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The Random Coin Method: Solution of the Problem of the Simulation of a Random Function in the Plane[1]

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This paper presents a new geostatistical method to obtain realizations of stationary random functions in the plane.

KEY WORDS: random function, geostatistics, simulation, covariance.

INTRODUCTION

Several authors (see Matèrn 1947, Whittle 1954, Guibal 1972, Chilés 1977) have attempted to solve the problem of the simulation of stationary random functions which satisfy a covariance model or a given semivariogram in two-dimensional space.

The method proposed in this paper gives an accurate solution—efficient from the practical point of view—to the problem of obtaining realizations of the classical covariance models used in Geostatistics.

Turning Band Method

The turning band method of Matheron consists of transforming the simulation in n-dimensional space $\mathcal{R}^n$ into several simulations of straight lines which rotate in $\mathcal{R}^n$.

Let us consider N straight lines, D_i, in the $\mathcal{R}^n$ space, each line defined by the unit vector e_i. Along each line D_i the realization of a random function $Z_i(u)$ with $C_1(h)$ covariance is drawn, independent from the other lines. Then, if x is a point in $\mathcal{R}^n$, the random function

$$Z(x) = 1/N^{1/2} \sum_{i=1}^{N} Z_i(\langle x, e_i \rangle)$$

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has a covariance

$$C_n(h) = 1/N \sum_{i=1}^{N} C_1(\langle h, e_i \rangle)$$

If $N \to \infty$ and if the unit vectors e_i are uniformly distributed in the unitary semisphere of $\mathbb{R}^n$, the following isotropic covariance is obtained (see Matheron, 1972)

$$C_n(h) = 2\Gamma(n/2)/[(\pi)^{1/2}\Gamma((n-1)/2)] \cdot (1/h^{n-2}) \int_0^h (h^2 - u^2)^{(n-3)/2} C_1(u)\, du \tag{1}$$

In practice, if we want to simulate a random function with known $C_n(h)$, the relationship provided by $C_1(h)$ in relation to $C_n(h)$ is the most important point.

If $n = 3$, it results from (1) that

$$C_1(h) = d/dh\ [hC_3(h)]$$

Nevertheless, when $n = 2$ the solution is not direct. As a matter of fact, it is obtained that

$$C_2(h) = 2/\pi \int_0^h [C_1(u)\, du]/(h^2 - u^2)^{1/2} \tag{2}$$

This integral equation is too difficult to be solved so we discard this method.

METHOD OF RANDOM COINS

General Formula

The geometric covariogram of a circle of diameter x, that is, the area of the intersection of a circle and its displacement over distance h is:

$$B(h,x) = \begin{cases} (x^2/2 \text{ Arc} \cos h/x) - [(h/2)(x^2 - h^2)^{1/2}] & \text{if } h \leqslant x \\ 0 & \text{if } h > x \end{cases} \tag{3}$$

If it is assumed that the diameter x is a random variable with distribution function $F(x)$, the following function is a covariance on $\mathbb{R}^2$ (i.e., a positive definite function)

$$C(h) = \int_h^{\infty} B(h,x)\, dF(x) \tag{4}$$

Taking successive derivatives of equation (4) it is found that

$$C'(h) = -\int_h^\infty (x^2 - h^2)^{1/2}\, dF(x) \tag{5}$$

$$C''(h) = \int_h^\infty h dF(x)/(x^2 - h^2)^{1/2} \tag{6}$$

We are interested in the inversion of equation (6), so we obtain

$$\int_r^\infty [C''(p)\, dp]/(p^2 - r^2)^{1/2} = \int_r^\infty dF(u) \int_r^u p\, dp/[(p^2 - r^2)(u^2 - p^2)]^{1/2}$$

The change of variables $x = (p^2 - r^2)/(u^2 - r^2)$ gives

$$\int_r^u p\, dp/[(p^2 - r^2)(u^2 - p^2)]^{1/2} = 1/2 \int_0^1 dx/[x(1 - x)]^{1/2} = \pi/2$$

Therefore, it follows that

$$\boxed{\int_r^\infty dF(u) = 2/\pi \int_r^\infty C''(p)\, dp/(p^2 - r^2)^{1/2}} \tag{7}$$

This fundamental formula (7) gives the probability $P(D \geqslant r)$ of the random diameter D.

Simulation of the Spherical Covariance

The spherical covariance of range a (one of the most used models in geostatistics) is given by

$$C(h) = \begin{cases} (\pi a^2/6)\,\{1 - [(3/2)(h/a)] + [(1/2)(h^3/a^3)]\} & \text{if } h \leqslant a \\ 0 & \text{if } h > a \end{cases} \tag{8}$$

(The factor $\pi a^2/6$ can be explained by the fact that $\pi/6$ is the volume of a sphere with a diameter equal to 1 and the factor a^2 has been taken to have a dimensionally correct equation; see formula (7))

This results in $C''(h) = \pi h/2a$ and formula (7) gives $(r \leqslant a)$

$$\int_r^\infty dF(u) = 1 - F(r) = 1/a \int_r^a p\, dp/(p^2 - r^2) = (1/a)(a^2 - r^2)^{1/2}$$

Therefore, the probability function associated with the random diameter of the circle is

$$F(r) = \begin{cases} [a - (a^2 - r^2)^{1/2}]/a & \text{if} \quad r \leqslant a \\ 1 & \text{if} \quad r > a \end{cases} \tag{9}$$

Simulation of the Exponential Covariance

For the exponential covariance $C(h) = a^2 e^{-b/a}$ we have

$$\int_r^\infty dF(u) = 1 - F(r) = 2/\pi \int_r^\infty (e^{(-p/a)}\, dp)/(p^2 - r^2)^{1/2}$$

$$= 2/\pi \int_{r/a}^\infty (e^{-t}\, dt)/[t^2 - (r/a)^2]^{1/2}$$

According to the Sonine-Schläfli relationship this implies

$$F(r) = 1 - (2/\pi)\, K_0(r/a) \tag{10}$$

where $K_0(x)$ is the modified Bessel function of second order of imaginary argument.

It can be observed that equation (10) does not define a probability distribution (since $K_0(0) = \infty$). This difficulty is not important because values that follow the distribution (10) can always be obtained using a suitable truncation of $K_0(r/a)$ near the origin.

The above mentioned mathematical exposition establishes the general formula of the method that I have named "the random coin method." Now we shall see a more intuitive exposition which gives us a practical method to obtain simulations.

Intuitive Exposition of the Method

The same results as above can be obtained through the following geometric probabilistic approach.

(a) The three-dimensional space $\mathcal{R}^3$ is filled with spheres of equal radius R setting the centers of the spheres in $\mathcal{R}^3$ according to a Poisson process. All the points inside the sphere i are associated with a random number ξ_i (following an independent way from one sphere to another). Each point x of the space is associated with a random variable

$$Z(x) = \sum_i \xi_i \tag{11}$$

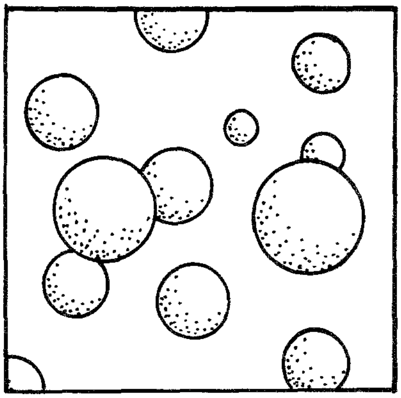

Fig. 1. Poisson spheres in $\mathcal{R}^3$.

in which the addition is carried out over all the spheres that contain the point x. So the random function establishes a diffusion process and its covariance in the three-dimensional space is a spherical covariance (see Fig. 1).

(b) Let us take any plane Q of the realization of $Z(x)$. This plane will have a "random coin" configuration (see Fig. 2).

(c) A specific sphere intercepts the plane Q if and only if (see Fig. 3):

$$-R \leqslant \zeta \leqslant R$$

Let R_x be the radius of the random circle on the plane Q. We have

$$R_x = (R^2 - \zeta^2)^{1/2} \tag{12}$$

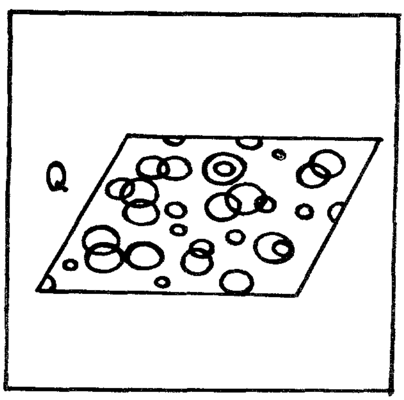

Fig. 2. Random coins.

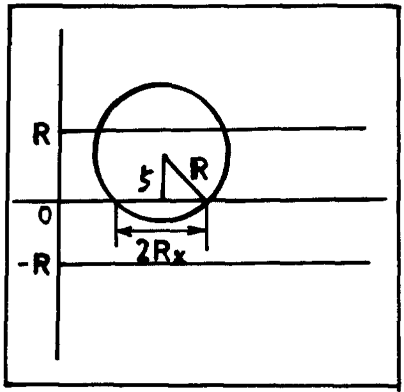

Fig. 3. Intersection of a sphere with Q.

but according to the localization of the sphere centers, ξ follows a uniform distribution in $[-R, R]$. Therefore, the radius R_x follows a density function

$$f_1(x) = x/[R(R^2 - x^2)^{1/2}], \qquad -R \leqslant x \leqslant R$$

(according to the formula of change of variables). Consequently, the diameter $D_x = 2R_x$ assumes the density

$$f_2(x) = x/[D(D^2 - x^2)^{1/2}], \qquad -D \leqslant x \leqslant D \tag{13}$$

that is, the density associated with the probability function (9). Therefore, to obtain a number D_x that follows the probability function (9) we must take a number x with uniform distribution in $[-D, D]$, and construct $D_x = (D^2 - x^2)^{1/2}$.

Simulation Method

The preceding paragraph provides us with the method to simulate a model (spherical, exponential, . . .) on the plane.

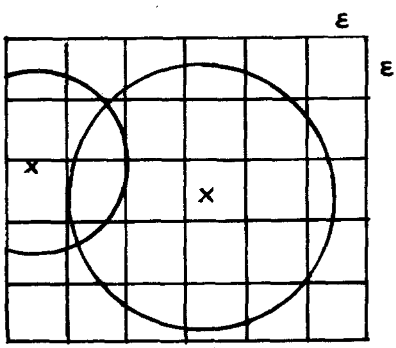

Fig. 4. Practical simulation.

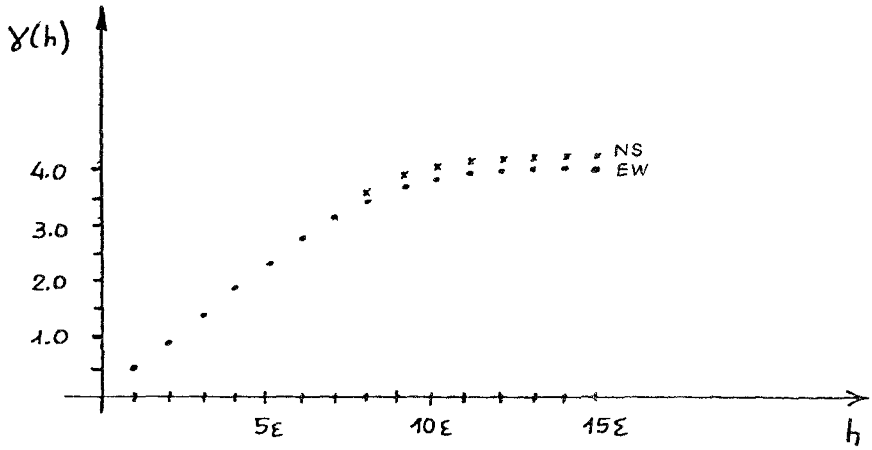

Fig. 5. NS and EW semivariograms.

Refill the space $\mathbb{R}^2$ with circles whose random diameters follow the law defined by equation (7). The centers of the circles are taken according to a Poisson process with parameter λ on the plane (in order to obtain a realization with good properties of continuity, it is convenient to choose a large density λ).

Let us define a grid $\epsilon \times \epsilon$ and associate with each point of this grid the contribution of all the circles containing that point (see Fig. 4).

This method is easily programmed on the computer and, if a display is available, the structure of the simulation can be observed as circles are added.

As an example, we have simulated a $55\epsilon \times 55\epsilon$ field of a spherical model with range $D = 11\epsilon$ and the semivariograms obtained are shown in Figure 5. The number of circles planted in the field is $N = 2500$.

Other Models

The random coin method can also be used to generate models in two-dimensional space. For example, with the probability function

$$f(x) = \begin{cases} \alpha x^{2p-1}/(1-x^2)^{1/2} & \text{if} \quad x \in [0,1] \\ 0 & \text{if} \quad x \notin [0,1] \end{cases}$$

where $p \in \mathbb{N}$ and α is a parameter such that $\int f(x)\,dx = 1$, it can be shown (see Alfaro, 1979) that formula (4) or (5) gives a covariance of the following type

$$C(h) = \begin{cases} \sum_{k=0}^{2p+1} a_k |h|^k & \text{if} \quad |h| \leq 1 \\ 0 & \text{if} \quad |h| > 1 \end{cases} \tag{14}$$

that is, a polynomial covariance on $\mathbb{R}^2$. Families of models can also be generated. These models are important in the theory of random functions of the kth order (see Chilés, 1977, and Matheron, 1973).

CONCLUSION

The random coin method is an efficient and accurate method of obtaining realizations of random functions in the plane, and has also the great advantage of not creating the problems of discretization presented by other methods (see Chilés, 1977; and Guibal, 1972).

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