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GENERAL THEORY OF INFINITESIMALS

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INTRODUCTION

The use of infinitesimals, it is well known, goes back to the beginning of the eighteenth century, when the first step were made in the calculus of infinitesimals, which was invented independently by Leibniz and Newton. In Leibniz' approach, the derivative at a point of a real function of a real argument is the ratio of two infinitesimals. Leibniz did not introduce a correct definition of these infinitesimals, but he did give the idea of these elements and described them as ideal numbers subject to the same laws as ordinary numbers. The absense of a rigorous definition leads to paradoxes; the following example is due to Pierre Cartier (from an unpublished article, cited with this kind consent):

"For calculation of the derivative of the function $y = x^2$, if an 'infinitesimal' quantity dx is added to x, then it turns out that $y + dy = (x + dx)^2 = x^2 + 2xdx + (dx)^2$ and finally $dy/dx = 2x + dx2x$. The result $dy/dx = 2x$ agrees well with orthodoxy, but the paradox is that it is necessary to assume that dx is not equal to zero in order to carry out the division, and

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at the same time it is necessary to assume that dx is equal to zero in order to write the desired equation $2x + dx = 2x$."

Subsequently, the ideas of Weierstrass, Abel, Dirichlet, and Kronecker concentrated all of analysis around the concept of the limit, defined by the ritual periphrasis: "$\forall\varepsilon > 0\ \exists\delta > 0\ldots$."

We note that physicists and engineers continue to use infinitesimal quantities in their arguments which, although condemned by mathematicians as not rigorous, work in the majority of cases. We can find as well from the pen of V. I. Arnol'd the following heuristic statement: "...We consider parallel displacement of vectors along a boundary of a 'small' region D on our surface. It is easy to see that the result of such a displacement is a rotation by a 'small' angle..." [1, p. 264] (quotation marks " not in original text). Further, we find: "...The value of the curvature form on a pair of 'infinitesimal' tangent vectors equals the rotation angle for displacement along the infinitesimal parallelogram constructed by these theorems."

The goal of the theory of infinitesimals is to permit us to omit the quotation marks " in the preceding excepts.

It can be considered that this goal was accomplished in the works of Abraham Robinson, taking as his initial point Skolem's discovery in the 1930s of nonstandard models of Peano arithmetic (see [2]).

Application of Robinson's method presents some difficulties caused by the necessity of the intrusion of logic into mathematical practice – in fact, it is well known that mathematicians do not like logic.

Fortunately, Nelson's article [3] referred to the logical problems in set theory. In this article theory of internal sets was outlined that reduced the use of infinitesimals to three simple laws – the principles of transfer, idealization, and standarization.

However, all modern theories of infinitesimals have deficiencies that are pointed out at the beginning of [4].

We shall attempt, beginning with a simple situation, to distinguish a theory of infinitesimals whose sphere of possible applications is wider.

Let (u_n) be a sequence of numbers such that $\lim\limits_{n\to\infty} u_n = 0$. Intuitively this fact can be expressed in the following way: u_n is arbitrarily small if n is chosen large enough. The mania peculiar to mathematicians of creating ideal objects in an insurmountable way requires us to write: if n is infinitely large, then u_n is infinitely small. At first glance, it seems that to attach a rigorous meaning to these two terms is rather difficult. What does "ω is infinitely large" mean, if ω is a natural number?

If by $\mathbf{N}$ we denote the set of all natural numbers, then one of the possibilities is the attempt to construct a set $^*\mathbf{N}$, containing $\mathbf{N}$, on which the relation > can be extended by the relation $^*>$ defined on all $^*\mathbf{N}$. Something similar became possible after Skolem's discovery; this attempt was continued by Robinson.

It can also be said that "infinitely large" means "larger than any ordinary (standard, intuitive) natural number," and recognized that the latter numbers do not fill up the entire set $\mathbf{N}$. In fact, again we mention the remarkable challenge of Georges Reeb and we write

$$\mathbf{N} \neq \{0, 1, 2, \ldots, n, \ldots\}$$

for the 0, 1, 2,..., that are perceived as ordinary natural numbers. This assertion, shocking to classical mathematics (which is not accustomed to external set theory), within the framework of Zermelo–Fraenkel set theory is no more unprovable than its negation. More than that, it simply cannot be stated in ZF language. For this we would need the infinitely long formula:

$$\omega \neq 0 \wedge \omega \neq 1 \wedge \ldots \wedge \omega \neq n \wedge \ldots .$$

In order to obtain a shorter statement, it is necessary to enrich the lexicon. Nelson's idea is enrichment of ZF language by a new work – a predicate denoted by st (read: standard). This makes it possible to write the following well-formed formula:

$$\exists \omega \in \mathbf{N}\ (\forall n \in \mathbf{N}\ (n\,\mathrm{st} \Rightarrow \omega \neq n)).$$

If "ordinary" natural numbers are considered to be standard, then the natural number ω from this formula can be considered infinitely large.

It seems important to us to note that the concepts of infinitely small and infinitely large quantities are mathematical abstractions from the intuitive concepts of a very small (to whatever degree necessary), or negligible quantity; very large,... etc.

We note equally the relative character of these definitions. Infinite smallness means in fact infinite smallness in comparison with the standard. We shall consider as an example the sequence $(u_n) = (1/\varepsilon_n)(-1)^n$, where ε is an infinitesimal. It is clear that (u_n) approaches zero when n approaches infinity, but if n is infinitely large but does not exceed $1/\varepsilon$, then u_n is not an infinitesimal.

For this reason the necessity arises for infinitely-infinitely large natural numbers (infinitely large in comparison with $1/\varepsilon$), or infinitely large of order 2. This goal is achieved in the present article. In a number of other applications we shall show that if $(x_n(t))$ is a sequence of continuous functions pointwise converging to x(t), and if $P_m(\varepsilon) = \{t: |x_m(t) - x(t)| < \varepsilon\}$, then the set of points of continuity of function x(t) can be near (in the sense which we shall refine below) to set $P_m(\varepsilon)$, where ε is "very large" (infinitely large), and m is "very very large."

In order to construct infinitely small and infinitely large quantities (and many other things in addition) of all orders completely rigorously, it is obvious that we shall need a richer lexicon.

I. GRADUATED INTERNAL SET THEORY

We shall put set theory into a formal framework constructed in the following manner. The language is the same as in the Zermelo-Fraenkel set theory, with its single binary predicate $\in$, enriched by a sequence of new predicates indexed by a sequence of ordinary natural numbers and denoted st, st 1/2,...,st 1/p,... (write: standard, 1/2-standard, etc.).

Among the well-formed formulas formed in the usual manner (see [2]) we distinguish those in which none of the predicates st, st 1/p play a part; these are called internal formulas.

The axioms include

(a) axioms of Zermelo-Fraenkel set theory with the axiom of choice (ZFC), associated with the internal formulas,

(b) three schemes of axioms - graduated analogues of Nelson's axioms:

Scheme of Transfer Axioms (T_p). If $\varnothing$ is an internal formula, all constants $a_1,\dots,a_n$ of which are 1/p-standard, then $\forall^{\mathrm{st}\,1/p}x \ \varnothing(x, a_1, \dots, a_n) \Leftrightarrow \forall x \ \varnothing(x, a_1, \dots, a_n)$.

Scheme of Idealization Axioms (I_p). If $\varnothing$ is an internal formula with two free variables x and y, and all of its constants are 1/p standard, then $\forall^{\mathrm{st}\,1/p\,\mathrm{fini}} z \exists y \forall x \in z \ \varnothing(x, y) \Rightarrow \exists^{\mathrm{st}\,1/p+1} \xi \forall^{\mathrm{st}\,1/p} x \ \varnothing(x, \xi)$.

Scheme of Standardization Axioms (S). If $\varnothing$ is a formula, then $\forall^{\mathrm{st}} z \exists^{\mathrm{st}} x \forall^{\mathrm{st}} t \ t \in x \Leftrightarrow (t \in z \wedge \varnothing(t))$.

In these formulations we use the abbreviations $\forall^{\mathrm{st}\,1/p}x$ for $\forall x \ \ x \,\mathrm{st}\, 1/p \Rightarrow \dots$, $\exists^{\mathrm{st}\,1/p}x$ for $\exists x \ \ x \,\mathrm{st}\, 1/p$ and $\dots$, $\forall^{\mathrm{st}\,1/p\,\mathrm{fini}}x$ for $\forall^{\mathrm{st}\,1/p}x$, x

(c) a new scheme of axioms relating the external predicates:

Scheme of Graduating Axioms (G_p). For each "ordinary" natural number p we have $\forall x \times x \,\mathrm{st}\, 1/p \Rightarrow x \,\mathrm{st}\, 1/p+1$.

In Appendix below we prove that our set theory is a conservative extension of ZFC.

The idealization axiom furnishes us with infinitesimals of order p. It is sufficient to apply it to the formula

$$\varnothing(x, y) = (x \in \mathbf{R}_+ \wedge y \in \mathbf{R} \wedge |y| < x).$$

Infinitely large quantities of order p are obtained in the same way.

We note that we do not give a graduated analogue of Nelson's principle of standardization. The justification of this is that axiom (S) postulates that for each formula $\varnothing$, not necessarily internal, there exists a standard set whose standard elements are exactly the

the standard elements of our z, satisfying $\emptyset$. In our set theory we do not have the right to write $E=\{t \in z: \varnothing(t)\}$ since the scheme of axioms of separation is applicable only to the internal formulas. Nevertheless, we shall allow such expressions; they are not sets of graduated internal set theory GIST; we shall say that these are external sets.

A set x from (S) is called a standardization of E and denoted by st E. If E is an external set, then there does not always exist a 1/p-standard set containing the same 1/p-standard elements as E, as the following example shall show. In cases when such a set exists, we shall denote it by $st_{1/p}E$, and we shall call it a standardization of order p of set E. When it exists, the standardization of order p is unique, since the transfer axiom permits us to assert that if two 1/p-standard sets have the same 1/p-standard elements, then they have the same elements.

Example. If $E=\{n \in \mathbf{N}: \neg(n \, \mathrm{st})\}$, then st E = $\emptyset$. Further, $st_{1/2}E$ does not exist, since otherwise it would be a nonempty subset of N and would have a least element, which is obviously impossible.

II. NONSTANDARD ANALYSIS

II.1. Relations of Infinite Nearness

In the following, $(X, \mathcal{O})$ denotes a Hausdorff topological space. For $a \in X$ we set $\mathcal{O}(a)=\{U \in \mathcal{O}: a \in U\}$

Definition 1. If X is a 1/p-standard space and $a \in X$ is a 1/p-standard point, then we say that x is infinitely near to a of order p if $\forall^{\mathrm{st}\,1/p} U \in \mathcal{O} \quad (a \in U \Rightarrow x \in U)$. Notation: $x \xrightarrow{p} a$

We shall write $x \xrightarrow{p} a$. with a bar behind the arrow if $x \overset{p}{\mapsto} a$ and x is (1/p + 1)-standard.

We note that in the case when X is R with the usual topology, $x \xrightarrow{p} a$ is equivalent to the fact that $\forall^{\mathrm{st}\,1/p} s \in \mathbf{R}_+^* \; |x-a|<s$.

Definition 2. If X is R and $x \xrightarrow{p} 0$, then we say that x is an infinitesimal of order p.

We shall study some deeper properties of the relations $\xrightarrow{p}$. It is clear that these are external relations, since their definition requires the use of predicates st 1/p.

Property 1. If p and p' are two ordinary natural numbers such that p' is greater than p, then $x \xrightarrow{p'} a \Rightarrow x \xrightarrow{p} a$.

Property 2. If a is st 1/p and b is st 1/p, then $b \xrightarrow{p} a$ implies b = a.

Property 3. If X is a 1/p-standard metric space with metric d and a is a 1/p-standard point of X, then $x \xrightarrow{p} a$ is equivalent to the fact that d(x, a) is an infinitesimal or order p.

Remark. If $X=\overline{\mathbf{R}}$ is the real line supplemented with points $\pm\infty$, and $x \in \mathbf{R}$, then $x \xrightarrow{p} +\infty$, if and only if x is infinitely large of order p.

Definition 3. If X is a 1/p-standard metric space with metric d, then we shall say that two points x and y are equivalent of order p if d(x, y) is an infinitesimal of order p. We denote the latter relation by $x \overset{p}{\simeq} y$.

Equivalence of order p and infinite nearness of order p differ by the absence in the first case of the requirements of type st 1/p in the argument.

Property 4. If $x \xrightarrow{p+1} b$ and $b \overset{p}{\mapsto} a$, then $x \xrightarrow{p} a$.

Property 5. If X is a 1/p-standard metric space, then $(x \overset{p}{\simeq} y$ and $y \xrightarrow{p} a) \Rightarrow x \xrightarrow{p} a$.

Property 6. If X is a regular topological space, $a, b \in X$, where X, a, and b are 1/p-standard and p, q are ordinary natural numbers, and q is greater than p, then $(x \xrightarrow{q} b \wedge x \xrightarrow{p} a) \Rightarrow b \xrightarrow{p} a$.

Property 7. If X is a 1/p-standard topological space and E is a 1/p-standard subset of X, then for $x \in X$

$$(\forall^{\mathrm{st}\,1/p+1} y \in E \;\; y \xrightarrow{p} x) \Leftrightarrow (\forall y \in E \;\; y \xrightarrow{p} x).$$

Property 8. If X is a 1/p-standard metric space, and F and G are 1/p-standard subsets of X, then

$$(\forall^{\mathrm{st}\,1/p+1}x\in F\ \forall^{\mathrm{st}\,1/p+1}y\in G\ x\overset{p}{\simeq}y)\Leftrightarrow(\forall x\in F\ \forall y\in G\ x\overset{p}{\simeq}y).$$

The reader will have no difficulty proving Properites 1-8.

II.2. Some Elementary Applications in Topology and Analysis

THEOREM 1. If a topological space $(X,\mathcal{O})$ and element $a\in X$ are 1/p-standard, then there exists $V\in\mathcal{O}(a)$ such that V is (1/p + 1)-standard and all points of V are infinitely near point a of order p.

Proof. Apply the axiom of idealization to the inclusion relation defined on $\mathcal{O}(a)$.

THEOREM 2. If X is 1/p-standard, A is a 1/p-standard subset of X, and a is a 1/p-standard point, then the following ar equivalent:

i) $a\in\mathring{A}$ (where $\mathring{A}$ is the interior of A),

ii) $x\overset{p}{\to}a$ implies $x\in A$,

iii) $x\overset{p}{\mapsto}a$ implies $x\in A$.

Proof. i) $\Rightarrow$ ii). We have $a\in\mathring{A}\Leftrightarrow\exists U\in\mathcal{O}(a)U\subset A\Leftrightarrow\exists^{\mathrm{st}\,1/p}U\in\mathcal{O}(a)\ \ U\subset A$ (transfer). If $x\overset{p}{\to}a$, then $x\in U$ and consequently $x\in A$.

ii) $\Rightarrow$ iii). Obvious.

iii) $\Rightarrow$ i). Let $V\in\mathcal{O}(a)$ be chosen as in Theorem 1, then $\forall^{\mathrm{st}\,1/p+1}x\in V\ x\in A$, since $x\overset{p}{\mapsto}a$; therefore $\forall x\in V\ \ x\in A$ (transfer), and therefore V $\subset$ A and $a\in\mathring{A}$.

THEOREM 3. With the assumptions of Theorem 2, the following are equivalent:

i) $a\in\bar{A}$ (where $\bar{A}$ is the closure of A),

ii) $\exists x\ x\overset{p}{\to}a$ and $x\in A$,

iii) $\exists x\ x\overset{p}{\mapsto}a$ and $x\in A$.

Proof. It is very easy to establish the equivalence of the conditions "not i)," "not ii)," and "not iii)." For this it is sufficient to apply the preceding theorem, replacing A with its complement in X.

THEOREM 4 (nonstandard characterization of compactness). Assume that a topological space X is 1/p-standard, A is its 1/p-standard part, q is a natural number greater than p, and consider the conditions:

i) $\forall x\in A\ \exists a\in A\ x\overset{p}{\to}a$,

ii) $\forall^{\mathrm{st}\,1/q}x\in A\ \exists a\in A\ x\overset{p}{\to}a$,

iii) A is compact.

Then i) $\Leftrightarrow$ ii) $\Rightarrow$ iii).

Proof. ii) $\Rightarrow$ i). Apply Property 7.

i) $\Rightarrow$ ii). Obvious.

ii) $\Rightarrow$ iii). Suppose not. Then there exists an open 1/p-standard cover Ω of space X from which it is impossible to derive a finite subcover; now applying the axiom of idealization to the formula

$$\varnothing(x,U)\equiv(U\in\Omega\wedge x\notin U),$$

we conclude that there exists a (1/p + 1)-standard x in X such that for all 1/p-standard U in Ω, we have $x\notin U$. This means that x is not infinitely near of order p to even one 1/p-standard point. Then ii) is not true.

Remark. In the case p = 1 it can be shown that in Nelson's IST theory, i) $\Leftrightarrow$ iii).

The interest presented by the characterizations given in Theorems 2-4 is obvious. As an example we introduce for 1/p-standard a and A the direct characterization of properties

$a \in \bar{A}$, $a \in \overset{\circ}{\bar{A}}$. In an obvious manner we obtain:

$$a \in \bar{A} \Leftrightarrow \left(x \overset{p}{\mapsto} a \Rightarrow \exists y \left(y \xrightarrow{p+1} x \wedge y \in A\right)\right),$$

$$a \in \overset{\circ}{\bar{A}} \Leftrightarrow \exists x \overset{p}{\mapsto} a \ \forall y \left(y \xrightarrow{p+1} x \Rightarrow y \in A\right).$$

It is clear that these characterizations are impossible within the framework of the theories of Robinson and Nelson, where p = 1.

Now we shall penetrate to the heat of the analysis with the help of the limit concept. The following theorem gives an infinitesimal characterization of the concept of the limit at a point.

THEOREM 5. Let X and X' be 1/p-standard topological spaces, $a \in X$ and $a' \in X'$ be 1/p-standard points, f be a 1/p-standard mapping from X to X', q a natural number larger than or equal to p. The following are equivalent:

i) $\lim_{x \to a} f(x) = a'$

ii) $x \overset{q}{\to} a \Rightarrow f(x) \overset{q}{\mapsto} a'$;

iii) $x \overset{q}{\mapsto} a \Rightarrow f(x) \overset{p}{\to} a'$.

In ii) and iii) we can replace $x \overset{q}{\mapsto} a$ with $x \overset{q}{\to} a$; in this case we must also replace $f(x) \overset{q}{\mapsto} a'$ with $f(x) \overset{q}{\to} a'$.

Proof. ii) ⇒ iii). Obvious.

iii) ⇒ ii). In the classical definition of i), all constants are 1/p-standard, therefore by the transfer axiom it is sufficient to show that for each open 1/p-standard set U containing a', there exists an open set V lying in X such that f(V) ⊂ U. Theorem 1 supplies us with an open set V, all of whose points are near to a of order q; now it is sufficient to apply iii) in order to get f(V) ⊂ U.

Finally, the proof of i) ⇒ ii) is analogous to that in the theories of Robinson and Nelson.

If X is the one-point compactification $\mathbf{N} \cup \{+\infty\}$ of the natural series, then the following theorem is obtained as a corollary to Theorem 5.

THEOREM 6. If (x_n) is a 1/p-standard sequence in a 1/p-standard topological space, a is a 1/p-standard point of this space, and q, p are natural numbers, $q \geqslant p$, then the following are equivalent:

i) $\lim_{n \to \infty} (x_n) = a$,

ii) $n \overset{q}{\to} +\infty \Rightarrow x_n \overset{q}{\to} a$,

iii) $n \overset{q}{\to} +\infty \Rightarrow x_n \overset{p}{\to} a$

In ii) and iii) we can replace $n \overset{q}{\mapsto} +\infty$ with $n \overset{q}{\to} +\infty$, and then we must replace $x_n \overset{q}{\mapsto} a$ with $x_n \overset{q}{\to} a$ in ii).

We can already characterize in a nonstandard way the limit points of the sequence. We have

THEOREM 7. In the assumptions of Theorem 6 the following are equivalent:

i) a is a limit point of the sequence (x_n),

ii) $\exists n \in \mathbf{N} \left(n \overset{q}{\mapsto} +\infty \text{ and } x_n \overset{q}{\to} a\right)$,

iii) $\exists n \in \mathbf{N} \left(n \overset{q}{\mapsto} +\infty \text{ and } x_n \overset{p}{\to} a\right)$.

In ii) and iii) we can replace $n \overset{q}{\to} +\infty$ with $n \overset{q}{\to} +\infty$, thus replacing $x_n \overset{q}{\mapsto} a$ with $x_n \overset{q}{\to} a$ in ii).

In Theorems 8 and 9, X and X' denote 1/p-standard topological spaces.

THEOREM 8. If q is a natural number greater than or equal to p, X and X' are metric spaces, and f: X → X' is a 1/p-standard function, then the following are equivalent:

i) f is uniformly continuous;

ii) $x \overset{q}{\sim} y \Rightarrow f(x) \overset{q}{\sim} f(y)$,

iii) $x \overset{q}{\sim} y \Rightarrow f(x) \overset{p}{\sim} f(y)$.

THEOREM 9. If q is a natural number greater than or equal to p, X is a metric space, and (f_n) is a 1/p-standard sequence of functions from X to X', then the following are equivalent:

i) (f_n) uniformly converges in X to a function f,

ii) $n \overset{q}{\mapsto} +\infty \Rightarrow (\forall x \in X\ f_n(x) \overset{q}{\sim} f(x))$,

iii) $n \overset{q}{\mapsto} +\infty \Rightarrow (\forall x \in X\ f_n(x) \overset{p}{\sim} f(x))$.

As an illustration of the preceding theorems we cite a nonclassical proof of the following theorem:

A uniform limit f of sequence (f_n) of continuous functions is a continuous function.

By the axiom of transfer, it is sufficient to carry out the proof for standard f and (f_n). Thus it is sufficient to establish the continuity of f at each standard point.

Let $a \in X$, a be standard. By Theorem 5 it is sufficent to show that

$$x \overset{2}{\mapsto} a \Rightarrow f(x) \overset{1}{\mapsto} f(a).$$

We take $x \overset{2}{\mapsto} a$, and let $n \overset{1}{\mapsto} +\infty$. From Theorems 7, 5, and 6 it follows that

$$f(x) \overset{1}{\sim} f_n(x) \overset{2}{\mapsto} f_n(a) \overset{1}{\mapsto} f(a),$$

hence $f(x) \overset{1}{\mapsto} f(a)$. Thus, f is continuous at point a.

If we compare this proof with that given by Robinson in [2, p. 117], then it is obvious that ours is much shorter because of the application of infinitesimals, because, as Robinson himself notes, his proof does not use the nonstandard condition of uniform convergence.

For those who are not familiar with infinitesimals, the latter proof can be shown in a complicated way, like the classical proof, and is not of theoretical interest. The application must lead to a development of this point of view.

We shall assume that the sequence (f_n) converges to f pointwise. It will be obvious that generalized nonstanard analysis permits us to give a nice description of the set of points of continuity of function f.

We denote by (f_n) the sequence of functions from X to X' which converges to a function f pointwise. For each natural m and each positive real ε we set

$$P_m(\varepsilon) = \{x \in X:\ d(f_m(x),\ f(x)) < \varepsilon\}$$

(X' is a metric space). If topological spaces X and X' and sequence (f_n) are standard (then f is automatically standard), then we have

THEOREM 10. For each positive real ε such that $\varepsilon \overset{1}{\to} 0$, and each natural number m such that $m \overset{2}{\to} +\infty$, the set E of points of continuity of function f equals $\mathrm{st}(\mathring{P}_m(\varepsilon))$.

Proof. It is sufficient to show that E and $\mathring{P}_m(\varepsilon)$ have the same standard points. Let x be a standard point of E, and let $y \overset{3}{\to} x$. From Theorems 5 and 6 we deduce:

$$f(y) \overset{3}{\longrightarrow} f(x) \overset{2}{\longleftarrow} f_m(x) \overset{3}{\longleftarrow} f_m(y).$$

We conclude that $f_m(y) \overset{2}{\sim} f(x)$, and since ε is 1/2-standard, then $d(f_m(y), f(y)) < \varepsilon$; therefore $y \in P_m(\varepsilon)$ is true for any $y \overset{3}{\to} x$. Since $P_m(\varepsilon)$ is 1/3-standard, then $x \in P_m(\varepsilon)$. Conversely, we shall work out standard x from $\mathring{P}_m(\varepsilon)$. Since $P_m(\varepsilon)$ is 1/3-standard, by Theorem 2 if $y \overset{3}{\to} x$, then $y \in P_m(\varepsilon)$. Then $f_m(y) \overset{1}{\sim} f(y)$, since $\varepsilon \sim 0$. Consequently, $f(y) \overset{1}{\sim} f_m(y) \overset{3}{\mapsto} f_m(x) \overset{2}{\mapsto} f(x)$, and therefore $f(y) \overset{1}{\sim} f(x)$. Thus, f is continuous at x by Theorem 5.

It is obvious that this proof, like the expression for the set of points of continuity, is very natural.

The classical analogue of Theorem 10 can be found in [10]. Yosida gives for E the expression

$$E=\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\mathring{P}_m(1/n).$$

The following theorem is proved in [2] by Robinson, however his proof is partially classical. We recall that an element x of metric space X is called limited if there exists a standard limited set containing x.

THEOREM 11. If X is a standard metric space, X' is a standard regular space, and f is a standard mapping from X to X', then f is compact if and only if for each limited x, f(x) is near-standard.

Proof. For proof of the necessity, we work out limited $x\in X$. By definition there exists a limited standard B containing x. Since $f(x) \in \overline{f(B)}$ and the latter set is compact, then the nonstandard criterion of compactness permits us to conclude the required result.

For proof of sufficiency we establish the property "$\overline{f(B)}$ is relatively compact" for limited standard B. Let x be a 1/2-standard element in $\overline{f(B)}$. By Theorem 3 there exists $y \in f(B)$ such that $y \overset{2}{\to} x$; on the other hand, y is near-standard by assumption; thus, there exists such that $y \overset{1}{\to} z$.

It is sufficient to apply Property 6 in order to observe that $x \overset{1}{\to} z$, and Theorem 3 to observe that $z \in \overline{f(B)}$. The proof is completed by the use of the nonstandard criterion of compactness.

II.3. Application to Differential Equations

Below we state the first applications of the method of infinitesimals to almost-periodic differential equations. This investigation will be continued and deepened in another article.

The Concept of Almost-Period. In [6] Sari showed that nonstandard analysis permits introduction of the concept of almost-period, which is lacking in the classical theory of almost-periodic functions.

If f is a standard mapping of $\mathbf{R}$ to a normed space E, then we shall say that τ is an almost-period for f if for all $t\in\mathbf{R}$ we have $f(t+\tau)\sim f(t)$. We denote by P(f) the set of all almost-periods of mapping f. We note that set P(f) is external (which is inconvenient).

If ε is a positive real number, then we set $P(f,\varepsilon)=\{\tau\in\mathbf{R}:\ \forall t\in\mathbf{R}\ \|f(t+\tau)-f(t)\|<\varepsilon\}$.

The classical definition asserts that a continuous function f is almost-periodic if for each ε, P(f, ε) is relatively dense in $\mathbf{R}$, i.e., a real ℓ(ε) can be found such that for any $a\in\mathbf{R}$, the set $P(f,\varepsilon)\cap[a,a+l(\varepsilon)]$ is nonempty.

THEOREM 12 [6]. A continuous function f is almost-periodic if and only if P(f) is relatively dense.

Remark. If ε is an arbitrary but fixed infinitesimal (of sufficient order), then it is clear that P(f, ε) is contained in P(f), from which we conclude that f is an almost-periodic function (APF) if and only if P(f, ε) is relatively dense.

P(f) carries the structure of a group (analogoues to the group of periods of a periodic function), which does not happen for P(f, ε).

Let $(f_x)_{x\in K}$ be a stanard family of PFs. We denote by K_S the internal set of standard elements of K.

THEOREM 13. Internal set $P_S(f)=\bigcap_{x\in K_S}P(f_x)$ is relatively dense in $\mathbf{R}$.

Proof. Let $\varepsilon>0$, $\varepsilon\overset{1}{\to}0$, and $P_S(f_x,\varepsilon)=\bigcap_{x\in K_S}P(f_x,\varepsilon)$; we shall consider the formula with two variables: $\varnothing(x,L)=(\forall a\in\mathbf{R}\ P(f_x,\varepsilon)\cap[a,a+L]\neq\varnothing)$. It is known [7] that every finite intersection of sets of type $P(f_x,\varepsilon)$ is relatively dense in $\mathbf{R}$; therefore it is sufficient to apply the axiom of idealization to formula ∅ in order to establish that $P_S(f,\varepsilon)$ is relatively dense. Since $P_S(f,\varepsilon)$ lies in $P_S(f)$, the proof is concluded.

Uniformly Almost-Periodic Functions. Let $f\colon K\times \mathbf{R}\to E$ be a continuous function; we set $f_x = f(x, \ )$. In the classical case it is said that f is a uniformly almost-periodic function (UAPF) on K if for each real $\varepsilon > 0$, $P(f, \varepsilon)=\bigcap_{x\in K} P(f_x, \varepsilon)$ is relatively dense in $\mathbf{R}$. The following nonstandard characterization of uniform almost-periodicity is true.

THEOREM 14. Function f is uniformly almost-periodic on K if and only if $P(f)=\bigcap_{x\in K} P(f_x)$ is relatively dense in $\mathbf{R}$.

THEOREM 15. If K is a compact topological space and all f_x are almost-periodic; then f is a UAPF if and only if f is continuous in x uniformly with respect to $t\in\mathbf{R}$.

Proof. Sufficiency. We shall establish that $P_s(f)$ lies in P(f), after which we can apply Theorem 11. We shall assume that all constants of the theorem are standard and we work out τ from $P_s(f)$. For each $y \in K$ there exists a standard $x \in K$ (its shadow) such that $y \to x$ (apply the nonstandard criterion of compactness). The condition of uniform continuity with respect to $t\in\mathbf{R}$ can be written thus: if $y \to x$, then for all t we have $f(y, t) \sim f(x, t)$. Thus, we have the sequence

$$f(y, t+\tau)\sim f(x, t+\tau)\sim f(x, t)\sim f(y, t).$$

The first and last equivalences follow from continuity in x uniformly with respect to t, the second from the fact that $\tau\in P(f_x)$.

In order to show necessity, it is sufficient to establish that for standard x we have $y\overset{2}{\to} x\Rightarrow(\forall t\in\mathbf{R}\ f(y, t)\sim f(x, t))$. Thus, let x be standard and let $y \overset{2}{\to} x$. With each real t we associate an almost-period $\tau(t)$ from $P(f)\cap[-t, -t+L]$, where L is a length assumed to be 1/2-standard (this is possible because of transfer). We have three equivalences:

(1) $f(y, t) \sim f(y, t + \tau(t))$ since $\tau(t)\in P(f)$,

(2) $f(y, t + \tau(t)) \sim f(x, t + \tau(t))$ since $t+\tau(t)\in[0, L]$ and f is uniformly continuous on 1/2-standard compactum $K \times [0, L]$,

(3) $f(x, t + \tau(t)) \sim f(x, t)$ since $\tau(t)\in P(f)$, from which it follows that $f(y, t) \sim f(x, t)$. This completes the proof.

Remark. If f is a UAPF on compactum K, then P(f) contains a halo of zero.

Almost-Periodic Differential Equations. Consider the differential equation

(E) $$\dot{x}=f(x, t), \text{ where } x\in\mathbf{R}^n,\ t\in\mathbf{R},\ f \text{ is an APF}$$

Does (E) admit an almost-periodic solution? In the general case, no, if no additional conditions are imposed. Below we shall attempt to use the concept of almost-periodic in order to find sufficient conditions for the existence of such solutions.

We shall introduce an auxiliary family of differential equations which we shall need in the following:

(E_τ) $$\dot{x}=f^\tau(x, t), \text{ where } \tau\in\mathbf{R} \text{ and } f^\tau(x, t)=f(x, t+\tau).$$

We shall assume that for each τ and $x_0\in\mathbf{R}^n$ the Cauchy problem

$$\dot{x}=f^\tau(x, t), \quad x(t_0)=x_0$$

admits a unique solution defined for all t. We denote this solution by $F^\tau_{t,t_0}x_0$ and write F_{t,t_0} without upper indices when $\tau = 0$. We recall the following classical properties, the proofs of which are obvious:

$$\text{for all } t,\ t_0, t_1 \quad F_{t,t_0}=F_{t,t_1}F_{t_1,t_0},$$

$$\text{for all } \tau \text{ and all } t,\ t_0 \quad F^\tau_{t,t_0}=F_{t+\tau,\, t_0+\tau}.$$

The next theorem expresses the properties of trajectories of almost-periodic differential equation. With the goal of simplifying the presentation, we shall everywhere assume f and t_0 are standard; generalization to the nonstandard case does not present any difficulties.

THEOREM 16. If f is Lipschitzian (with constant k) and uniformly almost-periodic, then there exists a real $S \overset{1}{\to} +\infty$ and relatively dense set P contained in P(f) such that

$$\forall x\in\mathbf{R}^n\ \forall\tau\in P\ \forall t\in[t_0-S, t_0+S]\ \left\|F_{t,t_0}x-F^\tau_{t,t_0}x\right\|\sim 0.$$

Proof. We work out $\varepsilon \overset{1}{\to} 0$ and take $P = P(f, \varepsilon)$. It is clear that P is 1/2-standard. For all $x \in \mathbf{R}^n$, $t \in {}^{\sigma}c$, and $\tau \in P$ we have $|f(x, t) - f^{\tau}(x, t)| < \varepsilon$. Applying the Gronwall lemma, we deduce from this the inequality

$$\forall x \in \mathbf{R}^n \ \forall t \in \mathbf{R} \ \forall \tau \in P \ \| F_{t,t_0}x - F^{\tau}_{t,t_0}x \| \leqslant |t - t_0| \varepsilon e^{k|t-t_0|}.$$

Set $g(s) = se^{ks}$. Function g increases for positive s, and when s approaches $+\infty$, then g(s) approaches $+\infty$; since g is standard, there exists a real 1/2-standard S such that if $0 \leqslant s \leqslant S$, then $g(s) \leqslant 1/\sqrt{\varepsilon}$. We conclude the proof by noting that for $|t - t_0| \leqslant S$ we have $|t - t_0| \times \varepsilon e^{k|t-t_0|} \leqslant \sqrt{\varepsilon} \sim 0$.

The following theorem is just one step in the investigation of almost-periodic solutions. If h is such a solution, then we know that it, as almost-periodic, must be limited; in addition, if t is infinitely large, then for each almost-periodic τ we must have $h(t + \tau) \sim h(t)$. Are these conditions sufficient?

We think that the answer to this question is negative; nevertheless for standard f we have

THEOREM 17. If f belongs to class C^1, is Lipschitzian and uniformly almost-periodic, and there exists a solution h(t) of Eq. (E) such that

(i) h(t) is limited and standard,

(ii) if $t \overset{2}{\to} +\infty$ then for all 1/2-standard $\tau \in P(f)$ we have $h(t + r) \sim h(t)$,

then there exists a standard solution k(t), an infinitely large number S, a relatively dense set P lying in P(f), and a standard real t_0 such that

$$\forall \tau \in P \ \forall t \in [t_0 - S, t_0 + S] \ k(t + \tau) \sim k(t).$$

Proof. We define $\varepsilon > 0$, $\varepsilon \overset{1}{\to} 0$ and choose S according to Theorem 16. We can write $h(t) = F_{t,t_0}x_0$ with standard t_0 and x_0. From assumption (i) we deduce the existence of almost periodic $\bar{\tau} \in P = P(f, \varepsilon)$ and standard x_1 such that $F_{t_0+\bar{\tau},t_0}x_0 \overset{2}{\to} x_1$ and $\bar{\tau} \overset{2}{\mapsto} +\infty$. In fact, let (τ_n) be a standard sequence such that for all n we have $\tau_n \geqslant n$ and $\tau_n \in P(f, 1/n)$. Standard sequence $(F_{t_0+\tau_n,t_0}x)$ is limited, therefore it admits a standard limit point x_1. This implies (Theorem 7) the existence of $\bar{n} \overset{2}{\to} +\infty$ such that $F_{t_0+\tau_{\bar{n}},t_0}x_0 \overset{2}{\mapsto} x_1$. Thus, it is sufficient to set $\tau = \tau_{\bar{n}}$; since $1/n < \varepsilon$, then $\bar{\tau} \in P(f, \varepsilon)$. On the other hand, $\bar{\tau} \overset{2}{\to} +\infty$, since $\tau \geqslant \bar{n}$.

For all 1/2-standard and all 1/2-standard $\tau \in P$ we have the equivalences:

$$F_{t+\tau,t_0} \cdot x_1 \overset{1}{\sim} F_{t+\tau,t_0} \cdot F_{t_0+\bar{\tau},t_0} \cdot x \overset{2}{\sim} F_{t+\tau+\bar{\tau},t_0+\bar{\tau}} \cdot F_{t_0+\bar{\tau},t_0} \cdot x_0 =$$
$$= F_{t+\tau+\bar{\tau},t_0} \cdot x_0 \overset{3}{\sim} F_{t+\bar{\tau},t_0} \cdot x_0 = F_{t+\bar{\tau},t_0+\bar{\tau}} \cdot F_{t_0+\bar{\tau},t_0} \cdot x_0 \overset{4}{\sim} F_{t,t_0} \cdot F_{t_0+\bar{\tau},t_0} \cdot x_0 \overset{5}{\sim}$$
$$\overset{5}{\sim} F_{t,t_0} \cdot x_1$$

[bases for the equivalences are (1) continuity/initial conditions; (2) Theorem 16; (3) $t + \bar{\tau} \overset{2}{\to} +\infty$, condition (i); (4) Theorem 16; (5) continuity/initial conditions].

The proof is concluded by an application of Property 8 from Sec. II.1.

In fact, we shall establish in another article the existence of an almost-automorphic solution.

APPENDIX

PROOF OF COINCIDENCE OF SYSTEM OF AXIOMS OF GRADUATED INTERNAL SET THEORY (GIST)

We shall follow Robinson [2] in terminology and notation. Let $M = \{B_\tau\}$ be some structure of higher order, based on infinite set $B_0 = A$. We shall work out an ordinary natural number ω, sufficiently large for all possible applications.

We define a sequence of structures ${}^1M, {}^2M, \ldots, {}^{\omega}M$ in the following manner.

(a) Set ${}^1M = \{{}^1B_\tau\} = M$.

(b) If K_1 is the stratified set of all expressions from higher order language Λ that are true in ${}^1M$ with respect to some defined injection C_1 of constants in Λ to the aggregate of all relations in ${}^1M$, then we form, following [2], the extension $K_2^!$ of set K_1 by means of enlargement of set H_2 of all expressions of the type $\varnothing_\tau(b, g, a_b)$, where b is a concurrent constant, g is a constant from the domain of definition of b, τ is the type of b, and a_b is the constant ascribed to each concurrent relation b such that $b_1 \neq b_2 \Rightarrow a_{b_1} \neq a_{b_2}$ and each a_b is not a constant in K_1. We denote by ${}^2M = \{{}^2B_\tau\}$ the model of $K_2^!$ with respect to some injection C_2 from the set of constants encountered in $K_2^!$ and ${}^2M$.

(c) We continue the mapping C_2 to an injection (denoted as well by C_2) from the set of constants of language Λ onto the family of all relations from ${}^2M$ (it is assumed that Λ contains sufficiently many constants for this goal). By K_2 we denote the stratified set of all expressions of language A true in ${}^2M$. It is obvious that K_2 contains K_2;

(d) We extend K_2 analogously to the previous, up to the previous, up to joint set $K_3^!$ of expressions, and we denote its model by ${}^3M = \{{}^3B_\tau\}$. As above, we define a set K_3 containing $K_3^!$ consisting of all expressions true in model ${}^3M$.

(e) Repeat the operation up to step ω.

Thus joint sets of expressions $K_1, K_2, \ldots, K_\omega$ arise such that:

i) $K_1, K_2, \ldots, K_\omega$;

ii) for each $i > 1$, K_i contains extension $K_i^!$ of set K_{i-1};

iii) models ${}^iM$ of sets K_i are contained in each other in a sense which we shall make more precise below.

Notation: if R is a relation of arbitrary type from structure M, we write $R \in M$; if $1 \leqslant p \leqslant q$ and $R \in {}^pM$, then by ${}^qR$ we denote the reinterpretation of R in ${}^qM$.

THEOREM A1. If R is a concurrent relation of structure M, then for any p, we have that ${}^pR$ is a concurrent relation of structure ${}^pM$.

Proof. It is sufficient to reinterpret in ${}^pM$ the assertion that R is a concurrent relation in M.

COROLLARY. The inclusions $K_1 \subset K_2 \subset \ldots \subset K_\omega$ are strict.

Definition A1. If p is a natural number and R is a relation of arbitrary type in ${}^pA$, then we shall say that relation R is internal in ${}^pA$ if $R \in {}^pM$. Otherwise R is an external relation in ${}^pA$.

Let $\tau = (\tau_1, \ldots, \tau_m, \tau_{m+1}, \ldots, \tau_n)$, $\tau' = (\tau_1, \ldots, \tau_m)$ be types and let $R \in {}^pM$ be a relation of type τ. We denote by $\Pi_{\tau'}(R)$ the image of R under projection on ${}^pB_{\tau_1} \times {}^pB_{\tau_2} \times \ldots \times {}^pB_{\tau_m}$, and by R^c the set ${}^pB_{\tau_1} \times \ldots \times {}^pB_{\tau_n} \setminus R$.

THEOREM A2. If R and R' are internal relations on ${}^pA$ of type τ, then $R \cap R'$, R^c and $\Pi_{\tau'}(R)$ are internal relations on ${}^pA$.

Remark. None of the structures ${}^2M, \ldots, {}^pM, \ldots, {}^\omega M$ generally speaking, is complete (external relations exist on them).

Let ω be a natural number and M be a structure based on set A; we define for each $q = 1, \ldots, \omega$ and $p \leqslant q$ the mappings $f_{q,p}\colon {}^pM \to {}^qM$ with ${}^1M = M$, $f_{q,p}\colon R \to {}^qR$. It is clear that for all p, q, and r, for which the relation $f_{q,p} \circ f_{p,r} = f_{q,r}$ has meaning, it is true. Also, if we set ${}^p\underline{B}_\tau = \{f_{\omega,p}(R)\colon R \in {}^pB_\tau\}$, then we have the inclusions ${}^1\underline{B}_\tau \subset {}^2\underline{B}_\tau \subset \ldots \subset {}^p\underline{B}_\tau \subset \ldots \subset {}^\omega\underline{B}_\tau$. Everything stated can be summed up in the following manner: ${}^1\underline{M} \subset \ldots \subset {}^p\underline{M} \subset \ldots \subset {}^\omega\underline{M}$.

Definition A2. If $p < \omega$, then we shall say that relation R is 1/p-standard or standard of order p, when $R \in {}^p\underline{M}$. The notation is $R \operatorname{st} 1/p$.

Property. For each integral p for which $p + 1 < \omega$, we have $R \operatorname{st} 1/p \Rightarrow R \operatorname{st}(1/p+1)$.

In the rest of the article we shall everywhere omit the upper index ω wherever it appears as a constant.

Now we shall study to recognize the internal relations. As it is already known, each standard relation is internal, the same as the relations obtained by idealization beginning with the concurrent relations. Starting with these easily recognized relations, we can describe all internal relations.

A formula all of whose constants are internal shall be called an internal formula in M. Further, we shall write x st 1/p if $x \in {}^{p}M$ or $x \in {}^{p}B_{\tau}$. Thus, a formula is internal if all its constants are internal and it does not contain any of the expressions st 1/p.

Proposition A1. Relation $R \in M$ is internal if and only if it can be separated by means of an internal formula.

Proof. The necessity is obvious. In order to establish the sufficiency, one must express the logical connectives by means of operations in a Boolean algebra, and the universal quantifiers by means of projections; then use Theorem A2 with p = ω.

THEOREM A3 (transfer). Let ∅ be an internal formula with one free variable x, all constants $a_1, a_2, \dots, a_n$ of which are 1/p-standard. Then

$$\forall^{st\,1/p} x \in M\ \varnothing(x, a_1, \dots, a_n) \Leftrightarrow \forall x \in M\ \varnothing(x, a_1, \dots, a_n).$$

Proof. Let $b_i \in {}^{p}M$ be such that $a_i = {}^{\omega}b_i$, $i \in \{1, 2, \dots, n\}$. We have

$$\forall^{st\,1/p} x \in M\ \varnothing(x, a_1, \dots, a_n) \Leftrightarrow \forall x \in {}^{p}M\ \varnothing(x, b_1, \dots, b_n) \Leftrightarrow$$

$$\Leftrightarrow \forall x \in M\ \varnothing(x, b_1, \dots, b_n) \Leftrightarrow \text{reinterpretation in } M) \Leftrightarrow \forall x \in M\ \varnothing(x, a_1, \dots, a_n).$$

THEOREM A4 (idealization). Let ∅ be an internal formula with two free variables x and y assuming values in a 1/p-standard set V, all constants $a_1, \dots, a_n$ of which are 1/p-standard. Then

$$\forall^{st\,1/p\,\mathrm{fini}} z \subset V\ \exists y \in V\ \forall x \in z\ \varnothing(x, y) \Rightarrow \exists^{st\,1/p+1} \xi \in V\ \forall^{st\,1/p} x \in V\ \varnothing(x, \xi).$$

Proof. Since ∅ is an internal formula, relation $R = \{(x, y) \in V \times V \colon \varnothing(x, y)\}$ is also internal. The fact that all constants in ∅ are 1/p-standard implies that R is 1/p-standard.

Let $S \in {}^{p}M$ be such that R = ${}^{\omega}S$, and $W \in {}^{p}M$ such that V = ${}^{\omega}W$. It is clear (using the transfer theorem) that S is a concurrent relation on ${}^{q}W$. Thus, there exists $x \in {}^{p+1}W$ such that $\forall z \in W\ (x, {}^{p+1}z) \in {}^{p+1}S$. Set $\xi = {}^{\omega}x$. This is the desired element. In fact,

a) $\forall z \in W \quad ((x, {}^{p+1}z) \in {}^{p+1}S \Leftrightarrow (\xi, {}^{\omega}z) \in {}^{\omega}S)$ and $x \in {}^{p+1}W \Rightarrow \xi = {}^{\omega}x$ is (1/p + 1)-standard.

b) $z \in W$ and $W \in {}^{p}M \Rightarrow z \in {}^{p}M$; consequently, when z runs through W, then z runs through all 1/p-standard elements of V.

For proof of relative coincidence of the GIST theory it is sufficient to show that GIST is a conservative extension of ZFC, i.e.

Theorem of Conservation. Each internal theoem of GIST is a theorem of ZFC.

For construction of a model we use the sets R(β) defined by induction on ordinals in the following manner; R(∅) = ∅ and for any ordinal β we have $R(\beta) = \bigcup_{\mu \in \beta} (R(\mu))_{(0)}$. We shall need the following properties of sets R(β).

Series of Properties 1. 1. If $\beta \in \beta'$, then $R(\beta) \subset R(\beta')$.

2. If β is a limit ordinal $\left(\beta = \bigcup_{\mu \in \beta} \mu\right)$, then $R(\beta) = \bigcup_{\mu \in \beta} R(\mu)$.

3. If $x \in R(\beta)$ and $t \in x$, then $t \in R(\beta)$.

4. If x ⊂ z and $x \in R(\beta)$, then $x \in R(\beta)$.

Property 2. The axiom of regularity is equivalent to the assertion that each set is contained in R(β).

The proof is in [8, p. 53].

Property 3. If $A_1, A_2, \dots, A_n$ are expressions in ZFC, then there exists a limit ordinal β such that

$$(A_1 \Leftrightarrow A_1^{R(\beta)}) \wedge \dots \wedge (A_n \Leftrightarrow A_n^{R(\beta)}).$$

[Here $A^{R(\beta)}$ denotes the relativization of expression A in R(β).]

The proof is in [3, Theorem 8.7].

Property 4. Let β be a limit ordinal and ${}^{2}R(\beta)$ be an extension of R(β). If z is a finite set and $z \in {}^{2}R(\beta)_{(0)}$, then $z \in {}^{2}R(\beta)$.

Proof of the Theorem. It follows almost step by step the proof of Powell given in [3, p. 1196].

Let A be an internal theorem of GIST; its proof uses the axioms of ZFC in a "naively" finite number, $A_1,\ldots,A_n$ as well as possibly axiom schemes (I_p), (S), (T_p), where p are natural numbers in a finite number such that $p + 1 < \omega$ for some natural number ω.

Let β be a limit ordinal such that

$$(A \Leftrightarrow A^{R(\beta)}) \wedge (A_1 \Leftrightarrow A_1^{R(\beta)}) \wedge \ldots \wedge (A_n \Leftrightarrow A_n^{R(\beta)}),$$

and let $R(\beta) \subset {}^2R(\beta) \subset \ldots \subset {}^pR(\beta) \subset \ldots \subset {}^\omega R(\beta)$ be sequential extensions of $R(\beta)$. Set $E(\beta) = \{(x, y) \in R(\beta) \times R(\beta)\colon x \in y\}$ and consider the structure

$$S_0 = ({}^\omega R(\beta), E(\beta), R(\beta), {}^2R(\beta), \ldots, {}^pR(\beta), \ldots)$$

with interpretations $(x, y) \in E(\beta)$. We shall show that S_0 is a model of the axiom system:

$$\{A_1, \ldots, A_n, (I_p), (S), (T_p), (G_p)\}_{p<\omega-1}.$$

Coincidence of (T_p). If $\emptyset$ is an internal formula with 1/p-standard constants, then for any interpretation $\bar{a}_1,\ldots,\bar{a}_n$ constant in ${}^pR(\beta)$, the interpretation $\overline{\varnothing}(x, \bar{a}_1, \ldots, \bar{a}_n)$ of formula $\varnothing(x, a_1, \ldots, a_n)$ satisfies the conditions of the transfer theorem, therefore the interpretation T_p:

$$\forall x \in {}^pR(\beta)\ \overline{\varnothing}(x, \bar{a}_1, \ldots, \bar{a}_n) \Leftrightarrow \forall x \in {}^\omega R(\beta)\ \varnothing(x, \bar{a}_1, \ldots, \bar{a}_n)$$

is a theorem of ZFC. We consider that (T_p) is true in S_0.

Coincidence of (I_p). For each interpretation in ${}^pR(\beta)$ of the constants of formula $\emptyset$, if we denote by $\bar{\emptyset}$ the corresponding interpretation of $\emptyset$ in structures, axiom I_p admits the following intepretation in S:

$$\forall^{\mathrm{fin}} z \in {}^pR(\beta)\ \exists y \in {}^\omega R(\beta)\ \forall x \in z\ \overline{\varnothing}(x, y) \Rightarrow \exists \xi \in {}^{p+1}R(\beta)\ \forall x \in {}^pR(\beta)\ \overline{\varnothing}(x, \xi).$$

Thus, any finite part of ${}^pR(\beta)$ is internal, since by Property 4 of set $R(\beta)$, we have the relation [z is finite and $z \subset {}^pR(\beta)) \Rightarrow z \in {}^pR(\beta)$. It remains to apply the idealization theorem with $V = {}^\omega R(\beta)$ in order to see that I_p is true in S_0.

Coincidence of (S). For each formula $\emptyset$ and each of its interpretations $\bar{\emptyset}$ in the structure S_0 for $z \in R(\beta)$ we set

$$x = \{t \in z\colon \overline{\varnothing}(t)\}. \qquad (*)$$

By definition x lies in z, consequently (Property 1, part 4), $x \in R(\beta)$. Then

$$\forall z \in R(\beta)\ \exists x \in R(\beta)\ \forall t \in R(\beta)\,(t \in x) \Leftrightarrow (t \in z \wedge \overline{\varnothing}(z)) \qquad (**)$$

[it is sufficient to take the element defined by (*) as x]. Since (**) is the interpretation of (S) corresponding to the interpretation $\bar{\emptyset}$ of formula $\emptyset$, then (S) is true in S_0.

Coincidence of (G_p). This follows from the inclusions ${}^pR(\beta) \subset {}^{p+1}R(\beta)$.

Now the proof is completed analogously to Nelson's proof in [3].

Let $\bar{A}$ be an interpretation of A in S_0; in the proof of A interpreted in S_0, the axioms $A_1,\ldots,A_n$, (I_p), (S), (T_p) and (G_p), $p + 1 < \omega$ take part, since in the proof of $\bar{A}$ the axioms $\bar{A}_1,\ldots,\bar{A}_n$, ($\bar{I}_p$), ($\bar{S}$), ($\bar{T}_p$) and ($\bar{G}_p$) take part. Since A is internal, by the transfer theorem $\bar{A} = A^{{}^\omega R(\beta)} \Leftrightarrow A^{R(\beta)}$. On the other hand, we can choose β for which $A \Leftrightarrow A^{R(\beta)}$; this proves that A is a theorem in ZFC.

CONCLUDING REMARKS

1. The set theory presented above must not be considered as a final system. Other axioms can be chosen as needed. For example, it is clear that it would be desirable to introduce a hierarchy of external sets. We do not do this since such a hierarchy seems to us of little use for applications at the present time; on the other hand, it would seriously complicate the formalism. The guides to our choice were many discussions between participants of the congress in Luminy cited in [4].

2. We did not succeed in working out a completely satisfactory graduated axiom of standardization. However, it can be shown that if $\emptyset$ is a formula in which no predicates

st 1/k enter for k strictly less than p, and if all constants admit a standardization of order 1/p, then the "external set" $\{t \in z \colon \varnothing(t)\}$ admits a standardization of order 1/p.

3. We note that Lemma 2 in [9] is mistaken; as a result, the axiomatics presented there are inconsistent.

4. In the statement of Nelson's principle of idealization (I) no assumptions are made with respect to constants taking part in formula $\emptyset$. Conversely, in the statement of (I_p) it is necessary to assume that the constants are 1/p-standard in order to make it possible to assert (consistently) that an "ideal element" is (1/p + 1)-standard.

5. We cannot show within the framework of GIST that if $n \in \mathbf{N}$ is a 1/p-standard element, then any natural number m < n is 1/p-standard, as it would be natural to expect. This follows from the fact that GIST formalizes the theory of sequential extensions of a structure, where extensions are understood in exactly the sense of Robinson.

In fact, let ^{2}M and ^{3}M be sequential extensions of structure M = ^{1}M containing $\mathbf{N}$. Let element $n \in {}^2\mathbf{N}$ be infinitely large; we define the internal relation R on $^2\mathbf{M}$ by condition $(p, q) \in R \Leftrightarrow (p \in {}^2\mathbf{N},\ q \in {}^2\mathbf{N},\ p^2 \leqslant n,\ q^2 \leqslant n$ and $p \neq q)$. It is concurrent, and therefore in $^3\mathbf{N}$ there exists an element $n_0{}^3 \leqslant n$ which is not 1/2-standard.

6. For analogous reasons we cannot prove in GIST that each 1/p-limited real number is near-1/p-standard. This follows from the fact that the preimage in ^{2}M of relation R from structure ^{3}M (i.e., an R-element of ^{3}M) is not an internal relation (element of ^{2}M); therefore, as we noted above, the graduated principle of standardization that is necessary for a proof is lacking.

Some of the difficulties indicated above are eliminated in [10].

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