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GENERALISED RAMSEY NUMBERS FOR SMALL GRAPHS★

Robert W. IRVING*

Department of Mathematics, University of Glasgow, Glasgow G12 8QW, United Kingdom

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Abstract. The generalised Ramsey number $R(G_1, G_2, ..., G_k)$ is defined as the smallest integer n such that, if the edges of K_n, the complete graph on n vertices, are coloured using k colours $C_1, C_2, ..., C_k$, then for some i $(1 \leq i \leq k)$ there is a subgraph G_i of K_n with all of its edges colour C_i. When $G_1 = G_2 = ... = G_k = G$, we use the more compact notation $R_k(G)$.

The generalised Ramsey numbers $R_k(G)$ are investigated for all graphs G having at most four vertices (and no isolates). This extends the work of Chvátal and Harary, who made this investigation in the case $k = 2$.

1. Statement of the problem

The following is a statement of the special case of Ramsey's theorem which can be stated in graph-theoretic terms.

Theorem 1.1 (Ramsey [25]). *Given integers $k \geq 1, m_1, m_2, ..., m_k \geq 2$, there exists a smallest integer $R = R(m_1, m_2, ..., m_k; 2)$ such that, if the edges of K_R, the complete graph on R vertices, are coloured using k colours $C_1, C_2, ..., C_k$, then for some i $(1 \leq i \leq k)$ there is a complete subgraph K_{m_i} of K_R with all of its edges colour C_i.*

The numbers $R(m_1, m_2, ..., m_k; 2)$ are called Ramsey numbers, and have been the subject of much investigation. Their complete determination appears to be hopelessly difficult, but some values are known and various estimates available.

An immediate corollary of the above can now be stated.

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* Present address: Department of Mathematics, University of Salford, Salford M5 4WT, U.K.

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Corollary 1.2. *Given an integer $k \geq 1$ and graphs $G_1, G_2, \ldots, G_k$, there exists a smallest integer $R = R(G_1, G_2, \ldots, G_k)$ such that, if the edges of K_R are coloured using k colours $C_1, C_2, \ldots, C_k$, then for some i $(1 \leq i \leq k)$ there is a subgraph G_i of K_R with all of its edges colour C_i.*

In this notation, the Ramsey number $R(m_1, m_2, \ldots, m_k; 2)$ becomes $R(K_{m_1}, K_{m_2}, \ldots, K_{m_k})$. In the case $G_1 = G_2 = \ldots = G_k = G$, we use the more compact notation $R_k(G)$. The $R(G_1, G_2, \ldots, G_k)$ are called generalised Ramsey numbers.

A $(G_1, G_2, \ldots, G_k)$*-edge-colouring* of a graph G is defined to be a colouring of the edges of G using k colours $C_1, C_2, \ldots, C_k$ such that, for all i $(1 \leq i \leq k)$ there is no subgraph G_i of G with all of its edges colour C_i. When $G_1 = G_2 = \ldots = G_k = G$, we call this a $(G)_k$*-edge-colouring*. Hence $R(G_1, G_2, \ldots, G_k)$ is the smallest value of n for which no $(G_1, G_2, \ldots, G_k)$-edge-colouring of K_n exists.

These generalised Ramsey numbers were apparently first investigated by Gerencsér and Gyárfás [18], who evaluated $R(P_m, P_n)$, where P_m is the path on m vertices. Many papers dealing with generalised Ramsey numbers have subsequently appeared (e.g. [5–14, 16, 17, 22–24]). A survey of many of these papers appears in [21]. However, all such papers deal with the special case $k = 2$, while many of the more interesting problems emerge in cases $k > 2$. In this paper, we shall examine the generalised Ramsey numbers $R_k(G)$ for various "small" graphs G.

2. Extremal graph theory

We shall require some known results from extremal graph theory. The central problem of this area of graph theory is the following: given a positive integer n and a graph G, what is the largest number $\alpha(n; G)$ of edges in any n-vertex graph which contains no copy of G?

We define

$$\beta(n; G) = \frac{2\alpha(n; G)}{n(n-1)},$$

so that $\beta(n; G)$ is the proportion of the total possible number of edges which are present in an n-vertex graph extremal with respect to G.

Theorem 2.1 (Erdős and Simonovits [15]).

$$\beta(G) = \lim_{n\to\infty} \beta(n; G) = 1 - \frac{1}{\chi(G)-1},$$

where $\chi(G)$ *is the chromatic number of* G.

Corollary 2.2. $\beta(G) = 0$ *if* G *contains no odd cycle. Otherwise* $\beta(G) \geq \frac{1}{2}$.

The following simple result indicates the relevance of the function $\beta(n; G)$ to the estimation of the generalised Ramsey number $R_k(G)$.

Theorem 2.3 (Chvátal and Harary [12]). $\beta(n; G) < 1/k$ *implies that* $R_k(G) \leq n$.

Hence, for the purpose of estimating generalised Ramsey numbers $R_k(G)$, graphs G fall naturally into two classes according as whether or not they posses an odd cycle. If G contains no odd cycle, Theorem 2.3 will yield an upper bound for $R_k(G)$, whereas if G does contain an odd cycle, other methods have to be found.

3. Small graphs

Chvátal and Harary [13] listed all the graphs G having no isolates and at most four vertices, and for each graph in turn, they evaluated $R_2(G)$. In [14], they then evaluated $R(G_1, G_2)$ for each pair of such graphs. Here, we shall extend the results of [13] by evaluating or estimating $R_k(G)$ for each such graph G.

We first list the ten graphs which qualify (Fig. 1). The notation used is that of Harary [20].

We summarise the results obtained by Chvátal and Harary in [13].

(i) $R_2(K_2) = 2$, (ii) $R_2(P_3) = 3$,
(iii) $R_2(2K_2) = 5$, (iv) $R_2(K_3) = 6$,
(v) $R_2(P_4) = 5$, (vi) $R_2(K_{1,3}) = 6$,
(vii) $R_2(C_4) = 6$, (viii) $R_2(K_{1,3}+x) = 7$,
(ix) $R_2(K_4-x) = 10$, (x) $R(K_4) = 18$.

We now examine $R_k(G)$ for each of these graphs G in turn in a series of theorems.

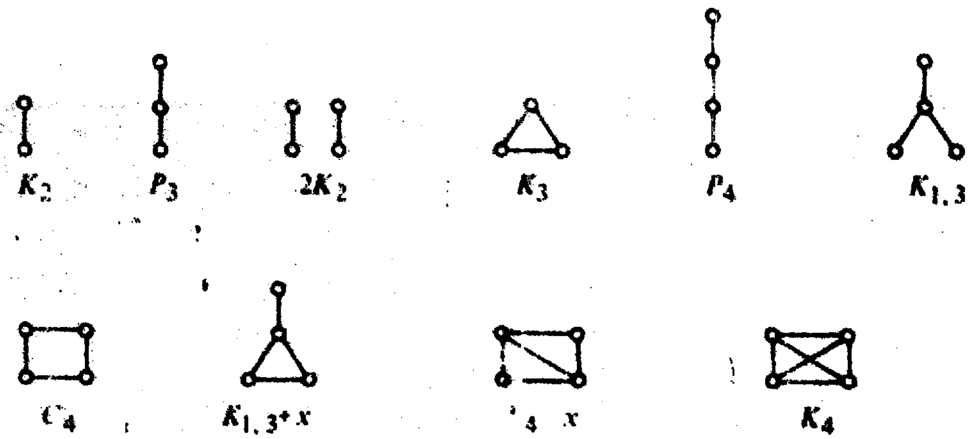

Fig. 1.

Theorem 3.1. $R_k(K_2) = 2$.

Proof. In a $(K_2)_k$-edge-colouring of K_n, no edge can be present. Hence $n \leq 1$.

Theorem 3.2.

$$R_k(P_3) = \begin{cases} k+2 & \text{if } k \text{ odd}, \\ k+1 & \text{if } k \text{ even}. \end{cases}$$

This theorem follows as an easy consequence of a result of Behzad, Chartrand and Cooper [4], which we state as a lemma. We first require two definitions.

An *independent set of edges* in a graph G is a subset of the edge set of G in which no two edges have an end-vertex in common.

The *edge-chromatic number* $\chi'(G)$ of a graph G is the minimum number of independent sets into which the edge set of G can be partitioned.

Lemma 3.3 (Behzad, Chartrand and Cooper [4]).

$$\chi'(K_n) = \begin{cases} n & \text{if } n \text{ odd}, n \geq 3, \\ n-1 & \text{if } n \text{ even}. \end{cases}$$

Proof of Theorem 3.2. (i) *k odd.* It is immediate that $R_k(P_3) \leq k+2$, since in a $(P_3)_k$-edge-colouring of K_n, each vertex can have at most one incident edge of each colour. On the other hand, Lemma 3.3 gives $\chi'(K_{k+1}) = k$, since $k + 1$ is even, and it follows at once that $R_k(P_3) = k + 2$.

(ii) *k even*. In a colouring of the edges of K_{k+1} with no monochromatic P_3, at most $\frac{1}{2}k$ of the edges can be of any one colour. Hence $R_k(P_3) \leq k + 1$. On the other hand, $\chi'(K_k) = k - 1$, since k is even, so that $R_k(P_3) \geq k + 1$, and the theorem is proved.

Theorem 3.4. $R_k(2K_2) = k + 3$.

Proof. To show $R_k(2K_2) \geq k + 3$, we construct a $(2K_2)_k$-edge-colouring of K_{k+2} as follows: Label the vertices of K_{k+2} by $V_1, V_2, \ldots, V_{k+2}$ in some order. The edge $V_i V_j$ $(1 \leq i < j \leq k + 2, i \leq k)$ is given the ith colour. Edge $V_{k+1} V_{k+2}$ is given the kth colour. The subgraph of colour i is a star graph for $i = 1, 2, \ldots, k-1$, while the subgraph of colour k is K_3. Hence, no $2K_2$ is monochromatic, and $R_k(2K_2) \geq k + 3$.

On the other hand, in a $(2K_2)_k$-edge-colouring of K_n, it is clear that the subgraph in any one colour must be either a star graph or K_3. Suppose that there exists such an edge-colouring of K_{k+3}, with the monochromatic subgraphs being m stars and $k-m$ triangles. Then the number of edges coloured is at most

$$\begin{aligned}(k + 2) + (k + 1) + k + \ldots + (k + 3 - m) + 3(k-m) = \\ = km + 3k - \tfrac{1}{2}m(m + 1) < \tfrac{1}{2}(k + 3)(k + 2) \quad \text{for all } m.\end{aligned}$$

This is a contradiction, and the result follows.

The case $G = K_3$ is that of the classical Ramsey number $R_k(3; 2)$. For the sake of completeness, we summarise the known information in this case.

Theorem 3.5. (i) $R_2(K_3) = 6$, $R_3(K_3) = 17$, $50 \leq R_4(K_3) \leq 65$ (*see* [19, 27]).

(ii) $\frac{1}{2}(3^k + 3) \leq R_k(K_3) \leq \lfloor k!(e - \frac{1}{24}) \rfloor + 1$ (*see* [2, 19, 27]).

(iii) $R_k(K_3) \geq c \cdot 89^{k/4}$ *for some absolute constant* c (*see* [1]).

Theorem 3.6.

$$R_k(P_4) = \begin{cases} 2k+2 & \textit{if } k \equiv 1 \pmod 3, \\ 2k+1 & \textit{if } k \equiv 2 \pmod 3, \\ 2k \textit{ or } 2k+1 & \textit{if } k \equiv 0 \pmod 3. \end{cases}$$

In order to prove this theorem, we shall require two lemmas.

Lemma 3.7.

$$\alpha(n; P_4) = \begin{cases} n & \textit{if } n \equiv 0 \pmod 3, \\ n-1 & \textit{if } n \equiv 1, 2 \pmod 3. \end{cases}$$

Proof. The only connected graphs which do not contain P_4 as a subgraph are star graphs and the triangle graph K_3. Hence, if G is an n-vertex graph extremal with respect to P_4, G must be the union of graphs of these types, and it is clear that G will contain as many triangles as possible, namely $[\frac{1}{3}n]$. This gives the result of the lemma.

The next lemma contains a recent well-known result of Ray-Chaudhuri and Wilson [26].

Lemma 3.8 (Ray-Chaudhuri and Wilson [26]). *There exists a Kirkman triple system of order v if and only if $v \equiv 3 \pmod 6$.*

Proof of Theorem 3.6. We first prove the upper bound in each case.

(i) $k \equiv 1 \pmod 3$.

$$\begin{aligned} \beta(2k+2; P_4) &= 2\alpha(2k+2; P_4)/(2k+2)(2k+1) \\ &= 1/(k+1) \quad \text{by Lemma 3.7,} \\ &< 1/k. \end{aligned}$$

This implies that $R_k(P_4) \leq 2k+2$ by Theorem 2.3

(ii), (iii) $k \equiv 0, 2 \pmod 3$.

$$\begin{aligned} \beta(2k+1; P_4) &= 2\alpha(2k+1; P_4)/(2k+1)2k \\ &= 2/(2k+1) \quad \text{by Lemma 3.7,} \\ &< 1/k. \end{aligned}$$

This implies that $R_k(P_4) \leq 2k+1$ by Theorem 2.3.

On the other hand, if $k \equiv 1 \pmod 3$, let $v = 2k+1$, so that $v \equiv 3 \pmod 6$. By Lemma 3.8, there exists a Kirkman triple system of order v. We label the vertices of K_{2k+1} with the elements of such a system. Now, any pair of elements of the system belong to exactly one triple, and this triple belongs to exactly one of the k sets of triples into which the system can be resolved, with each element in exactly one triple of the set. We colour the edge with end-vertices labelled x, y the ith colour according as the triple containing the pair (x, y) lies in the ith of the k sets of triples. Then we easily see that every monochromatic subgraph in such a colouring is a union of triangles, and so contains no P_4. Hence $R_k(P_4) \geq 2k+2$ if $k \equiv 1 \pmod 3$.

If $k \equiv 2 \pmod 3$, we proceed as above, using $k-1$ colours and K_{2k-1}. We then adjoin a $2k$th vertex and colour every edge incident on it with the kth colour, and this gives $R_k(P_4) \geq 2k+1$.

If $k \equiv 0 \pmod 3$, we proceed as above, using $k-2$ colours and K_{2k-3}. We then adjoin two additional vertices and colour every edge incident on the first with the $(k-1)$th colour and every remaining edge incident on the second with the kth colour. This gives $R_4(P_4) \geq 2k$, and the proof of the theorem is complete.

In the case $k \equiv 0 \pmod 3$ of Theorem 3.6, we have been unable to determine, in general, which of the two possible values is the correct one. It is a simple matter to show $R_3(P_4) = 6$, but we have been unable to show $R_k(P_4) = 2k$ in general, for $k \equiv 0 \pmod 3$. However, this leads us to an interesting problem of a structure very like a Kirkman triple system.

It is well-known that Steiner triple systems of order v exist for all $v \equiv 1, 3 \pmod 6$. The result of Ray-Chaudhuri and Wilson tells us that for all $v \equiv 3 \pmod 6$ there is a resolvable Steiner triple system of order v. A simple counting argument can be used to show that no Steiner triple system of order $v \equiv 1 \pmod 6$ is resolvable. However, if we remove one element from such a system, together with all triples containing that element, there is no simple counting argument which tells us that the resulting structure cannot be resolvable. We make this more precise.

We define a *quasi-Kirkman triple system* of order v to be a set of v paired elements $a_1, a_1', a_2, a_2', \ldots, a_u, a_u'$ $(u = \frac{1}{2}v)$, together with a set of

$\frac{1}{6}v(v-2)$ triples of these elements, such that

(i) if x, y are two elements with $x \neq y'$, x and y appear together in exactly one triple;

(ii) x and x' never appear together in a triple;

(iii) the $\frac{1}{6}v(v-2)$ triples can be split into $\frac{1}{2}(v-2)$ sets such that each set contains every element in exactly one of its triples.

It is immediate that a quasi-Kirkman triple system of order v can exist only if $v \equiv 0 \pmod 6$. Further, by arguments similar to those used in the proof of Theorem 3.6, we get the following proposition.

Proposition 3.9. *If there exists a quasi-Kirkman triple system of order* $2k$, *then* $R_k(P_4) = 2k + 1$.

We have been unable to find any quasi-Kirkman triple system, but except in the cases $v = 6, 12$, we have also been unable to show that quasi-Kirkman triple systems of order v do not exist.

Finally, in the spirit of the original Kirkman problem, we state our problem in more picturesque language.

Problem 3.10. Given a positive integer n, is it possible to split $3n$ married couples into $2n$ triples on each of $3n-1$ successive days, in such a way that each person appears together in a triple with every other person except his/her spouse?

Theorem 3.11. $R_k(K_{1,3}) = 2k + 2$.

Proof. Suppose that the edges of K_{2k+2} are coloured using k colours. Then any chosen vertex must have at least three incident edges of some colour, so that a monochromatic $K_{1,3}$ is present. Hence $R_k(K_{1,3}) \leq 2k + 2$.

On the other hand, if we number the vertices of K_{2k+1} by $1, 2, \ldots, 2k+1$, and colour an edge with the ith colour if the difference between its end-vertices is i or $(2k+1-i)$ $(1 \leq i \leq k)$, then it is easy to see that no monochromatic $K_{1,3}$ results, and so $R_k(K_{1,3}) \geq 2k + 2$.

Theorem 3.12. (i) $R_k(C_4) \leq k^2 + k + 1$.
(ii) *If* $k-1 = p^\alpha$, *a prime power*, $R_k(C_4) \geq k^2 - k + 2$.

Again, we require a lemma for the proof of this theorem.

Lemma 3.13. $\alpha(n: C_4) < \frac{1}{4}n(1 + \sqrt{4n-3})$.

Proof. Let G be an n-vertex graph extremal with respect to C_4, let $V_1, V_2, \ldots, V_n$ be its vertices with degrees $v_1, v_2, \ldots, v_n$ respectively, and let G have m edges. Then

$$2m = \sum_{i=1}^{n} v_i .$$

The number of distinct three-point paths in G is

$$\sum_{i=1}^{n} \binom{v_i}{2} = \frac{1}{2} \sum_{i=1}^{n} v_i(v_i - 1) \geq m((2m/n) - 1)$$

with equality possible only if $v_i = 2m/n$ for all i. Therefore,

$$m((2m/n) - 1) \leq \tfrac{1}{2}n(n-1) ,$$

since the number of such paths cannot exceed the number of pairs of vertices. Hence

$$m < \tfrac{1}{4}n(1 + \sqrt{4n-3}).$$

We have to show that equality is not possible, so we assume that equality holds. Then the adjacency matrix A of the graph G possesses the following properties:

(1) A is a symmetric $n \times n$ (0, 1)-matrix with zero main diagonal.

(2) Each row and each column of A contain $2m/n = \frac{1}{2}(1 + \sqrt{4n-3})$ ones.

(3) Any two rows (columns) have exactly one 1 in the same position.

Such is precisely the incidence matrix of a finite projective plane of order $\frac{1}{2}(\sqrt{4n-3} - 1) = s$, say.

Since A is symmetric, the mapping of points to lines and lines to points determined by relating corresponding rows and columns of A is a polarity of the plane, and the number of ones on the main diagonal is the number of absolute points of the polarity. However, it is well-known (see [3]) that a polarity of a finite projective plane of order s must have at least $s + 1$ absolute points, and this contradicts the fact that A has zero main diagonal. This proves the lemma.

Proof of Theorem 3.12. (i) By Lemma 3.13,

$$\beta(k^2 + k + 1; C_4) < (1 + \sqrt{4k^2 + 4k + 1})/2k(k + 1) = 1/k.$$

Hence $R_k(C_4) \leq k^2 + k + 1$ by Theorem 2.3.

(ii) $k - 1 = p^\alpha$. By Singer's theorem, there exists a cyclic projective plane of order $k - 1$. Its incidence matrix A is a $(k^2 - k + 1) \times (k^2 - k + 1)$ circulant with k ones in each row and column. Denote by $\hat{A}$ the matrix obtained from A by reversing the order of its rows. Then $\hat{A}$ is a back-circulant which is symmetric, and is clearly still an incidence matrix for the plane.

Denote by $A_1, A_2, \ldots, A_k$ the k back-circulants obtained from $\hat{A}$ in the following way. A_i $(i = 1, 2, \ldots, k)$ is the matrix obtained by permuting the rows of $\hat{A}$ cyclically, with first row the ith of those rows of $\hat{A}$ which have a 1 in the last position. Then each A_i has ones in its back-main diagonal, but apart from these, no two of the A_i have a 1 in the same position. For this would clearly imply that two rows of $\hat{A}$ have more than one 1 in the same position.

We use A_1 as an adjacency matrix for the subgraph of K_{k^2-k+1} having edges of colour 1, and for $i = 2, 3, \ldots, k$, we use $A_i - M$ as an adjacency matrix for the subgraph of colour i, where M is the matrix with ones in its back-main diagonal and zeros elsewhere. Ones on the main diagonal are ignored in these adjacency matrices.

In the resulting colouring, every edge is coloured, and no C_4 has all of its edges the same colour. For in any A_i $(1 \leq i \leq k)$, two rows have at most one 1 in the same position, and hence two vertices of K_{k^2-k+1} are both joined by an edge of colour i to at most one other vertex. This completes the proof.

We give no theorem concerning the number $R_k(K_{1,3} + x)$. For $k > 2$, it seems likely that $R_k(K_{1,3} + x) = R_k(K_3)$.

Theorem 3.14.

$$R_k(K_4 - x) \le \begin{cases} [k \cdot k!\, e + 2/k(k+1)] + 1 & \text{if } k \text{ odd,} \\ [k!(ke-1) + 2/k(k+1)] + 1 & \text{if } k \text{ even.} \end{cases}$$

In order to prove this result, we require some new notation and a lemma. Let

$$R([K_4 - x]^m, [P_3]^{k-m}) = R(\underbrace{K_4 - x, \dots, K_4 - x}_{m \text{ terms}}, \underbrace{P_3, \dots, P_3}_{k-m \text{ terms}}).$$

Lemma 3.15.

$$R([K_4 - x]^m, [P_3]^{k-m}) \le$$
$$\le m \cdot R([K_4 - x]^{m-1}, [P_3]^{k-m+1}) + (k - 2m + 2),$$

with strict inequality if $m < k$ and the expression on the right-hand side is even.

Proof. Suppose that there exists a colouring in k colours of the edges of K_n with no monochromatic $K_4 - x$ of one of the first m colours, and no monochromatic P_3 of one of the remaining $k-m$ colours. Suppose that there are a_i edges of colour i incident with some fixed vertex V $(1 \le i \le k)$. If $1 \le i \le m$, we clearly have

$$a_i \le R([K_4 - x]^{m-1}, [P_3]^{k-m+1}) - 1,$$

while if $m + 1 \le i \le k$, we have

$$a_i \le 1.$$

Hence

$$n \le \sum_{i=1}^{k} a_i + 1 \le m \cdot R([K_4 - x]^{m-1}, [P_3]^{k-m+1}) + (k - 2m + 1),$$

and the inequality of the lemma follows.

Suppose $m < k$, the right-hand side is even and equality holds. Then in an optimal colouring, n is odd, so that we cannot have $a_i = 1$ (for all i, $m + 1 \le i \le k$) for every vertex V, giving a contradiction.

Proof of Theorem 3.14. We use repeated applications of Lemma 3.15, but consider separately the cases k odd and k even.

(i) k odd.

$$\begin{aligned} R_k(K_4-x) &= R([K_4-x]^k, [P_3]^0) \\ &\leq k \cdot R([K_4-x]^{k-1}, [P_3]^1) - (k-2) \\ &\;\vdots \\ &\leq k! \cdot R([P_3]^k) - \{(k-2) + k(k-4) + k(k-1)(k-6) + \ldots \\ &\quad + k(k-1) \ldots (\tfrac{1}{2}(k+5)) \cdot 1\} + \{k(k-1) \ldots (\tfrac{1}{2}(k+3)) \cdot 1 + \\ &\quad \ldots + k!(k)\}. \end{aligned}$$

Hence, using Theorem 3.2,

$$\begin{aligned} R_k(K_4-x) &\leq k!(k+2) + k!(k+2)\{1 + 1/2! + \ldots + 1/k!\} \\ &\quad - k!\{2 + 4/2! + \ldots + 2k/k!\} \\ &< k!ke + 1 + 2/k(k+1). \end{aligned}$$

(ii) In case k even, the argument is similar to that above.

It seems likely that the upper bound of Theorem 3.14 is not very sharp. However, we have been unable to improve the trivial lower bound given by $R_k(K_4-x) \geq R_k(K_3)$ for $k \geq 3$. We have found the value $R([K_4-x]^1, [P_3]^2) = 7$, and the bounds $11 \leq R([K_4-x]^2, [P_3]^1) \leq 13$, which yields $R_3(K_4-x) \leq 38$. Theorem 3.14 gives $R_3(K_4-x) \leq 50$.

Finally, the case of $R_k(K_4)$ is that of the classical Ramsey number $R_k(4;2)$, and we summarise the known information on this number in our final theorem.

Theorem 3.16. (i) $R_k(K_4) \geq c \cdot 33^{k/2}$ *for some absolute constant* c *(see* [1]*).*

(ii) $R_k(K_4) < k(3k-1)(2k-2)!/2^{k-2}$.

The inequality (ii) follows as a fairly easy application of the recursive relation

$$R(m_1, m_2, ..., m_k; 2) \leq R(m_1-1, m_2, ..., m_k; 2) + R(m_1, m_2-1, ..., m_k; 2) + ... + R(m_1, m_2, ..., m_k-1; 2) - (k-2),$$

which appears in [19].

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