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Clifton R. Johnston and Marcelo Epstein

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On the exact amplitude, speed and shape of ion-acoustic waves

Clifton R. Johnston and Marcelo Epstein
Department of Mechanical and Manufacturing Engineering, University of Calgary, Alberta T2N 1N4, Canada

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Nonlinear ion-acoustic solitary waves in cold collisionless plasma are investigated by a direct analysis of the field equations. The exact amplitude is obtained by simply solving an algebraic equation. The results are compared to those of a third-order perturbation approach and the error associated with using the perturbation technique is determined. [S1070-664X(00)02303-X]

I. INTRODUCTION

In an article by Malfliet and Wieërs,[1] a study was undertaken to reexamine the prediction of solitary waves in a cold collisionless plasma. At lowest order, when dealing with fairly large waves, it is well known that a Korteweg–deVries (KdV) equation is obtained.[2–4] To study higher-order contributions, the reductive perturbation technique (RPT) has been used, by means of a stretching of the space-time variables. As pointed out by Malfliet and Wieërs,[1] however, the RPT is not without problems, arising primarily from the choice of the ''smallness'' parameter. In their systematic study of this problem, Malfliet and Wieërs[1] advocate the use of a traveling wave solution up to third-order perturbation and also suggest a modification to the RPT. As a result of their analysis, they show that, for a particular example, the first-order solution would underestimate the amplitude of the solitary wave by as much as 20%. Thus, Malfliet and Wieërs[1] clearly illustrate the dangers and limitations inherent in the use of *any* perturbation technique.

In this paper, we show that the *exact* amplitude of the ion-acoustic soliton can be obtained directly without resorting to any successive approximation technique. The exact solution involves only the finding of a root of a simple algebraic equation. This can be done easily to any degree of accuracy. Thus, with this method, not only is the obtained solution ''exact,'' but the computational effort involved in finding it is minimal. Moreover, the technique proposed is not limited to the particular equations at hand, but can, in principle, be applied to situations beyond the range of validity of the simplifying physical assumptions of the theory.

II. MATHEMATICAL PRELIMINARIES

The mathematical basis for the technique employed in this study is presented in detail in Epstein and Johnston.[5] For this study, we will present the key results.

A partial differential equation (PDE) for a function u of two independent variables, x and t, is said to admit solitary waves if a solution of the form $u=f(x-\nu t)=f(\eta)$ exists such that

(1) ν is a constant (the speed of propagation);
(2) f is bounded; and
(3) $\lim_{\eta\to\infty} f$ exists.

For our particular application, we will replace condition (3) by the more stringent condition:

(3′) f approaches its limits at infinity exponentially fast, from one side.

If we assume that the original equation was a quasilinear second-order PDE, with coefficients not explicitly dependent on x or t, the resulting ordinary differential equation (ODE) can be brought to the form

$$f''=F_\nu(f,f'),\tag{1}$$

where F_ν is a function whose form depends on the original PDE and on the parameter ν. The properties and long-term features of the solutions of this generic nonlinear equation are amenable to treatment by means of the theory of dynamical systems (see, e.g., Ref. 6). However, we will confine our attention to two particular cases corresponding to special forms of the right-hand side of (1).

Let us assume henceforth, for definiteness, that we are searching for a general solitary wave where a value of $f_{\max}$ is attained at the origin and approaches zero at infinity. If the first case we consider is the following:

$$f''=F_\nu(f),\tag{2}$$

then it is obvious that F_ν must satisfy the following conditions:

(i) it must have two roots, one at $f=0$ (behavior at infinity), and the other one at some (positive) finite value $f=f_1$ (point of inflection);
(ii) it must be positive in the interval $(0, f_1)$, and negative in the interval $(f_1, f_{\max})$; and
(iii) the integral of $F_\nu(f)$ between 0 and $f_{\max}$ must vanish, as it follows by integrating (2) between those limits and enforcing the vanishing thereat of the slope of f.

A different way to arrive at the above conditions is by noting that the first-order ODE

$$\frac{1}{2}(f')^2=\int_0^f F_\nu(\varphi)\,d\varphi+C,\tag{3}$$

where C is a constant, is a first integral of (2), as can be verified directly by differentiation of (3) with respect to η. The analysis then requires that the right-hand side of (3) have a double root at $f=0$ [which implies that $C=0$ and $F_\nu(0)=0$] and a single root at $f=f_{\max}$, and be positive in the interval (0, $f_{\max}$). One should consider, however, that although every solution of (2) satisfies (3), the converse may not necessarily be true. For example, when (3) is differentiated, we find

$$f'f''=F_\nu(f)f'.$$

In order to recover Eq. (2) we must divide both sides by f', which is only valid if $f'\neq 0$. This means that Eq. (3) can admit the solution $f=\text{const}$ and therefore Eq. (3) has a solution which Eq. (2) does not.

One should observe that if the initial conditions satisfy: $f'(0)=0$ and $0<f(0)<f_{\max}$, the behavior of the solution of Eq. (2) will be smoothly periodic, while if the initial conditions satisfy: $f'(0)=0$ and $f(0)>f_{\max}$, the behavior will drastically change and may become unbounded. Therefore, the solitary wave can also be identified, for a given ν, as that solution corresponding to a value of $f(0)$ situated exactly at the transition between those two modes of behavior. In a phase portrait of Eq. (2), therefore, the solitary wave will correspond to the separatrix between regions of closed and open orbits (see, Ref. 7, p. 19).

More general cases of (1), where first derivatives are present, can also be considered. For this study we will confine our attention to the particular form

$$f''=G_\nu(f)+H_\nu(f)f'^2, \tag{4}$$

where G_ν and H_ν are smooth functions. This case can be reduced to the previous one through the following.

Let

$$h(f)=\exp\left(-\int_0^f H_\nu(\varphi)d\varphi\right) \tag{5}$$

and

$$k(f)=2\int_0^f G_\nu(\varphi)h^2(\varphi)d\varphi. \tag{6}$$

Then: (a) the expression

$$h^2(f)f'^2-k(f)=D, \tag{7}$$

where D is a constant, is a first integral of (4); and

(b) every (nonconstant) solution of (4) is also a solution of

$$f''=G_\nu(f)+H_\nu(f)\frac{k(f)+D}{h^2(f)}, \tag{8}$$

and, vice-versa, among all the solutions of (8), that corresponding in (3) to $C=D/2$, is also a solution of (4).

We are then once again left with a problem of the form of (2) and consequently the right-hand side (rhs) of (8) must satisfy conditions (i)–(iii) for a solitary wave solution to exist.

III. THE GOVERNING EQUATIONS AND THEIR REDUCTION

We begin this analysis with the well-known dimensionless set of nonlinear equations describing a one-dimensional collisionless plasma (Davidson[8]),

$$\frac{\partial n_i}{\partial t}+\frac{\partial(n_i u)}{\partial x}=0, \tag{9}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{\partial\varphi}{\partial x}, \tag{10}$$

$$\frac{\partial^2\varphi}{\partial x^2}=\exp(\varphi)-n_i, \tag{11}$$

where n_i is the ion density, u is the flow velocity of the ions and φ is the electrostatic potential. At equilibrium $n_i=1$, $u=0$ and $\varphi=0$. Malfliet and Wieërs[1] introduce the substitution $n_i=1+n$, which we shall adopt henceforth.

Our first step in the search for solitary waves for the field equations (9)–(11) is to perform the substitutions implied in assuming the solution to be of the form $x-\nu t$. The approach of assuming the solution to be a function of $x-\nu t$ has been used as far back as Adlam and Allen[9] in a setting of simplified field equations. Since then it has also been applied to cases of multicomponent and relativistic plasmas[10–12] by using the pseudopotential approach. It is important to note that our method exploits this assumption $(x-\nu t)$ as a starting point in obtaining an exact solution of the complete field equations.

By substituting $\partial u/\partial x=f'$ and $\partial u/\partial t=-\nu f'$ into (9), (10) and (11), we obtain the following set of ODE's:

$$(u-\nu)n'+(1+n)u'=0, \tag{12}$$

$$\varphi''-\exp(\varphi)+(1+n)=0, \tag{13}$$

$$-\nu u'+\left(\frac{1}{2}u^2\right)'+\varphi'=0, \tag{14}$$

where, without risk of confusion, the original variable names are used to indicate corresponding functions of the single variable. Integrating (12) exactly yields the following link between u and n:

$$u-\nu=\frac{A}{(1+n)}, \tag{15}$$

where A is a constant of integration. If we then impose the initial equilibrium conditions $u=\varphi=0$, $n_i=1$ (and consequently $n=0$), the integration constant is found to be $A=-\nu$.

We now solve in terms of u to avoid the irrational expression (square root) which appears when solving for φ. Substituting (15) and the derivative of (14) into (13) results in the following single ODE governing the ion-acoustic solitary waves,

$$u''=\frac{\exp\left(\nu u-\frac{1}{2}u^2\right)-(u/(\nu-u))-1+(u')^2}{(\nu-u)}, \tag{16}$$

which is of the form of Eq. (4). Following the description in Sec. II above, we can transform (16) into an equivalent equation of the form of (2), allowing us to apply conditions (i)–(iii) directly.

By inspection of (16), we determine $H_\nu(u)$ and $G_\nu(u)$ to be the following:

$$H_\nu(u)=\frac{1}{\nu-u}, \tag{17}$$

$$G_\nu(u)=\frac{1}{\nu-u}\left(\exp\left(\nu u-\frac{1}{2}u^2\right)-\frac{u}{\nu-u}-1\right). \tag{18}$$

Performing the integrations described in (5) and (6) we determine that

$$h(u)=\frac{\nu-u}{\nu}, \tag{19}$$

$$k(u)=\frac{2}{\nu^2}\left(\exp\left(\nu u-\frac{1}{2}u^2\right)-1-\nu u\right). \tag{20}$$

We can now write (16) in the form of (8) by utilizing (19) and (20) from above. After rewriting (16) and performing some simplifications we are left with

$$u''=\frac{1}{(\nu-u)}\left(\exp\left(\nu u-\frac{1}{2}u^2\right)-\frac{\nu}{\nu-u}\right)+\frac{2}{(\nu-u)^3}\left(\exp\left(\nu u-\frac{1}{2}u^2\right)-1-\nu u+\frac{\nu^2 D}{2}\right). \tag{21}$$

By condition (i), establishing the behavior at infinity, we must have $D=0$, yielding

$$u''=\frac{1}{(\nu-u)^3}\left(((\nu-u)^2+2)\exp\left(\nu u-\frac{1}{2}u^2\right)-\nu(\nu+u)-2\right). \tag{22}$$

This ODE is equivalent to (16) but is in the form of (2), thus providing a new reduced form of the governing equation for ion-acoustic solitary waves in a cold collisionless plasma.

IV. SOLITARY WAVE SOLUTION

It is now necessary to verify whether and under what circumstances the rhs of (22) satisfies conditions (i)–(iii) for the existence of solitary waves. We notice that the root of (22) at $u=0$ always exists, regardless of the value of the wave speed ν. Our task now is to find, for any given value of ν, whether or not, in addition to the zero root, there exists a positive root, u_1, for the right-hand side of (22), and more importantly, prove the existence of a value $u_{\max}$ satisfying condition (iii).

While (22) permits a complete numerical analysis of the solitary waves, in this problem we have been able to extend the closed-form analysis by integrating (22) to obtain a first integral exactly. Indeed, employing integration by parts we find the first integral,

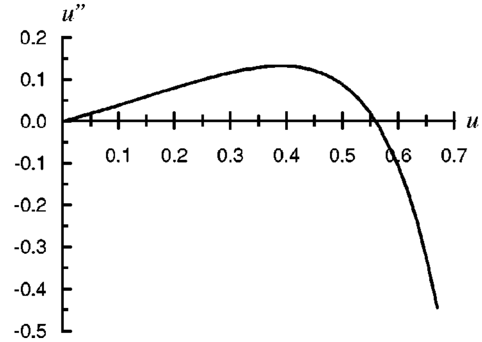

FIG. 1. Evaluation of the rhs of (22) for a wave speed of $\nu=1.25$.

$$\frac{1}{2}(u')^2=\frac{1}{(\nu-u)^2}\left(\exp\left(\nu u-\frac{1}{2}u^2\right)-\nu u-1\right)+B, \tag{23}$$

where B is the constant of integration (which is zero). Every solution of (22) will also satisfy (23). This allows us to find $u_{\max}$ exactly by solving a simple algebraic equation, namely,

$$\exp\left(\nu u-\frac{1}{2}u^2\right)-\nu u-1=0, \tag{24}$$

which can be easily solved numerically to any degree of precision. To obtain the shape of the solitary wave, we need only to perform a numerical integration of (23) with $u(0)=u_{\max}$. Finally, we must check that the solution satisfies the original equation (22).

In order to demonstrate the method used, we will present a detailed example of one particular case. We will choose a value of $\nu=1.25$ for the example case presented here. This value of ν corresponds to a case presented in Malfliet and Wieërs,[1] permitting us to compare the different approaches.

Figure 1 shows the right-hand side of (22) evaluated for the value of ν given above. We can determine that in addition to the root at zero, a second positive root does exist at $u_1=0.560\,18$, which can be found to any specified degree of precision. Figure 2 shows the right-hand side of (23), also evaluated with the value of ν given above. The value of $u_{\max}=0.7116\,03$, satisfying condition (iii), was determined by numerically solving (24). Again, this root can easily be found to any specified degree of accuracy.

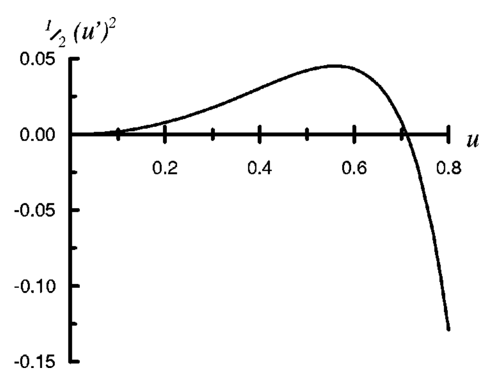

FIG. 2. Evaluation of the rhs of (23) for a wave speed of $\nu=1.25$.

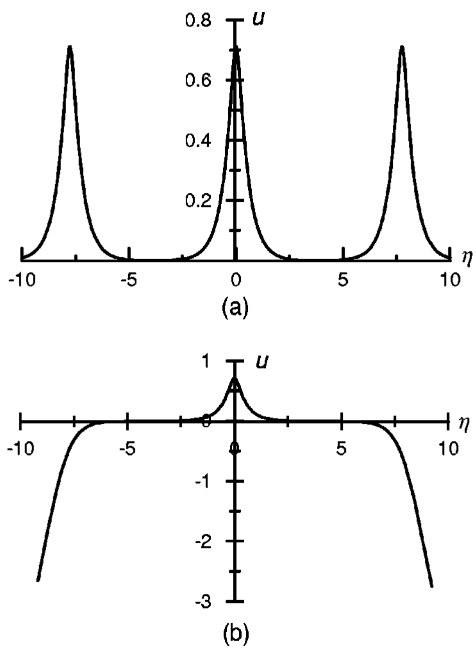

FIG. 3. Solution of (22) for a wave speed of $\nu = 1.25$ showing (a) periodic behavior for $u(0) = 0.711\,603$ and (b) divergent behavior for $u(0) = 0.711\,604$.

By way of illustration, Figs. 3(a) and 3(b), obtained directly from the Mathematica®[13] differential equation solver, show the solutions of (22) for the initial conditions $u(0) = 0.711\,603$ and $u(0) = 0.711\,604$. The dramatic difference in solutions for a change in initial conditions of only one-one millionth, poignantly illustrates the existence of a solitary wave solution. The shape of the solitary wave can be accurately represented by the shape of any one period, just before the transition, as shown in Fig. 3(a).

An alternate, albeit less appealing method of searching for the solitary wave, consists of specifying a very small initial value $u(0)$, rather than $u_{\max}$. As the value of $u(0)$ becomes vanishingly small, the solution will approach the solitary wave with an amplitude of $u_{\max}$. In our example, for the values of $u(0) = 10^{-6}, 10^{-9}$, we obtain the amplitudes 0.711 601 and 0.711 602.

V. DISCUSSION

The solitary wave found is of moderate amplitude, and a comparison with the results of Malfliet and Wieërs' [1] perturbation reduction technique would be expected to produce satisfactory results. In order to make a comparison we recall that Malfliet and Wieërs[1] introduced a wave-number-like parameter, c, such that $u = c(x - \nu t)$. To determine a value of c, we use Eq. (14) from Malfliet and Wieërs[1] where

$$\nu = \frac{1}{(1-4c^2)^{1/2}}, \tag{25}$$

and calculate that $c = 0.3$ for $\nu = 1.25$.

Substituting the above value of c into (31) of Malfliet and Wieërs[1] for $\eta = 0$ (initial shape at $t = 0$) yields $u_{\max} = 0.699\,925$ and a wave speed of $\nu = 1.243\,18$ (up to the c^6 approximation). For the ''exact'' solution, we recall from above that $u_{\max} = 0.711\,603$ and $\nu = 1.25$. The error in the perturbation solution is then approximately 1.6% in $u_{\max}$ and 0.55% in ν. To illustrate these differences, Fig. 4 shows the comparison of the predicted solitary wave from the ''exact'' solution and Malfliet and Wieërs.[1] Although these differences do not appear significant, it should be pointed out that a third-order perturbation was required to approach the ''exact'' solution for a case of moderate amplitude. The perturbation approach would require an undetermined number of additional perturbations for waves of higher amplitude, yet it would still not be clear whether and in what sense the solution had converged.

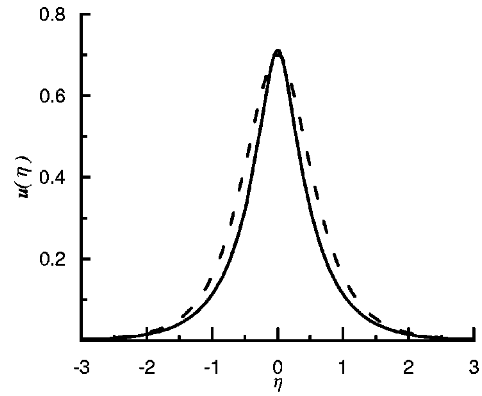

FIG. 4. Comparison between the ''exact'' solution (——) and the perturbation solution of Malfliet and Wieërs (Ref. 1) (— — —) with $\nu = 1.25$ and $c = 0.30$.

It is interesting to note that Malfliet and Wieërs[1] observed that with successive (higher order) approximations, the predicted solitary wave became larger, taller and moved faster. In Fig. 4 this trend continues from the highest order perturbation solution to the ''exact'' solution, where the ''exact'' solution is larger, taller and faster than the c^6 perturbation approximation.

While it may be argued that the perturbation approach could eventually approximate the ''exact'' solution by including additional perturbations, there exist three major advantages to the procedure developed here. First, the approach used in this study is not limited to small amplitude waves, unlike the reductive perturbation approach. The relatively close agreement between the ''exact'' solution and the perturbation solution was expected because the example considered was of moderate amplitude. If larger amplitudes (or faster waves) had been considered, the discrepancy between the two solutions would have widened. Evidence of this is suggested by Malfliet and Wieërs.[1] They point out that for waves with $c \leqslant 0.2$ (i.e., $\nu \to 0$), no significant difference is observed between successive approximations ($c^2 \to c^6$), yet for $c \geqslant 0.3$ (i.e., increasing ν) they observe significant differences. Second, the proposed technique offers the advantage of yielding a highly accurate solution (one might say an exact solution) of the exact equations, with no intermediary equations. Third, and perhaps most importantly, the ''exact'' solution is obtained with less effort than the perturbation solution. This is especially true for large amplitude waves where an unspecified number of additional perturbations would be required.

VI. CONCLUSIONS

Based on theoretical considerations implicit in the very definition of a solitary wave, an analysis has been presented which predicts with any desired degree of accuracy the amplitude and shape of ion-acoustic solitary waves for any given wave speed in a cold collisionless plasma. In fact the amplitude is obtained by just solving the simple algebraic equation (24).

Using this technique it is possible to determine the magnitude of errors incurred through use of the reductive perturbation techniques. As expected, the errors are relatively small for cases of moderate amplitude, but would be expected to increase with the increase in amplitude (and speed) of the wave. While the retention of higher-order terms will, to some degree, reduce these differences, the advantages of the proposed technique are that it is not limited to small amplitude waves, it yields a solution for the exact equations and the effort required to determine a solution does not vary with amplitude. Finally, the method proposed may be applicable to more general physical models.

ACKNOWLEDGMENTS

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