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The Structure of Continuous-Valued Neutral Monotonic Social Functions

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Abstract. This article consists of several results characterizing neutral monotonic social functions over alternative sets that are compact, convex subsets of Euclidean space. One major result is that a neutral monotonic social function is continuous-valued (i.e., does not make abrupt reversals) for all profiles of continuous-valued weak orderings if and only if it is simple (i.e., completely determined by a single set of decisive coalitions). A second major result is that a continuous-valued neutral monotonic social function will guarantee the existence of a socially undominated alternative for all profiles of continuous-valued, almost convex weak orderings if and only if the smallest empty intersection within the set of decisive coalitions is at least $m+2$ in size where m is the dimension of the alternative set.

I. Introduction

Blau and Brown (1978) characterize neutral monotonic social functions defined on finite alternative sets in terms of direct sums of simple games, i.e., families of conditionally decisive sets. In this paper we extend their characterization to continuous-valued neutral monotonic social functions defined on compact and convex subsets of a Euclidean space $\mathbb{R}^m$. A social function is continuous-valued if

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for any alternative, x, the set of all points socially preferred to x and the set that x is socially preferred to are both open sets. This property ensures that the social function will not make abrupt reversals and thus appears to be an attractive property to impose on social functions.

We show, however, that despite its attractiveness, imposing it places severe restrictions on the form of the direct sums of simple games that generate the social functions. In particular, in Sect. III we show (Theorem 2) that a necessary and sufficient condition for a social function to be continuous-valued for all profiles of continuous-valued weak orderings is that it be simple (i.e., completely determined by a single set of decisive coalitions). This contrasts with the finite case studied by Blau and Brown in which a coalition that is not decisive within the society at large may nevertheless become decisive in some proper subset of "concerned" (i.e., nonindifferent) members.

Section IV considers social *decision* functions, i.e., social functions that always yield an undominated alternative. The main result, Theorem 3, asserts that a continuous-valued neutral social function is a social decision function (for all profiles of continuous-valued weak orderings) if and only if the smallest empty intersection within the set of decisive coalitions has at least $m+2$ members where m is the dimension of the alternative set. This result is parallel to Blau and Brown's Theorem 5 (1978, Theorem 5, p. 18) which asserts that an acyclic neutral monotonic social function over a finite alternative set A need not have veto players when the number of concerned individuals is greater than $|A|$. Under Theorem 3 a neutral monotonic social decision function over a nonempty, compact and convex subset of $\mathbb{R}^m$ need not have veto players when the number of concerned individuals is greater than $m+1$. Theorem 3 also generalizes the main result of Greenberg (1979) which states that in a society with n members a necessary and sufficient condition for the existence of an undominated alternative in any nonempty, compact and convex subset of $\mathbb{R}^m$ under k-majority rule is that $k>(m/(m+1))n$. Theorem 4 in Sect. IV shows that this result is the special case of Theorem 3 that arises if anonymity is required.

II. Definitions and Notation

Take the set of social alternatives to be A. When A is finite, $|A|$ is the number of members in A. When A is a subset of a Euclidean space, $d(A)$ denotes the dimension of A.

A *CCE subset* is a nonempty, compact, convex subset of a Euclidean space. When A is a CCE subset with $d(A)=m$, then A is an *m-CCE subset*. It is assumed that A contains more than one alternative, so that there is a nontrivial social choice problem.

P is a *preference relation* on A if P is an asymmetric binary relation on A, i.e., xPy and yPx cannot both be true for x, $y\in A$. If they are both false then xIy, and xRy means that xPy or xIy. When an individual i has tastes characterized by a preference relation, that preference relation shall be denoted by P_i, R_i and I_i.

A preference relation is a *weak ordering* if R is transitive and complete. A preference relation is a *strict ordering* if P is transitive and complete. A preference relation is *acyclic* if for all $x_1, x_2, \ldots, x_n \in A$ $x_1 P x_2, x_2 P x_3, \ldots, x_{n-1} P x_n \Rightarrow x_1 R x_n$.

A preference relation, P is *continuous-valued* over a set of alternatives that is a subset, S, of Euclidean space if and only if for every $x \in S$:

(A) the set $\{y \in S | xPy\}$ is an open set;

and *(B)* the set $\{y \in S | yPx\}$ is an open set.

When condition (A) holds but condition (B) may or may not hold, the preference relation is *upper semicontinuous*. When condition (B) holds but condition (A) may or may not hold, the preference relation is *lower semicontinuous*. A preference relation is *almost convex* over a set of alternatives that is a subset, S, of Euclidean space if and only if for every $x \in S$, $x \notin H(\{y \in S | yPx\})$ where $H(C)$ denotes the convex hull of the set C. A preference relation is *convex* over a set of alternatives that is a subset, S, of Euclidean space if for all $x \in S$, $\{y \in S | yPx\}$ is a convex set.

Suppose that A is a *CCE* subset of a Euclidean space, S. Then an individual, i, has *type I preferences* if there exists a "bliss point" x (in S but not necessarily in A) such that yP_iz if and only if alternative y is closer in Euclidean distance to x than alternative z.

The set of all individuals in society is I. The set I will be taken to be finite with n members. When A is finite, W^I is the product space of individual weak orderings for the individuals in society. When A is a *CCE*-subset, W^I is the product space of continuous-valued, individual weak orderings for the individuals in society. A *social function* is a mapping from W^I into the set of all preference relations on A. An *acyclic social function* is a mapping from W^I into the set of all acyclic preference relations on A. The set $U(A,P) = \{x \in A | xRy \forall y \in A\}$ is the set of *undominated alternatives* for A under P. A *social decision function* is a mapping from W^I into the set of all preference relations P on A such that $U(A,P)$ is nonempty.

A *profile* is a member of W^I. For a profile p and $a, b \in A$, define $p(a>b)$ as the set of individuals who prefer a to b. The *concerned set* for p and the pair of alternatives $\{a,b\}$ is $p(a>b) \cup p(b>a)$. Let p and q be profiles and let a, b, c, $d \in A$. A social function, σ, is *neutral and monotonic* when the following condition holds: if $p(a>b) \subseteq q(c>d)$ and $q(d>c) \subseteq p(b>a)$, then $a\sigma(p)b$ implies $c\sigma(q)d$. If the condition holds with equalities rather than inclusions, then σ is *neutral*. A social function, σ, is *semimonotonic* when the following condition holds: if $p(a>b) = q(c>d)$ and $q(d>c) \subseteq p(b>a)$, then $a\sigma(p)b$ implies $c\sigma(q)d$. A *binary decision rule*, σ, is a social function that satisfies the following property of *binaryness*: if $p(a>b) = q(a>b)$ and $p(b>a) = q(b>a)$ then $a\sigma(p)b$ implies $a\sigma(q)b$. A binary decision rule, σ, is *strong without individual indifference* when for any profile p where $p(a>b) \cup p(b>a) = I$ either $a\sigma(p)b$ or $b\sigma(p)a$. That is, there will not be social indifference if everyone is concerned. A social function, σ, is *anonymous* if and only if for any permutation γ of $(1,2,\ldots,n)$, $\sigma(p_1,p_2,\ldots,p_n) = \sigma(p_{\gamma(1)},p_{\gamma(2)},\ldots,p_{\gamma(n)})$ where p_i is the weak ordering of individual i.

If J is a nonempty subset of I, then a *simple game* on J is a collection of subsets of J, Γ_J, such that:

(*a*) $A \in \Gamma_J$, $A \subseteq B \Rightarrow B \in \Gamma_J$;

(*b*) $A \in \Gamma_J \Rightarrow A^c \notin \Gamma_J$, where A^c is the complement of A in J.

When property (a) is true, but property (b) may or not be true, Γ_J is called a *monotonic game*. When property (b) is true, but property (a) may or may not be true, Γ_J is called a *proper game*. The *null (simple) game* on J is the empty collection of subsets of J. If Γ_J is a simple game on J, then $\Gamma_J^* = \{E \subseteq J | E^c \in \Gamma_J\}$ where E^c is the complement of E in J. Γ_J often is called the family of *winning coalitions* and Γ_J^* the family of *losing coalitions*. Note that under these definitions in a given simple game Γ_J, it may be true that a coalition $E \subseteq J$ is neither winning nor losing.

A *direct sum of simple games* is an indexed family $[\Gamma_J]_{J \in 2^I}$ such that:

(i) Γ_J is a simple game for J (possibly null);
(ii) for all K, $L \in 2^I$, if $K \subseteq L$, then $\Gamma_L \cap 2^K \subseteq \Gamma_K$;
(iii) for all K, $L \in 2^I$, if $K \subseteq L$, then $\Gamma_K^* \subseteq \Gamma_L^*$.

Every direct sum of simple games, $\Gamma = [\Gamma_J]_{J \in 2^I}$ *generates an aggregation rule*, μ_Γ, where for every profile p and alternatives a, $b \in A$, $a\mu_\Gamma(p)b$ if and only if $p(a>b) \in \Gamma_{p(a>b) \cup p(b>a)}$. In other words, the set of individuals who prefer a over b is a winning coalition in the simple game defined on the set of individuals concerned about a versus b. If σ is a social function and $J \subseteq K$ then J is said to be a *decisive subset for K* (with respect to σ) if for all profiles p where K is the concerned set, $J \subseteq p(a > b) \Rightarrow a\sigma(p)b$ for all a, $b \in A$. The set of all decisive subsets for K is the *decisive set for K*. When a social function is generated by a direct sum of simple games, Γ, then Γ_J is the decisive set for $J \subseteq I$. A social function is *simple* if for any $x, y \in A$, xPy if and only if all the members of at least one of the coalitions in Γ_I prefer x to y. When a simple social function is generated by Γ, a direct sum of simple games, then for $J \subseteq I \Gamma_J = \Gamma_I \cap 2^J$.

When A is a finite set, then an *acyclic game* on $J \subseteq I$ is a simple game, Γ_J, on J such that any empty intersection of coalitions in Γ_J has at least $|A|+1$ members. When A is an *m-CCE* subset, then a *maximal game* on $J \subseteq I$ is a simple game, Γ_J, on J such that any empty intersection of coalitions in Γ_J has at least $m+2$ members. A *prefilter* on J is a simple game, Γ_J, on J such that $\cap \Gamma_J \neq \emptyset$ and Γ_J is not a null game.

Suppose A is an *m-CCE* subset, and $C \subseteq N$ is a set with $|C| = c$. An *r(C)-majority rule* is an integer, $r(C)$, greater than $((m/(m+1))c$ such that for any x, $y \in A$ for which C is the concerned set xPy if and only if some set of persons $D \subseteq C$ with $|D| \geqq r(C)$ all prefer x to y.

III. The Link between Simplicity and Continuous-Valuedness

Throughout this paper the underlying individual preferences are profiles of individual weak orderings that are continuous-valued. Sometimes we also require that these weak orderings be almost convex. An important first step is to characterize neutral monotonic social functions in a way that is valid under any of these restrictions on preferences when the alternative set is a *CCE* subset. Blau and Brown (1978, Theorem 1, p. 8) found that for finite alternative sets and the case where strict preferences are admissible, neutral monotonic social functions are characterized by direct sums of simple games. Using a proof that is similar to their proof, the following result is true.[1]

Theorem 1. *Suppose that the alternative set is a CCE subset and that individual type I preferences are admissible. Then*

(a) The aggregation rule generated by a direct sum of simple games is a neutral monotonic social function.

(b) For any neutral monotonic social function there exists a unique direct sum of simple games with an aggregation rule identical to the social function.

This result is exactly the starting point that we need. Type I preferences are continuous-valued and almost convex and are the most restricted class of preferences that we consider.

The following theorem states restrictions on direct sums of simple games that are necessary and sufficient for the neutral monotonic social functions generated by the direct sums of simple games to be continuous-valued whenever all individuals' preferences are continuous-valued.

Theorem 2. *A direct sum of simple games will generate a continuous-valued neutral monotonic social function over a CCE-subset for all profiles of continuous-valued individual weak orderings if and only if the social function is simple.*

Proof: (*a*) *Sufficiency:* Suppose that Γ, a direct sum of simple games, generates a simple neutral monotonic social function. Then there is a set W composed of winning coalitions for concerned set I such that for $x, y \in A$, xPy if and only if $C = \{i \in I | xP_i y\} \in W$. To show that social preferences are continuous-valued, it is necessary to demonstrate that for any $z \in A$:

(*1*) $\{x \in A | xPz\}$ is an open set;
and (*2*) $\{y \in A | zPy\}$ is an open set.

The proofs for these two properties are nearly identical. As a result, only property (*2*) is proven here.

Since individual preferences are continuous-valued, the sets $D_i = \{x \in A | zP_i x\}$ are open sets for all $z \in A$. A coalition C prefers z to x if and only if every member of the coalition does. A coalition's preferences clearly form a preference relation since a unanimous preference within the coalition for one alternative over another precludes the opposite unanimous preference when individual preferences are preference relations. Denoting the preference relation for coalition C as P_C we have

[1] Blau and Brown's paper assumes that the alternative set is finite, and they use profiles of strict orderings in part of their proof. The assumption that the alternative set is finite is no problem since their proof does not rely on it. However, important subclasses of continuous-valued, almost convex weak orderings do not admit strict orderings over some *CCE* subsets. For example, if A is an *m-CCE* subset with $m \geq 2$ and an entity has type *I* preferences defined in the m-dimensional space containing A, then the entity will be indifferent between at least two alternatives in A. Intuitively, A must contain portions of at least one of the entity's hyperspherical indifference shells. It is straightforward to extend Blau and Brown's proof to one that works as long as individuals may have type *I* preferences. The proof is so similar to Blau and Brown's that it is not repeated here. But the proof can be found in Strnad (1982, pp 67–71)

$C(z)=\{x\in A|zP_Cx\}=\bigcap_{i\in C}\{x\in A|zP_ix\}$. Since the number of individuals in society is finite, the number in any coalition C must be finite. $C(z)$ is thus an intersection of a finite number of open sets and is therefore an open set.

Alternative z is socially preferred to a set of alternatives $Q(z)\subset A$. Because the social preference relation is simple, $x\in Q(z)$ if and only if at least one coalition $C_j\in W$ prefers z to x. Thus, $Q(z)=\{x\in A|zPx\}=\bigcup_{C_j\in W}\{y\in A|zP_{C_j}y\}$, and $Q(z)$ is an open set because it is a union of open sets.

(*b*) *Necessity:* Suppose that Δ, a direct sum of simple games, does not generate a simple social function. Then there is a concerned set $E\subset I$ and a coalition $B\subseteq E$ such that $B\in\Delta_E$ but $B\notin\Delta_I$. Since Δ is a direct sum of simple games, $B\in\Delta_B$. Let $C=I/B$.

Consider first the case where A is a one dimensional CCE subset. Since A is convex and compact, it must consist of a single line segment that contains its endpoints r and s. Suppose b, b', c, c' and z are five points on the line segment between r and s as in Fig. 1. The point c is the midpoint of the segment $c'z$ and the point b is the midpoint of the segment $b'z$.

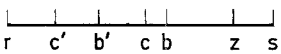

Fig. 1.

Consider a profile of type I individual preferences where all individuals in B have bliss point b, and all individuals in C have bliss point c.

Since the aggregation rule generated by Δ is monotonic, it follows from $B\in\Delta_B$ that $I\in\Delta_I$. There are three cases to consider:

(*1*) $C\in\Delta_I$ (and thus $C\in\Delta_C$);
(*2*) $C\notin\Delta_I$ and $C\in\Delta_C$;
(*3*) $C\notin\Delta_I$ and $C\notin\Delta_C$.

For these three cases, social preferences are as follows when points on the line segment rs are compared to z:

Set containing point x	Social Preference		
	Case (1)	Case (2)	Case (3)
segment rc' without c'	zPx	zPx	zPx
c'	zPx	zPx	zPx
segment $c'b'$ without b', c'	xPz	xIz	xIz
b'	xPz	xPz	xIz
segment $b'z$ without b', z	xPz	xPz	xPz
z	xIz	xIz	xIz
segment zs without z	zPx	zPx	zPx

In all three cases, $R(z)=\{y\in A|yRz\}$ is segment $c'z$ without point c' and is not a closed set.

Continuous-valuedness requires that $P^*(z)=\{x\in A|zPx\}$ be an open set and, therefore, that $R(z)$, the complement of $P^*(z)$, be a closed set. Thus, social preferences are not continuous-valued in any of the three cases.

This counterexample can be extended from the case where A is a 1-*CCE* subset to cases where A is an *m-CCE* subset with $m > 1$. Given that A contains more than one point, choose r and s to be any two points in A. Since A is convex, the line segment rs is in A. Set up points b, b', c, c' and z on rs as in the diagram above.

A is embedded in an m-dimensional Euclidean space, S. Let the line containing the segment rs be one coordinate axis in S. Label the coordinates on that axis as w. For a point $x \in A$ let x_w be the w coordinate of x. Let individual preferences be defined as follows: for $x, y \in A$ and for $i \in I$ x is preferred to y if and only if $|x_w - w_i| < |y_w - w_i|$ where w_i is the bliss w coordinate for i. I.e., individual i prefers x to y if and only if x's w coordinate is closer to w_i than y's w coordinate. Those individual preferences are not type I preferences but they are continuous-valued, almost convex weak orderings over A.

Now, choose $w_i = b$ for $i \in B$ and $w_i = c$ for $i \in C$. The proof by counterexample above then applies with three modifications. First, the "set containing point x" column of the table includes not only the portions of the line rs indicated in that column but all points in A with the same w coordinates as that portion of rs. Second, the set $R(z)$ is not the segment $c'z$ without point c' but all points in A with the same w coordinates as that portion of line rs. Finally, change "segment zs without z" to "all alternatives $y \in A$ such that $y_w > z_w$" and change "segment rc' without c'" to "all alternatives $y \in A$ such that $y_w < c'_w$."[2] QED

Ferejohn, Grether, Matthews and Packel (1980) prove results that are related to Theorem 2 here. Instead of focusing on neutral monotonic social functions, they focus on semimonotonic binary decision rules, social functions that satisfy the properties of binaryness and semimonotonicity. These two properties are weaker than neutrality and monotonicity. The "main result" of their paper (1980, Theorem 3, p. 793) establishes some necessary conditions for a binary decision rule to be simple. The necessary conditions are that the binary decision rule is continuous-valued, strong without individual indifference (SWII), and semimonotonic. Two of their other results (1980, Theorem 2 and Lemma 4, pp. 792–793) indicate that these three conditions together imply that the binary decision rule will be neutral.

However, it is apparent from Corollary 1 proven below that if we assume neutrality we can dispense with SWII as a necessary condition. This is interesting because SWII is an extremely restrictive condition. For example, anonymous voting rules that require more than a majority or that require a majority when the number of voters is even may have social indifference as an outcome in a binary choice although no individual is indifferent.

Corollary 1 shows that Theorem 2 survives the weakening of neutrality and monotonicity to neutrality and semimonotonicity.

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2 The proof of part (b) shows that upper semicontinuity, one out of two aspects each necessary for social preferences to be continuous-valued, is not present for some profiles of continuous-valued weak orderings. Specifically, the proof gives an example where the set of points that a given point is socially preferred to is not open.

A similar example could be constructed that would violate lower semicontinuity, the other aspect of continuous-valuedness. In particular, if points b and c are switched, points b' and c' are switched and the profile is the same, then $R^*(z) = \{y \in A \mid zRy\}$ = segment rc' without c' plus segment zs, and $R^*(z)$ is not a closed set. But lower semicontinuity requires that $P(z) = \{x \in A \mid xPz\}$ be an open set and, therefore, that $R^*(z)$, the complement of $P(z)$, be a closed set

Corollary 1. *A neutral semimonotonic binary decision rule will be a continuous-valued social function over a CCE subset for all profiles of continuous-valued individual weak orderings if and only if the binary decision rule is simple.*

Proof: (*a*) *Sufficiency:* The proof is almost identical to the proof of sufficiency for Theorem 2. That proof does not use neutrality or monotonicity of the social function, but it does use the characterization of the social function by a set of winning coalitions for each concerned set. It is only necessary to show, then, that use of that characterization is justified for neutral binary decision rules given that they may not be characterized by direct sums of simple games since the social function may not be monotonic.

In fact, Lemma 1 in Strnad (1983) establishes a stronger characterization result than is needed here. By using Theorem 1 in Ferejohn and Fishburn (1979) that Lemma shows that neutral social functions are characterized by collections of proper games. In other words, for each concerned set there is a set of winning coalitions and that set is a proper game. Since neutrality implies binaryness, a social function is neutral if and only if it is a neutral binary decision rule.

(*b*) *Necessity:* The proof of part (*b*) of Theorem 2 applies with some modifications. Δ is a collection of proper games rather than a direct sum of simple games. If Δ is not simple, then there is a concerned set $E \subset I$ and a coalition $B \subseteq E$ such that $B \in \Delta_E$ but $B \notin \Delta_I$. By semimonotonicity $B \in \Delta_E \Rightarrow B \in \Delta_B$ and $C \in \Delta_I \Rightarrow C \in \Delta_C$.

The only place where monotonicity instead of semimonotonicity is used in the proof is where $B \in \Delta_B \Rightarrow I \in \Delta_I$ is asserted. In order to complete the proof using only semimonotonicity, it is necessary to consider $I \notin \Delta_I$ as a separate case. In that case, the table in part (*b*) of the proof of Theorem 2 becomes:

Set containing point x	Social Preference		
	Case (1)	Case (2)	Case (3)
Segment rc' without c'	xIz	xIz	xIz
c'	zPx	zPx	zPx
Segment $c'b'$ without b', c'	xPz	xIz	xIz
b'	xPz	xPz	xIz
Segment $b'z$ without b', z	xIz	xIz	xIz
z	xIz	xIz	xIz
Segment zs without z	xIz	xIz	xIz

In all three cases, $R(z) = \{y \in A \mid yRz\}$ is the whole set rs without point c' and is not a closed set. QED

IV. Characterizing Continuous-Valued Neutral Monotonic Social Functions Over CCE Subsets

When will a continuous-valued neutral monotonic social function defined over a *CCE* subset alternative set guarantee the existence of a socially undominated alternative? Theorem 3, the main result of this section, answers that question by providing necessary and sufficient conditions for the existence of a socially undominated alternative for all profiles of continuous-valued weak orderings. We know from Theorem 2 in the previous section that a continuous-valued neutral monotonic social function is simple. Thus, we need only concern ourselves with a single set of winning coalitions rather than with sets of winning coalitions that differ depending on which members of society are indifferent in each particular binary choice. Theorem 3 uses the condition that this single set of winning coalitions be a maximal game. That is, if the dimension of the *CCE* subset is m, then the intersection of any collection of $m+1$ or fewer winning coalitions must be nonempty. Thus, Theorem 3 asserts that a continuous-valued neutral monotonic social function defined over a *CCE* subset will guarantee the existence of a socially undominated alternative if and only if the set of winning coalitions is a maximal game. This Theorem therefore characterizes continuous-valued neutral monotonic social *decision* functions.[3]

Theorem 3 generalizes a result in Greenberg (1979). Greenberg found that an anonymous social function is guaranteed to produce a socially undominated alternative if and only if it is of the following form:

> xPy for $x, y \in A$ if and only if the number of persons preferring x to y exceeds some number r with $r > (m/(m+1))n$ where $d(A)=m$ and $|I|=n$, the number of persons in society.

Social functions of this form are simple because the required number of votes is independent of the concerned set. Theorem 4 below shows that Greenberg's result is the special case of Theorem 3 that occurs when social functions are required to be anonymous. The reason for the connection is apparent from Lemma 3 below: Imposing anonymity makes the class of rules stated by Greenberg and the class of maximal games coextensive.

Theorem 3 also provides parallels to results for acyclic social functions when the alternative set is finite. Let us first review those results and their extension to *CCE* subset alternative sets. Blau and Brown (1978, Theorem 5, p. 18) show that when $|I|$, the number of people in society, exceeds $|A|$, and a neutral monotonic social function is required to be acyclic, then the associated direct sum of simple games may contain simple games without veto players. More precisely, their Theorem 5 states that acyclic neutral monotonic social functions are characterized by direct sums of acyclic games. An acyclic game is a simple game where any empty intersection within the set of winning coalitions has more than $|A|$ members. If the concerned set (containing nonindifferent individuals for a particular binary choice) is larger than $|A|$, then there are acyclic games without veto players. For example, if the concerned set is C, let the winning coalitions be all coalitions of size $|C|-1$ and greater. This simple game is an acyclic game since the smallest empty intersection has $|C|$ members and $|C|>|A|$. At the same time no individual is in all winning coalitions.[4]

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[3] Schofield (1984) proves a result similar to the necessity part of Theorem 3. He essentially begins with simple social functions on *CCE* subsets and shows that the set of winning coalitions must be a maximal game

[4] Craven (1971) shows that in order to guarantee acyclicity for an anonymous proportional voting rule over a finite alternative set A, the minimum-sized winning coalition must be of size greater than $[(|A|-1)/|A|]|I|$ when $|A|<|I|$, the number of individuals in society. Blau and Brown (1978, Proposition 4, pp. 19–20) show that Craven's result is the special case of their Theorem 5 that arises when anonymity is imposed

Blau and Brown (1978, Theorem 4, p. 14) also show that when $|I|$ does not exceed $|A|$ and a neutral monotonic social function is required to be acyclic, all of the simple games in the associated direct sum of simple games must be prefilters. That is, the set of winning coalitions for each concerned set has nonempty intersection, and thus the corresponding simple game has at least one veto player. When A is a *CCE* subset containing more than one point, it will be true that $|A| > |I|$ given that I is finite. Thus, for *CCE* subset alternative sets, acyclic neutral monotonic social functions must be characterized by direct sums of prefilters.

Theorem 3 restores the possibility of having no veto players by replacing acyclicity with the requirement that neutral monotonic social functions always yield at least one socially undominated alternative in each *CCE* subset. In that case the possibility of no veto players exists if the dimension of the alternative set is at least two less than the number of concerned individuals. This parallels Blau and Brown's Theorem 5 which requires the number of alternatives to be at least one less than the number of concerned individuals in order to have the possibility of choice rules without veto players.

In order to prove Theorem 3 we begin with a result due to Sonnenschein (1971, Theorem 4, p. 219): When an individual has preferences that are complete, upper semicontinuous and almost convex[5] over a *CCE* subset of alternatives, A, then there will be an undominated alternative in A under the individual's preferences. This result can be used to prove that particular social functions will yield an undominated alternative in any *CCE* subset by showing that the *social* preferences represented by the functions are upper semicontinuous and almost convex.[6]

The following lemma provides a sufficient condition for social preferences to be almost convex.

Lemma 1. *Suppose the alternative set is an m-CCE subset. If individual preference relations are almost convex and if the set of all coalitions that are winning coalitions in at least one simple game in a direct sum of simple games, Γ, is a maximal game then the social preferences generated by Γ are almost convex.*

Proof:[7] Let $P_i(z)=\{x\in A|xP_iz\}$, let $P(z)=\{x\in A|xPz\}$ where P is the social preference relation, and let $H(B)$ stand for the convex hull of a set of points B. Suppose that all the conditions of the lemma are met except that for some alternative z, $z\subseteq H(P(z))$.

By Caratheodory's Theorem,[8] there exist $m+1$ points in the alternative set (which is an m-dimensional subset of Euclidean space) such that each of the points is

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[5] Sonnenschein assumes convexity of preferences where convex preferences are defined in the same way as in this article. Nonetheless, in proving his Theorem 4 he actually only uses the weaker property (implied by the convexity of preferences) that preferences are almost convex

[6] Greenber (1979) used this approach to prove his main result, discussed at pp. 3–4

[7] This proof is almost identical to one in Greenberg (1979, pp. 630–31). This lemma simply generalizes part (*ii*) of his Theorem 2 to the case where the social preference relation need not be anonymous

[8] Caratheodory's theorem states that the convex hull of $X\subseteq \mathbb{R}^m$ equals the set of all points represented by $\sum_i a_ix_i$ where $\sum_i a_i=1$ and $a_i\geqq 0$ for all i as the $m+1$ x_i independently range over X and the weights a_i take on all possible values. See Nikaido (1968, Theorem 2.4, p. 19)

in $P(z)$ and z is a linear combination of the $m+1$ points with nonnegative coefficients that sum to one. For each of these points x_i there is some winning coalition all of whose members prefer x_i to z. Choose a set, S, of $m+1$ coalitions such that the jth coalition prefers x_j to z. Since the set of all winning coalitions is a maximal game, the coalitions in S must have at least one member in common. This member, h, prefers each of the x_j to z. Thus, it is the case that $H(P_h(z))$ contains z since z is the convex hull of the $m+1$ points x_j. This contradicts the assumption that individual preferences are almost convex. QED

Given Theorem 2 and Lemma 1 it is easy to state sufficient conditions for a direct sum of simple games to generate an aggregation rule that is a continuous-valued neutral monotonic social decision function. Lemma 2 provides the machinery to show that the sufficient conditions are also necessary.

Lemma 2. *If Γ, a direct sum of simple games, generates a neutral monotonic social decision function over a CCE subset alternative set, then Γ must be a direct sum of maximal games.*

Proof:[9] Let $d(A)=m$. Suppose that for some concerned set S the set, Γ_S, of winning coalitions is not a maximal game. Then there is at least one subset $W \subseteq \Gamma_S$ that contains at most $m+1$ coalitions, and those coalitions have empty intersection. Reduce W by removing coalitions until removal of any additional coalition will result in W having nonempty intersection. The resulting set, W', will contain at most $m+1$ coalitions and will have empty intersection.

Since the alternative set is an m-dimensional subset of $\mathbb{R}^m$, it contains simplices of every dimension up to m. An n-dimensional simplex has $n+1$ $(n-1)$-dimensional faces. (An s-dimensional face of a simplex is the convex hull of $s+1$ vertices of the simplex.) Suppose that the set W' contains s coalitions. Then take an $(s-1)$-dimensional simplex in the alternative space. This simplex has $s(s-2)$-dimensional faces. For each coalition C_j $(j=1,2,\ldots,s)$ in W' there is a corresponding jth $(s-2)$-dimensional face, F_j, of the simplex. (Henceforth refer to that face as the "jth facet" where a facet is a face with a dimension one lower than the simplex in which it is embedded.)

$(s-1)$-dimensional simplices have the property that an intersection of up to $s-1$ facets is nonempty but an intersection of all s facets is empty. Let $W_i' = \{C_j \in W' | i \in C_j\}$ and let $T=\{i \in I | W_i' \neq \emptyset\}$. Because Γ is a direct sum of simple games, $C_j \in \Gamma_S \Rightarrow C_j \in \Gamma_T$ follows from $T \subseteq S$. For concerned set T, Γ_T is not a maximal game since W' is a set of $s<m+2$ winning coalitions with empty intersection. For $i \in T$, then, W_i' has at least one member and at most $s-1$ members. For $i \in T$ let $Q_i = \bigcap_{j | C_j \in W_i'} F_j$ which is a face of the simplex and is therefore a nonempty closed set in Euclidean m-space. If $d(Q_i)>0$, then there is a unique hyperplane G_i with $d(G_i)=d(Q_i)$ such that $Q_i \subset G_i$. Define $\operatorname{int}(Q_i)$ as the union of all open sets in G_i contained in Q_i. Consider a profile p with type I preferences for $i \in T$ and with all $i \in I/T$ indifferent between all alternatives in A. Choose as a bliss point for each $i \in T$

[9] This proof is inspired by Greenberg's proof of his Theorem 1 (1979, pp. 629–30). The proof here is similar in its basic idea but more fully exploits the properties of simplices to get a much more general result

some $i' \in \mathrm{int}(Q_i)$ if $d(Q_i) > 0$ or $i' = Q_i$ if $d(Q_i) = 0$. Let $H_j = H(\{i' | i \in C_j\})$ where $H(B)$ is the convex hull of the set B. That is, H_j is the convex hull of the bliss points for members of coalition C_j. By the choice of i' for each $i \in T$, $H_j \subseteq F_j$.

With this profile of continuous-valued, convex weak orderings Greenberg's proof of case (*i*) for his Theorem 1 (1979, pp. 629–30) can be used to show that there is no alternative in A socially preferred to or socially indifferent to all others. Rather than exactly duplicating Greenberg's proof, this proof merely sketches his arguments using the terminology of this paper.

Consider a point, z, outside of the simplex. Since the simplex is compact and convex, there exists a unique point, z', in the simplex closest to z. This point z' is closer to the bliss points of all members of T. As a result, z' is preferred to z by all coalitions in W' and thus is socially preferred to z.

Consider a point, y, in the simplex. There is at least one facet of the simplex that does not contain y. Suppose that one of the facets that does not contain y is the kth facet. This facet contains H_k, the convex hull of the bliss points for all members of coalition $C_k \in W'$. H_k is compact and convex so that there is a point, y', in H_k closest to y, and every point in H_k is closer to y' than to y. Since H_k contains the bliss point of each member of coalition C_k, all the members of coalition C_k must prefer y' to y. Since C_k is a winning coalition, y' is socially preferred to y.[10]

We have constructed a profile of individual preferences such that there are no points in the simplex or outside of it that are socially preferred or indifferent to every other alternative. Thus, the social function generated by the direct sum of simple games does not guarantee an undominated element for all possible sets of continuous valued and almost convex individual preferences where any one simple game in the direct sum is not a maximal game. QED

Using Theorems 1 and 2 and Lemmas 1 and 2 it is possible to prove the major result of this section.

Theorem 3. *For a CCE subset alternative set, a continuous-valued neutral monotonic social function σ generated by Γ, a direct sum of simple games, will produce at least one socially undominated alternative for all profiles of continuous-valued, almost convex weak orderings if and only if Γ_I, the set of all winning coalitions, is a maximal game.*[11]

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[10] The possibility that some members in T but not in C_k are indifferent between y and y' is not a problem. Because we are dealing with a direct sum of simple games, if C_k wins when T is the concerned set, it wins when any smaller set containing C_k is the concerned set

[11] A more general result could be obtained if the converse of Lemma 2 were true, i.e., if it were true that each direct sum of maximal games generates a neutral monotonic social decision function. Then neutral monotonic social decision functions would be characterized by direct sums of maximal games. The restriction in Theorem 3 to continuous-valued (and thus simple) social functions could be removed. Unfortunately, there are examples of direct sums of maximal games that do not generate social decision functions. See Strnad (1982, Theorem 5, p. 100). Thus, a direct sum of maximal games structure and continuous-valuedness (simplicity) are jointly sufficient to guarantee a socially undominated alternative, and neither condition is sufficient by itself.

It is also true that continuous-valuedness is not a necessary condition for there to be a socially undominated alternative for all profiles of continuous-valued, almost convex weak orderings. For example, the Pareto extension rule (xPy if xR_iy for all i and xP_jy for some j) is a social decision function over *CCE* subset alternative sets, but the rule is not simple and therefore is not necessarily continuous-valued. See Strnad (1982, pp. 103–111)

Proof: By Theorem 1 Γ generates an aggregation rule that is a neutral monotonic social function σ. Because σ is continuous-valued, it is a simple social preference relation and the set of all coalitions that are winning coalitions in at least one simple game in Γ is Γ_I. It remains to show that σ is a social decision function if and only if Γ_I is a maximal game.

(a) Sufficiency: Since Γ_I is a maximal game and contains all coalitions that win in at least one simple game in Γ, by Lemma 1 σ represents social preferences that are almost convex as well as continuous-valued. That σ is a social decision function then follows from Sonnenschein's result (1971, Theorem 4, p. 219).

(b) Necessity: Since σ is a neutral monotonic social decision function, Lemma 2 implies that Γ is a direct sum of maximal games. But then Γ_I is a maximal game. QED

Recall that Greenberg (1979) showed that an anonymous social function is a weak social decision function if and only if it is of the following form:

> xPy for $x, y \in A$ if and only if the number of persons preferring x to y exceeds some number r with $r > (m/(m+1))n$ where $d(A)=m$ and $|\mathrm{I}|=n$, the number of persons in society.

A social function of this form specifies that for each concerned set, C, the social choice rule is $r(C)$-majority rule with $r(C)=r$. Each $r(C)$-majority rule is a simple game, and, as the following Lemma indicates, imposing anonymity makes the class of maximal games and the class of $r(C)$-majority rules coextensive.

Lemma 3. *Suppose that Γ, a direct sum of simple games, generates an anonymous social function. Then Γ_J, a simple game in Γ, is a maximal game if and only if it is an $r(J)$-majority rule.*

Proof: (a) Sufficiency. Suppose that the set of winning coalitions in the simple game for concerned set J conforms to an $r(J)$-majority rule. Then the minimal winning coalition size must be some number d greater than $(m/(m+1))t$ where m is the dimension of the alternative set and $|J|=t$. Let the number of individuals be $t=a(m+1)+b$ where $0 \leqq b < m+1$.

Consider the case where $b=0$. Then $(m/(m+1))t=am$. For the social decision rule to be an $r(J)$-majority rule, we must have $d > am$. When $d > am$, each winning coalition lacks at most $a-1$ members from the whole set of $a(m+1)$ members. The smallest possible winning coalition size that could define the $r(J)$-majority rule therefore would be $t-(a-1)$ members. Since there are no smaller coalitions that could be winning coalitions, the minimum-sized empty intersection has number of members equal to the smallest integer greater than or equal to $t/(a-1)$. (For $a \geqq 2$. For $a=1$, there is no empty intersection possible since the only winning coalition under the $r(J)$-majority rule is the coalition of the whole.) But $t/(a-1)= (m+1)(a/(a-1))$ so that the smallest integer greater than or equal to $t/(a-1)$ must be at least $m+2$. Thus, in the case of $b=0$ every $r(J)$-majority rule generates a maximal game.

Now consider the case where $b \neq 0$. The minimum-sized winning coalition will be of size equal to the smallest integer greater than $am+(m/(m+1))b$. Such coalitions can exclude at most a number of members of society equal to the largest integer smaller than $a(m+1)+b-am-(m/(m+1))b=a+(1/(m+1))b$. But a is that integer since b is less than $m+1$. Therefore the minimum-sized empty intersection has number of members $m+2$ since the fastest one can eliminate members of the set from an intersection is a members for each coalition added to the intersection. (One can always exclude any particular a members since anonymity guarantees that every coalition that excludes a members will be a winning coalition.) Since the total number of individuals is $a(m+1)+b$, one "uses up" $m+1$ coalitions eliminating the $m+1$ groups of a members from the intersection and then uses a final coalition to eliminate the remaining b members from the intersection.

(b) Necessity: Given that the social function is anonymous, only the size of coalitions determines whether or not they are winning coalitions. In a simple game, there will be a minimal size, e, required for a coalition to win since adding to a winning coalition does not change the fact that it wins.

Now suppose that a simple game for concerned set J with $|J|=t$ is a maximal game and $d(A)=m$. Define $s=t-e$. Then the smallest number of winning coalitions that comprise an empty intersection must be the smallest integer greater than or equal to t/s since there is an appropriate coalition of size $t-s$ to eliminate any particular s persons from an intersection of coalitions. If the set of winning coalitions is to be a maximal game, it must be true that $t/s>m+1$. Rearranging that inequality in terms of e yields the requirement that $e>(m/(m+1))t$. But that requirement is the definition of an $r(J)$-majority rule. QED

The following theorem shows the sense in which Greenberg's result is a special case of Theorem 3.

Theorem 4. *Let A be a CCE subset, let $d(A)=m$, let $|I|=n$. A neutral monotonic social function σ generated by Γ, a direct sum of simple games, is a continuous-valued, anonymous social decision function for all profiles of continuous-valued weak orderings if and only if there exists an integer $r>(m/(m+1))n$ such that $E\in\Gamma_J$ for any $J\subseteq I$ if and only if $|E|\geqq r$.*

Proof: (a) Sufficiency: The rule, Q, that $E\in\Gamma_J$ for any $J\subseteq I$ if and only if $|E|\geqq r$ for some fixed r with $r>(m/(m+1))n$ is a simple social function. It is simple because there is a single set of decisive coalitions regardless of concerned set. It is a social function because $m\geqq 1 \Rightarrow r>0.5n$ so that xPy and yPx is impossible.

The rule Q is generated by a direct sum of simple games. For a concerned set J, Q is an $r(J)$-majority rule and is therefore a simple game. Q is a direct sum of simple games because reducing the concerned set cannot make a winning coalition lose and increasing the concerned set cannot make a losing coalition win. In each case there is a critical size that determines whether a coalition wins or loses and this size is independent of the concerned set.

By Lemma 3, under anonymity direct sums of $r(C)$-majority rules are direct sums of maximal games. Thus, Q is a simple social function generated by a direct sum of maximal games. By Theorem 3 it is therefore a continuous-valued weak social decision function. Q is clearly anonymous since coalition size and not the identity of coalition members determines whether a coalition is in Γ_J for any given $J \subseteq I$.

(b) Necessity: By Theorem 2 if a neutral monotonic social function σ is a continuous-valued weak social decision function for all profiles of continuous-valued weak orderings, it must be a simple social function, and it must be generated by a direct sum of maximal games. Since σ is anonymous, each maximal game must be an $r(C)$-majority rule. Since σ is simple, there must be a single set S of decisive coalitions regardless of the concerned set and S must equal Γ_I. But Γ_I is an $r(I)$-majority rule, and consequently there exists an integer $r > (m/(m+1))n$ such that $E \in \Gamma_J$ for any $J \subseteq I$ if and only if $|E| \geqq r$. QED

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