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PAIRING ENERGY EFFECTS IN EXCITED NUCLEI

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Abstract: The nucleus is considered as a Fermi gas with pairs of degenerate single particle states and a coupling energy Δ between the two states of a pair. A statistical treatment gives expressions for the density of excited states and the expected value of the total angular momentum. Suitable approximations for purposes of calculation are indicated. All the formulae become equivalent to those of a simple Fermi gas at high enough excitations.

1. Introduction

Measurement of level densities in excited nuclei by different methods have become accurate enough in recent years for some conflicts to be noted [1,2,3]) in the nuclear parameters on which the densities depend. Many of these conflicts can be resolved if the level density at low excitations, particularly in even nuclei, is assumed to depart from the simplest form which is adequate to explain high energy behaviour. It has been suggested that the discrepancy may be due to

(1) a gap in the levels available to a single particle at the highest occupied level [4]),

(2) shell effects [5,6]),

(3) pairing of nucleons whose states differ only in having opposite projections of angular momentum on the symmetry axis of a non-spherical nucleus.

The third hypothesis has the advantage that it brings in the even-odd effects in the ground state energy of the nucleus and in the level density in a natural way. It has been particularly considered by Ericson [7]) but the results obtained still require awkward numerical calculations before they can be compared with experiment. It is the purpose of this paper to obtain simpler forms for numerical calculation. By a more direct approach some unification is achieved between excited pair states and rotational states, and the moment of inertia of the nucleus in such states is calculated. A reduction of the pairing energy with temperature may be required to give a reasonable fit to experiment.

2. The Partition Function with Pairing Energy

For a given pair $s+$, $s-$ of degenerate single particle states, such as occur in the unified nuclear model [8,9,10]) according to Nilsson's [11]) level scheme with components $\pm m_s$ of angular momentum along the symmetry axis of the nucleus, there are four possible configurations, viz. both empty, one or other filled, both filled. Assuming that residual interactions within the nucleus lead to a coupling energy of magnitude Δ_s between members of a pair, these four configurations have energies 0, $a_s+\frac{1}{2}\Delta_s$ and $2a_s$, respectively, for neutrons and likewise 0, $b_r+\frac{1}{2}\Delta_r$ and $2b_r$ for protons. The pairing energy leads to characteristic differences $\frac{1}{2}\Delta$, Δ between the ground state energies of even and of odd mass and odd nuclei, and the magnitude of Δ may be taken from Stolovy and Harvey's [12]) empirical estimate of these differences:

$$\Delta_0 = 3.36-0.0084A \text{ MeV} \quad \text{for } A > 40. \tag{2.1}$$

The partition function for the neutron-proton assembly is then given by

$$\begin{aligned} e^{-\beta\Phi} = \prod\nolimits_{s,r} & \{1+2e^{\beta(\mu_1-a_s-\frac{1}{2}\Delta_s)}+e^{2\beta(\mu_1-a_s)}\} \\ & \times \{1+2e^{\beta(\mu_2-b_r-\frac{1}{2}\Delta_r)}+e^{2\beta(\mu_2-b_r)}\}, \end{aligned} \tag{2.2}$$

where μ_1 and μ_2 are the Fermi levels for neutrons and protons and $\beta = 1/t$ is the reciprocal of the thermodynamic temperature. This expression must be identical with

$$e^{-\beta\Phi} = \sum_{N'Z'} \int_0^\infty dE'\, P(N', Z', E') e^{\beta(\mu_1 N'+\mu_2 Z'-E')} \tag{2.3}$$

if $P(N', Z', E')$ is the density of excited levels of a nucleus of N' neutrons and Z' protons in the neighbourhood of energy E'.

The neutron contribution to $\beta\Phi$ in (2.2) is

$$-\beta\Phi_n = \sum_s \log\,[1+2e^{\beta(\mu_1-a_s-\frac{1}{2}\Delta_s)}+e^{2\beta(\mu_1-a_s)}] \tag{2.4}$$

$$\begin{aligned} &= \sum_{a_s \leqq \mu_1} \{2\beta(\mu_1-a_s)+\log\,[1+2e^{\beta(a_s-\mu_1-\frac{1}{2}\Delta_s)}+e^{2\beta(a_s-\mu_1)}]\} \\ &+ \sum_{a_s>\mu_1} \log\,[1+2e^{\beta(\mu_1-a_s-\frac{1}{2}\Delta_s)}+e^{2\beta(\mu_1-a_s)}]. \end{aligned} \tag{2.5}$$

We will assume that the value of Δ_s is the same for all states and that the densities of pairs of states available to neutrons and protons can be described by two continuous functions $g_n(a)$ and $g_p(b)$ respectively.

Equation (2.5) can be written more simply if g_n is symmetric about μ_1 as

$$-\beta\Phi_n = \sum_{a_s\leqq\mu_1} 2\beta(\mu_1-a_s)+\frac{1}{\beta}I_n \tag{2.6}$$

where

$$I_n = 2\int_0^\infty g_n(x) \log\,(1+2\cos\lambda e^{-x}+e^{-2x})dx, \tag{2.7}$$

$$\cos\lambda = e^{-\frac{1}{2}\beta\Delta}; \tag{2.8}$$

and x is the magnitude $|\beta(a_s-\mu_1)|$.

We define I_p similarly and $I = (I_n + I_p)$. If $g_n(x)$ and $g_p(x)$ are constant for the range of orbits whose occupation state is likely to change, the integral is

$$I = 2(g_n + g_p)(\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\lambda^2) \tag{2.9}$$

and we have, collecting results

$$e^{-\beta\Phi} = \exp\left\{\frac{I}{\beta} + \sum_{a_s \leqq \mu_1} 2\beta(\mu_1 - a_s) + \sum_{b_r \leqq \mu_2} 2\beta(\mu_2 - b_r)\right\}. \tag{2.10}$$

At high temperatures $\lambda \to 0$ and (2.10) reduces to the result of the single particle treatment [13] because

$$2g_n + 2g_p = g \tag{2.11}$$

is the total density of single particle states.

3. Reversing the Summation

To evaluate $P(N, Z, E')$ for a particular N and Z we evaluate the residues of $\exp(-\beta\Phi - \mu_1\beta(N+1))$ and $\exp(-\beta\Phi - \mu_1\beta(N+1) - \mu_2\beta(Z+1))$ in turn. The saddle point for the first is given by

$$\frac{\partial\Phi}{\partial\mu_1} = -N = \sum_{a_s \leqq \mu_1} 2. \tag{3.1}$$

If N is odd we must adopt the convention that μ_1 is equal to the energy of the last occupied level and take half the last term of the summation. This follows from the original definition of Φ whence

$$\frac{\partial\Phi}{\partial\mu_1} = \sum_s \frac{2e^{\beta(\mu_1 - a_s - \frac{1}{2}\Delta)} + 2e^{\beta(\mu_1 - a_s)}}{1 + 2e^{\beta(\mu_1 - a_s - \frac{1}{2}\Delta)} + e^{2\beta(\mu_1 - a_s)}} \tag{3.2}$$

$$= \sum_s \frac{2\{1 + e^{\beta(a_s - \mu_1 - \frac{1}{2}\Delta)}\}}{1 + 2e^{\beta(a_s - \mu_1 - \frac{1}{2}\Delta)} + e^{2\beta(a_s - \mu_1)}} = \sum_s n_s. \tag{3.3}$$

If $\beta \to \infty$, i.e. at low excitation, each term
$\to 2$ for $a_s < \mu_1$
$\to 0$ for $a_s > \mu_1$
and for $a_s = \mu_1$ the term $= \dfrac{2(1+\exp(-\frac{1}{2}\Delta))}{2(1+\exp(-\frac{1}{2}\Delta))} = 1$,
i.e. for N even μ_1 must lie between the last occupied level and the next unoccupied, and for N odd μ_1 is the last (singly) occupied level.

Then, if the residues for N and Z are evaluated by saddle point integration, we have

$$\int_0^\infty dE' P(N, Z, E') e^{-\beta E'} = \frac{\beta}{2\pi\left(\dfrac{\partial N}{\partial\mu_1}\dfrac{\partial Z}{\partial\mu_2}\right)^{\frac{1}{2}}} e^{-\beta(\Phi + \mu_1 N + \mu_2 Z)} \tag{3.4}$$

The total energy of neutrons is

$$E_n = \sum_{n_s=2} 2a_s + \sum_{n_s=1} (a_s+\tfrac{1}{2}\Delta), \tag{3.5}$$

and similarly for the energy E_p of the protons. Using the above convention about the contribution of the highest occupied level, the summation in (2.10) reduces to

$$\sum_{a_s \leq \mu_1} 2\beta(\mu_1-a_s)+\sum_{b_r \leq \mu_2} 2\beta(\mu_2-b_r) = \beta(\mu_1 N+\mu_2 Z-E^*), \tag{3.6}$$

where $E^* = E_0$, the ground state energy for even nuclei
$= E_0-\tfrac{1}{2}\Delta_0$ for odd mass nuclei (3.7)
$= E_0-\Delta_0$ for odd nuclei

and we write Δ_0, in case Δ should subsequently become temperature dependent, because in (3.7) we want the value for the ground state.

We can now simplify (3.4) by replacing the energy E by the excitation energy $U = E-E_0$, and, using (2.10),

$$\int_0^\infty P(N, Z, U')e^{-\beta U'}dU' = \exp\left\{\frac{I}{\beta}+\log\frac{\beta}{2\pi}-\tfrac{1}{2}\log 4g_n g_p+\beta(E_0-E^*)\right\} \tag{3.8}$$

since

$$\frac{\partial N}{\partial \mu_1} = 2g_n \quad \text{and} \quad \frac{\partial Z}{\partial \mu_2} = 2g_p. \tag{3.9}$$

We define (3.8) to be $e^{-\beta F}$ for the system with exactly N neutrons and Z protons. Reversing the Laplace transform over the saddle point gives

$$P(N, Z, U) = e^S/(-2\pi\partial U/\partial\beta)^{\frac{1}{2}} \tag{3.10}$$

where $S = \beta(U-F)$ is the entropy of the system. Then

$$\begin{aligned} U = \partial(\beta F)/\partial\beta &= It^2-t\partial I/\partial\beta-t-(E_0-E^*) \\ &= gt^2(\tfrac{1}{6}\pi^2-\tfrac{1}{2}\lambda^2)+\tfrac{1}{2}tg\Delta\lambda\cot\lambda-t-(E_0-E^*) \end{aligned} \tag{3.11}$$

and

$$P(N, Z, U) = \frac{\beta^2 \exp\{2It-\partial I/\partial\beta-1\}}{(2\pi)^{\frac{1}{2}}\{4g_n g_p(2It-2\partial I/\partial\beta+\beta\partial^2 I/\partial\beta^2-1\}^{\frac{1}{2}}}. \tag{3.12}$$

4. Distribution of Levels over Angular Momentum

The interpretation of the first of equations (3.2) is that

$$p_s(+\text{or}-) = \frac{2e^{\beta(\mu_1-a_s-\frac{1}{2}\Delta)}}{1+2e^{\beta(\mu_1-a_s-\frac{1}{2}\Delta)}+e^{2\beta(\mu_1-a_s)}} \tag{4.1}$$

is the probability that one or other, but not both, of the states a_s is occupied and

$$p_s(+-) = \frac{e^{2\beta(\mu_1-a_s)}}{1+2e^{\beta(\mu_1-a_s-\frac{1}{2}\Delta)}+e^{2\beta(\mu_1-a_s)}} \tag{4.2}$$

is the probability that both are occupied.

These formulae remain true if $\Delta = 0$ and, in order to recover some familiar results, we first consider the simple case of a Fermi gas of independent particles in which the states a_s may be labelled $j \pm m$ *with m measured in an arbitrary direction of quantization*. For the pair a_s the average over the probability distribution (4.1), (4.2) is

$$\langle \sum m \rangle = 0, \ \langle (\sum m)^2 \rangle = m_s^2 p_s(+ \text{ or } -). \tag{4.3}$$

Let $M_N = \sum m$, taken over all neutrons, then $\langle M_N \rangle = 0$ and, since the different levels a_s are uncorrelated,

$$\langle M_N^2 \rangle = \sum_s m_s^2 p_s(+ \text{ or } -). \tag{4.4}$$

Thus for $\Delta = 0$,

$$\langle M_N^2 \rangle = 2 \sum_s \frac{m_s^2 e^{\beta(\mu_1-a_s)}}{[1+e^{\beta(\mu_1-a_s)}]^2} = 2 \sum_m m^2 g_n(m) \int_{-\infty}^{\infty} \frac{e^{\beta x} dx}{[1+e^{\beta x}]^2}, \tag{4.5}$$

where $g_n(m)$ is the density of neutron levels with quantum number m in the neighbourhood of the Fermi level, and so

$$\langle M_N^2 \rangle = 2 g_n \overline{m^2}/\beta \tag{4.6}$$

where $\overline{m^2}$ is the average m^2 value for single particles near the Fermi level.

Combining effects of neutrons and protons, the mean square component of the resultant angular momentum of the nucleus is

$$\langle M^2 \rangle = \overline{m^2} g t = ct \quad \text{where} \quad c\hbar^2 = \overline{m^2} g \hbar^2 \tag{4.7}$$

is equal to the moment of inertia of the nucleus, considered as a rigid body.

The distribution of M is

$$p(M) = (2\pi ct)^{-\frac{1}{2}} e^{-M^2/2ct} \tag{4.8}$$

and the other components of total angular momentum have the same distribution, since M is measured in an arbitrary direction. Therefore the probability of a state with total angular momentum J is

$$\frac{4\pi(J+\frac{1}{2})^2}{(2\pi ct)^{\frac{3}{2}}} e^{-J(J+1)/2ct} \tag{4.9}$$

and the probability of a nuclear level with angular momentum J is accordingly

$$\frac{2J+1}{\pi^{\frac{1}{2}}(2ct)^{\frac{3}{2}}} e^{-J(J+1)/2ct}, \tag{4.10}$$

which are well known results.

Now consider $\Delta > 0$. The pairing of states $\pm m$ now has important dynamical effects and the nuclear model to which it seems most appropriate is the unified model of Bohr and Mottelson [8, 9, 10]) and Nilsson [11]) in which m may be measured along the axis S of a non-spherical nucleus with axial symmetry. (See fig. 1.)

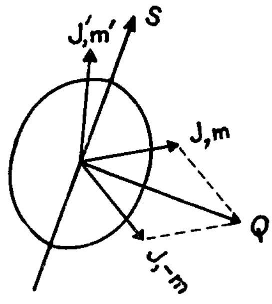

Fig. 1. The symmetry axis S of a deformed nucleus, with two paired particles j, $\pm m$; and the unpaired single particle j', m', angular momentum vectors. The resultant angular momentum Q of the pair is non zero, although the projection on S cancels.

The resultant angular momentum, in the direction S, of the neutrons has mean zero but the mean square is still determined by (4.4) which now gives

$$\begin{aligned}\langle M_N{}^2\rangle &= 2\sum_s \frac{m_s{}^2 e^{\beta(\mu_1-a_s-\frac{1}{2}\Delta)}}{1+2e^{\beta(\mu_1-a_s-\frac{1}{2}\Delta)}+e^{2\beta(\mu_1-a_s)}}\\ &= 2\overline{m^2}\sum_s \frac{e^{\beta(\mu_1-a_s-\frac{1}{2}\Delta)}}{1+2e^{\beta(\mu_1-a_s-\frac{1}{2}\Delta)}+e^{2\beta(\mu_1-a_s)}}\\ &= 2\overline{m^2}\partial(\beta\Phi_N)/\partial(\beta\Delta),\end{aligned}\tag{4.11}$$

from (2.4). For even nuclei this comes to

$$\begin{aligned}\langle M_N{}^2\rangle &= 2\overline{m^2}g_n t\partial\lambda^2/\partial(\beta\Delta)\\ &= 2\overline{m^2}g_n t\lambda\cot\lambda.\end{aligned}\tag{4.12}$$

Combining the contributions of neutrons and protons, the mean square component of the resultant angular momentum in the direction S is, *for even nuclei*,

$$\langle M^2\rangle = \lambda\cot\lambda\overline{m^2}gt = \lambda\cot\lambda\ ct = c't.\tag{4.13}$$

At high temperatures $\lambda\to 0$ as $(\Delta/t)^{\frac{1}{2}}$ and $c'\to c$ so that (4.7) and (4.10) hold approximately. But at low temperatures, the expression (4.13) is less than (4.7) by the factor $\lambda/\tan\lambda$ which can be very small because the pairing energy inhibits the excitation of single particles when $\Delta \gg t$. In such circumstances it is worthwhile to note that if the total number of neutrons is odd there must be at least one unpaired neutron which contributes $\overline{m^2}$ to $\langle M_N{}^2\rangle$. Accordingly (4.13) should be generalized to

$$\begin{aligned}\langle M^2\rangle &= \lambda \cot \lambda \overline{m^2} gt + \begin{cases} 0 & \text{for even nuclei} \\ \overline{m^2} & \text{for odd mass nuclei} \\ 2\overline{m^2} & \text{for odd nuclei} \end{cases} \\ &= c' t. \end{aligned} \tag{4.14}$$

We must now evaluate the mean particle angular momentum in the plane x, y perpendicular to S. First considering the neutron contribution, we have obviously

$$\langle J_{Nx}\rangle = \langle J_{Ny}\rangle = 0$$

because the direction is random.

For an even nucleus we can assume that if all the occupation numbers n_{s+}, n_{s-} simultaneously have precisely their mean values in the Gibbs ensemble, the total angular momentum has magnitude zero and this includes in particular the assumption that the ground state has total angular momentum zero. Fluctuations of the occupation numbers away from their mean values give rise to a non-zero magnitude for the total angular momentum determined by

$$\langle J_{Nx}^2 + J_{Ny}^2\rangle = \sum_s (j_x{}^2 + j_y{}^2)\{\langle n_{s+}^2\rangle - \langle n_{s-}\rangle^2 + \langle n_{s+}^2\rangle - \langle n_{s-}\rangle^2\}, \tag{4.15}$$

for the components of angular momentum in the x, y plane are not paired and the states $s+$ and $s-$ contribute independently to the mean square. Accordingly for n_{s+} we may take the mean occupation numbers from (3.2) or (4.1, 4.2) as

$$\begin{aligned}\langle n_{s+}\rangle &= \langle n_{s+}^2\rangle = \langle n_{s-}\rangle = \langle n_{s-}^2\rangle \\ &= \frac{e^{\beta(\mu_1 - a_s - \frac{1}{2}\Delta)} + e^{2\beta(\mu_1 - a_s)}}{1 + 2e^{\beta(\mu_1 - a_s - \frac{1}{2}\Delta)} + e^{2\beta(\mu_1 - a_s)}}.\end{aligned} \tag{4.16}$$

Then from (3.2) and (4.11) we can evaluate terms of (4.15) as,

$$2\sum_s \{\langle n_{s+}^2\rangle - \langle n_{s+}\rangle^2\} = \tfrac{1}{2}\frac{\partial^2(\beta\Phi_N)}{\partial(\beta\mu_1)^2} + \frac{\partial(\beta\Phi_N)}{\partial(\beta\Delta)} = g_n t(1 + \lambda \cot \lambda) \tag{4.17}$$

after use of (2.10) and (4.12).

Now unless the shape of the nucleus is very far from spherical, we may take $\langle j_x{}^2 + j_y{}^2\rangle = 2\overline{m^2}$ and find, after combining neutron and proton contributions,

$$\langle J_x{}^2 + J_y{}^2\rangle = \overline{m^2} gt(1 + \lambda \cot \lambda) = (c + c')t \tag{4.18}$$

in the notation of (4.7) and (4.14).

If Δ/t is large, that is at low excitation energies, excitation of pairs is more important than excitation of single particles and $\lambda \cot \lambda$ and c' are small. Then most of the nuclear angular momentum is carried by excited pairs of particles or pairs of holes, which may have a resultant angular momentum Q, perpendicular to S, as indicated in fig. 1. It is noteworthy that in this case (4.18) gives

$$\langle J^2\rangle = ct \tag{4.19}$$

which is the mean square angular momentum at temperature t appropriate to a rotator with moment of inertia $\frac{1}{2}c$, just half the rigid body value. Furthermore the exclusion principle requires the resultant angular momentum of the pairs to be even and, as remarked above, symmetries and residual interactions between particles produce the value zero in the ground state.

Now it is known [9]) that the collective rotation of a distorted nucleus must be about an axis like Q perpendicular to S and that the spacings of levels of spin 0, 2, 4 correspond to nuclear moments of inertia ranging up to about half the rigid body value. These results suggest that it may be possible to associate the rotational angular momentum of the collective model with the angular momentum of excited pairs in a pairing model. Elliott [14]) has previously shown how the collective rotational bands in some light nuclei are built up from the individual particle motions. This point of view also relieves a difficulty in the work of Bohr, Mottelson and Pines [4]), who introduced a pairing energy to account for the difficulty of exciting single particles in even nuclei, but observed that this does not prevent excitation of pairs. The excited pairs need not be suppressed if they can be used to account for low energy rotational levels.

Of course the pairing model will not account for the observed regularities of rotational energy spectra without further elaboration and assumptions about the interaction between particles. But the correspondence seems close enough to justify the belief that in counting up the number of excited pair states, we are counting up the rotational states.

In view of these results and Elliott's work, we consider that the level density P of (3.12), which certainly contains all possible arrangements of the particles, must include all the rotational states. This point of view differs from Ericson's [7]) who used a formula equivalent to (3.12) to determine the number of "intrinsic states", each of which was considered as the parent of a rotational band, and so multiplied the total number of states. By this procedure, using conventional values of g and Δ, he was able to find sufficient levels of low angular momentum to agree with the experimental values at neutron binding energy. But in § 6 it will be shown that if we are limited to a total of P states, the number of levels at neutron binding energy is too small to agree with experiment unless Δ is decreased below its value for the ground state.

If the direction of the symmetry axis is not specified, equations (4.14) and (4.18) imply that the expression (4.10) for the probability of a level with total angular momentum J is modified to

$$\frac{2J+1}{\pi^{\frac{1}{2}}(2c''t)^{\frac{3}{2}}}\,\mathrm{e}^{-J(J+1)/2c''t} \qquad (4.20)$$

with c'' determined by

$$(2c'')^{\frac{3}{2}} = (2c')^{\frac{1}{2}}(c+c') \quad \text{or} \quad c'' = (c')^{\frac{1}{3}}(\tfrac{1}{2}c+\tfrac{1}{2}c')^{\frac{2}{3}}, \tag{4.21}$$

since the normalisation constant is fixed by the product of the standard deviations of the three components of J. At $t = \frac{1}{2}\Delta$, we find, from § 5, $c'' \approx \frac{1}{2}c$ and the distribution (4.20) slightly increases the proportion of states with low angular momentum in comparison with the distribution (4.10).

For comparison we quote Ericson's [7]) formula

$$\rho(U, N, Z, J) = \frac{(J+\frac{1}{2})}{K_{\mathrm{m}}}\left(\frac{3}{\pi\nu}\right)^{\frac{1}{2}} \exp - \left\{\frac{2(J+\frac{1}{2})^2}{3\tau}\left(\frac{\hbar^2}{\mathscr{I}}+\frac{3\tau}{2\nu K_{\mathrm{m}}{}^2}\right)\right\} P(U, N, Z, 0) \tag{4.22}$$

in which $\mathscr{I}$ is the moment of inertia about an axis perpendicular to the symmetry axis, K_{m} the highest value expected of the angular momentum of a single particle and ν the number of unpaired particles.

We have discussed the effects of pairing energy in relation to non-spherical nuclei because that seems its most natural application. However, *for spherical nuclei*, the dependence of the total energy on the 'seniority' quantum number may be crudely represented as a tendency for like nucleons to couple together in pairs with *total angular momentum zero*. If we represent the coupling energy by Δ, eq. (4.14) gives an estimate of the mean square angular momentum in *any* direction, not as before only in the direction of the symmetry axis S. Therefore, the probability of a nuclear level with total angular momentum J is represented approximately by (4.10) or (4.20) using the smaller moment of inertia c' for spherical nuclei.

5. Approximate Forms

Solving the equation $e^{\frac{1}{2}\Delta\beta} = \cos\lambda$ we obtain

$$\lambda^2 = \Delta\beta[1-\tfrac{1}{3}(\tfrac{1}{2}\Delta\beta)+\tfrac{2}{45}(\tfrac{1}{2}\Delta\beta)^2+\tfrac{1}{315}(\tfrac{1}{2}\Delta\beta)^3-\tfrac{2}{1575}(\tfrac{1}{2}\Delta\beta)^4\ldots] \tag{5.1a}$$

or

$$\Delta\beta = \lambda^2[1+\tfrac{1}{6}\lambda^2+\tfrac{2}{45}\lambda^4\ldots]. \tag{5.1b}$$

This will be a good approximation at high excitation and $\lambda = \frac{1}{2}\pi-\cos\lambda-\frac{1}{6}\cos^3\lambda$ is valid at low excitations. From these, at high excitations,

$$U = \tfrac{1}{6}\pi^2 g t^2 - t - \frac{g\Delta^2}{12} + \frac{g\Delta^3}{90t} + \frac{g\Delta^4}{1680t^2} + \ldots - (E_0 - E^*) \tag{5.2}$$

and

$$P(N, Z, U) = \frac{\exp\left\{\frac{1}{3}\pi^2 g t - 1 - \frac{1}{2}g\Delta + \frac{g\Delta^3}{180t^2} + \frac{g\Delta^4}{2520t^3}\cdots\right\}}{12(\pi t)^{\frac{1}{2}}(\frac{1}{6}g)^{\frac{3}{2}}} \tag{5.3}$$

or

$$P(N, Z, U) \approx \frac{\exp\{2(\frac{1}{6}\pi^2 gU')^{\frac{1}{2}} - \frac{1}{2}g\Delta\}}{12(\frac{1}{6}g)^{\frac{1}{4}}(U'+t)^{\frac{5}{4}}}, \tag{5.4}$$

where

$$U' = U + (E_0 - E^*) + \tfrac{1}{12}g\Delta^2. \tag{5.5}$$

For equations (4.20) and (4.21) a simple expression for $c''t$ will be required. From (4.14) we see that at high temperatures the even-odd dependence of c' is slight and find approximately

$$c' = \overline{m^2}gt[1-(\Delta/3t)]. \tag{5.6}$$

Any too rapid variation of the bracketted term indicates that the range of the approximation has been exceeded. It would be expected that $t > \frac{1}{2}\Delta$ would be a suitable limit for most purposes. This in turn gives the first three terms of (5.2), the first three terms of the exponential in (5.3), or (5.4) as sufficient approximation wherever the approximation is valid. Comparison of (5.4) with (A-10) of ref.[13]) shows that at high temperature the level density is like that of a free Fermi gas except that U' replaces U.

The approximation to (2.8) and (2.9)

$$I = g(\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\lambda^2) \approx \tfrac{1}{24}g\pi^2(1+3\mathrm{e}^{-0.437\beta\Delta}) \tag{5.7}$$

has been found to be very close over almost the whole range of Δ and directly shows the transition between the low temperature and high temperature limits. From this formula more widely valid formulae for U, P, and $c't$ in terms of t can be found:

$$U = \tfrac{1}{6}g\pi^2 t^2\{\tfrac{1}{4} + \tfrac{3}{4}(1+a/t)\mathrm{e}^{-a/t}\} - t; \tag{5.8}$$

$$P = \frac{\exp\{\frac{1}{3}\pi^2 gt[\frac{1}{4}+\frac{3}{4}(1+a/2t)\mathrm{e}^{-a/t}] - 1\}}{12(\pi t)^{\frac{5}{2}}(\frac{1}{6}g)^{\frac{3}{2}}}, \tag{5.9}$$

where

$$a = 0.437\Delta; \tag{5.10}$$

then, from (4.12, 4.13), for even nuclei,

$$c't = \overline{m^2}g\tfrac{1}{4}\mu\pi^2 t(0.437\mathrm{e}^{-a/t}) \approx ct\mathrm{e}^{-a/t}, \tag{5.11}$$

and according to (4.14) $c't$ must be increased by $\overline{m^2}$ for odd mass and $2\overline{m^2}$ for odd nuclei. These formulae have the advantage of being accurate over a wide range of t.

6. Comparison with Experiment

Results of scattering and evaporation experiments depend on the temperature $\tau = \mathrm{d}U/\mathrm{d}(\log P)$ and are thus much more sensitive to the form of the entropy than to any of the slowly varying factors in the expression for the level density. From (5.3) we find that τ is related to t by $\tau \approx t+\frac{1}{2}g$ so that (5.2) can be written

$$U = \tfrac{1}{6}\pi^2 g\tau^2 - \tfrac{5}{2}\tau - \tfrac{1}{12}g\Delta^2 - (E_0 - E^*); \tag{6.1}$$

although this approximation would be expected to be accurate only for energies at which $t > \frac{1}{2}\Delta$, i.e. $U > \frac{1}{3}g\Delta^2$, it should nevertheless give a useful indication in predicting the variation of τ with U. The results of computation from the accurate forms displayed in fig. 2 confirm this.

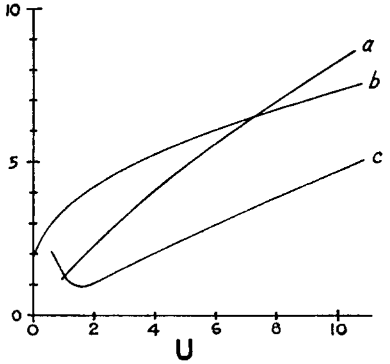

Fig. 2. a) $\log_{10}$ (level density) plotted on unit scale against U in MeV. b) Nuclear temperature τ in units 0.1 MeV plotted against U in MeV. c) Square of nuclear temperature τ^2 plotted in units 0.1 MeV^2 against $(U+\frac{5}{2}\tau)$ in MeV. All three graphs are for $g = 15\,\text{MeV}^{-1}, \Delta = 1.145$ MeV.

It is seen that some estimates of the pairing energies required can be obtained from the intercept of a graph of τ^2 against U. An attempt was made to carry out this measurement from the results of J. S. Levin and L. Cranberg[15]), but the accuracy of the experiment is still insufficient to give conclusive evidence either for or against an intercept of the required magnitude for a pairing energy constant with respect to t. It should be noted that any system whose free energy can be approximated by a form like $F = at^2 e^{-\Delta/kt}$ will give a positive intercept on the τ^2 axis.

Measurements of total absorption cross section for 1 MeV neutrons and individual resonances with thermal neutrons give an absolute knowledge of the level density of a compound nucleus near the binding energy of the last neutron added. To predict this accurately requires that all the factors involved in (4.20) and (4.21) must be evaluated. From ref. [16]) we obtain $\overline{m^2} \approx 0.146A^{\frac{2}{3}}$ and hence, using (5.4) and (5.11),

$$\rho(N, Z, U, J) = (J+\tfrac{1}{2})e^{-J(J+1)/2c''t}\,\frac{\exp\{2(\frac{1}{6}\pi^2 gU')^{\frac{1}{2}} - \frac{1}{2}g\Delta + a/t\}}{0.0636\,A^2(U'+t)^2} \tag{6.2}$$

if we use the dispersion c'' of angular momentum. The last term in the exponential is omitted if we use c. We have here assumed that approximations to Δ, g in the denominator will suffice to determine the value of the exponential term if ρ is known experimentally. As a result of Ericson's work, however, it was suspected that the value of g required would have to be large and the value of Δ smaller than that used in correcting the ground state energies. Some calculations were therefore done using values of level spacing, pairing energy and neutron binding energy taken from Stolovy and Harvey[12]) and assuming [1]) that $\frac{1}{6}g\pi^2 = \frac{1}{7}\Delta$. Despite the fact that this is an upper limit on g [1]) the values obtained for the level spacing were still too small by a factor of about 3 using c'' and about 10 using the wider angular momentum distribution c. Further, with the value of Δ assumed and the consequent value of t, the entropy is insensitive to the value of Δ (5.4). To introduce a factor 3 in the level density will then require the assumption of a pairing energy differing by thirty percent from that to correct the odd-even effect in the ground state. It was therefore not considered worth while to make a least squares fit to the data, especially as it would not be surprising if the correlation Δ between states of individual particles were less in excited states than in the ground state of the nucleus.

References

1) K. J. Le Couteur and D. W. Lang, Nuclear Physics **13** (1959) 32
2) D. L. Livesey, Can. J. Phys. **33** (1955) 391
3) G. Igo and H. E. Wegner, Phys. Rev. **102** (1956) 1364 (L)
4) A. Bohr, B. R. Mottelson and D. Pines, Phys. Rev. **110** (1958) 936
5) Torleif Ericson, Nuclear Physics **8** (1958) 265
6) N. Rosenzweig, Phys. Rev. **108** (1957) 817
7) Torleif Ericson, Nuclear Physics **6** (1958) 62
8) A. Bohr and B. R. Mottelson, Mat. Fys. Medd. Dan. Vid. Selsk. **27**, No. 16 (1953)
9) A. Bohr, Rotational States of Atomic Nuclei (København, Ejnar Munksgaards Forlag, 1954)
10) A. Bohr, Nuovo Cimento, Supplemento, **4** (1956) 1091
11) S. G. Nilsson, Mat. Fys. Medd. Dan. Vid. Selsk. **29**, No. 16 (1955)
12) A. Stolovy and J. A. Harvey, Phys. Rev. **108** (1957) 353
13) J. M. B. Lang and K. J. Le Couteur, Proc. Phys. Soc. A **67** (1954) 585
14) J. P. Elliott, Proc. Roy. Soc. A **245** (1958) 128, 562
15) J. S. Levin and L. Cranberg, O.R.N.L. 2309 (1957) 84
16) J. H. D. Jensen and J. M. Luttinger, Phys. Rev. **86** (1952) 907