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On the sensitivity of multiple eigenvalues of nonsymmetric matrix pencils ☆

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Abstract

This paper considers the sensitivity of semisimple multiple eigenvalues and corresponding generalized eigenvector matrices of a nonsymmetric matrix pencil analytically dependent on several parameters. The directional derivatives of the multiple eigenvalues are obtained, and the average of eigenvalues and corresponding generalized eigenvector matrices are proved to be analytic. The results can be used to define the sensitivity of the semisimple multiple eigenvalues and corresponding generalized eigenvector matrices. These are useful for investigating structural optimal design, model updating, and structural damage detection.

Keywords: Nonsymmetric matrix pencil; Semisimple multiple eigenvalue; Generalized eigenvector matrix; Sensitivity analysis

1. Introduction

Throughout this paper we use the following notation. $C^{m\times n}$ denotes the set of complex $m\times n$ matrices, $C^n = C^{n\times 1}$, $C = C^1$ and $C_r^{m\times n} = \left\{A \in C^{m\times n} | \mathrm{rank}(A) = r\right\}$. $N_k = \{A = (a_{ij}) \in C^{k\times k} | a_{j,j+1} = 1\ (j = 1, \ldots, k-1)$; otherwise $a_{ij} = 0$ $(i, j = 1, \ldots, k)\}$. I_n is the identity matrix of order n. $\|\cdot\|_2$ denotes the Euclidean vector norm and the spectral norm, and $\|\cdot\|_F$ denotes the Frobenius norm. For

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$A, B \in C^{n\times n}$, the set of the eigenvalues of $Ax = \lambda x$ and $Ax = \lambda Bx$ are respectively denoted by $\lambda(A)$ and $\lambda(A, B)$, $\lambda_1(A), \ldots, \lambda_n(A)$ denote the eigenvalues of an $n \times n$ matrix A, $\rho(A)$ denotes the spectral radius of A. $R(A)$ and $\mathrm{Ker}(A)$ denote the range and the kernel of the matrix A, respectively. If $A = (a_{ij}) \in C^{m\times m}$, $B \in C^{n\times n}$, then $A \otimes B = (a_{ij}B) \in C^{mn\times mn}$ is called the Kronecker product of A and B.

Definition 1.1. Let $A,B \in C^{n\times n}$. If $\det(A + \lambda B) \not\equiv 0$ for $\lambda \in C$, then $\{A, B\}$ is called a regular matrix pencil or nonsymmetric matrix pencil.

If A and B are Hermitian matrices and B is positive definite, then $\{A, B\}$ is a regular matrix pencil and is called the symmetric matrix pencil.

Definition 1.2. Let $A, B \in C^{n\times n}$. The algebraic multiplicity of the eigenvalue $\lambda_i \in \lambda(A, B)$ is its multiplicity as a zero of the characteristic polynomial $\det(A - \lambda B)$. The geometric multiplicity of the eigenvalue $\lambda_i \in \lambda(A, B)$ is defined as the dimension of the subspace $\mathrm{Ker}(A - \lambda_i B)$. An eigenvalue λ_i of the matrix pencil $\{A, B\}$ is called semisimple if the algebraic multiplicity of the eigenvalue λ_i is equal to its geometric multiplicity.

Definition 1.3. Let $A, B \in C^{n\times n}$. Suppose there exist $X_1, Y_1 \in C_r^{n\times r}$, $C_1 \in C^{r\times r}$ such that

$$AX_1 = BX_1C_1, \quad Y_1^{\mathrm{T}}A = C_1Y_1^{\mathrm{T}}B, \quad Y_1^{\mathrm{T}}BX_1 = I_r.$$

If $\lambda(C_1) = \{\lambda_1, \ldots, \lambda_r\}$ (obviously, $\lambda_1, \ldots, \lambda_r$ are eigenvalues of $\{A, B\}$), then X_1 and Y_1 are respectively called the right and left generalized eigenvector matrices of $Ax = \lambda Bx$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_r$; $R(X_1)$ and $R(Y_1)$ are respectively called the right and left generalized invariant subspaces corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_r$.

Suppose that L is an open set of C^N, $p = (p_1, \ldots, p_N)^{\mathrm{T}} \in L$, $A(p), B(p) \in C^{n\times n}$ are analytic matrix-valued functions in some neighborhood $N(p^*)$ of the point $p^* \in L$. Without loss of generality we may assume that the point p^* is the origin of C^n. In this paper we study sensitivity analysis of the following generalized eigenvalue problems

$$A(p)x(p) = \lambda(p)B(p)x(p), \quad \lambda(p) \in C, \quad x(p) \in C^n, \quad p \in N(0). \tag{1.1}$$

Except where stated otherwise, we consider the case in which

(i) $\{A(p), B(p)\}$ is a regular matrix pencil for $p \in N(0)$.
(ii) Eq. (1.1) has an eigenvalue, $\lambda_1 = \lambda_1(0)$, of multiplicity $r > 1$ when $p = 0$ and λ_1 is a semisimple multiple eigenvalue of the matrix pencil $\{A(0), B(0)\}$.

Sensitivity analysis of the eigenvalue problems is of great importance in some engineering applications, for example, structural optimal design [1,2], model updating [3–5] and structural damage detection [6]. The derivatives of the eigenvalues and eigenvectors of a matrix or matrix pencil depending on one parameter have been studied by Rellich [7], Lancaster [8], Kato [9], Meyer and Stewart [10]. The derivatives of simple eigenvalues and corresponding eigenvectors of a matrix or matrix pencil depending on several parameters have been considered by many authors [11–17]. Sensitivity analysis of multiple eigenvalues of a general matrix or symmetric matrix pencil depending on several parameters have been investigated by Sun [18–22], Simpson [24], Haug and Rousselet [25], Chu [26], Dailey [27]. A study is made of the derivatives of eigenvalues and corresponding eigenvectors of matrix-valued functions depending on several parameters by Andrew et al. [28]. However, the situation of the nonsymmetric matrix pencils depending on several parameters when the eigenvalues are multiple is rarely treated in the literature. In this paper we consider the sensitivity of multiple eigenvalues and corresponding generalized eigenvector matrices of the nonsymmetric eigenproblems (1.1) with respect to the parameters $p_1, \dots, p_N$.

The sensitivity analysis of the semisimple multiple eigenvalues and corresponding generalized eigenvector matrices of the nonsymmetric matrix pencils is investigated in Section 2. The directional derivatives of the semisimple multiple eigenvalues are given, and the average of the semisimple multiple eigenvalues and corresponding generalized eigenvector matrices are proved to be analytic. These results were derived by using the techniques described by Sun [17–22] and Chu [26]. In Section 3, we define the sensitivity of the semisimple multiple eigenvalues and corresponding generalized eigenvector matrices, and give some formulas for computing the sensitivities. In Section 4, we discuss the determination of the sensitive elements.

2. Some results on partial derivatives

Before giving the main results, we cite the Implicit Function Theorem and give the related concepts.

Theorem 2.1 (Implicit Function Theorem [29]). *Let Ω be a neighborhood of the origin of C^{s+t}. If the complex-valued functions $f_j(\xi_1, \dots, \xi_s, \eta_1, \dots, \eta_t)$ $(j = 1, \dots, s)$ satisfy*

(i) $f_j(0, \dots, 0) = 0$ $(j = 1, \dots, s)$;
(ii) $f_j(\xi_1, \dots, \xi_s, \eta_1, \dots, \eta_t)$ $(j = 1, \dots, s)$ *are analytic on* Ω;
(iii) $\det \frac{\partial(f_1,\dots,f_s)}{\partial(\xi_1,\dots,\xi_s)} \neq 0$ *for* $\xi_1 = \cdots = \xi_s = \eta_1 = \cdots = \eta_t = 0$,

then there exists a neighborhood U of the origin of C^t such that the equations

$$f_j(\xi_1, \dots, \xi_s, \eta_1, \dots, \eta_t) = 0, \quad j = 1, \dots, s$$

have a unique solution $\xi_j = g_j(\eta_1, \dots, \eta_t)$ $(j = 1, \dots, s)$ which satisfy

(1) $g_j(0, 0, \ldots, 0) = 0$ $(j = 1, \ldots, s)$;
(2) $g_j(\eta_1, \ldots, \eta_t)$ $(j = 1, \ldots, s)$ *are analytic on* U.

Definition 2.1. Let $\mathscr{B}$ be an open set of C^N, $u(p)$ be a function defined on $\mathscr{B}$, $p^* \in \mathscr{B}$, $v \in C^N$ with $\|v\|_2 = 1$. If

$$\lim_{t \to 0+} \frac{u(p^* + tv) - u(p^*)}{t}$$

exists, then the limit value is called the directional derivative of $u(p)$ in the direction v at p^*, denoted by $D_v u(p^*)$.

Definition 2.2. Let $p = (p_1, \ldots, p_N)^{\mathrm{T}} \in C^N$, $A(p)$ and $B(p)$ be complex $n \times n$ matrix-valued functions analytic at $p^* \in C^N$. Fox any fixed $j \in \{1, \ldots, N\}$, $\lambda \in C$, define a matrix of order n

$$S_j(p^*, \lambda) = \left.\frac{\partial A(p)}{\partial p_j}\right|_{p=p^*} - \lambda \left.\frac{\partial B(p)}{\partial p_j}\right|_{p=p^*}.$$

Lemma 2.1 [23]. *Let* $A, B \in C^{n \times n}$. *If* $\{A, B\}$ *is a regular matrix pencil, then there exist invertible matrices* $P, Q \in C^{n \times n}$, *such that*

$$PAQ = \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad PBQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix},$$

where $J = \mathrm{diag}(J_1(\lambda_1), \ldots, J_r(\lambda_r)) \in C^{n_1 \times n_1}$ $(\lambda_i \neq \lambda_j$ *for* $i \neq j)$, $J_i(\lambda_i) = \mathrm{diag}(J_i^{(1)}(\lambda_i), \ldots, J_i^{(k_i)}(\lambda_i)) \in C^{n(\lambda_i) \times n(\lambda_i)}$, $\sum_{i=1}^r n(\lambda_i) = n_1$,

$$J_i^{(k)}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \lambda_i \end{pmatrix} \in C^{n_k(\lambda_i) \times n_k(\lambda_i)},$$

$\sum_{k=1}^{k_i} n_k(\lambda_i) = n(\lambda_i)$, $N = \mathrm{diag}(N_{l_1}, \ldots, N_{l_s}) \in C^{n_2 \times n_2}$, $\sum_{j=1}^{s} l_j = n_2$, $n_1 + n_2 = n$.

If $\{A, B\}$ is a regular matrix pencil of order n and λ_1 is a semisimple multiple eigenvalue of $\{A, B\}$ with multiplicity r, then it follows from Lemma 2.1 that there exist invertible matrices P and Q, such that

$$PAQ = \begin{bmatrix} \lambda_1 I_r & 0 \\ 0 & A_2 \end{bmatrix}, \quad PBQ = \begin{bmatrix} I_r & 0 \\ 0 & B_2 \end{bmatrix},$$

where $A_2, B_2 \in C^{(n-r) \times (n-r)}$, $\lambda_1 \notin \lambda(A_2, B_2)$.

Using the techniques in [17–22] and [26], we obtain the following results.

Theorem 2.2. *Let* $p = (p_1, \ldots, p_N)^{\mathrm{T}} \in C^N$, $U(0)$ *be a neighborhood of the origin of* C^N, $A(p), B(p) \in C^{n \times n}$ *be analytic on* $U(0)$. *Assume that* $\{A(0), B(0)\}$ *is a* *regular matrix pencil, and* λ_1 *is a semisimple multiple eigenvalue of* (1.1) *at the origin with multiplicity* r $(r > 1)$, *i.e., there exist invertible matrices* $X = [X_1, X_2] \in C^{n\times n}$, $Y = [Y_1, Y_2] \in C^{n\times n}$, $X_1, Y_1 \in C^{n\times r}$ *such that*

$$Y^{\mathrm{T}}A(0)X = \begin{bmatrix} \lambda_1 I_r & 0 \\ 0 & A_2 \end{bmatrix}, \quad Y^{\mathrm{T}}B(0)X = \begin{bmatrix} I_r & 0 \\ 0 & B_2 \end{bmatrix},$$
$$\lambda_1 \notin \lambda(A_2, B_2), \tag{2.1}$$

then

(1) *there exist a neighborhood of the origin* $U_1(0) \subseteq U(0)$ *and* r *functions* $\lambda_1(p), \ldots, \lambda_r(p)$ *defined on* $U_1(0)$, *such that* $\lambda_1(p), \ldots, \lambda_r(p)$ *are eigenvalues of* $\{A(p), B(p)\}$, $\lambda_i(p)$ $(i = 1, \ldots, r)$ *are continuous at the origin, and* $\lambda_i(0) = \lambda_1$ $(i = 1, \ldots, r)$;

(2) *define* $\hat{\lambda}(p) = \left[\sum_{i=1}^{r} \lambda_i(p)\right]\big/ r$, *where* $\lambda_i(p)$ $(i = 1, \ldots, r)$ *are as in* (1), *then* $\hat{\lambda}(p)$ *is analytic on* $U_1(0)$, *and*

$$\left.\frac{\partial\hat{\lambda}(p)}{\partial p_i}\right|_{p=0} = \frac{1}{r}\mathrm{tr}\left(Y_1^{\mathrm{T}} S_i(0, \lambda_1) X_1\right), \quad i = 1, \ldots, N.$$

(3) *for any fixed direction* $v \in C^N$ *with* $\|v\|_2 = 1$, *there exist* $\beta > 0$ *and* r *single valued continuous functions* $u_1(tv), \ldots, u_r(tv)$ *defined on* $[-\beta, \beta]$, *such that*
   (i) $u_1(tv), \ldots, u_r(tv)$ *are* r *eigenvalues of* $\{A(tv), B(tv)\}$;
   (ii) $\{u_s(tv)\}_{s=1}^{r} = \{\lambda_s(tv)\}_{s=1}^{r}$ *for each* $t \in [-\beta, \beta]$, *and there is a one-to-one correspondence between the elements of* $\{u_s(tv)\}_{s=1}^{r}$ *and* $\{\lambda_s(tv)\}_{s=1}^{r}$, *where* $\lambda_1(p), \ldots, \lambda_r(p)$ *are as in* (1);
   (iii) *there exists a permutation* π *of* $\{1, \ldots, r\}$ *dependent on* v *such that*

$$D_v u_k(0) = \lambda_{\pi(k)}\left(\sum_{j=1}^{N} v_j Y_1^{\mathrm{T}} S_j(0, \lambda_1) X_1\right), \qquad k = 1, \ldots, r.$$

(4) *there exist a neighborhood* $U_1(0) \subseteq U(0)$ *of the origin and analytic matrix-valued functions* $X_1(p), Y_1(p) \in C^{n\times r}$ *defined on* $U_1(0)$, *such that* $X_1(p), Y_1(p)$ *are respectively the right and left generalized eigenvector matrices of* (1.1) *corresponding to* $\lambda_1(p), \ldots, \lambda_r(p)$, *where* $\lambda_1(p), \ldots, \lambda_r(p)$ *are as in* (1). *Moreover*, $X_1(0) = X_1$, $Y_1(0) = Y_1$,

$$\left.\frac{\partial X_1(p)}{\partial p_j}\right|_{p=0} = X_2\left(\lambda_1 B_2 - A_2\right)^{-1} Y_2^{\mathrm{T}} S_j(0, \lambda_1) X_1, \quad j = 1, \ldots, N,$$
$$\left.\frac{\partial Y_1(p)}{\partial p_j}\right|_{p=0} = Y_2(\lambda_1 B_2 - A_2)^{-\mathrm{T}} X_2^{\mathrm{T}} S_j^{\mathrm{T}}(0, \lambda_1) Y_1 - Y_1 X_1^{\mathrm{T}} \left.\frac{\partial B^{\mathrm{T}}(p)}{\partial p_j}\right|_{p=0} Y_1,$$
$$j = 1, \ldots, N.$$

Proof. Let

$$\begin{aligned}\tilde{A}(p) &= Y^{\mathrm{T}}A(p)X = \begin{bmatrix}\tilde{A}_{11}(p) & \tilde{A}_{12}(p)\\ \tilde{A}_{21}(p) & \tilde{A}_{22}(p)\end{bmatrix},\\ \tilde{B}(p) &= Y^{\mathrm{T}}B(p)X = \begin{bmatrix}\tilde{B}_{11}(p) & \tilde{B}_{12}(p)\\ \tilde{B}_{21}(p) & \tilde{B}_{22}(p)\end{bmatrix},\end{aligned} \tag{2.2}$$

where $\tilde{A}_{11}(p), \tilde{B}_{11}(p) \in C^{r\times r}$.

We introduce the following matrix-valued functions

$$\begin{aligned}F(Z_1, W_1, Z_2, W_2, p) &= \tilde{A}_{11}(p)W_1 + Z_1\tilde{A}_{21}(p)W_1 + \tilde{A}_{12}(p) + Z_1\tilde{A}_{22}(p),\\ G(Z_1, W_1, Z_2, W_2, p) &= \tilde{B}_{11}(p)W_1 + Z_1\tilde{B}_{21}(p)W_1 + \tilde{B}_{12}(p) + Z_1\tilde{B}_{22}(p),\\ H(Z_1, W_1, Z_2, W_2, p) &= Z_2\tilde{A}_{11}(p) + \tilde{A}_{21}(p) + Z_2\tilde{A}_{12}(p)W_2 + \tilde{A}_{22}(p)W_2,\\ K(Z_1, W_1, Z_2, W_2, p) &= Z_2\tilde{B}_{11}(p) + \tilde{B}_{21}(p) + Z_2\tilde{B}_{12}(p)W_2 + \tilde{B}_{22}(p)W_2,\end{aligned}$$

where $Z_1, W_1 \in C^{r\times(n-r)}$, $Z_2, W_2 \in C^{(n-r)\times r}$.

Observe that the functions $F(Z_1, W_1, Z_2, W_2, p)$, $G(Z_1, W_1, Z_2, W_2, p)$, $H(Z_1, W_1, Z_2, W_2, p)$, $K(Z_1, W_1, Z_2, W_2, p)$ are analytic for $Z_1, W_1 \in C^{r\times(n-r)}$, $Z_2, W_2 \in C^{(n-r)\times r}$, $p \in U(0)$, and $F(Z_1, W_1, Z_2, W_2, p) = G(Z_1, W_1, Z_2, W_2, p) = H(Z_1, W_1, Z_2, W_2, p) = K(Z_1, W_1, Z_2, W_2, p) = 0$ for $(Z_1, W_1, Z_2, W_2, p) = 0$. Moreover,

$$\begin{aligned}&\det \left.\frac{\partial(F, G, H, K)}{\partial(Z_1, W_1, Z_2, W_2)}\right|_{(Z_1, W_1, Z_2, W_2, p)=0}\\ &= \det \left.\begin{bmatrix}\frac{\partial F}{\partial Z_1} & \frac{\partial G}{\partial Z_1} & \frac{\partial H}{\partial Z_1} & \frac{\partial K}{\partial Z_1}\\ \frac{\partial F}{\partial W_1} & \frac{\partial G}{\partial W_1} & \frac{\partial H}{\partial W_1} & \frac{\partial K}{\partial W_1}\\ \frac{\partial F}{\partial Z_2} & \frac{\partial G}{\partial Z_2} & \frac{\partial H}{\partial Z_2} & \frac{\partial K}{\partial Z_2}\\ \frac{\partial F}{\partial W_2} & \frac{\partial G}{\partial W_2} & \frac{\partial H}{\partial W_2} & \frac{\partial K}{\partial W_2}\end{bmatrix}\right|_{(Z_1, W_1, Z_2, W_2, p)=0}\\ &= \det \begin{bmatrix} I_r \otimes \tilde{A}_{22}(0) & I_r \otimes \tilde{B}_{22}(0) & 0 & 0\\ \tilde{A}_{11}^{\mathrm{T}}(0) \otimes I_{n-r} & \tilde{B}_{11}^{\mathrm{T}}(0) \otimes I_{n-r} & 0 & 0\\ 0 & 0 & I_{n-r} \otimes \tilde{A}_{11}(0) & I_{n-r} \otimes \tilde{B}_{11}(0)\\ 0 & 0 & \tilde{A}_{22}^{\mathrm{T}}(0) \otimes I_r & \tilde{B}_{22}^{\mathrm{T}}(0) \otimes I_r\end{bmatrix}\\ &= \det \begin{bmatrix} I_r \otimes A_2 & I_r \otimes B_2 & 0 & 0\\ \lambda_1 I_r \otimes I_{n-r} & I_r \otimes I_{n-r} & 0 & 0\\ 0 & 0 & I_{n-r} \otimes \lambda_1 I_r & I_{n-r} \otimes I_r\\ 0 & 0 & A_2^{\mathrm{T}} \otimes I_r & B_2^{\mathrm{T}} \otimes I_r\end{bmatrix}\\ &= \det \begin{bmatrix} I_r \otimes A_2 & I_r \otimes B_2\\ \lambda_1 I_r \otimes I_{n-r} & I_r \otimes I_{n-r}\end{bmatrix} \det \begin{bmatrix} I_{n-r} \otimes \lambda_1 I_r & I_{n-r} \otimes I_r\\ A_2^{\mathrm{T}} \otimes I_r & B_2^{\mathrm{T}} \otimes I_r\end{bmatrix}\\ &= (-1)^{r(n-r)}[\det(A_2 - \lambda_1 B_2)]^{2r}.\end{aligned}$$

Since $\lambda_1 \notin \lambda(A_2, B_2)$, we have $\det(A_2 - \lambda_1 B_2) \neq 0$ and

$$\det \left.\frac{\partial(F, G, H, K)}{\partial(Z_1, W_1, Z_2, W_2)}\right|_{(Z_1, W_1, Z_2, W_2, p)=0} \neq 0.$$

It follows from Theorem 2.1 that the equations

$$\begin{cases} F(Z_1, W_1, Z_2, W_2, p) = 0, \\ G(Z_1, W_1, Z_2, W_2, p) = 0, \\ H(Z_1, W_1, Z_2, W_2, p) = 0, \\ K(Z_1, W_1, Z_2, W_2, p) = 0, \end{cases}$$

have unique solutions $Z_1 = Z_1(p)$, $W_1 = W_1(p)$, $Z_2 = Z_2(p)$, $W_2 = W_2(p)$ in a neighborhood $\hat{U}(0) \subseteq U(0)$ of the origin, which satisfy

(1) $Z_1(0) = 0$, $W_1(0) = 0$, $Z_2(0) = 0$, $W_2(0) = 0$;
(2) $Z_1(p)$, $W_1(p)$, $Z_2(p)$, $W_2(p)$ are analytic on $\hat{U}(0)$.

Thus, we have

$$\begin{bmatrix} I_r & Z_1(p) \\ Z_2(p) & I_{n-r} \end{bmatrix} \begin{bmatrix} \tilde{A}_{11}(p) & \tilde{A}_{12}(p) \\ \tilde{A}_{21}(p) & \tilde{A}_{22}(p) \end{bmatrix} \begin{bmatrix} I_r & W_1(p) \\ W_2(p) & I_{n-r} \end{bmatrix} = \begin{bmatrix} A_1(p) & 0 \\ 0 & A_2(p) \end{bmatrix}, \tag{2.3}$$

$$\begin{bmatrix} I_r & Z_1(p) \\ Z_2(p) & I_{n-r} \end{bmatrix} \begin{bmatrix} \tilde{B}_{11}(p) & \tilde{B}_{12}(p) \\ \tilde{B}_{21}(p) & \tilde{B}_{22}(p) \end{bmatrix} \begin{bmatrix} I_r & W_1(p) \\ W_2(p) & I_{n-r} \end{bmatrix} = \begin{bmatrix} B_1(p) & 0 \\ 0 & B_2(p) \end{bmatrix}, \tag{2.4}$$

where

$$\begin{cases} A_1(p) = \tilde{A}_{11}(p) + Z_1(p)\tilde{A}_{21}(p) + \tilde{A}_{12}(p)W_2(p) + Z_1(p)\tilde{A}_{22}(p)W_2(p), \\ A_2(p) = \tilde{A}_{22}(p) + Z_2(p)\tilde{A}_{12}(p) + \tilde{A}_{21}(p)W_1(p) + Z_2(p)\tilde{A}_{11}(p)W_1(p), \\ B_1(p) = \tilde{B}_{11}(p) + Z_1(p)\tilde{B}_{21}(p) + \tilde{B}_{12}(p)W_2(p) + Z_1(p)\tilde{B}_{22}(p)W_2(p), \\ B_2(p) = \tilde{B}_{22}(p) + Z_2(p)\tilde{B}_{12}(p) + \tilde{B}_{21}(p)W_1(p) + Z_2(p)\tilde{B}_{11}(p)W_1(p). \end{cases} \tag{2.5}$$

It is easily proved that

$$\left.\frac{\partial A_1(p)}{\partial p_i}\right|_{p=0} = Y_1^{\mathrm{T}} \left.\frac{\partial A(p)}{\partial p_i}\right|_{p=0} X_1, \quad \left.\frac{\partial B_1(p)}{\partial p_i}\right|_{p=0} = Y_1^{\mathrm{T}} \left.\frac{\partial B(p)}{\partial p_i}\right|_{p=0} X_1, \qquad i = 1, \ldots, N. \tag{2.6}$$

From (2.3) and (2.4), we obtain

$$\begin{bmatrix} I_r & Z_1(p) \\ Z_2(p) & I_{n-r} \end{bmatrix} \tilde{A}(p) \begin{bmatrix} I_r \\ W_2(p) \end{bmatrix} = \begin{bmatrix} A_1(p) \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} I_r & Z_1(p) \\ Z_2(p) & I_{n-r} \end{bmatrix} \tilde{B}(p) \begin{bmatrix} I_r \\ W_2(p) \end{bmatrix} = \begin{bmatrix} B_1(p) \\ 0 \end{bmatrix}.$$

According to the analyticity of $B_1(p)$, from $B_1(0) = I_r$ we know that $B_1(p)$ is nonsingular on $\hat{U}(0)$ provided that the neighborhood $\hat{U}(0)$ is sufficiently small. It follows from (1) and (2) that

$$\begin{bmatrix} I_r & Z_1(p) \\ Z_2(p) & I_{n-r} \end{bmatrix}, \quad \begin{bmatrix} I_r & W_1(p) \\ W_2(p) & I_{n-r} \end{bmatrix}$$

are nonsingular if p belongs to the sufficiently small neighborhood $\hat{U}(0)$. Thus, we get

$$\tilde{A}(p)\begin{bmatrix} I_r \\ W_2(p) \end{bmatrix} = \tilde{B}(p)\begin{bmatrix} I_r \\ W_2(p) \end{bmatrix} B_1(p)^{-1}A_1(p). \tag{2.7}$$

Similarly, it follows that

$$[I_r, \ \ Z_1(p)]\tilde{A}(p) = A_1(p)B_1(p)^{-1}[I_r, \ \ Z_1(p)]\tilde{B}(p). \tag{2.8}$$

Let

$$X_1(p) = X\begin{bmatrix} I_r \\ W_2(p) \end{bmatrix}, \quad Y_1(p) = Y\begin{bmatrix} I_r \\ Z_1(p)^{\mathrm{T}} \end{bmatrix} B_1^{-T}(p). \tag{2.9}$$

From (2.2), (2.4), (2.7)–(2.9), we obtain

$$A(p)X_1(p) = B(p)X_1(p)B_1(p)^{-1}A_1(p), \tag{2.10}$$

$$Y_1(p)^{\mathrm{T}}A(p) = B_1(p)^{-1}A_1(p)Y_1(p)^{\mathrm{T}}B(p), \tag{2.11}$$

$$Y_1(p)^{\mathrm{T}}B(p)X_1(p) = I_r, \tag{2.12}$$

and

$$A_1(0) = \lambda_1 I_r, \quad B_1(0) = I_r, \tag{2.13}$$

$$X_1(0) = X_1, \quad Y_1(0) = Y_1. \tag{2.14}$$

Let $C_1(p) = B_1(p)^{-1}A_1(p)$, then $\lambda(C_1(p)) \subseteq \lambda(A(p), B(p))$.

(1) Since $C_1(p)$ is analytic on $\hat{U}(0)$, combining with $C_1(0) = B_1(0)^{-1}A_1(0) = \lambda_1 I_r$, by Theorem 2.1 in [19] we know that there exist a neighborhood $U_1(0) \subseteq \hat{U}(0)$ of the origin and r eigenvalue functions $\lambda_1(p), \ldots, \lambda_r(p)$ of $C_1(p)$ defined on $U_1(0)$, which are continuous at $p = 0$ and satisfy $\lambda_s(0) = \lambda_1$ $(s = 1, \ldots, r)$. From (2.10), we know that $\lambda_1(p), \ldots, \lambda_r(p)$ are also eigenvalue functions of $\{A(p), B(p)\}$. Hence, the first result is proved.

(2) Since $\hat{\lambda}(p) = \left[\sum_{i=1}^{r} \lambda_i(p)\right] / r = [\mathrm{tr}(C_1(p))]/r$ and $C_1(p)$ is analytic on $U_1(0)$, then $\hat{\lambda}(p)$ is analytic on $U_1(0)$, moreover,

$$\left.\frac{\partial \hat{\lambda}(p)}{\partial p_i}\right|_{p=0} = \frac{1}{r}\mathrm{tr}\left(\left.\frac{\partial C_1(p)}{\partial p_i}\right|_{p=0}\right).$$

Utilizing (2.6), we have

$$\begin{aligned}\left.\frac{\partial C_1(p)}{\partial p_i}\right|_{p=0} &= Y_1^{\mathrm{T}} \left.\frac{\partial A\,(p)}{\partial p_i}\right|_{p=0} X_1 - \lambda_1 Y_1^{\mathrm{T}} \left.\frac{\partial B(p)}{\partial p_i}\right|_{p=0} X_1 \\ &= Y_1^{\mathrm{T}} S_i(0,\lambda_1) X_1. \end{aligned} \tag{2.15}$$

Hence,

$$\left.\frac{\partial\hat{\lambda}(p)}{\partial p_i}\right|_{p=0} = \frac{1}{r}\mathrm{tr}\left[Y_1^{\mathrm{T}} S_i(0,\lambda_1)X_1\right] \quad (i=1,\dots,N).$$

(3) The proof of the first result of the theorem shows that $\lambda_1(p),\dots,\lambda_r(p)$ are eigenvalue functions of $C_1(p)$ defined on $U_1(0)$ and $\lambda_1(0)=\cdots=\lambda_r(0)=\lambda_1$. Let $\tilde{X}=I_r$, $\tilde{Y}=I_r$, we have $\tilde{Y}^{\mathrm{T}}C_1(0)\tilde{X}=\lambda_1 I_r$ from (2.13). By Theorem 1.1 in [22], we know that for any fixed direction $v\in C^N$ with $\|v\|_2=1$, there are a positive scalar β and r single-valued continuous functions $u_1(tv),\dots,u_r(tv)$ for $t\in[-\beta,\beta]$ such that $u_1(tv),\dots,u_r(tv)$ are r eigenvalues of $C_1(tv)$, the set $\{u_s(tv)\}_{s=1}^r$ and the set $\{\lambda_s(tv)\}_{s=1}^r$ are identical, and there is a one-to-one correspondence between the elements of the two sets. Moreover, we have

$$\{D_v u_s(0)\}_{s=1}^r = \lambda\left(\sum_{j=1}^N v_j \left.\frac{\partial C_1(p)}{\partial p_j}\right|_{p=0}\right).$$

From (2.10), we have $\lambda(C_1(tv))\subseteq\lambda(A(tv),B(tv))$. Hence, $u_1(tv),\dots,u_r(tv)$ are eigenvalue functions of $\{A(tv),B(tv)\}$. Combining with (2.15), we obtain

$$\{D_v u_s(0)\}_{s=1}^r = \lambda\left(\sum_{j=1}^N v_j Y_1^{\mathrm{T}} S_j(0,\lambda_1)X_1\right).$$

Consequently, the proof of the third result is completed.

(4) The relations (2.10)–(2.12) show that $X_1(p)$, $Y_1(p)$ are respectively the right and left generalized eigenvector matrices of (1.1) corresponding to the eigenvalues $\lambda_1(p),\dots,\lambda_r(p)$. Moreover, since $Z_1(p)$, $W_2(p)$, $B_1(p)$ are analytic on $U_1(0)$, then $X_1(p)$ and $Y_1(p)$ are analytic on $U_1(0)$ and $X_1(0)=X_1$, $Y_1(0)=Y_1$. Differentiating (2.10), we get

$$[A(0)-\lambda_1 B(0)]\frac{\partial X_1(0)}{\partial p_i} = \lambda_1\frac{\partial B(0)}{\partial p_i}X_1 - \frac{\partial A(0)}{\partial p_i}X_1 + B(0)X_1\frac{\partial C_1(0)}{\partial p_i}.$$

From (2.9), then

$$\begin{aligned}&[A(0)-\lambda_1 B(0)]X\begin{bmatrix}0\\ \frac{\partial W_2(0)}{\partial p_i}\end{bmatrix} \\ &= \lambda_1\frac{\partial B(0)}{\partial p_i}X_1 - \frac{\partial A(0)}{\partial p_i}X_1 + B(0)X_1\frac{\partial C_1(0)}{\partial p_i}.\end{aligned} \tag{2.16}$$

Combining (2.16) and (2.1), we have

$$\begin{bmatrix} 0 & 0 \\ 0 & A_2 - \lambda_1 B_2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{\partial W_2(0)}{\partial p_i} \end{bmatrix} = \begin{bmatrix} -Y_1^{\mathrm{T}} S_i(0, \lambda_1) X_1 \\ -Y_2^{\mathrm{T}} S_i(0, \lambda_1) X_1 \end{bmatrix} + \begin{bmatrix} \frac{\partial C_1(0)}{\partial p_i} \\ 0 \end{bmatrix}.$$

From (2.15) and $\lambda_1 \notin \lambda(A_2, B_2)$, it follows that

$$\frac{\partial W_2(0)}{\partial p_i} = -(A_2 - \lambda_1 B_2)^{-1} Y_2^{\mathrm{T}} S_i(0, \lambda_1) X_1.$$

From (2.9) we obtain

$$\frac{\partial X_1(0)}{\partial p_i} = X_2 \frac{\partial W_2(0)}{\partial p_i} = X_2(\lambda_1 B_2 - A_2)^{-1} Y_2^{\mathrm{T}} S_i(0, \lambda_1) X_1. \tag{2.17}$$

Since $\frac{\partial B_1^{-1}(p)}{\partial p_i} B_1(p) + B_1^{-1}(p) \frac{\partial B_1(p)}{\partial p_i} = 0$, it follows from (2.6), (2.9) and (2.13) that

$$\left. \frac{\partial Y_1^{\mathrm{T}}(p)}{\partial p_i} \right|_{p=0} = \left[-Y_1^{\mathrm{T}} \frac{\partial B(0)}{\partial p_i} X_1, \quad \frac{\partial Z_1(0)}{\partial p_i} \right] Y^{\mathrm{T}}. \tag{2.18}$$

Differentiating (2.11), we have

$$\begin{aligned} &\frac{\partial Y_1^{\mathrm{T}}(0)}{\partial p_i} [A(0) - \lambda_1 B(0)] \\ &\quad = -Y_1^{\mathrm{T}} \left[\frac{\partial A(0)}{\partial p_i} - \lambda_1 \frac{\partial B(0)}{\partial p_i} \right] + \frac{\partial C_1(0)}{\partial p_i} Y_1^{\mathrm{T}} B(0). \end{aligned} \tag{2.19}$$

Substituting (2.18) into (2.19), postmultiplying by X, we get

$$\begin{aligned} &\left[-Y_1^{\mathrm{T}} \frac{\partial B(0)}{\partial p_i} X_1, \quad \frac{\partial Z_1(0)}{\partial p_i} \right] \begin{bmatrix} 0 & 0 \\ 0 & A_2 - \lambda_1 B_2 \end{bmatrix} \\ &= -\left[Y_1^{\mathrm{T}} S_i(0, \lambda_1) X_1, \quad Y_1^{\mathrm{T}} S_i(0, \lambda_1) X_2 \right] + \left[\frac{\partial C_1(0)}{\partial p_i}, \quad 0 \right]. \end{aligned}$$

Combining with (2.15), we have

$$\frac{\partial Z_1(0)}{\partial p_i} (A_2 - \lambda_1 B_2) = -Y_1^{\mathrm{T}} S_i(0, \lambda_1) X_2.$$

Since $\lambda_1 \notin \lambda(A_2, B_2)$, then

$$\frac{\partial Z_1(0)}{\partial p_i} = -Y_1^{\mathrm{T}} S_i(0, \lambda_1) X_2 (A_2 - \lambda_1 B_2)^{-1}.$$

From (2.18), we obtain

$$\left. \frac{\partial Y_1(p)}{\partial p_i} \right|_{p=0} = Y_2 (\lambda_1 B_2 - A_2)^{-T} X_2^{\mathrm{T}} S_i^{\mathrm{T}}(0, \lambda_1) Y_1 - Y_1 X_1^{\mathrm{T}} \left. \frac{\partial B^{\mathrm{T}}(p)}{\partial p_i} \right|_{p=0} Y_1.$$

So the proof of the theorem is completed. □

3. Sensitivity of the semisimple multiple eigenvalues

According to Theorem 2.2 we may introduce the following definition.

Definition 3.1. Let $p = (p_1, \ldots, p_N)^{\mathrm{T}} \in C^N$, $U(0)$ be a neighborhood of the origin of C^N, $A(p)$, $B(p) \in C^{n\times n}$ be analytic on $U(0)$. Assume that $\{A(0), B(0)\}$ is a regular matrix pencil, and there exist invertible matrices $X = [X_1, X_2] \in C^{n\times n}$, $Y = [Y_1, Y_2] \in C^{n\times n}(X_1, Y_1 \in C^{n\times r})$ such that

$$\begin{aligned} &Y^{\mathrm{T}} A(0) X = \begin{bmatrix} \lambda_1 I_r & 0 \\ 0 & A_2 \end{bmatrix}, \quad Y^{\mathrm{T}} B(0) X = \begin{bmatrix} I_r & 0 \\ 0 & B_2 \end{bmatrix}, \\ &\lambda_1 \notin \lambda(A_2, B_2). \end{aligned} \tag{3.1}$$

For any fixed direction $v \in C^N$ with $\|v\|_2 = 1$, the quantity

$$S_p^v(\lambda_1) = \rho\left(\sum_{j=1}^{N} v_j Y_1^{\mathrm{T}} S_j(0, \lambda_1) X_1\right)$$

is called the sensitivity of the semisimple multiple eigenvalue λ_1 of (1.1) in the direction v. The quantity

$$S_p(\lambda_1) = \max_{\substack{v\in C^N \\ \|v\|_2=1}} S_p^v(\lambda_1)$$

is called the sensitivity of the semisimple multiple eigenvalue λ_1 of (1.1) at $p = 0$. Moreover, the quantities

$$S_p^v(X_1) = \left\| \sum_{j=1}^{N} v_j X_2(\lambda_1 B_2 - A_2)^{-1} Y_2^{\mathrm{T}} S_j(0, \lambda_1) X_1 \right\|_{\mathrm{F}},$$

$$S_p^v(Y_1) = \left\| \sum_{j=1}^{N} v_j \left[Y_2(\lambda_1 B_2 - A_2)^{-T} X_2^{\mathrm{T}} S_j^{\mathrm{T}}(0, \lambda_1) Y_1 - Y_1 X_1^{\mathrm{T}} \frac{\partial B^{\mathrm{T}}(0)}{\partial p_j} Y_1 \right] \right\|_{\mathrm{F}}$$

are called the sensitivity of the right and left generalized eigenvector matrices $X_1(p)$ and $Y_1(p)$ of (1.1) in the direction v at $p = 0$, respectively. The quantities

$$S_p(X_1) = \max_{\substack{v\in C^N \\ \|v\|_2=1}} S_p^v(X_1), \quad S_p(Y_1) = \max_{\substack{v\in C^N \\ \|v\|_2=1}} S_p^v(Y_1)$$

are called the sensitivity of the right and left generalized eigenvector matrices $X_1(p)$ and $Y_1\,(p)$ of (1.1) at $p = 0$, respectively.

Let $v = e_i$, from Definition 3.1, we get

$$\begin{aligned} S_p^v(\lambda_1) &= \rho\big(Y_1^{\mathrm{T}} S_i(0, \lambda_1) X_1\big), \\ S_p^v(X_1) &= \left\| X_2(\lambda_1 B_2 - A_2)^{-1} Y_2^{\mathrm{T}} S_i(0, \lambda_1) X_1 \right\|_{\mathrm{F}}, \\ S_p^v(Y_1) &= \left\| Y_2(\lambda_1 B_2 - A_2)^{-T} X_2^{\mathrm{T}} S_i^{\mathrm{T}}(0, \lambda_1) Y_1 - Y_1 X_1^{\mathrm{T}} \frac{\partial B^{\mathrm{T}}(0)}{\partial p_i} Y_1 \right\|_{\mathrm{F}}. \end{aligned} \tag{3.2}$$

Definition 3.2. Let $A(p), B(p), X, Y, \lambda_1$ be as in Definition 3.1. The quantity $S_p^v(\lambda_1)$ in (3.2) is called the sensitivity of the semisimple multiple eigenvalue λ_1 of (1.1) with respect to the parameter p_i, denoted by $S_{p_i}(\lambda_1)$. The quantities $S_p^v(X_1)$ and $S_p^v(Y_1)$ in (3.2) are called the sensitivity of the right and left generalized eigenvector matrices X_1 and Y_1 of (1.1) with respect to the parameter p_i, respectively, denoted by $S_{p_i}(X_1)$ and $S_{p_i}(Y_1)$. The quantity

$$S_{p_{i_1},\dots,p_{i_m}}(\lambda_1) = \max_{\sum_{j=1}^m |p_{i_j}|^2=1} \rho\left(\sum_{j=1}^m p_{i_j} Y_1^{\mathrm{T}} S_{i_j}(0,\lambda_1) X_1\right)$$

is called the sensitivity of the semisimple multiple eigenvalue λ_1 of (1.1) with respect to the parameters $p_{i_1},\dots,p_{i_m}$. The quantities

$$S_{p_{i_1},\dots,p_{i_m}}(X_1) = \max_{\sum_{j=1}^m |p_{i_j}|^2=1} \left\| \sum_{j=1}^m p_{i_j} X_2(\lambda_1 B_2 - A_2)^{-1} Y_2^{\mathrm{T}} S_{i_j}(0,\lambda_1) X_1 \right\|_{\mathrm{F}},$$

$$S_{p_{i_1},\dots,p_{i_m}}(Y_1) = \max_{\sum_{j=1}^m |p_{i_j}|^2=1} \left\| \sum_{j=1}^m p_{i_j} \left[Y_2(\lambda_1 B_2 - A_2)^{-T} X_2^{\mathrm{T}} S_{i_j}^{\mathrm{T}}(0,\lambda_1) Y_1 - Y_1 X_1^{\mathrm{T}} \frac{\partial B^{\mathrm{T}}(0)}{\partial p_{i_j}} Y_1 \right] \right\|_{\mathrm{F}}$$

are called the sensitivity of the right and left generalized eigenvector matrices X_1 and Y_1 of (1.1) with respect to the parameters $p_{i_1},\dots,p_{i_m}$, respectively.

Example 3.1. Let

$$A(p) = \begin{bmatrix} 2p_1+p_2+1 & p_1+p_2 & p_2 \\ 2p_2 & p_2+1 & p_1+2p_2 \\ p_1 & p_2 & 2 \end{bmatrix}, \quad B(p) = \begin{bmatrix} 1 & p_1 & 0 \\ p_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $p = (p_1, p_2)^{\mathrm{T}} \in C^2$.

Obviously,

$$A(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the eigenvalues of $\{A(0), B(0)\}$ are $\lambda_1 = 1$ with multiplicity 2 and $\lambda_2 = 2$. Furthermore, let $X = I_3$, $Y = I_3$, then

$$Y^{\mathrm{T}} A(0) X = \begin{bmatrix} \lambda_1 I_2 & 0 \\ 0 & 2 \end{bmatrix}, \quad Y^{\mathrm{T}} B(0) X = \begin{bmatrix} I_2 & 0 \\ 0 & 1 \end{bmatrix}.$$

For any fixed $v \in (v_1, v_2)^{\mathrm{T}} \in C^2$ with $\|v\|_2 = 1$, then

$$S_p^v(\lambda_1) = \max\left\{ \left| v_1 + v_2 + \sqrt{v_1^2 + v_2^2} \right|, \left| v_1 + v_2 - \sqrt{v_1^2 + v_2^2} \right| \right\},$$

$$\begin{aligned}
S_p(\lambda_1) &= \max_{|v_1|^2+|v_2|^2=1} \max\left\{\left|v_1+v_2+\sqrt{v_1^2+v_2^2}\right|, \left|v_1+v_2-\sqrt{v_1^2+v_2^2}\right|\right\},\\
&= \max_{|v_1|^2+|v_2|^2=1} \{|v_1|+|v_2|+1\} = \sqrt{2}+1,\\
S_p^v(X_1) &= \left\|\begin{bmatrix} 0 & 0\\ 0 & 0\\ -v_1 & -v_2\end{bmatrix}\right\|_{\mathrm{F}} = \sqrt{|v_1|^2+|v_2|^2} = 1,\\
S_p(X_1) &= 1,\\
S_p^v(Y_1) &= \left\|\begin{bmatrix} 0 & -v_2\\ -v_1 & 0\\ -v_2 & -v_1-2v_2\end{bmatrix}\right\|_{\mathrm{F}} = \sqrt{2|v_2|^2+|v_1+2v_2|^2+|v_1|^2},\\
S_p(Y_1) &= \max_{|v_1|^2+|v_2|^2=1} \sqrt{2|v_2|^2+|v_1+2v_2|^2+|v_1|^2}\\
&= \max_{\substack{t_1^2+t_2^2=1\\ 1\geqslant t_1,t_2\geqslant 0}} \sqrt{t_2^2+(t_1+2t_2)^2+1} = \sqrt{4+2\sqrt{2}}.
\end{aligned}$$

4. Determination of sensitive elements

Determination of sensitive locations of a structure is an important problem in structural damage detection. Mathematically, this problem is to determine the sensitive elements of a matrix pencil.

Let $A=(a_{ij})$, $B=(b_{ij})\in C^{n\times n}$, $\{A,B\}$ be a regular matrix pencil, λ_1 be a semi-simple multiple eigenvalue of $\{A,B\}$ with multiplicity r, i.e., there exist invertible matrices $X=[X_1,X_2]$, $Y=[Y_1,Y_2]$($X_1,Y_1\in C^{n\times r}$) such that

$$Y^{\mathrm{T}}AX=\begin{bmatrix}\lambda_1 I_r & 0\\ 0 & A_2\end{bmatrix},\quad Y^{\mathrm{T}}BX=\begin{bmatrix} I_r & 0\\ 0 & B_2\end{bmatrix},\quad \lambda_1\notin\lambda(A_2,B_2).$$

Regarding the elements a_{ij} and b_{ij} as parameters, utilizing (3.2), we have

$$S_{a_{ij}}(\lambda_1)=\rho\left(Y_1^{\mathrm{T}}\frac{\partial A}{\partial a_{ij}}X_1\right),\quad S_{b_{kl}}(\lambda_1)=|\lambda_1|\rho\left(Y_1^{\mathrm{T}}\frac{\partial B}{\partial b_{kl}}X_1\right).$$

Let $X_1=\begin{pmatrix}x_1'^T\\ \vdots\\ x_n'^T\end{pmatrix}$, $Y_1=\begin{pmatrix}y_1'^T\\ \vdots\\ y_n'^T\end{pmatrix}$, $x_j', y_j'\in C^r$ $(j=1,\dots,n)$, then

$$S_{a_{ij}}(\lambda_1)=\rho\big(y_i'x_j'^T\big)=\big|x_j'^T y_i'\big|,\quad i,j=1,\dots,n,$$

$$S_{b_{kl}}(\lambda_1)=|\lambda_1|\rho\big(y_k x_l'^T\big)=|\lambda_1|\big|x_l'^T y_k'\big|,\quad l,k=1,\dots,n.$$

Let $M = \max_{1\leqslant i,j\leqslant n}\left\{S_{a_{ij}}(\lambda_1), S_{b_{ij}}(\lambda_1)\right\}$, $E_A = \left\{a_{ij}|S_{a_{ij}}(\lambda_1) = M, i, j = 1, \ldots, n\right\}$, $E_B = \left\{b_{ij}|S_{b_{ij}}(\lambda_1) = M, i, j = 1, \ldots, n\right\}$. The elements in E_A and E_B are said to be sensitive.

By Definition 3.2, we get

$$S_A(\lambda_1) = \max_{\sum_{i,j=1}^{n}|a_{ij}|^2=1} \rho\left(\sum_{i,j=1}^{n} a_{ij} Y_1^{\mathrm{T}} S_{ij}(0,\lambda_1) X_1\right) = \max_{\substack{A\in C^{n\times n}\\ \|A\|_{\mathrm{F}}=1}} \rho\left(Y_1^{\mathrm{T}} A X_1\right),$$

$$S_B(\lambda_1) = |\lambda_1| \max_{\substack{B\in C^{n\times n}\\ \|B\|_{\mathrm{F}}=1}} \rho\left(Y_1^{\mathrm{T}} B X_1\right).$$

Example 4.1. Let

$$A = (a_{ij}) = \begin{bmatrix} 5 & 4 & 6 \\ 4 & 2 & 4 \\ -8 & -8 & -9 \end{bmatrix}, \quad B = (b_{ij}) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix},$$
$$X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 4/3 & 1/3 & -1/3 \\ -7/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \end{bmatrix},$$

then

$$Y^{\mathrm{T}} A X = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad Y^{\mathrm{T}} B X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, $\lambda_1 = -1$ is an eigenvalue of $\{A, B\}$ with multiplicity 2. Obviously, $S_{a_{ij}}(\lambda_1) = S_{b_{ij}}(\lambda_1)$. The sensitivities of the eigenvalue λ_1 with respect to the elements of the matrix pencil $\{A, B\}$ are presented with the following matrix $S = (s_{ij})$, where s_{ij} denotes the sensitivity of λ_1 with respect to $a_{ij}(b_{ij})$,

$$S = \begin{bmatrix} 4/3 & 1/3 & 5/3 \\ 7/6 & 1/3 & 5/6 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}.$$

The matrix S shows that $a_{13}(b_{13})$ is the sensitive element. In fact, perturbing one element of A and B at one time, we verify that the eigenvalue λ_1 is sensitive to the perturbation of a_{13} and b_{13}.

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