# A Generalization of a Matrix Trace Inequality

**ARTICLE** *in* JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS · FEBRUARY 1995

Impact Factor: 1.12 · DOI: 10.1006/jmaa.1995.1056



**1 AUTHOR:**

Xin Min Yang
Chongqing Normal University

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **189**, 897–900 (1995)

# Note

# A Generalization of a Matrix Trace Inequality

XIN-MIN YANG

*Department of Mathematics, Chongqing Normal University, Chongqing 630047, People's Republic of China*

*Submitted by E. R. Love*

Received June 24, 1993

In an earlier paper, Yang solved a problem set by R. Bellman. Recently Neudecker (*J. Math. Anal. Appl.* **166** (1992), 302–303) proved Yang's result using a different method. In this note, we give a new proof of Yang's result and generalize it to a generalized positive semidefinite matrix. 

First, we give a new proof of Yang's result.

LEMMA 1. *If $A$ is a positive semidefinite matrix, then*

$$\operatorname{tr}(A^2) \le (\operatorname{tr} A)^2.$$

Lemma 1 is obvious.

LEMMA 2. *If $C$ and $D$ are two positive semidefinite matrices of the same order, then*

$$2 \operatorname{tr}(CD) \le \operatorname{tr}(C^2) + \operatorname{tr}(D^2).$$

Obviously $\operatorname{tr}((C - D)^2) = \operatorname{Tr}(C^2) - \operatorname{tr}(CD) - \operatorname{tr}(DC) + \operatorname{tr}(D^2) = \operatorname{tr}(C^2) - 2\operatorname{tr}(CD) + \operatorname{tr}(D^2)$ and $\operatorname{tr}((C - D)^2) \ge 0$, so Lemma 2 is true.

THEOREM 1. *If $A$ and $B$ are two positive semidefinite matrices of the same order, then*

(i) $\operatorname{tr} AB \ge 0$



0022-247X/95 $6.00

*and*

(ii) $(\operatorname{tr} AB)^{1/2} \leq (\operatorname{tr} A + \operatorname{tr} B)/2$.

*Proof.* Let $A^{1/2}$ be a positive semidefinite square root of $A$ (see [3]). Since $B$ is a positive semidefinite matrix, we know that $A^{1/2}BA^{1/2}$ is a positive semidefinite matrix. Hence,

$$\operatorname{tr}(AB) = \operatorname{tr}(A^{1/2}BA^{1/2}) \geq 0.$$

If $A = 0$ or $B = 0$, (ii) trivially holds.

If $A \neq 0$ and $B \neq 0$, let $C = A/(\operatorname{tr} A)$ and $D = B/(\operatorname{tr} B)$, the denominators being non-zero.

From Lemma 2 we obtain

$$2 \operatorname{tr}(AB)/((\operatorname{tr} A)\ (tr\ B)) \leq \operatorname{tr}(A)^2/(\operatorname{tr} A)^2 + \operatorname{tr}(B)^2/(\operatorname{tr} B)^2. \tag{1}$$

By Lemma 1 and (1), we have

$$\operatorname{tr}(AB) \leq (\operatorname{tr} A)\ (\operatorname{tr} B). \tag{2}$$

Finally, $(\operatorname{tr} A + \operatorname{tr} B)^2 - 4(\operatorname{tr} A)\ (tr\ B) = (\operatorname{tr} A - \operatorname{tr} B)^2 \geq 0$, gives

$$\operatorname{tr}(AB) \leq (\operatorname{tr} A)\ (\operatorname{tr} B) \leq (\operatorname{tr} A + \operatorname{tr} B)^2/4,$$

which proves (ii).

Next, we generalize Yang's result to a generalized positive semidefinite matrix.

DEFINITION. Let $A \in R^{n\times n}$. If for all $x \in R^n\backslash\{0\}$,

$$x'\ Ax \geq 0 \qquad (>0),$$

then $A$ is said to be a generalized positive semidefinite (definite) matrix.

LEMMA 3. *Let* $A \in R^{n\times n}$, $S(A) := (A + A')/2$, *and* $T(A) = (A - A')/2$, *then* $\operatorname{tr} S(A) = \operatorname{tr} A$, $\operatorname{tr} T(A) = 0$.

Lemma 3 is obvious.

LEMMA 4. *Let* $A \in R^{n\times n}$ *be a symmetric matrix*, $B \in R^{n\times n}$, *then*

$$\operatorname{tr}(AB) = \operatorname{tr} A[S(B)] \tag{3}$$

$$\operatorname{tr}[AT(B)] = 0. \tag{4}$$

*Proof.*
$$\begin{aligned}\operatorname{tr}[AT(B)] &= \operatorname{tr}[A\,(B - B')/2]\\ &= [\operatorname{tr} AB - \operatorname{tr} AB']/2\\ &= [\operatorname{tr} AB - \operatorname{tr} (AB')']/2\\ &= [\operatorname{tr} AB - \operatorname{tr} BA']/2\\ &= [\operatorname{tr} AB - \operatorname{tr} BA]/2\\ &= 0,\end{aligned}$$

which proves (4).

$$\begin{aligned}\operatorname{tr} AB &= \operatorname{tr}\{A\,[S(B) + T(B)]\}\\ &= \operatorname{tr} AS(B) + \operatorname{tr} AT(B)\\ &= \operatorname{tr} AS(B) \qquad \text{(from (4))},\end{aligned}$$

which proves (3).

THEOREM 2. *If $A \in R^{n\times n}$ is a positive semidefinite (symmetric) matrix and $B \in R^{n\times n}$ is a generalized positive semidefinite matrix, then*

$$\text{(a)} \quad \operatorname{tr}(AB) \geq 0$$

*and*

$$\text{(b)} \quad (\operatorname{tr} AB)^{1/2} \leq (\operatorname{tr} A + \operatorname{tr} B)/2.$$

*Proof.* Since $B$ is a generalized positive semidefinite matrix, for all $x \in R^n\backslash\{0\}$, $x'Bx \geq 0$. From $(x'\,Bx)' = x'B'x$ we have $x'B'x \geq 0$. Therefore

$$x'S(B)\,x = x'\,((B + B')/2)\,x = (x'Bx + x'B'x)/2 \geq 0, \qquad \text{for all } x \in R^n\backslash\{0\}.$$

That is, $S(B)$ is a positive semidefinite matrix.
Lemma 4(3) and Theorem 1(i) yield that

$$\operatorname{tr}(AB) \geq 0.$$

From Lemma 4, Theorem 1 (ii), and Lemma 3 we have

$$\begin{aligned}(\operatorname{tr} AB)^{1/2} &= [\operatorname{tr} AS(B)]^{1/2}\\ &\leq [\operatorname{tr} A + \operatorname{tr} S(B)]/2\\ &= (\operatorname{tr} A + \operatorname{tr} B)/2.\end{aligned}$$

This completes the proof of Theorem 2.

*Remark.* If $A$ and $B$ are two generalized positive semidefinite matrices of the same order, Theorem 2 is not true.

EXAMPLE. This example illustrates that Theorem 2 cannot be extended to two matrices which are both merely generalized positive semidefinite.

Let

$$A = \begin{pmatrix} 1 & 4 \\ 0 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}.$$

Then both $A$ and $B$ are generalized positive definite matrices of the same order, but

$$(\operatorname{tr} AB)^{1/2} \not\leq (\operatorname{tr} A + \operatorname{tr} B)/2.$$

## References

1. R. Bellman, Some inequalities for positive definite matrices, *in* "General Inequalities 2: Proceedings, 2nd International Conference on General Inequalities" (E. F. Beckenbach, Ed.), pp. 89–90, Birkhäuser, Basel, 1980.
2. Y. Yang, A matrix trace inequality, *J. Math. Anal. Appl.* **133** (1988), 573–574.
3. H. Neudecker, A matrix trace inequality, *J. Math. Anal. Appl.* **166** (1992), 302–303.
4. H. R. Johnson, An inequality for matrices whose symmetric pair is positive definite, *Linear Algebra Appl.* **6** (1973), 13–18.