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Transformation methods for finding multiple roots of nonlinear equations ☆

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ABSTRACT

In this paper, to estimate a multiple root p of an equation $f(x) = 0$, we transform the function $f(x)$ to a hyper tangent function combined with a simple difference formula whose value changes from -1 to 1 as x passes through the root p. Then we apply the so-called numerical integration method to the transformed equation, which may result in a specious approximate root. Furthermore, in order to enhance the accuracy of the approximation we propose a Steffensen-type iterative method, which does not require any derivatives of $f(x)$ nor is quite affected by an initial approximation. It is shown that the convergence order of the proposed method becomes cubic by simultaneous approximation to the root and its multiplicity. Results for some numerical examples show the efficiency of the new method.

1. Introduction

We consider the problem of finding a root p of multiplicity $m \geqslant 1$, unknown, of a nonlinear equation

$$f(x) = 0, \tag{1}$$

on an interval (a, b). There is vast literature on the numerical methods for finding multiple roots [1–12]. As long as the author is aware, even when the multiplicity m is known, most of the higher order methods have a defect that they need derivatives of $f(x)$ (see for example [4,7–9]). For the case that the multiplicity m is unknown, several iterative methods which are based on some transformations of $f(x)$ have been developed [5,6,10,11].

In particular, we notice the literature [10] which used a quadratically convergent Newton-like method without derivatives as follows:

$$p_{k+1} = p_k - \frac{G(p_k)^2}{tG(p_k)^2 + G(p_k) - G(p_k - G(p_k))}, \quad k \geqslant 0, \tag{2}$$

where t should be chosen such that the denominator is largest in magnitude and $G(x)$ is a transformation of $f(x)$ so that the multiple zero p of $f(x)$ reduces to a simple zero of $G(x)$. In fact, therein $G(x)$ is defined by:

$$G(x) = \begin{cases} \frac{f(x)^2}{\delta + f(x+f(x)) - f(x)}, & \text{if } f(x) \neq 0, \\ 0, & \text{if } f(x) = 0, \end{cases} \tag{3}$$

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where $\delta = \text{sgn}(f(x + f(x)) - f(x))f(x)^2$. It should be noted that every Newton-like iterative methods such as the aforementioned method will depend on the initial approximation, and thus choosing a proper initial approximation is a very important problem in numerical implementation.

Motivated by the method proposed by Parida and Gupta [10], this paper aims at developing a new transformation method for finding multiple roots which does not require any derivatives and is free from the problem of choosing an appropriate initial approximation.

The paper is organized as follows. In the next section we define a hyper tangent function combined with a difference formula for $f(x)$ as given in (6). The transformed function $H(\epsilon;x)$ becomes closer to the signum function $\text{sgn}(x-p)$ as ϵ goes to 0. Thus the numerical integration method (NIM) proposed in [13] is applicable to $H(\epsilon;x)$, which directly produces an approximate root q by the form of (9).

Since the accuracy of the approximation q may not be sufficient in general, in Section 3 we use another transformation $K(\epsilon;x)$ in (10) so that the multiple zero of $f(x)$ reduces to a simple zero of $K(\epsilon;x)$. Then we propose an iteration formula, by a similar form to the Steffensen's method [10,14], which includes a parameter $\mu \neq 0$ and has a quadratic convergence order regardless of μ. Moreover, employing an approximation m_k to the multiplicity m, we develop a third order iterative method.

In Section 4 we take some examples to show the efficiency of the new method. One can see that numerical results of the proposed methods are consistent with the convergence analysis investigated in Section 3.

2. A hyper tangent transformation for the NIM

It can be noted that the existing Newton-like iterative methods, such as that in (2), would fail to converge to the original root p when the initial guess is not properly chosen or some iterate p_k is located on a troublesome region within the given interval. Moreover, the transformation $G(x)$ defined by (3) may have unexpected zeros besides the searched zero of $f(x)$. See for example Fig. 1, where $f(x)$ is chosen as

$$f(x) = e^x - 1 - x + \frac{x^2}{2}, \quad -5 \leqslant x \leqslant 2, \tag{4}$$

which has a zero $p = 0$ with the multiplicity $m = 2$ (for this case numerical implementation of finding the zero and its multiplicity is investigated by Example 1 in Section 4). Fig. 1 shows troublesome regions near the endpoints -5 and 2, where the Newton-like method will fail to converge to the root $p = 0$. This problem results from the fact that the values of $f(x + f(x))$ are extremely large there. In addition, unexpected discontinuity of $G(x)$ is observed because of the rapid change of the signs of $f(x + f(x)) - f(x)$ near the point $x = -2.255$.

To overcome the problems mentioned above, we propose a simple difference formula of $f(x)$ as:

$$f_\epsilon(x) = f(x + \epsilon f(x)) - f(x), \tag{5}$$

for $\epsilon > 0$. Some basic properties of $f_\epsilon(x)$ are included in the following theorem.

Theorem 1. *Let an equation $f(x) = 0$ have a root p of multiplicity $m \geqslant 1$ on an interval (a,b) and assume that $f(x) \in C^1(c,d)$ and $f'(x) \neq 0$ on $(c,d)\backslash\{p\}$ for some c and d such that $c < a < b < d$.*

Then we have:

(1) *For any $\epsilon > 0$, $f_\epsilon(x) = O((x-p)^{2m-1})$ near $x = p$*
(2) *For a sufficiently small $\epsilon > 0$, $f_\epsilon(x) \in C^1(a,b)$ and*

$$\text{sgn}(f_\epsilon(x)) = \text{sgn}(x-p), \quad a < x < b.$$

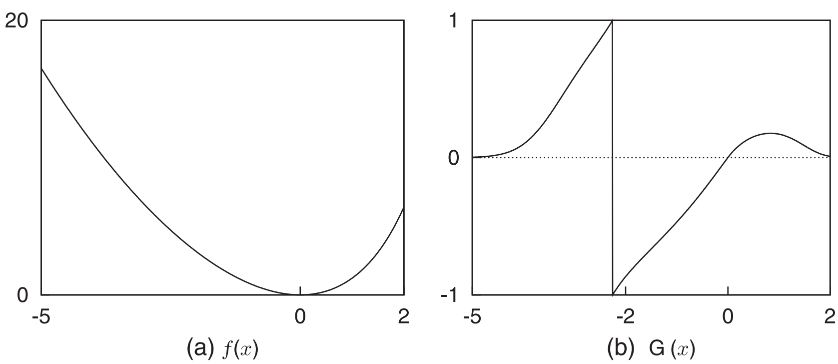

Fig. 1. Graphs of a function $f(x)$ having a multiple zero $p = 0$ and its transformation $G(x)$ defined in (3) for the existing method (2).

Proof. From the assumptions we can write $f(x)$ as

$$f(x) = (x-p)^m g(x),$$

where $g(x) \neq 0$ on (c,d). On the other hand $f_\epsilon(x)$ can be represented by:

$$f_\epsilon(x) = \epsilon f(x) \frac{f(x+\epsilon f(x)) - f(x)}{\epsilon f(x)}.$$

For x near the root p, since $\epsilon f(x) \sim 0$ regardless of ϵ fixed, we have

$$f_\epsilon(x) \sim \epsilon f(x) f'(x) = O((x-p)^{2m-1}),$$

as asserted in (1).

For the property (2) if ϵ is sufficiently small such that $x + \epsilon f(x) \in (c,d)$ for all $x \in (a,b)$ then $f_\epsilon(x) \in C^1(a,b)$ from the definition of $f_\epsilon(x)$ in (5). Moreover, we assume $\epsilon f(x)$ is so small that $f_\epsilon(x) \approx \epsilon f(x) f'(x)$ for any $x \in (a,b)$. Thus, the assumptions that $f(x) \in C^1(c,d)$ and $f'(x) \neq 0$ on $(c,d)\backslash\{p\}$ imply that $\mathrm{sgn}\,(f_\epsilon(x)) = \mathrm{sgn}(x-p)$. □

From the property (2) in Theorem 1 one can see that the role of ϵ in $f_\epsilon(x)$ is to assure the uniqueness of the root on the given interval (a,b).

To adjust the function $f_\epsilon(x)$ to the non-iterative root finding method [13] which we call the numerical integration method (NIM), we consider a transformation based on the hyper tangent function as follows:

$$H(x) = H(\epsilon; x) = \begin{cases} \tanh[1/f_\epsilon(x)], & \text{if } f(x) \neq 0, \\ 0, & \text{if } f(x) = 0. \end{cases} \tag{6}$$

For sufficiently small $\epsilon > 0$, since

$$f_\epsilon(x) \approx \epsilon f(x) f'(x), \quad a < x < b \tag{7}$$

and $\mathrm{sgn}(f_\epsilon(x)) = \mathrm{sgn}(x-p)$ from Theorem 1(2), we have

$$H(x) \approx \mathrm{sgn}(x-p), \quad a < x < b. \tag{8}$$

Thus the transformation $H(x)$ becomes closer to the signum function $\mathrm{sgn}(x-p)$ as ϵ goes to 0.

We now suppose that ϵ is so small that $H(x)$ is sufficiently close to $\mathrm{sgn}(x-p)$. Then using the NIM proposed in [13] and noting that $H(a) < 0$, we may obtain an approximation to the root of the transformed equation $H(x) = 0$ as below.

$$q = \frac{a+b}{2} - \frac{1}{2}\widetilde{I}(H(x)), \tag{9}$$

where $\widetilde{I}(H(x))$ denotes numerical evaluation of an integral

$$I(H(x)) = \int_a^b H(x)dx.$$

Since the root p satisfies

$$p = \frac{a+b}{2} - \frac{1}{2}I(\mathrm{sgn}(x-p)),$$

the accuracy of the approximation q to the root p depends on the accuracy of the numerical integration $\widetilde{I}(H(x))$ with sufficiently small $\epsilon > 0$.

Fig. 2(a)–(c) include, for example, graphs of the function $f(x)$ defined in (4) and its transformations $f_\epsilon(x)$ and $H(x)$.

3. Iterative methods for finding roots and multiplicities

Since the integrand $H(x)$ in (9) is a step-like function for small ϵ, traditional quadrature rules may not be sufficiently accurate. Thus to obtain further accurate approximation to the root we first consider another transformation of $f(x)$ introduced in [15] as:

$$K(x) = K(\epsilon; x) = \begin{cases} \frac{\epsilon f(x)^2}{f_\epsilon(x)}, & \text{if } f(x) \neq 0, \\ 0, & \text{if } f(x) = 0. \end{cases} \tag{10}$$

Since $f_\epsilon(x) \sim \epsilon f(x) f'(x)$ near the root p of multiplicity $m \geqslant 1$ of the original equation $f(x) = 0$, one can see that

$$K(x) \sim \frac{f(x)}{f'(x)} = \frac{x-p}{m} + O((x-p)^2), \tag{11}$$

for x near p. Therefore, the multiple zero p of $f(x)$ has been transformed to a simple zero of the function $K(x)$ regardless of ϵ (see Fig. 2(d) for example).

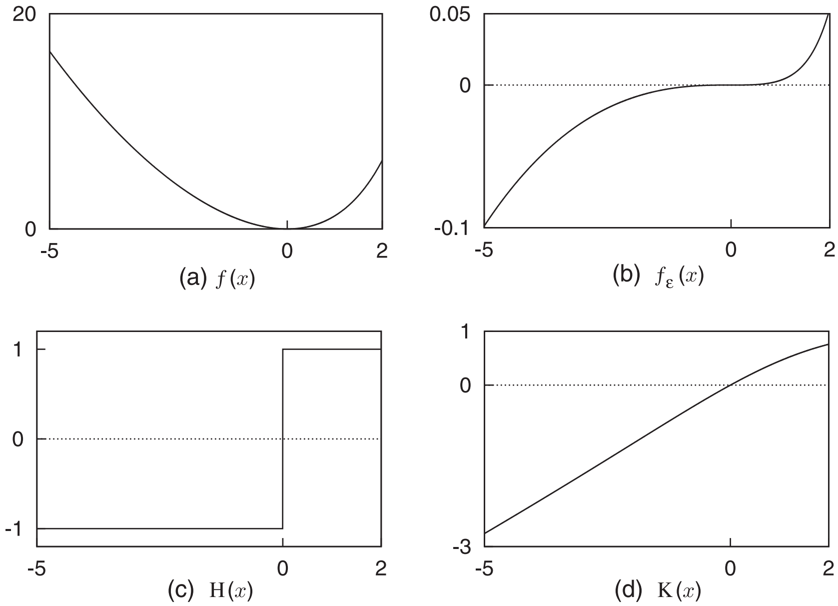

Fig. 2. Graphs of a function $f(x)$ having a double zero $p = 0$ and its transformations, $f_\epsilon(x)$, $H(x)$ and $K(x)$ with $\epsilon = 0.001$.

Next, for the transformed equation $K(x) = 0$ having a simple root, we propose a Steffensen-type iterative formula

$$p_{k+1} = p_k - \frac{\mu K(p_k)^2}{K(p_k + \mu K(p_k)) - K(p_k)}, \quad k \geqslant 0, \tag{12}$$

for a real number $\mu \neq 0$. Of course, the initial approximation may be chosen as $p_0 = q$ via the NIM (9) because q will be rather close to the root p.

Theorem 2. *For a function $f(x)$ having a root p of multiplicity $m \geqslant 1$, assume that $f''(x)$ is continuous and $f'''(x)$ exists on a neighborhood of p and let $e_k = p_k - p$ denote the error of the iterate p_k obtained from (12). Then for p_k which is sufficiently close to p we have*

$$e_{k+1} = \left[\frac{m+\mu}{2} K''(p)\right] e_k^2 + O(e_k^3).$$

Proof. Suppose that p_k is sufficiently close to p such that both p_k and $p_k + \mu K(p_k)$ are included in the neighborhood of p where $f''(x)$ is continuous and $f'''(x)$ exists. Subtracting the root p from both sides of (12) and using the Taylor series expansion of the denominator, we have

$$\begin{aligned} e_{k+1} &= e_k - \frac{\mu K(p_k)^2}{K'(p_k)\mu K(p_k) + \frac{1}{2}K''(p_k)\mu^2 K(p_k)^2 + O(K(p_k)^3)} = e_k - \frac{K(p_k)}{K'(p_k) + \frac{1}{2}K''(p_k)\mu K(p_k) + O(K(p_k)^2)} \\ &= e_k - \frac{K(p+e_k)}{K'(p+e_k) + \frac{1}{2}K''(p+e_k)\mu K(p+e_k) + O(K(p+e_k)^2)}. \end{aligned}$$

Since $K(p+e_k) = K'(p)e_k + \frac{1}{2}K''(p)e_k^2 + O(e_k^3)$, $K'(p+e_k) = K'(p) + K''(p)e_k + O(e_k^2)$ and $K''(p+e_k)\mu K(p+e_k) = \mu K'(p)K''(p)e_k + O(e_k^2)$, it follows that

$$\begin{aligned} e_{k+1} &= e_k - \frac{K'(p)e_k + \frac{1}{2}K''(p)e_k^2 + O(e_k^3)}{K'(p) + \{\frac{1}{2}\mu K'(p) + 1\}K''(p)e_k + O(e_k^2)} = e_k - \frac{e_k + \frac{K''(p)}{2K'(p)}e_k^2 + O(e_k^3)}{1 + \{\frac{1}{2}\mu K'(p) + 1\}\frac{K''(p)}{K'(p)}e_k + O(e_k^2)} \\ &= e_k - \left\{e_k + \frac{K''(p)}{2K'(p)}e_k^2 + O(e_k^3)\right\}\left\{1 - \left(\frac{1}{2}\mu K'(p) + 1\right)\frac{K''(p)}{K'(p)}e_k + O(e_k^2)\right\} = \{\mu K'(p) + 1\}\frac{K''(p)}{2K'(p)}e_k^2 + O(e_k^3). \end{aligned}$$

In the third equation we used the series expansion $\frac{1}{1+\xi} = 1 - \xi + O(\xi^2)$ since e_k is sufficiently small by the assumption. Finally, $K'(p) = \frac{1}{m}$ from (11), and thus we have

$$e_{k+1} = \left[\frac{m+\mu}{2} K''(p)\right] e_k^2 + O(e_k^3). \qquad \square$$

Corollary 2.1. *Let $f(x) = (x-p)^m g(x)$ with $g(x) \neq 0$ on a neighborhood of p. Then we have*

$$e_{k+1} = -\left[\frac{m+\mu}{m^2}\frac{g'(p)}{g(p)}\right]e_k^2 + O(e_k^3).$$

Proof. Since

$$K(x) \sim \frac{f(x)}{f'(x)} = \frac{(x-p)g(x)}{mg(x)+(x-p)g'(x)} = \frac{x-p}{m}\left\{1 - \frac{(x-p)g'(x)}{mg(x)} + O((x-p)^2)\right\},$$

for x near p, it follows that

$$K'(x) \sim \frac{1}{m} - \frac{2(x-p)g'(x)}{m^2 g(x)} + O((x-p)^2)$$

and

$$K''(x) \sim -\frac{2g'(x)}{m^2 g(x)} + O(x-p),$$

so that $K''(p) = -2g'(p)/(m^2 g(p))$. Substituting this into Theorem 2, we have the required formula above. □

Corollary 2.2. *If the multiplicity m of the root p is known, then the convergence order of the iterative method (12) with $\mu = -m$ is at least cubic.*

Proof. When $\mu = -m$ it is clear that $e_{k+1} = O(e_k^3)$ from Theorem 2. □

Therefore, when the multiplicity m is known, the formula (12) with $\mu = -m$ results in a cubic iterative formula as below.

$$p_{k+1} = p_k - \frac{mK(p_k)^2}{K(p_k) - K(p_k - mK(p_k))}, \quad k \geqslant 0. \tag{13}$$

When the multiplicity $m \geqslant 1$ of the root p is unknown in general, referring to the equation for $K'(x)$ in the proof of Corollary 2.1 which implies that $K'(p) = \frac{1}{m}$, we have

$$m = \frac{1}{K'(p)} \approx \frac{K(p_k)}{K(p_k) - K(P_k - K(p_k))},$$

for p_k which is sufficiently close to p. Thus we can use the following approximation m_k to m.

$$m_k = \frac{K(p_k)}{K(p_k) - K(p_k - K(p_k))}, \quad k \geqslant 0. \tag{14}$$

Then substituting m_k into (13) for m, we have

$$p_{k+1} = p_k - \frac{m_k K(p_k)^2}{K(p_k) - K(p_k - m_k K(p_k))}, \quad k \geqslant 0, \tag{15}$$

which can be seen as the result obtained by taking $\mu = -m_k$ in (12).

Theorem 3. *Under the assumptions for f(x) given in Theorem 2, the iterates $\{m_k\}_{k=0}^{\infty}$ and $\{p_k\}_{k=0}^{\infty}$ obtained by (14) and (15) at least cubically converge to the multiplicity m and the root p, respectively.*

Proof. Since $m = 1/K'(p)$, from (14) and the Taylor's theorem for $K(p_k - K(p_k))$ it follows that

$$m - m_k = \frac{1}{K'(p)} - \frac{K(p_k)}{K(p_k) - K(p_k - K(p_k))} = \frac{1}{K'(p)} - \frac{K(p_k)}{K'(p_k)K(p_k) - \frac{1}{2}K''(\xi)K(p_k)^2} = \frac{K'(p_k) - K'(p) - \frac{1}{2}K''(\xi)K(p_k)}{K'(p)\{K'(p_k) - \frac{1}{2}K''(\xi)K(p_k)\}},$$

where ξ is a point between p_k and $p_k - K(p_k)$. But, by the mean value theorem in the numerator, $K'(p_k) - K'(p) = K''(\eta)e_k$ and $K(p_k) = K'(\zeta)e_k$ for some η and ζ between p and p_k. Therein $e_k = p_k - p$ and it is assumed to be sufficiently small.

Thus we have

$$m - m_k = \frac{K''(\eta) - \frac{1}{2}K''(\xi)K'(\zeta)}{K'(p)\{K'(p_k) - \frac{1}{2}K''(\xi)K'(\zeta)e_k\}}e_k = Ce_k,$$

for a constant C. Therefore, from Theorem 2 with $\mu = -m_k$,

$$e_{k+1} = \left[\frac{m-m_k}{2}K''(p)\right]e_k^2 + O(e_k^3) = O(e_k^3).$$

In addition, under the assumption that $C \neq 0$

$$m - m_{k+1} = Ce_{k+1} = C'e_k^3 = C''(m-m_k)^3,$$

for some constants C' and C''. This completes the proof. □

On the other hand, we may consider the iteration (14) combined with the iteration (12) for p_k which does not include m_k. In this case we can see that the convergence order of m_k is quadratic, that is, $m - m_{k+1} = O((m-m_k)^2)$ as $e_{k+1} = O(e_k^2)$, for any $\mu \neq 0$, from Theorem 2.

In order to avoid the ambiguity in choosing the parameter ϵ in the transformation $K(x)$, we suggest a choice of ϵ as below.

$$\epsilon = \beta e^{-\alpha}, \tag{16}$$

where

$$\alpha = \max\{|f(a)|, |f(b)|\}, \quad \beta = \min\{|f(a)|, |f(b)|\},$$

which is automatically set with respect to the given function $f(x)$. In practice, we can see that this choice of ϵ is effective for all the examples given in the next section.

4. Numerical examples

In this paper we have proposed two iterative methods for finding both multiple roots and related multiplicities without any derivatives. One is the second order iterative method using (12) for p_k and (14) for m_k, which we call the *parallel* method. The other is the third order method using (15) and (14) for estimating p_k and m_k recursively, which we call the *correlated* method.

In numerical implementation for the following examples we have used a programming package *Mathematica* V.7. Moreover we take ϵ given in (16) to define the functions $H(x)$ and $K(x)$, and set $\mu = \epsilon$ in the iteration (12).

We consider following four examples, where each multiplicity m is assumed to be unknown.

Example 1.

$$f(x) = e^x - 1 - x + \frac{x^2}{2} = 0, \quad -5 < x < 2,$$

with a root $p = 0$ of multiplicity $m = 2$.

Example 2.

$$f_2(x) = (x-1)^3(x^2 - 5x + 6) = 0, \quad 0 < x < 1.5,$$

with a root $p = 1$ of multiplicity $m = 3$.

Table 1
Numerical errors for the root p and the multiplicity m of the parallel method (12) with $\mu = \epsilon$ and the correlated method (15). The initial approximation is chosen by $p_0 = q$ based on the NIM in (9).

Examples	Iterations	Parallel method ($\mu = \epsilon$)		Correlated method	
	k	$\lvert p_k - p\rvert$	$\lvert m_k - m\rvert$	$\lvert p_k - p\rvert$	$\lvert m_k - m\rvert$
$f_1(x) = 0$	0	1.5×10^{-15}	3.9×10^{-16}	1.5×10^{-15}	3.9×10^{-16}
	1	2.0×10^{-31}	5.0×10^{-32}	1.3×10^{-47}	3.3×10^{-48}
	2	3.4×10^{-63}	8.4×10^{-64}	7.7×10^{-144}	1.9×10^{-144}
	3	9.5×10^{-127}	2.4×10^{-127}	1.6×10^{-432}	3.6×10^{-433}
$f_2(x) = 0$	0	1.2×10^{-4}	3.0×10^{-4}	1.2×10^{-4}	3.0×10^{-4}
	1	7.0×10^{-9}	1.8×10^{-8}	2.8×10^{-13}	7.0×10^{-13}
	2	2.5×10^{-17}	6.2×10^{-17}	3.6×10^{-39}	9.0×10^{-39}
	3	3.1×10^{-34}	7.7×10^{-34}	7.8×10^{-117}	1.9×10^{-116}
$f_3(x) = 0$	0	2.4×10^{-4}	1.2×10^{-8}	2.4×10^{-4}	1.2×10^{-8}
	1	6.4×10^{-13}	9.1×10^{-26}	5.6×10^{-22}	6.9×10^{-44}
	2	1.3×10^{-38}	3.5×10^{-77}	4.1×10^{-110}	3.7×10^{-220}
	3	9.7×10^{-116}	2.1×10^{-231}	8.7×10^{-551}	–
$f_4(x) = 0$	0	1.1×10^{-5}	1.5×10^{-5}	1.1×10^{-5}	1.5×10^{-5}
	1	1.9×10^{-11}	2.6×10^{-11}	2.6×10^{-17}	3.5×10^{-17}
	2	5.4×10^{-23}	7.5×10^{-23}	3.1×10^{-52}	4.3×10^{-52}
	3	4.5×10^{-46}	6.2×10^{-46}	5.8×10^{-157}	8.0×10^{-157}

Table 2
Numerical errors for the root p and the multiplicity m of the presented methods. The initial approximation is chosen by $p_0 = b$, the right end-point of each given interval.

Examples	Iterations	Parallel method ($\mu = \epsilon$)		Correlated method	
	k	$\lvert p_k - p\rvert$	$\lvert m_k - m\rvert$	$\lvert p_k - p\rvert$	$\lvert m_k - m\rvert$
$f_1(x) = 0$	2	5.0×10^{-2}	1.2×10^{-2}	1.4×10^{-5}	3.5×10^{-6}
	3	2.0×10^{-4}	5.0×10^{-5}	9.4×10^{-18}	2.3×10^{-18}
	4	3.3×10^{-9}	8.3×10^{-10}	2.9×10^{-54}	7.1×10^{-55}
	5	9.3×10^{-19}	2.3×10^{-19}	8.1×10^{-164}	2.0×10^{-164}
	6	7.2×10^{-38}	1.8×10^{-38}	1.8×10^{-492}	4.6×10^{-492}
$f_2(x) = 0$	2	7.5×10^{-2}	2.0×10^{-1}	1.8×10^{-5}	4.5×10^{-5}
	3	3.2×10^{-3}	8.0×10^{-3}	9.4×10^{-16}	2.4×10^{-15}
	4	5.1×10^{-6}	1.3×10^{-5}	1.4×10^{-46}	3.5×10^{-46}
	5	1.3×10^{-11}	3.2×10^{-11}	4.6×10^{-139}	1.2×10^{-138}
	6	8.3×10^{-23}	2.1×10^{-22}	1.6×10^{-416}	4.1×10^{-416}
$f_3(x) = 0$	2	4.7×10^{-2}	4.9×10^{-5}	3.3×10^{-10}	2.3×10^{-20}
	3	5.1×10^{-6}	5.7×10^{-12}	2.7×10^{-51}	1.6×10^{-102}
	4	6.3×10^{-18}	8.9×10^{-36}	1.1×10^{-256}	2.9×10^{-513}
	5	1.2×10^{-53}	3.3×10^{-107}	–	–
	6	8.9×10^{-161}	1.7×10^{-321}	–	–
$f_4(x) = 0$	2	1.8×10^{-3}	2.5×10^{-3}	2.3×10^{-8}	3.2×10^{-8}
	3	5.0×10^{-7}	6.9×10^{-7}	2.4×10^{-25}	3.3×10^{-25}
	4	3.8×10^{-14}	5.2×10^{-14}	2.5×10^{-76}	3.5×10^{-76}
	5	2.2×10^{-28}	3.0×10^{-28}	3.0×10^{-229}	4.1×10^{-229}
	6	7.4×10^{-57}	1.0×10^{-56}	–	–

Example 3.

$$f_3(x) = \frac{(x-1)^4}{20 + 2x - x^2} = 0, \quad 0 < x < 3,$$

with a root $p = 1$ of multiplicity $m = 4$.

Example 4.

$$f_4(x) = \left\{x - \frac{\pi}{3} e^{\pi/3 - x}\right\}^3 \sin^2\left(\frac{x}{2} - \frac{\pi}{6}\right) = 0, \quad 0 < x < 2,$$

with a root $p = \pi/3$ of multiplicity $m = 5$.

Table 1 shows numerical results of two presented methods, named the parallel method and the correlated method, where the initial approximation is chosen as $p_0 = q$ by the NIM in (9). In practice, we have used the Mathematica command, **NIntegrate**[H (x), {x, a, b}] to estimate the initial approximation q. The sign "–" in Table 1 (and in Table 2) indicates that the corresponding iteration is stopped as the approximation error is less than a tolerance error or the function value $f(p_k)$ is so small that it is considered as 0. We can observe that the convergence orders of the parallel method and correlated method are at least quadratic and cubic, respectively, as shown in Theorems 2 and 3.

On the other hand, Table 2 includes numerical results using the initial approximation $p_0 = b$, the right end-point of the given interval (a, b) without efforts to obtain a proper initial approximation. Of course, though the results are worse than

Table 3
Numerical errors of the existing transformation method [10], where the initial approximation is chosen by $p_0 = b$, the right end-point of each given interval.

Examples	Iterations	Existing method [10]	
	k	$\lvert p_k - p\rvert$	$\lvert m_k - m\rvert$
$f_1(x) = 0$	k_{max}	NC	NC
$f_2(x) = 0$	k_{max}	NC	NC
$f_3(x) = 0$	k_{max}	NC	NC
$f_4(x) = 0$	8	4.4×10^{-4}	1.0×10^{-4}
	9	1.8×10^{-7}	4.3×10^{-8}
	10	3.2×10^{-14}	7.4×10^{-15}
	11	9.4×10^{-28}	2.2×10^{-28}
	12	8.4×10^{-55}	1.9×10^{-55}

those in Table 1, convergence orders are consistent with Theorems 2 and 3. This implies that the availability of the new methods does not depend on the initial approximation.

It should be noted that from Corollary 2.1 the parallel method based on (12) and (14) becomes cubic in a particular case of $g'(p) = 0$. In fact, $f_3(x) = (x-1)^4 g_3(x)$, where $g_3(x) = 1/(20+2x-x^2)$ and thus

$$g_3'(x) = \frac{2(x-1)}{(20+2x-x^2)^2},$$

which satisfies $g_3'(p) = 0$. Therefore, in this case we can obtain the cubic convergence of the parallel method as shown in Tables 1 and 2.

To compare the presented methods with the existing method, we consider the second order transformation method (2) proposed by Pariad and Gupta [10]. Numerical results of the method are given in Table 3, where the term "NC" means that the iterates do not converge to the root until the maximum iteration number $k_{\max} = 100$. Referring to the results in Table 2, one can see that the presented methods are superior to the existing transformation method.

5. Conclusions

In this paper we have proposed the transformation $H(x)$ in (6), combined with the difference formula $f_\epsilon(x)$ in (5), for finding a multiple root p of an equation $f(x) = 0$ via the non-iterative method, say, NIM given in (9).

On the other hand, based on another transformation $K(x)$ in (10), we have developed a new Steffensen-type iterative method (12). In addition, the unknown multiplicity m of the root p can also be estimated by the proposed iteration (14). This method to approximate the root and the multiplicity, respectively, is named the parallel method. We have proposed another iterative method, named the correlated method, which consists of the iteration (15) and (14) for approximating p and m, recursively.

The methods developed in this paper are summarized as follows:

i. The NIM (9) using the transformation $H(x)$ in (6) is, for itself, available for finding a root p as long as the numerical integration $\widetilde{I}(H(x))$ is sufficiently accurate.
ii. When the accuracy of q obtained by the NIM is not satisfactory, it can be further improved by one of the presented iterative methods (the parallel method and the correlated method) with the initial approximation $p_0 = q$. The convergence orders of the parallel method and the correlated method have been proved to be quadratic and cubic, respectively and these methods do not require any derivatives of the function $f(x)$.
iii. As shown by the numerical examples, the new iterative methods are still useful without any prior process such as, for example, the NIM to obtain an appropriate initial approximation.

References

[1] K.E. Atkinson, An Introduction to Numerical Analysis, second ed., John Wiley & Sons, Singapore, 1988.
[2] C. Chun, H.J. Bae, B. Neta, New families of nonlinear third-order solvers for finding multiple roots, Comput. Math. Appl. 57 (2009) 1574–1582.
[3] C. Chun, B. Neta, A third-order modification of Newton's method for multiple roots, Appl. Math. Comput. 211 (2009) 474–479.
[4] C. Dong, A family of multipoint iterative formula of the higher order for computing multiple roots of an equation, Int. J. Comput. Math. 21 (1987) 363–367.
[5] R.F. King, A secant method for multiple roots, BIT 17 (1977) 321–328.
[6] J.B. Kioustelidis, A derivative-free transformation preserving the order of convergence of iteration methods in case of multiple zeros, Numer. Math. 33 (1979) 385–389.
[7] B. Neta, New third order nonlinear solvers for multiple roots, Appl. Math. Comput. 202 (2008) 162–170.
[8] B. Neta, A.N. Johnson, High order nonlinear solvers for multiple roots, Comput. Math. Appl. 55 (2008) 2012–2017.
[9] N. Osada, An optimal multiple root finding method of order three, J. Comput. Appl. Math. 51 (1994) 131–133.
[10] P.K. Parida, D.K. Gupta, An improved method for finding multiple roots and it's multiplicity of nonlinear equations in $\mathbb{R}$, Appl. Math. Comput. 202 (2008) 498–503.
[11] J.F. Traub, Iterative Methods for the Solution of Equations, Prentice Hall, Englewood, 1964.
[12] X.Y. Wu, J.L. Xia, R. Shao, Quadratically convergent multiple roots finding method without derivatives, Comput. Math. Appl. 42 (2001) 115–119.
[13] B.I. Yun, A non-iterative method for solving non-linear equations, Appl. Math. Comput. 198 (2008) 691–699.
[14] L.F. Steffensen, Remark on Iteration, vol. 16, Skand, Aktuarietidskr, 1933.
[15] B.I. Yun, A derivative free iterative method for finding multiple roots of nonlinear equations, Appl. Math. Lett. 22 (2009) 1859–1863.