Higher Order Finite Difference Method for the Reaction and Anomalous-Diffusion Equation

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Higher order finite difference method for the reaction and anomalous-diffusion equation ☆,☆☆

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ABSTRACT

In this paper, our aim is to study the high order finite difference method for the reaction and anomalous-diffusion equation. According to the equivalence of the Riemann–Liouville and Grünwald–Letnikov derivatives under the suitable smooth condition, a second-order difference approximation for the Riemann–Liouville fractional derivative is derived. A fourth-order compact difference approximation for second-order derivative in spatial is used. We analyze the solvability, conditional stability and convergence of the proposed scheme by using the Fourier method. Then we obtain that the convergence order is $O(\tau^2 + h^4)$, where τ is the temporal step length and h is the spatial step length. Finally, numerical experiments are presented to show that the numerical results are in good agreement with the theoretical analysis.

1. Introduction

Fractional differential equations have attracted increasing interest due to its playing a significant role in neurons, control, electromagnetism, biophysics, physics, regular variation in thermodynamics, mathematical, mechanics, signal and image processing, blood flow phenomena, etc. [1–5].

It is well known that it is difficult to find the analytical solutions of the fractional differential equations. Therefore, seeking numerical methods is an important task in the studies of fractional differential equations. In recent years, there have existed various numerical methods for fractional differential equations, for instance, finite difference method [6–14], finite element method [15–17], and so on.

In this paper, we numerically study the following reaction and anomalous-diffusion equation [18,19]:

$$\frac{\partial u(x,t)}{\partial t} = {}_{RL}D_{0,t}^{1-\alpha}\left(K_\alpha \frac{\partial^2 u(x,t)}{\partial x^2} - C_\alpha u(x,t)\right) + f(x,t), \quad \alpha \in (0,1),\ 0 < t \leqslant T,\ 0 < x < L, \tag{1}$$

subject to the initial, boundary value conditions

$$u(x,0) = 0, \quad 0 \leqslant x \leqslant L,$$

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$$u(0,t) = \varphi_1(t), \quad 0 \leqslant t \leqslant T,$$

$$u(L,t) = \varphi_2(t), \quad 0 \leqslant t \leqslant T,$$

where ${}_{RL}D_{0,t}^{1-\alpha}$ denotes the Riemann–Liouville derivative of order $1-\alpha$ defined by Podlubny [4]

$${}_{RL}D_{0,t}^{1-\alpha}u(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,s)}{(t-s)^{1-\alpha}}ds,$$

K_α and C_α are two positive constants, $f(x,t)$, $\varphi_1(t)$ and $\varphi_2(t)$ are sufficiently smooth functions.

Upto now, some numerical methods are available for the reaction and anomalous-diffusion equation with the case $C_\alpha = 0$. For example, Yuste and Acedo [20] proposed an explicit finite difference method, where the order of convergence was $O(\tau + h^2)$. In [21], Yuste proposed the weighted average finite difference method, where for different weighted parameter λ, he got different convergence order. Chen et al. [22] presented an implicit scheme, in which the order of convergence was $O(\tau + h^2)$. Cui [23] obtained an unconditionally stable finite difference scheme, in which the order of convergence was $O(\tau + h^4)$. For the above Eq. (1), Chen et al. [24] proposed the implicit and explicit finite difference schemes, and got the convergence with the order $O(\tau + h^2)$. Very recently, Ding and Li [25] constructed a class of numerical methods and obtained different convergence orders by choosing different spline parameters. From the references available, it seems easy to increase the accuracy in spatial direction but difficult to increase the accuracy in time direction. In the present paper, a higher order approach in time direction for the numerical treatment of Eq. (1) is derived.

The outline of the rest of this paper is organized as follows. In Section 2, a numerical method for solving the reaction and anomalous-diffusion equation is proposed. The solvability, stability and convergence are analyzed in Sections 3 and 4, respectively. In Section 5, numerical experiments are carried out to support the theoretical analysis. And the conclusion is included in the last section.

2. Numerical method

In this section, we present an effective numerical method to simulate the solution of the reaction and anomalous-diffusion (1).

To establish the numerical scheme for the above Eq. (1), we let $x_i = ih$ $(i = 0, 1, \ldots, M)$ and $t_k = k\tau$ $(k = 0, 1, \ldots, N)$, where $h = \frac{L}{M}$ and $\tau = \frac{T}{N}$ are the uniform spatial and temporal step sizes respectively, and M, N are two positive integers.

Firstly, using the Taylor series expansion at point (x_i, t_k), one gets

$$u(x_i,t_{k+1}) = u(x_i,t_k) + \tau\frac{\partial u(x_i,t_k)}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2 u(x_i,t_k)}{\partial t^2} + \cdots = \left(\mathcal{I} + \tau\frac{\partial}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2}{\partial t^2} + \ldots\right)u(x_i,t_k) = \exp\left(\tau\frac{\partial}{\partial t}\right)u(x_i,t_k), \tag{2}$$

where $\mathcal{I}$ is the identity operator.

For $\exp\left(\tau\frac{\partial}{\partial t}\right)$, we can use the following $(1,1)$ Padé approximation

$$\left\|\exp\left(\tau\frac{\partial}{\partial t}\right)\right\| = \left\|\left(2\mathcal{I} + \tau\frac{\partial}{\partial t}\right)\left(2\mathcal{I} - \tau\frac{\partial}{\partial t}\right)^{-1}\right\| + \mathcal{O}(\tau^3). \tag{3}$$

Applying the Taylor series expansion at point (x_i, t_k) (or using (2)), one can obtain

$$\frac{1}{2}\left[\frac{\partial u(x_i,t_k)}{\partial t} + \frac{\partial u(x_i,t_{k+1})}{\partial t}\right] = \frac{1}{\tau}[u(x_i,t_{k+1}) - u(x_i,t_k)] + \mathcal{O}(\tau^2),$$

which can be rewritten as the following compact form

$$\left(\mathcal{I} + \frac{1}{2}\Delta_t\right)\frac{\partial u(x_i,t_k)}{\partial t} = \frac{1}{\tau}\Delta_t u(x_i,t_k) + \mathcal{O}(\tau^2), \tag{4}$$

where Δ_t denote forward difference operator with respect to t, defined by $\Delta_t u(x_i,t_k) = u(x_i,t_{k+1}) - u(x_i,t_k)$.

Secondly, we focus on an approximation for the Riemann–Liouville derivative. Due to the equivalence between Riemann–Liouville derivative and Grünwald–Letnikov derivative under smooth condition, we usually approximate the Riemann–Liouville derivative by using the following Grünwald–Letnikov formula if the homogeneous condition satisfies:

$${}_{RL}D_{0,t}^{1-\alpha}u(x,t) = \frac{1}{\tau^{1-\alpha}}\sum_{j=0}^{\left[\frac{t}{\tau}\right]}\varpi_{1,j}^{(1-\alpha)}u(x,t-j\tau) + \mathcal{O}(\tau), \tag{5}$$

where $\left[\frac{t}{\tau}\right]$ denotes the integer part of $\frac{t}{\tau}$, the coefficients are

$$\varpi_{1,j}^{(1-\alpha)} = (-1)^j\binom{1-\alpha}{j} = (-1)^j\frac{\Gamma(2-\alpha)}{\Gamma(j+1)\Gamma(2-\alpha-j)}, \quad j = 0, 1, \ldots$$

In fact, the generating functions of the coefficients $\varpi_{1,j}^{(1-\alpha)}$ in Eq. (5) is $(1-z)^{1-\alpha}$, i.e.,

$$(1-z)^{1-\alpha} = \sum_{j=0}^{\infty} \varpi_{1,j}^{(1-\alpha)} z^j. \tag{6}$$

Next, we will give a second numerical formula for the Riemann–Liouville derivative with the help of the following Lemma.

Lemma 1. *Assume $u^*(x,t) \in L_t^1(\mathbb{R})$ for any given x, ${}_LD_{-\infty,t}^{3-\alpha}u^*(x,t)$ for any x and its Fourier transform with respect to t belong to $L_t^1(\mathbb{R})$, where*

$$L_t^1(\mathbb{R}) = \left\{ u^*(x,t) \middle| \int_{-\infty}^{\infty} |u^*(x,t)| dt < \infty \right\}.$$

Let

$$\mathcal{D}_t^{1-\alpha} u^*(x,t) = \frac{1}{\tau^{1-\alpha}} \sum_{j=0}^{\infty} \varpi_{2,j}^{(1-\alpha)} u^*(x, t - j\tau), \tag{7}$$

in which the coefficients $\varpi_{2,j}^{(1-\alpha)}$ $(j = 0, 1, \ldots, [\frac{t}{\tau}])$ are given by the following equation [26]

$$\left(\frac{3}{2} - 2z + \frac{1}{2}z^2\right)^{1-\alpha} = \sum_{j=0}^{\infty} \varpi_{2,j}^{(1-\alpha)} z^j. \tag{8}$$

Suppose that the series $\sum_{j=0}^{\infty} \varpi_{2,j}^{(1-\alpha)} u^*(x, t-j\tau)$ uniformly converges on $\mathbb{R}$, then we have

$${}_LD_{-\infty,t}^{1-\alpha} u^*(x,t) = \frac{1}{\tau^{1-\alpha}} \sum_{j=0}^{\infty} \varpi_{2,j}^{(1-\alpha)} u^*(x, t-j\tau) + \mathcal{O}(\tau^2),$$

where ${}_LD_{-\infty,t}^{1-\alpha} u^*(x,t)$ is the left Liouvile fractional derivative and defined by Miller and Ross [3]

$${}_LD_{-\infty,t}^{1-\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_{-\infty}^{t} \frac{u(x,s)}{(t-s)^{1-\alpha}} ds.$$

Proof. Let

$$\hat{u}^*(x,\omega) = \mathcal{F}_t\{u^*(x,t);\omega\} = \int_{-\infty}^{\infty} \exp(I\omega t) u^*(x,t) dt,$$

which is the Fourier transform of function $u^*(x,t)$ with respect to t and $I^2 = -1$. Then we have

$$\mathcal{F}_t\{u^*(x, t-j\tau);\omega\} = \exp(I\omega j\tau)\hat{u}^*(x,\omega).$$

Taking the Fourier transform in both sides of (7) with respect to t and using (8) yields

$$\begin{aligned}\mathcal{F}_t\{\mathcal{D}_t^{1-\alpha}u^*(x,t);\omega\} &= \frac{1}{\tau^{1-\alpha}} \sum_{j=0}^{\infty} \varpi_{2,j}^{(\alpha)} \exp(I\omega j\tau)\hat{u}^*(x,\omega) = \frac{1}{\tau^{1-\alpha}} \left(\frac{3}{2} - 2\exp(I\omega\tau) + \frac{1}{2}\exp(2I\omega\tau)\right)^{1-\alpha} \hat{u}^*(x,\omega) \\ &= (-I\omega)^{1-\alpha} \left(\frac{3 - 4\exp(I\omega\tau) + \exp(2I\omega\tau)}{-2I\omega\tau}\right)^{1-\alpha} \hat{u}^*(x,\omega) = (-I\omega)^{1-\alpha} \mathcal{Q}(I\omega\tau)\hat{u}^*(x,\omega),\end{aligned}$$

where

$$\mathcal{Q}(I\omega\tau) = \left(\frac{3 - 4\exp(I\omega\tau) + \exp(2I\omega\tau)}{-2I\omega\tau}\right)^{1-\alpha} = 1 - \frac{1}{3}(1-\alpha)(I\omega\tau)^2 - \frac{1}{4}(1-\alpha)(I\omega\tau)^3 + \mathcal{O}(|\omega\tau|^4).$$

Denote

$$\hat{\mathcal{R}}(\omega,\tau) = \mathcal{F}_t\{\mathcal{D}_t^{1-\alpha}u^*(x,t);\omega\} - \mathcal{F}_t\left\{{}_LD_{-\infty,t}^{1-\alpha}u^*(x,t);\omega\right\},$$

then there exists a constant $\mathcal{C}_1$ such that

$$\begin{aligned}\left|\hat{\mathcal{R}}(\omega,\tau)\right| &= \left|\mathcal{F}_t\{\mathcal{D}_t^{1-\alpha}u^*(x,t);\omega\} - \mathcal{F}_t\left\{{}_LD_{-\infty,t}^{1-\alpha}u^*(x,t);\omega\right\}\right| = \left|(-I\omega)^{1-\alpha}\mathcal{Q}(I\omega\tau)\hat{u}^*(x,\omega) - (-I\omega)^{1-\alpha}\hat{u}^*(x,\omega)\right| \\ &\leqslant \mathcal{C}_1\tau^2|(-I\omega)^{3-\alpha}\hat{u}^*(x,\omega)|.\end{aligned}$$

By the assumption $\mathcal{F}_t\left\{{}_LD_{-\infty,t}^{3-\alpha}u^*(x,t);\omega\right\} \in L_t^1(\mathbb{R})$, one gets

$$\int_{-\infty}^{\infty}\left|\mathcal{F}_t\left\{{}_LD_{-\infty,t}^{3-\alpha}u^*(x,t);\omega\right\}\right|dw = \int_{-\infty}^{\infty}|(-I\omega)^{3-\alpha}\hat{u}^*(x,\omega)|dw = \mathcal{C}_2 < \infty.$$

It immediately follows that

$$\left|\mathcal{D}_t^{1-\alpha}u^*(x,t) - {}_LD_{-\infty,t}^{1-\alpha}u^*(x,t)\right| = |\mathcal{R}(\omega,\tau)| = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}\exp(-I\omega\tau)\hat{\mathcal{R}}(\omega,\tau)d\omega\right| \leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{\mathcal{R}}(\omega,\tau)|d\omega \leqslant \mathcal{C}\tau^2 = \mathcal{O}(\tau^2),$$

where $\mathcal{C} = \frac{\mathcal{C}_1\mathcal{C}_2}{2\pi}$. That is to say,

$$\mathcal{D}_t^{1-\alpha}u^*(x,t) = {}_LD_{-\infty,t}^{1-\alpha}u^*(x,t) + \mathcal{O}(\tau^2).$$

Therefore,

$${}_LD_{-\infty,t}^{1-\alpha}u^*(x,t) = \frac{1}{\tau^{1-\alpha}}\sum_{j=0}^{\infty}\varpi_{2,j}^{(1-\alpha)}u^*(x,t-j\tau) + \mathcal{O}(\tau^2). \tag{9}$$

This finishes the proof. □

From the initial condition $u(x,0) = 0$, we define

$$u^*(x,t) = \begin{cases} u(x,t), & \text{when } t \in [0,T], \\ 0, & \text{when } t \notin [0,T], \end{cases}$$

Owing to $u^*(x,t) = 0$ for $t \notin [0,T]$, we see that the Liouvile derivative ${}_LD_{-\infty,t}^{1-\alpha}u^*(x,t)$ coincides with the Riemann–Liouvile derivative ${}_{RL}D_{0,t}^{1-\alpha}u^*(x,t)$. According to (9), we have

$${}_{RL}D_{0,t}^{1-\alpha}u(x,t) = \frac{1}{\tau^{1-\alpha}}\sum_{j=0}^{[\frac{t}{\tau}]}\varpi_{2,j}^{(1-\alpha)}u(x,t-j\tau) + \mathcal{O}(\tau^2). \tag{10}$$

The key question now is: how can we to obtain the coefficients $\varpi_{2,j}^{(1-\alpha)}$ $(j = 0,1,\ldots)$ of Eq. (10)? The common calculation method is by using the Fast Fourier Transform [4]. In [27], we give a simple and effective way to calculate them as follows.

Using (6), we can get

$$\left(\frac{3}{2} - 2z + \frac{1}{2}z^2\right)^{1-\alpha} = \left(\frac{3}{2}\right)^{1-\alpha}(1-z)^{1-\alpha}\left(1-\frac{1}{3}z\right)^{1-\alpha} = \left(\frac{3}{2}\right)^{1-\alpha}\left[\sum_{j=0}^{\infty}(-1)^j\binom{1-\alpha}{j}z^j\right]\left[\sum_{j=0}^{\infty}\left(-\frac{1}{3}\right)^j\binom{1-\alpha}{j}z^j\right]$$
$$= \sum_{j=0}^{\infty}\left[\left(\frac{3}{2}\right)^{1-\alpha}\sum_{m=0}^{j}\left(\frac{1}{3}\right)^m\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j-m}^{(1-\alpha)}\right]z^j. \tag{11}$$

Comparing (8) and (11) yields

$$\varpi_{2,j}^{(1-\alpha)} = \left(\frac{3}{2}\right)^{1-\alpha}\sum_{m=0}^{j}\left(\frac{1}{3}\right)^m\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j-m}^{(1-\alpha)} = \left(\frac{3}{2}\right)^{1-\alpha}\sum_{m=0}^{j}(-1)^j\left(\frac{1}{3}\right)^m\binom{1-\alpha}{m}\binom{1-\alpha}{j-m}, \quad j = 0,1,\ldots \tag{12}$$

It immediately follows that

$$\varpi_{2,0}^{(1-\alpha)} = \left(\frac{3}{2}\right)^{1-\alpha},$$

$$\varpi_{2,j+1}^{(1-\alpha)} = \varpi_{2,j}^{(1-\alpha)} - \Gamma(2-\alpha)\left(\frac{3}{2}\right)^{1-\alpha}\left(-\frac{1}{3}\right)^j\mathcal{S}_j^{(\alpha)}, \quad j = 0,1,\ldots,$$

where

$$\mathcal{S}_j^{(\alpha)} = \frac{1}{\Gamma(j+2)\Gamma(1-\alpha-j)} + \sum_{m=0}^{j}\frac{3^m\Gamma(3-\alpha)}{\Gamma(m+2)\Gamma(j-m+1)\Gamma(2-\alpha-m)\Gamma(2-\alpha-j+m)}.$$

Usually, we approximate the second derivative $\frac{\partial^2u(x,t)}{\partial x^2}$ at points (x_i,t_k) $(i = 0,1,\ldots,M,\ k = 0,1,\ldots,N)$ by the following fourth-order compact difference scheme [28]

$$\left(\mathcal{I} + \frac{1}{12}\delta_x^2\right)\frac{\partial^2u(x_i,t_k)}{\partial x^2} = \frac{1}{h^2}\delta_x^2u(x_i,t_k) + \mathcal{O}\left(h^4\right), \tag{13}$$

where δ_x^2 is the second-order central difference operator with respect to x and defined by $\delta_x^2u(x_i,t_k) = u(x_{i+1},t_k) - 2u(x_i,t_k) + u(x_{i-1},t_k)$.

Naturally, Eq. (13) can be changed into the following form

$$\frac{1}{12}\frac{\partial^2 u(x_{i-1},t_k)}{\partial x^2}+\frac{5}{6}\frac{\partial^2 u(x_i,t_k)}{\partial x^2}+\frac{1}{12}\frac{\partial u(x_{i+1},t_k)}{\partial x^2}=\frac{1}{h^2}[u(x_{i-1},t_k)-2u(x_i,t_k)+u(x_{i+1},t_k)]+\mathcal{O}(h^4). \tag{14}$$

Considering the Eq. (1) at the point (x_i, t_k), one gets

$$\frac{\partial u(x_i,t_k)}{\partial t}={}_{RL}D^{1-\alpha}_{0,t_k}\left(K_\alpha\frac{\partial^2 u(x_i,t_k)}{\partial x^2}-C_\alpha u(x_i,t_k)\right)+f(x_i,t_k),\quad i=0,1,\ldots,M,\ k=0,1,\ldots,N. \tag{15}$$

Define the average operator $\mathcal{L}$ in the x direction as follows:

$$\mathcal{L}v_i=\frac{1}{12}v_{i-1}+\frac{5}{6}v_i+\frac{1}{12}v_{i+1}. \tag{16}$$

Then (14) can be written as

$$\mathcal{L}\frac{\partial^2 u(x_i,t_k)}{\partial x^2}=\frac{1}{h^2}[u(x_{i-1},t_k)-2u(x_i,t_k)+u(x_{i+1},t_k)]+\mathcal{O}(h^4). \tag{17}$$

Acting (15) with $\mathcal{L}$, one gets

$$\mathcal{L}\frac{\partial u(x_i,t_k)}{\partial t}={}_{RL}D^{1-\alpha}_{0,t_k}\left(K_\alpha\mathcal{L}\frac{\partial^2 u(x_i,t_k)}{\partial x^2}-C_\alpha\mathcal{L}u(x_i,t_k)\right)+\mathcal{L}f(x_i,t_k),\quad i=0,1,\ldots,M,\ k=0,1,\ldots,N. \tag{18}$$

Let u_i^k be the numerical approximation of $u(x_i,t_k)$. Substituting (4), (10), (16) and (17) into (18) and omitting the high-order infinitesimal terms, and through a series of algebraic operation, we can obtain the following finite difference scheme for the Eq. (1),

$$\begin{aligned}
&\left[\frac{1}{6}-\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi^{(1-\alpha)}_{2,0}\right]u^{k+1}_{i+1}+\left[\frac{5}{3}+\left(2\mu_1+\frac{5}{6}\mu_2\right)\varpi^{(1-\alpha)}_{2,0}\right]u^{k+1}_{i}+\left[\frac{1}{6}-\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi^{(1-\alpha)}_{2,0}\right]u^{k+1}_{i-1}\\
&\quad=\left[\frac{1}{6}+\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi^{(1-\alpha)}_{2,0}\right]u^{k}_{i+1}+\left[\frac{5}{3}-\left(2\mu_1+\frac{5}{6}\mu_2\right)\varpi^{(1-\alpha)}_{2,0}\right]u^{k}_{i}+\left[\frac{1}{6}+\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi^{(1-\alpha)}_{2,0}\right]u^{k}_{i-1}\\
&\quad+\mu_1\sum_{j=1}^{k}\varpi^{(1-\alpha)}_{2,j}\left(u^{k+1-j}_{i+1}-2u^{k+1-j}_{i}+u^{k+1-j}_{i-1}\right)-\mu_2\sum_{j=1}^{k}\varpi^{(1-\alpha)}_{2,j}\left(\frac{1}{12}u^{k+1-j}_{i+1}+\frac{5}{6}u^{k+1-j}_{i}+\frac{1}{12}u^{k+1-j}_{i-1}\right)\\
&\quad+\mu_1\sum_{j=1}^{k}\varpi^{(1-\alpha)}_{2,j}\left(u^{k-j}_{i+1}-2u^{k-j}_{i}+u^{k-j}_{i-1}\right)-\mu_2\sum_{j=1}^{k}\varpi^{(1-\alpha)}_{2,j}\left(\frac{1}{12}u^{k-j}_{i+1}+\frac{5}{6}u^{k-j}_{i}+\frac{1}{12}u^{k-j}_{i-1}\right)+\frac{1}{12}\tau f^k_{i+1}+\frac{5}{6}\tau f^k_i+\frac{1}{12}\tau f^k_{i-1}\\
&\quad+\frac{1}{12}\tau f^{k+1}_{i+1}+\frac{5}{6}\tau f^{k+1}_i+\frac{1}{12}\tau f^{k+1}_{i-1},\quad i=1,\ldots,M-1,\ k=0,1,\ldots,N-1,
\end{aligned} \tag{19}$$

where $f_i^k=f(x_i,t_k)$, $\mu_1=\frac{K_\alpha\tau^\alpha}{h^2}$, $\mu_2=C_\alpha\tau^\alpha$.

The initial and boundary value conditions can be discretized by

$$u_i^0=0,\quad i=0,1,\ldots,M,$$

$$u_0^k=\varphi_1(k\tau),\quad u_M^k=\varphi_2(k\tau),\quad k=0,1,\ldots,N.$$

From the above tedious computations, the truncation error reads as

$$R_i^{k+1}=\mathcal{O}\left(\tau^2+h^4\right).$$

3. The solvability of the finite difference scheme

For convenience, denote

$$\mathbf{U}^0=(\phi(x_1),\phi(x_2),\ldots,\phi(x_{M-1}))^T,$$

$$\mathbf{U}^k=\left(u_1^k,u_2^k,\ldots,u_{M-1}^k\right)^T,\quad k=1,2,\ldots,N,$$

$$\mathbf{F}^k=\left(f_1^k,f_2^k,\ldots,f_{M-1}^k\right)^T,\quad k=0,1,\ldots,N,$$

$$\Lambda_1 = \begin{pmatrix} \frac{5}{6} & \frac{1}{12} & & & \\ \frac{1}{12} & \frac{5}{6} & \frac{1}{12} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{12} & \frac{5}{6} & \frac{1}{12} \\ & & & \frac{1}{12} & \frac{5}{6} \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}.$$

Then the matrix form of the difference scheme (19) can be written as

$$A\mathbf{U}^{k+1} = B\mathbf{U}^k + \sum_{j=1}^{k} P_j\left(\mathbf{U}^{k-j} + \mathbf{U}^{k+1-j}\right) + D\mathbf{F}^k + D\mathbf{F}^{k+1} + G, \tag{20}$$

where

$$A = \left(2 + \mu_2 \varpi_{2,0}^{(1-\alpha)}\right)\Lambda_1 - \mu_1 \varpi_{2,0}^{(1-\alpha)}\Lambda_2,$$

$$B = \left(2 - \mu_2 \varpi_{2,0}^{(1-\alpha)}\right)\Lambda_1 + \mu_1 \varpi_{2,0}^{(1-\alpha)}\Lambda_2,$$

$$P_j = \varpi_{2,j}^{(1-\alpha)}(\mu_1\Lambda_2 - \mu_2\Lambda_1), \quad j = 1, 2, \ldots, k,$$

$$D = \tau\Lambda_1, \quad G = \Big(\Psi_1, \underbrace{0, \ldots, 0}_{M-3}, \Psi_{M-1}\Big)^T,$$

in which

$$\begin{aligned}\Psi_1 = &-\left[\frac{1}{6} - \left(\mu_1 - \frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]\varphi_1(t_{k+1}) + \left[\frac{1}{6} + \left(\mu_1 - \frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]\varphi_1(t_k) \\ &+ \left(\mu_1 - \frac{1}{12}\mu_2\right)\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\left(\varphi_1(t_{k-j}) + \varphi_1(t_{k+1-j})\right) + \frac{1}{12}\tau f(x_0, t_k) + \frac{1}{12}\tau f(x_0, t_{k+1}),\\ \Psi_{M-1} = &-\left[\frac{1}{6} - \left(\mu_1 - \frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]\varphi_2(t_{k+1}) + \left[\frac{1}{6} + \left(\mu_1 - \frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]\varphi_2(t_k) \\ &+ \left(\mu_1 - \frac{1}{12}\mu_2\right)\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\left(\varphi_2(t_{k-j}) + \varphi_2(t_{k+1-j})\right) + \frac{1}{12}\tau f(x_M, t_k) + \frac{1}{12}\tau f(x_M, t_{k+1}).\end{aligned}$$

Lemma 2 [29]. *Let S be an $(M-1)\times(M-1)$ tridiagonal matrix*

$$S = \begin{pmatrix} b & a & & & \\ c & b & a & & \\ & \ddots & \ddots & \ddots & \\ & & c & b & a \\ & & & c & b \end{pmatrix},$$

then the eigenvalues of the tridiagonal matrix S are

$$\kappa_i = b + 2a\sqrt{\frac{c}{a}}\cos\left(\frac{\pi i}{M}\right), \quad i = 1, 2, \ldots, M-1.$$

Theorem 1. *The difference Eq. (20) is uniquely solvable.*

Proof. By Lemma 2, it is easy to see that the eigenvalues of the matrix A are

$$\kappa_i = \left(2 + \mu_2\varpi_{2,0}^{(1-\alpha)}\right)\left[1 - \frac{1}{3}\sin^2\left(\frac{\pi i}{2M}\right)\right] + 4\mu_1\varpi_{2,0}^{(1-\alpha)}\sin^2\left(\frac{\pi i}{2M}\right), \quad i = 1, 2, \ldots, M-1.$$

Because $\mu_1 > 0$ and $\mu_2 > 0$, it holds that $\kappa_i > 0$.

Hence,

$$\det(A) = \prod_{i=1}^{M-1}\kappa_i > 0,$$

that is to say, the matrix A is nonsingular, thus the difference scheme (20) has a unique solution. □

4. Stability and convergence analysis

In this section, we analyze the stability and convergence of the difference scheme (19) by using the Fourier method [22,24,23,25]. In order to obtain the stability and convergence of the difference scheme (19), we introduce the following four lemmas.

Lemma 3 (Raabe Criterion [30]). *Let $\sum_{j=1}^{\infty} a_j$ be a positive series where $a_j > 0$ for all j. Define*

$$\theta = \lim_{j\to+\infty} j\left(1 - \frac{a_{j+1}}{a_j}\right).$$

If $\theta > 1$, then $\sum_{j=1}^{\infty} a_j$ converges.

Lemma 4 (Comparison Criterion [30]). *Suppose $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ are positive series with $a_j \leqslant b_j$ for all j.*

(1) *If $\sum_{j=1}^{\infty} b_j$ converges, then so does $\sum_{j=1}^{\infty} a_j$,*
(2) *If $\sum_{j=1}^{\infty} a_j$ diverges, then so does $\sum_{j=1}^{\infty} b_j$,*

Lemma 5. *The coefficients $\varpi_{1,j}^{(1-\alpha)}$ $(j = 0, 1, \ldots)$ $(0 < \alpha < 1)$ satisfy* [23]

$$(1)\ \varpi_{1,j}^{(1-\alpha)} = \left(1 - \tfrac{2-\alpha}{j}\right)\varpi_{1,j-1}^{(1-\alpha)}, \quad \varpi_{1,0}^{(1-\alpha)} = 1, \quad \varpi_{1,j}^{(1-\alpha)} < 0, \quad j \geqslant 1,$$

$$(2)\ \sum_{j=0}^{\infty} \varpi_{1,j}^{(1-\alpha)} = 0, \quad \lim_{j\to+\infty} \varpi_{1,j}^{(1-\alpha)} = 0.$$

Proof. For the proof of (1), one can refer to [22,23]. Here, we only show (2) which has not been rigorously proved yet. As for the positive series $\sum_{j=1}^{\infty}\left(-\varpi_{1,j}^{(1-\alpha)}\right)$, one has

$$\theta = \lim_{j\to+\infty} j\left(1 - \frac{\varpi_{1,j+1}^{(1-\alpha)}}{\varpi_{1,j}^{(1-\alpha)}}\right) = (2-\alpha)\lim_{j\to+\infty}\frac{j}{j+1} = (2-\alpha) > 1.$$

According to Lemma 3, the series $\sum_{j=1}^{\infty}\left(-\varpi_{1,j}^{(1-\alpha)}\right)$ is convergent, so $\lim_{j\to+\infty}\varpi_{1,j}^{(1-\alpha)} = 0$.

On the other hand, it is easy to show that the convergent radius of the functional series $\sum_{j=0}^{\infty}\varpi_{1,j}^{(1-\alpha)}z^j$ is one, that is $\sum_{j=0}^{\infty}\varpi_{1,j}^{(1-\alpha)}z^j$ is convergent if $|z| < 1$. Therefore, simply letting $z = 1$ in (6) yields $\sum_{j=0}^{\infty}\varpi_{1,j}^{(1-\alpha)} = 0$. All this ends the proof. □

Lemma 6. . *The coefficients $\varpi_{2,j}^{(1-\alpha)}$ $(j = 0, 1, \ldots)$ $(0 < \alpha < 1)$ satisfy*

$$(1)\ \varpi_{2,0}^{(1-\alpha)} = \left(\frac{3}{2}\right)^{1-\alpha} > 0,$$

$$\varpi_{2,1}^{(1-\alpha)} = \frac{4(\alpha-1)}{3}\left(\frac{3}{2}\right)^{1-\alpha} < 0,$$

$$\varpi_{2,2}^{(1-\alpha)} = \frac{(\alpha-1)(8\alpha-3)}{9}\left(\frac{3}{2}\right)^{1-\alpha}, \varpi_{2,2}^{(1-\alpha)} \geqslant 0 \quad \text{for} \quad 0 < \alpha \leqslant \frac{3}{8} \quad \text{and} \quad \varpi_{2,2}^{(1-\alpha)} \leq 0 \quad \text{for} \quad \frac{3}{8} \leqslant \alpha < 1,$$

$$\varpi_{2,3}^{(1-\alpha)} = \frac{4\alpha(\alpha-1)(8\alpha-1)}{81}\left(\frac{3}{2}\right)^{1-\alpha}, \varpi_{2,2}^{(1-\alpha)} \geqslant 0 \quad \text{for } 0 < \alpha \leqslant \frac{1}{8} \quad \text{and} \quad \varpi_{2,2}^{(1-\alpha)} \leq 0 \quad \text{for} \frac{1}{8} \leqslant \alpha < 1;$$

$$\varpi_{2,4}^{(1-\alpha)} = \frac{1}{4!}\times\frac{4}{3^4}(\alpha-1)\alpha(64\alpha^2+48\alpha+11)\left(\frac{3}{2}\right)^{1-\alpha} < 0,$$

$$\varpi_{2,5}^{(1-\alpha)} = \frac{1}{5!}\times\frac{16}{3^5}(\alpha-1)\alpha(\alpha+1)(64\alpha^2+80\alpha+39)\left(\frac{3}{2}\right)^{1-\alpha} < 0,$$

$$\varpi_{2,6}^{(1-\alpha)} = \frac{1}{6!} \times \frac{8}{3^6}(\alpha-1)\alpha(\alpha+1)\left(512\alpha^3+1728\alpha^2+2152\alpha+1083\right)\left(\frac{3}{2}\right)^{1-\alpha} < 0,$$

$$\varpi_{2,7}^{(1-\alpha)} = \frac{1}{7!} \times \frac{32}{3^7}(\alpha-1)\alpha(\alpha+1)(\alpha+2)\left(512\alpha^3+2112\alpha^2+3400\alpha+2181\right)\left(\frac{3}{2}\right)^{1-\alpha} < 0,$$

$$\varpi_{2,8}^{(1-\alpha)} = \frac{1}{8!} \times \frac{16}{3^8}(\alpha-1)\alpha(\alpha+1)(\alpha+2)\left(4096\alpha^4+30720\alpha^3+92288\alpha^2+135456\alpha+81945\right)\left(\frac{3}{2}\right)^{1-\alpha} < 0,$$

$$\varpi_{2,9}^{(1-\alpha)} = \frac{1}{9!} \times \frac{64}{3^9}(\alpha-1)\alpha(\alpha+1)(\alpha+2)(\alpha+3)\left(4096\alpha^4+34816\alpha^3+121472\alpha^2+208736\alpha+147585\right)\left(\frac{3}{2}\right)^{1-\alpha} < 0,$$

$$\begin{aligned}\varpi_{2,10}^{(1-\alpha)} = {} & \frac{1}{10!} \times \frac{32}{3^{10}}(\alpha-1)\alpha(\alpha+1)(\alpha+2)(\alpha+3)\big(32768\alpha^5+430080\alpha^4+2370560\alpha^3+6952320\alpha^2+10882472\alpha \\ & +7232925\big)\left(\frac{3}{2}\right)^{1-\alpha} < 0,\end{aligned}$$

$$\begin{aligned}\varpi_{2,11}^{(1-\alpha)} = {} & \frac{1}{11!} \times \frac{128}{3^{11}}(\alpha-1)\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\big(32768\alpha^5+471040\alpha^4+2882560\alpha^3+9457280\alpha^2 \\ & +16607592\alpha+12399975\big)\left(\frac{3}{2}\right)^{1-\alpha} < 0,\end{aligned}$$

$$\begin{aligned}\varpi_{2,12}^{(1-\alpha)} = {} & \frac{1}{12!} \times \frac{64}{3^{12}}(\alpha-1)\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\big(262144\alpha^6+5308416\alpha^5+46551040\alpha^4+228034560\alpha^3 \\ & +660016576\alpha^2+1068685584\alpha+753311475\big)\left(\frac{3}{2}\right)^{1-\alpha} < 0,\end{aligned}$$

$$\varpi_{2,j}^{(1-\alpha)} < 0, \quad j = 13, 14, \ldots$$

$$(2)\ \sum_{j=0}^{\infty}\varpi_{2,j}^{(1-\alpha)} = 0, \quad \lim_{j\to+\infty}\varpi_{2,j}^{(1-\alpha)} = 0.$$

Proof. (1) In view of Eq. (12), we easily obtain the analytical expressions of the coefficients $\varpi_{2,j}^{(1-\alpha)}$, $j = 0, 1, \ldots, 12$. These coefficients are often used in numerical calculations so are listed here.

In the following, we show that $\varpi_{2,j}^{(1-\alpha)} < 0$ if $j \geqslant 4$.

$$\varpi_{2,j}^{(1-\alpha)} = \left(\frac{3}{2}\right)^{1-\alpha}\sum_{m=0}^{j}\left(\frac{1}{3}\right)^m\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j-m}^{(1-\alpha)} = \left(\frac{3}{2}\right)^{1-\alpha}\left\{\left[1+\left(\frac{1}{3}\right)^j\right]\varpi_{1,j}^{(1-\alpha)} + \sum_{m=1}^{j-1}\left(\frac{1}{3}\right)^m\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j-m}^{(1-\alpha)}\right\}.$$

Note that

$$\frac{\varpi_{1,s}^{(1-\alpha)}\varpi_{1,j-s}^{(1-\alpha)}}{\varpi_{1,s+1}^{(1-\alpha)}\varpi_{1,j-s-1}^{(1-\alpha)}} = \frac{\left(1-\frac{2-\alpha}{j-s}\right)\varpi_{1,s}^{(1-\alpha)}\varpi_{1,j-s-1}^{(1-\alpha)}}{\left(1-\frac{2-\alpha}{s+1}\right)\varpi_{1,s}^{(1-\alpha)}\varpi_{1,j-s-1}^{(1-\alpha)}} = \frac{\left(1-\frac{2-\alpha}{j-s}\right)}{\left(1-\frac{2-\alpha}{s+1}\right)} \geqslant 1, \quad s = 1, 2, \ldots, \left[\frac{j}{2}\right].$$

So one gets

$$\begin{aligned}\varpi_{2,j}^{(1-\alpha)} &\leqslant \left(\frac{3}{2}\right)^{1-\alpha}\left\{\left[1+\left(\frac{1}{3}\right)^j\right]\varpi_{1,j}^{(1-\alpha)} + \varpi_{1,1}^{(1-\alpha)}\varpi_{1,j-1}^{(1-\alpha)}\sum_{m=1}^{j-1}\left(\frac{1}{3}\right)^m\right\} \leqslant \left(\frac{3}{2}\right)^{1-\alpha}\left\{\left[1+\left(\frac{1}{3}\right)^j\right]\varpi_{1,j}^{(1-\alpha)} + \varpi_{1,1}^{(1-\alpha)}\varpi_{1,j-1}^{(1-\alpha)}\sum_{m=1}^{\infty}\left(\frac{1}{3}\right)^m\right\} \\ &= \left(\frac{3}{2}\right)^{1-\alpha}\left\{\left[1+\left(\frac{1}{3}\right)^j\right]\varpi_{1,j}^{(1-\alpha)} + \frac{1}{2}\varpi_{1,1}^{(1-\alpha)}\varpi_{1,j-1}^{(1-\alpha)}\right\} = \left(\frac{3}{2}\right)^{1-\alpha}\left\{\left[1+\left(\frac{1}{3}\right)^j\right]\left(1-\frac{2-\alpha}{j}\right)+\frac{\alpha-1}{2}\right\}\varpi_{1,j-1}^{(1-\alpha)} \\ &= \left(\frac{3}{2}\right)^{1-\alpha}\left[\frac{(j-4)+\alpha(j+2)}{2j} + \frac{(j-2)+\alpha}{j}\left(\frac{1}{3}\right)^j\right]\varpi_{1,j-1}^{(1-\alpha)}, \quad j \geqslant 4.\end{aligned}$$

From Lemma 5, $\varpi_{1,j}^{(1-\alpha)} < 0$ for $j \geqslant 1$. So we easily known that

$$\varpi_{2,j}^{(1-\alpha)} < 0, j = 4, 5, \ldots.$$

(2) Let

$$\vartheta_1 = \left(1 - \frac{2-\alpha}{j}\right)\left(1 - \frac{2-\alpha}{j-1}\right)\cdots\left(1 - \frac{2-\alpha}{m+1}\right), \quad j \geqslant m+1$$

and

$$\vartheta_2 = \left(1 - \frac{2-\alpha}{j-1}\right)\left(1 - \frac{2-\alpha}{j-2}\right)\cdots\left(1 - \frac{2-\alpha}{j-(m-1)}\right) \quad m \geqslant 2.$$

Obviously, $\vartheta_1 < 1$ and $\vartheta_2 < 1$. From Lemma 5, one has

$$\varpi_{1,j}^{(1-\alpha)} = \vartheta_1 \varpi_{1,m}^{(1-\alpha)}, \quad \varpi_{1,j-1}^{(1-\alpha)} = \vartheta_2 \varpi_{1,j-m}^{(1-\alpha)}.$$

Furthermore,

$$\varpi_{1,j}^{(1-\alpha)} \varpi_{1,j-1}^{(1-\alpha)} = \vartheta_1\vartheta_2 \varpi_{1,m}^{(1-\alpha)} \varpi_{1,j-m}^{(1-\alpha)} < \varpi_{1,m}^{(1-\alpha)} \varpi_{1,j-m}^{(1-\alpha)}.$$

So,

$$\begin{aligned}
\varpi_{2,j}^{(1-\alpha)} &= \left(\frac{3}{2}\right)^{1-\alpha} \sum_{m=0}^{j} \left(\frac{1}{3}\right)^m \varpi_{1,m}^{(1-\alpha)} \varpi_{1,j-m}^{(1-\alpha)} = \left(\frac{3}{2}\right)^{1-\alpha} \left\{ \left[1 + \left(\frac{1}{3}\right)^j\right] \varpi_{1,j}^{(1-\alpha)} + \sum_{m=1}^{j-1} \left(\frac{1}{3}\right)^m \varpi_{1,m}^{(1-\alpha)} \varpi_{1,j-m}^{(1-\alpha)} \right\} \\
&\geqslant \left(\frac{3}{2}\right)^{1-\alpha} \left\{ \left[1 + \left(\frac{1}{3}\right)^j\right] \varpi_{1,j}^{(1-\alpha)} + \varpi_{1,j}^{(1-\alpha)} \varpi_{1,j-1}^{(1-\alpha)} \sum_{m=1}^{j-1} \left(\frac{1}{3}\right)^m \right\} \\
&= \left(\frac{3}{2}\right)^{1-\alpha} \left\{ \left[1 + \frac{1}{2}\varpi_{1,j-1}^{(1-\alpha)}\right] + \left(\frac{1}{3}\right)^j \left[1 - \frac{3}{2}\varpi_{1,j-1}^{(1-\alpha)}\right] \right\} \varpi_{1,j}^{(1-\alpha)}, \quad j \geqslant 4.
\end{aligned}$$

Let

$$\chi_j = \left[1 + \frac{1}{2}\varpi_{1,j-1}^{(1-\alpha)}\right] + \left(\frac{1}{3}\right)^j \left[1 - \frac{3}{2}\varpi_{1,j-1}^{(1-\alpha)}\right], \quad j = 4, 5, \ldots$$

Because $\chi_j > 0$ and

$$\chi = \lim_{j \to +\infty} \chi_j = 1, \quad j = 4, 5, \ldots,$$

then there exists a positive constant $\mathcal{N}$, such that

$$\chi_j \leqslant \mathcal{N}, \quad j = 4, 5, \ldots$$

In this situation, one gets

$$\left(-\varpi_{2,j}^{(1-\alpha)}\right) \leqslant \widetilde{\mathcal{N}}\left(-\varpi_{1,j}^{(1-\alpha)}\right), \quad j = 4, 5, \ldots,$$

where $\widetilde{\mathcal{N}} = \mathcal{N}\left(\frac{3}{2}\right)^{1-\alpha} > 0$.

It follows form Lemma 4 that series $\sum_{j=4}^{\infty} \varpi_{2,j}^{(1-\alpha)}$ is convergent, so $\sum_{j=0}^{\infty} \varpi_{2,j}^{(1-\alpha)}$ is also convergent, therefore $\lim_{j \to +\infty} \varpi_{2,j}^{(1-\alpha)} = 0$. It is easy to know that the convergent radius of the functional series $\sum_{j=0}^{\infty} \varpi_{2,j}^{(1-\alpha)} z^j$ is one, that is $\sum_{j=0}^{\infty} \varpi_{2,j}^{(1-\alpha)} z^j$ is convergent if $|z| < 1$. Hence one can set $z = 1$ in (8) and gets $\sum_{j=0}^{\infty} \varpi_{2,j}^{(1-\alpha)} = 0$. All this completes the proof. □

Besides the above properties which will be used in the proofs of the following theorems, $\varpi_{2,j}^{(1-\alpha)}$ is rigorously monotonic increasing when $j \geqslant 4$. See Appendix for more details.

Let U_i^k be the approximate solution of (19), and define

$$\rho_i^k = u_i^k - U_i^k, \quad i = 0, 1, \ldots, M, \ k = 0, 1, \ldots, N$$

and

$$\rho^k = \left(\rho_1^k, \rho_2^k, \ldots, \rho_{M-1}^k\right)^T.$$

So, it is no difficult to obtain the following roundoff error equation

$$\left[\frac{1}{6}-\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]\rho_{i+1}^{k+1}+\left[\frac{5}{3}+\left(2\mu_1+\frac{5}{6}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]\rho_{i}^{k+1}+\left[\frac{1}{6}-\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]\rho_{i-1}^{k+1}$$
$$=\left[\frac{1}{6}+\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]\rho_{i+1}^{k}+\left[\frac{5}{3}-\left(2\mu_1+\frac{5}{6}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]\rho_{i}^{k}+\left[\frac{1}{6}+\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]\rho_{i-1}^{k}$$
$$+\mu_1\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\left(\rho_{i+1}^{k+1-j}-2\rho_{i}^{k+1-j}+\rho_{i-1}^{k+1-j}\right)-\mu_2\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\left(\frac{1}{12}\rho_{i+1}^{k+1-j}+\frac{5}{6}\rho_{i}^{k+1-j}+\frac{1}{12}\rho_{i-1}^{k+1-j}\right)$$
$$+\mu_1\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\left(\rho_{i+1}^{k-j}-2\rho_{i}^{k-j}+\rho_{i-1}^{k-j}\right)-\mu_2\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\left(\frac{1}{12}\rho_{i+1}^{k-j}+\frac{5}{6}\rho_{i}^{k-j}+\frac{1}{12}\rho_{i-1}^{k-j}\right), \tag{21}$$

$$\rho_0^k=\rho_M^k=0,\quad k=0,1,\ldots,N.$$

Now, define the grid function as follows [22,23]

$$\rho^k(x)=\begin{cases}\rho_i^k, & \text{when } x_i-\frac{h}{2}<x\leqslant x_i+\frac{h}{2},\ i=1,2,\ldots,M-1,\\ 0, & \text{when } 0\leqslant x\leqslant\frac{h}{2} \text{ or } L-\frac{h}{2}<x\leqslant L,\end{cases}$$

where $\rho^k(x)$ can be expanded in a Fourier series

$$\rho^k(x)=\sum_{l=-\infty}^{\infty}\xi_k(l)\exp\left(\frac{2\pi lx}{L}I\right),$$

in which

$$\xi_k(l)=\frac{1}{L}\int_0^L\rho^k(x)\exp\left(-\frac{2\pi lx}{L}I\right)dx.$$

We define the following discrete 2-norm

$$\|\rho^k\|_2=\left(\sum_{i=1}^{M-1}h|\rho_i^k|^2\right)^{\frac{1}{2}}=\left[\int_0^L|\rho^k(x)|^2dx\right]^{\frac{1}{2}}.$$

According to the Parseval equality

$$\int_0^L|\rho^k(x)|^2dx=\sum_{l=-\infty}^{\infty}|\xi_k(l)|^2,$$

we obtain

$$\|\rho^k\|_2=\left(\sum_{l=-\infty}^{\infty}|\xi_k(l)|^2\right)^{\frac{1}{2}}. \tag{22}$$

Suppose ρ_i^k in Eq. (21) has the form

$$\rho_i^k=\xi_k\exp(\beta ihI), \tag{23}$$

where $\beta=2\pi l/L$ is a real spatial wave number. Substituting (23) into (21) leads to

$$\left\{\left(2+\mu_2\varpi_{2,0}^{(1-\alpha)}\right)\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]+4\mu_1\varpi_{2,0}^{(1-\alpha)}\sin^2\left(\frac{1}{2}\beta h\right)\right\}\xi_{k+1}$$
$$=\left\{\left(2-\mu_2\varpi_{2,0}^{(1-\alpha)}\right)\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-4\mu_1\varpi_{2,0}^{(1-\alpha)}\sin^2\left(\frac{1}{2}\beta h\right)\right\}\xi_k-\mu_2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\xi_{k+1-j}$$
$$-4\mu_1\sin^2\left(\frac{1}{2}\beta h\right)\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\xi_{k+1-j}-\mu_2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\xi_{k-j}-4\mu_1\sin^2\left(\frac{1}{2}\beta h\right)\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\xi_{k-j}. \tag{24}$$

Denote

$$\mathcal{A}=\mu_2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]+4\mu_1\sin^2\left(\frac{1}{2}\beta h\right),$$

then Eq. (24) is changed into

$$\left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]+\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right\}\xi_{k+1}=\left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right\}\xi_k-\mathcal{A}\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}(\xi_{k-j}+\xi_{k+1-j}). \tag{25}$$

Lemma 7. *If*

$$\frac{\tau^\alpha}{h^2}\left(6K_\alpha+C_\alpha h^2\right)\le 2\cdot\left(\frac{2}{3}\right)^{1-\alpha}, \tag{26}$$

then

$$2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\geqslant 0.$$

Proof. First of all, note that $K_\alpha \geqslant 0$, then from (26), we easily get

$$C_\alpha\tau^\alpha\le 2\cdot\left(\frac{2}{3}\right)^{1-\alpha},$$

i.e.,

$$2-\mu_2\varpi_{2,0}^{(1-\alpha)}\geqslant 0. \tag{27}$$

In addition, in view of (26), we arrive at

$$(6\mu_1+\mu_2)\varpi_{2,0}^{(1-\alpha)}\le 2, \tag{28}$$

which can be rewritten as

$$\frac{2}{3}+\left(4\mu_1-\frac{1}{3}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\le 2-\mu_2\varpi_{2,0}^{(1-\alpha)}. \tag{29}$$

Combining (27) and (29) yields

$$\left[\frac{2}{3}+\left(4\mu_1-\frac{1}{3}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]\sin^2\left(\frac{1}{2}\beta h\right)\le 2-\mu_2\varpi_{2,0}^{(1-\alpha)}. \tag{30}$$

Furthermore, inequality (30) can be rewritten in the following equivalent form

$$2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\geqslant 0.$$

This finishes the proof. □

Lemma 8. *Let* ξ_{k+1} $(k=0,1,\ldots,N-1)$ *be the solution of Eq.* (25), *when* $\alpha\in\left[\frac{3}{8},1\right)$, *if temporal and spatial step sizes* τ *and* h *satisfy* (26), *then it holds that*

$$|\xi_{k+1}|\leqslant|\xi_0|,\quad k=0,1,\ldots,N-1.$$

Proof. Here, we use the mathematical induction to prove it. When $k=0$ in (25), we can get

$$|\xi_1|=\left|\frac{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}}{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]+\varpi_{2,0}^{(1-\alpha)}\mathcal{A}}\right||\xi_0|\leqslant|\xi_0|.$$

For $\alpha\in\left[\frac{3}{8},1\right)$, it can be seen that

$$\varpi_{2,j}^{(1-\alpha)}\le 0,\quad j=1,\ldots, \tag{31}$$

Now, we suppose that

$$|\xi_\ell|\leqslant|\xi_0|\quad(\ell=1,2,\ldots,k).$$

For $k>0$, if the temporal and spatial step sizes τ and h satisfy (26), from Eqs. (24) and (31), and Lemma 7, we have

$$\begin{aligned}
\left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]+\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right\}|\xi_{k+1}| &= \left|\left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right\}\xi_k-\mathcal{A}\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}(\xi_{k-j}+\xi_{k+1-j})\right|\\
&\leqslant \left|2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right||\xi_k|+\left|\mathcal{A}\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}(\xi_{k-j}+\xi_{k+1-j})\right|\\
&\leqslant \left|2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right||\xi_k|+\mathcal{A}\sum_{j=1}^{k}\left|\varpi_{2,j}^{(1-\alpha)}\right|(|\xi_{k-j}|+|\xi_{k+1-j}|)\\
&\leqslant \left\{\left|2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right|+2\mathcal{A}\sum_{j=1}^{\infty}\left|\varpi_{2,j}^{(1-\alpha)}\right|\right\}|\xi_0|\\
&= \left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}-2\mathcal{A}\sum_{j=1}^{\infty}\varpi_{2,j}^{(1-\alpha)}\right\}|\xi_0|\\
&= \left[2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}-2\mathcal{A}\left(\sum_{j=0}^{\infty}\varpi_{2,j}^{(1-\alpha)}-\varpi_{2,0}^{(1-\alpha)}\right)\right]|\xi_0|\\
&= \left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]+\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right\}|\xi_0|,
\end{aligned}$$

i.e,

$$|\xi_{k+1}|\leqslant|\xi_0|.$$

This ends the proof. □

Theorem 2. *When $\alpha\in[\frac{3}{8},1)$, if temporal and spatial step sizes τ and h satisfy (26), then the difference scheme (19) is stable with respect to the initial values.*

Proof. On the basis of Lemma 8 and (22), it is clear that the solution of the difference Eq. (19) satisfies

$$\|\rho^{k+1}\|_2\leqslant\|\rho^0\|_2,\quad k=0,1,\ldots,N-1,$$

which means that the difference schemes (19) is stable. □

Next, we give the convergence analysis of the finite difference scheme (19).

Suppose that $u(x_i,t_k)$ is the exact solution of Eq. 1 at point $(x_i,t_k),\ i=0,1,\ldots,M;\ k=0,1,\ldots,N$. Define

$$E_i^k=u(x_i,t_k)-u_i^k,\quad i=0,1,\ldots,M;\ k=0,1,\ldots,N$$

and denote

$$E^k=\left(E_0^k,E_1^k,\ldots,E_{M-1}^k,E_M^k\right)^T,\quad k=1,\ldots,N,$$

$$R^k=\left(R_0^k,R_1^k,\ldots,R_{M-1}^k,R_M^k\right)^T,\quad k=1,\ldots,N.$$

By the above analysis, we obtain

$$\begin{aligned}
&\left[\frac{1}{6}-\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]E_{i+1}^{k+1}+\left[\frac{5}{3}+\left(2\mu_1+\frac{5}{6}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]E_i^{k+1}+\left[\frac{1}{6}-\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]E_{i-1}^{k+1}\\
&\quad=\left[\frac{1}{6}+\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]E_{i+1}^{k}+\left[\frac{5}{3}-\left(2\mu_1+\frac{5}{6}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]E_i^{k}+\left[\frac{1}{6}+\left(\mu_1-\frac{1}{12}\mu_2\right)\varpi_{2,0}^{(1-\alpha)}\right]E_{i-1}^{k}\\
&\qquad+\mu_1\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\left(E_{i+1}^{k+1-j}-2E_i^{k+1-j}+E_{i-1}^{k+1-j}\right)-\mu_2\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\left(\frac{1}{12}E_{i+1}^{k+1-j}+\frac{5}{6}E_i^{k+1-j}+\frac{1}{12}E_{i-1}^{k+1-j}\right)\\
&\qquad+\mu_1\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\left(E_{i+1}^{k-j}-2E_i^{k-j}+E_{i-1}^{k-j}\right)-\mu_2\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}\left(\frac{1}{12}E_{i+1}^{k-j}+\frac{5}{6}E_i^{k-j}+\frac{1}{12}E_{i-1}^{k-j}\right)+\frac{1}{12}\tau R_{i+1}^{k+1}+\frac{5}{6}\tau R_i^{k+1}\\
&\qquad+\frac{1}{12}\tau R_{i-1}^{k+1},\quad i=1,\ldots,M-1;\ k=0,1,\ldots,N-1.
\end{aligned}\tag{32}$$

Similar to the stability analysis as above, we define the grid functions:

$$E^k(x) = \begin{cases} E_i^k, & \text{when } x_i - \frac{h}{2} < x \leqslant x_i + \frac{h}{2},\ i = 1, 2, \ldots, M-1, \\ 0, & \text{when } 0 \leqslant x \leqslant \frac{h}{2} \text{ or } L - \frac{h}{2} < x \leqslant L \end{cases}$$

and

$$R^k(x) = \begin{cases} R_i^k, & \text{when } x_i - \frac{h}{2} < x \leqslant x_i + \frac{h}{2},\ i = 1, 2, \ldots, M-1, \\ 0, & \text{when } 0 \leqslant x \leqslant \frac{h}{2} \text{ or } L - \frac{h}{2} < x \leqslant L, \end{cases}$$

respectively.

Expand $E^k(x)$ and $R^k(x)$ into the following Fourier series,

$$E^k(x) = \sum_{l=-\infty}^{\infty} \zeta_k(l) e^{I2\pi lx/L}$$

and

$$R^k(x) = \sum_{l=-\infty}^{\infty} \eta_k(l) e^{I2\pi lx/L},$$

respectively, where

$$\zeta_k(l) = \frac{1}{L}\int_0^L E^k(x) e^{-I2\pi lx/L} dx, \quad \eta_k(l) = \frac{1}{L}\int_0^L R^k(x) e^{-I2\pi lx/L} dx.$$

Similar to the discussions in [22–24], one has

$$\|E^k\|_2 = \left(\sum_{i=1}^{M-1} h\left|E_i^k\right|^2\right)^{\frac{1}{2}} = \left(\sum_{l=-\infty}^{\infty} |\zeta_k(l)|^2\right)^{\frac{1}{2}} \tag{33}$$

and

$$\|R^k\|_2 = \left(\sum_{i=1}^{M-1} h\left|R_i^k\right|^2\right)^{\frac{1}{2}} = \left(\sum_{l=-\infty}^{\infty} |\eta_k(l)|^2\right)^{\frac{1}{2}}, \tag{34}$$

respectively.

Suppose E_i^k and R_i^k have the forms

$$E_i^k = \zeta_k e^{I\beta ih}, \quad R_i^k = \eta_k e^{I\beta ih}.$$

Substituting the above relations into (32) results in

$$\begin{aligned} &\left\{2\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] + \varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right\}\zeta_{k+1} = \left\{2\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] - \varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right\}\zeta_k - \mathcal{A}\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}(\zeta_{k-j} + \zeta_{k+1-j}) \\ &\quad + \tau\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]\eta_{k+1}, \quad k = 0, 1, \ldots, N-1. \end{aligned} \tag{35}$$

Lemma 9. *Suppose that ζ_{k+1} $(k = 0, 1, \cdots, N-1)$ is the solution of Eq. (35). When $\alpha \in [\frac{3}{8}, 1)$, if temporal and spatial step sizes τ and h satisfy (26), then there exists a positive constant $\mathcal{C}^*$, such that*

$$|\zeta_{k+1}| \leqslant \mathcal{C}^*(k+1)\tau|\eta_1|, \quad k = 0, 1, \ldots, N-1.$$

Proof. Noticing that $E^0 = 0$, we have

$$\zeta_0 \equiv \zeta_0(l) = 0.$$

Using the convergence of the series in the right-hand side of (34), there is a positive constant $\mathcal{C}^*$ such that

$$|\eta_{k+1}| \equiv |\eta_{k+1}(l)| \leqslant \mathcal{C}^*|\eta_1| \equiv \mathcal{C}^*|\eta_1(l)|, \quad k = 0, 1, \ldots, N-1. \tag{36}$$

Again, we use the mathematical induction to show it. From (35), when $k = 0$, we obtain

$$|\zeta_1| \leqslant \frac{2\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] - \varpi_{2,0}^{(1-\alpha)}\mathcal{A}}{2\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] + \varpi_{2,0}^{(1-\alpha)}\mathcal{A}}|\zeta_0| + \frac{\tau\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]}{2\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] + \varpi_{2,0}^{(1-\alpha)}\mathcal{A}}|\eta_1| = \frac{\tau\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]}{2\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] + \varpi_{2,0}^{(1-\alpha)}\mathcal{A}}|\eta_1| \leqslant \mathcal{C}^*\tau|\eta_1|.$$

Now suppose that

$$|\zeta_\ell| \leqslant \mathcal{C}^*\ell\tau|\eta_1|, \quad \ell = 1,\dots,k.$$

When $k > 0$, from (35), (36) and Lemma 4, we have

$$\begin{aligned}
\left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]+\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right\}|\zeta_{k+1}| &= \left|\left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right\}\zeta_k-\mathcal{A}\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}(\zeta_{k-j}+\zeta_{k+1-j})+\tau\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]\eta_{k+1}\right| \\
&\leqslant \left|2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right||\zeta_k|+\left|\mathcal{A}\sum_{j=1}^{k}\varpi_{2,j}^{(1-\alpha)}(\zeta_{k-j}+\zeta_{k+1-j})\right|+\tau\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]|\eta_{k+1}| \\
&\leqslant \left|2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right||\zeta_k|+\mathcal{A}\sum_{j=1}^{k}\left|\varpi_{2,j}^{(1-\alpha)}\right|(|\zeta_{k-j}|+|\zeta_{k+1-j}|)+\tau\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]|\eta_{k+1}| \\
&\leqslant \mathcal{C}^*k\tau\left(\left|2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right|+2\mathcal{A}\sum_{j=1}^{\infty}\left|\varpi_{2,j}^{(1-\alpha)}\right|\right)|\eta_1|+\mathcal{C}^*\tau\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]|\eta_1| \\
&= \mathcal{C}^*k\tau\left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}-2\mathcal{A}\sum_{j=1}^{\infty}\varpi_{2,j}^{(1-\alpha)}\right\}|\eta_1|+\mathcal{C}^*\tau\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]|\eta_1| \\
&= \mathcal{C}^*k\tau\left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-\varpi_{2,0}^{(1-\alpha)}\mathcal{A}-2\mathcal{A}\left(\sum_{j=0}^{\infty}\varpi_{2,j}^{(1-\alpha)}-\varpi_{2,0}^{(1-\alpha)}\right)\right\}|\eta_1| \\
&\quad +\mathcal{C}^*\tau\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]|\eta_1| \\
&= \mathcal{C}^*k\tau\left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]+\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right\}|\eta_1|+\mathcal{C}^*\tau\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]|\eta_1| \\
&\leqslant \mathcal{C}^*(k+1)\tau\left\{2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]+\varpi_{2,0}^{(1-\alpha)}\mathcal{A}\right\}|\eta_1|,
\end{aligned}$$

that is

$$|\zeta_{k+1}| \leqslant \mathcal{C}^*(k+1)\tau|\eta_1|, \quad k = 0,1,\dots,N-1. \qquad \square$$

Theorem 3. *When $\alpha \in [\frac{3}{8},1)$, the finite difference scheme (19) is L_2-convergent if temporal and spatial step sizes τ and h satisfy (26). And there exists a positive constant $\mathcal{C}^{**}$ such that*

$$|u(x_i,t_k)-u_i^k| \leqslant \mathcal{C}^{**}\left(\tau^2+h^4\right), \quad i = 1,\dots,M-1;\ k = 1,\dots,N.$$

Proof. Firstly, according to the analysis of the truncation error and the left-hand equality of (34), we have

$$\|R^{k+1}\|_2 \leqslant \mathcal{C}\sqrt{(M-1)h}\left(\tau^2+h^4\right) \leqslant \mathcal{C}\sqrt{L}\left(\tau^2+h^4\right). \tag{37}$$

Additionally, using (33), (34), (37) and Lemma 9, we can obtain

$$\left\|E^{k+1}\right\|_2 \leqslant \mathcal{C}^*(k+1)\tau\left\|R^1\right\|_2 \leqslant \mathcal{C}^*\mathcal{C}\sqrt{L}(k+1)\tau\left(\tau^2+h^4\right).$$

Noticing $(k+1)\tau \leqslant T$, therefore

$$\left\|E^{k+1}\right\|_2 \leqslant \mathcal{C}^{**}\left(\tau^2+h^4\right), \quad k = 0,1,\dots,N-1,$$

in which $\mathcal{C}^{**} = \mathcal{C}^*\mathcal{C}T\sqrt{L}$. It immediately follows that

$$|u(x_i,t_k)-u_i^k| \leqslant \mathcal{C}^{**}\left(\tau^2+h^4\right), \quad i = 1,\dots,M-1;\ k = 1,\dots,N. \qquad \square$$

From the above discussions, the condition $\alpha \in [\frac{3}{8},1)$ in Lemmas 8, 9 and Theorems 2, Theorem 3, is sufficient. However, from many numerical simulations, the results in these lemmas and theorems still hold for $\alpha \in (0,\frac{3}{8})$. Using the present analytical method, we can not get the same conclusions as in Lemmas 8, 9 and Theorems 2, 3 for $\alpha \in (0,\frac{3}{8})$. We conjecture that *the conclusions in Lemmas 8, 9 and Theorems 2, 3 still hold for $\alpha \in (0,\frac{3}{8})$*. We hope that such a problem can be solved somewhere in the near future.

Table 1
The maximum error, temporal convergence orders for $\alpha \in (0, \frac{3}{8})$.

α	$h = \frac{1}{80}$	The maximum error	The temporal convergence order
0.1	$\tau = \frac{1}{20}$	2.033801e−008	–
	$\tau = \frac{1}{40}$	5.364865e−009	1.9226
	$\tau = \frac{1}{80}$	1.408689e−009	1.9292
	$\tau = \frac{1}{160}$	3.716875e−010	1.9222
	$\tau = \frac{1}{320}$	1.014558e−010	1.8732
0.2	$\tau = \frac{1}{20}$	5.189840e−008	–
	$\tau = \frac{1}{40}$	1.377237e−008	1.9139
	$\tau = \frac{1}{80}$	3.613549e−009	1.9303
	$\tau = \frac{1}{160}$	9.433469e−010	1.9376
	$\tau = \frac{1}{320}$	2.481223e−010	1.9267
0.3	$\tau = \frac{1}{20}$	9.981453e−008	–
	$\tau = \frac{1}{40}$	2.639489e−008	1.9190
	$\tau = \frac{1}{80}$	6.880066e−009	1.9398
	$\tau = \frac{1}{160}$	1.778744e−009	1.9516
	$\tau = \frac{1}{320}$	4.597640e−010	1.9519

Table 2
The maximum error, temporal convergence orders for $\alpha \in [\frac{3}{8}, 1)$.

α	$h = \frac{1}{80}$	The maximum error	The temporal convergence order
0.4	$\tau = \frac{1}{20}$	1.535929e−007	–
	$\tau = \frac{1}{40}$	4.037862e−008	1.9274
	$\tau = \frac{1}{80}$	1.045006e−008	1.9501
	$\tau = \frac{1}{160}$	2.680152e−009	1.9631
	$\tau = \frac{1}{320}$	6.855737e−010	1.9669
0.5	$\tau = \frac{1}{20}$	2.004939e−007	–
	$\tau = \frac{1}{40}$	5.239550e−008	1.9360
	$\tau = \frac{1}{80}$	1.347505e−008	1.9592
	$\tau = \frac{1}{160}$	3.434627e−009	1.9721
	$\tau = \frac{1}{320}$	8.726464e−010	1.9767
0.6	$\tau = \frac{1}{20}$	2.285883e−007	–
	$\tau = \frac{1}{40}$	5.942417e−008	1.9436
	$\tau = \frac{1}{80}$	1.520555e−008	1.9665
	$\tau = \frac{1}{160}$	3.858091e−009	1.9786
	$\tau = \frac{1}{320}$	9.759422e−010	1.9830
0.7	$\tau = \frac{1}{20}$	2.282270e−007	–
	$\tau = \frac{1}{40}$	5.907547e−008	1.9498
	$\tau = \frac{1}{80}$	1.505833e−008	1.9720
	$\tau = \frac{1}{160}$	3.808558e−009	1.9832
	$\tau = \frac{1}{320}$	9.607957e−010	1.9869
0.8	$\tau = \frac{1}{20}$	1.924883e−007	–
	$\tau = \frac{1}{40}$	4.965990e−008	1.9546
	$\tau = \frac{1}{80}$	1.262333e−008	1.9760
	$\tau = \frac{1}{160}$	3.186039e−009	1.9863
	$\tau = \frac{1}{320}$	8.026355e−010	1.9889
0.9	$\tau = \frac{1}{20}$	1.170772e−007	–
	$\tau = \frac{1}{40}$	3.013306e−008	1.9580
	$\tau = \frac{1}{80}$	7.645766e−009	1.9786
	$\tau = \frac{1}{160}$	1.927723e−009	1.9878
	$\tau = \frac{1}{320}$	4.858249e−010	1.9884

Table 3
The maximum error, spatial convergence orders for $\alpha \in (0, \frac{3}{8})$.

α	$\tau = \frac{1}{800}$	The maximum error	The spatial convergence order
0.1	$h = \frac{1}{20}$	8.916692e−009	–
	$h = \frac{1}{25}$	3.731808e−009	3.9035
	$h = \frac{1}{30}$	1.803335e−009	3.9889
	$h = \frac{1}{35}$	9.644074e−010	4.0602
	$h = \frac{1}{40}$	5.561657e−010	4.1222
0.2	$h = \frac{1}{20}$	8.085284e−009	–
	$h = \frac{1}{25}$	3.391930e−009	3.8928
	$h = \frac{1}{30}$	1.643157e−009	3.9753
	$h = \frac{1}{35}$	8.809168e−010	4.0442
	$h = \frac{1}{40}$	5.092362e−010	4.1043
0.3	$h = \frac{1}{20}$	7.288021e−009	–
	$h = \frac{1}{25}$	3.064796e−009	3.8820
	$h = \frac{1}{30}$	1.408689e−009	3.9615
	$h = \frac{1}{35}$	7.999651e−010	4.0278
	$h = \frac{1}{40}$	4.635763e−010	4.0859

Table 4
The maximum error, spatial convergence orders for $\alpha \in [\frac{3}{8}, 1)$.

α	$\tau = \frac{1}{800}$	The maximum error	The spatial convergence order
0.4	$h = \frac{1}{20}$	6.538064e−009	–
	$h = \frac{1}{25}$	2.755853e−009	3.8716
	$h = \frac{1}{30}$	1.341687e−009	3.9480
	$h = \frac{1}{35}$	7.229200e−010	4.0116
	$h = \frac{1}{40}$	4.199649e−010	4.0674
0.5	$h = \frac{1}{20}$	5.843012e−009	–
	$h = \frac{1}{25}$	2.468371e−009	3.8616
	$h = \frac{1}{30}$	1.204599e−009	3.9349
	$h = \frac{1}{35}$	6.506498e−010	3.9956
	$h = \frac{1}{40}$	3.789053e−010	4.0491
0.6	$h = \frac{1}{20}$	5.206024e−009	–
	$h = \frac{1}{25}$	2.203854e−009	3.8523
	$h = \frac{1}{30}$	1.077946e−009	3.9225
	$h = \frac{1}{35}$	5.836122e−010	3.9804
	$h = \frac{1}{40}$	3.406741e−010	4.0313
0.7	$h = \frac{1}{20}$	4.627048e−009	–
	$h = \frac{1}{25}$	1.962504e−009	3.8437
	$h = \frac{1}{30}$	9.619205e−010	3.9109
	$h = \frac{1}{35}$	5.219543e−010	3.9659
	$h = \frac{1}{40}$	3.053758e−010	4.0143
0.8	$h = \frac{1}{20}$	4.103924e−009	–
	$h = \frac{1}{25}$	1.743646e−009	3.8359
	$h = \frac{1}{30}$	8.563009e−010	3.9003
	$h = \frac{1}{35}$	4.656058e−010	3.9525
	$h = \frac{1}{40}$	2.729934e−010	3.9982
0.9	$h = \frac{1}{20}$	3.633252e−009	–
	$h = \frac{1}{25}$	1.546068e−009	3.8290
	$h = \frac{1}{30}$	7.606019e−010	3.8907
	$h = \frac{1}{35}$	4.143566e−010	3.9402
	$h = \frac{1}{40}$	2.434303e−010	3.9833

5. Numerical example

In this section, a numerical example will be provided to demonstrate the convergence order of the numerical method.

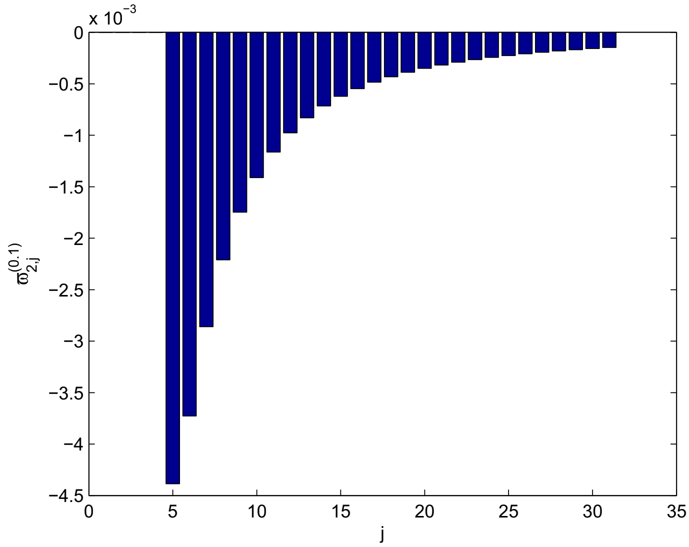

Fig. 1. The coefficients $\varpi_{2,j}^{(0.1)}$ for $j = 5, 6, \ldots$

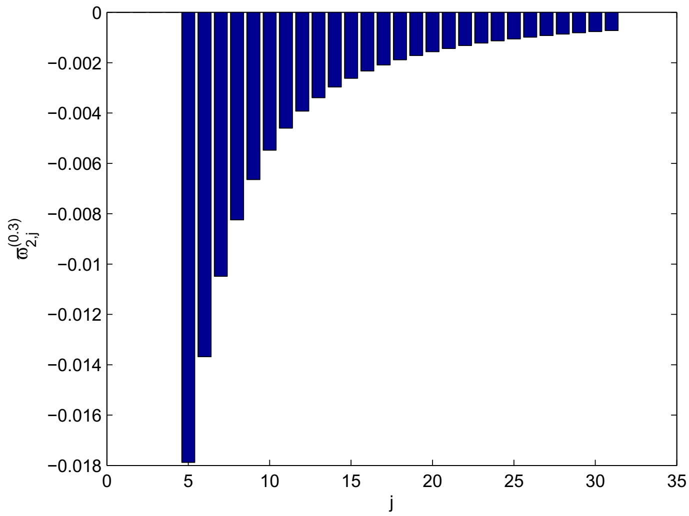

Fig. 2. The coefficients $\varpi_{2,j}^{(0.3)}$ for $j = 5, 6, \ldots$

Example 1. Consider the following equation

$$\frac{\partial u(x,t)}{\partial t} = {}_{RL}D_{0,t}^{1-\alpha}\left[\frac{1}{\pi^8}\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\alpha^2}{4}u(x,t)\right] + f(x,t), \quad 0 < x < 1,\ 0 < t \leqslant 1,$$

together with the following initial and boundary conditions

$$u(x,0) = 0, \quad 0 \leqslant x \leq 1,$$

$$u(0,t) = u(1,t) = 0, \quad 0 \leqslant t \leqslant 1,$$

where

$$f(x,t) = 2tx^2(1-x)^2\left[x^2(1-x)^2\left(1 + \frac{\alpha^2 t^\alpha}{4\Gamma(2+\alpha)}\right) - \frac{4t^\alpha}{\pi^8\Gamma(2+\alpha)}(14x^2 - 14x + 3)\right].$$

The exact solution of the above problem is $u(x,t) = t^2x^4(1-x)^4$.

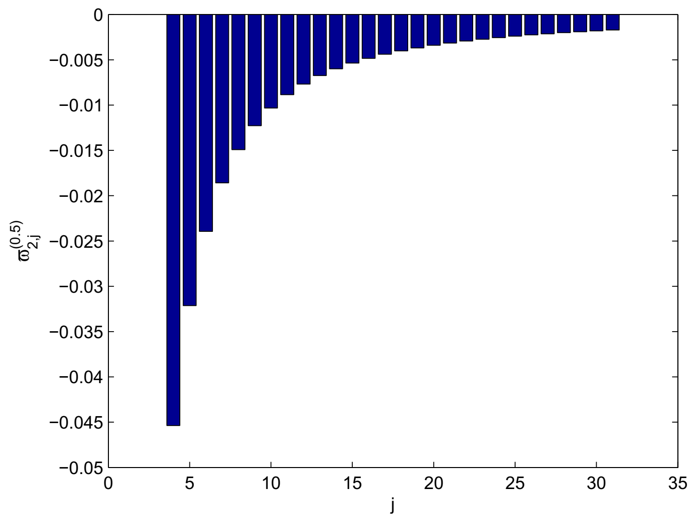

Fig. 3. The coefficients $\varpi_{2,j}^{(0.5)}$ for $j = 5, 6, \ldots$

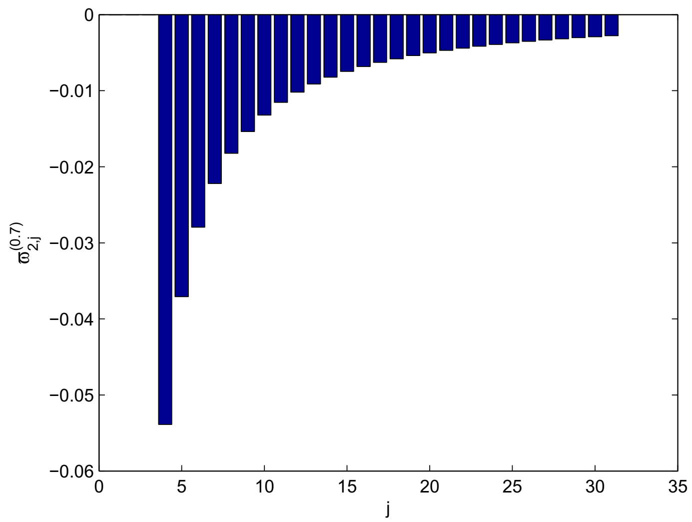

Fig. 4. The coefficients $\varpi_{2,j}^{(0.7)}$ for $j = 5, 6, \ldots$

The maximum error is defined as follows

$$\mathcal{E}_\infty = \max_{0\leqslant k\leqslant N} \max_{0\leqslant i\leqslant M} \left\{ \left| u_i^k - u(x_i, t_k) \right| \right\}.$$

Tables 1–4, presents the maximum errors of the numerical solution and the exact solution at all mesh points, and the convergence orders in the temporal, spatial direction, respectively. Evidently, for different values of $\alpha \in \left(0, \frac{3}{8}\right)$ and $\alpha \in \left[\frac{3}{8}, 1\right)$, we see that the convergence orders in the temporal and spatial direction are almost 2 and 4, which is in accordance with the theoretical analysis.

6. Conclusion

In the present paper, we construct an effective difference scheme for the reaction and anomalous-diffusion equation. The solvability, local truncation error are analyzed. And the stability, convergence are proved by the Fourier method. Finally, the numerical experiments show that the scheme is convergent with order $\mathcal{O}(\tau^2 + h^4)$, which support the theoretical results. As far as we know, the order of convergence in the temporal direction is less than 2 for the subdiffusion equations except the

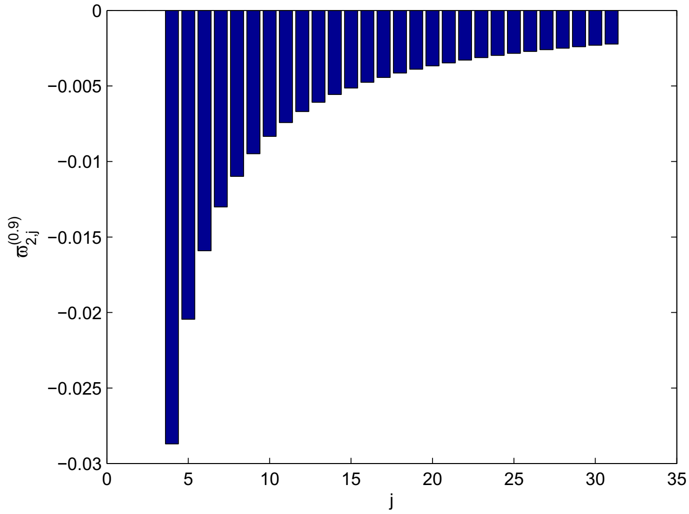

Fig. 5. The coefficients $\varpi_{2,j}^{(0.9)}$ for $j = 5, 6, \ldots$

results reported in this paper. Our future duty is to construct much higher order (>2) difference schemes in the temporal direction for the reaction and anomalous-diffusion equations.

Appendix A

If $\alpha \in (0,1)$, then $\varpi_{2,j}^{(1-\alpha)} < \varpi_{2,j+1}^{(1-\alpha)}$, $j = 4, 5, 6, \ldots$ where $\varpi_{2,j}^{(1-\alpha)}$ is defined by Eq. (12).

Proof. From (12), one has

$$\begin{aligned}\varpi_{2,j+1}^{(1-\alpha)} &= \left(\frac{3}{2}\right)^{1-\alpha}\sum_{m=0}^{j+1}\left(\frac{1}{3}\right)^{m}\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j+1-m}^{(1-\alpha)}\\ &= \varpi_{2,j}^{(1-\alpha)} - (2-\alpha)\left(\frac{3}{2}\right)^{1-\alpha}\left(\frac{1}{3}\right)^{j}\sum_{m=1}^{j-1}\frac{3^m}{m+1}\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j-m}^{(1-\alpha)} - \left(\frac{3}{2}\right)^{1-\alpha}\left[\frac{5-3\alpha}{3}\left(\frac{1}{3}\right)^{j} + \frac{2-\alpha}{j+1} + \frac{2-\alpha}{j+1}\left(\frac{1}{3}\right)^{j+1}\right]\varpi_{1,j}^{(1-\alpha)}.\end{aligned}$$

Let

$$f(x) = \frac{3^x}{x+1}, \quad x \in [1, j-1],\ j \geqslant 4.$$

Then

$$f'(x) = \frac{3^x[(x+1)\ln 3 - 1]}{(x+1)^2} > 0,$$

which follows that $f(x)$ is an increasing function and

$$f_{\max}(x) = f(j-1) = \frac{3^{j-1}}{j}.$$

Denote

$$S_1 = \sum_{m=1}^{j-1}\frac{3^m}{m+1}\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j-m}^{(1-\alpha)}.$$

One gets

$$S_1 = \sum_{m=1}^{j-1}\frac{3^m}{m+1}\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j-m}^{(1-\alpha)} \leqslant \frac{3^{j-1}}{j}\sum_{m=1}^{j-1}\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j-m}^{(1-\alpha)} = \frac{3^{j-1}}{j}\left[\sum_{m=0}^{j}\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j-m}^{(1-\alpha)} - 2\varpi_{1,j}^{(1-\alpha)}\right].$$

So,

$$\varpi_{2,j+1}^{(1-\alpha)} - \varpi_{2,j}^{(1-\alpha)} = -(2-\alpha)\left(\frac{3}{2}\right)^{1-\alpha}\left(\frac{1}{3}\right)^{j} S_1 - \left(\frac{3}{2}\right)^{1-\alpha}\left[\frac{5-3\alpha}{3}\left(\frac{1}{3}\right)^{j} + \frac{2-\alpha}{j+1} + \frac{2-\alpha}{j+1}\left(\frac{1}{3}\right)^{j+1}\right]\varpi_{1,j}^{(1-\alpha)}$$

$$\geqslant -\frac{2-\alpha}{3j}\left(\frac{3}{2}\right)^{1-\alpha}\sum_{m=0}^{j}\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j-m}^{(1-\alpha)} - \left(\frac{3}{2}\right)^{1-\alpha}\left[\frac{5-3\alpha}{3}\left(\frac{1}{3}\right)^{j} + \frac{(j-2)(2-\alpha)}{3j(j+1)} + \frac{2-\alpha}{j+1}\left(\frac{1}{3}\right)^{j+1}\right]\varpi_{1,j}^{(1-\alpha)}.$$

Set

$$S_2 = -\left(\frac{3}{2}\right)^{1-\alpha}\left[\frac{5-3\alpha}{3}\left(\frac{1}{3}\right)^{j} + \frac{(j-2)(2-\alpha)}{3j(j+1)} + \frac{2-\alpha}{j+1}\left(\frac{1}{3}\right)^{j+1}\right]\varpi_{1,j}^{(1-\alpha)}. \qquad \square$$

It is obvious that $S_2 > 0$ due to Lemma 5.

On the other hand,

$$S_3 \triangleq -\frac{2-\alpha}{3j}\left(\frac{3}{2}\right)^{1-\alpha}\sum_{m=0}^{j}\varpi_{1,m}^{(1-\alpha)}\varpi_{1,j-m}^{(1-\alpha)} = -\frac{2-\alpha}{3j}\left(\frac{3}{2}\right)^{1-\alpha}\sum_{m=0}^{j}(-1)^m\binom{1-\alpha}{m}(-1)^{j-m}\binom{1-\alpha}{j-m} = -\frac{2-\alpha}{3j}\left(\frac{3}{2}\right)^{1-\alpha}$$

$$\sum_{m=0}^{j}\binom{m+\alpha-2}{m}\binom{j-m+\alpha-2}{j-m} = -\frac{2-\alpha}{3j}\left(\frac{3}{2}\right)^{1-\alpha}\binom{j+2\alpha-4}{j} = -\frac{2-\alpha}{3j}\left(\frac{3}{2}\right)^{1-\alpha}\frac{\Gamma(j+2\alpha-3)}{\Gamma(j+1)\Gamma(2\alpha-3)} > 0, \quad j \geqslant 4.$$

Therefore one obtains $\varpi_{2,j+1}^{(1-\alpha)} - \varpi_{2,j}^{(1-\alpha)} = S_2 + S_3 > 0$ if $j \geqslant 4$, which gives

$\varpi_{2,j}^{(1-\alpha)} < \varpi_{2,j+1}^{(1-\alpha)}, \quad j \geqslant 4$. The proof is finished.

In the following, we plot Figs. 1–5 which display the rigorous monotonicity.

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