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On Endomorphism Rings of Non-separable Abelian p-Groups

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0. INTRODUCTION

There have been spectacular advances in recent years in the so-called realization problem for certain classes of abelian groups and modules. The advances have derived mainly from powerful combinatorial set-theoretic tools pioneered by Shelah. In the case of separable abelian p-groups these results appear, e.g., in [1, 4, 5]. For non-separable abelian p-groups the most significant contribution has been [2]. Our objective in this paper is to extend the results in [2] and derive analogous results to those obtained in [4, 5]. It is perhaps worth pointing out that our methods are entirely algebraic; the necessary set-theoretical work has been carried out in the separable case and no further set-theoretical arguments are needed. In this the work is reminiscent of [8]. All our set theory (with one exception) is in ZFC and this is not acknowledged in the statement of individual results. In the exceptional case where we work in Gödel's Constructible Universe $(V=L)$ this is acknowledged by appending $(V=L)$ after the statement number. The principal realization result can be stated as:

THEOREM 0.1. *If A is a p-adic algebra whose underlying module is the completion of a free p-adic module then if λ is an infinite cardinal such that $\lambda^{\aleph_0}=\lambda \geqslant |A|$, there exist 2^λ non-separable p-groups G_α, each of cardinality λ, such that*

(i) $E(G_\alpha)=A \oplus E_\theta(G_\alpha)$;

(ii) *every homomorphism from G_α to G_β $(\alpha \neq \beta)$ is thin;*

(iii) *$p^\omega G$ is an elementary p-group.*

In our final section we modify our results to make them suitable for dealing with transitive and fully transitive p-groups. In particular we establish the existence of proper classes (i.e., not sets) of groups having one transitivity property but not the other.

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We conclude this introduction by noting that all groups (except automorphism groups) are additively written abelian groups and all unexplained terms may be found in the standard works of Fuchs [9]; our notation is largely in accord with [9] except that maps are written on the right. In particular $E(G)$ and Aut G will, respectively, denote the endomorphism ring and automorphism group of the abelian group G, while $U(-)$ will denote the group of units of $(-)$.

1. Preliminaries

A key role in all of our subsequent discussions will be played by the so-called "thin homomorphisms" introduced by Corner in [2]. Recall that if G and H are p-groups then a homomorphism $\zeta\colon G \to H$ is said to be thin if for each e there is an n such that $(p^nG[p^e])\zeta \leqslant p^\omega H$. The set $\mathrm{Hom}_\theta(G, H)$ of all thin homomorphisms $G \to H$ is a subgroup of $\mathrm{Hom}(G, H)$ and $E_\theta G = \mathrm{Hom}_\theta(G, G)$ is a 2-sided ideal in $E(G)$. This is an obvious extension of the concept of small homomorphism introduced by Pierce [12]. Drawing on terminology widely used recently in a variety of realization problems (see, e.g., [4, 6, 10]) we shall say that a collection of p-groups $\{G_i\}$ is essentially-rigid if $E(G_i) = A \oplus E_\theta(G_i)$ for some ring A and every homomorphism from G_i to G_j $(i \neq j)$ is thin: should we need to distinguish this concept from the situation where thin is replaced by small we shall say essentially-rigid (w.r.t. small homomorphisms). The following proposition contains some basic properties relating to thin homomorphisms:

Proposition 1.1. *Let G, H be reduced p-groups, then*

(i) *the quotient ring $E(G)/E_\theta(G)$ is torsion-free, and complete and Hausdorff in its p-adic topology*;

(ii) *if $\zeta\colon G \to H$ induces a small homomorphism $G/p^\omega G \to H/p^\omega H$ then ζ is thin*;

(iii) *if $\zeta\colon G \to H$ is thin then $p^\alpha G \leqslant p^{\omega+\alpha} H$ for every infinite ordinal α*;

(iv) *if $\pi\colon G \to G$ is thin and $p^\omega G$ is bounded then* Im π *is bounded.*

Proof. Parts (i)–(iii) are in Corner [2] while (iv) is immediate.

Finally, we recall the notions of transitive and fully transitive p-groups introduced by Kaplansky [11]. If G is a reduced p-group then the height $\mathrm{ht}_G(x)$ of an element $x \in G$ is defined to be an ordinal or ∞ in the usual way (see [11] or [9]) and we recall that the Ulm sequence of x in G is the sequence $U_G(x) = (\alpha_0, \alpha_1, \ldots)$, where $\alpha_n = \mathrm{ht}_G(p^n x)$; Ulm sequences are partially ordered component-wise. The group G is then said to be transitive if, for any $x, y \in G$ such that $U_G(x) = U_G(y)$, there is an automorphism ϕ

of G such that $x\phi = y$; G is fully transitive if, for any $x, y \in G$ such that $U_G(x) \leqslant U_G(y)$, there is an endomorphism ϕ of G such that $x\phi = y$. Further details of such groups may be found in, e.g., [3, 11]; we merely note that a transitive group which is not fully transitive is necessarily a 2-group [11, Theorem 26].

2. THE MAIN REALIZATION THEOREM

Our fundamental method of lifting results from separable to non-separable p-groups is illustrated in the following result.

THEOREM 2.1. *Let $\{K_i\}_{i \in I}$ be a family of separable p-groups such that*

(i) *for each $i \in I$, $E(K_i) = A \oplus E_s(K_i)$ for some p-adic algebra A;*

(ii) *each group K_i has a pure, dense A-invariant subgroup B_i which satisfies the Crawley condition*

(C) *if $a \in A$ and $a(p^n B_i)[p] = 0$ for some n, then $a \in pA$.*

Then there exists a family $\{X_i\}_{i \in I}$ of p-groups such that (a) $X_i/p^\omega X_i \cong K_i$ *and* $|X_i| = |K_i|$, (b) $B_i \cap p^\omega X_i = 0$, (c) $E(X_i) \cong A \oplus E_\theta(X_i)$, *and* (d) $p^\omega X_i \cong \operatorname{Ker} \alpha$, *where α is any preassigned epimorphism from $K_i/B_i \to K_i/B_i$ which is an A-homomorphism. (In particular $p^\omega X_i$ may be chosen to be an elementary p-group.) Moreover if, for $i \neq j \in I$, we have that every homomorphism $K_i \to K_j$ is small, then every homomorphism $X_i \to X_j$ is thin.*

Proof. Set $D = K_i/B_i$ and let $\alpha\colon D \to K_i/B_i$ be any epimorphism which is an A-homomorphism. (Notice that we may take α to be multiplication by p if we so desire.) Then, writing $C = \operatorname{Ker} \alpha$ (and observe if α is multiplication by p then C is elementary), we can form the pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\
0 \arrow[r] & B_i \arrow[r] \arrow[d, equal] & X_i \arrow[r, "\gamma"] \arrow[d, "\delta"] & D \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & B_i \arrow[r] & K_i \arrow[r, "\beta"'] \arrow[d] & K_i/B_i \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where β is the canonical projection. It is immediate that $|X_i| = |K_i|$ and the remaining properties (a), (b), and (d) follow from well-known results on such diagrams (see, e.g., Salce [13, Sect. 44].) Notice that β is also an A-homomorphism and so X_i is an A-module in a natural way and all maps are A-homomorphisms. Consequently multiplication in X_i by $a \in A$ induces multiplication by a on the quotient $X_i/p^\omega X_i \cong K_i$. But then if $\phi \in E(X_i)$, the induced map $\bar{\phi}$ on $X_i/p^\omega X_i \cong K_i$ is an endomorphism of K_i and so there is an $a \in A$ with $\bar{\phi} - a$ small. Since $\overline{\phi - a} = \bar{\phi} - \bar{a} = \bar{\phi} - a$ is small we conclude from Proposition 1.1 that $\phi - a$ is thin. Thus $E(X_i) \leqslant A + E_\theta(X_i)$; the reverse inclusion is immediate so we have equality. Hence to prove (c) it only remains to show $A \cap E_\theta(X_i) = 0$.

Suppose, then, that $\psi \in A$ is thin. Then for each positive integer e, ψ maps $(p^n X_i)[p^e]$ into $p^\omega X_i$ for some $n < \omega$. Thus, since B_i is A-invariant, $\psi(p^n B_i[p^e]) \leqslant B_i \cap p^\omega X_i = 0$. But then by a standard extension of the Crawley condition (see, e.g., [1, Lemma 2.4]) we have that $\psi = p^e a$ for some $a \in A$. Thus ψ vanishes on $X_i[p^e]$ for each e and so $\psi = 0$.

Finally, suppose that every homomorphism $K_i \to K_j$ $(i \neq j \in I)$ is small and consider any homomorphism ϕ from $X_i \to X_j$. Then ϕ induces a homomorphism $\bar{\phi}$ from $X_i/p^\omega X_i \to X_j/p^\omega X_j$ and so $\bar{\phi}$ is small. It follows immediately from Proposition 1.1 that ϕ is thin.

Note. Although Theorem 2.1 has been stated for a single fixed algebra A, it follows readily from an examination of the proof that A could be replaced in (i) and (ii) by algebras A_i, leaving conclusions (a), (b), and (d) unchanged and changing the algebra A in conclusion (c) to A_i.

COROLLARY 2.2. *If λ is an infinite cardinal such that $\lambda^{\aleph_0} = \lambda \geqslant |A|$, where the underlying group of the ring A is the completion of a free p-adic module, then there exists an essentially-rigid family of 2^λ non-separable p-groups G_α $(\alpha < 2^\lambda)$, each of cardinality λ, with $p^\omega G_\alpha$ a direct sum of cyclic groups of the same order p^n for any preassigned positive integer n.*

Proof. Under the stated conditions on λ it follows from [4] that there exists a family of separable p-groups which is essentially-rigid (w.r.t. small homomorphisms). Moreover it is possible to choose basic subgroups B_α of the form $\oplus_n \oplus_\lambda A/p^n A$ and these of course will automatically satisfy the Crawley condition (C) (cf. [1, Theorem 1.1.]). The corollary now follows immediately from Theorem 2.1 by taking α to be multiplication by p^n since this is clearly an A-homomorphism. Observe that Theorem 0.1 is a special case.

COROLLARY 2.3. $(V = L)$. *There exists an essentially-rigid proper class of non-separable p-groups each of which is essentially indecomposable.*

Proof. It follows from [7] that, assuming $V=L$, there exists a proper class of separable p-groups which is essentially rigid (w.r.t. small homomorphisms) and for each such group K, $E(K)=\hat{Z}_p \oplus E_s(K)$. Moreover the groups so constructed have basic subgroups satisfying the Crawley condition (C). The corresponding class of groups obtained by using Theorem 2.1 is essentially rigid and for each group X in the class, $E(X)=\hat{Z}_p \oplus E_\theta(X)$. Finally, since $p^\omega X$ is bounded it follows from the standard argument of Corner [1, Proposition 5.1] and Proposition 1.1 that each X is essentially indecomposable.

Remarks. (1) The construction used in Theorem 2.1 is, in fact, categorical and the results obtained could be rephrased in terms of full embeddings of appropriate categories. There is thus, clearly, a family resemblance to results of Franzen and Goldsmith [8]. It follows, moreover, that Corollaries 2.2 and 2.3 are by no means an exhaustive list of results on non-separable groups which can be obtained using Theorem 2.1. In particular all the standard results on pathological decompositions can be so obtained.

(2) The pullback construction which yielded our non-separable groups can easily be modified to enable one to obtain groups X in which $p^\alpha X$ rather than $p^\omega X$ is obtained as the appropriate kernel. Such a construction is most interesting when the ordinal α satisfies $\alpha<\omega+\omega$ (cf. Corner [2]).

3. Prescribing Actions of $E(G)$ and Aut G

Our objective in this section is to achieve results analogous to chose of Corner [2] but without the essentially countable restrictions of that work.

Lemma 3.1. *Let Φ be a reduced p-adic algebra which is separable and complete in its own p-adic topology, then there exists a reduced, torsion-free, separable, complete p-adic algebra Ψ and an epimorphism $\sigma\colon \Psi \to \Phi$ which maps $U(\Psi)$ onto $U(\Phi)$.*

Proof. See the proof of Theorem 9.1 in [2].

Our main results of this section is:

Theorem 3.2. *Let C be any separable p-group and Φ any separable, complete (in its own p-adic topology) p-adic subalgebra of $E(C)$, then there exist arbitrarily large p-groups G with $p^\omega G=C$ and $E(G)|_C=\Phi$,* Aut $G|_C=U(\Phi)$.

Proof. Observe first that in the proof of Theorem 2.1 the group $D = K_i/B_i$ may be replaced by any divisible p-group having K_i/B_i as an epic image under an A-homomorphism. (The only "casualty" among the conclusions is the cardinality of X_i which now obeys $|X_i| \leqslant |D|$.)

Let C be any separable p-group and $\Phi \leqslant E(C)$ an appropriate p-adic subalgebra. Let Ψ be the corresponding torsion-free algebra constructed in Lemma 3.1. Now C becomes an Ψ-module via $c\psi = c(\psi\sigma)$, $c \in C$, $\psi \in \Psi$. Let D_0 be the Ψ-divisible hull of C; observe that, qua groups, D_0/C is a divisible p-group. Now choose B to be an unbounded direct sum of cyclics of the form $B = \bigoplus_n \bigoplus_\lambda \Psi/p^n\Psi$. Further choose λ so that $\lambda^{\aleph_0} = \lambda \geqslant |\Psi|$ and $|B| \geqslant |D_0|$; this is always possible and such λ may be chosen arbitrarily large. Then it follows from [4] that there exists a p-group K having B as basic subgroup with $E(K) = \Psi \oplus E_s(K)$. Moreover, as observed in the proof of Corollary 2.2, this B will automatically satisfy condition (ii) of Theorem 2.1. Now define $\alpha: (K/B \oplus D_0) \to K/B$ by $\alpha = 1 \oplus \eta$, where $\eta: D_0 \to D_0/C \leqslant K/B$ is the canonic projection. Clearly α is a Ψ-homomorphism and $\operatorname{Ker} \alpha = C$. It follows from Theorem 2.1 that there exists a p-group X with $p^\omega X = C$ and $E(X) = \Psi \oplus E_\theta(X)$. But then any thin endomorphism of X vanishes on $p^\omega X$ since $p^{\omega+\omega}X = 0$ and so $E(X)|_{p^\omega X} = \Psi|_C = \Phi$, $\operatorname{Aut} X|_{p^\omega X} = U(\Psi)|_C = U(\Phi)$.

There is, of course, no difficulty in replacing the group G constructed in Theorem 3.2 by a family of 2^λ groups G_α (for λ sufficiently large) which is essentially rigid.

COROLLARY 3.3. *There exists a proper class of non-transitive fully transitive p-groups.*

Proof. Choose C and Φ to be, respectively, the group and algebra constructed by Corner in [3, Sect. 3]. Then, using the proof of Theorem 3.2 one can construct for each λ sufficiently large (i.e., λ must satisfy $\lambda^{\aleph_0} = \lambda \geqslant |\Psi|$ and $|B| \geqslant |D_0|$, where B, D_0, and Ψ are as before) a p-group G_λ of cardinality λ with $p^\omega G_\lambda = C$ and $E(G_\lambda)$ acts on C as Φ while $\operatorname{Aut} G_\lambda$ acts on C as $U(\Phi)$. Then, as proved by Corner [3], the group G_λ will be fully transitive but not transitive. The existence of a proper class of values λ follows since the cardinalities of the Ψ and D_0 used are fixed.

Let $\mathscr{C}$ denote the class of cardinals λ which satisfy the conditions in the above proof. Then as observed above we have:

COROLLARY 3.4. *For each $\lambda \in \mathscr{C}$ there is an essentially-rigid family of 2^λ fully transitive p-groups none of which is transitive.*

We can also use Corner's construction in [3, Sect. 4] to get:

COROLLARY 3.5. *There exists a proper class of transitive 2-groups none of which is fully transitive.*

Proof. Let $C = \langle a \rangle \oplus \langle b \rangle$, where a, b are of orders 2 and 8, respectively. Let Φ be the subring of $E(C)$ generated by Aut C. Then, exactly as in the proof of Corollary 3.3, if λ is sufficiently large, there exists a 2-group G_λ with $2^\omega G_\lambda = C$ such that $E(G_\lambda)$ acts on C as Φ and Aut G_λ acts on C as $U(\Phi)$. It follows from Corner's argument [3, Sect. 4] that each G_λ is transitive but not fully transitive.

It is clear that a result analogous to Corollary 3.4 will hold with transitivity and full transitivity interchanged.

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