A geometry of the correlation space and a nonlocal degenerate parabolic equation from isotropic turbulence

ARTICLE *in* ZAMM JOURNAL OF APPLIED MATHEMATICS AND MECHANICS: ZEITSCHRIFT FÜR ANGEWANDTE MATHEMATIK UND MECHANIK · MARCH 2012

Impact Factor: 1.16 · DOI: 10.1002/zamm.201100021

2 AUTHORS:

Vladimir Nikolaevich Grebenev
Institute of Computational Technologies

Martin Oberlack
Technical University Darmstadt

Journal of Applied Mathematics and Mechanics

ZAMM

Zeitschrift für Angewandte Mathematik und Mechanik
Founded by Richard von Mises in 1921

Edited in cooperation with Martin-Luther-Universität Halle-Wittenberg and Gesellschaft für Angewandte Mathematik und Mechanik e. V. (GAMM)

Editors-in-Chief: H. Altenbach, S. Odenbach, G. Schneider, C. Wieners
Managing Editor: H. Altenbach

www.zamm-journal.org

WILEY

REPRINT

ZAMM · Z. Angew. Math. Mech. **92**, No. 3, 179 – 195 (2012) / **DOI** 10.1002/zamm.201100021

A geometry of the correlation space and a nonlocal degenerate parabolic equation from isotropic turbulence

V. N. Grebenev[1,*] **and M. Oberlack**[2,3,4,**]

[1] Institute of Computational Technologies, Russian Academy of Science, Novosibirsk 630090, Russia
[2] Technische Universität Darmstadt, Chair of Fluid Dynamics, Department of Mechanical Engineering, Petersenstr. 30, 64287 Darmstadt, Germany
[3] Center of Smart Interfaces, TU Darmstadt, Petersenstr. 32, 64287 Darmstadt, Germany
[4] GS Computational Engineering, TU Darmstadt, Dolivostr. 15, 64293 Darmstadt, Germany

Received 23 February 2011, revised 26 July 2011, accepted 9 September 2011
Published online 24 October 2011

Key words Two-point correlation tensor, Lagrangian, von Kármán-Howarth equation, initial-boundary value problem, solvability, asymptotic behavior.

Considering the metric tensor $ds^2(t)$, associated with the two-point velocity correlation tensor field (parametrized by the time variable t) in the space $\mathcal{K}^3$ of correlation vectors, at the first part of the paper we construct the Lagrangian system $(M^t, ds^2(t))$ in the extended space $\mathcal{K}^3 \times R_+$ for homogeneous isotropic turbulence. This allows to introduce into the consideration common concept and technics of Lagrangian mechanics for the application in turbulence. Dynamics in time of $(M^t, ds^2(t))$ (a singled out fluid volume equipped with a family of pseudo-Riemannian metrics) is described in the frame of the geometry generated by the 1-parameter family of metrics $ds^2(t)$ whose components are the correlation functions that evolve according to the von Kármán-Howarth equation. This is the first step one needs to get in a future analysis the physical realization of the evolution of this volume. It means that we have to construct isometric embedding of the manifold M^t equipped with metric $ds^2(t)$ into R^3 with the Euclidean metric. In order to specify the correlation functions, at the second part of this paper we study in details an initial-boundary value problem to the closure model [19,26] for the von Kármán-Howarth equation in the case of large Reynolds numbers limit.

Introduction

We deal with homogeneous isotropic turbulence and emphasis is placed on the use of the two-point velocity correlation tensor field of the velocity fluctuations to equip the correlation space $\mathcal{K}^3$ (an affine space of the correlation vectors) by a family of the pseudo-Riemannian metrics (parametrized by the time variable). It was shown in [9] that a special form of this tensor field generates the so-called semi-reducible pseudo-Riemannian metrics [13]. Moreover, they admit a 1-parameter group of (isometric) motion and the conserved quantities can be derived in the frame of the corresponding Lagrangian system constructed. This is the approach one needs to adopt with physical realization of the 1-parameter family of metrics $ds^2(t)$ under the consideration, i.e. to construct a 1-parameter family of isometric embedding $\iota(t) = \iota_t$ into the Euclidean space R^3 such that $\iota_t{}^* h^2 = ds^2(t)$ where h denotes the Euclidian metric. If we will do it then we can answer on the question how the shape of eddies are deformed in time. This is conceptually similar to visualization of Ricci flows (see, for an example [32]). In this paper, we restrict yourself by studying the evolution of a singled out fluid volume of turbulent flow where the distance between points of the space $\mathcal{K}^3$ at each fixed time is calculated by the above-mentioned pseudo-Riemannian metric. The construction of embedding ι_t (physical realization or visualization) presents a separate investigation and it will be given elsewhere. The advantage of using the 1-parameter family of metrics generated by this tensor field is that $ds^2(t)$ is deformed in time due to the evolution of the correlation functions.

The present knowledge about geometry of homogeneous isotropic turbulence comes mainly from DNS coupled with the Lagrangian description of turbulence. This is a typical direct method of modeling and it generates such complicated problems as the statistical description of trajectories, the definition of the overall size of the cloud of particles and its shape, the cooperation behavior of the particles, the geometry of their configuration and so on. An exhaustive review on this topic can be found in [29, 30]. Note that our approach is free from these introduced peculiarities by the above-mentioned

* Corresponding author E-mail: vngrebenev@gmail.com, Phone: +7 383 330 8570, Fax: +7 383 330 6342
** E-mail: oberlack@fdy.tu-darmstadt.de

Lagrangian method. We do not look on how a marked fluid particle or an ensemble of marked fluid particles (the separation distance between marked particles or the length scale of turbulent motion in a singled out direction) is traveled in turbulent flow but we prefer to observe entirely the deformation of length scales of turbulent motion localized within a singled out fluid volume of this flow in time.

The paper is organized as follows. Section 1 is devoted to studying the geometry of a domain $M^t \subset \mathcal{K}^3$ associated with a singled out volume of homogeneous isotropic fluid. Using the results of [9], we equip the correlation space $\mathcal{K}^3$ by the structure of a pseudo-Riemannian manifold. The two-point velocity correlation tensor of the velocity fluctuations is applied to generate a family of these pseudo-Riemannian metrics $ds^2(t)$.

In order to avoid a redundant complexity of the exposition of material, we use only the elementary information about the structure of the two-point velocity correlation tensor of the velocity fluctuations for homogeneous isotropic flows. The modern theory of the properties and structure of second-order (Cartesian) correlation tensors is given in [28]. We only mention that in the course of this development, the authors examined carefully several important misleading or incorrect statements that have remained uncorrected in the literature of the theory of correlation tensors for homogeneous (or isotropic turbulence). Most of these problems arisen because of confusion over the circumstances in which the generating scalar functions can be, or must, pseudoscalar, see for details [28]. What about a fine structure of the third-order correlation tensor of the velocity fluctuations and the connection with velocity helicity, we referee to [3, 4] wherein the spectral scalings are presented for two scenarios of turbulent cascades, see also [16].

In Sect. 1 we also show that the couple $(M^t, ds^2(t))$ is reduced to a Lagrangian system of the one-degree of freedom for each fixed time t due to the second conservation law obtained. The first integrals of the equations of geodesic curves form the "kinematic" conservation laws for $(M^t, ds^2(t))$. Note that the dynamics in time of $(M^t, ds^2(t))$ is described in terms of the correlation functions. In order to specify these correlation functions and find out additional information about these functions, in Sect. 2 we study a closure model of the von Kármán-Howarth equation. At first, we review the results of group analysis of the von Kármán-Howarth equation and indicate that the closure relationship suggested in [19, 27] is a differential invariant of the two-parametric scaling groups admitted by the von Kármán-Howarth equation in the limit of large Reynolds numbers. Then we exhibit a 1-parameter family (parametrized by the quantity σ related to the decay of correlation functions) of the self-similar solutions [27] (see also [22]) of the closure model for the von Kármán-Howarth equation and present several interesting geometric properties [8] of the self-similar solution that was obtained in [19, 26] for a particular case of the parameter σ determined by Symmetry Analysis. We fix the decay law of isotropic turbulence at higher Reynolds number to introduce the so-called similarity representation of solutions to the initial-boundary value problem under consideration and then we establish the well-posedness of this problem. Finally, both the large time behavior of the solution obtained and the asymptotic as the spatial self-similar variable tends to infinity are given .

1 Geometry of a singled out isotropic fluid domain

We start by reviewing elementary notions of homogeneous isotropic turbulence. Then we demonstrate how to equip the correlation space by a pseudo-Riemannian metric (which depends on time) on the base of the two-point correlation tensor that makes it possible to determine the length scales along both meridians of a cylindrical-type domain (a singled out fluid volume) of the correlation space and the radius of each cross-section of this domain. Moreover, we construct the Lagrangian system using the above-mentioned metrics. The dynamics of all quantities introduced above depends on the longitudinal and transversal correlation functions that leads to investigation of the von Kármán-Howarth equation.

1.1 Two-point velocity correlation tensor

The statistical description of fluid turbulence employ the Reynolds decomposition to separate the fluid velocity $\vec{u}$ at a point $\vec{x}$ into its mean and fluctuating components as $\vec{u} = \bar{\vec{u}} + \vec{u}'$. Here $(\bar{\cdot})$ is the mean velocity, while $\vec{u}$ is the corresponding fluctuating quantity, usually interpreted as representing turbulence. The two-point correlation tensor is defined by

$$B_{ij}(\vec{x}, \vec{x}'; t_c) = \overline{(u_i'(\vec{x}; t_c))(u_j'(\vec{x}'; t_c))}, \tag{1}$$

where $\vec{u}'(\vec{x}; t_c)$ and $\vec{u}'(\vec{x}'; t_c)$ are fluctuating velocities at the points $(\vec{x}; t_c)$ and $(\vec{x}'; t_c)$ for each fixed $t_c \in R_+$. Therefore, $B_{ij}(\vec{x}, \vec{x}'; t)$ defines a tensor field of the independent variables $\vec{x}$, $\vec{x}'$, and t on a domain D of the Euclidian space $R_+ \times R^6$.

The assumption of homogeneity and isotropy of a turbulent flow (invariance with respect to rotation, reflection, and translation) implies that this tensor may be written in the form

$$B_{ij}(\vec{r}, t_c) = \overline{u_i'(\vec{x}; t_c) u_j'(\vec{x} + \vec{r}; t_c)}, \tag{2}$$

which is defined in the so-called correlation (affine) space $\mathcal{K}^3 \equiv \{\vec{r} = (r_1, r_2, r_3)\}$, $\mathcal{K}^3 \simeq R^3$ for each t_c, where $\vec{r} = \vec{x} - \vec{x}'$. Moreover for isotropic turbulence, $B_{ij}(\vec{r}, t_c)$ is a symmetric tensor which depends on the length $|\vec{r}|$ of the vector $\vec{r} =$

$\vec{r}(x, x', t_c)$, $(x, x') \in R^6$, and the correlations B_{ij} can be expressed by using only the longitudinal correlational function $B_{LL}(|\vec{r}|, t_c)$ and the transversal correlation function $B_{NN}(|\vec{r}|, t_c)$ [25] i.e. the correlation tensor B_{ij} takes the diagonal form with the components $B_{11} = B_{LL}$ and $B_{22} \equiv B_{33} = B_{NN}$ in the system of coordinates where the vector $\vec{r}$ goes along the $\vec{r}_1$-axes. Here we note that now $|\vec{r}| = |\vec{r}_1|$ $(\vec{r} = |\vec{r}_1|\vec{e}_1)$. Further instead of directly employing the correlation function B_{LL} and B_{NN}, we use their normalized representations f and g where $B_{LL} = \overline{u'^2(t)} f(|\vec{r}_1|, t)$, $B_{NN} = \overline{u'^2(t)} g(|\vec{r}_1|, t)$ with the turbulence intensity equals $\overline{u'^2(t)} = B_{LL}(0, t)$. Due to the symmetry of B_{ij} with respect to permutation of the indexes, we can equip the correlation space $\mathcal{K}^3$ by the following family of (pseudo-)Riemannian metrics

$$dl^2(t) = \overline{u'^2(t)} f(|\vec{r}_1|, t) dr_1^2 + \overline{u'^2(t)} g(|\vec{r}_1|, t)(dr_2^2 + dr_3^2),$$

where $dl^2(t)$ are indefinite quadratic forms in general. If we consider in the correlation space an infinite Euclidean cylindrical domain (a singled out fluid tube at some fixed time) then a metric of the surface which bounds this domain can be defined by the formula

$$ds^2(t) = \overline{u'^2(t)} f(|\vec{r}_1|, t) dr_1^2 + \overline{u'^2(t)} g(|\vec{r}_1|, t) \rho^2 d\phi^2, \tag{3}$$

where ρ denotes the Euclidean radius of the cross-section $\{a\} \times S^1(\rho)$ of the surface $R \times S^1(\rho)$, $a \in R$. We can account that $\rho = 1$ and identify this manifold with $R \times S^1(1)$. The following two sizes of this manifold presents interest: the length $\mathcal{L}(t)$ along the meridian and the radius $\mathcal{R}_a(t)$ of the cross-section $\{a\} \times S^1(1)$ which are determined by the formulas

$$\mathcal{L}(t) = 2\int_0^\infty \sqrt{\overline{u'^2(t)} f(|\vec{r}_1|, t)} d|\vec{r}_1|, \qquad \mathcal{R}_a(t) = \sqrt{|\overline{u'^2(t)} g(|a|, t)|}. \tag{4}$$

The functions f, g are non-dimensional with $f(0, t) = g(0, t) = 1$ and physically f is a positive function such that $f \to 0$ $(g \to 0)$ as $|\vec{r}_1|$ tends to infinite. Moreover, f and g are bounded even functions such that $f \leq 1$, $|g| \leq 1$. For all correlation interval $[0, \infty)$ the integral length scale (see [25] or other handbooks on turbulence) is defined respectively by the formula

$$\ell_t = \int_0^\infty f(|\vec{r}_1|, t) d|\vec{r}_1|$$

and physically ℓ_t is a bounded quantity for each time t provided f goes faster to zero than $|\vec{r}_1^{\,-2}|$, when $|\vec{r}_1|$ tends to infinity [31]. It means that the integral in (4) converges as $|\vec{r}_1| \to \infty$ for each time $t \in R_+$ and the map

$$q(x, t) = \int_0^x \sqrt{\overline{u'^2(t)} f(|\vec{r}_1|, t)} d|\vec{r}_1|, \quad x \in R$$

acts as $(-\infty, \infty) \mapsto [-\mathcal{L}(t), \mathcal{L}(t)]$. The normalized longitudinal correlational function $f(|\vec{r}_1|, t)$ is dynamically evolved due to the von Kármán-Howarth equation [14]

$$\frac{\partial \overline{u'^2(t)} f(|r_1|, t)}{\partial t} = \frac{1}{r_1^4} \frac{\partial}{\partial r_1} r_1^4 \left(\overline{u'^2(t)}^{3/2} h(|r_1|, t) + 2\nu \frac{\partial}{\partial r_1} \overline{u'^2(t)} f(|r_1|, t) \right). \tag{5}$$

h is the normalized triple-correlation function and $\overline{u'^2(t)}$ is the turbulence intensity (a positive everywhere function that vanishes on infinity) or the velocity scale for the turbulent kinetic energy, $\overline{u'^2(t)}^{3/2}$ determines the scale for the turbulence transfer. This single equation directly follows from the Navier-Stokes equation (see, e.g. [25]) and contains two unknowns f, h with the turbulence intensity equals $\overline{u'^2(t)} = B_{LL}(0, t)$ which cannot be defined from (5) without the use of additional hypothesis. The normalized transversal correlation function g satisfies the relation (taken from the continuity) [31]

$$g(|\vec{r}_1|, t) = f(|\vec{r}_1|, t) + \frac{r_1}{2} \frac{\partial}{\partial r_1} f(|\vec{r}_1|, t). \tag{6}$$

The physical property that f decays faster that $|\vec{r}_1|^{-2}$ on infinity together with Eq. (6) yields [31]

$$\int_0^\infty |\vec{r}_1| g(|\vec{r}_1|, t) d|\vec{r}_1| = 0. \tag{7}$$

Hence $g(|\vec{r}_1|, t)$ is an alternative sign function. Typical forms of experimentally measured functions f and g are given on Fig. 1. The data presented we use to determine the qualitative behaviors of f and g, in particular, the algebraic properties of these correlation functions. Thus, we will assume that f is a positive everywhere function, g changes sign only in intervals $(-\varepsilon + |\vec{r}_1^*|, |\vec{r}_1^*| + \varepsilon)$, $\varepsilon > 0$. g is a positive function on $[0, \pm|\vec{r}_1^*|)$ and therefore $g < 0$ outside of $[-|\vec{r}_1^*|, |\vec{r}_1^*|]$. The change sign of g means that the quadratic forms $dl^2(t)$ (therefore $ds^2(t)$) have a variable signature.

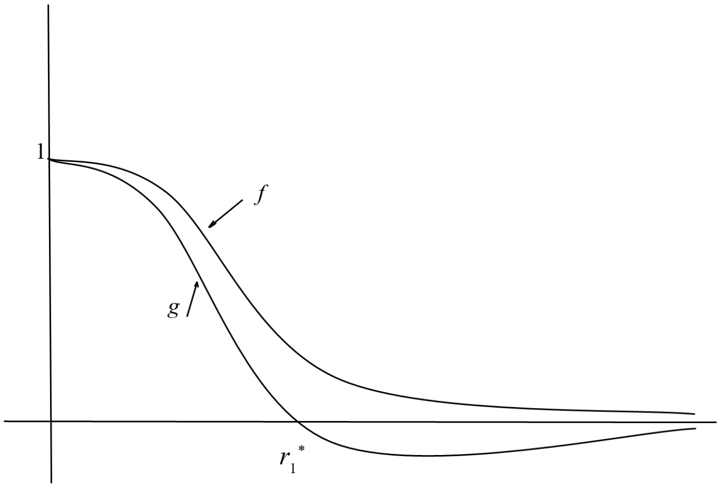

Fig. 1 Typical forms of the normalized longitudinal and transversal correlation functions.

1.2 Lagrangian system generated by $ds^2(t)$

We rewrite the metric $ds^2(t)$ in terms of the variable q

$$ds^2(t) = \overline{u'^2(t)}\{dq^2 + G(q,t)d\phi^2\}, \quad G(q,t) = g(|r_1|,t). \tag{8}$$

We need the following definition [13]: *a point p_0 is called the pole of a (pseudo-)Riemannian manifold M if p_0 is a fixed point of a group of diffeomorphisms $\mathbf{g}(\vec{x},\vec{a})$, $\vec{a} = (a_1,\ldots,a_r)$ which acts on M.*

The metric (8) admits an one-parametric group of (isometric) motion $\mathbf{g}_\tau(\vec{p}) \equiv \mathbf{g}(\vec{p},a_1)$, $\vec{p} = (q,\phi)$ of the form

$$\mathbf{g}_\tau \,:\, (q,\phi) \mapsto (q,\phi+\chi\tau), \quad \chi = \text{const}$$

with the generator

$$X = \xi^i \frac{\partial}{\partial p^i} \equiv \chi \frac{\partial}{\partial \phi}.$$

The scalar product of the generator X equals

$$X^2 = \langle X, X\rangle = \left\langle \chi\frac{\partial}{\partial\phi}, \chi\frac{\partial}{\partial\phi}\right\rangle \equiv \chi^2 \left\langle \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right\rangle = \overline{u'^2(t)}\chi^2 G(q,t) \tag{9}$$

for each time t. We note that if $p = p_0$ (p_0 is the pole of $\mathbf{g}_\tau$) then $X^2(p_0) = 0$ and due to (9) p_0 coincides with the roots of the equation $G(q,\cdot) = 0$. Therefore the points $q^* \in [-\mathcal{L}(t),\mathcal{L}(t)]$ wherein G vanishes are the poles of $\mathbf{g}_\tau$. In view of our assumption on $g(|r_1|,t)$, the equation $G(q,\cdot) = 0$ has only 4 roots q_i^*, $i = 1,\ldots,4$ such that $|q_1^*| = q_4^* = \mathcal{L}(t)$ and $|q_2^*| = q_3^*$. Thus the metric (8) has the different signature for $q \in I_1 = (q_2^*, q_3^*)$ and $q \in I_2 = (-\mathcal{L}(t), q_2^*)$, $q \in I_3 = (q_3^*, \mathcal{L}(t))$, respectively, where q_i^* depends on t. This metric determines for $q \in I_1$ the element of length of the surface of revolution in R^3 and the radius-vector $\vec{R} = (q,\phi,\cdot)$ of this surface is given by

$$\vec{R}(q,\phi,\cdot) = \left(q, \{\overline{u'^2(t)}G\}^{1/2}(q,\cdot)\cos\phi, \{\overline{u'^2(t)}G\}^{1/2}(q,\cdot)\sin\phi\right).$$

Therefore the model manifold defined by (8) for $q \in I_1$ is a cylindrical-type surface $M^t_{I_1} = (q_2^*, q_3^*) \times S^1(1)$ wherein the radius of the cross-section $\{q\} \times S^1(1)$ equals $G^{1/2}(q,t)$. For the $q \in I_i$, $i = 2,3$ where G is negative the positive defined metric

$$ds^2(t) = \overline{u'^2(t)}\{dq^2 - |G(q,t)|d\phi^2\}, \quad G(q,t) = g(|\vec{r}_1|,t) \tag{10}$$

can be realized (see for details [9]) as a surface of revolution (for each fixed time) embedded in the Minkowski space $R^3_{1,2}$ with the element of length

$$d\rho^2 = dx_1^2 - dx_2^2 - dx_3^2$$

when the form $ds^2(t)$ is of a fixed sign [7]. Here the rotation presents the motion along the pseudo-circle of the radius $|G(q,\cdot)|^{1/2}$, $q \in I_i$. Indeed, let us fix the point $p_a = (q_a,\phi_a)$ on the cross-section $\{q_a\} \times S^1(1)$ and consider the action of the group $\mathbf{g}_\tau$ on p_a i.e. the orbit $\mathbf{O}_{p_a}\,:\,\tau\mapsto\mathbf{g}_\tau(p_a)$. This action is a motion along $\{q_a\}\times S^1(1)$ and if p_a does not coincide with the poles $\mathbf{g}_\tau$ then $\mathbf{O}_{p_a}$ is a not compact set [13]. In particular, $\mathbf{O}_{p_a}\subseteq\{(x_1,x_2)\,:\,x_1^2-x_2^2=\overline{u'^2(t)}|G(q_a,t)|\}$ for each fixed time t which coincides with the so-called pseudo-circle under the embedding $M^t_{I_3}$ ($M^t_{I_2}$) into the Minkowski space $R^3_{1,2}$. Moreover, the poles are saddle points of a negative index for the orbits $\mathbf{O}_p$, $p\in M^t_{I_3}$ ($M^t_{I_2}$). The cross-sections $\{q_a\}\times S^1(1)$ of $M^t_{I_3}$ ($M^t_{I_2}$) for $q_a\in\{q_3^*,\mathcal{L}(t)\}$ (respectively $q_a\in\{-\mathcal{L}(t),q_2^*\}$) are the pseudo-circles of zero radius and consist of the isotropic rays with the initial points q_3^* and $\mathcal{L}(t)$ (respectively $-\mathcal{L}(t)$ and q_2^*). The action of $\mathbf{g}_\tau$ on the point p is a motion along these piecewise linear isotropic curves when $p\in\{-\mathcal{L}(t),q_2^*,q_3^*,\mathcal{L}(t)\}$. We can identify $M^t_{I_3}$ ($M^t_{I_2}$) with the foliation space of orbits $M^t_{I_j}=\bigcup_p\mathbf{O}_p$ and associate the modulus of the transversal correlation function $G(q,t)$ with the length of the velocity vector $\vec{\xi}(p)$ of the orbit $\mathbf{O}_p$ by $|\vec{\xi}(p)|=\overline{u'^2(t)}\chi\sqrt{|G(q,t)|}$. The length of displacement of the point p (or the length of arch), with respect to the vector field generated by $\mathbf{g}_\tau(p)$, is determined by the formula

$$\lambda(\phi_a,\phi_b)=\int_{\phi_a}^{\phi_b}\chi\sqrt{\overline{u'^2(t)}|G(q,t)|}d\phi\equiv\chi\sqrt{\overline{u'^2(t)}|G(q,t_c)|}(\phi_b-\phi_a),\quad \chi=\text{const}$$

for each fixed time that defines the following length scale along the orbit $\mathbf{O}_p$

$$\lambda_{\mathbf{O}_p}=\chi\sqrt{\overline{u'^2(t)}|G(q,t)|}2\pi\phi,\quad q\in(q_3^*,\mathcal{L}(t)),\quad \chi=1.$$

The constant χ can be fixed by normalizing the velocity vector $\vec{\xi}(p)$.

The Gaussian curvature K of the manifold $M^t_{I_1}$ admits a singular behavior at the poles q_2^* and q_3^*, where G vanishes for $G_q(q_i^*,t)\neq 0$, $i=2,3$. In the case when the poles are multiplicative zeroes of some finite order i.e. $G(q_i^*,t)=G_q(q_i^*,t)=\ldots G_{q\ldots q}(q_i^*,t_c)=0$ then the direct calculations show that again K is a singular function at q_i^* for all t. If zero is of infinite order then $G(q,t)\equiv 0$ in a neighborhood of q_2^* (q_3^*) under the assumption that G is an analytical function. The same argument we can apply to investigation of the behavior of the Gaussian curvature of the manifold $M^t_{I_3}$ ($M^t_{I_2}$) for the poles g_3^* (g_2^*). In the case of the pole $q_4^*=\mathcal{L}(t)$ ($q_1^*=-\mathcal{L}(t)$) we use that f and therefore g have to go faster to zero than $|\vec{r}_1|^{-2}$ when $|\vec{r}_1|\to\infty$. Employing the formula $G_q=f^{-1/2}g_r$, $r=|\vec{r}_1|$ and the above-mentioned assumption about the behavior f (and G) as $|\vec{r}_1|\to\infty$ (and $q\to\pm\mathcal{L}(t)$), we derive in terms of the variable $|\vec{r}_1|$ that $|K|$ have to go faster to infinity than $|\vec{r}_1|^2$ when $|\vec{r}_1|\to\infty$ ($q\to\pm\mathcal{L}(t)$).

Remark 1.1. The peculiarity of the metric presented consists in the arising a singularity of "the shrinking cylinder type" due to alternative sign of the transversal correlation function B_{NN}. It means that we can describe "shrinking phenomenon" for a singled out fluid tube in terms of singularity points of the metric (8). In Sect. 2 we cover this result by exposing B_{NN} of the same behavior as the self-similar solution [26] of a closure model for the von Kármán-Howarth equation.

To accomplish our analysis, we present the metric $ds^2(t)$ in the so-called conformal form at first for $q\in I_1$. Let us consider on the interval I_1 a new measure dz with the density $\sigma(q,t)=1/\sqrt{\overline{u'^2(t)}G(q,t)}$ for each fixed time and rewrite $ds^2(t)$ in the form

$$ds^2(t)=F(z,t)\left(dz^2+d\phi^2\right),\tag{11}$$

where $F(z,t)=\overline{u'^2(t)}G(q,t)$. For $q\in I_i$, $i=2,3$ this metric takes the form

$$ds^2(t)=F(z,t)\left(dz^2-d\phi^2\right).\tag{12}$$

Here $F(z,t)=\overline{u'^2(t)}|G(q,t)|$ and $dz=\sqrt{\overline{u'^2(t)}|G(q,t)|}dq$. The equations for geodesic curves on the manifold $M^t_{I_1}$ in the frame of (11) read

$$\frac{d^2z}{d\theta^2}+\frac{F_z}{2F}\left[\left(\frac{dz}{d\theta}\right)^2-\left(\frac{d\phi}{d\theta}\right)^2\right]=0,\quad \frac{d^2\phi}{d\theta^2}+\frac{F_z}{F}\frac{dz}{d\theta}\frac{d\phi}{d\theta}=0.\tag{13}$$

While for the metric (12) given on $M^t_{I_i}$, $i=2,3$ we have

$$\frac{d^2z}{d\theta^2}+\frac{F_z}{2F}\left[\left(\frac{dz}{d\theta}\right)^2+\left(\frac{d\phi}{d\theta}\right)^2\right]=0,\quad \frac{d^2\phi}{d\theta^2}+\frac{F_z}{F}\frac{dz}{d\theta}\frac{d\phi}{d\theta}=0.\tag{14}$$

The systems of Eqs. (13) and (14) admit the following first integrals

$$F(z,t)\phi_\theta=\mathcal{M},\quad F(z,t)(z_\theta^2+\phi_\theta^2)=\mathcal{N}$$

and

$$-F(z,t)\phi_\theta = \mathcal{M}, \quad F(z,t)(z_\theta^2 - \phi_\theta^2) = -\mathcal{N}$$

correspondingly where $\mathcal{M}$ and $\mathcal{N}$ depend on the time t in general. We can account that $\mathcal{N} = 1$ if we take θ as the canonical parameter for geodesic curves. Let us consider now the vector $\vec{M} = F(z,t)\phi_\theta[dz, d\phi]_{dl^2}$. This is the so-called momentum vector from the cotangent vector space to the one-parametric group of (isometric) motion $\mathbf{g}_\tau$. Here $[\cdot,\cdot]_{dl^2}$ denotes the vector product with respect to the metric $dz^2 \pm d\phi^2$. The equality $\pm F(z,t_c)\phi_\theta \equiv \mathcal{M} = \text{const}$ means that the momentum vector $\vec{M}$ is a conserved quantity: $d\vec{M}/d\theta = 0$ for each fixed time t. On the plane $(dz, d\phi)$ the momentum vector $\vec{M}$ is calculated by the formula $\vec{M} = [\vec{p}, \vec{p}_\theta]_{dl^2}$, $\vec{p}(\theta,t) = (z(\theta,t), \phi(\theta,t))$. Without loss of generality (in view of the invariance of $ds^2(t)$ with respect to $\mathbf{g}_\tau$), we can write $\vec{p}(\theta,t) = F^{1/2}(z,t)dz$ and $d\vec{p}/d\theta = (F^{1/2})_\theta dz + F^{1/2}dz_\theta = (F^{1/2})_\theta dz + F^{1/2}\phi_\theta d\phi$. Therefore

$$\vec{M} = [\vec{p}, (F^{1/2})_\theta dz]_{dl^2} + [\vec{p}, F^{1/2}\phi_\theta d\phi]_{dl^2} = F\phi_\theta [dz, d\phi]_{dl^2},$$

where the first bracket vanishes and the length of the vector $[dz, d\phi]_{dl^2}$ equals -1 (1) for the signature $(+-)$ $((++))$. Using the above-mentioned first integrals or the compatible differential constraints to (13), (14), the reduction of (13) on invariant manifolds defined by these constraints leads to the following equation

$$\frac{d^2 z}{d\theta^2} + \frac{F_z}{2F}\left(\frac{\pm\mathcal{N} \mp 2\mathcal{M}^2 F^{-1}}{F}\right) = 0,$$

where the upper sign in $\pm \cdots \mp$ corresponds to the signature $(++)$ and the lower symbols are reserved for the signature $(+-)$. The second equation of this system is satisfied identically on the constraint $\pm F(z,t)\phi_\theta = \mathcal{M}$. As a result, we obtain

$$\frac{d^2 z}{d\theta^2} = -\frac{dV}{dz}, \quad \text{where } V = \mp\frac{\mathcal{N}}{2F(z,t)} \pm \frac{\mathcal{M}^2}{2F^2(z,t)} + \text{const} \tag{15}$$

considering t as a parameter. Equation (15) coincides with the well-known equation of the motion of a unit mass point in the potential field with the effective potential energy V (the terminology of Newtonian mechanics is used). Then the Lagrangian $\mathcal{S}_{M^t_{I_i}}$ and the "total mechanical" energy $E_{M^t_{I_i}}$ read

$$\mathcal{S}_{M^t_{I_i}} = \frac{1}{2}\left(\frac{dz}{d\theta}\right)^2 - V, \quad E_{M^t_{I_i}} = \frac{1}{2}\left(\frac{dz}{d\theta}\right)^2 + V.$$

Here $E_{M^t_{I_i}}$ is the conserved quantity on solutions of Eq. (15) for each fixed time t. Therefore the couple $(M^t_{I_i}, ds^2(t))$ generates a Lagrangian system of the one-degree of freedom. Hence

$$d\theta = \frac{dz}{\sqrt{2\left(E_{M^t_{I_i}} - V\right)}}, \quad d\phi = \pm\frac{\mathcal{M}d\theta}{F(z,t)} \tag{16}$$

and the standard excepting $d\theta$ leads to the following equation for $z = z(\phi)$

$$\frac{dz}{d\phi} = \frac{F(z,t_c)}{\pm\mathcal{M}}\sqrt{2\left(E_{M^t_{I_i}} - V\right)}.$$

Geometrically, the parameter θ is the length of arch of a non-isotropic geodesic curve with the initial position at the corresponding pole. If this length is changed then in during this process the "total mechanical" energy and momentum are preserved quantities along geodesic curves. The angle α between the vector

$$\vec{\xi}(p) = \left.\frac{d\mathbf{g}_\tau(p)}{d\tau}\right|_{\tau=0} \quad \text{and the velocity vector } \vec{v}_{\vec{\gamma}} = (z_\theta, \phi_\theta) \quad \text{of the geodesic curve } \vec{\gamma}$$

equals

$$\cos\alpha = \frac{\langle \vec{v}_{\vec{\gamma}}, \vec{\xi}(p)\rangle_{ds^2}}{\sqrt{|\langle \vec{v}_{\vec{\gamma}}, \vec{v}_{\vec{\gamma}}\rangle_{ds^2}\langle \vec{\xi}(p), \vec{\xi}(p)\rangle_{ds^2}|}} = \frac{\phi_\theta F}{\sqrt{|F(z_\theta^2 \pm \phi_\theta^2)|}} = \frac{\phi_\theta F}{\sqrt{|\pm\mathcal{N}F|}} = \frac{\pm\mathcal{M}}{\sqrt{|\pm\mathcal{N}F|}}$$

or

$$\sqrt{F}\cos\alpha = \pm\mathcal{M}/\sqrt{|\pm\mathcal{N}|}.$$

The case of $\cos\alpha = 0$ corresponds to the motion along the orbit $\mathbf{O}_p$, where $\vec{\xi}^2(\mathbf{g}_\tau(p)) = \text{const}$ ($F \equiv \text{const}$) and this orbit presents unclosed curve in the case of $(M^t_{I_i}, ds^2(t))$ for $i = 2, 3$. Notice that in the sharp contrast to the previous consideration about the dynamics of a fluid particle on the surface of revolution $M^t_{I_1}$ the zero value of $|\vec{\xi}^2(\mathbf{g}_\tau(p))|$ means that $\mathbf{O}_p$ belongs to the so-called isotropic curve ($ds^2(t) = 0$) which is given by $\phi = z - z_0$ (or $\phi = -(z - z_0)$). This curve coincides with the pseudo-circle of zero radius. The canonical parameter θ (the length of the corresponding isotropic curve) is defined by the formula

$$\theta = \int_{z_0}^{z} F(z,t)dz \quad \text{or in the variable } q \quad \theta = \int_{q_0}^{q} \sqrt{\overline{u'^2(t)}|G(q,t)|}dq. \tag{17}$$

Here $(z_0, 0)$ is the coordinate of the corresponding pole of $M^{t_c}_{I_3}$ ($M^t_{I_2}$). The existence of closed geodesic curves depends on the form of the potential V.

To find the quantities $\mathcal{L}(t)$, $\mathcal{R}_a(t)$, the effective potential energy V and therefore the Lagrangian $\mathcal{S}_{M^t_{I_i}}$, we have to specify the correlation functions B_{LL} and B_{NN}. Below we consider a closure model that connects $f(|\vec{r}_1|, t)$ and $h(|\vec{r}_1|, t)$. For brevity we set $r = |\vec{r}_1|$.

Remark 1.2. The above-mentioned conservation laws are standard consequence of invariance of the flow with respect to the translation and rotation groups. Note that the quantity $F(z,t)(z^2_\theta \pm \phi^2_\theta)$ means geometrically that along geodesic curves the length of tangent vectors is conserved. Together with the length of the momentum vector $\vec{M}$ they present new conserved quantities arising in homogeneous isotropic turbulence.

2 The closed von Kármán-Howarth equation

In this section we consider the closure model for the von Kármán-Howarth equation [14] suggested in [19, 26]. First, we prove the existence and uniqueness theorem to the initial-boundary value problem posed. Then, we investigate both the large-time behavior of solutions of the problem and their asymptotic properties as $r \to \infty$.

2.1 Basic equations

We review shortly the closure approximations suggested and expose some properties of exact solutions obtained for the closure approximation [19, 26].

The von Kármán-Howarth equation for homogeneous isotropic turbulence given by

$$\frac{\partial B_{LL}}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r} r^4 \left(B_{LL,L} + 2\nu\frac{\partial B_{LL}}{\partial r}\right) \tag{18}$$

or in the normalized form

$$\frac{\partial \overline{u'^2(t)} f(r,t)}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\left(\overline{u'^2(t)}^{3/2} h(r,t) + 2\nu\frac{\partial \overline{u'^2(t)} f(r,t)}{\partial r}\right)\right] \tag{19}$$

is an exact analytical relation between the time rate of change of the second-order two-point velocity correlation function B_{LL} and the gradient of the third-order two-point velocity correlation function $B_{LL,L}$. Equation (18) (or (19)) is the lowest-order two-point statistical equation for turbulence dynamics and may be understood as a relationship between the rate of change of energy in scales $\sim r$ to the flux of energy through scales $\sim r$. The gradient of the third-order two-point velocity correlation function arises from the nonlinear term in the Navier-Stokes equations and presents the extra unknown function to be modeled which leads to the so-called closure problem.

Starting with Heisenberg [12] many gradient-type closures have been proposed for this problem but much less attention has been devoted to closures in real space coordinates. Hasselmann [11] and Domoradzki [6] are concentrated on modeling the inertial subrange but beside minor deficiencies overlooked the serious shortcoming not to fulfill continuity under all initial conditions. Millionshtchikov in [23] outlined a more general hypotheses which produces parametric models of isotropic turbulence. The essence of these hypotheses is that $B_{LL,L}$ is modeled by the following relation of gradient-type

$$B_{LL,L} = 2\nu_T\frac{\partial B_{LL}}{\partial r}, \tag{20}$$

where ν_T has the dimension of the turbulent kinematic viscosity which is characterized by a single length and velocity scale. Millionshtchikov's hypotheses [23] assumes that

$$\nu_T = \kappa_1 \overline{u'^2}^{1/2} r, \qquad \overline{u'^2} = B_{LL}(0,t), \tag{21}$$

where κ_1 denotes an empirical constant. The further requirement that the power law of the inertial subrange is reproduced, is essential of the model [19, 26]. We reproduce this approximation following Oberlack–Peters model [26]

$$\nu_T = \kappa_2 r D_{LL}^{1/2}, \quad D_{LL} = 2[\overline{u'^2} - B_{LL}(r,t)], \quad \kappa_2 = \frac{\sqrt{2}}{5C^{3/2}}, \tag{22}$$

where C is the Kolmogorov constant. In [26] a closure model connecting f and h was developed to capture all properties of the Kolmogorov turbulence theory for f and h and also the proper limiting behavior for $r \to 0$ and $r \to \infty$. Comparison with experimental data was done calculating the triple correlation h (the normalized triple-correlation function) out of measured values of the normalized double correlation function f using the model (20), (22). The normalized double correlation function f was recovered simultaneously with the triple correlation h in Stewart/Taunsend experiments [33]. Good agreement between measured and computed values of h was achieved within the range of the reliable data [26]. Note that the Millionshtchikov closure model can be considered as the asymptotic approximation of (22) for $r \gg 1$, because continuity requires B_{LL} to decay faster than r^{-2} in infinity [31].

In [27] it was recognized that in zero limit of ν (or alternatively in the limit of large Reynolds number) the unclosed Eq. (18) admits the following two scaling groups

$$G^{a_1} \,:\, t^* = t, \quad r^* = e^{a_1} r, \quad \overline{u'^2}^* = e^{2a_1}\overline{u'^2}, \quad f^* = f, \quad h^* = h, \tag{23}$$

$$G^{a_2} \,:\, t^* = e^{a_2} t, \quad r^* = r, \quad \overline{u'^2}^* = e^{-2a_2}\overline{u'^2}, \quad f^* = f, \quad h^* = h, \tag{24}$$

and a translation group in t

$$G^{a_3} \,:\, t^* = t + a_3, \quad r^* = r, \quad \overline{u'^2}^* = \overline{u'^2}, \quad f^* = f, \quad h^* = h, \tag{25}$$

with $(a_1, a_2, a_3) \in R$ are the group parameters. These scaling groups are generated by

$$X_{a_1} = r\frac{\partial}{\partial r} + 2\overline{u'^2}\frac{\partial}{\partial \overline{u'^2}}, \quad X_{a_2} = t\frac{\partial}{\partial t} - 2\overline{u'^2}\frac{\partial}{\partial \overline{u'^2}}.$$

The two groups X_{a_1} and X_{a_2} may be combined to generate the two-parametric Lie scaling group

$$G^{a_1,a_2} \,:\, t^* = e^{a_2} t, \quad r^* = e^{a_1} r, \quad \overline{u'^2}^* = e^{2(a_1 - a_2)}\overline{u'^2}, \quad f^* = f, \quad h^* = h.$$

We note that

$$\hat{K} = \frac{r\overline{u'^2}^{3/2}(1-f)^{1/2}\partial f/\partial r}{t^{-3(\sigma+1)/(\sigma+3)}}$$

is a differential invariant of G^{a_1,a_2}. It means that the closure relationships (20), (22) are compatible with the two-parametric scaling group G^{a_1,a_2}. To this end, the closure model connecting f and h was developed which captures all properties of the Kolmogorov turbulence theory for f and h and also the proper limiting behavior for $r \to 0$ and $r \to \infty$ i.e.

- Kolmogorov 2/3-law: $\overline{u'^2}(1-f) = C(\varepsilon r)^{\frac{2}{3}}$ with the Kolmogorov constant $C \approx 2.3$ in the range $\eta \ll r \ll \ell_t$
- Kolmogorov 4/5-law: $\overline{u'^2}^{\frac{3}{2}} h = \frac{2}{15}\varepsilon r$ in the range $\eta \ll r \ll \ell_t$ (the name 4/5-law emerged from a somewhat different notation)
- $r \to 0$: $\overline{u'^2}(1-f) = \frac{1}{30}\frac{\varepsilon r^2}{\nu}$ and $h = \frac{1}{3}h_0''' r^3$
- $r \to \infty$: f decays faster than r^{-2}

where

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} \quad \text{and} \quad \ell_t = \int_0^\infty f \mathrm{d}r \tag{26}$$

are the Kolmogorov and the integral length scales, respectively, and ε is the dissipation rate of the turbulent kinetic energy. Though not discussed in [26] the closure model has to obey the full group properties of the von Kármán-Howarth equation and hence has to be conformal to (23)–(25).

The resulting equation that we use in order to study homogeneous isotropic dynamics (therefore the evolution of a singled out tube domain i.e. $\mathcal{L}(t)$, $\mathcal{R}_a(t)$) reads

$$\frac{\partial B_{LL}(r,t)}{\partial t} = \frac{2\kappa_2}{r^4}\frac{\partial}{\partial r}\left[r^4\left(r\sqrt{B_{LL}(0,t)-B_{LL}(r,t)}\frac{\partial B_{LL}(r,t)}{\partial r} + 2\nu\frac{\partial B_{LL}(r,t)}{\partial r}\right)\right] \tag{27}$$

which subject to the following initial-boundary value conditions

$$B_{LL}(r,0) = B_{0LL}(r), \quad r \geq 0 \tag{28}$$

$$B_{LL,L}(r,t) \equiv 2\kappa_2 r\frac{\partial B_{LL}(r,t)}{\partial r} = 0, \quad \text{for } r=0, \quad t\geq 0 \tag{29}$$

$$B_{LL}(r,t) \to 0, \quad \text{as } r\to\infty, \quad t \geq 0. \tag{30}$$

2.2 Self-similar solutions

In the case of large Reynolds numbers limit or in the limit of $\nu \to 0$ Eq. (27) can be written in the following inviscid form

$$\frac{\partial \overline{u'^2(t)}f(r,t)}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4 2\kappa_2 r\sqrt{\overline{u'^2}(1-f)}\frac{\partial\overline{u'^2(t)}f(r,t)}{\partial r}\right] \tag{31}$$

and invariants of the scaling group G^{a_1,a_2} are

$$\xi = \frac{r}{t^{2/(\sigma+3)}}, \qquad \hat{f} = \frac{\overline{u'^2}f}{t^{-2(\sigma+1)/(\sigma+3)}}, \qquad \sigma = \frac{2a_2-3a_1}{a_1}. \tag{32}$$

Here r is scaled by the integral length scale $\ell_t \propto t^{2/(\sigma+3)}$ and $\overline{u'^2}$ accordingly $\overline{u'^2} \propto t^{-2(\sigma+1)/(\sigma+3)}$, where a_i, $i=1,2$ as given above. Note that the translation group in time t may also be invoked which in fact gives a time shift $t \to t+a_3$.

The invariants above enable us to reduce the number of variables in Eq. (31) and as a result, to get the ordinary differential equation

$$\frac{2\kappa_2}{\xi^4}\frac{d}{d\xi}\left[\xi^5(1-\hat{f})^{1/2}\frac{d\hat{f}}{d\xi}\right] + \delta\xi\frac{d\hat{f}}{d\xi} + \gamma\hat{f} = 0, \tag{33}$$

subject to the boundary conditions

$$\xi = 0 : \hat{f}(\xi) = 1 \quad \text{and} \quad \xi\to+\infty : \hat{f}(\xi) \to 0, \tag{34}$$

where

$$\delta = \frac{2}{\sigma+3}, \quad \gamma = 2\frac{\sigma+1}{\sigma+3}, \tag{35}$$

Here σ is undetermined and hence (33) together with (34) generates an one-parameter family of boundary value problems. We note also that in the limit of large Reynolds numbers that σ [31] intimately related to the spatial decay of the normalized longitudinal correlation function f or the low wave-number dependence of the energy spectrum. The solvability of the corresponding nonlinear eigenvalue problem (33)–(35) was studied in [22].

Considering the asymptotic behavior of $\hat{f}$ for large values $\xi\to\infty$ we can rewrite formally Eq. (33) for large values $\xi\to+\infty$ in the following asymptotic form

$$\frac{2\kappa_2}{\xi^4}\frac{d}{d\xi}\left[\xi^5\frac{d\hat{f}}{d\xi}\right] + \delta\xi\frac{d\hat{f}}{d\xi} + \gamma\hat{f} = 0, \tag{36}$$

using that $(1-\hat{f})^{1/2} \to 1$ as $\xi\to+\infty$. In the rescaled variable $\hat{\xi} = (\delta/2\kappa_2)\xi$ this equation is of the canonical form

$$\frac{1}{\breve{\xi}^9}\frac{d}{d\breve{\xi}}\left[\breve{\xi}^9\frac{d\breve{f}}{d\breve{\xi}}\right] + \frac{\breve{\xi}}{2}\frac{d\breve{f}}{d\breve{\xi}} + \frac{\gamma}{\delta}\breve{f} = 0, \qquad \breve{f}(\breve{\xi}) = \hat{f}(\hat{\xi})$$

or

$$\frac{1}{\breve{\xi}^9}\frac{d}{d\breve{\xi}}\left[\breve{\xi}^9\frac{d\breve{f}}{d\breve{\xi}}\right] + \frac{\breve{\xi}}{2}\frac{d\breve{f}}{d\breve{\xi}} + (\sigma+1)\breve{f} = 0, \qquad \breve{\xi} = 2\hat{\xi}^{1/2} \tag{37}$$

and it is well-known that for $\sigma + 1 \in (0, 5)$ admits decreasing positive everywhere solutions with the algebraic asymptotic:

$$\breve{f}(\breve{\xi}) = M\breve{\xi}^{-2(\sigma+1)} + \ldots, \qquad M = \mathrm{const} > 0;$$

if $\sigma + 1 = 5$, then

$$\breve{f}(\breve{\xi}) = Me^{-\breve{\xi}^2/4};$$

for $\sigma + 1 > 5$ the equation has no positive everywhere solutions.

Let us consider again the problem (33)–(35). In the case of $\sigma = 4$ Eq. (33) is integrated explicitly [19,26]. The exponent $\sigma = 4$ corresponds to the Loitsyansky integral arising in isotropic turbulence. A solution of (33), (34) for $\sigma = 4$ reads

$$\xi = 7\kappa_2 \left(\ln \left[\frac{1 + (1 - \hat{f})^{1/2}}{1 - (1 - \hat{f})^{1/2}} \right] - 2(1 - \hat{f})^{1/2} \right). \tag{38}$$

It follows from the formula (38) that

$$\hat{f}(\xi) \approx \exp\left(-\frac{\xi}{7\kappa_2} \right) \quad \text{for } \xi \gg 1, \qquad \hat{f}(\xi) \approx 1 - \frac{3}{14\kappa_2}\xi^{2/3}, \quad \xi \ll 1. \tag{39}$$

This guarantee that the Loitsyansky integral

$$\Lambda = \overline{u'^2} \int_0^\infty r^4 f(r,t)dr = \int_0^\infty \xi^4 \hat{f}(\xi)d\xi$$

exists. The computed evolution of $\overline{u'^2}(t)$ and the integral length scale $\ell_t(t)$ are

$$\overline{u'^2}(t) \propto (t + a)^{-10/7}, \quad \ell_t \propto (t + a)^{2/7}, \quad a \in R. \tag{40}$$

The formulas (40) coincide with the well-known Kolmogorov laws [15] and present an example of such realization.

This solution admits the following geometric interpretation [8]: let us rewrite (38) in the form

$$\frac{1}{14\kappa_2}\xi = -(1 - \hat{f})^{1/2} + \frac{1}{2} \ln \left[\frac{1 + (1 - \hat{f})^{1/2}}{1 - (1 - \hat{f})^{1/2}} \right]$$

and introduce the new variables

$$x = \xi/14\kappa_2, \quad y = \hat{f}^{1/2}.$$

Then the equation is transformed to the well-known *tractrix equation* [24]

$$x = x(y) = -(a^2 - y^2)^{1/2} + \frac{a}{2} \ln \left[\frac{a + (a^2 - y^2)^{1/2}}{a - (a^2 - y^2)^{1/2}} \right], \qquad a = 1 \tag{41}$$

arising in differential geometry. The curve $x = x(y)$ is the element of Beltarmi surface. This is a remarkable fact since Beltrami surface is a canonical surface of the constant (Gaussian) negative curvature equals -1. Reflecting this surface with respect to the plane yOz of the Cartesian space R^3, we obtain the so-called pseudo-sphere: a hyperbolic manifold of the constant negative curvature. This manifold has singular points at $x = 0$ which forms the so-called break circle where the manifold loses smoothness. The parametric equations of the curve $x = x(y)$ (or the graphic of $\hat{f}$) read

$$x = \ln \cot \frac{1}{\theta} - \cos\theta, \quad y = \sin\theta, \quad 0 < \theta < \frac{\pi}{2}.$$

Revolving this curve about Ox-axis, we obtain the following family of curves

$$x_\omega \equiv x, \quad y_\omega = \sin\theta \cos\omega, \quad z_\omega = \sin\theta \sin\omega, \quad -\infty < \omega < \infty \tag{42}$$

where ω is an angle of rotation of the plane XY and Eqs. (42) present the so-called universal covering of Beltrami surface. Here the value of angle $\theta = \pi/2$ corresponds the singular points of the pseudo-sphere.

Therefore the self-similar solution (38) looks as the Beltrami surface for each fixed time embedded in the correlation space $\mathcal{K}^3$ and generates a two-dimensional hyperbolic manifold.

Remark 2.1. There are some other approaches to find self-similar solutions of the von Kármán-Howarth equation. These approaches and the results obtained have been analyzed in [2, 5, 10, 17, 21, 34] and others.

Remark 2.2. Asymptotic behavior of solutions to the problem (27)–(30) is formally described for $r \gg 1$ by the Millionshtchikov closure model

$$\frac{\partial B_{LL}(r,t)}{\partial t} = \frac{2\kappa_2}{r^4}\frac{\partial}{\partial r}\left[r^4\left(r\sqrt{B_{LL}(0,t)}\frac{\partial B_{LL}(r,t)}{\partial r} + 2\nu\frac{\partial B_{LL}(r,t)}{\partial r}\right)\right] \tag{43}$$

which is supplemented by the same initial-boundary conditions (28)–(30). This problem was investigated in details in [22] wherein the existence and uniqueness of solution to the above-mentioned initial-boundary value problem was proven. The theory of contractive linear semigroups combined with the iterative process (the Trotter-Kato product formula) was applied to find a solution of this problem. Then, we proved that the continuous semigroup constructed admits the Loitsyansky integral as an integral invariant for a suitable initial date. However, this assertion is incorrect in general, depending on how the homogeneous isotropic turbulence is created at initial time. Dependence of solution obtained on the laminar viscosity (in the limit of large Reynolds numbers) it was also studied in [22] together with other accompanying questions including the asymptotic stability of the Millionshtchikov self-similar solution (compare with (39), (40))

$$B_{LL} = \text{const}\cdot(t+a_3)^{-10/7}\exp\left(-\frac{r}{7\kappa_2(t+a_3)^{2/7}}\right).$$

2.3 Initial-boundary value problem in the limit of large Reynolds numbers

In this section, we prove the existence and uniqueness results for solutions to an initial-boundary value problem for the closure model (20), (22), associated with the similarity representation of solutions, defined at each moment of time in accordance to the form of the self-similar solution (parametrized by σ) (33), (34)

$$B_{LL}(r,t) \equiv \overline{u'^2(t)}f(r,t) = (t+a_3)^{-2(\sigma+1)/(\sigma+3)}f(\xi\cdot t^{2/(\sigma+3)},t), \tag{44}$$

where $\overline{u'^2(t)} = (t+a_3)^{-2(\sigma+1)/(\sigma+3)}$ and $\xi = r/(t+a_3)^{2/(\sigma+3)}$. Then Eq. (31) similarity "transforms" into

$$\frac{\partial f(\xi,\tau)}{\partial\tau} = \frac{2\kappa_2}{\xi^4}\frac{\partial}{\partial\xi}\left[\xi^5\sqrt{1-f}\frac{\partial f(\xi,\tau)}{\partial\xi}\right] + \frac{2}{\sigma+4}\xi\frac{\partial f}{\partial\xi} + 2\frac{\sigma+1}{\sigma+3}f, \tag{45}$$

where $\tau = \ln t$. Introducing the new variable $q = 2\xi^{1/2}$, we can rewrite Eq. (45) in the following form

$$\frac{\partial f(q,\tau)}{\partial\tau} = \frac{2\kappa_2}{q^9}\frac{\partial}{\partial q}\left[q^9\sqrt{1-f}\frac{\partial f(q,\tau)}{\partial q}\right] + \frac{1}{\sigma+4}q\frac{\partial f}{\partial q} + 2\frac{\sigma+1}{\sigma+3}f. \tag{46}$$

Also we will use the following representation of (46) for $u = 1-f$

$$\frac{\partial u(\xi,\tau)}{\partial\tau} = \frac{2\kappa_2}{q^9}\frac{\partial}{\partial q}\left[q^9u^{1/2}\frac{\partial u(\xi,\tau)}{\partial q}\right] + \frac{1}{\sigma+4}q\frac{\partial u}{\partial q} + 2\frac{\sigma+1}{\sigma+3}(u-1). \tag{47}$$

This equation looks like the porous media type equation. Due to the self-similar form of $B_{LL}(0,t)$ the following initial and boundary conditions are posed

$$f(q,t_0) = f_0(q), \quad q \ge 0 \tag{48}$$

$$f(0,t) = 1, \quad \text{for } q = 0, \quad t \ge 0 \tag{49}$$

$$f(q,t) \to 0, \quad \text{as } q \to \infty, \quad t \ge 0. \tag{50}$$

Considering $f(q,\tau)$ more from a physical point of view the function f is a positive at $(q,\tau) \in [0,\infty)\times[0,\infty)$ and satisfies the inequality $f < 1$ for $q > 0$.

At first, we prove the existence and uniqueness theorem for weak solutions of the problem (46), (48)–(50). Then we show that in the open domain $Q = (q > 0, t > 0)$ the weak solution obtained satisfies Eq. (46) in the classical sense.

To introduce the notion of a weak solution of the problem, we write (46) as follows

$$\frac{\partial f}{\partial\tau} = \frac{2\kappa_2}{q^9}\frac{\partial}{\partial q}\left[q^9\frac{2}{3}\frac{\partial(1-f)^{3/2}}{\partial q}\right] + \frac{1}{\sigma+4}\frac{\partial(qf)}{\partial q} - \frac{(\sigma+3)+2(\sigma+1)(\sigma+4)}{(\sigma+3)(\sigma+4)}f \tag{51}$$

and state what is understood by a weak solution.

Definition 2.3. A function $f(q,\tau)$ defined on Q is said to be a weak solution of the problem (46), (48)–(50) if for any $0<\tau_1$ the function f is nonnegative, bounded above by 1, $f\in C([0,\tau_1],C[0,\infty))$ and satisfies the integral identity

$$\int_0^{\tau_1}\int_{q_1}^{q_2}\left\{q^9 f\frac{\partial\phi}{\partial\tau}+2\kappa_2\frac{2}{3}(1-f)^{3/2}\frac{\partial}{\partial q}\left(q^9\frac{\partial\phi}{\partial q}\right)\right.$$
$$\left.-\frac{q}{\sigma+1}f\frac{\partial(q^9\phi)}{\partial q}-\frac{(\sigma+3)+2(\sigma+1)(\sigma+4)}{(\sigma+3)(\sigma+4)}fq^9\phi\right\}dqd\tau=0 \tag{52}$$

for all nonnegative $\phi\in C^{2,1}([q_1,q_2]\times[0,\tau])$ with a compact support in $(q_1,q_2)\times(0,\tau_1)$ for any $0<q_1<q_2<\infty$, $0<\tau_1<\infty$ and the initial-boundary conditions (48)–(50).

If we replace above the sign of equality on $\leq$ $(\geq)$ then f is called a weak supersolution (subsolution) of the problem.

So, we assume that $f_0(q)$ is a continuous positive function such that $f_0(q)<1$ for all $q>0$.

2.4 Existence and uniqueness results

Lemma 2.4. *Under the above-mentioned assumption the weak solution of the problem* (46), (48)*–*(50) *exists.*

We give the proof of this Lemma using the variables $y=2r^{1/2}$ and $U=\overline{u'^2(t)}-B_{LL}(y,t)$, where for $U(y,t)$ we have the equation

$$\frac{\partial U}{\partial t}=\frac{2\kappa_2}{y^9}\frac{\partial}{\partial y}\left[y^9U^{1/2}\frac{\partial U}{\partial y}\right]+\frac{\partial\overline{u'^2(t)}}{\partial t}. \tag{53}$$

Here the function $\overline{u'^2(t)}$ coincides with the self-similar representation obtained above. This equation is supplemented by the initial-boundary conditions

$$U(y,t_0)=U_0(y),\quad y\geq 0, \tag{54}$$

$$U(0,t)=0,\quad t\geq t_0, \tag{55}$$

$$U(y,t)\to\overline{u'^2(t)},\quad \text{as } y\to\infty. \tag{56}$$

Let $U_0(y)\in C[0,\infty)$ and $U_0(y)\geq U^s(y,t_0)$, $U_0(y)\leq\overline{u'^2(t_0)}$. Here $U^s(y,t)\equiv\overline{u'^2(t)}-B^s_{LL}(y,t)$ where $B^s_{LL}(y,t)=\overline{u'^2(t)}\hat{f}(\xi)$ is the self-similar solution of (33), (34). We approximate (53)–(56) by a family of problems

$$\frac{\partial U_n}{\partial t}=\frac{2\kappa_2}{y^9}\frac{\partial}{\partial y}\left[y^9U_n^{1/2}\frac{\partial U_n}{\partial y}\right]+\frac{\partial\overline{u'^2(t)}}{\partial t} \tag{57}$$

in $Q_n=(n^{-1},\infty)\times(0,T)$, $T>0$, $n=2,3,\dots$ with the conditions

$$U_n(y,t_0)=U_0(y),\quad y\geq n^{-1}, \tag{58}$$

$$U_n(n^{-1},t)=U_0(n^{-1}),\quad t\geq t_0, \tag{59}$$

$$U_n(y,t)\to\overline{u'^2(t)},\quad \text{as } y\to\infty. \tag{60}$$

Due to the assumption that $\overline{u'^2(t_0)}\geq U_0(y)\geq U^s(y,t_0)$ where U^s is the positive everywhere function and the comparison theorem we obtain that U^s is a subsolution of the problem for each n and $U_n(y,t)\leq\overline{u'^2(t)}$ where $\overline{u'^2(t)}$ is a suppersolution correspondingly. We do not repeat here the standard argument of the Theory of Porous Medium Equations [36] to prove the existence result of $U_n(y,t)\in C^{2,1}(Q_n)\cap C(\bar{Q}_n)$ to the problem (57)–(60) only mention that the use of the self-similar solution U^s of the problem and the comparison theorem guarantees that $U_n(y,t)\geq\delta_n>0$ in Q_n. On the lateral boundary (n^{-1},t), $t\geq t_0$ where $U_n(n^{-1},t)=U_0(n^{-1})\geq U^s(n^{-1},t_0)$, we have $U_n(n^{-1},t)\geq U^s(n^{-1},t)$ since $U^s(n^{-1},t)\leq U^s(n^{-1},t_0)$. Hence $U_n(y,t)$ is the classical solution of (57)–(60). The interior Schauder estimates [18] obey the boundedness of the derivatives $U_{nt}(y,t)$ and $U_{nyy}(y,t)$ inside of Q_n. Therefore there exists a subsequence n_k such that $U_{n_k}\to U$ as $n_k\to\infty$ uniformly on each compact subsets of Q. Moreover in a neighborhood of any points $(y,t)\in Q$, where $U(y,t)>0$ (i.e. the equation is not degenerated near by this point) we get the convergence of derivatives $U_{n_kt}\to U_t$, $U_{n_ky}\to U_y$ and $U_{n_kyy}\to U_{yy}$ uniformly on each compact subsets. By continuity we can extend the function obtained $U(y,t)$ up to $y=0$ and $y\to\infty$. Indeed, taking into account that $\lim_{n\to\infty}U_n(n^{-1},t)=0$, we get that $U(0,t)=0$ and $U(y,t)\to\overline{u'^2(t)}$ as $y\to\infty$ in view of the inequalities $U^s(y,t)\leq U_n(y,t)\leq\overline{u'^2(t)}$ and $\lim_{y\to\infty}U^s(y,t)=\overline{u'^2(t)}$. We determine $U(y,t)$ at $t=t_0$ choosing the diagonal subsequence $\{U_{n_k}(y,t_s)\}$, $t_s\to t_0$ that convergence to $U_0(y)$.

Therefore $U(y,t) \in C([0,T], C(R^+))$ for $T < \infty$ that enables us to determine the function $f(q,\tau) \in C([0,\tau_1], C(R^+))$ according to the formula $f(q,\tau) = 1 - \overline{u'^2(t)}^{-1} U(y,t)$. The estimates obtained for $U(y,t)$ mean that $0 \leq f(q,\tau) \leq 1$. The check that $f(q,\tau)$ satisfies the integral identity (52) is elementary.

Lemma 2.5. *The weak solution of the problem* (46)*,* (48)*–*(50) *is unique.*

The proof is based on the well-worked out technics of the Theory of Porous Medium Equations applied to the problem (53)–(56). Indeed, we write the weak equalities (the corresponding integral identities) which are satisfied by U_1 and U_2 with respect to a test function and then subtract their. As a result, we get the integral identity for the difference $U_1 - U_2$ of the same form as for radially symmetric solutions of the porous medium equation. The accompany details here are omitted.

Lemma 2.6. *Under the above-mentioned assumption on the initial data the weak solution of the problem* (46)*,* (48)*–*(50) *satisfies Eq.* (46) *in the classical sense.*

We need only to establish the following inequality $f(q,\tau) < 1$ for $q > 0$ or to show that $U(y,t) > 0$, $y > 0$. Let us assume that there exists the point (y_0,t_0) such that $U(y_0,t_0) = 0$. Then due to Lemma 2.5 we can approximate $U(y,t)$ by the sequence of $U_n(y,t)$ where $U_n(y,t)$ are the solutions of the corresponding problem (57)–(60). Choosing n sufficiently large, in view of the assumption above, we have that $U_n(y_0,t_0) < \epsilon$ for arbitrary small ϵ. The subsolution U^s for U_n in any interior point of Q is a positive function, it means that $U^s(y_0,t_0) \geq \delta > 0$ for some δ. Since $U_n(y,t) \geq U^s(y,t)$ for all n and ϵ is arbitrary small we get the contradiction.

2.5 Asymptotic behaviors

Now we study the large-time behavior of the weak solution of the problem (46), (48)–(50). The stationary form of Eq. (46) reads

$$\frac{2\kappa_2}{q^9}\frac{\partial}{\partial q}\left[q^9\sqrt{1-f}\,\frac{\partial f(q,\tau)}{\partial q}\right] + \frac{1}{\sigma+4}q\frac{\partial f}{\partial q} + 2\frac{\sigma+1}{\sigma+3}f = 0. \tag{61}$$

The weak steady-state solution of the corresponding problem is determined similarly as in Definition 2.3 wherein the integral identity is replaced on

$$\int_{q_1}^{q_2}\left\{2\kappa_2\frac{2}{3}(1-f)^{3/2}\frac{\partial}{\partial q}\left(q^9\frac{\partial\psi}{\partial q}\right)\right. \tag{62}$$
$$\left. -\frac{q}{\sigma+1}f\frac{\partial(q^9\psi)}{\partial q} - \frac{(\sigma+3)+2(\sigma+1)(\sigma+4)}{(\sigma+3)(\sigma+4)}fq^9\psi\right\}dq = 0,$$

where $\psi \in C^{2,1}([q_1,q_2])$ with a compact support in (q_1,q_2) for any $0 < q_1 < q_2 < \infty$. Note that the self-similar solution $\breve{f}(q) = \hat{f}(\xi)$ is also the weak solution.

Lemma 2.7. *The weak solution obtained convergence to a steady-state solution which coincides with the self-similar solution $\hat{f}$ of the problem as $t \to \infty$ in the norm of $L^1(q_1,q_2)$.*

We denote by $f(\tau) = f(\tau,\cdot)$ the orbit of f. The orbit $f(\tau)$ is bounded uniformly in $L^1(R^+) \cap L^\infty(R^+)$ Recall that Eq. (46) admits the translation group in τ. Hence it follows immediately that the integral identity (52) we can write as

$$\int_0^{\tau_1}\int_{q_1}^{q_2}\left\{q^9 f(q,\tau+a_n)\frac{\partial\phi(q,\tau)}{\partial\tau} + 2\kappa_2\frac{2}{3}(1-f(q,\tau+a_n))^{3/2}\frac{\partial}{\partial q}\left(q^9\frac{\partial\phi(q,\tau)}{\partial q}\right)\right. \tag{63}$$
$$\left. -\frac{q}{\sigma+1}f(q,\tau+a_n)\frac{\partial(q^9\phi(q,\tau)}{\partial q} - \frac{(\sigma+3)+2(\sigma+1)(\sigma+4)}{(\sigma+3)(\sigma+4)}f(q,\tau+a_n)q^9\phi(q,\tau)\right\}dqd\tau = 0,$$

where we set $\phi(q,\tau) = \psi(q)\eta(\tau)$. Let $a_n \to \infty$. Then the sequence $f(\tau_n) = f(q,\tau+a_n)$ is relatively compact locally in $L^1(R^+)$ and also in $C_{loc}(R^+)$ due to the compactness arguments developed for the Porous Medium Equations, taking into account that

$$U_t \leq \frac{2\kappa_2}{y^9}\frac{\partial}{\partial y}\left[y^9U^{1/2}\frac{\partial U}{\partial y}\right].$$

Therefore we may pass to the limit and form the so-called ω-limit, which is the set

$$\omega(f) = \{f^* \in L^1(R^+) \cap C_{loc}(R^+)\ :\ \exists\tau_{n_j} \to \infty \quad \text{such that} \quad f(\tau_{n_j}) \to f^*\}.$$

Hence passing to limit in (63) as $a_{n_j} \to \infty$, we get

$$\eta(\tau)\int_{q_1}^{q_2}\Big\{2\kappa_2\frac{2}{3}(1-f^*)^{3/2}\frac{\partial}{\partial q}\left(q^9\frac{\partial\psi}{\partial q}\right) \tag{64}$$
$$-\frac{q}{\sigma+1}f^*\frac{\partial(q^9\psi)}{\partial q}-\frac{(\sigma+3)+2(\sigma+1)(\sigma+4)}{(\sigma+3)(\sigma+4)}f^*q^9\psi\Big\}\,dq=0.$$

Here the integral

$$\int_0^{\tau_1}\int_{q_1}^{q_2}\{q^9 f(q,\tau+a_{n_j})\frac{\partial\phi(q,\tau)}{\partial\tau}\}dqd\tau\to 0, \qquad \text{as } a_{n_j}\to\infty$$

due to the vanishing $\eta(\tau)$ at $\tau=0$, $\tau=\tau_1$ and the theorem of repeated integrals. The (nonnegative bounded) function obtained $f^*(q)$ satisfies Eq. (61) in the weak sense. Moreover the limit function $u^*(q)=1-f^*(q)$ is positive for $q>0$. It means that we can differentiate $f^*(q)$ for $q>0$. This function vanishes on infinity due to the inequality $f(q,\tau)\le\breve{f}(q)$. Then Lemma 2.5 guarantees that f^* coincides with $\breve{f}$.

Let us consider the asymptotic behavior of the solution obtained $f(q,\tau)$ as $q\to\infty$. Since $\breve{f}(q)$ is a supersolution the result follows immediately from $f(q,\tau)\le\breve{f}(q)$ and the formal asymptotic

$$\breve{f}(q)=M_1q^{-2(\sigma+1)}+\dots,\quad q\gg 1,\quad \text{for } \sigma+1\in(0,5);$$

if $\sigma+1=5$, then

$$\breve{f}(q)=M_2e^{-q^2/4}+\dots,\quad q\gg1,\quad \breve{f}(q)=\hat{f}(\xi),$$

where the constants depends on κ_2. Considering σ more from a physical point of view, we restrict the variation of values σ by the interval $[2,4]$. Then the function f in the variables (ξ,τ) decay at least as $\xi^{-(\sigma+1)}$ for $\sigma<4$ and $\exp(-\xi)$ when $\sigma=4$ as $\xi\to\infty$. Therefore for $\sigma\in[2,4]$ we can see that $B_{LL}(r,t)=\overline{u'^2(t)}f(r,t)$ for each fixed time $t=t^*$ decays faster than r^{-2} as $r\to\infty$ that coincides with the physical claim of decaying the correlation function. Thus the estimates obtained for the correlation function determined in the frame of the problem (46), (48)–(50) allow to prove the convergence of the integral

$$\mathcal{L}(t)=2\int_0^\infty\sqrt{\overline{u'^2(t)}f(|r_1|,t)}d|\vec{r}_1|$$

and estimate the rate of convergence.

Let $\sigma=4$. Then for the self-similar solution $B^s_{LL}(r,t)=\overline{u'^2(t)}\hat{f}(\xi)$ the Loitsyansky integral Λ is defined for each $t\ge t_0$ and this integral is independent on time i.e. invariant. Now, observe that $\hat{f}(\xi)$ is an upper solution for $f(r,t)$. Hence the Loitsyansky integral

$$\Lambda=\overline{u'^2(t)}\int_0^\infty r^4 f(r,t)dr \tag{65}$$

exists and depends on the time t in general. We prove that Λ is invariant for $B_{LL}(r,t)$ constructed under the assumption that $B_{LL}(r,t_0)$ is a monotonically decreasing function. First, we prove the property of monotonicity.

Lemma 2.8. *Let $B_{LL}(r,t)$ be the solution constructed in Lemma* 2.4 *and assume that $\partial B_{LL}(r,t_0)/\partial r<0$. Then $B_{LL}(r,t)$ is decreasing in r.*

It is not hard to verify that U_n takes minimal values on the lateral boundary (n^{-1},t) of the rectangle Q_n. Since the equation

$$\frac{\partial U_n}{\partial t}=\frac{2\kappa_2}{y^9}\frac{\partial}{\partial y}\left[y^9U_n^{1/2}\frac{\partial U_n}{\partial y}\right]+\frac{\partial\overline{u'^2(t)}}{\partial t}$$

for U_n admits differentiation in y then the maximum principal for $U_{ny}(y,t)$ immediately completes the proof.

We are now in a position to show the invariance of Λ.

Lemma 2.9. *Let $\Lambda(0)=\int_0^\infty r^4B_{LL}(r,0)dr<\infty$. Then $\mathrm{d}\Lambda(t)/\mathrm{d}t=0$ for all $t\in[0,\infty)$.*

Multiplying on r^4 and integrating in r the equation

$$\frac{\partial B_{LL}(r,t)}{\partial t} = \frac{2\kappa_2}{r^4}\frac{\partial}{\partial r}\left[r^4\left(r\sqrt{u'^2(t) - B_{LL}(r,t)}\frac{\partial B_{LL}(r,t)}{\partial r}\right)\right]$$

in Q, we obtain

$$\frac{\partial}{\partial t}\int_0^r s^4 B_{LL}(s,t)ds = s^5\sqrt{u'^2(t) - B_{LL}(s,t)}\frac{\partial B_{LL}(s,t)}{\partial s}\bigg|_{s=r}.$$

To prove the assertion, we show that the right-hand side convergence to zero as $r \to \infty$. Since $B_{LL}(r,t) \to 0$ as $r \to \infty$ we can write

$$B_{LL}(r,t) = -\int_r^\infty \frac{\partial B_{LL}(s,t)}{\partial s}ds.$$

Multiplying this equality on r^4, we obtain

$$r^4 B_{LL}(r,t) = -r^4\int_r^\infty \frac{\partial B_{LL}(s,t)}{\partial s}ds \to 0, \qquad \text{as } r \to \infty$$

uniformly in t. Using that $\partial B_{LL}(s,t)/\partial s \le 0$, we get

$$-r^4\int_r^{2r}\frac{\partial B_{LL}(s,t)}{\partial s}ds = -r^5\,\frac{\partial B_{LL}(r,t)}{\partial r}\bigg|_{r=s^*} < \epsilon$$

for arbitrary small ϵ, $r \gg 1$ and $s^* \in (r, 2r)$. We also have that $-(2r)^5\,\partial B_{LL}(r,t)/\partial r|_{r=s^*} < 2^5\epsilon$. Therefore in view of arbitrary large r, we obtain that $s^{*5}\,|\partial B_{LL}(s,t)/\partial s||_{s=s^*} \to 0$ as $r \to \infty$ ($s^* \to \infty$) and the following equality takes place

$$\frac{\partial}{\partial t}\int_0^\infty r^4 B_{LL}(r,t)dr = 0$$

that completes the proof.

Let us show that the so-called flux $\Phi = U^{1/2}\partial U/\partial y$ is a bounded function in $Q^T = [0,\infty) \times [0,T]$, $0 < T < \infty$.

2.6 Boundedness of Φ

Let the conditions of Lemmas 2.4, 2.9 be hold. Recall that $\Phi(y,t)$ is defined into Q^T due to the positivity of $U(y,t)$. The solution U of the problem (53)–(56) admits the differentiation with respect to y and t in Q. Therefore the derivative $\partial U/\partial y$ satisfies the parabolic equation which is nondegenerate for $y > 0$. Hence the boundedness of $\partial U/\partial y$ in each interior domain $Q^T_\delta = [\delta,\infty) \times [\delta,T]$, $\delta > 0$ is a consequence of the classical results [18]. Note that the estimate of

$$\Phi(y,t) = \frac{3}{2}U^{1/2}\frac{\partial U}{\partial y} \quad \text{in } Q^T_\delta$$

depends on the distance to the lateral boundary $(0,t)$ of Q^T and $\max\Phi(y,t_0)$, $y \in [0,\infty)$. In order to prove the boundedness up to $(0,t)$, we have to prove that the flux is defined on the boundary $(0,t)$ and then to estimate Φ on the boundary $(0,t)$. First, we evaluate U near by $(0,t)$. In order to do this, we consider instead of U the function $V = 3U^{1/2}$ which presents the so-called scaled pressure in terms of the porous medium equation. Denote by U^+ a solution of the equation

$$\frac{\partial U^+}{\partial t} = \frac{2\kappa_2}{y^9}\frac{\partial}{\partial y}\left[y^9 {U^+}^{1/2}\frac{\partial U^+}{\partial y}\right]. \tag{66}$$

Then $U^+ \ge U$ and therefore $V^+ \ge V$ due to the comparison theorem under a suitable initial-boundary value conditions for U^+. Here V^+ satisfies

$$\frac{\partial V^+}{\partial t} = \kappa_2 V^+\left(\frac{\partial^2 V^+}{\partial y^2} + \frac{9}{y}\right) + \left(\frac{\partial V^+}{\partial y}\right)^2. \tag{67}$$

Equation (67) has in Q^T the following one-parameter self-similar solutions $\{g_c(y,t)\}$ [1]:

$$g_c(y,t) = y^2\varphi(c\eta)/(T-t), \quad \eta = (T-t)y^\alpha, \quad 0 < t < T,$$

where c is an arbitrary positive constant. The function $\varphi = \varphi(\eta)$ is obtained by solving the nonlinear degenerate ordinary differential equation

$$\frac{\alpha^2}{2}\varphi\varphi'' + \alpha^2(\varphi')^2 - \frac{\alpha}{2}(20-\alpha)\eta^{-1}\varphi\varphi' + 14\eta^{-2}\varphi^2 - \eta^{-2}\varphi + \eta^{-1}\varphi' = 0$$

subjects to the initial conditions $\varphi(0) = 0$ and $\varphi'(0) = 1$ with a positive parameter α such that $g_c(y,t)$ is an increasing function. It was proven in [1] that there exists α^* such that $\varphi(\eta) > 0$, $\eta > 0$ for $\alpha < \alpha^*$ where α^* satisfies $7/5 < \alpha^* < 2$. Hence $0 < 2 - \alpha^* < 3/5$. Therefore $g_c(0,t) = 0$ for $t \in [0,T]$ and $g_c(y,T) = cy^{2-\alpha}$ due to the boundary conditions for $\varphi(\eta)$ at $\eta = 0$. Moreover $g_c(y,t_0) = c_{t_0}y^{2-\alpha}\{1 + o(1)\}$ as $y \to 0$. Note that the self-similar solution $U^s(y,t)$ for $\sigma = 4$, rewritten in the pressure variable V^s, behaviors itself as $Cy^{2/3}$ near by $y = 0$ where C is a positive constant. If we take α at least in $[7/5, \alpha^*)$ then there exists c^* such that $g_{c^*}(y,t_0) \geq V^s(y,t_0)$ for $y \ll 1$. Since V^s is a bounded function ($V^s \leq 3\sqrt{\overline{u'^2(t)}}$) and $g_c(y,t_0) \to \infty$ as $y \to \infty$ then there exists $c^{**} \geq c^*$ such that $g_{c^{**}}(y,t_0) \geq V^s(y,t_0)$ for $y \in [0,\infty)$. In order to use $g_{c^{**}}(y,t)$ to estimate the solution $V(y,t)$ (obtained in Lemma 2.4) near by $y = 0$, we assume in addition to the conditions of Lemma 2.4 that $V(y,t_0) \leq g_{c^{**}}(y,t_0)$ for $y \in [0,\infty)$. This inequality is compatible with the assumption that $V(y,t_0) \geq V^s(y,t_0)$ as we showed above. Our aim is to prove the estimate $V(y,t) \leq g_{c^{**}}(y,t)$ for $t \in (t_0,T]$ and $y \ll 1$. It follows from the comparison theorem. Therefore, as a result, we obtain

$$V \leq c^{**}y^{2-\alpha} \leq c^{**}y^{3/5} \quad \text{and} \quad 0 \leq U \leq \frac{c^{**2}}{9}\,y^{6/5}. \tag{68}$$

Hence the derivative $\partial U/\partial y$ is determined up to $y = 0$ and $\partial U/\partial y \leq const \cdot y^{1/5}$ near by the line $(0,t)$. For the flux Φ the following estimate holds

$$0 \leq \Phi(y,t) \leq \text{const} \cdot y^{4/5}$$

for $y \ll 1$ and $\Phi(0,t) = 0$.

Thus we proved

Lemma 2.10. *Let the conditions of Lemmas* 2.4, 2.9 *be hold, and let the flux* $\Phi(y,t_0) < \infty$ *for* $y \in [0,\infty)$. *Assume that* $V(y,t_0) \leq g_{c^{**}}(y,t_0)$, $y \in [0,\infty)$ *and* $\sigma = 4$. *Then* $0 \leq \Phi(y,t) \leq K < \infty$ *for* $(y,t) \in Q_T$ *where* K *depends on* $\max_{y \geq 0}\Phi(y,t_0)$. *Moreover* $\Phi(y,t) \leq const \cdot y^{4/5}$ *for* $y \ll 1$ *and* $\Phi(0,t) = 0$.

Remark 2.11. The physical admitted values of σ belong to the interval $[2,4]$. Denote by f_σ a family of the solutions of the initial-boundary value problem (46), (48)–(50) for $\sigma \in [2,4]$. Applying the comparison theorem to the solutions $f_{\sigma'}$ and $f_{\sigma''}$, we obtain that for a suitable initial-boundary value conditions $f_{\sigma'}(y,t) \leq f_{\sigma''}(y,t)$ where $\sigma' \leq \sigma''$. This inequality enables us to get the estimation of $U_\sigma(y,t)$ for $\sigma < 4$ using the results obtained in Lemma 2.10.

3 Concluding remarks

The conclusions of this work read as: in the case of homogeneous isotropic turbulence the two-point correlation tensor of the velocity fluctuating can be used to equip the correlation space $\mathcal{K}^3$ by the structure of a pseudo-Riemannian manifold. It makes it possible to describe the evolution of a singled out fluid volume in the frame of deformation of the 1-parameter family of metrics $ds^2(t)$. To find the link with the physics of isotropic turbulence one needs physical realization of $(M^t, ds^2(t))$ i.e. the construction of a family of embedding $(M^t, ds^2(t))$ into (R^3, h^2) where h is the Euclidean metric. This method is used for visualizing Ricci flow of low-dimensional examples. Therefore the problem of shape deformations of eddies was reduced to a pure geometric task which presents the subject for future investigations. The quantities $\mathcal{L}(t)$ and $\mathcal{R}_a(t)$ characterize the metric sizes of a singled out fluid volume in terms of components of the metric tensor being presented by the correlation functions. It effects for instance the way to control the deformation of the shape of eddies in time. The next novelty is using the concept of the variational principle for the application in turbulence that enables to derive conservation laws associated with symmetries admitted by homogeneous isotropic turbulent flows.

At the second part of the paper, we established the well-posedness of an initial-boundary value problem for the closure model suggested in [19,20,26] arising in the context of the von Kármán-Howarth equation. It peculiarities are the nonlocal diffusion term arising in the principal part of the equation which is caused by the closure relationship and the degeneracy of this spatial differential operator.

Acknowledgements This work was supported by DFG (grant No OB 96/29-01) and RFBR (grant No 11-01-12075-OFIM-2011).

References

[1] D. G. Aronson and J. A. Graveleau, Self-similar solution to the focusing problem for the porous medium equation, Eur. J. Appl. Math. **4**(1), 65–81 (1993).
[2] G. I. Barenblatt and A. Gavrilov, On the thory of self-similar degeneracy of homogeneous isotropic turbulence, Sov. Phys. J. Exp. Theor. Phys. **38**, 399–402 (1974).
[3] O. Chkhetiani, On third-orders moments arising in helical turbulence, JETP Lett. **63**, 768–777 (1996).
[4] O. Chkhetiani, On the local structures of helical turbulence, Dokl. Phys. **422**, 618–621 (2008).
[5] J. Chasnov, Computational of the Loitsiansky integral in decaying isotropic turbulence, Phys. Fluids A **5**, 2579–2581 (1993).
[6] J. A. Domoradzki, Nonlocal triad interactions and the dissipation range of isotropic turbulence, Phys. Fluids **4**, 2037–2045 (1992).
[7] L. P. Eisenhart, Riemannian Geometry (Princeton University Press, Princeton, 1926).
[8] V. N. Grebenev and M. Oberlack, A geometric interpretation of the second-order structure function arising in turbulence, Math. Phys. Anal. Geom. **12**, 1–18 (2009).
[9] V. N. Grebenev and M. Oberlack, Geometric realization of the two-point correlation tensor for isotropic turbulence, J. Nonlinear Math. Phys. **18**, 109–120 (2011).
[10] W. George, The decay of homogeneous isotropic turbulence, Phys. Fluids A **4**, 1492–1509 (1992).
[11] K. Hasselmann, Zur Deutung der dreifachen Geschwindigkeitskorrelationen der isotropen Turbulenz, Dtsch. Hydrogr. Zeitschr. **11**, 207–211 (1958).
[12] W. Heisenberg, On the theory of statistical and isotropic turbulence, Proc. R. Soc. Lond. A **195**, 402–406 (1948).
[13] N. R. Kamyshanskij and A. S. Solodovnikov, Semireducible analytic spaces “in the large”, Russ. Math. Surv. **35**(5), 1–56 (1980).
[14] Th. von Kármán and L. Howarth, On the statistical theory of isotropic turbulence, Proc. R. Soc. Lond. A **164**, 192–215 (1938).
[15] A. N. Kolmogorov, Dissipation of energy in locally isotropic turbulence, Dokl. Akad. Nauk SSSR. **32**, 19–21 (1941).
[16] B. M. Koprov, V. M. Koprov, V. M. Ponomarev, and O. Chkhetiani, An experimental study of turbulent helicity and its spectrum in the atmospheric boundary layer, Dokl. Phys. **403**, 627–630 (2005).
[17] A. Korneev and L. Sedov, Theory of isotropic turbulence and its comparison with experimental data, Fluid Mech.-Sov. Res. **5**, 37–48 (1976).
[18] O. A. Ladyzhenskaya, V. Solonnikov, and N. Ural’ceva, Linear and Quasilinear Equations of Parabolic Type (AMS, Providence, Rhode Island, 1968).
[19] Y. M. Lytkin and G. G. Chernykh, About one way of the von Kármán-Howarth equation closure (in Russian), Dyn. Cont. Medium **27**, 124–130 (1976).
[20] Y. M. Lytkin and G. G. Chernykh, Calculation of correlation functions in isotropic turbulence (in Russian), Dyn. Cont. Medium **35**, 74–88 (1978).
[21] N. Mansour and A. Wray, Decay of isotropic turbulence in viscous at low Reynolds number, Phys. Fluids A **6**, 808–814 (1994).
[22] Z. Liu, M. Oberlack, V. N. Grebenev, and S. Liao, Explicit series solution of a closure model for the von Kármán-Howarth equation by means of the homotopy analysis method, ANZIAM J. **52**, 179–202 (2011).
[23] M. Millionshtchikov, Isotropic turbulence in the field of turbulente viscosity, JETP Lett. **8**, 406–411 (1969).
[24] A. S. Michenko and A. T. Fomenko, Lectures on Differential Geometry and Topology (Factorial Press, Moscow, 2000).
[25] A. S. Monin and A. M. Yaglom, Statistical Hydromechanics (Gidrometeoizdat, St.-Petersburg, 1994).
[26] M. Oberlack and N. Peters, Closure of the two-point correlation equation as a basis for Reynolds stress models, Appl. Sci. Res. **55**, 533–538 (1993).
[27] M. Oberlack, On the decay exponent of isotropic turbulence, Proc. Appl. Math. Mech. **1**, 294–297 (2000).
[28] S. Oughton, K.-H. Rädler, and W. H. Matthaeus, Phys. Rev. E **56**, 2857–2888, (1997).
[29] S. B. Pope, Lagrangian PDF methods for turbulent flows, Annu. Rev. Fluid Mech. **26**, 23–63 (1994).
[30] S. B. Pope, On the relationship between stochastic Lagrangian models of turbulence second-moment closure, Phys. Fluids **6**, 973–985 (1994).
[31] J. C. Rotta, Turbulente Strömungen (Teubner, Stuttgart, 1972).
[32] J. H. Rubinstein and R. Sinclair, Visualizing Ricci flow of manifolds of revolution, Exp. Math. **14**, 285–298, (2005).
[33] R. W. Stewart and A. A. Townsend, Similarity and self-preservation in isitropic turbulence, Philos. Trans. R. Soc. A **243**, 359–386 (1951).
[34] C. Spezialy and P. Bernard, The theory decay in self-preserving isotropic turbulence revisited, J. Fluid Mechan. **241**, 645–667 (1992).
[35] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. D. Mikhailov, Blow-up in Quasilinear Parabolic Equation (Walter de Gruyter, Berlin, New York, 1995).
[36] J. L. Vazquez, Porous Medium Equation (Oxford Science Publications, Oxford, 2007).