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Free vibration analysis of third-order shear deformable composite beams using dynamic stiffness method

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Abstract The dynamic stiffness method is introduced to investigate the free vibration of laminated composite beams based on a third-order shear deformation theory which accounts for parabolic distribution of the transverse shear strain through the thickness of the beam. The exact dynamic stiffness matrix is found directly from the analytical solutions of the basic governing differential equations of motion. The Poisson effect, shear deformation, rotary inertia, in-plane deformation are considered in the analysis. Application of the derived dynamic stiffness matrix to several particular laminated beams is discussed. The influences of Poisson effect, material anisotropy, slenderness and end condition on the natural frequencies of the beams are investigated. The numerical results are compared with the existing solutions in literature whenever possible to demonstrate and validate the present method.

Keywords Generally laminated beams · Third-order shear deformation theory · Free vibration · Dynamic stiffness matrix · Poisson effect

1 Introduction

The composite beams are common structural components when dealing with a variety of engineering structures such as airplane wings, helicopter blades as well as many others in the aerospace, mechanical, and civil industries. Due to the outstanding engineering properties, such as high strength/stiffness to weight ratios, the laminated composite beams are likely to play an increasing role in the design of various engineering type structures and partially replace the conventional isotropic beam structures. Most prominent characteristic for employing the composite structures is the property of tailoring the material for each particular application. The free vibration analysis of the composite beams which are generally used as structural elements in the idealization of light-weight heavy-load bearing components, both from theoretical and practical points of view, is fundamental to the better understanding of the dynamic behavior of the composite beam structures. This is important as the composite beam structures often provide higher flexibility in order to meet the increasing performance requirements and operate in complex environmental conditions. The free vibration characteristics of the composite beams have inspired continuing research interest in recent decades, and a large number of research works address this problem.

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When the Euler–Bernoulli theory for beams is used for the dynamic analysis of laminated beams, the natural frequencies are overestimated, which is the consequence of neglecting the transverse shear strain and assuming that the plane normal to the beam axis remains normal even after the deformation. Due to high ratio of extensional modulus to transverse shear modulus for composite beams, the deformation due to shear cannot be ignored even for reasonably large slenderness ratio. Timoshenko beam theory is then developed to account for shear deformation with the assumption that the displacement field through the beam thickness does not restrict plane section to remain perpendicular to the deformed centroidal line.

However, the first-order shear deformation theory has some limitations such as inaccurate constant transverse shear distribution through the beam thickness necessitating the need for a shear correction factor. The value of this factor has been the subject of considerable discussions. To capture the shear deformation more accurately, some higher-order beam theories have been reported.

Krishna Murty [1] developed the higher-order shear deformation theories based on the elementary theory of beam and applied these theories to the vibration analysis of laminated beams. Bhimaraddi and Chandrashekhara [2] modeled the laminated beams by a systematic reduction of the constitutive relations of the three-dimensional anisotropic body and obtained the basic equations of the beam theory based on parabolic shear deformation theory. Soldatos and Elishakoff [3] developed a third-order shear deformation theory for static and dynamic analysis of an orthotropic beam incorporating the effects of transverse shear and transverse normal deformations. Chandrashekhara and Bangera [4] studied the free vibration characteristics of laminated composite beams using a third-order beam theory. Singh and Abdelnaser [5] analyzed the equations of motion of a cross-ply symmetric laminated composite beam, using a third-order shear deformation theory. Khdeir [6] developed an analytical solution of the classical, first- and third-order laminated beam theories to study the dynamic response of antisymmetric cross-ply laminated beams with generalized boundary conditions and for arbitrary loadings. Marur and Kant [7] compared the higher-order refined theories on the transient response of laminated composite beams, showing the effects of shear deformation and rotary inertia. Kant et al. [8] presented an analytical method for the dynamic analysis of laminated beams using higher-order refined theory. Karama et al. [9] presented a new laminated composite beam model based on discrete layer theory for the static and dynamic analysis of thin and thick beams. Shimpi and Ainapure [10] studied the free vibration of two-layered laminated cross-ply beams using the variationally consistent layerwise trigonometric shear deformation theory. Shi and Lam [11] presented a new finite element formulation for the free vibration analysis of composite beams based on the third-order beam theory. Shimpi and Ainapure [12] presented a simple one-dimensional beam finite element with two nodes and three degrees of freedom per node, based on layerwise trigonometric shear deformation theory. Ghugal and Shimpi [13] presented a review of displacement and stress based refined theories for isotropic and anisotropic laminated beams and discussed various equivalent single layer and layerwise theories for laminated beams. Rao et al. [14] proposed a higher-order mixed theory for determining the natural frequencies of a diversity of simply supported laminated beams. Arya et al. [15] developed a zigzag model for symmetric laminated beam in which a sine term was used to represent the nonlinear displacement field across the thickness. Karama et al. [16] presented a new multi-layer laminated composite structure model by using exponential function to predict the mechanical behavior of multi-layered laminated composite structures. Kapuria et al. [17] proposed an efficient zigzag one-dimensional theory of laminated beams by comparison of analytical solutions of simply supported beam with exact two-dimensional elasticity solutions, first-order and third-order shear deformation solutions for static patch load, natural frequencies, harmonic transverse load with sinusoidal longitudinal variation and buckling under axial load. Murthy et al. [18] developed a refined two-node, four DOF/node beam element based on higher-order shear deformation theory for axial–flexural–shear coupled deformation in asymmetrically stacked laminated composite beams. Aydogdu [19] and Aydogdu [20] carried out the vibration analysis of cross-ply laminated beams and angle-ply laminated beams with different sets of boundary conditions by use of the Ritz method, respectively. The analysis was based on a three-degree-of-freedom shear deformable beam theory. Subramanian [21] performed the free vibration analysis of laminated composite beams using two higher-order displacement based shear deformation theories. Both theories assumed a quintic and quartic variation of in-plane and transverse displacements in the thickness coordinates of the beams, respectively.

A review of the literature indicates that the dynamic analysis of laminated composite beams with cross-ply lay-up has received considerable attention, whereas the literature available for vibration analysis of composite beams with arbitrary lay-up, to the author's knowledge, is rather small in number. Compared to the first-order shear deformation theory, the dynamic analysis of generally layered composite beams based on higher-order shear deformation theory is also limited in the literature. The objective of this paper is to formulate an exact dynamic stiffness matrix for a generally laminated beam based on third-order shear deformation theory which accounts for a parabolic variation of the transverse shear strain through beam thickness and there is no need to use the shear correction coefficient. The method of dynamic stiffness matrix in vibration analysis has certain advantages over the traditional finite element method, particularly when higher frequencies and better accuracies of results are required. This is because, the traditional finite element method only accounts for a finite number of degrees of freedom of a structure, while the dynamic stiffness method accounts for an infinite number of degrees of freedom of a vibrating structure.

An exact analytical method called the dynamic stiffness matrix method to determine the free vibration characteristics of generally layered composite beams is investigated in this paper. The equations of motion of laminated beams are derived via Hamilton's principle, where the Poisson effect and third-order shear deformation theory are considered. From an analytical standpoint, the third-order shear deformation theory results in additional complexity in the formulation of the governing equations and solution procedures. Although the displacement field used in the present work is the same as that of [4], the derived equations of motion slightly differ from those of [4]. The dynamic stiffness matrix is derived directly from the analytical solutions of the basic governing differential equations of motion. The application of the introduced method to free vibration analysis of several illustrative examples is discussed. A series of numerical results are presented to confirm the correctness and accuracy of the present method and the influences of Poisson effect, material anisotropy, slenderness and end condition on the natural frequencies of laminated beams are investigated.

2 Governing equations

Considering a laminated composite beam, illustrated in Fig. 1. The laminated beam is made of layers of orthotropic material in which the orthotropic axes of layer may be oriented at an arbitrary angle with respect to the x-axis. In the right-handed Cartesian coordinate system, the x-axis is coincident with the beam axis and the origin is on the mid-plane of the beam. The length, breadth and thickness of the beam are represented by L, b and h, respectively.

The assumed displacement field for a laminated composite beam based on third-order shear deformation theory is as

$$u_1(x, z, t) = u(x, t) + z\left[\phi - (4/3)\,(z/h)^2\,(\phi + \partial w/\partial x)\right] \tag{1a}$$
$$u_2(x, z, t) = 0 \tag{1b}$$
$$u_3(x, z, t) = w(x, t) \tag{1c}$$

where $u(x, t)$ and $w(x, t)$ are the mid-plane displacements of the beam in the x and z directions, $\phi(x, t)$ is the rotation of the normal to the mid-plane about the y axis, t is time. The displacement field chosen in which the longitudinal displacement is cubic function of the thickness coordinate and the transverse deflection is constant through beam thickness meets the condition that the transverse shear stresses vanish on the beam surfaces and are nonzero elsewhere.

The strain–displacement relations can be expressed as

$$\varepsilon_x^0 = \partial u/\partial x, \quad \kappa_x^0 = \partial\phi/\partial x, \quad \kappa_x^2 = -\left(4/3h^2\right)\left(\partial\phi/\partial x + \partial^2 w/\partial x^2\right) \tag{2a}$$
$$\varepsilon_{xz}^0 = (\phi + \partial w/\partial x), \quad \kappa_{xz}^2 = -\left(4/h^2\right)(\phi + \partial w/\partial x) \tag{2b}$$

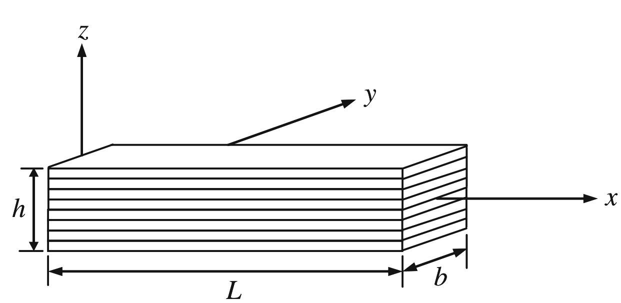

Fig. 1 Geometry of a laminated composite beam

The constitutive equations of the laminated plate based on third-order shear deformation theory can be expressed as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ P_x \\ P_y \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & E_{11} & E_{12} & E_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & E_{12} & E_{22} & E_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & E_{16} & E_{26} & E_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & F_{11} & F_{12} & F_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & F_{12} & F_{22} & F_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & F_{16} & F_{26} & F_{66} \\ E_{11} & E_{12} & E_{16} & F_{11} & F_{12} & F_{16} & H_{11} & H_{12} & H_{16} \\ E_{12} & E_{22} & E_{26} & F_{12} & F_{22} & F_{26} & H_{12} & H_{22} & H_{26} \\ E_{16} & E_{26} & E_{66} & F_{16} & F_{26} & F_{66} & H_{16} & H_{26} & H_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \\ \kappa_x^0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \\ \kappa_x^2 \\ \kappa_y^2 \\ \kappa_{xy}^2 \end{Bmatrix} \tag{3}$$

where N_x, N_y and N_{xy} are the in-plane forces, M_x, M_y and M_{xy} the bending and twisting moments, P_x, P_y and P_{xy} the higher-order bending and twisting moments, ε_x^0, ε_y^0 and ε_{xy}^0 the mid-plane strains, κ_x^0, κ_y^0 and κ_{xy}^0 the bending and twisting curvatures, κ_x^2, κ_y^2 and κ_{xy}^2 the higher-order bending and twisting curvatures, A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, and H_{ij} $(i, j = 1, 2, 6)$, are stiffness coefficients.

For a laminated beam the width direction is free of stresses, so N_y, N_{xy}, M_y, M_{xy}, P_y and P_{xy} equal to zero while ε_y^0, ε_{xy}^0, κ_y^0, κ_{xy}^0, κ_y^2, κ_{xy}^2 are assumed to be nonzero. Thus, Eq. (3) can be rewritten as

$$\begin{Bmatrix} N_x \\ M_x \\ P_x \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{B}_{11} & \bar{E}_{11} \\ \bar{B}_{11} & \bar{D}_{11} & \bar{F}_{11} \\ \bar{E}_{11} & \bar{F}_{11} & \bar{H}_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \kappa_x^0 \\ \kappa_x^2 \end{Bmatrix} \tag{4}$$

where

$$\begin{aligned} \begin{bmatrix} \bar{A}_{11} & \bar{B}_{11} & \bar{E}_{11} \\ \bar{B}_{11} & \bar{D}_{11} & \bar{F}_{11} \\ \bar{E}_{11} & \bar{F}_{11} & \bar{H}_{11} \end{bmatrix} &= \begin{bmatrix} A_{11} & B_{11} & E_{11} \\ B_{11} & D_{11} & F_{11} \\ E_{11} & F_{11} & H_{11} \end{bmatrix} - \begin{bmatrix} A_{12} & A_{16} & B_{12} & B_{16} & E_{12} & E_{16} \\ B_{12} & B_{16} & D_{12} & D_{16} & F_{12} & F_{16} \\ E_{12} & E_{16} & F_{12} & F_{16} & H_{12} & H_{16} \end{bmatrix} \\ &\times \begin{bmatrix} A_{22} & A_{26} & B_{22} & B_{26} & E_{22} & E_{26} \\ A_{26} & A_{66} & B_{26} & B_{66} & E_{26} & E_{66} \\ B_{22} & B_{26} & D_{22} & D_{26} & F_{22} & F_{26} \\ B_{26} & B_{66} & D_{26} & D_{66} & F_{26} & F_{66} \\ E_{22} & E_{26} & F_{22} & F_{26} & H_{22} & H_{26} \\ E_{26} & E_{66} & F_{26} & F_{66} & H_{26} & H_{66} \end{bmatrix}^{-1} \begin{bmatrix} A_{12} & B_{12} & E_{12} \\ A_{16} & B_{16} & E_{16} \\ B_{12} & D_{12} & F_{12} \\ B_{16} & D_{16} & F_{16} \\ E_{12} & F_{12} & H_{12} \\ E_{16} & F_{16} & H_{16} \end{bmatrix} \end{aligned}$$

The transverse shear force–strain relation for the laminated beam can be expressed as

$$\begin{Bmatrix} Q_{xz} \\ R_{xz} \end{Bmatrix} = \begin{bmatrix} A_{55} & D_{55} \\ D_{55} & F_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xz}^0 \\ \kappa_{xz}^2 \end{Bmatrix} \tag{5}$$

Equations (4) and (5) are the one-dimensional laminated beam constitutive equations that account for the third-order shear deformation and Poisson effect.

The laminate stiffness coefficients A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} $(i, j = 1, 2, 6)$ and the transverse shear stiffnesses A_{55}, D_{55}, F_{55}, which are functions of laminate ply orientation, material property and stack sequence, are defined by

$$\left(A_{ij} \quad B_{ij} \quad D_{ij} \quad E_{ij} \quad F_{ij} \quad H_{ij} \right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij} \left(1, z, z^2, z^3, z^4, z^6\right) \mathrm{d}z \tag{6a}$$

$$\left(A_{55} \quad D_{55} \quad F_{55} \right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{55} \left(1, z^2, z^4\right) \mathrm{d}z \tag{6b}$$

The transformed reduced stiffness constants $\bar{Q}_{ij}$ $(i, j = 1, 2, 6)$ and $\bar{Q}_{55}$ are given by

$$\bar{Q}_{11} = Q_{11} \cos^4 \vartheta + 2(Q_{12} + 2Q_{66}) \sin^2 \vartheta \cos^2 \vartheta + Q_{22} \sin^4 \vartheta \tag{7a}$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \vartheta \cos^2 \vartheta + Q_{12}\left(\sin^4 \vartheta + \cos^4 \vartheta\right) \tag{7b}$$

$$\bar{Q}_{22} = Q_{11} \sin^4 \vartheta + 2(Q_{12} + 2Q_{66}) \sin^2 \vartheta \cos^2 \vartheta + Q_{22} \cos^4 \vartheta \tag{7c}$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin \vartheta \cos^3 \vartheta + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \vartheta \cos \vartheta \tag{7d}$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \vartheta \cos \vartheta + (Q_{12} - Q_{22} + 2Q_{66}) \sin \vartheta \cos^3 \vartheta \tag{7e}$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \vartheta \cos^2 \vartheta + Q_{66}\left(\sin^4 \vartheta + \cos^4 \vartheta\right) \tag{7f}$$

$$\bar{Q}_{55} = G_{13} \cos^2 \vartheta + G_{23} \sin^2 \vartheta \tag{7g}$$

where ϑ is the angle between the fiber direction and the beam axis, the reduced stiffness constants Q_{11}, Q_{22}, Q_{12} and Q_{66} can be obtained in terms of the engineering constants [22]

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12} E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21} E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}$$

The total strain energy V of laminated composite beam shown in Fig. 1 is given by

$$V = \frac{b}{2} \int_0^L \left[N_x \varepsilon_x^0 + M_x \kappa_x^0 + P_x \kappa_x^2 + Q_{xz} \varepsilon_{xz}^0 + R_{xz} \kappa_{xz}^2 \right] \mathrm{d}x \tag{8}$$

Substituting N_x, M_x, P_x, Q_{xz} and R_{xz} from Eqs. (4) and (5) into Eq. (8) results in

$$V = \frac{b}{2} \int_0^L \begin{bmatrix} \bar{A}_{11} \left(\varepsilon_x^0\right)^2 + 2\bar{B}_{11} \varepsilon_x^0 \kappa_x^0 + 2\bar{E}_{11} \varepsilon_x^0 \kappa_x^2 + 2\bar{F}_{11} \kappa_x^0 \kappa_x^2 + \bar{D}_{11} \left(\kappa_x^0\right)^2 \\ + \bar{H}_{11} \left(\kappa_x^2\right)^2 + A_{55} \left(\varepsilon_{xz}^0\right)^2 + 2D_{55} \varepsilon_{xz}^0 \kappa_{xz}^2 + F_{55} \left(\kappa_{xz}^2\right)^2 \end{bmatrix} \mathrm{d}x \tag{9}$$

The total kinetic energy T of laminated composite beam is given by

$$T = \frac{b}{2} \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left[(\dot{u}_1)^2 + (\dot{u}_2)^2 + (\dot{u}_3)^2 \right] \mathrm{d}z \mathrm{d}x \tag{10}$$

where ρ is the mass density per unit volume, the superscript dot indicates the differentiation with respect to time t.

Substituting u_1, u_2 and u_3 from Eqs. (1) into Eq. (10) yields

$$T = \frac{b}{2} \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left\{ \left(\dot{u} + z \left[\dot{\phi} - (4/3)\,(z/h)^2 \left(\dot{\phi} + \partial \dot{w} / \partial x \right) \right] \right)^2 + \dot{w}^2 \right\} \mathrm{d}z \mathrm{d}x \tag{11}$$

Here we use Hamilton's principle to derive the governing equations of motion and end conditions of laminated composite beam appropriate for the displacement field in Eqs. (1) and constitutive equations in Eqs. (4) and (5). The Hamilton's principle can be stated as

$$\int_{t_1}^{t_2} (\delta T - \delta V)\mathrm{d}t = 0 \tag{12}$$
$$\delta u = \delta w = \delta\phi = \delta(\partial w/\partial x) = 0 \quad \text{at} \quad t = t_1, t_2$$

where δ represents the first variation, t_1 and t_2 are two arbitrary time variables.

Substituting for T and V from Eqs. (11) to (9), noting that the strains can be written in terms of the displacements using Eqs. (2), integrating by parts and using the fundamental lemma of calculus of variations, the following governing equations of motion for laminated composite beam are obtained

$$\begin{aligned} &-I_1\ddot{u} + \left[\left(4/3h^2\right) I_4 - I_2\right]\ddot{\phi} + \left(4/3h^2\right) I_4\partial\ddot{w}/\partial x \\ &+\bar{A}_{11}\partial^2 u/\partial x^2 + \left[\bar{B}_{11} - \left(4/3h^2\right)\bar{E}_{11}\right]\partial^2\phi/\partial x^2 - \left(4/3h^2\right)\bar{E}_{11}\partial^3 w/\partial x^3 = 0 \end{aligned} \tag{13a}$$

$$\begin{aligned} &-I_1\ddot{w} - \left(4/3h^2\right) I_4\partial\ddot{u}/\partial x + \left[\left(16/9h^4\right) I_7 - \left(4/3h^2\right) I_5\right]\partial\ddot{\phi}/\partial x + \left(16/9h^4\right) I_7\partial^2\ddot{w}/\partial x^2 \\ &+ \left(4/3h^2\right)\bar{E}_{11}\partial^3 u/\partial x^3 + \left[\left(4/3h^2\right)\bar{F}_{11} - \left(16/9h^4\right)\bar{H}_{11}\right]\partial^3\phi/\partial x^3 \\ &- \left(16/9h^4\right)\bar{H}_{11}\partial^4 w/\partial x^4 + \left[A_{55} - \left(8/h^2\right) D_{55} + \left(16/h^4\right) F_{55}\right]\left(\partial\phi/\partial x + \partial^2 w/\partial x^2\right) = 0 \end{aligned} \tag{13b}$$

$$\begin{aligned} &-\left[I_2 - \left(4/3h^2\right) I_4\right]\ddot{u} - \left[I_3 + \left(16/9h^4\right) I_7 - \left(8/3h^2\right) I_5\right]\ddot{\phi} + \left[\left(4/3h^2\right) I_5 - \left(16/9h^4\right) I_7\right]\partial\ddot{w}/\partial x \\ &+\left[\bar{B}_{11} - \left(4/3h^2\right)\bar{E}_{11}\right]\partial^2 u/\partial x^2 + \left[\bar{D}_{11} - \left(8/3h^2\right)\bar{F}_{11} + \left(16/9h^4\right)\bar{H}_{11}\right]\partial^2\phi/\partial x^2 \\ &+\left[\left(16/9h^4\right)\bar{H}_{11} - \left(4/3h^2\right)\bar{F}_{11}\right]\partial^3 w/\partial x^3 \\ &+\left[\left(8/h^2\right) D_{55} - A_{55} - \left(16/h^4\right) F_{55}\right](\phi + \partial w/\partial x) = 0 \end{aligned} \tag{13c}$$

where

$$(I_1, I_2, I_3, I_4, I_5, I_7) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\left(1, z, z^2, z^3, z^4, z^6\right)\mathrm{d}z$$

The appropriate end conditions at the beam ends ($x = 0, L$) are

$$\left\{-\bar{A}_{11}\partial u/\partial x + \left[\left(4/3h^2\right)\bar{E}_{11} - \bar{B}_{11}\right]\partial\phi/\partial x + \left(4/3h^2\right)\bar{E}_{11}\partial^2 w/\partial x^2\right\}\delta u = 0 \tag{14a}$$

$$\begin{aligned} &\left\{\left(4/3h^2\right) I_4\ddot{u} + \left[\left(4/3h^2\right) I_5 - \left(16/9h^4\right) I_7\right]\ddot{\phi} - \left(16/9h^4\right) I_7\partial\ddot{w}/\partial x \right.\\ &- \left(4/3h^2\right)\bar{E}_{11}\partial^2 u/\partial x^2 + \left[\left(16/9h^4\right)\bar{H}_{11} - \left(4/3h^2\right)\bar{F}_{11}\right]\partial^2\phi/\partial x^2 \\ &\left.+ \left(16/9h^4\right)\bar{H}_{11}\partial^3 w/\partial x^3 + \left[\left(8/h^2\right) D_{55} - A_{55} - 16/h^4 F_{55}\right](\phi + \partial w/\partial x)\right\}\delta w = 0 \end{aligned} \tag{14b}$$

$$\begin{aligned} &\{\left[\left(4/3h^2\right)\bar{E}_{11} - \bar{B}_{11}\right]\partial u/\partial x + \left[\left(8/3h^2\right)\bar{F}_{11} - \bar{D}_{11} - \left(16/9h^4\right)\bar{H}_{11}\right]\partial\phi/\partial x \\ &+\left[\left(4/3h^2\right)\bar{F}_{11} - \left(16/9h^4\right)\bar{H}_{11}\right]\partial^2 w/\partial x^2\}\delta\phi = 0 \end{aligned} \tag{14c}$$

$$\begin{aligned} &\left\{\left(4/3h^2\right)\bar{E}_{11}\partial u/\partial x + \left[\left(4/3h^2\right)\bar{F}_{11} - \left(16/9h^4\right)\bar{H}_{11}\right]\partial\phi/\partial x \right.\\ &\left.- \left(16/9h^4\right)\bar{H}_{11}\partial^2 w/\partial x^2\right\}\delta w' = 0 \end{aligned} \tag{14d}$$

3 Dynamic stiffness method

It can be shown that Eqs. (13) have solutions that are separable in time and space, and that the time dependence is harmonic. Letting

$$u(x,t) = U(x)\mathrm{e}^{i\omega t}, \quad w(x,t) = W(x)\mathrm{e}^{i\omega t}, \quad \phi(x,t) = \Phi(x)\mathrm{e}^{i\omega t} \tag{15}$$

where ω is the angular frequency, $U(x)$, $W(x)$ and $\Phi(x)$ are the amplitudes of the harmonically varying longitudinal displacement, bending displacement and normal rotation, respectively.

Introducing Eqs. (15) into Eqs. (13), the following differential eigenvalue problem is obtained

$$\begin{aligned}&I_1\omega^2 U-\left[\left(4/3h^2\right)I_4-I_2\right]\omega^2\Phi-\left(4/3h^2\right)I_4\omega^2\partial W/\partial x+\bar{A}_{11}\partial^2U/\partial x^2\\&\quad+\left[\bar{B}_{11}-\left(4/3h^2\right)\bar{E}_{11}\right]\partial^2\Phi/\partial x^2-\left(4/3h^2\right)\bar{E}_{11}\partial^3W/\partial x^3=0\end{aligned}\tag{16a}$$

$$\begin{aligned}&I_1\omega^2W+\left(4/3h^2\right)I_4\omega^2\partial U/\partial x-\left[\left(16/9h^4\right)I_7-\left(4/3h^2\right)I_5\right]\omega^2\partial\Phi/\partial x\\&\quad-\left(16/9h^4\right)I_7\omega^2\partial^2W/\partial x^2+\left(4/3h^2\right)\bar{E}_{11}\partial^3U/\partial x^3\\&\quad+\left[\left(4/3h^2\right)\bar{F}_{11}-\left(16/9h^4\right)\bar{H}_{11}\right]\partial^3\Phi/\partial x^3-\left(16/9h^4\right)\bar{H}_{11}\partial^4W/\partial x^4\\&\quad+\left[A_{55}-\left(8/h^2\right)D_{55}+\left(16/h^4\right)F_{55}\right]\left(\partial\Phi/\partial x+\partial^2W/\partial x^2\right)=0\end{aligned}\tag{16b}$$

$$\begin{aligned}&\left[I_2-\left(4/3h^2\right)I_4\right]\omega^2U+\left[I_3+\left(16/9h^4\right)I_7-\left(8/3h^2\right)I_5\right]\omega^2\Phi\\&\quad-\left[\left(4/3h^2\right)I_5-\left(16/9h^4\right)I_7\right]\omega^2\partial W/\partial x+\left[\bar{B}_{11}-\left(4/3h^2\right)\bar{E}_{11}\right]\partial^2U/\partial x^2\\&\quad+\left[\bar{D}_{11}-\left(8/3h^2\right)\bar{F}_{11}+\left(16/9h^4\right)\bar{H}_{11}\right]\partial^2\Phi/\partial x^2\\&\quad+\left[\left(16/9h^4\right)\bar{H}_{11}-\left(4/3h^2\right)\bar{F}_{11}\right]\partial^3W/\partial x^3\\&\quad+\left[\left(8/h^2\right)D_{55}-A_{55}-\left(16/h^4\right)F_{55}\right]\left(\Phi+\partial W/\partial x\right)=0\end{aligned}\tag{16c}$$

The solutions to Eqs. (16) are given by

$$U(x)=\tilde{A}e^{\kappa x},\quad W(x)=\tilde{B}e^{\kappa x},\quad \Phi(x)=\tilde{C}e^{\kappa x}\tag{17}$$

Substituting Eqs. (17) into Eqs. (16), the equivalent algebraic eigenvalue equations are obtained and the equations have nontrivial solutions when the determinant of the coefficient matrix of $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ vanishes. Setting the determinant equal to zero yields the characteristics equation, which is a eighth-order polynomial equation in κ:

$$\eta_4\kappa^8+\eta_3\kappa^6+\eta_2\kappa^4+\eta_1\kappa^2+\eta_0=0\tag{18}$$

where

$$\eta_4=16\left(-2\bar{B}_{11}\bar{E}_{11}\bar{F}_{11}+\bar{A}_{11}\bar{F}_{11}^2+\bar{B}_{11}^2\bar{H}_{11}+\bar{D}_{11}\left(\bar{E}_{11}^2-\bar{A}_{11}\bar{H}_{11}\right)\right)$$

$$\begin{aligned}\eta_3=&-9\left(\bar{B}_{11}^2-\bar{A}_{11}\bar{D}_{11}\right)\left(16F_{55}-8D_{55}h^2+A_{55}h^4\right)+16\left(\bar{F}_{11}^2I_1+2\bar{B}_{11}\bar{H}_{11}I_2+\bar{E}_{11}^2I_3\right.\\&-\bar{A}_{11}\bar{H}_{11}I_3-2\bar{B}_{11}\bar{E}_{11}I_5-2\bar{F}_{11}\left(\bar{E}_{11}I_2+\bar{B}_{11}I_4-\bar{A}_{11}I_5\right)+\bar{B}_{11}^2I_7\\&\left.-\bar{D}_{11}\left(\bar{H}_{11}I_1-2\bar{E}_{11}I_4+\bar{A}_{11}I_7\right)\right)\omega^2\end{aligned}$$

$$\begin{aligned}\eta_2=&\,\omega^2\left(-16\bar{E}_{11}^2I_1+24\bar{B}_{11}\bar{E}_{11}h^2I_1-24\bar{A}_{11}\bar{F}_{11}h^2I_1-9\bar{B}_{11}^2h^4I_1+9\bar{D}_{11}\left(16F_{55}\right.\right.\\&\left.-8D_{55}h^2+\left(\bar{A}_{11}+A_{55}\right)h^4\right)I_1+16\bar{A}_{11}\bar{H}_{11}I_1-288\bar{B}_{11}F_{55}I_2+144\bar{B}_{11}D_{55}h^2I_2\\&-18A_{55}\bar{B}_{11}h^4I_2+144\bar{A}_{11}F_{55}I_3-72\bar{A}_{11}D_{55}h^2I_3+9\bar{A}_{11}A_{55}h^4I_3\\&+32\bar{E}_{11}\left(I_3I_4-I_2I_5\right)\omega^2+16\bar{D}_{11}\left(I_4^2-I_1I_7\right)\omega^2+16\left(\bar{H}_{11}I_2^2-\bar{H}_{11}I_1I_3\right.\\&\left.\left.-2\bar{F}_{11}I_2I_4+2\bar{F}_{11}I_1I_5-2\bar{B}_{11}I_4I_5+\bar{A}_{11}I_5^2+2\bar{B}_{11}I_2I_7-\bar{A}_{11}I_3I_7\right)\omega^2\right)\end{aligned}$$

$$\begin{aligned}\eta_1=&-9\bar{A}_{11}\left(16F_{55}-8D_{55}h^2+A_{55}h^4\right)I_1\omega^2+\left(-24\bar{F}_{11}h^2I_1^2+9\bar{D}_{11}h^4I_1^2+16\bar{H}_{11}I_1^2\right.\\&+24\bar{E}_{11}h^2I_1I_2-18\bar{B}_{11}h^4I_1I_2-144F_{55}I_2^2+72D_{55}h^2I_2^2-9A_{55}h^4I_2^2+144F_{55}I_1I_3\\&-72D_{55}h^2I_1I_3+9\bar{A}_{11}h^4I_1I_3+9A_{55}h^4I_1I_3-32\bar{E}_{11}I_1I_4+24\bar{B}_{11}h^2I_1I_4\\&\left.-24\bar{A}_{11}h^2I_1I_5+16\bar{A}_{11}I_1I_7\right)\omega^4+16\left(-2I_2I_4I_5+I_1I_5^2+I_2^2I_7+I_3\left(I_4^2-I_1I_7\right)\right)\omega^6\end{aligned}$$

$$\begin{aligned}\eta_0=&\,I_1\omega^4\left(-9\left(16F_{55}-8D_{55}h^2+A_{55}h^4\right)I_1+\left(-9h^4\left(I_2^2-I_1I_3\right)+24h^2\left(I_2I_4-I_1I_5\right)\right.\right.\\&\left.\left.+16\left(-I_4^2+I_1I_7\right)\right)\omega^2\right)\end{aligned}$$

Equation (18) can be rewritten as

$$\chi^4 + a_1\chi^3 + a_2\chi^2 + a_3\chi + a_4 = 0 \tag{19}$$

where

$$\chi = \kappa^2, \quad a_1 = \eta_3/\eta_4, \quad a_2 = \eta_2/\eta_4, \quad a_3 = \eta_1/\eta_4, \quad a_4 = \eta_0/\eta_4$$

The fourth-order Eq. (19) can be factorized as

$$\left(\chi^2 + p_1\chi + q_1\right)\left(\chi^2 + p_2\chi + q_2\right) = 0 \tag{20}$$

where

$$\left\{ \begin{matrix} p_1 \\ p_2 \end{matrix} \right\} = \frac{1}{2}\left[a_1 \pm \sqrt{a_1^2 - 4a_2 + 4\lambda_1}\right] \quad \left\{ \begin{matrix} q_1 \\ q_2 \end{matrix} \right\} = \frac{1}{2}\left[\lambda_1 \pm \frac{a_1\lambda_1 - 2a_3}{\sqrt{a_1^2 - 4a_2 + 4\lambda_1}}\right] \tag{21}$$

and λ_1 is a real root of the following cubic equation

$$\lambda^3 - a_2\lambda^2 + (a_1a_3 - 4a_4)\lambda + \left(4a_2a_4 - a_3^2 - a_1^2a_4\right) = 0 \tag{22}$$

Then the four roots of Eq. (19) can be written as

$$\left\{ \begin{matrix} \chi_1 \\ \chi_2 \end{matrix} \right\} = -\frac{p_1}{2} \pm \sqrt{\frac{p_1^2}{4} - q_1} \quad \left\{ \begin{matrix} \chi_3 \\ \chi_4 \end{matrix} \right\} = -\frac{p_2}{2} \pm \sqrt{\frac{p_2^2}{4} - q_2} \tag{23}$$

The three roots of Eq. (22) can be written as

$$\lambda_1 = a_2/3 + 2\sqrt{-\bar{Q}}\cos\left(\bar{\vartheta}/3\right), \quad \lambda_2 = a_2/3 + 2\sqrt{-\bar{Q}}\cos\left(\left(\bar{\vartheta} + 2\pi\right)/3\right)$$
$$\lambda_3 = a_2/3 + 2\sqrt{-\bar{Q}}\cos\left(\left(\bar{\vartheta} + 4\pi\right)/3\right)$$

where

$$\bar{\vartheta} = \cos^{-1}\left(\bar{R}/\sqrt{-\bar{Q}^3}\right), \quad \bar{Q} = -\left(a_2^2 - 3a_1a_3 + 12a_4\right)/9$$
$$\bar{R} = \left(2a_2^3 - 9a_1a_2a_3 + 27a_3^2 + 27a_1^2a_4 - 72a_2a_4\right)/54$$

Whichever of the three roots is chosen, the same solutions (23) are obtained.

The general solutions to Eqs. (16) are given by

$$U(x) = A_1e^{\kappa_1x} + A_2e^{-\kappa_1x} + A_3e^{\kappa_2x} + A_4e^{-\kappa_2x} + A_5e^{\kappa_3x} + A_6e^{-\kappa_3x} + A_7e^{\kappa_4x} + A_8e^{-\kappa_4x}$$
$$= \sum_{j=1}^{4}(A_{2j-1}e^{\kappa_jx} + A_{2j}e^{-\kappa_jx}) \tag{24a}$$
$$W(x) = B_1e^{\kappa_1x} + B_2e^{-\kappa_1x} + B_3e^{\kappa_2x} + B_4e^{-\kappa_2x} + B_5e^{\kappa_3x} + B_6e^{-\kappa_3x} + B_7e^{\kappa_4x} + B_8e^{-\kappa_4x}$$
$$= \sum_{j=1}^{4}(B_{2j-1}e^{\kappa_jx} + B_{2j}e^{-\kappa_jx}) \tag{24b}$$
$$\Phi(x) = C_1e^{\kappa_1x} + C_2e^{-\kappa_1x} + C_3e^{\kappa_2x} + C_4e^{-\kappa_2x} + C_5e^{\kappa_3x} + C_6e^{-\kappa_3x} + C_7e^{\kappa_4x} + C_8e^{-\kappa_4x}$$
$$= \sum_{j=1}^{4}(C_{2j-1}e^{\kappa_jx} + C_{2j}e^{-\kappa_jx}) \tag{24c}$$

where $\kappa_1 = \sqrt{\chi_1}$, $\kappa_2 = \sqrt{\chi_2}$, $\kappa_3 = \sqrt{\chi_3}$, $\kappa_4 = \sqrt{\chi_4}$. $A_1 \sim A_8$, $B_1 \sim B_8$ and $C_1 \sim C_8$ are three different sets of constants. If any of the χ_j's are zero or are repeated in the solution of Eq. (19), the solutions (24) have to be modified according to the well-known methods of ordinary differential equations with constant coefficients.

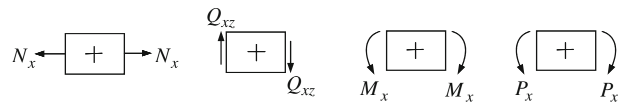

Fig. 2 Sign convention for positive normal force $N_x(x)$, shear force $Q_{xz}(x)$, bending moment $M_x(x)$ and higher-order moment $P_x(x)$

From Eqs. (16), only eight of the 24 constants are independent. The relationship among the constants is given by

$$\begin{aligned} A_{2j-1} &= t_j C_{2j-1}, \quad A_{2j} = t_j C_{2j} \\ B_{2j-1} &= \bar{t}_j C_{2j-1}, \quad B_{2j} = -\bar{t}_j C_{2j} \end{aligned}$$

where

$$\begin{aligned} t_j = &\big(-9\bar{B}_{11}\big(16F_{55} - 8D_{55}h^2 + A_{55}h^4\big)\kappa_j^4 - 16\big(\bar{E}_{11}\bar{F}_{11} - \bar{B}_{11}\bar{H}_{11}\big)\kappa_j^6 + \kappa_j^2\big(-144F_{55}I_2 \\ &+3h^2\big(4\bar{E}_{11}I_1 + 24D_{55}I_2 - 3h^2\big(\bar{B}_{11}I_1 + A_{55}I_2\big)\big) + 16\big(\bar{H}_{11}I_2 - \bar{F}_{11}I_4 - \bar{E}_{11}I_5 \\ &+\bar{B}_{11}I_7\big)\kappa_j^2\big)\omega^2 - \big(9h^4I_1I_2 - 12h^2I_1I_4 + 16\big(I_4I_5 - I_2I_7\big)\kappa_j^2\big)\omega^4\big)/\Delta_j \end{aligned}$$

$$\begin{aligned} \bar{t}_j = &\kappa_j\big(-9\bar{A}_{11}\big(16F_{55} - 8D_{55}h^2 + A_{55}h^4\big)\kappa_j^2 - 4\big(4\bar{E}_{11}^2 - 3\bar{B}_{11}\bar{E}_{11}h^2 + 3\bar{A}_{11}\bar{F}_{11}h^2 \\ &-4\bar{A}_{11}\bar{H}_{11}\big)\kappa_j^4 - \big(9\big(16F_{55} - 8D_{55}h^2 + A_{55}h^4\big)I_1 + 4\big(3\bar{F}_{11}h^2I_1 - 4\bar{H}_{11}I_1 \\ &-3\bar{E}_{11}h^2I_2 + 8\bar{E}_{11}I_4 - 3\bar{B}_{11}h^2I_4 + 3\bar{A}_{11}h^2I_5 - 4\bar{A}_{11}I_7\big)\kappa_j^2\big)\omega^2 \\ &+4\big(-4I_4^2 + 3h^2\big(I_2I_4 - I_1I_5\big) + 4I_1I_7\big)\omega^4\big)/\Delta_j \end{aligned}$$

$$\begin{aligned} \Delta_j = &9\bar{A}_{11}\big(16F_{55} - 8D_{55}h^2 + A_{55}h^4\big)\kappa_j^4 + 16\big(\bar{E}_{11}^2 - \bar{A}_{11}\bar{H}_{11}\big)\kappa_j^6 + \kappa_j^2\big(9\big(16F_{55} \\ &-8D_{55}h^2 + \big(\bar{A}_{11} + A_{55}\big)h^4\big)I_1 - 16\big(\bar{H}_{11}I_1 - 2\bar{E}_{11}I_4 + \bar{A}_{11}I_7\big)\kappa_j^2\big)\omega^2 \qquad (j = 1 \sim 4) \\ &+\big(9h^4I_1^2 + 16\big(I_4^2 - I_1I_7\big)\kappa_j^2\big)\omega^4 \end{aligned}$$

Following the sign convention shown in Fig. 2, the expressions for the normal force $N_x(x)$, shear force $Q_{xz}(x)$, bending moment $M_x(x)$ and higher-order moment $P_x(x)$ can be obtained from Eqs. (14) and (24) as

$$N_x(x) = \sum_{j=1}^{4}\left\{\bar{A}_{11}\kappa_j t_j - \left[\left(4/3h^2\right)\bar{E}_{11} - \bar{B}_{11}\right]\kappa_j - \left(4/3h^2\right)\bar{E}_{11}\kappa_j^2\bar{t}_j\right\}\left(C_{2j-1}e^{\kappa_j x} - C_{2j}e^{-\kappa_j x}\right) \tag{25a}$$

$$\begin{aligned} Q_{xz}(x) = \sum_{j=1}^{4}&\left\{-\left(4/3h^2\right)I_4\omega^2 t_j + \left[-\left(4/3h^2\right)I_5 + \left(16/9h^4\right)I_7\right]\omega^2 + \left(16/9h^4\right)I_7\omega^2\kappa_j\bar{t}_j\right. \\ &-\left(4/3h^2\right)\bar{E}_{11}\kappa_j^2 t_j + \left[\left(16/9h^4\right)\bar{H}_{11} - \left(4/3h^2\right)\bar{F}_{11}\right]\kappa_j^2 + \left(16/9h^4\right)\bar{H}_{11}\kappa_j^3\bar{t}_j \\ &\left.+\left[\left(8/h^2\right)D_{55} - A_{55} - \left(16/h^4\right)F_{55}\right]\left(1 + \kappa_j\bar{t}_j\right)\right\}\left(C_{2j-1}e^{\kappa_j x} + C_{2j}e^{-\kappa_j x}\right) \end{aligned} \tag{25b}$$

$$\begin{aligned} M_x(x) = \sum_{j=1}^{4}&\left\{\left[\left(4/3h^2\right)\bar{E}_{11} - \bar{B}_{11}\right]\kappa_j t_j + \left[\left(8/3h^2\right)\bar{F}_{11} - \bar{D}_{11} - \left(16/9h^4\right)\bar{H}_{11}\right]\kappa_j\right. \\ &\left.+\left[\left(4/3h^2\right)\bar{F}_{11} - \left(16/9h^4\right)\bar{H}_{11}\right]\kappa_j^2\bar{t}_j\right\}\left(C_{2j-1}e^{\kappa_j x} - C_{2j}e^{-\kappa_j x}\right) \end{aligned} \tag{25c}$$

$$\begin{aligned} P_x(x) = \sum_{j=1}^{4}&\left\{\left(4/3h^2\right)\bar{E}_{11}\kappa_j t_j + \left[\left(4/3h^2\right)\bar{F}_{11} - \left(16/9h^4\right)\bar{H}_{11}\right]\kappa_j - \left(16/9h^4\right)\bar{H}_{11}\kappa_j^2\bar{t}_j\right\} \\ &\times\left(C_{2j-1}e^{\kappa_j x} - C_{2j}e^{-\kappa_j x}\right) \end{aligned} \tag{25d}$$

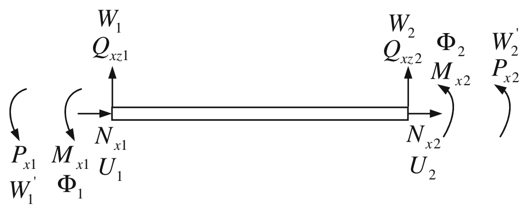

Fig. 3 End conditions for displacements and forces of composite beam

With reference to Fig. 3, the end conditions for displacements and forces of the composite beam are, respectively,

$$\begin{aligned} x=0: &\quad U=U_1, \quad W=W_1, \quad \Phi=\Phi_1, \quad W'=W_1' \\ x=L: &\quad U=U_2, \quad W=W_2, \quad \Phi=\Phi_2, \quad W'=W_2' \end{aligned} \tag{26a}$$

$$\begin{aligned} x=0: &\quad N_x=-N_{x1}, \quad Q_{xz}=Q_{xz1}, \quad M_x=M_{x1}, \quad P_x=P_{x1} \\ x=L: &\quad N_x=N_{x2}, \quad Q_{xz}=-Q_{xz2}, \quad M_x=-M_{x2}, \quad P_x=-P_{x2} \end{aligned} \tag{26b}$$

where the superscript prime denotes the partial derivative with respect to x.

Substituting Eqs. (26a) into Eqs. (24), the nodal degrees of freedom defined in Fig. 3 can be expressed in terms of C_i as

$$\{D\}=[R]\{C\} \tag{27}$$

where $\{D\}$ is the nodal degree-of-freedom vector defined by

$$\{D\}=\left\{U_1 \quad W_1 \quad \Phi_1 \quad W_1' \quad U_2 \quad W_2 \quad \Phi_2 \quad W_2'\right\}^{\mathrm{T}}$$

$$\{C\}=\left\{C_1 \quad C_3 \quad C_5 \quad C_7 \quad C_2 \quad C_4 \quad C_6 \quad C_8\right\}^{\mathrm{T}}$$

$$[R]=\begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_1 & t_2 & t_3 & t_4 \\ \bar{t}_1 & \bar{t}_2 & \bar{t}_3 & \bar{t}_4 & -\bar{t}_1 & -\bar{t}_2 & -\bar{t}_3 & -\bar{t}_4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \kappa_1\bar{t}_1 & \kappa_2\bar{t}_2 & \kappa_3\bar{t}_3 & \kappa_4\bar{t}_4 & \kappa_1\bar{t}_1 & \kappa_2\bar{t}_2 & \kappa_3\bar{t}_3 & \kappa_4\bar{t}_4 \\ t_1\mathrm{e}^{\kappa_1 L} & t_2\mathrm{e}^{\kappa_2 L} & t_3\mathrm{e}^{\kappa_3 L} & t_4\mathrm{e}^{\kappa_4 L} & t_1\mathrm{e}^{-\kappa_1 L} & t_2\mathrm{e}^{-\kappa_2 L} & t_3\mathrm{e}^{-\kappa_3 L} & t_4\mathrm{e}^{-\kappa_4 L} \\ \bar{t}_1\mathrm{e}^{\kappa_1 L} & \bar{t}_2\mathrm{e}^{\kappa_2 L} & \bar{t}_3\mathrm{e}^{\kappa_3 L} & \bar{t}_4\mathrm{e}^{\kappa_4 L} & -\bar{t}_1\mathrm{e}^{-\kappa_1 L} & -\bar{t}_2\mathrm{e}^{-\kappa_2 L} & -\bar{t}_3\mathrm{e}^{-\kappa_3 L} & -\bar{t}_4\mathrm{e}^{-\kappa_4 L} \\ \mathrm{e}^{\kappa_1 L} & \mathrm{e}^{\kappa_2 L} & \mathrm{e}^{\kappa_3 L} & \mathrm{e}^{\kappa_4 L} & \mathrm{e}^{-\kappa_1 L} & \mathrm{e}^{-\kappa_2 L} & \mathrm{e}^{-\kappa_3 L} & \mathrm{e}^{\kappa_4 L} \\ \kappa_1\bar{t}_1\mathrm{e}^{\kappa_1 L} & \kappa_2\bar{t}_2\mathrm{e}^{\kappa_2 L} & \kappa_3\bar{t}_3\mathrm{e}^{\kappa_3 L} & \kappa_4\bar{t}_4\mathrm{e}^{\kappa_4 L} & \kappa_1\bar{t}_1\mathrm{e}^{-\kappa_1 L} & \kappa_2\bar{t}_2\mathrm{e}^{-\kappa_2 L} & \kappa_3\bar{t}_3\mathrm{e}^{-\kappa_3 L} & \kappa_4\bar{t}_4\mathrm{e}^{-\kappa_4 L} \end{bmatrix}$$

Substituting Eqs. (26b) into Eqs. (25), the nodal forces defined in Fig. 3 can be expressed in terms of C_i as

$$\{F\}=[H]\{C\} \tag{28}$$

where $\{F\}$ is the nodal force vector defined by

$$\{F\} = \left\{ N_{x1} \quad Q_{xz1} \quad M_{x1} \quad P_{x1} \quad N_{x2} \quad Q_{xz2} \quad M_{x2} \quad P_{x2} \right\}^{\mathrm{T}}$$

$$[H] = \begin{bmatrix} -\hat{t}_1 & -\hat{t}_2 & -\hat{t}_3 & -\hat{t}_4 & \hat{t}_1 & \hat{t}_2 & \hat{t}_3 & \hat{t}_4 \\ \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 & \tilde{t}_4 & \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 & \tilde{t}_4 \\ \bar{\bar{t}}_1 & \bar{\bar{t}}_2 & \bar{\bar{t}}_3 & \bar{\bar{t}}_4 & -\bar{\bar{t}}_1 & -\bar{\bar{t}}_2 & -\bar{\bar{t}}_3 & -\bar{\bar{t}}_4 \\ \hat{\hat{t}}_1 & \hat{\hat{t}}_2 & \hat{\hat{t}}_3 & \hat{\hat{t}}_4 & -\hat{\hat{t}}_1 & -\hat{\hat{t}}_2 & -\hat{\hat{t}}_3 & -\hat{\hat{t}}_4 \\ \hat{t}_1 e^{\kappa_1 L} & \hat{t}_2 e^{\kappa_2 L} & \hat{t}_3 e^{\kappa_3 L} & \hat{t}_4 e^{\kappa_4 L} & -\hat{t}_1 e^{-\kappa_1 L} & -\hat{t}_2 e^{-\kappa_2 L} & -\hat{t}_3 e^{-\kappa_3 L} & -\hat{t}_4 e^{-\kappa_4 L} \\ -\tilde{t}_1 e^{\kappa_1 L} & -\tilde{t}_2 e^{\kappa_2 L} & -\tilde{t}_3 e^{\kappa_3 L} & -\tilde{t}_4 e^{\kappa_4 L} & -\tilde{t}_1 e^{-\kappa_1 L} & -\tilde{t}_2 e^{-\kappa_2 L} & -\tilde{t}_3 e^{-\kappa_3 L} & -\tilde{t}_4 e^{-\kappa_4 L} \\ -\bar{\bar{t}}_1 e^{\kappa_1 L} & -\bar{\bar{t}}_2 e^{\kappa_2 L} & -\bar{\bar{t}}_3 e^{\kappa_3 L} & -\bar{\bar{t}}_4 e^{\kappa_4 L} & \bar{\bar{t}}_1 e^{-\kappa_1 L} & \bar{\bar{t}}_2 e^{-\kappa_2 L} & \bar{\bar{t}}_3 e^{-\kappa_3 L} & \bar{\bar{t}}_4 e^{-\kappa_4 L} \\ -\hat{\hat{t}}_1 e^{\kappa_1 L} & -\hat{\hat{t}}_2 e^{\kappa_2 L} & -\hat{\hat{t}}_3 e^{\kappa_3 L} & -\hat{\hat{t}}_4 e^{\kappa_4 L} & \hat{\hat{t}}_1 e^{-\kappa_1 L} & \hat{\hat{t}}_2 e^{-\kappa_2 L} & \hat{\hat{t}}_3 e^{-\kappa_3 L} & \hat{\hat{t}}_4 e^{-\kappa_4 L} \end{bmatrix}$$

in which

$$\hat{t}_j = \left\{ \bar{A}_{11}\kappa_j t_j - \left[\left(4/3h^2\right)\bar{E}_{11} - \bar{B}_{11}\right]\kappa_j - \left(4/3h^2\right)\bar{E}_{11}\kappa_j^2\bar{t}_j \right\}$$

$$\begin{aligned} \tilde{t}_j = \{ & -\left(4/3h^2\right) I_4\omega^2 t_j + \left[-\left(4/3h^2\right) I_5 + \left(16/9h^4\right) I_7\right]\omega^2 + \left(16/9h^4\right) I_7\omega^2\kappa_j\bar{t}_j \\ & - \left(4/3h^2\right)\bar{E}_{11}\kappa_j^2 t_j + \left[\left(16/9h^4\right)\bar{H}_{11} - \left(4/3h^2\right)\bar{F}_{11}\right]\kappa_j^2 + \left(16/9h^4\right)\bar{H}_{11}\kappa_j^3\bar{t}_j \\ & + \left[\left(8/h^2\right) D_{55} - A_{55} - \left(16/h^4\right) F_{55}\right]\left(1 + \kappa_j\bar{t}_j\right) \} \end{aligned}$$

$$\begin{aligned} \bar{\bar{t}}_j = \{ & \left[\left(4/3h^2\right)\bar{E}_{11} - \bar{B}_{11}\right]\kappa_j t_j + \left[\left(8/3h^2\right)\bar{F}_{11} - \bar{D}_{11} - \left(16/9h^4\right)\bar{H}_{11}\right]\kappa_j \\ & + \left[\left(4/3h^2\right)\bar{F}_{11} - \left(16/9h^4\right)\bar{H}_{11}\right]\kappa_j^2\bar{t}_j \} \end{aligned}$$

$$\hat{\hat{t}}_j = \left\{ \left(4/3h^2\right)\bar{E}_{11}\kappa_j t_j + \left[\left(4/3h^2\right)\bar{F}_{11} - \left(16/9h^4\right)\bar{H}_{11}\right]\kappa_j - \left(16/9h^4\right)\bar{H}_{11}\kappa_j^2\bar{t}_j \right\} \quad (j = 1 \sim 4)$$

Eliminating the coefficients C_i from Eqs. (27) and (28) gives the following relationship between the nodal forces and nodal displacements

$$\{F\} = [H][R]^{-1}\{D\} = [K]\{D\} \tag{29}$$

where $[K]$ is the frequency-dependent dynamic stiffness matrix. Lengthy expressions of the elements of dynamic stiffness matrix, which may be obtained using Mathematica [23] via symbolic computation, are not given here due to space limitations. Unlike the traditional finite element method in which the mass and stiffness matrices of a structural element are obtained separately, the dynamic stiffness method involves only one frequency-dependent matrix called the dynamic stiffness matrix which contains both the mass and stiffness properties of the structural element.

If the dynamic stiffness matrix for each element of the composite beam is known, the global dynamic stiffness matrix for the entire beam structure can be assembled in a completely analogous way to that used for the traditional finite element method. Once the global dynamic stiffness matrix is obtained, a reliable and accurate method, i.e. Wittrick–Williams algorithm [24] is adopted in the present study to determine the natural frequencies of the composite beam. Since the Wittrick–Williams algorithm has featured in numerous papers the procedure is not explained here for brevity. The algorithm uses the Sturm sequence property of the dynamic stiffness matrix and ensures that there is no possibility of missing a nature frequency of the structure. The algorithm is easy to use and interested researchers are encouraged to read the original work of Wittrick and Williams [24] on how to apply the Wittrick–Williams algorithm to the dynamic stiffness matrix. The mode shapes corresponding to the natural frequencies can be found in the usual way by making an arbitrary assumption about one unknown variable of the composite beam and then calculating the remaining variables in terms of the arbitrarily chosen one.

Table 1 Natural frequencies (in Hz) of glass–polyester beam

Mode no.	Clamped–Clamped	Clamped–Simply supported	Free–Free			Clamped–Free	Simply–Supported
			Present	Ref. [25]	Ref. [26]		
1	763.0	527.0	764.9	765.4	767.9	120.5	338.0
2	2,087.6	1,698.1	2,096.4	2,097.3	2,091.9	752.2	1,346.4
3	4,052.2	3,513.1	4,075.6	4,075.5	4,155.9	2,092.8	3,008.2
4	6,616.7	5,942.0	6,664.3	6,662.1	6,762.7	4,063.8	5,296.7

Table 2 Natural frequencies (in Hz) of glass–polyester beam without Poisson effect included

Mode no.	Clamped–Clamped	Clamped–Simply supported	Free–Free		Clamped–Free	Simply–Supported
			Present	Ref. [25]		
1	895.9	619.4	899.6	823.7	141.7	397.5
2	2,445.6	1,992.3	2,462.1	2,255.0	883.9	1,581.6
3	4,733.5	4,111.7	4,776.7	4,375.4	2,454.9	3,526.9
4	7,702.9	6,933.4	7,789.9	7,140.8	4,754.9	6,193.9

4 Numerical results

In order to assess the dynamic stiffness behavior of laminated composite beams for free vibration studies, several examples are considered. The dynamic stiffness matrix formulated previously is directly applied to the illustrative examples to obtain the numerical results.

For each example, the natural frequencies are determined for Clamped–Clamped, Clamped–Simply supported, Free–Free, Clamped–Free and Simply–Supported end conditions.
For the clamped end: $U = W = \Phi = W' = 0$
For the simply supported end: $U = W = M_x = P_x = 0$
For the free end: $N_x = Q_{xz} = M_x = P_x = 0$

A glass–polyester laminated beam of rectangular cross-section having [45°/45°/45°/45°] stacking sequence is considered first. This problem is also considered by Teboub and Hajela [25]. The properties of the composite beam are given as follows

$$E_1 = 37.41 \times 10^9\,\text{Pa}, \quad E_2 = 13.67 \times 10^9\,\text{Pa}, \quad G_{12} = 5.478 \times 10^9\,\text{Pa}, \quad G_{13} = 6.03 \times 10^9\,\text{Pa}$$
$$G_{23} = 6.666 \times 10^9\,\text{Pa}, \quad \nu_{12} = 0.3 \quad \rho = 1,968.9\,\text{kg/m}^3$$
$$L = 0.11179\,\text{m}, \quad b = 12.7 \times 10^{-3}\,\text{m}, \quad h = 3.38 \times 10^{-3}\,\text{m}$$

The first four natural frequencies of the beam are calculated for various end conditions using the dynamic stiffness approach. Table 1 shows the numerical results with the Poisson effect included. In order to gain the insight into the Poisson effect upon the natural frequencies, the solutions with the Poisson effect excluded are given in Table 2.

It is seen from Table 1 that the present results are excellent agreement with the solutions of first-order shear deformation theory by Teboub and Hajela [25] and the experimental results of Ritchie et al. [26]. A comparison Table 1 with Table 2, it can be found that the absolute relative errors for the first four frequencies with the Poisson effect neglected are about 16% for all end conditions. For each support condition, the relative error slightly decreases with the increase of mode number. The importance of considering the Poisson effect is evident from this example. It is interesting to find that the present results of third-order shear deformation model are significantly different from the solutions of first-order shear deformation model when the Poisson effect is ignored.

To identify the effect of material anisotropy on the natural frequencies of laminated beams, free vibration of a graphite–epoxy composite beam with lay-up [30°/50°/30°/50°] and material properties from [4] is investigated next. For convenience, the geometric and material properties of the beam are given as follows

$$E_2 = 9.65 \times 10^9\,\text{Pa}, \quad G_{12} = G_{13} = 4.14 \times 10^9\,\text{Pa}, \quad G_{23} = 3.45 \times 10^9\,\text{Pa}, \quad \nu_{12} = 0.3$$
$$\rho = 1,389.23\,\text{kg/m}^3, \quad L = 0.381\,\text{m}, \quad b = 25.4 \times 10^{-3}\,\text{m}, \quad h = 25.4 \times 10^{-3}\,\text{m}$$

Table 3 shows the natural frequencies of the first five modes of vibration for above laminated beam with $E_1 = 144.80 \times 10^9\,\text{Pa}$. The whole beam is idealized with one element. In order to validate the accuracy of

Table 3 Natural frequencies (in Hz) of [30°/50°/30°/50°] composite beam with $E_1 = 144.80 \times 10^9$ Pa

Mode no.	Clamped–Clamped		Clamped–Simply supported	Free–Free	Clamped–Free	Simply–Supported
	Present	Ref. [4]				
1	638.5	640.5	450.5	659.4	105.3	294.9
2	1,657.1	1,666.8	1,389.9	1,738.8	637.6	1,132.6
3	3,033.1	3,059.5	2,723.9	3,213.7	1,697.9	2,414.6
4	4,658.7	3,397.8	4,339.5	4,798.4	2,399.1	4,012.8
5	4,798.2	4,712.5	4,797.8	4,961.2	3,120.6	4,797.7

Table 4 Natural frequencies (in Hz) of [30°/50°/30°/50°] composite beam with $E_1 = 232.3 \times 10^9$ Pa

Mode no.	Clamped–Clamped	Clamped–Simply supported	Free–Free	Clamped–Free	Simply–Supported
1	687.7	487.0	714.9	114.3	319.7
2	1,770.1	1,492.2	1,874.2	688.8	1,221.8
3	3,214.9	2,901.9	3,439.4	1,821.8	2,586.0
4	4,904.0	4,589.6	5,167.9	2,583.9	4,265.5
5	5,167.5	5,166.6	5,270.1	3,323.2	5,166.3

the procedure proposed, a comparison with traditional finite element results is performed for the beam in a Clamped–Clamped configuration.

Table 3 shows a very good agreement of the present results with the converged results given by the finite element model based on third-order shear deformation theory [4]. It is noted that the first three and fifth natural frequencies are in excellent agreement with the solutions presented in [4]. However, the fourth natural frequency deviates significantly from the solution in [4]. It is thought that the fourth frequency given by Chandrashekhara and Bangera [4] is incorrect because this frequency for Clamped–Clamped end condition is smaller than the one for Simply–Supported end condition. It is worth emphasizing that the present results are exact, whereas those given by Chandrashekhara and Bangera [4] depend on the number of element used.

The first five natural frequencies of the composite beam with $E_1 = 232.3 \times 10^9$ Pa are displayed in Table 4. The effect of material anisotropy on the natural frequencies is illustrated in Tables 3 and 4 for different end conditions. It should be mentioned that only the value of E_1 is varied, the other material properties keep invariant. The effect of increasing extensional modulus E_1 is to increase the natural frequencies of the beam and it affects considerably on the higher frequencies.

For further demonstration of the proficiency of dynamic stiffness approach, more numerical calculations for generally laminated beams are preformed. Results are compared to those obtained by other numerical methods in the published references. Unless mentioned otherwise, the mechanical properties of the composite beam with lay-up sequence [30°/−60°/30°/−60°] adopted from [25] are used for the analysis hereafter.

$$E_1 = 221 \times 10^9\,\text{Pa}, \quad E_2 = 6.9 \times 10^9\,\text{Pa}, \quad G_{12} = 4.8 \times 10^9\,\text{Pa}, \quad G_{13} = 4.14 \times 10^9\,\text{Pa}$$
$$G_{23} = 3.45 \times 10^9\,\text{Pa}, \quad \nu_{12} = 0.3, \quad \rho = 1550.1\,\text{kg/m}^3$$
$$b = 25.4 \times 10^{-3}\,\text{m}, \quad h = 25.4 \times 10^{-3}\,\text{m}$$

In order to illustrate the effect of beam length on the natural frequencies and mode shapes of the composite beam, two different numerical values of beam length L, i.e. $L = 0.381$ m and $L = 0.572$ m are investigated.

One element is taken to model the beam. Tables 5 and 6 show the comparison of the first six natural frequencies of the composite beam with two different values of beam length. It can be seen that the longer the beam length, the lower the natural frequencies are obtained.

The frequencies obtained by first-order shear deformation theory are also given in Table 5 in a comparative manner. A good agreement between present results and published values is observed from Table 5. The differences between the frequencies calculated by the present third-order theory and those by the first-order theory [25] are very small (less than 0.1%) for the fundamental frequency, with increasing differences (less than 2%) for higher modes.

For the case of cantilever beam, which is the beam configuration for most aerospace applications and many robots, the first six mode shapes for two different beam lengths are shown in Figs. 4 and 5. In each mode the amplitudes along the beam length are normalized with respect to the maximum amplitude for that mode. Clear differences are seen between the first and fourth mode shapes obtained for two different lengths.

Table 5 Natural frequencies (in Hz) of composite beam with $L = 0.381$ m

Mode no.	Clamped–Clamped		Clamped–Simply supported	Free–Free	Clamped–Free	Simply–Supported
	Present	Ref. [25]				
1	635.2	635.8	449.0	658.9	105.3	293.9
2	1,639.2	1,633.0	1,379.5	1,730.7	635.5	1,128.0
3	2,984.5	2,957.5	2,689.7	3,183.6	1,684.7	2,392.3
4	4,562.9	4,496.4	4,265.3	4,860.3	2,430.1	3,959.3
5	4,860.2	N/A	4,859.8	4,890.0	3,080.6	4,859.7
6	6,302.0	N/A	6,020.3	6,757.3	4,721.0	5,728.8

Note N/A denotes the result not available

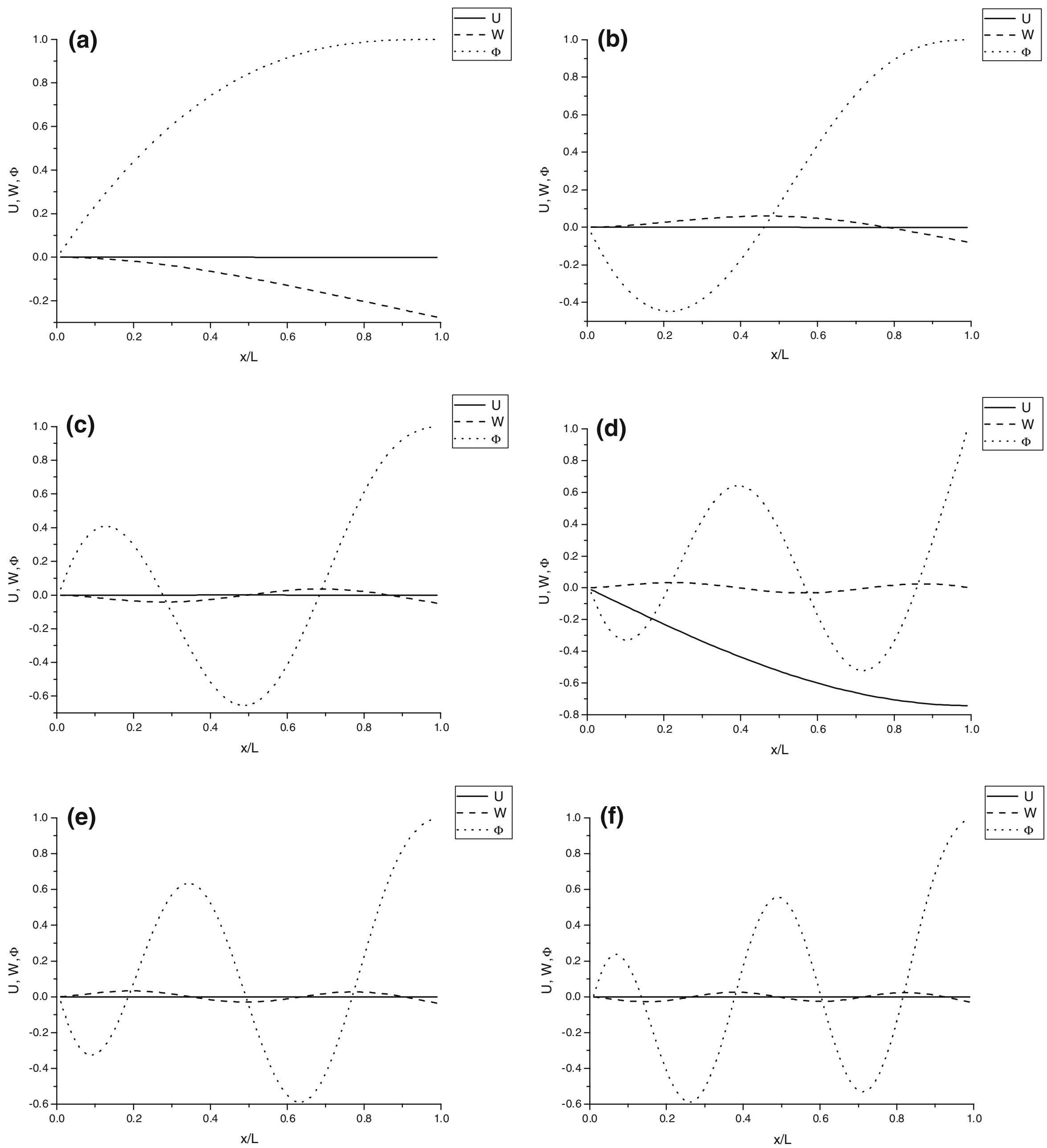

Fig. 4 First six normal mode shapes of Clamped–Free composite beam with $L = 0.381$ m **a** mode 1; **b** mode 2; **c** mode 3; **d** mode 4; **e** mode 5; **f** mode 6

Table 6 Natural frequencies (in Hz) of composite beam with $L = 0.572\,\mathrm{m}$

Mode no.	Clamped–Clamped	Clamped–Simply supported	Free–Free	Clamped–Free	Simply–Supported
1	291.1	203.1	296.2	46.9	131.4
2	776.7	641.5	798.1	288.9	515.3
3	1,463.1	1,292.3	1,514.9	787.6	1,126.3
4	2,310.6	2,118.9	2,405.9	1,488.4	1,927.7
5	3,237.3	3,084.6	3,237.2	1,618.7	2,882.4
6	3,285.7	3,237.5	3,434.0	2,357.2	3,236.1

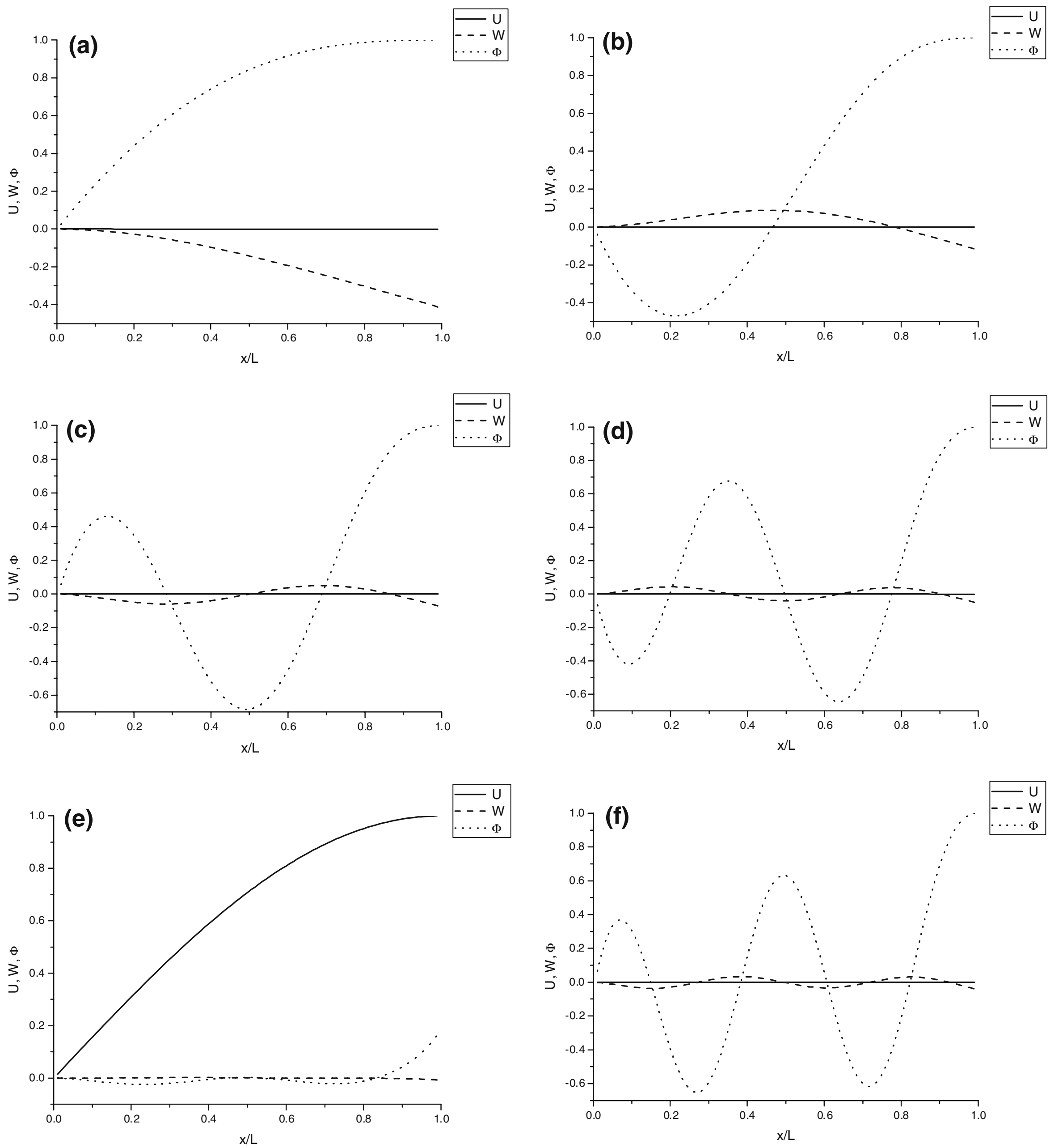

Fig. 5 First six normal mode shapes of Clamped–Free composite beam with $L = 0.572\,\mathrm{m}$ **a** mode 1; **b** mode 2; **c** mode 3; **d** mode 4; **e** mode 5; **f** mode 6

5 Concluding remarks

In this paper, a dynamic stiffness method for determining the natural frequencies and mode shapes of laminated composite beams based on third-order shear deformation theory is introduced. The exact dynamic stiffness matrix of a uniform laminated beam is derived using the analytical solutions of the governing differential equations of the beam in free vibration. The applicability of the introduced approach is demonstrated by a series of numerical results, which show good agreement with the published results. A parametric study of the influences of Poisson effect, material anisotropy, slenderness and end condition on the natural frequencies of the composite beam is also performed. It is expected that the method can be used to compute the natural frequencies and mode shapes of simple structures constructed from laminated beams.

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