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A Note on Finite Strain Theory of Elasto-Plasticity

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Summary

An alternative derivation of the elastic-plastic strain decomposition recently introduced by Lee is discussed. References to papers in which this decomposition has been effectively utilized in working out a general computer program are given.

In three recently published articles [1], [2], [3] an incremental version of finite strain theory of elasto-plasticity has been presented and thoroughly discussed. The theory is based on finite deformation nonlinear kinematics according to which the total strain rate is not equal to the sum of the elastic and plastic strain rates. In these interesting articles one can find a repeatedly expressed opinion that all analytical and computer codes for elastic-plastic analysis (including those specifically devised for finite deformation problems) are based upon the assumption of the summability of the elastic and plastic strain rates and, as such, are not general enough for use in finite strain elastic-plastic problems.

In the present note we aim at showing that a correct finite strain formulation which effectively uses the stress-free configuration concept has been devised in [4], [5], [6] and later taken to form a basis for the finite element computer program LARSTRAN. This program has been developed at the Institute of Static and Dynamic (ISD) of University of Stuttgart and has been in widespread use since 1977 at both ISD and the author's home institution in Warsaw.

Even if to some analysts details of the approach as presented in [5]—[8] might seem obscured by a special, "natural" finite element formalism (incidently, a very effective numerical technique), the theory includes essentially all the characteristic features of the recent Prof. Lee's approach and ultimately uses exactly the same equations. In fact, this is by no means strange as both the present Prof. Lee's and our approaches were strongly motivated by the previous fundamental publication of the former author [9].

The purpose of this note is twofold:

- to make the continuum mechanics community aware of some apparently overlooked theoretical developments which led to working out a very efficient computer program,
- to show an alternate derivation of Prof. Lee's recent theory.

In the following we use Prof. Lee's tensorial notation with some minor modifications to facilitate referencing to the formulations devised in [4], [5], [6]. Only a very short presentation is given but this is thought to be entirely sufficient to illustrate the full equivalence of both approaches in question.

Accepting the multiplicative gradient devomposition

$$\boldsymbol{F} = \boldsymbol{F}^e \boldsymbol{F}^p \tag{1}$$

our approach starts with the basic additive relation for the strain rates defined with respect to the stress-free configuration (see Appendix for a one-dimensional illustration):

$$\overset{\nabla_P}{\boldsymbol{G}} = \overset{\nabla_P}{\boldsymbol{G}^e} + \boldsymbol{D}^p, \tag{2}$$

cf. Eqs. (4.13) of [4], Eqs. (84) of [5], where

$$\boldsymbol{G} = \frac{1}{2}\,(\boldsymbol{C}^e - \overset{-1}{\boldsymbol{B}^p}) \tag{3}$$

cf. Table 1 of [4], Eqs. (75) of [5],

$$\boldsymbol{C}^e = \boldsymbol{F}^{e^t}\boldsymbol{F}^e, \qquad \boldsymbol{B}^p = \boldsymbol{F}^p\boldsymbol{F}^{p^t}, \tag{4}$$

$$\boldsymbol{G}^e = \frac{1}{2}\,(\boldsymbol{C}^{(e)} - \boldsymbol{1}), \tag{5}$$

$$\overset{\nabla_P}{(\ldots)} = \dot{(\ldots)} + \boldsymbol{L}^{p^t}(\ldots) + (\ldots)\,\boldsymbol{L}^p, \tag{6}$$

$$\boldsymbol{D}^p = \frac{1}{2}\,(\boldsymbol{L}^{p^t} + \boldsymbol{L}^p), \tag{7}$$

$$\boldsymbol{L}^p = \dot{\boldsymbol{F}}^p\overset{-1}{\boldsymbol{F}^p}. \tag{8}$$

The symbol $\overset{\nabla_P}{(\ldots)}$ indicates the convective time differentiation with respect to the plastic deformation, cf. [4]. We note that the strain rate decomposition (2) has a very clear kinematic interpretation (cf. Table 3 of [4], text below Eq. (89) in [5], and Appendix) and essentially decouples the elastic and plastic rate effects. It can readily be shown that

$$\boldsymbol{D} = \boldsymbol{F}^{e^{-t}}\overset{\nabla_P}{\boldsymbol{G}}\boldsymbol{F}^{e^{-1}} \tag{9}$$

cf. Eq. (4.12) of [4], Eqs. (90) of [5], so that Eq. (2) yields

$$\boldsymbol{D} = \boldsymbol{F}^{e^{-t}}\overset{\nabla_P}{\boldsymbol{G}^e}\boldsymbol{F}^{e^{-1}} + \boldsymbol{F}^{e^{-t}}\boldsymbol{D}^p\boldsymbol{F}^{e^{-1}} \tag{10}$$

cf. Eqs. (4.13) of [4], Eqs. (83) of [5]. From Eqs. (5), (6) we obtain

$$\overset{\nabla_P}{\boldsymbol{G}^e} = \frac{1}{2}\,(\dot{\boldsymbol{C}}^e + \boldsymbol{L}^{p^t}\boldsymbol{C}^e + \boldsymbol{C}^e\boldsymbol{L}^p) - \frac{1}{2}\,(\boldsymbol{L}^{p^t} + \boldsymbol{L}^p) \tag{11}$$

or, by Eq. (7)

$$\overset{\nabla_P}{G^e} = \frac{1}{2}\,(\dot{C}^e + L^{P^t}C^e + C^eL^P) - D^P. \tag{12}$$

Now, introducing Eq. (11) into Eq. (10) yields

$$D = \frac{1}{2}\,F^{e^{-t}}(\dot{C}^e + L^{P^t}C^e + C^eL^P)\,F^{e^{-1}} \tag{13}$$

or

$$D = \frac{1}{2}\,F^{e^{-t}}\overset{\nabla_P}{C^e}F^{e^{-1}} \tag{14}$$

because the terms involving D^P cancel. On account of the relations

$$W^P = \frac{1}{2}\,(L^P - L^{P^t}), \tag{15}$$

$$L^P = D^P + W^P, \tag{16}$$

it is now easy to show that

$$\overset{\nabla_P}{C^e} = \overset{\nabla}{C^e} + D^PC^e + C^eD^P \tag{17}$$

where $\overset{\nabla}{C^e}$ is the Jaumann derivative of C^e for axes rotating with spin W^P, i.e., cf. [2]

$$\overset{\nabla}{C^e} = \dot{C}^e - W^PC^e + C^eW^P. \tag{18}$$

Eqs. (14), (17) give

$$D = \frac{1}{2}\,F^{e^{-t}}\overset{\nabla}{C^e}F^{e^{-1}} + \frac{1}{2}\,F^{e^{-t}}(D^PC^e + C^eD^P)\,F^{e^{-1}}. \tag{19}$$

Now, Prof. Lee assumes $F^e = V^e$, a symmetric matrix, which corresponds to the rotation free destressing. As a matter of fact, exactly the same assumption was made in [5], [6] as the natural finite element formulation eliminates any rotation during destressing automatically (only rotationless element deformations are considered on the constitutive level per definition). Therefore assuming additionally the full isotropy of the problem, using the relation

$$C^e = V^{e^2} \tag{20}$$

and noting that multiplications in terms of the form $V^eD^PV^{e^{-1}}$ are commutative, we end up with

$$D = \frac{1}{2}\,V^{e^{-1}}\overset{\nabla}{C^e}V^{e^{-1}} + D^P = D^e + D^P \tag{21}$$

which coincides exactly with Eq. (21) of [2], and gives a final proof of the equivalence of both kinematic considerations. The further development of [5], [6] is also exactly the same and uses independently postulated constitutive laws for elastic and plastic party of the total strain rate. This ultimately leads to elasto-plastic (tangent) constitutive matrix for large strain theory.

In the computer program two equivalent formulations were alternatively implemented. The first uses explicitly the stress-free configuration as the reference state, takes Eq. (2) as the kinematic background and involves as basic dual variables stresses $\bar{\sigma}_{P-K}$ and strain rates $\overset{\nabla_P}{\boldsymbol{G}}$, $\overset{\nabla_P}{\boldsymbol{G}^e}$, $\boldsymbol{D}^p$, where $\bar{\sigma}_{P-K}$ is the second Piola-Kirchhoff stress tensor with respect to the reference configuration. The second approach is a current configuration approach based on the decomposition (21) and the dual variables σ_{P-K} and $\boldsymbol{D}$, $\boldsymbol{D}^e$, $\boldsymbol{D}^p$, where σ_{P-K} is now the second Piola-Kirchhoff stress tensor on the current configuration.

The following relation is noted with regard to the equivalence of both the above formulations

$$\frac{1}{\varrho}\,\mathrm{tr}\,(\sigma_{P-K}\,\boldsymbol{D}) = \frac{1}{\bar{\varrho}}\,\mathrm{tr}\,(\bar{\sigma}_{P-K}\,\overset{\nabla_P}{\boldsymbol{G}}) \tag{22}$$

with the similar relations for the elastic and plastic parts of the deformation valid as well, cf. Eqs. (62) of [4], Eqs. (95) of [5]. The symbols ϱ and $\bar{\varrho}$ stand here for mass densities in the current and stress-free configurations, respectively, while

$$\bar{\sigma}_{P-K} = \frac{\bar{\varrho}}{\varrho}\,\boldsymbol{F}^{e^{-1}}\sigma_{P-K}\boldsymbol{F}^{e^{-t}} \tag{23}$$

cf. Eq. (97) of [5], Eq. (6.1) of [4].

The constitutive equation for the elastic part of the deformation has been postulated in the form

$$\boldsymbol{D}^e = \boldsymbol{M}^e : \dot{\sigma}_{P-K} \tag{24}$$

cf. Eq. (99) or [5], Eq. (26) of [2], while the plastic constitutive law has been simply taken as a finite strain generalization of the Prandtl-Reuss flow rule leading (in general terms) to the expression

$$\boldsymbol{D}^p = \boldsymbol{M}^p : \dot{\sigma}_{P-K} \tag{25}$$

cf. Eqs. (109) or (148) of [5], Eq. (28) of [2]. The notation $(\boldsymbol{M} : \sigma)_{ij} = M_{ijkl}\,\sigma_{kl}$ is here employed. Using Eq. (21) we can finally obtain

$$\dot{\sigma}_{P-K} = (\boldsymbol{M}^e + \boldsymbol{M}^p)^{-1} : \boldsymbol{D} \tag{26}$$

cf. Eqs. (119) of (166) of [5], Eq. (31) of [2], which is the basis of the incremental finite element formulation of [5], [6] and also the final result given in [2]. We emphasize that the time derivative of the second Piola-Kirchhoff stress tensor used above equals the so-called Truesdell rate of the Cauchy stress tensor and is nothing else but the Jaumann derivative of the Kirchhoff stress tensor employed in [2].

The theory has been applied to a number of numerical analyses as documented in recent ISD publications. As expected, for statically loaded metals, when elastic strains are expected to be small, the differences between the present results and those obtained from classical finite deformation codes are mostly negligible. However, for dynamic problems, and for some static problems of soils and void-containing metals some essential differences can be observed.

It should certainly be emphasized in closing that the approach discussed here seems to be that simple for purely isotropic situations only and, as indicated in [3], descritpion of finite elastic-plastic strains in anisotropic materials still creates a challenging problem.

Appendix

It can be shown that, cf. Eq. (4.10) of [4]

$$\boldsymbol{D}^P = \overset{\nabla_P}{\boldsymbol{G}^P} \tag{A 1}$$

where

$$\boldsymbol{G}^P = \frac{1}{2}\,(\boldsymbol{1} - \overset{-1}{\boldsymbol{B}^P}), \tag{A 2}$$

see also Eqs. (2), (3), (5). For a one-dimensional example of a bar under extension we denote: l = current length of the bar, l_0 = initial length of the bar, $\bar{l}$ = unstressed length of the bar.

We have then

$$G^e = \frac{1}{2}\,\frac{l^2 - \bar{l}^2}{\bar{l}^2}, \quad G^P = \frac{1}{2}\,\frac{\bar{l}^2 - l_0^2}{\bar{l}^2}, \quad G = \frac{1}{2}\,\frac{l^2 - l_0^2}{\bar{l}^2}, \tag{A 3}$$

$$\dot{G}^e = \frac{l\bar{l}\dot{l} - l^2\dot{\bar{l}}}{\bar{l}^3}, \quad \dot{G}^P = \frac{l_0^2\dot{\bar{l}}}{\bar{l}^3}, \quad \dot{G} = \frac{l\bar{l}\dot{l} - l^2\dot{\bar{l}} + l_0^2\dot{\bar{l}}}{\bar{l}^3}, \tag{A 4}$$

$$F^P = \frac{\bar{l}}{l_0}, \quad L^P = \frac{\dot{\bar{l}}}{\bar{l}}, \tag{A 5}$$

$$\overset{\nabla_P}{G^e} = \dot{G}^e + 2L^P G^e = \frac{l\dot{l} - \bar{l}\dot{\bar{l}}}{\bar{l}^2}, \tag{A 6}$$

$$D^P = \dot{G}^P + 2L^P G^P = \frac{\dot{\bar{l}}}{\bar{l}} = \frac{\bar{l}\dot{\bar{l}}}{\bar{l}^2}, \tag{A 7}$$

$$\overset{\nabla_P}{G} = \dot{G} + 2L^P G = \frac{l\dot{l}}{\bar{l}^2}. \tag{A 8}$$

The three-dimensional strain rate measures $\overset{\nabla}{\boldsymbol{G}}$, $\overset{\nabla_P}{\boldsymbol{G}^e}$, $\boldsymbol{D}^P$ can be given the following kinematic interpretation, cf. Table 3 of [4]:

$$\frac{1}{2}\,\frac{d}{dt}\,(ds^2 - ds_0{}^2) = d\bar{\boldsymbol{x}}\,\overset{\nabla_P}{\boldsymbol{G}}\,d\bar{\boldsymbol{x}} \tag{A 9}$$

$$\frac{1}{2}\frac{d}{dt}(ds^2 - d\bar{s}^2) = d\bar{\boldsymbol{x}}\,\overset{\nabla_P}{\boldsymbol{G}^e}\,d\bar{\boldsymbol{x}}, \tag{A 10}$$

$$\frac{1}{2}\frac{d}{dt}(d\bar{s}^2 - ds_0{}^2) = d\bar{\boldsymbol{x}}\,\boldsymbol{D}^P\,d\bar{\boldsymbol{x}}, \tag{A 11}$$

where $d\bar{x}$ stands for an infinitesimal element in the stress-free configuration while $ds, d\bar{s}, ds_0$ denote the arc lengths of this element in the current, stress-free and initial configurations, respectively. For the uniforme one-dimensional deformation Eqs. (A 9—A 11) read

$$\frac{1}{2}\frac{d}{dt}(l^2 - l_0{}^2) = \bar{l}^2\overset{\nabla_P}{G}, \tag{A 12}$$

$$\frac{1}{2}\frac{d}{dt}(l^2 - \bar{l}^2) = \bar{l}^2\overset{\nabla_P}{G^e}, \tag{A 13}$$

$$\frac{1}{2}\frac{d}{dt}(\bar{l}^2 - l_0{}^2) = \bar{l}^2D^P. \tag{A 14}$$

the scalars $\overset{\nabla_P}{G}$, $\overset{\nabla_P}{G^e}$ and D^P being given by Eqs. (A 6)—(A 8). It is seen that:

— the plastic strain rate D^P describes the plastic part of the deformation process with respect to the stress-free configuration exactly as the classical Eulerian approach to rigid plasticity describes plastic deformation with respect to the actual configuration,

— the elastic strain rate $\overset{\nabla_P}{G^e}$ describes the elastic part of the deformation process with respect to the stress-free configuration exactly as the classical Lagrangian approach to nonlinear elasticity describes elastic deformation with respect to the initial (now moving!) configuration.

The above assertions have been accepted in [4], [5], [6] as a justification for using Eq. (2) (or Eq. (21), which is equivalent as shown above) as the kinematic basis for the finite strain theory of elasto-plasticity.

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