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# SOLUTIONS OF EULER-POISSON EQUATIONS IN $R^n$*

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**Abstract** In this article, the authors study the structure of the solutions for the Euler-Poisson equations in a bounded domain of $R^n$ with the given angular velocity and $n$ is an odd number. For a ball domain and a constant angular velocity, both existence and non-existence theorem are obtained depending on the adiabatic gas constant $\gamma$. In addition, they obtain the monotonicity of the radius of the star with both angular velocity and center density. They also prove that the radius of a rotating spherically symmetric star, with given constant angular velocity and constant entropy, is uniformly bounded independent of the central density. This is different to the case of the non-rotating star.

**Key words** Euler-Poisson equations, existence

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## 1 Introduction

We consider the following Euler-Poisson equations:

$$\begin{cases} \rho_t + \mathrm{div}_x(\rho \mathbf{v}) = 0, \\ \rho \mathbf{v}_t + (\rho \mathbf{v} \cdot \nabla)\mathbf{v} + \nabla P + \rho \nabla \Phi = 0, \\ S_t + \mathbf{v} \cdot \nabla S = 0, \\ \Delta \Phi = n(n-2)\omega_n G \rho, \end{cases} \tag{1.1}$$

where $\rho$, $\mathbf{v}$, $S$, and $\Phi$ denote the density, velocity, entropy, and gravitational potential, respectively. Here, $(t,x) \in \mathbf{R}^+ \times \mathbf{R^n}$, $x \in D$, and $D$ being a bounded domain in $R^n$, $n \geq 3$. $\omega_n$ is the measure of the unit ball in $R^n$, and $P$ is the pressure satisfying the following state equation

$$P = P(\rho, S) = e^S \rho^\gamma, \tag{1.2}$$

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where $\gamma > 1$ is the adiabatic exponent.

Suppose the star rotates about the $x_n$-axis. In which cuse, we are interested in finding an axi-symmetric solution, $(\rho, \mathbf{v}, S, \Phi)(x,t) = (\rho, \mathbf{v}, S, \Phi)(\eta(x), x_n, t)$, of (1.1) with given time-independent angular velocity $\Omega(\eta)$, where

$$\eta = \eta(x) = \sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}.$$

In this case, the velocity field is given by $\mathbf{v} = (-x_{n-1}\Omega, \cdots, -x_{\frac{n+1}{2}}\Omega, x_{\frac{n-1}{2}}\Omega, \cdots, x_1\Omega, 0)$ ($n$ is an odd number) and thus $v_t = 0$, $\mathrm{div}_x(\rho\mathbf{v}) = 0$ and $\mathbf{v}\cdot\nabla S = 0$. By $(1.1)_1$ and $(1.1)_3$, $\rho_t = 0$ and $S_t = 0$. Thus, the solution $(\rho, \mathbf{v}, S, \Phi)$ of (1.1) is time-independent and satisfies the following system of equations.

$$\begin{cases} \nabla P(\rho, S) + \rho\nabla\Phi - \rho\nabla J(\eta) = 0, \\ \Delta\Phi = n(n-2)\omega_n G\rho, \end{cases} \tag{1.3}$$

where

$$J(\eta) = \int_0^\eta s\Omega^2(s)\mathrm{d}s, \tag{1.4}$$

and the entropy $S$ is a given bounded $C^1$ function. Gravitation potential, $\Phi$, is given by

$$\Phi(x) = -G\int_{R^n} \frac{\rho(y)}{|\, x-y \,|^{n-2}}\mathrm{d}y =: -B_\rho(x).$$

The problem then reduces to finding the solution of the following equation,

$$\nabla\Big[\frac{\gamma\rho^{\gamma-1}}{\gamma-1} - B_\rho - J(\eta)\Big](x) = 0, \tag{1.5}$$

where $\rho > 0$. We formulated this as a variational problem; namely, minimizing

$$E(\rho) = \int_{R^n} \Big[\frac{\rho^\gamma}{\gamma-1} - \rho\cdot B_\rho - \rho J(\eta)\Big](x)\mathrm{d}x, \tag{1.6}$$

in the class

$$W_M = \Big\{\rho \geq 0 \Big| \int_{R^n} \rho(x)\mathrm{d}x = M\Big\}, \tag{1.7}$$

where $M$ is the given total mass of the star.

When $n = 3$, by assuming that the angular velocity, $\Omega$, satisfies the following decay properties

$$\begin{aligned} &J(+\infty) < +\infty,\ J \in C^1[0, +\infty), \\ &\eta(J(+\infty) - J(\eta)) \to 0,\ \text{as } \eta \to +\infty, \end{aligned} \tag{1.8}$$

and that the adiabatic exponent, $\gamma$, satisfies $\gamma > 4/3$, Auchmuty and Beals [1] proved the existence of a minimizer of the functional $E(\rho)$ in the class of functions (1.7). Moreover, this minimizer has compact support and satisfies equation (1.5) wherever it is positive. In [3] the shape of the free boundary which separates the vacuum and fluid was studied for the Auchmuty-Beals solutions. In [9] the case of an isentropic uniformly rotating star (that is the angular velocity, $\Omega$, is constant) was discussed by Li. He proved the existence of a minimizer of the functional (1.6) in the class $W_M$ , under the assumption $\gamma > 4/3$. In [4] the diameter of the support of the density, $\rho$, was studied for the solution obtained in [9].

In the proof of the above results, the prescribed total mass serves as a constraint on these variational problems and without this constraint, it is not clear that the minimizer exists. In [1], the angular velocity is given in the entire space, $R^3$ (even in the vacuum region), and is assumed to satisfy the decay property (1.8). All of the above results are for isentropic fluids and $\gamma$ is required to be greater than 4/3. Furthermore, Tao Luo and Smoller considered (1.3) when $n = 3$. Some interesting existence and non-existence results are obtained when $S(x)$ and $\gamma$ satisfy different conditions. The main purpose of this article is to extend all the existence results and properties of solutions [10] to the case when $n \geq 3$ and $n$ is odd.

Now, we consider the solution of (1.3) in a bounded domain $D \subset R^n$ , with $\rho(x) > 0$ for $x \in D$ and $\rho(x) = 0$ if $x \in \partial D$. From the first equation in system (1.3), we have

$$\frac{1}{\rho}\nabla P = \nabla(J - \Phi), \tag{1.9}$$

for $x \in D$. Thus, by the second equation of (1.3), we obtain

$$\mathrm{div}(\frac{1}{\rho}\nabla P) = \Delta(J - \Phi) = \Delta J - n(n-2)\omega_n G\rho, \tag{1.10}$$

for $x \in D$. Set

$$\omega = \frac{\gamma}{\gamma - 1}(e^{\frac{S}{\gamma}}\rho)^{\gamma-1}. \tag{1.11}$$

Using (1.2), we can verify that

$$\frac{1}{\rho}\nabla P = e^{\frac{S}{\gamma}}\nabla\omega. \tag{1.12}$$

Substituting this into (1.10), we obtain the following elliptic equation

$$\mathrm{div}(e^{\alpha S}\nabla\omega) + Ke^{-\alpha S}\omega^q - \Omega(\eta)\big[(n-1)\Omega(\eta) + 2\eta\Omega'(\eta)\big] = 0, \tag{1.13}$$

where

$$q = \frac{1}{\gamma - 1},\ \alpha = \frac{1}{\gamma} \tag{1.14}$$

and $K = n(n-2)\omega_n G(\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}$, and for simplicity, we can normalize $K$ to make $K = 1$. We want to look for the solutions of (1.13) satisfying

$$\rho(x) > 0, x \in D;\ \rho(x) = 0, x \in \partial D, \tag{1.15}$$

or equivalently,

$$\omega(x) > 0, x \in D;\ \omega(x) = 0, x \in \partial D. \tag{1.16}$$

In this article, we only consider the case when

$$1 < \gamma < 2,$$

because when $\gamma > 2$ and $0 < q < 1$, the equation (1.13) becomes sublinear, and this situation was studied completely in [17]. If $\gamma = 2$, equation (1.13) is linear, and there is a complete theory for linear elliptic equations (cf.[8]).

We study the following two cases respectively.

**Case 1** Suppose $\Omega(\eta) = \Omega$ =constant, $S(x) = S(r), r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ and the domain $D$ is a ball $B_R(0)$.

In this case, we seek the spherically symmetric solutions of problem (1.13) and (1.16), that is, $\omega(x) = \omega(r)$ and $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Then $\omega(r)$ satisfies the following equation

$$\omega''(r) + \Big[\frac{n-1}{r} + \alpha S'(r)\Big]\omega'(r) + e^{-\alpha S(r)}\big[e^{-\alpha S(r)}\omega^q - \sigma\big] = 0, \tag{1.17}$$

and boundary conditions

$$\omega'(0) = 0,\ \omega(R) = 0, \tag{1.18}$$

where we have set

$$\sigma = (n-1)\Omega^2. \tag{1.19}$$

We also want to use the "time-map" method used by Luo Tao and J. Smoller (cf.[10]). For this purpose, we consider problem (1.17) with initial data

$$\omega(0) = p > 0 \quad \text{and} \quad \omega'(0) = 0, \tag{1.20}$$

where $p$ is a free parameter. Let $\omega(r,p,\sigma)$ be the solution of this problem and

$$R(p,\sigma) =: \inf\{R \mid R > 0, \omega(r,p,\sigma) > 0 \ \text{ if } \ 0 \le r < R, \ \text{ and } \ \omega(R,p,\sigma) = 0\}, \tag{1.21}$$

so $R(p,\sigma)$ is the "first time" at which $\omega$ is 0 (we will write $R(p,\sigma) = \infty$ if $\omega(r,p,\sigma) > 0$ for all $r \ge 0$). Thus, $R(p,\sigma)$ is the radius of a rotating star with the given center density (cf.(1.11)).

$$\rho(0) = (\frac{\gamma-1}{\gamma}p)^{\frac{1}{\gamma-1}} e^{-\frac{S(0)}{\gamma}}$$

and angular velocity

$$\Omega = \sqrt{\frac{\sigma}{n-1}}.$$

To state our first theorem, we denote

$$\bar{S} = \sup_{r\ge0} S(r) \quad \text{and} \quad \underline{S} = \inf_{r\ge0} S(r) \tag{1.22}$$

and assume

$$-\infty < \underline{S} \le \bar{S} < \infty. \tag{1.23}$$

We define the constant $b$ by

$$b =: \frac{(n-1)[(n^2+2n-4)q + n^2 - 4]\sigma^{1-\frac{1}{q}} e^{-\frac{\alpha\bar{S}}{q}}}{(1+(n-1)e^{\alpha\bar{S}})[(n+2)-q(n-2)]^{1-\frac{1}{q}}[(n-1)(n+2)(q+1)]^{\frac{1}{q}}}. \tag{1.24}$$

Obviously, $b > 0$ if $1 < q < \frac{n+2}{n-2}$ .

**Theorem 1.1** Assume that D is a ball in $\mathbf{R^n}$ and $\Omega$ is a nonzero constants.

1) If $1 < \gamma \le \frac{2n}{n+2}$ and $S'(r) \ge 0$ for $r \ge 0$, then

$$\omega(r,p,\sigma) > 0,$$

for every $r \ge 0$ and $p > 0$.

2) If $\frac{2(n-1)}{n} < \gamma < 2$, then

$$R(p,\sigma) < +\infty, \tag{1.25}$$

for sufficiently large $p$, provided that $S(r)$ satisfies the following condition

$$\alpha S'(r)e^{\alpha S}z' > -\frac{bz}{n-1}, \tag{1.26}$$

for $0 < r < z_1$ , where $z(r)$ is the solution of the initial value problem of ordinary differential equation

$$z'' + \frac{n-1}{r}z' + bz = 0, \ \ z(0) = 1, z'(0) = 0, \tag{1.27}$$

and $z_1$ is the first zero point of $z(r)$; and $\sup\limits_{0\le r<1+z_1} |S'(r)|$ is sufficiently small.

3) If $\frac{2n}{n+2} < \gamma < 2$, assume that the conditions in 2) hold, and also assume that the entropy satisfies the following condition

$$S'(r) \le 0 \text{ for } \ 0 \le r \le z_1, \tag{1.28}$$

where b is given by (1.24). Then

$$R(p, \sigma) < +\infty \tag{1.29}$$

for sufficiently large $p$.

4) For the solutions $\omega(r, p, \sigma) =: \omega(r)$ in 2) and 3), we have the following estimates on the mass, $M(r) = \int_0^r \rho(\tau)n\omega_n\tau^{n-1}\mathrm{d}\tau$, and the average density, $\bar\rho(r) = \frac{1}{\mathrm{Vol}B_r(0)}M(r)$, in the ball $B_r(0)$ (here $\rho = [\frac{\gamma-1}{\gamma}\omega]^{\frac{1}{\gamma-1}}e^{-\frac{S}{\gamma}}$ is the density (cf.(1.11)) and $\mathrm{Vol}B_r(0) = \int_0^r n\omega_n\tau^{n-1}\mathrm{d}\tau$):

$$M(r) \ge \frac{(n-1)\Omega^2 r^n}{n(n-2)G}, \tag{1.30}$$

$$\bar\rho(r) \ge \frac{(n-1)\Omega^2}{n(n-2)G\omega_n}, \tag{1.31}$$

for $r \le R(p, \sigma)$, where G is the Newtonian gravitational constant.

**Remark 1** The conditions imposed on the entropy $S(r)$ in Theorem 1.1 are automatically satisfied in the isentropic case: $S(r)$ =constant.

**Case 2** Suppose $S$ =constant, $\Omega(\eta) = \Omega$ =constant, and $D$ is a ball.

In the case of isentropic fluids (that is, $S$=constant), we can obtain further results on the qualitative properties of the solutions if the angular velocity $\Omega(\eta) = \Omega$ =constant and the domain $D$ is a ball. Without loss of generality, we may assume

$$S = 0. \tag{1.32}$$

In this case, it follows from the celebrated Gidas, Ni, and Nirenberg results ([7]) that positive solutions to (1.13) and (1.16) must be spherically symmetric. Substituting $S = 0$ into (1.17), we obtain the following equation:

$$\omega''(r) + \frac{n-1}{r}\omega'(r) + \omega^q - \sigma = 0. \tag{1.33}$$

We consider the problem (1.33) with initial data

$$\omega(0) = p > 0, \ \omega'(0) = 0. \tag{1.34}$$

We will again use $\omega(r,p,\sigma)$ to denote the solution of the above problem. The following theorem gives some interesting properties of $R(p,\sigma)$, which is the radius of the star with central density (cf.(1.11) with $S=0$)

$$\rho(0)=\Big(\frac{\gamma-1}{\gamma}p\Big)^{\frac{1}{\gamma-1}}.$$

**Theorem 1.2** Assume that $D$ is a ball in $\mathbf{R^n}$, $\Omega$ is a nonzero constant, and the entropy S is constant (we set $S=0$ for convenience). Then the following statements hold.

1) If $\frac{2n}{n+2}<\gamma<2$, there exists a constant $p_0>0$ depending only on $\gamma$ and $\sigma$ such that

$$R(p,\sigma)<+\infty \Leftrightarrow p\geq p_0. \tag{1.35}$$

Moreover, we can estimate $p_0$ from below ,

$$p_0\geq\Big[\frac{(n+2)q+n+2}{n+2-(n-2)q}\Big]^{\frac{1}{q}}=\Big[\frac{(n+2)(n-1)\gamma\Omega^2}{(n+2)\gamma-2n}\Big]^{\gamma-1}. \tag{1.36}$$

2) If $\frac{2n}{n+2}<\gamma<2$,

$$R(p,\sigma_1)\geq R(p,\sigma_2), \tag{1.37}$$

provided $\sigma_1>\sigma_2>0$, and

$$R(p_1,\sigma)\geq R(p_2,\sigma), \tag{1.38}$$

if $p_2>p_1\geq p_0$.

In addition, we have the following theorem, in which $p_0$ is the positive constant given in Theorem 1.2.

**Theorem 1.3** Assume that D is a ball in $\mathbf{R^n}$ , $\Omega$ is a nonzero constant, and the entropy S is constant (we set $S=0$ for convenience). If $\frac{2n}{n+2}<\gamma<2$, then there exists a positive constant, $C$, depending only on $p_0$ and $\Omega$ such that

$$R(p,\sigma)\leq Cp^{\frac{\gamma-2}{2(\gamma-1)}}\leq Cp_0^{\frac{\gamma-2}{2(\gamma-1)}}, \tag{1.39}$$

for $p\geq p_0$. In particular, this implies

$$R(p,\sigma)\to 0,\ \text{as}\ p\to\infty. \tag{1.40}$$

For the physical meaning of Theorem 1.2 and Theorem 1.3 in three dimensions, please refer to the article [10].

The rest of this article is organized as follows. In Section 2, we prove Theorem 1.1. Theorems 1.2 and 1.3 are proved in Section 3.

## 2 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1 by "time-map" method. For this purpose, we define

$$f(r,\omega)=\begin{cases} e^{-\alpha S(r)}\omega^q-\sigma, & \text{if}\ \ \omega\geq 0,\\ -\sigma, & \text{if}\ \ \omega<0\end{cases} \tag{2.1}$$

and
$$F(r,\omega)=\int_0^{\omega} f(r,z)\mathrm{d}z. \tag{2.2}$$

Then (1.17) can be rewritten as
$$\omega''(r)+\Big[\frac{n-1}{r}+\alpha S'(r)\Big]\omega'(r)+e^{-\alpha S}f(r,\omega)=0. \tag{2.3}$$

As in Section 1, we use the notation $\omega(r,p,\sigma)$ to denote the solution of (2.3) and
$$\omega(0)=p>0 \quad \text{and} \quad \omega'(0)=0. \tag{2.4}$$

Let $R(p,\sigma)$ be as in (1.21), that is, $R(p,\sigma)$ is the first point at which $\omega$ is 0. We define function $G(r)$ by
$$G(r):=r^n\Big[e^{\alpha S(r)}\frac{\omega'^2}{2}+F(r,\omega)\Big]+\frac{n-2}{2}r^{n-1}e^{\alpha S(r)}\omega\omega'(r)+\frac{\alpha}{2}\int_0^r t^n e^{\alpha S(t)}S'\omega'^2\mathrm{d}t. \tag{2.5}$$

We compute that
$$G'(r)=nr^{n-1}F(r,\omega)-\frac{n-2}{2}r^{n-1}\omega f(r,\omega)+r^n g(r,\omega), \tag{2.6}$$

where
$$g(r,\omega)=\int_0^{\omega}\frac{\partial f(r,z)}{\partial r}\mathrm{d}z=\begin{cases}-\dfrac{\alpha S'(r)e^{-\alpha S}\omega^{q+1}}{q+1}, & \omega\ge 0,\\ 0, & \omega<0.\end{cases} \tag{2.7}$$

Thus
$$G(r)=\int_0^r\Big\{nt^{n-1}F(t,\omega)-\frac{n-2}{2}t^{n-1}\omega f(t,\omega)+t^n g(t,\omega)\Big\}\mathrm{d}t. \tag{2.8}$$

We now prove that if $1<\gamma<\frac{2n}{n+2}$ ($q\ge\frac{n+2}{n-2}$) and $S'(r)\ge 0$ for $r\ge 0$, then
$$R(p,\sigma)=+\infty \tag{2.9}$$

for any $p>0$ and $\sigma>0$.

**Proof of part 1) of Theorem 1.1.**

The proof is by contradiction. Suppose $R(p,\sigma)<+\infty$, and let $\bar{R}=R(p,\sigma)$. Thus,
$$\omega(\bar{R})=0 \quad \text{and} \quad \omega(r)>0 \text{ for } 0\le r<\bar{R}. \tag{2.10}$$

By the definition of $G(r)$, we have
$$G(\bar{R})=\frac{\bar{R}^n e^{\alpha S(\bar{R})(\omega'(\bar{R}))^2}}{2}+\frac{\alpha}{2}\int_0^{\bar{R}} r^n e^{\alpha S}S'(r)\omega'^2\mathrm{d}r.$$

Applying this to (2.8), we obtain, with the help of (2.1), (2.2), (2.7), and (2.10),
$$\begin{aligned}&\frac{\bar{R}^n e^{\alpha S(\bar{R})(\omega'(\bar{R}))^2}}{2}+\frac{\alpha}{2}\int_0^{\bar{R}} r^n e^{\alpha S}S'(r)\omega'^2\mathrm{d}r\\ =&\int_0^{\bar{R}} r^{n-1}\Big[\Big(\frac{n}{q+1}-\frac{n-2}{2}\Big)e^{-\alpha S}\omega^{q+1}-\frac{n+2}{2}\sigma\omega\Big]\mathrm{d}r-\int_0^{\bar{R}} r^n\frac{\alpha S'(r)e^{-\alpha S}\omega^{q+1}}{q+1}\mathrm{d}r.\end{aligned} \tag{2.11}$$

Since $\frac{n}{q+1}-\frac{n-2}{2}\le 0$ for $q\ge\frac{n+2}{n-2}$ and $S'(r)\ge 0$, we can see that the left hand side of (2.11) is positive, while the right hand side of (2.11) is negative and thus, (2.11) gives a contradiction. Thus, (2.9) holds and Part 1) in Theorem 1.1 is proved.

To prove Parts 2) and 3) in Theorem 1.1, we need a few Lemmas as follows. First, it follows from (2.3) that

$$(r^{n-1}e^{\alpha S}\omega')'+r^{n-1}f(r,\omega)=0. \tag{2.12}$$

Thus, from (2.12) and (2.4), we obtain

$$\omega'(r)=-\frac{e^{-\alpha S(r)}}{r^{n-1}}\int_0^r t^{n-1}f(t,\omega)\mathrm{d}t,\ r\ge 0. \tag{2.13}$$

For fixed $p>0$, since $\omega(0)=p$, by (2.1), (2.2), and (2.13), we have $\omega'(r)<0$ for small $r$, $r>0$, if $p$ is sufficiently large. So we can define $r_1$ to be the point such that

$$\omega(r_1)=2p/3,\ \omega(r)>2p/3\ \text{ for }\ 0<r<r_1. \tag{2.14}$$

Then we have the following lemma which estimates $r_1$ and $G(r_1)$ in terms of $p$.

**Lemma 2.1** For sufficiently large $p$, if $1<q<\frac{n+2}{n-2}$ ($\frac{2n}{n+2}<\gamma<2$), then there are positive constants $c_1$, $c_2$, and $c_3$ independent of $p$ such that

$$c_1p^{-(q-1)/2}\le r_1\le c_2p^{-(q-1)/2}, \tag{2.15}$$

and

$$G(r_1)\ge c_3p^{\frac{n+2-q(n-2)}{2}}, \tag{2.16}$$

provided that $\sup\limits_{0\le r\le r_1}|S'(r)|$ is sufficiently small.

**Proof** By the definition of $r_1$ , we have

$$2p/3\le\omega(r)\le p,\quad 0\le r\le r_1. \tag{2.17}$$

Thus, from (2.1), (2.13), and (2.17),

$$\begin{aligned}2p/3=\omega(r_1)&=p-\int_0^{r_1}\frac{e^{-\alpha S(r)}}{r^{n-1}}\int_0^r t^{n-1}\big[e^{-\alpha S(t)}\omega^q(t)-\sigma\big]\mathrm{d}t\mathrm{d}r\\&\ge p-\int_0^{r_1}\frac{e^{-\alpha\underline{S}}}{r^{n-1}}\int_0^r t^{n-1}\big(e^{-\alpha\underline{S}}p^q-\sigma\big)\mathrm{d}t\mathrm{d}r\\&=p-\frac{r_1^2}{2n}\big(e^{-2\alpha\underline{S}}p^q-e^{-\alpha\underline{S}}\sigma\big),\end{aligned} \tag{2.18}$$

if $p$ is sufficiently large, where $\underline{S}=\inf\limits_{0\le r<+\infty}S(r)$ and by (1.23) $\underline{S}>-\infty$. From (2.18), we have

$$\frac{3r_1^2}{2n}\big(e^{-2\alpha\underline{S}}p^q-e^{-\alpha\underline{S}}\sigma\big)\ge p. \tag{2.19}$$

This leads to $r_1\ge c_1p^{-(q-1)/2}$ for some positive constant $c_1$, if $p$ is sufficiently large. Similar to the argument in (2.18), we can obtain

$$2p/3\le p-\frac{r_1^2}{2n}\big[e^{-2\alpha\bar{S}}(2p/3)^q-e^{-\alpha\bar{S}}\sigma\big], \tag{2.20}$$

where $\bar{S} = \sup\limits_{0\le r<+\infty} S(r)$, and $\bar{S} < \infty$. So from (2.20), $r_1 \le c_2 p^{-(q-1)/2}$ for some positive constant $c_2$ if $p$ is sufficiently large. This proves (2.15). To prove (2.16), in view of (2.8) and by virtue of (2.1), (2.2), and (2.7), we obtain

$$G(r_1) = \int_0^{r_1} r^{n-1}\Big\{\Big[\frac{n}{1+q} - \frac{n-2}{2} - \frac{\alpha r S'(r)}{q+1}\Big]e^{-\alpha S(r)}\omega^{q+1} - \frac{n+2}{2}\sigma\omega\Big\}\mathrm{d}r. \tag{2.21}$$

Since $1 < q < \frac{n+2}{n-2}$ $(q+1 < \frac{2n}{n-2})$, we have

$$\frac{n}{1+q} - \frac{n-2}{2} - \frac{r\alpha}{q+1}S'(r) > c_3,\ \ 0 \le r \le r_1, \tag{2.22}$$

if $\sup\limits_{0\le r\le r_1} |S'(r)|$ is sufficiently small, where $c_3$ is some positive constant independent of $p$. From (2.14), (2.21), and (2.22), we have

$$G(r_1) \ge \frac{r_1^n}{n}\Big[c_3 e^{-\alpha\bar{S}}(2p/3)^{q+1} - \frac{n+2}{2}\sigma p\Big]. \tag{2.23}$$

Therefore, if $p$ is sufficiently large, (2.23) and (2.15) imply (2.16). This proves Lemma 2.1.

By (2.1), we have

$$f(r,\omega) > 0,\ \text{for}\ \omega > [e^{\alpha\bar{S}}\sigma]^{1/q}. \tag{2.24}$$

Therefore, in view of (2.13), we have $\omega'(r) < 0$ for small $r$ if $p$ is sufficiently large, so $\omega(r)$ decreases for small $r$. Using (2.13) and (2.24), we can see that $\omega'(r) < 0$ and thus $\omega(r)$ decreases as long as $\omega > [e^{\alpha\bar{S}}\sigma]^{1/q}$. Because of this, we can define $T =: T(p)$ as the point such that

$$\omega(T) = \Big[(n-1)\frac{(n+2)(q+1)e^{\alpha\bar{S}}\sigma}{(n+2)-q(n-2)}\Big]^{1/q} =: A,\ \omega(r) > A,\ \text{for}\ 0 \le r < T, \tag{2.25}$$

since

$$\omega(T) = A > [e^{\alpha\bar{S}}\sigma]^{1/q}, \tag{2.26}$$

when $1 < q < \frac{n+2}{n-2}$. So by (2.1) and (2.25), we have $f(r,\omega(r)) > 0, 0 \le r \le T$, and thus,

$$\omega'(r) < 0,\ \ \text{for}\ 0 < r \le T. \tag{2.27}$$

**Remark 2** The existence of such a $T$ follows by a similar argument as in [17].

The next lemma gives an upper bound for $T$, where the constant $b$ is given by (1.24).

**Lemma 2.2** Assume that $1 < q < \frac{n+2}{n-2}$ $(\frac{2n}{n+2} < \gamma < 2)$ and

$$\alpha S'(r)e^{\alpha S}z' > \frac{b}{1-n}z, \tag{2.28}$$

for $0 < r < z_1$, where $z(r)$ is the solution of the initial value problem of second order linear equation

$$z'' + \frac{n-1}{r}z' + bz = 0,\ \ z(0) = 1, z'(0) = 0, \tag{2.29}$$

and $z_1$ is the first zero point of $z(r)$, then

$$T \le z_1 < +\infty. \tag{2.30}$$

**Proof** By (2.12) and (2.29), we have

$$[r^{n-1}e^{\alpha S(r)}(z'\omega-\omega'z)]'=r^{n-1}[f(r,\omega)z-bz\omega e^{\alpha S}+\alpha e^{\alpha S}S'(r)z'\omega]. \tag{2.31}$$

We prove (2.30) by contradiction. Suppose $T>z_1$, in view of (2.1), (2.26), and (2.27), we have

$$\frac{f(r,\omega(r))}{\omega(r)}-be^{\alpha S(r)}\geq e^{-\alpha\bar{S}}(\omega(T))^{q-1}-\frac{\sigma}{\omega(T)}-be^{\alpha\bar{S}}=\frac{b}{n-1}. \tag{2.32}$$

Integrating (2.31) over the interval $[0,z_1]$ and noticing the fact that $z(z_1)=0$, $z'(0)=0$, and $\omega'(0)=0$, we obtain

$$r^{n-1}e^{\alpha S(r)}z'\omega\mid_{r=z_1}=\int_0^{z_1}r^{n-1}\omega z\Big[\frac{f(r,\omega)}{\omega}-be^{\alpha S}+\frac{\alpha e^{\alpha S}S'(r)z'}{z}\Big]\mathrm{d}r. \tag{2.33}$$

By (2.27), we have $\omega(z_1)>\omega(T)>0$ if $T>z_1$. Moreover, from (2.29), $z(0)=1>0$ and $z_1$ is the first zero point of $z(r)$, so $z'(z_1)<0$. Thus the left hand side of (2.33) is negative. On the other hand, by (2.28) and (2.32), we can see that the right hand side of (2.33) is positive. This is a contradiction and thus, (2.30) is proved.

Next, in view of (2.1) and (2.2), there exists a positive constant $B$ such that

$$\mid e^{-\alpha S(r)}F(r,\omega)\mid\leq B,\ \text{for } 0\leq\omega\leq A, \tag{2.34}$$

$$e^{-\alpha S(r)}\mid g(r,\omega)-\alpha S'(r)F(r,\omega)\mid\leq B,\ \text{for } 0\leq\omega\leq A, \tag{2.35}$$

where $A=\omega(T)$ is defined in (2.25) and $g(r,\omega)$ is defined in (2.7). In the following, we denote

$$Q:=Q(T)=\omega'(T), \tag{2.36}$$

where $T=T(p)$ is defined in (2.25). By (2.27), we have

$$Q<0. \tag{2.37}$$

**Lemma 2.3** If the entropy S satisfies the hypothesis in Lemma 2.2, and

$$\frac{4B}{Q^2}(1+\frac{A}{|Q|})+\frac{4nA}{|Q|T}(1+\frac{4B}{Q^2}+\frac{4AB}{|Q|^3})<\frac{1}{2}, \tag{2.38}$$

and

$$0\leq\omega(r)\leq A=\omega(T),\ \text{for } T\leq r\leq T+L, \tag{2.39}$$

for any $L$ satisfying

$$0\leq L\leq\min\Big\{\frac{2A}{|Q|},1\Big\}, \tag{2.40}$$

then

$$\omega'(r)\leq-\frac{|Q|}{\sqrt{2}},\ \text{for } T\leq r\leq T+L, \tag{2.41}$$

provided that $\sup\limits_{0\leq r<1+z_1}|S'(r)|$ is sufficiently small.

**Proof** First, by (2.3), we have

$$\Big[\frac{\omega'^2}{2}+e^{-\alpha S}F(r,\omega)\Big]'=-\Big[\frac{n-1}{r}+\alpha S'(r)\Big]\omega'^2+e^{-\alpha S(r)}[g(r,\omega)-\alpha S'(r)F(r,\omega)], \tag{2.42}$$

where $g(r,\omega)$ is given by (2.7). Notice that $T$ is bounded by $z_1$ (cf. (2.30)), where $b$ is given by (1.24), so

$$[T,T+1]\subset[0,z_1+1]. \tag{2.43}$$

This, together with (2.40), implies

$$[T,T+L]\subset[0,z_1+1]. \tag{2.44}$$

Therefore, if $\sup\limits_{0\le r<1+z_1}|S'(r)|$ is sufficiently small, then $|rS'(r)|$ is also small for $T\le r\le T+L$. Thus,

$$0<n-1+\alpha rS'(r)<n, \tag{2.45}$$

for $T\le r\le T+L$. It follows from (2.42) and (2.45) that

$$\frac{\omega'^2}{2}+e^{-\alpha S}F(r,\omega)\le\frac{Q^2}{2}+e^{-\alpha S(T)}F(T,\omega(T))+\int_T^r e^{-\alpha S(t)}[g(t,\omega)-\alpha S'(t)F(t,\omega)]\mathrm{d}t, \tag{2.46}$$

for $T\le r\le T+L$. Using (2.34), (2.35), and (2.46), we obtain

$$\frac{\omega'^2}{2}\le\frac{Q^2}{2}+2B+BL\le\frac{Q^2}{2}+2B+\frac{2BA}{|Q|}, \tag{2.47}$$

for $T\le r\le T+L$. On the other hand, we have, from (2.35) and (2.42), that

$$\frac{\omega'^2}{2}+e^{-\alpha S}F(r,\omega)\ge\frac{Q^2}{2}+e^{-\alpha S(T)}F(T,\omega(T))-\int_T^r\Big[\frac{n-1}{t}+\alpha S'(t)\Big](\omega'(t))^2\mathrm{d}t-BL, \tag{2.48}$$

for $T\le r\le T+L$. By (2.34), (2.45), and (2.48), we obtain

$$\frac{\omega'^2}{2}\ge\frac{Q^2}{2}-2B-BL-\int_T^{T+L}\frac{n}{t}(\omega'(t))^2\mathrm{d}t\ge\frac{Q^2}{2}-2B(1+L/2)-\int_T^{T+L}\frac{n}{T}(\omega'(t))^2\mathrm{d}t, \tag{2.49}$$

for $T\le r\le T+L$. So, if $L$ satisfies (2.40), by virtue of (2.49) and (2.47), we have

$$\begin{aligned}\frac{\omega'^2}{2}&\ge\frac{Q^2}{2}-2B(1+\frac{A}{|Q|})-\int_T^{T+\frac{2A}{|Q|}}\frac{n}{T}(\omega'(t))^2\mathrm{d}t\\&\ge\frac{Q^2}{2}-2B(1+\frac{A}{|Q|})-\int_T^{T+\frac{2A}{|Q|}}\frac{n}{T}(Q^2+4B+\frac{4BA}{|Q|})\mathrm{d}t\\&=\frac{Q^2}{2}-2B(1+\frac{A}{|Q|})-\frac{2nA}{T|Q|}(Q^2+4B+\frac{4BA}{|Q|})\end{aligned} \tag{2.50}$$

for $T\le r\le T+L$. Hence,

$$\frac{\omega'^2}{Q^2}\ge1-\frac{4B}{Q^2}(1+\frac{A}{|Q|})-\frac{4nA}{T|Q|}(1+\frac{4B}{Q^2}+\frac{4BA}{(|Q|)^3}), \tag{2.51}$$

for $T\le r\le T+L$. Therefore, if (2.38) holds, then we have

$$(\omega'(r))^2\ge\frac{Q^2}{2}, \tag{2.52}$$

for $T\le r\le T+L$. This implies $\omega'(r)$ does not change sign for $T\le r\le T+L$. Since $Q=\omega'(T)<0$ (cf.(2.27)),

$$\omega'(r)\le-\frac{|Q|}{\sqrt{2}}, \tag{2.53}$$

for $T \leq r \leq T + L$.

**Lemma 2.4** If $\omega(r) > 0$ for $r \in [T, T+L]$, where L satisfies (2.40), then

$$0 < \omega(r) < A, \text{ for } T < r \leq T + L. \tag{2.54}$$

**Proof** Since $\omega(T) = A$ and $\omega'(T) = Q < 0$ as we have shown before, then $\omega'(r) < 0$, and thus, $\omega(r) < A$ for $r > T$, $(r - T)$ small. We prove (2.54) by contradiction. If (2.54) were false, then there exists $r_2 \in (T, T+L]$ such that

$$\omega(r) < A, \text{ for } r \in (T, r_2), \ \omega(r_2) = A. \tag{2.55}$$

Since $\omega(T) = \omega(r_2) = A$ and $r_2 > T$, by Rolle's Theorem, we have

$$\omega'(\tau) = 0, \tag{2.56}$$

for some $\tau \in (T, r_2)$. This contradicts (2.53) and thus, completes the proof of the lemma.

**Lemma 2.5** Assume that the entropy S satisfies the hypothesis in Lemma 2.2. Let $T(p) = T$ be the point defined in (2.25). If

$$\omega'(T)T \to -\infty \text{ as } p \to +\infty, \tag{2.57}$$

then $R(p, \sigma) < +\infty$ if p is sufficiently large and $\sup\limits_{0 \leq r \leq 1+z_1} |S'(r)|$ is sufficiently small.

**Proof** First, in view of (2.30), (2.57) implies

$$Q(T) = \omega'(T) \to -\infty \text{ as } p \to +\infty. \tag{2.58}$$

Hence, there exists $p_0 > 0$ such that (2.38) holds for every $p \geq p_0$. Now, for any fixed $p \in [p_0, +\infty)$, we show that there exists $r^* \in [T, T + \frac{2A}{|Q|}]$ such that

$$\omega(r^*) \leq 0, \tag{2.59}$$

and this implies $R(p, \sigma) < +\infty$. We prove (2.59) by contradiction. Suppose

$$\omega(r) > 0, \quad \text{for} \quad r \in \Big[T, T + \frac{2A}{|Q|}\Big]. \tag{2.60}$$

Then by Lemma 2.4, we have

$$0 < \omega(r) < A, \quad \text{for} \quad r \in \Big[T, T + \frac{2A}{|Q|}\Big]. \tag{2.61}$$

Thus, we can apply Lemma 2.3 with $L = \frac{2A}{|Q|}$ to obtain

$$\omega'(r) \leq -\frac{|Q|}{\sqrt{2}}, \quad \text{for} \quad T \leq r \leq T + \frac{2A}{|Q|}. \tag{2.62}$$

Therefore, since $\omega(T) = A$, we have

$$\omega(T + \frac{2A}{|Q|}) = A + \int_T^{T + \frac{2A}{|Q|}} \omega'(r)\mathrm{d}r \leq A - \frac{|Q|}{\sqrt{2}}\frac{2A}{|Q|} < 0. \tag{2.63}$$

This contradicts (2.60). The proof of the lemma is completed.

Now, we come to prove parts 2) and 3) of Theorem 1.1.

**Proof of part 2) of Theorem 1.1.**

To prove this part, it suffices to verify (2.57) in Lemma 2.5. This follows by the following argument. Set

$$\omega'(T) = Q.$$

By (2.13), we have

$$-e^{\alpha S(T)} T^{n-1} Q = \int_0^T r^{n-1} f(r, \omega(r)) \mathrm{d}r. \tag{2.64}$$

We estimate $QT^{n-1}$ as follows. For $0 \le r \le T$, by (2.27), we have

$$\omega(r) \ge \omega(T). \tag{2.65}$$

Therefore, by (2.1) and (2.25), we obtain

$$\begin{aligned} f(r, \omega(r)) &= e^{-\alpha S(r)} \omega^q(r) - \sigma \ge e^{-\alpha \bar{S}} \omega^q(r) - \sigma \\ &\ge e^{-\alpha \bar{S}} \omega^q(T) - \sigma \ge \frac{(n^2+2n-4)q + n^2 - 4}{n+2-q(n-2)} \sigma > 0, \end{aligned} \tag{2.66}$$

for $0 \le r \le T$, since $1 < q < \frac{n}{n-2}$ $(\frac{2n-2}{n} < \gamma < 2)$. On the other hand, for $r_1$ defined in (2.14), since $2p/3 > A = \omega(T)$ if $p$ is large enough, then (2.25) and (2.27) show

$$T > r_1, \tag{2.67}$$

for large $p$. Hence, it follows from (2.64)–(2.67) that

$$-e^{\alpha S(T)} T^{n-1} Q \ge \int_0^{r_1} r^{n-1} f(r, \omega(r)) \mathrm{d}r. \tag{2.68}$$

By (2.14), we have

$$\omega(r) \ge \omega(r_1) = 2p/3\,, \text{ for } \quad 0 \le r \le r_1. \tag{2.69}$$

Thus by (2.1), there exists a positive constant $C$ such that

$$f(r, \omega(r)) \ge e^{-\alpha \bar{S}} (2p/3)^q - \sigma \ge C \cdot p^q, \ \text{ for } 0 \le r \le r_1, \tag{2.70}$$

if $p$ is sufficiently large. Therefore, we obtain, by (2.15), (2.68), and (2.70), that

$$-e^{\alpha S(T)} T^{n-1} Q \ge C \cdot p^q r_1^n \ge C \cdot p^{\frac{(2-n)q+n}{2}}. \tag{2.71}$$

This implies $T^{n-1} Q \to -\infty$ as $p \to +\infty$ if $q < \frac{n}{n-2}$. Condition (2.57) is, thus, verified in view of (2.30) in Lemma 2.2.

**Proof of part 3) of Theorem 1.1.**

For this case, it suffices to verify (2.57) in Lemma 2.5. First, in view of (2.14) and (2.25), we have

$$\omega(r_1) > \omega(T), \tag{2.72}$$

if $p$ is sufficiently large. This, together with (2.27), implies that

$$r_1 < T. \tag{2.73}$$

Once again, by (2.25), we obtain

$$\omega(r) \geq \omega(T), \tag{2.74}$$

for $r_1 \leq r \leq T$. It follows from (2.6) that

$$G(T) = G(r_1) + \int_{r_1}^{T} r^{n-1}\Big[nF(r,\omega) - \frac{n-2}{2}\omega f(r,\omega) + rg(r,\omega)\Big]\mathrm{d}r. \tag{2.75}$$

By (2.1), (2.2), and (2.7), we have

$$\begin{aligned} &nF(r,\omega) - \frac{n-2}{2}\omega f(r,\omega) + rg(r,\omega) \\ &= \Big[\frac{n}{1+q} - \frac{n-2}{2} - \frac{r\alpha S'(r)}{q+1}\Big]e^{-\alpha S}\omega(r)^{q+1} - \frac{n+2}{2}\sigma\omega, \end{aligned} \tag{2.76}$$

for $r_1 \leq r \leq T$. If $\sup\limits_{0\leq r<1+z_1} |S'(r)|$ is sufficiently small, in view of (2.30), we have

$$\frac{n}{1+q} - \frac{n-2}{2} - \frac{r\alpha S'(r)}{q+1} > \frac{1}{2}\Big(\frac{n}{1+q} - \frac{n-2}{2}\Big) > 0, \tag{2.77}$$

for $r \leq T$, since $1 < q < \frac{n+2}{n-2}$ ($\frac{2n}{n+2} < \gamma < 2$). Moreover, by (2.25), (2.74), (2.76), and (2.77), we obtain

$$nF(r,\omega) - \frac{n-2}{2}\omega f(r,\omega) + rg(r,\omega) > \Big\{\frac{1}{2}\Big[\frac{n}{1+q} - \frac{n-2}{2}\Big]e^{-\alpha\bar{S}}\omega(T)^q - \frac{n+2}{2}\sigma\Big\}\omega \geq 0, \tag{2.78}$$

for $r_1 \leq r \leq T$. This, together with (2.16) and (2.75), implies that

$$G(T) \geq G(r_1) \geq c_3 p^{\frac{q(2-n)+n+2}{2}}. \tag{2.79}$$

By (2.5) and (2.79), we have

$$\begin{aligned} &T^n\Big[e^{\alpha S(T)}\frac{Q^2}{2} + F(T,\omega(T))\Big] + \frac{n-2}{2}T^{n-1}e^{\alpha S(T)}AQ \\ &+\frac{\alpha}{2}\int_0^T r^n e^{\alpha S(r)}S'(r)(\omega'(r))^2\mathrm{d}r \geq c_3 p^{\frac{q(2-n)+n+2}{2}}. \end{aligned} \tag{2.80}$$

where $Q = \omega'(T)$ and $\omega(T) = A$. So, if $S'(r) \leq 0$ for $0 \leq r \leq z_1$, we have, in view of (2.30),

$$T^n\Big[e^{\alpha S(T)}\frac{Q^2}{2} + F(T,\omega(T))\Big] + \frac{n-2}{2}T^{n-1}e^{\alpha S(T)}AQ \geq c_3 p^{\frac{q(2-n)+n+2}{2}}. \tag{2.81}$$

Since $T \leq z_1$, $Q = \omega'(T)$, and $1 < q < \frac{n+2}{n-2}$ , we have

$$TQ \to -\infty, \text{ as } p \to +\infty. \tag{2.82}$$

By Lemma 2.5, part 3) is proved.

**Proof of part 4) of Theorem 1.1**

Now, we prove part 4) of Theorem 1.1. For the solutions $\omega(r,p,\sigma) =: \omega(r)$ in 2) and 3) of Theorem 1.1, let $\rho = [\frac{\gamma-1}{\gamma}\omega]^{\frac{1}{\gamma-1}}e^{-\frac{S(r)}{\gamma}}$ with $\rho$ being the density function (cf.(1.11)). Set

$$M(r) = \int_0^r \omega_n n\rho(\tau)\tau^{n-1}\mathrm{d}\tau,$$

the mass in the ball $B_r(0)$. We calculate each term in (1.13) as follows, by the fact that $\omega$ and $S$ are spherically symmetric and $\Omega$ is a constant. First, for $r=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$, we have

$$\nabla\omega=\omega_r(x_1/r,x_2/r,\cdots,x_n/r),\nabla S=S_r(x_1/r,x_2/r,\cdots,x_n/r), \tag{2.83}$$

and thus,

$$\operatorname{div}(e^{\alpha S}\nabla\omega)=\nabla e^{\alpha S}\cdot\nabla\omega+e^{\alpha S}\Delta\omega=\alpha e^{\alpha S}S_r\omega_r+e^{\alpha S}(\omega_{rr}+\frac{n-1}{r}\omega_r). \tag{2.84}$$

Noticing (1.10), (1.12), and that $\Omega$ is a constant, we can obtain

$$e^{\alpha S}\omega_{rr}+e^{\alpha S}(\frac{n-1}{r}+\alpha S_r)\omega_r+n(n-2)\omega_n G\rho-(n-1)\Omega^2=0. \tag{2.85}$$

Thus,

$$(r^{n-1}e^{\alpha S}\omega'(r))'+r^{n-1}[n(n-2)\omega_n G\rho-(n-1)\Omega^2]=0, \tag{2.86}$$

where $G$ is the Newtonian gravitational constant. Therefore,

$$r^{n-1}e^{\alpha S}\omega'(r)=\frac{n-1}{n}\Omega^2r^n-GM(r)(n-2), \tag{2.87}$$

for $r\leq R(p,\sigma)$, since $w'(0)=0$. By the proofs of parts 2) and 3) of Theorem 1.1, we have $\omega'(r)\leq 0$ for $r\leq R(p,\sigma)$. Thus, $M(r)\geq\frac{(n-1)\Omega^2r^n}{n(n-2)G}$, for $r\leq R(p,\sigma)$. The estimate (1.31) follows immediately from (1.30).

## 3 Proof of Theorem 1.2 and 1.3

In this section, we consider the isentropic case, $S=$constant. Theorem 1.1 can be applied in this case because the conditions imposed on the entropy $S$ in Theorem 1.1 are automatically satisfied when $S$ is constant. Without loss of generality, throughout this section, we assume

$$S=0, \tag{3.1}$$

for convenience. We prove Theorem 1.2 as follows. First, we define

$$f(\omega)=\omega^q-\sigma,\ \omega\geq 0 \tag{3.2}$$

and

$$F(\omega)=\frac{\omega^{q+1}}{q+1}-\sigma\omega,\ \omega\geq 0. \tag{3.3}$$

We still use $\omega(r,p,\sigma)$ to denote the solution of the problem

$$\omega''(r)+\frac{n-1}{r}\omega'(r)+f(\omega)=0, \tag{3.4}$$

and

$$\omega'(0)=0,\omega(0)=p>0. \tag{3.5}$$

Let $R(p,\sigma)$ be defined as in (1.21), and $R(p,\sigma)$ is the first point at which $\omega$ is 0. Then we have the following proposition.

**Proposition 3.1** If $\sigma > 0$ and $1 < q < \frac{n+2}{n-2}$ $(\frac{2n}{n+2} < \gamma < 2)$, then

$$R(p,\sigma) = +\infty, \text{ if } 0 \le p < \underline{p}, \tag{3.6}$$

where

$$\underline{p} = \Big[\sigma\frac{(n+2)q+n+2}{n+2-(n-2)q}\Big]^{\frac{1}{q}} = \Big[\frac{(n+2)(n-1)\gamma\Omega^2}{(n+2)\gamma-2n}\Big]^{\gamma-1}. \tag{3.7}$$

**Proof** It is easy to verify that $[\sigma\frac{(n+2)q+n+2}{n+2-(n-2)q}]^{\frac{1}{q}}$ is the unique positive zero of $nF(\omega) - \frac{n-2}{2}\omega f(\omega)$. If $R(p,\sigma) < +\infty$, we let $R(p,\sigma) = R > 0$. Then $G(R) \ge 0$, where the function $G$ is defined as in (2.5) with $S = 0$. On the other hand, $0 \le \omega(r) \le p$, for $0 \le r \le R$. If $p \le \underline{p} = [\sigma\frac{(n+2)q+n+2}{n+2-(n-2)q}]^{\frac{1}{q}}$, then

$$nF(\omega) - \frac{n-2}{2}\omega f(\omega) \le 0, \quad \text{for} \quad 0 \le r \le R.$$

Hence, (2.8) implies $G(R) < 0$. This is a contradiction and thus (3.6) holds.

For fixed $\sigma > 0$, let

$$p_0 = \inf\{p \mid R(p,\sigma) < +\infty\}. \tag{3.8}$$

By part 3) of Theorem 1.1 and Proposition 3.1, we know that $\underline{p} \le p_0 < +\infty$, where $\underline{p}$ is given by (3.7). If we can show

$$R(p_0,\sigma) < \infty, \tag{3.9}$$

then part 1) in Theorem 1.2 will be proved. Now, (3.9) can be shown by the following argument. (3.4) can be written as a first order system, namely,

$$\omega' = \upsilon, \ \ \upsilon' = -\frac{n-1}{r}\upsilon + \sigma - \omega^q \tag{3.10}$$

with initial conditions

$$\omega(0) = p_0 > 0, \ \ \upsilon(0) = 0. \tag{3.11}$$

Define the Hamiltonian $H(\omega,\upsilon)$ by

$$H(\omega,\upsilon) = \frac{\upsilon^2}{2} + \frac{\omega^{q+1}}{q+1} - \sigma\omega. \tag{3.12}$$

Then $H' = -\frac{n-1}{r}\upsilon^2$, so that $H$ decreases on orbits of (3.10). This implies that the solution of (3.10)–(3.11) must have $\upsilon$ bounded from below in the region $S = \{\omega \ge 0, \upsilon \le 0\}$. This solution also can not exit $S$ via $\omega = 0$ at some $\upsilon < 0$ for some $R > 0$; Otherwise, since $\upsilon'(R) = -\frac{n-1}{R}\upsilon(R) + \sigma > 0$, the solution crosses the line $\omega = 0$ transversally, so by continuity, there would be a neighborhood $N$ of $p_0$ on the $\omega$-axis, such that for $p \in N$, the orbit of (3.10) satisfying $\omega(0) = p, \upsilon(0) = 0$ would also exit $S$ at a point near $\{\omega = 0, \upsilon(R) < 0\}$. This would contradict the definition of $p_0$. Similarly, the $p_0$ orbit can not exit $S$ via some point $(\omega, 0)$, with $0 < \omega < p_0$. Thus, the $p_0$ orbit exits $S$ via $\omega = 0$ and $\upsilon = 0$ , so (3.9) holds, and this proves part 1) of Theorem 1.2.

**Remark 3** It is not hard to show that every solution of (3.10) satisfying $\omega(0) = p$ and $\upsilon(0) = 0$ tends towards the rest point $(\omega = \sigma^{1/q}, \upsilon = 0)$ as $r \to \infty$.

**Proof of part 2) of Theorem 1.2.**

First, we show that the radius of the star increases with the increasing angular velocity. We use $\omega_i(r)$ $(i = 1, 2)$ to denote the solution of the following initial value problem

$$\begin{cases} \omega_i''(r) + \dfrac{n-1}{r}\omega_i'(r) + \omega_i^q - \sigma_i = 0, \ \ r > 0 \\ \omega_i(0) = p, \ \omega_i'(0) = 0. \end{cases} \tag{3.13}$$

We assume

$$\sigma_1 > \sigma_2 > 0, \tag{3.14}$$

and we want to show that

$$R(p, \sigma_1) \geq R(p, \sigma_2), \quad \text{for} \quad p \geq p_0. \tag{3.15}$$

From (3.13), we have

$$\omega_i' = \frac{1}{r^{n-1}} \int_0^r s^{n-1} [\sigma_i - \omega_i^q(s)] \mathrm{d}s, \quad i = 1, 2. \tag{3.16}$$

It is easy to verify, by using L'Hospital's rule, that

$$\lim_{r \to 0^+} \frac{n-1}{r} \omega'(r) = \frac{n-1}{n}(\sigma_i - p^q). \tag{3.17}$$

It follows from (3.13) and (3.17) that

$$\lim_{r \to 0^+} \omega_i''(r) = \frac{1}{n}(\sigma_i - p^q). \tag{3.18}$$

Since $\sigma_1 > \sigma_2$, (3.18) implies

$$\omega_1''(0^+) > \omega_2''(0^+). \tag{3.19}$$

This, together with the fact that $\omega_1(0) = \omega_2(0) = p$ and $\omega_1'(0) = \omega_2'(0) = 0$, implies

$$\omega_1 > \omega_2, \ \ \text{for small } r > 0. \tag{3.20}$$

We shall show (3.15) by contradiction. If (3.15) were false, then there exists an $r_0$, $0 < r_0 < R(p, \sigma_1) < R(p, \sigma_2)$, such that

$$\omega_1(r) > \omega_2(r), \ \ \text{for } 0 < r < r_0, \ \text{and } \omega_1(r_0) = \omega_2(r_0). \tag{3.21}$$

Let

$$y(r) = \omega_1(r) - \omega_2(r).$$

We then have, from (3.13), that

$$y''(r) + \frac{n-1}{r} y'(r) + yB(r) - (\sigma_1 - \sigma_2) = 0. \tag{3.22}$$

Here,

$$B(r) = q \int_0^1 [\lambda\omega_1 + (1-\lambda)\omega_2]^{q-1}(r) \mathrm{d}\lambda. \tag{3.23}$$

From (3.13) and (3.21), we have

$$y(0) = y'(0) = 0, y(r) > 0, \quad \text{for} \quad 0 < r < r_0 \quad \text{and} \quad y(r_0) = 0. \tag{3.24}$$

Multiplying (3.22) by $y'$ and integrating the resulting equation over the interval $[0, r_0]$, we obtain, since $y'(0) = 0$,

$$\frac{(y'(r_0))^2}{2} + \int_0^{r_0} \frac{n-1}{r}(y'(r))^2 \mathrm{d}r + \int_0^{r_0} B(r)yy'(r)\mathrm{d}r - \int_0^{r_0} (\sigma_1 - \sigma_2)y'(r)\mathrm{d}r = 0. \tag{3.25}$$

With the help of (3.24) and integration by parts, we have

$$\int_0^{r_0} B(r)yy'(r)\mathrm{d}r = \int_0^{r_0} B(r)(\frac{y^2}{2})' dr = -\frac{1}{2}\int_0^{r_0} B'(r)y^2(r)\mathrm{d}r \tag{3.26}$$

and

$$\int_0^{r_0} (\sigma_1 - \sigma_2)y'(r)\mathrm{d}r = 0. \tag{3.27}$$

Substituting (3.26) and (3.27) into (3.25), we have

$$\frac{(y'(r_0))^2}{2} + \int_0^{r_0} \frac{n-1}{r}(y'(r))^2 \mathrm{d}r - \frac{1}{2}\int_0^{r_0} B'(r)y^2(r)\mathrm{d}r = 0. \tag{3.28}$$

By the definition of $B(r)$ (see (3.23)), we have

$$B'(r) = q(q-1)\int_0^1 [\lambda\omega_1 + (1-\lambda)\omega_2]^{q-2}[\lambda\omega_1'(r) + (1-\lambda)\omega_2'(r)]\mathrm{d}\lambda. \tag{3.29}$$

Since $\omega_i'(r) < 0$ and $\omega_i(r) > 0$, for $0 < r < R(p, \sigma_i)$ $(i = 1, 2)$ ([GNN]), we thus have

$$B'(r) < 0 \text{ for } 0 < r \le r_0, \tag{3.30}$$

when $\gamma < 2$, that is, $q > 1$. Hence, each term in (3.28) must be zero. This contradicts (3.24) and so proves (3.15).

Now, we show that the radius of the star decreases with the increasing central density, that is,

$$R(p_1, \sigma) \ge R(p_2, \sigma) \tag{3.31}$$

if $p_0 \le p_1 < p_2$ and $\sigma > 0$. For this purpose, let $\omega(r, p, \sigma)$ be the solution of the following initial value problem

$$\begin{cases} \omega''(r) + \dfrac{n-1}{r}\omega'(r) + \omega^q - \sigma = 0, \\ \omega(0) = p,\ \omega'(0) = 0, \end{cases} \tag{3.32}$$

for $p \ge p_0$. We introduce the following rescaling,

$$\lambda = rp^{(q-1)/2} \quad \text{and} \quad \theta(\lambda) = \omega/p, \tag{3.33}$$

then $\theta(\lambda)$ is the solution of the following initial value problem:

$$\begin{cases} \theta_{\lambda\lambda} + \dfrac{n-1}{\lambda}\theta_\lambda + \theta^q - \dfrac{\sigma}{p^q} = 0, \lambda > 0, \\ \theta(0) = 1,\ \theta'(0) = 0. \end{cases} \tag{3.34}$$

The first zero point of $\theta$ depends only on the parameter $\frac{\sigma}{p^q}$. We use $\lambda(\frac{\sigma}{p^q})$ to denote this first zero point. Then by (3.33), we have

$$R(p, \sigma) = p^{(1-q)/2}\lambda(\frac{\sigma}{p^q}), \tag{3.35}$$

for $p \geq p_0$ . Similar to the argument in the proof of (3.15), we can show that $\lambda(\frac{\sigma}{p^q})$ increases with the parameter $\frac{\sigma}{p^q}$. Thus, for a fixed $\sigma$, it decreases with $p$ for $p \geq p_0$. This, together with (3.35) and the fact $q > 1$, implies (3.31). This completes the proof of part 2) of Theorem 1.2.

**Remark 4** The above scaling argument also works for non-rotating star, that is, the case where $\sigma = 0$. For the non–rotating star, the radius of the star, $R(p, 0)$, is always finite for $p > 0$, if $\frac{2n}{n+2} < \gamma < 2$ (see [C]). For the non–rotating star, (3.35) becomes

$$R(p,0) = \lambda_0 p^{(1-q)/2},\ q = \frac{1}{\gamma - 1}, \tag{3.36}$$

where $\lambda_0$ is the first zero of the function $\theta(\lambda)$, which is the solution of the following initial value problem:

$$\begin{cases} \theta_{\lambda\lambda} + \dfrac{n-1}{\lambda}\theta_\lambda + \theta^q = 0, \lambda > 0, \\ \theta(0) = 1,\ \theta'(0) = 0. \end{cases} \tag{3.37}$$

From (3.35), we see that the radius of a non-rotating star is proportional to $p^{(1-q)/2}$.

**Proof of Theorem 1.3.**

In (3.35), Since $\lambda(\frac{\sigma}{p^q})$ increases with the parameter $\frac{\sigma}{p^q}$, as we have already shown, we have

$$\lambda(\frac{\sigma}{p^q}) \leq \lambda(\frac{\sigma}{p_0^q}), \tag{3.38}$$

for $p \geq p_0$. We apply (3.35) to the case of $p = p_0$ and obtain that

$$R(p_0, \sigma) = p_0^{(1-q)/2} \lambda(\frac{\sigma}{p_0^q}). \tag{3.39}$$

Now, $R(p_0, \sigma) < +\infty$ from part 1) of Theorem 1.2, and $p_0$ is a positive constant depending only on $\gamma$ and $\sigma$. Thus, $\lambda(\frac{\sigma}{p_0^q})$ is a positive constant determined also only by $\gamma$ and $\sigma$. We denote this positive constant by $C$ in (3.39), then (1.39) follows, since $\gamma < 2$, (1.39) implies (1.40). The proof of Theorem 1.3 is completed.

## References


1 Auchmuty G, Beals R. Variational solutions of some nonlinear free boundary problems. Arch Rational Mech Anal, 1971, **43**: 255–271

2 Chandrasekhar S. Introduction to the Stellar Structure. University of Chicago Press, 1939

3 Caffarelli L, Friedman A. The shape of axi-symmetric rotating fluid. J Funct Anal, 1980, **694**: 109–142

4 Chanillo, Sagun, Li Yanyan. On diameters of uniformly rotating stars. Comm Math Phys, 1994, **166**(2): 417–430

5 Deng Y, Liu T P, Yang T, Yao Z. Solutions of Euler-Possion equations for gaseous stars. Arch Ration Mech Anal, 2002, **164**(3): 261–285

6 Evans L. Partial Differential Equations. Graduate Studies in Math. AMS, Providence, Rhode Island, 1998

7 Gidas B, Ni W, Nirenberg L. Symmetry of positive solutions of nonlinear elliptic equations in $R^n$. Comm Math Phys, 1975, **68**: 209–243

8 Gilbarg D, Trudinger N. Elliptic Partial Differential Equations of Second Order (2nded.). Springer, 1983

9 Li Yanyan. On uniformly rotating stars. Arch Rational Mech Anal, 1991, **115**(4): 367–393

10 Tao Luo, Joel Smoller. Rotating fluids with self-gravitation in bounded domains. Arch Ration Mech Anal, 2004, **173**(3): 345–377

11 Rabinowitz P. Minimax methods in critical point theory with applications to differential equations. CBMS #65, AMS, 1986

12 Smoller J. Shock Waves and Reaction-Diffusion Equations. Springer, Berlin, New York, 1983

13 Smoller J, Wasserman A. Existence of positive solutions for semilinear elliptic equations in general domains. Arch Rational Mech Anal, 1987, **98**: 229–249