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# Bilinearity and Multilinearity over Arbitrary Rings with Units

To Professor B. L. VAN DER WAERDEN in gratitude and esteem

By

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All rings here dealt with are understood to possess units, and all modules over them to be unitary. Zero-modules are not excluded. The additive group of a ring $A$ is written $\check{A}$; the natural $A$-left module on this is $\check{A}^L$, etc.; the opposite ring is $\tilde{A}$; the center of $A$ and of $\tilde{A}$ is $\mathring{A}$. The ring of integers is indicated by $Z$, and a ring written $C$ is always commutative. The tensorial product-ring $A \otimes B$ is understood to be $A_Z \otimes B_Z$ unless specifically noted otherwise as $A_C \otimes B_C$; this convention applies also when $B = A$.

Two modules on the same additive group are said to be *fully equivalent* when every transformation of that group obtainable in the one module is obtainable in the other and vice versa.

## § 1. Left-right bilinearity over a single ring

Let $A$ be a ring, $E^L$ an $A$-left module, $F_R$ an $A$-right module, $M^L_R$ a two-sided $A$-module, $\Phi$ a mapping of the Cartesian product $E \times F$ into $M^L_R$. What requirements must $\Phi$ fulfill in order to be called $A$-bilinear?

For commutative $A$, equal to its own center $\mathring{A}$, these are[1])

$$\left.\begin{array}{lll} P1) & \Phi(x+x', y) = \Phi(x, y) + \Phi(x', y) \\ P2) & \Phi(x, y+y') = \Phi(x, y) + \Phi(x, y') \\ \mathring{A}1) & \Phi(\mathring{\alpha} x, y) = \mathring{\alpha}(\Phi(x, y)) \\ \mathring{A}2) & \Phi(x, y\mathring{\alpha}) = (\Phi(x, y))\mathring{\alpha} \\ \mathring{X}) & \Phi(\mathring{\alpha} x, y) = \Phi(x, y\mathring{\alpha}) \end{array}\right\} \begin{array}{l} x, x' \in E^L \\ y, y' \in F_R \\ \mathring{\alpha} \in \mathring{A}. \end{array}$$

The bilinearity so defined can be embodied in a tensorial product-module $E^L \otimes F_R$. It is then easily shown that to every bilinear mapping

$$\Phi : E \times F \to M^L_R$$

[1]) The last three of these together amount to

$$\Phi(\mathring{\alpha} x, y) = \Phi(x, y\mathring{\alpha}) = \mathring{\alpha}(\Phi(x, y)) = (\Phi(x, y))\mathring{\alpha},$$

of which the first three terms, together with $P1$ and $P2$, constitute the traditional definition of a bilinear mapping into an $\mathring{A}$-left module $M^L$. Since any such $M^L$ can be made into a two-sided $\mathring{A}$-module with which $\mathring{A}$ commutes, to every $\mathring{A}$-bilinear mapping as traditionally defined exists one as defined here — and even more obviously vice versa — with the same values for the same arguments.

corresponds a unique homomorphism

$$\Psi : E^L \otimes F_R \to M^L_R ,$$

and vice versa, such that

$$\Phi(x, y) = \Psi(x \otimes y) , \quad x \in E^L, y \in F_R, x \otimes y \in E^L \otimes F_R .$$

This correspondence is an isomorphism of the two-sided $\mathring{A}$-modules constituted by the $\Phi$'s and the $\Psi$'s.

As soon as we have agreed on a definition of left-right bilinearity over a not necessarily commutative ring $A$, we can proceed to form a corresponding tensorial product, with corresponding results. As criteria for a "reasonable" definition, the following[2]) are here proposed:

1) The definition reduces to the above when $A$ $(= \mathring{A})$ is in fact commutative.

2) It is capable of being fulfilled in a non-trivial way for non-commutative rings of the kinds commonly dealt with (for example, for the skewfield of quaternions).

3) P1 and P2 remain as they stand.

4) $\mathring{A}1$ and $\mathring{A}2$ are replaced, if at all, by requirements exhibiting the same symmetry.

Additional "desirable" properties include:

5) If $\Phi$ is $A$-bilinear it is (two-sidedly) $A$-linear.

All of these conditions can be met by axioms within the set consisting of those above and the following:

$$\left.\begin{array}{lll} A1) & \Phi(\alpha x, y) = \alpha(\Phi(x, y)) \\ A2) & \Phi(x, y\alpha) = (\Phi(x, y))\,\alpha \\ X) & \Phi(\alpha x, y) = \Phi(x, y\alpha) \end{array}\right\} \begin{array}{l} x \in E^L \\ y \in F_R \\ \alpha \in A . \end{array}$$

Some of the non-solutions are (in effect) in use; others present a *prima facie* suitability, and still others are of inherent interest. For this reason, all the essentially distinct systems which fulfill (3), except some of those which violate (4), are given below, each indicated by its contents in addition to $P1$ and $P2$:

i) $A1$, $A2$, $X$. This violates (2): if $A$ is a simple non-commutative ring or the direct sum of such, there do not exist any bilinear mappings in this sense except the zero-mapping. For the image-set $\Phi(E \times F)$ is a two-sided module with which all of $A$ commutes, so that

$$\alpha\alpha'\Phi = \alpha(\alpha'\Phi) = (\alpha'\Phi)\alpha = \alpha'(\Phi\alpha) = \alpha'(\alpha\Phi) = \alpha'\alpha\Phi ;$$

the annulling ideal of $\Phi(E \times F)$ contains all elements of the form $\alpha\alpha' - \alpha'\alpha$, and in the cases mentioned this ideal must be all of $A$.

ii) $A1$ and $X$. An unsuccessful attempt at evasion: the violation of (4) does not prevent the violation of (2) in the same way as in (i). For in this system

$$\begin{aligned}(\alpha\alpha')\,\Phi(x, y) &= \alpha(\alpha'\Phi(x, y)) = \alpha(\Phi(\alpha' x, y)) = \alpha(\Phi(x, y\alpha')) = \Phi(\alpha x, y\alpha') = \Phi(\alpha'\alpha x, y) \\ &= (\alpha'\alpha)\,\Phi(x, y) .\end{aligned}$$

iii) X alone. This might be called the "inferable received definition" of $A$-bilinearity, since it is the one which is embodied in the received definition of the tensorial product for

[2]) No particular attempt has been made to select independent conditions.

non-commutative $A$[3]). It violates condition (1) (as well as (5)). This may be ameliorated by going over to

iv) $\mathring{A}1$, $\mathring{A}2$, $X$. This is "reasonable" if not "desirable", in that it satisfies the first four conditions but not the fifth: the image-set is a two-sided $\mathring{A}$-module, but cannot be asked to be a two-sided or even a one-sided $A$-module without returning to (i) or (ii) above. The corresponding tensorial product[4]) suffers the same disadvantage of putting the commutative cart before the operative horse.

v) $\mathring{A}1$, $\mathring{A}2$, $\mathring{X}$. Again satisfies (1)—(4) but not (5), with the opposite defect to the above: too unrestrictive. There is no point in introducing the term "$A$-bilinear" for what is already understood as "$\mathring{A}$-bilinear".

vi) $A1$, $\mathring{X}$ (or $A1$, $\mathring{A}2$, $\mathring{X}$). Leads to a tensorial product of tempting simplicity but doubtful significance. Violates (4) as well as (5).

vii) $A1$, $A2$. Violates (1), otherwise admirable (fulfills (2)—(5)).

viii) $A1$, $A2$, $\mathring{X}$. Fulfills (1)—(5).

Accordingly

**1.1 Definition:** A mapping $\Phi : E \times F \to M_R^L$ will be called *bilinear*, and specifically *left-right $A$-bilinear*, if it fulfills $P1$, $P2$, $A1$, $A2$, $\mathring{X}$.

In this definition $\mathring{X}$ may be replaced by

$$\mathring{X}1) \quad \mathring{\alpha}(\Phi(x, y)) = (\Phi(x, y))\mathring{\alpha}\,, \qquad \mathring{\alpha} \in \mathring{A}\,,$$

since the trio $A1$, $A2$, $\mathring{X}$ leads to and follows from $A1$, $A2$, $\mathring{X}1$.

The near-solution (vii) deserves special attention:

**1.2 Definition:** A mapping $\Phi : E \times F \to M_R^L$ will be called *prebilinear*, and specifically *left-right $A$-prebilinear* if it fulfills $P1$, $P2$, $A1$ and $A2$.

It is obvious that every bilinear mapping is prebilinear, and also (from $\mathring{X}1$) that

**1.3 Proposition:** *If $\mathring{A}$ commutes with $M_R^L$, every prebilinear mapping into $M_R^L$ is bilinear.*

In particular, every left-right prebilinear mapping into $A$ itself is bilinear[5]).

**1.4 Definition:** Where $S_R^L$ is the formal two-sided $A$-module generated by $E \times F$, and $T_R^L$ the submodule generated by all elements of the forms

$$\left.\begin{array}{ll} P_1) & (x + x', y) - (x, y) - (x', y) \\ P_2) & (x, y + y') - (x, y) - (x, y') \\ A_1) & \quad (\alpha x, y) - \alpha(x, y) \\ A_2) & \quad (x, y\alpha) - (x, y)\alpha \\ \mathring{X}_1) & \quad \mathring{\alpha}(x, y) - (x, y)\mathring{\alpha} \end{array}\right\} \begin{array}{c} x, x' \in E^L \\ y, y' \in F_R \\ \alpha \in A \\ \mathring{\alpha} \in \mathring{A}\,, \end{array}$$

the factor-module $S_R^L / T_R^L$ will be called the *tensorial product* of $E^L$ and $F_R$, and written $E^L \ominus F_R$. Where $V_R^L$ is the submodule of $S_R^L$ generated by the forms $P_1$, $P_2$, $A_1$ and $A_2$, the factor-module $S_R^L / V_R^L$ will be called the *pretensorial product* of $E^L$ and $F_R$, and written $E^L \oplus F_R$.

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[3]) BOURBAKI [3] App. II, No 1, p. 120; JACOBSON [5] Ch. V, 1, p. 95; CHEVALLEY [4] Ch. III, 8, p. 74 (where a mapping with these properties is called a "balanced map").

[4]) BOURBAKI [3] App. II, No 3, p. 124.

[5]) The present definitions are therefore in accord with at least one received definition of a "bilinear form": JACOBSON [5] Ch. IV, 6, Def. 1, pp. 69—70.

In the case of commutative $A = \mathring{A}$, the $\mathring{A}$-left module on the underlying group of $E^L \ominus F_R$ is fully equivalent to $E^L \ominus F_R$ itself. This single module (or the equivalent $\mathring{A}$-right module — the distinction between left and right is not usually maintained consistently with commutative rings) is the tensorial product $E \otimes F$ according to the received definitions of the literature[6]).

Thus the present definition of $\ominus$ coincides essentially with the received one of $\otimes$ for commutative $A$. For non-commutative $A$, on the other hand, the additive group called[7]) $E \otimes_A F$ is distinct from the underlying group of $E^L \ominus F_R$, as can be seen from the discussion under (iv) above.

Where $I$ and $K$ are index-sets for $E$ and $F$, the pairs $(x^\iota, y_\varkappa)$, $(\iota, \varkappa) \in I \times K$. clearly generate $S^L_R$: for every $\sigma$ in $S^L_R$ there is some set $N$ such that $\sigma$ has at least one expression as a weak sum (sum with a finite number of non-zero terms) $\sum_{(\iota,\varkappa)\in I\times K} \sum_{\nu\in N} \alpha^\nu_\iota (x^\iota, y_\varkappa) \beta^\varkappa_\nu$; $\alpha^\nu_\iota, \beta^\varkappa_\nu \in A$. These generators have furthermore an independence property which permits the following lemma:

**1.5 Proposition:** *If $\Phi$ is any mapping of $E \times F$ into an $A$-left $A$-right module $M^L_R$, then the homomorphism*

$$\overline{\Phi} : \sigma = \sum_{(\iota,\varkappa)\in I\times K} \sum_{\nu\in N} \alpha^\nu_\iota (x^\iota, y_\varkappa) \beta^\varkappa_\nu \to \sum_{(\iota,\varkappa)\in I\times K} \sum_{\nu\in N} \alpha^\nu_\iota (\Phi(x^\iota, y_\varkappa)) \beta^\varkappa_\nu$$

*is well-defined.*

*Proof:* We consider the formal $(A \otimes \tilde{A})$-left module $S^{L\tilde{L}}$ generated by $E \times F$. As a single module, it possesses[8]) a base composed of the elements $(x^\iota, y_\varkappa)$, $(\iota, \varkappa) \in I \times K$, left-linearly independent over $A \otimes \tilde{A}$. Now, $S^{L\tilde{L}}$ is the corresponding module to $S^L_R$, given by

$$(\alpha \otimes \tilde{\beta})\,(x, y) = \alpha(x, y)\,\beta\,, \quad \alpha \in A, \tilde{A} \ni \tilde{\beta} = \beta \in A, (x, y) \in E \times F\,,$$

[6]) Interpreting two received definitions in the present terminology and notation, we find:

1) In Jacobson [5] Ch. V, 1, p. 96, the additive group $S$ generated by $E \times F$ is in effect factored by the subgroup generated by $P_1$, $P_2$ and

$X$) $(\alpha x, y) - (x, y\alpha)$

(actually $F \times E$ factored by $P_1$, $P_2$ and

$X'$) $(y\alpha, x) - (y, \alpha x)$)

and then (Ch. V, 2, p. 101) for commutative $A = \mathring{A}$, the natural $\mathring{A}$-module is formed. It is remarked that single $\mathring{A}$-modules are indifferent as to side, so that this amounts to forming $\overline{S}^L_R = (S/(P_1, P_2, \mathring{X}))^L_R$ and setting the rule $\mathring{\overline{A}}_1 = \mathring{\overline{A}}_2 = 0$. Since the submodules generated by $\mathring{A}_1$, $\mathring{A}_2$, $\mathring{X}$ and by $\mathring{A}_1$, $\mathring{A}_2$, $\mathring{X}_1$ are the same, the result is naturally isomorphic to the present one.

2) In Bourbaki [3] § 1, No 2, pp. 4—6, $F_R$ is replaced by the naturally equivalent $\mathring{A}$-left module $F^L$ (given by $\mathring{\alpha} y = y \mathring{\alpha}$), and the formal Å-left module $\overline{S}^L$ generated by $E \times F$ is factored by the submodule $\overline{T}^L$ generated by the forms $\overline{P}_1$, $\overline{P}_2$, $\mathring{\overline{A}}_1$ and

$\mathring{\overline{A}}{}'_2$) $\overline{(x, \mathring{\alpha} y)} - \mathring{\alpha}\overline{(x, y)}$.

Now, the natural two-sided commutative $\mathring{A}$-module $\overline{S}^L_R$ on $\overline{S}$ is naturally isomorphic to $S^L_R/(\mathring{X}_1)^L_R$, and $\overline{S}^L$ and $\overline{S}^L_R$ are fully equivalent. Thus $\overline{S}^L/\overline{T}^L \cong S^L/T^L$, which is equivalent to $S^L_R/T^L_R$.

[7]) Bourbaki [3] App. II, No 1, Def. 1, p. 120.

[8]) Bourbaki [2] 1, No 8, pp. 15—16.

and $S^{L\tilde{L}}$ and $S^L_R$ are fully equivalent[9]). Thus if $\sigma = 0$, then for each $(\iota, \varkappa) \in I \times K$

$$\sum_{\nu \in N} (\alpha^\nu_\iota \otimes \tilde{\beta}^\varkappa_\nu) = 0 .$$

If this is the case, and if $\{m^\iota_\varkappa\}$ is any collection of elements (with or without repetitions) in $M$, then in $M^{L\tilde{L}}$ and hence in $M^L_R$

$$0 = \sum_{(\iota,\varkappa)\in I\times K} \sum_{\nu\in N} (\alpha^\nu_\iota \otimes \tilde{\beta}^\varkappa_\nu)\, m^\iota_\varkappa = \sum_{(\iota,\varkappa)\in I\times K} \sum_{\nu\in N} \alpha^\nu_\iota\, m^\iota_\varkappa\, \beta^\varkappa_\nu .$$

Thus if $\sigma'$, $\sigma''$ are two expressions for the same element of $S^L_R$, we have

$$\bar{\Phi}\sigma' - \bar{\Phi}\sigma'' = \bar{\Phi}(\sigma' - \sigma'') = 0 . \qquad \text{q.e.d.}$$

**1.6 Proposition** *("Embodiment theorem"): To every left-right $A$-bilinear (resp. $A$-prebilinear) mapping $\Phi$ of $E \times F$ into any two-sided $A$-module $M^L_R$ whatever, there exists a unique homomorphism $\Psi$ of $E^L \ominus F_R$ (resp. $E^L \oplus F_R$) into $M^L_R$, and vice versa, such that*

$$\Psi(x \ominus y) = \Phi(x, y)$$

$$(\textit{resp. } \Psi(x \oplus y) = \Phi(x, y)) , \qquad x \in E^L, y \in F_R .$$

*Proof:* Where $\Phi$ is any mapping whatsoever of $E \times F$ into an $M^L_R$, there exists a unique homomorphism $\bar{\Phi}$ of $S^L_R$ which extends $\Phi$, namely that given in the previous proposition. From $P1$, $P2$, $A1$, $A2$, $\mathring{X}1$ follow that $\bar{\Phi}$ is zero respectively on $P_1$, $P_2$, $A_1$, $A_2$, $\mathring{X}_1$; and conversely from $\bar{\Phi} P_1 = 0$, etc., follow $P1$, etc. Thus necessary and sufficient that $\Phi$ be bilinear (resp. prebilinear) is that the kernel of $\bar{\Phi}$ contain $T^L_R$ (resp. $V^L_R$). Hence if $\Phi$ is bilinear (resp. prebilinear), then $\bar{\Phi}$ has a unique decomposition $\bar{\Phi} = \Psi \cdot \bar{\Phi}_0$, where $\bar{\Phi}_0$ is the natural mapping of $S^L_R$ onto $S^L_R/T^L_R$ (resp. $S^L_R/V^L_R$), and $\Psi$ is a homomorphism of $S/^L_R T^L_R = E^L \ominus F_R$ (resp. $S^L_R/V^L_R = E^L \oplus F_R$) into $M^L_R$. This $\Psi$ clearly meets the proposition. Conversely, given any homomorphism $\Psi$, there is a unique $\Phi$ which meets the proposition, namely the restriction to $E \times F$ of $\bar{\Phi} = \Psi \cdot \bar{\Phi}_0$. q.e.d.

## § 2. Left-right bilinearity over two rings

Prop. 1.6 may be interpreted as stating that $E^L \oplus F_R$ is the most general module onto which $E^L$ and $F_R$ have *associable* linear mappings, and that $E^L \ominus F_R$ is the most general such module with which $\mathring{A}$ commutes — in other words, which is an algebra-double module for the natural $\mathring{A}$-algebra $A_{\mathring{A}}$ on $A$. The definition of pretensorial product can be extended immediately to modules over any pair of rings with units, and the definition of tensorial product to modules over any pair of rings with units which underlie $C$-algebras. Pursuing the analogy, the corresponding notions of mappings which are "associatively $A$-left-linear, $B$-right-linear" and "associatively $A$-left-linear, $B$-right-linear, $C$-loosely-commutative", are abbreviated as follows:

**2.1 Definition:** Where $E^L$ is an $A$-left module, $F_R$ a $B$-right module, $M^L_R$ an $A$-left $B$-right module, a mapping $\Phi: E \times F \to M^L_R$ will be called ***A-left B-right prebilinear*** if it fulfills $P1$, $P2$, $A1$ and

$B2)$ $$\Phi(x, y\beta) = (\Phi(x, y))\, \beta , \qquad x \in E^L, y \in F_R, \beta \in B .$$

[9]) JACOBSON [5] Ch. V, 2, pp. 102—103 or BANDLER [1] § 6.

Where $S_R^L$ is the formal $A$-left $B$-right module generated by $E \times F$, and $V_R^L$ the submodule generated by $P_1$, $P_2$, $A_1$ and

$$B_2) \qquad (x, y\,\beta) - (x, y)\,\beta\,, \qquad \beta \in B,$$

the factor-module $S_R^L / V_R^L = E^L \oplus F_R$ will be called the *pretensorial product* of $E^L$ and $F_R$.

If $A$ and $B$ underlie $C$-algebras (defined by $\gamma \to \gamma\varepsilon$, $\gamma \to \zeta\gamma$ for $\gamma \in C$; $\varepsilon$, $\zeta$ the units of $A$, $B$ respectively), $\Phi$ will be called *$A_C$-left $B_C$-right bilinear* if it fulfills $P1$, $P2$, $A1$, $B2$ and

$$\mathring{X}1') \qquad (\gamma\varepsilon)\,(\Phi(x, y)) = (\Phi(x, y))\,(\zeta\,\gamma)\,, \qquad \gamma \in C\,.$$

Where $T_R^L$ is the submodule of $S_R^L$ generated by $P_1$, $P_2$, $A_1$, $B_2$ and

$$\mathring{X}_1') \qquad (\gamma\varepsilon)\,(x, y) - (x, y)\,(\zeta\gamma)\,, \qquad \gamma \in C\,,$$

the factor-module $S_R^L / T_R^L = E^L \ominus F_R = E^{Lc} \ominus F_{Rc}$ will be called the *tensorial product* of $E^L$ and $F_R$.

It is clear that, in the more elaborate notation, $E^{Lz} \ominus F_{Rz} = E^L \oplus F_R$, and that the tensorial product of Def. 1.4 is $E^{L\mathring{A}} \ominus F_{R\mathring{A}}$. Also immediate is

**2.2 Proposition:** *If $C$ commutes loosely*[10]) *with $M_R^L$ (that is, if $M_R^L$ is an algebra-double module $M_{Rc}^{Lc}$), then every prebilinear mapping into $M_R^L$ is bilinear.*

Rings $A$ and $B$ (resp. algebras $A_C$ and $B_C$) are *compatible*[11]) if they possess a non-trivial $M_R^L$ (resp. $M_{Rc}^{Lc}$); equivalent to this is the condition that $A \otimes B$ (resp. $A_C \otimes B_C$) be non-trivial[12]). Compatibility of the rings (resp. algebras) is of course a necessary condition that $E^L \oplus F_R$ (resp. $E^L \ominus F_R$) be non-trivial. It is also a sufficient condition that there exist a non-trivial product for some $E^L$ and $F_R$ (for example, $\check{A}^L$ and $\check{B}_R$), but it is not sufficient to guarantee this for every (non-trivial) $E^L$ and $F_R$, because (and this is so whether or not $A$ and $B$ are corelevant[13]), or in particular $B = A$) the annulling ideals of $E^L$ and $F_R$ may so combine as to absorb $A$ and $B$. (A simple example is: $A = B = Z$, $E$ the additive group of $Z/(2)$, $F$ the additive group of $Z/(3)$, with the obvious module-structures; the annulling ideal of $E^L \oplus F_R$ is $((2), (3)) = (1)$ in $A$ and $B$, and $E^L \oplus F_R$ is the zero-module.) An additional condition on $E^L$ and $F_R$ which is sufficient, but by no means necessary, is given in Cor. 5.6 below.

Substituting $B$ and $\tilde{B}$ for the right-operating $A$ and its $\tilde{A}$, and $B2$ etc. for $A2$ etc. in Props. 1.5 and 1.6, we obtain what it will be convenient to refer to as

**2.3 Proposition** – *the left-right embodiment theorem* for the two-ring (pre-) tensorial product with respect to two-ring (pre-)bilinear mappings. The content of this is clear from Prop. 1.6.

Where $E^{L'}$, $F_{R'}$ indicate the natural $C$-modules on $E$, $F$ (given by $\gamma x = \gamma\varepsilon \cdot x$, $y\gamma = y \cdot \zeta\gamma$, and *not* in general equivalent to $E^L = E^{Lc}$, $F_R = F_{Rc}$), we may

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10) $(\gamma\varepsilon)x = x(\zeta\gamma)$, $x \in M_R^L$, $\gamma \in C$.

11) Bandler [1] § 1.

12) Op. cit. Cor. 5.7 (resp. Props. 5.9, 4.1).

13) Op. cit. § 1.

let $E^{L'} \ominus F_{R'}$ generate a formal $A$-left $B$-right module $\bar{S}_R^L$ (isomorphic to $S_R^L/(P_1, P_2, \mathring{X}_1')_R^L$) and then factor by $\bar{A}_1$ and $\bar{B}_2$ to obtain (within an isomorphism) $E^L \ominus F_R$. Thus the tensorial product over $A$ and $B$ is obtainable from that over $C$ by *extending the rings of operators to $A$ and $B$ on their appropriate sides*, that is, by "introducing the rules" (i.e. adjoining the endomorphisms):

$$\alpha(x \ominus y) = (\alpha x \ominus y); \quad (x \ominus y)\,\beta = (x \ominus y\,\beta)\,.$$

Thus the theory of $\ominus$-products can doubtless be worked out as an annex to the well-known theory of $\otimes$-products over a single commutative ring, by checking the admissibility of the extra operators in each case, and demonstrating the carry-over of the principal theorems. The present paper takes the contrary course of developing the essentials of $\ominus$-theory *ex novo*; the $\otimes$-theorems follow as special cases, and are never assumed as known for purposes of proof[14]).

## § 3. Left-left (and right-right) bilinearity

Let $E^L$ be an $A$-left module, $F^L$ a $B$-left module (both, as always here, assumed to be unitary), and wherever appropriate assume $A$ and $B$ to underlie $C$-algebras $A_C$, $B_C$, given by $\gamma \to \gamma\varepsilon$, $\gamma \to \zeta\,\gamma$. Translation leads from Def. 2.1 and in the special cases from Defs. 1.1 and 1.2 to

**3.1 Definition:** A mapping $\Phi$ of $E \times F$ into an $A_Z \otimes B_Z$-left module $M^{LL}$ will be called *$A_C$-left $B_C$-left bilinear* if it fulfills $P1$, $P2$ and

$$\left.\begin{array}{ll} A\,1') & \Phi(\alpha x, y) = (\alpha \otimes \zeta)\,\Phi(x, y) \\ B2') & \Phi(x, \beta y) = (\varepsilon \otimes \beta)\,\Phi(x, y) \\ \mathring{X}1'') & (\gamma\varepsilon \otimes \zeta)\,\Phi(x, y) = (\varepsilon \otimes \zeta\,\gamma)\,\Phi(x, y) \end{array}\right\} \begin{array}{l} x \in E^L,\ y \in F^L \\ \alpha \in A,\ \beta \in B \\ \varepsilon \text{ unit of } A,\ \zeta \text{ unit of } B\,. \end{array}$$

It will be called *$A$-left $B$-left prebilinear* if it fulfills $P1$, $P2$, $A\,1'$ and $B2'$.

In particular, an $A$-left $A$-left prebilinear mapping will be called *left-left $A$-prebilinear*, and an $A^\circ_A$-left $A^\circ_A$-left bilinear mapping *left-left $A$-bilinear*.

It should be noted that the trio $A\,1'$, $B2'$, $\mathring{X}1''$ together is equivalent to the trio of $A1'$, $B2'$ and the more "intrinsic"

$$\mathring{X}'')\ \Phi(\gamma\varepsilon \cdot x, y) = \Phi(x, y \cdot \zeta\,\gamma)\,.$$

Corresponding to Props. 1.3 and 2.2 we have

**3.2 Proposition:** *If $M^{LL}$ is also an $A_C \otimes B_C$-left module (strictly: if $M$ naturally underlies an $A_C \otimes B_C$-left module equivalent to $M^{LL}$), then every prebilinear mapping into $M^{LL}$ is bilinear.*

*Proof:* The premise means that the kernel of the natural homomorphism $\chi_{\bar{\mathfrak{x}}}$ of $A_Z \otimes B_Z$ onto $A_C \otimes B_C$ is contained in the annulling ideal of $M^{LL}$. This

[14]) All properties of the tensorial product-*rings* $A \otimes B$, $A_C \otimes B_C$ used hitherto have been demonstrated elsewhere (Op. cit.) without the citation of any $\otimes$-module theorem. The same is true of all properties used below, with one exception here removed: the substantial identity of these product-rings with those traditionally defined (employed in Part II of Prop. 5.1 below) was shown (Op. cit., first footnote following Cor. 5.10) by citing the embodiment theorem for the commutative case; this is now a special instance of Prop. 1.6, itself a special case of Prop. 2.3.

kernel $\mathfrak{k}$ is generated in $A_Z \otimes B_Z$ by all forms

$$(\gamma\varepsilon \otimes \zeta) - (\varepsilon \otimes \zeta\, \gamma)\,, \qquad \gamma \in C\,,$$

which accordingly annul $\Phi(E \times F)$, so that $\mathring{X}1''$ is fulfilled. q.e.d.

**3.3 Definition:** Where $S^{LL}$ is the formal $A_Z \otimes B_Z$-left module generated by $E \times F$, $T^{LL}$ the submodule generated by the forms $P_1$, $P_2$ and

$$\left.\begin{array}{ll} A_1') & (\alpha x, y) - (\alpha \otimes \zeta)(x, y) \\ B_2') & (x, \beta y) - (\varepsilon \otimes \beta)(x, y) \\ \mathring{X}_1'') & (\gamma\varepsilon \otimes \zeta)(x, y) - (\varepsilon \otimes \zeta\gamma)(x, y) \end{array}\right\} \begin{array}{l} x \in E^L,\ y \in F^L \\ \alpha \in A,\ \ \beta \in B\,,\ \gamma \in C\,, \end{array}$$

and $V^{LL}$ the submodule generated by $P_1$, $P_2$, $A_1'$ and $B_2'$, the factor-module $S^{LL}/T^{LL}$ is the *tensorial product* $E^L \ominus F^L$, and the factor-module $S^{LL}/V^{LL}$ is the *pretensorial product* $E^L \oplus F^L$, of $E^L$ and $F^L$.

**3.4 Proposition:** *If $E^L$ is an $A$-left module, $F_R$ a $B$-right module, and $F^{\tilde{L}}$ the $\tilde{B}$-left module on $F$ given by*

$$\tilde{\beta} y = y\,\beta\,, \qquad y \in F\,,\quad B \ni \beta = \tilde{\beta} \in \tilde{B}\,,$$

*then $E^L \ominus F_R$ and $E^L \ominus F^{\tilde{L}}$ are fully equivalent corresponding modules. To every $A_C$-left $B_C$-right bilinear mapping $\Phi$ of $E \times F$ into any $M^L_R$ corresponds an $A_C$-left $\tilde{B}_C$-left bilinear mapping $\Phi'$ of $E \times F$ into the corresponding $M^{L\tilde{L}}$, and vice versa, with the same values on the same arguments.*

*Proof:* This follows by checking the following facts[15]: the formal $A_Z \otimes \tilde{B}_Z$-left module generated by $E \times F$ is the corresponding module $S^{L\tilde{L}}$ to $S^L_R$; the defining kernel of the tensorial product is the corresponding module $T^{L\tilde{L}}$ to $T^L_R$; the image module $\Phi'(E \times F) \subseteq M^{L\tilde{L}}$ is the corresponding module to $\Phi(E \times F) \subseteq M^L_R$. q.e.d.

For $C = Z$ we have the analagous statements for the pretensorial product and prebilinear mappings.

Necessary that the pretensorial product (or for that matter $S^{LL}$) be non-trivial is that $A$ and $B$ be compatible rings; necessary that the tensorial product be non-trivial is that $A_C$, $B_C$ be compatible algebras; these conditions are sufficient that the corresponding products be non-trivial for some choices of $E^L$ and $F^L$. This is an obvious consequence of the hemianti-isomorphy[16]) between $A_C \otimes B_C$ and $A_C \otimes \tilde{B}_C$ and of the above.

**3.5 Proposition** — *the left-left embodiment theorem* follows from Props. 2.3 and 3.4. It may of course also be proved directly, and is indeed the simplest form of the theorem, since the $A \otimes B$-left linear independence of the generators of $S^{LL}$ is immediate; Prop. 2.3 then follows from this and 3.4, and one can dispense with any form of 1.5.

The right-right case is of course entirely analagous (dual) to the left-left one.

In the case of the tensorial product for $A = B = C = \mathring{A}$, which embodies left-left $\mathring{A}$-bilinearity, the mapping $\mathring{\alpha} \otimes \mathring{\alpha}' \to \mathring{\alpha}\mathring{\alpha}'$ is a natural isomorphism of

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[15]) Cf. discussion of corresponding modules in op. cit. § 6.

[16]) Loc. cit.

$\mathring{A}^{\circ}_{\mathring{A}} \otimes \mathring{A}^{\circ}_{\mathring{A}}$ onto $\mathring{A}$, and the natural $\mathring{A}$-module on the underlying group of $E^L \ominus F^L$ is fully equivalent to $E^L \ominus F^L$ itself. This single module is the tensorial $\otimes$-product of the received definition[17]), so that again the present definition of $\ominus$ coincides essentially with that of $\otimes$ for the case of a single commutative ring.

## § 4. Multilinearity

Where $A_1, \ldots, A_l$ are rings underlying $C$-algebras $A_{1C}, \ldots, A_{lC}$, the symbol $\bigotimes\limits_{l} A_{\lambda C}$ or $\bigotimes\limits_{\lambda=1}^{l} A_{\lambda C}$ idicates $A_1$ if $l = 1$, and $\left(\bigotimes\limits_{l-1} A_{\lambda C}\right) \otimes A_{lC}$ if $l > 1$. For present purposes, the symbol may be taken to represent $Z$ for $l = 0$. In the following definition, the results of Prop. 5.1 below are anticipated to the extent of writing the generators of this ring as $\alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes \ldots, (\alpha_\lambda \in A_\lambda)$ rather than $\big(((\alpha_1 \otimes \alpha_2) \otimes \alpha_3) \otimes \ldots\big)$.

**4.1 Definition:** Where $A_\lambda$ $(\lambda = 1, \ldots, l)$ and $B^\varrho$ $(\varrho = 1, \ldots, r)$ are rings with units $\varepsilon_\lambda$, $\zeta^\varrho$, and underlie $C$-algebras $A_{\lambda C}$, $B^\varrho_C$ and (for each $\lambda$, $\varrho$) $E^\lambda$ is a unitary $A_\lambda$-left module, $F_\varrho$ a unitary $B^\varrho$-right module, a mapping $\Phi$ of $\left(\mathop{\times}\limits_{\lambda=1}^{l} E^\lambda\right) \times \left(\mathop{\times}\limits_{\varrho=1}^{r} F_\varrho\right)$ into a $\left(\bigotimes\limits_{l} A_{\lambda Z}\right)$-left, $\left(\bigotimes\limits_{r} B^\varrho_Z\right)$-right module $M^{1,\ldots,l}_{1,\ldots,r}$ will be called *$(A_{1C}, \ldots, A_{lC})$-left, $(B^1_C, \ldots, B^r_C)$-right multilinear* if it fulfills

$$
\begin{aligned}
P'')\quad & \Phi(\ldots, x^\lambda + x'^\lambda, \ldots; \ldots, y_\varrho + y'_\varrho, \ldots) \\
& = \Phi(\ldots, x^\lambda, \ldots; \ldots, y_\varrho, \ldots) + \Phi(\ldots, x'^\lambda, \ldots; \ldots, y_\varrho, \ldots) + \\
& + \Phi(\ldots, x^\lambda, \ldots; \ldots, y'_\varrho, \ldots) + \Phi(\ldots, x'^\lambda, \ldots; \ldots, y'_\varrho, \ldots) \\
A'')\quad & \Phi(\alpha_1 x^1, \ldots, \alpha_l x^l; y_1 \beta^1, \ldots, y_r \beta^r) \\
& = (\alpha_1 \otimes \cdots \otimes \alpha_l)\, \Phi(x^1, \ldots, x^l; y_1, \ldots, y_r)\, (\beta^1 \otimes \cdots \otimes \beta^r) \\
\mathring{X}''')\quad & (\gamma_1 \varepsilon_1 \otimes \cdots \otimes \gamma_l \varepsilon_l)\, (\Phi)\, (\zeta^1 \gamma^1 \otimes \cdots \otimes \zeta^r \gamma^r) = (\gamma \varepsilon_1 \otimes \varepsilon_2 \otimes \cdots \otimes \varepsilon_l)\, (\Phi) \\
& = (\Phi)\, (\zeta^1 \otimes \cdots \otimes \zeta^{r-1} \otimes \zeta^r \gamma)
\end{aligned}
$$

for

$$
\left.
\begin{array}{l}
x^\lambda, x'^\lambda \in E^\lambda; y_\varrho, y'_\varrho \in F_\varrho \\
\alpha_\lambda \in A_\lambda; \beta^\varrho \in B^\varrho \\
\gamma_\lambda, \gamma^\varrho \in C \\
\gamma = \prod\limits_{\lambda=1}^{l} \gamma_\lambda \cdot \prod\limits_{\varrho=1}^{r} \gamma^\varrho
\end{array}
\right\}
\begin{array}{l}
\lambda = 1, \ldots, l \\
\varrho = 1, \ldots, r.
\end{array}
$$

If $\Phi$ fulfills $P''$ and $A''$ it will be called *$(A_1, \ldots, A_l)$-left $(B^1, \ldots, B^r)$-right premultilinear*. If (for all $\lambda$, $\varrho$) $A_{\lambda C} = A_C$, $B^\varrho_C = B_C$ (resp. $A_\lambda = A$, $B^\varrho = B$), the mapping will be called *$l$-fold $A_C$-left, $r$-fold $B_C$-right multilinear* (resp. *$l$-fold $A$-left, $r$-fold $B$-right premultilinear*). If furthermore $A_C = B_C = A^{\circ}_{\mathring{A}}$ (resp. $A = B$), then *$l$-fold left, $r$-fold right $A$-multilinear* (resp. . . . *$A$-premultilinear*).

17) Cf. footnote 6 above.

The necessary modifications of this definition for $l = 0$ or $r = 0$ are obvious, as are also the forms which generate the defining kernels of the *tensorial* and *pretensorial products* which embody these mapping concepts. It is also clear that we have, as usual,

**4.2 Proposition:** *If $M^{1,\ldots,l}_{1,\ldots,r}$ is also a (i.e. if $M$ naturally underlies an equivalent) $\bigotimes_{l} A_{\lambda C}$-left, $\bigotimes_{r} B^{\varrho}_{C}$-right module, then every premultilinear mapping into $M^{1,\ldots,l}_{1,\ldots,r}$ is multilinear.*

## § 5. Some properties of the tensorial and pretensorial products

In this section where we have to deal with several distinct modules over (in general) as many distinct rings, we adopt the notation of § 4, and furthermore indicate by $\tilde{B}_\varrho$ the opposite ring to $B^\varrho$, and by $F^{\tilde{\varrho}}$ the $\tilde{B}_\varrho$-left module on $F$ naturally equivalent, or corresponding, to $F_\varrho$, namely with

$$\tilde{\beta}_\varrho y^\varrho = y_\varrho \beta^\varrho, \quad y_\varrho = y^\varrho \in F, B^\varrho \ni \beta^\varrho = \tilde{\beta}_\varrho \in \tilde{B}_\varrho.$$

**5.1 Proposition** *(Associativity of $\ominus$ and of $\oplus$ for left-modules): $E^1 \ominus (E^2 \ominus E^3)$ is naturally isomorphic to $(E^1 \ominus E^2) \ominus E^3$. Consequently $E^1 \oplus (E^2 \oplus E^3)$ is naturally isomorphic to $(E^1 \oplus E^2) \oplus E^3$.*

*Proof:* I. We make the assumption that, for the given rings $A_\lambda$ $(\lambda = 1, 2, 3)$ and $C$ (including the case $C = Z$), $A_{1C} \otimes (A_{2C} \otimes A_{3C})$ is naturally isomorphic to $(A_{1C} \otimes A_{2C}) \otimes A_{3C}$, and accordingly write these products indifferently as $A_{1C} \otimes A_{2C} \otimes A_{3C}$, with generators $\alpha_1 \otimes \alpha_2 \otimes \alpha_3$. Then the natural mappings

$$\varphi_1 : \alpha_1 \to \alpha_1 \otimes \varepsilon_2 \otimes \varepsilon_3$$

$$\varphi_2 : \alpha_2 \to \varepsilon_1 \otimes \alpha_2 \otimes \varepsilon_3$$

$$\varphi_3 : \alpha_3 \to \varepsilon_1 \otimes \varepsilon_2 \otimes \alpha_3$$

are homomorphisms of $A_1$, $A_2$, $A_3$ respectively into their product, with kernels $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$. Since both $E^1 \ominus (E^2 \ominus E^3)$ and $(E^1 \ominus E^2) \ominus E^3$ are $A_{1C} \otimes A_{2C} \otimes A_{3C}$-left modules, $\mathfrak{a}_\lambda E^\lambda$ is contained in the kernels of the natural mappings of $E^\lambda$ into each of these product-modules. There is accordingly no loss in replacing $A_{\lambda C}$ by $A_{\lambda C}/\mathfrak{a}_\lambda$ and $E^\lambda$ by $E^\lambda/(\mathfrak{a}_\lambda E^\lambda)$; in other words we may take the $\varphi_\lambda$ to be isomorphisms of the (new) rings $A_\lambda$. Then

$$\varphi_2' : \alpha_2 \to \alpha_2 \otimes \varepsilon_3$$

is also an isomorphism, and the $(A_{2C} \otimes \varepsilon_3)$-left module $E^{2'}$ on the underlying group of (the new) $E^2$, given by

$$(\alpha_2 \otimes \varepsilon_3)\, x^2 = \alpha_2 x^2, \qquad E^{2'} \ni x^2 \in E^2$$

is fully equivalent to (the new) $E^2$. For any fixed $x^3 \in E^3$, the mapping

$$\Phi_{x^3} : (x^1, x^2) \to x^1 \ominus (x^2 \ominus x^3)$$

of $E^1 \times E^{2'}$ into $E^1 \ominus (E^2 \ominus E^3)$ is $A_{1C}$-left, $(A_{2C} \otimes \varepsilon_3)_C$-left bilinear, for it clearly fulfills $P1$ and $P2$, and since it takes

$$(\alpha_1 x^1, (\alpha_2 \otimes \varepsilon_3)\, x^2)$$

into

$$\alpha_1 x^1 \ominus ((\alpha_2 \otimes \varepsilon_3)(x^2 \ominus x^3)) = \alpha_1 x^1 \ominus (\alpha_2 x^2 \ominus x^3)$$
$$= (\alpha_1 \otimes \alpha_2 \otimes \varepsilon_3)(x^1 \ominus (x^2 \ominus x^3)),$$

it also fulfills $A1'$, $B2'$ and $\mathring{X}''$, the last because

$$\gamma\varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_3 = \varepsilon_1 \otimes \gamma\varepsilon_2 \otimes \varepsilon_3, \qquad \gamma \in C.$$

Accordingly, by the embodiment theorem 3.5,

$$\Psi_{x^3}: (x^1 \ominus x^2) \to x^1 \ominus (x^2 \ominus x^3)$$

defines a homomorphism of $E^1 \ominus E^{2'}$ – and hence a single-valued mapping of $E^1 \ominus E^2$ – into $E^1 \ominus (E^2 \ominus E^3)$. Therefore a single-valued mapping of $(E^1 \ominus E^2) \times E^3$ into $E^1 \ominus (E^2 \ominus E^3)$ is defined by

$$\Phi: (x^1 \ominus x^2, x^3) \to \Psi_{x^3}(x^1 \ominus x^2) = x^1 \ominus (x^2 \ominus x^3).$$

But this is in fact $(A_{1C} \otimes A_{2C})_C$-left $A_{3C}$-left bilinear, for besides $P1$ and $P2$ it fulfills $A1'$, $B2'$, $\mathring{X}1''$, since it takes

$$((\alpha_1 \otimes \alpha_2)(x^1 \ominus x^2), \alpha_3 x^3) = (\alpha_1 x^1 \ominus \alpha_2 x^2, \alpha_3 x^3)$$

into

$$\alpha_1 x^1 \ominus (\alpha_2 x^2 \ominus \alpha_3 x^3) = (\alpha_1 \otimes \alpha_2 \otimes \alpha_3)(x^1 \ominus (x^2 \ominus x^3)).$$

Every element of $E^1 \ominus (E^2 \ominus E^3)$ can be obtained as a sum of elements of this last form. Therefore the corresponding homomorphism $\Psi$ given by

$$\Psi: (x^1 \ominus x^2) \ominus x^3 \to x^1 \ominus (x^2 \ominus x^3)$$

is a natural homomorphism of $(E^1 \ominus E^2) \ominus E^3$ onto $E^1 \ominus (E^2 \ominus E^3)$, if these are considered as $A_{1C} \otimes A_{2C} \otimes A_{3C}$-left modules, and *a fortiori* if considered as $A_{1Z} \otimes A_{2Z} \otimes A_{3Z}$-left modules. Similarly, beginning with an $(\varepsilon_1 \otimes A_{2C})$-left module $E^{2''}$ and a fixed $x^1 \in E^1$, we arrive at a natural homomorphism $\Psi'$ of $E^1 \ominus (E^2 \ominus E^3)$ onto $(E^1 \ominus E^2) \ominus E^3$. Then $\Psi\Psi'$ and $\Psi'\Psi$ are the identities of these two modules, and hence $\Psi$ and $\Psi'$ are isomorphisms, establishing the first and (for $C = Z$) second isomorphies of the proposition on the initial assumption.

II) Where $\eta$ is the unit of $C$, $C_C \otimes C_C$ is naturally isomorphic to $C$ under the correspondence

$$\gamma \otimes \gamma' = \gamma\gamma' \otimes \eta = \eta \otimes \gamma\gamma' \leftrightarrow \gamma\gamma'.$$

Thus the case where all three rings are $C$ fulfills the initial assumption of (I). For any three $C$-algebras we then have from the proposition a natural isomorphy between the modules $\check{A}_1 \ominus (\check{A}_2 \ominus \check{A}_3)$ and $(\check{A}_1 \ominus \check{A}_2) \ominus \check{A}_3$. But[18]) the rings $A_{1C} \otimes (A_{2C} \otimes A_{3C})$ and $(A_{1C} \otimes A_{2C}) \otimes A_{3C}$ are the underlying groups of these modules equipped respectively with the multiplications

$$(\alpha_1 \ominus (\alpha_2 \ominus \alpha_3))(\alpha_1' \ominus (\alpha_2' \ominus \alpha_3')) = \alpha_1\alpha_1' \ominus ((\alpha_2 \ominus \alpha_3)(\alpha_2' \ominus \alpha_3'))$$
$$= \alpha_1\alpha_1' \ominus (\alpha_2\alpha_2' \ominus \alpha_3\alpha_3'),$$
$$((\alpha_1 \ominus \alpha_2) \ominus \alpha_3)((\alpha_1' \ominus \alpha_2') \ominus \alpha_3') = ((\alpha_1 \ominus \alpha_2)(\alpha_1' \ominus \alpha_2')) \ominus \alpha_3\alpha_3'$$
$$= (\alpha_1\alpha_1' \ominus \alpha_2\alpha_2') \ominus \alpha_3\alpha_3'.$$

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[18]) Cf. footnote 14 above.

Thus the natural module-isomorphism is also a ring-isomorphism and an algebra-isomorphism; in other words, the initial assumption of (I) is always fulfilled. This completes the proof.

Where $M^2_3$ is an $A_{2C}$-left $B^3_C$-right algebra-double module, and $M^{2\tilde{3}}$ the corresponding $A_{2C} \otimes \tilde{B}_{3C}$-left module, the tensorial product $E^1 \ominus M^{2\tilde{3}}$ is (by part II of the proof of the above) a left-module over $A_{1C} \otimes A_{2C} \otimes \tilde{B}_{3C} = (A_{1C} \otimes A_{2C}) \otimes \tilde{B}_{3C}$. Accordingly, the first of the objects in the following definition exists; by duality so does the second, while the existence of the third follows from[19])

$$\tilde{B}_C \otimes A_C = \psi \tilde{B} \vee \varphi A = \varphi A \vee \psi \tilde{B} = A_C \otimes \tilde{B}_C\,.$$

**5.2 Definition:** 1) $E^1 \ominus M^2_3$ is the $A_{1C} \otimes A_{2C}$-left, $B^3_C$-right module corresponding to $E^1 \ominus M^{2\tilde{3}}$.

2) $M^1_2 \ominus F_3$ is the $A_{1C}$-left, $B^2_C \otimes B^3_C$-right module corresponding to $M^{1\tilde{2}} \ominus F^{\tilde{3}}$ (or equivalently and dually, to $M_{\tilde{1}2} \ominus F_3$).

3) $F_R \ominus E^L$ is the $A$-left $B$-right module corresponding to $F^{\tilde{L}} \ominus E^L$.

**5.3 Corollary** *(Associativity of $\ominus$ and hence of $\oplus$ for arbitrary groupings of left and right modules): Where, independently, $J$ is either $E^1$ or $F_1$, $K$ is either $E^2$ or $F_2$, and $L$ is either $E^3$ or $F_3$, there is a natural module-isomorphism of*

$$J \ominus (K \ominus L) \quad \textit{onto} \quad (J \ominus K) \ominus L\,.$$

*Proof:* The statement is valid by the above proposition when all three are left-modules, hence by duality when all three are right-modules. It will suffice to prove it for the cases when one of the three is a right-module, whence it will follow by duality for the remaining possibilities. Writing $=$ for natural isomorphisms and $\leftrightarrow$ for correspondences, we have by Def. 5.2, Prop. 3.4 and Prop. 5.1:

$$\begin{aligned} E^1 \ominus (E^2 \ominus F_3) \leftrightarrow E^1 \ominus (E^2 \ominus F^{\tilde{3}}) = (E^1 \ominus E^2) \ominus F^{\tilde{3}} \leftrightarrow \\ = (E^1 \ominus E^2) \ominus F_3\,. \end{aligned}$$

The first and last members are naturally isomorphic because the correspondences $\leftrightarrow$ involved are isomorphisms of the underlying groups and natural matchings of the operators, and because both members are $A_{1C} \otimes A_{2C}$-left, $B_{3C}$-right modules. Similarly

$$\begin{aligned} E^1 \ominus (F_2 \ominus E^3) \leftrightarrow E^1 \ominus (F^{\tilde{2}} \ominus E^3) = (E^1 \ominus F^{\tilde{2}}) \ominus E^3 \leftrightarrow \\ = (E^1 \ominus F_2) \ominus E^3, \\ F_1 \ominus (E^2 \ominus E^3) \leftrightarrow F^{\tilde{1}} \ominus (E^2 \ominus E^3) = (F^{\tilde{1}} \ominus E^2) \ominus E^3 \leftrightarrow \\ = (F_1 \ominus E^2) \ominus E^3\,. \qquad \text{q. e. d.} \end{aligned}$$

It is clear that this associative property can be extended to any number of factors by induction[20]). The natural isomorphisms involved will here be

[19]) Bandler [1] § 5. Here $\vee$ indicates "ring generated by union"; $\varphi$ and $\psi$ are the natural mappings of $A$ and $\tilde{B}$ into their tensorial product-ring.

[20]) Thus generalizing a property of the $\otimes$-product over a single commutative ring, sketched in Bourbaki [3] § 1, No 7, p. 18.

considered as equalities, and the products written without parentheses. It is also clear that $\left(\bigoplus_{l} E^{\lambda}\right) \ominus \left(\bigoplus_{r} F_{\varrho}\right)$ meets the embodiment theorem for the multilinear mappings of Def. 4.1.

**5.4 Proposition** *(Commutativity of* $\ominus$ *and consequently of* $\oplus$*)*[21]*: The following natural isomorphisms hold:*

$$E^1 \ominus E^2 \cong E^2 \ominus E^1$$

$$E^1 \ominus F_2 \cong F_2 \ominus E^1$$

$$F_1 \ominus F_2 \cong F_2 \ominus F_1 \,.$$

*Proof:* From the symmetry of the definition (Def. 3.3) the underlying group of $E^1 \ominus E^2$ is naturally isomorphic to that of $E^2 \ominus E^1$ under the mapping

$$\Sigma x^1 \ominus x^2 \to \Sigma x^2 \ominus x^1 \,.$$

This takes

$$\Sigma(\varphi\alpha_1 \cdot \psi\alpha_2)\,(x^1 \ominus x^2) = \Sigma \alpha_1 x^1 \ominus \alpha_2 x^2,\ \Sigma\varphi\alpha_1 \cdot \psi\alpha_2 \in A_{1\,C} \otimes A_{2\,C}\,,$$

into

$$\Sigma \alpha_2 x^2 \ominus \alpha_1 x^1 = \Sigma(\psi\alpha_2 \cdot \varphi\alpha_1)\,(x^2 \ominus x^1) = \Sigma(\varphi\alpha_1 \cdot \psi\alpha_2)\,(x^2 \ominus x^1)\,,$$

and is accordingly a module-homomorphism. By duality we have the same result for $F_1 \ominus F_2$, and using Def. 5.2 and Prop. 3.4 for

$$E^1 \ominus F_2 \leftrightarrow E^1 \ominus \tilde{F^2} \cong \tilde{F^2} \ominus E^1 \leftrightarrow$$

$$\cong F_2 \ominus E^1 \,.$$ q. e. d.

It is of course legitimate to use the isomorphisms of this proposition as identifications, but then a better notation than $x^1 \ominus x^2$ is needed, such as

$$\varkappa_1 x^1 \cdot \varkappa_2 x^2 = \varkappa_2 x^2 \cdot \varkappa_1 x^1 \,,$$

where $\varkappa_1$, $\varkappa_2$ are the natural mappings of $E^1$, $E^2$ into their product-module, accompanied (as in the proof) by the notation $\varphi\alpha_1 \cdot \psi\alpha_2$ or $\varphi_1\alpha_1 \cdot \varphi_2\alpha_2$ for the generators of the operating ring. Otherwise there is danger of setting, say, $(\alpha_1 \otimes \alpha_2)\,(x^2 \ominus x^1)$ equal to $\alpha_1 x^2 \ominus \alpha_2 x^1$, which is in general meaningless if $A_1$ and $A_2$ are not the same ring, and erroneous if they are. A good standard practice for ordinary purposes, permitting the retention of the positional $\ominus$-notation, is to use the commutativity to "comb out" mixed products of left and right (or of left and left-wave) modules into an ordered product of left-modules times an ordered product of right (or left-wave) modules, leaving the orders within these two factors undisturbed.

**5.5 Proposition** *(Distributivity of* $\ominus$ *and hence of* $\oplus$ *with respect to* $\oplus$*)*[22]*: The (pre-) tensorial product of the direct sum of any collection of A-left or of A-right modules with the direct sum of any collection of B-left or of B-right modules is the direct sum of the (pre-) tensorial products of each A-module with each B-module.*

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[21]) Generalization of Prop. 4 in op. cit. § 1, No 3, p. 7.

[22]) Generalization of Prop. 7, loc. cit. p. 9.

*Proof:* For definiteness, suppose $E^L = \bigoplus_{\iota \in I} E^\iota$ an $A$-left module and $F^L = \bigoplus_{\varkappa \in K} F^\varkappa$ a $B$-left module, and let $S^{\iota\varkappa}$ be the formal $A_Z \otimes B_Z$-left module generated by $E^\iota \times F^\varkappa$, $T^{\iota\varkappa}$ the submodule generated by those forms $P_1, \ldots, \mathring{X}_1''$ which lie in $S^{\iota\varkappa}$,

$$G^{\iota\varkappa} = S^{\iota\varkappa}/T^{\iota\varkappa} = E^\iota \ominus F^\varkappa,$$

$$G^{LL} = \bigoplus_{(\iota,\varkappa) \in I \times K} G^{\iota\varkappa},$$

and

$$Q^{LL} = S^{LL}/T^{LL} = E^L \ominus F^L.$$

The elements of $S^{LL}$, $G^{LL}$ and $Q^{LL}$ are finite sums which may be written respectively

$$\Sigma(\alpha_\iota \otimes \beta_\varkappa)(x^\iota, y^\varkappa), \quad \Sigma(\alpha_\iota \otimes \beta_\varkappa)(x^\iota, y^\varkappa)', \quad \Sigma(\alpha_\iota \otimes \beta_\varkappa)(x^\iota, y^\varkappa)''.$$

Since each $T^{\iota\varkappa}$ lies in $T^{LL}$, the mapping

$$(x^\iota, y^\varkappa)' \to (x^\iota, y^\varkappa)''$$

defines a homomorphism of $G^{LL}$ onto $Q^{LL}$; but the mapping

$$(x^\iota, y^\varkappa) \to (x^\iota, y^\varkappa)'$$

defines a bilinear mapping $\Phi$ of $S^{LL}$ onto $G^{LL}$, so that by the embodiment theorem

$$\Psi: (x^\iota, y^\varkappa)'' \to (x^\iota, y^\varkappa)'$$

defines a homomorphism of $Q^{LL}$ onto $G^{LL}$. Whence

$$(x^\iota, y^\varkappa)'' \leftrightarrow (x^\iota, y^\varkappa)'$$

is one-to-one and defines a natural isomorphism between $Q^{LL}$ and $G^{LL}$, which may be used as an identity. The left-right and right-right cases are precisely analogous. q. e. d.

**5.6 Corollary:** *If $E^L$ has a base of $\overline{\overline{I}}$ elements over $A$, and $F^L$ (resp. $F_R$) has a base of $\overline{\overline{K}}$ elements over $B$, then $E^L \ominus F^L$ is isomorphic to $((\check{A}_C \otimes \check{B}_C)^{I \times K})^{LL}$ (resp. $E^L \ominus F_R$ is isomorphic to $((\check{A}_C \otimes \check{B}_C)^{I \times K})^L_R$), where $I$, $K$ are any sets of powers $\overline{\overline{I}}$, $\overline{\overline{K}}$.*

*Proof:* By a well-known theorem for left-modules[23]), $E^L$ is, on the hypothesis above, isomorphic to $(\check{A}^I)^L = \bigoplus_{\iota \in I} \check{A}^\iota$, under the mapping

$$e^\iota \to \varepsilon^\iota,$$

where $\{e^\iota\}$, $\iota \in I$, is the base of $E^L$ and $\varepsilon^\iota$ the unit of $\check{A}^\iota$. Similarly $F^L$ is isomorphic to $\bigoplus_{\varkappa \in K} \check{B}^\varkappa$, and $F_R$ to $\bigoplus_{\varkappa \in K} \check{B}_\varkappa$. By Prop. 5.5, $E^L \ominus F^L$ is thus isomorphic to[24])

$$\bigoplus_{(\iota,\varkappa) \in I \times K} \check{A}^\iota \ominus \check{B}^\varkappa = \bigoplus_{(\iota,\varkappa) \in I \times K} (\check{A}_C \otimes \check{B}_C)^{\iota\varkappa} = ((\check{A}_C \otimes \check{B}_C)^{I \times K})^{LL},$$

[23]) Bourbaki [2] § 1, No 8, Prop. 9, p. 15.
[24]) Cf. proof of part II of Prop. 5.1.

and $E^L \ominus F_R$ to

$$\bigoplus_{(\iota,\varkappa)\in I\times K} \check{A}^\iota \ominus \check{B}_\varkappa = \bigoplus_{(\iota,\varkappa)\in I\times K} (\check{A}_C \otimes \check{B}_C)^\iota_\varkappa = ((\check{A}_C \otimes \check{B}_C)^{I\times K})^L_R . \qquad \text{q. e. d.}$$

**5.7 Corollary**[25]: *If* $\{e^\iota\}$, $\iota \in I$, *is a base of* $E^L$, *and* $\{f^\varkappa\}$, $\varkappa \in K$, *is a base of* $F^L$, *and if* $A_C$ *and* $B_C$ *are compatible, then* $\{e^\iota \ominus f^\varkappa\}$, $(\iota, \varkappa) \in I \times K$, *is a base of* $E^L \ominus F^L$.

*Proof:* By hypothesis $I \times K$ is non-empty and $A_C \otimes B_C$ non-trivial, so that $((\check{A}_C \otimes \check{B}_C)^{I\times K})^{LL}$ is non-trivial and (by the theorem on single modules cited in the previous corollary) possesses a base $\{\eta^{\iota\varkappa}\}$, where $\eta^{\iota\varkappa}$ is the unit of $(A_C \otimes B_C)^{\iota\varkappa}$. But (where $\varepsilon^\iota$ is the unit of $A^\iota$ and $\zeta^\varkappa$ the unit of $B^\varkappa$) $\eta^{\iota\varkappa} = \varepsilon^\iota \otimes \zeta^\varkappa$, and this is the image of $e^\iota \ominus f^\varkappa$ by the isomorphism of the previous corollary.

q. e. d.

*Remark:* This corollary extends immediately to the case of $E^L \ominus F_R$ if we define a base of a double module as a set of generators satisfying the following independence criterion[26]).

**5.8 Definition:** A family $\{x^\iota_\varkappa\}$, $(\iota, \varkappa) \in I \times K$, in $M^L_R$ is $A_C$*-left* $B_C$*-right bilinearly independent (*resp. *A-left B-right prebilinearly independent)* if and only if, for all (finite) sets $N$, from the vanishing of the weak sum

$$\sum_{(\iota,\varkappa)\in I\times K} \sum_{\nu\in N} \alpha^\nu_\iota x^\iota_\varkappa \beta^\varkappa_\nu = 0$$

follows

$$\sum_{\nu\in N} \alpha^\nu_\iota \otimes \beta^\varkappa_\nu = 0$$

in $A_C \otimes B_C$ (resp. in $A \otimes B$).

This criterion of bilinear independence includes the weaker one of $A/\mathfrak{a}_{\mathfrak{u}+\mathfrak{x}}$-left linear independence plus $B/\mathfrak{b}_{\mathfrak{u}+\mathfrak{x}}$-right linear independence (where $\mathfrak{a}_{\mathfrak{u}+\mathfrak{x}}$, $\mathfrak{b}_{\mathfrak{u}+\mathfrak{x}}$ are the kernels of the natural mappings of $A$, $B$ into $A_C \otimes B_C$)[27]). It does not, in general, guarantee either $A$-left linear or $B$-right linear independence.

A family is prebilinearly independent in $M^L_R$ if and only if it is $A \otimes \tilde{B}$-left linearly independent in $M^{L\tilde{L}}$. Thus this criterion completes the full-equivalence relationship between corresponding modules.

## Literature


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[25]) Generalization of Cor. 2 in BOURBAKI [3] § 1, No 3, p. 11.

[26]) Cf. BANDLER [1] § 6.

[27]) Loc. cit.