On the Evolution of Particle Size Distribution in Pipe Flow of Dispersions Undergoing Breakage

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The problem of a flowing dispersion in a pipe with particles undergoing breakage and radial diffusion is considered in this note. This problem has been addressed in an earlier work [Kostoglou and Karabelas, *AIChE J.*, **2004**, *50*, 1746] where it is solved analytically for the case of radially uniform parameters, while the case of turbulent flow (with radially dependent parameters) was treated by other researchers [Nere and Ramkrishna, *Ind. Eng. Chem. Res*. **2005**, *44*, 1187]. Here, simplified versions of the extended mathematical problem in the limits of small and large diffusion influence are presented. The conditions under which self-similar particle size distributions can be developed are determined. Finally, a closed-form solution for the total particle number concentration for linear breakage rate with radial dependence and radial uniform velocity and diffusivity is derived.

Introduction

The combination of breakage with convection and diffusion is encountered in several industrial processes, e.g., the transport of gas−liquid and liquid−liquid dispersions, or particulate matter transport with simultaneous particle attrition. A very general form of the equation that describes steady-state convective diffusion and fragmentation in cylindrical geometry is presented as follows:

$$u(r)\frac{\partial f(x,r,z)}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}D(x,r)r\frac{\partial f(x,r,z)}{\partial r} + \int_x^\infty p(x,y;r)b(y,r)f(y,r,z)\,\mathrm{d}y - b(x;r)f(x,r,z) \quad (1)$$

where f is the particle number density function, u the velocity, D the diffusion coefficient of the particles, b the breakage rate, and p the fragment size distribution resulting from a particle of volume y. The parameters x, r, and z are the independent coordinates, i.e., particle volume, radial and axial coordinate, respectively. The dependencies of the above quantities on the independent variables are evident in their arguments.

The case of constant velocity ($u(r) = u$) with the diffusivity being dependent only on particle size ($D(x,r) = D(x)$) and space-independent breakage functions ($b(y;r) = b(y)$ and $p(x,y;r) = p(x,y)$ has been examined in detail for the cylindrical geometry (among other geometries), using the separation of variables technique, by Kostoglou and Karabelas.[1] These authors argued that the aforementioned form of the problem parameters (velocity, diffusivity, breakage rate, and kernel) is the most general form that allows application of the separation of variables technique, or eigenfunction expansion, or finite integral transform,[2] or linear operator expansion;[3] all these procedures should lead to the same result.

Recently, Nere and Ramkrishna[4] considered a case more general than that of Kostoglou and Karabelas,[1] concerning droplet breakage and diffusion in a turbulent pipe flow. The radial fluid velocity profile and the radial distribution of energy dissipation rate ϵ are obtained from a turbulent $k-\epsilon$ model. The profile of turbulent particle diffusivity is determined from a $k-\epsilon$ model as well, and it is assumed to be independent of the particle size. Thus, the breakage rate is r-dependent through its dependency on ϵ. In addition, the breakage functions considered are based on a particular physical system that has been studied previously.[5,6] The generalization, with respect to the work of Kostoglou and Karabelas,[1] is that the velocity is r-dependent, the diffusivity is r-dependent (although the size dependency has been omitted), and the breakage rate is r-dependent (having a product-type dependence on its parameters, i.e., $b(x;r) = b_1(r)b_2(x)$). The resulting equation takes the form

$$u(r)\frac{\partial f(x,r,z)}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}D(r)r\frac{\partial f(x,r,z)}{\partial r} + \int_x^\infty p(x,y)b_1(r)b_2(y)f(y,r,z)\,\mathrm{d}y - b_1(r)b_2(x)f(x,r,z) \quad (2)$$

with a boundary condition of zero flux at the wall (no deposition).

Nere and Ramkrishna[4] have reported a solution to the aforementioned equation, using the linear operator expansion technique (equivalent to the separation of variables technique used by Kostoglou and Karabelas[1]). However, it can be shown that the general solution of eq 2 by linear operator expansion is not possible as suggested by them, and we focus here on assessing the limiting cases when it can be applied.

Problem Analysis

An effort is made here to clarify the behavior of the physical system described by eq 2 for the case of a spatially uniform inlet size distribution, i.e., $f(x,r,0) = f_o(x)$. The particle size distribution (PSD) in each radial location is different, because of the different residence time (due to the velocity profile) and different breakage rate (due to the energy dissipation rate profile). The effect of diffusion is to eliminate the radial nonuniformities in the particle concentration that are produced by breakage. The key parameter for analyzing the system behavior is the ratio of diffusion to breakage rate ($\Lambda = D(r)/b(r,x)R^2$) for particle sizes of interest in this process.

Fast Diffusion. If this ratio is (everywhere and for every size of existing particles) very large ($\Lambda \gg 1$), then the fragments

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will instantly diffuse after their formation, to eliminate the driving force for diffusion. Thus, the PSD will be uniform across the pipe cross section (i.e., $f(x,r,z) = f(x,z)$) and will be given as

$$f(x,z) = f_{1D}\left(x, \frac{\int_0^R rb_1(r)\,dr}{\int_0^R ru(r)\,dr} z\right) \tag{3}$$

where $f_{1D}(x,t)$ is the solution of the following well-known batch breakage equation:

$$\frac{\partial f_{1D}(x,t)}{\partial t} = \int_x^\infty p(x,y)b_2(y)f_{1D}(y,t)\,dy - b_2(x)f_{1D}(x,t) \tag{4}$$

with $f_{1D}(x,0) = f_o(x)$.This analysis leads to the conclusion that, for $\Lambda \gg 1$, the usual approach of simulating the turbulent breakage problem using a one-dimensional (1-D) breakage equation (plug flow and cross-sectional average of the ϵ-profile) is quite correct.

Slow Diffusion. In case of $\Lambda \ll 1$, the diffusivity term in eq 2 can be ignored, so that there is no interaction between the droplets in the r-direction. The PSD distribution in each r position evolves independently from those in other r positions, determined by its own velocity and breakage rate. Under these conditions (an absence of diffusion), a single variable transformation[4] can be correctly used, leading to the final solution:

$$f(x,r,z) = f_{1D}\left(x, \frac{b_1(r)}{u(r)} z\right) \tag{5}$$

It is clear that Λ represents, in essence, the degree of lateral mixing. To determine a cross-sectional average PSD in the case of $\Lambda = 0$, the PSD for batch breakage under the local (in r) condition (residence time and breakage rate) must be obtained and then averaged over the cross section, whereas, for $\Lambda \gg 1$, the batch breakage equation can be solved directly for the average PSD, using the cross-sectional averages of breakage rate and residence time. For any other Λ values, the average PSD is intermediate between the two extremes previously described.

It must be stressed that the aforementioned guidelines, in regard to the significance of Λ for the solution of eq 2, are only of general value. In practice, the situation can be more complex, especially in the case of the small Λ limit. For example, even if the condition $\Lambda \ll 1$ holds over some length of the flow, the reduction of the particle size leads to a reduction of $b(x)$, so that Λ increases along the flow path, thus rendering the approximate solution (eq 5) invalid. Another complexity is introduced by the fact that the magnitude of the diffusion term is dependent not only on the diffusivity but also on the radial derivative of $f(x,r,z)$. However, it can be shown, based on eq 5, that, for $\Lambda \ll 1$, this radial derivative is proportional to z. This means that no matter how small Λ is, at some point along the flow, the driving force for radial diffusion will take a finite value and the diffusion will be important for the evolution of the particle concentration profile.

On the Self-Similarity Distribution. Now let us consider a function representing the local cumulative volume fraction:

$$F(x,r,z) = \frac{\int_0^x yf(y,r,z)\,dy}{\int_0^\infty yf(y,r,z)\,dy} \tag{6}$$

It is well-known[5,6] that, for the batch breakage problem and for a function $G(x,y) = 1/y \int_0^x zp(z,y)\,dz$, which can be written as $G(x,y) = g(b_2(x)/b_2(y))$, the function $F_{1D}(x,t)$ (corresponding to batch eq 4) evolves asymptotically to a so-called self-similar form (denoted by the subscript ss), which has no explicit dependence on time; i.e., $F_{1D}(x,t) = F_{ss}(\eta)$, where η is the so-called "similarity variable" ($\eta = b_2(x)t$).

In regard to the problem of breakage in pipe flow for $\Lambda = 0$, it is obvious that the use of a local (in r) similarity variable $\eta(r) = b_1(x)\, b_2(r)z/u(r)$ leads to the following asymptotic form: $F(x,r,z) = F_{ss}(\eta(r))$; i.e., the function F versus $\eta(r)$ for various values of x, r, and z always falls on the same curve. However, if Λ takes finite values, the above behavior disappears and the function $F(\eta(r))$ becomes dependent explicitly on r. Finally, for $\Lambda \gg 1$, a different self-similar behavior is observed. The PSD now is not dependent on r and shows self-similarity, with respect to a similarity variable independent of r; i.e., $F(x,r,z) = F_{ss}(\eta)$, with

$$\eta = b_2(x)\frac{\int_0^R rb_1(r)\,dr}{\int_0^R ru(r)\,dr} z \tag{7}$$

Spatial Distribution of Total Droplet Number Concentration for Linear Breakage Rate. Let us assume a breakage rate that has a linear dependence on x (i.e., $b_2(x) = B_2x$) and a binary breakage kernel (no restriction to a particular shape is needed). The total droplet number and mass concentrations are denoted as N and M, respectively; i.e.,

$$N(r,z) = \int_0^\infty f(x,r,z)\,dx$$

$$M(r,z) = \int_0^\infty xf(x,r,z)\,dx$$

Multiplying eq 2 with x and taking the integral with respect to x from zero (0) to infinity (∞) leads to an equation for total droplet mass concentration that must be solved combined with the symmetry and no wall penetration boundary conditions. The problem admits the trivial solution

$$M(r,z) = M_o = \int_0^\infty xf_o(x)\,dx$$

(the total particle mass concentration is the same everywhere). Then, taking the same integral for eq 2 directly (no multiplication by x) leads to the following evolution equation for $N(r,z)$:

$$u(r)\frac{\partial N}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}rD(r)\frac{\partial N}{\partial r} + b_1(r)M_o \tag{8}$$

The zero diffusion ($\Lambda = 0$) solution for the total number concentration N is as follows:

$$N(r,z) = N_o + \frac{b_1(r)}{u(r)}M_o z \tag{9}$$

where

$$N_o = \int_0^\infty f_o(x)\,dx$$

The only case for which an analytical solution is possible for nonzero Λ corresponds to uniform velocity and diffusivity profiles, i.e., $D(r) = D$, $u(r) = u$.

To proceed further, the function $b_1(r)$ is decomposed to an average value and a radial-dependent function, $b_1(r) = B_1\bar{b}_1(r)$, where

$$2\int_0^R r\bar{b}_1(r)\,\mathrm{d}r = 1$$

and the following nondimensionalization is introduced:

$$\bar{r} = \frac{r}{R}, \bar{N} = \frac{N}{N_o}, \Lambda = \frac{DN_o}{R^2B_1B_2M_o}, \bar{z} = \frac{zB_1B_2M_o}{uN_o}$$

Equation 8 takes the form

$$\frac{\partial\bar{N}}{\partial\bar{z}} = \Lambda\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\bar{r}\frac{\partial\bar{N}}{\partial\bar{r}} + \bar{b}_1(\bar{r}) \quad (10)$$

with the initial condition $\bar{N}(\bar{r},0) = 1$ and boundary conditions of zero radial flux at $\bar{r} = 0, 1$.

The solution of this problem can be derived by finding the Green function of the corresponding homogeneous problem and integrating it over the source function. The final result is

$$\bar{N}(\bar{r},\bar{z}) = 1 + \bar{z} + \sum_{i=2}^{\infty}\frac{2J_o(\lambda_i\bar{r})}{J_o^{\,2}(\lambda)}\frac{1-\mathrm{e}^{-\Lambda\lambda_i^2\bar{z}}}{\Lambda\lambda_i^{\,2}}\int_0^1 \bar{r}'\bar{b}_1(\bar{r}')J_o(\lambda_i\bar{r}')\,\mathrm{d}\bar{r}' \quad (11)$$

where λ_i is the ith root of equation $J_1(\lambda) = 0$ (counting $\lambda_1 = 0$ as the first root), and J_o, J_1 are the first type, zeroth- and first-order Bessel functions, respectively.

From the above exact result, one can confirm the already-stated conjectures regarding the dependence of solutions of eq 2 on the parameters Λ and z. It can be shown that, in the limit $\Lambda\bar{z} \ll 1$, eq 11 is simplified to $\bar{N}(\bar{r},\bar{z}) = 1 + \bar{b}_1(\bar{r})\bar{z}$, which is the result of no lateral diffusion approximation. In the case of $\Lambda\bar{z} \gg 1$, eq 11 takes the following general form:

$$\bar{N}(\bar{r},\bar{z}) = 1 + \bar{z} + \frac{1}{\Lambda}\Phi(\bar{r}) \quad (12)$$

As expected, in the limit of $\Lambda \gg 1$, it corresponds to the (radially uniform) complete mixing solution. However, the important observation is that, for finite values of Λ (even for $\Lambda \ll 1$), the diffusion determines the radial profile of particle concentration through the function $\Phi(r)$, which is dependent only on $\bar{b}_1(r)$. This dependence can be found by comparing eqs 11 and 12; however, to avoid the use of Bessel functions, it is much better to insert eq 12 in eq 10 and solve the resulting equation for Φ to obtain (after some algebra) the following result:

$$\Phi(\bar{r}) = \int_0^{\bar{r}}\frac{1}{x}\int_0^x y[1-\bar{b}_1(y)]\,\mathrm{d}y\,\mathrm{d}x - \int_0^1\bar{r}\int_0^{\bar{r}}\frac{1}{x}\int_0^x y[1-\bar{b}_1(y)]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\bar{r} \quad (13)$$

Explicit results for the function $\Phi(r)$ will be given for two functional forms of $b_1(r)$. In the first case (case I), $\bar{b}_1(\bar{r}) = (n + 2)/(2\bar{r}^n)$, i.e., the breakage rate increases from the pipe centerline outward, being zero at the center and maximum at the pipe periphery. The corresponding function Φ is

$$\Phi(\bar{r}) = \frac{1}{(n+2)(n+4)} - \frac{1}{8} + \frac{1}{4}\bar{r}^2 - \frac{1}{2}\frac{\bar{r}^{n+2}}{n+2}$$

In the second case (case II), $\bar{b}_1(\bar{r}) = [(n + 2)/n](1 - \bar{r}^n)$, i.e., the breakage rate increases outward, being maximum at the axis of the pipe and zero at the periphery. The corresponding function Φ is

$$\Phi(\bar{r}) = \frac{1}{4n} - \frac{2}{n(n+2)(n+4)} - \frac{\bar{r}^2}{2n} + \frac{\bar{r}^{n+2}}{n(n+2)}$$

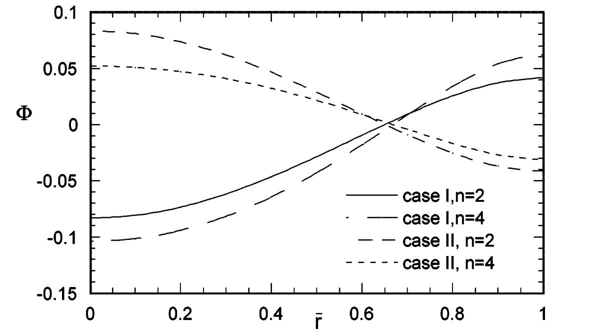

Figure 1. Plot of the function $\Phi(\bar{r})$ for two forms of $b_1(r)$ (cases I and II) and two values of n ($n = 2$ and 4).

The function Φ, corresponding to the radial profile of particle concentration, is shown in Figure 1 for both forms of the function $b_1(r)$ and $n = 2$ and 4. In summary, the solutions of eq 2 exhibit a rich variety of behavior, including global (with respect to r) self-similarity, local self-similarity, constant radial profile shape, etc. and deserve more study using theoretical and numerical tools in order to reveal the complete picture.

Conclusions

The mathematical problem of the pipe flow for a dispersion undergoing breakage is considered in this Note, under conditions that allow simplified treatment. The ratio between diffusion and breakage rates essentially determines the degree of lateral mixing in the pipe. It is shown that, in both cases of extremely slow and extremely fast diffusion (*no* and *complete* lateral mixing), the problem is reduced to the well-known batch breakage equation (for breakage rate with separable spatial and particle size dependencies). Furthermore, in both cases, a self-similar solution is possible with the self-similarity variable being radially dependent in the no-mixing case. In addition, by obtaining an exact solution (a total particle number concentration) for the case of linear breakage rate with uniform velocity and diffusivity, it is shown that the radial profile of constant shape along the flow is also possible for the total particle number concentration.

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