IMPULSIVE BOUNDARY VALUE PROBLEMS FOR STURM-LIOUVILLE TYPE DIFFERENTIAL INCLUSIONS

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Abstract In this paper, the authors investigate the existence of solutions of impulsive boundary value problems for Sturm-Liouville type differential inclusions which admit non-convex-valued multifunctions on right hand side. Two results under weaker conditions are presented. The methods rely on a fixed point theorem for contraction multi-valued maps due to Covitz and Nadler and Schaefer's fixed point theorem combined with lower semi-continuous multi-valued operators with decomposable values.

Key words Boundary value problems, contraction multi-valued map, impulsive differential inclusions, measurable selection.

1 Introduction

In this paper, we consider the existence of solutions for the following second-order ordinary differential inclusions with the form

$$(p(t)u'(t))' \in F(t, u(t)), \quad \text{a.e.} \quad t \in [0, T] \setminus \{t_1, t_2, \cdots, t_m\}, \tag{1}$$

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)), \quad \Delta u'|_{t=t_k} = J_k(u(t_k^-)), \ k = 1, 2, \cdots, m, \tag{2}$$

$$\alpha u(0) - \beta \lim_{t\to 0^+} p(t)u'(t) = 0, \qquad \gamma u(T) + \delta \lim_{t\to T^-} p(t)u'(t) = 0, \tag{3}$$

where $F : [0, T] \times R^n \to P(R^n)$ is a multi-valued map, $I_k, J_k \in C(R^n, R^n)$, $p \in C([0, T], R_+)$, α, β, γ and δ are nonnegative reals, $\alpha\delta + \beta\gamma + \gamma\alpha \int_0^T \frac{dr}{p(r)} \neq 0$, $P(R^n)$ is the family of all nonempty subsets of R^n, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively. $\Delta u'|_{t=t_k}$ is defined similarly.

Recently, the impulsive ordinary differential equations or inclusions were considered by Nieto[1], Cai[2], Benchohra et al.[3], Chang and Li[4], and Liu et al.[5] using various tools such as

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fixed point theorem, the Leray-Schauder alternative and the lower and upper solutions method. In [6], by using the fixed point index theory, Sun et al. studied the existence results for the following boundary value problems

$$\begin{cases} (p(t)u'(t))' + g(t)F(t,u(t)) = 0, \quad t \in (0,1), \\ \alpha u(0) - \beta \lim\limits_{t\to 0^+} p(t)u'(t) = 0, \\ \gamma u(1) + \delta \lim\limits_{t\to 1^-} p(t)u'(t) = 0. \end{cases}$$

In this paper, we establish the existence for problem (1)–(3) by using the Schaefer's fixed point theorem[7] under some weaker conditions than those in [1,3,4].

The study of impulsive differential equations and inclusions is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their durations are negligible in comparison with the total duration of the processes and phenomena. That is why the perturbations are considered to take place in the form of impulses. The theory of impulsive differential equations has seen considerable development; see [8], the monographs of Lakshmikantham et al.

This paper will be divided into three sections. In Section 2 we will recall some brief basic definitions and preliminary facts which will be used in the following sections. In Sections 3 we shall establish existence theorems for (1)–(3). Our method involves reducing the existence of solutions to problem (1)–(3) to a search for fixed points of suitable multi-valued maps on an appropriate Banach space. In order to prove the existence of fixed points, we shall rely on a fixed point theorem for contraction multi-valued maps due to [9] and the Schaefer's fixed point theorem.

2 Preliminaries

In this section, we list the notations, definitions, and preliminary facts which are prepared for the rest of this paper.

By $C([0,T],R^n)$ we denote the Banach space of all continuous functions u from $[0,T]$ into R^n with the norm

$$\|u\|_\infty := \sup\{|u(t)| : t \in [0,T]\}.$$

$L^1([0,T],R^n)$ denotes the Banach space of all measurable functions $y : [0,T] \to R^n$ which are Lebesgue integrable with the norm

$$\|y\|_{L^1} := \int_0^T |y(t)|dt.$$

Let E be a nonempty subset of R^n and $N : E \to \mathsf{P}(R^n)$ be a multi-valued map. We say N is lower semi-continuous on E (in brief l.s.c.) if the set $\{x \in E; N(x) \cap C \neq \phi\}$ is open for any open set C in R^n. We call N is closed on E if its graph $G(N) = \{(x,y) \in E \times R^n : y \in N(x)\}$ is closed, N is compact if for each bounded set $B \subset E$, $N(B)$ is relatively compact, and N is bounded if for each bounded set $B \subset E$, $N(B)$ is bounded. The multi-valued map N has a fixed point if there exists $x \in X$ such that $x \in N(x)$.

Let A denote a subset of $[0,T] \times R^n$. We call A is L-B measurable if A belongs to the σ-algebra generated by all sets of the form $L \times B$ where L is Lebesgue measurable in $[0,T]$ and B is Borel measurable in R^n. A subset S of $L^1([0,T],R^n)$ is decomposable if for all $u, v \in S$ and all measurable subset E of $[0,T]$, the function $u\chi_E + v_{[0,T]\setminus E} \in S$, where χ denotes the characteristic function.

For a metric space (X,d), let $P_{cl}(X)=\{Y\in P(X): Y \text{ is closed}\}$, $P_{cp}(X)=\{Y\in P(X): Y \text{ is compact}\}$. We introduce the Hausdorff metric H_d with respect to d, i.e.,

$$H_d(A,B)=\max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $A,B\in P(X)$, $d(a,B)=\inf\limits_{b\in B} d(a,b)$. Then $(P_{cl}(X),H_d)$ is a complete metric space. For a function u defined on $[0,T]$, we define $S_{F,u}$ by

$$S_{F,u}=\big\{v\in L^1([0,T],R^n): v(t)\in F(t,u(t))\big\}.$$

In order to define the solution to (1)–(3), we introduce the space $\Omega=\{u:[0,T]\to R^n: u\in C((t_k,t_{k+1}),R^n),\ u(t_k^-)$ and $u(t_k^+)$ exist and $u(t_k)=u(t_k^-),\ k=1,2,\cdots,m\}$. Then Ω is a Banach space with the norm $\|u\|_\Omega:=\sup\{|u(t)|: t\in[0,T]\}$. Set $\Omega_1=\{u: u\in\Omega \text{ and } u'\in\Omega\}$.

Definition 1 A function $u\in\Omega$ is said to be a solution to (1)–(3) if $pu'\in\Omega_1$ and u satisfy the differential inclusion (1) a.e. on $[0,T]\setminus\{t_1,t_2,\cdots,t_m\}$ and the conditions (2)–(3).

Let Π denote the set of all mappings $\varphi:[0,+\infty)\to[0,+\infty)$, where $\varphi(r)$ satisfies that there exists a strictly increasing mapping $\psi:[0,+\infty)\to[0,+\infty)$ such that $\psi(0)=0$, $\lim\limits_{r\to\infty}\psi(r)=+\infty$, $\psi(r)\le r-\varphi(r)$ for $r>0$.

Definition 2[10] Let (X,d) be a metric space and $f:X\to X$ is said to be a separate contraction mapping if there exists a function $\varphi\in\Pi$ such that $d(f(x),f(y))\le\varphi(d(x,y))$.

If $f:X\to X$ is a contraction mapping with Lipschitz constant k, then f is a separate contraction mapping with $\varphi\in\Pi$ defined by $\varphi(r)=kr$. For more results on separate contraction maps we refer the readers to the paper [10].

Definition 3 Let Y be a separable metric space and $N:Y\to P(L^1([0,T],R^n))$ be a multi-valued operator. We say N has property (BC) if

i) N is lower semi-continuous;

ii) N has nonempty closed and decomposable values.

The following lemmas will be used in the sequel.

Lemma 1[9] *Let (X,d) be a complete metric space. If $N:X\to P_{cl}(X)$ is a contraction mapping, then N admits a fixed point in X.*

Lemma 2[11] *Let Y be a separable metric space and $N:Y\to P(L^1([0,T],R^n))$ be a multi-valued map with the property (BC). Then N has a continuous selection, i.e., there exists a continuous function (single-value) $g:Y\to L^1([0,T],R^n)$ such that $g(u)\in N(u)$ for every $u\in Y$.*

Lemma 3[7] *Let $(X,||\cdot||)$ be a normed space, H be a continuous mapping of X into X which is compact on each bounded subset D of X. Then either*

i) *$x=\lambda Hx$ has a solution in X for $\lambda=1$, or*

ii) *the set $\{x: x=\lambda H(x) \text{ for some } \lambda\in(0,1)\}$ is unbounded.*

3 Existence Results

In this section, we shall prove our main results. Due to the fixed point theorem of Covitz and Nadler[9] for contraction multi-valued maps, we establish the existence for the problem (1)–(3). We give the following assumptions:

(H_1) $F(\cdot,u):[0,T]\to P_{cp}(R^n)$ is measurable for each $u\in R^n$;

(H_2) There exist two constants c_k and c_k' satisfying

$$|I_k(x)-I_k(y)|\le c_k|x-y|,\quad |J_k(x)-J_k(y)|\le c_k'|x-y|,$$

for each $k = 1, 2, \cdots, m$ and all $x, y \in R^n$;

(H$_3$) There exists a function $l \in L^1([0,T], R_+)$ satisfying

$$H_d(F(t,u), F(t,v)) \leq l(t)|u - v|,$$

for a.e. $t \in [0,T]$ and all $u, v \in R^n$, and $d(0, F(t,0)) \leq l(t)$.

Let

$$G(t,s) = \frac{1}{\rho}\begin{cases} \left(\beta + \alpha\displaystyle\int_0^s \frac{dr}{p(r)}\right)\left(\delta + \gamma\displaystyle\int_t^T \frac{dr}{p(r)}\right), & 0 \leq s \leq t \leq T, \\ \left(\beta + \alpha\displaystyle\int_0^t \frac{dr}{p(r)}\right)\left(\delta + \gamma\displaystyle\int_s^T \frac{dr}{p(r)}\right), & 0 \leq t < s \leq T, \end{cases}$$

where

$$\rho = \alpha\delta + \beta\gamma + \gamma\alpha\int_0^T \frac{dr}{p(r)}.$$

Set

$$p_0 = \min_{t\in[0,T]} |p(t)|, \quad \widetilde{p} = \max_{t\in[0,T]} |p(t)|, \quad \text{and} \quad C = \max_{(t,s)\in[0,T]\times[0,T]} \{|G(t,s)|\}.$$

Theorem 1 *Assume that* (H$_1$)–(H$_3$) *hold. Then the problem* (1)–(3) *has a solution in Ω, provided*

$$C\|l\|_{L^1} + \sum_{k=1}^m \left(c_k + \frac{T\widetilde{p}c'_k}{p_0}\right) < 1.$$

Proof We transform the problem (1)–(3) into a fixed point problem. Consider the multi-valued map $M : \Omega \to P(\Omega)$, defined by $M(u) = \{h \in \Omega\}$, where for $t \in [0,T]$, $g \in S_{F,u}$,

$$h(t) = \int_0^T G(t,s)g(s)ds + \sum_{0<t_k<t}\left[I_k(u(t_k^-)) + \int_{t_k}^t \frac{p(t_k)}{p(r)}dr J_k(u(t_k^-))\right].$$

It is clear that the fixed points of M are solutions to the problem (1)–(3). For each $u \in \Omega$, since the set $S_{F,u}$ is nonempty, by assumption (H$_1$), F has a measurable selection. We shall prove that M fulfills the assumptions of Lemma 1. The proof should be given by two steps as below.

Step 1 $M(u) \in P_{cl}(\Omega)$ for each $u \in \Omega$.

In fact, let $\{u_n\}_{n\geq 0} \subset M(u)$ such that $u_n \to u^*$ in Ω and there exists $g_n \in S_{F,u}$ such that for each $t \in [0,T]$,

$$u_n(t) = \int_0^T G(t,s)g_n(s)ds + \sum_{0<t_k<t}\left[I_k(u(t_k^-)) + \int_{t_k}^t \frac{p(t_k)}{p(r)}dr J_k(u(t_k^-))\right].$$

Since F has compact values and (H$_3$) holds, we may pass to a subsequence if necessary to get that g_n converges to $g \in L^1([0,T], R^n)$ and hence $g \in S_{F,u}$. Thus for each $t \in [0,T]$,

$$u_n(t) \to u^* = \int_0^T G(t,s)g(s)ds + \sum_{0<t_k<t}\left[I_k(u(t_k^-)) + \int_{t_k}^t \frac{p(t_k)}{p(r)}dr J_k(u(t_k^-))\right].$$

So $u^* \in M(u)$ and $M(u)$ is closed.

Step 2 M is a contraction map, i.e., there exists a constant $\gamma < 1$ such that for all $u, v \in \Omega$

$$H_d(M(u), M(v)) \leq \gamma \|u - v\|_\Omega.$$

Indeed, let $u, v \in \Omega$ and $h \in M(u)$. Then there exists $g(t) \in F(t, u(t))$ such that for each $t \in [0, T]$,

$$h(t) = \int_0^T G(t,s)g(s)ds + \sum_{0<t_k<t} \left[I_k(u(t_k^-)) + \int_{t_k}^t \frac{p(t_k)}{p(r)} dr J_k(u(t_k^-)) \right].$$

From (H_3) it follows that for each $t \in [0, T]$

$$H_d(F(t, u(t)), F(t, v(t))) \leq l(t) \|u - v\|_\Omega.$$

Hence there exists $w \in F(t, v(t))$ such that

$$|g(t) - w| \leq l(t) \|u - v\|_\Omega, \quad t \in [0, T].$$

Consider $U : [0, T] \to P(R^n)$, given by

$$U(t) = \{ w \in R^n : |g(t) - w| \leq l(t) \|u - v\|_\Omega \}.$$

Since the multi-valued map $V(t) = U(t) \cap F(t, v(t))$ is measurable[12, Theorem III.4], there exists a function $\overline{g}(t)$, which is a measurable selection for V. So $\overline{g}(t) \in F(t, v(t))$ and

$$|g(t) - \overline{g}(t)| \leq l(t) \|u - v\|_\Omega, \quad t \in [0, T].$$

For each $t \in [0, T]$, we define

$$\overline{h}(t) = \int_0^T G(t,s)\overline{g}(s)ds + \sum_{0<t_k<t} \left[I_k(v(t_k^-)) + \int_{t_k}^t \frac{p(t_k)}{p(r)} dr J_k(v(t_k^-)) \right].$$

Then, we obtain

$$\begin{aligned}
|h(t) - \overline{h}(t)| &\leq \int_0^T |G(t,s)||g(s) - \overline{g}(s)|ds + \sum_{0<t_k<t} |I_k(u(t_k)) - I_k(v(t_k))| \\
&\quad + \frac{T\widetilde{p}}{p_0} \sum_{0<t_k<t} |J_k(u(t_k)) - J_k(v(t_k))| \\
&\leq C \int_0^T |g(s) - \overline{g}(s)|ds + \sum_{k=1}^m c_k |u(t_k) - v(t_k)| + \frac{T\widetilde{p}}{p_0} \sum_{k=1}^m c_k' |u(t_k) - v(t_k)| \\
&\leq C \int_0^T l(s) \|u - v\|_\Omega ds + \sum_{k=1}^m c_k \|u - v\|_\Omega + \frac{T\widetilde{p}}{p_0} \sum_{k=1}^m c_k' \|u - v\|_\Omega \\
&\leq \left[C\|l\|_{L^1} + \sum_{k=1}^m \left(c_k + \frac{T\widetilde{p}c_k'}{p_0} \right) \right] \|u - v\|_\Omega.
\end{aligned}$$

Thus

$$\|h - \overline{h}\|_\Omega \leq \left[C\|l\|_{L^1} + \sum_{k=1}^m \left(c_k + \frac{T\widetilde{p}c_k'}{p_0} \right) \right] \|u - v\|_\Omega.$$

From an analogous reasoning by interchanging the roles of u and v, it follows that

$$H_d(M(u), M(v)) \leq \left[C\|l\|_{L^1} + \sum_{k=1}^{m}\left(c_k + \frac{T\widetilde{p}c_k'}{p_0}\right)\right]\|u - v\|_{\Omega}.$$

Thus, M is a contraction map and by Lemma 1, M admits a fixed point which is a solution to the problem (1)–(3). ∎

Based on Schaefer's theorem combined with the selection theorem of Bressan and Colombo for semi-continuous maps with decomposable values, we will establish some new existence results for problem (1)–(3). We consider the following assumptions:

(H_4) Assume that $F : [0, T] \times R^n \to P_{cp}(R^n)$ satisfy

1) $(t, u) \to F(t, u)$ is L-B measurable;

2) $u \to F(t, u)$ is l.s.c. for a.e. $t \in [0, T]$;

(H_5) There exists a function $D \in L^1([0, T], R_+)$ such that for a.e. $t \in [0, T]$ and $u \in \Omega$

$$\|F(t, u(t))\| = \sup\{|v(t)| : v(t) \in F(t, u(t))\} \leq D(t);$$

(H_6) There exists a function $\varphi \in \Pi$ and a constant $b > 0$ such that

$$\|B(u)\|_{\Omega} \leq \varphi(\|u\|_{\Omega}) + b,$$

where $u \in \Omega$ and $B : \Omega \to \Omega$ defined by

$$B(u)(t) = \sum_{0<t_k<t}\left[I_k(u(t_k^-)) + \int_{t_k}^{t}\frac{p(t_k)}{p(r)}dr J_k(u(t_k^-))\right].$$

Lemma 4[13] *Let $F : [0, T] \times R^n \to P_{cp}(R^n)$ be a multi-valued map and the assumptions* (H_4) *and* (H_5) *hold. Then F is lower semi-continuous.*

Theorem 2 *Assume that* (H_4)–(H_6) *hold. Then the problem* (1)–(3) *admits a solution in Ω.*

Proof For a.e. $t \in [0, T]$, set

$$\mathsf{F}(u) = \{w \in L^1([0, T], R^n) : w(t) \in F(t, u(t))\}.$$

Note that (H_4), (H_5), and Lemma 4 imply that F is lower semi-continuous. By Lemma 2, there exists a continuous function $f : \Omega \to L^1([0, T], R^n)$ such that $f(u) \in \mathsf{F}(u)$ for all $u \in \Omega$.

We consider the corresponding problem

$$(p(t)u'(t))' = f(u)(t), \quad \text{a.e.} \quad t \in [0, T] \setminus \{t_1, t_2, \cdots, t_m\}, \tag{4}$$

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)), \quad \Delta u'|_{t=t_k} = J_k(u(t_k^-)), \quad k = 1, 2, \cdots, m, \tag{5}$$

$$\alpha u(0) - \beta \lim_{t\to 0^+} p(t)u'(t) = 0, \quad \gamma u(T) + \delta \lim_{t\to T^-} p(t)u'(t) = 0, \tag{6}$$

It is clear that if $u \in \Omega$ is a solution to (4)–(6), then u is a solution to problem (1)–(3). Transform the problem (4)–(6) into a fixed point problem. Consider the operators $\Lambda : \Omega \to \Omega$, defined by

$$\Lambda(u)(t) = \int_0^T G(t, s)f(u)(s)ds + \sum_{0<t_k<t}\left[I_k(u(t_k^-)) + \int_{t_k}^{t}\frac{p(t_k)}{p(r)}dr J_k(u(t_k^-))\right].$$

We shall prove that Λ fulfills the assumptions of Lemma 3. It is clear that Λ is continuous by the continuousness of functions f, I_k, and J_k. The rest of proof will be given by the following two steps.

Step 1 Λ maps bounded sets into relatively compact sets in Ω.

Let $B_q = \{u \in \Omega : \|u\|_\Omega \le q\}$ be a bounded set in Ω and $u \in B_q$, we obtain, based upon the assumptions (H_5) and (H_6),

$$\begin{aligned}|\Lambda(u)(t)| &\le \int_0^T |G(t,s)||f(u)(s)|ds + \left|\sum_{0<t_k<t}\left[I_k(u(t_k^-)) + \int_{t_k}^t \frac{p(t_k)}{p(r)}dr J_k(u(t_k^-))\right]\right| \\ &\le C\int_0^T |f(u)(s)|ds + \varphi(\|u\|_\Omega) + b \le C\int_0^T D(s)ds + q + b \\ &= C\|D\|_{L^1} + q + b.\end{aligned}$$

Thus, for each $u \in B_q$, we have

$$\|\Lambda(u)\|_\Omega \le C\|D\|_{L^1} + q + b.$$

On the other hand, note that

$$\frac{d}{dt}G(t,s) = \frac{1}{p(t)\rho}\begin{cases} -\gamma\left(\beta + \alpha\displaystyle\int_0^s \frac{dr}{p(r)}\right), & 0 \le s \le t \le T, \\ \alpha\left(\delta + \gamma\displaystyle\int_s^T \frac{dr}{p(r)}\right), & 0 \le t < s \le T. \end{cases}$$

Set

$$J_q = \max\{|J_k(x)| : |x| \le q,\ k = 1, 2, \cdots, m\}, \quad \widetilde{C} = \max_{(t,s)\in[0,T]\times[0,T]}\left\{\left|\frac{d}{dt}G(t,s)\right|\right\}.$$

Then, for $\tau_1, \tau_2 \in [0,T]$, $\tau_1 < \tau_2$ and $\tau_1 \ne t_k$, $k = 1, 2, \cdots, m$, we have

$$\begin{aligned}|\Lambda(u)(\tau_2) - \Lambda(u)(\tau_1)| &\le \int_0^T |G(\tau_2,s) - G(\tau_1,s)||g(s)|ds \\ &\quad + \left|\sum_{\tau_1<t_k<\tau_2} I_k(u(t_k^-))\right| + \frac{\widetilde{p}}{p_0}J_q|\tau_2 - \tau_1| \\ &\le \widetilde{C}\|D\|_{L^1}|\tau_2 - \tau_1| + \left|\sum_{\tau_1<t_k<\tau_2} I_k(u(t_k^-))\right| + \frac{\widetilde{p}}{p_0}J_q|\tau_2 - \tau_1|.\end{aligned}$$

As $\tau_2 \to \tau_1$, the right hand side of the above inequality tends to zero. By Ascoli-Arzela theorem, we easily deduce that $\Lambda(B_q)$ is a relatively compact set.

Step 2 The set $E = \{u \in \Omega : \exists\,\theta \in (0,1), u = \theta\Lambda(u)\}$ is bounded.

Indeed, let $u \in E$, then $u(t) = \theta\Lambda(u)(t)$ for some $\theta \in (0,1)$. Noting the assumption (H_6), we obtain that

$$\begin{aligned}|u(t)| &\le \theta\int_0^T |G(t,s)||f(u)(s)|ds + \theta\varphi(\|u\|_\Omega) + \theta b \\ &\le C\int_0^T |f(u)(s)|ds + \varphi(\|u\|_\Omega) + b \\ &\le C\int_0^T D(s)ds + \varphi(\|u\|_\Omega) + b \\ &= C\|D\|_{L^1} + \varphi(\|u\|_\Omega) + b.\end{aligned}$$

Thus we have

$$\|u\|_\Omega \le C\|D\|_{L^1} + \varphi(\|u\|_\Omega) + b.$$

Since $\varphi \in \Pi$, we can achieve

$$\psi(\|u\|_\Omega) \le \|u\|_\Omega - \varphi(\|u\|_\Omega) \le C\|D\|_{L^1} + b.$$

Hence, from the monotonousness of $\psi(\cdot)$, we get

$$\|u\|_\Omega \le \psi^{-1}(C\|D\|_{L^1} + b) < \infty.$$

This implies that the set E is bounded.

In view of Lemma 3, we deduce that Λ has a fixed point which is a solution to the problem (1)–(3). ∎

Corollary 1 *Assume that* (H$_4$) *and* (H$_5$) *hold. In addition, if one of following conditions holds:*

(A$_1$) *B is a separate contraction mapping;*

(A$_2$) *there exist constants c_k, c'_k, d_k, and b_k such that for each $x \in R^n$*

$$|I_k(x)| \le c_k|x| + b_k, \quad |J_k(x)| \le c'_k|x| + d_k, \quad \sum_{k=1}^{m}\left(c_k + \frac{T\widetilde{p}}{p_0}c'_k\right) < 1.$$

Then the problem (1)–(3) *admits a solution in Ω.*

Proof We only show the conditions (A$_1$) and (A$_2$) imply the assumption (H$_6$) respectively. First, if the condition (A$_1$) holds, then there exists a function $\varphi \in \Pi$ such that

$$\|B(u) - B(v)\|_\Omega \le \varphi(\|u - v\|_\Omega),$$

where $u, v \in \Omega$. Let $v = 0$ and $b := \sum_{k=1}^{m}(|I_k(0)| + \frac{T\tilde{p}}{p_0}|J_k(0)|)$, we arrive at the assumption (H$_6$) immediately.

If the condition (A$_2$) holds, for each $t \in [0, T]$, we have

$$\begin{aligned}
|B(u)(t)| &= \left|\sum_{0<t_k<t}\left[I_k(u(t_k^-)) + \int_{t_k}^{t}\frac{p(t_k)}{p(r)}dr J_k(u(t_k^-))\right]\right| \\
&\le \sum_{k=1}^{m}\left(c_k|u(t_k^-)| + \frac{T\widetilde{p}}{p_0}c'_k|u(t_k^-)|\right) + \sum_{k=1}^{m}\left(b_k + \frac{T\widetilde{p}}{p_0}d_k\right) \\
&\le \sum_{k=1}^{m}\left(c_k + \frac{T\widetilde{p}}{p_0}c'_k\right)\|u\|_\Omega + \sum_{k=1}^{m}\left(b_k + \frac{T\widetilde{p}}{p_0}d_k\right).
\end{aligned}$$

Thus

$$\|B(u)\|_\Omega \le \sum_{k=1}^{m}\left(c_k + \frac{T\widetilde{p}}{p_0}c'_k\right)\|u\|_\Omega + \sum_{k=1}^{m}\left(b_k + \frac{T\widetilde{p}}{p_0}d_k\right).$$

By $\sum_{k=1}^{m}(c_k + \frac{T\tilde{p}}{p_0}c'_k) < 1$, we define a function as $\varphi(r) = \sum_{k=1}^{m}(c_k + \frac{T\tilde{p}}{p_0}c'_k)r$ for $r > 0$. Then φ fulfills the assumption (H$_6$). Thus, from Theorem 2 it follows that the problem (1)–(3) admits a solution in Ω. The proof of Corollary 1 is complete. ∎

From the above proof of Theorem 2, we immediately obtain the following corollaries.

Corollary 2 *Assume that* (H$_2$), (H$_4$), *and* (H$_5$) *hold. Then the problem* (1)–(3) *admits a solution in Ω, provided*

$$\sum_{k=1}^{m}\left(c_k+\frac{T\widetilde{p}}{p_0}c'_k\right)<1.$$

Corollary 3 *Assume that* (H$_4$) *and* (H$_5$) *hold. Then the problem* (1)–(3) *admits a solution in Ω, provided $I_k, J_k (k=1,2,\cdots,m)$ are bounded.*

Theorem 3 *Assume that* (H$_4$), (H$_5$), *and the following conditions hold:*

(H$_7$) $0\le \overline{\lim\limits_{|x|\to\infty}}\frac{|I_k(x)|}{|x|}\le c_k,\ k=1,2,\cdots,m;$

(H$_8$) $0\le \overline{\lim\limits_{|x|\to\infty}}\frac{|J_k(x)|}{|x|}\le c'_k,\ k=1,2,\cdots,m.$

Then the problem (1)–(3) *admits a solution in Ω, provided*

$$\sum_{k=1}^{m}\left(c_k+\frac{T\widetilde{p}}{p_0}c'_k\right)<1.$$

Proof Firstly, we transform the problem (4)–(6) into a fixed point problem. Consider the operators $A:\Omega\to\Omega$ defined by

$$A(u)(t)=\int_0^T G(t,s)f(u)(s)ds+\sum_{0<t_k<t}\left[I_k(u(t_k))+\int_{t_k}^{t}\frac{p(t_k)}{p(r)}drJ_k(u(t_k))\right].$$

We shall prove that A fulfills the assumptions of Lemma 3. Based on the step 1 in the proof of Theorem 2, we only need to prove the set $E=\{u\in\Omega:\exists\lambda\in(0,1),u=\lambda A(u)\}$ is bounded. From conditions (H$_7$) and (H$_8$), we conclude that there exist positive constants ε_k, ε'_k $(k=1,2,\cdots,m)$ and C such that for all $|u|>C$

$$|I_k(u)|\le(c_k+\varepsilon_k)|u|,\quad |J_k(u)|\le(c'_k+\varepsilon'_k)|u|,$$

and

$$\sum_{k=1}^{m}\left(c_k+\varepsilon_k+\frac{T\widetilde{p}}{p_0}(c'_k+\varepsilon'_k)\right)<1.$$

We prove E is bounded by contradiction. Suppose that there exists a sequence $u_n\in E$ such that $\|u_n\|_\Omega\to+\infty$ as $n\to+\infty$. For any constant $C_1>C$ and each n, we define

$$\begin{aligned}
&D_n=\{t:t\in[0,T],|u_n(t)|\le C_1\}, && L=\max\{|I_k(x)|:|x|\le C_1\},\\
&L'=\max\{|J_k(x)|:|x|\le C_1\}, && L_n=\max\{|I_k(u_n(t))|:t\in D_n\},\\
&L'_n=\max\{|J_k(u_n(t))|:t\in D_n\}.
\end{aligned}$$

It is clear that $L_n\le L$ and $L'_n\le L'$ for each n. Then, for each $u\in E$ and n, we have

$$\begin{aligned}
|u_n(t)|&\le\int_0^T|G(t,s)||f(u_n)(s)|ds+\left|\sum_{0<t_k<t}\left[I_k(u_n(t_k))+\int_{t_k}^{t}\frac{p(t_k)}{p(r)}drJ_k(u_n(t_k))\right]\right|\\
&\le C\int_0^T|f(u_n)(s)|ds+\sum_{t_k\in D_n}|I_k(u_n(t_k^-))|+\sum_{t_k\in[0,T]\setminus D_n}|I_k(u_n(t_k^-))|\\
&\quad+\frac{T\widetilde{p}}{p_0}\left[\sum_{t_k\in D_n}|J_k(u_n(t_k^-))|+\sum_{t_k\in[0,T]\setminus D_n}|J_k(u_n(t_k^-))|\right]
\end{aligned}$$

$$\leq C\int_0^T D(s)ds + mL + \frac{T\widetilde{p}}{p_0}mL' + \sum_{t_k\in[0,T]\setminus D_n} |I_k(u_n(t_k^-))|$$
$$+\frac{T\widetilde{p}}{p_0}\sum_{t_k\in[0,T]\setminus D_n} |J_k(u_n(t_k^-))|$$
$$\leq C\|D\|_{L^1} + mL + \frac{T\widetilde{p}}{p_0}mL' + \sum_{k=1}^m \left(c_k + \varepsilon_k + \frac{T\widetilde{p}}{p_0}(c_k' + \varepsilon_k')\right)\|u_n\|_\Omega.$$

Thus we have

$$\|u_n\|_\Omega \leq C\|D\|_{L^1} + mL + \frac{T\tilde{p}}{p_0}mL' + \sum_{k=1}^m \left(c_k + \varepsilon_k + \frac{T\tilde{p}}{p_0}(c_k' + \varepsilon_k')\right)\|u_n\|_\Omega.$$

This implies

$$\|u_n\|_\Omega \leq \frac{C\|D\|_{L^1} + mL + \frac{T\tilde{p}}{p_0}mL'}{1-\sum_{k=1}^m\left(c_k + \varepsilon_k + \frac{T\tilde{p}}{p_0}(c_k'+\varepsilon_k')\right)} < \infty.$$

Since $\|u_n\|_\Omega \to +\infty$ as n goes to infinity, we arrive at a contradiction. Thus set E is bounded. The proof of Theorem 3 is complete. ▮

Corollary 4 *Assume that* (H_4) *and* (H_5) *hold. In addition, if one of following conditions holds:*

(A_3) $\lim\limits_{|x|\to\infty}\frac{|I_k(x)|}{|x|} = 0,\ \lim\limits_{|x|\to\infty}\frac{|J_k(x)|}{|x|} = 0,\ k = 1, 2, \cdots, m;$

(A_4) *there exist constants* $c_k, c_k', b_k, b_k' \in R$ *and* $\alpha_k, \alpha_k' \in [0, 1)$ *such that for each* $x \in R^n$

$$|I_k(x)| \leq c_k|x|^{\alpha_k} + b_k, \quad |J_k(x)| \leq c_k'|x|^{\alpha_k'} + b_k',$$

then the problem (1)–(3) *admits a solution in* Ω*.*

4 Example

Our results may be applied to a class of nonlinearities.

Example 1 Consider the following scholar differential inclusions:

$$u''(t) \in \left[-\frac{1+t^2}{9}|u(t)| - \frac{1}{4},\ \ 0\right], \quad t \in [0,1]\setminus\left\{\frac{1}{2}\right\}, \tag{7}$$
$$\Delta u|_{t=\frac{1}{2}} = I_1,\ \Delta u'|_{t=\frac{1}{2}} = J_1, \tag{8}$$
$$u(0) - 3u'(0) = 0,\ \ u'(1) = 0, \tag{9}$$

where I_1 and J_1 are two constants.

Conclusion 1 The problem (7)–(9) has a solution.

In fact, take $\alpha = 1$, $\beta = 3$, $\gamma = 0$, and $\delta = 1$. Let $l(t) = \frac{1+t^2}{9}$ and the multi-valued function F be defined by

$$F(t,x) = \left[-\frac{1+t^2}{9}|x| - \frac{1}{4},\ 0\right], \quad \text{for } (t,x) \in [0,1]\times R.$$

By direct computations, we see that the assumptions (H_1) and (H_3) fulfill. Furthermore, the condition (H_2) holds with $c_1 = c_1' = 0$. On the other hand,

$$G(t,s) = \begin{cases} 3+s, & 0 \leq s \leq t \leq 1, \\ 3+t, & 0 \leq t < s \leq 1, \end{cases}$$

then $C = \max\limits_{(t,s)\in[0,1]\times[0,1]}\{|G(t,s)|\} = 4$ and $\|l\|_{L^1} = \frac{4}{27}$. Thus $C\|l\|_{L^1} < 1$. From Theorem 1 it follows that the problem (7)–(9) has a solution. Moreover, let

$$u(t) = \begin{cases} \sin\left(\dfrac{t}{3}\right) + 1, & t \in \left(0, \dfrac{1}{2}\right), \\ \dfrac{1}{8}t(2-t), & t \in \left(\dfrac{1}{2}, 1\right). \end{cases}$$

By direct computations, we see that $u(t)$ is a solution to problem (7)–(9) with $I_1 = -(\frac{29}{32} + \sin(\frac{1}{6}))$ and $J_1 = \frac{1}{8} - \frac{1}{3}\cos(\frac{1}{6})$.

Remark 1 Impulsive differential equations and inclusions under the conditions (H_2), (A_2)–(A_4) have been widely studied by many authors, see for instance [1,3,4]. Obviously, (A_2) and (H_2) are special cases of (H_6); (A_3) and (A_4) are special cases of (H_7) and (H_8).

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