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PARTITIONS OF VECTOR SPACES

P. KOMJÁTH (Budapest)

0. Introduction

Let V be a vector space over $\mathbf{Q}$, the rationals, and consider

$$\sigma : \sum_{j=1}^{n} \lambda_{ij} x_j = 0 \qquad (1 \le i \le m), \tag{0.1}$$

a system of homogeneous linear equations over $\mathbf{Q}$. Is it true, that if V is colored with countably many colors (is *ω-colored*, in short), then there is a monocolored solution of σ satisfying

$$x_j \ne x_{j'} \quad (j \ne j')? \tag{0.2}$$

Clearly, the answer depends on the dimension of V which is the same as the cardinal of V assuming V is uncountable. (For countable V, the answer is obviously "no".) By a theorem of Erdős–Kakutani [5], every vector space of size $\le \omega_1$ is the union of countably many linearly independent sets, so the dimension in question must be at least ω_2. Let $\lambda(\sigma)$ be the least dimension such that the answer to our question is "yes". If no such cardinal exists, we write $\lambda(\sigma) = \infty$. An old observation of Rado's (?) is that $\lambda(x + y = 2z) = \infty$. P. Erdős showed many years ago, that $\lambda(x + y = z + t) = \omega_2$. The question, therefore, arises, how to determine $\lambda(\sigma)$, or at least, which cardinals occur as the values of λ. As there are only countably many systems of linear equations, clearly there is a cardinal bigger than every $\lambda(\sigma)$ with $\lambda(\sigma) < \infty$. In Theorem 1.3. we prove that the supremum of $2^\omega, 2^{2^\omega}, \dots$ is such a bound. This statement is a special case of a two-cardinal theorem of Vaught (see [2], [8]), but we think that our proof is simpler in this case. It is not clear, if that is the least upper bound, we have only been able to find an example with $\lambda(\sigma) = (2^\omega)^+$. It is, however, the least upper bound, if GCH is assumed, as we show that every $\omega_n (2 \le n < \infty)$ is $\lambda(\sigma)$ for some σ. We further show that every system σ with

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$\lambda(\sigma) \leq 2^\omega$ satisfies a certain property which in turn implies that $\lambda(\sigma) < \aleph_\omega$. An interesting corollary of this result is that if $\lambda(\sigma) \leq 2^\omega$, then $\lambda(\sigma) < \aleph_\omega$. This also follows from S. Shelah's two-cardinal theorem [7].

NOTATION. We use the standard set theory notation. The well-ordering theorem is assumed. Cardinals are identified with initial ordinals. If $\kappa = \aleph_\alpha$, then $\kappa^+ = \aleph_{\alpha+1}$. 2^ω is the cardinal of the continuum.

1. A bound for $\lambda(\sigma)$

THEOREM 1.1. $\lambda(x + y = z) = (2^\omega)^+$.

PROOF. To show that $\lambda(x + y = z) > 2^\omega$, it suffices to give a decomposition of $\mathbf{R}$ into countably many pieces with no solution of the above mentioned equation. Decompose in such a way that the ratio of two numbers in the same class should always be between 1 and 2.

For the other direction, let $B = \{b_\alpha : \alpha < (2^\omega)^+\}$ be a basis in a vector space of dimension $(2^\omega)^+$. If we color the differences $\{b_\alpha - b_\beta : \alpha < \beta < (2^\omega)^+\}$ with countably many colors, then, by the Erdős–Rado theorem [6], there exist $\alpha < \beta < \gamma$ such that $b_\alpha - b_\beta$, $b_\beta - b_\gamma$ and $b_\alpha - b_\gamma$ get the same color, thereby giving a monocolored solution of our equation. ■

DEFINITION 1.2. $\exp_0(\omega) = \aleph_0$, $\exp_{n+1}(\omega) = 2^{\exp_n(\omega)}$ and let the cardinal $\exp_\omega(\omega)$ be the supremum of the cardinals $\exp_n(\omega)$ $(n < \omega)$.

THEOREM 1.3. *If $\lambda(\sigma) < \infty$ for some system σ of equations, then $\lambda(\sigma) <$ $\exp_\omega(\omega)$.*

PROOF. Assume that there is a vector space, V with basis $\{b_\alpha : \alpha < \kappa\}$ witnessing $\lambda(\sigma) < \omega$. Color the nonzero vectors of V as follows. If

$$v = \sum_{i=1}^{k} \gamma_i b_{\alpha_i}, \qquad (\alpha_1 < \cdots < \alpha_k) \tag{1.1}$$

then assign the ordered sequence of rationals $(\gamma_1, \ldots, \gamma_k)$ to v. Give a separate color to 0. Glearly, this is a coloring with countably many colors. By our assumptions, there are n different vectors, getting the same color, which form a solution of σ. This means, that there are $K < \omega$ ordinals, $\alpha_1 < \cdots < \alpha_K < \kappa$ such that among the vectors which can be written in the form

$$\sum_{i=1}^{k} \gamma_i b_{\alpha_{j_i}} \tag{1.2}$$

with $1 \leq j_1 < \cdots < j_k \leq K$, there are some n which solve σ. Now assume that a vector space with basis $\{b_\alpha : \alpha < (\exp_k(\omega))^+\}$ is colored with countably many colors. This specifically colors all the vectors of the form

$$\sum_{i=1}^{k} \gamma_i b_{\alpha_i} \tag{1.3}$$

where $\alpha_1 < \cdots < \alpha_k < (\exp_k(\omega))^+$. By the Erdős–Rado theorem, there are $\alpha_1 < \cdots < \alpha_K$ such that all subsums of the above form get the same color. But some n of them give a solution of σ. ■

2. Regular systems

DEFINITION 2.1. Given a system of linear equations $\sigma : \sum_{j=1}^{n} \lambda_{ij} x_j = 0$ $(1 \leq i \leq m)$, we call a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of the set $\{1, \ldots, n\}$ *balanced*, if $\sum\{\lambda_{ij} : j \in P_k\} = 0$ holds for every $1 \leq i \leq m, 1 \leq k \leq t$. A collection $\{\mathcal{P}_1, \ldots, \mathcal{P}_s\}$ of balanced partitions is *separative* if whenever $1 \leq x < y \leq n$ then there is a $\mathcal{P}_j$ with x, y in different classes. σ is *regular* if it possesses a separative collection of balanced partitions. It is *k-regular,* if the number of partitions is $\leq k$. (Notice that k must be at least 2.)

THEOREM 2.2. *If $\lambda(\sigma) \leq 2^\omega$, then σ is regular.*

PROOF. Take a basis $B = \{b_j : j \in J\}$ of $\mathbf{R}$. For $v \in \mathbf{R}$, $v \neq 0$, $v = \sum_{i=1}^{k} \gamma_i b_{j_i}, b_{j_1} < \cdots < b_{j_k}$, color v by $(q_1, \ldots, q_{k-1}; \gamma_1, \ldots, \gamma_k)$, where the q_i's are rational numbers with $b_{j_1} < q_1 < b_{j_2} < \cdots < b_{j_{k-1}} < q_{k-1} < b_{j_k}$. An important property of this coloring is that if v, v' get the same color and some $b \in B$ appears in both, then b gets the same index in them. Assume that $\sum_{j=1}^{n} \lambda_{ij} x_j = 0$ and the x_j's get the same color, $(q_1, \ldots, q_{k-1}; \gamma_1, \ldots, \gamma_k)$. Given $1 \leq r \leq k$, put $j \sim j'$ for $1 \leq j, j' \leq n$ if the r'th basis vector of $x_j, x_{j'}$ coincide. This gives r partitions of $\{1, \ldots, n\}$ and, as the x_j's are different, they constitute a separative system. If b is the basis vector corresponding to class P_k of one of the partitions, then, by the above mentioned property of the coloring, the co-ordinate of b in $\sum \lambda_{ij} x_j = 0$ is $\sum\{\lambda_{ij} : j \in P_k\} = 0$, so the partitions are balanced. ■

THEOREM 2.3. *If σ is k-regular, then $\lambda(\sigma) \leq \omega_k$.*

PROOF. We need a polarized partition relation of Elekes–Erdős–Hajnal (see [3] where it is stated and [4] where the case $k = 2$ is proved).

LEMMA 2.4. *If $t, k < \omega, |B_i| = \aleph_i$ for $i = 1, \ldots, k$, $f : B_1 \times \cdots \times B_k \to \omega$, then f is constant on $C_1 \times \cdots \times C_k$ for some $C_i \subset B_i$, $|C_i| = t$.*

PROOF. By induction on k. The case $k = 1$ is trivial. If the Lemma holds for $k - 1$ and is not true for k, and f is a counter-example, then for all selections

$C_1 \subseteq B_1, \ldots, C_{k-1} \subseteq B_{k-1}, |C_i| = t$, $(1 \leq i < n)$, $j < \omega$, there are at most $t-1$ different $x \in B_k$ such that $f(C_1 \times \cdots \times C_{k-1} \times \{x\}) = j$. As there are at most $\aleph_{k-1}$ such selections, and $|B_k| = \aleph_k$, there is an $x \in B_k$ not in the union of the above mentioned sets, i.e., the coloring restricted to $B_1 \times \cdots \times B_{k-1} \times \{x\}$ is a coloring of $B_1 \times \cdots \times B_{k-1}$ with no monochromatic set as claimed, a contradiction to our inductive hypothesis. ■

DEFINITION 2.5. If k, t are natural numbers, $\{y_{rj} : 1 \leq r \leq k, 1 \leq j \leq t\}$ are vectors, for $f : \{1, \ldots, k\} \to \{1, \ldots, t\}$ put

$$x(f) = \sum \{y_{rf(r)} : 1 \leq r \leq k\}. \tag{2.1}$$

If these t^k vectors are different, we call their set a (k,t)-cube.

LEMMA 2.6. *If σ is k-regular, then there is a natural number t, such that every (k,t)-cube contains a solution of σ.*

PROOF. Assume that $\mathcal{P}_r = \{P_{r1}, \ldots, P_{rt_r}\}$ are the partitions witnessing k-regularity, with $t_r \leq t$ for $1 \leq r \leq k$. Let $\{x(f)\}$ be some (k,t)-cube, determined by $\{y_{rs} : 1 \leq r \leq k, 1 \leq s \leq t\}$. For $1 \leq j \leq n$ put $x_j = x(f_j)$, where f_j is constructed the following way. $f_j(r) = s$ iff $j \in P_{rs}$. We claim that the x_j's solve σ:

$$\sum_{j=1}^{n} \lambda_{ij} x_j = \sum_{j=1}^{n} \lambda_{ij} \sum_{r=1}^{k} \{y_{rs} : j \in P_{rs}\} = \sum_{r=1}^{k} \sum_{s=1}^{t} y_{rs} \sum \{\lambda_{ij} : j \in P_{rs}\} = 0 \tag{2.2}$$

for any $1 \leq i \leq m$. ■

To conclude the proof of Theorem 2.3. assume that $|V| = \omega_k$ and $D_1 \cup \cdots \cup D_k$ is a basis with $D_1, \ldots, D_k$ pairwise disjoint, and $|D_i| = \omega_i$. If V is ω-colored, all the vectors of the form $d_1 + \cdots + d_k$ $(d_i \in D_i)$ are ω-colored, as well, so by Lemma 2.4. there is a monochromatic (k,t)-cube for every $t < \omega$, and one of them contains a solution of σ.

COROLLARY 2.7. *If $\lambda(\sigma) \leq 2^\omega$, then $\lambda(\sigma) = \omega_k$ for some $k < \omega$.*

PROOF. From Theorems 2.2. and 2.3. ■

We now turn to the problem whether there are regular systems σ of equations with $\lambda(\sigma)$ arbitrarily high below $\aleph_\omega$.

THEOREM 2.8. *For $k = 2, 3, \ldots$, there is a k-regular system σ_k with $\lambda(\sigma_k) = \omega_k$.*

We first show that there is a k-regular system of equations σ_k such that the solutions of σ_k are exactly the $(k,2)$-cubes. If $1 \leq r \leq k$, f, g, f', g' are $\{1, 2, \ldots, k\} \to \{1, 2\}$ functions such that $f(r) = 1$, $f'(r) = 2$, $g(r) = 1$, $g'(r) = 2$, but for $r' \neq r$, $f(r') = f'(r')$ and $g(r') = g'(r')$ hold, then clearly $x(f) - x(f') = x(g) - x(g')$ $(= y_{r1} - y_{r2})$. Let these equations (for all possible choices of f, f', g, g') form our system σ_k. Given a solution $x(f)$, we can find appropriate y's as follows. Put

$y_{r1} = 0$ $(1 \le r < k)$, $y_{k1} = x(1)$, $y_{r2} = x(f^r) + y_{r1}$ (where $f^r(r') = 2$ iff $r' = r$, and 1 is the constant 1 function). Given $g : \{1, 2, \ldots, k\} \to \{1, 2\}$, we get

$$(2.3)\qquad x(g) = \sum\{x(f^r) + y_{r1} : 1 \le r \le k, g(r) = 2\} = \sum\{y_{rg(r)} : 1 \le r \le k\}.$$

To show that σ_k is k-regular, let $\mathcal{P}_r$ put $f : \{1, \ldots, k\} \to \{1, 2\}$ into classes 1 and 2 according to the value of $f(r)$.

The statement that if $|V| = \omega_k$ is ω-colored, then there is a monocolored $(k, 2)$-cube, is shown in the proof of Theorem 2.3. For the lower bound for $\lambda(\sigma_k)$ we need the following result of Alon–Füredi.

LEMMA 2.9. (N. Alon, Z. Füredi, [1]) *There are no $k - 1$ hyperplanes which cover all but exactly one node of the $(k, 2)$-cube.*

PROOF OF THEOREM 2.8. We start with the well known observation that if V is a vector space with $|V| = \kappa > \omega$, then there is a disjoint decomposition $V = \bigcup\{V(\alpha) : \alpha < \kappa\}$ such that the sets $W(\alpha) = \bigcup\{V(\beta) : \beta < \alpha\}$ are subspaces of size $< \kappa$. (One can select a basis $B = \{b_\alpha : \alpha < \kappa\}$ and put v into $V(\alpha)$ iff its largest nonzero co-ordinate is at b_α.) If X is a subset of size κ in a vector space V, let Y be the linear hull of X, decompose $Y = \bigcup\{Y(\alpha) : \alpha < \kappa\}$, as above, and take $X(\alpha) = Y(\alpha) \cap X$. Though it is no longer true that the sets $Z(\alpha) = \bigcup\{X(\beta) : \beta < \alpha\}$ will be subspaces, nevertheless they will be enough closed in the following sense: if some $y \in X$ linearly depends from $x_1, \ldots, x_t \in Z(\alpha)$ then $y \in Z(\alpha)$.

To prove the Theorem, we need to show that every V with $|V| \le \omega_{k-1}$ has an ω-coloring with no monochromatic $(k, 2)$-cube. Decompose V by our remarks above as $V = \bigcup\{V(\alpha_{k-1}) : \alpha_{k-1} < \omega_{k-1}\}$, and put $W(\alpha_{k-1}) = \bigcup\{V(\beta) < \alpha_{k-1}\}$. Notice that $|V(\alpha_{k-1})| \le \omega_{k-2}$, so we can decompose it as $V(\alpha_{k-1}) = \bigcup\{V(\alpha_{k-1}, \alpha_{k-2}) : \alpha_{k-2} < \omega_{k-2}\}$. Put also $W(\alpha_{k-1}, \alpha_{k-2}) = \bigcup\{V(\alpha_{k-1}, \beta) : \beta < \alpha_{k-2}\}$. Keep continuing. In the last step, we decompose $V(\alpha_{k-1}, \ldots, \alpha_2)$ as $\bigcup\{V(\alpha_{k-1}, \ldots, \alpha_1) : \alpha_1 < \omega_1\}$, into the union of countable sets, and also define $W(\alpha_{k-1}, \ldots, \alpha_1) = \bigcup\{V(\alpha_{k-1}, \ldots \beta) : \beta < \alpha_1\}$. The procedure decomposes V into the *disjoint* union of the countable sets

$$\{V(\alpha_{k-1}, \ldots, \alpha_1) : \alpha_{k-1} < \omega_{k-1}, \ldots, \alpha_1 < \omega_1\}.$$

Color V in such a way that points in the same set of the decomposition get different colors. In order to show that this coloration works, assume that K is a monochromatic $(k, 2)$-cube. Let $\alpha_{k-1} < \omega_{k-1}$ be the largest ordinal with a node of K in $V(\alpha_{k-1})$ (exists, as K is finite). Then let $\alpha_{k-2} < \omega_{k-2}$ be the largest ordinal with a node of K in $V(\alpha_{k-1}, \alpha_{k-2})$, etc. Finally, $\alpha_1 < \omega_1$ is the largest ordinal with a node of K in $V(\alpha_{k-1}, \ldots, \alpha_1)$. As the points in this latter set get different colors, there is just one node of K in it. The rest of K cannot be covered by $W(\alpha_{k-1}) \cup \cdots \cup W(\alpha_{k-1}, \ldots, \alpha_1)$, the union of $k - 1$ subspaces, by Lemma 2.9. ∎

3. Infinite cubes

DEFINITION 3.1. If $\{y, y_i < \omega\}$ are vectors in a vector space, and the vectors of the form $y + \sum\{y_i : i \in S\}$ (S finite) are different, then we call the set of those vectors an $(\omega, 2)$-*cube*.

THEOREM 3.2. *Every vector space V has a coloring with natural numbers such that there is no $(n, 2)$-cube in color n.*

PROOF. It suffices to give a coloring that for any color class there exists a number $t < \omega$ such that no $(t, 2)$-cube can be found in that class; we can get the result by re-indexing the classes.

We apply the coloring used in Theorem 1.3. Let $B = \{b_\alpha : \alpha < \kappa\}$ be a basis of V. If $v = \sum\{\gamma_i b_{\alpha_i} : 1 \leq i \leq n\}$ with $\alpha_1 < \cdots < \alpha_n$, let the color of v be the ordered sequence $(\gamma_1, \ldots, \gamma_n)$. Put also $\operatorname{supp}(v) = \{\alpha_1, \ldots, \alpha_n\}$.

Assume that $\{y, y_i : i < t\}$ generate a $(t, 2)$-cube monochromatic in the color determined by the sequence $(\gamma_1, \ldots, \gamma_n)$. By the Δ-system lemma of Erdős and Rado, if $t = t(n)$ is large enough, our system can be thinned out to $\{y, y_i : 1 \leq i \leq 2n + 1\}$ such that $\operatorname{supp}(y + y_i) = d \cup b_i$ $(1 \leq i \leq 2n + 1)$, where the sets $\{d, b_i : 1 \leq i \leq 2n + 1\}$ are pairwise disjoint, and the points in b_i are in the same intervals of d. Clearly, the b_i's are nonempty. If we select $i_1 < \cdots < i_{n+1}$ such that $b_{i_t} \cap \operatorname{supp}(y)$ is empty for $1 \leq t \leq n + 1$, then

$$\operatorname{supp}(y + y_{i_1} + \cdots + y_{i_{n+1}}) \supseteq b_{i_1} \cup \cdots \cup b_{i_{n+1}} \tag{3.1}$$

which is impossible, as the left hand side has size n, and the right hand side has size at least $n + 1$. ■

COROLLARY 3.3. *Every vector space has an ω-coloring with no monochromatic $(\omega, 2)$-cube.*

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DEPARTMENT OF MATHEMATICS AND STATISTICS
SIMON FRASER UNIVERSITY
BURNABY, B. C. V5A 1S6, CANADA

DEPARTMENT OF COMPUTER SCIENCE
R. EÖTVÖS UNIVERSITY
BUDAPEST, MÚZEUM KRT 6–8.
1088, HUNGARY