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# Bounces and the calculation of quantum tunneling effects for the asymmetric double-well potential

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**Abstract**

The imaginary part of the energy of the metastable state for the asymmetric double-well potential is calculated by using the path-integral method. The tunneling process is dominated by bounces. 

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## 1. Introduction

Quantum effects on the macroscopic scale have attracted considerable attention in recent years owing mainly to the development of technology in mesoscopic physics. The most intriguing quantum effect which could take place on the macroscopic scale is quantum tunneling [1,2]. The instanton method is by now well known as a powerful tool for dealing with quantum tunneling phenomena. The splitting $\Delta E$ of asymptotically degenerate energy levels of the double-well potential can be obtained with instanton methods [3,4]. The splitting results from the tunneling between two neighboring vacua. On the other hand, in a study of the decay of metastable states by quantum tunneling, Coleman and Callan [5,6] introduced the concept of the bounce, and they showed that the quantum tunneling process is dominated by a bounce in this case. In particular Coleman and Callan pointed out that the bounce is not a minimum of action but a saddle point and accordingly the second variation of the action at the bounce leads to a divergent Gaussian integral as a result of an associated negative eigenvalue. The purpose of this paper is to investigate the quantum tunneling effects for the asymmetric double-well potential. In the double-well potential (or called symmetric double-well potential) two minima correspond to two degenerate vacua, while in asymmetric double-well potential one of two minima corresponds to metastable state (cf. Ref. [7] for a spin system). The finiteness of the barrier between the two minima leads to the quantum decay of the metastable state. The process of the decay is expected to be dominated by bounces and to be

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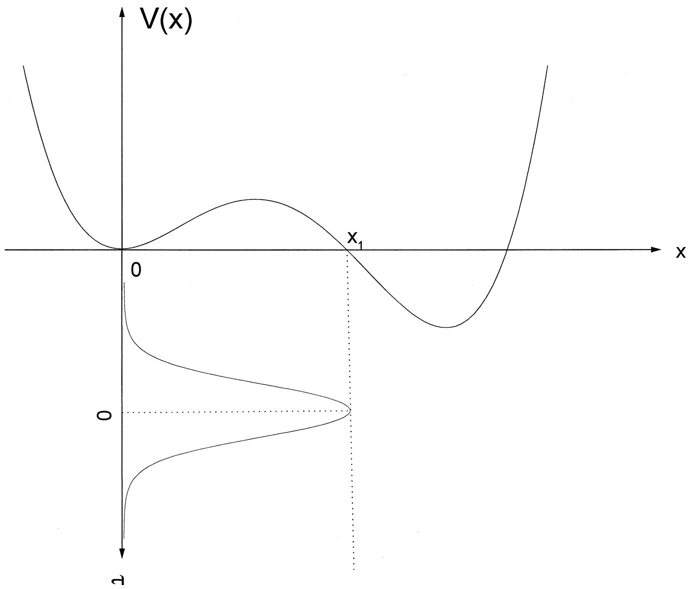


Fig. 1. The asymmetric double-well potential and the bounce solution.

described formally by a complex energy. In order to calculate the imaginary part of an energy eigenvalue, we adopt the path-integral method. The level splitting of the (symmetric) double-well potential is obtained by expanding the action about instanton configuration and taking into account instanton-anti-instanton contributions, correspondingly in this Letter the imaginary part of the eigenvalue for the asymmetric double-well potential is obtained by expanding the action about the bounce configuration and taking into account all bounce configurations. The imaginary part of the energy of the metastable ground state results from the antisymmetry of the first derivative of the bounce solution under time reversal, in agreement with Coleman's argument [8].

## 2. Bounces for the asymmetric double-well potential

We consider a scalar field $x(t)$ in one time and zero space dimensions. The Lagrangian is

$$L=\frac{1}{2}\left[\frac{dx}{dt}\right]^2-V(x) \tag{1}$$

(mass $m_0=1$ and $\hbar=1$ being used throughout), where

$$V(x)=\tfrac{1}{2}Ax^2-\tfrac{1}{3}Bx^3+\tfrac{1}{4}Cx^4 \tag{2}$$

is a double-well potential. If $B^2=\frac{9}{2}AC$, the $V(x)$ is a symmetric potential with double minima, while for $B^2\neq\frac{9}{2}AC$, $V(x)$ is an asymmetric potential with double minima. The tunneling effects for the symmetric double-well potential have been investigated numerously [3,4,9–11]. We are interested in the asymmetric double-well potential. In this Letter, we choose in the asymmetric case, $B^2>\frac{9}{2}AC$ (see Fig. 1). The classical solution, which minimizes the action with Euclidean time $\tau=it$, satisfies the equation

$$\frac{1}{2}\left[\frac{dx_c}{d\tau}\right]^2-V(x_c(\tau))=0 \tag{3}$$

with zero energy at $x=0$ is

$$x_c(\tau)=\frac{3A}{B\left(1+\xi\cosh\sqrt{A}\,\tau\right)} \tag{4}$$

with

$$\frac{dx_c(\tau)}{d\tau} = -\frac{3A^{3/2}\xi\sinh\sqrt{A}\,\tau}{B\left(1+\xi\cosh\sqrt{A}\,\tau\right)^2} \tag{5}$$

where $\xi = \sqrt{1-9AC/2B^2}$. It is seen that $x(\tau), \dot{x}(\tau)$ have the characteristic properties of a bounce as enunciated by Coleman [8]:

$$x_c|_{\tau\to\pm\infty} = 0, \quad x_c|_{\tau\to 0} = \text{const.},$$

$$x_{\dot{c}}|_{\tau\to\pm\infty} = 0, \quad x_{\dot{c}}|_{\tau\to 0} = 0 \tag{6}$$

The solution (4) shown in Fig. 1 has an obvious interpretation: the classical equilibrium point $x=0$ can only be reached asymptotically, as $\tau$ approaches plus or minus infinity, and it reaches turning point $x_1 = 2B(1-\xi)/3C$ at $\tau=0$ with a vanishing velocity $dx_c/d\tau|_{\tau=0} = 0$. The motion of the particle for positive $\tau$ is just the time reversal of its motion for negative $\tau$ ; the particle simply bounces off $x_1 = 2B(1-\xi)/3C$ and returns to $x=0$ at $\tau=+\infty$.

The Euclidean action at the bounce is

$$S_E(x_c) = \int_{-\infty}^{+\infty} 2V(x_c)\,d\tau = \frac{6A^{3/2}}{5B^2}\xi^2 F \tag{7}$$

where $F = F(5/2,2;7/2;9AC/(2B^2))$ is a hypergeometric function. What we need to know is the effect induced by the motion of the bounce on the metastable ground state energy. To this end we consider the Feynman kernel. For the ground state dominated by the zero-energy bounce, the kernel reduces to [12]

$$\langle x_f, T | x_i, -T\rangle = \langle x_f | 0\rangle\langle 0 | x_i\rangle e^{-2ET} \tag{8}$$

where $x_i$ and $x_f$ are initial and final positions. Our purpose in the following is to calculate the Feynman kernel and to obtain the 'false' ground-state energy $E$.

## 3. Imaginary part of the energy

Starting from the standard path integral the kernel is defined by

$$\langle x_f, T | x_i, -T\rangle = \int D\{x(\tau)\}\exp(-S_E[x]) \tag{9}$$

Setting $x(\tau) = x_c(\tau) + y(\tau)$, the Euclidean action at the bounce can be factored out so that

$$\langle x_f, T | x_i, -T\rangle = \exp(-S_E[x_c])\int D\{y\} e^{-\delta^2 S_E}$$

$$\equiv \exp(-S_E[x_c]) I \tag{10}$$

where

$$\delta^2 S_E = \int_{-T}^{T} d\tau (yMy) \tag{11}$$

and

$$M = -\frac{1}{2}\frac{d^2}{d\tau^2} + V''(x_c) \tag{12}$$

is the operator of the second variation of the action at the bounce in which we restrict ourselves to the one-loop approximation.

Setting $y(\tau) = \sum C_n \Psi_n(\tau)$, where $\{\Psi_n(\tau)\}$ is the complete set of eigenstates of the operator $M$ i.e.

$$M\Psi_n = E_n \Psi_n \tag{13}$$

The integral in Eq. (10) can be evaluated as

$$I = \left|\frac{D\{y(\tau)\}}{DC_n}\right| \prod_n \left[\frac{\pi}{E_n}\right]^{\frac{1}{2}} \tag{14}$$

It is noted that zero is one of eigenvalues of Eq. (13), corresponding to eigenfunction $dx_c/d\tau$ (the Goldstone mode), i.e. $Mdx_c/d\tau = 0$ [13,14]. Since $dx_c/d\tau$ (see Eq. (5)) has got one node, there must exist a nodeless eigenfunction of $M$ corresponding to an eigenvalue lower than zero. The existence of the one negative eigenvalue makes $I$ of Eq. (10) imaginary [5,6,15]. The zero mode of Eq. (13) can be removed by using the Faddeev–Popov technique [11,12,16]. The contribution of the one bounce considered yields

$$\langle x_f, T | x_i, -T\rangle = 2T\exp\left[-S_E(x_c)\right] \times \left|\frac{D\{y(\tau)\}}{DC_n}\right| \prod_{n\neq 0}\left[\frac{\pi}{E_n}\right]^{\frac{1}{2}} \times \left[\frac{6A^{3/2}}{5B^2}\xi^2 F\right]^{\frac{1}{2}} \tag{15}$$

Using a method used by Dashen et al. [12,17] to calculate directly the path integral $I$ in Eq. (10), we give

$$I=\left[\frac{1}{2\pi}\right]\left[N(T)N(-T)\right]^{-\frac{1}{2}}\left[\int_{-T}^{T}\frac{d\tau}{N^{2}(\tau)}\right]^{-\frac{1}{2}} \tag{16}$$

where $N(\tau)=dx_{c}/d\tau$. Since the characteristic of bounce solution $x_{c}$ is its time-reversal symmetry, the function $N(\tau)$ is antisymmetric under time reversal, and square root of $N(T)N(-T)$ implies that $I$ is imaginary. For $T\to\infty$, we find that

$$I=-i\left(\frac{\sqrt{A}}{2\pi}\right)^{\frac{1}{2}} \tag{17}$$

Instead of the infinite time interval, we now consider a large but finite time $T$. In this case $E_{0}$ is not zero, comparing Eq. (14) and (17) we have

$$I_{0}=-i\left(\frac{\sqrt{A}}{2\pi}\right)^{\frac{1}{2}}\left(\frac{E_{0}}{\pi}\right)^{\frac{1}{2}} \tag{18}$$

$E_{0}$ can be evaluated by a formula proposed in Ref. [4]. For large $T$, we obtain

$$E_{0}=\frac{60A^{2}}{\xi^{5}F}\mathrm{e}^{-2\sqrt{A}T} \tag{19}$$

Substituting $E_{0}$ back into (18), we have the desired result for one bounce contribution to the Feynman kernel, i.e.

$$\langle x_{\mathrm{f}},T|x_{\mathrm{i}},-T\rangle=-i2T\exp\left[-\frac{6A^{3/2}}{5B^{2}}\xi^{2}F\right]\times\frac{6A^{2}}{B\pi\xi^{3/2}}\mathrm{e}^{-\sqrt{A}T} \tag{20}$$

It is noted that, in (10), the classical action $S_{E}(x_{\mathrm{c}})$ is from $-T$ to $T$ and $T$-dependent, while in (20) the action (7) evaluated from $-\infty$ to $+\infty$ is used, which is $T$-independent. The reason is that $S_{E}(x_{\mathrm{c}})$ is evaluated along bounce trajectory (cf. Fig. 1), and the classical equilibrium point $x=0$ can only be reached asymptotically, as $\tau$ approaches plus or minus infinity. It follows that $\pm T\to\pm\infty$ for the classical action $S_{E}(x_{\mathrm{c}})$, so the action (7) is evaluated from $-\infty$ to $+\infty$, and is $T$-independent.

The path-integral calculation require a sum over all possible paths. In our case, the path fall into an infinite number of classes according to the number of bounces (tunneling times) for a given interval, i.e.

$$\langle x_{\mathrm{f}},T|x_{\mathrm{i}},-T\rangle=\sum_{n=0}^{\infty}K_{n}(x_{\mathrm{f}},T;x_{\mathrm{i}},-T) \tag{21}$$

where $K_{n}$ is the kernel of the contribution of $n$ bounces. Using the Faddeev–Popov technique [11,12,16] for each bounce and remembering that only the classical action for bounce depends on the end points, we have

$$K_{n}=(-i)^{n}\frac{(2T)^{n}}{n!}\left(\exp\left[-\frac{6A^{3/2}}{5B^{2}}\xi^{2}F\right]\right)^{n}\times\left(\frac{\pi}{\sqrt{A}}\right)^{\frac{n-1}{2}}\left(\frac{6A^{2}}{B\pi\xi^{3/2}}\right)^{n}\mathrm{e}^{-\sqrt{A}T} \tag{22}$$

The final result for the Feynman kernel is therefore

$$\langle x_{\mathrm{f}},T|x_{\mathrm{i}},-T\rangle=\left(\frac{\sqrt{A}}{\pi}\right)^{\frac{1}{2}}\mathrm{e}^{-\sqrt{A}T}\times\exp\left\{-i2T\frac{6A}{B}\left(\frac{A^{3/2}}{\pi\xi^{3}}\right)^{\frac{1}{2}}\times\exp\left[-\frac{6A^{3/2}}{5B^{2}}\xi^{2}F\right]\right\} \tag{23}$$

Comparing Eq. (23) with Eq. (8), we obtain the imaginary part of the energy as

$$\mathrm{Im}\,E=\frac{6A}{B}\left(\frac{A^{3/2}}{\pi\xi^{3}}\right)^{\frac{1}{2}}\exp\left[-\frac{6A^{3/2}}{5B^{2}}\xi^{2}F\right] \tag{24}$$

In the above, quantum tunneling effect for the asymmetric double-well potential is investigated. Differing from (symmetric) double-well potential, in which the tunneling process is dominated by instanton and leads to the level splitting, for asymmetric double-well potential, bounces are responsible for the quantum tunneling for the decay of a metastable ground state. The imaginary part of the energy is calculated by using the path-integral method, and we show explicitly that the imaginary part of the energy of the metastable ground state results from the antisymmetry of its first derivative of the bounce solution under time reversal. This is in agreement with a general argument given by Coleman [8].

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