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A mathematical formalism for the Kondo effect in Wess-Zumino-Witten branes

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In the paper, we adapt our previous formalism for a mathematical treatment of branes to include processes, specifically the Kondo flow for Wess-Zumino-Witten (WZW) branes. In this framework, we give the precise mathematical definitions and formulate a mathematical conjecture relating WZW branes to nonequivariant twisted K theory in the case of the group SU(n). We also discuss regularization of the Kondo flow, thereby giving a first step toward proving our conjecture. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2746133]

I. INTRODUCTION

The aim of this paper is to give a setting which would allow a mathematically rigorous treatment of the theory of Wess-Zumino-Witten (WZW) D-branes. In particular, we apply (with some changes) the formalism developed in Ref. 26 to capturing the WZW D-brane picture. The theory of WZW branes has several components and has been previously worked out quite satisfactorily physically (see, e.g., Refs. 1, 2, 6, 12, 14, 16, 20, 32, 33, 37, and 39). The physical answer is that stable D-branes (at least on CFT level) in the level k WZW model corresponding to a group G are classified by a suitable twisted K-theory group[1]

$$K_\tau(G). \tag{1}$$

However, as is to be expected, it is not trivial to interpret this classification result mathematically. In fact, the present paper was motivated by a question of Mike Hopkins whether there is an analogous geometric interpretation of the twisted K-group $K_\tau(G)$ as the previous complete calculation of the equivariant twisted K-group $K^G_\tau(G)$ as the Verlinde algebra by Hopkins, Freed and Teleman[21] (the twisting τ corresponds to the level or the level shifted by the dual Coxeter number, depending whether or not supersymmetry is involved, respectively). Although physically the answer to this question is known as to be given by conformal field theory (CFT)-stable WZW D-branes, for the purposes of a mathematician, the answer must be revisited and stated in precise mathematical terms.

The goal of the present paper is to advance this program by stating at least a precise mathematical conjecture: We would like to conjecture that the group of charges of WZW branes is equal to the twisted K-group proposed, but at present there is no precise mathematical definition of one side of the equation, namely the group of charges of D-branes (other than, of course, a definition which would input the answer desired). In the previous paper,[26] the authors proposed a mathematical definition of D-brane charges, but this definition does not capture an important physical aspect of the WZW situation, namely, the Kondo flow of Affleck and Ludwig.[1,2] and

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[1]The groups (1) are now completely known for G a simple Lie group: Braun (Ref. 13), using an idea of Schafer-Nameki (Ref. 36), calculated the answer with the assumption that the Verlinde algera is a complete intersection ring. A proof removing that assumption was independently obtained by Douglas (Ref. 15).

Bachas and Gaberdiel.[7] These flows mean that we must allow, as a part of the formalism, transition theories which break conformal invariance on the boundary. Without such flows (or some other additional input, such as space time symmetries), the physical answer would not be the nonequivariant twisted K-group conjectured.

In the present paper, then, we develop a variant of the formalism of Ref. 26 which allows an axiomatization of the flows. Therefore, we present here a setting in which the group of D-brane charges is well defined. We do not prove in this paper that the group of charges is equal to the twisted K-group. Thus, our main result is formulating the conjecture. However, there is one concrete piece of the puzzle which we do fill in: even physically, currently little is known about the renormalization of the Kondo flows. In other words, not much is known about the nature of the flow. It is, for example, not known that the theories with broken boundary conformal invariance which arise along the flow are well defined field theories with convergent amplitudes. This question, in and of itself, is hard, and will also remain conjectural in the present paper. However, as evidence of the conjecture, we will construct rigorously mathematically one element of the conjectured structure of the theories along the flow, namely, the "deforming field" at each point of the flow. Because we do not construct the whole deformed theory, the precise setting in which this field will be constructed will have to be specified later. We should mention, however, that this does provide a firm starting point for a mathematical foundation of calculations, as with knowing the deforming field, vacua of the deformed theory are obtained as solutions of differential equations, and hence our result essentially puts the Kondo flow on equal footing with a renormalized perturbative quantum field theory (although the formalism is somewhat different).

To be even more concrete about what is to be done, we must understand the elements of the WZW model in more detail. First, there is the question of a mathematical formalism for CFT. In the present paper, we use the formalism proposed by Segal,[38] and written down in detail in Refs. 18, 26, and 27 using our formalism of stacks of lax commutative monoids with cancellation (SLCMC) (explained below in Sec. II). (Actually, for a CFT with one-dimensional anomaly, the formalism of Ref. 38 is sufficient; subtleties arise in the case of chiral CFT, which will come into play later.) The reason we choose the Segal formalism is that it is in some sense the most "maximalistic," i.e., it models most of the structure desired, and the axioms can be traced most naively, i.e., with least input, to fundamental physical principles. The reason the work of Refs. 18, 26, and 27 became necessary is that if one wants to capture rational conformal field theory and modular functors (which is relevant here), the notion of "equality" of some of the spaces involved becomes delicate and leads to two-category theory. In fact, Fiore[19] recently showed that even a simplified "cobordism" approach to CFT proposed in Ref. 38 leads to two-category theory when completely rigorized on this level. The axiomatizations of Refs. 18, 19, 26, and 27 proceed in the direction of the same philosophy, i.e., the axiomatization is "zero input," it uses only Segal's geometric principles and generic two-category formalism. Since developing this theory, the authors discovered that an alternate rigorous approach to modular functors was previously developed by Bakalov-Kirillov[8] following Turaev[42] (for specific work related to the present context, see also Andersen and Ueno.[3,4]). These formalisms, however, use more input in the sense that more sophisticated concrete axioms have to be known and stated upfront.

A major alternative of Segal's axioms are the axioms of vertex operator algebra, which is originally designed to model a part of the structure of the genus 0 part of a chiral CFT. Actually the axioms (and also terminology) vary somewhat, but the basic mathematical references are Refs. 11 and 22. Recently,[28,29] this formalism has been adapted to work for the genus 0 part of a conformal field theory with both chiralities. Extensions of information to higher genus has also been discussed in Refs. 24, 30, and 40. Other references include Refs. 9, 29, and 31 and many others. These works include discussions of boundary sectors, although we are not aware of any references discussing flows in this axiomatic setting. An important point is that all of these approaches set out to capture only parts of the "naive" structure one sees in the Segal approach. With all the recent advances mentioned, it is quite possible that in a suitable setting, both approaches can be proved to be equivalent, but as far as we are aware, no such theorems are known at present. On the other hand, it is also important to mention that the vertex operator algebra

(VOA) approach captures certain situations the Segal approach cannot, for basic substantive reasons. For example, the Segal approach, following the idea that CFT should be a "quantum mechanics of strings," uses state spaces which are Hilbert spaces, but if one wants to discuss CFT on spacetime with indefinite signature, and the Becchi-Rouet-Stora-Tyutin formalism (BRST), one definitely cannot use Hilbert spaces, and the "naive" conformal invariance of the Segal approach is too much to expect. This is the mathematical explanation of the one-loop divergence of bosonic string theory. On the other hand, these questions can be certainly discussed in the vertex operator formalism. Additionally, the vertex operator formalism is generally much more successful for the purpose of concrete calculations. It is also not known whether a conformal field theory in the sense of Segal always gives rise to a vertex operator algebra: this requires taking limits of "distributional type," whose existence is difficult to prove without additional assumptions. In this paper, we will also resort to the vertex operator formalism at one point, namely, when constructing the deforming field of the Kondo flow. Proving that the flow exists in the Hilbert space setting would require additional work which at the moment is conjectural.

Next thing to discuss is the WZW model, which will be described in Sec. IV, and which is to be our main example. As a physical theory, this CFT has been constructed in Ref. 44. There have been many developments since. Mathematically, the corresponding vertex operator algebra has been constructed by Frenkel and Zhu,[23] and the conformal blocks were constructed by Tsuchiya-Ueno-Yamada.[41] Physically, the spectrum of primary fields was derived by Felder-Gawedzki-Kupiainen.[17] Another important contribution is the free field realization of the WZW model, which is, including the nonzero labels, is described well in Ref. 25 The modular functor of the WZW model (in a sense equivalent to ours) have been formalized in Ref. 8. Although a formal proof that the WZW model satisfies Segal's axioms, as far as we know, has never been published, we believe that a proof can be pieced together from the sources cited, since the main point particular to the Segal approach is convergence of the vacua in the Hilbert space setting, which actually follows from the Coulomb gas realization (in fact, it shows that the vacua are "smooth" elements, which is a much stronger statement than Hilbert space convergence). We will return to this point in Sec. IV below. (Note: Here by "vacua" we mean vacua from the point of view of two-dimensional quantum field theory (QFT) on each individual worldsheet; from a string point of view, it is more common to call these elements "amplitudes," and reserve the term "vacuum" for the vacuum associated with the unit disk.)

Nevertheless, checking all the details of the comparisons of the relevant setups in all of the points needed to show that the WZW model is a Segal CFT is a subtle proposition, which we feel is somewhat independent of the main subject of this paper, which is a mathematical axiomatization of the flows. For this reason, the discussion of the Segal axioms for the WZW in this paper stops short of a mathematical theorem, and technically, the statement remains a conjecture. We believe that the remaining steps are routine, and, additionally, really belong to other authors. On the other hand, we do describe, in any rational conformal field theory (RCFT), a construction of open sector Ishibashi and Cardy vacua, by adapting the constructions of Refs. 10 and 16 to the worldsheets considered in the Segal approach. In the context of general RCFT, Hilbert space convergence of these vacua follows from the convergence of the closed chiral vacua, which again in the case of the WZW follows from the Coulomb gas realization. Therefore, having constructed the Hilbert vacua, together with the published results[16] on open/closed modular functors, we have all the ingredients for proving that the Cardy branes form a consistent D-brane category in our precise mathematical sense. Again, however, we do not state this as a theorem, and technically only conjecture that the construction we describe satisfies all the axioms stated.

Next, one must discuss mathematically the axiomatization of the notion of D-brane. A mathematical notion of D-brane category from our viewpoint, extending Segal's axioms, has been given in Ref. 26 (there are variants, we shall explain in Sec. III below which variant we use here and why). An additional complication is that we do not have a mathematical interpretation for the question what *are* the D-branes of a given CFT. It is in principle possible that even if we fix a set of D-branes a and know each state space of open strings beginning and ending on a, the state spaces of open strings stretched between different branes could vary: in physical language, this is expressed by the statement that branes violate locality somewhat. Instead of conjecturing that the WZW D-brane charges are classified by the twisted K-theory (1), one must therefore really conjecture that there exists a consistent system of D-branes whose charges are classified by the conjectured group.

To be precise, we must again distinguish whether supersymmetry (SUSY) is present. When we work with the supersymmetric WZW model, the twisting in Eq. (1) is simply by k times the canonical generator of the group of twistings. This case is more interesting physically, as one can then (at least for appropriate values of the central charge) embed the WZW model into an actual critical superstring theory and consider nonanomalous string branes. However, axiomatizing supersymmetric CFT and superstring theory mathematically brings even more complications (although we do not feel substantial new concepts are needed); for this reason, in the present paper, we suppress SUSY and consider only the bosonic WZW model. The conjectured groups of charges in the supersymmetric and nonsupersymmetric cases run in parallel, with the twisting shifted. Thus, in the bosonic case (which we consider for reasons of simplicity), the twisting τ will again be by $k+h^{\vee}$ the canonical generator where $h^{\vee}$ is the dual Coxeter number [equal to n when $G=\mathrm{SU}(n)$].

To go further, we shall introduce yet another restriction: instead of trying to interpret the entire group (1), we will only attempt to interpret its subgroup corresponding to "Cardy branes." In the case of $G=\mathrm{SU}(n)$, the answer should then be

$$\mathbb{Z}/((k+n)/gcd(k+n,lcm(1,\ldots,n-1))), \tag{2}$$

where k is the level. The full group (1) is Eq. (2) tensored with an exterior algebra, the augmentation ideal of which conjecturally corresponds to "higher" branes. In the Cardy case, however, a much more explicit physical theory is available, and so more concrete mathematical conjectures can be stated (for example, explicit state spaces of open CFT string sectors are known in the Cardy case; in the higher case, one encounters questions such as whether higher homology classes are represented by submanifolds).

To be more precise about the sense in which we are using the term Cardy brane, as already hinted, a part of the statement about the WZW model being a CFT is that it is, in fact, a RCFT, i.e., is obtained by a certain canonical procedure from a unitary chiral CFT with modular functor. Now Cardy[14] has proposed a general approach to $D0$-branes in RCFT. Mathematically, Cardy has not defined all the structure which is required by the Segal formalism described in Sec. III, but has defined enough of it to make it convincing that his proposal is undoubtedly right. In fact, as already mentioned, we show in Sec. IV that the Cardy branes do, in fact, fit our formalism by constructing the open string vacua, so the situation is still quite satisfactory so far.

Looking at Cardy branes only, however, one does not obtain the group (2); rather, Cardy's branes are classified by the Verlinde algebra. To obtain the answer (2), one must consider *continuous deformation* of Cardy branes. The mathematical situation is substantially less satisfactory here. What we can describe rigorously mathematically (at least modulo proving convergence of certain integrals) is *infinitesimal* deformation along primary fields in the open sector. This is analogous to infinitesimal deformation of bulk CFT as described by Segal in Ref. 38 (see also Ref. 26). We describe this construction in Sec. V below. This construction actually has the remarkable property that when we infinitesimally deform a brane b, it automatically updates also all the open string sectors, including mixed sector between b and other branes. Therefore, although we still do not have a precise interpretation of the question as to what "are" branes, the corresponding linearized infinitesimal question does make sense.

Physically, however, infinitesimal deformations are not enough. One is interested in finite deformations; intermediate steps are deformations "to perturbative level," which means given by Taylor expansion in the deformation parameter. Here, however, one hits an obstruction. Namely, it turns out that the infinitesimal boundary deformations of CFT D-branes cannot be always be continued while preserving full conformal invariance of the boundary sectors. Therefore, we need to look for ways to break conformal symmetry on the boundary. There is, fortunately, a relatively easy way of doing so, by defining worldsheets where D-brane components are also parametrized like string components (with some minor modifications). The set of such worldsheets has the same formal properties as the set of closed/open CFT worldsheets, and our formalism therefore allows us to define CFT with such source worldsheets (i.e., with conformal invariance broken on the boundary). In this more relaxed setting, one can again ask if infinitesimal deformations of a brane a by fields in the open string sector K_{aa} can be "exponentiated," i.e., if there exist paths of such theories where at a given point derivatives of the vacua by the path parameter are equal to the values prescribed by the infinitesimal deformation.

Affleck and Ludwig[1,2] conjectured that a particular such paths connects certain Cardy D-branes in the G-WZW model. More specifically, for example, if we consider an irreducible representation V with weight λ, the corresponding brane should be connected with the sum of $|V|$ copies of the brane corresponding to the trivial irreducible representation by exponentiating, in the latter boundary sector, the field

$$\sum_{\alpha} S_{\alpha} J_{-1}^{\alpha}$$

where α are generators of the Lie algebra g corresponding to G and S_{α} are the corresponding elements of $Hom(V,V)$ given by the representation. This is related to the Kondo effect modeling magnetic impurities in superconductors and other metallic materials. This proposal is the basis of the answer (2). It is interesting to note that the Affleck-Ludwig proposal actually models an effect which can be observed in the laboratory.

Although we do not prove the Affleck-Ludwig conjecture here, we work out this example in substantial detail, and actually make progress in substantiating the process mathematically, by constructing rigorously the whole perturbative expansion of the deformation field of the process. To make treatment of convergence easier, we do this calculation in the vertex operator setting and not the Hilbert space setting. We work by induction on degree of the perturbative parameter, and inductively prove an estimate for the fields involved. The induction is complete in the sense that we are able to then prove the same estimate at the next power, so the perturbative expansion of the deformation field is rigorously defined. We do not, however, prove that the resulting vacua are represented by trace class elements in Hilbert spaces, and do not prove that the process has the λ-brane limit predicted by the Kondo effect.

Our main point, then, is to propose in Sec. VII an axiomatization of *topological space D-brane category*, which is compatible with the formalism described in Sec. III. This way, one can conjecture that there exists such category whose set of objects has the structure of a manifold whose tangent space is isomorphic to the space of weight one fields considered in Sec. VI, and that contains the Cardy branes, and its set of connected components is Eq. (2).

The present paper is organized as follows. In Sec. II, we review our formalism for axiomatizing CFT. In Sec. III, we review how this formalism applies to branes. In Sec. IV, we review Cardy branes, boundary states, and the case of the WZW model. In Sec. V, we describe infinitesimal deformation of branes. In Sec. VI, we consider the Kondo effect, conformal symmetry breaking on the boundary, and deformations to perturbative level. In Sec. VII, we give our formalism for topological D-brane category and state a rigorous conjecture regarding the group (2).

II. PRELIMINARIES: OUR FORMALISM FOR CONFORMAL FIELD THEORY

Our formalism for conformal field theory, rigorizing all aspects of the definition given in Ref. 38, can be found in Refs. 18, 26, and 27. The whole treatment requires a substantial amount of detail which we do not want to reproduce in the present paper. Because of this, we limit ourselves to an informal review. The reader may refer to Refs. 18, 26, and 27 for a fully detailed discussion.

The main idea of our approach to quantum conformal field theory[26,27] is that we noticed that conformal field theory, which one usually thinks of as a fundamental object, is actually a morphism of two specimens of the same structure:

$$\mathcal{C} \to \mathcal{V}. \tag{3}$$

The structure is called SLCMC (the acronym and its meaning will be explained below). The source SLCMC $\mathcal{C}$ in Eq. (3) consists usually (but not necessarily) of some type of "surfaces with additional data," while the target SLCMC $\mathcal{V}$ in Eq. (3) consists of some type of "state spaces," usually vector spaces with some additional data.

It should be pointed out that Eq. (3) axiomatizes "conformal field theories" in a fairly general sense, which exceeds the generality of the meaning in which the term is used physically: the same formalism indeed describes conformal field theories in the classical sense, modular functors, conformal field theories with modular functors, closed/open conformal field theories with various grades of anomaly allowed, etc. In other words, it is an axiomatic setting where conformal invariance (if it makes sense at all in the context considered) can certainly be broken. We shall find that capability useful later in this paper. A few words are in order, however, on why use the attribute "conformal" at all? It may seem that an arbitrary quantum field theory is roughly of the form (3). The main restriction on our formalism is that we wish to use the language of *stacks*. This is a categorical mechanism which allows us to introduce families of objects, and thereby notions such as continuity, analycity, and holomorphy into the formalism in a fairly easy way, without leaving the realm of algebra. The use of stacks, if we want to use it without further elaboration, however, requires that the modulus space be roughly, at least locally a quotient of some (possibly infinite-dimensional) maniflod by a discrete group. This is typically not true when we break conformal invariance substantially (e.g., in general QFT). On the other hand, we will see that breaking conformal invariance on the *boundary* of open worldsheets only does not cause such difficulties, and hence can still be captured by our formalism. This is essential for describing the Kondo effect in WZW branes, see Secs. VI and VII below.

Let us be more specific about what we mean by SLCMC, then. The acronym stands for stack of lax commutative monoids with cancellation. As already hinted above, the "stack" part is used to capture continuous families only, so let us discuss that last. If we drop the stack attribute, we will see precisely its sections over a point, i.e., the algebraic structure alone (e.g., the set of all worldsheets), without looking at continuous families. Now such section of a stack over a point is thought of as discrete topologically. It is not, however, a set, but a category. Indeed, when looking at worldsheets which are, say, Riemann surfaces with parametrized boundary components (as will be understood when applying our approach to closed sector CFT), we must consider not only the worldsheets but their isomorphisms, which are holomorphic diffeomorphisms compatible with boundary parametrizations. Discrete automorphism groups occur.

This is related to what "lax" means: when discussing any algebraic structure in the context of categories, it is generally unreasonable to assume that algebraic identities (such as commutativity, associativity, etc.) hold precisely. They generally hold only up to natural isomorphisms, called *coherence isomorphisms*, but those must in turn satisfy certain commutative diagrams (called *coherence diagrams*). There is a general formalism for how. to form such diagrams, which is discussed in detail in Refs. 18, 26, and 27. This is in general what we mean by the word lax: it means up to natural isomorphisms, satisfying canonical coherence diagrams. It should be pointed out that terminology unfortunately varies somewhat. In the context of category theory, what we call lax is usually labeled by the prefixes "pseudo" or "bi" (see Ref. 18 for a dictionary of terms).

Let us now, finally, discuss the algebraic structure we introduce, commutative monoids with cancellation. What is this structure, and why do we introduce it? The answer is that this is precisely the structure which describes the "stringy" aspect of quantum conformal field theory, which means, on the closed worldsheet level, disjoint union and gluing of outbound and inbound boundary components, and on the state space level usually some type of tensor product and trace. This algebraic structure is perhaps somewhat unusual to algebraists, in that it has "dynamically indexed" operations: instead of a fixed set of operations, we have variable "sets of inbound and outbound boundary components," and the gluing operations depend on these sets. Algebraically, the best approach to this is to keep the concepts involved as abstract as possible. This means, a commutative monoid with cancellation consists of a commutative monoid T (often, on the lax level which we are interested in, it will simply be the category of finite sets), and for each pair of elements $s,t \in T$ (sets of inbound and outbound boundary components), a set $X_{s,t}$. The disjoint union operation is then a product

$$X_{s,t} \times X_{u,v} \to X_{s+u,t+v}, \tag{4}$$

and the gluing is a "unary" operation of the form

$$X_{s+u,t+u} \to X_{s,t}. \tag{5}$$

A unit (empty worldsheet) $0 \in X_{0,0}$ is also an operation (constant). These operations satisfy obvious axioms (commutativity, associativity, distributivity), which are described in detail in Refs. 26 and 27. There, we also list the procedure for obtaining coherence diagrams on the lax level.

An important observation to make is that lax algebraic structures form a two-category. This means that we have the objects, one-morphisms, which are functors laxly preserving the algebraic structure, and two-morphisms, which are natural transformations of such functors, still compatible with the algebraic structure. The simplest example of a two-category is the category of categories (which is the same thing as "lax sets"): objects then are categories, one-morphisms are functors and two-morphisms are natural transformations.

Let us now return to the stacks. Stacks are "lax sheaves." This means that we must specify a site, which is a category $\mathcal{G}$, with a Grothendieck topology, which means certain tuples of morphisms called coverings (generalizing colimits). The category specifies objects b over which we wish to index "continuous families." The stack then specifies the "category of sections" over each object b. A typical example for chiral CFT is the category of complex manifolds where coverings are open covers. Sections over a complex manifold b are then holomorphic families of worldsheets, parametrized over b. In nonchiral cases (such as physical CFT or closed/open CFT), we do not have a notion of holomorphy, so we must "weaken" the category $\mathcal{G}$. For the purposes of this paper, we shall then consider just the category of real-analytic manifolds with Grothendieck topology by open covers.

Now given a site, stacks can be considered with values in any two-category which has lax limits (see Ref. 18 for details). But this is true for any category of lax algebras, such as lax commutative monoids with cancellation. This is why the notion of SLCMC is possible. The main axiom of a stack is that it be a lax contravariant functor form $\mathcal{G}$ into the two-category, which takes Grothendieck covers to lax limits (a coherence diagram condition is also needed).

III. THE FORMALISM FOR STATIC BRANE THEORY

Again, this section is a review of the axiomatization of D-brane categories given in Ref. 26, and the reader should refer to that paper for a full discussion of points which are only outlined here.

There are two boundary CFT formalisms discussed in Ref. 26. In the more general setup, we discussed a notion of D-brane category where both the closed and open sectors can have a finite-dimensional anomalies, which means that there is a finite-dimensional vector space of vacua. This means that we had to introduce *labels* in both the open and closed string sectors. Labels in the open string sector are not the same as D-branes; rather, each D-brane can have multiple labels. This ultimately led to a three-vector space formalism for D-brane categories in Ref. 26.

From the point of view of physics, however, the smaller the anomaly, the more interesting the model is. In fact, ultimately, we would like the anomaly to vanish. This, however, can be only discussed in the framework of full-fledged superstring theory. In conformal field theory alone, the best we can hope for is *one-dimensional anomaly* in both the open and closed sectors. All the examples discussed in the present paper will have one-dimensional anomaly in both sectors.

Now one-dimensional anomaly allows us to decrease the level of category theory involved by 1, if we apply a "fermionic" approach to the anomaly, i.e., we consider the vacuum as an element of the corresponding projective space of the state space involved. In the closed sector, then, no two-category theory is needed (no labels); in the open sector, we can make do with a two-vector space $\mathcal{B}$ of D-branes. To be more precise, let us first assume $\mathcal{B}$ is a finite-dimensional free two-vector space. Let us recall that a two-vector space is a lax module (in the sense outlined in the previous section, see Refs. 18 and 26 for details) over the lax commutative semiring (which is the same thing as a symmetric bimonoidal category) of finite-dimensional vector spaces and homomorphisms. One can then, by virtue of a general formalism in two-category theory,[18] form lax functors such as $?\otimes_{\mathbb{C}_2}?$, $Hom_{\mathbb{C}_2}(?,?)$. Since in CFT one needs to talk about Hilbert spaces, one can also form the lax commutative algebra $\mathbb{C}_2^{Hilb}$ over $\mathbb{C}_2$ of Hilbert spaces, where the product is the Hilbert tensor product. The notation $(?)^{Hilb}$ then means $?\otimes_{\mathbb{C}_2}\mathbb{C}_2^{Hilb}$.

A finite-dimensional free two-vector space (which is what we assumed about $\mathcal{B}$ here) can be visualized simply as a product of finitely many copies of $\mathbb{C}_2$: this is the "alias," or fixed base, interpretation of the "alibi," or functorial definition we are giving here (see more comments below).

Now we shall have a *closed string state space* $\mathcal{H}$ and an *open string state space* $\mathcal{K}\in \mathrm{Obj}(\mathcal{B}\otimes_{\mathbb{C}_2}\mathcal{B}^*)_{Hilb}$. Here

$$\mathcal{B}^* = Hom_{\mathbb{C}_2}(\mathcal{B},\mathbb{C}_2),$$

and the fact that $\mathcal{B}$ is finite and free implies a functor

$$\mathrm{tr}:\mathcal{B}\otimes_{\mathbb{C}_2}\mathcal{B}^* \to \mathbb{C}_2.$$

(Note that it is a matter of discussion if the open sectors should be Hilbert spaces, or if some other type of Banach space models them better; however, we shall not go into that here.)

Now we define a closed/open CFT as a morphism of SLCMC's:

$$\mathcal{S}\to \mathcal{C}(\mathcal{B},\mathcal{K},\mathcal{H}). \tag{6}$$

Here all SLCMC's are stacks over the site of real-analytic manifolds and open covers. Over a point, the LCMC's have the underlying lax commutative monoid the category of finite sets labeled by two labels *closed*, *open*, i.e., maps of finite sets

$$S\to \{\mathrm{closed},\mathrm{open}\}.$$

When referring to an object $X_{b,c}$ of such SLCMC, we shall call b respectively, c the set of inbound respectively outbound boundary components. To make the definition (6) correct, there is one other subtlety (aside from the fact, that we, of course, are yet to define the right hand side). The point is, there are different types of arrangements of open and closed boundary components, and the morphism (6) must remember this data. This is taken care of by defining[26] an auxiliary SLCMC Γ. For lack of a better term, we call its objects *graphs*, although they are, in fact, a rather special kind of graphs: these graphs are only allowed to have discrete vertices and components which are circles. The discrete vertices are labeled closed string inbound, outbound, and brane, and circles can have any number of vertices $\geqslant 1$, are oriented, and their edges (corresponding to open string boundary components) are additionally labeled inbound and outbound. (Note that what we refer to as D-brane boundary component is sometimes in the literature referred to as free boundary.) Now it is clear how Γ is an SLCMC over the same lax commutative monoid of sets with elements labeled *inbound, outbound*, i.e., how the information it encodes behaves under gluing (just imagine some abstract worldsheet with boundary described by the graph). Consequently, we get a SLCMC.[26] Now the additional requirement on Eq. (6) is that both sides be equipped with morphisms of SLCMC's into Γ, and that the morphism (6) be *over* Γ, i.e., that the diagram formed by Eq. (6) and the two morphisms into Γ strictly commute.

Now the anomaly-free SLCMC $\mathcal{C}(\mathcal{B},\mathcal{K},\mathcal{H})$ over the SLCMC of graphs Γ has sections over a point and over p_{in}, p_{out} closed inbound and outboud components, q pure D-brane components, $s_1,\ldots,s_\ell$ open string components on ℓ mixed boundary components are points of

$$\bigotimes_{i=1}^{p_{\mathrm{in}}} \mathcal{H}^* \otimes \bigotimes_{i=1}^{p_{\mathrm{out}}} \mathcal{H} \otimes \bigotimes_{i=1}^{\ell} \mathrm{tr}_{\mathrm{cyc}}\left(\bigotimes_{j=1}^{s_i} \mathcal{K}^{(*)} \right). \tag{7}$$

Here we supress completion from the notation (always taking trace class elements in the sense of Ref. 26 in the Hilbert tensor product), and the $^{(*)}$ superscript means that dual is assigned to those string components which correspond to inbound open string components. The symbol $\mathrm{tr}_{\mathrm{cyc}}$ means that we take trace as many times as we have D-brane components on each mixed boundary component; in the end, Eq. (7) is just a Banach space (subset of a Hilbert space), and does not have any labels. Actually, the dual in the open sector requires further discussion. Let us, for simplicity, work with objects of two-vector spaces, the Hilbertization case follows analogously. Assume, therefore, we have

$$x \in \mathrm{Obj}(\mathcal{B}). \tag{8}$$

We claim that we have a canonical object

$$x^* \in \mathrm{Obj}(\mathcal{B}^*) \tag{9}$$

together with a canonical morphism

$$x \otimes x^* \to 1, \tag{10}$$

where

$$1 \in \mathrm{Obj}\, \mathcal{B} \otimes \mathcal{B}^* \simeq Hom(\mathcal{B},\mathcal{B})$$

is the identity (note that the second equivalence follows from the assumption that $\mathcal{B}$ is finite and free).

Assuming for the moment that for Eq. (8), we have Eq. (9) with Eq. (10), by $\mathcal{K}^* \in \mathrm{Obj}(\mathcal{B} \otimes \mathcal{B}^*)$ we mean the coordinatewise dual; this would lie in $\mathrm{Obj}(\mathcal{B}^* \otimes \mathcal{B})$, so we switch the coordinates, so they appear in the same order as in the inbound components: $\mathrm{Obj}(\mathcal{B} \otimes \mathcal{B}^*)$. Then, we can take the cyclic trace regardless on which open string components are inbound and which are outbound.

The sections (7) are "stacked" in the usual way, we skip that discussion (see Ref. 26). The only thing to point out is that, as usual, open worldsheets do not form a complex manifold, so we cannot have stacks of holomorphic sections, the best we can do is *real-analytic* sections.

A note is, however, in order on gluing of the sections (7). That requires the following additional construction. Consider the following diagram:

$$\begin{array}{ccccc} \mathcal{B} & & & & \mathcal{B}^* \\ & \mathcal{B}^* & & \mathcal{B} & \\ & \Big|\ \mathcal{K} & & \mathcal{K}^*\ \Big| & \\ & \mathcal{B} & & \mathcal{B}^* & \\ \mathcal{B}^* & & & & \mathcal{B}. \end{array} \tag{11}$$

The diagram is to be read as follows: Along each solid line, we take the object denoted, and along each dotted line, we take the trace (=evaluation). We need to explain how the diagram (11) maps naturally to the diagram

$$\begin{array}{c} \mathcal{B} - - - - - - - \mathcal{B}^* \\ \\ \mathcal{B}^* - - - - - - - \mathcal{B}. \end{array} \tag{12}$$

Note that by the triangle identity,

$$\mathcal{B} - - - \mathcal{B}^* \overset{1}{\text{——}} \mathcal{B} - - - \mathcal{B}^*$$

is canonically isomorphic to

$$\mathcal{B} - - - \mathcal{B}^*.$$

Therefore, it suffices to exhibit a canonical map from

$$\begin{array}{ccc} \mathcal{B}^* & & \mathcal{B} \\ \Big| \mathcal{K} & & \mathcal{K}^* \Big| \\ \mathcal{B} & & \mathcal{B}^*, \end{array}$$

into

$$\begin{array}{c} \mathcal{B}^* \overset{1}{\text{——}} \mathcal{B} \\ \\ \mathcal{B} \overset{1}{\text{——}} \mathcal{B}^* \end{array}$$

but that follows from Eq. (10).

To construct, from Eqs. (8) and (9) with Eq. (10), we consider x as a functor

$$x:\mathbb{C}_2 \to \mathcal{B},$$

and let

$$x^*:\mathcal{B} \to \mathbb{C}_2$$

be its right adjoint. Recall here that a functor G is right adjoint to F or equivalently F is left adjoint to G if we have a natural bijection

$$Hom(x,Gy) \cong Hom(Fx,y). \tag{13}$$

Equation (10) then follows from the so called triangle identities for adjoint functor, which are direct consequences of Eq. (13). To be more specific, the counit of the adjunction is of the form

$$x \otimes x^*(y) \to y,$$

which, under the equivalence

$$\mathcal{B} \otimes \mathcal{B}^* \simeq Hom_{\mathbb{C}_2}(\mathcal{B},\mathcal{B}), \tag{14}$$

is

$$x \otimes x^* \to Id,$$

which is Eq. (10).

As pointed out above, in order to have CFT examples, we need to allow one-dimensional anomaly. This is done by introducing the SLCMC

$$\tilde{\mathcal{C}}(\mathcal{B},\mathcal{K},\mathcal{H}),$$

whose sections over a point consist of a complex line L and a linear map from L to the space of sections of $\mathcal{C}(\mathcal{B},\mathcal{K},\mathcal{H})$. Since gluing is linear, this behaves well with respect to gluing. Sections over an open set in the category of real-analytic manifolds are again defined in the usual way (see Ref. 26). (There is also an adjoint approach where one sticks to the nonanomalous SLCMC in the target, and uses a "$\mathbb{C}^{\times}$-central extension" of the source SLCMC, see Ref. 27.)

Up to now in this section, and also in Ref. 26, we have tried to express the formalism for closed and open CFT's in alibi form, with as much functoriality as possible. However, for various reasons, it is also useful to have an alias interpretation, i.e., to express everything in terms of bases. In the current setting, this means that we write $\mathcal{B}$ explicitly as a free two-vector space on a finite set of elementary D-branes B. We can then think of the open state space as simply a set of Hilbert spaces

$$\mathcal{K}_{ab} \tag{15}$$

of states of outbound open string beginning on the D-brane a and ending on the D-brane b. One finds that this is not necessarily symmetric in a and b. The state space of an inbound open string whose string beginning point is on a and string endpoint is on b is then

$$\mathcal{K}^{*}_{ab}. \tag{16}$$

This is easily seen to coincide with the above interpretation; in particular, it makes gluing work.

One may wonder why introduce the more complicated alibi interpretation above. The answer is that it is appealing to follow the philosophy of "replacing sets with k-vector spaces with $k \in \{0,1,2,\ldots\}$ as low as possible. In mathematics, it is known that a stable theory k-vector spaces is v_k-periodic.[5] For $k=0$, it means that the theory is a kind of ordinary cohomology, for $k=1$ a kind of K-theory, and for $k=2$ a kind of elliptic cohomology. The functoriality we detect therefore predicts that "charges" of D-branes in a D-brane category lie in a kind of K-theory, as discovered by Witten (see Ref. 45). The functoriality discovered in Ref. 26, for example, predicts that a D-brane category with higher-dimensional anomaly (a modular functor associated with open string sectors) has charges in a kind of elliptic cohomology.

Unfortunately, this kind of connection is hard to make precise without somehow "group completing" the category of k-vector spaces. This is because for $k \geqslant 2$, the category itself has too few isomorphisms, as is well known (see Ref. 5). In Ref. 26, we propose a way of group completing the category of vector spaces using topology. This predicts that modular functors with values in supervector spaces will be needed to fully understand the connection of CFT's with elliptic cohomology. One already knows that in the open sector, such formalism will be needed to fully understand anti-D-branes.

In this paper, we do not discuss the group completion of the category of vector spaces, but topology does come into play when we introduce a mathematical formalism for the Kondo effect in WZW branes.

IV. AN EXAMPLE: CARDY BRANES

Cardy's theory[14] considers a RCFT, which is a conformal field theory of the form

$$\mathcal{H} = \bigoplus_{\lambda} \mathcal{H}_{\lambda} \otimes \bar{\mathcal{H}}_{\lambda}. \tag{17}$$

Here, the tensor product is Hilbert tensor product, and $\bar{\mathcal{H}}$ denotes the complex conjugate Hilbert space (which is canonically isomorphic to the dual Hilbert space $\mathcal{H}^{*}$). The collection of Hilbert spaces $\mathcal{H}_{\lambda}$ is a closed chiral CFT in alias notation, i.e., the two-vector space of labels is

$$\mathcal{M} = \bigoplus_{\lambda} \mathbb{C}_2.$$

Recall from Ref. 26 that our formalism for modular functor and closed chiral CFT is given by morphisms of SLCMC's,

$$\mathcal{C} \to \mathcal{C}(\mathcal{M}),$$

$$\mathcal{C} \to \mathcal{C}(\mathcal{M},\mathcal{H}),$$

respectively, where $\mathcal{C}$ is the SLCMC of closed worldsheets (over the site of complex manifolds and open covers; over a point, in the closed case, all SLCMC's can be taken to have as the underlying lax commutative monoid the category of finite sets. It becomes more precisely covering spaces in the stack context.) Here, $\mathcal{M}$ is a finite free two-vector space of *labels*. The category of sections of the SLCMC $\mathcal{C}(\mathcal{M})$ over a point and over sets of p inbound and q outbound components is

$$\mathcal{M}^{*\otimes p} \otimes \mathcal{M}^{\otimes q}.$$

Now choose $\mathcal{H} \in \mathrm{Obj}(\mathcal{M}^{Hilb})$. To define the SLCMC $\mathcal{C}(\mathcal{M},\mathcal{H})$, we let the sections over a point and over p inbound and q outbound elements be the category of sections M of $\mathcal{C}(\mathcal{M})^{p,q}$ together with trace class morphisms

$$M \to \mathcal{H}^{*\otimes p} \otimes \mathcal{H}^{\otimes q}.$$

Here, the tensor product on the right means Hilbert tensor product, and trace class means that there exist bases in all tensor factors with respect to which, when we expand the given element, we obtain a convergent norm sum (as opposed to just quadratic convergent). See Ref. 26

We may then consider

$$(\mathcal{H}_\lambda) \in \mathrm{Obj}(\mathcal{M}^{Hilb})$$

as in Ref. 26 We assume that the the modular functor satisfies, in the alias notation,

$$M_{\bar{X}} \cong \overline{M_X} \tag{18}$$

(where M_X is the value of the modular functor on a worldsheet X) subject to the condition that Eq. (18) is an isomorphism of modular functors; here X is a closed worldsheet with labeled boundary components, and $\bar{X}$ is the complex-conjugate worldsheet with labels replaced by contragredient labels. We also assume that there is a condition on vacua. Namely, we also require "reflection positivity," i.e., that, in the alias notation, for $\mu \in M_X$,

$$U_{X,\mu} = \overline{U_{\bar{X},\mu}} \tag{19}$$

where $U_{X,\mu}$ is the vacuum of the labeled worldsheet X assigned to a given element of the modular functor M_X. (Note that in $\bar{X}$, the outbound boundary components of X become inbound and vice versa, which makes the target Hilbert spaces on both sides of Eq. (19) dual, hence complex conjugate.) In Eq. (17), we consider $M_{\bar{X}}$, $\bar{\mathcal{H}}_\lambda$ functors not in $\bar{X}$ but in X, thereby making them antiholomorphically dependent on X (or "antichiral," or "right moving"). To make Eq. (17) a CFT with one-dimensional anomaly, more is, however, needed. Namely, one needs a nondegenerate pairing

$$M_X \otimes \overline{M_X} \to L_X, \tag{20}$$

where L_X is a one-dimensional modular functor, subject to the condition that Eq. (20) is a morphism of modular functors (see Ref. 38).

It is conjectured that the level k WZW model on SU (n) is an example. Although a proof has not been published, the authors believe that a proof is essentially contained in the literature. Looking at the chiral part, for example, a vertex operator algebra version of this theory has been constructed by Frenkel and Zhu.[23] The modular functor, in their own formalism, has been treated in detail by Bakalov-Kirillov.[8] The main question is whether the worldsheet vacua converge in Segal's formalism which we use here. However, the level k WZW model is a subquotient of a free field theory by Ref. 25. For free field theories, it is known that the scaled vertex operators corresponding to worldsheets are smooth (Ref. 27), i.e., decay exponentially with weight. Using gluing, this can be used to show that the worldsheet vacua, in fact, converge in the sense of the Segal model. Nevertheless, all the details related mostly to compatibility of the various approaches to CFT have not been checked, and we do not prove this conjecture here.

In the case of the level k WZW model for SU (n), the labels are indexed by $\lambda \in P_k$, where P_k is a set enumerating level k irreducible lowest weight representations of the universal central extension $\widetilde{L}\mathrm{SU}(n)$ of the loop group LSU (n), and $\mathcal{H}_\lambda$ is the representation corresponding to λ. We refer the reader to Ref. 35 for details on loop groups.

Now Cardy[14,37,20,32,33] gives arguments in special cases which can be generalized to say that a RCFT arising in the sense of Segal[38] from a chiral sector which is a CFT with (finite) modular functor, always has the following D-brane category: the two-vector spaces $\mathcal{B}$ of branes is equal to the two-vector space of the closed labels λ of its chiral theory. $\mathcal{B}$ is free on a set of "elementary branes" B (in the case of the WZW model, $B=P_k$). Furthermore, using the formalism of boundary states, Cardy states that the open sectors in alias notation should be

$$\mathcal{K}_{\lambda\mu} = \bigoplus_{\kappa} N^{\kappa}_{\lambda\mu^*}\mathcal{H}_{\kappa}. \tag{21}$$

Here, μ^* denotes the contragredient label to μ, $N^{\gamma}_{\alpha\beta}$ is the fusion rule, i.e., the dimension of the chiral modular functor on a pair of pants with two inbound boundary components labeled by α,β and one outbound boundary component labeled by γ.

Again, however, a construction of the open vacua in Segal's approach has not been published in the literature, although related constructions are given in Ref. 10 and 16 in Ref. 10 the construction of vacua is approached physically in the language of Polyakov path integrals and string determinants, Ref. 16 relates rigorously mathematically boundary charges to three-dimensional topological quantum field theory (following the closed worldsheet treatment of Ref. 42). Ref. 16 however, does not discuss CFT vacua. The construction of the open vacua can be outlined as follows: Consider an open worldsheet Σ. First, assume for simplicity that there are no closed string boundary components. Although the D-brane components are not parametrized, we can nevertheless use them to glue canonically Σ to its complex conjugate $\bar{\Sigma}$, thus turning it into a closed worldsheet $\Sigma\bar{\Sigma}$ (the D-brane boundary components are used up in the gluing, the open string components turn into closed ones). To be more precise, if we denote by $\partial_b(\Sigma)$ the union of the D-brane components of Σ, then there is a canonical diffeomorphism

$$\partial_b(\Sigma) \cong \partial_b(\bar{\Sigma}) \tag{22}$$

[recalling the construction of the complex conjugate complex manifold, the underlying manifold of $\bar{\Sigma}$ can, in fact, be taken to be equal to the underlying manifold of Σ, in which case the map (22) is the identity]. Thus, Eq. (22) provides a canonical recipe for gluing Σ and $\bar{\Sigma}$. Now specifying the gluing map endows $\Sigma\bar{\Sigma}$ with a canonical complex structure. In the case of analytic boundary components (which we can assume here), this is an elementary direct construction, which can be extended to nonanalytic smooth boundary components (and even further) by Ahlfors-Bers theory: this is the basic gluing operation of Segal's approach to CFT, although we are using it in a different way here, gluing the brane rather than string components on the boundary.

Remark: A reader familiar with Ref. 16 may find this confusing, since the construction given there appears slightly different. However, the point is that Ref. 16 proceeds in the language of correlation functions, which means that string components of the boundary are not present at all, and are replaced by punctures, which are labeled points on the worldsheet. Thus, the entire boundary of the worldsheet as considered in Ref. 16 consists in fact, of brane components. A labeled point in the interior of the worldsheet in Ref. 16 corresponds in our language to a closed string component, a labeled point in on the boundary in Ref. 16 corresponds to an open string component. With this translation, our construction is exactly the same as the doubling construction of Ref. 16 in the limit where string components degenerate to points. Even more directly, one can relate our construction to that of Ref. 16 without taking limits by gluing (in our sense, following Segal) standard disks (respectively, half-disks) with labeled point in the origin to all closed (resp. open) string components of our worldsheet, thus obtaining a worldsheet in the sense of Ref. 16 This operation then converts our doubling construction exactly to the doubling construction of Ref. 16.

In any case, this gives us a map

$$M_{\Sigma\bar{\Sigma}} \to \otimes \mathcal{H}_{\kappa_i}^{(*)}. \tag{23}$$

Here κ_i, $i=1,\ldots,s$, are labels ($\in\mathcal{B}$) of the string components of Σ (also $\Sigma\bar{\Sigma}$), $^{(*)}$ as usual denotes dualizing wherever an inbound string component occurs, and M, as above, is the modular functor of the corresponding chiral theory.

The idea is to use the map (23) for defining open string vacua, and to show that when we take the particular linear combinations (21) of the $\mathcal{H}_\kappa$'s, the source of Eq. (23) can somehow be reduced canonically, compatibly with gluing, to a one-dimensional subspace. To this end, it is quite clear that we should not interpret the fusion coefficients in Eq. (21) as numbers, but as actual values of the chiral modular functor corresponding to a particular ("reference") pair of pants. Actually, we do not want an arbitrary pair of pants, we want it to be of the form $P\bar{P}$, i.e., sewn together from two complex conjugate "front" and "back" pieces, each of which is an open worldsheet P of genus 0 with one boundary component and three string components, labeled by κ, λ, μ. The components labeled by κ, μ have the same orientation, the component labeled by λ opposite. In our case, we will therefore have s such parts P_i, $i=1,\ldots,s$. Here, P_i has one string component c labeled by κ_i of orientation opposite to κ_i (so they can be glued), and two string components d,e which are labeled by the D-branes of Σ to which they are adjacent. Let us assume the d component is adjacent to the brane of Σ which comes before c in the clockwise order. Then d has the orientation opposite to c in P_i, and e has the same orientation as c.

Now sew the open worldsheets Σ, P_i, $i=1,\ldots,s$ to obtain a new worldsheet Θ, and then sew the front and back to obtain a closed worldsheet $\Theta\bar{\Theta}$. This worldsheet $\Theta\bar{\Theta}$ now has closed string components, labeled by the D-branes λ of Σ. In fact, each D-brane component α of Σ corresponds to *two* boundary components c_α, $\bar{c}_\alpha$ of $\Theta\bar{\Theta}$ with the same label λ_α as the D-brane corresponding to α, of opposite orientations. Also, it is symmetrical with respect to complex conjugation. We want to show that the modular functor $M_{\Theta\bar{\Theta}}$ contains a canonical line.

The idea is that although c_α, $\bar{c}_\alpha$ are on the "plane of symmetry" of $\Theta\bar{\Theta}$ (the symmetry being complex conjugation), since they have opposite orientations, they can be moved to opposite sides (into the interior of Θ, $\bar{\Theta}$ without breaking the symmetry of $\Theta\bar{\Theta}$). In other words, the worldsheet $\Theta\bar{\Theta}$ is perturbed into a new worldsheet $\Theta'\overline{\Theta'}$ where now Θ' is a closed worldsheet. (Further, the perturbation move can be expressed by tensoring with a known line, since we know the modular functor of cylinders.)

Now in alias notation, let γ be a set of labels to put on the new closed string components of Θ', via which it is attached to $\overline{\Theta'}$ (the other string components are already labeled). Then we have

$$M_{\Theta'\overline{\Theta'}} = \bigoplus_{\gamma} M_{\Theta'_\kappa} \otimes \overline{M_{\Theta'_\kappa}} \tag{24}$$

by Eq. (18). But the right hand side of Eq. (24) contains a canonical line by Eq. (20).

The case when the original worldsheet Σ has closed string components is handled in a trivially modified manner: one repeats verbatim the construction of the worldsheet Θ and uses the vacuum in the chiral theory. If there are p closed string components, then the state space of the worldsheet $\Sigma\bar{\Sigma}$ picks up a factor

$$\bigotimes_{i=1}^{p} \mathcal{H}$$

(summing over all the possible choices of labels on the closed string components, which we require to be the same in Σ and $\bar{\Sigma}$). Additionally, the modular functor of $\Sigma\bar{\Sigma}$ (and also $\Theta\bar{\Theta}$) picks up an additional factor of

$$\bigotimes_{i=1}^{p} \left(\bigoplus_{\lambda \in B} M_\lambda \otimes \overline{M_\lambda} \right) \tag{25}$$

which, as we know from Eq. (20), again contains a canonical line. Compatibility under gluing is not difficult to verify.

Another subtlety in the above discussion is the case of closed D-brane components c. Strictly speaking, the construction we just described only gives a *sum* of states over all D-brane labels in this case, because the labeling of the closed D-brane component disappears during the doubling. To remedy this in a way which is most convenient for proving consistency under gluing, we may cut c by a short open string component s, and endow it with a reference half-pair of pants as before. The labels this way will be retained in the doubling, and we may recover the state corresponding to c by gluing back in the open string vacuum corresponding to s. This is consistent, since we know how to move strings around in $\Theta\bar{\Theta}$ using the Virasoro algebra, and s can always be moved out of the way of any cuts to prove consistency under gluing.

However, there is a better approach which may be preferrable calculationaly: we may replace the closed D-brane component by a closed string component, and then glue in the "boundary state," corresponding to the cylinder A_τ with one closed D-brane and one closed string component (parametrized in the standard way), obtained by gluing a parallelogram Q with one side 1 and one side τ in $\mathbb{C}$. It is convenient to assume that $\tau=0$, i.e., the cylinder A_0 has width 0, i.e., is degenerate. This will cause the state to be divergent from in the Hilbert space (i.e., distributional), but that does not matter: we may always glue in the vacuum of a string annulus to get a true trace class Hilbert state. Then the boundary state corresponding to brane $\lambda \in B$ must have the form

$$b_\lambda = \sum_{\mu \in B} \alpha_{\lambda\mu} 1_\mu, \tag{26}$$

where $1_\mu : \mathcal{H}_\mu \to \mathcal{H}_\mu$ is the identity [called the 'Ishibashi state,' see Eq. (17), Sec. 5.1 of Ref. 32], and $\alpha_{\lambda\mu}$ are some coefficients.

The trick for calculating the coefficients $\alpha_{\lambda\mu}$ (see, e.g., Ref. 32 and 37) is to actually consider the cylinder $A_{\tau/2}$ with finite width $\tau/2$ and glue it with the opposite cylinder $A'_{\tau/2}$ where the string boundary component is oriented the opposite way. Let C_τ be the cylinder with two D-brane components obtained by gluing $A_{\tau/2}$ and $A'_{\tau/2}$, with D-brane component labeled by another D-brane λ'. Then, using the usual anomaly trivialization on rigid cylinders, the vacuum corresponding to C_τ is

$$U_{A_\tau} = \sum_{\mu \in B} \alpha_{\lambda\mu} \alpha_{\lambda'^*\mu} Z_\mu(q), \tag{27}$$

where Z_μ is the partition function of the chiral sector $\mathcal{H}_\mu$, and $q = e^{2\pi i \tau}$.

On the other hand, however, U_{A_τ} can be computed by cutting A_τ into a parallelogram Q with two open D-brane components parallel to the vector $\tau \in \mathbb{Z}$, and two open string components parallel to the vector $1 \in \mathbb{C}$ which is conformally isomorphic to the parallelogram where the open

D-brane components are parallel to 1 and the open string components are parallel to $\tau' = -1/\tau$. We can calculate this trace using the "doubling" described above. The expression one obtains is

$$\operatorname{tr} U_q = \sum_{\mu \in B} N^{\mu}_{\lambda\lambda'^*} Z_\mu(q'), \tag{28}$$

where $q' = e^{2\pi i \tau'}$. Now consider the modular S-matrix, i.e., the unitary matrix corresponding to the modular transformation $-1/\tau$, in the basis B (the elementary closed labels). Then, using the Verlinde conjecture[43] (the proof of which in the present context is outlined in Ref. 38 an earlier proof in more physical setting appearing in Ref. 34), we compute

$$\sum_{\mu} N^{\mu}_{\lambda\lambda'^*} Z_\mu(q') = \sum_{\kappa} \frac{S^{\kappa}_{\lambda} S^{\kappa}_{\lambda'^*} (S^{-1})^{\kappa}_{\mu}}{S^{\kappa}_{0}} Z_\mu(q') = \sum_{\kappa} \frac{S^{\kappa}_{\lambda} S^{\kappa}_{\lambda'^*}}{S^{\kappa}_{0}} Z_\kappa(q) \tag{29}$$

(where 0 is the zero label). Now plugging Eq. (29) into Eq. (28) and using the equality with Eq. (27), we obtain the equations

$$\alpha_{\lambda\mu} \alpha_{\lambda'^*} = \frac{S^{\mu}_{\lambda} S^{\mu}_{\lambda'^*}}{S^{\mu}_{0}}, \tag{30}$$

which has, up to possible multiple of -1, a unique solution

$$\alpha_{\lambda\mu} = \frac{S^{\mu}_{\lambda}}{\sqrt{S^{\mu}_{0}}} \tag{31}$$

(for example, set $\lambda'^* = \lambda$). Plugging back into Eq. (26) gives the formula[37,32] for the boundary state,

$$b_\lambda = \sum_{\mu \in B} \frac{S^{\mu}_{\lambda}}{\sqrt{S^{\mu}_{0}}} 1_\mu. \tag{32}$$

A note is due on the status of the claim that the level k SU (n)-WZW model is a RCFT: as far as we know, a rigorous proof written in the present setting is not anywhere in the literature. However, it seems that a rigorous proof can be obtained by combining the results of Bakalov and Kirillov[8] with the technique for proving convergence results by the boson-fermion correspondence.[27] Therefore, modulo writing down certain tedious details, the WZW RCFT, and its theory of Cardy branes is on fairly solid ground.

Additionally, it is worth pointing out that in the WZW model of a compact Lie group G, the brane vacua have the following geometrical interpretation. For simplicity, instead of the Cardy vacua, we consider the Ishibashi vacua, i.e., open worldsheets where branes are unlabeled and open strings are labeled by $\lambda \in P_k$. It is then appropriate to consider the state space $\mathcal{H}_\lambda$ as a representation of a particular loop group as follows: the closed state space $\mathcal{H}$ is a representation of a central extension of the loop group

$$LG \times LG. \tag{33}$$

Consider now the unit disk D and let its boundary be the source of the loops in Eq. (33). Consider now the open worldsheet $D^+ = \{z \in D \mid \operatorname{Im}(z) \geq 0\}$ where the D-brane component is $D^+ \cap \mathbb{R}$. We may then consider the Ishibashi open string sector $\mathcal{H}_\lambda$ as an irreducible representation of the subgroup

$$LG_\Delta = \{(f,g) \in LG \times LG \mid f(z) = g(\bar{z}) \quad \text{for } z \in S^1\} \tag{34}$$

of Eq. (33). There are other variations of this definition, for example we may look at the group LG_o of pairs of functions (f,g) from a neighborhood U of $S^1 \cap D^+$ into $G_{\mathbb{C}}$ where f is holomorphic, g is antiholomorphic, and $f \equiv g$ on the real line. The advantage of the definition of LG_o is that it is more "local." Indeed, for an Ishibashi open worldsheet Σ (which, for simplicity, we assume contains no closed string components), we may look at the subgroup

$$LG_{\Sigma} \subset \prod LG_0 \tag{35}$$

(the product is over boundary components) given by pairs (f,g) of functions

$$\Sigma \to G_{\mathbb{C}},$$

where f is holomorphic, g is antiholomorphic and $f \equiv g$ on D-brane components of Σ. Then using standard methods (actually in this case easier for the WZW model than the lattice CFT), one shows that the cocycle defining the universal central extension of ΠLG_o vanishes when restricted to LG_{Σ}. The Ishibashi vacuum on Σ is then defined as the fixed subspace

$$\left(\prod \mathcal{H}_{\kappa_i}\right)^{LG_{\Sigma}} \tag{36}$$

of the state space. Of course, Eq. (36) agrees with the above "doubling" construction for general RCFT.

V. BRANE DYNAMICS: INFINITESIMAL DEFORMATIONS

It is well known that a closed CFT with one-dimensional anomaly can be infinitesimally deformed by integrating a fixed primary field of weight (1, 1) over the worldsheet (see Refs. 27 and 38). The Kondo effect, in its abstract mathematical form, is a similar construction allowing us to modify infinitesimally the open sector of a fixed closed CFT $\mathcal{H}$. To be more precise, suppose we have an closed/open CFT

$$\Phi: \mathcal{D} \to \tilde{\mathcal{C}}(\mathcal{B},\mathcal{K},\mathcal{H}), \tag{37}$$

where $\tilde{\mathcal{C}}(\mathcal{B},\mathcal{K},\mathcal{H})$ is as in Sec. III as above, and $\mathcal{D}$ is the SLCMC of closed/open worldsheets as in Ref. 26.

Now suppose we have a brane $a \in \mathrm{Obj}(\mathcal{B})$. Consider $a^* \in Obj(\mathcal{B}^*)$ as in Sec. III above. Then we may consider

$$a \otimes a^* \in \mathrm{Obj}(\mathcal{B} \otimes \mathcal{B}^*). \tag{38}$$

But we have

$$\mathcal{B} \otimes \mathcal{B}^* \simeq Hom_{\mathbb{C}_2}(\mathcal{B} \otimes \mathcal{B}^*, \mathbb{C}_2), \tag{39}$$

so we may also consider $a \otimes a^*$ as a morphism of two-vector spaces:

$$a \otimes a^*: \mathcal{B} \otimes \mathcal{B}^* \to \mathbb{C}_2. \tag{40}$$

This way, we may define the open sector

$$\mathcal{K}_{aa} = (a \otimes a^*)(\mathcal{K}). \tag{41}$$

Now the semigroup $\mathbb{C}^{\times<1}$ of rigid annuli does not act on $\mathcal{K}_{aa}$, but its subsemigroup $(0,1) = \mathbb{R}_+^{\times<1}$ (the semigroup of scale transformations) does. An element of $\mathcal{K}_{aa}$ is said to have *weight k* if $t \in (0,1)$ acts by t^k on x. In fact, more generally, we shall consider the semigroup $\mathcal{G}$ of holomorphic maps $f: D \to D$ (where D is the unit disk in $\mathbb{C}$) with the property that $f(0)=0$, $f([0,1]) \subseteq [0,1]$, $f([-1,0]) \subseteq [-1,0]$. Note that every such map f necessarily has real derivative at 0. Note also that such map necessarily defines an open worldsheet, and hence a map

$$f: \mathcal{K}_{aa} \to \mathcal{K}_{aa}. \tag{42}$$

(The central extension necessarily splits canonically on worldsheets of this type, similarly as in the closed sector.) We shall call $x \in \mathcal{K}_{aa}$ a *primary field of weight* k if for every $f \in \mathcal{G}$,

$$f(x) = f'(0)^k \cdot x. \tag{43}$$

We claim that then it is possible to construct an *infinitesimal deformation* a_x of the brane a along the element x when x is a primary field of weight 1. This is more easily explained in the alias interpretation, i.e., with the choice of a closed/open worldsheet Σ with p closed string, q closed D-brane and s mixed boundary components. Assume the mth mixed boundary component, $m = 1, \ldots, s$, has ℓ_m open D-brane components, to which we assign D-branes a_{mi}, $i=1,\ldots,\ell_m$ (we shall set $a_{m0}=a_{m,\ell_m}$). Assume also to the jth closed D-brane boundary component we assign the D-brane a_j. Then for this choice c, our formalism gives us a vacuum vector

$$U_c \in \bigotimes_{i=1}^{p} \mathcal{H}^{(*)} \otimes \bigotimes_{m=1}^{s} \left(\bigotimes_{i=0}^{\ell_m - 1} \mathcal{K}^{(*)}_{a_{mi} a_{m,i+1}} \right), \tag{44}$$

where, as usual, $^{(*)}$ denotes dual if the corresponding string component is inbound, and notation for Hilbert completing the tensor products and then passing to trace class elements is omitted from the notation.

The infinitesimal deformation is described as follows: suppose that on finitely many D-brane boundary components $c_1, \ldots, c_r$ of Σ (open or closed) the D-brane assigned to each c_i in the choice c is a. (But we allow a to be also possibly assigned to other D-brane components of Σ.) Then we may define a deformed choice c' which is the same as c except that the D-brane assigned to each c_i is the "infinitesimally deformed D-brane a_x." The "vacuum deformation" is defined simply by

$$\Delta U_c = \int_{c_1 \cup \ldots \cup c_r} \Psi_t(x). \tag{45}$$

This is to be read as follows: t is an arbitrary point of $c_1 \cup \ldots \cup c_r$. The element $\Psi_t(x)$ is defined by choosing a holomorphic embedding $g: D^+ \to \Sigma$ where

$$D^+ = \{z \in \mathbb{C} \| \|z\| \leq 1, \ \mathrm{Im}(z) \geq 0\},$$

such that $g(0)=t$. Consider a new worldsheet Σ_g obtained from Σ by cutting out $\mathrm{Im}(g)$ (the new boundary component is an open string component d parametrized outbound by arc length scaled to 1). The element $\Psi_t(x)$ of the right hand side of Eq. (44) is now defined by taking the vacuum element of Σ_g (with D-brane choices induced by c), and "inserting" (i.e., plugging in) the element $x \in \mathcal{K}_{aa}$ to d. The main point of the operator $\Psi_t(x)$ is that its dependence on g is expressed by the formula

$$\Psi_{t,gf}(x) = \Psi_{t,g}(x) f'(0). \tag{46}$$

This follows from the fact that x is a primary field of weight 1. But note that Eq. (46) means that the element $\Psi_t(x)$ of the right hand side of Eq. (44) transforms as a 1-dimensional measure, which means precisely that it can be integrated without "multiplying by dt." We must, of course, assume that the integral (45) converges in the subspace of trace class elements of the Hilbert tensor product.

Given that, however, Eq. (45) can be considered an "infinitesimal deformation" in the following sense: Suppose we form the formal expression

$$U_{c'} = U_c + \epsilon(\Delta U_c). \tag{47}$$

Then $U_{c'}$ will satisfy gluing identities up to linear terms in ϵ. This is simply a consequence of additivity of integration with respect to the integration locus.

It is important to note that one-dimensional anomaly does not spoil the above construction. It is because, in the language of central extensions of the worldsheet SLCMC, the line corresponding to D^+ (with the standard parametrization of the open string component) is canonically isomorphic to $\mathbb{C}$, and therefore the lines corresponding to the worldsheets Σ, Σ' are canonically isomorphic by gluing.

VI. DEFORMATIONS OF BRANES TO PERTURBATIVE LEVEL: BREAKING BOUNDARY CONFORMAL INVARIANCE AND THE KONDO FLOW

Having an infinitesimal deformation does not guarantee that it can be exponentiated to a finite deformation, or even to perturbative level (i.e., to a Taylor series expansion in a formal deformation parameter u). Let us look at the example of the Kondo effect in WZW branes. For this example, consider the level k WZW model for $SU(n)$. In the simplest case (sufficient for our purposes), one considers the Cardy brane a corresponding to the 0 label (we denote it by a instead of 0 to avoid confusion due to overuse of that symbol). Now the primary fields we wish to consider are in the state space $\mathcal{K}_{Va,Va}$ where V is the target vector space of an irreducible representation of $SU(n)$ which corresponds to a label $\lambda \in P_k$. We then have

$$\mathcal{K}_{Va,Va} = Hom(V,V) \otimes \mathcal{H}_0. \tag{48}$$

We shall sometimes write $\mathcal{K}_{aa}$ instead of $\mathcal{K}_{V_a,V_a}$ for brevity. The primary fields in Eq. (48) we wish to use for deformation are of the form

$$\phi_\lambda = \sum_\alpha S_\alpha J^\alpha_{-1}, \tag{49}$$

where α runs through a basis of the Lie algebra $su(n)$, $S: su(n) \to Hom(V,V)$ is the representation corresponding to λ on the level of Lie algebras, and J^α_{-1} is the weight 1 primary field in $\mathcal{H}_0 = \mathcal{K}_{00}$ corresponding to α [realizing the action of $su(n)$ on $\mathcal{H}_0$ by vertex operators].

Let us compute some information about this infinitesimal deformation explicitly. Specifically, let us consider the infinitesimally deformed operator $L_p(\epsilon)$ where L_p corresponds to the vector field $z^{p-1}\partial/\partial z$ in the upper semicircle $S^{1+}=\{z \in \mathbb{C} \| |z\| = 1, \mathrm{Im}(z) \geqslant 0\}$. But this deformation is easy to compute, since we must simply add two summands, one of which is insertion of the vertex operator corresponding to Eq. (49) at $z=1$, and the other at $z=-1$; the second term must have the sign $(-1)^{p-1}$. One thing to note is that we must be careful with the order of matrix multiplicaton due to the fact that $Hom(V,V)$ is noncommutative. In fact, neglecting terms of higher than linear order in ϵ, we get

$$L_p(\epsilon) = \sum_{n \in \mathbb{Z}, \alpha} :J^\alpha_n J^\alpha_{-n+p}: + \epsilon(S^L_\alpha J^\alpha_n + (-1)^{p+n+1} S^R_\alpha J^\alpha_n), \tag{50}$$

where the superscripts L,R indicate that the matrix in question should multiply the $Hom(V,V)$ factor from the left (right). As usual, $::$ denotes normal ordering. Another comment is that in the first summand of Eq. (50), one of the α indexes should be lowered. However, it is customary to write the term in the present form. This is correct if the α's form an orthonormal basis.

Now consider the level one primary field ϕ_λ in $\mathcal{H}_0$. Looking for a primary field of general weight w in the infinitesimally deformed sector $\mathcal{K}_{\alpha(\epsilon),\alpha(\epsilon)}$ of the form

$$\phi_\lambda(\epsilon) = \phi_\lambda + \epsilon \psi_\lambda, \tag{51}$$

we get the equation

$$\begin{aligned} L_p(\epsilon)\phi_\lambda(\epsilon) &= 0 \quad \text{for } p > 0 \\ &= w\phi_\lambda(\epsilon) \quad \text{for } p = 0, \end{aligned} \tag{52}$$

which must be satisfied up to (and including) linear order in ϵ. Expanding Eq. (52) using Eq. (51), we see that we must have $w = 1 + v\epsilon$ to get the ϵ-constant terms to match, and the ϵ-linear terms give that

$$\sum_{n\in\mathbb{Z},\alpha} :J_n^\alpha J_{-n+p}^\alpha:\psi_\lambda + \sum_{n\in\mathbb{Z},\alpha} (S_\alpha^L J_n^\alpha + (-1)^{p+n+1} S_\alpha^R J_n^\alpha)\phi_\lambda$$

is equal to 0 for $p>0$ and to

$$v\phi_\lambda + \psi_\lambda$$

for $p=0$. Now the condition for $p=0$ gives

$$\sum_{n\in\mathbb{Z},\alpha} :J_n^\alpha J_{-n}^\alpha:\psi_\lambda + \sum_{n\in\mathbb{Z},\alpha,\beta} (S_\alpha^L J_n^\alpha + (-1)^{n+1} S_\alpha^R J_n^\alpha) S_\beta J_{-1}^\beta = v\phi_\lambda + \psi_\lambda. \tag{53}$$

Putting

$$\psi_\lambda = \sum_{n>0} \psi_\lambda(n),$$

where $\psi_\lambda(n)$ is homogeneous of weight n, we get from Eq. (53) the equation

$$\begin{aligned}(n-1)\psi_\lambda(n) + \sum_{\alpha,\beta} (S_\alpha^L J_{n-1}^\alpha + (-1)^n S_\alpha^R J_{n-1}^\alpha) S_\beta J_{-1}^\beta = 0 \quad \text{if } n \neq 1 \\ = v \sum S_\alpha J_{-1}^\alpha \quad \text{if } n=1.\end{aligned} \tag{54}$$

We see that Eq. (54) for $n\neq 1$ can be solved, yielding

$$\psi_\lambda(n) = \sum_{\alpha,\beta} (S_\alpha^L J_{n-1}^\alpha + (-1)^n S_\alpha^R) J_{-1}^\alpha/(1-n). \tag{55}$$

For $n=1$, however, we get

$$\sum_{\alpha,\beta} S_{[\alpha,\beta]} J_{-1}^{[\alpha,\beta]} = v \sum S_\alpha J_{-1}^\alpha. \tag{56}$$

This is a condition on v, which we easily see can force it to be nonzero. For a precise evaluation of v, we must look more closely at Eq. (49). To lower one of the α indices in that formula, we are using the Killing form in the Lie algebra g scaled by some positive real factor. In other words, if

$$\iota \in g \otimes_{\mathbb{R}} g$$

is the inverse of the scaled Killing form, then we have

$$\phi_\lambda = (S\otimes 1)\iota.$$

Therefore, v depends inverse linearly on the scaling factor of the Killing form. For $g=\mathrm{su}(n)$, we may as well complexify. We have

$$\mathrm{su}(n)_{\mathbb{C}} = \mathrm{sl}(n,\mathbb{C}).$$

Upon complexification, the Killing form extends to a complex symmetric bilinear form (not a Hermitian form). Let us normalize by taking the symmetric bilinear form on $\mathrm{sl}(n,\mathbb{C})$ which is the restriction of the symmetric bilinear form B on $\mathrm{gl}(n,\mathbb{C})$ defined by

$$B(e_{ij},e_{k\ell}) = -\delta_i^\ell \delta_j^k.$$

We are trying to calculate

$$\begin{pmatrix} [,] \\ \otimes \\ [,] \end{pmatrix} (B^{-1} \otimes B^{-1}).$$

We know this is invariant, hence equal to vB^{-1} for some v [see Eq. (56)]. To calculate v, we can then select a basis element x of $\mathrm{sl}(n)$ with

$$B(x,x) = 1,$$

say,

$$x = \frac{i}{\sqrt{2}}(e_{11} - e_{22}). \tag{57}$$

Then we have

$$v = \sum_{i,j,k,\ell} B([e_{ij}, e_{k\ell}], x)^2. \tag{58}$$

The nonzero contributions to Eq. (58) are only when $\ell = j$, $k = i$, and the contribution is 2 when $\{i,j\} = \{1,2\}$ and $1/2$ when $|\{i,j\} \cap \{1,2\}| = 1$ (and 0 otherwise). Thus,

$$v = 2\left(2 + 2(n-2)\frac{1}{2}\right) = 2n.$$

Now it is easily seen that the condition (55) indeed implies Eq. (52) for all p, so we obtain a primary field $\phi_\lambda(\epsilon)$ of weight $1 + \epsilon v$ to linear order in ϵ. In other words, a primary field of weight 1 is not possible.

Before saying anything about higher powers of ϵ, however, let us ask what would be the effect of deforming boundary sectors $\mathcal{K}_{aa}$ of CFT's defined on the SLCMC of closed/open worldsheets along arbitrary fields. Just as in the last section, we have an equation of the form

$$\frac{dU_{\Sigma(\epsilon)}}{d\epsilon} = \int_c J(u). \tag{59}$$

Here $\Sigma(\epsilon)$ is the worldsheet Σ with brane labels a replaced by $a(\epsilon)$, c is again the union of brane components labeled by $a(\epsilon)$. $J(u)$ is again the element obtained by cutting out a holomorphic image, under a function f, of a standard half-disk D^+ from $\Sigma(\epsilon)$ such that the brane component of D^+ lies on c and is mapped isometrically. (To be precise, near a boundary point of c, only a neighborhood of 0 in D^+ will lie on c, and the condition must be modified accordingly, using analytic continuation.) Assuming all necessary convergence works, we obtain a theory which breaks conformal invariance only on c, i.e., is a CFT defined on a modification of the SLCMC of worldsheets $\mathcal{D}$ where now each worldsheet is endowed with an analytic metric on the brane components.

One can attempt to write perturbative formulas for the deforming field and calculate the deformed worldsheet vacua by continuing the method just outlined. We have to realize, however, that there are delicate convergence questions. For example, the field ψ_λ is not an element of the Hilbert space $\mathcal{K}_{aa}$ (has "infinite norm"), so the norm would have to be also updated along with the deformation. Another issue (as we will see) is that the wrong choice of the deformed field ϕ_λ (for example, making it constant) will lead to nonconvergence even in a more profound sense, i.e., in the product of the weight-homogeneous summands of $\mathcal{K}_{aa}$. One therefore needs a method for keeping track, at least to some degree, of the convergence properties of the deformed vacua.

Before starting this discussion, it is helpful to realize that there is at least one simplification in the boundary sector deformation questions which is not available in the case of bulk deformations (which we do not discuss here). Namely, to characterize vacua of all worldsheets whose brane components, in the alias notation, are labeled by known branes and $a(\epsilon)$, it suffices to know the vacuum $u_\epsilon(t)$ in (some completion of) $\mathcal{K}_{aa}$ corresponding to the worldsheet $D^+_{a(\epsilon),t}$, which is the standard half-disk D^+ with brane component labeled by $a(\epsilon)$ and metric which is scaled length in such a way that the total length of the brane component is $2t$. The reason this suffices is that every worldsheet $B(\epsilon)$ whose brane components are labeled by known branes and $a(\epsilon)$ can be glued from copies of $D^+_{a(\epsilon),t}$ and a worldsheet $\Sigma(\epsilon)$ which is labeled by known branes and $a(\epsilon)$ and has the property that all branes labeled by $a(\epsilon)$ have length 0 (are single points). But the vacuum associated with $\Sigma(\epsilon)$ should not be affected by the deformation, i.e., we should have

$$U_{\Sigma(\epsilon)} = U_{\Sigma(0)}.$$

Therefore, U_B can be calculated from $U_{\Sigma(0)}$ and $U_{D^+_{a(\epsilon),t}} = u_\epsilon(t)$. Now the worldsheets $\Sigma(\epsilon)$ are not really elements of our moduli space, so the vacua $U_{\Sigma(\epsilon)}$ may really not be well defined as trace class elements of the corresponding Hilbert spaces (additional completion may be required). Nevertheless, U_B should still be determined by limit arguments. We will not treat this in detail in this paper.

Nevertheless, the idea one gets from this is to rewrite the deformation process in such a way that it refers only to the elements $u_\epsilon(t)$. As it turns out, it is actually more advantageous to work with the element

$$v_\epsilon(t) = A_t^+ u_\epsilon(t), \tag{60}$$

where A_t^+ is the worldsheet obtained by cutting tD_a^+ out of D_a^+.

We pause here to note that the a limit of Eq. (60) is not expected to (and, in fact, does not) exist as $t \to 0$. This can be seen from the following physical argument: The element (60) corresponds to a worldsheet which has two brane components labeled by a, touching at one point each a brane component labeled by $a(\epsilon)$ (this is the one of length t). When computing the vacuum of this wordlsheet, boundary condition changing fields ψ are inserted at each point where the different branes touch. As $t \to 0$, the two points where the fields ψ are inserted approach, which causes the limit to be singular.

This is why the $t \to 0$ limit is *not* considered in the subsequent argument; in fact, the main contribution of the present method is the idea of confining our recursive procedure for obtaining the ϵ expansion to $v_\epsilon(t)$ for "finite" (i.e., positive) values of t.

Now for brevity, we use the same notation A_t^+ for the associated operator $\mathcal{K}_{aa} \to \mathcal{K}_{aa}$ (which would be more precisely denoted by $U_{A_t^+}$, which however is ambigous, since it can be understood as an operator or an element of $\mathcal{K}^*_{aa} \hat{\otimes} \mathcal{K}_{aa}$). Note that in the operator A_t^+, we use the operator corresponding to the brane a, not the deformed brane $a(\epsilon)$. Because of this discrepancy, note that we will need to discuss convergence: let $K(n) \subset \mathcal{K}_{aa}$ be the subspace of elements of weight n. Then we have

$$K = \bigoplus_{n \geq 0} K(n) \subset \mathcal{K}_{aa} \subset \hat{K} = \prod_{n \geq 0} K(n).$$

Operators defined on K may not extend to $\hat{K}$. Note that by Eq. (60), $v_\epsilon(t)$ is obtained from $u_\epsilon(t)$ by multiplying the weight n summand by t^n. Our approach here will be the same as in Remark 2 of Sec. IV. For the remainder of this section, then, the elements discussed will be elements of $\hat{K}$, and convergence of operators applied to each individual element will be treated by means of analysis. In this setting, we will construct rigorously the elements

$$v_\epsilon(t) \in \hat{K}[[\epsilon]],$$

i.e., the element $v_\epsilon(t)$ is an actual function of t, but is perturbatively expanded in ϵ.

Now the Kondo flow, as reviewed, say, in (Ref. 33), (note, however, that in the present paper we do not consider supersymmetry), one expects the equation

$$\frac{dv_\epsilon(t)}{dt} = \epsilon(Y(\phi_\lambda,t)_L + Y(\phi_\lambda,-t)_R)v_\epsilon(t) \tag{61}$$

[as before, the susbcripts L and R refer to whether the $Hom(V,V)$ matrix from ϕ_λ multiplies the one from $v_\epsilon(t)$ from the left or right and where $Y(\phi_\lambda,t)$ is the vertex operator corresponding to ϕ_λ]. The equation (61) is a direct analog of the equation (5.30) from Ref. 33 [which refers to the unit disk with a closed brane boundary component labeled by $a(\epsilon)$, while Eq. (61) refers to $D^+_{a(\epsilon),t}$]. From our point of view, this corresponds to exponentiating the infintesimal deformations by the *constant* field $\phi_\lambda(\epsilon)=\phi_\lambda$. We shall discuss this point in more detail shortly.

Of course, we know ϕ_λ and its vertex operator, so Eq. (61) becomes

$$\frac{dv_\epsilon(t)}{dt} = \epsilon\sum_{n\in\mathbb{Z}} (S^L_\alpha - (-1)^{n-1}S^R_\alpha)J^\alpha_{-n}v_\epsilon(t)t^{n-1}. \tag{62}$$

Instead of writing the solution to Eq. (62) formally using the path-ordered exponentiation $P\exp$, let us try to expand the solution perturbatively in ϵ. Setting

$$v_\epsilon(t) = \sum_{n\geq 0} v_n(t)\epsilon^n, \tag{63}$$

we get from Eq. (63)

$$\frac{dv_\epsilon(t)}{dt} = \sum_{n\in\mathbb{Z}} (S^L_\alpha - (-1)^{n-1}S^R_\alpha)J^\alpha_{-n}v_{n-1}(t)\cdot t^{n-1}. \tag{64}$$

Assuming

$$v_0(t) = 1, \tag{65}$$

the equation for $v_1(t)$ from Eq. (64) works out well. We get

$$v_1(t) = 2\sum_{n\geq 0} S_\alpha J^\alpha_{-2n-1}t^{2n+1}/(2n+1), \tag{66}$$

which agrees with what we already know. The problem is, however, that the equation (64) for v_2 diverges. Therefore, we are faced with the problem of regularization.

We proceed as follows: first note that the equation (62) takes the infinitesimal approach, composing $v_\epsilon(t)$ with "$v_\epsilon(1/\infty)$," or more precisely with

$$\left.\frac{\partial v_\epsilon(t)}{\partial t}\right|_{t=0^+}. \tag{67}$$

This should be $2\phi_\lambda\epsilon$. In fact, the idea is that the derivative of $v_\epsilon(t)$ at $t=0$ should have the meaning of a worldsheet with "infinitesimal" $a(\epsilon)$-brane component. Therefore, it should be deformed by simply integrating the field $\phi_\lambda(\epsilon)$, so in general, Eq. (67) should be equal to

$$2\int_0^\epsilon \phi_\lambda(x)dx.$$

When $\phi_\lambda(\epsilon)=\phi_\lambda$ is constant, we get $2\phi_\lambda\epsilon$. Composition of $D^+_{a(\epsilon),t}$ with the "infinitesimal worldsheet" $D^+_{a(\epsilon),dt}$ then gives the equation (61). But we saw that assuming this leads to a divergent formula for $v_\epsilon(t)$. Therefore, if the formula is to be regularized, we must assume that $\phi_\lambda(\epsilon)$ cannot be constant in ϵ and that moreover its coefficients at ϵ^1 (which we formerly denoted by ψ_λ) must be divergent. In other words, to regularize rigorously, we must consider the case when the limit Eq. (67) does not exist at all. To proceed, we must replace Eq. (60) by an equation involving the finite worldsheet $B_{s,t,a(\epsilon)}$ obtained from $D^{+}_{a(\epsilon),s+t}$ by cutting out $-sD^{+}_{a(\epsilon),t}$ and $tD^{+}_{a(\epsilon),s}$. Identifying, again, notationally $B_{s,t,a(\epsilon)}$ with the corresponding operator

$$K \otimes K \to \hat{K},$$

we have

$$B_{s,t,a}(u_s(\epsilon),u_t(\epsilon)) = u_{s+t}(\epsilon),$$

which, in terms of vertex operators in K, can be written as

$$\exp(-sL_{-1})Y(v_s(\epsilon),s+t)v_t(\epsilon) = v_{s+t}(\epsilon). \tag{68}$$

[Recall again that this operator is independent of brane deformation because the $a(\epsilon)$-brane component is degenerate to a discrete points; in other words, the operators corresponding to $B_{s,t,a}$, $B_{s,t,a(\epsilon)}$ coincide.] Let us attempt to solve Eq. (68) by perturbative expansion in ϵ, i.e., by setting Eq. (63), and assuming Eqs. (65) and (66). [Actually, one has to verify that Eq. (66) is consistent, i.e., makes Eq. (68) satisfied to linear order in ϵ, but that is easily done.] In order to make this work, we need an inductive convergence assumption. Recall that for an element $u \in \hat{K}$, we denote by $u(n)$ its homogeneous summand of weight n. Our assumption is

$$\begin{gathered}\text{for all } m, v_n(t)(m) \text{ is a } Hom(V,V)\text{-matrix of polynomials of degree} \leqslant n \\ \text{in } \mathcal{H}_0 \text{ and } \partial_t^k \text{ of all its coefficients are of order } o((t')^m) \text{ for every } t' > t,\ k \geqslant 0.\end{gathered} \tag{69}$$

By way of explanation, recall that $\mathcal{H}_0$ is a Hilbert completion of an algebra with generators J^{α}_{-n}. The algebra is not commutative, but by a polynomial of degree $\leqslant n$ we mean simply a sum of finitely many elements, each of which is a product of $\leqslant n$ of the generators.

We emphasize that Eq. (69) is an induction hypothesis, i.e., a statement which will be completely proved in the course of the construction. For $n=0,1$, the statement is trivial [see Eq. (65) and (66)]. The induction step, i.e., the fact that assuming Eq. (69) with n replaced by $n-1$ in fact implies the statement Eq. (69) for a given n, will be given at the end of the construction.

Now assume $v_1(t),\ldots,v_{n-1}(t)$ have been constructed and Eq. (69) is satisfied with n replaced by each of the numbers $1,\ldots,n-1$. Then the coefficient at ϵ^n of Eq. (68) gives the equation

$$\exp(-sL_{-1})v_n(t) + \exp(tL_{-1})v_n(s) = v_n(s+t) + C(s,t). \tag{70}$$

The element $C(s,t)$ is known and converges by Eq. (69), and further satisfies

$$\begin{gathered}\text{All the } C(s,t)(m)\text{'s are } Hom(V,V)\text{-matrices of polynomials in } \mathcal{H}_0 \text{ of degree} \\ \leqslant n \text{ and all } \partial_t^k\partial_s^\ell\text{'s of all their coefficients are } o((t')^m) \text{ for every } t' > s+t,\ k,\ \ell \geqslant 0.\end{gathered} \tag{71}$$

The functional equation (70) is solved as follows: first notice that

$$\left(\frac{\partial}{\partial s} + L_{-1}\right)\exp(-sL_{-1})v_n(t) = 0, \tag{72}$$

$$\left(\frac{\partial}{\partial t} - L_{-1}\right)\exp(tL_{-1})v_n(s) = 0. \tag{73}$$

Further, the operators

$$\frac{\partial}{\partial s}+L_{-1}, \frac{\partial}{\partial t}-L_{-1}$$

commute, so we get from Eqs. (72) and (73)

$$\left(\frac{\partial}{\partial s}+L_{-1}\right)\left(\frac{\partial}{\partial t}-L_{-1}\right)v_n(s+t) = -\left(\frac{\partial}{\partial s}+L_{-1}\right)\left(\frac{\partial}{\partial t}-L_{-1}\right)C(s,t). \tag{74}$$

But

$$\frac{\partial}{\partial s}f(s+t) = \frac{\partial}{\partial t}f(s+t) = f'(s+t),$$

so from Eq. (74),

$$-\left(\frac{\partial}{\partial s}+L_{-1}\right)\left(\frac{\partial}{\partial t}-L_{-1}\right)C(s,t) = h(s+t)$$

for some function h. If we denote, for the moment, ordinary differentiation by D, Eq. (74) also gives

$$(D+L_{-1})(D-L_{-1})v(s) = h(s) \tag{75}$$

[where $v(s)=v_n(s)$], or

$$v''(s) - L_{-1}^2 v(s) = h(s). \tag{76}$$

The basis of solutions of the homogeneous equation corresponding to Eq. (76) are the solutions to the equations

$$v'(s) = L_{-1}v(s), \tag{77}$$

$$v'(s) = -L_{-1}v(s), \tag{78}$$

which would be

$$v(s) = \exp(L_{-1}s)u, \tag{79}$$

$$v(s) = \exp(-L_{-1}s)u \tag{80}$$

for an arbitrary field u, respectively, if L_{-1} were invertible. In fact, note that

$$v(s) = \frac{\exp(L_{-1}s) - \exp(-L_{-1}s)}{2L_{-1}}u \tag{81}$$

spans the space of solutions to the homogeneous functional equation corresponding to Eq. (70). Thus, the actual basis of solutions consists of one of the functions (79) and (80), and the function (81). We may now explicitly solve the nonhomogeneous differential equation (76) by variation of constants: if

$$Y'(s) = A(s)Y(s)$$

is a basis of solutions for a homogeneous first order system, then

$$Z'(s) = A(s)Z(s) + K(s)$$

is solved by

$$Z(s) = Y(s) \int Y(s)^{-1} K(s) ds. \tag{82}$$

In any case, we know how to solve the non-homogeneous differential equation (75). To solve the nonhomogeneous functional equation (70), we first must check that a solution of Eq. (70) exists. For that, there is a necessary and sufficient *cocycle condition* on $C(s,t)$:

$$\exp(uL_{-1})C(s,t) + C(s+t,u) = C(s,t+u) + \exp(-sL_{-1})C(t,u). \tag{83}$$

There is a standard reason why the cocycle condition is always satisfied in this situation: Consider the element

$$y(s) = \sum_{m=0}^{n-1} v_m(s)\epsilon^m.$$

Now consider

$$D(s,t) = B^{+}_{s,t,a}(y(s),y(t)) - y(s+t).$$

Then $D(s,t)$ satisfies a cocycle condition coming from the associativity of composition:

$$B^{+}_{s+t,u,a}(B^{+}_{s,t,a}(u_1,u_2),u_3) = B^{+}_{s,t+u,a}(u_1,B^{+}_{t,u,a}(u_2,u_3)). \tag{84}$$

But the ϵ^n-coefficient of $D(s,t)$ is $C(s,t)$, the ϵ^n-coefficient of Eq. (84) is Eq. (83).

Therefore, the functional equation (70) has a solution v_n. If v is a solution of the differential equation (75), then $v_n - v$ is a solution of the corresponding homogeneous differential equation. But as we saw, we know all such solutions, and we further saw that the solutions (81) satisfy the homogeneous functional equation corresponding to Eq. (70). Therefore, to solve Eq. (70), plug in a general solution (82) of Eqs. (75), (74), (73), (72), (71), and (70) and calculate [up to the indeterminacy (81)] the constants from the resulting system of consistent (underdetermined) linear equations.

The induction step. There remains the subtle question whether the condition (69) is satisfied for any of the solutions $v_n(t)$ just constructed. This can be shown as follows. First, call any function of $t>0$ satisfying Eq. (69) *properly bounded*. We shall prove that several functions have this property. First, we know from Eq. (71) that

$$(D^2 - L_{-1}^2)v_n(t) \text{ is properly bounded} \tag{85}$$

for any solution $v_n(t)$ of Eq. (70). Next, consider

$$(D + L_{-1})v_n(t) =: w(t). \tag{86}$$

Applying

$$\frac{\partial}{\partial s} + L_{-1}$$

to Eq. (70) [or alternately, exponentiating Eq. (85)], we see that

$$\exp(tL_{-1})w(s) - w(s+t) \text{ is properly bounded}. \tag{87}$$

It follows from Eq. (87) that if $w(s_0)$ is properly bounded, then so is $w(s)$ for all $s > s_0$. Unfortunately, that is not enough. However, we can remedy the situation by using the following trick: $w(t)$ is a solution to the first order system of differential equations

$$(D - L_{-1})w(t) = h(t). \tag{88}$$

But Eq. (88) is partially decoupled with respect to conformal weight: For each weight n, we have an equation of the form

$$Dw(t)(n) = L_{-1}w(t)(n-1) + \phi(t)(n).$$

This means that we can apply initial conditions successively at weights $0,1,2,\ldots$ at points $t_0 > t_1 > t_2 > \ldots$, where $t_n \to 0$. It follows from the definition and Eq. (87) that choosing the initial conditions

$$w(t_n)(n) = 0 \tag{89}$$

implies that

$$w(t) \text{ is properly bounded for all } t > 0. \tag{90}$$

[Note that plugging in Eq. (81) for $v_n(t)$ gives $w_{\text{homog}}(t) = \exp(tL_{-1})u$, thus showing that the initial conditions for w can indeed be chosen arbitrarily for a solution $v_n(t)$ of the nonhomogeneous functional equation (70).] Now suppose $v_n(t_0)$ is not properly bounded where $v_n(t)$ solves Eqs. (70) and (86). Then by Eq. (86),

$$\exp(-s(D+L_{-1}))v_n(t) = \exp(-sL_{-1})v_n(t_0 - s) - v_n(t_0) \text{ is properly bounded for all } 0 < s < t_0 \tag{91}$$

which means

$$v_n(t) \text{ is not properly bounded for any } 0 < t < t_0. \tag{92}$$

Choosing $0 < s, t$ such that $s + t < t_0$, Eq. (70) can be rewritten as

$$\exp(tL_{-1})v_n(s) = -\exp(-s(D+L_{-1}))v_n(s+t) + C(s,t). \tag{93}$$

The right hand side is properly bounded by Eqs. (86) and (90), while the left hand side is not by Eq. (92)—a contradiction.

[In more detail, the first term of the right hand side of Eq. (93) is

$$\exp(-sL_{-1})v_n(t) - v_n(s+t). \tag{94}$$

Treating $r = s + t$ as a constant, Eq. (94) is

$$\exp(-sL_{-1})v_n(r-s) - v_n(r). \tag{95}$$

Applying $\partial/\partial s$, we get

$$\exp(-sL_{-1})\left(-L_{-1} - \frac{\partial}{\partial s}\right)v_n(r-s),$$

which is properly bounded for all s by Eq. (90). Now integrate from 0 to s by ds.]

Thus, we have defined a process which constructs the vacua $u_\epsilon(t)$ expanded in the perturbative parameter ϵ. The process is convergent in the sense of QFT. We also saw that the process has infinitely many degrees of freedom due to the indeterminacy in solving the equation (70) at every step (power of ϵ). It is possible that some natural normalization of this process exists, but it cannot be obtained by considering the limit (67), which does not exist. It is possible that the limit brane $\mathcal{H}_\lambda$ can be obtained as a nonperturbative limit of this process, i.e., as an analytic continuation of one of the theories constructed, at a particular value of ϵ. To that end, however, several steps are still missing. Let us recapitulate what we constructed. The vacua $u_\epsilon(t)$ which we did construct determine all vacua corresponding to worldsheets X_ϵ with brane components labeled by $a(\epsilon)$ or known (Cardy) branes. Further, the elements $u_\epsilon(t)$ converge as formal series in the parameter ϵ.

This is what we mean by convergence in the sense of QFT. What we have not shown is that the vacua corresponding to all the worldsheets X_ϵ converge and in what sense. To do so, an inner product dependent on ϵ would have to be constructed on K. We would then define $\mathcal{K}_{a(\epsilon),a(\epsilon)}$ as the Hilbert completion of K under this inner product, and then we would have to show that our recipe for calculating vacua indeed gives trace class elements in the appropriate Hilbert tensor products. All this would be needed to actually prove that we have constructed a D-brane category over worldsheets breaking conformal invariance on brane components labeled by $a(\epsilon)$. Nevertheless, all these statements, which we do not prove here, make plausible conjectures.

In the next section, we describe a formalism of topological D-brane categories, which captures such conjectured process. At the end of the next section, we shall state the Kondo effect conjecture (or more precisely one of its consequences) in a more precise way.

VII. A GENERAL MATHEMATICAL FORMALISM FOR DESCRIBING KONDO-LIKE EFFECTS

The point of this section is mostly to introduce a *topological* version of the concepts introduced in Sec. III.

First, we let the topology on $\mathbb{C}_2$, the category of finite-dimensional $\mathbb{C}$-vector spaces, be discrete on objects, and the standard topology on morphisms (induced by the product topology of copies of $\mathbb{C}$).

Now the D-brane space is a two-vector space $\mathcal{B}$, which is free on a set B, but we can no longer assume that it is finite dimensional (i.e., that B is finite), as we clearly need infinitely many nonisomorphic branes to reproduce a Kondo-like effect. Now we further assume that $\mathcal{B}$ is *topologized*. By this we will mean that $\mathcal{B}$ has a topology on its sets of objects, and morphisms, which makes all the categorical operations continuous, and also the $\mathbb{C}_2$-module structure continuous. Note that (and this is important) we do not assume that $B=B\cdot\mathbb{C}$ would be a closed subset of $\mathcal{B}$, i.e., we allow (as in the Kondo-effect) that "elementary" branes (by which we mean branes in B) converge to linear combinations of elementary branes.

The first thing to notice is that since $\mathcal{B}$ is infinite dimensional, we do not have an equivalence $\mathcal{B}^{**}\simeq\mathcal{B}$ or Eq. (14). We will, therefore, as usual instead use the embedding

$$\mathcal{B}\to\mathcal{B}^{*} \tag{96}$$

using the basis B of "elementary branes" [the map (96) is given by sending $b\in B$ to the map which assigns to a linear combination its coefficient at b]. Now clearly introducing the map (96) breaks (some) functoriality, and hence, it will also break continuity, since we are not assuming that B is a closed set in $\mathrm{Obj}(\mathcal{B})$.

Next, we must then discuss the open string sector $\mathcal{K}$. According to the convention just described, we will simply specify, for each pair of elementary branes $a,b\in B$, a Hilbert space $\mathcal{K}_{ab}$. We can extend this to a map of $\mathbb{C}_2$-modules

$$\mathcal{K}:\mathcal{B}\otimes\mathcal{B}\to\mathbb{C}_2^{Hilb} \tag{97}$$

in the canonical way. It is, however, too strong to assume that the functor (97) is continuous on the nose, since we have a discrete topology on the target. A similar issue arose in Ref. 26 when we proposed a topological group completion of $\mathbb{C}_2$. Here, however, we shall deal with it in a more straightforward way. We simply require that for each object $a\in\mathrm{Obj}(\mathcal{B})$, we are given an open neighborhood U_a of a in $\mathrm{Obj}(\mathcal{B})$, and for each $x\in U_a$, $y\in U_b$ an isomorphism of vector spaces

$$\phi_{xy}^{ab}:\mathcal{K}_{ab}\to\mathcal{K}_{xy}. \tag{98}$$

Furthermore, when $z\in U_x\cap U_a$, $t\in U_y\cap U_b$, we have

$$\phi_{zt}^{ab} = \phi_{zt}^{xy}\phi_{xy}^{ab}. \tag{99}$$

The morphisms ϕ_{xy}^{ab} are also required to be preserved by addition and scalar multiplication in $\mathcal{B}$, and to form commutative diagrams with coherence isomorphisms of those operations, as usual.

We want to introduce the SLCMC $\mathcal{C}(\mathcal{B},\mathcal{K},\mathcal{H})$ in this topological case, and also its one-dimensional anomaly version $\tilde{\mathcal{C}}(\mathcal{B},\mathcal{K},\mathcal{H})$. $\mathcal{H}$ is as usual, $\mathcal{B},\mathcal{K}$ are as above. We shall only explain what changes in the definition of sections of $\mathcal{C}(\mathcal{B},\mathcal{K},\mathcal{H})$ over a point, since the other points are routine. A section over a point over a graph Γ with p closed components, q open pure D-brane components and s open components with ℓ_j open brane components, $j=1,\ldots,s$, specifies for each choice c of elementary branes $a_{jm}\in B$, where $m\in\{1,\ldots,\ell_j\}$, (as usual, we put $a_{j0}=a_{jm}$) and elementary branes $a_i\in B$ for $i=1,\ldots q$ an element

$$U_c \in \bigotimes_{i=1}^{p}\mathcal{H}^{(*)} \otimes \bigotimes_{j=1}^{s}\left(\bigotimes_{m=0}^{\ell_j-1}\mathcal{K}^{(*)}_{a_{jm},a_{j,m+1}}\right), \tag{100}$$

subject to certain conditions [again, completions of tensor products by taking Hilbert tensor product and then specializing to trace class elements are suppressed from the notation, and $^{(*)}$ means that Hilbert dual is taken on inbound string components, as is Sec. (3) above].

The condition we need on Eq. (100) is continuity. To formulate that, we first note that we can extend the choices c allowed in Eq. (100) to $a_{jm},a_i\in \mathrm{Obj}(\mathcal{B})$ as follows: each time we multiply an elementary brane corresponding to an open brane component by a vector space V, the corresponding target space of Eq. (100) is multiplied by $V\otimes V^*$ because two open string components end on that brane component. Simply then multiply the vacuum vector U_c by the identity in $V\otimes V^*$. An analogous prescription works also for forming sums, which allows us to plug in any linear combinations of elementary branes. For consistency reasons, closed brane components also enter the picture: when multiplying a brane a_i, $i=1,\ldots,q$, by V, the vacuum vector is multiplied by $\dim(V)$ (and is also additive on taking sums of branes in the place of a_i).

Now the continuity condition on vacua simply states that for any convergent net $x_{jm}\to a_{jm}$, $x_i\to a_i$, the vacua U_{c_x} converge to U_{c_a} using, where necessary, the transformations ϕ of Eq. (98) to identify target spaces.

We may now state formally the conjecture about the WZW model. Unfortunately, the formulation still is not completely satisfactory, due to the fact that as commented in Ref. 26, we still have no mathematical formalism for asking what *are* the branes of a closed CFT: even if we add a brane a, and specify the sector of open strings ending on that brane a, it will not explain the sectors of strings beginning on a and ending on another brane. From a physical point of view, this is an example of *locality violation* (see comments at the end of Sec. 5.5 in Ref. 33).

Remarkably, however, we saw in Sec. V that at least infinitesimally, the physical recipe for *deformations* to a brane update automatically all sectors of strings whose one or both ends lie on the deformed brane, and that local deformations (on CFT's with source $\mathcal{S}_b$) correspond to weight 1 fields. Further, this effect is maintained by exponentiation, at least to perturbative level. Therefore, we can state our conjecture as follows.

Conjecture: Let $\mathcal{H}$ be the closed sector of the level k WZW model for SU(n). Then there exists an SLCMC $\mathcal{C}(\mathcal{B},\mathcal{K},\mathcal{H})$ and a CFT

$$\mathcal{S}_b \to \mathcal{C}(\mathcal{B},\mathcal{K},\mathcal{H})$$

with the following properties:

(1) Every object of $\mathcal{B}$ is (isomorphic to an object) in the path component of a (finite vector space-valued) linear combination of Cardy branes.
(2) The neighborhoods U_a for those $a\in \mathrm{Obj}\,\mathcal{B}$ which are Cardy branes are homeomorphic to the (real parts of) vector spaces of weight 1 vectors in $\mathcal{K}_{aa}$, such that for $x\in U_a$, the transition functions between $U_a\cap U_x$ and U_a are smooth real-analytic (with a convergent exponentiation map described by the procedure in the last section).
(3) The set of (path)-connected components of $\mathcal{B}$ is isomorphic to

$$\mathbb{Z}/((k+n)/gcd(k+n,lcm(1,\ldots,n-1)))$$

via the map which sends a to the dimension of the bottom conformal weight space of the corresponding irreducible level k representation of $\widetilde{L}\mathrm{SU}(n)$.

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