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Thermal stress analysis of a trilayer film/substrate system with weak interfaces

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ABSTRACT

Trilayer film/substrate systems with weak interfaces are investigated in this article, and the closed-form analytical solution is presented, which can be readily used to evaluate the thermal residual stresses induced by temperature change. A fourth-order differential equation governing the curvature is derived and its solution is obtained. The present analytical results can be degenerated to the ones for a system with perfect interfaces, as well as those for a bilayer system with weak interface as reported previously. A quite different feature from our previous work is that certain force equilibrium conditions in the longitudinal direction must be used for trilayer systems, while they are satisfied automatically in the analysis of bilayer systems.

1. Introduction

Film/substrate systems have been widely used in many different engineering systems, electronic packaging, surface coatings, and flexible electronics [1–5]. Thermal stresses in film/substrate systems play an important role in the design and service lifetime [6–8]. When the systems are subjected to an ambient temperature change, thermal stresses will be induced due to different thermal expansion coefficients of the films and substrate, which may cause a large deformation of the system or even delamination along the interface. Thermal stresses are almost unavoidable in all deposition methods, and are one of the major causes of mechanical failures in film/substrate systems.

Over the last several decades, numerous theoretical studies have been performed to investigate the mechanical behavior of film/substrate systems. Stoney [9] first studied the mechanical behavior of film/substrate systems, and derived the relation between the stress in the film and the curvature of the substrate. Stoney's formula has been widely used as the basis of experimental techniques for measuring the stress in thin films. However, several assumptions (e.g. both the film and substrate thicknesses are small compared to the lateral dimensions, the thickness of the film is much less than that of the substrate, and the stresses in the film are invariant under the change in position) have been made in the analysis so that its applicability is quite limited. Timoshenko [10] examined bimetal thermostats submitted to a uniform heating using an elementary beam theory, and assumed that the displacement is continuous across the interfaces. This assumption has evolved into a basic hypothesis of numerous other theories. Hsueh [11] derived a general closed-form solution for elastic deformation of multilayers due to residual stresses. Zhang [12] developed an alternative analytical model in terms of the curvature radius of the neutral axis (for zero normal strain) and the normal strain at the interface between the substrate and the films, and presented a thermoelastic stress analysis of multilayered beams. The above-mentioned theoretical solutions all employed Timoshenko's method, which makes use of the force equilibrium equations to calculate the thermal stresses in bonded joints.

The aforementioned works are based on the assumption that the interfaces are perfect, which means no interfacial slip. As a matter of fact, when the films are ordinarily deposited/bonded onto the substrate by physical or chemical methods, the effect of interface between the constituents should be considered in order to better describe the mechanical properties of the systems [13]. Many efforts have been devoted to the influence of weak interface in layered structures. Chen and Nelson [14] applied the shear-lag theories [15,16] to investigate the stress distributions in bonded materials. In the study of multilayered plates with weak interfaces, Suhir [17–19] developed two models to calculate the shearing stress and peeling stress between the constituents of bimetal thermostats. Pao and Eisele [20] analyzed multilayer structures based on Suhir's model. Ru [21] presented a simple non-local modified beam theory to evaluate interfacial thermal stresses in bimaterial elastic beams. Cheng et al. [22] adopted the spring model which assumes that the normal and tangential displacement jumps occur when crossing the

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interface from one side to another while the tractions remain continuous and are proportional to the displacement jumps. The difference between the spring model and the shear-lag model is that the thickness of the interface vanishes in the spring model. Chen et al. [23–27] presented some exact solutions of elastic laminated beams/plates/shells with bonding imperfections via the state-space approach. A simple model, which assumes the deflections of both components identical along the beam axis and allows only interface slip, was employed by Newmark et al. [28] in the analysis of composite beams with incomplete interaction. Newmark's model has been widely employed by researchers in the area of civil engineering, see Goodman and Popov [29,30], Girhammar and Gopu [31], and Nie and Cai [32], to name a few. It is also noted that Chen et al. [33] developed a state-space formulation, which is very effective in the analysis of composite beam-columns with partial interaction. Recently, the formulation has been extended to dynamic problems and a new route has been suggested to establish the orthogonality of vibrational normal modes [34].

Trilayer film/substrate systems are widely used in ball-grid-array structures, thin small outline package designs, laser package assemblies, etc. A closed-form model of trilayered assembly subjected to uniform temperature change was proposed by Schmidt [35]. Suhir [36] analyzed the interfacial thermal stresses in a trimaterial assembly based on an approximate structure model. Sujan et al. [37] developed a model which accounts for different temperature conditions in each layer of a trilayer structure. It is noted that, these authors all adopted the same assumption that all components of the system have an equal deflection during deformation; the interfacial slips between components are not considered.

Recently, we obtained a closed-form solution of a bilayer system with weak interface [38]. It is shown that the presence of weak interface complicates the problem and makes the curvature of the system no longer constant. In this paper, we extend the above analysis to a trilayer structure consisting of different materials subjected to a common temperature rise. This problem is more complex than that of the bilayer system. To begin with, we adopt the aforementioned assumption that any two adjacent components of the trilayer system are always in contact (no normal displacement jump), but can slip with respect to each other along the interface (tangential displacement jump can occur). In this analysis, we shall make use of certain longitudinal force equilibrium conditions, which however are satisfied automatically in our previous analysis of bilayer systems. A fourth-order differential equation governing the curvature of the system is thus derived, in accordance with the Euler–Bernoulli beam theory. The exact solution to the equation is presented and the residual stresses in the films and the deflection of the system are obtained for illustration.

2. Basic formulations

Consider a trilayer beam system, as shown in Fig. 1. Let $2l$ be the length of the beam, b be the width and h be the total thickness, consisting of the thicknesses of two films h_1, h_2 and that of the elastic substrate h_s. The thermoelastic properties of the films and the substrate are generally different, and denoted by E_i, α_i ($i = 1, 2$) and E_s, α_s, respectively, where E is the Young's modulus and α the thermal expansion coefficient. The beam is initially free from stress, but residual thermal stresses will be induced when submitted to a uniform temperature variation T.

The global coordinate system is selected such that the axis x is located at the interface between the substrate and film 1, and the axis y placed at the mid-span of the beam. Local coordinate systems are used in each layer, with the longitudinal coordinate axes x_1, x_2 and x_s being in accordance with the centroidal principal axes (these are indicated in Fig. 1 as cg, s, cg, 1 and cg, 2), respectively, and the transverse coordinate axes y_1, y_2 and y_s still placed at the mid-span of the beam. We have the following relations between different coordinate systems:

$$y_1 = y + d_1, \quad y_2 = y + d_2, \quad y_s = y - d_s \tag{1}$$

in which $d_1 = h_1/2$, $d_2 = h_1 + h_2/2$ and $d_s = h_s/2$ indicate the distances of the centroidal principal axes of two films and the substrate from the x-axis of the whole section, as shown in Fig. 2.

With the assumption that the films and the substrate have the same curvature κ during bending, and based on the Euler–Bernoulli beam theory, we can write down the normal strains in all components as

$$\begin{aligned} &\varepsilon_1 = \kappa(y + d_1) + \varepsilon_{01}, \quad \varepsilon_2 = \kappa(y + d_2) + \varepsilon_{02}, \\ &\varepsilon_s = \kappa(y - d_s) + \varepsilon_{0s} \end{aligned} \tag{2}$$

where $\varepsilon_{01}, \varepsilon_{02}$ and ε_{0s} are the normal strains at the respective centroidal principal axes.

In view of Eq. (2), and making use of the one-dimensional constitutive relation for thermoelastic materials, we obtain the following normal stresses in the films and the substrate

$$\begin{aligned} \sigma_1 &= E_1\kappa(y + d_1) + E_1\varepsilon_{01} - E_1\alpha_1 T \\ \sigma_2 &= E_2\kappa(y + d_2) + E_2\varepsilon_{02} - E_2\alpha_2 T \\ \sigma_s &= E_s\kappa(y - d_s) + E_s\varepsilon_{0s} - E_s\alpha_s T \end{aligned} \tag{3}$$

When there are no external axial force and moment, the equilibrium of the trilayer system leads to

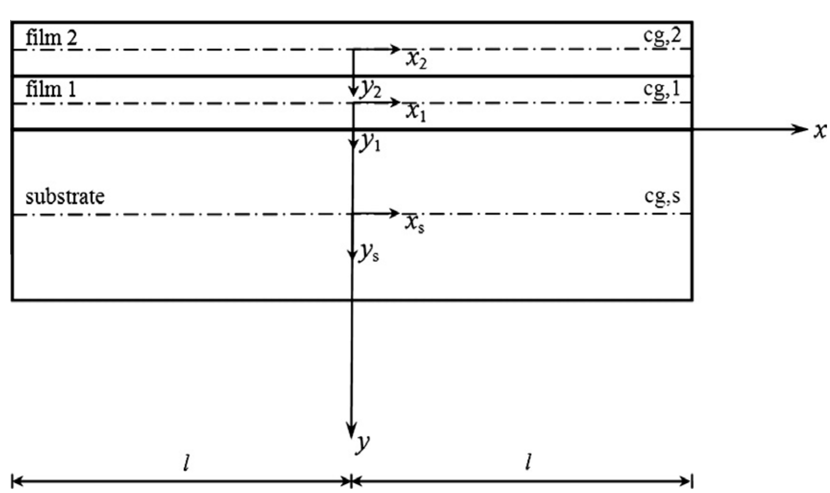

Fig. 1. Trilayer beam and the coordinate system.

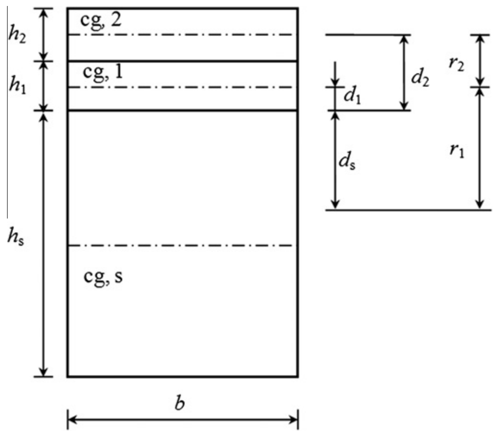

Fig. 2. Geometric parameters of the trilayer structure.

$$\begin{aligned}&\int_{A_1}\sigma_1 dA_1+\int_{A_2}\sigma_2 dA_2+\int_{A_s}\sigma_s dA_s=0\\&\int_{A_1}\sigma_1 y dA_1+\int_{A_2}\sigma_2 y dA_2+\int_{A_s}\sigma_s y dA_s=0\end{aligned}\tag{4}$$

where A_1, A_1 and A_s are the cross-sectional areas of the films and the substrate. Substituting Eq. (3) into Eq. (4) yields

$$\begin{aligned}&E_sA_s\varepsilon_{0s}+E_1A_1\varepsilon_{01}+E_2A_2\varepsilon_{02}-F^T=0\\&E_sA_sd_s\varepsilon_{0s}-E_1A_1d_1\varepsilon_{01}-E_2A_2d_2\varepsilon_{02}+EI_0\kappa+M^T=0\end{aligned}\tag{5}$$

where $EI_0=E_1I_1+E_2I_2+E_sI_s$, with $E_1I_1=E_1bh_1^3/12, E_2I_2=E_2bh_2^3/12$ and $E_sI_s=E_sbh_s^3/12$ being the bending rigidities with respect to the centroidal principal axes of the films and the substrate, and

$$\begin{aligned}&F^T=(E_sA_s\alpha_s+E_1A_1\alpha_1+E_2A_2\alpha_2)T\\&M^T=(E_1A_1d_1\alpha_1+E_2A_2d_2\alpha_2-E_sA_sd_s\alpha_s)T\end{aligned}\tag{6}$$

Consider the free body of a differential element of the trilayer system, as shown in Fig. 3. The moment, shear force, longitudinal force, and slip force per unit length are denoted by M, V, N, and τ, respectively. From the global equilibrium consideration, we know

$$\frac{dV}{dx}=-q,\quad \frac{dM}{dx}=V\tag{7}$$

which in turn give

$$\frac{d^2M}{dx^2}=-q\tag{8}$$

In this study, we assume that no external transverse force (i.e., $q=0$) is applied on the system.

Considering the equivalence of the internal and external actions on the left side of the free-body diagram in Fig. 3, we can obtain the following relations between the resultant forces applied on the whole beam section and those on the film and substrate layers:

$$\begin{aligned}&F=F_s+F_1+F_2\equiv 0\\&V=V_s+V_1+V_2\\&M=M_s+M_1+M_2+Fd_s-F_1r_1-F_2(r_1+r_2)\end{aligned}\tag{9}$$

where $r_1=(h_s+h_1)/2$ is the distance between the centroidal axes of film 1 and the substrate, $r_2=(h_1+h_2)/2$ is the distance between the centroidal axes of film 1 and film 2.

The moment equilibrium of each layer yields

$$\begin{aligned}&V_2=\frac{dM_2}{dx}+\frac{1}{2}\tau_2h_2,\quad V_1=\frac{dM_1}{dx}+\frac{1}{2}(\tau_1+\tau_2)h_1,\\&\quad V_s=\frac{dM_s}{dx}+\frac{1}{2}\tau_1h_s\end{aligned}\tag{10}$$

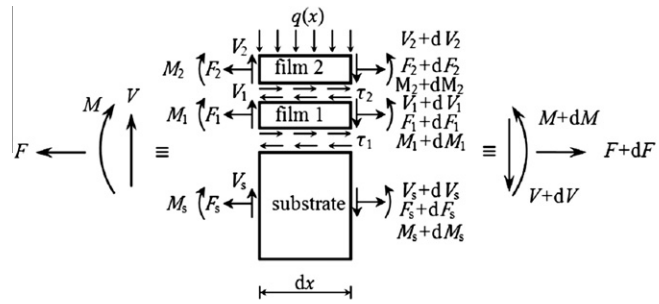

Fig. 3. Differential element in the trilayer system.

Substituting into the second in Eq. (9) yields

$$V=\frac{dM_s}{dx}+\frac{dM_1}{dx}+\frac{dM_2}{dx}+r_1\tau_1+r_2\tau_2\tag{11}$$

By differentiation, we obtain

$$\frac{dV}{dx}=\frac{d^2M_s}{dx^2}+\frac{d^2M_1}{dx^2}+\frac{d^2M_2}{dx^2}+r_1\frac{d\tau_1}{dx}+r_2\frac{d\tau_2}{dx}\equiv 0\tag{12}$$

The slip force at the weak interface is assumed to be proportional to the displacement jump, given by

$$\begin{aligned}&\tau_1=K_1\Delta u_1=K_1(u_s-u_1+r_1dw/dx)\\&\tau_2=K_2\Delta u_2=K_2(u_1-u_2+r_2dw/dx)\end{aligned}\tag{13}$$

where u_1, u_2 and u_s are the axial displacements at the centroidal principal axes of the films and the substrate, respectively, w is the deflection of the trilayer system, and K_1 and K_2 are the stiffness constants of the weak interfaces.

According to the elementary bending theory of beams, we have

$$\kappa=\frac{M_1}{E_1I_1},\quad \kappa=\frac{M_2}{E_2I_2},\quad \kappa=\frac{M_s}{E_sI_s},\quad \kappa=-\frac{d^2w}{dx^2}\tag{14}$$

Combining it with Eqs. (7), (12), and (13), yields

$$\begin{aligned}&EI_0\frac{d^2\kappa}{dx^2}-(K_1r_1^2+K_2r_2^2)\kappa+K_1r_1(\varepsilon_{0s}-\varepsilon_{01})+K_2r_2(\varepsilon_{01}-\varepsilon_{02})\\&\quad =0\end{aligned}\tag{15}$$

From Eqs. (5) and (15), we can express the normal strains in term of the curvature κ as

$$\begin{aligned}&\varepsilon_{01}=\frac{-1}{E_1A_1r_2}[E_sA_s(r_1+r_2)\varepsilon_{0s}+EI_0\kappa+M^T-F^Td_2]\\&\varepsilon_{02}=\frac{1}{E_2A_2r_2}(E_sA_sr_1\varepsilon_{0s}+EI_0\kappa+M^T-F^Td_1)\\&\varepsilon_{0s}=-\frac{1}{C_1}\frac{d^2\kappa}{dx^2}+\frac{C_2}{C_1}\kappa+\frac{C_3}{C_1}\end{aligned}\tag{16}$$

where

$$\begin{aligned}&C_1=\frac{K_1r_1}{EI_0}\left(1+\frac{E_sA_sr_1}{E_1A_1r_2}+\frac{E_sA_s}{E_1A_1}\right)-\frac{K_2E_sA_s}{EI_0}\left(\frac{r_1+r_2}{E_1A_1}+\frac{r_1}{E_2A_2}\right)\\&C_2=\frac{K_1r_1^2}{EI_0}+\frac{K_2r_2^2}{EI_0}+\frac{K_2}{E_2A_2}-\frac{K_1r_1}{E_1A_1r_2}+\frac{K_2}{E_1A_1}\\&C_3=\frac{1}{EI_0}\left\{K_1\left[\frac{E_sA_s\alpha_s(r_1+r_2)r_1}{E_1A_1r_2}+\alpha_1r_1\right]\right.\\&\qquad\left.+K_2\left[r_2(\alpha_2-\alpha_1)-E_sA_s\alpha_s\left(\frac{r_1}{E_2A_2}+\frac{r_1+r_2}{E_1A_1}\right)\right]\right\}T\end{aligned}\tag{17}$$

The force equilibrium in the longitudinal direction in each layer requires that

$$\frac{dF_2}{dx}=-\tau_2,\quad \frac{dF_1}{dx}=\tau_2-\tau_1,\quad \frac{dF_s}{dx}=\tau_1 \tag{18}$$

where the axial forces are given by

$$F_s=E_sA_s(\varepsilon_{0s}-\alpha_sT),\quad F_1=E_1A_1(\varepsilon_{01}-\alpha_1T),\\ F_2=E_2A_2(\varepsilon_{02}-\alpha_2T) \tag{19}$$

Combining it with Eqs. (13), (16), and (18), we can obtain a fourth-order ordinary differential equation in term of the curvature κ

$$\frac{d^4\kappa}{dx^4}-\mu\frac{d^2\kappa}{dx^2}+\eta\kappa+\gamma=0 \tag{20}$$

where

$$\begin{aligned}\mu&=\frac{K_1}{EI_0}r_1^2+\frac{K_2}{EI_0}r_2^2+\frac{K_1}{E_sA_s}+\frac{K_1}{E_1A_1}+\frac{K_2}{E_1A_1}+\frac{K_2}{E_2A_2}\\ \eta&=K_1K_2\left[\frac{r_2^2}{EI_0E_sA_s}+\frac{r_1^2}{EI_0E_2A_2}+\frac{(r_1+r_2)^2}{EI_0E_1A_1}\right.\\&\quad\left.+\frac{1}{E_sA_s}\frac{1}{E_2A_2}+\frac{1}{E_sA_s}\frac{1}{E_1A_1}+\frac{1}{E_1A_1E_2A_2}\right]\\ \gamma&=\frac{K_1K_2}{EI_0}\left[(\alpha_1-\alpha_s)\frac{r_1}{E_2A_2}+(\alpha_2-\alpha_1)\frac{r_2}{E_sA_s}+(\alpha_2-\alpha_s)\frac{r_1+r_2}{E_1A_1}\right]T\end{aligned} \tag{21}$$

Thus the general solution to Eq. (20) may be written as follows

$$\kappa(x)=A\cosh(\lambda_Hx)+B\cosh(\lambda_Vx)+C\sinh(\lambda_Hx)\\ +D\sinh(\lambda_Vx)-\frac{\gamma}{\eta} \tag{24}$$

where A, B, C and D are integral constants. The symmetry about the plane $x = 0$ immediately reduces the solution to

$$\kappa(x)=A\cosh(\lambda_Hx)+B\cosh(\lambda_Vx)-\frac{\gamma}{\eta} \tag{25}$$

Since no external force applied to the trilayer beam, the moments and axial force vanish at the ends, i.e.,

$$\begin{aligned}&M_s(\pm l)=M_1(\pm l)=M_2(\pm l)=0\\ &F_s(\pm l)=F_2(\pm l)=F_1(\pm l)=0\end{aligned} \tag{26}$$

Taking advantage of Eq. (14), we obtain

$$\kappa(\pm l)=0 \tag{27}$$

Making use of Eqs. (16) and (19), we obtain from the second in Eq. (26)

$$\frac{d^2\kappa(\pm l)}{dx^2}=\frac{1}{EI_0}[K_1r_1(\alpha_1-\alpha_s)+K_2r_2(\alpha_2-\alpha_1)]T \tag{28}$$

Thus the integral constants A and B can be determined as

$$\begin{aligned}A&=\frac{(\lambda_H^2-K_2/E_2A_2)K_1r_1(\alpha_1-\alpha_s)+(\lambda_H^2-K_1/E_sA_s)K_2r_2(\alpha_2-\alpha_1)+K_1K_2(\alpha_s-\alpha_2)(r_1+r_2)/E_1A_1}{\lambda_H^2(\lambda_H^2-\lambda_V^2)EI_0\cosh(\lambda_Hl)}T\\ B&=\frac{(\lambda_V^2-K_2/E_2A_2)K_1r_1(\alpha_1-\alpha_s)+(\lambda_V^2-K_1/E_sA_s)K_2r_2(\alpha_2-\alpha_1)+K_1K_2(\alpha_s-\alpha_2)(r_1+r_2)/E_1A_1}{\lambda_V^2(\lambda_V^2-\lambda_H^2)EI_0\cosh(\lambda_Vl)}T\end{aligned} \tag{29}$$

It should be pointed out that, there are three equations in Eq. (18), which, when combined with Eq. (16), will give rise to three equations about κ, among which only one equation is new and independent as shown in Eq. (20). This can be readily explained by the following two facts: (a) any two equations in Eq. (18) will lead to the third equation because of the first in Eq. (9); and (b) The combination of the sum of the first two in Eq. (18) and the third in Eq. (9) will give Eq. (12). Thus, only one in Eq. (18) is independent. It is noted that, in our previous analysis of bilayer systems, there is no need to consider the equilibrium conditions of longitudinal forces in the layers, like those given in Eq. (18), which actually can be satisfied automatically. This is a distinct feature and such equilibrium conditions shall play a role when the number of imperfect interfaces exceeds 2 in a multilayered system.

3. Exact solution

The eigenvalues of the fourth-order ordinary differential Eq. (20) can be determined as

$$\lambda_1=-\lambda_H,\quad \lambda_2=\lambda_H,\quad \lambda_3=-\lambda_V,\quad \lambda_4=\lambda_V \tag{22}$$

where

$$\lambda_H=\sqrt{\frac{1}{2}\mu-\frac{1}{2}\sqrt{\mu^2-4\eta}},\quad \lambda_V=\sqrt{\frac{\mu}{2}+\frac{1}{2}\sqrt{\mu^2-4\eta}} \tag{23}$$

Then the curvature $\kappa(x)$ of the trilayer system with weak interfaces submitted to a uniform temperature rise is completely determined, which varies with the position, just as the case of a bilayer system with weak interface [38]. From the curvature, the expressions for any physical quantities of interest can be easily derived.

For example, by substituting Eqs. (16) and (25) into Eq. (3), we can obtain the stresses in the films

$$\begin{aligned}\sigma_1&=E_1\{[(y+d_1)A+a_1]\cosh(\lambda_Hx)+[(y+d_1)B+b_1]\cosh(\lambda_Vx)\}\\&\quad+E_1\left[c_1-(y+d_1)\frac{\gamma}{\eta}-\alpha_1T\right]\\ \sigma_2&=E_2\{[(y+d_2)A+a_2]\cosh(\lambda_Hx)+[(y+d_2)B+b_2]\cosh(\lambda_Vx)\}\\&\quad+E_2\left[c_2-(y+d_2)\frac{\gamma}{\eta}-\alpha_2T\right]\end{aligned} \tag{30}$$

The average normal stress in the film is defined by

$$\bar{\sigma}_i=\frac{1}{h_i}\int_{-(d_i+\frac{h_i}{2})}^{-(d_i-\frac{h_i}{2})}\sigma_i(x,y)dy\quad (i=1,2) \tag{31}$$

Thus

$$\begin{aligned}\bar{\sigma}_1&=E_1[a_1\cosh(\lambda_Hx)+b_1\cosh(\lambda_Vx)]+E_1(c_1-\alpha_1T)\\ \bar{\sigma}_2&=E_2[a_2\cosh(\lambda_Hx)+b_2\cosh(\lambda_Vx)]+E_2(c_2-\alpha_2T)\end{aligned} \tag{32}$$

where

$$
\begin{aligned}
a_1 &= \frac{EI_0[E_sA_sK_2r_2 + E_2A_2K_1r_1 - E_sA_sE_2A_2(r_1+r_2)\lambda_H^2] + E_sA_sE_2A_2(r_1+r_2)(K_1r_1^2+K_2r_2^2)}{E_1A_1E_sA_sK_2r_1r_2 - E_1A_1E_2A_2K_1r_1r_2 - E_sA_sE_2A_2(r_1+r_2)(K_1r_1-K_2r_2)}A \\
b_1 &= \frac{EI_0[E_sA_sK_2r_2 + E_2A_2K_1r_1 - E_sA_sE_2A_2(r_1+r_2)\lambda_V^2] + E_sA_sE_2A_2(r_1+r_2)(K_1r_1^2+K_2r_2^2)}{E_1A_1E_sA_sK_2r_1r_2 - E_1A_1E_2A_2K_1r_1r_2 - E_sA_sE_2A_2(r_1+r_2)(K_1r_1-K_2r_2)}B \\
c_1 &= \frac{EI_0(E_sA_s\alpha_s + E_1A_1\alpha_1 + E_2A_2\alpha_2) + E_sA_s[E_1A_1\alpha_1r_1^2 + E_2A_2(r_1+r_2)(\alpha_2r_1+\alpha_sr_2)] + E_1A_1E_2A_2\alpha_1r_2^2}{EI_0E_sA_s + E_1A_1(EI_0+E_sA_sr_1^2+E_2A_2r_2^2) + E_2A_2[EI_0+E_sA_s(r_1+r_2)^2]}T \\
a_2 &= \frac{EI_0[E_sA_sK_2r_2 - (E_sA_s+E_1A_1)K_1r_1 + E_1A_1E_sA_s\lambda_H^2r_1] - E_1A_1E_sA_sr_1(K_1r_1^2+K_2r_2^2)}{E_1A_1E_sA_sK_2r_1r_2 - E_1A_1E_2A_2K_1r_1r_2 - E_sA_sE_2A_2(r_1+r_2)(K_1r_1-K_2r_2)}A \\
b_2 &= \frac{EI_0[E_sA_sK_2r_2 - (E_sA_s+E_1A_1)K_1r_1 + E_1A_1E_sA_sr_1\lambda_V^2] - E_1A_1E_sA_sr_1(K_1r_1^2+K_2r_2^2)}{E_1A_1E_sA_sK_2r_1r_2 - E_1A_1E_2A_2K_1r_1r_2 - E_sA_sE_2A_2(r_1+r_2)(K_1r_1-K_2r_2)}B \\
c_2 &= \frac{EI_0(E_2A_2\alpha_2 + E_sA_s\alpha_s + E_1A_1\alpha_1) + E_sA_s[E_1A_1r_1(\alpha_1r_1+\alpha_1r_2-\alpha_sr_2) + E_2A_2\alpha_2(r_1+r_2)^2] + E_1A_1E_2A_2\alpha_2r_2^2}{EI_0E_sA_s + E_1A_1(EI_0+E_sA_sr_1^2+E_2A_2r_2^2) + E_2A_2[EI_0+E_sA_s(r_1+r_2)^2]}T
\end{aligned}
\tag{33}
$$

These also correspond to the normal stresses at the centroidal axes according to the classical beam bending theory. In view of the relation $\kappa = -d^2w/dx^2$, the deflection $w(x)$ can be obtained as

$$w(x) = -\frac{A}{\lambda_H^2}\cosh(\lambda_Hx) - \frac{B}{\lambda_V^2}\cosh(\lambda_Vx) + \frac{\gamma}{2\eta}x^2 + c_1'x + c_2' \tag{34}$$

where the integral constants c_1', c_2' correspond to the rigid body rotation and translation, respectively. Setting $c_1' = 0$ and $w(0) = 0$, we obtain

$$c_2' = \frac{A}{\lambda_H^2} + \frac{B}{\lambda_V^2} \tag{35}$$

Thus, the deflection $w(x)$ of the trilayer system is given by

$$w(x) = \frac{A}{\lambda_H^2}[1-\cosh(\lambda_Hx)] + \frac{B}{\lambda_V^2}[1-\cosh(\lambda_Vx)] + \frac{\gamma}{2\eta}x^2 \tag{36}$$

4. Discussion and numerical illustration

When $K_2 \to \infty$, but K_1 is finite (i.e. the interface between the two films is perfect, while that between film 1 and the substrate is imperfect), Eq. (20) reduces to

$$\mu'\frac{d^2\kappa}{dx^2} - \eta'\kappa - \gamma' = 0 \tag{37}$$

where

$$
\begin{aligned}
\mu' &= \frac{1}{EI_0}r_2^2 + \frac{1}{E_1A_1} + \frac{1}{E_2A_2} \\
\eta' &= K_1\left[\frac{r_2^2}{EI_0E_sA_s} + \frac{r_1^2}{EI_0E_2A_2} + \frac{(r_1+r_2)^2}{EI_0E_1A_1} + \frac{1}{E_sA_s}\frac{1}{E_2A_2} + \frac{1}{E_sA_s}\frac{1}{E_1A_1} + \frac{1}{E_1A_1E_2A_2}\right]
\end{aligned}
\tag{38}
$$

$$\gamma' = \frac{K_1}{EI_0}\left[(\alpha_1-\alpha_s)\frac{r_1}{E_2A_2} + (\alpha_2-\alpha_1)\frac{r_2}{E_sA_s} + (\alpha_2-\alpha_s)\frac{r_1+r_2}{E_1A_1}\right]T$$

By further assuming that the two films have identical material properties, we find that Eq. (37) is the same as that for a bilayer system with weak interface as derived by the authors [38].

When both interfaces are perfect (i.e. $K_1, K_2 \to 0$), Eq. (37) further becomes

$$
\begin{aligned}
\kappa(x) &= -\frac{\gamma'}{\eta'} \\
&= -\frac{(\alpha_1-\alpha_s)\frac{r_1}{E_2A_2} + (\alpha_2-\alpha_1)\frac{r_2}{E_sA_s} + (\alpha_2-\alpha_s)\frac{r_1+r_2}{E_1A_1}}{\frac{r_2^2}{E_sA_s} + \frac{r_1^2}{E_2A_2} + \frac{(r_1+r_2)^2}{E_1A_1} + EI_0\left(\frac{1}{E_sA_s}\frac{1}{E_2A_2} + \frac{1}{E_sA_s}\frac{1}{E_1A_1} + \frac{1}{E_1A_1E_2A_2}\right)}T
\end{aligned}
\tag{39}
$$

which indicates that the curvature of the perfect trilayer system is constant. This formula is the same as that derived by Zhang [12].

If the beam length $2l$ approaches infinity, then each section of the system will experience the same deformation. From Eq. (29), we know that $A = B = 0$. Thus we have

$$\kappa = -\frac{\gamma}{\eta} \tag{40}$$

It is the same as Eq. (39), indicating that the weak interfaces have no effect on the curvature. This may be impractical, and to obtain a physically more reliable solution, nonlinear deformation analysis shall be performed since, with a constant curvature, the infinitely long trilayer system will evolve into an infinite number of circles of finite radius.

In the numerical computations, a simple cross-section of the laser diode is considered, with a 2 μm active GaAs layer and a 4 μm AlAs cap deposited onto a 100 μm Si substrate in turn. The length of the system is 2000 μm, and a uniform temperature rise T = 100 °C is assumed. The related material properties are given in Table 1.

The curvature $\kappa(x)$ of the trilayer system with weak interfaces are shown in Fig. 4, which indicates that the stiffness of the weak interface impacts upon the curvature significantly. The curvature tends to a constant with the increases of both stiffnesses of the two weak interfaces, as predicated by Eq. (39). Close to the edge, the curvature will approach to zero, as required by the stress-free boundary. The average stresses in the two films are given in Figs. 5 and 6, which also imply the stress-free boundary condition at the edge. It is seen that the stiffness of weak interfaces plays an important role in the trilayer system. Figs. 7 and 8 depict the shear stresses calculated from Eqs. (18) and (19). It is seen that the shear stresses vanish at both ends of the system.

Table 1
Material properties.

Properties	Si	GaAs	AlAs
E (GPa)	130.2	100	83.5
α (10^{-6}/°C)	2.59	6.86	5.2

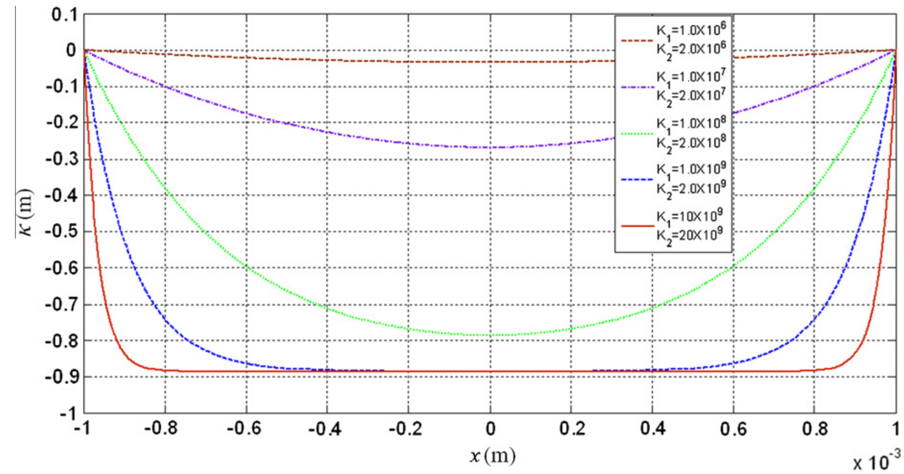

Fig. 4. Variation of curvature along the beam axis for different weak interfaces.

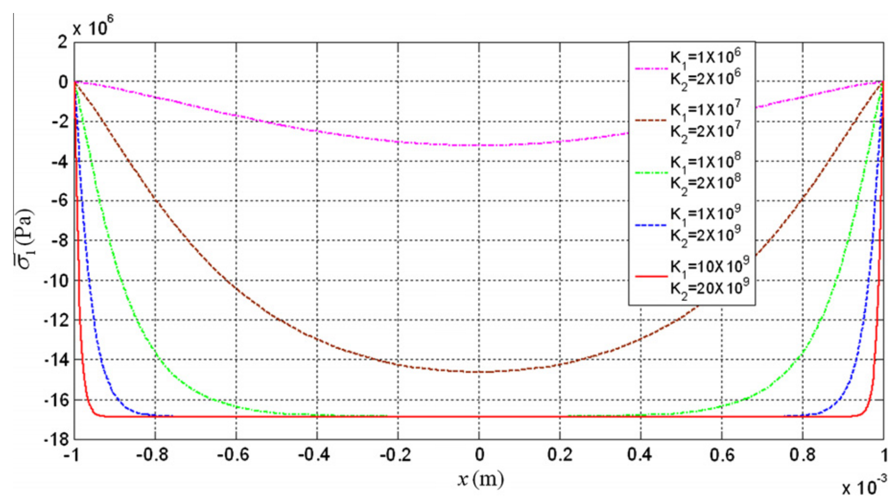

Fig. 5. Variation of average normal stress in film 1 along the beam axis for different weak interfaces.

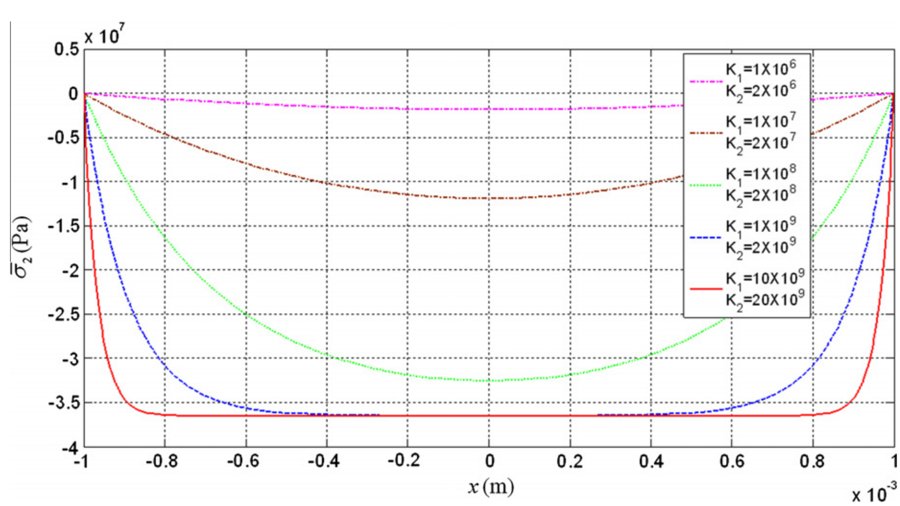

Fig. 6. Variation of average normal stress in film 2 along the beam axis for different weak interfaces.

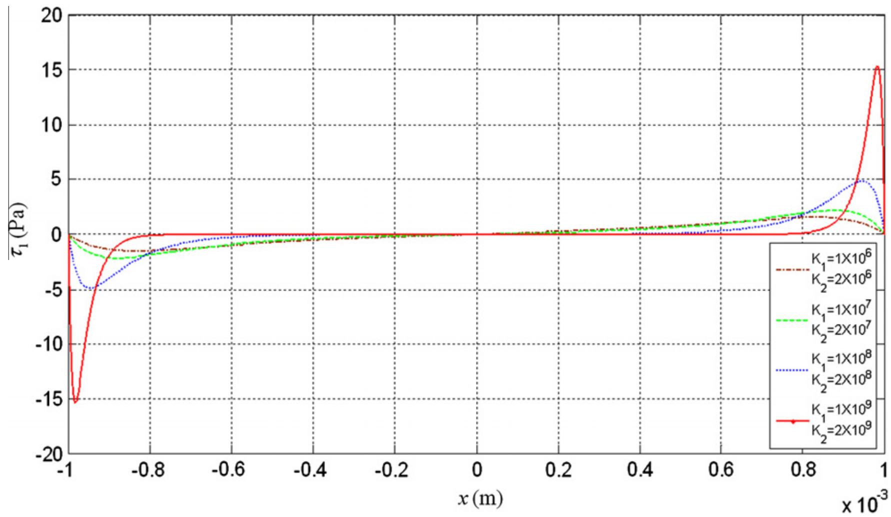

Fig. 7. Variation of interfacial shear stress τ_1 along the beam axis for different weak interfaces.

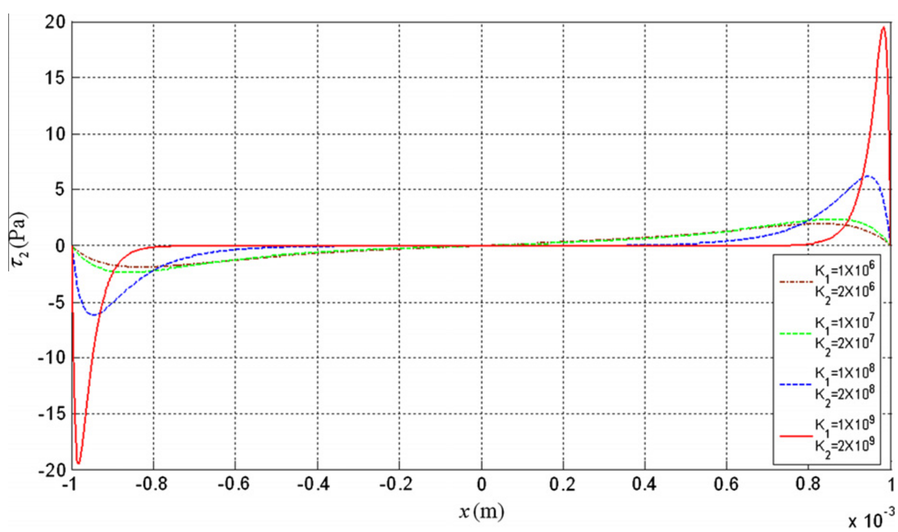

Fig. 8. Variation of interfacial shear stress τ_2 along the beam axis for different weak interfaces.

5. Conclusions

In this paper, an exact closed-form solution is derived for the residual stress problem of trilayer systems with weak interfaces subjected to temperature variation. The differential governing equation is expressed in terms of the curvature, and is of fourth-order, as compared to the second-order one for the bilayer system [38]. The solution of the equation is obtained by taking account of symmetric and boundary conditions. It is shown that the results can be readily degenerated to the trilayer system with perfect interfaces or a bilayer system with weak interface.

It is finally pointed out here that the present analysis is also extendable to multilayer structures, but tedious and complicated mathematical details will be involved inevitably. Thus, we are working on an alternative and more unified method to treat multilayer systems, which will be reported in the near future.

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