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A simple method for constructing the inhomogeneous quantum group $IGL_q(n)$ and its universal enveloping algebra $U_q(\mathrm{igl}(n))$

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We propose a simple and concise method to construct the inhomogeneous quantum group $\mathrm{IGL}_q(n)$ and its universal enveloping algebra $\mathrm{U}_q(\mathrm{igl}(n))$. Our technique is based on embedding an n-dimensional quantum space in an $n+1$-dimensional one as the set $x_{n+1}=1$. This is possible only if one considers the multiparametric quantum space whose parameters are fixed in a specific way. The quantum group $\mathrm{IGL}_q(n)$ is then the subset of $\mathrm{GL}_q(n+1)$, which leaves the $x_{n+1}=1$ subset invariant. For the deformed universal enveloping algebra $\mathrm{U}_q(\mathrm{igl}(n))$, we will show that it can also be embedded in $\mathrm{U}_q(\mathrm{gl}(n+1))$, provided one uses the multiparametric deformation of $\mathrm{U}(\mathrm{gl}(n+1))$ with a specific choice of its parameters. © *1995 American Institute of Physics.*

I. INTRODUCTION

The quantization of inhomogeneous groups is not well stablished. These nonsemisimple groups are important, both from a purely mathematical point of view, and also because one of the most important groups appearing in physics, viz. the Poincaré group, is inhomogeneous. In the last few years various authors studied the quantization of these objects from different viewpoints and by different methods.[1–8]

Our method is based on embedding an n-dimensional quantum space in an $n+1$-dimensional quantum space. This idea becomes clear when one considers $\mathrm{IGL}(n)$ as an inhomogeneous transformation on the $x_{n+1}=1$ subspace of R^{n+1}. However, it cannot be applied to the standard quantum space, because in the standard case, all the coordinates are noncommutative and none of the coordinates lie in the center. But in the case of multiparametric quantum space, there is freedom for fixing the parameters in such a way to set one of the coordinates, e.g., x_{n+1} in the center. This observation makes the meaning of the embedding more transparent and gives us the ability to construct the inhomogeneous quantum group $\mathrm{IGL}(n)$. We will show that $\mathrm{IGL}(n)$ can be embedded in the multiparametric deformation of $\mathrm{GL}(n+1)$, while the same constraints on the parameters have been imposed.

We will also construct the $\mathrm{U}_q(\mathrm{igl}(n))$ á la Drinfeld-Jimbo. It is worthy of mention that in the classical case the universal enveloping algebra as an algebra is the same as the algebra of vector fields, however, in the quantum case these two algebras are different.[9] The deformed algebra of vector fields may be constructed by the Woronowicz's method. The differential calculus and the algebra of vector fields for $\mathrm{IGL}(n)$ has been constructed in Ref. 10. Unfortunately, the algebra of vector fields is not equipped with a coproduct, and so it is not a Hopf algebra.

In the next two sections we deal with $\mathrm{IGL}_q(n)$ and $\mathrm{U}_q(\mathrm{igl}(n))$, respectively. To clarify our approach, in each section we will first consider the two-dimensional case $n=2$, and then we will consider the higher dimensions. Throughout this paper, when we use a subscript q for an object, we mean the multiparametric deformation of that object. The number of and the conditions on the parameters will be clear from the context. Multiparametric quantum groups have been studied by various authors.[11–17] In this paper we will use the notations of Ref. 17.

II. EMBEDDING $\mathrm{IGL}_q(N)$ IN $\mathrm{GL}_q(N+1)$

Let us consider the two-dimensional inhomogeneous group IGL(2). Its action on the two-dimensional plane R^2 is

$$x \mapsto x' = ax + by + u, \quad y \mapsto y' = cx + dy + v. \tag{1}$$

This action may be represented on the $z=1$ subset of R^3 by means of the following matrix:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \tag{2}$$

We would like to study the quantum deformed version of this embedding. First of all, we have to consider a two-dimensional quantum space R_q^2 spanned by (x,y). If this noncommutative space is to be embedded in a three-dimensional quantum space R_q^3 with coordinates (x,y,z) by setting $z=1$, then z must commute with everything, otherwise it would be meaningless to set $z=1$. The embedding is possible only if we use the multiparameteric quantum space R_q^3 with a specific fixing of the parameters. To see this we write the commutation relations of the coordinates of R_q^3:

$$x_i x_j = q p_{ij} x_j x_i, \quad i<j. \tag{3}$$

Therefore, $x_3 = z$ is central only if $p_{13} = p_{23} = q^{-1}$.

The multiparameteric quantum group $\mathrm{GL}_q(3)$[17] is generated by the entries of the following T matrix:

$$T = \begin{pmatrix} a & b & u \\ c & d & v \\ e & f & w \end{pmatrix}. \tag{4}$$

If we set $e=f=0$, then w will automatically commute with a, b, c, and d. However, w has nontrivial commutation relations with u and v. To make w central, we have to impose the same condition $p_{13} = p_{23} = q^{-1}$. We set $w=1$ and call the resulting quantum group, $\mathrm{IGL}_q(2)$.

For completeness we write the commutation relations among the generators,

$$\begin{aligned}
&ab = qp^{-1}ba, \quad au = q^2 ua, \quad bu = q^2 ub, \\
&cd = qp^{-1}\, dc, \quad cv = q^2 vc, \quad dv = q^2 vd, \\
&ac = qpca, \quad bd = qp\, db, \quad uv = qpvu, \\
&bc = p^2 cb, \quad uc = q^{-1} pcu, \quad ud = q^{-1} p\, du, \\
&ad - da = \lambda p^{-1} bc, \quad av - qpva = q\lambda uc, \quad bv - qpvb = q\lambda ud,
\end{aligned} \tag{5}$$

where $p := p_{12}$. The coproduct, counity, and the antipode for this quantum group are also inherited from $\mathrm{GL}_q(3)$.

Generalizing this to n dimensions is straightforward. Using the multiparameteric quantum space R_q^{n+1} and the corresponding quantum group $\mathrm{GL}_q(n+1)$, we set $p_{i\,n+1} = q^{-1}$ for $i = 1,2,\ldots,n$; and $T_{n+1\ i} = 0$ for $i = 1,2,\ldots,n$, and then, because of the fixing of parameters, $x_{n+1} \in R_q^{n+1}$ and $w := T_{n+1\ n+1} \in \mathrm{GL}_q(n+1)$ become central. Setting $x_{n+1} = 1$ and $w = 1$, one gets the quantum group $\mathrm{IGL}_q(n)$.

III. EMBEDDING THE UNIVERSAL ENVELOPING ALGEBRAS

Now we want to consider the universal enveloping algebra $U_q(\mathrm{igl}(n))$ á la Drinfeld-Jimbo. Classically $U(igl(n))$ is a sub-Hopf algebra, but the quantum analog of this is not true for the standard deformation $U_q(gl(n+1))$. In fact $U_q(igl(n))$ is a subalgebra of $U_q(gl(n+1))$, but it is not a sub-Hopf algebra of $U_q(gl(n+1))$. We show that this difficulty can be overcome using the multiparametric deformation of $U_q(gl(n+1))$ with a specific choice of parameters.

In the classical case, one can consider the infinitesimal form of the transformation (2), and obtain the generators of the infinitesimal transformations of igl(2) as follows:

$$\begin{aligned} &H_1=e_{11}-e_{22}, \quad X_1^+=e_{12} \quad X_1^-=e_{21}, \\ &X_2^+=e_{23}, \quad X_{12}^+=e_{13} \quad K=e_{11}+e_{22}, \end{aligned} \tag{6}$$

where $[e_{ij}]_{kl}=\delta_{ik}\delta jl$. The set $\{H_1,X_1^+,X_1^-\}$ generates $SL(2)$; K generates dilatations (scaling), and, X_2^+ and X_{12}^+ are the generators of the translations. Note that this Lie algebra, igl(2), is a subalgebra of the gl(3), with the other generators of gl(3) being

$$H_2=e_{22}-e_{33}, \quad X_2^-=e_{32}, \quad X_{12}^-=e_{31}. \tag{7}$$

The generator I, which is a generator of gl(3), and commutes with everything, is not an independent generator,

$$I=\tfrac{3}{2}K-\tfrac{1}{2}H_1-H_2. \tag{8}$$

Now we turn to the quantization of the universal enveloping algebra U(igl(2)). There exists a multiparameteric deformation of U(gl(3)) that we name U_q(gl(3)). The deformation parameters are q, s_{12}, s_1, and s_2. It is known that in the multiparametric deformations the algebra is actually one parametric and the extra parameters enter in the coproduct. This Hopf algebra is generated by nine generators H_1, H_2, $X_1^\pm$, $X_2^\pm$, $X_{12}^\pm$ and I with the following Hopf structure:

$$\begin{aligned} &[I,\cdots]=0, \quad [H_1,H_2]=0, \\ &[H_i,X_j^\pm]=\pm a_{ij}X_j^\pm, \quad [X_i^+,X_j^-]=\delta_{ij}[H_i]_q, \\ &[X_{12}^+,X_{12}^-]=[H_1+H_2]_q, \quad [X_2^\pm,X_1^\pm]_{q^{\pm1}}=\pm X_{12}^\pm, \\ &[X_1^+,X_{12}^+]_q=0, \quad [X_1^-,X_{12}^-]_{q^{-1}}=0, \\ &[X_2^+,X_{12}^+]_{q^{-1}}=0, \quad [X_2^-,X_{12}^-]_q=0, \\ &[X_1^-,X_{12}^+]=q^{1/2}q^{H_1}X_2^+, \quad [X_1^+,X_{12}^-]=-q^{1/2}X_2^-q^{H_1}, \\ &[X_2^-,X_{12}^+]=q^{1/2}q^{H_2}X_1^+, \quad [X_2^+,X_{12}^-]=-q^{1/2}X_1^-q^{H_2}, \end{aligned} \tag{9}$$

$$\begin{aligned} &\Delta(X_1^\pm)=X_1^\pm\otimes q^{(1/2)H_1}s_{12}^{\pm[(2/3)H_2+(1/3)H_1)}s_1^{\pm I}+q^{(-1/2)H_1}s_{12}^{\mp[(2/3)H_2+(1/3)H_1]}s_1^{\mp I}\otimes X_1^\pm, \\ &\Delta(X_2^\pm)=X_2^\pm\otimes q^{(1/2)H_2}s_{12}^{\mp[(1/3)H_2+(2/3)H_1)}s_2^{\pm I}+q^{-(1/2)H_2}s_{12}^{\pm[(1/3)H_2+(2/3)H_1]}s_1^{\pm I}\otimes X_2^\pm, \\ &\Delta(X_{12}^\pm)=\pm[\Delta(X_2^\pm),\Delta(X_1^\pm)]_{q^{\pm1}}, \\ &\Delta(H_i)=H_i\otimes1+1\otimes H_i, \\ &\Delta(I)=I\otimes1+1\otimes I, \end{aligned} \tag{10}$$

$$\begin{aligned} &S(H_i)=-H_i, \quad S(I)=-I, \\ &S(X_i^\pm)=-q^{\pm2}X_i^\pm, \quad S(X_{12}^\pm)=-q^{\pm4}X_{12}^\pm, \end{aligned} \tag{11}$$

where $[a_{ij}]$ is the Cartan matrix of sl(3) and

$$[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{12}$$

$$[x,y]_q := q^{1/2}xy - q^{-1/2}yx. \tag{13}$$

Following the analogy with the classical case, we expect that also in the quantum case, the subalgebra generated by $\{H_1,K,X_1^{\pm},X_2^{+},X_{12}^{+}\}$, where K is given by (8), is the quantum group $U_q(\mathrm{igl}(2))$. From the commutation relations (9) it is obvious that the algebra generated by the set $\{H_1,K,X_1^{\pm},X_2^{+},X_{12}^{+}\}$, is closed, i.e. X_2^{-}, X_{12}^{-}, I, and H_2 do not appear in the commutation relations. However, as we see the generators H_2 and I appear in the coproduct of $X_1^{\pm}$ and X_2^{+}. For the standard one-parametric deformation it is not possible to embed $U_q(\mathrm{igl}(2))$ in $U_q(\mathrm{gl}(3))$. But if we use the multiparametric deformation, it is possible to construct $\mathrm{U}_q(\mathrm{igl}(2))$ and embed it in $\mathrm{U}_q(\mathrm{gl}(3))$ with its full Hopf structure, provided we fix the parameters in a specific way such that the extra generators in the coproduct disappear,

$$s_{12} := s, \quad s_1 = s^{2/3}, \quad s_2 = q^{1/2}s^{-1/3}. \tag{14}$$

Now the co-products of the generators of $\mathrm{U}_q(\mathrm{igl}(2))$ are

$$\Delta(X_1^{\pm}) = X_1^{\pm} \otimes q^{(1/2)H_1} s^{\pm(1/2)(H_1-K)} + q^{-(1/2)H_1} s^{\mp(1/2)(H_1-K)} \otimes X_1^{\pm},$$

$$\Delta(X_2^{+}) = X_2^{+} \otimes (q^{1/2}s)^{-(1/2)H_1}(q^{3/2}s^{-1})^{(1/2)K} + (q^{1/2}s)^{(1/2)H_1}(q^{3/2}s^{-1})^{-(1/2)K} \otimes X_2^{+},$$

$$\Delta(H_1) = H_1 \otimes 1 + 1 \otimes H_1, \quad \Delta(K) = K \otimes 1 + 1 \otimes K. \tag{15}$$

The coproduct of X_{12}^{+} may be obtained from these relations, commutation relations, and the fact that Δ is a homomorphism of algebra.

At this point it is convenient to introduce another notation for the generators that reflects their ‘‘geometric’’ meaning better,

$$(K,J_3,J_+,J_-,P_1,P_2) := (K,H_1,X_1^{+},X_1^{-},X_{12}^{+},X_2^{+}). \tag{16}$$

Here $J_{\pm}$ and J_3 together generate an sl(2), P_1 and P_2 are generators of translations, and K is the generator of dilatation.

To generalize this method to higher dimensions, we use the multiparametric deformation of $\mathrm{U}(\mathrm{gl}(n+1))$, where s_{ij} and s_i are extra parameters of deformation, $s_{ij} = s_{ji}{}^{-1}$ and $s_{ii}=1$. Therefore, the number of parameters is $1+\frac{1}{2}n(n+1)$,

$$\{q,s_i,s_{ij},i,j=1,2,\ldots,n\}, \quad \text{where } s_{ij}=s_{ji}^{-1}, \quad \text{and} \quad s_{ii}=1, \tag{17}$$

and n^2 generators,

$$\{H_i,I,X_{ij}^{\pm},i \leqslant j=1,2,\ldots,n\}. \tag{18}$$

Here, for notational convenience, we have named $X_{ii}^{\pm} := X_i^{\pm}$, which are simple roots, and $X_{ij}^{\pm}$ for $i \neq j$ are composite roots, e.g., for positive composite roots we have

$$X_{ij}^{+} = [\cdots[Xs+j,[Xs+j-1,[\cdots,Xs+i,], \tag{19}$$

$\mathrm{U}_q(\mathrm{gl}(n))$ is generated by the following subset of generators:

$$\{H_i,K,X_{ij}^{\pm}, \quad i \leqslant j=1,2,\ldots,n-1\}, \tag{20}$$

where

$$K = \frac{1}{n+1} \sum_{i=1}^{n} iH_i + \frac{n}{n+1} I. \tag{21}$$

A basis for $\mathrm{igl}(n)$ is obtained by adding to the set (20) the following generators for translations:

$$P_i := X_{in}^{+}, \quad i=1,2,\ldots,n. \tag{22}$$

In the quantum case too, we introduce this set as the generating set of the multiparametric $\mathrm{U}_q(\mathrm{igl}(n))$,

$$\mathscr{B}_n = \{K, H_i, X_{ij}^{\pm}, \quad i \leqslant j = 1,2,\ldots,n-1, P_k, \quad k \leqslant n\}; \tag{23}$$

$\mathscr{B}$ is a subset of the multiparametric $\mathrm{U}_q(\mathrm{gl}(n+1))$. It is straightforward to check that this set is closed under "commutation." We see that one cannot embed igl(2) as a Hopf algebra in the gl(3), this difficulty is related to the appearance of generators, which are not in B_2, in the coproducts. The same problem occure for the n-dimensional case. It is sufficient to consider the coproduct of simple roots, because the composite roots are q commutations of the simple roots. Let us consider the coproduct of simple roots of the multiparametric $\mathrm{U}_q(\mathrm{gl}(n+1))$,[17]

$$\Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{(1/2)H_i} \left(\prod_{j=1}^{n} s_{ij}^{\pm \Sigma_{k=1}^{n} a_{jk}^{-1} H_k} \right) s_i^{\pm I} + q^{-(1/2)H_i} \left(\prod_{j=1}^{n} s_{ij}^{\mp \Sigma_{k=1}^{n} a_{jk}^{-1} H_k} \right) s_i^{\mp I} \otimes X_i^{\pm}, \tag{24}$$

where $[a_{ij}]$ is the Cartan matrix of $\mathrm{sl}(n+1)$, and a_{ij}^{-1} stands for ijth element of the inverse of the Cartan matrix and their explicit form are as below:

$$a_{ij} = 2\,\delta_{ij} - \delta_{i+1,j} - \delta_{i-1,j}, \tag{25}$$

$$a_{ij}^{-1} = \frac{j(n+1-i)}{n+1} - (j-1)\,\theta_{ij}, \quad \theta_{ij} = \begin{cases} 1, & i<j, \\ 0, & i \geqslant j. \end{cases} \tag{26}$$

Looking at the coproduct one sees that H_n and I appear in some of the terms, e.g. $\Delta(X_i^{\pm})$. However, we have the freedom of fixing the parameters s_{ij} and s_i in such a way that only H_j for $j<n+1$ and K appear in these expressions. If we fix the parameters s_{ij} and s_i as below,

$$s_i = q^{(1/2)\delta_{in}} \prod_{j=1}^{n} s_{ij}^{a_{jn}^{-1}}, \tag{27}$$

the generators H_n and I disappear from the coproducts. This will fix n parameters and we are left with $1+(1/2)n(n-1)$ free parameters. One can check that counity and the antipode also respect this splitting of generators of $\mathrm{U}_q(\mathrm{gl}(n+1))$ for the above mentioned fixing of parameters.

In summary, using the multiparametric deformation of $\mathrm{U}(\mathrm{gl}(n+1))$, a specific fixing of the parameters leads to a Hopf algebra that has a sub-Hopf-algebra generated by $\mathscr{B}$, which we call it $\mathrm{U}_q(\mathrm{igl}(n))$. It depends on $1+(1/2)n(n-1)$ parameters. Here, $\mathrm{U}_q(\mathrm{igl}(n))$ is generated by the set of generators of its subalgebra, $\mathrm{U}_q(\mathrm{gl}(n-1))$ and n translations, $P_i := X_{in}^{+}$. The commutation relations are the followings:

$$\begin{aligned} &[H_i, H_j] = 0, \quad [K, \cdots] = 0, \\ &[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm}, \quad [X_i^{+}, X_j^{-}] = \delta_{ij}[H_i], \end{aligned} \tag{28}$$

$$(X_{i\pm1}^{+})^2 X_i^{\pm} - (q+q^{-1}) X_{i\pm1}^{\pm} X_i^{\pm} X_{i\pm1}^{\pm} + X_i^{\pm} (X_{i\pm1}^{\pm})^2 = 0, \tag{29}$$

$$
\begin{aligned}
&[H_i,P_j]=\sum_{k=j}^{n} a_{ik}P_j, \quad [K,P_i]=P_i \\
&[X_i^+,P_i]_q=0, \quad [P_i,X_j^-]=\delta_{ij}q^{-1/2}P_{i+1}q^{H_i}, \\
&[P_i,X_{i-1}^+]_q=P_{i-1}, \quad [P_i,X_j^+]=0, \quad |i-j|\leqslant 2.
\end{aligned} \tag{30}
$$

Note that we have written the subalgebra $\mathrm{gl}(n)$ in the Chevalley basis. The coproducts are

$$\Delta(H_i)=H_i\otimes 1+1\otimes H_i, \tag{31}$$

$$\Delta(K)=K\otimes 1+1\otimes K, \tag{32}$$

$$
\begin{aligned}
\Delta(X_i^{\pm})=X_i^{\pm}\otimes q^{(1/2)H_i}\left(\prod_{j=1}^{n} s_{ij}^{\pm\{\Sigma_{k=1}^{n-1}[k(n-1)/n-(k-1)\theta_{jk}]H_k+(1-j+1/n)K\}}\right) \\
+q^{-(1/2)H_i}\left(\prod_{j=1}^{n} s_{ij}^{\mp[\Sigma_{k=1}^{n-1}[k(n-1)/n-(k-1)\theta_{jk}]H_k+(1-j+1/n)K]}\right)\otimes X_i^{\pm},
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
\Delta(P_n)=P_n\otimes q^{-1/2\Sigma_{k=1}^{n-1}(k/n)H_n}\left(\prod_{j=1}^{n} s_{nj}^{\Sigma_{k=1}^{n-1}[k(n-1)/n-(k-1)\theta_{jk}]}\right)^{H_k}\left(q^{(n+1)/2n}\left(\prod_{j=1}^{n} s_{nj}^{1-j+1/n}\right)\right)^{K} \\
+q^{1/2\Sigma_{k=1}^{n-1}(k/n)H_n}\left(\prod_{j=1}^{n} s_{nj}^{\Sigma_{k=1}^{n-1}[-k(n-1)/n+(k-1)\theta_{jk}]}\right)^{H_k}\left(q^{(-n+1)/2n}\left(\prod_{j=1}^{n} s_{nj}^{-1+j-1/n}\right)\right)^{K}\otimes P_n,
\end{aligned} \tag{34}
$$

for computing these coproducts one must use (26). The coproducts of the other P_i's can be obtained by using the fact that Δ is a homomorphism of the algebra and using the following relation:

$$P_i=[\cdots[[[P_n,X_{n-1}^+]_q,X_{n-2}^+]_q,X_{n-3}^+]_q,\ldots,X_i^+]_q. \tag{35}$$

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