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Message survival and decision dynamics in a class of reactive complex systems subject to external fields

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HIGHLIGHTS

- The dynamics of decisions in complex networks with external fields are analyzed.
- Sufficient conditions for opinion extinction are presented.
- Analytic bounds for the effects of exogenous perturbations are established.
- Numerical simulations are presented for a power law degree distribution.

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ABSTRACT

In this study, the dynamics of decisions in complex networks subject to external fields are studied within a Markov process framework using nonlinear dynamical systems theory. A mathematical discrete-time model is derived using a set of basic assumptions regarding the convincement mechanisms associated with two competing opinions. The model is analyzed with respect to the multiplicity of critical points and the stability of extinction states. Sufficient conditions for extinction are derived in terms of the convincement probabilities and the maximum eigenvalues of the associated connectivity matrices. The influences of exogenous (e.g., mass media-based) effects on decision behavior are analyzed qualitatively. The current analysis predicts: (i) the presence of fixed-point multiplicity (with a maximum number of four different fixed points), multi-stability, and sensitivity with respect to the process parameters; and (ii) the bounded but significant impact of exogenous perturbations on the decision behavior. These predictions were verified using a set of numerical simulations based on a scale-free network topology.

1. Introduction

Recently, the problem of modeling, analyzing, and simulating rumor and decision dynamics in complex networks has attracted increasing attention, including studies of rumor spread in social and small-world networks [1], and decision dynamics in scale-free networks [2]. The problem of rumor spreading [1] considers the time evolution of rumor spreaders and stiflers, but the decision dynamics [2] are concerned with the time evolution of different decisions in terms of their associated competitive interplay, which is similar to that observed in biological studies of different predators competing for the same prey, chemical species competing for the same reactant, and a virus spreading via reinfection.

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The main background from related research areas is summarized next, without any claim of completeness.

1.1. Axelrod's cultural dissemination model and its extensions

Axelrod's seminal article on culture dissemination is included among previous studies that have provided inspiration for further research into opinion spreading dynamics [3]. In this study, the author proposed a model for the mechanism of how people become more similar in terms of their interaction and how this convergence stops before reaching completion. In his initial claim, Axelrod stated that agents which can interact and be influenced in this manner share common cultural attributes, whereas those that cannot interact retain their differences. The methodology utilized in Ref. [3] is based on the following three principles: (i) agent-based modeling, (ii) no central authority, and (iii) adaptive rather than rational agents. Axelrod [3] also defined the concept of a degree of cultural similarity among agents and simulated the dynamic process of social influence by repeating the following steps: (i) randomly select a site to be active and choose one of its neighbors, and (ii) these two sites interact with a probability that is equal to their cultural similarity. An interaction comprises the random selection of a feature where the active site and its neighbor differ (if there is a difference) and changing the active site's trait for this feature into the neighbor's trait for this feature.

It was observed in Ref. [3] that other factors such as the range of interaction affected the number of stable regions and this effect attracted our attention in a previous study [4,5], where we considered the emergence of nontrivial collective behavior. This study considered a lattice of locally interacting elements with an absorbing order–disorder phase transition and the behavior of this system under a field influence, which was based on the model of cultural dissemination proposed by Axelrod in Ref. [3]. In Ref. [4], exogenous and endogenous effects were integrated into the model proposed in Ref. [3] to produce a unified framework. In this framework, the agents can interact with their neighbors in the system and via mass media according to their cultural similarities, where their cultural attributes can be changed by the influence of culturally compatible neighbors as well as by their exposure to mass media. For further details see Refs. [4,5].

An attribute of the system is that it exhibits a phase because the system can be varied externally, e.g., its amplitude and frequency. We assume that B is uniform, i.e., the mass B. If this is the case, then the system can be estimated as $q_0 \approx 55$ in two dimensions [6] with a message intensity B_c and an order parameter g, where $g(B, q) \sim [B - k_2(\mathrm{e}^{-k_1 q} - \mathrm{e}^{-k_1 q_0})^{k_3(q_0 - q)}]$.

1.2. Sociophysics and voter models

Another phenomenon related to spreading over networks that has attracted the interest of many researchers in the last decade is the propagation of political opinions in a social network [7–11]. In Ref. [7], the authors presented the results of a dual model of opinion networks, which complements agent-based opinion models by attaching a social agent (voters) network to a political opinion (party) network, which has its own intrinsic mechanisms of evolution. The evolution of the voter network involves adding and deleting links, where the changes in opinions are governed by social influences, political climate, attraction to a particular party, and interactions. For further details, see Ref. [7].

Another very interesting approach to the mathematical modeling of political opinions in a social network is that proposed in Ref. [8]. In Ref. [8], a modified version of a finite random field Ising ferromagnetic model in an external magnetic field at zero temperature was proposed for describing group decision making, where the fields could have a non-zero average. It was assumed that inter-individual conflicts were minimal. In these conditions, interactions led to group polarization. For further details, see Refs. [8–11].

1.3. Virus spreading in complex networks

The most notable differences in terms of these classic fields of competition dynamics are the social system dynamics in the topologies of the underlying contact networks. For example, social phenomena may involve different personalities where each has a different number of contacts, thus the dynamics have an intrinsic distributed character, where the emergence of nearly homogeneous groups (so-called clusters) is possible. These differences mean that it is necessary to analyze social behavior from the viewpoint of dynamic networks. It is particularly noteworthy that that the underlying network topology, unless it changes over time, always fulfills the same characteristics in terms of its node degree distribution (e.g., see Refs. [12–14]). Formal mathematical models can be developed and analyzed to understand the mechanisms that underlie the dynamic phenomena that occur in dynamic networks, as well as using numerical simulation studies.

In Ref. [2], the regularity of the spread of information and public opinions toward two competing products was analyzed in complex networks, with a particular emphasis on the dynamics of decision competition in scale-free networks. A simple linear model was proposed and simulated over time, which showed that in contrast to most previous models based on modified SIS and SIR models [15–23], various types of information frequently spread simultaneously (different viruses, multiple opinions, rumors, etc.) in real-life settings and these may be mutually strengthened or even annihilated, unlike the dynamics of single opinion spread. The competition-based dynamics of two opinions that flowed freely in a complex network were studied and some interesting characteristics of the behavior of decision competition were determined.

However, studying the underlying mechanisms of the mutual influence between nodes in a network is expected to lead to nonlinear dynamics, as found in the models based on SIS and SIR mentioned above. Thus, it may be assumed that inherent competition mechanisms lead to classical nonlinear phenomena such as multiple attractors and parameter sensitivity.

1.4. Similarities and differences compared with our model

The model proposed in the present study is similar to the dynamical systems models proposed in Refs. [19,23], where the phenomena we address are similar to those treated in Refs. [7,8] and Refs. [4,5]. A difference between our model and that presented in Ref. [4] is that the latter was based on the model one proposed in Ref. [3] whereas ours was not. Another aspect that makes different our model from those based on Axelrod's model is that we consider scale-free probability degree distributions whereas the studies reported in Refs. [3–5] considered lattice topologies. This is an important difference because we are interested in testing the impact of the network topology as well as the type of contact (contact process, reactive process, or intermediate cases [22]) on the prevalence of one opinion compared with another in the presence or absence of exogenous messages. Another difference between our model and those presented in Refs. [4,5] is that we model the exogenous influence of the mass media messages as an event that is independent of the interaction messages between agents, whereas previous studies modeled a parameter B that represented the interaction intensity. The final difference between our model and that in Ref. [4] is that we consider each agent as a Markov chain with three states. The model in Ref. [4] is strongly related to that proposed in Ref. [3], which was used in Ref. [4] to determine the phase transition from pluriculturalism to monoculturalism in terms of parameters such as the number of cultural features, cultural traits, and interaction intensity. Our model aims to study phase transitions in terms of parameters such as the eigenvalues of the adjacency matrix of the system or the type of contact (which can be viewed as an interaction intensity) to analyze the stability of the system.

A major difference between our model and that used in Refs. [8–11] is that we do not employ a ferromagnetic model. Our model is very similar to that proposed in Ref. [2], except our model is not linear. This is an important difference in terms of the stability analysis performed using our model.

In the present study, we introduce a mathematical model and analyze the dynamics of two competing opinions (with a neutral intermediate state), which explicitly account for the type of nonlinear interactions that are inherent aspects of the dynamics of competing opinions in complex social networks. A generalized model of the interaction mechanisms is employed, which was proposed recently in Ref. [22], and we consider intermediate cases between the classical contact and reactive process, where the influence of exogenous mass media opinion propagation in the network is considered explicitly. The mathematical model is analyzed with respect to attractor multiplicity, thereby delimiting some inherent behavioral characteristics of the system. In particular, we focus on the effects of opinion adaptation through: (i) contacts and (ii) exogenous (e.g., mass media) influences. To analyze these effects, sufficient conditions for opinion extinction are derived in terms of the maximum eigenvalue of the connectivity matrix and the state transition probabilities. A series of numerical simulations is provided for the two extreme cases of contact and the reactive process [22], thereby illustrating the theoretically predicted behavior for static and dynamic contacts, as well as the presence of exogenous mass media propagation for one or both opinions. The results obtained illustrate: (i) attractor multiplicity, (ii) sensitivity to process parameters, and (iii) the strong (but bounded) impact of exogenous perturbations.

From a methodological viewpoint, we combine the quantitative and qualitative potentials of mathematical modeling and simulation with some basic concepts of dynamical systems theory related to attractor multiplicity and stability.

This paper is organized as follows. In Section 2, we derive the nonlinear opinion dynamics model for dynamic networks based on a set of basic assumptions. In Section 3, the fixed points are established for the corresponding static and unperturbed versions. In addition, the sufficient conditions for opinion extinction are derived and the qualitative bounds for the effects of exogenous influences are established in terms of the state transition probabilities and the spectrum of the connectivity matrix. In Section 4, numerical simulations are presented to illustrate the theoretical predictions in Section 3. In Section 5, we discuss the possible impacts of this model in various application areas. In Section 6, the main contributions of this study are summarized.

2. Decision dynamics model

In this section, we derive a mathematical model of the prevalence of two messages (*opinions, etc.*) with the associated states $\mathcal{A}$ and $\mathcal{B}$, and neutral intermediate state $\mathcal{N}$ based on the Markov process assumption [24], such that the state at any time instant $t \geq 1$ ($t \in \mathbb{N}$) depends exclusively on the state at the immediately preceding time instant $t-1$. This assumption requires the presence of reactive agents in the network that only act in response to external stimuli and their actual state, i.e., memory has no effect on their decisions. The result is a discrete time $2N$-dimensional nonlinear model, where N is the number of nodes in the network.

2.1. Basic assumptions and nomenclature

To derive the model, we introduce the following basic assumptions:

(A1) Two messages $\mathcal{A}$ and $\mathcal{B}$ are propagated in a social network with N nodes according to a power law or binomial node (i) degree (k_i) distribution [25]

$$P[k_i] \sim k_i^{-\gamma}, \quad \text{or} \quad P[k_i] \sim \binom{N-1}{k} p^k (1-p)^{N-1-k} \tag{1}$$

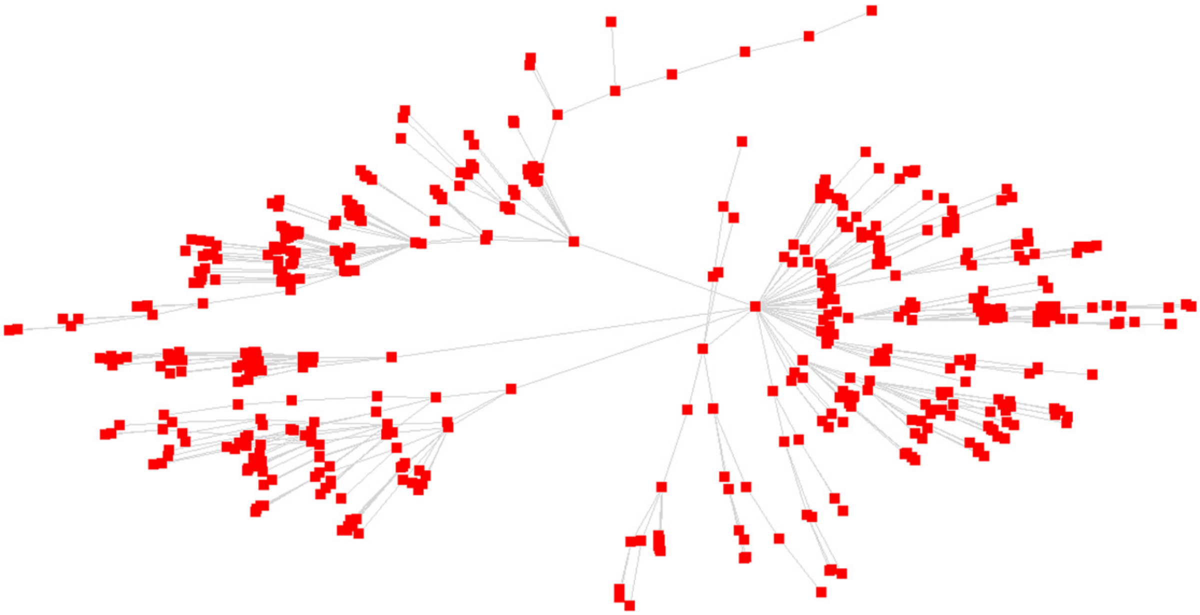

Fig. 1. Network distributed according to a power law degree with $N = 500$ nodes and $P[k] \sim k^{-2.7}$.

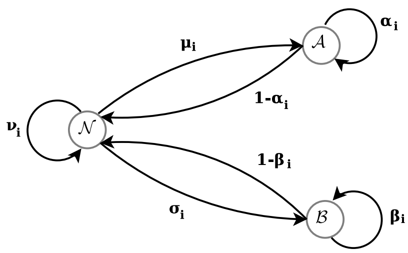

Fig. 2. State transition diagram.

(see Fig. 1 for the power law distribution with $N = 100$ nodes). The corresponding network topology is reflected in the square adjacency matrix $A(t)$ with time-varying entries $A_{ij}(t) \in \{0, 1\}$.

(A2) The total number of nodes N is constant.

(A3) Any node i can be in one of three states: $\mathcal{A}$ if it has message $\mathcal{A}$, $\mathcal{B}$ if it has message $\mathcal{B}$, and $\mathcal{N}$ if it is neutral with respect to messages $\mathcal{A}$ and $\mathcal{B}$.

(A4) To change from message $\mathcal{A}$ to $\mathcal{B}$ (or *vice versa*), the node has to pass through the neutral state $\mathcal{N}$, i.e., there is no direct connection between states $\mathcal{A}$ and $\mathcal{B}$. The associated state diagram is shown in Fig. 2.

(A5) The probabilities that a node i is in states $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{N}$ are denoted by a_i, b_i, and n_i, respectively. It holds that $a_i, b_i, n_i \in [0, 1]$ for any $i = 1, \ldots, N$.

(A6) If node i has message $\mathcal{A}$ (or $\mathcal{B}$), it will convince message $\mathcal{A}$ (or $\mathcal{B}$) via its connected nodes with probability κ_A (or κ_B).

(A7) A neutral node i (i.e., a node in state $\mathcal{N}$) does not convince any neighboring nodes.

(A8) The impact of any neighbor node $j \neq i$ on node i is independent of j.

(A9) The influences of any two nodes $j \neq k \neq i$ on node i are mutually independent. The total effect of all neighbors of node i is the mean influence.

(A10) Opinion $\mathcal{A}$ (or $\mathcal{B}$) interchanges between two nodes i and j a total of λ_A (or λ_B) times during each time step [22].

(A11) The influence of an exogenous signal can be modeled in terms of a perturbation in the period of appearance T, while considering the underlying automata transition possibilities (see assumption (A4) and Fig. 2).

A direct consequence of assumptions (A2) and (A4) is that

$$a_i(t) + b_i(t) + n_i(t) = 1. \tag{2}$$

Assumption (A8) is motivated by the fact that any node j has the same impact on i (assumption (A7)) and the final state of node i after contacting its neighbors will be the weighted sum of all the specific contacts. Assumption (A9) is associated with the type of interchange mechanism [22]. Two extreme cases have been reported previously: the contact process (with

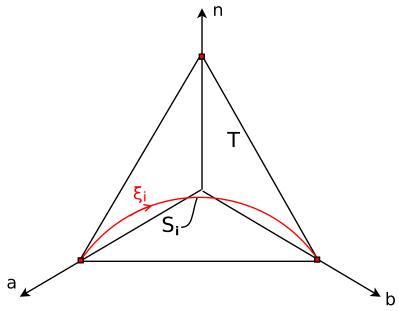

Fig. 3. Outline of the triangle set T (5) where the decision probabilities of each node i and the mean probabilities ρ evolve, and the curve S is parameterized by the scalar ξ (18).

$\lambda_k = 1,\ k = A, B$) and the reactive process (with $\lambda_k \to \infty,\ k = A, B$). The probability that one of λ intended connections is successful is given by the binomial distribution equation (cp. Ref. [22])

$$r_{ij}^{A}(t) = 1 - \left(1 - \frac{A_{i,j}(t)}{N_i(t)}\right)^{\lambda_A}, \qquad r_{ij}^{B}(t) = 1 - \left(1 - \frac{A_{i,j}(t)}{N_i(t)}\right)^{\lambda_B} \tag{3}$$

for messages A and B, respectively, where $N_i(t)$ is the total number of neighbors of node i at time t, i.e.,

$$N_i(t) = \sum_{j \neq i} A_{i,j}(t). \tag{4}$$

Note that for $A_{i,j} = 0$ (i.e., nodes i and j are not connected), the connectivity is also $r_{ij} = 0$. For the reactive process ($\lambda_k \to \infty,\ k = A, B$) $r_{ij}^{k} = 1,\ k = A, B$. The transition probabilities associated with the process are denoted as follows.

- If node i is in states $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{N}$, the probabilities of node i remaining in $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{N}$ are denoted by α_i, β_i, or ν_i, respectively.
- If node i is in the neutral state $\mathcal{N}$, the probability of being convinced by message $\mathcal{A}$ (or $\mathcal{B}$) is denoted by μ_i (or σ_i).

2.2. Markov process model

In this subsection, we present the Markov process model. A detailed derivation of the model based on the above assumptions is presented in Appendix A.

The algebraic constraint $1 = a_i + b_i + n_i$ restricts the state space to a two-dimensional linear triangle manifold $T \subset \mathbb{R}^{3N}$ (see Fig. 3)

$$2T = \{z = [z_1, z_2, z_3]' \in [0, 1]^3 \subset \mathbb{R}^3 \mid z_1 + z_2, +z_3 = 1\}. \tag{5}$$

The dynamics of the trajectories on this manifold T are given by

$$\begin{aligned} a_i(t+1) &= \alpha_i(t)a_i(t) + [1 - \beta_i(t)][1 - a_i(t) - b_i(t)], \qquad a_i(0) = a_{i0} \\ b_i(t+1) &= \beta_i(t)b_i(t) + [1 - \alpha_i(t)][1 - a_i(t) - b_i(t)], \qquad b_i(0) = b_{i0} \\ n_i(t) &= 1 - a_i(t) - b_i(t) \end{aligned} \tag{6}$$

with

$$\alpha_i(t) = \frac{1}{N_i(t)} \sum_{j \neq i} [A_{i,j}(t) - r_{ij}^{B} \kappa_B b_j(t)], \tag{7}$$

$$\beta_i(t) = \frac{1}{N_i(t)} \sum_{j \neq i} [A_{i,j}(t) - r_{ij}^{A}(t) \kappa_A a_j(t)]. \tag{8}$$

In vector notation, the preceding dynamics are written as

$$x(t+1) = f[x(t)], \qquad x(0) = x_0, \qquad x = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad a = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}. \tag{9}$$

We introduce the mean decision probabilities

$$\rho_a = \frac{1}{N} \sum_{i=1}^{\mathcal{N}} a_i, \qquad \rho_b = \frac{1}{N} \sum_{i=1}^{\mathcal{N}} b_i, \qquad \rho_n = \frac{1}{N} \sum_{i=1}^{\mathcal{N}} n_i = 1 - \rho_a - \rho_b, \tag{10}$$

and the associated mean probability vector $\rho(t) = [\rho_a(t), \rho_b(t), \rho_n(t)]'$. It is clear that

$$\rho(t) \in T, \quad \forall t \geq 0 \tag{11}$$

where T is the triangle set defined in (5).

Exogenous fields that act on some of the states of the nodes in the network, such as external signals, mass-media mechanisms, publicity, cultural pressure, and social pressure, are modeled as (time-varying) perturbations δ_{Ai}, δ_{Bi} on each node, which lead to the following non-autonomous dynamics

$$\begin{aligned} a_i(t+1) &= \alpha_i(t)a_i(t) + [1-\beta_i(t)][1-a_i(t)-b_i(t)] + \delta_{Ai}(t) - \delta_{Bi}(t) \\ b_i(t+1) &= \beta_i(t)b_i(t) + [1-\alpha_i(t)][1-a_i(t)-b_i(t)] - \delta_{Ai}(t) + \delta_{Bi}(t) \end{aligned} \tag{12}$$

where the amplitudes of δ_{Ai} and δ_{Bi} are bounded by the constants Δ_A, Δ_B, respectively, i.e.,

$$\sup_{t\geq 0} \|\delta_{ki}\| \leq \Delta_k, \quad k = A, B \tag{13}$$

where $\Delta_k,\ k = A, B$, is sufficiently small that $0 \leq a_i, b_i \leq 1$ at any time instant t.

3. System dynamics

In this section, the dynamical behavior of the decision dynamics model (A.6) (or equivalently Eq. (6)) is analyzed with respect to fixed point multiplicity and parameter dependency, as well as the stability of the extinction states. The effects of perturbations on the dynamics are studied by deriving the analytic bounds for the trajectory deviations from the unperturbed behavior.

3.1. Fixed point multiplicity

Assume that the connections between nodes do not vary over time (i.e., the adjacency matrix A is constant, thus N_i is constant for any node i). Based on this assumption, the fixed points associated with the dynamics (A.6) can be determined to establish limit case conditions for the time evolution of opinions $\mathcal{A}$ and $\mathcal{B}$ in the network.

The fixed point conditions are

$$a_i(t+1) = a_i(t) = a_i, \qquad b_i(t+1) = b_i(t) = b_i, \qquad n_i(t+1) = n_i(t) = n_i \tag{14}$$

where a_i, b_i, n_i are the fixed point values of the stochastic variables $a_i(t)$, $b_i(t)$, and n_i, respectively. It follows that

$$\begin{aligned} 0 &= (\alpha_i - 1)a_i + (1-\beta_i)n_i, \\ 0 &= (\beta_i - 1)b_i + (1-\alpha_i)n_i, \\ 0 &= -(1-\alpha_i)a_i - (1-\beta_i)b_i + (2-\alpha_i-\beta_i)n_i. \end{aligned} \tag{15}$$

Given that for $a_i = 0$ (or $b_i = 0$) for all i it holds that $\beta_i = 1$ (or $\alpha_i = 1$), it follows that the three vertices

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

of the triangle set T (5) (Fig. 3) are particular fixed points. Furthermore, solving the equation set (15) in general with respect to a_i, b_i, and n_i yields the following set of implicit solutions

$$a_i = \frac{1}{\xi_i^2 + \xi_i + 1}, \qquad b_i = \frac{\xi_i^2}{\xi_i^2 + \xi_i + 1}, \qquad n_i = 1 - a_i - b_i = \frac{\xi_i}{\xi_i^2 + \xi_i + 1}, \quad i = 1, \ldots, N, \tag{17}$$

which are parameterized by the scalar

$$\xi_i = \frac{1-\alpha_i}{1-\beta_i} \in \mathbb{R}, \qquad \alpha_i = \frac{1}{N_i}\sum_{j\neq i}(1-\kappa_b b_j), \qquad \beta_i = \frac{1}{N_i}\sum_{j\neq i}(1-\kappa_a a_j). \tag{18}$$

By introducing the solution in vector notation

$$s_i = [a_i, b_i, n_i]' \in T, \tag{19}$$

the following limit cases can be derived directly:

$$\kappa_A \to 0, \Rightarrow \xi_i \to \infty \quad \text{and} \quad s_i \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \kappa_B \to 0 \Rightarrow \xi_i \to 0 \quad \text{and} \quad s_i \to \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{20}$$

which correspond to the two bottom vertices of the triangle set T (5) (Fig. 3). In terms of the opinion density ρ (11), four fixed points are associated:

$$\rho \in \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{N} \begin{bmatrix} \sum_{i=1}^{N} a_i \\ \sum_{i=1}^{N} b_i \\ \sum_{i=1}^{N} n_i \end{bmatrix} \right\}, \tag{21}$$

where a_i, b_i, and n_i are given in (17). Naturally, the limit cases (20) can be expressed in terms of the mean probability solution s_ρ as

$$\kappa_A \to 0, \Rightarrow s_\rho \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \kappa_B \to 0 \Rightarrow s_\rho \to \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad s_\rho = \frac{1}{N} \sum_{j \neq i} s_i. \tag{22}$$

In terms of the vector x (9), the extinction states are given by

$$x_{e1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad x_{e1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad x_{e1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{23}$$

In summary, there are three extinction fixed points and there may be additional intermediate fixed points. Next, the stability of the extinction states is analyzed in terms of the probabilities κ_A, κ_B and the maximum eigenvalues of the connectivity matrices R_A and R_B.

3.2. Extinction conditions

To delimit the basic behavior of the decision dynamics in complex networks based on our proposed basic Markov model (6), it is sufficient to establish conditions for the extreme cases, which are determined by the extinction of: (i) both messages, (ii) message A, or (iii) message B. In the present section, a simple dynamical *boundary layer* system for the mean probabilities is presented for this purpose, where the states z_a and z_b are such that if $z_k \to 0$ ($k = A$ or B) it also holds that $\rho_k \to 0$ (and thus $a_k, b_k \to 0$ for all $i = 1, \ldots, N$). The local stability of these boundary layer dynamics are studied and sufficient conditions are stated for the extinction of a message.

By summing up over every node i for the equations of messages A and B (6), then dividing them by the number of nodes in the network N, and applying some bounds that stem from the probabilistic nature of states a_i, b_i, the following *boundary layer* dynamics are obtained (see the derivation in Appendix B)

$$\begin{bmatrix} z_a(t+1) \\ z_b(t+1) \end{bmatrix} = \begin{bmatrix} \left(1 + \kappa_A \lambda^+(R_A)\right) z_a(t) - \kappa_B \lambda^+(R_B) z_a(t) z_b(t) - \kappa_A \lambda^+(R_A) z_a(t)(z_a(t) + z_b(t)) \\ \left(1 + \kappa_B \lambda^+(R_B)\right) z_b(t) - \kappa_A \lambda^+(R_A) z_a(t) z_b(t) - \kappa_B \lambda^+(R_B) z_b(t)(z_a(t) + z_b(t)) \end{bmatrix}. \tag{24}$$

The states z_a and z_b are such that if $z_k \to 0$, the corresponding mean probability is also $\rho_k \to 0$ (see Appendix B). Based on this fact, the following main result is obtained for the stability properties of the mean probabilities ρ_a and ρ_b.

Theorem 1 (*Proof in* Appendix B)**.** *Consider the dynamics* (6) *with the (boundary layer) dynamics for the mean probabilities* (24)*. Let $\lambda^+(R_A)$, $\lambda^+(R_B)$ denote the maximum eigenvalue of the connectivity matrices R_A and R_B, respectively. The following results hold:*

(I) *The extinction fixed point $x_{e1} = (0', 0')'$ is a repulsor node for any value of κ_A and κ_B.*

(II) *The extinction state $x_{e2} = (1', 0')'$ is a local attractor if*

$$\kappa_A > \kappa_{Ac} = \frac{1 + \kappa_B \lambda^+(R_B)}{\lambda^+(R_A)} \tag{25}$$

where κ_{Ac} denotes the threshold value for κ_A.

(III) *The extinction state $x_{e3} = (0, 1)'$ is a local attractor if*

$$\kappa_B > \kappa_{Bc} = \frac{1 + \kappa_A \lambda^+(R_A)}{\lambda^+(R_B)} \tag{26}$$

where κ_{Bc} denotes the threshold value for κ_B.

This result generalizes those presented in previous studies of a virus spreading in complex networks with a single species (or disease) [20,22] to the case with two species, which is associated with the dynamics of decisions between two mutually excluding messages. Condition (25) states that in order to achieve the extinction of one message, there is an intrinsic tradeoff between the connectivity, which is reflected in the associated maximum eigenvalues λ^+, and the infection probabilities κ. In particular, even if $\kappa_B > \kappa_A$, the state B may converge to extinction if the nodes in state A have a higher connectivity, like those in state B. Effectively, the stability can be reformulated in terms of the connectivity as follows

$$\lambda^+(R_A) > \lambda_A^c = \frac{1 + \kappa_B \lambda^+(R_B)}{\kappa_A}, \tag{27}$$

which means that for a given maximum eigenvalue $\lambda^+(R_B)$ of connectivity matrix R_B and given infection probabilities κ_A and κ_B, the extinction state x_{e2} is a local attractor if the maximum eigenvalue of connectivity matrix R_A is sufficiently large. In the same manner, the condition (27) can be reformulated in terms of a critical maximum eigenvalue for connectivity matrix R_B.

It should be noted that the statement in Theorem 1 does not necessarily rely on static interconnections because the the maximum eigenvalues can be replaced by the upper bounds for the given network scale-free topology (e.g., see Ref. [26]).

3.3. Bounding the influence of exogenous perturbations

The extinction conditions presented in Theorem 1 ensure the existence of a constant rate $\lambda_2 < 1$ (or $\lambda_3 < 1$) at which the deviation dynamics will converge to the fixed point x_{e2}, or x_{e3} in the absence of exogenous perturbations

$$\|x(k) - x_{ei}\| \le \lambda_i^k \|x_0 - x_{ei}\|, \quad i = 2 \text{ or } 3. \tag{28}$$

In the presence of exogenous perturbations, their effect will be bounded in the following sense

$$\|x(k) - x_{ei}\| \le \lambda_i^k \left(\|x_0 - x_{ei}\| + \sum_{l=1}^{k-1} \frac{\lambda_i^l}{\lambda_i^k} \|\delta(l-1)\| \right), \quad i = 2 \text{ or } 3, \tag{29}$$

which implies that the initial deviations and exogenous perturbations are attenuated over time. In particular, if at any time instant, the exogenous perturbations vanish, their effects will disappear after sufficient time. It should be noted that realistic perturbations do not exceed a critical norm value, which is determined by the requirement that $x \in [0, 1]^{2N}$, thus (29) makes sense.

The inequality (29) states that bounded perturbations produce bounded deviations from a fixed point. As a consequence, if the extinction condition (25) (or Eq. (26)) is satisfied, the trajectories converge to a bounded set about the extinction fixed point, the size of which is determined by the perturbation amplitude.

4. Simulation results

In this section, the theoretical results derived in the preceding section are verified and illustrated using numerical simulation studies. Thus, the following different scenarios are considered.

- Scale-free dynamic network without external fields.
- Scale-free dynamic network where the exogenous signal of message $\mathcal{A}$ influences 18% of the population with a period of $T_A = 8$ time units.
- Scale-free dynamic network where the exogenous signal of both messages influences 10% of the population with periods of $T_A = 8$ time units and $T_B = 16$ time units.

The simulations were performed using a total population size of $N = 10{,}000$ nodes and a power law node distribution degree of k (1) with

$$\gamma = 2.16. \tag{30}$$

4.1. Dynamics without external fields

To illustrate the main (nonlinear) behavior of the decision dynamics process according to the fixed point multiplicity discussed earlier, two limit cases where $\lambda_k = 1, k = A, B$ (contact process) and $\lambda_k \to \infty, k = A, B$ (reactive process) are illustrated for three different parameter scenarios. In particular, the projection of the trajectories in the triangle set T (5) onto the (ρ_a, ρ_b)-plane is presented. Note that it is possible that the trajectories may intersect in the projection.

We consider the following three parameter sets:

$$\kappa_A = 0.1, \quad \kappa_B = 0.9\,(\kappa_A < \kappa_B), \quad \kappa_A = \kappa_B = 0.5\,(\kappa_A = \kappa_B), \quad \kappa_A = 0.9, \quad \kappa_B = 0.1\,(\kappa_A > \kappa_B) \tag{31}$$

and $\mathcal{A}$ and $\mathcal{B}$ are propagated via contact processes (i.e., $\lambda_A = \lambda_B = 1$). The corresponding simulation results are presented in Fig. 4.

In Fig. 4, it can be seen that:

(i) for the first parameter set ($\kappa_A < \kappa_B$), all the trajectories move toward an attractor in the upper left corner of the (ρ_A, ρ_B)-plane.

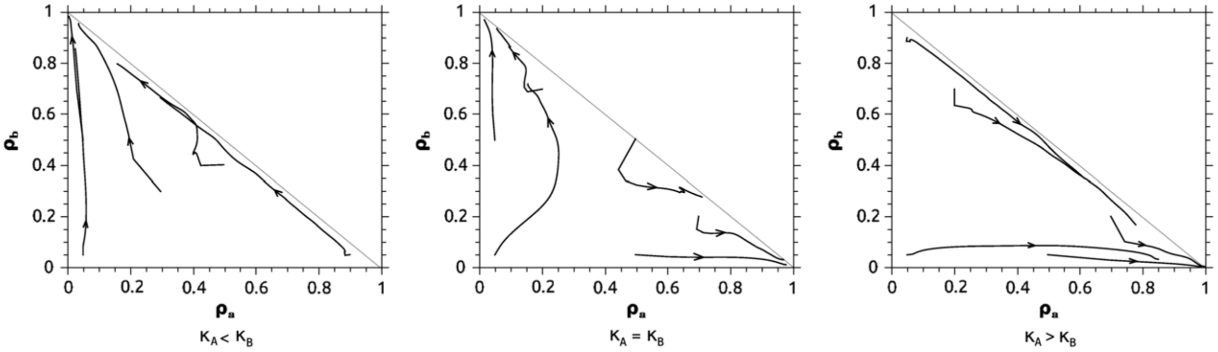

Fig. 4. Projection onto the (ρ_a, ρ_b)-plane of the trajectories of the mean decision probability ρ (11) for the three different parameter sets given in (31) and a contact transmission process.

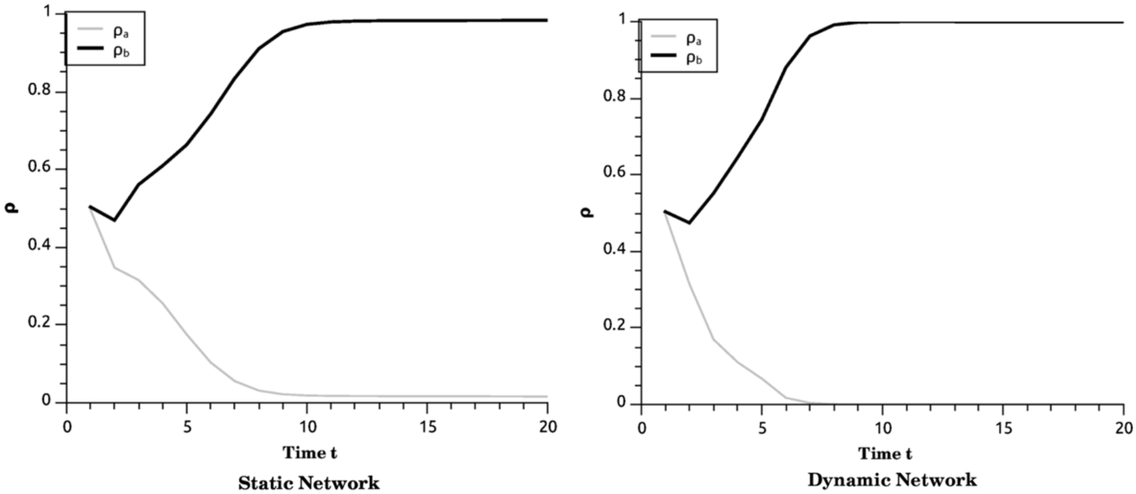

Fig. 5. Time evolution of the mean decision trajectories associated with decisions $\mathcal{A}$ (thin gray line) and $\mathcal{B}$ (thick black line) for the parameter values and initial conditions set (33) with a reactive transmission process in a scale-free dynamic network.

(ii) for the second parameter set ($\kappa_A = \kappa_B$), there are two attractors (one in the upper left corner and one in the lower right corner) toward which the decision trajectory ρ may converge, depending on the initial value.
(iii) for the third parameter set ($\kappa_A > \kappa_B$), all the trajectories move toward an attractor in the lower right corner of the (ρ_A, ρ_B)-plane.

According to these results, there is a strong sensitivity in terms of the type of structural instability [27] of the decision behavior in the network that depends on the convincing parameter pair (κ_A, κ_B), which manifests itself mainly in the passage from the first to the third parameter set in (31). Initially, the attractor in the lower right corner is unique and it coexists with a second attractor in the upper left corner, but the former finally disappears leaving only the attractor in the upper left corner. Thus, for some parameter combinations, the final decision state will depend strongly on the initial conditions.

It should be noted that the dynamics become very slow close to the fixed-point, which corresponds to the fact that almost the whole network has a single opinion, so the existing contacts only confirm the present decision state. This behavior is predicted by the model (6), given that close to the fixed point, the functions $\alpha_i, \beta_i \approx 1$ for any node i, so the system attains a slowly varying (i.e., quasi steady-state) behavior according to

$$a_i(t+1) \approx a_i(t), \qquad b_i(t+1) \approx b_i(t), \qquad n_i(t+1) \approx n_i(t). \tag{32}$$

We conjecture that this fact is also related to the generation of decision clusters within the network.

To compare these results for a static network with those for a dynamically changing network topology, the time evolution associated with both cases is compared in Fig. 5, with the parameter values and the initial condition set

$$\kappa_A = 0.1, \qquad \kappa_B = 0.9, \qquad \rho_a(0) = \rho_b(0) = 0.5, \qquad \rho_n(0) = 0 \tag{33}$$

and a reactive process (i.e., $r_{ij}^k = 1, k = A, B$). It can be seen that the overall behavior is quite similar, but the important difference is that the trajectories are smoother in the dynamic network and they converge within about nine time steps, whereas the fixed point is still not reached in the static network setup. These observations agree with the results presented

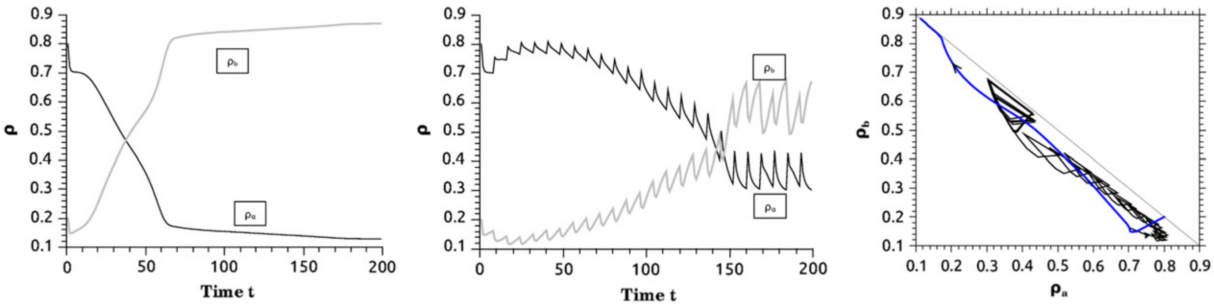

Fig. 6. Left: Time evolution of the unperturbed decision dynamics in a dynamic network with the parameter values and initial conditions given in (35), and a reactive transmission process. Center: Time evolution of the associated decision dynamics with an exogenous influence on 18% of the population each $T_A = 8$ time units. Right: Projection onto the (ρ_a, ρ_b)-plane of the trajectories of the mean decision probability ρ (11) for the unperturbed dynamics (thick blue line) and that with exogenous perturbation (black line). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

in Ref. [28], which are probably related to the dynamic network set-up, i.e., the contacts change at each time instant, so the formation of decision clusters is less likely than that in the static network setup. This conjecture should be studied carefully in future research.

4.2. Dynamics with the external field of message $\mathcal{A}$

To analyze the impact of mass media on the decision behavior of a dynamic network, the exogenous propagation of opinion $\mathcal{A}$ is studied based on the model (12) with a periodicity of

$$T_A = 8 \text{ time units} . \tag{34}$$

Fig. 6 presents the simulation results with the parameter values and initial conditions

$$\kappa_A = 0.3, \qquad \kappa_B = 0.6, \qquad \rho_A(0) = 0.8, \qquad \rho_B(0) = 0.2, \qquad \rho_N(0) = 0. \tag{35}$$

If there are no external forces, the trajectory converges toward the upper left corner, which corresponds to decision $\mathcal{B}$, as shown by the time evolution in the left sub-figure of Fig. 6 and in the projection onto the (ρ_a, ρ_b)-plane indicated by the thick blue line in the right of the sub-figure. In the case where $\mathcal{A}$ is propagated by an exogenous force where 18% of the population are influenced arbitrarily each $T_A = 8$ time units, the corresponding time evolution is shown in the central sub-figure of Fig. 6, while the projection onto the (ρ_a, ρ_b)-plane is indicated in the right sub-figure by the black line. During each period, a force drives 18% of the population toward decision $\mathcal{A}$. Therefore, the trajectory does not reach the attractor (which corresponds to decision $\mathcal{B}$) but it is maintained in an oscillatory regime about some intermediate decision distribution. This illustrates the impact of exogenous forces on the dynamic behavior within the network and the influence on the overall (mean) decision vector ρ, which agrees with the qualitative analytic bound (29) derived in Section 3.3.

4.3. Dynamics with the external fields of both messages

To illustrate the competition of two external forces, we consider the case where both decisions are propagated through external forces into the network (e.g., by exogenous mass media) but with different periods

$$T_A = 8 \text{ time units}, \qquad T_B = 16 \text{ time units}. \tag{36}$$

It is considered that both mechanisms affect 10% of the total population to switch to their opinion. The parameter values and initial conditions were set to

$$\kappa_A = \kappa_B = 0.5, \qquad \rho_1(0) = [0.25, 0.15, 0.6]', \qquad \rho_2(0) = [0.15, 0.25, 0.6]', \tag{37}$$

which correspond to the two representative cases shown in Fig. 7. As illustrated by the thick blue lines for the unperturbed case, the trajectory that starts at $\rho_1(0)$ (37) converges toward the attractor in the upper left corner and the trajectory that starts at $\rho_2(0)$ (37) converges toward the attractor at the lower right corner. In the presence of exogenous perturbations with the periods given in (36), which influence 10% of the population to switch their opinion, the trajectories are indicated by the black lines. The trajectory that starts at $\rho_1(0)$ no longer converges toward the attractor in the upper left corner because it enters an oscillatory regime about some intermediate mean decision value ρ, while the trajectory that starts at $\rho_2(0)$ almost converges toward the lower right corner but it enters into an oscillatory regime close to the corresponding attractor for the unperturbed trajectory.

These results illustrate the differences that may be produced by exogenous propagation mechanisms on the decision dynamics within complex networks. In particular, they show that the final decision state will depend strongly on the initial conditions, as well as the specific process and the node degree distribution parameters.

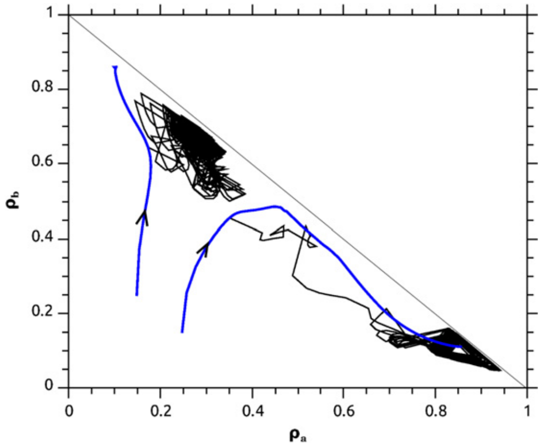

Fig. 7. Projection onto the (ρ_a, ρ_b)-plane of two representative trajectories of the mean decision probability ρ (11) for the parameter values and initial conditions given in (37) with a reactive transmission process. The unperturbed trajectories are represented by the thick blue lines and those with exogenous perturbation by the black lines. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

5. Discussion

It should be mentioned that the network nodes are modeled with a very simple behavior in the present study. They do not possess a memory and their decision capacity is constrained to a state transition based on stimuli transmitted by their contacts with neighbors and an exogenous source. This assumption is similar to those considered in previous studies of cultural dissemination (e.g., Ref. [3]). Another assumption is that the nodes are synchronous, where the transitions occur according to a common clock step. These assumptions are simplistic but similar behaviors can be observed in some circumstances, e.g., in biological cell networks or in social networks with short time intervals. However, these assumptions allow the derivation of a mathematical model under which we can predict a given system state in time based on some initial conditions and parameters. Furthermore, it allows us to identify the nonlinear mechanisms that are inherent in these types of processes and to discuss their possible implications.

In future studies, the possibility of providing each node with a bigger decision capacity and memory should be considered to generate a more realistic scenario, as well as the possibility of prediction over longer time horizons.

6. Conclusions

In this study, we analyzed decision dynamics in complex networks based on considerations of inherently nonlinear competition mechanisms. Sufficient conditions were derived for the extinction of opinions in terms of the convincement probabilities and the maximum eigenvalues of the connectivity (or transmission) matrices. We showed that the dynamics exhibit fixed-point multiplicity, possible multi-stability, and sensitivity to the process parameters. The effects of exogenous perturbations, such as mass media propagation mechanisms, were delimited analytically in the sense that bounded perturbations lead to bounded deviations from natural behavior. A set of numerical simulation was presented to illustrate the theoretical results.

Appendix A. Derivation of the Markov process model

As a point of departure, the assumptions (A1)–(A11) (Section 2), the transition probabilities α_i, β_i, ν_i, μ_i, and σ_i are determined as follows.

Suppose that a node i has message $\mathcal{A}$ and it is in contact with a node $j \neq i$. Then, the following events are possible:

- Node j is in state $\mathcal{A}$ (or $\mathcal{N}$) with a probability a_j (or n_j), thus it will not convince node i to change its message.
- Node j is in state $\mathcal{B}$ with a probability b_j, thus it tries to convince node i, where the probability of success is given by $r_{ij}^B \kappa_B$, and the failure to convince i to change to $\mathcal{B}$ has a probability of $1 - r_{ij}^B \kappa_B$, where r_{ij}^B given in (3).

Based on these considerations, the probability of node i not being convinced by node j to change from state $\mathcal{A}$ to $\mathcal{N}$ (the only alternative is to remain in state $\mathcal{A}$) is given by the sum

$$\alpha_{ij} = a_j + n_j + (1 - r_{ij}^B \kappa_B) b_j = 1 - r_{ij}^B \kappa_B b_j, \tag{A.1}$$

where the last equality is a consequence of the property (2).

The same reasoning applies to the probability β_{ij} of node i being in state $\mathcal{B}$ and not being convinced to change from state $\mathcal{B}$ to state $\mathcal{N}$ by its neighbor $j \neq i$ (while considering the property (2)):

$$\beta_{ij} = b_j + n_j + (1 - r_{ij}^A \kappa_A) a_j = 1 - r_{ij}^A \kappa_A a_j. \tag{A.2}$$

If node i is in state $\mathcal{N}$ and in contact with node $j \neq i$, the following events imply that i remains in $\mathcal{N}$:

- Node j has the opinion $\mathcal{N}$ with a probability n_j and it does not convince i to change its message.
- Node j has message $\mathcal{A}$ (or $\mathcal{B}$) with a probability of a_j (or b_j) and it does convince i to change to $\mathcal{A}$ (or $\mathcal{B}$) with a probability of $r_{ij}^A \kappa_A$ (or $r_{ij}^B \kappa_B$), whereas it fails with a probability of $1 - r_{ij}^A \kappa_A$ (or $1 - r_{ij}^B \kappa_B$).

Thus, while considering the property (2), the probability of node i not being convinced by node $j \neq i$ is given by

$$\nu_{ij} = n_j + (1 - r_{ij}^A \kappa_A) a_j + (1 - r_{ij}^B \kappa_B) b_j = 1 - r_{ij}^A \kappa_A a_j - r_{ij}^B \kappa_B b_j, \tag{A.3}$$

and the probabilities of node $j \neq i$ convincing node i to change to $\mathcal{A}$ or $\mathcal{B}$ are given, respectively, by

$$\mu_{ij} = r_{ij}^A \kappa_A a_j, \qquad \sigma_{ij} = r_{ij}^B \kappa_B b_j. \tag{A.4}$$

Next, to consider the combined effect of all the neighbors of node i on the transition probabilities, we recall assumption (A8). The overall transition probabilities for node i at time t are then given by:

$$\begin{aligned}
\alpha_i(t) &= \frac{1}{N_i(t)} \sum_{j \neq i} \alpha_{ij}(t) = \frac{1}{N_i(t)} \sum_{j \neq i} [A_{i,j}(t) - r_{ij}^B \kappa_B b_j(t)], \\
\beta_i(t) &= \frac{1}{N_i(t)} \sum_{j \neq i} \beta_{ij}(t) = \frac{1}{N_i(t)} \sum_{j \neq i} [A_{i,j}(t) - r_{ij}^A(t) \kappa_A a_j(t)], \\
\nu_i(t) &= \frac{1}{N_i(t)} \sum_{j \neq i} \nu_{ij}(t) = \frac{1}{N_i(t)} \sum_{j \neq i} [A_{i,j}(t) - r_{ij}^A(t) \kappa_A a_j(t) - r_{ij}^B(t) \kappa_B b_j(t)], \\
\mu_i(t) &= \frac{1}{N_i(t)} \sum_{j \neq i} \mu_{ij}(t) = \frac{1}{N_i(t)} \sum_{j \neq i} r_{ij}^A(t) \kappa_A a_j(t), \\
\sigma_i(t) &= \frac{1}{N_i(t)} \sum_{j \neq i} \sigma_{ij}(t) = \frac{1}{N_i(t)} \sum_{j \neq i} r_{ij}^B(t) \kappa_B b_j(t)
\end{aligned} \tag{A.5}$$

where $A_{i,j}$ and N_i are the adjacency matrix entry (i, j) and the degree of node i, which are defined in assumption (A1) and Eq. (4), respectively.

The associated discrete time model for the dynamics of the decisions between $\mathcal{A}$ and $\mathcal{B}$ in the network is given by

$$\begin{aligned}
& a_i(t+1) = \alpha_i(t) a_i(t) + \mu_i(t) n_i(t), \qquad a_i(0) = a_{i0} \\
& n_i(t+1) = \nu_i(t) n_i(t) + [1 - \alpha_i(t)] a_i(t) + [1 - \beta_i(t)] b_i(t), \qquad n_i(0) = n_{i0} \\
& b_i(t+1) = \beta_i(t) b_i(t) + \sigma_i(t) n_i(t), \qquad b_i(0) = b_{i0} \\
& 0 = 1 - [a_i(t) + b_i(t) + n_i(t)].
\end{aligned} \tag{A.6}$$

Considering the identities (see Eq. (A.5))

$$\mu_i(t) = 1 - \beta_i(t), \qquad \sigma_i(t) = 1 - \alpha_i(t) \tag{A.7}$$

model (A.6) is equivalent to

$$\begin{aligned}
& a_i(t+1) = \alpha_i(t) a_i(t) + [1 - \beta_i(t)][1 - a_i(t) - b_i(t)], \qquad a_i(0) = a_{i0} \\
& b_i(t+1) = \beta_i(t) b_i(t) + [1 - \alpha_i(t)][1 - a_i(t) - b_i(t)], \qquad b_i(0) = b_{i0} \\
& n_i(t) = 1 - a_i(t) - b_i(t)
\end{aligned} \tag{A.8}$$

where $\alpha_i(t)$ and $\beta_i(t)$ are defined in (A.5).

Appendix B. Proof of Theorem 1

Recall the definition of the mean probabilities ρ_a and ρ_b (10). After some algebraic manipulation, their dynamics can be written as

$$\begin{aligned}
\rho_a(t+1) &= \rho_a - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{N_i(t)} \sum_{j \neq i} a_i r_{ij}^B \kappa_B b_j + \frac{1}{N} \sum_{i=1}^{N} \frac{1}{N_i(t)} \sum_{j \neq i} r_{ij}^A \kappa_A a_j (1 - a_i - b_i) \\
&\leq \rho_a - \frac{\kappa_B}{N^2} a^T R_B b + \frac{\kappa_A}{N} \sum_{i=1}^{N} e_i^T R_A a - \frac{\kappa_A}{N^2} (a^T + b^T) R_A a \\
\rho_b(t+1) &\leq \rho_b - \frac{\kappa_A}{N^2} b^T R_A a + \frac{\kappa_B}{N} \sum_{i=1}^{N} e_i^T R_B b - \frac{\kappa_B}{N^2} (a^T + b^T) R_B b
\end{aligned} \tag{B.1}$$

where e_i is the ith eigenvector of the $2N$-dimensional Euclidean space. These dynamics can be interpreted as a two-dimensional non-autonomous system with exogenous inputs. For the quadratic terms in (B.1), we have the following lemma

Lemma 1. *For $a_i, b_i \in [0, 1]$, $i = 1, \ldots, N$ and any matrix R with positive eigenvalues $\lambda_i(R)$ and maximum eigenvalue $\lambda^+(R)$, the following holds:*

$$\frac{1}{N^2} a^T R b \leq \lambda^+(R) \rho_a \rho_b, \qquad \rho_l = \frac{1}{N} \sum_{i=1}^{N} l_i, \quad l = a, b. \tag{B.2}$$

Proof. Given that all the product terms satisfy $0 \leq a_i b_j \leq 1$ and the eigenvalues of the matrix R are positive, it follows that

$$\begin{aligned} \frac{1}{N^2} a^T R b &\leq \frac{1}{N^2} \lambda^+(R) a^T b = \frac{\lambda^+(R)}{N^2} \sum_{i=1}^{N} a_i b_i \\ &\leq \frac{\lambda^+(R)}{N^2} \sum_{i=1}^{N} \sum_{k=0}^{i} a_{k+1} b_{i-k} = \lambda^+(R) \rho_a \rho_b \end{aligned}$$

where the inequality holds given that only positive terms are added to the original expression. □

According to the preceding lemma, the right-hand side of the dynamics equations (B.1) is bounded as follows

$$\begin{aligned} \rho_a(t+1) &\leq \left(1 + \kappa_A \lambda^+(R_A)\right) \rho_a(t) - \kappa_B \lambda^+(R_B) \rho_a(t) \rho_b(t) - \kappa_A \lambda^+(R_A) \rho_a(t) [\rho_a(t) + \rho_b(t)] \\ \rho_b(t+1) &\leq \left(1 + \kappa_B \lambda^+(R_B)\right) \rho_b(t) - \kappa_A \lambda^+(R_A) \rho_a(t) \rho_b(t) - \kappa_B \lambda^+(R_B) \rho_b(t) [\rho_a(t) + \rho_b(t)]. \end{aligned} \tag{B.3}$$

In allusion to singular perturbation theory [29], the preceding bound for the dynamics of the mean probabilities are denoted by the *boundary layer dynamics* and its solution is denoted by $z = (z_a, z_b)'$

$$\begin{bmatrix} z_a(t+1) \\ z_b(t+1) \end{bmatrix} = \begin{bmatrix} \left(1 + \kappa_A \lambda^+(R_A)\right) z_a(t) - \kappa_B \lambda^+(R_B) z_a(t) z_b(t) - \kappa_A \lambda^+(R_A) z_a(t) (z_a(t) + z_b(t)) \\ \left(1 + \kappa_B \lambda^+(R_B)\right) z_b(t) - \kappa_A \lambda^+(R_A) z_a(t) z_b(t) - \kappa_B \lambda^+(R_B) z_b(t) (z_a(t) + z_b(t)) \end{bmatrix}. \tag{B.4}$$

In vector notation, these dynamics are written as

$$z(t+1) = F(z(t)), \qquad z(0) = z_0, \qquad z = \begin{bmatrix} z_a \\ z_b \end{bmatrix}, \qquad F((z_a, z_b)') = \begin{bmatrix} F_a(z_a, z_b) \\ F_b(z_a, z_b) \end{bmatrix}. \tag{B.5}$$

It holds that the solutions $\rho(t)$ of the dynamics (B.1) and the solutions $z(t)$ of the boundary layer dynamics (B.4) are related in the following manner:

$$\rho_{r0} = z_0 \Rightarrow \|\rho_r(t)\| \leq \|z(t)\|, \quad \forall\, t \geq 0,\ r = a, b \tag{B.6}$$

where $\|\cdot\|$ denotes any norm in the Euclidean space $[0, 1]^{2N}$.

The linear approximation for the boundary layer dynamics (B.5) about the extinction fixed point $x_{e1} = (0, 0)'$ is given by

$$\begin{bmatrix} \rho_a(t+1) \\ \rho_b(t+1) \end{bmatrix} = \begin{bmatrix} 1 + \kappa_A \lambda^+(R_A) & 0 \\ 0 & 1 + \kappa_B \lambda^+(R_B) \end{bmatrix} \begin{bmatrix} \rho_a(t) \\ \rho_b(t) \end{bmatrix}. \tag{B.7}$$

The associated eigenvalues are given by the diagonal elements and they are larger than 1, which implies that x_{e1} is unstable for any values of κ_A and κ_B. The same result can be obtained using the original dynamics in the (a, b)-coordinates (6) written in vector notation, where instead of $1 + \kappa_l \lambda^+(R_l)$, the matrix term $I + \kappa_l R_l$, $l = A, B$ appears. Given that an unstable linear approximation about the fixed point implies the instability of the fixed point for the nonlinear exact dynamics (e.g., see Ref. [30]), the extinction fixed point $x_{e1} = (0, 0)'$ is unstable.

To analyze the stability of the second extinction fixed point $x_{e2} = (1, 0)'$, we introduce the deviation coordinate

$$\epsilon_a = 1 - z_a \tag{B.8}$$

for the boundary layer state z_a and write the dynamics of ϵ_a and z_b (F_a and F_b are defined in (B.5))

$$\begin{aligned} \epsilon_a(t+1) &= 1 - z_a(t+1) = F_a(1, 0) - F_a[1 - \epsilon_a(t), z_b(t)] \\ &= -\left(1 - \kappa_A \lambda^+(R_A)\right) \epsilon_a(t) - (\kappa_A \lambda^+(R_A) + \kappa_B \lambda^+(R_B)) z_b(t) (1 - \epsilon_a(t)) - \kappa_A \lambda^+(R_A) \epsilon_a^2(t) \\ z_b(t+1) &= F_b(1 - \epsilon_a(t), z_b(t)) \\ &= \left(1 + \kappa_B \lambda^+(R_B) - \kappa_A \lambda^+(R_A)\right) z_b(t) + (\kappa_A \lambda^+(R_A) - \kappa_B \lambda^+(R_B)) \epsilon_a(t) z_b(t) + \kappa_B \lambda^+(R_B) z_b^2(t). \end{aligned} \tag{B.9}$$

In the neighborhood of the fixed point x_{e2}, the corresponding dynamics can be approximated by the linear dynamics

$$\begin{bmatrix} \epsilon_a(t+1) \\ z_b(t+1) \end{bmatrix} = \begin{bmatrix} -1 + \kappa_A \lambda^+(R_A) & -\kappa_A \lambda^+(R_A) - \kappa_B \lambda^+(R_B) \\ 0 & 1 + \kappa_B \lambda^+(R_B) - \kappa_A \lambda^+(R_A) \end{bmatrix} \begin{bmatrix} \epsilon_a(t) \\ z_b(t) \end{bmatrix} \tag{B.10}$$

and they are stable if the associated eigenvalues of the system matrix are less than one. In turn, this holds if

$$\kappa_A > \kappa_{Ac} = \frac{1 + \kappa_B \lambda^+(R_B)}{\lambda^+(R_A)}. \tag{B.11}$$

If this condition is satisfied, it follows that (locally) $\|(\epsilon_a(t), z_b(t))'\| \to 0$, which means that the boundary layer trajectory $z(t) \to (1, 0)'$. Given that the local deviation dynamics about x_{e2} are the same for the boundary layer (ϵ_a, z_b) and the original $(1 - \rho_a, \rho_b)$ system, the same result applies to the mean probability vector $\rho(t)$, i.e., $\rho(t) \to (1, 0)'$.

By virtue of the symmetry of the automata (see Fig. 2) for the two options $\mathcal{A}$ and $\mathcal{B}$, which are reflected in the system dynamics (6), the corresponding stability conditions for the third extinction state $x_{e3} = (0, 1)'$ can be derived similarly and read

$$\kappa_B > \kappa_{Bc} = \frac{1 + \kappa_A \lambda^+(R_A)}{\lambda^+(R_B)}. \tag{B.12}$$

This concludes the proof. □

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