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AN EMBEDDING THEOREM FOR A WEIGHTED SPACE OF SOBOLEV TYPE AND CORRECT SOLVABILITY OF THE STURM-LIOUVILLE EQUATION

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Abstract. We consider the weighted space $W_1^{(2)}(\mathbb{R},q)$ of Sobolev type

$$W_1^{(2)}(\mathbb{R},q)=\Big\{y\in A_{\rm loc}^{(1)}(\mathbb{R})\colon\ \|y''\|_{L_1(\mathbb{R})}+\|qy\|_{L_1(\mathbb{R})}<\infty\Big\}$$

and the equation

$$-y''(x)+q(x)y(x)=f(x),\quad x\in\mathbb{R}. \tag{1}$$

Here $f\in L_1(\mathbb{R})$ and $0\leqslant q\in L_1^{\rm loc}(\mathbb{R})$.

We prove the following:

1) The problems of embedding $W_1^{(2)}(\mathbb{R},q)\hookrightarrow L_1(\mathbb{R})$ and of correct solvability of (1) in $L_1(\mathbb{R})$ are equivalent;
2) an embedding $W_1^{(2)}(\mathbb{R},q)\hookrightarrow L_1(\mathbb{R})$ exists if and only if

$$\exists a>0\colon\ \inf_{x\in\mathbb{R}}\int_{x-a}^{x+a}q(t)\,{\rm d}t>0.$$

Keywords: Sobolev space, embedding theorem, Sturm-Liouville equation

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1. Introduction

In the present paper, we consider the weighted functional space $W_1^{(2)}(\mathbb{R},q)$ of Sobolev type (see [5]):

$$W_1^{(2)}(\mathbb{R},q)=\{y\in AC_{\rm loc}^{(1)}(\mathbb{R})\colon\ \|y''\|_{L_1(\mathbb{R})}+\|qy\|_{L_1(\mathbb{R})}<\infty\} \tag{1.1}$$

and the Sturm-Louiville equation

$$-y''(x)+q(x)y(x)=f(x),\quad x\in\mathbb{R}. \tag{1.2}$$

Here $f\in L_1$ ($L_1(\mathbb{R}):=L_1$, $\|f\|_{L_1}:=\|f\|_1$), $AC^{(1)}_{\rm loc}(\mathbb{R})$ is the set of functions absolutely continuous together with their first derivative on any finite interval, and

$$0\leqslant q\in L_1^{\rm loc}(\mathbb{R}). \tag{1.3}$$

Our general goal is to reveal the relationship between the problem of the existence of an embedding $W_1^{(2)}(\mathbb{R},q)\hookrightarrow L_1$ (see [5]) and the problem of correct solvability of the equation (1.2) in the space L_1 (see [2]). To be more precise, let us introduce the following definitions.

Definition 1.1 [5]**.** We say that the space $W_1^{(2)}(\mathbb{R},q)$ is embedded into the space L_1 (and write $W_1^{(2)}(\mathbb{R},q)\hookrightarrow L_1$) if $W_1^{(2)}(\mathbb{R},q)\subseteq L_1$ and

$$\|y\|_1\leqslant c\{\|y''\|_1+\|qy\|_1\},\quad \forall y\in W_1^{(2)}(\mathbb{R},q). \tag{1.4}$$

Our general convention is to denote by the letter c absolute positive constants which are not essential for exposition and may differ even within a single chain of calculations.

Definition 1.2. By a solution of (1.2) we mean any function $y\in AC^{(1)}_{\rm loc}(\mathbb{R})$ satisfying the equation (1.2) almost everywhere in $\mathbb{R}$.

Definition 1.3. The equation (1.2) is called correctly solvable in the space L_1 if the following assertions hold:

I) for any function $f\in L_1$ there exists a unique solution of (1.2) $y\in L_1$;

II) for the solution of (1.2) $y\in L_1$ we have the inequality

$$\|y\|_1\leqslant c\|f\|_1,\quad \forall f\in L_1. \tag{1.5}$$

We can now formulate our main results.

Theorem 1.4. *An embedding $W_1^{(2)}(\mathbb{R},q)\hookrightarrow L_1$ exists if and only if the equation* (1.2) *is correctly solvable in the space L_1.*

Corollary 1.5. *An embedding* $W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_1$ *exists if and only if*

$$\exists a > 0\colon \inf_{x\in\mathbb{R}} \int_{x-a}^{x+a} q(t)\,\mathrm{d}t > 0. \tag{1.6}$$

In connection with this corollary, note that under the restrictions (1.3), the condition (1.6) is a minimal requirement guaranteeing the embedding $W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_p(\mathbb{R})$, $p \in (1, \infty)$. This assertion will be proved in a forthcoming paper.

Corollary 1.6. *Denote*

$$\mathscr{D} = \{y \in L_1\colon\ y \in AC^{(1)}_{\mathrm{loc}}(\mathbb{R}),\ -y'' + qy \in L_1\}. \tag{1.7}$$

Then under the condition (1.6) *we have the equality*

$$\mathscr{D} = W_1^{(2)}(\mathbb{R}, q).$$

In the monograph [5], a well-known handbook on the theory of weighted Sobolev spaces, conditions for the embedding of such spaces into the spaces L_p, $p \in [1, \infty]$, are expressed in terms different from (1.6). Therefore, to complete the picture, we reformulate Corollary 1.5 using the language adopted in [5]. Towards this end, suppose that in addition to (1.3) the following requirement holds:

$$\int_{-\infty}^{x} q(t)\,\mathrm{d}t > 0, \qquad \int_{x}^{\infty} q(t)\,\mathrm{d}t > 0, \qquad \forall x \in \mathbb{R} \tag{1.8}$$

(see [1], [2]). We now fix $x \in \mathbb{R}$ and consider the equation in $d \geqslant 0$:

$$d \int_{x-d}^{x+d} q(t)\,\mathrm{d}t = 2. \tag{1.9}$$

It is known (see [1]) that under the conditions (1.3) and (1.8), (1.9) has a unique finite positive solution. Denote it by $d(x)$, $x \in \mathbb{R}$. The function d was introduced by M. Otelbaev (see [5]). Note that the function

$$q^*(x) = d^{-2}(x), \qquad x \in \mathbb{R}$$

is the Steklov average with step $d(x)$ of the function $q(t)$, $t \in \mathbb{R}$ at the point $t = x$ (see (1.9)):

$$q^*(x) = \frac{1}{d^2(x)} = \frac{1}{2d(x)} \int_{x-d(x)}^{x+d(x)} q(\xi)\,\mathrm{d}\xi.$$

In [2] it is shown that the condition (1.6) holds if and only if we have (1.8) and $q_0^* > 0$ where

$$q_0^* = \inf_{x\in\mathbb{R}} q^*(x).$$

Together with Corollary 1.5, this implies the following assertion.

Corollary 1.7. *There exists an embedding* $W_1^{(2)}(\mathbb{R},q) \hookrightarrow L_1$ *if and only if the condition* (1.8) *and the inequality* $q_0^* > 0$ *hold.*

Example. It is known (see [2]) that if

$$q(x) = 1 + \cos(|x|^\theta), \quad \theta > 0, \ x \in \mathbb{R}, \tag{1.10}$$

then the equation (1.2) is correctly solvable in L_1 if and only if $\theta \geqslant 1$. Therefore for (1.10), Theorem 1.4 implies that $W_1^{(2)}(\mathbb{R},q) \hookrightarrow L_1$ only for $\theta \geqslant 1$.

2. Prelimnaries

Lemma 2.1 [1]**.** *Suppose that* (1.3) *and* (2.1) *hold:*

$$\int_{-\infty}^{x} q(t)\,\mathrm{d}t > 0, \quad \int_{x}^{\infty} q(t)\,\mathrm{d}t > 0, \quad \forall x \in \mathbb{R}. \tag{2.1}$$

Then there exists a fundamental system of solutions (FSS) $\{u, v\}$ *of the equation*

$$z''(x) = q(x)z(x), \quad x \in \mathbb{R} \tag{2.2}$$

which has the following properties:

$$u(x) > 0, \quad v(x) > 0, \quad u'(x) \leqslant 0, \quad v'(x) \geqslant 0, \quad \forall x \in \mathbb{R}, \tag{2.3}$$

$$v'(x)u(x) - u'(x)v(x) = 1, \quad x \in \mathbb{R}, \tag{2.4}$$

$$\lim_{x\to-\infty} \frac{v(x)}{u(x)} = \lim_{x\to\infty} \frac{u(x)}{v(x)} = 0. \tag{2.5}$$

Throughout the sequel we reserve the symbol $\{u, v\}$ for denoting a FSS of (2.2) with the properties (2.3)–(2.5).

Let us introduce the Green function corresponding to (1.2)

$$G(x,t) = \begin{cases} u(x)v(t), & x \geqslant t, \\ u(t)v(x), & x \leqslant t. \end{cases} \tag{2.6}$$

Lemma 2.2 [1]**.** *Suppose that* (1.3) *and* (2.1) *hold. Then*

$$\sup_{x\in\mathbb{R}} T(x) \leqslant 1, \quad T(x) = \int_{-\infty}^{\infty} q(t)G(x,t)\,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{2.7}$$

Lemma 2.3 [1]**.** *Suppose that* (1.3) *and* (2.1) *hold. Then* (2.2) *has no solutions* $z \in L_1$ *except for* $z \equiv 0$.

Theorem 2.4 [2]**.** *Under the condition* (1.3), *the equation* (1.2) *is correctly solvable in* L_1 *if and only if* (1.6) *holds.*

Theorem 2.5 [3]**.** *Suppose that* (1.3) *and* (2.1) *hold. Then the equation* (1.2) *is correctly solvable in* L_1 *if and only if the Green operator* $G\colon L_1 \to L_1$ *is bounded. Here*

$$(2.8)\qquad (Gf)(x) = \int_{-\infty}^{\infty} G(x,t) f(t)\, \mathrm{d}t, \quad x \in \mathbb{R}, \quad \forall f \in L_1.$$

If, in addition, $\|G\|_{1\to 1} < \infty$, then the solution $y \in L_1$ of (1.2) is of the form $y = Gf$.

Theorem 2.6 [3]. *Suppose that* (1.1) *is correctly solvable in* L_1. *Then its solution* $y \in L_1$ *satisfies the inequality*

$$(2.9)\qquad \|y''\|_1 + \|qy\|_1 \leqslant 3\|f\|_1.$$

Remark 2.7. The inequality (2.9) was stated as a conjecture by R. Oinarov (see [5, p. 259]) and proved in [4] under the requirement $q \geqslant 1$, in addition to (1.3). See [1], [6], [7] for various generalizations of this result from [4]. Theorem 2.6 provides another assumption for a version of (2.9) to hold, which is especially adjusted to our terminology. See §3 for the proof of Theorem 2.6. In connection with inequalities of type (2.9), see also the papers [8], [9].

3. Proofs

Lemma 3.1. *Suppose that* (1.3) *holds and* $W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_1$. *Then* (2.1) *also holds.*

Proof. Assume the contrary, say

$$\int_{x_0}^{\infty} q(t)\, \mathrm{d}t = 0$$

for some $x_0 \in \mathbb{R}$. Let φ be such that $\varphi \in C^\infty(\mathbb{R})$, $\operatorname{supp}\varphi = [x_0, \infty)$, $0 \leqslant \varphi \leqslant 1$ for $x \in \mathbb{R}$ and $\varphi(x) \equiv 1$ for $x \geqslant x_0 + 1$. Clearly, $\varphi \in AC^{(1)}_{\mathrm{loc}}(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} |q(t)\varphi(t)|\,\mathrm{d}t = \int_{x_0}^{\infty} q(t)\varphi(t)\,\mathrm{d}t = 0$$
$$0 < \int_{-\infty}^{\infty} |\varphi''(t)|\,\mathrm{d}t = \int_{x_0}^{x_0+1} |\varphi''(t)|\,\mathrm{d}t = c < \infty.$$

Hence $\varphi \in W_1^{(2)}(\mathbb{R}, q)$ and since $W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_1$, we have

$$\infty > c\Big\{\int_{-\infty}^{\infty} |\varphi''(t)|\,\mathrm{d}t + \int_{-\infty}^{\infty} |q(t)\varphi(t)|\,\mathrm{d}t\Big\} \geqslant \int_{-\infty}^{\infty} |\varphi(t)|\,\mathrm{d}t \geqslant \int_{x_0+1}^{\infty} 1\,\mathrm{d}t = \infty,$$

a contradiction. □

Lemma 3.2. *Suppose that* (1.3) *and* (2.1) *hold. Set* $y = Gf$ *where* $f \in L_1$ *(see* (2.8)*). Then* y *is a solution of* (1.2) *which satisfies* (2.9)*. In particular,* $y \in W_1^{(2)}(\mathbb{R}, q)$.

Proof. Lemma 2.1 implies the inequalities

$$\int_{-\infty}^{x} v(t)|f(t)|\,\mathrm{d}t \leqslant v(x)\|f\|_1, \qquad \int_{x}^{\infty} u(t)|f(t)|\,\mathrm{d}t \leqslant u(x)\|f\|_1, \quad x \in \mathbb{R}.$$

Hence the function $y(x) = (Gf)(x)$, $x \in \mathbb{R}$, is well-defined. Since

$$y(x) = (Gf)(x) = u(x)\int_{-\infty}^{x} v(t)f(t)\,\mathrm{d}t + v(x)\int_{x}^{\infty} u(t)f(t)\,\mathrm{d}t, \quad x \in \mathbb{R}, \tag{3.1}$$

from (3.1) and Lemma 2.1 we immediately obtain

$$y'(x) = u'(x)\int_{-\infty}^{x} v(t)f(t)\,\mathrm{d}t + v'(x)\int_{x}^{\infty} u(t)f(t)\,\mathrm{d}t, \quad x \in \mathbb{R},$$
$$y''(x) = q(x)y(x) - f(x), \quad x \in \mathbb{R}.$$

Hence $y \in AC^{(1)}_{\mathrm{loc}}(\mathbb{R})$. Further, by Fubini's theorem and (2.7), we have

$$\begin{aligned}\|qy\|_1 &= \int_{-\infty}^{\infty} q(x)\Big|\int_{-\infty}^{\infty} G(x,t)f(t)\,\mathrm{d}t\Big|\,\mathrm{d}x \leqslant \int_{-\infty}^{\infty} q(x)\int_{-\infty}^{\infty} G(x,t)|f(t)|\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} |f(t)|\Big(\int_{-\infty}^{\infty} q(x)G(x,t)\,\mathrm{d}x\Big)\,\mathrm{d}t \leqslant \int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t = \|f\|_1.\end{aligned} \tag{3.2}$$

From (3.2), (1.1) and the triangle inequality, we obtain (2.9), and hence $y \in W_1^{(2)}(\mathbb{R}, q)$. □

Proof of Theorem 2.6. Since the equation (1.1) is correctly solvable in L_1, $y = Gf$ is a unique solution of (1.1) from the class L_1 (see Theorems 2.4 and 2.5). It remains to refer to Lemma 3.2. $\square$

Proof of Theorem 1.4. Necessity. Suppose that $W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_1$. Since (2.1) holds due to Lemma 3.1, and $Gf \in W_1^{(2)}(\mathbb{R}, q)$ for $f \in L_1$ due to Lemma 3.2, then by (1.4) and (2.9) we have

$$\|(Gf\|_1 \leqslant c\big\{\|(Gf)''\|_1 + \|qGf\|_1\big\} \leqslant c\|f\|_1, \quad \forall f \in L_1.$$

Thus the operator $G\colon L_1 \to L_1$ is bounded, and by Theorem 2.5 the equation (1.1) is correctly solvable in L_1. $\square$

Proof of Theorem 1.4. Sufficiency. Suppose that the equation (1.1) is correctly solvable in L_1 and $\tilde{y} \in W_1^{(2)}(\mathbb{R}, q)$. Then from the triangle inequality it follows that $f(\tilde{y}) \in L_1$ where $f(\tilde{y}) = -\tilde{y}'' + q\tilde{y}$. Denote by y the solution of the equation (1.1) with $f = f(\tilde{y})$ from the class L_1. It is easy to see that the function $z = y - \tilde{y}$ is a solution of (2.2), and by (2.9) and (1.1) we have

$$\|qz\|_1 \leqslant \|qy\|_1 + \|q\tilde{y}\|_1 \leqslant 3\|f(\tilde{y})\|_1 + \|q\tilde{y}\|_1 \leqslant 3\|\tilde{y}''\|_1 + 4\|q\tilde{y}\|_1 < \infty. \tag{3.3}$$

Since z is a solution of (2.2), it is of the form

$$z(x) = c_1 u(x) + c_2 v(x) \tag{3.4}$$

(c_1, c_2 are arbitrary constants). Let us show that $c_1 = c_2 = 0$. Assume the contrary, say, $c_2 \neq 0$. From (2.5) it follows that there is $x_0 \gg 1$ such that for all $x \geqslant x_0$ we have the estimates

$$|z(x)| = |c_1 u(x) + c_2 v(x)| \geqslant |c_2| v(x)\Big[1 - \Big|\frac{c_1}{c_2}\Big|\frac{u(x)}{v(x)}\Big] \geqslant \frac{|c_2|}{2} v(x). \tag{3.5}$$

Note that from Theorem 2.4 and (1.6) it follows that

$$\int_{-\infty}^{0} q(t)\,\mathrm{d}t = \int_{0}^{\infty} q(t)\,\mathrm{d}t = \infty. \tag{3.6}$$

Therefore by (3.3), (3.4), (3.5), (3.6) and (2.3), we have

$$\begin{aligned}\infty > \int_{-\infty}^{\infty} q(t)|z(t)|\,\mathrm{d}t \geqslant \int_{x_0}^{\infty} q(t)|z(t)|\,\mathrm{d}t \geqslant \frac{|c_2|}{2}\int_{x_0}^{\infty} q(t)v(t)\,\mathrm{d}t \\ \geqslant \frac{|c_2|}{2}\, v(x_0)\cdot\int_{x_0}^{\infty} q(t)\,\mathrm{d}t = \infty,\end{aligned}$$

a contradiction. Hence $c_2 = 0$, and, similarly, $c_1 = 0$. Thus $\tilde{y} = y \in L_1$ and therefore by (1.5), we have

$$\|\tilde{y}\|_1 = \|y\|_1 \leqslant c\|f(\tilde{y})\|_1 = c\| - \tilde{y}_1'' + q\tilde{y}\|_1 \leqslant c\{\|\tilde{y}''\|_1 + \|q\tilde{y}\|_1\}, \quad \forall \tilde{y} \in W_1^{(2)}(\mathbb{R}, q).$$

□

P r o o f of Corollary 1.5. It follows from Theorems 1.4 and 2.4. □

P r o o f of Corollary 1.6. By Theorem 2.4, the equation (1.2) is correctly solvable in L_1. Let $y \in D$. Then $y \in L_1$ and $f \in L_1$ where $f = -y'' + qy$. Hence $y'' \in L_1$, $qy \in L_1$ (see (2.9)), and therefore $y \in W_1^{(2)}(\mathbb{R}, q)$, i.e., $D \subseteq W_1^{(2)}(\mathbb{R}, q)$. Conversely, let $y \in W_1^{(2)}(\mathbb{R}, q)$. Since $W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_1$, due to Theorem 1.4, we have $y \in L_1$, i.e., $y \in D$, and therefore $W_1^{(2)}(\mathbb{R}, q) \subset D$. Thus $D = W_1^{(2)}(\mathbb{R}, q)$. □

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