Lists of Face-Regular Polyhedra

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We introduce a new notion that connects the combinatorial concept of regularity with the geometrical notion of face transitivity. This new notion implies finiteness results in the case of bounded maximal face size. We give lists of structures for some classes and investigate polyhedra with constant vertex degrees and faces of only two sizes.

INTRODUCTION

Polyhedra are well-studied objects in mathematics[1,2] and also play an important role as models for chemical molecules.[3,4] The discovery of the polyhedral fullerene C_{60}[5] increased the interest in polyhedra more (see ref 6 for a survey on this topic).

A planar (finite or infinite) graph is called *face transitive* if the automorphism group acts transitively on the set of faces. For finite polyhedra (see ref 7) as well as for infinite graphs in the plane with finite faces and finite vertex degrees (that is, tilings, see refs 8 and 9), it is well-known that the graph can be realized with its full combinatorial automorphism group as its group of geometrical symmetries. Restricting our attention to polyhedra with constant vertex degrees, up to combinatorial equivalence, only the five *platonic solids* have an automorphism group acting transitively on their faces. In the remaining text, we will restrict our attention to polyhedra with constant vertex degrees.

A natural generalization of this concept—let us call it *weakly face transitive*—is to require that only faces of the same size are equivalent under the automorphism group. This concept can be further relaxed by using the notion of a *corona* (see ref 10 for an example where coronas are used for graph invariants): If we define the 0th corona of a face to be the face itself and the nth corona to be the set of all those faces that are contained in the $(n - 1)$th corona or share an edge with it, we can require only some coronas of fixed size to be isomorphic by an isomorphism mapping of the central faces onto each other. A polyhedron with all n coronas of faces of the same size isomorphic is called *weakly n-transitive*. Obviously, all polyhedra are weakly 0-transitive, and if a polyhedron is weakly $(n + 1)$-transitive, it is also weakly n-transitive. So the first interesting case to study is the case of weakly 1-transitive polyhedra. Relaxing this condition by not requiring the first coronas to be isomorphic, but just to be isomorphic as multisets (that is, every face of a given size i must have the same number of neighbors of size i' for every i'), still gives a very restrictive condition, and as we will see, it already implies finiteness in the case where the maximal size of a face is bounded. We call this condition (strong) *face regularity*. Therefore, the class of all face-regular polyhedra contains all weakly n-transitive polyhedra for any $n \geq 1$ and also the weakly face-transitive or even face-transitive ones.

The same concept can be reached by strengthening the notion of a regular dual graph induced by faces of a certain size: Let p_i denote the number of i -gons in a given polyhedron. We use the notation $p = (p_3, p_4, ..., p_i, ...p_b)$ for the face vector (or p-vector) of a polyhedron; b is the maximal number for which a face with size b exists.

A less restrictive definition of face regularity, but only for bifaced polyhedra, was considered in ref 11. Namely, if only p_a and p_b are nonzero and $a < b$, then the number of i faces, edge-adjacent to any given i face, was required to be independent of the choice of the i face, for $i = a$ or $i = b$. For a k-valent polyhedron, we write aR_f or bR_f, if this partial (or *weak*) face regularity holds for a-gonal or, respectively, b-gonal faces. All such simple polyhedra with $b \leq 6$, as well as all four-valent ones with $b = 4$, except the cases $4R_0$ for $(k;a,b) = (3;4,6)$ and aR_0 and aR_1 for $(k;a,b) \in\{(4;3,4), (3;5,6)\}$, were found in ref 11. For example, all 12 (respectively 6, 4, 10, and 26) polyhedra bR_f for all 5 possible cases—$k = 4$; $k = 3$, $b < 6$ and $k = 3$, $b = 6$, $a \in\{3,4,5\}$—are listed there. (The graphs of all 26 $6R_f$ *fullerenes* (i.e., $(k;a,b)=(3;5,6)$) are given in list 6 (Figure 6) below.) In these cases, eight (respectively 6, 4, 9, and 12) polyhedra are also aR_f, i.e., face regular in the sense of the present paper.

The case $5R_0$ represents the chemically especially interesting set of fullerenes that fulfill the *isolated pentagon rule* (see refs 12−14).

The face regularity which we consider is a purely combinatorial property of the skeleton of a polyhedron. It is different from the affine notion of *regular-faced* (i.e., all faces being regular polygons) polyhedra.

We use the abbreviation *frp* for *face-regular polyhedron*. An frp in one of the lists below is described by i_j, where j is

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the number of the list and i is its number in list j. We also use the notation i for i_1.

We call two frp *fr isomers* if they have the same parameters as frp, i.e., v, the p-vector and the numbers $f(a,b)$ (i.e., the number of b-faces, edge-adjacent to each a-face for any a,b) coincide.

All fr isomers in list 1 are bifaced. They are 11, 12(v = 16); 20, 21 (v = 32); 32, 33 (v = 80); and 3-faced 49, 50 (v = 20).

All fr isomers in list 2 are:

for $v = 20$: $10_2, 11_2$
for $v = 24$: $61_2, 62_2$
for $v = 26$: 16_2-19_2
for $v = 28$: $66_2, 67_2$; $69_2, 70_2$; 72_2-74_2; $75_2, 77_2$; $76_2, 78_2$
for $v = 32$: 28_2-31_2; 32_2-34_2; $87_2, 88_2$; $89_2, 90_2$
for $v = 36$: $2_2, 3_2$; $95_2, 96_2$; $102_2, 103_2$
for $v = 40$: $42_2, 43_2$
for $v = 44$: $5_2, 6_2$; 49_2-51_2; $119_2, 120_2$; 137_2-139_2

All fr isomers in lists 4 and 5 are 6_4, 7_4 with $v = 14$.

Considering the polyhedra of lists 1−3 with respect to collapsing all triangular faces to points (i.e., the inverse to vertex truncation), we see that in list 1, any such collapsing gives a member of list 1. But in list 2 there are polyhedra, such that this collapsing does not give an frp. The smallest one is 116_2.

Examples of sequences of frp, such that each of them comes from the previous one by one-edge truncation, are 1, 4, 2, 6, 7, 8, 9; 1, 4, 35, 36, 59, 39, 11; and 1, 4, 2, 6, 14, 4_2.

Some infinite families of three-valent frp:

bifaced: Prism$_n$ and Barrel$_n$ (i.e., two n-gons separated by two layers of 5-gons)

three-faced: Prism$_n$; Barrel$_n$, truncated on all $2n$ vertices of both n-gons; Prism$_{2n}$, edge-truncated on n disjoint edges of only one n-gon; Prism$_{3n}$, edge-truncated on n edges, separated by at least two edges, of only one n-gon

four-faced: Prism$_n$, (vertex) truncated on all vertices of only one n-gon

five-faced: Barrel$_n$, truncated on all vertices of only one n-gon

In fact, many of the frp in the lists are some partial truncations of prism$_n$ and barrel$_n$. For example, there are exactly 10 frp, which are partial truncations of the cube: There is 1 (respectively 3,1,3,1,1) possibility for truncations on 1 (respectively 2,3,4,6,8) vertex.

Remarks. (i) Among the chiral polyhedra in the lists are, for example, 41, 61, 62, 63, 100, and 104 in list 2; 9 in list 3; and, especially, 13, 22, and 34 in list 1 and 9 in list 4 with symmetry T, O, I, and O, respectively.

(ii) None of the polyhedra in any of our lists have a trivial symmetry group.

2. FINITENESS OF CLASSES WITH BOUNDED FACE SIZE

Theorem 1. *For every n there is only a finite number of face-regular polyhedra with constant vertex degrees and face sizes not exceeding n.*

Proof. We will assume that the polyhedra in question all contain an n-gon. The total number can be obtained by summing over all $m \le n$.

Recall that for i, j, the number $f(i,j)$ denotes the number of neighboring j-gons of an i-gon. So $f(i,j)p_i = f(j,i)p_j$ is the number of edges between i-gons and j-gons, and we can express p_j as $p_j = [f(i,j)/f(j,i)]p_i$ in the case where i-gonal and j-gonal faces share at least one edge.

Look at the graph G with vertex set $V = \{i|p_i > 0\}$ and edge set $E = \{\{i,j\}|f(i,j) > 0\}$. This graph is connected since the dual of the underlying polyhedron is connected. We can express every other value p_i by a formula of the kind

$$\frac{f(i_1,i)}{f(i,i_1)}\frac{f(i_2,i_1)}{f(i_1,i_2)}\cdots\frac{f(i_b,i_k)}{f(i_k,i_b)}p_n =: g(i)p_n$$

if for i, i_1, ..., i_k, n is a (e.g., shortest) path from i to n in G.

Since for fixed n all the $f(i,j)$ as well as the length of the path are bounded and since the number of graphs on n vertexes is also finite, we have only a finite number of possible sets of equations $p_i = g(i)p_n$ ($3 \le i \le n$).

As a well-known consequence of Euler's formula, we get $\sum_{i=3}^{n} (6 - i)p_i = 12$ in the three-valent case, $\sum_{i=3}^{n} (4 - i)p_i = 8$ for four-valent polyhedra, and $\sum_{i=3}^{n} (10 - 3i)p_i = 20$ for five-valent polyhedra.

Substituting p_i by $g(i)p_n$ in this formula, every set of equations gives exactly one solution for p_n and, therefore, also for each p_i. So for every set of equations, there is a well-determined number of faces and, therefore, there is a maximum number of faces that is possible.

Corollary 1. *If in the cubic case the number of nonhexagons is bounded or in the quartic case the number of nonsquares is bounded, then there is only a finite number of face-regular polyhedra.*

Proof. The fact that the number of faces smaller than six (respectively four) is bounded gives an upper bound on the maximum face size, implying the result by the previous theorem.

3. STATISTICS

In this section, we will give some statistics about the number of face-regular polyhedra compared to the number of all polyhedra for some classes. These are given in Table 1−3.

4. LIST 1: ALL 64 FACE-REGULAR SIMPLE POLYHEDRA WITH $b \le 6$

Among the 64 polyhedra of list 1 (Figure 1), the first 3 are regular and then there are 31 bifaced ones: 6 with $b \le 5$, 4 3_n (for n = 12, 16, 16, 26), 9 4_n (for n = 12, 14, 20, 20, 24, 26, 32, 32, 36), and 12 fullerenes 5_n (which are $F_{24}(D_{6d})$, $F_{28}(T_d)$, $F_{32}(D_{3h})$, $F_{38}(C_{3v})$, $F_{44}(T)$, $F_{52}(T)$, $F_{56}(T_d)$, $F_{60}(I_h)$,

Table 1. Cubic Polyhedra

vertices	polyhedra	face-reg polyhedra
4	1	1
6	1	1
8	2	2
10	5	4
12	14	7
14	50	5
16	233	15
18	1 249	9
20	7 595	33
22	49 566	11
24	339 722	58
26	2 406 841	29
28	17 490 241	99
30	129 664 753	44
32	977 526 957	194
34	7 475 907 149	25
36	57 896 349 553	318

Table 2. Cubic Polyhedra without Triangles

vertices	polyhedra	face-reg polyhedra
8	1	1
10	1	1
12	2	2
14	5	3
16	12	3
18	34	1
20	130	10
22	525	2
24	2 472	8
26	12 400	5
28	65 619	10
30	357 504	7
32	1 992 985	30
34	11 284 042	1
36	64 719 885	22
38	375 126 827	16
40	2 194 439 398	18

Table 3. Cubic Polyhedra without Faces Larger Than a Hexagon[a]

vertices	polyhedra	face-reg polyhedra
4	1	1
6	1	1
8	2	2
10	5	4
12	10	7
14	15	3
16	30	7
18	44	2
20	77	10
24	184	6
26	267	2
28	420	3
30	595	1
32	883	5
38	2 445	1
44	6 319	1
52	19 345	1
56	32 219	2
60	52 293	1
68	128 343	1
80	425 998	2
140	???	1

[a] For all vertex numbers not mentioned, no face-regular polyhedra exist.

$F_{68}(T_d)$, $F_{80}(I_h)$, $F_{80}(D_{5h})$, $F_{140}(I)$). Numbers 35−57 have 3 types of faces and the last 7 polyhedra, 58−64, have 4 types of faces. 3_n, 4_n, 5_n above are simple polyhedra with only 3-, 4-, 5-yons besides 6-yons.

Among the 64 polyhedra of list 1, 3 are regular ones (tetrahedron, cube, and dodecahedron), 5 are semiregular (3-, 5-, 6-gonal prisms, truncated octahehedron, and truncated icosahedron), and none coincides with an affinely regular-faced polyhedron, such as are comprehensively identified in ref 15 including the list of 92 exceptional ones. But there are three that are dual to the regular-faced snub disphenoid, 3-augmented 3-gonal prism, and gyroelongated square dipyramid (with the last three having numbers 84, 51, and 17, respectively, in the list of ref 15). Together with three regular ones and 3- and 5-gonal prisms, it gives the duals of all eight convex deltahedra.

5. LIST 2: ALL 160 FACE-REGULAR SIMPLE POLYHEDRA WITH $b = 7$ AND UP TO 24 FACES

Figure 2 lists all 160 face-regular simple polyhedra with $b = 7$ and up to 24 faces.

6. LIST 3: SELECTED FACE-REGULAR SIMPLE POLYHEDRA WITH $b \geq 8$

Figure 3 lists selected face-regular simple polyhedra with $b = 8$.

7. LIST 4: ALL NINE FACE-REGULAR 4-VALENT POLYHEDRA WITH $b = 4$

For the polyhedra in this list (Figure 4), the graph induced by the 4-gons is interesting: in 6, 7, and 9, the graphs are two C_4, C_8 and the truncated octahedron, respectively.

8. LIST 5: ALL FACE-REGULAR FOUR-VALENT POLYHEDRA WITH $b = 5$ AND UP TO 24 FACES

Among the polyhedra of lists 4 and 5 (Figure 5), there are the octahedron, three semiregular ones (4-, 5-gonal antiprisms and the cuboctahedron), and three regular-faced (number 3 of list 4 and 4 and 6 of list 5, which are the elongated square dipyramid, the pentagonal gyrobicupola, and the pentagonal orthobicupola, having numbers 15, 31 and 30, respectively, in the list of 92 polyhedra in ref 15).

Number 2 in list 5 is the octahedron truncated and capped on four vertices of an induced C_4. Number 3 is the elongated antiprism. Number 5 is the dual rhombic icosahedron (2-elongated 5-gonal antiprism).

9. LIST 6: ALL 26 $6R_J$ FULLERENES

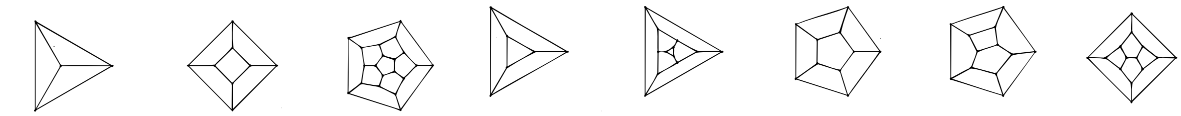
The polyhedra 1, 3, 6, 12, 16, 18−21, and 24−26 of this list are face-regular (Figure 6). They are the polyhedra 23−34 of list 1, respectively.

10. FACE-REGULAR k-VALENT BIFACED POLYHEDRA

In this section, we will study the case where faces of exactly two sizes $a < b$ occur.

Bifaced polyhedra and similar concepts are well-studied, e.g., in refs 16−29.

Clearly, in this case the V graph from the proof of theorem 1 is K_2, so we get the equation $pb = pa[f(a,b)/f(b,a)]$.



Nr.1 $v = 4$
$p_3 = 4 : 3$
Groupsize: 24
Group: T_d

Nr.2 $v = 8$
$p_4 = 6 : 0,4$
Groupsize: 48
Group: O_h

Nr.3 $v = 20$
$p_5 = 12 : 0,0,5$
Groupsize: 120
Group: I_h

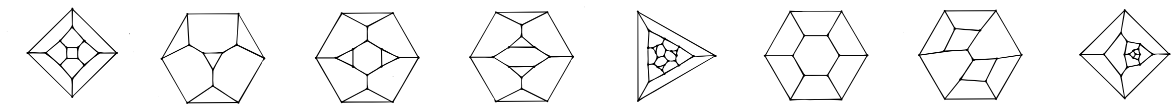
Nr.4 $v = 6$
$p_4 = 3 : 2,2$
$p_3 = 2 : 0,3$
Groupsize: 12
Group: D_{3h}

Nr.5 $v = 12$
$p_5 = 6 : 1,0,4$
$p_3 = 2 : 0,0,3$
Groupsize: 12
Group: D_{3d}

Nr.6 $v = 10$
$p_5 = 2 : 0,5,0$
$p_4 = 5 : 0,2,2$
Groupsize: 20
Group: D_{5h}

Nr.7 $v = 12$
$p_5 = 4 : 0,3,2$
$p_4 = 4 : 0,1,3$
Groupsize: 8
Group: D_{2d}

Nr.8 $v = 14$
$p_5 = 6 : 0,2,3$
$p_4 = 3 : 0,0,4$
Groupsize: 12
Group: D_{3h}



Nr.9 $v = 16$
$p_5 = 8 : 0,1,4$
$p_4 = 2 : 0,0,4$
Groupsize: 16
Group: D_{4d}

Nr.10 $v = 12$
$p_6 = 4 : 3,0,0,3$
$p_3 = 4 : 0,0,0,3$
Groupsize: 24
Group: T_d

Nr.11 $v = 16$
$p_6 = 6 : 2,0,0,4$
$p_3 = 4 : 0,0,0,3$
Groupsize: 24
Group: T_d

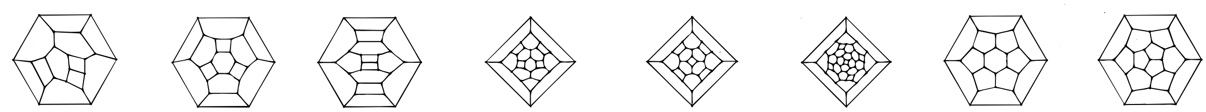
Nr.12 $v = 16$
$p_6 = 6 : 2,0,0,4$
$p_3 = 4 : 0,0,0,3$
Groupsize: 8
Group: D_{2h}

Nr.13 $v = 28$
$p_6 = 12 : 1,0,0,5$
$p_3 = 4 : 0,0,0,3$
Groupsize: 12
Group: T

Nr.14 $v = 12$
$p_6 = 2 : 0,6,0,0$
$p_4 = 6 : 0,2,0,2$
Groupsize: 24
Group: D_{6h}

Nr.15 $v = 14$
$p_6 = 3 : 0,4,0,2$
$p_4 = 6 : 0,2,0,2$
Groupsize: 12
Group: D_{3h}

Nr.16 $v = 20$
$p_6 = 6 : 0,2,0,4$
$p_4 = 6 : 0,2,0,2$
Groupsize: 12
Group: D_{3d}



Nr.17 $v = 20$
$p_6 = 6 : 0,3,0,3$
$p_4 = 6 : 0,1,0,3$
Groupsize: 6
Group: D_3

Nr.18 $v = 24$
$p_6 = 8 : 0,3,0,3$
$p_4 = 6 : 0,0,0,4$
Groupsize: 48
Group: O_h

Nr.19 $v = 26$
$p_6 = 9 : 0,2,0,4$
$p_4 = 6 : 0,1,0,3$
Groupsize: 12
Group: D_{3h}

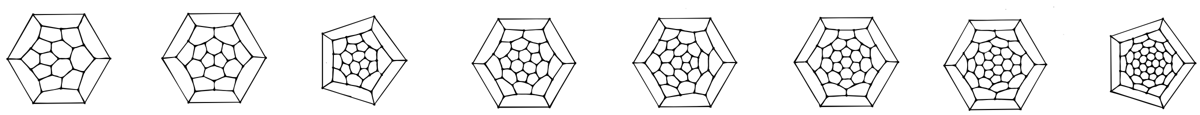
Nr.20 $v = 32$
$p_6 = 12 : 0,2,0,4$
$p_4 = 6 : 0,0,0,4$
Groupsize: 12
Group: D_{3r}

Nr.21 $v = 32$
$p_6 = 12 : 0,2,0,4$
$p_4 = 6 : 0,0,0,4$
Groupsize: 48
Group: O_h

Nr.22 $v = 56$
$p_6 = 24 : 0,1,0,5$
$p_4 = 6 : 0,0,0,4$
Groupsize: 24
Group: O

Nr.23 $v = 24$
$p_6 = 2 : 0,0,6,0$
$p_5 = 12 : 0,0,4,1$
Groupsize: 24
Group: D_{6d}

Nr.24 $v = 28$
$p_6 = 4 : 0,0,6,0$
$p_5 = 12 : 0,0,3,2$
Groupsize: 24
Group: T_d



Nr.25 $v = 32$
$p_6 = 6 : 0,0,4,2$
$p_5 = 12 : 0,0,3,2$
Groupsize: 12
Group: D_{3h}

Nr.26 $v = 38$
$p_6 = 9 : 0,0,4,2$
$p_5 = 12 : 0,0,2,3$
Groupsize: 6
Group: C_{3v}

Nr.27 $v = 44$
$p_6 = 12 : 0,0,3,3$
$p_5 = 12 : 0,0,2,3$
Groupsize: 12
Group: T

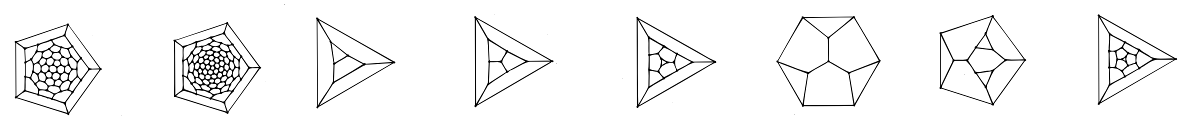
Nr.28 $v = 52$
$p_6 = 16 : 0,0,3,3$
$p_5 = 12 : 0,0,1,4$
Groupsize: 12
Group: T

Nr.29 $v = 56$
$p_6 = 18 : 0,0,2,4$
$p_5 = 12 : 0,0,2,3$
Groupsize: 24
Group: T_d

Nr.30 $v = 60$
$p_6 = 20 : 0,0,3,3$
$p_5 = 12 : 0,0,0,5$
Groupsize: 120
Group: I_h

Nr.31 $v = 68$
$p_6 = 24 : 0,0,2,4$
$p_5 = 12 : 0,0,1,4$
Groupsize: 24
Group: T_d

Nr.32 $v = 80$
$p_6 = 30 : 0,0,2,4$
$p_5 = 12 : 0,0,0,5$
Groupsize: 120
Group: I_h



Nr.33 $v = 80$
$p_6 = 30 : 0,0,2,4$
$p_5 = 12 : 0,0,0,5$
Groupsize: 20
Group: D_{5h}

Nr.34 $v = 140$
$p_6 = 60 : 0,0,1,5$
$p_5 = 12 : 0,0,0,5$
Groupsize: 60
Group: I

Nr.35 $v = 8$
$p_5 = 2 : 2,2,1$
$p_4 = 2 : 1,1,2$
$p_3 = 2 : 0,1,2$
Groupsize: 4
Group: C_{2v}

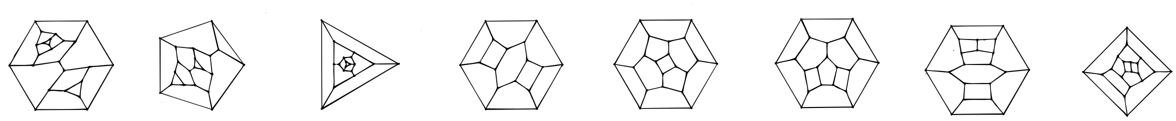
Nr.36 $v = 10$
$p_5 = 3 : 1,2,2$
$p_4 = 3 : 0,2,2$
$p_3 = 1 : 0,0,3$
Groupsize: 6
Group: C_{3v}

Nr.37 $v = 18$
$p_6 = 6 : 1,2,0,3$
$p_4 = 3 : 0,0,0,4$
$p_3 = 2 : 0,0,0,3$
Groupsize: 12
Group: D_{3h}

Nr.38 $v = 10$
$p_6 = 1 : 3,0,3,0$
$p_5 = 3 : 2,0,2,1$
$p_3 = 3 : 0,0,2,1$
Groupsize: 6
Group: C_{3v}

Nr.39 $v = 14$
$p_6 = 3 : 2,0,2,2$
$p_5 = 3 : 1,0,2,2$
$p_3 = 3 : 0,0,1,2$
Groupsize: 6
Group: C_{3v}

Nr.40 $v = 24$
$p_6 = 6 : 1,0,3,2$
$p_5 = 6 : 0,0,2,3$
$p_3 = 2 : 0,0,0,3$
Groupsize: 12
Group: D_{3d}



Nr.41 $v = 24$
$p_6 = 6 : 0,0,2,4$
$p_5 = 6 : 1,0,2,2$
$p_3 = 2 : 0,0,3,0$
Groupsize: 12
Group: D_{3h}

Nr.42 $v = 20$
$p_6 = 6 : 1,0,2,3$
$p_5 = 3 : 1,0,0,4$
$p_3 = 3 : 0,0,1,2$
Groupsize: 6
Group: C_{3h}

Nr.43 $v = 18$
$p_6 = 3 : 0,0,4,2$
$p_5 = 6 : 1,0,2,2$
$p_3 = 2 : 0,0,3,0$
Groupsize: 12
Group: D_{3h}

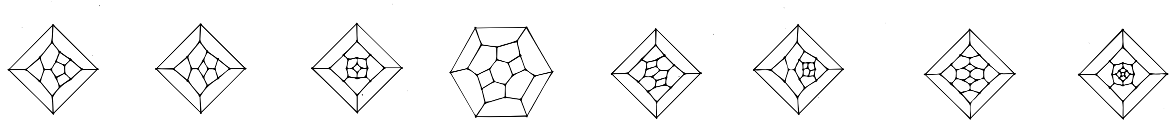
Nr.44 $v = 16$
$p_6 = 2 : 0,2,4,0$
$p_5 = 4 : 0,2,1,2$
$p_4 = 4 : 0,1,2,1$
Groupsize: 8
Group: D_{2h}

Nr.45 $v = 20$
$p_6 = 3 : 0,2,4,0$
$p_5 = 6 : 0,1,2,2$
$p_4 = 3 : 0,0,2,2$
Groupsize: 12
Group: D_{3h}

Nr.46 $v = 20$
$p_6 = 4 : 0,2,3,1$
$p_5 = 4 : 0,2,0,3$
$p_4 = 4 : 0,0,2,2$
Groupsize: 8
Group: D_{2h}

Nr.47 $v = 20$
$p_6 = 4 : 0,1,3,2$
$p_5 = 4 : 0,2,0,3$
$p_4 = 4 : 0,1,2,1$
Groupsize: 8
Group: D_{2d}

Nr.48 $v = 20$
$p_6 = 4 : 0,2,3,1$
$p_5 = 4 : 0,1,1,3$
$p_4 = 4 : 0,1,1,2$
Groupsize: 4
Group: C_{2v}



Nr.49 $v = 20$
$p_6 = 4 : 0,2,2,2$
$p_5 = 4 : 0,2,1,2$
$p_4 = 4 : 0,0,2,2$
Groupsize: 8
Group: D_{2d}

Nr.50 $v = 20$
$p_6 = 4 : 0,2,2,2$
$p_5 = 4 : 0,2,1,2$
$p_4 = 4 : 0,0,2,2$
Groupsize: 8
Group: D_{2h}

Nr.51 $v = 24$
$p_6 = 4 : 0,0,4,2$
$p_5 = 8 : 0,1,2,2$
$p_4 = 2 : 0,0,4,0$
Groupsize: 16
Group: D_{4h}

Nr.52 $v = 24$
$p_6 = 6 : 0,2,2,2$
$p_5 = 4 : 0,1,1,3$
$p_4 = 4 : 0,0,1,3$
Groupsize: 4
Group: D_2

Nr.53 $v = 26$
$p_6 = 6 : 0,1,3,2$
$p_5 = 6 : 0,1,1,3$
$p_4 = 3 : 0,0,2,2$
Groupsize: 6
Group: D_3

Nr.54 $v = 28$
$p_6 = 8 : 0,1,1,4$
$p_5 = 4 : 0,2,1,2$
$p_4 = 4 : 0,0,2,2$
Groupsize: 4
Group: D_2

Nr.55 $v = 30$
$p_6 = 10 : 0,2,1,3$
$p_5 = 2 : 0,0,0,5$
$p_4 = 5 : 0,0,0,4$
Groupsize: 20
Group: D_{5h}

Nr.56 $v = 32$
$p_6 = 8 : 0,0,2,4$
$p_5 = 8 : 0,1,2,2$
$p_4 = 2 : 0,0,4,0$
Groupsize: 16
Group: D_{4h}

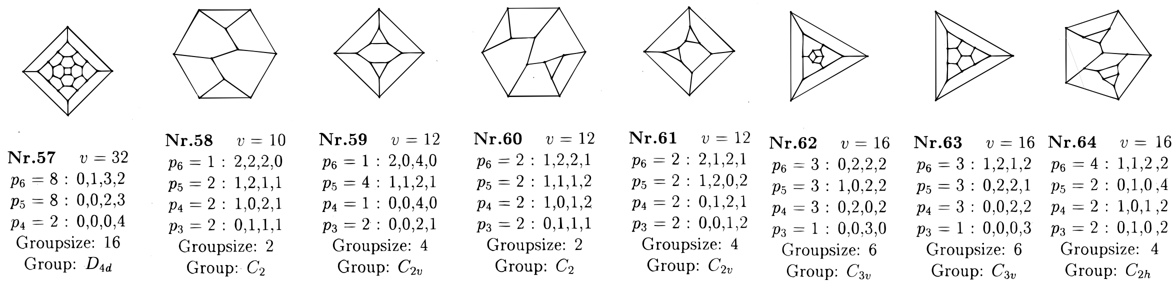


Figure 1. Sixty-four polyhedra of list 1.

By using $kv = 2e = ap_a + bp_b$ and the Euler formula $v - e + (p_a + p_b) = 2$, we get

$$p_a = \frac{4k}{(2k + 2a - ak) - (bk - 2b - 2k)\dfrac{\mathrm{f}(a,b)}{\mathrm{f}(b,a)}} \qquad (1)$$

We will use the following notation for operations on polyhedra:

Four-Cap. The 4-cap of a polyhedron is obtained by putting a pyramid on 4-gonal faces.

Four-Triakon. The 4-triakon of a polyhedron is obtained by partitioning each triangle into a ring of three 4-gons by putting a vertex in the middle and connecting it to the midpoint of every edge in the boundary.

Five-Triakon. A 5-triakon of a polyhedron is obtained by partitioning each hexagon into a ring of three pentagons by putting a vertex in the middle and connecting it to the midpoint of every second edge in the boundary. (An example of two different face-regular bifaced polyhedra, both coming as a 5-triakon of the truncated octahedron, is given in remark 1 after theorem 5 below.)

Theorem 2. *For $k > 3$, there is only one infinite series of face-regular (a,b)-polyhedra, that is, the antiprisms $APrism_b$ for any $b > 3$. Apart from this, all face-regular k -valent (a,b) -polyhedra have* $(k; a) = (4; 3)$ *and* $b = 4, 5, 6$. *They are:* $b = 4$, *seven polyhedra, given as 3−9 in list 4;* $b = 5$, *the icosidodecahedron and number 2 in list 5 (the 4-cap of the octahedron, truncated on all but two opposite vertices); and* $b = 6$, *the 4-cap of the (fully) truncated octahedron.*

Theorem 2 will follow from the following four Lemmata.

Remark. Theorem 2 shows that the largest 4-valent face-regular (a,b)-polyhedra have 30 vertexes (i.e., 32 faces) and $a = 3$. They have $(p_3, b) \in \{(8,4), (20,5), (24,6)\}$ and are number 9 in list 4, the icosidodecahedron, and the 4-cap of the truncated octahedron, respectively. The largest 3-valent face-regular $(3, b)$-polyhedron also has 32 faces. It is the fully truncated dodecahedron with $(p_3, b) = (20,10)$.

As we will see below, all three largest 3-valent face-regular polyhedra have 140 vertexes. They are the unique largest $(4,b)$-polyhedron (the 4-triakon of the truncated dodecahedron, so $p = (p_4 = 60, p_{15} = 12)$; its symmetry is I_h) and the two largest $(5,b)$-polyhedra (a 5-triakon of the truncated icosahedron, so $p = (p_5 = 60, p_{10} = 12)$; its symmetry is I_h, see Figure 6 in ref 30) and the fullerene $C_{140}(I)$ (the truncation of the dual snub dodecahedron on all 12 5-valent vertexes, so $p = (p_5 = 12, p_6 = 60)$).

Lemma 1. *The only possibilities for (k;a,b) are* $(5;3,4)$, $(4;3,b)$, $(3;3,b)$, $(3;4,b)$, and $(3;5,b)$.

In all other cases, the denominator in eq 1 will be nonpositive even for $\mathrm{f}(a,b)/\mathrm{f}(b,a) = 1/b$, the smallest possible value.

Lemma 2. *The case* $(k;a,b) = (5;3,4)$ *is not possible, so there is no face-regular 5-valent polyhedron.*

Proof. The denominator in eq 1 is positive only for $(\mathrm{f}(a,b), \mathrm{f}(b,a)) = (1,4), (1,3)$.

In the first case, any 4-gon is surrounded by 12 3-gons, which implies neighboring 4-gons in the next layer—a contradiction. In the case (1,3), any pair of adjacent 4-gons is surrounded by 16 3-gons, so the next layer contains a 4-gon with 2 4-gonal neighbors—again a contradiction.

In the following lemma, we will exclude some of the theoretically possible parameters for $k = 4$.

Lemma 3. *A comprehensive list of all cases for (k; a) =* $(4; 3)$ *is as follows*: (a) $b = 4$, *the 8 polyhedra 2−9 in list 4;* (b) $b = 5$, *the icosidodecahedron with* $\mathrm{f}(3,5) = 3$, $\mathrm{f}(5,3) = 5$, $v = 30$; (c) $b > 3$, *the infinite class of antiprisms* $APrism_b$; (d) $b > 3$, $(\mathrm{f}(3,b), \mathrm{f}(b,3)) = (1, b - 3)$, $(p_3, p_b) = (8b - 24, 8)$, $v = 8b - 18$; (e) $(\mathrm{f}(3,b), \mathrm{f}(b,3)) = (1, b - 2)$, $(p_3, p_b) = (4b - 8,4)$, $v = 4b - 6$.

Proof. For $b > 4$, the denominator in (eq 1) is positive only if (1) $(\mathrm{f}(3,b), \mathrm{f}(b,3)) \in \{(1, b - 3), (1, b - 2), (1, b - 1), (1, b)\}$; (2) $b \in \{5,6,7\}$ and $(\mathrm{f}(3,b), \mathrm{f}(b,3)) = (2, b)$; (3) $b \in \{5,6\}$ and $(\mathrm{f}(3,b), \mathrm{f}(b,3)) = (2, b - 1)$; or (4) $b = 5$ and $(\mathrm{f}(3,b), \mathrm{f}(b,3)) \in \{(2,3), (3,4), (3,5)\}$.

The subcase $(1, b - 1)$ in case 1 is not possible, because $p_3 = 8(b - 1)/3$ and $p_b = 8/3$. The subcases $(1, b)$, $(1, b - 3)$, *and* $(1, b - 2)$ of case 1 are, respectively, cases c, d, and e of lemma 3.

Cases 2 and 3 are not possible, because we get three 3-gons on three consecutive edges of each b-gon, so the 3-gonal neighbor of the 3-gon in the middle will be adjacent to four 3-gons, a contradiction.

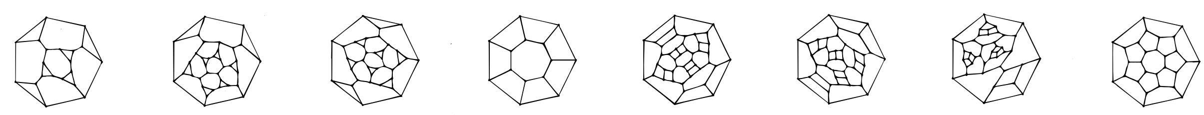
The subcase (3, 4) of 4 is not possible, since the 3-gonal neighbors of two adjacent 5-gons containing a vertex of the intersection would share an edge.

In the subcase (2, 3) of 4, all three triangles neighboring a pentagon in a row would imply a triangle neighboring two other ones, so we assume we have a 5-gon and three neighboring 3-gons not all in a row. But then one of the 5-gonal neighbors has all 3-gonal neighbors in a row—again a contradiction.

The remaining subcase (3, 5) of 4 is case b of lemma 3.

Lemma 4. *Cases d and e of lemma 3 are realized only by polyhedron 3 in list 4 and the 4-cap of suitably truncated octahedra, given in theorem 2.*

Proof. In both cases, all 3-gons are organized in four cycles, surrounded by b-gons, since otherwise we will get



Nr.1 $v = 20$
$p_7 = 6 : 3,0,0,0,4$
$p_3 = 6 : 0,0,0,0,3$
Groupsize: 12

Nr.2 $v = 36$
$p_7 = 12 : 2,0,0,0,5$
$p_3 = 8 : 0,0,0,0,3$
Groupsize: 6

Nr.3 $v = 36$
$p_7 = 12 : 2,0,0,0,5$
$p_3 = 8 : 0,0,0,0,3$
Groupsize: 24

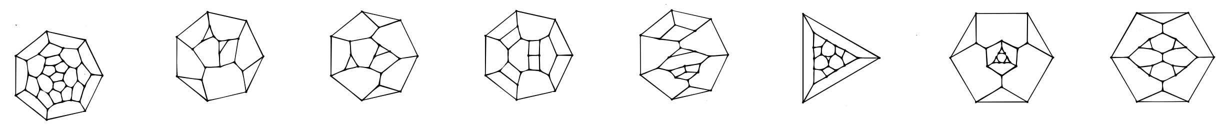
Nr.4 $v = 14$
$p_7 = 2 : 0,7,0,0,0$
$p_4 = 7 : 0,2,0,0,2$
Groupsize: 28

Nr.5 $v = 44$
$p_7 = 12 : 0,3,0,0,4$
$p_4 = 12 : 0,1,0,0,3$
Groupsize: 24

Nr.6 $v = 44$
$p_7 = 12 : 0,3,0,0,4$
$p_4 = 12 : 0,1,0,0,3$
Groupsize: 6

Nr.7 $v = 44$
$p_7 = 12 : 0,2,0,0,5$
$p_4 = 12 : 0,2,0,0,2$
Groupsize: 12

Nr.8 $v = 28$
$p_7 = 2 : 0,0,7,0,0$
$p_5 = 14 : 0,0,4,0,1$
Groupsize: 28



Nr.9 $v = 44$
$p_7 = 6 : 0,0,6,0,1$
$p_5 = 18 : 0,0,3,0,2$
Groupsize: 12

Nr.10 $v = 20$
$p_7 = 4 : 2,0,3,0,2$
$p_5 = 4 : 1,0,1,0,3$
$p_3 = 4 : 0,0,1,0,2$
Groupsize: 4

Nr.11 $v = 20$
$p_7 = 4 : 2,0,3,0,2$
$p_5 = 4 : 1,0,1,0,3$
$p_3 = 4 : 0,0,1,0,2$
Groupsize: 4

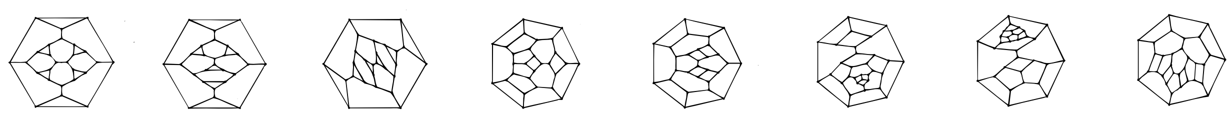
Nr.12 $v = 24$
$p_7 = 4 : 0,4,0,1,2$
$p_6 = 2 : 0,4,0,0,2$
$p_4 = 8 : 0,1,0,1,2$
Groupsize: 8

Nr.13 $v = 24$
$p_7 = 6 : 1,2,0,0,4$
$p_4 = 6 : 0,2,0,0,2$
$p_3 = 2 : 0,0,0,0,3$
Groupsize: 4

Nr.14 $v = 24$
$p_7 = 6 : 1,3,0,0,3$
$p_4 = 6 : 0,1,0,0,3$
$p_3 = 2 : 0,0,0,0,3$
Groupsize: 6

Nr.15 $v = 24$
$p_7 = 6 : 2,0,0,1,4$
$p_6 = 2 : 3,0,0,0,3$
$p_3 = 6 : 0,0,0,1,2$
Groupsize: 12

Nr.16 $v = 26$
$p_7 = 6 : 2,0,0,2,3$
$p_6 = 3 : 2,0,0,0,4$
$p_3 = 6 : 0,0,0,1,2$
Groupsize: 4



Nr.17 $v = 26$
$p_7 = 6 : 2,0,0,2,3$
$p_6 = 3 : 2,0,0,0,4$
$p_3 = 6 : 0,0,0,1,2$
Groupsize: 12

Nr.18 $v = 26$
$p_7 = 6 : 2,0,0,2,3$
$p_6 = 3 : 2,0,0,0,4$
$p_3 = 6 : 0,0,0,1,2$
Groupsize: 4

Nr.19 $v = 26$
$p_7 = 6 : 2,0,0,2,3$
$p_6 = 3 : 2,0,0,0,4$
$p_3 = 6 : 0,0,0,1,2$
Groupsize: 12

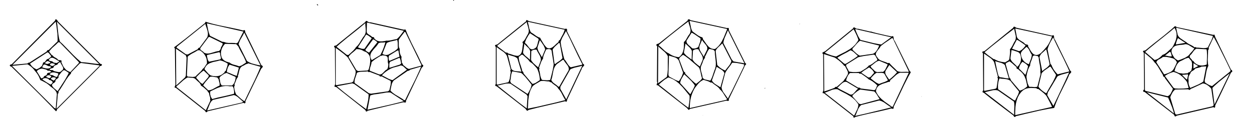
Nr.20 $v = 28$
$p_7 = 4 : 0,2,4,0,1$
$p_5 = 8 : 0,1,2,0,2$
$p_4 = 4 : 0,0,2,0,2$
Groupsize: 8

Nr.21 $v = 28$
$p_7 = 4 : 0,4,0,3,0$
$p_6 = 4 : 0,2,0,1,3$
$p_4 = 8 : 0,1,0,1,2$
Groupsize: 8

Nr.22 $v = 32$
$p_7 = 6 : 0,2,3,0,2$
$p_5 = 6 : 0,0,2,0,3$
$p_4 = 6 : 0,2,0,0,2$
Groupsize: 12

Nr.23 $v = 32$
$p_7 = 6 : 0,2,1,0,4$
$p_5 = 6 : 0,2,2,0,1$
$p_4 = 6 : 0,0,2,0,2$
Groupsize: 12

Nr.24 $v = 32$
$p_7 = 6 : 0,2,3,0,2$
$p_5 = 6 : 0,2,0,0,3$
$p_4 = 6 : 0,0,2,0,2$
Groupsize: 12



Nr.25 $v = 32$
$p_7 = 6 : 0,1,3,0,3$
$p_5 = 6 : 0,2,0,0,3$
$p_4 = 6 : 0,1,2,0,1$
Groupsize: 6

Nr.26 $v = 32$
$p_7 = 6 : 0,3,2,0,2$
$p_5 = 6 : 0,1,2,0,2$
$p_4 = 6 : 0,0,1,0,3$
Groupsize: 12

Nr.27 $v = 32$
$p_7 = 6 : 0,2,3,0,2$
$p_5 = 6 : 0,1,1,0,3$
$p_4 = 6 : 0,1,1,0,2$
Groupsize: 6

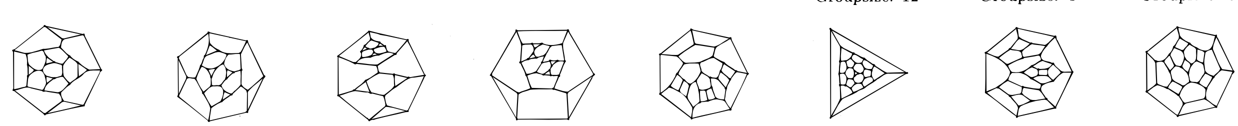
Nr.28 $v = 32$
$p_7 = 6 : 0,2,2,0,3$
$p_5 = 6 : 0,2,1,0,2$
$p_4 = 6 : 0,0,2,0,2$
Groupsize: 12

Nr.29 $v = 32$
$p_7 = 6 : 0,2,2,0,3$
$p_5 = 6 : 0,2,1,0,2$
$p_4 = 6 : 0,0,2,0,2$
Groupsize: 4

Nr.30 $v = 32$
$p_7 = 6 : 0,2,2,0,3$
$p_5 = 6 : 0,2,1,0,2$
$p_4 = 6 : 0,0,2,0,2$
Groupsize: 12

Nr.31 $v = 32$
$p_7 = 6 : 0,2,2,0,3$
$p_5 = 6 : 0,2,1,0,2$
$p_4 = 6 : 0,0,2,0,2$
Groupsize: 4

Nr.32 $v = 32$
$p_7 = 6 : 2,0,0,3,2$
$p_6 = 6 : 1,0,0,2,3$
$p_3 = 6 : 0,0,0,1,2$
Groupsize: 6



Nr.33 $v = 32$
$p_7 = 6 : 2,0,0,3,2$
$p_6 = 6 : 1,0,0,2,3$
$p_3 = 6 : 0,0,0,1,2$
Groupsize: 12

Nr.34 $v = 32$
$p_7 = 6 : 2,0,0,3,2$
$p_6 = 6 : 1,0,0,2,3$
$p_3 = 6 : 0,0,0,1,2$
Groupsize: 6

Nr.35 $v = 32$
$p_7 = 6 : 1,0,0,2,4$
$p_6 = 6 : 2,0,0,2,2$
$p_3 = 6 : 0,0,0,2,1$
Groupsize: 12

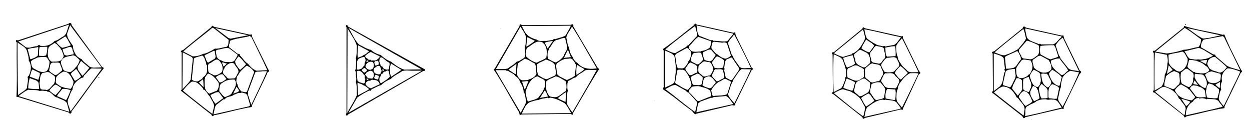
Nr.36 $v = 32$
$p_7 = 6 : 1,0,0,3,3$
$p_6 = 6 : 2,0,0,1,3$
$p_3 = 6 : 0,0,0,2,1$
Groupsize: 6

Nr.37 $v = 36$
$p_7 = 4 : 0,2,0,4,1$
$p_6 = 8 : 0,2,0,2,2$
$p_4 = 8 : 0,1,0,2,1$
Groupsize: 8

Nr.38 $v = 36$
$p_7 = 6 : 1,0,4,0,2$
$p_5 = 12 : 0,0,3,0,2$
$p_3 = 2 : 0,0,0,0,3$
Groupsize: 12

Nr.39 $v = 38$
$p_7 = 6 : 0,0,4,0,3$
$p_5 = 12 : 0,1,2,0,2$
$p_4 = 3 : 0,0,4,0,0$
Groupsize: 12

Nr.40 $v = 38$
$p_7 = 6 : 0,3,0,2,2$
$p_6 = 6 : 0,3,0,1,2$
$p_4 = 9 : 0,0,0,2,2$
Groupsize: 12



Nr.41 $v = 40$
$p_7 = 10 : 0,3,1,0,3$
$p_5 = 2 : 0,0,0,0,5$
$p_4 = 10 : 0,1,0,0,3$
Groupsize: 10

Nr.42 $v = 40$
$p_7 = 12 : 1,2,0,0,4$
$p_4 = 6 : 0,0,0,0,4$
$p_3 = 4 : 0,0,0,0,3$
Groupsize: 6

Nr.43 $v = 40$
$p_7 = 12 : 1,2,0,0,4$
$p_4 = 6 : 0,0,0,0,4$
$p_3 = 4 : 0,0,0,0,3$
Groupsize: 24

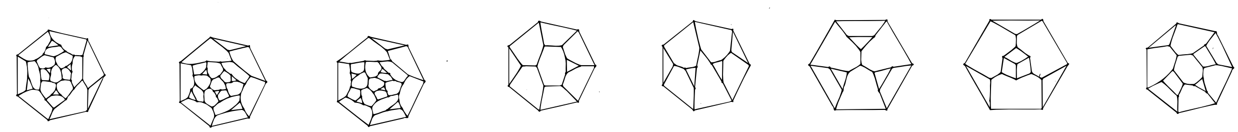
Nr.44 $v = 40$
$p_7 = 12 : 2,0,0,1,4$
$p_6 = 2 : 0,0,0,0,6$
$p_3 = 8 : 0,0,0,0,3$
Groupsize: 8

Nr.45 $v = 42$
$p_7 = 2 : 0,0,7,0,0$
$p_6 = 7 : 0,0,4,2,0$
$p_5 = 14 : 0,0,2,2,1$
Groupsize: 28

Nr.46 $v = 42$
$p_7 = 2 : 0,0,0,7,0$
$p_6 = 14 : 0,2,0,3,1$
$p_4 = 7 : 0,0,0,4,0$
Groupsize: 28

Nr.47 $v = 44$
$p_7 = 4 : 0,0,4,2,1$
$p_6 = 4 : 0,0,4,0,2$
$p_5 = 16 : 0,0,3,1,1$
Groupsize: 8

Nr.48 $v = 44$
$p_7 = 6 : 1,0,0,4,2$
$p_6 = 12 : 1,0,0,3,2$
$p_3 = 6 : 0,0,0,2,1$
Groupsize: 6



Nr.49 $v = 44$
$p_7 = 12 : 1,0,2,0,4$
$p_5 = 6 : 1,0,0,0,4$
$p_3 = 6 : 0,0,1,0,2$
Groupsize: 6

Nr.50 $v = 44$
$p_7 = 12 : 1,0,2,0,4$
$p_5 = 6 : 1,0,0,0,4$
$p_3 = 6 : 0,0,1,0,2$
Groupsize: 6

Nr.51 $v = 44$
$p_7 = 12 : 1,0,2,0,4$
$p_5 = 6 : 1,0,0,0,4$
$p_3 = 6 : 0,0,1,0,2$
Groupsize: 2

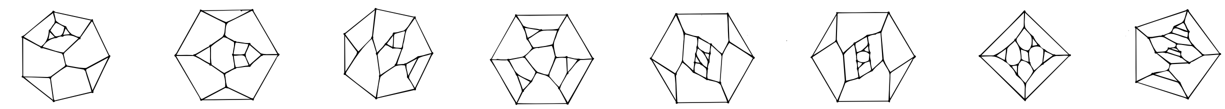
Nr.52 $v = 16$
$p_7 = 2 : 1,2,4,0,0$
$p_5 = 4 : 1,1,1,0,2$
$p_4 = 2 : 0,0,2,0,2$
$p_3 = 2 : 0,0,2,0,1$
Groupsize: 4

Nr.53 $v = 16$
$p_7 = 2 : 2,0,2,2,1$
$p_6 = 2 : 2,0,1,1,2$
$p_5 = 2 : 2,0,0,1,2$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

Nr.54 $v = 16$
$p_7 = 3 : 2,2,0,1,2$
$p_6 = 1 : 0,3,0,0,3$
$p_4 = 3 : 1,0,0,1,2$
$p_3 = 3 : 0,1,0,0,2$
Groupsize: 6

Nr.55 $v = 16$
$p_7 = 3 : 2,2,0,1,2$
$p_6 = 1 : 3,0,0,0,3$
$p_4 = 3 : 0,2,0,0,2$
$p_3 = 3 : 0,0,0,1,2$
Groupsize: 6

Nr.56 $v = 20$
$p_7 = 2 : 1,2,0,4,0$
$p_6 = 4 : 1,2,0,1,2$
$p_4 = 4 : 0,1,0,2,1$
$p_3 = 2 : 0,0,0,2,1$
Groupsize: 4



Nr.57 $v = 20$
$p_7 = 2 : 2,0,1,4,0$
$p_6 = 4 : 1,0,1,2,2$
$p_5 = 2 : 2,0,0,2,1$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

Nr.58 $v = 20$
$p_7 = 3 : 1,2,0,2,2$
$p_6 = 3 : 2,0,0,2,2$
$p_4 = 3 : 0,2,0,0,2$
$p_3 = 3 : 0,0,0,2,1$
Groupsize: 6

Nr.59 $v = 20$
$p_7 = 3 : 1,2,0,2,2$
$p_6 = 3 : 1,1,0,2,2$
$p_4 = 3 : 1,0,0,1,2$
$p_3 = 3 : 0,1,0,1,1$
Groupsize: 3

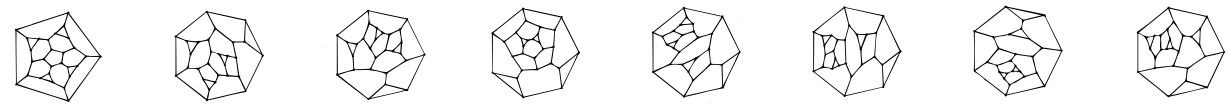
Nr.60 $v = 24$
$p_7 = 3 : 1,0,4,2,0$
$p_6 = 2 : 0,0,3,0,3$
$p_5 = 6 : 1,0,1,1,2$
$p_3 = 3 : 0,0,2,0,1$
Groupsize: 6

Nr.61 $v = 24$
$p_7 = 4 : 1,0,3,1,2$
$p_6 = 2 : 2,0,2,0,2$
$p_5 = 4 : 1,0,0,1,3$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

Nr.62 $v = 24$
$p_7 = 4 : 1,0,3,1,2$
$p_6 = 2 : 2,0,2,0,2$
$p_5 = 4 : 1,0,0,1,3$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

Nr.63 $v = 24$
$p_7 = 4 : 2,1,0,3,1$
$p_6 = 4 : 1,1,0,1,3$
$p_4 = 2 : 0,0,0,2,2$
$p_3 = 4 : 0,0,0,1,2$
Groupsize: 4

Nr.64 $v = 26$
$p_7 = 6 : 1,1,2,0,3$
$p_5 = 3 : 0,1,0,0,4$
$p_4 = 3 : 1,0,1,0,2$
$p_3 = 3 : 0,1,0,0,2$
Groupsize: 6



Nr.65 $v = 28$
$p_7 = 2 : 2,0,1,4,0$
$p_6 = 8 : 1,0,1,3,1$
$p_5 = 2 : 0,0,0,4,1$
$p_3 = 4 : 0,0,0,2,1$
Groupsize: 4

Nr.66 $v = 28$
$p_7 = 4 : 1,0,2,3,1$
$p_6 = 4 : 1,0,2,0,3$
$p_5 = 4 : 1,0,0,2,2$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

Nr.67 $v = 28$
$p_7 = 4 : 1,0,2,3,1$
$p_6 = 4 : 1,0,2,0,3$
$p_5 = 4 : 1,0,0,2,2$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

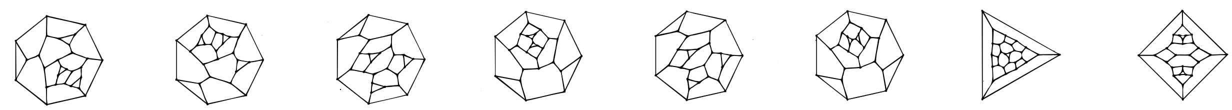
Nr.68 $v = 28$
$p_7 = 4 : 2,0,2,2,1$
$p_6 = 4 : 1,0,2,1,2$
$p_5 = 4 : 0,0,1,2,2$
$p_3 = 4 : 0,0,0,1,2$
Groupsize: 8

Nr.69 $v = 28$
$p_7 = 4 : 1,0,1,3,2$
$p_6 = 4 : 1,0,2,0,3$
$p_5 = 4 : 1,0,1,2,1$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

Nr.70 $v = 28$
$p_7 = 4 : 1,0,1,3,2$
$p_6 = 4 : 1,0,2,0,3$
$p_5 = 4 : 1,0,1,2,1$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

Nr.71 $v = 28$
$p_7 = 4 : 1,0,3,2,1$
$p_6 = 4 : 2,0,1,1,2$
$p_5 = 4 : 0,0,1,1,3$
$p_3 = 4 : 0,0,0,2,1$
Groupsize: 4

Nr.72 $v = 28$
$p_7 = 4 : 1,0,2,3,1$
$p_6 = 4 : 1,0,1,1,3$
$p_5 = 4 : 1,0,1,1,2$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 2



Nr.73 $v = 28$
$p_7 = 4 : 1,0,2,3,1$
$p_6 = 4 : 1,0,1,1,3$
$p_5 = 4 : 1,0,1,1,2$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

Nr.74 $v = 28$
$p_7 = 4 : 1,0,2,3,1$
$p_6 = 4 : 1,0,1,1,3$
$p_5 = 4 : 1,0,1,1,2$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

Nr.75 $v = 28$
$p_7 = 4 : 1,0,2,2,2$
$p_6 = 4 : 1,0,2,1,2$
$p_5 = 4 : 1,0,0,2,2$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

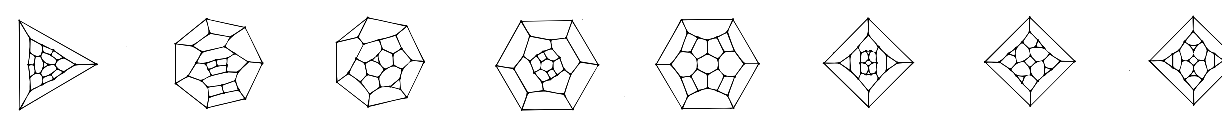
Nr.76 $v = 28$
$p_7 = 4 : 2,0,2,2,1$
$p_6 = 4 : 0,0,2,2,2$
$p_5 = 4 : 1,0,0,2,2$
$p_3 = 4 : 0,0,1,0,2$
Groupsize: 8

Nr.77 $v = 28$
$p_7 = 4 : 1,0,2,2,2$
$p_6 = 4 : 1,0,2,1,2$
$p_5 = 4 : 1,0,0,2,2$
$p_3 = 4 : 0,0,1,1,1$
Groupsize: 4

Nr.78 $v = 28$
$p_7 = 4 : 2,0,2,2,1$
$p_6 = 4 : 0,0,2,2,2$
$p_5 = 4 : 1,0,0,2,2$
$p_3 = 4 : 0,0,1,0,2$
Groupsize: 8

Nr.79 $v = 30$
$p_7 = 6 : 1,1,3,0,2$
$p_5 = 6 : 0,1,1,0,3$
$p_4 = 3 : 0,0,2,0,2$
$p_3 = 2 : 0,0,0,0,3$
Groupsize: 6

Nr.80 $v = 30$
$p_7 = 6 : 0,2,2,0,3$
$p_5 = 6 : 1,0,2,0,2$
$p_4 = 3 : 0,0,0,0,4$
$p_3 = 2 : 0,0,3,0,0$
Groupsize: 12



Nr.81 $v = 30$
$p_7 = 6 : 1,2,0,2,2$
$p_6 = 3 : 0,2,0,0,4$
$p_4 = 6 : 0,1,0,1,2$
$p_3 = 2 : 0,0,0,0,3$
Groupsize: 12

Nr.82 $v = 32$
$p_7 = 3 : 0,2,1,4,0$
$p_6 = 6 : 0,1,1,2,2$
$p_5 = 3 : 0,2,0,2,1$
$p_4 = 6 : 0,1,1,1,1$
Groupsize: 6

Nr.83 $v = 32$
$p_7 = 3 : 1,0,2,4,0$
$p_6 = 6 : 1,0,2,1,2$
$p_5 = 6 : 0,0,2,2,1$
$p_3 = 3 : 0,0,0,2,1$
Groupsize: 6

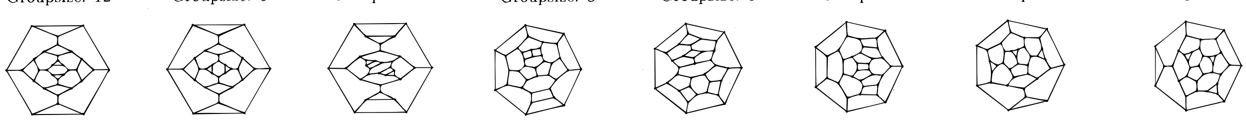
Nr.84 $v = 32$
$p_7 = 4 : 0,1,4,0,2$
$p_6 = 2 : 0,2,4,0,0$
$p_5 = 8 : 0,1,1,1,2$
$p_4 = 4 : 0,0,2,1,1$
Groupsize: 8

Nr.85 $v = 32$
$p_7 = 4 : 0,2,4,1,0$
$p_6 = 2 : 0,0,4,0,2$
$p_5 = 8 : 0,1,1,1,2$
$p_4 = 4 : 0,0,2,0,2$
Groupsize: 8

Nr.86 $v = 32$
$p_7 = 4 : 1,0,0,4,2$
$p_6 = 8 : 1,1,0,2,2$
$p_4 = 2 : 0,0,0,4,0$
$p_3 = 4 : 0,0,0,2,1$
Groupsize: 8

Nr.87 $v = 32$
$p_7 = 8 : 1,1,2,0,3$
$p_5 = 4 : 1,0,0,0,4$
$p_4 = 2 : 0,0,0,0,4$
$p_3 = 4 : 0,0,1,0,2$
Groupsize: 8

Nr.88 $v = 32$
$p_7 = 8 : 1,1,2,0,3$
$p_5 = 4 : 1,0,0,0,4$
$p_4 = 2 : 0,0,0,0,4$
$p_3 = 4 : 0,0,1,0,2$
Groupsize: 8



Nr.89 $v = 32$
$p_7 = 8 : 1,2,0,1,3$
$p_6 = 2 : 2,0,0,0,4$
$p_4 = 4 : 0,0,0,0,4$
$p_3 = 4 : 0,0,0,1,2$
Groupsize: 8

Nr.90 $v = 32$
$p_7 = 8 : 1,2,0,1,3$
$p_6 = 2 : 2,0,0,0,4$
$p_4 = 4 : 0,0,0,0,4$
$p_3 = 4 : 0,0,0,1,2$
Groupsize: 8

Nr.91 $v = 32$
$p_7 = 8 : 1,1,0,1,4$
$p_6 = 2 : 0,2,0,0,4$
$p_4 = 4 : 1,0,0,1,2$
$p_3 = 4 : 0,1,0,0,2$
Groupsize: 4

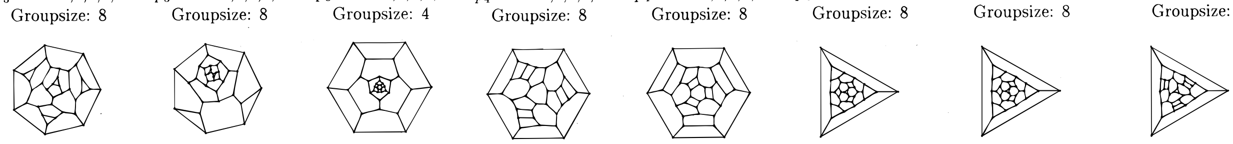
Nr.92 $v = 36$
$p_7 = 4 : 0,1,4,2,0$
$p_6 = 4 : 0,2,2,0,2$
$p_5 = 8 : 0,0,2,1,2$
$p_4 = 4 : 0,1,0,2,1$
Groupsize: 8

Nr.93 $v = 36$
$p_7 = 4 : 0,0,4,1,2$
$p_6 = 4 : 0,2,2,1,1$
$p_5 = 8 : 0,1,1,1,2$
$p_4 = 4 : 0,0,2,2,0$
Groupsize: 8

Nr.94 $v = 36$
$p_7 = 4 : 0,1,4,2,0$
$p_6 = 4 : 0,1,2,1,2$
$p_5 = 8 : 0,1,1,1,2$
$p_4 = 4 : 0,0,2,1,1$
Groupsize: 8

Nr.95 $v = 36$
$p_7 = 4 : 1,0,2,4,0$
$p_6 = 8 : 1,0,1,2,2$
$p_5 = 4 : 0,0,1,2,2$
$p_3 = 4 : 0,0,0,2,1$
Groupsize: 8

Nr.96 $v = 36$
$p_7 = 4 : 1,0,2,4,0$
$p_6 = 8 : 1,0,1,2,2$
$p_5 = 4 : 0,0,1,2,2$
$p_3 = 4 : 0,0,0,2,1$
Groupsize: 4



Nr.97 $v = 36$
$p_7 = 4 : 0,0,2,4,1$
$p_6 = 8 : 1,0,1,2,2$
$p_5 = 4 : 1,0,0,2,2$
$p_3 = 4 : 0,0,1,2,0$
Groupsize: 8

Nr.98 $v = 36$
$p_7 = 4 : 2,0,2,2,1$
$p_6 = 8 : 0,0,1,4,1$
$p_5 = 4 : 1,0,0,2,2$
$p_3 = 4 : 0,0,1,0,2$
Groupsize: 4

Nr.99 $v = 36$
$p_7 = 6 : 0,1,2,0,4$
$p_6 = 2 : 0,3,3,0,0$
$p_5 = 6 : 0,2,0,1,2$
$p_4 = 6 : 0,0,2,1,1$
Groupsize: 12

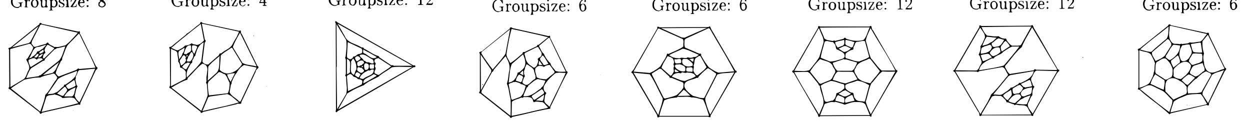
Nr.100 $v = 36$
$p_7 = 6 : 0,2,3,1,1$
$p_6 = 2 : 0,0,3,0,3$
$p_5 = 6 : 0,1,0,1,3$
$p_4 = 6 : 0,1,1,0,2$
Groupsize: 6

Nr.101 $v = 36$
$p_7 = 6 : 0,2,3,1,1$
$p_6 = 2 : 0,3,0,0,3$
$p_5 = 6 : 0,1,1,0,3$
$p_4 = 6 : 0,0,1,1,2$
Groupsize: 6

Nr.102 $v = 36$
$p_7 = 6 : 1,2,0,2,2$
$p_6 = 6 : 0,2,0,2,2$
$p_4 = 6 : 0,0,0,2,2$
$p_3 = 2 : 0,0,0,0,3$
Groupsize: 12

Nr.103 $v = 36$
$p_7 = 6 : 1,2,0,2,2$
$p_6 = 6 : 0,2,0,2,2$
$p_4 = 6 : 0,0,0,2,2$
$p_3 = 2 : 0,0,0,0,3$
Groupsize: 12

Nr.104 $v = 36$
$p_7 = 6 : 1,1,0,3,2$
$p_6 = 6 : 0,2,0,1,3$
$p_4 = 6 : 0,1,0,2,1$
$p_3 = 2 : 0,0,0,0,3$
Groupsize: 6



Nr.105 $v = 36$
$p_7 = 6 : 1,0,0,2,4$
$p_6 = 6 : 0,2,0,2,2$
$p_4 = 6 : 0,2,0,2,0$
$p_3 = 2 : 0,0,0,0,3$
Groupsize: 4

Nr.106 $v = 36$
$p_7 = 6 : 0,2,0,1,4$
$p_6 = 6 : 1,2,0,2,1$
$p_4 = 6 : 0,0,0,2,2$
$p_3 = 2 : 0,0,0,3,0$
Groupsize: 12

Nr.107 $v = 36$
$p_7 = 6 : 0,3,0,2,2$
$p_6 = 6 : 1,1,0,2,2$
$p_4 = 6 : 0,0,0,1,3$
$p_3 = 2 : 0,0,0,3,0$
Groupsize: 12

Nr.108 $v = 36$
$p_7 = 8 : 1,1,0,2,3$
$p_6 = 4 : 0,1,0,1,4$
$p_4 = 4 : 1,0,0,1,2$
$p_3 = 4 : 0,1,0,0,2$
Groupsize: 4

Nr.109 $v = 36$
$p_7 = 8 : 1,1,0,1,4$
$p_6 = 4 : 1,2,0,1,2$
$p_4 = 4 : 0,0,0,2,2$
$p_3 = 4 : 0,0,0,1,2$
Groupsize: 4

Nr.110 $v = 38$
$p_7 = 6 : 0,2,2,1,2$
$p_6 = 3 : 0,0,4,0,2$
$p_5 = 6 : 0,0,1,2,2$
$p_4 = 6 : 0,2,0,0,2$
Groupsize: 12

Nr.111 $v = 38$
$p_7 = 6 : 0,2,1,2,2$
$p_6 = 3 : 0,0,0,2,4$
$p_5 = 6 : 0,2,2,0,1$
$p_4 = 6 : 0,0,2,0,2$
Groupsize: 12

Nr.112 $v = 38$
$p_7 = 6 : 0,2,2,1,2$
$p_6 = 3 : 0,2,2,0,2$
$p_5 = 6 : 0,1,1,1,2$
$p_4 = 6 : 0,0,1,1,2$
Groupsize: 6

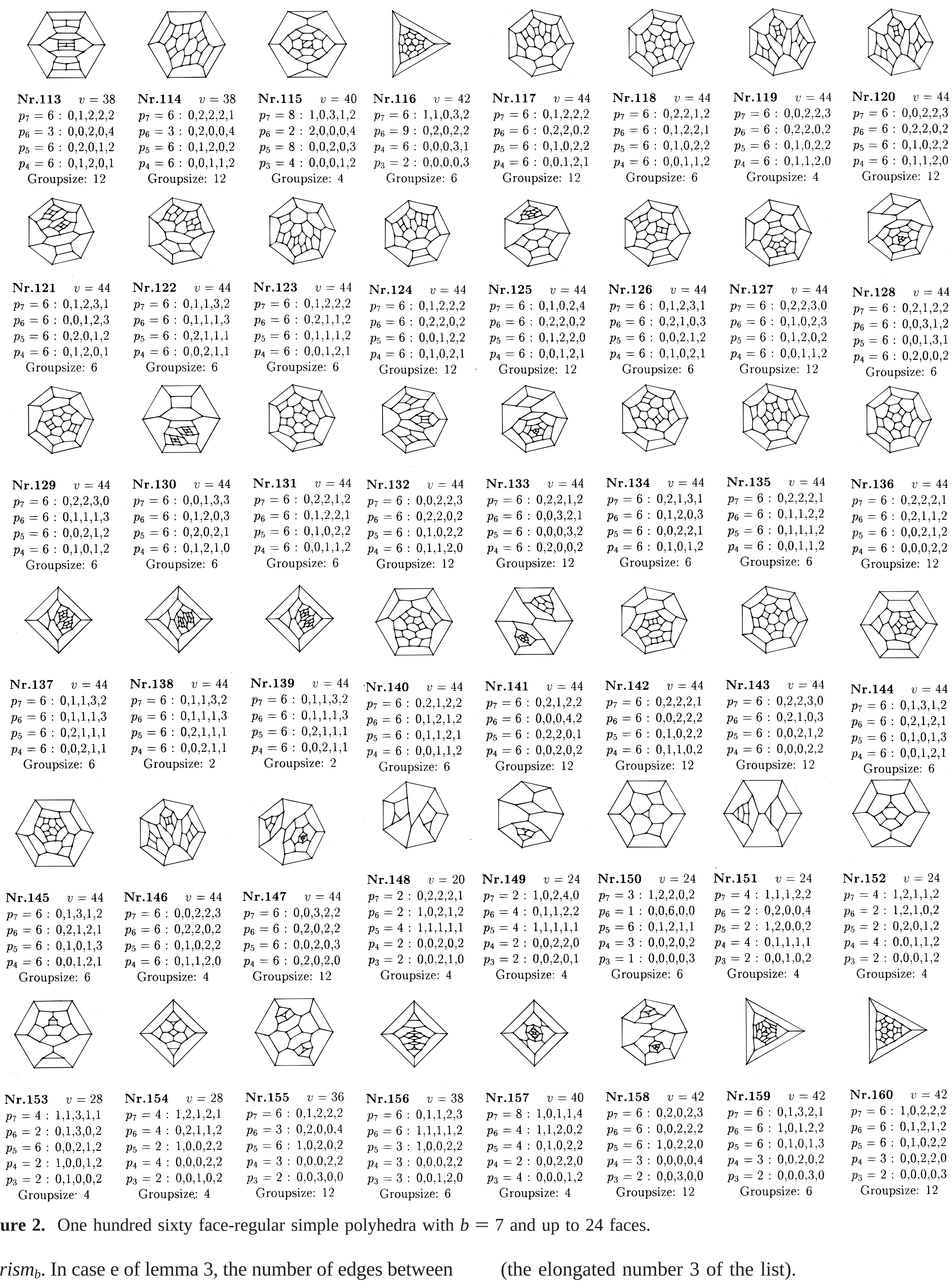


Figure 2. One hundred sixty face-regular simple polyhedra with $b = 7$ and up to 24 faces.

$APrism_b$. In case e of lemma 3, the number of edges between two b-gons is $p_b(b - f(b,3))/2 = 4$. So the only possibility is $b = 4$, and number 3 of list 4 is a unique realization (it is the dual of the octahedron, truncated on two opposite vertexes). In case d of lemma 3, the number of $(b - b)$ edges is 12. This implies $b \in \{4,5,6\}$, and we get the 4-cap of three suitably truncated octahedra, the first one being 5 of list 4 (the elongated number 3 of the list).

Theorem 3. *All face-regular cubic* $(3, b)$*-polyhedra have* $b \leq 10$. *They are 14 special truncations of the tetrahedron, the cube, and the dodecahedron:* (a) *the 1- and 4-truncated tetrahedron*; (b) $2(b - 4)$*-truncated cubes (one for* $b \in \{5,7,8\}$ *and two for* $b = 6$*)*; (c) $4(b - 5)$*-truncated dodecahedra (one for* $b = 6$, 9, 10 *and two for* $b = 7$, 8).

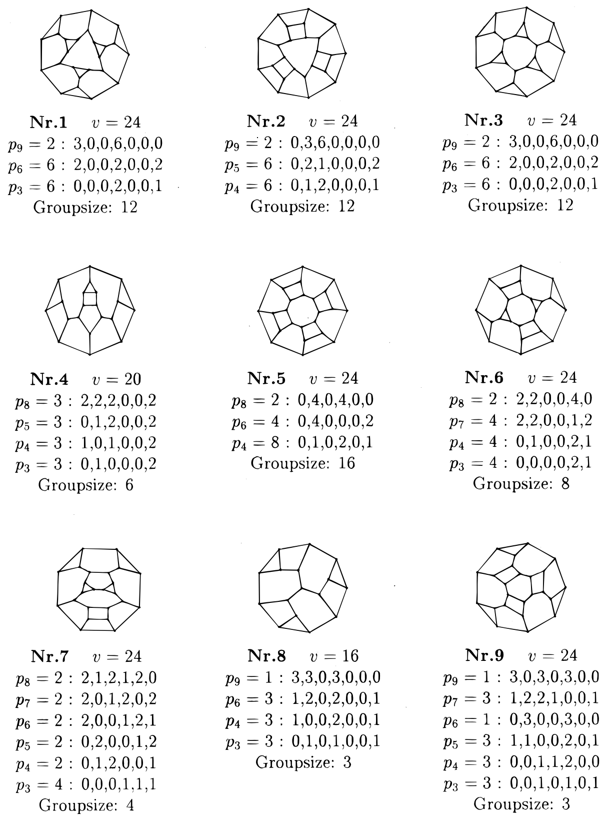


Figure 3. Selected face-regular simple polyhedra with $b \geq 8$.

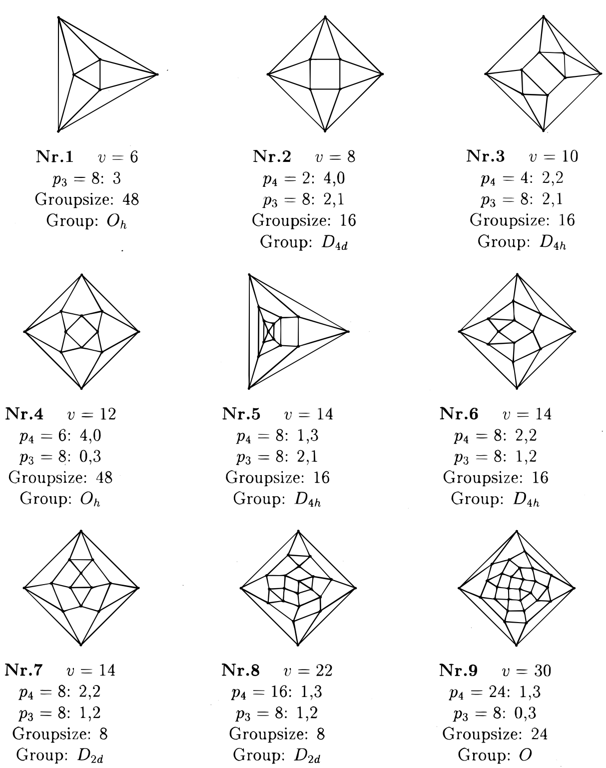


Figure 4. All nine face-regular four-valent polyhedra with $b = 4$.

Proof. Due to the three-connectedness of polyhedra, we get f(3,b) = 3 for all cubic (3, b)-polyhedra. So each triangle is isolated and $3p_3 \leq v = p_3(2 + 6/\mathrm{f}(b,3)) - 4$. Together with equality 1 and $p_3 > 0$, we get f(b,3) $\geq b - 5$ and f(b,3) $\leq$ min(5, b/2). So $b \leq 10$.

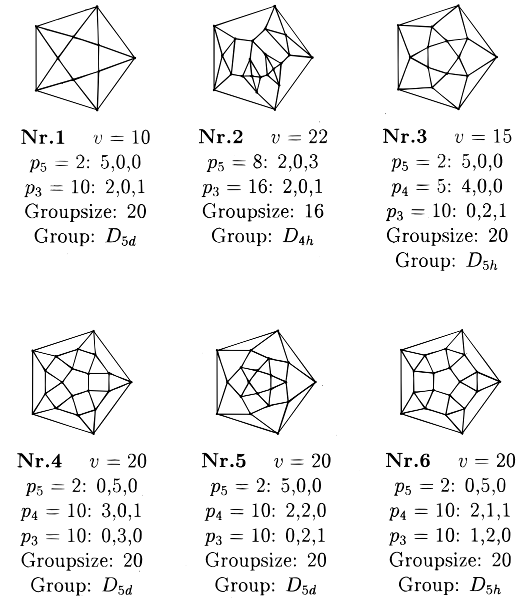


Figure 5. All face-regular four-valent polyhedra with b = 5 and up to 24 faces.

Actually this is a result by Malkevitch[16] for general (that is, not only face-regular) cubic (3, b)-polyhedra.

The remaining possibilities for $b > 6$ are (b, f(b,3); v) $\in$ {(10,5;60), (9,4;52), (8,3;44), (7,2;36), (8,4;24), and (7,3;20). The first four cases are realized by truncations of the dodecahedron, giving only one polyhedron in the first two cases and two nonisomorphic polyhedra in the others. The last two cases are realized by truncations of the cube (giving a unique polyhedron in every case). For $b \leq 6$, all wanted polyhedra are 4, 5, and 10–13 of list 1.

Numbers 1–3 of list 2 are the 6-truncated cube and two 8-truncated dodecahedra. For $b > 6$, there remain three (3,8)-polyhedra, one (3,9)-polyhedron, and one (3,10)-polyhedron.

Remark. If we do not require three-connectedness in theorem 3, more graphs exist, e.g., one for every $b \geq 9$, divisible by 3 with $p_b = 2$, $p_3 = 2b/3$, and $v = 4b/3$.

Theorem 4. *There is only one infinite family of cubic* (4, *b*)*-polyhedra, that is, Prism$_b$ for any $b \geq 3$. The finite families are as follows: (1) 2 80-vertex (4, 7)-polyhedra, coming as the truncation of the dual rombicuboctahedron or dual Miller's solid (its twist) on all 18 4-valent vertexes; (2) 14 polyhedra, coming from those of theorem 3 by the 4-triakon decoration; they have (b, v) ∈{(15,140), (13,116), (11,92), (9,68), (7,44), (12,56), (10,44), (8,32), (6,20); (9,28) (6,14)}; there are exactly two nonisomorphic polyhedra for the third, fourth, and eighth case and 1 for the others; (3) 3 polyhedra (besides the prism) for b = 5 and 6 for b = 6 not covered by previous cases (7–9 and 17–22 of list 1); (4) 2 44-vertex (4,7)-polyhedra 5 and 6 of list 2 (coming by suitable doubling of all 6 isolated 4-gons in 21 and 20 of list 1); (5) the 80-vertex (4,8)-polyhedron of symmetry D_3 (coming by suitable doubling of all 12 isolated 4-gons of the truncation on all 12 4-valent vertices of the dual of the unique 18-vertex 4-valent (3,4)-polyhedron given below, the 3-gons of which are organized into 2 isolated ones and 3 isolated pairs) (Figure 7).*

Proof. The possible values for f(4,b) are 2, 3, and 4. The case f(4,b) = 2 is possible only if the 4-gons form a ring

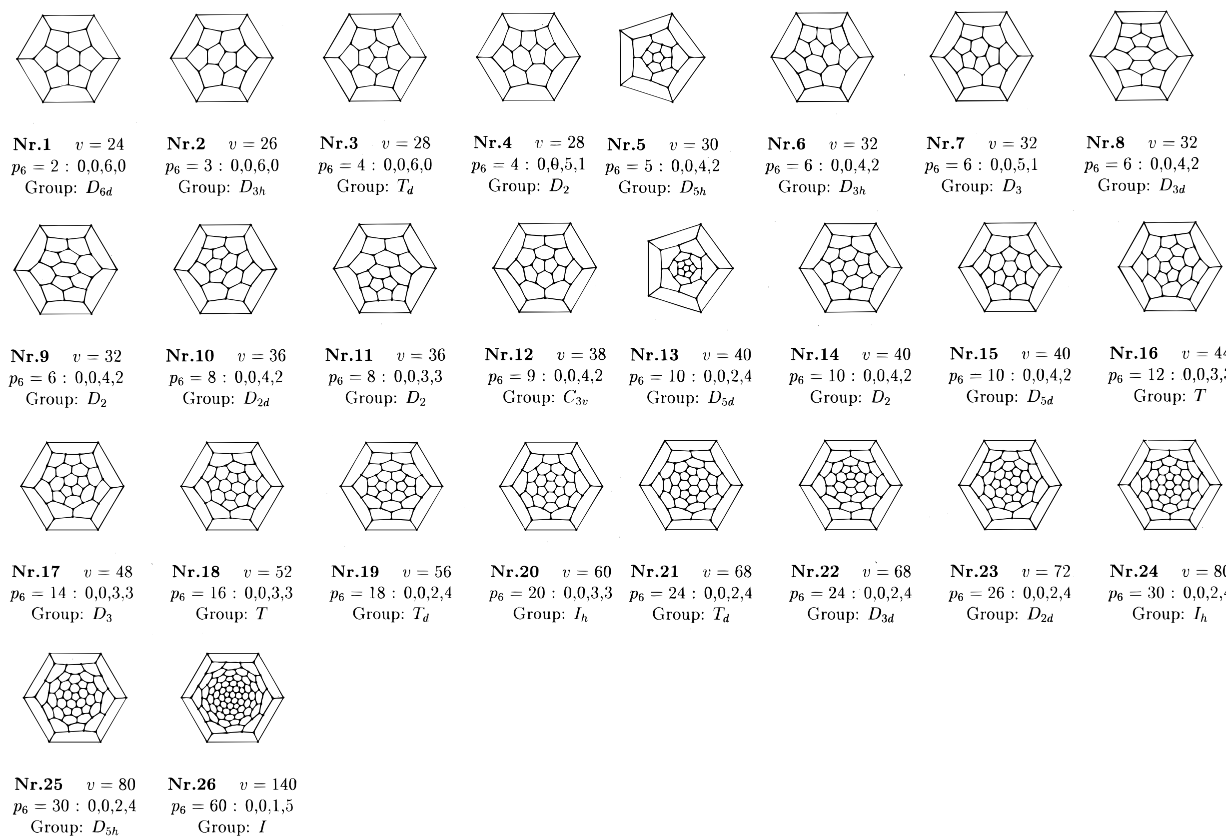


Figure 6. All 26 $6R_j$ fullerenes.

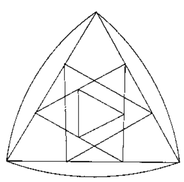


Figure 7. Eighteen-vertex (3,4)-polyhedron.

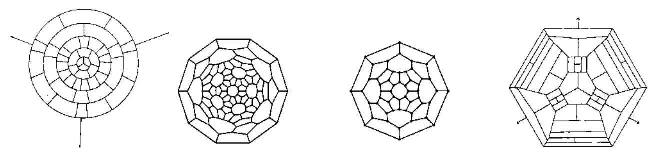


Figure 8. Unique (5,7)-, (5,10)-, (5,8)-, (5,8)-polyhedron with b = 92, 140, 56, 42 and symmetry C_{3b}, I_h, D_h, T_d.

(giving a prism) or isolated 3-rings of 4-gons, which is exactly case 2 of theorem 4.

If $f(4,b) = 4$, all 4-gons are isolated. So $4p_4 \le v$. Equality 1 and $p_4 > 0$ imply $2(b - 6) < f(4,b) \le \min(3, b/2)$. So $b \le 7$ and the only possibility for $b = 7$ is $f(b,4) = 3$, $p_4 = 18$, $p_7 = 24$; $v = 80$. It is exactly case 1 of theorem 4.

In the remaining case $f(4,b) = 3$, the 4-gons are organized into isolated adjacent pairs. So, $6(p_4/2) \le v$ and, using eq 1 and $p_4 > 0$, we get $[3(b - 6)]/2 < f(b,4) \le \min(5, 3b/5)$, which implies $b \le 9$. Moreover, the only possible values for $(b, f(b,4))$ are (9,5), (8,4), (7,2), (7,3), and (7,4). The last subcase is not possible, because it gives $p_4 = 48/5$. The subcase (7,3) gives $v = 44$. A computer search showed that it is exactly case 4 of the theorem. The remaining subcases leave three possibilities: $(b, f(b,4); p_4, p_b; v) \in \{(7,2;24,36;116), (8,4;24,18;80), (9,5;60,36;188)\}$. The first of them can easily be removed by a geometric argument. For the last one, there are eight vertices contained only in 9-gons. It is easy to show that there is only one out of two possible ways in which the pairs of 4-gons can neighbor a 9-gon containing such a vertex. Using this, geometric arguments give a contradiction when trying to construct the polyhedron. The middle one is realized by the polyhedron of case 5 of the theorem. The uniqueness follows from its construction.

Clearly, all remaining wanted polyhedra have $b \le 6$, so they are covered by list 1; it gives the last entry of case 3 in theorem 4.

Remark. The largest face-regular cubic (4,6)- and (5,6)-polyhedra are also (just like in case 1 of Theorem 4) the dual 4-cap or the dual snub cube and the dual 5-cap (putting pyramids on all 5-gonal faces) of the snub dodecahedron.

Theorem 5. *There is only one infinite family of face-regular cubic (5, b)-polyhedra with $b \ge 6$: $Barrel_b$ (two b-gons, separated by two rows of b 5-gons) for any b. The finite families are as follows: (1) 12 (5, 6)-polyhedra, 23-34 in list 1; (2) a unique 92-vertex (5, 7)-polyhedron, organized into concentric 3-, 15-, and 12-ring of 5-gons, separated by 6-, 9-, and 3-ring of 7-gons (as in Figure 8); (3) a unique 44-vertex (5,7)-polyhedron (number 9 in list 2) and 3 polyhedra with $f(5,b) = 2$, also given in Figure 8; the unique 140-vertex (5,10)-polyhedron, the unique 56-vertex (5,8)-polyhedron, and the unique 92-vertex (5,8)-polyhedron with $(p_5, p_8; f(8,5)) = (36,12;6)$.*

Proof. The case $f(5,b) = 1$ gives exactly $Barrel_b$. If $f(5,b) = 5$, then all 5-gons are isolated, so $5p_5 \le v$. So eq 1 and $p_5 > 0$ imply $5(b - 6) < f(b,5) \le 3$, giving $b < 7$.

If $f(5,b) = 4$, then all 5-gons are organized into isolated pairs, so $8(p_5/2) \le v$. Again we get $4(b - 6) < f(b,5) \le 3$ and $b < 7$.

The case f(5,b) = 3 has 5-gons organized in disjoint rings. Let t denote the number of 3-rings among them, so $3p_5 + t \leq v$. The same count as above gives $3(b - 6) < f(b,5) \leq \min(5, (3b(t + 4) - 18)/(t + 16))$. So $b = 7$ is the only possibility for $b > 6$. In the case $b = 7$, we have either f(7,5) = 4 and $t \leq 20$ or f(7,5) = 5 and $t \leq 2$. The first subcase gives $p_5 = 48$ and $p_7 = 36$, and it should be the 164-vertex (5,7)-polyhedron with all 48 5-gons organized into isolated rings. V. P. Grishukhin (private communication) established, case by case, the nonexistence in this subcase. The second subcase is f(7,5) = 5; i.e., 7-gons also should be organized in isolated rings. We get $p_5 = 30$ and $p_7 = 18$; $v = 92$. So 5- and 7-gons should be organized in concentric alternating rings, and only two vertices belong to three faces of the same type. It is easy to show that case 2 of the theorem is the unique possibility for such a polyhedron.

All possibilities with $b = 6$ are covered by list 1 (it is case 1 of the theorem). The only remaining case is f(5,b) = 2, $b > 6$. By using eq 1, we get $2(b - 6) < f(b,5) \leq b$. Clearly, $p_b = p_5(f(5,b)/f(b,5)) = 24/(f(b,5) - 2(b - 6))$ and $v = 2p_5 + 2p_b - 4 = (8b + 20\, f(b,5))/(f(b,5) - 2(b - 6))$. So all possibilities with positive integer v are given by f(b,5) = $b-i$, $b \leq 11 - i$ for $0 \leq i \leq 4$. In the subcase f(b,5) = b, the b-gons are all isolated. It is easy to see that b should be even and that for $b \in \{8,10\}$ the only possibilities are the 140- and 56-vertex polyhedra in case 3 of the theorem. In subcase f(b,5) = $b - 1$, an attempt to construct the structures easily gives the impossibility for $b \in \{9,8\}$ and unicity (the 44-vertex polyhedron of case 3 of the theorem) for $b = 7$. The impossibility of cases (b, f(b,5); v) = (7,5;52), (7,4;68) was checked with the help of a computer. The remaining four cases should be (5,b)-polyhedra with f(5,b) = 2, having (b, f(b,5); p_5, p_b; v) = (7,3;36,24;116), (8,5;60,24;162), (8,6;36,12;92), (9,7;84,24;212). The third possibility is realized, uniquely, by the 92-vertex polyhedron of case 3 in the theorem. It and the nonexistence in the other three subcases can be checked in the following easy way. Let us fix a b-gon $A_0 = (1, 2, ..., b)$ and, without loss of generality, suppose that the other b-gon, say A_1, is adjacent to A_0 by the edge (1,2). In all four subcases, it is not possible that A_0 was adjacent to a 5-gon by the edge (2,3) and to a b-gon by the edge (3,4), because this 5-gon would have 3, not 2 b-gons as neighbors. Now consider the situation when A_0 is adjacent to 5-gons by (2,3) and by (3,4), but not by (4,5). An attempt to construct a polyhedron, respecting our conditions on f(5,b), f(b,5), will continue uniquely and always end in a situation where the construction cannot proceed except in subcase 3. (The most difficult situation is when all b − f(b,5) b neighbors of the original b-gon A_0 are adjacent to it in a row: by (1,2), (2,3), and so on.)

Remark 1. The polyhedron 24 of list 1 (i.e., the fullerene $F_{28}(T_d)$) and the third and second polyhedron in case 3 of theorem 5 come by a 5-triakon decoration of the (fully) truncated tetrahedron, octahedron, and icosahedron, respectively.

The fourth polyhedron of case 3 of theorem 5 (92-vertex (5,8)-polyhedron) comes from the face-regular fullerene $F_{56}(T_d)$ (number 29 of list 1) by the following decoration of (all six) of its hexagons, having two adjacent 5-gons being adjacent on opposite edges: put some "H" with sides parallel to the above opposite edges so that the hexagon will be partitioned into four pentagons. The above face-regular fullerene comes by a 5-triakon decoration (another one, than the one, producing the above 56-vertex (5,8)-polyhedron) of a face-regular 24-vertex (4,6)-polyhedron: the truncated octahedron.

Remark 2. The above theorems 3, 4, and 5 together give the following classification:

Besides the two infinite families, Prism_b and Barrel_b, there are exactly 55 3-valent face-regular bifaced polyhedra. With respect to the number v vertices, they have the face sizes (a, b) as follows.

for $v = 140$:	(4,15), (5,6), (5,10)
for v=116:	(4,13)
for v=92:	two (4,11), (5,7), (5,8)
for v=80:	two (4,7), (4,8), two (5,6)
for v=68:	two (4,9), (5,6)
for v=60:	(3,10), (5,6)
for v=56:	(4,6), (4,12), (5,6), (5,8)
for v=52:	(3,9), (5,6)
for v=44:	two (3,8), three (4,7), (4,10), (5,6), (5,7)
for v=38:	(5,6)
for v=36:	two (3,7)
for v=32:	two (4,6), two (4,8), (5,6)
for v=28:	(3,6), (4,9), (5,6)
for v=26:	(4,6)
for v=24:	(3,8), (4,6)
for v=20:	(3,7), two (4,6)
for v=16:	two (3,6), (4,5)
for v=14:	(4,5), (4,6)
for v=12:	(3,6), (4,5)

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