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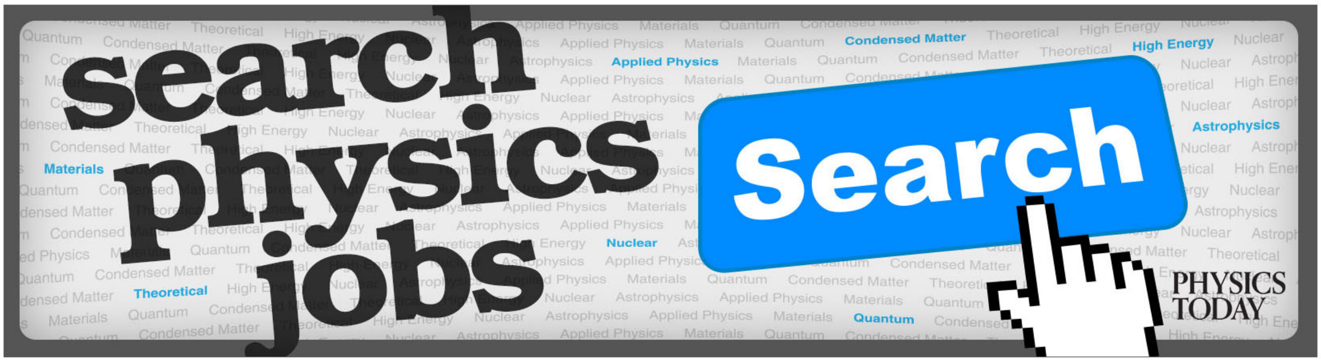

The state labeling problems for SO(N) in U(N) and U(M) in Sp($2M$)

H. S. Green, C. A. Hurst, and Y. Ilamed*

Department of Mathematical Physics, University of Adelaide, S. Australia 5001
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It is shown that, in a boson representation, the operators whose eigenvalues serve to label representations of SO(N) in U(N) also serve to label representations of U(M) in Sp($2M$). The problem of labeling U(2) in Sp(4) is considered in detail, and it is shown how to find labeling operators with rational eigenvalues, depending, however, on the representation. The solution of this problem is shown to provide a solution of the equivalent problem of the labeling of SO(3) in U(3).

1. INTRODUCTION

The study of the representations of the classical groups has, in recent years, been motivated as much by important physical applications as by its intrinsic mathematical interest. Most of the associated problems have been solved, in principle at least, but several problems remain which, in spite of their apparent mathematical simplicity and physical importance, have not yielded to persistent attack. One of these, formulated by Racah[1] and Ilamed,[2] concerns the definition of an operator with known eigenvalues, to complete the labeling of irreducible representations of SO(3) within U(3). Hughes[3] and Judd, Miller, Patera, and Winternitz[4] have recently shown how to determine the eigenvalues of two different operators, which however turn out to be irrational in general, and no general formula for the eigenvalues is known. Green and Bracken,[5] on the other hand, have introduced an operator with integral eigenvalues, probably related to the integral parameter of Bargmann and Moshinsky[6]; but so far no explicit definition of this operator has been found. There is a similar problem, of some importance in relativistic quantum mechanics, concerning the labeling of irreducible representations of SO(4) within U(4); this has been considered in a similar way by Jarvis,[7] but again only a partial solution has been obtained. This is, of course, true also of the more general problem of labeling representations of SO(N) withwithin U(N).

An apparently unrelated problem concerns the labeling of the states of Sp($2M$) with operators related to the integral parameters of the Weyl or Gel'fand bases. Using boson representations of the generators, Lohe and Hurst[8] have considered the problem of labeling Sp($2M-2$) in Sp($2M$), but again no explicit labeling operators have been found in general.

Finally, it may be mentioned that Govorkov[9] has encountered an apparently intractable problem (for $p \geqslant 3$) when seeking labeling operators for representations of U(N) within generalized parafermion algebras of order p. There is, besides, an analogous problem, not yet discussed in the literature, associated with the labeling of representations of generalized paraboson algebras.

Our intention is to show that all the problems mentioned above are closely related, and, as usual, progress made in the solution of one of the problems is an important aid to the solution of the others. We shall make use of boson realizations of the generators of the algebras (or, equivalently, of differential operators), and the reciprocal relation between representations of Sp($2M$) and U(N) which has been exploited recently by Quesne and Moshinsky.[10] It is easy, in this formalism, to show that the problem of labeling irreducible representations of SO(N) within U(N) is equivalent, for a suitable choice of M, to the problem of labeling irreducible representations of U(M) within Sp($2M$). We establish simple relations between the invariants of the algebras concerned, and show that, for $N=3$ or 4 and $M=2$, the solution of the problem may be found within an interesting algebra, which is not a finite dimensional Lie algebra, though it has finite-dimensional representations. It is shown how to compute matrix elements of all the invariants of SO(3) or SO(4) within U(3) or U(4), equivalently of U(2) within Sp(4), and hence to determine their eigenvalues.

2. TENSOR REPRESENTATIONS OF U(N) AND Sp($2M$)

In canonical form, the generators b_{ij} of U(N) satisfy the commutation relations

$$[b_{ij}, b_{kl}] = \delta_{kj} b_{il} - \delta_{il} b_{kj}, \quad (i,j,k,l = 1, \ldots, N). \tag{1}$$

Irreducible representations may be labeled by eigenvalues of the first N of the invariants

$$\langle b \rangle = b_{ii}, \quad \langle b^2 \rangle = b_{ij} b_{ji}, \qquad \langle b^3 \rangle = b_{ij} b_{jk} b_{ki}, \tag{2}$$

etc., or of the set of invariants $(L_1, L_2, \ldots, L_N)$ whose eigenvalues in finite dimensional representations are the highest weights. They are related to the $\langle b^r \rangle$ by

$$\sum_{r=1}^{N} L_r = \langle b \rangle, \qquad \sum_{r=1}^{N} L_r (L_r + N + 1 - 2r) = \langle b^2 \rangle, \tag{3}$$

and similar but more complicated identities of higher degree. The (eigenvalues of the) L_r differ by integers, and $L_1 \geqslant L_2 \geqslant \cdots \geqslant L_N$. If

$$C_{ij} = b_{ij} + c\delta_{ij}, \tag{4}$$

where c is a constant, the C_{ij} are also generators of

U(N), with representations labeled $(L_1+c,\ L_2+c,\ \cdots,\ L_N+c)$.

The generators l_{ij} of the orthogonal subgroup SO(N) of U(N) may be defined by

$$l_{ij}=b_{ij}-b_{ji}, \tag{5}$$

or $l=b-\bar{b}$, where $\bar{b}$ denotes the transpose of the matrix b whose elements are b_{ij}. If $H=[\frac{1}{2}N]$ (i.e., $\frac{1}{2}N$ when N is even, but $\frac{1}{2}N-\frac{1}{2}$ when N is odd), irreducible representations of SO(N) may be labeled by eigenvalues of the first H of the invariants

$$\begin{aligned}\langle l^2\rangle &= 2\langle b^2-b\bar{b}\rangle = l_{ij}l_{ji},\\ \langle l^4\rangle &= \langle(\bar{b}-b)^4\rangle = l_{ij}l_{jk}l_{kp}l_{pi},\end{aligned} \tag{6}$$

etc., or of the set of invariants $(l_1,l_2,\ldots,l_H)$ corresponding to the highest weights. They are related to the $\langle l^{2s}\rangle$ by

$$2\sum_{s=1}^{H} l_s(l_s+N-2s)=\langle l^2\rangle \tag{7}$$

and similar identities of higher degree.[11]

Next we consider Sp($2M$). If we denote the generators by S_{PQ} $(P,Q=1,2,\ldots,2M)$ the commutation relations

$$[S_{PQ},S_{UV}]=g_{UP}S_{QV}+g_{UQ}S_{PV}+g_{VP}S_{QU}+g_{VQ}S_{PU} \tag{8}$$

are satisfied, where $S_{PQ}=S_{QP}$ but $g_{PQ}=-g_{QP}$. If $S^P{}_Q=g^{PR}S_{RQ}$, where $g^{PR}g_{RQ}=\delta^P{}_Q$, the invariants $\langle S^2\rangle,\ldots,\langle S^{2M}\rangle$ defined by

$$\begin{aligned}\langle S^2\rangle &= S^P{}_Q S^Q{}_P,\\ \langle S^4\rangle &= S^P{}_Q S^Q{}_U S^U{}_V S^V{}_P,\end{aligned} \tag{9}$$

etc., may serve to label irreducible representations. Alternatively, the set of invariants $(\Lambda_1,\Lambda_2,\ldots,\Lambda_M)$. whose eigenvalues in finite dimensional irreducible representations are the highest weights, may be used: these are related to the $\langle S^{2p}\rangle$ by[11]

$$2\sum_{S=1}^{M}\Lambda_S(\Lambda_S+2M+2-2S)=\langle S^2\rangle \tag{10}$$

and similar identities of higher degree. We may choose

$$g_{PQ}=g^{QP}=\delta_{P\ Q+M}-\delta_{P+M\ Q}. \tag{11}$$

If λ and μ take integral values between 1 and M, we define

$$\begin{aligned}&\alpha_{\lambda\mu}=S_{\lambda\mu},\quad \alpha^{\lambda\mu}=S_{\lambda+M\ \mu+M},\\ &\alpha^\lambda{}_\mu=S_{\lambda+M\ \mu},\end{aligned} \tag{12}$$

so that the commutation relations (8) reduce to

$$\begin{aligned}&[\alpha^\lambda{}_\mu,\alpha^\nu{}_\rho]=\delta^\nu{}_\mu\alpha^\lambda{}_\rho-\delta^\lambda{}_\rho\alpha^\nu{}_\mu,\\ &[\alpha^\lambda{}_\mu,\alpha^{\nu\rho}]=\delta^\nu{}_\mu\alpha^{\lambda\rho}+\delta^\rho{}_\mu\alpha^{\lambda\nu},\\ &[\alpha_{\lambda\mu},\alpha^\nu{}_\rho]=\delta^\nu{}_\lambda\alpha_{\mu\rho}+\delta^\nu{}_\mu\alpha_{\lambda\rho},\\ &[\alpha_{\lambda\mu},\alpha^{\nu\rho}]=\delta^\nu{}_\lambda\alpha^\rho{}_\mu+\delta^\rho{}_\lambda\alpha^\nu{}_\mu+\delta^\nu{}_\mu\alpha^\rho{}_\lambda+\delta^\nu{}_\lambda\alpha^\rho{}_\mu.\end{aligned} \tag{13}$$

The elements $\alpha^\lambda{}_\mu$ are evidently generators of the unitary subgroup U(M) of Sp($2M$), and the invariants of this subgroup analogous to the $\langle b^r\rangle$ are $\langle\alpha\rangle,\ldots,\langle\alpha^M\rangle$, where

$$\begin{aligned}&\langle\alpha\rangle=\alpha^\lambda{}_\lambda,\quad \langle\alpha^2\rangle=\alpha^\lambda{}_\mu\alpha^\mu{}_\lambda,\\ &\langle\alpha^3\rangle=\alpha^\lambda{}_\mu\alpha^\mu{}_\nu\alpha^\nu{}_\lambda,\end{aligned} \tag{14}$$

etc. The set of invariants $(\lambda_1,\lambda_2,\ldots,\lambda_M)$, analogous to $(l_1,l_2,\ldots,l_N)$, are given by

$$\begin{aligned}&\sum_{r=1}^{M}\lambda_r=\langle\alpha\rangle,\\ &\sum_{r=1}^{M}\lambda_r(\lambda_r+M+1-2r)=\langle\alpha^2\rangle\end{aligned} \tag{15}$$

and similar identities of higher degree.[11]

We now introduce a set of boson creation and annihilation operators [or coordinates and differential operators], denoted by a_{Pi} $(P=1,\ldots,2M;\ i=1,\ldots,N)$, and satisfying

$$[a_{Pi},a_{Qj}]=g_{QP}\delta_{ij}. \tag{16}$$

We may suppose that

$$\begin{aligned}&a_{\lambda+Mi}=a^*_{\lambda i}\ \ [\text{or }\partial/\partial a_{\lambda i}],\quad \lambda\le M'\le M,\\ &a_{\lambda i}=-a^*_{\lambda+Mi}\ \ [\text{or }-\partial/\partial a_{\lambda+Mi}],\quad M'<\lambda,\end{aligned} \tag{17}$$

where a^*_{Pi} now means the Hermitian conjugate of a_{Pi}. Then, if

$$a^\lambda{}_i=a_{\lambda+Mi},$$

$a^\lambda{}_i$ will be a creation operator [or coordinate] for $\lambda\le M'$, and an annihilation or differential operator for $\lambda>M'$.

Products of creation operators like $a^\lambda{}_i a^\mu{}_j\cdots a_{\nu k}a_{\rho l}\cdots$ (where $\lambda,\mu,\cdots\le M'$ and $\nu,\rho,\cdots>M'$) can be regarded as vectors of (reducible) representations of either U(N) or Sp($2M$). In such representations, the generators of U(N) are

$$b_{ij}=a^\lambda{}_i a_{\lambda j}+\tfrac{1}{2}M\delta_{ij}, \tag{18}$$

with an appropriate choice of the constant c in (4), and those of Sp($2M$) are

$$S_{PQ}=a_{Pi}a_{Qi}+\tfrac{1}{2}Ng_{PQ}, \tag{19}$$

$$\alpha^\lambda{}_\mu=a^\lambda{}_i a_{\mu i}+\tfrac{1}{2}N\delta^\lambda{}_\mu, \tag{20}$$

$$\alpha^{\lambda\mu}=a^\lambda{}_i a^\mu{}_i,\quad \alpha_{\lambda\mu}=a_{\lambda i}a_{\mu i}.$$

The commutation relations (1) and (13) are satisfied on account of (16), or

$$[a_{\lambda i},a^\mu{}_j]=\delta^\mu{}_\lambda\delta_{ij}. \tag{21}$$

Since each generator b_{ij} of U(N) commutes with each generator $\alpha^\lambda{}_\mu$ of the unitary subgroup U(M) of Sp($2M$), the invariants of U(M) are also invariants of U(N) in this representation. It is, indeed, easy to see that

$$\begin{aligned}&\langle\alpha\rangle=a^\lambda{}_i a_{\lambda i}+\tfrac{1}{2}MN=\langle b\rangle,\\ &\langle(\alpha-\tfrac{1}{2}N)(\alpha+\tfrac{1}{2}N)\rangle=a^\lambda{}_i a_{\mu i}a_{\lambda j}a^\mu{}_j=\langle(b-\tfrac{1}{2}M)(b+\tfrac{1}{2}M)\rangle,\end{aligned}$$

and, more generally, it has been shown by M.C.K. Aguilera-Navarro and V.C. Aguilera-Navarro

$$\begin{aligned}&\langle(\alpha-\tfrac{1}{2}N)(\alpha+\tfrac{1}{2}N)^n\rangle=\langle(b-\tfrac{1}{2}M)(b+\tfrac{1}{2}M)^n\rangle,\\ &\langle(\bar{\alpha}+\tfrac{1}{2}N)(\bar{\alpha}-\tfrac{1}{2}N)^n\rangle=\langle(\bar{b}+\tfrac{1}{2}M)(\bar{b}-\tfrac{1}{2}M)^n\rangle,\end{aligned} \tag{22}$$

for $n=0,1,2,\cdots$, where $\bar{\alpha}$ again means the transpose of the matrix α with elements $\alpha^\lambda{}_\mu$ [so that, e.g., $(\bar{\alpha}^2)^\lambda{}_\mu=\bar{\alpha}^\lambda{}_\nu\bar{\alpha}^\nu{}_\mu=\alpha^\nu{}_\lambda\alpha^\mu{}_\nu$]. A short proof of the results of (22), and others needed below, is given for convenience in the Appendix to this paper. It follows from (22), together with (15) and (3), that the invariants $(\lambda_1,\lambda_2,\ldots,\lambda_M)$ of U(M) are connected with those, $(L_1,L_2,\ldots,L_N)$, of U(N) by

$$\begin{aligned}&\lambda_r' - \tfrac{1}{2}N = L_r' - \tfrac{1}{2}M, \quad r \leq \min(M,N),\\ &\lambda_r' - \tfrac{1}{2}N = 0, \quad N < r \leq M, \\ &L_r' - \tfrac{1}{2}M = 0, \quad M < r \leq N,\end{aligned} \tag{23}$$

where λ_r' and L_r' are defined in terms of the λ_r and L_r by relations of the type $\lambda_r' - r = \lambda_s - s$, $L_r' - r = L_s - s$, chosen so that the eigenvalues of both $(\lambda_1,\lambda_2,\ldots,\lambda_M)$ and $(L_1,L_2,\ldots,L_N)$ are in decreasing order.

Similarly, since each generator S_{PQ} of Sp($2M$) commutes with each generator l_{ij} of the orthogonal subgroup SO(N) of U(N) in the representation considered, the invariants of SO(N) are also invariants of Sp($2M$). It may be verified explicitly that

$$\langle (S - \tfrac{1}{2}N)(S + \tfrac{1}{2}N)\rangle = a^P{}_i a_{Qi} a_{Pj} a^Q{}_j = \langle (l - M)(l + M + 1)\rangle$$

and, as shown in the Appendix,

$$\langle (S - \tfrac{1}{2}N)(S + \tfrac{1}{2}N)^n\rangle = \langle (l - M)(l + M + 1)^n\rangle \tag{24}$$

for $n = 0, 1, 2, \cdots$. It follows from this, together with (10) and (17), that the invariants $(\Lambda_1,\Lambda_2,\ldots,\Lambda_M)$ of Sp($2M$) and those, $(l_1,l_2,\ldots,l_H)$, of O(N) are connected by

$$\begin{aligned}&\Lambda_s' + M + 1 - \tfrac{1}{2}N = l_s, \quad s \leq \min(H,M),\\ &\Lambda_s' + M + 1 - \tfrac{1}{2}N = 0, \quad H < s \leq M,\\ &l_s' = 0, \quad M < s \leq H,\end{aligned} \tag{25}$$

where Λ_s' and l_s' are defined in terms of the Λ_s and l_s by relations of the type $\Lambda_s' - s = \Lambda_r - r$, $l_s' - s = l_r - r$, chosen so that the eigenvalues of both $(\Lambda_1,\Lambda_2,\ldots,\Lambda_M)$ and $(l_1,l_2,\ldots,l_H)$ are in decreasing order.

The operators whose eigenvalues could serve to label equivalent irreducible representations of SO(N) within an irreducible representation of U(N) are also related to the operators whose eigenvalues could serve to label equivalent representations of U(M) within an irreducible representation of Sp($2M$). To construct such operators, we introduce linear operators $A, \bar{A}, B$, and $\bar{B}$, defined on arbitrary symmetric tensors $\phi_{\lambda\mu}$, $\phi^{\lambda\mu}$, and ψ_{ij} by

$$\begin{aligned}&\phi_{\lambda\mu}A = \phi_{\lambda\nu}\alpha^\nu{}_\mu + \phi_{\mu\nu}\alpha^\nu{}_\lambda,\\ &\phi^{\lambda\mu}\bar{A} = \phi^{\lambda\nu}\alpha^\mu{}_\nu + \phi^{\mu\nu}\alpha^\lambda{}_\nu,\\ &B\psi_{ij} = b_{ik}\psi_{kj} + b_{jk}\psi_{ki},\\ &\bar{B}\psi_{ij} = b_{ki}\psi_{kj} + b_{kj}\psi_{ki}.\end{aligned} \tag{26}$$

The tensor $(\bar{b}b + \bar{b})_{ij} = b_{ki}b_{kj} + b_{ji}$ is symmetric, and it is readily verified that

$$\begin{aligned}\langle \bar{\alpha}_c\alpha_c\rangle &= \alpha^{\lambda\mu}\alpha_{\lambda\mu} = \langle (\bar{b} - \tfrac{1}{2}M - 1)(b - \tfrac{1}{2}M)\rangle,\\ \langle \bar{\alpha}_c\alpha_c(A + N)\rangle &= \alpha^{\lambda\mu}(2\alpha_{\lambda\nu}\alpha^\nu{}_\mu + N\alpha_{\lambda\mu})\\ &= \langle (\bar{b} - \tfrac{1}{2}M - 1)(b - \tfrac{1}{2}M)(2b + M)\rangle\\ &= \langle (\bar{b} - \tfrac{1}{2}M - 1)(b - \tfrac{1}{2}M)(B + M)\rangle,\end{aligned}$$

and, more generally, as shown in the Appendix,

$$\langle \bar{\alpha}_c\alpha_c(A + N)^n\rangle = \langle (\bar{b} - \tfrac{1}{2}M - 1)(b - \tfrac{1}{2}M)(B + M)^n\rangle. \tag{27}$$

Similarly,

$$\langle \alpha_c\bar{\alpha}_c\rangle = \alpha_{\lambda\mu}\alpha^{\lambda\mu} = \langle (b + \tfrac{1}{2}M + 1)(\bar{b} + \tfrac{1}{2}M)\rangle$$

and, as shown in the Appendix,

$$\langle \alpha_c\bar{\alpha}_c(\bar{A} - N)^n\rangle = \langle (\bar{b} + \tfrac{1}{2}M + 1)(\bar{b} + \tfrac{1}{2}M)(\bar{B} - M)^n\rangle. \tag{28}$$

These, and similar results which can be derived by the same method for expressions involving more than one pair of factors $\alpha_{\lambda\mu}, \alpha^{\nu\rho}$, show that a solution to the problem of labeling representations of SO(N) within an irreducible representation of SU(N) is also a solution to the problem labeling representations of U(M) within a corresponding irreducible representation of Sp($2M$), and conversely. However, as can be seen from (23) and (25), the most general representation of SU(N) can only be realized by taking $M \geq N - 1$, while the most general representation of Sp($2M$) can only be realized by taking $H = [\tfrac{1}{2}N] \geq M$. In particular, the solution of the problem of labeling general representations of SU(3) is solved by taking $M = 2$, but the solution of the problem of labeling general representations of Sp(4) is solved by taking $N = 4$.

The representations of Sp($2M$) in terms of boson operators, constructed in this section, are necessarily infinite dimensional; finite dimensional representations could be obtained, if desired, by using fermion or parafermion operators[13] instead. The infinite dimensionality may be thought of as associated with the multiplicity of *distinct* representations of U(M) contained within an irreducible representation of Sp($2M$). The operators whose eigenvalues serve to label *equivalent* representations of U(M) within Sp($2M$) are, as we have seen, the same as those which label equivalent representations of SO(N) within U(N), and generate an algebra $\underline{A}$ with finite dimensional representations. For $M = 2$, we shall investigate the structure of this algebra below. For the sake of symmetry, we eventually choose $M' = 1$ in (17), so that U(2) is strictly replaced by the noncompact form U(1,1). However, analogous results hold also for $M' = 0$ and $M' = 2$, so that the conclusions do not depend in any essential way on this choice.

3. LABELING OF U(2) IN Sp(4)

An irreducible representation of Sp(4) is labeled by the eigenvalues of the invariants (Λ_1,Λ_2), related to those defined in (9) by

$$\begin{aligned}&\langle S^2\rangle = 2(\Lambda^2 + \Lambda'^2 - 5),\\ &\langle S^4\rangle = 2(\Lambda^4 + \Lambda'^4 + 3\Lambda^2 + 3\Lambda'^2 - 32),\\ &\Lambda = \Lambda_1 + 2 = l_1 + \tfrac{1}{2}N - 1,\\ &\Lambda' = \Lambda_2 + 1 = l_2 + \tfrac{1}{2}N - 2,\end{aligned} \tag{29}$$

where l_1 and l_2 take nonnegative integral eigenvalues. Distinct representations of U(2) in an irreducible representation of Sp(4) are labeled by eigenvalues of (λ_1,λ_2), related to the invariants defined in (14) by

$$\begin{aligned}&\langle \alpha\rangle = \lambda - \lambda', \quad \langle \alpha^2\rangle = \lambda^2 + \lambda'^2 - \tfrac{1}{2},\\ &\lambda = \lambda_1 + \tfrac{1}{2}, \quad \lambda' = -\lambda_2 + \tfrac{1}{2}.\end{aligned} \tag{30}$$

If $M' = 1$ in (17), the following inequalities are satis-satisfied:

$$\lambda \geqslant \tfrac{1}{2}(N-1), \quad \lambda' \geqslant \tfrac{1}{2}(N-1),$$
$$\lambda+\lambda' \geqslant l' = l_1 + l_2 + N - 1, \tag{31}$$

so that if $N=3$ and $l_2=0$, λ and λ' are positive integers such that $\lambda+\lambda' \geqslant l_1+2$, but if $N=4$ they are half-odd-integers $\geqslant \frac{3}{2}$ such that $\lambda+\lambda' \geqslant l_1+l_2+3$.

Equivalent representations of U(2) within an irreducible representation of Sp(4) are distinguished by eigenvalues of elements of the algebra $\underline{A}$ of invariants of U(2), constructed from the tensors $\alpha^{\lambda\mu}$ and $\alpha_{\lambda\mu}$, as well as the generators of U(2). In order to determine the representations of the algebra $\underline{A}$, it is helpful to introduce also a set of operators $(P,\bar{P},Q,\bar{Q},R,\bar{R},S,\bar{S})$ which shift from one irreducible representation of U(2) to another. The simplest of these operators can be defined directly by

$$R = \alpha^{\lambda}{}_1\alpha_{\lambda 2} - \alpha^{\lambda}{}_2\alpha_{\lambda 1}, \quad \bar{R} = \alpha^{\lambda 2}\alpha^1{}_{\lambda} - \alpha^{\lambda 1}\alpha^2{}_{\lambda},$$
$$S = \alpha_{11}\alpha_{22} - (\alpha_{12})^2 = \epsilon^{\lambda\mu}\epsilon^{\rho\sigma}\alpha_{\lambda\rho}\alpha_{\mu\sigma},$$
$$\bar{S} = \alpha^{11}\alpha^{22} - (\alpha^{12})^2 = \epsilon_{\lambda\mu}\epsilon_{\rho\sigma}\alpha^{\lambda\rho}\alpha^{\mu\sigma}. \tag{32}$$

They can be regarded as nonvanishing elements of antisymmetric tensors, and therefore change the eigenvalues of (λ,λ'), as defined in (30), by $(-1,+1)$, $(+1,-1)$, $(-2,+2)$, and $(+2,-2)$, respectively. The remaining shift operators can be constructed from symmetric tensors, and are easily derived by making use of the characteristic identities [11]

$$(A-\lambda+\lambda'-2)(A-2\lambda-1)(A+2\lambda'-1)=0,$$
$$(\bar{A}-\lambda+\lambda'+2)(\bar{A}-2\lambda+1)(\bar{A}+2\lambda'+1)=0, \tag{33}$$

satisfied by the linear operators A and $\bar{A}$ introduced in (26), when $M=2$. By the omission of one of the factors of the left sides of these identities, we obtain projection operators which, applied to $\alpha_{\lambda\mu}$ and $\alpha^{\lambda\mu}$, isolate the required shift operators. Thus, if

$$\rho_j = \lambda+\lambda'+j \tag{34}$$

and

$$\alpha_{\lambda\mu} = \alpha^{+}_{\lambda\mu}(\rho_{-1}\rho_0)^{-1} - 2\alpha^{0}_{\lambda\mu}(\rho_{-1}\rho_1)^{-1} + \alpha^{-1}_{\lambda\mu}(\rho_0\rho_1)^{-1},$$
$$\alpha^{+}_{\lambda\mu} = \tfrac{1}{2}\alpha_{\lambda\mu}(A-\lambda+\lambda'-2)(A+2\lambda'-1),$$
$$\alpha^{0}_{\lambda\mu} = \tfrac{1}{2}\alpha_{\lambda\mu}(A-2\lambda-1)(A+2\lambda'-1), \tag{35}$$
$$\alpha^{-}_{\lambda\mu} = \tfrac{1}{2}\alpha_{\lambda\mu}(A-\lambda+\lambda'-2)(A-2\lambda-1),$$

then any component of the symmetric tensors $\alpha^{+}_{\lambda\mu}, \alpha^{0}_{\lambda\mu}$, and $\alpha^{-}_{\lambda\mu}$ will change (λ,λ') by $(-2,0)$, $(-1,+1)$, and $(0,+2)$, respectively. Similarly, if

$$\alpha^{\lambda\mu} = \alpha^{+\mu}_{\lambda}(\rho_{-1}\rho_0)^{-1} - 2\alpha^{\lambda\mu}_{0}(\rho_{-1}\rho_1)^{-1} + \alpha^{\lambda\mu}_{-}(\rho_0\rho_1)^{-1},$$
$$\alpha^{\lambda\mu}_{+} = \tfrac{1}{2}\alpha^{\lambda\mu}(\bar{A}-\lambda+\lambda'+2)(\bar{A}-2\lambda+1),$$
$$\alpha^{\lambda\mu}_{0} = \tfrac{1}{2}\alpha^{\lambda\mu}(\bar{A}+2\lambda'+1)(\bar{A}+2\lambda+1),$$
$$\alpha^{\lambda\mu}_{-} = \tfrac{1}{2}\alpha^{\lambda\mu}(\bar{A}-\lambda+\lambda'+2)(\bar{A}+2\lambda+1), \tag{36}$$

then any component of $\alpha^{\lambda\mu}_{+}$, $\alpha^{\lambda\mu}_{0}$, and $\alpha^{\lambda\mu}_{-}$ will change (λ,λ') by $(0,-2)$, $(+1,-1)$, and $(+2,0)$, respectively. Clearly $\alpha^{0}_{\lambda\mu}$ and $\alpha^{\lambda\mu}_{0}$ must differ from R and $\bar{R}$ in (32) only by factors depending on the U(2) generators; and all the required shift operators, except S and $\bar{S}$ can be defined by

$$\alpha^{+}_{11} = P(\gamma_0\gamma_1\rho_0/\rho_2)^{1/2}, \quad \alpha^{11}_{-} = \bar{P}(\gamma_{-2}\gamma_{-1}\rho_0/\rho_{-2})^{1/2},$$
$$\alpha^{0}_{11} = -R\alpha^2{}_1, \quad \alpha^{11}_{0} = -\bar{R}\alpha^1{}_2,$$
$$\alpha^{-}_{11} = Q(\alpha^2{}_1)^2(\rho_0/\gamma_0\gamma_1\rho_{-2})^{1/2}, \quad \alpha^{11}_{+} = \bar{Q}(\alpha^1{}_2)^2(\rho_0/\gamma_{-2}\gamma_{-1}\rho_2)^{1/2}$$
$$\gamma_j = \lambda - \alpha^1{}_1 + j + \tfrac{1}{2} = \lambda' + \alpha^2{}_2 + j + \tfrac{1}{2}, \tag{37}$$

with normalization factors chosen so that

$$P\bar{P} = \alpha_{\lambda\mu}\alpha^{\lambda\mu}_{+}, \quad \bar{P}P = \alpha^{\lambda\mu}\alpha^{-}_{\lambda\mu},$$
$$R\bar{R} = \alpha_{\lambda\mu}\alpha^{\lambda\mu}_{0}, \quad \bar{R}R = \alpha^{\lambda\mu}\alpha^{0}_{\lambda\mu}, \tag{38}$$
$$Q\bar{Q} \quad \alpha_{\lambda\mu}\alpha^{\lambda\mu}_{-}, \quad \bar{Q}Q = \alpha^{\lambda\mu}\alpha^{+}_{\lambda\mu}.$$

Commutation relations for the shift operators so defined can be obtained from the commutation relations satisfied by $\alpha_{\lambda\mu}$ and $\alpha^{\lambda\mu}$, by substituting from (33) and (34) into (13) and separating terms which shift between one irreducible representation of U(2) and another. Thus we obtain

$$[P,Q] = 4\rho_0 S, \quad [P,\bar{Q}] = 0,$$
$$[P,R] = 0, \quad [P,\bar{R}] = [\bar{Q},R],$$
$$[P,S] = [Q,S] = [R,S] = 0,$$
$$[S,\bar{R}] = 2(\lambda-\lambda'+1)R,$$
$$[S,\bar{P}] = 4\lambda Q, \quad [S,\bar{Q}] = 4\lambda' P,$$
$$QP = R^2 + \rho_1{}^2 S, \tag{39}$$

together with conjugate relations, like $[\bar{Q},\bar{P}] = 4\rho_0\bar{S}$.

All elements of the algebra A can be expressed as functions of $\lambda,\lambda',\Lambda,\Lambda'$, and two independent invariants

$$X = \tfrac{1}{2}[R,\bar{R}], \quad Y = \tfrac{1}{2}(R\bar{R}+\bar{R}R). \tag{40}$$

Obviously $R\bar{R} = X+Y$ and $\bar{R}R = Y-X$, but if we make use of the explicit expressions for $\langle S^2\rangle$ and $\langle S^4\rangle$ in (9), and the above relations (39), we also obtain

$$\bar{P}P = Y + \rho_{-2}X + \rho_{-1}{}^2\phi_{-1}, \quad P\bar{P} = Y + \rho_2 X + \rho_1{}^2\phi_1,$$
$$\bar{Q}Q = Y - \rho_2 X + \rho_1{}^2\phi_1{}', \quad Q\bar{Q} = Y - \rho_{-2}X + \rho_{-1}{}^2\phi'_{-1},$$
$$\bar{S}S = Y - (\sigma-2)X + \phi_{-1}\phi_1{}' + K, \quad S\bar{S} = Y - (\sigma+2)X + \phi_1\phi'_{-1} + K,$$
$$\bar{S}R^2 + \bar{R}^2 S = (\phi_{-2} + \phi'_2 - \rho_0{}^2 + 1)Y - X^2 + (\sigma-4)X + (1-\rho_0{}^2)K,$$
$$S\bar{R}^2 + R^2\bar{S} = (\phi_2 + \phi'_{-2} - \rho_0{}^2 + 1)Y - X^2 + (\sigma+4)X + (1-\rho_0{}^2)K, \tag{41}$$

where

$$\phi_j = (\lambda+j)^2 - \tfrac{1}{2}(\Lambda^2 + \Lambda'^2 - \tfrac{1}{2}),$$
$$\phi'_j = (\lambda'+j)^2 - \tfrac{1}{2}(\Lambda^2 + \Lambda'^2 - \tfrac{1}{2}), \quad \sigma = \lambda - \lambda',$$
$$K = \tfrac{1}{2}(\Lambda^2 + \Lambda'^2 - \tfrac{1}{2}) - \tfrac{1}{4}(\Lambda^2 - \Lambda'^2)^2. \tag{42}$$

It also follows from the commutation relations that

$$[X,Y] = 2(\bar{R}^2 S - \bar{S}R^2) = 2(S\bar{R}^2 - R^2\bar{S}). \tag{43}$$

Clearly the operators X and Y do not commute in general, and are not elements of any finitely generated Lie algebra. However, there are several special classes of representations of U(2) in which X and Y commute, and in fact have unique eigenvalues. Since for $M'=1$, $\lambda+\lambda' \geqslant l'$, according to (31), if $|r\rangle$ is a vector belonging to a representation of U(2) such that $\lambda+\lambda'=l'$ or $l'+1$, we must have $P|r\rangle = \bar{Q}|r\rangle = 0$; it then follows from (41) that X and Y have eigenvalues

$$X = -\tfrac{1}{2}\rho_{-1}{}^2\sigma, \quad Y = -\tfrac{1}{2}\rho_{-1}{}^2(\dot{\phi}_{-1} + \phi'_{-1})$$
$$(\lambda + \lambda' = l' \text{ or } l' + 1), \tag{44}$$

in such representations. Also, when λ has its minimum value $\frac{1}{2}(N-1)$, so that $\bar{P}P = \bar{S}S = 0$, and when λ' has its minimum value $\frac{1}{2}(N-1)$, or next to minimum value $\frac{1}{2}(N+1)$, so that $Q\bar{Q} = S\bar{S} = 0$, X and Y have unique eigenvalues given by

$$\begin{aligned}
&X = Y = -\rho_{-1}\phi_{-1}, && [\lambda = \tfrac{1}{2}(N-1)],\\
&X - 2\phi_{-2} = Y + 2\rho_{-2}\phi_{-2} = -\rho_{-1}\phi_{-1}, && [\lambda = \tfrac{1}{2}(N+1)],\\
&X = -Y = \rho_{-1}\phi'_{-1}, && [\lambda' = \tfrac{1}{2}(N-1)],\\
&X + 2\phi'_{-2} = Y + 2\rho_{-2}\phi'_{-2} = -\rho_{-1}\phi'_{-1} && [\lambda' = \tfrac{1}{2}(N+1)].
\end{aligned} \tag{45}$$

To construct a general matrix representation for the operators $\bar{S}S$ and $S\bar{S}$, and hence for X and Y, we note that $\mu_1 = \lambda - \frac{1}{2}N + \frac{1}{2}$ and $\mu_2 = \lambda' - \frac{1}{2}N + \frac{1}{2}$ have nonnegative integral eigenvalues in irreducible representations of U(2) within the irreducible representation (Λ, Λ') of Sp(4), and that $\mu_1 + \mu_2$ is odd or even according as $\Lambda + \Lambda'$ is odd or even. Let us introduce a set of eigenvectors $|r\rangle_{p,q}$ of μ_1 and μ_2 such that

$$\mu_1 |r\rangle_{p,q} = (r + 2p)|r\rangle_{p,q}, \quad \mu_2 |r\rangle_{p,q} = (s + 2q)|r\rangle_{p,q}, \tag{46}$$
$$r + s = \Lambda + \Lambda', \quad |r\rangle_{p,q} = \bar{P}^p \bar{Q}^q |r\rangle_{0,0},$$
$$|r+2\rangle_{0,0} = [(r - \Lambda + 1)^2 - 1/4]\bar{S}|r\rangle_{0,0}.$$

Clearly, in a representation of U(2) in which μ_1 and μ_2 have fixed eigenvalues, states are sufficiently labeled by r, which takes even or odd values ranging from $\max(\eta, \Lambda + \Lambda' - \mu_2)$ to $\min(\mu_1, \Lambda + \Lambda' - \eta)$ where $\eta = 0$ or 1 according as μ_1 is even or odd. From the commutation relations (39), and the known eigenvalues of X and Y in the states $|r\rangle_{0,0}$, given by (44), we find that

$$\begin{aligned}
\bar{S}S|r\rangle_{p,q} = {}& \{g(r-1)g(s+1) + (\mu_1{}^2 - r^2)[(\mu_2+2)^2 - s^2]\}|r\rangle_{p,q}\\
&+ (\mu_1{}^2 - r^2)g(s-1)|r+2\rangle\\
&+ [(\mu_2+2)^2 - (s+2)^2]g(r-1)|r-2\rangle,\\
S\bar{S}|r\rangle_{p,q} = {}& \{g(r+1)g(s-1) + [(\mu_1+2)^2 - r^2](\mu_2{}^2 - s^2)\}\\
&\times |r\rangle_{p,q} + (\mu_2{}^2 - s^2)g(r-1)|r-2\rangle\\
&+ [(\mu_1+2)^2 - (r+2)^2]g(s-1)|r+2\rangle,
\end{aligned} \tag{47}$$

where

$$g(x) = (x - \Lambda') - \tfrac{1}{4}. \tag{48}$$

These formulas provide an explicit matrix representation for $\bar{S}S$ and $S\bar{S}$, and hence for X and Y, in an irreducible representation of U(2), labeled by $\lambda = \mu_1 + \frac{1}{2}N - \frac{1}{2}$ and $\lambda' = \mu_2 + \frac{1}{2}N - \frac{1}{2}$, within an irreducible representation of Sp(4), labeled by Λ and Λ'. From the matrices, eigenvalues can of course be computed without difficulty; they are irrational in general. In the next section we shall discuss the definition of an operator with eigenvalues corresponding to the integral parameter r.

4. LABELING OF SO(3) IN U(3)

The problem of finding an operator with known eigenvalues to label representations of SO(3) within an arbitrary irreducible representation of U(3) is the simplest and best known of the problems under consideration.

According to (23), if a representation of U(2) is labeled $(\lambda_1, \lambda_2) = (\mu_1 + N/2 - 1, -\mu_2 - N/2 + 1)$, the corresponding representation of U(N) will be labeled $(L_1, L_2, \ldots, L_N)$, where $L_1 = \mu_1$, $L_N = -\mu_2$, and the other L_r vanish. We therefore consider a representation of U(3) labelled $(\mu_1, 0, -\mu_2)$, with invariants

$$\begin{aligned}
&\langle b\rangle = \mu_1 - \mu_2 = \sigma,\\
&\langle b^2\rangle = \mu_1(\mu_1+2) + \mu_2(\mu_2+2).
\end{aligned} \tag{49}$$

The corresponding representation of SU(3) is labeled (μ_1, μ_2) and the invariants of SU(3) defined by Racah[1] and Ilamed[2] are

$$\begin{aligned}
g = {}& 2[\mu_1(\mu_1+2) + \mu_2(\mu_2+2) - (\mu_1-\mu_2)^2/3],\\
g_3 = {}& 4(\mu_1-\mu_2)[5(\mu_1-\mu_2)^2/9 - \mu_1(\mu_1+2)\\
&- \mu_2(\mu_2+2) - 2]/3.
\end{aligned} \tag{50}$$

Different representations of SO(3) contained within an irreducible representation of U(3) of the type considered are labeled by the operator l, defined by

$$l(l+1) = \langle b^2 - b\bar{b}\rangle; \tag{51}$$

this is, of course, the angular momentum in quantum-mechanical applications. In the corresponding representations of Sp(4), the invariants defined in (9) are given by

$$\begin{aligned}
&\langle S^2\rangle = 2l(l+1) - 9,\\
&\langle S^4\rangle = 2[(l+\tfrac{1}{2})^4 + 3(l+\tfrac{1}{2})^2 - 32 + \tfrac{13}{16}],
\end{aligned} \tag{52}$$

and the representations are labeled by

$$(\Lambda_1, \Lambda_2) = (l - \tfrac{3}{2}, -1 \pm \tfrac{1}{2}).$$

Operators whose eigenvalues, if they could be found, would serve to label equivalent representations of SO(3) in an irreducible representation of U(3), have been defined by Racah[1] and Ilamed[2]; given by

$$\begin{aligned}
x = {}& \langle b\bar{b}b + \bar{b}b\bar{b} - 2b\rangle + \langle b\rangle^3 + \langle b\rangle(4l(l+1)/3 - 3\langle b^2\rangle),\\
y = {}& 8\langle b\rangle x/3 - 4\langle b^2\bar{b}^2\rangle - (16\langle b^2\rangle/9 + 9)l(l+1)\\
&+ 2\langle b^2\rangle(\langle b^2\rangle + 2\langle b\rangle^2 + 4) + \langle b\rangle^2 - 4
\end{aligned} \tag{53}$$

in the present notation. The operators $0_l{}^0$ and $Q_l{}^0$ introduced by Hughes[3] are related to them by

$$\begin{aligned}
&0_l{}^0 = -3\sqrt{6}x,\\
&Q_l{}^0 = -18y + 12l(l+1)[2\langle b^2\rangle - 2\langle b\rangle^2/3 - l(l+1) - 3],
\end{aligned}$$

while those considered by Green and Bracken[5] are

$$\begin{aligned}
S_3 &= \langle b\bar{b}b + \bar{b}b\bar{b}\rangle,\\
S_4 &= \langle b\bar{b}^2 b + \bar{b}b^2\bar{b}\rangle\\
&= 2\langle b^2\bar{b}^2\rangle + 9\langle b^2\rangle - 3\langle b\rangle^2 - 12l(l+1).
\end{aligned}$$

Although the eigenvalues of these operators have been obtained in special representations,[3] they are irrational numbers in general, and no general formula is known, The algebra of the operators x and y was studied by Ilamed,[2] who has shown how to derive commutation relations of the type

$$[x,[x,y]] = 24y^2 + \zeta_1 y + \zeta_1 y + \zeta_2 x^2 + \zeta_3 x + \zeta_4,$$
$$[y,[y,x]] = 32x^3 + \zeta_2(xy+yx) + \zeta_3 y + \zeta_5 x^2 + \zeta_6 x + \zeta_7,$$

$$[x,y]^2 = 16x^4 + \zeta_8 x^3 + 16y^3 + \zeta_9(x^2y + yx^2) + \zeta_{10}(xy + yx) + \zeta_{11}x^2 + \zeta_{12}y^2 + \zeta_{13}x + \zeta_{14}y + \zeta_{15}, \tag{54}$$

where the ζ_i are given as simple polynomials in g, g_3, and

$$s = l(l+1). \tag{55}$$

Our object in this paper is to show how the results obtained in the previous section can be used to solve the above equations. As all the operators to be used are SO(3) invariants, we may consider an irreducible representation in which l has a fixed integral value. However, we shall find it convenient to consider a variety of representations of U(3), in which μ_1 and μ_2 take different nonnegative integral eigenvalues. The dimensions of the irreducible representations of x and y depend on the eigenvalues of μ_1 and μ_2 and are, as one can see from (46), never greater than $\frac{1}{2}(\mu_1 + \mu_2 - l) + 1$.

To solve (54), we first consider the representation for which $[x,y] = 0$. These can be found by setting the right sides of (54) equal to zero; as we have already seen in the last section, the solutions fall into six classes, all included in (44) or (45): (i) $\mu_1 + \mu_2 = l$, (ii) $\mu_1 + \mu_2 = l + 1$, (iii) $\mu_1 = 0$, (iv) $\mu_1 = 1$, (v) $\mu_2 = 0$, and (vi) $\mu_2 = 1$. The eigenvalues of x and y in these representations are related to those of X and Y found in the previous section, the precise relationship between the operators being given by

$$x = 2X + \sigma s/3, \qquad y = -4Y + 4\sigma X/3 + (2\langle b^2\rangle + 3 - 8\sigma^2/9)s. \tag{56}$$

Now, a general representation of U(3) is related to a corresponding representation of one of the classes (i) and (ii) listed above by a shift operator of the type $\bar{P}^{\rho-r} Q^{\sigma+r}$, with $\bar{P}$ and Q defined as in the previous section. Thus, to determine a general representation of x and y, labeled by s, μ_1, μ_2, and r, it is sufficient to define $\bar{P}$ and Q. The considerations of the previous section show that the following factorizations are possible:

$$\begin{aligned}
&Y + (\mu_1 + \mu_2)X + (\mu_1 + \mu_2 + 1)^2\phi_{-1} = \bar{P}P,\\
&Y + (\mu_1 + \mu_2 + 4)X + (\mu_1 + \mu_2 + 3)^2\phi_1 = P\bar{P},\\
&Y - (\mu_1 + \mu_2 + 4)X + (\mu_1 + \mu_2 + 3)^2\phi_1' = \bar{Q}Q,\\
&Y - (\mu_1 + \mu_2)X + (\mu_1 + \mu_2 + 1)^2\phi_{-1}' = Q\bar{Q},\\
&\phi_j = (\mu_1 + j + 1)^2 - \tfrac{1}{2}s, \quad \phi_j' = (\mu_2 + j + 1)^2 - \tfrac{1}{2}s,
\end{aligned} \tag{57}$$

where P and $\bar{Q}$ are represented by rectangular matrices with two more rows than columns, and $\bar{P}$ and Q are their adjoints. By different factorizations,

$$\begin{aligned}
&Y + X = R\bar{R}, \quad Y - X = \bar{R}R,\\
&Y - (\sigma - 2)X + \phi_{-1}\phi_1' + \tfrac{1}{2}s(1 - \tfrac{1}{2}s) = \bar{S}S,\\
&Y - (\sigma + 2)X + \phi_1\phi_{-1}' + \tfrac{1}{2}s(1 - \tfrac{1}{2}s) = S\bar{S},
\end{aligned} \tag{58}$$

we can define the shift operators R and S and their adjoints. As shown by (47), a representation can be found in which $\bar{S}S$ and $S\bar{S}$ have codiagonal form. In such a representation, certain linear combinations of $\bar{S}S$ and $S\bar{S}$ can be found with upper and lower triangular form, and these have rational eigenvalues. But the eigenvalues of other linear combinations of X and Y are in general irrational.

We now recapitulate by stating explicitly a simple solution of the classical problem of defining an operator or operators with integral eigenvalues, which can be used to label representations of SO(3) in SU(3).

Let

$$\begin{aligned}
&W_1(\nu_1) = (\mu_1 + \nu_1 + 4)\bar{S}S - (\mu_1 + \nu_1)S\bar{S},\\
&W_2(\nu_2) = (\mu_2 + \nu_2 + 4)S\bar{S} - (\mu_2 + \nu_2)\bar{S}S,
\end{aligned} \tag{59}$$

where ν_1 and ν_2 are operators whose eigenvalues are the parameters r and s in the states defined by (47), which are related by

$$r + s = \Lambda + \Lambda'. \tag{60}$$

Then it follows directly from (47) that if

$$\begin{aligned}
w_1(\nu_1) = {}&(\mu_1 + \nu_1 + 4)g(\nu_1 - 1)g(\nu_2 + 1)\\
&- (\mu_1 + \nu_1)g(\nu_1 + 1)g(\nu_2 - 1) + 4(\mu_1 + \nu_1)\{(\mu_2 + 1)[(\mu_2 + 2)^2\\
&- (\nu_2 + 2)^2] - (\nu_1 + 1)(\mu_2{}^2 - \nu_2{}^2)\},\\
w_2(\nu_2) = {}&(\mu_2 + \nu_2 + 4)g(\nu_1 + 1)g(\nu_2 - 1) - (\mu_2 + \nu_2)g(\nu_1 - 1)\\
&\times g(\nu_2 + 1) + 4(\mu_2 + \nu_2)\{(\mu_1 + 1)[(\mu_1 + 2)^2\\
&- (\nu_1 + 2)^2] - (\nu_2 + 1)(\mu_1{}^2 - \nu_1{}^2)\},
\end{aligned}$$

then

$$\begin{aligned}
&[W_1(r) - w_1(r)]\,|r\rangle\\
&\quad = 4(\mu_2 - s)(\mu_1 + \mu_2 + r + s + 4)g(r-1)\,|r-2\rangle,\\
&[W_2(s) - w_2(s)]\,|r\rangle = 4(\mu_1 - r)(\mu_1 + \mu_2 + r + s + 4)g(r-1)\,|r+2\rangle
\end{aligned} \tag{61}$$

It follows that $w_1(r)$ is an eigenvalue of $W_1(r)$ in a nonorthogonal basis, and that $w_2(r)$ is an eigenvalue of $W_2(r)$ in a different nonorthogonal basis, for each value of r between $\min(\eta, \Lambda + \Lambda' - \mu_2)$ and $\max(\mu_1, \Lambda + \Lambda' - \eta)$. Hence if we define ν_1 and ν_2 by means of the algebraic equations

$$W_1(\nu_1) = w_1(\nu_1), \quad W_2(\nu_2) = w_2(\nu_2), \tag{62}$$

ν_1 and ν_2 will have integral eigenvalues. This may be compared with the definition of l by means of the algebraic equation (51). Either of the operators ν_1 and ν_2 defined by (62) may be used as a labeling operator for the representation of SO(3) in SU(3).

APPENDIX

We now provide a proof, in the notation of this paper, of the results stated in Eqs. (22), (24), (27), and (28).

First we note that, since

$$(\alpha + \tfrac{1}{2}N)^\lambda{}_\mu = a_{\mu i}\alpha^\lambda{}_i, \quad (b + \tfrac{1}{2}M)_{ij} = a_{\lambda j}\alpha^\lambda{}_i, \tag{A1}$$

the relation

$$a_{\lambda j}[(\alpha + \tfrac{1}{2}N)^n]^\lambda{}_\mu = a_{\mu i}[(b + \tfrac{1}{2}M)^n]_{ij} \tag{A2}$$

is trivially satisfied for $n = 0$ or 1. Also, since $\alpha^\lambda{}_\mu$ and b_{ij} commute, it follows from (A2) that

$$\begin{aligned}
a_{\lambda j}[(\alpha + \tfrac{1}{2}N)^{n+1}]^\lambda{}_\mu &= a_{\nu i}(a + \tfrac{1}{2}N)^\nu{}_\mu[(b + \tfrac{1}{2}M)^n]_{ij}\\
&= a_{\mu k}(b + \tfrac{1}{2}M)_{ki}[(b + \tfrac{1}{2}M)^n]_{ij};
\end{aligned}$$

so (A2) is true for $n = 2, 3, \cdots$, by induction. If we multiply (A2) on the left by $a^\mu{}_j$, we obtain the first relation of (22). The second relation of (22) is obtained in a similar way by multiplying

$$\alpha^{\lambda}{}_{j}[(\bar{\alpha}-\tfrac{1}{2}N)^{n}]^{\lambda}{}_{\mu}=a^{\mu}{}_{i}[(\bar{b}-\tfrac{1}{2}M)^{n}]_{ij} \tag{A3}$$

on the left by $a_{\mu j}$. Results of this type have been already obtained by the Aguilera-Navarros.[12]

To prove (24), we proceed in a similar way. We note that

$$(S+\tfrac{1}{2}N)^{P}{}_{Q}=a_{Qi}a^{P}{}_{i}, \quad (l+M)_{ij}=a_{Pi}a^{P}{}_{i}, \tag{A4}$$

so that

$$a_{Qi}[(l+M+1)^{n}]_{ij}=a_{Pj}[(S+\tfrac{1}{2}N)^{n}]^{P}{}_{Q} \tag{A5}$$

is easily verified for $n=0$ or 1. Since l_{ij} and $S^{P}{}_{Q}$ commute, it follows by induction that (A5) is true also for $n=2,3,\cdots$. The desired relation (24) is obtained from (A5) by multiplying with $a^{Q}{}_{j}$ on the left.

Finally we prove the results (27) and (28) by a similar method. We use the representations

$$A^{\rho\sigma}{}_{\lambda\mu}=\alpha^{\sigma}{}_{\mu}\delta^{\rho}{}_{\lambda}+\alpha^{\sigma}{}_{\lambda}\delta^{\rho}{}_{\lambda},\ B_{ijkl}=b_{ik}\delta_{jl}+b_{jk}\delta_{il} \tag{A6}$$

for the linear operators A and B, and note that the relation

$$a_{\mu k}a_{\lambda l}[(B+M)^{n}]_{klij}=a_{\rho j}a_{\sigma i}[(A+N)^{n}]^{\rho\sigma}{}_{\lambda\mu} \tag{A7}$$

follows immediately from (A1) and (A6), for $n=0$ or 1. As B_{klij} and $A^{\rho\sigma}{}_{\lambda\mu}$ commute, this result follows also by induction for $n=2,3,\cdots$. If we set $i=j$ in (A7) and multiply on the left by $\alpha^{\lambda\mu}$, the result (27) follows, with the help of

$$\alpha^{\lambda\mu}a_{\mu k}a_{\lambda l}=[(\bar{b}-\tfrac{1}{2}M-1)(b-\tfrac{1}{2}M)], \text{ etc.} \tag{A8}$$

In a similar way, we use the representations

$$\bar{A}^{\rho\sigma}{}_{\lambda\mu}=\bar{\alpha}^{\sigma}{}_{\mu}\delta^{\rho}{}_{\lambda}+\bar{\alpha}^{\sigma}{}_{\lambda}\delta^{\rho}{}_{\mu}, \quad \bar{B}_{ijkl}=b_{ki}\delta_{jl}+\bar{b}_{li}\delta_{jk} \tag{A9}$$

for $\bar{A}$ and $\bar{B}$ and, establish the relation

$$a^{\mu}{}_{k}a^{\lambda}{}_{l}[(\bar{B}-M)^{n}]_{ijkl}=a^{\rho}{}_{j}a^{\sigma}{}_{i}[(\bar{A}-N)^{n}]^{\rho\sigma}{}_{\lambda\mu}. \tag{A10}$$

Then we set $i=j$ and multiply this relation on the left by $\alpha_{\lambda\mu}$ to obtain the result (28), with the help of

$$\alpha_{\lambda\mu}a^{\mu}{}_{k}a^{\lambda}{}_{l}=[(b+\tfrac{1}{2}M+1)(b+\tfrac{1}{2}M)]_{lk}. \tag{A11}$$

A wide variety of interesting and useful identities can be established by the use of this technique.

*Permanent address: Soreq Institute, Tel Aviv, Israel.

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