SEMICLASSICAL QUANTIZATION OF RELATIVE SKYRMION–SKYRMION MOTION

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The relative motion of the two-skyrmion system is quantized semiclassically. The reduced mass is calculated as a function of the relative distance and a large enhancement is seen at short distances. An effective momentum-dependent interaction is discussed quantitatively.

The $1/N_c$ expansion of quantum chromodynamics has led a renewed interest in the Skyrme soliton model for the baryon [1,2]. Much work has been done on the single-soliton system with a considerable phenomenological success [3]. The model has also been applied to multi-baryon systems. Pioneering calculations of the adiabatic potential between two solitons show a qualitative success [4]. The shape and size of interacting solitons have also been discussed in the adiabatic approximation [5].

In this letter, we discuss semiclassical quantization of the relative motion in the two-skyrmion system. The problem is not trivial because of the nonlocality of the soliton system. One may expect a reduced mass which depends on the relative distance of the two solitons. The contribution of the kinetic energy to the intersolitonic interaction will indicate the size of higher order corrections to the adiabatic treatment. This correction is of the same order in $1/N_c$ as that due to quantization of spin and isospin (or relative orientation of the two solitons), which has already been discussed in comparing the adiabatic potential with a realistic nuclear force [4].

The skyrme lagrangian for the $SU(2)\times SU(2)$ chiral theory is given in terms of the SU(2) matrix U.

$$\mathcal{L} = -\tfrac{1}{16}F_\pi^2 \operatorname{Tr}\{U^\dagger \partial_\mu U U^\dagger \partial_\mu U\} + (1/32e^2)\operatorname{Tr}\{[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2\} . \tag{1}$$

The second term is Skyrme's choice for the quartic derivative term, which is introduced to stabilize a finite-size soliton. The skyrmion is a chiral soliton for the lagrangian (1), whose topological current is given by

$$B^\mu = (1/24\pi^2)\epsilon^{\mu\nu\sigma\rho}\operatorname{Tr}\{U^\dagger \partial_\nu U U^\dagger \partial_\sigma U U^\dagger \partial_\rho U\} , \tag{2}$$

which is trivially conserved regardless of the dynamics, and B^0 is identified as the baryon density. A static $B = 1$ solution is obtained from the hedgehog ansatz

$$U(\boldsymbol{r}) = \exp[\mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{F}(\boldsymbol{r})] = \exp[\mathrm{i}\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}F(r)] , \tag{3}$$

with boundary conditions for the chiral angle, $F(0) = \pi$ and $F(\infty) = 0$.

In order to quantize the translational motion of the soliton (3), we introduce a time-dependent collective coordinate $\boldsymbol{R}(t)$ which represents the center of the soliton, i.e., the solution (3) is replaced by

$$U(\boldsymbol{r}) = \exp[\mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{R})] . \tag{4}$$

Computing the lagrangian up to $O(\dot{R}^2)$, we obtain

$$L \equiv \int \mathrm{d}^3\boldsymbol{r}\, \mathcal{L} = L(\text{static}) + \Delta L , \tag{5}$$

$$\begin{aligned}\Delta L &= \int \mathrm{d}^3\boldsymbol{r}\big[\tfrac{1}{16}F_\pi^2 \operatorname{Tr}\{\partial_t U^\dagger \partial_t U\} \\ &\quad - (1/16e^2)\operatorname{Tr}\{[U^\dagger \partial_t U, U^\dagger \nabla_i U]^2\}\big] \\ &= \tfrac{1}{2}\mathcal{M}_{kl}\dot{R}_k\dot{R}_l + O(\dot{R}^3) \\ &= \tfrac{1}{2}M\dot{R}^2 ,\end{aligned} \tag{6}$$

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where $\dot{R}_k \equiv \mathrm{d}R_k/\mathrm{d}t$ and in the last line we have used the fact that $\mathcal{M}_{kl} = \delta_{kl}M$ for a single spherical soliton. In particular, for the hedgehog solution (3), the inertia M is given by

$$M = 2(\tfrac{1}{3}E_2 + \tfrac{2}{3}E_4)\,, \tag{7}$$

with

$$E_2 = \int \tfrac{1}{8}F_\pi^2[(F')^2 + 2r^{-2}\sin^2 F]\,\mathrm{d}^3\boldsymbol{r}\,, \tag{8}$$

$$E_4 = \int e^{-2}r^{-2}\sin^2 F\,[(F')^2 + \tfrac{1}{2}r^{-2}\sin^2 F]\,\mathrm{d}^3\boldsymbol{r}, \tag{9}$$

where E_2 (E_4) is nothing but the contribution to the static energy from the quadratic (quartic) derivative term of the lagrangian. According to the virial theorem, the static solution $F(r)$ satisfies $E_2 = E_4 = E(\text{static})/2$. (Using a length scale λ, one can express $E_2 = \epsilon_2\lambda$ and $E_4 = \epsilon_4/\lambda$ with positive constants ϵ_2 and ϵ_4. Choosing λ so as to minimize the total energy $E_2 + E_4$, we obtain $E_2 = E_4$. It is also proved that $(E_2 + 2E_4)/3 = E(\text{static})/2$ even if the lagrangian contains the pion mass term.) Finally, we get a reasonable relation between the inertia M and the static energy $E(\text{static})$, i.e., $M = E(\text{static})$. The corresponding effective hamiltonian is given by

$$H = M + P^2/2M\,, \tag{10}$$

where P is the conjugate momentum of the collective coordinate R. It should be noted that this trivial-looking result is derived only for a solution $F(r)$ which minimizes the energy.

The two-soliton configuration will be approximated by the product of two solitons, i.e.,

$$U_2(\boldsymbol{r}) = U_1(\boldsymbol{r}_1)\,U_1(\boldsymbol{r}_2)\,, \tag{11}$$

where $U_1(\boldsymbol{r}_i)$ denotes a $B = 1$ soliton centered at $\pm\boldsymbol{R}/2$ ($\boldsymbol{r}_1 \equiv \boldsymbol{r} - \boldsymbol{R}/2$ and $\boldsymbol{r}_2 \equiv \boldsymbol{r} + \boldsymbol{R}/2$). In the present study we adopt a spherical approximation, assuming $U_1(\boldsymbol{r})$ to have the spherical hedgehog form (3) like the free solution. This approximation is taken for simplicity, even though geometrical deformation of U_1 may be important at short distances [6]. The ansatz was first proposed by Skyrme [1] and has been used in some previous studies [4,5].

Quantization of the relative motion is similar to the previous problem. Making the relative coordinate $\boldsymbol{R} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ time dependent, the same procedure leads us to the lagrangian

$$L = L(\text{static}) + \Delta L\,, \tag{12}$$

$$\Delta L = \tfrac{1}{2}\mathcal{M}_{kl}\dot{R}_k\dot{R}_l + \mathrm{O}(\dot{R}^3)\,. \tag{13}$$

The mass tensor $\mathcal{M}_{kl}$ has two terms,

$$\mathcal{M}_{kl} = M_\mathrm{A}\delta_{kl} + M_\mathrm{B}\hat{R}_k\hat{R}_l\,, \tag{14}$$

where $\hat{\boldsymbol{R}} \equiv \boldsymbol{R}/R$, and M_A and M_B are functions of R, which are expressed by integrals involving F and F'. We obtain

$$\Delta L = \tfrac{1}{2}M_\mathrm{A}\dot{R}^2 + \tfrac{1}{2}M_\mathrm{B}(\dot{\boldsymbol{R}}\cdot\hat{\boldsymbol{R}})^2$$
$$= \tfrac{1}{2}\mu_\mathrm{R}\dot{R}_\mathrm{R}^2 + \tfrac{1}{2}\mu_\theta\dot{R}_\theta^2\,. \tag{15}$$

Thus the inertia for the radial motion is given by

$$\mu_\mathrm{R} = M_\mathrm{A} + M_\mathrm{B} = \hat{R}_k\hat{R}_l\mathcal{M}_{kl}\,, \tag{16}$$

and that for the rotation by

$$\mu_\theta = M_\mathrm{A} = \tfrac{1}{2}(\delta_{kl} - \hat{R}_k\hat{R}_l)\mathcal{M}_{kl} = \mathcal{I}/R^2\,, \tag{17}$$

where $\mathcal{I}$ is the moment of inertia for the rotation.

The static energy of the total system is given as a function of the separation R by

$$E = -L(\text{static}) = 2M + V(R)\,, \tag{18}$$

where the last equality is the definition of the adiabatic potential, $V(R)$. Taking the variation of E with respect to F leads to a second-order integro-differential equation for F, which has been solved in ref. [5]. Using the solution $F(r)$ for each R, we can calculate $\mu_\mathrm{R}(R)$ and $\mu_\theta(R)$.

Two limiting cases are interesting, i.e., $R \to \infty$ and $R = 0$. For $R \to \infty$, $\mu_\mathrm{R} = \mu_\rho \to M/2$, which is clearly consistent with the inertia of a single free soliton. The two-soliton system is spherical for $R = 0$ and therefore $M_\mathrm{B} = 0$, which implies $\mu_\mathrm{R} = \mu_\theta$. This is not necessarily the case in general. At $R = 0$, we have $F(\boldsymbol{r}_1) = F(\boldsymbol{r}_2)$ and (using the dimensionless variable $x = rF_\pi e$)

$$\mu_\mathrm{R} = \mu_\theta = \frac{F_\pi}{e}\int \mathrm{d}^3\boldsymbol{x}\,\tfrac{1}{6}x^{-2}\sin^4 F$$
$$\times\{1 + 16[(F')^2 + x^{-2}\sin^2 F\cos^2 F]\}\,, \tag{19}$$

which no longer has a trivial relation to the energy (18).

We consider two cases. calculation with the free

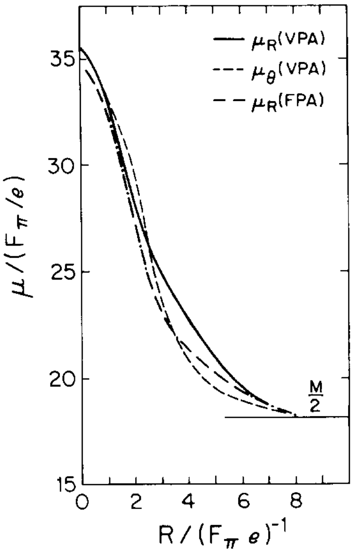

Fig. 1. Reduced masses of the two-skyrmion system. The solid (μ_R) and the dashed (μ_θ) curves have been calculated by VPA, while the dash-dotted curve (μ_R) by FPA. (See the text for details.) Dimensionless variables are used. A typical value, F_π = 186 MeV and e = 4.76, gives the energy unit F_π/e = 39.1 MeV and the length unit $(F_\pi e)^{-1}$ = 0.223 fm.

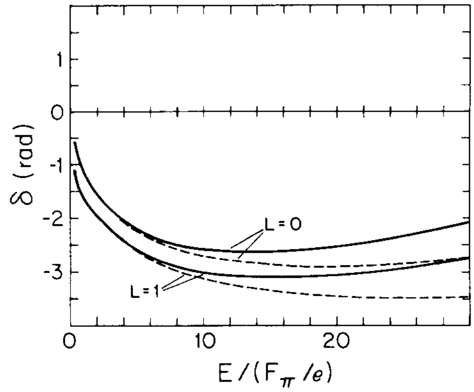

Fig. 2. Phase shifts calculated with (solid) and without (dashed) the R-dependence of μ_R and μ_θ. e = 4.76 is used in this calculation.

product approximation (FPA) and that with the variational product approximation (VPA). The former uses the free skyrmion solution for the chiral angle $F(r)$ at any R. For the latter, we instead use the variational solution which minimizes the total energy for fixed R [5]. Fig. 1 shows the results of numerical calculations. Large enhancement of the reduced mass for small R (say, $R < 1$ fm) is observed. One sees that the three curves in fig. 1 show the same qualitative behavior. The difference between μ_R and μ_θ is less than 10% (maximum at $RF_\pi e \approx 4$). The result for FPA shows little difference from VPA. At $R = 0$, μ_R is about twice as large as the asymptotic value, $M/2$.

In terms of the conjugate momentum P_R and the relative angular momentum L, the effective hamiltonian is written as

$$\begin{aligned} H &= 2M + V(R) + P_R^2/2\mu_R + L^2/2\mu_\theta R^2 \\ &= 2M + P_R^2/M + V(R) + (1/2\mu_R - 1/M)P_R^2 \\ &\quad + L^2/2\mu_\theta R^2 \,. \end{aligned} \tag{20}$$

Note that μ_R and μ_θ are R dependent. The fourth term of the last line is interpreted as an effective momentum-dependent potential (MDP). The qualitative effect of the enhancement of μ_R and μ_θ would be (1) an attractive velocity-dependent interaction, and (2) suppression of the centrifugal barrier.

In order to investigate the quantitative significance of MDP, we solve the scattering problem for the effective hamiltonian (20). As is well known, the quantization of the hamiltonian is not unique due to the noncommutability of $\mu_R(R)$ and P_R. Here we use a simple prescription, replacing $P_R^2/2\mu_R$ by a hermitian operator

$$(\overleftarrow{\nabla}\cdot\hat{R})\,[1/2\mu_R(R)]\,(\hat{R}\cdot\overrightarrow{\nabla}) \,. \tag{21}$$

Fig. 2 shows the phase shifts obtained by solving the Schrodinger equation with and without MDP. As is expected, we observe moderate attraction of MDP at higher energies, although the whole skyrmion–skyrmion interaction remains strongly repulsive. A similar effect as on the S-wave phase shift is seen on the P-wave one, where the suppression of the centrifugal barrier is expected. Clearly the strong short-range repulsion suppresses the contribution of the momentum-dependent attraction at low energy.

In summary, we discussed a semiclassical quantization of the relative motion of the two skyrmions. The result is not trivial and shows numerically an enhancement of the reduced mass at short distances.

The contribution of the induced momentum-dependent interaction was studied in the scattering problem. We can conclude that the adiabatic approximation used in the calculation of the potential is only slightly modified by this correction.

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