ORIGIN AND DEVELOPMENT OF THE CONCEPT OF WAVE*

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ABSTRACT. The modern notion of wave is the conclusion of a long intellectual process performed during two centuries. The first concept of wave was suggested by the configurations of a vibrating string, but later it was discovered that these configurations may admit singularities, opening, in this wave, the route to the theory of schock waves. The most recent results consist in the characterization of schock fronts in materials with memory.

SOMMARIO. Si espone la storia della evoluzione del concetto di onda dalle prime richerche sulle vibrazioni dei fili alla scoperta dell'esistenza di onde di urto, dando anche un cenno sulle richerche più recenti riguardo alle onde di discontinuità nei solidi.

KEY WORDS. Wave, history of physics.

It is not an easy task to discover the date of birth of the concept of 'wave', and to follow its development up to the present day. In fact, for more than two thousand years the ancient investigators were only concerned with the vibrations of strings on account of their interest in stringed musical instruments. The notion of wave, as we are accustomed to it, was developed from those investigations only by means of a long and laborious process of abstraction.

The first research in this field goes back to Greek antiquity: the names of the greek scientists, Euclid, Theon of Smyrna, Filon of Bysantium, Eron of Alexandrya can be quoted in connection with the ancient researches on the string's vibrations. In the Middle Ages and in years less remote from our days than Greek antiquity, the names of Jordan of Nemore, Leonardo da Vinci and Galileo must be taken into account.

But if we want to follow the development of the studies in this field from a rational point of view rather than from an empirical one it is convenient, I think, to start from that 18th century which saw the birth of the modern mathematics owing to the work of such great scientists as John and Daniel Bernoulli, Euler and d'Alembert. As far as this development is concerned, it must be borne in mind that after the publication of Newton's *Principia* (1687) more than sixty years elapsed before the general equations of motion of a system of particles were known. In fact, they were established by Euler in 1750.

This is the reason why, in the first half of the 18th Century, scientists dealt with particular problems on the basis of *ad hoc* assumptions, without reference to general and well established principles.

Among the problems that scientists dealt with in that period I shall select two problems which, I think, are of interest for our purposes. The first one is the problem of the law of the speed of sound in the atmosphere. Newton suggested, giving some justification for his conjecture, that the law the square of the speed of sound in an ideal fluid at rest is given by p/ρ, p being the pressure and ρ the density; that is to say, assuming Boyle's law, C^2 = const. Some years later, Euler obtained the same law on the basis of his own general equations of motion of a fluid.

Many years later, Laplace suggested multiplying p/ρ by the ratio C_p/C_v between the specific heat at constant pressure and the specific heat at constant volume. It must be borne in mind that at the time of Newton, Euler and Laplace, thermodynamics was only taking its first steps, so that thermodynamical considerations were initially excluded. (Note that Laplace's suggestion follows Euler's theory by almost fifty years, when thermodynamics had begun to be taken into consideration).

Poisson improved the theoretical deduction of the law relating sound speed, pressure, and density, but only in the year 1885 was Hugoniot able to give a single and rigorous proof of that law. In fact he could prove that, for an ideal barotropic fluid, C^2 is given by $(\mathrm{d}p/\mathrm{d}\rho)_S$, a law today universally accepted.

Much more important for our aim is the long and harsh controversy about the small transverse vibration of a string with fixed ends. The first theoretical investigations go back to Brook Taylor (1713) who determined the frequency of the principal mode as a function of the string's length. He could not appeal to the general equations of motion of a string, since at that time they had not yet been discovered. John Bernoulli succeeded (1727) in determining the principal frequencies of a massless string, loaded by n equal masses equally spaced along the string, a model which had been already proposed (but not exploited) by Huygens. Some years later (1733) Daniel Bernoulli (to whom we owe the principle of superposition) discovered simple modes of vibrations and the principal frequencies.

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Though Euler succeeded in writing the general equation of motion of a string, it was John Le Rond d'Alembert who first established (1747–1749) the general equation of small transverse vibrations of a stretched homogeneous string, in two memoirs on this subject published by the Berlin Academy of Sciences. Thus, after a long period of laborious research and investigation, the one-dimensional wave equation at last made his appearance in the domain mathematics, thus giving birth to the rigorous treatment of the motion of a wave.

Euler returned to this subject in the year 1750 (one year after d'Alembert's last paper) in the course of a violent controversy, which, during the last half of the century, involved D. Bernoulli, as well as the young Lagrange.

Indeed, d'Alembert had proved that each solution of the one-dimensional wave equation is the sum of a progressive wave and of a regressive one. The conditions that the ends of the string must be fixed reduce the two independent functions to only one arbitrary function. The initial condition had still to be satisfied, and this was the subject of the controversy, which was made worse because D. Bernoulli claimed to have established the equation of the motion of a string before Euler. Indeed, it seemed impossible with such regular functions, as they were in d'Alembert's opinion, to fit any arbitrary initial shape the string could assume, which must be given 'de nôtre bon plaisir', as Euler wrote. The difficulty essentially originated in the different concept of function that d'Alembert and Euler had in mind.

In Euler's first great treatise on analysis, *Introductio in analysin infinitorum*, the function is defined in this way: 'A function of variable quantities is an analytic expression made in any arbitrary way from these quantities and of numbers or constant quantities". In short, a function has a unique analytic form over the whole domain of definition. Using this definition of a function, it will be clearly impossible that the function which describes the motion of the string can fit, in the initial time, an arbitrarily given shape of the string, perhaps with some discontinuities of the tangent line, as in the case of a pinched string. This was strongly remarked by d'Alembert, who rejected Euler's proposal of resorting to both regular and mechanical (that is non-regular) curves to fit the initial, arbitrarily given shape of the string. It was exactly this problem of the string's vibrations that led Euler to modify his definition of function. In his treatise *Institutiones calculi differentialis* (1755) he writes: "If some quantities depend on some others so that, from their variation, the first ones also change, it is customary to name them functions of the former". This definition made it possible to take account of irregular curves.

Daniel Bernoulli intervened in the controversy, with the remark that, owing to the linearity of the wave equation, the sum of any number of sinusoidal functions was a solution too, so that it could be possible that, with a suitable choice of the coefficients of any linear combination of elementary solutions, one would be able to satisfy any initial condition whatsoever.

However, D. Bernoulli's remark was simultaneously criticized by Euler and d'Alembert: how could it be possible, using such regular functions as the trigonometric ones, to represent such an irregular curve as the initial shape of a plucked string? We are now aware of the difficulty of the problem, which could be solved only more than a century later.

The young Lagrange contributed to the discussion with a long paper in which, after having strongly criticized the methods adopted by d'Alembert, Euler and Bernoulli, he claimed to have found a general and completely rigorous theory of the motion of a string. Assuming the model already considered by John Bernoulli, of a massless string loaded by n equal masses, with a passage to the limit, Lagrange obtained what he claimed to be the general equations of motion of a homogeneous string. While the equations appeared to be correct, the method was criticized by Euler, Bernoulli and d'Alembert, because some points in the passage to the limit appeared to be insufficiently justified. At that time, the concept of the limit was not well established.

I have dwelt on this controversy for such a long time, not only because it involved the greatest mathematicians of the century, and not only because the occurrence of the polemic displays the difficulties the mathematicians encountered in the course of their investigations, but mostly because it marks the birth of the theoretical concept of a wave. From now on, every solution of d'Alembert's equation is a wave. Therefore, studying waves is just the same as studying the properties of the solutions of d'Alembert's equation.

By that time, the interest in wave phenomena was continuously increasing, which is also testified to by the numerous competitions announced by the French Academy of Sciences (or Institut Nationale des Arts et Sciences, as it was the name of the Académie from 1793 to 1815 when, on account of the Restoration, the Institut again took on its old denomination). In 1815 the Institut announced a competition on the theory of waves on the surface of a heavy infinitely deep liquid. The competition was won by a young engineer endowed with a strong mathematical aptitude: Augustin Louis Cauchy.

Cauchy's memoir was published only in 1827 much enriched in comparison to the original version. In the meantime, Poisson, who as a member of the board of the examiners was prevented from taking part in the competition, published, as was his practice, a long paper on the subject in the memoirs of the Academy (1818), preceded by a short summary in 1815. Afterwards, in 1819, Poisson published a long memoir on the propagation of three-dimensional waves in a fluid followed by two papers by Cauchy on the same topic, which appeared in the years 1821 and 1822.

Owing to the great interest in the research on wave propagation, the Academy announced a competition on the mathematical theory of the vibration of an elastic shell, a natural generalization of the studies on the vibrating string. In that time great interest had been aroused among the French scientists by the research of Ernst Florens Friedrich Chladni on the vibrations of the bells, presented in a celebrated book published in Leipzig in the year 1802.

The competition was won by the French scientist Sophie Germain, whose contributions to differential geometry and to the theory of numbers are well known. Her memoir appeared in the year 1821.

Sophie Germain was self-educated since, at her times, women were forbidden to attend to the lectures delivered at the *École Polytechnique*, but in some manner she succeeded in obtaining the texts of Lagrange's lectures. More than this, she dared to address some remarks and suggestions to him but, fearing that she would not be taken seriously, she signed her letters with a male name, such as Mr. Leblanc. Lagrange much appreciated the remarks of his correspondent, even when he discovered that he was a woman. In fact, many of the details about the life and work of Sophie Germain are known to us from Lagrange's writings.

In her memoir Sophie Germain discovered, among other things, the existence of nodal lines. The method was founded on particular assumptions, such as the dependence of the stress on the curvature of the surface. It must be borne in mind that no general theory for elastic materials was then known.

In effect, the general theory for an elastic isotropic material was to be established by Navier in 1827; while the theory for anisotropic materials was to be proposed by Cauchy in 1828. Since these years, the mathematical theory of elasticity developed rapidly, mainly because of the work of Poisson and Cauchy. D'Alembert's equation was written down for a three-dimensional fluid by Poisson in 1819. In 1829, Poisson discovered that two kinds of waves can propagate in elastic isotropic solids. This result may be connected with the so called multi-constants or rari-constants controversy; that is, the question whether the elasticity of an isotropic solid must depend on two constants, as asserted by Navier, or on only one, as asserted by Cauchy.

In 1821 one more reason for polemic was added to the subject, when Fresnel announced that the observed properties of the interference of polarized light can be explained only with the existence of transverse waves. Accustomed as they were to the propagation of waves in fluids, it seemed hardly understandable that transverse waves could exist. Cauchy devoted himself to this problem (*Exercises de Mathématique*), and also George Green achieved an accurate study of the propagation of waves in an isotropic and in an anisotropic elastic solid.

At that time, only the solutions to d'Alembert's equations were considered to be and were called waves. In this order of ideas, Stokes proved that the two kinds of wave discovered by Poisson were respectively dilatational and distortional.

Once again it occurred to Stokes to discover a new kind of wave. In fact, a difficulty noticed by Challis in the problem of the theoretical determination of the speed of sound gave Stokes the opportunity to make some remarks on the propagation of one-dimensional waves in a brief note published in 1848 in the *Philosophical Magazine*. Resorting to a function found by Poisson as a solution of an "accurate equation of motion", Stokes proved that the slope of the curve of the velocity as a function of the abscissa at different instants, can go to infinity in a finite time at some particular points. This occurrence led Stokes to suggest that a discontinuity in the velocity could appear in a finite time, even if the initial conditions are completely regular. In modern terms, Stokes had discovered the possibility of shock waves. However, some years later, Stoke's suggestion was strongly criticized, first by Lord Kelvin, and later on by Lord Rayleigh. This latter, in a private letter, pointed out that the discontinuity conditions across the wave-front for the mass and the linear momentum would be in conflict with the principle of energy conservation. It must be noticed that at that time no clear ideas on the influence of thermodynamics on wave propagation were available. Lord Rayleigh assumed that the mechanical energy was conserved across the wave, which is wrong.

However, it must be recalled that it would be necessary to wait until 1905 (Zemplen) for a remark about the increase of entropy across a shock wave. In any case, Stokes was convinced by the arguments of his critics, so he added a brief section to the reprint of his paper in 1883 in which he explained why his own suggestion about the formation of a shock wave could not be accepted. In spite of Lord Rayleigh's opinion, the existence of a new kind of waves, the so-called discontinuity waves, had been discovered, which gave rise to a new, very important field of research.

At last we have reached our own age: the concept of a wave as a solution of the d'Alembert equation is quite clear, and that a wave must propagate is firmly established. By contrast, the new concept of a discontinuity wave, especially in the case of a shock wave, is not yet completely clear. A great deal of research was necessary to cast light on the matter: among the numerous papers on the subject, a great memoir by Riemann ought to be quoted as a fundamental paper in the field. This was published in 1860 and is still admirable for its physical meaning, clarity, and brilliance of exposition. Riemann studies the one-dimensional motion of a barotropic fluid. The problem is nonlinear, but it is rendered linear with the introduction of the variables r and s which today are named 'Riemann's variables'. But the most astonishing result is that a shock wave can develop from very regular initial conditions in a finite time, as Stokes conjectured by intuition. More than this, Riemann had the idea that, across a discontinuity wave, some compatibility conditions must be satisfied, and he wrote down these conditions for the mass and the linear momentum (as Stokes had already done before), but Riemann interpreted them as necessary conditions for the existence of shock waves. Particularly noticeable are the bounds for the displacement speed of the wave as functions of the speed of the fluid, which recalls the well known property of the fluid's speed behind and in front of the wave.

Riemann's paper was followed by two great memoirs by Christoffel, published in the *Annali di Matematica Pura e Applicata* in the year 1877. In the first paper, Christoffel generalized Riemann's results to the case of a three-dimensional motion of an ideal fluid, while in the second paper he investigated the propagation of a shock wave in a linearly elastic solid. Here, too, thermodynamic effects are not taken into account and, owing to the difficulties of the problem, only weak waves are studied. Moreover, the only case investigated is that in which straight lines orthogonal to the successive configurations of the wave surface are constant. In spite of these restrictions, Christoffel's papers are very important as they gave the first rigorous treatment of shock propagation in a solid.

Christoffel introduced for the first time the notion of displacement speed in the three-dimensional case, and realized the necessity of kinematical conditions of compatibility across the wave (*phoronomische Unstetigkeitbedingungen*). A curiosity: it was Christoffel who introduced the symbol, now universally adopted, to indicate the discontinuity of a quantity across the wave, the square bracket. He also wrote down explicitly the compatibility condition for mass and linear momentum for a three-dimensional continuum.

The notion of a discontinuity wave of first order is now clearly rendered precise. Foreseen by Stokes, rigorously introduced by Riemann in the case of one-dimensional motion of a fluid, it was generalized by Christoffel to three-dimensional motions, and, what is much more important, to the motion of an elastic solid.

In this domain, the last decades of the century were dominated by the work of Pierre Henri Hugoniot. His contributions to gas dynamics are too well known for us to discuss them here. In regard to our subject, it is universally acknowledged that Hugoniot explicitly introduced for the first time the notion of compatibility conditions, though restricted to kinematical conditions. He also solved the problem of determining what the evolution of a wave surface would be if it did not satisfy the compatibility conditions. His merit is having introduced, in a quite general manner, the notion of propagation speed as an alternative to that of displacement speed, and, as a consequence, that of local speed of propagation. However, it is certain that the most important of Hugoniot's contributions to the mathematical theory of discontinuity waves is the introduction of the notion of discontinuities of order greater than one. I have already said that Hugoniot determined the law for the speed of sound in an ideal fluid; he succeeded in this task just making use of the notion of second order discontinuity waves.

During his brief life, Hugoniot produced essential contributions to the theory of discontinuity waves. Without his work, it is certain that the theory would not have developed to so great an extent in just a few years. He brought the theory from a semi-empirical state to a rigorous system, firmly rooted on mathematical grounds. It is certain that, without Hugoniot's discoveries, an exhaustive book such as the celebrated treatise by Hadamard would never have been written.

Hugoniot's researches were extended by Duhem in several papers. In particular, the first investigation on the propagation of discontinuity waves in a linearly viscous fluid is due to Duhem.

The appearance of Hadamard's famous treatise (1903) had a stimulating effect on the research on discontinuity waves. Over a period of years a lot of papers were published on such subjects as balance equations for continua, thermodynamic influences on wave propagation, and so on. There followed a long period (from the twenties to the fifties of this century) during which very few papers on discontinuity waves appeared. But, from the 1950s a new period of flourishing research commenced; one which still persists. Perhaps the research which started this period is best represented by a paper by Eriksen (1953) who investigated in the spirit of Hadamard's treatise, the propagation of second-order discontinuity waves (acceleration waves) in an incompressible, isotropic, perfectly elastic solid. After the publication of Eriksen's paper, several other papers were published on the matter, giving rise to a renewed flourishing of studies in continuum mechanics. I am not able to give even a reduced list of such papers. However, I cannot shrink from citing a fundamental paper by Truesdell and those by Coleman, Noll, Gurtin and others.

A very exhaustive account of the research in this domain is contained in the volumes of the *Encyclopaedia of Physics*; and in a lot of books which have been continuously published to the present day.

Notwithstanding the great development of the theory some problems still remain unsolved; among them, the connection between discontinuities and the characteristic surfaces of a differential system. But I fear I may have taken undue advantage of your patience, and therefore I shall stop here.

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