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FAR-SIGHTED EQUILIBRIA IN 2×2, NON-COOPERATIVE, REPEATED GAMES

ABSTRACT. Consider a two-person simultaneous-move game in strategic form. Suppose this game is played over and over at discrete points in time. Suppose, furthermore, that communication is not possible, but nevertheless we observe some regularity in the sequence of outcomes. The aim of this paper is to provide an explanation for the question why such regularity might persist for many (i.e., infinite) periods.

Each player, when contemplating a deviation, considers a sequential-move game, roughly speaking of the following form: "if I change my strategy this period, then in the next my opponent will take his strategy 'b' and afterwards I can switch to my strategy 'a', but then I am worse off since at that outcome my opponent has no incentive to change anymore, whatever I do". Theoretically, however, there is no end to such reaction chains. In case that deviating by some player gives him less utility in the long run than before deviation, we say that the original regular sequence of outcomes is far-sighted stable for that player. It is a far-sighted equilibrium if it is far-sighted stable for both players.

Keywords: repeated games, departure games, far-sighted equilibria.

1. INTRODUCTION

In Brams and Wittman (1981) an interesting solution concept for 2×2, simultaneous-move games is presented. They write:

> [it is] based on the idea that players can look ahead and anticipate where a process might end up if they are allowed to make an indefinite number of sequential moves and countermoves from any outcome in the game.

Given a strategy for each player, and therefore an outcome, a player contemplating a change of strategy knows that he will invoke a sequence of moves and countermoves. If, after considering the possible consequences of change, a player finds that he is better off at the initial outcome before changing, then this outcome is declared to be non-myopically stable for that player. An outcome is a non-myopic equilibrium if it is non-myopically stable for both players.

The actual game is played once with the players simultaneously choosing a strategy. The sequence of moves and countermoves in

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determining whether an outcome is non-myopically stable for a particular player is called the departure game for that player. So, different players have different departure games, since the first move in a departure game for a particular player is made by that player.

The rules of play are the following (from Brams and Hessel (1983), with slight modification in rule (4)):

(1) Both players simultaneously choose a strategy, thereby defining an initial outcome of the game.
(2) Once at an initial outcome, either player can unilaterally switch his strategy and change that outcome to a subsequent outcome.
(3) The other player can respond by unilaterally switching his strategy, thereby moving the game to a new outcome.
(4) These strictly alternating moves continue until the player whose turn it is, chooses not to switch, or until the game returns to the initial outcome. When either of these two possibilities happens, the departure game ends, and the outcome reached is the final outcome of the departure game.

If the final outcome is the initial outcome then this outcome is non-myopically stable for the player whose departure game it is.

Notice that in 2×2 games the maximal length of a departure game is four. The Nash equilibrium concept can be viewed as having departure games of maximal length one.

Kilgour (1984) considers departure games of length n, for $n = 1, 2, \ldots$, respectively. Let $F(n)$ be the final outcome of the departure game (for some player) of length n. The initial outcome is declared to be extended non-myopically stable if there exists an $\bar{n}$ such that: (a) for all $n \geqslant \bar{n}$ we have $F(n) = F(\bar{n})$, and (b) $F(\bar{n})$ is the initial outcome.

The moves and countermoves in a departure game are announced, not physically played, under the assumption that each player is committed to his announcement. Since the players cannot sign binding agreements, the departure game is evaluated by using the subgame perfect equilibrium concept (Selten, 1973, 1975). However, the actual game is played only once. Hence, in our opinion, only one-move departure games should be considered (as in the Nash equilibrium concept).

In this paper we will assume that the actual game is played repeatedly with no limit to the number of plays. Suppose that after a while we observe a certain periodicity in the outcomes. For the moment assume that the length of this period is 1, i.e., one particular strategy pair is chosen time after time by the players. We will not attempt to explain how this outcome is arrived at, but rather make plausible why the players have no incentive to change their choice.

Fix any strategy pair, and thereby an outcome, in the underlying one-shot game. Suppose this strategy pair is chosen over and over. Suppose one of the players, say i, is contemplating a deviation from his strategy. He cannot expect that the other player will not react to this. But it does not stop there. Player i himself can react to this reaction, etc. For example, player i might think something like this: "if I change my strategy this period, then in the next my opponent will take his strategy 'b' and afterwards I can switch to my strategy 'a', but then I am worse off since at that outcome my opponent has no incentive to change anymore, whatever I do". Theoretically, however, there is no end to such reaction chains.

Player i is 'playing' *in his mind* a sequential-move infinite game. This game is called the departure game for player i at the given outcome (i.e., the strategy pair repeatedly chosen in the underlying one-shot game). If the evaluation of this departure game is such that player i cannot gain by deviating, the original strategy bundle in the one-shot game is said to be far-sighted stable for i. This strategy bundle is a far-sighted equilibrium if it is far-sighted stable for both players.

Notice that each player has a different departure game at a particular strategy pair of the underlying one-shot game. Furthermore, each strategy pair in the underlying one-shot game induces a different departure game. (That is, the structure of each departure game is the same, but the nodes are labeled differently, and therefore the utilities of each branch are different.)

We want to emphasize that departure games are not actually played. They are only 'played' in the minds of the players.

Intuitive plausible results are obtained in the Prisoners' Dilemma game and the zero-sum game Matching Pennies. In the Prisoners'

Dilemma game the only far-sighted equilibrium is the symmetric Pareto optimal one. In the Matching Pennies game the value of any departure game is zero, i.e., any departure game induces cycling over all four outcomes. Thus, this game does not have any far-sighted equilibrium.

In Section 3 we consider the case that the observed sequence of outcomes has a period of length greater than 1. Define such a sequence to be a correlated outcome of the underlying one-shot game. Define the utility of a correlated outcome as the mean utility of the sequence in that outcome. In the Matching Pennies game, *only* correlated outcomes that give a utility of zero to both players are far-sighted equilibria.

2. THE MODEL

Let $G = (\{i, j\}, X, Y, u, v)$ be a 2-person, simultaneous-move, one-shot game in strategic form, in which cooperation is not allowed. For simplicity we will assume that each player has only two strategies available, and we write $X = \{T, B\}$ and $Y = \{L, R\}$. Let $X \times Y$ be the set of strategy pairs, called *G-outcomes* (outcomes of the game G). The function $u, v: X \times Y \rightarrow R$ is player i's, j's Von Neumann–Morgenstern utility function. (That is, utility scales are invariant only under linear transformations. Thus, we are not considering purely ordinal utilities.) When player i chooses $x \in \{T, B\}$ and player j chooses $y \in \{L, R\}$, then player i enjoys utility $\mu_{xy} = u(x, y)$ and player j enjoys utility $\nu_{xy} = v(x, y)$.

The data contained in G can be conveniently summarized in a matrix as in Figure 1.

Utility bundles will always be in square brackets; the first number refers to player i's utility and the second to player j's, whether row or column vector.

The game is simultaneous move, meaning that players make a choice of strategy in ignorance of what the other has or will choose. One-shot means that the game is played only once. No cooperation means that the players cannot form a coalition.

The data in G are common knowledge in the sense that each player not only knows all data in G, but also that the other knows that he knows, etc.

		j	
		L	R
i	T	$[\mu_{TL}, \nu_{TL}]$	$[\mu_{TR}, \nu_{TR}]$
	B	$[\mu_{BL}, \nu_{BL}]$	$[\mu_{BR}, \nu_{BR}]$

Fig. 1.

The question now is: can we predict which outcome will be chosen by the players. Any solution concept must choose among the set of G-outcomes that are 'self-enforced'. Since we excluded the possibility of cooperation, a G-outcome is *self-enforced* if no player has an incentive to unilaterally deviate from that G-outcome. Note that the player who contemplates a deviation need not take into account possible retaliation of the other player, since the game is one-shot. The set of self-enforced G-outcomes is precisely the set of Nash equilibria (Nash, 1951).

Now assume that the game G is played repeatedly at discrete points in time. The players believe that they will be engaged in this game forever. A more or less straightforward approach is to view the repeated play of the game G as one (large) game; one constructs the so-called supergame (see, e.g., Aumann and Shapley, 1978) based on G. This supergame is itself simultaneous-move and one-shot. One then investigates the self-enforced outcomes (Nash equilibria) in this supergame.

In this paper we assume that communication between the players is limited. The absence of cooperation does not imply absence of communication. Since there are an infinite number of (subgame perfect) Nash equilibria in the repeated game (e.g., Rubinstein, 1980), there clearly is coordination problem. If communication would be possible, the players could 'agree' on some Nash equilibrium strategy bundle in the supergame (recall that a Nash equilibrium is self-enforcing), although it is not at all clear how this could be done. In Kalai (1981) communication is modeled as part of the game, but the same coordination problem arises in the enlarged communication game.

We, therefore, have to look for some other form of self-enforcement. Suppose that the game has already been played many times. Suppose furthermore that we observe a certain periodicity in the sequence of G-outcomes. The aim of the paper is not to explain how the players arrive at such periodicity, but rather to explain why such behavior might persist for longer periods of time (i.e., infinitely long if there are no outside disturbances).

For the moment assume that this periodicity is of length 1, i.e., one particular G-outcome is played over and over each period. (In Section 3 we consider a more general situation.) Suppose (T, L) is the G-outcome that is played repeatedly. Suppose player i contemplates a switch to his other strategy B at some point in time. Now it is not sufficient to only compare $u(B, L)$ and $u(T, L)$ as in the case of the one-shot game G. Player i cannot expect that player j will not react. But it does not end there; player i can react to the reaction of player j, and so forth. There is no end to this. Figure 2 depicts the possible reaction chains.

Notice that Figure 2 is a game in extensive form with complete and perfect information. This game is called the *departure game* for player i at G-outcome (T, L), and is denoted by $D(i, (T, L))$. At the first node i compares non-deviating (STAY) and deviating.

We want to emphasize that this sequential-move game is 'played' only in the mind of player i. The actual game is simultaneous-move each period. Player j at node (T, L) has a different departure game, since in $D(j, (T, L))$ player j is the first to move.

We think that this is not an unrealistic description of how players might behave in the absence of communication. Note that we are not developing a normative theory, at least not at this stage, but rather try to describe what might be going on in the minds of the players, and what are the logical consequences of those processes.

Consider again the tree $D(i, (T, L))$ in Figure 2. Player i has no incentive to deviate from repeatedly playing (T, L) in the actual game if his evaluation of $D(i, (T, L))$ results in choosing 'STAY' at the first node. (We will discuss the evaluation of the departure games momentarily.) In that case we say that the G-outcome (T, L) is *far-sighted*

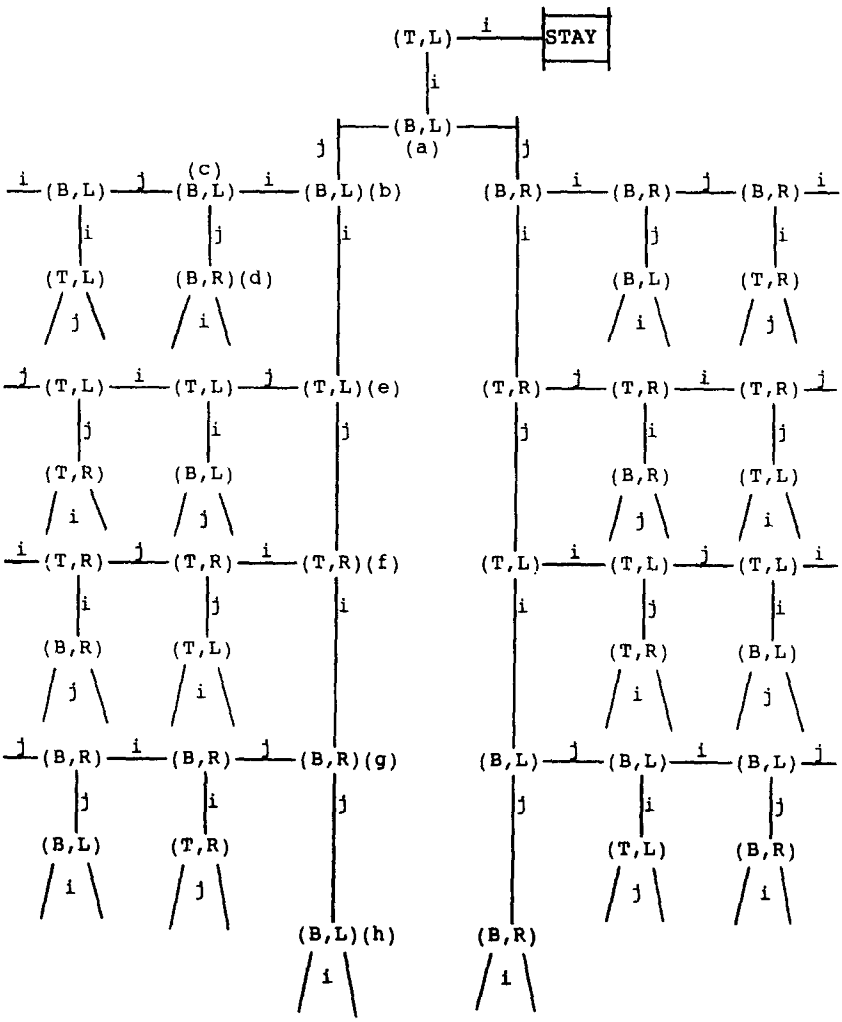

Fig. 2.

stable for i. A G-outcome is called a *far-sighted equilibrium* if it is far-sighted stable for both players.

We will now discuss the evaluation of a departure game, in particular $D(i, (T, L))$. Solutions to any other departure game can be obtained by relabeling players and/or G-outcomes. First of all we need to define the preferences of the players over infinite sequences of

G-outcomes. (These preferences are the same in both the actual game and the departure games.) By abuse of notation, let $(x, y) = (x_t, y_t)_{t=1}^{\infty} \in (X \times Y)^{\infty}$ denote an infinite sequence of G-outcomes. At date t, the G-outcome (x_t, y_t) has been chosen. The associated stream of utility will be denoted by

$$[\mu, \nu] = [\mu_t, \nu_t]_{t=1}^{\infty},$$

where $\mu_t = u(x_t, y_t)$ and $\nu_t = v(x_t, y_t)$.

We assume the following strict preferences P^i and P^j for player i and j respectively. These are based on the so-called Ramsey–Von Weizaecker criterion (Ramsey, 1928; Von Weizaecker, 1965). The sequence of G-outcomes $(\hat{x}, \hat{y})$ is strictly preferred by a player to the sequence of G-outcomes $(\bar{x}, \bar{y})$ if and only if there exists a time t_0 such that the mean utility for that player generated by $(\hat{x}, \hat{y})$ is strictly larger than the mean utility generated by $(\bar{x}, \bar{y})$ in each period from t_0 on. Formally, define the mean utility at period t of the stream of utilities $\mu = (\mu_t)_{t=1}^{\infty}$ by

$$m_t(\mu) = (1/t) \sum_{q=1}^{t} \mu_t .$$

Then $(\hat{x}, \hat{y}) P^i (\bar{x}, \bar{y})$ iff

$$\exists t_0 \in N,\ \forall t \geqslant t_0 : m_t(\hat{\mu}) > m_t(\bar{\mu}),$$

and similarly for player j.

Thus, players are interested in the long-run consequences of their actions. We will see, however, that only regular sequences (repetitions of a finite sequence) of G-outcomes need to be considered. The preference relations P^i and P^j then reduce to comparing the utilities of these generating finite sequences. This is intuitively appealing since such finite sequences can be considered as short run consequences.

Consider the departure game $D(i, (T, L))$. It is a game with perfect and complete information. A natural solution concept to such a game is the subgame perfect Nash equilibrium (Selten, 1973, 1975). For finite games in the class of complete and perfect information games, such subgame perfect Nash equilibrium can be obtained through backwards induction. This procedure cannot be applied right away to the game $D(i, (T, L))$, since this game is not finite. However, we have the following observation. Consider game $D(i, (T, L))$ in Figure 2. Starting with node (a), each branch has the property that at a certain stage there will be a node at which one of the players faces the exact same decision problem as at some node before in that branch. For example, consider the branch starting with nodes (a), (b), (c). At node (c), player j faces the same decision problem as at node (a). Since the game is infinite, the subgame starting at node (c) is a copy of the subgame starting at node (a). Another example is the branch starting with nodes (a), (b), (e), (f), (g), (h). At node (h) player i faces the exact same decision problem as at node (b). The subgame starting at node (h) is a copy of the subgame starting at node (b). It can be shown that taking the same alternative at identical decision problems is at least as good, if not better, than taking different alternatives. A formal proof of this statement can be obtained from the author upon request. With this in mind we now cut off branches that are suboptimal.

Consider node (c) again. At node (a), player j chose L. Hence, at node (c) he will again choose L. Now cut off the subtree that starts with node (d). At the node (B, L) following (c), player i has the same decision problem as at node (b). At (b) he chose B, so he will again choose B. Cut off the subtree that follows when he would choose T.

Proceeding in this way we are left with a finite number (31) of infinite branches. These branches are pairwise disjoint from a certain stage on. See Figure 3. The numbers in the leafs of this tree refer to the numbers in Figure 4. They indicate the sequence of nodes that is repeated all the time after we do the cutting as explained above.

The evaluation of infinite branches in which a particular sequence of nodes is repeated (possibly preceded by some finite sequence of different nodes) is not difficult. It can easily be shown that in comparing two such infinite branches, all we have to do is to compare the mean utility of the generating sequence. The initial finite sequence is irrelevant. A formal proof of this statement can be obtained from the author upon request. Thus, in Figure 3 we can replace the numbers in the leafs by the mean utility (one number for each player) of its

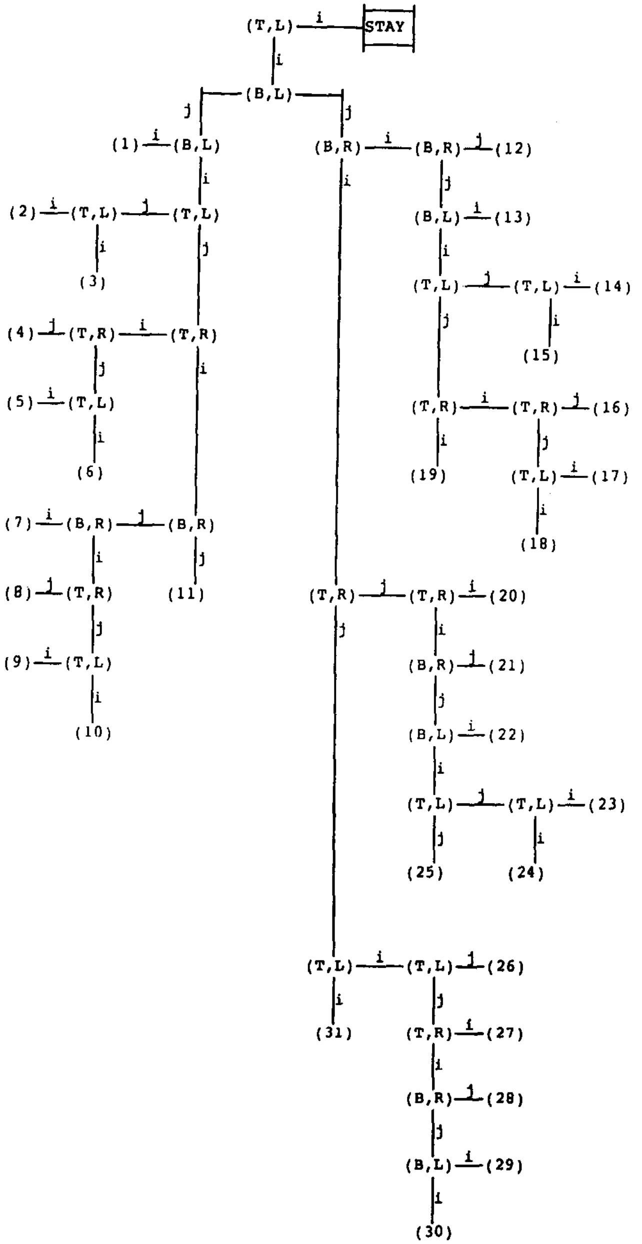
(T,L)
STAY
(B,L)
(1)
(B,L)
(B,R)
(B,R)
(12)
(2)
(T,L)
(T,L)
(B,L)
(13)
(3)
(T,L)
(T,L)
(14)
(4)
(T,R)
(T,R)
(15)
(5)
(T,L)
(T,R)
(T,R)
(16)
(6)
(19)
(T,L)
(17)
(7)
(B,R)
(B,R)
(18)
(8)
(T,R)
(11)
(T,R)
(T,R)
(20)
(9)
(T,L)
(B,R)
(21)
(10)
(B,L)
(22)
(T,L)
(T,L)
(23)
(25)
(24)
(T,L)
(T,L)
(26)
(31)
(T,R)
(27)
(B,R)
(28)
(B,L)
(29)
(30)

Fig. 3.

1. $\underline{\;i\;}(B,L)\underline{\;j\;}(B,L)\underline{\;i\;}$

2. $\underline{\;i\;}(T,L)\underline{\;j\;}(T,L)\underline{\;i\;}$

3. $\underline{\;i\;}(B,L)\underline{\;j\;}(B,L)\underline{\;i\;}(T,L)\underline{\;j\;}(T,L)\underline{\;i\;}$

4. $\underline{\;j\;}(T,R)\underline{\;i\;}(T,R)\underline{\;j\;}$

5. $\underline{\;i\;}(T,L)\underline{\;j\;}(T,R)\underline{\;i\;}(T,R)\underline{\;j\;}(T,L)\underline{\;i\;}$

6. $\underline{\;i\;}(B,L)\underline{\;j\;}(B,L)\underline{\;i\;}(T,L)\underline{\;j\;}(T,R)\underline{\;i\;}(T,R)\underline{\;j\;}(T,L)\underline{\;i\;}$

7. $\underline{\;i\;}(B,R)\underline{\;j\;}(B,R)\underline{\;i\;}$

8. $\underline{\;j\;}(T,R)\underline{\;i\;}(B,R)\underline{\;j\;}(B,R)\underline{\;i\;}(T,R)\underline{\;j\;}$

9. $\underline{\;i\;}(T,L)\underline{\;j\;}(T,R)\underline{\;i\;}(B,R)\underline{\;j\;}(B,R)\underline{\;i\;}(T,R)\underline{\;j\;}(T,L)\underline{\;i\;}$

10. $\underline{\;i\;}(B,L)\underline{\;j\;}(B,L)\underline{\;i\;}(T,L)\underline{\;j\;}(T,R)\underline{\;i\;}(B,R)\underline{\;j\;}$

$\underline{\;j\;}(B,R)\underline{\;i\;}(T,R)\underline{\;j\;}(T,L)\underline{\;i\;}$

11. $\underline{\;j\;}(B,L)\underline{\;i\;}(T,L)\underline{\;j\;}(T,R)\underline{\;i\;}(B,R)\underline{\;j\;}$

12. $\underline{\;j\;}(B,R)\underline{\;i\;}(B,R)\underline{\;j\;}$

13. $\underline{\;i\;}(B,L)\underline{\;j\;}(B,R)\underline{\;i\;}(B,R)\underline{\;j\;}(B,L)\underline{\;i\;}$

14. $\underline{\;i\;}(T,L)\underline{\;j\;}(T,L)\underline{\;i\;}$

15. $\underline{\;i\;}(B,L)\underline{\;j\;}(B,R)\underline{\;i\;}(B,R)\underline{\;j\;}(B,L)\underline{\;i\;}(T,L)\underline{\;j\;}(T,L)\underline{\;i\;}$

16. $\underline{\;j\;}(T,R)\underline{\;i\;}(T,R)\underline{\;j\;}$

17. $\underline{\;i\;}(T,L)\underline{\;j\;}(T,R)\underline{\;i\;}(T,R)\underline{\;j\;}(T,L)\underline{\;i\;}$

18. $\underline{\;i\;}(B,L)\underline{\;j\;}(B,R)\underline{\;i\;}(B,R)\underline{\;j\;}(B,L)\underline{\;i\;}(T,L)\underline{\;j\;}$

$\underline{\;j\;}(T,R)\underline{\;i\;}(T,R)\underline{\;j\;}(T,L)\underline{\;i\;}$

19. $\underline{\;i\;}(B,R)\underline{\;j\;}(B,L)\underline{\;i\;}(T,L)\underline{\;j\;}(T,R)\underline{\;i\;}$

Fig. 4a.

20. $\underline{\;i\;}$ (T,R) $\underline{\;j\;}$ (T,R) $\underline{\;i\;}$

21. $\underline{\;j\;}$ (B,R) $\underline{\;i\;}$ (T,R) $\underline{\;j\;}$ (T,R) $\underline{\;i\;}$ (B,R) $\underline{\;j\;}$

22. $\underline{\;i\;}$ (B,L) $\underline{\;j\;}$ (B,R) $\underline{\;i\;}$ (T,R) $\underline{\;j\;}$ (T,R) $\underline{\;i\;}$ (B,R) $\underline{\;j\;}$ (B,L) $\underline{\;i\;}$

23. $\underline{\;i\;}$ (T,L) $\underline{\;j\;}$ (T,L) $\underline{\;i\;}$

24. $\underline{\;i\;}$ (B,L) $\underline{\;j\;}$ (B,R) $\underline{\;i\;}$ (T,R) $\underline{\;j\;}$ (T,R) $\underline{\;i\;}$ (B,R) $\underline{\;j\;}$
$\underline{\;j\;}$ (B,L) $\underline{\;i\;}$ (T,L) $\underline{\;j\;}$ (T,L) $\underline{\;i\;}$

25. $\underline{\;j\;}$ (T,R) $\underline{\;i\;}$ (B,R) $\underline{\;j\;}$ (B,L) $\underline{\;i\;}$ (T,L) $\underline{\;j\;}$

26. $\underline{\;j\;}$ (T,L) $\underline{\;i\;}$ (T,L) $\underline{\;j\;}$

27. $\underline{\;i\;}$ (T,R) $\underline{\;j\;}$ (T,L) $\underline{\;i\;}$ (T,L) $\underline{\;j\;}$ (T,R) $\underline{\;i\;}$

28. $\underline{\;j\;}$ (B,R) $\underline{\;i\;}$ (T,R) $\underline{\;j\;}$ (T,L) $\underline{\;i\;}$ (T,L) $\underline{\;j\;}$ (T,R) $\underline{\;i\;}$ (B,R) $\underline{\;j\;}$

29. $\underline{\;i\;}$ (B,L) $\underline{\;j\;}$ (B,R) $\underline{\;i\;}$ (T,R) $\underline{\;j\;}$ (T,L) $\underline{\;i\;}$ (T,L) $\underline{\;j\;}$
$\underline{\;j\;}$ (T,R) $\underline{\;i\;}$ (B,R) $\underline{\;j\;}$ (B,L) $\underline{\;i\;}$

30. $\underline{\;i\;}$ (T,L) $\underline{\;j\;}$ (T,R) $\underline{\;i\;}$ (B,R) $\underline{\;j\;}$ (B,L) $\underline{\;i\;}$

31. $\underline{\;i\;}$ (B,L) $\underline{\;j\;}$ (B,R) $\underline{\;i\;}$ (T,R) $\underline{\;j\;}$ (T,L) $\underline{\;i\;}$

Fig. 4b.

generating sequence. The tree game in Figure 3 then reduces to a finite game in extensive form with perfect and complete information. We now apply backwards induction to evaluate this tree. We assume that players resolve possible ties at certain nodes in favor of the other player (any other tie-breaking rule is of course just as plausible). See Figure 5.

EXAMPLE 1. *Prisoners' Dilemma*

Substituting the utilities of Figure 6 into Figure 5, we get Figure 7.

	Mean utility of generating sequence	
leaf	i	j
1	μ_{BL}	ν_{BL}
2	μ_{TL}	ν_{TL}
3	$(\mu_{BL} + \mu_{TL})/2$	$(\nu_{BL} + \nu_{TL})/2$
4	μ_{TR}	ν_{TR}
5	$(\mu_{TL} + \mu_{TR})/2$	$(\nu_{TL} + \nu_{TR})/2$
6	$(\mu_{BL} + \mu_{TL} + \mu_{TR})/3$	$(\nu_{BL} + \nu_{TL} + \nu_{TR})/3$
7	μ_{BR}	ν_{BR}
8	$(\mu_{TR} + \mu_{BR})/2$	$(\nu_{TR} + \nu_{BR})/2$
9	$(\mu_{TL} + \mu_{TR} + \mu_{BR})/3$	$(\nu_{TL} + \nu_{TR} + \nu_{BR})/3$
10	$(\mu_{BL} + \mu_{TL} + \mu_{TR} + \mu_{BR})/4$	$(\nu_{BL} + \nu_{TL} + \nu_{TR} + \nu_{BR})/4$
11	$(\mu_{BL} + \mu_{TL} + \mu_{TR} + \mu_{BR})/4$	$(\nu_{BL} + \nu_{TL} + \nu_{TR} + \nu_{BR})/4$
12	μ_{BR}	ν_{BR}
13	$(\mu_{BL} + \mu_{BR})/2$	$(\nu_{BL} + \nu_{BR})/2$
14	μ_{TL}	ν_{TL}
15	$(\mu_{BL} + \mu_{BR} + \mu_{TL})/3$	$(\nu_{BL} + \nu_{BR} + \nu_{TL})/3$
16	μ_{TR}	ν_{TR}
17	$(\mu_{TL} + \mu_{TR})/2$	$(\nu_{TL} + \nu_{TR})/2$
18	$(\mu_{BL} + \mu_{BR} + \mu_{TL} + \mu_{TR})/4$	$(\nu_{BL} + \nu_{BR} + \nu_{TL} + \nu_{TR})/4$
19	$(\mu_{BR} + \mu_{BL} + \mu_{TL} + \mu_{TR})/4$	$(\nu_{BR} + \nu_{BL} + \nu_{TL} + \nu_{TR})/4$
20	μ_{TR}	ν_{TR}
21	$(\mu_{BR} + \mu_{TR})/2$	$(\nu_{BR} + \nu_{TR})/2$
22	$(\mu_{BL} + \mu_{BR} + \mu_{TR})/3$	$(\nu_{BL} + \nu_{BR} + \nu_{TR})/3$
23	μ_{TL}	ν_{TL}
24	$(\mu_{BL} + \mu_{BR} + \mu_{TR} + \mu_{TL})/4$	$(\nu_{BL} + \nu_{BR} + \nu_{TR} + \nu_{TL})/4$
25	$(\mu_{TR} + \mu_{BR} + \mu_{BL} + \mu_{TL})/4$	$(\nu_{TR} + \nu_{BR} + \nu_{BL} + \nu_{TL})/4$
26	μ_{TL}	ν_{TL}
27	$(\mu_{TR} + \mu_{TL})/2$	$(\nu_{TR} + \nu_{TL})/2$
28	$(\mu_{BR} + \mu_{TR} + \mu_{TL})/3$	$(\nu_{BR} + \nu_{TR} + \nu_{TL})/3$
29	$(\mu_{BL} + \mu_{BR} + \mu_{TR} + \mu_{TL})/4$	$(\nu_{BL} + \nu_{BR} + \nu_{TR} + \nu_{TL})/4$
30	$(\mu_{TL} + \mu_{TR} + \mu_{BR} + \mu_{BL})/4$	$(\nu_{TL} + \nu_{TR} + \nu_{BR} + \nu_{BL})/4$
31	$(\mu_{BL} + \mu_{BR} + \mu_{TR} + \mu_{TL})/4$	$(\nu_{BL} + \nu_{BR} + \nu_{TR} + \nu_{TL})/4$

Fig. 5.

Using backwards induction in the departure game $D(i, (T, L))$ in Figure 3, it can be seen that the utility for player i of deviating is $\mu = 3$. An optimal path (i.e., the result of a subgame perfect Nash equilibrium) is the following. At the first node (B, L) player j will not choose L, since then player i will choose B and the result is leaf (1)

		j	
		L	R
i	T	[3 , 3]	[1 , 4]
	B	[4 , 1]	[2 , 2]

Fig. 6.

leaf	i	j
1	4	1
2	3	3
3	7/2	2
4	1	4
5	2	7/2
6	8/3	8/3
7	2	2
8	3/2	3
9	2	3
10	5/2	5/2
11	5/2	5/2
12	2	2
13	3	3/2
14	3	3
15	3	2
16	1	4

leaf	i	j
17	2	7/2
18	5/2	5/2
19	5/2	5/2
20	1	4
21	3/2	3
22	7/3	7/3
23	3	3
24	5/2	5/2
25	5/2	5/2
26	3	3
27	2	7/2
28	2	3
29	5/2	5/2
30	5/2	5/2
31	5/2	5/2

Fig. 7.

with utility bundle [4, 1]. This cannot be optimal for player j. Thus, j will choose R at the first node. At node (B, R) player i is indifferent between B and T. Suppose he chooses B. At node (B, R) player j will choose L. At node (T, L) player i will choose T. At node (T, L) player j will choose L. At the next node (T, L) player i chooses T. At the resulting node (T, L) player j faces the same decision problem as at some node before in the branch. He chose L. Therefore, we get the 'sequence' $((T, L))$. See leaf (14). The mean utility for player i of this leaf is 3. The utility of 'STAY' for player i at the first node also equals 3. Hence, player i thinks that deviating will not improve his utility. Therefore, (T, L) is far-sighted stable for player i. By symmetry it is also far-sighted stable for player j. Hence, G-outcome (T, L) with utility [3, 3] is a far-sighted equilibrium.

Now relabel the strategies of both players so that utility bundle [2, 2] is now associated with strategy bundle (T, L). See Figure 8.

Again substitute the utilities in Figure 5. Solving the departure game in Figure 3, we now find that the utility for player i of deviating is 5/2. The utility of 'STAY' is 2. Hence, this G-outcome is not far-sighted stable for i, and thus is not a far-sighted equilibrium.

The strategy bundle with associated utility bundle [1, 4] is obviously far-sighted stable for player j, but not for player i, and hence is not a far-sighted equilibrium. Thus, the only far-sighted equilibrium is the symmetric Pareto optimal one, with utility bundle [3, 3].

		j	
		L	R
i	T	[2 , 2]	[4 , 1]
	B	[1 , 4]	[3 , 3]

Fig. 8.

		j	
		L	R
i	T	[-1 , 1]	[1 ,-1]
	B	[1 ,-1]	[-1 , 1]

Fig. 9.

EXAMPLE 2. *Matching Pennies*

In this game there are no far-sighted equilibria. The utility of deviating in any departure game is zero. Intuitively this can be seen as follows. Consider repeated play of G-outcome (T, L). Suppose player i were to start his departure game $D(i, (T, L))$, i.e., he moves to (B, L). This G-outcome is unfavorable for player j; he will move to (B, R). But this G-outcome is very unfavorable for player i; he will move to (T, R). And so forth. Any departure game will induce cycling over all four G-outcomes. Hence, the utility of deviating in any departure game is zero.

In the Matching Pennies game we expect to observe that players alternate over all four G-outcomes, i.e., the sequence $((T, L), (B, L), (B, R), (T, R))$ will be played repeatedly. Player j cannot hope to get G-outcome (T, L) or (B, R) all the time, and similarly player i cannot hope to get G-outcome (B, L) or (T, R) all the time. Games like this will be the topic of the next section.

3. CORRELATED G-OUTCOMES

In this section we relax the assumption that the period of the sequence of observed G-outcomes is 1. For example, in the Matching Pennies game (Figure 9) we expect to observe that players repeatedly alternate between all four G-outcomes (i.e., the period is 4). Call such an observed sequence of G-outcomes a *correlated* G-*outcome*. Now evaluate the departure games for each player starting with any of the G-outcomes in the observed correlated G-outcome. Let the utility of 'STAY' in each such departure game be the mean utility of the correlated G-outcome.

In the Matching Pennies game the utility of 'STAY' of the 4-period correlated G-outcome $((T, L), (T, R), (B, R), (B, L))$ is 0 for both players. In the previous section we argued that the utility of deviating in any departure game is 0. Hence, the 4-period correlated G-outcome is a far-sighted equilibrium. (In fact, any correlated G-outcome with a 'STAY'-utility of 0 for both players is a far-sighted equilibrium.) Thus, although the Matching Pennies game has no 1-period far-sighted equilibrium, it does have multi-period far-sighted equilibria.

We conjecture that any game has multi-period far-sighted equilibria. We have not been able to prove this, however.

4. CONCLUSIONS

We presented a theory of behavior in 2×2, simultaneous-move, repeated games in which communication is limited, but in which we nevertheless observe a certain periodicity in the sequence of outcomes of the underlying one-shot game. In evaluating the consequences of a deviation of such a periodic repetition of outcomes, players think of alternating moves rather than simultaneous ones.

The logical consequences of this theory were investigated for two games: Prisoners' Dilemma and Matching Pennies. In both games, our theory gives not unrealistic conclusions. Although communication is limited, repetition still induces 'cooperation'. (The so-called Folk Theorem in repeated game theory.) At this moment we do not have any general result with regard to the existence of far-sighted (correlated) equilibria.

To determine whether a correlated G-outcome of length T is a far-sighted equilibrium or not, is a simple task. There are $2T$ finite games in extensive form to consider, each easy to solve.

We have restricted ourselves to 2×2 games. Extensions to 2-person, many-(but finite)-strategy games is straightforward. The extension to many-player games is not at all obvious. The problem is how the player who contemplates a deviation thinks the other players will react.

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