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Spin and Statistics with an Electromagnetic Field

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Because of the impossibility of simultaneously satisfying the requirements of manifest Lorentz covariance and positive-definite (finite) norm in the Hilbert space, no simple proof of the connection between spin and statistics with an electromagnetic field has been given. This note is to point out that it is indeed not necessary to have manifest Lorentz covariance in the full $3 + 1$ space to show such a connection. Using the axial gauge $A_3 = 0$, we have succeeded in constructing a simple straightforward proof.

INTRODUCTION

In the proofs of the connection between spin and statistics, it is generally assumed that[1,2]

(1) There is a unique invariant vacuum, the lowest energy state.

(2) The spectral density is semipositive-definite, so that probability interpretation exists in the theory.

(3) The theory is manifestly Lorentz covariant.

These assumptions are not entirely met by the electromagnetic field; we have the annoying situation that, in any manifestly Lorentz-covariant gauges, (2) is violated; while, in the radiation guage, (2) is restored at the expense of (3). The relationship between spin and statistics is known to hold, however. It is therefore urgent to re-examine whether all these three assumptions are necessary to construct a proof.

In the axial gauge, $A_3 = 0$,[3,4] Assumption (3) is given up. We shall use this fact to show that (3) is actually not needed to show the connection between spin and statistics.

The proof given here may be considered as an extension of the work of Brown and Schwinger.

There has been a paper by D. Boulware,[5] which deals with the same problem we are considering here. However, he had to work in both the radiation guage and the Lorentz gauge in order to make use of all the three requirements stated above. We are of the opinion that our proof is more straightforward and can be easily generalized.

SPIN AND STATISTICS

Let χ be a Hermitian field of finite multiplicity. In the proof of the connection between spin and statistics by Brown and Schwinger, it is essential that the spectral density $m(p, \kappa^2)$ (see later for definition) depends on p only algebraically. For this reason, their proof cannot be carried through to include electrodynamics in the radiation gauge,

$$\nabla \cdot \mathbf{A} = 0.$$

Similarly, in the axial gauge, $A_3 = 0$, we fail to establish that $m(p, \kappa^2)$, depends on p_3 only algebraically, as the second-order mass operator indicates.[4] It is then necessary to modify the proof given in I somewhat.

We use the notations

$$\bar{p}^\mu = (p^0, p^1, p^2, 0), \qquad \bar{x}^\mu = (x^0, x^1, x^2, 0),$$

and

$$\bar{p} \cdot \bar{x} = p^1 x^1 + p^2 x^2 - p^0 x^0,$$

i.e., the Lorentz metric is $(-1, 1, 1, 1)$.

Using the conditions that the vacuum is invariant under translations in space and time and that only positive frequency timelike energy momentum vectors $|p>$ can exist in the physical quantum vector space, we write

$$\begin{aligned}\langle \chi(x) \chi(x') \rangle &= \langle \chi e^{iP(x-x')} \chi \rangle \\ &= \int_0^\infty d\kappa^2 \frac{1}{2L} \left(\sum_{p_3 \neq 0} + \sum_{p_3 = 0} e^{ip_3(x_3 - x_3')} \right) \\ &\quad \times \int \frac{d\bar{p}}{(2\pi)^2} e^{i\bar{p} \cdot (\bar{x} - \bar{x}')} \eta_+(p)\, \delta(p^2 + \kappa^2) m(p, \kappa^2), \quad (1)\end{aligned}$$

where P are the translation (energy–momentum)

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[1] N. Burgoyne, Nuovo Cimento 8, 607, (1958); G. Luders and B. Zumino, Phys. Rev. 110, 1450 (1958).

[2] L. Brown and J. Schwinger, Progr. Theoret. Phys., (Kyoto) 26, 917 (1961). This paper will be referred to as I. It is suggested that the readers go through this paper first, since we shall follow the notations and the basic concepts presented there.

[3] Y. P. Yao, J. Math. Phys. 5, 1319 (1964) (previous paper).

[4] Y. P. Yao, Ph. D. thesis (1964), Department of Physics, Harvard University. Consistency, Lorentz invariance, etc. of quantization in a finite volume have been verified here.

[5] D. Boulware, J. Math. Phys. 3, 50 (1962).

operators, and

$$\eta_+(p) = \eta_-(-p) = \begin{cases} 1 & p^0 > 0, \\ 0 & p^0 < 0. \end{cases}$$

The Hermitian spectral density

$$m(p, \kappa^2) = m^{*T}(p, \kappa^2) = (2\pi)^2(2L)\langle\chi M(p)\chi\rangle \tag{2}$$

must be a positive matrix over some region of the momentum space; at least over $p^\circ \geq 0$, in order to have probability interpretation in the theory. $2L$ is the extension along the third direction in which we enclose the system and $M(p)$ is a nonnegative Hermitian projection operator. (With proper normalization, it is $|p\rangle\langle p|$.)

The complex conjugate of (1) is

$$\begin{aligned}\langle\chi(x')\chi(x)\rangle &= \langle\chi(x)\chi(x')\rangle^* \\ &= \int_0^\infty d\kappa^2 \frac{1}{2L}\left(\sum_{p_3\neq 0} + \sum_{p_3=0} e^{ip_3(x_3-x_3')}\right) \\ &\times \int \frac{d\bar{p}}{(2\pi)^2} e^{i\bar{p}\cdot(\bar{x}-\bar{x}')}\eta_-(p)\, \delta(p^2+\kappa^2)m^*(-p, \kappa^2). \end{aligned}\tag{3}$$

Using

$$\eta_\pm(p) = \tfrac{1}{2}(1 \pm \epsilon(p)),$$

we combine (1) and (3) to yield the commutator $[\ \ ]_-$ or the anticommutator $[\ \ ]_+$

$$\begin{aligned}\langle[\chi(x), \chi(x')]_\pm\rangle &= \left(\sum_{p_3=0} + \sum_{p_3\neq 0}\frac{1}{2L}e^{ip(x_3-x_3')}\right) \\ &\times \int \frac{d\bar{p}}{(2\pi)^2} d\kappa^2\, e^{i\bar{p}\cdot(\bar{x}-\bar{x}')}\, \delta(p^2+\kappa^2)\tfrac{1}{2}[m(p, \kappa^2) \\ &\pm m^*(-p, \kappa^2) + \epsilon(p)(m(p, \kappa^2) \pm m^*(-p, \kappa^2))]. \end{aligned}\tag{4}$$

We shall examine the structure of $m(p, \kappa^2)$.

To each homogeneous, proper, orthochronous Lorentz transformation

$$x' = lx, \qquad \det l = 1, \qquad l_0^0 \geq 1,$$

there correspond a unitary operator $U(l)$ and a finite real matrix $L(l)$, such that

$$\bar{\chi}(x') = L(l)\chi(x)$$

and

$$\bar{\chi}(x) = U^{-1}(l)\chi(x)U(l). \tag{5}$$

For an infinitesimal transformation

$$l_\mu^\nu = \delta_\mu^\nu + \delta\omega_\mu^\nu, \qquad \delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu},$$

we have

$$U(\delta\omega) = 1 + \tfrac{1}{2}i\delta\omega^{\mu\nu}J_{\mu\nu}$$

and

$$L(\delta\omega) = 1 + \tfrac{1}{2}i\delta\omega^{\mu\nu}S_{\mu\nu} + \delta\omega^{\mu\nu}O_{\mu\nu}(\Lambda),$$

$$O_{\mu\nu}(\Lambda) = 0, \qquad \mu, \nu = 0, 1, 2,$$

where $O_{\mu\nu}(\Lambda)$ is a function of the gauge transformation Λ, which must accompany the Lorentz transformation l to maintain the gauge condition in the new coordinate system. For example, in electrodynamics, expanding

$$A(x) = \int \frac{dp}{(2\pi)^4} e^{ip\cdot x} A(p)$$

and

$$\Psi(x) = \int \frac{dp}{(2\pi)^4} e^{ip\cdot x}\, \Psi(p),$$

we have[4]

$$(LA(p))_\mu = l_\mu^\nu A_\nu(p) + ip_\mu\lambda(p)$$

and

$$\begin{aligned}L\Psi(p) &= (1 + \tfrac{1}{2}i\delta\omega_{\mu\nu}\sigma^{\mu\nu})\Psi(p) \\ &\quad + ieq\int \frac{dk}{(2\pi)^4}\lambda(p-k)\Psi(k), \end{aligned}\tag{6}$$

where

$$\lambda(p) = \frac{-1}{ip_3}\delta\omega^{3\kappa}A_\kappa(p)$$

and

$$\sigma^{\mu\nu} = i\gamma^\mu\gamma^\nu, \qquad \mu \neq \nu.$$

Equation (6) can be integrated to a finite rotation. Let us consider a specific case when we make a rotation in the 1–3 plane through an angle α. It can be shown that[6]

$$\begin{aligned}\bar{A}^0(x') &= A^0(x) + \partial^0\lambda, \\ \bar{A}^1(x') &= (A^1(x) + \partial^1\lambda)\cos\alpha + (A^3(x) + \partial^3\lambda)\sin\alpha, \\ \bar{A}^2(x') &= A^2(x) + \partial^2\lambda, \\ \bar{A}^3(x') &= -(A^1(x) + \partial^1\lambda)\sin\alpha + (A^3(x) + \partial^3\lambda)\cos\alpha,\end{aligned}$$

and

$$\bar{\Psi}(x') = e^{\frac{1}{2}i\alpha\sigma^{13}}e^{ieq\lambda}\Psi(x), \tag{7}$$

where the accompanying gauge transformation is

$$\lambda = -A^1(x)\sin\alpha/\partial_3' \tag{8}$$

and

$$\begin{aligned}x^{1\prime} &= x^1\cos\alpha + x^3\sin\alpha, \\ x^{3\prime} &= -x^1\sin\alpha + x^3\cos\alpha.\end{aligned}$$

(We have tacitly avoided mentioning the complica-

[6] D. Boulware, Ph.D. thesis (1962), Department of Physics, Harvard University.

tion caused by $p_3=0$. This does not, however, cause any limitation on the following proof.[7])

The differential characterization of (5) is (See I, p. 920)

$$\frac{1}{i}\left(p_\mu\frac{\partial}{\partial p^\nu}-p_\nu\frac{\partial}{\partial p^\mu}\right)m(p,\kappa^2)+S_{\mu\nu}m(p,\kappa^2)$$
$$+\,m(p,\kappa^2)S^T_{\mu\nu}=0,\qquad \mu,\nu=0,1,2,\qquad(9)$$

since no gauge transformation is needed if we do not disturb the third axis. After we introduce $p_4=ip_0$ and $S_{k4}=iS_{0k}$, as far as the dependence on $\bar p$ is concerned, (9) is a statement of the rotational invariance of a system in the 0–1–2 Euclidean subspace. $m(p,\kappa^2)$ is therefore a sum of products of some arbitrary functions of κ^2 and p_3, finite numerical matrices, and three-dimensional spherical harmonics. These spherical functions are homogeneous algebraic functions of $\bar p$, of degrees no greater than $2S$, S being the highest spin value associated with χ, and irreducible with respect to the contraction $-p^2=\kappa^2$.

When $\alpha=\pi$, (8) gives $\lambda=0$. This result is very important, because it means that no gauge transformation is needed if we rotate the coordinate axes in the 1–3 plane by π,[8] (although this result is proved for spin-$\frac{1}{2}$ electrodynamics, we believe it to be true in all other cases.) If, after this operation, we further rotate the coordinate system in the 3–4 plane by π, the combined effect is the total reflection $x^\mu\to-x^\mu$ and the corresponding matrix $L(l)$ is[9]

$$R_{st}=e^{\pi iS_{13}}e^{\pi iS_{34}}.$$

Using the properties

$$R^2_{st}=1,$$

iS_{13} being real, and $iS_{24}=-S_{02}$ being imaginary, we have

$$R^*_{st}=(-1)^{2S}R_{st}.$$

This last relation and (5) lead to

$$m(-p,\kappa^2)=R_{st}m(p,\kappa^2)R^T_{st}=(-1)^{2S}R_{st}m(p,\kappa^2)R^{*T}_{st}.$$

On the other hand, $m(p,\kappa^2)\geq 0$ for $p^0\geq 0$, and therefore we have analytically continued $m(p,\kappa^2)$ to negative value of p^0, supplying the property

$$p^0>0:(-1)^{2S}m(p,\kappa^2)\geq 0.\qquad(10)$$

We return to (4). We have just shown that $m(p,\kappa^2)$ depends on $\bar p$ algebraically. For this reason, it is legitimate to write (4) as [in other words, we replace $\bar p$ by $1/i\ \bar\partial$ to emphasize the algebraic structure]

$$\langle[\chi(x),\chi(x')]_\pm\rangle$$
$$=\int_0^\infty d\kappa^2\frac{1}{2L}\sum_{p_3\neq 0}e^{ip_3(x_3-x_3')}\frac{1}{2}\left[m\left(\frac{1}{i}\bar\partial,p_3,\kappa^2\right)\right.$$
$$\left.\pm\, m^*\left(\frac{1}{i}\bar\partial,-p_3,\kappa^2\right)\right]\int\frac{d\bar p}{(2\pi)^2}e^{i\bar p\cdot(\bar x-\bar x')}\ \delta(p^2+\kappa^2)$$
$$+\int_0^\infty d\kappa^2\frac{1}{2L}\sum_{p_3\neq 0}e^{ip_3(x_3-x_3')}\frac{1}{2}\left[m\left(\frac{1}{i}\bar\partial,p_3,\kappa^2\right)\right.$$
$$\left.\mp\, m^*\left(\frac{1}{i}\bar\partial,-p_3,\kappa^2\right)\right]\int\frac{d\bar p}{(2\pi)^2}e^{i\bar p\cdot(\bar x-\bar x')}\ \delta(p^2+\kappa^2)\epsilon(p)$$
$$+\frac{1}{2L}\int\frac{d\bar p}{(2\pi)^2}d\kappa^2\,e^{i\bar p\cdot(\bar x-\bar x')}\ \delta(p^2+\kappa^2)\tfrac{1}{2}[m(p,\kappa^2)$$
$$\pm\, m^*(-p,\kappa^2)+\epsilon(p)(m(p,\kappa^2)\mp m^*(-p,\kappa^2))]_{p_3}=0.\qquad(11)$$

The function, designated as

$$\Delta(\bar x-\bar x',\kappa^2+p_3^2)=\int\frac{d\bar p}{(2\pi)^2}e^{i\bar p\cdot(\bar x-\bar x')}\delta(p^2+\kappa^2)\epsilon(p)$$

is odd with respect to x_0-x_0', because of $\epsilon(p)$, i.e.,

$$\Delta(-(x_0-x_0'),\mathbf{x}-\mathbf{x}',\kappa^2)=-\Delta(\bar x-\bar x',\kappa^2).$$

This results in

$$\Delta(\bar x-\bar x',\kappa^2)=0\quad\text{when}\quad x^0=x^{0\prime}.$$

But $\Delta(\bar x-\bar x',\kappa^2)$ is an invariant function in the 0–1–2 subspace; the invariant statement is then

$$\Delta(\bar x-\bar x',\kappa^2)=0,\qquad(\bar x-\bar x')^2>0.$$

It is at this kind of spatial separation from each other, namely,

$$(x-x')^2\geq(\bar x-\bar x')^2=(x_1-x_1')^2$$
$$+\,(x_2-x_2')^2-(x_0-x_0')^2>a^2,$$

that we assign the two χ fields in what follows. a is some arbitrary but sufficiently large real number.

The vanishing of the commutator or the anticommutator for spatial separations is now assumed. Because x_3-x_3' can run over all values, the uniqueness of Fourier expansion ensures that each individual component of the Fourier transform in (11) with respect to p_3 is null. Thus, when $p_3\neq 0$,

$$\int_0^\infty d\kappa^2\left[m\left(\frac{1}{i}\bar\partial,p_3,\kappa^2\right)\pm m^*\left(\frac{1}{i}\bar\partial,-p_3,\kappa^2\right)\right]$$
$$\times\int\frac{d\bar p}{(2\pi)^2}e^{i\bar p\cdot(\bar x-\bar x')}\ \delta(p^2+\kappa^2)=0,\ (\bar x-\bar x')^2>a^2.\qquad(12)$$

[7] As we shall see later, if the system is enclosed in a finite volume, we shall avoid using the spectral density around $p_3=0$. When the volume becomes infinite, then points with $p_3=0$ have measure zero and thus should not complicate matters.

[8] We believe that this result is also true, if we have a guage condition which distinguishes two of the spatial axes from the other one (but does not involve the time component.)

[9] J. Schwinger, Lecture Notes, Summer Institute in Theoretical Physics, Brandeis University (1959).

We have the representation, for $(\bar{x} - \bar{x}')^2 > a^2$

$$\Delta^1(\bar{x} - \bar{x}', \kappa^2 + p_3^2) \equiv \int \frac{d\bar{p}}{(2\pi)^2} e^{i\bar{p}\cdot(\bar{x}-\bar{x}')} \delta(p^2 + \kappa^2) = \frac{1}{(\pi)^{\frac{3}{2}}} \int_0^\infty \frac{d\lambda}{(\lambda)^{\frac{1}{2}}} \exp\left(-\lambda \bar{x}^2 - \frac{\kappa^2 + p_3^2}{4\lambda}\right),$$

and (12) becomes

$$\int_0^\infty d\kappa^2 \left[m\left(\frac{1}{i}\bar{\partial}, p_3, \kappa^2\right) \pm m^*\left(\frac{1}{i}\bar{\partial}, -p_3, \kappa^2\right)\right] \times \int_0^\infty \frac{d\lambda}{(\lambda)^{\frac{1}{2}}} \exp\left(-\lambda\bar{x}^2 - \frac{\kappa^2 + p_3^2}{4\lambda}\right) = 0. \quad (13)$$

The proof from here on duplicates that of I. Because of what we said about the structure of $m[(1/i)\bar{\partial}, p_3, \kappa^2]$ after Eq. (9) and because $\Delta^1(\bar{x} - \bar{x}', \kappa^2 + p_3^2)$ is a function of $(\bar{x} - \bar{x}')^2$ only, we expand

$$\left[m\left(\frac{1}{i}\bar{\partial}, p_3, \kappa^2\right) \pm m^*\left(\frac{1}{i}\bar{\partial}, -p_3, \kappa^2\right)\right] \times \Delta^1(\bar{x} - \bar{x}', \kappa^2 + p_3^2) = \sum_{l=1}^{2S} m_{l\pm}(p_3, \kappa^2) Y_l(\bar{x} - \bar{x}') \times \left(\frac{\partial}{\partial(\bar{x} - \bar{x}')^2}\right)^l \Delta^1(\bar{x} - \bar{x}', \kappa^2 + p_3^2) \quad (14)$$

into harmonic series where $Y_l(\bar{x} - \bar{x}')$ are three-dimensional spherical harmonics, and $m_{l\pm}(p_3, \kappa^2)$ are the associated expansion coefficients, being scalar functions in p_3 and κ^2.

Because $(\partial/\partial(\bar{x} - \bar{x}')^2)^l \Delta^1(\bar{x} - \bar{x}', \kappa^2 + p_3^2) \neq 0$, we must have for each $l = 0, 2S$

$$\int_0^\infty d\lambda\, e^{-\lambda x^2} \int_0^\infty d\kappa^2 \times \exp\left[-(\kappa^2 + p_3^2)/4\lambda\right](\lambda)^{-\frac{1}{2}} m_{l\pm}(p_3, \kappa^2) = 0.$$

We restrict ourselves to the class of functions $m_{l\pm}(p_3, \kappa^2)$ which can grow at most algebraically with κ^2.[10] We now invoke Laplace's lemma,[2] which states that if

$$\int_0^\infty (dx)\, e^{-tx} f(x) = 0 \qquad (t \geq 1)$$

for $f(x)$ such that

$$\int_0^\infty (dx)\, e^{-x}\, |f(x)|^2 < \infty,$$

then

$$f(x) = 0.$$

We apply this lemma first with respect to the λ transform and then the κ^2 transform. We come up with the conclusion

$$m_{l\pm}(p_3, \kappa^2) \equiv 0, \qquad l = 0, 2S.$$

Therefore it follows from (4) that the vanishing of the commutator $(-)$ or the anticommutator $(+)$ for sufficient spacelike separation requires

$$m(p, \kappa^2) \mp m^*(-p, \kappa^2) = 0, \quad \text{respectively.}$$

Comparing this with (2) and (4), we see that $m(p, \kappa^2) \equiv 0$, if we assume the wrong connection between spin and statistics. This cannot be so, for $m(p, \kappa^2) \equiv 0$ implies $\chi \equiv 0$. Thus, we must demand the commutator condition (Bose–Einstein statistics) to go with integral spin fields, and the anticommutator conditions (Fermi–Dirac statistics) to go with half-integral spin fields.

We also see that the same proof can be carried out, if we have a theory which is manifestly Lorentz-covariant in a plane containing the 0 axis and one of the spatial axes.

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APPENDIX I[4]

The one-photon Green's function has the spectral representation

$$\langle A_\mu(x) A_\nu(x')\rangle = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2L} \sum_{p_3} \int_0^\infty d\kappa^2\, e^{ip\cdot(x-x')} \times \eta_+(p)\, \delta(p^2 + \kappa^2) A_{\mu\nu}(p),$$

where

$$p_3 = 0 : A_{\mu\nu}(p) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu - (n\cdot p)(n_\mu p_\nu + n_\nu p_\mu)}{(p + n(n\cdot p))^2}\right) A_1(\kappa^2),$$

$n = (1, 0, 0, 0)$ a timelike unit vector;

$$p_3 \neq 0 : A_{\mu\nu}(p) = \left(g_{\mu\nu} + \frac{p_\mu p_\nu}{2}\left(\frac{1}{(m\cdot p - i\epsilon)^2} + \frac{1}{(m\cdot p + i\epsilon)^2}\right) - \frac{m_\mu p_\nu + m_\nu p_\mu}{2}\left(\frac{1}{(m\cdot p - i\epsilon)} + \frac{1}{(m\cdot p + i\epsilon)}\right)\right) A_2(\kappa^2)$$

---

[10] This condition can always be satisfied, if we have some finite equal-time commutation relations of the χ [i. e., finite coefficients multiplied by $\delta(\mathbf{x} - \mathbf{x}')$], because then we can show that there are finite sum rules (integrals with respect to κ^2) that $m(p, \kappa^2)$ have to obey. See, for instance, the Appendix.

($\lim \epsilon \to 0$). $m = (0, 0, 0, 1)$ a spacelike unit vector, with

$$A_1(\kappa^2) \geq 0, \qquad A_2(\kappa^2) \geq 0,$$

$$\int_0^\infty A_1(\kappa^2)\, d\kappa^2 = \int_0^\infty A_2(\kappa^2)\, d\kappa^2 = 1;$$

it is apparent that

$$A_{\mu\mu}(p) \geq 0 \quad \begin{array}{l}\text{(semipositive-definite)}\\ \text{(no sum).}\end{array}$$

The one-electron Green's function has the spectral representation

$$\langle\psi(x)\psi(x')\rangle = -\int \frac{d^3\bar{p}}{(2\pi)^2} \int_0^\infty d\kappa^2 \frac{1}{2L} \left\{ \sum_{p_3} e^{ip\cdot(x-x')} \right.$$
$$\times\ \eta_+(p)\ \delta(p^2 + \kappa^2) i\beta[(\gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2)$$
$$\times\ A(p_3^2, \kappa^2) + \gamma^3 p_3 B(p_3^2, \kappa^2) + mC(p_3^2, \kappa^2)].$$

We have the properties

$$A(p_3^2, \kappa^2) \geq 0 \quad \text{(semipositive-definite)}$$

and

$$\int_0^\infty A(p_3^2, \kappa^2)\, d\kappa^2 = 1.$$

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The Wave Equation and the Green's Dyadic for Bounded Magnetoplasmas

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In studies of electromagnetic wave propagation and radiation in magnetoplasmas, the wave equation takes the form of a dyadic–vector Helmholtz equation. The investigation here shows that the dyadic–vector Helmholtz equation is solvable by the separation method in four cylindrical coordinate systems. Solutions in the form of complete sets of eigenfunctions are possible when boundary surfaces are present. For problems involving current sources in the plasma, the Green's dyadics for finite or semifinite domains can be constructed from the complete sets of eigenfunctions which are solutions to the homogeneous equation. The Green's dyadic for infinite domain is also shown to be obtained from that for a semifinite domain through a limiting process.

INTRODUCTION

THE presence of a static magnetic field in a plasma region results in an effective electric conductivity which is of dyadic form. Assuming monochromatic waves, the equation describing the waves, generated by a source, $\mathbf{J}_s$, in such an anisotropic medium may be written as

$$\nabla\times\nabla\times\mathbf{E} - \mathbf{k}\cdot\mathbf{E} = \mathbf{J}_s. \tag{1}$$

Written in matrix form, the dyadic $\mathbf{k}$ is

$$\mathbf{k} = \begin{bmatrix} k_\perp & k_T & 0 \\ -k_T & k_\perp & 0 \\ 0 & 0 & k_\parallel \end{bmatrix}. \tag{2}$$

Assuming spatial homogeneity, the parameters $k_\perp$, k_T, and $k_\parallel$ are constants with respect to time and space coordinates.

Solutions for Eq. (1) in terms of auxiliary Green's function for infinite domain have been discussed by Bunkin[1] and subsequently extended by Chow[2] and Brandstater.[3] However, the solutions of Eq. (1) for a bounded region have proved to be more difficult to obtain. The studies dealt with here reveal that, in order to solve for a finite-domain or semifinite-domain Green's function, a better understanding of the free wave equation, $\mathbf{J}_s = 0$ in Eq. (1), is needed, and that the Green's function may be constructed from the solutions of the homogeneous equation.

THE HOMOGENEOUS EQUATION

The homogeneous equation describing free wave propagation is

[1] F. V. Bunkin, Zh. Eksperim. i Teor. Fiz. **32**, 338 (1957) [English transl.: Soviet Phys.—JETP **5**, 277 (1957)].
[2] Y. Chow, Trans. IRE trans. Antennas Propagation **10**, 464 (1962).
[3] J. J. Brandstater, *An Introduction to Waves, Rays, and Radiation in Plasma* (McGraw-Hill Book Company, Inc., New York, 1963) Chap. 9.