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Controller Design for Nonlinear Process Systems via Variable Transformations

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Departing from the traditional approach of local linearization followed by linear controller design for the thus "linearized" system, it is shown in this paper how transformations may be found which transform the nonlinear system into one that is exactly linear. The new approach is based on the hypothesis that a system which is nonlinear in its original variables is linear in some transformation of these original variables. Constructive methods for finding the appropriate transformations are presented, and practical guidelines to facilitate design are included. The design of a controller for the linear transformed system may then be carried out with a great deal of facility (the resulting controller will of course be nonlinear when recast in terms of the original system variables). The specific problem of level control in process flow systems having various geometric configurations is used to illustrate the potentials of this approach. An example simulation is used to demonstrate the design and performance of these controllers.

1. Introduction

It is well-known that the great majority of important chemical processes exhibit nonlinear dynamic behaviors. Thus, most of the practical process control problems will involve nonlinear systems. This fact notwithstanding, the bulk of existing control theory involves the design of linear controllers for linear systems (cf.: Ray, 1981; Stephanopoulos, 1984). Only a modest collection of results is available which may be directly applied to nonlinear systems.

The traditional and easiest approach to the controller design problem for nonlinear systems involves linearizing the modeling equation around a steady state and applying linear control theory results. It is obvious that the controller performance in this case will deteriorate as the process moves further away from the steady state around which the model was linearized.

Apart from the "local linearization" approach, there are a few other "special-purpose" design procedures (cf.: Ray, 1981) which may be applied directly to nonlinear systems. However, as noted by Ray (1981), these usually have limited applicability and are often based on accumulated experience with a special type of nonlinear system.

When the controller design problem for nonlinear systems is approached in a more general fashion, even fewer results are available. These techniques are usually quite complicated (e.g.: Sommer, 1980; Watanabe and Himmelblau, 1982) and often require the use of sophisticated mathematical tools which most practitioners are not familiar with (cf. for example: Sommer, 1980; Marino, 1984; Hoo and Kantor, 1984).

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It is obvious that because of the very nature of nonlinear systems, controller design from a general viewpoint will be inherently more difficult than for linear systems. Nevertheless, the objective of this paper is to present a technique which attempts to strike a balance between the generality of treatment on the one hand and the usual accompanying complexity of actual design and implementation on the other.

The approach is based on the hypothesis that if the proper transformation can be found, a system which is nonlinear in its original variables can be made into a linear system in a new set of variables. Controller design for the transformed system may then be based on the powerful body of results available for linear systems. When the controller is implemented in the original system variables, it will of course be nonlinear. Although the principle behind this approach is not new (for example, this is a frequently used strategy for solving certain nonlinear differential equations), the usual obstacle consists in developing constructive methods for finding the appropriate transformations which are general enough without sacrificing simplicity and transparency. Such a method is developed in this paper.

The nonlinear control problem is first treated in a general fashion, and the general principles are applied to the problem of level control in process flow systems with cylindrical, conical, and spherical configurations. A simulation example is then presented to illustrate the design procedure and the controller performance.

2. A Special Linear System

Consider the system described by

$$\frac{dz}{dt} = a + bv \tag{1}$$

where z is the system state variable, $v(t)$ is the system input variable, a and b are system parameters. The general state space representation for linear systems is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + Bu + \Gamma d \tag{2}$$

Note that apart from the different notation used to represent the state and control variables in the two equations, (1) is a special form of (2) with $A = 0$, $B = b$, $\Gamma = 1$, and $d = a$, a constant. We aim to show first that irrespective of the values of a and b, the system in (1) may be driven to *any* desired state z^* with a regular PI controller. The reason for introducing this particular system and discussing its controllability with PI controllers will become obvious in the next section.

A PI controller for the system in (1) calculates the control action $v(t)$ from

$$v(t) = K_c\left\{(z^* - z) + \frac{1}{\tau_I}\int_0^t (z^* - z)\,\mathrm{d}t\right\} \tag{3}$$

where K_c and $1/\tau_I$ are the usual PI controller parameters. Introducing (3) into (1) results in a representation of the combined system/controller dynamic behavior:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = a + bK_c(z^* - z) + \frac{bK_c}{\tau_I}\int_0^t (z^* - z)\,\mathrm{d}t \tag{4}$$

This is an integrodifferential equation which may be differentiated once to give, upon some slight rearrangements,

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2} + bK_c\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{bK_c}{\tau_I}z = \frac{bK_c}{\tau_I}z^* \tag{5}$$

a second-order differential equation in z. Provided K_c and τ_I are chosen such that the two roots of the characteristic equation for the homogeneous part of (5) both have negative real parts, the overall closed-loop system is *stable*, and the steady state achieved after a disturbance may be evaluated by setting all the time derivatives to zero in (5). We observe from the exercise that irrespective of a or b, z equals z^* at steady state. We may therefore conclude that the system in (1) is "PI controllable" in the above sense.

3. A General Treatment of the Control Problem.

3.1. Transformation of Nonlinear Models. The general, single-input, single-output nonlinear control system will be represented by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,u) \tag{6}$$

where $F(\cdot)$ is an arbitrary nonlinear function of x, the system state variable, and u, the control variable. Throughout this discussion, we shall consider that the system state variable is directly available for measurement. (If this were not the case, it is of course possible to take advantage of the existence of nonlinear filters (cf.: Jazwinski, 1970; Ray, 1981) which can be used for providing the required state estimates for such nonlinear systems.)

We start by noting the very crucial fact that it is *always* possible to break $F(x,u)$ up as follows:

$$F(x,u) = c_1f_1(x) + c_2f_2(x,u) \tag{7}$$

where $f_1(x)$ is a function of x, and x alone, and $f_2(x,u)$ is a function of both x and u. Both f_1 and f_2 are taken to be nonlinear, and no restrictions are placed on their functional forms. c_1 and c_2 are constants.

Now consider the transformation

$$z = g(x) \tag{8}$$

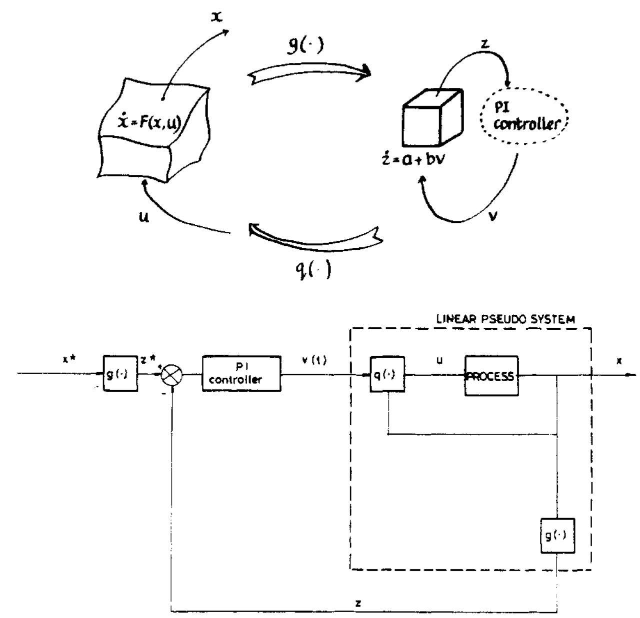

Figure 1. (a, top) Conceptual configuration of the transformation controller. (b, bottom) Block diagram configuration of the nonlinear transformation controller.

where $g(\cdot)$ is a yet undetermined function of x and x alone. Differentiating (8) with respect to t gives

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}g}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \tag{9}$$

From (6) and (7),

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c_1f_1(x) + c_2f_2(x,u) \tag{10}$$

and introducing (10) into (9) gives

$$\frac{\mathrm{d}z}{\mathrm{d}t} = c_1f_1(x)\frac{\mathrm{d}g}{\mathrm{d}x} + c_2f_2(x,u)\frac{\mathrm{d}g}{\mathrm{d}x} \tag{11}$$

Observe that we may now choose $g(x)$ such that

$$c_1f_1(x)\frac{\mathrm{d}g}{\mathrm{d}x} = a \tag{12}$$

and

$$c_2f_2(x,u)\frac{\mathrm{d}g}{\mathrm{d}x} = bv \tag{13}$$

and the nonlinear system is tranformed to

$$\frac{\mathrm{d}z}{\mathrm{d}t} = a + bv \tag{14}$$

a linear system which may be efectively controlled with a PI-type controller (as was shown in the preceding section).

We shall shortly discuss strategies for choosing $g(x)$ such that (12) and (13) are satisfied. For now, we note that for a given $g(x)$, a PI controller may be designed with a great deal of facility, which receives information about z and outputs v. The actually implemented u, however, may be evaluated by solving (13) for u, i.e., given $g(\cdot)$ and f_2, solving for u as

$$u = q(x,v) \tag{15}$$

Equation 15 is the nonlinear control law *actually* implemented on the nonlinear system, but observe that we only have to design a PI controller for the linear system in (14), a very straightforward task. Figure 1 shows conceptual configurations of this design strategy. As far as the controller is concerned, it is interfaced with the linear pseudosystem given in (1) or (14). This is made possible by the use of the transformations $g(\cdot)$ and $q(\cdot)$. Thus, the entire design procedure boils down simply to the determination of $g(\cdot)$ (which transforms the system to the linear form in (14)) and given $g(\cdot)$ to obtain the corresponding $q(\cdot)$ which recovers the actual u to be implemented from the output of the transformed system controller.

We shall now discuss how to obtain $g(\cdot)$ and $q(\cdot)$.

3.2. Obtaining the Required Transformations. From (12)

$$f_1(x)\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{a}{c_1} = k_1 \text{ (another constant)}$$

we may obtain g in any of two ways:

(a) By Direct Integration. Since f_1 is a function of x alone, the variables in (12) are separable and we obtain

$$g(x) = k_1 \int \frac{\mathrm{d}x}{f_1(x)} \tag{16}$$

Remarks. (i) While we recognize the fact that in general arbitrary f_1 functions may be constructed which may make the integral in (16) difficult to evaluate, we note that the functions which occur in chemical process models are usually powers of x, exponential, logarithmic, or possibly sinusoidal functions, and these are such that the integral is readily evaluated in *closed form.* (This is demonstrated in the next section for process flow system models.)

(ii) In situations where the integral might be difficult to evaluate, it is possible to obtain $g(\cdot)$ by inspection (this may sometimes be easier when the integral is difficult to evaluate).

(b) By Inspection. In finding particular integrands for nonhomogeneous differential equations, the method of undetermined coefficients (cf. Mickley et al. (1957) for example) involves entertaining a trial solution whose form is dictated by inspection of the forcing function. The unknown coefficients of the trial solution are determined such that the two sides of the equation match identically. This same strategy may be used in (12). Given the form of $f_1(x)$ (recall that this is part of the system model), a trial functional form is entertained for $g(\cdot)$ whose coefficients are to be determined such that (12) is exactly satisfied.

We shall demonstrate in the next section how each of these approaches may be used in example situations.

To obtain $q(\cdot)$, we recall that from (12)

$$\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{k_1}{f_1(x)}$$

which implies that from (13),

$$f_2(x,u) = \frac{bvf_1(x)}{c_2k_1} = k_2vf_1(x) \tag{17}$$

This is the equation to be solved for u. Because we cannot specify a *general* functional form for $f_2(\cdot)$, it is not possible to carry this any further without loss of generality. However, we may make the following remarks.

(i) No matter how complicated the functional form of $f_2(\cdot)$ is, given x (available system information) and v (the PI controller output), (17) can *always* be solved for u: *if not in closed form, then numerically.* (Computer programs are of course available that routinely solve nonlinear equations very rapidly.)

(ii) Most process models of practical importance are *control linear*; i.e., $f_2(x,u)$ is of the form

$$f_2(x,u) = uf_3(x) \tag{18}$$

Where this is the case, (17) merely reduces to

$$u = \frac{k_2vf_1(x)}{f_3(x)} = q(v,x) \tag{19}$$

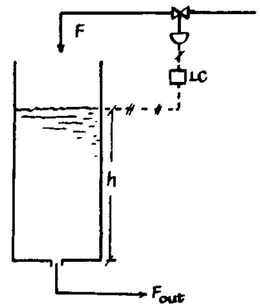

Figure 2. Level control in a cylindrical receiver.

4. Application to Level Control in Process Flow Systems

We shall now apply the foregoing principles to the specific problem of level control in flow systems having various geometric configurations.

4.1. Level Control in a Cylindrical Receiver. Consider the level control system shown in Figure 2. An overall material balance around the system yields the mathematical model

$$A_c\frac{\mathrm{d}h}{\mathrm{d}t} = F - F_{\text{out}} \tag{20}$$

Now, it is known that F_{out} is given by

$$F_{\text{out}} = Ch^{1/2} \tag{21}$$

where C is a constant related to the discharge coefficient and other properties of the outlet orifice.

Thus, (20) becomes

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{A_c}F - Kh^{1/2} \tag{22}$$

where $K = C/A_c$

Note the following: (i) The state variable of interest is h; the control variable is F. (ii) The model is nonlinear in h because of the presence of $h^{1/2}$. We may recast this as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -Kx^{1/2} + \frac{1}{A_c}u \tag{23}$$

where we have merely replaced h with x and F with u to conform to the notation of sections 2 and 3.

Observe that in accordance with (7), we have for this system that

$$f_1(x) = x^{1/2} \qquad c_1 = -K \tag{24}$$

$$f_2(x,u) = u \qquad c_2 = \frac{1}{A_c} \tag{25}$$

(Note that $f_2(x,u)$ is of the form $uf_3(x)$, where $f_3(x) = 1$.) We now need to obtain the transformation $g(\cdot)$, and subsequently $q(\cdot)$, such that in accordance with the preceding discussion, (23) becomes linear in the new variable, $z = g(x)$.

From (12), $g(\cdot)$ is required to be chosen such that

$$x^{1/2}\frac{\mathrm{d}g}{\mathrm{d}x} = k_1 \tag{26}$$

(where $k_1 = -a/K$) and, by direct integration, observe that the required $g(\cdot)$ is given by

$$g(x) = k_1\int\frac{\mathrm{d}x}{x^{1/2}}$$

$$g(x) = 2k_1x^{1/2} \tag{27}$$

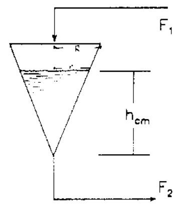

Figure 3. Level control in a conical tank.

On the other hand, by inspection of (26), we may postulate a trial function

$$g(x) = Ax^n \tag{28}$$

motivated by the fact that $f_1(x)$ in this case is a power of x.

Differentiating (28) and introducing this into (26) gives

$$-KAnx^{n-1/2} = a \tag{29}$$

which will be satisfied if and only if

$$n = \tfrac{1}{2}$$

and

$$A = \frac{-2a}{K} = 2k_1$$

as earlier obtained by direct integration. Thus, the required linearizing transformation is

$$z = 2k_1x^{1/2} \tag{30}$$

or merely $z = x^{1/2}$, since the constants are merely scaling factors.

From (13) or (17), we have

$$f_2(x,u) = k_2vf_1(x)$$

Since $f_2(x,u)$ is given by (25), this implies

$$u = k_2vx^{1/2} \tag{31}$$

Remarks. The implications of the above are as follows: (i) In terms of the variable z (which from (30) is merely a scalar multiple of the square root of the height), the modeling equations describing the system in Figure 2 is LINEAR.

(ii) A simple PI controller receiving information based on this square root transformation is all that is required. The controller is easily tuned since the system will now be modeled by

$$\frac{dz}{dt} = a + bv \tag{32}$$

and the closed-loop response is as shown in eq 5, i.e.,

$$\frac{d^2z}{dt^2} + bK_c\frac{dz}{dt} + \frac{bK_c}{\tau_I}z = \frac{bK_c}{\tau_I}z^*$$

where z^* is the desired value for the transformed variable z. Note that since the values which a and b take are entirely up to the designer, these values, as well as K_c and τ_I values, may be chosen to give any desired closed-loop response. (Note how transparent the choice of parameters is.)

(iii) The output of the PI controller is $v(t)$, but the actually implemented control action (the inlet flow rate, in this case) is obtained as a scalar multiple of the product of v and the square root of the height, according to (31).

(iv) It is of course obvious that the original variable x must attain its desired set point when $z = x^{1/2}$ attains its desired set point, z^*.

In the next section, a simulation example is used to test the performance of this controller.

4.2. Level Control in a Conical Receiver. If the liquid level in a conical tank as shown in Figure 3 is now to be controlled, the problem takes on a new configuration.

The mathematical model is

$$\frac{d}{dt}(V) = F_1 - F_2 \tag{33}$$

where the liquid volume in the tank V is given by

$$V = \tfrac{1}{3}\pi r^2h$$

and eliminating r, using properties of similar triangles, i.e.,

$$\frac{r}{h} = \frac{R}{H}$$

we have

$$V = \frac{1}{3}\frac{R}{H}\pi h^3 = \frac{1}{3}\alpha h^3$$

R and H being known tank dimensions. Equation 33 therefore becomes

$$\tfrac{1}{3}\alpha\frac{d}{dt}(h^3) = F_1 - Ch^{1/2} \tag{34}$$

since the outflow rate is proportional to $h^{1/2}$. Simplifying further,

$$\frac{dh}{dt} = \frac{F_1}{\alpha h^2} - Kh^{-3/2} \tag{35}$$

In terms of x and u, this becomes

$$\frac{dx}{dt} = -Kx^{-3/2} + \alpha ux^{-2} \tag{36}$$

Note how severely nonlinear this model is. Yet, according to our previous discussion, (36) may easily be transformed to

$$\frac{dz}{dt} = a + bv$$

The transformation required in this case is obtained as follows. In accordance, once again, with (7), for the system in (36)

$$f_1(x) = x^{-3/2} \qquad c_1 = -K \tag{37}$$

$$f_2(x,u) = ux^{-2} \qquad c_2 = \alpha \tag{38}$$

$f_2(\cdot)$ is of the form $uf_3(x)$ with $f_3(x) = x^{-2}$. The required $g(\cdot)$ is obtained from

$$x^{-3/2}\frac{dg}{dx} = k_1 \qquad (k_1 = -a/K) \tag{39}$$

By direct integration, we obtain

$$z = g(x) = \tfrac{2}{5}k_1x^{5/2} \tag{40}$$

The same result is of course obtainable by inspection as demonstrated in the preceding subsection. The control implementation scheme is obtained from

$$f_2(x,u) = k_2vf_1(x)$$

In this case

$$ux^{-2} = k_2vx^{-3/2}$$

or

$$u = k_2vx^{1/2} \tag{41}$$

Thus, for the conical tank system, (i) the variable

$$z = (\tfrac{2}{5}k_1)x^{5/2}$$

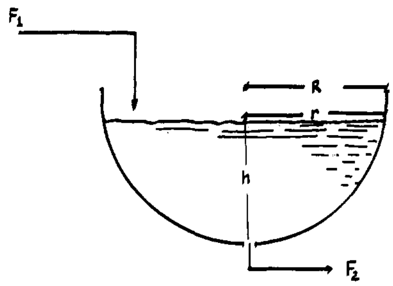

Figure 4. Level control in a hemispherical tank.

is what is required to transform the original severely nonlinear equation to the linear.

$$\frac{dz}{dt} = a + bv \tag{42}$$

(ii) A regular PI controller gives an output $v(t)$ and effectively controls z, the state variable of the linear pseudosystem in (42). $v(t)$ is converted to the flow rate to be implemented on the *real* system using (41).

$$u = k_2 v x^{1/2}$$

4.3. Level Control in a Hemispherical Receiver. For the hemispherical receiver shown in Figure 4, the problem takes on yet another different configuration.

The mathematical model is given by

$$\frac{d}{dt}(V) = F_1 - Ch^{1/2} \tag{43}$$

where

$$V = \tfrac{1}{6}\pi h(3r^2 + h^2)$$

is the liquid volume in the tank. We may eliminate r by using the Pythagorean relationship that

$$R^2 = r^2 + (R - h)^2$$

Further elementary algebraic manipulations yield

$$V = \pi(Rh^2 - \tfrac{1}{3}h^3) \tag{44}$$

such that (43) becomes

$$\frac{dh}{dt} = \frac{1}{\pi}\frac{1}{(2Rh - h^2)}(F_1 - Ch^{1/2}) \tag{45}$$

or, in terms of x and u,

$$\frac{dx}{dt} = \frac{-Kx^{1/2}}{(2Rx - x^2)} + \frac{1}{\pi}\frac{u}{(2Rx - x^2)} \tag{46}$$

Again, note how severely nonlinear this model is. For this system,

$$f_1(x) = \frac{x^{1/2}}{(2Rx - x^2)} \qquad c_1 = -K$$

$$f_2(x,u) = \frac{u}{(2Rx - x^2)} \qquad c_2 = \frac{1}{\pi}$$

Again, $f_2(\cdot)$ is of the form $uf_3(x)$.

The transformation

$$z = g(x)$$

required to transform (46) into the linear form in (42) is obtained from

$$\frac{x^{1/2}}{(2Rx - x^2)}\frac{dg}{dx} = k_1 \tag{47}$$

Despite the formidable appearance of (47), $g(\cdot)$ is easily obtained by direct integration:

$$g(x) = k_1 \int \frac{2Rx - x^2}{x^{1/2}}\,dx = k_1\frac{4R}{3}x^{3/2} - \frac{2}{5}x^{5/2}$$

$$g(x) = k_1 x^{3/2}\left[\frac{4R}{3} - \frac{2}{5}x\right] \tag{48}$$

And from

$$f_2(x,u) = k_2 v f_1(x)$$

i.e.,

$$\frac{u}{(2Rx - x^2)} = k_2 v\,\frac{x^{1/2}}{(2Rx - x^2)}$$

we obtain the expression for the control implementation, again,

$$u = k_2 v x^{1/2} \tag{49}$$

Thus, for the hemispherical tank, (i) we require the use of the variable

$$z = k_1 x^{3/2}\left[\frac{4R}{3} - \frac{2}{5}x\right]$$

to transform the model in (46) to the linear form given in (42). Note that although this is not as simple an expression as with the other flow systems, given x, the actual height in the tank, it is quite straightforward to calculate z.

(ii) The PI controller output is converted to the actual flow rate by using (49). It is interesting to note that irrespective of the geometrical configuration of the flow system, and irrespective of the derived function $z = g(x)$ required for transforming the nonlinear model to the linear PI controllable one in (42), the obtained control implementation laws are IDENTICAL (eq 31, 41, and 49). Notice therefore how this technique uncovers fundamental properties of systems in question.

4.4. Summary. We now present a summary of the procedure for the design and implementation of these controllers.

(a) Design. 1. Obtain nonlinear model

$$\frac{dx}{dt} = F(x,u)$$

and break $F(x,u)$ up to obtain

$$\frac{dx}{dt} = c_1 f_1(x) + c_2 f_2(x,u)$$

2. To transform the nonlinear model into

$$\frac{dz}{dt} = a + bv$$

obtain the z required to achieve the transformation from

$$z = g(x) = k_1 \int \frac{dx}{f_1(x)} \qquad k_1 = a/c_1$$

(or by direct inspection using (12)).

3. Obtain the control implementation law by solving

$$f_2(x,u) = k_2 v f_1(x)$$

for u. Usually $f_2(x,u) = uf_3(x)$; thus,

$$u = \frac{k_2 v f_1(x)}{f_3(x)}$$

(b) Implementation. 1. Obtain actual x measurement from the system.

2. Calculate z from the functional form of $g(x)$ (e.g., for the conical tank system from (40), obtain z as ${}^2/_5 k_1 x^{5/2}$).

3. The tuned PI controller receives z, compares this with the desired value z^*, and prescribes the control action $v(t)$ for the linear pseudosystem.

4. The control action which is implemented on the actual system is obtained from

$$u = k_2 v(t) x^{1/2}$$

a control law nonlinear in the original system variable x.

Let us conclude this section by noting that the reader may want to verify, as an exercise, the fact that the derived functions $z = g(x)$ actually transform the system models as we have asserted. This may be done by starting with the given $z = g(x)$ and carrying out a variable transformation on the original model. In each case, it would be observed that the result will always be

$$\frac{dz}{dt} = a + bv$$

5. Some Practical Design Guidelines

5.1. Breaking up $F(x,u)$ into $c_1 f_1(x)$ and $c_2 f_2(x,u)$. Ordinarily, this involves collecting together whatever portion of $F(x,u)$ is a function of x *alone* as $c_1 f_1(x)$, while the remainder is simply assigned to $c_2 f_2(x,u)$, in which case $c_2 f_2(x,u)$ will contain functions of u alone and/or functions of x and u in inextricable combinations.

However, there is no reason why a part of $c_2 f_2(x,u)$ cannot contain a function of x alone, since in principle $f_2(x,u)$, by its strictest definition as a function of x and u, can contain functions of x alone along with those listed above. Clearly then, the choice of $c_1 f_1(x)$ and $c_2 f_2(x,u)$ for each $F(x,u)$ may not always be unique.

The following guidelines may be employed in making decisions as to what may be assigned to $c_1 f_1(x)$ and $c_2 f_2(x,u)$.

(i) When $F(x,u)$ is such that it can only be broken up in one way, obviously the question of choice does not arise. Thus, first ascertain if there indeed is more than one choice possible.

(ii) When there are multiple possible selections, a choice of an $f_1(x)$ whose reciprocal is easily integrable (recall (16)) and an $f_2(x,u)$ which is easily invertible in order to yield u is to be preferred. A note of caution is in order here: in some cases, it may not be possible to achieve both objectives simultaneously in any one choice. In such cases, the recommendation is to aim for an integrable reciprocal of $f_1(x)$ and rely upon numerical techniques for recovering u from the resulting $c_2 f_2(x,u)$. This is a question of the degrees of freedom; whatever is not in $c_1 f_1(x)$ is automatically in $c_2 f_2(x,u)$. The following are guidelines for selecting each one in its own right.

5.2. Choosing $f_1(x)$. (i) The only hard-and-fast constraint is that this must be a function of x alone. Even though *any* function of x satisfies this condition, we know that obtaining the required transformation $g(x)$ involves integrating the reciprocal of $f_1(x)$. If we have a choice, we must therefore opt for an $f_1(x)$ which will make this operation easier to perform.

(ii) Some functions whose reciprocals are easy to integrate (and which occur more frequently than others in chemical process models) include, for example, $\exp(x)$, $\ln x$, x^n, etc., and products thereof. (The products may be dealt with via integration by parts.)

5.3. Choosing $f_2(x,u)$. (i) The major design objective to be put in consideration here is the recovery of u, the actual control to be implemented on the system. Since this is done by solving (17) for u explicitly (if possible), then any choice of $f_2(x,u)$ which makes this maneuver easy to perform will be favored.

(ii) Unlike their linear counterparts, however, nonlinear functions can occur in far too many variations that an exhaustive treatment of the possible forms which $f_2(x,u)$ can assume, and how to solve (17) in each case is obviously impossible.

Nevertheless, certain easily identifiable forms which $f_2(x,u)$ can take such that (17) will be easily solved for u are shown below:

(a) $\quad f_2(x,u) = h(u)$ (a function of u alone)

(b) $f_2(x,u) = h(u)p(x)$ (i.e., an $f_2(\cdot)$ with a separable kernel; note that $uf_3(x)$ is a special case of this form)

(c) $\quad f_2(x,u) = h(u)p_1(x) + p_2(x)$

with $h(u)$ in each situation being an invertible function of u, say u^n, e^u, $\ln u$, $\sin ku$, etc. The functions of x have no restrictions whatever on the forms they can take.

Observe now that in each case, (17) becomes, respectively,

for (a) $\quad h(u) = k_2 v f_1(x)$

for (b) $\quad h(u) = k_2 v f_1(x)/p(x)$

for (c) $\quad h(u) = (k_2 v f_1(x) - p_2(x))/p_1(x)$

from which u is straightforwardly obtained from the inverse of the $h(\cdot)$ function.

To further clarify this last point, let us take as an example, the situation with $h(u) = e^u$, and $f_2(x,u)$ of type (b). Then, (17) becomes

$$e^u = k_2 v f_1(x)/p(x)$$

clearly giving $u = \ln\,[k_2 v f_1(x)/p(x)]$.

(iii) Many more *specific* cases (not listed above) exist for which ingenious methods can be devised for solving for u in (17). These are too arbitrary to be catalogued here. The control engineer must exercise judgement and may have to apply a great deal of ingenuity with this question of choosing $f_2(x,u)$ and inverting (17) for u.

(iv) Finally, where no closed form solution exists for u in (17) (or where one fails to see the means of obtaining such solutions even though they exist), one can always fall back on numerical techniques.

6. Simulation Results

The main aim of this section is not so much to *compare* the performance of these nonlinear transformation controllers with regular linear control strategies. It should be abundantly clear that because it is easier to design a controller for the system

$$\frac{dz}{dt} = a + bv$$

than for

$$\frac{dx}{dt} = F(x,u)$$

we have a better handle on the performance of the nonlinear controller, and it is as such not a difficult task finding such a nonlinear controller which would perform better than a regular linear controller. This section is therefore more concerned with demonstrating through example how such controllers are designed and implemented.

We chose for our example the control of water level in a cylindrical receiver.

The cylindrical pipe has an internal diameter of 5.3 cm (i.e., cross-sectional area = 22.062 cm^2), and from previous

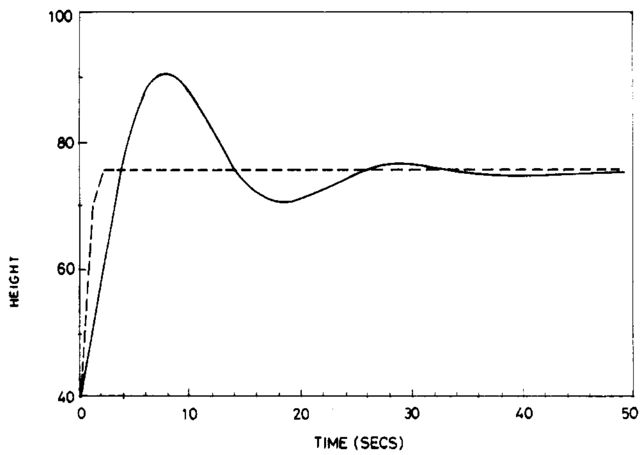

Figure 5. System response to set-point change: (- - -) nonlinear controller, (—) linear controller.

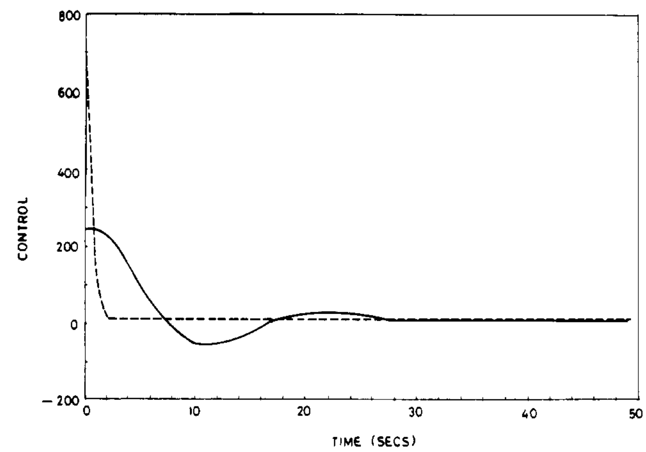

Figure 6. Control action for system response in Figure 5: (- - -) nonlinear controller, (—) linear controller.

experience (through experimentation), it is known that the dependence of the out flow rate on the water level in the tank is given by

$$F_{out} = A_c \times 0.1176h^{1/2}\ \mathrm{cm^3/s} \qquad (A_c = \text{cross-sectional area})$$

(These are the actual specifications for the level control system used for instructional purposes in the Process Control Laboratory at the Chemical Engineering Department, University of Wisconsin, Madison.) A control valve regulates F, the inlet flow rate. The mathematical model for this system is

$$\frac{dh}{dt} = \frac{F}{22.062} - 0.1176h^{1/2} \tag{50}$$

or

$$\frac{dx}{dt} = -0.1176x^{1/2} + \frac{u}{22.062};\ \text{i.e., } c_1 = -0.1176,\quad c_2 = 1/22{\cdot}062$$

Choosing a such that $k_1 = -a/0.1176 = {}^1/_2$ then from (27),

$$z = x^{1/2} \tag{51}$$

and choosing b such that $k_2 = b/k_1c_2$ is unity (i.e., b = 0.0225), then the control law is given by

$$u = vx^{1/2} \tag{52}$$

where v is the output of a PI controller.

The characteristic equation for the closed-loop linear pseudosystem and PI controller from (5) is given by

$$m^2 + bK_cm + bK_I = 0 \tag{53}$$

where $K_I = K_c/\tau_I$. For our choice of $b = 0.0225$, we may now choose K_c and K_I to give us the type of closed-loop response we desire. But keep in mind that *this is for z, the square root of x the water level*, and not x directly. A choice of $K_c = 90$ and $K_I = 0.2$, for example, gives a characteristic equation whose two roots may be verified as being

$$r_{1,2} = -4.0736, -0.0044$$

For a step change in the system set point from 40 to 75 cm, a PI controller, using the above given settings, accepts information $x^{1/2}$ and provides v, but the implemented flow rate is $vx^{1/2}$ in accordance with (51). The dashed line in Figure 5 shows the performance of this controller. A regular PI controller using *the same* controller settings resulted in an almost unstable system. Upon further reduction, the controller settings $K_c = 9.0$ and $K_I = 1.0$, the behavior of the regular PI controller (which takes decisions on the inlet flow rate using direct level measurements) is shown in solid lines. The corresponding control actions are shown in Figure 6. Again the main point is not merely to demonstrate that the nonlinear controller is more effective. While this may indeed be shown to be so, we wish to emphasize more the fact that the nonlinear controller is designed with better understanding and more insight, whereas the linear controller design must permanently remain a trial and error procedure.

7. Conclusions

We have shown that SISO nonlinear control systems may be transformed quite easily to exactly linear systems that can be effectively controlled with PI-type controllers. Constructive methods for finding the required transformations have been presented, thus converting the usually ill-defined problem of controller design for nonlinear systems to a more transparent, well defined problem. The actual control action is implemented by a control law (derived from the PI controller output and the transformation) which is nonlinear in the original system variables. Several examples have been used to show that this approach is general enough but at the same time retains the desirable features of simplicity and transparency of design and implementation. It offers the practicing control systems designer a method for getting a better understanding and clearer insight into the nonlinear system to be controlled.

Some important questions still remain to be answered, however. At this stage, we are not yet able to discuss the robustness properties of this technique in any detail. It is, however, obvious that since, like all modern control techniques, this is also model dependent, the issue of robustness is an important one. Neither have we discussed how time delays are to be handled.

These questions, along with questions concerning possible extensions of this technique to multivariable systems, are currently under investigation.

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Kinetics of the Reaction of Hydrogen Sulfide and Sulfur Dioxide in Organic Solvents

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Calorimetry was used to study the kinetics of the irreversible reaction between hydrogen sulfide and sulfur dioxide in mixtures of *N,N*-dimethylaniline (DMA) and diethylene glycol monomethyl ether (DGM) and of DMA and triethylene glycol dimethyl ether (triglyme). The reaction was found to be first order in both H_2S and SO_2 in the presence of DMA. The approximate heat of reaction is 28 kcal/mol of sulfur dioxide. The addition of DMA accelerates the reaction by an order of magnitude over that obtained in the glycol ethers alone. Rate constants are in the range of 1-20 L/(mol s). Hydroxylated species such as water, methanol, and other alcohols increase the rate still more dramatically when added to the DMA/ether mixtures. The results of these experiments show some of the effects of solvent composition on the kinetics of the reaction.

Hydrogen sulfide is an undesirable component found in many industrial process streams. Traditionally, its removal and recovery have been accomplished in an absorber-stripper operation, followed by the Claus process in which the H_2S is reacted over alumina catalyst with SO_2 obtained by burning a portion of the inlet stream. The gas-phase reaction to produce sulfur is

$$2H_2S + SO_2 \rightarrow 3/xS_x + 2H_2O$$

This reaction is equilibrium-limited to 95-97% conversion in two to four stages because the temperature must be kept above the dew point of sulfur. Additional processing must be provided to reduce the concentration of sulfurous compounds in the effluent stack gas to environmentally acceptable levels.

When the reaction between H_2S and SO_2 is carried out in organic liquids at temperatures below 150 °C, it is irreversible and goes essentially to completion. This shift in the equilibrium state is caused both by the lower temperature and by the reduced activity of the reaction products when in solution. In many organic solvents, such as triethylene glycol dimethyl ether (triglyme) and diethylene glycol methyl ether (DGM), the reaction is impractically slow. Moreover, at room temperature, the sulfur formed is too finely divided to be readily separated. Urban (1961) found that the presence of *N,N*-dimethylaniline (DMA) increased the crystal size of the precipitated sulfur. Furthermore, DMA accelerates the reaction to the extent that 99+% removal of H_2S is possible with careful selection of the solvent mixture.

The present work was undertaken to study the reaction between H_2S and SO_2 in mixtures of DMA/triglyme and DMA/DGM. Experiments performed by monitoring the temperature rise of this exothermic reaction in an adiabatic calorimeter show the effects of various solvent compositions on the kinetics of the reaction. Solvent selection criteria for an appropriate process scheme can then be set. This reaction system is suitable for application to a process currently being studied in this laboratory for the removal of hydrogen sulfide from industrial gas streams. The primary purpose of this investigation was to obtain design data for the process. The process itself will be described in a future publication.

Experimental Methods

All rate measurements were made in a 50-mL Erlenmeyer flask that contained a magnetic stir bar and was sealed by a septum cap (see Figure 1). Insulation was provided by a styrofoam block with a hole drilled out for the reactor. The reaction progress was monitored by recording the temperature rise as indicated by the change in the millivolt-level potential produced by a bare, type T thermocouple connected to a chart recorder. A measured quantity of a solution of one reactant was first placed in the reaction vessel. The reaction was initiated when a measured quantity of a solution of the second reactant at the same temperature was injected quickly by syringe into the stirred vessel.

The range of initial reactant concentrations was limited to 0.05-0.25 M SO_2 and 0.1-0.5 M H_2S. These compositions were high enough to produce sufficient reaction to permit accurate measurement of the temperature rise without exceeding an arbitrarily selected maximum temperature increase of 5 °C. Since the temperature coefficient of the reaction rate constant was found to be relatively low (see below), the rate constants obtained from these adiabatic experiments are very nearly equal to those for the reaction carried out under isothermal conditions, i.e., at the average temperature of a run.

Methods of Analysis

Acid-Base Experiments. Reactions between NaOH and HCl were carried out in the calorimeter to determine the rate of mixing, the thermal mass of the apparatus, and the rate of heat loss from the system. The experimental method was that used in the H_2S-SO_2 reactions, and the quantities of reactants were chosen to give similar tem-

