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2 AUTHORS, INCLUDING:

Magdi S Mahmoud
King Fahd University of Petroleum and M…

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Robust stability, stabilization and $\mathscr{H}_\infty$ control of time-delay systems with Markovian jump parameters

Magdi S. Mahmoud[1,*,†,‡] and Peng Shi[2,§,¶]

[1] *College of Engineering, UAE University, P.O. Box 17555, Al-Ain, United Arab Emirates*
[2] *Land Operations Division, Defence Science and Technology Organisation, P.O. Box 1500, Edinburgh 5111 SA, Australia*

SUMMARY

In this paper, the problems of stochastic stability and stabilization for a class of uncertain time-delay systems with Markovian jump parameters are investigated. The jumping parameters are modelled as a continuous-time, discrete-state Markov process. The parametric uncertainties are assumed to be real, time-varying and norm-bounded that appear in the state, input and delayed-state matrices. The time-delay factor is constant and unknown with a known bound. Complete results for both delay-independent and delay-dependent stochastic stability criteria for the nominal and uncertain time-delay jumping systems are developed. The control objective is to design a state feedback controller such that stochastic stability and a prescribed $\mathscr{H}_\infty$-performance are guaranteed. We establish that the control problem for the time-delay Markovian jump systems with and without uncertain parameters can be essentially solved in terms of the solutions of a finite set of coupled algebraic Riccati inequalities or linear matrix inequalities. Extension of the developed results to the case of uncertain jumping rates is also provided. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: time-delay systems; Markovian jump parameters; $\mathscr{H}_\infty$ state feedback; stochastic stability; uncertain parameters

1. INTRODUCTION

Dynamical systems whose structures vary in response to random changes, which may result from abrupt phenomena such as parameter shifting, component and interconnection failures, are frequently occurring in practical situations. Such systems can be modelled by combined continuous and discrete states such as fault-tolerant systems [1–3]. Fault-tolerant control systems (FTCSs) have been developed in order to achieve high levels of reliability and performance in situations where the controlled system can have potentially damaging effects on the environment in case component failures take place. A major part of the design objective is to

*Correspondence to: Magdi S. Mahmoud, College of Engineering, UAE University, P.O. Box 17555, Al-Ain, United Arab Emirates.
† E-mail: goudims@hotmail.com
‡ Magdi S. Mahmoud was working with the Faculty of Engineering, Arab Academy for Science and Technology, Egypt.
§ E-mail: peng.shi@dsto.defence.gov.au
¶ Peng Shi was working with the University of South Australia.

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retain mainly a portion of its control integrity by automatically detecting and identifying the failed components and then reconfiguring the control function on-line in response to these decisions. Dynamical models of such active (FTCSs) are represented by stochastic differential equations (SDEs). An important class of SDEs is the one with Markovian jumping parameters, which has been widely used in jumping systems (JSs). In the linear case, the underlying dynamics is governed by different forms depending on the value of an associated finite-state Markov process. Research into this class of systems and their applications span several decades [3–5]. Basic results on jumping linear quadratic control (JLQC) theory can be found in References [6–10]. For some representative related work on this general topic, we refer the reader to References [6, 1, 2, 11–15] and references therein. In particular, the problems of controllability, stabilizability, JLQ control and stochastic stability properties have been theoretically addressed [13, 14].

When the plant modelling uncertainty or external disturbance uncertainty is of major concern in control systems, robust control theory provides tractable design tools using the time domain and the frequency domain; see Reference [16] and references cited therein. For Markov jumping linear continuous-time systems, the issue of robust stability and control has been investigated in References [17, 18] and the counterpart of $\mathscr{H}_\infty$-control has been developed in Reference [19].

On another research front, dynamical systems with state-delay have been the subject of extensive research during the part two-decades; for a recent coverage on the available results the reader is referred to Reference [15].

The purpose of this paper is to extend the results of [14, 19, 20] further by initiating the study and developing criteria of stochastic stability and stabilization of a class of uncertain time-delay systems with Markovian jump parameters. The jumping parameters are treated as continuous-time, discrete-state Markov process. The parametric uncertainties are assumed to be real, time-varying and norm-bounded that appear in the state, input and delayed-state matrices. The time-delay factor is treated as an unknown constant within a prespecified range. Complete results of delay-independent and delay-dependent stochastic stability criteria are developed for both the nominal and uncertain jumping system. We establish that the control problem for the time-delay Markovian jump systems with and without uncertain parameters can be essentially solved in terms of the solutions of a finite set of coupled algebraic Riccati inequalities or linear matrix inequalities. Extension of the developed results to the case of uncertain jumping rates is provided.

Notations and facts: In the sequel, the Euclidean norm is used for vectors. We use W^{t}, W^{-1}, $\lambda(W)$ and $\|W\|$ to denote, respectively, the transpose of, the inverse of, the eigenvalues of and the induced norm of any square matrix W. We use $W > 0$ $(\geqslant, <, \leqslant 0)$ to denote a symmetric positive definite (positive semidefinite, negative, negative semidefinite matrix W) matrices $\lambda_{\mathrm{m}}(W)$ and $\lambda_{\mathrm{M}}(W)$ denote the minimum and maximum eigenvalues of W and I denotes the $n \times n$ identity matrix. The Lebesgue space $\mathscr{L}_2[0, T]$ consists of square-integrable functions on the interval $[0, T]$ equipped with the norm $\|.\|_2$. $\mathbb{E}[\cdot]$ stands for mathematical expectation. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

Fact 1: For any real matrices Σ_1, Σ_2 and Σ_3 with appropriate dimensions and $\Sigma_3^{\mathrm{t}}\Sigma_3 \leqslant I$, it follows that

$$\Sigma_1\Sigma_3\Sigma_2 + \Sigma_2^{\mathrm{t}}\Sigma_3^{\mathrm{t}}\Sigma_1^{\mathrm{t}} \leqslant \alpha^{-1}\Sigma_1\Sigma_1^{\mathrm{t}} + \alpha\,\Sigma_2^{\mathrm{t}}\Sigma_2, \quad \forall \alpha > 0$$

Fact 2: Let $\Sigma_1, \Sigma_2, \Sigma_3$ and $0 < R = R^{\mathrm{t}}$ be real constant matrices of compatible dimensions and let $H(t)$ be a real matrix function satisfying $H^{\mathrm{t}}(t)H(t) \leqslant I$. Then for any $\rho > 0$ satisfying $\rho\Sigma_2^{\mathrm{t}}\Sigma_2$

$<R$, the following matrix inequality holds:

$$(\Sigma_3 + \Sigma_1 H(t)\Sigma_2)R^{-1}(\Sigma_3^t + \Sigma_2^t H^t(t)\Sigma_1^t) \leqslant \rho^{-1}\Sigma_1\Sigma_1^t + \Sigma_3(R - \rho\Sigma_2^t\Sigma_2)^{-1}\Sigma_3^t$$

Fact 3 (Schur complement): Given constant matrices $\Omega_1, \Omega_2, \Omega_3$ where $\Omega_1 = \Omega_1^t$ and $0 < \Omega_2 = \Omega_2^t$ then $\Omega_1 + \Omega_3^t\Omega_2^{-1}\Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^t \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^t & \Omega_1 \end{bmatrix} < 0$$

2. PROBLEM STATEMENT

Given a probability space $(\Omega, \mathscr{F}, \mathbf{P})$, Ω is the sample space, $\mathscr{F}$ is the algebra of events and $\mathbf{P}$ is the probability measure defined on $\mathscr{F}$. Let the random form process $\{\eta_t,\ t \in [0, \mathscr{T}]\}$ be a homogeneous, finite-state Markovian process with right continuous trajectories and taking values in a finite set $\mathscr{S} = \{1, 2, \ldots, s\}$ with generator $\mathfrak{I} = (\alpha_{ij})$ and transition probability from mode i at time t to mode j at time $t + \delta, i, j \in \mathscr{S}$,

$$\begin{aligned} p_{ij} &= Pr(\eta_{t+\delta} = j \mid \eta_t = i) \\ &= \begin{cases} \alpha_{ij}\delta + o(\delta) & \text{if } i \neq j \\ 1 + \alpha_{ij}\delta + o(\delta) & \text{if } i = j \end{cases} \end{aligned} \tag{1}$$

with transition probability rates $\alpha_{ij} \geqslant 0$ for $i, j \in \mathscr{S}, i \neq j$ and

$$\alpha_{ii} = -\sum_{m=1, m\neq i}^{s} \alpha_{im} \tag{2}$$

where $\delta > 0$ and $\lim_{\delta\downarrow 0} o(\delta)/\delta = 0$. The set $\mathscr{S}$ comprises the various operational modes of the system under study.

We consider a class of stochastic uncertain time-delay systems with Markovian jump parameters described over the space $(\Omega, \mathscr{F}, \mathbf{P})$ by

$$\begin{aligned} (\Sigma_J) : \dot{x}(t) &= [A_o(\eta_t) + \Delta A_o(t, \eta_t)]x(t) + [A_d(\eta_t) + \Delta A_d(t, \eta_t)]x(t - \tau) \\ &\quad + [B_o(\eta_t) + \Delta B_o(t, \eta_t)]u(t) + \Gamma(\eta_t)w(t), \quad t \geqslant 0 \\ &= A_{\Delta o}(t, \eta_t)x(t) + A_{\Delta d}(t, \eta_t)x(t - \tau) + B_{\Delta o}(t, \eta_t)u(t) \\ &\quad + \Gamma(\eta_t)w(t), \quad x(t) = \phi(t),\ t \in [-\tau, 0],\ \eta_o = i,\ t \geqslant 0 \end{aligned} \tag{3}$$

$$y(t) = x(t) \tag{4}$$

$$z(t) = G(\eta_t)x(t) + \Phi(\eta_t)w(t) + F(\eta_t)u(t) \tag{5}$$

where $x(t) \in \mathfrak{R}^n$ is the state vector; $u(t) \in \mathfrak{R}^m$ is the control input; $w(t) \in \mathfrak{R}^q$ is the disturbance input which belongs to $\mathscr{L}_2[0, \mathscr{T}]$; $y(t) \in \mathfrak{R}^p$ is the measured output; $z(t) \in \mathfrak{R}^r$ is the controlled output which belongs to $\mathscr{L}_2[(\Omega, \mathscr{F}, \mathbf{P}), [0, \mathscr{T}]]$ and $\tau \in [0, \tau^*]$ is an unknown time-varying delay factor satisfying $0 \leqslant \tau \leqslant \tau^*, 0 \leqslant \dot{\tau} \leqslant \tau^+$ where the bounds $\tau^*, \tau^+ < 1$ are known constants.

For each possible value $\eta_t = i,\ i \in \mathscr{S}$, we will denote the system matrices of (Σ_J) associated with mode i by

$$A_o(\eta_t) \triangleq A_o(i), \quad B_o(\eta_t) \triangleq B_o(i), \quad \Gamma(\eta_t) \triangleq \Gamma(i), \quad G(\eta_t) \triangleq G(i)$$
$$F(\eta_t) \triangleq F_j(i), \quad A_d(\eta_t) \triangleq A_d(i), \quad \Phi(\eta_t) \triangleq \Phi(i) \tag{6}$$

where $A_o(i)$, $B_o(i)$, $A_d(i)$, $G(i)$, $F(i)$, $\Gamma(i)$ and $\Phi(i)$ are known real constant matrices of appropriate dimensions which describe the nominal system of (Σ_J). The matrices $\Delta A_o(t,\eta_t)$, $\Delta A_d(t,\eta_t)$ and $\Delta B_o(t,\eta_t)$ are real, time-varying matrix functions representing the norm-bounded parameter uncertainties. For $\eta_t = i$, the admissible uncertainties are assumed to be modelled in the form:

$$\Delta A_o(t,i) = M_a(i)\Delta_a(t,i)N_a(i), \quad \Delta B_o(t,i) = M_a(i)\Delta_a(t,i)N_b(i)$$
$$\Delta A_d(t,i) = M_a(i)\Delta_d(t,i)N_d(i) \tag{7}$$

where $M_a(i) \in \Re^{n\times\alpha}$, $N_a(i) \in \Re^{\beta\times n}$, $N_b(i) \in \Re^{\beta\times m}$, and $N_d(i) \in \Re^{\beta\times n}$ are known real constant matrices, with $\Delta_a(t,i)$ and $\Delta_d(t,)$ being unknown, time-varying matrix functions satisfying

$$\|\Delta_a(t,i)\|_2 \leqslant 1, \quad \|\Delta_d(t,i)\|_2 \leqslant 1 \tag{8}$$

where the elements of $\Delta_a(t,i)$ and $\Delta_d(t,i)$ are Lebesgue measurable for any $i \in \mathscr{S}$.

Our purpose in this paper is to develop criteria for stochastic stability and stabilization of system (Σ_J) and examine their robustness, then design appropriate $\mathscr{H}_\infty$ feedback controllers that guarantee stochastic stability with a prescribed performance.

3. STOCHASTIC STABILITY RESULTS

The theorems established in the sequel show that the stability behaviour of system (Σ_J) is related to the existence of a positive definite solution of a family of algebraic Riccati inequalities (ARIs) or linear matrix inequalities (LMIs) thereby providing a clear key to designing the state feedback controller.

3.1. Case 1: without uncertainties

In the absence of uncertainties ($\Delta_a \equiv 0,\ \Delta_d \equiv 0$), we extract from system (Σ_J) for $\eta_t = i \in \mathscr{S}$ the following systems:

(1) The free nominal jump system:

$$\begin{aligned}(\Sigma_{\rm f}): \dot{x}(t) &= A_o(i)x(t) + A_d(i)x(t-\tau), \quad t \geqslant 0\\ x(t) &= \phi(t), \quad t \in [-\tau, 0], \quad \eta_o = i\end{aligned} \tag{9}$$

(2) The controlled nominal jump system:

$$\begin{aligned}(\Sigma_{\rm c}): \dot{x}(t) &= A_o(i)x(t) + A_d(i)x(t-\tau) + B_o(i)u(t), \quad t \geqslant 0\\ x(t) &= \phi(t), \quad t \in [-\tau, 0], \quad \eta_o = i\end{aligned} \tag{10}$$

(3) The nominal jump system:

$$(\Sigma_{\rm n}): \dot{x}(t) = A_o(i)x(t) + A_d(i)x(t-\tau) + \Gamma(i)w(t), \quad t \geqslant 0$$
$$x(t) = \phi(t), \quad t \in [-\tau, 0], \quad \eta_o = i, \tag{11}$$

$$z(t) = G(i)x(t) + \Phi(i)w(t) \tag{12}$$

For the above systems, we have the following definitions and results.

Definition 3.1
System $\Sigma_{\rm f}$ is said to be *stochastically stable independent of delay* (SSID) if, for all finite initial vector function $\phi \in \Re^n$ defined on the interval $[-\tau, 0]$ and initial mode $\eta_o \in \mathscr{S}$,

$$\lim_{\mathscr{T}\to\infty} \left\{ \int_0^{\mathscr{T}} \mathbb{E}\{\|x(t,\phi)\|^2\}\,{\rm d}t \right\} < +\infty$$

Theorem 3.1
System $(\Sigma_{\rm f})$ is SSID if, given matrix sequence $Q(i) = Q^{\rm t}(i) > 0,\ i \in \mathscr{S}$, and letting

$$\bar{Q}(i) = (1-\tau^+)Q(i) - \xi^{-1}(i)\sum_{m=1}^{s} \alpha_{im}Q(m),$$

$$\hat{Q}(i) = Q(i) + \xi(i)\sum_{m=1}^{s} \alpha_{im}Q(m), \quad i \in \mathscr{S}$$

for some scalars $\xi(i) > 0,\ i \in \mathscr{S}$, there exist matrices $P(i) = P^{\rm t}(i) > 0,\ i \in \mathscr{S}$, satisfying the system of ARIs

$$\Pi(i) \triangleq A_o^{\rm t}(i)P(i) + P(i)A_o(i) + \sum_{m=1}^{s} \alpha_{im}P(m) + \hat{Q}(i)$$
$$+\ P(i)A_d(i)\bar{Q}^{-1}(i)A_d^{\rm t}(i)P(i) < 0 \tag{13}$$

Proof
Let $\mathbf{x}_s(t) \triangleq x(s+t),\ t-\tau_{\eta_t} \leqslant s \leqslant t$ and define the process $\{(\mathbf{x}(t), \eta_t),\ t \geqslant 0\}$ over the state space $\bar{\mathscr{C}}$. It should be observed that $\{(\mathbf{x}(t), \eta_t),\ t \geqslant 0\}$ is strong Markovian [4]. For $\eta_t = i,\ i \in \mathscr{S}$, and given $Q(i) = Q^{\rm t}(i) > 0$, let the Lyapunov functional $V(\cdot): \Re^n \times \Re_+ \times \mathscr{S} \to \Re_+$ be selected as

$$V(t, x, \eta_t = i) = V(t, x, i) = x^{\rm t}(t)P(i)x(t) + \int_{-\tau}^{0} x^{\rm t}(t+\theta)Q(i)x(t+\theta)\,{\rm d}\theta \tag{14}$$

The weak infinitesimal operator $\Im_1^x[\cdot]$ of the process $\{x(t), \eta_t, t \geqslant 0\}$ for system (9) at the point $\{t, x, \eta_t\}$ is given by [11, 14]

$$\Im_1^x[V] = \partial V/\partial t + \dot{x}^{\rm t}(t)\partial V/\partial x|_{\eta_t = i} + \sum_{m=1}^{s} \alpha_{im}V(t, x, i, m) \tag{15}$$

Using (9) in (14)–(15), manipulating the terms, applying the argument of completing the squares and over-bounding the result using Fact 1, we get

$$
\begin{aligned}
\Im_1^x[V] = {} & x^t(t)\Bigg\{A_o^t(i)P(i) + P(i)A_o(i) + \sum_{m=1}^{s} \alpha_{im}P(m) + Q(i) \\
& - (1-\dot{\tau})x^t(t-\tau)Q(i)x(t-\tau) \\
& + \sum_{m=1}^{s} \alpha_{im}\int_{-\tau}^{0} x^t(t+\theta)Q(i)x(t+\theta)\,\mathrm{d}\theta\Bigg\}x(t) + x^t(t-\tau)A_d^t(i)P(i)x(t) \\
& + x^t(t)P(i)A_d(i)x(t-\tau) \\
\leqslant {} & x^t(t)\Bigg\{A_o^t(i)P(i) + P(i)A_o(i) + \sum_{m=1}^{s} \alpha_{im}P(m) + \hat{Q}(i) \\
& + P(i)A_d(i)\bar{Q}^{-1}(i)A_d^t(i)P(i)\Big\}x(t) \\
\triangleq {} & x^t(t)\Pi(i)x(t)
\end{aligned}
\tag{16}
$$

where $\bar{Q}(i) > 0,\ i \in \mathscr{S}$, by selection of $\xi(i),\ i \in \mathscr{S}$. In view of (13), $\Pi(i) < 0\ \forall i \in \mathscr{S}$, we conclude that $\Im_1^x[V] < 0$ for all $x \neq 0$ and $\Im_1^x[V] \leqslant 0$ for all x. Since $\|x(t+\beta)\| \leqslant \varphi\|x(t)\|,\ \forall \beta \in [-\tau, 0]$ and some $\varphi > 0$ [21], it follows from (14) that $V(x,i) \leqslant x^t(t)P(i)x(t) + \mu\|x\|^2$ where $\mu = \varphi\tau(\max_i \lambda_M \times [P(i)] + \lambda_M[Q(i)])$. Therefore, for all $x \neq 0$, we have

$$
\frac{\Im_1^x[V]}{V(x,i)} \leqslant \frac{x^t\Pi(i)x}{x^tP(i)x + \mu\|x\|^2}; \leqslant -\zeta \triangleq -\min_{i\in\mathscr{S}} \left\{\frac{\lambda_m[-\Pi(i)]}{\lambda_M[P(i)] + \mu}\right\}
\tag{17}
$$

It is readily seen from (17) that $\zeta > 0$ and hence we get $\Im_1^x[V] \leqslant -\zeta V(t,x,i)$. From Reference [11] by using the Gronwall–Bellman lemma [15] and letting $x(t=0,\phi,\eta_o) = x_o$, one has

$$
\mathbb{E}[V(x,i)|\phi,\eta_o] \leqslant \mathrm{e}^{-\zeta t}V(x_o,i)
\tag{18}
$$

Since $\mathbb{E}\{\int_{-\tau}^{0} x^t(t+\theta)Q(i)x(t+\theta)\,\mathrm{d}\theta|\phi,\eta_o\} \geqslant 0$ it is easy to see from (14) that

$$
\begin{aligned}
\mathbb{E}\{x^t(t)P(i)x(t)|\phi,\eta_o\} \leqslant \mathrm{e}^{-\zeta t}V(x_o,i) \Rightarrow {} & \mathbb{E}\left\{\int_0^{\mathscr{T}} x^t(t)P(i)x(t)\,\mathrm{d}t|\phi,\eta_o = i\right\} \\
\leqslant {} & \left[\int_o^{\mathscr{T}} \mathrm{e}^{-\zeta t}\,\mathrm{d}t\right]V(x_o,i) = \frac{1}{\zeta}[\mathrm{e}^{-\zeta\mathscr{T}} - 1]V(x_o,i) \\
\Rightarrow {} & \lim_{\mathscr{T}\to\infty} \mathbb{E}\left\{\int_0^{\mathscr{T}} x^t(t)P(i)x(t)\,\mathrm{d}t|\phi,\eta_o = i\right\} \\
\leqslant {} & \frac{1}{\zeta}x_o^tP(\eta_o)x_o + \frac{\tau^*}{\zeta}[Q(\eta_o)]\|x(t+\theta)\|_o^2, \quad \forall\theta \in [\tau, 0]
\end{aligned}
\tag{19}
$$

where $\|x(t+\theta)\|_*^2 \triangleq \sup_{\theta\in[-\tau,0]}\|x(t+\theta)\|_2^2$. Let

$$
\bar{P}(i) = \max_{i\in\mathscr{S}} \left\{\frac{P(\eta_o)\|x_o\|^2 + \tau^*[Q(\eta_o)]\|x(t+\theta)\|_*^2}{\zeta[P(\eta_o)]\|x_o\|^2}\right\}
$$

it follows from (19) for $i \in \mathscr{S}$ that

$$\lim_{\mathscr{T}\to\infty} \mathbb{E}\left\{\int_0^{\mathscr{T}} x^{\mathrm{t}}(t)x(t)\,\mathrm{d}t|\phi, \eta_o = i\right\} \leqslant x_o^{\mathrm{t}}\lambda_{\mathrm{M}}(\bar{P}(i))x_o < +\infty$$

which, in the light of Definition 3.1, shows that system (Σ_{f}) is SSID. □

Remark 3.1
It is interesting to observe that in the delayless case ($A_d(.) \equiv 0$), Theorem 3.1 recovers the result of Reference [14] and thus our work generalizes the results of Reference [14] to time-delay systems.

Definition 3.2
System Σ_{n} is said to be *SSID with a disturbance attenuation* γ if, for zero initial vector function $\phi \equiv 0$ defined on the interval $[-\tau, 0]$ and initial mode $\eta_o \in \mathscr{S}$ the following inequalities hold:

$$\lim_{\mathscr{T}\to\infty} \left\{\int_0^{\mathscr{T}} \mathbb{E}\{||x(t,\phi)||^2\}\,\mathrm{d}t\right\} < +\infty$$

$$||z(t)||_{E_2} \triangleq \mathbb{E}\left[\int_0^{\infty} z^{\mathrm{t}}(t)z(t)\,\mathrm{d}t\right]^{1/2} < \gamma||w(t)||_2$$

for all $0 \neq w(t) \in \mathscr{L}_2[0,\infty)$, where $\gamma > 0$ is a prescribed level of disturbance attenuation and $||.||_{E_2}$ denotes the norm in $\mathscr{L}_2((\Omega, \mathscr{F}, \mathbf{P}), [0,\infty))$.

Theorem 3.2
System (Σ_{n}) is *SSID with a disturbance attenuation* γ, if, for given matrices $Q(i) = Q^{\mathrm{t}}(i) > 0,\ i \in \mathscr{S}$, there exist matrices $P(i) = P^{\mathrm{t}}(i) > 0,\ i \in \mathscr{S}$, satisfying the system of ARIs

$$\begin{aligned} &A^{\mathrm{t}}(i)P(i) + P(i)A(i) + \sum_{m=1}^{s} \alpha_{im}P(m) + P(i)A_d(i)\bar{Q}^{-1}(i)A_d^{\mathrm{t}}(i)P(i) \\ &+ G^{\mathrm{t}}(i)G(i) + \gamma^{-2}[P(i)\Gamma(i) + G^{\mathrm{t}}(i)\Phi(i)][\Gamma^{\mathrm{t}}(i)P(i) + \Phi^{\mathrm{t}}(i)G(i)] + \hat{Q}(i) < 0 \end{aligned} \tag{20}$$

where $\bar{Q}(i),\ \hat{Q}(i)$ are as stated in Theorem 3.1.

Proof
The stochastic stability of system (Σ_{n}) follows as a result of Theorem 3.1. What we need to show here is that system (Σ_{n}) has a disturbance attenuation γ. Let the Lyapunov functional $V(t,x,\eta_t)$, for $\eta_t = i$ be given by (14). By evaluating the weak infinitesimal operator $\mathfrak{I}_2^x[\cdot]$ of the process $\{x(t), \eta_t, t \geqslant 0\}$ for system (11)–(12) at the point $\{t, x, \eta_t\}$ using (15) and manipulating we get

$$\mathfrak{I}_2^x[V] \leqslant \mathfrak{I}_1^x[V] + x^{\mathrm{t}}(t)P(i)\Gamma(i)w(t) + w^{\mathrm{t}}(t)\Gamma^{\mathrm{t}}(i)P(i)x(t) \triangleq \mathfrak{I}_+^x[V] \tag{21}$$

Now, we introduce $\mathscr{J}(x) := \mathbb{E}\{\int_0^\infty [z^{\mathrm{t}}(t)z(t) - \gamma^2 w^{\mathrm{t}}(t)w(t)]\,\mathrm{d}t\}$. By Dynkin's formula [11], one has $\mathbb{E}\{\int_0^\infty \mathfrak{I}_+^x[V]\,\mathrm{d}t\} = \mathbb{E}\{V(t,x,\eta_t)|_{t=\infty}\} - V(t,x,\eta_t)|_{t=0} \geqslant 0$. With some manipulations using

(12), we obtain

$$
\begin{aligned}
\mathscr{J}(x) &= \mathbb{E}\left\{ \int_0^\infty [z^t(t)z(t) - \gamma^2 w^t(t)w(t) + \Im_+^x[V] - \Im_+^x[V]]\,\mathrm{d}t \right\} \\
&\leqslant \mathbb{E}\left\{ \int_0^\infty [z^t(t)z(t) - \gamma^2 w^t(t)w(t) + \Im_+^x[V]]\,\mathrm{d}t \right\} \\
&< \mathbb{E}\left\{ \int_0^{\mathscr{T}} x^t(t)\left\{ A^t(i)P(i) + P(i)A(i) + \sum_{m=1}^{s} \alpha_{im}P(m) \right.\right. \\
&\quad + P(i)A_d(i)Q^{-1}(i)A_d^t(i)P(i) + G^t(i)G(i) + \gamma^{-2}[P(i)\Gamma(i) \\
&\quad \left.\left. + G^t(i)\Phi(i)][\Gamma^t(i)P(i) + \Phi^t(i)G(i)] + Q(i) \right\} x(t) \right\}
\end{aligned} \tag{22}
$$

By using (20) and the results of Theorem 3.1, it follows from inequality (22) that $\mathscr{T}(x) < 0$ and by Definition 3.2, the proof is completed. □

3.2. Case 2: with uncertainties

In this section, we consider the robust counterparts of Theorems 3.1 and 3.2, that is, the problems of robust stability and robust disturbance attenuation of the uncertain system (Σ_J). In a similar way for $\eta_t = i \in \mathscr{S}$, we deal with the following related systems:

(1) The free uncertain jump system:

$$
\begin{aligned}
(\Sigma_{\Delta f}): \dot{x}(t) &= [A_o(\eta_t) + \Delta A_o(t,\eta_t)]x(t) + [A_d(\eta_t) + \Delta A_d(t,\eta_t)]x(t-\tau) \\
&= A_{\Delta o}(t,\eta_t)x(t) + A_{\Delta d}(t,\eta_t)x(t-\tau), \quad t \geqslant 0 \\
x(t) &= \phi(t), \quad t \in [-\tau, 0], \quad \eta_o = i
\end{aligned} \tag{23}
$$

(2) The controlled uncertain jump system:

$$
\begin{aligned}
(\Sigma_{\Delta c}): \dot{x}(t) &= [A_o(\eta_t) + \Delta A_o(t,\eta_t)]x(t) + [A_d(\eta_t) + \Delta A_d(t,\eta_t)]x(t-\tau) \\
&\quad + [B_o(\eta_t) + \Delta B_o(t,\eta_t)]u(t) \\
&= A_{\Delta o}(t,\eta_t)x(t) + A_{\Delta d}(t,\eta_t)x(t-\tau) + B_{\Delta o}(t,\eta_t)u(t), \quad t \geqslant 0
\end{aligned}
$$

$$
x(t) = \phi(t), \quad t \in [-\tau, 0], \quad \eta_o = i \tag{24}
$$

(3) The uncertain jump system:

$$
\begin{aligned}
(\Sigma_{\Delta n}): \dot{x}(t) &= [A_o(\eta_t) + \Delta A_o(t,\eta_t)]x(t) + [A_d(\eta_t) + \Delta A_d(t,\eta_t)]x(t-\tau) \\
&\quad + \Gamma(\eta_t)w(t) \\
&= A_{\Delta o}(t,\eta_t)x(t) + A_{\Delta d}(t,\eta_t)x(t-\tau) + \Gamma(t,\eta_t)w(t), \quad t \geqslant 0
\end{aligned}
$$

$$
x(t) = \phi(t), \quad t \in [-\tau, 0], \quad \eta_o = i \tag{25}
$$

$$
z(t) = G(\eta_t)x(t) + \Phi(\eta_t)w(t) \tag{26}
$$

We have the following definitions and results:

Definition 3.3
System $\Sigma_{\Delta f}$ is said to be *robustly stochastically stable independent of delay* (RSSID) if for zero initial vector function $\phi \equiv 0$ defined on the interval $[-\tau, 0]$ and initial mode $\eta_o \in \mathscr{S}$

$$\lim_{\mathscr{T}\to\infty} \left\{ \int_0^{\mathscr{T}} \mathbb{E}\{\|x(t,\phi)\|^2\}\, \mathrm{d}t \right\} < +\infty$$

for all admissible uncertainties satisfying (7)–(8)

Theorem 3.3
System $(\Sigma_{\Delta f})$ is RSSID if given matrices $Q(i) = Q^{\mathrm{t}}(i) > 0,\ i \in \mathscr{S}$, and letting

$$\bar{Q}(i) = (1-\tau^+)Q(i) - \xi^{-1}(i)\sum_{m=1}^{s} \alpha_{im}Q(m)$$

$$\hat{Q}(i) = Q(i) + \xi(i)\sum_{m=1}^{s} \alpha_{im}Q(m), \quad i \in \mathscr{S}$$

for some scalars $\xi(i) > 0,\ i \in \mathscr{S}$, there exist matrices $P(i) = P^{\mathrm{t}}(i) > 0$ and scalars $\varepsilon(i) > 0,\ \rho(i) > 0,\ i \in \mathscr{S}$, satisfying the system of ARIs

$$\begin{aligned} &A_o^{\mathrm{t}}(i)P(i) + P(i)A_o(i) + \sum_{m=1}^{s} \alpha_{im}P(m) + \varepsilon^{-1}(i)N_a^{\mathrm{t}}(i)N_a(i) \\ &\quad + \varepsilon(i)P(i)M_a(i)M_a^{\mathrm{t}}(i)P(i) + \hat{Q}(i) + \rho(i)P(i)M_a(i)M_a^{\mathrm{t}}(i)P(i) \\ &\quad + P(i)A_d(i)[\bar{Q}(i) - \rho^{-1}(i)N_d^{\mathrm{t}}(i)N_d(i)]^{-1}A_d^{\mathrm{t}}(i)P(i) < 0 \end{aligned} \tag{27}$$

Proof
For $\eta_t = i,\ i \in \mathscr{S}$, given $Q(i) = Q^{\mathrm{t}}(i) > 0$, let the Lyapunov functional $V(\cdot): \Re^n \times \Re_+ \times \mathscr{S} \to \Re_+$ be selected as (14). By similarity to Theorem 3.1, the weak infinitesimal operator $\Im_3^x[\cdot]$ of the process $\{x(t), \eta_t, t \geqslant 0\}$ for system (23) at the point $\{t, x, \eta_t\}$ is given by

$$\begin{aligned} \Im_3^x[V] &= \Im_1^x[V] + x^{\mathrm{t}}(t)\{\Delta A_o^{\mathrm{t}}(i)P(i) + P(i)\Delta A_o(i)\}x(t) + x^{\mathrm{t}}(t-\tau)\Delta A_d^{\mathrm{t}}(i)P(i)x(t) \\ &\quad + x^{\mathrm{t}}(t)P(i)\Delta A_d(i)x(t-\tau) \leqslant x^{\mathrm{t}}(t)\left\{ A_o^{\mathrm{t}}(i)P(i) + P(i)A_o(i) + \sum_{m=1}^{s} \alpha_{im}P(m) \right. \\ &\quad + \dot{Q}(i) + P(i)[A_d(i) + \Delta A_d(i)]\bar{Q}^{-1}(i)[A_d(i) + \Delta A_d(i)]^{\mathrm{t}}P(i)\Delta A_o^{\mathrm{t}}(i)P(i) \\ &\quad + P(i)\Delta A_o(i)\}x(t) \end{aligned} \tag{28}$$

Using (7), Facts 1 and 2 and manipulating, we obtain for some $\varepsilon(i) > 0$ and $\mu(i) > 0$, $i \in \mathscr{S}$:

$$\begin{aligned}\Im_3^x[V] \leqslant x^{\mathrm{t}}(t)\Bigg\{ & A_o^{\mathrm{t}}(i)P(i) + P(i)A_o(i) + \sum_{m=1}^{s} \alpha_{im}P(m) \\ & + \varepsilon^{-1}(i)N_a^{\mathrm{t}}(i)N_a + \varepsilon(i)P(i)M_a(i)M_a^{\mathrm{t}}(i)P(i) \\ & + \hat{Q}(i) + \rho(i)P(i)M_a(i)M_a^{\mathrm{t}}(i)P(i) + P(i)A_d(i)[\bar{Q}(i) \\ & - \rho^{-1}(i)N_d^{\mathrm{t}}(i)N_d(i)]^{-1}A_d^{\mathrm{t}}(i)P(i)\Big\}x(t)\end{aligned} \tag{29}$$

If $\Im_3^x[V] < 0, \ \forall i \in \mathscr{S}$, when $x \neq 0$ then $x(t) \to 0$ as $t \to \infty$ and hence using the results of Theorem 3.1, system $(\Sigma_{\Delta \mathrm{f}})$ is RSSID. □

Definition 3.4
System $(\Sigma_{\Delta \mathrm{n}})$ is said to be RSSID with a disturbance attenuation γ if for zero initial vector function $\phi \equiv 0$ defined on the interval $[-\tau, 0]$ and initial mode $\eta_o \in \mathscr{S}$

$$\lim_{\mathscr{T} \to \infty} \left\{ \int_0^{\mathscr{T}} \mathbb{E}\{\|x(t,\phi)\|^2\} \, \mathrm{d}t \right\} < +\infty$$

and

$$\|z(t)\|_{E_2} := \mathbb{E}\left[\int_0^{\infty} z^{\mathrm{t}}(t)z(t) \, \mathrm{d}t\right]^{1/2} < \gamma \|w(t)\|_2$$

for all $0 \neq w(t) \in \mathscr{L}_2[0, \infty)$ and for all admissible uncertainties satisfying (7)–(8).

Theorem 3.4
System $(\Sigma_{\Delta \mathrm{n}})$ is *RSSID with a disturbance attenuation* γ if, given matrices $Q(i) = Q^{\mathrm{t}}(i) > 0$, $i \in \mathscr{S}$, there exist matrices $P(i) = P^{\mathrm{t}}(i) > 0$, $i \in \mathscr{S}$ and scalars $\varepsilon(i) > 0$, $\rho(i) > 0$, $i \in \mathscr{S}$, satisfying the system of ARIs:

$$\begin{aligned}& P(i)A_o(i) + A_o^{\mathrm{t}}(i)P(i) + \sum_{m=1}^{s} \alpha_{im}P(m) + \varepsilon(i)N_a^{\mathrm{t}}(i)N_a(i) + \hat{Q}(i) + G^{\mathrm{t}}(i)G(i) \\ & [\varepsilon^{-1}(i) + \rho^{-1}(i)]P(i)M_a(i)M_a^{\mathrm{t}}P(i) + P(i)A_d(i)[\bar{Q}(i) \\ & \quad - \rho(i)N_d^{\mathrm{t}}(i)N_d(i)]^{-1}A_d^{\mathrm{t}}(i)P(i) + [P(i)\Gamma(i) + G^{\mathrm{t}}(i)\Phi(i)] \ (\gamma^2 I \\ & \quad - \Phi^{\mathrm{t}}(i)\Phi(i))^{-1}[\Gamma^{\mathrm{t}}(i)P(i) + \Phi^{\mathrm{t}}(i)G(i)] < 0\end{aligned} \tag{30}$$

where $\bar{Q}(i), \hat{Q}(i)$ are as stated in Theorem 3.3.

Proof
It can be worked out by using the same technique as that used in Theorem 3.2. □

Remark 3.2
We note that the foregoing theorems provide sufficient stability conditions which can be checked by solving the resulting ARIs for $i \in \mathscr{S}$ by employing suitable computational schemes while tuning the scaling parameters $\mu(i)$, $\varepsilon(i)$ using appropriate gridding methods [15]. An alternative route will now be presented by considering Theorem 3.3. Given the data $Q(i)$, $i \in \mathscr{S}$,

we select the scalars $\xi(i)$, $i \in \mathscr{S}$, to ensure that $\bar{Q}(i) = Q(i) - \xi^{-1}(i) \sum_{m=1}^{s} \alpha_{im} Q(m) > 0$, $i \in \mathscr{S}$. This can be simply achieved by choosing large values for $\xi(i)$, $i \in \mathscr{S}$. Then direct application of Fact 3 to ARI (12) yields

$$\begin{bmatrix} \begin{array}{c} A_o^{\mathrm{t}} P(i) + P(i) A_o(i) \\ + \sum_{m=1}^{s} \alpha_{im} P(m) + \hat{Q}(i) \end{array} & P(i) A_d(i) & N_a^{\mathrm{t}}(i) & P(i) M_a(i) & P(i) A_d(i) \\ A_d^{\mathrm{t}}(i) P(i) & -\bar{Q}(i) & 0 & 0 & 0 \\ N_a(i) & 0 & -\varepsilon(i) I & 0 & 0 \\ M_a^{\mathrm{t}}(i) P(i) & 0 & 0 & -\mu(i) I & 0 \\ A_d^{\mathrm{t}}(i) P(i) & 0 & 0 & 0 & -\bar{Q}(i) + \mu(i) N_d^{\mathrm{t}}(i) N_d(i) \end{bmatrix} < 0$$

which is a standard LMI in the variables $P(i)$, $\mu(i)$, $\varepsilon(i)$, $i \in \mathscr{S}$ [22]. It can be conveniently solved by the software environment [23] while looking for feasible solutions. In those cases when the result gives infeasible solution, we adjust $Q(i)$, $\xi(i)$, $i \in \mathscr{S}$, and repeat the process. Indeed, similar LMIs can be readily constructed for other stability theorems.

4. DELAY-DEPENDENT STABILITY RESULTS

It can be argued that delay-independent criteria of stability, stabilization and disturbance attenuation are generally conservative [15] since they do not include any information about the delay factor τ or its upper bound τ^*. For this purpose, we focus attention here on the delay-dependent stochastic stability and control conditions of system (Σ_J) and develop corresponding criteria. To this end, we require the following assumption.

Assumption 4.1

$$\lambda[A_o(i) + A_d(i)] \subset \mathbb{C}, \quad \forall i \in \mathscr{S}$$

Note that Assumption 4.1 corresponds to the stability condition when $\tau = 0$. Hence, it is necessary for system (Σ_J) to be stable for any $\tau \geqslant 0$.

Theorem 4.1

Consider system (Σ_{f}) satisfying Assumption 4.1. Then given a scalar $\tau^* > 0$, this system is *stochastically stable* (SS) for any constant time-delay τ *satisfying* $0 \leqslant \tau \leqslant \tau^*$ if there exist matrices $0 < P(i) = P^{\mathrm{t}}(i) \in \Re^{n \times n}$, $i \in \mathscr{S}$, and scalars $r_1(i) > 0$, $r_2(i) > 0$, $i \in \mathscr{S}$, satisfying the system of ARIs

$$\begin{aligned} P(i)[A_o(i) + A_d(i)] + [A_o(i) + A_d(i)]^{\mathrm{t}} P(i) + \tau r_1(i) A_o^{\mathrm{t}}(i) A_o(i) + \sum_{m=1}^{s} \alpha_{im} P(m) \\ + \tau r_2(i) A_d^{\mathrm{t}}(i) A_d(i) + \tau r_1^{-1}(i) P(i) A_d(i) A_d^{\mathrm{t}}(i) P(i) \\ + \tau r_2^{-1}(i) P(i) A_d(i) A_d^{\mathrm{t}}(i) P(i) < 0 \end{aligned} \tag{31}$$

Proof
Introduce the Lyapunov functional

$$
\begin{aligned}
W(x,i) &= x^{\mathrm{t}}(t)P(i)x(t) + \int_{t-\tau}^{\mathrm{t}} \int_{t+\theta}^{\mathrm{t}} r_1(i)[x^{\mathrm{t}}(s)A_o^{\mathrm{t}}(i)A_o(i)x(s)]\,\mathrm{d}s\,\mathrm{d}\theta \\
&+ \int_{t-\tau}^{\mathrm{t}} \int_{t-\tau+\theta}^{\mathrm{t}} r_2(i)[x^{\mathrm{t}}(s)A_d^{\mathrm{t}}(i)A_d(i)x(s)]\,\mathrm{d}s\,\mathrm{d}\theta
\end{aligned} \tag{32}
$$

where $0 < P(i) = P^{\mathrm{t}}(i) \in \Re^{n\times n},\ i \in \mathscr{S}$, and $r_1(i) > 0,\ r_2(i) > 0,\ i \in \mathscr{S}$, are appropriate weighting factors. First from (9) we have

$$
\begin{aligned}
x(t-\tau) &= x(t) - \int_{-\tau}^{0} \dot{x}(t+\theta)\,\mathrm{d}\theta \\
&= x(t) - \int_{-\tau}^{0} A_o(i)x(t+\theta)\,\mathrm{d}\theta - \int_{-\tau}^{0} A_d(i)x(t-\tau+\theta)\,\mathrm{d}\theta
\end{aligned} \tag{33}
$$

Substituting (33) back into (9) we get

$$
\begin{aligned}
\dot{x}(t) &= [A_o(i) + A_d(i)]x(t) \\
&- A_d(i)\left\{\int_{-\tau}^{0} A_o(i)x(t+\theta)\,\mathrm{d}\theta + \int_{-\tau}^{0} A_d(i)x(t-\tau+\theta)\,\mathrm{d}\theta\right\}
\end{aligned} \tag{34}
$$

Evaluating the weak infinitesimal operator $\Im^x[\cdot]$ of the process $\{x(t), \eta_t, t \geqslant 0\}$ for system (34) at the point $\{t, x, \eta_t\}$, we get

$$
\begin{aligned}
\Im^x[W] &= x^{\mathrm{t}}(t)\left\{P(i)[A_o(i) + A_d(i)] + [A_o(i) + A_d(i)]^{\mathrm{t}}P(i) + \sum_{m=1}^{s} \alpha_{im}P(m)\right\}x(t) \\
&- 2x^{\mathrm{t}}(t)P(i)A_d(i)\int_{-\tau}^{0} A_o(i)x(t+\theta)\,\mathrm{d}\theta + \tau r_2(i)x^{\mathrm{t}}(t)A_d^{\mathrm{t}}(i)A_d(i)x(t) \\
&- 2x^{\mathrm{t}}(t)P(i)A_d(i)\int_{-\tau}^{0} A_d(i)x(t-\tau+\theta)\,\mathrm{d}\theta + \tau r_1(i)x^{\mathrm{t}}(t)A_o^{\mathrm{t}}(i)A_o(i)x(t) \\
&- \int_{-\tau}^{0} r_1(i)[x^{\mathrm{t}}(t+\theta)A_o^{\mathrm{t}}(i)A_o(i)x(t+\theta)]\,\mathrm{d}\theta \\
&- \int_{-\tau}^{0} r_2[x^{\mathrm{t}}(t-\tau+\theta)A_d^{\mathrm{t}}(i)A_d(i)x(t-\tau+\theta)]\,\mathrm{d}\theta
\end{aligned} \tag{35}
$$

Note that for $i \in \mathscr{S}$,

$$
\begin{aligned}
&- 2x^{\mathrm{t}}(t)P(i)A_d(i)\int_{-\tau}^{0} A_o(i)x(t+\theta)\,\mathrm{d}\theta \\
&\leqslant r_1^{-1}(i)\int_{-\tau}^{0}[x^{\mathrm{t}}(t)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i)x(t)]\,\mathrm{d}\theta \\
&\quad + r_1(i)\int_{-\tau}^{0}[x^{\mathrm{t}}(t+\theta)A_o^{\mathrm{t}}(i)A_o(i)x(t+\theta)]\,\mathrm{d}\theta \\
&= \tau r_1^{-1}(i)x^{\mathrm{t}}(t)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i)x(t) \\
&\quad + r_1(i)\int_{-\tau}^{0}[x^{\mathrm{t}}(t+\theta)A_o^{\mathrm{t}}(i)A_o(i)x(t+\theta)]\,\mathrm{d}\theta
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
&- 2x^{\mathrm{t}}(t)P(i)A_d(i)\int_{-\tau}^{0} A_d(i)x(t-\tau+\theta)\,\mathrm{d}\theta \\
&\leqslant \tau r_2^{-1}(i)x^{\mathrm{t}}(t)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i)x(t) \\
&\quad + r_2(i)\int_{-\tau}^{0}[x^{\mathrm{t}}(t-\tau+\theta)A_d^{\mathrm{t}}(i)A_d(i)x(t-\tau+\theta)]\,\mathrm{d}\theta
\end{aligned}
\tag{37}
$$

It then follows from (35)–(37) that

$$
\begin{aligned}
\mathfrak{I}^x[W] \leqslant\, & x^{\mathrm{t}}(t)\{P(i)[A_o(i)+A_d(i)] + [A_o(i)+A_d(i)]^{\mathrm{t}}P(i) + \tau r_1(i)A_o^{\mathrm{t}}(i)A_o(i) \\
& + \sum_{m=1}^{s}\alpha_{im}P(m) + \tau r_2(i)A_d^{\mathrm{t}}(i)A_d(i) + \tau r_1^{-1}(i)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i) \\
& + \tau r_2^{-1}(i)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i)\}x(t)
\end{aligned}
\tag{38}
$$

By (31), $\mathfrak{I}^x[W] < 0$ when $x \neq 0$. Then $x(t) \to 0$ as $t \to \infty$ which, in view of Theorem 3.1, implies that system (Σ_{f}) is stochastically stable. Since (38) is monotonically dependent on τ, we conclude that the delay-dependent stochastic stability is guaranteed for any $\tau \in [0, \tau^*]$. □

Theorem 4.2
Consider system (Σ_{n}) satisfying Assumption 4.1. Then given a scalar $\tau^* > 0$, this system is *SS for any constant time-delay* τ *satisfying* $0 \leqslant \tau \leqslant \tau^*$ *with a disturbance attenuation* γ if there exist scalars $\sigma(i) > 0,\ i \in \mathscr{S}$, such that

$$
\mathscr{R} := \gamma^2 I - \Phi^{\mathrm{t}}(i)\Phi(i) - \tau^*\sigma(i)\Gamma^{\mathrm{t}}(i)\Gamma(i) > 0 \tag{39}
$$

and there exist matrices $0 < P(i) = P^{\mathrm{t}}(i) \in \Re^{n\times n},\ i \in \mathscr{S}$, and scalars $\varepsilon(i) > 0, \mu(i) > 0,\ i \in \mathscr{S}$ satisfying the set of ARIs:

$$
\begin{aligned}
& P(i)[A_o(i)+A_d(i)] + [A_o(i)+A_d(i)]^{\mathrm{t}}P(i) + \sum_{m=1}^{s}\alpha_{im}P(m) + G^{\mathrm{t}}(i)G(i) \\
& + \tau^*(\varepsilon(i)+\mu(i))P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i) + \tau^*\varepsilon^{-1}(i)A_o^{\mathrm{t}}(i)A_o(i) + \tau^*\mu^{-1}(i)A_d^{\mathrm{t}}(i)A_d(i) \\
& + G^{\mathrm{t}}(i)\Phi(i)[\gamma^2 I - \Phi^{\mathrm{t}}(i)\Phi(i) - \tau^*\sigma(i)\Gamma^{\mathrm{t}}(i)\Gamma(i)]^{-1}\Phi^{\mathrm{t}}(i)G(i) < 0
\end{aligned}
\tag{40}
$$

Proof
Initially, we get from (11) the delayed state and the delayed dynamics:

$$x(t-\tau) = x(t) - \int_{-\tau}^{0} \dot{x}(t+\theta)\,\mathrm{d}\theta = x(t) - \int_{-\tau}^{0} A_o(i)x(t+\theta)\,\mathrm{d}\theta - \int_{-\tau}^{0} A_d(i)x(t-\tau+\theta)\,\mathrm{d}\theta - \int_{-\tau}^{0} \Gamma(i)w(t+\theta)\,\mathrm{d}\theta \tag{41}$$

$$\dot{x}(t) = [A_o(i) + A_d(i)]x(t) - A_d(i)\left\{\int_{-\tau}^{0} A_o(i)x(t+\theta)\,\mathrm{d}\theta + \int_{-\tau}^{0} A_d(i)x(t-\tau+\theta)\,\mathrm{d}\theta + \int_{-\tau}^{0} \Gamma(i)w(t+\theta)\,\mathrm{d}\theta\right\} \tag{42}$$

Introduce the Lyapunov functional

$$W_w(x,i) = W(x,i) + \int_{t-\tau}^{\mathrm{t}} \int_{t+\theta}^{\mathrm{t}} r_3(i)[w^{\mathrm{t}}(s)\Gamma^{\mathrm{t}}(i)\Gamma(i)w(s)]\,\mathrm{d}s\,\mathrm{d}\theta \tag{43}$$

for some $r_3(i) > 0,\ i \in \mathscr{S}$. In the present case, the weak infinitesimal operator $\Im^x[W_w]$ of the process $\{x(t), \eta_t, t \geqslant 0\}$ for system (43) at the point $\{t, x, \eta_t\}$ is evaluated

$$\Im^x[W_w] = \Im^x[W] - 2x^{\mathrm{t}}(t)P(i)A_d(i)\int_{-\tau}^{0} \Gamma(i)w(t+\theta)\,\mathrm{d}\theta + \tau r_3(i)w^{\mathrm{t}}(t)\Gamma^{\mathrm{t}}(i)\Gamma(i)w(t) - \int_{-\tau}^{0} r_3(i)[w^{\mathrm{t}}(t+\theta)\Gamma^{\mathrm{t}}(i)\Gamma(i)w(t+\theta)]\,\mathrm{d}\theta \tag{44}$$

For $i \in \mathscr{S}$, we have

$$\begin{aligned}
&-2x^{\mathrm{t}}(t)P(i)A_d(i)\int_{-\tau}^{0} \Gamma(i)w(t+\theta)\,\mathrm{d}\theta \\
&\leqslant r_3^{-1}(i)\int_{-\tau}^{0} [x^{\mathrm{t}}(t)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i)x(t)]\,\mathrm{d}\theta \\
&\quad + r_3(i)\int_{-\tau}^{0} [w^{\mathrm{t}}(t+\theta)\Gamma^{\mathrm{t}}(i)\Gamma(i)w(t+\theta)]\,\mathrm{d}\theta \\
&= \tau r_3^{-1}(i)x^{\mathrm{t}}(t)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i)x(t) \\
&\quad + r_3(i)\int_{-\tau}^{0} [w^{\mathrm{t}}(t+\theta)\Gamma^{\mathrm{t}}(i)\Gamma(i)w(t+\theta)]\,\mathrm{d}\theta
\end{aligned} \tag{45}$$

Combining (45)–(48) and (45), we obtain

$$
\begin{aligned}
\Im^x[W_w] \leqslant\, & x^{\mathrm{t}}(t)\{P(i)[A_o(i)+A_d(i)] + [A_o(i)+A_d(i)]^{\mathrm{t}}P(i) \\
& + \sum_{m=1}^{s} \alpha_{im}P(m) + \tau r_1(i)A_o^{\mathrm{t}}(i)A_o(i) \\
& + \tau r_2(i)A_d^{\mathrm{t}}(i)A_d(i) + \tau r_3(i)A_d^{\mathrm{t}}(i)A_d(i) + \tau r_1^{-1}(i)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i) \\
& + \tau r_2^{-1}(i)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i)\}x(t) + \tau r_3(i)w^{\mathrm{t}}(t)\Gamma^{\mathrm{t}}(i)\Gamma(i)w(t)
\end{aligned} \tag{46}
$$

One should note that $\Im^x[W_w]\leqslant 0$. To assess the performance of system (Σ_{n}), we use the measure $\mathscr{J}(x)$. It follows from (46) with standard manipulations that (here, $\mathscr{T}\to\infty$)

$$
\begin{aligned}
\mathscr{J}(x) :=\, & \mathbb{E}\bigg\{\int_0^{\mathscr{T}} x^{\mathrm{t}}(t)[P(i)[A_o(i)+A_d(i)] + [A_o(i)+A_d(i)]^{\mathrm{t}}P(i) \\
& + \sum_{m=1}^{s} \alpha_{im}P(m) + \tau r_1(i)A_o^{\mathrm{t}}(i)A_o(i) + \tau r_2(i)A_d^{\mathrm{t}}(i)A_d(i) \\
& + \tau r_3(i)A_d^{\mathrm{t}}(i)A_d(i) + \tau r_1^{-1}(i)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i) \\
& + \tau r_2^{-1}(i)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i)]x(t) + \tau r_3(i)w^{\mathrm{t}}(t)\Gamma^{\mathrm{t}}(i)\Gamma(i)w(t) \\
& + \{x^{\mathrm{t}}(t)G^{\mathrm{t}}(i) + w^{\mathrm{t}}(t)\Phi^{\mathrm{t}}(i)\}\{G(i)x(t)+\Phi(i)w(t)\} - \gamma^2 w^{\mathrm{t}}(t)w(t)\} \\
& - \mathbb{E}\bigg\{\int_0^{\mathscr{T}} \Im^x[W_w]\,\mathrm{d}t\bigg\} \\
\leqslant\, & \mathbb{E}\bigg\{\int_0^{\mathscr{T}} x^{\mathrm{t}}(t)[P(i)[A_o(i)+A_d(i)] + [A_o(i) \\
& + A_d(i)]^{\mathrm{t}}P(i) + \sum_{m=1}^{s} \alpha_{im}P(m) + \tau r_1(i)A_o^{\mathrm{t}}(i)A_o(i) + \tau r_2(i)A_d^{\mathrm{t}}(i)A_d(i) \\
& + \tau r_3(i)A_d^{\mathrm{t}}(i)A_d(i) + \tau r_1^{-1}(i)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i) \\
& + \tau r_2^{-1}(i)P(i)A_d(i)A_d^{\mathrm{t}}(i)P(i) + G^{\mathrm{t}}(i)G(i) \\
& + \gamma^{-2}G^{\mathrm{t}}(i)\Phi(i)\bar{\mathscr{R}}^{-1}\Phi^{\mathrm{t}}(i)G(i)]x(t)\,\mathrm{d}t\}
\end{aligned} \tag{47}
$$

where $\bar{\mathscr{R}} := I - \gamma^{-2}\Phi^{\mathrm{t}}(i)\Phi(i) - \gamma^{-2}\tau r_3(i)\Gamma^{\mathrm{t}}(i)\Gamma(i)$. Based on the results of Theorems 3.1 and 4.1, it follows from inequality (47) that $\mathscr{T}(x)<0$ which, in the light of Definition 3.4, means that the uncertain system (Σ_{n}) has a disturbance attenuation γ. By defining $r_1(i)=\varepsilon^{-1}(i), r_2(i)=\mu^{-1}\times(i), r_3(i)=\sigma(i),\ i\in\mathscr{S}$, it follows that for any $\tau\in[0,\tau^*]$ the stochastic stability condition can be cast into the ARIs (40) as desired. □

Theorem 4.3

Consider system $(\Sigma_{\Delta\mathrm{f}})$ satisfying Assumption 4.1. Then given a scalar $\tau^*>0$, the system (Σ_{f}) *is robustly stochastically stable* (*RSS*) *for any constant time-delay* τ *satisfying* $0\leqslant\tau\leqslant\tau^*$ *and for all admissible uncertainties satisfying* (7)–(8) if there exist matrices $0<X(i)=X^{\mathrm{t}}(i)\in\Re^{n\times n},\ i\in\mathscr{S}$, and scalars $\beta_1(i)>0, \beta_2(i)>0, \beta_3(i)>0, \beta_4(i)>0, \beta_5(i)>0, \beta_6(i)>0, \beta_7(i)>0$,

$\delta_5(i) > 0,\ \delta_7(i) > 0,\ i \in \mathscr{S}$ satisfying the system of ARIs

$$\begin{aligned}
&[A_o(i) + A_d(i)]X(i) + X(i)[A_o(i) + A_d(i)]^{\mathrm{t}} + X(i)\sum_{m=1}^{s} \alpha_{im}X(m)X(i) \\
&+ \beta_1^{-1}(i)X(i)[N_a(i) + N_d(i)]^{\mathrm{t}}[N_a(i) + N_d(i)]X(i) + \beta_3^{-1}(i)X(i)N_a^{\mathrm{t}}(i)N_a(i)X(i) \\
&+ \tau X(i)A_o^{\mathrm{t}}(i)[\beta_2 I - \beta_3(i)M_a(i)M_a^{\mathrm{t}}(i)]^{-1}A_o(i)X(i) + \beta_1(i)M_a(i)M_a^{\mathrm{t}}(i) \\
&+ \tau X(i)\{\delta_5^{-1}(i)N_d^{\mathrm{t}}(i)N_d(i) + A_d^{\mathrm{t}}(i)[\beta_4 I - \delta_5(i)M_a(i)M_a^{\mathrm{t}}(i)]^{-1}A_d(i)\}X(i) \\
&+ \tau\beta_7^{-1}(i)M_a(i)M_a^{\mathrm{t}}(i) + \tau A_d(i)[\beta_6 I - \delta_7(i)N_d^{\mathrm{t}}(i)N_d(i)]^{-1}A_d^{\mathrm{t}}(i) < 0
\end{aligned} \tag{48}$$

Proof
In line of Theorem 4.1, we obtain from (25) the dynamics

$$\dot{x}(t) = [A_{\Delta o}(i) + A_{\Delta d}(i)]x(t) - A_{\Delta d}(i)\zeta(x,i)$$

$$\begin{aligned}
\zeta(x,i) = &-\int_{-\tau}^{0} A_{\Delta o}(t+\theta, i)x(t+\theta)\,\mathrm{d}\theta \\
&-\int_{-\tau}^{0} A_{\Delta d}(t-\tau+\theta, i)x(t-\tau+\theta)\,\mathrm{d}\theta
\end{aligned} \tag{49}$$

Introduce the Lyapunov functional $W_\Delta(x,i) = x^{\mathrm{t}}(t)P(i)x(t) + \hat{W}_\Delta(x,i)$ with

$$\begin{aligned}
\hat{W}_\Delta(x,i) = &\int_{-\tau}^{0} \rho(i)\varphi^{-1}(i)\int_{t+\theta}^{\mathrm{t}} \|A_{\Delta o}(s,i)x(s)\|^2\,\mathrm{d}s\,\mathrm{d}\theta \\
&+ \int_{-\tau}^{0} \rho(i)\varphi(i)\int_{t-\tau+\theta}^{\mathrm{t}} \|A_{\Delta d}^{\mathrm{t}}(s-\tau, i)x(s)\|^2\,\mathrm{d}s\,\mathrm{d}\theta
\end{aligned} \tag{50}$$

where $0 < P(i) = P^{\mathrm{t}}(i) \in \Re^{n\times n}, i \in \mathscr{S}$, and $\rho(i) > 0,\ \varphi(i) > 0,\ i \in \mathscr{S}$, are appropriate weighting factors. Evaluating the weak infinitesimal operator $\Im^x[\cdot]$ of the process $\{x(t), \eta_t,\ t \geqslant 0\}$ for system (49) at the point $\{t, x, \eta_t\}$, we have

$$\begin{aligned}
\Im^x[W_\Delta] = x^{\mathrm{t}}(t)\Big\{&P(i)[A_{\Delta o}(i) + A_{\Delta d}(i)] + [A_{\Delta o}(i) + A_{\Delta d}(i)]^{\mathrm{t}}P(i) \\
&+ \sum_{m=1}^{s} \alpha_{im}P(m)\Big\}x(t) + 2x^{\mathrm{t}}(t)P(i)A_{\Delta d}(i)\xi(x,i) + \dot{\hat{W}}_\Delta(x,i)
\end{aligned} \tag{51}$$

Observe that for $i \in \mathscr{S}$, we have

$$\begin{aligned}
2x^{\mathrm{t}}(t)P(i)A_{\Delta d}(i)\zeta(x,i) \leqslant\ &\tau\rho^{-1}(i)x^{\mathrm{t}}(t)P(i)A_{\Delta d}(i)A_{\Delta d}^{\mathrm{t}}(i)P(i)x(t) \\
&+ \tau^{-1}\rho(i)\zeta^{\mathrm{t}}(x,i)\zeta(x,i)
\end{aligned} \tag{52}$$

$$\begin{aligned}\zeta^{\mathrm{t}}(x,i)\zeta(x,i) &\leqslant \varphi^{-1}(i)\left|\left|\int_{-\tau}^{0} A_{\Delta o}(t+\theta,i)x(t+\theta)\,\mathrm{d}\theta\right|\right|^2 \\ &\quad + \varphi(i)\left|\left|\int_{-\tau}^{0} A_{\Delta d}(t-\tau+\theta,i)x(t-\tau+\theta)\,\mathrm{d}\theta\right|\right|^2 \\ &\leqslant \tau\varphi^{-1}(i)\int_{-\tau}^{0} \|A_{\Delta o}(t+\theta,i)x(t)\|^2\,\mathrm{d}\theta \\ &\quad + \tau\varphi(i)\int_{-\tau}^{0} \|A_{\Delta d}(t-\tau+\theta,i)x(t-\tau+\theta)\|^2\,\mathrm{d}\theta \end{aligned} \tag{53}$$

and from (50), we get

$$\begin{aligned}\dot{W}_\Delta(x,i) &= \tau\rho(i)[1+\varphi^{-1}(i)]\|A_{\Delta o}(t,i)x(t)\|^2 + \tau\rho(i)[1+\varphi(i)]\|A_{\Delta d}(t-\tau,i)x(t)\|^2 \\ &\quad - \rho(i)\varphi^{-1}(i)\int_{-\tau}^{0} \|A_{\Delta o}(t+\theta,i)x(t+\theta)\|^2\,\mathrm{d}\theta \\ &\quad - \rho(i)\varphi(i)\int_{-\tau}^{0} \|A_{\Delta d}^{\mathrm{t}}(t-\tau,i)x(t-\tau+\theta)\|^2\,\mathrm{d}s \end{aligned} \tag{54}$$

By grouping (52)–(54) into (51) and arranging terms, we arrive at

$$\begin{aligned}\Im^x[W_\Delta] \leqslant x^{\mathrm{t}}(t)\big\{ &P(i)[A_{\Delta o}(i)+A_{\Delta d}(i)] + [A_{\Delta o}(i)+A_{\Delta d}(i)]^{\mathrm{t}}P(i) \\ &+ \tau\rho^{-1}(i)P(i)A_{\Delta d}(i)A_{\Delta d}^{\mathrm{t}}(i)P(i) + \sum_{m=1}^{s}\alpha_{im}P(m) \\ &+ \tau\rho(i)\varphi^{-1}(i)A_{\Delta o}^{\mathrm{t}}(t,i)A_{\Delta o}(i) + \tau\rho(i)\varphi(i)A_{\Delta d}^{\mathrm{t}}(i)A_{\Delta d}(i)\big\}x(t) \end{aligned} \tag{55}$$

If $\Im^x[W_\Delta]<0$ when $x\neq 0$, then $x(t)\to 0$ as $t\to\infty$. Hence, in view of Theorem 3.1, system $(\Sigma_{\Delta\mathrm{f}})$ is stochastically stable. The stochastic stability condition is met if

$$\begin{aligned}&P(i)[A_{\Delta o}(i)+A_{\Delta d}(i)] + [A_{\Delta o}(i)+A_{\Delta d}(i)]^{\mathrm{t}}P(i) + \tau\rho^{-1}(i)P(i)A_{\Delta d}(i)A_{\Delta d}^{\mathrm{t}}(i)P(i) \\ &+ \sum_{m=1}^{s}\alpha_{im}P(m) + \tau\rho(i)[1+\varphi^{-1}(i)]A_{\Delta o}^{\mathrm{t}}(t,i)A_{\Delta o}(i) \\ &+ \tau\rho(i)[1+\varphi(i)]A_{\Delta d}^{\mathrm{t}}(i)A_{\Delta d}(i) < 0 \end{aligned} \tag{56}$$

for all admissible uncertainties satisfying (7)–(8). Using Facts 1 and 2, one obtains the following:

$$\begin{aligned}&P(i)[A_{\Delta o}(i)+A_{\Delta d}(i)] + [A_{\Delta o}(i)+A_{\Delta d}(i)]^{\mathrm{t}}P(i) \\ &\leqslant P(i)[A_o(i)+A_d(i)] + [A_o(i)+A_d(i)]^{\mathrm{t}}P(i) \\ &+ \delta_1(i)P(i)M_a(i)M_a^{\mathrm{t}}(i)P(i) + \delta_1^{-1}(i)[N_a(i)+N_d(i)]^{\mathrm{t}}[N_a(i)+N_d(i)] \end{aligned} \tag{57}$$

$$\begin{aligned}P(i)A_{\Delta d}(i)A_{\Delta d}^{\mathrm{t}}(i)P(i) &\leqslant \delta_2^{-1}(i)P(i)M_a(i)M_a^{\mathrm{t}}(i)P(i) \\ &\quad + P(i)A_d(i)[I-\delta_2(i)N_d^{\mathrm{t}}(i)N_d(i)]^{-1}A_d^{\mathrm{t}}(i)P(i) \end{aligned} \tag{58}$$

$$A_{\Delta o}^{\mathrm{t}}(t,i)A_{\Delta o}(i) \leqslant \delta_3^{-1}(i)N_a^{\mathrm{t}}(i)N_a(i) + A_o^{\mathrm{t}}(i)[I-\delta_3(i)M_a(i)M_a^{\mathrm{t}}(i)]^{-1}A_o(i) \tag{59}$$

$$A^{\mathrm{t}}_{\Delta d}(i)A_{\Delta d}(i) \leqslant \delta_4^{-1}(i)N_d^{\mathrm{t}}(i)N_d(i) + A_d^{\mathrm{t}}(i)[I - \delta_4(i)M_a(i)M_a^{\mathrm{t}}(i)]^{-1}A_d(i) \tag{60}$$

for some scalars $\delta_1(i) > 0,\ \delta_2(i) > 0,\ \delta_3(i) > 0,\ \delta_4(i) > 0,\ i \in \mathcal{S}$, such that

$$\begin{aligned}&[I - \delta_2(i)N_d^{\mathrm{t}}(i)N_d(i)] > 0, \quad [I - \delta_3(i)M_a(i)M_a^{\mathrm{t}}(i)] > 0,\\ &[I - \delta_4(i)M_a(i)M_a^{\mathrm{t}}(i)] > 0.\end{aligned}$$

From (57)–(60) into (56), it follows that

$$\begin{aligned}&P(i)[A_o(i) + A_d(i)] + [A_o(i) + A_d(i)]^{\mathrm{t}}P(i) + \delta_1^{-1}(i)[N_a(i) + N_d(i)]^{\mathrm{t}}[N_a(i) + N_d(i)]\\ &+ \sum_{m=1}^{s} \alpha_{im}P(m) + \tau\rho(i)\varphi^{-1}(i)A_o^{\mathrm{t}}(i)[I - \delta_3(i)M_a(i)M_a^{\mathrm{t}}(i)]^{-1}A_o(i)\\ &+ \tau\rho(i)\varphi^{-1}(i)\delta_3^{-1}(i)N_a^{\mathrm{t}}(i)N_a(i) + \tau\rho(i)\varphi(i)\{\delta_4^{-1}(i)N_d^{\mathrm{t}}(i)N_d(i)\\ &+ A_d^{\mathrm{t}}(i)[I - \delta_4(i)M_a(i)M_a^{\mathrm{t}}(i)]^{-1}A_d(i)\} + P(i)\delta_1(i)M_a(i)M_a^{\mathrm{t}}(i)P(i)\\ &+ P(i)\{\tau\rho^{-1}(i)\delta_2^{-1}(i)M_a(i)M_a^{\mathrm{t}}(i)\\ &+ \tau\rho^{-1}(i)A_d(i)[I - \delta_2(i)N_d^{\mathrm{t}}(i)N_d(i)]^{-1}A_d^{\mathrm{t}}(i)\}P(i) < 0\end{aligned} \tag{61}$$

for some scalars $\rho(i) > 0,\ \varphi(i) > 0,\ \delta_1(i) > 0,\ \delta_2(i) > 0,\ \delta_3(i) > 0,\ \delta_4(i) > 0,\ i \in \mathcal{S}$, such that

$$\begin{aligned}&[I - \delta_2(i)N_d^{\mathrm{t}}(i)N_d(i)] > 0, \quad [I - \delta_3(i)M_a(i)M_a^{\mathrm{t}}(i)] > 0,\\ &[I - \delta_4(i)M_a(i)M_a^{\mathrm{t}}(i)] > 0\end{aligned}$$

Since (61) is monotonically non-decreasing with respect to τ for any $\tau \in [0, \tau^*]$ the stochastic stability is guaranteed. Let $X(i) = P^{-1}(i)$ and introduce a vector $\beta(i) := [\beta_1(i), \ldots, \beta_7(i)]^{\mathrm{t}}$ which is related to the vector $[\delta_1(i), \ldots, \delta_4(i), \rho, \varphi(i), \phi(i)]^{\mathrm{t}}$. Then for scalars $\beta_1(i) > 0,\ \beta_2(i) > 0,\ \beta_3 \times (i) > 0,\ \beta_4(i) > 0,\ \beta_5(i) > 0,\ \beta_6(i) > 0,\ \beta_7(i) > 0,\ \beta_8(i) > 0,\ \beta_9(i) > 0,\ i \in \mathcal{S}$, such that $[\beta_2 I - \beta_3(i)M_a(i)M_a^{\mathrm{t}}(i)] > 0,\ [\beta_4 I - \delta_5(i)M_a(i)M_a^{\mathrm{t}}(i)] > 0,\ [\beta_6 I - \delta_7(i)N_d^{\mathrm{t}}(i)N_d(i)] > 0$, it follows that stochastic stability condition can be put into the ARIs (47). □

Theorem 4.4

Consider system $(\Sigma_{\Delta n})$ satisfying Assumption 4.1. Then given a scalar $\tau^* > 0$, this system is *RSS for any constant time-delay τ satisfying $0 \leqslant \tau \leqslant \tau^*$ with a disturbance attenuation γ and for all admissible uncertainties satisfying* (7)–(8) if there exist scalars $\sigma(i) > 0,\ i \in \mathcal{S}$, such that

$$\mathcal{M} := \gamma^2 I - \Phi^{\mathrm{t}}(i)\Phi(i) - \tau^*\phi(i)\Gamma^{\mathrm{t}}(i)\Gamma(i) > 0 \tag{62}$$

and matrices $0 < X(i) = X^t(i) \in \Re^{n\times n}$, $i \in \mathscr{S}$, *and scalars* $\beta_1(i) > 0$, $\beta_2(i) > 0$, $\beta_3(i) > 0$, $\beta_4(i) > 0, \beta_5(i) > 0, \beta_6(i) > 0$, $\beta_7(i) > 0, \delta_5(i) > 0$, $\delta_7(i) > 0$, $i \in \mathscr{S}$ *satisfying the system of ARIs*

$$\begin{aligned}
&[A_o(i) + A_d(i)]X(i) + X(i)[A_o(i) + A_d(i)]^t + X(i)\sum_{m=1}^{s} \alpha_{im}X(m)X(i) \\
&+ \beta_1^{-1}(i)X(i)[N_a(i) + N_d(i)]^t[N_a(i) + N_d(i)]X(i) + \beta_3^{-1}(i)X(i)N_a^t(i)N_a(i)X(i) \\
&+ \tau X(i)A_o^t(i)[\beta_2 I - \beta_3(i)M_a(i)M_a^t(i)]^{-1}A_o(i)X(i) \\
&+ \tau A_d(i)[\beta_8 I - \delta_9(i)N_d^t(i)N_d(i)]^{-1}A_d^t(i) + \tau X(i)\{\delta_5^{-1}(i)N_d^t(i)N_d(i) \\
&+ A_d^t(i)[\beta_4 I - \delta_5(i)M_a(i)M_a^t(i)]^{-1}A_d(i)\}X(i) \\
&+ \beta_1(i)M_a(i)M_a^t(i) + \tau\beta_7^{-1}(i)M_a(i)M_a^t(i) + \tau A_d(i)[\beta_6 I - \delta_7(i)N_d^t(i)N_d(i)]^{-1}A_d^t(i) \\
&+ X(i)\{G^t(i)G(i) + \gamma^{-2}G^t(i)\Phi(i)\bar{\mathscr{M}}^{-1}\Phi^t(i)G(i)\}X(i) < 0
\end{aligned} \tag{63}$$

Proof
From (25) we obtain the dynamics

$$\dot{x}(t) = [A_{\Delta o}(i) + A_{\Delta d}(i)]x(t) - A_{\Delta d}(i)\pi(x, i)$$

$$\begin{aligned}
\pi(x,i) = -\bigg\{&\int_{-\tau}^{0} A_{\Delta o}(t+\theta, i)x(t+\theta)\,\mathrm{d}\theta + \int_{-\tau}^{0} A_{\Delta d}(t-\tau+\theta, i)x(t-\tau+\theta)\,\mathrm{d}\theta \\
&+ \int_{-\tau}^{0} \Gamma(i)w(t+\theta)\,\mathrm{d}\theta\bigg\}
\end{aligned} \tag{64}$$

Introduce the Lyapunov functional $W_\Delta(x,i) = W_\Delta(x,i) + W_{\Delta w}(i)$ with

$$W_{\Delta w}(i) = \int_{-\tau}^{0} \phi(i)\int_{t+\theta}^{t} \|\Gamma(s,i)w(s)\|^2\,\mathrm{d}s\,\mathrm{d}\theta \tag{65}$$

where $\hat{W}_\Delta(x,i)$ is given by (50) and $\phi(i) > 0$, $i \in \mathscr{S}$, is a weighting factor. The weak infinitesimal operator $\Im^x[W_{\Delta t}]$ of the process $\{x(t), \eta_t, t \geqslant 0\}$ for system (64) at the point $\{t, x, \eta_t\}$ is evaluated as

$$\begin{aligned}
\Im^x[W_{\Delta t}] = \Im^x[W_\Delta] &- 2x^t(t)P(i)A_{\Delta d}(i)\int_{-\tau}^{0}\Gamma(i)w(t+\theta)\,\mathrm{d}\theta \\
&+ \tau\phi(i)w^t(t)\Gamma^t(i)\Gamma(i)w(t) \\
&- \int_{-\tau}^{0}\phi(i)[w^t(t+\theta)\Gamma^t(i)\Gamma(i)w(t+\theta)]\,\mathrm{d}\theta
\end{aligned} \tag{66}$$

For $i \in \mathscr{S}$, we have

$$
\begin{aligned}
&- 2x^{t}(t)P(i)A_{\Delta d}(i)\int_{-\tau}^{0}\Gamma(i)w(t+\theta)\,\mathrm{d}\theta \\
&\leqslant \phi^{-1}(i)\int_{-\tau}^{0}[x^{t}(t)P(i)A_{\Delta d}(i)A_{\Delta d}^{t}(i)P(i)x(t)]\,\mathrm{d}\theta \\
&+\ \phi(i)\int_{-\tau}^{0}[w^{t}(t+\theta)\Gamma^{t}(i)\Gamma(i)w(t+\theta)]\,\mathrm{d}\theta \\
&=\ \tau\phi^{-1}(i)x^{t}(t)P(i)A_{\Delta d}(i)A_{\Delta d}^{t}(i)P(i)x(t) \\
&+\ \phi(i)\int_{-\tau}^{0}[w^{t}(t+\theta)\Gamma^{t}(i)\Gamma(i)w(t+\theta)]\,\mathrm{d}\theta
\end{aligned} \tag{67}
$$

Using (55) and (67) in (66), we have

$$
\begin{aligned}
\Im^{x}[W_{\Delta t}] \leqslant x^{t}(t)\Bigg\{ & P(i)[A_{\Delta o}(i)+A_{\Delta d}(i)]+[A_{\Delta o}(i)+A_{\Delta d}(i)]^{t}P(i)+\sum_{m=1}^{s}\alpha_{im}P(m) \\
&+\tau\rho^{-1}(i)P(i)A_{\Delta d}(i)A_{\Delta d}^{t}(i)P(i)+\tau\rho(i)[1+\varphi^{-1}(i)]A_{\Delta o}^{t}(t,i)A_{\Delta o}(i) \\
&+\tau\rho(i)[1+\varphi(i)]A_{\Delta d}^{t}(i)A_{\Delta d}(i)+\tau\phi^{-1}(i)P(i)A_{\Delta d}(i)A_{\Delta d}^{t}(i)P(i)\Big\}x(t) \\
&+\tau\phi(i)w^{t}(t)\Gamma^{t}(i)\Gamma(i)w(t)
\end{aligned} \tag{68}
$$

It is readily seen from (68) that $\Im^{x}[W_{\Delta t}]\leqslant 0$. To assess the performance of system $(\Sigma_{\Delta n})$, we use the measure $\mathscr{J}(x)\triangleq\mathbb{E}\{\int_0^{\mathscr{T}}[z^{t}(t)z(t)-\gamma^{2}w^{t}(t)w(t)]\,\mathrm{d}t\}\leqslant\mathbb{E}\{\int_0^{\mathscr{T}}[z^{t}(t)z(t)-\gamma^{2}w^{t}(t)w(t)]\,\mathrm{d}t+\Im^{x}[W_{\Delta t}]\}$. It follows with standard matrix manipulations that

$$
\begin{aligned}
\mathscr{J}(x) \leqslant\ & \mathbb{E}\Bigg\{\int_0^{\mathscr{T}} x^{t}(t)\big\{P(i)[A_{\Delta o}(i)+A_{\Delta d}(i)]+[A_{\Delta o}(i)+A_{\Delta d}(i)]^{t}P(i) \\
&+\sum_{m=1}^{s}\alpha_{im}P(m)+\tau\rho^{-1}(i)P(i)A_{\Delta d}(i)A_{\Delta d}^{t}(i)P(i)+\tau\rho(i)\varphi^{-1}(i)A_{\Delta o}^{t}(t,i)A_{\Delta o}(i) \\
&+\tau\rho(i)\varphi(i)A_{\Delta d}^{t}(i)A_{\Delta d}(i)+\tau\phi^{-1}(i)P(i)A_{\Delta d}(i)A_{\Delta d}^{t}(i)P(i)\big\}x(t) \\
&+\tau\phi(i)w^{t}(t)\Gamma^{t}(i)\Gamma(i)w(t)+\{x^{t}(t)G^{t}(i)+w^{t}(t)\Phi^{t}(i)\} \\
&\times\{G(i)x(t)+\Phi(i)w(t)\}-\gamma^{2}w^{t}(t)w(t)\Big\} \\
\leqslant\ & \mathbb{E}\Bigg\{\int_0^{\mathscr{T}} x^{t}(t)[P(i)[A_{\Delta o}(i)+A_{\Delta d}(i)]+[A_{\Delta o}(i)+A_{\Delta d}(i)]^{t}P(i)+\sum_{m=1}^{s}\alpha_{im}P(m)
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
&+\ \tau\rho^{-1}(i)P(i)A_{\Delta d}(i)A^{\mathrm{t}}_{\Delta d}(i)P(i) + \tau\rho(i)\varphi^{-1}(i)A^{\mathrm{t}}_{\Delta o}(t,i)A_{\Delta o}(i)\\
&+\ \tau\rho(i)\varphi(i)A^{\mathrm{t}}_{\Delta d}(i)A_{\Delta d}(i) +\ \tau\phi^{-1}(i)P(i)A_{\Delta d}(i)A^{\mathrm{t}}_{\Delta d}(i)P(i)\\
&+ G^{\mathrm{t}}(i)G(i) + \gamma^{-2}G^{\mathrm{t}}(i)\Phi(i)\bar{\mathscr{M}}^{-1}\Phi^{\mathrm{t}}(i)G(i)]x(t)\}
\end{aligned}
$$

where $\bar{\mathscr{M}} := I - \gamma^{-2}\Phi^{\mathrm{t}}(i)\Phi(i) - \gamma^{-2}\tau\phi(i)\Gamma^{\mathrm{t}}(i)\Gamma(i)$. Using (57)–(60) in (69) and rearranging, it follows that

$$
\begin{aligned}
&P(i)[A_o(i)+A_d(i)] + [A_o(i)+A_d(i)]^{\mathrm{t}}P(i) + \delta_1^{-1}(i)[N_a(i)+N_d(i)]^{\mathrm{t}}\\
&[N_a(i)+N_d(i)] + \tau\rho(i)\varphi^{-1}(i)\delta_3^{-1}(i)N^{\mathrm{t}}_a(i)N_a(i) + \tau\rho(i)\varphi^{-1}(i)A^{\mathrm{t}}_o(i)\\
&[I-\delta_3(i)M_a(i)M^{\mathrm{t}}_a(i)]^{-1}A_o(i) + \tau\rho(i)\varphi(i)\{\delta_4^{-1}(i)N^{\mathrm{t}}_d(i)N_d(i)\\
&\quad +\ A^{\mathrm{t}}_d(i)[I-\delta_4(i)M_a(i)M^{\mathrm{t}}_a(i)]^{-1}A_d(i)\} + \sum_{m=1}^{s}\alpha_{im}P(m) + G^{\mathrm{t}}(i)G(i)\\
&\quad +\ P(i)\{\tau[\rho^{-1}+\phi^{-1}(i)]\delta_2^{-1}(i)M_a(i)M^{\mathrm{t}}_a(i) + \tau[\rho^{-1}+\phi^{-1}(i)]A_d(i)\\
&[I-\delta_2(i)N^{\mathrm{t}}_d(i)N_d(i)]^{-1}A^{\mathrm{t}}_d(i)\\
&\quad +\ \delta_1(i)M_a(i)M^{\mathrm{t}}_a(i)\}P(i) + \gamma^{-2}G^{\mathrm{t}}(i)\Phi(i)\bar{\mathscr{M}}^{-1}\Phi^{\mathrm{t}}(i)G(i) < 0
\end{aligned}
\tag{70}
$$

for some scalars $\rho(i)>0,\ \varphi(i)>0,\ \delta_1(i)>0,\ \delta_2(i)>0,\ \delta_3(i)>0,\ \delta_4(i)>0,\ i\in\mathscr{S}$, such that $[I-\delta_2(i)N^{\mathrm{t}}_d(i)N_d(i)0$, $[I-\delta_3(i)M_a(i)M^{\mathrm{t}}_a(i)0$, $[I-\delta_4(i)M_a(i)M^{\mathrm{t}}_a(i)0$. Since (70) is monotonically non-decreasing with respect to τ for any $\tau\in[0,\tau^*]$ the stochastic stability is guaranteed. Let $X(i) = P^{-1}(i)$ and introduce a vector $\beta(i) := [\beta_1(i),\ldots,\beta_7(i)]^{\mathrm{t}}$ which is related to the vector $[\delta_1(i),\ldots,\delta_4(i),\rho,\varphi(i),\phi(i)]^{\mathrm{t}}$. Then for some scalars $\beta_1(i)>0,\ldots,\beta_7(i)>0,\ i\in\mathscr{S}$, such that $[\beta_2 I - \beta_3(i)M_a(i)M^{\mathrm{t}}_a(i)0$, $[\beta_4 I - \delta_5(i)M_a(i)M^{\mathrm{t}}_a(i)0$, $[\beta_6 I - \delta_7(i)N^{\mathrm{t}}_d(i)N_d(i)0$, we express the stability condition as the system of ARIs (62)–(63). □

5. EXAMPLES

In order to illustrate the theoretical results of this paper, we provide some numerical examples.

5.1. Example 1

We consider a pilot-scale single-reach water quality system which can fall into type (3)–(5) with $\tau^* = 0.95,\ \tau^+ = 0.5$. Let the Markov process governing the mode switching has generator

$$
\mathfrak{I} = \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}
$$

For the two operating conditions (modes), the associated date are:

Mode 1:

$$A_o(1) = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad A_d(1) = \begin{bmatrix} 0 & 0.3 \\ -0.3 & -0.2 \end{bmatrix}, \quad B_o(1) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$F_o(1) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$\Gamma(1) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad \Phi(1) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad H_a(1) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$$

$$G(1) = [0.1 \quad 0.1], \quad E_a(1) = [0.2 \;\; 0.4], \quad E_d(1) = [0.1 \;\; 0.3], \quad E_b(1) = [0.3 \;\; 0.3]$$

Mode 2:

$$A_o(2) = \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix}, \quad A_d(2) = \begin{bmatrix} -0.5 & -0.6 \\ -0.2 & -0.1 \end{bmatrix}, \quad B_o(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_o(2) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$\Gamma(2) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad \Phi(2) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad H_a(2) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

$$G(2) = [0.1 \;\; 0.1], \quad E_a(2) = [0.4 \;\; 0.2], \quad E_d(2) = [0.3 \;\; 0.1], \quad E_b(2) = [0.1 \;\; 0.1]$$

Initially note that Assumption 4.1 is met for both modes. Using the initial data for $i = 1, 2$

$$Q(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and selecting $\xi(1) = 20$, $\xi(2) = 25$ ensures that $\bar{Q}(1) > 0$, $\bar{Q}(2) > 0$. Invoking the software environment [23], we convert inequalities (13) into LMIS whose feasible solutions are given by

$$P(1) = \begin{bmatrix} 0.5032 & 0.5920 \\ 0.5920 & 2.2628 \end{bmatrix}, \quad P(2) = \begin{bmatrix} 0.5149 & 0.5128 \\ 0.5128 & 1.9912 \end{bmatrix}$$

This verifies Theorem 3.1 and in turn confirms the robust stochastic stability independent of delay. Next, we solve the LMIs equivalent of ARIs (20) to get

$$P(1) = \begin{bmatrix} 0.4574 & 0.4180 \\ 0.4180 & 1.9415 \end{bmatrix}, \quad P(2) = \begin{bmatrix} 0.5636 & 0.5628 \\ 0.5628 & 2.2182 \end{bmatrix}, \quad \gamma = 3.1428$$

Since $P(1) > 0$, $P(2) > 0$, Theorem 3.2 is validated. Turning to ARIs (27), in view of Remark 3.2, we solve the resulting LMIs with

$$Q(1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q(2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and selecting $\xi(1) = 20,\ \xi(2) = 25$ ensures that $\bar{Q}(1) > 0,\ \bar{Q}(2) > 0$. We obtain

$$P(1) = \begin{bmatrix} 2.9930 & 2.3143 \\ 2.3143 & 5.2018 \end{bmatrix}, \quad \varepsilon(1) = 0.3112, \quad \rho(1) = 0.5009$$

$$P(2) = \begin{bmatrix} 3.1172 & 1.2844 \\ 1.2844 & 6.0789 \end{bmatrix}, \quad \varepsilon(2) = 1.2632, \quad \rho(2) = 1.9743$$

Once again, since $P(1) > 0,\ P(2) > 0$, Theorem 3.3 is validated. On solving the LMIs equivalent to (30), we get

$$P(1) = \begin{bmatrix} 3.0124 & 2.1850 \\ 2.1850 & 7.3122 \end{bmatrix}, \quad \varepsilon(1) = 1.2548, \quad \rho(1) = 0.7598$$

$$P(2) = \begin{bmatrix} 3.4520 & 2.0843 \\ 2.0843 & 8.1065 \end{bmatrix}, \quad \varepsilon(2) = 3.4398, \quad \rho(2) = 2.4784$$

which verifies Theorem 3.4.

Focusing on the delay-dependent stability, we solve inequality (31) with $r_1(1) = 1,\ r_2(1) = 2$, and $r_2(2) = 2$. The feasible results are

$$P(1) = \begin{bmatrix} 4.9377 & 2.2076 \\ 2.2076 & 2.1177 \end{bmatrix}, \quad P(2) = \begin{bmatrix} 5.5537 & 0.6743 \\ 0.6743 & 1.8289 \end{bmatrix}, \quad \tau^* = 0.2105$$

This reads that the water quality system is stochastically stable for any constant time-delay τ satisfying $0 \leqslant \tau \leqslant 0.2105$. Next, we solve inequality (48) with $r_1(1) = 1, r_2(1) = 2, r_1(2) = 1, r_2(2) = 2$. The feasible results are:

$$\begin{aligned}
P(1) &= \begin{bmatrix} 4.7985 & 2.1054 \\ 2.1054 & 2.1168 \end{bmatrix}, \quad \beta_1(1) = 0.2987, \ \ \beta_2(1) = 1.0374, \ \ \beta_3(1) = 0.8117 \\
\beta_4(1) &= 1.1577, \quad \beta_5(1) = 2.9487, \quad \beta_6(1) = 5.2066 \\
\beta_7(1) &= 0.2987, \quad \delta_5(1) = 11.0321, \quad \delta_7(1) = 9.6108 \\
P(2) &= \begin{bmatrix} 5.4325 & 0.5546 \\ 0.5546 & 1.9065 \end{bmatrix}, \quad \beta_1(2) = 0.3164, \quad \beta_2(2) = 0.7374, \\
\beta_3(2) &= 1.2870 \\
\beta_4(2) &= 1.5063, \quad \beta_5(2) = 6.0094, \quad \beta_6(2) = 4.1286 \\
\beta_7(2) &= 0.3358, \quad \delta_5(2) = 1.9987, \quad \delta_7(1) = 7.5671, \quad \tau^* = 0.3098
\end{aligned}$$

On comparing these results with the foregoing ones, one finds that the upper bound on τ for delay-independent stability was set at 0.95, where it is determined for delay-dependent stability as 0.2105 for the nominal case and as 0.3098 for the uncertain case. This clearly emphasizes the fact that stochastic stability independent of delay is more conservative than delay-dependent stochastic stability.

6. $\mathscr{H}_\infty$-STATE FEEDBACK CONTROLLER

In this section, we consider the design of an $\mathscr{H}_\infty$-state feedback controller for system (Σ_J) subject to (7)–(8) and initially we focus on the nominal system for $\eta_t = i \in \mathscr{S}$.

$$(\Sigma_{Jo}) : \dot{x}(t) = A_o(i)x(t) + A_d(i)x(t-\tau) + B_o(i)u(t) + \Gamma(i)w(t), \quad t \geqslant 0$$

$$x(t) = \phi(t), \quad t \in [-\tau, 0], \quad \eta_o = i \tag{71}$$

$$y(t) = x(t) \tag{72}$$

$$z(t) = G(i)x(t) + \Phi(i)w(t) + F_o(i)u(t) \tag{73}$$

Define the following matrix expressions for $i \in \mathscr{S}$:

$$\Lambda(i) = \gamma^2 I - \Phi^{\mathrm{t}}(i)\ \Phi(i), \quad \Pi(i) = I - \gamma^2\Phi(i)\Phi^{\mathrm{t}}(i)$$

$$\Xi(i) = F_o^{\mathrm{t}}(i)\Pi(i)F_o(t), \quad \Omega(i) = P(i)\Gamma(i) + G^{\mathrm{t}}(i)\Phi(i)$$

$$\Theta_o(i) = P(i)A_o(i) + A_o^{\mathrm{t}}(i)P(i) + \sum_{m=1}^{s} \alpha_{im}P(m) + G^{\mathrm{t}}(i)G(i) + \hat{Q}(i)$$

$$\Sigma(i) = [P(i)B_o(i) + \Omega(i)\Lambda(i)\Phi^{\mathrm{t}}(i)F_o(i) + G^{\mathrm{t}}(i)F_o(i)]\{F_o^{\mathrm{t}}(i)\Pi(i)F_o(i)\}^{-1}$$

$$[B_o^{\mathrm{t}}(i)P(i) + F_o^{\mathrm{t}}(i)G(i) + F_o^{\mathrm{t}}(i)\Phi(i)\Lambda(i)\Omega^{\mathrm{t}}(i)] \tag{74}$$

The main result is established by the following theorem.

Theorem 6.1

Consider system (Σ_{Jo}). Then, for a given $\gamma > 0$, there exists a state-feedback controller $u(t)$ such that the closed-loop system is SS and

$$\|z(t)\|_{E_2} < \gamma \|w(t)\|_2$$

for all non-zero $w(t) \in \mathscr{L}_2[0, \infty)$, if for given matrix $Q(i) = Q^{\mathrm{t}}(i) > 0$, $i \in \mathscr{S}$, there exist matrices $P(i) = P^{\mathrm{t}}(i) > 0$, $i \in \mathscr{S}$, satisfying the system of ARIs

$$\begin{aligned} P(i)A_o(i) + A_o^{\mathrm{t}}(i)P(i) + \sum_{m=1}^{s} \alpha_{im}P(m) + \hat{Q}(i) + G^{\mathrm{t}}(i)G(i) \\ + P(i)A_d(i)\bar{Q}^{-1}(i)A_d^{\mathrm{t}}(i)P(i) - \Sigma(i) < 0, \quad i \in \mathscr{S} \end{aligned} \tag{75}$$

Moreover, the feedback controller is $u(t) = K^*(i)x(t)$ has the gain

$$K^*(i) = -\{F_o^t(i)\Pi(i)F_o(i)\}^{-1}[B_o^t(i)P(i) + F_o^t(i)\Pi(i)G(i) + F_o^t(i)\Phi(i)\Lambda(i)\Omega^t(i)], \quad i \in \mathscr{S} \tag{76}$$

where $\bar{Q}(i)$, $\hat{Q}(i)$ are as stated in Theorem 3.3.

Proof

Consider the Lyapunov functional $V(\cdot)$ as the form of (14). Evaluating the weak infinitesimal operator $\mathfrak{I}_f^x[\cdot]$ of the process $\{x(t), \eta_t,\ t \geqslant 0\}$ for system (71)–(73) at the point $\{t, x, \eta_t = i\}$, we get

$$\begin{aligned}
\mathfrak{I}_f^x[V] &= x^t(t)\left\{A_o^t(i)P(i) + P(i)A_o(i) + \sum_{m=1}^{s} \alpha_{im}P(m) + Q(i)\right. \\
&\quad \left. + \sum_{m=1}^{s} \alpha_{im}\int_{-\tau}^{0} x^t(t+\theta)Q(i)x(t+\theta)\,\mathrm{d}\theta\right\}x(t) + x^t(t-\tau)A_d^t(i)P(i)x(t) \\
&\quad + x^t(t)P(i)A_d(i)x(t-\tau) - x^t(t-\tau)Q(i)x(t-\tau) + x^t(t)P(i)\Gamma(i)w(t) \\
&\quad + w^t(t)\Gamma^t(i)P(i)x(t) + x^t(t)P(i)B_o(i)u(t) + u^t(t)B_o^t(i)P(i)x(t) \\
&\leqslant x^t(t)\left\{A_o^t(i)P(i) + P(i)A_o(i) + \sum_{m=1}^{s} \alpha_{im}P(m) + \hat{Q}(i)\right\}x(t) \\
&\quad + x^t(t-\tau)A_d^t(i)P(i)x(t) + x^t(t)P(i)A_d(i)x(t-\tau) \\
&\quad - x^t(t-\tau)\bar{Q}(i)x(t-\tau) + x^t(t)P(i)\Gamma(i)w(t) + w^t(t)\Gamma^t(i)P(i)x(t) \\
&\quad + x^t(t)P(i)B_o(i)u(t) + u^t(t)B_o^t(i)P(i)x(t)
\end{aligned} \tag{77}$$

Given the performance measure $\mathscr{J}$ with $\mathscr{T} \to \infty$, algebraic manipulation in the manner of Theorem 4.4 using (73)–(74), (77) and the control law $u(i) = K^*(i)x(i)$ yields

$$\mathscr{J}(x) \leqslant \mathbb{E}\int_0^\infty \chi^t(i)\Upsilon_f(i)\chi(i)\,\mathrm{d}t \tag{78}$$

where

$$\Upsilon_f(i) = \begin{bmatrix} \Theta_o(i) + \Theta_k(i) & P(i)A_d(i) & \Omega(i) + K^{*t}(i)F_o^t\Phi(i) \\ A_d^t(i)P(i) & -\bar{Q}(i) & 0 \\ \Omega^t(i) + \Phi^t(i)F_o(i)K^*(i) & 0 & -\Lambda(i) \end{bmatrix}$$

$$\chi(i) = [x^t(i) \quad x^t(t-\tau) \quad w^t(i)]$$

$$\begin{aligned}
\Theta_k(i) &= G^t(i)F_o(i)K^*(i) + K^{*t}(i)F_o^t(i)G(i) + K^{*t}(i)F_o^t(i)F_o(i)K^*(i) \\
&\quad + K^{*t}(i)B_o^t(i)P(i) + P(i)B_o(i)K^*(i)
\end{aligned} \tag{79}$$

Application of Fact 1 to (78)–(79) gives

$$\begin{aligned}\mathcal{J}(x) \leqslant \mathbb{E}\Big\{\int_0^\infty & x^t(t)[\Theta_o(i) + \Theta_k(i) + P(i)A_d(i)\bar{Q}^{-1}(i)A_d^t(i)P(i) \\ & + (\Omega(i) + K^{*t}(i)F_o^t(i)\Phi(i)]\Lambda^{-1}(i)[\Omega^t(i) + \Phi^t(i)F_o(i)K^*(i))]x(i)\,dt\Big\}\end{aligned} \quad (80)$$

Finally, using the matrix inversion lemma [15] and the controller gain (76), it follows from (74), (75), (80) and the results of Theorem 3.1, that $\mathcal{J}(x) < 0$ and hence the resulting closed-loop system is stochastically stable with disturbance attenuation γ, which completes the proof. □

The next theorem provides an expression of $\mathcal{H}_\infty$ state feedback controller for system (Σ_J).

Theorem 6.2

Consider system (Σ_J). Then, for a given $\gamma > 0$, there exists a state-feedback controller $u \times (t) = K^+(i)x(t)$ such that the closed-loop system is robustly stochastically stable and

$$||z(t)||_{E_2} < \gamma||w(t)||_2$$

for all non-zero $w(t) \in \mathcal{L}_2[0,\infty)$, and all admissible parameter uncertainties satisfying (7)–(8), if given matrices $Q(i) = Q^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $P(i) = P^t(i) > 0$, $i \in \mathcal{S}$, and scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\mu(i) > 0$, $i \in \mathcal{S}$ satisfying the system of LMIs

$$\begin{bmatrix} \Theta_o(i) - \Sigma(i) + \varepsilon_1(i)N_a^t(i)N_a(i) & P(i)M_a(i) & P(i)M_a(i) & P(i)M_a(i) & P(i)A_d(i) \\ M_a^t(i)P(i) & -\varepsilon_1(i)I & 0 & 0 & 0 \\ M_a^t(i)P(i) & 0 & -\varepsilon_2(i)I & 0 & 0 \\ M_a^t(i)P(i) & 0 & 0 & -\mu(i)I & \\ A_d^t(i)P(i) & 0 & 0 & 0 & -\bar{Q}(i) + \varepsilon_2(i)N_d^t(i)N_d(i) \end{bmatrix} < 0$$

$$\begin{bmatrix} -\bar{Q}(i) & N_d^t(i) \\ N_d(i) & -\varepsilon_2(i)I \end{bmatrix} < 0, \quad i \in \mathcal{S} \quad (81)$$

where $\bar{Q}(i)$, $\hat{Q}(i)$ are as stated in Theorem 3.3. Moreover, the feedback gain is given by

$$\begin{aligned}K^+(i) = & -\{F_o^t(i)\Pi(i)F_o(i) + \mu(i)N_b^t(i)N_b(i)\}^{-1}[B_o^t(i)P(i) + F_o^t(i)\Pi(i)G(i) \\ & + F_o^t(i)\Phi(i)\Lambda(i)\Omega^t(i)]\end{aligned} \quad (82)$$

Proof

Consider the Lyapunov functional $V(\cdot)$ as the form of (14). Evaluating the weak infinitesimal operator $\mathfrak{I}_t^x[\cdot]$ of the process $\{x(t), \eta_t, t \geqslant 0\}$ for system (3)–(5) at the point $\{t, x, \eta_t = i\}$ using

(77), we get

$$\begin{aligned}\Im_t^x[V] &\leqslant \Im_f^x[V] + x^t(t)\{\Delta A_o^t(i)P(i) + P(i)\Delta A_o(i)\}x(t)\\ &+ x^t(t-\tau)\Delta A_d^t(i)P(i)x(t) + x^t(t)P(i)\Delta A_d(i)x(t-\tau)\\ &+ u^t(t)\Delta B_o^t(i)P(i)x(i) + x^t(i)P(i)\Delta B_o(i)u(t)\\ &\leqslant \Im_f^x[V] + x^t(t)[\varepsilon_1^{-1}(i) + \varepsilon_2^{-1}(i) + \mu^{-1}(i)]P(i)M_a(i)M_a^t(i)P(i)x(i)\\ &+ \varepsilon_1(i)x^t(t)N_a^t(i)N_a(i)x(t) + \varepsilon_2(i)x^t(t-\tau)N_d^t(i)N_d(i)x(t-\tau)\\ &+ \mu(i)u^t(t)N_b^t(i)N_b(i)u(t)\end{aligned} \tag{83}$$

for some scalars $\varepsilon_1(i) > 0$, $\varepsilon_2 > 0$, $\mu(i) > 0$, $i \in \mathscr{S}$. On substituting $u(t) = K^+(i)x(i)$, we evaluate the performance measure $\mathscr{T} \to \infty$ in the manner of Theorem 6.1 to get

$$\mathscr{J}(x) \leqslant \mathbb{E}\int_0^\infty \chi^t(i)\Upsilon_t(i)\chi(i)\,\mathrm{d}t \tag{84}$$

where

$$\Upsilon_t(i) = \begin{bmatrix} \Theta_o(i) + \Theta_t(i) & P(i)A_d(i) & \Omega(i) + K^{+t}(i)F_o^t(i)\Phi(i) \\ A_d^t(i)P(i) & -\bar{Q}(i) + \varepsilon_2(i)N_d^t(i)N_d(i) & 0 \\ \Omega^t(i) + \Phi^t(i)F_o(i)K^+(i) & 0 & -\Lambda(i)\end{bmatrix}$$

$$\begin{aligned}\Theta_t(i) = &[\varepsilon_1^{-1}(i) + \varepsilon_2^{-1}(i) + \mu^{-1}(i)]P(i)M_a(i)M_a^t(i)P(i) + G^t(i)F_o(i)K^+(i)\\ &+ K^{+t}(i)F_o^t(i)G(i) + K^{+t}(i)F_o^t(i)F_o(i)K^+(i) + K^{+t}(i)B_o^t(i)P(i)\\ &+ P(i)B_o(i)K^+(i) + \varepsilon_1(i)N_a^t(i)N_a(i) + \mu(i)K^{+t}(i)N_b^t(i)N_b(i)K^+(i)\end{aligned} \tag{85}$$

Application of Fact 1 to (85) using (74) gives

$$\begin{aligned}\mathscr{J}(x) \leqslant \mathbb{E}\Big\{\int_0^\infty & x^t(t)[\Theta_o(i) + \Theta_t(i) + P(i)A_d(i)[\bar{Q}(i) - \varepsilon_2(i)N_d^t(i)N_d(i)]^{-1}A_d^t(i)P(i)\\ &+ (\Omega(i) + K^{+t}(i)F_o^t(i)\Phi(i)]\Lambda^{-1}(i)[\Omega^t(i) + \Phi^t(i)F_o(i)K^+(i))]x(i)\,\mathrm{d}t\Big\}.\end{aligned} \tag{86}$$

Substituting the controller gain (82) and arranging terms into the LMIs (81), it follows in the manner of Theorem 6.1 that $\mathscr{J}(x) < 0$ and hence the resulting closed-loop system is stochastically stable with disturbance attenuation γ for all admissible parameter uncertainties, which completes the proof. □

Remark 6.1
It should be emphasized that inequalities (81) are standard LMIs in the variables $P(i)$, $\mu(i)$, $\varepsilon_1(i)$, $\varepsilon_2(i)$, $i \in \mathscr{S}$ [22].

Remark 6.2
Extension of the developed robustness results can be made to the case where the jumping rates are subject to uncertainties. Specifically, we consider the transition probability from mode i at

time t to mode j at time $t+\delta$, $i, j \in \mathscr{S}$, to be

$$p_{ij} = Pr(\eta_{t+\delta} = j \mid \eta_t = i) = \begin{cases} (\alpha_{ij} + \Delta\alpha_{ij})\delta + o(\delta) & \text{if } i \neq j \\ 1 + (\alpha_{ij} + \Delta\alpha_{ij})\delta + o(\delta) & \text{if } i = j \end{cases} \tag{87}$$

with transition probability rates $(\alpha_{ij} + \Delta\alpha_{ij}) \geqslant 0$ for $i, j \in \mathscr{S}$, $i \neq j$ and

$$\alpha_{ii} + \Delta\alpha_{ii} = -\sum_{m=1, m\neq i}^{s} (\alpha_{im} + \Delta\alpha_{im}) \tag{88}$$

We assume that the uncertainties $\Delta\alpha_{ij}$ satisfy

$$\|\Delta\alpha_{ij}\| \leqslant \beta_{ij}, \quad \forall i, j \in \mathscr{S} \tag{89}$$

where β_{ij} are known scalars, $\forall i, j \in \mathscr{S}$.

In line of Theorems 6.1 and 6.2, we have the following robustness results:

Theorem 6.3
Consider system (Σ_{Jo}) over the space $(\Omega, \mathscr{F}, \mathbf{P})$ where $\mathbf{P}$ is described by (87)–(89). Then, for a given $\gamma > 0$, there exists a state-feedback controller $u(t) = K^*(i)x(t)$ such that the closed-loop system is stochastically stable and

$$\|z(t)\|_{E_2} < \gamma \|w(t)\|_2$$

for all non-zero $w(t) \in \mathscr{L}_2[0, \infty)$, if for given matrices $Q(i) = Q^{\mathrm{t}}(i) > 0$, $i \in \mathscr{S}$, there exist matrices $P(i) = P^{\mathrm{t}}(i) > 0$, satisfying the system of ARIs

$$\begin{aligned} P(i)A_o(i) + A_o^{\mathrm{t}}(i)P(i) + \sum_{m=1}^{s} (\alpha_{im} + \beta_{im})P(m) + \hat{Q}(i) + G^{\mathrm{t}}(i)G(i) \\ + P(i)A_d(i)\bar{Q}^{-1}(i)A_d^{\mathrm{t}}(i)P(i) - \Sigma(i) < 0, \quad i \in \mathscr{S} \end{aligned} \tag{90}$$

where $\bar{Q}(i)$, $\hat{Q}(i)$, $i \in \mathscr{S}$ are as stated in Theorem 3.3. Moreover, the feedback gain $K^*(i)$ is given by (76).

Theorem 6.4
Consider system (Σ_J) over the space $(\Omega, \mathscr{F}, \mathbf{P})$ where $\mathbf{P}$ is described by (87)–(89). Then, for a given $\gamma > 0$, there exists a state-feedback controller $u(t) = K^+(i)x(t)$ such that the closed-loop system is stochastically stable and

$$\|z(t)\|_{E_2} < \gamma \|w(t)\|_2$$

for all non-zero $w(t) \in \mathscr{L}_2[0, \infty)$, and all admissible parameter uncertainties, if for given matrices $Q(i) = Q^{\mathrm{t}}(i) > 0$, $i \in \mathscr{S}$, there exist matrices $P(i) = P^{\mathrm{t}}(i) > 0$, $i \in \mathscr{S}$, and scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$,

$\mu(i) > 0$, $i \in \mathscr{S}$ satisfying the system of LMIs

$$\begin{bmatrix} \bar{\Theta}_o(i) - \Sigma(i) & & & & \\ + \varepsilon_1(i)N_a^t(i)N_a(i) & P(i)M_a(i) & P(i)M_a(i) & P(i)M_a(i) & P(i)A_d(i) \\ M_a^t(i)P(i) & -\varepsilon_1(i)I & 0 & 0 & 0 \\ M_a^t(i)P(i) & 0 & -\varepsilon_2(i)I & 0 & 0 \\ M_a^t(i)P(i) & 0 & 0 & -\mu(i)I & \\ A_d^t(i)P(i) & 0 & 0 & 0 & -\bar{Q}(i) \\ & & & & + \varepsilon_2(i)N_d^t(i)N_d(i) \end{bmatrix} < 0$$

$$\begin{bmatrix} -\bar{Q}(i) & N_d^t(i) \\ N_d(i) & \varepsilon_2(i)I \end{bmatrix} < 0, \quad i \in \mathscr{S} \tag{91}$$

where $\bar{Q}(i)$, $\hat{Q}(i)$ are as stated in Theorem 3.3, the feedback gain $K^+(i)$ is given by (82) and

$$\bar{\Theta}_o(i) = P(i)A_o(i) + A_o^t(i)P(i) + \sum_{m=1}^{s} (\alpha_{im} + \beta_{im})P(m) + G^t(i)G(i) + \hat{Q}(i)$$

7. CONCLUSIONS

We have investigated the problems of stochastic stability and stabilization for a class of linear time-delay systems with Markovian jump parameters. Then, we have designed a state feedback controller such that stochastic stability and a prescribed $\mathscr{H}_\infty$-performance are guaranteed. Complete results for both delay-independent and delay-dependent stochastic stability criteria for the nominal and uncertain time-delay jumping systems have been developed. Robust $\mathscr{H}_\infty$-control problems for linear systems with Markovian jump parameters and parametric uncertainties have also been studied where the parametric uncertainties are assumed to be real, time-varying and norm-bounded that appear in the state, input and delayed-state matrices. We have established that the control problem for the system under consideration with and without uncertain parameters can be essentially solved in terms of the solutions of a finite set of coupled algebraic Riccati or linear matrix inequalities. The robust $\mathscr{H}_\infty$-control problem with uncertain jumping rates has also been examined.

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