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Lower hybrid wave scattering by density fluctuations

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A wave kinetic equation is formulated for lower hybrid scattering by low frequency density fluctuations (such as those recently observed in tokamaks). Implications for heating of tokamak plasmas are discussed.

I. INTRODUCTION

Recently, density fluctuation levels in tokamaks have been measured by laser[1,2] and microwave[3] scattering techniques. Such fluctuations (presumably drift waves) are of obvious interest for confinement. Another important issue which we address here is the effect that wave scattering by these fluctuations will have on radio frequency plasma heating by lower hybrid waves. Previous studies include those of Bellan and Wong,[4] and Sen,[5] and Berger *et al.*[6] Bellan and Wong use the geometrical optics approximation to compute the diffusive effects of wave scattering. However, because of the large off-diagonal components of the lower hybrid wave dielectric tensor (due to wave induced $\mathbf{E}\times\mathbf{B}$ drifts), the conditions for the validity of the geometric optics approximation become very restrictive and are not satisfied in typical tokamak situations. Sen uses a formalism developed by Keller[7] to obtain the attenuation of the phase coherent part of the wave due to scattering. Calculation of this quantity, however, does not give information concerning the fate of the scattered energy. Berger *et al.* obtain an estimate for the lower hybrid wave scattering length for the case in which the wavelengths of the low frequency density fluctuation and the lower hybrid wave are comparable, i.e., large angle scattering. More recently, work on the subject has been presented by Sen and Fisch[8] and by Hsu *et al.*[9] Sen and Fisch utilize the diffusion approximation [as in our Eq. (4′)] and given an estimate of the wave diffusion coefficient due to wave induced $\mathbf{E}\times\mathbf{B}$ drifts. Hsu *et al.*[9] present a wave Fokker–Plank equation approach which utilizes geometrical optics to describe the scattering process.

In Sec. II we formulate a wave kinetic equation for the propagation of lower hybrid waves in an inhomogeneous plasma column with density fluctuations. Our formulation includes as limits both the $\mathbf{E}\times\mathbf{B}$ contribution (usually dominant) and the geometrical optics contribution and is applicable for aribitrary lower hybrid wavelengths (long wavelengths are usually most relevant.) It is found that scattering can be important for present-day and reactor-size tokamaks. Asymptotic expansions of the wave kinetic equation for the strong (Sec. III) and weak scattering (Sec. IV) cases are made, resulting in a spatial diffusion equation for strong scattering and resonance cone spreading for weak scattering.

II. WAVE KINETIC EQUATION AND SCATTERING

We consider the wave kinetic equation in a radially inhomogeneous cylindrical plasma with coordinates r, θ, z (toricity is neglected). The propagation in the averaged medium (i.e., with fluctuations suppressed) is well-described by the ray approximation. (Note, however, that the ray approximation may not be valid for computing the scattering.) We introduce a wave distribution function $F(r,\theta,z;k_r,m,k_z)$, where $m=rk_\theta$, such that $\int F\,dk_r\,dm\,dk_z$ is the wave energy density. (k_r, k_θ and k_z are the r, θ, and z components of the local wavenumber.) Note that m is used as an independent variable (instead of k_θ), since this choice puts the ray equations in Hamiltonian form (i.e., m is the variable which is canonically conjugate to θ). Thus, if we express the local dispersion relation as $\omega=\bar{\omega}(m,k_r,k_z,r)$, the ray equations become $dr/dt=\partial\bar{\omega}/\partial k_r$, $d\theta/dt=\partial\bar{\omega}/\partial m$, $dz/dt=\partial\bar{\omega}/\partial k_z$, $dk_r/dt=-\partial\bar{\omega}/\partial r$, $dm/dt=0$, and $dk_z/dt=0$, where we make use of the assumed homogeneity of $\bar{\omega}$ in θ and z. The wave kinetic equation can then be written

$$(dF/dt)_r=(dF/dt)_s, \tag{1}$$

where $(dF/dt)_s$ is the rate of change of F due to scattering by the low frequency fluctuations, and $(dF/dt)_r$ is the time derivative of F following the path of a ray

$$\left(\frac{dF}{dt}\right)_r\equiv\frac{\partial F}{\partial t}+\frac{\partial\bar{\omega}}{\partial m}\frac{\partial F}{\partial\theta}+\frac{\partial\bar{\omega}}{\partial k_r}\frac{\partial F}{\partial r}+\frac{\partial\bar{\omega}}{\partial k_z}\frac{\partial F}{\partial z}-\frac{\partial\bar{\omega}}{\partial r}\frac{\partial F}{\partial k_r}.$$

We average (1) over θ and z; take the magnetic field to be purely in the z direction; introduce a change of variables: $(m/r)=k_\perp\sin\phi$, $k_r=k_\perp\cos\phi$, $\hat{\omega}(k_\perp,k_z,r)\equiv\bar{\omega}(m,k_r,k_z,r)$ ($\bar{\omega}$ does not depend on ϕ for $\mathbf{B}=B_0\mathbf{z}_0$); and define $\hat{F}(r,k_\perp,k_z,\phi)=k_\perp^{-1}\bar{F}(r,\theta)\delta[k_\perp-k_0(r)]\delta(k_z-k_{z0})$ (appropriate for an incident wave with well-defined k_z and narrow frequency bandwidth). Here, $k_0(r)$ is determined from the local dispersion relation $[\omega=\hat{\omega}(k_0,k_{z0};r)]$, and $\hat{F}\equiv(2\pi L)^{-1}\int_0^L\int_0^{2\pi}F\,d\theta\,dz$ with L a periodicity length in z (similar to the major circumference in a torus). Equation (1) then becomes (cf. Appendix A)

$$\frac{\partial\bar{F}}{\partial t}-\frac{\partial}{\partial r}(v\cos\phi\bar{F})+\sin\phi\frac{v}{r}\frac{\partial\bar{F}}{\partial\phi}+\frac{\omega_0'}{k_0}\frac{\partial}{\partial\phi}(\sin\phi\bar{F})=\left(\frac{d\bar{F}}{dt}\right)_s. \tag{2}$$

In (2), $v(r)$ and $\omega_0'(r)$ are $-\partial\hat{\omega}/\partial k_\perp$ and $\partial\hat{\omega}/\partial r$ evaluated at $k_\perp=k_0(r)$. Note that from the definition of $\bar{F}$, the total θ, z-averaged energy density of lower hybrid waves at r is $\int_0^{2\pi}\bar{F}(r,\phi)\,d\phi$.

We now calculate $(d\bar{F}/dt)_s$, using the weak turbulence approximation. (Subsequently, we examine the validity of this approximation.) The level of low frequency turbulence is taken to be fixed. Also, the low frequency wave frequency and wavenumber along the magnetic field ($\mathbf{B}\equiv B_0\mathbf{z}_0$) are neglected compared with k_z and ω of the lower hybrid wave (a good approximation for the ob-

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served fluctuations).[1-3] Thus, k_z and ω are conserved on scattering. In addition, since $\hat{\omega}=\hat{\omega}(k_\perp, k_z; r)$, $k_\perp$ is also conserved on scattering, and the only scattering is in ϕ. Let $(\omega_{\mathbf{k}},\mathbf{k})$ and $(\omega_{\mathbf{k}'},\mathbf{k}')$ be two lower hybrid waves interacting through a density fluctuation $(0,\mathbf{k}-\mathbf{k}')$. Then, for small amplitudes the wave evolution is described by the mode coupling equation

$$i\left(\frac{\partial\epsilon}{\partial\omega}\right)_{\omega_{\mathbf{k}}}\frac{\partial\hat{\phi}(\mathbf{k})}{\partial t}=V\hat{n}(\mathbf{k}-\mathbf{k}')\phi(\mathbf{k}')\exp[-i(\omega_{\mathbf{k}'}-\omega_{\mathbf{k}})t\,], \tag{3}$$

where (cf. Appendix B)

$$V=\frac{\omega_p^2}{\omega^2}\frac{k_zk_z'}{k^2}+\left(\frac{\omega_{pi}^2}{\omega^2}-\frac{\omega_{pe}^2}{\Omega_{ce}^2}\right)\frac{k_\perp k_\perp'}{k^2}\cos(\phi'-\phi)$$
$$+i\,\frac{\omega_{pe}^2}{\omega\Omega_{ce}}\,\frac{k_\perp k_\perp'}{k^2}\sin(\phi'-\phi)\,;$$

$\hat{n}$ is the Fourier amplitude of the low frequency density fluctuation normalized to the ambient density, and the cold plasma equations have been used in calculating V. Also, $\epsilon=1+(k_\perp/k)^2\epsilon_\perp+(k_\parallel/k)^2\epsilon_\parallel$, $\epsilon_\perp=(\omega_{pe}^2/\Omega_{ce}^2)-(\omega_{pi}^2/\omega^2)$, and $\epsilon_\parallel=-\omega_p^2/\omega^2$. $\omega_{pe}(\omega_{pi})$, Ω_{ce}, and ω_p are the electron (ion) plasma frequency, the electron cyclotron frequency, and $\omega_p^2=\omega_{pe}^2+\omega_{pi}^2$. Applying the weak turbulence approximation[10] to (3) we obtain (cf. Appendix C for details)

$$\left(\frac{d\overline{F}}{dt}\right)_s=\frac{2\pi k_0}{v(\partial\epsilon/\partial\omega)^2}\int_0^{2\pi}J(\beta)S\left(2k_0\sin\frac{\beta}{2}\right)$$
$$\times[\overline{F}(\phi+\beta)-\overline{F}(\phi)]\,d\beta\,, \tag{4}$$

$$J(\beta)=\left[1-2\left(\frac{k_0}{k}\right)^2\epsilon_\perp\sin^2\,\frac{\beta}{2}\right]^{\,2}+\sin^2\beta\frac{\omega_{pe}^4}{(\omega\Omega_{ce})^2}\,,$$

where we have taken the density fluctuation wavenumber spectrum, $S(\zeta)[\langle(\Delta n/N)^2\rangle\equiv\int_0^\infty S(\zeta)2\pi\zeta\,d\zeta]$, to be isotropic perpendicular to $\mathbf{B}$ (this assumption can easily be dropped). The full wave kinetic equation is given by (2) and (4). In Ref. 2 it is found that $S(\zeta)$ can be approximated by a Gaussian

$$S(\zeta)=[\pi(\zeta_0)^2]^{-1}\langle(\Delta n/N)^2\rangle\exp(-\zeta^2/\zeta_0^2)\,. \tag{5}$$

In the integrand of (4) the term $\overline{F}(\phi+\beta)$ represents the increase of $\overline{F}(\phi)$ due to scattering of waves at $\phi+\beta$ into ϕ, while the term in the integrand, $\overline{F}(\phi)$, represents the attenuation of $\overline{F}(\phi)$ due to scattering.

In the case $k_0^2\gg\zeta_0^2$, we can make the approximation, $\overline{F}(\phi+\beta)-\overline{F}(\phi)\cong\beta\,\partial\overline{F}/\partial\phi+(\beta^2/2)\,\partial^2\overline{F}/\partial\phi^2$. Equations (4) and (5) then yield

$$(d\overline{F}/dt)_s=D_{\phi\phi}\,\partial^2\overline{F}/\partial\phi^2\,, \tag{4'}$$

$$D_{\phi\phi}=D_0[1+\tfrac{3}{2}(\omega_{pe}^2/\omega\Omega_{ce})^2(\zeta_0/k_0)^2]\,,$$
$$D_0=\pi^{1/2}\zeta_0\langle(\Delta n/N)^2\rangle[2k_0^2v(\partial\epsilon/\partial\omega)^2_{\omega_{\mathbf{k}}}]^{-1}\,. \tag{6}$$

D_0 is the result that would be obtained if one calculated $D_{\phi\phi}$ assuming that the ray equations can be used to describe the scattering process.[4,9] Here we see, however, that even if $\zeta_0^2\ll k_0^2$, the ray equation result may fail because the second term in (6) can still be significant [in typical situations $(\omega_{pe}^2/\omega\Omega_{ce})^2\sim10^2$ to 10^3]. This second term is due to the coupling of the $\mathbf{E}\times\mathbf{B}$ drift produced by the lower hybrid wave with the low frequency density perturbation. The large term, $(\omega_{pe}^2/\omega\Omega_{ce})^2$, reflects the fact that the electron velocity perturbation produced by a lower hybrid wave in the direction transverse to the wave vector (the $\mathbf{E}\times\mathbf{B}$ drift) is much larger than the corresponding longitudinal velocity perturbation. In order to see if it is to be expected that $k_0^2\gg\zeta_0^2$, we follow Ref. 6 and write the ratio k_0^2/ζ_0^2 in the form

$$\frac{k_0^2}{\zeta_0^2}=\frac{N_\parallel^2}{1-(\omega_{lh}/\omega)^2}\,\frac{m_i}{m_e}\,\frac{T_i}{m_ec^2}\left(1+\frac{\Omega_{ce}^2}{\omega_{pe}^2}\right)^{-1}(\zeta_0\rho_i)^{-2}\,,$$

where we have made use of the dispersion relation for lower hybrid waves in the cold plasma approximation. Here, $N_\parallel=k_zc/\omega$ is the parallel index of refraction, T_i is the ion temperature, $m_ec^2=511$ keV, $m_i(m_e)$ is the ion (electron) mass, $\omega_{lh}=\omega_{pi}[1+(\omega_{pe}/\Omega_{ce})^2]^{-1/2}$ is the lower hybrid frequency, and ρ_i is the ion Larmor radius. For drift waves we expect $\zeta_0\rho_i\sim1$ or less. As an example, first consider a reactor-type plasma with $N_\parallel^2=10$, $m_i/m_e=4000$ (a deuterium-tritium plasma), $\omega_{lh}^2=\frac{1}{2}\omega^2$, $T_i=8$ keV, and $\omega_{pe}^2=\Omega_{ce}^2$. For this case $k_0^2/\zeta_0^2=640/(\zeta\rho_i)^2\gg1$. For an example typical of present-day machines consider the following numbers: $N_\parallel^2=10$, $m_i/m_e=1600$, $\omega_{lh}^2=\frac{1}{2}\omega^2$, $T_i=250$ eV, $\omega_{pe}^2=0.3\,\Omega_{ce}^2$, in which case $k_0^2/\zeta_0^2\lesssim15$ for $\zeta_0\rho_i\gtrsim0.5$. On the other hand, k_0^2/ζ_0^2 can easily be of the order of one if $\zeta_0\rho_i$ is larger or if we consider a region closer to the edge where T_i and $\omega_{pe}^2/\Omega_{ce}^2$ are sufficiently small [cf. Eq. (6)], in which case (4) must then be used instead of (4′).

We define the 90° scattering length

$$l_s=v(\tfrac{1}{2}\pi)^2/(2D_{\phi\phi})\,, \tag{7}$$

for $k_0^2\gg\zeta_0^2$. For the reactor numbers previously cited and with $B\cong50$ kG, $[\langle(\Delta n/N)^2\rangle]^{1/2}\cong0.03$, and $\zeta_0\rho_i\sim1$, we obtain $l_s\sim120$ cm which is much less than the expected minor radius of a reactor (~400 cm). (On the other hand, scattering is weak for fluctuation levels less than 1%.) For the numbers previously cited for a typical present-day machine and with $B\cong17$ kG, $[\langle(\Delta n/N)^2\rangle]^{1/2}\cong0.03$, and $\zeta_0\rho_i\sim0.5$, we obtain $l_s\sim14$ cm. In both of these examples the term proportional to ζ_0^2/k_0^2 in (6) is much larger than one, i.e., the $\mathbf{E}\times\mathbf{B}$ coupling dominates the geometrical optics result.

Another definition of the scattering length which does not depend on the approximation $\zeta_0^2\ll k_0^2$ can be obtained as follows. Noting that $(d\overline{F}/dt)_s$ is a linear operator in ϕ on $\overline{F}$, we represent it symbolically as $L_\phi(\overline{F})$. By direct substitution in (4) it is readily verified that $\cos p\phi$ and $\sin p\phi$ $(p=0,1,2,\ldots)$ are eigenfunctions of the operator L_ϕ. For example, substituting $\cos p\phi$ for $\overline{F}(\phi)$ and $\cos[p(\phi+\beta)]=\cos p\phi\cos p\beta-\sin p\phi\sin p\beta$ for $\overline{F}(\phi+\beta)$ in (4) we obtain $L_\phi(\cos\phi)=-\nu_p\cos\phi$, where the coefficient of $\sin p\phi$ vanishes due to the oddness of $\sin p\beta$ (η and S are even in β), and

$$\nu_p=\frac{4\pi k_0}{v(\partial\epsilon/\partial\omega)^2}\int_0^{2\pi}J(\beta)S\;\,2k_0\,\sin\frac{\beta}{2}\left(\sin^2\,\frac{p\beta}{2}\right)d\beta\,. \tag{8}$$

The quantity ν_p may be interpreted as the damping rate of the pth cosine harmonic of $\overline{F}$ due to scattering. A reasonable definition of the scattering length could then be based on the damping of the lowest-order, ϕ-symmetric, directional component of $\overline{F}$

$$L_s=v/\nu_1\,. \tag{7'}$$

Note that for $\zeta_0^2\ll k_0^2$, we obtain [either from (4′) or (8)] the result $\nu_p \cong p^2 D_{\phi\phi}$. In this case the estimates (7) and (7′) coincide except for a numerical factor $\pi^2/8 = 1.23$.

We have applied Kraichnan's direct interaction approximation[11] to the wave scattering problem. Although we have not been able to solve the resulting equations, they yield Eq. (4) as a limit. In this way a condition for the applicability of the weak turbulence random phase approximation can be obtained. The condition for the validity of (4′) is

$$\zeta_0 l_s > k_0^2/\zeta_0^2 .$$

This condition is easily satisfied for the numerical example utilizing present-day tokamak parameters, and is marginally satisfied for the reactor example.

The effect of weak scattering, $l_s \sim a$ or less (where a is the minor radius), may be beneficial, in that the intense concentration of wave energy in resonance cones is mitigated, and certain parametric instabilities may be suppressed. Furthermore, coherent three-wave parametric processes become less of a problem since the pump loses coherence. On the other hand, for $l_s \ll a$ the wave energy transport becomes a spatial diffusion process, and as we shall show, the wave energy density increases with decreasing (l_s/a). This may increase the possibility of nonresonant parametric instabilities[12] (particularly those for decay waves with large parallel phase velocities which do not experience excessive electron Landau damping).

Finally, we note several effects which have been neglected: poloidal magnetic field, toricity, the finite frequency of the density fluctuations, thermal effects on $\omega(\mathbf{k})$ (important near the mode conversion layer), and electromagnetic effects on $\omega(\mathbf{k})$ (important very close to the plasma edge). These effects can be included within the framework of the present theory and are problems for future study.

III. STRONG SCATTERING

We now consider the wave kinetic equation for the case of strong scattering $l_s \ll a$, and again represent $(d\overline{F}/dt)_s$ symbolically as $L_\phi(\overline{F})$. The wave kinetic equation now becomes [cf. (2)]

$$L_\phi(\overline{F}) = \delta\left[-\frac{\partial}{\partial r}(v\cos\phi\overline{F}) + \sin\phi\frac{v}{r}\frac{\partial\overline{F}}{\partial\phi} + \frac{\omega_0'}{k_0}\frac{\partial}{\partial\phi}(\sin\phi\overline{F}) + \delta\left(\frac{\partial\overline{F}}{\partial t} + \gamma\overline{F}\right)\right], \tag{9}$$

where we have introduced a linear wave damping, γ, and a *formal* expansion parameter δ for strong scattering (actually $\delta \equiv 1$). Expanding $\overline{F} = f_0(r,\phi) + \delta f_1(r,\phi) + \delta^2 f_2(r,\phi) + O(\delta^3)$, we can solve (9) in a way analogous to the well-known Chapman–Engskog method. The lowest-order equation $L_\phi(f_0) = 0$ shows that f_0 is independent of ϕ. Using the relation $\omega_0' = v\,dk_0/dr$, the $O(\delta)$ equation is $L_\phi(f_1) = -k_0\cos\phi\,\partial(vf_0/k_0)/\partial r$, which is solved by $f_1(r,\phi) = \tilde{f}_1(r)\nu_1^{-1}\cos\phi$, where $\tilde{f}_1(r) = \partial(vf_0)/\partial r$, and ν_1 is defined by Eq. (8). Proceeding to $O(\delta^2)$ we obtain

$$L_\phi(f_2) = \frac{\partial}{\partial r}\left(\frac{v}{\nu_1}\cos^2\phi\tilde{f}_1\right) - \frac{v\tilde{f}_1}{r\nu_1}\sin^2\phi + \frac{\omega_0'}{k_0\nu_1}\tilde{f}_1 \times(\cos^2\phi - \sin^2\phi) + \frac{\partial f_0}{\partial t} + \gamma f_0 .$$

Noting that $\int_0^{2\pi} L_\phi(\overline{F})\,d\phi \equiv 0$ (conservation of wave action on scattering), we see that the solubility condition for f_2 is

$$\frac{\partial f_0}{\partial t} + \gamma f_0 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{k_0 v}{2\nu_1}\frac{\partial}{\partial r}\left(\frac{v}{k_0}f_0\right)\right]. \tag{10}$$

Equation (10) is the desired diffusion equation for the lower hybrid wave energy density. As an example, consider $\gamma = \partial/\partial t = 0$, in which case (10) has the solution (C_1 and C_2 are constants)

$$k_0^{-1}vf_0 = C_1 + C_2\int^r \frac{2\nu_1(r_1)}{v(r_1)r_1k_0(r_1)}\,dr_1 . \tag{11}$$

Assume that the total power injected at the plasma edge, $r = a$, is P and that any energy backscattered is eventually reflected inward. Thus, the power flux at any given point r' is $P/(2\pi r')$, for $\gamma(r) \cong 0$ in $r' \leq r \leq a$. If the absorption (γ) becomes strong in some region $r < r_d$ (i.e., ion damping increases rapidly as the mode conversion layer is approached), then (10) shows that f_0 becomes small there. Thus, we have an approximate boundary condition for (11), $f_0(r_d) \cong 0$ (this is justified in Appendix D). On the other hand, the power flux is

$$\frac{P}{2\pi r} = \int_0^{2\pi} v\cos\phi\,\overline{F}\,d\phi = \pi v\frac{\tilde{f}_1}{v_1} .$$

Thus, (11) becomes

$$f_0 = Pk_0(16\pi^2 v)^{-1}\int_{r_d}^r \frac{2\nu_1}{k_0 v r_1}\,dr_1 . \tag{12}$$

Comparing this result with the value of $\overline{F}$ that would obtain if there were no scattering [namely, $\overline{F} = P/(2\pi rv)$] we see that scattering enhances the θ, z-averaged energy density by a factor of the order of a/l_s.

IV. WEAK SCATTERING

Even in the case $l_s \gtrsim a$ the effect of scattering can be significant. Although the wave vector is not deflected sufficiently to alter the inward propagation of the wave, the intense concentration of energy in resonance cones can be mitigated. Since this effect may have implications for parametric instabilities, we consider it here. As an illustration consider a model problem in Cartesian coordinates (x, y, z), for which the average properties of the medium and the fluctuations are homogeneous. Taking the magnetic field to be along z, $\mathbf{B} = B_0\mathbf{z}_0$, the derivative of F following a ray is

$$\left(\frac{dF}{dt}\right)_r = \frac{\partial F}{\partial t} + \frac{\partial\overline{\omega}}{\partial k_x}\frac{\partial F}{\partial x} + \frac{\partial\overline{\omega}}{\partial k_y}\frac{\partial F}{\partial y} + \frac{\partial\overline{\omega}}{\partial k_z}\frac{\partial F}{\partial z} ,$$

where $\omega = \overline{\omega}(k_x, k_y, k_z)$ is the dispersion relation. Introducing $k_x = k_\perp\cos\phi$, $k_y = k_\perp\sin\phi$, assuming $\partial/\partial t = \partial/\partial y = 0$ and $k_\perp^2 \gg \zeta_0^2$, (1) and (4′) yield

$$v\cos(\phi - \pi)\frac{\partial F}{\partial x} + v_z\frac{\partial F}{\partial z} = D_{\phi\phi}\frac{\partial^2 F}{\partial\phi^2} ,$$

where $v_z = \partial\overline{\omega}/\partial k_z$ and $v = -\partial\overline{\omega}/\partial k_\perp$. For forward propagation (i.e., $\mathbf{v}_g\cdot\mathbf{x}_0 > 0$ with $\mathbf{v}_g = \partial\overline{\omega}/\partial\mathbf{k}$) with the group velocity in the x, z plane (i.e., $\mathbf{v}_g\cdot\mathbf{y}_0 = 0$), $\phi = \pi$ (recall that $\partial\overline{\omega}/\partial k_\perp < 0$ for lower hybrid waves). Since $l_s < a$, we use the approximation $\phi - \pi \ll 1$ and obtain

$$V\frac{\partial F}{\partial \chi} \cong D_{\phi\phi}\frac{\partial^2 F}{\partial \psi^2} + \frac{1}{2}\bar{v}\psi^2\frac{\partial F}{\partial \eta}, \tag{13}$$

where $V^2 = v^2 + v_z^2$, $\psi = \phi - \pi$, $\bar{v} = v_z v/V$, $\chi = (v/V)x + (v_z/V)z$, and $\eta = (v_z/V)x - (v/V)z$. In the absence of scattering, (13) yields the solution $F(\chi,\eta,\psi) = G(\eta)\delta(\psi)$; thus, χ represents the direction along the resonance cone, and the variation of F in the direction transverse to χ (i.e., η) does not change as the wave propagates. In the presence of scattering an initial delta function in ψ at $\chi = 0$ is broadened as χ increases. This spreading in ψ then leads to spreading in η. To illustrate these essential features consider the boundary condition $F(0,\eta,\psi) = G(\eta)\delta(\psi)$, where $\int_{-\infty}^{+\infty}\eta^2 G\,d\eta < \infty$. Multiplying (13) successively by 1, ψ^2, η, ψ^4, $\psi^2\eta$, and η^2, and then integrating over all ψ and η we obtain the following exact *closed* set of moment equations:

$$V\,dI/d\chi = 0, \tag{14a}$$

$$V\,d\langle\psi^2\rangle/d\chi = 2D_{\phi\phi}, \tag{14b}$$

$$V\,d\langle\eta\rangle/d\chi = -(\bar{v}/2)\langle\psi^2\rangle, \tag{14c}$$

$$V\,d\langle\psi^4\rangle/d\chi = 12D_{\phi\phi}\langle\psi^2\rangle, \tag{14d}$$

$$V\,d\langle\psi^2\eta\rangle/d\chi = 2D_{\phi\phi}\langle\eta\rangle - (\bar{v}/2)\langle\psi^4\rangle. \tag{14e}$$

$$V\,d\langle\eta^2\rangle/d\chi = -\bar{v}\langle\psi^2\eta\rangle, \tag{14f}$$

where $I \equiv \iint F\,d\psi\,d\eta$, $\langle Q\rangle \equiv I^{-1}\iint QF\,d\psi\,d\eta$, and (14a) has been used in obtaining (14b)–(14f). Solving (14) we readily obtain

$$\langle\psi^2\rangle = 2D_{\phi\phi}\chi/V, \tag{15a}$$

$$\langle\eta\rangle = \langle\eta\rangle_0 - [\bar{v}/(2V^2)]D_{\phi\phi}\chi^2, \tag{15b}$$

$$\langle(\delta\eta)^2\rangle = \langle(\delta\eta)^2\rangle_0 + (\bar{v}D_{\phi\phi}\chi^2)^2/(3V^4), \tag{15c}$$

where $\langle(\delta\eta)^2\rangle \equiv \langle(\eta - \langle\eta\rangle)^2\rangle$ and $\langle Q\rangle_0$ denotes the value of $\langle Q\rangle$ at $\chi = 0$. Thus, we see that the resonance cone experiences a transverse displacement [Eq. (15b)] and spreading [Eq. (15c)].

From (15c) it results that, even for $\chi < l_s$, it is possible to have appreciable spreading of the resonance cone [i.e., the second term on the right-hand side of (15c) is larger than the first term] for parameters of practical interest in tokamaks.

V. CONCLUSION

In conclusion, we have formulated the wave kinetic equation for lower hybrid wave scattering in an inhomogeneous plasma column [Eqs. (2) and (4)]. For long wavelength density fluctuations the scattering becomes diffusive in wave vector angle [Eqs. (4′) and (6)]. For typical cases, our result for the scattering length [Eq. (7)] shows that scattering can be significant in present-day and future tokamaks. For strong scattering the wave kinetic equation asymptotically reduces to a diffusion equation in space [Eq. (10)] which is easily solved [Eq. (12)]. In this case the θ, z-averaged lower hybrid wave energy density is enhanced, and nonresonant parametric decay may become more likely. For weak scattering the principal effect is spreading of the resonance cones [Eq. (15c)].

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APPENDIX A

Averaging $(dF/dt)_r$ over θ and z we obtain

$$\left(\frac{d\hat{F}}{dt}\right)_r = \frac{\partial\hat{F}}{\partial t} + \frac{\partial\bar{\omega}}{\partial k_r}\frac{\partial\hat{F}}{\partial r} - \frac{\partial\bar{\omega}}{\partial r}\frac{\partial\hat{F}}{\partial k_r} = \frac{\partial\hat{F}}{\partial t} + \frac{\partial}{\partial r}\left(\hat{F}\frac{\partial\bar{\omega}}{\partial k_r}\right) - \frac{\partial}{\partial k_r}\left(\hat{F}\frac{\partial\omega}{\partial r}\right).$$

From $k_r = k_\perp\cos\phi$ and $(m/r) = k_\perp\sin\phi$, we can express $k_\perp$ and ϕ in terms of m, k_r, and r: $k_\perp = [(m/r)^2 + k_r^2]^{1/2}$ and $\phi = \arctan[m/(rk_r)]$. Now, from the chain rule for partial differentiation

$$\left.\frac{\partial}{\partial r}\right)_{k_r,m} = \left.\frac{\partial}{\partial r}\right)_{k_\perp,\phi} - \frac{k_\perp}{r}\sin^2\phi\frac{\partial}{\partial k_\perp} - \sin\phi\cos\phi\frac{1}{r}\frac{\partial}{\partial\phi},$$

$$\frac{\partial}{\partial k_r} = \cos\phi\frac{\partial}{\partial k_\perp} - \frac{\sin\phi}{k_\perp}\frac{\partial}{\partial\phi}.$$

Using these expressions for $\partial/\partial k_r$ and $\partial/\partial r$, and simplifying, we obtain

$$\left(\frac{d\bar{F}}{dt}\right)_r = \frac{\partial\hat{F}}{\partial t} + \frac{\partial}{\partial r}\left(\hat{F}\cos\phi\frac{\partial\bar{\omega}}{\partial k_\perp}\right) - \frac{\sin\phi}{r}\frac{\partial\hat{\omega}}{\partial k_\perp}\frac{\partial\hat{F}}{\partial\phi} - \cos\phi\frac{\partial}{\partial k_\perp}\left(\hat{F}\frac{\partial\hat{\omega}}{\partial r}\right) + \frac{\sin\phi}{k_\perp}\frac{\partial\hat{\omega}}{\partial r}\frac{\partial\hat{F}}{\partial\phi}.$$

Now substitute $\hat{F} = k_\perp^{-1}\bar{F}\delta(k_\perp - k_0)\delta(k_z - k_{z0})$, multiply by $k_\perp dk_\perp$, and integrate $(d\bar{F}/dt)_r$ over all $k_\perp$. The result, Eq. (2), readily follows.

APPENDIX B

To obtain the coupling coefficient V in Eq. (3) we utilize the cold plasma equations

$$\partial n_\sigma/\partial t + \nabla\cdot n_\sigma\mathbf{v}_\sigma = 0, \quad \sigma = e, i,$$

$$\partial\mathbf{v}_i/\partial t + \mathbf{v}_i\cdot\nabla\mathbf{v}_i = -(e/m_i)\nabla\hat{\phi},$$

$$\partial\mathbf{v}_e/\partial t + \mathbf{v}_e\cdot\nabla\mathbf{v}_e = -(e/m_e)[-\nabla\hat{\phi} + \mathbf{v}_e\times(B_0\mathbf{z}_0)],$$

where m_σ is the particle mass and $\hat{\phi}$ is the electrostatic potential. From the continuity equation, the second-order density perturbation $n_\sigma^{(2)}(\mathbf{k})$ produced by the coupling of a lower hybrid wave at $(\omega, \mathbf{k}')$ with a (quasi-neutral) density perturbation $n(\mathbf{k} - \mathbf{k}')$ is

$$n_\sigma^{(2)}(\mathbf{k}) = n(\mathbf{k} - \mathbf{k}')[\mathbf{k}\cdot\mathbf{v}_\sigma^{(1)}(\mathbf{k}')/\omega],$$

where $v_\sigma^{(1)}$ denotes the linear velocity perturbation. The ion and electron momentum equations yield

$$\mathbf{v}_i^{(1)}(\mathbf{k}') = (e/m_i)\mathbf{k}'\phi(\mathbf{k}')/\omega,$$

$$v_{ex}^{(1)}(\mathbf{k}') \cong B_0^{-1}[-ik_y' + (\omega/\Omega_{ce})k_x']\hat{\phi}(\mathbf{k}'),$$

$$v_{ey}^{(1)}(\mathbf{k}') \cong B_0^{-1}[ik_x' + (\omega/\Omega_{ce})k_y']\hat{\phi}(\mathbf{k}'),$$

$$v_{ez}^{(1)}(\mathbf{k}') = -(e/m_e)(k_z'/\omega)\hat{\phi}(\mathbf{k}'),$$

where we have utulized $\omega^2 \ll \Omega_{ce}^2$ for lower hybrid waves. Applying the preceding results and the defining equation[13] for V

$$k^2 V\hat{n}(\mathbf{k}-\mathbf{k}')\phi(\mathbf{k}') = 4\pi e[n_i^{(2)}(\mathbf{k}) - n_e^{(2)}(\mathbf{k})],$$

we obtain the result following Eq. (3).

APPENDIX C

Here, we provide additional information concerning the derivation of Eq. (4) from Eq. (3). From (3) and following the procedure in Ref. 10, we obtain

$$\left(\frac{dN_{\mathbf{k}}}{dt}\right)_s = \sum_{\mathbf{k}'_\perp} \alpha(\mathbf{k},\mathbf{k}')|\hat{n}(\mathbf{k}-\mathbf{k}')|^2(N_{\mathbf{k}'} - N_{\mathbf{k}})\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}),$$

where $\alpha(\mathbf{k},\mathbf{k}') = 2\pi|V|^2(\partial\epsilon/\partial\omega)^{-2}$, and $N_{\mathbf{k}}$ is the wave action density, $N_{\mathbf{k}} = k^2|\phi(\mathbf{k})|^2(\partial\epsilon/\partial\omega)$. Passing from a sum over discrete wavenumbers ($\Delta k = n\pi/L_0$, where L_0 is the box size) to an integral over continuous $\mathbf{k}(L_0 \to \infty)$, we obtain

$$\left(\frac{dF}{dt}\right)_s = \int d^2\mathbf{k}'_\perp\, \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})\alpha(\mathbf{k},\mathbf{k}')S(\mathbf{k}-\mathbf{k}')$$
$$\times [F(\mathbf{x},\mathbf{k}',t) - F(\mathbf{x},\mathbf{k},t)],$$

where $S(\zeta)$ is the limit as $L_0 \to \infty$ of $|\hat{n}(\zeta)|^2(\pi/L_0)^{-2}$ and we have replaced $\omega_{\mathbf{k}}N_{\mathbf{k}}(\mathbf{x},t)$ by $F(\mathbf{x},\mathbf{k},t)$. Now introducing the polar angles ϕ and β (where β is defined by $\mathbf{k}'_\perp \cdot \mathbf{k}_\perp = k'_\perp k_\perp \cos\beta$), defining $l = |\mathbf{k}'_\perp - \mathbf{k}_\perp|$, and making a change of integration variables from $\mathbf{k}'_\perp$ to (l,β), we have

$$\left(\frac{dF}{dt}\right)_s = \frac{1}{2}\int_0^{2\pi}\int_0^\infty l\, d\beta\, dl\, \alpha S(l)\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})[F(\phi+\beta) - F(\phi)],$$

where

$$\alpha = \frac{2\pi}{(\partial\epsilon/\partial\omega)^2}\left[\left(1 - \epsilon_\perp \frac{lk_\perp}{k^2}\sin\frac{\beta}{2}\right)^2 + \left(\frac{\omega_{pe}^2}{\omega\Omega_{ce}}\frac{lk_\perp}{k^2}\cos\frac{\beta}{2}\right)^2\right],$$

and the delta function becomes

$$\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}) = \delta(l - 2k_\perp \sin\beta/2)/(v\sin\beta/2),$$

from which (4) follows.

APPENDIX D

In order to see that $f_0(r_d) \cong 0$ on the edge of a region with large γ in $r < r_d$, consider a model problem for $\partial/\partial t \equiv 0$, with k_0, v, and ν_1 independent of r, and define a wave energy conductivity $K = v^2/(2\nu_1)$. Also consider that the region where f_0 attenuates to a small value is narrow compared with r_d, so that cylindrical effects can be ignored in $r < r_d$. For $\gamma(r) = 0$ in $r > r_d$ and $\gamma(r) = \gamma_0$ in $r < r_d$ (with γ_0 a constant), Eq. (10) is then easily solved subject to continuity conditions on f_0 and $\partial f_0/\partial r$ at $r = r_d$. The result is $f_0 = G[(K/\gamma_0)^{1/2}r_d^{-1} + \ln(r/r_d)]$ in $r > r_d$ and $f_0 = (K/\gamma_0)^{1/2}r_d^{-1}G\exp[-(r_d - r)(\gamma_0/K)^{1/2}]$ in $r < r_d$ (where G is a constant). Evidently, $f_0 \cong G\ln(r/r_d)$ is a good approximation for $\ln(r/r_d) \gg (K/\gamma_0)^{1/2}r_d^{-1}$, which is a large region if $(K/\gamma_0)^{1/2} \ll r_d$. Thus, the approximation is valid if γ_0^{-1} is short compared with a typical conductivity time scale, r_d^2/K.

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