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1 AUTHOR:

Erhard Neher
University of Ottawa

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On Triagonal Elements and Centralizers in Jordan Triple Systems

ERHARD NEHER*,†

Mathematical Institute, University of Münster, 4400 Münster, West Germany

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Let V be a Jordan triple system over a field k of characteristic $\neq 2$ and let $\varnothing \neq X \subset V$ be a subset. The left multiplication of V is denoted by L. One of the two topics of this paper is the subspace

$$C_V(X) = \{v \in V; L(v, x) = L(x, v) \text{ for all } x \in X\},$$

the *centralizer of X in V*. For Jordan algebras centralizers were studied by Jacobson in [3] (characteristic 0) and afterwards by Harris in [2] (characteristic $\neq 2$). In this paper we generalize the results in [3] and some of the results in [2] to Jordan triple systems. We point out that the proofs of [2] depend on the classification of separable Jordan algebras whereas our proofs are classification free.

The main tool in our analysis of centralizers is triagonal elements, and—more generally—separable elements. The former are by definition linear combination of orthogonal tripotents whereas the latter become so after extending the base field to an algebraic closure. In practice we always deal with separable elements in their split form, to have available the tripotents and their Peirce decomposition.

We characterize the triagonal and separable elements via their minimal decompositions and minimum polynomials. We show two commuting separable elements generate a separable subsystem. Although the centralizers $C_V(X)$ need not in general be triple subsystems, this holds if X is spanned by separable elements; if V is also separable then we show $C_V(X)$ is a separable subsystem if characteristic $k = 0$ or if X consists of a single element. Over perfect fields we obtain a Jordan–Chevalley decomposition.

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† *Present address*: Department of Mathematics, University of Ottawa, Ottawa, Ontario K1N 9B4, Canada.

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1. Triagonal Elements

Throughout we will be working with Jordan triple systems, usually denoted by V, over a field k of scalars. In the first three sections k can be arbitrary, but eventually we will assume characteristic $k \neq 2$.

Thus, we are given a quadratic map $P: V \to \operatorname{End} V: x \to P(x)$ such that the following identities hold in all scalar extensions

$$L(x, y) P(x) = P(x) L(y, x) \tag{1.1}$$

$$L(P(x) y, y) = L(x, P(y) x) \tag{1.2}$$

$$P(P(x) y) = P(x) P(y) P(x) \tag{1.3}$$

where $L(x, y) z := P(x, z) y := P(x+z) y - P(x) y - P(z) y = \{xyz\}$. Such triple systems are treated in [5, 6, 8]. In particular, one can derive the following identity (see, e.g., [6, Sect. 2.1]):

$$[L(x, y), L(u, v)] = L(\{xyu\}, v) - L(u, \{yxv\}). \tag{1.4}$$

From this one easily sees $D(x, y) = L(x, y) - L(y, x)$ is an *inner derivation* of V. We say x, y *commute on* V if

$$D(x, y) = 0, \qquad \text{i.e.,} \qquad L(x, y) = L(y, x). \tag{1.5}$$

A particular example of commuting elements are *orthogonal* elements $x \perp y$, where $L(x, y) = L(y, x) = 0$. Note these concepts are not intrinsic to x and y; they depend on V: if x, y commute or are orthogonal on V they need not continue to commute or be orthogonal on some larger system $W \supset V$.

A triple system is *flat* [5, 3.1] if all elements commute, i.e., $L(u, v) = L(v, u)$ for all u, v. From [5, 2.8]: if V is flat, then

$$\begin{aligned} &\text{(a)} \quad [L(u, v), L(x, y)] = [P(u, v), P(x, y)] = 0 \\ &\text{(b)} \quad \{x_1 x_2 \{x_3 x_4 x_5\}\} = \{x_1 \{x_2 x_3 x_4\} x_5\} = \{\{x_1 x_2 x_3\} x_4 x_5\} \end{aligned} \tag{1.6}$$

is a symmetric function of the variables x_i in V.

Since we are dealing with triple systems and not with algebras, there is—at least as first glance—no natural way to associate with x an element x^2. However, we have a natural "cubing": we set $x^3 := P(x) x$. In general the odd *powers* of $x \in V$ are defined inductively by

$$x^1 = x, \qquad x^{2n+1} := P(x) x^{2n-1} = P(x)^n x \qquad \text{for} \quad n \geqslant 1.$$

The subspace of V spanned by all the odd powers of x is denoted by $k\{x\}$; it is a flat subsystem [8, (13.32)].

An element $e \in V$ with $P(e)e = e$ is called a *tripotent*. Every tripotent e induces a *Peirce decomposition* of V [8, XV], which we write in the following form:

$$V = V_2(e) \oplus V_1(e) \oplus V_0(e), \qquad \text{where} \quad V_\mu(e) = \{x \in V; L(e, e)\, x = \mu x\}$$

$\mu = 0, 1, 2$. Moreover, for any tripotent $e \in V$ the Peirce space $V_2(e)$ is invariant under $P(e)$ and $P(e) \mid V_2(e)$ is an involutive automorphism. We denote by $V_2^{\pm}(e)$ the eigenspaces of $P(e)$:

$$V_2^{\pm}(e) := \{x \in V_2(e); P(e)\, x = \pm x\}.$$

One says that $(e_1, ..., e_r)$ is an *orthogonal system* if all e_j are nonzero tripotents and $e_i \in V_0(e_j)$ for all $j \neq i$. Every orthogonal system $(e_1, ..., e_r)$ induces a simultaneous Peirce decomposition of V (see [8, XV] for details):

$$V = \bigoplus_{0 \leqslant i \leqslant j \leqslant r} V_{ij} \tag{1.7}$$

where $V_{00} = \bigcap_i V_0(e_i)$, $V_{ii} = V_2(e_i)$ $(1 \leqslant i \leqslant r)$, $V_{ij} = V_1(e_i) \cap V_1(e_j)$ $(1 \leqslant i < j \leqslant r)$, and $V_{0j} = V_1(e_j) \cap \bigcap_{i \neq j} V_0(e_i)$.

A Jordan algebra $\mathscr{J}$ is called *diagonal* if $\mathscr{J} = kd_1 \oplus \cdots \oplus kd_r$ is a direct sum of copies of k, and an element is *diagonal* if it is a linear combination $x = \alpha_1 d_1 + \cdots + \alpha_r d_r$ of orthogonal idempotents d_i. In the same way we call a triple system V *triagonal* if $V = ke_1 \oplus \cdots \oplus ke_r$ for an orthogonal system $(e_1, ..., e_r)$, and we call an element x *triagonal* if it may be written as a linear combination $x = \alpha_1 e_1 + \cdots + \alpha_r e_r$ of orthogonal tripotents (we call such a decomposition a *triagonalization* of x). Obviously, among all such triagonalizations we may choose one of minimal length r; we call this a *minimum triagonalization* of x. This might also be called a "semisimple minimum decomposition." Note that a general minimum decomposition contains a nilpotent part (see Corollary 6.8). Observe that by Peirce orthogonality any triagonal system is flat.

In the algebra case a diagonalization $x = \sum \alpha_i d_i \neq 0$ is minimal iff the α_i are distinct and nonzero, in which case it is unique; in the triple case a triagonalization is minimal iff the α_i^2 are distinct and nonzero, in which case it is unique up to sign (of the α_i and the e_i).

THEOREM 1.1 (Minimality Criterion). *The following conditions are equivalent for a triagonalization*

$$x = \alpha_1 e_1 + \cdots + \alpha_r e_r \qquad (\alpha_j \in k) \tag{1.8}$$

of a nonzero element x in V:

(1) (1.8) *is minimal*;

(2) *all* α_j^2 *are nonzero and distinct* (*i.e., all* α_j *are nonzero and* $\alpha_j \neq \pm\alpha_k$ *for* $j \neq k$);

(3) $k\{x\} = ke_1 \oplus \cdots \oplus ke_r$;

(4) $\dim k\{x\} = r$,

and in this case the $\alpha_j \in k$ *and tripotents* e_j *are—up to enumeration and signs—uniquely determined: The* e_j *are the minimal tripotents of* $k\{x\}$ *and the eigenvalues* α_j *are the values* $\lambda(x)$ *of all characters* (*nonzero homomorphisms* $k\{x\} \to^{\lambda} k$). *Thus* x *is a triagonal element iff* $k\{x\}$ *is a triagonal subsystem.*

Proof. (1) $\Rightarrow$ (2) since if r is minimal then there is no summand which does not contribute to the sum, so $\alpha_j \neq 0$, and no two e_j, e_k can be coalesced, so $\alpha_j \neq \pm\alpha_k$ (if $\alpha_j = \varepsilon\alpha_k$ for $\varepsilon = \pm 1$ then $\alpha_j e_j + \alpha_k e_k = \alpha_k(\varepsilon e_j + e_k)$, where $\varepsilon e_j + e_k$ is again a tripotent).

The Peirce multiplication rules imply $x^m = \alpha_1^m e_1 + \cdots + \alpha_r^m e_r$ for odd m in (1.8), hence

$$k\{x\} \subset ke_1 \oplus \cdots \oplus ke_r.$$

This immediately shows (3) $\Leftrightarrow$ (4), and (4) $\Rightarrow$ (1) since if $x = \beta_1 d_1 + \cdots + \beta_s d_s$ is any other triagonalization then $\dim k\{x\} \leqslant s$, and it also shows (2) $\Rightarrow$ (3) since $x, \ldots, x^{2r-1}$ are represented by $e_1, \ldots, e_r$ through a Vandermonde-type matrix whose determinant is $\alpha_1 \cdot \cdots \cdot \alpha_r \prod_{j>k}(\alpha_j^2 - \alpha_k^2)$, which is nonzero by (2), so the e_j can be expressed in terms of the x^m.

To see the e_j are minimal in $W = k\{x\}$, note $d = \beta_1 e_1 + \cdots + \beta_r e_r$ is tripotent iff all $\beta_j = \pm 1$ or 0, and if $\beta_j \neq 0$ then $e_j \in W_2(d)$, so if d is minimal it cannot have two $\beta_j, \beta_k \neq 0$ (two orthogonal nonzero e_j, e_k in $W_2(d)$ contradicts the definition of minimality of d), so $d = \pm e_j$. By (3) any character has $\lambda(e_i) = \beta_i$ with $\beta_i^3 = \beta_i$, $\beta_i^2\beta_j = 0$, $i \neq j$, so if $\lambda \neq 0$ exactly one $\beta_i = \pm 1$, i.e., $\lambda = \pm\lambda_i$ for $\lambda_i(x^m) = \alpha_i^m$. ∎

An important feature of Jordan triple systems is that they come along with a lot of Jordan algebras: To every $u \in V$ one associates the Jordan algebra $V^{(u)}$ on the vector space V which has the quadratic representation $U^{(u)}(x) = P(x)P(u)$ and the squaring operation $x^{(2,u)} = P(x)u$ (see [8, 10.2]). In general $V^{(u)}$ does not contain a unit. We denote by $\hat{V}^{(u)}$ the Jordan algebra which arises from $V^{(u)}$ by adjoining a unit.

There is a well-known theory of minimum decompositions for elements in a power-associative algebra [1, 5.4]. The following theorem compares the minimum triagonalization of x in V and in $\hat{V}^{(x)}$:

THEOREM 1.2 (Comparison Theorem). *Let* x *be a nonzero element of a Jordan triple system* V. *If* x *has minimum triagonalization*

(a) $x = \alpha_1 e_1 + \cdots + \alpha_r e_r$

in V *then*

(b) $x = \alpha_1^2 d_1 + \cdots + \alpha_r^2 d_r$

with $d_j = \alpha_j^{-1} e_j$ *is the minimum diagonalization of* x *in* $\hat{V}^{(x)}$. *Conversely, if* x *has minimum diagonalization* (b) *in* $\hat{V}^{(x)}$ (*with all eigenvalues in* k^{*2}) *then* (a) *with* $e_j = \alpha_j d_j$ *is the minimum triagonalization of* x *in* V.

Thus x *is triagonal in* V *iff it is diagonal in* $\hat{V}^{(x)}$ *with all its eigenvalues squares in* k^*.

Proof. It is easy to check that whenever $x = \sum \alpha_j e_j$ for orthogonal tripotents e_j in V then $e_j^{(2,x)} = \alpha_j e_j$ and $e_j \circ e_k = P(e_j)\,P(x)\,e_k = 0$ for $k \neq j$, so when all $\alpha_j \neq 0$ the $\alpha_j^{-1} e_j$ are orthogonal idempotents in $\hat{V}^{(x)}$. Conversely, if we set $W = k\{x\}$, $W^{(x)} = k[x]^{(x)}$ then by "power-associativity" $P(x)$ commutes with all $P(w)$; when (b) holds for distinct nonzero α_j^2 we have all d_j in W and $P(x)$ is invertible on W, so $P(x)\,P(d_i)\,d_j = P(d_i)\,P(x)\,d_j = U(d_i)\,d_j = 0$, $P(x)\{d_i d_j W\} = P(d_i, W)\,P(x)\,d_j = U(d_i, W)\,d_j = 0$ by Peirce orthogonality in $W^{(x)}$ implies $P(d_i)\,d_j = \{d_i d_j W\} = 0$ in W, from which it is easily checked that $e_i = \alpha_i d_i$ are orthogonal tripotents in $W \subset V$. ∎

We remark that it is possible that x has a diagonalization in $\hat{V}^{(x)}$ but that because its eigenvalues are not squares may have to triagonalization in V. As an example one can take $V = k$ over the rationals $k = \mathbb{Q}$, with the Jordan product $P(u)\,v = \frac{1}{2}u^2 v$, $\{uvw\} = uvw$. Here V has no nonzero tripotents, yet $x = 2$ has $x = 2d$ for $d = 1$ idempotent in $\hat{V}^{(x)}$ (note its eigenvalue 2 is not a square in k).

We recall the notions of minimum and reduced minimum polynomial for elements of a Jordan algebra, for which we take $V^{(y)}$ since this is the case we are interested in: To every polynomial $f = \sum \alpha_i \tau^i$ we associate the element $f_{x,y} = \sum \alpha_i x^{(i,y)} \in \hat{V}^{(y)}$. We obtain the ideal $\{f \in k[\tau]; f_{x,y} = 0\}$, which is generated by a unique monic polynomial $\mu_{x,y}$, and the ideal $\{f \in k[\tau]; f_{x,y}$ is nilpotent in $\hat{V}^{(y)}\}$, generated by a unique monic polynomial $\mu^0_{x,y}$. The polynomial $\mu_{x,y}$ is called the *minimum polynomial* and $\mu^0_{x,y}$ the *reduced minimum polynomial* (of x in $\hat{V}^{(y)}$). Since an element of a Jordan algebra is diagonal iff the minimum polynomial splits over k and coincides with the reduced minimum polynomial, Theorem 1.2 implies

COROLLARY 1.3. *Let* $x \in V$. *Then* x *is triagonal iff* $\mu_{x,x}$ *splits over* k *with every root of* $\mu_{x,x}$ *a square and* $\mu_{x,x} = \mu^0_{x,x}$.

This criterion shows that it is easier to find triagonal elements if the ground field k is algebraically closed. Indeed, the following theorem, which is an easy consequence of results of H. Braun, M. Koecher, and O. Loos, says that they are available in abundance as long as V is semisimple. We define the *degree* of a triagonal x to be $r = \dim k\{x\}$; we say x has *maximal degree* if its degree equals the generic degree of V (the degree of the generic minimum polynomial m).

THEOREM 1.4. *Let V be semisimple and finite-dimensional over the algebraically closed field k. Then the triagonal elements are dense in V (relative to the Zariski topology). Indeed, the set of triagonal elements of maximal degree is nonempty and open, hence dense in V.*

Proof. It is shown in the proof of [5, Theorem 5.2] that $k\{x\}$ has maximal degree iff $\mu_{x,x} = m_{x,x}$, where m is the generic minimum polynomial of V. Thus the set under consideration is $T := \{x \in V;\ \mu^0_{x,x} = \mu_{x,x} = m_{x,x}\}$. Since m and the reduced generic minimum polynomial m^0 of V coincide by semisimplicity [5, Theorem 6.5] we have $T = \{x \in V; \mu^0_{x,x} = m^0_{x,x}\} \cap \{x \in V;$ $\mu_{x,x} = m_{x,x}\}$, i.e., T is the intersection of the diagonal in $V \times V$ with $U \cap U^0$, where $U^{(0)} = \{(x, y) \in V \times V;\ \mu^{(0)}_{x,y} = m^{(0)}_{x,y}\}$. Because both U and U^0 are open and dense in $V \times V$ [5, Theorems 5.2 and 6.2] the assertion follows. ∎

The following proposition is needed later:

PROPOSITION 1.5. (a) *If x is triagonal in V, then $\bar{x}$ is triagonal in any homomorphic image $\bar{V}$.*

(b) *$x = x_1 \oplus \cdots \oplus x_r$ is triagonal in $V = V_1 \oplus \cdots \oplus V_r$ iff each x_i is triagonal in V_i.*

Proof. (a) If $\bar{x} = \sum \alpha_i \bar{e_i}$ in $\bar{V}$, where the $\bar{e_i}$ are still orthogonal tripotents (but may be zero).

(b) If x is triagonal so is each x_i by (a). Conversely, if each $x_i = \sum_j \alpha_{ij} e_{ij}$ for orthogonal tripotents e_{ij} in V_i, then $x = \sum_{i,j} \alpha_{ij} e_{ij}$, where $\{e_{ij}\}$ forms an orthogonal family of tripotents in V. ∎

2. SEPARABLE ELEMENTS

We want to introduce a slightly more general notion than triagonal elements. For this purpose we need the notion of separability, which depends on the following fact: If V is a finite-dimensional Jordan triple system over k and $K \supset k$ is an extension field, then [6, 15.2]

$$K \otimes \operatorname{Rad} V \subset \operatorname{Rad}(K \otimes V) \tag{2.1}$$

where in general, there is no equality. This leads to the definition: A (perhaps infinite-dimensional) V is called *separable* if $K \otimes V$ is semisimple for every extension field $K \supset k$. It is also shown in [6, 15.2] that

$$K \otimes \operatorname{Rad} V = \operatorname{Rad}(K \otimes V) \qquad \text{if} \quad K \supset k \text{ is separable.} \tag{2.2}$$

Consequently, every semisimple finite-dimensional Jordan triple system over a perfect field is separable. Moreover, V is separable iff $\bar{k} \otimes V$ is semisimple where $\bar{k}$ is an algebraic closure of k [6, 15.2].

One calls $x \in V$ *algebraic* if $k\{x\}$ is finite dimensional (equivalently, x satisfies some nontrivial polynomial $\sum_{i=1}^{r} \alpha_i x^{2i-1} = 0$), and an algebraic element is said to be *semisimple* if $k\{x\}$ is semisimple, and *separable* if $k\{x\}$ is separable.

LEMMA 2.1. *Any triagonal element is separable algebraic.*

Proof. Let $K \supset k$ be an extension field. Then Lemma 1.1(3) shows $K\{x\} = K \otimes k\{x\} = Ke_1 \oplus \cdots \oplus Ke_r$, which is a semisimple subsystem of $K \otimes V$. ∎

It is easy to construct examples of separable elements which are not triagonal. However, we will see that at least after a finite Galois extension the two notions coincide. For this we need:

LEMMA 2.2. (a) *x is algebraic in V iff x is algebraic in $V^{(x)}$ iff x is algebraic in $\hat{V}^{(x)}$.*

(b) *Let $x \in V$ be algebraic. Then $k\{x\}$ is semisimple iff $k\{x\}^{(x)}$ is semisimple iff $(k[x]^{(x)})\hat{}$ is semisimple.*

Proof. (a) As vector spaces $k\{x\} = k[x]^{(x)}$, which is finite dimensional iff $(k[x]^{(x)})\hat{} = k1 + k[x]^{(x)}$ is finite dimensional.

(b) By [8, 13 Theorem 8] we have

$$\operatorname{Rad}(k\{x\}^{(x)}) = \{u \in k\{x\};\, P(x)\, u \in \operatorname{Rad} k\{x\}\} \supset \operatorname{Rad} k\{x\} \qquad (*)$$

which immediately implies "$\Leftarrow$." Conversely, if $k\{x\}$ is semisimple it is von Neumann regular [8, 14 Theorem 3]. Therefore $P(x) \mid k\{x\}$ is surjective and hence bijective. But then $(*)$ shows $\operatorname{Rad} k\{x\}^{(x)} = 0$. ∎

THEOREM 2.3. *For $x \in V$ the following are equivalent:*

(1) *x is separable algebraic in V,*

(2) *x is separable algebraic in $\hat{V}^{(x)}$,*

(3) *x is algebraic with separable minimum polynomial $\mu_{x,x} = \mu^0_{x,x}$,*

(4) *there exists a finite extension K of k such that x becomes triagonal in $K \otimes V$.*

If char $k \neq 2$ *the extension $K \supset k$ in (4) can be chosen to be Galois.*

Proof. (1) $\Leftrightarrow$ (2) by Lemma 2.2 and (2) $\Rightarrow$ (3) by standard results on quadratic Jordan algebras. For (3) $\Rightarrow$ (4) let K be the splitting field of

$g \in k[\tau]$, where $g(\tau) = \mu_{x,x}(\tau^2)$. Then $\mu_{x,x}$ splits over K with every root a square, hence x is triagonal in $K \otimes V$ by Corollary 1.3. If in addition char $k \neq 2$, g is separable and $K \supset k$ is Galois.

Finally, for (4) $\Rightarrow$ (1) it is enough to prove that $\bar{k} \otimes k\{x\} = \bar{k}\{x\}$ is semisimple. We may assume $k \subset K \subset \bar{k}$ and thus $\bar{k}\{x\} = \bar{k} \otimes K\{x\}$. But $K\{x\}$ is separable by Lemma 2.1 and therefore $\bar{k}\{x\}$ is semisimple. ∎

The following example shows that in characteristic 2 there need not exist a finite Galois extension $K \supset k$ such that a given separable algebraic element of V becomes triagonal in $K \otimes V$: Let k be a field with char $k = 2$ and $\alpha \in k$, but $\alpha \notin k^2$. We consider the Jordan triple system V on the vector space k with product $P(u)\, v = \alpha u^2 v$. Then V is separable and $x = \alpha \in V$ is separable algebraic, but does not become triagonal in $K \otimes V$ for any Galois extension $K \supset k$ because $\sqrt{\alpha}$ is not separable over k.

3. Simple Properties of Centralizers. Examples

For a subset $X \subset V$ we define the *centralizer of X in V* to be the subspace

$$\begin{aligned} C_V(X) &:= \{v \in V; L(v, x) = L(x, v) \text{ for all } x \in X\} \\ &= \{v \in V; D(v, X) = 0\} = \{v \in V; D(X, v) = 0\} \end{aligned}$$

of all elements v which commute with all elements of X as defined in (1.5). If no ambiguity is possible we will write $C(X)$ instead of $C_V(X)$.

We begin our investigations with some easy properties of centralizers:

$$C(X) = \bigcap_{x \in X} C(\{x\}) = C(kX) \tag{3.1}$$

where kX denotes the subspace of V generated by X.

$$\text{If } \dim_k V < \infty, \text{ then } C(X) = C(\bar{X}) \tag{3.2}$$

where $\bar{X}$ is the closure of X in the Zariski topology of V.

$$\begin{gathered} \text{If } K \supset k \text{ is a field extension, then} \\ K \otimes C_V(X) = C_{K \otimes V}(X) \end{gathered} \tag{3.3}$$

where we identify V with $1 \otimes V$.

$$\begin{gathered} X \subset C(C(X)),\, X \subset Y \Rightarrow C(X) \supset C(Y), \\ \text{consequently } C(X) = C(C(C(X))). \end{gathered} \tag{3.4}$$

We now look at a special class of Jordan triple systems: Every Jordan algebra $\mathscr{J}$ can be viewed as a Jordan triple system by simply forgetting the squaring operation (or the unit element, if $\mathscr{J}$ has one). The triple system arising in this way is denoted by $V(\mathscr{J})$. Since $D(x, y) = L(x, y) - L(y, x) = [V(x), V(y)]$ we get

$$C_{V(\mathscr{J})}(X) = C_{\mathscr{J}}(X) = \{a \in \mathscr{J};\ [V(a), V(x)] = 0 \text{ for all } x \in X\}. \quad (3.5)$$

This means that in the case $V = V(\mathscr{J})$ of a linear Jordan algebra $\mathscr{J}$ (char $k \neq 2$) our notion of a centralizer coincides with the one studied by Jacobson in [3] and Harris in [2] (see also [4, VIII, 3]).

EXAMPLE 3.1. Let $\mathscr{A}$ be an associative algebra. Every subspace V of $\mathscr{A}$ which is closed under $P(x)\,y = xyx$ is a Jordan triple system, as can be easily checked. Since $\{xyz\} = xyz + zyx$ the following equivalence for $x, y \in V$ is readily seen:

$$D(x, y) = 0 \Leftrightarrow [[xy]\,z] = 0 \qquad \text{for all } z \in V \quad (3.6)$$

where $[uv] = uv - vu$. If $\mathscr{A}$ is an envelope (i.e., is generated as associative algebra by V) then

$$D(x, y) = 0 \qquad \text{iff} \qquad [xy] \in C(\mathscr{A}) \quad (3.7)$$

which is weaker than the associative commutativity condition $[xy] = 0$.

EXAMPLE 3.2. We can easily construct an infinite-dimensional associative algebra over any field k generated by 1, a and b with $[ab] = 1$ (e.g., a twisted polynomial ring). Clearly by (3.7) a and b commute in the Jordan sense, but a and b^2 do not Jordan-commute if char $k \neq 2$ since $[ab^2] = [ab]\,b + b[ab] = 2b \notin C(\mathscr{A})$. This shows that $C(V)$, $V = \mathscr{A}$, need not be a subsystem in general: $1 \in C(V)$, $b \in C(V)$, but $b^2 = P(b)\,1 \notin C(V)$.

The same phenomenon also occurs in finite dimensions: We can construct a 10-dimensional associative algebra $\mathscr{A}$ generated by 1, a and b with

$$[ab] = c, \qquad [ac] = [bc] = 0, \qquad c^2 \neq 0.$$

(Let $\mathscr{A}$ be the free algebra in a, b modulo the ideal I generated by all monomials of degree $\geqslant 3$ in a or in b, together with a^2b, ab^2, $ba^2 - 2aba$, $b^2a - 2bab$, a^2b^2, ba^2b, $abab$, ab^2a, $b^2a^2 - 2baba$; $\mathscr{A}$ has basis 1, a, b, a^2, ab, ba, b^2, aba, bab, $baba$.) From (3.7) we deduce $1 \in C(a)$ and $b \in C(a)$ since $[[ab]\,a] = 0 = [[ab]\,b]$ implies $[ab] \in C(\mathscr{A})$ if 1, a and b generate $\mathscr{A}$, but $P(b)\,1 \notin C(a)$ if char $k \neq 2$, since $[[ab^2]\,a] = [2bc, a] = -2c^2 \neq 0$. Thus $C(a)$ need not be a subsystem of V in general.

EXAMPLE 3.3. The following is an example of a Jordan triple system which in general does not arise from a Jordan algebra: Let q be a quadratic form on a vector space V over the field k. Then V together with P becomes a Jordan triple system if we put

$$P(x)\,y = q(x, y)\,x - q(x)\,y$$

where $q(x, y) = q(x + y) - q(x) - q(y)$, $x, y \in V$. One knows that the radical of V coincides with the radical of q. It is readily checked that if char $k \neq 2$ then $y \in C(x)$ iff $q(z, y)\,x = q(x, z)\,y$ for all $z \in V$. Thus

$$\begin{aligned} C(x) &= \operatorname{Rad} q = \operatorname{Rad} V && \text{for} \quad 0 \neq x \in \operatorname{Rad} q \\ C(x) &= kx && \text{for} \quad 0 \neq x \notin \operatorname{Rad} q. \end{aligned}$$

In both cases $C(x)$ is a subsystem of V.

We specialize to the case char $k \neq 2$, q nondegenerate and $V = V_1 \oplus V_2$ with dim $V_i \geqslant i$. Then V and V_i are central simple. We have $C(V_2) = 0$ since dim $V_2 \geqslant 2$ and hence $C(C(V_2)) = V \supsetneqq V_2$. This shows that in general the inclusion in (3.4) is strict: There is no double centralizer theorem.

Another consequence we can draw here is $C_V(V) = 0$. This is contrary to the Jordan algebra situation, where the unit element centralizes everything. The following theorem is proven in [11]:

THEOREM 3.4. *Let V be central and non-degenerate (i.e., $P(x) = 0 \Rightarrow x = 0$) and* char $k \neq 2$. *Then $C_V(V) \neq 0$ iff there exists a unital Jordan algebra $\mathcal{J}$ on the underlying vector space of V with quadratic representation U and a scalar $\zeta \in k^*$ such that $P(x) = \zeta U(x)$ for all $x \in V$.*

4. CENTRALIZERS OF A SINGLE ELEMENT

From now on we will assume char $k \neq 2$.

In this section we study centralizers of a single element. For easier notation we write $C(x) := C(\{x\})$. The formula (13.32) in [8] says $L(u, v) = L(v, u)$ for $u, v \in k\{x\}$ which gives

$$k\{x\} \subset C(k\{x\}) \subset C(x). \tag{4.1}$$

From Example 3.3 we see that in general we do not have equality. We also note that equality would imply that $C(x)$ is a subsystem of V, which, as we know, is not the case in general. However, for triagonal elements $C(x)$ is a subsystem.

THEOREM 4.1 (Triagonal Centralizing Criterion). *Let x be triagonal with minimal triagonalization $x=\sum_{i=1}^{r}\alpha_i e_i$. Then for v in V the following are equivalent*:

(1) $v \in C(k\{x\})$,

(2) $v \in C(x)$,

(3) $D(x, v)x = 0$,

(4) $v \in V_{00} + (\sum_{i=1}^{r} V_2^+(e_i))$.

In particular, $C(x)$ is a subsystem for triagonal x. Further, if e is tripotent then

$$C(e) = V_0(e) \oplus V_2^+(e).$$

Proof. From the definition we see $(1) \Rightarrow (2) \Rightarrow (3)$. Let $V = \sum V_{ij}$ be the Peirce decomposition of V. Then, using the Peirce multiplication rules proved in [8, 15] and the notation $\bar{v}_{ij} = P(e_1 + \cdots + e_r)\, v_{ij}$ we get

$$\{xxv\} = 2\sum_{i>0} \alpha_i^2 v_{ii} + \sum_{j>i>0} (\alpha_i^2 + \alpha_j^2)\, v_{ij} + \sum_{i>0} \alpha_i^2 v_{i0}$$

$$\{xvx\} = 2\sum_{i>0} \alpha_i^2 \bar{v}_{ii} + 2\sum_{j>i>0} \alpha_i \alpha_j \bar{v}_{ij}.$$

Since the α_j are all distinct and nonzero and char $k \neq 2$ we have $(3) \Leftrightarrow (4)$ since $D(x, v)x = 0$ iff $\{xxv\} = \{xvx\}$ iff all $v_{0i} = 0$, $\bar{v}_{ii} = v_{ii}$, and $v_{ij} = 0$ (initially we get $(\alpha_i^2 + \alpha_j^2)\, v_{ij} = 2\alpha_i \alpha_j \bar{v}_{ij}$, which implies $v_{ij} = \lambda \bar{v}_{ij}$, hence $v_{ij} = \pm \bar{v}_{ij}$, and $(\alpha_i \pm \alpha_j)^2\, v_{ij} = 0$; because $\alpha_i \pm \alpha_j \neq 0$ this forces v_{ij} to be 0). Thus it remains to show $(4) \Rightarrow (1)$. Since $k\{x\} = \bigoplus_{j=1}^{r} ke_j$ we get $C(k\{x\}) = \bigcap_j C(e_j)$ by (3.1). But $C(e_j) = V_2^+(e_j) \oplus V_0(e_j)$ by what we have shown up to now. Hence $\bigcap_j C(e_j) = V_{00} \oplus (\bigoplus_{j=1}^{r} V_2^+(e_j))$ and $(4) \Rightarrow (1)$ follows. ∎

The following theorem says that $C(x)$ is a subsystem for the much broader class of separable elements of V:

THEOREM 4.2. *Let $x \in V$ be separable. Then*

(a) $C(x) = C(k\{x\}) = \{v \in V;\ \{xxv\} = \{xvx\}\}$ *is a subsystem.*

(b) *If W is a subsystem of V containing x, then $C_W(x) = W \cap C_V(x)$.*

(c) *If in addition V itself is separable, then $C(x)$ is a separable subsystem.*

Proof. Using (3.3) it is easy to see that it is enough to show

(a) in case k is algebraically closed. But in this case x is triagonal and so the assertions follow from Lemma 4.1.

(b) Since x is also separable in W, we get $C_W(x) \subset W \cap C_V(x)$ from (a); the other inclusion is always true.

(c) We have to show that $\bar{k} \oplus C_V(x) = C_{\bar{k} \oplus V}(x)$ is semisimple, where $\bar{k}$ is an algebraic closure of k. This follows from Theorem 4.1(4) since we will see below in Corollary 4.4 that all Peirce spaces V_{00}, $V_2^+(e_i)$ (and hence their direct sum) inherit semisimplicity.

For semisimplicity of Peirce spaces we need the following lemma, which supplements the investigations in [10]. (Actually (4.3) is true for an involutive grading of any Jordan structure over an arbitrary ring of scalars.)

LEMMA 4.3. *Let V be a Jordan triple system and $V = V_+ \oplus V_-$ the eigenspace decomposition of an involutive automorphism of V. Then*

$$\operatorname{Rad} V_\varepsilon = V_\varepsilon \cap \operatorname{Rad} V, \qquad \varepsilon = \pm \tag{4.3}$$

where Rad *denotes the Jacobson-radical*; *in particular*

$$V \text{ semisimple} \Leftrightarrow V_\pm \text{ semisimple.}$$

Proof. For $x, y \in V_\varepsilon$ we have according to [6, 3.2]: (x, y) is quasi-invertible in V iff (x, y) is quasi-invertible in V_ε. This already implies $V_\varepsilon \cap \operatorname{Rad} V \subset \operatorname{Rad} V_\varepsilon$. Let x be in $\operatorname{Rad} V_\varepsilon$. For the reverse inclusion in (4.3) we must show $(x, y+z)$ is quasi-invertible for all z in V_ε, y in $V_{-\varepsilon}$, which by [6, (3.7)] is equivalent to (x, z) and (x^z, y) being quasi-invertible. Now by definition of $x \in \operatorname{Rad} V_\varepsilon$ we have (x, z) quasi-invertible for z in V_ε. Let $a := x^z$ in V_ε. Then $x - P(x)z = B(x, z)a = a - \{xza\} + P(x)P(z)a$ for x in the ideal $\operatorname{Rad} V_\varepsilon$ shows a also is in $\operatorname{Rad} V_\varepsilon$. Because $P(y)a \in V_\varepsilon$ we have $(a, P(y)a)$ quasi-invertible in V_ε, hence $B(a, P(y)a) = B(a, y)B(a, -y)$ (by [6, JP24]) is invertible, so $B(a, y)$ is surjective and (by [6, 3.2]) (a, y) is quasi-invertible. Thus (x, z) and (x^z, y) are quasi-invertible. ∎

COROLLARY 4.4. (a) ([6, 5.8; 12, Lemma 6]). *Let $V = \bigoplus V_{ij}$ be the Peirce decomposition of V relative to an orthogonal system $(e_1, \dots, e_r)$. Then*

$$\operatorname{Rad} V_{ij} = V_{ij} \cap \operatorname{Rad} V \qquad \text{for} \quad 0 \leqslant i \leqslant j \leqslant r. \tag{4.4}$$

(b) *For any tripotent e we have*

$$\operatorname{Rad} V_2^\varepsilon(e) = V_2^\varepsilon(e) \cap \operatorname{Rad} V, \qquad \varepsilon = \pm.$$

In particular, if V is semisimple, then so are all V_{ij} and $V_2^\varepsilon(e)$.

Proof. (a) We apply 4.3 to the automorphism $B(e, 2e)$. Here $V_+ = V_2(e) + V_0(e)$, $V_- = V_1(e)$, so $\operatorname{Rad} V_\mu(e) = V_\mu(e) \cap \operatorname{Rad} V$ for $\mu = 0, 1, 2$.

Applying this to $e_0 = e_1 + \cdots + e_r$ ($\mu = 0$) and to e_i ($\mu = 2$) gives (4.4) for V_{00}, V_{ii}. Restricting $B(e_i, 2e_i)$ to $V_1(e_0)$ and $V_1(e_j)$ with $\mu = 1$ gives (4.4) for $V_{i0} = V_1(e_0) \cap V_1(e_i)$ and $V_{ij} = V_1(e_j) \cap V_1(e_i)$.

(b) We already know Rad $V_2(e) = V_2(e) \cap \operatorname{Rad} V$. Since $V_2^{\pm}(e)$ are the eigenspaces of the involutive automorphism $P(e) \mid V_2(e)$, the result follows from (4.3). ∎

As another application of Lemma 4.1 we will prove that the sum of two commuting separable elements is separable. We begin with a preparatory lemma:

LEMMA 4.5. *Let $e \in V$ be a tripotent and $y \in V_2^+(e)$. Then*

(a) *$k\{e, y\}$ is flat.*

(b) *If y is triagonal, so is $k\{e, y\}$.*

Proof. (a) Since $P(e) = \mathrm{Id}$ on $\mathcal{T} = V_2^+(e)$ we see $\mathcal{T}$ is a Jordan algebra with unit e, $k\{e, y\} = k[y]$; it is well known that any Jordan subalgebra $k[y]$ is flat.

(b) By Theorem 1.1 $k\{y\} = kf_1 \oplus \cdots \oplus kf_r$ is triagonal if y is. Since $y \in V_2^+(e)$ we get $y \in C(e)$ from Theorem 4.1. Then $e \in C(y)$ and we apply once more Theorem 4.1, this time to $x = y$, $v = e$. We obtain $e = e_0 + e_1 + \cdots + e_r$ for elements $e_0 \in V_0(f_i)$, $e_i \in V_2^+(f_i)$. Because e is tripotent the e_i are orthogonal tripotents. We claim

$$k\{e, y\} = ke_0 \oplus (kg_1 \oplus kh_1) \oplus \cdots \oplus (kg_r \oplus kh_r)$$
$$(g_i = \tfrac{1}{2}(f_i + e_i),\ h_i = \tfrac{1}{2}(f_i - e_i)).$$

It suffices to prove the g_i, h_j are orthogonal tripotents; then they span a subsystem containing e_i, f_i, hence e, y, yet are contained in $k\{e, y\}$, so they span $k\{e, y\}$. But orthogonal tripotency is easily verified using (1) $P(f_i)\, e_i = e_i$ $(e_i \in V_2^+(f_i))$, (2) $P(e_i) f_i = f_i$ $(f_i \in V_2^+(e))$, (3) $L(f_i, e_i) = L(e_i, f_i)$, (4) g_i, h_i, $e_i, f_i \in V_2^+(f_i)$, $e_0 \in V_0(f_i)$. ∎

THEOREM 4.6. *Assume that x and y commute and x is separable. Then*

(a) *$k\{x, y\}$ is flat.*

(b) *If y is also separable then $k\{x, y\}$ is separable.*

(c) *If x, y are triagonal then $k\{x, y\}$ is triagonal.*

Proof. It is enough to prove the assertions (a), (c) when x is triagonal, since these will imply the assertions (a), (b), (c) for separable x by passing to the algebraic closure (using Theorem 2.3(4)). Let $x = \alpha_1 e_1 + \cdots + \alpha_r e_r$ be the minimal triagonalization of x. By Theorem 4.1 $y = y_0 + y_1 + \cdots + y_r$, where $y_0 \in V_{00}$, $y_i \in V_2^+(e_i)$. Since the Peirce projections in (1.7) are given by multiplication operators by the e_i, we have a direct sum decomposition

$$k\{x, y\} = k\{y_0\} \oplus k\{e_1, y_1\} \oplus \cdots \oplus k\{e_r, y_r\}.$$

By 4.5(a) each $k\{e_i, y_i\}$ is flat, hence so is their direct sum, establishing (a). Now assume y is triagonal. By Proposition 1.5 each y_i is triagonal too, and by Lemma 4.5(b) each $k\{e_i, y_i\}$ is triagonal, so their direct sum $k\{x, y\}$ is triagonal, too. ▮

5. Arbitrary Centralizers

We turn to the description of arbitrary centralizers.

THEOREM 5.1. *Let X be a finite-dimensional separable subsystem of V, or more generally any subset such that KX is spanned by separable elements for some field extension $K \supset k$. Then*

(a) *$C_V(X)$ is a subsystem of V.*

(b) *If W is a subsystem of V containing X, then*

$$C_W(X) = W \cap C_V(X).$$

Proof. Let $Y \subset KX$ consist of separable elements with $KY = KX$.

(a) It is enough to show that $K \otimes C_V(X)$ is a subsystem of $K \otimes V$, which follows from $K \otimes C_V(X) = C_{K \otimes V}(KX) = C_{K \otimes V}(Y) = \bigcap_{y \in Y} C_{K \otimes V}(y)$ and Theorem 4.2(a).

(b) We may assume $KX \subset K \otimes W \subset K \otimes V$. Then Theorem 4.2(b) implies $K \otimes C_W(X) = C_{K \otimes W}(KX) = \bigcap_{y \in Y} C_{K \otimes W}(y) = K \otimes W \cap \bigcap_{y \in Y} C_{K \otimes V}(y) = (K \otimes W) \cap K \otimes C_V(X) = K \otimes (W \cap C_V(X))$, which shows (b). The case when X is a separable subsystem follows from Theorem 1.4 (the separable elements are dense, hence span X). ▮

If we specialize Theorem 5.1 to the situation where V is the Jordan triple system of a Jordan algebra we get

COROLLARY 5.2. *Let $\mathcal{J}$ be an arbitrary Jordan algebra and $\mathcal{B}$ a finite-dimensional separable subalgebra. Then the centralizer $C_{\mathcal{J}}(\mathcal{B})$ is a subalgebra, and if $\mathcal{K}$ is a subalgebra of $\mathcal{J}$ containing $\mathcal{B}$ then $C_{\mathcal{K}}(\mathcal{B}) = \mathcal{K} \cap C_{\mathcal{J}}(\mathcal{B})$.*

This corollary was first proved by N. Jacobson in characteristic 0 using the theory of "associates" [3, Theorem 2] and afterwards generalized as stated above by Harris [2, Proposition 1.1] using the classification of separable algebras.

The proof of the following theorem uses a technique which is "Weyl's unitary trick for Jordan simple systems."

THEOREM 5.3. *Let V be a finite-dimensional semisimple Jordan triple system over a field of characteristic* 0, *W a semisimple subsystem over V. Then $C_V(W)$ is a semisimple subsystem.*

Proof. We begin by recalling that in characteristic zero semisimplicity is a "linear property": V (over k) is semisimple

$\Leftrightarrow K \otimes V$ is semisimple for every extension field $K \supset k$

$\Leftrightarrow L \otimes V$ is semisimple for some extension field $L \supset k$.

Furthermore, by Theorem 5.1 we only need to show "semisimplicity" of $C_V(W)$.

(a) Let V be compact (see [9, 3.1(c)] for a definition). Since every subsystem of a compact Jordan triple system is again compact [9, Satz, 3.6(b)] and therefore semisimple, the theorem holds in this case.

(b) Assume $k = \mathbb{C}$. By [9, Satz 3.5] we can choose a Cartan involution of W and extend it to a Cartan involution of V. Since Cartan involutions of semisimple complex Jordan triple systems are just conjugations relative to compact forms [9, Satz 2.8], this means that there exists a compact Jordan triple system U such that $V = \mathbb{C} \otimes U$ and $W = \mathbb{C} \times (W \cap U)$. Because $C_V(W) = \mathbb{C} \otimes C_U(U \cap W)$ and (a) holds, the theorem follows in this case.

(c) If $\mathbb{Q} \subset k \subset \mathbb{C}$ the theorem follows from (b) and the remark at the beginning of the proof.

(d) Let k be arbitrary, $(x_1, ..., x_m)$ a basis of W over k, $(x_1, ..., x_m, x_{m+1}, ..., x_n)$ a basis of V over k, c^l_{ijr} the structure constants of V relative to $x_1, ..., x_n$, i.e., $\{x_i x_j x_r\} = \sum_{l=1}^n c^l_{ijr} x_l$, and let k_0 be the smallest subfield of k containing all the c^l_{ijr}. Define $V_0 = \bigoplus_{j=1}^n k_0 x_j$ and $W_0 = V_0 \cap W = \bigoplus_{j=1}^m k_0 x_j$. Obviously, V_0 is a Jordan triple system over k_0 such that $k \otimes V_0 = V$ and W_0 is a subsystem of V_0 with $k \otimes W_0 = W$. Hence W_0 and V_0 are semisimple and the theorem follows in general if we can prove it for V_0 and W_0. But since k_0 is finitely generated over the prime field of k_0, there exists a field isomorphism of k_0 onto a subfield k_1 of $\mathbb{C}$. This can be extended to an isomorphism (over the integers) of V_0 onto $V_1 = \bigoplus_{j=1}^n k_1 x_j$ sending W_0 onto $W_1 = \bigoplus_{j=1}^m k_1 x_j$. Because the radical does not change by restricting scalars to $\mathbb{Z}$ the triple systems V_1 and W_1 are semisimple. By (c) $C_{V_1}(W_1)$ is semisimple and thus also $C_{V_0}(W_0)$. ∎

We point out that Theorem 5.3 generalizes Theorem 4.2 at the expense of restricting the characteristic of the base field. Further, analogous to Corollary 5.2 one gets the corresponding result for Jordan algebras which was proved by N. Jacobson [3, Theorem 3]. This result was extended by Harris [2, Theorem 3.1] to characteristic $\neq 2$ using the classification of separable Jordan algebras.

6. Jordan–Chevalley Decomposition

In this section we will show the existence of a Jordan–Chevalley decomposition for elements in a Jordan triple system over a perfect field. To this end we need to investigate Jordan triple systems which are flat, i.e., $L(u, v) = L(v, u)$ for all $u, v \in V$. It was proved in [5, 2.8] that the flat Jordan triple systems are precisely the subsystems $V = \mathscr{A}_-$ of elements in a unital commutative associative algebra $\mathscr{A}$ which are skew under an involutorial automorphism σ. Indeed, setting

$$\mathscr{A}_+ := k\,\mathrm{Id}_V + \left\{\sum L(x_i, y_i); x_i, y_i \in V\right\}$$

$$\mathscr{A}_- := V \text{ (as vector space)}$$

$$XY = X \circ Y, \qquad uv = L(u, v), \qquad Xv = vX = X(v)$$

makes $\mathscr{A} = \mathscr{A}_+ \oplus \mathscr{A}_-$ into a unital commutative and associative algebra with involutorial automorphism $\sigma(a_+ + a_-) = a_+ - a_-$ and

$$P(u)\,v = uvu, \qquad \{uvw\} = 2uvw. \tag{6.1}$$

We point out that the odd powers of $v \in V$ in V and $\mathscr{A}$ coincide. This reduction to commutative associative systems makes it easy to establish results about flat systems.

Lemma 6.1. *Let V be flat, $u, v \in V$ and n odd. Then*

(a) $(P(u)\,v)^n = P(u^n)\,v^n$,

(b) $(u+v)^n = \sum_{\substack{i=0 \\ i \equiv 1(2)}}^{n} \binom{n}{i} P(v)^{(1/2)(n-i)}\,u^i + \sum_{\substack{i=0 \\ i \equiv 0(2)}}^{n} \binom{n}{i} P(u)^{(1/2)i}\,v^{n-i}$.

Proof. (a) Using (6.1) we get $(P(u)\,v)^n = (uvu)^n = u^n v^n u^n = P(u^n)\,v^n$.

(b) Since $\mathscr{A}$ is commutative and associative, we can apply the usual binomial formula and get

$$(u+v)^n = \sum_{i=0}^{n} \binom{n}{i} u^i v^{n-i} = \sum_{\substack{i=0 \\ i\equiv 1(2)}}^{n} \binom{n}{i} v^{(1/2)(n-i)} u^i v^{(1/2)(n-i)}$$
$$+ \sum_{\substack{i=0 \\ i\equiv 0(2)}}^{n} \binom{n}{i} u^{(1/2)i} v^{n-i} u^{(1/2)i}$$

which is just our assertion. ∎

For a Jordan triple system V in general, one calls $v \in V$ *nilpotent* if $v^n = 0$ for some n. Moreover, v is called *properly nilpotent* if v is nilpotent in all Jordan algebras $V^{(u)}$, $u \in V$.

COROLLARY 6.2. *Let V be flat and N the set of nilpotent elements of V. Then N is an ideal of V consisting of properly nilpotent elements.*

Proof. By Lemma 6.1(b) N is a subspace of V and by Lemma 6.1(a) we have $P(V)N + P(N)V \subset N$. But $P(V)N \subset N$ also implies $P(V, V)N = L(V, V)N \subset N$, thus N is an ideal of V. For $x \in N$ and $u \in V$ we get by induction: $x^{(2n+1,u)} = P(u)^n x^{2n+1}$, because $x^{(2n+1,u)} = P(x)P(u)x^{(2n-1,u)} = P(x)P(u)P(u)^{n-1}x^{2n-1} = P(u)^n x^{2n+1}$, which shows that x is properly nilpotent. ∎

COROLLARY 6.3. *Let V be flat and finite dimensional. Then the radical of V is the set of nilpotent elements of V. In particular, if V is semisimple so is any subsystem.*

Proof. This is a consequence of Corollary 6.2 and [8, XIII, Theorem 12]. ∎

Besides nilpotent elements the other ingredient of Jordan–Chevalley decompositions is separable elements. We put

$$S(V) := \{x \in V;\ x \text{ separable}\}.$$

As an immediate consequence of Theorem 4.6 we get:

LEMMA 6.4. *Let V be flat. Then $S(V)$ is a subsystem of V, i.e., $S(V)$ is a subspace of V and $x, y \in S(V)$ implies $P(x)y \in S(V)$.*

LEMMA 6.5. *Let V be flat and finite dimensional.*

(a) *If V is separable, then every element of V is separable.*

(b) *Conversely, if k is perfect and every element of V is separable, then V is separable.*

Proof. (a) Since $K \otimes V$ is again flat and semisimple, it is enough to show that for $x \in V$ $k\{x\}$ is semisimple. This follows from Corollary 6.3 because the only separable and nilpotent element is zero.

(b) We have:

$$K \otimes S(V) = S(K \otimes V) \qquad \text{if} \quad \begin{array}{l} K/k \text{ is Galois} \\ V/k \text{ finite dimensional} \end{array} \tag{$*$}$$

since $\operatorname{Rad}(K\{g(x)\}) = g(\operatorname{Rad} K\{x\})$ shows $S(K \otimes V)$ is stable under $g \in \operatorname{Gal}(K/k)$, hence by [6, 15.1] it is defined over k, so $S(K \otimes V) = K \otimes \{S(K \otimes V) \cap V\} = K \otimes S(V)$. In particular, if k is perfect all K/k are separable and have a Galois closure $\bar{K}$, and if $S(V) = V$, then $S(\bar{K} \otimes V) = \bar{K} \otimes V$ shows $\operatorname{Rad}(\bar{K} \otimes V) = 0$, hence $\operatorname{Rad}(K \otimes V) = 0$, too, in view of (2.1), and V is separable. ∎

THEOREM 6.6. *Let V be a finite-dimensional and flat Jordan triple system over a perfect field k. Then V is the direct sum of the separable subsystem $S(V) = \{x \in V;\ x \text{ separable}\}$ and the ideal* $\operatorname{Rad} V = \{x \in V;\ x$ *nilpotent*$\}$:

$$V = S(V) \oplus \operatorname{Rad} V. \tag{6.2}$$

Moreover, the decomposition (6.2) *is unique: Every separable subsystem of V is contained in $S(V)$.*

Proof. By Lemma 6.5(b) we already know that $S(V)$ is a separable subsystem of V. Further, $S(V) \cap \operatorname{Rad} V = \{0\}$.

Since $\bar{k} \otimes \operatorname{Rad} V = \operatorname{Rad} \bar{k} \otimes V$ and $\bar{k} \otimes S(V) = S(\bar{k} \otimes V)$ by formula $(*)$ in the proof of Lemma 6.5(b) it is enough to show $V = S(V) + \operatorname{Rad} V$ for an algebraically closed k. We denote by $x \to \bar{x}$ the canonical projection of V onto $\bar{V} = V/\operatorname{Rad} V$. For an arbitrary $x \in V$ we have that $\bar{x}$ is separable by Lemma 6.5(a).

Therefore Theorem 2.3 implies the existence of an orthogonal system $(\bar{e}_1,..., \bar{e}_r)$ in $\bar{V}$ with $x = \alpha\bar{e}_1 + \cdots + \alpha_r\bar{e}_r$. By [7, 4.9] we can lift $(\bar{e}_1,..., \bar{e}_r)$ to an orthogonal system $(e_1,..., e_r)$ of tripotents in V. Because $\bar{x} = \overline{(\alpha_1 e_1 + \cdots + \alpha_r e_r)}$ and $\alpha_1 e_1 + \cdots + \alpha_r e_r$ is triagonal we get $V = S(V) \oplus \operatorname{Rad} V$.

The uniqueness of the decomposition follows from Lemma 6.5(a). ∎

Remark 6.7. Another way of phrasing Theorem 6.6 is to say that V has a unique Wedderburn decomposition. This puts Theorem 6.6 into a wider context. Actually, it is possible to derive this theorem from the classical Wedderburn decomposition theorem and Malcev's theorem about the conjugacy of such decompositions by making use of the imbedding of V into an associative and commutative algebra as described in the beginning of this section.

COROLLARY 6.8. *Let V be a Jordan triple system over a perfect field k and let $x \in V$ be algebraic. Then there exists a unique decomposition*

$$x = x_s + x_n \tag{6.3}$$

with the properties

(1) *x_s and x_n commute,*

(2) *x_s is separable and x_n is nilpotent.*

Moreover, $x_s, x_n \in k\{x\}$ are polynomials in x.

Proof. We can apply Theorem 6.6 to $k\{x\}$ and get $k\{x\} = S(k\{x\}) \oplus \operatorname{Rad} k\{x\}$ and thus $x = x_s + x_n$, where $x_s \in S(k\{x\})$ is separable and $x_n \in \operatorname{Rad} k\{x\}$ is nilpotent.

Let $x = x_s' + x_n'$, where x_s', x_n' satisfy (1) and (2). Then $W = k\{x_s', x_n'\}$ is flat by Theorem 4.6(a), so by Theorem 6.6 $W = S(W) \oplus \operatorname{Rad} W$. Here $x = x_s + x_n = x_s' + x_n'$ for $x_s, x_n \in k\{x\} \subset W$ with $x_n, x_n' \in N = \operatorname{Rad} W$ by (2) and Corollary 6.3. Moreover, $x_s, x_s' \in S(W)$ by (2), so by uniqueness of the direct decomposition we have $x_s = x_s'$, $x_n = x_n'$. ∎

Generalizing the notion in associative and Lie situations we call (6.3) the *Jordan–Chevalley decomposition.*

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