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Gentzen-Type Methods for Bilattice Negation

Abstract. A general Gentzen-style framework for handling both bilattice (or strong) negation and usual negation is introduced based on the characterization of negation by a modal-like operator. This framework is regarded as an extension, generalization or refinement of not only bilattice logics and logics with strong negation, but also traditional logics including classical logic LK, classical modal logic S4 and classical linear logic CL. Cut-elimination theorems are proved for a variety of proposed sequent calculi including CLS (a conservative extension of CL) and $\mathrm{CLS}_{\mathrm{cw}}$ (a conservative extension of some bilattice logics, LK and S4). Completeness theorems are given for these calculi with respect to phase semantics, for SLK (a conservative extension and fragment of LK and $\mathrm{CLS}_{\mathrm{cw}}$, respectively) with respect to a classical-like semantics, and for SS4 (a conservative extension and fragment of S4 and $\mathrm{CLS}_{\mathrm{cw}}$, respectively) with respect to a Kripke-type semantics. The proposed framework allows for an embedding of the proposed calculi into LK, S4 and CL.

Keywords: Bilattice negation, strong negation, sequent calculus, substructural logic, completeness.

1. Introduction

The notion of *bilattices* as a generalization of Belnap's four-valued logic [5] was first introduced by Ginsberg [9] as a tool of knowledge representation in AI, and has since been further applied by Fitting as a framework for logic programming semantics [7]. Theories and applications of various kinds of bilattices have been proposed by many researchers, as discussed in detail in a comprehensive review by Gargov [8][1].

Logics of logical bilattices were introduced by Arieli and Avron [2, 3], and were formulated using Gentzen-type cut-free sequent calculi that correspond to certain bilattices. A number of bilattice logics or four-valued logics, which are natural extensions of Belnap's four-valued logic, have also been introduced and studied comprehensively by Pynko [20]. This gives a number of cut-free sequent calculi with not only bilattice negation but also

[1]Generalized versions of bilattices, called "trilattices", were studied comprehensively by Shramko, Dunn and Takenaka [23] as constructive truth value spaces.

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usual classical negation[2]. The logics which were investigated by Arieli and Avron [2, 3] and also by Pynko [20] have a novel kind of bilattice negation that is known to have the desirable feature of *paraconsistency* [21]. This property is known to be useful to discuss inconsistency-tolerant reasoning and non-monotonic reasoning in AI. It was also pointed out by Gargov [8], in a historical account, that this kind of bilattice negation is very similar to a kind of strong negation discussed in the area of traditional mathematical logic[3].

The notion of *strong negation* was first introduced by Nelson [16], and a number of studies in connection with logics with strong negation have since been developed by many logicians and computer scientists (see e.g. [1, 24, 25]). In particular, intuitionistic substructural logics with strong negation were presented by Wansing [25], and further extended by Kamide [12]. These intuitionistic substructural logics have a characteristic property called *constructible falsity*, and can be applied to logic programming languages and concurrent process calculi. The main difference between bilattice negation and strong negation is the base logic: although bilattice logics by Arieli and Avron [2, 3] and by Pynko [20] are based on classical (substructural) positive logics, various versions of strong negation logics are based on intuitionistic (substructural) positive logics. Thus, the kind of negation analogous to bilattice negation is called strong negation in this paper.

The motivation of the present study is to give a general and widely applicable Gentzen-type framework for handling both strong negation and usual classical negation. Various kinds of cut-free sequent calculi (or extended classical substructural logics) are introduced based on the concept of strong negation as a modal-like operator. In these sequent calculi, a kind of strong negation is introduced as a new modal-like operator, representing conservative extensions of known logics including some of Arieli and Avron's bilattice logics, some of Pynko's bilattice logics, classical linear logic CL [10], classical logic LK and modal logic S4. The idea of controlling or characterizing negation using a modal-like operator is regarded as an analogue of the idea in linear logics, where structural rules are controlled by exponential modal operators. A basic sequent calculus CLS (classical linear logic with strong negation), an extended classical linear logic with two kinds of

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[2]An alternative approach to bilattice logic was also proposed by Schöter [22], called "evidential bilattice logic", which has a number of applications in knowledge representation and natural language semantics, but does not have a sequent calculus formulation.

[3]It is shown in [15] that Nelson's logic N4, Rauszer's H-B logic, Belnap's four-valued logic, dual-intuitionistic logic and Anderson-Belnap's first-degree entailment represent essentially the same meaning of "falsification" in certain restricted languages.

negation, is introduced as an example of using such negation. This extended logic is considered to be a resource-conscious refinement of Arieli and Avron's bilattice logics, Pynko's bilattice logics and the strong-negation logics mentioned above[4] because CLS has not only paraconsistency and constructible falsity but also resource-sensitivity.

This paper is organized as follows.

In Section 2, some classical substructural logics with two kinds of negation (including CLS and an extended classical logic CLS_{cw}) are introduced as two-sided sequent calculi, and the relationships between these proposed logics, Arieli and Avron's logics, Pynko's logics and other logics are clarified. A faithful embedding of the proposed logics into logics without strong negation (i.e. standard logics such as S4, LK and CL) is also presented.

In Section 3, a novel sequent calculus SLK is introduced as a conservative extension of LK and a fragment of CLS_{cw}. The completeness theorem (with respect to a simple classical-like semantics) for SLK is proved using a method by Maehara that allows the cut-elimination theorem to be derived at the same time.

In Section 4, a sequent calculus SS4, also a fragment of CLS_{cw} and a conservative extension of S4 and LK, is introduced, and the completeness theorem (with respect to a Kripke semantics) for SS4 is presented.

In Section 5, using a modified version of the phase semantic proof method posed by Okada [18], the completeness theorems and cut-elimination theorems are proved uniformly for one-sided versions of the proposed sequent calculi including CLS and CLS_{cw}.

In Section 6, two versions of cut-free one-sided calculi for the proposed calculi, called *subformula calculi* and *dual calculi*, are presented as appendices. An intuitive meaning of strong negation is clarified using these calculi.

2. Two-Sided Calculi

Before the precise discussion, we introduce the language used in this paper. *Formulas* are constructed from propositional variables, constants $\mathbf{1}, \bot$ (multiplicatives), $\mathbf{0}, \top$ (additives), $\to$ (implication), $\wedge$ (additive conjunction), $*$ (fusion or multiplicative conjunction), $\vee$ (additive disjunction), $+$ (fission, par or multiplicative disjunction), $\cdot^{\perp}$ (negation), $\sim$ (strong negation) and modal operators ! (of course), ? (why not). Lower-case letters $p, q, \ldots$ are used to denote propositional variables, Greek lower-case letters $\alpha, \beta, \ldots$ are

[4]This fact is similar to the fact that linear logic is regarded as a refinement or fine-graind form of classical and intuitionistic logics with the help of exponential modalities.

used to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used to represent finite (possibly empty) multisets of formulas. $!\Gamma$ ($?\Gamma$ or $\sim\Gamma$) denotes the multiset $\{!\gamma \mid \gamma \in \Gamma\}$ ($\{?\gamma \mid \gamma \in \Gamma\}$ or $\{\sim\gamma \mid \gamma \in \Gamma\}$ respectively). A *sequent* of two-sided calculi is an expression of the form $\Gamma \Rightarrow \Delta$. The symbol $\equiv$ is used to denote equality as sequences (or multisets) of symbols. The expression of the form $L \vdash \Gamma \Rightarrow \Delta$ means that the sequent $\Gamma \Rightarrow \Delta$ is provable in a two-sided sequent calculus L. We will sometimes omit L in this expression. Since all logics discussed in this paper are formulated as sequent calculi, we will occasionally identify a sequent calculus with the logic determined by it.

DEFINITION 2.1 (CLS). We define a basic two-sided sequent calculus CLS for the classical linear logic with strong negation.

The initial sequents of CLS are of the forms:

$$\alpha \Rightarrow \alpha \qquad \Rightarrow \mathbf{1} \qquad \bot \Rightarrow \qquad \Gamma \Rightarrow \Delta, \top \qquad \mathbf{0}, \Gamma \Rightarrow \Delta$$

$$\sim\mathbf{1} \Rightarrow \qquad \Rightarrow \sim\bot \qquad \sim\top, \Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \sim\mathbf{0}$$

The rules of inferences of CLS are of the forms:

$$\frac{\Delta \Rightarrow \Pi, \alpha \quad \alpha, \Sigma \Rightarrow \Gamma}{\Delta, \Sigma \Rightarrow \Pi, \Gamma}\ (\text{cut}) \quad \frac{\Gamma \Rightarrow \Delta}{\mathbf{1}, \Gamma \Rightarrow \Delta}\ (\mathbf{1}\text{we}) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \bot}\ (\bot\text{we})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha{\rightarrow}\beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({\rightarrow}\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha{\rightarrow}\beta}\ ({\rightarrow}\text{right})$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha * \beta, \Gamma \Rightarrow \Delta}\ ({*}\text{left}) \qquad \frac{\Gamma \Rightarrow \Sigma, \alpha \quad \Delta \Rightarrow \Pi, \beta}{\Gamma, \Delta \Rightarrow \Sigma, \Pi, \alpha * \beta}\ ({*}\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge\text{left1}) \quad \frac{\beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge\text{left2}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ (\wedge\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}\ (\vee\text{right1}) \quad \frac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}\ (\vee\text{right2})$$

$$\frac{\alpha, \Gamma \Rightarrow \Sigma \quad \beta, \Delta \Rightarrow \Pi}{\alpha + \beta, \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ ({+}\text{left}) \qquad \frac{\Gamma \Rightarrow \Pi, \alpha, \beta}{\Gamma \Rightarrow \Pi, \alpha + \beta}\ ({+}\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\alpha^{\perp}, \Gamma \Rightarrow \Delta}\ (\perp\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha^{\perp}}\ (\perp\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim\sim\alpha, \Gamma \Rightarrow \Delta}\ (\sim\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \sim\sim\alpha}\ (\sim\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta}{\sim\bot, \Gamma \Rightarrow \Delta}\ (\sim\bot\text{we}) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim\mathbf{1}}\ (\sim\mathbf{1}\text{we})$$

$$\frac{\Gamma \Rightarrow \Delta, \sim\alpha \quad \sim\beta, \Sigma \Rightarrow \Pi}{\sim(\alpha{\rightarrow}\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim{\rightarrow}\text{left}) \quad \frac{\sim\alpha, \Gamma \Rightarrow \Delta, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha{\rightarrow}\beta)}\ (\sim{\rightarrow}\text{right})$$

$$\frac{{\sim}\alpha, {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha * \beta), \Gamma \Rightarrow \Delta}\ ({\sim}{*}\text{left}) \quad \frac{\Gamma \Rightarrow \Sigma, {\sim}\alpha \quad \Delta \Rightarrow \Pi, {\sim}\beta}{\Gamma, \Delta \Rightarrow \Sigma, \Pi, {\sim}(\alpha * \beta)}\ ({\sim}{*}\text{right})$$

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \Delta \quad {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ ({\sim} \wedge \text{left})$$

$$\frac{\Gamma \Rightarrow \Delta, {\sim}\alpha}{\Gamma \Rightarrow \Delta, {\sim}(\alpha \wedge \beta)}\ ({\sim} \wedge \text{right1}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha \wedge \beta)}\ ({\sim} \wedge \text{right2})$$

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \Delta}{{\sim}(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ ({\sim} \vee \text{left1}) \quad \frac{{\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ ({\sim} \vee \text{left2})$$

$$\frac{\Gamma \Rightarrow \Delta, {\sim}\alpha \quad \Gamma \Rightarrow \Delta, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha \vee \beta)}\ ({\sim} \vee \text{right})$$

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \Sigma \quad {\sim}\beta, \Delta \Rightarrow \Pi}{{\sim}(\alpha + \beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ ({\sim}{+}\text{left}) \qquad \frac{\Gamma \Rightarrow \Pi, {\sim}\alpha, {\sim}\beta}{\Gamma \Rightarrow \Pi, {\sim}(\alpha + \beta)}\ ({\sim}{+}\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, {\sim}\alpha}{{\sim}(\alpha^{\perp}), \Gamma \Rightarrow \Delta}\ ({\sim}{\perp}\text{left}) \quad \frac{{\sim}\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha^{\perp})}\ ({\sim}{\perp}\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{!\alpha, \Gamma \Rightarrow \Delta}\ (!\text{left}) \quad \frac{!\Gamma_1, {\sim}!\Gamma_2 \Rightarrow ?\Delta_1, {\sim}?\Delta_2, \alpha}{!\Gamma_1, {\sim}!\Gamma_2 \Rightarrow ?\Delta_1, {\sim}?\Delta_2, !\alpha}\ (!\text{right})$$

$$\frac{\alpha, !\Gamma_1, {\sim}!\Gamma_2 \Rightarrow ?\Delta_1, {\sim}?\Delta_2}{?\alpha, !\Gamma_1, {\sim}!\Gamma_2 \Rightarrow ?\Delta_1, {\sim}?\Delta_2}\ (?\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, ?\alpha}\ (?\text{right})$$

$$\frac{!\alpha, !\alpha, \Gamma \Rightarrow \Delta}{!\alpha, \Gamma \Rightarrow \Delta}\ (!\text{co}) \qquad \frac{\Gamma \Rightarrow \Delta, ?\alpha, ?\alpha}{\Gamma \Rightarrow \Delta, ?\alpha}\ (?\text{co})$$

$$\frac{\Gamma \Rightarrow \Delta}{!\alpha, \Gamma \Rightarrow \Delta}\ (!\text{we}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, ?\alpha}\ (?\text{we})$$

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \Delta}{{\sim}!\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}!\text{left}) \quad \frac{!\Gamma_1, {\sim}!\Gamma_2 \Rightarrow ?\Delta_1, {\sim}?\Delta_2, {\sim}\alpha}{!\Gamma_1, {\sim}!\Gamma_2 \Rightarrow ?\Delta_1, {\sim}?\Delta_2, {\sim}!\alpha}\ ({\sim}!\text{right})$$

$$\frac{{\sim}\alpha, !\Gamma_1, {\sim}!\Gamma_2 \Rightarrow ?\Delta_1, {\sim}?\Delta_2}{{\sim}?\alpha, !\Gamma_1, {\sim}!\Gamma_2 \Rightarrow ?\Delta_1, {\sim}?\Delta_2}\ ({\sim}?\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim}\alpha}{\Gamma \Rightarrow \Delta, {\sim}?\alpha}\ ({\sim}?\text{right})$$

$$\frac{{\sim}!\alpha, {\sim}!\alpha, \Gamma \Rightarrow \Delta}{{\sim}!\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}!\text{co}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}?\alpha, {\sim}?\alpha}{\Gamma \Rightarrow \Delta, {\sim}?\alpha}\ ({\sim}?\text{co})$$

$$\frac{\Gamma \Rightarrow \Delta}{{\sim}!\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}!\text{we}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}?\alpha}\ ({\sim}?\text{we})$$

We remark that the exchange rules are omitted since we have agreed that the antecedents and succeedents of the sequents in this system are multisets.

We consider the following structural rules:

$$\frac{\alpha, \alpha, \Pi \Rightarrow \Delta}{\alpha, \Pi \Rightarrow \Delta}\ (\text{co-left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \alpha}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{co-right})$$

$$\frac{\Pi \Rightarrow \Delta}{\alpha, \Pi \Rightarrow \Delta}\ (\text{we-left}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{we-right})$$

We define the calculi:

$\mathrm{CLS_w}$ = CLS+(we-left)+(we-right),

$\mathrm{CLS_c}$ = CLS+(co-left)+(co-right),

$\mathrm{CLS_{cw}}$ = $\mathrm{CLS_w}$+(co-left)+(co-right).

The $\{!, ?, \cdot^{\perp}, \rightarrow\}$-free part of $\mathrm{CLS_{cw}}$ is equivalent to GBL(4), which is a logic of logical bilattices by Arieli and Avron, and the constant-free fragment of GBL(4) is called GBL (see [2, 3]). The $\{\wedge, \vee, \sim\}$-part of CLS is Avron's basic system BS [4]. The rules ($\sim\perp$left) and ($\sim\perp$right) are from Pynko, and the $\{!, ?, \rightarrow\}$-free part of $\mathrm{CLS_{cw}}$ is a logic by Pynko [20], referred to here as PL. The $\{\wedge, \vee, \sim\}$-part of $\mathrm{CLS_{cw}}$ is Belnap's four-valued logic, called BL here. The $\{\wedge, \vee, \rightarrow, \cdot^{\perp}\}$-part of $\mathrm{CLS_{cw}}$ is equivalent to a sequent calculus for classical logic without constants, and is called LK in this paper. The $\{\sim, *, +, \mathit{constants}\}$-free part of $\mathrm{CLS_{cw}}$ is equivalent to a sequent calculus for the bi-modal S4 logic without constants, and is called Bi-S4 here. The $\sim$-free part of CLS is the classical linear logic CL[5].

For any formulas α and β, the expression $\alpha \leftrightarrow \beta$ means that both the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$ are provable in CLS. Then, we have the following laws with respect to $\sim$[6]: $\sim(\alpha{\rightarrow}\beta) \leftrightarrow (\sim\alpha{\rightarrow}\sim\beta)$, $\sim(\alpha^{\perp}) \leftrightarrow (\sim\alpha)^{\perp}$, $\sim(\alpha \wedge \beta) \leftrightarrow \sim\alpha \vee \sim\beta$, $\sim(\alpha \vee \beta) \leftrightarrow \sim\alpha \wedge \sim\beta$, $\sim(\alpha * \beta) \leftrightarrow \sim\alpha * \sim\beta$, $\sim(\alpha + \beta) \leftrightarrow \sim\alpha + \sim\beta$, $\sim(!\alpha) \leftrightarrow !(\sim\alpha)$, and $\sim(?\alpha) \leftrightarrow ?(\sim\alpha)$.

The self-duality laws $\sim(!\alpha) \leftrightarrow\ !(\sim\alpha)$ and $\sim(?\alpha) \leftrightarrow\ ?(\sim\alpha)$, which correspond to postulating the inference rules concerning $\{\sim, !, ?\}$, seem to be unusual, because the standard duality laws are $\sim(!\alpha) \leftrightarrow\ ?(\sim\alpha)$ and $\sim(?\alpha) \leftrightarrow\ !(\sim\alpha)$, which are modal versions of the laws $\sim(\alpha \wedge \beta) \leftrightarrow \sim\alpha \vee \sim\beta$ and $\sim(\alpha \vee \beta) \leftrightarrow \sim\alpha \wedge \sim\beta$, respectively[7]. However, since the operators ! and ? are known as infinite versions of the multiplicative connectives $*$ and $+$, respectively, the self-duality laws with respect to ! and ? are regarded as natural modal versions of the laws $\sim(\alpha{*}\beta) \leftrightarrow \sim\alpha{*}\sim\beta$ and $\sim(\alpha{+}\beta) \leftrightarrow \sim\alpha{+}\sim\beta$, respectively. The standard duality laws can also be adopted for the underlying logic as a natural choice. But, in this paper, we adopt the self-duality laws with motivations of linear logic, e.g. concurrency theory. As mentioned in [12], a formula $\sim\alpha$ can read as "action (or process) α is stopped or

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[5]Strictly speaking, in the case of Bi-S4 and CL, the rules (!right) and (?left) need to modified by deleting $\sim!\Gamma_2$ and $\sim?\Delta_2$.

[6]Of course, we also have the standard laws (in the classical linear logic) with respect to $\cdot^{\perp}$: $(\alpha \wedge \beta)^{\perp} \leftrightarrow \alpha^{\perp} \vee \beta^{\perp}$, $(\alpha \vee \beta)^{\perp} \leftrightarrow \alpha^{\perp} \wedge \beta^{\perp}$, $(\alpha * \beta)^{\perp} \leftrightarrow \alpha^{\perp} + \beta^{\perp}$, $(\alpha + \beta)^{\perp} \leftrightarrow \alpha^{\perp} * \beta^{\perp}$, $(\alpha{\rightarrow}\beta)^{\perp} \leftrightarrow \alpha^{\perp} + \beta$, $(!\alpha)^{\perp} \leftrightarrow ?(\alpha^{\perp})$ and $(?\alpha)^{\perp} \leftrightarrow !(\alpha^{\perp})$.

[7]As a motivation of intuitionistic modal logic with strong negation, to analyze the duality laws is known to be an important issue [17].

suspended" as a concurrency-theoretic reading. For example, a concurrency-theoretic meaning of $\sim(!\alpha) \leftrightarrow\ !(\sim\alpha)$ is that the fact "infinitely duplicated actions $!\alpha$ are stopped" is equivalent to the fact "stop action $\sim\alpha$ is executed for each (infinitely) duplicated action α".

The law $\sim(\alpha{\rightarrow}\beta) \leftrightarrow (\sim\alpha{\rightarrow}\sim\beta)$, which corresponds to the inference rules ($\sim\rightarrow$left) and ($\sim\rightarrow$right), means that the strong negation connective $\sim$ is introduced as a modal-like operator, since the modal logic K has the law $K(\alpha{\rightarrow}\beta){\rightarrow}(K\alpha{\rightarrow}K\beta)$. On the other hand, the law $(K\alpha{\rightarrow}K\beta){\rightarrow}K(\alpha{\rightarrow}\beta)$ is not a law of K, and hence $\sim$ is different from the K-type operator. The operator $\sim$ is, indeed, a kind of *spatial modal operators* introduced in [14] rather than the standard modal operator. We explain the analogy between $\sim$ and these spatial modal operators. Let S be a non-empty set of locations (or spaces). Then, the spatial modal operators $[l_i]$ ($l_i \in S$), which are introduced in [14] based on a linear logic, are, roughly speaking, axiomatized as follows: for any $l, k \in S$, $[l](\alpha{\rightarrow}\beta) \leftrightarrow [l]\alpha{\rightarrow}[l]\beta$, $[l](\alpha * \beta) \leftrightarrow [l]\alpha * [l]\beta$, $[l]!\alpha \leftrightarrow ![l]\alpha$, $[l][k]\alpha \leftrightarrow [k]\alpha$,

$$\frac{\alpha}{[l]\alpha}\ , \qquad\qquad \frac{\forall s \in S\ ([s]\alpha)}{\alpha}\ .$$

In this axiomatization, the law $[l](\alpha{\rightarrow}\beta) \leftrightarrow [l]\alpha{\rightarrow}[l]\beta$ is the same setting as $\sim(\alpha{\rightarrow}\beta) \leftrightarrow \sim\alpha{\rightarrow}\sim\beta$. The axiomatization derives a possible reading of a formula $[l]\alpha$ as "α is true at location l". Analogously, we may read $\sim\alpha$ as "α is negated at a location". A possible reading of $\sim(\alpha{\rightarrow}\beta) \leftrightarrow \sim\alpha{\rightarrow}\sim\beta$ is thus "if $\alpha{\rightarrow}\beta$ is negated at a location, then α is negated at the location implies β is negated at the location, and vice versa". Furthermore, a special case of the spatial operators is regarded as the involution operator which appears in the algebraic structures of *involutive quantales* introduced by Mulvey and Pelletier. Roughly speaking, the involution operator $\cdot^{\bullet}$ in a logic corresponding to the involutive quantales is considered to be a single domain version (i.e. S is singleton) of the spatial operators (see e.g. [13]). The setting of $\sim(\alpha{\rightarrow}\beta) \leftrightarrow \sim\alpha{\rightarrow}\sim\beta$ for $\sim$ is thus also the same as that for $\cdot^{\bullet}$. In addition, another possible reading of $\sim(\alpha{\rightarrow}\beta) \leftrightarrow \sim\alpha{\rightarrow}\sim\beta$ is motivated in concurrency theory: "to stop the action $\alpha{\rightarrow}\beta$ (first α invokes, then β invokes)" is equivalent to "first α halts, then β halts".

The law $\sim(\alpha{\rightarrow}\beta) \leftrightarrow (\sim\alpha{\rightarrow}\sim\beta)$ corresponds to postulating ($\sim\rightarrow$left) and ($\sim\rightarrow$right), although, other settings are considered. For example, we can consider the following inference rules corresponding to $\sim(\alpha{\rightarrow}\beta) \leftrightarrow \alpha * \sim\beta$.

$$\frac{\alpha, \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Sigma, \alpha \quad \Delta \Rightarrow \Pi, \sim\beta}{\Gamma, \Delta \Rightarrow \Sigma, \Pi, \sim(\alpha{\rightarrow}\beta)}$$

A number of $\{$constants$, !, ?, \cdot^{\perp}\}$-free sequent calculi that have these inference rules instead of ($\sim\!\to$left) and ($\sim\!\to$right) are discussed in [3, 11, 20][8]. The following inference rules were presented by Crolard [6].

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\alpha - \beta, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Sigma, \alpha \quad \beta, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Sigma, \Pi, \alpha - \beta}$$

where the connective "$-$" denotes the "subtraction operator". Since these rules by Crolard are similar to the rules presented above, we can approximately regard $\sim(\alpha\to\beta)$ as "subtraction". Alternative inference rules concerning $\sim$ and $\to$ have also been proposed in [3]. These inference rules can provide many different meanings for knowledge representation in AI.

It is also remarked that we can also consider the following inference rules corresponding to the axiom scheme $\sim(\alpha\to\beta) \leftrightarrow (\alpha\to\sim\beta)$, which is introduced by Wansing [28] in order to axiomatize a basic system of *connexive modal logic*.

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha\to\beta)} \qquad \frac{\Gamma \Rightarrow \Sigma, \alpha \quad \sim\beta, \Delta \Rightarrow \Pi}{\sim(\alpha\to\beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi}$$

The setting of the rules ($\sim\!\to$left) and ($\sim\!\to$right) provides very simple and uniform versions of one-sided calculi, phase semantics and two-valued like semantics, as well as Kripke semantics[9]. This is a merit of introducing the setting.

Various other negation rules can be considered. For example, the following inference rules by Arieli and Avron [3], called the inference rules of *conflation*[10], can be introduced.

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\sim(\alpha^{\perp}), \Gamma \Rightarrow \Delta} \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim(\alpha^{\perp})}$$

These rules correspond to the law $\sim(\alpha^{\perp}) \leftrightarrow \alpha^{\perp}$. The following inference rules can also be considered.

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim(\alpha^{\perp}), \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \sim(\alpha^{\perp})}$$

[8] These calculi have the cut-elimination property, and the inference rules corresponding to $\sim(\alpha\to\beta) \leftrightarrow \alpha + \sim\beta$ can be considered. The proposed framework can deal with such variations in a similar way.

[9] We can give the cut-elimination theorems and the completeness theorems (with respect to phase semantics) for the setting of rules corresponding to $\sim(\alpha\to\beta) \leftrightarrow \alpha * \sim\beta$. In the SLK/SS4-like frameworks presented in Sections 3 and 4, we can also give the cut-elimination theorems and the completeness theorems (with respect to classical-like semantics and Kripke semantics) for the setting of rules corresponding to $\sim(\alpha\to\beta) \leftrightarrow \alpha \wedge \sim\beta$.

[10] Strictly speaking, in [3], $\sim$ is $-$ (conflation), and $\cdot^{\perp}$ is $\sim$ (bilattice negation).

These rules correspond to the law $\sim(\alpha^{\perp}) \leftrightarrow \alpha$. We can deal with these kinds of negation using the proposed framework in a similar way with some appropriate modifications, however, we will discuss only the first setting in detail here.

The following theorem is derived from the phase semantic completeness proof in a later section.

THEOREM 2.2 (Cut-Elimination). *Let L be* CLS, $\mathrm{CLS_w}$, $\mathrm{CLS_c}$ *or* $\mathrm{CLS_{cw}}$. *The rule* (cut) *is admissible in cut-free L.*

As a corollary of Theorem 2.2, we can obtain various basic results.

COROLLARY 2.3 (Conservativity). *Let* Bi-S4, LK, CL, GB(4), PK, BL *and* BS *be the sequent calculi of bi-modal logic, classical logic, classical linear logic, Arieri and Avron's bilattice logic, Pynko's bilattice logic, Belnap's four-valued logic, and Avron's basic system, respectively.* (1) $\mathrm{CLS_{cw}}$ *is a conservative extension of* Bi-S4, LK, GB(4), PK *and* BL. (2) CLS *is a conservative extension of* CL *and* BS.

Since this corollary shows the importance of discussing the logics $\mathrm{CLS_{cw}}$ and CLS, we will focus on here the logics CLS and $\mathrm{CLS_{cw}}$, along with two remarkable fragments of $\mathrm{CLS_{cw}}$: SLK and SS4.

Paraconsistency is usually defined with respect to consequence relations [21]. Here, however, it is defined with respect to sequents. Let $\neg$ be a negation connective. A sequent calculus L is called *explosive* with respect to $\neg$ if for any formulas α and β, the sequent $\alpha, \neg\alpha \Rightarrow \beta$ is provable in L. It is called *paraconsistent* with respect to $\neg$ if it is not explosive with respect to $\neg$.

COROLLARY 2.4 (Paraconsistency). (1) $\mathrm{CLS_{cw}}$ *is paraconsistent with respect to* $\sim$. (2) $\mathrm{CLS_{cw}}$ *is explosive with respect to* $\cdot^{\perp}$.

The subsystems of $\mathrm{CLS_{cw}}$ discussed above have the same property.

The following property, called "constructible falsity", is an important property for logics with strong negation (see e.g. [16, 25]).

COROLLARY 2.5 (Constructible Falsity for CLS). *For any formulas α and β, if $\Rightarrow \sim(\alpha \wedge \beta)$ is provable in* CLS, *then so is $\Rightarrow \sim\alpha$ or $\Rightarrow \sim\beta$.*

We explain some of the differences between $\sim$ and $\cdot^{\perp}$ as follows. Let p and q be distinct propositional variables. The sequents $p, \sim p \Rightarrow q$ and $\Rightarrow p \vee \sim p$ are not provable in $\mathrm{CLS_{cw}}$, but $p, p^{\perp} \Rightarrow q$ and $\Rightarrow p \vee p^{\perp}$ are provable in

$\mathrm{CLS}_{\mathrm{cw}}$. Similarly, $p, {\sim}p \Rightarrow ?q$ and $\Rightarrow ?(p \vee {\sim}p)$ are not provable in CLS, but $p, p^{\perp} \Rightarrow ?q$ and $\Rightarrow ?(p \vee p^{\perp})$ are provable in CLS.

Next we give an embedding of CLS into CL, which is a slight modification of the embedding presented by Wansing [27] and originally by Rautenberg. We fix a set $PROP$ of propositional variables, used as a component of the language of CLS, and define the set $PROP' := \{p' \mid p \in PROP\}$ of propositional variables. The language L_{CLS} of CLS is defined by using $PROP$, $\mathbf{1}, \top, \bot, \mathbf{0}$, $\rightarrow, \wedge, \vee, *, +, !, ?, \cdot^{\perp}$ and ${\sim}$. The language L_{CL} of CL is obtained from L_{CLS} by adding $PROP'$ and by deleting ${\sim}$.

DEFINITION 2.6 (Embedding). A mapping f from L_{CLS} to L_{CL} is defined inductively as follows: $f(p) := p$ and $f({\sim}p) := p' \in PROP'$ for any $p \in PROP$, $f(\mathbf{1}) := \mathbf{1}$, $f(\mathbf{0}) := \mathbf{0}$, $f(\top) := \top$, $f(\bot) := \bot$, $f(\alpha \circ \beta) := f(\alpha) \circ f(\beta)$ where $\circ \in \{\rightarrow, *, +, \wedge, \vee\}$, $f(\alpha^{\perp}) := (f(\alpha))^{\perp}$, $f(\circ\alpha) := \circ f(\alpha)$ where $\circ \in \{!, ?\}$, $f({\sim}{\sim}\alpha) := f(\alpha)$, $f({\sim}\mathbf{1}) := \bot$, $f({\sim}\mathbf{0}) := \top$, $f({\sim}\top) := \mathbf{0}$, $f({\sim}\bot) := \mathbf{1}$, $f({\sim}(\alpha \circ \beta)) := f({\sim}\alpha) \circ f({\sim}\beta)$ where $\circ \in \{\rightarrow, *, +\}$, $f({\sim}(\alpha \wedge \beta)) := f({\sim}\alpha) \vee f({\sim}\beta)$, $f({\sim}(\alpha \vee \beta)) := f({\sim}\alpha) \wedge f({\sim}\beta)$, $f({\sim}(\alpha^{\perp})) := (f({\sim}\alpha))^{\perp}$, and $f({\sim}(\circ\alpha)) := \circ f({\sim}\alpha)$ where $\circ \in \{!, ?\}$.

Let Γ be a multiset of formulas in L_{CLS}. Then, $f(\Gamma)$ denotes the result of replacing every occurence of a formula α in Γ by an occurence of $f(\alpha)$.

PROPOSITION 2.7. *Let Γ and Δ be a multisets of formulas in L_{CLS}.* $\mathrm{CLS} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathrm{CL} \vdash f(\Gamma) \Rightarrow f(\Delta)$.

With some appropriate modifications, this type of the embedding is applicable to other calculi discussed in this paper.

3. SLK and Completeness

In the following, we consider formulas constructed from propositional variables, $\bot, \top, \rightarrow, \wedge, \vee, \cdot^{\perp}$ and ${\sim}$. The following discussion is mainly based on Ono's textbook [19].

DEFINITION 3.1 (SLK). A sequent calculus SLK is obtained from the $\{\rightarrow, \wedge, \vee, \cdot^{\perp}, {\sim}\}$-fragment of $\mathrm{CLS}_{\mathrm{cw}}$ by adding the initial sequents of the forms $(\bot \Rightarrow), (\Rightarrow \top)$, $({\sim}\top \Rightarrow)$ and $(\Rightarrow {\sim}\bot)$[11].

[11] We remark that in $\mathrm{CLS}_{\mathrm{cw}}$, the formulas $\mathbf{1}, \top, {\sim}\bot$ and ${\sim}\mathbf{0}$ are logically equivalent, and also so are $\mathbf{0}, \bot, {\sim}\top$ and ${\sim}\mathbf{1}$. Thus, in $\mathrm{CLS}_{\mathrm{cw}}$, we can adopt the initial sequents (w.r.t. the constants) of the forms $(\Rightarrow \top)$, $(\bot \Rightarrow)$, $({\sim}\top \Rightarrow)$, $(\Rightarrow {\sim}\bot)$, and can delete the rules ($\mathbf{1}$we), ($\bot$we), (${\sim}\bot$we), (${\sim}\mathbf{1}$we), and the initial sequents of the forms $(\Gamma \Rightarrow \Delta, \top)$, $(\mathbf{0}, \Gamma \Rightarrow \Delta)$, $({\sim}\mathbf{1} \Rightarrow)$, $({\sim}\top, \Gamma \Rightarrow \Delta)$, $(\Gamma \Rightarrow \Delta, {\sim}\mathbf{0})$. Hence, SLK is in fact the $\{\top, \bot, \rightarrow, \wedge, \vee, \cdot^{\perp}, {\sim}\}$-fragment of $\mathrm{CLS}_{\mathrm{cw}}$.

Let Γ be a multiset $\{\alpha_1, ..., \alpha_m\}$ $(m \geq 0)$. Then Γ^* is defined as $\alpha_1 \vee \cdots \vee \alpha_m$ if $m \geq 1$, and $\perp$ if $m = 0$. Also Γ_* is defined as $\alpha_1 \wedge \cdots \wedge \alpha_m$ if $m \geq 1$, and $\top$ if $m = 0$[12].

DEFINITION 3.2. Valuations v^+ and v^- are mappings from the set of all propositional variables to the set $\{t, f\}$. These valuations v^+ and v^- are extended to mappings from the set of all formulas to $\{t, f\}$ by

1. $v^+(\top) = t$ and $v^+(\perp) = f$,
2. $v^+(\alpha \wedge \beta) = t$ iff $v^+(\alpha) = v^+(\beta) = t$,
3. $v^+(\alpha \vee \beta) = t$ iff $v^+(\alpha) = t$ or $v^+(\beta) = t$,
4. $v^+(\alpha{\rightarrow}\beta) = t$ iff $v^+(\alpha) = f$ or $v^+(\beta) = t$,
5. $v^+(\alpha^\perp) = t$ iff $v^+(\alpha) = f$,
6. $v^+({\sim}\alpha) = t$ iff $v^-(\alpha) = t$,
7. $v^-(\top) = f$ and $v^-(\perp) = t$,
8. $v^-(\alpha \wedge \beta) = t$ iff $v^-(\alpha) = t$ or $v^-(\beta) = t$,
9. $v^-(\alpha \vee \beta) = t$ iff $v^-(\alpha) = v^-(\beta) = t$,
10. $v^-(\alpha{\rightarrow}\beta) = t$ iff $v^-(\alpha) = f$ or $v^-(\beta) = t$,
11. $v^-(\alpha^\perp) = t$ iff $v^-(\alpha) = f$,
12. $v^-({\sim}\alpha) = t$ iff $v^+(\alpha) = t$.

A formula α is called *tautology* if $v^+(\alpha) = t$ holds for any valuations v^+ and v^-. A sequent of the form $\Gamma \Rightarrow \Delta$ is called tautology if so is the formula $\Gamma_*{\rightarrow}\Delta^*$.

THEOREM 3.3 (Soundness for SLK). *For any sequent S, if* SLK $\vdash S$, *then S is tautology.*

THEOREM 3.4 (Completeness for SLK). *For any sequent S, if S is tautology, then* SLK$-$(cut) $\vdash S$.

Combining this theorem with the soundness theorem, we can obtain the cut-elimination theorem for SLK[13].

[12]In the following discussion, the commutativity of $\wedge$ or $\vee$ is assumed. We have the following fact: for any formulas $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n$, the sequent $\alpha_1, ..., \alpha_m \Rightarrow \beta_1, ..., \beta_n$ is provable in SLK if and only if so is $\alpha_1 \wedge \cdots \wedge \alpha_m \Rightarrow \beta_1 \vee \cdots \vee \beta_n$. Also, in the sequent expression of the form $\Gamma \Rightarrow \Delta$, the expressions Γ and Δ are considered as sets of formulas.

[13]Of course, the cut-elimination theorem for SLK is also an immediate consequence of that for $\mathrm{CLS_{cw}}$.

THEOREM 3.5 (Cut-Elimination for SLK). *The rule* (cut) *is admissible in cut-free* SLK.

We note that SLK is a conservative extension of LK and Belnap's logic BL.

In the following, we will prove Theorem 3.4 by using the method by Maehara, which is presented in [19].

DEFINITION 3.6. A *decomposition* of a sequent S is defined as of the form S' or $S'; S''$ by

(1a) $\Gamma \Rightarrow \Delta, \alpha$; $\Gamma \Rightarrow \Delta, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha \wedge \beta$,

(1b) $\alpha, \beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha \wedge \beta, \Gamma \Rightarrow \Delta$,

(2a) $\Gamma \Rightarrow \Delta, \alpha, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha \vee \beta$,

(2b) $\alpha, \Gamma \Rightarrow \Delta$; $\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha \vee \beta, \Gamma \Rightarrow \Delta$,

(3a) $\alpha, \Gamma \Rightarrow \Delta, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha \rightarrow \beta$,

(3b) $\Gamma \Rightarrow \Delta, \alpha$; $\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha \rightarrow \beta, \Gamma \Rightarrow \Delta$,

(4a) $\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha^{\perp}$,

(4b) $\Gamma \Rightarrow \Delta, \alpha$ is a decomposition of $\alpha^{\perp}, \Gamma \Rightarrow \Delta$,

(5a) $\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim\sim\alpha, \Gamma \Rightarrow \Delta$,

(5b) $\Gamma \Rightarrow \Delta, \alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim\sim\alpha$,

(6a) $\Gamma \Rightarrow \Delta, \sim\alpha, \sim\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim(\alpha \wedge \beta)$,

(6b) $\sim\alpha, \Gamma \Rightarrow \Delta$; $\sim\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim(\alpha \wedge \beta), \Gamma \Rightarrow \Delta$,

(7a) $\Gamma \Rightarrow \Delta, \sim\alpha$; $\Gamma \Rightarrow \Delta, \sim\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim(\alpha \vee \beta)$,

(7b) $\sim\alpha, \sim\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim(\alpha \vee \beta), \Gamma \Rightarrow \Delta$,

(8a) $\sim\alpha, \Gamma \Rightarrow \Delta, \sim\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim(\alpha \rightarrow \beta)$,

(8b) $\Gamma \Rightarrow \Delta, \sim\alpha$; $\sim\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta$,

(9a) $\sim\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim(\alpha^{\perp})$,

(9b) $\Gamma \Rightarrow \Delta, \sim\alpha$ is a decomposition of $\sim(\alpha^{\perp}), \Gamma \Rightarrow \Delta$.

A *decomposition tree of S* is a tree which expresses a process of some repeated decomposition of S. In other words, a decomposition tree corresponds to a bottom up proof seach tree. We remark that any repeated decomposition process terminates by the definition of decomposition. A *complete decomposition tree of S* is a decomposition tree of S in which all the formulas occuring in all the leaves of the tree are one of the following forms: $p, \sim p, \top, \perp, \sim\top$ and $\sim\perp$.

LEMMA 3.7. *Let S_1 or S_1 ; S_2 be a decomposition of S. If S is tautology, then so are S_1 and S_2.*

PROOF. We show only (8b).

(8b): Suppose that $\sim(\alpha\to\beta) \wedge \Gamma_*\to\Delta^*$ is tautology. First, we show that $\Gamma_*\to\Delta^* \vee \sim\alpha$ is tautology. Suppose that (1) $v^+(\Gamma_*) = t$. We show $v^+(\Delta^* \vee \sim\alpha) = t$. If $v^+(\sim\alpha) = t$, then $v^+(\Delta^* \vee \sim\alpha) = t$. Thus, suppose that $v^+(\sim\alpha) = v^-(\alpha) = f$. Then, $v^-(\alpha\to\beta) = t$, and hence (2) $v^+(\sim(\alpha\to\beta)) = v^-(\alpha\to\beta) = t$. On the other hand, we have (3) $v^+(\sim(\alpha\to\beta)\wedge\Gamma_*\to\Delta^*) = t$ by the hypothesis. Thus, we obtain $v^+(\Delta^*) = t$ by (1), (2) and (3). Therefore $v^+(\Delta^* \vee \sim\alpha) = t$. Second, we show that $\sim\beta \wedge \Gamma_*\to\Delta^*$ is tautology. Suppose that (4) $v^+(\sim\beta\wedge\Gamma_*) = t$. Then, (5) $v^+(\sim\beta) = v^-(\beta) = t$ and (6) $v^+(\Gamma_*) = t$. By (5), we have $v^-(\alpha\to\beta) = t$, and hence (7) $v^+(\sim(\alpha\to\beta)) = v^-(\alpha\to\beta) = t$. By (3), (6) and (7), we obtain the required fact $v^+(\Delta^*) = t$. ■

LEMMA 3.8. (1) *suppose that each α_i or β_j in $\{\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n\}$ is a propositional constant, a propositional variable or the formula of the form $\sim\gamma$ where γ is a propositional variable or constant. Then, the sequent $\alpha_1, ..., \alpha_m \Rightarrow \beta_1, ..., \beta_n$ is tautology if and only if* (a) *there is α_i ($i \leq m$) such that $\alpha_i \equiv \bot$ or $\alpha_i \equiv \sim\top$,* (b) *there is β_i ($i \leq n$) such that $\beta_i \equiv \top$ or $\beta_i \equiv \sim\bot$, or* (c) *there are α_i ($i \leq m$) and β_j ($j \leq n$) such that $\alpha_i \equiv \beta_j$.* (2) *the sequents of the forms $(\alpha, \Gamma \Rightarrow \Delta, \alpha)$, $(\bot, \Gamma \Rightarrow \Delta), (\Gamma \Rightarrow \Delta, \top)$, $(\sim\top, \Gamma \Rightarrow \Delta)$ and $(\Gamma \Rightarrow \Delta, \sim\bot)$ are provable in cut-free* SLK.

LEMMA 3.9. *Let S_1 (or S_1 ; S_2) be a decomposition of S. If S_1 (or S_2) is provable in cut-free* SLK, *then so is S.*

Now we prove the completeness theorem. Suppose that a sequent S is tautology. Then all the leaves of a complete decomposition tree of S are tautologies by using Lemma 3.7 repeatedly. Then, these leaves are provable in cut-free SLK by Lemma 3.8 (1) (2). By using Lemma 3.9 repeatedly for the complete decomposition tree of S, all the sequents in the tree are proveble in cut-free SLK. Therefore, in particular, S is provable in cut-free SLK.

4. SS4 and Completeness

In the following, we consider formulas constructed from propositional variables, $\to, \wedge, \vee, \cdot^{\bot}, \sim$ and $\heartsuit$. Also, $\heartsuit\Gamma$ denotes the multiset $\{\heartsuit\gamma \mid \gamma \in \Gamma\}$. Suppose that Γ is a multiset $\{\alpha_1, ..., \alpha_m\}$ ($m \geq 0$), and that p is a fixed propositional variable. Then Γ^* is defined as $\alpha_1 \vee \cdots \vee \alpha_m$ if $m \geq 1$, and $(p\to p)^{\bot}$ if $m = 0$. Also Γ_* is defined as $\alpha_1 \wedge \cdots \wedge \alpha_m$ if $m \geq 1$, and $p\to p$ if $m = 0$. In the following discussion, we consult [19].

Definition 4.1 (SS4). A sequent calculus SS4 is obtained from the $\{\top, \bot\}$-free SLK[14] by adding the inference rules of the forms:

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\heartsuit\alpha, \Gamma \Rightarrow \Delta}\ (\heartsuit\text{left}), \qquad \frac{\heartsuit\Gamma, {\sim}\heartsuit\Delta \Rightarrow \alpha}{\heartsuit\Gamma, {\sim}\heartsuit\Delta \Rightarrow \heartsuit\alpha}\ (\heartsuit\text{right}),$$

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \Delta}{{\sim}\heartsuit\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\heartsuit\text{left}), \qquad \frac{\heartsuit\Gamma, {\sim}\heartsuit\Delta \Rightarrow {\sim}\alpha}{\heartsuit\Gamma, {\sim}\heartsuit\Delta \Rightarrow {\sim}\heartsuit\alpha}\ ({\sim}\heartsuit\text{right}).$$

Theorem 4.2 (Cut-Elimination for SS4). *The rule* (cut) *is admissible in cut-free* SS4.

Definition 4.3. A structure $\langle M, R\rangle$ is called a Kripke frame if (1) M is a non-empty set and (2) R is a transitive and reflexive binary relation on M.

Valuations v^+ and v^- are mappings from the set of all propositional variables to the power set of M. These valuations v^+ and v^- are extended to mappings from the set of all formulas to the power set of M by

1. ($x \models^+ p$ iff $x \in v^+(p)$) and ($x \models^- p$ iff $x \in v^-(p)$) for any propositional variable p,
2. $x \models^+ \alpha \wedge \beta$ iff $x \models^+ \alpha$ and $x \models^+ \beta$,
3. $x \models^+ \alpha \vee \beta$ iff $x \models^+ \alpha$ or $x \models^+ \beta$,
4. $x \models^+ \alpha \rightarrow \beta$ iff not ($x \models^+ \alpha$) or $x \models^+ \beta$,
5. $x \models^+ \alpha^\perp$ iff not ($x \models^+ \alpha$),
6. $x \models^+ \heartsuit\alpha$ iff $\forall y \in M\ (xRy \Longrightarrow y \models^+ \alpha)$,
7. $x \models^+ {\sim}\alpha$ iff $x \models^- \alpha$,
8. $x \models^- \alpha \wedge \beta$ iff $x \models^- \alpha$ or $x \models^- \beta$,
9. $x \models^- \alpha \vee \beta$ iff $x \models^- \alpha$ and $x \models^- \beta$,
10. $x \models^- \alpha \rightarrow \beta$ iff not ($x \models^- \alpha$) or $x \models^- \beta$,
11. $x \models^- \alpha^\perp$ iff not ($x \models^- \alpha$),
12. $x \models^- \heartsuit\alpha$ iff $\forall y \in M\ (xRy \Longrightarrow y \models^- \alpha)$,
13. $x \models^- {\sim}\alpha$ iff $x \models^+ \alpha$.

[14] We can consider the logic SS4 with the addition of $\top$ and $\bot$, however, for the sake of simplicity of the discussion, we adopt the $\{\top, \bot\}$-less logic. We can also introduce alternative modal logics based on the standard logics such as K, KT and S5, and can prove the completeness theorems for such logics, with some appropriate modifications of the present framework.

A formula α is *valid in a Kripke frame* $\langle M, R\rangle$ if $x \models^{+} \alpha$ holds for any valuations $\models^{+}$, $\models^{-}$ and any $x \in M$. A sequent $\Gamma \Rightarrow \Delta$ is valid in a Kripke frame if so is the formula $\Gamma_{*} \rightarrow \Delta^{*}$.

We can give the following theorem by using a standard method with some appropriate modification.

THEOREM 4.4 (Soundness and Completeness for SS4). *For any sequent S, $\mathrm{SS4} \vdash S$ if and only if S is valid in any Kripke frame.*

We give a proof of this theorem. Since the soundness part is obvious, we consider the completeness part in the following.

DEFINITION 4.5. Let Φ be the set of all formulas of SS4, and $U, V \subseteq \Phi$. A pair (U, V) is called *consistent* if for any $\alpha_1, ..., \alpha_m \in U$ and any $\beta_1, ..., \beta_n \in V$ (m and n are arbitrary finite fixed integers and $m, n \geq 0$), the sequent $\alpha_1, ..., \alpha_m \Rightarrow \beta_1, ..., \beta_n$ is not provable in SS4. A pair (U, V) is called *maximal consistent* if (1) (U, V) is consistent and (2) $U \cup V = \Phi$.

LEMMA 4.6. *If a pair (U_0, V_0) is consistent, then there are $U, V \in \Phi$ such that $U_0 \subseteq U$, $V_0 \subseteq V$ and (U, V) is maximal consistent.*

We define a canonical model $\langle M_L, R_L, \models_L^{+}, \models_L^{-}\rangle$ in the following. We define $M_L := \{U(\subseteq \Phi) \mid (U, \Phi - U)$ is maximal consistent$\}$ and $U_{\heartsuit} := \{\alpha \mid \heartsuit\alpha \in U\} \cup \{{\sim}\alpha \mid {\sim}\heartsuit\alpha \in U\}$ for any $U \in M_L$. For any $U_1, U_2 \in M_L$, $U_1 R_L U_2$ is defined as $(U_1)_{\heartsuit} \subseteq (U_2)_{\heartsuit}$. For any $U \in M_L$ and any propositional variable p, $U \models_L^{+} p$ is defined as $p \in U$ and $U \models_L^{-} p$ is defined as ${\sim}p \in U$.

LEMMA 4.7. *Let $U \in M_L$. (1) if $\alpha_1, ..., \alpha_m \in U$ and $\mathrm{SS4} \vdash \alpha_1, ..., \alpha_m \Rightarrow \beta$, then $\beta \in U$. (2) for any formula α, either $\alpha \in U$ or $\alpha^{\perp} \in U$.*

By using this lemma, we can show the following.

COROLLARY 4.8. *Let $U \in M_L$. (1) $\alpha \wedge \beta \in U$ iff $\alpha \in U$ and $\beta \in U$. (2) $\alpha \vee \beta \in U$ iff $\alpha \in U$ or $\beta \in U$. (3) $\alpha \rightarrow \beta \in U$ iff $\alpha \notin U$ or $\beta \in U$. (4) $\alpha^{\perp} \in U$ iff $\alpha \notin U$. (5) $\heartsuit\alpha \in U$ iff $\forall W \in M_L$ $[U R_L W \Longrightarrow \alpha \in W]$. (6) $\alpha \in U$ iff ${\sim}{\sim}\alpha \in U$. (7) ${\sim}(\alpha \wedge \beta) \in U$ iff ${\sim}\alpha \in U$ or ${\sim}\beta \in U$. (8) ${\sim}(\alpha \vee \beta) \in U$ iff ${\sim}\alpha \in U$ and ${\sim}\beta \in U$. (9) ${\sim}(\alpha \rightarrow \beta) \in U$ iff ${\sim}\alpha \notin U$ or ${\sim}\beta \in U$. (10) ${\sim}(\alpha^{\perp}) \in U$ iff ${\sim}\alpha \notin U$. (11) ${\sim}\heartsuit\alpha \in U$ iff $\forall W \in M_L$ $[U R_L W \Longrightarrow {\sim}\alpha \in W]$.*

PROOF. We show only (11).

(11): ($\Longrightarrow$) Suppose ${\sim}\heartsuit\alpha \in U$, $U R_L W$ and $W \in M_L$. Then we have ${\sim}\alpha \in U_{\heartsuit} \subseteq W_{\heartsuit}$, and hence $\heartsuit{\sim}\alpha \in W$ or ${\sim}\heartsuit\alpha \in W$. In the former case, by using Lemma 4.7 (1) and the fact $\vdash \heartsuit{\sim}\alpha \Rightarrow {\sim}\alpha$, we obtain ${\sim}\alpha \in W$. In the latter case, by using Lemma 4.7 (1) and the fact $\vdash {\sim}\heartsuit\alpha \Rightarrow {\sim}\alpha$, we obtain ${\sim}\alpha \in W$. ($\Longleftarrow$) We show the contraposition. Suppose ${\sim}\heartsuit\alpha \notin U$. Then (*): $(U_\heartsuit, \{{\sim}\alpha\})$ is consistent (this is proved later). By using Lemma 4.6, we have that there is a maximal consistent pair (W, V) such that $U_\heartsuit \subseteq W$ and ${\sim}\alpha \in V$. Then we have $W \in M_L$ and ${\sim}\alpha \notin W$. Moreover we have (**): $U_\heartsuit \subseteq W$ implies $U_\heartsuit \subseteq W_\heartsuit$ (this is proved later). Therefore we have the required fact that there is $W \in M_L$ such that UR_LW and ${\sim}\alpha \notin W$.

We show the remained fact (*). Suppose that $(U_\heartsuit, \{{\sim}\alpha\})$ is not consistent. Then there are $\beta_1, ..., \beta_n, {\sim}\delta_1, ..., {\sim}\delta_o \in U_\heartsuit$ (and $\heartsuit\beta_1, ..., \heartsuit\beta_n, {\sim}\heartsuit\delta_1, ..., {\sim}\heartsuit\delta_o \in U$) such that $\vdash \beta_1, ..., \beta_n, {\sim}\delta_1, ..., {\sim}\delta_o \Rightarrow {\sim}\alpha$. Applying ($\heartsuit$left), (${\sim}\heartsuit$left), (${\sim}\heartsuit$right), we obtain $\vdash \heartsuit\beta_1, ..., \heartsuit\beta_n, {\sim}\heartsuit\delta_1, ..., {\sim}\heartsuit\delta_o \Rightarrow {\sim}\heartsuit\alpha$. By Lemma 4.7 (1), and $\heartsuit\beta_1, ..., \heartsuit\beta_n, {\sim}\delta_1, ..., {\sim}\heartsuit\delta_o \in U$, we obtain ${\sim}\heartsuit\alpha \in U$. This contradicts for the assumption ${\sim}\heartsuit\alpha \notin U$. Therefore $(U_\heartsuit, \{{\sim}\alpha\})$ is consistent.

We show the remained fact that (**): $U_\heartsuit \subseteq W$ implies $U_\heartsuit \subseteq W_\heartsuit$. Suppose $\gamma \in U_\heartsuit$. Then $\heartsuit\gamma \in U$ or ($\gamma \equiv {\sim}\beta$ and ${\sim}\heartsuit\beta \in U$). By Lemma 4.7 and the facts $\vdash \heartsuit\gamma \Rightarrow \heartsuit\heartsuit\gamma$ and $\vdash {\sim}\heartsuit\beta \Rightarrow \heartsuit{\sim}\heartsuit\beta$, we obtain $\heartsuit\heartsuit\gamma \in U$ or $\heartsuit{\sim}\heartsuit\beta \in U$. In the latter case, by using Lemma 4.7 (1) and the fact $\vdash \heartsuit{\sim}\heartsuit\beta \Rightarrow \heartsuit\heartsuit{\sim}\beta$, we also obtain $\heartsuit\heartsuit\gamma \in U$. Thus, we have $\heartsuit\gamma \in U_\heartsuit \subseteq W$ by the assumption. Therefore $\gamma \in W_\heartsuit$. ∎

By using Corollary 4.8, we can prove the following statement: for any $U \in M_L$ and any formula α, (1) ($U \models^+_L \alpha$ iff $\alpha \in U$) and (2) ($U \models^-_L \alpha$ iff ${\sim}\alpha \in U$). This can be proved by (simaltaneous) induction on the complexity of the formula α. We show only the case $\alpha \equiv \heartsuit\beta$ for (2) as follows: $U \models^-_L \heartsuit\beta$ iff $\forall W \in M_L$ $[(U)_\heartsuit \subseteq (W)_\heartsuit \Longrightarrow W \models^-_L \beta]$ iff $\forall W \in M_L$ $[(U)_\heartsuit \subseteq (W)_\heartsuit \Longrightarrow {\sim}\beta \in W]$ (by the induction hypothesis for (2)) iff ${\sim}\heartsuit\beta \in U$ (by Corollary 4.8 (11)).

Now, we prove the completeness theorem by using the canonical model. Suppose that a sequent $\Gamma \Rightarrow \Delta$ is valid in every Kripke frame. Moreover suppose that $\Gamma \Rightarrow \Delta$ is not provable in SS4. Let $\Gamma \equiv \{\alpha_1, ..., \alpha_m\}$, $\Delta \equiv \{\beta_1, ..., \beta_n\}$ and $m, n \geq 0$. Then, the pair $(\{\alpha_1, ..., \alpha_m\}, \{\beta_1, ..., \beta_n\})$ is consistent. By using Lemma 4.6, we have that there is a maximal consistent pair (U, V) such that $\{\alpha_1, ..., \alpha_m\} \subseteq U$ and $\{\beta_1, ..., \beta_n\} \subseteq V$. Then, we have $U \in M_L$. Taking the statement (1) proved above, we obtain that $U \models^+_L \alpha_i$ ($i = 1, ..., m$) and not $[U \models^+_L \beta_j]$ ($j = 1, ..., n$). Hence, we have that not $[U \models^+_L (\alpha_1 \wedge \cdots \wedge \alpha_m) \rightarrow (\beta_1 \vee \cdots \vee \beta_n)]$. This contradicts for the hypothesis. Therefore SS4 $\vdash \Gamma \Rightarrow \Delta$.

5. One-Sided Calculi, Cut-Elimination and Completeness

For the sake of simplicity of the completeness and cut-elimination proofs, we introduce one-sided calculi for CLS, $\mathrm{CLS_w}$, $\mathrm{CLS_c}$ and $\mathrm{CLS_{cw}}$. These calculi and the two-sided calculi are essentially the same thing by assuming the De Morgan and other laws w.r.t. $\cdot^{\perp}$ (see [10]). We thus use the same names CLS, $\mathrm{CLS_w}$, $\mathrm{CLS_c}$ and $\mathrm{CLS_{cw}}$ for the one-sided calculi. An expression $\vdash \Gamma$ is a sequent of one-sided calculi where Γ denotes the multiset of formulas. We remark that $\Gamma \Rightarrow \Delta$ of the two-sided calculi corresponds to $\vdash \Gamma^{\perp}, \Delta$ in the one-sided calculi where $\Gamma^{\perp} \equiv \{\gamma_1^{\perp}, \cdots, \gamma_n^{\perp}\}$ if $\Gamma \equiv \{\gamma_1, \cdots, \gamma_n\}$. Moreover remark that if the cut-elimination theorem holds for a one-sided calculus, then the theorem also holds for the corresponding two-sided calculus.

Definition 5.1 (One-Sided CLS). We define a one-sided calculus CLS. The initial sequents of CLS are of the forms:

$$\vdash \alpha, \alpha^{\perp} \qquad \vdash \mathbf{1} \qquad \vdash \Gamma, \top \qquad \vdash \Gamma, {\sim}\mathbf{0} \qquad \vdash {\sim}\bot$$

The rules of inferences of CLS are of the forms:

$$\frac{\vdash \Gamma, \alpha \quad \vdash \Delta, \alpha^{\perp}}{\vdash \Gamma, \Delta}\ (\mathrm{cut}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ (\bot) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, {\sim}\mathbf{1}}\ ({\sim}\mathbf{1})$$

$$\frac{\vdash \Gamma, \alpha \quad \vdash \Delta, \beta}{\vdash \Gamma, \Delta, \alpha * \beta}\ (*) \qquad \frac{\vdash \Gamma, \alpha, \beta}{\vdash \Gamma, \alpha + \beta}\ (+)$$

$$\frac{\vdash \Gamma, \alpha \quad \vdash \Gamma, \beta}{\vdash \Gamma, \alpha \wedge \beta}\ (\wedge) \qquad \frac{\vdash \Gamma, \alpha}{\vdash \Gamma, \alpha \vee \beta}\ (\vee 1) \qquad \frac{\vdash \Gamma, \beta}{\vdash \Gamma, \alpha \vee \beta}\ (\vee 2)$$

$$\frac{\vdash \Gamma, \alpha}{\vdash \Gamma, {\sim}{\sim}\alpha}\ ({\sim}) \qquad \frac{\vdash \Gamma, {\sim}\alpha \quad \vdash \Delta, {\sim}\beta}{\vdash \Gamma, \Delta, {\sim}(\alpha * \beta)}\ ({\sim}*) \qquad \frac{\vdash \Gamma, {\sim}\alpha, {\sim}\beta}{\vdash \Gamma, {\sim}(\alpha + \beta)}\ ({\sim}+)$$

$$\frac{\vdash \Gamma, {\sim}\alpha \quad \vdash \Gamma, {\sim}\beta}{\vdash \Gamma, {\sim}(\alpha \vee \beta)}\ ({\sim}\vee) \qquad \frac{\vdash \Gamma, {\sim}\alpha}{\vdash \Gamma, {\sim}(\alpha \wedge \beta)}\ ({\sim} \wedge 1) \qquad \frac{\vdash \Gamma, {\sim}\beta}{\vdash \Gamma, {\sim}(\alpha \wedge \beta)}\ ({\sim} \wedge 2)$$

$$\frac{\vdash ?\Gamma_1, {\sim}?\Gamma_2, \alpha}{\vdash ?\Gamma_1, {\sim}?\Gamma_2, !\alpha}\ (!) \qquad \frac{\vdash \Gamma, \alpha}{\vdash \Gamma, ?\alpha}\ (?) \qquad \frac{\vdash \Gamma, ?\alpha, ?\alpha}{\vdash \Gamma, ?\alpha}\ (?\mathrm{co}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?\alpha}\ (?\mathrm{we})$$

$$\frac{\vdash ?\Gamma_1, {\sim}?\Gamma_2, {\sim}\alpha}{\vdash ?\Gamma_1, {\sim}?\Gamma_2, {\sim}!\alpha}\ ({\sim}!) \qquad \frac{\vdash \Gamma, {\sim}\alpha}{\vdash \Gamma, {\sim}?\alpha}\ ({\sim}?) \qquad \frac{\vdash \Gamma, {\sim}?\alpha, {\sim}?\alpha}{\vdash \Gamma, {\sim}?\alpha}\ ({\sim}?\mathrm{co}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, {\sim}?\alpha}\ ({\sim}?\mathrm{we})$$

We define the structural rules:

$$\frac{\vdash \Gamma, \alpha, \alpha}{\vdash \Gamma, \alpha}\ (\mathrm{co}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \alpha}\ (\mathrm{we})$$

and the calculi:

$\text{CLS}_\text{w} = \text{CLS}+(\text{we})$,

$\text{CLS}_\text{c} = \text{CLS}+(\text{co})$,

$\text{CLS}_\text{cw} = \text{CLS}+(\text{co})+(\text{we})$.

We remark that the one-sided CL of the original classical linear logic is obtained from the one-sided CLS by deleting the initial sequents and rules with respect to $\sim$, and by deleting $\sim ?\Gamma_2$ in the rule (!).

Next, we define phase semantics for these logics. The difference between such semantics and the original semantics is only the definition of the valuations: whereas the original semantics has a valuation ϕ, our semantics has two kinds of valuations ϕ^+ and ϕ^-, where ϕ^+ is the same as ϕ.

Definition 5.2. Let $\langle M, \cdot, 1\rangle$ be a commutative monoid with the unit 1. If $X, Y \subseteq M$, we define $X \circ Y := \{x \cdot y \mid x \in X \text{ and } y \in Y\}$[15]. A *phase spase* is a structure $\langle M, \hat{\bot}, \hat{I}\rangle$ where $\hat{\bot}$ is a fixed subset of M. For $X \subseteq M$, we define $X^{\hat{\bot}} := \{y \mid \forall x \in X\ (x \cdot y \in \hat{\bot})\}$. We define $\hat{I} := \{x \in M \mid x \cdot x = x\} \cap \hat{\bot}^{\hat{\bot}}$. X $(\subseteq M)$ is called a *fact* if $X^{\hat{\bot}\hat{\bot}} = X$. The set of facts is denoted by D_M.

Definition 5.3. A phase space $\langle M, \hat{\bot}, \hat{I}\rangle$ is called a *weakening phase space* if the following weakening-condition holds: $\hat{\bot}^{\hat{\bot}} = M$. A phase space $\langle M, \hat{\bot}, \hat{I}\rangle$ is called a *contraction phase space* if the following contraction-condition holds: $z \cdot x \cdot x \in \hat{\bot}$ implies $z \cdot x \in \hat{\bot}$ for all $x, z \in M$. A phase space $\langle M, \hat{\bot}, \hat{I}\rangle$ is called a *contraction-weakening phase space* if both the contraction-condition and the weakening-condition hold.

Proposition 5.4. *Let $X, Y \subseteq M$. (1) $X \subseteq Y^{\hat{\bot}}$ iff $X \circ Y \subseteq \hat{\bot}$. (2) if $X \subseteq Y$ then $X \circ Y^{\hat{\bot}} \subseteq \hat{\bot}$. (3) $X \circ X^{\hat{\bot}} \subseteq \hat{\bot}$. (4) if $X \subseteq Y$ then $Y^{\hat{\bot}} \subseteq X^{\hat{\bot}}$. (5) if $X \subseteq Y$ then $X^{\hat{\bot}\hat{\bot}} \subseteq Y^{\hat{\bot}\hat{\bot}}$. (6) $X \subseteq X^{\hat{\bot}\hat{\bot}}$. (7) $(X^{\hat{\bot}\hat{\bot}})^{\hat{\bot}\hat{\bot}} \subseteq X^{\hat{\bot}\hat{\bot}}$. (8) $X^{\hat{\bot}\hat{\bot}} \circ Y^{\hat{\bot}\hat{\bot}} \subseteq (X \circ Y)^{\hat{\bot}\hat{\bot}}$. (9) $x \in X^{\hat{\bot}}$ iff $\{x\} \circ X \subseteq \hat{\bot}$. (10) if $X \circ Y \subseteq \hat{\bot}$ then $X \circ Y^{\hat{\bot}\hat{\bot}} \subseteq \hat{\bot}$.*

Proposition 5.5. *Let $X, Y \subseteq M$. (1) $X^{\hat{\bot}}$ is a fact. (2) $X^{\hat{\bot}\hat{\bot}}$ is the smallest fact that includes X. (3) if X and Y are facts, then so is $X \cap Y$.*

Definition 5.6. Let $A, B \subseteq M$. We define the following operators and constants: $\hat{\bot} := \{1\}^{\hat{\bot}}$, $\hat{\mathbf{1}} := \hat{\bot}^{\hat{\bot}} = \{1\}^{\hat{\bot}\hat{\bot}}$, $\hat{\top} := M = \emptyset^{\hat{\bot}}$, $\hat{\mathbf{0}} := \hat{\top}^{\hat{\bot}} = M^{\hat{\bot}} = \emptyset^{\hat{\bot}\hat{\bot}}$, $A \mathbin{\hat{\wedge}} B := A \cap B$, $A \mathbin{\hat{\vee}} B := (A \cup B)^{\hat{\bot}\hat{\bot}}$, $A \mathbin{\hat{*}} B := (A \circ B)^{\hat{\bot}\hat{\bot}}$, $A \mathbin{\hat{+}} B := (A^{\hat{\bot}} \circ B^{\hat{\bot}})^{\hat{\bot}}$, $\hat{!}A := (A \cap \hat{I})^{\hat{\bot}\hat{\bot}}$, and $\hat{?}A := (A^{\hat{\bot}} \cap \hat{I})^{\hat{\bot}}$.

[15] We remark that the operation $\circ$ is commutative and assosiative, and has the monotonicity property: $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$ imply $X_1 \circ X_2 \subseteq Y_1 \circ Y_2$ for any X_1, X_2, Y_1, Y_2 $(\subseteq M)$.

We can show that, by Proposition 5.5, the constants defined above are facts and the operators defined above are closed under D_M.

DEFINITION 5.7. A valuation ϕ^+ (ϕ^-) on a phase space $\langle M, \hat{\perp}, \hat{I}\rangle$ is a mapping which assigns a fact to each propositional variables. Each valuation ϕ^+ (ϕ^-) can be extended to a mapping $\cdot^+$ ($\cdot^-$) from the set Φ of all formulas to D_M by $p^+ := \phi^+(p)$ for any propositional variable p, $\perp^+ := \hat{\perp}$, $\mathbf{1}^+ := \hat{\mathbf{1}}$, $\top^+ := \hat{\top}$, $\mathbf{0}^+ := \hat{\mathbf{0}}$, $(\alpha^{\perp})^+ := (\alpha^+)^{\hat{\perp}}$, $(\alpha \wedge \beta)^+ := \alpha^+ \mathbin{\hat{\wedge}} \beta^+$, $(\alpha \vee \beta)^+ := \alpha^+ \mathbin{\hat{\vee}} \beta^+$, $(\alpha * \beta)^+ := \alpha^+ \mathbin{\hat{*}} \beta^+$, $(\alpha + \beta)^+ := \alpha^+ \mathbin{\hat{+}} \beta^+$, $(!\alpha)^+ := \hat{!}(\alpha^+)$, $(?\alpha)^+ := \hat{?}(\alpha^+)$, $(\sim\alpha)^+ := \alpha^-$, $p^- := \phi^-(p)$ for any propositional variable p, $\perp^- := \hat{\mathbf{1}}$, $\mathbf{1}^- := \hat{\perp}$, $\top^- := \hat{\mathbf{0}}$, $\mathbf{0}^- := \hat{\top}$, $(\alpha^{\perp})^- := (\alpha^-)^{\hat{\perp}}$, $(\alpha \wedge \beta)^- := \alpha^- \mathbin{\hat{\vee}} \beta^-$, $(\alpha \vee \beta)^- := \alpha^- \mathbin{\hat{\wedge}} \beta^-$, $(\alpha * \beta)^- := \alpha^- \mathbin{\hat{*}} \beta^-$, $(\alpha + \beta)^- := \alpha^- \mathbin{\hat{+}} \beta^-$, $(!\alpha)^- := \hat{!}(\alpha^-)$, $(?\alpha)^- := \hat{?}(\alpha^-)$, and $(\sim\alpha)^- := \alpha^+$.

We call the values α^+ and α^- the *inner-values* of α ($\in \Phi$).

DEFINITION 5.8. $\langle M, \hat{\perp}, \hat{I}, \phi^+, \phi^-\rangle$ is a *phase (weakening phase, contraction phase or contraction-weakening phase) model* if $\langle M, \hat{\perp}, \hat{I}\rangle$ is a phase (weakening phase, contraction phase or contraction-weakening phase respectively) space and ϕ^+ and ϕ^- are valuations on $\langle M, \hat{\perp}, \hat{I}\rangle$. A sequent $\vdash \alpha$ is *true* in a phase (weakening phase, contraction phase or contraction-weakening phase) model $\langle M, \hat{\perp}, \hat{I}, \phi^+, \phi^-\rangle$ if $\alpha^{+\hat{\perp}} \subseteq \hat{\perp}$ (or equivalently $1 \in \alpha^+$), and *valid* in a phase (weakening phase, contraction phase or contraction-weakening phase respectively) space $\langle M, \hat{\perp}, \hat{I}\rangle$ if it is true for any valuations ϕ^+ and ϕ^- on the phase space. A sequent $\vdash \alpha_1, \cdots, \alpha_n$ is *true* in a phase (weakening phase, contraction phase or contraction-weakening phase) model $\langle M, \hat{\perp}, \hat{I}, \phi^+, \phi^-\rangle$ if $\vdash \alpha_1 + \cdots + \alpha_n$ is true in the model, and *valid* in a phase (weakening phase, contraction phase or contraction-weakening phase respectively) space $\langle M, \hat{\perp}, \hat{I}\rangle$ if it is true for any valuation ϕ^+ and ϕ^- on the phase space.

THEOREM 5.9 (Soundness). *Let L_1, L_2, L_3 and L_4 be* CLS, CLS$_{\mathrm{w}}$, CLS$_{\mathrm{c}}$ *and* CLS$_{\mathrm{cw}}$ *respectively. If a sequent S is provable in L_1 (L_2, L_3 or L_4) then the sequent S is valid for any phase (weakening phase, contraction phase or contraction-weakening phase respectively) space.*

THEOREM 5.10 (Completeness). *Let L_1, L_2, L_3 and L_4 be* CLS, CLS$_{\mathrm{w}}$, CLS$_{\mathrm{c}}$ *and* CLS$_{\mathrm{cw}}$ *respectively. If a sequent S is valid for any phase (weakening phase, contraction phase or contraction-weakening phase respectively) space, then the sequent S is cut-free provable in L_1 (L_2, L_3 or L_4 respectively).*

In the following, we only prove this theorem for CLS. To prove this theorem, we construct a canonical phase model $\langle M, \hat{\perp}, \hat{I}, \phi^+, \phi^-\rangle$. Here M

is the set of all multisets of formulas where multiple occurrence of a formula of the form $?\alpha$ (or $\sim ?\alpha$) in the multisets counts only once. $\langle M, \cdot, 1\rangle$ is a commutavive monoid where $\Delta \cdot \Gamma := \Delta \cup \Gamma$ (the multiset union) for all $\Delta, \Gamma \in M$, and 1 ($\in M$) is $\emptyset$ (the empty multiset). For any formula α, we define $[\alpha]_+ := \{\Delta \mid \vdash_{cf} \Delta, \alpha\}$ and $[\alpha]_- := \{\Delta \mid \vdash_{cf} \Delta, \sim\alpha\}$ where $\vdash_{cf} \Delta, \alpha$ means that $\vdash \Delta, \alpha$ is cut-free provable. We call $[\alpha]_*$ ($* \in \{+, -\}$) the *outer-values* of α ($\in \Phi$). We define $\hat{\bot} := [\bot]_+ = \{\Delta \mid \vdash_{cf} \Delta\}$. Moreover $\hat{I}$ is defined as $\{?\Gamma \cup \sim ?\Delta \in 2^{\Phi} \mid \Gamma \cup \Delta \in M\}$. We show that the set $\hat{I}$ defined is equivalent to the set $\dot{I} = \{\Delta \in M \mid \Delta \cup \Delta = \Delta\} \cap \hat{\bot}^{\hat{\bot}}$. First, we show $\hat{I} \subseteq \dot{I}$. Suppose $\Sigma \in \hat{I}$. Then, this means that Σ is a set of formulas of the forms $?\gamma$ and $\sim ?\beta$, because the multiple occurrence of a formula of the form $?\gamma$ or $\sim ?\gamma$ in the multisets in M counts only once. We can verify $\Sigma \in M$ and $\Sigma \cup \Sigma = \Sigma$, because Σ is a set and $\cup$ is regarded as the set union. We can also show $\Sigma \in \hat{\bot}^{\hat{\bot}} = \{\Delta \mid \forall \Gamma \in \hat{\bot}\ (\Delta \cup \Gamma \in \hat{\bot})\}$, i.e. $\forall \Gamma\ [\vdash_{cf} \Gamma \Longrightarrow\ \vdash_{cf} \Gamma, \Sigma]$, by using (?we) and/or ($\sim$?we). Thus, we have $\Sigma \in \dot{I}$. Next, we show $\dot{I} \subseteq \hat{I}$. Suppose $\Sigma \in \dot{I}$. Then, we have $\Sigma \in M$, $\Sigma \cup \Sigma = \Sigma$ and $\forall \Gamma\ [\vdash_{cf} \Gamma \Longrightarrow\ \vdash_{cf} \Gamma, \Sigma]$. These mean that Σ is a set of formulas of the forms $?\gamma$ and $\sim ?\beta$, and hence $\Sigma \in \hat{I}$. Therfore $\dot{I} = \hat{I}$. Using $M, \hat{\bot}$ and $\hat{I}$ defined above, we have the fact that $\langle M, \hat{\bot}, \hat{I}\rangle$ is a phase space. The valuations ϕ^+ and ϕ^- of the canonical model are defined as $\phi^+(p) := [p]_+$ and $\phi^-(p) := [p]_-$ for any propositional variable p.

To prove the completeness theorem, we must prove some lemmas which are analogous to the lemma established by Okada in [18].

LEMMA 5.11. *Let α be any formula.* (1) *if $\alpha^+ \subseteq [\alpha]_+$ then $\{\alpha\} \in \alpha^{+\hat{\bot}}$.* (2) *if $\alpha^- \subseteq [\alpha]_-$ then $\{\sim\alpha\} \in \alpha^{-\hat{\bot}}$.*

LEMMA 5.12. *Let α be any formula.* (1) $[\alpha]_+^{\hat{\bot}\hat{\bot}} = [\alpha]_+$. (2) $[\alpha]_-^{\hat{\bot}\hat{\bot}} = [\alpha]_-$. (3) $[\sim\alpha]_+ = [\alpha]_-$. (4) $[\sim\alpha]_- = [\alpha]_+$.

PROOF. We show only (4).

(4): We first show $[\alpha]_+ \subseteq [\sim\alpha]_-$. Suppose $\Delta \in [\alpha]_+$, i.e., $\vdash_{cf} \alpha, \Delta$. Then, by using the rule ($\sim$), we obtain $\vdash_{cf} \sim\sim\alpha, \Delta$, and hence $\Delta \in [\sim\alpha]_-$. Next, we show the converse, i.e., $\Delta \in [\sim\alpha]_-$ implies $\Delta \in [\alpha]_+$. To prove the fact, we must use the fact that the rule

$$\frac{\vdash \Gamma, \sim\sim\alpha}{\vdash \Gamma, \alpha}\ (\sim^{-1})$$

is admissible in cut-free CLS. This fact can be proved easily by induction on the cut-free proof in CLS. Now suppose $\Delta \in [\sim\alpha]_-$, i.e., $\vdash_{cf} \sim\sim\alpha, \Delta$. By applying ($\sim^{-1}$) to $\vdash_{cf} \sim\sim\alpha, \Delta$, we obtain $\vdash_{cf} \alpha, \Delta$, and hence $\Delta \in [\alpha]_+$. ∎

By Lemmas 5.11 and 5.12, we can prove the following main lemma. The lemma directly implies the completeness theorem. If the sequent $\vdash \alpha$ is true, then $\emptyset \in \alpha^+$. On the other hand $\alpha^+ \subseteq [\alpha]_+$, and hence $\emptyset \in [\alpha]_+$. This means that $\vdash \alpha$ is cut-free provable.

LEMMA 5.13. *Let α be any formula.* (1) $\alpha^+ \subseteq [\alpha]_+$, *and* (2) $\alpha^- \subseteq [\alpha]_-$.

PROOF. We prove this lemma by (simultaneous) induction on the complexity of the formula α.

• Base step: Obvious by the definitions.

• Induction step for (1): We show only the following case. The other cases are the same as that in [18].

(Case $\alpha \equiv {\sim}\beta$ for (1)): We show $({\sim}\beta)^+ \subseteq [{\sim}\beta]_+$. Suppose $\Gamma \in ({\sim}\beta)^+$, i.e., $\Gamma \in \beta^-$. Then we obtain $\Gamma \in \beta^- \subseteq [\beta]_- = [{\sim}\beta]_+$ by the induction hypothesis for (2) and Lemma 5.12 (3).

• Induction step for (2): We show some cases.

(Case $\alpha \equiv p^\perp$ for (2) where p is a propositional variable):[16] We show $p^{\perp -} \subseteq [p^\perp]_-$. Suppose $\Gamma \in p^{-\hat{\perp}}$. Then we have $\Gamma \in [p]_-^{\hat{\perp}}$ by the definition $p^- := [p]_-$, and hence we have (*): $\forall \Pi$ [$\Pi \in [p]_-$ implies $\vdash_{cf} \Gamma, \Pi$]. Now we have $\{({\sim}p)^\perp\} \in [p]_-$, i.e., $\vdash_{cf} ({\sim}p)^\perp, {\sim}p$, as an initial sequent. Thus, we obtain $\vdash_{cf} \Gamma, {\sim}(p^\perp)$ by (*) and the law $({\sim}p)^\perp \leftrightarrow {\sim}(p^\perp)$. Therefore $\Gamma \in [p^\perp]_-$.

(Case $\alpha \equiv \beta \wedge \gamma$ for (2)): We show $(\beta \wedge \gamma)^- \subseteq [\beta \wedge \gamma]_-$. Suppose $\Gamma \in (\beta \wedge \gamma)^- = \beta^- \hat{\vee} \gamma^- = (\beta^- \cup \gamma^-)^{\hat{\perp}\hat{\perp}}$. Now we have $\beta^- \subseteq [\beta]_-$ and $\gamma^- \subseteq [\gamma]_-$ by the induction hypothesis. Suppose (*): $[\beta]_- \cup [\gamma]_- \subseteq [\beta \wedge \gamma]_-$ (this will be proved). Then we have $\beta^- \cup \gamma^- \subseteq [\beta]_- \cup [\gamma]_- \subseteq [\beta \wedge \gamma]_-$, and hence (**): $\beta^- \cup \gamma^- \subseteq [\beta \wedge \gamma]_-$. By the hypothesis, (**), Proposition 5.4 (5) and Lemma 5.12 (2), we obtain $\Gamma \in (\beta^- \cup \gamma^-)^{\hat{\perp}\hat{\perp}} \subseteq [\beta \wedge \gamma]_-^{\hat{\perp}\hat{\perp}} = [\beta \wedge \gamma]_-$. This means $\Gamma \in [\beta \wedge \gamma]_-$. We show the remained fact (*). Suppose $\Delta \in [\beta]_- \cup [\gamma]_-$. Then we have $\vdash_{cf} \Delta, {\sim}\beta$ or $\vdash_{cf} \Delta, {\sim}\gamma$. By applying (${\sim}\wedge 1$) or (${\sim}\wedge 2$), we obtain $\vdash_{cf} \Delta, {\sim}(\beta \wedge \gamma)$, and hence $\Delta \in [\beta \wedge \gamma]_-$.

(Case $\alpha \equiv !\beta$ for (2)): We show $(!\beta)^- \subseteq [!\beta]^-$. Suppose (*): $\Gamma \in (!\beta)^- = \hat{!}(\beta^-) = (\beta^- \cap \hat{I})^{\hat{\perp}\hat{\perp}}$. Also, suppose (**): $\beta^- \cap \hat{I} \subseteq [!\beta]_-$ (this will be proved). By (*) and (**), Proposition 5.4 (5) and Lemma 5.12 (2), we obtain $\Gamma \in (\beta^- \cap \hat{I})^{\hat{\perp}\hat{\perp}} \subseteq [!\beta]_-^{\hat{\perp}\hat{\perp}} = [!\beta]_-$. Therefore $\Gamma \in [!\beta]_-$. Next, we

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[16] We do not have to consider the other cases such as $\alpha \equiv \gamma^\perp$ where γ is any formula except propositional variable. Because we have the De Morgan and other laws for $\cdot^\perp$. For example, in the case $\alpha \equiv ({\sim}\beta)^\perp$, we consider $({\sim}\beta)^{\perp -} \subseteq [({\sim}\beta)^\perp]_-$. Then it is enough to consider $({\sim}(\beta^\perp))^- \subseteq [{\sim}(\beta^\perp)]_-$ by the law $({\sim}\beta)^\perp \leftrightarrow {\sim}(\beta^\perp)$. Here taking $\gamma \equiv \beta^\perp$, we see $({\sim}\gamma)^- \subseteq [{\sim}\gamma]_-$. Therefore this case is included the case $\alpha \equiv {\sim}\gamma$ for (2).

show the remained fact (**). Suppose $\Delta \in \beta^- \cap \hat{I}$, i.e., $\Delta \in \beta^-$ and $\Delta \in \hat{I}$. By the induction hypothesis $\beta^- \subseteq [\beta]_-$, we have $\Delta \in [\beta]_-$, and hence (1): $\vdash_{cf} \Delta, \sim\beta$. On the other hand, $\Delta \in \hat{I} = \{?\Sigma \cup \sim ?\Pi \in 2^{\Phi} \mid \Sigma \cup \Pi \in M\}$ and hence (2): Δ is of the form $?\Delta_1 \cup \sim ?\Delta_2$. By applying $(\sim !)$ to (1), we obtain $\vdash_{cf} ?\Delta_1, \sim ?\Delta_2, \sim !\beta$, i.e., $\vdash_{cf} \Delta, \sim !\beta$. This means $\Delta \in [!\beta]_-$. ■

By combining this main lemma with the soundness theorem, we can derive the cut-elimination theorem for (one-sided and two-sided) CLS (i.e. Theorem 2.2). Of course, we can also show the cut-elimination and completeness theorems for $\mathrm{CLS_w}$, $\mathrm{CLS_c}$ and $\mathrm{CLS_{cw}}$ with some appropriate modifications of the proof for CLS.

6. Appendix: Subformula Calculus and Dual Calculus

We introduce two alternative cut-free sequent calculi for the one-sided CLS, which are regarded as classical versions of the calculi proposed in [12]. These calculi can be modified for the other logics mentioned in this paper, and can also be obtained for the two-sided versions in a similar way. Thus, we only focus on the case of the one-sided CLS.

First, we introduce a subformula calculus SC, which has the subformula property. This sort of sequent calculus was also presented as a display calculus by Wansing [26]. The sequents of SC are of the forms $\vdash \Gamma : \Delta$ where Γ, Δ are multisets of formulas. We call Γ and Δ in the expression *negative (or refutability) context* and *positive (or provability) context* respectively. The sequent $\vdash \Gamma : \Delta$ in SC intuitively means $\vdash \sim\Gamma, \Delta$ in CLS.

Definition 6.1 (SC). We define a (one-sided) subformula calculus SC. The initial sequents of SC are of the forms:

$$\vdash \emptyset : \alpha, \alpha^{\perp} \quad \vdash \alpha, \alpha^{\perp} : \emptyset \quad \vdash \emptyset : \mathbf{1} \quad \vdash \Gamma_1 : \Gamma_2, \top \quad \vdash \mathbf{0}, \Gamma_1 : \Gamma_2 \quad \vdash \perp : \emptyset$$

The inference rules of SC are of the forms:

$$\frac{\vdash \alpha, \Gamma_1 : \Gamma_2 \quad \vdash \alpha^{\perp}, \Delta_1 : \Delta_2}{\vdash \Gamma_1, \Delta_1 : \Gamma_2, \Delta_2} \qquad \frac{\vdash \Gamma_1 : \Gamma_2, \alpha \quad \vdash \Delta_1 : \Delta_2, \alpha^{\perp}}{\vdash \Gamma_1, \Delta_1 : \Gamma_2, \Delta_2}$$

$$\frac{\vdash \Gamma : \Delta, \alpha}{\vdash \sim\alpha, \Gamma : \Delta} \qquad \frac{\vdash \alpha, \Gamma : \Delta}{\vdash \Gamma : \Delta, \sim\alpha} \qquad \frac{\vdash \Gamma : \Delta}{\vdash \Gamma : \Delta, \perp} \qquad \frac{\vdash \Gamma_1 : \Gamma_2, \alpha \quad \vdash \Delta_1 : \Delta_2, \beta}{\vdash \Gamma_1, \Delta_1 : \Gamma_2, \Delta_2, \alpha * \beta}$$

$$\frac{\vdash \Gamma : \Delta, \alpha, \beta}{\vdash \Gamma : \Delta, \alpha + \beta} \qquad \frac{\vdash \Gamma : \Delta, \alpha \quad \vdash \Gamma : \Delta, \beta}{\vdash \Gamma : \Delta, \alpha \wedge \beta} \qquad \frac{\vdash \Gamma : \Delta, \alpha}{\vdash \Gamma : \Delta, \alpha \vee \beta} \qquad \frac{\vdash \Gamma : \Delta, \beta}{\vdash \Gamma : \Delta, \alpha \vee \beta}$$

$$\frac{\vdash ?\Gamma_1 : ?\Gamma_2, \alpha}{\vdash ?\Gamma_1 : ?\Gamma_2, !\alpha} \qquad \frac{\vdash \Gamma : \Delta, \alpha}{\vdash \Gamma : \Delta, ?\alpha} \qquad \frac{\vdash \Gamma : \Delta, ?\alpha, ?\alpha}{\vdash \Gamma : \Delta, ?\alpha} \qquad \frac{\vdash \Gamma : \Delta}{\vdash \Gamma : \Delta, ?\alpha} \qquad \frac{\vdash \Gamma : \Delta}{\vdash \mathbf{1}, \Gamma : \Delta}$$

$$\frac{\vdash \alpha, \Gamma_1 : \Gamma_2 \quad \vdash \beta, \Delta_1 : \Delta_2}{\vdash \alpha * \beta, \Gamma_1, \Delta_1 : \Gamma_2, \Delta_2} \quad \frac{\vdash \alpha, \beta, \Gamma : \Delta}{\vdash \alpha + \beta, \Gamma : \Delta} \quad \frac{\vdash \alpha, \Gamma : \Delta \quad \vdash \beta, \Gamma : \Delta}{\vdash \alpha \vee \beta, \Gamma : \Delta} \quad \frac{\vdash \alpha, \Gamma : \Delta}{\vdash \alpha \wedge \beta, \Gamma : \Delta}$$

$$\frac{\vdash \beta, \Gamma : \Delta}{\vdash \alpha \wedge \beta, \Gamma : \Delta} \quad \frac{\vdash \alpha, ?\Gamma_1 : ?\Gamma_2}{\vdash !\alpha, ?\Gamma_1 : ?\Gamma_2} \quad \frac{\vdash \alpha, \Gamma : \Delta}{\vdash ?\alpha, \Gamma : \Delta} \quad \frac{\vdash ?\alpha, ?\alpha, \Gamma : \Delta}{\vdash ?\alpha, \Gamma : \Delta} \quad \frac{\vdash \Gamma : \Delta}{\vdash ?\alpha, \Gamma : \Delta}$$

We can obtain the equivalence between SC and CLS by (1) if $\vdash \Gamma : \Delta$ is provable in SC, then $\vdash {\sim}\Gamma, \Delta$ is provable in CLS, and (2) if $\vdash {\sim}\Gamma, \Delta$ is cut-free provable in CLS, then $\vdash \Gamma : \Delta$ is cut-free provable in SC. We can also obtain the cut-elimination and subformula properties for SC.

Next, we introduce a dual calculus DC. A sequent of the form $\vdash^+ \Gamma$ is called a *positive (or provability) sequent*, and a sequent of the form $\vdash^- \Gamma$ is called a *negative (or refutability) sequent*.

DEFINITION 6.2 (DC). We define a (one-sided) dual calculus DC. The initial sequents of DC are of the forms:

$$\vdash^+ \alpha, \alpha^\perp \qquad \vdash^+ \mathbf{1} \qquad \vdash^+ \Gamma, \top \qquad \vdash^- \alpha, \alpha^\perp \qquad \vdash^- \Gamma, \mathbf{0} \qquad \vdash^- \perp$$

The inference rules of DC are of the forms:

$$\frac{\vdash^- \Gamma, \alpha \quad \vdash^- \Delta, \alpha^\perp}{\vdash^- \Gamma, \Delta} \quad \frac{\vdash^+ \Gamma, \alpha \quad \vdash^+ \Delta, \alpha^\perp}{\vdash^+ \Gamma, \Delta} \quad \frac{\vdash^- {\sim}\Gamma, \Delta}{\vdash^+ \Gamma, {\sim}\Delta} \quad \frac{\vdash^+ {\sim}\Gamma, \Delta}{\vdash^- \Gamma, {\sim}\Delta}$$

$$\frac{\vdash^+ \Gamma}{\vdash^+ \Gamma, \perp} \quad \frac{\vdash^+ \Gamma, \alpha \quad \vdash^+ \Delta, \beta}{\vdash^+ \Gamma, \Delta, \alpha * \beta} \quad \frac{\vdash^+ \Gamma, \alpha, \beta}{\vdash^+ \Gamma, \alpha + \beta} \quad \frac{\vdash^+ \Gamma, \alpha \quad \vdash^+ \Gamma, \beta}{\vdash^+ \Gamma, \alpha \wedge \beta}$$

$$\frac{\vdash^+ \Gamma, \alpha}{\vdash^+ \Gamma, \alpha \vee \beta} \quad \frac{\vdash^+ \Gamma, \beta}{\vdash^+ \Gamma, \alpha \vee \beta} \quad \frac{\vdash^+ ?\Gamma_1, {\sim}?\Gamma_2, \alpha}{\vdash^+ ?\Gamma_1, {\sim}?\Gamma_2, !\alpha} \quad \frac{\vdash^+ \Gamma, \alpha}{\vdash^+ \Gamma, ?\alpha} \quad \frac{\vdash^+ \Gamma, ?\alpha, ?\alpha}{\vdash^+ \Gamma, ?\alpha}$$

$$\frac{\vdash^+ \Gamma}{\vdash^+ \Gamma, ?\alpha} \quad \frac{\vdash^- \Gamma}{\vdash^- \Gamma, \mathbf{1}} \quad \frac{\vdash^- \Gamma, \alpha \quad \vdash^- \Delta, \beta}{\vdash^- \Gamma, \Delta, \alpha * \beta} \quad \frac{\vdash^- \Gamma, \alpha, \beta}{\vdash^- \Gamma, \alpha + \beta} \quad \frac{\vdash^- \Gamma, \alpha \quad \vdash^- \Gamma, \beta}{\vdash^- \Gamma, \alpha \vee \beta}$$

$$\frac{\vdash^- \Gamma, \alpha}{\vdash^- \Gamma, \alpha \wedge \beta} \quad \frac{\vdash^- \Gamma, \beta}{\vdash^- \Gamma, \alpha \wedge \beta} \quad \frac{\vdash^- ?\Gamma_1, {\sim}?\Gamma_2, \alpha}{\vdash^- ?\Gamma_1, {\sim}?\Gamma_2, !\alpha} \quad \frac{\vdash^- \Gamma, \alpha}{\vdash^- \Gamma, ?\alpha} \quad \frac{\vdash^- \Gamma, ?\alpha, ?\alpha}{\vdash^- \Gamma, ?\alpha} \quad \frac{\vdash^- \Gamma}{\vdash^- \Gamma, ?\alpha}$$

We can obtain the equivalence between DC and CLS by (1) if $\vdash^* \Gamma$ ($* \in \{+, -\}$) is provable in DC, then the sequent $\vdash \Gamma$ is provable in CLS if $* = +$, or the sequent $\vdash {\sim}\Gamma$ is provable in CLS if $* = -$, and (2) if $\vdash \Gamma$ is cut-free provable in CLS, then the sequent $\vdash^+ \Gamma$ is cut-free provable in DC. We can also obtain the cut-elimination property for DC.

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