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Selecting a unique competitive equilibrium with default penalties

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Abstract The enlargement of the general-equilibrium structure to allow for default subject to penalties results in a construction of a simple mechanism for selecting a unique competitive equilibrium. We consider economies for which a common credit money can be applied to uniquely select each of the competitive equilibria with suitable default penalties. We identify two classes of such economies.

Keywords Competitive equilibrium · Credit mechanism · Marginal utility of income · Saddle-point characterization · Welfare economics

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1 Introduction

The problem of finding the most general conditions required to guarantee a unique competitive equilibrium (CE in short) for a general-equilibrium system is complex

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and challenging mathematically. By enlarging the problem, an approach is proposed that both offers a solution and facilitates an interesting selection.[1]

There appear to be two basic ways to obtain the uniqueness of equilibrium in economic models. The first is to seek reasonable conditions within the mathematical structure of a given economic model. A good example of this is the explorations of the conditions for an exchange economy that guarantee gross substitutability. The second is to extend or modify the actual model. In this paper we are concerned with the latter approach. Morris and Shin (2000) have approached the obtaining of the uniqueness via uncertainty of the information conditions, utilizing the global games formulation of Carlsson and van Damme (1993).

Our approach is based on a different fact. The general-equilibrium model does not utilize credit or financial institutions because trust is implicitly perfect. All trade is balanced at the end of the market. It is as if each individual at the start of the model has available implicitly a credit line equal to the worth of the individual's initial wealth at the final market price. When trust is imperfect and credit is introduced, a mechanism is needed to predict and determine the worth of an individual at the end of trade. This includes the possibility of having credit left over and the need for a penalty for those ending up in debt. We introduce these default conditions and argue the uniqueness and other properties from them. Economic dynamics require institutions to carry process. As such, it appears to be quite plausible that there are many contributing factors that can lead to the same phenomenon. We believe that there may be several different economic and political factors that contribute to the uniqueness of equilibrium. However, institutionally, the presence of default conditions is certainly of importance.

To investigate the possibility of obtaining the uniqueness of CE from default conditions, we consider a simple mechanism for an economy that requires trade be in a credit money issued by a government bank. Before trading begins, traders exchange personal IOUs for the credit money with the bank charging them an interest rate of zero. Each trader may exchange personal IOUs for the credit money without limit. But, after he has bought and received income from selling, he goes to the bank to settle up all outstanding credit.[2] Ending up with net credit is worthless for him, while a default penalty is levied against him for ending as a net debtor.[3]

By including an option to default, the mechanism enlarges the traders' budget sets. As a consequence, the mechanism puts stronger conditions on CEs. We say that a CE for an economy is a selection by the credit mechanism if the CE allocation remains as

[1] The problem of the uniqueness of competitive equilibrium is not a mere mathematical curiosity. It raises basic questions in macroeconomics in the employment of any dynamic model. It can be avoided by limiting oneself to sufficiently low dimensional models as has been done by those using dynamic programming models of macro–micro economic phenomena, where a single good is put up for sale and bought for cash (see Shubik and Whitt 1973; Lucas 1978; Karatzas et al. 1994, for examples). However, the difficulties remain for higher dimensional problems.

[2] One way to play the model in a classroom is at the beginning to give each student a large stack of banknotes and inform him that at the end of the game, after he has bought and received income from selling, he has to return exactly the amount he started with initially or he will have to pay a default penalty. For discussions on alternative credit mechanisms for the competitive model, the reader is referred to Shubik (1999).

[3] For example, default penalties may be in the form of asset confiscation from the debtors or jail sentences or other societal punishments.

a CE allocation in the presence of the option to default subject to penalties. We refer to CEs for the economy that can be selected by the credit mechanism as default-qualified competitive equilibria.

We begin investigating the credit mechanism with a useful property of general-equilibrium analysis. Namely, under some mild conditions, a CE corresponds to a saddle-point for each trader. This saddle-point characterization of a CE has useful applications to the design of default penalties towards the selection of a unique CE, as well as to the study of the already familiar welfare properties of CE allocations. We then consider economies for which a uniform credit money (i.e., a uniform unit of account) can be applied, such that each CE of the economy can be a unique default-qualified CE with suitable default penalties.[4] We identify two classes of such economies. One class consists of economies with homogeneous of degree 1 utility functions, while the other class consists of economies with quasi-linear utility functions. Our analysis is carried out for pure exchange economies. However, we discuss extensions to production economies via Rader's *equivalence principle*.

The rest of the paper is organized as follows. The next section briefly discusses saddle-point characterization of CEs. Section 3 presents results for pure exchange economies. Section 4 discusses extensions to economies with production and Sect. 5 concludes the paper.

2 Saddle-point characterization

Consider an exchange economy $\mathcal{E} = \{X^i, u^i, a^i\}_{i=1}^n$ with consumption set X^i, utility function u^i, and endowment a^i for trader i. The number of commodities m and the number of traders n are both finite. Set $N = \{1, 2, \ldots, n\}$. We assume for all $i \in N$, A1: $X^i = \Re^m_+$; A2: u^i *is locally non-satiated, continuous, and concave*; and A3: $a^i \in \Re^m_+$ *with* $a^i \neq 0$ and $u^i(a^i) > 0$. We also assume A4: *for each commodity* $1 \leq h \leq m$, *there is a trader* i *such that* $u^i(x^i + \delta e^h) > u^i(x^i)$ *for all* $x^i \in \Re^m_+$ *and for all* $\delta > 0$, *where* $e^h \in \Re^m$ *is the bundle with* $e^h_h = 1$ *and* $e^h_k = 0$ *for all* $k \neq h$.[5] These are familiar assumptions in general-equilibrium analysis. Assumption A4 requires that each commodity h be desirable by some trader, in the sense that an increase in the amount of commodity h from any bundle increases his utility. Local non-satiation of a trader's utility function is automatic if there is a commodity which is desirable by him. Assumption A4 guarantees that no commodity can be free in CE. By local non-satiation, Lagrangian multipliers associated with traders' utility maximization problems are strictly positive.

A CE for economy $\mathcal{E}$ is a pair $(\bar{x}, \bar{p})$ with allocation $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$ and price vector $\bar{p}$ such that for all i, $\bar{x}^i$ solves

$$\max u^i(x^i) \ \text{ s.t. } \ \bar{p} \cdot (a^i - x^i) \geq 0, \ x^i \geq 0. \tag{1}$$

[4] This will include CEs with the *minimal cash flow* property. Such CEs are special for the reason that they minimize the need for a substitute-for-trust in trade.

[5] Given a positive integer l, $\Re^l_+$ denotes the non-negative orthant of the l-dimensional Euclidean space and $\Re^l_{++}$ denotes the subset of $\Re^l_+$ containing vectors in $\Re^l_+$ with positive components.

and

$$\sum_{i \in N} \bar{x}^i = \sum_{i \in N} a^i. \tag{2}$$

The market clearance condition (2) is stronger than the clearance condition allowing for surplus. However, A4 implies that no commodity can be free in CE; hence, there cannot be any surplus.

The saddle-point characterization of CEs in Theorem 1 below is well-known. A proof can be established by applying the saddle-point characterization of solutions for non-linear programming problems.[6]

Theorem 1 (Saddle-point characterization) *Let* $\mathcal{E} = \{X^i, u^i, a^i\}_{i \in N}$ *be an exchange economy satisfying* A1–A4. *Then, a pair* $(\bar{x}, \bar{p}) \in \Re^{mn}_{+} \times \Re^{m}_{++}$ *is a CE if and only if* $\bar{x}$ *satisfies* (2) *and there exists a vector* $\bar{\lambda} = (\bar{\lambda}^i)_{i=1}^n \in \Re^n_{++}$ *such that for all* i, *the triplet* $(\bar{x}^i, \bar{p}, \bar{\lambda}^i)$ *satisfies*

$$u^i(x^i) + \bar{\lambda}^i \bar{p} \cdot (a^i - x^i) \leq u^i(\bar{x}^i) + \bar{\lambda}^i \bar{p} \cdot (a^i - \bar{x}^i) \leq u^i(\bar{x}^i) + \lambda^i \bar{p} \cdot (a^i - \bar{x}^i) \tag{3}$$

for all $x^i \in \Re^m_+$ *and* $\lambda^i \in \Re_+$.

Condition (3) is equivalent to $(\bar{x}^i, \bar{\lambda}^i)$ being a saddle-point of the *Lagrangian function* $\mathcal{L}^i(x^i, \lambda^i) = u^i(x^i) + \lambda^i p \cdot (a^i - x^i)$ for utility maximization problem (1). When a triplet $(\bar{x}, \bar{p}, \bar{\lambda})$ satisfies (2) and (3) for all i, we call it *a competitive triplet* and we call $\bar{x}$, $\bar{p}$, and $\bar{\lambda}$, respectively, a *competitive allocation*, a *competitive price vector*, and a *competitive multiplier vector*.

3 Results

The unit of account is arbitrary in competitive equilibrium. By normalizing the prices, a particular unit of count is imposed. In what follows, we consider normalized prices so as to fix a unit of account to measure credit and debt.

Suppose that traders use a credit money to buy or sell goods. A bank provides the credit money of zero interest. Before trading begins, traders exchange personal IOUs for the credit money with the bank for as much as he wants. But, after he has bought and received income from selling, trader i settles up all outstanding credit with the bank. There is a penalty equal to $\mu^i > 0$ amount of reduction in i's utility for each unit of debt he is unable to repay, while it is of no value to him for ending as a net creditor.

The credit money and default penalties together transforms economy $\mathcal{E}$ into $\mathcal{E}_\mu$, in which trader i has utility function

$$U^i(x^i, p) = u^i(x^i) + \mu^i \min[p \cdot a^i - p \cdot x^i, 0]. \tag{4}$$

[6] See, for example, Takayama (1985, p. 75) for the saddle-point characterization of solutions for non-linear programming problems.

A CE for $\mathcal{E}_\mu$ is a pair $(\bar{x}, \bar{p})$ such that for all i, $\bar{x}^i$ solves

$$\max_{x^i \in \Re^m_+} U^i(x^i, \bar{p}) \tag{1$'$}$$

and

$$\sum_{i \in N} \bar{x}^i = \sum_{i \in N} a^i. \tag{2$'$}$$

By (4), it is un-optimal for trader i not to incur debt unless his marginal utility of income is less than or equal to μ^i.

Definition 1 Let $\mathcal{E} = \{X^i, u^i, a^i\}_{i \in N}$ be an exchange economy. We say that a CE for $\mathcal{E}$ with competitive allocation $\bar{x}$ and price vector $\bar{p}$ is selected with default penalty vector μ if $(\bar{x}, \bar{p})$ is also a CE for $\mathcal{E}_\mu$.

Given default penalty vector μ, we call CEs for economy $\mathcal{E}$ that can be selected default-qualified CEs (DCEs in short). In the law, the default penalties are written as broad categories, not as individual penalties. In practice, the litigation that often accompanies default individualizes the penalties. Thus, including a vector of individual penalties is reasonable. The following theorem provides a connection between DCEs and CEs.

Theorem 2 *Let $\mathcal{E} = \{X^i, u^i, a^i\}_{i \in N}$ be an exchange economy and $\mu = (\mu^i)_{i=1}^n$ a vector of default penalties. Assume $\mathcal{E}$ satisfies* A1-A4. *If $(\bar{x}, \bar{p})$ is a CE for $\mathcal{E}_\mu$, then there exists a vector $\bar{\lambda} \in \Re^n_{++}$ with $\bar{\lambda} \leq \mu$ such that $(\bar{x}, \bar{p}, \bar{\lambda})$ is a competitive triplet for $\mathcal{E}$.*

Proof Let $(\bar{x}, \bar{p})$ be a CE for $\mathcal{E}_\mu$. By A4 and (1$'$), $\bar{p} >> 0$ (i.e., $\bar{p}_h > 0$ for all $h = 1, 2, \ldots, m$). From (4) and the local non-satiation of u^i it follows that $\bar{x}^i_{m+1} = \bar{p} \cdot a^i - \bar{p} \cdot \bar{x}^i \leq 0$ for all $i \in N$.[7] Thus, $(\bar{x}^i, \bar{x}^i_{m+1})$ solves

$$\begin{aligned} \max & \left\{ u^i(x^i) + \mu^i x^i_{m+1} \right\} \\ \text{s.t. } & \bar{p} \cdot a^i - \bar{p} \cdot x^i \geq x^i_{m+1} \quad \text{and} \\ & \bar{p} \cdot a^i - \bar{p} \cdot \sum_{j \in N} a^j \leq x^i_{m+1} \leq 0, \;\; x^i \in \Re^m_+. \end{aligned}$$

By A1–A4, the preceding constrained maximization problem satisfies conditions for the saddle-point characterization of solutions for non-linear programming problems. Thus, there exists a number $\bar{\lambda}^i \geq 0$ such that the triplet $(\bar{x}^i, \bar{x}^i_{m+1}, \bar{\lambda}^i)$ satisfies

[7] This can be verified as follows. Suppose on the contrary $\bar{p} \cdot \bar{x}^i < \bar{p} \cdot a^i$. Then, by the local non-satiation of u^i, there exists a bundle $y^i \in \Re^m_+$ such that $\bar{p} \cdot y^i < \bar{p} \cdot a^i$ and $u^i(y^i) > u^i(\bar{x}^i)$. Notice $\min[\bar{p} \cdot a^i - \bar{p} \cdot y^i, 0] = \min[\bar{p} \cdot a^i - \bar{p} \cdot \bar{x}^i, 0] = 0$. Hence, by (4), $U^i(\bar{x}^i, \bar{p}) = u^i(\bar{x}^i) < u^i(y^i) = U^i(y^i, \bar{p})$. This contradicts $\bar{x}^i$ being the equilibrium bundle for i in $\mathcal{E}_\mu$.

$$\begin{aligned} &u^i(x^i) + \mu^i x^i_{m+1} + \bar{\lambda}^i(\bar{p} \cdot a^i - \bar{p} \cdot x^i - x^i_{m+1}) \\ &\quad \le u^i(\bar{x}^i) + \mu^i \bar{x}^i_{m+1} + \bar{\lambda}^i(\bar{p} \cdot a^i - \bar{p} \cdot \bar{x}^i - \bar{x}^i_{m+1}) \\ &\quad \le u^i(\bar{x}^i) + \mu^i \bar{x}^i_{m+1} + \lambda^i(\bar{p} \cdot a^i - \bar{p} \cdot \bar{x}^i - \bar{x}^i_{m+1}) \end{aligned} \tag{5}$$

for all $x^i \in \Re^m_+$, $\bar{p} \cdot a^i - \bar{p} \cdot \sum_{j \in N} a^j \le x^i_{m+1} \le 0$, and for all $\lambda^i \in \Re_+$.

The local non-satiation of u^i together with the first inequality in (5) implies $\bar{\lambda}^i > 0$. This in turn with the second inequality in (5) implies

$$\bar{p} \cdot \bar{x}^i + \bar{x}^i_{m+1} = \bar{p} \cdot a^i. \tag{6}$$

By (2′), $\sum_{i \in N} \bar{x}^i = \sum_{i \in N} a^i$ and as noticed before, $\bar{x}^i_{m+1} \le 0$ for all i. Hence, it follows from (6) that $\bar{x}^i_{m+1} = 0$ for all i.

Next, by (5) and (6), $\bar{x}^i_{m+1} = 0$ implies that $(\bar{x}^i, \bar{p}, \bar{\lambda}^i)$ satisfies (3). Thus, $(\bar{x}^i, \bar{p}, \bar{\lambda}^i)$ is a competitive triplet for $\mathcal{E}$. By the first inequality in (5),

$$(\mu^i - \bar{\lambda}^i) x^i_{m+1} \le 0, \quad \forall x^i_{m+1} \text{ s.t. } \bar{p} \cdot a^i - \bar{p} \cdot \sum_{j \in N} a^j \le x^i_{m+1} \le 0.$$

Since $\bar{p} \cdot a^i < \bar{p} \cdot \sum_{j \in N} a^j$ by A3, the preceding inequality implies $\bar{\lambda}^i \le \mu^i$. □

If a competitive multiplier vector $\bar{\lambda}$ associated with a CE of $\mathcal{E}$ is such that $\bar{\lambda}_i > \mu_i$ for some trader i, then the per-unit penalty on trader i is not severe enough in the sense that on the margin trader i gains from being in debt. When this occurs, the budget constraint will be violated. Since no one ends as a net creditor, the market for commodities will be imbalanced. Hence, such a CE cannot be a DCE. A direct application of Theorem 2 implies:

Corollary 1 *Let $\mathcal{E} = \{X^i, u^i, a^i\}_{i \in N}$ be an exchange economy. Assume $\mathcal{E}$ satisfies* A1–A4. *Then, given default penalties μ, only those CEs of $\mathcal{E}$ with multiplier vectors $\bar{\lambda} \le \mu$ are selected.*

3.1 Selection of a unique CE

Let λ' and λ'' be two competitive multiplier vectors. We say that λ' Pareto dominates λ'' if $\lambda' \ge \lambda''$. Suppose that a CE of $\mathcal{E}$ has an associated competitive multiplier vector which does not Pareto dominate any other competitive multiplier vector. Then, by Corollary 1, this CE can be uniquely selected with penalty vector equal to the associated competitive multiplier vector. For later reference, we summarize this result in the following theorem whose proof is omitted.

Theorem 3 *Let $\mathcal{E} = \{X^i, u^i, a^i\}_{i \in N}$ be an exchange economy. Assume $\mathcal{E}$ satisfies* A1–A4. *Assume further $\mathcal{E}$ satisfies* A5*: Pareto dominance between any two competitive multiplier vectors does not hold. Then, given any competitive triplet $(\bar{x}, \bar{p}, \bar{\lambda})$ of $\mathcal{E}$, $(\bar{x}, \bar{p})$ is a unique selection of the credit mechanism with $\mu = \bar{\lambda}$.*

We now present two examples which satisfies A5 with a proper choice of price normalization. Later, we show that these are examples of economies from two classes satisfying A5.

Example 1 (Shapley and Shubik 1977) There are two goods and two traders with endowments $a^1 = (40, 0)$, $a^2 = (0, 50)$, and utility functions $u^1(x^1) = x_1^1 + 100(1 - e^{-x_2^1/10})$, $u^2(x^2) = 110(1 - e^{-x_1^2/10}) + x_2^2$ on $\Re_+^2$. Traders 1 and 2 are, respectively, named Ivan and John in Shapley and Shubik (1977); goods 1 and 2 are, respectively, called rubles and dollars in that paper. There are three CEs all with interior allocations in this economy. Notice the interior Pareto optimal allocations satisfy

$$x_2^2 = x_1^2 + 50 - 10 \ln 110. \tag{7}$$

Since $\bigtriangledown u^1(x^1) = (1, 10e^{-x_2^1/10})$ and $\bigtriangledown u^2(x^2) = (11e^{-x_1^2/10}, 1)$, Eq. (7) implies that to be Pareto optimal, trader 2's consumption of good 2 increases with his consumption of good 1. Since trader 1's marginal utility of good 1 is constant and his marginal utility of good 2 is decreasing in x_2^1, while trader 2's marginal utility of good 2 is constant and his marginal utility of good 1 is decreasing in x_1^2, from (7) we have that at any two interior Pareto optimal allocations $\bar{x}$ and $\tilde{x}$

$$\bigtriangledown u^1(\bar{x}^1) \cdot \bar{a} > \bigtriangledown u^1(\tilde{x}^1) \cdot \bar{a} \Leftrightarrow \bigtriangledown u^2(\bar{x}^2) \cdot \bar{a} < \bigtriangledown u^2(\tilde{x}^2) \cdot \bar{a} \tag{8}$$

where $\bar{a} = a^1 + a^2$ denotes the total endowment. Since u^1 and u^2 are differentiable and CE bundles are all interior bundles, it follows that at any competitive triplet $(\bar{x}, \bar{p}, \bar{\lambda})$,

$$\bigtriangledown u^1(\bar{x}^1) = \bar{\lambda}^1 \bar{p} \quad \text{and} \quad \bigtriangledown u^2(\bar{x}^2) = \bar{\lambda}^2 \bar{p}. \tag{9}$$

Suppose price normalization is such that the wealth of total endowment $\bar{a}$ is constant:

$$p \cdot \bar{a} \equiv \text{constant}.$$

Then, this price normalization together with (8) and (9) implies that A5 holds for this economy. □

Example 2 There are two goods and two traders with endowments $a^1 \in \Re_+^2$, $a^2 \in \Re_+^2$, and utility functions $u^1(x^1) = \left[(x_1^1)^{-2} + \alpha(x_2^1)^{-2}\right]^{-\frac{1}{2}}$, $u^2(x^2) = \left[\beta(x_1^2)^{-2} + (x_2^2)^{-2}\right]^{-\frac{1}{2}}$. Assume

$$a_1^1 + a_1^2 = a_2^1 + a_2^2 > 0 \quad \text{and} \quad \beta < \alpha^{-1}. \tag{10}$$

When $a^1 = (1, 0)$, $a^2 = (0, 1)$, and $\alpha = \beta = (12/37)^2$, this example is the same as exercise 15.B.6 in Mas-Colell et al. (1995). As with that exercise, the economy has three CEs, all with interior bundles. Notice that at interior bundles,

$$\text{MRS}^1(x^1) = \alpha^{-1}\left(\frac{x_2^1}{x_1^1}\right)^3 \quad \text{and} \quad \text{MRS}^2(x^2) = \beta\left(\frac{x_2^2}{x_1^2}\right)^3.$$

It follows that interior Pareto optimal allocations satisfy

$$\alpha^{-1}\left(\frac{x_2^1}{x_1^1}\right)^3 = \beta\left(\frac{x_2^2}{x_1^2}\right)^3 = K \tag{11}$$

for some $K > 0$. Since $\alpha^{-1} > \beta$, it follows from (11) that $K > \alpha^{-1}$ implies

$$x_2^1 + x_2^2 > x_1^1 + x_1^2.$$

By (10) and the functional forms of traders' utility functions, the above inequality contradicts to the allocation being Pareto optimal. This shows $K \leq \alpha^{-1}$. A similar argument shows $K \geq \beta$.

Notice also at interior bundles,

$$\bigtriangledown u^1(x^1) = \frac{\left[\alpha^{-1}\left(\frac{x_2^1}{x_1^1}\right)^3, 1\right]}{\sqrt{\alpha}\left[1+\alpha^{-1}\left(\frac{x_2^1}{x_1^1}\right)^2\right]^{\frac{3}{2}}} \quad \text{and} \quad \bigtriangledown u^2(x^2) = \frac{\left[\beta\left(\frac{x_2^2}{x_1^2}\right)^3, 1\right]}{\left[1+\beta\left(\frac{x_2^2}{x_1^2}\right)^2\right]^{\frac{3}{2}}}.$$

It follows from the preceding gradients that at allocations satisfying (11)

$$\bigtriangledown u^1(x^1)\cdot\bar{a} = \frac{\left[1+\alpha^{-1}\left(\frac{x_2^1}{x_1^1}\right)^3\right]\bar{a}_1}{\sqrt{\alpha}\left[1+\alpha^{-1}\left(\frac{x_2^1}{x_1^1}\right)^2\right]^{\frac{3}{2}}} \quad \text{and} \quad \bigtriangledown u^2(x^2)\cdot\bar{a} = \frac{\left[1+\beta\left(\frac{x_2^2}{x_1^2}\right)^3\right]\bar{a}_1}{\left[1+\beta\left(\frac{x_2^2}{x_1^2}\right)^2\right]^{\frac{3}{2}}},$$

where $\bar{a} = a^1 + a^2$ and $\bar{a}_1 = a_1^1 + a_1^2$ (the total endowment of commodity 1). Hence, at interior Pareto optimal allocations with traders' common MRS equal to K, we have

$$\bigtriangledown u^1(x^1)\cdot\bar{a} = \frac{[1+K]\bar{a}_1}{\sqrt{\alpha}\left[1+\alpha^{-\frac{1}{3}}K^{\frac{2}{3}}\right]^{\frac{3}{2}}} \quad \text{and} \quad \bigtriangledown u^2(x^2)\cdot\bar{a} = \frac{[1+K]\bar{a}_1}{\left[1+\beta^{\frac{1}{3}}K^{\frac{2}{3}}\right]^{\frac{3}{2}}}.$$

Simple calculation shows that $\bigtriangledown u^1(x^1)\cdot\bar{a}$ is decreasing while $\bigtriangledown u^2(x^2)\cdot\bar{a}$ is increasing in $K \in [\beta, \alpha^{-1}]$. Thus, as with the previous example, these monotonicity properties together with the differentiability of traders' utility functions, the interiority of CE bundles, and the price normalization in Example 1 imply that A5 holds. □

3.2 Two classes of economies satisfying A1–A5

The quasi-linear economy in Example 1 and CES economy in Example 2 belong to two classes of economies. One class satisfies A1–A4 and A6: *u^i is homogenous of* *degree 1 for all* i. The other class satisfies A1, A3, A4, A7: *for each trader* i, $u^i(x^i) = v^i(x^i_{-h(i)}) + \theta^i x^i_{h(i)}$ *where* $v^i : \Re^{n-1}_+ \longrightarrow \Re$ *is continuous concave,* $\theta^i > 0$ *is a constant, and* $1 \le h(i) \le m$ *is a commodity*, and A8: *for each trader* i, *the non-negativity constraint on commodity* $h(i)$ *is not binding in CE.*[8] We now show that both classes satisfy all A1–A5 under some proper choices of price normalization.

Theorem 4 *Assumption* A5 *is implied by* A2, A3, *and* A6 *with price normalization*

$$p \cdot \sum_{i \in N} a^i \equiv constant. \tag{12}$$

Assumptions A2 *and* A5 *are implied by* A4, A7, *and* A8 *with price normalization*

$$\sum_{i \in N} p_{h(i)} \equiv constant. \tag{13}$$

Before proving Theorem 4, we wish to make the following observations. First, the usual case of an economy with quasi-linear utility functions corresponds to $h(i) = h(j)$ for all $i, j \in N$. In that case, the common commodity with respect to which traders' utility functions are quasi-linear is regarded as the numeraire. Second, assumption A4 is implied by A7 if for each commodity h there is a trader i such that $h(i) = h$. Third, in the economy in Example 1, $m = n = 2$ and $h(1) = 1$ and $h(2) = 2$. The price normalization we applied in that example is different from (13). Thus, different credit monies may work for the selection of a unique CE.

Proof Suppose that A2, A3, and A6 hold. Then, bundle x^i is optimal for trader i at price-income pair (p, w^i) if and only if tx^i is optimal at (p, tw^i) for all $t > 0$ due to A6. Thus, trader i's indirect utility function satisfies $V^i(p, w^i) = \tilde{V}^i(p)w^i$ for some function $\tilde{V}^i$ of the price vector.[9] It follows that

$$\frac{\partial V^i(p, w^i)}{\partial w^i} = \tilde{V}^i(p). \tag{14}$$

Let $(\bar{x}, \bar{p}, \bar{\lambda})$ be a competitive triplet. Then, by A2, $\bar{\lambda}^i > 0$. Since $\bar{\lambda}^i$ equals trader i's marginal utility of income at $\bar{p}$ and $\bar{w}^i = \bar{p} \cdot a^i$, (14) implies $\bar{\lambda}^i = \tilde{V}^i(\bar{p})$. Hence,

$$\sum_{i \in N} \frac{V^i(\bar{p}, \bar{w}^i)}{\bar{\lambda}^i} = \bar{p} \cdot \sum_{i \in N} a^i. \tag{15}$$

Next, from the saddle-point characterization (3) in Theorem 1 it follows easily that the left-hand-side of (15) is the maximum total weighted welfare, with $1/\bar{\lambda}^i$ as the

[8] Given $1 \le h \le m$ and given a bundle x^i, x^i_{-h} is the sub-bundle obtained from x^i by excluding quantity x^i_h.

[9] See, for example, Exercise 3.D.3 in Mas-Colell et al. (1995). Notice that differentiability of u^i is not required to have this functional form for trader i's indirect utility function.

welfare weight for trader i. By A3, $V^i(\bar{p}, \bar{w}^i) > 0$ for all i. Consequently, the preceding welfare property together with (15) and price normalization (12) implies that A5 holds.

Suppose now A4, A7, and A8 hold. The continuity and concavity of v^i together with the linearity of u^i in the quantity of the $h(i)$-th commodity imply that A2 holds. Furthermore, trader i's indirect utility function satisfies[10]

$$V^i(p, w^i) = \tilde{V}^i(p) + \frac{\theta^i w^i}{p_{h(i)}}$$

for all price-income pairs (p, w^i) such that $p_{h(i)} > 0$ and the non-negativity constraint on commodity $h(i)$ is not binding for trader i. Given any competitive triplet $(\bar{x}, \bar{p}, \bar{\lambda})$, A4 and A7 imply $\bar{p} >> 0$ and $\bar{\lambda} >> 0$. Hence, by A8 and the functional form of i's indirect utility function,

$$\bar{\lambda}^i = \frac{\partial V^i(\bar{p}, \bar{w}^i)}{\partial w^i} = \frac{\theta^i}{\bar{p}_{h(i)}}$$

where $\bar{w}^i = \bar{p} \cdot a^i$. It follows that

$$\sum_{i \in N} \frac{\theta^i}{\bar{\lambda}^i} = \sum_{i \in N} \bar{p}_{h(i)}. \tag{16}$$

Assumption A5 follows easily from price normalization (13) and (16). □

Putting Theorem 3 and Theorem 4 together, a uniform credit money exists such that each CE of $\mathcal{E}$ is a unique selection of the credit mechanism with default penalties equal to competitive multipliers under either A1–A4 and A6 or A1, A3, A4, A7, and A8. We end this section with an example to demonstrate the total cash flows of CEs.

Example 3 Consider the 2-person economy of Shapley and Shubik (1977). If we normalize the prices by the condition $p \cdot (a^1 + a^2) = 1{,}000$, so that the economy's total wealth is always 1,000, then from Table 1 in Shapley and Shubik (1977, p. 875) it follows that the competitive price vectors, competitive multiplier vectors, total cash flows are as in the following table, all with a two-digit decimal rounding off:

	$\bar{x}$	$\bar{p}$	$\bar{\lambda}$	TW	TCF
CE1	((32.26, 39.26), (7.74, 10.74))	(3.4, 17.27)	(0.29, 0.06)	1000	704.34
CE2	((13.17, 20.18), (26.83, 29.82))	(12.9, 9.68)	(0.08, 0.1)	1000	541.45
CE3	((3.22, 10.23), (36.78, 39.77))	(18.5, 5.19)	(0.05, 0.19)	1000	733.52

In this table, TW stands for the total wealth of the economy and TCF for the total cash flow. The cash flow required from trader i at prices p_1, p_2 and bundle x^i is given by $p_1 \max\{0, x_1^i - a_1^i) + p_2 \max\{0, x_2^i - a_2^i\}$ and the total cash flow required in a CE is the sum of the cash flows required from both traders at their respective equilibrium

[10] See, for example, Exercise 3.D.4. in Mas-Colell et al. (1995).

bundles and the equilibrium prices. Notice that the middle CE (CE2) is the only CE with minimum cash flow. To uniquely select it, we can set the per-unit default penalties equal to the traders' competitive multipliers 0.08 and 0.1. Alternatively, we can also choose a non-discriminatory per-unit default penalty equal to 0.1. In fact, it follows from the proof of Theorem 2 that any non-discriminatory per-unit default penalties between 0.1 and the next highest maximum competitive multiplier which is equal to 0.19 would work. □

4 Selection with production

An *economy* with l goods, n traders, and household production is an array $\mathcal{E} = \{(X^i, u^i, a^i, Y^i)\}_{i \in N}$, where N is the trader set, $X^i \subseteq \Re^m$ is the consumption set of trader i, u^i is i's utility function, a^i is his endowment bundle, and $Y^i \subseteq \Re^m$ is his household production possibility set.[11] An element y^i in Y^i represents a production plan that i can carry out. As usual, inputs into production appear as negative components of y^i and outputs as positive components. For all $i \in N$, X^i and Y^i are closed and convex.

4.1 Competitive allocations

A production plan changes a trader's initial endowment before trading. Hence, the selection of a production plan by an individual is guided by utility maximization instead of profit maximization. However, with price-taking traders, utility maximization implies profit maximization.

Definition 2 A CE for economy $\mathcal{E} = \{(X^i, u^i, a^i, Y^i)\}_{i \in N}$ is a point

$$((x^{*i}, y^{*i})_{i \in N}, p^*) \in (\Pi_{i \in N}(X^i \times Y^i)) \times \Re^m_+$$

such that

(i) For $i \in N$, $p^* \cdot x^{*i} = p^* \cdot a^i + p^* \cdot y^{*i}$ and $u^i(x^i) > u^i(x^{*i})$ implies $p^* \cdot x^i > p^* \cdot a^i + p^* \cdot y^i$ for all $y^i \in Y^i$;
(ii) $\sum_{i \in N} x^{*i} = \sum_{i \in N} a^i + \sum_{i \in N} y^{*i}$.

4.2 Arrow–Debreu economy

In the Arrow–Debreu model of an economy with $m < \infty$ goods, there are a finite set N of traders with trader $i \in N$ characterized by a triplet (X^i, u^i, a^i) and a finite set J of producers with producer $j \in J$ characterized by a production possibility set Y^j. In addition, for all $i \in N$ and $j \in J$, trader i is endowed with a relative share

[11] This model of an economy was considered in Hurwicz (1960), Rader (1964), Shapley (1973), Billera (1974), among others. Qin (1993) applies this model to study competitive outcomes in the cores of NTU market games.

θ_{ij} of firm j's profit such that $0 \leq \theta_{ij} \leq 1$ and $\sum_{i' \in N} \theta_{i'j} = 1$ (see Arrow and Debreu 1954; Debreu 1959). Symbolically, an *Arrow–Debreu economy* is an array $\mathcal{E} = \{\{(X^i, u^i, a^i)\}_{i \in N}, \{Y^j\}_{j \in J}, \{\theta_{ij}\}_{i \in N, j \in J}\}$. For all $i \in N$ and all $j \in J$, X^i and Y^j are closed and convex.

Definition 3 A CE for $\mathcal{E} = \{\{(X^i, u^i, a^i)\}_{i \in N}, \{Y^j\}_{j \in J}, \{\theta_{ij}\}_{i \in N, j \in J}\}$ is a point

$$((x^{*i})_{i \in N}, (y^{*j})_{j \in J}, p^*) \in (\Pi_{i \in N} X^i) \times (\Pi_{j \in J} Y^j) \times \Re^m_+$$

such that

(i′a) For $i \in N$, $p^* \cdot x^{*i} = p^* \cdot a^i + \sum_{j \in J} \theta_{ij} p^* \cdot y^{*j}$ and $u^i(x^i) > u^i(x^{*i})$ implies $p^* \cdot x^i > p^* \cdot a^i + \sum_{j \in J} \theta_{ij} p^* \cdot y^{*j}$;

(i′b) For $j \in J$, $p^* \cdot y^{*j} \geq p^* \cdot y^j$, for $y^j \in Y^j$;

(ii′) $\sum_{i \in N} x^{*i} = \sum_{i \in N} a^i + \sum_{j \in J} y^{*j}$.

The relative shares θ_{ij} may be interpreted as representing private proprietorships of the production possibilities and facilities. With this interpretation, we can think of trader i as owning the technology set $\theta_{ij} Y^j$ at his disposal in firm j. Consequently, we may think of trader i as owning the following production possibility set in the Arrow–Debreu economy:

$$\tilde{Y}^i = \sum_{j \in J} \theta_{ij} Y^j. \tag{17}$$

We denote elements in $\tilde{Y}^i$ by $\tilde{y}^i = \sum_{j \in J} \theta_{ij} y^{ij}$ for some $y^{ij} \in Y^j$, $j \in J$. The reader is referred to Rader (1964, pp. 160–163) and Nikaido (1968, p. 285) for a justification of this understanding of the traders' ownership shares. With Eq. (17), the Arrow–Debreu economy $\mathcal{E}$ is converted into an economy with household production which we denote by $\tilde{\mathcal{E}} = \{(X^i, u^i, a^i, \tilde{Y}^i)\}_{i \in N}$.

Rader shows that an Arrow–Debreu economy $\mathcal{E}$ with convex production possibility sets is equivalent to economy $\tilde{\mathcal{E}}$, in the sense that the competitive allocations are the same across the two economies (see Rader 1964, pp. 160–163):

Theorem 5 *Let* $\mathcal{E} = \{\{(X^i, u^i, a^i)\}_{i \in N}, \{Y^j\}_{j \in J}, \{\theta_{ij}\}_{i \in N, j \in J}\}$ *be an Arrow–Debreu economy and let* $\tilde{\mathcal{E}} = \{(X^i, u^i, a^i, \tilde{Y}^i)\}_{i \in N}$ *with* $\tilde{Y}^i$ *given in* (17). *Then, for any CE* $((x^{*i})_{i \in N}, (y^{*j})_{j \in J}, p^*)$ *of* $\mathcal{E}$, *there are production plans* $\tilde{y}^{*i} \in \tilde{Y}^i$, $i \in N$, *such that* $((x^{*i}, \tilde{y}^{*i})_{i \in N}, p^*)$ *is a CE of* $\tilde{\mathcal{E}}$. *Conversely, for any CE* $((x^{*i}, \tilde{y}^{*i})_{i \in N}, p^*)$ *of* $\tilde{\mathcal{E}}$, *there are production plans* y^{*j}, $j \in J$, *such that* $((x^{*i})_{i \in N}, (y^{*j})_{j \in J}, p^*)$ *is a CE of* $\mathcal{E}$.

4.3 *Rader's equivalence principle*

Rader (1964) considers how to transform an economy with household production into an exchange economy using *induced preferences*. He shows that all the properties pertaining to the traders' characteristics in a production economy go over to the induced exchange economy. Furthermore, the CEs of the original economy and those of the induced exchange economy are equivalent (see Rader 1964, pp. 155–157). It follows that our credit mechanism and results in the previous sections can be extended to a production economy via its induced exchange economy.

5 Conclusion

In this paper we investigated the possibilities to enlarge the general-equilibrium structure by allowing default subject to appropriate default penalties. The enlargement of the general equilibrium structure results in a construction of a simple mechanism for a credit using society to select a unique CE under certain conditions.[12]

The implementation of the mechanism involves a bank providing a credit money that traders use as a direct and anonymous means of payment. The traders exchange personal IOUs for the credit money without limit. But, they settle up all outstanding credits with the bank at the end of the market. Ending as a net debtor is penalized while ending as a net creditor is worthless.

We characterized the CEs that will be selected by the mechanism. They are those CEs with traders' marginal utilities of income dominated by the corresponding per-unit default penalties. Applying this result, we showed that for two classes of economies, price normalization (credit money) exists with which each CE can be uniquely selected with default penalties equal to the associated competitive multipliers.

This essay is the first step in a projected exploration of the relationship between a monetary economy and transferable and non-transferable utility games. We conjecture that the monetary economy can be considered as a quasi-transferable utility game where, as in this essay when the penalties are "just right", budgets are balanced and the "money" is strictly credit as it nets precisely to zero. This relates to the λ-transfer in non-transferable utility games. When the penalties are set otherwise someone will opt for default. In a subsequent essay we intend to explore the introduction of an "outside money" that does not necessarily net to zero. If this money were quasi-linear, the presence of it in an appropriate amount appropriately distributed would permit the conventional budget conditions to be abrogated without bankruptcy, producing a quasi-transferable utility game where the money serves as providing sufficient liquidity to avoid bankruptcy.

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[12] The explicit introduction of default penalties allows the traders the option to break the usual budget constraints whenever desirable. This option turns out to be essential for eliminating all but one CEs as DCE.

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