Foundations of Physics, Vol. 34, No. 4, April 2004 (© 2004)

Radiation Reaction of a Nonrelativistic Quantum Charged Particle

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Received July 10, 2003; revised September 18, 2003

An alternative approach to analyze the nonrelativistic quantum dynamics of a rigid and extended charged particle taking into account the radiation reaction is discussed with detail. Interpretation of the field operators as annihilation and creation ones, theory of perturbations and renormalization are not used. The analysis is carried out in the Heisenberg picture with the electromagnetic field expanded in a complete orthogonal basis set of functions which allows the electromagnetic field to satisfy arbitrary boundary conditions. The corresponding coefficients are the field operators which satisfy the usual commutation relations. A nonlinear equation of motion for the charged particle is obtained. A careful consideration of the quantum effects allows the derivation of a linear equation of motion which is free of both runaway solutions and preacceleration, even for a point charge. Also, the electromagnetic mass, which is defined as the coefficient of the acceleration operator, vanishes for a point particle. However, this does not mean that the results are free of ambiguities which are exhibited and discussed.

KEY WORDS: quantum radiation reaction; nonlinear equation of motion; quantum electromagnetic mass; preacceleration and causality; extended charged particle; nonrelativistic quantum electrodynamics.

1. INTRODUCTION

In quantum electrodynamics a lot of success has been achieved, however some loose ends still remain.[(1–5)] It is important to exhibit the difficulties of

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0015-9018/04/0400-0547/0

the theoretical structure[2, 3] in order to establish its validity. The problems in the foundations of the theory have seldom been recognized as such, rather they are usually ignored, except in very few publications.[2–5] However, these problems have been inherited by the standard model, i.e., the theory of the electroweak and the strong interactions, since it was built following quantum electrodynamics.[6–9] Therefore, if an analysis of the foundations of quantum electrodynamics allows a way to be found to solve the problems then it would be possible to have a better quantum field theory.

The radiation reaction problem arises when it is desired to determine the motion of a charged particle taking into account the radiation emitted. Until now there has been no satisfactory solution to this fundamental problem in the realm of usual quantum electrodynamics. The aim of this paper is to explore an alternative approach. A novel method will lead to intriguing results, entirely different from the ones obtained through the usual method in quantum electrodynamics. However, the analysis presented here only extends the perspective of the problem which otherwise will still have some loose ends.

In the usual quantum electrodynamics, the theoretical structure is explicitly covariant and based on use of annihilation and creation operators, perturbation theory and renormalization.[6–11] Notwithstanding, some results are worked out in a noncovariant way[10, 11] in order to establish the foundations of the covariant method and to point out its advantages.

A noncovariant and nonperturbative method will be presented in this work. Also, this method requires neither interpretation of the field operators as annihilation and creation ones nor renormalization. Even so, the field variables are represented by operators, therefore the electromagnetic field is quantized and the whole method represents a new alternative in the realm of nonrelativistic quantum electrodynamics.[4, 5, 12–14]

Moreover, the analysis will be carried out by considering that the particle is extended. In this way, the interaction between the particle and the electromagnetic field is always present and the radiation reaction will be correctly and fully treated. It is convenient to point out that the analysis of the dynamics of this kind of particle is absent in usual quantum electrodynamics. In the realm of nonrelativistic quantum electrodynamics such an analysis has been carried out[2, 4, 5, 14] but the results obtained deserve a careful discussion. Precisely, the ideas presented in this work go in this direction.

However, such studies provide a basis to consider that in the realm of classical electrodynamics a charged particle must be extended, with a size greater than the Compton wavelength.[15] In fact, the results obtained in this work can also support such interpretation. It has been shown that the classical dynamics of a quasi-point charged particle has an adequate physical meaning. The classical dynamics of a rigid and nonrotating and charged particle has been widely analyzed elsewhere.[16–23] The retarded interaction between different elements can be taken into account in a natural way. This means that an extended particle introduces a non-locality. The result is that the classical dynamics of a rigid and nonrotating extended charged particle is free of difficulties. The same must be true for the quantum dynamics of this kind of particles.

The quantum dynamics of a point charged particle will be considered as the point particle limit of a theory of extended particles. However, this limit is only taken after retardation and quantum behavior are properly considered.[3, 5] The results obtained are therefore different from those arising from usual quantum electrodynamics.

The method will be developed in the Heisenberg picture. The fundamental equations of motion are the Heisenberg ones. The deduction of a very general equation of motion for the charged particle will be carried out in a similar way as is done in the classical case.[16–18] An advantage arises from this procedure. It is possible to define clearly a classical limit in each step.

The outline of the paper is as follows. In usual quantum electrodynamics the vector potential is expanded in a complete orthonormal set of plane modes.[6–14, 24] In the free case, each coefficient satisfies an equation of motion which corresponds to a harmonic oscillator. The usual interpretation is that each coefficient is either a creation operator of photons or an annihilation one. In this paper it will be explicitly exhibited that the potential vector can be expanded in an arbitrary complete orthogonal basis set of functions. The physical meaning is that the analysis of the electromagnetic field-matter interaction can be carried out by considering that the electromagnetic field can satisfy arbitrary boundary conditions. Then, in Sec. 2 it is established that two paths can be followed: (1) specify the charge distribution, keeping the boundary conditions arbitrary; (2) specify the boundary conditions, keeping the charge distribution arbitrary. However, the derivation of the set of coupled equations which will determine the dynamical evolution of the system electromagnetic field-charged particle can be carried out without making a specific choice of any of these two possibilities. The outstanding feature of this set of equations is that each coefficient of the expansion of the vector potential satisfies an equation of motion which corresponds to a forced harmonic oscillator, since the field-particle interaction is always present. It is convenient to remark that in usual quantum electrodynamics the existence of the two possibilities is not mentioned at all. This kind of analysis has scarcely been considered in the realm of classical electrodynamics.[19, 25]

It will be shown, in Sec. 3, that from the set of dynamical equations for the composed system it is possible to derive an integrodifferential and highly nonlinear equation of motion for the extended charged particle. This implies that to take into account the radiation emitted by the charged particle in the analysis of its dynamics unavoidably leads to a nonlinear problem. This feature is only mentioned in some analyses of usual quantum electrodynamics but the nonlinear equation of motion is never exhibited.[6,7] In many of the classical analyses the goal is to find a linear equation of motion for the charged particle, some approximations are therefore imposed from the beginning.[4,10,20,21] However, a better understanding of the radiation reaction problem is achieved when a classical nonlinear equation of motion is deduced and analyzed.[16–19,22,23] The quantum nonlinear equation of motion derived in this paper is quite general. It applies for an electromagnetic field satisfying arbitrary boundary conditions and for a rigid and nonrotating extended particle with an arbitrary charge configuration on it. However, in order to compare with the results obtained in other works,[4–14] it is convenient to consider both that the charge is spherically spread throughout the particle and that the electromagnetic field is expanded in plane modes. Even so, the equation of motion for the particle is nonlinear and allows a deep analysis of the radiation reaction problem in quantum physics.

The high nonlinearity of the radiation reaction problem, in the realm of either classical or quantum electrodynamics, makes it necessary to impose approximations. Perturbation theory allows the derivation of results, in usual quantum electrodynamics,[6–12] which can be compared with those obtained experimentally. This approach works very well for scattering experiments, but the radiation reaction is handled in a very unsatisfactory way.[1,2–5] Besides, the point particle model is used from the beginning and no other model of particle is discussed. An alternative approach is explored in Sec. 4 by imposing a linear approximation after the quantum effects are considered in a consistent way. The result is an integrodifferential equation of motion for the particle with structure which is very similar to the classical one.[16,17,19,21] However, it will be shown that a linear contribution of the magnetic self interaction arises as a new quantum effect. This result is in disagreement with that obtained in other works,[2,4,5] but in agreement with the result of a perturbative method.[14]

The analysis of the electromagnetic mass is carried out in Sec. 5. It is shown that the linear integrodifferential equation of motion for the particle can be equivalently written as an infinite-order differential equation. The coefficient of the acceleration operator must be interpreted as the electromagnetic mass, which depends on the model of charged particle, as is expected.[4,14] For many choices of the charge density the electromagnetic mass must be finite and it can be calculated in a closed way. However, the core is that the problem of the electromagnetic mass becomes enriched when the quantum effects are taken into consideration. The point particle model leads to a vanishing electromagnetic mass. This result is completely different from the one obtained in usual quantum electrodynamics.[6–14] The variety of ways that the electromagnetic mass can be defined shows that the problem is far from a closed one. It is necessary to carry out a deep analysis of each definition. It will be exhibited that the point particle model inserts some ambiguities in the approach employed in this work, as also happens with the other approaches. The point particle model is a very strong approximation in electrodynamics, either classical or quantum.

Despite the ambiguities related to the point particle model it is possible to derive an equation of motion for it. This is done in Sec. 6 and it is shown that the solution agrees well with causality (there is no preacceleration) and the runaway solution is absent when the external force is zero. The usual analyses in perturbative nonrelativistic quantum electrodynamics do not discuss this point.[11, 12, 14] In the nonperturbative analysis carried out by Moniz and Sharp[4] a similar result is achieved. However, their linear equation of motion and the one obtained in this work are not the same. The differences lie in the way that the linear contributions are taken into account.

It is worthwhile mentioning that in the classical analyses of the dynamics of a point charged particle several difficulties arise.[16–22] In the realm of classical electrodynamics the only way to get an equation of motion for a point charged particle consistent with the usual notion of causality is to choose appropriate boundary conditions for the electromagnetic field.[19, 25] The same is true for the runaway solution.

2. A NONPERTURBATIVE APPROACH

The use of explicit covariance is a way to ensure that any physical theory satisfies the fundamental principles established by the special theory of relativity. Quantum electrodynamics was built following this type of requirement. The usual formulation of this theory is based on perturbation theory, second quantization and renormalization.[6–11, 24, 26]

A complete analysis of the dynamics of an extended charged particle cannot be achieved by quantum electrodynamics since it is not possible to define a relativistic rigid structure. However, some years ago it was shown that the dynamics of this kind of particle can be studied in the framework of nonrelativistic quantum electrodynamics.[2–5, 14] Recovering the discussion of many fundamental questions, which fell into oblivion after the success of quantum electrodynamics, is an additional bonus.

In this work a quite general quantum equation of motion for a rigid and nonrotating extended charged particle will be derived from the basic quantum equations. It is therefore important to remark on the differences between the approach followed here and the perturbative one. In the latter it is assumed that initially and finally there is no interaction between the electromagnetic field and the particle field. Therefore, the analysis of the free cases is an important piece of the formalism.

However, the electromagnetic field produced by a charged particle cannot be considered alien to the particle itself. This is the reason to build a formalism where such as interaction is always present. It will be shown that the evolution of the field operators is coupled with the evolution of the particle operators. The analysis becomes more general by considering that the electromagnetic field satisfies arbitrary boundary conditions. Thus, the expansion of the vector potential will be carried out in any complete orthonormal basis set of functions instead of the plane modes employed in usual quantum electrodynamics.

Except for the explicit covariance requirement both nonrelativistic quantum electrodynamics and quantum electrodynamics share the same formalism. For the sake of comparison, it is convenient to have a glance at the starting point of such theories, which is the following plane mode expansion for the vector potential[24]

$$A_i(\mathbf{x}, t) = \sum_{\mathbf{k}, \sigma}{}' \{Q_{\mathbf{k}\sigma}(t)[u_i(\mathbf{x})]_{\mathbf{k}\sigma} + Q^\dagger_{\mathbf{k}\sigma}(t)[u_i^*(\mathbf{x})]_{\mathbf{k}\sigma}\}, \tag{2.1}$$

where

$$[u_i(\mathbf{x})]_{\mathbf{k}\sigma} = L^{-3/2} e^{i\mathbf{k}\cdot\mathbf{x}} [\hat{\varepsilon}_i]_{\mathbf{k}\sigma}, \tag{2.2}$$

with $\sigma = 1, 2$. The unit vectors $[\hat{\varepsilon}_i]_{\mathbf{k}1}$, $[\hat{\varepsilon}_i]_{\mathbf{k}2}$, and $\hat{k}_i$ form a right handed set and vectors $k_i = k\hat{k}_i$ satisfy periodic boundary conditions at the walls of a large cubical box of volume L^3. The prime in the summation implies that only the half of the $\mathbf{k}$ space is considered, so that the function $[u_i^*(\mathbf{x})]_{\mathbf{k}\sigma}$ does not duplicate $[u_i(\mathbf{x})]_{-\mathbf{k}\sigma}$.[24]

For a free electromagnetic field the Hamiltonian operator is given by

$$H(Q_{\mathbf{k}\sigma}, \Pi^\dagger_{\mathbf{k}\sigma}, Q^\dagger_{\mathbf{k}\sigma}, \Pi_{\mathbf{k}\sigma}) = \sum_{\mathbf{k}\sigma}{}' 4\pi c^2 \Pi^\dagger_{\mathbf{k}\sigma}(t)\, \Pi_{\mathbf{k}\sigma}(t) + \sum_{\mathbf{k}\sigma}{}' \frac{k^2}{4\pi c^2} Q^\dagger_{\mathbf{k}\sigma}(t)\, Q_{\mathbf{k}\sigma}(t). \tag{2.3}$$

The pairs of canonical variables are $Q_{\mathbf{k}\sigma}$, $\Pi^\dagger_{\mathbf{k}\sigma}$ and $Q^\dagger_{\mathbf{k}\sigma}$, $\Pi_{\mathbf{k}\sigma}$. Therefore the commutation relations are

$$[Q^{\dagger}_{\mathbf{k}\sigma}, \Pi_{\mathbf{k}'\sigma'}] = i\hbar\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$$

$$[Q_{\mathbf{k}\sigma}, \Pi^{\dagger}_{\mathbf{k}'\sigma'}] = i\hbar\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} \tag{2.4}$$

$$[Q_{\mathbf{k}\sigma}, \Pi_{\mathbf{k}'\sigma'}] = [Q^{\dagger}_{\mathbf{k}\sigma}, \Pi^{\dagger}_{\mathbf{k}'\sigma'}] = [\Pi_{\mathbf{k}\sigma}, \Pi^{\dagger}_{\mathbf{k}'\sigma'}] = [Q_{\mathbf{k}\sigma}, Q^{\dagger}_{\mathbf{k}'\sigma'}] = 0.$$

The use of the Heisenberg equations leads to the following equations of motion

$$\ddot{Q}_{\mathbf{k}\sigma} + \omega^2_{\mathbf{k}\sigma} Q_{\mathbf{k}\sigma} = 0, \tag{2.5}$$

which are easily integrated to give

$$Q_{\mathbf{k}\sigma}(t) = a_{\mathbf{k}\sigma}(0)\, e^{-i\omega_{\mathbf{k}\sigma}t} + a'^{\dagger}_{\mathbf{k}\sigma}(0)\, e^{i\omega_{\mathbf{k}\sigma}t}. \tag{2.6}$$

From here it is usual to identify[24] $a'_{\mathbf{k}\sigma}$ with $a_{-\mathbf{k}\sigma}$ and get the very well known Hamiltonian

$$H(a_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}\sigma}) = \sum_{\mathbf{k}\sigma} \hbar\omega_{\mathbf{k}}(a_{\mathbf{k}\sigma}(t)\, a^{\dagger}_{\mathbf{k}\sigma}(t) + \tfrac{1}{2}), \tag{2.7}$$

where the summation is over all $\mathbf{k}$ space.

In this work, the starting point is the generalization of Eq. (2.1) in order to consider any complete orthonormal basis set of functions $\{[D_i(\mathbf{x})]_{\mathbf{k}\sigma}\}$. Each element of the basis satisfies the Helmholtz equation

$$\nabla^2[D_i(\mathbf{x})]_{\mathbf{k}\sigma} + k^2[D_i(\mathbf{x})]_{\mathbf{k}\sigma} = 0. \tag{2.8}$$

The calculation will be carried out in the Coulomb gauge, therefore the condition $\partial_i[D_i(\mathbf{x})]_{\mathbf{k}\sigma} = 0$ must be satisfied. It is important to point out that the functions $[D_i(\mathbf{x})]_{\mathbf{k}\sigma}$ can satisfy very general boundary conditions. The plane modes, Eq. (2.2), are only a possible choice among many others. Once the effects due to retardation and quantum dynamics are taken into account in a full way the plane modes will be reconsidered in order to establish a comparison with known results.

The vector potential is now an operator given by

$$A_i(\mathbf{x}, t) = \sum_{\mathbf{k},\sigma}{}' \alpha_{\mathbf{k}}\{Q_{\mathbf{k}\sigma}(t)[D_i(\mathbf{x})]_{\mathbf{k}\sigma} + Q^{\dagger}_{\mathbf{k}\sigma}(t)[D^*_i(\mathbf{x})]_{\mathbf{k}\sigma}\}, \tag{2.9}$$

where $\alpha_{\mathbf{k}}$ will be fixed below. The Hamiltonian operator for the composed system: electromagnetic field-charged particle is

$$
\begin{aligned}
H(\mathbf{R}, \mathbf{P}, Q_{\mathbf{k}\sigma}, \Pi^{\dagger}_{\mathbf{k}\sigma}, Q^{\dagger}_{\mathbf{k}\sigma}, \Pi_{\mathbf{k}\sigma})
&= \frac{1}{2m} P_l(t)\, P_l(t) - \frac{e}{2mc} P_l(t) \sum_{\mathbf{k},\sigma}{}' \alpha_{\mathbf{k}} \left\{ Q_{\mathbf{k}\sigma}(t) \int d^3x\, \rho(\mathbf{x}, t)[D_l(\mathbf{x})]_{\mathbf{k}\sigma} \right. \\
&\quad \left. + Q^{\dagger}_{\mathbf{k}\sigma}(t) \int d^3x\, \rho(\mathbf{x}, t)[D^*_l(\mathbf{x})]_{\mathbf{k}\sigma} \right\} \\
&\quad - \frac{e}{2mc} \sum_{\mathbf{k},\sigma}{}' \alpha_{\mathbf{k}} \left\{ Q_{\mathbf{k}\sigma}(t) \int d^3x\, \rho(\mathbf{x}, t)[D_l(\mathbf{x})]_{\mathbf{k}\sigma} \right. \\
&\quad \left. + Q^{\dagger}_{\mathbf{k}\sigma}(t) \int d^3x\, \rho(\mathbf{x}, t)[D^*_l(\mathbf{x})]_{\mathbf{k}\sigma} \right\} P_l(t) \\
&\quad + \frac{e^2}{2mc^2} \left[\sum_{\mathbf{k},\sigma}{}' \alpha_{\mathbf{k}} \left\{ Q_{\mathbf{k}\sigma}(t) \int d^3x\, \rho(\mathbf{x}, t)[D_l(\mathbf{x})]_{\mathbf{k}\sigma} \right.\right. \\
&\quad \left.\left. + Q^{\dagger}_{\mathbf{k}\sigma}(t) \int d^3x\, \rho(\mathbf{x}, t)[D^*_l(\mathbf{x})]_{\mathbf{k}\sigma} \right\} \right]^2 \\
&\quad + \sum_{\mathbf{k}\sigma}{}' \frac{1}{\beta_{\mathbf{k}}} \Pi^{\dagger}_{\mathbf{k}\sigma}(t)\, \Pi_{\mathbf{k}\sigma}(t) + \sum_{\mathbf{k}\sigma}{}' \beta_{\mathbf{k}} \omega^2_{\mathbf{k}\sigma} Q^{\dagger}_{\mathbf{k}\sigma}(t)\, Q_{\mathbf{k}\sigma}(t),
\end{aligned}
\tag{2.10}
$$

where $\rho(\mathbf{x}, t)$ is normalized to unity and $\beta_{\mathbf{k}}$ will be fixed below. In the following it will be assumed that the extended particle is rigid and non-rotating.

As usual $\mathbf{R}$ and $\mathbf{P}$ are the canonical variables, in the Heisenberg picture, for the particle, therefore these variables are referred to the center of mass. The commutation relations for the operators $\mathbf{R}$ and $\mathbf{P}$ are the usual ones

$$
[R_i, P_j] = i\hbar\delta_{ij}, \qquad [R_i, R_j] = [P_i, P_j] = 0, \tag{2.11}
$$

while the field variables obey the commutation relations given in Eqs. (2.4). It is important to establish that it will be assumed, as usual, that each canonical variable of the particle commutes with any canonical variable of the field. The Heisenberg equations lead to the following set of equations of motion for the field variables

$$
\begin{aligned}
\ddot{Q}_{\mathbf{k}\sigma} + \omega^2_{\mathbf{k}\sigma} Q_{\mathbf{k}\sigma} &= \frac{e}{2c} \frac{\alpha_{\mathbf{k}}}{\beta_{\mathbf{k}}} \left\{ \dot{R}_l \int d^3x\, \rho(\mathbf{x}, t)[D_l(\mathbf{x})]_{\mathbf{k}\sigma} \right. \\
&\quad \left. + \int d^3x\, \rho(\mathbf{x}, t)[D_l(\mathbf{x})]_{\mathbf{k}\sigma}\, \dot{R}_l \right\}.
\end{aligned}
\tag{2.12}
$$

This means that the electromagnetic field is a set of forced harmonic oscillators when the interaction with matter is considered. The same behavior is obtained in usual quantum electrodynamics[(6–11)] The homogeneous solution of Eq. (2.12) is given in Eq. (2.6)

$$(Q_{\mathbf{k}\sigma})_h = q_{\mathbf{k}\sigma}(0)\, e^{-i\omega_{\mathbf{k}} t} + q'_{\mathbf{k}\sigma}(0)\, e^{i\omega_{\mathbf{k}} t} \tag{2.13}$$

and also

$$(\dot{Q}_{\mathbf{k}\sigma})_h = \left(\frac{\Pi_{\mathbf{k}\sigma}}{\beta_{\mathbf{k}}}\right)_h, \tag{2.14}$$

therefore

$$(\Pi_{\mathbf{k}\sigma})_h = -i\omega\beta_{\mathbf{k}}(q_{\mathbf{k}\sigma}(0)\, e^{-i\omega_{\mathbf{k}} t} - q'_{\mathbf{k}\sigma}(0)\, e^{i\omega_{\mathbf{k}} t}). \tag{2.15}$$

These results correspond to the free case, as they should (see Eq. (2.6) and Ref. 24).

A complete solution of Eq. (2.12) consists of the general solution to the homogeneous equation plus a particular solution to the inhomogeneous equation. Precisely, the inhomogeneous solution is the relevant one for the method that will be carried out below.

As usual, it is convenient to make the following transformation

$$Q_{\mathbf{k}\sigma}(t) = a_{\mathbf{k}\sigma}(t) + a'^{\dagger}_{\mathbf{k}\sigma}(t), \qquad \Pi_{\mathbf{k}\sigma}(t) = -i\omega_{\mathbf{k}}\beta_{\mathbf{k}}(a_{\mathbf{k}\sigma}(t) - a'^{\dagger}_{\mathbf{k}\sigma}(t)). \tag{2.16}$$

The commutation relations for these new operators can be obtained from Eqs. (2.4):

$$[a_{\mathbf{k}\sigma}(t), a^{\dagger}_{\mathbf{k}'\sigma'}(t)] = [a'_{\mathbf{k}\sigma}(t), a'^{\dagger}_{\mathbf{k}'\sigma'}(t)] = \frac{\hbar}{2\omega_{\mathbf{k}}\beta_{\mathbf{k}}}\,\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} \tag{2.17}$$

with all other pairs commuting. The vector potential written in terms of the new field variables is

$$A_i(\mathbf{x}, t) = \sum_{\mathbf{k},\sigma} \alpha_{\mathbf{k}}(a_{\mathbf{k}\sigma}(t)[D_i(\mathbf{x})]_{\mathbf{k}\sigma} + a^{\dagger}_{\mathbf{k}\sigma}(t)[D^*_i(\mathbf{x})]_{\mathbf{k}\sigma}), \tag{2.18}$$

where the summation is now over all **k** space.[(24)] This expansion for the vector potential becomes the usual one in quantum electrodynamics when each function $D_i(\mathbf{x})$ is a plane mode.[(6–14)] The new Hamiltonian operator is obtained substituting Eqs. (2.16) into Eq. (2.10), however, for the purposes of the method presented in this work it is better to write the Hamiltonian operator in an explicitly hermitian way

$$
\begin{aligned}
&H(\mathbf{R}, \mathbf{P}, a_{\mathbf{k}\sigma}, a_{\mathbf{k}\sigma}^{\dagger}) \\
&= \frac{1}{2m} P_l(t)\, P_l(t) - \frac{e}{4mc} P_l(t) \sum_{\mathbf{k},\sigma} \alpha_{\mathbf{k}} \int d^3x (a_{\mathbf{k}\sigma}(t)\, \rho(\mathbf{x}, t) + \rho(\mathbf{x}, t)\, a_{\mathbf{k}\sigma}(t))[D_l(\mathbf{x})]_{\mathbf{k}\sigma} \\
&\quad - \frac{e}{4mc} P_l(t) \sum_{\mathbf{k},\sigma} \alpha_{\mathbf{k}} \int d^3x (a_{\mathbf{k}\sigma}^{\dagger}(t)\, \rho(\mathbf{x}, t) + \rho(\mathbf{x}, t)\, a_{\mathbf{k}\sigma}^{\dagger}(t))[D_l^*(\mathbf{x})]_{\mathbf{k}\sigma} \\
&\quad - \frac{e}{4mc} \sum_{\mathbf{k},\sigma} \alpha_{\mathbf{k}} \int d^3x (a_{\mathbf{k}\sigma}(t)\, \rho(\mathbf{x}, t) + \rho(\mathbf{x}, t)\, a_{\mathbf{k}\sigma}(t))[D_l(\mathbf{x})]_{\mathbf{k}\sigma}\, P_l(t) \\
&\quad - \frac{e}{4mc} \sum_{\mathbf{k},\sigma} \alpha_{\mathbf{k}} \int d^3x (a_{\mathbf{k}\sigma}^{\dagger}(t)\, \rho(\mathbf{x}, t) + \rho(\mathbf{x}, t)\, a_{\mathbf{k}\sigma}^{\dagger}(t))[D_l^*(\mathbf{x})]_{\mathbf{k}\sigma}\, P_l(t) \\
&\quad + \frac{e^2}{8mc^2} \Bigg[\sum_{\mathbf{k},\sigma} \alpha_{\mathbf{k}} \int d^3x \{ (a_{\mathbf{k}\sigma}(t)\, \rho(\mathbf{x}, t) + \rho(\mathbf{x}, t)\, a_{\mathbf{k}\sigma}(t))[D_l(\mathbf{x})]_{\mathbf{k}\sigma} \\
&\quad + (a_{\mathbf{k}\sigma}^{\dagger}(t)\, \rho(\mathbf{x}, t) + \rho(\mathbf{x}, t)\, a_{\mathbf{k}\sigma}^{\dagger}(t))[D_l^*(\mathbf{x})]_{\mathbf{k}\sigma} \} \Bigg]^2 \\
&\quad + \sum_{\mathbf{k}\sigma} \beta_{\mathbf{k}} \omega_{\mathbf{k}}^2 (a_{\mathbf{k}\sigma}^{\dagger}(t)\, a_{\mathbf{k}\sigma}(t) + a_{\mathbf{k}\sigma}(t)\, a_{\mathbf{k}\sigma}^{\dagger}(t)); \qquad (2.19)
\end{aligned}
$$

the last term corresponds to a set of harmonic oscillators if $\beta_{\mathbf{k}} = \hbar/(2\omega_{\mathbf{k}})$, and also $\alpha_{\mathbf{k}} = ((2\pi c^2 \hbar)/(\omega_{\mathbf{k}}))^{1/2}$ is the usual choice. The Hamiltonian operator used in nonrelativistic quantum electrodynamics[11–13] is recovered if the point particle model is considered. A particle with structure has been considered only rarely.[14] In the perturbative analyses it is considered that initially there is no interaction, the operators $a_{\mathbf{k}\sigma}^{\dagger}$ and $a_{\mathbf{k}\sigma}$ are interpreted as creation and annihilation operators, respectively and the matter-electromagnetic field interaction is analyzed considering that the terms which depend on e in Eq. (2.19), are a perturbation. However, the perturbation series is only asymptotically convergent.[6, 7]

It should be physically meaningless to assume that initially there is no interaction in order to determine the motion of a charged particle considering the radiation emitted by itself. For this reason, it is important to build a new method in nonrelativistic quantum electrodynamics, founded on two underlying ideas: the charged particle and its electromagnetic field must be taken into account together and the retardation of the electromagnetic field must be considered explicitly in each step.

In order to continue with the construction of a method with the features mentioned above it is necessary to bear in mind that in the Heisenberg picture the evolution of a physical system is determined by the temporal dependence of the dynamical variables which are represented by operators.

This means that both the particle and its electromagnetic field are quantum entities. In the coordinate representation the use of the Heisenberg equations leads to the following set of coupled equations

$$\dot{a}_{\mathbf{k}\sigma} = -i\omega_{\mathbf{k}} a_{\mathbf{k}\sigma} + \frac{ie}{\hbar c}\left(\frac{2\pi c^2\hbar}{\omega_{\mathbf{k}}}\right)^{1/2} \int d^3x \, \frac{1}{2}(\rho\dot{R}_l + \dot{R}_l\rho)[D_l^*]_{\mathbf{k}\sigma} \tag{2.20}$$

$$\dot{a}^\dagger_{\mathbf{k}\sigma} = i\omega_{\mathbf{k}} a^\dagger_{\mathbf{k}\sigma} - \frac{ie}{\hbar c}\left(\frac{2\pi c^2\hbar}{\omega_{\mathbf{k}}}\right)^{1/2} \int d^3x \, \frac{1}{2}(\rho\dot{R}_l + \dot{R}_l\rho)[D_l]_{\mathbf{k}\sigma} \tag{2.21}$$

$$\begin{aligned} \dot{R}_i &= \frac{P_i}{m} - \frac{e}{2mc}\sum_{\mathbf{k},\sigma}\left(\frac{2\pi c^2\hbar}{\omega_{\mathbf{k}}}\right)^{1/2} \int d^3x \\ &\quad \times \{(a_{\mathbf{k}\sigma}\rho + \rho a_{\mathbf{k}\sigma})[D_i]_{\mathbf{k}\sigma} + (a^\dagger_{\mathbf{k}\sigma}\rho + \rho a^\dagger_{\mathbf{k}\sigma})[D_i^*]_{\mathbf{k}\sigma}\} \end{aligned} \tag{2.22}$$

$$\begin{aligned} \dot{P}_i &= \frac{e}{4c}\sum_{\mathbf{k},\sigma}\left(\frac{2\pi c^2\hbar}{\omega_{\mathbf{k}}}\right)^{1/2} \int d^3x \{(a_{\mathbf{k}\sigma}\dot{R}_l(\nabla_i\rho) + a_{\mathbf{k}\sigma}(\nabla_i\rho)\,\dot{R}_l)[D_l]_{\mathbf{k}\sigma} \\ &\quad + (\dot{R}_l(\nabla_i\rho)\,a_{\mathbf{k}\sigma} + (\nabla_i\rho)\,\dot{R}_l a_{\mathbf{k}\sigma})[D_l]_{\mathbf{k}\sigma} \\ &\quad + (a^\dagger_{\mathbf{k}\sigma}\dot{R}_l(\nabla_i\rho) + a^\dagger_{\mathbf{k}\sigma}(\nabla_i\rho)\,\dot{R}_l)[D_l^*]_{\mathbf{k}\sigma} \\ &\quad + (\dot{R}_l(\nabla_i\rho)\,a^\dagger_{\mathbf{k}\sigma} + (\nabla_i\rho)\,\dot{R}_l a^\dagger_{\mathbf{k}\sigma})[D_l^*]_{\mathbf{k}\sigma}\}, \end{aligned} \tag{2.23}$$

where $[D_l]_{\mathbf{k}\sigma} = [D_l(\mathbf{x})]_{\mathbf{k}\sigma}$, $\rho = \rho(\mathbf{x}, t)$, and $R_l = R_l(t)$. Equations (2.20) and (2.21) are equivalent to Eq. (2.12). In fact, they are the Maxwell–Lorentz equations. It is interesting to note that the hermitian operator

$$J_l(\mathbf{x}, t) = \tfrac{1}{2}(\rho(\mathbf{x}, t)\,\dot{R}_l(t) + \dot{R}_l(t)\,\rho(\mathbf{x}, t)) \tag{2.24}$$

appears in Eqs. (2.20) and (2.21). This is the current density operator. The law of charge conservation is readily written as an equation between operators as follows

$$\frac{1}{i\hbar}[\rho(\mathbf{x}, t), H] = \frac{1}{2}(\nabla_l\rho(\mathbf{x}, t)\,\dot{R}_l + \dot{R}_l\nabla_l\rho(\mathbf{x}, t)) = \frac{\partial\rho(\mathbf{x}, t)}{\partial t}. \tag{2.25}$$

This shows that the Hamiltonian operator given by Eq. (2.19) leads to the expected results.

3. NONLINEAR EQUATION OF MOTION

The set of coupled dynamical equations, Eqs. (2.20)–(2.23), is quite general for the determination of the dynamical behavior of both the electromagnetic field and the charged particle. It is important to remark that Eqs. (2.20) and (2.21) indicate that the electromagnetic field is a set of forced harmonic oscillators. The quantum differential equation for a forced harmonic oscillator can be solved,[27] but it is necessary to establish the physical grounds of the solution since it can be retarded, advanced or a linear superposition of both. In this paper it is considered that the fundamental laws relate cause and effect, where the former always precedes or is simultaneous to the latter. This is known as the principle of antecedence.[28] Therefore, only the retarded solutions are considered, as in many classical analyses of the radiation reaction[4, 16–23, 25, 29, 30] and they are given by

$$a_{\mathbf{k}\sigma}(t) = a_{\mathbf{k}\sigma}(t_0)\, e^{-i\omega_{\mathbf{k}}(t-t_0)} + e^{-i\omega_{\mathbf{k}}(t-t_0)} \frac{ie}{\hbar c}\left(\frac{2\pi c^2\hbar}{\omega_{\mathbf{k}}}\right)^{1/2} \int_{t_0}^{t} dt' \int d^3x' \\ \times e^{i\omega_{\mathbf{k}}t'} J_s(\mathbf{x}', t')[D_s^*(\mathbf{x}')]_{\mathbf{k}\sigma}, \tag{3.1}$$

$$a^\dagger_{\mathbf{k}\sigma}(t) = a^\dagger_{\mathbf{k}\sigma}(t_0)\, e^{i\omega_{\mathbf{k}}(t-t_0)} - e^{i\omega_{\mathbf{k}}(t-t_0)} \frac{ie}{\hbar c}\left(\frac{2\pi c^2\hbar}{\omega_{\mathbf{k}}}\right)^{1/2} \int_{t_0}^{t} dt' \int d^3x' \\ \times e^{-i\omega_{\mathbf{k}}t'} J_s(\mathbf{x}', t')[D_s(\mathbf{x}')]_{\mathbf{k}\sigma}. \tag{3.2}$$

The first terms on the right-hand side of these equations represent the free field solutions and the second terms take into account the interaction field-matter. The next step consists in the substitution of these results in Eqs. (2.22) and (2.23) in order to obtain an equation of motion for the charged particle which will result in a nonlinear equation, as will be shown below. Therefore, the introduction of some approximations is unavoidable in order to obtain tangible results.

In the usual formulation of both nonrelativistic quantum electrodynamics and quantum electrodynamics it is assumed that initially the electromagnetic field is free, therefore $a_{\mathbf{k}\sigma}(t) = a_{\mathbf{k}\sigma}(0)\, e^{-i\omega_{\mathbf{k}}t}$ and $a^\dagger_{\mathbf{k}\sigma}(t) = a^\dagger_{\mathbf{k}\sigma}(0)\, e^{i\omega_{\mathbf{k}}t}$. The consequence is that the quantum field variables $a_{\mathbf{k}\sigma}(t)$ and $a^\dagger_{\mathbf{k}\sigma}(t)$ are interpreted as annihilation and creation operators, respectively. Theoretical results, which are in complete agreement with the experimental ones, are obtained using perturbation theory and renormalization. The latter is necessary because some quantities diverge; the electromagnetic mass is an example.

Despite the success of quantum electrodynamics for analyzing scattering processes, there is some dissatisfaction with its foundations. The radiation reaction problem does not have a true solution in the formalism.

A more systematic analysis of the radiation reaction problem in nonrelativistic quantum electrodynamics is necessary. This is the aim of this work. In conventional quantum electrodynamics it is accepted that $a_{\mathbf{k}\sigma}(t)$ and $a^{\dagger}_{\mathbf{k}\sigma}(t)$ can no longer be considered as annihilation and creation operators when the interaction with matter is present.[6] Therefore, the usual interpretation of the field operators as annihilation and creation will not be used here. Besides, the method is nonperturbative and it is not necessary to use renormalization.

The fundamental idea is to work with the retarded solutions, given in Eqs. (3.1) and (3.2), to exhibit the nonlinear equation of motion for the charged particle. Afterwards, a linear equation of motion will be obtained by a suitable and systematic procedure. But the outstanding feature is that the retardation effect will be taken into account in an exact way when the interaction of a charged particle with its own field is analyzed. It has been pointed out elsewhere[2, 3, 5] that the retardation is not treated in a correct way in conventional quantum electrodynamics. Therefore the analysis presented here makes an improvement on the study of the quantum dynamics of charged particles.

In fact, the result is amazing since the linear equation of motion leads to a physically acceptable dynamical behavior for a point charged particle. This result significantly differs from the results obtained in both classical theory and conventional quantum electrodynamics. Another nonperturbative analysis has appeared elsewhere.[4] The fundamental idea is the same as the one used in this paper but the method is cumbersome and it is not mentioned that the electromagnetic field could satisfy other types of boundary conditions.

The equation of motion for the charged particle is obtained straightforwardly by considering that $\ddot{R}_i = \frac{1}{i\hbar}[\dot{R}_i, H]$ and by substituting the retarded solutions, Eqs. (3.1) and (3.2), in Eqs. (2.22) and (2.23). It is convenient to point out that the first terms in Eqs. (3.1) and (3.2) can be interpreted as the contribution of an external electromagnetic field. The contribution of these terms to the equation of motion for the charged particle is a term proportional to e which is called the external force operator and denoted by $(F_{\text{ext}})_i$. On the other hand, the second terms in Eqs. (3.1) and (3.2) correspond to the electromagnetic field produced by the charged particle itself. This means that in the equation of motion for the charged particle a term appears which takes into account the interaction of the charged particle with the electromagnetic field produced by itself; this self-interaction is proportional to e^2. Therefore, in the equation of motion for the charged particle only these two powers of e can appear, there is no place for any other powers. Such an equation is

$$
\begin{aligned}
m\ddot{R}_i = (F_{\text{ext}})_i - \pi e^2 \sum_{\mathbf{k}\sigma} \int_{t_0}^{t} dt' \int d^3x \int d^3x' \\
\times \Big\{ &\cos kc(t-t')[[D_s^{*\prime}]_{\mathbf{k}\sigma}\,[D_i]_{\mathbf{k}\sigma} + [D_s']_{\mathbf{k}\sigma}\,[D_i^*]_{\mathbf{k}\sigma}](J_s'\rho + \rho J_s') \\
&- i \sin kc(t-t')[[D_s^{*\prime}]_{\mathbf{k}\sigma}\,[D_i]_{\mathbf{k}\sigma} - [D_s']_{\mathbf{k}\sigma}\,[D_i^*]_{\mathbf{k}\sigma}](J_s'\rho + \rho J_s') \\
&- i\,\frac{e^{-i\omega_{\mathbf{k}}(t-t')}}{2kc}\, J_s'[D_s^{*\prime}]_{\mathbf{k}\sigma}\,[(\dot{R}_l(\nabla_i\rho) + (\nabla_i\rho)\,\dot{R}_l)[D_l]_{\mathbf{k}\sigma} \\
&- (\dot{R}_l(\nabla_l\rho) + (\nabla_l\rho)\,\dot{R}_l)[D_i]_{\mathbf{k}\sigma}] \\
&+ i\,\frac{e^{i\omega_{\mathbf{k}}(t-t')}}{2kc}\, J_s'[D_s']_{\mathbf{k}\sigma}\,[(\dot{R}_l(\nabla_i\rho) + (\nabla_i\rho)\,\dot{R}_l)[D_l^*]_{\mathbf{k}\sigma} \\
&- (\dot{R}_l(\nabla_l\rho) + (\nabla_l\rho)\,\dot{R}_l)[D_i^*]_{\mathbf{k}\sigma}] \\
&- i\,\frac{e^{-i\omega_{\mathbf{k}}(t-t')}}{2kc}\,[(\dot{R}_l(\nabla_i\rho) + (\nabla_i\rho)\,\dot{R}_l)[D_l]_{\mathbf{k}\sigma} \\
&- (\dot{R}_l(\nabla_l\rho) + (\nabla_l\rho)\,\dot{R}_l)[D_i]_{\mathbf{k}\sigma}]\, J_s'[D_s^{*\prime}]_{\mathbf{k}\sigma} \\
&+ i\,\frac{e^{i\omega_{\mathbf{k}}(t-t')}}{2kc}\,[(\dot{R}_l(\nabla_i\rho) + (\nabla_i\rho)\,\dot{R}_l)[D_l^*]_{\mathbf{k}\sigma} \\
&- (\dot{R}_l(\nabla_l\rho) + (\nabla_l\rho)\,\dot{R}_l)[D_i^*]_{\mathbf{k}\sigma}]\, J_s'[D_s']_{\mathbf{k}\sigma} \Big\},
\end{aligned}
\tag{3.3}
$$

where $[D_s']_{\mathbf{k}\sigma} = [D_s(\mathbf{x}')]_{\mathbf{k}\sigma}$, $J_s = J_s(\mathbf{x}, t)$, and $J_s' = J_s(\mathbf{x}', t')$. Up to this step, neither boundary conditions for the electromagnetic field nor a charge density on the particle have been specified. Only two conditions have been assumed in order to get this nonlinear equation of motion: the motion is translational and the particle is rigid. Thus, this is a quite general equation of motion for the charged particle. It is important to remember that the mentioned approximations do not lead to difficulties in the classical analysis. The same should happen in the quantum analysis.

The self-interaction leads to nonlinear terms in the equation of motion. The same general behavior is obtained in conventional quantum electrodynamics.[6] This means that the radiation reaction problem is inherently nonlinear and it has a vast complexity.

The charge density can be written as

$$
\rho(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3q\, \tilde{\rho}(q)\, e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{R}(t))}
\tag{3.4}
$$

and the current density as

$$J_i(\mathbf{x}, t) = \frac{1}{2(2\pi)^{3/2}} \int d^3q\, \tilde{\rho}(q)\, e^{i\mathbf{q}\cdot\mathbf{x}}(e^{-i\mathbf{q}\cdot\mathbf{R}(t)}\dot{R}_i(t) + \dot{R}_i(t)\, e^{-i\mathbf{q}\cdot\mathbf{R}(t)}). \tag{3.5}$$

It is important to emphasize that a nonlocality is being introduced. It has been shown, in the realm of classical electrodynamics, that when either a rigid and nonrotating extended charged particle[16–23] or a quasi-point charged particle[15] is considered, the corresponding equation of motion is free of unphysical behavior. It is expected that the same result will be obtained in the quantum analysis.

It is convenient to take $[D_l]_{\mathbf{k}\sigma} = D_{\mathbf{k}}\hat{\varepsilon}_{l\sigma}$, where $\sigma = 1, 2$. The unit vectors $\hat{\varepsilon}_{l1}$ and $\hat{\varepsilon}_{l2}$ are mutually perpendicular. The most general nonlinear quantum equation of motion becomes

$$\begin{aligned}
m\ddot{R}_i = (F_{\text{ext}})_i - \frac{e^2}{4(2\pi)^2} \sum_{\mathbf{k}\sigma} \int_{t_0}^{t} dt' \int d^3x \int d^3x' \int d^3q \int d^3q' \\
\times \Big\{ &\cos kc(t-t')\, \tilde{\rho}\tilde{\rho}' e^{i\mathbf{q}'\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{x}'} \hat{\varepsilon}_{s\sigma}\hat{\varepsilon}_{i\sigma} [D_{\mathbf{k}}^{*\prime} D_{\mathbf{k}} + D_{\mathbf{k}}' D_{\mathbf{k}}^{*}] \\
&\times [(e^{-i\mathbf{q}\cdot\mathbf{R}'}\dot{R}_s' + \dot{R}_s' e^{-i\mathbf{q}\cdot\mathbf{R}'})\, e^{-i\mathbf{q}'\cdot\mathbf{R}} + e^{-i\mathbf{q}'\cdot\mathbf{R}}(e^{-i\mathbf{q}\cdot\mathbf{R}'}\dot{R}_s' + \dot{R}_s' e^{-i\mathbf{q}\cdot\mathbf{R}'})] \\
&- i \sin kc(t-t')\, \tilde{\rho}\tilde{\rho}' e^{i\mathbf{q}'\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{x}'} \hat{\varepsilon}_{s\sigma}\hat{\varepsilon}_{i\sigma} [D_{\mathbf{k}}^{*\prime} D_{\mathbf{k}} - D_{\mathbf{k}}' D_{\mathbf{k}}^{*}] \\
&\times [(e^{-i\mathbf{q}\cdot\mathbf{R}'}\dot{R}_s' + \dot{R}_s' e^{-i\mathbf{q}\cdot\mathbf{R}'})\, e^{-i\mathbf{q}'\cdot\mathbf{R}} + e^{-i\mathbf{q}'\cdot\mathbf{R}}(e^{-i\mathbf{q}\cdot\mathbf{R}'}\dot{R}_s' + \dot{R}_s' e^{-i\mathbf{q}\cdot\mathbf{R}'})] \\
&- \frac{1}{2kc} \sin kc(t-t')\, \tilde{\rho}\tilde{\rho}' e^{i\mathbf{q}'\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{x}'} \hat{\varepsilon}_{s\sigma} \\
&\times [D_{\mathbf{k}}'(\hat{\varepsilon}_{l\sigma}\partial_i D_{\mathbf{k}}^{*} - \hat{\varepsilon}_{i\sigma}\partial_l D_{\mathbf{k}}^{*}) + D_{\mathbf{k}}^{*\prime}(\hat{\varepsilon}_{l\sigma}\partial_i D_{\mathbf{k}} - \hat{\varepsilon}_{i\sigma}\partial_l D_{\mathbf{k}})] \\
&\times [(e^{-i\mathbf{q}\cdot\mathbf{R}'}\dot{R}_s' + \dot{R}_s' e^{-i\mathbf{q}\cdot\mathbf{R}'})(e^{-i\mathbf{q}'\cdot\mathbf{R}}\dot{R}_l + \dot{R}_l e^{-i\mathbf{q}'\cdot\mathbf{R}}) \\
&+ (e^{-i\mathbf{q}'\cdot\mathbf{R}}\dot{R}_l + \dot{R}_l e^{-i\mathbf{q}'\cdot\mathbf{R}})(e^{-i\mathbf{q}\cdot\mathbf{R}'}\dot{R}_s' + \dot{R}_s' e^{-i\mathbf{q}\cdot\mathbf{R}'})] \\
&+ \frac{i}{2kc} \cos kc(t-t')\, \tilde{\rho}\tilde{\rho}' e^{i\mathbf{q}'\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{x}'} \hat{\varepsilon}_{s\sigma} \\
&\times [D_{\mathbf{k}}'(\hat{\varepsilon}_{l\sigma}\partial_i D_{\mathbf{k}}^{*} - \hat{\varepsilon}_{i\sigma}\partial_l D_{\mathbf{k}}^{*}) + D_{\mathbf{k}}^{*\prime}(\hat{\varepsilon}_{l\sigma}\partial_i D_{\mathbf{k}} - \hat{\varepsilon}_{i\sigma}\partial_l D_{\mathbf{k}})] \\
&\times [(e^{-i\mathbf{q}\cdot\mathbf{R}'}\dot{R}_s' + \dot{R}_s' e^{-i\mathbf{q}\cdot\mathbf{R}'})(e^{-i\mathbf{q}'\cdot\mathbf{R}}\dot{R}_l + \dot{R}_l e^{-i\mathbf{q}'\cdot\mathbf{R}}) \\
&+ (e^{-i\mathbf{q}'\cdot\mathbf{R}}\dot{R}_l + \dot{R}_l e^{-i\mathbf{q}'\cdot\mathbf{R}})(e^{-i\mathbf{q}\cdot\mathbf{R}'}\dot{R}_s' + \dot{R}_s' e^{-i\mathbf{q}\cdot\mathbf{R}'})] \Big\}.
\end{aligned} \tag{3.6}$$

This integrodifferential equation for the operators of the particle should be the starting point for any further analysis of the radiation reaction problem, due to its general validity. It is still possible to choose any boundary conditions for the electromagnetic field. It was pointed out that in conventional quantum electrodynamics the electromagnetic field is only expanded in plane modes. In a nonperturbative analysis[4] of quantum radiation reaction the retarded potentials are used, thus the development of the method is carried out as in the classical analysis. This means that in most analyses of the radiation reaction problem it is not mentioned that an equation of motion like Eq. (3.6) could be obtained.

It is convenient to remark on the differences between the perturbative method and the nonperturbative one which has been built to study the quantum dynamics of a charged particle. For this reason it is adequate to consider an expansion in plane modes for the electromagnetic field. The equation of motion for the charged particle turns out to be

$$\begin{aligned}
m\ddot{R}_i = (F_{\text{ext}})_i - \frac{e^2(2\pi)^4}{2L^3} \sum_{\mathbf{k}} \int_{t_0}^{t} dt' \Big\{ &\cos kc(t-t')\, |\tilde{\rho}|^2 \left(\delta_{si} - \frac{k_s k_i}{k^2} \right) \\
&\times [(e^{i\mathbf{k}\cdot\mathbf{R}'} \dot{R}'_s + \dot{R}'_s e^{i\mathbf{k}\cdot\mathbf{R}'})\, e^{-i\mathbf{k}\cdot\mathbf{R}} + e^{-i\mathbf{k}\cdot\mathbf{R}} (e^{i\mathbf{k}\cdot\mathbf{R}'} \dot{R}'_s + \dot{R}'_s e^{i\mathbf{k}\cdot\mathbf{R}'})] \\
&+ \frac{i}{2kc} \sin kc(t-t')\, |\tilde{\rho}|^2\, (\delta_{sl} k_i - \delta_{si} k_l) \\
&\times [(e^{i\mathbf{k}\cdot\mathbf{R}'} \dot{R}'_s + \dot{R}'_s e^{i\mathbf{k}\cdot\mathbf{R}'})(e^{-i\mathbf{k}\cdot\mathbf{R}} \dot{R}_l + \dot{R}_l e^{-i\mathbf{k}\cdot\mathbf{R}}) \\
&+ (e^{-i\mathbf{k}\cdot\mathbf{R}} \dot{R}_l + \dot{R}_l e^{-i\mathbf{k}\cdot\mathbf{R}})(e^{i\mathbf{k}\cdot\mathbf{R}'} \dot{R}'_s + \dot{R}'_s e^{i\mathbf{k}\cdot\mathbf{R}'})] \Big\},
\end{aligned} \tag{3.7}$$

with

$$\sum_{\sigma} \hat{\varepsilon}_{s\sigma} \hat{\varepsilon}_{l\sigma} = \delta_{sl} - \frac{k_s k_l}{k^2}. \tag{3.8}$$

The analysis is simpler when it is considered that the normalization box is very large; then the discrete basis of functions becomes a continuous one and the equation of motion for the charged particle is

$$\begin{aligned}
m\ddot{R}_i = (F_{\text{ext}})_i - \pi e^2 \int d^3k \int_{t_0}^{t} dt' \Big\{ &\cos kc(t-t')\, |\tilde{\rho}|^2 \left(\delta_{si} - \frac{k_s k_i}{k^2} \right) \\
&\times [(e^{i\mathbf{k}\cdot\mathbf{R}'} \dot{R}'_s + \dot{R}'_s e^{i\mathbf{k}\cdot\mathbf{R}'})\, e^{-i\mathbf{k}\cdot\mathbf{R}} + e^{-i\mathbf{k}\cdot\mathbf{R}} (e^{i\mathbf{k}\cdot\mathbf{R}'} \dot{R}'_s + \dot{R}'_s e^{i\mathbf{k}\cdot\mathbf{R}'})] \\
&+ \frac{i}{2kc} \sin kc(t-t')\, |\tilde{\rho}|^2\, (\delta_{sl} k_i - \delta_{si} k_l) \\
&\times [(e^{i\mathbf{k}\cdot\mathbf{R}'} \dot{R}'_s + \dot{R}'_s e^{i\mathbf{k}\cdot\mathbf{R}'})(e^{-i\mathbf{k}\cdot\mathbf{R}} \dot{R}_l + \dot{R}_l e^{-i\mathbf{k}\cdot\mathbf{R}}) \\
&+ (e^{-i\mathbf{k}\cdot\mathbf{R}} \dot{R}_l + \dot{R}_l e^{-i\mathbf{k}\cdot\mathbf{R}})(e^{i\mathbf{k}\cdot\mathbf{R}'} \dot{R}'_s + \dot{R}'_s e^{i\mathbf{k}\cdot\mathbf{R}'})] \Big\}.
\end{aligned} \tag{3.9}$$

This equation is very similar to that obtained in the classical analysis.[18] The difference lies in the non commuting operators which appear in Eq. (3.9). A possible definition of the classical limit consists in transforming the operators into functions. As there are no commuting rules for functions then the classical equation of motion is recovered.

4. A LINEAR EQUATION OF MOTION

It is not possible to solve a nonlinear integrodifferential equation in a closed way; then it is absolutely necessary to make approximations. In conventional nonrelativistic quantum electrodynamics[12–14] perturbation theory is the way to overcome the difficulty. The linear approximation is imposed in almost all classical analyses.[16, 17, 20, 21, 25, 28, 29]

However, when a nonlinear dynamical problem is under study the consequences of each approximation must be carefully analyzed since meaningless physical results could be obtained.[4, 19–23, 28, 31] It is hoped that the nonlinear equation of motion does not lead to difficulties because it was directly obtained from the fundamental laws of both electrodynamics and nonrelativistic quantum mechanics. It is not possible to guarantee the same for any equation of motion obtained after the imposition of some approximations.

The subsequent analysis is exclusively devoted to the linear approximation. The purpose is to establish if the linear equation of motion for the charged particle is compatible or not with an acceptable dynamical behavior. At first sight it would seem that the omission of the nonlinear terms in Eq. (3.9) leads to the following linear equation of motion

$$m\ddot{R}_i = (F_{\text{ext}})_i - 4\pi e^2 \int d^3k \int_{t_0}^{t} dt' \cos kc(t-t')\, |\tilde{\rho}|^2 \left(\delta_{si} - \frac{k_s k_i}{k^2}\right) \dot{R}'_s. \tag{4.1}$$

This equation is formally identical to the classical linear equation of motion.[16, 17, 21] But now this is a linear integrodifferential equation among operators. Considering a point particle the equation of motion is

$$ma_i = (F_{\text{ext}})_i - (\delta m)\, a_i + \frac{2e^2}{3c^3}\, \dot{a}_i, \tag{4.2}$$

which is the Abraham–Lorentz equation for operators. However, the richness of quantum mechanics has not been completely displayed. Years ago, Moniz and Sharp[4] built a more systematic procedure to obtain a linear equation of motion for the charged particle. A similar method will be followed in this work. The clue is to use the basic commutation relations in order to obtain the expressions

$$e^{i\mathbf{k}\cdot\mathbf{R}}\dot{R}_s + \dot{R}_s e^{i\mathbf{k}\cdot\mathbf{R}} = 2e^{i\mathbf{k}\cdot\mathbf{R}}\dot{R}_s + \frac{\hbar}{m} k_s e^{i\mathbf{k}\cdot\mathbf{R}}, \tag{4.3}$$

$$e^{i\mathbf{k}\cdot\mathbf{R}}\dot{R}_s + \dot{R}_s e^{i\mathbf{k}\cdot\mathbf{R}} = 2\dot{R}_s e^{i\mathbf{k}\cdot\mathbf{R}} - \frac{\hbar}{m} k_s e^{i\mathbf{k}\cdot\mathbf{R}}, \tag{4.4}$$

where two possible orderings have been defined. In the right-ordering the operator $(d^n/dt^n)\, R$ stands to the right of $e^{i\mathbf{k}\cdot\mathbf{R}}$ and in the the left-ordering the operator $(d^n/dt^n)\, R$ stands to the left of $e^{i\mathbf{k}\cdot\mathbf{R}}$. These orderings are related by reversing the order of all operators and by changing the sign of $\hbar$. In order to have hermitian expressions it is necessary to use both orderings. Therefore the equation of motion is

$$\begin{aligned}
m\ddot{R}_i = (F_{\text{ext}})_i - \pi e^2 \int d^3k \int_{t_0}^{t} dt' \Bigg\{ &\cos kc(t-t')\, |\tilde{\rho}|^2 \left(\delta_{si} - \frac{k_s k_i}{k^2}\right) \\
\times \Bigg[&\left(2\dot{R}'_s e^{i\mathbf{k}\cdot\mathbf{R}'} - \frac{\hbar}{m} k_s e^{i\mathbf{k}\cdot\mathbf{R}'}\right) e^{-i\mathbf{k}\cdot\mathbf{R}} + e^{-i\mathbf{k}\cdot\mathbf{R}} \left(2e^{i\mathbf{k}\cdot\mathbf{R}'}\dot{R}'_s + \frac{\hbar}{m} k_s e^{i\mathbf{k}\cdot\mathbf{R}'}\right)\Bigg] \\
+ &\frac{i}{2kc} \sin kc(t-t')\, |\tilde{\rho}|^2\, (\delta_{sl} k_i - \delta_{si} k_l) \\
\times \Bigg[&\left(2\dot{R}'_s e^{i\mathbf{k}\cdot\mathbf{R}'} - \frac{\hbar}{m} k_s e^{i\mathbf{k}\cdot\mathbf{R}'}\right)\left(2e^{i\mathbf{k}\cdot\mathbf{R}}\dot{R}'_l + \frac{\hbar}{m} k_l e^{i\mathbf{k}\cdot\mathbf{R}}\right) \\
+ &\left(2\dot{R}_l e^{-i\mathbf{k}\cdot\mathbf{R}} - \frac{\hbar}{m} k_l e^{-i\mathbf{k}\cdot\mathbf{R}}\right)\left(2e^{i\mathbf{k}\cdot\mathbf{R}'}\dot{R}'_s + \frac{\hbar}{m} k_s e^{i\mathbf{k}\cdot\mathbf{R}'}\right)\Bigg]\Bigg\}.
\end{aligned} \tag{4.5}$$

It is important to emphasize that the use of commutation relations leads to the explicit appearance of $\hbar$ in the equations.

The next clue is to remember that the evolution operator is defined as

$$U = \exp\left(-\frac{H}{i\hbar}(t'-t)\right) \tag{4.6}$$

and then two operators evaluated at different times can be related as follows:

$$O(t') = \exp\left(-\frac{H}{i\hbar}(t'-t)\right) O(t) \exp\left(\frac{H}{i\hbar}(t'-t)\right), \tag{4.7}$$

which is equivalent to

$$O(t') = \sum_{n=0}^{\infty} \frac{(-1)^n (t-t')^n}{n!} (ad^n H)\, O(t), \tag{4.8}$$

where[4]

$$(ad^1 H)\, O(t) = (i\hbar)^{-1} [O(t), H],\ (ad^2 H)\, \hat{O}(t) = (i\hbar)^{-2} [[O(t), H], H], \text{ etc.,} \tag{4.9}$$

then the equation of motion becomes

$$\begin{aligned} m\ddot{R}_i = {} & (F_{\text{ext}})_i - 2\pi e^2 \int d^3k \int_{t_0}^{t} dt' \left\{ \cos kc(t-t')\, |\tilde{\rho}|^2 \left(\delta_{si} - \frac{k_s k_i}{k^2} \right) \right. \\ & \times \sum_{n=0}^{\infty} \frac{(-1)^n (t-t')^n}{n!} \left[\left((ad^n H) \left(\dot{R}_s e^{i\mathbf{k}\cdot\mathbf{R}} - \frac{\hbar}{2m} k_s e^{i\mathbf{k}\cdot\mathbf{R}} \right) \right) e^{-i\mathbf{k}\cdot\mathbf{R}} \right. \\ & \left. + e^{-i\mathbf{k}\cdot\mathbf{R}} \left((ad^n H) \left(e^{i\mathbf{k}\cdot\mathbf{R}} \dot{R}_s + \frac{\hbar}{2m} k_s e^{i\mathbf{k}\cdot\mathbf{R}} \right) \right) \right] \\ & + \frac{i}{kc} \sin kc(t-t')\, |\tilde{\rho}|^2 (\delta_{sl} k_i - \delta_{si} k_l) \sum_{n=0}^{\infty} \frac{(-1)^n (t-t')^n}{n!} \\ & \times \left[\left((ad^n H) \left(\dot{R}_s e^{i\mathbf{k}\cdot\mathbf{R}} - \frac{\hbar}{2m} k_s e^{i\mathbf{k}\cdot\mathbf{R}} \right) \right) \left(e^{-i\mathbf{k}\cdot\mathbf{R}} \dot{R}_l + \frac{\hbar}{2m} k_l e^{-i\mathbf{k}\cdot\mathbf{R}} \right) \right. \\ & \left. \left. + \left(\dot{R}_l e^{-i\mathbf{k}\cdot\mathbf{R}} - \frac{\hbar}{2m} k_l e^{-i\mathbf{k}\cdot\mathbf{R}} \right) \left((ad^n H) \left(e^{i\mathbf{k}\cdot\mathbf{R}} \dot{R}_s + \frac{\hbar}{2m} k_s e^{i\mathbf{k}\cdot\mathbf{R}} \right) \right) \right] \right\}. \end{aligned} \tag{4.10}$$

The next step is the evaluation of $(ad^n H)(\dot{R}_s e^{i\mathbf{k}\cdot\mathbf{R}} - \frac{\hbar}{2m} k_s e^{i\mathbf{k}\cdot\mathbf{R}})$. For $n = 1$ it is straightforward to obtain

$$(ad^1 H)\, e^{i\mathbf{k}\cdot\mathbf{R}} \left(\dot{R}_s + \frac{\hbar}{2m} k_s \right) = e^{i\mathbf{k}\cdot\mathbf{R}} \ddot{R}_s + i e^{i\mathbf{k}\cdot\mathbf{R}} \left(k_j \dot{R}_j + \frac{\hbar}{2m} k^2 \right) \left(\dot{R}_s + \frac{\hbar}{2m} k_s \right) \tag{4.11}$$

when the right-ordering is chosen. The expression for the left-ordering can also be obtained in the same way. As expected, the difference between the expressions lies in the order followed by the operators and in the sign of $\hbar$.

Only the right-ordering expressions will be explicitly written. For $n=2$ the result is

$$(ad^2H)\, e^{i\mathbf{k}\cdot\mathbf{R}}\left(\dot{R}_s+\frac{\hbar}{2m}k_s\right)$$
$$=e^{i\mathbf{k}\cdot\mathbf{R}}R_s^{(3)}+2ie^{i\mathbf{k}\cdot\mathbf{R}}\left(k_j\dot{R}_j+\frac{\hbar}{2m}k^2\right)\ddot{R}_s+ie^{i\mathbf{k}\cdot\mathbf{R}}k_j\ddot{R}_j\left(\dot{R}_s+\frac{\hbar}{2m}k_s\right)$$
$$-e^{i\mathbf{k}\cdot\mathbf{R}}\left(k_j\dot{R}_j+\frac{\hbar}{2m}k^2\right)\left(k_r\dot{R}_r+\frac{\hbar}{2m}k^2\right)\left(\dot{R}_s+\frac{\hbar}{2m}k_s\right), \tag{4.12}$$

and for any n

$$(ad^nH)\, e^{i\mathbf{k}\cdot\mathbf{R}}\left(\dot{R}_s+\frac{\hbar}{2m}k_s\right)$$
$$=e^{i\mathbf{k}\cdot\mathbf{R}}(ad^nH)(\dot{R}_s)$$
$$+\sum_{r=1}^{n}\binom{n}{r}(ad^{(r-1)}H)\left\{ie^{i\mathbf{k}\cdot\mathbf{R}}\left(\dot{R}_s+\frac{\hbar}{2m}k_s\right)\right\}(ad^{(n-r)}H)\left(\dot{R}_s+\frac{\hbar}{2m}k_s\right). \tag{4.13}$$

The use of commutation relations shows that the handling of nonlinear terms involving operators deserves a special treatment. In this sense Eqs. (4.1) and (4.2) come from a naive handling of nonlinear terms.

Since the correct quantum effects have been exhibited, it is time to omit the remaining nonlinear terms. For any n the operator $(ad^nH)(R_s)$ becomes

$$(ad^nH)\, e^{i\mathbf{k}\cdot\mathbf{R}}\left(\dot{R}_s+\frac{\hbar}{2m}k_s\right)$$
$$=e^{i\mathbf{k}\cdot\mathbf{R}}\left\{\sum_{r=0}^{n}\binom{n}{r}\left(i\frac{\hbar}{2m}k^2\right)^r(\dot{R}_s)^{(n-r)}\right.$$
$$\left.+k_s\left(\frac{i\hbar}{2m}\right)\left(\left[\sum_{r=1}^{n}\binom{n}{r-1}\left(i\frac{\hbar}{2m}k^2\right)^{r-1}(k_j\dot{R}_j)^{(n-r)}\right]-i\left(i\frac{\hbar}{2m}k^2\right)^n\right)\right\}. \tag{4.14}$$

This equation and the corresponding left-ordering one must be inserted in Eq. (4.10). The resulting linear equation of motion for the particle is

$$
\begin{aligned}
m\ddot{R}_i = (F_{\text{ext}})_i - 2\pi e^2 \int d^3k \int_{t_0}^{t} dt' \Big\{ & \cos kc(t-t')\, |\tilde{\rho}|^2 \left(\delta_{si} - \frac{k_s k_i}{k^2}\right) \\
& \times \sum_{n=0}^{\infty} \frac{(-1)^n (t-t')^n}{n!} \Big[\Big(\sum_{r=0}^{n} \binom{n}{r} \left(\frac{-i\hbar}{2m} k^2\right)^r (\dot{R}_s)^{(n-r)} \Big) \\
& + k_s \left(\frac{-i\hbar}{2m}\right) \Big(\Big[\sum_{r=1}^{n} \binom{n}{r-1} \left(\frac{-i\hbar}{2m} k^2\right)^{r-1} (k_j \dot{R}_j)^{(n-r)} \Big] - i \left(\frac{-i\hbar}{2m} k^2\right)^n \Big) \\
& + \sum_{r=0}^{n} \binom{n}{r} \left(\frac{i\hbar}{2m} k^2\right)^r (\dot{R}_s)^{(n-r)} \\
& + k_s \left(\frac{i\hbar}{2m}\right) \Big(\Big[\sum_{r=1}^{n} \binom{n}{r-1} \left(\frac{i\hbar}{2m} k^2\right)^{r-1} (k_j \dot{R}_j)^{(n-r)} \Big] - i \left(\frac{i\hbar}{2m} k^2\right)^n \Big) \\
& + \frac{i}{kc} \sin kc(t-t')\, |\tilde{\rho}|^2 \left(\delta_{sl} k_i - \delta_{si} k_l\right) \sum_{n=0}^{\infty} \frac{(-1)^n (t-t')^n}{n!} \\
& \times \Big[-ik_s \left(\frac{-i\hbar}{2m}\right)\left(\frac{-i\hbar}{2m} k^2\right)^n \dot{R}_l + k_l \frac{\hbar}{2m} \sum_{r=0}^{n} \binom{n}{r} \left(\frac{-i\hbar}{2m} k^2\right)^r (\dot{R}_s)^{(n-r)} \\
& + k_s k_l \frac{\hbar}{2m} \left(\frac{-i\hbar}{2m}\right) \Big[\sum_{r=1}^{n} \binom{n}{r-1} \left(\frac{-i\hbar}{2m} k^2\right)^{r-1} (k_j \dot{R}_j)^{(n-r)} \Big] \\
& - ik_s k_l \left(\frac{\hbar}{2m}\right)\left(\frac{-i\hbar}{2m}\right)\left(\frac{-i\hbar}{2m} k^2\right)^n \\
& - ik_s \left(\frac{i\hbar}{2m}\right)\left(\frac{i\hbar}{2m} k^2\right)^n \dot{R}_l + k_l \left(-\frac{\hbar}{2m}\right) \sum_{r=0}^{n} \binom{n}{r} \left(\frac{i\hbar}{2m} k^2\right)^r (\dot{R}_s)^{(n-r)} \\
& + k_l k_s \left(-\frac{\hbar}{2m}\right)\left(\frac{i\hbar}{2m}\right) \sum_{r=1}^{n} \binom{n}{r-1} \left(\frac{i\hbar}{2m} k^2\right)^{r-1} (k_j \dot{R}_j)^{(n-r)} \\
& - ik_l k_s \left(-\frac{\hbar}{2m}\right)\left(\frac{i\hbar}{2m}\right)\left(\frac{i\hbar}{2m} k^2\right)^n \Big]\Big\}. \qquad (4.15)
\end{aligned}
$$

The relevant point is that the magnetic self-interaction contributes to the linear equation of motion. This does not happen in the classical analysis, therefore it is a direct consequence of the commutation relations and the use of Eqs. (4.3) and (4.4). In Eq. (3.7) it would seem that the magnetic self-interaction is inherently nonlinear, however the rules established above transform the nonlinear terms into a combination of linear and other nonlinear terms. The result is that a consistent consideration of the quantum behavior unavoidably leads to a linear contribution of the magnetic self-interaction.

The analysis has been carried out following the underlying ideas settled by Moniz and Sharp.[4] However, in their method the linear contribution of the magnetic self-interaction is omitted from the beginning by analogy with the classical reasoning. Although they use the same commutation relations and an equivalent form for Eqs. (4.3) and (4.4), their full consequences are underestimated.

A linear equation of motion has already been obtained, but the features of the spherically symmetric distribution of charge allows a simplification since $\tilde{\rho}(k)$ does not depend on angular variables, therefore

$$\int d\Omega_{\mathrm{k}} \left(\delta_{si} - \frac{k_s k_i}{k^2} \right) = \frac{8\pi}{3} \, \delta_{si} . \tag{4.16}$$

Also the following identities must be considered:

$$\begin{aligned}
&\left(\delta_{si} - \frac{k_s k_i}{k^2} \right) k_s = 0, \\
&(\delta_{sl} k_i - \delta_{si} k_l) \, k_s = k_l k_i - k_i k_l = 0, \\
&(\delta_{sl} k_i - \delta_{si} k_l) \, k_s k_l = 0, \\
&(\delta_{sl} k_i - \delta_{si} k_l) \, k_l = k_s k_i - \delta_{si} k_l k_l = -k^2 \left(\delta_{si} - \frac{k_s k_i}{k^2} \right).
\end{aligned} \tag{4.17}$$

In consequence the linear equation of motion for the charged particle is

$$\begin{aligned}
m\ddot{R}_i = (F_{\mathrm{ext}})_i &- \frac{16\pi^2 e^2}{3} \int_0^\infty dk \, k^2 \int_{t_0}^t dt' \Bigg\{ \cos kc(t-t') \, |\tilde{\rho}|^2 \\
&\times \sum_{l=0}^\infty \frac{(-1)^l \, (t-t')^l}{l!} \sum_{p=0}^l \binom{l}{p} \left[\left(\frac{-i\hbar}{2m} k^2 \right)^p + \left(\frac{i\hbar}{2m} k^2 \right)^p \right] (\dot{R}_i)^{(l-p)} \\
&+ \frac{1}{kc} \sin kc(t-t') \, |\tilde{\rho}|^2 \sum_{s=0}^\infty \frac{(-1)^s \, (t-t')^s}{s!} \\
&\times \sum_{p=0}^s \binom{s}{p} \left[\left(\frac{-i\hbar}{2m} k^2 \right)^{p+1} + \left(\frac{i\hbar}{2m} k^2 \right)^{p+1} \right] (\dot{R}_i)^{(s-p)} \Bigg\},
\end{aligned} \tag{4.18}$$

where the indices have been relabeled for further usefulness. In the electric part of the self-interaction only the terms with $p = 2r$ contribute, meanwhile in the magnetic part of the self-interaction the contribution comes from terms with $p+1 = 2r$. It is convenient to introduce the Compton wavelength, defined as $\lambda = \frac{\hbar}{mc}$, in order to write the equation of motion for the particle as follows:

$$m\ddot{R}_i = (F_{\text{ext}})_i - \frac{16\pi^2 e^2}{3}\int_0^\infty dk\, k^2 \int_{t_0}^t dt' \left\{ \cos kc(t-t')\, |\tilde{\rho}|^2 \right.$$
$$\times \sum_{l=0}^\infty \frac{(-1)^l\,(t-t')^l}{l!} \sum_{r=0}^{\left[\frac{l}{2}\right]} \binom{l}{2r} 2c^{2r} \left(-\frac{\lambda^2}{4}k^4\right)^r (\dot{R}_i)^{(l-2r)}$$
$$+\frac{1}{kc}\sin kc(t-t')\, |\tilde{\rho}|^2 \sum_{s=0}^\infty \frac{(-1)^s\,(t-t')^s}{s!}$$
$$\left. \times \sum_{q=0}^{\left[\frac{s-1}{2}\right]} \binom{s}{2q+1} 2c^{2q+2} \left(-\frac{\lambda^2}{4}k^4\right)^{q+1} \dot{R}_i^{\,(s-2q-1)} \right\}, \tag{4.19}$$

where

$$\left[\frac{u}{2}\right] = \begin{cases} \frac{u}{2} & \text{if } u \text{ is even} \\ \frac{u-1}{2} & \text{if } u \text{ is odd} \end{cases}$$

and $r = q+1$ in the last summation of Eq. (4.18).

If it is assumed that the summation converges, the following change of indices

$$l - 2r = n \geqslant 0, \tag{4.20}$$

$$r = p \geqslant 0, \tag{4.21}$$

$$s - 1 - 2q = j \geqslant 0, \tag{4.22}$$

and

$$q = p \geqslant 0 \tag{4.23}$$

leads to the equation of motion

$$m\ddot{R}_i = (F_{\text{ext}})_i - \frac{32\pi^2 e^2}{3}\int_0^\infty dk\, k^2\, |\tilde{\rho}|^2 \int_{t_0}^t dt' \left\{ \cos kc(t-t') \right.$$
$$\times \sum_{n=0}^\infty \sum_{p=0}^\infty \frac{(-1)^{n+2p}\,(t-t')^{n+2p}}{(n+2p)!} \binom{n+2p}{2p} c^{2p} \left(-\frac{\lambda^2}{4}k^4\right)^p \dot{R}_i^{(n)}$$
$$+\frac{1}{k}\sin kc(t-t') \sum_{j=0}^\infty \sum_{p=0}^\infty \frac{(-1)^{j+2p+1}\,(t-t')^{j+2p+1}}{(j+2p+1)!}$$
$$\left. \times \binom{j+2p+1}{2p+1} c^{2p+1} \left(-\frac{\lambda^2}{4}k^4\right)^{p+1} \dot{R}_i^{(j)} \right\}, \tag{4.24}$$

and, by performing the sum, an integro-differential equation is obtained, which reads as

$$
\begin{aligned}
m\ddot{R}_i = (F_{\text{ext}})_i - \frac{32\pi^2 e^2}{3} \int_0^\infty dk\, k^2\, |\tilde{\rho}|^2 \\
\times \int_{t_0}^{t} dt' \left\{ \cos kc(t-t') \cos \frac{\lambda k^2 c(t-t')}{2} \right. \\
\left. + \frac{\lambda k}{2} \sin kc(t-t') \sin \frac{\lambda k^2 c(t-t')}{2} \right\} \dot{R}_i(t'). \qquad (4.25)
\end{aligned}
$$

This is the linear quantum equation of motion for an extended charged particle with a spherically symmetric charge distribution on it. The first term of the integrand is the electric contribution to self-interaction while the second one is the magnetic contribution. The linear classical equation of motion for a charged particle which takes into account the radiation reaction[16, 17, 19, 21, 29] can be recovered, from Eq. (4.25), in the limit $\lambda \to 0$ and by considering that the operators turn out to be functions. Only the electric part of the self-interaction contributes to the classical equation of motion. In this sense, the Compton wavelength can be considered as a measure of the quantum effects and the magnetic contribution is one of them.

5. THE ELECTROMAGNETIC MASS

A linear integrodifferential equation of motion can be converted into an infinite-order differential one. The latter is more suitable to carry out a comparison with other analyses.[4, 5, 14] The coefficient of the operator $\ddot{R}_i$ will be interpreted as the electromagnetic mass. This is a natural generalization of the corresponding classical definition. However, in the quantum realm the concept of electromagnetic mass acquires a vast complexity and its classical limit is very intricate, as will be shown. Both the spreading of the charge on the particle and the structure-like contribution coming from the Compton wavelength contribute to determine the electromagnetic mass. The result is that it is possible to get a vanishing electromagnetic mass for a point charged particle.

In order to derive the infinite-order differential equation it is necessary to go back to Eq. (4.25). With the change of variable $\tau = t - t'$ and by taking $t_0 \to -\infty$ the equation of motion is

$$m\ddot{R}_i = (F_{\text{ext}})_i - \frac{32\pi^2 e^2}{3} \int_0^\infty dk\, k^2\, |\tilde{\rho}|^2 \int_0^\infty d\tau$$
$$\times \left\{ \sum_{s=0}^\infty \frac{(-1)^{2s+1}\, \tau^{2s+1}}{(2s+1)!} \left[\cos kc\tau \cos \frac{\lambda k^2 c\tau}{2} + \frac{\lambda k}{2} \sin kc\tau \sin \frac{\lambda k^2 c\tau}{2} \right] (\dot{R}_i)^{(2s+1)} \right.$$
$$\left. + \sum_{s=0}^\infty \frac{(-1)^{2s}\, \tau^{2s}}{(2s)!} \left[\cos kc\tau \cos \frac{\lambda k^2 c\tau}{2} + \frac{\lambda k}{2} \sin kc\tau \sin \frac{\lambda k^2 c\tau}{2} \right] (\dot{R}_i)^{(2s)} \right\} \tag{5.1}$$

where the terms with the odd powers of τ have been separated from the terms with the even powers of τ. The use of the identities

$$\tau^{2s+1} \cos \frac{\lambda k^2 c\tau}{2} = (-1)^s \left(\frac{2}{k^2 c}\right)^{2s+1} \frac{\partial^{2s+1}}{\partial \lambda^{2s+1}} \sin \frac{\lambda k^2 c\tau}{2}, \tag{5.2}$$

$$\tau^{2s+1} \sin \frac{\lambda k^2 c\tau}{2} = (-1)^{s+1} \left(\frac{2}{k^2 c}\right)^{2s+1} \frac{\partial^{2s+1}}{\partial \lambda^{2s+1}} \cos \frac{\lambda k^2 c\tau}{2}, \tag{5.3}$$

$$\tau^{2s} \cos \frac{\lambda k^2 c\tau}{2} = (-1)^s \left(\frac{2}{k^2 c}\right)^{2s} \frac{\partial^{2s}}{\partial \lambda^{2s}} \cos \frac{\lambda k^2 c\tau}{2}, \tag{5.4}$$

$$\tau^{2s} \sin \frac{\lambda k^2 c\tau}{2} = (-1)^s \left(\frac{2}{k^2 c}\right)^{2s} \frac{\partial^{2s}}{\partial \lambda^{2s}} \sin \frac{\lambda k^2 c\tau}{2} \tag{5.5}$$

and

$$\int_0^\infty d\tau\, e^{-i\tau z} = \pi\delta(z) + i\, \frac{P}{z}, \tag{5.6}$$

allows an equation of motion for the charged particle to be obtained, which reads

$$m\ddot{R}_i = (F_{\text{ext}})_i - \frac{16\pi^2 e^2}{3c} \sum_{s=0}^\infty \frac{(-1)^s}{(2s+1)!} \int_0^\infty dk\, k^2\, |\tilde{\rho}|^2 \left(\frac{2}{k^2 c}\right)^{2s+1}$$
$$\times \left[\frac{\partial^{2s+1}}{\partial \lambda^{2s+1}}\, P\, \frac{\lambda}{(1 - \frac{\lambda^2 k^2}{4})} + \lambda\, \frac{\partial^{2s+1}}{\partial \lambda^{2s+1}}\, P \left(\frac{1}{(1 - \frac{\lambda^2 k^2}{4})} \right) \right] \ddot{R}_i^{(2s)}$$
$$- \frac{16\pi^3 e^2}{3c} \sum_{s=0}^\infty \frac{(-1)^s}{(2s)!} \int_0^\infty dk\, k^2\, |\tilde{\rho}|^2 \left(\frac{2}{k^2 c}\right)^{2s}$$
$$\times \left[\frac{\partial^{2s}}{\partial \lambda^{2s}}\, \delta\left(k - \frac{\lambda}{2}\right) + \frac{\lambda k}{2} \frac{\partial^{2s}}{\partial \lambda^{2s}}\, \delta\left(k - \frac{\lambda}{2}\right) \right] \dot{R}_i^{(2s)}. \tag{5.7}$$

This is the desired infinite-order differential equation. It is important to remark that each coefficient of the derivatives of $R_i(t)$ is given as a closed expression. The validity of Eq. (5.7) lies in the convergence of the series, as was mentioned in the deduction of Eq. (4.24). For many choices of $\tilde{\rho}(k)$ there will be no difficulties.

In Eq. (5.7) there is a term proportional to the acceleration operator $\ddot{R}_i$, then it must be interpreted as a mass which is usually called the electromagnetic mass m_e and is given by

$$m_e = \frac{32\pi^2 e^2}{3c^2}\left(\frac{\partial}{\partial\lambda}\lambda P\int_0^\infty dk\,\frac{|\tilde{\rho}(k)|^2}{1-\frac{\lambda^2k^2}{4}} + \lambda\frac{\partial}{\partial\lambda}P\int_0^\infty dk\,\frac{|\tilde{\rho}(k)|^2}{1-\frac{\lambda^2k^2}{4}}\right). \tag{5.8}$$

The second term on the right-hand side of the equation is the contribution to the magnetic self-interaction. In the nonperturbative analysis developed by Moniz and Sharp[(4)] this part of the self-interaction is neglected. On the other hand, Grotch and Kazes[(14)] developed an analysis of radiation reaction in nonrelativistic quantum electrodynamics and they show that there is a magnetic contribution to the electromagnetic mass. However, their result is completely different from Eq. (5.8). Therefore, there are three different expressions for the electromagnetic mass of an extended charged particle. All of them must be finite for many choices of $\tilde{\rho}(k)$. The reason is clear, although the starting point is shared by all the analyses, the framework is different in each one.

The electromagnetic mass depends on the particular choice for $\tilde{\rho}(k)$. The same happens in the classical analyses,[(4, 16–23, 29)] where it is also shown that many difficulties appear when the point particle model is under study. Therefore, it is very interesting to investigate if the point particle model leads to the same difficulties when the quantum equation of motion, given by (5.7), is used.

For a point particle

$$\tilde{\rho}(k) = \frac{1}{(2\pi)^{\frac{3}{2}}}, \tag{5.9}$$

and the electromagnetic mass is

$$m_e = \frac{4e^2}{3\pi c^2}\left(\frac{\partial}{\partial\lambda}\lambda P\int_0^\infty dk\,\frac{1}{1-\frac{\lambda^2k^2}{4}} + \lambda\frac{\partial}{\partial\lambda}P\int_0^\infty dk\,\frac{1}{1-\frac{\lambda^2k^2}{4}}\right) = 0. \tag{5.10}$$

This is a striking result which notably differs from that obtained in the classical analyses, where a linearly divergent electromagnetic mass is obtained. The same result is obtained by Moniz and Sharp.[4] In perturbative nonrelativistic quantum electrodynamics,[14] where it is assumed that initially and finally the particle and its own electromagnetic field do not interact, a logarithmically divergent electromagnetic mass is obtained.

A close inspection will exhibit the existence of many intriguing points. An alternative expression for the electromagnetic mass is

$$m_e = -\frac{32\pi^2 e^2}{3} \int_0^\infty dk\, k^2\, |\tilde{\rho}|^2 \int_{t_0}^t dt' \left\{ \cos kc(t-t') \right.$$
$$\times \sum_{p=0}^\infty \frac{(t-t')^{2p+1}}{(2p)!} c^{2p} \left(-\frac{\lambda^2}{4} k^4\right)^p$$
$$\left. + \frac{1}{k} \sin kc(t-t') \sum_{p=0}^\infty \frac{(t-t')^{2p+2}}{(2p+1)!} c^{2p+1} \left(-\frac{\lambda^2}{4} k^4\right)^{p+1} \right\}, \tag{5.11}$$

as can be seen from Eq. (4.24). The first term of each summation is

$$m_e' = -\frac{32\pi^2 e^2}{3} \int_0^\infty dk\, k^2\, |\tilde{\rho}|^2 \int_{t_0}^t dt' \cos kc(t-t')(t-t') \tag{5.12}$$

and

$$m_e'' = -\frac{32\pi^2 e^2}{3} \left(\frac{\lambda^2}{2}\right) c \int_0^\infty dk\, k^5\, |\tilde{\rho}|^2 \int_{t_0}^t dt' \sin kc(t-t')(t-t')^2, \tag{5.13}$$

for the electric and the magnetic contributions, respectively. For a point particle

$$m_e' = \frac{4e^2}{3c^2} \left(\frac{1}{\pi} \int_0^\infty dk\right) \tag{5.14}$$

and

$$m_e'' = \frac{4e^2}{3c^2} \lambda^2 \left(\frac{1}{\pi} \int_0^\infty dk\, k^2\right). \tag{5.15}$$

In fact $m_e' = (m_e)_{\text{classical}}$, as has been noted by Rohrlich,[2] and it diverges linearly. On the other hand, m_e'' does not have a classical counterpart and it diverges cubically. This shows that the first terms of the expansion for the electromagnetic mass diverge. The conditions for the convergence of the summations (4.18) through (4.24) are not strictly satisfied when a point particle is considered. The difficulty was hidden in many of the equations obtained above because for an extended charged particle m_e' and m_e'' are finite for many choices of $\tilde{\rho}(k)$. The summations are evaluated formally, with $\lambda \neq 0$, and Eqs. (4.25), (5.7), and (5.8) are free of infinities. The last step is to take the point particle limit and Eq. (5.10) is obtained. Therefore, if it is assumed that the approximation of point particle is taken after effects due to both retardation and quantum behavior have been considered, the result is a vanishing electromagnetic mass for a point particle. The physical meaning of this is that the complete accounting of retardation and all quantum effects, by considering all the powers of λ, and the limit of a point charged particle are not interchangeable. This point of view is also considered in the work of Moniz and Sharp[2, 4] and they obtain the same result. The problem of the quantum electromagnetic mass is still under discussion and remains as a loose end.[2–5]

However, the relevant point here is that for an extended charged particle the electromagnetic mass can be finite and positive. The result depends on the specific choice of $\tilde{\rho}(k)$. A complete discussion of this has been carried out by Moniz and Sharp.[2, 4] Also, it is possible that other physical effects, which are taken into account in perturbative quantum electrodynamics, contribute to the electromagnetic mass.

The classical limit of the electromagnetic mass defined above is also intriguing since from Eq. (5.10) it must be zero. Therefore, the classical electromagnetic mass of a point particle is zero as a result of considering that the classical dynamics of a point charged particle is an approximation of the quantum dynamics of a point charged particle, which is also an approximation of the quantum dynamics of an extended charged particle. But if in Eq. (5.10) the classical limit, $\lambda \to 0$, is taken first, and then the point particle limit is considered, the result is a linearly divergent electromagnetic mass. This means that in this case the classical dynamics of a point charged particle is an approximation of the classical dynamics of an extended charged particle, which is also an approximation of the quantum dynamics of an extended charged particle. These limits are not interchangeable. This is an additional indication that the point particle model is a very strong approximation in both classical or quantum electrodynamics. Always, some kind of ambiguities arise.

6. ANALYSIS OF CAUSALITY

Other difficulties for a point charged particle also arise in the classical analyses: the preacceleration and the runaway solution. In usual quantum electrodynamics the prime concern is the study of scattering phenomena. The discussion of these questions is therefore lacking in the general formalism. In the analysis of the radiation reaction problem in the realm of perturbative nonrelativistic quantum electrodynamics[14] the discussion of either preacceleration or runaway solutions is not carried out.

The analysis of this problem arises in a natural way in the method developed in this paper. The result is again amazing. Neither preacceleration nor runaway solutions are present as a consequence of taking into account retardation and all the quantum effects.

A deeper analysis of the equation of motion given by Eq. (5.7) requires a decision about the electromagnetic mass and the evaluation of the remaining coefficients. In the subsequent analysis it is considered that the electromagnetic mass of a point particle is zero. Also, for this type of particle[4]

$$P\int_0^\infty dk\,\frac{k^{-4s}}{1-\frac{\lambda^2 k^2}{4}}=0. \tag{6.1}$$

An additional approximation is to consider that the particle is instantaneously at rest,[4] as is done in many classical analyses.[16, 17, 20, 21] Then the equation of motion is

$$\begin{aligned} m\ddot{R}_i &= (F_{\text{ext}})_i + \frac{2e^2}{3c^3}\dot{R}_i^{(2)} \\ &\quad -\frac{2e^2}{3c}\sum_{s=2}^{\infty}\frac{(-1)^s}{(2s)!}\left(\frac{2}{c}\right)^{2s}\left(\frac{1}{2^{4s-2}}\right)\left[1-\frac{2s-2}{4s-2}\right]\frac{(4s-2)!}{(2s-2)!}\lambda^{2s-2}\dot{R}_i^{(2s)}, \end{aligned} \tag{6.2}$$

where the external force is a c-number. The first term in the square bracket represents the electric contribution and the second one is the magnetic contribution. A similar infinite-order differential equation has been obtained elsewhere[4] through a cumbersome derivation. That equation is different from Eq. (6.2) not only because there are terms coming from the magnetic self-interaction but because the electrical terms do not coincide. The use of identities given in Eq. (4.17) allowed handling of the different contributions to the linear equation of motion in the correct way.

In a stronger approximation, the terms going as $(\frac{1}{c})^5$ or superior could be dropped out. Then only the first two terms in the right-hand side of Eq. (6.2) remain and the Abraham–Lorentz equation for operators (see Eq. (4.2)) is reobtained again, but with $m_e=0$. However this approximation does not consider the quantum behavior in a complete way. In fact, in Eq. (6.2) the existence of a new quantum effect is revealed. As this equation has the same form as the classical linear equation of motion for a particle with structure,[16, 17, 21, 29] it is possible to interpret this coincidence by saying that the point particle acquires a quantum apparent structure with the

Compton wavelength playing the role of an apparent radius. This interpretation must be valid for any particle model; from a quantum point of view a particle has therefore two structures, one proper and the other apparent.

With $n = 2s - 1$ the equation of motion takes the form

$$m\ddot{\mathbf{R}}(t) = \mathbf{F}_{\text{ext}}(t) - \frac{2e^2}{3c^2} \sum_{n(\text{odd})}^{\infty} \frac{(-1)^n}{n!\, c^n} [(-1)^{\frac{n-1}{2}} (2n-1)!!]\, \lambda^{n-1} \ddot{\mathbf{R}}(t)^{(n)}, \tag{6.3}$$

which will be very useful to investigate the existence of both preacceleration and runaway solution. It must be remembered that these difficulties arise in the classical analysis of the radiation reaction when the point particle model is used.

The Fourier transform is given as

$$\ddot{\mathbf{R}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \tilde{\ddot{\mathbf{R}}}(\omega)\, e^{i\omega t}, \tag{6.4}$$

then the equation of motion in Fourier space is

$$m\tilde{\ddot{\mathbf{R}}}(\omega) = \tilde{\mathbf{F}}_{\text{ext}}(\omega) + m\frac{2}{3}\alpha \sum_{n(\text{odd})}^{\infty} \frac{1}{n!} [(-1)^{\frac{n-1}{2}} (2n-1)!!]\, \eta^n \tilde{\ddot{\mathbf{R}}}(\omega), \tag{6.5}$$

with $\eta = (i\omega\lambda)/c$, and $\alpha = (e^2)/(\hbar c)$ is the fine structure constant. The series

$$\sum_{n(\text{odd})}^{\infty} \frac{1}{n!} [(-1)^{\frac{n-1}{2}} (2n-1)!!]\, \eta^n$$

converges for $|\eta| < \frac{1}{2}$ and it can be summed inside its circle of convergence. Therefore Eq. (6.5) becomes

$$\tilde{\ddot{\mathbf{R}}}(\omega) = \frac{\tilde{\mathbf{F}}_{\text{ext}}(\omega)}{m(1 - \frac{2}{3}\alpha f(\eta))}, \tag{6.6}$$

where

$$f(\eta) = \left(\frac{i}{2}\right) [(1 + 2i\eta)^{-\frac{1}{2}} - (1 - 2i\eta)^{-\frac{1}{2}}].$$

This result implies that Eq. (6.3) has the form

$$\ddot{\mathbf{R}}(t) = \int_{-\infty}^{\infty} dt'\, G(t - t')\, \mathbf{F}_{\text{ext}}(t'), \tag{6.7}$$

where

$$G(t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\frac{e^{i\omega(t-t')}}{1-\frac{2}{3}\alpha f\left(\frac{i\omega\lambda}{c}\right)}. \tag{6.8}$$

The retarded solutions for the field operators given in Eqs. (3.1) and (3.2) are consistent with the principle of antecedence.[28] This is the cornerstone of the method presented here. The consistency will be preserved if in Eq. (6.7) the acceleration operator does not depend on the values of $\mathbf{F}_{\text{ext}}(t')$ for $t<t'$. This implies that $G(t-t')=0$ if $t<t'$, otherwise preacceleration will appear. The integrand in Eq. (6.8) cannot, therefore, have poles in the lower half of the ω plane.

The poles in Eq. (6.8) are the solution to

$$1-\tfrac{2}{3}\alpha f(\eta)=0, \tag{6.9}$$

which is equivalent to

$$1+4\eta^2=\tfrac{1}{81}\left(\alpha^2\pm\alpha\sqrt{\alpha^2-18}\right)^2. \tag{6.10}$$

Then it is necessary to consider the cases $\alpha>\sqrt{18}$ and $\alpha<\sqrt{18}$. The known value for α corresponds to the second one. Therefore, it must be noted that

$$|\eta|^4=\frac{1}{16}\left(1+\frac{4\alpha^2}{9}\right); \tag{6.11}$$

but, since $|\eta|<\frac{1}{2}$,

$$\left(1+\frac{4\alpha^2}{9}\right)^{\frac{1}{4}}<1, \tag{6.12}$$

which can not be satisfied because α is a real number. In consequence there are no poles in the lower half of the ω plane and Eq. (6.3) does not lead to preacceleration. This is a sharp difference with the classical result.

The existence of a runaway solution must be investigated by putting $\mathbf{F}_{\text{ext}}(t)=0$ and by noting that

$$\ddot{\mathbf{R}}(t)=\exp\left(\frac{iH}{\hbar}(t)\right)\ddot{\mathbf{R}}(0)\exp\left(-\frac{iH}{\hbar}(t)\right). \tag{6.13}$$

When stationary states, defined by $H|l\rangle=E_l|l\rangle$, are considered, the following relation for the matrix elements is obtained

$$\ddot{\mathbf{R}}_{jl}(t)=e^{i\omega_{jl}t}\ddot{\mathbf{R}}_{jl}(0), \tag{6.14}$$

where $\omega_{jl} = (E_j - E_l)/\hbar$ and $\ddot{\mathbf{R}}_{jl}(t) = \langle j|\, \ddot{\mathbf{R}}(t)\, |l\rangle$. On the other hand the matrix elements satisfy

$$m\ddot{\mathbf{R}}_{jl}(t) = -\frac{2e^2}{3c^2} \sum_{n(\text{odd})}^{\infty} \frac{(-1)^n}{n!\, c^n} \left[(-1)^{\frac{n-1}{2}} (2n-1)!!\right] \lambda^{n-1} (\ddot{\mathbf{R}}_{jl}(t))^{(n)}. \qquad (6.15)$$

The insertion of Eq. (6.14) to Eq. (6.15) implies that Eq. (6.9), with $\eta = (i\lambda\omega_{jl})/c$, must be satisfied. But it was already shown that there are no solutions for it when $|\eta| < \frac{1}{2}$, or equivalently $(E_j - E_l)/(mc^2) < \frac{1}{2}$, in complete agreement with a nonrelativistic approach. Therefore Eq. (6.3) has neither preacceleration nor runaway solutions for a point charged particle.

The case $\alpha > \sqrt{18}$ corresponds to large values of α which is equivalent to $\hbar \to 0$, with e and c fixed, therefore this case corresponds to the classical limit. In fact, $\eta_1 = \alpha^2/9$ and $\eta_2 = 3/(2\alpha)$ for large α. The latter implies that $3/(2\alpha) = (\beta\lambda)/c$, then $\beta = (3mc^3)/(2e^2) = \tau^{-1}$, which represents the existence of a runaway solution or preacceleration. This is the classical result.

7. CONCLUDING REMARKS

A quite general analysis of the quantum dynamics of an extended charged particle has been worked out. It has been shown that a nonlinear problem arises when the radiation reaction is taken into account.

In the approach explored here the field operators are not interpreted as annihilation and creation ones and the self-interaction is always present. Tangible results are obtained after a suitable linear approximation is imposed without using perturbation theory and renormalization. This procedure allows a consistent theoretical structure to be built to study the nonrelativistic quantum dynamics of a rigid and nonrotating extended charged particle where it was exhibited that the particle has a proper structure, but there is also another one due to the quantum effects. The nonlocality introduced by considering a charged particle with finite size allowed an exact treatment of the retardation when different elements of the particle interact. This was the essential step to reach the goal. Difficulties will not arise with this kind of particles. The difficulty of defining a rigid structure in a relativistic way is the great obstacle to build a relativistic generalization of the ideas presented in this paper. Also, the electromagnetic stability of the extended particle remains as another loose end.

The internal consistency of the dynamics of an extended charged particle is preserved when the point charged particle limit is considered.

However, this does not mean that the dynamics of a point particle is free of ambiguities since if the point particle limit is introduced before taking into account all the quantum effects, then the infinities would appear again.

It is also important to mention that the procedure followed here allows a classical limit to be established in each step. In this way the link between the classical and the quantum realms is better understood. The results obtained here could be interpreted as an indication that a point charged particle is outside of the domain of the classical theory, as it is asserted in other works.(2–5) This interpretation has been argued to build a classical dynamics of a quasi-point charged particle,(15) where it is assumed that the size of this kind of particle must be greater than the Compton wavelength. It has been shown that the equation of motion of a quasi-point particle is free of unphysical behavior, as a consequence of the nonlocality introduced by the finite size.

On the other hand the quantum dynamics of a rigid but rotating charged particle can be built by following the classical method(32) and the procedure presented here.

ACKNOWLEDGMENTS

We are grateful to Prof. F. Rohrlich for his valuable comments to improve this work. We express our gratitude to E. Ley Koo, I. Campos, M. Villavicencio, and Neil Bruce for the careful reading of the manuscript, for the continued interest in this work and for helpful discussions. The authors thank the referees for their valuable suggestions.

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