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Self-formation and Evolution of Singletons

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Abstract

New solutions for reaction–diffusion equations, called singletons and polytons, are found to be of fundamental significance. They exhibit properties of self-formation. In the presence of nonlinear saturation effects their evolutions in time have pronounced maxima. Information about details of an initial spatial distribution is found to be carried along in time with the dynamic process of forming a growing singleton and to be restored at a later stage.

1. Introduction

In many fields of science, physics, and chemistry, as well as biology, phenomena governed by reaction–diffusion equations are attracting considerable attention [1–4]. The area of research covering these basic equations, which are generally nonlinear partial differential equations, has therefore become a field of growing interest [1–17]. Progress in this field benefits from the developments of new tools of analytic theory, varying from group theoretical methods [7] and scaling techniques [8] to specific methods of finding particular solutions [9–17]. The use of modern computational techniques, offering new possibilities of sophisticated simulation studies, has also greatly extended the scope of the field. It is characteristic for the solutions of the reaction–diffusion equations that the solutions depend on the simultaneous presence of reactions as well as diffusion, both of which are in general nonlinear. The presence of nonlinear reaction terms of a creative type causes the amplitude to grow drastically, whereas the contrary is true for a steep gradient diffusive term. Interesting situations may accordingly develop where the various processes compete with each other and partially balance each other.

Explicit solutions of nonlinear growth, so-called explosive solutions, exist for reaction–diffusion equations and describe simultaneous reaction and nonlinear diffusion processes [13–16]. These solutions are characterized by the fact that nonlinear growth occurs in time while the shape of certain particular "bell-shaped" density profiles are preserved in space [13–17]. The solutions are of the form of single localized bumps, "singletons," or periodic structures, "polytons," depending on the spatial ini-

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tial conditions. They can be described in one, two, and three dimensions. Saturation effects may lead to the formation of a density maximum in time for a singleton, accompanied by a nonlinear decay which, symmetrically in time, mirrors the explosive growth of the initial phase of evolution. Repetitive "explosions" in time of singletons (or polytons) might also occur as a result of nonlinear saturation effects. The new solutions are of fundamental significance in that states which are "nearby" are attracted to the singletons. The singletons accordingly exhibit the property of self-formation. They are dynamically stable against small scale ($\lesssim 1/4$ singleton width) perturbations. A variety of spatial distributions tend to approach a singleton form when time tends toward the time of explosion. The amplitude value at this time becomes finite when nonlinear saturation effects are included. Details of the initial spatial density distribution are reconstructed as the amplitude returns from the peak value to levels of the initial state.

2. Basic Equations

Consider the following reaction–diffusion equation:

$$\frac{\partial n}{\partial t} = x^{-\gamma}\frac{\partial}{\partial x}\left[x^{\gamma}D\frac{\partial n}{\partial x}\right] - bn + cn^{p}, \tag{1}$$

where $\gamma = 0, 1, 2$ for the dimensions $d = 1, 2, 3$, $(d = \gamma + 1)$; n describes a population density, which is assumed to be radially symmetric for cylindrical and sperical cases ($\gamma = 1, 2$); p is a positive quantity, $p > 1$; and b and c are constant coefficients with $b > 0$ for linear dissipation and $c > 0$ for creative reactions (explosive cases). D is a diffusion coefficient, which we assume to be of the form $D = an^{\delta}$, where a and δ are constants.

The cases where $p = \delta + 1$ are of particular interest and comprise the significant case where $p = 2$, $\delta = 1$. (For fully ionized gases one often has $D = an$). For $p = \delta + 1$ Eq. (1) is separable in space and time. This fact is essential for the analysis and has important consequences for the results. For $p = \delta + 1$ the term in Eq. (1) which is linear in n, $(-bn)$, and which describes linear dissipation for $b > 0$ (linear growth for $b < 0$) can be easily eliminated by means of the transformation

$$N = n\exp(bt), \tag{2}$$

$$\tau = \frac{1}{(p-1)b}\{1 - \exp[-(p-1)bt]\}, \tag{3}$$

which introduces the variables N and τ, in terms of which Eq. (1) takes the form

$$\frac{\partial N}{\partial \tau} = x^{-\gamma}\frac{\partial}{\partial x}\left[x^{\gamma}D_N\frac{\partial N}{\partial x}\right] + cN^{p}, \tag{4}$$

with $D_N = aN$, which is formally identical to Eq. (1) with n substituted by N, t by τ, and b absent. Therefore it suffices for $p = \delta + 1$ to study Eq. (1) with $b = 0$. Returning to Eq. (1) with $b = 0$, it is convenient to introduce a renormalization of variables of space and time; accordingly, $(c/a)^{1/2}x \rightarrow x$, $ct \rightarrow t$.

The remaining equation becomes

$$\frac{\partial n}{\partial t} = x^{-\gamma}\frac{\partial}{\partial x}\left[x^{\gamma}n^{\delta}\frac{\partial n}{\partial x}\right] + n^{p}. \tag{5}$$

3. Solutions for $p = \delta + 1$

The solution for Eq. (5), assuming $p = \delta + 1$, can be written in the form [15, 16]

$$n(x,t) = (t_0 - t)^{-(p-1)^{-1}}\phi(x)\,, \tag{6}$$

$$\phi(x) = \phi_0 \cos^{\lambda}[x/L - \varepsilon(x/L)^3]\,, \tag{7}$$

with

$$\phi_0 = \left[\frac{2p + \gamma(p - 1)}{p^2 - 1}\right]^{(p-1)^{-1}}, \tag{7a}$$

$$L^2 = \frac{2}{(p - 1)^2}[2p + \gamma(p - 1)]\,, \tag{7b}$$

$$\lambda = \frac{2}{p - 1}, \tag{7c}$$

$$\varepsilon = \frac{\gamma}{12}\frac{1}{3 + \gamma} \tag{7d}$$

The solution expressed by relations (6) and (7) is an exact solution of Eq. (5) for $\gamma = 0$, whereas for $\gamma = 1, 2$ it satisfies Eq. (5) to order x^2 for small x. For the one-dimensional case, $\gamma = 0$, and assuming $p = 2$ ($\delta = 1$), one obtains [13–15]

$$\phi(x) = \frac{4}{3}\cos^2[x/(2\sqrt{2})]\,. \tag{8}$$

If initially at $t = 0$ the spatial distribution of density has the form (8), the density will develop in time according to relation (6). It will simply preserve its initial spatial form. This form could, by definition, be localized in space so that $\phi(x)$ is determined by relation (8) for $|x| < \pi\sqrt{2}$, but $\phi(x) = 0$ otherwise. The solution will then stay localized in space for all time. Or, the initial condition could be periodic in space, i.e., $\phi(x)$ given by relation (8) for all x initially and then consequently for all time [13, 14]. It has been shown recently by extensive analysis [15–17] and, furthermore, confirmed by computer experiments [17], which also extend the results of the analysis, that states which are initially different from the form (8), and, in fact, even quite a lot different, develop in time in such a way that the solution approaches the form (8) as $t \to t_0$, the time of explosion, when the density n tends toward infinite values, provided higher order saturation effects are neglected. As a result of the remarkable fact that "nearby" states are attracted to the explosive localized solution (ELS), which therefore plays a unique role for the type of reaction–diffusion equation (5), it has been named a "singleton" for the case of a localized single bump and a "polyton" for the spatially periodic configuration [14, 15].

4. Influence of Nonlinear Saturation Effects

Nature offers various possibilities by which explosively growing amplitudes may become limited. In order to elucidate the effects of nonlinear saturation, Eq. (5) has to be modified. Considering the case $p = 2$, $\delta = 1$, and making use of a saturation model previously considered for three wave interaction [10], the modified equation may be written

$$\frac{\partial n}{\partial t} = \pm\Lambda_s\left\{\frac{1}{2}\frac{\partial^2 n^2}{\partial x^2} + n^2\right\}, \tag{9}$$

where the saturation factor

$$\Lambda_s = \left\{1 - n^2\left(a_1 - \frac{a_2}{n^4}\right)^2\right\}^{1/2} \tag{10}$$

contains two constant "control" parameters a_1 and a_2.

The solution of Eq. (9) (considering first $a_2 = 0$), can be written

$$n = [(t_1 - t)^2 + a_1^2]^{-1/2}\phi(x), \tag{11}$$

with

$$t_1 = [n(x)]_{t=0}^{-1}\{\phi^2(x) - a_1^2[n^2(x)]_{t=0}\}^{1/2}, \tag{12}$$

and where

$$n_{\max} = [n(x)]_{t=t_1} = a_1^{-1}\phi(x). \tag{13}$$

The results (11)–(13) mean that the solution continues to develop in time after a maximum (13), i.e., for $t > t_1$. The density n has the same value for $t = t_1 \pm |\Delta t|$, where $|\Delta t|$ denotes the deviation in time from the time for maximum density $n_{\max}$.

Considering next $a_2 \neq 0$ as well as $a_1 \neq 0$, the peaks in the solutions of Eq. (9) will be repetitive in time [10].

5. Evolution of Narrow Perturbations in Density of the Singleton

The evolution in time of narrow perturbations of the singleton, which are assumed to be present in the initial state, is now considered, taking into account effects of nonlinear saturation.

The initial form of the perturbation is taken as

$$\Delta n(x, 0) = \varepsilon n_0 \cos[(x - x_1)/L_1], \tag{14}$$

where $\varepsilon \ll 1$, $n_0 = n(0,0)$, $L_1 \ll L = 2\sqrt{2}\,(p = 2, \delta = 1)$, and $|x - x_1|L_1^{-1} \ll \pi/2$. Introducing expression (14) into Eq. (9), neglecting $(\Delta n)^2$ terms and considering L_1 as a constant, assuming $x \approx x_1$, one obtains for $a_2 = 0$ in (10)

$$\frac{\Delta n(x_1, t)}{\Delta n(x_1, 0)} = \{[(t_1 - t)^2 + a_1^2]/[t_1^2 + a_1^2]\}^{q/2}, \tag{15}$$

where

$$q \cong \left\{ L_1^{-2} - 2 - \frac{d^2\phi}{dx^2} \Big/ \phi \right\}_{x=x_1} \phi(x_1) . \tag{16}$$

From the expression (15) and (16) it follows that the perturbation grows in time if q is negative, whereas, if q is positive it becomes diminished by a factor of about $(a_1/t_1)^q$ as $t \to t_1$. From the expression (16) it follows [15] that the singleton is dynamically "stable" for narrow perturbations if $L_1/L \leq 1/4$, i.e., if the width of the perturbation is less than a quarter of the width of the singleton. Otherwise the perturbation tends to grow with the bulk of the profile. These results are valid on the whole also for two and three dimensions.

6. Attraction to the Form of a Singleton for States of Various Initial Widths and Amplitudes

By means of a new technique recently developed [15], it has been possible to show that states, which initially differ in form from that of a singleton, are attracted to the singleton form and become confluent with it asymptotically for large n, i.e., when t tends toward the time of explosion.

The technique consists in studying two coupled nonlinear differential equations in time, for the amplitude and width of the profiles [15, 16]. This turns out to be a useful and efficient technique that enables one to study also various cases where $p \neq \delta + 1$, which exhibit a wide variety of different features of solutions, including collapse and anticollapse solutions as well as some types of reversed-collapse solutions [16, 17].

The introduction, presented in Section 4, of a nonlinear saturation factor for the case $p = \delta + 1$ connects the solutions in time before and after the "explosive" maximum. Such transitions occur even without considering nonlinear saturation effects for cases where $p \neq \delta + 1$.

The interplay between theoretical and computational analysis has proved extremely useful in studying these questions, as can be seen from extensive reports published elsewhere [17].

7. Concluding Remarks

Spatially structured profiles which initially deviate from the singleton form develop in such a way that finer details (widths $\leq 1/4$ singleton width) decay according to expression (15) with $q > 0$, whereas broader deviations in the spectrum of deviations grow in time as seen from relations (15) and (16) with $q < 0$. In the process of approaching the maximum in time, the evolution of the structure becomes confluent with that of a pure singleton.

As a result of the finiteness of the singleton maximum amplitude value, caused by the nonlinear saturation, reminiscences are preserved from the finer details of the initial spatial distribution. In accordance with expression (15), these reminiscences, which are diminished to extremely small contributions near the peak value ($t = t_1$), grow again in time for $t > t_1$, i.e., during the all-over singleton decay period following the maximum. Due to the symmetry in time around $t = t_1$ of expression (15), the details in the spatial spectrum would be expected to become fully restored as the all-over singleton amplitude reaches initial levels. The procedure might then repeat itself in time.

The self-formation of a singleton in a structure of background density distribution therefore occurs in such a way that information about the initial state is carried along with the process of evolution in time. It will escape being lost at the critical time t_1 due to the finiteness of the maximum amplitude. It thus remains available for automatic reconstruction in space of the initial distribution at a proper time ($2t_1$). A full restoration requires that no influences from external sources or any additional filtering mechanisms are active in the process.

The recovery of the initial state at a later time is particularly interesting in view of the fact that the reaction–diffusion equation generally describes irreversible processes. This apparent paradox is a consequence of the fact that the singleton solution corresponds to a particular balance of two simultaneous processes: reaction and diffusion.

The above results should be considered as indicative of fundamental inherent properties of a dynamic system described by the reaction–diffusion equation for creative reactions. The results should be a challenge for continued studies of related questions using more general concepts of theoretical analysis [18–21] and further simulation studies.

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