Running for the Exit: Distressed Selling and Endogenous Correlation in Financial Markets

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Running for the Exit: distressed selling and endogenous correlation in financial markets

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Abstract

We propose a simple multiperiod model of price impact in a market with multiple assets, which illustrates how feedback effects due to distressed selling and short selling lead to endogenous correlations between asset classes. We show that distressed selling by investors exiting a fund and short selling of the fund's positions by traders may have non-negligible impact on the realized correlations between returns of assets held by the fund. These feedback effects may lead to positive realized correlations between fundamentally uncorrelated assets, as well as an increase in correlations across all asset classes and in the fund's volatility which is exacerbated in scenarios in which the fund undergoes large losses. By studying the diffusion limit of our discrete time model, we obtain analytical expressions for the realized covariance and show that the realized covariance may be decomposed as the sum of a fundamental covariance and a liquidity-dependent 'excess' covariance. Finally, we examine the impact of these feedback effects on the volatility of other funds. Our results provide insight into the nature of spikes in correlation associated with the failure or liquidation of large funds.

Contents

1 Introduction 3
1.1 Summary 4
1.2 Related Literature 5
1.3 Outline 6

2 A multi-asset model of price impact from distressed selling 6

3 Numerical experiments 9
3.1 Simulation procedure 9
3.2 Realized variance and realized correlations 9
3.3 Fund volatility 13

4 Diffusion limit 14

5 Realized correlations 16
5.1 Realized covariance 16
5.2 Case of zero fundamental correlations 17
5.3 The path-dependent nature of realized correlation 19
5.4 Liquidation impact 20

6 Endogenous risk and spillover effects 21
6.1 Realized variance of the fund 22
6.2 Fund volatility in the case of zero fundamental correlations 22
6.3 Spillover effects 23

7 Conclusion 26

Appendices 26

A Proof of lemmas for Theorem 4.1 26
A.1 Lemma 4.2 27
A.2 Lemma 4.3 27
A.3 Lemma 4.4 28

B Proof of Proposition 6.1 29

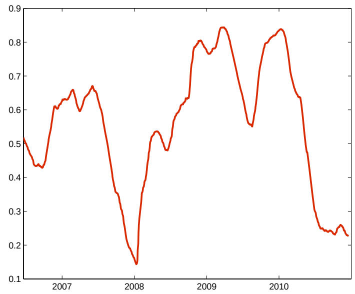
References 29

1 Introduction

Correlations in asset returns are a crucial ingredient for quantifying the risk of financial portfolios and a key input for asset allocation and trading. Correlations and covariances between returns of assets, indices and funds are routinely estimated from historical data and used by market participants as inputs for trading, portfolio optimization and risk management. Whereas sophisticated models –featuring stochastic volatility, conditional heteroskedasticity and jumps– have been proposed for univariate price dynamics, the dependence structure of returns is typically assumed to be stationary, either through a time-invariant correlation matrix or a copula, and estimated from historical time series of returns. For example, a popular method is to use (exponentially-weighted) moving average (EWMA) estimators of realized correlation.

However, empirical evidence shows that such estimators of *realized correlation* exhibit large variations in time [15]; in particular, estimators of realized correlation may exhibit large spikes or dips associated to market events. Figure 1 show examples of variability in time of realized correlations in equity indices; in particular, we observe a sharp increase in realized correlations associated with the collapse of Lehman Brothers on September 15th, 2008. More generally, unexpected correlation spikes are often associated with the liquidation of a large fund. For instance, in 1998, due to heavy losses in its investments in Russian bonds, LTCM was forced to liquidate its positions after a sudden increase in the correlations across its –previously uncorrelated– positions led to a sharp increase in its volatility [22]. Unexpected spikes of correlation arose between asset classes that used to be uncorrelated (Russian bonds and US equity for instance), leading to further losses for the fund and finally its collapse. A more complex phenomenon occured in August 2007: between August 7 and August 9 2007, all long-short equity market neutral hedge funds lost around 20% per day whereas major equity indices hardly moved. Khandani and Lo [19] suggest that this 'quant event' of August 2007 was due to the unwinding of a large long-short market neutral hedge fund's positions, that created extreme volatility on other funds with similar portfolios, while leaving index funds unaffected. These examples illustrate that "asset correlations can be different during a liquidity crisis because price movements are caused by distressed selling and predatory trading rather than fundamental news" [7].

The evidence for time-variation in the dependence structure of asset returns has motivated the development of new classes of stochastic models with time-dependent correlation structures [14, 13, 17, 24] in which the conditional distribution of asset returns is given by a multivariate distribution with a randomly evolving covariance structure whose evolution is specified exogenously. However, such models where correlation is represented as an exogenous factor do not explain the presence of spikes in correlations associated with market events such as the liquidation of large funds. These observations suggest the existence of an *endogenous* component in asset correlations, which should be modeled by taking into account the impact of supply and demand generated by investors, in particular in situations of market distress. Such endogenous variations in volatility and correlations, generated by systematic patterns in supply and demand linked to rule-



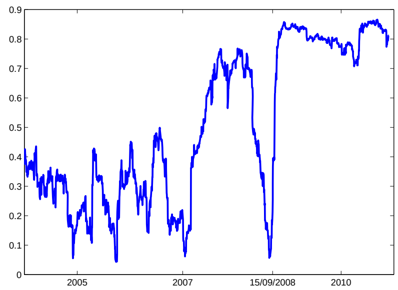


Figure 1: Left: EWMA estimator of average pairwise correlations of daily returns in EuroStoxx 50 index. Right: one year EMWA correlation between two ETF of the the S&P 500: SPDR XLE (energy) and SPDR XLK (technology)

based trading strategies, short selling or fire sales, have played an important role in past financial crises and have been the focus of several studies [1, 2, 6, 8, 9, 21, 23] which underline the link between liquidity and volatility in financial markets.

In the present paper, we establish a link between the economic literature on fire sales and endogenous risk and the quantitative/econometric literature on realized correlations by proposing a simple, analytically tractable model which shows how feedback effects from 'distressed selling' and short selling may lead to *endogenous correlations* between assets held by a fund, when investors facing losses simultaneously try to exit the fund. Our model framework is sufficiently general and allows for multiple assets and an arbitrary dependence structure in the exogenous economic factors driving these assets, yet simple enough to lead a detailed analytical study. This allows in particular to obtain *quantitative* results on the impact of distressed selling on realized variance and realized correlation.

1.1 Summary

We consider a fund investing in various asset classes/ strategies whose returns are decomposed into random components that represent exogenous economic factors (fundamentals) and a liquidity term, which is a function of aggregate excess demand for each asset generated either by investors liquidating their positions or by short sellers/speculators shorting the fund's position once the fund value drops below a threshold.

Simulations of this discrete-time model reveal that, even in the case of assets with zero fundamental correlation, one observes a significant positive level of realized correlation resulting in higher than expected fund volatility. Furthermore, this realized correlation is observed to be path-dependent.

These simulation results are confirmed by theoretical results on the continuous-time limit of our model. We exhibit conditions under which the discrete time model exhibits a diffusion limit and provide explicit expressions for the instantaneous covariance, correlation across assets and fund volatility for the limiting diffusion process. Our analytical results show that realized covariance is *path-dependent*. It is the sum of a fundamental covariance and a *liquidity-dependent excess covariance term*. Furthermore, the impact of the liquidation of a fund is *computable* under our model assumptions. Even in absence of correlations between fundamentals, asset returns may exhibit significant positive correlation, resulting in higher fund volatility. Even when liquidity is constant, the liquidation of large fund positions can generate significant positive correlation between the fund's assets. Distressed selling in a reference fund generates spillover effects. We can compute the effects of fund liquidation on the variance of other target funds. We show that when the target fund's strategy is 'orthogonal' to the reference fund's strategy, its variance is not modified.

Our results point to the limits of diversification, previously discussed by many authors, but also allow to quantify these limits. We show that a fund manager investing in apparently uncorrelated strategies may experience significant realized correlation across his/her strategies in the case of distressed selling by investors facing losses, thus losing the benefit of diversification exactly when it is needed. These results provide simple explanations for the sudden rise in correlations associated with the LTCM and the August 2007 events. Our study provides insight into the nature of spikes in correlation and fund volatility associated with the failure or liquidation of large funds and gives a quantitative framework to evaluate strategy crowding as a risk factor. In particular our model offers a simple explanation for the events of August 2007: our results show how the liquidation of a large long-short equity market neutral fund can generate high volatility for other quantitative funds with similar allocations while leaving index funds unaffected.

1.2 Related Literature

Empirical evidence of distressed selling and its impact on market dynamics has been documented by several previous studies. Funds experiencing large outflows sell their holdings, as documented by Coval and Stafford [12]. For regulatory reasons, after large losses, banks must sell risky assets, as discussed by Berndt et al. [5] for the corporate debt market. Khandani and Lo [19] describe how the need to reduce risk exposure compelled market-neutral long-short equity hedge funds to liquidate large position in equity markets in the second week of August 2007, generating a series of huge losses which are explained quantitatively by our model. Empirical studies by Jones et al. [10] show the importance of short selling in financial markets (40.2% and 39.2% of total dollar volume on the NYSE and Nasdaq, respectively). Haruvy and Noussair [18] examine empirically the effects of short selling restrictions finding that relaxing short selling constraints does not induce prices to track fundamentals. Our study provides a quantitative framework for analyzing these empirical observations.

Various theoretical models have been proposed for analyzing feedback effects resulting from fire sales in financial markets, mostly in a single-asset framework. Market losses in subprime mortgage-backed securities, largely seen as being uncorrelated with equity markets, led to huge falls in equity markets as explained by Brunnermeier [6]. Shin [23] describes the mechanisms which amplified the recent financial crisis and the systemic risk they generate. Investors 'running for the exit' can generate spirals in prices and spillovers to other asset classes as well as a crowding effect, as discussed by Pedersen [21]. Andrade et al. [3] show how trading imbalances in one asset class can lead to deviation of prices from fundamental value in other asset classes. Short selling by predators is described in [7], where Brunnermeier shows how shorting the portfolio of a fund approaching its liquidation value can lead to the collapse of the fund. Our detailed quantitative analysis confirms these predictions. Furthermore, while all those studies mainly focus on asset prices and fund value, our multi-asset framework allows for a computation of the impact of fund liquidation or short selling on realized correlation between assets and fund volatility.

Avellaneda and Lipkin [4] quantify the impact of short selling restrictions on price dynamics, in a single asset market. They conclude that short selling restrictions generate higher prices and higher price volatility and modify the Call-Put parity. Our model extends this analysis to a multi-asset framework.

1.3 Outline

The paper is organized as follows. Section 2 presents our discrete-time model. In particular, it characterizes the behavior of distressed sellers and short sellers and their effect on prices. Section 3 displays the results of the simulations of this model. In Section 4, we find the continuous-time limit of our discrete-time dynamics. Section 5 gives analytical expressions for the realized variance and covariance of asset returns in the continuous-time limit and uses these expressions to study the path-dependence of realized correlations and role of market depth. Using these analytical results, we show in Section 6 how feedback effects lead to endogenous volatility in a distressed fund and spillover effects across funds. Section 7 concludes.

2 A multi-asset model of price impact from distressed selling

Consider a market with n financial assets in which trading takes place at discrete dates $t_k = k\tau$ ($\tau = \frac{T}{M}$ and $0 \leq k \leq M$). The price of asset i at date t_k is denoted S_k^i and we denote $S_k = (S_k^1, ..., S_k^n)$. It is useful, in the examples, to think of S^i as the value of an index or ETF representing a sector or geographic zone. At each period, the value of the assets moves due to exogenous economic factors, whose impact on the return is modeled by a sequence $\xi_k = (\xi_k^1, ..., \xi_k^n)_{1\leq k\leq M}$ of iid n-dimensional centered random variables, with covariance matrix Σ: in absence of other effects, the return of asset i at period k would be $\sqrt{\tau}\xi_{k+1}^i$. We denote $(S_{k+1}^i)^* = S_k^i(1 + \sqrt{\tau}\xi_{k+1}^i)$.

We consider a large leveraged fund holding $\alpha_i \geq 0$ units of asset i with $1 \leq i \leq n$ between dates $t = 0$ and T. Thus, between t_k and t_{k+1}, price moves due to exogenous economic factors move the value of the fund from $V_k = \sum_{i=1}^{n} \alpha_i S_k^i$ to $V_{k+1}^* = \sum_{i=1}^{n} \alpha_i (S_{k+1}^i)^* = V_k + \sum_{i=1}^{n} \alpha_i S_k^i \sqrt{\tau}\xi_{k+1}^i$.

Investors enter the fund at $t = 0$ when the fund is valued at V_0. Like most investors in mutual funds, investors in the fund adopt a passive, buy and hold behavior as long as the fund is performing well. However, when the fund value drops below a threshold $\beta_0 V_0 < V_0$, investors progressively exit their positions, generating a negative demand across all assets held by the fund, proportionally to the positions held by the fund. Our purpose is to model the price impact of this *distressed selling* and investigate its effect on realized volatility and correlations of the assets held by the fund.

We model the supply/demand pattern generated by distressed selling by introducing a function $f : \mathbb{R} \to \mathbb{R}$ which measures the rate at which investors in the fund exit their positions: when fund value drops from V_k to V_{k+1}^*, investors redeem a fraction $f(\frac{V_k}{V_0}) - f(\frac{V_{k+1}^*}{V_0})$ of their position in the fund. Thus, the net supply in asset i due to distressed selling (or short selling) is equal to

$$-\alpha_i(f(\frac{V_{k+1}^*}{V_0}) - f(\frac{V_k}{V_0}))$$

The above assumptions on investor behavior imply that $f : \mathbb{R} \to \mathbb{R}$ is increasing, constant on $[\beta_0, +\infty[$.

We furthermore assume that the fund is liquidated when the value reaches $\beta_{liq} V_0$ where $\beta_{liq} < \beta_0$. In practice, as the fund loses value and approaches liquidation, distressed selling becomes more intense: this feature is captured by choosing f to be *concave*. Figure 2 gives an example of such a function f.

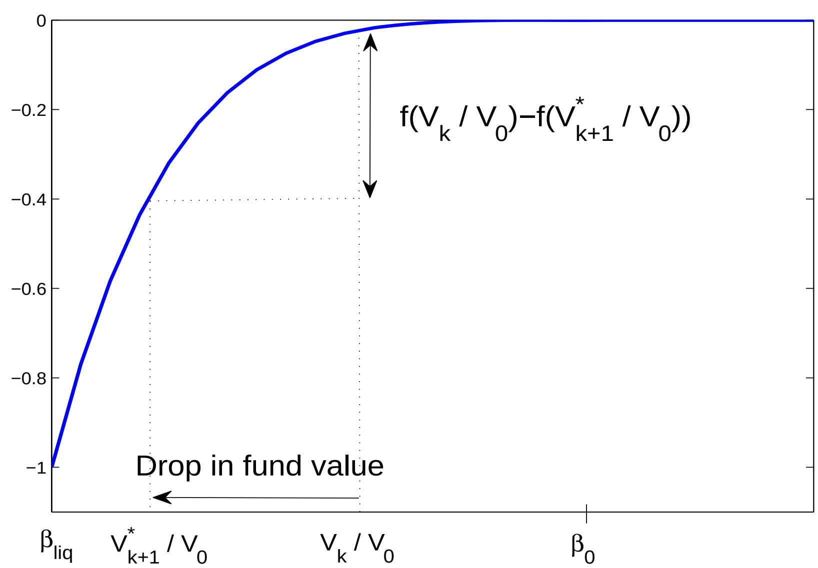


Figure 2: Net supply due to distressed selling and short selling is equal to $-\alpha_i(f(\frac{V_{k+1}^*}{V_0}) - f(\frac{V_k}{V_0}))$

This distressed selling activity impacts prices in a non-random manner. Empirical studies [20, 11] provide evidence for the linearity of this price impact at daily and intraday frequencies. Between t_k and t_{k+1}, given the net supply generated by short sellers and distressed sellers, market impact on asset i's return is equal to

$$\frac{\alpha_i}{\lambda_i}(f(\frac{V^*_{k+1}}{V_0}) - f(\frac{V_k}{V_0}))$$

where λ_i represent the *depth* of the market in asset i: a net demand of $\frac{\lambda_i}{100}$ shares for security i moves i's price by one percent. Obizhaeva [20] studies empirically the link between market depth and average daily volume (ADV) on NYSE and NASDAQ stocks, finding that $\frac{ADV_i\sqrt{250}}{\lambda_i\sigma_i}$ is close to 1. We will use this relation to choose reasonable simulation parameters for the size of the fund's position on i, α_i, compared to its market depth λ_i, in the examples in Section 3.

The supply/demand pattern generated by these distressed sellers exiting the fund may be amplified by short sellers or predatory trading: the presence of short sellers may result in scenarios where a fraction > 1 of the fund is exited/liquidated. From our perspective, their effect on price dynamics is similar and we will not distinguish between distressed (e.g. long) sellers and short sellers.

$$\begin{tikzcd}
t_k \arrow[d] & & t_{k+1} \arrow[d] \\
\boxed{S_k, V_k} \arrow[r, "\text{exogenous}", "\text{economic factors}"'] & \boxed{S^*_{k+1}, V^*_{k+1}} \arrow[r, "\text{distressed selling}", "\text{short selling}"'] & \boxed{S_{k+1}, V_{k+1}}
\end{tikzcd}$$

Summing up, the dynamics of asset prices is given by:

$$S^i_{k+1} = S^i_k \left(1 + \sqrt{\tau}\xi^i_{k+1} + \frac{\alpha_i}{\lambda_i}\left(f(\frac{V_k}{V_0} + \sum_{i=1}^{n} \frac{\alpha_i S^i_k}{V_0}\sqrt{\tau}\xi^i_{k+1}) - f(\frac{V_k}{V_0})\right)\right) \tag{1}$$

where

$$V_k = \sum_{i=1}^{n} \alpha_i S^i_k \tag{2}$$

Equations 2 and 2 show that S_{k+1} depends only on its value at t_k and on ξ_{k+1}, that is independent from what happened before t_k. The vector of prices $S = (S^1, ..., S^n)^t$ is a Markov Chain.

Using the dimensionless variables $\tilde{S}_k = (\frac{S^1_k}{S^1_0}, ..., \frac{S^n_k}{S^n_0})^t$ and $\tilde{V}_k = \frac{V_k}{V_0}$, we can rewrite 2–2 as

$$\tilde{S}^i_{k+1} = \tilde{S}^i_k \left(1 + \sqrt{\tau}\xi^i_{k+1} + \frac{\alpha_i}{\lambda_i}\left(f(\tilde{V}_k + \sum_{i=1}^{n} \frac{\alpha_i S^i_0}{V_0}\tilde{S}^i_k\sqrt{\tau}\xi^i_{k+1}) - f(\tilde{V}_k)\right)\right)$$

where

$$\tilde{V}_k = \sum_{i=1}^{n} \frac{\alpha_i S^i_0}{V_0}\tilde{S}^i_k$$

Hence the dynamics of $\tilde{S}_k$ is entirely determined by

- the fundamental covariance matrix Σ
- the vector $(\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n})$ which expresses the sizes of the fund's positions in each asset relative to the asset's market depth. This is a dimensionless measure of the size of positions, which is relevant for measuring market impact in case of liquidation.
- the dollar proportions $(\frac{\alpha_1 S_0^1}{V_0}, ..., \frac{\alpha_n S_0^n}{V_0})$ initially invested in each asset by the fund
- the function f which describes the supply generated by distressed/short selling

3 Numerical experiments

3.1 Simulation procedure

We perform a Monte Carlo simulation of the multiperiod model described above for a fund investing in two strategies/asset classes with zero fundamental correlation and volatilities respectively given by 30% and 20%. We assume that the volume held by the fund on each asset is of the order of 20 times average daily volume for the asset. In comparison, LTCM's on balance sheet assets totalled around \$125 billion, which represented 250 times average daily volume on the S&P 500 in 1998. We assume that the fund initially invests the same amount of cash in both assets and that distressed sellers can trade once a day. We use the following parameters in our simulations:

- $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$ with $\sigma_1 = 30\%\ year^{-\frac{1}{2}}$ and $\sigma_2 = 20\%\ year^{-\frac{1}{2}}$
- $\frac{\alpha_1}{\lambda_1} = \frac{\alpha_2}{\lambda_2} = \frac{1}{10}$: the fund's position on asset 1 (resp. on asset 2) is equal to 10% of asset 1's (resp. asset 2's) market depth, or, using [20], around 15 times average daily volume for asset 1 (resp. around 20 times average daily volume for asset 2)
- $\frac{\alpha_1 S_0^1}{V_0} = \frac{\alpha_2 S_0^2}{V_0} = \frac{1}{2}$: the fund initially invests the same amount of cash in 1 and 2
- We use the following choice for f: $f(x) = \frac{-1}{(\beta_{liq} - \beta_0)^4}(x - \beta_0)^4$ which satisfies the conditions described in Section 2, with $\beta_0 = 0.95$ and $\beta_{liq} = 0.55$,.

Figure 3 is an example of trajectory for the fund's relative value $(\frac{V_t}{V_0})$ and the price of each asset (we arbitrarily set initial asset prices to 100\$).

3.2 Realized variance and realized correlations

In each simulated path, we compute the log-returns of asset $r_k^i = \log(\frac{S_{k+1}^i}{S_k^i})$ for $i = 1, 2$.

Let $\overline{r^i}$ be the sample average of those returns: $\overline{r^i} = \frac{1}{M} \sum_{k=0}^{M-1} r_k^i$. For each sample path, we compute the realized covariance between assets i and j:

$$\widehat{C^{i,j}} = \frac{1}{T} \sum_{k=0}^{M-1} (r_k^i - \overline{r^i})(r_k^j - \overline{r^j})$$

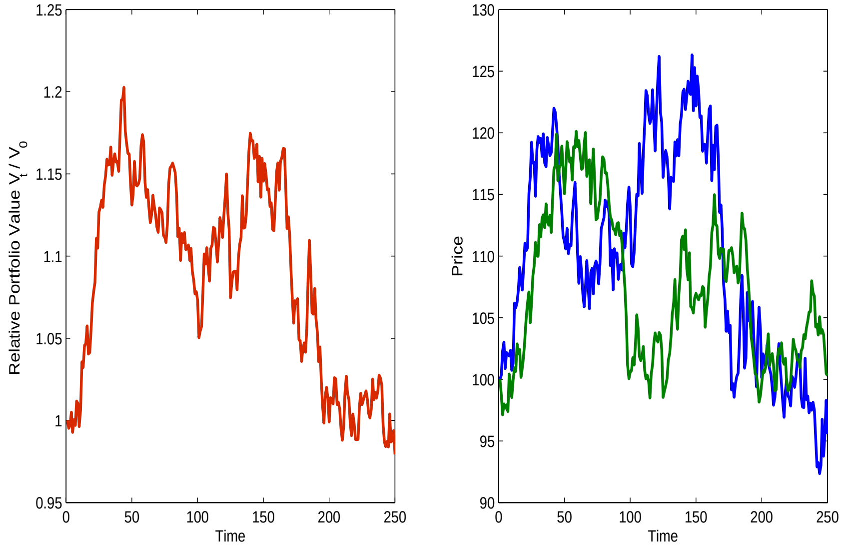


Figure 3: An example of trajectory: relative value of the fund ($\frac{V_t}{V_0}$) and price of each asset

and the realized correlation between i and j: $\frac{\widehat{C^{i,j}}}{(\widehat{C^{i,i}}\widehat{C^{j,j}})^{\frac{1}{2}}}$. The realized volatility for i is given by $(\widehat{C^{i,i}})^{\frac{1}{2}}$.

Figure 4 show the distribution of the one-year realized correlation for the two strategies across 10^6 independent scenarios. In each scenario, we also computed realized correlation without feedback effects. Figure 5 is a scatter plot of the one-year realized correlation with and without feedback effects from distressed selling/short selling. Each point of the graph corresponds to one trajectory (for clarity, we choose to display only 1000 trajectories on scatter plots). For each point of the graph and hence each trajectory, realized correlation in presence of feedback effects (resp. without feedback effects) can be read on the vertical axis (resp. the horizontal axis).

In presence of distressed selling, the distribution of realized correlation is significantly modified. Our simulations show that *distressed selling by investors exiting funds with similar portfolios and short selling can generate significant realized correlation, even between assets with zero fundamental correlation.* In Figure 4, the distribution of realized correlation without feedback effects reflects the statistical error in the estimation of correlation. Hence, the aspect of the distribution of realized correlation with feedback effects due to distressed selling or short selling reflects the effects of such trading on correlation between assets. Figure 4 shows that average correlation in presence of feedback effects is higher than its fundamental value $\rho = 0$ and the profile of its distribution presents a thick upper tail. In Figure 5, all points are above the Y=X axis, confirming the fact that distressed selling increases correlation between assets. This is due to the fact that when

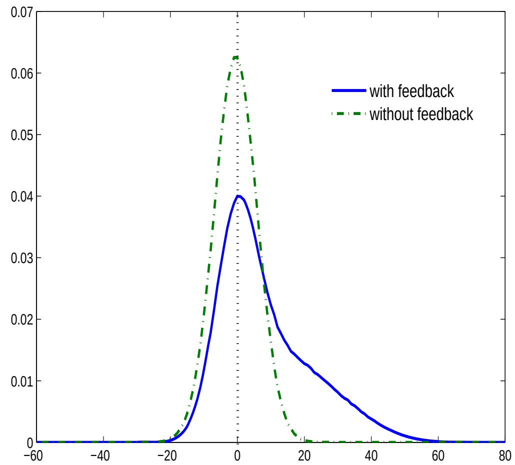


Figure 4: Distribution of realized correlation between the two securities (with $\rho = 0$) with and without feedback effects due to distressed selling

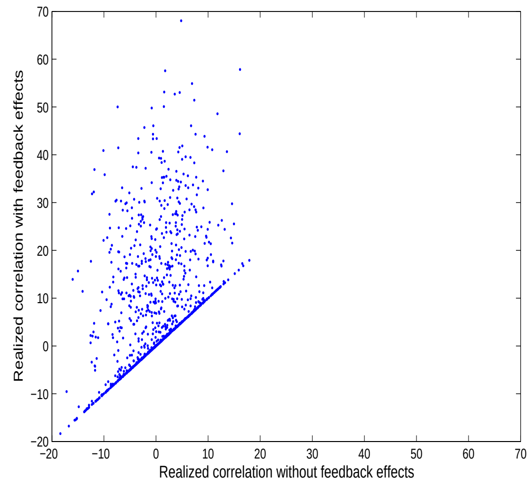


Figure 5: Scatter plot of realized correlation with and without feedback effects due to distressed selling (each data point represents one simulated scenario)

the fund's market value drops below $\beta_0 V_0$, all assets face either a positive demand or a negative demand. In presence of feedback effects, correlation becomes path dependant. It is interesting to examine the distribution of realized correlation in scenarios where fund value reaches $\beta_0 V_0$, triggering distressed selling/short selling.

Conditional correlation: In Figure 6, we simulate 10^6 trajectories of Equations 1 and 2 and we divide trajectories into two categories, whether fund value reaches $\beta_0 V_0$ between 0 and T or not. We display the distribution of realized correlation for those two categories: in plain line, the distribution of realized correlation in scenarios where fund value reaches $\beta_0 V_0$, triggering distressed selling; in dotted line, the distribution of realized correlation in scenarios where fund value remains above $\beta_0 V_0$ and there is no distressed selling or short selling. *Realized correlation conditional on the fact that distressed selling took place is significantly higher than realized correlation in scenarios where there was no distressed selling.* As we pointed out in the previous paragraph, in scenarios where distressed selling took place, price impact affects all assets of the fund in the same direction during the time the fund's market value is below the threshold $\beta_0 V_0$. This results in higher realized correlation in those scenarios.

Asset volatility: *In presence of feedback effects from distressed selling/short-selling, asset volatility increases.* Figure 7 shows that the distribution of realized volatility of each asset, in scenarios where there was distressed selling, is centered around a higher value than the asset's fundamental volatility and presents a thick upper tail. In such scenarios,

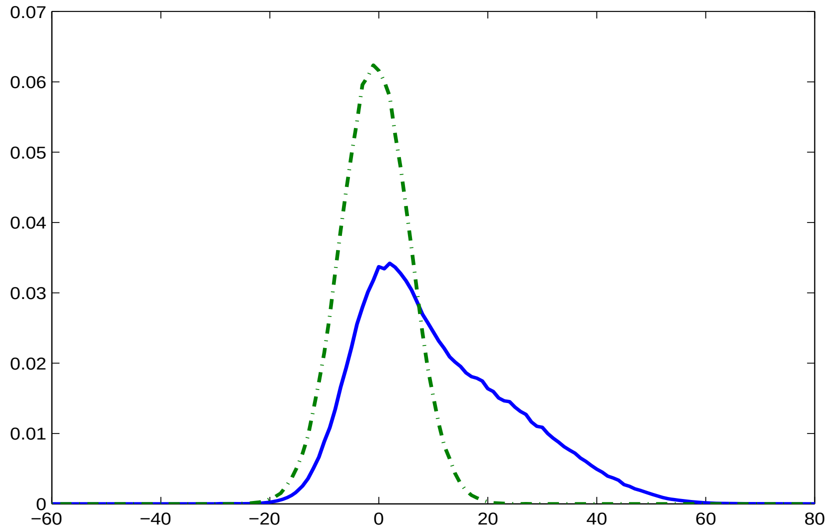


Figure 6: Distribution of realized correlation in scenarios where fund value reaches $\beta_0 V_0$ between 0 and T (plain line) and in scenarios where fund value remains above $\beta_0 V_0$ (dotted line)

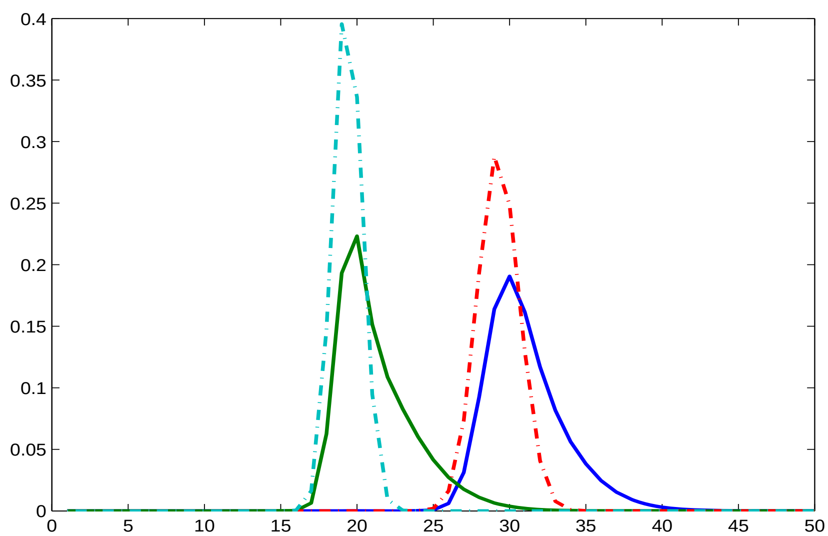


Figure 7: Distribution of realized volatilities for each security in scenarios where fund value reaches $\beta_0 V_0$ between 0 and T (plain lines) and in scenarios where fund value remains above $\beta_0 V_0$ (dotted lines) (with $\sigma_1 = 30\%$ and $\sigma_2 = 20\%$)

assets are more volatile than in scenarios without distressed selling. The action of distressed sellers (and short sellers) increases the amplitude of price moves and generates higher asset volatility. This should result in higher fund volatility.

3.3 Fund volatility

We simulated 10^6 trajectories of Equations 1 and 2 and obtained the following distributions (see Figure 8) for fund volatility. In each scenario, we also compute fund volatility without feedback effects. Figure 9 is a scatter plot of fund volatility, with and without feedback effects from distressed selling/short-selling.

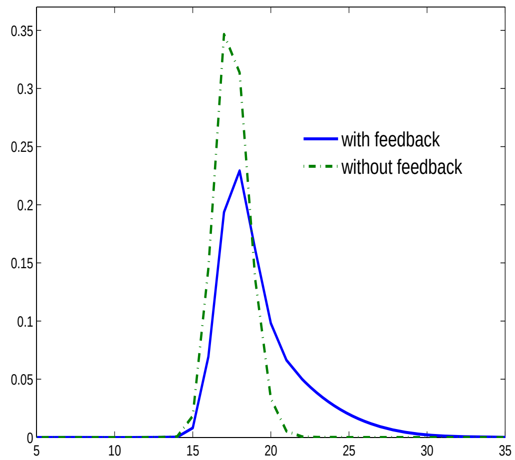


Figure 8: Distribution of realized volatility of the fund with and without feedback effects

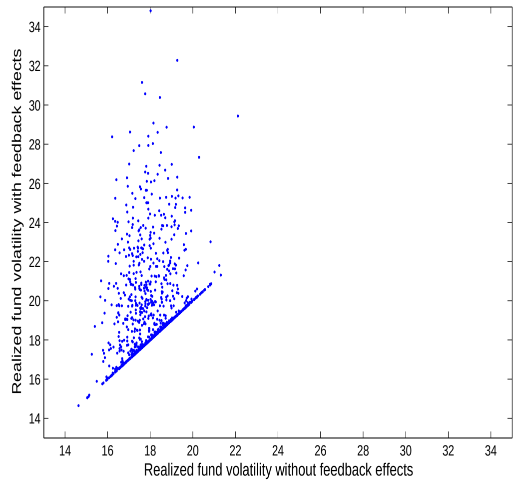


Figure 9: Scatter plot of realized volatility of the fund with and without feedback effects

Distressed selling increases the fund's volatility: the distribution of realized fund volatility presents a thick upper tail when there are feedback effects from short sellers or distressed sellers. Figures 8 and 9 underline the fact that feedback effects increase the fund's volatility compared to scenarios without such effects. As in Figure 6, we can compare the distribution of fund volatility in scenarios where the fund reaches $\beta_0 V_0$ or not. Figure 10 displays the distributions of fund volatility in those two scenarios: when there is distressed selling, the fund is more volatile than when fund value remains above $\beta_0 V_0$ and there is no distressed selling.

Our simulations show that, even in the case of assets with zero fundamental correlation, one observes a significant positive level of realized correlation resulting in higher than expected fund volatility.

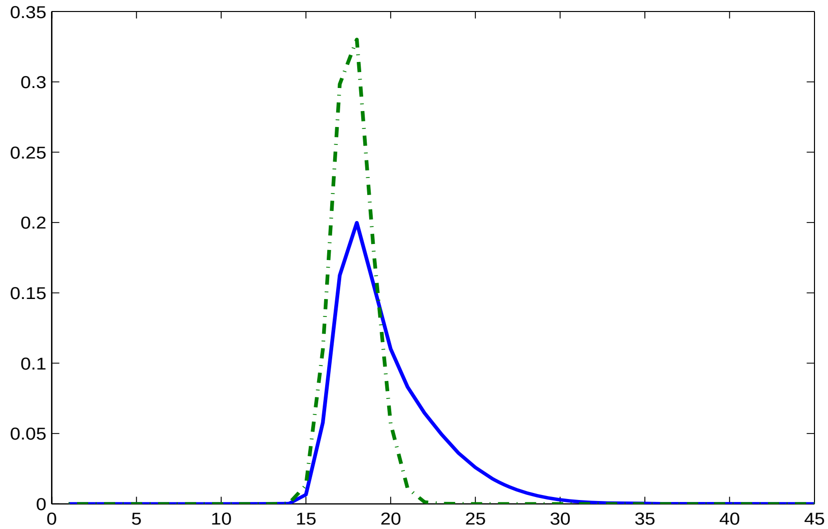


Figure 10: Distribution of realized fund volatility in scenarios where fund value reaches $\beta_0 V_0$ between 0 and T (plain line) and in scenarios where fund value remains above $\beta_0 V_0$ (dotted line)

4 Diffusion limit

To confirm that the phenomena observed in the numerical experiments are not restricted to particular parameter choices or a particular choice of the function f, we will now analyze the continuous-time limit of our discrete-time model: the study of this limit allows to obtain analytical formulas for realized correlation which confirm quantitatively the effects observed in the numerical experiments. Our main theoretical result is the following theorem which describes the diffusion limit of the price process:

Theorem 4.1 *Under the assumption that* $\mathbb{E}(|\xi|^4) < \infty$ *and that* $f \in C_b^3$ *such that* $\sup |xf'(x)| < \min \frac{\lambda_i}{\alpha_i}$, $S_{\lfloor t\tau \rfloor}$ *converges weakly towards a diffusion* $P_t = (P_t^1, ...P_t^n)^t$ *when* τ *goes to* 0 *where*

$$\frac{dP_t^i}{P_t^i} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i \qquad 1 \le i \le n$$

where μ *(resp.,* σ*) is a* $\mathbb{R}^n$*-valued (resp. matrix-valued) adapted process defined by*

$$\mu_i(P_t) = \frac{\alpha_i}{2\lambda_i} f''(\frac{V_t}{V_0}) \frac{< \pi_t, \Sigma \pi_t >}{V_0^2} \tag{3}$$

$$\sigma_{i,j}(P_t) = A_{i,j} + \frac{\alpha_i}{\lambda_i} f'(\frac{V_t}{V_0}) \frac{\left(A^t \pi_t\right)_j}{V_0} \tag{4}$$

Here W_t is a n-dimensional Brownian motion, $\pi_t = (\alpha_1 P_t^1, ..., \alpha_n P_t^n)^t$ is the (dollar) allocation of the fund, $V_t = \sum_{1\leq k\leq n} \alpha_k P_t^k$ the value of the fund and A is a square-root of the fundamental covariance matrix: $AA^t = \Sigma$.

Proof In order to study the continuous-time limit of the Markov chain $S_{\lfloor t\tau \rfloor}$, we first show (see Appendix) that, under the assumption that $f \in C_b^3$ and $\mathbb{E}(|\xi|^4) < \infty$, we have:

Lemma 4.2 *For all $\epsilon > 0$ and $r > 0$:*

$$\lim_{\tau\to 0} \sup_{\|S\|\leq r} \frac{1}{\tau}\mathbb{P}(|S_{k+1}^i - S_k^i| \geq \epsilon | S_k = S) = 0$$

Lemma 4.3 *For all $1 \leq i \leq n$ and $r > 0$:*

$$\lim_{\tau\to 0} \sup_{\|S\|\leq r} \left(\frac{1}{\tau}\mathbb{E}(S_{k+1}^i - S_k^i | S_k = S) - b_i(S)\right) = 0$$

Lemma 4.4 *For all $1 \leq i, j \leq n$ and $r > 0$:*

$$\lim_{\tau\to 0} \sup_{\|S\|\leq r} \left(\frac{1}{\tau}\mathbb{E}[(S_{k+1}^i - S_k^i)(S_{k+1}^j - S_k^j) | S_k = S] - a_{i,j}(S)\right) = 0$$

where $a : \mathbb{R}^n \mapsto \mathrm{Sym}_n(\mathbb{R})$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ are given by

$$a_{i,j}(S) = S^i S^j \left(\Sigma_{i,j} + \frac{\alpha_j}{\lambda_j} f'(\frac{<\alpha, S>}{V_0}) \sum_{1\leq l\leq n} \frac{\alpha_l}{V_0} S^l \Sigma_{i,l} + \frac{\alpha_i}{\lambda_i} f'(\frac{<\alpha, S>}{V_0}) \sum_{1\leq l\leq n} \frac{\alpha_l}{V_0} S^l \Sigma_{j,l}\right)$$

$$+S^i S^j \left(\frac{\alpha_i \alpha_j}{\lambda_i \lambda_j} (f')^2 (\frac{<\alpha, S>}{V_0}) \sum_{1\leq l,p\leq n} \frac{\alpha_l \alpha_p}{V_0^2} S^l S^p \Sigma_{l,p}\right)$$

$$b_i(S) = S^i \left(\frac{\alpha_i}{2\lambda_i} f''(\frac{<\alpha, S>}{V_0}) \sum_{1\leq j,l\leq n} \frac{\alpha_j \alpha_l}{V_0^2} S^j S^l \Sigma_{j,l}\right)$$

where $<\alpha, S> = \sum_{1\leq i\leq n} \alpha_i S^i$.

Define the differential operator $G : C_0^\infty(\mathbb{R}^n) \mapsto C_0^\infty(\mathbb{R}^n)$ by

$$Gh(x) = \frac{1}{2} \sum_{1\leq i,j\leq n} a_{i,j}(x) \partial_i \partial_j h + \sum_{1\leq i\leq n} b_i(x) \partial_i h$$

Let $\gamma_{i,j}(S) = S^i \left(A_{i,j} + \frac{\alpha_i}{\lambda_i} f'(\frac{V(S)}{V_0}) \sum_{1\leq k\leq n} \frac{\alpha_k}{V_0} S^k A_{k,j}\right) = S^i \sigma_{i,j}(S)$ for $S \in \mathbb{R}^n$, where $AA^t = \Sigma$. $a(S) = \gamma(S)\gamma^t(S)$, so a is a symmetric non negative function. As $f \in C_b^3$,

a and b are Lipschitz–continuous functions so the martingale problem for (G, δ_{S_0}) is well-posed.

So, by [16, Theorem 4.2, Ch.7], the Markov chain $S_{\lfloor t\tau \rfloor}$ converges in distribution to the solution $(\mathbb{P}, (P_t)_{t\in[0,T]})$ of the martingale problem for (G, δ_{S_0}) when $\tau \to 0$. $P_t = (P_t^1, ... P_t^n)$ is thus a weak solution of the stochastic differential equation

$$\frac{dP_t^i}{P_t^i} = \mu_t^i dt + (\sigma_t dW_t)_i$$

where μ, σ are given by (3) – (4).

* The result we obtain is valid for any $f \in C_b^3$ such that $\sup |xf'(x)| < \min \frac{\lambda_i}{\alpha_i}$.
* When market depth is infinite (i.e. price impact is negligible) the continuous-time limit is multivariate geometric Brownian motion

$$\frac{dP_t^i}{P_t^i} = (AdW_t)_i$$

and the covariance of the log-returns is given by the 'fundamental' covariance: $\text{cov}(\ln P_t^i, \ln P_t^j) = t\Sigma_{ij}$.

* As f is concave on $[\beta_{liq}, \beta_0]$, f'' is negative and the action of distressed sellers and short sellers pushes down the price as investors exit the fund i.e. when $\frac{V_t}{V_0} \in [\beta_{liq}, \beta_0]$.
* The expression of σ shows that distressed selling modifies correlation between assets, asset volatility and fund volatility. We will focus on this phenomenon in the next sections.

5 Realized correlations

5.1 Realized covariance

The following result follows by direct computation from Theorem 4.1:

Proposition 5.1 *The realized covariance between securities i and j between 0 and t is equal to $\frac{1}{t}\int_0^t C_s^{i,j}\,\mathrm{d}s$, where $C_s^{i,j}$, the instantaneous covariance between i and j, is given by:*

$$C_s = \Sigma + \frac{1}{V_0} f'(\frac{V_s}{V_0})[\Lambda \pi_s^t \Sigma + \Sigma \pi_s \Lambda^t] + \frac{1}{V_0^2}(f')^2(\frac{V_s}{V_0}) < \pi_s, \Sigma \pi_s > \Lambda\Lambda^t$$

where

- $\pi_t = (\alpha_1 P_t^1, ..., \alpha_n P_t^n)^t$ *denotes the (dollar) holdings of the reference fund*
- $\Lambda = (\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n})^t$ *represents the positions of the reference fund in each market as a fraction of the respective market depth.*

The explicit expression for the realized covariance of asset returns shows that *realized covariance is path dependant and has an additive decomposition: it is the sum of the fundamental covariance and a liquidity-dependent excess covariance term.* Excess covariance is composed of a term of order one in $\frac{\alpha}{\lambda}$ and a term of order two in $\frac{\alpha}{\lambda}$. When market depth is infinite, instantaneous covariance reduces to fundamental covariance. Moreover, the expression of instantaneous covariance shows that $C^{i,j}$ is a deterministic and continuous function of vector π_t, hence *the impact of the liquidation of a fund on realized correlation is computable* under our model assumptions. Realized covariance and correlation between assets depend on the first derivative of f which represents the rate at which investors exit their positions due to fund underperformance.

In scenarios where the fund value stays above $\beta_0 V_0$ realized covariances converge to their fundamental value. However, as soon as the fund value falls below the threshold $\beta_0 V_0$ which triggers distressed selling, excess covariance appears: in such distress scenarios, realized correlation and realized variance differ from the values implied by the 'fundamental covariance' Σ. In the case where fundamental correlation is positive between all pairs of assets, distressed selling increases realized covariance. As shown by Eq. (5.1), the magnitude of this effect is determined by the size α_i of the positions being liquidated relative to the depth λ_i of the market in these assets: this is further discussed in Section 5.4. It is also interesting to notice that when the fund invests significantly in an asset i compared to its market depth and when fund value drops below $\beta_0 V_0$, instantaneous covariance between i and any other asset j in the market is different from its fundamental value $\Sigma_{i,j}$ (as $\frac{\alpha_i}{\lambda_i} f'(\frac{V_s}{V_0}) \frac{(\Sigma \pi_t)_j}{V_0} \neq 0$).

5.2 Case of zero fundamental correlations

We now focus on the case of a diagonal covariance matrix Σ: the n assets are uncorrelated. We denote $\Sigma_{i,i} = \sigma_i^2$ (σ_i is asset i's volatility) and $\Sigma_{i,j} = 0$ for all $1 \leq i, j \leq n$ and $i \neq j$.

Corollary 5.2 *If the fundamental covariance matrix Σ is diagonal, then, for all $1 \leq i, j \leq n$, the instantaneous covariance between i and j ($i \neq j$) and the instantaneous variance of asset i are given by:*

$$C_t^{i,j} = \frac{\alpha_j}{\lambda_j} f'(\frac{V_t}{V_0}) \frac{\alpha_i}{V_0} P_t^i \sigma_i^2 + \frac{\alpha_i}{\lambda_i} f'(\frac{V_t}{V_0}) \frac{\alpha_j}{V_0} P_t^j \sigma_j^2 + \frac{\alpha_i \alpha_j}{\lambda_i \lambda_j} (f')^2 (\frac{V_t}{V_0}) \sum_{1 \leq l \leq n} (\frac{\alpha_l}{V_0} P_t^l \sigma_l)^2 \geq 0$$

and

$$C_t^{i,i} = \sigma_i^2 + 2 \frac{\alpha_i}{\lambda_i} f'(\frac{V_t}{V_0}) \frac{\alpha_i}{V_0} P_t^i \sigma_i^2 + (\frac{\alpha_i}{\lambda_i})^2 (f')^2 (\frac{V_t}{V_0}) \sum_{1 \leq l \leq n} (\frac{\alpha_l}{V_0} P_t^l \sigma_l)^2 \geq \sigma_i^2$$

Realized correlation between i and j (resp realized volatility for asset i) between 0 and T are equal to $\frac{\int_0^T C_t^{i,j} \, \mathrm{d}t}{(\int_0^T C_t^{i,i} \, \mathrm{d}t \int_0^T C_t^{j,j} \, \mathrm{d}t)^{\frac{1}{2}}}$ (resp. $(\frac{1}{T} \int_0^T C_t^{i,i} \, \mathrm{d}t)^{\frac{1}{2}}$).

Since f is an increasing function, $C_t^{i,j}$ is positive, implying that realized correlation between i and j is positive: *in absence of fundamental correlation, distressed selling/short selling creates positive realized correlation across the fund's strategies.* This positive correlation is due to the fact that when $V_t < \beta_0 V_0$, all strategies owned by the fund face a net demand of the same sign. In particular a large fall in fund value generates a negative demand by investors across all positions held by the fund. These results confirm our simulations.
Even if the fund invests in 'fundamentally' uncorrelated strategies, in scenarios where the fund experiences losses e.g. $V_t < \beta_0 V_0$ and approaches liquidation, distressed selling by investors leads to a positive realized correlation between the fund's strategies, reducing the benefit of diversification.
Volatility of asset i is greater than σ_i (because f is an increasing function). *Distressed selling increase the volatility of all assets detained by the fund*: if the value of the portfolio drops below $\beta_0 V_0$, distressed sellers start exiting their positions, increasing the amplitude of the price movements.

To check whether these asymptotic results are relevant in the case of daily rebalancing, we compare the theoretical formula for realized covariance in continuous time given in Corollary 5.2 and the realized covariance in a discrete-time market as calculated in 3.2. Figure 11 shows that the higher the trading frequency, the better the concordance

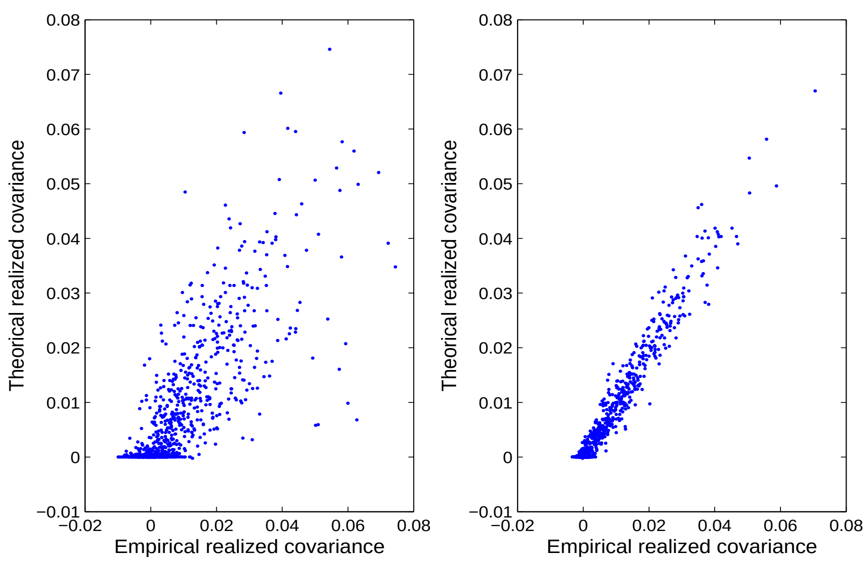


Figure 11: Scatter plot of theoretical realized covariance and empirical realized covariance for $\frac{T}{M} = \frac{1}{250}$ (left) and $\frac{T}{M} = \frac{1}{2500}$ (right)

between empirical realized covariance (calculated as in section 3.2) and the continuous-time result (given by Corollary 5.2). More precisely, a linear regression of the realized covariance with respect to the theoretical values computed using Corollary 5.2 shows good agreement between the empirical and theoretical values: the regression yields a slope of 0.95 ($R^2 = 0.63$) for $\frac{T}{M} = \frac{1}{250}$ and a slope of 0.99 ($R^2 = 0.96$) for $\frac{T}{M} = \frac{1}{2500}$.

5.3 The path-dependent nature of realized correlation

Proposition 5.1 shows that instantaneous covariance $C_t^{i,j}$ is a deterministic function of $\pi_t = (\alpha_1 P_t^1, ..., \alpha_n P_t^n)^t$. Figures 12 and 13 show examples of the evolution of correlation, given a trajectory π_t. We used the same parameters as in our simulations (section 3). In particular, we focus on two assets with zero fundamental correlation and we examine the case when $\beta_0 = 0.95$. Each Figure shows the evolution of $\frac{V_t}{V_0}$ on the graph at the left and the evolution of realized correlation since $t = 0$, $\frac{\frac{1}{t}\int_0^t C_s^{1,2},\mathrm{ds}}{\left((\frac{1}{t}\int_0^t C_s^{1,1},\mathrm{ds})(\frac{1}{t}\int_0^t C_s^{2,2},\mathrm{ds})\right)^{\frac{1}{2}}}$, on the graph at the right.

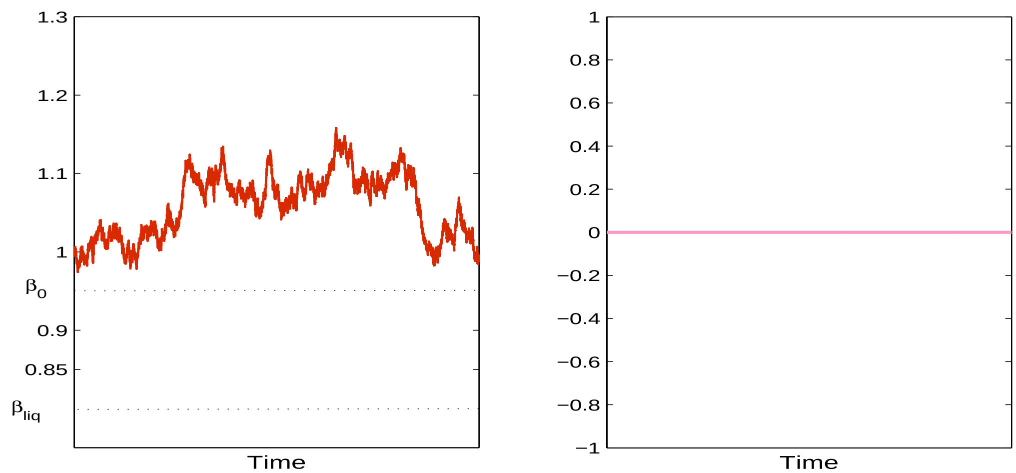


Figure 12: $\frac{V_t}{V_0}$ (left) and realized correlation on $[0, t]$ (right)

Figure 12 is an example of a price trajectory where $\frac{V_t}{V_0}$ never reaches β_0. In that case, there is no distressed selling or short selling and correlation is equal to its fundamental value (0 in our example). In Figure 13, the fund's value reaches $\beta_0 V_0$: distressed selling and short selling begin, creating a strictly positive correlation between the two assets. It is interesting to notice how correlation increases when there is distressed selling ($V_t < \beta_0 V_0$) and decreases when the fund value goes above that threshold and instantaneous covariance becomes zero.

Bad performance of the fund is amplified by investors exiting funds with similar strategies or players trading against the fund. This not only drives down the price of the fund but increases the correlation between the two strategies to unexpected levels. As a result, realized correlation among strategies will be much higher than the 'fundamental'

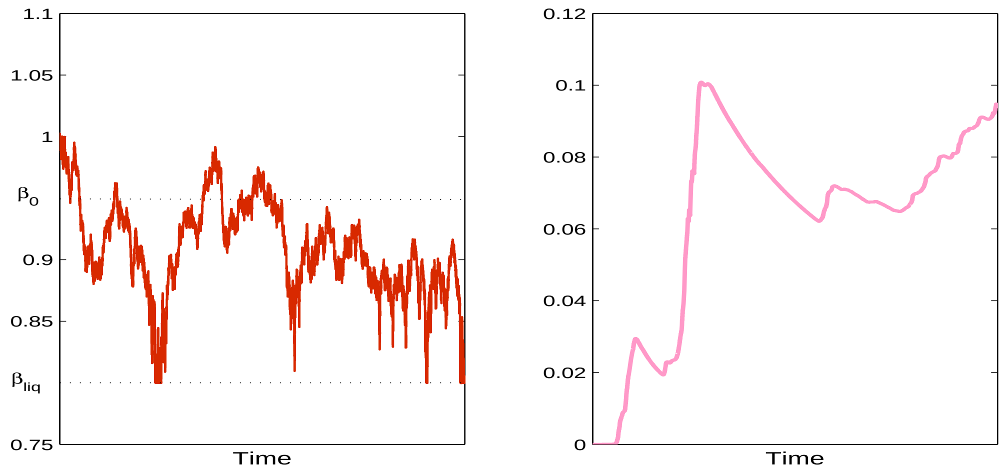


Figure 13: $\frac{V_t}{V_0}$ (left) and realized correlation on $[0, t]$ (right)

correlation, exactly when the fund is in dire need of diversification. The spiral can be triggered by a large loss in one of its strategies. This leads to investors exiting similar funds, others shorting its positions and thus generates a high correlation among all its positions.

5.4 Liquidation impact

Theorem 4.1 and Proposition 5.1 show that the price dynamics and correlation between assets are functions of the positions of the fund relative to the market depth of each asset: $\Lambda = (\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n})^t$. These results suggest that $\frac{\alpha_i}{\lambda_i}$ may be used as an indicator of the impact on asset i of the liquidation of the fund's position.

When $\frac{\alpha}{\lambda} \to 0$, we find, as expected, a Black-Scholes model with constant correlation between assets. Proposition 5.1 shows that the excess covariance tends to 0 when market depth goes to infinity. Corollary 5.2 proves that in the case of assets with zero fundamental correlation, *the bigger the fund's positions compared to the market depth of each asset, the more correlated its strategies will be*, as can be seen on Figure 14.

It is interesting to underline the fact that when all assets, except one, denoted i_0, have infinite market depths and when distressed selling/short selling occurred in the market ($\exists t_0 \tilde{V_{t_0}} \in [\beta_{liq}, \beta_{pred}]$), all strategies are positively correlated with strategy i_0 (as $\alpha_i \alpha_{i_0} C^{t_0}_{i,i_0} > 0$). Figure 15 shows that when one asset (asset 2) has finite market depth, the underperformance of other assets (asset 1) leads to strictly positive realized correlation.

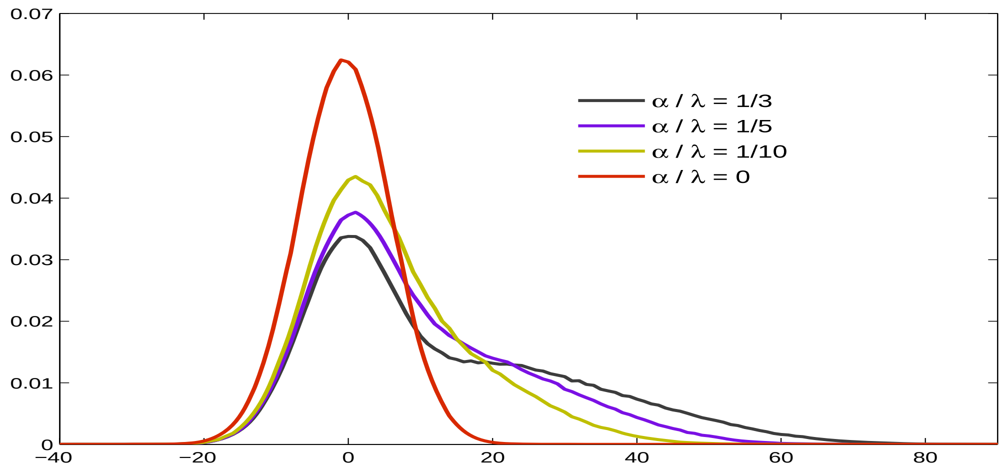


Figure 14: Distribution of realized correlation for different values of $\frac{\alpha}{\lambda}$

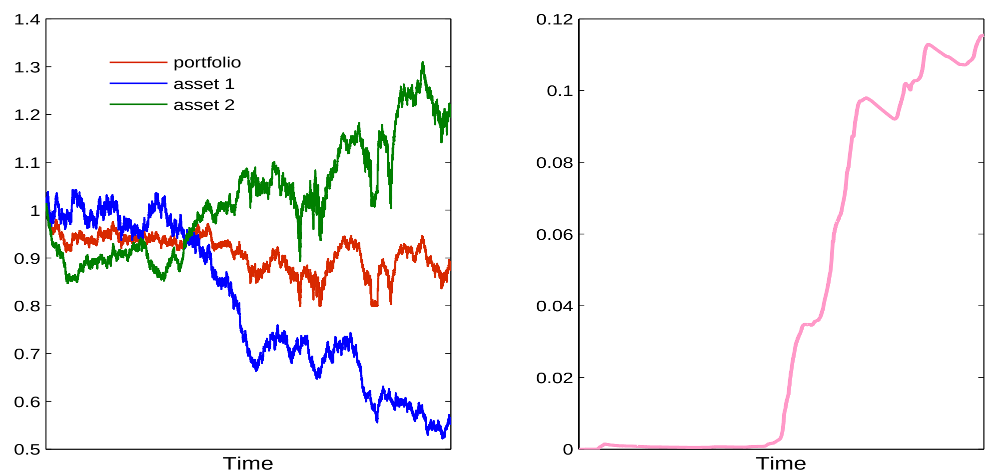


Figure 15: Fund value and asset value (left) and realized correlation (right) for $\frac{\alpha_1}{\lambda_1} = 0$ and $\frac{\alpha_2}{\lambda_2} = \frac{1}{5}$

6 Endogenous risk and spillover effects

The computation of realized correlations is relevant for the assessment of the (realized) volatility of portfolios: the explicit formulas obtained in Section 5 allow to quantify the impact of distressed selling on the volatility of the fund being exited/shorted by investors. Moreover, we will see that distressed selling on a reference fund affects other funds'volatility.

6.1 Realized variance of the fund

Proposition 6.1 *The fund's realized variance between 0 and t is equal to $\frac{1}{t}\int_0^t \Gamma_s \, \mathrm{d}s$ where Γ_s, the instantaneous variance of the fund, is given by:*

$$\Gamma_s V_s^2 = <\pi_s, \Sigma\pi_s>$$

$$+\frac{2}{V_0} f^{'}(\frac{V_s}{V_0}) <\pi_s, \Sigma\pi_s><\Lambda, \pi_s> + \frac{1}{V_0^2}(f^{'}(\frac{V_s}{V_0}))^2 <\pi_s, \Sigma\pi_s> (<\Lambda, \pi_s>)^2$$

where

- $\pi_t = (\alpha_1 P_t^1, ..., \alpha_n P_t^n)^t$ *denotes the (dollar) holdings of the reference fund*
- $\Lambda = (\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n})^t$ *represents the positions of the reference fund in each market as a fraction of the respective market depth.*

Proof See appendix

We note that distressed selling increases the fund's volatility:

$$\Gamma_s V_s^2 \geq <\pi_s, \Sigma\pi_s>$$

The fund's instantaneous variance is equal to its fundamental value $\frac{1}{V_s^2} <\pi_s, \Sigma\pi_s>$ plus a term of order one in $\frac{\alpha}{\lambda}$ $(<\Lambda, \pi_s>)$ and a term of order two in $\frac{\alpha}{\lambda}$ $((<\Lambda, \pi_s>)^2)$. In a market with infinite market depth, Γ_s is equal to its fundamental value. As in the case of instantaneous covariance between assets, Γ_s is a continuous and deterministic function of π_s and is a superposition of two regimes: a fundamental regime and an excess volatility regime, that is exacerbated with illiquidity (λ is small) or when the fund has big positions (α is high). Note that, even without liquidity drying up (λ constant), feedback effects increase significantly fund volatility when investors exit large positions. Also note that investors 'running for the exit' can account for dramatic changes in realized correlation between assets and realized fund volatility. In particular, spikes in correlation and fund volatility can be triggered by investors exiting their positions, even in absence of predatory trading by short sellers.

6.2 Fund volatility in the case of zero fundamental correlations

Corollary 6.2 *If the fundamental covariance matrix Σ is diagonal, the instantaneous variance of the fund's value is given by:*

$$\Gamma_t = (\sum_{1\leq j\leq n} (\frac{\alpha_j P_t^j}{V_t}\sigma_j)^2)(1 + \sum_{1\leq i\leq n} \frac{\alpha_i P_t^i}{V_0}\frac{\alpha_i}{\lambda_i} f^{'}(\frac{V_t}{V_0}))^2$$

Proof This is a direct consequence of Proposition 6.1 and the fact that Σ is a diagonal matrix.

The fund's realized volatility between 0 and T is equal to $(\frac{1}{T}\int_0^T \Gamma_t \, dt)^{\frac{1}{2}}$.

Corollary 6.2 explains the observations of section 3 and confirms that, when the size of the fund's positions are non-negligible with respect to market depth, distressed selling lead to an increase in fund volatility, even when the fund invests in assets with zero fundamental correlation. Similarly, fund volatility increases when there exists other funds with a similar portfolio, experiencing difficulties forcing them to liquidate part of their positions.
These results point to the limits of diversification: even if the fund manager invests in uncorrelated strategies, short-selling and liquidation by investors facing losses will correlate them positively, exactly in scenarios where the fund experiences difficulty, increasing the volatility of the portfolio and reducing the benefit of diversification.

6.3 Spillover effects

In this section, we examine the impact of such distressed selling and short selling concerning one reference fund on other funds.

Consider a (small) fund investing in the n securities and following a self-financing strategy. We denote by μ_t^i the number of units of i detained by the target fund at date t. Note that we allow for dynamic strategies. Its market value at t is $M_t = \sum_{1 \le i \le n} \mu_t^i P_t^i$. As the target fund's strategy is self-financing, we have $dM_t = \sum_{1 \le i \le n} \mu_t^i dP_t^i$.

In our framework, the target fund's strategy should impact prices and its action should modify the dynamics of P given by Theorem 4.1. However, when the target fund's positions are very small compared to the size of the reference fund, the impact of its trading strategy is negligible compared to feedback effects due to distressed selling and short selling. As M_t can take negative values, in order to quantify the risk of the portfolio μ, we study $d[M]_t$ (and no longer $d[\log(M)]_t$).
Under the assumption that the target fund's strategy does not impact prices, P still follows the dynamics given in Theorem 4.1 and we obtain:

$$d[M]_t =< \pi_t^\mu, C_t \pi_t^\mu > dt$$

where $\pi_t^\mu = (\mu_t^1 P_t^1, ..., \mu_t^n P_t^n)^t$ and C_t is the instantaneous covariance matrix of P given in Proposition 5.1. Finally, given 5.1, we find that:

Proposition 6.3 *Under the assumption that the target fund's strategy does not impact prices, the quadratic variation of the fund value is given by*

$$[M]_t = \int_0^t \gamma_s^M \quad ds \qquad \text{where}$$

$$\gamma_t^M =< \pi_t^\mu, \Sigma\pi_t^\mu > + \frac{2f'(\frac{V_t}{V_0})}{V_0} < \pi_t^\mu, \Sigma\pi_t^\alpha >< \Lambda, \pi_t^\mu > + \frac{f'(\frac{V_t}{V_0})^2}{V_0^2} < \pi_t^\alpha, \Sigma\pi_t^\alpha > (< \Lambda, \pi_t^\mu >)^2$$

where

- $M_t = \sum_{1 \le i \le n} \mu_t^i P_t^i$ *is the fund value*
- $\pi_t^\alpha = (\alpha_1 P_t^1, ..., \alpha_n P_t^n)$ *denotes the (dollar) holdings of the reference fund,*
- $\pi_t^\mu = (\mu_t^1 P_t^1, ..., \mu_t^n P_t^n)$ *denotes the (dollar) holdings of the target fund,*
- $\Lambda = (\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n})^t$ *represents the positions of the reference fund in each market as a fraction of the respective market depth.*

This result shows how distressed selling in a reference fund affects the volatility of other funds. In presence of feedback effects, the quadratic variation $[M]$ of the fund's value is given by its fundamental value $\int_0^t < \pi_s^\mu, \Sigma\pi_s^\mu > ds$ plus a term of order 1 in $\frac{\alpha}{\lambda}$ and a term of order 2 in $\frac{\alpha}{\lambda}$.

It is interesting to see if there exist strategies whose volatility is unaffected by distressed selling on the reference fund's positions. When strategies α and μ are orthogonal for Σ ($< \pi_t^\mu, \Sigma\pi_t^\alpha >= 0$), the term of order one in $\frac{\alpha}{\lambda}$ in Proposition 6.3 is equal to 0 and the target fund's absolute variance is equal to its fundamental value plus a term of order two in $\frac{\alpha}{\lambda}$.
More interestingly, if the allocations of the two funds verify the 'orthogonality' condition

$$< \Lambda, \pi_t^\mu >= \sum_{1 \le i \le n} \frac{\alpha_i}{\lambda_i} \mu_t^i P_t^i = 0 \tag{5}$$

distressed selling and short selling on the reference fund do not affect the target fund's variance:

$$[M]_t = \int_0^t < \pi_s^\mu, \Sigma\pi_s^\mu > \; ds$$

On the contrary, the excess volatility due to feedback is maximal when strategies μ and α are colinear (i.e. when the vectors π_t^μ and π_t^α are colinear). Figure 16 shows how excess quadratic variation is exacerbated when the target fund's allocations are orthogonal to the reference fund's, whereas it is close to zero for an orthogonal target fund.

These results shed some light on the 'quant event' of August 2007. In August 2007, long-short equity market-neutral funds experienced extreme volatility and large losses

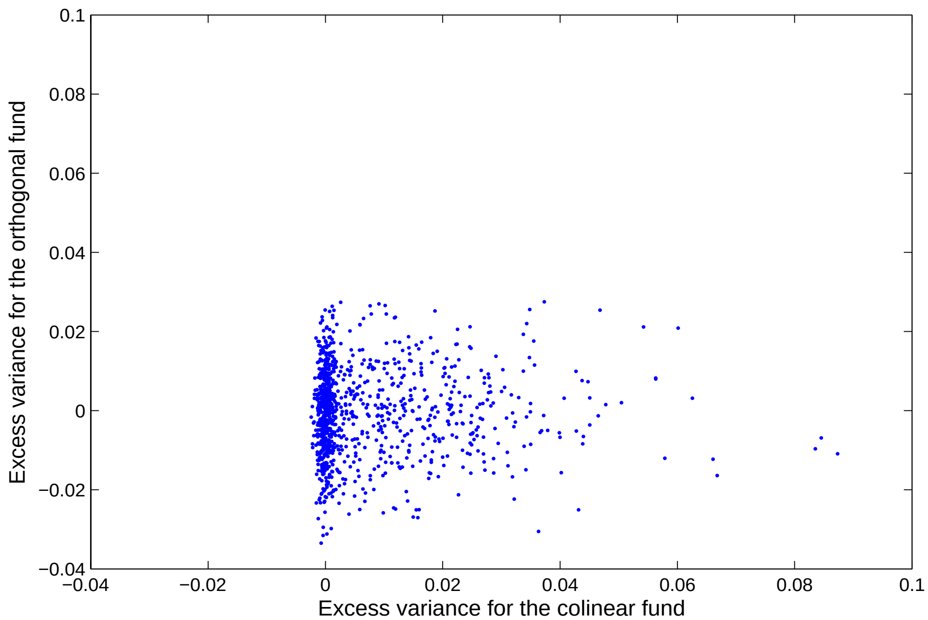


Figure 16: Scatter plot of the target fund's excess quadratic variation when the reference fund and the target fund are colinear (horizontal axis)/orthogonal (vertical axis)

during three days, whereas there was no tangible effect on major equity indices in the same period. An explanation which has been advanced is that a large position in such a market-neutral long-short fund, was liquidated by an investor in this 3 day period. Our model suggest that this rapid liquidation would then exacerbate the volatility of other long-short market-neutral funds following similar strategies (i.e. whose allocation vector has a positive projection on the allocation vector of the fund being exited). Since, by construction of market-neutral funds, the holdings of index funds are orthogonal to market-neutral funds in the sense of the orthogonality condition (5), our model predicts that index funds would not be subject to these feedback effects: indeed, they were insensitive to this event.

Alternative explanations advanced for the August 2007 events are sometimes based on a supposed dry-up of liquidity in equity markets. However, there is no evidence such a dry-up occured: in fact, trading in equity indices occured seamlessly during this period. In contrast, the mechanism underlying our model does not require any liquidity dry-up for these effects to occur: indeed, all these effects are present even when the market depth λ_i is constant. Also, our explanation entails that the population of long-short market-neutral funds affected by this event had allocations with substantial 'colinearity' i.e. that crowding was a major risk factor in this market.

More generally, the results of this section show the relevance of *strategy crowding* as a risk factor and represent a first step in quantifying this important risk factor.

Our analysis points in particular to the necessity of using indicators based on the *size* of positions when quantifying crowding effects, via proxies such as the market capitalization of various strategies. Clearly, factors based on returns cannot capture such size effects.

7 Conclusion

We have presented a simple and analytically tractable model for investigating the impact, on volatility and correlation of assets held by a fund, of fire sales by investors exiting the fund. Our model yields explicit results for the realized variance and realized correlations of assets held by the fund and shows that the realized covariance between returns of two assets may be decomposed into the sum of a 'fundamental' covariance and a liquidity-dependent 'excess covariance', which is found to be inversely proportional to the market depth of these assets.

We have shown that the presence of this excess covariance leads to endogenous risk for large portfolios –liquidating the positions of such a large portfolio entails a higher-than-expected volatility which may increase liquidation costs– as well as spillover effects: distressed selling of investors in a large fund may also exacerbate the volatility of funds with similar allocations, while leaving funds verifying an 'orthogonality' condition unaffected. This underlines the necessity of considering 'strategy crowding' as a risk factor and gives a quantitative framework to evaluate such risk.

More generally, our study shows that "liquidity risk" and "correlation risk", often treated as separate sources of risk, may be difficult to disentangle in practice: rather than being treated as an exogenous factor to be estimated using statistical methods, correlation risk needs to be modeled at its source, namely comovements in supply and demand across asset classes.

Each of these observations raises a point which merits an independent, in-depth study. We plan to pursue some of these research directions in a forthcoming work.

Appendices

A Proof of lemmas for Theorem 4.1

Before proving Lemmas 4.2, 4.3 and 4.3, we prove that a is inversible. As $a_{i,j}(S) = S^i S^j (\sigma\sigma^t)_{i,j}(S)$, we show that σ is inversible. Writing σ as a matrix:

$$\sigma(S) = A + \frac{1}{V_0} f'(\frac{< \alpha, S >}{V_0})\Lambda\pi(S)A = \left(I_d + \frac{1}{V_0} f'(\frac{< \alpha, S >}{V_0})\Lambda\pi(S)\right) A$$

where $\pi(S) = (\alpha_1 S^1, ..., \alpha_n S^n)^t$, and using the fact that A is inversible, it suffices to show that the norm of the matrix valued process $S \mapsto \frac{1}{V_0} f'(\frac{<\alpha,S>}{V_0})\Lambda\pi(S)$ is strictly below the norm of $S \mapsto S$. For a n dimensional matrix M, we define the norm of M:

$\|M\| = \max_i \sum_{j=1}^{n} |M_{i,j}|$. We then have $\|I_d\| = 1$. Furthermore, for all S, we have:

$$\|\frac{f'(<\alpha, S>)}{V_0}\Lambda\pi(S)\| = \frac{1}{V_0} f'(\frac{<\alpha, S>}{V_0}) \max_i \frac{\alpha_i}{\lambda_i} \sum_{j=1}^{n} \alpha_j S^j$$

$$= \frac{<\alpha, S>}{V_0} f'(\frac{<\alpha, S>}{V_0}) \max_i \frac{\alpha_i}{\lambda_i} \leq \sup |x f'(x)| \max_i \frac{\alpha_i}{\lambda_i}$$

which is stricly below 1 in our hypotheses.

A.1 Lemma 4.2

We show Lemma 4.2 with less restrictive assumptions: Lemma 4.2 holds if $f \in C_b^1$ and there exists $\beta > 2$ such that $\mathbb{E}(|\xi|^\beta) < \infty$.
We can write:

$$|S_{k+1}^i - S_k^i| \leq |S_k^i| \left(\sqrt{\tau} |\xi_{k+1}^i| + \frac{\alpha_i}{\lambda_i} \|f'\|_\infty (\sum_{j=1}^{n} \frac{\alpha_j}{V_0} S_k^j \sqrt{\tau} |\xi_{k+1}^j|) \right)$$

As a consequence, using the fact that $\mathbb{E}(|\xi|^\beta) < \infty$ and the convexity inequality:

$$(\frac{1}{n} \sum_{1 \leq i \leq n} \gamma_i)^\beta \leq \frac{1}{n} \sum_{1 \leq i \leq n} \gamma_i^\beta$$

we obtain:

$$\sup_{\|S\| \leq r} \mathbb{E}(|S_{k+1}^i - S_k^i|^\beta \ |S_k = S) \leq C \tau^{\frac{\beta}{2}}$$

where $r > 0$ and C is a constant that does not depend on τ. Chebyshev's inequality gives us:

$$\sup_{\|S\| \leq r} \mathbb{P}_k(|S_{k+1}^i - S_k^i| \geq \epsilon) \leq \sup_{\|S\| \leq r} \frac{\mathbb{E}_k(|S_{k+1}^i - S_k^i|^\beta)}{\epsilon^\beta} \leq C \frac{\tau^{\frac{\beta}{2}}}{\epsilon^\beta}$$

for all $\epsilon > 0$. Finally, as $\frac{\beta}{2} > 1$, we find that:

$$\lim_{\tau \to 0} \sup_{\|S\| \leq r} \frac{1}{\tau} \mathbb{P}_k(|S_{k+1}^i - S_k^i| \geq \epsilon) = 0$$

A.2 Lemma 4.3

Let's calculate, for $1 \leq i \leq n$:

$$\mathbb{E}(S_{k+1}^i - S_k^i | S_k = S) = \mathbb{E}\left(S^i (\sqrt{\tau} \xi_{k+1}^i + \frac{\alpha_i}{\lambda_i} (f(\frac{V_{k+1}^*}{V_0}) - f(\frac{V_k}{V_0}))) | S_k = S \right)$$

$$= S^i \mathbb{E}\left(\sqrt{\tau}\xi^i_{k+1} + \frac{\alpha_i}{\lambda_i}(f(\frac{V^*_{k+1}}{V_0}) - f(\frac{V_k}{V_0}))|S_k = S\right)$$

Writing $f(\frac{V^*_{k+1}}{V_0}) - f(\frac{V_k}{V_0}) =$

$$f^{'}(\frac{V_k}{V_0})(\sum_{j=1}^{n}\frac{\alpha_j}{V_0}S^j_k\sqrt{\tau}\xi^j_{k+1})+\frac{1}{2}f^{''}(\frac{V_k}{V_0})(\sum_{j=1}^{n}\frac{\alpha_j}{V_0}S^j_k\sqrt{\tau}\xi^j_{k+1})^2+\frac{1}{6}f^{'''}(\frac{D_{k+1}}{V_0})(\sum_{j=1}^{n}\frac{\alpha_j}{V_0}S^j_k\sqrt{\tau}\xi^j_{k+1})^3$$

and using the fact that f''' is bounded and $\mathbb{E}(|\xi|^3) < \infty$, we find that:

$$|\mathbb{E}(S^i_{k+1} - S^i_k|S_k = S) - \tau b_i(S)| \leq C'\tau^{\frac{3}{2}}$$

which gives us Lemma 4.3

A.3 Lemma 4.4

Let's calculate, for $1 \leq i, j \leq n$:

$$\mathbb{E}\left((S^i_{k+1} - S^i_k)(S^j_{k+1} - S^j_k)|S_k = S\right)$$

$$= S^i S^j \mathbb{E}\left((\sqrt{\tau}\xi^i_{k+1} + \frac{\alpha_i}{\lambda_i}(f(V^*_{k+1}) - f(V_k)))(\sqrt{\tau}\xi^j_{k+1} + \frac{\alpha_j}{\lambda_j}(f(V^*_{k+1}) - f(V_k)))|S_k = S\right)$$

Using the fact that $f(\frac{V^*_{k+1}}{V_0})-f(\frac{V_k}{V_0}) = f^{'}(\frac{V_k}{V_0})(\sum_{j=1}^{n}\frac{\alpha_j}{V_0}S^j_k\sqrt{\tau}\xi^j_{k+1})+\frac{1}{2}f^{''}(\frac{C_{k+1}}{V_0})(\sum_{j=1}^{n}\frac{\alpha_j}{V_0}S^j_k\sqrt{\tau}\xi^j_{k+1})^2$, that f' and f'' are bounded and that $\mathbb{E}(|\xi|^3) < \infty$ and $\mathbb{E}(|\xi|^4) < \infty$, we obtain that:

- $\mathbb{E}\left(\sqrt{\tau}\xi^i_{k+1}\sqrt{\tau}\xi^j_{k+1}|S_k = S\right) = \Sigma_{i,j}\tau$
- $\mathbb{E}\left(\sqrt{\tau}\xi^i_{k+1}\frac{\alpha_j}{\lambda_j}(f(\frac{V^*_{k+1}}{V_0}) - f(\frac{V_k}{V_0}))|S_k = S\right) = \frac{\alpha_j}{\lambda_j}f^{'}(\frac{<\alpha,S>}{V_0})\sum_{1\leq l\leq n}\frac{\alpha_l}{V_0}S^l\Sigma_{i,l}\tau + o(\tau)$
- $\mathbb{E}\left(\sqrt{\tau}\xi^j_{k+1}\frac{\alpha_i}{\lambda_i}(f(\frac{V^*_{k+1}}{V_0}) - f(\frac{V_k}{V_0}))|S_k = S\right) = \frac{\alpha_i}{\lambda_i}f^{'}(\frac{<\alpha,S>}{V_0})\sum_{1\leq l\leq n}\frac{\alpha_l}{V_0}S^l\Sigma_{j,l}\tau + o(\tau)$
- $\mathbb{E}\left(\frac{\alpha_i}{\lambda_i}(f(\frac{V^*_{k+1}}{V_0}) - f(\frac{V_k}{V_0}))\frac{\alpha_j}{\lambda_j}(f(\frac{V^*_{k+1}}{V_0}) - f(\frac{V_k}{V_0}))|S_k = S\right)$
  $= \frac{\alpha_i\alpha_j}{\lambda_i\lambda_j}(f^{'})^2(\frac{<\alpha,S>}{V_0})\sum_{1\leq l,p\leq n}\frac{\alpha_l\alpha_p}{V_0^2}S^lS^p\Sigma_{l,p}\tau + o(\tau)$

where $o(\tau)$ is uniform on every compact set K containing S. As a consequence, we have Lemma 4.4.

B Proof of Proposition 6.1

Theorem 4.1 gives us the dynamics of asset prices in the continuous-time limit:

$$\frac{dP_t^i}{P_t^i} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i$$

where W is a n dimensional Brownian motion, $(\sigma_{i,j}(P_t) = A_{i,j} + \frac{\alpha_i}{\lambda_i} f'(\frac{V_t}{V_0}) \sum_{1 \le k \le n} \frac{\alpha_k}{V_0} P_t^k A_{k,j}$ and $\mu_i(P_t) = \frac{\alpha_i}{2\lambda_i} f''(\frac{V_t}{V_0}) \sum_{1 \le j,l \le n} \frac{\alpha_j \alpha_l}{V_0^2} P_t^j P_t^l \Sigma_{j,l}$. As a consequence, for all $1 \le i \le n$, P_t^i is strictly positive. We also have for all $1 \le i \le n$, $\alpha_i > 0$ and so: $V_t = \sum_{1 \le k \le n} \alpha_k P_t^k$ is strictly positive. Let's focus on the dynamics of the fund's position.

$$dV_t = \sum_{1 \le i \le n} \alpha_i dP_t^i = \sum_{1 \le i \le n} \alpha_i P_t^i (\mu_i(P_t)dt + (\sigma(P_t)dW_t)_i)$$

Dividing this equality by V_t and denoting $x_t^i = \frac{\alpha_i P_t^i}{V_t}$, we have the following dynamics for V_t:

$$\begin{aligned}
\frac{dV_t}{V_t} &= \sum_{1 \le i \le n} x_t^i (\mu_i(P_t)dt + (\sigma(P_t)dW_t)_i) \\
&= \sum_{1 \le i \le n} x_t^i \mu_i(P_t)dt + \sum_{1 \le i \le n} x_t^i (\sigma(P_t)dW_t)_i \\
&= \sum_{1 \le i \le n} x_t^i \mu_i(P_t)dt + \sum_{1 \le i \le n} x_t^i (\sum_{1 \le j \le n} \sigma_{i,j}(P_t)dW_t^j) \\
&= \sum_{1 \le i \le n} x_t^i \mu_i(P_t)dt + \sum_{1 \le j \le n} \sum_{1 \le i \le n} x_t^i \sigma_{i,j}(P_t)dW_t^j
\end{aligned}$$

As a consequence, the instantaneous variance of the fund is $\sum_{1 \le j \le n} (\sum_{1 \le i \le n} x_t^i \sigma_{i,j}(P_t))^2$. The statement follows.

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