J. geom. 62 (1998) 190 - 199
0047-2468/98/020190-10 $ 1.50+0.20/0

Journal of Geometry

DESMIC SYSTEMS OF TETRAHEDRA ASSOCIATED WITH A MORLEY-PETERSEN-STUDY CONFIGURATION

Charles Thas

The configuration of ten lines in an Euclidean space R^3 such that each line intersects three others orthogonally was discovered in 1898 almost at the same time by F. Morley and J. Petersen (Hjelmslev). After an observation made by the famous German mathematician Study, Morley proved in 1899 the existence of an analogous configuration in a non-Euclidean space, which, considered in a projective space $\mathcal{P}^3$, consists of ten pairs of reciprocal polar lines with regard to a non-singular quadric $\mathcal{K}$ such that each pair are the common transversals of three pairs of the configuration. We call this a Morley-Petersen-Study configuration (an $\mathcal{M}$-$\mathcal{P}$-$\mathcal{S}$). In this paper we prove that for a general choice of a pair of reciprocal polar lines with respect to the quadric $\mathcal{K}$, five line congruences of class four and degree four are associated with an $\mathcal{M}$-$\mathcal{P}$-$\mathcal{S}$, such that each contains the chosen pair of polar lines and four pairs of reciprocal polar lines of the configuration and such that the intersection of the five congruences consists of eighteen lines which form the edges of two desmic systems of three tetrahedra.

1. INTRODUCTION

Assume that (a, a'), (b, b'), (c, c') are pairs of reciprocal polar lines with regard to a non-singular quadric $\mathcal{K}$ in a 3-dimensional projective space $\mathcal{P}^3$ over an algebraically closed field (for instance, over the complex field $\mathbf{C}$). For a general choice of the three pairs of lines, each of the four lines (b, b', c, c'), (c, c', a, a'), (a, a', b, b') has a pair of common transversals which are denoted by (d, d'). (e, e'), (f, f'), respectively. Finally, we call (l, l'), (m, m'), (n, n') the pairs of common transversals of the four lines (a, a', d, d'), (b, b', e, e'), $(c, c'f, f')$, respectively. Then (l, l'), (m, m') and (n, n') have a common pair of transversals, denoted by (s, s') (F. Morley).

With a general choice of (a, a'), (b, b'), (c, c') we have that the seven associated pairs of lines are uniquely determined and are pairs of reciprocal polar lines with respect to $\mathcal{K}$. A configuration of ten pairs of reciprocal polar lines of this kind is called a Morley-Petersen-Study configuration in $\mathcal{P}^3$ (an $\mathcal{M}$-$\mathcal{P}$-$\mathcal{S}$). Remark that in an elliptic space with an absolute quadric $\mathcal{K}$ of signature four, the twenty lines of the configuration are real if the starting lines (a, a'), (b, b'), (c, c') are real (in this case, for instance the lines (f, f') are the common orthogonal transversals of the lines (a, b), and also of (a', b'), (a, b') and (a', b)).
In a paper of J. Bilo ([3]), one finds an extensive bibliography about the $\mathcal{M}$-$\mathcal{P}$-$\mathcal{S}$ problem and several proofs (most of them original) of the configuration. The paper also contains an extension of the problem.

2. CONICS ASSOCIATED WITH A DESARGUES CONFIGURATION IN A COMPLEX PROJECTIVE PLANE $\mathcal{P}^2$

2.1. Assume that ABC, DEF are two triangles which form a Desargues pair, i.e. the six vertices are mutually different and AD, BE, CF are three mutually different lines with a common point S, called the center of the Desargues pair. The points $L = BC \cap EF$, $M = CA \cap FD$, $N = AB \cap DE$ are collinear on a line s, the axis of the Desargues pair.
A theorem of Chasles says that a polarity π with respect to a non-singular conic k exists, such that the vertices A, B, C and the sides a, b, c of the triangle ABC are mapped by π on the corresponding sides d, e, f and on the corresponding vertices D, E, F of DEF. Moreover, the points S, L, M, N are mapped by π on s, $l = AD$, $m = BE$, $n = CF$, respectively. The ten points and the ten lines form a Desargues $(10_3, 10_3)$-configuration $\mathcal{D}$, which is self-polar with regard to k. The configuration is uniquely determined when the triangle ABC and the conic k are given. Each point of $\mathcal{D}$ is the centre of a Desargues pair of triangles in $\mathcal{D}$. Such point and the vertices of one of the triangles of the corresponding Desargues pair, form the vertices of a (complete) quadrangle which is *apolar* with respect to k, i.e. it is a quadrangle such that two pairs of opposite sides (and then also the third pair) are conjugate with regard to k (theorem of Hesse).
It is easy to see that the $(10_3, 10_3)$-configuration $\mathcal{D}$ contains five quadrangles which are apolar with regard to k: $SABC$, $SDEF$, $ADMN$, $BENL$, $CFLM$. They are mapped by π on the five (complete) quadrilaterals $sabc$, $sdef$, $admn$, $benl$, $cflm$ in $\mathcal{D}$ which are apolar with respect to k, i.e. the opposite vertices of these quadrilaterals are conjugate with regard to k.

A punctual conic γ (a conic considered as a point set) is called *apolar* with regard to a conic k, if γ is circumscribed to a triangle which is a self-polar triangle of k. Then each point of γ is the vertex of a self-polar triangle of k inscribed in γ. Sometimes γ is called *harmonically circumscribed* to k (Samuel [5]).
The vertices of a quadrangle which is apolar with respect to k are the base points of a pencil of conics, all of which are apolar with regard to k. Conversely, the conics which are apolar with respect to k and which contain at least three vertices of an apolar quadrangle, also

contain the fourth vertex.
Of course we have the dual definitions and properties for a tangential conic (a conic considered as the set of its tangent lines) which is apolar with regard to a conic. If γ is apolar with respect to k, then the tangential conic k^* is apolar with regard to γ, i.e. k^* is inscribed in a self-polar triangle of γ.
Recall that if a conic is inscribed in two triangles, then there also exists a conic circumscribed to the two triangles.

Consider again the Desargues configuration $\mathcal{D}$, with its associated conic k.

2.2. LEMMA. The conics γ^i $(i = 1, ..., 5)$ through an arbitrary point X of the plane $\mathcal{P}^2$ and circumscribed to the apolar quadrangles $SABC$, $SDEF$, $ADMN$, $BENL$, $CFLM$ of the Desargues configuration $\mathcal{D}$, have two other points Y, Z in common, which form together with X a self-polar triangle of k.

Proof. If the point X is not chosen in a special way, we may assume that $X \notin k$ and that the polar line x of X with respect to k has two different points Y, Z in common with the conic γ^1 through the points S, A, B, C, X (see also remark 2.4). Since γ^1 contains the vertices of the apolar quadrangle $SABC$ with regard to k, the conic γ^1 is apolar with respect to k and thus XYZ is a self-polar triangle of k. The polarity π with regard to k maps γ^1 on the tangential conic $\pi(\gamma^1)$ which is inscribed in the apolar quadraliteral $(s,\ EF,\ FD,\ DE)$ and also in the self-polar triangle XYZ. This means that there exist four conics circumscribed to XYZ and to the triangles DEF, DMN, ENL, FLM, respectively (we get the sides of these triangles by omitting each time one side of the quadrilateral $(s,\ EF,\ FD,\ DE)$). These conics are apolar with respect to k, because they are circumscribed to the self-polar triangle XYZ of k, and consequently these conics also contain the vertex which forms together with the considered triangle an apolar quadrangle of k. The five conics which are associated in this way with the $(10_3, 10_3)$-configuration $\mathcal{D}$ for an arbitrary choice of the point X are: $\gamma^1(SABCXYZ)$, $\gamma^2(SDEFXYZ)$, $\gamma^3(ADMNXYZ)$, $\gamma^4(BENLXYZ)$ and $\gamma^5(CFLMXYZ)$.

2.3. REMARK. For any Desargues pair of triangles of the configuration $\mathcal{D}$, for instance ABC and DEF, and for a general choice of the point X, the points X, Y, Z are the invariant points of the projectivity f of $\mathcal{P}^2$, determined by:

$$f:\ A,\ B,\ C,\ X \longrightarrow D,\ E,\ F,\ X.$$

Proof. Since ABC is a self-polar triangle for the correlation $\pi \circ f$, this correlation is a polarity of $\mathcal{P}^2$. This means that $\pi \circ f = f^{-1} \circ \pi$ or $f = \pi \circ f^{-1} \circ \pi$.
Put $x = \pi(X)$, then $f(x) = \pi \circ f^{-1} \circ \pi(x) = \pi \circ f^{-1}(X) = \pi(X) = x$ and $x = YZ$ is an invariant line of f.
Moreover, working with the conics γ^1, γ^2, we get from Steiner's theorem (Samuel [5], pp. 66,67) the following projective connections:
$XA,\ XB,\ XC,\ XY,\ XZ \wedge SA,\ SB,\ SC,\ SY,\ SZ\ =\ SD,\ SE,\ SF,\ SY,\ SZ \wedge$

XD, XE, XF, XY, XZ.
This gives for the lines through X:

$$XA,\ XB,\ XC,\ XY,\ XZ \wedge XD,\ XE,\ XF,\ XY,\ XZ,$$

and XY, XZ are invariant lines of f. Therefore $XY \cap x = Y$ and $XZ \cap x = Z$ are invariant points of f.

2.4. REMARK. In the foregoing we only considered the case where $x = \pi(X)$ has two different common points with the conic γ^1. One can prove that the locus of the points X of $\mathcal{P}^2$ for which $x = \pi(X)$ is a tangent line of γ^1, is a rational curve of order six for which the ten points of the configuration $\mathcal{D}$ are nodes (Thas [7]).

2.5. REMARK. Let us suppose that the points Y, Z are the absolute (or cyclic) points of a complexified Euclidean plane. In that case ABC and DEF are direct similar triangles, the conics γ^i are circles, the conic k is an orthogonal hyperbola and X is the common point of Miquel of the five apolar quadrilaterals of the configuration $\mathcal{D}$.

3. THE $\mathcal{M}$-$\mathcal{P}$-$\mathcal{S}$ CONFIGURATION

3.1. A remarkable proof of the $\mathcal{M}$-$\mathcal{P}$-$\mathcal{S}$ configuration, determined by three pairs of reciprocal polar lines (a, a'), (b, b'), (c, c') with respect to a non-singular quadric $\mathcal{K}$, uses the classical method of Plücker-Klein for the bijective mapping of the line variety in $\mathcal{P}^3$ onto a hyperquadric $\mathcal{Q}$ in $\mathcal{P}^5$. The principles of this mapping of the lines in $\mathcal{P}^3$ on the points of Klein's hyperquadric $\mathcal{Q}$ in $\mathcal{P}^5$ can be found in several classical books on algebraic geometry (Baker [2], Semple and Roth [6]).
The images of intersecting lines l_1, l_2 are points P_1, P_2 of $\mathcal{Q}$ which are conjugate with respect to $\mathcal{Q}$, i.e. P_1 (P_2, respectively) belongs to the tangent hyperplane of $\mathcal{Q}$ at P_2 (P_1, respectively). The intersections of $\mathcal{Q}$ with linear spaces of dimension four, three, two, respectively, are the images of linear line complexes, linear line congruences and quadratic reguli.
The complementary reguli G and H of a non-singular quadric $\mathcal{K}$ of $\mathcal{P}^3$ are mapped onto conics k_g and k_h on $\mathcal{Q}$ in planes V_g and V_h ($V_g \cap V_h = \oslash$), which are reciprocal polar planes with regard to $\mathcal{Q}$. Reciprocal polar lines in $\mathcal{P}^3$ with respect to $\mathcal{K}$ are mapped on points of $\mathcal{Q}$ which are conjugate in the biaxial involution of $\mathcal{P}^5$ with axes (planes of invariant points) V_g and V_h. Conversely, if $P_1 \in V_g$ ($P_1 \notin k_g$) and $P_2 \in V_h$ ($P_2 \notin k_h$), the line P_1P_2 intersects the hyperquadric $\mathcal{Q}$ in the images P, P' of two reciprocal polar lines p, p' with regard to $\mathcal{K}$ in $\mathcal{P}^3$.
If $P_1 \in k_g$ and $P_2 \notin k_h$, then $P_1P_2 \cap Q = \{P_1\}$ and P_1 is the image of a line of G. If $P_1 \in k_g$ and $P_2 \in k_h$, then P_1P_2 is a line on $\mathcal{Q}$, which is the image of the pencil of tangent lines of $\mathcal{K}$ at the common point of the generators of $\mathcal{K}$ with images P_1 and P_2.

3.2. One can prove the existence of the $\mathcal{M}$-$\mathcal{P}$-$\mathcal{S}$ configuration determined in $\mathcal{P}^3$ by three pairs of reciprocal polar lines $(a,\ a')$, $(b,\ b')$, $(c,\ c')$ with regard to a non singular quadric $\mathcal{K}$ (cfr & 1), using two $(10_3, 10_3)$-configurations, which are obtained as follows. The images on $\mathcal{Q}$ of the given pairs of lines are called $(A,\ A')$, $(B,\ B')$, $(C,\ C')$. They are conjugate pairs in the biaxial involution in $\mathcal{P}^5$ with axes V_g and V_h. The lines AA', BB', CC' intersect V_g and V_h in A_g, B_g, C_g and A_h, B_h, C_h, respectively. Assume that $\mathcal{D}_i = (A_i,\ B_i,\ C_i,\ D_i,\ E_i,\ F_i,\ L_i,\ M_i,\ N_i,\ S_i)$, $i = g, h$ are the Desargues configurations which are determined in the planes V_g and V_h by the triangles $A_iB_iC_i$, $i = g, h$ and by the polarities π_g and π_h with respect to the conics k_g and k_h, respectively (cfr & 2). Then the intersection points of $\mathcal{Q}$ with the lines connecting corresponding points of $\mathcal{D}_g$ and $\mathcal{D}_h$ are the images of the ten pairs of reciprocal polar lines of the $\mathcal{M}$-$\mathcal{P}$-$\mathcal{S}$ configuration (Bilo [3]).

4. THE FIVE $(4,4)$-CONGRUENCES ASSOCIATED WITH AN $\mathcal{M}$-$\mathcal{P}$-$\mathcal{S}$ CONFIGURATION AND A GENERAL PAIR OF RECIPROCAL POLAR LINES $(x,\ x')$ WITH REGARD TO $\mathcal{K}$

Consider in $\mathcal{P}^5$ the common points X_g, X_h of the planes V_g and V_h with the line connecting the images $(X,\ X')$ in $\mathcal{P}^5$ of the lines $(x,\ x')$ in $\mathcal{P}^3$ and consider the conics $\gamma_i^1(S_iA_iB_iC_iX_iY_iZ_i)$ $i = g, h$ in V_g and in V_h, such as described in & 2.2. The intersections of $\mathcal{Q}$ with V_g and V_h are the conics k_g and k_h.

4.1. The quadratic hypercones $\mathcal{K}_g^1$ and $\mathcal{K}_h^1$ of $\mathcal{P}^5$, projecting the conic γ_g^1 from V_h and γ_h^1 from V_g, respectively, intersect $\mathcal{Q}$ in the images of quadratic line complexes $\mathcal{C}_g^1$ and $\mathcal{C}_h^1$ in $\mathcal{P}^3$. The intersections of the hypercones $\mathcal{K}_g^1 \cap \mathcal{K}_h^1$ is the 3-dimensional variety of order 4, which is generated in $\mathcal{P}^5$ by the lines connecting a variable point $P_1 \in \gamma_g^1$ with a variable point $P_2 \in \gamma_h^1$, and $\mathcal{K}_g^1 \cap \mathcal{K}_h^1 \cap Q$ is the image of a line congruence of order 4 and class 4 in $\mathcal{P}^3$, called Γ^1.
We can proceed in the same way with the five pairs of conics $(\gamma_g^j,\ \gamma_h^j)$, $j = 1, ..., 5$ associated with the Desargues configurations $\mathcal{D}_g$ and $\mathcal{D}_h$ (see lemma 2.2), and we obtain five congruences Γ^j, $j = 1, ..., 5$ in $\mathcal{P}^3$.

4.2. Construction in $\mathcal{P}^3$ of the quadratic line complex $\mathcal{C}_g^1$.
The polarity in V_g with respect to k_g is denoted by π_g. Suppose that in $\mathcal{P}^5$ the point P_1 is on the conic γ_g^1, then $< P_1,\ V_h > \cap Q$ is the image of the linear line congruence in $\mathcal{P}^3$ consisting of the lines meeting two lines (the axes of the congruence), which are mapped on the intersection points of k_g with the polar line $p_1 = \pi_g(P_1)$ of P_1 with respect to k_g. Thus these two lines are generators of the regulus G.
If in $\mathcal{P}^5$ the point P_1 is variable on γ_g^1, the locus of $p_1 = \pi_g(P_1)$ is the tangential polar conic of γ_g^1 with regard to k_g, i.e. it is the tangential conic $\pi_g(\gamma_g^1)$ inscribed in the triangles $D_gE_gF_g$, $X_gY_gZ_g$ and tangent at the line s_g of the configuration $\mathcal{D}_g$.
The variable point-pair $p_1 \cap k_g$ generates the symmetric $(2,2)$- correspondence on k_g, which is determined by the intersection point-pairs of k_g with the polar lines with regard to k_g of the points A_g, B_g, C_g, S_g, X_g, respectively (Samuel [5]).
This $(2,2)$-correspondence is the image in $\mathcal{P}^5$ of the $(2,2)$-correspondence in the regulus G in $\mathcal{P}^3$, determined by the pairs of generators of G which intersect the lines a, b, c, s, x,

respectively (and which also intersect their polar lines a', b', c', s', x' with respect to $\mathcal{K}$). The quadratic line complex $\mathcal{C}_g^1$ in $\mathcal{P}^3$ is generated by the lines intersecting both lines of a variable pair of this $(2,2)$-correspondence in G. For the line complex $\mathcal{C}_h^1$, we get an analogous construction from the symmetric $(2,2)$-correspondence obtained in the same way in the regulus H.

4.3. Construction in $\mathcal{P}^3$ of the line congruence $\Gamma^1 = \mathcal{C}_g^1 \cap \mathcal{C}_h^1$

The congruence Γ^1 is generated by the pairs of polar lines $(p,\ p')$ with regard to $\mathcal{K}$, which are the diagonals of skew quadrangles whose opposite sides are generators of G and H, respectively, corresponding in the symmetric $(2,2)$-relations in G and in H, which are uniquely determined by the pairs $(a,\ a')$, $(b,\ b')$, $(c,\ c')$, $(s,\ s')$, $(x,\ x')$, considered as pairs of diagonals of such quadrangles.

Remark that the first four pairs of lines correspond with the apolar quadrangles $A_iB_iC_iS_i$ with regard to k_i, $i = g, h$.

With the five pairs of conics $(\gamma_g^i,\ \gamma_h^i)$, $i = 1, ..., 5$ correspond in this way the five congruences Γ^i, $i = 1, ..., 5$.

4.4. Next, we give an interpretation in $\mathcal{P}^3$ of the projectivities

$$\phi_i :\ X_iA_i,\ X_iB_i,\ X_iC_i \wedge X_iD_i,\ X_iE_i,\ X_iF_i$$

with invariant lines X_iY_i, X_iZ_i, $i = g, h$ (cfr 2.3).

Because of the general position of $(x,\ x')$ with regard to $(a,\ a')$ in $\mathcal{P}^3$, we may assume that x, x', a, a' are not four lines of a quadratic regulus and that they have exactly two common transversals, denoted by t_a and $t_{a'}$. These two lines t_a, $t_{a'}$ are reciprocal polar lines with respect to $\mathcal{K}$ and are the diagonals of a skew quadrangle the sides of which are two generators g_a^1, g_a^2 of G and two generators h_a^1, h_a^2 of H.

The line connecting the images of t_a and $t_{a'}$ on $\mathcal{Q}$, intersects the planes V_g and V_h at points T_{ag} and T_{ah}, which are the poles of X_gA_g and X_hA_h with regard to k_g and k_h, respectively. Moreover, $T_{ai} \in Y_iZ_i$, $i = g, h$.

Using analogous notations in connection with the other lines through X_i, $i = g, h$ and denoting the polarities with regard to k_g and k_h by π_g and π_h, we get

$$\pi_i \circ \phi_i = \Phi_i : T_{ai},\ T_{bi},\ T_{ci},\ Z_i,\ Y_i \wedge T_{di},\ T_{ei},\ T_{fi},\ Z_i,\ Y_i,\ \ i = g, h.$$

The images in $\mathcal{P}^5$ of the pairs of generators $(g_a^1,\ g_a^2)$ and $(h_a^1,\ h_a^2)$ are $k_g \cap X_gA_g$ and $k_h \cap X_hA_h$. If X_iP_i is any line (in V_i) through X_i, the points $X_iP_i \cap k_i$ are conjugate in the involution I_i on k_i, determined by its invariant points $Y_iZ_i \cap k_i$, $i = g, h$. The involutions I_g on k_g and I_h on k_h in $\mathcal{P}^5$ correspond in $\mathcal{P}^3$ with the involutions i_g and i_h on the reguli G and H of $\mathcal{K}$ the fixed generators of which contain the intersection points of x and x' with $\mathcal{K}$.

4.5. Conclusion. With the projectivity ϕ_j corresponds in $\mathcal{P}^3$ the projectivity λ_j of the variety whose elements are the pairs of generators which are conjugate in the involution i_j, $j = g, h$ and λ_g, λ_h are determined by:

$$\lambda_g : (g_a^1,\ g_a^2),\ (g_b^1,\ g_b^2),\ (g_c^1,\ g_c^2) \wedge (g_d^1,\ g_d^2),\ (g_e^1,\ g_e^2),\ (g_f^1,\ g_f^2)$$

$$\lambda_h : (h_a^1,\ h_a^2),\ (h_b^1,\ h_b^2),\ (h_c^1,\ h_c^2) \wedge (h_d^1,\ h_d^2),\ (h_e^1,\ h_e^2),\ (h_f^1,\ h_f^2).$$

These projectivities determine special $(2,2)$-correspondences on G and on H (Samuel [5]). If $(g_y^1,\ g_y^2)$, $(g_z^1,\ g_z^2)$ and $(h_y^1,\ h_y^2)$, $(h_z^1,\ h_z^2)$ are the fixed pairs of the projectivities λ_g and λ_h, then the diagonals of the skew quadrangles $g_y^1 g_y^2 h_y^1 h_y^2$ and $g_z^1 g_z^2 h_z^1 h_z^2$ are the pairs of reciprocal polar lines $(y,\ y')$ and $(z,\ z')$ which are mapped in $\mathcal{P}^5$ on the pairs of intersection points of $Y_g Y_h$ and $Z_g Z_h$ with $\mathcal{Q}$.

4.6. REMARK. In the foregoing we may switch the notations Y, Z for the points Y_h, Z_h and we find two other (analogous) skew quadrangles $g_y^1 g_y^2 h_z^1 h_z^2$ and $g_z^1 g_z^2 h_y^1 h_y^2$ (cfr 5.5).

5. DESMIC TETRAHEDRA DETERMINED BY THE INTERSECTION $\cap_{i=1}^5 \Gamma_i$.

5.1. The eighteen lines of $\cap_{i=1}^5 \Gamma_i$ correspond in $\mathcal{P}^5$ with the common points of $\mathcal{Q}$ and the nine lines connecting the vertices of the triangle $X_g Y_g Z_g$ and the vertices of the triangle $X_h Y_h Z_h$. Consider for instance the three lines $X_g X_h$, $Y_g Y_h$, $Z_g Z_h$. Since $X_g Y_g Z_g$ and $X_h Y_h Z_h$ are self-polar triangles with respect to k_g and k_h, and since the conics k_h, k_g belong to polar planes V_g and V_h with respect to $\mathcal{Q}$, each of these three lines is the polar line with regard to $\mathcal{Q}$ of the 3-dimensional space spanned by the two other lines. Therefore, the pairs of intersection points of these three lines with $\mathcal{Q}$ are the images in $\mathcal{P}^5$ of the pairs of opposite edges of a self-polar tetrahedron of $\mathcal{K}$ in $\mathcal{P}^3$, denoted by $U_0U_1U_2U_3$. We assume that $(x = U_0U_1,\ x' = U_2U_3)$, $(U_0U_2,\ U_1U_3)$ and $(U_0U_3,\ U_1U_2)$ are mapped in $\mathcal{P}^5$ on the pairs of intersection points of $\mathcal{Q}$ with the lines $X_g X_h$, $Y_g Y_h$ and $Z_g Z_h$, respectively (cfr & 4).

5.2. Assume that we have a projective coordinatesystem in $\mathcal{P}^3$ such that $U_0(1,0,0,0)$, $U_1(0,1,0,0)$, $U_2(0,0,1,0)$, $U_3(0,0,0,1)$ and with unit point $V_0(1,1,1,1)$ one of the eight associated points for which the polar plane with respect to $\mathcal{K}$ coincide with the polar plane with regard to $U_0U_1U_2U_3$. The equation of the quadric $\mathcal{K}$ is: $x_0^2+x_1^2+x_2^2+x_3^2 = 0$. The reguli G and H of $\mathcal{K}$ are determined by:

$G:\ (\lambda_0(x_0+ix_1)+\lambda_1(x_2-ix_3)=0,\ \lambda_1(x_0-ix_1)-\lambda_0(x_2+ix_3)=0,\ (\lambda_0,\lambda_1)\neq(0,\ 0)),$

$H:\ (\mu_0(x_0+ix_1)-\mu_1(x_2+ix_3)=0,\ \mu_1(x_0-ix_1)+\mu_0(x_2-ix_3)=0,\ (\mu_0,\mu_1)\neq(0,\ 0)).$

For the Plücker coordinates (p_{ij}) of a line of $\mathcal{P}^3$, we have $p_{01}p_{23}+p_{02}p_{31}+p_{03}p_{12} = 0$, which actually is the equation of Klein's hyperquadric $\mathcal{Q}$ in projective coordinates $(p_{01},\ p_{02},\ p_{03},\ p_{23},\ p_{31},\ p_{12})$ in $\mathcal{P}^5$.

Recall that the projective coordinates of the intersection points of the line (p_{ij}) with the faces of the coordinate frame $U_0U_1U_2U_3$ are:

$(0,\ p_{01},\ p_{02},\ p_{03}),\ (-p_{01},\ 0,\ p_{12},\ -p_{31}),\ (-p_{02},\ -p_{12},\ 0,\ p_{23}),(-p_{03},\ p_{31},\ -p_{23},0).$

Next, we use (pseudo) Klein coordinates (ξ_i) defined by (the usual Klein coordinates (ζ_i) are defined by $\zeta_0 = p_{01} + ip_{23}, \ldots$):
$\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4 : \xi_5 =$
$p_{01} + p_{23} : p_{02} + p_{31} : p_{03} + p_{12} : p_{01} - p_{23} : p_{02} - p_{31} : p_{03} - p_{12}$.
The equation of $\mathcal{Q}$ becomes: $\xi_0^2 + \xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2 - \xi_5^2 = 0$.
The regulus G is mapped on the conic k_g : $(\xi_0^2 + \xi_1^2 + \xi_2^2 = 0,\ \xi_3 = \xi_4 = \xi_5 = 0)$,
and the image of the regulus H is the conic k_h : $(\xi_3^2 + \xi_4^2 + \xi_5^2 = 0,\ \xi_0 = \xi_1 = \xi_2 = 0)$.
The Plücker coordinates of the lines $x = U_0U_1$ and $x' = U_2U_3$ are $(1,0,0,0,0,0)$ and $(0,0,0,1,0,0)$, the (pseudo) Klein coordinates are $(1,0,0,1,0,0)$ and $(1,0,0,-1,0,0)$, respectively. Thus, in (pseudo) Klein coordinates we find $X_g(1,0,0,0,0,0)$ and $X_h(0,0,0,1,0,0)$, $Y_g(0,1,0,0,0,0)$ and $Y_h(0,0,0,0,1,0)$, $Z_g(0,0,1,0,0,0)$ and $Z_h(0,0,0,0,0,1)$.
Incidence of two lines (ξ_i), (ξ_i') is expressed by:
$\xi_0\xi_0' + \xi_1\xi_1' + \xi_2\xi_2' - \xi_3\xi_3' - \xi_4\xi_4' - \xi_5\xi_5' = 0$.

5.3. Two tetrahedra Θ_1 and Θ_2 in $\mathcal{P}^3$ are called *desmic* if each edge of Θ_1 meets two opposite edges of Θ_2, i.e. the edges in each face of Θ_1 are the diagonals of the complete quadrilateral which is the intersection of that face with Θ_2. It is obvious that *desmic* is a symmetric relation. Moreover, if Θ_1 and Θ_2 are desmic, they are perspective from four different centers, which are the vertices of a third tetrahedron Θ_3 and Θ_3 is desmic with Θ_1 and with Θ_2. Thus, desmic tetrahedra always occur in triples and we call this configuration a *desmic triple*. Any two tetrahedra of a desmic triple are desmic and are in perspective from any vertex of the third tetrahedron (Altshiller-Court [1], Hudson [4]).

5.4. We use the following notations for points in $\mathcal{P}^3$ (put $-1 = \bar{1}$):

$$\begin{array}{l}
U_0(1000),\ U_1(0100),\ U_2(0010),\ U_3(0001),\\
V_0(1111),\ V_1(11\bar{1}\bar{1}),\ V_2(1\bar{1}1\bar{1}),\ V_3(1\bar{1}\bar{1}1),\\
W_0(\bar{1}111),\ W_1(1\bar{1}11),\ W_2(11\bar{1}1),\ W_3(111\bar{1}),\\
U_{01}(1100),\ U_{23}(0011),\ U_{01}'(1\bar{1}00),\ U_{23}'(001\bar{1}),\\
U_{02}(1010),\ U_{13}(0101),\ U_{02}'(10\bar{1}0),\ U_{13}'(010\bar{1}),\\
U_{03}(1001),\ U_{12}(0110),\ U_{03}'(100\bar{1}),\ U_{12}'(01\bar{1}0).
\end{array}$$

The following table gives the (pseudo) Klein coordinates of the lines in $\mathcal{P}^3$ which correspond with the intersection points of $\mathcal{Q}$ and the lines connecting the vertices of the triangle $X_gY_gZ_g$ and the triangle $X_hY_hZ_h$:

$$\begin{array}{llll}
X_gX_h \cap \mathcal{Q}\ (100100) & x = U_0U_1 = U_{01}U_{01}' & (100\bar{1}00) & x' = U_2U_3 = U_{23}U_{23}'\\
Y_gY_h \cap \mathcal{Q}\ (010010) & y = U_0U_2 = U_{02}U_{02}' & (0100\bar{1}0) & y' = U_1U_3 = U_{13}U_{13}'\\
Z_gZ_h \cap \mathcal{Q}\ (001001) & z = U_0U_3 = U_{03}U_{03}' & (00100\bar{1}) & z' = U_1U_2 = U_{12}U_{12}'
\end{array}$$

$$\begin{array}{llll}
X_gY_h \cap \mathcal{Q}\ (100010) & W_0W_3 = U_{12}U_{03}' & (1000\bar{1}0) & W_1W_2 = U_{03}U_{12}'\\
Y_gZ_h \cap \mathcal{Q}\ (010001) & W_0W_1 = U_{23}U_{01}' & (01000\bar{1}) & W_2W_3 = U_{01}U_{23}'\\
Z_gX_h \cap \mathcal{Q}\ (001100) & W_0W_2 = U_{13}U_{02}' & (001\bar{1}00) & W_1W_3 = U_{02}U_{13}'
\end{array}$$

$X_gZ_h \cap Q$	(100001)	$V_0V_2 = U_{02}U_{13}$	$(10000\bar{1})$	$V_1V_3 = U'_{02}U'_{13}$
$Y_gX_h \cap Q$	(010100)	$V_0V_3 = U_{03}U_{12}$	$(010\bar{1}00)$	$V_1V_2 = U'_{03}U'_{12}$
$Z_gY_h \cap Q$	(001010)	$V_0V_1 = U_{01}U_{23}$	$(0010\bar{1}0)$	$V_2V_3 = U'_{01}U'_{23}$.

From this table it is easy to see that the tetrahedra $U_0U_1U_2U_3$, $V_0V_1V_2V_3$, $W_0W_1W_2W_3$ form a desmic triple. These tetrahedra correspond with the triples of lines of $\mathcal{P}^5$ which connect the points X_g, Y_g, Z_g with the points of the even permutations of $(X,\ Y,\ Z)$ in $(X_h,\ Y_h,\ Z_h)$. The three tetrahedra are self-polar with regard to $\mathcal{K}$.
The desmic triple is totally determined by the tetrahedra $U_0U_1U_2U_3$ and by the vertex V_0: V_1, V_2, V_3 are the images of V_0 in the biaxial involutions with axes $(U_0U_1,\ U_2U_3)$, $(U_0U_2,\ U_1U_3)$, $(U_0U_3,\ U_1U_2)$ and W_0, W_1, W_2, W_3 are the images of V_0 in the harmonic homologies with centers U_0, U_1, U_2, U_3 and axes (planes of invariant points) $U_1U_2U_3$, $U_0U_2U_3$, $U_0U_1U_3$, $U_0U_1U_2$, respectively.

5.5. With the odd permutations of (X, Y, Z) in $(X_h,\ Y_h,\ Z_h)$ correspond a second desmic triple: the three tetrahedra $U_{01}U_{23}U'_{01}U'_{23}$, $U_{02}U_{13}U'$]02U'_{13} and $U_{03}U_{12}U'_{03}U'_{12}$. The points U'_{ij} are the intersection points of U_iU_j with the polar plane of V_0 with respect to $\mathcal{K}$ and $U_{ij} = U_iU_j\cap$ plane (V_0, U_kU_l), where U_kU_l is the opposite edge of U_iU_j in $U_0U_1U_2U_3$.

ACKNOWLEDGEMENT

The author wishes to express his thanks to Prof. Dr. J. Bilo for his help in the elaboration of this paper.

REFERENCES

[1] ALTSCHILLER-COURT N.: Modern pure solid geometry. Chelsea publishing company 1964.

[2] BAKER H.F.: Principles of geometry, Vol. III. Higher geometry. Cambridge University Press 1925.

[3] BILO J.: Over een uitbreiding van de configuratie van Morley-Petersen in de niet-Euclidische meetkunde. With an English summary. Meded. Kon. Vl. Acad. Wet., Lett., Sch. Kunsten België. *Klasse Wetenschappen. Jaargang xxxi. 1969. Nr. 10.*

[4] HUDSON R.W.H.T.: Kummer's quartic surface. Cambridge University Press 1990.

[5] SAMUEL P.: Projective geometry. Springer-Verlag 1988.

[6] SEMPLE J.G. AND ROTH L.: Introduction to algebraic geometry. Oxford at the Clarendon Press 1949.

[7] THAS C.: A rational sextic associated with a Desargues configuration. Geometriae Dedicata 51, pp 163-180, 1994.

Department of Pure Mathematics
and Computer Algebra
University of Ghent
Krijgslaan 281
B-9000 Gent (Belgium)

Eingegangen am 15. Oktober 1992; in revidierter Fassung am 19. Oktober 1995