AIP Journal of Applied Physics

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Citation: Journal of Applied Physics **37**, 2937 (1966); doi: 10.1063/1.1703143
View online: http://dx.doi.org/10.1063/1.1703143
View Table of Contents: http://scitation.aip.org/content/aip/journal/jap/37/8?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF APPLIED PHYSICS VOLUME 37, NUMBER 8 JULY 1966

Surface-Wave Instability in Helicon Propagation. II: Effect of Collisional Losses

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(Received 19 January 1966)

The surface-wave instability analyzed by Baraff and Buchsbaum in the limit of infinite $\omega_c\tau$ is investigated here for $\omega_c\tau$ large but finite. In the previous analysis, the carrier drift velocity required to establish the instability was found to be proportional to (K_1-K_2), the effective dielectric mismatch across the interface at which the surface wave was excited. Here we find that with $\omega_c\tau$ large but finite, this threshold velocity is increased by an amount proportional to $(\omega_c\tau)^{-1}$ and inversely proportional to (K_1-K_2). For a given value of $\omega_c\tau$ there thus exists an optimum choice of (K_1-K_2) which minimizes the required threshold velocity. These new results can be obtained from heuristic arguments which do not require the calculation of field components beyond lowest nonvanishing order in $(\omega_c\tau)^{-1}$. The heuristic arguments are here justified rigorously for the physically interesting case of small (K_1-K_2).

I. INTRODUCTION

A RECENT analysis of helicon wave propagation along the composite structure shown in Fig. 1 led to the prediction that an instability would be produced if the carriers in either medium were given sufficient drift velocity.[1] An intriguing feature of the instability is that the drift velocity needed to activate the instability can be far less than the phase velocity of the wave propagating along the structure, in marked contrast to other instabilities which rely on drift currents interacting with traveling waves.[2] This feature is important because large currents in solids are accompanied by heat production which sets an upper limit on the drift velocities that can be achieved. The analysis presented in Ref. 1 was incomplete, however, in that collisional losses were ignored. This was a reasonable way to investigate the physics of the instability and to focus attention on the mechanism. It left open the important question of whether actual materials, in which the carriers always suffer some collisions, could be expected to support the instability. It is the purpose of this paper to complete that particular aspect of the analysis and to study the effects of collisions on the instability.

Before summarizing the effect of collisions we first recall the physical mechanism of the instability. When a helicon wave propagates along the structure depicted in Fig. 1, a "surface" wave, localized around the interface between the media, accompanies the propagation. It exists to satisfy boundary conditions which the helicon wave in the bulk (the bulk wave) alone cannot quite do. With the surface wave is associated loss and, although the loss is caused by collisions, it remains finite in the limit $\omega_c\tau \to \infty$ provided that $\omega\tau$ remains bounded.[3,4] In this limit, the bulk wave is lossless. It was shown in I that the surface wave can be made to disappear by making the carriers in one medium drift with a velocity such that the Doppler-shifted dielectric constants of the media are equal. At this, the threshold velocity, the loss previously associated with the surface wave vanishes. At velocities greater than the threshold velocity, the surface wave reappears but with its phase reversed. It was shown in I that this reversed phase was responsible for the instability.

The effect of collisions is not difficult to surmise. In the absence of drift, collisions will damp the bulk helicon wave and in addition, will increase the damping of the surface waves. A new threshold will have to be established such that the gain just balances the additional losses. There is also the possibility that the amount of gain itself may be changed by collisions. It turns out however, as is shown in Sec. III, that for the physically interesting limit of nearly equal dielectric constants in the two media, it is only the additional loss in the bulk wave that must be overcome. This is fortunate because the analysis is considerably simplified thereby.

The structure of this paper is as follows: In Sec. II, we derive the new threshold velocity using a heuristic argument in which we assume that the gain at the interface is unchanged by collisions and that only the additional losses of the bulk wave need be overcome. The

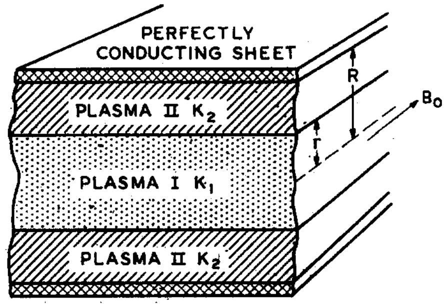

FIG. 1. Schematic of the sandwich structure analyzed in the text. Media I and II are assumed to contain different concentrations of free carriers leading to different effective dielectric constants K_1 and K_2.

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[1] G. A. Baraff and S. J. Buchsbaum, Phys. Rev. 144, 266 (1966). This reference will be referred to as I through the remainder of the paper.
[2] J. Bok and P. Nozieres, J. Phys. Chem. Solids 24, 709 (1963).
[3] ω_c is the cyclotron frequency of the carriers, τ is a phenomenological free time between collisions of the carriers with the medium.
[4] L. M. Saunders, G. A. Baraff, and S. J. Buchsbaum, J. Appl. Phys. 37, 2935 (1966) (preceding paper).

analysis in I indicated that the threshold velocity in the infinite $\omega_c\tau$ limit was proportional to the dielectric mismatch, the difference in dielectric constants of the two media. The analysis here shows that the increment in threshold velocity caused by collisional losses will be inversely proportional to the dielectric mismatch. Thus the problem of choosing the dielectric mismatch so as to minimize the actual threshold velocity arises. The resolution of this problem concludes Sec. II.

In Sec. III, we present the mathematical analysis on which the heuristic argument is based. The mathematics needed to prove the validity of the heuristic approach of Sec. II is unfortunately long, tedious, and routine. In point of fact, the heuristic arguments were devised to convey the content of the mathematics, and so where the heuristic explanations are not fully convincing, the mathematics of Sec. III may be used as alternative justification.

II. HEURISTIC DERIVATION OF THE EFFECTS OF COLLISIONAL LOSSES

We now consider propagation of a helicon wave along the structure shown in Fig. 1. The fields associated with the wave will, as shown in I, have a spatial dependence of the form $f(y)\exp i(k_z z-\omega t)$, where $f(y)$ is a sum of two "bulk" waves of the form $\exp\pm ik_y y$ and two surface waves of the form $\exp\pm\omega N_y y/c$. The two sign possibilities for N_y enter in such a manner that the surface waves have their greatest amplitude at the boundary and decay rapidly with distance away from the boundary. The relationship between k_y, k_z, and ω which exists because of Maxwell's equations and a transport equation for the carriers depends on certain parameters characterizing the medium in which the wave exists. In footnote 7 of Ref. 1, this relationship was shown to be

$$n_y^2=[X^2/(n_z^2-2iX/\omega_c\tau)]-n_z^2, \qquad (2.1a)$$

where

$$X=K(1-V_D/V_\varphi), \qquad (2.1b)$$

$$K=\omega_p^2/\omega\omega_c, \qquad (2.1c)$$

$$\mathbf{n}=c\mathbf{k}/\omega. \qquad (2.1d)$$

Here V_D is the drift velocity of the carriers, V_φ is the (as yet undetermined) phase velocity ω/k_z, ω_p is the plasma frequency of the carriers, and ω_c is their cyclotron frequency in the static magnetic field B_0. The quantity K serves as a dielectric constant in the sense that the phase velocity of a helicon wave propagating parallel to the static magnetic field, through an unbounded medium free of carrier drift, would be $c/K^{\frac{1}{2}}$. The quantity X, which plays the role of K when the carriers drift, is called the "Doppler-shifted dielectric constant" and is most easily derived by introducing the drift velocity via the transformation to a moving frame of reference. Equations (2.1) are valid to first order in $(\omega_c\tau)^{-1}$, which is the order to which we work here.

Within the structure shown in Fig. 1 there are two adjacent media, each with its own value of K, of X, of V_D, and of $\omega_c\tau$. Waves propagating through this structure must have the same value of n_z in both media if the boundary conditions on the electric and magnetic fields are to be satisfied. For the sake of simplifying the discussion, assume that only the carriers in medium I are drifting, $V_{D1}=V$, $V_{D2}=0$, and also assume that $\omega_c\tau$ has the same value in both media. Then if the drift velocity V is adjusted to that value V_T such that $X_1=X_2$, i.e., such that

$$K_1(1-k_zV_T/\omega)=K_2, \qquad (2.2)$$

it follows from (2.1) that n_y will have the same value in both media.[5] The helicon waves in both media have the same values of n_z and n_y, and hence have the same spatial dependence at all points in the structure. The boundary conditions at the interface will cause the waves in the two media to join smoothly, so that to all appearances, the helicon wave stretches across the entire cross section of the structure as if that cross section were all filled with one material.

The boundary condition that the tangential components of the electric field vanish at $y=\pm R$ now must be imposed. The x component of the electric field e_x can be made to vanish at $y=+R$ by combining the two bulk waves into a linear combination of the form $\sin[k_y(R-y)]$. This linear combination will again go to zero at $y=-R$, provided that $\sin(2Rk_y)=0$, or equivalently

$$2Rk_y=m\pi, \quad m=\text{integer}. \qquad (2.3)$$

The z component of the electric field will not vanish at $y=\pm R$ because the linear combination just chosen results in a z field of the form $\cos[k_y(R-y)]$. However as shown in I, the z component of the helicon electric field is smaller by one order of $(\omega_c\tau)^{-1}$ than the x component. This z component can be canceled at the outer boundary by the excitation of a surface wave. The z component of the surface wave electric field will thus

[5] The validity of the heuristic approach used in this section demands that the ratio K_1/K_2 be near unity. The flaw in the argument is that (assuming real ω) Eq. (2.2) cannot be satisfied unless k_z is real. The closest we can come to satisfying $X_1-X_2=0$ is to adjust V to a value such that $\mathrm{Re}(X_1-X_2)=0$, in which case $X_1-X_2=i[\mathrm{Im}(X_1)]$. But $\mathrm{Im}(X_1)=(-K_1V_T/\omega)\ \mathrm{Im}(k_z)=-(K_1V_T/\omega)\ \mathrm{Re}(k_z)[\mathrm{Im}(k_z)/\mathrm{Re}(k_z)]$. From $\mathrm{Re}(X_1-X_2)=0$, we have $-(K_1V_T/\omega)\ \mathrm{Re}(k_z)=(K_2-K_1)$. For helicons propagating along the z direction in unbounded media, $[\mathrm{Im}(k_z)/\mathrm{Re}(k_z)]=1/2\omega_c\tau$. This result will not be altered radically by the waveguide effects. Thus, the smallest we can make X_1-X_2 is about $i(K_2-K_1)/2\omega_c\tau$. The heuristic argument based on $X_1=X_2$ is thus strictly valid only when $(K_1-K_2)/\omega_c\tau$ vanishes. The development in this section is correct to lowest order in $(1-K_1/K_2)$, as comparison with the rigorous approach of Sec. III will verify.

be of order $(\omega_c\tau)^{-1}$ times the x component of the helicon field. The x component of the surface wave electric field is smaller by one order of $(\omega_c\tau)^{-1}$ than its z component, and hence is two orders of $(\omega_c\tau)^{-1}$ smaller than the x component of the helicon field. To first order in $(\omega_c\tau)^{-1}$ this does not spoil the vanishing of e_x at $y=\pm R$ which is achieved by satisfying Eq. (2.3).

The situation at threshold is essentially the same as that found in I. The bulk helicon wave has the same wave number throughout the structure, with the transverse wavelength determined by the condition that an integer number of transverse half-wavelengths be included between $y=+R$ and $y=-R$. The difference is that the situation held exactly in I where $\omega_c\tau$ was infinite, while here, it holds only up to and including terms of order $(\omega_c\tau)^{-1}$, and even then only in the limit of small $(1-K_1/K_2)$.

Now consider Eq. (2.1a) in medium II at threshold. Substituting the threshold value (2.3) gives

$$\left(\frac{m\pi c}{2R\omega}\right)^2 \equiv n_0^2 = \frac{K_2^2}{n_z^2-2iK_2/\omega_c\tau} - n_z^2. \qquad (2.4)$$

To lowest order in $(\omega_c\tau)^{-1}$, this gives

$$\mathrm{Im}n_z/n_z^0 = (\omega_c\tau)^{-1}/(\eta+\eta^3), \qquad (2.5a)$$

where

$$\eta = [n_z^0/(K_2)^{\frac{1}{2}}]^2. \qquad (2.5b)$$

The superscript zero in (2.5) means that n_z is to be evaluated from (2.4) with $(\omega_c\tau)^{-1}$ set equal to zero. This yields

$$\eta = (1+T^4/4)^{\frac{1}{2}} - T^2/2, \qquad (2.6a)$$

$$T = n_0/(K_2)^{\frac{1}{2}}. \qquad (2.6b)$$

When the over-all width of the structure becomes comparable to or greater than the infinite medium helicon wavelength ($T\lesssim\frac{1}{2}$), η approaches unity, and (2.5a) gives the familiar $1/2\omega_c\tau$ attenuation of the infinite medium helicon waves.

The expression (2.5) gives the value of $\mathrm{Im}n_z$ caused by collisional loss in the bulk when the drift velocity is V_T, i.e., where there is no power gain or loss caused by the surface wave at the interface. At other values of drift velocity, we should add to expression (2.5) that contribution to $\mathrm{Im}n_z$ produced by the surface waves. We use Eq. (3.5) of I for that contribution, namely,

$$\frac{\mathrm{Im}n_z}{n_z^0} = \frac{(V-V_T)(1-K_1/K_2)\varphi^2(r)}{2\omega R(4+T^4)^{\frac{1}{2}}}, \qquad (2.7a)$$

$$\varphi(r) = \sin\left[\frac{m\pi}{2}\left(1-\frac{r}{R}\right)\right]. \qquad (2.7b)$$

Adding (2.5) and (2.7) gives the final expression for $\mathrm{Im}n_z$, namely,[6]

$$\frac{\mathrm{Im}n_z}{n_z^0} = \frac{1}{(4+T^4)^{\frac{1}{2}}}\left[\frac{(\omega_c\tau)^{-1}}{\eta^2} + \frac{(V-V_T)(1-K_1/K_2)\varphi^2(r)}{2\omega R}\right]. \qquad (2.8)$$

The value of V_T, calculated from (2.2) is

$$V_T = c(1-K_2/K_1)/n_z^0 \approx -c(1-K_1/K_2)/n_z^0,$$

the approximation being valid to lowest order in K_1-K_2. The quantity

$$M \equiv (1-K_1/K_2) \qquad (2.9)$$

may be called the dielectric mismatch. In terms of this, Eq. (2.8) is

$$\frac{\mathrm{Im}n_z}{n_z^0} = \frac{1}{(4+T^4)^{\frac{1}{2}}}\left[\frac{(\omega_c\tau)^{-1}}{\eta^2} + \frac{(V+MV_\varphi)M\varphi^2(r)}{2\omega R}\right], \qquad (2.10a)$$

$$V_\varphi = c/n_z^0 = \omega/\mathrm{Re}(k_z). \qquad (2.10b)$$

This expression is the most important result of this work. It gives the growth or decay rate of the wave as a function of the drift velocity in medium I. The actual threshold, that value of drift velocity for which $\mathrm{Im}n_z$ vanishes, is

$$V_{\mathrm{th}} = -\{[2\omega R(\omega_c\tau)^{-1}/M\eta^2\varphi^2(r)] + MV_\varphi\}. \qquad (2.11)$$

One part of this expression, MV_φ, is the threshold velocity found in I in the $\omega_c\tau\to\infty$ limit. It can be made arbitrarily small by making the dielectric mismatch small. The other part of this expression is the increment in drift velocity needed to overcome the finite $\omega_c\tau$ losses. It is inversely proportional to the dielectric mismatch because, as pointed out in I, the power generated at the interface is, for a given incremental velocity, proportional to M. The other factors of the incremental drift velocity are also understood: $\varphi^2(r)$ is, as explained in I, a measure of the efficiency with which the interface generates power, relative to the power generated by an interface located at the position of the largest helicon fields; R is proportional to the volume in which collisional losses occur; and η^2 is a factor which goes to unity when the structure is wide enough to make $n_y/n_z \ll 1$, i.e., for the wave to be essentially a plane wave traveling along the z direction.

The choice of M which yields the smallest actual threshold velocity is easily obtained from (2.11). It is

$$M = \pm[2\omega R(\omega_c\tau)^{-1}/\eta^2\varphi^2(r)V_\varphi]^{\frac{1}{2}} \qquad (2.12a)$$

[6] The rationale for adding the various contributions to obtain $\mathrm{Im}n_z$ is that this quantity is proportional to the spatial growth rate which, alternatively could be calculated by considering the energy gain or loss in a thin length of material filling the entire cross section of the structure. The energy gain or loss is additive, consisting of a bulk wave loss and an interface contribution and hence $\mathrm{Im}n_z$ should be composed of additive contributions.

and the corresponding actual threshold velocity is

$$V_{\text{th}} = -2MV_\varphi. \qquad (2.12\text{b})$$

This threshold velocity becomes smaller as V_φ is reduced. From (2.5b) the phase velocity is given by $V_\varphi = V_H/\eta^{\frac{1}{2}}$, where $V_H = c/(K_2)^{\frac{1}{2}}$ is the helicon velocity in an infinite sample of medium II. The phase velocity is very nearly equal to V_H until the width of the structure becomes equal to or less than the infinite-medium helicon wavelength $\lambda_H = 2\pi V_H/\omega$. For narrower structures, the phase velocity rises rapidly. Therefore, one condition for a low actual threshold is $2R \geq \lambda_H$. On the other hand, a large thickness R means a large volume in which collisional losses can occur, hence, $2R = \lambda_H$ may be a reasonable choice for lowering the threshold velocity. This choice would give

$$M = \pm[2\pi(\omega_c\tau)^{-1}/\eta^{\frac{1}{2}}\varphi^2(r)]^{\frac{1}{2}},$$

and, via Eqs. (2.4,5,6), would give $T=\frac{1}{2}$, $\eta \approx \frac{7}{8}$. If the internal interfaces are located close to the center of the structure where the bulk helicon fields are largest, then φ^2 will be unity, giving

$$M = \pm(7.7/\omega_c\tau)^{\frac{1}{2}}, \qquad (2.13\text{a})$$

$$V_{th} = \mp 2V_\varphi(7.7/\omega_c\tau)^{\frac{1}{2}}. \qquad (2.13\text{b})$$

The heuristic analysis presented here has been valid to first order in M, so as a practical matter, $\omega_c\tau$ should be chosen larger than about 77. Terms of order M^2 would then give corrections of order 10%.

III. PROOF OF THE VALIDITY OF THE HEURISTIC DERIVATION

Although the discussion just given is plausible, it is by no means obvious that it is correct. There is no doubt of the validity of (2.1) to describe the bulk helicon attenuation. The doubtful feature of the heuristic argument involves the effect of collisions on the surface waves, whose collisional losses have so far apparently been ignored. In addition, there are corrections of order $(\omega_c\tau)^{-1}$ to both the surface wave and bulk wave fields, so it is *a priori* likely that the power generated at the interface will also change to first order in $(\omega_c\tau)^{-1}$. Thus, there may be other first-order contributions to (2.10). When $(\omega_c\tau)^{-1}$ has the same value in each of the two media, it turns out that the added terms will be proportional to $(1-K_1/K_2)$ and that the leading terms *are* correctly given by (2.10), but the only way to prove this is to solve the dispersion relation for the over-all structure to the next order in $(\omega_c\tau)^{-1}$.

The dispersion relation arises when we try to satisfy all of the boundary conditions in the structure of Fig. 1. There are four boundary conditions at each internal interface, namely, continuity of the four tangential fields at $y=\pm r$, and two boundary conditions at each outer boundary, namely the vanishing of the two tangential components of the electric field at $y=\pm R$. The fields in each medium are linear combinations of the four wave modes which the infinite medium will support, as shown in I. We use the convention that $e_\alpha{}^k$ and $b_\alpha{}^k$ are the α component of the electric and magnetic fields of the kth mode evaluated at $y=r$, and that $E_\alpha{}^k$ and $B_\alpha{}^k$ are the α component of the kth mode evaluated at $y=R$. The superscript k used as a mode label is a letter which identifies both the spatial dependence of the mode and the medium where it is to be used. When a linear combination of modes is taken to represent a field, the coefficient in that sum will be the capital letter which, as a lower case superscript, identifies the mode. By convention, the coefficients of modes used in medium I are prefixed by a minus sign. Referring to the first, second, and fourth columns of I Table III, the reader will see that the two boundary conditions at $y=R$ would be written

$$SE_x{}^s + TE_x{}^t + UE_x{}^u + VE_x{}^v = 0, \qquad (3.1\text{a})$$

$$SE_z{}^s + TE_z{}^t + UE_z{}^u + VE_z{}^v = 0. \qquad (3.1\text{b})$$

The information in the third column of I Table III, that v is a type-6 solution means, according to the last row of I Table I, that all field components in it decrease exponentially with increasing y. We always assume that $(R-r)$ is large enough so that the v mode excited at $y=r$ decays before reaching $y=R$. Hence we delete $E_x{}^v$ and $E_z{}^v$ from (3.1) and eliminate U from the resulting equations to get

$$T = -S[(E_z{}^uE_x{}^s - E_x{}^uE_z{}^s)/(E_z{}^uE_x{}^t - E_x{}^uE_z{}^t)]. \quad (3.2)$$

In I Table I, $E_\alpha{}^j$ or $B_\alpha{}^j$, where j is a number (not a letter), denote a constant which multiplies a function of y; e.g., the statement (I Table III) that S is a type-1 mode means (I Table I) that

$$e_x{}^s = E_x{}^1\cos(k_2r), \quad E_x{}^s = E_x{}^1\cos(k_2R).$$

A zero entry in I Table I means that the corresponding field component is smaller by order $(\omega_c\tau)^{-1}$ than the other field components in the same mode. We cannot in general ignore these small field components here. However, reference to I Tables I and III shows that the second term in both the numerator and denominator of Eq. (3.2) is the product of two fields, each of which is smaller by an order of $(\omega_c\tau)^{-1}$ than either of the fields in the first term. Thus each of these terms represents an $(\omega_c\tau)^{-2}$ correction to the first, which when dropped leaves

$$T = -SE_x{}^s/E_x{}^t, \qquad (3.3)$$

to first order, i.e., $(\omega_c\tau)^{-1}$, accuracy. This equation is identical to (IA.3).

Consider the four boundary conditions at $y=r$. One would expect all four modes in each medium to be involved in equating the tangential fields at this boundary. However, I Tables I and III indicate that the u mode is a growing exponential function of y. It is excited at $y=R$ but dies away before reaching $y=r$

just as the v mode, excited at $y=r$, died away before reaching $y=R$. This eliminates the u mode for boundary conditions at $y=r$. Furthermore, the symmetry of the structure about the plane $y=0$ results in fields whose z components are either even or odd functions of y. For this reason, it is useful in medium I, which contains the plane $y=0$, to choose modes whose e_z and b_z fields are both either even or odd functions of y. Then modes of one type or the other, but not both, will appear in the final solution. That is, again referring to I Table I, either solutions of type 1 and type 2 can be used in medium I, or solutions of type 3 and type 4 can be used. For definiteness, we shall here choose solutions of types 1 and 2. Referring to I Table III, we find that with this choice only modes Q and R will be present in medium I. The four boundary conditions at $y=r$ can thus be written:

$$Qe_x{}^q+Re_x{}^r+Se_x{}^s+Te_x{}^t+Ve_x{}^v=0, \tag{3.4a}$$

$$Qe_z{}^q+Re_z{}^r+Se_z{}^s+Te_z{}^t+Ve_z{}^v=0, \tag{3.4b}$$

$$Qb_x{}^q+Rb_x{}^r+Sb_x{}^s+Tb_x{}^t+Vb_x{}^v=0, \tag{3.4c}$$

$$Qb_z{}^q+Rb_z{}^r+Sb_z{}^s+Tb_z{}^t+Vb_z{}^v=0. \tag{3.4d}$$

T can be eliminated by the use of (3.3). It is convenient to carry out this elimination, as in I, by defining type W solutions.

$$We_\alpha{}^w=Se_\alpha{}^s+Te_\alpha{}^t,$$
$$Wb_\alpha{}^w=Sb_\alpha{}^s+Tb_\alpha{}^t.$$

This definition is identical to (IA.4). These particular linear combinations are, because of the identity in form between (3.3) and (IA.3), the same as those in (IA.6), namely the $\sin[k(r-R)]$ and $\cos[k(r-R)]$ combinations mentioned in Sec. II of this paper. In order for (3.4) to have a nontrivial solution, Δ, the determinant of coefficients of Q, R, W, and V, must vanish.

$$\Delta=\begin{vmatrix} e_x{}^q & e_x{}^r & e_x{}^w & e_x{}^v \\ e_z{}^q & e_z{}^r & e_z{}^w & e_z{}^v \\ b_x{}^q & b_x{}^r & b_x{}^w & b_x{}^v \\ b_z{}^q & b_z{}^r & b_z{}^w & b_z{}^v \end{vmatrix}.$$

It is convenient to alter the form, but not the value, of Δ by subtracting $e_z{}^r/e_z{}^v$ times the fourth column from the second column, so that the four entries in the second column are δe_x, 0, δb_x, and δb_z, where

$$\begin{aligned} \delta e_x&=e_x{}^r-e_x{}^v e_z{}^r/e_z{}^v, \\ \delta b_x&=b_x{}^r-b_x{}^v e_z{}^r/e_z{}^v, \\ \delta b_z&=b_z{}^r-b_z{}^v e_z{}^r/e_z{}^v. \end{aligned} \tag{3.5}$$

Then, instead of setting Δ to zero, we set $\Delta'=\Delta/(e_x{}^q\delta b_x e_x{}^w e_z{}^v)$ to zero. Evidently,

$$\Delta'=\begin{vmatrix} 1 & \delta e_x/\delta b_x & 1 & e_x{}^v/e_z{}^v \\ e_z{}^q/e_x{}^q & 0 & e_z{}^w/e_x{}^w & 1 \\ b_x{}^q/e_x{}^q & 1 & b_x{}^w/e_x{}^w & b_x{}^v/e_z{}^v \\ b_z{}^q/e_x{}^q & \delta b_z/\delta b_x & b_z{}^w/e_x{}^w & b_z{}^v/e_z{}^v \end{vmatrix} \tag{3.6}$$

which we represent symbolically as

$$\Delta'=\begin{vmatrix} 1 & a & 1 & b \\ c & 0 & d & 1 \\ E & 1 & F & G \\ H & J & K & L \end{vmatrix}. \tag{3.7}$$

A comparison between I Tables I and III and the elements of (3.6) reveals that each element designated by a lower case entry in (3.7) is one order of $(\omega_c\tau)^{-1}$ smaller than are the other elements in the same column. Hence, the expansion of Δ' to first order in $(\omega_c\tau)^{-1}$ will exclude terms which are products of lower case quantities. Setting Δ' to zero gives

$$\begin{aligned} &[H-HaF+c(JG-L)] \\ &\quad=[K-KaE+d(JG-L)]+J(E-F). \end{aligned} \tag{3.8}$$

It is instructive now to make contact with the results of I by evaluating (3.8) in the limit of infinite $\omega_c\tau$. This deletes all quantities containing a lower case factor, leaving

$$H=K+J(E-F). \tag{3.9}$$

Each of the quantities E, F, H, J, and K must be evaluated to zeroth order in $(\omega_c\tau)^{-1}$. This evaluation is simple, requiring only the use of I Tables I, II, and III, and I Eqs. (2.14bc) and (2.20c). The results are

$$E=X_1/n_z, \tag{3.10a}$$

$$F=X_2/n_z, \tag{3.10b}$$

$$J=i(K_1-K_2)/(K_1+K_2), \tag{3.10c}$$

$$H=n_1\cot(k_1r), \tag{3.11a}$$

$$K=n_2\cot[k_2(r-R)], \tag{3.11b}$$

and thus (3.9) is just (I3.1a), the dispersion relation of I.

In I, the threshold condition resulting in a real n_z was that normalized velocities U_1 and U_2 ($U_i=V_{Di}/c$; c = speed of light) be so adjusted that $X_1=X_2$, or

$$K_1(1-U_1n_t)=K_2(1-U_2n_t), \tag{3.12}$$

where n_t is the value of n_z which results when (2.1), (3.9), and (3.12) are solved simultaneously.

Now suppose that the currents are kept at these threshold values but that the collisions are turned on gradually, that is $(\omega_c\tau)^{-1}$ is made finite but small. We shall still refer to this situation as being "at threshold," because the currents still have their lowest-order threshold values. Clearly, n_z will no longer be real because adding collisions to the system without making other compensating changes can only result in attenuation of the wave.

Let us write the new (complex) value of n_z at threshold as

$$n_z=n_t+\gamma/\omega_c\tau. \tag{3.13}$$

Then at threshold,

$$X_1-X_2=K_1[1-U_1(n_t+\gamma/\omega_c\tau)]-K_2[1-U_2(n_t+\gamma/\omega_c\tau)]=-(K_1U_1-K_2U_2)\gamma/\omega_c\tau$$

which, using (3.12), is

$$X_1-X_2=-(K_1-K_2)\gamma/(n_t\omega_c\tau). \qquad (3.14)$$

This means, using (3.10ab) that the quantity $(E-F)$ is one order of $(\omega_c\tau)^{-1}$ smaller than either E or F. We denote this by writing $E-F$ in lower case, so that (3.8) becomes

$$[H-HaF+c(JG-L)]=[K-KaE+d(JG-L)]+J(e-f). \qquad (3.15)$$

The significance of this form is that all quantities appearing in it, except for the first quantity on each side, contain one lower case factor. To study (3.15) to first order, the terms multiplied by a lower case factor need be evaluated only to zeroth order, which can be done using the tables and equations of I. The lower case quantities themselves, being one order of $(\omega_c\tau)^{-1}$ smaller, need be calculated only to lowest nonvanishing order. Furthermore, H and K, the only quantities needed to higher order, are given exactly by (3.11) because the form (3.11) is a consequence only of $\nabla\times\mathbf{e}=-\dot{\mathbf{b}}$, and not of any expansion in powers of $(\omega_c\tau)^{-1}$. Of the various quantities, c and d are easily calculated using I Tables I, II, and III and recalling the convention that $E_z{}^i=1$ for all modes. The quantity a involves fields which were not needed in I, and the quantity $(e-f)$ requires that E and F be evaluated to next higher order to insure that there are no higher contributions to their difference. The evaluation of a and $(e-f)$ is simple and may be found in the Appendix. The needed results are

$$a=\frac{i}{\omega_c\tau}\frac{1}{n_z}, \qquad (3.16a)$$

$$c=\frac{-i}{\omega_c\tau}\frac{n_1}{n_z}\frac{X_1}{K_1}\cot(k_1r), \qquad (3.16b)$$

$$d=\frac{-i}{\omega_c\tau}\frac{n_2}{n_z}\frac{X_2}{K_2}\cot[k_2(r-R)], \qquad (3.16c)$$

$$e-f=\frac{X_1-X_2}{n_z}=-\frac{\gamma}{\omega_c\tau}\frac{(K_1-K_2)}{n_t^2}, \qquad (3.16d)$$

$$G=-iK_2/n_z, \qquad (3.16e)$$

$$L=-K_2/n_z, \qquad (3.16f)$$

where the second form in (3.16d) follows from (3.14). In evaluating these expressions, each magnetic field component was multiplied by the factor c/i. This procedure simplifies the expressions somewhat, and serves only to multiply the dispersion relation by a constant. Substituting Eqs. (3.10), (3.11), and (3.16) into (3.8) gives

$$\left\{n_1-\frac{in_1}{\omega_c\tau}\left[\frac{X_2}{n_z^2}+\frac{2X_1X_2}{n_z^2(K_1+K_2)}\right]\right\}\cot(k_1r)=\left\{n_2-\frac{in_2}{\omega_c\tau}\left[\frac{X_1}{n_z^2}+\frac{2X_2K_1}{n_z^2(K_1+K_2)}\right]\right\}\times\cot[k_2(r-R)]-\frac{i(K_1-K_2)^2\gamma}{(K_1+K_2)n_t^2\omega_c\tau}. \qquad (3.17)$$

To first-order accuracy, we regard n_1 and n_2 as having the form

$$n_j=n_0+i\gamma_j/\omega_c\tau, \qquad (3.18)$$

in the first term on each side of (3.17) and in the argument of the cotangent functions. Then (3.17) has the form

$$[n_0+i(A+\gamma_1)P]\cot(k_1r)=[n_0+i(B+\gamma_2)P]\cot[k_2(r-R)]+i\gamma CP, \qquad (3.19a)$$

where

$$P=1/\omega_c\tau, \qquad (3.19b)$$

$$A=-\frac{n_0}{n_t^2}\left[X_2+\frac{2X_1K_2}{K_1+K_2}\right], \qquad (3.19c)$$

$$B=-\frac{n_0}{n_t^2}\left[X_1+\frac{2X_2K_1}{K_1+K_2}\right], \qquad (3.19d)$$

$$C=-(K_1-K_2)^2/(K_1+K_2)n_t^2. \qquad (3.19e)$$

Finally, it is necessary to express γ_j in terms of γ. From (2.1) in the form

$$n_j^2=\frac{K_j^2(1-U_jn_z)^2}{n_z^2-2iK_j(1-U_jn_z)/\omega_c\tau}-n_z^2,$$

differentiation with respect to P at $P=0$, and use of (3.13) and (3.18) gives

$$2in_0\gamma_j=-\frac{2\gamma}{n_t}\left[\frac{X_j^2+X_jK_jU_jn_t}{n_t^2}+n_t^2\right]+\frac{2iX_j^3}{n_t^4}, \qquad (3.20)$$

which provides the needed relations. To solve (3.19a) for γ, differentiate it with respect to P at $P=0$. The trigonometric functions appearing on each side of the resulting expression are equal because, by setting $P=0$ in (3.19a), one is left with $n_0\cot(k_0r)=n_0\cot[k_0(r-R)]$. Thus, the differentiated form is

$$(A-B+\gamma_1-\gamma_2)\cot(Z)-(\gamma_1k_0r-\gamma_2Z)\csc^2(Z)=\gamma C, \qquad (3.21a)$$

where

$$Z = k_0(r-R), \tag{3.21b}$$

$$k_0 = n_0\omega/c. \tag{3.21c}$$

We have

$$\gamma_1 k_0 r - \gamma_2 Z = (\gamma_1-\gamma_2)k_0 r + \gamma_2 k_0 R. \tag{3.22}$$

From (3.19cd)

$$A-B = \frac{n_0}{n_t^2}\frac{(X_1+X_2)(K_1-K_2)}{(K_1+K_2)} = \frac{2n_0 X(K_1-K_2)}{n_t^2(K_1+K_2)}, \tag{3.23}$$

where $X_1 = X_2 = X$ at threshold. From (3.20)

$$\gamma_1-\gamma_2 = \frac{i\gamma X}{n_0 n_t}\frac{K_1U_1n_t - K_2U_2n_t}{n_t^2} = \frac{i\gamma X(K_1-K_2)}{n_0 n_t^3}. \tag{3.24}$$

Substituting Eqs. (3.19e), (3.22), (3.23), and (3.24) into (3.21a) gives an equation, all of whose terms are proportional to (K_1-K_2) except for one proportional to $(K_1-K_2)^2$, and one, $\gamma_2 k_0 R \csc^2(Z)$, which does not contain the factor (K_1-K_2). In the limit that K_1 approaches K_2, all these other terms go to zero, leaving

$$\gamma_2 k_0 R \csc^2(Z) = 0$$

which requires $\gamma_2 = 0$. But, $\gamma_2 = 0$ means that to first order, n_2 is unchanged. Hence, n_z can be calculated from (2.1) by using the value of n_y which occurs at threshold with $(\omega_c\tau)^{-1} = 0$. This is exactly the procedure used in (2.4) and subsequently. This shows that, if K_1 approaches K_2, then the dominant term in the derivative of n_z with respect to $(\omega_c\tau)^{-1}$, evaluated at threshold, is just the expression which was derived by using (2.4).

The calculation of $\partial n_z/\partial(\omega_c\tau)^{-1}$ just presented was performed at $(V-V_T)=0$. In the same spirit, the calculation of $\partial n_z/\partial(V-V_T)$ performed in I was performed at $(\omega_c\tau)^{-1}=0$. If we regard n_z as a function of the two independent variables $\alpha=(V-V_T)$ and $\beta=(\omega_c\tau)^{-1}$, then the Taylor series expansion of $\mathrm{Im}\,n_z(\alpha,\beta)$ about $\mathrm{Im}\,n_z(0,0)=0$, carried to first order in α and β, provides us with the justification for adding Eq. (2.5) to Eq. (2.7) to obtain Eq. (2.8).

APPENDIX

The expression (3.8), or alternatively (3.15), represents the dispersion relation for the composite structure shown in Fig. 1. In this appendix, we evaluate the terms which occur in the dispersion relation. For the sake of simplicity, each magnetic field component is multiplied by c/i, a procedure which serves only to multiply the dispersion relation by a constant.

Evaluation of a:

From (3.5), (3.6), and (3.7) we have

$$a = \frac{\delta e_x}{\delta b_x} = \frac{e_x^r - e_x^v e_z^r/e_z^v}{b_x^r - b_x^v e_z^r/e_z^v} = \frac{e_x^r e_z^v - e_x^v e_z^r}{b_x^r e_z^v - b_x^v e_z^r}. \tag{A1}$$

We use a numerical superscript to denote the field solution type, and subscripts I and II to denote the medium in which the solution is to be evaluated. (See I Table III.)

$$a = \frac{(e_x^2)_{\rm I}(e_z^6)_{\rm II} - (e_x^6)_{\rm II}(e_z^2)_{\rm I}}{(b_x^2)_{\rm I}(e_z^6)_{\rm II} - (b_x^6)_{\rm II}(e_z^2)_{\rm I}}. \tag{A2}$$

The use of lower case letters for the field components indicates that they are to be evaluated at $y=r$. With the exception of $(e_z^6)_{\rm II}$ and $(e_z^2)_{\rm I}$, these components can be evaluated directly from I Tables I and II. From (I-2.22ac), and the definitions of the type-2 and type-6 solutions in terms of elementary surface wave solutions, we obtain

$$(e_z^2)_{\rm I} = \frac{-K_1}{n_z^2\omega_c\tau}\sinh[\mathcal{K}_1 r], \tag{A3a}$$

$$(e_z^6)_{\rm II} = \frac{K_2}{n_z^2\omega_c\tau}\exp[-\mathcal{K}_2 r]. \tag{A3b}$$

Substituting these results together with those of I Tables I and II into (A2) yields

$$a = \frac{(-K_1/n_z^2\omega_c\tau) - (K_2/n_z^2\omega_c\tau)}{[N_y(S+iZn_z)/n_z^2]_{\rm I} - [-N_y(S+iZn_z)/n_z^2]_{\rm I}}$$
$$= -\frac{1}{\omega_c\tau}\left\{\frac{(K_1+K_2)}{[N_y(S+iZn_z)]_{\rm I} + [N_y(S+iZn_z)]_{\rm II}}\right\}. \tag{A4}$$

Using (I-2.14bc) and (I-2.20b) we find

$$N_y(S+iZn_z) = n_z\omega_c\tau(iK/\omega_c\tau) = iKn_z, \tag{A5}$$

so that

$$a = -\frac{1}{\omega_c\tau}\left\{\frac{(K_1+K_2)}{iK_1n_z + iK_2n_z}\right\} = \frac{i}{n_z\omega_c\tau}. \tag{A6}$$

This is the result given in (3.16a).

Evaluation of $(E-F)$:

From (3.6) and (3.7) we have

$$E = b_x^q/e_x^q, \tag{A7a}$$

$$F = b_x^w/e_x^w. \tag{A7b}$$

To lowest order in $(\omega_c\tau)^{-1}$ these expressions can be evaluated simply. For (A7a) we require only I Tables I, II, and III which give

$$E = X_1/n_z. \tag{A8}$$

For (A7b) we use (I-A.6) together with I Table II to obtain

$$F = X_2/n_z. \tag{A9}$$

As a consequence of (3.14), the difference of the lowest order expressions for E and F, $(E-F) = (X_1-X_2)/n_z$, is of order $(\omega_c\tau)^{-1}$ smaller than either E or F. However, there is the possibility that terms of order $(\omega_c\tau)^{-1}$ in E and F, which were previously neglected, could contribute to this difference. We now investigate such a possibility.

The ratios in (A7) involve bulk wave field components in such a way that any position dependence in the numerator is exactly cancelled by the same position dependence in the denominator. Therefore, we can calculate as though the fields varied simply as $\exp[ik_y y]$ with an error of at most an over-all sign. Comparison with the lowest-order results for E and F, (A8) and (A9), will determine the correct sign.

From the equation $\mathbf{k}\times\mathbf{e}=\omega\mathbf{b}$ we have

$$\begin{aligned} cb_x/e_x &= (n_y e_z - n_z e_y)/e_x = (n_y De_z - n_z De_y)/De_x \\ &= (n_y D - n_z De_y)/De_x, \end{aligned} \tag{A10}$$

where c is the speed of light, D is an arbitrary factor, and e_z is conventionally taken to be 1. The ratio (A.10) may be calculated exactly using (I2.16).

$$\frac{cb_x}{e_x}=\frac{n_y}{in_y}\left\{\frac{[S^2-S(2n_z^2+n_y^2)+n_z^2(n_z^2+n_y^2)-X^2]+n_z[(S-n_y^2-n_z^2)(iZ+n_z)-XY]}{Y(S-n_z^2)-X(iZ+n_z)}\right\}. \tag{A11}$$

The terms in the numerator of (A11) may be collected as

$$\begin{aligned}\mathfrak{N}=S(S+iZn_z)-(n_y^2+n_z^2)(S+iZn_z) \\ -X(X+n_zY),\end{aligned} \tag{A12}$$

which, using the definitions of (I2.14), is

$$\mathfrak{N}=-\frac{KX}{(\omega_c\tau)^2}-\frac{iK}{\omega_c\tau}(n_z^2+n_y^2)-XK. \tag{A13}$$

This will be correct to order $(\omega_c\tau)^{-1}$ if we drop the first term in (A13) and replace n_y^2 by its zeroth-order form using (I-2.20a). Thus,

$$\mathfrak{N}=-XK-\frac{iKX^2}{\omega_c\tau n_z^2}=-XK\left[1+\frac{iX}{n_z^2\omega_c\tau}\right]. \tag{A14}$$

In the denominator of (A11), we use (I-2.17b) to write

$$\begin{aligned}\mathfrak{D}&=Y(S-n_z^2)-X(iZ+n_z)=-n_z(X+Yn_z)\\ &=-n_zK,\end{aligned} \tag{A15}$$

where the last equality follows from (I-2.14). The result (A11) evaluated to first order in $(\omega_c\tau)^{-1}$ is thus

$$\frac{c}{i}\frac{b_x}{e_x}=\frac{X}{n_z}\left(1+\frac{iX}{n_z^2\omega_c\tau}\right). \tag{A16}$$

Suppressing the factor c/i, we obtain

$$E=(X_1/n_z)(1+iX_1/n_z^2\omega_c\tau), \tag{A17a}$$

$$F=(X_2/n_z)(1+iX_2/n_z^2\omega_c\tau) \tag{A17b}$$

when (A16) is evaluated in medium I and medium II, with reference to (A7ab).

The difference $(E-F)$ is then

$$\begin{aligned}(E-F)&=\frac{X_1-X_2}{n_z}+\frac{i(X_1^2-X_2^2)}{n_z^3\omega_c\tau}\\ &=\frac{X_1-X_2}{n_z}\left[1+\frac{i(X_1+X_2)}{n_z^2\omega_c\tau}\right].\end{aligned} \tag{A18}$$

Near threshold, this becomes

$$(e-f)=\frac{-(K_1-K_2)\gamma}{n_t^2(\omega_c\tau)}\left(1+\frac{2iX}{n_t^2\omega_c\tau}\right), \tag{A19}$$

where use has been made of (3.14). The importance of the result (A19) is that is shows the correction to $(e-f)$, from the form given in (3.16d), to be of higher order in $(\omega_c\tau)^{-1}$. Hence, the result (3.16d) is valid for an expansion which retains terms of order $(\omega_c\tau)^{-1}$ as was previously claimed.