A New Methodology for the General Multiparametric Mixed-Integer Linear Programming (MILP) Problems

ARTICLE *in* INDUSTRIAL & ENGINEERING CHEMISTRY RESEARCH · JULY 2007

Impact Factor: 2.59 · DOI: 10.1021/ie070148s

Marianthi Ierapetritou
Rutgers, The State University of New Jersey

A New Methodology for the General Multiparametric Mixed-Integer Linear Programming (MILP) Problems

Zukui Li and Marianthi G. Ierapetritou*

Dept. of Chemical and Biochemical Engineering, Rutgers University, Piscataway, New Jersey 08854

In this paper, a general algorithm is developed to address the multiparametric mixed-integer linear programming (mpMILP) problem with uncertain parameters in the left-hand side (LHS), right-hand side (RHS), and objective function coefficients simultaneously. The algorithm is based on a general multiparametric linear programming (mpLP) algorithm, which is derived using the optimality conditions of standard linear programming (LP) problem. The intent of the proposed framework is to propose a general framework to address different uncertainties in the process engineering problems. In addition, the paper also discusses the solution of the special problem where LHS uncertainty is included in the optimization model as a coefficient of continuous variable, and it notes the high computational complexity needed to retrieve the rigorous solution of large-scale problems, because of the nonlinearities of the objective function and nonconvex critical regions. Several numerical examples are presented to illustrate the effectiveness and applicability of the proposed method.

1. Introduction

Uncertainty appears in all the different levels of industry from the detailed process description to multisite manufacturing. The existence of parameter variability in most real process design and operation problems necessitates the consideration of uncertainties in the mathematical programming models used to simulate and optimize the design performance and process operations. The process system engineering community has proposed several ideas in the literature to address these process design and operations problems under uncertainty, which involve stochastic programming,[1] fuzzy programming,[2] robust optimization,[3] and reactive scheduling;[4] a thorough review can be found in the work by Sahinidis.[5]

An alternative approach to process design/operations optimization under uncertainty is either to utilize the information extracted from sensitivity analysis[6] or to use parametric programming. Although both approaches provide input regarding the effect of uncertainty variability in optimal solution and objective function value, parametric programming has a unique advantage, because it results in the full characterization of the entire range of solution space and provides a complete map of the various alternatives in the face of uncertainty.

Based on the number of the uncertain parameters, two categories can be distinguished: the single parametric programming methods, which address the presence of a one uncertain parameter; and the multiparametric programming, which considers more than two uncertain parameters. Original studies on parametric programming were initiated to address the single parametric case.

The first method for single parametric linear programming was published by Gass and Saaty,[7] where parametric objective function is analyzed using simplex algorithm. A vast number of publications followed this initial work, a detailed review of which can be found in the book by Gal.[8] Parametric integer programming problems that consider a single parameter variation are also widely studied. Those methods include implicit enumeration methods, branch and bound (B&B) methods, and cutting plane methods. A detailed review can be found in the work by Jenkins.[9]

* To whom correspondence should be addressed. Tel.: 732-445-2971. Fax: 732-445-2421. E-mail address: marianth@soemail.rutgers.edu.

Early studies on multiparametric linear programming (mpLP) include the work of Gal and Nedoma,[10] which addressed the right-hand side (RHS) mpLP problem based on the simplex algorithm for deterministic linear programming (LP) problems. They proposed to cover the parameter space by critical regions, where a critical region is defined to be the set of parameters such that a given basis is optimal for the parametric program. The proposed procedure is a dual pivoting method, which searches a connected component of the graph whose vertices are the dual feasible bases of the problem and whose edges are the dual pivots.

Recently, multiparametric programming has received more attention from the field of system theory and optimal control. A geometric algorithm for mpLP is proposed by Borrelli et al.,[11] which is based on a recursive subdivision of the parameter space around a critical region and requires that the critical region be convex and be described by a set of linear constraints. Filippi and Romanin-Jacur[12] proposed a lexicographic network pivoting method under which degeneracy is virtually eliminated. However, the characteristics of the method are not extendable to a general multiparametric linear program. Recently, Filippi[13] developed an algorithm to obtain the approximate solution of an mpLP problem. For a given full-dimensional simplex in the parameter space and an optimizer for each simplex vertex, the algorithm formulates the linear interpolation of the given solutions as an explicit function of the parameters and gives a primal feasible approximation of an optimizer in the simplex. However, this method still only considers the RHS uncertainty.

Motivated by the need for technology to implement constrained optimal feedback control with a minimal amount of real-time computations, multiparametric quadratic programming (mpQP) problem is studied by Seron et al.,[14] Bemporad et al.,[15] Baotic,[16] and Tøndel et al.[17] All these algorithms are based on an iterative procedure of generating new polyhedral regions in the parameter space at each step, and they differ in the way they explore the parameter space. As an extension, the more-general multiparametric nonlinear programming (mpNLP) problem is also studied recently. Johansen[18] used local mpQP to approximate the original nonlinear programming (NLP) problem. Bemporad and Filippi[19] proposed a recursive algorithm to approximate the optimal value function in a given suboptimality threshold. Acevedo and Salgueiro[20] studied the RHS multipara-

10.1021/ie070148s CCC: $37.00
Published on Web 06/23/2007

metric convex nonlinear problems based on an outer-approximation algorithm. Hale et al.[21] proposed a framework for numerically approximating mpNLP solutions based on numerical methods for single-parameter nonlinear programming.

Existing multiparametric mixed-integer linear programming (mpMILP) methods are generally based on B&B methods and the solution of mpLP subproblems at the B&B nodes. Such an approach is developed by Acevedo and Pistikopoulos[22] for the analysis of linear process engineering problems under uncertainty. Dua and Pistikopoulos[23] proposed an alternative algorithm by extending the idea of Pertsinidis et al.[24] The problem is decomposed into an mpLP and an MILP subproblem. When the values of the binary variable are fixed, an mpLP is solved, and its solution provides a parametric upper bound to the objective function value. When the parameters in the original problem are treated as free variables, an MILP is solved, which provides a new integer vector. The algorithm is composed of an initialization step, and a recursion between the solution of an mpLP subproblem and an MILP subproblem. Jia and Ierapetriou[25] recently proposed a systematic framework to solve the mpMILP problem, which is used in this work and will be described in detail in the next section; they also developed a new mpLP method to address RHS uncertain parameters, which solves many nonlinear programming (NLP) problems iteratively. All these existing mpMILP approaches only involve the uncertain parameters in RHS of the MILP problems. Crema[26] studied the multiparametric 0−1 integer programming problem, which considers the perturbation of the constraint matrix, the objective, and the RHS vector, but it is designed for the 0−1 integer programming and cannot address the general MILP problem.

In the direction of addressing more-general mixed-integer nonlinear problems, Dua et al.[27] studied the multiparametric mixed-integer quadratic programming (mpMIQP), using a sequential procedure iterating between a mpQP problem and a mixed-integer nonlinear programming (MINLP) problem until convergence is achieved. Bemporad[28] studied multiparametric nonlinear integer programming problems where the optimization variables belong to a finite set and where the cost function and the constraints are linear functions of the uncertain parameters and analyze the main theoretical properties of the optimizer and of the optimum as a function of the parameters.

Many process design and operation problems (e.g., plant synthesis, process scheduling) are commonly formulated as mixed-integer linear programming (MILP) problems. Thus, an efficient mpMILP algorithm can address different uncertainties emerging in process engineering problems. Many examples can be found in the literature, including the work of Acevedo and Pistikopoulos,[22] who formulated the plant synthesis problem as an mpMILP problem with demand uncertainty. Pistikopoulos and Dua[31] studied the uncertain process planning problem, which is also formulated as a mpMILP problem. Ryu and Pistikopoulos[32] studied the zero-wait batch process scheduling problem, considering processing time uncertainty, and formulated the problem as mpMILP with RHS uncertainty after linearization. In Jia and Ierapetritou,[6] the mpMILP problem with RHS uncertainty is generated from a scheduling problem and a framework is proposed that is extended in this paper. As the previous literature review illustrate, most existing mpLP/mpMILP approaches only involve uncertain parameters in RHS and objective coefficients of the LP/MILP problems. However, realistic problems require incorporating the left-hand side (LHS) uncertainty. For example, in the problem of process scheduling under uncertainty, processing time, demand, and price correspond to the LHS, RHS, and objective parameters, respectively. In this paper, we focus on developing a general algorithm to address the mpMILP problem with different types of uncertainty. Future publications will present the application of the algorithm in scheduling problems.

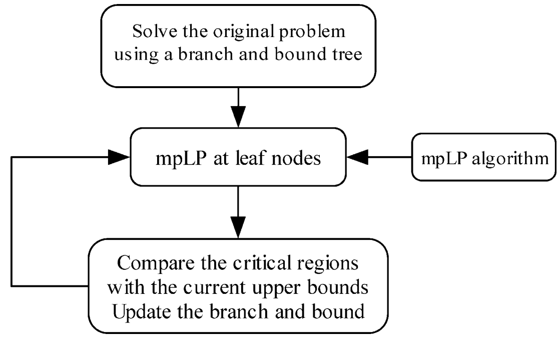

Figure 1. Flow chart for multiparametric mixed-integer linear programming (mpMILP). (From Jia and Ierapetritou.[25])

In particular, we propose a methodology for the mpMILP problems with uncertain parameters in the LHS, RHS, and objective function coefficients simultaneously. The algorithm is based on a B&B framework and a general mpLP algorithm, which is derived using the optimality conditions of a standard linear programming (LP) problem. The objective of the proposed framework is to propose a general framework to address different uncertainties in the process engineering problems. In addition, the paper also discusses the solution of the special problem where LHS uncertainty is included in the optimization model as a coefficient of continuous variable, and it notes the high computational complexity needed to retrieve the rigorous solution of large-scale problems, because of the nonlinearities of the objective function and nonconvex critical regions.

The article is organized as following: the general mpMILP framework is presented in next section, followed by the mpLP algorithm in section 3. A few numerical examples then are presented to illustrate the proposed method. The paper is summarized in the last section, including a discussion of future challenges.

2. mpMILP Framework

The algorithm proposed in this paper is based on the mpMILP framework proposed by Jia and Ierapetritou,[25] as shown in Figure 1, which addresses the mpMILP problem with RHS uncertain parameters. Here, we extend the methodology to cover the general case of mpMILP problems and present the basic principles and steps of the framework for completeness. A small example is also given in this section to illustrate the proposed framework.

Let us consider the general form of mpMILP problem:

$$\min z = c(\theta)x \qquad \text{(P1)}$$
$$\text{s.t. } A(\theta)x = b(\theta)$$

where x is the vector of decision variables that involve continuous variables and integer variables; θ is the parameter vector that generally varies within a fixed region, which is described using a set of linear constraints or, most commonly, a set of bounds. The objective of mpMILP is to determine the effect of the parameter variation on the optimal solution and the optimal objective value.

To address this problem, a general mpMILP algorithm is developed that follows these steps:

Step 1. The original problem is first solved at an initial set of uncertain parameters (θ_0), following a B&B solution procedure. The node that gets the optimal solution is flagged as the current optimal node.

Step 2. The mpLP is solved at each of the leaf nodes, which include infeasible nodes, because they are just infeasible for current parameter value. The details of the solution of the mpLP problem are given in the next section, because this is the main step of the proposed algorithm. By performing mpLP at leaf node p, several critical regions ($\mathrm{CR}_p^{(1)}$, $\mathrm{CR}_p^{(2)}$, ..., $\mathrm{CR}_p^{(K)}$) are identified, and in each critical region $\mathrm{CR}_p^{(k)}$, $k = 1, 2, ..., K$ (where K is the number of critical regions for this node), the corresponding optimal function ($z_p^{(k)}$) value is determined. The critical region is defined as the range of parameter values where the same basis remains optimal. If the leaf node is infeasible for all the parameter values, it is fathomed and will not be branched later. The optimal function value of the current optimal node is denoted as $z_{\mathrm{opt}}^{(k)}$ and its critical regions is denoted as $\mathrm{CR}_{\mathrm{opt}}^{(k)}$.

Step 3. The B&B tree is then updated. The main idea of this step is to compare the optimal functions between the current optimal nodes and other leaf nodes in the intersection of their critical regions to identify a set of new critical regions, their corresponding objective function values, and optimal integer values. To perform this comparison between the leaf node i and the current optimal node (opt), the following redundancy test problem (P2) is solved in the intersection between $\mathrm{CR}_i^{(k1)}$ and $\mathrm{CR}_{\mathrm{opt}}^{(k2)}$.

$$\max err = \pm(z_i^{(k1)} - z_{\mathrm{opt}}^{(k2)}) \qquad \text{(P2)}$$

$$\text{s.t. } \theta \in \mathrm{CR}_i^{(k1)} \cap \mathrm{CR}_{\mathrm{opt}}^{(k2)}$$

The outcome of this problem results in one of the following three cases.

Case I: $err^* = \max\{z_i^{(k1)} - z_{opt}^{(k2)}\} \leq 0$ *or* $err^* = \max\{z_{opt}^{(k2)} - z_i^{(k1)}\} \leq 0$*, where* err^* *is the optimal value of problem P2.* For this case, one optimal function value is not always greater than the function at the leaf node i at the intersection region. If $z_i^{(k1)}$ is also an integer node (all the integer variables get integer values), $z_i^{(k1)}$ is updated as the current optimal node in that region. If node i is not an integer node (only a subset of the integer variables get integer values), the node should be branched. If $z_{\mathrm{opt}}^{(k2)}$ is not greater than $z_i^{(k1)}$, $z_{\mathrm{opt}}^{(k2)}$ remains the current optimal at this intersection region.

Case II: $err^* = \max\{z_i^{(k1)} - z_{opt}^{(k2)}\} > 0$ *and* $err^* = \max\{z_{opt}^{(k2)} - z_i^{(k1)}\} > 0$. For this case, we divide the intersection region into two parts, using the constraint $z_i^{(k1)} = z_{\mathrm{opt}}^{(k2)}$. In the region$\{\mathrm{CR}_i^{(k1)} \cap \mathrm{CR}_{\mathrm{opt}}^{(k2)} \cap (z_i^{(k1)} \leq z_{\mathrm{opt}}^{(k2)})\}$, $z_i^{(k1)}$ is updated as current optimal; whereas, in the other part $\{\mathrm{CR}_i^{(k1)} \cap \mathrm{CR}_{\mathrm{opt}}^{(k2)} \cap (z_i^{(k1)} \geq z_{\mathrm{opt}}^{(k2)})\}$, $z_{\mathrm{opt}}^{(k2)}$ remains the current optimal.

Case III: Infeasible Problem. In this case, there is no intersection between the two critical regions $\mathrm{CR}_i^{(k1)}$ and $\mathrm{CR}_{\mathrm{opt}}^{(k2)}$, so $z_{\mathrm{opt}}^{(k2)}$ remains the current optimal solution at $\mathrm{CR}_{\mathrm{opt}}^{(k2)}$. $\mathrm{CR}_i^{(k1)}$ is updated as the current optimal in any part of critical regions of node i that have no intersection with the critical regions of current optimal nodes.

Step 4. The mpLP is performed at the new leaf nodes and the comparison procedure continues. In every iteration step of the procedure, the new leaf nodes in the updated B&B tree will be compared to the current optimal nodes, to determine the new optimal functions in the intersection regions. This procedure stops when no further branching is required.

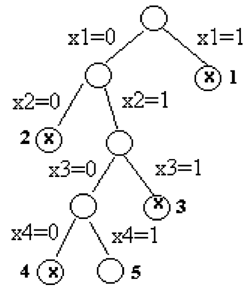

Figure 2. Branch and bound (B&B) tree of example 1. (Throughout the figures, the infeasible nodes are identified by the symbol "x".)

The global optimization solver should be used to solve problem P2 for the case that it is a nonconvex NLP problem. The final parametric solution at the uncertain range of interest can be represented by several critical regions that contain their corresponding optimal functions and integer solutions. The following example illustrates the mpMILP framework.

Example 1. mpMILP illustration.

$$\begin{aligned} &\max 2x_1 + x_3 \\ &\text{s.t. } \theta x_1 + 0.75x_2 \leq 0.75 \\ &\theta x_3 + 0.75x_4 \leq 0.75 \\ &-\theta x_1 - 0.75x_2 \leq -0.25 \\ &-\theta x_3 - 0.75x_4 \leq -0.25 \\ &x_1 - x_2 + x_3 - x_4 \leq 1 \\ &x_1 + x_2 \leq 1 \\ &x_3 + x_4 \leq 1 \\ &x_i \in \{0,1\} \qquad (\text{for } 0 \leq \theta \leq 1) \end{aligned}$$

The first step is to solve the problem using a B&B methodology at the nominal value of $\theta = 0$, as shown in Figure 2, which gives five leaf nodes, including four infeasible ones.

Based on the B&B tree in Figure 2, the next step is to solve the parametric programming problem at the five leaf nodes. The parametric LP at the leaf nodes (using the approach described in the next section) leads to the following results.

At node 1, the result is

$$z = 2.5 \qquad (\text{for } 0.25 \leq \theta \leq 0.75)$$

At nodes 2 and 4, the result is

$$\text{infeasible} \qquad (\text{for all } 0 \leq \theta \leq 1)$$

At node 3, the result is

$$z = 1 \qquad (\text{for } 0.25 \leq \theta \leq 0.75)$$

At node 5, the result is

$$z = 0 \qquad (\text{for } 0 \leq \theta \leq 1)$$

Thus, after this step, the current optimal node is node 1, because it has the maximum objective function value (2.5).

The third step is then to compare the optimal function values between nodes in the intersection of critical regions. We can fathom node 2 and node 4, because they are infeasible. We fathom node 3 because it is always less than node 1 in the intersection region of $0.25 \leq \theta \leq 0.75$. The comparison of the function values between the current optimal node 1 and node 5

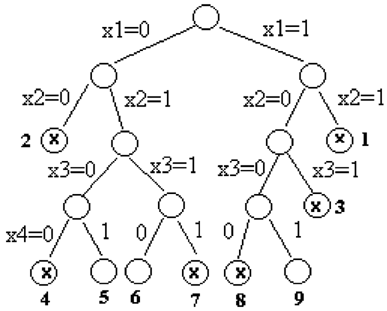

Figure 3. Updated B&B tree of example 2.

gives the following result: In the range of $0.25 \leq \theta \leq 0.75$, node 1 must be branched and then compared with node 5 $\{x = (0, 1, 0, 1), z = 0, \text{ for } 0.25 \leq \theta \leq 0.75\}$.

To update the B&B tree in the range of $0.25 \leq \theta \leq 0.75$, we select $\theta = 0.5$ and solve the B&B as shown in Figure 3. (Note that node 2 has been studied and is infeasible for $0 \leq \theta \leq 1$, so it is not necessary to resolve the entire B&B tree.)

Parametric programming at new leaf nodes is depicted as follows (node 2 and node 4 are already fathomed).

At nodes 1, 3, 7, and 8, the result is

$$\text{infeasible} \qquad (\text{for all } 0 \leq \theta \leq 1)$$

At node 6, the result is

$$z = 1 \qquad (\text{for } 0.25 \leq \theta \leq 0.75)$$

At node 9, the result is

$$z = 2 \qquad (\text{for } 0.25 \leq \theta \leq 0.75)$$

Comparing these results with $\{x = (0, 1, 0, 1), z = 0, \text{ for } 0.25 \leq \theta \leq 0.75\}$, we have node 9 as being current optimal. Until now, no further branch has been needed, so the final result is

$$x = (0, 1, 0, 1), z = 0 \qquad (\text{for } 0 \leq \theta < 0.25 \text{ or } 0.75 < \theta \leq 1)$$
$$x = (1, 0, 0, 1), z = 2 \qquad (\text{for } 0.25 \leq \theta \leq 0.75)$$

In the framework proposed by Jia and Ierapetritou,[25] a mpLP approach is developed for the RHS uncertainty, which solves many NLP problems iteratively. The solution of this optimization problem results in identification of a point in the uncertainty space where the objective value cannot be represented by the current optimal functions. The new optimal function is then determined and included in the next iteration. However, the method is limited to the cases where the optimal objectives are linear functions of the uncertain parameters. As shown in the next section however, when RHS and objective function uncertainties appear simultaneously, or when LHS uncertainty exists, the optimal value functions are not linear, which necessitates the need for a new methodology (which is presented in the next section). The new approach can address the general mpLP problem where uncertainties in the LHS, RHS, and objective function can appear simultaneously.

3. General mpLP Method

The main idea of the proposed mpLP method is based on the optimality conditions of the standard LP problem. Thus, we start by deriving the optimality condition of the standard LP problem:

$$\begin{aligned} \min z &= c^T x \\ \text{s.t. } Ax &= b \\ x &\geq 0 \end{aligned} \tag{P3}$$

The constraints $Ax = b$ can be partitioned in two parts by the basic and nonbasic variables:

$$Ax = (\mathbf{B} \ \ \mathbf{N})\begin{pmatrix} x_B \\ x_N \end{pmatrix} = Bx_B + Nx_N = b \tag{1}$$

where B forms a square matrix, x_B is the vector of basic variables, and x_N is the vector of nonbasic variables. If we fix x_N to some values, we can calculate x_B, provided that B is invertible:

$$x_B = \mathbf{B}^{-1}(b - \mathbf{N}x_N) \tag{2}$$

The nonbasic variables are normally fixed at one of their (noninfinite) bounds (for example, zero when the other bound is infinite). We can partition the objective similarly and derive the following equation:

$$z = c_B^T x_B + c_N^T x_N = c_B^T \mathbf{B}^{-1} b + (c_N^T - c_B^T \mathbf{B}^{-1}\mathbf{N})x_N = \text{constant} + \sum_{j \in N} \sigma_j x_j \tag{3}$$

where $\sigma_j = c_j - c_B^T \mathbf{B}^{-1} A_j$ represent the reduced costs. The optimality condition for the LP problem (P3) can be derived as follows:

$$x_B = \mathbf{B}^{-1} b \geq 0, \sigma_j = c_j - c_B^T \mathbf{B}^{-1} A_j \geq 0 \tag{4}$$

The optimal solution then is

$$x = (x_B \ \ x_N) = (\mathbf{B}^{-1} b \ \ 0)$$

and the optimal objective is

$$z^* = c_B^T \mathbf{B}^{-1} b$$

Defining $\mathbf{A}_B$ as the matrix composed by the columns corresponding to the basic variables, and $\mathbf{A}_N$ as the matrix corresponding to the nonbasic variables, the following result is obtained describing the optimality conditions and the optimal objective function:

$$\text{Optimality conditions:} \quad \mathbf{A}_B^{-1} b \geq 0 \tag{5}$$

$$c_N^T - c_B^T \mathbf{A}_B^{-1} A_N \geq 0 \tag{6}$$

$$\text{Optimal objective:} \quad z^* = c_B^T \mathbf{A}_B^{-1} b \tag{7}$$

For the general mpLP problem

$$\begin{aligned} \min z &= c(\theta)x \\ \text{s.t. } A(\theta)x &= b(\theta) \\ x &\geq 0 \end{aligned} \tag{P4}$$

the derived optimality conditions (eqs 5 and 6) still hold. A critical region of the mpLP problem is the set of parameters such that a given basis is optimal for the parametric program. Therefore, when the uncertain parameter θ is included in eqs 5

Table 1. Projection Process To Identify the Feasible Region

polytope	vertices, $[x_1\ x_2\ \theta_1\ \theta_2]$	projected vertice, $[\theta_1\ \theta_2]$	convex hull
$-x_1 - x_2 + \theta_1 + 3\theta_2 \le 0$	[0 2500 1000 500]	[1000 500]	$-\theta_1 - 1.27\theta_2 \le 0$
$-2x_1 + 3x_2 - 8\theta_1 + \theta_2 \le 0$	[0 2614 1000 157.89]	[1000 157.89]	$-\theta_1 - 11.25\theta_2 \le 0$
	[0 1000 1000 0]	[1000 0]	
$2x_1 + 3x_2 - 5\theta_1 - 18\theta_2 \le 0$	[0 1133 1000 −88.89]	[1000 −88.89]	$-\theta_1 \le 1000$
$x_1 - 2x_2 + 2\theta_1 - 3\theta_2 \le 0$	[0 0 0 0]	[0 0]	$\theta_1 \le 1000$
$-x_1 \le 0$	[3000 0 0 1000]	[0 1000]	$\theta_2 \le 1000$
	[2500 0 −500 1000]	[−500 1000]	
$-x_2 \le 0$	[7000 3000 1000 1000]	[1000 1000]	
$\theta_1 \le 1000$	[4000 5000 1000 1000]	[1000 1000]	
$-\theta_1 \le 1000$	[1000 3000 1000 1000]	[1000 1000]	
	[3000 1000 1000 1000]	[1000 1000]	
$\theta_2 \le 1000$	[4400 0 −1000 800]	[−1000 800]	
$-\theta_2 \le 1000$	[4489 64 −1000 787.2]	[−1000 787.2]	
	[5857 428.6 −1000 1000]	[−1000 1000]	
	[4500 0 −1000 1000]	[−1000 1000]	
	[5500 666.7 −1000 1000]	[−1000 1000]	
	[5000 0 −1000 1000]	[−1000 1000]	

and 6, the optimality conditions define the critical region of the mpLP problem and the optimal objective described by eq 7 provides the optimal objective function for the mpLP. Based on this idea, we develop the general mpLP algorithm, the objective of which is to explore the entire feasible region to identify all the critical regions.

The mpLP algorithm is composed of the following steps:

Step 1. Fix the parameter values and solve the relaxed LP problem. Determine the current basis from the solution with the idea that nonbasic variables must become zero (their lower bound). For the initial step, the uncertain parameters can be fixed at the nominal value.

Step 2. Identify the critical region from the optimality conditions described by eqs 5 and 6, and get the optimal value function using eq 7.

Step 3. Generate a new point outside the identified critical regions and go to step 1. This is achieved by generating evenly distributed points in the feasible region and testing whether they have already been covered by the existing critical regions.

For the case when LHS uncertainty is included, because the feasible region cannot be retrieved, the evenly distributed points are generated in the original boundary.

For the case without LHS uncertainty, the evenly distributed points are generated in the feasible region, which can be obtained using the following method. Because the critical regions are formed by linear constraints characterized by eqs 5 and 6, the entire feasible region of the original problem can be determined using a projection method. The feasible region for problem P4 is given by

$$\{\theta | A(\theta)x = b(\theta), x \ge 0\}$$

To get the feasible region of uncertain parameters, we can project the polytope formed by $\{\theta | A(\theta)x = b(\theta), x \ge 0\}$ to the θ space, using the projection algorithm. Many effective projection algorithms can be used; for example, a vertex enumeration/convex hull-based projection algorithm that was proposed by Avis and Fukuda,[29] which first enumerates the vertices of the polytope and then projects these vertices onto the sub-hyperplane. Finally, the projection is evaluated by calculating the convex hull from these projected vertices. These computations are completed via the double description method proposed by Fukuda and Prodon.[30] In this way, the complete description of the feasible region is determined.

The following is an example of identifying the feasible region for an mpLP problem:

$$\begin{aligned} \max\ & z = x_1 + 2x_2 \\ \text{s.t. } & x_1 + x_2 \ge \theta_1 + 3\theta_2 \\ & -2x_1 + 3x_2 \le 8\theta_1 - \theta_2 \\ & 2x_1 + 3x_2 \le 5\theta_1 + 18\theta_2 \\ & x_1 - 2x_2 \le -2\theta_1 + 3\theta_2 \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

To form a polytope that describes the feasible region based on the original constraints of the problem, we must add some additional bounds on the uncertain parameters, that are large enough so that they will not affect the final result. For this example, the following constraints are added: $-1000 \le \theta_1 \le 1000$, $-1000 \le \theta_2 \le 1000$. In addition, the feasible region is obtained as shown in Table 1.

The following example is used to illustrate, step by step, the proposed mpLP algorithm.

Example 2. mpLP illustration.

$$\begin{aligned} \min\ & z = \theta_1 x_1 + x_2 \\ \text{s.t. } & -x_1 + x_2 + x_3 = \theta_2 \\ & x_1 - \theta_3 x_2 + x_4 = 1 \\ & x_i \ge 0 \\ & -5 \le \theta_i \le 5 \qquad (\text{for } i = 1, 2, 3) \end{aligned}$$

This mpLP example involves three uncertain parameters, in the RHS, LHS, and objective functions. The optimal solution of the problem with $\theta = (0, 0, 0)$ is $x = (1, 0, 1, 0)$, where x_1 and x_3 are basic variables; then,

$$\mathbf{A}_B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{A}_N = \begin{bmatrix} 1 & 0 \\ -\theta_3 & 1 \end{bmatrix}$$

$$c_N^T = [1 \ \ 0]$$

$$c_B^T = [\theta_1 \ \ 0]$$

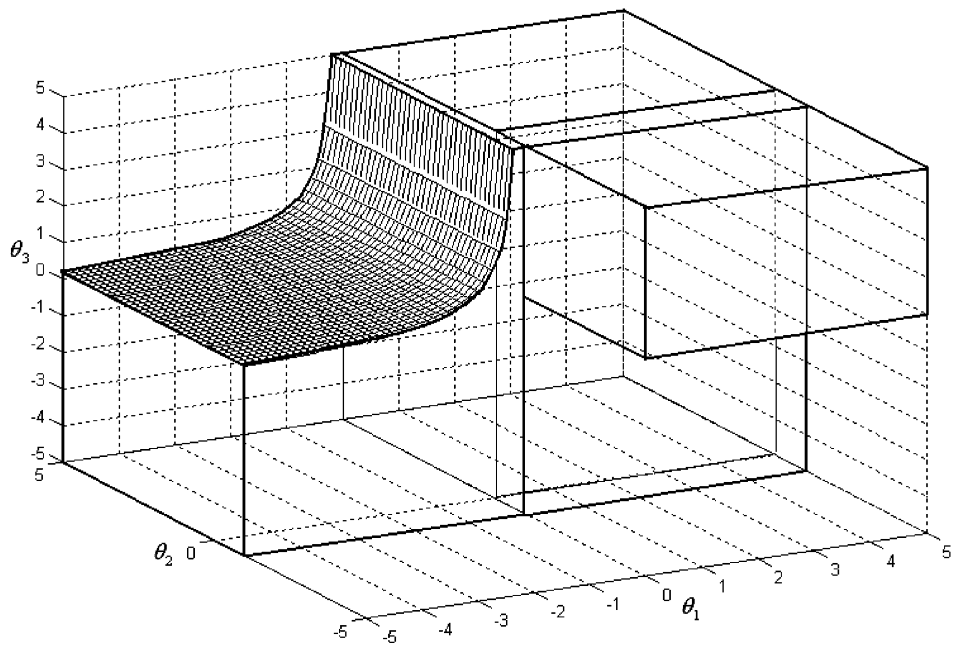

Figure 4. Critical regions of example 2.

Using the optimality conditions described by eqs 5 and 6, we get the following inequalities to describe the first critical region $\{\theta_2 \geq -1, \theta_1 \leq 0, \theta_1\theta_3 \geq -1\}$. The corresponding optimal function is derived from eq 7: $z_1 = \theta_1$. By generating evenly distributed points in the boundary of parameters and checking whether they are covered by the identified regions, we get the final solution:

$$z_1 = \theta_1, \quad \mathrm{CR}_1 = \{-5 \leq \theta_1 \leq 0, -1 \leq \theta_2 \leq 5, \theta_1\theta_3 \geq -1\}$$

$$z_2 = -\theta_1\theta_2, \quad \mathrm{CR}_2 = \{0 \leq \theta_1 \leq 5, -1 \leq \theta_2 \leq 0\}$$

$$z_3 = 0, \quad \mathrm{CR}_3 = \{0 \leq \theta_1 \leq 5, 0 \leq \theta_2 \leq 5\}$$

$$z_4 = \frac{-[\theta_1(\theta_2\theta_3 + 1) - (\theta_2 + 1)]}{\theta_3 - 1}, \quad \mathrm{CR}_4 = \{0 \leq \theta_1 \leq 5, -5 \leq \theta_2 \leq -1, 1 \leq \theta_3 \leq 5\}$$

It should be noticed from the result of example 2 that the optimal objective function z_4 takes the fractional nonlinear form, which is caused by the inverse of basis matrix, where uncertain parameters is included. We can predict that, for large-scale problems with LHS uncertainty, great computational effort is required to retrieve the rigorous solution of the problem. However, there is a special type of mpMILP with LHS uncertainty that can be addressed efficiently, which is the case when LHS uncertain parameters only appear as the coefficient of integer variables, because, in the relaxed mpLP problem, the LHS uncertainty is actually transformed to RHS. Also, from Figure 4, we can see that one critical region is nonlinear and actually nonconvex, which is due to the presence of LHS uncertainty and is a problem that is not addressed in most mpLP approaches in the literature. An intuitive method that addresses the problem of exploring the parameter space is the geometric method proposed by Borrelli et al.,[10] which, however, is only effective when the critical region is formed by linear constraints. Almost all the examples studied in the past involve RHS and objective uncertainties, where the critical region is formed by linear constraints. However, the critical region can be nonconvex when LHS uncertainties appear, which, thus, leads to a much more complex problem in regard to exploring the parameter space.

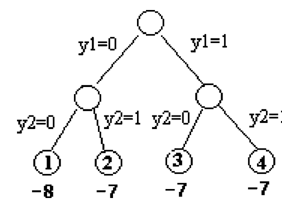

Figure 5. B&B tree of example 3.

4. Case Studies

In this section, we present several numerical examples to illustrate the effectiveness of the proposed general mpMILP algorithm. These examples involve parameter uncertainties in the LHS, RHS, and objective functions simultaneously.

Example 3. Objective Uncertainty. This example involves uncertainty in the objective function coefficients that is modeled with the introduction of two uncertain parameters, θ_1 and θ_2:

$$\begin{aligned} \min z = {} & (-2 + \theta_1)x_1 + (-1 + \theta_2)x_2 + y_1 + y_2 \\ \text{s.t. } & x_1 + 3x_2 - y_1 \leq 9 \\ & 2x_1 + x_2 - y_2 \leq 8 \\ & x_1 - y_1 + y_2 \leq 4 \\ & x_i \geq 0, y_i \in \{0,1\}, 0 \leq \theta_i \leq 10, \forall\, i = 1, 2 \end{aligned}$$

Step 1: The B&B solution. At $\theta = (0, 0)$, the B&B method results in the tree shown in Figure 5.

Step 2: Solving the mpLP. At the leaf nodes, following the general mpLP method proposed in the previous section, the following result is obtained. Let $f_1 = \theta_1 - 2$, $f_2 = \theta_2 - 1$, $f_3 = 3\theta_1 - \theta_2 - 5$, and $f_4 = \theta_1 - 2\theta_2$:

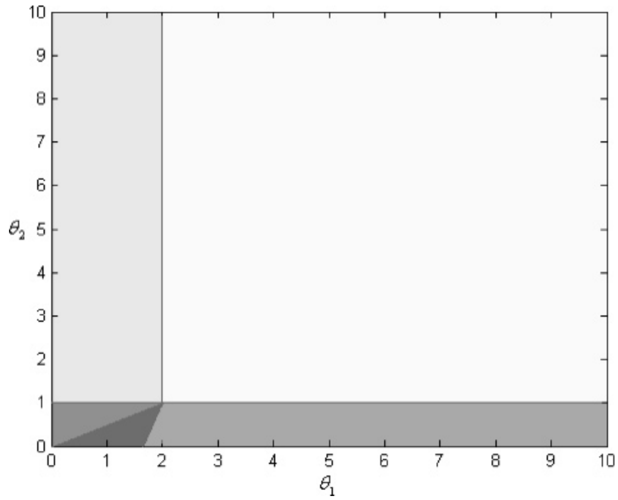

Figure 6. Critical regions for node 4.

At node 1, the mpLP result is

$$z_1^{(1)} = 0, \{f_1 \geq 0, f_2 \geq 0\}; \quad z_1^{(2)} = 3\theta_2 - 3, \{f_2 \leq 0, f_3 \geq 0\}$$

$$z_1^{(3)} = -8 + 3\theta_1 + 2\theta_2, \{f_3 \leq 0, f_4 \geq 0\};$$

$$z_1^{(4)} = -8 + 4\theta_1, \{f_1 \leq 0, f_4 \leq 0\}$$

At node 2, the mpLP result is

$$z_2^{(1)} = 1, \{f_1 \geq 0, f_2 \geq 0\}; \quad z_2^{(2)} = 3\theta_2 - 2, \{f_2 \leq 0, f_3 \geq 0\}$$

$$z_2^{(3)} = -5 + 3\theta_1, \{f_1 \leq 0, f_2 \geq 0\};$$

$$z_2^{(4)} = -7 + 3\theta_1 + 2\theta_2, \{f_2 \leq 0, f_3 \leq 0\}$$

At node 3, the mpLP result is

$$z_3^{(1)} = 1, \{f_1 \geq 0, f_2 \geq 0\};$$

$$z_3^{(2)} = -2.333 + 3.333\theta_2, \{f_2 \leq 0, f_3 \geq 0\}$$

$$z_3^{(3)} = -7 + 4\theta_1, \{f_1 \leq 0, f_4 \leq 0\};$$

$$z_3^{(4)} = -7 + 2.8\theta_1 + 2.4\theta_2, \{f_3 \leq 0, f_4 \geq 0\}$$

At node 4, the mpLP result is

$$z_4^{(1)} = 2, \{f_1 \geq 0, f_2 \geq 0\}; \quad z_4^{(2)} = 4\theta_1 - 6, \{f_1 \leq 0, f_2 \geq 0\}$$

$$z_4^{(3)} = -1.333 + 3.333\theta_2, \{f_2 \leq 0, f_3 \geq 0\}$$

$$z_4^{(4)} = 3.4\theta_1 + 2.2\theta_2 - 7; \{f_3 \leq 0, f_4 \geq 0\};$$

$$z_4^{(5)} = 4\theta_1 + \theta_2 - 7 \qquad \{f_2 \leq 0, f_4 \leq 0\}$$

(the critical regions are shown in Figure 6).

Step 3: Comparing the optimal functions. From the mpLP results, there are five intersection regions between the critical regions of the four nodes, which are the same as the critical region of node 4, as shown in Figure 6. We must compare the relevant optimal value functions in these regions. For example, in $CR_4^{(1)}$ of node 4, we must compare the optimal value functions $z_1^{(1)} = 0$, $z_2^{(1)} = 1$, $z_3^{(1)} = 1$, and $z_4^{(1)} = 2$; clearly, $z_1^{(1)} = 0$ is optimal. Similarly, we see that node 1 is optimal in the other four regions. Thus, finally, the integer solution of the problem is $(y_1,y_2) = (0, 0)$ and the critical regions and corresponding optimal value functions are the same as the mpLP result of node 4. Note that, for this problem, the redundancy test problem P2 is not solved, because the problem of comparing the critical regions is trivial.

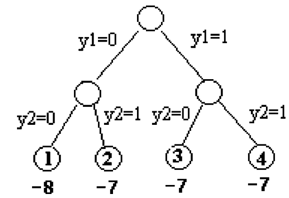

Figure 7. B&B tree of example 4.

Example 4: LHS Uncertainty. This example involves uncertainty in the constraint matrix coefficients, which is modeled with the introduction of two uncertain parameters, θ_1 and θ_2:

$$\begin{aligned}
\min z &= -2x_1 - x_2 + y_1 + y_2 \\
\text{s.t. } & x_1 + (3 + \theta_1)x_2 - y_1 \leq 9 \\
& (2 + \theta_2)x_1 + x_2 - y_2 \leq 8 \\
& x_1 - y_1 + y_2 \leq 4 \\
& x_i \geq 0, y_i \in \{0,1\}, 0 \leq \theta_i \leq 10, \forall\, i = 1, 2
\end{aligned}$$

Step 1: The B&B solution. Performing the B&B method at $\theta = (0, 0)$ results in the B&B tree shown in Figure 7. The leaf nodes for this problem are nodes 1, 2, 3, and 4. Node 1 has the minimum objective (−8) and corresponds to the current optimal node.

Step 2: Solving the mpLP. The mpLP is solved at the leaf nodes.

At node 1, the mpLP result is

$$z_1 = \frac{-40 - 16\theta_1 - 9\theta_2}{5 + 2\theta_1 + 3\theta_2 + \theta_1\theta_2}$$

$$\{0 \leq \theta_1 \leq 10, 0 \leq \theta_2 \leq 10\}$$

At node 2, the mpLP result is

$$z_2^{(1)} = \frac{-21 - 5\theta_1}{3 + \theta_1} \qquad \left\{0 \leq \theta_1 \leq 10, 0 \leq \theta_2 \leq \frac{1 + \theta_1}{3 + \theta_1}\right\}$$

$$z_2^{(2)} = \frac{-45 - 18\theta_1 - 9\theta_2}{5 + 2\theta_1 + 3\theta_2 + \theta_1\theta_2} + 1$$

$$\left\{0 \leq \theta_1 \leq 10, \frac{1 + \theta_1}{3 + \theta_1} \leq \theta_2 \leq 10\right\}$$

At node 3, the mpLP result is

$$z_3 = \frac{-40 - 16\theta_1 - 10\theta_2}{5 + 2\theta_1 + 3\theta_2 + \theta_1\theta_2} + 1$$

$$\{0 \leq \theta_1 \leq 10, 0 \leq \theta_2 \leq 10\}$$

At node 4, the mpLP result is

$$z_4^{(1)} = \frac{-24 - 6\theta_1}{3 + \theta_1} \qquad \left\{0 \leq \theta_1 \leq 10, 0 \leq \theta_2 \leq \frac{\theta_1 - 3}{12 + 4\theta_1}\right\}$$

$$z_4^{(2)} = \frac{-45 - 18\theta_1 - 10\theta_2}{5 + 2\theta_1 + 3\theta_2 + \theta_1\theta_2} + 2$$

$$\left\{0 \leq \theta_1 \leq 10, \frac{\theta_1 - 3}{12 + 4\theta_1} \leq \theta_2 \leq 10\right\}$$

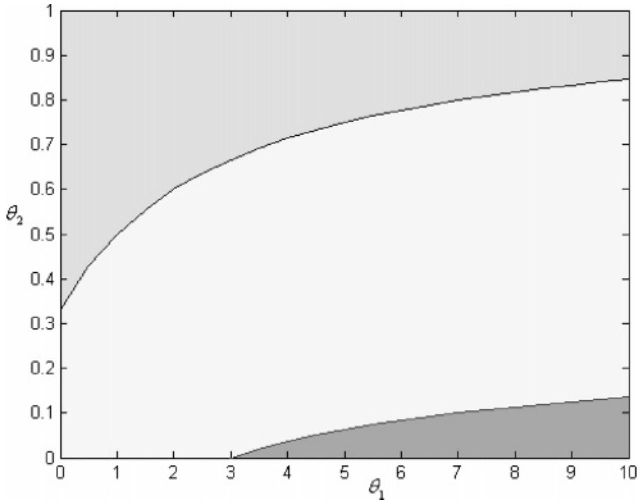

Figure 8. Intersection between critical regions of example 4.

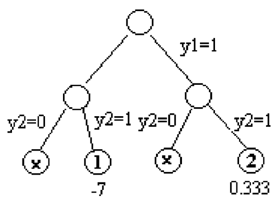

Figure 9. B&B tree of example 5.

Step 3: Comparison. We can see that there are three intersection regions between the critical regions of four leaf nodes, so we must compare the optimal value functions in these regions. After solving several redundancy test with problem P2, we found that node 1 is always the optimal node. Thus, the integer solution is (0, 0) and the critical region and optimal value functions are the same as the mpLP result at node 1.

As discussed previously, because of the existence of LHS uncertainty, the optimal value function takes a fractional form that involves a quadratic term in the denominator. Some of the critical regions in the mpLP results also become nonconvex, as shown in Figure 8.

Example 5: RHS and Objective Uncertainties. This example involves uncertainty in both the objective function coefficients and the RHS that is modeled with the introduction of two uncertain parameters, θ_1 and θ_2:

$$
\begin{aligned}
\min z = {} & (-3+\theta_1)x_1 + (-2+\theta_2)x_2 + 10y_1 + 5y_2 \\
\text{s.t. } & x_1 \le 10 + \theta_1 + 2\theta_2 \\
& x_2 \le 10 - \theta_1 + \theta_2 \\
& x_1 + x_2 \le 20 - \theta_2 \\
& x_1 + 2x_2 \le 12 + \theta_1 \\
& x_1 - 20y_1 \le 0 \\
& x_2 - 20y_2 \le 0 \\
& x_1 - x_2 \le -4 \\
& -y_1 - y_2 \le -1 \\
& x \ge 0, y \in \{0,1\}, 0 \le \theta \le 5
\end{aligned}
$$

Step 1: The B&B solution. Performing the B&B method at $\theta = (0, 0)$ results in the B&B tree shown in Figure 9. Node 1 is the current optimal node, and two leaf nodes get infeasible solutions.

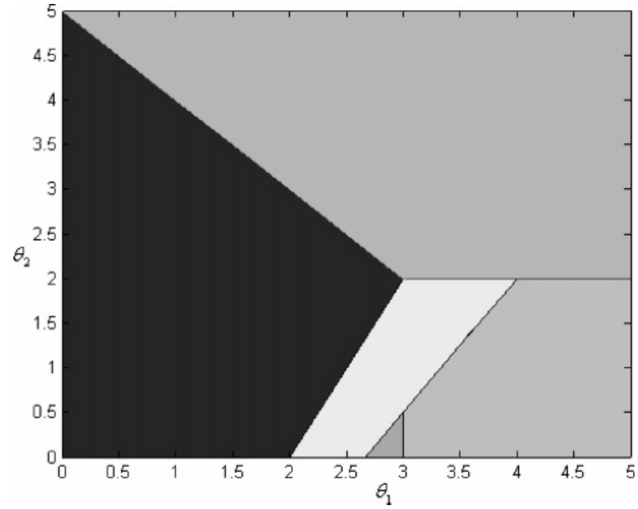

Figure 10. Critical regions of leaf node 2.

Step 2. Solving the mpLP. The mpLP are solved at the four leaf nodes. The two infeasible nodes prove to be infeasible for all $0 \le \theta \le 5$ values, so they are fathomed. For node 1 and node 2, we introduce several functions to describe their critical regions. Let $f_1 = 4 - 1.5\theta_1 + \theta_2$, $f_2 = \theta_2 - 2$, $f_3 = 4\theta_1 - 3\theta_2 - 14$, $f_4 = \theta_1 + \theta_2 - 5$, $f_5 = 2\theta_1 - \theta_2 - 4$, and $f_6 = \theta_1 - 3$; the following mpLP results are obtained.

For node 1, the mpLP result is

$$z_1^{(1)} = (0.5\theta_1 + 6)(\theta_2 - 2) + 5 \qquad \{f_1 \ge 0, f_2 \le 0\}$$

$$z_1^{(2)} = (-\theta_1 + \theta_2 + 10)(\theta_2 - 2) + 5 \qquad \{f_1 \le 0, f_2 \le 0\}$$

$$z_1^{(3)} = 4\theta_2 - 3 \qquad \{f_2 \ge 0\}$$

For node 2, the mpLP result is

$$z_2^{(1)} = (\theta_1 - 3)(1.333 + 0.333\theta_1) + (\theta_2 - 2)(5.333 + 0.333\theta_1) + 15 \qquad \{f_3 \le 0, f_4 \ge 0, f_5 \le 0\}$$

$$z_2^{(2)} = (\theta_2 - 2)(0.5\theta_1 + 6) + 15 \qquad \{f_1 \ge 0, f_2 \le 0, f_5 \ge 0\}$$

$$z_2^{(3)} = (\theta_1 - 3)(3\theta_1 - 2\theta_2 - 8) + (\theta_2 - 2)(-\theta_1 + \theta_2 + 10) + 15 \qquad \{f_1 \le 0, f_3 \le 0, f_5 \ge 0, f_6 \le 0\}$$

$$z_2^{(4)} = (\theta_2 - 2)(-\theta_1 + \theta_2 + 10) + 15 \qquad \{f_1 \le 0, f_2 \le 0, f_6 \ge 0\}$$

$$z_2^{(5)} = 4\theta_2 + 7 \qquad \{f_4 \ge 0, f_2 \ge 0\}$$

Step 3: Comparison. The critical regions of node 2 are plotted in Figure 10. There are a total of six intersection regions between the critical regions of node 1 and node 2. The redundancy test problem P2 is solved in these six regions. After comparison, the final solution of the mpMILP problem is

$$(y_1, y_2) = (0, 1)$$

$$z_1 = (0.5\theta_1 + 6)(\theta_2 - 2) + 5 \qquad \{4 - 1.5\theta_1 + \theta_2 \ge 0, 0 \le \theta_1 \le 5, 0 \le \theta_2 \le 2\}$$

$$z_2 = (-\theta_1 + \theta_2 + 10)(\theta_2 - 2) + 5 \qquad \{4 - 1.5\theta_1 + \theta_2 \le 0, 0 \le \theta_1 \le 5, 0 \le \theta_2 \le 2\}$$

$$z_3 = 4\theta_2 - 3 \qquad \{0 \le \theta_1 \le 5, 2 \le \theta_2 \le 5\}$$

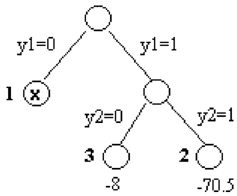

Figure 11. B&B tree of example 6.

As we discussed in the previous section, because the RHS and objective uncertainties appear simultaneously, the optimal value functions are nonlinear quadratic functions and the critical regions are formed by linear constraints and, thus, form convex regions.

Example 6: LHS, RHS, and Objective Uncertainties. This example involves the uncertainty in objective function coefficients, RHS, and constraint matrix coefficients simultaneously that is modeled with the introduction of three uncertain parameters, θ_1, θ_2, and θ_3.

$$\begin{aligned} \min z &= (-3+\theta_1)x_1 - 8x_2 + 4y_1 + 2y_2 \\ \text{s.t. } & x_1 + x_2 \le 13 + \theta_2 \\ & (5+\theta_3)x_1 - 4x_2 \le 20 \\ & -8x_1 + 22x_2 \le 121 \\ & -4x_1 - x_2 \le -8 \\ & x_1 - 10y_1 \le 0 \\ & x_2 - 15y_2 \le 0 \\ & x \ge 0, y \in \{0,1\}, 0 \le \theta \le 10 \end{aligned}$$

Step 1: The B&B Solution. The result from solving the problem using the B&B method at $\theta = (0,0,0)$ is shown in Figure 11; node 1 is infeasible, and node 3 is the current optimal node.

Step 2: Solving the mpLP. The mpLP at the leaf nodes can be solved as follows. Let $f_1 = \theta_1 - 5.909$, $f_2 = \theta_2 - 6.136$, $f_3 = \theta_3 - \{-78\theta_2 + 675\}/\{22\theta_2 + 165\}$, $f_4 = \theta_3 - 0.655$, $f_5 = \theta_1 - 3$, and $f_6 = \theta_3 - 5$. Node 1 is infeasible for all parameter values. Therefore, it is fathomed. The results of nodes 2 and 3 are as follows.

At node 2 ($(y_1, y_2) = (1, 1)$), the mpLP result is

$$z_1^{(1)} = (-3+\theta_1)(5.5 + 0.7333\theta_2) - 2.1333\theta_2 - 54 \qquad \{f_1 \le 0, f_2 \le 0, f_3 \le 0\}$$

$$z_1^{(2)} = \frac{-4446 + 462\theta_1 - 484\theta_3}{11\theta_3 + 39} + 6 \qquad \{f_1 \le 0, f_3 \ge 0\}$$

$$z_1^{(3)} = 0.573\theta_1 - 41.385 \qquad \{f_1 \ge 0\}$$

$$z_1^{(4)} = 10\theta_1 - 97.091 \qquad \{f_1 \le 0, f_2 \ge 0, f_4 \le 0\}$$

At node 3 ($(y_1, y_2) = (1, 0)$), the mpLP result is

$$z_2^{(1)} = \frac{20(\theta_1 - 3)}{\theta_3 + 5} + 4 \qquad \{f_5 \le 0, f_6 \le 0\}$$

$$z_2^{(2)} = -2 + 2\theta_1 \qquad \{f_5 \ge 0, f_6 \le 0\}$$

Step 3: Comparison. To compare the optimal value functions in the intersection between $CR_1^{(1)}$ and $CR_2^{(1)}$, the following redundancy test is solved, corresponding to problem P2:

$$\max \epsilon = z_1^{(1)} - z_2^{(1)} >$$

$$\text{s.t. } f_1 \le 0, f_2 \le 0, f_3 \le 0, f_5 \le 0, f_6 \le 0 \quad (\text{for } 0 \le \theta \le 10)$$

The optimal objective is -58, so $z_1^{(1)} \le z_2^{(1)}$. Similarly, we get

$$z_1^{(1)} \le z_2^{(2)} \text{ in } CR_1^{(1)} \cap CR_2^{(2)}$$

$$z_1^{(2)} \le z_2^{(1)} \text{ in } CR_1^{(2)} \cap CR_2^{(1)}$$

$$z_1^{(2)} \le z_2^{(2)} \text{ in } CR_1^{(2)} \cap CR_2^{(2)}$$

$$CR_1^{(3)} \cap CR_2^{(1)} = \phi$$

$$z_1^{(3)} \le z_2^{(2)} \text{ in } CR_1^{(3)} \cap CR_2^{(2)}$$

$$z_1^{(4)} \le z_2^{(1)} \text{ in } CR_1^{(4)} \cap CR_2^{(1)}$$

$$z_1^{(4)} \le z_2^{(2)} \text{ in } CR_1^{(4)} \cap CR_2^{(2)}$$

Therefore, the final integer solution of this problem is $(y_1, y_2) = (1, 1)$, and the critical regions and corresponding optimal value functions are the same as the result of node 2, as shown in Figure 12.

Example 7: LHS, RHS, and Objective Uncertainties. This example also involves the uncertainty in objective function coefficients, RHS, and constraint matrix coefficients. Three uncertain parameters (θ_1, θ_2, and θ_3) appear in the LHS, RHS, objective, respectively.

$$\begin{aligned} \max\ & 4x_1 + \theta_3 x_2 + x_3 \\ \text{s.t. } & 3x_1 + \theta_1 x_2 + x_3 \le \theta_2 \\ & x_1 - x_2 - x_3 \le 0 \\ & -x_1 - x_2 + x_3 \le 0 \\ & -x_1 - x_2 - x_3 \le -1 \\ & x_i \in \{0,1\}, \quad 4 \le \theta_1 \le 6 \\ & 3 \le \theta_2 \le 6, \quad 3 \le \theta_3 \le 6 \end{aligned}$$

Step 1: The B&B Solution. Solve the problem using the B&B method at $(\theta_1,\theta_2,\theta_3) = (4,3,3)$. Because all the leaf nodes are infeasible, as shown in Figure 13, we move the parameter to $(\theta_1,\theta_2,\theta_3) = (4,6,3)$ and get the B&B tree shown in Figure 14.

Step 2: Solving the mpLP. At the leaf nodes, node 1, node 2, and node 5 are infeasible for all parameter values, so they are fathomed. The mpLP result for node 3 is $z = \theta_3$ (for $\theta_1 - \theta_2 \le 0$); the mpLP result for node 4 is $z = \theta_3 + 1$ (for $\theta_1 - \theta_2 + 1 \le 0$); and for node 6, $z = 5$ (for $\theta_2 \ge 4$). The current optimal node is dependent on the value of parameter θ_3.

Step 3: Comparison. Because nodes 1, 2, and 5 are fathomed, no further branching is needed. The next step is then to compare the optimal value functions. The results between nodes 3 and 4 are

$$\theta_1 - \theta_2 + 1 \le 0, z = \theta_3 + 1, (x_1, x_2, x_3) = (0,1,1)$$

$$-1 < \theta_1 - \theta_2 \le 0, z = \theta_3, (x_1, x_2, x_3) = (0,1,0)$$

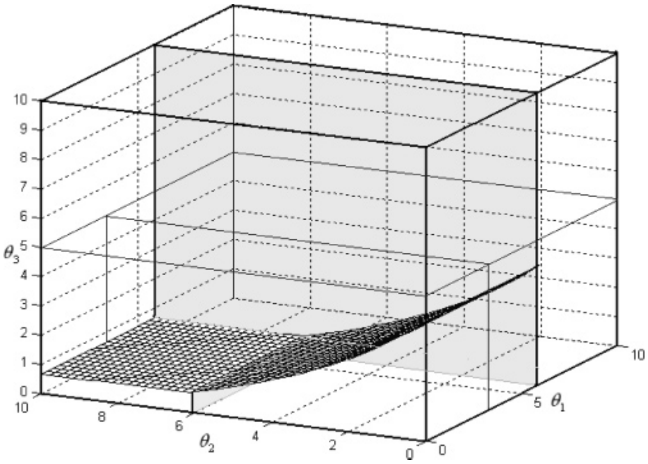

Figure 12. Critical regions of example 6.

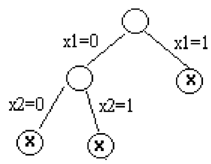

Figure 13. B&B tree of example 7 at $(\theta_1,\theta_2,\theta_3) = (4,3,3)$.

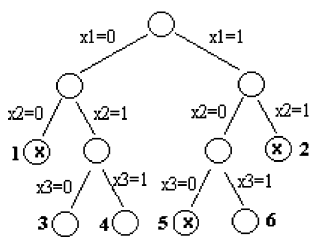

Figure 14. Updated B&B tree of example 7 at $(\theta_1,\theta_2,\theta_3) = (4,6,3)$.

Table 2. Solution of Example 7

$z = \theta_3 + 1$	$z_2 = 5$	$z = \theta_3$	$z_4 = 5$
$(x_1,x_2,x_3) = (0,1,1)$	$(x_1,x_2,x_3) = (1,0,1)$	$(x_1,x_2,x_3) = (0,1,0)$	$(x_1,x_2,x_3) = (1,0,1)$
$\begin{cases} \theta_1 - \theta_2 + 1 \le 0 \\ 4 \le \theta_1 \le 6 \\ 4 \le \theta_2 \le 6 \\ 4 \le \theta_3 \le 6 \end{cases}$	$\begin{cases} \theta_1 - \theta_2 + 1 \le 0 \\ 4 \le \theta_1 \le 6 \\ 4 \le \theta_2 \le 6 \\ 3 \le \theta_3 \le 4 \end{cases}$	$\begin{cases} -1 < \theta_1 - \theta_2 \le 0 \\ 4 \le \theta_1 \le 6 \\ 4 \le \theta_2 \le 6 \\ 5 \le \theta_3 \le 6 \end{cases}$	$\begin{cases} -1 < \theta_1 - \theta_2 \le 0 \\ 4 \le \theta_1 \le 6 \\ 4 \le \theta_2 \le 6 \\ 3 \le \theta_3 \le 5 \end{cases}$

Continuing the comparison of this result with node 6, we get the final result shown in Table 2.

5. Summary and Future Work

Motivated by the fact that, in real-world applications of process design and operations, uncertainty is involved in most parameters in model formulation, a multiparametric mixed-integer linear programming (mpMILP) method was proposed to examine the left-hand side (LHS), right-hand side (RHS), and objective function uncertainties based on a general multi-parametric linear programming (mpLP) approach that was developed based on standard linear programming (LP) optimality conditions. The objective of this paper was to present the algorithm such that it is only illustrated through a few numerical examples. Future publications will describe the application of the mpMILP approach for the solution of scheduling under uncertainty.

The framework proposed in this paper for general mpMILP problems is based on a branch and bound (B&B) framework, where, at the leaf nodes, the mpLP problem is solved using a new methodology that explores the primal and dual optimality conditions for the deterministic LP problems. The approach is general to address all different uncertainties and has been proved to work well for small problems.

The mpLP method with RHS and objective uncertainty is studied, mostly in the literature, because of the relative smaller complexity. In these cases, the critical region is formed by linear constraints, which can be directly predicted from the optimality conditions of the LP problem; thus, it is also a convex region. When LHS uncertainty is included in the mpLP problem, the constraints of critical region become nonlinear and the region becomes nonconvex. The optimal function also takes a fractional nonlinear form, which brings more complexity for mpLP problems in calculating the rigorous solution of the problem.

As illustrated in the paper, the proposed method can handle LHS uncertainty for small mpMILP problems. For the special case when the LHS uncertainty in the MILP formulation only appears as coefficients of integer variables, the problem can be transformed to one with RHS uncertainty and solved without the complications of the LHS case, as shown with example 7. However, for large-scale problems, the handling of LHS uncertainty to get an exact solution of the mpMILP problem is computationally complex, as shown with example 4.

Further improvement of the proposed method will focus on more-effective ways to address degenerated LP problems, which can result in an increase in computational effort, because of basis multiplicity. Among the work in the literature, most papers have addressed uncertainty in process design and relatively less attention has been devoted to the issue of uncertainty in process planning and scheduling, mainly because of the increased complexity of the deterministic problem. Therefore, another direction on improving the proposed approach is to develop computationally efficient numerical techniques for routinely solving large-scale problems.

Acknowledgment

The authors gratefully acknowledge financial support from the National Science Foundation (under Grant Nos. CTS 0625515 and 0224745).

Literature Cited

(1) Ierapetritou, M. G.; Pistikopoulos, E. N. Batch plant design and operations under un-certainty. *Ind. Eng. Chem. Res.* **1996**, *35*, 772.

(2) Balasubramanian, J.; Grossmann, I. E. Scheduling optimization under uncertainty − an alternative approach. *Comput. Chem. Eng.* **2003**, *27*, 469.

(3) Lin, X.; Janak, S. L.; Floudas, C. A. A new robust optimization approach for scheduling under uncertainty—I. bounded uncertainty. *Comput. Chem. Eng.* **2004**, *28*, 2109.

(4) Honkomp, S. J.; Mockus, L.; Reklaitis, G. V. A framework for schedule evaluation with processing uncertainty. *Comput. Chem. Eng.* **1999**, *23*, 595.

(5) Sahinidis, N. V. Optimization underuncertainty: State-of-the-art and opportunities. *Comput. Chem. Eng.* **2004**, 28, 971.

(6) Jia, Z.; Ierapetritou, M. G. Short-term Scheduling under Uncertainty Using MILP Sensitivity Analysis. *Ind. Eng. Chem. Res.* **2004**, *43*, 3782.

(7) Gass, S.; Saaty, T. The Computational Algorithm for the Parametric Objective Function. *Nav. Res. Logist. Q.* **1955**, *2*, 39.

(8) Gal, T. *Postoptimal Analyses, Parametric Programming, and Related Topics*, 2nd Edition; de Gruyter: Berlin, Germany, 1995.

(9) Jenkins, L. Parametric methods in integer linear programming. *Ann. Oper. Res.* **1990**, 27, 77.

(10) Gal, T.; Nedoma, J. Multiparametric Linear Programming, *Manage. Sci.* **1972**, *18*, 406.

(11) Borrelli, E.; Bemporad, A.; Morari, M. A geometric algorithm for multi-parametric linear programming. *J. Opt. Therm. Appl.* **2003**, *118*, 515.

(12) Filippi, C.; Romanin-Jacur, G. Multiparametric Demand Linear Transportation Problem. *Eur. J. Oper. Res.* **2002**, 139, 206.

(13) Filippi, C. An Algorithm for Approximate Multiparametric Linear Programming. *J. Opt. Therm. Appl.* **2004**, *120*, 73.

(14) Seron, M.; De Dona, J. A.; Goodwin, G. C. Global analytical model predictive control with input constraints. In *Proceedings of the IEEE Conference on Decision and Control*, Sydney, Australia, 2000; p TuA05-2.

(15) Bemporad, A.; Morari, M.; Dua, V.; Pistikopoulos, E. N. The explicit linear quadratic regulator for constrained systems. *Automatica* **2002**, *38*, 3.

(16) Baotic, M. An Efficient Algorithm for Multiparametric Quadratic Programming. Technical Report AUT02-04, Automatic Control Laboratory, ETH Zurich, Switzerland, February 2002.

(17) Tøndel, P.; Johansen, T. A.; Bemporad, A. An algorithm for multi-parametric quadratic programming and explicit MPC solutions, *Automatica* **2003**, *39*, 489.

(18) Johansen, T. A. On multi-parametric nonlinear programming and explicit nonlinear model predictive control. In *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, NV, 2002; pp 2768−2773.

(19) Bemporad, A.; Filippi, C. An Algorithm for Approximate Multi-parametric Convex Programming. *Comput. Optim. Appl*. **2006**, *35*, 87.

(20) Acevedo, J.; Salgueiro, M. An efficient algorithm for convex multiparametric nonlinear programming problems. *Ind. Eng. Chem. Res.* **2003**, *42*, 5883.

(21) Hale, T. E.; Qin, S. J. Multi-parametric nonlinear programming and the evaluation of implicit optimization model adequacy. In *Proceedings of the 7th International Symposium on the Dynamics and Control of Process Systems*, Cambridge, MA, 2004.

(22) Acevedo, J.; Pistikopoulos, E. N. A multiparametric programming approach for linear process engineering problems under uncertainty. *Ind. Eng. Chem. Res.* **1997**, *36*, 717.

(23) Dua, V.; Pistikopoulos, E. N. Algorithms for the solution of multiparametric mixed integer linear optimization problems. *Ann. Oper. Res.* **2000**, 99, 123.

(24) Pertsinidis, A.; Grossmann, I. E.; McRae, G. J. Parametric optimization of MILP programs and a framework for the parametric optimization of MINLPs. *Comput. Chem. Eng.* **1998**, *22*, S205.

(25) Crema, A. The multiparametric 0−1 integer linear programming problem: a unified approach. *Eur. J. Oper. Res.* **2002**, *139*, 511.

(26) Jia, Z.; Ierapetritou, M. G. Uncertainty Analysis on the Right-Hand-Side for MILP problems. *AIChE J.* **2006**, *52*, 2486.

(27) Dua, V.; Bozinis, N. A.; Pistikopoulos, E. N. A multiparametric programming approach for mixed-integer quadratic engineering problems. *Comput. Chem. Eng.* **2002**, *26*, 715.

(28) Bemporad, A. Multiparametric nonlinear integer programming and explicit quantized optimal control. In *Proceedings of the IEEE Conference on Decision and Control*, 2003; pp 3167−3172.

(29) Avis; Fukuda, K. A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra. *Discrete Comput. Geom.* **1992**, *8*, 295.

(30) Fukuda, K.; Prodon, A. Double description method revisited. In *Combinatorics and Computer Science*; Deza, M., Euler, R., Manoussakis, I., Eds.; Lecture Notes in Computer Science 1120; Springer: Berlin, New York, 1996; p 91.

(31) Pistikopoulos, E. N.; Dua, V. Planning under uncertainty; a parametric optimization approach. In *Proceedings of the Third International Conference on Foundations of Computer Aided Operations (FOCAPO)*, Snowbird, UT, July 1998; Pekny, J. F., Blau, G. E., Eds.

(32) Ryu, J. H.; Pistikopoulos, E. N. A novel approach to scheduling of zero-wait batch processes under processing time variations. *Comput. Chem. Eng.* **2007**, *31*, 101.

Received for review January 25, 2007
Revised manuscript received April 24, 2007
Accepted May 22, 2007

IE070148S