eta.\quad(5)$$

The following two lemmas will be used in the further proof.

Lemma 5.

$$\int\limits_0^\infty\Big[e^{-z^2\left(1+\sum\limits_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}\right)}\cos\Big(2z\eta-z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\Big)-e^{-pz^2}\cos 2z\eta\Big]\,dz\xrightarrow{t\to\infty}0$$

uniformly with respect to $\eta\in(-\infty,+\infty)$.

Proof. We fix an arbitrary positive ε and decompose the estimated integral into two terms:

$$\int\limits_0^\delta+\int\limits_\delta^\infty\overset{\text{def}}{=}I_{1,\delta}+I_{2,\delta}.$$

The modulus of the whole estimated integral is estimated from above by

$$2\int\limits_0^\infty e^{-Cz^2}\,dz;$$

therefore, there exists a positive δ such that $|I_{2,\delta}|\le\frac{\varepsilon}{2}$ for any $\eta\in(-\infty,+\infty)$, $t>0$. Fix this δ and consider $I_{1,\delta}$. Its integrand is equal to

$$e^{-pz^2}\Big[e^{z^2\sum\limits_{k=1}^m a_k\left(1-\cos\frac{h_kz}{\sqrt{t}}\right)}\cos\Big(2z\eta-z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\Big)-\cos 2z\eta\Big]$$

$$=e^{-pz^2}\Big(e^{2z^2\sum\limits_{k=1}^m a_k\sin^2\frac{h_kz}{2\sqrt{t}}}\Big[\cos 2z\eta\cos\Big(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\Big)+\sin 2z\eta\sin\Big(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\Big)\Big]-\cos 2z\eta\Big)$$

$$=e^{-pz^2}\Big(\cos 2z\eta\Big[e^{2z^2\sum\limits_{k=1}^m a_k\sin^2\frac{h_kz}{2\sqrt{t}}}\cos\Big(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\Big)-1\Big]$$

$$+e^{2z^2\sum\limits_{k=1}^m a_k\sin^2\frac{h_kz}{2\sqrt{t}}}\sin 2z\eta\sin\Big(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\Big)\Big)\overset{\text{def}}{=}A_1(\eta,t;z)+A_2(\eta,t;z).$$

The inequality

$$\left|\int_0^\delta A_2(\eta,t;z)\,dz\right| \le e^{2\delta^2\sum\limits_{k=1}^m |a_k|}\int_0^\delta \left|\sin\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)\right| dz$$

is valid for any η, t. Denote

$$\frac{16\delta^8\left(\sum\limits_{k=1}^m |a_k|\,|h_k|\right)^2 e^{4\delta^2\sum\limits_{k=1}^m |a_k|}}{\varepsilon^2}$$

by T_0. Then for any $t > T_0$ and any $k = \overline{1,m}$,

$$\left|\frac{h_kz}{\sqrt{t}}\right| \le \frac{\varepsilon}{4\delta^3 e^{2\delta^2\sum\limits_{k=1}^m |a_k|}}\,\frac{|h_k|}{\sum\limits_{k=1}^m |a_k|\,|h_k|} \implies \left|\sin\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)\right| \le \left|z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right| \le \frac{\varepsilon}{4\delta e^{2\delta^2\sum\limits_{k=1}^m |a_k|}}$$

(because $0 \le z \le \delta$). Thus,

$$\left|\int_0^\delta A_2(\eta,t;z)\,dz\right| \le \frac{\varepsilon}{4}$$

for any $t > T_0$ and any $\eta \in (-\infty,+\infty)$. It remains to estimate

$$\int_0^\delta A_1(\eta,t;z)\,dz.$$

Its modulus does not exceed

$$\int_0^\delta e^{-pz^2}\left|e^{2z^2\sum\limits_{k=1}^m a_k\sin^2\frac{h_kz}{2\sqrt{t}}}\cos\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)-1\right| dz.$$

The difference under the modulus of the integrand is represented as

$$\begin{aligned}
&e^{2z^2\sum\limits_{k=1}^m a_k\sin^2\frac{h_kz}{2\sqrt{t}}}\cos\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right) - \cos\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right) + \cos\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right) - 1\\
&\qquad = \cos\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)\left(e^{2z^2\sum\limits_{k=1}^m a_k\sin^2\frac{h_kz}{2\sqrt{t}}}-1\right) + \cos\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right) - 1.
\end{aligned}$$

Choose T_1 so large that

$$\left|e^{2z^2\sum\limits_{k=1}^m a_k\sin^2\frac{h_kz}{2\sqrt{t}}}-1\right| \le \frac{\varepsilon}{8\delta}$$

for any $t > T_1$, $z \in [0,\delta]$. This is possible because there exists a positive δ_1 such that $e^x \in \left(1-\frac{\varepsilon}{8\delta}, 1+\frac{\varepsilon}{8\delta}\right)$ for any $x \in (-\delta_1,\delta_1)$. Hence, we can, e.g., assign

$$T_1 \stackrel{\text{def}}{=} \frac{\delta^4\sum\limits_{k=1}^m |a_k|h_k^2}{2\delta_1}.$$

Further, there exists $\delta_2 \in (0,+\infty)$ such that $\cos x \in \left(1-\frac{\varepsilon}{8\delta}, 1+\frac{\varepsilon}{8\delta}\right)$ for any $x \in (-\delta_2,\delta_2)$. Assign

$$T_2 \stackrel{\text{def}}{=} \frac{\delta^6\left(\sum\limits_{k=1}^m |a_k|\,|h_k|\right)^2}{\delta_2^2}.$$

Then for any $t > T_2$, $z \in [0, \delta]$,

$$\left| z^2 \sum_{k=1}^{m} a_k \sin \frac{h_k z}{\sqrt{t}} \right| \le z^2 \sum_{k=1}^{m} \frac{|a_k|\,|h_k z|}{\sqrt{T_2}} \le \frac{\delta^3}{\sqrt{T_2}} \sum_{k=1}^{m} |a_k|\,|h_k| = \delta_2;$$

therefore,

$$\left| \cos\left(z^2 \sum_{k=1}^{m} a_k \sin \frac{h_k z}{\sqrt{t}} \right) - 1 \right| < \frac{\varepsilon}{8\delta}$$

for any $t > T_2, z \in [0, \delta]$. Hence,

$$\left| \int_0^{\delta} A_1(\eta, t; z)\, dz \right| < \frac{\varepsilon}{4} \implies |I_{1,\delta}| < \frac{\varepsilon}{2}$$

for any $t > \max\{T_0, T_1, T_2\}$ and any $\eta \in (-\infty, +\infty)$. $\square$

Lemma 6. *There exists a positive M depending only on a and h, such that*

$$\left| \int_0^{\infty} e^{-z^2\left(1 + \sum\limits_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}}\right)} \cos\left(2z\eta - z^2 \sum_{k=1}^{m} a_k \sin \frac{h_k z}{\sqrt{t}} \right) dz \right| \le \frac{M}{\eta^2}.$$

Proof. Represent the estimated integral as

$$\int_0^{\infty} e^{-z^2\left(1 + \sum\limits_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}}\right)} \cos 2z\eta \cos\left(z^2 \sum_{k=1}^{m} a_k \sin \frac{h_k z}{\sqrt{t}} \right) dz$$
$$+ \int_0^{\infty} e^{-z^2\left(1 + \sum\limits_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}}\right)} \sin 2z\eta \sin\left(z^2 \sum_{k=1}^{m} a_k \sin \frac{h_k z}{\sqrt{t}} \right) dz \quad (6)$$

and consider the first (for definiteness) term of the above sum.

Assuming that t is a parameter, we denote

$$e^{-z^2\left(1 + \sum\limits_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}}\right)} \cos\left(z^2 \sum_{k=1}^{m} a_k \sin \frac{h_k z}{\sqrt{t}} \right)$$

by $g(z)$ and integrate

$$\int_0^{\infty} g(z) \cos 2\eta z\, dz$$

by parts. We obtain that

$$g(z) \frac{\sin 2\eta z}{2\eta} \bigg|_{z=0}^{z=+\infty} - \frac{1}{2\eta} \int_0^{\infty} g'(z) \sin 2\eta z\, dz = -\frac{1}{2\eta} \int_0^{\infty} g'(z) \sin 2\eta z\, dz,$$

because $g(+\infty) = 0$ since

$$1 + \sum_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}} \ge C > 0.$$

Integrating by parts again, we get

$$g'(z) \frac{\cos 2\eta z}{4\eta^2} \bigg|_{z=0}^{z=+\infty} - \frac{1}{4\eta^2} \int_0^{\infty} g''(z) \cos 2\eta z\, dz.$$

We have

$$g'(z) = e^{-z^2\left(1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}\right)}\left[-2z\left(1+\sum_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}\right) + \frac{z^2}{\sqrt{t}}\sum_{k=1}^{m} a_k h_k \sin\frac{h_k z}{\sqrt{t}}\right]\cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)$$
$$- e^{-z^2\left(1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}\right)}\sin\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\left(2z\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}} + \frac{z^2}{\sqrt{t}}\sum_{k=1}^{m} a_k h_k \cos\frac{h_k z}{\sqrt{t}}\right)$$
$$= -e^{-z^2\left(1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}\right)}\left[\frac{z^2}{\sqrt{t}}\sin\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\sum_{k=1}^{m} a_k h_k \cos\frac{h_k z}{\sqrt{t}}\right.$$
$$- \frac{z^2}{\sqrt{t}}\cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\sum_{k=1}^{m} a_k h_k \sin\frac{h_k z}{\sqrt{t}} + 2z\cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\sum_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}$$
$$\left. + 2z\sin\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}} + 2z\cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\right]$$
$$= e^{-z^2\left(1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}\right)}\left[\frac{z^2}{\sqrt{t}}\sum_{k=1}^{m} a_k h_k \sin\left(\frac{h_k z}{\sqrt{t}} - z^2\sum_{l=1}^{m} a_l \sin\frac{h_l z}{\sqrt{t}}\right)\right.$$
$$\left. + 2z\sum_{k=1}^{m} a_k \cos\left(\frac{h_k z}{\sqrt{t}} - z^2\sum_{l=1}^{m} a_l \sin\frac{h_l z}{\sqrt{t}}\right) - 2z\cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\right].$$

Thus,

$$g'(0) = g'(+\infty) = 0 \implies \int\limits_0^\infty g(z)\cos 2\eta z\, dz = -\frac{1}{4\eta^2}\int\limits_0^\infty g''(z)\cos 2\eta z\, dz.$$

Obviously, there exists a polynomial $P(z)$ such that its positive coefficients depend only on a and h while the inequality

$$|g''(z)| \le e^{-z^2\left(1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}\right)} P(z)$$

holds in $[0,+\infty)$ for any $t \ge 1$. Therefore,

$$\left|\int\limits_0^\infty g(z)\cos 2\eta z\, dz\right| \le \frac{1}{4\eta^2}\int\limits_0^\infty e^{-Cz^2}P(z)\, dz$$

for any $t > 1$ and any $\eta \in \mathbb{R}^1 \setminus \{0\}$. Thus, the claimed estimate is fulfilled for the first term of (6). The second term is estimated in the same way. □

We are able to complete the proof of Theorem 2 now. Decompose (5) into the sum

$$\frac{2}{\pi}\left(\int\limits_{-\infty}^{-R} + \int\limits_{-R}^{R} + \int\limits_{R}^{+\infty}\right) \stackrel{\text{def}}{=} \frac{2}{\pi}[I_{3,R}(t) + I_{4,R}(t) + I_{5,R}(t)],$$

where R is a positive parameter. Assuming (without loss of generality) that $t > 1$, we have

$$|I_{5,R}(t)| \le \sup_{\mathbb{R}^1}|u_0(x)|\int\limits_R^{+\infty}\left(\frac{M}{\eta^2} + \frac{\sqrt{\pi}}{\sqrt{4p}}e^{-\frac{\eta^2}{p}}\right)d\eta$$

by virtue of Lemma 6 and the boundedness of u_0. The latter integral converges; hence, for any positive ε, there exists $R_0 \in (1,+\infty)$ such that

$$|I_{5,R_0}(t)| \le \frac{\pi\varepsilon}{6}$$

for any $t \in (1,+\infty)$. In this case, I_{3,R_0} obviously satisfies the same estimate.

Fix that R_0 (fixing ε beforehand) and consider $I_{4,R_0}(t)$. Its modulus does not exceed

$$\sup_{\mathbb{R}^1}|u_0(x)| \int\limits_{-R_0}^{+R_0} \left| \int\limits_0^{\infty} \left[e^{-z^2\left(1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}\right)} \cos\left(2z\eta - z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) - e^{-pz^2}\cos 2z\eta\right] dz \right| d\eta.$$

Due to Lemma 5, there exists $T^* > 1$ such that the modulus of the inner integral of the latter expression does not exceed

$$\frac{\pi\varepsilon}{12R_0 \sup\limits_{\mathbb{R}^1}|u_0(x)|}$$

for any $t > T^*$ and any real η. This implies that the modulus of expression (5) does not exceed ε for any $t > T^*$. Since the positive ε was chosen arbitrarily, this means that $\lim\limits_{t\to\infty}[u(x_0,t) - v(x_0,pt)] = 0$. This completes the proof of Theorem 2 because the real x_0 was chosen arbitrarily. $\square$

Corollary 1. *Let* $x, l \in (-\infty,+\infty)$. *Then*

$$\lim_{t\to\infty} u(x,t) = l \iff \lim_{R\to\infty} \frac{1}{2R}\int\limits_{-R}^{R} u_0(x)\,dx = l.$$

To prove this, it is enough to note that the above corollary is a classical theorem on pointwise stabilization (see [24]). It holds for the function $v(x,t)$; then it remains to apply Theorem 2 directly.

Remark 3. Note that, although Theorem 2 and Corollary 1 hold under the same assumptions, the assertion of Theorem 1 is stronger because it establishes the *closeness* of the solutions; unlike Corollary 1, which is a *stabilization* theorem, it informs us about the behavior of the solution also in the case where the (necessary and sufficient) condition of stabilization is not satisfied.

4. The Case of Several Spatial Variables

Let $n, m_1, \dots, m_n \in \mathbb{N}$, $a_i, b_i \in \mathbb{R}^{m_i}$, where $a_i = (a_{i1},\dots,a_{im_i})$, $b_i = (b_{i1},\dots,b_{im_i})$, $i = \overline{1,n}$. Consider the following equation in $\{x \in \mathbb{R}^n \mid t > 0\}$:

$$\frac{\partial u}{\partial t} = L_{(n)}u \stackrel{\text{def}}{=} \Delta u + \sum_{i=1}^{n}\sum_{j=1}^{m_i} a_{ij}\frac{\partial^2 u}{\partial x_i^2}(x_1,\dots,x_{i-1},x_i+b_{ij},x_{i+1},\dots,x_n,t). \tag{7}$$

We also consider initial-value condition (2), where u_0 is assumed to be continuous and bounded in $\mathbb{R}^n$.

As in Sec. 1 (see also [3, Sec. 8]), we impose the condition of positive definiteness on the symbol of the operator $-L_{(n)}$: there exists a positive constant C such that

$$-\operatorname{Re} L_{(n)}(\xi) = |\xi|^2 + \sum_{i=1}^{n}\xi_i^2\sum_{j=1}^{m_i} a_{ij}\cos b_{ij}\xi_i \ge C|\xi|^2$$

for any $\xi \in \mathbb{R}^n$.

As in the one-dimensional case, the operator $-L_{(n)}$ possessing the above property will be called *strongly elliptic* in the whole space.

Note that, as in the one-dimensional case (cf. also [3, Ex. 8.1]), the condition of strong ellipticity does not restrict the values of the coefficients of the equation.

Also note that, as in the case of a bounded domain (see [3, Sec. 9]), the strong ellipticity is substantially different for differential and differential-difference operators; thus, the influence of the difference terms has a fundamental importance.

Define the following function in $\mathbb{R}^n \times (0, \infty)$:

$$\mathcal{E}_{(n)}(x,t) \overset{\text{def}}{=} \frac{1}{2^n} \int\limits_{\mathbb{R}^n} e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right)} \cos\left(x\xi - t \sum_{i=1}^{n} \xi_i^2 \sum_{j=1}^{m_i} a_{ij} \sin b_{ij}\xi_i \right) d\xi. \tag{8}$$

The power of the latter exponent is equal to

$$-t \sum_{i=1}^{n} \xi_i^2 \left(1 + \sum_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i \right);$$

if $(\xi, t) \in \mathbb{R}^n \times (0, \infty)$, then it does not exceed

$$-t \sum_{i=1}^{n} C_i \xi_i^2,$$

where $C_1, \dots, C_n$ are positive. Indeed, take an arbitrary $i \in \overline{1,n}$ and apply the strong ellipticity condition taken for $\xi_1 = \dots = \xi_{i-1} = \xi_{i+1} = \dots = \xi_n = 0$. We obtain that

$$\xi_i^2 + \xi_i^2 \sum_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i \geq C\xi_i^2$$

for any real ξ_i. Hence,

$$1 + \sum_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i \geq C$$

for any $\xi_i \neq 0$. Let us prove that the latter inequality holds (maybe with another positive constant) for $\xi = 0$ as well, i.e., that

$$1 + \sum_{j=1}^{m_i} a_{ij} > 0.$$

To do this, we suppose, to the contrary, that

$$1 + \sum_{j=1}^{m_i} a_{ij} \leq 0.$$

Then for any $\xi_i \neq 0$,

$$\begin{aligned}
C &\leq 1 + \sum_{j=1}^{m_i} a_{ij} - \sum_{j=1}^{m_i} a_{ij} + \sum_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i \\
&= 1 + \sum_{j=1}^{m_i} a_{ij} + \sum_{j=1}^{m_i} a_{ij}(\cos b_{ij}\xi_i - 1) = 1 + \sum_{j=1}^{m_i} a_{ij} - 2\sum_{j=1}^{m_i} a_{ij} \sin^2 \frac{b_{ij}\xi_i}{2} \\
&= 1 + \sum_{j=1}^{m_i} a_{ij} - \frac{1}{2} \sum_{j=1}^{m_i} a_{ij} b_{ij}^2 \xi_i^2 \left(\frac{\sin \frac{b_{ij}\xi_i}{2}}{\frac{b_{ij}\xi_i}{2}} \right)^2 \leq -\frac{\xi_i^2}{2} \sum_{j=1}^{m_i} a_{ij} b_{ij}^2 \left(\frac{\sin b_{ij}\xi_i 2}{\frac{b_{ij}\xi_i}{2}} \right)^2.
\end{aligned}$$

Now, choosing a sufficiently small positive ξ_i, we obtain a contradiction to the positivity of the constant C.

Thus, integral (8) converges absolutely and uniformly with respect to $(x,t) \in \mathbb{R}^n \times [t_0, T]$ for any $[t_0, T] \subset (0, +\infty)$; therefore, $\mathcal{E}_{(n)}(x,t)$ is well defined.

Let us formally differentiate $\mathcal{E}_{(n)}$ with respect to t inside the integral; we have

$$2^n \frac{\partial \mathcal{E}_{(n)}}{\partial t} = \int\limits_{\mathbb{R}^n} e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right)} \sin\left(x\xi - t\sum_{i=1}^{n} \xi_i^2 \sum_{j=1}^{m_i} a_{ij} \sin b_{ij}\xi_i\right) \sum_{i=1}^{n} \xi_i^2 \sum_{j=1}^{m_i} a_{ij} \sin b_{ij}\xi_i \, d\xi$$

$$- \int\limits_{\mathbb{R}^n} \left(|\xi|^2 + \sum_{i=1}^{n} \xi_i^2 \sum_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right) e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right)} \cos\left(x\xi - t\sum_{i=1}^{n} \xi_i^2 \sum_{j=1}^{m_i} a_{ij} \sin b_{ij}\xi_i\right) d\xi$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m_i} a_{ij} \int\limits_{\mathbb{R}^n} e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right)}$$

$$\times \xi_i^2 \left[\sin b_{ij}\xi_i \sin\left(x\xi - t\sum_{k=1}^{n} \xi_k^2 \sum_{l=1}^{m_k} a_{kl} \sin b_{kl}\xi_k\right) - \cos b_{ij}\xi_i \cos\left(x\xi - t\sum_{k=1}^{n} \xi_k^2 \sum_{l=1}^{m_k} a_{kl} \sin b_{kl}\xi_k\right)\right] d\xi$$

$$- \int\limits_{\mathbb{R}^n} |\xi|^2 e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right)} \cos\left(x\xi - t\sum_{k=1}^{n} \xi_k^2 \sum_{l=1}^{m_k} a_{kl} \sin b_{kl}\xi_k\right) d\xi$$

$$= - \int\limits_{\mathbb{R}^n} |\xi|^2 e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right)} \cos\left(x\xi - t\sum_{k=1}^{n} \xi_k^2 \sum_{l=1}^{m_k} a_{kl} \sin b_{kl}\xi_k\right) d\xi$$

$$- \sum_{i=1}^{n} \sum_{j=1}^{m_i} a_{ij} \int\limits_{\mathbb{R}^n} \xi_i^2 e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right)} \cos\left(x\xi + b_{ij}\xi_i - t\sum_{k=1}^{n} \xi_k^2 \sum_{l=1}^{m_k} a_{kl} \sin b_{kl}\xi_k\right) d\xi.$$

Further, formally differentiating $\mathcal{E}_{(n)}$ with respect to x_i inside the integral, we have

$$2^n \frac{\partial^2 \mathcal{E}_{(n)}}{\partial x_i^2} = - \int\limits_{\mathbb{R}^n} \xi_i^2 e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right)} \cos\left(x\xi - t\sum_{k=1}^{n} \xi_k^2 \sum_{l=1}^{m_k} a_{kl} \sin b_{kl}\xi_k\right) d\xi,$$

$$2^n \frac{\partial^2 \mathcal{E}_{(n)}}{\partial x_i^2}(x_1, \ldots, x_{i-1}, x_i + b_{ij}, x_{i+1}, \ldots, x_n, t)$$
$$= - \int\limits_{\mathbb{R}^n} \xi_i^2 e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right)} \cos\left(x\xi + b_{ij}\xi_i - t\sum_{k=1}^{n} \xi_k^2 \sum_{l=1}^{m_k} a_{kl} \sin b_{kl}\xi_k\right) d\xi.$$

All of the above integrals converge absolutely and uniformly with respect to $(x, t) \in \mathbb{R}^n \times [t_0, T]$ for any $[t_0, T] \subset (0, +\infty)$; therefore, $\mathcal{E}_{(n)}(x, t)$ satisfies (in the classical sense) Eq. (7) in $\mathbb{R}^n \times (0, +\infty)$.

The following assertion is valid.

Lemma 7. *Let $x \in \mathbb{R}^n$, $t > 0$. Then*

$$\int\limits_{\mathbb{R}^n} u_0(x - \xi)\mathcal{E}_{(n)}(\xi, t)\, d\xi \tag{9}$$

converges absolutely.

Proof. Since (8) converges absolutely, we can apply the Fubini theorem to it; this means that $\mathcal{E}_{(n)}(x,t)$ is equal to

$$\frac{1}{2^n}\underbrace{\int\limits_{-\infty}^{+\infty}\dots\int\limits_{-\infty}^{+\infty}}_{n\text{ times}}\prod_{i=1}^{n}e^{-t\xi_i^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos b_{ij}\xi_i\right)}\cos\sum_{i=1}^{n}\left(x_i\xi_i-t\xi_i^2\sum_{j=1}^{m_i}a_{ij}\sin b_{ij}\xi_i\right)d\xi_1\dots d\xi_n.$$

The integrand of the latter integral can be decomposed into a finite sum of terms of the kind

$$\prod_{i=1}^{n}e^{-t\xi_i^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos b_{ij}\xi_i\right)}g_i\left(x_i\xi_i-t\xi_i^2\sum_{j=1}^{m_i}a_{ij}\sin b_{ij}\xi_i\right),$$

where either $g_i(\tau)=\cos\tau$ or $g_i(\tau)=\sin\tau$. Hence, the latter integral is a finite sum of terms of the kind

$$\prod_{i=1}^{n}\int\limits_{-\infty}^{+\infty}e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos b_{ij}\tau\right)}g_i\left(x_i\tau-t\tau^2\sum_{j=1}^{m_i}a_{ij}\sin b_{ij}\tau\right)d\tau.$$

Only one of the above terms contains no sine functions inside the integral. Therefore, the specified term is different from the identical zero; it is equal to

$$\prod_{i=1}^{n}\int\limits_{-\infty}^{+\infty}e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos b_{ij}\tau\right)}\cos\left(x_i\tau-t\tau^2\sum_{j=1}^{m_i}a_{ij}\sin b_{ij}\tau\right)d\tau$$
$$=2^n\prod_{i=1}^{n}\int\limits_{0}^{\infty}e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos b_{ij}\tau\right)}\cos\left(x_i\tau-t\tau^2\sum_{j=1}^{m_i}a_{ij}\sin b_{ij}\tau\right)d\tau.$$

All of the remaining terms vanish because each one contains at least one factor equal to zero, that is, an integral of the kind

$$\int\limits_{-\infty}^{+\infty}e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos b_{ij}\tau\right)}\sin\left(x_i\tau-t\tau^2\sum_{j=1}^{m_i}a_{ij}\sin b_{ij}\tau\right)d\tau$$

(its integrand is odd). Thus,

$$\mathcal{E}_{(n)}(x,t)=\prod_{i=1}^{n}\int\limits_{0}^{\infty}e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos b_{ij}\tau\right)}\cos\left(x_i\tau-t\tau^2\sum_{j=1}^{m_i}a_{ij}\sin b_{ij}\tau\right)d\tau.$$

Each factor of the latter product is a function $\mathcal{E}_{a_i,b_i}(x_i,t)=\mathcal{E}(x_i,t)$ of the kind (3). Fix an arbitrary positive t. Then the function $\mathcal{E}(x_i,t)$ is bounded on $\mathbb{R}^1$ for any $i=\overline{1,n}$. Further, the function $x_i^2\mathcal{E}(x_i,t)$ is bounded on $\mathbb{R}^1$ due to Lemma 1. Hence, function $(1+x_i^2)\mathcal{E}(x_i,t)$ is bounded on $\mathbb{R}^1$ as well. This means that there exists a positive M such that $|\mathcal{E}(x_i,t)|\le\frac{M}{1+x_i^2}$ on $\mathbb{R}^1$ $(i=\overline{1,n})$ implies

$$|\mathcal{E}_{(n)}(x,t)|\le\frac{(2M)^n}{\prod\limits_{i=1}^{n}(1+x_i^2)}$$

in $\mathbb{R}^n$.

Now let Ω be an arbitrary (arbitrarily large) domain in $\mathbb{R}^n$. There exists a positive A_0 such that $\Omega\subset Q(A_0)$, where $Q(A_0)=\{|x_i|<A_0\mid i=\overline{1,n}\}$. Then

$$\int\limits_{Q(A_0)} |u_0(x-\xi)\mathcal{E}_{(n)}(\xi,t)|\,d\xi \le (2M)^n \sup|u_0| \int\limits_{Q(A_0)} \frac{d\xi}{\prod\limits_{i=1}^{n}(1+\xi_i^2)}$$

$$= (2M)^n \sup|u_0| \biggl(\int\limits_{-A_0}^{A_0} \frac{d\eta}{1+\eta^2}\biggr)^n = (4M\arctan A_0)^n \sup|u_0| \le (2\pi M)^n \sup|u_0|.$$

Therefore, (9) converges absolutely and satisfies the same estimate. □

Thus, the following function is well defined in $\mathbb{R}^n \times (0,+\infty)$:

$$u(x,t) \overset{\text{def}}{=} \frac{1}{\pi^n} \int\limits_{\mathbb{R}^n} \mathcal{E}_{(n)}(x-\xi,t)u_0(\xi)\,d\xi. \tag{10}$$

Similarly to the representation of $\mathcal{E}_{(n)}$ in Lemma 7, we have

$$\frac{\partial^2 \mathcal{E}_{a_i,b_i}}{\partial x_i^2} \prod_{\substack{k=1\\k\ne i}}^{n} \mathcal{E}_{a_k,b_k}(x_k,t).$$

Combining Lemma 2 and the fact that $\mathcal{E}_{(n)}$ satisfies Eq. (7), we obtain that function (10) can be differentiated inside the integral. This yields

Theorem 3. *Let the operator $-L_{(n)}$ be strongly elliptic in $\mathbb{R}^n$. Then function* (10) *satisfies* (*in the classical sense*) *Eq.* (7) *in the half-space $\mathbb{R}^n \times (0,+\infty)$.*

Note that function (10) satisfies problem (7), (2) *in the sense of distributions* (see, e.g., [21]).

To prove the uniqueness of the found solution, we investigate, according to [21], the real part of the symbol of the elliptic operator $L_{(n)}$ contained in Eq. (7). We will denote the specified symbol by

$$\mathcal{P}(z_1,\dots,z_n) \overset{\text{def}}{=} \mathcal{P}(z) \overset{\text{def}}{=} \mathcal{P}(\sigma+i\tau) \overset{\text{def}}{=} \mathcal{P}(\sigma_1+i\tau_1,\dots,\sigma_n+i\tau_n);$$

it is equal to

$$-\sum_{k=1}^{n} z_k^2\biggl(1+\sum_{j=1}^{m_k} a_{kj}e^{-ib_{kj}z_k}\biggr) = \sum_{k=1}^{n}(\tau_k^2-\sigma_k^2-2i\sigma_k\tau_k)\biggl(1+\sum_{j=1}^{m_k} a_{kj}e^{-ib_{kj}z_k}\biggr)$$
$$= \sum_{k=1}^{n}(\tau_k^2-\sigma_k^2-2i\sigma_k\tau_k)\biggl(1+\sum_{j=1}^{m_k} a_{kj}e^{b_{kj}\tau_k-ib_{kj}\sigma_k}\biggr)$$
$$= \sum_{k=1}^{n}(\tau_k^2-\sigma_k^2-2i\sigma_k\tau_k)\biggl(1+\sum_{j=1}^{m_k} a_{kj}e^{b_{kj}\tau_k}\cos b_{kj}\sigma_k - i\sum_{j=1}^{m_k} a_{kj}e^{b_{kj}\tau_k}\sin b_{kj}\sigma_k\biggr).$$

Thus,

$$\operatorname{Re}\mathcal{P}(z) = \sum_{k=1}^{n}\biggl[(\tau_k^2-\sigma_k^2)\biggl(1+\sum_{j=1}^{m_k} a_{kj}e^{b_{kj}\tau_k}\cos b_{kj}\sigma_k\biggr) - 2\sigma_k\tau_k\sum_{j=1}^{m_k} a_{kj}e^{b_{kj}\tau_k}\sin b_{kj}\sigma_k\biggr]$$
$$= |\tau|^2-|\sigma|^2+\sum_{k=1}^{n}\biggl[(\tau_k^2-\sigma_k^2)\sum_{j=1}^{m_k} a_{kj}e^{b_{kj}\tau_k}\cos b_{kj}\sigma_k - 2\sigma_k\tau_k\sum_{j=1}^{m_k} a_{kj}e^{b_{kj}\tau_k}\sin b_{kj}\sigma_k\biggr].$$

Estimating the function $\mathcal{Q}(z,t_0,t) \overset{\text{def}}{=} e^{(t-t_0)\mathcal{P}(z)}$, we obtain that

$$|\mathcal{Q}(z,t_0,t)| \le e^{(t-t_0)[C_1(1+|\sigma|^4)+C_2e^{C_3|\tau|}]}.$$

It follows from the latter estimate (see [21, Chap. 2, Appendix 1]) that problem (7), (2) has no more than one solution in the sense of distributions.

Note that, as in the one-dimensional case, the uniqueness holds for much wider classes of initial-value functions (see Remark 2). However, we consider only continuous, bounded, initial-value functions for the same reason as in the one-dimensional case. Similarly to Lemma 4, we can compute the integral of $\mathcal{E}_{(n)}$ over the whole $\mathbb{R}^n$; it is equal to π^n.

5. Many-Dimensional Case: Stabilization of the Solution

The long-term behavior of $u(x,t)$ in the case of several spatial variables is studied in this section. To do this, we consider (together with Eq. (7)) the equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} p_i \frac{\partial^2 u}{\partial x_i^2}, \tag{11}$$

where $p_i = 1 + \sum\limits_{j=1}^{m_i} a_{ij}$, $i = \overline{1,n}$ (note that, as we proved above, all the constants p_i are positive).

Let $v(x,t)$ denote the classical bounded solution of problem (11), (2).

The following assertion is valid.

Theorem 4. $\lim\limits_{t\to\infty}[u(x,t) - v(x,t)] = 0$ *for any* $x \in \mathbb{R}^n$.

Proof. Let $x_0 \stackrel{\text{def}}{=} (x_1^0, \dots, x_n^0)$ be an arbitrary point of $\mathbb{R}^n$.

Let us change the variables in (10): $\eta_i \stackrel{\text{def}}{=} \frac{x_i^0 - \xi_i}{2\sqrt{t}}$ $(i = \overline{1,n})$. We obtain that

$$u(x_0,t) = \left(\frac{2\sqrt{t}}{\pi}\right)^n \int\limits_{\mathbb{R}^n} u_0(x_0 - 2\sqrt{t}\eta)\mathcal{E}_{(n)}(2\sqrt{t}\eta, t)\, d\eta. \tag{12}$$

We have

$$\begin{aligned} t^{\frac{n}{2}}\mathcal{E}_{(n)}(2\sqrt{t}\eta,t) &= t^{\frac{n}{2}} \prod_{i=1}^{n} \int\limits_0^\infty e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\tau\right)} \cos\left(2\eta_i\tau\sqrt{t} - t\tau^2\sum_{j=1}^{m_i} a_{ij}\sin b_{ij}\tau\right) d\tau \\ &= \prod_{i=1}^{n} \int\limits_0^\infty e^{-z^2\left(1+\sum\limits_{j=1}^{m_i} a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)} \cos\left(2z\eta_i - z^2\sum_{j=1}^{m_i} a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right) dz; \end{aligned}$$

therefore,

$$\begin{aligned} u(x_0,t) &= \left(\frac{2}{\pi}\right)^n \int\limits_{\mathbb{R}^n} u_0(x_0 - 2\sqrt{t}\eta) \prod_{i=1}^{n} \int\limits_0^\infty e^{-z^2\left(1+\sum\limits_{j=1}^{m_i} a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)} \cos\left(2z\eta_i - z^2\sum_{j=1}^{m_i} a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right) dz\, d\eta, \\ v(x_0,t) &= \frac{1}{\pi^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} u_0(x_1^0 - 2\sqrt{p_1 t}\xi_1, \dots, x_n^0 - 2\sqrt{p_n t}\xi_n) e^{-|\xi|^2}\, d\xi. \end{aligned}$$

The change of variables $\sqrt{p_i}\xi_i = \eta_i$ $(i = \overline{1,n})$ reduces the latter expression to

$$\frac{1}{\pi^{\frac{n}{2}} \prod\limits_{i=1}^{n} \sqrt{p_i}} \int\limits_{\mathbb{R}^n} u_0(x_1^0 - 2\sqrt{t}\eta_1, \dots, x_n^0 - 2\sqrt{t}\eta_n) e^{-\sum\limits_{i=1}^{n} \frac{\eta_i^2}{p_i}}\, d\eta_i.$$

Thus,

$$u(x_0,t)-v(x_0,t)=\left(\frac{2}{\pi}\right)^n\int\limits_{\mathbb{R}^n}u_0(x_0-2\sqrt{t}\eta)$$
$$\times\left[\prod_{i=1}^n\int\limits_0^\infty e^{-z^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)}\cos\left(2z\eta_i-z^2\sum_{j=1}^{m_i}a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right)dz-\prod_{i=1}^n\frac{\sqrt{\pi}}{2\sqrt{p_i}}e^{-\frac{\eta_i^2}{p_i}}\right]d\eta$$
$$=\left(\frac{2}{\pi}\right)^n\left(\int\limits_{Q(A)}+\int\limits_{\mathbb{R}^n\setminus Q(A)}\right)\overset{\text{def}}{=}\left(\frac{2}{\pi}\right)^n(J_1+J_2),\tag{13}$$

where A is a positive parameter and, as above, $Q(A)$ is the cube $\{|x_i|<A\mid i=\overline{1,n}\}$.

Let $\varepsilon>0$. Due to Lemma 6, for any $i=\overline{1,n}$, there exists a positive M_i such that for any $\eta_i\geq 1$, for any $t>1$,

$$\left|\int\limits_0^\infty e^{-z^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)}\cos\left(2z\eta_i-z^2\sum_{j=1}^{m_i}a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right)dz\right|\leq\frac{M_i}{\eta_i^2}\leq\frac{2M_i}{1+\eta_i^2}.$$

Further, if $\eta_i\in[0,1]$, then the left-hand side of the latter inequality does not exceed

$$\int\limits_0^\infty e^{-Cz^2}\,dz\overset{\text{def}}{=}\frac{\sqrt{\pi}}{2\sqrt{C}}\leq\frac{\sqrt{\pi}}{\sqrt{C}(1+\eta_i^2)};$$

therefore, for any real η_i,

$$\left|\int\limits_0^\infty e^{-z^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)}\cos\left(2z\eta_i-z^2\sum_{j=1}^{m_i}a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right)dz\right|\leq\frac{M_i^*}{1+\eta_i^2},$$

where $M_i^*=\max\left(2M_i,\sqrt{\frac{\pi}{C}}\right)$.

Thus, the absolute value of the integrand in (13) does not exceed

$$\sup|u_0|\left[\prod_{i=1}^n\frac{M_i^*}{1+\eta_i^2}+\left(\frac{\pi}{4}\right)^{\frac{n}{2}}\prod_{i=1}^n\frac{1}{\sqrt{p_i}}e^{-\frac{\eta_i^2}{p_i}}\right].$$

Hence, integral (13) converges absolutely and uniformly with respect to $t\in(1,+\infty)$; therefore, there exists a positive A such that $|J_2|<\frac{\varepsilon\pi^n}{2^{n+1}}$ for any $t>1$. Fix that A and consider J_1 for $t>1$.

Due to Lemma 5, for any $i=\overline{1,n}$,

$$\int\limits_0^\infty e^{-z^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)}\cos\left(2z\eta_i-z^2\sum_{j=1}^{m_i}a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right)dz\xrightarrow{t\to\infty}\sqrt{\frac{\pi}{4p_i}}\,e^{-\frac{\eta_i^2}{p_i}}$$

uniformly with respect to $\eta_i\in(-\infty,+\infty)$.

As we proved above, each inner (one-dimensional) integral of expression (13) is bounded (e.g., it is bounded by the constant M_i^*). This implies that

$$\prod_{i=1}^n\int\limits_0^\infty e^{-z^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)}\cos\left(2z\eta_i-z^2\sum_{j=1}^{m_i}a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right)dz\xrightarrow{t\to\infty}\prod_{i=1}^n\sqrt{\frac{\pi}{4p_i}}\,e^{-\frac{\eta_i^2}{p_i}}$$

uniformly with respect to $\eta \in \mathbb{R}^n$. Hence, there exists a positive T such that for any $t \in (T, +\infty)$,

$$\left| \prod_{i=1}^{n} \int_{0}^{\infty} e^{-z^2\left(1+\sum\limits_{j=1}^{m_i} a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)} \cos\left(2z\eta_i - z^2 \sum_{j=1}^{m_i} a_{ij} \sin\frac{b_{ij}z}{\sqrt{t}}\right) dz - \prod_{i=1}^{n} \sqrt{\frac{\pi}{4p_i}}\, e^{-\frac{\eta_i^2}{p_i}} \right| \le \frac{\varepsilon\pi^n}{2^{2n+1}A^n \sup|u_0|}.$$

This means that $|J_1| \le \frac{\varepsilon\pi^n}{2^{n+1}}$; therefore, $|u(x_0,t) - v(x_0,t)| < \varepsilon$.

A positive ε was chosen arbitrarily; hence,

$$\lim_{t\to\infty} [u(x_0,t) - v(x_0,t)] = 0.$$

Since $x_0 \in \mathbb{R}^n$ was chosen arbitrarily, this completes the proof of Theorem 4. $\square$

As in Sec. 3, this implies

Corollary 2. *Let $x \in \mathbb{R}^n$, $l \in (-\infty, +\infty)$. Then*

$$\lim_{t\to\infty} u(x,t) = l \iff \lim_{R\to\infty} \frac{1}{R^n} \int\limits_{B_R(p_1,\dots,p_n)} u_0(x)\,dx = \frac{2\pi^{\frac{n}{2}} \prod\limits_{i=1}^{n} \sqrt{p_i}}{n\Gamma(\frac{n}{2})}\, l,$$

where

$$B_R(p_1,\dots,p_n) = \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{x_1^2}{p_1} + \dots + \frac{x_n^2}{p_n} < R \right\}.$$

Remark 4. Theorem 4 also holds for the case where $n = 1$, i.e., we have the asymptotic closeness of the solutions of Eq. (1) and the equation $\frac{\partial u}{\partial t} = p\frac{\partial^2 u}{\partial x^2}$, However, this does not give us new information about the stabilization of the solution because the necessary and sufficient condition of the stabilization for the solution of problem (1), (2) provided by the above closeness theorem entirely coincides with the assertion of Corollary 1.

6. The General Case of Inhomogeneous Elliptic Operators

In this section, we extend the investigation to the case where the right-hand side of Eq. (7) contains low-order (nonlocal) terms as well. Our attention will be concentrated only on the aspects substantially different from the prototypical case of homogeneous elliptic operators, which is considered in Secs. 4 and 5 in detail. Thus, the following equation is considered instead of (7):

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{n}\sum_{j=1}^{m_{2,i}} a_{ij} \frac{\partial^2 u}{\partial x_i^2}\big(x + h_{ij}^{(2)} e_i, t\big) + \sum_{i=1}^{n}\sum_{j=1}^{m_{1,i}} b_{ij} \frac{\partial u}{\partial x_i}\big(x + h_{ij}^{(1)} e_i, t\big) + \sum_{i=1}^{n}\sum_{j=1}^{m_{0,i}} c_{ij} u\big(x + h_{ij}^{(0)} e_i, t\big). \quad (14)$$

Here e_i denotes the ith coordinate vector of the space $\mathbb{R}^n$, $m_{k,i} \in \mathbb{N}$ for $i = \overline{1,n}$, $k = \overline{0,2}$, and the coefficients a_{ij}, b_{ij}, c_{ij}, and $h_{ij}^{(k)}$ are assumed to be real for $i = \overline{1,n}$, $k = \overline{0,2}$, and $j = \overline{1,m_k}$.

The fundamental solution is introduced, instead of (8), as

$$\mathcal{E}_{(n)}(x,t) \stackrel{\text{def}}{=} \frac{1}{2^n} \int\limits_{\mathbb{R}^n} e^{-t[|\xi|^2 + G_1(\xi)]} \cos[x\xi - tG_2(\xi)]\,d\xi, \quad (8')$$

where

$$G_1(\xi) = \sum_{i=1}^{n} \xi_i^2 \sum_{j=1}^{m_{2,i}} a_{ij} \cos h_{ij}^{(2)}\xi_i + \sum_{i=1}^{n} \xi_i \sum_{j=1}^{m_{1,i}} b_{ij} \sin h_{ij}^{(1)}\xi_i - \sum_{i=1}^{n}\sum_{j=1}^{m_{0,i}} c_{ij}\cos h_{ij}^{(0)}\xi_i,$$

$$G_2(\xi) = \sum_{i=1}^{n} \xi_i^2 \sum_{j=1}^{m_{2,i}} a_{ij} \sin h_{ij}^{(2)}\xi_i - \sum_{i=1}^{n} \xi_i \sum_{j=1}^{m_{1,i}} b_{ij} \cos h_{ij}^{(1)}\xi_i - \sum_{i=1}^{n}\sum_{j=1}^{m_{0,i}} c_{ij}\sin h_{ij}^{(0)}\xi_i.$$

The following assertion is valid.

Theorem 5. *Let the operator $-L_{(n)}$ be strongly elliptic in $\mathbb{R}^n$. Then function* (10) *with $\mathcal{E}_{(n)}$ defined by* ($8'$) *satisfies* (*in the classical sense*) *Eq.* (14) *in $\mathbb{R}^n \times (0, +\infty)$ and is a unique solution* (*in the sense of distributions*) *of problem* (14), (2).

Proof. First we substitute function ($8'$) into Eq. (14):

$$\begin{aligned}
2^n \frac{\partial \mathcal{E}_n}{\partial t} \\
&= -\int\limits_{\mathbb{R}^n} [\,|\xi|^2 + G_1(\xi)] e^{-t[|\xi|^2 + G_1(\xi)]} \cos[x\xi - tG_2(\xi)]\, d\xi + \int\limits_{\mathbb{R}^n} G_2(\xi) e^{-t[|\xi|^2 + G_1(\xi)]} \sin[x\xi - tG_2(\xi)]\, d\xi \\
&= \int\limits_{\mathbb{R}^n} e^{-t[|\xi|^2 + G_1(\xi)]} \big(G_2(\xi) \sin[x\xi - tG_2(\xi)] - G_1(\xi) \cos[x\xi - tG_2(\xi)] - |\xi|^2 \cos[x\xi - tG_2(\xi)]\big)\, d\xi.
\end{aligned}$$

Further,

$$\begin{aligned}
\sin h_{ij}^{(2)} \xi_i \sin[x\xi - tG_2(\xi)] - \cos h_{ij}^{(2)} \xi_i \cos[x\xi - tG_2(\xi)] &= -\cos[x\xi + h_{ij}^{(2)} \xi - tG_2(\xi)], \\
-\cos h_{ij}^{(1)} \xi_i \sin[x\xi - tG_2(\xi)] - \sin h_{ij}^{(1)} \xi_i \cos[x\xi - tG_2(\xi)] &= -\sin[x\xi + h_{ij}^{(1)} \xi - tG_2(\xi)], \\
-\sin h_{ij}^{(0)} \xi_i \sin[x\xi - tG_2(\xi)] + \cos h_{ij}^{(0)} \xi_i \cos[x\xi - tG_2(\xi)] &= \cos[x\xi + h_{ij}^{(0)} \xi - tG_2(\xi)];
\end{aligned}$$

therefore,

$$\begin{aligned}
2^n \frac{\partial \mathcal{E}_n}{\partial t} &= \int\limits_{\mathbb{R}^n} e^{-t[|\xi|^2 + G_1(\xi)]} \Bigg(-|\xi|^2 \cos[x\xi - tG_2(\xi)] - \sum_{i=1}^n \xi_i^2 \sum_{j=1}^{m_{2,i}} a_{ij} \cos[x\xi + h_{ij}^{(2)} \xi - tG_2(\xi)] \\
&\quad - \sum_{i=1}^n \xi_i \sum_{j=1}^{m_{1,i}} b_{ij} \sin[x\xi + h_{ij}^{(1)} \xi - tG_2(\xi)] + \sum_{i=1}^n \sum_{j=1}^{m_{0,i}} c_{ij} \cos[x\xi + h_{ij}^{(0)} \xi - tG_2(\xi)] \Bigg)\, d\xi \\
&= -\sum_{i=1}^n \sum_{j=1}^{m_{2,i}} a_{ij} \int\limits_{\mathbb{R}^n} \xi_i^2 e^{-t[|\xi|^2 + G_1(\xi)]} \cos[(x + h_{ij}^{(2)} e_i)\xi - tG_2(\xi)]\, d\xi \\
&\quad - \sum_{i=1}^n \sum_{j=1}^{m_{1,i}} b_{ij} \int\limits_{\mathbb{R}^n} \xi_i e^{-t[|\xi|^2 + G_1(\xi)]} \sin[(x + h_{ij}^{(1)} e_i)\xi - tG_2(\xi)]\, d\xi \\
&\quad + \sum_{i=1}^n \sum_{j=1}^{m_{0,i}} c_{ij} \int\limits_{\mathbb{R}^n} e^{-t[|\xi|^2 + G_1(\xi)]} \cos[(x + h_{ij}^{(0)} e_i)\xi - tG_2(\xi)]\, d\xi - \int\limits_{\mathbb{R}^n} |\xi|^2 e^{-t[|\xi|^2 + G_1(\xi)]} \cos[x\xi - tG_2(\xi)]\, d\xi, \\
2^n \frac{\partial \mathcal{E}_n}{\partial x_i} &= -\int\limits_{\mathbb{R}^n} \xi_i e^{-t[|\xi|^2 + G_1(\xi)]} \sin[x\xi - tG_2(\xi)]\, d\xi, \\
2^n \frac{\partial^2 \mathcal{E}_n}{\partial x_i^2} &= -\int\limits_{\mathbb{R}^n} \xi_i^2 e^{-t[|\xi|^2 + G_1(\xi)]} \cos[x\xi - tG_2(\xi)]\, d\xi.
\end{aligned}$$

Thus, $\mathcal{E}_{(n)}(x,t)$ satisfies Eq. (14) in $\mathbb{R}^n \times (0, +\infty)$.

Now we note that $G_1(\xi)$ is equal to

$$\sum_{i=1}^n \Bigg(\xi_i^2 \sum_{j=1}^{m_{2,i}} a_{ij} \cos h_{ij}^{(2)} \xi_i + \xi_i \sum_{j=1}^{m_{1,i}} b_{ij} \sin h_{ij}^{(1)} \xi_i - \sum_{j=1}^{m_{0,i}} c_{ij} \cos h_{ij}^{(0)} \xi_i \Bigg) \stackrel{\text{def}}{=} \sum_{i=1}^n G_{1,i}(\xi_i),$$

and $G_2(\xi)$ is equal to

$$\sum_{i=1}^{n}\biggl(\xi_i^2\sum_{j=1}^{m_{2,i}} a_{ij}\sin h_{ij}^{(2)}\xi_i - \xi_i\sum_{j=1}^{m_{1,i}} b_{ij}\cos h_{ij}^{(1)}\xi_i - \sum_{j=1}^{m_{0,i}} c_{ij}\sin h_{ij}^{(0)}\xi_i\biggr) \stackrel{\text{def}}{=} \sum_{i=1}^{n} G_{2,i}(\xi_i).$$

Then function (8′) is equal to

$$\frac{1}{2^n}\underbrace{\int\limits_{-\infty}^{+\infty}\dots\int\limits_{-\infty}^{+\infty}}_{n\text{ times}}\prod_{i=1}^{n} e^{-t[\xi_i^2+G_{1,i}(\xi_i)]}\cos\sum_{i=1}^{n}[x_i\xi_i - tG_{2,i}(\xi_i)]\,d\xi_1\dots d\xi_n.$$

Taking into account that $G_{1,i}$ is even while $G_{2,i}$ is odd for any $i=\overline{1,n}$, we reduce the latter expression (similarly to the proof of Lemma 7) to

$$\prod_{i=1}^{n}\int\limits_{0}^{\infty} e^{-t[\tau^2+G_{1,i}(\tau)]}\cos[x_i\tau - tG_{2,i}(\tau)]\,d\tau. \tag{15}$$

This means that to prove the solvability it remains (see the proof of Lemma 7) to prove the analogs of Lemmas 1 and 2 for the case where the function (of one spatial variable) $\mathcal{E}(x,t)$ has the form

$$\int\limits_{0}^{\infty} e^{-t[\tau^2+G_1(\tau)]}\cos[x\tau - tG_2(\tau)]\,d\tau,$$

where $G_{1,i}$ and $G_{2,i}$ are taken instead of G_1 and G_2 respectively, $i=\overline{1,n}$. To do this, as in Lemma 1, we fix a positive t and consider

$$\begin{aligned}&\int\limits_{0}^{\infty} e^{-t[\tau^2+G_1(\tau)]}\cos[tG_2(\tau)]\cos x\tau\,d\tau\\ &\quad= \frac{\sin x\tau}{x}e^{-t[\tau^2+G_1(\tau)]}\cos[tG_2(\tau)]\bigg|_{\tau=0}^{\tau=\infty} - \frac{1}{x}\int\limits_{0}^{\infty}\sin x\tau\bigl(e^{-t[\tau^2+G_1(\tau)]}\cos[tG_2(\tau)]\bigr)'\,d\tau\\ &\quad= -\frac{1}{x}\int\limits_{0}^{\infty}\sin x\tau\bigl(e^{-t[\tau^2+G_1(\tau)]}\cos[tG_2(\tau)]\bigr)'\,d\tau\end{aligned}$$

(the first factor of the integrated term vanishes at the origin while the second one vanishes at infinity). We have

$$\bigl(e^{-t[\tau^2+G_1(\tau)]}\cos[tG_2(\tau)]\bigr)' = -e^{-t[\tau^2+G_1(\tau)]}\bigl([2\tau+G_1'(\tau)]\cos[tG_2(\tau)] + tG_2'(\tau)\sin[tG_2(\tau)]\bigr).$$

Obviously, $G_1'(0)=G_2(0)=0$, hence, integrating by parts again; we see that the integrated term vanishes again; therefore, we obtain

$$-\frac{1}{x^2}\int\limits_{0}^{\infty}\cos x\tau\bigl(e^{-t[\tau^2+G_1(\tau)]}\cos[tG_2(\tau)]\bigr)''\,d\tau.$$

The latter integral is a bounded function of the variable x, hence, Lemma 1 holds for the specified case. In the same way, we prove the boundedness of the functions $x^2\frac{\partial\mathcal{E}}{\partial x}$ and $x^2\frac{\partial^2\mathcal{E}}{\partial x^2}$ for any positive t.

Further, arguing exactly as in Theorem 3, we prove the solvability.

To prove the uniqueness, we consider, as above, the symbol of the corresponding elliptic operator:

$$\mathcal{P}(z) = -|z|^2 - \sum_{k=1}^{n} z_k^2 \sum_{j=1}^{m_{2,k}} a_{kj} e^{-ih_{kj}^{(2)} z_k} - i \sum_{k=1}^{n} z_k \sum_{j=1}^{m_{1,k}} b_{kj} e^{-ih_{kj}^{(1)} z_k} + \sum_{k=1}^{n} \sum_{j=1}^{m_{0,k}} c_{kj} e^{-ih_{kj}^{(0)} z_k}$$

$$= \sum_{k=1}^{n} (\tau_k^2 - \sigma_k^2 - 2i\sigma_k\tau_k)\Bigl(1 + \sum_{j=1}^{m_{2,k}} a_{kj} e^{h_{kj}^{(2)}\tau_k} \cos h_{kj}^{(2)}\sigma_k - i \sum_{j=1}^{m_{2,k}} a_{kj} e^{h_{kj}^{(2)}\tau_k} \sin h_{kj}^{(2)}\sigma_k\Bigr)$$

$$+ \sum_{k=1}^{n} (\tau_k - i\sigma_k)\Bigl(\sum_{j=1}^{m_{1,k}} b_{kj} e^{h_{kj}^{(1)}\tau_k} \cos h_{kj}^{(1)}\sigma_k - i \sum_{j=1}^{m_{1,k}} b_{kj} e^{h_{kj}^{(1)}\tau_k} \sin h_{kj}^{(1)}\sigma_k\Bigr)$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{m_{0,k}} c_{kj} e^{h_{kj}^{(0)}\tau_k} \cos h_{kj}^{(0)}\sigma_k - i \sum_{k=1}^{n} \sum_{j=1}^{m_{0,k}} c_{kj} e^{h_{kj}^{(0)}\tau_k} \sin h_{kj}^{(0)}\sigma_k.$$

This means that

$$\operatorname{Re}\mathcal{P}(z) = \sum_{k=1}^{n} \Bigl[(\tau_k^2 - \sigma_k^2)\Bigl(1 + \sum_{j=1}^{m_{2,k}} a_{kj} e^{h_{kj}^{(2)}\tau_k} \cos h_{kj}^{(2)}\sigma_k\Bigr) - 2\sigma_k\tau_k \sum_{j=1}^{m_{2,k}} a_{kj} e^{h_{kj}^{(2)}\tau_k} \sin h_{kj}^{(2)}\sigma_k$$

$$+ \tau_k \sum_{j=1}^{m_{1,k}} b_{kj} e^{h_{kj}^{(1)}\tau_k} \cos h_{kj}^{(1)}\sigma_k - \sigma_k \sum_{j=1}^{m_{1,k}} b_{kj} e^{h_{kj}^{(1)}\tau_k} \sin h_{kj}^{(1)}\sigma_k + \sum_{j=1}^{m_{0,k}} c_{kj} e^{h_{kj}^{(0)}\tau_k} \cos h_{kj}^{(0)}\sigma_k\Bigr]$$

$$= |\tau|^2 - |\sigma|^2 + \sum_{k=1}^{n} \Bigl[(\tau_k^2 - \sigma_k^2) \sum_{j=1}^{m_{2,k}} a_{kj} e^{h_{kj}^{(2)}\tau_k} \cos h_{kj}^{(2)}\sigma_k - 2\sigma_k\tau_k \sum_{j=1}^{m_{2,k}} a_{kj} e^{h_{kj}^{(2)}\tau_k} \sin h_{kj}^{(2)}\sigma_k$$

$$+ \tau_k \sum_{j=1}^{m_{1,k}} b_{kj} e^{h_{kj}^{(1)}\tau_k} \cos h_{kj}^{(1)}\sigma_k - \sigma_k \sum_{j=1}^{m_{1,k}} b_{kj} e^{h_{kj}^{(1)}\tau_k} \sin h_{kj}^{(1)}\sigma_k + \sum_{j=1}^{m_{0,k}} c_{kj} e^{h_{kj}^{(0)}\tau_k} \cos h_{kj}^{(0)}\sigma_k\Bigr].$$

Thus, the function $\mathcal{Q}(z, t_0, t)$ satisfies the same estimate as in Sec. 4 (generally, with other constants); this proves the uniqueness of the constructed solution. □

Investigating the long-term behavior of the solution of problem (14), (2), we consider, besides the specified problem, the problem

$$\frac{\partial w}{\partial t} = \Delta w, \quad x \in \mathbb{R}^n, \quad t > 0; \tag{16}$$

$$w|_{t=0} = w_0(x), \quad x \in \mathbb{R}^n, \tag{17}$$

where $w_0(x) = u_0(\sqrt{p_1}\, x_1, \dots, \sqrt{p_n}\, x_n)$, i.e., w_0 depends on positive parameters $p_1, \dots, p_n$.

The classical bounded solution of the latter problem exists and is unique because the function $w_0(x)$ is continuous and bounded; we will denote that solution by $w(x, t)$.

Below we consider, without loss of generality, that the finite sequences $\bigl\{b_{kj}h_{kj}^{(1)}\bigr\}_{j=1}^{m_{1,k}}$ and $\{c_{kj}\}_{j=1}^{m_{0,k}}$ are nonincreasing for any $k = \overline{1, n}$. We denote $\min\limits_{b_{kj}h_{kj}^{(1)} > 0} j$ and $\min\limits_{c_{kj} > 0} j$ by $\tilde{m}_{1,k}$ and $\tilde{m}_{0,k}$, respectively, $k = \overline{1, n}$. If $b_{kj}h_{kj}^{(1)} \leq 0$ ($c_{kj} \leq 0$) for any $j = \overline{1, m_{1,k}}$ ($j = \overline{1, m_{0,k}}$), then $\tilde{m}_{i,k} \overset{\text{def}}{=} m_{i,k} + 1$, $i = 0, 1$. We also denote the positive constant

$$1 + \sum_{j=1}^{m_{2,k}} a_{kj} + \sum_{j \geq \tilde{m}_{1,k}} b_{kj}h_{kj}^{(1)}$$

by σ_k, $k=\overline{1,n}$. Consider the operator $\mathcal{L}$ acting as follows:

$$\mathcal{L}u \overset{\text{def}}{=} \Delta u + \sum_{k=1}^{n}\biggl[\sum_{j<\tilde{m}_{0,k}} \frac{c_{kj}}{\sigma_k} u\bigl(x+h_{kj}^{(0)}e_k,t\bigr) - \sum_{j<\tilde{m}_{1,k}} \frac{2|b_{kj}|}{\sigma_k} u\Bigl(x+\sqrt{|h_{kj}^{(1)}|}\,e_k,t\Bigr)\biggr].$$

Note that, although the *differential-difference* operator $\mathcal{L}$ contains merely *low-order nonlocal* terms, it depends on the coefficients of the *high-order nonlocal* terms of the original operator $L_{(n)}$.

Finally, we denote

$$\sum_{k=1}^{n}\frac{1}{\sigma_k}\biggl(\sum_{j<\tilde{m}_{0,k}} c_{kj} - 2\sum_{j<\tilde{m}_{1,k}} |b_{kj}|\biggr)I - \mathcal{L}$$

by R.

The following assertion is valid.

Theorem 6. *Let $R(\xi)$ be positive definite. Then*

$$\lim_{t\to\infty}\biggl[e^{-t\sum\limits_{k=1}^{n}\sum\limits_{j=1}^{m_{0,k}} c_{kj}} u(x_0,t) - w\left(\frac{x_1^0+q_1t}{\sqrt{p_1}},\dots,\frac{x_n^0+q_nt}{\sqrt{p_n}},t\right)\biggr]=0$$

for any $x_0 \overset{\text{def}}{=} (x_1^0,\dots,x_n^0)\in\mathbb{R}^n$, where

$$p_i = 1+\sum_{j=1}^{m_{2,i}} a_{ij} + \sum_{j=1}^{m_{1,i}} b_{ij}h_{ij}^{(1)} + \frac{1}{2}\sum_{j=1}^{m_{0,i}} c_{ij}\bigl[h_{ij}^{(0)}\bigr]^2,\quad q_i = \sum_{j=1}^{m_{1,i}} b_{ij} + \sum_{j=1}^{m_{0,i}} c_{ij}h_{ij}^{(0)},\quad i=\overline{1,n}.$$

Proof. First we show that $p_1,\dots,p_n$ are positive under the assumption of the theorem. To do this, we take an arbitrary $k\in\overline{1,n}$ and assign $\xi_1=\dots=\xi_{k-1}=\xi_{k+1}=\dots=\xi_n=0$ in the condition of the positive definiteness of $R(\xi)$ given below:

$$\sum_{k=1}^{n}\frac{1}{\sigma_k}\biggl(\sum_{j<\tilde{m}_{0,k}} c_{kj} - 2\sum_{j<\tilde{m}_{1,k}} |b_{kj}|\biggr) + |\xi|^2 \\ - \sum_{k=1}^{n}\frac{1}{\sigma_k}\biggl(\sum_{j<\tilde{m}_{0,k}} c_{kj}\cos h_{kj}^{(0)}\xi_k - 2\sum_{j<\tilde{m}_{1,k}} |b_{kj}|\cos\sqrt{|h_{kj}^{(1)}|}\,\xi_k\biggr) \ge C|\xi|^2.$$

We obtain that

$$\sum_{j<\tilde{m}_{0,k}} c_{kj} - 2\sum_{j<\tilde{m}_{1,k}} |b_{kj}| + \sigma_k\xi_k^2 + 2\sum_{j<\tilde{m}_{1,k}} |b_{kj}|\cos\sqrt{|h_{kj}^{(1)}|}\,\xi_k - \sum_{j<\tilde{m}_{0,k}} c_{kj}\cos h_{kj}^{(0)}\xi_k \ge C\xi_k^2$$

for any positive ξ_k. This yields

$$\begin{aligned} C\xi_k^2 &\le \sigma_k\xi_k^2 - 2\sum_{j<\tilde{m}_{1,k}} |b_{kj}|\Bigl(1-\cos\sqrt{|h_{kj}^{(1)}|}\,\xi_k\Bigr) + \sum_{j<\tilde{m}_{0,k}} c_{kj}\bigl(1-\cos h_{kj}^{(0)}\xi_k\bigr) \\ &= \sigma_k\xi_k^2 - 4\sum_{j<\tilde{m}_{1,k}} |b_{kj}|\sin^2\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2} + 2\sum_{j<\tilde{m}_{0,k}} c_{kj}\sin^2\frac{h_{kj}^{(0)}\xi_k}{2} \\ &= \sigma_k\xi_k^2 - \xi_k^2\sum_{j<\tilde{m}_{1,k}} |b_{kj}|\,\bigl|h_{kj}^{(1)}\bigr|\left(\frac{\sin\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}{\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}\right)^2 + \frac{\xi_k^2}{2}\sum_{j<\tilde{m}_{0,k}} c_{kj}\bigl[h_{kj}^{(0)}\bigr]^2\left(\frac{\sin\frac{h_{kj}^{(0)}\xi_k}{2}}{\frac{h_{kj}^{(0)}\xi_k}{2}}\right)^2; \end{aligned}$$

hence,

$$\sigma_k - \sum_{j<\tilde m_{1,k}} |b_{kj}|\,\big|h_{kj}^{(1)}\big| \left(\frac{\sin\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}{\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}} \right)^2 + \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2 \left(\frac{\sin\frac{h_{kj}^{(0)}\,\xi_k}{2}}{\frac{h_{kj}^{(0)}\,\xi_k}{2}} \right)^2 \ge C$$

for any positive ξ_k.

This implies that

$$\sigma_k - \sum_{j<\tilde m_{1,k}} |b_{kj}|\,\big|h_{kj}^{(1)}\big| + \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2 > 0.$$

Indeed, suppose, to the contrary, that

$$\sigma_k - \sum_{j<\tilde m_{1,k}} |b_{kj}|\,\big|h_{kj}^{(1)}\big| + \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2 \le 0.$$

Let $\xi_k > 0$. Then

$$\begin{aligned}
C &\le \sigma_k + \sum_{j<\tilde m_{1,k}} |b_{kj}|\,\big|h_{kj}^{(1)}\big| - \sum_{j<\tilde m_{1,k}} |b_{kj}|\,\big|h_{kj}^{(1)}\big| + \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2 - \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2 \\
&\quad - \sum_{j<\tilde m_{1,k}} |b_{kj}|\,\big|h_{kj}^{(1)}\big| \left(\frac{\sin\frac{\sqrt{|h_{kj}^{(1)}|}\xi_k}{2}}{\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}} \right)^2 + \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2 \left(\frac{\sin\frac{h_{kj}^{(0)}\,\xi_k}{2}}{\frac{h_{kj}^{(0)}\,\xi_k}{2}} \right)^2 \\
&= \sigma_k - \sum_{j<\tilde m_{1,k}} |b_{kj}|\,\big|h_{kj}^{(1)}\big| + \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2 \\
&\quad - \sum_{j<\tilde m_{1,k}} |b_{kj}|\,\big|h_{kj}^{(1)}\big| \left[\left(\frac{\sin\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}{\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}} \right)^2 - 1 \right] + \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2 \left[\left(\frac{\sin\frac{h_{kj}^{(0)}\,\xi_k}{2}}{\frac{h_{kj}^{(0)}\,\xi_k}{2}} \right)^2 - 1 \right] \\
&\le \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2 \left[\left(\frac{\sin\frac{h_{kj}^{(0)}\,\xi_k}{2}}{\frac{h_{kj}^{(0)}\,\xi_k}{2}} \right)^2 - 1 \right] - \sum_{j<\tilde m_{1,k}} |b_{kj}|\,\big|h_{kj}^{(1)}\big| \left[\left(\frac{\sin\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}{\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}} \right)^2 - 1 \right].
\end{aligned}$$

However, by virtue of the finiteness of the above sums, we can choose a small positive ξ_k such that the last expression does not exceed $\frac{C}{2}$. The obtained contradiction proves the positivity of

$$\begin{aligned}
&\sigma_k - \sum_{j<\tilde m_{1,k}} |b_{kj}|\,\big|h_{kj}^{(1)}\big| + \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2 \\
&\quad = 1 + \sum_{j=1}^{m_{2,k}} a_{kj} + \sum_{j\ge\tilde m_{1,k}} b_{kj} h_{kj}^{(1)} - \sum_{j<\tilde m_{1,k}} |b_{kj}|\,\big|h_{kj}^{(1)}\big| + \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2 \\
&\quad = 1 + \sum_{j=1}^{m_{2,k}} a_{kj} + \sum_{j=1}^{m_{1,k}} b_{kj} h_{kj}^{(1)} + \frac{1}{2} \sum_{j<\tilde m_{0,k}} c_{kj} \big[h_{kj}^{(0)}\big]^2;
\end{aligned}$$

thus, p_k is positive a fortiori.

Let us fix an arbitrary x_0 from $\mathbb{R}^n$ now. We have

$$w\left(\frac{x_1^0+q_1t}{\sqrt{p_1}},\dots,\frac{x_n^0+q_nt}{\sqrt{p_n}},t\right)=\frac{1}{(2\sqrt{\pi t})^n}\int\limits_{\mathbb{R}^n}u_0(\sqrt{p_1}\,\xi_1,\dots,\sqrt{p_n}\,\xi_n)e^{-\frac{1}{4t}\sum\limits_{i=1}^n\left(\frac{x_i^0+q_it}{\sqrt{p_i}}-\xi_i\right)^2}d\xi$$

$$=\frac{1}{(2\sqrt{\pi t})^n\prod\limits_{i=1}^n\sqrt{p_i}}\int\limits_{\mathbb{R}^n}u_0(\eta)e^{-\frac{1}{t}\sum\limits_{i=1}^n\frac{(x_i^0+q_it-\eta_i)^2}{4p_i}}d\eta=\frac{1}{\pi^{\frac{n}{2}}\prod\limits_{i=1}^n\sqrt{p_i}}\int\limits_{\mathbb{R}^n}u_0(x_0-2\sqrt{t}\xi)e^{-\sum\limits_{i=1}^n\frac{(2\xi_i+q_i\sqrt{t})^2}{4p_i}}d\xi$$

$$=\frac{1}{2^n\pi^{\frac{n}{2}}\prod\limits_{i=1}^n\sqrt{p_i}}\int\limits_{\mathbb{R}^n}u_0(x_0+tq-\sqrt{t}y)e^{-\sum\limits_{i=1}^n\frac{y_i^2}{4p_i}}dy,$$

where q denotes the vector $(q_1,\dots,q_n)$.

Further, it follows from (12) and (15) that

$$u(x_0,t)=\left(\frac{2\sqrt{t}}{\pi}\right)^n\int\limits_{\mathbb{R}^n}u_0(x_0-2\sqrt{t}\eta)\prod_{i=1}^n\int\limits_0^\infty e^{-t[\tau^2+G_{1,i}(\tau)]}\cos[2\sqrt{t}\eta_i\tau-tG_{2,i}(\tau)]\,d\tau\,d\eta$$

$$=\left(\frac{\sqrt{t}}{\pi}\right)^n\int\limits_{\mathbb{R}^n}u_0(x_0+tq-\sqrt{t}y)\prod_{i=1}^n\int\limits_0^\infty e^{-t[\tau^2+G_{1,i}(\tau)]}\cos[y_i\tau\sqrt{t}-q_i\tau t-tG_{2,i}(\tau)]\,d\tau\,dy;$$

therefore,

$$e^{-t\sum\limits_{k=1}^n\sum\limits_{j=1}^{m_{0,k}}c_{kj}}u(x_0,t)$$

$$=\left(\frac{\sqrt{t}}{\pi}\right)^n\int\limits_{\mathbb{R}^n}u_0(x_0+tq-\sqrt{t}y)\prod_{i=1}^n\int\limits_0^\infty e^{-t[\tau^2+G_{1,i}(\tau)+\sum\limits_{j=1}^{m_{0,i}}c_{ij}]}\cos[y_i\tau\sqrt{t}-q_i\tau t-tG_{2,i}(\tau)]\,d\tau\,dy$$

$$=\left(\frac{1}{\pi}\right)^n\int\limits_{\mathbb{R}^n}u_0(x_0+tq-\sqrt{t}y)\prod_{i=1}^n\int\limits_0^\infty e^{-z^2-tG_{1,i}\left(\frac{z}{\sqrt{t}}\right)-t\sum\limits_{j=1}^{m_{0,i}}c_{ij}}\cos\left[y_iz-q_iz\sqrt{t}-tG_{2,i}\left(\frac{z}{\sqrt{t}}\right)\right]dz\,dy.$$

Thus,

$$e^{-t\sum\limits_{k=1}^n\sum\limits_{j=1}^{m_{0,k}}c_{kj}}u(x_0,t)-w\left(\frac{x_1^0+q_1t}{\sqrt{p_1}},\dots,\frac{x_n^0+q_nt}{\sqrt{p_n}},t\right)=\left(\frac{1}{\pi}\right)^n\int\limits_{\mathbb{R}^n}u_0(x_0+tq-\sqrt{t}y)$$

$$\times\left(\prod_{i=1}^n\int\limits_0^\infty e^{-z^2-tG_{1,i}(\frac{z}{\sqrt{t}})-t\sum\limits_{j=1}^{m_{0,i}}c_{ij}}\cos\left[y_iz-q_iz\sqrt{t}-tG_{2,i}\left(\frac{z}{\sqrt{t}}\right)\right]dz-\prod_{i=1}^n\frac{\sqrt{\pi}}{2\sqrt{p_i}}e^{-\frac{y_i^2}{4p_i}}\right)dy. \quad (18)$$

The following assertions are valid.

Lemma 8. *If the assumptions of Theorem 6 are fulfilled, then for any $i=\overline{1,n}$,*

$$\int\limits_0^\infty e^{-z^2-tG_{1,i}\left(\frac{z}{\sqrt{t}}\right)-t\sum\limits_{j=1}^{m_{0,i}}c_{ij}}\cos\left[yz-q_iz\sqrt{t}-tG_{2,i}\left(\frac{z}{\sqrt{t}}\right)\right]dz-\frac{\sqrt{\pi}}{2\sqrt{p_i}}e^{-\frac{y^2}{4p_i}}\xrightarrow{t\to\infty}0$$

uniformly with respect to $y\in(-\infty,+\infty)$.

Lemma 9. *If the assumptions of Theorem 6 are fulfilled, then for any $i = \overline{1, n}$, there exists M_i depending only on the coefficients of Eq.* (14) *such that*

$$\left| \int_0^\infty e^{-z^2 - tG_{1,i}\left(\frac{z}{\sqrt{t}}\right) - t\sum\limits_{j=1}^{m_{0,i}} c_{ij}} \cos\left[yz - q_i z\sqrt{t} - tG_{2,i}\left(\frac{z}{\sqrt{t}}\right)\right] dz \right| < \frac{M}{y^2}$$

for any $y \in \mathbb{R}^1 \setminus \{0\}$ and any $t \in [1, \infty)$.

Lemma 9 is proved in the same way as Lemma 6 (see also [18, Lemma 5]).

To prove Lemma 8, we represent the power of its integrand exponent as

$$-z^2 - z^2 \sum_{j=1}^{m_{2,i}} a_{ij} \cos\frac{h_{ij}^{(2)} z}{\sqrt{t}} - z\sqrt{t} \sum_{j=1}^{m_{1,i}} b_{ij} \sin\frac{h_{ij}^{(1)} z}{\sqrt{t}} + t \sum_{j=1}^{m_{0,i}} c_{ij} \left(\cos\frac{h_{ij}^{(0)} z}{\sqrt{t}} - 1\right)$$

$$= -z^2 \left(1 + \sum_{j=1}^{m_{2,i}} a_{ij} \cos\frac{h_{ij}^{(2)} z}{\sqrt{t}}\right) - z\sqrt{t} \sum_{j=1}^{m_{1,i}} b_{ij} \frac{\sin\frac{h_{ij}^{(1)} z}{\sqrt{t}}}{\frac{h_{ij}^{(1)} z}{\sqrt{t}}} \frac{h_{ij}^{(1)} z}{\sqrt{t}} - 2t \sum_{j=1}^{m_{0,i}} c_{ij} \sin^2\frac{h_{ij}^{(0)} z}{2\sqrt{t}}$$

$$= -z^2 \left[1 + \sum_{j=1}^{m_{2,i}} a_{ij} \cos\frac{h_{ij}^{(2)} z}{\sqrt{t}} + \sum_{j=1}^{m_{1,i}} b_{ij} h_{ij}^{(1)} \frac{\sin\frac{h_{ij}^{(1)} z}{\sqrt{t}}}{\frac{h_{ij}^{(1)} z}{\sqrt{t}}} + \frac{1}{2} \sum_{j=1}^{m_{0,i}} c_{ij} \left[h_{ij}^{(0)}\right]^2 \left(\frac{\sin\frac{h_{ij}^{(0)} z}{2\sqrt{t}}}{\frac{h_{ij}^{(0)} z}{2\sqrt{t}}}\right)^2\right].$$

The argument of its integrand cosine function is represented as

$$z\left(y - q_i\sqrt{t} + \sqrt{t} \sum_{j=1}^{m_{1,i}} b_{ij} \cos\frac{h_{ij}^{(1)} z}{\sqrt{t}}\right) - z^2 \sum_{j=1}^{m_{2,i}} a_{ij} \sin\frac{h_{ij}^{(2)} z}{\sqrt{t}} + t \sum_{j=1}^{m_{0,i}} c_{ij} \frac{\frac{\sin h_{ij}^{(0)} z}{\sqrt{t}}}{\frac{h_{ij}^{(0)} z}{\sqrt{t}}} \frac{h_{ij}^{(0)} z}{\sqrt{t}}$$

$$= z\left(y - q_i\sqrt{t} + \sqrt{t} \sum_{j=1}^{m_{1,i}} b_{ij} \cos\frac{h_{ij}^{(1)} z}{\sqrt{t}} + \sqrt{t} \sum_{j=1}^{m_{0,i}} c_{ij} h_{ij}^{(0)} \frac{\frac{\sin h_{ij}^{(0)} z}{\sqrt{t}}}{\frac{h_{ij}^{(0)} z}{\sqrt{t}}}\right) - z^2 \sum_{j=1}^{m_{2,i}} a_{ij} \sin\frac{h_{ij}^{(2)} z}{\sqrt{t}}.$$

The rest of the proof of Lemma 8 is similar to the proof of Lemma 5 (see also [18, Lemma 4]). Now we can decompose (18) into sum (13) and estimate it as in the proof of Theorem 4, using Lemmas 8 and 9 instead of Lemmas 5 and 6, respectively. This completes the proof of Theorem 6. □

Remark 5. Exponential weights arising in the above closeness theorems are not caused by the presence of nonlocal terms; the reason is the dissipativity of the problem. The specified weight is preserved in the classical case as well: if all the shifts $h_{ij}^{(k)}$ are equal to zero, then the difference, the limit of which is estimated in Theorem 6, is *identically* equal to zero.

Remark 6. It is easy to see that the function

$$\omega(x, t) \stackrel{\text{def}}{=} w\left(\frac{x_1 + q_1 t}{\sqrt{p_1}}, \dots, \frac{x_n + q_n t}{\sqrt{p_n}}, t\right)$$

is the classical bounded solution of the equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} p_i \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} q_i \frac{\partial u}{\partial x_i}, \tag{19}$$

satisfying initial-value condition (2); therefore, one can use problem (19), (2) instead of problem (16), (17) in the theorem on the (weighted) closeness of the solutions. Regarding this, we note that Theorem 6 presents a behavior of the solution qualitatively different from the prototypical case of a homogeneous elliptic operator on the right-hand side of the equation (cf. Theorem 4), but the specified type of that behavior is preserved even if the equation has no nonlocal *high-order* terms at all (cf. [18, Theorem 2]). Thus, we have a situation similar to the classical parabolic theory (see [26]): if we add low-order terms to a parabolic differential-difference equation, then qualitatively new effects can arise.

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