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Hongya Liu

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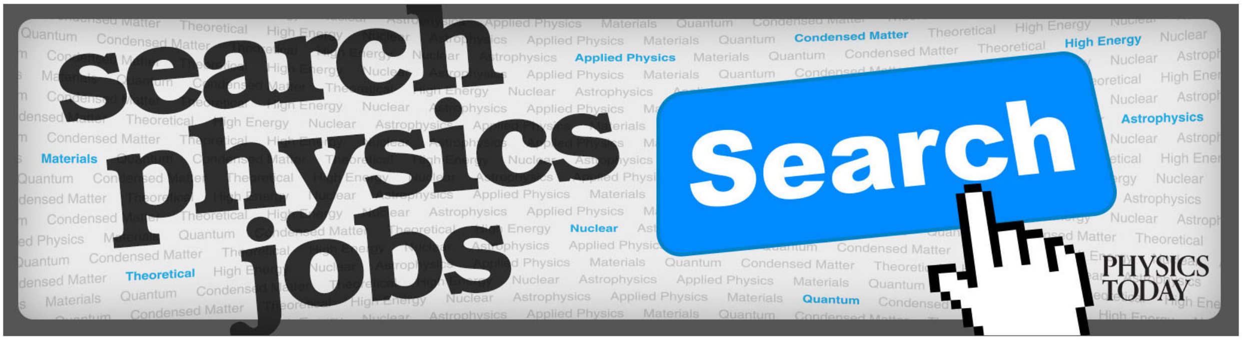

Spherically symmetric density perturbation in relativistic cosmology

Hongya Liu[a)]
Center of Theoretical Physics, CCAST (World Laboratory) and Department of Physics, Zhengzhou University, Zhengzhou, Henan, People's Republic of China

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Einstein's field equations for spherically symmetric perfect fluids are reduced to two equations plus three definitions, and from these the spherical density perturbation equation is derived. It is found that this perturbation equation can be separated into two equations by the method of separation of variables and can be solved explicitly in some special cases. An interesting result is that for matter-dominated era the perturbation equation is found to be of the Bonnor form of Newtonian theory. Another interesting result is that for zero-curvature universe and radiation-dominated era the density perturbation has two growing modes: one is something like a power of the proper time, and the other is of the exponential law of the conformal time, each corresponds to a different limit of approximation.

I. INTRODUCTION

Spherically symmetric perturbations are of particular significance in the theory of galaxy formation. In a previous paper[1] we have derived a perturbation equation from the general spherically symmetric model of dust universe (i.e., with $p = 0$). It is found that this equation is just a special case of the Bonnor perturbation equation of Newtonian theory with the influence of the pressure being ignored. The purpose of this paper is to take the pressure into account and to extend the work in Ref. 1 to the more general perfect fluid case. That is, we will study the Einstein field equations for spherically symmetric perfect fluids, and from these equations we will derive the density perturbation equation and to look for its solutions.

II. RIGOROUS FIELD EQUATIONS

The general spherically symmetric metric may be taken in the form

$$ds^2 = e^{2\sigma} dt^2 - e^{2\omega} dr^2 - Y^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{2.1}$$

where the metric parameters σ, ω, and Y are all functions of the time t and a comoving radial space coordinate r. With this metric and a perfect-fluid energy-momentum tensor, Einstein's field equations can be expressed as five relations.[2–4] These are

$$2\mu/Y = 1 + e^{-2\sigma}\dot{Y}^2 - e^{-2\omega}Y'^2, \tag{2.2a}$$

$$\mu' = 4\pi\rho Y^2 Y', \tag{2.2b}$$

$$\dot{\mu} = -4\pi p Y^2 \dot{Y}, \tag{2.2c}$$

$$\sigma' = -p'/(\rho + p), \tag{2.2d}$$

$$\dot{\omega} = -\dot{\rho}/(\rho + p) - 2\dot{Y}/Y. \tag{2.2e}$$

Here, ρ is the energy density, p is the pressure, $\mu = \mu(t,r)$ is the mass interior to r, and a dot and a prime denote partial derivatives with respect to t and r, respectively. Units are chosen so that $c = G = 1$.

We see that there are six unknown functions in Eqs. (2.2): σ, ω, μ, Y, ρ, and p, of which the pressure p can be determined from the equation of state. This set of equations (2.2) is a great simplification to the original Einstein field equations, but it remains in general difficult to solve unless additional restrictions on the metric coefficients are made.[5–7] Therefore some further simplifications to these equations are desirable and valuable. In fact, we find that the five equations (2.2) can be solved partly, giving two equations and three definitions in terms of three unknowns Y, ρ and p. The process is as follows.

Without loss of generality we set

$$e^{2\omega} = Y'^2 e^{2S(t,r)} A^{-1}(r). \tag{2.3}$$

Substituting this into Eq. (2.2e), we obtain

$$\dot{\rho} = -(\rho + p)(2\dot{Y}/Y + \dot{Y}'/Y' + \dot{S}). \tag{2.4}$$

From (2.2b) and (2.2c) we have

$$\frac{\partial}{\partial t}(\rho Y^2 Y') = -\frac{\partial}{\partial r}(p Y^2 \dot{Y}). \tag{2.5}$$

Substitution of (2.4) into (2.5) gives

$$\dot{S} = [p'/(\rho + p)](\dot{Y}/Y'). \tag{2.6}$$

Thus we find that three unknowns μ, σ, and S can be integrated explicitly from Eqs. (2.2b), (2.2d), and (2.6) in terms of the other three unknowns Y, ρ, and p. And then the five Eqs. (2.2) reduce to two equations and three definitions:

$$e^{2S}(1 - 2\mu/Y + e^{-2\sigma}\dot{Y}^2) = A(r), \tag{2.7a}$$

$$\dot{\rho} = -(\rho + p)\left(\frac{2\dot{Y}}{Y} + \frac{\dot{Y}'}{Y'} + \frac{p'}{\rho + p}\frac{\dot{Y}}{Y'}\right), \tag{2.7b}$$

with

$$\mu \equiv \int_0^r 4\pi\rho Y^2 Y'\, dr, \tag{2.8a}$$

$$\sigma \equiv -\int \frac{p'}{\rho + p}\, dr, \tag{2.8b}$$

$$S \equiv \int \frac{p'}{\rho + p}\frac{\dot{Y}}{Y'}\, dt, \tag{2.8c}$$

and from (2.3) and (2.7a) the metric (2.1) reduces to

[a)] Present address: Dept. of Physics and Astronomy, University of Missouri—Columbia, Columbia, MO 65211.

$$ds^2 = e^{2\sigma} dt^2 - Y'^2(1 - 2\mu/Y + e^{-2\sigma}\dot{Y}^2)^{-1} dr^2 - Y^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \quad (2.9)$$

From (2.8b) the parameter σ is defined just up to an arbitrary t-dependent constant of integration. This is because in the comoving coordinates (2.9) a scale transformation $\tilde{t} = \tilde{t}(t)$ is still permitted. So we can specify this t-dependent constant of integration arbitrarily. From (2.8c) the parameter S is defined just up to an arbitrary r-dependent constant of integration. This is associated with the arbitrary function $A(r)$ in (2.7a). So we can also specify it arbitrarily, leaving $A(r)$ to be determined by, for instance, the initial value of (2.7a).

The two equations in (2.7) together with the equation of state constitute the fundamental equations of the spherically symmetric cosmological models, and (2.7a) is the Einstein equation and (2.7b) is the energy-conservation equation.

III. SPECIAL CASES OF THE RIGOROUS FIELD EQUATION

Now we consider two special cases of the field equations (2.7) derived in Sec. II: the homogeneous and isotropic perfect-fluid model and the spherically symmetric dust model.

A. Homogeneous and isotropic perfect-fluid cosmological model

For the homogeneous and isotropic cosmological model we have

$$\rho = \rho(t), \quad p = p(t), \quad Y = ra(t). \quad (3.1)$$

Then Eqs. (2.8) give

$$\sigma = S = 0, \quad \mu = \tfrac{4}{3}\pi\rho a^3 r^3, \quad (3.2)$$

where we have set the two constants of integration to be zero. Equations (2.7) and (2.9) are then reduced to

$$\dot{a}^2 + K = \tfrac{8}{3}\pi\rho a^2, \quad (3.3a)$$

$$\dot{\rho} = -(\rho + p)(3\dot{a}/a), \quad (3.3b)$$

$$ds^2 = dt^2 - a^2[dr^2/(1 - Kr^2) + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)], \quad (3.4)$$

where K is a constant and

$$A(r) = 1 - Kr^2. \quad (3.5)$$

This is just the Friedmann–Robertson–Walker standard cosmological model.

B. Spherically symmetric dust universe

For the dust universe we have $p = 0$, and from (2.8b) and (2.8c),

$$\sigma = S = 0. \quad (3.6)$$

The integration of (2.7b) gives

$$\rho = \rho_i (Y_i^2 Y_i' / Y^2 Y'), \quad (3.7)$$

where ρ_i, Y_i, and Y_i' are the initial values of ρ, Y, and Y' respectively. Thus (2.8a) reduces to

$$\mu = \int_0^r 4\pi\rho_i Y_i^2 Y_i'\, dr = \mu(r) \quad (3.8)$$

and (2.7a) reduces to

$$\dot{Y}^2 + K(r) r^2 = 2\mu(r)/Y, \quad (3.9)$$

with

$$K(r) r^2 \equiv 1 - A = \frac{2\mu(r)}{Y_i} - \dot{Y}_i^2. \quad (3.10)$$

Thus we obtain the initial-value expressions of equations of the spherically symmetric dust universe.[8]

IV. PERTURBATION EQUATIONS IN HOMOGENEOUS AND ISOTROPIC UNIVERSE

Now we consider the behavior of spherically symmetric perturbations in the background of a homogeneous and isotropic universe. The spherically symmetric perturbation equations can be derived directly from the rigorous Eqs. (2.7) of the spherically symmetric perfect fluids. Clearly, this method will be much easier than those of deriving them from the original Einstein field equations $G\mu\nu = -8\pi T\mu\nu$.

Now both the perturbed and the background space-times must satisfy the rigorous Eqs. (2.7)–(2.9). We define small perturbation quantities as

$$\delta\rho = \rho_p(t,r) - \rho(t,r), \quad (4.1)$$
$$\delta Y = Y_p(t,r) - Y(t,r),$$

and so on. Here, ρ_p, Y_p, and so on represent quantities of the perturbed space-time, while ρ, Y, and so on quantities of the background space-time. Then the required perturbation equations can be derived from (2.7) and (2.8) by adding small perturbation quantities to the corresponding background quantities and by keeping only the first-order quantities in perturbation in (2.7) and (2.8). Denote

$$\delta Y = Y\alpha, \quad (4.2)$$

$$\delta\rho = (\rho + p)\beta. \quad (4.3)$$

For adiabatic perturbations we have

$$\delta p = v_s^2 \delta\rho \quad (4.4)$$

where v_s is the speed of sound. We choose the background space-time to be of the homogeneous and isotropic form of (3.1)–(3.5) and we suppose $v_s = v_s(t)$. Then Eqs. (2.7)–(2.8) give

$$\delta A = 2(1 - Kr^2)\delta S - 2\delta\mu/ar + \tfrac{8}{3}\pi\rho r^2 a^2 \alpha - 2r^2\dot{a}^2\delta\sigma + 2r^2\dot{a}(\dot{a}\alpha + a\dot{\alpha}), \quad (4.5)$$

$$\delta\dot{\rho} = -(\rho + p)\left[\frac{3\dot{a}\beta}{a} + \frac{\dot{a}v_s^2}{ar^2}(r^3\beta)' + \frac{(r^3\dot{\alpha})'}{r^2}\right], \quad (4.6)$$

$$\delta\mu = \int_0^r 4\pi a^3 r^2[(\rho + p)\beta + 3\rho\alpha + \rho r\alpha']dr, \quad (4.7)$$

$$\delta\sigma = -v_s^2\beta, \quad (4.8)$$

$$\delta S = \int v_s^2 a^{-1}\dot{a}r\beta'\, dt. \quad (4.9)$$

Substituting (4.3) into (4.6), we find

$$(r^3\dot{\alpha})' = -r^2[\dot{\beta} + [\dot{p}/(\rho + p)]\beta + (\dot{a}v_s^2/ar^2)(r^3\beta)']. \quad (4.10)$$

From (4.5) and using (4.7), (4.8), (3.3a), and (4.10), we obtain

$$(r\delta A)' = [2r(1-Kr^2)\delta S]' - 2K(r^3\alpha)' - 2a\dot{a}r^2\dot{\beta}$$
$$- 2[4\pi(\rho+p)a^2 + a\dot{a}\dot{p}/(\rho+p)]r^2\beta. \tag{4.11}$$

Differentiating this equation with respect to t and substituting Eqs. (4.9) and (4.10) into it and using (3.3), we finally obtain an equation with only one variable β,

$$(1-Kr^2)\beta'' + (1/r)(2-3Kr^2)\beta' + 3K\beta$$
$$= \frac{a^2}{v_s{}^2}\left[\ddot{\beta} + \left(\frac{2\dot{a}}{a} + \frac{\dot{p}}{\rho+p}\right)\dot{\beta} + B(t)\beta\right], \tag{4.12}$$

where

$$B(t) = -4\pi(\rho+p) + \frac{2\dot{a}}{a}\frac{\dot{p}}{\rho+p} + \frac{\partial}{\partial t}\left(\frac{\dot{p}}{\rho+p}\right). \tag{4.13}$$

Equation (4.12) is the fundamental differential equation that governs the growth or decay of the density contrast $\beta = \delta\rho/(\rho+p)$. Substituting this β into (4.10), we can obtain the perturbation $\alpha = \delta Y/Y$. Other perturbation quantities can be obtained by (4.5)–(4.9).

V. SOLUTIONS OF THE DENSITY PERTURBATION EQUATION

The fundamental perturbation equation (4.12) can be solved by the method of separation of variables. Let $\beta = R(r)\ \delta(t)$, then Eq. (4.12) separates into two equations:

$$(1-Kr^2)R'' + \frac{1}{r}(2-3Kr^2)R' + (3K+k^2)R = 0, \tag{5.1}$$

$$\ddot{\delta} + \left(\frac{2\dot{a}}{a} + \frac{\dot{p}}{\rho+p}\right)\dot{\delta} + [a^{-2}v_s{}^2k^2 + B(t)]\delta = 0, \tag{5.2}$$

where k is a separation constant.

A. Solution of the radial equation (5.1) for $K=0$

In the case of $K=0$, the radial equation (5.1) reduces to

$$R'' + (2/r)R' + k^2R = 0. \tag{5.3}$$

Since $R(r)$ must be finite for $r=0$, the desired solution of (5.3) is

$$R(r) \propto \sin kr/r. \tag{5.4}$$

B. Solutions of the evolution equation (5.2) for $p=0$

In the matter-dominated era we can set $p=0$, then the evolution equation (5.2) reduces to

$$\ddot{\delta} + (2\dot{a}/a)\dot{\delta} + (v_s{}^2k^2/a^2 - 4\pi\rho)\delta = 0. \tag{5.5}$$

This is just the same equation of perturbation presented by Bonnor based on the Newtonian theory.[9,10] Its zero-pressure solutions and zero-curvature solutions are well known.[10]

C. Solutions of the evolution equation (5.2) for $p=\rho/3$ and $K=0$

In the radiation-dominated era we have

$$p=\rho/3, \quad \rho \propto a^{-4}, \quad v_s{}^2 = 1/3. \tag{5.6}$$

So the evolution equation (5.2) reduces to

$$\ddot{\delta} + \frac{\dot{a}}{a}\dot{\delta} + \left(\frac{v_s{}^2k^2+K}{a^2} - \frac{16}{3}\pi\rho\right)\delta = 0. \tag{5.7}$$

Furthermore, we suppose $K=0$, then we have

$$a \propto t^{1/2}, \quad \rho = 3/32\pi t^2. \tag{5.8}$$

Thus (5.7) becomes

$$\ddot{\delta} + \frac{1}{2t}\dot{\delta} + \left(\frac{\Lambda^2}{t} - \frac{1}{2t^2}\right)\delta = 0, \tag{5.9}$$

where $\Lambda = (k/a)\sqrt{t/3}$ is a constant. The solutions of Eq. (5.9) are

$$\delta_\pm \propto t^{1/4}J_{\pm 3/2}(2\Lambda t^{1/2}), \tag{5.10}$$

where $J_{\pm 3/2}$ are the usual Bessel functions of order $\pm 3/2$. That is,

$$\delta_+ \propto \cos(2\Lambda t^{1/2}) - \sin(2\Lambda t^{1/2})/2\Lambda t^{1/2}, \tag{5.11a}$$

$$\delta_- \propto \sin(2\Lambda t^{1/2}) + \cos(2\Lambda t^{1/2})/2\Lambda t^{1/2}. \tag{5.11b}$$

In the low-frequency limit $2\Lambda t^{1/2} \ll 1$, (5.11) become

$$\delta_+ \propto t, \quad \delta_- \propto t^{-1/2}. \tag{5.12}$$

The high-frequency limit $2\Lambda t^{1/2} \gg 1$ gives

$$\delta_+ \propto \cos(2\Lambda t^{1/2}), \quad \delta_- \propto \sin(2\Lambda t^{1/2}). \tag{5.13}$$

From (5.12) we see that in the low-frequency limit δ_+ is the growing mode and δ_- is the decaying mode. From (5.13) and combining it with (5.4) we see that in the high-frequency limit the density contrast are of the form

$$\delta\rho_+/(\rho+p) \propto (\sin kr/r)\cos(2\Lambda t^{1/2}),$$
$$\delta\rho_-/(\rho+p) \propto (\sin kr/r)\sin(2\Lambda t^{1/2}), \tag{5.14}$$

which are clearly modes of spherical acoustic waves.

D. Exponentially growing mode

The separation constant k in Eqs. (5.1) and (5.2) may also be a purely imaginary constant. In this case the solution of the radial equation (5.3) becomes

$$R(r) \propto \sinh \tilde{k}r/r. \tag{5.15}$$

And instead of Eqs. (5.11), we have, for the $K=0$ radiation-dominated era,

$$\delta_1 \propto \cosh(2\Lambda t^{1/2}) - \sinh(2\Lambda t^{1/2})/2\Lambda t^{1/2}, \tag{5.16a}$$

$$\delta_2 \propto \sinh(2\Lambda t^{1/2}) - \cosh(2\Lambda t^{1/2})/2\Lambda t^{1/2}. \tag{5.16b}$$

The limit $2\Lambda t^{1/2} \gg 1$ gives

$$\delta_1 = \delta_2 \propto \exp(2\Lambda t^{1/2}). \tag{5.17}$$

From Eqs. (3.4) and (5.8) the conformal time $\tilde{t}$ relates to the proper time t by

$$\tilde{t} = t^{1/2}, \tag{5.18}$$

then

$$\delta\rho/(\rho+p) = \beta \propto \exp(2\Lambda\tilde{t}). \tag{5.19}$$

This is to say that the spherically symmetric density perturbation in $K=0$ radiation-dominated era may grow exponentially with the conformal time $\tilde{t}$.

VI. DISCUSSION

The five Eqs. (2.2) contain six unknowns and constitute a complete set of field equations describing the general spherically symmetric fields of perfect fluids. In this paper the five equations are partly solved and successfully reduced to two Eqs. (2.7) plus three definitions (2.8). We should say that generally two is the minimum number that Eqs. (2.2) can be reduced to, because even in the special case of homogeneous and isotropic cosmological model the number of independent field equations is not one but two as in Eqs. (3.3).

Starting from these two equations and three definitions we have derived the spherical density perturbation equation (4.12) which can be separated by the method of separation of variables into two equations (5.1) and (5.2). For some special cases (5.1) and (5.2) are solved explicitly. An interesting result is that for the matter-dominated era the equation (5.2) is found to be the Newtonian perturbation equation given by Bonnor. Another interesting result is that for the $K=0$ universe and the radiation-dominated era the density perturbation has two growing modes: one is something like a power of the proper time t as in (5.12), and another is of an exponential law of the conformal time $\tilde{t}$ as in (5.19), each corresponding to a different limit of approximation.

It is well known that density perturbations are gauge-dependent quantities.[11] Therefore, we should know whether the perturbation modes obtained in Sec. V are of physical reality. From Bardeen,[11] there are two independent gauge-invariant density perturbations:

$$\mathscr{E}_m = \delta_B + 3(1+\omega)\frac{1}{k_B}\frac{\dot{a}}{a}(v-B)\frac{dt}{d\tilde{t}}, \tag{6.1}$$

$$\mathscr{E}_g = \delta_B - 3(1+\omega)\frac{1}{k_B}\frac{\dot{a}}{a}\left(B-\frac{1}{k_B}\dot{H}_T\frac{dt}{d\tilde{t}}\right)\frac{dt}{d\tilde{t}}, \tag{6.2}$$

where $\omega = p/\rho$ and $\tilde{t}$ is the conformal time as mentioned in Eq. (5.18). Other quantities in (6.1) and (6.2) can be determined by comparing the corresponding expressions in the two papers. First, we point out that the gauge introduced in this paper is a comoving time-orthogonal one that differs from the synchronous gauge used by Lifshitz and others,[12,13,10] and so we have

$$v=0, \quad B=0. \tag{6.3}$$

Then we find that the radial equation (5.1) can be expressed in the form of the scalar Helmholtz equation,

$$^3g^{\alpha\beta}R_{;\alpha;\beta} = k^2 + 3K, \tag{6.4}$$

where $^3g^{\alpha\beta}$ is the metric tensor for the three-space in Eq. (3.4). Thus we get the following relations:

$$k_B = k(1+3K/k^2)^{1/2}, \tag{6.5}$$

$$\delta_B = (1+p/\rho)\delta, \tag{6.6}$$

$$\dot{H}_T = (1+3K/k^2)[\dot{\delta} + \dot{p}/(\rho+p)\delta], \tag{6.7}$$

where the derivation of (6.7) is very lengthy and the process is neglected. Now Eqs. (6.1) and (6.2) reduce to

$$\mathscr{E}_m = (1+p/\rho)\delta, \tag{6.8}$$

$$\mathscr{E}_g = \left(1+\frac{p}{\rho}\right)\delta + \frac{3}{k^2}\left(1+\frac{p}{\rho}\right)\frac{\dot{a}}{a}\left(\dot{\delta}+\frac{\dot{p}}{\rho+p}\delta\right)\left(\frac{dt}{d\tilde{t}}\right)^2. \tag{6.9}$$

From Eq. (6.8) we find that the gauge-invariant modes in $\mathscr{E}_m$ are just the same as the modes in δ as derived in Sec. V. The modes in $(\mathscr{E}_g - \mathscr{E}_m)$ for the radiation-dominated era are

$$(\mathscr{E}_g - \mathscr{E}_m)_{\delta_+} \propto \text{const}, \tag{6.10}$$

$$(\mathscr{E}_g - \mathscr{E}_m)_{\delta_-} \propto t^{-3/2}, \tag{6.11}$$

in the low-frequency limit (5.12), i.e., in the early universe. In the high-frequency limit (5.13) we find

$$(\mathscr{E}_g - \mathscr{E}_m)_{\delta_\pm} \to 0. \tag{6.12}$$

These results are in agreement with the analysis of Bardeen.[11]

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[1] H. Y. Liu, J. Math. Phys. **31**, 2462 (1990).
[2] C. W. Misner and D. H. Sharp, Phys. Rev. B **136**, 571 (1964).
[3] M. A. Podurets, Sov. Astron. **8**, 19 (1964).
[4] P. S. Wesson, J. Math. Phys. **25**, 3297 (1984).
[5] R. Kantowski and R. K. Sachs, J. Math. Phys. **7**, 443 (1966).
[6] R. N. Henriksen, A. G. Emslie, and P. S. Wesson, Phys. Rev. D **27**, 1219 (1983).
[7] P. S. Wesson, Astrophys. J. **336**, 58 (1989).
[8] H. Y. Liu, J. Math. Phys. **31**, 2459 (1990).
[9] W. B. Bonnor, Mon. Not. R. Astron. Soc. **117**, 111 (1957).
[10] See, for example, S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[11] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[12] E. M. Lifschitz and I. M. Khalatnikov, Adv. Phys. **12**, 185 (1963).
[13] P. J. E. Peebles, *The Large-Scale Structure of the Universe* (Princeton U.P., Princeton, NJ, 1980).