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Canonical quantization of a string describing N branes at angles

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Abstract

We study the canonical quantization of a bosonic string in presence of N twist fields. This generalizes the quantization of the twisted string in two ways: the in and out states are not necessarily twisted and the number of twist fields N can be bigger than 2.

In order to quantize the theory we need to find the normal modes. Then we need to define a product between two modes which is conserved. Because of this we need to use the Klein–Gordon product and to separate the string coordinate into the classical and the quantum part. The quantum part has different boundary conditions than the original string coordinates but these boundary conditions are precisely those which make the operator describing the equation of motion self adjoint.

The splitting of the string coordinates into a classical and quantum part allows the formulation of an improved overlap principle. Using this approach we then proceed in computing the generating function for the generic correlator with L untwisted operators and N (excited) twist fields for branes at angles. We recover as expected the results previously obtained using the path integral. This construction explains why these correlators are given by a generalization of the Wick theorem.
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1. Introduction and conclusions

Since their introduction, D-branes have been very important in the formal development of string theory as well as in attempts to apply string theory to particle phenomenology and cosmology. However, the requirement of chirality in any physically realistic model leads to a somewhat restricted number of possible D-brane set-ups. An important class is intersecting brane models where chiral fermions can arise at the intersection of two branes at angles. An important issue for these models is the computation of Yukawa couplings and flavor changing neutral currents.

Besides the previous computations many other computations often involve correlators of twist fields and excited twist fields. It is therefore important and interesting in its own right to be able to compute these correlators. As is known in the literature [1] and explicitly shown in [2] for the branes at angles case and in less precise way in [3] for the case of magnetized branes these computations boil down to the knowledge of the Green function in presence of twist fields and of the correlators of the plain twist fields. In many previous papers correlators with excited twisted fields have been computed on a case by case basis without a clear global picture, see for example [5–7].

In this technical paper we have analyzed the N excited twist fields amplitudes with L boundary vertices at tree level for open strings localized at D-branes intersections on R^2 (or T^2) and we have rederived the results of [2].

We will nevertheless follow a different approach from most of the literature in a twofold way. Firstly, we use the so-called Reggeon vertex [25], which allows to compute the generating function of all correlators, in particular we will use the formulation put forward in [26]. Secondly we use the canonical quantization approach while all the previous literature has used the classical path integral approach [1,8]. In the case at hand the path integral approach has proven to be more efficient than also the classical sewing approach [9,10]. In fact the path integral approach has been explored in many papers in the branes at angles setup as well as the T dual magnetic branes setup see for example [11–40].

At the heart of the path integral approach is the idea that the interaction of a string with twisted strings in the fundamental state can be replaced by a discontinuity on the string boundary conditions. This is pictured in Fig. 1. We use this idea as the starting point of our computation based on canonical formalism. The fact that we have boundary conditions which change with the worldsheet time implies, as we show, that we have a very mild worldsheet time dependent worldsheet metric. Hence the usual quantization cannot be applied in a straightforward way but we have to find a proper way to define the product between modes. This is done using the Klein–Gordon product used in General Relativity. To have a well defined, worldsheet time independent product

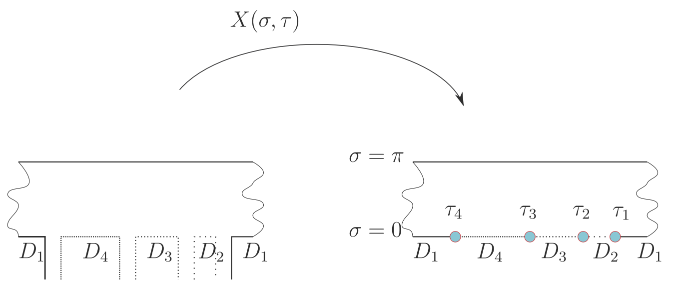

Fig. 1. The interacting strings are mapped into boundary condition discontinuities.

between two modes implies that we have to split the string into a classical and quantum part and quantize the quantum part only, exactly as in the path integral approach. Since this procedure is at variance with the usual one we check that we recover the standard results in the cases of the untwisted string and of the usual twisted ones. All this is done in Section 2. In this section we discuss also the expression of the Hamiltonian in term of oscillators. We find that it is quadratic in oscillator but not diagonal since we have an almost free theory with worldsheet time dependent background (this should not be a surprise since the system is worldsheet time dependent and hence it is not meaningful to diagonalize the Hamiltonian). We derive also the modes for the three twists case. We are able to find many orthogonal basis but all of them are missing of one mode with respect to the basis of the in string. This fact can be partially understood as the consequence of the boundary "interactions" which break some symmetries of the original in string. Since we have not understood this issue completely, we resort to using an improved version of the standard overlap approach which is also used in quantum mechanics in presence of discontinuities of the Hamiltonian.

In Section 3 we tackle the problem of computing the in and out vacua. They differ since we have a worldsheet time dependence in the boundary conditions. In principle it should be possible to compute them from the basic principles. Because of the not complete understanding of modes we compute the out vacuum as a kind of surface state (an exponential of an expression quadratic in the operators) of the in vacuum assuming the knowledge of the Green function. This assumption is however not a big issue since Green functions can be derived using the analytic properties and boundary conditions.

In Section 4 we perform the actual computations of the generating functions for amplitudes involving plain and excited twisted states. This is done in steps. First considering the amplitudes with plain unexcited twisted fields and arbitrary untwisted states. Then considering amplitudes with excited twisted states without untwisted ones and finally, assembling all.

Our main result is to be able to rederive in a different way the generating function of correlators with N excited twists and L untwisted states found in [2]. This generating function is given in Eq. (129) which shows that all correlators can be computed once the N plain twist operators correlator together with the Green function in presence of these N twists are known. This expression requires the precise knowledge of the Green function[1] and its regularized versions. Luckily these are well known [24,16]. From these expressions it is clear that the computation of amplitudes, i.e. moduli integrated correlators, with (untwisted) states carrying momenta are very unwieldy because Green functions can at best be expressed as sum of product of type D Lauricella functions. This should however not be a complete surprise since in [27] it was shown that twist fields correlators in orbifold setup are connected to loop amplitudes which, up to now, have not been expressed in term of simpler functions.

2. The setup of branes at angles

The Euclidean action for a string configuration is given by

$$S_E = \frac{1}{4\pi\alpha'}\int d\tau_E \int_0^\pi d\sigma \left(\partial_\alpha X^I\right)^2 = \frac{1}{4\pi\alpha'}\int_H d^2u \left(\partial_u X^z \bar\partial_{\bar u} X^{\bar z} + \bar\partial_{\bar u} X^z \partial_u X^{\bar z}\right) \tag{1}$$

[1] Note that the Green functions used in this paper are dimensionful and normalized as $\partial_u \bar\partial_{\bar u} G^{IJ}(u,\bar u; v,\bar v; \{\epsilon_t\}) = -\frac{\alpha'}{2}\delta^{IJ}\delta^2(u-v)$.

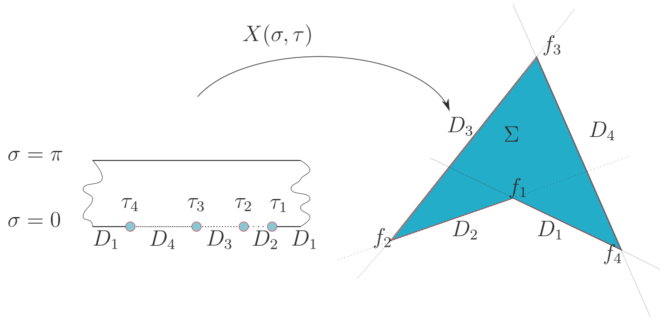

Fig. 2. Map from the Minkowskian worldsheet to the target polygon Σ with a plain in and out string. The map $X(\sigma, \tau)$ folds the $\sigma = 0$ starting from $\tau = -\infty$ in a counterclockwise direction.

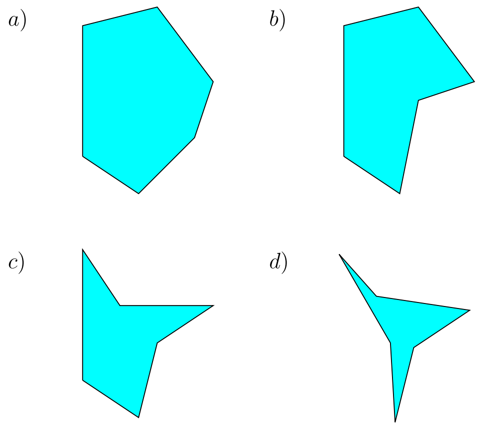

Fig. 3. The four different cases with $N = 6$. a) $M = 4$. b) $M = 3$. c) $M = 2$. d) $M = 1$.

where $u = e^{\tau_E + i\sigma} \in H$, the upper half plane, $d^2u = e^{2\tau_E} d\tau_E d\sigma = \frac{du\, d\bar{u}}{2i}$ and $I = 1, 2$ or $z, \bar{z}$ so that $X^z = \frac{1}{\sqrt{2}}(X^1 + iX^2)$, $X^{\bar{z}} = X^{z*}$. The complex string coordinate is a map from the upper half plane to a closed polygon Σ in $\mathbb{C}$, i.e. $X : H \to \Sigma \subset \mathbb{C}$. For example in Fig. 2 we have pictured the interaction of $N = 4$ branes at angles D_t with $t = 1, \ldots, N$. The interaction between brane D_t and D_{t+1} is at $f_t \in \mathbb{C}$. We use the rule that index t is defined modulo N. As shown in [3] given the number of twist fields N there are $N - 2$ different sectors. They are labeled by an integer M, $1 \leq M \leq N - 2$ which is in correspondence with the number of reflex angles (the interior angles bigger than π), more precisely M is $N - 2$ minus the number of reflex angles as shown in Fig. 3 in the case $N = 6$. The intuitive reason why these sectors are different is that we need go through the straight line, i.e. no twist, if we want to go from a reflex angles to a convex one.

2.1. *Splitting into classical and quantum part and Klein–Gordon metric*

In order to proceed with the canonical quantization we want to find the normal modes associated with the equations of motion

$$\partial_u \bar{\partial}_{\bar{u}} X^z = \partial_u \bar{\partial}_{\bar{u}} X^{\bar{z}} = 0 \quad u \in H \tag{2}$$

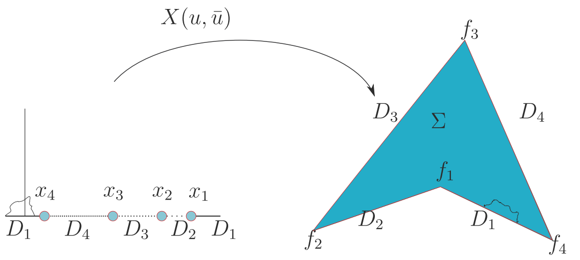

Fig. 4. Map from the upper half plane to the target polygon Σ with untwisted in and out strings. The map $X(u,\bar{u})$ folds the boundary of the upper half plane starting from $x=-\infty$ in a counterclockwise direction and preserves the orientation.

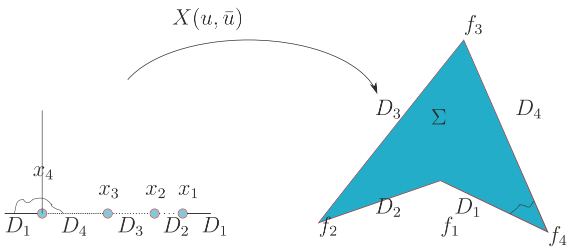

Fig. 5. Map from the upper half plane to the target polygon Σ with a twisted in and an untwisted out string.

and the boundary conditions

$$e^{-i\pi\alpha_t}\partial_y X^z(u,\bar{u})\big|_{u=x+i0^+} + e^{i\pi\alpha_t}\partial_y X^{\bar{z}}(u,\bar{u})\big|_{u=x+i0^+} = 0 \quad x_t < x < x_{t-1}$$
$$e^{-i\pi\alpha_t} X^z(u,\bar{u})\big|_{u=x+i0^+} - e^{i\pi\alpha_t} X^{\bar{z}}(u,\bar{u})\big|_{u=x+i0^+} = 2ig_t \quad x_t < x < x_{t-1}. \tag{3}$$

The previous constraints are simply stating that when $x_t < x < x_{t-1}$ a boundary of the string is on the brane D_t. The brane D_t is described in a well adapted coordinate system as $\sqrt{2}iX^{2_t} = e^{-i\pi\alpha_t}X^z - e^{i\pi\alpha_t}X^{\bar{z}} = 2ig_t \in i\mathbb{R}$, i.e. it extends along X^{1_t} with $\sqrt{2}X^{1_t} = e^{-i\pi\alpha_t}X^z + e^{i\pi\alpha_t}X^{\bar{z}}$. Therefore the string has Dirichlet boundary condition in the X^{2_t} direction and has Neumann boundary condition in the perpendicular direction X^{1_t}. In particular $\sqrt{2}|g_t|$ is the distance of the brane from the origin and

$$f_t = \frac{e^{i\pi\alpha_{t+1}}g_t - e^{i\pi\alpha_t}g_{t+1}}{\sin\pi(\alpha_{t+1}-\alpha_t)} \tag{4}$$

is the intersection point between D_t and D_{t+1}. The configuration in the Euclidean case can be pictured as in Fig. 4.

This configuration corresponds to the Minkowskian configuration of Fig. 2 where both the in and the out strings are untwisted. A similar configuration is drawn in Fig. 5. In this case the in string is twisted because one twist is sitting in the origin and the outgoing one is untwisted. Obviously there is also a third possibility pictured in Fig. 8 where both the in and out strings are twisted, this corresponds to the case where there is a twist at $x=0$ and one at $x=\infty$. When there are $N=2$ twists this is the usual configuration describing a twisted string.

The issue is now to find a (non-positive definite) product for the modes which is conserved in (Euclidean) time. This issue is less trivial than usual because our spacetime, i.e. the worldsheet, is changing with (Euclidean) time even if in a mild way through the change of the boundary conditions. The solution to this problem is well known in General Relativity. Since we are dealing with free fields satisfying the Klein–Gordon equation we know that there is a conserved current. Explicitly given any two solutions $F_i = (f_i^z, f_i^{\bar{z}})^T$ with $i = 1, 2$ the current

$$j_\alpha = i F_1^\dagger \overleftrightarrow{\partial}_\alpha F_2 \tag{5}$$

is conserved because of the equation of motion. This is nevertheless not sufficient to get a conserved (non-positive definite) product, in fact we must deal with boundary contributions. In the Euclidean case we consider the surface $S(r_0, r_1)$ in the upper half plane delimited by two semicircles of "time" r_0 and r_1 ($r_0 < r_1$) and by the two segments on the x axis $[r_0, r_1]$ and $[-r_1, -r_0]$ then we find[2]

$$0 = \int_{S(r_0,r_1)} d * j = \int_{|u|=r_1} *j - \int_{|u|=r_0} *j + \int_{[r_0,r_1]} *j + \int_{[-r_1,-r_0]} *j. \tag{6}$$

In order to get a metric independent on time r we need rewriting the two integrals along the x axis as a difference of a function which depends only on the background fields at the initial and final times. While we know that we can write a definite integral as a difference of a function evaluated at final and initial times what it is not certain is that this difference does depend only the final and initial background fields since we have boundary conditions with discontinuities.

The two integrals along the x axis can be expressed as a difference of a function which depends only on the background fields at the evaluation time and actually vanish only if we consider solutions F_q which satisfy the boundary conditions in Eq. (3), with $g_t = 0$, i.e.

$$\begin{aligned} &e^{-i\pi\alpha_t} \partial_y X^z(u, \bar{u})\big|_{u=x+i0^+} + e^{i\pi\alpha_t} \partial_y X^{\bar{z}}(u, \bar{u})\big|_{u=x+i0^+} = 0 \quad x_t < x < x_{t-1} \\ &e^{-i\pi\alpha_t} X^z(u, \bar{u})\big|_{u=x+i0^+} - e^{i\pi\alpha_t} X^{\bar{z}}(u, \bar{u})\big|_{u=x+i0^+} = 0 \quad x_t < x < x_{t-1}. \end{aligned} \tag{7}$$

In the following we call these boundary conditions quantum boundary conditions. The quantum boundary conditions can be also written as

$$(\mathbb{I} + R_t)\partial_y F_q|_{y=0} = 0, \qquad (\mathbb{I} - R_t) F_q|_{y=0} = 0 \quad x_t < x < x_{t-1} \tag{8}$$

with

$$R_t = R_t^\dagger = R_t^{-1} = \begin{pmatrix} & e^{i2\pi\alpha_t} \\ e^{-i2\pi\alpha_t} & \end{pmatrix}. \tag{9}$$

Consider then one of the x boundary contributions

$$-i \int_{[r_0,r_1]} *j = -\int_{[r_0,r_1]} dx\, j_y = \int_{[r_0,r_1]} dx \left(\partial_y F_1^\dagger F_2 - F_1^\dagger \partial_y F_2\right)\big|_{y=0}. \tag{10}$$

Let us consider the first term $\partial_y F_1^\dagger F_2$. We can split the integration interval into pieces where the boundary conditions are constant. Then using the quantum boundary conditions we have the identity $F_q = \frac{\mathbb{I}+R_t}{2} F_q$ and therefore we can write

$$\partial_y F_1^\dagger F_2 = \partial_y F_1^\dagger \frac{\mathbb{I} + R_t}{2} F_2 = \left(\frac{\mathbb{I} + R_t}{2} \partial_y F_1\right)^\dagger F_2 = 0. \tag{11}$$

[2] Our conventions are $*du = -i\, du$, $*d\bar{u} = i\, d\bar{u}$.

If we would not use the quantum boundary conditions but the original boundary conditions the second identity in (8) would read $(\mathbb{I} - R_t)F|_{y=0} = 2G_t = -2ig_t(e^{2\pi\alpha_t}, e^{-i\pi\alpha_t})^T$ then the contribution from a piece of the integration interval where the boundary conditions are constant would be $-\partial_y F_1^\dagger G_t + G_t^\dagger \partial_y F_2$. This term can be integrated explicitly using the fact that Fs split into a sum of left and right moving pieces and it is non vanishing. Hence the resulting boundary contribution (10) would depend not only on the fields at r_0 and r_1 but also on the fields at the discontinuities between r_0 and r_1. This means that the would be product does depend on the history and not only on the background fields at the time where the metric is computed and hence there is no time independent product of modes. We conclude therefore that only for solutions satisfying the quantum boundary conditions (8) we have the non positive definite metric (actually Hermitian form)

$$(F_1, F_2) = (F_2, F_1)^* = \int_{|u|=r} \left(i F_1^\dagger \overleftrightarrow{\partial_x} F_2\, dy - i F_1^\dagger \overleftrightarrow{\partial_y} F_2\, dx\right)$$

$$= \int_{0; r=const}^{\pi} d\theta\, r i \left(F_1^\dagger \overleftrightarrow{\partial_r} F_2\right). \tag{12}$$

This means that we must proceed as in the path integral approach and split the string coordinate into a classical and quantum part and then quantize only the quantum part. Explicitly we write

$$X^I(u, \bar{u}) = X_{cl}^I\left(u, \bar{u}; \{x_t, g_t, \alpha_t\}\right) + X_q^I\left(u, \bar{u}; \{x_t, \alpha_t\}\right) \tag{13}$$

with X_{cl} satisfying the original boundary conditions (3) and X_q satisfying the quantum conditions (7). Notice that only with the quantum boundary conditions (7) the two dimensional laplacian $\partial_u \bar{\partial}_{\bar{u}}$ is self-adjoint and it is certain to have a Green function. This is shown in Appendix A.

Finally, notice that the previous discussion can be and must be applied also to branes with magnetic field. In this case the Minkowskian boundary conditions read $X'^I - \mathcal{F}_{IJs}\dot{X}^J|_{\sigma=s} = 0$ with $s = 0, \pi$ and $\mathcal{F}_{IJs}$ the magnetic fields. Then the boundary contribution corresponding to (10) is not anymore zero but evaluates to

$$-i \int_{\tau_0}^{\tau_1} *j = i \int_{\tau_0}^{\tau_1} d\tau\, j_\sigma = -F_1^\dagger \mathcal{F}_0 F_2\big|_{\sigma=0,\tau_1} + F_1^\dagger \mathcal{F}_0 F_2\big|_{\sigma=0,\tau_0} \tag{14}$$

and therefore induces a product

$$(F_1, F_2) = \int_0^\pi i F_1^\dagger \overleftrightarrow{\partial_\tau} F_2\, d\sigma + i F_1^\dagger \mathcal{F}_0 F_2\big|_{\sigma=0} - i F_1^\dagger \mathcal{F}_\pi F_2\big|_{\sigma=\pi} \tag{15}$$

which is the "weird" metric used in [4].

2.2. Doubling trick and the metric

We have established that we must quantize the fluctuation around the classical solution. These fluctuations satisfy the quantum boundary conditions (7). We now look for solutions to equations of motion with these boundary conditions. As usual the general solution of Eq. (2) is given by

$X^I(u,\bar u) = X^I_L(u) + X^I_R(\bar u)$. Then because of the boundary conditions we are led to consider the two possible independent sets of quantum modes

$$X_{(c)}(u,\bar u) = \begin{pmatrix} X^z_{q,L}(u) \\ X^{\bar z}_{q,R}(\bar u) \end{pmatrix}, \qquad \bar X_{(a)}(u,\bar u) = \begin{pmatrix} X^z_{q,R}(\bar u) \\ X^{\bar z}_{q,L}(u) \end{pmatrix}, \tag{16}$$

where $(X^z_{q,L}(u), X^{\bar z}_{q,R}(\bar u))^T$ can be any element of the basis of solutions and is labeled by a further basis index which is suppressed in this section. Similarly for $(X^z_{q,R}(\bar u), X^{\bar z}_{q,L}(u))^T$. After this splitting the couple of quantum boundary conditions in Eq. (8) become simply one condition, one for each set, explicitly[3]

$$\begin{aligned} &X^z_{q,L}(x+i0^+) = e^{i2\pi\alpha_t} X^{\bar z}_{q,R}(x-i0^+), \quad x_t < x < x_{t-1} \\ &X^z_{q,R}(x-i0^+) = e^{i2\pi\alpha_t} X^{\bar z}_{q,L}(x+i0^+), \quad x_t < x < x_{t-1} \end{aligned} \tag{17}$$

since the derivative boundary condition follows from the previous because $\partial_y X_L|_{y=0^+} = -i\partial_x X_L|_{y=0^+}$ and $\partial_y X_R|_{y=0^+} = i\partial_x X_R|_{y=0^+}$. In the case of the classical part the previous equations would have been stated using the derivatives because there is no obvious way of splitting the constants g_t into a left and right part. Using derivatives we miss the information on the constants g_t which has to be kept by adding further conditions. Explicitly the boundary conditions for the classical part can be written as

$$\begin{aligned} &\partial X^z_{cl,L}(x+i0^+) = e^{i2\pi\alpha_t} \partial X^{\bar z}_{cl,R}(x-i0^+), \quad x_t < x < x_{t-1} \\ &\partial X^z_{cl,R}(x-i0^+) = e^{i2\pi\alpha_t} \partial X^{\bar z}_{cl,L}(x+i0^+), \quad x_t < x < x_{t-1} \\ &X^z_{cl}(x_t, x_t) = f_t. \end{aligned} \tag{18}$$

The $X_{(c)}(u,\bar u)$ and $\bar X_{(a)}(u,\bar u)$ can be combined into two sets of functions defined on the whole complex plane minus the cut $[x_N, x_1]$ using the doubling trick as[4]

$$\begin{aligned} \mathcal{X}(z) &= \begin{cases} X^z_{q,L}(u) & z = u \text{ with } Im\, z > 0 \text{ or } z \in \mathbb{R} - [x_N, x_1] \\ e^{i2\pi\alpha_1} X^{\bar z}_{q,R}(\bar u) & z = \bar u \text{ with } Im\, z < 0 \text{ or } z \in \mathbb{R} - [x_N, x_1] \end{cases} \\ \bar{\mathcal{X}}(z) &= \begin{cases} X^{\bar z}_{q,L}(u) & z = u \text{ with } Im\, z > 0 \text{ or } z \in \mathbb{R} - [x_N, x_1] \\ e^{-i2\pi\alpha_1} X^{z}_{q,R}(\bar u) & z = \bar u \text{ with } Im\, z < 0 \text{ or } z \in \mathbb{R} - [x_N, x_1] \end{cases} \end{aligned} \tag{19}$$

with $\bar{\mathcal{X}}(z) \neq (\mathcal{X}(z))^*$ and very simple boundary conditions

$$\begin{aligned} &\begin{cases} \mathcal{X}(x+i0^+) = \mathcal{X}(x+i0^-) & x < x_N \text{ and } x > x_1 \\ \mathcal{X}(x+i0^+) = e^{i2\pi(\alpha_t-\alpha_1)} \mathcal{X}(x+i0^-) & x_t < x < x_{t-1} \text{ for } t = 2\dots N, \end{cases} \\ &\begin{cases} \bar{\mathcal{X}}(x+i0^+) = \bar{\mathcal{X}}(x+i0^-) & x < x_N \text{ and } x > x_1 \\ \bar{\mathcal{X}}(x+i0^+) = e^{-i2\pi(\alpha_t-\alpha_1)} \bar{\mathcal{X}}(x+i0^-) & x_t < x < x_{t-1} \text{ for } t = 2\dots N. \end{cases} \end{aligned} \tag{20}$$

[3] The untwisted case requires a slightly more general solution because of the $\log|u|$. I.e. we could ask $X^z_{q,L}(x+i0^+) = e^{i2\pi\alpha_t} X^{\bar z}_{q,R}(x-i0^+) + \delta_{Lt}$, $X^z_{q,R}(x-i0^+) = e^{i2\pi\alpha_t} X^{\bar z}_{q,L}(x+i0^+) + \delta_{Rt}$ but then the non derivative boundary conditions imply that $\delta_{Lt} + \delta_{Rt} = 0$ and that there is a unique non chiral solution $X^I_L(u) + X^I_R(\bar u)$.

[4] Notice that in order to perform the gluing this way we need the D_1 to be the last brane on the real positive axis.

Obviously the gluing can be performed in different ways, i.e. we can glue along $D_{\bar{t}}$ brane instead of D_1 as

$$\mathcal{X}^{(\bar{t})}(z) = \begin{cases} X^z_{q,L}(u) & z = u \text{ with } Im\, z > 0 \text{ or } z \in \mathbb{R} - (-\infty, x_{\bar{t}}] - [x_{\bar{t}+1}, \infty) \\ e^{i2\pi\alpha_{\bar{t}}} X^{\bar{z}}_{q,R}(\bar{u}) & z = \bar{u} \text{ with } Im\, z < 0 \text{ or } z \in \mathbb{R} - (-\infty, x_{\bar{t}}] - [x_{\bar{t}+1}, \infty) \end{cases}$$
$$\bar{\mathcal{X}}^{(\bar{t})}(z) = \begin{cases} X^{\bar{z}}_{q,L}(u) & z = u \text{ with } Im\, z > 0 \text{ or } z \in \mathbb{R} - (-\infty, x_{\bar{t}}] - [x_{\bar{t}+1}, \infty) \\ e^{-i2\pi\alpha_{\bar{t}}} X^{z}_{q,R}(\bar{u}) & z = \bar{u} \text{ with } Im\, z < 0 \text{ or } z \in \mathbb{R} - (-\infty, x_{\bar{t}}] - [x_{\bar{t}+1}, \infty), \end{cases} \tag{21}$$

again with $\bar{\mathcal{X}}^{(\bar{t})}(z) \neq (\mathcal{X}^{(\bar{t})}(z))^*$ and boundary conditions

$$\begin{cases} \mathcal{X}(x + i0^+) = \mathcal{X}(x + i0^-) & x_{\bar{t}} < x < x_{\bar{t}-1} \\ \mathcal{X}(x + i0^+) = e^{i2\pi(\alpha_t - \alpha_{\bar{t}})} \mathcal{X}(x + i0^-) & x_t < x < x_{t-1} \text{ for } t = 1 \dots N,\ t \neq \bar{t}, \end{cases}$$
$$\begin{cases} \bar{\mathcal{X}}(x + i0^+) = \bar{\mathcal{X}}(x + i0^-) & x_{\bar{t}} < x < x_{\bar{t}-1} \\ \bar{\mathcal{X}}(x + i0^+) = e^{-i2\pi(\alpha_t - \alpha_1)} \bar{\mathcal{X}}(x + i0^-) & x_t < x < x_{t-1} \text{ for } t = 1 \dots N \quad t \neq \bar{t}. \end{cases} \tag{22}$$

In the following we use always the gluing along D_1 if not otherwise stated.

The metric (12) can then be calculated for any pairs of these functions using the doubled formalism as (for more details see Appendix B)

$$(X_{(c)1}, X_{(c)2}) = (\bar{X}_{(a)1}, \bar{X}_{(a)2}) = 0 \tag{23}$$

$$\begin{aligned} (X_{(c)}, \bar{X}_{(a)}) &= 2e^{i2\pi\alpha_1} \oint_{z = r_0 \exp i\theta; \theta \in [-\pi, \pi]} dz\, \big(\mathcal{X}(\bar{z})\big)^* \frac{d\bar{\mathcal{X}}(z)}{dz} \\ &= 2e^{i2\pi\alpha_1} \int_{-\pi}^{\pi} d\theta\, \big(\mathcal{X}\big(r_0 e^{-i\theta}\big)\big)^* \frac{d\bar{\mathcal{X}}(r_0 e^{i\theta})}{d\theta}, \end{aligned} \tag{24}$$

where the last equation is meaningful since the product $(\mathcal{X}(r_0 e^{-i\theta}))^* \frac{d\bar{\mathcal{X}}(r_0 e^{i\theta})}{d\theta}$ is continuous at $\theta = 0$ despite the fact its factors are not. A direct computation similar to the one done to get the previous Eq. (24) gives

$$(\bar{X}_{(a)}, X_{(c)}) = 2e^{-i2\pi\alpha_1} \int_{-\pi}^{\pi} d\theta\, \big(\bar{\mathcal{X}}\big(r_0 e^{-i\theta}\big)\big)^* \frac{d\mathcal{X}(r_0 e^{i\theta})}{d\theta} \tag{25}$$

which is obviously compatible with the Hermitian property of the form and the product $(X_{(c)}, \bar{X}_{(a)})$ in Eq. (24).

2.3. Radial canonical quantization

We want now quantize the Euclidean string action (1). In order to do so we split the string field into its classical and quantum part (13) so that the action becomes

$$S_E = S_{E,cl} + \frac{1}{4\pi\alpha'} \int_H d\theta\, dr r \big(\partial_r X^z_q \partial_r X^{\bar{z}}_q + r^{-2} \partial_\theta X^z_q \partial_\theta X^{\bar{z}}_q\big). \tag{26}$$

We take $\tau_E = \ln r$ to be the time but we write all expressions as functions of r, this means that the Hamiltonian rescales $r = |u| = e^{\tau_E}$ and not that it shifts r. The canonical momentum is then given by

$$P_q = \begin{pmatrix} P_{q\,\bar{z}} \\ P_{q\,z} \end{pmatrix} = \frac{r}{2\pi\alpha'} \begin{pmatrix} \partial_r X_q^z \\ \partial_r X_q^{\bar{z}} \end{pmatrix}, \tag{27}$$

and the Euclidean Hamiltonian is by definition

$$H = \int_0^\pi d\theta \left(\pi\alpha' P_{q\,\bar{z}} P_{q\,z} + \frac{1}{4\pi\alpha'} \partial_\theta X_q^z \partial_\theta X_q^{\bar{z}} \right). \tag{28}$$

From the canonical commutation relation

$$\left[X_q^I(\theta), P_{q\,J}(\theta') \right] = i\delta_J^I \delta(\theta - \theta') \tag{29}$$

with $\delta(\theta - \theta')$ the delta function with the appropriate boundary conditions, we get

$$\left[H, X_q(\theta) \right] = -ir\partial_r X_q(\theta), \qquad \left[H, P_q(\theta) \right] = -ir\partial_r P_q(\theta). \tag{30}$$

In order to write the Hamiltonian using the creation and annihilation operators we need a way to extract them from the quantum field X_q by mean of a product of the quantum fluctuation X_q with an appropriate solution F. Using Eq. (12) the product can be written as

$$(F, X_q) = i \int_0^\pi d\theta \left(2\pi\alpha' F^\dagger P_q - r\partial_r F^\dagger X_q \right). \tag{31}$$

In particular the commutation relation of two such products is given by

$$\left[(F_1, X_q), (F_2, X_q) \right] = -2\pi\alpha' i \left(F_1, \sigma_1 F_2^* \right), \tag{32}$$

where σ_1 is the Pauli matrix. When choosing the two solutions $F_{1,2}$ to be any of the basis elements the previous commutation relation become

$$\begin{aligned} &\left[(X_{(c)n}, X_q), (X_{(c)m}, X_q) \right] = \left[(\bar{X}_{(a)n}, X_q), (\bar{X}_{(a)m}, X_q) \right] = 0 \\ &\left[(X_{(c)n}, X_q), (\bar{X}_{(a)m}, X_q) \right] = -2\pi\alpha' i \left(X_{(c)n}, \sigma_1 \bar{X}^*_{(a)m} \right), \end{aligned} \tag{33}$$

where $X_{(c)n}$ and $\bar{X}_{(c)n}$ belong to a basis for the quantum modes. In deriving the first equation in the first line we used the fact that any $\sigma_1 X^*_{(c)n}$ has the same boundary conditions as any of $X_{(c)m}$ and hence they can be expanded on the $X_{(c)m}$ basis. Then we can use Eq. (23) to set to zero the commutation relations involving two $X_{(c)}$ modes. Similarly for $\bar{X}_{(a)n}$.

We can now simplify the previous Eq. (33) if we notice that the function $\bar{\chi}_{\bullet n}(z)$ associated with $\sigma_1 \bar{X}^*_{(c)n}$ by the doubling trick can be rewritten as

$$\bar{\chi}_{\bullet n}(z) = e^{-i2\pi\alpha_1} \left[\bar{\chi}_n(\bar{z}) \right]^*, \tag{34}$$

and similarly for $\chi_{\bullet n}(z) = e^{i2\pi\alpha_1}[\chi_n(\bar{z})]^*$ which is associated with $\sigma_1 X^*_{(c)n}$. Using these relations, the hermiticity property of the product and the explicit expression in Eq. (24) for the product in terms of the doubled mode functions we can then write the non vanishing commutation relations for the basis elements (33) as

$$\big[(\bar{X}_{(a)n}, X_q), (X_{(c)m}, X_q)\big] = -4\pi\alpha' i\bigg[\oint_{|z|=r_0} dz\, \bar{\mathcal{X}}_m(z)\frac{d\mathcal{X}_n(z)}{dz}\bigg]^*. \tag{35}$$

We now expand the quantum fluctuation as[5]

$$X_q(u,\bar{u}) = \sum_{n\in\mathbb{Z}}(x_n X_{(c)n} + \bar{x}_n \bar{X}_{(a)n}). \tag{36}$$

We suppose that the (doubling of) quantum modes satisfy a reality condition like

$$\big(\chi_n(\bar{z})\big)^* = e^{i\beta}\chi_n(z), \qquad \big(\bar{\chi}_n(\bar{z})\big)^* = e^{i\hat{\beta}}\bar{\chi}_n(z), \quad \beta,\hat{\beta}\in\mathbb{R} \tag{37}$$

and a normalization condition

$$\big(X_{(c)n}, \sigma_1 \bar{X}^*_{(a)m}\big) = -N_n\delta_{n+m,s} \tag{38}$$

with s an integer and $N_n^* = -N_n e^{-i\beta-i\hat{\beta}}$. It follows that the creation and annihilation operators can be obtained as

$$x_n = \frac{e^{i2\pi\alpha_1+i\beta}}{N_{s-n}}(\bar{X}_{(a)s-n}, X_q), \qquad \bar{x}_n = -\frac{e^{i2\pi\alpha_1+i\hat{\beta}}}{N_{s-n}}(X_{(c)s-n}, X_q), \tag{39}$$

and that they satisfy the commutation relations

$$[x_n, \bar{x}_m] = i2\pi\alpha'\frac{e^{i\beta+i\hat{\beta}}}{N_{s-n}}\delta_{n+m,s}. \tag{40}$$

On general ground the Euclidean Hamiltonian defined in Eq. (28) is generically *not* diagonal in the mode operators since the Lagrangian is time dependent or that is the in states are not equal to out states and hence they cannot be eigenstates of the Hamiltonian.[6] Another way to understand this is to notice that the Hamiltonian rescales u but the generic mode function is not a homogeneous function of u. Explicitly the Hamiltonian can be expanded in modes as

$$H = \sum_{n,m\in\mathbb{Z}} h_{nm}x_n\bar{x}_m, \qquad h_{nm} = \frac{1}{2\pi\alpha'}e^{i2\pi\alpha_1}(\bar{X}_{(a)n}, r\partial_r X_{(c)m}), \tag{41}$$

where h_{nm} is constant despite the Lagrangian is time dependent because of the boundary conditions. This happens because $r\partial_r X_{(c)m}$ are also solutions of the e.o.m. and therefore satisfy the same "selection rules" as $X_{(c)m}$.

H is not diagonal in modes even if it is self-adjoint w.r.t. the usual L^2 metric since the usual metric differs from the product used. In other words H is self-adjoint for any time τ w.r.t. the usual L^2 metric but the basis depend on time.

[5] Notice that generically neither X nor its derivatives $\partial_u X$ are conformal fields since they are not well behaved under time evolution $u \to \lambda u$.

[6] Since the system is time dependent it is not meaningful to consider eigenstates of the Hamiltonian and this is why the normal modes do not diagonalize the Hamiltonian while their commutation relations are time independent.

Finally, notice that we have not written any normal ordering since its definition depends on the vacuum and we have not specified any. Neither we will do it since we will use an overlap approach which uses different Hamiltonian for different worldsheet times (see Sections 2.7 and 3).

2.4. The $N = 0$ case: the usual untwisted string

Since the previous product for the modes is different from the normal one it is of interest to see how it works in the usual and simplest case with ND boundary conditions. It is also worth to check that we get the same commutation relations as in the quantization with the normal product.

We consider a single D1 brane D_t in $\mathbb{R}^2$. In this case the boundary conditions are simply for all $x \in \mathbb{R}$

$$e^{-i\pi\alpha_t} \partial_y X^z(u,\bar{u})\big|_{u=x+i0^+} + e^{i\pi\alpha_t} \partial_y X^{\bar{z}}(u,\bar{u})\big|_{u=x+i0^+} = 0,$$
$$e^{-i\pi\alpha_t} X^z(u,\bar{u})\big|_{u=x+i0^+} - e^{i\pi\alpha_t} X^{\bar{z}}(u,\bar{u})\big|_{u=x+i0^+} = 2ig_t. \tag{42}$$

The classical solution is simply given by

$$X_{cl}\big(u,\bar{u};\{g_t,\alpha_t\}\big) = e^{i\pi\alpha_t} i g_t \begin{pmatrix} 1 \\ -e^{-i2\pi\alpha_t} \end{pmatrix}. \tag{43}$$

The two sets of modes in Eq. (16) which obey the quantum boundary conditions are

$$X_{(c)n}\big(u,\bar{u};\{\alpha_t\}\big) = \begin{pmatrix} \frac{u^{-n}}{n} \\ e^{-i2\pi\alpha_t}\frac{\bar{u}^{-n}}{n} \end{pmatrix}, \qquad \bar{X}_{(a)n}\big(u,\bar{u};\{\alpha_t\}\big) = \begin{pmatrix} \frac{\bar{u}^{-n}}{n} \\ e^{-i2\pi\alpha_t}\frac{u^{-n}}{n} \end{pmatrix}, \quad n \neq 0 \tag{44}$$

and

$$\hat{X}_0\big(u,\bar{u};\{\alpha_t\}\big) = \begin{pmatrix} \log|u| \\ e^{-i2\pi\alpha_t}\log|u| \end{pmatrix}, \qquad X_*\big(\{\alpha_t\}\big) = \begin{pmatrix} 1 \\ e^{-i2\pi\alpha_t} \end{pmatrix}. \tag{45}$$

Notice however that $\hat{X}_0$ is different from all the others elements since its components are neither holomorphic nor antiholomorphic since neither the holomorphic nor the antiholomorphic parts satisfy the boundary conditions separately.

Furthermore X_* is not proportional to the classical solution X_{cl}, as it could at first glance look because of the sign of the second component.

Using the doubling trick the previous modes can be combined into the following functions defined in the whole complex plane

$$\mathcal{X}_n(z) = \frac{z^{-n}}{n}, \qquad \bar{\mathcal{X}}_n(z) = e^{-i2\pi\alpha_t}\frac{z^{-n}}{n}, \qquad \mathcal{X}_0(z,\bar{z}) = \log|z|, \qquad \mathcal{X}_* = 1. \tag{46}$$

Using these functions and Eqs. (23) and (24) we can compute the non vanishing products of these elements

$$(X_{(c)n}, \bar{X}_{(a)m}) = \big[(\bar{X}_{(a)m}, X_{(c)n})\big]^* = -\frac{4\pi i}{n}\delta_{n+m,0}$$
$$(X_*, \hat{X}_0) = \big[(\hat{X}_0, X_*)\big]^* = -2\pi i, \tag{47}$$

where all products involving $\hat{X}_0$ cannot be computed using Eq. (24). In fact this equation is derived under the hypothesis that the two functions can be assembled into (anti)holomorphic doubled functions defined on the complex plane. Therefore these products must be computed from the original definition given in Eq. (12).

We can now expand the quantum fluctuations as

$$X_q(u,\bar{u}) = x_0 X_* + \hat{x}_0 \hat{X}_0 + \sum_{n\neq 0}(x_n X_{(c)n} + \bar{x}_n \bar{X}_{(a)n})$$
$$= \begin{pmatrix} x_0 + \hat{x}_0 \ln|u| + \sum_{n\neq 0}(x_n \frac{u^{-n}}{n} + \bar{x}_n \frac{\bar{u}^{-n}}{n}) \\ e^{-i2\pi\alpha_t}(x_0 + \hat{x}_0 \ln|u| + \sum_{n\neq 0}(x_n \frac{\bar{u}^{-n}}{n} + \bar{x}_n \frac{u^{-n}}{n})) \end{pmatrix}. \tag{48}$$

The operators can be extracted from the previous expansion as

$$x_0 = \frac{1}{2\pi i}(\hat{X}_0, X_q)$$
$$\hat{x}_0 = -\frac{1}{2\pi i}(X_*, X_q)$$
$$x_n = \frac{-n}{4\pi i}(\bar{X}_{(a)-n}, X_q)$$
$$\bar{x}_n = \frac{n}{4\pi i}(X_{(c)-n}, X_q), \tag{49}$$

and then we can compute the non-vanishing commutation relations as

$$[x_n, \bar{x}_m] = e^{i2\pi\alpha_t}\frac{\alpha'}{2} m \delta_{n+m,0}$$
$$[x_0, \hat{x}_0] = \alpha' e^{i2\pi\alpha_t}. \tag{50}$$

These are exactly the usual commutation relations and expansion once we identify ($n > 0$)

$$x_n = i\frac{\sqrt{2\alpha'}}{2} e^{i\pi\alpha_t}\bar{\alpha}_n, \qquad x_{-n} = i\frac{\sqrt{2\alpha'}}{2} e^{i\pi\alpha_t}\alpha_{-n},$$
$$\bar{x}_n = i\frac{\sqrt{2\alpha'}}{2} e^{i\pi\alpha_t}\alpha_n, \qquad \bar{x}_{-n} = i\frac{\sqrt{2\alpha'}}{2} e^{i\pi\alpha_t}\bar{\alpha}_{-n},$$
$$x_0 = e^{i\pi\alpha_t}\frac{x^{\hat{1}}}{\sqrt{2}}, \qquad \hat{x}_0 = e^{i\pi\alpha_t}\frac{-i2\alpha' p^{\hat{1}}}{\sqrt{2}}. \tag{51}$$

As usual the vacuum is defined as

$$p^{\hat{1}}|0\rangle = \alpha_n|0\rangle = \bar{\alpha}_n|0\rangle = 0 \quad n > 0. \tag{52}$$

We can then compute the untwisted Green functions

$$G^{zz}_{U_t}(u,\bar{u};v,\bar{v};\alpha_t) = \left[X^{(+)}(u,\bar{u}), X^{(-)}(v,\bar{v})\right] = \left(-i\frac{1}{2}\sqrt{2\alpha'}e^{i\pi\alpha_t}\right)^2 \ln|u-\bar{v}|^2$$
$$G^{\bar{z}\bar{z}}_{U_t}(u,\bar{u};v,\bar{v};\alpha_t) = \left[\bar{X}^{(+)}(u,\bar{u}), \bar{X}^{(-)}(v,\bar{v})\right] = \left(-i\frac{1}{2}\sqrt{2\alpha'}e^{-i\pi\alpha_t}\right)^2 \ln|u-\bar{v}|^2$$
$$G^{z\bar{z}}_{U_t}(u,\bar{u};v,\bar{v};\alpha_t) = \left[X^{(+)}(u,\bar{u}), \bar{X}^{(-)}(v,\bar{v})\right] = \left(-i\frac{1}{2}\sqrt{2\alpha'}\right)^2 \ln|u-v|^2. \tag{53}$$

Notice that $G^{z\bar{z}}_U$ does not feel whether the brane is rotated while both G^{zz}_U and $G^{\bar{z}\bar{z}}_U$ do because of the phases.

Finally, we consider two parallel branes non-overlapping then the non-derivative boundary conditions become

$$e^{-i\pi\alpha_t} X^z(u,\bar u)\big|_{u=x+i0^+} - e^{i\pi\alpha_t} X^{\bar z}(u,\bar u)\big|_{u=x+i0^+} = \sqrt{2} i g_{t+1}, \quad x<0$$
$$e^{-i\pi\alpha_t} X^z(u,\bar u)\big|_{u=x+i0^+} - e^{i\pi\alpha_t} X^{\bar z}(u,\bar u)\big|_{u=x+i0^+} = \sqrt{2} i g_{t}, \quad x>0 \tag{54}$$

while the derivative one is left unchanged. The classical solution is then

$$X_{cl}\big(u,\bar u;\{g_t,\alpha_t\}\big) = e^{i\pi\alpha}\, i\, g_t \begin{pmatrix} 1 \\ -e^{-i2\pi\alpha_t} \end{pmatrix} + e^{i\pi\alpha} \frac{g_{t+1}-g_t}{\pi} \begin{pmatrix} \frac{1}{2}\ln\frac{u}{\bar u} \\ e^{-i2\pi\alpha_t}\frac{1}{2}\ln\frac{\bar u}{u} \end{pmatrix} \tag{55}$$

while the quantum fluctuations remain unchanged. Notice however that the classical solution has infinite action because of the finite and constant energy density of the stretched string.

2.5. An $N=2$ case: the usual twisted string

We now consider the quantization of two $D1$ branes at angles, D_t and D_{t+1}. This is the usual setup where the string describes a twisted in and out state. This means that the twist fields are located at $x=0$ and $x=\infty$ so that the boundary conditions read

$$\begin{cases} e^{-i\pi\alpha_{t+1}} \partial_y X^z(u,\bar u)\big|_{u=x+i0^+} + e^{i\pi\alpha_{t+1}} \partial_y X^{\bar z}(u,\bar u)\big|_{u=x+i0^+} = 0 \\ e^{-i\pi\alpha_{t+1}} X^z(u,\bar u)\big|_{u=x+i0^+} - e^{i\pi\alpha_{t+1}} X^{\bar z}(u,\bar u)\big|_{u=x+i0^+} = 2 i g_{t+1}, \end{cases} \quad x<0$$
$$\begin{cases} e^{-i\pi\alpha_{t}} \partial_y X^z(u,\bar u)\big|_{u=x+i0^+} + e^{i\pi\alpha_{t}} \partial_y X^{\bar z}(u,\bar u)\big|_{u=x+i0^+} = 0 \\ e^{-i\pi\alpha_{t}} X^z(u,\bar u)\big|_{u=x+i0^+} - e^{i\pi\alpha_{t}} X^{\bar z}(u,\bar u)\big|_{u=x+i0^+} = 2 i g_{t}, \end{cases} \quad x>0. \tag{56}$$

The classical solution is then simply given by the constant

$$X_{cl}\big(u,\bar u;\{g_t,\alpha_t;g_{t+1},\alpha_{t+1}\}\big) = \begin{pmatrix} f_t \\ f_t^* \end{pmatrix}, \qquad f_t = \frac{e^{i\pi\alpha_{t+1}} g_t - e^{i\pi\alpha_t} g_{t+1}}{\sin\pi(\alpha_{t+1}-\alpha_t)} \tag{57}$$

which is a special case of Eq. (4). The quantum fluctuations can be expanded on

$$X_{(c)n}\big(u,\bar u;\{\alpha_t,\alpha_{t+1}\}\big) = \begin{pmatrix} \frac{u^{-n-\bar\epsilon_t}}{n+\bar\epsilon_t} \\ e^{-i2\pi\alpha_t}\frac{\bar u^{-n-\bar\epsilon_t}}{n+\bar\epsilon_t} \end{pmatrix},$$
$$\bar X_{(a)n}\big(u,\bar u;\{\alpha_t,\alpha_{t+1}\}\big) = \begin{pmatrix} \frac{\bar u^{-n-\epsilon_t}}{n+\epsilon_t} \\ e^{-i2\pi\alpha_t}\frac{u^{-n-\epsilon_t}}{n+\epsilon_t} \end{pmatrix}. \tag{58}$$

These can be combined into the function defined on the whole complex plane minus the real negative axis as follows from Eq. (19)[7]

$$\mathcal{X}_n(z) = \frac{z^{-n-\bar\epsilon_t}}{n+\bar\epsilon_t}, \qquad \bar{\mathcal{X}}_n(z) = e^{-i2\pi\alpha_t}\frac{z^{-n-\epsilon_t}}{n+\epsilon_t}, \quad z\in\mathbb{C}-\mathbb{R}^- \tag{59}$$

with $\epsilon_t = \alpha_{t+1}-\alpha_t+\theta(\alpha_t-\alpha_{t+1})$ and $\bar\epsilon_t = 1-\epsilon_t$ so that $0<\epsilon_t,\bar\epsilon_t<1$.

The non-vanishing products of these elements are

$$(X_{(c)n}, \bar X_{(a)m}) = \big[(\bar X_{(a)m}, X_{(c)n})\big]^* = -\frac{4\pi i}{n+\bar\epsilon_t}\delta_{n+m+1,0}. \tag{60}$$

As in the previous case the quantum fluctuations can be expanded as

[7] We choose D_t to be on the real positive axis in order to be able to apply this general formula, in particular this means that the cut is on the negative real axis and $-\pi<\arg(z)<\pi$.

$$X_q(u,\bar u)=\sum_{n\in\mathbb{Z}}\big(x_n X_{(c)n}(u,\bar u)+\bar x_n \bar X_{(a)n}(u,\bar u)\big)$$
$$=\begin{pmatrix} \sum_{n\in\mathbb{Z}}(x_n\frac{u^{-n-\bar\epsilon_t}}{n+\bar\epsilon_t}+\bar x_n\frac{\bar u^{-n-\epsilon_t}}{n+\epsilon_t}) \\ e^{-i2\pi\alpha_t}\sum_{n\in\mathbb{Z}}(x_n\frac{\bar u^{-n-\bar\epsilon_t}}{n+\bar\epsilon_t}+\bar x_n\frac{u^{-n-\epsilon_t}}{n+\epsilon_t})\end{pmatrix}. \tag{61}$$

The coefficients can then be extracted as

$$x_n=\frac{n+\bar\epsilon_t}{4\pi i}(\bar X_{(a)-n-1},X_q)$$
$$\bar x_n=\frac{n+\epsilon_t}{4\pi i}(X_{(c)-n-1},X_q) \tag{62}$$

and their non-vanishing commutation relations are

$$[\bar x_n,x_m]=\frac{\alpha'}{2}e^{i2\pi\alpha_t}(m+\bar\epsilon_t)\delta_{m+n+1,0}. \tag{63}$$

With the identification ($n\ge 0$)

$$x_n=i\frac{\sqrt{2\alpha'}}{2}e^{i\pi\alpha_t}\bar\alpha_{n+\bar\epsilon_t},\qquad x_{-(n+1)}=i\frac{\sqrt{2\alpha'}}{2}e^{i\pi\alpha_t}\alpha^\dagger_{n+\epsilon_t},$$
$$\bar x_n=i\frac{\sqrt{2\alpha'}}{2}e^{i\pi\alpha_t}\alpha_{n+\epsilon_t},\qquad \bar x_{-(n+1)}=i\frac{\sqrt{2\alpha'}}{2}e^{i\pi\alpha_t}\bar\alpha^\dagger_{n+\bar\epsilon_t}, \tag{64}$$

we recover the usual commutation relations

$$\big[\alpha_{n+\epsilon_t},\alpha^\dagger_{m+\epsilon_t}\big]=(n+\epsilon_t)\delta_{m,n},\qquad \big[\bar\alpha_{n+\bar\epsilon_t},\bar\alpha^\dagger_{m+\bar\epsilon_t}\big]=(n+\bar\epsilon_t)\delta_{m,n}. \tag{65}$$

Finally, we can write the usual expansion for the quantum fluctuations as

$$X_q(u,\bar u)=i\frac{1}{2}\sqrt{2\alpha'}\sum_{n=0}^{\infty}\begin{pmatrix} e^{i\pi\alpha_t}[\frac{\bar\alpha_{n+\bar\epsilon_t}}{n+\bar\epsilon_t}u^{-(n+\bar\epsilon_t)}-\frac{\alpha^\dagger_{n+\epsilon_t}}{n+\epsilon_t}u^{n+\epsilon_t}] \\ e^{-i\pi\alpha_t}[-\frac{\bar\alpha^\dagger_{n+\bar\epsilon_t}}{n+\bar\epsilon_t}u^{n+\bar\epsilon_t}+\frac{\alpha_{n+\epsilon_t}}{n+\epsilon_t}u^{-(n+\epsilon_t)}]\end{pmatrix}$$
$$+i\frac{1}{2}\sqrt{2\alpha'}\sum_{n=0}^{\infty}\begin{pmatrix} e^{i\pi\alpha_t}[-\frac{\bar\alpha^\dagger_{n+\bar\epsilon_t}}{n+\bar\epsilon_t}\bar u^{n+\bar\epsilon_t}+\frac{\alpha_{n+\epsilon_t}}{n+\epsilon_t}\bar u^{-(n+\epsilon_t)}] \\ e^{-i\pi\alpha_t}[\frac{\bar\alpha_{n+\bar\epsilon_t}}{n+\bar\epsilon_t}\bar u^{-(n+\bar\epsilon_t)}-\frac{\alpha^\dagger_{n+\epsilon_t}}{n+\epsilon_t}\bar u^{n+\epsilon_t}]\end{pmatrix}. \tag{66}$$

The vacuum is defined in the usual way by

$$\alpha_{n+\epsilon_t}|T_t\rangle=\bar\alpha_{n+\bar\epsilon_t}|T_t\rangle=0\quad n\ge 0. \tag{67}$$

We can then compute the $N=2$ twisted Green functions

$$G^{zz}_{N=2,T}\big(u,\bar u;v,\bar v;\{0,\alpha_t;\infty,\alpha_{t+1}\}\big)$$
$$=\big[X_q^{z(+)}(u,\bar u),X_q^{z(-)}(v,\bar v)\big]$$
$$=-\left(-i\frac{1}{2}\sqrt{2\alpha'}e^{i\pi\alpha_t}\right)^2\left[\frac{1}{\epsilon_t}\left(\frac{v}{\bar u}\right)^{\epsilon_t}{}_2F_1\left(1,\epsilon_t;1+\epsilon_t;\frac{v}{\bar u}\right)\right.$$
$$\left.+\frac{1}{\bar\epsilon_t}\left(\frac{\bar v}{u}\right)^{\bar\epsilon_t}{}_2F_1\left(1,\bar\epsilon_t;1+\bar\epsilon_t;\frac{\bar v}{u}\right)\right]$$

$$
\begin{aligned}
&G^{\bar z\bar z}_{N=2,T}\big(u,\bar u; v,\bar v; \{0,\alpha_t;\infty,\alpha_{t+1}\}\big)\\
&\quad=\big[X_q^{\bar z(+)}(u,\bar u), X_q^{\bar z(-)}(v,\bar v)\big]\\
&\quad=-\left(-i\frac{1}{2}\sqrt{2\alpha'}e^{-i\pi\alpha_t}\right)^2\left[\frac{1}{\epsilon_t}\left(\frac{\bar v}{u}\right)^{\epsilon_t}{}_2F_1\left(1,\epsilon_t;1+\epsilon_t;\frac{\bar v}{u}\right)\right.\\
&\qquad\left.+\frac{1}{\bar\epsilon_t}\left(\frac{v}{\bar u}\right)^{\bar\epsilon_t}{}_2F_1\left(1,\bar\epsilon_t;1+\bar\epsilon_t;\frac{v}{\bar u}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&G^{z\bar z}_{N=2,T}\big(u,\bar u; v,\bar v; \{0,\alpha_t;\infty,\alpha_{t+1}\}\big)\\
&\quad=\big[X_q^{z(+)}(u,\bar u), X_q^{\bar z(-)}(v,\bar v)\big]\\
&\quad=G^{\bar z z}_{N=2}\big(v,\bar v;u,\bar u;\{0,\alpha_t;\infty,\alpha_{t+1}\}\big)=G^{\bar z z}_{N=2}\big(u,\bar u;v,\bar v;\{0,\alpha_t;\infty,\alpha_{t+1}\}\big)\\
&\quad=-\left(-i\frac{1}{2}\sqrt{2\alpha'}\right)^2\left[\frac{1}{\epsilon_t}\left(\frac{\bar v}{\bar u}\right)^{\epsilon_t}{}_2F_1\left(1,\epsilon_t;1+\epsilon_t;\frac{\bar v}{\bar u}\right)\right.\\
&\qquad\left.+\frac{1}{\bar\epsilon_t}\left(\frac{v}{u}\right)^{\bar\epsilon_t}{}_2F_1\left(1,\bar\epsilon_t;1+\bar\epsilon_t;\frac{v}{u}\right)\right],
\end{aligned}
\tag{68}
$$

where we have used

$$
{}_2F_1(1,\epsilon;1+\epsilon;x)=\sum_{n=0}^{\infty}\frac{\epsilon}{n+\epsilon}x^n\quad |x|<1 \tag{69}
$$

as follows from the general expression for the hypergeometric function ${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}x^n$ with $(a)_n=\Gamma(a+n)/\Gamma(a)$ the Pochhammer symbol. They have the following symmetry properties

$$
\begin{aligned}
G^{IJ}_{N=2,T}\big(u,\bar u;v,\bar v;\{0,\alpha_t;\infty,\alpha_{t+1}\}\big)&=G^{JI}_{N=2,T}\big(v,\bar v;u,\bar u;\{0,\alpha_t;\infty,\alpha_{t+1}\}\big)\\
&=G^{IJ}_{N=2,T}\big(v,\bar v;u,\bar u;\{0,\alpha_t;\infty,\alpha_{t+1}\}\big)
\end{aligned}
\tag{70}
$$

which follow from the hypergeometric transformation properties, in particular ${}_2F_1(1,\epsilon;1+\epsilon;x)=\frac{\epsilon}{\bar\epsilon x}{}_2F_1(1,\bar\epsilon;1+\bar\epsilon;1/x)$ implies the last equality.

2.6. An $N=3$ case: in and out twisted strings

In this section we want to exam the next simplest example which corresponds to the case of $N=3$ twists, one located in the origin, one at infinity and the third in an arbitrary point of the positive real axis. This example shows the main issues to be understood and solved. When we look after the classical solution with finite action it is naturale to consider a basis of the derivatives which can be used to compute the classical solution like $\partial_z\chi_{cl}(z)$. Nevertheless when we try to use their integrated expressions $\int dz\partial_z\chi_{cl}(z)$ as a basis for the quantum fluctuations we realize immediately that they do not satisfy all the boundary conditions. We are therefore forced to consider a combination of them but in doing so it seems naively that we cannot find all possible asymptotics behaviors for $u\to 0$. The apparent solution of this problem seems to be very simple. It amounts to consider from start a basis $\chi_q(z)$ and not a basis for derivatives. This basis is however not apt to easily find the classical solution since the configuration with finite action is the combination of infinite basis elements. Moreover the simplest basis is not orthogonal and therefore we must orthogonalize it. In doing so combinations of infinite basis elements must

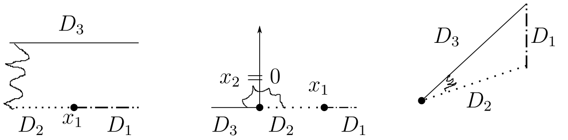

Fig. 6. Three different pictures of the setup with incoming and outgoing twisted strings with a discontinuity on the boundary at x_1. In the pictures an incoming string between D_2 and D_3 is pictured.

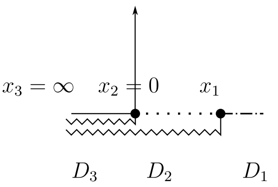

Fig. 7. The cuts of the classical and quantum basis.

be considered. It happens then that some new basis elements have a finite convergence radius and must be analytically continued. In performing the analytic continuation we find that these new basis elements do not satisfy anymore the original and required boundary conditions. Hence one of the new basis element must be used to make further combinations which respect the boundary conditions and in order to preserve orthogonality it must be dropped as independent basis element. In this way we find again the original problem, i.e. we cannot find all possible asymptotics behaviors for $u \to 0$. Since we are not yet in stand of understanding completely this issue we use the classical overlap approach to derive the correlators.

Let us now explain in more details the issues discussed above.

In order to apply the general formula (19) we need setting D_1 as the last brane on the real positive axis therefore our setup is with D_3 for $x_3 = -\infty < x < x_2 = 0$, D_2 for $x_2 = 0 < x < x_1$ and D_1 for $x_1 < x$, i.e. we have the boundary conditions

$$\begin{aligned}
&Re\left(e^{-\pi\alpha_3}\partial_y X^z\right) = 0, \qquad Im\left(e^{-\pi\alpha_3} X^z\right) = g_3, \quad x_3 = -\infty < x < x_2 = 0\\
&Re\left(e^{-\pi\alpha_2}\partial_y X^z\right) = 0, \qquad Im\left(e^{-\pi\alpha_2} X^z\right) = g_2, \quad x_2 = 0 < x < x_1\\
&Re\left(e^{-\pi\alpha_1}\partial_y X^z\right) = 0, \qquad Im\left(e^{-\pi\alpha_1} X^z\right) = g_1, \quad x_1 < x
\end{aligned} \tag{71}$$

The setup is shown in Fig. 6.

A basis for the derivative of the classical solution on $\mathbb{C} - [x_N, x_1] = \mathbb{C} - (-\infty, x_1]$ is given by

$$\begin{aligned}
&\partial\chi_{cl,n}(z) = z^{-\bar{\epsilon}_2 - n - 1}(z - x_1)^{-\bar{\epsilon}_1}\\
&\partial\bar{\chi}_{cl,m}(z) = e^{-i2\pi\alpha_1} z^{-\epsilon_2 - m - 1}(z - x_1)^{-\epsilon_1},
\end{aligned} \tag{72}$$

where the power of the cut at $z = x_1$ is dictated by the boundary conditions and the finiteness of the contribute from the area around $u = x_1$ to the Euclidean action S_E. The same requirement from $u = 0$ implies $n = m = 1$. Since $\sum_{t=1}^{3}\epsilon_t = M = 1$ only $\partial\chi_{cl,0}$ has then a finite Euclidean action. The cuts of these basis elements are pictured in Fig. 7.

Then the classical solution is given by[8]

$$\chi_{cl}(z) = f_2 + c_0 \int_{w=0}^{z} dw\, w^{-\bar\epsilon_2}(w - x_1)^{-\bar\epsilon_1}$$
$$= f_2 + c_0 \frac{c_0(-x_1)^{-\bar\epsilon_1}}{\epsilon_2} z^{\epsilon_2}{}_2F_1\left(\epsilon_2, \bar\epsilon_1; 1 + \epsilon_2; \frac{z}{x_1}\right), \quad (73)$$

where the constant c_0 is fixed by the condition $\chi_{cl}(x_1) = f_1$.

We can now try to use the integrated Eqs. (72) to find a basis for the in quantum fluctuations, i.e. for the fluctuations which behave as the solutions (59) as $z \to 0$. So we can for example write

$$\chi_n(z) \sim \int_{x_1}^{z} dw\, \partial\chi_{cl,n}(w), \quad (74)$$

where the lower integration extreme is chosen to fulfill the quantum boundary condition $\chi_{cl}(x_1) = 0$ which can be understood as the fact that when $g_t = 0$ for ts all branes intersect at the origin, i.e. $f_t = 0$ for the quantum fluctuations.

The integrals with $n \geq 1$ behave as $const_{n,0} + const_{n,1}\, z^{-n-\bar\epsilon_2}$ for $z \to 0$. This is not the desired behavior for $z \to 0$ because of $const_{n,0}$. The required behavior needs $const_{n,0} = 0$ as follows from Eq. (59) for $z \to 0$. Therefore we must subtract the constants $const_{n,0}$. This can be achieved by making linear combinations in many different ways, for example between $\partial\chi_{cl,n}(w)$ with $n > 1$ and $\partial\chi_{cl,n=1}(w)$. This approach produces a basis but gives the impression that we are missing the $n = 1$ mode. Even if we choose a different way of substracting the constants $const_{n,0}$ we get one constraint and hence one mode "less" than expected.

To investigate better this point we start directly with a basis[9] for the quantum fluctuations which satisfies all the boundary conditions given by

$$\xi_{q,n}(z) = \frac{1}{n + \bar\epsilon_2} z^{-\bar\epsilon_2 - n}\left(\frac{z}{x_1} - 1\right)^{\epsilon_1}$$
$$\bar\xi_{q,n}(z) = e^{-i2\pi\alpha_1} \frac{1}{n + \epsilon_2} z^{-\epsilon_2 - n}\left(\frac{z}{x_1} - 1\right)^{\bar\epsilon_1}, \quad (75)$$

where cuts are running from $-\infty$ to either $x_2 = 0$ or to x_1 as in Fig. 7. They are located in this way in order to satisfy the boundary conditions. To these functions defined on $\mathbb{C} - [x_N, x_1] = \mathbb{C} - (-\infty, x_1]$ correspond the modes defined on the upper half plane H

$$Y_{(c)n}(u, \bar u) = \begin{pmatrix} \frac{u^{-\bar\epsilon_2 - n}}{n + \bar\epsilon_2}(\frac{u}{x_1} - 1)^{\epsilon_1} \\ e^{i2\pi\alpha_1} \frac{\bar u^{-\bar\epsilon_2 - n}}{n + \bar\epsilon_2}(\frac{\bar u}{x_1} - 1)^{\epsilon_1} \end{pmatrix},$$
$$\bar Y_{(a)n}(u, \bar u) = \begin{pmatrix} \frac{\bar u^{-\epsilon_2 - n}}{n + \epsilon_2}(\frac{\bar u}{x_1} - 1)^{\bar\epsilon_1} \\ e^{i2\pi\alpha_1} \frac{u^{-\epsilon_2 - n}}{n + \epsilon_2}(\frac{u}{x_1} - 1)^{\bar\epsilon_1} \end{pmatrix}. \quad (76)$$

[8] The phase of $(-x_1)^{-\bar\epsilon_1}$ depends on the consider $z \in H$ in which case $(-x_1)^{-\bar\epsilon_1} = (x_1)^{-\bar\epsilon_1} e^{-i\pi\bar\epsilon_1}$ or $z \in H^-$ for which $(-x_1)^{-\bar\epsilon_1} = (x_1)^{-\bar\epsilon_1} e^{+i\pi\bar\epsilon_1}$.

[9] There are other possible basis but this seems to be the most natural.

Notice that it is not immediate to use this basis to find the classical solution since all these modes have infinite Euclidean action and only a combination of them reproduce the classical solution. In particular

$$\partial\xi_{q,n}(z) = x_1^{\bar\epsilon_1}\left(\partial\chi_{cl,n}(z) - \frac{n-\epsilon_3}{n+\bar\epsilon_2}\frac{1}{x_1}\partial\chi_{cl,n-1}(z)\right) \tag{77}$$

implies that the derivative of the classical solution $\partial\chi_{cl,1}(z)$ is given by an infinite sum of quantum basis elements. We now compute the non vanishing products of two quantum basis elements and get

$$(Y_{(c)n}, \bar Y_{(a)m}) = \left[(\bar Y_{(a)m}, Y_{(c)n})\right]^* = \frac{-4\pi i}{(n+\bar\epsilon_2)}\left[-\delta_{n+m,-1} + \frac{m+\epsilon_3}{m+\epsilon_2}\frac{1}{x_1}\delta_{n+m,0}\right]. \tag{78}$$

This means that these modes are not orthogonal and we need to find combinations which are orthogonal. For any $B\in\mathbb{Z}$ we can then search orthogonal solutions in the form

$$\chi_{q,n}(z) = \begin{cases} (\sum_{k=-B}^{n} c_{n\,k}\frac{1}{n+\bar\epsilon_2}z^{-\bar\epsilon_2-n})(\frac{z}{x_1}-1)^{\epsilon_1} & n\ge -B \\ \frac{1}{n+\bar\epsilon_2}z^{-\bar\epsilon_2-n}(\frac{z}{x_1}-1)^{\epsilon_1} & n\le -B-1 \end{cases}$$
$$\bar\chi_{q,n}(z) = \begin{cases} e^{-i2\pi\alpha_1}(\sum_{k=-\infty}^{n} \bar c_{n\,k}\frac{1}{n+\epsilon_2}z^{-\epsilon_2-n})(\frac{z}{x_1}-1)^{\bar\epsilon_1} & n\ge B \\ e^{-i2\pi\alpha_1}\frac{1}{n+\epsilon_2}z^{-\epsilon_2-n}(\frac{z}{x_1}-1)^{\bar\epsilon_1} & n\le B-1. \end{cases} \tag{79}$$

The reason of such an ansatz is that given the integer B, $\chi_{q,n} = \xi_{q,n}$ with $n\le -B-1$ and $\bar\chi_{q,m} = \bar\xi_{q,m}$ with $m\ge B-1$ are automatically orthogonal and are the biggest set of orthogonal elements among the original ξ and $\bar\xi$. The explicit solution is then

$$c_{n\,k} = \left(\frac{1}{x_1}\right)^{n-k}\frac{(-n-\epsilon_3)_{n-k}}{(-n-\bar\epsilon_2)_{n-k}}c_{n\,n}$$
$$\bar c_{n\,k} = \left(\frac{1}{x_1}\right)^{n-k}\frac{(-n+\epsilon_3)_{n-k}}{(-n-\epsilon_2)_{n-k}}\bar c_{n\,n}. \tag{80}$$

In particular we recognize that for $n\ge B$

$$\bar\chi_{q,n}(z) = e^{-i2\pi\alpha_1}\frac{1}{n+\epsilon_2}z^{-\epsilon_2-n}\left(\frac{z}{x_1}-1\right)^{\bar\epsilon_1}{}_2F_1\left(1, -n+\epsilon_3; -n+\bar\epsilon_2; \frac{z}{x_1}\right) \tag{81}$$

when we set $\bar c_{n\,n} = 1$. This is one of the solution associated with the Papperitz–Riemann symbol of the hypergeometric

$$z^{-n-\epsilon_2}\left(\frac{z}{x_1}-1\right)^{\bar\epsilon_1} P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & 1 & \frac{z}{x_1} \\ n+\epsilon_2 & -\bar\epsilon_1 & -n+\epsilon_3 & \end{Bmatrix}$$
$$= P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & 0 & \frac{z}{x_1} \\ -n-\epsilon_2 & \bar\epsilon_1 & n+1-\bar\epsilon_3 & \end{Bmatrix}. \tag{82}$$

The last form of the P symbol clearly shows that the indices around the singular points are the desired ones. The P symbol also shows that the other solution is simply the constant 1. This is important when performing the analytic continuation of the hypergeometric to the region $|z/x_1|>1$. In fact in the region $|z/x_1|>1$ we find that the two independent solutions mix and the analytically continued solution does not satisfy anymore the required boundary conditions.

This means that we need again to consider combinations of the solutions found above in the $|z/x_1| < 1$ so that their continuation satisfies the proper boundary conditions. Therefore we find again one mode less than expected.

Another way of seeing the problem is to notice that the hypergeometric equation associated with the symbol (82) has $a = 0$, $b = n + 1 - \bar{\epsilon}_3$ and $c = n + 1 + \epsilon_2$ so that the hypergeometric equation reads

$$w(1-w)\frac{d^2\bar{\chi}_{q,n}(w)}{dw^2} + \left[n + 1 + \epsilon_2 - (n - \bar{\epsilon}_3 + 2)w\right]\frac{d\bar{\chi}_{q,n}(w)}{dw} = 0 \tag{83}$$

with $w = z/x_1$. This can immediately integrated and gives

$$\bar{\chi}_{q,n}(z) = e^{-i2\pi\alpha_1}\int_{x_1}^{z} dw\, w^{-\epsilon_2-n-1}\left(\frac{w}{x_1} - 1\right)^{-\epsilon_1}, \quad n \geq B \tag{84}$$

exactly as for the classical solutions in Eq. (74). It has therefore the same problems with the boundary conditions as before. This problem can be generalized to the generic N and it is that one asymptotic mode (either for $u \to 0$ or equivalently for $u \to \infty$) is missing for any twist field we insert at $x_t \neq 0, \infty$. This is caused by the constraints $X_q(x_t, x_t) = 0$ we have to impose on the quantum fields. Physically we can partially understand this as a breaking of some symmetries, for example if the in string is untwisted the momentum conservation of the whole is broken because the other branes at angles. Another naive way of understanding is that the classical solution freezes these modes. It would however be nice to have a better understanding of this problem and of how to choose the basis.

From the previous discussion we learn however a rule how to write a system of orthogonal modes for a generic N. The rule is simple up to the further linear combinations/subtractions in order to satisfy the boundary conditions in the whole upper plane when performing the analytic continuation. We start from a maximally orthogonal subset of the would be quantum modes χ_q and $\bar{\chi}_q$. This subset corresponds to the modes $\chi_{q,n} = \xi_{q,n}$ with $n \leq -B - 1$ and $\bar{\chi}_{q,m} = \bar{\xi}_{q,m}$ with $m \geq B - 1$ in the case of this section. Since the product is roughly $\oint dz \chi_q(z)\frac{d\bar{\chi}_q}{dz}$, we can define the remaining orthogonal modes by choosing their derivatives $\frac{d\bar{\chi}_q}{dz}$ so that the products $\chi_q(z)\frac{d\bar{\chi}_q}{dz}$ have only simple poles. This practically means that $\frac{d\bar{\chi}_q}{dz} \sim \frac{d\bar{\chi}_{cl}}{dz}$. In the case of this section in the integrated form this corresponds to the modes given in Eq. (84).

2.7. Normal modes and improved overlap approach

The issue is how to cope with the problem in finding a basis of orthogonal modes with the proper boundary conditions and having all the possible asymptotic modes. From the discussion in the previous section this is probably impossible. The simplest way to proceed further is to consider the standard approach in quantum mechanics. The standard procedure in quantum mechanics in presence of discontinuities is using the overlap of the wave function and that the basis of the incoming wave functions is complete and orthogonal. This is essentially the same approach used by Cremmer, Gervais, Kato, Ogawa and Mandelstam in the early seventies in computing the three-string vertex [41,42] and the extended to the case of the D1 string in [43].

There is however a slight subtlety to be stressed and understood before we can apply the overlap method to our case. In previous papers there were three strings and the conditions imposed were the overlap of the string coordinates X and their momenta P. In our case we have

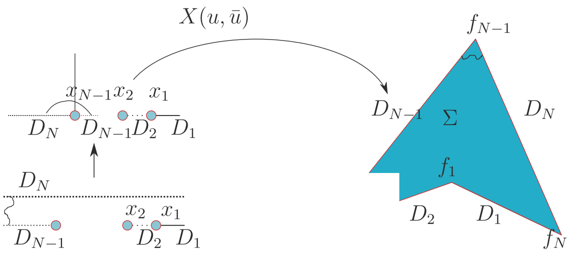

Fig. 8. Map from the stripe to the upper half plane to the target polygon Σ with twisted in and out strings.

only one string with discontinuous boundary conditions but we still want to impose the continuity of the coordinate. In the case $N = 3$ of Section 2.6 we could think of simply using the in and out expansions and require at the transition point $X^{(in)}(x_1, x_1) = X_{cl}^{(in)} + X_q^{(in)}(x_1, x_1) = X^{(out)}(x_1, x_1) = X_{cl}^{(out)} + X_q^{(out)}(x_1, x_1)$ with X_{cl} the constant given in Eq. (57) and $X_q(u, \bar{u})$ the quantum field given in Eq. (66) for in and out strings. Nevertheless in the case where multiple transitions are encountered as we discuss in the next section it is not clear which is the classical part of the intermediate strings. We are therefore led to proceed as follows. First we split $X(u, \bar{u})$ into the classical solution $X_{cl}(u, \bar{u})$ and the quantum part $X_q(u, \bar{u})$. The classical solution has a global nature and must be computed for the whole evolution before proceeding to the second step. Second we require that only the quantum fluctuations overlap. This approach is a refined version of the overlap approach and is consistent with the naive idea of overlap of the whole string coordinate.

We formulate therefore the overlap approach as[10]

$$X_q^{(t+1)}(u, \bar{u})\Big|_{|u|=x_t-0^+} = X_q^{(t)}(u, \bar{u})\Big|_{|u|=x_t+0^+} \tag{85}$$

where $X_q^{(t)}(u, \bar{u})$ is the quantum fluctuation of the string comprised in the half annulus with $x_t < |u| < x_{t-1}$ and having the appropriate boundary conditions. One could wonder whether this is the only way to define an improved overlap approach. There is not clear cut answer but the suggested one seems the simplest one which conjugate global and local aspects.

3. Green function, in and out vacuum

From now on we adopt the strategy outlined in the previous section and focus on the configuration pictured in Fig. 8 where both in and out states are twisted. This means that $x_{N-1} = 0$ and $x_N = \infty$.

In principle we should be able to derive the Green function from the canonical quantization. The derivative of the Green function is given in operatorial formalism by

$$\partial_u \partial_v G^{IJ}\big(u, \bar{u}; v, \bar{v}; \{x_t, \alpha_t\}\big) = \frac{\langle 0_{out}|\partial_u X_q^I(u, \bar{u})\partial_v X_q^J(v, \bar{v})|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} \tag{86}$$

but since we have to yet completely understood the global modes we assume we compute the Green function in the usual way i.e. using its singularities and boundary conditions. Notice that in the configuration we consider there is implicit the limit $x_N \to \infty$, however this limit does not

[10] As done in the whole paper we suppose that all $x_t > 0$.

require any special treatment or factor since the CFT definition of the derivative of the Green function gives

$$\partial_u \partial_v G^{IJ}\big(u, \bar{u}; v, \bar{v}; \{x_t, \alpha_t\}\big)\big|_{x_N=\infty} = \lim_{x_N \to \infty} \frac{x_N^{\epsilon_N \bar{\epsilon}_N} \langle \partial_u X_q^I(u, \bar{u}) \partial_v X_q^J(v, \bar{v}) \prod_{t=1}^N \sigma_{\epsilon_t}(x_t)\rangle}{x_N^{\epsilon_N \bar{\epsilon}_N} \langle \prod_{t=1}^N \sigma_{\epsilon_t}(x_t)\rangle}. \tag{87}$$

Comparison between the two previous expression (86) and (87) suggest that we can identify

$$\langle 0_{out} | 0_{in}\rangle = \lim_{x_N \to \infty} x_N^{\epsilon_N \bar{\epsilon}_N} \left\langle \prod_{t=1}^N \sigma_{\epsilon_t}(x_t) \right\rangle \tag{88}$$

for the configuration we consider. In the following we will assume such a natural identification.

Given the Green function we can consider the region $|u| < x_{N-2}$. Using the overlap approach we can identify $\partial_u X_q^I(u, \bar{u})$ with the twisted string $\partial_u X^I_{T(D_{N-1}\, D_N),q}(u, \bar{u})$. This twisted string has the boundary conditions which follow from being attached to the D_{N-1} brane on $x > 0$ and to the D_N brane on $x < 0$. The derivative of the Green function can then be written as

$$\partial_u \partial_v G^{IJ}\big(u, \bar{u}; v, \bar{v}; \{x_t, \alpha_t\}\big) = \frac{\langle 0_{out} | \partial_u X^I_{T(D_{N-1}\, D_N),q}(u, \bar{u}) \partial_v X^J_{T(D_{N-1}\, D_N),q}(v, \bar{v}) | 0_{in}\rangle}{\langle 0_{out} | 0_{in}\rangle}, \quad |u|, |v| < x_{N-2}. \tag{89}$$

In particular we can identify the in vacuum with the vacuum of the twisted Hilbert space $\mathcal{H}_{(D_{N-1} D_N)}$

$$|0_{in}\rangle = |T_{T(D_{N-1}\, D_N)}\rangle. \tag{90}$$

After performing the normal ordering with respect to this vacuum the previous equation can then be written as

$$\partial_u \partial_v \Delta^{IJ}_{(N,M)(t=N-1)}\big(u, \bar{u}; v, \bar{v}; \{x_t, \alpha_t\}\big) = \frac{\langle 0_{out} | : \partial_u X^I_{T(D_{N-1}\, D_N),q}(u, \bar{u}) \partial_v X^J_{T(D_{N-1}\, D_N),q}(v, \bar{v}) : | 0_{in}\rangle}{\langle 0_{out} | 0_{in}\rangle}, \tag{91}$$

where we have defined the Green function regularized at the twisted interaction point x_t with $t = N - 1$ (in the case at hand $x_{N-1} = 0$) in the sector $M = \sum_{t=1}^N \epsilon_t$ as

$$\begin{aligned} &\Delta^{IJ}_{(N,M)(t=N-1)}\big(u, \bar{u}; v, \bar{v}; \{x_t, \alpha_t\}\big) \\ &\quad = G^{IJ}_{(N,M)}\big(u, \bar{u}; v, \bar{v}; \{x_t, \alpha_t\}\big) \\ &\qquad - G^{IJ}_{N=2,T(D_{N-1}\, D_N)}\big(u, \bar{u}; v, \bar{v}; \{0, \alpha_{N-1}; \infty, \alpha_N\}\big), \end{aligned} \tag{92}$$

where $G^{IJ}_{N=2,T(D_{N-1}\, D_N)}$ is the twisted Green function given in Eq. (68) for the boundary conditions associated to the Hilbert space $\mathcal{H}_{(D_{N-1} D_N)}$, i.e. with the twist $\sigma_{\epsilon_{N-1}, f_{N-1}}$ at $x = 0$ and the anti-twist $\sigma_{\bar{\epsilon}_{N-1}, f_{N-1}}$ in $x = \infty$.

From Eq. (91) we can then determine the coefficients which enter the operatorial expression for the out vacuum

$$\langle 0_{out}| = \mathcal{N}\langle 0_{in}| \exp\left\{ \frac{1}{2} \sum_{n,m=0}^{\infty} [B_{n\bar{z},m\bar{z}} \alpha_{n+\epsilon_{N-1}} \alpha_{m+\epsilon_{N-1}} + B_{nz,mz} \bar{\alpha}_{n+\bar{\epsilon}_{N-1}} \bar{\alpha}_{m+\bar{\epsilon}_{N-1}} \right.$$
$$\left. + 2B_{n\bar{z},mz} \alpha_{n+\epsilon_{N-1}} \bar{\alpha}_{m+\bar{\epsilon}_{N-1}}] \right\} \tag{93}$$

with $B_{n\bar{z},mz} = B_{mz,n\bar{z}}$. A simple computation gives $\mathcal{N} = \langle 0_{out}|0_{in}\rangle$. If we take $IJ = zz$ in Eq. (91) we get the $\partial_u \partial_v$ derivative of

$$\sum_{n,m=0}^{\infty} \left[B_{n\bar{z},m\bar{z}} u^{n+\bar{\epsilon}_{N-1}} v^{m+\bar{\epsilon}_{N-1}} + B_{nz,mz} \bar{u}^{n+\bar{\epsilon}_{N-1}} \bar{v}^{m+\bar{\epsilon}_{N-1}} \right.$$
$$\left. + B_{n\bar{z},mz} \left(u^{n+\epsilon_{N-1}} \bar{v}^{m+\bar{\epsilon}_{N-1}} + v^{n+\epsilon_{N-1}} \bar{u}^{m+\bar{\epsilon}_{N-1}} \right) \right]$$
$$= \left(-i \frac{\sqrt{2\alpha'}}{2} e^{i\pi\alpha_{N-1}} \right)^{-2} \Delta^{zz}_{(N,M)(t=N-1)}\left(u, \bar{u}; v, \bar{v}; \{x_t, \alpha_t\}\right). \tag{94}$$

If we consider the analogous expressions of Eq. (91) with the derivatives $\partial_u \partial_{\bar{v}}$, $\partial_{\bar{u}} \partial_v$ and $\partial_{\bar{u}} \partial_{\bar{v}}$ we find the corresponding derivatives of the previous equation (94). The derivative expressions can be integrated to obtain Eq. (94) where the integration constant is fixed to be zero because of the cut structure. Similarly if we take $IJ = z\bar{z}$ we get the derivatives of

$$\sum_{n,m=0}^{\infty} \left[B_{n\bar{z},m\bar{z}} u^{n+\bar{\epsilon}_{N-1}} \bar{v}^{m+\bar{\epsilon}_{N-1}} + B_{nz,mz} \bar{u}^{n+\bar{\epsilon}_{N-1}} v^{m+\bar{\epsilon}_{N-1}} \right.$$
$$\left. + B_{n\bar{z},mz} \left(u^{n+\epsilon_{N-1}} v^{m+\bar{\epsilon}_{N-1}} + \bar{v}^{n+\epsilon_{N-1}} \bar{u}^{m+\bar{\epsilon}_{N-1}} \right) \right]$$
$$= \left(-i \frac{\sqrt{2\alpha'}}{2} \right)^{-2} \Delta^{z\bar{z}}_{(N,M)(t=N-1)}\left(u, \bar{u}; v, \bar{v}; \{x_t, \alpha_t\}\right), \tag{95}$$

where the integration constant is again fixed to be zero because of the cut structure. There is also the third possibility given by $IJ = \bar{z}\bar{z}$. Nevertheless all three expressions for the B coefficients are equivalent because of the Δ^{IJ} boundary conditions.

The previous equation can also be rewritten in a more compact form as

$$\langle 0_{out}| = \mathcal{N}\langle 0_{in}| \exp\left\{ \oint_{z=0} \frac{dz}{2\pi i} \oint_{w=0} \frac{dw}{2\pi i} \partial_z \mathcal{X}^{I(+)}_{T(D_{N-1}\, D_N),q}(z) \partial_w \mathcal{X}^{J(+)}_{T(D_{N-1}\, D_N),q}(w) \right.$$
$$\left. \times \Delta^{IJ}_{(N,M)(t=N-1)LL}\left(z; w; \{x_t, \alpha_t\}\right) \right\}, \tag{96}$$

where $\Delta^{IJ}_{(N,M)(t=N-1)LL}(z; w)$ is the holomorphic part in both u and v analytically continued in the complex plane minus cuts.

Finally let us comment on why we have not explicitly indicated the $x_N \to \infty$ limit in all the previous expressions. The reason is that we could repeat all the previous arguments for a configuration with a finite x_N. In particular also for an outgoing untwisted vacuum. There would be two minor differences. The one difference with the previous results is not taking the $x_N \to \infty$ limit. The second one is the relation between the correlator of the plain twist fields and the product of the in and out vacua which would read

$$\langle 0_{out}|0_{in}\rangle = \left\langle \prod_{t=1}^{N} \sigma_{\epsilon_t}(x_t) \right\rangle \quad \text{with finite } x_N. \tag{97}$$

4. The generating function of amplitudes

In this section we would like to perform the actual computations of the generating functions for amplitudes involving plain and excited twisted states. This is done in steps. First considering the amplitudes with plain unexcited twisted fields and arbitrary untwisted states. Then considering amplitudes with excited twisted states without untwisted ones and finally, assembling all.

4.1. Amplitudes with untwisted vertices and plain twist fields

Given the previous knowledge of the out vacuum as a sliver constructed on the in vacuum we can now easily compute boundary amplitudes with vertices in the $|u| < x_{N-1}$ region. In particular we want to compute the generating function of the amplitudes

$$\left\langle \prod_{i=1}^{L} V_{\xi_i}(\hat{x}_i) \prod_{t=1}^{N} \sigma_{\epsilon_t, f_t}(x_t) \right\rangle \tag{98}$$

with $|\hat{x}_i| < x_{N-1}$. ξ_i is a generic untwisted state living on D_{N-1} or D_N and $V_{\xi_i}(\hat{x}_i)$ its emission vertex. To this end we need mapping the abstract vertex operator $V_{\xi_i}(\hat{x}_i)$ into its operatorial realization in the twisted Hilbert space $\mathcal{H}_{T(D_{N-1}D_N)}$. This mapping is realized using the SDS vertex $\mathcal{S}(x; \{c^I\})$. The SDS vertex for the twisted Hilbert space $\mathcal{H}_{T(D_{N-1}D_N)}$ is explicitly given by

$$\begin{aligned}
&\mathcal{S}_{T(D_{N-1}D_N)}\big(x; \{c^I\}\big) \\
&\quad = :\exp\left\{ \sum_{n=0}^{\infty} c_{nI}\, \partial_x^n X^I_{T(D_{N-1}D_N)}\big(x + i0^+, x - i0^+\big) \right\}: \\
&\qquad \times \exp\left\{ \frac{1}{2} \sum_{n,m=0}^{\infty} c_{nI}\, c_{mJ}\, \partial_{x_1}^n \partial_{x_2}^m \Delta^{IJ}_{T(D_{N-1}D_N),bou(x)}\big(x_1; x_2; \{0, \alpha_{N-1}; \infty, \alpha_N\}\big)\big|_{x_1=x_2=x} \right\},
\end{aligned} \tag{99}$$

where $:\cdots:$ is the normal ordering in the twisted Hilbert space. We defined the boundary regularized Green function for the twisted Hilbert space at the point x

$$\begin{aligned}
&\Delta^{IJ}_{T(D_{N-1}D_N),bou(x)}\big(x_1; x_2; \{0, \alpha_{N-1}; \infty, \alpha_N\}\big) \\
&\quad = G^{IJ}_{N=2,T(D_{N-1}\,D_N)}\big(x_1 + i0^+, x_1 - i0^+; x_2 + i0^+, x_2 - i0^+; \{0, \alpha_{N-1}; \infty, \alpha_N\}\big) \\
&\qquad - G^{IJ}_{U(t_x)}\big(x_1 + i0^+, x_1 - i0^+; x_2 + i0^+, x_2 - i0^+; \alpha_{N-1}\big),
\end{aligned} \tag{100}$$

where t_x is the index of the brane on which the vertex with coordinate x is. For $x_t < x < x_{t-1}$ we have $t_x = t$. In the case at hand we have actually only two possibilities, either $0 < x < x_{N-1}$ then $t_x = N - 1$ or $-x_{N-1} < x < 0$ then $t_x = N$.

In this expression $G^{IJ}_{N=2,T(D_{N-1}\,D_N)}(u, \bar{u}; v, \bar{v}; \{0, \alpha_{N-1}; \infty, \alpha_N\})$ the same Green function used in Eq. (92) and given in Eq. (68). $G^{IJ}_{U(t_x)}(u, \bar{u}; v, \bar{v})$ is the Green function of the untwisted string with both ends on D_{t_x} and given in Eq. (53).

Then we can realize the mapping as

$$\left[e^{i\bar{k}X+ik\bar{X}}\prod_{n=1}^{\infty}\left(\partial_x^n X\right)^{N_n}\left(\partial_x^n \bar{X}\right)^{\bar{N}_n}\right]\left(x+i0^+,x-i0^+\right)$$
$$\leftrightarrow \prod_{n=1}^{\infty}\frac{\partial^{N_n}}{\partial\bar{c}_n^{N_n}}\frac{\partial^{\bar{N}_n}}{\partial c_n^{\bar{N}_n}}\mathcal{S}\left(x;\{c,\bar{c}\}\right)\big|_{c_{0I}=i\,k_I,c_{n\geq 1}=0} \tag{101}$$

when we identify $c_z = c^{\bar{z}} = c$ and $c_{\bar{z}} = c^z = \bar{c}$. This is the correct map because operatorial realizations have the same OPEs as the abstract operators despite the use of the twisted Green function associated with the twisted fields $X^I_{T(D_{N-1}D_N)}$. It is worth stressing for what follows that the fields $X^I_{T(D_{N-1}D_N)}$ are the full fields and not only the quantum fluctuations.

In order to compute the generating function of all the correlators like (98) it is then enough to insert an SDS for any untwisted operator into the radial ordered expression

$$V_{0+L}\left(\{\hat{x}_i;\{c_{(i)}\}\}\right) = \langle 0_{out}|R\left[\prod_{i=1}^{L}\mathcal{S}_{(D_{N-1}D_N)}\left(\hat{x}_i;\{c_{(i)}\}\right)\right]|0_{in}\rangle. \tag{102}$$

To compute explicitly the previous expression we have to split the full fields $X^I_{T(D_{N-1}D_N)}$ into classical and quantum parts and then normal order the quantum parts. After these operations we get

$$\begin{aligned}
&V_{0+L}\left(\{\hat{x}_i;\{c_{(i)}\}\}\right)\\
&\quad=\prod_{i=1}^{L}\Bigg\{\exp\left[\sum_{n=0}^{\infty}c_{(i)nI}\,\partial_x^n X^I_{cl}\left(\hat{x}_i+i0^+,\hat{x}_i-i0^+;\{x_t,\alpha_t,f_t\}\right)\right]\\
&\qquad\times\exp\Bigg[\frac{1}{2}\sum_{n,m=0}^{\infty}c_{(i)nI}c_{(i)mJ}\,\partial_{x_1}^n\partial_{x_2}^m\Delta^{IJ}_{T(D_{N-1}D_N),bou(i)}\\
&\qquad\times\left(x_1;x_2;\{0,\alpha_{N-1};\infty,\alpha_N\}\right)\big|_{x_1=x_2=\hat{x}_i}\Bigg]\Bigg\}\\
&\qquad\times\prod_{1\leq i<j\leq L}\exp\Bigg[\sum_{n,m=0}^{\infty}c_{(i)nI}c_{(j)mJ}\,\partial_{x_1}^n\partial_{x_2}^m G^{IJ}_{N=2,T(D_{N-1}\,D_N)}\\
&\qquad\times\left(x_1,x_1;x_2,x_2;\{0,\alpha_{N-1};\infty,\alpha_N\}\right)\big|_{x_1=\hat{x}_i;x_2=\hat{x}_j}\Bigg]\\
&\qquad\times\langle 0_{out}|\exp\left[\sum_{i=1}^{L}\sum_{n=0}^{\infty}c_{(i)nI}\,\partial_x^n X^{I(-)}_{T(D_{N-1}\,D_N),q}\left(\hat{x}_i+i0^+,\hat{x}_i-i0^+\right)\right]|0_{in}\rangle,
\end{aligned} \tag{103}$$

where $bou(i)$ is a short hand for $bou(\hat{x}_i)$. The term in the last line can be evaluated using the analogous expression of $\langle 0|e^{\beta a^2}e^{\alpha a^\dagger}|0\rangle = e^{\beta\alpha^2}$ for an infinite number of oscillators and gives

$$\langle 0_{out}|\exp\left[\sum_{i=1}^{L}\sum_{n=0}^{\infty}c_{(i)nI}\,\partial_x^n X^{I(-)}_{T(D_{N-1}\,D_N),q}\left(\hat{x}_i+i0^+,\hat{x}_i-i0^+\right)\right]|0_{in}\rangle$$

$$
\begin{aligned}
&= \lim_{x_N\to\infty} x_N^{\epsilon_N\bar\epsilon_N} \left\langle \prod_{t=1}^{N} \sigma_{\epsilon_t, f_t}(x_t) \right\rangle \\
&\quad \times \prod_{1\le i,j\le L} \exp\left[\frac{1}{2} \sum_{n,m=0}^{\infty} c_{(i)nI} c_{(j)mJ} \partial_{x_1}^n \partial_{x_2}^m \Delta^{IJ}_{(N,M)(t=N-1)} \right. \\
&\quad \left. \times \big(x_1, x_1; x_2, x_2; \{x_t, \alpha_t\}\big)\big|_{x_1=\hat x_i; x_2=\hat x_j} \right],
\end{aligned}
\tag{104}
$$

where $\Delta^{IJ}_{(N,M)(t=N-1)}$ is the expression in Eq. (92). When we assemble all contributions we find finally

$$
\begin{aligned}
&V_{0+L}\big(\{\hat x_i; \{c_{(i)}\}\}\big) \\
&= \lim_{x_N\to\infty} x_N^{\epsilon_N\bar\epsilon_N} \left\langle \prod_{t=1}^{N} \sigma_{\epsilon_t, f_t}(x_t) \right\rangle \\
&\quad \times \prod_{i=1}^{L} \left\{ \exp\left[\sum_{n=0}^{\infty} c_{(i)nI} \partial_x^n X^I_{cl}\big(\hat x_i + i0^+, \hat x_i - i0^+; \{x_t, \alpha_t, f_t\}\big) \right] \right. \\
&\quad \left. \times \exp\left[\frac{1}{2} \sum_{n,m=0}^{\infty} c_{(i)nI} c_{(i)mJ} \partial_{x_1}^n \partial_{x_2}^m \Delta^{IJ}_{(N,M)bou(i)}\big(x_1; x_2; \{x_t, \alpha_t\}\big)\big|_{x_1=x_2=\hat x_i} \right] \right\} \\
&\quad \times \prod_{1\le i<j\le L} \exp\left[\sum_{n,m=0}^{\infty} c_{(i)nI} c_{(j)mJ} \partial_{x_1}^n \partial_{x_2}^m G^{IJ}_{(N,M)} \right. \\
&\quad \left. \times \big(x_1, x_1; x_2, x_2; \{x_t, \alpha_t\}\big)\big|_{x_1=\hat x_i; x_2=\hat x_j} \right],
\end{aligned}
\tag{105}
$$

where we have introduced the boundary Green function in sector M regularized at the point $\hat x_i$

$$
\begin{aligned}
&\Delta^{IJ}_{(N,M)bou(i)}\big(x_1; x_2; \{x_t, \alpha_t\}\big) \\
&\quad = G^{IJ}_{(N,M)}\big(x_1 + i0^+, x_1 - i0^+; x_2 + i0^+, x_2 - i0^+; \{x_t, \alpha_t\}\big) \\
&\quad - G^{IJ}_{U(t_i)}\big(x_1 + i0^+, x_1 - i0^+; x_2 + i0^+, x_2 - i0^+; \alpha_{t_i}\big)
\end{aligned}
\tag{106}
$$

with $t_i = N-1$ for $\hat x_i > 0$ and $t_i = N$ for $\hat x_i < 0$. This is the result of the computation $\Delta^{IJ}_{(N,M)(t=N-1)} - \Delta^{IJ}_{T(D_{N-1}D_N),bou(\hat x_i)}$. The expressions of the two terms are given in Eqs. (92) and (100). We have now found the generating function for the emission of untwisted states from either D_{N-1} of D_N and with $|\hat x_i| < x_{N-1}$. Eq. (105) is the same expression found in [2] (Eq. (80) in Section 4.1) when we drop the $x_N \to \infty$ limit, we let the $\hat x_i$ be generic and not constrained by $|\hat x_i| < x_{N-2}$ and we substitute $\Delta^{IJ}_{(N,M)bou(i)}(x_1; x_2; \{x_t, \alpha_t\}) \to \Delta^{IJ}_{(N,M)bou(i)}(x_1; x_2; \{x_t, \alpha_t\})$. The expression in [2] is however valid without the constraint on $\hat x$. The reason for this substantial equality is quite simple. Suppose we want compute the generating function for the emission of untwisted states from any of the brane of the configuration we consider. We need therefore computing the analogous expression of Eq. (102) where the SDS vertices are the proper ones for the untwisted states. For example, for $x_t < \hat x_i < x_{t-1}$ the emission of an untwisted state is from the brane D_t. The corresponding operatorial vertex is realized in the twisted Hilbert space

$\mathcal{H}_{(D_t D_N)}$ using the full fields $X^I_{T(D_t D_N)}$. The map abstract from operator to its operatorial realization is then performed by an SDS vertex analogous to (99) but defined in the twisted Hilbert space $\mathcal{H}_{(D_t D_N)}$ using the fields $X^I_{T(D_t D_N)}$.

To compute the analogous expression of Eq. (102) but with no constraints on the location of the vertices we proceed as follows. We split all the full fields $X^I_{T(D_t D_N)}$ into classical and quantum parts. Then we can use the continuity equation (85) for the quantum part to perform an analytic continuation of the quantum part of any SDS vertex from an arbitrary region to the region $|u| < x_{N-2}$. This seems at first sight quite difficult because of the normal ordering which differs among the different twisted Hilbert spaces. Fortunately it is not so. The key observation is that the normal ordered SDS vertex in Eq. (99) is obtained from a non normal ordered vertex as

$$\begin{aligned}
&\mathcal{S}_{T(D_t D_N)}\big(x; \{c^I\}\big) \\
&\quad = \lim_{\eta\to 0^+} e^{\sum_{n=0}^\infty c_{nI}\partial_x^n [X^{I(+)}_{T(D_t D_N)}(x+i0^+, x-i0^+) + X^{I(-)}_{T(D_t D_N)}(xe^{-\eta}+i0^+, xe^{-\eta}-i0^+)]} \\
&\qquad \times \exp\left\{-\frac{1}{2}\sum_{n,m=0}^\infty c_{nI}c_{mJ}\partial_{x_1}^n\partial_{x_2}^m G^{IJ}_{U(t_x)}\big(x_1, x_1; x_2, x_2; \alpha_{t_x}\big)\Big|_{x_1 = x_2 e^\eta = x}\right\}
\end{aligned} \tag{107}$$

with $t_x = t$ when $x > 0$. We can then easily perform the analytic continuation and get

$$\begin{aligned}
&\mathcal{S}_{T(D_t D_N)}\big(x; \{c^I\}\big) \\
&\quad = \mathcal{S}_{T(D_{N-1} D_N)}\big(x; \{c^I\}\big) \\
&\qquad \times \lim_{\eta\to 0^+} e^{-\frac{1}{2}\sum_{n,m=0}^\infty c_{nI}c_{mJ}\partial_{x_1}^n\partial_{x_2}^m [G^{IJ}_{U(t_x)}(x_1,x_1;x_2,x_2;\alpha_{t_x}) - G^{IJ}_{U(N-1)}(x_1,x_1;x_2,x_2;\alpha_{N-1})]|_{x_1 = x_2 e^\eta = x}},
\end{aligned} \tag{108}$$

where it is necessary to take the $\eta \to 0^+$ limit in the last line only after we have continued back the result to the original position. Only then the difference $G^{IJ}_{U(t)} - G^{IJ}_{U(N-1)}$ which would otherwise be divergent in the limit $\eta \to 0^+$ combines with $\Delta^{IJ}_{(N,M)bou(i)}$ given in Eq. (106) and valid for $|\hat{x}| < x_{N-2}$ to give the finite boundary Green function regularized at the original point $x_t < \hat{x} < x_{t-1}$, i.e.

$$\begin{aligned}
&\Delta^{IJ}_{(N,M)bou(i)}\big(x_1; x_2; \{x_t, \alpha_t\}\big) \\
&\quad = G^{IJ}_{(N,M)}\big(x_1 + i0^+, x_1 - i0^+; x_2 + i0^+, x_2 - i0^+; \{0, \alpha_{N-1}; \infty, \alpha_N\}\big) \\
&\qquad - G^{IJ}_{U(t_i)}\big(x_1 + i0^+, x_1 - i0^+; x_2 + i0^+, x_2 - i0^+; \alpha_{t_i}\big).
\end{aligned} \tag{109}$$

Notice that this analytic continuation is performed only on the operators and not on the possible polarizations and momenta which are still the ones allowed in the original region.

4.2. Amplitudes with chiral vertices and plain twist fields

In this section we would like to compute the correlators of L_c chiral vertices as a warm up for the next section where we compute the correlators of N excited twists. In particular we want to compute the generating function of the amplitudes

$$\left\langle \prod_{c=1}^{L_c}\left[\prod_{n=1}^\infty \big(\partial_u^n X^z\big)^{N_{(c)n}}\big(\partial_u^n X^{\bar{z}}\big)^{\bar{N}_{(c)n}}\right](u_c)\prod_{t=1}^N \sigma_{\epsilon_t, f_t}(x_t)\right\rangle. \tag{110}$$

As in the previous section the SDS vertex for the emission of chiral untwisted states from a twisted string with ends on D_t and D_N is given by

$$\mathcal{S}_{T(D_t D_N)}\big(u;\{c_{(c)}\}\big) = \lim_{\eta\to 0^+} e^{\sum_{n=1}^{\infty} c_{(c)nI}\partial_u^n [X^{I(+)}_{T(D_t D_N)}(u,\bar u)+X^{I(-)}_{T(D_t D_N)}(ue^{-\eta},\bar u e^{-\eta})]} \times \exp\left\{-\sum_{n,m=1}^{\infty}\bar c_{(c)n} c_{(c)m}\partial_{u_1}^n\partial_{u_2}^m G^{z\bar z}_{U(t)}(u_1,\bar u_1;u_2,\bar u_2;\alpha_t)\big|_{u_1=u_2e^{\eta}=u}\right\}, \tag{111}$$

where we used the fact that $\partial_{u_1}^n\partial_{u_2}^m G^{IJ}_{U(t)}$ is different from zero only when $IJ = z\bar z$ or $IJ = \bar z z$. Moreover $G^{z\bar z}_{U(t)}$ is actually independent on t since it does not depend on the phase α_t. The previous equation can be written in a more compact way by normal ordering the operatorial part and performing the limit. It then reads as

$$\mathcal{S}_{T(D_t D_N)}\big(u;\{c_{(c)}\}\big) = :e^{\sum_{n=1}^{\infty} c_{(c)nI}\partial_u^n X^I_{T(D_t D_N)}(u,\bar u)}: \times \exp\left\{\sum_{n,m=1}^{\infty}\bar c_{(c)n} c_{(c)m}\partial_{u_1}^n\partial_{u_2}^m \Delta^{z\bar z}_{T(D_t D_N),chir} \times \big(u_1,\bar u_1;u_2,\bar u_2;\{0,\alpha_t;\infty,\alpha_N\}\big)\big|_{u_1=u_2=u}\right\}, \tag{112}$$

where we have introduced the derivative of the regularized chiral Green function as

$$\partial_u\partial_v \Delta^{z\bar z}_{T(D_t D_N),chir}\big(u,\bar u;v,\bar v;\{0,\alpha_t;\infty,\alpha_N\}\big) = \partial_u\partial_v\big[G^{z\bar z}_{T(D_t D_N)}\big(u,\bar u;v,\bar v;\{0,\alpha_t;\infty,\alpha_N\}\big) - G^{z\bar z}_{U(t)}(u,\bar u;v,\bar v;\alpha_t)\big]. \tag{113}$$

Performing the same steps as in the previous section we get

$$V_{N+L_c}\big(\{u_c,\{c_{(c)}\}\}\big) = \langle \sigma_{\epsilon_1,f_1}(x_1)\ldots\sigma_{\epsilon_N,f_N}(x_N)\rangle \times \prod_{c=1}^{L_c}\Big\{e^{\sum_{n=1}^{\infty} c_{(c)nI}\partial_{u_c}^{n-1}[\partial_u X^I_{cl}(u_c,\bar u_c;\{x_t,\alpha_t,f_t\})]} \times e^{\frac{1}{2}\sum_{n,m=1}^{\infty} c_{(c)nI}c_{(c)mJ}\partial_{u_c}^n\partial_{v_c}^m \Delta^{IJ}_{(N,M)(c)}(u_c,\bar u_c;v_c,\bar v_c;\{x_t,\alpha_t\})|_{v_c=u_c}]}\Big\} \times \prod_{1\le c<\hat c\le N} e^{\sum_{n,m=1}^{\infty} c_{(c)nI}c_{(\hat c)mJ}\partial_{u_t}^n\partial_{v_{\hat t}}^m G^{IJ}_{(N,M)}(u_c,\bar u_c;v_{\hat c},\bar v_{\hat c};\{x_t,\alpha_t\})}, \tag{114}$$

where we have written the dependence on the complex conjugate variables $\bar u_c$ even if the derivatives of $X^I_{cl}(u,\bar u)$ and $G^{IJ}_{(N,M)}(u,\bar u;v,\bar v;\{x_t,\alpha_t\})$ are independent of it. We have done this in order to be consistent with the notation used in the boundary case. The regularized chiral Green function is defined as expected as

$$\partial_u\partial_v \Delta^{IJ}_{(N,M)(c)}\big(u,\bar u;v,\bar v;\{x_t,\alpha_t\}\big) = \partial_u\partial_v\big[G^{IJ}_{(N,M)}\big(u,\bar u;v,\bar v;\{x_t,\alpha_t\}\big) - G^{IJ}_{U(t)}(u,\bar u;v,\bar v;\alpha_t)\big]. \tag{115}$$

Notice once more that the subtraction term is different from zero only when $IJ = z\bar z$ or $IJ = \bar z z$ because of the derivatives.

There is actually an extra bonus in this approach: we can also prove the explicit form of amplitudes involving closed string states that were conjectured in [2]. This happens using the vertices for the emission of closed string states written in open string formalism. These vertices are written as a product of a chiral times an antichiral part (up to cocycles). Then we can apply the same procedure described before to these products.

4.3. Amplitudes with excited twist fields

In this section we would like to compute the correlators of N excited twists. In particular we want to compute the generating function of the amplitudes

$$\left\langle \prod_{t=1}^{N} \left[\prod_{n=1}^{\infty} \left(\partial_u^n X^z\right)^{N_{n(t)}} \left(\partial_u^n X^{\bar z}\right)^{\bar N_{n(t)}} \sigma_{\epsilon_t, f_t} \right](x_t) \right\rangle. \tag{116}$$

The excited twists in the previous correlator can be described using the operator to state correspondence. According to the notation introduced in [2] the operator to state correspondence is given by

$$\begin{aligned} &\lim_{x\to 0} \left[\prod_{n=0}^{\infty} \left(\partial_u^{n+1} X^z\right)^{N_n} \left(\partial_u^{n+1} X^{\bar z}\right)^{\bar N_n} \sigma_{\epsilon_t, f_t} \right](x) |0\rangle_{SL(2)} \\ &\quad = \prod_{n=0}^{\infty} \left(k_{\epsilon_t} n! \alpha^\dagger_{n+\epsilon_t}\right)^{N_n} \left(k_{\bar\epsilon_t} n! \bar\alpha^\dagger_{n+\bar\epsilon_t}\right)^{\bar N_n} |T_{T(D_{t-1}\, D_t)}\rangle \end{aligned} \tag{117}$$

with $k_{\epsilon_t} = -i\frac{1}{2}\sqrt{2\alpha'} e^{i\pi\alpha_t}$ and $k_{\bar\epsilon_t} = -i\frac{1}{2}\sqrt{2\alpha'} e^{-i\pi\alpha_t}$.[11] These states are built on the twisted vacuum $|T_{T(D_{t-1}\, D_t)}\rangle = \sigma_{\epsilon_t, f_t}(0)|0\rangle_{SL(2)}$ therefore the excited twists are naturally described in the twisted Hilbert space $\mathcal{H}_{T(D_{t-1}\, D_t)}$. However the $\mathcal{H}_{T(D_{t-1} D_t)}$ space is generically not any of the twisted Hilbert spaces there are during the string propagation. They are $\mathcal{H}_{T(D_t\, D_N)}$ for $t = 1, \ldots, N-1$ since the $x < 0$ ($\sigma = \pi$ in Minkowskian version) boundary of the string is always attached on D_N. This is luckily not a problem since we can describe any excited twist as a limit of the product of a chiral operator and a plain twist field, for example

$$\begin{aligned} &\Big(\partial_u^{n-1}\big[(u-x_t)^{\bar\epsilon_t} \partial_u X^z\big] \partial_u^{m-1}\big[(u-x_t)^{\epsilon_t} \partial_u X^{\bar z}\big] - \partial_u^{n-1} \partial_v^{m-1}\big[(u-x_t)^{\bar\epsilon_t} (v-x_t)^{\epsilon_t} \\ &\qquad \times \partial_u \partial_v \Delta^{z\bar z}_{T(D_{t-1} D_t), chir}\big(u-x_t; v-x_t; \{0, \alpha_{t-1}; \infty, \alpha_t\}\big)\big]\Big) \sigma_{\epsilon_t, f_t}(x_t) \\ &\quad = \big(\partial^n X^z \partial^m X^{\bar z} \sigma_{\epsilon_t, f_t}\big)(x_t) + O(u-x_t). \end{aligned} \tag{118}$$

In the previous expression the term with $\partial_u \partial_v \Delta^{z\bar z}_{T(D_{t-1} D_t), chir}$ is necessary to cancel the term which arises from the equation analogous to Eq. (112) for the $\mathcal{H}_{T(D_{t-1}\, D_t)}$ twisted Hilbert space.

In our case the plain twist field $\sigma_{\epsilon_t, f_t}(x_t)$ is hidden into the boundary conditions. We can get any excited twist field at x_t by choosing the appropriate chiral operator at u and then take the limit $u \to x_t$. It follows then that it is enough to represent the abstract chiral operator needed to create the wanted excited twisted field in the desired twisted Hilbert space which can be either $\mathcal{H}_{T(D_t\, D_N)}$ or $\mathcal{H}_{T(D_{t-1}\, D_N)}$.

As discussed in [2] the generating function for the abstract operators which is needed for generating any excited twists is given by

[11] There is obviously also an analogous expression with the antichiral operators.

$$
\begin{aligned}
&\mathcal{T}_{abs(t)}\big(u;\{d_{(t)}\}\big)\\
&\quad = \exp\Bigg\{\sum_{n=1}^{\infty}\big[\bar d_{(t)n}\partial_u^{n-1}\big[(u-x_t)^{\bar\epsilon_t}\partial_u X^z(u,\bar u)\big]+d_{(t)n}\partial_u^{n-1}\big[(u-x_t)^{\epsilon_t}\partial_u X^{\bar z}(u,\bar u)\big]\big]\Bigg\}\\
&\qquad \times\exp\Bigg\{-\sum_{n,m=1}^{\infty}\bar d_{(t)n}\,d_{(t)m}\,\partial_u^{n-1}\partial_v^{m-1}\big[(u-x_t)^{\bar\epsilon_t}(v-x_t)^{\epsilon_t}\\
&\qquad \times\partial_u\partial_v\Delta^{z\bar z}_{T(D_{t-1}\,D_t),chir}\big(u-x_t,\bar u-x_t;v-x_t,\bar v-x_t;\{0,\alpha_t;\infty,\alpha_N\}\big)\big]\big|_{v=u}\Bigg\}. \qquad (119)
\end{aligned}
$$

Explicitly this means that

$$
\begin{aligned}
&\Bigg[\prod_{n=1}^{\infty}\big(\partial_u^n X^z\big)^{N_n}\big(\partial_u^n X^{\bar z}\big)^{\bar N_n}\sigma_{\epsilon_t,f_t}\Bigg](x_t)\\
&\quad=\lim_{u\to x_t}\prod_{n=1}^{\infty}\frac{\partial^{N_n}}{\partial\bar d_{(t)n}^{N_n}}\frac{\partial^{\bar N_n}}{\partial d_{(t)n}^{\bar N_n}}\mathcal{T}_{abs}\big(u;\{d_{(t)}\}\big)\big|_{d=0}\sigma_{\epsilon_t,f_t}(x_t). \qquad (120)
\end{aligned}
$$

The abstract generator (119) can be realized in any Hilbert space. In particular in the twisted Hilbert space $\mathcal{H}_{T(D_t\,D_N)}$ where $x_t<|u|<x_{t-1}$ it reads

$$
\begin{aligned}
&\mathcal{T}_{T(D_t D_N)}\big(u;\{d_{(t)}\}\big)\\
&\quad = :\exp\Bigg\{\sum_{n=1}^{\infty}d_{(t)nI}\,\partial_u^{n-1}\big[(u-x_t)^{\epsilon_{tI}}\partial_u X^I_{T(D_t\,D_N)}(u)\big]\Bigg\}:\\
&\qquad\times\exp\Bigg\{\sum_{n,m=1}^{\infty}\bar d_{(t)n}\,d_{(t)m}\,\partial_u^{n-1}\partial_v^{m-1}\big[(u-x_t)^{\bar\epsilon_t}(v-x_t)^{\epsilon_t}\\
&\qquad\times\partial_u\partial_v\Delta^{z\bar z}_{T(D_t\,D_N),chir}\big(u,\bar u;v,\bar v;\{0,\alpha_t;\infty,\alpha_N\}\big)\big]\big|_{v=u}\Bigg\}\\
&\qquad\times\exp\Bigg\{-\sum_{n,m=1}^{\infty}\bar d_{(t)n}\,d_{(t)m}\,\partial_u^{n-1}\partial_v^{m-1}\big[(u-x_t)^{\bar\epsilon_t}(v-x_t)^{\epsilon_t}\partial_u\partial_v\\
&\qquad\times\Delta^{z\bar z}_{T(D_{t-1}\,D_t),chir}\big(u-x_t,\bar u-x_t;v-x_t,\bar v-x_t;\{0,\alpha_{t-1};\infty,\alpha_t\}\big)\big]\big|_{v=u}\Bigg\}. \qquad (121)
\end{aligned}
$$

The first two factors on the right hand side are nothing else but the chiral SDS vertex with u dependent c_{nI}, roughly $c_{nI}\to d_{nI}(u-x_t)^{\epsilon_{tI}}$. Therefore the $\Delta^{z\bar z}$ in the quadratic term depends on u and v only and not on x_t. The last factor is the "normalization" of the abstract vertex, i.e. the second factor on the right hand side in Eq. (119) and therefore depends on $u-x_t$ and $v-x_t$.

It is now enough to insert an SDS for any untwisted operator into the radial ordered expression

$$
V_{0+N}\big(\{x_t;\{d_{(t)}\}\}\big)=\lim_{\{u_t\}\to\{x_t\}}\langle 0_{out}|R\Bigg[\prod_{t=1}^{N}\mathcal{T}_{T(D_tD_N)}\big(u_t;\{d_{(t)}\}\big)\Bigg]|0_{in}\rangle \qquad (122)
$$

in order to compute the generating function of all the correlators like (116).

The computation of the expectation value is the same as before for the chiral vertices. What is interesting is to trace the contributions to self interactions. For any t we get two contributions: one from the expectation value and the other from the "normalization" of the abstract vertex, i.e. the third factor in Eq. (121). We get therefore

$$\exp\left\{\frac{1}{2}\sum_{n,m=1}^{\infty} d_{(t)nI}\, d_{(t)mJ}\, \partial_u^{n-1}\partial_v^{m-1}\big[(u-x_t)^{\epsilon_{tI}}(v-x_t)^{\epsilon_{tJ}}\partial_u\partial_v \times \Delta^{IJ}_{(N,M)(c_t)}\big(u,\bar u; v,\bar v; \{x_{\hat t},\alpha_{\hat t}\}\big)\big]\big|_{v=u=u_t}\right\}$$
$$\times \exp\left\{-\sum_{n,m=1}^{\infty} \bar d_{(t)n}\, d_{(t)m}\, \partial_u^{n-1}\partial_v^{m-1}\big[(u-x_t)^{\bar\epsilon}(v-x_t)^{\epsilon}\partial_u\partial_v \Delta^{z\bar z}_{chir,T(D_{t-1}D_t)} \times \big(u-x_t,\bar u-x_t; v-x_t,\bar v-x_t; \{0,\alpha_{t-1};\infty,\alpha_t\}\big)\big]\big|_{v=u=u_t}\right\}, \tag{123}$$

where c_t is the index associated with the chiral vertex at $u_{c_t}=u_t$ and $\partial_u\partial_v\Delta^{IJ}_{(N,M)(c_t)}$ is the derivative of regularized Green function given in Eq. (115). This expression can be written in a more compact way as

$$\exp\left\{\frac{1}{2}\sum_{n,m=1}^{\infty} d_{(t)nI}\, d_{(t)mJ}\, \partial_u^{n-1}\partial_v^{m-1} \times \big[(u-x_t)^{\epsilon_{tI}}(v-x_t)^{\epsilon_{tJ}}\partial_u\partial_v \Delta^{IJ}_{(N,M)(t)}\big(u,\bar u; v,\bar v; \{x_{\hat t},\alpha_{\hat t}\}\big)\big]\big|_{v=u=u_t}\right\}, \tag{124}$$

where we have defined the (derivative of the) regularized Green function at the position x_t of the twist fields t to be

$$\begin{aligned}
&\partial_u\partial_v\Delta^{IJ}_{(N,M)(t)}\big(u,\bar u; v,\bar v; \{x_{\hat t},\alpha_{\hat t}\}\big)\\
&\quad= \partial_u\partial_v\big[\Delta^{IJ}_{(N,M)(c_t)}\big(u,\bar u; v,\bar v; \{x_{\hat t},\alpha_{\hat t}\}\big) - \Delta^{IJ}_{chir,T(D_{t-1}D_t)}\\
&\qquad\times \big(u-x_t,\bar u-x_t; v-x_t,\bar v-x_t; \{0,\alpha_{t-1};\infty,\alpha_t\}\big)\big]\\
&\quad= \partial_u\partial_v\big[G^{IJ}_{(N,M)}\big(u,\bar u; v,\bar v; \{x_{\hat t},\alpha_{\hat t}\}\big) - G^{IJ}_{N=2,T(D_tD_N)}\\
&\qquad\times \big(u-x_t,\bar u-x_t; v-x_t,\bar v-x_t; \{0,\alpha_t;\infty,\alpha_N\}\big)\big]
\end{aligned} \tag{125}$$

and we used $G^{IJ}_{U(t)}(u-x_t, v-x_t) = G^{IJ}_{U(t)}(u,v)$ to write the two last lines. Notice that when $x_{N-1}=0$ the previous equation becomes Eq. (92). Actually because of the chiral derivatives the previous expression simplifies for two combinations of indices IJ to

$$\begin{aligned}
\partial_u\partial_v\Delta^{zz}_{(N,M)(t)}\big(u,\bar u; v,\bar v; \{x_{\hat t},\alpha_{\hat t}\}\big) &= \partial_u\partial_v G^{zz}_{(N,M)}\big(u,\bar u; v,\bar v; \{x_{\hat t},\alpha_{\hat t}\}\big)\\
\partial_u\partial_v\Delta^{\bar z\bar z}_{(N,M)(t)}\big(u,\bar u; v,\bar v; \{x_{\hat t},\alpha_{\hat t}\}\big) &= \partial_u\partial_v G^{\bar z\bar z}_{(N,M)}\big(u,\bar u; v,\bar v; \{x_{\hat t},\alpha_{\hat t}\}\big)
\end{aligned} \tag{126}$$

The third combination corresponds to $\partial_u\partial_v\Delta^{z\bar z}_{(N,M)(t)}$ and does not simplify.

On general basis the regularized Green function is obtained by subtracting the divergent part with the proper monodromy at the point of regularization. At the point where a twist field is located the divergent part with the proper monodromy means $G_{N=2,T(D_{t-1}\,D_t)}$ while in all other points means G_U. In particular both $(u-x_t)^{\epsilon_t I}(v-x_t)^{\epsilon_t J}\partial_u\partial_v \Delta^{IJ}_{(N,M)(t)}$ and $(u-x_t)^{\epsilon_t I}(v-x_t)^{\epsilon_t J}\partial_u\partial_v G^{IJ}_{(N,M)}$ are analytic functions at $u=x_t$.

Assembling all pieces we get therefore the generating function for the excited twists correlator

$$\begin{aligned}
&V_{0+N}\big(\{x_t;\{d_{(t)}\}\}\big)\\
&\quad=\lim_{\{u_t\}\to\{x_t\}}\big\langle\sigma_{\epsilon_1,f_1}(x_1)\dots\sigma_{\epsilon_N,f_N}(x_N)\big\rangle\\
&\qquad\times\prod_{t=1}^{N}\big\{e^{\sum_{n=1}^{\infty} d_{(t)nI}\,\partial_{u_t}^{n-1}[(u_t-x_t)^{\epsilon_t I}\,\partial_u X^I_{cl}(u_t,\bar u_t;\{x_t,\alpha_t,f_t\})]}\\
&\qquad\times e^{\frac12\sum_{n,m=1}^{\infty} d_{(t)nI}d_{(t)mJ}\,\partial_{u_t}^{n-1}\partial_{v_t}^{m-1}[(u_t-x_t)^{\epsilon_t I}(v_t-x_t)^{\epsilon_t J}\,\partial_u\partial_v \Delta^{IJ}_{(N,M)(t)}(u_t,\bar u_t;v_t,\bar v_t;\{x_{\hat t},\alpha_{\hat t}\})]|_{v_t=u_t}}\big\}\\
&\qquad\times\prod_{1\le t<\hat t\le N} e^{\sum_{n,m=1}^{\infty} d_{(t)nI}d_{(\hat t)mJ}\,\partial_{u_t}^{n-1}\partial_{v_{\hat t}}^{m-1}[(u_t-x_t)^{\epsilon_t I}(v_{\hat t}-x_{\hat t})^{\epsilon_{\hat t} J}\,\partial_u\partial_v G^{IJ}_{(N,M)}(u_t,\bar u_t;v_{\hat t},\bar v_{\hat t};\{x_{\hat t},\alpha_{\hat t}\})]}.
\end{aligned}\tag{127}$$

4.4. *The generating function for N excited twist fields and L plain vertices*

We are now in the position of computing the desired generating function for N excited twist fields and L plain vertices. It simply amounts to the computation of

$$\begin{aligned}
&V_{L+N}\big(\{x_t;\{d_{(t)}\}\}\big)\\
&\quad=\lim_{\{u_t\}\to\{x_t\}}\langle 0_{out}|R\left[\prod_{i=1}^{L}\mathcal{S}_{T(D_tD_N)}\big(\hat x_i;\{c_{(i)}\}\big)\prod_{t=1}^{N}\mathcal{T}_{T(D_tD_N)}\big(u_t;\{d_{(t)}\}\big)\right]|0_{in}\rangle.
\end{aligned}\tag{128}$$

This computation can be done as explained in the previous sections. The result is made of the product of three blocks: interactions between two excited twists, interactions between two plain vertices and interactions between one excited twists and one plain vertex. This structure is evident in the final result

$$\begin{aligned}
&V_{N+L}(K_t,J_i)\\
&\quad=\lim_{\{u_t\}\to\{x_t\}}\big\langle\sigma_{\epsilon_1,f_1}(x_1)\dots\sigma_{\epsilon_N,f_N}(x_N)\big\rangle\\
&\qquad\times\prod_{t=1}^{N}\big\{e^{\sum_{n=1}^{\infty} d_{(t)nI}\,\partial_{u_t}^{n-1}[(u_t-x_t)^{\epsilon_t I}\,\partial_u X^I_{cl}(u_t,\bar u_t;\{x_t,\alpha_t,f_t\})]}\\
&\qquad\times e^{\frac12\sum_{n,m=1}^{\infty} d_{(t)nI}d_{(t)mJ}\,\partial_{u_t}^{n-1}\partial_{v_t}^{m-1}[(u_t-x_t)^{\epsilon_t I}(v_t-x_t)^{\epsilon_t J}\,\partial_u\partial_v \Delta^{IJ}_{(N,M)(t)}(u_t,\bar u_t;v_t,\bar v_t;\{x_{\hat t},\alpha_{\hat t}\})]|_{v_t=u_t}}\big\}\\
&\qquad\times\prod_{i=1}^{L}\big\{e^{\sum_{n=0}^{\infty} c_{(i)nI}\,\partial_{x_i}^{n} X^I_{cl}(x_i,x_i;\{x_t,\alpha_t,f_t\})}\\
&\qquad\times e^{\frac12\sum_{n=0}^{\infty} c_{(i)nI}\sum_{m=0}^{\infty} c_{(i)mJ}\,\partial_{x_i}^{n}\partial_{\hat x_i}^{m}\Delta^{IJ}_{(N,M),bou(i)}(x_i,\hat x_i;\{x_{\hat t},\alpha_{\hat t}\})|_{\hat x_i=x_i}}\big\}\\
&\qquad\times\prod_{1\le t<\hat t\le N} e^{\sum_{n,m=1}^{\infty} d_{(t)nI}d_{(\hat t)mJ}\,\partial_{u_t}^{n-1}\partial_{v_{\hat t}}^{m-1}[(u_t-x_t)^{\epsilon_t I}(v_{\hat t}-x_{\hat t})^{\epsilon_{\hat t} J}\,\partial_u\partial_v G^{IJ}_{(N,M)}(u_t,\bar u_t;v_{\hat t},\bar v_{\hat t};\{x_{\hat t},\alpha_{\hat t}\})]}
\end{aligned}$$

$$\times \prod_{1\le i<j\le L} e^{\sum_{n=0}^{\infty} c_{(i)nI} \sum_{m=0}^{\infty} c_{(j)mJ} \partial_{x_i}^n \partial_{x_j}^m G^{IJ}_{(N,M),bou}(x_i,x_j;\{x_{\hat t},\alpha_{\hat t}\})}$$

$$\times \prod_{1\le t\le N} \prod_{1\le j\le L} e^{\sum_{n=1}^{\infty} d_{(t)nI} c_{(j)mJ} \partial_{u_t}^{n-1} \partial_{x_j}^m [(u_t-x_t)^{\epsilon_t I} \partial_u G^{IJ}_{(N,M)}(u_t,\bar u_t;x_j,x_j;\{x_{\hat t},\alpha_{\hat t}\})]}, \tag{129}$$

where the last line is exactly due to the interactions between one excited twist and one plain vertex.

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Appendix A. Self-adjointness of the laplacian

We want to show that $\partial\bar\partial$ is a self-adjoint operator only if we use the quantum boundary conditions. In particular we define $\partial\bar\partial = \partial_x^2 + \partial_y^2$ as operator which acts on a couple of complex functions $f^I(u,\bar u)$ defined on the upper half plane. Then we take not only $f^I \in L^2(H)$ but we require that $\partial_x f^I$, $\partial_y f^I$, $\partial_x^2 f^I$ and $\partial_y^2 f^I$ be defined almost everywhere and that the action $\int_H dx\,dy\, f^{I*}(\partial_x^2+\partial_y^2) f^I$ be finite. Since we need to integrate by part we need

$$\int_a^b dx \partial_x^2 f^I(u,\bar u) = \partial_x f^I(b+iy,b-iy) - \partial_x f^I(a+iy,a-iy) \tag{130}$$

(and similarly for y) hence $\partial_x f^I$ and $\partial_y f^I$ must be absolutely continuous. The similar condition with a single derivative is a consequence of the existence almost everywhere of $\partial_x^2 f^I$, $\partial_y^2 f^I$ which imply that $\partial_x f^I$, $\partial_y f^I$ be almost everywhere continuous.

Finally we impose the boundary conditions

$$f^z(x,x) = e^{i2\pi\alpha_t} f^{\bar z}(x,x), \qquad \partial_y f^z(x,x) = -e^{i2\pi\alpha_t}\partial_y f^{\bar z}(x,x), \quad x\in(x_t,x_{t-1}) \tag{131}$$

and

$$f^I(u,\bar u)\to 0 \quad \text{as } u\to\infty. \tag{132}$$

Now we can determine the domain of the dual operator, i.e. we determine the conditions we must impose on an arbitrary vector g^I so that we can write $(g,\partial\bar\partial f) = (\partial\bar\partial g, f)$. In order to do so we compute using the previous boundary conditions

$$\int_H dx,\, dy\, g^{I*}\big(\partial_x^2+\partial_y^2\big) f^I$$

$$= \int_0^\infty dy \big[g^{z*}\partial_x f^z + g^{\bar z *}\partial_x f^{\bar z}\big]\Big|_{x=-\infty}^{x=+\infty}$$

$$+ \sum \int_{x_t}^{x_{t-1}} dx\,\big[g^z(x,x) - e^{i2\pi\alpha_t} g^{\bar z}(x,x)\big]^* \partial_y f^z$$

$$+\sum \int_{x_t}^{x_{t-1}} dx \left[\partial_y g^{z}(x,x) + e^{i2\pi\alpha_t}\partial_y g^{\bar z}(x,x)\right]^* f^z$$
$$+\int_H dx\,dy\,\left(\partial_x^2+\partial_y^2\right)g^{I*}f^I \tag{133}$$

from which we see that g^I must satisfy the same boundary conditions as f^I and hence the operator is not only Hermitian but self-adjoint.

Appendix B. Details on the metric for modes

Consider for example the computation $(\bar X_{(a)1}, \bar X_{(a)2})$. In the following we write $\bar{\mathcal{X}}_1 = \bar{\mathcal{G}}$ and $\bar{\mathcal{X}}_2 = \bar{\mathcal{F}}$ for notational simplicity. It is immediate to get

$$\begin{aligned}-i\int_{|u|=r_0} *j = r_0\int_0^{\pi} d\theta\,\Big[&\big(\bar{\mathcal{G}}\big(re^{i\theta}\big)\big)^*\partial_r\bar{\mathcal{F}}\big(re^{i\theta}\big)+\big(\bar{\mathcal{G}}\big(re^{-i\theta}\big)\big)^*\partial_r\bar{\mathcal{F}}\big(re^{-i\theta}\big)\\ &-\partial_r\big(\bar{\mathcal{G}}\big(re^{i\theta}\big)\big)^*\bar{\mathcal{F}}\big(re^{i\theta}\big)-\partial_r\big(\bar{\mathcal{G}}\big(re^{-i\theta}\big)\big)^*\bar{\mathcal{F}}\big(re^{-i\theta}\big)\Big]\Big|_{r=r_0}-\end{aligned} \tag{134}$$

Now we rewrite $\partial_r\bar{\mathcal{F}}(re^{i\theta}) = -\frac{i}{r}\partial_\theta\bar{\mathcal{F}}(re^{i\theta})$ and so on for all the other derivatives then we get

$$-i\int_{|u|=r_0} *j = i\int_0^{\pi} d\theta\,\big[\partial_\theta\big(\big(\bar{\mathcal{G}}\big(r_0e^{-i\theta}\big)\big)^*\bar{\mathcal{F}}\big(r_0e^{-i\theta}\big)\big)-\partial_\theta\big(\big(\bar{\mathcal{G}}\big(r_0e^{+i\theta}\big)\big)^*\bar{\mathcal{F}}\big(r_0e^{+i\theta}\big)\big)\big] \tag{135}$$

which vanishes because of the boundary conditions which ensure that $(\bar{\mathcal{G}}(r_0e^{i\theta}))^*\bar{\mathcal{F}}(r_0e^{i\theta})|_{\theta=0^+} = (\bar{\mathcal{G}}(r_0e^{-i\theta}))^*\bar{\mathcal{F}}(r_0e^{-i\theta})|_{\theta=0^+}$. Similarly for $(\bar X_{(a)}, X_{(c)})$ we arrive to

$$-i\int_{|u|=r_0} *j = ie^{-i2\pi\alpha_1}\int_{-\pi}^{\pi} d\theta\,\big[\big(\partial_\theta\bar{\mathcal{G}}\big(r_0e^{-i\theta}\big)\big)^*\mathcal{F}\big(r_0e^{i\theta}\big)-\big(\bar{\mathcal{G}}\big(r_0e^{-i\theta}\big)\big)^*\partial_\theta\mathcal{F}\big(r_0e^{+i\theta}\big)\big], \tag{136}$$

where it is meaningful write the integration interval as $[-\pi,\pi]$ since both the terms are continuous at $\theta=0$, i.e. for example $(\partial_\theta\bar{\mathcal{G}}(r_0e^{-i\theta}))^*\mathcal{F}(r_0e^{+i\theta})|_{\theta=0^+} = (\partial_\theta\bar{\mathcal{G}}(r_0e^{i\theta}))^*\mathcal{F}(r_0e^{-i\theta})|_{\theta=0^+}$. Now integrating by part the first and using the boundary conditions to evaluate to zero the constant obtained from the integration by part we find

$$-i\int_{|u|=r_0} *j = -2ie^{-i2\pi\alpha_1}\int_{-\pi}^{\pi} d\theta\,\big(\partial_\theta\bar{\mathcal{G}}\big(r_0e^{-i\theta}\big)\big)^*\mathcal{F}\big(r_0e^{+i\theta}\big). \tag{137}$$

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