AIP Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **50**, 102108 (2009); doi: 10.1063/1.3250164
View online: http://dx.doi.org/10.1063/1.3250164
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/50/10?ver=pdfcov
Published by the AIP Publishing

Explicit pure-state density operator structure for quantum tomography

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(Received 19 June 2009; accepted 3 September 2009; published online 29 October 2009)

The formulation of region operators named by D. Ellinas and A. J. Bracken [Phys. Rev. A **78**, 052106 (2008)], which appears as the phase-space integration corresponding to the straight line over the Wigner operator, is manifestly improved and generalized. By virtue of the technique of integration within ordered (both normally ordered and Weyl ordered) product of operators, we show that the integration involved in the generalized region operator can be directly carried through to completion that leads to the explicit pure-state density operator $|u\rangle_{\lambda,\tau\lambda,\tau}\langle u|$, where $|u\rangle_{\lambda,\tau}$ makes up the coordinate-momentum intermediate representation. This directly results in that the tomogram of a quantum state $|\psi\rangle$ is just proportional to $|_{\lambda,\tau}\langle u|\psi\rangle|^2$, where $_{\lambda,\tau}\langle u|\psi\rangle$ is the wave function of $|\psi\rangle$ in the coordinate-momentum intermediate representation. *© 2009 American Institute of Physics.*
[doi:10.1063/1.3250164]

I. INTRODUCTION

Phase-space formulation of quantum mechanics has brought increasing interests to physicists working in the fields of quantum optics, quantum statistical physics, quantum information, and quantum tomography since the pioneer works of Weyl[1] and Wigner.[2] In a recent paper,[3] Ellinas and Bracken proposed the so-called region operators, which result from quantization, by Weyl's quantization rule,[1] of characteristic functions of regions in phase space. Region operators stand as operator-valued measures for these regions and, given a state density operator of a quantum system, they can assign quasiprobabilities to these regions by means of integrals of systems' Wigner function[4–6] over those regions. Tomographic methods of reconstructing the Wigner function based on the Radon transform can be derived by using the region operators. Our aim in this work is to further develop strategies for the construction of region operators so that they appear as pure-state density operator and can be more conveniently applied in physics.

Our work is organized as follows. In Sec. II, we briefly introduce the region operator. In Secs. III and IV, by virtue of the technique of integration within ordered (normally ordered and Weyl ordered) product of operators,[7–14] we, respectively, carry through the integration involved in the generalized region operator to completion, which leads to the pure-state density operator $|u\rangle_{\lambda,\tau\lambda,\tau}\langle u|$, and we prove that $|u\rangle_{\lambda,\tau}$ makes up the coordinate-momentum intermediate representation. Consequently, the tomogram of a quantum state $|\psi\rangle$ is just proportional to $|_{\lambda,\tau}\langle u|\psi\rangle|^2$, where $_{\lambda,\tau}\langle u|\psi\rangle$ is the wave function of $|\psi\rangle$ in the coordinate-momentum intermediate representation. Some further discussions are presented in Sec. V.

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II. THE REGION OPERATOR NAMED BY ELLINAS AND BRACKEN

Let $\hat{Q}$ and $\hat{P}$ denote a pair of dimensionless Hermitian coordinate and momentum operators that satisfy the canonical commutation relation $[\hat{Q},\hat{P}]=i$, $\hbar=1$, and set

$$\hat{Q}=\frac{\hat{A}+\hat{A}^{\dagger}}{\sqrt{2}}, \quad \hat{P}=\frac{\hat{A}-\hat{A}^{\dagger}}{\sqrt{2}i}, \quad [\hat{A},\hat{A}^{\dagger}]=1, \tag{1}$$

where $\hat{A}$ and $\hat{A}^{\dagger}$ are Bose annihilation and creation operators, respectively. We consider the Hermitian Wigner operator $\hat{\Delta}(q,p)$ in the coordinate representation,

$$\hat{\Delta}(q,p)=\int_{-\infty}^{\infty}\frac{dx}{\pi}e^{2ipx}|q+x\rangle\langle q-x|. \tag{2}$$

Here, $|q\rangle$ is the coordinate eigenvector, $\hat{Q}|q\rangle=q|q\rangle$, in the Fock representation it is expressed as

$$|q\rangle=\pi^{-1/4}\exp\left\{-\frac{q^2}{2}+\sqrt{2}q\hat{A}^{\dagger}-\frac{\hat{A}^{\dagger 2}}{2}\right\}|0\rangle, \tag{3}$$

where $|0\rangle$ is annihilated by $\hat{A}$. When one directly performs this integration in (2), one can obtain $\hat{\Delta}(q,p)=\pi^{-1}:e^{-(q-Q)^2-(p-P)^2}:$,[7–9] where :: denotes normal ordering [see also Eq. (15), below]. Another form of $\hat{\Delta}(q,p)$ is

$$\hat{\Delta}(q,p)=\pi^{-1}\hat{D}(q,p)\hat{\Pi}\hat{D}^{\dagger}(q,p)=\pi^{-1}\hat{D}(2q,2p)\hat{\Pi}, \tag{4}$$

where $\hat{D}(q,p)$ is the unitary displacement operator, $\hat{\Pi}$ is the parity operator, and $\hat{N}$ is the number operator,

$$\hat{D}(q,p)=\exp(ip\hat{Q}-iq\hat{P}), \quad \hat{\Pi}=(-1)^{\hat{N}}, \quad \hat{N}=\hat{A}^{\dagger}\hat{A}. \tag{5}$$

So the Wigner function corresponding to a given pure-state $|\psi\rangle$ is simply expressed in terms of the Wigner operator as

$$W(q,p)=\frac{1}{\pi}\int\psi(x-y)\psi(x+y)^{*}e^{2ipy}dy=\langle\psi|\hat{\Delta}(q,p)|\psi\rangle \tag{6}$$

and can also be written in terms of the density operator $\hat{\rho}=|\psi\rangle\langle\psi|$ as

$$W(q,p)=\mathrm{Tr}[\hat{\Delta}(q,p)\hat{\rho}]. \tag{7}$$

More generally, the Weyl–Wigner transform of an arbitrary operator $\hat{F}$ is defined as

$$F(q,p)=2\pi\,\mathrm{Tr}[\hat{\Delta}(q,p)\hat{F}], \tag{8}$$

Conversely, the quantization of an arbitrary function F is given by the inverse Weyl–Wigner transform (equivalent to Weyl's quantization map) as

$$\hat{F}=\int\hat{\Delta}(q,p)F(q,p)dqdp. \tag{9}$$

For $-\infty<u<+\infty$ and $0\leq\theta<\pi$, the integration transform of the Wigner operator,

is named as Radon transform.[4,15,16] Here, the δ function is the characteristic (generalized) function of the straight line in the q-p plane that has perpendicular displacement u from the origin and whose normal makes an angle θ, measured in the counterclockwise sense, with the q axis. Ellinas and Bracken referred to $\hat{R}(u,\theta)$ as the region operator corresponding to the line (or region) in phase space. Alternately, they gave

$$\hat{R}(u,\theta) = \int_{-\infty}^{\infty} \hat{\Delta}(q(u,v,\theta),p(u,v,\theta))dv,$$

$$q(u,v,\theta) = u\cos\theta + v\sin\theta, \quad p(u,v,\theta) = u\sin\theta - v\cos\theta, \quad -\infty < v < +\infty, \tag{11}$$

From Eqs. (4) and (11), they also noticed

$$\hat{R}(u,\theta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \hat{D}(q(u,v,\theta),p(u,v,\theta))\hat{\Pi}\hat{D}^{\dagger}(q(u,v,\theta),p(u,v,\theta))dv = |u,\theta\rangle\langle u,\theta|. \tag{12}$$

So the region operator has three form-different expressions as Eqs. (10)–(12) in Ref. 3. All these three expressions are in integral form, whereas the explicit form of $|u,\theta\rangle$ in (12) is unknown. In this work, we try to improve Ellinas and Brackens' formulation of region operators significantly. We first generalize Eq. (10) to the integration

which includes two independent parameters λ and τ, especially when $\lambda=\cos\theta$, $\tau=\sin\theta$, and $\lambda^2+\tau^2=1$. Equation (13) reduces to (10); then we directly perform the integration (13) by two approaches, so the generalized region operator actually has explicit neat expression, which will certainly bring much convenience to study the theory of quantum tomography.[5,6] One approach is using the technique of integration within normally ordered product (IWOP) of operators,[10–12] by which we can see that the directly performed integration leads to $\hat{R}(u,\lambda,\tau)=|u\rangle_{\lambda,\tau\lambda,\tau}\langle u|$, a pure-state density operator, $|u\rangle_{\lambda,\tau}$ is just the coordinate-momentum intermediate representation; when $\lambda=\cos\theta$ and $\tau=\sin\theta$, then $|u\rangle_{\lambda,\tau\lambda,\tau}\langle u|$ reduces to $|u,\theta\rangle\langle u,\theta|$ in Eq. (12), which reveals the explicit form of $|u,\theta\rangle$. Another approach is using the technique of integration within Weyl ordered product,[13,14] by which we can see that Eq. (11) is a natural result of the Weyl ordered form of the Wigner operator $\hat{\Delta}(p,q)=\vdots\,\delta(p-\hat{P})\delta(q-\hat{Q})\,\vdots$, where the symbol $\vdots\;\vdots$ denotes Weyl ordering.

III. DOING INTEGRATION (13) BY THE IWOP (NORMALLY ORDERED) METHOD

Using the normal ordering form of the vacuum projector,

$$|0\rangle\langle 0| = :\exp(-\hat{A}^{\dagger}\hat{A}): \tag{14}$$

and the IWOP technique to explicitly perform the integration (2), we obtain[7–11]

$$\begin{aligned}\hat{\Delta}(q,p) &= \int_{-\infty}^{\infty}\frac{dx}{\pi^{3/2}}e^{2ipx}\exp\left\{-\frac{(q+x)^2}{2}+\sqrt{2}(q+x)\hat{A}^{\dagger}-\frac{\hat{A}^{\dagger 2}}{2}\right\}|0\rangle \\ &\quad\times\langle 0|\exp\left\{-\frac{(q-x)^2}{2}+\sqrt{2}(q-x)\hat{A}-\frac{\hat{A}^2}{2}\right\} \\ &= \int_{-\infty}^{\infty}\frac{dx}{\pi^{3/2}}e^{2ipx} \\ &\quad\times:\exp\left\{-q^2-x^2+\sqrt{2}q(\hat{A}^{\dagger}+\hat{A})+\sqrt{2}x(\hat{A}^{\dagger}-\hat{A})-\frac{(\hat{A}^{\dagger}+\hat{A})^2}{2}\right\}: \\ &= \frac{1}{\pi}:e^{-(\hat{Q}-q)^2-(p-\hat{P})^2}:.\end{aligned} \tag{15}$$

This is a normally ordered Gaussian form. We can also obtain this result by performing the following ket-bra integration in the coherent state representation of the Wigner operator,[7–9]

$$\hat{\Delta}(q,p)=\int \frac{d^2z}{\pi^2}|\alpha+z\rangle\langle\alpha-z|e^{\alpha z^*-\alpha^* z}, \quad \alpha=(q+ip)/\sqrt{2}, \tag{16}$$

where $|z\rangle=\exp(z\hat{A}^\dagger-z^*\hat{A})|0\rangle$ is the coherent state.[17,18] The marginal integration of $\hat{\Delta}(q,p)$ can be easily obtained

$$\int_{-\infty}^{\infty} dp\hat{\Delta}(q,p)=\frac{1}{\sqrt{\pi}}{:}e^{-(\hat{Q}-q)^2}{:}=|q\rangle\langle q|, \tag{17}$$

$$\int_{-\infty}^{\infty} dq\hat{\Delta}(q,p)=\frac{1}{\sqrt{\pi}}{:}e^{-(p-\hat{P})^2}{:}=|p\rangle\langle p|, \tag{18}$$

where $|p\rangle$ is the momentum eigenvector, $\hat{P}|p\rangle=p|p\rangle$. Formulas (17) and (18) also appear in Ref. 19. Now we substitute (15) into the generalized region operator (13) and using the IWOP technique to do the integration,

$$\begin{aligned}\hat{R}(u,\lambda,\tau)&=\frac{1}{\pi}\int_{-\infty}^{\infty} dqdp\,\delta(u-\lambda q-\tau p){:}e^{-(\hat{Q}-q)^2-(p-\hat{P})^2}{:}\\&=[\pi(\lambda^2+\tau^2)]^{-1/2}{:}\exp\left\{\frac{-1}{\lambda^2+\tau^2}(u-\lambda\hat{Q}-\tau\hat{P})^2\right\}{:}.\end{aligned} \tag{19}$$

Then, using $|0\rangle\langle 0|={:}\exp(-\hat{A}^\dagger\hat{A}){:}$ we can decompose the right-hand side of (19) as

$$[\pi(\lambda^2+\tau^2)]^{-1/2}{:}\exp\left\{\frac{-1}{\lambda^2+\tau^2}(u-\lambda\hat{Q}-\tau\hat{P})^2\right\}{:}=|u\rangle_{\lambda,\tau\lambda,\tau}\langle u|, \tag{20}$$

where the new state $|u\rangle_{\lambda,\tau}$ is

$$|u\rangle_{\lambda,\tau}=[\pi(\lambda^2+\tau^2)]^{-1/4}\exp\left\{-\frac{u^2}{2(\lambda^2+\tau^2)}+\frac{\sqrt{2}\hat{A}^\dagger u}{\lambda-i\tau}-\frac{\lambda+i\tau}{2(\lambda-i\tau)}\hat{A}^{\dagger 2}\right\}|0\rangle. \tag{21}$$

Thus, the region operator turns out to be a pure-state density operator, which provides with convenient route for understanding tomography. We can prove that $|u\rangle_{\lambda,\tau}$ is the eigenstate of $\lambda\hat{Q}+\tau\hat{P}$; in fact, acting $\hat{A}$ on $|u\rangle_{\lambda,\tau}$ yields

$$\hat{A}|u\rangle_{\lambda,\tau}=\frac{1}{\tau+i\lambda}[\sqrt{2}iu+(\tau-i\lambda)\hat{A}^\dagger]|u\rangle_{\lambda,\tau}, \tag{22}$$

which directly leads to

$$(\lambda\hat{Q}+\tau\hat{P})|u\rangle_{\lambda,\tau}=u|u\rangle_{\lambda,\tau}. \tag{23}$$

Here, $|u\rangle_{\lambda,\tau}$ is complete since

$$\int_{-\infty}^{\infty} du|u\rangle_{\lambda,\tau\lambda,\tau}\langle u|=[\pi(\lambda^2+\tau^2)]^{-1/2}\int_{-\infty}^{\infty} du{:}\exp\left\{\frac{-1}{\lambda^2+\tau^2}(u-\lambda\hat{Q}-\tau\hat{P})^2\right\}{:}=1. \tag{24}$$

Besides, it is orthonormal

$$_{\lambda,\tau}\langle u'|u\rangle_{\lambda,\tau}=\delta(u-u'), \tag{25}$$

so $|u\rangle_{\lambda,\tau}$ is qualified to make up a new quantum mechanical representation; by observing (23), we name it the coordinate-momentum intermediate representation.

Therefore, the Radon transform of Wigner function $W(p,q)=\langle\psi|\hat{\Delta}(p,q)|\psi\rangle$ is a marginal distribution on the line defined in the λ-τ direction of the phase space

$$\int_{-\infty}^{\infty} dqdp\,\delta(u-\lambda q-\tau p)W(p,q)=|_{\lambda,\tau}\langle u|\psi\rangle|^2. \tag{26}$$

We now realize that the tomogram of a state $|\psi\rangle$ is $|_{\lambda,\tau}\langle u|\psi\rangle|^2$, where $_{\lambda,\tau}\langle u|\psi\rangle$ is the wave function of $|\psi\rangle$ in the coordinate-momentum intermediate representation. The appearance of the state vector $|u\rangle_{\lambda,\tau}$ also brings convenience for the reconstruction of the Wigner operator; in fact, using the completeness relation (24) and the eigenvector Eq. (23), we have

$$\exp[-ig(\lambda\hat{Q}+\tau\hat{P})]=\int_{-\infty}^{\infty} du|u\rangle_{\lambda,\tau\,\lambda,\tau}\langle u|e^{-igu}. \tag{27}$$

On the other hand, using the Weyl map, we have

$$\exp[-ig(\lambda\hat{Q}+\tau\hat{P})]=\int_{-\infty}^{\infty} dqdp\hat{\Delta}(p,q)e^{-ig(\lambda q+\tau p)}. \tag{28}$$

Combining (27) and (28), we have

$$\int_{-\infty}^{\infty} du|u\rangle_{\lambda,\tau\,\lambda,\tau}\langle u|e^{-igu}=\int_{-\infty}^{\infty} dqdp\hat{\Delta}(p,q)e^{-ig(\lambda q+\tau p)}. \tag{29}$$

By viewing the right-hand side of (29) as a twofold Fourier transformation, then its inverse transform is

$$\hat{\Delta}(p,q)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty} du'\int_{-\infty}^{\infty} dg'|g'|\int_0^{\pi} d\varphi|u'\rangle_{\lambda,\tau\,\lambda,\tau}\langle u'|\times\exp\left[-ig'\left(\frac{u'}{\sqrt{\lambda^2+\tau^2}}-q\cos\varphi-p\sin\varphi\right)\right], \tag{30}$$

where

$$g'=g\sqrt{\lambda^2+\tau^2},\quad \cos\varphi=\frac{\lambda}{\sqrt{\lambda^2+\tau^2}},\quad \sin\varphi=\frac{\tau}{\sqrt{\lambda^2+\tau^2}}. \tag{31}$$

This is reconstructing the Wigner operator based on the Radon transform derived by using $|u'\rangle_{\lambda,\tau\,\lambda,\tau}\langle u'|$.

IV. DOING INTEGRATION (13) BY THE IWOP (WEYL ORDERED) METHOD

Before introducing the second method, based on the Weyl quantization rule and the Wigner operator, we briefly review the technique of integration within the Weyl ordered product of operators in Ref. 12, where we have explained that Weyl ordering symbol $\vdots\;\vdots$ possesses three remarkable properties.

(a) the order of Bose operators within a Weyl ordered product (or within the Weyl ordering symbol $\vdots\;\vdots$) can be permuted;
(b) a Weyl ordered product can be integrated with respect to a c-number provided that the integration is convergent; and
(c) the Weyl ordered form of the Wigner operator is[12]

$$\hat{\Delta}(p,q) = \vdots\, \delta(p-\hat{P})\,\delta(q-\hat{Q})\, \vdots = \vdots\, \delta(q-\hat{Q})\,\delta(p-\hat{P})\, \vdots, \tag{32}$$

which can be seen from the Weyl quantization rule for a classical function $h(p,q)$ transiting to its quantum operator $h(\hat{P},\hat{Q})$,

$$\vdots\, h(\hat{P},\hat{Q})\, \vdots = \int_{-\infty}^{\infty} dpdqh(p,q)\hat{\Delta}(p,q). \tag{33}$$

For example, Weyl ordering is defined through the Weyl quantization scheme of classical quantity $q^m p^n$,[1]

$$q^m p^n \rightarrow \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} ml\hat{Q}^{m-l}\hat{P}^n\hat{Q}^l. \tag{34}$$

The right-hand side exhibits the definition of Weyl ordering, so

$$\left(\frac{1}{2}\right)^m \sum_{l=0}^{m} ml\hat{Q}^{m-l}\hat{P}^n\hat{Q}^l = \vdots \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \frac{m!}{l\,!\,(m-l)!}\hat{Q}^{m-l}\hat{P}^n\hat{Q}^l\, \vdots = \vdots\, \hat{Q}^m\hat{P}^n\, \vdots, \tag{35}$$

which means that

$$\vdots\, \hat{Q}^m\hat{P}^n\, \vdots = \int_{-\infty}^{\infty} dpdqq^m p^n\hat{\Delta}(p,q), \tag{36}$$

in agreement with (33). Equation (32) can be confirmed by examining another original definition of the Wigner operator, i.e.,

$$\hat{\Delta}(p,q) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dudve^{iu(p-\hat{P})+iv(q-\hat{Q})} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dudv\, \vdots\, e^{iu(p-\hat{P})+iv(q-\hat{Q})}\, \vdots = \vdots\, \delta(q-\hat{Q})\,\delta(p-\hat{P})\, \vdots, \tag{37}$$

Then, using (13) and the IWOP technique (Weyl ordered), we have

$$\hat{R}(u,\lambda,\tau) = \int_{-\infty}^{\infty} \delta(u-q\lambda-p\tau)\hat{\Delta}(q,p)dqdp = \int_{-\infty}^{\infty} dqdp\,\delta(u-q\lambda-p\tau)\, \vdots\, \delta(p-\hat{P})\,\delta(q-\hat{Q})\, \vdots$$

$$= \vdots\, \delta(u-\hat{Q}\lambda-\hat{P}\tau)\, \vdots. \tag{38}$$

This is another neat expression of the region operator, so

$$|u\rangle_{\lambda,\tau\lambda,\tau}\langle u| = \vdots\, \delta(u-\hat{Q}\lambda-\hat{P}\tau)\, \vdots. \tag{39}$$

V. DISCUSSION

In the case of Ellinas and Bracken, Eq. (38) reduces to

$$\hat{R}(u,\theta) = \vdots\, \delta(u-\hat{Q}\cos\theta-\hat{P}\sin\theta)\, \vdots. \tag{40}$$

According to the definition of $q(u,v,\theta)$ and $p(u,v,\theta)$ in (11) and (32), we have

$$\hat{\Delta}(q(u,v,\theta),p(u,v,\theta)) = \vdots\, \delta(u\cos\theta+v\sin\theta-\hat{Q})\,\delta(u\sin\theta-v\cos\theta-\hat{P})\, \vdots, \tag{41}$$

so we immediately obtain

$$\int_{-\infty}^{\infty}\hat{\Delta}(q(u,v,\theta),p(u,v,\theta))dv=\int_{-\infty}^{\infty}\vdots\,\delta(u\cos\theta+v\sin\theta-\hat{Q})\delta(u\sin\theta-v\cos\theta-\hat{P})\vdots\,dv$$

$$=\sec\theta\vdots\,\delta[u\cos\theta+(u\tan\theta-\hat{P}\sec\theta)\sin\theta-\hat{Q}]\vdots$$

$$=\vdots\,\delta[u\cos^2\theta+(u\sin\theta-\hat{P})\sin\theta-\hat{Q}\cos\theta]\vdots$$

$$=\vdots\,\delta(u-\hat{Q}\cos\theta-\hat{P}\sin\theta)\vdots=\hat{R}(u,\theta), \tag{42}$$

this is just the first line of Eq. (11). Similarly, we have

$$\int_{-\infty}^{\infty}\hat{\Delta}(q(u,v,\theta),p(u,v,\theta))du=\vdots\,\delta(-v-\hat{Q}\sin\theta-\hat{P}\cos\theta)\vdots. \tag{43}$$

Combining (42) and (43), we also see

$$\hat{\Delta}(q(u,v,\theta),p(u,v,\theta))=\vdots\,\delta(u-\hat{Q}\cos\theta-\hat{P}\sin\theta)\delta(v+\hat{Q}\sin\theta+\hat{P}\cos\theta)\vdots. \tag{44}$$

From (44), we see that $\hat{\Delta}(q(u,v,\theta),p(u,v,\theta))=R(\theta)\hat{\Delta}(u,v)R^{-1}(\theta)$, where $R(\theta)$ is a rotation operator rotating $(\hat{Q},\hat{P})$ to $(\hat{Q}\cos\theta-\hat{P}\sin\theta,-\hat{P}\cos\theta-\hat{Q}\sin\theta)$ since the $R(\theta)\vdots\circ\circ\circ\vdots R^{-1}(\theta)=\vdots R(\theta)\circ\circ\circ R^{-1}(\theta)\vdots$; its proof can be seen in Refs. 12, 20, and 21. Comparing $\int\hat{\Delta}(q(u,v,\theta),p(u,v,\theta))dv=R(\theta)\int\hat{\Delta}(u,v)dvR^{-1}(\theta)$ with $\int_{-\infty}^{\infty}dqdp\,\delta(u-q\cos\theta-p\sin\theta)\hat{\Delta}(p,q)$ and using (17), we see that the Radon transformation of the Wigner operator is equal to

$$\int_{-\infty}^{\infty}dqdp\,\delta(u-q\cos\theta-p\sin\theta)\hat{\Delta}(p,q)=R(\theta)|q\rangle\langle q||_{q=u}R^{-1}(\theta) \tag{45}$$

a rotated pure-state density operator. In this way, we endow the Radon transformation of the Wigner operator with a new physical meaning.

In summary, the region operator theory can be developed in a more explicit way, all the integrations about it can be performed out by virtue of the IWOP technique, and the explicit pure-state density operator structure for quantum tomography is found. Thus, the formulation can be simplified significantly. In the coordinate-momentum intermediate representation, the physical meaning of the tomography is clearer, so the tomography theory may benefit from our discussion in this work.

[1] H. Weyl, Z. Phys. **46**, 1 (1927).
[2] E. P. Wigner, Phys. Rev. **40**, 749 (1932).
[3] D. Ellinas and A. J. Bracken, Phys. Rev. A **78**, 052106 (2008).
[4] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).
[5] U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, 1997).
[6] W. P. Schleich, *Quantum Optics in Phase Space* (Wiley-VCH, New York, 2001).
[7] H.-Y. Fan, J. Math. Phys. **30**, 1273 (1989).
[8] H.-Y. Fan and H.-G. Weng, J. Math. Phys. **31**, 257 (1990).
[9] H.-Y. Fan and H. R. Zaidi, Phys. Lett. A **124**, 303 (1987).
[10] H.-Y. Fan, J. Opt. B: Quantum Semiclassical Opt. **5**, R147 (2003).
[11] H.-Y. Fan, H.-L. Lu, and Y. Fan, Ann. Phys. **321**, 480 (2006).
[12] H.-Y. Fan, H. R. Zaidi, and J. R. Klauder, Phys. Rev. D **35**, 1831 (1987).
[13] H.-Y. Fan, J. Phys. A **25**, 3443 (1992).
[14] H.-Y. Fan, Ann. Phys. **323**, 50 (2008).
[15] S. R. Deans, *The Radon Transform and Some of Its Applications* (Wiley, New York, 1983).
[16] S. Helgason, *The Radon Transform* (Birkhaüser, Boston, 1980).
[17] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[18] J. R. Klauder and B. S. Skargerstam, *Coherent States* (World Scientific, Singapore, 1985).
[19] D. Ellinas and I. Tsohantjis, Rep. Math. Phys. **57**, 69 (2006).
[20] H.-Y. Fan, Mod. Phys. Lett. A **12**, 2325 (1997).
[21] H.-Y. Fan and Y. Fan, Int. J. Mod. Phys. A **17**, 701 (2002).