Similarity solution for an unsteady dry patch in a shear-stress-driven thin film of non-Newtonian power-law fluid

Siti Sabariah Abas and Yazariah Mohd Yatim

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Similarity Solution for an Unsteady Dry Patch in a Shear-Stress-Driven Thin Film of Non-Newtonian Power-Law Fluid

Siti Sabariah Abas and Yazariah Mohd Yatim

School of Mathematical Sciences, Universiti Sains Malaysia 11800 USM, Penang, Malaysia

Abstract. In this study, the lubrication approximation is used to derive the governing equations for the unsteady flow of non-Newtonian power-law fluid around a symmetric slender dry patch on an inclined plane. The flow is driven by a constant shear stress at the free surface, and we assume that surface-tension effect is neglected. The governing partial differential equation is transformed into the ordinary differential equation using a unique travelling-wave similarity transformation. We obtain a one-parameter family of solutions parameterised by the non-dimensional velocity of the dry patch, c. The similarity solution is independent of the power-law index, N and it predicts that the dry patch has a parabolic shape.

Keywords: unsteady thin-film flow; travelling-wave similarity solution; dry patch; non-Newtonian fluid
PACS: 47.50.Cd

INTRODUCTION

The flow of a thin liquid film is an important phenomenon in nature and industrial processes, in which the breakdown of the film may lead to the formation of dry patch: a non-wetted surface. One of the earliest studies of dry patch was performed by Hartley and Murgatroyd [1], who considered a flowing film draining under both gravity and surface shear stress. The influential factors for the stability of dry patch have been proposed. This work was extended by Murgatroyd [2] to the case of surface shear stress, and later by Ruckenstein [3] who considered the dynamic contact angle to study the stability of a dry patch. Wilson et al. [4] obtained two similarity solutions for steady flow around a thin, slender dry patch in the case of weak and strong surface-tension effects. In the case of weak surface-tension effect, the solution shows that the transverse profile of the free surface has a monotonically increasing shape, and the dry patch has a parabolic shape. In the case of strong surface-tension effect, it is found that the transverse profile of the free surface has an oscillatory shape with a characteristic "capillary ridge" near the contact line, and the dry patch has a quartic shape. Yatim et al. [5], on the other hand, obtained similarity solutions for the unsteady shear-stress-driven flow around a dry patch for both Newtonian and non-Newtonian power-law fluids. The influence of flow rate on the shape and stability of dry patches has been studied by Rio and Limat [6] experimentally. They investigated the effect of increasing and decreasing the flow rate, and obtained the minimum and maximum flow rates for the dry patch to persist. Using a similar approach to [5], Yatim et al. [7] found a different kind of similarity solution, namely a travelling-wave solution for unsteady flow of Newtonian fluid around a dry patch. Then, they extended their work [8] to flow driven by gravity and/or a prescribed constant shear stress on the free surface. In the present study, we consider unsteady shear-stress-driven flow of an infinitely wide thin film of non-Newtonian power-law fluid around a symmetric slender dry patch on an inclined plane. Motivated by our previous study on gravity-driven flow of non-Newtonian fluid around a dry patch [9], the present work is an extension of [7] and [8] to the flow of non-Newtonian fluid.

THE MATHEMATICAL MODEL

Consider a thin-film flow of non-Newtonian power-law fluid flowing around a dry patch on an inclined plane with angle of inclination α, as shown in Figure 1. The coordinate z is perpendicular to the plane and the x, y coordinates lie on the plane, with the y axis horizontal and the x axis in the downward direction. We denote the components of velocity of the fluid in the direction x, y and z as u, v and w, respectively. The free-surface profile of the film is denoted by $z = h(x, y, t)$, where t denotes time, with the thickness of the film h. We shall consider the

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flow to be symmetric about $y = 0$ with (unknown) semi-width $a = a(x, t)$, so that the film has zero thickness at its contact lines $y = \pm a$.

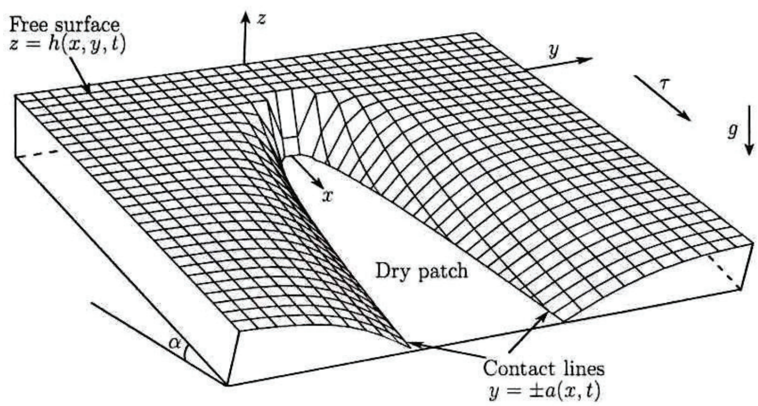


FIGURE 1. Geometry of the problem.

With the familiar lubrication approximation, the velocity (u, v, w), pressure p and the thickness h satisfy the governing equations

$$u_x + v_y + w_z = 0, \tag{1}$$
$$(\mu u_z)_z = 0, \tag{2}$$
$$(\mu v_z)_z - p_y = 0, \tag{3}$$
$$-p_z - \rho g \cos\alpha = 0, \tag{4}$$

where ρ denotes constant density, g denotes gravitational acceleration and μ is the variable viscosity given by $\mu = \mu_0 \gamma^{N-1}$, where μ_0 is a consistency coefficient of the fluid, γ is the local shear rate given by $\gamma = u_z$, approximately and $N (> 0)$ is the power-law index which determines the behavior of the flow. The fluid is characterised as shear thinning (pseudoplastic) for $0 < N < 1$, shear thickening (dilatant) for $N > 1$ and a Newtonian fluid for $N = 1$. By integrating equations (1)-(4) and imposing the boundary conditions on the substrate $z = 0$,

$$u = v = w = 0, \tag{5}$$

and on the free surface $z = h$,

$$p = p_a, \qquad \mu u_z = \tau, \qquad \mu v_z = 0, \tag{6}$$

the solutions for p, u and v are found to satisfy

$$p = p_a + \rho g \cos\alpha\,(h - z), \tag{7}$$
$$u = \left(\frac{\tau}{\mu_0}\right)^{\frac{1}{N}} z, \tag{8}$$
$$v = -\frac{p_y}{2\mu}(2hz - z^2), \tag{9}$$

where p_a denotes atmospheric pressure and $\tau\ (> 0)$ denotes the constant surface shear stress. We define a local flux as $\bar{u} = \bar{u}(x, y, t) = \int_0^h u\, dz$. Then, we have

$$\bar{u} = \frac{1}{2}\left(\frac{\tau}{\mu_0}\right)^{\frac{1}{N}} h^2. \tag{10}$$

Similarly, for $\bar{v} = \bar{v}(x, y, t) = \int_0^h v\, dz$, we have

$$\bar{v} = -\frac{p_y}{3\mu} h^3, \tag{11}$$

so that the kinematic condition on $z = h$ given by $h_t + \bar{u}_x + \bar{v}_y = 0$ can be written in the form

$$3\mu h_t = \rho g \cos\alpha \left[h^3 h_y\right]_y - \frac{3}{2}\tau[h^2]_x, \tag{12}$$

where $\mu = \mu_0(\tau/\mu_0)^{(N-1)/N}$, with the appropriate conditions at the contact lines, namely

$$h = 0 \quad \text{at} \quad y = \pm a, \qquad h^3 h_y \to 0 \quad \text{as} \quad y \to \pm a. \tag{13}$$

Now, we consider an unsteady travelling-wave similarity solution in the form

$$\begin{aligned} h &= h_\infty F(\eta), \qquad \eta = \frac{y}{[l(x-ct)]^{1/2}}, \qquad &&\text{if} \quad l(x-ct) \geq 0, \\ h &= h_\infty, &&\text{if} \quad l(x-ct) < 0, \end{aligned} \tag{14}$$

where h_∞ is the uniform thickness of the fluid far from the contact line, c is the velocity of the dry patch, the constant $l(> 0)$ is to be specified and the dimensionless function $F = F(\eta)(\geq 0)$ of the dimensionless similarity variable η is to be determined. By applying (14) to (12), we obtain

$$4\rho g \cos\alpha\, {h_\infty}^3 (F^3 F')' + l\eta(3\tau h_\infty F^2 - 6\mu F c)' = 0. \tag{15}$$

By choosing $l = 4\rho g \cos\alpha {h_\infty}^2 / 3\tau$, we have

$$(F^3 F')' + \eta \left(F^2 - \frac{c}{U} F\right)' = 0, \tag{16}$$

where $U = \tau h_\infty / 2\mu$ and a prime denotes differentiation with respect to η. We denote the unknown position of the contact line by $\eta = \eta_0$ (corresponding to the contact-line position $y = a$), so that the semi-width of dry patch is given by

$$a = \sqrt{l(x-ct)}\eta_0, \tag{17}$$

which varies with x and t, obviously identifying that the dry patch has a parabolic shape. From (13), we also have

$$F = 0 \quad \text{at} \quad \eta = \eta_0, \qquad F^3 F' \to 0 \quad \text{as} \quad \eta \to \eta_0, \tag{18}$$

and the far-field condition is given by

$$F \to 1 \quad \text{as} \quad \eta \to \infty. \tag{19}$$

In order to determine the cross-sectional area for the dry patch, we evaluate the difference between the cross-sectional area of the region where the film thickness is uniform, h_∞, and the cross-sectional area of the region where the film thickness is $h_\infty F(\eta)$, denoted by ΔA. We have

$$\Delta A = \lim_{y_\infty \to \infty} 2\left(h_\infty y_\infty - \int_a^{y_\infty} h\, dy\right) = 2h_\infty \sqrt{l(x-ct)}\, q_{area}, \tag{20}$$

where $q_{area} = \eta_0 + \int_{\eta_0}^{\eta_\infty} (1 - F) d\eta$. We apply the same approach to evaluate the volume flux of the fluid, denoted by ΔQ, to yield

$$\Delta Q = \lim_{y_\infty \to \infty} 2\left(U h_\infty y_\infty - \int_a^{y_\infty} \bar{u}\, dy\right) = \left(\frac{\tau}{\mu_0}\right)^{\frac{1}{N}} {h_\infty}^2 \sqrt{l(x-ct)} \left[\eta_0 + \int_{\eta_0}^{\eta_\infty} (1 - F^2)\, d\eta\right]. \tag{21}$$

By integrating (15) and using (18)-(21) we obtain a relationship between ΔA and ΔQ, namely

$$\Delta Q = c\Delta A = 2h_\infty\sqrt{l(x-ct)}cq_{area}. \tag{22}$$

We may also write (22) in the form

$$\Delta Q = c\Delta A = 2Uh_\infty\sqrt{l(x-ct)}q_{flux}, \qquad q_{flux} = \frac{c}{U}q_{area}. \tag{23}$$

Then, the introduction of non-dimensional variables in terms of

$$\begin{gathered} x = Xx^*, \qquad y = \sqrt{lX}y^*, \qquad z = h_\infty z^*, \qquad t = \frac{X}{U}t^*, \qquad c = Uc^*, \\ h = h_\infty h^*, \qquad a = \sqrt{lX}a^*, \qquad \Delta A = h_\infty\sqrt{lX}\Delta A^*, \qquad \Delta Q = Uh_\infty\sqrt{lX}\Delta Q^*, \end{gathered} \tag{24}$$

where X is a length scale in the x direction, which we may choose arbitrarily, transforms (16) into the ordinary differential equation

$$(F^3F')' + \eta(F^2 - cF)' = 0 \tag{25}$$

and we have dropped the star symbol for convenience. Note that the ordinary differential equation (25) is independent of N and therefore it is the same as the ordinary differential equation in [7]. Also, we now finally can rewrite (20) and (23) as

$$\Delta A = 2\sqrt{(x-ct)}q_{area}, \qquad \Delta Q = c\Delta A = 2\sqrt{(x-ct)}q_{flux}, \qquad q_{flux} = cq_{area}, \tag{26}$$

respectively.

Conditions for dry patch to be thin and slender are that the length scales in the x, y and z direction (namely X, $\sqrt{lX}$ and h_∞), satisfy $h_\infty \ll \sqrt{lX} \ll X$, so that $X \gg \tau/\rho g\cos\alpha$ and $X \gg {h_\infty}^2\rho g\cos\alpha/\tau$, respectively, showing that X must be sufficiently larger than h_∞ and that α cannot be close to $\pi/2$.

We find that

$$F \sim [3c\eta_0(\eta - \eta_0)]^{\frac{1}{3}} \tag{27}$$

near the contact line $\eta = \eta_0$, provided that the non-dimensional velocity of the dry patch c, is greater than zero. Then, we write $F = 1 + f$ with $|f| \ll 1$, to find that

$$F - 1 \propto \frac{1}{\eta}\exp\left[-\frac{2-c}{2}\eta^2\right] \tag{28}$$

in the limit $\eta \to \eta_\infty$, showing that c should be within the range $0 < c < 2$.

RESULTS AND DISCUSSION

A numerical solution of the ordinary differential equation (25) was computed using a shooting method via the computational software Mathematica, by shooting from the contact line position $\eta = \eta_0$ with an initial guess for c. The correct value of c is chosen so that the solution satisfies (19). However, due to the singularity given by (27), the computation cannot be started exactly at $\eta = \eta_0$, but it is instead begun at $\eta = \eta_0 + \delta$ for sufficiently small $\delta (\delta > 0)$, subject to the approximated boundary conditions given by

$$F(\eta_0 + \delta) = (3c\eta_0\delta)^{\frac{1}{3}}, \qquad F'(\eta_0 + \delta) = \left(\frac{c\eta_0}{9\delta^2}\right)^{\frac{1}{3}}. \tag{29}$$

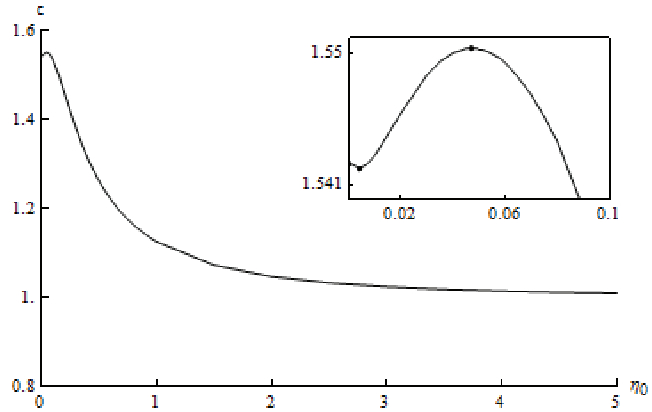


FIGURE 2. Plot of c as a function of η_0 with $\delta = 10^{-20}$. The inset shows an enlargement of the behavior near $\eta_0 = 0$.

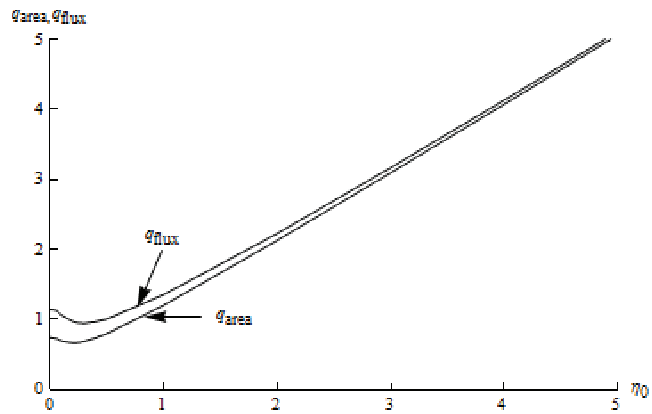


FIGURE 3. Plot of q_{area} and q_{flux} as functions of η_0.

The one-parameter family of solutions parameterised by the non-dimensional velocity of the dry patch c is shown in Figure 2. The plot behaves non-monotonically as c decreases from its value $c = c_0 \simeq 1.5424$ at $\eta_0 = 0$ to a local minimum value $c = c_{min} \simeq 1.5421$ at $\eta_0 \simeq 0.0040$. Then, it reaches its maximum value $c = c_{max} \simeq 1.5503$ at $\eta_0 \simeq 0.0470$ before it turns to decrease, approaching one as $\eta_0 \to \infty$. These results thus confirm the results obtained in [7]. Figure 2 also demonstrates as c decreases, the dry patch becomes wider. Figure 3 shows plots of q_{area} and q_{flux} $(= c q_{area})$ as functions of η_0, demonstrating that each of them decreases from its value at $\eta_0 = 0$ to a minimum value, and then increases monotonically with η_0. Figure 4 shows three-dimensional plots of the free-surface profiles h with different contact line positions, $\eta_0 = 0.5$, 1 and 1.5 at time $t = 1$, and Figure 5 illustrates three-dimensional plots at times $t = 0$, 1 and 3 with $\eta_0 = 0.5$. As time increases, the dry patch travels down the plane and becomes narrower, and then disappears with the flow.

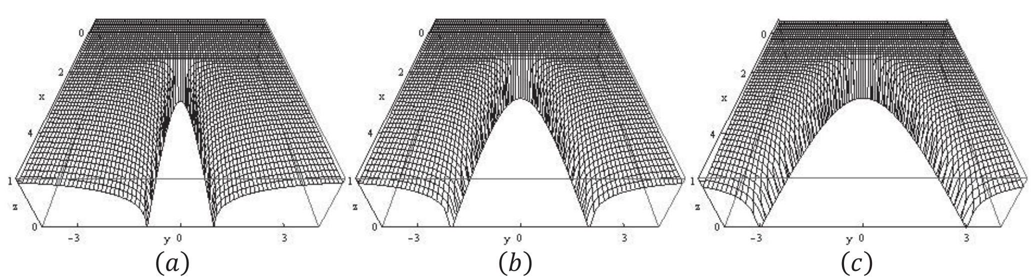
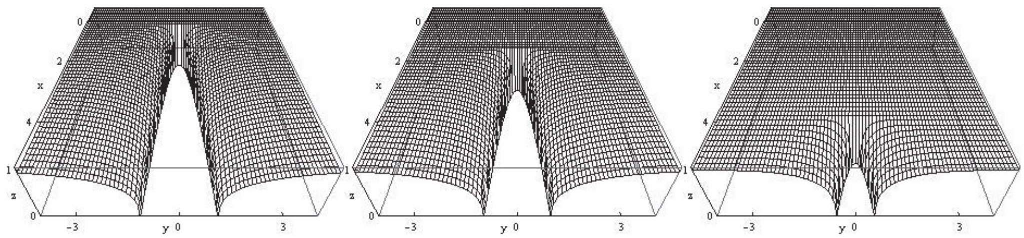


FIGURE 4. Three-dimensional plots of the free-surface profiles h with different contact line positions, $(a)\ \eta_0 = 0.5$, $(b)\ \eta_0 = 1$ and $(c)\ \eta_0 = 1.5$ at time $t = 1$.



(a) (b) (c)

FIGURE 5. Three-dimensional plots of the free-surface profiles h at different times, (a) $t = 0$, (b) $t = 1$ and (c) $t = 3$ with $\eta_0 = 0.5$.

CONCLUSIONS

The problem for unsteady shear-stress-driven flow around a symmetric slender dry patch has been investigated in the past by [7] and [8] but only for Newtonian fluid. Following their approaches, we obtain the travelling-wave similarity solution for the flow of non-Newtonian power-law fluid around a dry patch. The solution predicts that the dry patch travels down the plane at constant velocity c and that it has a parabolic shape, its scaled semi-width η_0 varying like $(x - ct)^{1/2}$, and the thickness of the fluid film increases monotonically away from the dry patch. The ordinary differential equation obtained here is found to be independent of the power-law index N and thus automatically describes the same solutions found in [7] and somewhat similar to [8] (with different scaling) in the case of purely shear-stress-driven flow. Using shooting methods, we find that the dimensionless velocity of the dry patch satisfies $1 \leq c \leq c_{max} \simeq 1.5503$, and this is valid for both Newtonian and non-Newtonian fluid. We observe that for any choice of η_0, there is a unique solution whose c takes values in a finite interval $1 < c \leq c_{max}$, but, for any choice of c, there can be zero, one, two or three dry patches which have different width that travel at the same velocity. In this paper, the evolutions of a dry patch due to the changes of η_0 and t are also observed. It is worth mentioning that there are significant differences between shear-stress-driven flow studied herein and the gravity-driven flow studied in [9]. The solutions obtained here are not influenced by the power-law index N, unlike in [9] in which the variation of N used is well illustrated.

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