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Extended hyperbolicity

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Abstract The concept of Brody hyperbolicity is interpreted in terms of homotopy theoretic structures. We extend the definition of Brody hyperbolicity to simplicial sheaves of sets over the site of complex spaces with the strong topology. Imitating one possible definition of homotopy groups for a topological space, we defined the *holotopy* groups for a simplicial sheaf and showed that their vanishing in "positive" degrees is a necessary condition for a sheaf to be Brody hyperbolic. A partial converse to this theorem is proved at the end of the paper. We deduce that if X is a complex space with a nonzero holotopy group in positive degree, then X cannot be weakly equivalent (in a particular sense) to a hyperbolic complex space (in particular is not itself hyperbolic). We finish the manuscript by applying these results along with a *topological realization functor*, constructed in the previous section, to prove that complex projective spaces cannot be weakly equivalent to hyperbolic complex spaces.

Keywords Kobayashi hyperbolic spaces · Simplicial sheaves · Homotopical algebra

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1 Introduction

We develop a theory in which the concept of Brody hyperbolicity of a reduced complex space is interpreted in terms of homotopy-theoretical structures, therefore opening a new link between two, a priori completely different, branches of mathematics (recall that a complex space X is called *Brody hyperbolic* if every holomorphic function $\mathbb{C} \to X$ is constant). Based on our current understanding, we contend that this interplay will be particularly useful if implemented by applying homotopy-theoretical techniques and constructions to get information on Brody hyperbolic spaces rather then following the opposite path.

The relation between homotopy theory and Brody hyperbolicity is explicited by certain groups associated with complex spaces which we called *holotopy groups*, that are defined by mimicking one of the constructions of the classical homotopy groups. What we change is the underlying homotopy category which is no longer locally trivial, its "homotopy" classes of objects are no longer homotopy invariant (in the classical sense) but they are still biholomorphism invariant. In fact, they are invariant under a more refined homotopy equivalence, called $\mathbb{A}^1$-*homotopy* after Morel and Voevodsky. In such a concept lie the ties with Brody hyperbolicity which are expressed in this manuscript by means of the Theorem 4.1 asserting the vanishing of the holotopy groups of Brody hyperbolic complex spaces.

The background to define holotopy groups involves a first stage in which the category $\mathsf{An}/\mathbb{C}$ of complex spaces and holomorphic maps is fully embedded into a new category. The objects of this bigger category are *simplicial sheaves of sets* on the site $(\mathsf{An}/\mathbb{C})_{str}$ of complex spaces with the strong topology.

Then we wish to construct a category obtained from the latter by "adding" the inverses of certain morphisms. Ideally, in the case of the category $\mathsf{An}/\mathbb{C}$, we would have liked to add the inverse to the canonical map $p : \mathbb{C} \to \mathsf{pt}$ along with all its base changed maps. The task of inverting a family S of morphisms in a category $\mathcal{C}$ can be accomplished by starting from an arbitrary category $\mathcal{C}$ with a given family S of morphisms satisfying suitable compatibility conditions. The category $S^{-1}\mathcal{C}$ that we obtain is called the *localization* of $\mathcal{C}$ with respect to S. In this kind of generality, $S^{-1}\mathcal{C}$ is not practical to work with. In this sense, reasonable categories are the "homotopy categories" associated with a *model structure* in the sense of Quillen (cfr. [9]). In general, deciding whether a localized category $S^{-1}\mathcal{C}$ is equivalent to the homotopy category associated with a model structure on $\mathcal{C}$ is a very complicated task. However, this can be done if $\mathcal{C} = \Delta^{\mathsf{op}}\mathcal{F}_T(\mathcal{S})$, the category of simplicial objects in the category of sheaves of sets over a small site $\mathcal{S}_T$. In the case $\mathcal{S}_T = (\mathsf{An}/\mathbb{C})_{str}$, the category $\mathcal{S}$ is fully embedded in $\Delta^{\mathsf{op}}\mathcal{F}_T(\mathcal{S})$. This structure is implemented through the Bousfield framework [1] to get a new model structure on $\Delta^{\mathsf{op}}\mathcal{F}_T(\mathcal{S})$ whose weak equivalences contain in an optimal way the morphism p and all its base changes. This yields an (unstable) homotopy category which we will denote by $\mathcal{H}^{olo}$, whose objects include $\mathbb{A}^1$-homotopy classes of complex spaces. Unlike the classical homotopy category of topological spaces, the category $\mathcal{H}^{olo}$ reflects the complex structure of the objects. These preliminary constructions are the content of Sect. 2: our approach uses the theory developed in [8], which works in a quite general context. It follows that almost all the results proved here are valid for the category Sch/B of the algebraic schemes of finite type over a noetherian base, as well.

Afterwards, we define a notion of hyperbolicity for simplicial sheaves which we prove it restricts to the Brody hyperbolicity for complex spaces (see Corollary 3.3). Hence, in the particular case $\mathcal{X} = X$ is a compact complex space, in view of Brody's Theorem (cfr. [3]) we conclude that X is hyperbolic according to our definition if and only if it is Kobayashi hyperbolic (see Corollary 3.2), effectively making the definition of Sect. 3.1 a generalization of the classical concept of hyperbolicity for complex spaces. By this definition of hyperbolicity for simplicial sheaves, a general result in Bousfield localized categories (cfr. [1]) implies that in each class of complex spaces lies a hyperbolic representative $\mathfrak{Ip}(X)$, which in general will not be a complex space. Such a correspondence is functorial and there exists a canonical morphism $c_X : X \to \mathfrak{Ip}(X)$ satisfying the universality property described in Corollary 3.1. $\mathfrak{Ip}(X)$ is given by a complicated construction, even when X is a complex space, although its *topological realization* (see Sect. 6) is a topological space homotopy equivalent to the topological space underlying X.

In Sect. 4 we define the aforementioned *holotopy sets* or *groups* when applicable (see (12), (13)). We prove that the vanishing of some of the holotopy groups of a complex space X is a necessary condition for the "hyperbolic model" $\mathfrak{Ip}(X)$ to be isomorphic in $\mathcal{H}^{olo}$ to some Brody hyperbolic complex space. The holotopy groups of a simplicial sheaf $\mathcal{X}$ are formally defined like in ordinary homotopy theory, i.e. as $\mathsf{Hom}_{\mathcal{H}^{olo}}$ sets from suitable models of spheres to the simplicial sheaf in question. Some of the spheres may be represented by complex spaces (e.g. $\mathbb{CP}^1$ or $\mathbb{C}^*$), but unlike in classical homotopy theory it is in general false that any holotopy class is represented by a map of simplicial sheaves from some sphere to $\mathcal{X}$. In the familiar case in which $\mathcal{X} = X$ is a complex space, thus with trivial simplicial structure, this means that some holotopy classes will be represented by a simplicial sheaf map from a sphere to $\widetilde{X}$, where $\widetilde{X}$ is a simplicial sheaf isomorphic to X in $\mathcal{H}^{olo}$, with possibly a nontrivial simplicial structure and satisfying a condition. Some of those holotopy classes should be "undetectable" in complex geometry, yet their existence gives information on a complex geometric invariant, namely the non-hyperbolicity of X. According to our definition of hyperbolicity and by general properties of Bousfield localized homotopy categories, the condition $\widetilde{X}$ must satisfy is precisely hyperbolicity.

In the following Sect. 5 we construct a functor that proves to be useful for explicit computations: the *topological realization functor*. To a simplicial sheaf it associates a topological space in such a way few reasonable properties are satisfied (see a), b), c)). It induces an homomorphism from the holotopy groups of a complex space to the homotopy groups of the underlying topological space. Such homomorphism in general is neither injective nor surjective and its image are the homotopy classes which have a "complex geometric origin" and have different flavors according to whether it is image of a parabolic circle or hyperbolic circle [see (a), (b), (c)].

In the last section, applying some of our results, we show that the hyperbolic model $\mathfrak{Ip}(\mathbb{CP}^n)$ of the n-th dimensional projective complex space $\mathbb{CP}^n$ is not $\mathbb{A}^1$-weakly equivalent to a Brody hyperbolic space for any $n > 0$ and that the same holds for complex spaces admitting a covering with a fiber intersecting a complex line in at least two points (see Proposition 6.1). Finally, a partial converse to Theorem 4.1 is derived, namely, the non-hyperbolicity of a complex space X implies the existence of a nonzero class in $\pi_{1,0}(X, x)$, under certain circumstances.

2 Basic constructions

The general problem we are dealing with is to modify the category of complex spaces to a category where the constant morphism $\mathsf{p} : \mathbb{C} \to \mathsf{pt}$ is invertible.

In this section we recall the main results of [8]. The constructions made there hold in the general context of a site with enough points in the sense of [4]. We restrict ourselves to the sites $(\mathsf{An}/\mathbb{C})_{str}$ of complex spaces with the strong topology or $(\mathsf{Sch}/B)_T$ of schemes of finite type over a noetherian scheme B of finite dimension, endowed with a Grothendieck topology T which is weaker or as fine as the quasi compact flat topology. Both these sites are denoted by $\mathcal{S}_T$ for short, $\mathcal{S}$ being the corresponding category.

2.1 Sheaves and simplicial objects: the categories $\mathcal{F}_T(\mathcal{S})$ and $\Delta^{\mathsf{op}}\mathcal{F}_T(\mathcal{S})$

Let $\mathcal{S}$ be the category of complex spaces or schemes of finite type over a noetherian scheme B. If we wish to do some kind of homotopy theory on it, we should check the shape of co-limits of certain diagrams. We are particularly interested in co-limits of diagrams of the kind

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & X \\ \mathsf{pt} & \end{tikzcd}$$

where i is an injection. In this paper, such co-limits will sometimes be called *quotient* of X by A along i. The examples below show that, in general, it may happen that the quotient does not exist in the category $\mathcal{S}$ or if it exists, it is different from the one taken in the underlying category of topological spaces.

(1) Let **D** be the diagrams

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "i"] \arrow[d, "f"] & \mathbb{C} \\ \mathsf{pt} & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{CP}^1 \arrow[r, "i"] \arrow[d, "f"] & \mathbb{CP}^2 \\ \mathsf{pt} & \end{tikzcd}$$

where i are the canonical embeddings. Then the co-limits of **D** in $\mathsf{An}/\mathbb{C}$ are just a point in both cases, unlike their respective co-limits in the category of topological spaces.

(2) Let **D** be the diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i"] \arrow[d, "f"] & \mathbb{C} \\ \mathsf{pt} & \end{tikzcd}$$

where i is the canonical injection. **D** has no co-limit in $(\mathsf{An}/\mathbb{C})_{str}$. Indeed, by contradiction, let $X = \mathsf{colim}_{\mathbf{D}}$ in $\mathcal{S}$, $p : \mathbb{C} \to X$ the corresponding canonical holomorphic map and $x = p(\mathbb{Z})$. Since there exists a nonconstant holomorphic function $h : \mathbb{C} \to \mathbb{C}$ such that $h(n) = 0$ for every $n \in \mathbb{Z}$, X cannot be just the point x, moreover $p^{-1}(x) = i(\mathbb{Z})$. Let U be a relatively compact neighborhood of x and $\{z^{(n)}\} \subset p^{-1}(U)$ a sequence with no accumulation points. If $h : \mathbb{C} \to \mathbb{C}$ is a holomorphic function satisfying $h(n) = 0$

and $h(z^{(n)}) = n$ for every $n \in \mathbb{Z}$, no holomorphic function $g : X \to \mathbb{C}$ exists such that $g \circ p = h$.

A similar argument can be used to prove that the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "i"] \arrow[d, "f"] & \mathbb{C} \times \mathbb{C} \\ \mathsf{pt} & \end{tikzcd}$$

where i is the injection $\mathbb{C} \to \{0\} \times \mathbb{C}$, has no co-limit in $\mathcal{S}$.

(3) Let $\mathbf{D}$ be the diagram

$$\begin{tikzcd} \{0\} \cup \{1\} \arrow[r, "i"] \arrow[d, "f"] & \mathbb{A}^1_k \\ \mathsf{pt} & \end{tikzcd}$$

where $\mathbb{A}^1_{\Bbbk}$ is the affine line over a field $\Bbbk$ and i is the embedding of the corresponding rational points. Then, since the $\Bbbk$-algebra of the polynomials $P(x)$ of the form $a + x(x-1)Q(x)$, $a \in k$, is not finitely generated, $\mathbf{D}$ has no co-limit in the category $\mathsf{Sch}/\Bbbk$

We therefore enlarge $\mathcal{S}$ to a category which contains the co-limits of all diagrams and, at the same time, have a "reasonably good" shape from our point of view. Such a category is $\mathcal{F}_T(\mathcal{S})$: the objects are sheaves of sets on the site $\mathcal{S}_T$ (namely, the site $(\mathsf{An}/\mathbb{C})_{str}$ in the case of the complex spaces, $(\mathsf{Sch}/\Bbbk)_T$ in the algebraic case) and morphisms are maps of sheaves of sets.

Let $\mathbf{Y}(X) := \mathsf{Hom}_{\mathcal{S}}(\cdot, X)$. The functorial equality

$$\mathsf{Hom}_{\mathcal{S}}(A, B) = \mathsf{Hom}_{Funt(\mathcal{S}^{op}, Sets)}(\mathbf{Y}(A), \mathbf{Y}(B))$$

is known as Yoneda Lemma. The Yoneda embedding is a faithfully full functor $\mathbf{Y} : \mathcal{S} \hookrightarrow Fun(\mathcal{S}^{op}, Sets)$. In view of our hypothesis on the Grothendieck site the image of $\mathbf{Y}$ is contained in the full subcategory $\mathcal{F}_T(\mathcal{S})$. Moreover,

Theorem 2.1 *For every $X \in S_T$ the functor $\mathsf{Hom}_{\mathcal{S}}(-, X)$ is a sheaf for the topology T.*

The category $\mathcal{F}_T(\mathcal{S})$ is complete and co-complete. Indeed, the limit of a diagram $\mathbf{D}$ in $\mathcal{F}_T(\mathcal{S})$ is the functor $U \rightsquigarrow \lim \mathbf{D}(U)$ which is a sheaf for the topology T. As for the co-limit, it is defined as $a_T(U \rightsquigarrow \operatorname{colim} \mathbf{D}(U))$ where a_T is the associated sheaf.

Let $\Delta^{\mathsf{op}}\mathcal{F}_T(\mathcal{S})$ be the category of simplicial objects in $\mathcal{F}_T(\mathcal{S})$ ([11]). For every integer $n \geq 0$, we denote by $\Delta[n]$ the simplicial object that at the level m has as many copies of points $*$ as nondecreasing monotone functions $[m] \to [n]$. The $m+1$ injective functions $[m-1] \to [m]$ induce the faces and the m surjective functions $[m] \to [m-1]$ induce the degeneracies of $\Delta[n]$. For every simplicial object $\mathcal{X}$, the pair of $*$ in degree 0 defines two morphisms ϵ_0 and $\epsilon_1 : \mathcal{X} \to \mathcal{X} \times \Delta[1]$.

Let $\mathcal{X}, \mathcal{Y}$ be two objects of $\Delta^{\mathsf{op}}\mathcal{F}_T(\mathcal{S})$. A *homotopy* between two morphisms $f, g : \mathcal{X} \to \mathcal{Y}$ is a morphism $H : \mathcal{X} \times \Delta[1] \to \mathcal{Y}$ such that $H \circ \epsilon_0 = f$, $H \circ \epsilon_1 = g$.

In particular, this definition gives a notion of homotopy for objects and morphisms of $\mathcal{F}_T(\mathcal{S})$.

Finally, we recall that a morphism $\phi : A_\bullet \to B_\bullet$ of simplicial sets is said to be a *weak equivalence* if its topological realization $|\phi| : |A_\bullet| \to |B_\bullet|$ is a weak equivalence, i.e. the homomorphisms $|\phi|_* : \pi_k(|A|, a) \to \pi_k(|B|, |\phi|(a))$, between the homotopy groups are isomorphisms, for all $k > 0$ and a bijection for $k = 0$.

2.2 Simplicial localization

The following will endow $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$ with a model structure in the sense of Quillen: a morphism $f : \mathcal{G} \to \mathcal{F}$ of simplicial sheaves is a *weak equivalence* if for every point x of a complex space or a scheme over B, the morphism $f_x : \mathcal{G}_x \to \mathcal{F}_x$ of the respective stalks over x is a weak equivalence of simplicial sets.

An injective morphism $f : \mathcal{X} \to \mathcal{Y}$ is said to be a *simplicial co-fibration*.

A *lifting* in a commutative square of morphisms

$$\begin{tikzcd} \mathcal{A} \arrow[r, "q"] \arrow[d, "j"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{B} \arrow[r, "r"] & \mathcal{Y} \end{tikzcd} \tag{1}$$

is a morphism $h : \mathcal{B} \to \mathcal{X}$ which makes the diagram commutative. In such situation we say that j has the *left lifting property* with respect to f and f has the *right lifting property* with respect to j.

A morphism $f : \mathcal{X} \to \mathcal{Y}$ is called a *fibration* if all diagrams (1) admit a lifting, for all *acyclic co-fibrations* j (co-fibration and weak equivalence simultaneously).

An object $\mathcal{X}$ of $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$

(1) is called *co-fibrant* if $\emptyset \to \mathcal{X}$ is a co-fibration;
(2) is called *fibrant* if $\mathcal{X} \to \mathsf{pt}$ is a fibration.

The classes of weak equivalences, co-fibrations and fibrations give $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$ a structure of *simplicial* model category as shown in [6]. Under these assumptions, there exists a localization of $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$ with respect of the weak equivalences.

2.3 Notation

(1) We denote pt the simplicial constant sheaf defined as the associated sheaf to the presheaf which associates with an object of $\mathcal{S}$ the set consisting of one element. The *pointed category* associated with $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$ is the category $\Delta^{\mathrm{op}}_\bullet\mathcal{F}_T(\mathcal{S})$ whose objects are the pairs $(\mathcal{X}, x)$ where $\mathcal{X} \in \Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$ and $x : \mathsf{pt} \to \mathsf{X}$ is a morphism; a morphism of pairs $(\mathcal{X}, x) \to (\mathcal{Y}, y)$ is a morphism $f : \mathcal{X} \to \mathcal{Y}$ sending x to y.
(2) Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of (pointed) simplicial sheaves. The symbol $\mathsf{cof}(f)$ denotes the co-limit of the diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "f"] \arrow[d] & \mathcal{X} \\ \mathsf{pt} & \end{tikzcd}$$

(pointed by the image of $\mathcal{Y}$) where pt is a point. $\mathsf{cof}(f)$ is called the *co-fiber* of f. If f is a co-fibration the co-fiber of f is sometimes denoted by $\mathcal{X}/\mathcal{Y}$.
(3) Let $\mathcal{X}$ and $\mathcal{Y}$ be pointed simplicial sheaves. The sheaf $\mathcal{X} \vee \mathcal{Y}$ is, by definition, the co-limit of

$$\begin{tikzcd} \mathsf{pt} \arrow[r] \arrow[d] & \mathcal{X} \\ \mathcal{Y} & \end{tikzcd}$$

pointed by the image of pt.

(4) The pointed simplicial sheaf $\mathcal{X} \wedge \mathcal{Y}$ is defined by $\mathcal{X} \times \mathcal{Y}/\mathcal{X} \vee \mathcal{Y}$.

(5) The simplicial pointed constant sheaf S_s^1 is defined by $\Delta[1]/\partial\Delta[1]$ where $\partial\Delta[1]$ is the simplicial subsheaf of $\Delta[1]$ consisting in the union of the images of the face morphisms of $\Delta[1]$. For $n \in \mathbb{N}$ we set $S_s^n = S_s^1 \wedge \overset{n}{\cdots} \wedge S_s^1$.

Performing the same constructions as for $\Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$ we obtain a homotopy category $\mathcal{H}_{s\bullet}$.

For a description of the main properties of $\mathcal{H}_s$ and $\mathcal{H}_{s\bullet}$ we refer to [8,9]. Here we only recall the following result which will be used in the sequel: given a simplicial co-fibration $i : \mathcal{Y} \to \mathcal{X}$ of pointed simplicial sheaves and a pointed simplicial sheaf $\mathcal{Z}$ the morphism i induces a long exact sequence of pointed sets and groups

$$\begin{aligned} &\mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathcal{Y}, \mathcal{Z}) \xleftarrow{i^*} \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathcal{X}, \mathcal{Z}) \xleftarrow{\pi^*} \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathcal{X}/\mathcal{Y}, \mathcal{Z}) \\ &\quad \leftarrow \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathcal{Y} \wedge S_s^1, \mathcal{Z}) \xleftarrow{i^*} \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathcal{X} \wedge S_s^1) \xleftarrow{\pi^*} \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathcal{X}/\mathcal{Y} \wedge S_s^1, \mathcal{Z}) \cdots . \end{aligned} \quad (2)$$

2.4 Affine localization

In this section we will use the simplicial model structure on $\Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$ to give a new model structure on the same category. The notion of weak equivalences involves the concept of *interval* ([10]). We will denote by $\mathbb{A}^1$ the intervals in our sites $\mathcal{S}_T$, namely $\mathbb{C}$ in the complex analytic case and *Spec* $\mathbb{k}[x] = \mathbb{A}^1_{\mathbb{k}}$ in the algebraic case. Affine weak equivalences contain $p : \mathbb{A}^1 \to \mathsf{pt}$, and are in a sense the "smallest" class containing all the base changes of p, as well. Such weak equivalences are written in terms of morphisms in $\mathcal{H}_s$ and the homotopy category associated with this model structure is the localization of $\mathcal{H}_s$ with respect to the weak equivalences. Unless otherwise mentioned, the results presented in this subsection are taken from Sect. 3.2 of [8].

A simplicial sheaf $\mathcal{X} \in \Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$ is said to be $\mathbb{A}^1$-*local* if it is simplicially fibrant and the projection $\mathcal{Y} \times \mathbb{A}^1 \to \mathcal{Y}$ induces a bijection of sets

$$\mathsf{Hom}_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X}) \to \mathsf{Hom}_{\mathcal{H}_s}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$$

for every $\mathcal{Y} \in \Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$.

In what follows we describe a new structure of models on $\Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$, which we will call *affine*.

A morphism $f : \mathcal{X} \to \mathcal{Y}$ is called:

(1) an *affine weak equivalence* if, for every $\mathbb{A}^1$-local simplicial sheaf $\mathcal{Z} \in \Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$

$$f^* : \mathsf{Hom}_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{Z}) \to \mathsf{Hom}_{\mathcal{H}_s}(\mathcal{X}, \mathcal{Z})$$

is a bijection;

(2) an *affine co-fibration* if it is injective;

(3) an *affine fibration* if all diagrams (1) admit a lifting, where j is any affine co-fibration and affine weak equivalence.

$\mathbb{A}^1$-fibrant and co-fibrant objects are defined as usual.

In this manuscript we are going to use an equivalent notion of affine weak equivalences, which explicits the Bousfield machinery that lies behind (cfr. [5]). Let $Map(\mathcal{X}, \mathcal{Y})$ be a homotopy function complex of the category $\Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$; in particular this is a simplicial set. We define $\mathcal{Y}$ a simplicial sheaf to be B-$\mathbb{A}^1$-*local* if it is simplicially fibrant and $Map(\mathcal{X}, \mathcal{Z}) \to Map(\mathcal{X} \times \mathbb{A}^1, \mathcal{Z})$ is a weak equivalence for any $\mathcal{X} \in \Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$. A map $f : \mathcal{X} \to \mathcal{Y}$ is a B-$\mathbb{A}^1$-*weak equivalence* if and only if $f^* : Map(\mathcal{Y}, \mathcal{Z}) \to Map(\mathcal{X}, \mathcal{Z})$ is a simplicial weak equivalence for any B-$\mathbb{A}^1$-local $\mathcal{Z}$.

Lemma 2.1 *$\mathcal{X}$ is B-$\mathbb{A}^1$-local if and only if it is $\mathbb{A}^1$-local. A morphism $f : \mathcal{X} \to \mathcal{Y}$ is a B-$\mathbb{A}^1$-weak equivalence if and only if it is an affine weak equivalence*

A remarkable fact from the Bousfield theory is that $\mathbb{A}^1$-local objects are the $\mathbb{A}^1$-fibrant objects.

The structures listed above endow $\Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$ of a model structure, which will be called *affine model structure* or $\mathbb{A}^1$-*model structure* [8, Theorem 3.2].

The localized category with respect of the affine weak equivalences is denoted as $\mathcal{H}$ and its pointed version as $\mathcal{H}_\bullet$.

Remark 2.1
1. Any object of $\Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$ is both (simplicially) co-fibrant and $\mathbb{A}^1$-co-fibrant.
2. If $f : \mathcal{Y} \to \mathcal{X}$ is a simplicial weak equivalence (respectively a simplicial co-fibration) then it is an affine weak equivalence (respectively an affine co-fibration). Therefore, the affine localization functor $\Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S}) \to \mathcal{H}$ factors as $\Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S}) \to \mathcal{H}_s \to \mathcal{H}$, where the first functor is the simplicial localization and the second is the identity on objects. However, the functor $\mathcal{H}_s \to \mathcal{H}$ is *not* an equivalence of categories.
3. The same classes of pointed morphisms, give $\Delta^{\mathsf{op}}_\bullet\boldsymbol{\mathcal{F}}_T(\mathcal{S})$ a model structure.

From [9, Proposition 4'] we derive the following fundamental result: given an affine co-fibration $j : Y \to \mathcal{X}$ (i.e., an injection of simplicial pointed sheaves) and a pointed simplicial sheaf $\mathcal{Z}$ the morphism j induces a long exact sequence of pointed sets and groups

$$\begin{aligned}&\mathsf{Hom}_{\mathcal{H}_\bullet}(\mathcal{Y}, \mathcal{Z}) \xleftarrow{j^*} \mathsf{Hom}_{\mathcal{H}_\bullet}(\mathcal{X}, \mathcal{Z}) \xleftarrow{\pi^*} \mathsf{Hom}_{\mathcal{H}_\bullet}(\mathcal{X}/\mathcal{Y}, \mathcal{Z})\\ &\leftarrow \mathsf{Hom}_{\mathcal{H}_\bullet}(\mathcal{Y} \wedge S^1_s, \mathcal{Z}) \xleftarrow{j^*} \mathsf{Hom}_{\mathcal{H}_\bullet}(\mathcal{X} \wedge S^1_s) \xleftarrow{\pi^*} \mathsf{Hom}_{\mathcal{H}_\bullet}(\mathcal{X}/\mathcal{Y} \wedge S^1_s, \mathcal{Z}) \cdots . \qquad (3)\end{aligned}$$

3 Hyperbolicity and Brody hyperbolicity

3.1 Hyperbolic simplicial sheaves

Let us go back to the concept of hyperbolicity. A simplicial sheaf $\mathcal{X}$ is said to be *hyperbolic* if it is $\mathbb{A}^1$-local. Let $\mathcal{Y}$ be a simplicial subsheaf of $\mathcal{X}$. The simplicial sheaf $\mathcal{X}$ is said to be *hyperbolic mod* $\mathcal{Y}$ if $\mathcal{X}/\mathcal{Y}$ is hyperbolic.

A *hyperbolic resolution* of $\mathcal{X}$ is a morphism of simplicial sheaves $\mathfrak{r} : \mathcal{X} \to \widetilde{\mathcal{X}}$ where $\widetilde{\mathcal{X}}$ is a hyperbolic simplicial sheaf and $\mathfrak{r}$ is an affine weak equivalence.

A *hyperbolic resolution functor* is a pair $(\mathfrak{I}, \mathfrak{r})$ where $\mathfrak{I}$ is a functor

$$\Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S}) \to \Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$$

and $\mathfrak{r}$ is a natural transformation $\mathsf{Id} \to \mathfrak{I}$ such that every morphism $\mathcal{X} \to \mathfrak{I}(\mathcal{X})$ is a hyperbolic resolution.

Proposition 2.19 of [8] applied in our situation gives the following: there exists a hyperbolic resolution functor $(\mathfrak{I}\mathfrak{p}, \mathfrak{r})$ with the following properties:

(i) for every $\mathcal{X} \in \Delta^{\mathrm{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$ the simplicial sheaf $\mathfrak{Jp}(\mathcal{X})$ is hyperbolic and (simplicially) fibrant;
(ii) $\mathfrak{r}$ is an affine equivalence and a co-fibration;
(iii) let $\mathcal{H}_{s,\mathbb{A}^1}$ be the full subcategory in $\mathcal{H}_s$ of $\mathbb{A}^1$-local (hyperbolic) objects. $\mathfrak{Jp}$ sends an affine weak equivalence to a simplicial weak equivalence, hence it induces a functor $L : \mathcal{H}_s \to \mathcal{H}_{s,\mathbb{A}^1}$, that factors as $\mathcal{H}_s \to \mathcal{H} \to \mathcal{H}_{s,\mathbb{A}^1}$, where the first functor is the identity on objects (see also Remark 2.1 (2));
(iv) the canonical immersion $I : \mathcal{H}_{s,\mathbb{A}^1} \hookrightarrow \mathcal{H}_s$ is a right adjoint of L.

Furthermore, $\mathcal{H}_{s,\mathbb{A}^1}$ is a category equivalent to $\mathcal{H}$. Given $X = \mathcal{X} \in \boldsymbol{\mathcal{F}}_T$, $\mathfrak{Jp}(X)$ is the hyperbolic simplicial sheaf associated with the simplicially constant sheaf X. However, due to its rather involved construction, the use of $\mathfrak{Jp}(\mathcal{X})$ is problematic even in the case when X is a complex space or a scheme over $\mathbb{k}$. Therefore, in general, the previous result shall be considered as an existence theorem. Nevertheless, it may occur that, in some particular cases, the class of $\mathfrak{Jp}(\mathcal{X})$ in $\mathcal{H}$ could be represented by an understandable object, or even by a hyperbolic space (see Sect. 3.2).

Lemma 3.1 *For any simplicial sheaf $\mathcal{X}$, the morphism $\mathfrak{r} : \mathcal{X} \to \mathfrak{Jp}(\mathcal{X})$ is universal in the category $\mathcal{H}$ (respectively in the category $\mathcal{H}_s$) in the following sense: for any hyperbolic object $\mathcal{Y}$ and morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathcal{H}$ (respectively in the category $\mathcal{H}_s$), there exists a unique morphism $\tilde{f} : \mathfrak{Jp}(\mathcal{X}) \to \mathcal{Y}$ in $\mathcal{H}_s$ factoring f as $\tilde{f} \circ \mathfrak{r}$.*

Proof Consider the commutative square

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\mathfrak{r}_{\mathcal{X}}"] \arrow[d, "f"] & \mathfrak{Jp}(\mathcal{X}) \arrow[d, "\mathfrak{Jp}(f)"] \\ \mathcal{Y} \arrow[r, "\mathfrak{r}_{\mathcal{Y}}", "\cong"'] & \mathfrak{Jp}(\mathcal{Y}). \end{tikzcd} \tag{4}$$

By definition of $\mathbb{A}^1$ weak equivalence, $\mathfrak{r}_{\mathcal{Y}}$ is a simplicial weak equivalence, since both $\mathcal{Y}$ and $\mathfrak{Jp}(\mathcal{Y})$ are hyperbolic. The map $\tilde{f}$ is defined as $\mathfrak{r}_{\mathcal{Y}}^{-1} \circ \mathfrak{Jp}(f)$. Notice that $\mathfrak{Jp}(f)$ is a morphism in $\mathcal{H}_s$. □

Corollary 3.1 *Let X and Y be sheaves with Y hyperbolic and $f : X \to Y$ be a morphism of sheaves. Then the composition $\tilde{f} \circ \mathfrak{r}$ is a morphism of sheaves and the commutativity of the diagram*

$$\begin{tikzcd} X \arrow[r, "\mathfrak{r}"] \arrow[d, "f"'] & \mathfrak{Jp}(X) \arrow[dl, "\tilde{f}"] \\ Y & \end{tikzcd} \tag{5}$$

is in the category of simplicial sheaves, i.e., it is strictly commutative and not only "up to homotopy" in $\mathcal{H}_s$.

Proof By the previous lemma, we have commutativity in the category $\mathcal{H}_s$. Remark 1.14 of [8] implies that $\mathsf{Hom}(X, Y) = \mathsf{Hom}_{\mathcal{H}_s}(X, Y)$ since both X and Y have simplicial dimension zero. Therefore, equality of the morphisms f and $\tilde{f} \circ \mathfrak{r}$ in $\mathcal{H}_s$ is an equality of morphisms of sheaves. □

3.2 Relations with Brody hyperbolicity

In this subsection we will compare the different notions of hyperbolicity that we have introduced above. In particular, we prove that a simplicial sheaf represented by a complex space X is hyperbolic if and only if X is Brody hyperbolic. This is a corollary of the following

Theorem 3.1 *A sheaf $X \in \mathcal{F}_T(\mathcal{S})$ is a hyperbolic sheaf if and only if the projection $U \times \mathbb{A}^1 \to U$ induces a bijection*

$$\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U, X) \to \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U \times \mathbb{A}^1, X)$$

for every object $U \in \mathcal{S}_T$. Moreover, under this hypothesis, for every $Y \in \mathcal{F}_T(\mathcal{S})$ there exists a bijection

$$\mathsf{Hom}_{\mathcal{H}}(Y, X) \cong \mathsf{Hom}_{\mathcal{H}_s}(Y, X) \cong \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(Y, X). \tag{6}$$

Proof We first prove the second part of the theorem by using the definition of hyperbolicity as being B-$\mathbb{A}^1$ locality (see Lemma 2.1): the set $\mathsf{Hom}(\mathcal{Y}, \mathcal{X})$ can be recovered from the homotopy function complex $Map\,(\mathcal{X}, \mathcal{Y})$ by applying the functor of path components π_0. On the other hand, since we are assuming that $\mathcal{X}$ is $\mathbb{A}^1$-fibrant (i.e. $\mathbb{A}^1$-local) and that $\mathcal{X} = X$ and $\mathcal{Y} = Y$ are sheaves, the simplicial set $Map\,(\mathcal{X}, \mathcal{Y})$ is homotopy equivalent to the constant simplicial set $\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(Y, X)$.

To prove the first part it suffices to show that $Map\,(\mathcal{Y}, X) \to Map\,(\mathcal{Y} \times \mathbb{A}^1, X)$ is a weak equivalence for any $\mathcal{Y} \in \Delta^{\mathsf{op}}\mathcal{F}_T(\mathcal{S})$ if

$$\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U, X) \to \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U \times \mathbb{A}^1, U)$$

is a bijection for any $U \in \mathcal{S}_T$. By the small object argument, there is a simplicial weak equivalence $\mathcal{Y}' \to \mathcal{Y}$ where $\mathcal{Y}'_n$ are a direct sum of object of $\mathcal{S}_T$. Moreover, since for any simplicial sheaf $\mathcal{Z}$ the canonical map hocolim $\mathcal{Z}_n \to \mathcal{Z}$ is a simplicial weak equivalence, we have that

$$Map\,(\mathcal{Y}, \mathcal{X}) \cong Map\,(\text{hocolim}\,\mathcal{Y}'_n, \mathcal{X}) \cong \text{holim}\,Map\,(\mathcal{Y}'_n, \mathcal{X})$$

The same argument works for $Map\,(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X}) \cong \text{holim}\,Map\,(\mathcal{Y}'_n \times \mathbb{A}^1, \mathcal{X})$, as well. Since $\mathcal{Y}'_n$ are objects of the site and holim sends weak equivalences to weak equivalences it suffices to show that $Map\,(\mathcal{Y}'_n, \mathcal{X}) \to Map\,(\mathcal{Y}'_n \times \mathbb{A}^1, \mathcal{X})$ are weak equivalences of simplicial sets. This holds if and only if by applying π_i we get group isomorphisms for $i \geq 1$ and a bijection on π_0. Assuming that $\mathcal{X}$ is a sheaf X, we have that $Map\,(\mathcal{Y}'_n, \mathcal{X})$ is the constant simplicial set, hence this reduces to having a bijection on π_0, but $\pi_0(Map\,(\mathcal{Y}'_n, \mathcal{X})) = \mathsf{Hom}_{\mathcal{H}_s}(\mathcal{Y}'_n, \mathcal{X})$. Therefore, $\mathsf{Hom}_{\mathcal{H}_s}(U, X) \to \mathsf{Hom}_{\mathcal{H}_s}(U \times \mathbb{A}^1, X)$ implies hyperbolicity of X. These sets are the $\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}$ of the same objects, because of the remark in the proof of the second part of the theorem. □

Remark 3.1 If in (6) the sets have a group structure induced (up to homotopy) by a group structure on Y or by a co-group structure (up to homotopy) on X, the bijection is a group isomorphism.

We now use this theorem to compare the notion of Brody hyperbolicity for simplicial sheaves with the classical one for complex spaces.

Lemma 3.2 *Let $X \in \mathcal{S}_T$ and $\mathsf{p} : U \times \mathbb{A}^1 \to U$ be the projection. Then the map*

$$\mathsf{p}^* : \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U, X) \to \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U \times \mathbb{A}^1, X)$$

is bijective for every smooth scheme U if and only if

$$\mathsf{p}^*_{\Bbbk(u)} : \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(Spec\ \Bbbk(u), X) \to \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{A}^1_{k(u)}, X) \tag{7}$$

is, for every finite field extension $Spec\ L \to Spec\ \Bbbk$, $\mathsf{p}_L : \mathbb{A}^1_L \to Spec\ L$ being the projection.

Proof The complex analytic case is trivial. In the algebraic site we have just to prove that the bijectivity of $\mathsf{p}_{\mathsf{L}}^*$ for every L finite extension of $\Bbbk$ implies the bijectivity of p^* for every smooth scheme U. The morphism $\mathsf{p} : U \times \mathbb{A}^1 \to U$ is a faithfully flat covering, thus, by faithfully flat descent we have the following exact sequence of sets

$$0 \to \mathsf{Hom}(U, X) \xrightarrow{\mathsf{p}^*} \mathsf{Hom}(U \times \mathbb{A}^1, X) \underset{p_2^*}{\overset{\mathsf{p}_1^*}{\rightrightarrows}} \mathsf{Hom}((U \times \mathbb{A}^1) \times_U (U \times \mathbb{A}^1), X).$$

In order to prove the surjectivity of p^*, we have to show that $\mathsf{p}_1^* = \mathsf{p}_2^*$. Notice that

$$(U \times \mathbb{A}^1) \times_U (U \times \mathbb{A}^1) = U \times \mathbb{A}^2$$

and p_1^* and p_2^* are induced by the projections on the factors of $\mathbb{A}^2$ to $\mathbb{A}^1$. Thus, given $\alpha \in \mathsf{Hom}(U \times \mathbb{A}^1, X)$, we prove that

$$\alpha \circ \mathsf{p}_1 = \alpha \circ \mathsf{p}_2 : U \times \mathbb{A}^2 \to X.$$

By hypothesis, any map $\mathbb{A}^1_L \to X$ factors through $Spec\ L$ for any finite extension L/k. In particular, $\alpha \circ \mathsf{p}_1$ and $\alpha \circ \mathsf{p}_2$ coincide on the closed points of $U \times \mathbb{A}^2$. Since the union of all closed points of $U \times \mathbb{A}^2$ is an everywhere dense subset for the Zariski topology, we conclude that $\alpha \circ \mathsf{p}_1 = \alpha \circ \mathsf{p}_2$. □

Corollary 3.2 *Let X be a compact complex space. Then X is Kobayashi hyperbolic if and only if it is hyperbolic according to the definition of Sect. 3.1.*

Proof Consequence of Theorem 3.1, Lemma 3.2 and Brody's Theorem. □

Corollary 3.3 *Let X be a complex space, C a closed complex subspace of X. Then X is hyperbolic modulo C in the sense of Brody if and only if X/C is a hyperbolic sheaf according to the definition of Sect. 3.1.*

Proof Let $\mathcal{S}_T$ be the site of complex spaces. By definition, the sheaf of $\mathcal{S}_T$ given by $Y \rightsquigarrow \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(Y, X/C)$ is the associated sheaf for the strong topology to the presheaf, which associates with a complex space Y the co-limit of

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{S}}(Y, C) \arrow[r] \arrow[d] & \mathsf{Hom}_{\mathcal{S}}(Y, X) \\ \mathsf{Hom}_{\mathcal{S}}(Y, \mathsf{pt}). & \end{tikzcd}$$

If X/C is a hyperbolic sheaf, then, by Theorem 3.1, we obtain that the morphism

$$\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathsf{pt}, X/C) \to \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{C}, X/C)$$

is a bijection. Assume, by contradiction, that there exists a nonconstant holomorphic map $f : \mathbb{C} \to X$ such that $f(\mathbb{C}) \not\subset C$. Then f represents an element in $\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{C}, X/C)$

which is not in the image of $\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathsf{pt}, X/C)$ which is absurd. Conversely, if X is Brody-hyperbolic modulo C, one has

$$\mathsf{Hom}_{\mathcal{S}}(\mathbb{C}, X) = (X \backslash C) \amalg \mathsf{Hom}_{\mathcal{S}}(\mathbb{C}, C).$$

On the other hand, we observe that $\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{C}, X/C)$ is precisely equal to the co-limit of

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{S}}(\mathbb{C}, C) \arrow[r] \arrow[d] & \mathsf{Hom}_{\mathcal{S}}(\mathbb{C}, X) \\ \mathsf{Hom}_{\mathcal{S}}(\mathbb{C}, \mathsf{pt}). & \end{tikzcd}$$

This follows from the fact that the new sections that we would get by taking the associated sheaf are of the form (f, g) where $f : U \to X$, $g : V \to X$ are holomorphic maps, $\{U,\ V\}$ is an open covering of $\mathbb{C}$ (we may assume both U and V to be connected) and $f(U \cap V)$, $g(U \cap V)$ are contained in C. In this situation, we have that both $f(U)$ and $f(V)$ are contained in C, as well. Therefore, $(f, g) = (U \to \mathsf{pt}, V \to \mathsf{pt}) = \mathbb{C} \to \mathsf{pt}$ and we already have this section in $\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{C}, X/C)$. Consequently,

$$\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{C}, X/C) = (X - C) \amalg \mathsf{Hom}_{\mathcal{S}}(\mathbb{C}, \mathsf{pt})$$

but the latter set is $\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathsf{pt}, X/C)$, thus X/C is a hyperbolic sheaf by Theorem 3.1. □

Let us discuss some examples of hyperbolic resolutions of complex spaces. Roughly speaking, $\mathfrak{Ip}(X)$ "enlarges" X by adding a simplicial structure which trivializes passing from $\mathcal{H}_s$ to $\mathcal{H}$. If X is a Brody hyperbolic complex space, $\mathfrak{Ip}(X)$ is isomorphic to X in the category $\mathcal{H}_s$. If X is not Brody hyperbolic the simplicial structures added to $\mathfrak{Ip}(X)$ have the task to "make constant" (up to simplicial homotopy, hence in $\mathcal{H}_s$) all morphisms $\mathbb{C} \to X$. Passing from $\mathcal{H}_s$ to $\mathcal{H}$, X and $\mathfrak{Ip}(X)$ become isomorphic objects.

(1) $\mathfrak{Ip}(\mathbb{C})$ is a simplicial sheaf isomorphic to a point in $\mathcal{H}$. Indeed, $\mathbb{C} \cong \mathsf{pt}$ in $\mathcal{H}$ and the hyperbolic resolutions preserve affine equivalences. This fact is not surprising because if we want to make all morphisms $\mathbb{C} \to \mathbb{C}$ homotopically constant, in particular this must be true for the identity $\mathbb{C} \to \mathbb{C}$. For the same reason, $\mathfrak{Ip}(\mathbb{C}^n) \cong \mathsf{pt}$ in $\mathcal{H}$ for every $n \in \mathbb{N}$.
(2) More generally, if $p : V \to X$ is a vector bundle, $\mathfrak{Ip}(p) : \mathfrak{Ip}(V) \xrightarrow{\sim} \mathfrak{Ip}(X)$ in $\mathcal{H}_s$ because p is a $\mathbb{C}$-weak equivalence. Therefore, if X is a hyperbolic complex space, then $\mathfrak{Ip}(V) \cong X$ in $\mathcal{H}_s$ and hence in $\mathcal{H}$.
(3) If X is a complex space and $\mathfrak{Ip}(X)$ is represented by a hyperbolic complex space Y, then Y is unique up to isomorphisms (cfr. the lemma below). In general, this is not the case; e.g. in the next section we will show that $\mathfrak{Ip}(\mathbb{CP}^n)$ cannot be $\mathbb{C}$-equivalent to a hyperbolic complex space.

In the case $\mathfrak{Ip}(\mathcal{X})$ admits a hyperbolic complex space as representative, then such a space is unique up to biholomorphism:

Lemma 3.3 *Let $\mathcal{X}$ be a simplicial sheaf, Y, Y' hyperbolic complex spaces such that*

$$\mathfrak{Ip}(\mathcal{X}) = [Y]_{\mathcal{H}} = [Y']_{\mathcal{H}}.$$

Then Y' and Y are isomorphic complex spaces.

Proof Let $\mathcal{S}$ be the category of complex spaces. By hypothesis, there exists an isomorphism $\phi : Y \cong Y'$ in $\mathcal{H}$, namely a morphism $\psi : Y' \to Y$ in $\mathcal{H}$ such that $\psi \circ \phi = \mathsf{id}_Y$ and $\phi \circ \psi = \mathsf{id}_{Y'}$ in $\mathcal{H}$. Since Y, Y' are complex hyperbolic spaces, and in particular $\mathbb{C}$-fibrant objects by Corollary 3.2 (see also the end of Sect. 2), ϕ and ψ can be represented by morphisms $\phi' : Y \to Y'$ and $\psi' : Y' \to Y$ in $\Delta^{\mathsf{op}}\mathcal{F}_T(\mathcal{S})$. More precisely, we may suppose that ϕ' and ψ' are holomorphic maps, Y, Y' being complex spaces and $\mathcal{S} \hookrightarrow \Delta^{\mathsf{op}}\mathcal{F}_T(\mathcal{S})$ being a full immersion. Moreover, the fact that ϕ, ψ are inverse to each other means that $\psi' \circ \phi' \sim \mathsf{id}_Y, \phi' \circ \psi' \sim \mathsf{id}_{Y'}$ as holomorphic maps, where $f \sim g$ if and only if there exists a holomorphic map $H : W \times \mathbb{C} \to V$ such that $H|_{W\times 0} = f$ e $H|_{W\times 1} = g$. Since both Y, Y' are hyperbolic, H must be constant along the fibers which are isomorphic to $\mathbb{C}$, thus $f \sim g$ if and only if $f = g$ as maps. In particular, $\psi' \circ \phi' = \mathsf{id}_Y$ and $\phi' \circ \psi' = \mathsf{id}_{Y'}$. □

In some cases, we can extend some results known for hyperbolic complex spaces to hyperbolic sheaves:

Proposition 3.1 *Let $\mathcal{F}_T(\mathcal{S})$ be the category of sheaves of sets on the site of complex spaces with the strong topology and F be a hyperbolic sheaf. Then*

$$\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{CP}^n, F) = F(\mathsf{pt})$$

for any $n \geq 1$. In other words, any sheaf map from $\mathbb{CP}^n$ to a hyperbolic sheaf F must be constant.

Proof Consider the case $n = 1$ first. Let $\mathbb{CP}^1 = U_0 \cup U_1$ be an open covering with $U_0 = \mathbb{CP}^1\backslash\{0\}$ and $U_1 = \mathbb{CP}^1\backslash\{\infty\}$. Then the square

$$\begin{tikzcd} U_0 \cap U_1 \arrow[r, hook, "i_0"] \arrow[d, hook, "i_1"] & U_0 \arrow[d] \\ U_1 \arrow[r] & \mathbb{CP}^1 \end{tikzcd} \tag{8}$$

is co-Cartesian in the category of sheaves. Thus

$$\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{CP}^1, F) = \lim \left(\begin{tikzcd} \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U_0 \cap U_1, F) & \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U_0, F) \arrow[l, "i_0^*"'] \\ \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U_1, F) \arrow[u, "i_1^*"] & \end{tikzcd} \right). \tag{9}$$

Since $U_0 \cong U_1 \cong \mathbb{C}$, we have that

$$\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U_j, F) = \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathsf{pt}, F) = F(\mathsf{pt})$$

for $j = 0, 1$ because of the Theorem 3.1. Moreover, i_j^* are injective because they have a retraction given by f^* where $f : \mathsf{pt} \to U_0 \cap U_1$ is any point. We conclude the statement of the lemma in the case of $\mathbb{CP}^1$ by noticing that the image of i_0^* coincides with the one of i_1^*. Consider now the open covering of $\mathbb{CP}^n$ given by $U_0 = \mathbb{CP}^n\backslash\mathbb{CP}^{n-1} \cong \mathbb{C}^n$ and $U_1 = \mathbb{CP}^n\backslash\{\infty\}$, where ∞ coincides with the point $(0, 0, \ldots, 0) \in U_0 = \mathbb{C}^n$. We get a co-Cartesian square like (8) with $\mathbb{CP}^n$ replacing $\mathbb{CP}^1$. The previous argument carries through in the general case. The only thing to check is that $\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U_1, F) = F(\mathsf{pt})$. Notice that the canonical projection $p : U_1 \to \mathbb{CP}^{n-1}$ is a rank one vector bundle. Locally on $\mathbb{CP}^{n-1}$ (for the strong topology) it is $V \times \mathbb{C}$, where V is an open affine of $\mathbb{P}^{n-1}$. Hence

$$p_V^* : \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(V, F) \to \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(V \times \mathbb{C}, F) \tag{10}$$

are bijections for all V, since F is hyperbolic. Glueing these data for V ranging on an open affine covering of $\mathbb{CP}^{n-1}$, we get that

$$p^* : \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{CP}^{n-1}, F) \to \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U_1, F) \tag{11}$$

is a bijection. By inductive assumption, we conclude that

$$\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(U_1, F) = F(\mathsf{pt}).$$

□

4 Holotopy groups

Throughout this section, $\mathcal{S}_T$ will denote the site of complex spaces endowed with the strong topology. A simplicial object of $\mathcal{S}_T$ is, by definition, a *simplicial complex space*. If we forget the complex structure, we could study the objects of $\mathcal{S}_T$ by means of the classical homotopy groups. Isomorphism classes of homotopy groups are invariant under homeomorphisms hence, a fortiori, under biholomorphisms, however, they do not reflect the existence and the properties of the complex structure. A rather natural modification of the definition of homotopy enables us to attach to every simplicial sheaf on $\mathcal{S}_T$ two families $\{\pi_{i,j}^{\mathsf{par}}(\mathcal{X})\}_{i,j}$, $\{\pi_{n,m}^{\mathsf{iper}}(z_1, z_2)(\mathcal{X})\}_{m,n}$ of sets [see (12), (13)] which, for positive simplicial degrees, have a canonical group structure and are invariant under biholomorphisms. We will use these groups in Sect. 6 to show that there exist complex spaces (e.g $\mathbb{CP}^n$) whose hyperbolic resolutions (see Sect. 3.1) are not isomorphic to the class of hyperbolic complex spaces, not even in the category $\mathcal{H}$.

Define the *parabolic circle* by

$$S^1_{\mathsf{par}} = \mathbb{C}/(0 \amalg 1),$$

and we denote by S^n_{par} the sheaf $S^1_{\mathsf{par}} \wedge \overset{n}{\cdots} \wedge S^1_{\mathsf{par}}$.

Let $D \subset \mathbb{C}$ be the unit disc and $z_1 \neq z_2$ two points of D. We define the *hyperbolic circle* $S^1_{\mathsf{iper}}(z_1, z_2)$ by

$$S^1_{\mathsf{iper}}(z_1, z_2) = D/(z_1 \amalg z_2)$$

and we denote by $S^n_{\mathsf{iper}}(z_1, z_2)$ the sheaf $S^1_{\mathsf{iper}}(z_1, z_2) \wedge \overset{n}{\cdots} \wedge S^1_{\mathsf{iper}}(z_1, z_2)$.

The quotients defining parabolic and hyperbolic circles are taken in the category $\mathcal{F}_T(\mathcal{S})$, even though the set theoretic quotients have a complex structure.

Let $\mathcal{X}$ be a simplicial sheaf on $\mathcal{S}_T$. Define

$$\pi_{i,j}^{\mathsf{par}}(\mathcal{X}, x) = \mathsf{Hom}_{\mathcal{H}_\bullet}((\mathbb{C}^*)^{\wedge j} \wedge S^{i-j}_{\mathsf{par}}, (\mathcal{X}, x)) \tag{12}$$

for $i \geq j \geq 0$,

$$\pi_{n,m}^{\mathsf{iper}}(z_1, z_2)(\mathcal{X}, x) = \mathsf{Hom}_{\mathcal{H}_\bullet}(S^n_{\mathsf{iper}}(z_1, z_2) \wedge S^m_{\mathsf{par}}, (\mathcal{X}, x)) \tag{13}$$

for $n, m \geq 0$.

These sets are called respectively *parabolic holotopy pointed sets* of $\mathcal{X}$ (or groups in the case they are) of bidegree (i, j) and *hyperbolic holotopy pointed sets* of $\mathcal{X}$ (or groups in the case they are) of bidegree (n, m).

Remark 4.1 The definitions above are compatible with the classical ones of algebraic topology. More precisely, let $\mathcal{H}^{\text{top}}$ be the (unstable) homotopy category of topological spaces (i.e. the localization of the category of topological spaces with respect to the usual weak equivalences); then we have $\pi_n(X, x) = \mathsf{Hom}_{\mathcal{H}^{\text{top}}}((S^n, p), (X, x))$ for every topological space X. Moreover, the topological realization functor (cfr. Sect. 5) provides functorial group homomorphisms

$$\pi_{i,j}^{\mathsf{par}}(X, x) \to \pi_i(X, x), \quad \pi_{n,m}^{\mathsf{iper}}(z_1, z_2)(\mathcal{X}, x) \to \pi_{n+m}(X, x)$$

for any complex space X.

Lemma 4.1 *The sets $\pi_{i,j}^{\mathsf{par}}$, $\pi_{n,m}^{\mathsf{iper}}$ have a canonical group structure for $i > j > 0$ and $m > 0$.*

Proof The first step consists in proving that $S^1_{\mathsf{par}} \cong S^1_s$ in $\mathcal{H}$.

Consider the co-fibration sequence

$$0 \amalg 1 \hookrightarrow \mathbb{C} \to S^1_{\mathsf{par}} \to S^1_s \to \mathbb{C} \wedge S^1_s \to \cdots \tag{14}$$

where $0 \amalg 1$ and $\mathbb{C}$ are pointed by 0. Since $\mathbb{C} \cong \mathsf{pt}$ in $\mathcal{H}$, we have $\mathbb{C} \wedge S^1_s \cong \mathsf{pt}$ in $\mathcal{H}$. Applying the functor $\mathsf{Hom}_{\mathcal{H}_\bullet}(\ \ , \mathcal{Z})$, in view of (3), we obtain long exact sequences of sets and, from these, the isomorphism

$$\mathsf{Hom}_{\mathcal{H}_\bullet}(S^1_{\mathsf{par}}, \mathcal{Z}) \cong \mathsf{Hom}_{\mathcal{H}_\bullet}(S^1_s, \mathcal{Z})$$

for every $\mathcal{Z} \in \Delta^{\mathsf{op}}_\bullet \mathcal{F}_T(\mathcal{S})$. It follows that $S^1_{\mathsf{par}} \cong S^1_s$ in $\mathcal{H}$. The simplicial object S^1_s is a co-group (object) in $\mathcal{H}_s$ (and consequently in $\mathcal{H}$). It is sufficient to observe that, if a_{str} is the associated sheaf for the strong topology, $S^1_s \cong a_{\mathsf{str}}(\mathsf{Sing}(S^1))$ in $\mathcal{H}_s$ and that S^1 is a co-group in $\mathcal{H}^{\text{top}}$ with projection

$$\mathsf{p} : S^1 \to S^1/(\{i\} \amalg \{-i\}) \overset{\mathsf{homeo}}{\cong} S^1 \vee S^1$$

as structural map. Then, applying to p the functor $a_{\mathsf{str}}(\mathsf{Sing}(\ -\ ))$ we get a morphism

$$[S^1_s] \to [S^1_s \vee S^1_s] = [S^1_s] \vee [S^1_s]$$

in $\mathcal{H}_s$ which satisfies the properties making it a co-multiplication. These properties are formulated in such a way to induce on the sets $\mathsf{Hom}_{\mathcal{H}_s}(S^1_s, \mathcal{Z})$ a natural group structure. The same holds for $\mathsf{Hom}_{\mathcal{H}}(S^1_s, \mathcal{Z})$. □

Theorem 4.1 *Let X be a hyperbolic sheaf. Then the groups $\pi_{i,j}^{\mathsf{par}}(X, x)$, $\pi_{n,m}^{\mathsf{iper}}(X, x)$ vanish for $i - j > 0$ and any $m > 0$.*

Proof We begin by proving that $\mathsf{Hom}_{\mathcal{H}_\bullet}(Y \wedge S^1_{\mathsf{par}}, X) = 0$ for every pointed complex space $(Y, \{y\})$. By definition (see Sect. 2.3),

$$Y \wedge S^1_{\mathsf{par}} = (Y \times \mathbb{C})/R$$

where R is the complex space $Y \times (0 \amalg 1) \cup y \times \mathbb{C}$. Since $Y \times \mathbb{C}/R$ is a sheaf and X is a fibrant space, by Theorem 3.1 we conclude that

$$\begin{aligned}\mathsf{Hom}_{\mathcal{H}}((Y \times \mathbb{C})/R, X) &= \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}((Y \times \mathbb{C})/R, X)\\ &= \{f \in \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(Y \times \mathbb{C}, X) : f_{|R} = \mathsf{const}\}.\end{aligned} \tag{15}$$

Moreover, since $Y \times \mathbb{C}$ and X are complex spaces, we have

$$\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}(Y \times \mathbb{C}, X) = \mathsf{Hom}_{\mathsf{olom}}(Y \times \mathbb{C}, X).$$

X is Brody hyperbolic hence, for every $y \in Y$, the restriction of a holomorphic map $f : Y \times \mathbb{C} \to X$ to $y \times \mathbb{C}$ is constant. Furthermore, if $f \in \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})}((Y \times \mathbb{C})/R, X)$ then f is constant on $Y \times \{0\} \subset R$ and consequently on the whole $Y \times \mathbb{C}$. It follows that, if f is pointed, then f must be constant with image x, the base point of X. This shows that $\mathsf{Hom}_{\mathcal{H}_\bullet}(Y \wedge S^1_s, X) = x$ for any pointed complex space Y and any hyperbolic pointed sheaf X.

We would like now to prove the same result with Y being replaced by a quotient sheaf $W = Y/Z$. Consider the following commutative diagram

$$\begin{tikzcd} & R \arrow[r] \arrow[d] & \mathsf{pt} \arrow[d] \\ Z \times \mathbb{C} \arrow[r] \arrow[d] & Y \times \mathbb{C} \arrow[r] \arrow[d] & (Y \times \mathbb{C})/R \\ \mathbb{C} \arrow[r] & (Y/Z) \times \mathbb{C} & \end{tikzcd} \tag{16}$$

where the two squares are co-Cartesian. Consider now the two new co-Cartesian squares

$$\begin{tikzcd} Z \times \mathbb{C} \arrow[r] \arrow[d] & (Y \times \mathbb{C})/R \arrow[d] \\ \mathbb{C} \arrow[r] & P \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r] \arrow[d] & \mathsf{pt} \arrow[d] \\ (Y/Z) \times \mathbb{C} \arrow[r] & (W \times \mathbb{C})/R = W \wedge S^1_{\mathsf{par}}. \end{tikzcd} \tag{17}$$

By chasing the diagram (16) and using that $(Y/Z) \times \mathbb{C}$ and $(Y \times \mathbb{C})/R$ are co-limits of the relevant diagrams, we find two sheaf maps $P \to W \wedge S^1_{\mathsf{par}}$ and $W \wedge S^1_{\mathsf{par}} \to P$ that are mutually inverses. By definition of P we have

$$\begin{aligned} \mathsf{Hom}_{\mathcal{H}_\bullet}(W \wedge S^1_{\mathsf{par}}, X) &= \mathsf{Hom}_{\mathcal{H}_\bullet}(P, X) = \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})_\bullet}(P, X) \\ &= \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})_\bullet}((Y \times \mathbb{C})/R, X) \times_{\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})_\bullet}(Z \times \mathbb{C}, X)} \mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})_\bullet}(\mathbb{C}, X). \end{aligned} \tag{18}$$

Recall that $(Y \times \mathbb{C})/R = Y \wedge S^1_{\mathsf{par}}$, thus, by the first part of the proof of the proposition, $\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})_\bullet}((Y \times \mathbb{C})/R, X) = x$, the base point of X. The same holds for $\mathsf{Hom}_{\mathcal{F}_T(\mathcal{S})_\bullet}(\mathbb{C}, X)$ because, by assumption, X is Brody hyperbolic. Therefore,

$$\mathsf{Hom}_{\mathcal{H}_\bullet}(W \wedge S^1_{\mathsf{par}}, X) = x$$

for any quotient sheaf W, and in particular for

$$W = (\mathbb{C}^*)^{\wedge j} \wedge S^{i-j-1}_{\mathsf{par}} \quad \text{and} \quad W = S^n_{\mathsf{iper}}(z_1, z_2) \wedge S^{m-1}_{\mathsf{par}}$$

[see (12), (13)]. □

Corollary 4.1 *Let $\mathcal{X}$ be a simplicial sheaf. Assume that $\pi^{\mathsf{par}}_{i,j}(\mathcal{X}, x) \neq 0$ for $i - j > 0$ or $\pi^{\mathsf{iper}}_{n,m}(z_1, z_2)(\mathcal{X}, x) \neq 0$ for $m > 0$. Then $\mathfrak{Ip}(\mathcal{X})$ is not $\mathbb{C}$-weakly equivalent to a hyperbolic sheaf. In particular, if X is a complex space such that $\pi^{\mathsf{par}}_{i,j}(X, x) \neq 0$ for $i - j > 0$ or $\pi^{\mathsf{iper}}_{n,m}(z_1, z_2)(X, x) \neq 0$ for $m > 0$, then X is not a Brody hyperbolic complex space.*

Proof The proofs for the two cases are similar so we consider only the case of the parabolic holotopy groups. By definition,

$$\pi_{i,j}^{\mathsf{par}}(\mathcal{X},x) = \mathsf{Hom}_{\mathcal{H}_\bullet}((\mathbb{C}^*)^{\wedge j} \wedge S_s^{i-j}, (\mathcal{X},x))$$

and this set is a quotient of

$$\mathsf{Hom}_{\Delta_\bullet^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})}((\mathbb{C}^*)^{\wedge j} \wedge S_s^{i-j}, (\widetilde{\mathcal{X}}, \widetilde{x}))$$

where $(\widetilde{\mathcal{X}}, \widetilde{x})$ is a $\mathbb{C}$-fibrant pointed simplicial sheaf $\mathbb{C}$-weakly equivalent to $(\mathcal{X}, x)$. In particular, we may assume that $\widetilde{\mathcal{X}}$ is the hyperbolic resolution $\mathfrak{Jp}(\mathcal{X})$ of $\mathcal{X}$. If $\mathfrak{Jp}(\mathcal{X})$ were $\mathbb{C}$-weakly equivalent to a Brody hyperbolic complex space X', then $\pi_{i,j}^{\mathsf{par}}(\mathcal{X},x)$ would be a quotient of

$$\mathsf{Hom}_{\Delta_\bullet^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})}((\mathbb{C}^*)^{\wedge j} \wedge S_s^{i-j}, (X', x'))$$

which for $i - j > 0$ consists only of the constant map with value x (see Theorem 4.1). □

Remark 4.2 As mentioned in Sect. 2, to relate holotopy groups of a complex space X with morphisms in $\Delta^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})$ it is necessary to replace X with its hyperbolic model $\mathfrak{Jp}(X)$. Then we know that $\pi_{i,j}(X,x)$ will be a quotient of the set $\mathsf{Hom}_{\Delta_\bullet^{\mathsf{op}}\boldsymbol{\mathcal{F}}_T(\mathcal{S})}(\mathbf{S}^{i,j}, \mathfrak{Jp}(X))$, where $\mathbf{S}^{i,j}$ is a pointed model of the relevant sphere.

5 The topological realization functor

From now on, $\mathbb{CP}^n$ will denote the complex projective space seen as topological space. We would like to compare objects in $\mathcal{H}$ with the topological spaces, objects of the topological (unstable) homotopy category $\mathcal{H}^{\mathrm{top}}$. We will show that there exists a functor $t^{olo} : \mathcal{H} \to \mathcal{H}^{\mathrm{top}}$ which extends the functor, which associates the underlying topological space with a complex space. In the algebraic case extends the corresponding functor which associates with an algebraic variety over $\mathbb{C}$, the topological space of its (Zariski) closed points. The general case only applies to the site of smooth varieties over a field $\Bbbk$ which admits an embedding i in $\mathbb{C}$. It involves passing from a simplicial sheaf over $\Bbbk$ to a simplicial sheaf over $\mathbb{C}$ by means of i^* (or, more precisely, by means of its total left derived functor). Recall that for a sheaf F and a morphism of sites $\phi : \mathfrak{S}_1 \to \mathfrak{S}_2$, the sheaf $\phi^* F$ on $\mathfrak{S}_1$ is defined as the associated sheaf to the presheaf whose sections are $(\phi^* F)(U) = \mathsf{colim}_V F(V)$, where the co-limit is taken over all the morphisms $U \to \phi^{-1} V$ for $U \in \mathfrak{S}_1$ and any $V \in \mathfrak{S}_2$.

Let $(\mathfrak{S}, I)$ be a site with interval [8, Sect. 2.3] equipped with a realization functor $r : \mathfrak{S} \to Top$ to the category of topological spaces. Denote by $\mathcal{H}(\mathfrak{S})$ the I-homotopy category whose objects are simplicial sheaves over $\mathfrak{S}$. Then a functor $t_r : \mathcal{H}(\mathfrak{S}) \to \mathcal{H}^{\mathrm{top}}$ with values in the unstable homotopy category of topological spaces is called a *topological realization functor* if the following properties are satisfied:

a) if $X \in \Delta^{\mathsf{op}}\mathcal{F}(\mathfrak{S})$ is a simplicial set, then the class $t_r(X)$ can be represented by the geometric realization $|X|$;
b) if F is the sheaf $\mathsf{Hom}_{\mathfrak{S}}(\ , X)$, where $X \in \mathfrak{S}$, then $t_r(F)$ can be represented by $r(X)$;
c) t_r commutes with direct products and homotopy co-limits.

Theorem 5.1 *The sites with interval* $((\mathsf{An}/\mathbb{C})_{str}, \mathbb{C})$ *and* $((\mathsf{Sch}/\Bbbk)_T, \mathbb{A}^1_{\Bbbk})$ *admit a topological realization functor, provided that* $\Bbbk$ *can be embedded in* $\mathbb{C}$ *and* T *is not finer than the flat topology. In general, in the algebraic case, this functor will depend on the embedding of* $\Bbbk$ *in* $\mathbb{C}$.

Proof Let $\phi : (\mathfrak{S}_1, I_1) \to (\mathfrak{S}_2, I_2)$ be a reasonable continuous map of sites with interval (cfr. Definition 1.49 [8]). Consider the functor $\phi^* : \Delta^{\mathsf{op}}\mathcal{F}(\mathfrak{S}_2) \to \Delta^{\mathsf{op}}\mathcal{F}(\mathfrak{S}_1)$ obtained by applying the inverse image functor on each component of the simplicial sheaf on $\mathfrak{S}_2$. A classical result in model categories assures the existence of the total left derivative between homotopy categories of a functor, provided that such a functor sends weak equivalences between co-fibrant objects to weak equivalences. In the case of ϕ^*, we will not be able to prove this for every simplicial sheaf on $\mathfrak{S}_2$ and the relevant I model categories. However, we can get the same result in the following way. We consider the full category of I_2 local objects $\mathcal{H}_{s,I_2} \subset \mathcal{H}_s$ introduced in the Sect. 3.1, which is equivalent to the I_2 homotopy category $\mathcal{H}(\mathfrak{S}_2, I_2)$. Such a category has the property that a morphism is an I_2 weak equivalence if and only if it is a simplicial weak equivalence. Thus, to show that ϕ^* admits a total left derived functor between the I homotopy categories, it is sufficient to show that ϕ^* sends simplicial weak equivalences between I_2 local objects (since every object is co-fibrant) to simplicial weak equivalences. Actually, since the property for a simplicial sheaf $\mathcal{X}$ to be I_2 local is invariant under simplicial weak equivalences on $\mathcal{X}$ (see Sect. 2.4) to validate the same conclusion it suffices to show a weaker condition: there exists a (simplicial) resolution functor Φ and a natural transformation $\Phi \to \mathsf{id}$ with the property that ϕ^* sends simplicial weak equivalences between simplicial shaves of the kind $\Phi(\mathcal{X}) \to \Phi(\mathcal{Y})$ to simplicial weak equivalences for all I_2 local simplicial sheaves $\mathcal{X}$ and $\mathcal{Y}$. But this is precisely the statement of [8, Proposition 1.57.2] where Φ is taken to be Φ_Σ introduced in [8, Lemma 1.16]. This shows the existence of the total left derived functor $\mathbb{L}\phi^* : \mathcal{H}(\mathfrak{S}_2, I_2) \to \mathcal{H}(\mathfrak{S}_1, I_1)$ of ϕ^*. Explicitly, it is defined as follows: let $\mathcal{X}$ be a simplicial sheaf over $\mathfrak{S}_2$, then $\mathbb{L}\phi^*(\mathcal{X})$ is represented by the simplicial sheaf $\phi^*(\Phi_\Sigma(\mathfrak{Jp}(\mathcal{X})))$, where $\mathfrak{Jp}(\mathcal{X})$ is the I_2 local simplicial sheaf mentioned in Sect. 3.1. This definition is well posed on $\mathcal{H}(\mathfrak{S}_2, I_2)$ because of the above remarks and the fact that, if $\mathcal{X}$ and $\mathcal{X}'$ represent the same class in $\mathcal{H}(\mathfrak{S}_2, I_2)$, then $\mathfrak{Jp}(\mathcal{X})$ and $\mathfrak{Jp}(\mathcal{X}')$ are simplicially weak equivalent. □

We will now consider the case of the site with interval $(\mathsf{An}/\mathbb{C}, \mathbb{C})$, the algebraic case over $\mathbb{C}$ being the same. For the general algebraic case of a simplicial sheaf on the site of schemes over a field $\mathbb{k}$ embedded in the complex numbers, the topological realization functor involves first base changing the object over $\mathbb{C}$, by means of the chosen field embedding, and then realizing over $\mathbb{C}$.

We set the realization functor $r : \mathfrak{S} \to Top$ to be the one which associates the underlying topological space X^{top} with a complex space X. Let pt be the site with interval whose only nonempty object is the final object pt and ψ be the trivial morphism of sites with interval $\mathsf{pt} \to \mathsf{An}/\mathbb{C}$. Notice that a simplicial sheaf on pt is just a simplicial set. We take the interval I in pt to be the constant simplicial set $\mathsf{Hom}(\mathsf{pt}, \mathbb{C})$. The functor ψ^* sends a simplicial sheaf $\mathcal{X}$ on $\mathsf{An}/\mathbb{C}$ to the simplicial set $\mathcal{X}(\mathsf{pt})$. Thus, $\psi^*(\mathbb{C}) = I$ so that, in particular, it is I contractible. Because of this, the functor ψ is said to be a *reasonable* continuous map of sites with interval (cfr. [8, Definition 3.16]) and $\mathbb{L}\psi^*$ has a particularly nice description: $\mathbb{L}\psi^*(\mathcal{X})$ is represented by the simplicial sheaf $\psi^*\Phi_\Sigma(\mathcal{X})$ where Σ is the class of representable sheaves on $\mathsf{An}/\mathbb{C}$ (cfr. [8, Lemma 3.15]). Let Tlc_{open} be the category of locally contractible topological spaces. We now endow the images of $\mathbb{L}\psi^*$ by a structure of topological spaces in order to obtain a functor $\mathcal{H}(\Delta^{\mathsf{op}}\mathcal{F}(\mathsf{An}/\mathbb{C})) \to \mathcal{H}(\Delta^{\mathsf{op}}Tlc_{\mathrm{open}})$. If $\mathcal{Y}$ is a simplicial sheaf that in each degree is a disjoint union of representable sheaves $\amalg_{j\in J} Y_j$, then we set $\theta\mathcal{Y} := \mathcal{Y}^{\mathrm{top}}$, where $\mathcal{Y}^{\mathrm{top}}$ is the simplicial topological space having the topological space $\amalg_{j\in J}\mathcal{Y}_j^{\mathrm{top}}$ in the corresponding degree. Since ψ^* is reasonable, $\mathbb{L}\psi^*(\mathcal{X}) = [\psi^*\mathcal{Y}]$ where $\mathcal{Y}$ is any simplicial sheaf as above, equipped with a simplicial weak equivalence $\mathcal{Y} \to \mathcal{X}$. Any two such models will give rise to simplicially weak equivalent inverse images by [8, Proposition 1.57.2], thus, in particular, I weak equivalent. This shows that the definition of θ induces a functor

$$\mathcal{H} = \mathcal{H}(\Delta^{\mathrm{op}}\mathcal{F}(\mathsf{An}/\mathbb{C})) \to \mathcal{H}(\Delta^{\mathrm{op}}Tlc_{\mathrm{open}})$$

which we will call θ, as well.

Remark 5.1 $\mathcal{H}(\Delta^{\mathrm{op}}Tlc_{\mathrm{open}})$ is a full subcategory of $\mathcal{H}(\Delta^{\mathrm{op}}\mathcal{F}_{\mathrm{open}}(Tlc_{\mathrm{open}}))$. The latter category is the I homotopy category taking as interval the sheaf $I = \mathsf{Hom}_{\mathrm{cont}}(\ ,\mathbb{C})$. Such an interval is an object of $\Delta^{\mathrm{op}}Tlc_{\mathrm{open}}$, thus we can see $\mathcal{H}(\Delta^{\mathrm{op}}Tlc_{\mathrm{open}})$ as the localized category with respect to the $I = \mathbb{C}$-weak equivalences, considering $\mathbb{C}$ as constant simplicial topological space and no longer only as constant simplicial set.

Proposition 5.1 *There is an equivalence of categories $\gamma : \mathcal{H}(\Delta^{\mathrm{op}}Tlc_{\mathrm{open}}) \cong \mathcal{H}^{\mathrm{top}}$.*

Proof (sketch) It is a particular case of [8, Proposition 3.3]. Here we write the definition of the functor $\gamma : \mathcal{H}(\Delta^{\mathrm{op}}Tlc_{\mathrm{open}}) \to \mathcal{H}^{\mathrm{top}}$ which gives the equivalence of categories. Let $\mathcal{X}$ be a simplicial locally contractible topological space. Since for any topological space Z in Tlc_{open} there is an open covering $\amalg_i U_i \to Z$, with U_i contractible for all i, by [8, Lemma 3.15], $\mathcal{X}$ admits a (simplicial) weak equivalence $\widetilde{\mathcal{X}} \to \mathcal{X}$ with $\widetilde{\mathcal{X}}_j = \amalg_{i_j} U_{i_j}$. In turn, $\widetilde{\mathcal{X}}$ is I-weakly equivalent to $\mathcal{X}'$, where $\mathcal{X}'$ is the *simplicial set* with $\mathcal{X}'_j = \amalg_{i_j}\mathsf{pt}$, because $\widetilde{\mathcal{X}}$ and $\mathcal{X}'$ are termwise weakly equivalent and of [8, Proposition 2.14]. The equivalence of categories is defined as $[\mathcal{X}] \rightsquigarrow [|\mathcal{X}'|]$ where $|\mathcal{X}'|$ is the geometric realization of the simplicial set $\mathcal{X}'$. □

Remark 5.2 If X is a topological space in Tlc_{open}, then $|\mathcal{X}'|$ is weakly equivalent to $|Sing_\bullet(X)|$. But this topological space is weakly equivalent to X itself, thus the constant simplicial topological space X is sent by γ to a topological space weakly equivalent to X in the classical sense of homotopy theory.

Let $\mathbf{D}$ be a small category and $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})^{\mathbf{D}}$ be the category of functors from $\mathbf{D}$ to $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$. We will denote by hocolim$(\mathbf{D})$ a *homotopy co-limit* of $\mathbf{D}$ on the category $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$. That is a pair (k, a) consisting in a functor $k : \Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})^{\mathbf{D}} \to \Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$ which takes objectwise weak equivalences in $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})^{\mathbf{D}}$ to weak equivalences in $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$ and a natural transformation $a : k \to \mathrm{colim}_{\mathbf{D}}$. Such functor can be obtained by first taking a suitable co-fibrant diagram replacement of an element in $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})^{\mathbf{D}}$ and composed with the ordinary co-limit functor (cfr. [2]).

We set $t^{olo} : \mathcal{H} \to \mathcal{H}^{\mathrm{top}}$ to be the functor $\gamma \circ \theta$. Property a) follows by definition of γ. Property b) is a consequence of Remark 5.2. As for the property c), we have that ψ^* commutes with limits by definition. Since direct products in the homotopy categories are represented by direct products of objects, we have that t^{olo} commutes with direct products. ψ^* has a right adjoint, namely ψ_*, thus it is right exact. Moreover, ψ^*f sends co-fibrations (sectionwise injections) to co-fibrations. On the other hand, the same holds for the resolution functor of [8, Lemma 3.15] Φ: if i is a sectionwise injection, then $\Phi(i)$ is a sectionwise injection by definition of Φ; furthermore, Φ commutes with co-limits, since its value on objects has been defined as a co-limit. In particular, if $\mathbf{D}$ is a co-fibrant diagram in $\Delta^{\mathrm{op}}\mathcal{F}(\mathsf{An}/\mathbb{C})$, $\Phi(\mathbf{D})$ is co-fibrant and we conclude that $\psi^*\Phi(\mathbf{D})$ is co-fibrant as well and also that $\mathrm{colim}(\psi^*\Phi(\mathbf{D})) \cong \psi^*\Phi^*(\mathrm{colim}(\mathbf{D}))$. This shows that, for any diagram $\mathbf{D}$, $\mathbb{L}\psi^*(\mathrm{hocolim}(\mathbf{D})) = \mathrm{hocolim}(\mathbb{L}\psi^*(\mathbf{D}))$, since the former class can be represented by $\mathrm{colim}(\psi^*\Phi(\mathbf{D}'))$ for any co-fibrant replacement $\mathbf{D}' \xrightarrow{\sim} \mathbf{D}$ because $\psi^*\Phi(\mathbf{D}')$ is a co-fibrant diagram.

Therefore, θ commutes with homotopy co-limits. Recall that the equivalence γ is defined to be the functor that, to a class represented by a simplicial topological space $\mathcal{X}$, associates the class in $\mathcal{H}^{\mathrm{top}}$ represented by $|(\Phi_S(\mathcal{X}))^{\sim}|$ where S is the class of contractible topological spaces and the operation $\sim$ replaces each contractible topological space with a point. Because of the definition of Φ_S we see that $\sim$ sends injections to injections and commutes with co-limits. Before proceeding to investigate the properties of the functor $|\ \ |$, we need to recall the model structures involved in the categories. The functor $|\ \ |$ is defined on the category of simplicial sets and takes values in the category of topological spaces. The model structure for the category of simplicial sets is: let $f : X \to Y$ be a map of simplicial sets, then f is:

(1) a weak equivalence if $|f|$ is a weak homotopy equivalence (see below);
(2) a co-fibration if it is an injection;
(3) a fibration if f has the right lifting property with respect to acyclic co-fibrations.

Let $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \to \cdots$ be a sequential direct system of topological spaces such that for each n, (X_n, X_{n+1}) is a relative CW complex. Then we will say that the canonical function $X_0 \hookrightarrow \operatorname{colim} X_i$ is a *generalized relative CW inclusion*. A continuous function between topological spaces $f : X \to Y$ is

(1) a weak equivalence if $f_* : \pi_*(X, x) \to \pi_*(Y, f(x))$ is a group isomorphism for $* \geq 1$ and a bijection of pointed sets if $* = 0$;
(2) a co-fibration if it is a retract of a generalized relative CW inclusion;
(3) a fibration if it is a Serre fibration.

The functor $|\ \ |$ preserves co-fibrations and also it commutes with co-limits, because it has a right adjoint, namely the functor $\mathsf{Sing}(\ -\ )$. In conclusion, the functor γ commutes with homotopy co-limits, and so does the topological realization functor t^{olo}. □

5.1 Remarks on homotopy co-limits

The practical use of the topological realization functor requires few remarks on the differences between (homotopy) co-limits of diagrams in the category $\mathcal{H}^{\mathrm{top}}$ and the category $\mathcal{H}$. Let us consider the co-limit of the diagram

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, hook] \arrow[d] & \mathbb{C} \\ \mathrm{pt.} & \end{tikzcd} \tag{19}$$

In the category of complex spaces, this is just a point. However, we have previously inferred in this manuscript that the co-limit of such a diagram in the category of sheaves on $\mathsf{An}/\mathbb{C}$ is not (weakly equivalent to) the constant sheaf to a point. Indeed, its class in the respective homotopy categories plays the role of the two-dimensional sphere $S^2 = \mathbb{CP}^1$, or, more precisely, of the sheaf represented by $\mathbb{CP}^1$, whose class is by no means isomorphic to the one of the point. As a diagram of topological spaces, its co-limit is not a point, but it is not an appropriate model for S^2. We should point out that the diagram (19) is a co-fibrant diagram for the affine model structure in the category $\Delta^{\mathsf{op}}\mathcal{F}(\mathsf{An}/\mathbb{C})$, but it is not co-fibrant in the category of topological spaces for the model structure defined above. This apparent oddness disappears if we consider *homotopy* co-limits instead. For instance,

$$t^{olo}(S^n_{\mathsf{par}}) \cong t^{olo}(S^n_{\mathsf{iper}}(z_1, z_2)) \cong S^n$$

for any $n \geq 0$ and $z_i \in D$, $t^{olo}(\mathbb{C}^*) \cong S^1$ and we have natural maps of pointed sets (respectively of groups)

$$\pi^{\mathsf{par}}_{i,j}(\mathcal{X}, x) \to \pi_i(t^{olo}(\mathcal{X}), t^{olo}(x))$$

and

$$\pi^{\mathsf{iper}}_{n,m}(\mathcal{X}, x) \to \pi_{n+m}(t^{olo}(\mathcal{X}), t^{olo}(x)).$$

6 Some applications

In this last section we are going to consider few applications of the theory developed so far. We will begin with examples of complex spaces that are not $\mathbb{C}$-weakly equivalent to any complex hyperbolic space.

We will say that a complex space is *weakly hyperbolic* if is $\mathbb{C}$-weakly equivalent to a Brody hyperbolic complex space.

We recall a preliminary result (cfr. [8, Lemma 2.15]):

Lemma 6.1 *The pointed simplicial sheaf* $(\mathbb{C}^*) \wedge S^1_{\mathsf{par}}$ *is canonically weakly equivalent to* $\mathbb{CP}^1$.

Proof Consider the diagram $\mathbf{D}$

$$\begin{tikzcd} (\mathbb{C}^*, \{1\}) \arrow[r] \arrow[d] & (\mathbb{C}, \{1\}) \\ (\mathbb{C}^*, \{1\}) \wedge \Delta[1] & \end{tikzcd} \tag{20}$$

If $\mathbf{D}'$ is another diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "i"] & Y \\ \mathcal{Z} & \end{tikzcd} \tag{21}$$

in $\mathcal{H}$ then $\mathrm{colim}_{\mathbf{D}} \cong \mathrm{colim}_{\mathbf{D}'}$ in $\mathcal{H}$ if there exists a morphism of diagrams $\mathbf{D} \to \mathbf{D}'$ such that the morphisms are weak affine equivalences. Consider the diagrams $\mathbf{D}'$ and $\mathbf{D}''$

$$\begin{tikzcd} (\mathbb{C}^*, \{1\}) \arrow[r] \arrow[d] & \mathsf{pt} \\ (\mathbb{C}^*, \{1\}) \wedge \Delta[1] & \end{tikzcd} \qquad \begin{tikzcd} (\mathbb{C}^*, \{1\}) \arrow[r] \arrow[d] & (\mathbb{C}, \{1\}) \\ \mathsf{pt} & \end{tikzcd} \tag{22}$$

and the morphisms

$$f = ((\mathbb{C}, \{1\}) \to \mathsf{pt}, \mathsf{id}) : \mathbf{D} \to \mathbf{D}' \quad g = (\mathsf{id}, (\mathbb{C}^*, \{1\}) \wedge \Delta[1] \to \mathsf{pt}) : \mathbf{D} \to \mathbf{D}''.$$

The morphisms f and g induce affine weak equivalences $\mathrm{colim}_{\mathbf{D}} \to \mathrm{colim}_{\mathbf{D}'}$ and $\mathrm{colim}_{\mathbf{D}} \to \mathrm{colim}_{\mathbf{D}''}$. Identifying $\mathrm{colim}_{\mathbf{D}'}$ with $(\mathbb{C}^*, \{1\}) \wedge S^1_s$ and $\mathrm{colim}_{\mathbf{D}''}$ with $\mathbb{C}/(\mathbb{C}^*)$, we conclude that

$$(\mathbb{C}^*, \{1\}) \wedge S^1_s \cong \mathbb{C}/\mathbb{C}^*.$$

The square

$$\begin{array}{ccc} \mathbb{C}^* & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{CP}^1\backslash\{\infty\} & \longrightarrow & \mathbb{CP}^1 \end{array} \tag{23}$$

is co-Cartesian in $\mathcal{F}_T(\mathcal{S})$, hence the co-fibers of horizontal morphisms are isomorphic. We derive

$$\mathbb{C}/\mathbb{C}^* \cong \mathbb{CP}^1/(\mathbb{CP}^1\backslash\{\infty\})$$

in $\mathcal{F}_T(\mathcal{S})$. But

$$\mathbb{CP}^1/(\mathbb{CP}^1\backslash\{\infty\}) \cong \mathbb{CP}^1$$

in $\mathcal{H}$, since $\mathbb{CP}^1\backslash\{\infty\} \cong \mathsf{pt}$ in $\mathcal{H}$.

Remark 6.1 In the proof of Lemma 4.1 we have already seen that S^1_{par} is weakly equivalent to S^1_s.

We are now going to apply the theory developed so far to prove that

Theorem 6.1 *For any $n > 0$, $\mathbb{CP}^n$ is not weakly hyperbolic. In other words, $\mathfrak{Ip}(\mathbb{CP}^n)$ cannot be represented in $\mathcal{H}$ by a Brody hyperbolic complex space.*

Proof In view of Corollary 4.1, it is sufficient to show that

$$\pi^{\mathsf{par}}_{2,1}(\mathbb{P}^n, \infty) = \mathsf{Hom}_{\mathcal{H}_\bullet}(\mathbb{C}^* \wedge S^1_{\mathsf{par}}, (\mathbb{P}^n, \{\infty\})) \neq 0$$

or equivalently, by Lemma 6.1 and Remark 6.1, that

$$\mathsf{Hom}_{\mathcal{H}_\bullet}(\mathbb{CP}^1, (\mathbb{CP}^n, \{\infty\})) \neq 0.$$

Our candidate to represent a nonzero class is the canonical embedding $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$. The topological realization yields a group homomorphism

$$t : \pi^{\mathsf{par}}_{2,1}(\mathbb{CP}^n, \infty) \to \pi_2(\mathbb{CP}^n, \infty).$$

$t^{olo}(i) : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^n$ is the canonical inclusion and not null homotopic, since $\mathbb{CP}^n$ is obtained by $\mathbb{CP}^1$ by attaching cells of dimension 4 and above, hence it is an equivalence up to dimension 2 and in particular

$$t^{olo}(i)_* : \mathbb{Z} = \pi_2(\mathbb{CP}^1, \infty) \to \pi_2(\mathbb{CP}^n, \infty)$$

is an isomorphism. In conclusion $t[i] \neq 0$, thus $[i] \neq 0 \in \pi^{\mathsf{par}}_{2,1}(\mathbb{CP}^n, \infty)$. □

Proposition 6.1 *Let X be a complex space and $p : \widetilde{X} \to X$ a connected covering complex space. Assume that X is weakly hyperbolic and let $f : \mathbb{C} \to X$ be a nonconstant holomorphic function. Then for any lifting $\tilde{f}$ of f to $\widetilde{X}$, $\tilde{f}(\mathbb{C})$ contains just one point in each fiber of p or equivalently $p|_{\tilde{f}(\mathbb{C})}$ is a biholomorphism for any such f and $\tilde{f}$.*

Proof Let X be weakly hyperbolic. Assume, by contradiction, that there exists a nonconstant holomorphic function $f : \mathbb{C} \to X$ and a lifting $\tilde{f} : \mathbb{C} \to \widetilde{X}$ such that $a \neq b \in p^{-1}(x)$,

$x \in X$, $a, b \in \tilde{f}(\mathbb{C})$. For the purposes of this proof, we can assume that $\tilde{f}(0) = a$ and $\tilde{f}(1) = b$. Then we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tilde{f}"] \arrow[d, "q"] & \widetilde{X} \arrow[d, "p"] \\ \mathbb{C}/\{0\} \amalg \{1\} \arrow[r, dotted, "\alpha"] & X \end{tikzcd} \tag{24}$$

where α sends the class of $\{0\} \amalg \{1\}$ to $x \in X$. We have that $[\alpha] \neq 0 \in \pi_{1,0}^{\text{par}}(X, x)$. Indeed, $[\alpha^{\text{top}}] \neq 0 \in \pi_1(X^{\text{top}}, x)$. Consider the composition

$$[0, 1] \xrightarrow{g} \mathbb{C}/\{0\} \amalg \{1\} \xrightarrow{\alpha^{\text{top}}} X^{\text{top}},$$

where g is a path from 0 to 1 in $\mathbb{C}$. If $\alpha^{\text{top}} \circ g$ is not homotopic to a constant relatively to $\{0, 1\}$, then α^{top} is not homotopic to a constant. But, by construction, $\alpha^{\text{top}} \circ g$ lifts uniquely to a path in $\widetilde{X}^{\text{top}}$ starting from a and ending in b, hence $\alpha^{\text{top}} \circ g$ cannot be homotopic to a constant relatively to $\{0, 1\}$. This shows that $\pi_1(X^{\text{top}}, x) \neq 0$ which is absurd since X is weak hyperbolic. □

The Proposition 6.1 in particular implies the following

Corollary 6.1 *Any complex space X whose universal covering space is $\mathbb{C}^n$ for some $n \geq 1$, is not weakly hyperbolic.*

Proof Let $p : \mathbb{C}^n \to X$ be the universal covering of X. Let $a \neq b \in p^{-1}(x)$, $x \in X$. A complex line $l \subset \mathbb{C}^n$ passing through a, b provides a homomorphic map $f : \mathbb{C} \to X$ which does not satisfy the conclusion of Proposition 6.1. □

Knowing that a nonzero holotopy group implies that the complex space is not (weakly) hyperbolic, we may ask if the opposite implication holds, as well. In general the answer is negative; however, by a different rephrasing of the previous proposition, we conclude:

Proposition 6.2 *Let X be a non-Brody hyperbolic complex space admitting a covering $p : Y \to X$ with a fiber $p^{-1}(x)$ intersecting the image of $\mathbb{C} \to Y$ in at least two distinct points. Then $\pi_{1,0}^{\text{par}}(X, x) \neq 0$.*

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