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HEDGE ALGEBRAS: AN ALGEBRAIC APPROACH TO STRUCTURE OF SETS OF LINGUISTIC TRUTH VALUES

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Abstract: It is shown that any sets of linguistic values of linguistic variables can be axiomatized, which leads to a notion of hedge algebras. Some intuitive properties of linguistic hedges, which are a basis of the axiomatization of the hedge algebras, are discussed informally. The characteristics of the hedge algebras depend on such a discussion and they seem to reflect the natural structure of sets of linguistic values of linguistic variables. Some fundamental properties of hedge algebras are examined. They constitute a foundation of further development of our theory of hedge algebras and of some kind of fuzzy logic.

Keywords: Linguistic variable; linguistic value; hedge algebra; hedge operation; ordering operation; converse operation; comparable operation; canonical representation of elements; primary generators.

Introduction

Linguistic hedges interpreted by fuzzy sets were examined by G. Lakoff in [5]. Based on this idea, L.A. Zadeh introduced and investigated a fuzzy approach to human reasoning containing vague concepts. This approach seems very useful and applicable, since in daily life we often make and formulate decisions based on such concepts. We quote some examples of this kind of reasoning introduced in [12]:

Example 1. *Premise* 1: u is small. *Premise* 2: u and v are approximately equal. *Conclusion*: v is more or less small.

Example 2. *Premise* 1: (u is small) is very true. *Premise* 2: (u and v are approximately equal) is very true. *Conclusion*: (v is more or less small) is true.

Based on fuzzy set theory and fuzzy logic, Zadeh [12] introduced a computational way to approximate human reasoning. Zadeh's main ideas, roughly, are as follows:

Vague concepts, including the linguistic truth values, are interpreted as fuzzy sets taking values in the closed unit interval [0, 1]. The logical connectives and the linguistic hedges are interpreted as functors on the fuzzy sets. The quantitative truth values of sentences of the form given in Example 2 are computed by the so-called extension principle (see [11]).

Analyzing this approach, we see that the effectiveness of the method depends, on the one hand, on the interpretation of the primary vague concepts like 'small', 'large', 'true', 'false', . . . , and of the functors such as 'very', 'approximately', 'more or less', . . . , and, on the other hand, on the quantitative relationships between them. However, these interpretations do not always correspond to our intuition, properly. For example, suppose that the meanings of the vague concepts 'true', 'very true' and 'approximately true' are interpreted as fuzzy sets depicted in Figure 1. where 'very' is interpreted as the concentration operator CON introduced in [12]. We can adopt intuitively the assumption: 'true' is more true than '(very)^n approximately true', for any natural number n.

By such an interpretation, the function f_n with the label '(very)^n approximately true' is greater than the function f with the label 'true'. But, when we interpret 'very' as the operator CON, then f is greater than f_n, for a sufficiently great number n. This contradicts the intuitive meanings of the vague concepts 'true' and '(very)^n approximately true'.

Furthermore, one of our main aims is the comparison of linguistic truth values. In the traditional approach, such comparisons are based on the fundamental structure of the Łukasiewicz's logic based on [0, 1]. These observations suggest us to look for a fundamental structure which can model the natural structure of linguistic truth values. It is important, because a correctly examined structure of the set of vague concepts of the linguistic truth variable could lead to a suitable fuzzy logic of fuzzy reasoning.

Our idea is as follows: we shall ignore the separate interpretation of the meanings of vague concepts, but focus our attention to the intuitive ordering relationship between vague concepts of a linguistic variable. We observe that the set of linguistic values, or the domain of a linguistic variable, can be represented as a formal algebra with one-argument operations being hedges under con-

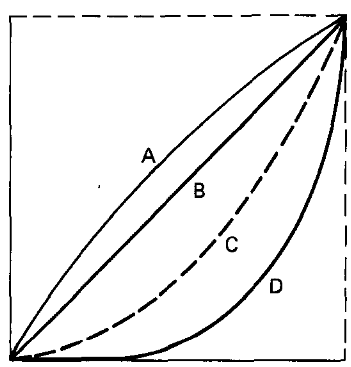

Fig. 1. Examples of fuzzy sets. (A) Approximately True; (B) True; (C) Very True = (True)^2; (D) f_n = (Very)^n Approximately True.

sideration and generators being the primary vague concepts of this linguistic variable. For example, the set of all possible truth values $T = \{$true, false, very true, very false, approximately true, ...$\}$ can be considered as an algebra with the generators 'true' and 'false' and with operations 'very', 'approximately', ... Furthermore, the meanings of hedges can be expressed by an ordering relation, e.g. very true > true, less false > false, approximately true < true, ... Based on some properties of hedges we shall show that T is a partially ordered set or, briefly, a poset. This set will have some interesting properties. Therefore, T becomes a powerful algebraic structure.

We would like to emphasize that the meanings of vague concepts can be expressed by their relative position in this structure. Notice that the meanings of 'true' and 'false' in the classical logic may be expressed by the relationship between the elements in a two-elements Boolean algebra, i.e. in a suitable algebraic structure.

In the traditional approach to human reasoning, vague concepts are expressed by fuzzy sets. The reasonability of this interpretation lies in the fact, which can explicitly be recognized, that fuzzy sets are impressive computational images of vague concepts. It is a computational aspect of this method. However, the reasonability of this method mainly lies in the fact, which is implicitly recognized, that the meanings of vague concepts are expressed by the relative positions of fuzzy sets representing vague concepts in the set of all fuzzy sets. Of course, the set of such fuzzy sets is not a good algebraic structure to describe the semantical relationship between vague concepts. Nevertheless, this observation gives us a natural viewpoint on our approach: the meanings of vague concepts can be expressed by elements in a suitable algebraic structure.

If the algebraic structure which we shall look for were able to model the semantic relationships of vague concepts, it would be a fundamental algebraic structure for some kind of fuzzy logic and fuzzy reasoning.

The paper is organized as follows. In the second section, we shall discuss some basical intuitive properties of linguistic hedges, which are a foundation of an axiomatization of domains of some linguistic variables. In the third section, we shall give an axiomatization for the so-called hedge algebras. Some properties of these algebras will be examined. They are a basis for a development of our theory in a next paper. Some conclusions will be given in the fourth section.

2. Intuitive properties of linguistic hedges

Linguistic hedges were first investigated by Lakoff in [5]. In [11], Zadeh pointed out that the set of linguistic values of linguistic variables can be regarded as a formal language generated by a context-free grammar and, for computing their meanings, hedges could be viewed as operators on fuzzy sets. These ideas suggest us to consider the sets of such linguistic values as algebras with operations to be linguistic hedges. Moreover, these sets have a natural partial ordering. We denote these algebras by $\boldsymbol{X} = (X, H, \leqslant)$, where X is the underlying set of $\boldsymbol{X}$, H is a set of hedges under consideration and $\leqslant$ is a partial ordering on X. The axioms for these algebras will be formulated later on. In this section we shall discuss some fundamental properties of linguistic hedges.

Let us consider e.g. hedges such as Very, More, Less, Approximately and Not, where Not is characterized by the meaning of 'not' in the following sentence: "John is not very tall". It means that he is, probably, a little very tall, but certainly not so very tall. Therefore, Not is different from the logical negation and it seems a 'local' negation and modifies only 'slightly' the meaning of an operant and hence it could be interpreted as hedge.

The hedges More and Less play similar roles as the operators 'plus' and 'minus' introduced by Zadeh in [10]. For instance, Plus Very Old can be understood as More Very Old, but in the latter case we can say Very More Very Old, which gives us the same intuitive meaning as Very Plus Very Old. In general, the meanings of linguistic hedges depend strongly on the context. However, some important general characteristics of hedges can be examined by means of an ordering relation.

We observe first that many linguistic variables have some primary vague concepts. For instance, the primary vague concepts of the linguistic variables AGE are 'old' and 'young', of TRUTH are 'true' and 'false', of HIGH are 'tall' and 'small' and so on. We think of the primary vague concepts 'true', 'old' and 'tall' as having 'positive meaning' and the others as having 'negative meaning'. In this sense, the hedges Very, More have the property that they strengthen the positive or negative meanings and the hedges Less, Approximately and Not (so) weaken them.

The interpretation of the hedge Approximately needs some explanation. Consider the following sentence: "This fact is approximately true". In our opinion, it means that we have not enough information to conclude certainly that this fact is true and hence the hedge Approximately weakens the positive meaning of 'true', and does not strengthen it. Therefore, the following property of hedges should be clear intuitively:

Hypothesis 1. Every hedge under consideration either strengthens or weakens the positive or negative meaning of the primary vague concepts.

By this hypothesis, if the domain of a linguistic variable is embedded into a suitable poset, i.e. a partially ordered set, $(X, \leqslant)$, then the inequality $hx \leqslant x$ or $hx \geqslant x$, where h is a hedge and x is a vague concept, represents some aspect of the meaning of the hedge h with respect to x.

Now we investigate some relationships between hedges. Similarly as above, we can observe that the meaning of each hedge can be strengthened or weakened by the others and by itself. For example, the hedge Very strengthens the meaning of the hedges More, and Less, but it weakens the meaning of the hedge Approximately, since Approximately decreases the meaning of True, while Very increases the meaning of Approximately True. This means also that Very decreases the effect of the hedge Approximately. Here, the relative positions of the concepts True, Approximately True, Very Approximately True in a suitable poset $(X, \leqslant)$ express some other aspects of the meanings of the hedges Very and Approximately. By this observation, we can adopt:

Hypothesis 2. Every hedge either strengthens or weakens the meanings of any other hedges. Especially:

(i) the hedges Very, More strengthen the meanings of the hedges Very, More, Less and Not, and they weaken the meaning of the hedge Approximately;

(ii) The hedges Less, Approximately and Not strengthen the meaning of the hedge Approximately and weaken the meanings of the hedges Very, More, Less and Not.

Many of our axioms for hedge algebras will be motivated by the following consideration about the 'concept category'. The primary concepts like True and False involve some intuitive meanings. It is known that the linguistic hedges modify only these meanings, and do not change them essentially. This means that if σ is an arbitrary string of hedges, say $\sigma = h_n \cdots h_1$, then σTrue and σFalse express something about the meaning of True and of False, respectively. For example, Very Very Less True expresses something about True but not about False and Very Very Less False expresses something about False but not about True. This suggests us to introduce the notion of 'concept category' of a vague concept x, denoted by $\textsc{conc}(x)$. For a given vague concept x, $\textsc{conc}(x)$ consists of all vague concepts expressing something about the meaning of the concept x. This means that $\textsc{conc}(x)$ is the set of all terms generated from x by means of hedges, called the generated terms. The term x is called the generating term of the concept category of x.

Since the meaning of each element in a concept category stems from the generating term, in comparing two arbitrary terms we can refer to the meaning of their generating term. For instance, if σ and σ' are any two strings of hedges, then we can conclude that σTrue $\geqslant \sigma'$False, since True $\geqslant$ False. A more difficult example is to compare, e.g. the two following terms: $x = \sigma$ More Very Less True and $y = \sigma'$ Very Very Less True. Since, as we can observe, More Very Less True $\geqslant$ Very Very Less True, we infer that $x \geqslant y$. Consequently, it can be intuitively seen that if h and k are two distinct hedges and x is an arbitrary vague concept, then $\textsc{conc}(hx)$ and $\textsc{conc}(kx)$ have no common meanings in the sense that two terms, one in $\textsc{conc}(hx)$ and the other in $\textsc{conc}(kx)$, do not have the same meaning, i.e. they are separated. We say that h and k have local and separated meanings. This observation seems to be natural, since every hedge determines an own meaning.

On account of this consideration, we can adopt:

Hypothesis 3. All hedges under consideration have a local and separated meaning and the meaning of a term generated from a vague concept x stems from the meaning of the concept x.

Hypotheses 1–3 explain more precisely the intuitive meanings of linguistic hedges, that seem to be, in the authors' opinion, easily accepted, because they describe some general characteristics of hedges. Here, we have focused our attention on the relative relationships between hedges, and we do not separately describe the meanings of hedges.

The axiomatization for hedge algebras will be based on these hypotheses.

3. Hedge algebras: Definitions and properties

Let us consider an algebra structure $\boldsymbol{X} = (X, H, \leqslant)$, where the underlying set X is a poset with the partial ordering relation $\leqslant$, H is a set of unary operations corresponding to the linguistic hedges under consideration. The operations in H will somtimes be called hedge operations or, simply, hedges. We denote by I the identification operation defined by the rule $Ix = x$, for all $x \in X$. The identification I can be regarded as an artificial linguistic hedge having no meaning and no another hedge can be applied to I, except just I. By our convention, if there is in a formulation an expression of the form $h_n \cdots h_1 Ix$, then we mean by $h_n \cdots h_1$ an empty string of hedges, i.e. this string has no effect.

For simplicity, we denote the set $H \cup \{I\}$ by $H + I$. Since every hedge h is a mapping from X into X, the image of an element $x \in X$ under h will be denoted by hx, instead of $h(x)$.

An operation h is said to be an ordering operation if either $hx \leqslant x$ or $hx \geqslant x$, for all $x \in X$. Throughout this paper we shall only consider ordering operations or, in other words, we shall assume that every hedge operation is an ordering operation.

The hedge operations will be denoted by h, k and the elements in X will be denoted by $u, v, x, y, z, \ldots,$ with indexes if necessary.

To define hedge algebras, we need some preparations.

As usual, the notation $x < y$ means that $x \leqslant y$ and $x \neq y$. We denote the set of all non-negative integers by Nat.

Definition 1. (i) Let h, k be two hedges in H. Then k is said to be positive (negative) w.r.t. h if for every $x \in X$, $hx \geqslant x$ implies $khx \geqslant hx$ ($khx \leqslant hx$) or, conversely, $hx \leqslant x$ implies $khx \leqslant hx$ ($khx \geqslant hx$).

(ii) Let σ and σ' be two strings of hedges in H. Then, $\sigma \leqslant \sigma'$ if for every $x \in X$, $x \leqslant \sigma x$ or $x \leqslant \sigma' x$ implies $x \leqslant \sigma x \leqslant \sigma' x$ and $x \geqslant \sigma x$ or $x \geqslant \sigma' x$ implies $x \geqslant \sigma x \geqslant \sigma' x$.

(iii) Two hedges h and k are said to be converse or we say one is a converse operation of the other, provided for every $x \in X$, $hx \leqslant x$ iff $kx \geqslant x$. Further, h and k are said to be compatible provided for every $x \in X$, $x \leqslant hx$ iff $x \leqslant kx$.

Since, in practice, we can compare the effects of linguistic hedges, e.g. Very is stronger than More, Definition 1(ii) induces an ordering relation on H, that is denoted also by $\leqslant$. This causes no confusion, because X and H are, of course, assumed to be disjoint. Therefore, we can assume that H is a poset. For example, if H consists of the operations V, M, L, A and N corresponding to the linguistic hedges 'very', 'more', 'less' or 'little', 'approximately' and 'not so', then we can naturally assume that $M \leqslant V$, $A \leqslant L$ and $N \leqslant L$. Furthermore, each of V and M is a converse operation of L, A and N.

We assume that H can be decomposed into two non-empty disjoint sets H^+ and H^- so that each element in H^+ is a converse operation of the operations in H^-. This implies that the operations in each of the sets H^+ and H^- are compatible. In addition, there exists in every natural language a hedge which most strongly strengthens a vague concept and another which most strongly weakens it; we can also assume that there is a greatest element in both H^+ and H^-, called the unit operation and denoted respectively by V and L. Obviously, the identification operation I is the zero element in $H^+ + I$ and in $H^- + I$.

In the sequel, we shall always adopt the assumption that $H^+ + I$ and $H^- + I$ are distributive lattices with the unit and the zero elements. We shall denote by UOS the set of the unit operations in H.

Definition 2. For any hedge operations h and k, we shall write $hx \lll kx$ ($hx \lll Ix$), if for any h' and k' in UOS and any n, m in Nat, $V^n h' hx \leqslant V^m k' kx$ ($V^n h' hx \leqslant Ix$). If the latter inequalities are always strict, then we shall write $hx \ll kx$ ($hx \ll Ix$).

The intention of Definition 2 is as follows. If $hx \lll kx$, for some vague concept x, then any vague concept deduced from hx by means of hedges is less than those deduced from kx.

Let Y be a subset of X and H' be a subset of H. We denote by $H'(Y)$ the smallest subset of X including Y, that is closed w.r.t. the operations in H'. Two elements x and y in X are said to be independent if $x \notin H(y)$ and $y \notin H(x)$. Since $x' \in H(y)$ means that x' can be deduced from y by means of hedges, this means that x and y determine somewhat essentially different relative meanings.

Now we introduce an axiomatization of domains of linguistic variables.

Definition 3. An algebraic structure $\boldsymbol{X} = (X, H, \leqslant)$, where $\leqslant$ is a partial ordering relation on X and H is a set of unary ordering operations, is said to be a hedge algebra if it satisfies the following axioms:

(A1) Every operation in H^+ is a converse operation of the operations in H^-.

(A2) The unit operation V in H^+ is either positive or negative w.r.t. any (hedge) operation. Especially, V is positive w.r.t. V in H^+ and the unit operation L in H^-.

(A3) If u and v are independent, i.e. $u \notin H(v)$ and $v \notin H(u)$, then $x \notin H(v)$, for every $x \in H(u)$. In addition, if u and v are incomparable, then so are x and y, for any x in $H(u)$ and y in $H(v)$.

(A4) If $x \neq hx$, then $x \notin H(hx)$. If $h \neq k$ and $hx \leqslant kx$, then $h'hx \leqslant k'kx$, for any h' and k' in UOS. Moreover, if $hx \neq kx$, then hx and kx are independent.

(A5) If $u \notin H(v)$ and $u \leqslant v$ ($u \geqslant v$), then $u \leqslant hv$ ($u \geqslant hv$), for each $h \in$ UOS.

We give some explanation of the axioms (A3)–(A5). Here, the notion of concept categories is important to describe properties of hedges.

(A3) means, that if two vague concepts are really different (independent), then their two concept categories are separated, i.e. they have no common meanings.

(A4) means that each hedge has its meaning and hence it determines its own concept category. So, for example, if hx is different from kx, then two categories of these vague concepts are separated.

(A5) describes the fact that h modifies only the meaning of a vague concept. It means that the meaning of hv stems from the concept v. Therefore, if the relative meanings of the concepts u and v are represented by an ordering relationship, then the hedge h preserves this meaning. The fact that u does not belong to $H(v)$ means that the meaning of u is really different from the meaning of v.

Example 1. Consider $\boldsymbol{X} = (X, H, \leqslant)$, where $X = \{0, 1\}$ with the usual ordering: $0 < 1$. The operations in H are defined as follows: $hx = x$, for every $x \in X$. Obviously, $\boldsymbol{X}$ satisfies the axioms in Definition 3.

Example 2. Let X be a poset as represented in Figure 2 and consider an algebraic structure $\boldsymbol{X} = (X, H, \leqslant)$, where $H = \{V, M, L, A, N\}$ as previously considered. For every hedge operation h in H, hTrue and hFalse are the elements represented in Figure 2. For $x \neq$ True and $x \neq$ False, we define $hx = x$. It can easily be seen that the operations are well defined and $\boldsymbol{X}$ satisfies the axioms in Definition 3.

Example 3. Let T be the set of all terms of the forms σTrue and σFalse, where σ is an arbitrary string of operations in the set H defined as above, i.e. $\sigma \in (H)^*$. The ordering relation on T is defined as follows: False < True, $\{V$ True, M True$\}$ > True, $\{V$ False, M False$\}$ < False, $\{L$ True, A True, N True$\}$ < True and $\{L$ False, A False, N False$\}$ > False, where $X \leqslant Y$ means that $x \leq y$ for any $x \in X$ and $y \in Y$. Now let us consider two terms $x = \sigma u = h \cdots h'u$ and $y = \sigma' u = k \cdots k'u$, where u is the maximal common subterm of x and y. Notice that $h' \neq k'$. Then, we can define by induction on the length of u, that $x < y$ iff $h'u < k'u$. Under such ordering, it can be verified that $\boldsymbol{T} = (T, H, \leqslant)$ is a hedge algebra.

Definition 4. Let x and u be two elements in a hedge algebra $\boldsymbol{X} = (X, H, \leqslant)$. The expression $h_n \cdots h_1 u$ is said to be a canonical representation of x w.r.t. u in $\boldsymbol{X}$ if (i) $x = h_n \cdots h_1 u$; (ii) $h_i \cdots h_1 u \neq h_{i-1} \cdots h_1 u$, for every $i \leqslant n$.

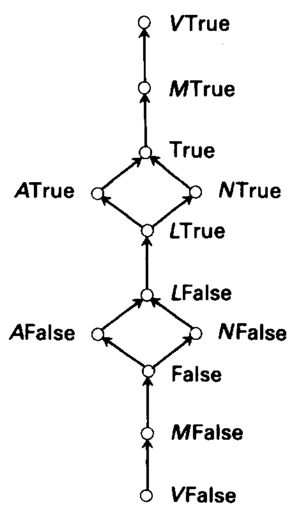

Fig. 2. Poset of Example 2.

Theorem 1. *Let $X = (X, H, \leqslant)$ be a hedge algebra. Then the following statements hold*:

(o) *If $hx \lll kx$, then $hx \leqslant kx$.*

(i) *The operations in H^+ or the ones in H^- are compatible.*

(ii) *If $x \in X$ is a fixed point of an operation h in H, i.e. $hx = x$, then it is a fixed point of the others.*

(iii) *If $x = h_n \cdots h_1 u$, then there exists an index i such that the suffix $h_i \cdots h_1 u$ of x is a canonical representation of x w.r.t. u and $h_j x = x$, for all $j \geqslant i$.*

(iv) *For any $h, k \in H$, if $x \leq hx$ $(x \geqslant hx)$, then $Ix \lll hx$ $(Ix \ggg hx)$ and if $hx \leqslant kx$ and $h \neq k$, then $hx \lll kx$.*

Proof. To prove (o), we suppose that $hx \lll kx$. By Definition 2, $h'hx \leqslant k'kx$, for any $h', k' \in \text{UOS}$. On account of the axiom (A1), we can choose a unit operation h' such that $h'hx \geqslant hx$ and a unit operation k' such that $k'kx \leqslant kx$. This implies that $hx \leqslant kx$.

It can be seen that (i) is a direct consequence of the axioms (A1) and (A2).

To prove (ii), suppose that $hx = x$ and k is an arbitrary operation in H. By (A2), k is either positive or negative w.r.t. h. Assume that k is positive w.r.t. h. Since $hx = x$ and so $hx \leqslant x$, it follows that $khx \leqslant hx$, which means that $kx \leqslant x$. On the other hand, we have also $hx \geqslant x$, and hence, by the same argument, $kx \geqslant x$ holds. Thus, $kx = x$. For the case where k is negative w.r.t. h, the proof is similar.

Since (iii) is, evidently, an immediate consequence of (ii), it remains to prove (iv). Suppose that $x \leqslant hx$. If $x = hx$, then x is a fixed point and so, by (ii), $x \leqslant V^n h'hx$. If $x < hx$, then, by (A4) and (A5), $x \notin H(hx)$ and $x \leqslant h'hx$, for any $h' \in \text{UOS}$. Since $x \notin H(h'hx)$, applying (A5) repeatedly, we obtain $x \leqslant V^n h'hx$, i.e. $Ix \lll hx$, by Definition 2.

Now suppose that $hx \leqslant kx$ and $h \neq k$. If $hx = kx$, then it follows from (A1) and (A4) that $hx \leqslant h'hx \leqslant k'kx \leqslant kx$, for suitable unit operations h' and k'. Hence, $h'hx = hx$ and so hx is a fixed point of the operations in H, by (ii) of the theorem. Thus, $V^n h'hx = V^m k'kx$, for all $h', k' \in \text{UOS}$ and $n, m \in \text{Nat}$.

Assume that $hx < kx$. By (A4), hx and kx are independent and hence, by the axiom (A5) for $u = hx$ and $v = kx$, we obtain $hx \leqslant k'kx$, for $k' \in \text{UOS}$. Since, in view of (A3), $hx \notin H(k'kx)$, this implies again by (A5) that $hx \leqslant Vk'kx$. Repeating this argument, we obtain $hx \leqslant V^m k'kx$. It is obvious, by virtue of (A3), that $u = V^m k'kx \notin H(hx)$. Hence, by the same argument as above, it follows that $V^n h'hx \leqslant V^m k'kx$, for any $h', k' \in \text{UOS}$ and $n, m \in \text{Nat}$. Consequently, we have proved that $hx \lll kx$. The theorem is completely proved.

The following theorem is important in the sequel.

Theorem 2. *For every operation $h \in H$, there exist two unit operations h^- and h^+ such that h^- is negative and h^+ is positive w.r.t. h and for any $h_1, \ldots, h_n \in H$,*

$$V^n h^- h \leq h_n \cdots h_1 h \leqslant V^n h^+ h.$$

Proof. We shall prove the theorem by induction on the number n of hedge operations.

Assume $n=1$. If h_1 is positive w.r.t. h and $hx \leqslant x$, then it follows from the axioms (A1), (A2) and from the assumptions on the hedges h^- and h^+, that $Vh^+hx \leqslant h^+hx \leqslant h^-hx \leqslant Vh^-hx$.

If h_1 is negative w.r.t. h and $hx \leqslant x$, then $hx \leqslant h_1hx$. Since both h^- and h_1 are negative w.r.t. h and since h^- is a unit operation, it follows that they together belong to H^+ or H^- and $h_1 \leqslant h^-$. Since, by the assumption in the theorem, h^+ is positive w.r.t. h and, by the axiom (A2), V is positive w.r.t. h^+, it can be seen that $Vh^+hx \leqslant hx \leqslant h_1hx \leqslant Vh^-hx$. For the case $hx \geqslant x$, the proof is similar. Consequently, it has been proved that the inequalities in theorem hold for $n=1$.

Assuming that the assertion holds for $n=i$, we have the induction hypothesis as follows: if $h_1hx \leqslant hx$, then

$$V^ih_1^+h_1hx \leqslant h_{i+1}h_i \cdots h_1hx \leqslant V^ih_1^-h_1hx$$

and if $h_1hx \geqslant hx$, then

$$V^ih_1^+h_1hx \geqslant h_{i+1}h_1 \cdots h_1hx \geqslant V^ih_1^-h_1hx,$$

where h_1^+, h_1^- and h_1 satisfy the assumption analogous as that on h^+, h^- and h.

Now we shall prove the induction conclusion for the case $hx \leqslant x$. For the opposite case, the proof is similar.

Suppose first that h_1 is positive w.r.t. h, and so $h_1hx \leqslant hx$. From the induction hypothesis and Theorem 1(iv), it follows that $h_{i+1} \cdots h_1hx \leqslant V^ih_1^-h_1hx \leqslant hx \leqslant h^-hx \leqslant V^{i+1}h^-hx$, since h^- is negative w.r.t. h and V is positive w.r.t. h^- and V.

We need to prove that $h_{i+1} \cdots h_1hx \geqslant V^{i+1}h^+hx$. Indeed, since both h_1 and h^+ are positive w.r.t. h, it follows that they together belong to H^+ or H^-. So $h^+ \geqslant h_1$. If $h^+ \neq h_1$, then $h^+hx \leqslant h_1hx$ and hence $h^+hx \lll h_1hx$, by Theorem 1(iv). According to the induction hypothesis and Definition 2, we have $h_{i+1} \cdots h_1hx \geqslant V^ih_1^+h_1hx \geqslant V^iVh^+hx$. If $h_1 = h^+$, then h_1 is either V or L. In both cases, $h_1^+ = V$ and hence from the induction hypothesis it follows $h_{i+1} \cdots h_1hx \geqslant V^iVh_1hx = V^{i+1}h^+hx$. Thus, for the case where h_1 is positive w.r.t. h, the induction conclusion follows. For the opposite case, it can be proved by an analogous argument. Hence, the theorem is completely proved.

Corollary 1. (i) *If $h<k$, then for any two strings of hedges σ and σ', the inequality $\sigma h \leqslant \sigma' k$ holds.*

(ii) *Let u be an arbitrary element in X. For every $x \in H(u)$, there exist $y, z \in \mathrm{UOS}(u)$, i.e. z and y are generated from u by means of the unit operations, such that $y \geqslant x \geqslant z$. Furthermore, the fact that either $u \leqslant x \leqslant V^nhu$ or $u \geqslant x \geqslant V^nhu$ holds for a suitable $h \in H$ and for sufficiently great numbers $n \in$* Nat.

Proof. We shall prove (i) for the case $x \leqslant hx$. Since $h<k$, we have $hx \leqslant kx$. According to Theorem 1(iv) and Definition 2, it follows from Theorem 2, that $\sigma hx \leqslant V^nh^+hx \leqslant V^mk^-kx \leqslant \sigma' kx$, which is the required inequality.

To prove (ii), suppose that $x = \sigma hu$ is a representation of x w.r.t. u, where σ is a string of hedges of the length m. There are two alternative cases: $u \leqslant hu$ and $u \geqslant hu$. Assuming that $u \leqslant hu$, it follows from Theorem 2 and Theorem 1(iv), that $x = \sigma hu \leqslant V^mh^+hu \leqslant V^{m+1}h'u$, where $h' \in \mathrm{UOS}$ and $h \leqslant h'$. On the other hand, again by Theorem 2 and Theorem 1(iv), we get $u \leq V^m h^- hu \leq \sigma hu = x$. Since the proof for the case $u \geq hu$ is similar, the corollary is completely proved.

To simplify the formulation of the following theorem, we introduce a formal convention about the occurrences of the operation I in a representation of elements: if the operation I occurs in an expression, it stands only in the prefix of this expression for example, $I \cdots I\sigma u$. Notice that $I \leq h$, for every h in $H^+ + I$ or in $H^- + I$.

Theorem 3. *Let $x = h_n \cdots h_1 u$ and $y = k_m \cdots k_1 u$ be two arbitrary canonical representations of x and y w.r.t. u, respectively. Then there exists an index $j \leq \min\{n, m\} + 1$ such that $h_i = k_i$, for all $i < j$, and*

(i) *$x < y$ iff $h_j x_j < k_j x_j$, where $x_j = h_{j-1} \cdots h_1 u$;*

(ii) *$x = y$ iff $n = m = j$ and $h_j x_j = k_j x_j$;*

(iii) *x and y are incomparable iff $h_j x_j$ and $k_j x_j$ are incomparable.*

Proof. Let j be the least index such that $h_j \neq k_j$. It can be seen that $j \leq \min(n, m) + 1$, since $I \neq h$ for $h \in H$.

Sufficiency: Suppose that $h_j x_j < k_j x_j$. By virtue of Theorem 1(iv) and the axiom (A3), it follows that $h_j x_j <\leq k_j x_j$, i.e. $V^p h\, h_j x_j < V^q k\, k_j x_j$, for all $h, k \in$ UOS and $p, q \in$ Nat. Note that this inequality is also true when one of h_j and k_j is the identification I, by Theorem 1(iv) and our formal convention on the occurrences of I (i.e. if e.g. $h_j = I$, then the string $V^p h$ is regarded as the empty). In view of Theorem 2, there are h' and k' in UOS such that $h_n \cdots h_j x_j \leq V^{n-j-1} h' h_j x_j$ and $k_m \cdots k_j x_j \geq V^{m-j-1} k'\, k_j x_j$, which implies $x < y$.

The sufficiency of (ii) is evident.

To prove the sufficiency of (iii), suppose that $h_j x_j$ and $k_j x_j$ are incomparable. If $h_j x_j \in H(k_j x_j)$ or $k_j x_j \in H(h_j x_j)$, then, on account of Corollary 1(ii) $h_j x_j$ and $k_j x_j$ are comparable, which contradicts the assumption. Thus, $h_j x_j \notin H(k_j x_i)$ and $k_j x_j \notin H(h_j x_j)$. From (A3) it follows that x and y are incomparable.

Necessity: Suppose that there is no index j such that $h_j \neq k_j$. Note that one of h_j and k_j may be the operation I. Then, it is evident that the two canonical representations of x and y are identical and hence $x = y$. Therefore, we can assume that these two canonical representations are different. Thus, there exists the least index j such that $h_j \neq k_j$. Obviously, $j \leq \min\{n, m\} + 1$. Between $h_j x_j$ and $k_j x_j$ there are the following ordering relationships: $h_j x_j = k_j x_j$, $h_j x_j < k_j x_j$, $h_j x_j > k_j x_j$ and $h_j x_j$ and $k_j x_j$ are incomparable. From the proof of the sufficiency, it follows that $x < y$ implies $h_j x_j < k_j x_j$, the incomparability of x and y implies that of $h_j x_j$ and $k_j x_j$ and $x = y$ implies $h_j x_j = k_j x_j$. Also, we have to show that if $h_j x_j = k_j x_j$, then $j = n = m$. In fact, set $u = h_j x_j = k_j x_j$. Since $h_j \neq k_j$, it follows from the axioms (A3) and (A4), that $h_j x_j \leq h' h_j x_j = k' k_j x_j \leq k_j x_j$, for some $h', k' \in$ UOS. Hence, u is a fixed point of the hedge h' and, then, of all hedges in H, by Theorem 1(ii). From the definition of the canonical representations it follows $j = n = m$.

Corollary 2 (uniqueness of the canonical representations). *If x is not a fixed point and u is an arbitrary element in $\mathbf{X}$, then the canonical representation of x w.r.t. u, if it exists, is unique, i.e. if $h_n \cdots h_1 u$ and $k_m \cdots k_1 u$ are two canonical representations of x w.r.t. u, then $n = m$ and $h_i = k_i$, for all $i \leq n$.*

Corollary 3. *If $y \notin H(u)$ and $y \geqslant x$ $(y \leqslant x)$, for some $x \in H(u)$, then $y > H(u)$ $(y < H(u))$.*

Proof. We shall prove the corollary for the case $y \geqslant x$. Suppose first that $u \notin H(y)$. So, u and y are independent. In view of (A3), it follows that u and y are comparable. If $y < u$, then, by applying the axiom (A4) repeatedly, we obtain $y < V^n h'u$, for any $n \in \mathrm{Nat}$ and $h' \in \mathrm{UOS}$. Let $x = h_n \cdots h_1 u$ be a canonical representation of x w.r.t. u. Assuming $h_1 u \geqslant u$, it follows from Theorem 1(iv) and Theorem 2, that $x \geqslant u > y$, which is contrary to the assumption. Thus, we can now suppose that $h_1 u \leqslant u$. If $h_1 \in \mathrm{UOS}$, then $h_1^+ = V$ and, by Theorem 2, $V^m h_1 u \leqslant x$. If $h_1 \notin \mathrm{UOS}$, then $h_1 < h$, for some $h \in \mathrm{UOS}$ and, hence, from Corollary 1(i) it follows also, that $V^m hu \leqslant x$. Consequently, we get $y < x$, which contradicts again the assumption. Thus, $u < y$. By the same argument, we can prove that $z < y$, for all $z \in H(u)$.

Now, suppose that $u \in H(y)$ and $u = h_n \cdots h_1 y$ is the canonical representation of u w.r.t. y. From Corollary 1(ii), it follows that either $u < y$ or $u > y$. The equality of u and y does not arise, since $y \notin H(u)$. On account of Theorem 3, if $u > y$, then $h_1 y > Iy = y$. Because $x \in H(u)$, x has a canonical representation w.r.t. u, $x = k_m \cdots k_1 u$. It is clear that $k_m \cdots k_1 h_n \cdots h_1 y$ is the canonical representation of x w.r.t. y. By Theorem 3, $h_1 y > y$ implies that $x > y$, which is contrary to the assumption. Therefore, we must have $u < y$ and $h_1 y < y$. Consider an element $z' = V^n hu = V^n h\, h_n \cdots h_1 y$, where $h \in \mathrm{UOS}$.

According to Theorem 1, $h_1 y$ is a suffix of a canonical representation of z' w.r.t. y and hence, again by Theorem 3, $z' = V^n hu < y$. By the same argument as above, we can prove that for every $z \in H(u)$, $z < y$. This completes the proof.

Definition 5. Let $\boldsymbol{X} = (X, H, \leqslant)$ be a hedge algebra. An element $a \in X$ is said to be a primary generator of $\boldsymbol{X}$ if $a \notin H(b)$, for every $b \in X$ and $a \neq b$. A primary generator a is said to be positive (negative) if $Va > a$ $(Va < a)$. If G is the set of all generators of $\boldsymbol{X}$ and $H(G) = X$, then $\boldsymbol{X}$ is called a primarily generated hedge algebra.

Theorem 4. *Let the sets H^+ and H^- of a hedge algebra $\boldsymbol{X} = (X, H, \leqslant)$ be linearly ordered. Then, the following statements hold:*

(i) *For every $u \in X$, $H(u)$ is a linearly ordered set.*

(ii) *If $\boldsymbol{X}$ is a primarily generated hedge algebra and the set G of the primary generators of $\boldsymbol{X}$ is linearly ordered, then so is the set $H(G)$. Furthermore, if $u \leqslant v$ and u and v are independent, then $H(u) < H(v)$.*

Proof. By virtue of the axioms (A1) and (A2) and Corollary 2, it can easily be seen that (i) is a consequence of Theorem 3.

To prove (ii), take any two elements x, y in X and suppose that $x \in H(u)$ and $y \in H(v)$, where u and v are independent. From Corollary 3, it follows that $u < H(v)$. Take any element $y \in H(v)$. From (A3) it follows $y \notin H(u)$ and hence, again by Corollary 3, $H(u) < y$, since $u < y$. This shows the validity of (ii).

4. Conclusions

In this paper, we have investigated hedge algebras in order to model the structure of sets of linguistic truth values. It is shown that ordering structures can be taken as a fundamental structure to represent the meanings of vague concepts. Based on these structures, many properties of linguistic hedges can be examined. Since the axiomatization of hedge algebras is based on an intuitive study of linguistic hedges, hedge algebras may reflect the natural structures of domains of linguistic variables. Therefore, in the authors' opinion, they can be taken as a fundamental structure of some fuzzy logic and fuzzy reasoning.

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