## Commentary

# What's Been Happening to Undergraduate Mathematics

by David M. Bressoud

For over a decade, the mathematical community has been experimenting with its undergraduate courses and curricula. Creative ideas have been tried. Debates have been generated. Improvements have been discovered, tested, and refined. As a result, there have been fundamental shifts in the way many of us teach mathematics.

The Mathematical Association of America (MAA) through its Committee on the Undergraduate Program in Mathematics (CUPM) is in the process of compiling curricular recommendations that will be communicated to all undergraduate mathematics programs in the United States. Many of the best innovations of the past decade have arisen from collaborations with client disciplines. For this reason, the CUPM has recognized that before it makes any recommendations, it needs to gather information from the chemists, biologists, physicists, engineers, and others who look to mathematics departments to help prepare their students.

To begin this process, a series of Curricular Foundations Workshops were held during 2000. The Chemistry Workshop met concurrently with one of the Biology Workshops at Macalester College in Minnesota, November 2–5. The participants were charged with providing responses to a series of questions asking what chemists expect their students to learn from supporting undergraduate mathematics courses in terms of understanding, content, and use of technology.

This article is an overview of what has been happening in undergraduate mathematics.[1] It summarizes some of the changes that are occurring, changes which the CUPM may choose to further encourage. The companion article by Craig (*1*) is an edited version of the report to the CUPM that was generated by the chemistry workshop. It is our hope that these articles will increase awareness among chemists of what is happening within mathematics and foster greater communication among educators in these disciplines.

The CUPM recommendations are scheduled to appear in draft form in late summer 2001 and the final report to appear in 2002.[2] Reactions from chemists to these articles and to the draft recommendations are eagerly sought.[3]

### An Overview of Recent Reform

The CUPM was established in 1953 to "modernize and upgrade" the mathematics curriculum. At roughly ten-year intervals it has issued curricular recommendations. Into the early 1970s, the focus was on the traditional program designed to prepare students for doctoral study in mathematics, but a sharp decline in mathematics majors during the 1970s led the CUPM to shift its attention to the average students who might choose to major in mathematics. This shift led to its 1981 report, Recommendations for a General Mathematical Sciences Program (*2*). The name itself reflects the recognition that departments should no longer think of themselves as keepers of a single, linear discipline called "Mathematics" but rather as centers for a broad collection of "Mathematical Sciences."

Among the recommendations in the 1981 report:

- The mathematical sciences curriculum should be designed around the abilities and academic needs of the average mathematical sciences student.
- A mathematical sciences program should use interactive classroom teaching to involve students actively in the development of new material.
- Applications should be used to illustrate and motivate material in abstract and applied courses.
- Students should have an opportunity to undertake "real-world" mathematical modeling projects.
- All Mathematical Science majors should be required to take a course in probability and statistics and a course in mathematical modeling.

Initially, this report had very little impact. In fact, it was not formally published until 1989. The pressure to undertake some sort of reform of mathematics education continued to build throughout the 1980s, much of it coming from engineering departments who complained loudly about calculus instruction: math departments were failing too many prospective engineers (typically, one-third of first-semester calculus students would drop out of the course or earn a grade lower than a C), and those who passed did not have the knowledge and skills the engineers required.

In the late 1980s, the National Science Foundation launched a calculus initiative, pumping money into innovative approaches to calculus instruction. This generated both heat and light as the mathematical community debated the merits of what came to be known as the Calculus Reform Movement (*3*).

Most of the experimentation took place within existing courses, especially calculus, but some institutions took a more radical approach that questioned the basic structure of the mathematics curriculum. In 1990, West Point implemented a new curriculum of four mathematics courses taken by all students during the first two years (*4*). The first course studies discrete dynamical systems including matrix algebra and the analysis of systems of equations. This is followed by a two-course sequence of calculus including differential equations, vector calculus, and multivariable calculus. The fourth course is probability and statistics. The University of Tennessee, Knoxville, and the University of Minnesota have developed two-course mathematics sequences for biology majors, courses that shift the focus from traditional calculus to modeling discrete and continuous dynamical systems and put much more emphasis on data analysis and statistics (*5*).

Although some of the projects funded by the NSF have faded from view and others have only survived by moderating their approach, collectively these efforts did create a noticeable shift in mainstream calculus instruction, a shift that has had repercussions throughout the mathematics curriculum. There are several enduring lessons of the Calculus Reform movement that will be elaborated in the next section:

- Key ideas should be treated graphically, numerically, and symbolically.
- Technology should be used as appropriate.
- Applications and interdisciplinary projects should be incorporated.
- Writing should be used to foster critical thinking skills.
- Cooperative learning should be encouraged.
- Appropriate assessment tools should be used to continue to improve teaching and learning.

## What We Have Learned

### *Graphical, Numerical, Symbolic*

One of the simplest yet most important insights of the Calculus Reform movement was that if we want students to be able to handle different representations of functions (graphs, tables of numerical values, or symbolic formulas), then we must supply examples, practice, and evaluations that use each of these approaches. A good example of the questions we now ask that were never asked of calculus students before is the second problem of the free response portion of The College Board's AP Calculus Examination (*6*) given in May 2000, as shown in the box below.

It was instructive to read the solutions. On all three questions, students were much more successful in finding the correct answer for Runner B than for Runner A. In particular, many students showed they knew that distance is obtained by integrating velocity, and they did so correctly for Runner B, but they had no idea how to accomplish this when the velocity was given graphically.

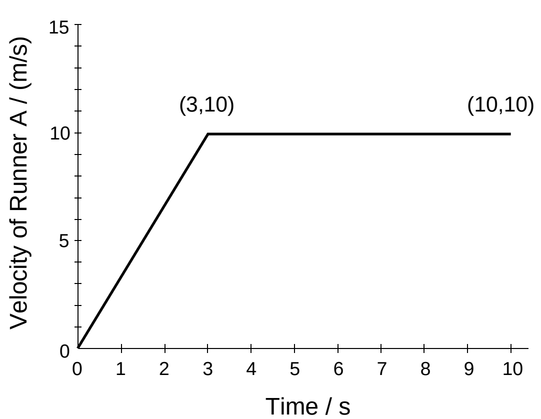
Mathematicians would expect all students who have learned calculus to be able to answer all parts of this question. We have always talked about the graphical meaning of the integral. But it has not been traditional to give students practice with or to test for student understanding of this graphical meaning. This is now changing.

### *Technology*

Few mathematicians hold to the belief that computers and calculators have no place in mathematics instruction, but there is still uncertainty about when and how they can best be used. In particular, there is concern about the investment of time and energy required to overcome a steep learning curve. We have discovered that it is very easy to misuse computers and calculators, but we've also learned how to avoid the most egregious examples of misuse. Some successful programs such as Calculus & Mathematica (*7*) out of the University of Illinois build a curriculum centered in computer explorations. Most of the calculus initiatives use computers or calculators in fundamental but more modest ways. A common example of what has been enabled by technology is the exploration of numerical and graphical solutions to nontrivial differential equations within the first calculus course. Even mainstream texts now provide opportunities for the incorporation of computer-assisted explorations of mathematical ideas, and many if not most undergraduate math courses take some advantage of computers. Graphing calculators are now used in most high schools and are required for students taking Advanced Placement Calculus.

### *Applications*

A common theme in most of the Calculus Reform projects was the use of applications to motivate mathematical ideas and the requirement that students learn how to use their mathematical knowledge for problem solving in such situations. Several of the Reform curricula, such as Project




Two runners, *A* and *B*, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph at the left, which consists of two line segments, shows the velocity, in meters per second, of Runner *A*. The velocity, in meters per second, of Runner *B* is given by the function $v$ defined by $v(t) = 24t/(2t + 3)$.

(a) Find the velocity of Runner *A* and the velocity of Runner *B* at time $t = 2$ seconds. Indicate units of measure.

(b) Find the acceleration of Runner *A* and the acceleration of Runner *B* at time $t = 2$ seconds. Indicate units of measure.

(c) Find the total distance run by Runner *A* and the total distance run by Runner *B* over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

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CALC (*8*) out of Duke University, are structured around application projects. The Consortium for Mathematics and Its Applications (COMAP) (*9*) has been an important source of projects. Writers of mathematics textbooks have become aware of the need for motivating examples and opportunities for in-depth projects, but these are only just beginning to gain wide acceptance.

### *Writing*

Mathematicians are still uncomfortable giving and grading assignments that require a great deal of writing, but such assignments are generally recognized as one of the best tools for measuring what a student has truly assimilated. Writing assignments can be used to teach critical thinking skills and to force students to clarify and so cement their understanding of concepts. In addition, the ability to write crisply and precisely should be a skill that we want mathematics students to acquire (*10*).

### *Cooperative Learning*

The importance of group projects is not universally accepted among mathematicians. Much less is there any consensus on how cooperative learning should be used. But there is considerable evidence that when students wrestle with the concepts of mathematics on their own and then come together to share their questions, insights, and understandings, a great deal of learning takes place. Group projects can also be an effective tool for teaching students how to tackle unfamiliar and complex problems. Watching peers unravel and solve a difficult problem can help create understanding and build confidence.

### *Assessment*

Assessment has many levels and many meanings. One insight is that what we do not test, students will not learn, even though it is as basic as the graphical interpretation of the integral. Another aspect of assessment is the research now being done in mathematics education to analyze how students process and build a working understanding of the concepts of undergraduate mathematics. Part of assessment is learning how to determine, day by day, where students are having difficulties. Assessment also encompasses the determination of the effectiveness of entire programs, an issue that has been pushed to the front by the proliferation of experimental approaches to mathematics education.

These many facets of assessment can be summarized in the recognition that we are still imperfect teachers. In the great balancing act between what is desirable and what is practicable, there are always opportunities for us to do a better job within the range of parameters in which we must work. Assessment is the continuing process of finding that better balance point (*11*).

### Summary

Change does not come easily, especially at large institutions. The American Mathematical Society (AMS)—whose primary focus is on research mathematicians and research universities—has come to recognize the critical role of service and undergraduate teaching, especially in these times of tight budgets and strict accountability. In 1999 it published Towards Excellence: Leading a Doctoral Mathematics Department in the 21st Century (*12*). The report summarized its advice in nine recommendations, of which two are immediately relevant to this article:

2. Make a commitment to quality undergraduate instruction.
6. Build strong relationships on campus. Faculty should make building strong relations with other departments and the campus administration a conscious departmental goal.

The report also describes five mathematics departments that have done an excellent, innovative job of reforming undergraduate mathematics education: University of Michigan, Oklahoma State University, University of Chicago, University of Arizona, and University of Texas at Austin.

Change is happening. The mathematical community realizes that it cannot afford to ignore undergraduate education or the needs of client disciplines. The Curriculum Foundations Workshops were a first step in establishing ongoing and diverse lines of communications (*13*). It is our hope that articles such as this will help further encourage this dialog.

### Notes

1. This article does not necessarily represent the views of the Mathematical Association of America or its Committee on the Undergraduate Program in Mathematics.

2. The CUPM recommendations will be available from the MAA at *www.maa.org*.

3. To contribute to this curriculum discussion, send an email message to the CUPM at *cupm-curric@maa.org* or by U.S. mail to Harriet Pollatsek, Chair of CUPM, Department of Mathematics and Statistics, Mt. Holyoke College, South Hadley, MA 01075.

### Literature Cited

1. Craig, N. C. *J. Chem. Educ.* **2001,** *78*, 582–586.
2. Recommendations for a General Mathematical Sciences Program was published by the MAA in 1989 in *Reshaping College Mathematics*. It is available at *http://www.maa.org/books/nte13.html*. Other MAA resources for undergraduate mathematics education can be found at *http://www.maa.org/pubs/books/teaching.html*.
3. A survey of the Calculus Reform Movement and its accomplishments is given in the report *Assessing Calculus Reform Efforts*, by Alan C. Tucker and James R. C. Leitzel, MAA, 1995. Available at *http://www.maa.org/pubs/books/acre.html*.
4. A description of West Point's Core Curriculum in Mathematics can be found at *http://www.dean.usma.edu/math/core/*.
5. A description of the Quantitative Curriculum for Life Science Students at the University of Tennessee, Knoxville is at *http://ecology.tiem.utk.edu/~gross/quant.lifesci.html*. The textbook written for the University of Minnesota course is *Calculus for Biology and Medicine*, by Claudia Neuhauser, Prentice Hall, 2000.

6. Descriptions of The College Board's AP Calculus Program and examples of recent exams can be found at *http://www.collegeboard.org/ap/calculus/*.
7. The Calculus & Mathematica homepage is at *http://www-cm.math.uiuc.edu/*.
8. The Project CALC homepage is at *http://www.math.duke.edu/education/proj_calc/index.html*.
9. The COMAP homepage is at *http://www.comap.com*. Among their publications is a textbook based on the West Point curriculum, *Principles and Practice of Mathematics*, by W. Meyer et al., 1997.
10. The MAA has published *Writing in the Teaching and Learning of Mathematics*, by John Meier and Thomas Rishel, 1998. Available at *http://www.maa.org/pubs/books/nte48.html*.
11. An MAA survey of techniques for assessment is published in *Assessment Practices in Undergraduate Mathematics*, by Bonnie Gold, Sandra Keith, and William Marion, 1999. Available at *http://www.maa.org/pubs/books/nte49.html*.
12. An electronic version of *Towards Excellence: Leading a Doctoral Mathematics Department in the 21st Century* is available at *http://www.ams.org/towardsexcellence*.
13. The report from the Chemistry workshop as well as reports from other workshops can be found at *http://academic.bowdoin.edu/math/faculty/barker/dissemination*; first choose *Curriculum_Foundations*, then choose *CF_Chemistry.doc*.


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