BREAKING OF DYNAMICAL SYMMETRIES AND PHASE SHIFT

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The concept of a broken dynamical symmetry is introduced by specifying operators that generalize the Runge–Lenz vector for the hydrogen atom. The relationship between the commutators of these operators and the Hamiltonian and phase shifts is established. The specification of symmetry on the discrete and continuous spectra are considered separately.

1. Introduction

Group-theoretical methods are now widely used in different branches of physics. In some exactly solvable problems of classical and quantum mechanics such as the Coulomb potential, magnetic monopole, and harmonic oscillator [1-6, 10], this has led to an understanding at a new, much deeper level of the connection between the invariance properties of a system and its physical characteristics, although technically new results have not been obtained. Subsequently, the algebraic scheme was imposed on systems that have no exact solutions but have certain invariance properties. Certain physical information was extracted, as in the case of the exactly solvable problems.

This program was applied most consistently by Barut et al. [7], who constructed a model for the electromagnetic interaction of elementary particles by taking as their point of departure the group-theoretical properties of the hydrogen atom.

Besides systems that have exact symmetries, one also considers systems whose symmetries are in a certain sense broken but are reestablished when a certain parameter vanishes. Such entities arise in current algebra and also in S-matrix theory and the Bethe–Salpeter equation.

Knowing the properties of the system with the exact symmetry and also the character of the symmetry breaking, one can obtain information about the system with the broken symmetry.

In this paper we investigate systems whose symmetry properties differ somewhat from those of the hydrogen atom. The difference is formulated algebraically. The phase shifts of scattering on a Coulomb potential have algebraic properties and can be obtained from purely group considerations. For a system with broken symmetries one can also obtain expressions for the phase shifts. Such systems include a particle moving in the field of the potential $U(r) = A/r + F(L^2)/r^2$, the relativistic hydrogen atom, and also certain models with a magnetic monopole.

2. Symmetries of the Coulomb Potential and Properties of the Scattering States

It is well known that the Coulomb potential is not only spherically symmetric but has an additional symmetry. This symmetry is called dynamical. In the case of quantum mechanics it is expressed by the fact that besides the operators of the angular momentum **L**, which commute with the Hamiltonian

$$H=\frac{1}{2m}p^2+\frac{\alpha}{r},\quad [\mathbf{L},H]=0 \tag{1}$$

there are a further three operators A_1, A_2, A_3, which form the so-called Runge–Lenz vector:

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$$\mathbf{A}={}^1/_2(\mathbf{L}\times\mathbf{p}-\mathbf{p}\times\mathbf{L})+\alpha\frac{\mathbf{r}}{r}, \tag{2}$$

and these also commute with the Hamiltonian:

$$[H, \mathbf{A}]=0. \tag{3}$$

If we make the operator substitution $\mathbf{A} \to \mathbf{N} = \mathbf{A}\sqrt{-m/2E}$, we obtain the commutation relations

$$[L_i, L_j]=i\varepsilon_{ijk}L_k, \quad [L_i, N_j]=i\varepsilon_{ijk}N_k,$$
$$[N_i, N_j]=i\varepsilon_{ijk}L_k. \tag{4}$$

We see that the operators **L** and **N** form the algebra of the group O(4) or O(3, 1) depending on the sign of E. One can calculate the Casimir operators

$$K_1=\mathbf{L}^2+\mathbf{N}^2+1=\rho^2, \quad K_2=(\mathbf{L}\cdot\mathbf{N})=0, \quad \rho=\sqrt{m/2E}.$$

This means [8] that the states corresponding to the energy E in the discrete spectrum form an irreducible representation of O(4) with the Casimir operators

$$K_1=+\rho^2=-n^2, \quad n=0, 1, 2, \ldots; \quad K_2=0,$$

while the states corresponding to E in the continuous spectrum form an irreducible representation of O(3, 1) with the Casimir operators

$$K_1=+\rho^2, \quad -\infty<\rho<\infty; \quad K_2=0.$$

States with fixed E, angular momentum l, and projection m of the angular momentum onto the Oz axis form the standard basis $|E, l, m\rangle$, in which the operators **L** and **N** are expressed in the usual manner:

$$L_3|E, l, m\rangle=m|E, l, m\rangle,$$
$$L_+|E, l, m\rangle=\sqrt{(l-m)(l+m+1)}\,|E, l, m+1\rangle,$$
$$L_-|E, l, m\rangle=\sqrt{(l+m)(l-m+1)}\,|E, l, m-1\rangle,$$
$$N_3|E, l, m\rangle=C_l\sqrt{l^2-m^2}\,|E, l-1, m\rangle-C_{l+1}\sqrt{(l+1)^2-m^2}\,|E, l+1, m\rangle, \tag{5}$$
$$N_+|E, l, m\rangle=C_l\sqrt{(l-m)(l-m-1)}\,|E, l-1, m+1\rangle+C_{l+1}\sqrt{(l+m+1)(l+m+2)}\,|E, l+1, m+1\rangle,$$
$$N_-|E, l, m\rangle=-C_l\sqrt{(l+m)(l+m-1)}\,|E, l-1, m-1\rangle-C_{l+1}\sqrt{(l-m+1)(l-m+2)}\,|E, l+1, m-1\rangle,$$
$$C_l=i\sqrt{\frac{l^2-\rho^2}{4l^2-1}}.$$

The scattering states in this basis have the usual form:

$$\Psi_E^{\pm}(\theta)=\sum_{l=0}^{\infty}\sqrt{2l+1}\,e^{\pm i\delta_l}|E, l, 0\rangle, \tag{6}$$

where the Coulomb phase shift is

$$e^{2i\delta_l}=\Gamma(l+1+i\rho)/\Gamma(l+1-i\rho). \tag{7}$$

Zwanziger [4, 9] noted that the vector $\Psi_E^{\pm}$ satisfies the two equations

$$L_3\Psi_E^{\pm}=0, \quad N_3\Psi_E^{\pm}=(1\pm i\rho)\Psi_E^{\pm}. \tag{8}$$

Subsequently, the invariance properties of the scattering states were investigated in [10], where a new basis was constructed in a representation space whose elements are the states $\Psi_E^{\pm}$, and the relationship to the scattering amplitude was also established.

In this paper we only require the fact that the form of $\Psi_E^{\pm}$ and the phase shifts (7) be uniquely determined from Eqs. (8) to within a factor $B(K_1, K_2)$ that depends only on the invariants of the representation and is such that $|B(K_1, K_2)| = 1$.

3. Breaking of Dynamical Symmetries

The relationship between the dynamical symmetry of the problem and the phase shifts shows that for systems that differ from the Coulomb potential operators with the properties (2), (3), (8) do not exist.

We shall attempt to modify somewhat the algebraic properties of the Coulomb potential and see what are then the consequences for the phase shifts.

Thus, suppose we have a quantum mechanical system described by a Hamiltonian H such that H in the limit $r \to \infty$ goes over into $H_0 = p^2/2m$, the Hamiltonian of a free particle. We require first of all that the Hamiltonian commute with the operator of the angular momentum $\mathbf{L}$: $[H, \mathbf{L}] = 0$. For what follows, we go over from the energy E to the quantity $\rho = \pm\sqrt{m/2E}$. In the case of complete dynamical symmetry, we could restrict ourselves to one value of the root — the positive (for $E > 0$); now we shall find it convenient to consider negative values of ρ as well. This means that we add to the previous quantum numbers E, l, m a new internal quantum number that takes the two values ± 1:

$$|E, l, m\rangle \to |E, \pm, l, m\rangle.$$

The number of states is doubled. In what follows we shall use the notation

$$|E, \pm, l, m\rangle \equiv |\rho, l, m\rangle,$$

assuming that $|\rho|$ and sign ρ are two different quantum numbers combined in the single symbol ρ:

$$|E, l, m\rangle \to |\rho, l, m\rangle, \quad 0 \leqslant E < \infty, \ -\infty < \rho < +\infty. \tag{9}$$

Thus, the model considered below will describe the scattering of a particle that can be in two different "internal" states ($\pm$). We shall obtain formulas that give the phase shifts in these two states. A situation of this kind is not encountered in the description of the scattering of an ordinary nonrelativistic spinless particle on a potential, but it may be of interest from the point of view of the theory of elementary particles.

Suppose that there also exist three operators N_1, N_2, N_3 such that together with the operators $\mathbf{L}$ they form the algebra O(3, 1), i.e., (4) is satisfied. However, instead of the property (5) we require that the operators N carry states with given values of ρ into states with other values $\rho' = \rho + \Delta\rho$:

$$\begin{aligned}
&L_3|\rho, l, m\rangle = m|\rho, l, m\rangle,\\
&L_+|\rho, l, m\rangle = \sqrt{(l-m)(l+m+1)}\,|\rho, l, m+1\rangle,\\
&L_-|\rho, l, m\rangle = \sqrt{(l+m)(l-m+1)}\,|\rho, l, m-1\rangle,\\
&N_3|\rho, l, m\rangle = C_l\sqrt{l^2-m^2}\,|\rho+\Delta\rho(l), l-1, m\rangle\\
&-C_{l+1}\sqrt{(l+1)^2-m^2}\,|\rho-\Delta\rho(l+1), l+1, m\rangle,\\
&N_+|\rho, l, m\rangle = C_l\sqrt{(l-m)(l-m-1)}\,|\rho+\Delta\rho(l), l-1, m+1\rangle\\
&+C_{l+1}\sqrt{(l+m+1)(l+m+2)}\,|\rho-\Delta\rho(l+1), l+1, m+1\rangle,\\
&N_-|\rho, l, m\rangle = -C_l\sqrt{(l+m)(l+m-1)}\,|\rho+\Delta\rho(l), l-1, m-1\rangle\\
&-C_{l+1}\sqrt{(l-m+1)(l-m+2)}\,|\rho-\Delta\rho(l+1), l+1, m-1\rangle,\\
&C_l = i\sqrt{\frac{l^2+\rho^2}{4l^2-1}}.
\end{aligned} \tag{10}$$

In general, the shift $\Delta\rho$ depends on l but not on ρ: $\Delta\rho = \Delta\rho(l)$. It follows from the form of Eqs. (10) that there exists a set of states

$$\begin{gathered}
|\rho_0, 0, 0\rangle, |\rho_0+\Delta\rho(1), 1, m\rangle, \ldots, \left|\rho_0 + \sum_{i=1}^{l}\Delta\rho(i), l, m\right\rangle, \ldots,\\
-\infty < \rho < +\infty,
\end{gathered} \tag{11}$$

on each of which the operators (10) act transitively. These sets are uniquely determined by the choice of ρ_0 and the form of $\Delta\rho(l)$. On the sets (11) a representation of the algebra of O(3, 1) is realized. Its Casimir operators are $K_1 = +\rho_0^2$, $K_2 = 0$. These representations are shown in Fig. 1. States that lie on the horizontal lines correspond to one value of the energy (expressed in terms of ρ). States that lie on inclined lines form an irreducible representation of O(3, 1) characterized by the Casimir operators K_1 at the point of intersection of this line with the ρ axis and $K_2 = 0$. In this representation one can construct invariant states $\Phi^\pm$ that satisfy the equations (8):

$$L_3\Phi^\pm = 0, \quad N_3\Phi^\pm = (1 \pm i\rho_0)\Phi^\pm. \tag{8'}$$

These states $\Phi^\pm$ can be expanded with respect to states with fixed angular momentum:

$$\begin{gathered}
\Phi^\pm = B^{\pm 1}(\rho_0)\sum_{l=0}^{\infty}\sqrt{2l+1}\,\xi_l^{\pm 1}(\rho_0)\,V_{\rho_0, l, 0},\\
V_{\rho_0, l, 0} = \left|\rho_0 + \sum_{i=1}^{l}\Delta\rho(i), l, 0\right\rangle.
\end{gathered} \tag{12}$$

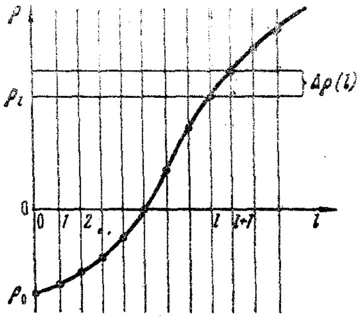

Fig. 1

From the algebraic point of view, the expression (12) is identical with the expression (6), but the parameters E and ρ_0 have a different meaning.

For $\xi_l(\rho_0)$ we obtain the ordinary Coulomb form

$$\xi_l^2(\rho_0)=\Gamma(l+1+i\rho_0)/\Gamma(l+1-i\rho_0). \tag{13}$$

All that we have done hitherto was simply an algebraic scheme imposed formally on the system and it is in no way related to a physical origin. In order to obtain such a relationship, we made additional assumptions. To each vector $V_{\rho_0,l,0}$ there corresponds a coefficient in the expansion (12). On the other hand, in the expansion of the scattering states with respect to the partial waves:

$$\Psi^\pm=\sum_{l=0}^{\infty}\sqrt{2l+1}\,e^{\pm i\delta_l(\rho)}|\rho,l,0\rangle, \tag{14}$$

the phase shifts $e^{\pm i\delta_l(\rho)}$ occur as coefficients of $|\rho,\ l,\ 0\rangle$. Our additional assumption, which relates physics to algebra, is that these coefficients are equal to the coefficient of $V_{\rho_0,l,0}$ in (12):

$$B(\rho_0)\xi_l(\rho_0)=e^{i\delta_l(\rho)}.$$

We can now calculate the phase shifts. We first establish how ρ and ρ_0 are related:

$$\rho=\rho_0+\Delta\rho(1)+\Delta\rho(2)+\ldots+\Delta\rho(l)=\rho_0+\sum_{i=1}^{l}\Delta\rho(i)=\rho_0-F(l)=\Phi(l), \tag{15}$$

$$e^{2i\delta_l(\rho)}=B^2(\rho_0)\xi_l^2(\rho_0)=B^2(\rho_2+F(l))\xi_l^2(\rho+F(l))=B^2(\rho+F(l))\Gamma(l+1+i\rho+iF(l))/\Gamma(l+1-i\rho-iF(l)). \tag{16}$$

Since $F(0)=0$, the factor $B(\rho)$ shows how much the zero phase $e^{i\delta_0(\rho)}$ differs from the Coulomb phase $e^{i\delta_0c(\rho)}$: $B(\rho)=e^{i(\delta_0(\rho)-\delta_0c(\rho))}$.

Hitherto we have operated with $\rho=\pm\sqrt{m/2E}$. When E varies from $E=0$ to $+\infty$, ρ varies from $-\infty$ to $+\infty$. The irreducible representations of the algebra (10), to which inclined lines correspond in Fig. 1, include, in general, states $|\rho,\ l,\ m\rangle$ with both $\rho>0$ and $\rho<0$. The curves in Fig. 1 cover the entire half-plane $\{\rho\times l\colon\ -\infty<\rho<+\infty;\ 0\le E<+\infty\}$ and do not intersect. Therefore, the spectrum of the partial-wave amplitudes $A_l(\rho)=e^{i\delta_l(\rho)}$ with respect to ρ extend from $-\infty$ to $+\infty$ and with respect to E from 0 to $+\infty$.

We now examine this picture in terms of the energy. For simplicity, we set $F(l)=-al$, $\rho_l=\rho_0+al$, $E_l=1/(\rho_0+al)^2$. It can be seen that whereas the shifts $\Delta\rho$ depend solely on l and not on ρ_0 the energy shifts ΔE depend on both l and ρ_0: $\Delta E=\Delta E(l,\ \rho_0)$.

Let us represent the dependence $E=E(l,\ \rho_0)$ graphically. In Fig. 2 the curves $E=1/(\rho_0+al)^2$ correspond to representations with $\rho_0\ge 0$. In these representations only states for which $E_l\le 1/a^2l^2$ are combined. For all these states $\rho_l\ge 0$. The remaining states with $E_l>1/a^2l^2$ and $\rho_l\ge 0$ lie on the curves $E=1/(\rho+al)^2$, where $\rho_0<0$ (see Fig. 3). They form irreducible representations only with states for which $\rho<0$. In Fig. 3 the states with $\rho<0$ lie to the left of the asymptote while states with $\rho>0$ lie to the right.

Hitherto, in the representations of O(3, 1) that we have considered the Casimir operator $K_2=(\mathbf{L}\cdot\mathbf{N})$ has been zero, $K_2=0$, as is the case for the hydrogen atom. In the case of other systems, for example, a magnetic monopole [4, 5, 6], $K_2=\mu$, where $\mu=0,\ \pm1/2,\ \pm1,\ \ldots$ The angular momentum l takes the values $l=|\mu|,\ |\mu|+1,\ \ldots$ In (15) the summation is from $|\mu|$ to l:

$$e^{2i\delta_l(\rho)}=B^2(\rho+F(l))\Gamma(l+1+\rho+F(l))/\Gamma(l+1-\rho-F(l)),$$

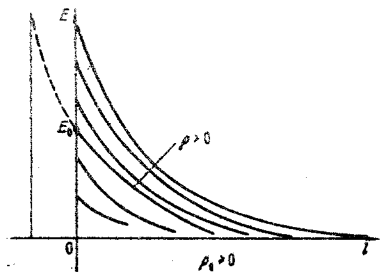

Fig. 2

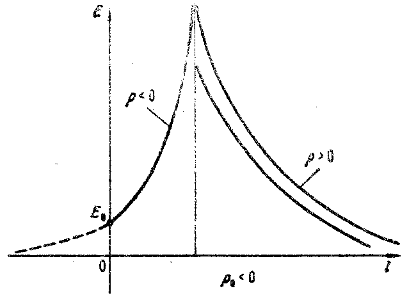

Fig. 3

$$\rho=\rho_0+\sum_{i=|\mu|}\Delta\rho(i)=\Phi(l), \tag{17}$$

where F(l) is determined by the differential equation

$$d\Phi/dl=\varphi(l),\quad \Phi(K_2)=K_1,$$
$$\Delta\rho(l)=\varphi(l),\quad F(l)=\Phi(l)+\rho_0,$$

i.e., the Casimir operators have entered the initial conditions for the equations that determine $\Phi(l)$.

4. Symmetries of the Discrete Spectrum

In the case of a continuous spectrum we have considered operators **N** that, when applied to states with the energy parameter $\rho = \sqrt{m/2E}$, carry them into states with $\rho-\Delta\rho(l)$, ρ, $\rho+\Delta\rho(l+1)$. Since $\Delta\rho$ in these formulas is real, the operators **N** carry states of the continuous spectrum into one another; on states of the discrete spectrum they were not defined. Therefore, one cannot draw any conclusions about the character of the discrete spectrum from (17) without making additional assumptions.

Suppose that our system has a discrete spectrum whose states are specified by three numbers $|k, l, m\rangle$, i.e., there are infinitely many states with any value of the angular momentum l: $k = 0, 1, 2, \ldots, l = 0, 1, 2, \ldots, -l \le m \le l$. The energy of the states depends only on k and l: $E = E_{k,l}$. We now split all the states into sets characterized by the principal quantum number $n = k + l$:

$$|k, l, m\rangle\rightarrow|n, l, m\rangle,\quad E_{k,l}\rightarrow E_{n,l}.$$

We now define the operators **N** by means of the relations (10), in which we take $\Delta\rho$ to be purely imaginary in such a way that the states

$$|n, 0, 0\rangle,\ |n, 1, m\rangle, \ldots, |n, l, m\rangle, \ldots, |n, n-1, m\rangle$$

form a representation of the group O(4) with the Casimir operators $K_1 = n^2$ and $K_2 = 0$. We can proceed in the same way if the discrete spectrum is finite but constructed in such a way that states with $l = 0$ occur P times and states with $l = 1$ occur P$-$1 times, etc., to states with $l = P-1$, which occur once. Since the operators **N** in this case are defined only on the states of the discrete spectrum, we can say nothing about the phase shifts. However, if the dependence of the energy on n and l has the form $i\rho_{n,l} = n + f(l) = n + l + F(l)$, $\rho = \sqrt{m/2E}$, then, assuming that the partial-wave amplitude is an analytic function with a cut on the (real) positive half-axis and poles on the negative half-axis corresponding to the discrete spectrum and no other singularities, then for it we can postulate the formula

$$e^{2i\delta_l}=B(f(l)+i\rho)\Gamma(l+1+F(l)+i\rho)/\Gamma(l+1+F(l)-i\rho).$$

If the discrete spectrum is finite, $l < P$, then $B(i\rho + F(l))$ must be chosen in the form

$$B(i\rho+F(l))=\Gamma(P+2+F(l)-i\rho)/\Gamma(P+2+F(l)-i\rho).$$

Such properties are possessed by potentials of the form

$$U(r)=-\frac{\alpha}{r}+\frac{f(L^2)}{r^2}.$$

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