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Stability of Calderón inverse conductivity problem in the plane

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Abstract

It is proved that, in two dimensions, the Calderón inverse conductivity problem in Lipschitz domains is stable when the conductivities are Hölder continuous with any exponent $\alpha > 0$.

Résumé

Nous démontrons, qu'en deux dimensions, le problème inverse de conductivité de Calderón est stable dans les domaines de Lipschitz, quand les conductivités sont hölderiennes pour un exposant quelconque.

MSC: 35R30; 35J15; 30C62

Keywords: Conductivity inverse problem; Quasiconformal mappings; Stability

1. Introduction

We consider a bounded domain Ω in $\mathbb{C}$ with connected complement. A well known problem that A.P. Calderón proposed was the determination of an isotropic L^∞ conductivity coefficient γ on Ω from boundary measurements.

These measurements are given by the Dirichlet to Neumann map defined for a function f on $\partial\Omega$ as the Neumann value $\Lambda_\gamma(f) = \gamma \frac{\partial}{\partial\nu} u$, where u is the solution of the Dirichlet boundary value problem:

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0, \\ u|_{\partial\Omega} = f, \end{cases} \tag{1.1}$$

and $\partial/\partial\nu$ denotes the outer normal derivative. The Dirichlet to Neumann map,

$$\Lambda_\gamma : H^{1/2} \to H^{-1/2}, \tag{1.2}$$

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can be defined for such general domain and conductivities as

$$\langle \Lambda_\gamma(f), \varphi_0\rangle = \int_\Omega \gamma \nabla u \cdot \nabla \varphi, \tag{1.3}$$

where $\varphi \in W^{1,2}(\Omega)$ is such that $\varphi|_{\partial\Omega} = \varphi_0$. The uniqueness of the inverse problem, that means the injectivity of the map

$$\gamma \to \Lambda_\gamma$$

has been completely solved in the two-dimensional case by K. Astala and L. Päivärinta in [7] improving previous results [29] and [34]. In higher dimension the known results require some extra a priori regularity on γ, basically some control on $3/2$ derivatives of γ, see [33,13,30,16].

A relevant question (specially in applications) is the stability of the inverse problem, that is the continuity of the inverse map:

$$\Lambda_\gamma \to \gamma.$$

For dimension $n > 2$ the known results are due to Alessandrini [2,3], who proved stability in for $\gamma \in W^{2,\infty}$. In the planar case, $n = 2$ the situation is different. Liu proved stability for conductivities in $W^{2,p}$ with $p > 1$ in [26]. Recently, stability was obtained for $\gamma \in C^{1+\alpha}$ with $\alpha > 0$ in [10].

In this paper we prove that Hölder regularity of γ is enough to obtain stability:

Theorem 1.1. *Let Ω be a Lipschitz domain in the plane. Let $\gamma = \gamma_1, \gamma_2$ be two planar conductivities satisfying*:

(I) *Ellipticity*: $\|\frac{1-\gamma}{1+\gamma}\|_{L^\infty} < \kappa < 1$.
(II) *Hölder-regularity*: $\gamma \in C^\alpha(\bar{\Omega})$ *with* $\alpha > 0$ *and with a priori bound* $\|\gamma\|_{C^\alpha} < \Gamma_0$.

Then there exists a nondecreasing continuous function $V : \mathbb{R} \to \mathbb{R}$ with $V(0) = 0$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leqslant V\big(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\big). \tag{1.4}$$

The function V can be taken

$$V(\rho) = C\log(\rho)^{-a}. \tag{1.5}$$

Where C and a depend on κ, α and Γ_0. An expression for a is given in (3.34).

Observe that the usual ellipticity condition for γ is:

$$1/C \leqslant \gamma(x) \leqslant C,$$

for a certain constant $C > 0$. This is equivalent to condition (I). We use this formulation because the distortion $\mu = (1-\gamma)/(1+\gamma)$ will be the coefficient of a Beltrami equation to be introduced later.

An example, due to Alessandrini [2], shows that some extra regularity on γ is necessary to obtain stability. Alessandrini gives noncontinuous conductivities in L^∞ such that $\|\Lambda_1 - \Lambda_2\|_{H^{1/2}\to H^{-1/2}} \to 0$, meanwhile $\|\gamma_1 - \gamma_2\|_{L^\infty} = 1$. Namely, if we denote by $B_{r_o} = \{x \in \mathbf{R}^2, |x| < r_o\}$ the ball centered at the origin with radius r_o, take $\Omega = B_1$ the unit ball in $\mathbf{R}^2$, $\gamma_1 = 1$ and $\gamma_2 = 1 + \chi_{B_{r_o}}$, then $\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} = 1$, but $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}} \leqslant 2r_o \to 0$ as $r_0 \to 0$. The details are given in [2]. Observe that in this case these conductivities actually lie in $W^{\varepsilon,2}$ for any $\varepsilon < 1/2$. This example suggests that some control on the modulus of continuity of the conductivities is necessary to prove stability in the L^∞ norm. A natural choice is the Hölder continuity condition (II).

The stability we obtain is just logarithmic. Unfortunately, an argument of A. Mandache [28] shows that even for C^∞ conductivities this is the best one can expect. Therefore the regularity of γ is just reflected on the constant a, see (3.34). An interesting problem is to determine what additional conditions on γ would imply a better stability, like Lipschitz or Hölder stability. Some answers in this direction are given in [4].

Finally, the Lipschitz regularity of the domain Ω is used to recover the boundary values of γ and then to reduce the problem to $\Omega = \mathbb{D}$ and conductivities compactly supported there, see Section 6.

Since the foundational paper of Calderón [18] there has been an intensive research on this problem [22,23]. It turns out that dimension $n=2$ is very different from higher dimensions and special techniques have been developed to treat this case yielding better results than in the higher-dimensional case. There has been several approaches to the Calderón problem in the plane, all of which are based on the construction of the approximated exponential solutions, the so-called complex geometric optics solutions, which have asymptotics e^{ikz} depending on the complex frequency k.

The first approach reduces the conductivity equation (1.1) to the Schrödinger equation:

$$\Delta v + qv = 0,$$

where $q(x) = \Delta\gamma^{1/2}/\gamma^{1/2}$. This works in any dimensions, however in the plane the problem is no longer overdetermined and it cannot be treated in the same way. Local uniqueness was obtained in [34] by using, as in higher dimensions, only bounds for the geometric optics solutions for large frequencies, but requiring the potential to be sufficiently close to 1 in $W^{3,\infty}$. Nachman [29] obtained uniqueness for conductivities in $W^{2,p}$, for $p>1$, by studying the so-called scattering transform $\mathbf{t}(\mathbf{q},\mathbf{k})$ of the potential q. This transform is given essentially by the behavior at infinity of the complex geometric optics solutions. Using this method Liu, [26], obtained stability for γ with the same regularity.

The next approach is due to Brown and Uhlmann [17]. They use the inverse scattering method of Beals and Coifman [11] for 2×2 matrices used before in the study of well posedness of some nonlinear systems. The conductivity equation is reduced to the $\partial\bar{\partial}$ system. This reduction has the advantage of requiring only one derivative of the conductivity. Introducing the scattering transform of the matrix Q, $S(Q,k)$, the uniqueness is obtained for conductivities in $W^{1,p}$ for $p>2$. In [10] it was shown that it is possible to quantify this approach and obtain stability for $\gamma\in\mathcal{C}^{1+\alpha}$.

The last approach is that of Astala–Päivärinta. The conductivity equation is then reduced to a complex Beltrami type equation:

$$\partial_{\bar{z}} f = \mu\overline{\partial_z f},$$

where $\mu = (\gamma-1)/(\gamma+1)$ has also its correspondent scattering transform $\tau(\mu,k)$. The lack of regularity for μ in L^∞ in this case makes the situation very delicate and topological arguments in both variables z and k are needed. This approach have proved to be effective also when dealing with the anisotropic case, see [9].

These three approaches share essentially the same philosophy. First the Dirichlet to Neumann data determine the corresponding scattering transform. Second the complex geometric optics are solutions in the k variable to the so-called $\bar{\partial}$-equation which depends on the scattering transform. If there is uniqueness for this equation, the problem is solved. In the previous works in stability one looks first for an explicit formula relating the difference of Dirichlet to Neumann maps to the difference of scattering transforms. Then one needs that the corresponding $\bar{\partial}$-equation in the k variable enjoys suitable a priori estimates. This cannot be achieved in the case of C^α conductivities and hence a different argument is needed.

Our proof of Theorem 1.1 tries to follow the scheme in the Astala–Päiväirinta setting. However, many difficulties arise. Notice that this is natural since uniqueness is valid for coefficients in L^∞, meanwhile stability is false by Alessandrini example. Let us outline the two main steps in the proof:

For the first step, the stability from the Dirichlet to Neumann map to the scattering transform, we only need the conductivities to satisfy the ellipticity condition. In [7] they start by showing that the Dirichlet to Neumann map determines the values of the geometric optic solutions in the exterior of $\mathbb{D}$ and hence the scattering transform. We prove that this recovery can be done in a stable manner. We control the errors by mean of new estimates for nonhomogeneous Beltrami equations. We give also an alternative way to achieve step one by obtaining an explicit formula for the difference of scattering transforms, parallel to that in [10] for $C^{1+\alpha}$ conductivities. Namely, if μ_1 and μ_2 are two Beltrami coefficients compactly supported in the unit disc $\mathbb{D}$, it holds that

$$\tau(\mu_1,k) - \tau(\mu_2,k) = \frac{-1}{8\pi\bar{k}}\int_{\partial\mathbb{D}} \bar{u}(\mu_1,z,-k)(\Lambda_\gamma - \Lambda_{\gamma_2})u(\mu_2,z,k)\,d\sigma, \tag{1.6}$$

where $u(\mu_i,z,k) = u_{\gamma_i}$ is the complex geometric optics solution given by (2.41).

The formula is valid for conductivities just in L^∞. This type of formulas have proved to be useful for the understanding of the scattering transforms [24,25]. In our setting if:

$$\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}},$$

we obtain that

$$\left|\tau(\mu_1, k) - \tau(\mu_2, k)\right| \leqslant \rho e^{c(1+|k|)}. \tag{1.7}$$

The regularity of γ is needed in the second step, the stability from the scattering transform to the conductivity. Several obstacles must be overcome to make quantitative the arguments in [7]. Let us mention two of them:

Firstly, a key ingredient in the proof of uniqueness is the subexponential decay of the complex geometric optics in the k variable. Namely $u(z,k) = e^{ik(z+\varepsilon_\mu(k))}$ where ε_μ is an unknown function depending on μ such that

$$\lim_{|k|\to\infty} \varepsilon_\mu(k) = 0.$$

Under the regularity assumption (II) it can be shown that $\varepsilon_\mu(k) = C|k|^{-a}$ where a and C depend just on Λ and α, see Theorem 3.36.

However, opposite to previous works in stability, this decay it is still not enough to prove that the corresponding pseudoanalytic equation enjoys good uniqueness properties or a priori estimates. What saves the day is that we also have information on the z variable. Namely, $u(z,k)$ is a function of two variables with controlled asymptotics in z and k and solving the pseudoanalytic equation in the k variable,

$$\partial_{\bar{k}} u(z,k) = \tau_\mu \bar{u}. \tag{1.8}$$

The results in [7] can be reinterpreted as a uniqueness result for the family $u(z,k)$ of solutions to Eq. (1.8) enjoying the correct asymptotics (actually they work with the so-called transport equation which solves a Beltrami type equation in the k variable, but their ideas work directly with the pseudoanalytic equation, see remark after Lemma 5.5).

What we need for stability is to prove a priori estimates for Eq. (1.8) for solutions $u(z,k)$ with uniformly controlled asymptotics (a precise expression for the asymptotics is given in (5.4)). We show that if (1.7) holds, then

$$\left\|u_1(z,k) - u_2(z,k)\right\|_{W^{1,\infty}(\mathbb{D},dz)} \leqslant C\left(|k|\right)\left|\log(\rho)\right|^{-a}, \tag{1.9}$$

see Proposition 6.3 and Theorem 5.1.

Notice that we obtain an estimate in the z variable but we use the equation solved by u in the k variable.

The uniqueness in [7] is proved by observing that $g(z,w,k) = \log(u_1(z,k)) - \log(u_2(w,k))$ solves a Beltrami type equation in the k variable and has asymptotics $ik(z-w)$. Thus, by a version of the argument principle, $g(z,w,k) = 0$ iff $k = 0$ for $z \neq w$. Hence it holds that $u_1(z,k) \neq u_2(w,k)$ if $z \neq w$. But the asymptotics in z imply that the u_i are surjective and hence $u_1(z,k) = u_2(w,k)$.

In the stability setting $g(z,w,k)$ solves a pseudoanalytic equation in k but with an extra error function E which is bounded by the difference of the scattering transforms. The game is to show now that $g(z,w,k) = 0$ implies that $|z-w| \leqslant C(|k|)|\log(\rho)|^{-a}$. This takes quite a lot of effort (all Section 5 in the paper) and it requires a deep understanding on the behavior of the solutions to these equations which might be of its own interest. After this is proved, the estimate (1.9) follows by the uniform Hölder regularity of the solution in the z variable and an interpolation argument to pass from the functions to the derivatives. Here it is also used that since γ is in C^α the solutions are in $C^{1,\alpha}$ by an appropriate version of Schauder's estimates.

An ordered description of the paper is as follows: In Section 2 we collect some results from Beltrami equations which are essential to our work and obtain some a priori estimates for nonhomogeneous Beltrami type equations. In Section 3 we prove some additional properties of the geometric optic solutions in the k and z variables. In the z variable we obtain Schauder type estimates and uniform lower bounds for the Jacobian. In the k variable we obtain an uniform decay as $k \to \infty$. In this section we also prove that the complex geometric optics are uniformly Lipschitz with respect to k, necessary for the arguments in Section 5.

In Section 4 we study the stability from the Dirichlet to Neumann map to the scattering transform. This section does not require the C^α-regularity of γ_j. In Section 5 we prove the stability from scattering transform to the geometric optic solutions. It is needed to remark that from Section 2 to Section 5 we study the particular case $\Omega = \mathbb{D}$ and conductivities to be 1 in a neighborhood of $\partial\Omega$. Theorem 1.1 is proved in Section 6. The proof includes the reduction to this particular case $\Omega = \mathbb{D}$ and compactly supported conductivities, see Proposition 6.1. We make this reduction in a different and more direct way than in previous works on the Calderón problem.

Let us finish the introduction with some further comments on the result and possibilities for future research. As opposite to [10], in the first step we do not need a global control of the scattering transform, which in the mentioned work was essential to assure uniqueness of the $\bar{\partial}$-equation in the second step. This gives Hölder type stability from the Dirichlet to Neumann map to local L^∞ norm of the scattering transform. The logarithmic stability appears in the second part from the scattering transform to the conductivity.

Concerning future research, let us remark that besides the example of Alessandrini, the lack of regularity of the solutions for just L^∞ conductivities yields another reason why it would not be expectable to obtain L^∞ stability. In fact, for a given $\kappa < 1$ there exists a conductivity γ satisfying the ellipticity condition $(\gamma - 1)/(\gamma + 1) < \kappa$, such that the corresponding solution satisfies that for every disc B in Ω,

$$\int_B |\nabla u|^{1+1/\kappa} = \infty.$$

For a reference see [5,19]. Under this viewpoint it is plausible that L^p stability holds for much more irregular conductivities, say in some Sobolev space. We will investigate this in a forthcoming work.

The scattering transform gives a nonlinear Fourier transform of the unknown coefficient. Each approach extends the previous transform to a wider class of functions. To see the relation between the Schrödinger and the Beals–Coifman scattering transform, see [24]. It will be of interest to see how the different definitions of scattering transform relate and which are the properties of this nonlinear Fourier transform. In this direction some results were obtained in [15] and [25].

Notation

- Differential operators:

$$\partial_{\bar z} u = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y) \quad \text{and} \quad \partial_z u = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y).$$

- Spaces:

$$W^{\alpha,p}(\mathbb{C}) = \bigl\{f\colon \bigl\|\bigl[\bigl(1+|\cdot|^2\bigr)^{\alpha/2}\hat{f}(\cdot)\bigr]^{\wedge}\bigr\|_{L^p} < \infty\bigr\}, \qquad C^\alpha(\Omega) = \left\{f\colon \|f\|_{L^\infty} + \sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty\right\},$$

$$H^1(\Omega) = W^{1,2}(\Omega), \qquad H_0^1 = W_0^{1,2}(\Omega), \qquad H^{1/2}(\partial\Omega) = H^1(\Omega) \setminus H_0(\Omega).$$

- We define the L^p difference of a function f by

$$\omega_p(f)(y) = \bigl\|f(\cdot + y) - f(\cdot)\bigr\|_{L^p(\mathbb{C})}. \tag{1.10}$$

- Then the L^p modulus of continuity of f is given for $t > 0$ by

$$\boldsymbol{\omega}_p(f)(t) = \sup_{|y|<t} \omega_p(f)(y). \tag{1.11}$$

- Constants: We remark that C or a denote constants which may change at each occurrence. We will indicate the dependence of the constants on parameters $\kappa, \Gamma, \ldots,$ by writing $C = C(\kappa, \Gamma, \ldots)$
- Finally, $e_k(z)$ denotes the unitary $\mathrm{e}^{\mathrm{i}kz+\mathrm{i}\bar{k}\bar{z}}$ and, for two conductivities γ_1 and γ_2, we write:

$$\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}.$$

2. Preliminaries

Our starting point is the work of K. Astala and L. Päivärinta [7]. In this section, we will make an overview of their beautiful use of quasiconformal mapping methods in the Calderón inverse problem, and we will collect some facts, to be used in our proof of stability. Some of the results that we remark were only implicitly stated in their work.

The relation between elliptic equations and quasiconformal mappings generalizes the fact that a harmonic function is the real part of a holomorphic mapping. That is, given a real solution u in $W^{1,2}$ of the elliptic equation:

$$\nabla \cdot (\gamma \nabla u) = 0, \tag{2.1}$$

one can find a real v, unique modulo constants, such that $f = u + \mathrm{i}v$ is a solution of the $\mathbb{R}$-linear Beltrami equation:

$$\partial_{\bar z} f = \mu \overline{\partial_z f}, \tag{2.2}$$

where the distortion or Beltrami coefficient μ is given for $z \in \mathbb{C}$ as

$$\mu(z) = (1-\gamma)/(1+\gamma). \tag{2.3}$$

The ellipticity condition is then equivalent to the existence of $\kappa < 1$ such that

$$\|\mu\|_{L^\infty} < \kappa. \tag{2.4}$$

For the detailed argument see [7, Lemma 2.1]. Then, Astala and Paivärinta worked with Eq. (2.2) instead of with Eq. (2.1). They were able to prove the existence of complex geometric optic solutions $f(z,k)$, control their asymptotics when $k \to \infty$, find the appropriate equations in the k variable and conclude their proof of uniqueness by an ingenious argument, combining the behavior of the complex geometric optic solutions in the k and z variables.

In Section 2.1 we will recall results in the theory of Beltrami equations required to treat this type of equations and also we will prove some further properties needed in our work. Then, in Section 2.2 we will gather all the results from [7] needed for our approach.

2.1. Beltrami equations

The theory of Beltrami equations and planar quasiregular and quasiconformal mappings generalizes many aspects of classical geometric function theory and is the key for a very rich theory of planar elliptic systems. We refer to the classical monographs [1,21,27,31,36]. See also the recent [6] were the last advances in the planar theory are collected (or proved). Key tools in this theory are the following operators:

- The solid Cauchy transform P:

$$P(g) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(w)}{w-z}\,\mathrm{d}m(w). \tag{2.5}$$

  The Beurling transform T:

$$T(g) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(w)}{(w-z)^2}\,\mathrm{d}m(w). \tag{2.6}$$

- The Beltrami operator B:

$$B = I - \mu T - \nu \overline{T}, \tag{2.7}$$

  where

$$\left|\mu(z)\right| + \left|\nu(z)\right| < \kappa.$$

The basic properties of these operators can be found in [7, Section 3]. Several of our arguments are based on the following estimates for nonhomogeneous Beltrami type of differential inequalities. The next theorem extends Proposition 3.3 in [7] to the nonhomogeneous case.

Theorem 2.1. *Let $\kappa < 1$, $M > 0$ and $2 < p < \infty$. Let $F \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ satisfy the differential inequality*:

$$|\partial_{\bar z} F| \leqslant \chi_{\mathbb{D}}\big(\kappa|\partial_z F| + M|F| + E\big), \tag{2.8}$$

with $E \in L^2(\mathbb{D})$. Suppose in addition that

$$\lim_{z\to\infty} F(z) = 0. \tag{2.9}$$

Then it holds that

(a) *There exists a constant C_1 depending on (p,κ) such that*

$$\|F\|_{L^p(\mathbb{C})} \leqslant \mathrm{e}^{C_1(1+M)}\|E\|_{L^2(\mathbb{D})}. \tag{2.10}$$

(b) *Suppose that $E \in L^p(\mathbb{D})$. Then there exists constant C_2 depending on p and κ such that*

$$\|F\|_{L^\infty(\mathbb{C})} + \|F\|_{W^{1,p}(\mathbb{C})} \leqslant e^{C_2(1+M)} \|E\|_{L^p(\mathbb{D})}. \tag{2.11}$$

Proof of Theorem 2.1. We reformulate the differential inequality (2.8) as a Beltrami type equation. Namely, it can be proved that there exists complex valued functions $\nu : \mathbb{D} \to \mathbb{D}(0,\kappa)$, $\gamma : \mathbb{D} \to \mathbb{D}(0, M)$ and $\tilde{E} \in L^2(\mathbb{D}, \mathbb{C})$ with $|\tilde{E}| = |E|$, such that

$$\overline{\partial_z} F = \chi_{\mathbb{D}}(-\nu \partial_z F + \gamma F + \tilde{E}). \tag{2.12}$$

Formally we declare, $\eta = P(I - \nu T)^{-1}(\gamma)$, $g = e^\eta$, $s = P(I - \nu T)^{-1}(\tilde{E} e^{-\eta})$ and $H = sg$. Then we have that

(1) $\overline{\partial_z}\eta - \nu \partial_z \eta = \gamma$.
(2) $\overline{\partial_z} g - \nu \partial_z g = \gamma g$.
(3) $\overline{\partial_z} s - \nu \partial_z s = \tilde{E} e^{-\eta}$.
(4) $\overline{\partial_z} H - \nu \partial_z H = \gamma H + \tilde{E}$.

Let us check that we can use all these operators. Since $\gamma \in L^\infty(\mathbb{C})$, $(I - \nu T)^{-1}(\gamma) \in L^p(\mathbb{D})$ for every,

$$1 + \kappa < p < 1 + \frac{1}{\kappa}. \tag{2.13}$$

Since for $p > 2$, $P : L^p(\mathbb{D}) \to W^{1,p}(\mathbb{C})$ is bounded, $\eta \in W^{1,p}(\mathbb{C})$ for every $2 < p < 1 + 1/\kappa$. Moreover, for $p > 2$, $W^{1,p}(\mathbb{C}) \hookrightarrow L^\infty(\mathbb{C})$ and hence we have that $\eta \in L^\infty(\mathbb{C})$ with

$$\|\eta\|_{L^\infty(\mathbb{C})} \leqslant CM. \tag{2.14}$$

Thus, g is a well defined solution of (2) and $\tilde{E} e^{-\eta} \in L^2(\mathbb{D})$. Then, $(I - \nu T)^{-1}(\tilde{E} e^{-\eta}) \in L^2(\mathbb{D})$ and s is in $L^p(\mathbb{C}) \cap BMO$ for every $p > 2$ with the estimate

$$\|s\|_{L^p(\mathbb{C})} \leqslant C(p) \|E\|_{L^2(\mathbb{D})} e^{CM}. \tag{2.15}$$

In particular, $s \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ and since $\eta \in W^{1,p}(\mathbb{C})$, $H \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ as well. Now $H = sg$, thus (2.15) implies that $H \in L^p(\mathbb{C})$ for $p > 2$. On the other hand since γ, ν and $\tilde{E}$ have compact support H is holomorphic out of $\mathbb{D}$. Thus we deduce that

$$\lim_{z \to \infty} H(z) = 0. \tag{2.16}$$

Now consider the function $R = F - H$. Clearly it solves the homogeneous equation,

$$\overline{\partial_z} R - \nu \partial_z R = \gamma R, \tag{2.17}$$

and by (2.16) and the assumptions on F:

$$\lim_{|z| \to \infty} R(z) = 0. \tag{2.18}$$

Now, we argue as in [7, Proposition 3.3]. Consider $G = e^{-\eta} R$. It follows from (2.17) that G solves:

$$\overline{\partial_z} G - \nu \partial_z G = 0. \tag{2.19}$$

Being $G \in W^{1,2}_{\mathrm{loc}}$, this implies that G is quasiregular mapping. Since $\eta \in L^\infty(\mathbb{C})$ and R satisfies (2.18) it follows that $\lim_{|z| \to \infty} G(z) = 0$. But then, the representation theorem of quasiregular mappings and Liouville theorem imply that $G = 0$. Thus $R = 0$ as well and,

$$F = H = s e^\eta. \tag{2.20}$$

With this representation of F, (2.10) follows from (2.15) and (2.14). If $E \in L^p(\mathbb{D})$ for $p > 2$ we have that

$$\left\| \partial_{\bar{z}}(s) \right\|_{L^p(\mathbb{D})} \leqslant C_p e^M \|E\|_{L^p(\mathbb{D})}$$

and arguing exactly as for η, we conclude that $s \in W^{1,p}(\mathbb{C})$ and therefore in L^∞. This yields the required estimate (2.11) for H and thus for F. $\square$

2.2. Astala–Päivärinta approach

The strategy of Astala and Päivärinta's work is the construction of the approximate complex exponential solution to the conductivity equation, as in previous works, by means of the corresponding Beltrami equations. This allows them to avoid any a priori regularity assumption on the coefficient γ.

The first step is to find a map associated to Eq. (2.2), which contains the information of the Dirichlet to Neumann map Λ_γ. This is done by the Hilbert transform $\mathcal{H}_\mu$ with respect to the distortion μ. For u real as above we define $\mathcal{H}_\mu : H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})$ as

$$\mathcal{H}_\mu(u|_{\partial\mathbb{D}}) = v|_{\partial\mathbb{D}}. \tag{2.21}$$

Since

$$\mathcal{H}_\mu \circ \mathcal{H}_{-\mu}(u) = \mathcal{H}_{-\mu} \circ \mathcal{H}_\mu(u) = -u + \frac{1}{2\pi}\int_{\partial\mathbb{D}} u, \tag{2.22}$$

it is natural to extend $\mathcal{H}_\mu$ $\mathbb{R}$-linearly by defining:

$$\mathcal{H}_\mu(\mathrm{i}u) = \mathrm{i}\mathcal{H}_{-\mu}(u).$$

We have (Proposition 2.3 in [7]):

Proposition 2.2. *The Dirichlet to Neumann map Λ_γ uniquely determines $\mathcal{H}_\mu$, $\mathcal{H}_{-\mu}$ and $\Lambda_{\gamma^{-1}}$. For u and v real valued, the following identity holds*:

$$\partial_T \mathcal{H}_\mu(u + \mathrm{i}v) = \Lambda_\gamma(u) + \mathrm{i}\Lambda_{\gamma^{-1}}(v). \tag{2.23}$$

Furthermore, for two given conductivities γ_1 and γ_2 we have:

$$\|\mathcal{H}_{\mu_1} - \mathcal{H}_{\mu_2}\|_{H^{1/2}(\partial\mathbb{D})\to H^{1/2}(\partial\mathbb{D})} \leqslant C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\mathbb{D})\to H^{-1/2}(\partial\mathbb{D})}. \tag{2.24}$$

Above, by an abuse of language, we have identified u and v with their restrictions to $\partial\mathbb{D}$.

The new Hilbert transforms give rise to the corresponding Riesz projections onto the μ Hardy spaces on ∂D. Namely consider the operators:

$$\begin{aligned} P_\mu(g) &= \frac{1}{2}(I + \mathrm{i}\mathcal{H}_\mu)(g) + \frac{1}{2}\oint_{\partial\mathbb{D}} g\,\mathrm{d}s, \\ Q_\mu(g) &= \frac{1}{2}(I - \mathrm{i}\mathcal{H}_\mu)(g) - \frac{1}{2}\oint g\,\mathrm{d}s. \end{aligned} \tag{2.25}$$

Then P_μ is the analogous of the Riesz projections in the following sense.

Lemma 2.3. *(See [8].) Let $g \in H^{1/2}(\partial\mathbb{D})$. Then*

(i) $P_\mu(g) + Q_\mu(g) = I$.
(ii) $P_\mu^2 = P_\mu$.
(iii) $P_\mu(g) = g \Leftrightarrow g = f|_{\partial\mathbb{D}}$ *for some* $f \in W^{1,2}$, *with*

$$\partial_{\bar z} f_\mu = \mu\overline{\partial_z f_\mu}. \tag{2.26}$$

(iv) *The range of P_μ consists of boundary values of solutions to* (2.26).

The following theorem gives the existence and properties of the approximate complex exponential solutions of Eq. (2.2). We look for perturbations of the solutions $\mathrm{e}^{\mathrm{i}kz}$ of the equation when $\mu = 0$. It relies on the following lemma.

Lemma 2.4. *Let $2 < p < p_\kappa = 1 + 1/\kappa$, $\alpha \in L^\infty$ supported on $\mathbb{D}$ and $|\nu(z)| \leqslant \kappa\chi_\mathbb{D}$ for almost every $z \in \mathbb{D}$. Then, the operator $K = K(\nu, \alpha)$ defined by*:

$$K(g) = P(I - \nu\overline{T})^{-1}(\alpha\bar g), \tag{2.27}$$

as well as $(I - K)^{-1}$ are bounded operators from $L^p(\mathbb{C}) \to L^p(\mathbb{C})$. Also $K : L^p \to W^{1,p}$.

Theorem 2.5. *For each $k \in \mathbb{C}$ and each $2 < p < 1 + 1/\kappa$ there exists a unique solution $f_\mu \in W^{1,p}_{\mathrm{loc}}$ of* (2.2) *such that*

$$f_\mu(z,k) = \mathrm{e}^{\mathrm{i}kz} M_\mu(z,k), \tag{2.28}$$

with

$$M_\mu(z,k) = 1 + \mathrm{O}(1/z) \quad \text{as } z \to \infty, \tag{2.29}$$

and

$$f_\mu(z,0) = 1, \tag{2.30}$$

$$M_\mu(z,k) = \mathrm{e}^{\eta(z,k)}, \tag{2.31}$$

$$\Re\left(\frac{M_\mu(z,k)}{M_{-\mu}(z,k)}\right) > 0. \tag{2.32}$$

The solution in the above theorem is constructed as

$$M_\mu(z,k) = \big(1 + \omega_\mu(z,k)\big) \quad \text{where } \omega_\mu = (I-K)^{-1}\big(K(\chi_{\mathbb{D}})\big), \tag{2.33}$$

for a suitable choice of α. The function $\omega(z,k)$ is actually in $W^{1,p}$ for $2 < p < 1 + 1/\kappa$. In fact we have the following estimate.

Proposition 2.6. *Let $\omega(z,k) = M_\mu(z,k) - 1$. Then $\omega(z,k) \in W^{1,p}(\mathbb{C}, \mathrm{d}z)$ is a solution to the equation*:

$$\partial_{\bar z}\omega - \mu e_{-k}\overline{\partial_z \omega} = -\mathrm{i}\bar{k}\mu e_{-k}(\bar\omega + 1). \tag{2.34}$$

Moreover, there exists a constant $C(\kappa, p)$ such that for every $2 < p < 1 + 1/\kappa$,

$$\|\omega\|_{W^{1,p}(\mathbb{C},\mathrm{d}z)} \leqslant \mathrm{e}^{C(1+|k|)}. \tag{2.35}$$

Proof. The fact that ω is a $W^{1,p}(\mathbb{C}, \mathrm{d}z)$ solution to (2.34) follows from the definition and properties of the operator K with $\alpha = -\mathrm{i}\bar{k}\mu e_{-k}$. Once we know that ω satisfies Eq. (2.34) the estimate (2.35) follows from Theorem 2.1. □

The other key ingredient in the proof of uniqueness is the so-called $\bar\partial_k$-equation which we state as

Theorem 2.7. *Let us define*:

$$F_+(z,k) = \frac{1}{2}(M_\mu + M_{-\mu}), \tag{2.36}$$

$$F_-(z,k) = \frac{\mathrm{i}e_{-k}}{2}(\overline{M_\mu} - \overline{M_{-\mu}}). \tag{2.37}$$

Then the functions $k \to F_\pm(z,k)$ have continuous derivatives with values in $W^{1,p}(\mathbb{C})$ in the norm sense. They satisfy

$$\partial_{\bar k} F_+(z,k) = \tau(\mu,k) e_{-k}(z)\overline{F_-(z,k)}, \tag{2.38}$$

$$\partial_{\bar k} F_-(z,k) = \tau(\mu,k) e_{-k}(z)\overline{F_+(z,k)}. \tag{2.39}$$

Where the scattering transform $\tau(\mu,\cdot)$, defined as

$$\overline{\tau(\mu,k)} = \frac{1}{4\pi\mathrm{i}} \int_{\mathbb{D}} \partial_{\bar z}\big(M_\mu(z,k) - M_{-\mu}(z,k)\big)\,\mathrm{d}m(z), \tag{2.40}$$

satisfies $|\tau(\mu,k)| < 1$.

We need to work with solutions of the original elliptic equation (2.1). The following definitions, [7, Section 1], give an extension to the case $\gamma \in L^\infty$ of the so-called complex geometric optics solutions of Nachman and Sylvester–Uhlmann:

$$u_\gamma(z,k) = Re\, f_\mu(z,k) + \mathrm{i} Im\, f_{-\mu}(z,k). \tag{2.41}$$

We also need the solutions of the γ^{-1}-conductivity equation $\nabla \cdot (\gamma^{-1}\nabla u) = 0$ defined as

$$-u_{\gamma^{-1}}(z) = Im\, f_\mu - \mathrm{i}Re\, f_{-\mu}. \tag{2.42}$$

We collect some of the properties of u_γ.

Proposition 2.8. *The complex geometric optics solutions $u = u_\gamma(z,\cdot)$ satisfy*:

$$u(z,k) = \mathrm{e}^{\mathrm{i}kz}\big(1 + R(z,k)\big) \quad \text{with } R(z,k) \in W^{1,p}(\mathbb{C}, \mathrm{d}z), \tag{2.43}$$

$$u(z,k) = \mathrm{e}^{\delta(z,k)}, \tag{2.44}$$

with

$$\delta(z,k) = \mathrm{i}k\big(z + \mathrm{O}(1/z)\big) \quad \text{for fixed } k \text{ as } z \to \infty, \tag{2.45}$$

and

$$\delta(\cdot,k)(\mathbb{C}) = \mathbb{C} \quad \text{for } k \neq 0. \tag{2.46}$$

Moreover, $u(z,\cdot) = u$ satisfies the equation:

$$\partial_{\bar{k}} u = -\mathrm{i}\tau(\mu,k)\bar{u}. \tag{2.47}$$

This proposition follows from the previous results for f_μ. In fact (2.43) is a consequence of (2.33) and the definition (2.41), with

$$R(z,k) = \omega_\mu + \omega_{-\mu} + e_{-k}\big(\overline{\omega_\mu - \omega_{-\mu}}\big); \tag{2.48}$$

(2.44) follows from (2.32). (2.45) follows from (2.29). Eq. (2.46) from (2.45) and an homotopy argument. Finally (2.47) is a consequence of (2.38) and (2.39).

To end this section we recall Theorem 7.1 in [7].

Theorem 2.9. *Let us define $f_{\lambda\mu}$ the solution of Theorem* 2.5 *with μ substituted by $\lambda\mu$ where $\lambda \in \partial\mathbb{D}$. Then $f_{\lambda\mu}$ admits a representation,*

$$f_{\lambda\mu}(z,k) = \mathrm{e}^{\mathrm{i}k\phi_\lambda(z,k)}, \tag{2.49}$$

where, for fixed $k \in \mathbb{C}\setminus 0$ and $\lambda \in \partial\mathbb{D}$, the function $\phi_\lambda(\cdot,k) : \mathbb{C} \to \mathbb{C}$ is a quasiconformal homeomorphism that satisfies:

$$\phi_\lambda(z,k) = z + \mathrm{O}(1/z) \quad \text{as } z \to \infty, \tag{2.50}$$

and

$$\bar{\partial}\phi_\lambda(z,k) = -\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}\big(\phi_\lambda(z,k)\big)\overline{\partial\phi_\lambda(z,k)}. \tag{2.51}$$

3. Further properties of the complex geometric optic solutions

In this section we study several properties of f_μ and $u = u_\gamma(z,k)$ when we assume the coefficient μ to be in the Hölder class $\mathcal{C}^\alpha$ for some fixed $\alpha > 0$. We start by stating a more precise asymptotic expansion of the "logarithmic" functions of the geometric optics solutions in the k variable and finally we prove some facts about their derivatives in the k variable. We also analyze the extra regularity that we gain in the z variable.

3.1. Uniform subexponential growth of f_μ

Let $\mu \leqslant \kappa\chi_{\mathbb{D}}$. It is proved in [7] that the Jost functions satisfy that

$$M_\mu = \mathrm{e}^{\mathrm{i}k(z-\varphi_\mu(z,k))},$$

where

$$z - \varphi_\mu(z,k) \to 0,$$

uniformly in $z \in \mathbb{C}$ as $k \to \infty$. However the rate of convergence depends on μ. The aim of this section is to show that these rate of converge is uniform for Beltrami coefficients such that the Hölder norm is bounded by some Γ_0. We start with some elementary properties of Hölder functions.

Lemma 3.1. *Let $\mu \in \mathcal{C}^\alpha$, supported on $\mathbb{D}(0, R)$, $\beta < \alpha$ and $1 \leqslant p < \infty$. Then*:

(i) *There exists a constant $C = C(p, R)$ such that*

$$\|\mu\|_{W^{\beta,p}} \leqslant \frac{C}{\alpha - \beta}\|\mu\|_{C^\alpha}. \tag{3.1}$$

(ii) *Let $f \in W^{\alpha,2}(\mathbb{C})$ then there exists a constant $C = C(R)$ such that*

$$\|\mu f\|_{W^{\beta,2}} \leqslant \frac{C}{\alpha - \beta}\|\mu\|_{C^\alpha}\|f\|_{W^{\beta,2}}. \tag{3.2}$$

Proof. We first prove the second claim. Recall that by [32, Section 3.5.2, Chapter V] we have:

$$\|\mu f\|_{W^{\alpha,2}} \sim \int_{\mathbb{C}} \frac{\omega_2(\mu f)(y)^2}{|y|^{2(1+\alpha)}}\,\mathrm{d}y + \|\mu f\|_{L^2}, \tag{3.3}$$

where ω_2 denotes the modulus of continuity, defined in the introduction. It is easy to see that

$$\omega_2(\mu f)(y) \leqslant \|f\|_{L^2}\|\mu\|_{C^\alpha}|y|^\alpha + \|\mu\|_{L^\infty}\omega_2(f)(y). \tag{3.4}$$

Thus, the claim follows by plugging (3.4) into (3.3) and integrating.

The first claim with $p = 2$ follows by taking f a compactly supported function which is 1 on the support of μ. If $p \neq 2$ the claim is a consequence of sufficient conditions, in terms of the L^p-modulus of continuity, for a function to be in $W^{\beta,p}$. See [32, Section 3.5.2, Chapter V]. □

Lemma 3.2. *Let $f \in W^{\beta,2}(\mathbb{C})$. Then for $R_0 > 1$*:

$$\|\chi_{\{|(.)|>R_0\}}\hat{f}\|_{L^2} \leqslant C R_0^{-\beta}.$$

Proof.

$$\int_{|\xi|>R_0} |\hat{f}|^2 \leqslant R_0^{-2\beta}\int \left(1 + |\xi|^2\right)^\beta |\hat{f}|^2 \leqslant R_0^{-2\beta}\|f\|_{W^{\beta,2}}. \qquad \square$$

Our task is to control the asymptotics of the exponent functions of

$$f_{\lambda\mu} = \mathrm{e}^{\mathrm{i}k\phi_\lambda(z,k)}, \tag{3.5}$$

with respect to the norm in the Hölder class $\mathcal{C}^\alpha$ of μ. Theorem 2.9 states the function $\phi_\lambda(z,k)$ in (3.5) for each k fixed in $\mathbb{C} \setminus \{0\}$, and $\lambda \in \partial\mathbb{D}$, is a quasiconformal homeomorphism on $\mathbb{C}$ that satisfies the nonlinear equation (2.51) and, for fixed k:

$$\phi_\lambda(z,k) = z + \mathrm{O}_k(1/z) \quad \text{as } z \to \infty. \tag{3.6}$$

We start by understanding the behavior of solutions to the linear equation related to (2.51).

Proposition 3.3. *Let $\psi(z,k)$ be the solution in $W^{1,2}_{\mathrm{loc}}$ of the equation,*

$$\partial_{\bar{z}}\psi(z) = -\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z)\partial_z\psi(z), \tag{3.7}$$

such that

$$\psi(z,k) = z + \mathrm{O}(1/z) \quad \textit{as } z \to \infty. \tag{3.8}$$

Let $R, \alpha, \Gamma_0 > 0$, $0 < \kappa < 1$. Then, there exists constants $a > 0$ and $C > 0$ depending on these parameters such that for any $z \in \mathbb{C}$, $\lambda \in \partial\mathbb{D}$ and any μ which satisfies $\|\mu\|_{C^\alpha} < \Gamma_0$ and $|\mu| < \kappa\chi_{\mathbb{D}(0,R)}$ it holds that,

$$\left|\psi(z,k) - z\right| \leqslant C|k|^{-a}. \tag{3.9}$$

An expression for a is given in (3.34).

We may reduce to $R = 1$. Recall that by the Cauchy formula we have:

$$\psi(z,k) - z = C\int_{\mathbb{D}} \Phi(w,z)\partial_{\bar{z}}\psi(w,k)\,\mathrm{d}m(w), \tag{3.10}$$

where $\Phi(w,z) = \chi_{\mathbb{D}}/(w-z)$. In the next lemmas we analyze more in detail the two factors $\Phi(w,z)$ and $\partial_{\bar{z}}\psi$. We will write $K_z(w) = 1/(z-w)$.

Lemma 3.4. *Let $1 < q < 2$ and let $s < (2-q)/(2q)$. Then the following properties hold*:

(i) *There exists $C = C(q,R)$ such that for every $z \in \mathbb{C}$, $\|K_z\|_{L^q(\mathbb{D}(0,R))} \leqslant C$.*
(ii) *There exists $C = C(q,s)$ such that for every $z \in \mathbb{C}$,*

$$\boldsymbol{\omega}_q(\Phi_z)(t) \leqslant Ct^s. \tag{3.11}$$

Proof. The first claim follows by direct calculation. For the second, take $q < \tilde{q} < 2$. Then, if $t < 1$ it holds that

$$\boldsymbol{\omega}_q^q(\Phi_z)(t) \leqslant \boldsymbol{\omega}_q^q(K_z)(t) + \|K_z\|^q_{L^{\tilde{q}}(\mathbb{D}(0,2))}\boldsymbol{\omega}_{\tilde{q}/(\tilde{q}-q)}(\chi_{\mathbb{D}})(t). \tag{3.12}$$

Now on one hand, for example by [1, p. 86] there exists C independent of z such that

$$\boldsymbol{\omega}_q^q(K_z)(t) \leqslant C|t|^{2-q}. \tag{3.13}$$

On the other for $p \geqslant 1$, a direct calculation yields that

$$\boldsymbol{\omega}_p(\chi_{\mathbb{D}})(t) = t^{1/p}. \tag{3.14}$$

Since for $1 < q < \tilde{q} < 2$, $2 - q > (\tilde{q} - q)/\tilde{q}$, plugging (3.13) and (3.14) into (3.12) we obtain that

$$\boldsymbol{\omega}_q(\Phi_z)(t) \leqslant C(q,\tilde{q})t^{(\tilde{q}-q)/(\tilde{q}q)}.$$

By taking the limit when $\tilde{q} \to 2$ we prove the second claim. □

The uniform control on the modulus of continuity of $\Phi_z = \Phi(z,\cdot)$ translates into a uniform control of the speed of converge of a suitable mollification of Φ_z.

Lemma 3.5. *Let $\delta > 0$. Then there exists a function $\Phi_\delta(z,w)$ such that for every $1 < q < 2$ the following properties hold*:

(i) *For every $s < (2-q)/(2q)$ there exists $C = C(s,q)$ such that*

$$\left\|\Phi_\delta(z,w) - \Phi_z(\omega)\right\|_{L^q} \leqslant C\delta^s.$$

(ii) *There exists $C = C(q)$ such that*

$$\left\|\Phi_\delta(z,\cdot)\right\|_{L^2} \leqslant C(q)\delta^{1-2/q}.$$

(iii) *Let $\delta R_0 > 1, m > 0$. Then there exists $C(q,m)$ such that*

$$\left\|\widehat{\Phi_\delta(z,\cdot)}\right\|_{L^2(\mathbb{C}\setminus\mathbb{D}(R_0))} \leqslant C(q,m)\delta^{1-2/q}(R_0\delta)^{-m}.$$

Proof. Take a compactly supported smooth function $\phi(w) = \phi_0(|w|)$ and let us consider the mollification $\Phi_\delta(z,w)$ defined by:

$$\widehat{\Phi_\delta}(z,\cdot)(\xi) = \widehat{\Phi}(\xi,z)\hat{\phi}(\delta\xi).$$

The first claim follows from Lemma 3.4, since,

$$\begin{aligned}\big\|\Phi_z(\cdot) - \Phi_\delta(z,\cdot)\big\|_{L^q} &\leqslant \int_{\mathbb{C}} \omega_q(\Phi_z)(w)\phi_\delta(w)\,\mathrm{d}w \\ &\leqslant \int \boldsymbol{\omega}_q(\Phi_z)\big(|w|\big)\big|\phi_\delta(w)\big|\,\mathrm{d}w \leqslant \int \boldsymbol{\omega}_q(\Phi_z)\big(\delta|y|\big)\big|\phi_0\big(|y|\big)\big|\,\mathrm{d}y \leqslant C(s)\delta^s,\end{aligned}$$

where $\phi_\delta(w) = \delta^{-2}\phi(\delta^{-1}w)$.

For the second, using Plancherel, Hölder and Hausdorff–Young inequalities and Lemma 3.4 we obtain, for $1/q - 1/p = 1/2$, that

$$\|\Phi_\delta\|_{L^2} \leqslant \|\Phi_z\|_{L^q}\big\|\hat{\phi}(\delta\chi)\big\|_{L^p} \leqslant C\delta^{1-2/q}.$$

For the third claim write again:

$$\begin{aligned}\|\widehat{\Phi_\delta}\|_{L^2(|\xi|>R_0)} &\leqslant \|\Phi_z\|_{L^q}\big\|\hat{\phi}(\delta\xi)\big\|_{L^p(|\xi|>R_0)} \\ &\leqslant \|\Phi_z\|_{L^q}\delta^{1-2/q}\big\|\hat{\phi}(\xi)\big\|_{L^p(|\xi|>\delta R_0)}.\end{aligned}$$

Since $\hat{\phi}$ is rapidly decreasing for any m there exists $C(m)$ such that

$$\big\|\hat{\phi}(\xi)\big\|_{L^p(|\xi|>\delta R_0)} \leqslant C(m)(\delta R_0)^{-m},$$

from where (iii) follows. □

Next, we analyze $\partial_{\bar{z}}\psi$. We use a decomposition lemma as in [7]. However, the extra Hölder regularity allows us to simplify the proof obtaining a uniform subexponential decay.

Lemma 3.6. *Let $\psi(z,k)$ be the solution of* (3.7) *and* (3.8). *Choose $2 \leqslant p_\kappa$ such that*

$$\kappa\|T\|_{L^{p_\kappa}\to L^{p_\kappa}} = \kappa_1 < 1. \tag{3.15}$$

Let $2 \leqslant p \leqslant p_\kappa$. Then there exist a constant $C_1 = C(\kappa,p)$, C_2 such that for every $n_0 \in \mathcal{N}$, $R_0 \in \mathbb{R}$, we can decompose $\bar{\partial}\psi = g + h$, where $g = g_{\lambda\mu}(z,k)$ and $h = h_{\lambda\mu}(z,k)$ satisfy:

- (i) $\sup_{k\in\mathbb{C}}\|h(\cdot,k)\|_{L^p} < C_1(\kappa,p)(\kappa\|T\|_{L^p})^{n_0}$;
- (ii) $\sup_{k\in\mathbb{C}}\|g(\cdot,k)\|_{L^p} < C_1$;
- (iii) *Let $|k| \geqslant 2R_0$ and $\beta < \alpha$. Then*, $\int_{|\xi|<R_0}|\hat{g}|^2 \leqslant (C_2\frac{\Gamma_0}{\alpha-\beta})^{n_0+1}|k|^{-\beta}$.

Proof. The function $\partial_{\bar{z}}\psi(z,k)$ is the solution in L^p of the equation:

$$\left(I + \frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z)T\right)\partial_{\bar{z}}\psi(z,k) = -\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z).$$

The condition (3.15) guarantees the solution to this equation can be expressed by the Born series:

$$\partial_{\bar{z}}\psi(z,k) = \sum_{n=0}^{\infty}\left(-\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z)T\right)^n\left(-\frac{\bar{k}}{k}\lambda\mu e_{-k}\right). \tag{3.16}$$

If we define h to be the tail,

$$h(z,k) = \sum_{n=n_0}^{\infty}\left(-\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z)T\right)^n\left(-\frac{\bar{k}}{k}\lambda\mu e_{-k}\right).$$

Then (i) follows easily.

For this choice, $g(z,k)$ can be expressed by:

$$g(z,k)=\sum_{n=0}^{n_0} G_n(z,k), \tag{3.17}$$

where

$$G_n(z,k)=\left(-\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z)T\right)^n\left(-\frac{\bar{k}}{k}\lambda\mu e_{-k}\right). \tag{3.18}$$

Then g satisfies condition (ii):

$$\big\|g(\cdot,k)\big\|_{L^p}\leqslant \pi^{1/p}\frac{\kappa_1}{1-\kappa_1}=C_0(\kappa).$$

Only (iii) remains to be proved. We write $G_n(z,k)$ as

$$G_n=\left(\frac{-\bar{k}}{k}\lambda\right)^{n+1}e_{-(n+1)k}f_n, \tag{3.19}$$

where

$$\begin{cases} f_n(z)=\mu T_n\mu T_{n-1}\mu\ldots\mu T_1(\mu), & \text{for } n>0,\\ f_0=\mu. \end{cases} \tag{3.20}$$

The operator $T_j=e_{jk}Te_{-jk}$ is a Fourier multiplier with unimodular symbol $(\xi-jk)/(\overline{\xi-jk})$. Therefore, $\|T_j\|_{W^{\alpha,2}\to W^{\alpha,2}}=1$. Combining this with Lemma 3.1 we arrive at

$$\|f_n\|_{W^{\beta,2}}\leqslant\left(C\frac{\Gamma_0}{\alpha-\beta}\right)^{n+1}. \tag{3.21}$$

Thus,

$$\int\limits_{|\xi|<R_0}|\hat{g}|^2\leqslant\sum_{n=0}^{n_0}\int\limits_{|\xi|<R_0}\big|\hat{f}_n\big(\xi-(n+1)k\big)\big|^2.$$

If $|k|\geqslant 2R_0$ this is bounded by:

$$\sum_{n=0}^{n_0}\int\limits_{|\xi|>(n+1)|k|-R_0}|\hat{f}_n|^2\leqslant\left(C\frac{\Gamma_0}{\alpha-\beta}\right)^{n_0+1}\sum_{n=0}^{n_0}\big((n+1)|k|-R_0\big)^{-\beta},$$

where the last inequality follows from Lemma 3.2. Therefore

$$\int\limits_{|\xi|<R_0}|\hat{g}|^2\leqslant\left(C\frac{\Gamma_0}{\alpha-\beta}\right)^{n_0+1}|k|^{-\beta}(n_0+1)^{1-\beta}\leqslant\left(\frac{C\Gamma_0}{\alpha-\beta}\right)^{n_0+1}|k|^{-\beta}.$$

The proof is concluded. □

Proof of Proposition 3.3. By (3.10) and Lemma 3.6,

$$\psi(z,k)-z=C\int\limits_{\mathbb{D}}\Phi(w,z)(g+h)\,\mathrm{d}m(w). \tag{3.22}$$

Let $2<p$ be such that

$$\kappa\|T\|_{L^p\to L^p}<1, \tag{3.23}$$

and let q be its dual. Then, Hölder's, Lemmas 3.4 and 3.6 imply that,

$$\left|\int h\Phi(w,z)\right|\leqslant\big\|\Phi(w,z)\big\|_{L^q}\|h\|_{L^p}\leqslant C(\kappa,q)\big(\kappa\|T\|_{L^p}\big)^{n_0+1}<\frac{\varepsilon}{4}. \tag{3.24}$$

The last inequality follows for any $n_0 \in \mathcal{N}$ such $n_0 \geqslant \frac{\log(\varepsilon C)}{\log(\kappa \|T\|_{L^p \to L^p})}$, where $C = C(\kappa, p) > 0$.

On the other hand by Lemma 3.5 for every $s < (2-q)/(2q)$,

$$\left|\int g\Phi\right| \leqslant \left|\int g\Phi_\delta\right| + \|g\|_{L^p}\|\Phi - \Phi_\delta\|_{L^q} \leqslant \left|\int g\Phi_\delta\right| + C(\kappa, s, q)\delta^s \leqslant \left|\int g\Phi_\delta\right| + \frac{\varepsilon}{4}, \tag{3.25}$$

where the last inequality is obtained if $\delta = 1/2C(\kappa, s, q)\varepsilon^{1/s}$.

It only remains to estimate the term $\int g\Phi_\delta$. We do this on the Fourier transform side:

$$\left|\int g\Phi_\delta\right| \leqslant \left|\int_{|\xi|<R_0} \hat{g}\hat{\Phi}_\delta\right| + \left|\int_{|\xi|\geqslant R_0} \hat{g}\hat{\Phi}_\delta\right|.$$

But, by Lemma 3.5 if $R_0 \geqslant \delta^{-1} \geqslant C(\kappa, s, q)\varepsilon^{-1/s}$,

$$\left|\int_{|\xi|\geqslant R_0} \hat{g}\hat{\Phi}_\delta\right| \leqslant \|g\|_{L^2}\|\hat{\Phi}_\delta\|_{L^2(|\xi|\geqslant R_0)} \leqslant C(q, m)\delta^{1-2/q}(R_0\delta)^{-m} < \frac{\varepsilon}{4},$$

the last inequality being if we further require $R_0 \geqslant C(q, m)\varepsilon^{-(m-1+2/q+s)/(sm)}$. Thus, by taking the limit when $m \to \infty$, we see, for s fixed in the open range $s < (2-q)/(2q)$, that there exists $C(\kappa, s, q)$ such that if $R_0 \geqslant C(\kappa, s, q)\varepsilon^{-1/s}$

$$\left|\int_{|\xi|\geqslant R_0} \hat{g}\hat{\Phi}_\delta\right| \leqslant \frac{\varepsilon}{4}. \tag{3.26}$$

Finally, we need to estimate $|\int_{|\xi|<R_0} \hat{g}\hat{\Phi}_\delta|$, where R_0, δ and n_0 have been fixed satisfying $R_0 \geqslant C(\kappa, s, q)\varepsilon^{-1/s}$, $\delta \leqslant C(\kappa, s, q)\varepsilon^{1/s}$ and $n_0 \geqslant \frac{\log(\varepsilon C)}{\log(\kappa\|T\|_{L^p\to L^p})}$. By Hölder:

$$\left|\int_{|\xi|\leqslant R_0} \hat{g}\hat{\Phi}_\delta\right| \leqslant \biggl(\int_{|\xi|\leqslant R_0} |\hat{g}|^2\biggr)^{1/2} \|\Phi_\delta\|_{L^2}. \tag{3.27}$$

We want to apply Lemma 3.5 and Lemma 3.6(iii). Thus, we need to assume that

$$|k| \geqslant 2R_0 \geqslant C(\kappa, s, q)\varepsilon^{-1/s}. \tag{3.28}$$

We obtain that

$$\left|\int_{|\xi|\leqslant R_0} \hat{g}\hat{\Phi}_\delta\right| \leqslant \left(C\frac{\Gamma_0}{\alpha-\beta}\right)^{1/2(n_0+1)} |k|^{-\beta/2}\delta^{1/2-1/q}, \tag{3.29}$$

which, after plugging in the optimal values of δ and n_0, is dominated by $C(\kappa, \alpha, q)\varepsilon^{-C(\kappa,q)\log(\frac{\Gamma_0}{\alpha-\beta}+\mathrm{e})-1/s(1/q-1/2)} \times |k|^{-\beta/2}$. Now we impose this quantity to be bounded by $\varepsilon/4$ which can be attained if,

$$|k| \geqslant C\varepsilon^{-(1/\beta)(2+C(\kappa,q)\log(\frac{\Gamma_0}{\alpha-\beta}+\mathrm{e})+1/s(2/q-1))}, \tag{3.30}$$

to arrive at

$$\left|\int_{|\xi|\leqslant R_0} \hat{g}\hat{\Phi}_\delta\right| \leqslant \frac{\varepsilon}{4} \tag{3.31}$$

as desired. Putting together (3.28) and (3.30), we have proved the following claim: For any $1 < q < 2$ such that $p = q/(q-1)$ satisfies $\kappa\|T\|_{L^p\to L^p} < 1$, any $s < (2-q)/(2q)$, there exists positive constants C and b depending on κ, Γ_0, α, s and q such that

$$|k| \geqslant C\varepsilon^{-b} \tag{3.32}$$

implies $|\phi(z,k)-z| \leqslant \varepsilon$. The constant b above is given by:

$$\max\left\{\frac{1}{s}, \frac{2 + C\log(\frac{\Gamma_0}{\alpha-\beta}+\mathrm{e}) + 1/s(2/q-1)}{\beta}\right\}.$$

Therefore we have obtained the claim of the proposition with:

$$a = 1/b = \min\left\{s, \beta\left(2 + C\log\left(\frac{\Gamma_0}{\alpha-\beta}+\mathrm{e}\right) + 1/s(2/q-1)\right)^{-1}\right\}. \tag{3.33}$$

By choosing appropriated s, q and β, a possible value of the exponent is

$$a = \min\left\{\frac{\delta(\kappa)}{4}, \frac{\alpha}{2}\big(6 + C(\kappa)\log(2\Gamma_0/\alpha+\mathrm{e})\big)^{-1}\right\}. \qquad \square \tag{3.34}$$

Unlike in [7], the corresponding theorem for the nonlinear equation follows easily thanks to the uniformity of the estimates in the case C^α.

Theorem 3.7. *Let ϕ_λ be the solution of* (2.51) *which satisfies* (3.6). *Let $R, \alpha, \Gamma_0 > 0$, $0 < \kappa < 1$. Then, there exists constants a and C depending on these parameters such that for any $z \in \mathbb{C}$, $\lambda \in \partial\mathbb{D}$ and any μ which satisfies $\|\mu\|_{C^\alpha} < \Gamma_0$ and $|\mu| < \kappa\chi_{\mathbb{D}(0,R)}$ it holds that*

$$\left|\phi_\lambda(z,k) - z\right| \leqslant C|k|^{-a}. \tag{3.35}$$

Proof. We observe as in [7] that since the estimates are uniform in z it is equivalent to prove similar asymptotics for the inverse function. $\psi_\lambda = \phi_\lambda^{-1}$. This satisfies the equation,

$$\bar\partial\psi_\lambda(z) = -\frac{\bar k}{k}\lambda\mu\big(\psi_\lambda(z)\big)e_{-k}(z)\partial\psi_\lambda(z),$$

under the condition,

$$\psi_\lambda(z,k) = z + \mathrm{O}_k(1/z) \quad \text{as } z \to \infty.$$

The theorem is a corollary of Proposition 3.3. We just need to prove that the coefficient $\nu(z) = \mu(\psi_\lambda(z,k))$ satisfies:

(a) For some $\beta = \beta(\alpha,\kappa)$ $[\nu]_C^\beta(4\mathbb{D}) \leqslant C(\kappa,\alpha)$.
(b) Its support is contained in $4\mathbb{D}$.

Condition (b) follows from $1/4$-Koebe theorem. Condition (a) follows because ψ_λ is a normalized quasiconformal mappings and therefore $1/K$ Hölder continuous on domains D with a constant $C = C(\kappa, D)$. Therefore for $\beta = 1/K$,

$$\|\nu\|_{C^{\alpha\beta}} \leqslant \|\mu\|_{C^\alpha}\|\psi_\lambda\|_{C^\beta}^\alpha \leqslant C(\Gamma_0,\alpha,\kappa). \qquad \square$$

3.2. *Uniform subexponential growth of u*

Theorem 3.8. *Let u be the solution of the equation,*

$$\operatorname{div}(\gamma\nabla u) = 0,$$

constructed as $u(z) = \Re f_\mu + \mathrm{i}\Im f_{-\mu}$, where μ, supported on $\mathbb{D}$ is such that $\|\mu\|_\infty < \kappa$ and $\|\mu\|_{C^\alpha} < \Lambda_0$. Then there exists $a = a(\kappa,\alpha,\Gamma_0)$ and $C = C(\kappa,\alpha,\Gamma_0)$ such that for every $z \in \mathbb{C}$, we may write:

$$u(z,k) = \mathrm{e}^{\mathrm{i}k(z+\varepsilon_z(k))}, \quad \textit{where } \left|\varepsilon_z(k)\right| \leqslant C|k|^{-a}. \tag{3.36}$$

Remark. A similar estimate can be proved for the solution $u = u_{\gamma^{-1}}$ of

$$\operatorname{div}(\gamma^{-1}\nabla u) = 0,$$

given in (2.42).

Proof. We can write:

$$u = f_\mu \left(1 + \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}}\right)^{-1} \left(1 + \frac{\overline{f_\mu} - \overline{f_{-\mu}}}{f_\mu + f_{-\mu}}\right). \tag{3.37}$$

Let

$$\alpha(z) = \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}}.$$

Then Theorem 3.7 implies that we may reduce the proof of (3.36) to prove

$$\left|\alpha(z)\right| \leqslant 1 - \mathrm{e}^{-k\varepsilon}, \tag{3.38}$$

where $|\varepsilon(z,k)| \leqslant C|k|^{-a}$. In [7, Lemma 8.2] it is shown that estimating α reduces to estimate $f_{\lambda\mu}$ for $\lambda \in S^1$. Since in our case the control of $f_{\lambda\mu}$ is uniform in our class of $\mu, z \in \mathbb{C}$ and $\lambda \in S^1$, (3.38) follows. □

3.3. Regularity of the complex geometric optic solutions

The C^α regularity that we assume on γ (equivalently on μ) it is also crucial because it implies extra regularity on the complex geometric solutions.

Theorem 3.9. *There exists constants $C_1 = C_1(\kappa, \Gamma_0, |k|, \alpha)$, and $C_2 = C_2(\kappa, \Gamma_0, |k|, \alpha) > 0$ such that for any μ with $|\mu| \leqslant \chi_{\mathbb{D}}\kappa$ and $\|\mu\|_{\mathcal{C}^\alpha} \leqslant \Gamma_0$:*

(i)
$$\left\| f_\mu(\cdot, k)\right\|_{\mathcal{C}^{1+\alpha}(\mathbb{D},\mathrm{d}z)} \leqslant C_1; \tag{3.39}$$

(ii)
$$\inf_{z\in\mathbb{D}} \left|J_{f_\mu}(z,k)\right| \geqslant C_2, \tag{3.40}$$

where J_{f_μ} is the Jacobian in the z variable of f_μ.

The proof is based on the following application of perturbation arguments of Schauder type. An elegant proof using the special structure of the Beurling transform will appear in [6].

Theorem 3.10. *Let $G \subset \mathbb{C}$ be a domain and let $\mu, \nu \in \mathcal{C}^\alpha(G)$ with $\|\mu\|_{\mathcal{C}^\alpha} + \|\nu\|_{\mathcal{C}^\alpha} < \Gamma_0$ and $|\mu(z)| + |\nu(z)| \leqslant \kappa < 1$, for $z \in G$. Let ω be a $W^{1,2}_{\mathrm{loc}}$ solution to the equation,*

$$\partial_{\bar z}\omega = \nu \partial_z \omega + \mu \overline{\partial_z \omega}, \tag{3.41}$$

and for any $G' \Subset G$ define $M = M(G')$:

$$\sup_{z\in G'} \left|\omega(z)\right| = M. \tag{3.42}$$

Then the following properties hold:

(I) *Let $D \Subset G'$. Then there exists a $\Lambda_1 = \Lambda_1(\kappa, M, D, \alpha, \Gamma_0)$ such that*

$$\|\partial_{\bar z}\omega\|_{\mathcal{C}^\alpha(D)} + \|\partial_z\omega\|_{\mathcal{C}^\alpha(D)} \leqslant \Lambda_1. \tag{3.43}$$

(II) *Assume further that ω is a quasiconformal homeomorphism in $\mathbb{C}$. Then for any $D \Subset \mathbb{C}$, there exists a constant $J = J(M, \kappa, \Gamma_0, D, \Gamma_0)$ such that*

$$\inf_{z\in D}\left(\left|\partial_z\omega(z)\right|^2 - \left|\partial_{\bar z}\omega(z)\right|^2\right) = \inf_{z\in D}\left(J_\omega(z)\right) \geqslant (1-\kappa^2)|\partial_z\omega|^2(z) > J. \tag{3.44}$$

Proof. For the proof of (I) we refer to [6].

Claim (II) follows by applying (I) to the inverse of ω, ω^{-1} defined on $\omega(D)$. The first observation is that

$$\max_{\xi\in\omega(D)} \left|\omega^{-1}(\xi)\right| = \max_{z\in D}\left|\omega(z)\right| = c_1. \tag{3.45}$$

Next, it is well known that ω^{-1} solves the following Beltrami equation [6],

$$\partial_{\bar{\xi}}\omega^{-1} = -(\mu \circ \omega^{-1})\overline{\partial_{\xi}\omega^{-1}} - (\nu \circ \omega^{-1})\partial_{\xi}\omega^{-1}. \tag{3.46}$$

Since ω^{-1} is also a quasiconformal mapping with the L^{∞} bound (3.45), it is $1/K$ Hölder continuous with constant $C = C(D)$, $K = (\kappa + 1)/(1 - \kappa)$. Thus, the coefficients in (3.46) are $C^{\alpha/K}$ with an uniform bound $C = C(\kappa, D, \Gamma_0)$. Therefore, ω^{-1} fulfills all the properties to apply claim (I). For example we obtain that for $z \in D$,

$$\left|\partial_{\xi}\omega^{-1} \circ \omega(z)\right| \leqslant \Lambda_1, \tag{3.47}$$

with $\Lambda_1 = \Lambda_1(\kappa, D, M, \alpha, \Gamma_0)$.

Now we can conclude using the chain rule and algebraic relations. Since ω is quasiregular,

$$J_{\omega} = \left|\partial_z\omega(z)\right|^2 - \left|\partial_{\bar{z}}\omega(z)\right|^2 \geqslant (1 - \kappa^2)\left|\partial_z\omega(z)\right|^2 \geqslant (1 - \kappa^2)\frac{|J_{\omega}|^2}{|\partial_z\omega|^2}. \tag{3.48}$$

The chain rule says that for $\xi = \omega(z)$,

$$|\partial_{\xi}\omega^{-1} \circ \omega| = \frac{|\partial_z\omega|}{|J_{\omega}|}.$$

Thus, (3.48) and (3.47) yield that for $z \in D$,

$$J_{\omega}(z) \geqslant \frac{(1 - \kappa^2)}{\Lambda_1^2}. \qquad \square$$

Proof of Theorem 3.9. (i) is an immediate consequence of Theorem 3.10, once we prove the existence of the constant $M(2\mathbb{D})$ in (3.42) for $f_{\mu}(\cdot, k)$. This follows from (k is fixed) $f_{\mu}(z, k) = \mathrm{e}^{\mathrm{i}k\phi(z,k)}$, where $\phi(z, k) = z + \varepsilon(z, k)$ with $\varepsilon(z, k)$ uniformly bounded for fixed k. Thus we obtain that

$$\max_{z\in\mathbb{D}}\left|\phi(z, k)\right| = c_1,$$

where $c_1 = c_1(k, \kappa, \mathbb{D})$ and hence for $z \in \mathbb{D}$,

$$\frac{1}{c_1} \leqslant \left|f_{\mu}(z, k)\right| \leqslant c_1, \tag{3.49}$$

which yields (i).

For (ii) we observe that by quasiregularity it is enough to obtain a lower bound for $\partial_z f_{\mu}$. Recall that ϕ is a quasiconformal mapping which solves equation (2.51). Since ϕ is normalized it is $1/K$ Hölder continuous with constant $C(\kappa)$. Therefore, the coefficient in Eq. (2.51), $\frac{\bar{k}}{k}\mu e_{-k}(\phi)\mu$, is Hölder on $\mathbb{D}$ with exponent $\min\{\alpha, 1/K\}$ and a constant depending just on κ. Thus, we can apply Theorem 3.10 to obtain the existence of a constant $J = J(\kappa, \alpha, \Gamma_0, \mathbb{D})$ such that

$$\inf_{z\in\mathbb{D}}|\partial_z\phi| \geqslant J. \tag{3.50}$$

Since, $\partial_z f_m u = k\partial_z\phi f_{\mu}$ (ii) follows form (3.49) and (3.50). $\square$

3.4. *Growth of the k derivatives of the complex geometric optic solutions*

In this section we study the behavior of the complex geometric optic solutions respect to the k variable. It is proved in Theorem 2.5 that the Jost functions $M_{\mu}(k, z)$ are C^{∞}-smooth in the k variable. As with many other properties, for stability it is needed a quantitative version of this fact. Bounds are provided by the fact that the derivatives in the k plane of $M_{\mu}(k, z)$ solve corresponding nonhomogeneous Beltrami equations. The results in this section just assume that $|\mu| \leqslant \kappa\chi_{\mathbb{D}}$.

Lemma 3.11. *Let $2 < p < p_k$, $e_t \in S^1$ and $h(z, k) = \partial_{e_t}\omega_{\mu}(z, k)$. Then $h(z, k) \in W^{1,p}(\mathbb{C})$ and satisfies the equation*:

$$\partial_{\bar{z}}h - \mu e_{-k}\overline{\partial_z h} = \mu(\gamma h + E). \tag{3.51}$$

Here

$$\gamma = -\mathrm{i}\bar{k}e_{-k}$$

and E is an error term given by:

$$E = \partial_{e_t} e_{-k}\bar{\partial}_z\omega + (\overline{\omega}+1)(-\mathrm{i})\partial_{e_t}(\bar{k}e_{-k}).$$

Proof. We start by finding a unique solution $h \in W^{1,p}$ of Eq. (3.51). Since $\omega \in W^{1,p}(\mathbb{C}, \mathrm{d}z)$ and μ has compact support , it follows that $E \in L^p(\mathbb{C})$ and $\gamma \in L^\infty$. Declare $\nu = \mu e_{-k}$. We will find h with the help of the operators K introduced in Lemma 2.4. Recall that for any L^∞ functions ν, α supported in $\mathbb{D}$ and $|\nu| \leqslant \kappa$ the operator $K(\nu, \alpha): L^p(\mathbb{C}) \to W^{1,p}(\mathbb{C})$ was defined by:

$$K(g) = P(I - \nu\bar{S})^{-1}(\alpha g).$$

Since by Lemma 2.4, $(I - K)$ is invertible in L^p we can find $h \in L^p$ solving the equation,

$$h - K(\nu, \mu\gamma)(h) = K(\nu, \mu)(E). \tag{3.52}$$

Since for $\eta \in L^p$, $\partial_{\bar{z}} P(\eta) = \eta$ if we take the distributional $\partial_{\bar{z}}$ derivative in (3.52), we obtain that

$$\partial_{\bar{z}} h = (I - \nu S)^{-1}(\mu\gamma h + \mu E). \tag{3.53}$$

Now h and E are in $L^p(\mathbb{C})$ for the required range of exponents. By the boundedness of the Beltrami operators, $\partial_{\bar{z}} h \in L^p(\mathbb{C})$. By means of the Beurling transform we achieve that $h \in W^{1,p}(\mathbb{C})$. Moreover, if h solves (3.53) then h solves (3.51) as well.

To see that in fact $h = \partial_{e_t}\omega$, we observe that Eq. (3.51) is obtained by formally differentiating in the k variable respect to the t direction the equation satisfied by ω (2.34). Standard arguments using difference quotients show that $h = \partial_{e_t}\omega$ as desired. □

Now we implement our knowledge of estimates for nonhomogeneous Beltrami equations to achieve the required control on the derivatives of the Jost functions and of the scattering transform.

Theorem 3.12. *There exists a constant $C = C(\kappa, p)$ such that for $2 < p < 1 + 1/\kappa$ it holds that*

$$\left\| \nabla_k M_\mu(k, z) \right\|_{W^{1,p}(\mathrm{d}z)} \leqslant \mathrm{e}^{(1+C)|k|}, \tag{3.54}$$

and that for every $k \in \mathbb{C}$,

$$\left| \nabla_k \tau(\mu, k) \right| \leqslant \mathrm{e}^{(1+C)|k|}. \tag{3.55}$$

Proof. Observe that independently of the direction $e_t \in S^1$,

$$\left\| \gamma(z, k) \right\|_{L^\infty(\mathbb{D}, \mathrm{d}z)} \leqslant |k|, \tag{3.56}$$

and for $2 < p < 1 + 1/\kappa$ by Proposition 2.6:

$$\| E \|_{L^p(\mathbb{D}, \mathrm{d}z)} \leqslant 6|k| \|\omega\|_{W^{1,p}(\mathbb{D}, \mathrm{d}z)} \leqslant C\mathrm{e}^{C|k|}. \tag{3.57}$$

Thus, the claim follows from Eq. (3.51) and Theorem 2.1. Now, from the formula:

$$\overline{\tau(\mu, k)} = \frac{1}{4\pi \mathrm{i}} \int_{\mathbb{D}} \partial_{\bar{z}}(M_\mu - M_{-\mu})\, \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

we can take derivatives with respect to k under the integral sign and conclude by Holder's inequality. □

4. Stability from the Dirichlet to Neumann map to the scattering transform

The main result in this section is Corollary 4.5. We obtain it in two different ways. In Section 4.1, following the lines of [7,8], we recover with stability the values of the geometric optics solutions in the exterior of $\mathbb{D}$. In Section 4.2 we prove a formula relating the differences of scattering transforms to the differences of Dirichlet to Neumann maps. Similar expressions were essential in previous works about stability, See [2,29,26,10]. Let us remark that in both subsections no extra regularity on γ is required.

4.1. Values of M_μ on $\mathbb{C}\setminus\mathbb{D}$

Theorem 4.1. *Let $2<p<\infty$. Then there exists an uniform constant $c=c(\kappa,p)$ such that if μ_1,μ_2 are complex with $|\mu_1|,|\mu_2|<\chi_{\mathbb{D}}$. Then,*

$$\|M_{\mu_1}-M_{\mu_2}\|_{L^p(\mathbb{C}\setminus\mathbb{D})}\leqslant \mathrm{e}^{c|k|}\rho,$$

where

$$\rho=\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}.$$

The proof of Theorem 4.1 is decomposed in the following lemmas:

Lemma 4.2. *There exists $C=C(\kappa)$, such that for any pair f_{μ_1},f_{μ_2} there exists two other functions $f_1,f_2\in H^{1/2}(\partial\mathbb{D})$ such that*

(1) $(f_{\mu_1}-f_{\mu_2})|_{\partial\mathbb{D}}=f_1+f_2$,
(2) $\|f_2\|_{H^{1/2}}\leqslant C\rho\|f_{\mu_2}\|_{H^{1/2}}$,
(3) $f_1=g|_{\partial\mathbb{D}}$, *with g such that*

$$\partial_{\bar z}g=\mu_1\overline{\partial g}.$$

Proof. The claim follows from Lemma 2.3. Let $g=f_{\mu_1}-f_{\mu_2}$. We set $f_1=P_{\mu_1}(g)$, $f_2=Q_{\mu_1}(g)$. Claims (1) and (3) are straightforward. For claim (2) observe that $Q_{\mu_1}(f_{\mu_1})=0$ and

$$Q_{\mu_1}(f_{\mu_2})=Q_{\mu_2}(f_{\mu_2})+\mathrm{i}(\mathcal{H}_{\mu_2}-\mathcal{H}_{\mu_1})(f_{\mu_2})=\mathrm{i}(\mathcal{H}_{\mu_2}-\mathcal{H}_{\mu_1})(f_{\mu_2}).$$

Therefore,

$$\|f_2\|_{H^{1/2}}\leqslant\|f_{\mu_2}\|_{H^{1/2}}\|\mathcal{H}_{\mu_2}-\mathcal{H}_{\mu_1}\|_{H^{1/2}\to H^{1/2}}\leqslant\rho\|f_{\mu_2}\|_{H^{1/2}}$$

as desired. □

Lemma 4.3. *There exists a constant $C(\kappa)$ such that*

$$\|f_\mu\|_{H^{1/2}(\partial\mathbb{D})}\leqslant \mathrm{e}^{C|k|}.$$

Proof. From (2.35) we control $\|\omega\|_{H^1(\mathbb{D})}$. Thus, the claim follows from the trace lemma. □

Lemma 4.4. *Let $E:H^{1/2}(\partial\mathbb{D})\to W^{1,2}(\mathbb{D})/W_0^{1,2}(\mathbb{D})$ be the extension operator. Let us define $G:\mathbb{C}\to\mathbb{C}$ by*:

$$G(z,k)=(f_{\mu_1}-f_{\mu_2})(1-\chi_{\mathbb{D}})+g+E(f_2)\chi_D.$$

Then G solves the equation:

$$\partial_{\bar z}G-\mu_1\overline{\partial_z}G=\big(\partial_{\bar z}E(f_2)-\mu_1\overline{\partial_z}E(f_2)\big)\chi_D. \tag{4.1}$$

Furthermore let $G_0=\mathrm{e}^{-\mathrm{i}kz}G$ and p be the right-hand side of (4.1)*, i.e.*

$$p(f_2)=\big(\partial_{\bar z}E(f_2)-\mu_1\overline{\partial_z}E(f_2)\big)\chi_D.$$

Then

$$\partial_{\bar z}G_0=\mathrm{e}^{-(\mathrm{i}kz+\mathrm{i}\bar k\bar z)}\mu_1\overline{\partial_z}G_0-\mathrm{i}k\mu_1\overline{G_0}+\mathrm{e}^{-(\mathrm{i}kz)}p(f_2). \tag{4.2}$$

Proof of the Theorem 4.1. Let G_0 as in Lemma 4.4. Then G_0 satisfies the differential inequality:

$$|\partial_{\bar z}G_0|\leqslant\kappa|\partial_zG_0|+\gamma|G_0|+h, \tag{4.3}$$

with $\kappa = \|\mu_1\|_{\infty,\mathbb{D}}$, $\gamma = k\kappa\chi_D$, $h = |p(f_2)|\mathrm{e}^{|k|}$. In particular we see that $\gamma \in L^\infty(\mathbb{C})$ and is compactly supported. Therefore we can apply Theorem 2.1 to obtain the bound:

$$\|G_0\|_{L^p(\mathbb{C})} \leqslant \mathrm{e}^{C|k|}\|p\|_{L^2}.$$

Finally we conclude the proof observing that on one hand, $\|p(f_2)\|_{L^2} \leqslant \|f_2\|_{H^{1/2}(\partial\mathbb{D})} \leqslant \mathrm{e}^{c|k|}\rho$, where we have used Lemmas 4.2 and 4.3. On the other hand $G_0(1-\chi_{\mathbb{D}}) = \mathrm{e}^{-\mathrm{i}kz}(f_{\mu_1} - f_{\mu_2})(1-\chi_{\mathbb{D}}) = M_{\mu_1} - M_{\mu_2}$. □

Corollary 4.5. *For $i = 1, 2$, let μ_i be as in Theorem* 4.1 *and let $\tau(\mu_i, k)$ be the corresponding scattering transforms defined in* (2.40). *Then there exists an uniform constant $c = c(\kappa)$ such that for every $k \in \mathbb{C}$,*

$$\left|\tau(\mu_1, k) - \tau(\mu_2, k)\right| \leqslant c\rho\mathrm{e}^{c|k|}. \tag{4.4}$$

Proof. The scattering transform $\tau(\mu, k)$ can be also defined in terms of the asymptotics in z of M_μ. Namely,

$$M_\mu - M_{-\mu} = \frac{\tau(\mu, k)}{z} + \mathrm{O}\left(\frac{1}{|z|^2}\right).$$

Therefore, we have that

$$D(z,k) = (M_{\mu_1} - M_{-\mu_1} - M_{\mu_2} + M_{-\mu_2})(z,k) = \frac{\tau(\mu_1,k) - \tau(\mu_2,k)}{z} + \mathrm{O}\left(\frac{1}{|z|^2}\right).$$

Now $D(z,k)$ is analytic in $\mathbb{C}\setminus\mathbb{D}$. Thus, for every $r > 1$ it holds that

$$\left|\tau(\mu_1,k) - \tau(\mu_2,k)\right| = \left|\int_{\partial\mathbb{D}_r} D(z,k)\,\mathrm{d}z\right|.$$

Integrating this expression respect to r yields that

$$\begin{aligned}\left|\tau(\mu_1,k) - \tau(\mu_2,k)\right| &\leqslant \frac{1}{2}\int_1^2\int_{\partial\mathbb{D}_r}\left|D(z,k)\right||\mathrm{d}z|\,\mathrm{d}r \\ &\leqslant \int_{1\leqslant|z|\leqslant 2}\left|D(z,k)\right|\mathrm{d}z \\ &\leqslant c\left\|D(z,k)\right\|_{L^p(\mathbb{C}\setminus\mathbb{D})} \leqslant \rho\mathrm{e}^{c(1+|k|)}.\end{aligned}$$

The last inequality follows from Theorem 4.1. The corollary is proved. □

4.2. An explicit formula

We will denote the dependencies by writing $u(\mu, k, z)$, etc. The scattering transform of μ is defined by:

$$\tau(\mu,k) = \frac{\mathrm{i}}{4\pi}\int_{\mathbb{D}}\partial_z\big(\bar{M}(\mu,z,k) - \bar{M}(-\mu,z,k)\big)\,\mathrm{d}m(z). \tag{4.5}$$

Since $f_\mu(z,k) = \mathrm{e}^{\mathrm{i}kz}M_\mu(z,k)$ we also have:

$$\tau(\mu,k) = \frac{\mathrm{i}}{4\pi}\int_{\mathbb{D}}\partial_z\big(\mathrm{e}^{\mathrm{i}\bar{k}\bar{z}}(\bar{f}_\mu - \bar{f}_{-\mu})\big)\,\mathrm{d}m(z). \tag{4.6}$$

Theorem 4.6. *Let γ_1 and γ_2 be L^∞ conductivities which are identically* 1 *in a neighborhood of* $\mathbb{C}\setminus\mathbb{D}$ *and μ_1 and μ_2 the corresponding Beltrami coefficients. We have that*

$$\tau(\mu_1,k) - \tau(\mu_2,k) = \frac{-1}{8\pi\bar{k}}\int_{\partial\mathbb{D}}\bar{u}(\mu_1,z,-k)(\Lambda_\gamma - \Lambda_{\gamma_2})u(\mu_2,z,k)\,\mathrm{d}\sigma, \tag{4.7}$$

where $u(\mu_i,z,k) = u_{\gamma_i}$ is the complex geometric optics solution given by (2.41).

To prove the theorem we need an alternative formula for the scattering transform.

Lemma 4.7.

$$\tau(\mu,k)=\frac{-1}{4\pi\bar{k}}\int_{\partial\mathbb{D}}\bar{z}e^{i\bar{k}\bar{z}}\partial_{\bar{z}}u\,d\sigma. \tag{4.8}$$

Proof. Since the normal at any point $z\in\partial\mathbb{D}$ can be identified with z, by Green Formula:

$$\tau(\mu,k)=\frac{i}{4\pi}\int_{\partial\mathbb{D}}\bar{z}e^{i\bar{k}\bar{z}}(\bar{f}_\mu-\bar{f}_{-\mu})\,d\sigma. \tag{4.9}$$

Notice that for $z\in\mathbb{D}$ the tangential derivative is:

$$\partial_T(e^{i\bar{k}\bar{z}})=\bar{k}\bar{z}e^{i\bar{k}\bar{z}}.$$

Thus, by integration by parts it follows that

$$\tau(\mu,k)=\frac{-i}{4\pi\bar{k}}\int_{\partial\mathbb{D}}e^{i\bar{k}\bar{z}}\partial_T(\bar{f}_\mu-\bar{f}_{-\mu})\,d\sigma. \tag{4.10}$$

Now we write $f_\mu-f_{-\mu}$ in terms of u and its Hilbert transform:

$$f_\mu-f_{-\mu}=\Re(u)+i\mathcal{H}_\mu\big(\Re(u)\big)+\mathcal{H}_\mu\big(\Im(u)\big)-i\Im(u).$$

Thus,

$$\partial_T(\bar{f}_\mu-\bar{f}_{-\mu})=\partial_T\big(\Re(u)\big)-i\Lambda_\gamma\big(\Re(u)\big)+\Lambda_\gamma\big(\Im(u)\big)+i\partial_T\big(\Im(u)\big).$$

Since $\gamma=1$ on $\partial\mathbb{D}$, we have that (in the weak sense) $\Lambda_\gamma(\varphi)=\partial_\nu(\varphi)$ for every $\varphi\in H^{1/2}(\partial\mathbb{D})$. Hence,

$$\partial_T(\bar{f}_\mu-\bar{f}_{-\mu})=\partial_T(u)-i\partial_\nu(u). \tag{4.11}$$

Since,

$$\bar{z}\partial_{\bar{z}}u=\partial_\nu u+i\partial_T u,$$

(4.11) implies that

$$i\partial_T(\bar{f}_\mu-\bar{f}_{-\mu})=\bar{z}\partial_{\bar{z}}u,$$

which plugged into (4.10) yields the claim. □

Proof of Theorem 4.6. The theorem will follow from the following claim.

Claim.

$$\begin{aligned}\int_{\partial\mathbb{D}}ze^{ikz}\partial_z\bar{u}(\mu_1,z,-k)\,d\sigma-\int_{\partial\mathbb{D}}\bar{z}e^{i\bar{k}\bar{z}}\partial_{\bar{z}}\bar{u}(\mu_2,z,k)\,d\sigma\\ =\frac{1}{2}\int_{\partial\mathbb{D}}\bar{u}(\mu_1,z,-k)(\Lambda_{\gamma_1}-\Lambda_{\gamma_2})u(\mu_2,z,k)\,d\sigma.\end{aligned} \tag{4.12}$$

Assume the claim, then take $\gamma_1=\gamma_2$ to obtain,

$$\tau(\mu_1,k)=\frac{-1}{4\pi\bar{k}}\int_{\partial\mathbb{D}}ze^{ikz}\partial_z\bar{u}(\mu_1,z,-k)\,d\sigma,$$

and hence (4.7).

Proof of Claim. From Proposition 2.8, we have:

$$\mathrm{e}^{\mathrm{i}kz} = u(\mu_2, z, k) - \mathrm{e}^{\mathrm{i}kz} R(\mu_2, k, z), \tag{4.13}$$

$$\mathrm{e}^{\mathrm{i}\bar{k}\bar{z}} = \bar{u}(\mu_1, z, k) - \mathrm{e}^{\mathrm{i}\bar{k}\bar{z}} \bar{R}(\mu_1, z, -k), \tag{4.14}$$

where $R(\mu_1, k, z) \in W^{1,p}(\mathbb{C})$ and $R(\mu_1, k, z) = \mathrm{O}(1/|z|)$. Inserting these expressions in the first member of (4.12) we obtain:

$$\begin{aligned}
&\int_{\partial\mathbb{D}} z\mathrm{e}^{\mathrm{i}kz} \partial_z \bar{u}(\mu_1, z, -k)\,\mathrm{d}\sigma - \int_{\partial\mathbb{D}} \bar{z}\mathrm{e}^{\mathrm{i}\bar{k}\bar{z}} \partial_{\bar{z}} \bar{u}(\mu_2, z, k)\,\mathrm{d}\sigma \\
&\quad = \int_{\partial\mathbb{D}} u(\mu_2, z, k) z\partial_z \bar{u}(\mu_1, z, -k)\,\mathrm{d}\sigma - \int_{\partial\mathbb{D}} \bar{u}(\mu_1, z, -k)\bar{z}\partial_{\bar{z}} u(\mu_2, z, k)\,\mathrm{d}\sigma \\
&\quad\quad - \int_{\partial\mathbb{D}} \mathrm{e}^{\mathrm{i}kz} R(\mu_2, z, k) z\partial_z \bar{u}(\mu_1, z, -k)\,\mathrm{d}\sigma + \int_{\partial\mathbb{D}} \mathrm{e}^{\mathrm{i}\bar{k}\bar{z}} \bar{R}(\mu_1, z, -k)\bar{z}\partial_{\bar{z}} u(\mu_2, z, k)\,\mathrm{d}\sigma. \qquad \square
\end{aligned}$$

We divide the proof of the claim in two lemmas:

Lemma 4.8.

$$\begin{aligned}
&\int_{\partial\mathbb{D}} u(\mu_2, z, k) z\partial_z \bar{u}(\mu_1, z, -k)\,\mathrm{d}\sigma - \int_{\partial\mathbb{D}} \bar{u}(\mu_1, z, -k)\bar{z}\partial_{\bar{z}} u(\mu_2, z, k)\,\mathrm{d}\sigma \\
&\quad = \frac{1}{2}\int_{\partial\mathbb{D}} \bar{u}(\mu_1, z, -k)(\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) u(\mu_2, z, k)\,\mathrm{d}\sigma.
\end{aligned}$$

Lemma 4.9.

$$-\int_{\partial\mathbb{D}} \mathrm{e}^{\mathrm{i}kz} R(\mu_2, z, k) z\partial_z \bar{u}(\mu_1, z, -k)\,\mathrm{d}\sigma + \int_{\partial\mathbb{D}} \mathrm{e}^{\mathrm{i}\bar{k}\bar{z}} \bar{R}(\mu_1, z, -k)\bar{z}\partial_{\bar{z}} u(\mu_2, z, k)\,\mathrm{d}\sigma = 0.$$

Proof of Lemma 4.8. Since,

$$\bar{z}\partial_{\bar{z}} = 1/2\partial_\nu + \mathrm{i}/2\partial_T,$$

and

$$z\partial_z = 1/2\partial_\nu - \mathrm{i}/2\partial_T,$$

we have, integrating by parts:

$$\begin{aligned}
&\int_{\partial\mathbb{D}} u(\mu_2, z, k) z\partial_z \bar{u}(\mu_1, z, -k)\,\mathrm{d}\sigma - \int_{\partial\mathbb{D}} \bar{u}(\mu_1, z, -k)\bar{z}\partial_{\bar{z}} u(\mu_2, z, k)\,\mathrm{d}\sigma \\
&\quad = \frac{1}{2}\int_{\partial\mathbb{D}} u(\mu_2, z, k)\Lambda_{\gamma_1} \bar{u}(\mu_1, z, -k) - \frac{1}{2}\int_{\partial\mathbb{D}} \bar{u}(\mu_1, z, k)\Lambda_{\gamma_2} u(\mu_2, z, -k) \\
&\quad\quad - \frac{\mathrm{i}}{2}\int_{\partial\mathbb{D}} \partial_T\big(u(\mu_1, z, -k) u(\mu_2, z, k)\big)\,\mathrm{d}\sigma.
\end{aligned}$$

The last integral vanishes from the fundamental theorem of calculus. Since Λ_γ is selfadjoint the lemma follows. $\square$

Proof of Lemma 4.9. Let $\mathbb{D}_R$ be the disc of radius R centered at the origin. Integration by parts in $\mathbb{D}_R \setminus \mathbb{D}$ gives:

$$-\int_{\partial\mathbb{D}} e^{ikz} R(\mu_2, z, k) z\partial_z \bar{u}(\mu_1, z, -k)\,d\sigma + \int_{\partial\mathbb{D}} e^{i\bar{k}\bar{z}} \bar{R}(\mu_1, z, -k)\bar{z}\partial_{\bar{z}} u(\mu_2, z, k)\,d\sigma$$
$$= \int_{\partial\mathbb{D}_R} \frac{z}{|z|} e^{ikz} R(\mu_2, z, k)\partial_z \bar{u}(\mu_1, z, -k)\,d\sigma - \int_{\partial\mathbb{D}_R} \frac{\bar{z}}{|z|} e^{i\bar{k}\bar{z}} \bar{R}(\mu_1, z, -k)\partial_{\bar{z}} u(\mu_2, z, k)\,d\sigma$$
$$+ \int_{\mathbb{D}_R\setminus\mathbb{D}} \big(-\partial_{\bar{z}}\big(e^{ikz} R(\mu_2, z, k)\partial_z \bar{u}(\mu_1, z, -k)\big) + \partial_z\big(e^{i\bar{k}\bar{z}} \bar{R}(\mu_1, z, -k)\partial_{\bar{z}} u(\mu_2, z, k)\big)\big)\,dm. \tag{4.15}$$

Since u is harmonic on the exterior of $\mathbb{D}$, $\partial_{\bar{z}}\partial_z u = 0$, this together with expressions (4.13) and (4.14) gives:

$$\partial_{\bar{z}}\big(e^{ikz} R(\mu_2, z, k)\partial_z \bar{u}(\mu_1, z, -k)\big) = \partial_{\bar{z}} u(\mu_2, z, k)\partial_z \bar{u}(\mu_1, z, -k)$$
$$= \partial_z\big(e^{i\bar{k}\bar{z}} \bar{R}(\mu_1, z, -k)\partial_{\bar{z}} u(\mu_2, z, k)\big).$$

Hence the last term in (4.15) vanishes. To finish the proof we estimate:

$$\int_{\partial\mathbb{D}_R} \frac{z}{|z|} e^{ikz} R(\mu_2, z, k)\partial_z \bar{u}(\mu_1, z, -k)\,d\sigma = o(R), \tag{4.16}$$

and

$$\int_{\partial\mathbb{D}_R} \frac{\bar{z}}{|z|} e^{i\bar{k}\bar{z}} \bar{R}(\mu_1, z, -k)\partial_{\bar{z}} u(\mu_2, z, k)\,d\sigma = o(R), \tag{4.17}$$

as $R \to \infty$. We know from (2.48) that

$$R(\mu_1, z, k) = \omega_{\mu_1} + \omega_{-\mu_1} + e_{-k}(\overline{\omega_{\mu_1} - \omega_{-\mu_1}}) \in W^{1,p}(\mathbb{C}).$$

We also have, since $\bar{f}_{\mu_1}$ is antianalytic on the exterior of $\mathbb{D}$,

$$\partial_z \bar{u}(\mu_1, z, k) = 1/2(\partial_z \bar{f}_{\mu_1} + \partial_z \bar{f}_{-\mu_1} + \partial_z f_{\mu_1} - \partial_z f_{-\mu_1}) = 1/2(\partial_z f_{\mu_1} - \partial_z f_{-\mu_1})$$
$$= 1/2 e^{-ikz}\partial_z(\tilde{w}_{\mu_{11}} - \tilde{w}_{\mu_2}).$$

Since $\partial_z(\tilde{w}_{\mu_1} - \tilde{w}_{\mu_2})$ is analytic in the exterior of $\mathbb{D}$ and is in $L^p(\mathbb{C})$, it decays as $O(1/|z|)$. Hence

$$\left|\int_{\partial\mathbb{D}_R} \frac{z}{|z|} e^{ikz} R(\mu_2, z, k)\partial_z \bar{u}(\mu_1, z, -k)\,d\sigma\right| \leqslant C\bigg(\int_{\partial\mathbb{D}_R} \big|R(\mu_2, z, k)\big|^p\bigg)^{1/p}\bigg(\int_{\partial\mathbb{D}_R} \frac{1}{|z|^q}\bigg)^{1/q}$$
$$\leqslant C R^{-1/p}\big\|R(\mu_2, \cdot, k)\big\|_{L^p(\partial\mathbb{D}_R)}.$$

From the trace theorem:

$$\big\|R(\mu_2, \cdot, k)\big\|_{L^p(\partial\mathbb{D}_R)} \leqslant C\|R\|_{W^{1,p}(\mathbb{C})},$$

and (4.16) is proved. (4.17) can be proved in a similar way. □

Remark. It follows from (4.8) that for every $k \in \mathbb{C}$,

$$\big|\tau(\mu_1, k) - \tau(\mu_2, k)\big| \leqslant \frac{1}{|k|}\rho\big\|u(\mu_1)\big\|^2_{H^{1/2}(\partial\mathbb{D})}. \tag{4.18}$$

On the other hand $\tau(\mu_i, 0) = 0$ and the Lipschitz bound given in Theorem 3.12 imply that for every k:

$$\big|\tau(\mu_i, k)\big| \leqslant e^{c(1+|k|}|k|. \tag{4.19}$$

Thus, we can use (4.19) for $|k| \leqslant \rho^{1/2}$ and (4.18) for $|k| \geqslant \rho^{1/2}$ to obtain the Hölder stability:

$$\big|\tau(\mu_1, k) - \tau(\mu_2, k)\big| \leqslant \rho^{1/2} e^{c(1+|k|)}. \tag{4.20}$$

Comparing this expression with Corollary 4.5 seems to indicate that there is room to improvement in (4.4) for small $|k|$.

5. Stability of the complex geometric optics solutions $u(z,k)$

In this section we consider two Beltrami coefficients μ_1, μ_2. Throughout the section we will use $j=1,2$ and assume $|\mu_j| \leqslant \kappa\chi_{\mathbb{D}}$ and that $[\mu_j]_{C^\alpha} \leqslant \Gamma_0$. We will assume that $\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\| \leqslant 1/2$, which is not a loss of generality as shown in the next section. The constant C may change at each occurrence, if the change involves new parameters we will write them explicitly.

Theorem 5.1. *Let us denote $u_j(z,k) = u_{\gamma_j}(z,k)$, given in* (2.41). *Then for each k there exists a constant C such that*

$$\left\|u_1(z,k) - u_2(z,k)\right\|_{L^\infty(\mathbb{D},\mathrm{d}z)} \leqslant \frac{C(k)}{|\log(\rho)|^a}. \tag{5.1}$$

The exponent $a = a(\kappa, \Gamma_0, \alpha)$ is given in (3.36).

Remark. A similar estimate can be proved for the solutions $u_{\gamma^{-1}}$ of:

$$\operatorname{div}(\gamma^{-1}\nabla u) = 0,$$

which are constructed as $u = \mathrm{i}\Re f_{-\mu} - \Im f_\mu$, see (2.42).

From Proposition 2.8, we know that for a conductivity γ_j the corresponding geometric optics solutions to the conductivity equation $u_j = u_{\gamma_j}(z,k)$ satisfies the pseudoanalytic equation in the k variable:

$$\partial_{\bar{k}} u_j = -\mathrm{i}\tau_j \overline{u_j}, \tag{5.2}$$

and can be written as

$$u_j(z,k) = \mathrm{e}^{\delta_j(z,k)}. \tag{5.3}$$

Moreover it follows from Theorem 3.8 that under our regularity conditions on μ the exponent functions have asymptotics:

$$\delta_j(z,k) = \mathrm{i}k\big(z + v_j(z,k)\big) \quad \text{for fixed } k, \tag{5.4}$$

$$\delta_j(z,k) = \mathrm{i}k\big(z + \varepsilon_j(z,k)\big) \quad \text{for fixed } z, \tag{5.5}$$

where $v_j(z,k)$ is in L^∞ and $|\varepsilon_j(z,k)| \leqslant C|k|^{-a}$.

The proof of Theorem 5.1 is inspired by the uniqueness proof in [7]. First we notice that the arguments leading to the uniqueness of the so-called transport matrix in [7], which satisfies a Beltrami type equation in k, can be applied directly to the functions u, that satisfy the pseudoanalytic equation (5.2). This allows us to avoid the use of the transport equation.

It is not known whether uniqueness might be derived directly from Eq. (5.2) in k. However, there is additional information to exploit, since the u_j are functions of z,k with controlled asymptotics in both variables. The asymptotics in z imply that the functions $\delta_j(z,k)$ have range $\mathbb{C}$, see Proposition 2.8. From the asymptotics in k and Eq. (5.2) we obtain that if $\tau_{\mu_1} = \tau_{\mu_2}$, for $k \neq 0$, then $z \neq w$ implies that $\delta_1(z,k) \neq \delta_2(w,k)$. Uniqueness follows then easily from these two facts. For stability we need also the first fact, which we restate as Proposition 5.2. However, we will obtain an appropriate quantitative version of the second fact in Proposition 5.3 below.

Proposition 5.2. *(See [7].) The functions $\delta_j(\cdot,k)$ for $k \neq 0$ have range $\mathbb{C}$.*

Proposition 5.3. *Let $g(z,w,k) = \delta_1(z,k) - \delta_2(w,k)$. Then there exists a constant C such that if $k \neq 0$ and $|z-w| \geqslant \frac{C}{|\log(\rho)|^a}$, then $g(z,w,k) \neq 0$. In particular, this gives that the conditions $g(z,w,k) = 0$ and $k \neq 0$ imply that $|z-w| \leqslant C|\log(\rho)|^{-a}$. The constant $a = a(\kappa, \Gamma_0) > 0$ is given in* (3.36).

We postpone the proof of Proposition 5.3 and prove first Theorem 5.1.

Proof of Theorem 5.1. Let $z \in \mathbb{D}$. We want to estimate:

$$\left|u_1(z,k) - u_2(z,k)\right|.$$

We write $u_j(z,k) = e^{\delta_j(z,k)}$. Since δ_1 is onto there exists $\omega \in \mathbb{C}$ such that $\delta_1(\omega,k) = \delta_2(z,k)$ and hence $u_1(\omega,k) = u_2(z,k)$. Moreover, by Proposition 5.3 $|\omega - z| \leqslant C|\log(\rho)|^{-a}$. Then by Theorem 3.9 u_1 is Lipschitz in $\mathbb{D}(0,2)$ with constant $C(|k|)$. Thus,

$$\left|u_1(z,k) - u_2(z,k)\right| = \left|u_1(z,k) - u_1(\omega,k)\right| \leqslant C\big(|k|\big)C\left|\log(\rho)\right|^{-a},$$

and we are done. □

Let us turn to the proof of Proposition 5.3, which is the heart of the matter. From (5.4) we can write for $\lambda = z - w$:

$$g(z,w,k) = \mathrm{i}\lambda k + k\varepsilon(k), \tag{5.6}$$

where $|\varepsilon(k)| \leqslant 2C|k|^{-a}$. We split the proof of Proposition 5.3 in several lemmas. First we find the equation satisfied by g in the k variable.

Lemma 5.4. *g satisfies the equation*:

$$\partial_{\bar{k}} g = \sigma g + E. \tag{5.7}$$

Where, if τ denotes the scattering transform,

$$\left\|\sigma(z,\cdot)\right\|_{L^\infty(\mathbb{C})} \leqslant 2\left\|\tau(\mu_2)\right\|_{L^\infty(\mathbb{C})} \leqslant 2, \tag{5.8}$$

and E and its derivatives satisfy:

$$\left|E(k)\right| \leqslant \rho e^{C(1+|k|)}, \qquad \left|DE(k)\right| \leqslant e^{C(1+|k|)}. \tag{5.9}$$

Proof. Since,

$$\partial_{\bar{k}}\delta_j = \partial_{\bar{k}}(\log u_j) = \frac{\partial_{\bar{k}} u_j}{u_j} = -\mathrm{i}\tau(\mu_j,k)\frac{\bar{u}_j}{u_j} = -i\tau(\mu_j,k)e^{\bar{\delta}_j - \delta_j},$$

then the $\partial_{\bar{k}}$ derivative of the function g satisfies the equation:

$$\partial_{\bar{k}} g = -\mathrm{i}\big(\tau(\mu_1) - \tau(\mu_2)\big)e^{\bar{\delta}_1 - \delta_1} - \mathrm{i}\tau(\mu_2)\big(e^{\bar{\delta}_1 - \delta_1} - e^{\overline{\delta_2} - \delta_2}\big),$$

which can be written as

$$\partial_{\bar{k}} g = \sigma g + E, \tag{5.10}$$

where

$$\sigma = -\mathrm{i}\tau_{\mu_2}(k)\left[\frac{e^{\bar{\delta}_1 - \delta_1} - e^{\bar{\delta}_2 - \delta_2}}{\delta_1 - \delta_2}\right],$$
$$E(k) = -\mathrm{i}(\tau_{\mu_1} - \tau_{\mu_2})e^{\bar{\delta}_1 - \delta_1}.$$

Since the function $e^{\mathrm{i}\theta}$ is globally Lipschitz for $\theta \in \mathbb{R}$, we can bound the coefficients of this equation as $|\sigma| \leqslant 2|\tau(\mu_2)| < 2$ and since, as we know from the previous section (4.4) $|\tau(\mu_1) - \tau(\mu_2)| \leqslant \rho e^{c|k|}$, we have also $|E(k)| \leqslant \rho e^{ck}$, where $\rho = \|\Lambda_{\sigma_1} - \Lambda_{\sigma_2}\|$ only depending on $\|\mu\|_\infty \leqslant \kappa < 1$. Finally, since $\tau(\mu_1), \tau(\mu_2), \delta_1, \delta_2$ are C^∞ functions on the k variable with bounds given by Theorem 3.12, (5.9) follows. □

In the following lemma we use Eq. (5.7) to decompose g suitably. Let a and C given in (3.36). Then we define a function $R : \mathbb{C} \to R$ by:

$$R(\lambda) = \begin{cases} |\frac{\lambda}{4C}|^{-1/a} & \text{if } |\lambda| \leqslant 4C, \\ 1 & \text{otherwise.} \end{cases} \tag{5.11}$$

This choice guarantees for $|k| \geqslant R(\lambda)$ that in (5.6) we have $|\varepsilon(k)| \leqslant |\lambda|/2$. We emphasize that R depends only on Γ_0 and κ.

Lemma 5.5. *There exists complex valued functions $\eta = \eta(z, w, k)$ and $S = S(z, w, k)$ such that the function,*

$$F = \mathrm{e}^{-\eta} g - S, \tag{5.12}$$

is analytic for $k \in \mathbb{D}(0, R(\lambda))$ and for any $\theta < 1$ there exists a constant C such that

$$\|\eta\|_{L^\infty(\mathbb{C},\mathrm{d}k)} \leqslant CR(\lambda), \tag{5.13}$$

$$\|S\|_{L^\infty(\mathbb{C},\mathrm{d}k)} \leqslant \rho R(\lambda)\mathrm{e}^{R(\lambda)} \leqslant \rho \mathrm{e}^{CR(\lambda)}, \tag{5.14}$$

$$\|\nabla S\|_{L^\infty(\mathbb{C},\mathrm{d}k)} \leqslant \rho^\theta \mathrm{e}^{CR(\lambda)}. \tag{5.15}$$

Proof. By Lemma 5.4 for fixed z and w, g satisfies in k the equation:

$$\partial_{\bar{k}} g = \sigma g + E. \tag{5.16}$$

Let us consider the disc $\mathbb{D}(0, R)$ centered at the origin 0 and radius R. Let us define:

$$\eta = P(\sigma\varphi_R),$$
$$S = P(\mathrm{e}^\eta E\varphi_R),$$

where P is the Cauchy transform and $\varphi_R \in C_0^\infty(\mathbb{D}(0, 2R))$ is a cut-off function such that $\varphi_R = 1$ on $\mathbb{D}(0, R)$. Then the equation $\partial_{\bar{k}} g = \sigma g + E$ is equivalent to

$$\partial_{\bar{k}}(\mathrm{e}^{-\eta} g - S) = 0.$$

Therefore by Weyl's lemma, the function,

$$F(\omega, z, k) = \mathrm{e}^{-\eta} g - S,$$

is holomorphic in $\mathbb{D}(0, R)$. Now, since $\eta = P(\sigma\varphi_R)$,

$$\left|\eta(k)\right| = \left|P(\sigma\varphi_R)\right| \leqslant \left|\int_{\mathbb{D}_{2R}} \frac{\sigma(t)}{k-t}\,\mathrm{d}t\right| \leqslant \int_{\mathbb{D}_{2R}} \frac{2}{|k-t|}\,\mathrm{d}t \leqslant 8\pi R. \tag{5.17}$$

Moreover, since the Beurling transform T maps $L^p \to L^p$, we have that

$$\|\partial_{\bar{k}}\eta\|_{L^p} \leqslant C(p)\|\sigma\varphi_R\|_{L^p} \leqslant C(p)\pi R^{2/p}. \tag{5.18}$$

Now we turn to S. For the L^∞ norm, since $S = P(\mathrm{e}^{-\eta} E\varphi_R)$ it follows that

$$\left|S(k)\right| \leqslant \int_{\mathbb{D}_{2R}} \left|\frac{\mathrm{e}^{-\eta(t)}E(t)}{k-t}\right| \mathrm{d}t \leqslant \rho \mathrm{e}^{c(1+R)}\mathrm{e}^{8\pi R} \int_{\mathbb{D}_{2R}} \frac{\mathrm{d}t}{|k-t|} \leqslant \rho \mathrm{e}^{CR}\rho,$$

where we have used (5.9). To bound the derivatives of S recall that since $S = P(\mathrm{e}^{-\eta} E\varphi_R)$,

$$\partial_{\bar{k}} S = \mathrm{e}^{-\eta} E\varphi_R.$$

Now, let us first notice that by (5.17), (5.9) and the compact support of φ_R,

$$\|\mathrm{e}^{-\eta} E\varphi_R\|_{L^\infty(\mathbb{C})} \leqslant \mathrm{e}^{4\pi R}\rho \mathrm{e}^{CR}. \tag{5.19}$$

Direct application of the boundedness of T would yield L^p (or BMO) bounds for $\partial_k S$. In particular, for $1 < p < \infty$,

$$\|\partial_k S\|_{L^p} = \left\|T(\mathrm{e}^{-\eta} E\varphi_R)\right\|_{L^p} \leqslant C\|\mathrm{e}^{-\eta} E\varphi_R\|_{L^p} \leqslant \rho R^{2/p}\mathrm{e}^{CR} \leqslant \rho \mathrm{e}^{CR}. \tag{5.20}$$

However we need an L^∞ bound. The crucial observation is that combining estimates (5.17), (5.18), (5.19) with (5.9), we obtain that for every $1 < p < \infty$,

$$\|\mathrm{e}^{-\eta} E\varphi_R\|_{W^{1,p}(\mathbb{C})} \leqslant R^{2/p}\left(\mathrm{e}^{4\pi R}\mathrm{e}^{CR} + \rho C(p)\pi\right) \leqslant \mathrm{e}^{CR}, \tag{5.21}$$

where $C = C(p, \kappa)$. Since T is a Calderón–Zygmund integral and also a Fourier multiplier it preserves the spaces $W^{1,p}$ and hence

$$\|\partial_k S\|_{W^{1,p}} \leqslant \mathrm{e}^{CR}. \tag{5.22}$$

Now we intend to use an interpolation argument to combine estimates (5.20) and (5.22). Let $0 \leqslant \theta \leqslant 1$ and $1 < p < \infty$. Then, for example, by complex interpolation [12] or by the wavelets characterization of Sobolev spaces (see [20]) we have that

$$\|\partial_k S\|_{W^{\theta,p}} \leqslant \rho^{(1-\theta)} \mathrm{e}^{\theta CR}.$$

Finally choosing $\theta > 2/p$, $W^{\theta,p} \hookrightarrow L^\infty$ and estimate (5.15) follows with any $\theta < 1$. The proof is concluded. □

Remark. Let us go back for a moment to the uniqueness proof in [7]. If $\rho = 0$, $S = 0$ and we would have $g = \mathrm{e}^\eta F$. Furthermore by the asymptotic behavior of g in (5.6) if $z \neq \omega$,

$$F(k) \sim (z-\omega) k \mathrm{e}^\eta \quad \text{in } \partial\mathbb{D}\big(0, R(\lambda)\big). \tag{5.23}$$

Thus, by the argument principle F has a unique 0. Since the zeros of g are those of F the proof of uniqueness is concluded. This strategy faces two obstacles in the stability setting. First, to have something like (5.23) we need $|z-\omega|$ to be sufficiently big in comparison to the size of S (and hence to the size of ρ). The second obstacle is how to pass information from F to $g = \mathrm{e}^\eta(F+S)$.

The next proposition handles the first obstacle:

Proposition 5.6. *Let a given in* (3.36). *Then there exists a constant C_1 such that if $|\lambda| \geqslant C_1 |\log(\rho)|^{-a}$, then $F(w,z,k) = 0$ only when $k = 0$.*

Proof. We start by proving that $F(k)$ cannot vanish in the set $|k| \leqslant R(\lambda)$ where it is holomorphic.

We will characterize the zeros of F by proving that F is homotopic in the k variable to $\mathrm{e}^{-\eta} \mathrm{i}\lambda k$ in $\partial\mathbb{D}(0, R(\lambda))$, where $R(\lambda)$ was defined in Lemma 5.5. Let $|k| = R(\lambda)$, and $0 \leqslant t \leqslant 1$, then

$$\begin{aligned} \big|tF(k) + (1-t)\mathrm{e}^{-\eta}\mathrm{i}\lambda k\big| &= \big|t\mathrm{e}^{-\eta}(g - \mathrm{i}\lambda k) + \mathrm{e}^{-\eta}\mathrm{i}\lambda k - tS\big| \\ &\geqslant \mathrm{e}^{-\|\eta\|_\infty}\left|\frac{\lambda}{2}k\right| - \max_{|k|=R}\big|S(k)\big|. \end{aligned} \tag{5.24}$$

Suppose that (5.24) is strictly positive.

Then $\deg(F, \mathbb{D}(0, R(\lambda)), 0) = \deg(\mathrm{e}^\eta \lambda k, \mathbb{D}(0, R(\lambda)), 0) = 1$ and being holomorphic, F would have a unique zero at $k = 0$. Therefore, the proof of the proposition will be finished if the following claim holds:

Claim. *Under the assumptions of the proposition*:

$$4\Big(\max_{|k|=R}\big|S(k)\big|\Big)\mathrm{e}^{\|\eta\|_\infty} \leqslant \big|\lambda R(\lambda)\big|. \tag{5.25}$$

To see that this is the case, we observe that Lemma 5.5 implies that

$$4\Big(\max_{|k|=R}\big|S(k)\big|\Big)\mathrm{e}^{\|\eta\|_\infty} \leqslant \rho \mathrm{e}^{CR}\mathrm{e}^{CR} = \mathrm{e}^{CR}\rho.$$

Thus, to attain (5.25) it suffices that

$$\rho \leqslant |\lambda| \mathrm{e}^{-CR(\lambda)}. \tag{5.26}$$

Since $\rho \leqslant 1/2$ and for an appropriate constant C we have that $|\lambda| \geqslant \mathrm{e}^{-CR(\lambda)}$. Therefore the claim follows for a suitable constant C_1.

Finally if $|k| \geqslant R(\lambda)$, $|F(k)| \geqslant \mathrm{e}^{-CR}|\lambda k/2| - \rho \mathrm{e}^{CR}$ which again by (5.26) never vanishes. □

To handle the second obstacle, how to pass information from F to $g = \mathrm{e}^\eta(F+S)$, it turns out that since F is analytic, from its asymptotics we gain more precise information that just the number of zeroes. Namely, in the following two lemmas we prove that, near the origin, it behaves basically as λk.

Lemma 5.7. *Let λ be such that $|\lambda| = |z - w| \geqslant C_1|\log\rho|^{-a}$, for C_1 given in Proposition 5.6. There exists a constant $M_0 > 0$ such that any function $F(w,z,k)$ as in Lemma 5.5, can be written as*

$$F(w,z,k) = \lambda k \mathrm{e}^{\nu(k)}, \tag{5.27}$$

for some function $\nu(k) = \nu(w,z,k)$ which on $|k| \leqslant R(\lambda)$ is holomorphic and satisfies:

$$\left|\nu(k)\right| \leqslant M_0 R(\lambda). \tag{5.28}$$

Proof. The function $F(w,z,k) = \mathrm{e}^{-\eta} g - S$ is analytic in $\mathbb{D}(0,R)$, therefore we might use the maximum principle. We will use the bounds:

$$\|\eta\|_{L^\infty(\mathbb{D}(0,R))} \leqslant CR(\lambda), \tag{5.29}$$

$$\|S\|_{L^\infty(\mathbb{D}(0,R))} \leqslant \rho \mathrm{e}^{CR(\lambda)}. \tag{5.30}$$

On one hand, if $|k| = R(\lambda)$, $g(w,z,k) = \mathrm{i}\lambda k + k\varepsilon(k)$ with $|\varepsilon(k)| < |\lambda|/2$ and we obtain (see the claim in the proof of Proposition 5.6):

$$\left|F(k)\right| \leqslant \frac{3}{2}|\lambda||k|\mathrm{e}^{\|\eta\|_\infty} + \left|S(k)\right| \leqslant |\lambda|R(\lambda)\mathrm{e}^{M_0 R(\lambda)}.$$

On the other hand F is analytic and from Proposition 5.6, only vanishes at $k = 0$ for $|\lambda| > \frac{C_1}{|\log(\rho)|^a}$. Then for such a λ, $\frac{F(k)}{k}$ is holomorphic as well and $\frac{F(k)}{k} \neq 0$ for every k. Therefore there exists a $\nu(k)$ analytic, such that

$$F(k) = \lambda k \mathrm{e}^{\nu(k)}.$$

Moreover, by the maximum principle,

$$\sup_{k\in\mathbb{D}(0,R)}\left|\frac{F(k)}{k}\right| = \sup_{|k|=R}\left|\frac{F(k)}{k}\right| \leqslant |\lambda|\mathrm{e}^{M_0 R(\lambda)}.$$

This proves that $|\nu(k)| \leqslant CR$ if $|k| < R$. □

Let us denote

$$\mathcal{F}_\lambda = \left\{F \in \mathcal{H}\big(\mathbb{D}\big(0, R(\lambda)\big)\big):\ F(k) = \lambda k \mathrm{e}^{\nu(k)},\ \left|\nu(k)\right| < M_0 R(\lambda)\right\}. \tag{5.31}$$

The next lemma describes two key properties of the class $\mathcal{F}_\lambda$. □

Lemma 5.8. *Let $F \in \mathcal{F}_\lambda$, then there exists a constant d such that*

(i) $F^{-1}(\mathbb{D}(0,\delta)) \subset \mathbb{D}(0, \frac{\delta \mathrm{e}^{M_0 R}}{|\lambda|})$;
(ii) $\inf_{|k|<d}|F'(k)| > \frac{1}{2}|\lambda|\mathrm{e}^{-M_0 R(\lambda)}$.

Proof. Part (i) follows directly from the definition of $\mathcal{F}_\lambda$. Let k be such that $|F(k)| < \delta$. Then,

$$\left|F(k)\right| = \left|\lambda k \mathrm{e}^{\nu(k)}\right| < \delta \quad \Rightarrow \quad |k| < \left|\frac{\delta \mathrm{e}^{-\nu(k)}}{\lambda}\right| \leqslant \frac{\delta \mathrm{e}^{M_0 R}}{|\lambda|}.$$

For part (ii) the definition of $\mathcal{F}_\lambda$ implies that

$$\left|F'(k)\right| = \left|\lambda \mathrm{e}^{\nu(k)} + \lambda k \mathrm{e}^{\nu(k)}\nu'(k)\right| \geqslant \mathrm{e}^{-M_0 R}|\lambda|\left|1 + \nu'(k)k\right|. \tag{5.32}$$

Let $|k| \leqslant d_1 \leqslant R(\lambda)/2$. Since $\nu(k)$ is analytic we can use the Cauchy integral formula to estimate:

$$\left|\nu'(k)\right| \leqslant \left|\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathbb{D}(0,R)} \frac{\nu(\omega)}{(\omega - k)^2}\right| \leqslant \frac{4M_0(R)R}{R^2} = 4M_0.$$

Then $|\nu'(k)k| < 4M_0|k| < 1/2$ if $|k| < 1/(8M_0)$ and this implies:

$$\left|1 + \nu'(k)k\right| \geqslant \frac{1}{2}. \tag{5.33}$$

We define $d = \inf\{d_1, 1/(8M_0)\}$.

Inserting (5.33) into (5.32) yields that for $|k| \leqslant d$, $|F'(k)| > \frac{1}{2}|\lambda|\mathrm{e}^{-M_0 R(\lambda)}$. □

The idea to conclude the proof comes from the fact that a linear function λk cannot be intersected twice by a function S, which is sufficiently small in $W^{1,\infty}$ and with $S(0) = 0$. By Lemma 5.8 one expects the same for $F \in \mathcal{F}_\lambda$, to see this we show first that if $(-S)$ and F meet they must do it in a neighborhood of the origin.

Lemma 5.9. *Let $H = F + S$. Then there exists C_2 such that if $|\lambda| > C_2|\log\rho|^{-a}$ then the set*

$$Z(H) = \{k\colon\ H(k) = 0\}$$

is contained in $\mathbb{D}(0, d)$, d is given in the previous lemma.

Proof. If k is a zero of H, then $F(k) = -S(k)$. But we know from Lemma 5.5 that

$$\|S(k)\|_{L^\infty(\mathbb{C},\mathrm{d}k)} \leqslant \rho\mathrm{e}^{CR}.$$

Thus, it follows that

$$Z(F+S) \subset F^{-1}\big(\mathbb{D}(0, \rho\mathrm{e}^{CR})\big) \subset \mathbb{D}\left(0, \frac{\rho}{|\lambda|}\mathrm{e}^{CR}\right),$$

where the last inequality follows from Lemma 5.8 part (i). Therefore we need λ to satisfy that

$$\rho \leqslant d|\lambda|\mathrm{e}^{-CR}, \tag{5.34}$$

which can be attained as (5.26). □

Next, we prove that the Jacobians DF and $(-DS)$ cannot meet near 0.

Proposition 5.10. *There exists C_3 such that for $|\lambda| > C_3|\log(\rho)|^{-a}$, $\det DH(k) \neq 0$ for every $|k| < d$.*

Proof. Choose first $|\lambda| > C_2|\log(\rho)|^{-a}$. We have:

$$\det DH(k) = |\partial_k H|^2 - |\partial_{\bar{k}} H|^2 \geqslant |F'|^2 - |DS|^2.$$

Let $|k| < d$. Then, Lemma 5.8(ii) and Lemma 5.5 condition (5.15) imply that

$$\det DH(k) \geqslant |\lambda|\mathrm{e}^{-M_0 R(\lambda)} - \rho^{2\theta}\mathrm{e}^{CR(\lambda)}, \tag{5.35}$$

which is not zero if $\rho^{2\theta} < |\lambda|\mathrm{e}^{-(M_0+C)R(\lambda)}$. By choosing a new constant C_3 the proposition is proved. □

Let us put all our knowledge together and conclude by a degree argument:

End of the proof of Proposition 5.3. Choose $\lambda \geqslant \frac{C_3}{|\log(\rho)|^a}$. By Lemma 5.9 the zeros of $g = \mathrm{e}^\eta H$ belong also to $\mathbb{D}(0, d)$. Now, since

$$g(w, z, k) = \mathrm{i}\lambda k + \varepsilon(k)k,$$

for $|k| = R(\lambda)$, $g(w, z, k)$ is homotopic to λk, then

$$\deg\big(g, \mathbb{D}\big(0, R(\lambda)\big), 0\big) = 1,$$

and since $g = \mathrm{e}^\eta(F + S)$ with e^η continuous,

$$\deg\big(F + S, \mathbb{D}\big(0, R(\lambda)\big), 0\big) = 1.$$

From Lemma 5.9 the zeroes of $H = F + S$ are in $\mathbb{D}(0, d)$ where $\det(H) \neq 0$. Thus, since $H \in C^1$ we can express the Brouwder degree by the formula:

$$1 = \deg\big(H, \partial B\big(0, R(\lambda)\big), 0\big) = \sum_{k_i \in Z(H)} \operatorname{Ind}(H, k_i) = \sum_{k_i \in Z(H)} \operatorname{sign}\det DH(k_i).$$

Suppose that there exists more that one zero. This would mean, that there exists $k_i \in Z(H) \setminus \{0\}$ such that $\det DH(k_i) < 0$ and, by continuity of the determinant, there would exists $t \in (0,1)$ such that $\det(H(tk_2)) = 0$. We arrive to a contradiction with Proposition 5.10. Thus if $|\lambda| > C_3|\log\rho|^{-a}$ there is a unique zero of g and the proof is concluded with $C = C_3$ $\square$

6. The proof of Theorem 1.1

Let γ_j be two conductivities satisfying the ellipticity (I) and regularity conditions (II) in Theorem 1.1. We start by showing that there is no loss of generality assuming that the $\gamma_j - 1$ are compactly supported in Ω and that $\Omega = \mathbb{D}$. The key point is the stable recovery of the values of the conductivities on $\partial\Omega$ from the Dirichlet to Neumann map. This is the content of the following proposition which follows easily from [14]. We would like to thank R. Brown for this personal communication. The theorem is stated for hypothesis adapted to our conductivities.

Proposition 6.1. *(See [14].) Let Ω be a Lipschitz domain and γ_1, γ_2 two conductivities in $\mathcal{C}(\bar{\Omega})$ Then there exists $C = C(\Omega, \kappa)$ such that*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leqslant C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}. \tag{6.1}$$

The stable boundary recovery has been studied in several works, for instance for $\gamma_j \in W^{1,p}$, some $p > 2$ and Ω a Lipschitz domain, see [3] and [29] and for continuous conductivities and smooth domains in [35]. By a combination of Proposition 6.1 and the bilinear weak formulation of the Dirichlet to Neumann map we prove that we can reduce to the case where $\gamma_j - 1$ compactly supported in $\mathbb{D}$.

Theorem 6.2. *Let Ω be a Lipschitz domain, $\Omega \Subset \mathbb{D}$. Let us denote*:

$$\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}.$$

There exist extensions $\tilde{\gamma}_j$ of γ_j such that

$$\|\gamma_j\|_{\mathcal{C}^\alpha(\mathbb{C})} < C\Gamma_0, \tag{6.2}$$

$$\|\Lambda_{\tilde{\gamma}_1} - \Lambda_{\tilde{\gamma}_2}\|_{H^{1/2}(\partial\mathbb{D})\to H^{-1/2}(\partial\mathbb{D})} \leqslant C\rho, \tag{6.3}$$

$$\operatorname{supp}(\tilde{\gamma}_j - 1) \subset \mathbb{D}. \tag{6.4}$$

Proof. We use the Whitney extension operator $\mathcal{E}_0$, see [32, p. 174]. We consider the closed set $F = \partial\Omega \cup (\mathbb{C} \setminus \mathbb{D}(r_0))$ for some $r_0 < 1$ such that $\Omega \Subset \mathbb{D}(r_0)$ and take the functions, defined on F, given by:

$$f_j(z) = \begin{cases} \gamma_j & \text{for } z \in \partial\Omega, \\ 1 & \text{for } z \in \mathbb{C} \setminus \mathbb{D}(r_0). \end{cases}$$

We take the Whitney extensions $\mathcal{E}_0(f_j)$ and define the extended conductivities as

$$\tilde{\gamma}_j(z) = \begin{cases} \gamma_j(z) & \text{for } z \in \Omega, \\ \mathcal{E}_0(f_j) & \text{for } z \in \mathbb{C} \setminus \Omega. \end{cases} \tag{6.5}$$

The condition (6.2) follows from the continuity of Whitney extension on $\mathcal{C}^\alpha$, $0 < \alpha < 1$.

From the linearity of Whitney extension we have:

$$\|\tilde{\gamma}_1 - \tilde{\gamma}_2\|_{L^\infty(\mathbb{D}\setminus\Omega)} = \big\|\mathcal{E}_0(f_1 - f_2)\big\|_{L^\infty(\mathbb{D}\setminus\Omega)} \leqslant \|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)}. \tag{6.6}$$

By (6.1) we have

$$\|\tilde{\gamma}_1 - \tilde{\gamma}_2\|_{L^\infty(\mathbb{D}\setminus\Omega)} \leqslant C\rho. \tag{6.7}$$

Now we are in conditions to prove (6.3). Let $\varphi_0 \in H^{1/2}(\partial\mathbb{D})$ and let $\tilde{u}_j \in H^1(\mathbb{D})$, for $j = 1, 2$, be the solutions to,

$$\begin{cases} \nabla\cdot(\tilde{\gamma}_j\nabla\tilde{u}_j) = 0, \\ \tilde{u}_j|_{\partial\mathbb{D}} = \varphi_0. \end{cases} \tag{6.8}$$

Consider u_2 be the solution to,

$$\begin{cases} \nabla\cdot(\gamma_2\nabla u_2)=0, \\ u_2|_{\partial\Omega}=\tilde u_1, \end{cases} \tag{6.9}$$

and declare:

$$\tilde v_2 = u_2\chi_\Omega + \tilde u_1\chi_{\mathbb{D}\setminus\Omega}.$$

The idea is that if $\rho=0$, in fact $\tilde v_2=\tilde u_2$. Thus, it is natural to conceive that $\int_{\mathbb{D}}|\nabla(\tilde v_2-\tilde u_2)|^2$ might be controlled in terms of ρ. In fact,

$$\int_{\mathbb{D}}\left|\nabla(\tilde v_2-\tilde u_2)\right|^2 \leqslant c\int_{\mathbb{D}}\tilde\gamma_2\left\langle\nabla(\tilde v_2-\tilde u_2),\nabla(\tilde v_2-\tilde u_2)\right\rangle = \int_{\mathbb{D}}\tilde\gamma_2\left\langle\nabla\tilde v_2,\nabla(\tilde v_2-\tilde u_2)\right\rangle.$$

Adding and subtracting $\tilde{\gamma_1}\nabla\tilde{u_1}$ we get:

$$\begin{aligned}\int_{\mathbb{D}}\left|\nabla(\tilde v_2-\tilde u_2)\right|^2 \leqslant{}& \left|\int_{\mathbb{D}}\left\langle\tilde\gamma_1\nabla\tilde u_1,\nabla(\tilde v_2-\tilde u_2)\right\rangle\right| + \left|\int_{\Omega}\left\langle\gamma_1\nabla\tilde u_1-\gamma_2\nabla u_2,\nabla(\tilde v_2-\tilde u_2)\right\rangle\right| \\ &+\left|\int_{\mathbb{D}\setminus\Omega}(\tilde\gamma_2-\tilde\gamma_1)\left\langle\nabla\tilde v_2,\nabla(\tilde v_2-\tilde u_2)\right\rangle\right|.\end{aligned} \tag{6.10}$$

The first term vanishes by (6.8). For the second we use the definition of ρ. Namely,

$$\begin{aligned}\left|\int_{\Omega}\left\langle\gamma_1\nabla\tilde u_1-\gamma_2\nabla u_2,\nabla(\tilde v_2-\tilde u_2)\right\rangle\right| &= \left|\left\langle(\Lambda_{\gamma_1}-\Lambda_{\gamma_2})(\tilde u_1|_{\partial\Omega}),(\tilde v_2-\tilde u_2)|_{\partial\Omega}\right\rangle\right| \\ &\leqslant \rho\|\tilde v_2-\tilde u_2\|_{H^1(\Omega)}\|\tilde u_1\|_{H^1(\mathbb{D})} \\ &\leqslant \rho\left\|\nabla(\tilde v_2-\tilde u_2)\right\|_{L^2(\mathbb{D})}\|\tilde u_1\|_{H^1(\mathbb{D})}.\end{aligned} \tag{6.11}$$

Here the Dirichlet to Neumann mappings Λ_{γ_j} are taken on $\partial\Omega$. Finally, from (6.7),

$$\begin{aligned}\int_{\mathbb{D}\setminus\Omega}(\tilde\gamma_2-\tilde\gamma_1)\left\langle\nabla\tilde v_2,\nabla(\tilde v_2-\tilde u_2)\right\rangle &\leqslant \|\tilde\gamma_2-\tilde\gamma_1\|_{L^\infty(\mathbb{D}\setminus\Omega)}\|\nabla\tilde v_2\|_{L^2(\mathbb{D}\setminus\Omega)}\|\nabla(\tilde v_2-\tilde u_2)\|_{L^2(\mathbb{D})} \\ &\leqslant \rho\left\|\nabla(\tilde v_2-\tilde u_2)\right\|_{L^2(\mathbb{D})}\|\tilde u_1\|_{H^1(\mathbb{D})}.\end{aligned} \tag{6.12}$$

Then (6.10), (6.11), (6.12) together yield that

$$\left(\int_{\mathbb{D}}\left|\nabla(\tilde v_2-\tilde u_2)\right|^2\right)^{1/2} \leqslant \rho\|\tilde u_1\|_{H^1(\mathbb{D})} \leqslant \rho\|\varphi_0\|_{H^{1/2}(\partial\mathbb{D})}. \tag{6.13}$$

Our last task is to compare the Dirichlet to Neumann mappings on $\partial\mathbb{D}$, $\Lambda_{\tilde\gamma_j}$. Let $\varphi_0,\psi_0\in H^{1/2}(\partial\mathbb{D})$ and $\psi\in H^1(\partial\mathbb{D})$ an extension of ψ_0. Then

$$\left\langle(\Lambda_{\tilde\gamma_1}-\Lambda_{\tilde\gamma_2})(\varphi_0),\psi_0\right\rangle = \int_{\mathbb{D}}\langle\tilde\gamma_1\nabla\tilde u_1-\tilde\gamma_2\nabla\tilde u_2,\nabla\psi\rangle. \tag{6.14}$$

Now we want to add and subtract $(\gamma_2\chi_\Omega+\tilde\gamma_1\chi_{\mathbb{D}\setminus\Omega})\nabla\tilde v_2$. Since we have:

$$\begin{aligned}\left|\int_{\mathbb{D}}\left\langle\tilde\gamma_1\nabla\tilde u_1-(\gamma_2\chi_\Omega+\tilde\gamma_1\chi_{\mathbb{D}\setminus\Omega})\nabla\tilde v_2,\nabla\psi\right\rangle\right| &= \left|\int_{\Omega}\langle\gamma_1\nabla\tilde u_1-\gamma_2\nabla u_2,\nabla\psi\rangle\right| \\ &= \left|\left\langle(\Lambda_{\gamma_1}-\Lambda_{\gamma_2})(u_1|_{\partial\Omega}),\psi|_{\partial\Omega}\right\rangle\right| \leqslant \rho\|\tilde u_1\|_{H^1(\mathbb{D})}\|\psi\|_{H^1(\mathbb{D})} \\ &\leqslant \rho\|\varphi_0\|_{H^{1/2}(\partial\mathbb{D})}\|\psi_0\|_{H^{1/2}(\partial\mathbb{D})}.\end{aligned} \tag{6.15}$$

We can obtain:

$$\left|\left\langle(\Lambda_{\tilde\gamma_1}-\Lambda_{\tilde\gamma_2})(\varphi_0),\psi_0\right\rangle\right| \leqslant \rho\|\tilde u_1\|_{H^1(\mathbb{D})}\|\psi\|_{H^1(\mathbb{D})}+\left|\int_{\mathbb{D}}\left\langle(\gamma_2\chi_\Omega+\tilde\gamma_1\chi_{\mathbb{D}\setminus\Omega})\nabla\tilde v_2-\tilde\gamma_2\nabla\tilde u_2,\nabla\psi\right\rangle\right|. \tag{6.16}$$

The second term is majorized by:

$$\left|\int_{\Omega}\left\langle\gamma_2\nabla(\tilde v_2-\tilde u_2),\nabla\psi\right\rangle\right|+\left|\int_{\mathbb{D}\setminus\Omega}\left\langle\tilde\gamma_2\nabla\tilde u_2-\tilde\gamma_1\nabla\tilde u_1,\nabla\psi\right\rangle\right|, \tag{6.17}$$

which using (6.7) and (6.13) is controlled by:

$$C\rho\|\tilde u_1\|_{H^1(\mathbb{D})}\|\psi\|_{H^1(\mathbb{D})}\leqslant C\rho\|\varphi_0\|_{H^{1/2}(\partial\mathbb{D})}\|\psi_0\|_{H^{1/2}(\partial\mathbb{D})}. \tag{6.18}$$

Therefore we arrive to,

$$\left|\left\langle(\Lambda_{\tilde\gamma_1}-\Lambda_{\tilde\gamma_2})(\varphi_0),\psi_0\right\rangle\right|\leqslant C\rho\|\varphi_0\|_{H^{1/2}(\partial\mathbb{D})}\|\psi_0\|_{H^{1/2}(\partial\mathbb{D})}, \tag{6.19}$$

which is equivalent to (6.3). The proof is concluded □

We now come back to the complex geometric optic solutions arising from μ_j compactly supported in $\mathbb{D}=\Omega$. So far we have obtained the stability of the complex geometric optics solutions, this suffices for uniqueness. But for stability we need to go further and obtain the stability of the derivatives of the functions f_μ, since the Beltrami coefficients are:

$$\mu=\partial_{\bar z}f_\mu/\overline{\partial_z f_\mu}.$$

By using an interpolation argument we extend the stability in the L^∞ norm proved in Theorem 5.1 to stability in the $W^{1,\infty}$ norm. We still require $\rho\leqslant 1/2$. This requirement will be removed at the end of the proof.

Proposition 6.3. *Let f_{μ_1}, f_{μ_2} be the complex geometric optic solutions. Then there exist constants $a=a(\kappa,\Gamma_0,\alpha)$ and $C(|k|)=C(\kappa,\Gamma_0,\alpha,|k|)$ such that*

$$\left\|f_{\mu_1}(z,k)-f_{\mu_2}(z,k)\right\|_{W^{1,\infty}(\mathbb{D},\mathrm{d}z)}\leqslant C\big(|k|\big)|\log\rho|^{-a}.$$

Proof. Let us start by noticing that, by (2.41), (2.42), the stability of the geometric optic solutions for both Beltrami and conductivity equations are equivalent. Thus, Theorem 5.1 implies that

$$\left\|f_{\mu_1}(z,k)-f_{\mu_2}(z,k)\right\|_{L^\infty(\mathbb{D},\mathrm{d}z)}\leqslant C\left|\log(\rho)\right|^{-a}. \tag{6.20}$$

On the other hand by Theorem 3.9 we know that for $\varepsilon<\alpha$:

$$\left\|f_{\mu_1}(z,k)-f_{\mu_2}(z,k)\right\|_{C^{1+\varepsilon}(\mathbb{D},\mathrm{d}z)}\leqslant C\big(|k|\big). \tag{6.21}$$

Let us consider the function $U=(f_{\mu_1}(z,k)-f_{\mu_2}(z,k))\varphi$ where $\varphi\in C_0^\infty(\mathbb{D}(0,2))$ is a cut-off function $\varphi\chi_{\mathbb{D}}=1$. Then for every $1<p<\infty$, (6.21) and (6.20) imply that

$$\|U\|_{W^{1+\varepsilon,p}(\mathbb{C},\mathrm{d}z)}\leqslant C\big(|k|\big)\quad\text{and}\quad\|U\|_{L^p(\mathbb{C},\mathrm{d}z)}\leqslant C\big(|k|\big)|\log\rho|^{-a}, \tag{6.22}$$

which calls for an interpolation argument. By interpolation between L^p and $W^{1+\varepsilon,p}$ we can obtain estimates for $\|U\|_{W^{\theta(1+\varepsilon),p}}$ and consequently for $\|DU\|_{W^{\theta(1+\varepsilon)-1,p}}$. If we further require that $\theta(1+\varepsilon)-1>2/p$, by Sobolev embedding we also control U in L^∞. Namely under these conditions we have:

$$\begin{aligned}\|DU\|_{L^\infty(\mathbb{D},\mathrm{d}z)}&\leqslant C\|DU\|_{W^{\theta(1+\varepsilon)-1,p}(\mathbb{C},\mathrm{d}z)}\leqslant C\|U\|_{W^{\theta(1+\varepsilon),p}(\mathbb{C},\mathrm{d}z)}\\&\leqslant\|U\|_{L^p}^{1-\theta}\|U\|_{W^{1+\varepsilon,p}}^{\theta}\leqslant\left|\log(\rho)\right|^{-a(1-\theta)}C\big(|k|\big).\end{aligned} \tag{6.23}$$

Since we can take any $1<p<\infty$ the estimate is valid for any $\theta<1/(1+\varepsilon)$. □

Proof of Theorem 1.1. By Theorem 6.2 we can reduce to conductivities γ_j such that $\gamma_j - 1$ are compactly supported in $\Omega = \mathbb{D}$. It is enough to control the difference of the Beltrami coefficients μ_j, since $\gamma = (\mu+1)/(1-\mu)$ implies that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\mathbb{D})} \leqslant \frac{4}{1-\kappa^2}\|\mu_1 - \mu_2\|_{L^\infty(\mathbb{D})}. \tag{6.24}$$

We can also assume that $\rho \leqslant 1/2$. Otherwise $\rho \geqslant \frac{1}{4\kappa}\|\mu_1 - \mu_2\|_{L^\infty(\mathbb{D})}$ and the claim follows. Therefore Proposition 6.3 can be applied. We only need the complex geometric optics solutions with fixed $k = 1$. Thus in the rest of the proof we take $f_\mu(z) = f_\mu(z, 1)$. The stability of the Beltrami coefficients is reduced to the stability of the derivatives. Since, from the definition of the Beltrami coefficients,

$$\|\mu_1 - \mu_2\|_{L^\infty(\mathbb{D})} = \|\partial_{\bar z} f_{\mu_1}/\overline{\partial_z f_{\mu_1}} - \partial_{\bar z} f_{\mu_2}/\overline{\partial_z f_{\mu_2}}\|_{L^\infty(\mathbb{D})}. \tag{6.25}$$

Now we use the regularity of the solutions. By Theorem 3.9 there exists a uniform constant $m > 0$ depending on Γ_0 such that for every μ,

$$\inf_{\mathbb{D}} |\partial_z f_\mu| \geqslant m,$$

and by (6.21) there exists another uniform constant,

$$\max_{\mathbb{D}} |Df_\mu| \leqslant M$$

Thus,

$$\begin{aligned}\|\partial_{\bar z} f_{\mu_1}/\overline{\partial_z f_{\mu_1}} - \partial_{\bar z} f_{\mu_2}/\overline{\partial_z f_{\mu_2}}\|_{L^\infty(\mathbb{D})} &\leqslant \frac{M}{m}\|Df_{\mu_1} - Df_{\mu_2}\|_{L^\infty(\mathbb{D})} \\ &\leqslant C\big|\log(\rho)\big|^{-a},\end{aligned} \tag{6.26}$$

and the theorem is proved. □

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