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An Integral Equation for the Associated Legendre Function of the First Kind

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The solution of the Fredholm homogeneous equation

$$\psi(\xi) = \lambda \int_0^1 M(\xi, \xi')\psi(\xi')\, d\xi',$$

where

$$M(\xi, \xi') = \int_0^\pi \frac{\cos m\phi\, d\phi}{(x^2 - 2xx' \cos\phi + x'^2)^{\frac{1}{2}}}$$

and $x = (1 - \xi^2)^{\frac{1}{2}}$ is found to be the associated Legendre function $P_n^m(\xi)$, $n + m$ even, and the characteristic numbers of this kernel are obtained. The solution of the corresponding equation of the second kind is also found. The kernel of the homogeneous equation whose solution is $P_n^m(\xi)$, $n + m$ odd, is obtained.

1. INTRODUCTION

MANY of the integral equations arising in electrostatics (Collins[1]), electromagnetic induction (Ashour[2,3]) and in diffraction theory (Noble[4]), have as kernel either the function

$$K(x, x') = \int_0^\pi \frac{\cos m\phi\, d\phi}{(x^2 - 2xx' \cos\phi + x'^2)^{\frac{1}{2}}} \quad (1)$$

(m integer)

or a function which does not differ from $K(x, x')$ except by a simple factor (depending on x and x'). Hence, it is of interest to find an exact solution for an integral equation whose kernel is simply related to that given by Eq. (1). It can easily be seen that $K(x, x')$ has an infinity of order $\log |x - x'|$ when $|x - x'| \to 0$. This kernel may also be expressed (Eason, Noble, and Sneddon[5]) as

$$K(x, x') = \pi \int_0^\infty J_m(x\, u)J_m(u\, x')\, du, \quad (2)$$

where $J_m(x)$ is the Bessel function of order m in x. In this paper, the solution of the integral equation

$$\psi(\xi) = f(\xi) + \lambda \int_0^1 M(\xi, \xi')\psi(\xi')\, d\xi' \quad (0 \le \xi \le 1), \quad (3)$$

when $f(\xi)$ is even, and

$$\xi = (1 - x^2)^{\frac{1}{2}}, \qquad M(\xi, \xi') = K(x, x'), \quad (4)$$

is obtained. In particular, the solution of Eq. (3) when $f(\xi) = 0$ is found to be simply the associated Legendre function of the first kind $P_n^m(\xi)$ with $n + m$ even, and the characteristic numbers λ_n of the kernel are obtained. The kernel of the homogeneous equation whose solution is $P_n^m(\xi)$, $n + m$ odd, is also obtained and the solution of the corresponding nonhomogeneous equation with $f(\xi)$ an odd function of ξ is found.

2. AN EXPANSION FOR $M(\xi, \xi')$

We first obtain a Fourier series expansion for $(x^2 - 2xx' \cos\phi + x'^2)^{-\frac{1}{2}}$ symmetrical in x and x'. Hobson[6] gave a Fourier series expression which is not symmetrical in x and x'. Sack[7] obtained an expansion as a series in the Legendre functions $P_l(\cos\phi)$, the coefficients being symmetrical in x and x'. Neither of these expressions is suitable for the present purpose.

Let R denote the distance between two points (ξ, ζ, ϕ) and (ξ', ζ', ϕ'), where ξ, ζ, ϕ are oblate spheroidal coordinates given in terms of cylindrical polar coordinates z, ρ, ϕ as

$$\left.\begin{aligned} z &= a\xi\zeta \\ \rho &= a\{(1 - \xi^2)(1 + \zeta^2)\}^{\frac{1}{2}} \end{aligned}\right\} \begin{aligned} 0 \le \zeta \le \infty, \\ -1 \le \xi \le 1. \end{aligned} \quad (5)$$

Hence,

$$\begin{aligned} a/R = [&2 + \zeta^2 + \zeta'^2 - \xi^2 - \xi'^2 - 2\xi\xi'\zeta\zeta' \\ &- 2\{(1 - \xi^2)(1 - \xi'^2)(1 + \zeta^2)(1 + \zeta'^2)\}^{\frac{1}{2}} \\ &\times \cos(\phi - \phi')]^{-\frac{1}{2}}. \end{aligned} \quad (6)$$

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[1] W. D. Collins, Proc. Cambridge Phil. Soc. 55, 377 (1955).

[2] A. A. Ashour, Quart. J. Mech. Appl. Math. 3, 119 (1950).

[3] A. A. Ashour, Quart. J. Mech. Appl. Math. (to be published).

[4] B. Noble, in *Electromagnetic Waves*, edited by Rudolf E. Langer, University of Wisconsin Press, Madison, Wisconsin (1962), p. 323.

[5] G. Eason, B. Noble, and I. N. Sneddon, Phil. Trans. Roy. Soc. (London) A247, 529 (1955).

[6] E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, (Cambridge University Press, New York 1951), p. 443.

[7] R. A. Sack, J. Math. Phys. 5, 245 (1964).

The expression for a/R as a series of oblate spheroidal harmonics has been given by Hobson (Ref. 6, p. 430). With the present notation (which is the same as that used by Smythe[8]), this expression is

$$\frac{a}{R} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} N_{mn} Q_n^m(i\zeta) P_n^m(\xi) \cos m(\phi - \varphi') \qquad \zeta \geq \zeta' \geq 0, \tag{7}$$

where

$$N_{mn} = i(2 - \delta_{m0})(-1)^m(2n+1) \times \{(n-m)!/(n+m)!\}^2 P_n^m(i\zeta') P_n^m(\xi'), \tag{8}$$

δ_{m0} being the Kronecker symbol and $P_n^m(u)$, $Q_n^m(u)$ are the associated Legendre functions of the first and second kinds in u. If now in Eqs. (6), (7), and (8) we take $\zeta = \zeta' = \phi' = 0$ and note that

$$\begin{aligned} P_n^m(i\cdot 0) &= 0 && n+m \text{ odd} \\ &= (-1)^{n/2} \frac{1\cdot 3\cdot 5 \cdots (n+m-1)}{2\cdot 4\cdot 6 \cdots (n-m)} && n+m \text{ even} \end{aligned} \tag{9}$$

$$Q_n^m(i\cdot 0) = -\tfrac{1}{2} i\pi P_n^m(i\cdot 0) \qquad n+m \text{ even}$$

it is found from (6), (7), (8), and (9) that

$$\{x^2 - 2xx'\cos\phi + x'^2\}^{-\frac{1}{2}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2-\delta_{m0}) \times (4n+2m+1)\left\{\frac{1\cdot 3\cdot 5\cdots 2n-1}{2\cdot 4\cdot 6\cdots 2n+2m}\right\}^2 P_{2n+m}^m(\xi) \times P_{2n+m}^m(\xi')\cos m\phi. \tag{10}$$

This is the required expression. From Eqs. (1) and (10) we now readily obtain

$$M(\xi,\xi') = \frac{\pi^2}{2}\sum_{n=0}^{\infty}(4n+2m+1) \times \left\{\frac{1\cdot 3\cdot 5\cdots 2n-1}{2\cdot 4\cdot 6\cdots 2n+2m}\right\}^2 P_{2n+m}^m(\xi) P_{2n+m}^m(\xi'). \tag{11}$$

If in (10), ϕ is replaced by ω where

$$\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'), \tag{12}$$

we obtain at once an expression (which appears to be new) for the reciprocal of the distance between the two points whose spherical polar coordinates are (ax, θ, ϕ), (ax', θ', ϕ') $(x, x' \leq 1)$.

3. SOLUTION OF THE FREDHOLM INTEGRAL EQUATION

We first consider the integral equation (3) with $f(\xi) = 0$. From Eq. (11), and the orthogonality[9] of the Legendre functions, we immediately see that the solution in this case is

$$\psi(\xi) = P_{2n+m}^m(\xi) \tag{13}$$

provided that λ is one of the numbers λ_n (n a positive integer or zero) given by

$$\lambda_n = \frac{2}{\pi^2}\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots 2n-1}\cdot\frac{2\cdot 4\cdot 6\cdots 2n+2m}{1\cdot 3\cdot 5\cdots 2n+2m-1}. \tag{14}$$

To obtain the solution of the general equation (3), we assume that $f(\xi)$ may be expanded as a series of Legendre functions of even parity all of the same order m [note that $f(\xi)$ is even], i.e.,

$$f(\xi) = \sum_{n=0}^{\infty} A_n P_{2n+m}^m(\xi), \tag{15}$$

where

$$A_n = \frac{(4n+2m+1)(2n)!}{(2n+2m)!}\int_0^1 f(\xi) P_{2n+m}^m(\xi)\, d\xi. \tag{16}$$

Then, using Eqs. (13), (14), and (15) we obtain the solution of Eq. (3) as

$$\psi(\xi) = \sum_{n=0}^{\infty}\frac{A_n P_{2n+m}^m(\xi)}{(1-\lambda/\lambda_n)}. \tag{17}$$

In the same way, it can be proved that the equation

$$\int_0^1 \psi(\xi') M(\xi,\xi')\, d\xi' = f(\xi), \tag{18}$$

where $f(\xi)$ is given by Eq. (15), has the solution

$$\psi(\xi) = \sum_0^{\infty}\lambda_n A_n P_{2n+m}^m(\xi). \tag{19}$$

4. AN INTEGRAL EQUATION FOR $P_n^m(\xi)$, $n+m$ ODD

If $\{\partial^2 R^{-1}/\partial\zeta\ \partial\zeta'\}_{\zeta=\zeta'=\phi'=0}$ is found using the two expressions (6) and (7) for R^{-1}, we obtain after inserting the numerical values for $P_n^{m\prime}(i\cdot 0)$ and $Q_n^{m\prime}(i\cdot 0)$:

$$\xi\xi'(x^2 - 2xx'\cos\phi + x'^2)^{-\frac{3}{2}} = -\frac{\pi}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2-\delta_{m0}) \times (4n+2m+3)\left\{\frac{1\cdot 3\cdot 5\cdots 2n+1}{2\cdot 4\cdot 6\cdots 2n+2m}\right\}^2 \times P_{2n+m+1}^m(\xi) P_{2n+m+1}^m(\xi')\cos m\phi. \tag{20}$$

[8] W. R. Smythe, *Static and Dynamic Electricity*, McGraw-Hill Book Company, Inc., New York (1950), pp. 146, 148, 152, 158, 166.

[9] $$\begin{aligned}\int_0^1 P_{2n+m}^m(\xi)\, P_{2n'+m}^m(\xi)\, d\xi &= 0 && (n \neq n') \\ &= \frac{(2n+2m)!}{(2n)!\,(4n+2m+1)} && (n = n').\end{aligned}$$

If in Eq. (20) ϕ is replaced by ω [given by Eq. (12)], we obtain a Fourier series expansion symmetrical in x, x' for the inverse third power of the distance between two points in spherical polar coordinates, which again appears to be new.[10]

We now define a new kernel $G(\xi, \xi')$ as

$$G(\xi, \xi') = \xi\xi' \int_0^\pi \frac{\cos m\phi \, d\phi}{(x^2 - 2xx' \cos \phi + x'^2)^{\frac{3}{2}}}. \tag{21}$$

From Eqs. (20) and (21) we obtain

$$G(\xi, \xi') = -\frac{\pi^2}{2} \sum_{n=0}^{\infty} (4n + 2m + 3) \left\{\frac{1\cdot 3\cdot 5 \cdots 2n + 1}{2\cdot 4\cdot 6 \cdots 2n + 2m}\right\}^2 P^m_{2n+m+1}(\xi) P^m_{2n+m+1}(\xi'). \tag{22}$$

Hence, $P^m_{2n+m+1}(\xi)$ satisfies the integral equation

$$\psi(\xi) = \mu \int_0^1 \psi(\xi') G(\xi, \xi') \, d\xi', \tag{23}$$

provided that μ equals one of the characteristic numbers μ_n given by

$$\mu_n = -\frac{2}{\pi^2} \frac{2\cdot 4\cdot 6 \cdots 2n}{1\cdot 3\cdot 5 \cdots 2n+1} \cdot \frac{2\cdot 4\cdot 6 \cdots 2n+2m}{1\cdot 3\cdot 5 \cdots 2n+2m+1}. \tag{24}$$

The solutions of the equations

$$\psi(\xi) = g(\xi) + \mu \int_0^1 \psi(\xi') G(\xi, \xi') \, d\xi' \tag{25}$$

and

$$\int_0^1 \psi(\xi') G(\xi, \xi') \, d\xi' = g(\xi), \tag{26}$$

where $g(\xi)$ is odd and can be expanded in the form

$$g(\xi) = \sum_{n=0}^{\infty} B_n P^m_{2n+m+1}(\xi), \tag{27}$$

can be found in the same way as before. They are

$$\sum_0^\infty \frac{B_n}{(1 - \mu/\mu_n)} P^m_{2n+m+1}(\xi) \quad \text{and} \quad \sum_{n=0}^{\infty} \mu_n B_n P^m_{2n+m+1}(\xi),$$

respectively.

Note added in proof: Another integral equation satisfied by $P^m_n(\xi)$ (n, m integers) is

$$\psi(\xi) = \nu \int_{-1}^{+1} H(\xi, \xi') \psi(\xi') \, d\xi' \tag{28}$$

where

$$H(\xi, \xi') = \int_0^\pi \frac{\cos m\phi \, d\phi}{(1 - \xi\xi' - xx' \cos \phi)^{\frac{1}{2}}}$$
$$= \pi \sqrt{2} \sum_{n=m}^{\infty} \frac{(n - m)!}{(n + m)!} P^m_n(\xi) P^m_n(\xi'). \tag{29}$$

This can be proved by taking $r = r' = 1$, $\phi' = 0$, $\cos \theta = \xi$, $\cos \theta' = \xi'$ in the expression for the inverse distance R^{-1} between two points r, θ, ϕ and r', θ', ϕ'.[11] The characteristic numbers ν_n are then readily found to be

$$\nu_n = (2n + 1)/2 \sqrt{2}\, \pi, \; n = 0, 1, \cdots. \tag{30}$$

The solutions of equations of the form (3) or (18), with kernel $H(\xi, \xi')$ and integration interval $-1 \to 1$, can be obtained in exactly the same manner as before.

[10] The Fourier series expansion of the general power of the distance between two points, symmetrical in x and x' will be discussed in another communication [J. Math. Phys. (to be published)].

[11] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), p. 1274, No. 10.3.3.37.