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Power series of the operators U_n^ϱ

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Abstract We study power series of members of a class of positive linear operators reproducing linear function constituting a link between genuine Bernstein-Durrmeyer and classical Bernstein operators. Using the eigenstructure of the operators we give a non-quantitative convergence result towards the inverse Voronovskaya operators. We include a quantitative statement via a smoothing approach.

Keywords Power series · Geometric series · Positive linear operator · Bernstein-type operator · Genuine Bernstein-Durrmeyer operator · Degree of approximation · Eigenstructure · Moduli of continuity

Mathematics Subject Classification (2000) 41A10 · 41A17 · 41A25 · 41A36

1 Introduction

The present note is essentially motivated by two key papers of Păltănea which both appeared in two hardly known local Romanian journals. In the first article mentioned [9] Păltănea defined power series of Bernstein operators (with n fixed) and studied their approximation behaviour for functions defined on the space $C_0[0,1] := \{f \mid f(x) =$

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$x(1-x)h(x), h \in C[0,1]\}$ to some extent. This article motivated a number of authors to study similar problems or give different proofs of Păltănea's main result. See [1], [2], [3], [11]. In one more and most significant article Păltănea [10] introduced a very interesting link between the classical Bernstein operators B_n and the so-called "genuine Bernstein-Durrmeyer operators" U_n, thus also bridging the gap between U_n and piecewise linear interpolation in a most elegant way for the cases $0 < \varrho \le 1$. The operators U_n^ρ also attracted several authors to study them further. See, for example, [6], [7]. In the present note we combine both approaches of Păltănea and study power (geometric) series of the operators U_n^ϱ, thus bridging the gap between power series of Bernstein operators and such of the genuine operators U_n mentioned above.

Our main results will concern the convergence of the series as n (the degree of the polynomials inside the series) tends to infinity. The first non-quantitative theorem will essentially use the eigenstructure of the U_n^ϱ which was recently studied in [8].

The second result describes the degree of convergence to the "inverse Voronovskaya operators" $-A_\varrho^{-1}$ using a smoothing (K- functional) approach and makes use of exact representations of the moments as presented in [6].

The quantitative statement also holds in the limiting case of Bernstein operators, thus supplementing the original work of Păltănea.

2 The operators U_n^ϱ and their eigenstructure

Denote by $C[0,1]$ the space of continuous, real-valued functions on $[0,1]$ and by Π_n the space of polynomials of degree at most $n \in \mathbb{N}_0 := \{0, 1, 2...\}$.

Definition 1 Let $\varrho > 0$ and $n \in \mathbb{N}_0$, $n \ge 1$. Define the operator $U_n^\varrho : C[0,1] \to \Pi_n$ by

$$\begin{aligned} U_n^\varrho(f,x) &:= \sum_{k=0}^{n} F_{n,k}^\varrho(f) p_{n,k}(x) \\ &:= \sum_{k=1}^{n-1} \left(\int_0^1 \frac{t^{k\varrho-1}(1-t)^{(n-k)\varrho-1}}{B(k\varrho,(n-k)\varrho)} f(t)dt \right) p_{n,k}(x) \\ &\quad + f(0)(1-x)^n + f(1)x^n, \end{aligned}$$

$f \in C[0,1]$, $x \in [0,1]$ and $B(\cdot,\cdot)$ is Euler's Beta function. The fundamental functions $p_{n,k}$ are defined by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \le k \le n, \quad x \in [0,1].$$

For $\varrho = 1$ and $f \in C[0,1]$, we obtain

$$\begin{aligned} U_n^1(f,x) = U_n(f,x) = (n-1) \sum_{k=1}^{n-1} \left(\int_0^1 f(t) p_{n-2,k-1}(t)dt \right) p_{n,k}(x) \\ +(1-x)^n f(0) + x^n f(1), \end{aligned}$$

where U_n are the "genuine" Bernstein-Durrmeyer operators (see [6, Th. 2.3]), while for $\varrho \to \infty$, for each $f \in C[0,1]$ the sequence $U_n^\varrho(f,x)$ converges uniformly to the Bernstein polynomial

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x).$$

Moreover, for n fixed and $\varrho \to 0$ one has uniform convergence of $U_n^\varrho f$ towards the first Bernstein polynomial $B_1 f$, i.e., linear interpolation at 0 and 1 (see [7, Th. 3.2]).

The eigenstructure of U_n^ϱ is described in [8]. The numbers

$$\lambda_{\varrho,j}^{(n)} := \frac{\varrho^j n!}{(n\varrho)^{\overline{j}}(n-j)!}, \quad j = 0, 1, ..., n, \tag{1}$$

are eigenvalues of U_n^ϱ. To each of them there corresponds a monic eigenpolynomial $p_{\varrho,j}^{(n)}$ such that $deg\ p_{\varrho,j}^{(n)} = j,\ \ j = 0, 1, ..., n$. In particular,

$$p_{\varrho,0}^{(n)}(x) = 1,\ p_{\varrho,1}^{(n)}(x) = x - \frac{1}{2},\ x \in [0,1]. \tag{2}$$

A complete description of $p_{\varrho,j}^{(n)}(x)$, $j = 2, ..., n$, can be found in [8]. From ([8], (3.14)) we get

$$p_{\varrho,j}^{(n)}(0) = p_{\varrho,j}^{(n)}(1) = 0,\ j = 2, ..., n. \tag{3}$$

Obviously $U_n^\varrho f$ can be decomposed with respect to the basis $\{p_{\varrho,0}^{(n)}, p_{\varrho,1}^{(n)}, ..., p_{\varrho,n}^{(n)}\}$ of Π_n; this allows us to introduce the dual functionals $\mu_{\varrho,j}^{(n)} : C[0,1] \to \mathbb{R}$, $j = 0, 1, ..., n$, by means of the formula

$$U_n^\varrho f = \sum_{j=0}^{n} \lambda_{\varrho,j}^{(n)} \mu_{\varrho,j}^{(n)}(f) p_{\varrho,j}^{(n)},\ f \in C[0,1]. \tag{4}$$

In particular, since U_n^ϱ restricted to Π_n is bijective, we have

$$p = \sum_{j=0}^{n} \mu_{\varrho,j}^{(n)}(p) p_{\varrho,j}^{(n)},\ p \in \Pi_n. \tag{5}$$

Now consider the numbers

$$\lambda_{\varrho,j} := -\frac{\varrho+1}{2\varrho}(j-1)j,\ j = 0, 1, ... \tag{6}$$

and the monic polynomials

$$p_0^*(x) = 1,\ p_1^*(x) = x - \frac{1}{2},\ p_j^*(x) = x(x-1)P_{j-2}^{(1,1)}(2x-1),\ j \geq 2, \tag{7}$$

where $P_i^{(1,1)}(x)$ are Jacobi polynomials, orthogonal with respect to the weight $(1-x)(1+x)$ on $[-1,1], i \geq 0$. Moreover, consider the linear functionals $\mu_j^* : C[0,1] \to \mathbb{R}$, defined as

$$\mu_0^*(f) = \frac{f(0)+f(1)}{2},\ \mu_1^*(f) = f(1) - f(0), \tag{8}$$

$$\mu_j^*(f) = \frac{1}{2}\binom{2j}{j}[(-1)^j f(0) + f(1) - j\int_0^1 f(x)P_{j-2}^{(1,1)}(2x-1)dx],\ j \geq 2. \tag{9}$$

It is easy to verify that

$$\lim_{n\to\infty} n(\lambda_{\varrho,j}^{(n)} - 1) = \lambda_{\varrho,j},\ j \geq 0. \tag{10}$$

The following result can be found in [8].

Theorem 1 ([8]) *For each $j \geq 0$ we have*

$$\lim_{n\to\infty} p_{\varrho,j}^{(n)} = p_j^*, \text{ uniformly on } [0,1], \tag{11}$$

$$\lim_{n\to\infty} \mu_{\varrho,j}^{(n)}(p) = \mu_j^*(p),\ p \in \Pi. \tag{12}$$

3 The power series A_n^ϱ

Consider the space

$$C_0[0,1] := \{f \mid f(x) = x(1-x)h(x), h \in C[0,1]\}. \tag{13}$$

For $f \in C_0[0,1]$, $f(x) = x(1-x)h(x)$, define the norm

$$||f||_0 := ||h||_\infty. \tag{14}$$

Endowed with the norm $||\cdot||_0$, $C_0[0,1]$ is a Banach space. Obviously,

$$||f||_\infty \leq \frac{1}{4}||f||_0,\ f \in C_0[0,1]. \tag{15}$$

Lemma 1 *As a linear operator on* $(C_0[0,1], ||\cdot||_0)$, U_n^{ϱ} *has the norm*

$$||U_n^{\varrho}||_0 = \frac{(n-1)\varrho}{n\varrho+1} < 1. \tag{16}$$

Proof Let $f \in C_0[0,1]$, $f(x) = x(1-x)h(x)$, $h \in C[0,1]$. By straightforward computation we get $U_n^{\varrho} f(x) = x(1-x)u(x)$, where

$$u(x)(n-1)\sum\nolimits_{k=1}^{n-1} \frac{\int_0^1 t^{k\varrho}(1-t)^{(n-k)\varrho}h(t)dt}{k(n-k)B(k\varrho,(n-k)\varrho)} p_{n-2,k-1}(x).$$

It follows immediately that $U_n^{\varrho} f \in C_0[0,1]$ and

$$||U_n^{\varrho} f||_0 = ||u||_\infty \leq \frac{(n-1)\varrho}{n\varrho+1}||h||_\infty = \frac{(n-1)\varrho}{n\varrho+1}||f||_0.$$

Thus

$$||U_n^{\varrho}||_0 \leq \frac{(n-1)\varrho}{n\varrho+1}. \tag{17}$$

On the other hand, let $g(x) = x(1-x)$, $x \in [0,1]$. Then $||g||_0 = 1$ and $U_n^{\varrho} g(x) = x(1-x)\frac{(n-1)\varrho}{n\varrho+1}$, which entails $||U_n^{\varrho} g||_0 = \frac{(n-1)\varrho}{n\varrho+1}$ and so

$$||U_n^{\varrho}||_0 \geq \frac{(n-1)\varrho}{n\varrho+1}. \tag{18}$$

Now (16) is a consequence of (17) and (18). □

According to Lemma 1, it is possible to consider the operator $A_n^{\varrho} : C_0[0,1] \to C_0[0,1]$,

$$A_n^{\varrho} := \frac{\varrho}{n\varrho+1}\sum_{k=0}^{\infty}(U_n^{\varrho})^k, n \geq 1. \tag{19}$$

For later purposes we also introduce the notation

$$A_n^{\infty} := \frac{1}{n}\sum_{k=0}^{\infty}(B_n)^k,\ n \geq 1,$$

in order to have Păltănea's power series included.

By using (16) we get $||A_n^{\varrho}||_0 \leq \dfrac{\varrho}{\varrho+1}$, and with the same function $g(x) = x(1-x)$ we find

$$||A_n^{\varrho}||_0 = \frac{\varrho}{\varrho+1}, n \geq 1. \tag{20}$$

Let $p \in \Pi_m \cap C_0[0,1]$, i.e., $p(0) = p(1) = 0$. Then $m \geq 2$. Let $n \geq m$. From (2), (3) and (5) we derive

$$p = \sum_{j=2}^{m} \mu_{\varrho,j}^{(n)}(p) p_{\varrho,j}^{(n)} \tag{21}$$

and, moreover,

$$(U_n^{\varrho})^k p = \sum_{j=2}^{m} (\lambda_{\varrho,j}^{(n)})^k \mu_{\varrho,j}^{(n)}(p) p_{\varrho,j}^{(n)},\ k \geq 0, \text{ for all } n \geq m. \tag{22}$$

According to (19), for all $p \in \Pi_m \cap C_0[0,1]$ and $n \geq m$,

$$A_n^{\varrho} p = \frac{\varrho}{n\varrho + 1} \sum_{j=2}^{m} \frac{1}{1 - \lambda_{\varrho,j}^{(n)}} \mu_{\varrho,j}^{(n)}(p) p_{\varrho,j}^{(n)}. \tag{23}$$

By using (10), (11) and (12) we get

$$\lim_{n\to\infty} A_n^{\varrho} p = \frac{\varrho}{\varrho + 1} \sum_{j=2}^{m} \frac{2}{j(j-1)} \mu_j^*(p) p_j^*, \tag{24}$$

uniformly on $[0, 1]$, for all $p \in \Pi_m \cap C_0[0,1]$.

4 The Voronovskaya operator A_ϱ

It was proved in [7, p. 918] that

$$\lim_{n\to\infty} n(U_n^{\varrho} g(x) - g(x)) = \frac{\varrho + 1}{2\varrho} x(1-x) g''(x),\ g \in C^2[0,1],$$

uniformly on $[0, 1]$. We need the following result.

Theorem 2 *The operator* $\{y \in C^2[0,1] \mid y(0) = y(1) = 0\} \to C_0[0,1]$ *defined by*

$$A_\varrho y(x) := \frac{\varrho + 1}{2\varrho} x(1-x) y''(x),\ x \in [0,1], \tag{25}$$

is bijective, and

$$||A_\varrho^{-1} f||_\infty \leq \frac{\varrho}{4(\varrho + 1)} ||f||_0,\ f \in C_0[0,1]. \tag{26}$$

Proof Obviously A_ϱ is injective. To prove the surjectivity, let $f \in C_0[0,1]$, $f(x) = x(1-x)h(x)$, $h \in C[0,1]$. It is a matter of calculus to verify that the function

$$-\frac{2\varrho}{\varrho+1}F_\infty(h;x) = y(x) := -\frac{2\varrho}{\varrho+1}\left[(1-x)\int_0^x th(t)dt + x\int_x^1 (1-t)h(t)dt\right],$$

for $x \in [0,1]$ is in $C^2[0,1]$, $y(0) = y(1) = 0$, and $A_\varrho y = f$. Therefore A_ϱ is bijective. Moreover, for $x \in [0,1]$, $y = A_\varrho^{-1}(f)$, i.e., $-y(x) = -A_\varrho^{-1}(f;x) = +\frac{2\varrho}{\varrho+1}F_\infty(h;x)$. Consequently,

$$|A_\varrho^{-1}f(x)| \le \frac{2\varrho}{\varrho+1}\left[(1-x)\int_0^x tdt + x\int_x^1 (1-t)dt\right]||h||_\infty \tag{27}$$

$$= \frac{\varrho}{\varrho+1}x(1-x)||h||_\infty \le \frac{\varrho}{4(\varrho+1)}||f||_0, \tag{28}$$

and this leads to (26). □

Remark 1 Further below we will use the notation $\Psi(x) := x(1-x)$, and

$$-A_\infty^{-1}(\Psi h) := 2\cdot F_\infty(h),\ \ h \in C[0,1],$$

in order to also cover the Bernstein case.

Another useful result reads as follows.

Lemma 2 *For all* $p \in \Pi \cap C_0[0,1]$ *we have*

$$\lim_{n\to\infty} A_n^{\varrho} p = -A_\varrho^{-1}p, \tag{29}$$

uniformly on $[0,1]$.

Proof The polynomials p_j^* from (7) satisfy

$$x(1-x)(p_j^*)''(x) = -j(j-1)p_j^*(x),\, x\in[0,1],\, j\ge 0 \tag{30}$$

(see, e.g., [4], p.155). This yields $A_\varrho p_j^* = -\dfrac{\varrho+1}{2\varrho}j(j-1)p_j^*$, $j \ge 0$, and, moreover,

$$A_\varrho\left(\sum_{j=2}^m \frac{2}{j(j-1)}\mu_j^*(p)p_j^*\right) = -\frac{\varrho+1}{\varrho}\sum_{j=2}^m \mu_j^*(p)p_j^* \tag{31}$$

for all $p \in \Pi_m \cap C_0[0,1]$. According to ([4], (4.18)), $\sum_{j=2}^{m} \mu_j^*(p) p_j^* = p$, so that (31) yields

$$\frac{\varrho}{\varrho+1} \sum_{j=2}^{m} \frac{2}{j(j-1)} \mu_j^*(p) p_j^* = -A_\varrho^{-1} p, \tag{32}$$

for all $p \in \Pi_m \cap C_0[0,1]$. Now (29) is a consequence of (24) and (32). $\square$

5 The convergence of A_n^ϱ on $C_0[0,1]$

One main result of the paper is contained in

Theorem 3 *For all* $f \in C_0[0,1]$,

$$\lim_{n\to\infty} A_n^\varrho f = -A_\varrho^{-1} f, \tag{33}$$

uniformly on $[0,1]$.

Proof Let $f \in C_0[0,1]$, $f(x) = x(1-x)h(x)$, $h \in C[0,1]$. Consider the polynomials $p_i(x) := x(1-x)B_i h(x)$, where B_i are the classical Bernstein operators, $i \geq 1$. Then $p_i \in C_0[0,1]$, $i \geq 1$, and $\lim_{i\to\infty} ||p_i - f||_0 = \lim_{i\to\infty} ||B_i h - h||_\infty = 0$. Let $\varepsilon > 0$ and fix $i \geq 1$ such that

$$||p_i - f||_0 \leq \frac{2\varrho+2}{3\varrho+2}\varepsilon. \tag{34}$$

Then, according to Lemma 2, there exists n_ε such that

$$||A_n^\varrho p_i + A_\varrho^{-1} p_i||_\infty \leq \frac{2\varrho+2}{3\varrho+2}\varepsilon, n \geq n_\varepsilon. \tag{35}$$

Now using (15) and (20) we infer

$$||A_n^\varrho f - A_n^\varrho p_i||_\infty \leq \frac{1}{4}||A_n^\varrho f - A_n^\varrho p_i||_0 \leq \frac{1}{4}||A_n^\varrho||_0 ||f - p_i||_0 \leq \frac{\varrho}{4(\varrho+1)} \frac{2\varrho+2}{3\varrho+2}\varepsilon,$$

so that

$$||A_n^\varrho f - A_n^\varrho p_i||_\infty \leq \frac{\varrho}{2(3\varrho+2)}\varepsilon. \tag{36}$$

On the other hand, (26) and (34) yield

$$||A_\varrho^{-1} f - A_\varrho^{-1} p_i||_\infty \leq \frac{\varrho}{4(\varrho+1)}||f - p_i||_0 \leq \frac{\varrho}{2(3\varrho+2)}\varepsilon. \tag{37}$$

Finally, using (35), (36) and (37) we obtain, for all $n \geq n_\varepsilon$,

$$||A_n^\varrho f + A_\varrho^{-1} f||_\infty \leq ||A_n^\varrho f - A_n^\varrho p_i||_\infty + ||A_n^\varrho p_i + A_\varrho^{-1} p_i||_\infty$$
$$+||A_\varrho^{-1} f - A_\varrho^{-1} p_i||_\infty \leq \varepsilon,$$

and this concludes the proof. □

On $(C[0,1], ||\cdot||_\infty)$ consider the linear operator $H_n^\varrho := A_n^\varrho - (-A_\varrho^{-1})$ given by

$$C[0,1] \ni h \mapsto A_n^\varrho(\Psi h; x) = \frac{\varrho}{n\varrho+1}\sum_{k=0}^{\infty}(U_n^\varrho)^k(\Psi h; x) \in C_0[0,1]$$

$$C[0,1] \ni h \mapsto -A_\varrho^{-1}(\Psi h; x) = \frac{2\varrho}{\varrho+1}\left[(1-x)\int_0^x th(t)dt + x\int_x^1 (1-t)h(t)dt\right]$$
$$= \frac{2\varrho}{\varrho+1}F_\infty(h; x) \in C_0[0,1]$$

Theorem 4 *Let* $h \in C[0,1], \varrho > 0, n \geq \frac{4\varrho+6}{\varrho}, \varepsilon = \sqrt{\frac{\varrho+2}{n\varrho+2}} \leq \frac{1}{2}$ *and* $\Psi(x) = x(1-x)$. *Then*

$$|H_n^\varrho(h; x)| \leq \Psi(x)\left[\frac{2\varrho}{3(\varrho+1)}\sqrt{\frac{\varrho+2}{n\varrho+2}}\omega_1(h;\varepsilon)\right. \tag{38}$$
$$\left. +\frac{3}{4}\left(\frac{2\varrho}{\varrho+1} + \frac{2\varrho}{3(\varrho+1)}\sqrt{\frac{\varrho+2}{n\varrho+2}} + \frac{7(\varrho+3)}{6(\varrho+1)}\right)\omega_2(h;\varepsilon)\right].$$

Proof Let $h \in C[0,1]$ be fixed, and $g \in C^2[0,1]$ be arbitrary.

Then $|H_n^\varrho(h; x)| \leq |H_n^\varrho(h-g; x)| + |H_n^\varrho(g; x)| = |E_1| + |E_2|$. Here

$$\begin{aligned}|E_1| &= |A_n^\varrho(\Psi(h-g); x) - (-A_\varrho^{-1}(\Psi(h-g); x))| \\ &= |A_n^\varrho(\Psi(h-g); x) - \frac{2\varrho}{\varrho+1}F_\infty(h-g; x)| \\ &\leq ||h-g||_\infty A_n^\varrho(\Psi; x) + \frac{2\varrho}{\varrho+1}|F_\infty(h-g; x)| \\ &= ||h-g||_\infty \frac{\varrho}{\varrho+1}\Psi(x) + \frac{2\varrho}{\varrho+1}||h-g||_\infty \frac{1}{2}\Psi(x) \\ &= \frac{2\varrho}{\varrho+1}\Psi(x)|h-g||_\infty\end{aligned}$$

and

$$|E_2| = |A_n^\varrho(\Psi g; x) - (-A_\varrho^{-1}(\Psi g; x))|.$$

For $g \in C^2[0,1]$ one has $F_\infty := F_\infty(g) \in C^4[0,1]$, $F''_\infty = -g$, $F'''_\infty = -g'$, $F^{(4)}_\infty = -g''$. Moreover, by Taylor's formula we obtain for any points $y, t \in [0,1]$:

$$F_\infty(t) = F_\infty(y) + F'_\infty(y)(t-y) + \frac{1}{2}F''_\infty(y)(t-y)^2 + \frac{1}{6}F'''_\infty(y)(t-y)^3 + \Theta_y(t) \tag{39}$$

where

$$\Theta_y(t) := \frac{1}{6}\int_y^t (t-u)^3 F^{(4)}_\infty(u)du.$$

Fix y and consider (39) as an equality between two functions in the variable t. Applying to this equality the operator $U_n^\varrho(\cdot, y)$ one arrives at

$$\begin{aligned} U_n^\varrho(F_\infty, y) &= F_\infty(y) + \frac{1}{2}F''_\infty(y)U_n^\varrho((t-y)^2; y) + \frac{1}{6}F'''_\infty(y)U_n^\varrho((t-y)^3; y) \\ &\quad + U_n^\varrho(\Theta_y; y) \\ &= F_\infty(y) - \frac{1}{2}g(y)U_n^\varrho((t-y)^2; y) - \frac{1}{6}g'(y)(y)U_n^\varrho((t-y)^3; y) \\ &\quad + U_n^\varrho(\Theta_y; y). \end{aligned}$$

This implies

$$\begin{aligned} &\frac{1}{2}g(y)U_n^\varrho((e_1-y)^2; y) - F_\infty(y) + U_n^\varrho(F_\infty, y) = -\frac{1}{6}g'(y)U_n^\varrho((e_1-y)^3; y) \\ &\quad + U_n^\varrho(\Theta_y; y). \end{aligned}$$

In the above equality we rewrite the left hand side as $\frac{1}{2}g(y)U_n^\varrho((e_1-y)^2; y) - (I - U_n^\varrho)(F_\infty, y)$. Thus we have

$$\begin{aligned} &g(y)U_n^\varrho((e_1-y)^2; y) - 2(I - U_n^\varrho)(F_\infty, y) = -\frac{1}{3}g'(y)(y)U_n^\varrho((e_1-y)^3; y) \\ &\quad + 2U_n^\varrho(\Theta_y; y). \end{aligned}$$

Application of A_n^ϱ yields

$$\begin{aligned} &A_n^\varrho(g(\cdot)U_n^\varrho((e_1-\cdot)^2; \cdot); x) - 2A_n^\varrho \circ (I - U_n^\varrho)(F_\infty, x) \\ &\quad = -\frac{1}{3}A_n^\varrho(g'(\cdot)U_n^\varrho((e_1-\cdot)^3; \cdot); x) + 2A_n^\varrho(Q; x) \end{aligned} \tag{40}$$

where $Q(y) := U_n^\varrho(\Theta_y; y)$. The first five moments are given by (see [7, Cor. 2.1]

$$U_n^\varrho(e_0; y) = 1,$$

$$U_n^\varrho(e_1 - y; y) = 0,$$
$$U_n^\varrho((e_1 - y)^2; y) = \frac{(\varrho+1)\Psi(y)}{n\varrho+1},$$
$$U_n^\varrho((e_1 - y)^3; y) = \frac{(\varrho+1)(\varrho+2)\Psi(y)\Psi'(y)}{(n\varrho+1)(n\varrho+2)},$$
$$U_n^\varrho((e_1 - y)^4; y) = \frac{3\varrho(\varrho+1)^2\Psi^2(y)n}{(n\varrho+1)(n\varrho+2)(n\varrho+3)} + \frac{-6(\varrho+1)(\varrho^2+3\varrho+3)\Psi^2(y)+(\varrho+1)(\varrho+2)(\varrho+3)\Psi(y)}{(n\varrho+1)(n\varrho+2)(n\varrho+3)}.$$

In the above expression we have $2A_n^\varrho \circ (I - U_n^\varrho)(F_\infty, x) = \frac{2\varrho}{n\varrho+1}F_\infty(x) = \frac{2\varrho}{n\varrho+1}F_\infty(g; x)$.

Also $A_n^\varrho(g(\cdot)U_n^\varrho((e_1 - \cdot)^2; \cdot); x) = A_n^\varrho(g(\cdot)\frac{\varrho+1}{n\varrho+1}\Psi(\cdot); x) = \frac{\varrho+1}{n\varrho+1}A_n^\varrho(\Psi g; x)$. Hence (40) can be written as

$$\begin{aligned}
&\left|\frac{\varrho+1}{n\varrho+1}A_n^\varrho(\Psi g; x) - \frac{2\varrho}{n\varrho+1}F_\infty(g; x)\right| \\
&\quad = \left|-\frac{1}{3}A_n^\varrho(g'(\cdot)U_n^\varrho((e_1 - \cdot)^3; \cdot); x) + 2A_n^\varrho(Q; x)\right| \\
&\quad \le \frac{1}{3}\left|A_n^\varrho\left(g'(\cdot)\frac{(\varrho+1)(\varrho+2)}{(n\varrho+1)(n\varrho+2)}\Psi'(\cdot)\Psi(\cdot); x\right)\right| + |2A_n^\varrho(Q; x)| \\
&\quad \le \frac{1}{3}\frac{(\varrho+1)(\varrho+2)}{(n\varrho+1)(n\varrho+2)}||g'||_\infty\frac{\varrho}{\varrho+1}\Psi(x) + |2A_n^\varrho(Q; x)|.
\end{aligned}$$

Multiplying the outermost sides of the latter inequality by $\frac{n\varrho+1}{\varrho+1}$ gives

$$\begin{aligned}
|E_2| &= \left|A_n^\varrho(\Psi g; x) - \frac{2\varrho}{\varrho+1}F_\infty(g; x)\right| \\
&\le \frac{\varrho(\varrho+2)}{3(n\varrho+2)(\varrho+1)}\Psi(x)||g'||_\infty + 2\frac{n\varrho+1}{\varrho+1}|A_n^\varrho(Q; x)|.
\end{aligned}$$

In the last summand we have $Q(y) = U_n^\varrho(\Theta_y; y)$ thus

$$\begin{aligned}
|U_n^\varrho(\Theta_y; y)| &\le \frac{1}{6}U_n^\varrho((e_1 - y)^4; y)||g''||_\infty \\
&\le \frac{1}{6}\cdot\frac{7}{4}\cdot\frac{(\varrho+1)(\varrho+2)(\varrho+3)}{\varrho(n\varrho+1)(n\varrho+2)}\Psi(y)||g''||_\infty.
\end{aligned}$$

Hence

$$\begin{aligned}\frac{2(n\varrho+1)}{\varrho+1}|A_n^{\varrho}(Q;x)| &\le \frac{2(n\varrho+1)}{\varrho+1}\cdot\frac{7}{24}\cdot\frac{(\varrho+1)(\varrho+2)(\varrho+3)}{\varrho(n\varrho+1)(n\varrho+2)}A_n^{\varrho}(\Psi;x)||g''||_\infty\\ &= \frac{7}{12}\cdot\frac{(\varrho+2)(\varrho+3)}{(\varrho+1)(n\varrho+2)}\Psi(x)||g''||_\infty.\end{aligned}$$

This leads to

$$\begin{aligned}|E_2| &\le \frac{\varrho(\varrho+2)}{3(n\varrho+2)(\varrho+1)}\Psi(x)||g'||_\infty + \frac{7}{12}\cdot\frac{(\varrho+2)(\varrho+3)}{(\varrho+1)(n\varrho+2)}\Psi(x)||g''||_\infty\\ &= \frac{(\varrho+2)}{3(n\varrho+2)(\varrho+1)}\Psi(x)\left\{\varrho||g'||_\infty + \frac{7}{4}(\varrho+3)||g''||_\infty\right\}.\end{aligned}$$

Hence for $h \in C[0,1]$ fixed, $g \in C^2[0,1]$ arbitrary we have

$$\begin{aligned}|H_n^{\varrho}(h;x)| &\le |E_1| + |E_2|\\ &\le \frac{2\varrho}{\varrho+1}\Psi(x)||h-g||_\infty + \frac{(\varrho+2)}{3(n\varrho+2)(\varrho+1)}\Psi(x)\left\{\varrho||g'||_\infty + \frac{7}{4}(\varrho+3)||g''||_\infty\right\}\end{aligned}$$

Next we choose $g = h_\varepsilon$, $0 < \varepsilon = \sqrt{\frac{\varrho+2}{n\varrho+2}} \le \frac{1}{2}$. This notation was used in [5]. By applying Lemmas 2.1 and 2.4 in [5] we obtain

$$\begin{aligned}||h-g||_\infty &\le \frac{3}{4}\omega_2(h;\varepsilon)\\ ||g'|| &\le \frac{1}{\varepsilon}[2\omega_1(h;\varepsilon) + \frac{3}{2}\omega_2(h;\varepsilon)]\\ ||g''|| &\le \frac{3}{2\varepsilon^2}\omega_2(h;\varepsilon).\end{aligned}$$

Thus

$$\begin{aligned}|H_n^{\varrho}(h;x)| \le \Psi(x)\Bigg[&\frac{2\varrho}{3(\varrho+1)}\sqrt{\frac{\varrho+2}{n\varrho+2}}\omega_1(h;\varepsilon)\\ &+\frac{3}{4}\left(\frac{2\varrho}{\varrho+1} + \frac{2\varrho}{3(\varrho+1)}\sqrt{\frac{\varrho+2}{n\varrho+2}} + \frac{7(\varrho+3)}{6(\varrho+1)}\right)\omega_2(h;\varepsilon)\Bigg].\end{aligned}$$

□

Corollary 1 *Recalling the above definition of H_n^{ϱ}, the inequality of Theorem 4 shows that*

$$\lim_{n\to\infty}||A_n^{\varrho}f - (-A_{\varrho}^{-1}f)||_0 = 0, \textit{ for all } f \in C_0[0,1].$$

Remark 2 If we let $1 \leq \varrho \to \infty$, then for all $n \geq 10$

$$\begin{aligned}\lim_{\varrho\to\infty} |H_n^\varrho(h;x)| &= \lim_{\varrho\to\infty} |A_n^\varrho(\Psi h;x) - (-A_\varrho^{-1})(\Psi h;x)| \\ &= |A_n^\infty(\Psi h;x) - (-A_\infty^{-1})(\Psi h;x)| \\ &\leq 3\Psi(x)\left[\frac{1}{\sqrt{n}}\omega_1\left(h;\frac{1}{\sqrt{n}}\right) + \omega_2\left(h;\frac{1}{\sqrt{n}}\right)\right].\end{aligned}$$

This is a quantitative form of Păltănea's convergence result in [9, Th. 3.2].

References

1. Abel, U.: Geometric series of Bernstein-Durrmeyer operators. East J. Approx. **15**, 439–450 (2009)
2. Abel, U., Ivan, M., Păltănea, R.: Geometric series of Bernstein operators revisited. J. Math. Anal. Appl. **400**, 22–24 (2013)
3. Abel, U., Ivan, M., Păltănea, R.: Geometric series of positive linear operators and inverse Voronovskaya theorem, arXiv:1304.5721, 21 Apr. 2013
4. Cooper, S., Waldron, S.: The eigenstructure of the Bernstein operator. J. Approx. Theory **105**, 133–165 (2000)
5. Gonska, H., Kovacheva, R.K.: The second order modulus revisited: remarks, applications, problems. Conferenze del seminario di matematica dell'universita di Bari, vol. 257 (1994)
6. Gonska, H., Păltănea, R.: Simultaneous approximation by a class of Bernstein-Durrmeyer operators preserving linear functions. Czechoslovak Math. J. **60**, 783–799 (2010)
7. Gonska, H., Păltănea, R.: Quantitative convergence theorems for a class of Bernstein-Durrmeyer operators preserving linear functions. Ukrainian Math. J. **62**, 913–922 (2010)
8. Gonska, H., Raşa, I., Stănilă, E.: The eigenstructure of operators linking the Bernstein and the genuine Bernstein-Durrmeyer operators. Mediterr. J. Math. (2013). doi:10.1007/s00009-013-0347-0
9. Păltănea, R.: The power series of Bernstein operators. Autom. Comput. Appl. Math. **15**, 247–253 (2006)
10. Păltănea, R.: A class of Durrmeyer type operators preserving linear functions. Ann. Tiberiu Popoviciu Sem. Funct. Equat. Approxim. Convex. (Cluj-Napoca) **5**, 109–117 (2007)
11. Raşa, I.: Power series of Bernstein operators and approximation of resolvents. Mediterr. J. Math. **9**, 635–644 (2012)