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CONVECTIVE INSTABILITY OF A BINARY MIXTURE WITH ALLOWANCE FOR THE SORET EFFECT

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The effect of weak mixture concentration on the threshold of convective instability of a binary mixture filling a cavity of arbitrary shape is investigated. In the case of thermally insulated boundaries in the neighborhood of the critical Rayleigh number monotonicity of perturbations is proved. This makes it possible to express the critical Rayleigh number for the mixture in terms of its analog for a single-component fluid at any values of the Soret parameter. In the general case of boundaries of arbitrary thermal conductivity an estimate of the critical Rayleigh number is obtained for small values of the Soret parameter.

1. Let us consider a cavity G in a solid mass filled with a viscous incompressible binary mixture. The inhomogeneous temperature distribution in the mixture leads to temperature and concentration (due to the thermodiffusion effect) density inhomogeneity and, as a consequence, to thermal and thermoconcentration convection [1]. When the temperature distribution in the solid mass is so specified that a vertical temperature gradient exists in the cavity, the mixture may be in mechanical equilibrium, which may be unstable for certain values of the parameters.

In the present paper, assuming a low concentration of one of the components and neglecting the cross effect of diffusion heat conduction (Dufour effect), we will consider the effect of an admixture on the threshold of convective stability of mechanical equilibrium.

Within the framework of above-mentioned assumptions, the convection equations of the mixture in the Boussinesq approximation and the heat transfer equations in the solid mass have the form [1]:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathrm{Pr}\,(\mathbf{v}\nabla)\mathbf{v} = -\nabla p + \Delta \mathbf{v} + \mathrm{Ra}\,(T(1+\varepsilon) + H)\boldsymbol{\gamma}$$

$$\mathrm{Pr}\frac{\partial H}{\partial t} + \mathbf{v}\nabla T = \mathrm{Le}\Delta H - \varepsilon\Delta T \tag{1.1}$$

$$\mathrm{Pr}\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \Delta T$$

$$\mathrm{Pr}\frac{\partial T_m}{\partial t} = c\Delta T_m, \qquad \mathrm{div}\,\mathbf{v} = 0$$

Here, $\mathbf{v}$ is the velocity, p is the pressure, T and T_m are the temperatures of the mixture and the solid mass. Moreover, we have introduced the auxiliary function $H = C + \alpha T$, where C is the concentration of the lighter component and α is the thermodiffusion coefficient.

We choose as units of time, distance, velocity, pressure, temperature and concentration h^2/ν, h, χ/h, $\rho\nu\chi/h^2$, Θ, and $\Theta\beta_1/\beta_2$, respectively, where h is the characteristic dimension of the cavity, ρ is the density of the mixture, β_1, β_2, χ, and ν are the thermal and "concentration" expansion coefficients (β_1 and $\beta_2 > 0$), thermal diffusivity, and viscosity coefficient, respectively. The unit vector γ is directed vertically upward.

On the boundary of the cavity the viscous no-slip conditions, temperature and heat flux continuity, and the no-flow condition for the light component are assigned:

$$\Gamma: \quad \mathbf{v} = 0, \qquad T = T_m, \qquad \frac{\partial T}{\partial n} = k\frac{\partial T_m}{\partial n}, \qquad \frac{\partial H}{\partial n} = 0 \tag{1.2}$$

Here, $\partial/\partial n$ is the derivative along the outward normal to the boundary.

The problem (1.1)—(1.2) contains six dimensionless parameters: the Rayleigh, Prandtl, Lewis, and Soret numbers and the ratios of the thermal diffusivity and conductivity coefficients of the solid mass and the mixture

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$$\mathrm{Ra} = \frac{g\beta_1\Theta h^3}{\nu\chi}, \quad \mathrm{Pr} = \frac{\nu}{\chi}, \quad \mathrm{Le} = \frac{D}{\chi}, \quad \varepsilon = \frac{\beta_1\alpha}{\beta_2}, \quad c = \frac{\chi_m}{\chi}, \quad k = \frac{\varkappa_m}{\varkappa}$$

where D is the diffusion coefficient.

For a certain assignment of the temperature field T_m in the solid mass the system of equations and boundary conditions (1.1), (1.2) admits the equilibrium solution

$$\mathrm{v} \equiv 0, \quad H \equiv \mathrm{const}_1, \quad p = \mathrm{Ra}\,(1 + \varepsilon)z + \mathrm{const}_2, \quad T = -z + \mathrm{const}_3 \tag{1.3}$$

The problem of determining this temperature distribution in the solid mass is not trivial. If the thermal conductivities of the fluid and the solid mass are different, the corresponding formulas can be written only in the case of a fairly high degree of symmetry of the cavity. In [1] solutions are given for certain specific shapes (plane horizontal layer, ellipsoid of revolution, vertical and horizontal elliptic cylinders).

We will consider the problem of the stability of the equilibrium distribution of v, H, p, T, and T_m with respect to small normal (exponentially dependent on time) perturbations. Using the same notation for the perturbation amplitudes as for the basic functions, we obtain the following system of equations (for brevity, we will not write down the continuity equation):

$$-\lambda \mathrm{v} = -\nabla p + \Delta \mathrm{v} + \mathrm{Ra}\,(T(1 + \varepsilon) + H) + \gamma \tag{1.4}$$

$$-\lambda \mathrm{Pr}\, T = \Delta T + v_z \tag{1.5}$$

$$-\lambda \mathrm{Pr} H = \mathrm{Le}\,\Delta H - \varepsilon \Delta t \tag{1.6}$$

$$-\lambda \mathrm{Pr}\, T_m = c \Delta T_m \tag{1.7}$$

Here, $\lambda = \lambda r + i\lambda_i$ is the perturbation growth rate.

On the boundary Γ of the cavity the conditions for the perturbations have the form:

$$\Gamma\colon \quad \mathrm{v} = 0, \quad T = T_m, \quad \frac{\partial T}{\partial n} = k\frac{\partial T_m}{\partial n}, \quad \frac{\partial H}{\partial n} = 0 \tag{1.8}$$

$$\Gamma_1\colon \quad T_m = 0 \tag{1.9}$$

Here, Γ_1 is the outer boundary of the solid mass.

2. We will consider the limiting case of thermally insulated boundaries $k=0$, i.e., $\partial T/\partial n = 0$. Then for the solid mass the heat equation (1.7) can be omitted.

We will show that in this case for a small supercriticality loss of stability is related with an increase in monotonic perturbations, and the critical Rayleigh number for the mixture can be expressed in terms of its analog for a single-component fluid.

We multiply Eqs. (1.5) and (1.6) by the complex conjugate functions H^* and T^*, respectively, and integrate over the volume of the cavity G

$$-\lambda \mathrm{Pr}\langle TH^*\rangle = \langle \Delta T H^*\rangle + \langle v_z H^*\rangle \tag{2.1}$$

$$-\lambda \mathrm{Pr}\,\langle T^* H\rangle = \mathrm{Le}\,\langle \Delta H T^*\rangle - \varepsilon\langle \Delta T T^*\rangle$$
$$\langle\cdot\rangle = \int_G dG \tag{2.2}$$

Summing (2.1) and the complex conjugate to (2.2) and integrating by parts with allowance for (1.2), we obtain

$$-2\lambda_r \mathrm{Pr}\,\langle TH^*\rangle = (\mathrm{Le} + 1)\langle \Delta T H^*\rangle + \langle v_z H^*\rangle + \varepsilon\langle |\nabla T|^2\rangle \tag{2.3}$$

We now multiply Eqs. (1.4), (1.5), and (1.6) by v^*, T^*, and H^*, respectively, integrate over the region and substitute the expression for $\langle \Delta TH^*\rangle$ from (2.3)

$$-\lambda\langle|\mathbf{v}|^2\rangle = -\langle|\mathrm{rot}\,\mathbf{v}|^2\rangle + \mathrm{Ra}\Big(\langle Hv_z^*\rangle + (\varepsilon + 1)\langle Tv_z^*\rangle\Big) \tag{2.4}$$

$$-\lambda\,\mathrm{Pr}\langle|T|^2\rangle = -\langle|\nabla T|^2\rangle + \langle v_z T^*\rangle \tag{2.5}$$

$$-\lambda\,\mathrm{Pr}\langle|H|^2\rangle = -\mathrm{Le}\,\langle|\nabla H|^2\rangle + \frac{\varepsilon}{\mathrm{Le} + 1}\Big(\langle v_z H^*\rangle + \varepsilon\langle|\nabla T|^2\rangle + 2\lambda_r \mathrm{Pr}\langle TH^*\rangle\Big) \tag{2.6}$$

Adding Eqs. (2.5) and (2.6), multiplied by $\mathrm{Ra}(\varepsilon + 1)$ and $\mathrm{Ra}(\mathrm{Le} + 1)/\varepsilon$, respectively, to Eq. (2.4), we obtain

$$\begin{gathered}-\lambda\left\{\langle|\mathbf{v}|^2\rangle + \mathrm{Ra}\,\mathrm{Pr}\,(\varepsilon + 1)\langle|T|^2\rangle + \frac{\mathrm{Ra}}{\varepsilon}\mathrm{Le}(\mathrm{Le} + 1)\langle|H|^2\rangle\right\} = \\ -\langle|\mathrm{rot}\,\mathbf{v}|^2\rangle - \mathrm{Ra}\langle|\nabla T|^2\rangle - \frac{\mathrm{Ra}}{\varepsilon}\mathrm{Le}(\mathrm{Le} + 1)\langle|\nabla H|^2\rangle + \\ 2\mathrm{Ra}\,\mathrm{Re}\Big\{\langle Hv_z^*\rangle + (\varepsilon + 1)\langle Tv_z^*\rangle\Big\} + 2\lambda_r \mathrm{Pr}\langle TH^*\rangle\end{gathered} \tag{2.7}$$

Here, $\mathrm{Re}\{\cdot\}$ denotes the real part.

At the critical point $\mathrm{Ra}=\mathrm{Ra}_c$ the real part of the decay rate $\lambda_r=0$ and, since the right side of (2.7) is real, the imaginary part of the decay rate λ_i is also equal to zero. Consequently, the amplitudes of the neutral perturbations $\mathbf{v}_c$, T_c, and H_c are real.

We can readily show that the decay rate λ and the amplitudes $\mathbf{v}$, T, and H are real at least in a finite neighborhood of the critical number Ra_c. For this purpose we expand λ, $\mathbf{v}$, T, and H in a power series in a small parameter, namely, in the supercriticality $(\mathrm{Ra} - \mathrm{Ra}_c)$. Substituting the expressions in (2.4)–(2.7), we find that in each order the corrections $\lambda^{(n)}$, $\mathbf{v}^{(n)}$, $T^{(n)}$, and $H^{(n)}$ can be expressed in terms of the corrections in the previous orders and the neutral perturbation amplitudes $\mathbf{v}_c$, T_c, and H_c. From the reality of these amplitudes there follows the reality of λ, $\mathbf{v}$, T, and H in the region of convergence of the expansion.

The proven monotonicity of the neutral perturbations makes it possible to obtain the following simple expression relating the critical Rayleigh number for the mixture Ra_c with its analog for a single-component fluid Ra_0:

$$\mathrm{Ra}_c = \frac{\mathrm{Ra}_0}{1 + \varepsilon + \varepsilon/\mathrm{Le}} \tag{2.8}$$

Relation (2.8) was first obtained in [2] on the assumption of monotonicity of the neutral perturbations.

3. We will now consider the general case of arbitrary thermal conductivity of the solid mass. Assuming the Soret parameter ε to be small, we expand the critical Rayleigh number $\mathrm{Ra}_c(\lambda)$, the decay rate $\lambda=i\lambda_i$, and the neutral perturbation amplitudes $\mathbf{v}_c$, T_c, T_{mc}, and H_c in a power series in ε which has the form:

$$f=f_0 + \varepsilon f_1 + \dots \tag{3.1}$$

Substituting expansions (3.1) in (1.4)–(1.9), in the zeroth order in ε we obtain the problem of determining the critical Rayleigh number Ra_0 and the neutral perturbation amplitudes $\mathbf{v}_0$, p_0, T_0, and H_0 in a single-component fluid. From the monotonicity of the amplitudes it follows [3] that

$$\lambda_{0i}=0, \qquad H_0=0, \qquad -\nabla p_0 + \Delta\mathbf{v}_0 + \mathrm{Ra}_0 T_0 \boldsymbol{\gamma} = 0 \tag{3.2}$$

$$\Delta T_0 = -v_{0z} \tag{3.3}$$

$$\mathrm{Le}\,\Delta H_1 = \Delta T_0, \qquad \Delta T_{m0} = 0 \tag{3.4}$$

In the first order in ε the inhomogeneous equation for the momentum has the form:

$$-\nabla p_1 + \Delta\mathbf{v}_1 + \mathrm{Ra}_0 T_1 \boldsymbol{\gamma} = -\big[\mathrm{Ra}_1 T_0 + \mathrm{Ra}_0(T_0 + H_1)\big]\boldsymbol{\gamma} - i\lambda_{1i}\mathbf{v}_0 \tag{3.5}$$

With allowance for the self-conjugacy of the problem and the reality of the amplitudes in the zeroth order the condition of solvability of Eq. (3.5) has the form:

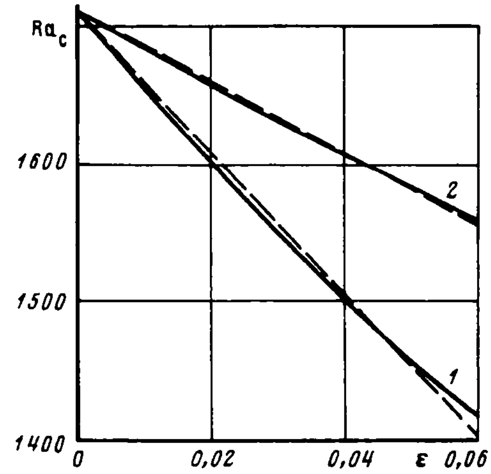

Fig. 1. Critical Rayleigh number as a function of the Soret parameter for Pr = 0.75. The continuous curves correspond to the numerical solution of the problem (1.4)—(1.8), the broken curves are the graph of the right side of inequality (3.11). Curves *1* and *2* correspond to Le = 0.5 and 2, respectively.

$$\lambda_{1i} = 0, \qquad \mathrm{Ra}_1 \langle T_0 v_{0z} \rangle = -\mathrm{Ra}_0 \langle T_0 + H_1, v_{0z} \rangle \tag{3.6}$$

Using (3.3), (3.4), and (3.6), correct to the first order in ε we obtain:

$$\mathrm{Ra}_c = \mathrm{Ra}_0 [1 - \varepsilon(1 + K)], \qquad K = \mathrm{Le} \frac{\langle H_1 \Delta H_1 \rangle}{\langle T_0 \Delta T_0 \rangle} \tag{3.7}$$

Denoting the integral over the solid mass by $<\cdot>_m$ and integrating (3.7) by parts, we obtain

$$K = \frac{\mathrm{Le} \langle |\nabla H_1|^2 \rangle}{\langle |\nabla T_0|^2 \rangle + k \langle |\nabla T_{m0}|^2 \rangle_m} > 0 \tag{3.8}$$

We can refine estimate (3.8). For this purpose we multiply the equation (3.4) for H_1 by T_0 and integrate by parts. Then, using the Schwartz inequality, we obtain

$$-\mathrm{Le}^{-1} \langle \Delta T_0 T_0 \rangle = \langle \nabla H_1 \nabla T_0 \rangle \le \langle |\nabla H_1|^2 \rangle^{1/2} \langle |\nabla T_0|^2 \rangle^{1/2}$$

$$\langle |\nabla H_1|^2 \rangle \ge \mathrm{Le}^{-2} \frac{\langle \Delta T_0 T_0 \rangle^2}{\langle |\nabla T_0|^2 \rangle} \tag{3.9}$$

From (3.8) and (3.9) there follows the estimate $K > \mathrm{Le}^{-1}$, with allowance for which from (3.7) we obtain

$$\mathrm{Ra}_c \le \mathrm{Ra}_0 [1 - \varepsilon - \varepsilon/\mathrm{Le}] \tag{3.10}$$

In order to verify this estimate of Ra_c the problem (1.4)—(1.8) was solved for a plane horizontal layer with ideally thermally-conducting boundaries. The Galerkin method with nine basic functions was used. In Fig. 1 the calculated dependence $\mathrm{Ra}_c(\varepsilon)$ is represented by continuous curves *1* and *2* for the two values of the Lewis number Le = 0.5 and Le = 2, respectively, and fixed Pr = 0.75. The broken curves correspond to the estimate (3.10).

The lower the relative thermal conductivity k of the solid mass, the more accurate the estimates obtained. In the limit $k = 0$ inequality (3.10) transforms to an equality that coincides with (2.8) correct to $O(\varepsilon)$.

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Summary. In binary mixture convection taking the Soret effect into account reduces the practical value of the Rayleigh number as compared with its single-component analog.

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