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Generically rational polynomials of quasi-simple type

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Abstract

A polynomial $f \in k[x, y]$ is generically rational if the curve $f = \alpha$ is rational for almost all $\alpha \in k$. It is of quasi-simple type if the generic member of the pencil $\Lambda(f)$ generated by f has all rational points at infinity except for one non-rational point of degree a. We classify all generically rational polynomials of quasi-simple type with $a = 2$.

0. Introduction

Let $k[x, y]$ be a polynomial ring in two variables over an algebraically closed field k of characteristic zero. The affine plane $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$ is identified with the complement of the line at infinity ℓ_∞ of the projective plane $\mathbb{P}^2$. For a polynomial $f \in k[x, y]$ of degree d, we denote by C_α the curve on $\mathbb{A}^2$ defined by $f = \alpha$ for $\alpha \in k$ and by $\bar{C}_\alpha$ the closure of C_α in $\mathbb{P}^2$. We denote by $\Lambda(f)$ the linear pencil $\{\bar{C}_\alpha;\ \alpha \in k \cup (\infty)\}$ and by $\Lambda_0(f)$ the restriction of $\Lambda(f)$ on $\mathbb{A}^2$, i.e., $\Lambda_0(f) = \{C_\alpha;\ \alpha \in k\}$, where we set $\bar{C}_\infty = d\ell_\infty$. We say that the polynomial f is *generically rational* if general members of $\Lambda(f)$ are rational curves. Since $\Lambda(f)$ has no base points on $\mathbb{A}^2$, it follows from the second theorem of Bertini that general members of $\Lambda(f)$ are smooth on $\mathbb{A}^2$. Let $\varphi : \bar{V} \to \mathbb{P}^2$ be a minimal sequence of blowing-

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ups which eliminate the base points of $\Lambda(f)$ and let $\Lambda'(f)$ be the proper transform of $\Lambda(f)$ by φ. Let $\bar{\rho}_1 : \bar{V} \to \mathbb{P}^1$ be a surjective morphism defined by the linear pencil $\Lambda'(f)$. Then $\bar{\rho}_1$ is a $\mathbb{P}^1$-fibration. Denote by S a general fiber of $\bar{\rho}_1$. Let $S_1, \ldots, S_r$ exhaust all singular fibers of $\bar{\rho}_1$ defined by $f = \alpha_i$ with $\alpha_i \in k$ $(1 \leqslant i \leqslant r)$ and let S_∞ be the fiber of $\bar{\rho}_1$ corresponding to the value ∞ of $\mathbb{P}^1$.

An irreducible curve Γ on $\bar{V}$ is called a *cross-section* (respectively *quasi-section*) if $(\Gamma \cdot S) = 1$ (respectively $(\Gamma \cdot S) \geqslant 1$). Then a generically rational polynomial f is said to be of *simple* type (respectively *quasi-simple* type) if all the quasi-sections of $\bar{\rho}_1$ arising in the process φ are cross-sections (respectively if one of the quasi-sections, say Δ, is an a-section with $a \geqslant 2$ and all the others are cross-sections, which we denote by $\Gamma_1, \ldots, \Gamma_\pi$).

The objective of the present paper is to classify generically rational polynomials of quasi-simple type with $a = 2$. We refer to Saito [9] for the case $\pi = 1$.

We shall explain our idea of writing down the *forms* of polynomials in question. The $\mathbb{P}^1$-fibration $\bar{\rho}_1 : \bar{V} \to \mathbb{P}^1$ has the singular fiber S_∞ which contains the proper transform of ℓ_∞ as a component which is necessarily a (-1) curve. The fiber S_∞ has to contain a component with multiplicity 1 because $\bar{\rho}_1$ has several cross-sections arising from the elimination of the base points of $\Lambda(f)$. Hence, leaving one such component of S_∞, say $S_{\infty 1}$, untouched, we can contract all the other components of S_∞ to obtain a surface $\tilde{V}$. By making use of the fact that $\varphi'(\ell_\infty)$ is a *unique* (-1) component in S_∞, we then restrict ourselves to the two cases (types (A) and (B), see Figs. 1, 2). Then the affine plane $\mathbb{A}^2$ is an open set of $\bar{V}$ and $\tilde{V}$. Now we blow up minimally the points on the boundary $\tilde{V} \setminus \mathbb{A}^2$ to make a divisor with simple normal crossings outside of $\mathbb{A}^2$. Then we obtain a smooth surface V and a $\mathbb{P}^1$-fibration $\rho_1 : V \to \mathbb{P}^1$ induced by $\bar{\rho}_1$.

We would like to determine one fiber of ρ_1 which intersects the open set $\mathbb{A}^2$. Then we can write down a polynomial f. In the subsequent arguments, whenever we have a $\mathbb{P}^1$-fibration on V or its blow-up and a fiber of the $\mathbb{P}^1$-fibration, any irreducible component of a fiber is called a *hidden component* if it meets the open set $\mathbb{A}^2$. In fact, we have to determine all the hidden components of one fiber of ρ_1. This can be done by means of the auxiliary $\mathbb{P}^1$-fibrations ρ_2, ρ_3 (and ρ_1', ρ_3' in certain cases) on V or its blow-up which are all naturally introduced.

On the other hand, the boundary divisor $V \setminus \mathbb{A}^2$ is brought to a *minimal* divisor with simple normal crossings by contracting (-1) components. Then its dual graph is a linear chain by Ramanujam [6] and completely classified in Morrow [5] (cf. Kishimoto [2] for a combined new proof of the results of Ramanujam–Morrow). This fact that the dual graph of a minimal normal compactification of $\mathbb{A}^2$ is a linear chain provides us with the restrictions on the fibers of ρ_1.

Then our main result is stated as follows, where we use the notation $f \sim g$ if $g = cf$ with $c \in k^*$ and where we define $Q(x, y) := x^t y + P(x)$ with $t > 0$, $P(x) \in k[x]$, $\deg P(x) < t$ and $P(0) \neq 0$ or $t = 0$ and $P(x) = 0$.

Theorem. *Let f be a generically rational polynomial of quasi-simple type with one 2-section and π cross-sections in the boundary at infinity. Then f has one of the following forms after a change of coordinates*:

(1)
$$f \sim x^{a_2} Q(x,y)^{a_1} \prod_{l=1}^{p} \bigl(x^{b_2} Q(x,y)^{b_1} + \beta_l\bigr)^{c_l+1} + \gamma \bigl(x^{b_2} Q(x,y)^{b_1} + \beta_i\bigr)$$
$$\times \bigl(x^{b_2} Q(x,y)^{b_1} + \beta_j\bigr) + \gamma' \bigl(x^{b_2} Q(x,y)^{b_1} + \beta_i\bigr) + \alpha,$$

where $\alpha \in k^*$, $\beta_k \neq \beta_l$ *if* $k \neq l$, $\gamma \in k^*$, $\gamma' \in k$, $-2 + \sum_{l=1}^{p}(c_l+1) \geqslant 1$, $1 \leqslant i, j \leqslant p$, *and*

(i) $a_2 > a_1 > 0$, $b_2 > b_1 > 0$, $b_1 > a_1$, $b_2 > a_2$ *and* $a_1 b_2 - a_2 b_1 = \pm 1$ (called the *unimodularity condition*), *or*
(ii) $a_1 = 1$, $a_2 = h$, $b_1 = 1$, $b_2 = h+1$ *with* $h \geqslant 0$; $Q(x,y) \neq y$ *if* $h = 0$, *or*
(iii) $a_1 = 0$, $a_2 = 1$, $b_1 = 1$, $b_2 = h+1$ *with* $h \geqslant 0$.

(2)
$$f \sim x^{b_2 - 2a_2} Q(x,y)^{b_1 - 2a_1} \bigl(x^{b_2} Q(x,y)^{b_1} + \gamma\bigr)^{c}$$
$$\times \bigl\{\bigl(x^{b_2} Q(x,y)^{b_1} + \gamma\bigr)^{c+1} + \gamma' x^{a_2} Q(x,y)^{a_1}\bigr\} + \alpha,$$

where $\alpha \in k^*$, $\gamma \in k^*$, $\gamma' \in k$, $c \geqslant 0$ *and*

(i) $a_2 > a_1 > 0$, $b_2 > b_1 > 0$, $b_1 > 2a_1$, $b_2 > 2a_2$ *and* $a_1 b_2 - a_2 b_1 = \pm 1$ (called the *semi-unimodularity condition*), *or*
(ii) $a_1 = 0$, $a_2 = 1$, $b_1 = 1$, $b_2 = h+1$ *with* $h \geqslant 1$; $Q(x,y) \neq y$ *if* $h = 1$, *or*
(iii) $a_1 = 1$, $a_2 = h$, $b_1 = 2$, $b_2 = 2h+1$ *with* $h \geqslant 0$.

(2′)
$$f \sim x^{-1} \bigl\{Q(x,y)\bigl(xQ(x,y) + \gamma\bigr)^{c} + \beta\bigr\} \bigl\{\bigl(xQ(x,y) + \gamma\bigr)^{c+1} + \gamma' x\bigr\} + \alpha,$$

where $\alpha \in k^*$, $\beta \in k^*$, $\gamma \in k^*$, $\gamma' \in k$, $c \geqslant 0$ *and* $Q(x,y) \neq y$ *such that* $Q(x,y) \times (xQ(x,y) + \gamma)^c + \beta$ *is divisible by* x.

(3)
$$f \sim \bigl(x^{b_2} Q(x,y)^{b_1} + \eta\bigr) \bigl\{x^{a_2 + cb_2} Q(x,y)^{a_1 + cb_1} \bigl(x^{b_2} Q(x,y)^{b_1} + \gamma\bigr)^{c'} + \beta_1\bigr\}$$
$$\times \bigl\{x^{a_2 + cb_2} Q(x,y)^{a_1 + cb_1} \bigl(x^{b_2} Q(x,y)^{b_1} + \gamma\bigr)^{c'} + \beta_2\bigr\}$$
$$+ \gamma' \bigl\{x^{a_2 + cb_2} Q(x,y)^{a_1 + cb_1} \bigl(x^{b_2} Q(x,y)^{b_1} + \gamma\bigr)^{c'} + \beta_1\bigr\} + \alpha,$$

where $\alpha \in k^*$, $\beta_1, \beta_2 \in k$, $\gamma \in k^*$, $\gamma' \in k$, $c \geqslant 0$, $c' \geqslant 0$ *and* $c + c' \geqslant 1$, $\eta = 0$ *or* γ; $c' \geqslant 1$ *if* $\eta = 0$ *and*

(i) a_1, a_2, b_1, b_2 *are satisfying the unimodularity condition, or*
(ii) $a_1 = 1$, $a_2 = h$, $b_1 = 1$, $b_2 = h+1$ *with* $h \geqslant 0$; $Q(x,y) \neq y$ *if* $h = 0$, *or*
(iii) $c \geqslant 1$ *if* $\eta = \gamma$, *and* $a_1 = 0$, $a_2 = 1$, $b_1 = 1$, $b_2 = h+1$ *with* $h \geqslant 0$.

(3′)
$$f \sim \bigl(x^{b} Q(x,y) + \gamma_1\bigr) \bigl\{x\bigl(x^{b} Q(x,y) + \gamma_1\bigr)^{c} \bigl(x^{b} Q(x,y) + \gamma_2\bigr)^{c'} + \beta_1\bigr\}$$
$$\times \bigl\{x\bigl(x^{b} Q(x,y) + \gamma_1\bigr)^{c} \bigl(x^{b} Q(x,y) + \gamma_2\bigr)^{c'} + \beta_2\bigr\}$$
$$+ \gamma' \bigl\{x\bigl(x^{b} Q(x,y) + \gamma_1\bigr)^{c} \bigl(x^{b} Q(x,y) + \gamma_2\bigr)^{c'} + \beta_1\bigr\} + \alpha,$$

where $\alpha \in k^*$, $\beta_1, \beta_2 \in k$, $\gamma_1, \gamma_2 \in k^*$ *with* $\gamma_1 \neq \gamma_2$, $\gamma' \in k$, $c \geqslant 0$, $c' \geqslant 1$ *and* $b \geqslant 1$.

(4)
$$f \sim \left\{x^{a_2+(c+1)b_2} Q(x,y)^{a_1+(c+1)b_1}\left(x^{b_2} Q(x,y)^{b_1}+\gamma\right)^{c'}+\beta\right\} \times \left\{x^{a_2+cb_2} Q(x,y)^{a_1+cb_1}\left(x^{b_2} Q(x,y)^{b_1}+\gamma\right)^{c'+1}+\gamma'\right\}+\alpha,$$

where $\alpha \in k^*$, $\beta \in k$, $\gamma \in k^*$, $\gamma' \in k$, $c \geqslant 0$, $c' \geqslant 0$ and
(i) *a_1, a_2, b_1, b_2 are satisfying the unimodularity condition, or*
(ii) *$a_1 = 1$, $a_2 = h$, $b_1 = 1$, $b_2 = h + 1$ with $h \geqslant 0$; $Q(x, y) \neq y$ if $h = 0$, or*
(iii) *$a_1 = 0$, $a_2 = 1$, $b_1 = 1$, $b_2 = h + 1$ with $h \geqslant 0$.*

(4′)
$$f \sim \left\{x\left(x^b Q(x,y)+\gamma_1\right)^{c+1}\left(x^b Q(x,y)+\gamma_2\right)^{c'}+\beta\right\} \times \left\{x\left(x^b Q(x,y)+\gamma_1\right)^{c}\left(x^b Q(x,y)+\gamma_2\right)^{c'+1}+\gamma'\right\}+\alpha,$$

where $\alpha \in k^*$, $\beta \in k$, $\gamma_1, \gamma_2 \in k^*$ with $\gamma_1 \neq \gamma_2$, $\gamma' \in k$, $c \geqslant 0$, $c' \geqslant 0$ and $b \geqslant 1$.

(5)
$$f \sim x^{b_2-2a_2} Q(x,y)^{b_1-2a_1}\left\{\left(x^{b_2} Q(x,y)^{b_1}+\gamma\right)^{c}+\beta_1 x^{a_2} Q(x,y)^{a_1}\right\} \times \left\{\left(x^{b_2} Q(x,y)^{b_1}+\gamma\right)^{c}+\beta_2 x^{a_2} Q(x,y)^{a_1}\right\}+\alpha,$$

where $\alpha \in k^*$, $\beta_1, \beta_2 \in k$, $\gamma \in k^*$, $c \geqslant 1$ and
(i) *a_1, a_2, b_1, b_2 are satisfying the semi-unimodularity condition, or*
(ii) *$a_1 = 0$, $a_2 = 1$, $b_1 = 1$, $b_2 = h + 1$ with $h \geqslant 1$; $Q(x, y) \neq y$ if $h = 1$, or*
(iii) *$a_1 = 1$, $a_2 = h$, $b_1 = 2$, $b_2 = 2h + 1$ with $h \geqslant 0$.*

(5′)
$$f \sim x^{-1}\left[Q(x,y)\left\{\left(xQ(x,y)+\gamma\right)^{c}+\beta_2 x\right\}+\gamma'\right] \times \left\{\left(xQ(x,y)+\gamma\right)^{c}+\beta_1 x\right\}+\alpha,$$

where $\alpha \in k^*$, $\beta_1, \beta_2 \in k$, $\gamma \in k^*$, $\gamma' \in k^*$, $c \geqslant 1$ and $Q(x, y) \neq y$ such that $Q(x,y)\{(xQ(x,y)+\gamma)^c+\beta_2 x\}+\gamma'$ is divisible by x.

(6)
$$f \sim \left\{\left(x^{b_2} Q(x,y)^{b_1}+\gamma\right)+\beta_1 x^{a_2} Q(x,y)^{a_1}\right\} \times \left\{x^{b_2-2a_2} Q(x,y)^{b_1-2a_1}\left(x^{b_2} Q(x,y)^{b_1}+\gamma\right) + (\beta_2+\beta_3) x^{b_2-a_2} Q(x,y)^{b_1-a_1}+\beta_2\beta_3\right\}+\alpha,$$

where $\alpha \in k^*$, $\beta_1, \beta_2, \beta_3 \in k^*$, $\gamma \in k^*$ and
(i) *a_1, a_2, b_1, b_2 are satisfying the semi-unimodularity condition, or*
(ii) *$a_1 = 0, a_2 = 1, b_1 = 1, b_2 = h + 1$ with $h \geqslant 1$; $Q(x, y) \neq y$ if $h = 1$, or*
(iii) *$a_1 = 1, a_2 = h, b_1 = 2, b_2 = 2h + 1$ with $h \geqslant 0$.*

(7)
$$f \sim (x+\gamma_1)\left\{(x+\gamma_1)^{a_1}(x+\gamma_2)^{a_2} y+(x+\gamma_1)^{a_1'}(x+\gamma_2)^{a_2'} R(x)+\beta_1\right\} \times \left\{(x+\gamma_1)^{a_1}(x+\gamma_2)^{a_2} y+(x+\gamma_1)^{a_1'}(x+\gamma_2)^{a_2'} R(x)+\beta_2\right\} + \gamma'\left\{(x+\gamma_1)^{a_1}(x+\gamma_2)^{a_2} y+(x+\gamma_1)^{a_1'}(x+\gamma_2)^{a_2'} R(x)+\beta_1\right\}+\alpha,$$

where $\alpha \in k^*$, $\beta_1, \beta_2 \in k$, $\gamma_1, \gamma_2 \in k$ with $\gamma_1 \neq \gamma_2$, $\gamma' \in k$, $a_1 \geqslant a_1' \geqslant 0$, $a_2 \geqslant a_2' \geqslant 0$, $a_2 \geqslant 1$ and $R(x) \in k[x]$; $R(-\gamma_i) \neq 0$ if $a_i > a_i'$ for $i = 1, 2$.

(8)

$$\begin{aligned} f \sim &\left\{(x+\gamma_1)^{a_1+1}(x+\gamma_2)^{a_2}y + (x+\gamma_1)^{a_1'+1}(x+\gamma_2)^{a_2'}R(x) + \beta\right\} \\ &\times \left\{(x+\gamma_1)^{a_1}(x+\gamma_2)^{a_2+1}y + (x+\gamma_1)^{a_1'}(x+\gamma_2)^{a_2'+1}R(x) + \gamma'\right\} + \alpha, \end{aligned}$$

where $\alpha \in k^*$, $\beta \in k$, $\gamma_1, \gamma_2 \in k$ with $\gamma_1 \neq \gamma_2$, $\gamma' \in k$, $a_1 \geqslant a_1' \geqslant 0$, $a_2 \geqslant a_2' \geqslant 0$ and $R(x) \in k[x]$; $R(-\gamma_i) \neq 0$ if $a_i > a_i'$ for $i = 1, 2$.

1. Generalities

We shall set up the necessary circumstances with which we consider the classification of generically rational polynomials of quasi-simple type. For the moment, we assume $a \geqslant 2$.

Lemma 1.1. *Let $\sigma : V \to B$ be a $\mathbb{P}^1$-fibration from a smooth projective surface V onto a smooth projective curve B. Let $F = \kappa_1 C_1 + \cdots + \kappa_s C_s$ be a singular fiber of σ, where the C_i are irreducible curves, $C_i \neq C_j$ if $i \neq j$, and $\kappa_i > 0$. Then the following assertions hold*:

(1) $\gcd(\kappa_1, \ldots, \kappa_s) = 1$ *and* $\operatorname{Supp}(F) = \bigcup_{i=1}^s C_i$ *is connected.*
(2) *For $1 \leqslant i \leqslant s$, $C_i \cong \mathbb{P}^1$ and $(C_i^2) < 0$.*
(3) *For $i \neq j$, $(C_i \cdot C_j) = 0$ or 1.*
(4) *For three distinct indices i, j and k, $C_i \cap C_j \cap C_k = \emptyset$.*
(5) *One of the C_i, say C_1, is a (-1)-curve, i.e., an exceptional curve of the first kind. If $\tau : V \to V_1$ is the contraction of C_1, then σ factors as $\sigma : V \xrightarrow{\tau} V_1 \xrightarrow{\sigma_1} B$, where $\sigma_1 : V_1 \to B$ is a $\mathbb{P}^1$-fibration.*
(6) *If one of the κ_i, say κ_1, equals 1, then there is a (-1)-curve among the components C_i with $2 \leqslant i \leqslant s$.*

Proof. See Miyanishi and Sugie [4, Lemma 1.1]. □

Lemma 1.2. *Let ℓ'_∞ be the proper transform of ℓ_∞ by φ which is an irreducible component of the member S_∞ of $\Lambda'(f)$. If S_∞ is irreducible, then $(\ell'^2_\infty) = 0$. If S_∞ is reducible, ℓ'_∞ is a unique (-1) curve in S_∞.*

Proof. If S_∞ is irreducible then it is obvious that $(\ell'^2_\infty) = 0$. Suppose S_∞ is a reducible fiber of $\bar{\rho}_1$. Since all the (-1) curves arising from the minimal elimination φ of the base points of $\Lambda(f)$ appear among the quasi-sections $\Gamma_1, \ldots, \Gamma_\pi$ and Δ and since S_∞ contains at least one (-1) component by Lemma 1.1, it follows that ℓ'_∞ is a unique (-1) component of S_∞. □

Corollary 1.3. (Cf. Russell [7].) *Let f be a generically rational polynomial. Then general members of $\Lambda(f)$ have at most two points outside of $\mathbb{A}^2$.*

Proof. The points on general members of $\Lambda(f)$ lying outside $\mathbb{A}^2$ must be the base points of $\Lambda(f)$. If general members of $\Lambda(f)$ have more than two points outside $\mathbb{A}^2$, we must perform the blowing-ups on ℓ_∞ at least three times in the process of eliminating the base points of $\Lambda(f)$. Then the self-intersection number of ℓ'_∞ is $\leqslant -2$. This contradicts Lemma 1.2. □

Let $\mathrm{Supp}(S_i) = \bigcup_{l=1}^{\nu_i} S_{il}$ be the irreducible decomposition of the singular fiber S_i for $1 \leqslant i \leqslant r$ and we assume, after a change of ordering, that $S_{il} \cap \mathbb{A}^2 \neq \emptyset$ for $1 \leqslant l \leqslant \nu'_i$ and $S_{il} \cap \mathbb{A}^2 = \emptyset$ for $\nu'_i + 1 \leqslant l \leqslant \nu_i$, where $\nu'_i \leqslant \nu_i$.

Lemma 1.4. *Let π' be the number of quasi-sections of $\bar{\rho}_1$. Then we have $\pi' - 1 = \sum_{i=1}^{r}(\nu'_i - 1)$, where $\nu'_i \geqslant 2$ for $1 \leqslant i \leqslant r$.*

Proof. The same arguments as in Miyanishi and Sugie [4, Lemma 1.6]. Note that $\pi' = \pi + 1$ in the present case since Δ is counted as one quasi-section. □

Lemma 1.5. *Let f be a generically rational polynomial of quasi-simple type. Then we have always $\pi \geqslant 1$.*

Proof. Suppose $\pi = 0$. Then Δ is a unique quasi-section of $\bar{\rho}_1$ lying in the boundary $\bar{V} - \mathbb{A}^2$. Since $\mathrm{Pic}(\mathbb{A}^2) = (0)$, it follows that $\mathrm{Pic}(\bar{V})$ is generated by Δ and the boundary components contained in $\bigcup_{i=1}^{r} \mathrm{Supp}(S_i) \cup \mathrm{Supp}(S_\infty)$. Meanwhile, since $\bar{\rho}_1 : \bar{V} \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration, there is a cross-section of $\bar{\rho}_1$, say M. Then we have

$$M \sim b\Delta + \left\{ \begin{array}{l} \text{divisors supported by the boundary} \\ \text{fiber components of } S_1, \dots, S_r, S_\infty \end{array} \right\},$$

where $b \in \mathbb{Z}$ and where the symbol $\sim$ stands for the linear equivalence. By computing the intersection number with a general fiber S of $\bar{\rho}_1$, we have

$$1 = (M \cdot S) = b(\Delta \cdot S) = ab.$$

This is a contradiction because $a \geqslant 2$ by our convention. □

2. Boundary of the affine plane

We recall that $\Gamma_1, \dots, \Gamma_\pi$ are defined to be the cross-sections of $\bar{\rho}_1$ in introduction. Let $\mathrm{Supp}(S_\infty) = \bigcup_{l=1}^{\nu} S_{\infty l}$ be the irreducible decomposition such that $S_{\infty 1} \cap \Gamma_1 \neq \emptyset$ and $S_{\infty l} \cap \Gamma_1 = \emptyset$ for $2 \leqslant l \leqslant \nu$. Then the multiplicity of $S_{\infty 1}$ in the fiber S_∞ is 1. Hence we can contract $S_{\infty 2}, \dots, S_{\infty \nu}$ to obtain a smooth surface $\tilde{V}$. Let $\bar{\psi} : \bar{V} \to \tilde{V}$ be the contraction. Then the image $\bar{\psi}(\Delta)$ may have singular points. If so, there is only one singular point of $\bar{\psi}(\Delta)$ which lies on the curve $\bar{\psi}(S_{\infty 1})$ and which is a cuspidal point because $\bar{V}$ is a normal compactification of $\mathbb{A}^2$ and hence the dual graph of $\bar{V} - \mathbb{A}^2$ contains no loops. Moreover, we have $(\bar{\psi}(S_{\infty 1}) \cdot \bar{\psi}(\Gamma_j)) = 1$ for $1 \leqslant j \leqslant \pi$ and $(\bar{\psi}(S_{\infty 1}) \cdot \bar{\psi}(\Delta)) = a$. From these assertions, the boundary $\tilde{V} - \mathbb{A}^2$ has one of the configurations of types (A), (B), (C) and (D) (see Figs. 1–4). For the sake of convenience, let $\Gamma_1, \ldots, \Gamma_p$ be the cross-sections of $\bar{\rho}_1$ whose images of $\bar{\psi}$ do not intersect $\bar{\psi}(\Delta)$, and let $\Gamma_1', \ldots, \Gamma_q'$ be the cross-sections

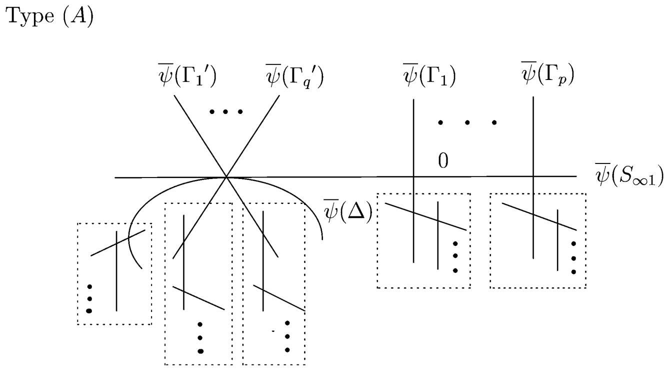

Fig. 1.

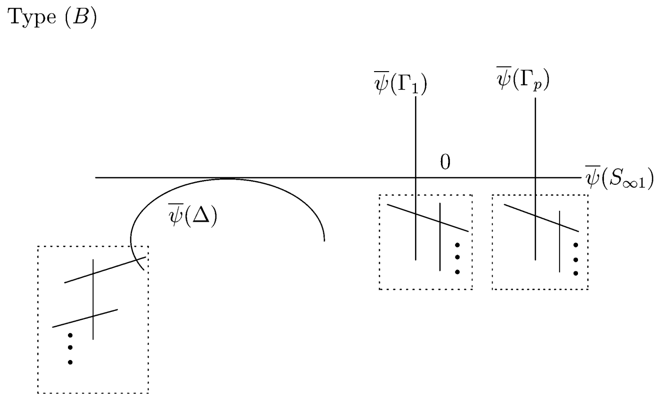

Fig. 2.

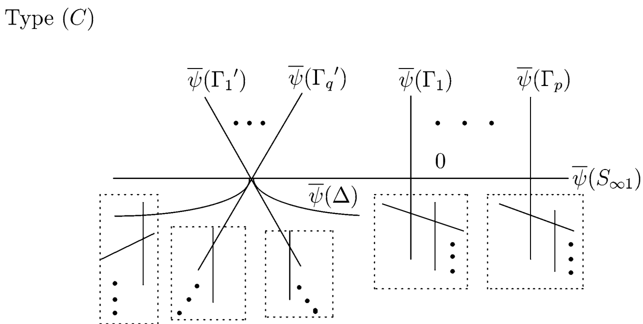

Fig. 3.

Type (D)

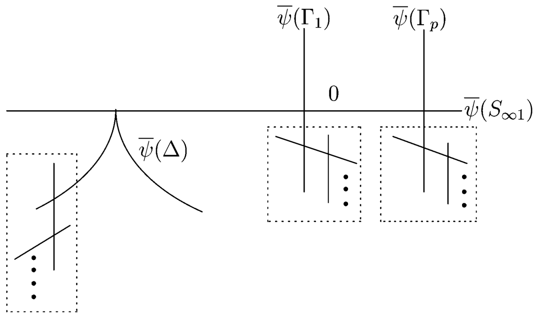

Fig. 4.

of $\bar{\rho}_1$ whose images of $\bar{\psi}$ intersect $\bar{\psi}(\varDelta)$. Note that $\pi = p + q$. In Figs. 2, 4, $\pi = p$ since $q = 0$. In Figs. 1–4, the parts in dotted frame boxes represent the components of $\tilde{V} - \mathbb{A}^2$ which are the components of the singular fibers of the $\mathbb{P}^1$-fibration induced by $\bar{\rho}_1$.

In what follows, we employ the following notation. Let $\phi : V_1 \to V_2$ be a birational morphism of smooth projective surfaces. Let D_i be a divisor on V_i for $i = 1, 2$. Then $\phi'(D_2)$ denotes the proper transform of D_2 by ϕ. If there is no fear of confusion, we identify $\phi'(D_2)$ with D_2 and the image $\phi(D_1)$ with D_1.

Lemma 2.1. *If $a = 2$, the above cases (C) and (D) do not occur.*

Proof. In the cases (C) and (D) where $\bar{\psi}(S_{\infty 1})$ is not tangent to $\bar{\psi}(\varDelta)$, we perform blowing-ups as far as one of the following two cases occurs with the irreducible exceptional curves $E_1, \ldots, E_n$ arising from the preceding blowing-ups:

(1) Among $\varDelta, \varGamma'_1, \ldots, \varGamma'_q, S_{\infty 1}, E_1, \ldots, E_n$, there exist two curves meeting in a point with local intersection multiplicity $\geqslant 2$.
(2) Among $\varDelta, \varGamma'_1, \ldots, \varGamma'_q, S_{\infty 1}, E_1, \ldots, E_n$, there exist more than two curves meeting in one point.

Let $\theta : V \to \tilde{V}$ be the composite of the above blowing-ups. Cleary there exists a birational morphism $\psi : \bar{V} \to V$ such that $\bar{\psi} = \theta \cdot \psi$. Then there exists a $\mathbb{P}^1$-fibration $\rho_1 : V \to \mathbb{P}^1$ such that $\bar{\rho}_1 = \rho_1 \cdot \psi$. After the composite of blowing-ups θ, the dual graphs of $V - \mathbb{A}^2$ in the cases (C) and (D) are given as indicated in Figs. 5 and 6. In both of the cases, the singular fiber S_∞ will then contain at least two (-1) curves. This contradicts Lemma 1.2. $\square$

We note, however, that the cases (C) and (D) exist if $a \geqslant 3$. *From now on, we deal with only the case $a = 2$.*

By virtue of Lemma 2.1, it suffices to consider the cases (A) and (B). First of all, we discuss the case (A). By the composite of blowing-ups θ as given in the proof of Lemma 2.1, we have the dual graph of $V - \mathbb{A}^2$ in Fig. 7. Here note that we can exchange

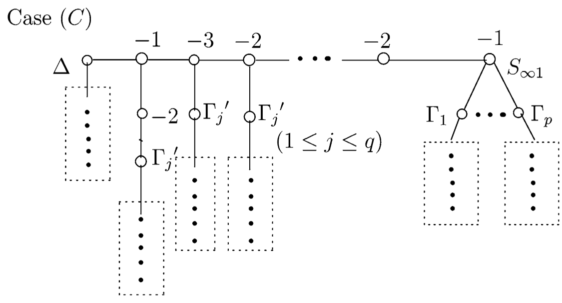

Fig. 5.

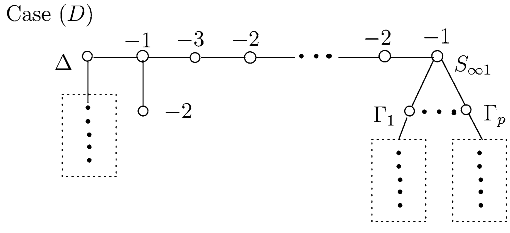

Fig. 6.

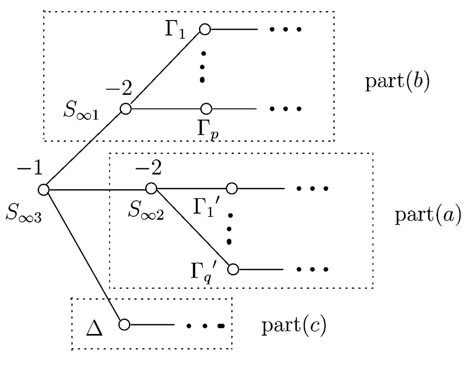

Fig. 7.

the roles of the part (a) and the part (b) in Fig. 7 by contracting $S_{\infty 3}$ and $S_{\infty 1}$ instead of $S_{\infty 3}$ and $S_{\infty 2}$. In the parts (a)–(c) enclosed by dotted lines in Fig. 7, the components except for $S_{\infty 1}, S_{\infty 2}, \Delta, \Gamma_1, \ldots, \Gamma_p, \Gamma_1', \ldots, \Gamma_q'$, are all contained in the fibers of ρ_1 other than S_∞.

Lemma 2.2. *In the case* (A)*, it suffices to consider each of the following four cases*:

$(A1)$ *The part* (a) *is contractible, but the parts* (b) *and* (c) *are not.*
$(A2)$ *The part* (c) *is contractible, but the parts* (a) *and* (b) *are not.*
$(A3)$ *The parts* (a) *and* (c) *are contractible, but the part* (b) *is not.*
$(A4)$ *The parts* (a) *and* (b) *are contractible, but the part* (c) *is not.*

Proof. Since any minimal normal compactification (abbreviated as M.N.C.) of $\mathbb{A}^2$ has a linear chain as the boundary dual graph, one of the parts (a)–(c) must be contractible, where we say that a set of curves is contractible if it is contractible to a smooth point. If all of the parts (a)–(c) are contractible, then $S_{\infty 3}$ will become a unique component of a new M.N.C. of $\mathbb{A}^2$ with self-intersection number $\geqslant 2$. But this is impossible by Morrow [5]. Hence one of the following cases occurs:

$(A1)$ The part (a) is contractible, but the parts (b) and (c) are not.
$(A1')$ The part (b) is contractible, but the parts (a) and (c) are not.
$(A2)$ The part (c) is contractible, but the parts (a) and (b) are not.
$(A3)$ The parts (a) and (c) are contractible, but the part (b) is not.
$(A3')$ The parts (b) and (c) are contractible, but the part (a) is not.
$(A4)$ The parts (a) and (b) are contractible, but the part (c) is not.

By the above remark about the exchangeability of the parts (a) and (b), the cases $(A1')$ and $(A3')$ are reduced to the cases $(A1)$ and $(A3)$, respectively. So, we have only to consider the cases $(A1)$–$(A4)$. □

In the case (B), we have a dual graph similar to the one in Fig. 7, where the part (a) consists only of $S_{\infty 2}$. Then we have the following result.

Lemma 2.3. *In the case (B), it suffices to consider each of the following two cases*:

$(B1)$ *The part (b) is contractible, but the part (c) is not.*
$(B2)$ *The part (c) is contractible, but the part (b) is not.*

Proof. By the same reasoning as in the case (A), one of the parts (b) and (c) is contractible. If both of the parts (b) and (c) are contractible, the image of $S_{\infty 3}$ will have self-intersection number $\geqslant 1$ after these contractions and it is connected to the image of $S_{\infty 2}$ whose self-intersection number is -2. But this case is excluded in the list of Morrow. □

In the subsequent sections, we shall deal with each of the above six cases $(A1)$–$(A4)$, $(B1)$, $(B2)$.

3. Case $(A1)$

We first discuss the case $(A1)$. Since there are no (-1)-curves in the components of $V - \mathbb{A}^2$ which are contained in the singular fibers of ρ_1 other than S_∞ and since any M.N.C. of $\mathbb{A}^2$ has a linear chain as the boundary graph, the part (a) of Fig. 7 has only one of the dual graphs in Fig. 8, where $n_1' \geqslant 0$. Hence $q = 1$ or 2 in Fig. 7. For the sake of subsequent argument, we denote by Γ_L the component adjacent to Γ_1' in the case of type $(a1)$. After the contraction of the part (a), the self-intersection number of $S_{\infty 3}$ becomes $\geqslant 0$.

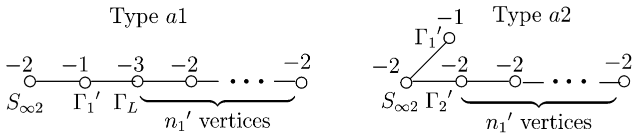

Fig. 8.

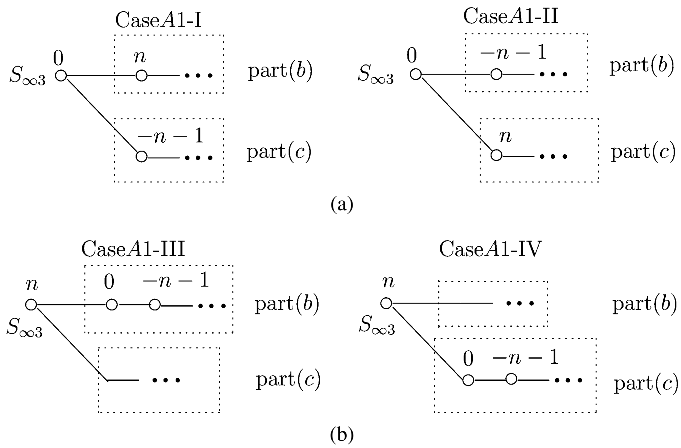

Fig. 9.

Next, we contract all the contractible components of the parts (b) and (c) in Fig. 7 and have one of the four cases $A1$-I, ..., $A1$-IV in Figs. 9(a) and 9(b) by Morrow [5], where $n \geqslant 1$. Note here that the case $n_1' > 0$ in the dual graph of Fig. 8 necessarily leads to the case $A1$-III or $A1$-IV.

Before looking into details in the cases $A1$-I, ..., $A1$-IV, we introduce the dual graphs $G_1, G_1', G_1'', G_2, G_2', G_2'', G_3, G_4, G_5, G_5', G_5''$ as shown in Fig. 10, where $l_i \geqslant 0$ and $k_i \geqslant 2$ for $1 \leqslant i \leqslant u$, and $m \geqslant 0$ in the dual graphs $G_1, G_1', G_1'', G_2, G_2', G_2'', G_3$ and G_4, and $m \geqslant 1$ in the dual graphs G_5, G_5', G_5''. Adhere these linear graphs to either (or both) side (the part abbreviated by dots) of the graph in Figs. 9(a) and 9(b) in such a way that one obtains a boundary dual graph of $\mathbb{A}^2$ in Morrow's list.

By the convention that there are no (-1)-curves in the components of $V - \mathbb{A}^2$ which are contained in the singular fibers of ρ_1 other than S_∞ and by the fact that any M.N.C. of $\mathbb{A}^2$ has a linear chain as the boundary graph, we obtain the accurate dual graphs of $V - \mathbb{A}^2$ in the cases $A1$-I, ..., $A1$-IV (cf. Morrow [5]). Note that the left end of the graphs introduced in Fig. 10 is to be connected to a component adjacent to the encircled portions $D_1, \ldots, D_5$ in Figs. 11–16.

Case $A1$-I. The part (a) of Fig. 7 has the graph of type $(a1)$ or $(a2)$ with $n_1' = 0$ because the self intersection number of $S_{\infty 3}$ becomes 0 after the contraction of the part (a) of Fig. 7. We have two different graphs of the part (b) as shown in Figs. 11 and 12. We have $n = -2 + (n_1 + 1) + \cdots + (n_p + 1) \geqslant 1$ in Fig. 11 and $n = -2 + (n_1 + 1) + \cdots + (n_{p-1} + 1) \geqslant 1$ in Fig. 12 with Γ_p becoming the leftmost component of the graph to be put in the place of D_2.

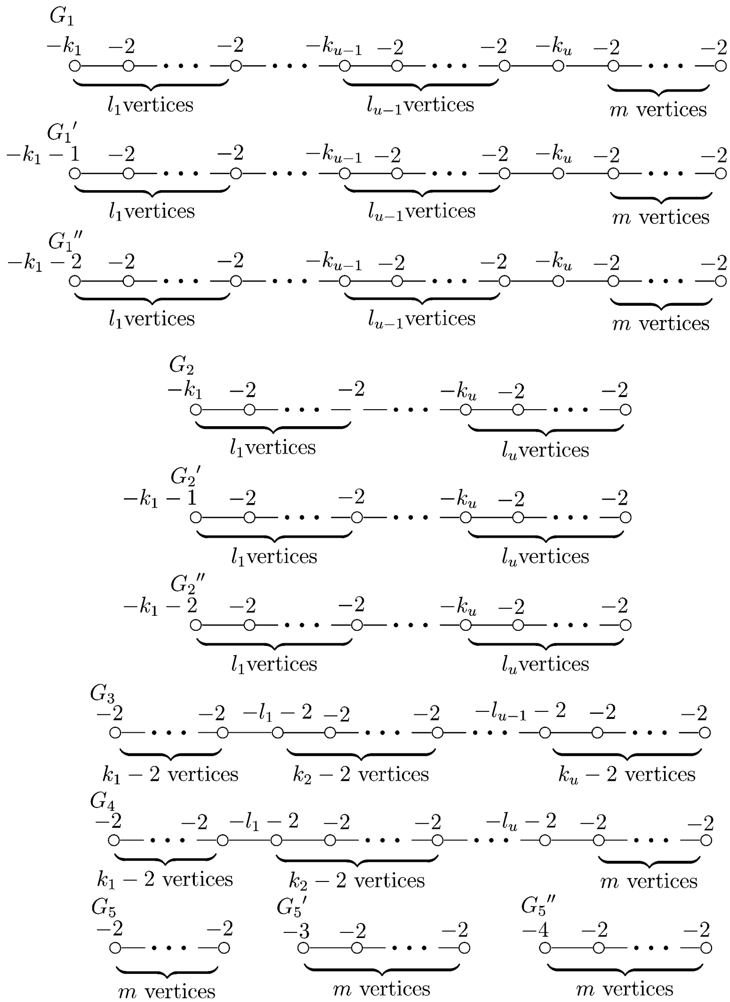
G_1
$-k_1$ -2 -2 $-k_{u-1}$ -2 -2 $-k_u$ -2 -2
$l_1$vertices l_{u-1}vertices m vertices
$G_1{}'$
$-k_1-1$ -2 -2 $-k_{u-1}$ -2 -2 $-k_u$ -2 -2
$l_1$vertices l_{u-1}vertices m vertices
$G_1{}''$
$-k_1-2$ -2 -2 $-k_{u-1}$ -2 -2 $-k_u$ -2 -2
$l_1$vertices l_{u-1}vertices m vertices
G_2
$-k_1$ -2 -2 $-k_u$ -2 -2
$l_1$vertices $l_u$vertices
$G_2{}'$
$-k_1-1$ -2 -2 $-k_u$ -2 -2
$l_1$vertices $l_u$vertices
$G_2{}''$
$-k_1-2$ -2 -2 $-k_u$ -2 -2
$l_1$vertices $l_u$vertices
G_3
-2 -2 $-l_1-2$ -2 -2 $-l_{u-1}-2$ -2 -2
k_1-2 vertices k_2-2 vertices k_u-2 vertices
G_4
-2 -2 $-l_1-2$ -2 -2 $-l_u-2$ -2 -2
k_1-2 vertices k_2-2 vertices m vertices
G_5 $G_5{}'$ $G_5{}''$
-2 -2 -3 -2 -2 -4 -2 -2
m vertices m vertices m vertices

Fig. 10.

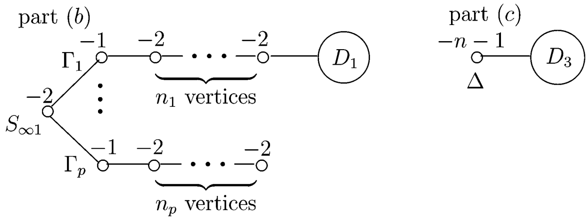
part (b)
-1 -2 -2
Γ_1
D_1
-2
n_1 vertices
$S_{\infty 1}$
-1 -2 -2
Γ_p
n_p vertices
part (c)
$-n-1$
D_3
Δ

Fig. 11.

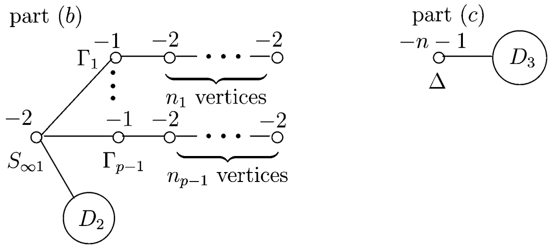

Fig. 12.

Case $A1$-II. The part (a) of Fig. 7 has the graph of type $(a1)$ or $(a2)$ with $n_1' = 0$ for the same reason as in the case $A1$-I. We have necessarily $p = 1$, $q = 1$ and Fig. 13, where the cross-section Γ_1 is the leftmost component of D_1.

Case $A1$-III. We have four possibilities for the part (b) (top left and right graphs, and middle left and right graphs in Fig. 14). We have $n_1 \geqslant 0$ and $n = n_1 + n_1' \geqslant 1$. Δ is the leftmost component of D_5 if $n_1 = 0$.

We divide the case $A1$-IV into the cases $A1$-IVa and $A1$-IVb.

Case $A1$-IVa. In this case, we set $(\Gamma_1^2) = -1$. Then we have two possibilities for the part (b) as shown in Fig. 15 (top left and bottom left), where Γ_2 is the leftmost component of D_2 if $n_1 = 0$. Here we have $n_1 \geqslant 0$ and $n = n_1 + n_1' + 1$.

Case $A1$-IVb. In this case, we set $(\Gamma_1^2) \leqslant -2$. The parts (b) and (c) are shown in Fig. 16. Furthermore, D_1 is considered, by convention in this case, to be adjacent to $S_{\infty 3}$, and we have $k_1 = 2$ since $S_{\infty 1}$ is the leftmost component of D_1 (cf. Fig. 10). Here we have $n = n_1' \geqslant 1$.

3.1. Cases A1-I and A1-II

To begin with, we discuss the cases $A1$-I and $A1$-II. Let L be the linear pencil on V defined by $|S_{\infty 2} + S_{\infty 3} + \Gamma_1'|$ and let ρ_2 be the rational map associated with L.

Lemma 3.1. *Let L and $\rho_2 : V \cdots\to \mathbb{P}^1$ be the linear pencil and the rational map defined as above. Then the following assertions hold*:

(1) *L has no base points.*
(2) *There exists a generically rational polynomial v of simple type with two places at infinity such that $\Lambda_0(v)$ ($= \Lambda(v)$ restricted onto $\mathbb{A}^2$) gives rise to ρ_2.*
(3) *The $\mathbb{P}^1$-fibration ρ_2 has $S_{\infty 1}$ and Δ as cross-sections lying outside of $\mathbb{A}^2$.*
(4) *$\Gamma_1, \ldots, \Gamma_p$ are contained in mutually distinct members of L.*
(5) *$v - \beta$ is irreducible except for one value, say β_0, and $v - \beta_0$ has exactly two irreducible factors.*

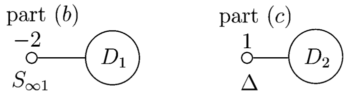

Fig. 13.

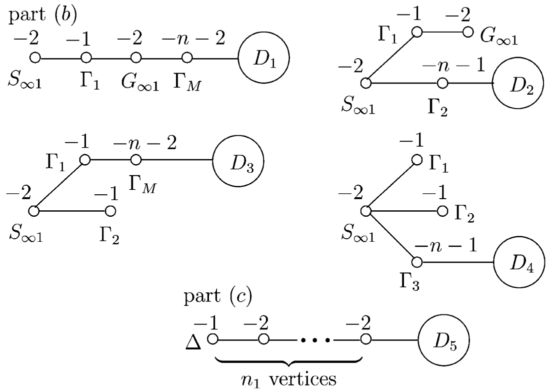

Fig. 14.

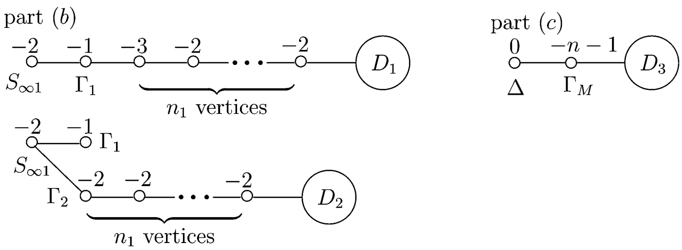

Fig. 15.

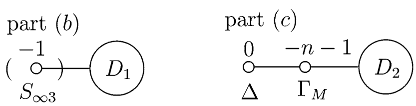

Fig. 16.

Proof. Since $(S_{\infty 2} + S_{\infty 3} + \Gamma_1')^2 = 0$, L has no base points. Furthermore, since $S_{\infty 2} + S_{\infty 3} + \Gamma_1'$ lies outside of $\mathbb{A}^2$, the restriction of ρ_2 on $\mathbb{A}^2$ induces a morphism onto $\mathbb{A}^1$. Hence it is defined by a polynomial $v \in k[x, y]$. Since $(S_{\infty 1} \cdot (S_{\infty 2} + S_{\infty 3} + \Gamma_1')) = 1$ and $(\Delta \cdot (S_{\infty 2} + S_{\infty 3} + \Gamma_1')) = 1$, $S_{\infty 1}$ and Δ are cross-sections of ρ_2. It is obvious that they exhaust all quasi-sections of ρ_2 contained in $V - \mathbb{A}^2$. Hence, we conclude that v is a generically rational polynomial of simple type with two places at infinity. Thus the assertions (1), (2) and (3) are satisfied. The assertion (4) follows if one notes $(\Gamma_i \cdot S_{\infty 1}) = 1$ for $1 \leqslant i \leqslant p$.

We shall show that the assertion (5) is satisfied. In fact, with the notations of Lemma 1.4, we have $r \geqslant 1$, while $r \leqslant 1$ by classification of v (cf. [8]). Hence $r = 1$ and $\nu_1' = 2$. $\square$

Remark. The assertion (5) above shows that there exists a unique fiber of ρ_2 which has two hidden components. Call these hidden components C_1 and C_2. Then we may assume that $C_1 \cap \mathbb{A}^2 \cong C_2 \cap \mathbb{A}^2 \cong \mathbb{A}^1$ ($C_1 \cap C_2 \neq \emptyset$) or $C_1 \cap \mathbb{A}^2 \cong \mathbb{A}^1_*$ and $C_2 \cap \mathbb{A}^2 \cong \mathbb{A}^1$ ($C_1 \cap C_2 = \emptyset$) (cf. [4, Lemma 2.1]).

Note that a general member F of L, restricted onto $\mathbb{A}^2$, is isomorphic to $\mathbb{A}^1_*$. In order to classify the singular fibers, we introduce the dual graphs $H_1, H_2, H_1', H_2', H_1''$ and H_2'' as shown in Fig. 17.

To exhibit our idea, we shall pick up two representative cases, $A1$-I with $D_1 = G_5'$ and $D_3 = \emptyset$, and $A1$-II with $D_1 = G_5$ and $D_2 = \emptyset$. The remaining cases can be treated in fashions similar to one of these two cases. Though the arguments are made for these representative cases, all lemmas to be stated below hold in general in the cases $A1$-I and $A1$-II.

Case *A1-I* [with $D_1 = G_5'$ and $D_3 = \emptyset$]. The boundary graph is as in Fig. 7, where the part (a) is replaced by the graph of type $(a1)$ or $(a2)$ with $n_1' = 0$ (see Fig. 8) and the parts (b) and (c) by the graphs as shown above. Set $\rho_2(\Gamma_l) = \beta_l \in k$ for $1 \leqslant l \leqslant p$. By Lemma 3.1(4), $\beta_l \neq \beta_{l'}$ if $l \neq l'$.

Lemmas 1.1 and 3.1 show that $v - \beta_1$ is reducible and $v - \beta_l$ is irreducible for $2 \leqslant l \leqslant p$. Namely, the fiber $\rho_2^{-1}(\beta_l)$ $(2 \leqslant l \leqslant p)$ has one hidden (-1) component, say F_{l1}, and the fiber $\rho_2^{-1}(\beta_1)$ has two hidden components, where we set $F_{11} = C_1$ and $F_{12} = C_2$. Then we have the dual graph of $S_{\infty 1} + \Delta + \sum_{l=1}^{p} \rho_2^{-1}(\beta_l)$ as in Fig. 18, where either the graph H_1' or the graph H_2' is to be entered in the portion denoted E_1. In the case $A1$-I with $E_1 = H_1'$ (respectively $E_2 = H_2'$), F_{11} denotes the vertex with weight $-\mu - 1$ (respectively $-\mu_1 - 1$), and F_{11} is connected to the component Δ.

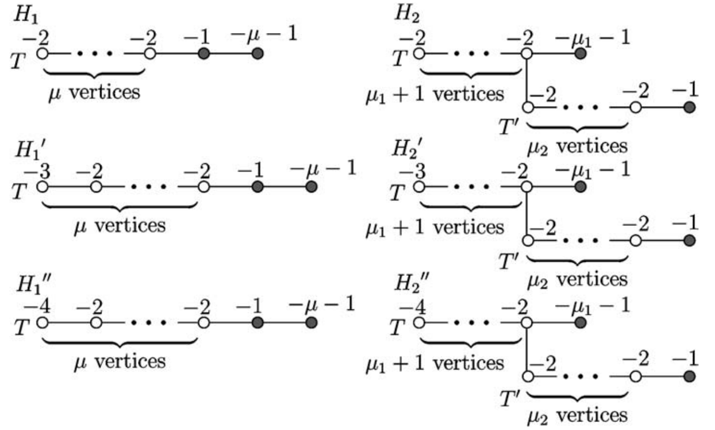

Fig. 17.

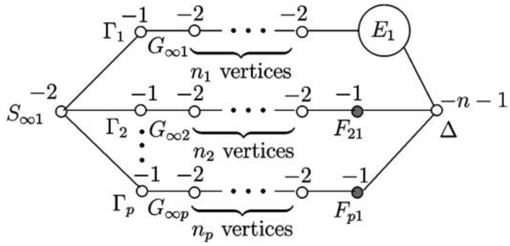

Fig. 18.

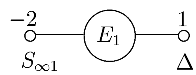

Fig. 19.

Case *A*1-II [with $D_1 = G_5$ and $D_2 = \emptyset$]. By a similar argument, we know that ρ_2 has two singular fibers $\rho_2^{-1}(\infty)$, and $\rho_2^{-1}(\beta_1)$ which has two hidden components, where we set $F_{11} = C_1$ and $F_{12} = C_2$. The graph of $\rho_2^{-1}(\beta_1) + S_{\infty 1} + \Delta$ is as given in Fig. 19, where either the graph H_1 or the graph H_2 is to be inserted into the portion E_1. F_{11} is chosen in the same way as in the case A1-I with $D_1 = G'_5$ and $D_3 = \emptyset$.

At this stage, we know what all singular fibers of the $\mathbb{P}^1$-fibration ρ_2 look like. We do not know, however, singular fibers of ρ_1. For this purpose, we need one more $\mathbb{P}^1$-fibration ρ_3 on V or a one-point blow-up V' of V. In each of the two cases A1-I and A1-II, we define a linear pencil M on V (or V') and ρ_3 as the $\mathbb{P}^1$-fibration associated with M as follows.

Case *A*1-I. We define the pencil $M_{ii} = |S_{\infty 1} + 2\Gamma_i + G_{\infty i}|$ if $n_i > 0$ for $1 \leqslant i \leqslant p$, and the pencil $M_{ij} = |S_{\infty 1} + \Gamma_i + \Gamma_j|$ if $i \neq j$ for $1 \leqslant i, j \leqslant p$, where $G_{\infty i}$ denotes the component right-adjacent to Γ_i (cf. Fig. 18). We take one of M_{ii} and M_{ij} as M. Then M_{ii} and M_{ij} have no base points.

Case *A*1-II. We have $(\Delta^2) = 1$ and hence $\dim |\Delta| = 2$ by Riemann–Roch theorem. Then there exists a unique base points P of the linear pencil $|\Delta|$. Let $\sigma : V' \to V$ be the blowing-up of P and $M = |\sigma' \Delta|$, where $\sigma' \Delta$ is the proper transform of Δ. Then M has no base points. Let E be the exceptional curve of σ. We write V' as V anew by abuse of notations.

Lemma 3.2. *Let M and $\rho_3 : V \to \mathbb{P}^1$ be the linear pencil and the $\mathbb{P}^1$-fibration defined as above. Then the following assertions hold*:

(1) *There exists a generically rational polynomial w of simple type such that $\Lambda_0(w)$ ($= \Lambda(w)$ restricted onto $\mathbb{A}^2$) gives rise to ρ_3.*
(2) *$S_{\infty 3}$ is a cross-section of ρ_3, and hence $S_{\infty 1}$, $S_{\infty 2}$ and Δ belong to mutually distinct fibers of ρ_3.*

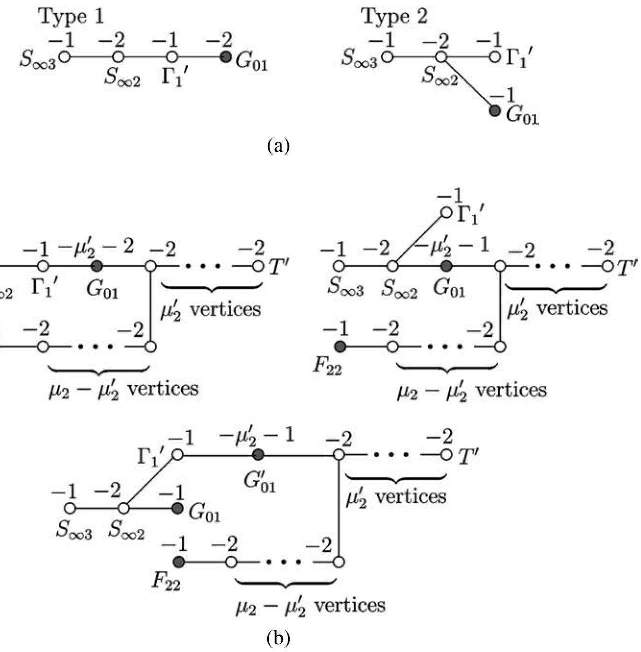

Fig. 20.

(3) *The singular fiber containing $S_{\infty 2}$, which we call G_0 by putting $w(S_{\infty 2})=0$, has one or two hidden components.*

(4) *If $G_0\cap\mathbb{A}^2$ is irreducible, then G_0 consists of three irreducible components $S_{\infty 2}$, Γ_1' and G_{01} whose dual graph is as in Fig.* 20(a) (*types* 1 *and* 2).

Proof. For the sake of simplicity, we prove the assertions for the representative cases $A1$-I with $E_1=H_1'$ or H_2' where $p\geqslant 2$ and $M=M_{12}$, and $A1$-II with $E_1=H_1$ or H_2. The other cases can be proved with minor changes:

(1) We take $w\in k(x,y)$ so that the inclusion $k(w)\hookrightarrow k(x,y)$ defines the fibration ρ_3 and that the member used explicitly in the above definition of M corresponds to $w=\infty$. Then $w\in k[x,y]$. It follows from the construction of M that all the quasi-sections of ρ_3 lying outside of $\mathbb{A}^2$ are cross-sections. Thus w is a generically rational polynomial of simple type.

(2) Since $S_{\infty 3}$ is a cross-section and since $S_{\infty 3}$ meets $S_{\infty 1}$, $S_{\infty 2}$ and Δ, the assertion follows immediately.

(3) We shall apply Lemma 1.4 to count the number of irreducible components of $G_0\cap\mathbb{A}^2$. In the case $A1$-II with $E_1=H_1$ or H_2, w has two places at infinity. If $P=\Delta\cap\rho_2^{-1}(\beta_1)$, F_{11} and F_{12} are contained in the fiber $\rho_3^{-1}(\rho_3(S_{\infty 1}))$. If $P=\Delta\cap\rho_2^{-1}(\beta_2)$ with $\beta_2\neq\beta_1$, F_{11} is a cross-section of ρ_3 since $(F_{11}\cdot\Delta)=1$. Then F_{12} and F_{21}, which is a hidden component of the fiber $\rho_2^{-1}(\beta_2)$, are contained in the fiber $\rho_3^{-1}(\rho_3(S_{\infty 1}))$.

In the case $A1$-I with $E_1=H_1'$ or H_2', we look at the singular fiber G_γ containing Δ, where we put $w(\Delta)=\gamma\in k^*$. Then G_γ contains $F_{31},\dots,F_{p1}$ and F_{11},F_{12} (respectively F_{11}) in the case $A1$-I with $E_1 = H_1'$ (respectively $E_1 = H_2'$) since $F_{11}, F_{31}, \dots, F_{p1}$ are connected to Δ and since the component right-adjacent to Γ_1 is a cross-section of ρ_3. The component F_{12} in the case $A1$-I with $E_1 = H_2'$ belongs to G_γ unless the component T with weight -3 in the graph H_2' is a cross-section of ρ_3, and $\mu_1 = 0$. In fact, T is a cross-section of ρ_3 if and only if $n_1 = 0$. The component F_{21} also belongs to G_γ unless $n_2 = 0$. In fact, F_{21} is a cross-section of ρ_3 if and only if $n_2 = 0$. Lemma 1.4 implies that $G_0 \cap \mathbb{A}^2$ is irreducible except for the case $A1$-I with $E_1 = H_2'$ and $n_1 = \mu_1 = 0$. If $n_1 = \mu_1 = 0$ in the case $A1$-I with $E_1 = H_2'$, a linear chain connecting the component T' in the graph H_2' and F_{12} may be contained in the fiber G_0. Then $G_0 \cap \mathbb{A}^2$ consists of two hidden components, G_{01} and F_{12} (cf. Figs. 17 and 20(b)). Note here that the component T' is next to the component T if $\mu_1 = 0$. We omit the argument of this case because we can deal with this case in mostly the same fashion as in the case where $G_0 \cap \mathbb{A}^2$ is irreducible.

(4) In fact, if G_0 has components other than $S_{\infty 2}$, Γ_1 and G_{01} ($G_{01} \cap \mathbb{A}^2 \neq \emptyset$), then they must be the boundary components. However, it is readily verified that they are contained in either the singular fibers of ρ_3 other than $\rho_3^{-1}(0)$ or cross-sections of ρ_3. □

The next step is to write down the polynomial f explicitly. We first consider the case of type 1 in Lemma 3.2. We may assume that $G_0 \cap \mathbb{A}^2$ is irreducible. Our strategy is explained as follows:

(1) In Fig. 7, fill the parts (a)–(c) with correct graphs and add further vertices and edges, i.e., the segments to connect two vertices, so that all the singular fibers of ρ_2 and ρ_3 can be read off. We call this graph a *complete graph* and denote it by CG.
(2) Note that $(S_\infty \cdot G_{01}) = 0$, where $S_\infty = S_{\infty 1} + 2S_{\infty 3} + S_{\infty 2}$. So, G_{01} is a fiber component of ρ_1. Determine the fiber, say Σ, of ρ_1 containing G_{01}. For this purpose, find all components in CG which are cross-sections of ρ_3 except for $S_{\infty 3}$ and connect G_{01} to all these components by edges. Then Σ can be written down by picking up all the fiber components of ρ_1 in CG so that they belong to the same fiber.
(3) There is a singular fiber of ρ_2 which has two hidden components. Then we can find the coordinates x, y such that $C_1 \cap \mathbb{A}^2$ and $C_2 \cap \mathbb{A}^2$ are defined by $y = 0$ and $x = 0$, respectively, in the first case and that $C_2 \cap \mathbb{A}^2$ is defined by $x = 0$ and $C_1 \cap \mathbb{A}^2$ is defined by a polynomial of the form $x^t y + P(x)$, in the second case, where $t > 0$, $P(x) \in k[x]$ with $\deg P(x) < t$ and $P(0) \neq 0$ (cf. [4, Lemma 2.2]).
(4) Let $\alpha = \rho_1(\Sigma) \in k^*$. Then $f - \alpha$ is decomposed into distinct irreducible factors with suitable multiplicities. These irreducible factors correspond to the hidden components of Σ and their multiplicities are the same as those of the singular fiber Σ.
(5) We can write down the irreducible factors as polynomials in x, y by looking at the singular fibers of ρ_2 or ρ_3 which contain the corresponding hidden components.

Lemma 3.3. *Suppose that the fiber G_0 of ρ_3 has type* 1 *in Lemma* 3.2. *Then the following assertions hold*:

(1) *In the case $A1$-I, f is written in the form* (1) *in the main theorem, where $\gamma' = 0$.*
(2) *In the case $A1$-*II *with $P = \Delta \cap \rho_2^{-1}(\beta_1)$* (*respectively $P = \Delta \cap \rho_2^{-1}(\beta_2)$*), *$f$ is written in the forms* (2)(i), (ii) (*respectively* (2′)) *in the main theorem, where $\gamma' = 0$ and $c = 0$.*

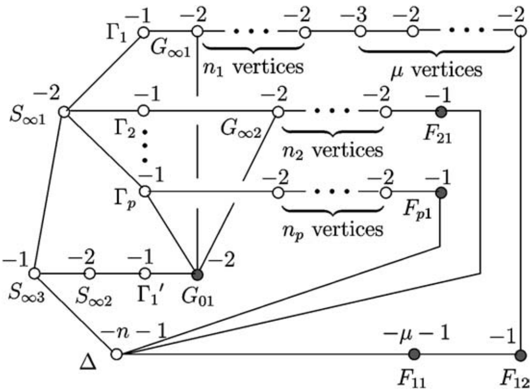

Fig. 21.

Proof. We shall determine the explicit forms of f in the case $A1$-I with $E_1 = H_1'$ where $p \geqslant 2$ and $M = M_{12}$, and the case $A1$-II with $E_1 = H_2$, where all particularities appear.

Case *A1*-I [with $E_1 = H_1'$ (cf. Fig. 21)]. We named the components right-adjacent to Γ_1, Γ_2 as $G_{\infty 1}, G_{\infty 2}$, respectively, where $G_{\infty 2} = F_{21}$ if $n_2 = 0$. Then all the cross-sections of ρ_3 in CG are $G_{\infty 1}, G_{\infty 2}, \Gamma_3, \ldots, \Gamma_p$ and $S_{\infty 3}$. Since $\Gamma_3, \ldots, \Gamma_p$ are cross-sections of ρ_1, the singular fiber Σ is a linear chain consisting of a linear chain connecting F_{21} and $G_{\infty 2}$, G_{01}, a linear chain connecting $G_{\infty 1}$ and F_{12}, and F_{11}. Let v_{l1} (respectively w_{01}) be the defining polynomial of F_{l1} (respectively G_{01}). Then f is written as

$$f \sim v_{21} w_{01} x^{\mu+1} y + \alpha, \quad \alpha \in k^*.$$

Next look at the singular fibers of ρ_2 to determine v_{l1} $(2 \leqslant l \leqslant p)$. Note that F_{l1} appears in the fiber $\rho_2^{-1}(\rho_2(\Gamma_l))$ with multiplicity one and that F_{11}, F_{12} appear in $\rho_2^{-1}(\rho_2(\Gamma_1))$ with respective multiplicity $1, \mu+1$. Hence we have

$$v_{l1} \sim x^{\mu+1} y + \beta_l, \quad \beta_l \in k^*.$$

In order to determine w_{01}, look at the fibers G_γ and G_0 of the fibration ρ_3, where $\gamma = \rho_3(\Delta) \in k^*$. Then G_γ contains $F_{11}, F_{12}, F_{21}, F_{31}, \ldots, F_{p1}$ with respective multiplicities $n_1+1, (\mu+1)n_1+\mu, n_2, n_3+1, \ldots, n_p+1$. Hence we have

$$w_{01} \sim x^{(\mu+1)n_1+\mu} y^{n_1+1} v_{21}^{n_2} \prod_{l=3}^{p} v_{l1}^{n_l+1} + \gamma, \quad \gamma \in k^*.$$

Combining these expressions together, we have an explicit form of f which is

$$f \sim x^\mu y \prod_{l=1}^{p} \bigl(x^{\mu+1} y + \beta_l\bigr)^{n_l+1} + \gamma \bigl(x^{\mu+1} y + \beta_1\bigr)\bigl(x^{\mu+1} y + \beta_2\bigr) + \alpha,$$

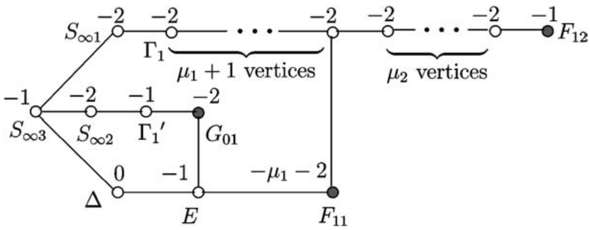

Fig. 22.

which has the form of type (1)(ii) with $\beta_1 = 0$, $\gamma' = 0$, $c_l = n_l$ for $1 \leqslant l \leqslant p$, $h = \mu$, and $Q(x, y) = y$ in the main theorem.

Case *A*1-II [with $E_1 = H_2$]. Suppose that $P = \Delta \cap \rho_2^{-1}(\beta_1)$ (cf. Fig. 22). Since all the cross-sections of ρ_3 in CG are $S_{\infty 3}$ and E, the singular fiber Σ of ρ_1 is a chain consisting of a linear chain connecting the component right-adjacent to Γ_1 and F_{12}, F_{11}, E and G_{01}. Note that $(F_{11}^2) = -\mu_1 - 2$ since we perform the blowing-up σ on the point P. By looking at ρ_2 which induces an $\mathbb{A}^1_*$-fibration on $\mathbb{A}^2$, we can choose the coordinates x, y of $\mathbb{A}^2$ so that F_{12} and F_{11} are respectively defined by $x = 0$ and $x^t y + P(x) = 0$, where t and $P(x)$ are as specified in the strategy (cf. [4, Lemma 2.2]). Then f is written as

$$f \sim w_{01} x^{\mu_1}\bigl(x^t y + P(x)\bigr) + \alpha, \quad \text{for } \alpha \in k^*.$$

In order to determine w_{01}, look at the fiber $\rho_3^{-1}(\rho_3(S_{\infty 1}))$ which contains F_{11}, F_{12} with respective multiplicities $1, \mu_1 + 2$. Hence we have

$$w_{01} \sim x^{\mu_1+2}\bigl(x^t y + P(x)\bigr) + \gamma, \quad \gamma \in k^*.$$

Combining these expressions together, we can write an explicit form of f as

$$f \sim x^{\mu_1}\bigl(x^t y + P(x)\bigr)\bigl\{x^{\mu_1+2}\bigl(x^t y + P(x)\bigr) + \gamma\bigr\} + \alpha,$$

which has type (2)(ii) with $\gamma' = 0$, $c = 0$ and $h = \mu_1 + 1$ in the main theorem.

Next, suppose that $P = \Delta \cap \rho_2^{-1}(\beta_2)$ (cf. Fig. 23). Then all the cross-sections of ρ_3 in CG are $S_{\infty 3}$ and E. Therefore the singular fiber Σ of ρ_1 is a chain consisting of E, G_{01}, F_{11}, and a linear chain connecting F_{12} and a component right-adjacent to Γ_1. Note that $(F_{21}^2) = -1$ since we perform the blowing-up σ on the point P, and that F_{11} is a cross-section of ρ_3. In the same fashion as in the case with $P = \Delta \cap \rho_2^{-1}(\beta_1)$, we have

$$f \sim w_{01} x^{\mu_1}\bigl(x^t y + P(x)\bigr) + \alpha, \quad \text{for } \alpha \in k^*, \quad \text{and}$$
$$w_{01} \sim x v_{21} + \gamma, \quad \gamma \in k^*, \quad \text{and}$$
$$v_{21} \sim x^{\mu_1+1}\bigl(x^t y + P(x)\bigr) + \beta_2, \quad \beta_2 \in k^*,$$

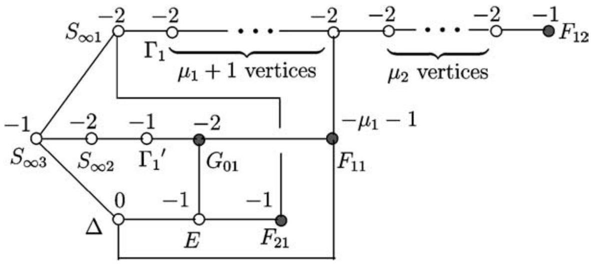

Fig. 23.

where v_{21} is a defining polynomial of F_{21}. Combining these expressions together, we obtain an explicit form of f which has type $(2')$ with $\beta = -\beta_2$, $\gamma' = 0$, $c = 0$, and $Q(x, y) = x^{\mu_1+1}(x^t y + P(x)) + \beta_2$ in the main theorem. $\square$

Next, we consider the case of type 2 in Lemma 3.2. In this case, in each of the two cases $A1$-I, $A1$-II, we define a new linear pencil M' on V and ρ_3' as the $\mathbb{P}^1$-fibration associated with M' as follows.

Case *A*1-I. Set $M_i' = |\Gamma_i + S_{\infty 1} + S_{\infty 3}|$ for $1 \leqslant i \leqslant p$. If $M = M_{ij}$ for some i and j, we take $M' = M_i'$ with the same i. Then M' has no base points.

Case *A*1-II. The exceptional curve E, which appears from the blowing-up of the point P, is a cross-section of ρ_3. The component G_{01} in the fiber G_0 in Lemma 3.2 meets the section E. So, we set $M' = |G_{01} + E|$. Then M' has no base points.

Lemma 3.4. *In the case of type* 2 *in Lemma* 3.2, *let M' and $\rho_3' : V \to \mathbb{P}^1$ be the linear pencil and the $\mathbb{P}^1$-fibration defined as above. Then the following assertions hold*:

(1) *There exists an element $w' \in k(x, y)$ such that the inclusion $k(w') \hookrightarrow k(x, y)$ gives rise to ρ_3'. In the case* $A1$-I, *w' is a generically rational polynomial of simple type, whereas, in the case* $A1$-II *as well, all the quasi-sections of ρ_3' contained in $V - \mathbb{A}^2$ are cross-sections.*
(2) *$S_{\infty 2}$ is a cross-section of ρ_3', and hence G_{01}, $S_{\infty 3}$ and Γ_1' belong to mutually distinct fibers of ρ_3'.*
(3) *The singular fiber containing Γ_1', which we call G_0' by putting $w'(\Gamma_1') = 0$, has one, two or three irreducible hidden components.*

Proof. For the sake of simplicity, we prove the assertions for the representative cases $A1$-I with $E_1 = H_1'$ or H_2' where $p \geqslant 2$, $M = M_{12}$ and $M' = M_1'$, and $A1$-II with $E_1 = H_1$ or H_2. The other cases can be proved with minor changes:

(1) In the case $A1$-I with $E_1 = H_1'$ or H_2', the proof is completely similar to that of Lemma 3.2. In the cases $A1$-II with $E_1 = H_1$ or H_2, since all the members of M' have hidden components, we have $w' \notin k[x, y]$. But it follows from the construction of M' that all the quasi-sections of ρ_3' contained in $V - \mathbb{A}^2$ are cross-sections.

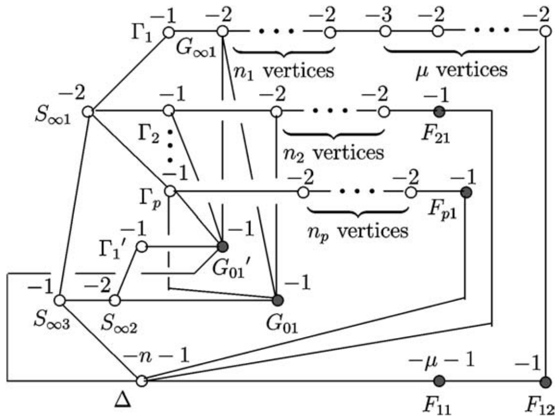

Fig. 24.

(2) Since $S_{\infty 2}$ is a cross-section and since $S_{\infty 2}$ meets $S_{\infty 3}$, Γ_1' and G_{01}, the assertion follows immediately.

(3) We shall apply Lemma 1.4 to count the number of irreducible hidden components of G_0'. In the case $A1$-I with $E_1 = H_1'$ or H_2', we look at the singular fibers $\rho_3'^{-1}(\rho_3'(F_{l1}))$ of ρ_3' for $1 \leqslant l \leqslant p$.

Suppose that $G_0' \neq \rho_3'^{-1}(\rho_3'(F_{l1}))$ for $1 \leqslant \forall l \leqslant p$. F_{11} and F_{12} are contained in the same fiber of ρ_3' unless $\mu_1 = n_1 = 0$ in the case $A1$-I with $E_1 = H_2'$. Then Lemma 1.1 shows that $\rho_3'^{-1}(\rho_3'(F_{11}))$ has at least three hidden components, F_{11}, F_{12} and the other hidden components, and that $\rho_3'^{-1}(\rho_3'(F_{l1}))$ has at least two hidden components, F_{l1} and the other hidden components, for $2 \leqslant \forall l \leqslant p$. Here note that ρ_3' has $p+2$ quasi-sections contained in $V - \mathbb{A}^2$, all of which are cross-sections. Substituting $\pi' = p+2$, $\nu_1' \geqslant 3$ and $\nu_i' \geqslant 2$ for $2 \leqslant i \leqslant p$ into the equation in Lemma 1.4, we obtain $p+1 = \pi' - 1 = \sum_{i=1}^{r}(\nu_i' - 1) \geqslant 2 + p - 1 = p + 1$. Therefore we know that G_0' has one hidden component, say G_{01}'. We have $G_0' = \Gamma_1' + G_{01}'$ with $(G_{01}'^2) = -1$ by Lemma 1.1 (see Fig. 24). If $\mu_1 = n_1 = 0$ and $G_0' \neq \rho_3'^{-1}(\rho_3'(F_{12}))$ in the case $A1$-I with $E_1 = H_2'$, we have the same conclusion as above in the same fashion. If $\mu_1 = n_1 = 0$ and $G_0' = \rho_3'^{-1}(\rho_3'(F_{12}))$ in the case $(A1)$ with $E_1 = H_2'$, a linear chain connecting the component T' in the graph H_2' and F_{12} is contained in the fiber G_0'. Then $G_0' \cap \mathbb{A}^2$ consists of two hidden components, G_{01}' and F_{11} (cf. Figs. 17 and 20(b)). We omit the argument of this case because we can deal with this case in mostly the same fashion as in the case where $G_0' \cap \mathbb{A}^2$ is irreducible. On the other hand, we know that the component $G_{\infty 2}$, which is a cross-section of ρ_3, is a fiber component of ρ_3' by virtue of the construction of M and M'. Hence G_{01} is a fiber component of $\rho_3'^{-1}(\rho_3'(F_{21}))$. It follows from the assertion (2) that $G_0' \neq \rho_3'^{-1}(\rho_3'(F_{21}))$.

If $G_0' = \rho_3'^{-1}(\rho_3'(F_{l1}))$ for $3 \leqslant \exists l \leqslant p$, Lemmas 1.1 and 1.4 show that there is a hidden component, say G_{01}', such that $(G_{01}'^2) = -2$, that G_{01}' meets $G_{\infty l}$, and that G_0' consists of Γ_1', G_{01}' and a linear chain between $G_{\infty l}$ and F_{l1} (see Fig. 18). By replacing M by M_{1l}, we are reduced to the case of type 1 in Lemma 3.2.

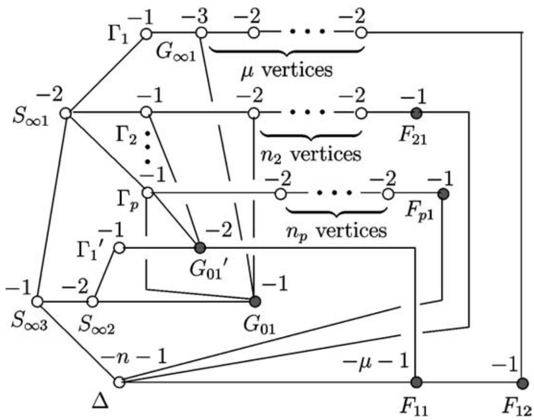

Fig. 25.

If $G_0' = \rho_3'^{-1}(\rho_3'(F_{11}))$ and $n_1 > 0$, we are reduced to the case of type 1 in Lemma 3.2 by replacing M by M_{11}. If $G_0' = \rho_3'^{-1}(\rho_3'(F_{11}))$ and $n_1 = 0$, Lemmas 1.1 and 1.4 show that there is a hidden component of G_0', say G_{01}', such that $(G_{01}'^2) = -2$, that G_{01}' meets F_{11}, and that G_0' consists of Γ_1', G_{01}', F_{11} and a linear chain between F_{12} and $G_{\infty 1}$ (see Fig. 25). We call this case the *atypical case*.

In the case $A1$-II with $E_1 = H_1$ or H_2, $S_{\infty 2}$ and Δ exhaust all quasi-sections (cross-sections, in fact) of ρ_3' lying in $V - \mathbb{A}^2$. Meanwhile, $\rho_3'|_{\mathbb{A}^2}$ is an $\mathbb{A}^1_*$-fibration parametrized by $\mathbb{P}^1$. Then, by the count of Picard rank, it is easy to show that every fiber of $\rho_3'|_{\mathbb{A}^2}$ is irreducible (cf. [3]). Hence $G_0' \cap \mathbb{A}^2$ is irreducible. $\square$

In the case of type 2 in Lemma 3.2, our strategy is then explained as follows. We just point out necessary modifications to be made on the previous strategy:

(0) Look at the singular fiber G_0' of ρ_3'. In the case of type 2, the component G_{01}' plays the same role as G_{01} in the case of type 1 when we determine f.
(1) Write a complete graph CG by means of ρ_2, ρ_3 and ρ_3'.
(2) Note that G_{01}' is a fiber component of ρ_1. Connect G_{01}' to all components of CG which are cross-sections of ρ_3' except for $S_{\infty 2}$. Then one can figure out the singular fiber of ρ_1 containing G_{01}', say Σ', from the complete graph.
(3) This step is the same as the previous strategy.
(4′) The defining polynomial w_{01}' of G_{01}' is written down if one knows the defining polynomial w_{01} of G_{01} because G_{01}' and G_{01} belong to distinct fibers of ρ_3'. The other hidden components of $\rho_3'^{-1}(\rho_3'(G_{01}))$ are already known.
(5′) To determine the polynomial w_{01}, look at the singular fibers of ρ_3. This step is the same as the one for type 1.

Here we determine f concretely in the case of type 2.

Lemma 3.5. *Suppose that the fiber G_0 of ρ_3 has type* 2 *in Lemma* 3.2. *Then the following assertions hold*:

(1) *In the case A*1-I, *f is written in the form* (1) *in the main theorem, where $\gamma' \neq 0$.*
(2) *In the case A*1-II *with $P = \Delta \cap \rho_2^{-1}(\beta_1)$ (respectively $P = \Delta \cap \rho_2^{-1}(\beta_2)$), f is written in the forms* (2)(i), (ii) (*respectively* (2′)) *in the main theorem, where $\gamma' \neq 0$ and $c = 0$.*

Proof. We shall determine the explicit forms of f in the cases A1-I with $E_1 = H_1'$ where $p \geqslant 2$, $M = M_{12}$ and $M' = M_1{}'$, and A1-II with $E_1 = H_2$, where all particularities appear.

Case *A*1-I [with $E_1 = H_1'$]. $G_{\infty 1}, \Gamma_2, \dots, \Gamma_p$ and Δ are cross-sections of ρ_3' other than $S_{\infty 2}$.

We first consider the typical case (cf. Fig. 24). Then Σ' consists of G_{01}', a linear chain between $G_{\infty 1}$ and F_{12}, and F_{11}. Choose the coordinates x, y as in the corresponding case of Lemma 3.3. Then we have

$$f \sim w_{01}' x^{\mu+1} y + \alpha, \quad \alpha \in k^*.$$

Now by considering the fibers of ρ_3' which contain G_{01}' and G_{01}, we obtain

$$G_{01}' + \Gamma_1' \sim G_{01} + G_{\infty 2} + \cdots + F_{21},$$

where the omitted components are those consisting of a linear chain connecting $G_{\infty 2}$ and F_{21}. Hence we have

$$w_{01}' \sim w_{01} v_{21} + \gamma', \qquad v_{21} \sim x^{\mu+1} y + \beta_2, \quad \gamma', \beta_2 \in k^*,$$

where v_{21} is the defining polynomial of F_{21}. By applying the step (5′) of the strategy, we have

$$G_{01} + S_{\infty 2} + \Gamma_1' \sim \Delta + (n_1+1) F_{11} + \big((\mu+1) n_1 + \mu\big) F_{12} + n_2 F_{21} + \sum_{l=3}^{p} (n_l + 1) F_{l1} + \cdots.$$

Hence we have the same expression of w_{01} as in the corresponding case of Lemma 3.3. So we have

$$f \sim x^\mu y \prod_{l=1}^{p} \big(x^{\mu+1} y + \beta_l\big)^{n_l+1} + \gamma \big(x^{\mu+1} y + \beta_1\big)\big(x^{\mu+1} y + \beta_2\big) + \gamma' \big(x^{\mu+1} y + \beta_1\big) + \alpha,$$

which has the form (1)(ii) with $\beta_1 = 0$, $\gamma' \neq 0$, $c_l = n_l$ for $1 \leqslant l \leqslant p$, $h = \mu$ and $Q(x, y) = y$ in the main theorem.

We next consider the atypical case (cf. Fig. 25). The fiber Σ' of ρ_1 consists of a linear chain connecting $G_{\infty 1}$ and F_{12}, F_{11} and G'_{01}. Then we have

$$f \sim w'_{01} x^{2\mu+1} y^2 + \alpha, \quad \alpha \in k^*.$$

We can determine v_{21}, v_{22}, v_{31} and w_{01} in the same way as in the typical case. In order to determine w'_{01}, we consider the fibers of ρ'_3 which contain G'_{01} and G_{01}. Hence we have

$$\Gamma'_1 + G'_{01} + F_{11} + \mu F_{12} + \cdots \sim G_{01} + G_{\infty 2} + \cdots + F_{21},$$

where the omitted components of the left side of the above equation are those consisting of a linear chain connecting F_{12} and the component right-adjacent to $G_{\infty 1}$ and where the omitted components of the right side of the above equation is the same as those in the typical case. Hence we obtain

$$w'_{01} x^{\mu} y \sim w_{01} v_{21} + \gamma', \quad \gamma' \in k^*.$$

These expression combined together provide the same description of f as given in the typical case.

Case *A*1-II [with $E_1 = H_2$]. Suppose that $P = \Delta \cap \rho_2^{-1}(\beta_1)$ (cf. Fig. 26). Since ρ'_3 is defined by $|G_{01} + E|$, $S_{\infty 2}$, Δ and F_{11} are cross-sections of ρ'_3. Hence connect G'_{01} to Δ and F_{11}. The singular fiber Σ' of ρ_1 consists of G'_{01}, E, F_{11} and a linear chain between F_{12} and a component right-adjacent to Γ_1. Hence we have

$$f \sim w'_{01} x^{\mu_1} \bigl(x^t y + P(x)\bigr) + \alpha, \quad \alpha \in k^*.$$

Now considering the fibers of ρ'_3, we have

$$G'_{01} + \Gamma'_1 \sim G_{01} + E \sim S_{\infty 3} + S_{\infty 1} + \cdots,$$

where the omitted components signify those between Γ_1 and F_{12}.

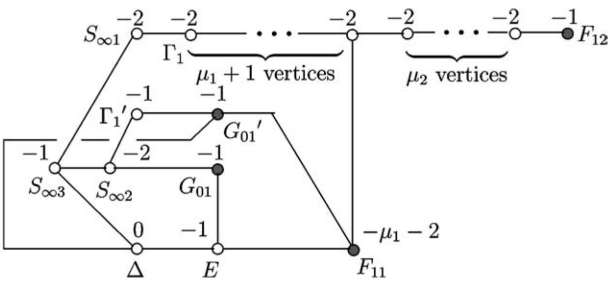

Fig. 26.

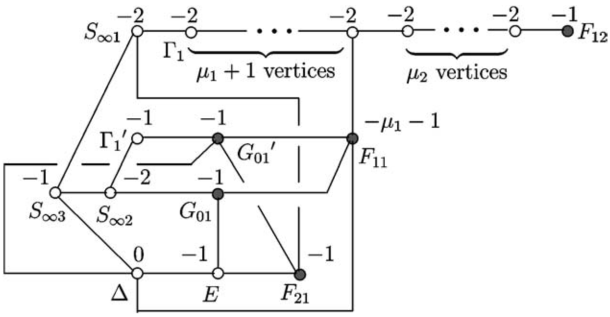

Fig. 27.

Hence we have

$$\frac{w'_{01}}{x} \sim \frac{w_{01}}{x} + \gamma', \quad \gamma' \in k^*.$$

Next considering the fibers of ρ_3, we have

$$G_{01} + S_{\infty 2} + \Gamma'_1 \sim S_{\infty 1} + F_{11} + \cdots,$$

where the omitted components are the same as above.

So we have

$$w_{01} \sim x^{\mu_1+2}\big(x^t y + P(x)\big) + \gamma, \quad \gamma \in k^*.$$

Putting all these expressions together, we have

$$f \sim x^{\mu_1}\big(x^t y + P(x)\big)\big[\big\{x^{\mu_1+2}\big(x^t y + P(x)\big) + \gamma\big\} + \gamma' x\big] + \alpha,$$

which has the form (2)(ii) with $\gamma' \neq 0$, $c = 0$, $h = \mu_1 + 1$ and $Q(x, y) = x^t y + P(x)$ in the main theorem.

Next, suppose that $P = \Delta \cap \rho_2^{-1}(\beta_2)$ (cf. Fig. 27). Since ρ'_3 is defined by $|G_{01} + E|$, $S_{\infty 2}$, Δ, F_{11} and F_{12} are cross-section of ρ'_3. Hence connect G'_{01} to Δ, F_{11} and F_{21}. The singular fiber Σ' of ρ_1 consists of G'_{01}, F_{11} and a linear chain between F_{12} and a component right-adjacent to Γ_1. In the same fashion as in the case with $P = \Delta \cap \rho_2^{-1}(\beta_1)$, we have

$$\begin{aligned}
&f \sim w'_{01} x^{\mu_1}\big(x^t y + P(x)\big) + \alpha, \quad \alpha \in k^*, \quad \text{and}\\
&\frac{w'_{01}}{x} \sim \frac{w_{01}}{x} + \gamma', \quad \gamma' \in k^*, \quad \text{and}\\
&w_{01} \sim x v_{21} + \gamma, \quad \gamma \in k^*, \quad \text{and}\\
&v_{21} \sim x^{\mu_1+1}\big(x^t y + P(x)\big) + \beta_2, \quad \beta_2 \in k^*,
\end{aligned}$$

where v_{21} is a defining polynomial of F_{21}. Combining these expressions together, we obtain an explicit form of f which has the form $(2')$ with $\beta = -\beta_2$, $\gamma' \neq 0$, $c = 0$ and $Q(x, y) = x^{\mu_1+1}(x^t y + P(x)) + \beta_2$ in the main theorem. $\square$

3.2. Cases A1-III and A1-IV

Since we can deal with the cases $A1$-III and $A1$-IV with $n'_1 = 0$ in the same manner as in the cases $A1$-I and $A1$-II, we consider only the cases $A1$-III and $A1$-IV with $n'_1 > 0$. We have two possibilities of the part (a) by choosing the type $(a1)$ or the type $(a2)$.

We define a linear pencil L and a rational map ρ_2 as in the cases $A1$-I and $A1$-II. Then the following result corresponds to Lemma 3.1.

Lemma 3.6. *In the present cases, the following assertions hold*:

(1) *L has no base points.*
(2) *There exists a generically rational polynomial v of simple type with three places at infinity such that $\Lambda_0(v)$ gives rise to ρ_2.*
(3) *In the case of type* $(a1)$ *in Fig.* 8 (*respectively type* $(a2)$), *the $\mathbb{P}^1$-fibration ρ_2 defined by L has $S_{\infty 1}$, Δ and Γ_L* (*respectively Γ'_2*) *as cross-sections lying outside of $\mathbb{A}^2$.*
(4) *$\Gamma_1, \dots, \Gamma_p$ are contained in mutually distinct members of L, where $p \leqslant 3$.*
(5) *$v - \beta$ is irreducible except for at most two exceptional values of β. If there is only one exceptional value of β, say β_0, then $v - \beta_0$ has three irreducible factors, and if there are two of them, say β_0 and β'_0, then $v - \beta_0$ and $v - \beta'_0$ have respectively two irreducible factors.*

Proof. We can demonstrate the assertions (1), (2) and (3) in the same way as in Lemma 3.1. The assertion (4) follows from the classified cases $A1$-III and $A1$-IV. The assertion (5) follows from Lemma 1.4. $\square$

We define the linear pencil M to be $|S_{\infty 1} + 2\Gamma_1 + G_{\infty 1}|$ or $|S_{\infty 1} + \Gamma_1 + \Gamma_2|$ so that M has no base points in the case $A1$-III, and $|\Delta|$ in the case $A1$-IV. Then in both cases, M has no base points. Let ρ_3 be the associated $\mathbb{P}^1$-fibration.

Lemma 3.7. *With the notations as above, the following assertions hold*:

(1) *There exists a generically rational polynomial w of simple type in $k[x, y]$ such that $\Lambda_0(w)$ gives rise to ρ_3. Furthermore, ρ_3 has two cross-sections in the boundary $V - \mathbb{A}^2$, one of which is $S_{\infty 3}$.*
(2) *Let G_0 be the singular fiber of ρ_3 which contains $S_{\infty 2}$. Let G_γ be the singular fiber of ρ_3 containing Δ* (*respectively $S_{\infty 1}$*) *in the case* $A1$-III (*respectively* $A1$-IV). *Then G_0 has a unique hidden component G_{01} whose dual graph is as in Fig.* 31 (*types* 1 *and* 2), *and G_γ has two hidden components $G_{\gamma 1}$ and $G_{\gamma 2}$, where $G_{\gamma 1} \cap \mathbb{A}^2 \cong G_{\gamma 2} \cap \mathbb{A}^2 \cong \mathbb{A}^1$ $(G_{\gamma 1} \cap G_{\gamma 2} \neq \emptyset)$ or $G_{\gamma 1} \cap \mathbb{A}^2 \cong \mathbb{A}^1_*$ and $G_{\gamma 2} \cap \mathbb{A}^2 \cong \mathbb{A}^1$ $(G_{\gamma 1} \cap G_{\gamma 2} = \emptyset)$* (*cf.* [4, Lemma 2.1]).

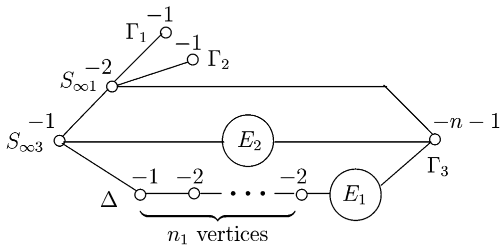

Fig. 28.

Proof. Straightforward. □

Here we shall take up the cases $A1$-III with $D_4 = \emptyset$ and $D_5 = G_5'$, $A1$-IVa with $n_1 \geqslant 1$, $D_1 = G_5'$ and $D_3 = \emptyset$, and $A1$-IVb with $D_1 = G_5$ and $D_2 = \emptyset$ as concrete examples. Note that the following lemmas in this section hold in general.

Case *A*1-III [with $D_4 = \emptyset$ and $D_5 = G_5'$]. The singular fibers of ρ_3 are read off from Fig. 28, where we insert in the place E_1 the graph H_1' or the graph H_2' in Fig. 17. Note that in each of these two cases, we have two possibilities for the place E_2, either type 1 or type 2 in Fig. 31.

Set $\rho_2(\Gamma_l) = \beta_l \in k$ for $1 \leqslant l \leqslant 3$. By Lemma 3.6(4), $\beta_l \neq \beta_{l'}$ if $l \neq l'$. By Lemma 3.6(5), $v - \beta_3$ is a unique reducible fiber of $\rho_2|_{\mathbb{A}^2}$ consisting of three hidden components, $G_{\gamma 1}$, $G_{\gamma 2}$ and G_{01} except for the case $A1$-III with $E_1 = H_2'$ and $\mu_1 = n_1 = 0$ where $G_{\gamma 1}$ and $G_{\gamma 2}$ belong to two different fibers of ρ_2 (cf. Figs. 17, 28). We omit the argument of this case because we can deal with this case in mostly the same fashion as in the case where $G_{\gamma 1}$ and $G_{\gamma 2}$ belong to one and the same fiber of ρ_2. In the case $A1$-III with $E_1 = H_2$ (respectively $E_1 = H_2'$), $G_{\gamma 1}$ denotes the vertex with $-\mu - 1$ (respectively $-\mu_1 - 1$) and is connected to the component Γ_3.

Case *A*1-IVa [with $n_1 \geqslant 1$, $D_1 = G_5'$ and $D_3 = \emptyset$]. As in the case $A1$-III, we know from Fig. 29 what the singular fibers of ρ_3 look like, where we insert in the place E_1 the graph H_1' or the graph H_2' in Fig. 17. $G_{\gamma 1}$ is also chosen as in the case $A1$-III and is connected to Γ_M. Set $\rho_2(\Gamma_1) = \beta_1 \in k$. By Lemma 3.6(5), $v - \beta_1$ is a unique reducible fiber of $\rho_2|\mathbb{A}^2$ consisting of three hidden components, $G_{\gamma 1}$, $G_{\gamma 2}$ and G_{01}.

Case *A*1-IVb [with $D_1 = G_5$ and $D_2 = \emptyset$]. As in the case $A1$-III, we know from Fig. 30 what the singular fibers of ρ_3 look like, where we insert in the place E_1 the graph H_1 or the graph H_2 in Fig. 17. Note here that the component T in Fig. 17 is $S_{\infty 1}$ and a boundary component right-adjacent to $S_{\infty 1}$ is Γ_1. $G_{\gamma 1}$ is chosen as in the case $A1$-IVa.

Set $\rho_2(\Gamma_1) = \beta_1 \in k$. By Lemma 3.6(5), $v - \beta_1$ is a unique reducible fiber of $\rho_2|\mathbb{A}^2$ consisting of three hidden components, $G_{\gamma 1}$, $G_{\gamma 2}$ and G_{01} except for the case $A1$-IVb with $E_1 = H_2$ and $\mu_1 = 0$ (cf. Figs. 17 and 30). In the case $A1$-IVb with $E_1 = H_2$ and $\mu_1 = 0$, set $\rho_2(G_{\gamma_1}) = \beta_2 \in k$. Clearly $\beta_1 \neq \beta_2$. The fiber $\rho_2^{-1}(\beta_2)$ consists of $G_{\gamma 1}$, Γ_M, G_{01} and a linear chain consisting of $n_1' - 1$ (-2) components (cf. Figs. 31 and 35). By Lemma 3.6(5), we know that the fiber $\rho_2^{-1}(\beta_1)$ has two hidden components, one of which

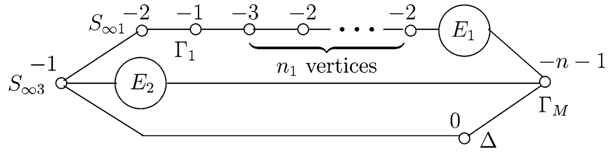

Fig. 29.

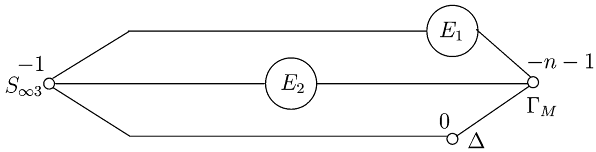

Fig. 30.

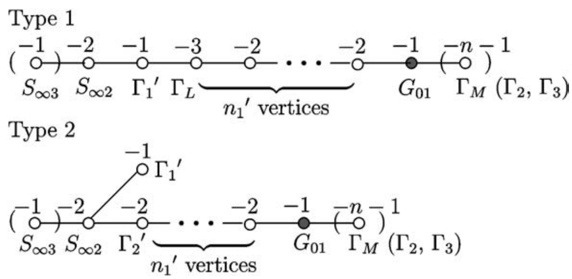

Fig. 31.

is $G_{\gamma 2}$. We name F_{11} another hidden component of the fiber $\rho_2^{-1}(\beta_1)$. Then the fiber $\rho_2^{-1}(\beta_1)$ consists of F_{11} and a linear chain between $G_{\gamma 2}$ and Γ_1 (cf. Fig. 35). We call this case the case $A1$-IVb of *irregular type* and distinguish regular type from irregular one.

In the case of type 2 of Lemma 3.7, we define the pencil $M' = |\Gamma_1 + S_{\infty 1} + S_{\infty 3}|$ in the cases $A1$-III and $A1$-IVa, and $|\Gamma_2' + \cdots + \Gamma_M|$ in the case $A1$-IVb, where the omitted components signify the components consisting the linear chain connecting Γ_2' and Γ_M (cf. Fig. 31). Note that $n = n_1'$ in the case $A1$-IVb. Then M' has no base points. Let ρ_3' be the $\mathbb{P}^1$-fibration associated with M' and w' be an element of $k(x, y)$ such that the inclusion $k(w') \hookrightarrow k(x, y)$ gives rise to ρ_3'.

Lemma 3.8. *In the case of type* 2 *in Lemma* 3.7*, let M' and $\rho_3' : V \to \mathbb{P}^1$ be the linear pencil and the $\mathbb{P}^1$-fibration defined as above. Then the following assertions hold*:

(1) *All the quasi-sections of ρ_3' contained in $V - \mathbb{A}^2$ are cross-sections.*
(2) *$S_{\infty 2}$ is a cross-section of ρ_3', and hence $S_{\infty 3}$, Γ_1' and Γ_2' belong to mutually distinct fibers of ρ_3'.*
(3) *Let G_0' denote the singular fiber of ρ_3' containing Γ_1' with $w'(\Gamma_1') = 0$. Then G_0' has one, two or three irreducible hidden components.*

Proof. We prove the assertions for the representative cases $A1$-III with $E_1 = H_1'$ or H_2', $A1$-IVa with $E_1 = H_1'$ or H_2', and $A1$-IVb with $E_1 = H_1$ or H_2. The other cases can be proved with minor changes. We obtain the assertions (1) and (2) in the same way as in Lemma 3.4. We prove the assertion (3) in the following way:

We first consider the case $A1$-III. If $G_0' \neq \rho_3'^{-1}(\rho_3'(G_{\gamma 1}))$, Lemmas 1.1 and 1.4 show that $G_0' = \Gamma_1' + G_{01}'$ with $G_{01}' \cap \mathbb{A}^2 \neq \emptyset$ and $(G_{01}'^2) = -1$ except for the case $A1$-III with $E_1 = H_2'$ and $\mu_1 = n_1 = 0$. In the case $A1$-III with $E_1 = H_2'$ and $\mu_1 = n_1 = 0$, a linear chain between $G_{\gamma 2}$ and the component T' in Fig. 17 may be contained in the fiber G_0'. Then $G_0' \cap \mathbb{A}^2$ has two hidden components by Lemma 1.4. We omit the argument of this case because we can deal with this case in mostly the same fashion as in the case where $G_0' \cap \mathbb{A}^2$ has one hidden component. If $G_0' = \rho_3'^{-1}(\rho_3'(G_{\gamma 1}))$, Lemmas 1.1 and 1.4 show that there is a hidden component of G_0', say G_{01}', such that $(G_{01}'^2) = -2$ and that G_{01}' meets Γ_1'. Then we have two cases: $n_1 > 0$ or $n_1 = 0$. In the case $n_1 > 0$, G_{01}' meets the component right-adjacent to Δ, which we call Δ', and G_0' consists of Γ_1', G_{01}', a linear chain connecting Δ' and $G_{\gamma 2}$, and $G_{\gamma 1}$. In the case $n_1 = 0$, G_{01}' meets $G_{\gamma 1}$, and G_0' consists of a linear chain connecting Δ' and $G_{\gamma 2}$, $G_{\gamma 1}$, G_{01}' and Γ_1'. Then we can see that these two cases are reduced to the case where G_0' has one hidden component as in the atypical case in Lemma 3.4.

Next, we consider the cases $A1$-IVa and $A1$-IVb. In the case $A1$-IVa, we know that $G_0' = \Gamma_1 + G_{01}'$ with $G_{01}' \cap \mathbb{A}^2 \neq \emptyset$ and $(G_{01}'^2) = -1$. In the case $A1$-IVb, it is easy to derive the same conclusion from the count of Picard rank. □

We do not explain the strategy for writing down the explicit form of f because the strategy is the same as the one in cases $A1$-I and $A1$-II. Now we can determine the polynomial f concretely.

Lemma 3.9. *Suppose that $n_1' > 0$ in the graphs of type $(a1)$ or $(a2)$. Then following assertions hold*:

(1) *In the case $A1$-III, f is written in the forms* (3)(i), (ii) *in the main theorem, where $\eta \in k^*$.*
(2) *In the case $A1$-IVa, f is written in the forms* (4)(i), (ii) *in the main theorem.*
(3) *In the case $A1$-IVb of regular type* (*respectively irregular type*)*, f is written in the forms* (2)(i), (ii) (*respectively* (2′)) *in the main theorem, where $c \geqslant 1$.*

Proof. We prove the assertion in the cases $A1$-III with $E_1 = H_1'$ of type 2 in Lemma 3.7, $A1$-IVa with $n_1 > 0$ and $E_1 = H_2'$ of type 1 in Lemma 3.7, and $A1$-IVb with $E_1 = H_2$ of type 1 in Lemma 3.7. The other cases can be treated in similar fashions.

Case *A1*-III [with $E_1 = H_1'$ of type 2 in Lemma 3.7 (cf. Fig. 32)]. We may assume that the fiber $\rho_2^{-1}(\beta_3)$ is a unique reducible fiber which has plural hidden components. Then Lemma 3.6(5) implies that the fiber $\rho_2^{-1}(\beta_1)$ (respectively $\rho_2^{-1}(\beta_2)$) has one hidden component, say F_{11} (respectively F_{21}). Since F_{11}, Γ_2, Γ_3 and Δ are cross-sections of ρ_3' other

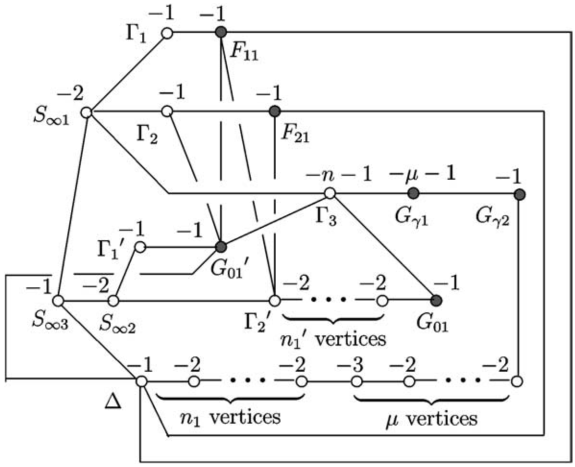

Fig. 32.

than $S_{\infty 2}$, connect G'_{01} to those components by edges. Let Σ' be the fiber of ρ_1 containing G'_{01}. Then $\Sigma' = G'_{01} + F_{11}$. Hence

$$f \sim w'_{01} v_{11} + \alpha, \quad \alpha \in k^*,$$

where w'_{01} and v_{11} are respectively the defining polynomials of G'_{01} and F_{11}. Now consider the fiber of $\rho_2^{-1}(\beta_3)$ which is a linear chain starting from a (-2) component next to Γ'_2, passing through $G_{01}, \Gamma_3, G_{\gamma 1}, G_{\gamma 2}$ and ending at the component Δ'. Since this fiber is linearly equivalent to the fibers $\rho_2^{-1}(\beta_1)$ and $\rho_2^{-1}(\beta_2)$ which are $\Gamma_1 + F_{11}$ and $\Gamma_2 + F_{21}$, respectively, we have

$$v_{l1} \sim x^{(\mu+1)n_1+\mu} y^{n_1+1} w_{01}^{n'_1} + \beta_l, \quad \beta_l \in k^* \text{ for } l = 1, 2,$$

where w_{01} is the defining polynomial of G_{01} and where we choose the coordinates x, y in such a way that $G_{\gamma 1}$ and $G_{\gamma 2}$ are defined respectively by $y = 0$ and $x = 0$. We have to determine w_{01} and w'_{01}. For this purpose, compare first the fibers of ρ_3 to obtain

$$G_{\gamma 1} + (\mu + 1)G_{\gamma 2} + \cdots + \Delta \sim S_{\infty 2} + \Gamma'_1 + \Gamma_2{}' + \cdots + G_{01},$$

which yields

$$w_{01} \sim x^{\mu+1} y + \gamma, \quad \gamma \in k^*.$$

To determine w'_{01}, we compare G'_0 with $\rho_3'^{-1}(\rho'_3(\Gamma_2{}'))$. Then we obtain

$$\Gamma'_1 + G'_{01} \sim F_{21} + \Gamma'_2 + \cdots + G_{01}.$$

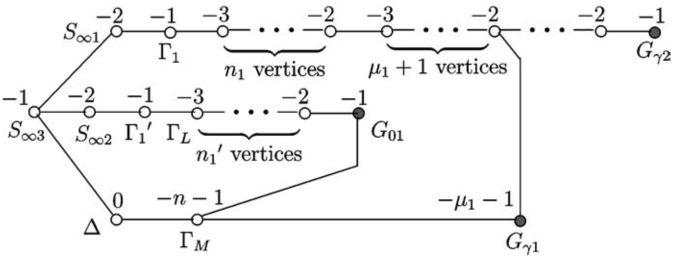

Fig. 33.

Hence we have

$$w'_{01} \sim w_{01}v_{21} + \gamma', \quad \gamma' \in k^*.$$

Combining these expressions together, we have

$$\begin{aligned} f \sim{}& \left(x^{\mu+1}y+\gamma\right)\left\{x^{(\mu+1)n_1+\mu}y^{n_1+1}\left(x^{\mu+1}y+\gamma\right)^{n'_1}+\beta_1\right\} \\ &\times\left\{x^{(\mu+1)n_1+\mu}y^{n_1+1}\left(x^{\mu+1}y+\gamma\right)^{n'_1}+\beta_2\right\} \\ &+\gamma'\left\{x^{(\mu+1)n_1+\mu}y^{n_1+1}\left(x^{\mu+1}y+\gamma\right)^{n'_1}+\beta_1\right\}+\alpha, \end{aligned}$$

which has the form (3)(ii) with $\eta = \gamma$, $c = n_1$, $c' = n'_1$, $h = \mu$ and $Q(x, y) = y$ in the main theorem.

Case *A*1-IVa [with $E_1 = H'_2$ of type 1 in Lemma 3.7 (cf. Fig. 33)]. Let Σ be the fiber of ρ_1 containing G_{01}. Then Σ consists of a linear chain between Γ_L and G_{01}, Γ_M, $G_{\gamma 1}$, and a linear chain from $G_{\gamma 2}$ to a (-3) component next to Γ_1. Hence, by choosing the coordinates x, y so that $G_{\gamma 2}$ and $G_{\gamma 1}$ are defined by $x = 0$ and $x^t y + P(x) = 0$ with $t > 0$, $P(x) \in k[x]$, $\deg P(x) < t$ and $P(0) \neq 0$, we have

$$f \sim x^{2\mu_1+(2n_1+1)(\mu_1+1)}\left(x^t y + P(x)\right)^{2+(2n_1+1)} w_{01}^{2n'_1+1} + \alpha, \quad \alpha \in k^*,$$

where w_{01} is the defining polynomial of G_{01}. Now compare G_0 with G_γ to obtain

$$S_{\infty 2} + 2\Gamma'_1 + \Gamma_L + \cdots + G_{01} \sim S_{\infty 1} + 2\Gamma_1 + \cdots + (\mu_1+1)G_{\gamma 2} + G_{\gamma 1},$$

which yields

$$w_{01} \sim x^{\mu_1+1}\left(x^t y + P(x)\right) + \gamma, \quad \gamma \in k^*.$$

So, we have an expression of f which has the form (4)(ii) with $\beta = 0$, $\gamma' = 0$, $c = n_1$, $c' = n'_1$, $h = \mu_1$ and $Q(x, y) = x^t y + P(x)$ in the main theorem.

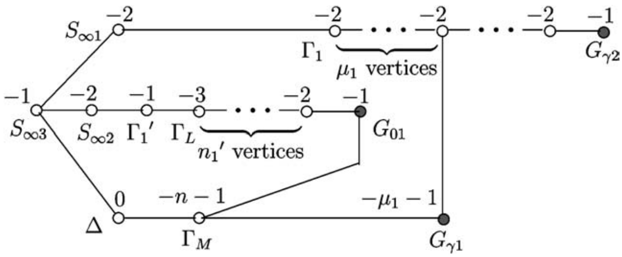

Fig. 34.

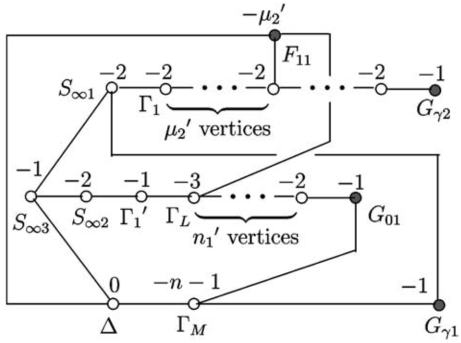

Fig. 35.

Case *A*1-IVb [with $E_1 = H_2$ of type 1 in Lemma 3.7]. First we consider the regular type (cf. Fig. 34). Let Σ be the fiber of ρ_1 containing G_{01}. Then Σ consists of a linear chain between Γ_L and G_{01}, Γ_M, $G_{\gamma 1}$ and a linear chain between $G_{\gamma 2}$ and a (-2) component next to Γ_1. Then we have

$$f \sim x^{\mu_1 - 1}\bigl(x^t y + P(x)\bigr) w_{01}^{2n_1'+1} + \alpha, \quad \alpha \in k^*,$$

where w_{01} is the defining polynomial of G_{01}. Note that $\mu_1 \geqslant 1$ in the case $A1$-IVb of regular type. We can determine w_{01} in the same way as in the case $A1$-IVa. Hence we have an expression of f which has the form (2)(ii) in the main theorem with $\gamma' = 0$, $c = n_1' \geqslant 1$ and $h = \mu_1$.

Finally, we consider the irregular type (cf. Fig. 35). We already know that the fiber $\rho_2^{-1}(\beta_1)$ has two hidden components, F_{11} and $G_{\gamma 2}$. Set $(F_{11}^2) = -\mu_2'$. Applying Lemma 1.1 to the fiber $\rho_2^{-1}(\beta_1)$, we obtain $1 \leqslant \mu_2' \leqslant \mu_2 + 1$. Furthermore, F_{11} meets Γ_L and Δ since Γ_L and Δ are cross-sections of ρ_2 other than $S_{\infty 1}$. Then Σ consists of a linear chain between $G_{\gamma 2}$ and a (-2) component next to Γ_1, F_{11}, a linear chain between Γ_L and G_{01}, and Γ_M. Note that $G_{\gamma 1}$ is not a fiber component of ρ_1 because $(G_{\gamma 1} \cdot S_{\infty 1}) = 1$. Hence we have

$$f \sim x^{\mu_2' - 1} v_{11} w_{01}^{n_1'+1} + \alpha, \quad \alpha \in k^*,$$

where v_{11} (respectively w_{01}) is a defining polynomial of F_{11} (respectively G_{01}).

Next look at the singular fibers of ρ_2 to determine v_{11}. Note that F_{11} and $G_{\gamma 2}$ appear in the fiber $\rho_2^{-1}(\beta_1)$ with respective multiplicities 1 and μ_2', and that G_{01} and $G_{\gamma 1}$ appear in the fiber $\rho_2^{-1}(\beta_2)$ with respective multiplicities n_1' and 1. Hence we have

$$x^{\mu_2'}v_{11} \sim \bigl(x^t y + P(x)\bigr)w_{01}^{n_1'} + \beta', \quad \beta' \in k^*,$$

which yields

$$x^{\mu_2'-1}v_{11} \sim x^{-1}\bigl\{\bigl(x^t y + P(x)\bigr)w_{01}^{n_1'} + \beta'\bigr\}, \quad \beta' \in k^*,$$

where $(x^t y + P(x))w_{01}^{n_1'} + \beta'$ is divisible by x. We determine w_{01} in the same way as in the case $A1$-IVa.

Combining these expressions together, we obtain an explicit form of f which has the form $(2')$ with $\beta = \beta'$, $\gamma' = 0$ and $c = n_1' \geqslant 1$ in the main theorem. □

4. Case ($A2$)

We consider the case ($A2$), where the part (c) is contractible to a smooth point but the parts (a) and (b) are not. The part (c) then has Δ as a unique (-1) curve with the other boundary components having self-intersection numbers $\leqslant -2$. This implies that the part (c) is a linear chain having Δ as an end component and $n_1 - 1$ (-2) components as the other components. Contracting the part (c) we conclude that the image of $S_{\infty 3}$, which we denote by the same letter, has self-intersection number $\geqslant 0$.

As explained in the case ($A1$), we will have a linear graph in Morrow's list after contracting all contractible components in the parts (a) and (b). According as $(S_{\infty 3}^2) = 0$ or $(S_{\infty 3}^2) = n > 0$, we have one of the graphs in Figs. 36(a) and (b). Since the parts (a) and (b) are exchangeable to each other, we have only to consider the cases $A2$-I and $A2$-II.

We divide the case $A2$-I into the cases $A2$-Ia and $A2$-Ib.

Case $A2$-Ia. The part (c) of Fig. 7 consists of only the component Δ because the self-intersection number of $S_{\infty 3}$ becomes 0 after the contraction of the part (c). The dual graphs of the parts (a) and (b) are shown in Fig. 37. We insert one of the graphs in Fig. 10 into the places named $D_1, \dots, D_4$ and D_8 in such a way that we have a boundary dual graph of $\mathbb{A}^2$ in Morrow's list (cf. Morrow [5]). The places D_5, D_6 and D_7 must be empty in this case.

Case $A2$-Ib. The part (c) of Fig. 7 is the same as in the case $A2$-Ia. The dual graphs of the parts (a) and (b) are shown in Fig. 37. We insert an appropriate dual graph in Fig. 10 into the places $D_5, \dots, D_7$ and D_8 in the same fashion as in the case $A2$-Ia. The places $D_1, \dots, D_4$ must be empty in this case. Note that the leftmost component of $D_5, \dots, D_7$ is a cross-section of ρ_1.

We also divide the case $A2$-II into the cases $A2$-IIa and $A2$-IIb.

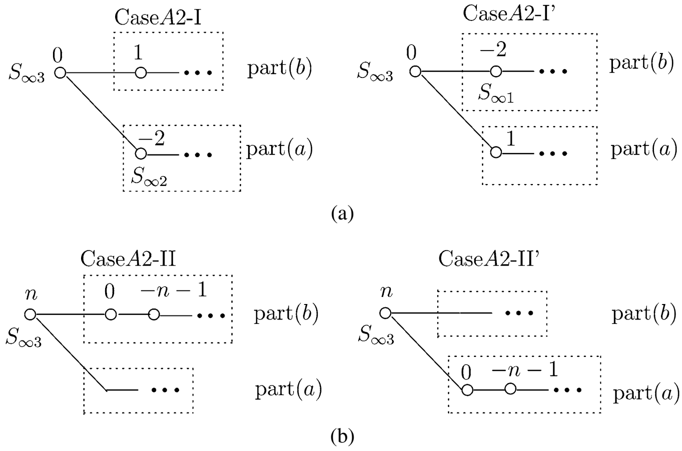

Fig. 36.

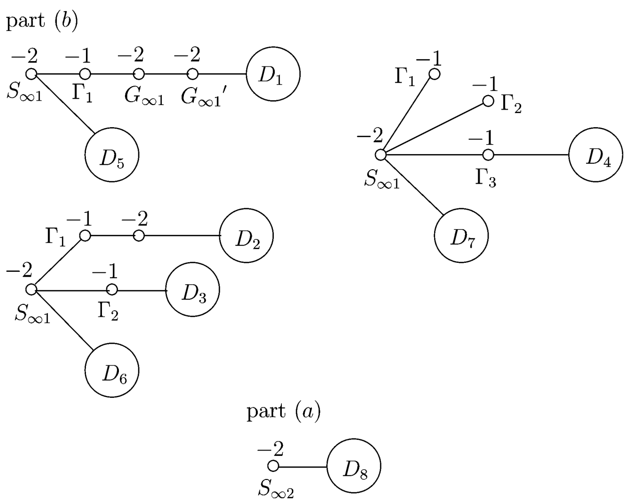

Fig. 37.

Case $A2$-IIa. In this case, we set $(\Gamma_1'^2) = -1$. Then we have four possibilities for the part (b) as shown in Fig. 14 (top left and right, and middle left and right). For the part (a), we have two possibilities (top left and right in Fig. 38). We insert an appropriate dual graph in Fig. 10 into the places $D_1, \ldots, D_6$ in the same fashion as in the case $A2$-Ia. Note that the leftmost component of D_6 is Γ_2' if $n_1' = 0$. Here we have $n = n_1 + n_1' \geqslant 1$.

Case $A2$-IIb. In this case, we set $(\Gamma_1'^2) \leqslant -2$. The part (b) is the same as in the case $A2$-IIa. For the part (a), we have a bottom one in Fig. 38. We select the graphs to be inserted into the places $D_1, \ldots, D_4$ and D_7 in the same fashion as in the case $A2$-Ia. Furthermore, D_7

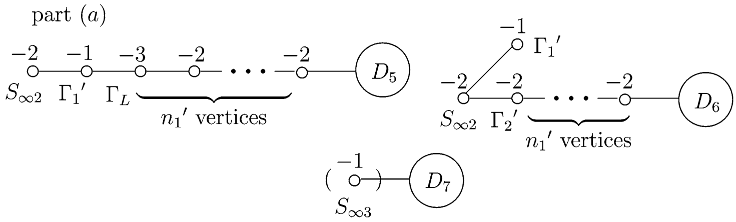

Fig. 38.

is considered, by convention in this case, to be adjacent to $S_{\infty 3}$, and we have $k_1 = 2$ since $S_{\infty 2}$ is the leftmost component of D_7 (cf. Fig. 10). Here we have $n = n_1 \geqslant 1$.

We omit the argument of the case $A2$-IIa because we can treat this case in mostly the same fashion as in the case $A1$-III. The final description of f in this case is shown as follows.

Lemma 4.1. *In the case* $A2$-IIa*, the polynomial* f *is written in the forms* (3)(i), (ii) *in the main theorem, where* $\eta = 0$.

Proof. Straightforward. □

Next, we consider the cases $A2$-Ia, $A2$-Ib and $A2$-IIb. Since the procedures we follow in these cases are mostly the same as those in the case ($A1$), we just point out the outlines of the procedures and make the differences clear:

(1) Find the $\mathbb{P}^1$-fibrations ρ_2 and ρ_3 such that $S_{\infty 1}$ (respectively $S_{\infty 3}$) is a cross-section of ρ_2 (respectively ρ_3) and that all quasi-sections in the boundary $V - \mathbb{A}^2$ are cross-sections and that ρ_3 has two cross-sections in $V - \mathbb{A}^2$. Except for the case $A2$-Ib, ρ_3 can be found out easily.
(2) By making use of ρ_3, find all hidden (fiber) components of ρ_3. Then one can define the linear pencil to define ρ_2 by choosing the boundary or hidden components. Once ρ_2 is defined, we can detect further hidden (fiber) components of ρ_2.
(3) In the case $A2$-Ib, the fibration ρ_3 is defined on a one-point blow-up V' of V. Let $\sigma : V' \to V$ be the blowing-up of the point $S_{\infty 1} \cap D_5$ ($S_{\infty 1} \cap D_6$ or $S_{\infty 1} \cap D_7$) and let E be the exceptional curve. Then ρ_3 is defined by the components of the part (b) except for E and $\sigma'(D_5)$ ($\sigma'(D_6)$ or $\sigma'(D_7)$), where we note that the portions D_1, D_2, D_3 and D_4 are all empty and where $\sigma'(D_5)$ ($\sigma'(D_6)$ or $\sigma'(D_7)$) is the graph on V' consisting of the proper transforms of the components of D_5 (D_6 or D_7). Then quasi-sections of ρ_3 in $V - \mathbb{A}^2$ are exhausted by two cross-sections $S_{\infty 3}$ and E. By abuse of notations, we denote V' by V and $\sigma'(D_5)$ by D_5, etc. Then the fiber of ρ_3 containing Δ (which is the unique component of the part (c)) has one hidden (-1) component G_{01}. We define ρ_2 as the $\mathbb{P}^1$-fibration associated with $|G_{01} + E|$ and find further hidden components. We need one more $\mathbb{P}^1$-fibration ρ_1' defined by $|\Gamma_1 + S_{\infty 1} + 2S_{\infty 3} + S_{\infty 2}|$ which differs from the fibration ρ_1 by the point that we take Γ_1 instead of E. Note that the fibration ρ_1' has therefore Δ as a 2-section.

(4) With all these boundary and hidden components and with further edges added so that the fibers and the cross-sections of ρ_2, ρ_3 and ρ_1' can be read off, we have a *complete graph*. Hence, one can figure out one singular fiber of ρ_1 other than $S_{\infty 1} + S_{\infty 2} + 2S_{\infty 3}$. Then one can write down the defining polynomials of the hidden components of this singular fiber in terms of the coordinates x, y.

Our result in the present case is:

Lemma 4.2. *The following assertions hold*:

(1) *In the case* $A2$-Ia*, the polynomial* f *is written in the forms* (5)(i), (ii) *or* (5′) *in the main theorem, where* $c = 1$.
(2) *In the case* $A2$-Ib*, the polynomial* f *is written in the forms* (6)(i), (ii) *in the main theorem.*
(3) *In the case* $A2$-IIb*, the polynomial* f *is written in the forms* (5)(i), (ii) *or* (5′) *in the main theorem, where* $c \geqslant 2$.

Proof. We shall pick up the cases $A2$-Ia with $D_4 = \emptyset$ and $D_8 = G_5$, and $A2$-Ib with $D_5 = G_1$ and $D_8 = G_3$ to show the procedures as explained before. The other cases can be handled in a similar way with the procedures in the case $(A1)$ conjoined. The case $A2$-IIb can be treated in the same fashion as in the case $A2$-Ia.

Case $A2$-Ia [with $D_4 = \emptyset$ and $D_8 = G_5$ (cf. Figs. 39, 40)]. We take the linear pencil $|S_{\infty 1} + \Gamma_1 + \Gamma_2|$ to define the fibration ρ_3, which has $S_{\infty 3}$ and Γ_3 as cross-sections. Then the linear chain G_5 with G_5 representing a linear chain of length m of (-2) components is contained in the fiber $\rho_3^{-1}(\rho_3(S_{\infty 2}))$. By virtue of Lemmas 1.1 and 1.4, putting $\rho_3(S_{\infty 2}) = \gamma \in k^*$, the fiber $\rho_3^{-1}(\gamma)$ has two hidden components $G_{\gamma 1}$ and $G_{\gamma 2}$, and all the other fibers of ρ_3 have only one hidden component. Then H_1 or H_2 in Fig. 17 is the graph of the fiber $\rho_3^{-1}(\gamma)$, where the component T in the graphs H_1 and H_2 is $S_{\infty 2}$ and where the boundary component next to $S_{\infty 2}$ is Γ_1'. Here, as in the case $(A1)$, we have two subcases according as $G_{\gamma 1} \cap G_{\gamma 2} \neq \emptyset$ (the graph H_1) or $G_{\gamma 1} \cap G_{\gamma 2} = \emptyset$ (the graph H_2). In the former case, $\mu = m + 1 \geqslant 2$, $(G_{\gamma 1}^2) = -\mu - 1$ and $(G_{\gamma 2}^2) = -1$. In the latter case, $\mu_1 + \mu_2 = m \geqslant 1$, $G_{\gamma 1}$ meets some (-2) component of H_2, say the $(\mu_1 + 1)$st component of H_2 from the left, and $(G_{\gamma 2}^2) = -1$. On the other hand, putting $\rho_3(\Delta) = 0$, we have $\rho_3^{-1}(0) = \Delta + G_{01}$, where $G_{01} \cap \mathbb{A}^2 \neq \emptyset$, $(G_{01}^2) = -1$ and $(G_{01} \cdot \Gamma_3) = 1$.

Now look at the fibration ρ_2 defined by $|G_{01} + \Gamma_3|$. Then ρ_2 has $S_{\infty 1}$, Δ and $G_{\gamma 1}$ as cross-sections. The fiber $\rho_2^{-1}(\rho_2(S_{\infty 3}))$ is a linear chain consisting of $S_{\infty 3}$, $S_{\infty 2}$ and a linear chain between Γ_1' and $G_{\gamma 2}$. The fiber $\rho_2^{-1}(\rho_2(\Gamma_1))$ (respectively $\rho_2^{-1}(\rho_2(\Gamma_2))$ is $\Gamma_1 + F_{11}$ (respectively $\Gamma_2 + F_{21}$), where F_{11} (respectively F_{21}) is a hidden (-1) component connected to Γ_1 (respectively Γ_2), $G_{\gamma 1}$ and Δ. Note that $\rho_2|_{\mathbb{A}^2}$ is parametrized by $\mathbb{P}^1$. Hence, by the count of Picard rank, every fiber of $\rho_2|_{\mathbb{A}^2}$ has only one hidden component.

Suppose that $G_{\gamma 1} \cap G_{\gamma 2} \neq \emptyset$ or $G_{\gamma 1} \cap G_{\gamma 2} = \emptyset$ with $\mu_1 > 0$. Note that $G_{\gamma 1}$ appears in a fiber of ρ_1 because $(G_{\gamma 1} \cdot S_\infty) = 0$. Let Σ be the fiber of ρ_1 containing $G_{\gamma 1}$. Then we

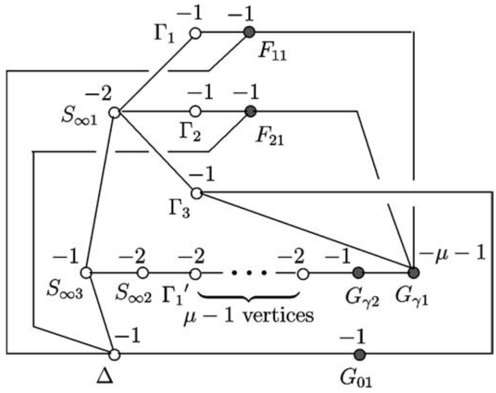

Fig. 39.

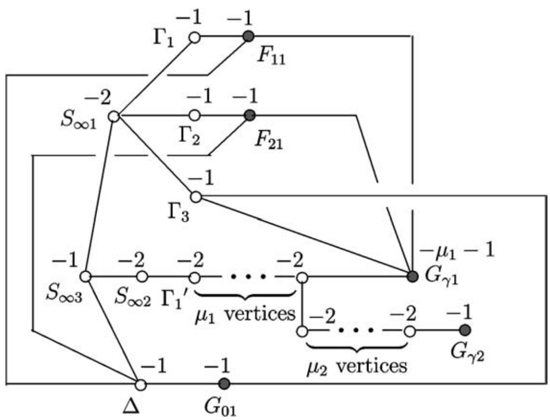

Fig. 40.

have

$$\Sigma = F_{11} + F_{21} + G_{\gamma 1} + \begin{cases} (\mu - 1)G_{\gamma 2} + \cdots & \text{(linear case)}, \\ (\mu_1 - 1)G_{\gamma 2} + \cdots & \text{(hooked case)}. \end{cases}$$

Now choose the coordinates x, y of $\mathbb{A}^2$ in such a way that the defining polynomials of $G_{\gamma 2}$ and $G_{\gamma 1}$ are respectively x and y (or $x^t y + P(x)$) in the linear (or hooked) case, where $t > 0$, $P(x) \in k[x]$, $\deg P(x) < t$ and $P(0) \neq 0$. Hence we have

$$f \sim \begin{cases} x^{\mu-1} y v_{11} v_{21} + \alpha, & \alpha \in k^* \quad \text{(linear case)}, \\ x^{\mu_1 - 1}(x^t y + P(x)) v_{11} v_{21} + \alpha, & \alpha \in k^* \quad \text{(hooked case)}, \end{cases}$$

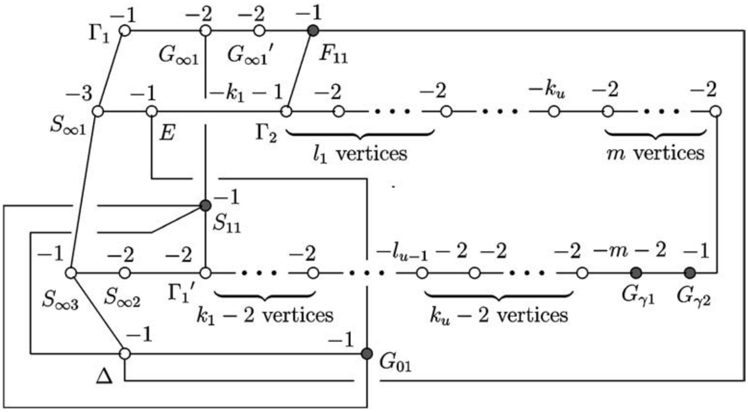

Fig. 41.

where v_{11} (respectively v_{21}) is the defining polynomial of F_{11} (respectively F_{21}). Furthermore, comparison of the fibers $\rho_2^{-1}(\rho_2(G_{01}))$, $\rho_2^{-1}(\rho_2(S_{\infty 3}))$, $\rho_2^{-1}(\rho_2(\Gamma_1))$ and $\rho_2^{-1}(\rho_2(\Gamma_2))$ yields a relation

$$v_{11} \sim w_{01} + \beta_1 x, \qquad v_{21} \sim w_{01} + \beta_2 x, \quad \beta_1, \beta_2 \in k^*, \ \beta_1 \neq \beta_2,$$

where w_{01} is the defining polynomial of G_{01}. Comparing next the two fibers $\rho_3^{-1}(\gamma)$ and $\rho_3^{-1}(0)$, we can determine w_{01} as follows:

$$w_{01} \sim \begin{cases} x^{\mu+1} y + \gamma, & \gamma \in k^* \quad \text{(linear case)}, \\ x^{\mu_1+1}(x^t y + P(x)) + \gamma, & \gamma \in k^* \quad \text{(hooked case)}. \end{cases}$$

Combining these expressions together, we have the polynomial f of the form (5)(ii) with $c = 1$, and $h = \mu$ and $Q(x, y) = y$ in the linear case, and $h = \mu_1$ and $Q(x, y) = x^t y + P(x)$ in the hooked case.

Suppose that $G_{\gamma 1} \cap G_{\gamma 2} = \emptyset$ with $\mu_1 = 0$. By arguments similar to those in the case $A1$-IVb of irregular type, the explicit form of f is $(5')$ with $c = 1$ in the main theorem.

Case *A2*-Ib [with $D_5 = G_1$ and $D_8 = G_3$ (cf. Fig. 41)]. In this case, we pass to the one-point blow-up of V, which we denote by the same letter. We have a (-1) curve E meeting $S_{\infty 1}$ and the leftmost component Γ_2 of $D_5 = G_1$, where $(S_{\infty 1}^2) = -3$ and $(\Gamma_2^2) = -k_1 - 1$ (cf. Fig. 10). The fibration ρ_3 is defined by $|S_{\infty 1} + 3\Gamma_1 + 2G_{\infty 1} + G_{\infty 1}'|$. Then $S_{\infty 3}$ and E are cross-sections of ρ_3. Hence $\rho_3|_{\mathbb{A}^2}$ is an $\mathbb{A}^1_*$-fibration over the affine line. Here we put $\rho_3(\Delta) = 0$ and $\rho_3(\Gamma_2) = \gamma \in k^*$. The fiber $\rho_3^{-1}(\gamma)$ is either a linear chain or a hooked chain starting from Γ_2 and ending at $S_{\infty 2}$ with two hidden components $G_{\gamma 1}$ and $G_{\gamma 2}$, where, in the hooked case, $G_{\gamma 1}$ is on the main line and $G_{\gamma 2}$ is hooked out. For the sake of simplicity, we only treat the linear case. Furthermore, $\rho_3^{-1}(0) = \Delta + G_{01}$ with a hidden (-1) component G_{01}, which is connected to the cross-section E of ρ_3.

Now look at the fibration ρ_2 defined by $|G_{01} + E|$. Then ρ_2 has $S_{\infty 1}, \Gamma_2$ and Δ as cross-sections. Here we put $\rho_2(G_{01}) = 0$, $\rho_2(S_{\infty 3}) = \infty$ and $\rho_2(\Gamma_1) = \beta_1 \in k^*$. The fiber $\rho_2^{-1}(\infty)$ is a linear chain consisting of $S_{\infty 3}, S_{\infty 2}, G_3, G_{\gamma 1}, G_{\gamma 2}$ and ${}^tG'_1 \setminus \Gamma_2$, where ${}^tG'_1$ signifies that the graph G'_1 turned leftside-right and where ${}^tG'_1 \setminus \Gamma_2$ signifies that the component Γ_2 drops out from ${}^tG'_1$. The fiber $\rho_2^{-1}(\beta_1)$ is $\Gamma_1 + G_{\infty 1} + G'_{\infty 1} + F_{11}$, where F_{11} is a hidden (-1) component connected to Γ_2 and Δ. Note that $\rho_2|_{\mathbb{A}^2}$ is parametrized by $\mathbb{P}^1$. Hence, by the count of Picard rank, every fiber of $\rho_2|_{\mathbb{A}^2}$ has one hidden component except for the fiber $\rho_2^{-1}(\infty)$.

Now we define the fibration ρ'_1 as given in the explanations of the procedures. Then $G_{\infty 1}, E, \Gamma'_1$ and Δ exhaust all quasi-sections in $V - \mathbb{A}^2$, and Δ is a 2-section, while the others are cross-sections. Here we put $\rho'_1(F_{11}) = \alpha' \in k^*$ and $\rho'_1(G_{01}) = 0$. The fiber ${\rho'_1}^{-1}(\alpha')$ consists of $G'_{\infty 1}, F_{11}, G'_1, G_{\gamma 2}, G_{\gamma 1}$ and ${}^tG_3 \setminus \Gamma'_1$, which has three hidden components. Now consider the fiber ${\rho'_1}^{-1}(0)$. Since $\rho'_1|_{\mathbb{A}^2}$ is parametrized by the affine line and hence defined by a generically rational polynomial with four quasi-sections, Lemma 1.4 implies that there is a fiber of ρ'_1 which has two hidden components. It is the fiber ${\rho'_1}^{-1}(0)$ which is written as $G_{01} + S_{11}$ with $S_{11} \cap \mathbb{A}^2 \neq \emptyset$ and $(S_{11}^2) = -1$, where S_{11} is connected to Δ, Γ'_1 and $G_{\infty 1}$ with $(S_{11} \cdot \Delta) = 1$.

Let Σ be the fiber of ρ_1 containing S_{11}, which is $S_{11} + G_{\infty 1} + G'_{\infty 1} + F_{11}$. So, we have

$$f \sim v_{11} f_{11} + \alpha, \quad \alpha \in k^*,$$

where v_{11} and f_{11} are the defining polynomials of F_{11} and S_{11}, respectively.

Meanwhile, for $\delta_1, \varepsilon_1 \in \mathbb{Z}$, we define $\delta_2(\delta_1, \varepsilon_1), \ldots, \delta_u(\delta_1, \varepsilon_1), \delta_{u+1}(\delta_1, \varepsilon_1), \varepsilon_2(\delta_1, \varepsilon_1), \ldots, \varepsilon_u(\delta_1, \varepsilon_1)$ in the following way:

$$\begin{cases} \delta_i(\delta_1, \varepsilon_1) := \delta_{i-1}(\delta_1, \varepsilon_1) + l_{i-1}\varepsilon_{i-1}(\delta_1, \varepsilon_1), \\ \varepsilon_i(\delta_1, \varepsilon_1) := (k_i - 2)\delta_{i-1}(\delta_1, \varepsilon_1) + ((k_i - 2)l_{i-1} + 1)\varepsilon_{i-1}(\delta_1, \varepsilon_1) \end{cases}$$

for $2 \leqslant i \leqslant u$ and $\delta_{u+1}(\delta_1, \varepsilon_1) := \delta_u(\delta_1, \varepsilon_1) + (m+1)\varepsilon_u(\delta_1, \varepsilon_1)$ (cf. Fig. 10).

On the other hand, comparison of the fibers $\rho_3^{-1}(\gamma)$ and $\rho_3^{-1}(0)$ provides a relation of the form

$$w_{01} \sim x^{e_2} y^{e_1} + \gamma, \quad \gamma \in k^*,$$

where w_{01} is the defining polynomial of G_{01}, and where the coordinates x and y are so chosen that x and y are respectively defining polynomials of $G_{\gamma 2}$ and $G_{\gamma 1}$, and where $e_1 = \varepsilon_u(1, k_1)$ and $e_2 = \delta_{u+1}(1, k_1)$.

Furthermore, comparison of the fibers $\rho_2^{-1}(0)$, $\rho_2^{-1}(\infty)$ and $\rho_2^{-1}(\beta_1)$ yields a relation

$$v_{11} \sim w_{01} + \beta_1 x^{d_1} y^{d_2}, \quad \beta_1 \in k^*,$$

where $d_1 = \varepsilon_u(0, 1)$ and $d_2 = \delta_{u+1}(0, 1)$.

Finally, comparison of the fibers ${\rho'_1}^{-1}(0)$ and ${\rho'_1}^{-1}(\alpha')$ yields a relation

$$w_{01} f_{11} \sim v_{11}^2 x^{c_2} y^{c_1} + \alpha', \quad \alpha' \in k^*,$$

where $c_1 = \varepsilon_u(1, k_1 - 2)$ and $c_2 = \delta_{u+1}(1, k_1 - 2)$. Here note that $c_1 + 2d_1 = e_1$ and $c_2 + 2d_2 = e_2$ by a straightforward computation. Now, plugging the expression of v_{11}, we have

$$v_{11}^2 x^{c_2} y^{c_1} + \alpha' = w_{01}^2 x^{c_2} y^{c_1} + 2\beta_1 w_{01} x^{c_2+d_2} y^{c_1+d_1} + \beta_1^2 x^{e_2} y^{e_1} + \alpha'.$$

This implies that $\beta_1^2 x^{e_2} y^{e_1} + \alpha'$ is divisible by w_{01}. Hence $\alpha' = \beta_1^2 \gamma$, and we obtain an expression of f_{11} in terms of x, y and w_{01}. All these expressions put together provide the polynomial f which turns out to have the form (6)(i) with $\beta_1 = \beta_2 = \beta_3 \in k^*$ in the main theorem. □

5. Cases $(A3)$ and $(A4)$

We consider the cases $(A3)$ and $(A4)$. In the case $(A3)$, the parts (a) and (c) are contractible to smooth points but the part (b) is not. Then the part (a) has type $(a1)$ or $(a2)$ in Fig. 8 and the part (c) is a linear chain having Δ as an end (-1) component and (-2) components as the other components. In the case $(A4)$, the parts (a) and (b) are contractible to smooth points but the part (c) is not. Then the parts (a) and (b) have type $(a1)$ or $(a2)$ in Fig. 8.

As explained in the previous cases, contracting all contractible components in the parts (a), (b) and (c) in the cases $(A3)$ and $(A4)$, we have one of four cases $A3$-I, $A3$-II, $A4$-I and $A4$-II in Fig. 42 by Morrow [5], where $n \geqslant 1$. The part (b) in the cases $A3$-I and $A3$-II has one of the graphs in Fig. 14 (top left and right, and middle left and right) and 43, respectively. The part (c) in the cases $A4$-I and $A4$-II has the graph in Fig. 43. Each of the places $D_1, \ldots, D_5$ is the graph G_5 in Fig. 10 or empty.

Since the procedures we are going to follow in the cases $(A3)$ and $(A4)$ are mostly the same as those in the cases $(A1)$ and $(A2)$, we just point out an outline.

(1) First, we consider the case $(A3)$. We define the linear pencil M to be $|S_{\infty 1} + 2\Gamma_1 + G_{\infty 1}|$ or $|S_{\infty 1} + \Gamma_1 + \Gamma_2|$ so that M has no base points. Let ρ_3 be the fibration associated with M. Then, in the case $A3$-I, there exists a unique fiber of ρ_3 which has two hidden components, and the other fibers of ρ_3 have a single hidden component since ρ_3 is an $\mathbb{A}^1_*$-fibration over the affine line. Note that $\rho_3^{-1}(\rho_3(S_{\infty 2}))$, $\rho_3^{-1}(\rho_3(\Delta))$ or a third fiber of ρ_3 can be this unique fiber of ρ_3 with two hidden components. In the case $A3$-II, all the fibers of ρ_3 except for $\rho_3^{-1}(\rho_3(S_{\infty 1}))$ have only one hidden component.
(2) We take the linear pencil $|S_{\infty 2} + S_{\infty 3} + \Gamma'_1|$ to define ρ_2 and detect the hidden components of the fibers of ρ_2. If the part (a) has type $(a1)$ in Fig. 8, then the detection of hidden components of ρ_2 is enough to write an explicit form of f. If the part (a) has type $(a2)$ in Fig. 8, then we take the linear pencil $|\Gamma_1 + S_{\infty 1} + S_{\infty 3}|$ to define ρ'_3 and detect the hidden component of the fiber of ρ'_3 containing Γ'_1. Using these hidden components, we obtain an explicit form of f.
(3) Next, we consider the case $(A4)$. In order to define the fibration ρ_3, we take the linear pencil $|\Delta|$, where $(\Delta^2) = 0$. Then, in the case $A4$-I, there exists a unique fiber which

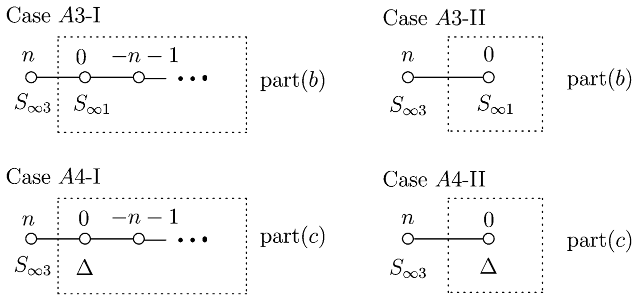

Fig. 42.

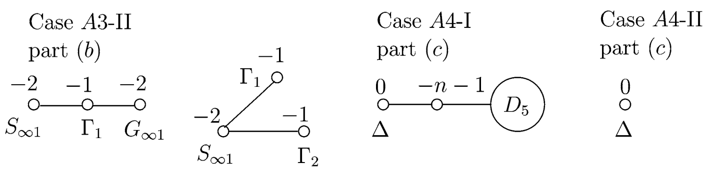

Fig. 43.

has two hidden components and the other fibers have a single hidden component as in the case $(A3)$. Note that $\rho_3^{-1}(\rho_3(S_{\infty2}))$, $\rho_3^{-1}(\rho_3(S_{\infty1}))$ or a third fiber of ρ_3 can be this unique fiber of ρ_3 with two hidden components. In the case $A4$-II, all the fibers of ρ_3 except for $\rho_3^{-1}(\rho_3(\Delta))$ have only one hidden component. The definition of ρ_2 and ρ_3' is the same as in the case $(A3)$, where we replace the components Γ_1' and Γ_2' of the part (b) by Γ_1 and Γ_2, respectively. Using the hidden components of the fibrations ρ_2, ρ_3 and ρ_3', we obtain an explicit form of f.

Our result is stated as follows:

Lemma 5.1. *The following assertions hold*:

(1) *In the case* $A3$-I, *the polynomial f is written in the form* (3)(iii) *or* (3′) *in the main theorem. In the case* $A3$-II, *written in the form* (7) *in the main theorem.*
(2) *In the case* $A4$-I, *the polynomial f is written in the form* (4)(iii) *or* (4′) *in the main theorem. In the case* $A4$-II, *written in the form* (8) *in the main theorem.*

Proof. Let $w \in k(x, y)$ be an element such that the inclusion $k(w) \hookrightarrow k(x, y)$ gives rise to ρ_3. In the cases $A3$-II and $A4$-II, we may assume that $w = x$ (cf. [1]), and we know expressions of all the hidden components of the fibers of ρ_2 due to Miyanishi and Sugie [4, Lemmas 3.2(1) and 3.6(2)]. The remains of the proof follow from the procedures. □

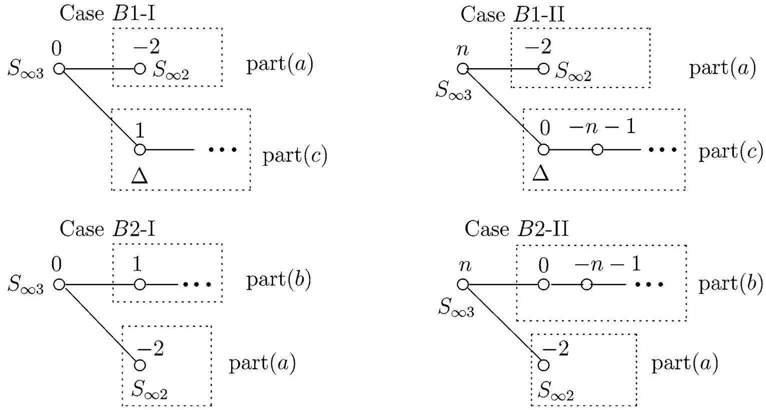

Fig. 44.

6. Cases ($B1$) and ($B2$)

We consider the cases ($B1$) and ($B2$). In the case ($B1$), the part (b) of Fig. 7 is contractible to a smooth point but the part (c) of Fig. 7 is not. Then the part (b) has type ($a1$) or ($a2$) in Fig. 8. In the case ($B2$), the part (c) is contractible to a smooth point but the part (b) is not. Then the part (c) is a linear chain having Δ as an end (-1) component and $n_1 - 1$ (-2) components as the other components. Note that the cases ($B1$) and ($B2$) with $\pi = p = 1$ are those treated in Saito [9]. As explained in the case ($A1$), contracting all contractible components in the part (b) (respectively (c)) in the case ($B1$) (respectively ($B2$)), we have the graphs in Fig. 44.

Case *B*1-I. The part (b) has type ($a1$) or ($a2$) with $n'_1 = 0$ since the self-intersection number of $S_{\infty 3}$ becomes 0 after the contraction of the part (b). Let P' be a point of intersection of Δ and a component right-adjacent to Δ, let σ' be a blowing-up at P', and let E' be an exceptional curve of σ'. Carrying out the blowing-up σ', we have the dual graph of the part (c) in Fig. 45.

Case *B*1-II. The part (b) has type ($a1$) or ($a2$) with $n'_1 > 0$ since the self-intersection number of $S_{\infty 3}$ becomes $n = n'_1 > 0$ after the contraction of the part (b). The dual graph of the part (c) is shown in Fig. 45.

We divide the case $B2$-I into the cases $B2$-Ia and $B2$-Ib.

Case *B*2-Ia. We have $n_1 = 1$ because the self-intersection number of $S_{\infty 3}$ becomes 0 after the contraction of the part (c). The dual graphs of the part (b) are shown in Fig. 37. Each of the places $D_1, \dots, D_4$ is the graph G'_5 in Fig. 10 or empty, but we do not insert G'_5 into D_2 and D_3 simultaneously. The places $D_5, \dots, D_7$ must be empty in this case.

Case *B*2-Ib. We also have $n_1 = 1$ for the same reason as in the case $B2$-Ia. The dual graphs of the part (b) are also the same as in the case $B2$-Ia. We insert the graph G_5 in Fig. 10 into

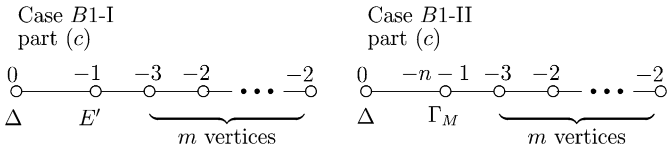

Fig. 45.

the places $D_5, \ldots, D_7$, but the places $D_1, \ldots, D_4$ must be empty. Note that the leftmost component of $D_5, \ldots, D_7$ is a cross-section of ρ_1.

Case *B*2-II. We have $n_1 > 1$ because the self-intersection number of $S_{\infty 3}$ becomes $n = n_1 - 1 > 0$ after the contraction of the part (c). The dual graphs of the part (b) are shown in Fig. 14 (top left and right, and middle left and right). Each of the places $D_1, \ldots, D_4$ is the graph G'_5 or empty.

We just point out an outline since the procedures we are going to follow in the cases $B1$-I, $B1$-II, $B2$-Ia, $B2$-Ib and $B2$-II are mostly the same as those in the cases $A1$-II, $A1$-IV, $A2$-Ia, $A2$-Ib and $A2$-IIb, respectively.

(1) In order to define the fibration ρ_3, we define the linear pencil M to be $|\varDelta|$ in the cases $B1$-I and $B1$-II, and $|S_{\infty 1} + 2\varGamma_1 + G_{\infty 1}|$ or $|S_{\infty 1} + \varGamma_1 + \varGamma_2|$ so that M has no base points in the cases $B2$-Ia and $B2$-II. In the cases $B2$-Ib, we define the linear pencil M in the same fashion as in the case $A2$-Ib. Then there exists a unique fiber of ρ_3 which has two hidden components, and the other fibers of ρ_3 have a single hidden component since ρ_3 is an $\mathbb{A}^1_*$-fibration over the affine line. Note that $\rho_3^{-1}(\rho_3(S_{\infty 2}))$ is the unique fiber of ρ_3 with two hidden components, which we call $G_{\gamma 1}$ and $G_{\gamma 2}$, and that $\rho_3^{-1}(\rho_3(\varDelta))$ (respectively $\rho_3^{-1}(\rho_3(S_{\infty 1}))$) is a fiber of ρ_3 with one hidden component, which we call G_{01}, in the case $(B2)$ (respectively $(B1)$). We may assume that $G_{\gamma 1} \cap \mathbb{A}^2 \cong G_{\gamma 2} \cap \mathbb{A}^2 \cong \mathbb{A}^1$ and $(G_{\gamma 1} \cdot S_{\infty 2}) = 1$ $(G_{\gamma 1} \cap G_{\gamma 2} \neq \emptyset)$ or $G_{\gamma 1} \cap \mathbb{A}^2 \cong \mathbb{A}^1_*$, $G_{\gamma 2} \cap \mathbb{A}^2 \cong \mathbb{A}^1$ and $(G_{\gamma 1} \cdot S_{\infty 2}) = 1$ $(G_{\gamma 1} \cap G_{\gamma 2} = \emptyset)$.
(2) We take the linear pencil L to define the fibration ρ_2 in the following way: Suppose that $\rho_3^{-1}(\rho_3(S_{\infty 2}))$ is a linear chain or a hooked chain with $m = m_1 + m_2 + 1 > 0$ (cf. top left and right graphs in Fig. 46). We define the linear pencil L to be $|S_{\infty 3} + S_{\infty 2} + G_{\gamma 1} + \cdots|$, where the omitted part signifies $G_{\gamma 2}$ and a linear chain consisting of $m-1$ (-2) boundary components. It follows that $S_{\infty 1}$, $\varDelta$ and the rightmost component of the fiber $\rho_3^{-1}(\rho_3(S_{\infty 2}))$, which is a (-3) boundary component, are cross-sections of ρ_3. Next suppose that $\rho_3^{-1}(\rho_3(S_{\infty 2}))$ is a hooked chain with $m = 0$ (cf. bottom graph in Fig. 46). We define the linear pencil L to be $|S_{\infty 3} + S_{\infty 2} + G_{\gamma 2}|$. It follows that $S_{\infty 1}$, $\varDelta$ and $G_{\gamma 1}$ are cross-sections of ρ_2. Then L has no base points. By the count of Picard rank, the fibers of ρ_2 other than $\rho_2^{-1}(\rho_2(S_{\infty 3}))$ have a single hidden component. Hence in the cases $B1$-I, $B2$-II, $B2$-Ia and $B2$-II where $\rho_3^{-1}(\rho_3(S_{\infty 2}))$ is a linear chain or a hooked chain with $m > 0$, we can write down an explicit form of f.
(3) In the case $B2$-Ib, we take the linear pencil $|\varGamma_1 + S_{\infty 1} + 2S_{\infty 3} + S_{\infty 2}|$ to define the fibration ρ'_1. Then Lemmas 1.1 and 1.4 imply that the fiber $\rho_1'^{-1}(\rho'_1(G_{01}))$ is written

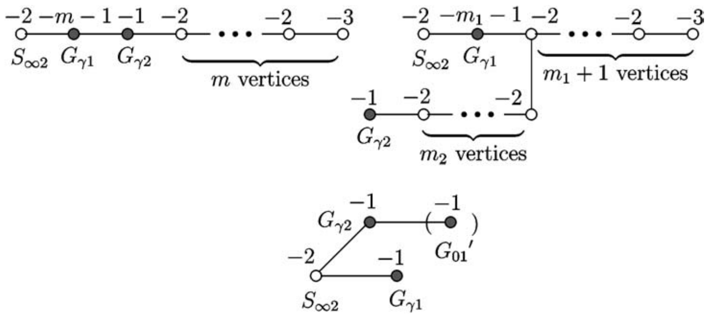

Fig. 46.

as $G_{01} + S_{11}$ with $S_{11} \cap \mathbb{A}^2 \neq \emptyset$, $(S_{11}^2) = -1$ and $(S_{11} \cdot \Delta) = 1$. Using the hidden components of the fibration ρ_1', ρ_2 and ρ_3, we obtain an explicit form of f.

(4) In the cases $B1$-I, $B1$-II, $B2$-Ia and $B2$-II where $\rho_3^{-1}(\rho_3(S_{\infty 2}))$ is a hooked chain with $m = 0$, we define the linear pencil M' to be $|\Gamma_1 + S_{\infty 1} + S_{\infty 3}|$. Then M' has no base points. Let ρ_3' be the fibration associated with M'. Lemmas 1.1 and 1.4 imply that the fiber $\rho_3'^{-1}(\rho_3'(G_{\gamma 2}))$ is written as $G_{\gamma 2} + G_{01}'$ with $G_{01}' \cap \mathbb{A}^2 \neq \emptyset$ and $(G_{01}'^2) = -1$. Using the hidden components of the fibration ρ_2, ρ_3 and ρ_3', we obtain an explicit form of f.

Our result is stated as follows.

Lemma 6.1. *The following assertions hold.*

(1) *In the cases* $B1$-I *and* $B1$-II *where* $\rho_3^{-1}(\rho_3(S_{\infty 2}))$ *is a linear chain or a hooked chain with* $m > 0$ (*respectively a hooked chain with* $m = 0$)*, the polynomial* f *is written in the form* (2)(iii) (*respectively* (2′)) *in the main theorem.*
(2) *In the cases* $B2$-Ia *and* $B2$-II *where* $\rho_3^{-1}(\rho_3(S_{\infty 2}))$ *is a linear chain or a hooked chain with* $m > 0$ (*respectively a hooked chain with* $m = 0$)*, the polynomial* f *is written in the form* (5)(iii) (*respectively* (5′)) *in the main theorem.*
(3) *In the cases* $B2$-Ib*, the polynomial* f *is written in the form* (6)(iii) *in the main theorem.*

Proof. Straightforward from the procedures. □

As all the above results, we have Tables 1–4.

Hence, when a generically rational polynomial of quasi-simple type is given, we can see what form of polynomial it belongs to by way of a boundary dual graph $V - \mathbb{A}^2$ obtained from the polynomial.

Finally, we remark that the polynomials in the main theorem correspond to mutually distinct complete graphs. Therefore it is obvious that they differ from each other.

Table 1

Case	Form of polynomial
$A1$-I	(1)
$A1$-II	(2)(i), (ii) or (2′) ($c = 0$)
$A1$-III	(3)(i), (ii) ($\eta = \gamma$)
$A1$-IVa	(4)(i), (ii)
$A1$-IVb	(2)(i), (ii) or (2′) ($c \geqslant 1$)

Table 2

Case	Form of polynomial
$A2$-Ia	(5)(i), (ii) or (5′) ($c = 1$)
$A2$-Ib	(6)(i), (ii)
$A2$-IIa	(3)(i), (ii) ($\eta = 0$)
$A2$-IIb	(5)(i), (ii) or (5′) ($c \geqslant 2$)

Table 3

Case	Form of polynomial
$A3$-I	(3)(iii) or (3′)
$A3$-II	(7)
$A4$-I	(4)(iii) or (4′)
$A4$-II	(8)

Table 4

Case	Form of polynomial
$B1$-I	(2)(iii) or (2′) ($c = 0$)
$B1$-II	(2)(iii) or (2′) ($c \geqslant 1$)
$B2$-Ia	(5)(iii) or (5′) ($c = 1$)
$B2$-Ib	(6)(iii)
$B2$-II	(5)(iii) or (5′) ($c \geqslant 2$)

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