Theory of generalized tautology in revised Kleene system

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Abstract This paper is a complement and extension of the theory of generalized tautology which was first proposed by Wang Guojun in revised Kleene system, Some interesting results are obtained: (i) accessible α^+-tautology and generalized contradiction which are dual theory to generalized tautology have been introduced; (ii) congruence partition about $\neg$ has been given in logic system $\overline{W}$, W, W_k; (iii) in logic system W_k, tautologies can be obtained by employing the upgrade algorithm at most $\left[\frac{k+1}{2}\right]$ times to an arbitrary formula; (iv) in logic system $\overline{W}(W)$, tautologies cannot be obtained by employing upgrade algorithm to non-tautologies within finitely many times; (v) the deduction rule $\left(\left[\left(\frac{1}{2}\right)^+\right]-MP\right)$ holds in logic system $\overline{W}(W)$.

Keywords: **logic system, accessible α^+-tautology, upgrade algorithm, α-contradiction, partition.**

The research and application of fuzzy theory have made much headway since 1965, the year the concept of fuzzy set was introduced by Zadeh. The fuzzy inference is the most active research topic[1]. To lay a neccessary foundation for lattice-valued logic system in semantics, Xu[2] and Qin[3] have founded the lattice implication algebra by combining lattice with implication algebra and investigated the propositional logic that uses the lattice implication algebra as its truth value domain. To establish the strict logic foundation for fuzzy modus ponens and fuzzy modus tollens, Wang has presented a kind of formal deductive system $\mathscr{L}^*$ for fuzzy propositional calculus[4]. And to establish the theory of semantics relevant to the system $\mathscr{L}^*$, Wang has put forth the revised Kleene logic system $\overline{W}$, W, W_k by introducing the implication operator R_0[3—7] and introduced the theory of Σ-(α-tautology). This theory offers a powerful tool for distinguishing quasi-tautologies. This paper aims to make some complement and extension to the theory of generalized tautologies in refs. [5,6] so that in semantics, we can get a deepgoing knowledge of the formula set $F(S)$ in the revised Kleene system.

1 Fundamental knowledge

Definition 1.1[5]. Let $S=\{p_1,p_2,\cdots\}$ be a countable set, let $\neg$ be a unary operation, and let $\vee\rightarrow$ be binary operations. The free algebra generated by S is denoted by $F(S)$, which has the type $(\neg,\vee,\rightarrow)$. An element of $F(S)$ is called a formula or proposition, and an element of S is called an atomic formula or atomic proposition.

Definition 1.2[5]. In $[0,1]$, we define $\alpha\vee\beta=\max\{\alpha,\beta\}$, $\neg\alpha=1-\alpha$, $\alpha\rightarrow\beta=R_0(\alpha,\beta)=\begin{cases}1, & \alpha\leqslant\beta\\ \neg\alpha\vee\beta, & \alpha>\beta\end{cases}$. Then $[0,1]$ becomes an algebra of type $(\neg,\vee,\rightarrow)$. We call it revised continuous Kleene system and denote it by $\overline{W}$. If $[0,1]\cap Q$ is in the place of $[0,1]$, we call the relative algebra revised rational Kleene system and denote it by W, where Q is representative of the rational number set.

Definition 1.3[5]. Let $n \in N$. When $k = 2n$, set $W_k = \{F = -n, \cdots, -1, 1, \cdots, n = T\}$; when $k = 2n+1$, set $W_k = \{F = -n, \cdots, -1, 0, 1, \cdots, n = T\}$. In W_k, we define $\neg \alpha = -\alpha$, $\alpha \vee \beta = \max\{\alpha, \beta\}$, $\alpha \rightarrow \beta = R_0(\alpha, \beta) = \begin{cases} T, & \alpha \leqslant \beta \\ \neg \alpha \vee \beta, & \alpha > \beta \end{cases}$. Then W_k becomes an algebra of type $(\neg, \vee, \rightarrow)$. We call it revised finite Kleene system, and denote it by W_k.

Definition 1.4[5]. Let $R \in \{\overline{W}, W, W_k\}$, $v: F(S) \rightarrow R$ be a homomorphism between algebras $F(S)$ and R. We call v an R evaluation of $F(S)$. The set consisting of them is denoted by Ω_R.

In the following we allow $\alpha = 0$ to be a complement to the concepts of generalized tautologies and introduce the concept of accessible α^+ - tautology.

Definition 1.5. Let $A \in F(S)$, $\alpha \in [0,1]$, $R \in \{\overline{W}, W\}$. If for each $v \in \Omega_R$, $v(A) \geqslant \alpha$, we call A Ω_R-(α- tautology), briefly, α-tautology. The set consisting of them is denoted by α-$T(R)$. Moreover, if there exists a $v_\alpha \in \Omega_R$ such that $v_\alpha(A) = \alpha$, we call A accessible α-tautology. The set consisting of them is denoted by $[\alpha]$-$T(R)$. Especially, 1-tautology is called tautology, and the set consisting of them is denoted by $T(R)$.

Definition 1.6. Let $A \in F(S)$, $\alpha \in [0,1)$, $R \in \{\overline{W}, W\}$. If for each $v \in \Omega_R$, $v(A) > \alpha$, we call A Ω_R-(α^+ - tautology), or briefly α^+-tautology. The set consisting of them is denoted by α^+-$T(R)$. Moreover, if for every $\varepsilon > 0$ there exists a $v_\varepsilon \in \Omega_R$ such that $\alpha < v_\varepsilon(A) < \alpha + \varepsilon$, we call A accessible α^+-tautology. The set consisting of them is denoted by $[\alpha^+]$-$T(R)$.

Lemma 1.1[5]. Let $A \in F(S)$, $\alpha \in \left(0, \frac{1}{2}\right]$. Then α-$T(\overline{W}) = \frac{1}{2}$-$T(\overline{W})$; that is, when $0 < \alpha \leqslant \frac{1}{2}$, A is α-tautology in logic system $\overline{W}$ if and only if A is $\frac{1}{2}$-tautology in logic system $\overline{W}$.

Lemma 1.2[5]. Let $A \in F(S)$, $\alpha \in \left(\frac{1}{2}, 1\right]$. Then α-$T(\overline{W}) = T(\overline{W})$. In other words, when $\frac{1}{2} < \alpha \leqslant 1$, A is α- tautology in logic system $\overline{W}$ if and only if A is tautology in logic system $\overline{W}$.

2 A partition of $F(S)$ in logic system $\overline{W}$

Theorem 2.1. Let $A \in F(S)$. Then 0^+-$T(\overline{W}) = \frac{1}{2}$-$T(\overline{W})$. That is, A is 0^+ - tautology in logic system $\overline{W}$ if and only if A is $\frac{1}{2}$-tautology in logic system $\overline{W}$.

Proof. If $A \in \frac{1}{2}$-$T(\overline{W})$, then naturally $A \in 0^+$-$T(\overline{W})$. Conversely, let $\overline{W}_3 = \left\{0, \frac{1}{2}, 1\right\}$. Obviously $\overline{W}_3$ is a subalgebra of $\overline{W}$. We define a mapping $\varphi: \overline{W} \rightarrow \overline{W}_3$, $\varphi(\alpha) = \frac{1}{2} + \frac{1}{2}\left(\operatorname{sgn}\left(\alpha - \frac{1}{2}\right) + \operatorname{sgn}\left(\frac{1}{2} - \alpha\right)\right)$. We can prove that φ is a homomorphism of type $(\neg, \vee, \rightarrow)$. In fact, φ preserves operations $\neg$ and $\vee$ is easily verified. In the following, we verify that φ also preserves the operation $\rightarrow$.

If $\alpha \leqslant \beta$, then $\varphi(\alpha) \leqslant \varphi(\beta)$; therefore, $\varphi(\alpha \rightarrow \beta) = \varphi(1) = 1 = \varphi(\alpha) \rightarrow \varphi(\beta)$; if $\alpha > \beta$, then $\varphi(\alpha) \geqslant \varphi(\beta)$, because $\varphi(\alpha \rightarrow \beta) = \varphi(\neg \alpha \vee \beta) = \varphi(\neg \alpha) \vee \varphi(\beta) = \neg \varphi(\alpha) \vee \varphi(\beta)$. Therefore, (i) when $\varphi(\alpha) > \varphi(\beta)$, $\varphi(\alpha) \rightarrow \varphi(\beta) = \neg \varphi(\alpha) \vee \varphi(\beta) = \varphi(\alpha \rightarrow \beta)$; (ii) when $\varphi(\alpha) = \varphi(\beta)$, with the definition of φ and $\alpha > \beta$, we see that $\varphi(\alpha) = \varphi(\beta) = \frac{1}{2}$ does not hold; therefore $\varphi(\alpha) = \varphi(\beta) = 0$ or $\varphi(\alpha) = \varphi(\beta) = 1$. Thus, $\varphi(\alpha \rightarrow \beta) = \neg \varphi(\alpha) \vee \varphi(\beta) = 1 = \varphi(\alpha) \rightarrow \varphi(\beta)$.

Now let $A \in 0^+\text{-}T(\overline{W})$. Then for each $v \in \Omega_{\overline{W}}, v(A) > 0$. Because $\varphi v: F(S) \to \overline{W}_3$ is a composition of homomorphisms, $\varphi v \in \Omega_{\overline{W}}$, and $\varphi v(A) > 0$; therefore, $\varphi v(a) \geqslant \frac{1}{2}$. By the constructure of φ, $v(A) \geqslant \frac{1}{2}$. Therefore $A \in \frac{1}{2}\text{-}T(\overline{W})$.

By synthesis of the above statements, we get $0^+\text{-}T(\overline{W}) = \frac{1}{2}\text{-}T(\overline{W})$.

Proposition 2.1. Let $\alpha \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$. Then $[\alpha]\text{-}T(\overline{W}) = \varnothing$.

Proof. If $\alpha \in \left(\frac{1}{2}, 1\right)$, by Definition 1.5 and Lemma 1.2 we get $[\alpha]\text{-}T(\overline{W}) \subset \alpha\text{-}T(\overline{W} = T(\overline{W})$. If there exists an $A \in [\alpha]\text{-}T(\overline{W})$, by Definition 1.5, there is a $v_\alpha \in \Omega_{\overline{W}}$ such that $v_\alpha(A) = \alpha < 1$; thus $A \,\bar{\in}\, T(\overline{W})$. This is in contradiction with $[\alpha]\text{-}T(\overline{W}) \subset T(\overline{W})$. Therefore, when $\alpha \in \left(\frac{1}{2}, 1\right)$, $[\alpha]\text{-}T(\overline{W}) = \varnothing$. If $\alpha \in \left(0, \frac{1}{2}\right)$, by combining Definition 1.5 and Lemma 1.1 in an analogous way, we can prove $[\alpha]\text{-}T(\overline{W}) = \varnothing$.

Proposition 2.2 Let $\alpha \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$. Then $[\alpha^+]\text{-}T(\overline{W}) = \varnothing$.

Proof. If $\alpha \in \left[0, \frac{1}{2}\right)$, by Definition 1.6, Lemma 1.1 and Theoem 2.1, we get $[\alpha^+]\text{-}T(\overline{W} \subset \alpha\text{-}T(\overline{W}) = \frac{1}{2}\text{-}T(\overline{W})$. If there exists an $A \in [\alpha^+]\text{-}T(\overline{W})$, letting $\varepsilon = \frac{1}{2} - \alpha$, we have $\varepsilon > 0$. By Definition 1.6, there is a $v_\varepsilon \in \Omega_{\overline{W}}$ such that $\alpha < v_\varepsilon(A) < \alpha + \varepsilon = \frac{1}{2}$; thus $A \,\bar{\in}\, \frac{1}{2}\text{-}T(\overline{W})$. This is in contradiction with $[\alpha^+]\text{-}\mathrm{T}(\overline{\mathrm{W}}) \subset \frac{1}{2}\text{-}\mathrm{T}(\overline{\mathrm{W}})$. Therefore, when $\alpha \in \left[0, \frac{1}{2}\right)$, $[\alpha^+]\text{-}T(\overline{W}) = \varnothing$.

When $\alpha \in \left(\frac{1}{2}, 1\right)$, via combination of Definition 1.6 and Lemma 1.2 in an analogous way we can prove $[\alpha^+]\text{-}T(\overline{W}) = \varnothing$.

Theorem 2.2. In logic system $\overline{W}$, $\left\{[0]\text{-}T(\overline{W}), \left[\frac{1}{2}\right]\text{-}T(\overline{W}), \left[(\frac{1}{2})^+\right]\text{-}T(\overline{W}), T(\overline{W})\right\}$ is a partition of $F(S)$.

Proof. By Definitions 1.5 and 1.6, the intersection of arbitrary 2 members in this family is empty. By Proposition 2.1 and 2.2, the union of this family is $F(S)$. In the following, we verify that each member of this family is not empty. Letting $A = p_1 \to p_1$, $B = p_1$, $C = \neg p_1 \vee p_1$, we easily verify that $A \in T(\overline{W})$, $B \in [0]\text{-}T(\overline{W})$, $C \in \left[\frac{1}{2}\right]\text{-}T(\overline{W})$. Again, let $D = (p_2 \to C) \vee p_2$. Then for each $v \in \Omega_{\overline{W}}$, $v(D) = (v(p_2) \to v(C)) \vee v(p_2)$. $D \in \left(\frac{1}{2}\right)^+\text{-}T(\overline{W})$ is easily verified. Moreover, for every $\varepsilon > 0$, take $v_\varepsilon \in \Omega_{\overline{W}}$ such that $v_\varepsilon(p_1) = \frac{1}{2}$, $v_\varepsilon(p_2) = \frac{1}{2} + \frac{1}{2}\varepsilon$; that is, v_ε is generated by mapping $v_\varepsilon': S \to \overline{W}$ satisfying conditions $v_\varepsilon'(p_1) = \frac{1}{2}$, $v'_\varepsilon(p_2) = \frac{1}{2} + \frac{1}{2}\varepsilon$. Then $v_\varepsilon(D) = \frac{1}{2} + \frac{1}{2}\varepsilon$. Thus $\frac{1}{2} < v_\varepsilon(D) < \frac{1}{2} + \varepsilon$. Therefore $D \in \left[(\frac{1}{2})^+\right]\text{-}T(\overline{W})$.

3 Upgrade algorithm in logic system $\overline{W}$

Let $A = A(p_{t_1}, \cdots, p_{t_i}) \in F(S)$, $q \in S - \{p_{t_1}, \cdots, p_{t_i}\}$. $B = (q \to A) \vee q$. Ref. [5] has introduced this structure in logic system W_k and called it upgrade algorithm from A to B. In the following, this upgrade algorithm will be generalized to the logic system $\overline{W}(W)$.

Theorem 3.1. If $A \in [0]\text{-}T(\overline{W})$, then $B \in [\frac{1}{2}]\text{-}T(\overline{W})$; if $A \in [\frac{1}{2}]\text{-}T(\overline{W})$, then $B \in [(\frac{1}{2})^+]\text{-}T(\overline{W})$.

Proof. As an example, we only prove the second conclusion.

If $A \in \left[\frac{1}{2}\right]\text{-}T(\overline{W})$, it is easily verified that $B \in \left(\frac{1}{2}\right)^+\text{-}T(\overline{W})$. Moreover, for every $\varepsilon > 0$, because $A \in \left[\frac{1}{2}\right]\text{-}T(\overline{W})$, there exists a $v_{\frac{1}{2}} \in \Omega_{\overline{W}}$ such that $v_{\frac{1}{2}}(A) = \frac{1}{2}$. We define mapping $v'_\varepsilon : S \to \overline{W}$ satisfying conditions $v'_\varepsilon \mid \{P_{t_1}, \cdots, p_{t_i}\} = v_{\frac{1}{2}} \mid \{P_{t_1}, \cdots, P_{t_i}\}$, $v'_\varepsilon(q) = \frac{1}{2} + \frac{1}{2}\varepsilon$. The evaluation generated by v_ε' is denoted by v_ε; thus $v_\varepsilon(B) = \frac{1}{2} + \frac{1}{2}\varepsilon$, and obviously $\frac{1}{2} < v_\varepsilon(B) < \frac{1}{2} + \varepsilon$. Therefore $B \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$.

Theorem 3.2. If $A \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$, then $B_n \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$. ($B_n$ is a result obtained by using upgrade algorithm to A for n times).

Proof. This is easily proved by mathematical deduction method. We only prove $B_1 \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$. Because $A \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$, it is easily verified $B_1 \in \left(\frac{1}{2}\right)^+\text{-}T(\overline{W})$. Moreover, for every $\varepsilon > 0$, from $A \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$, we know that there is a $v_A \in \Omega_{\overline{W}}$ such that $\frac{1}{2} < v_A(A) < \frac{1}{2} + \frac{1}{3}\varepsilon$. Define mapping $v'_\varepsilon : S \to \overline{W}$ satisfying $v'_\varepsilon \mid \{P_{t_1}, \cdots, P_{t_i}\} = v_A \mid \{P_{t_1}, \cdots, P_{t_i}\}$, $v'_\varepsilon(q) = \frac{1}{2} + \frac{2}{3}\varepsilon$. Then $v_\varepsilon(A) = v_A(A), v_\varepsilon(q) = v'_\varepsilon(q) = \frac{1}{2} + \frac{2}{3}\varepsilon$, where v_ε is the evaluation generated by v'_ε; thus $v_\varepsilon(B_1) = \frac{1}{2} + \frac{2}{3}\varepsilon$, so $\frac{1}{2} < v_\varepsilon(B_1) < \frac{1}{2} + \varepsilon$. Therefore, $B_1 \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$.

Corollary 3.1. Let $A \bar{\in} T(\overline{W})$. B_n is a result obtained by using upgrade algorithm to A for n times. Then $A \bar{\in} T(\overline{W})$. In other words, in logic system $\overline{W}$, tautologies cannot be gained by using upgrade algorithm to non-tautology within finitely many times.

Proof. It is trivial by Theorems 2.2, 3.1 and 3.2.

4 Congruence partition about $\neg$ of $F(S)$ in logic system $\overline{W}$

Definition 4.1. Let $\alpha \in [0,1]$, $R \in \{\overline{W}, W\}$, $A \in [0]\text{-}T(R)$. If for each $v \in \Omega_R$, $v(A) \leqslant \alpha$, we call A Ω_R-(α- contradiction) or briefly, α-contradiction. The set consisting of them is denoted by $\alpha\text{-}C(R)$. Moreover, if there exists a $v_\alpha \in \Omega_R$ such that $v_\alpha(A) = \alpha$, we call A accessible α- contradiction. The set consisting of them is denoted by $[\alpha]\text{-}C(R)$. Especially, 0-contradiction is called contradiction, and the set consisting of them is denoted by C(R).

Definition 4.2. Let $\alpha \in (0,1]$, $R \in \{\overline{W}, W\}$, $A \in T(R)$. If for each $v \in \Omega_R$, $v(A) < \alpha$, we call A Ω_R-(α^- - contradiction), briefly, α^--contradiction, and the set consisting of them is denoted by $\alpha^-\text{-}C(R)$. Moreover, if for every $\varepsilon > 0$, there exists a $v_\varepsilon \in \Omega_R$ such that $\alpha - \varepsilon < v_\varepsilon(A) < \alpha$, we call A accessible α^--contradiction. The set consisting of them is denoted by $[\alpha^-]\text{-}C(R)$.

Proposition 4.1. Let $\alpha \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$. Then $[\alpha] - C(\overline{W}) = \varnothing$.

Proof. Let $\alpha \in \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$. If there exists $A \in [\alpha]\text{-}C(\overline{W})$, then for each $v \in \Omega_{\overline{W}}$, $v(A) \leqslant \alpha$, and there exists a $v_\alpha \in \Omega_{\overline{W}}$ such that $v_\alpha(A) = \alpha$; thus for each $v \in \Omega_{\overline{W}}, v(\neg A) \geqslant 1 - \alpha$ and there exists $v_\alpha \in \Omega_{\overline{W}}$ such that $v_\alpha(\neg A) = 1 - \alpha$. Therefore $\neg A \in [1-\alpha]\text{-}T(\overline{W})$ and $1 - \alpha \in \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$. This is in contradiction with Proposition 2.1. Therefore, when $\alpha \in \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$, $[\alpha]\text{-}C(\overline{W}) = \varnothing$.

Proposition 4.2. Let $\alpha \in \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right]$. Then $[\alpha^-]\text{-}C(\overline{W}) = \varnothing$.

Proof. Let $\alpha \in \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right]$. Then $1 - \alpha \in \left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$. If there exists $A \in [\alpha^-]\text{-}C(\overline{W})$, then for each $v \in \Omega_{\overline{W}}$, $v(A) < \alpha$; thus for each $v \in \Omega_{\overline{W}}$, $v(\neg A) > 1 - \alpha$. That is, $A \in (1-\alpha)^+\text{-}T(\overline{W})$. Moreover, for every $\varepsilon > 0$, there exists $v_\varepsilon \in \Omega_{\overline{W}}$ such that $\alpha - \varepsilon < v_\varepsilon(A) < \alpha$. That is, $1 - \alpha < v(\neg A) < (1-\alpha) + \varepsilon$. Therefore, $\neg A \in [(1-\alpha)^+]\text{-}T(\overline{W})$ and $1 - \alpha \in \left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$. This is in contradiction with Proposition 2.2. Therefore, when $\alpha \in \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right]$, $[\alpha^-]\text{-}C(\overline{W}) = \varnothing$.

Proposition 4.3. Let $A \in F(S)$, if for any $v \in \Omega_{\overline{W}}$, $v(A) > 0$, there must exist a $v_1 \in \Omega_{\overline{W}}$ such that $v_1(A) = 1$.

Proof. Let $A = A(P_{t_1}, \cdots, P_{t_i})$, evaluation $v_1 : F(S) \to \overline{W}$ be generated by mapping $v_1' : S \to \overline{W}$ satisfying condition $v_1'(P_{t_k}) = 1$ $(k = 1, \cdots, i)$. By definition of $\neg$, $\vee$, $\to$, we can easily verify $v_1(A) = 0$ or $v_1(A) = 1$. Because $v_1(A) > 0$, it is certain that $v_1(A) = 1$.

Corollary 4.1. Let $A \in F(S)$, if for every $v \in \Omega_{\overline{W}}$, $v(A) < 1$, there must be a $v_0 \in \Omega_{\overline{W}}$ such that $v_0(A) = 0$.

Proposition 4.4 Let $A \in F(S)$. If there is a $v \in \Omega_{\overline{W}}$ such that $v(A) > \frac{1}{2}$, there must exist $v_1 \in \Omega_{\overline{W}}$ such that $v_1(A) = 1$.

Proof. If for every $v \in \Omega_{\overline{W}}$, $v(A) < 1$ holds, then for each $v \in \Omega_{\overline{W}}$, $v(\neg A) > 0$. By Theorem 2.1 $v(\neg A) \geqslant \frac{1}{2}$; that is, $v(A) \leqslant \frac{1}{2}$. This is in contradiction with the fact that there is a $v \in \Omega_{\overline{W}}$ such that $v(A) > \frac{1}{2}$. Therefore, there must exist $v_1 \in \Omega_{\overline{W}}$ such that $v_1(A) = 1$.

Corollary 4.2. Let $A \in F(S)$, if there is a$v \in \Omega_{\overline{W}}$ such that $v(A) < \frac{1}{2}$, there must exist $v_0 \in \Omega_{\overline{W}}$ such that $v_0(A) = 0$.

Theorem 4.1. In logic system $\overline{W}$, $\left\{C(\overline{W}), \left[\left(\frac{1}{2}\right)^-\right]\text{-}C(\overline{W}), \left[\frac{1}{2}\right]\text{-}C(\overline{W}), [1]\text{-}C(\overline{W}), \left[\frac{1}{2}\right]\text{-}T(\overline{W}), \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W}), T(\overline{W})\right\}$ is a congruence partition about $\neg$ of $F(S)$.

Proof. This can be directly verified by Definitions 4.1, 4.2, Theorem 2.2 and the propositions and corollaries in sec. 4.

5 Semantic MP rule and semantic HS rule

Definition 5.1[6]. Let $\alpha \in [0,1)$, $R \in \{\overline{W}, W\}$. The $(\alpha^+\text{-}MP)$ rule means that from $A \in \alpha^+\text{-}T(R)$ and $A \to B \in \alpha^+\text{-}T(R)$, we can get $B \in \alpha^+\text{-}T(R)$. The $(\alpha^+\text{-}HS)$ rule means that from $A \to B \in \alpha^+\text{-}T(R)$, $B \to C \in \alpha^+\text{-}T(R)$ we can get $A \to C \in \alpha^+\text{-}T(R)$.

Definition 5.2. Let $\alpha \in [0,1)$, $R \in \{\overline{W}, W\}$. The ($[\alpha^+]$-*MP*) rule means that from $A \in [\alpha^+]\text{-}T(R)$. $A \to B \in [\alpha^+]\text{-}T(R)$ we can get $B \in [\alpha^+]\text{-}T(R)$. The ($[\alpha^+]$-*HS*) rule means that from $A \to B \in [\alpha^+]\text{-}T(R)$, $B \to C \in [\alpha^+]\text{-}T(R)$ we can get $A \to C \in [\alpha^+]\text{-}T(R)$.

Lemma 5.1[6]. In logic system $\overline{W}$, $\left(\left(\frac{1}{2}\right)^+\text{-}MP\right)$ rule and $\left(\left(\frac{1}{2}\right)^+\text{-}HS\right)$ rule hold.

Theorem 5.1. In logic system $\overline{W}$, $\left(\left[\left(\frac{1}{2}\right)^+\right]\text{-}MP\right)$ rule holds.

Proof. Let $A \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$, $A \to B \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$. By Definition 1.6 and Lemma 5.1, $B \in \left(\frac{1}{2}\right)^+\text{-}T(\overline{W})$. If $B \notin \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$ holds, by Theorem 2.2. $B \in T(\overline{W})$. Thus, for each $v \in \Omega_{\overline{W}}$, $v(B) = 1$; for each $v \in \Omega_{\overline{W}}, v(A) \leqslant v(B)$; therefore $A \to B \in T(\overline{W})$. This is in contradiction with $A \to B \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$. So, $B \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$.

Proposition 5.1. In logic system $\overline{W}$, $\left(\left[\left(\frac{1}{2}\right)^+\right]\text{-}HS\right)$ rule does not hold.

Proof. We take an example to explain this conclusion.

Let $A = p_4$, $B = (p_2 \to \neg p_1 \vee p_1) \vee p_2$, $C = (p_4 \to \neg p_3 \vee p_3) \vee p_4$. By Theorem 3.1 we can verify $B, C \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$, and further, we can easily verify $A \to B \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$, $B \to C \in \left[\left(\frac{1}{2}\right)^+\right]\text{-}T(\overline{W})$. Obviously, $A \to C \in T(\overline{W})$. Therefore, $\left(\left[\left(\frac{1}{2}\right)^+\right]\text{-}HS\right)$ rule does not hold.

6 Theory of generalized tautology in logic system $W(W_k)$

(1) In the theory of generalized tautology in logic system $\overline{W}$, by putting W in the place of $\overline{W}$ and in an analogous way we can prove that the relative conclusions also hold in logic system W.

(2) Using a method analogous to the above, we can get the following conclusions.

Theorem 6.1. In logic system W_k, $\{C(W_k), [i]\text{-}T(W_k), [i]\text{-}C(W_k)] \mid 0 \leqslant i \leqslant T\}$ is a congruence partition about $\neg$ of $F(S)$ in logic system W_k.

Theorem 6.2. Let B be a formula obtained from A through upgrade algorithm given in sec. 3. Then if $A \in [F]\text{-}T(W_k)$, when $0 \in W_k$, $B \in [0]\text{-}T(W_k)$, and when $0 \notin W_k$, $B \in [1]\text{-}T(W_k)$; if $A \in [i]\text{-}T(W_k)$, then $B \in [i+1]\text{-}T(W_k)(0 \leqslant i < T)$.

Theorem 6.3. In logic system W_k, let $A \in F(S)$. Using upgrade algorithm to A repeatedly, we can get a tautology from A at most $\left[\frac{k+1}{2}\right]$ times, where $\left[\frac{k+1}{2}\right]$ means the maximal integer of $\frac{k+1}{2}$.

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