Alternative Identification Algorithms for Obtaining a First-Order Stable/Unstable Process Model From a Single Relay Feedback Test

ARTICLE *in* INDUSTRIAL & ENGINEERING CHEMISTRY RESEARCH · JANUARY 2008

Impact Factor: 2.59 · DOI: 10.1021/ie070856d

2 AUTHORS:

Tao Liu
Dalian University of Technology
72 PUBLICATIONS **1,078** CITATIONS

Furong Gao
The Hong Kong University of Science and Tec...
267 PUBLICATIONS **3,781** CITATIONS

Alternative Identification Algorithms for Obtaining a First-Order Stable/Unstable Process Model from a Single Relay Feedback Test

Tao Liu and Furong Gao*

Department of Chemical Engineering, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong

A first-order-plus-dead-time (FOPDT) model is most commonly used for control system design and on-line tuning. By using a single biased/unbiased relay feedback test, four alternative identification algorithms are proposed for obtaining a FOPDT model for a stable process and four alternative FOPDT algorithms for unstable processes. By deriving the exact relay response expressions for FOPDT stable/unstable processes, two of the above algorithms are proposed for implementation simplicity based on some measurable parameters of the relay response, while the other two algorithms are developed with more computation effort for obtaining enhanced accuracy for identification of high-order processes, according to whether a single biased or unbiased relay feedback test is used. A denoising method based on a low-pass Butterworth filter is presented to guarantee identification robustness in the presence of measurement noises. Illustrative examples are given to demonstrate the effectiveness and merits of the proposed identification algorithms.

1. Introduction

Process identification from relay feedback test has received much attention in the recent years, since the pioneering work of Åström and Hägglund[1] and Luyben.[2] Recent reviews on relay feedback identification development have been made by Atherton[3] and Hang et al.[4] An amount of research effort has been focused on using efficiently the ultimate frequency and ultimate gain for process regulation.[5−9] Multiple frequency response points of the process to be identified have also been exploited for improving control performance.[10−14] Low-order process models thus obtained have been effectively demonstrated for obtaining enhanced system performance.[15−19] It has been widely recognized that improved fitting for the process frequency response, in particular for the low-frequency range with a phase change from 0 to $-\pi$ that reflects the fundamental process response characteristics, contributes to enhanced controller design.[4,8] Presently, much research effort has been devoted to using a single relay feedback test to improve identification efficiency. There are in general two types of relay feedback tests, unbiased (symmetrical) and biased (asymmetrical). The process gain can be obtained separately from a biased relay test, but error may be resulted from the load disturbances that cannot be perceived. Under an unbiased relay test, the influence of load disturbances can be intuitively detected. On the basis of a single run of unbiased relay feedback, Luyben[20] proposed a first-order-plus-dead-time (FOPDT) modeling method by defining curvature factors for the relay response shapes of stable and unstable processes; Vivek and Chidambaram[21,22] reported two FOPDT identification algorithms for stable and unstable processes, respectively, based on the Fourier analysis of the process relay response; Panda and Yu[23] gave a FOPDT identification algorithm for stable processes by deriving an analytical expression for unbiased relay response; According to the ultimate frequency, Huang et al.[24] developed a simple formulation of FOPDT model for on-line tuning of stable processes; By using a proportional controller to stabilize an unstable process for relay identification, Majhi and Atherton[25] proposed a FOPDT modeling method for unstable processes. On the basis of a single run of biased relay test, Shen et al.[26] presented a FOPDT modeling method for stable processes according to the sustained oscillation conditions from the describing function analysis; Wang et al.[27] derived a FOPDT algorithm for stable processes by using the algebra properties of periodic oscillation; Kaya and Atherton[28] developed a FOPDT identification method based on the so-called A-locus analysis; Srinivasan and Chidambaram[29] proposed a united FOPDT identification algorithm for stable and unstable processes by using two fitting conditions for the relay response; Park et al.[30] presented a FOPDT identification algorithm for unstable processes by using the magnitude/phase oscillation conditions. In addition, Marchetti et al.[31] gave a modified autotune variation (ATV)[2] method for FOPDT identification of unstable processes by using two different unbiased relay tests. Relay-based identification algorithms for obtaining second- or even higher order models have also been explored in the recent literature,[32,33] including some of the aforementioned references.[5−9,23,24,26,28,31]

In view of the fact that the FOPDT model is the most commonly used low-order process model for controller tuning,[3,4] eight alternative identification algorithms are proposed in this paper for obtaining a FOPDT model from a single biased/unbiased relay test for stable and unstable processes. The general relay response expression for a FOPDT stable/unstable process is analytically derived for assessing the fluctuation range of process response under relay feedback test. Four identification algorithms are then derived according to the relay expressions, respectively for stable and unstable processes, each of which has the property of computational simplicity for engineering practice and can obtain good accuracy in the sense that the model structure matches well with the process to be identified. By using the fitting conditions established for the relay response, four alternative identification algorithms are proposed to improve frequency response fitting for high-order processes over the referred low-frequency range. A denoising method based on a low-pass Butterworth filter is given to guarantee identification robustness against measurement noises. The paper is organized as such: Section 2 presents the general relay response expression for a FOPDT stable/unstable process under relay feedback test. The identification algorithms are subsequently

* To whom correspondence should be addressed. Tel.: +852-2358-7139. Fax: +852-2358-0054. E-mail: kefgao@ust.hk.

10.1021/ie070856d CCC: $40.75
Published on Web 01/18/2008

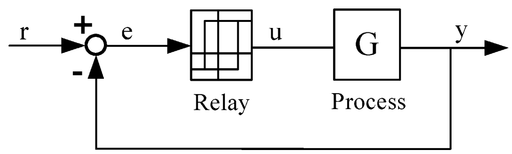

Figure 1. Block diagram of relay feedback identification.

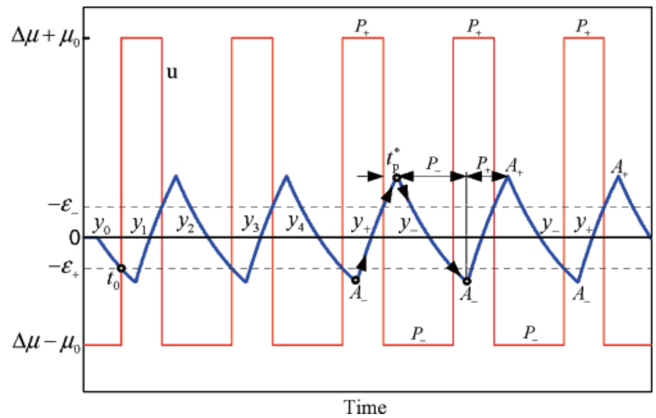

Figure 2. Limit cycle analysis for FOPDT process.

proposed in section 3. A denoising method based on a low-pass Butterworth filter is given in section 4. Illustrative examples are shown in section 5 to demonstrate the effectiveness and merits of the proposed identification algorithms. Finally, conclusions are drawn in section 6.

2. Relay Response Expression

Generally, a first-order stable process is described in the form of

$$G_1 = \frac{k_p e^{-\theta s}}{\tau_p s + 1} \tag{1}$$

and correspondingly, a first-order unstable process is described as

$$G_2 = \frac{k_p e^{-\theta s}}{\tau_p s - 1} \tag{2}$$

where k_p denotes the process gain, τ_p the time constant, and θ the time delay.

A general relay feedback structure for identification is shown in Figure 1, where the relay function is usually specified as

$$u(t) = \begin{cases} u_+ & \text{for } \{e(t) > \epsilon_+\} \text{ or } \{e(t) \geq \epsilon_- \text{ and } u(t_-) = u_+\} \\ u_- & \text{for } \{e(t) < \epsilon_-\} \text{ or } \{e(t) \leq \epsilon_+ \text{ and } u(t_-) = u_-\} \end{cases} \tag{3}$$

where $u_+ = \Delta\mu + \mu_0$ and $u_- = \Delta\mu - \mu_0$ denote, respectively, the positive and negative relay magnitudes. ϵ_+ and ϵ_- denote, respectively, the positive and negative relay switch hystereses. Note that letting $u_+ = -u_-$ and $\epsilon_+ = -\epsilon_-$ leads to an unbiased relay function. The choice of the relay magnitudes depends on the permissible fluctuation range of the process output around its steady-state operation level and usually can be taken as 5−20% of the maximal range. The relay switch hystereses may be set at least twice over the noise band, together with an upper limit almost equal to 0.95 times of the absolute minimum of u_+ and u_-.[34]

Without loss of generality, a biased relay feedback test is used herein to derive the relay response expression for a FOPDT stable/unstable process. The following proposition gives an exact relay response expression for a FOPDT stable process.

Proposition 1. For a first-order stable process modeled by eq 1 under a biased relay feedback test as shown in Figure 2, the resulting limit cycle of the process output response is characterized by

$$y_+(t) = k_p(\Delta\mu + \mu_0) - 2k_p\mu_0 E e^{-t/\tau_p}, \quad t \in [0, P_+] \tag{4}$$

$$y_-(t) = k_p(\Delta\mu - \mu_0) - 2k_p\mu_0 F e^{-t/\tau_p}, \quad t \in (P_+, P_u] \tag{5}$$

where $y_+(t)$ is the monotonously ascending part for $t \in [0, P_+]$, while $y_-(t)$ is the monotonously descending part for $t \in (P_+, P_u]$ and

$$E = \frac{1 - e^{-P_-/\tau_p}}{1 - e^{-P_u/\tau_p}} \tag{6a}$$

$$F = -\frac{1 - e^{-P_+/\tau_p}}{1 - e^{-P_u/\tau_p}} \tag{6b}$$

Proof. See the Appendix.

Following a similar analysis, we can obtain the relay response expression for a FOPDT unstable process given in the following corollary.

Corollary 1. For a first-order unstable process modeled by eq 2 under a biased relay feedback test, the resulting limit cycle of the process output response is characterized by

$$y_+(t) = -k_p(\Delta\mu + \mu_0) + 2k_p\mu_0 E e^{t/\tau_p}, \quad t \in [0, P_+] \tag{7}$$

$$y_-(t) = k_p(\mu_0 - \Delta\mu) + 2k_p\mu_0 F e^{t/\tau_p}, \quad t \in (P_+, P_u] \tag{8}$$

where $y_+(t)$ increases monotonously for $t \in [0, P_+]$ while $y_-(t)$ decreases monotonously for $t \in (P_+, P_u]$ and

$$E = \frac{1 - e^{P_-/\tau_p}}{1 - e^{P_u/\tau_p}} \tag{9a}$$

$$F = -\frac{1 - e^{P_+/\tau_p}}{1 - e^{P_u/\tau_p}} \tag{9b}$$

Remark 1. By substituting $\Delta\mu = 0$ and $P_+ = P_- = P_u/2$ into eqs 4−9, the relay response expression for a FOPDT stable/unstable process under unbiased relay test can be obtained. Therefore, the general formulation given herein may be used to summarize those relay expressions derived in the recent literature[35,36] for an unbiased relay test. It should be noted that the limiting condition given in Lin et al.[37] for a FOPDT unstable process to move into steady oscillation under relay feedback, which was derived in terms of the state-space representation of the process model and relay response, can be equivalently obtained in light of the analytical derivation for the relay expression of corollary 1, and thus is omitted from discussion.

3. Identification Algorithms

In the limit cycle, the process output is a periodic function with the oscillation angular frequency, $\omega_u = 2\pi/P_u$. By using the idea of time shift, we may view it as a periodic signal from the very beginning, so its Fourier transform can be derived as

$$Y(j\omega_u) = \lim_{N\to\infty} N \int_0^{P_u} y_{os}(t) e^{-j\omega_u t}\, dt = \lim_{N\to\infty} N \int_{t_{os}}^{t_{os}+P_u} y(t) e^{-j\omega_u t}\, dt \tag{10}$$

where $y_{os}(t) = y(t)$ for $t \in [t_{os},\infty)$ and t_{os} may be taken as any relay switch point in steady oscillation, such that the influence from the initial process response can be excluded.

Similarly, it follows that

$$U(j\omega_u) = \lim_{N\to\infty} N \int_{t_{os}}^{t_{os}+P_u} u(t)e^{-j\omega_u t}\, dt \tag{11}$$

Thereby, the process frequency response at ω_u can be obtained as

$$G(j\omega_u) = \frac{Y(j\omega_u)}{U(j\omega_u)} = \frac{\int_{t_{os}}^{t_{os}+P_u} y(t)e^{-j\omega_u t}\, dt}{\int_{t_{os}}^{t_{os}+P_u} u(t)e^{-j\omega_u t}\, dt} = A_u e^{j\varphi_u} \tag{12}$$

It should be noted that the integral in eq 12 may be numerically computed using the trapezoidal rule or the fast Fourier transform (FFT) for $Y(j\omega_u)$ and $U(j\omega_u)$.

When a biased relay test is used, the gain of a stable process can be similarly derived as

$$k_p = G(0) = \frac{\int_{t_{os}}^{t_{os}+P_u} y(t)\, dt}{\int_{t_{os}}^{t_{os}+P_u} u(t)\, dt} \tag{13a}$$

For an unstable process modeled by eq 2, the process gain can be derived accordingly as

$$k_p = -G(0) = -\frac{\int_{t_{os}}^{t_{os}+P_u} y(t)\, dt}{\int_{t_{os}}^{t_{os}+P_u} u(t)\, dt} \tag{13b}$$

Thereby, the identification algorithms developed according to whether a biased or an unbiased relay test is used are evidently different from each other, which are respectively given in the following two subsections.

3.1. Biased Relay Test for FOPDT Model. For a FOPDT stable process, it follows from proposition 1 that the process time delay can be visually measured as the time taken to reach the peak of the process output response from the initial relay switch point in a half period of the relay, which is denoted as t_p^* as shown in Figure 2. Correspondingly, it can be obtained from eqs 4 and 5 that

$$y_+(P_+) = k_p(\Delta\mu + \mu_0) - 2k_p\mu_0 E e^{-P_+/\tau_p} = A_+ \tag{14}$$

$$y_-(P_-) = k_p(\Delta\mu - \mu_0) - 2k_p\mu_0 F e^{-(P_-/\tau_p)} = A_- \tag{15}$$

$$y_+(P_+ - \theta) = k_p(\Delta\mu + \mu_0) - 2k_p\mu_0 E e^{-(P_+-\theta)/\tau_p} = -\epsilon_- \tag{16}$$

$$y_-(P_- - \theta) = k_p(\Delta\mu - \mu_0) - 2k_p\mu_0 F e^{-(P_--\theta)/\tau_p} = -\epsilon_+ \tag{17}$$

Substituting eq 14 into eq 16 yields

$$\tau_p = \frac{\theta}{\ln \dfrac{k_p(\Delta\mu + \mu_0) + \epsilon_-}{k_p(\Delta\mu + \mu_0) - A_+}} \tag{18}$$

Alternatively, substituting eq 15 into eq 17 yields

$$\tau_p = \frac{\theta}{\ln \dfrac{k_p(\Delta\mu - \mu_0) + \epsilon_+}{k_p(\Delta\mu - \mu_0) - A_-}} \tag{19}$$

Therefore, the process time constant can be derived from eq 18 or eq 19. It is preferred in practice to use eq 18 for better accuracy, in view of the fact that the positive fitting part in a half period of the limit cycle occupies a larger percentage compared to the negative fitting part in the other half period.

Hence, the identification algorithm for obtaining a FOPDT stable process model from a biased relay test can be summarized in the following algorithm named Algorithm-1-A1:

(i) Measure P_u and A_+ from the limit cycle.
(ii) Measure the process time delay as $\theta = t_p^*$.
(iii) Compute the process gain, k_p, from eq 13a.
(iv) Compute the process time constant, τ_p, from eq 18.

Remark 2. By using different fitting conditions of the limit cycle, a similar identification algorithm was developed by Wang et al.,[27] which, however, may give a FOPDT model different from the proposed to some extend for a real high-order process, as illustrated by example 3 in section 5.

Note that by substituting the FOPDT model of eq 1 into eq 12, we can obtain the process response fitting conditions at the oscillation frequency as the following

$$\frac{k_p}{\sqrt{\tau_p^{\,2}\omega_u^{\,2} + 1}} = A_u \tag{20}$$

$$-\theta\omega_u - \arctan(\tau_p\omega_u) = \varphi_u \tag{21}$$

It should be noted that $\varphi_u \in (-\pi, -\pi/2)$ rather than $\varphi_u = -\pi$, which had been conventionally used in the describing function analysis.[3,8]

Accordingly, the process time constant and time delay can also be derived from eqs 20 and 21 as

$$\tau_p = \frac{1}{\omega_u}\sqrt{\frac{k_p^{\,2}}{A_u^{\,2}} - 1} \tag{22}$$

$$\theta = -\frac{1}{\omega_u}[\varphi_u + \arctan(\tau_p\omega_u)] \tag{23}$$

Hence, an alternative identification algorithm for obtaining a FOPDT stable process model from a biased relay test can be summarized in the following algorithm named Algorithm-1-A2:

(i) Measure P_u from the limit cycle.
(i) Compute $G(j\omega_u)$ from eq 12.
(ii) Compute the process gain, k_p, from eq 13a.
(iii) Compute the process time constant, τ_p, from eq 22.
(iv) Compute the process time delay, θ, from eq 23.

Remark 3. Because the process response at the oscillation frequency is precisely represented by the FOPDT model derived from Algorithm-1-A2, it may be utilized to obtain enhanced identification accuracy for a real high-order process compared to Algorithm-1-A1, but at the cost of more computation effort. Besides, it should be noted that Shen et al.[26] adopted $\angle G(j\omega_u) = -\pi$ as in the describing function methodology[3] to derive the process time delay from eq 21, which inevitably results in degraded identification accuracy.

For a FOPDT unstable process, it follows from corollary 1 that the process time delay can also be directly measured as the time taken to reach the peak of the process output response from the initial relay switch point in a half period of the relay, i.e., t_p^*, the same as shown in Figure 2. Correspondingly, it can be obtained from eqs 7 and 8 that

$$y_+(P_+) = -k_p(\Delta\mu + \mu_0) + 2k_p\mu_0 E e^{P_+/\tau_p} = A_+ \quad (24)$$

$$y_+(P_+ - \theta) = -k_p(\Delta\mu + \mu_0) + 2k_p\mu_0 E e^{(P_+-\theta)/\tau_p} = -\epsilon_- \quad (25)$$

By substituting eq 24 into eq 25, we obtain

$$\tau_p = \frac{\theta}{\ln \dfrac{k_p(\mu_0 + \Delta\mu) + A_+}{k_p(\mu_0 + \Delta\mu) - \epsilon_-}} \quad (26)$$

Hence, the identification algorithm for obtaining a FOPDT unstable process model from a biased relay test can be summarized in the following algorithm named Algorithm-2-A1:

(i) Measure P_u and A_+ from the limit cycle.

(ii) Measure the process time delay as $\theta = t_p^*$.

(iii) Compute the process gain, k_p, from eq 13b.

(iv) Compute the process time constant, τ_p, from eq 26.

By substituting the FOPDT unstable process model of eq 2 into eq 12, we can derive the process time constant and time delay from the fitting conditions at the oscillation frequency as

$$\tau_p = \frac{1}{\omega_u}\sqrt{\frac{k_p^{\,2}}{A_u^{\,2}} - 1} \quad (27)$$

$$\theta = -\frac{1}{\omega_u}[\varphi_u + \pi - \arctan(\tau_p\omega_u)] \quad (28)$$

Therefore, an alternative identification of a FOPDT unstable process model from a biased relay test can be summarized in the following algorithm named Algorithm-2-A2:

(i) Measure P_u from the limit cycle.

(ii) Compute $G(j\omega_u)$ from eq 12.

(iii) Compute the process gain, k_p, from eq 13b.

(iv) Compute the process time constant, τ_p, from eq 27.

(v) Compute the process time delay, θ, from eq 28.

3.2. Unbiased Relay Test for FOPDT Model. For a FOPDT stable process, it can be verified by substituting $\Delta\mu = 0$ and $P_+ = P_- = P_u/2$ into proposition 1 that the process time delay can also be directly measured as the time taken to reach the peak of the process output response from the initial relay switch point in a half period of the relay, i.e., t_p^*, as shown in Figure 2. Accordingly, solving eqs 14 and 16 together to eliminate k_p yields

$$\epsilon_+(1 - e^{-p_u/(2\tau_p)}) = A_+(1 + e^{-p_u/(2\tau_p)} - 2e^{-(p_u-2\theta)/(2\tau_p)}) \quad (29)$$

Note that the left-hand side of eq 29 is monotonously decreasing with respect to τ_p, and there usually is a physical constraint of $0 < \tau_p < P_u$ in practice. Thus, only finite solutions of τ_p may exist for eq 29, which can be obtained using any iterative algorithm such as the Newton−Raphson method. The initial estimation for iteration may be taken as $\hat{\tau}_p = P_u/2 - \theta$, in view of that the influence of the process time constant to the relay response corresponds to this time. The process gain can then be derived from eq 14 as

$$k_p = \frac{A_+(1 + e^{-p_u/(2\tau_p)})}{\mu_0(1 - e^{-p_u/(2\tau_p)})} \quad (30)$$

Note that the suitable solution pair of τ_p and k_p can be determined by comparing the relay responses of the resulting model and the process, or using the conventional oscillation conditions of the describing function analysis.[3,8]

Therefore, the identification of a FOPDT stable process model from an unbiased relay test can be summarized in the following algorithm named Algorithm-1-B1:

(i) Measure P_u and A_+ from the limit cycle.

(ii) Measure the process time delay as $\theta = t_p^*$.

(iii) Compute the process time constant, τ_p, from eq 29 by using the Newton−Raphson iteration method. The initial estimation for iteration may be taken as $\hat{\tau}_p = P_u/2 - \theta$.

(iv) Compute the process gain, k_p, from eq 30.

(v) Determine the suitable solution pair of τ_p and k_p by comparing the relay responses of the resulting model and the process, or checking if $|N(A_+)\hat{G}(j\omega_u)| \to 1$ and $\angle N(A_+) + \angle\hat{G}(j\omega_u) \to -\pi$ are satisfied, where $N(A_+) = 4\mu_0 e^{-j\arcsin(\epsilon_+/A_+)}/(\pi A_+)$ denotes the describing function of the unbiased relay and $\hat{G}(j\omega_u)$ represents the FOPDT model response at the oscillation frequency.

Remark 4. Based on the general relay function as shown in eq 3, the proposed Algorithm-1-B1 can be used to summarize two similar algorithms given in the recent literature[23,33] for an ideal unbiased relay test with $\epsilon_+ = -\epsilon_- = 0$.

To enhance the fitting effect for a real high-order process, another rigorous fitting condition is proposed herein for identification. Note that the Laplace transform for the relay response can be decomposed as

$$Y(s) = \int_0^{t_{os1}} y(t)e^{-st}\,dt + \int_{t_{os1}}^{\infty} y(t)e^{-st}\,dt \quad (31)$$

where t_{os1} denotes the moment after which $y(t)$ becomes a periodic signal. The second integral in eq 31 can be derived as

$$\begin{aligned}\int_{t_{os1}}^{\infty} y(t)e^{-st}\,dt &= \int_{t_{os1}}^{t_{os1}+P_u} y(t)e^{-st}\,dt + \int_{t_{os1}+P_u}^{t_{os1}+2P_u} y(t)e^{-st}\,dt + \cdots \\ &= (1 + e^{-P_u s} + e^{-2P_u s} + \cdots)\int_{t_{os1}}^{t_{os1}+P_u} y(t)e^{-st}\,dt \\ &= \lim_{M\to\infty}\frac{1 - e^{-MP_u s}}{1 - e^{-P_u s}}\int_{t_{os1}}^{t_{os1}+P_u} y(t)e^{-st}\,dt\end{aligned}$$

Thereby, we can obtain for $Re(s) > 0$ that

$$Y(s) = \int_0^{t_{os1}} y(t)e^{-st}\,dt + \frac{1}{1 - e^{-P_u s}}\int_{t_{os1}}^{t_{os1}+P_u} y(t)e^{-st}\,dt \quad (32)$$

Likewise, the Laplace transform of the relay output for $Re(s) > 0$ can be derived as

$$U(s) = \int_0^{t_{os1}} u(t)e^{-st}\,dt + \frac{1}{1 - e^{-P_u s}}\int_{t_{os1}}^{t_{os1}+P_u} u(t)e^{-st}\,dt \quad (33)$$

Hence, the process transfer function for $Re(s) > 0$ can be obtained as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{(1 - e^{-P_u s})\int_0^{t_{os1}} y(t)e^{-st}\,dt + \int_{t_{os1}}^{t_{os1}+P_u} y(t)e^{-st}\,dt}{(1 - e^{-P_u s})\int_0^{t_{os1}} u(t)e^{-st}\,dt + \int_{t_{os1}}^{t_{os1}+P_u} u(t)e^{-st}\,dt} \quad (34)$$

By substituting $s = \alpha + j\omega_u$ into eq 34, we can obtain

$$G(j\omega_u + \alpha) = \frac{(1 - e^{-\alpha P_u}) \int_0^{t_{os1}} [y(t)e^{-\alpha t}]e^{-j\omega_u t}\, dt + \int_{t_{os1}}^{t_{os1}+P_u} [y(t)e^{-\alpha t}]e^{-j\omega_u t}\, dt}{(1 - e^{-\alpha P_u}) \int_0^{t_{os1}} [u(t)e^{-\alpha t}]e^{-j\omega_u t}\, dt + \int_{t_{os1}}^{t_{os1}+P_u} [u(t)e^{-\alpha t}]e^{-j\omega_u t}\, dt} \tag{35}$$

where $\alpha \in (0,\infty)$ may be viewed as a constant shift operator of Laplace transform. Note that $G(j\omega_u + \alpha) \to 0/0$ as $\alpha \to \infty$. It is therefore suggested to choose α with a numerical constraint of $\min\{|u(t_{os1} + P_u)e^{-\alpha(t_{os1} + P_u)}|, y(t_{os1} + P_u)e^{-\alpha(t_{os1} + P_u)}|\} > 10^{-4}$, such that both the initial and steady relay responses of $y(t)$ and $u(t)$ can be effectively included in the computation of eq 35 while maintaining the difference between $G(j\omega_u + \alpha)$ and $G(j\omega_u)$ for parameter derivation.

For the convenience of computation, it is suggested to choose t_{os1} as a minimal multiple of P_u, i.e., $t_{os1} = N_{os}P_u$, where N_{os} is the minimal integer. Thus, $G(j\omega_u + \alpha)$ can be simply computed as

$$\begin{aligned} G(j\omega_u + \alpha) &= \frac{(1 - e^{-\alpha P_u})\mathrm{FFT}\{y(kT_s)e^{-\alpha kT_s}\}_{I=N_{os}} + \mathrm{FFT}\{y(kT_s)e^{-\alpha kT_s}\}_{I=1}}{(1 - e^{-\alpha P_u})\mathrm{FFT}\{u(kT_s)e^{-\alpha kT_s}\}_{I=N_{os}} + \mathrm{FFT}\{u(kT_s)e^{-\alpha kT_s}\}_{I=1}} \\ &= A_\alpha e^{j\varphi_\alpha} \end{aligned} \tag{36}$$

where $\mathrm{FFT}\{\cdot\}_{I=i}$ denotes the ith element of the resulting FFT for its time series $\{\cdot\}$ ($k = 0, 1, 2, ..., N - 1$), and $T_s = t_{os1}/N$ (or P_u/N for the second FFT in the numerator or the denominator) is the sampling period for FFT computation. The reason lies in

$$\frac{2\pi}{T_s}\frac{N_{os}}{N} = \omega_u \Leftrightarrow N_{os} = \frac{t_{os1}}{P_u} \tag{37}$$

By substituting the FOPDT model of eq 1 into eq 36, we can obtain the fitting condition

$$\frac{k_p e^{-\alpha\theta}}{\sqrt{(\alpha\tau_p + 1)^2 + \tau_p^{\ 2}\omega_u^{\ 2}}} = A_\alpha \tag{38}$$

Consequently, by substituting eqs 22 and 23 into eq 38, we can obtain a transcendental equation with respect to k_p,

$$F_1(k_p) = 0 \tag{39}$$

This equation can be numerically solved using the Newton–Raphson iteration method. The initial value of k_p for iteration may be roughly estimated from the initial process response for a step change of the relay.

The process time constant and time delay can then be derived from eqs 22 and 23, respectively.

Hence, an alternative identification of a FOPDT stable process model from an unbiased relay test can be summarized in the following algorithm named Algorithm-1-B2:

(i) Measure P_u from the limit cycle.

(ii) Compute $G(j\omega_u)$ from eq 12.

(iii) Compute $G(j\omega_u + \alpha)$ from eq 36.

(iv) Compute the process gain, k_p, from eq 39 by using the Newton–Raphson iteration method. The initial value of k_p for iteration may be roughly estimated from the initial step response of the relay test.

(v) Compute the process time constant, τ_p, from eq 22.

(vi) Compute the process time delay, θ, from eq 23.

(vii) End the algorithm if the fitting constraint of relay response, $\sum_{k=1}^{N}[\hat{y}(kT_s + t_{os}) - y(kT_s + t_{os})]^2/N < \epsilon$, is satisfied, where $\hat{y}(kT_s + t_{os})$ and $y(kT_s + t_{os})$ denote respectively the model and process responses in the limit cycle, T_s is the sampling period with $N = P_u/T_s$, and ϵ is a user-specified fitting threshold that may be set between 0.01–1% . Otherwise, change the initial estimation of k_p and go to step iv.

For a FOPDT unstable process, it can be verified by substituting $\Delta\mu = 0$ and $P_+ = P_- = P_u/2$ into corollary 1 that the process time delay can also be directly measured as the time taken to reach the peak of the process output response from the initial relay switch point in a half period of the relay. Accordingly, solving eqs 24 and 25 together to eliminate k_p yields

$$\epsilon_+(1 - e^{-p_u/(2\tau_p)}) = A_+(2e^{-\theta/\tau_p} - e^{-p_u/(2\tau_p)} - 1) \tag{40}$$

Similar to the analysis of eq 29, it can be ascertained that only finite solutions of τ_p may exist for eq 40, which can be solved using the Newton–Raphson iteration method. The process gain can then be derived from eq 24 as

$$k_p = \frac{A_+(e^{p_u/(2\tau_p)} + 1)}{\mu_0(e^{p_u/(2\tau_p)} - 1)} \tag{41}$$

which is equivalent to eq 30.

Therefore, the identification of a FOPDT unstable process model from an unbiased relay test can be summarized in the following algorithm named Algorithm-2-B1:

(i) Measure P_u and A_+ from the limit cycle.

(ii) Measure the process time delay as $\theta = t_p^*$.

(iii) Compute the process time constant, τ_p, from eq 40 by using the Newton–Raphson iteration method. The initial estimation for iteration may be taken as $\hat{\tau}_p = P_u/2 - \theta$.

(iv) Compute the process gain, k_p, from eq 41.

(v) Determine the suitable solution pair of τ_p and k_p by comparing the relay responses of the resulting model and the process, or checking if $|N(A_+)\hat{G}(j\omega_u)| \to 1$ and $\angle N(A_+) + \angle\hat{G}(j\omega_u) \to -\pi$ are satisfied.

By substituting the FOPDT model of eq 2 into the fitting condition of eq 36, we obtain

$$\frac{k_p e^{-\alpha\theta}}{\sqrt{(\alpha\tau_p - 1)^2 + \tau_p^{\ 2}\omega_u^{\ 2}}} = A_\alpha \tag{42}$$

Accordingly, substituting eqs 27 and 28 into eq 42 yields a transcendental equation with respect to k_p,

$$F_2(k_p) = 0 \tag{43}$$

which may be numerically solved using the Newton–Raphson iteration method. The initial value of k_p for iteration may be roughly estimated from the initial step response of the relay test.

The process time constant and time delay can then be derived from eqs 27 and 28, respectively.

Hence, an alternative identification of a FOPDT unstable process model from an unbiased relay test can be summarized in the following algorithm named Algorithm-2-B2:

(i) Measure P_u from the limit cycle.

(ii) Compute $G(j\omega_u)$ from eq 12.

(iii) Compute $G(j\omega_u + \alpha)$ from eq 36.

(iv) Compute the process gain, k_p, from eq 43 by using the Newton–Raphson iteration method. The initial value of k_p for iteration may be roughly estimated from the initial step response of the relay test.

(v) Compute the process time constant, τ_p, from eq 27.

(vi) Compute the process time delay, θ, from eq 28.

(vii) End the algorithm if the fitting constraint of relay response, $\sum_{k=1}^{N}[\hat{y}(kT_s + t_{os}) - y(kT_s + t_{os})]^2/N < \epsilon$, is satisfied. Otherwise, change the initial estimation of k_p and go to step iv.

4. Measurement Noise Issue

It is common to assess identification robustness in the presence of measurement noises. The noise level is usually evaluated as the noise-to-signal ratio (NSR), i.e.,

$$\mathrm{NSR} = \frac{\mathrm{mean(abs(noise))}}{\mathrm{mean(abs(signal))}} \quad (44)$$

For a low noise level, e.g., NSR < 5%, it is suggested that the measured values of P_+, P_-, A_+, and A_- for 10−20 periods in steady oscillation may be respectively averaged as $\overline{P_+}$, $\overline{P_-}$, $\overline{A_+}$, and $\overline{A_-}$ for computation in the proposed identification algorithms. Such an exercise is based on the statistical principle for eliminating the random measurement errors and, thus, is capable of identification robustness for low noise level. It should be noted that the relay output, $u(t)$, remains as a constant for each half periods, and therefore, may be used for measurement of the oscillation period.

To cope with measurement noises causing higher NSR, a low-pass digital Butterworth filter is proposed to recover the corrupted limit cycle data, which may be determined by specifying the filter order, n_f, the cutoff angular frequency, ω_c, i.e.,

$$\mathrm{Butter}(n_f,\omega_c) = \frac{b_1 + b_2 z^{-1} + b_3 z^{-2} + \cdots + b_{n_f+1} z^{-n_f}}{1 + a_2 z^{-1} + a_3 z^{-2} + \cdots + a_{n_f+1} z^{-n_f}} \quad (45)$$

where $\mathrm{Butter}(n_f,\omega_c)$ denotes the filter function with two input parameters of n_f and ω_c. In view of the fact that measurement noises are mainly of high frequency, the guideline for choosing the cutoff angular frequency is given as

$$\omega_c \geq 5\omega_u = \frac{10\pi}{P_u} \quad (46)$$

Thereby, only the signal components within the frequency band around ω_u can be passed through. Note that the phase lag caused by the low-pass filter almost does not affect measurement of the oscillation amplitude and period of the limit cycle, because the relay output has a similar phase lag under the filtered feedback.

5. Illustration

Here, six examples from the recent literature are used to illustrate the effectiveness and merits of the proposed identification algorithms. Examples 1 and 2 are given to demonstrate the achievable accuracy of the proposed algorithms for identifying first-order stable and unstable processes, respectively. Measurement noises are also introduced in example 1 to illustrate identification robustness of the proposed algorithms. Examples 3−5 are given to show the achievable fitting effect of the proposed algorithms for applications to higher-order stable and unstable processes.

Example 1. Consider the first-order process widely studied in the literature:[21,26,29]

Table 1. Identified Models for Six Processes from a Single Relay Feedback Test

		intermediate values of the limit cycle							algorithm-1/2			
process	relay	P_+	P_-	A_+	A_-	A_u	φ_u	A_α	A1	A2	B1	B2
$\frac{(e^{-2s})}{10s+1}$	biased	5.69	9.88	0.3995	−0.2906	0.2405	−2.137		$\frac{1.0001e^{-2s}}{9.9954s+1}$	$\frac{1.0001e^{-2.005s}}{10.001s+1}$		
	unbiased	7.2	7.2	0.3452	−0.3452	0.2234	−2.2203	0.1706			$\frac{1.0052e^{-2s}}{10.0561s+1}$	$\frac{1.0048e^{-2.0024s}}{10.049s+1}$
$\frac{e^{-0.4s}}{s-1}$	biased	1.17	2.25	0.7413	−0.5975	0.4778	−2.8107		$\frac{1.0001e^{-0.4s}}{0.9976s-1}$	$\frac{1.0001e^{-0.4038s}}{1.0009s-1}$		
	unbiased	1.54	1.54	0.6472	−0.6457	0.4396	−2.8514	0.4326			$\frac{0.9877e^{-0.4s}}{0.9815s-1}$	$\frac{0.99e^{-0.4022s}}{0.9891s-1}$
$\frac{(-s+1)e^{-s}}{(s+1)^5}$	biased	6.3	8.08	1.1509	−0.6918	0.7051	−2.9108		$\frac{1.0001e^{-3.53s}}{1.7664s+1}$	$\frac{1.0001e^{-4.8578s}}{2.3017s+1}$		
$\frac{2.4(s+5)e^{-s}}{(s+1)^3(s+3)(s+4)}$	unbiased	4.41	4.41	0.6872	−0.6872	0.5227	−2.8408	0.3754			$\frac{0.9811e^{-1.87s}}{2.54s+1}$	$\frac{1.02e^{-2.538s}}{2.352s+1}$
$\frac{e^{-s}}{8(s+1)^3}$	unbiased	6.32	6.32	0.1143	−0.1143	0.0898	−1.8835	0.0643			$\frac{0.1369e^{-1.28s}}{2.6253s+1}$	$\frac{0.12e^{-2.3301s}}{1.7832s+1}$
$\frac{(e^{-0.5s})}{(2s-1)(0.5s+1)}$	biased	2.9	9.25	0.8581	−0.5451	0.6726	−2.8534		$\frac{1.0001e^{-1.08s}}{2.9336s-1}$	$\frac{1.0001e^{-1.0538s}}{2.1279s-1}$		
	unbiased	4.1	4.07	0.6435	−0.6486	0.5088	−2.9029	0.4907			$\frac{1.0097e^{-1.36s}}{2.74s-1}$	$\frac{0.99e^{-1.0303s}}{2.1703s-1}$

$$G = \frac{e^{-2s}}{10s + 1}$$

By using a biased relay test, Shen et al.[26] derived the process model, $G_m = 0.999e^{-2.005s}/(8.118s + 1)$, and Srinivasan and Chidambaram[29] gave the process model as $G_m = 1.03e^{-2.3s}/(10.3s + 1)$. Vivek and Chidambaram[21] obtained the process model, $G_m = 0.9467e^{-2s}/(9.5028s + 1)$, from an unbiased relay feedback test. By using the proposed Algorithm-1-A1, Algorithm-1-A2, Algorithm-1-B1, and Algorithm-1-B2, the process models obtained from a biased ($u_+ = 1.3$ and $u_- = -0.7$) or unbiased ($u_+ = -u_- = 1.0$) relay test with $\epsilon_+ = -\epsilon_- = 0.2$ and $\alpha = 0.1$ are respectively listed in Table 1, together with the intermediate values of the limit cycle for computation. It can be seen that the proposed algorithms result in good accuracy.

Now suppose that a random noise of $N(0,\sigma_N^2 = 0.0112\%)$ is added to the process output measurement which is used for relay feedback, causing NSR = 5%. It can be seen from Table 2 that by using the statistical averaging method for 10 steady oscillation periods, e.g., in the time interval of [60, 220](*s*), both the proposed Algorithm-1-A1 and Algorithm-1-A2 perform good identification robustness. It should be noted that the similar identification results can also be obtained by the proposed Algorithm-1-B1 and Algorithm-1-B2, and thus are omitted. When the noise level rises to NSR = 10%, it can be seen from Table 2 that the process time constant is somewhat underestimated by Algorithm-1-A1 in comparison with Algorithm-1-A2. To enhance identification accuracy, a first-order Butterworth filter with a cutoff angular frequency $\omega_c = 4.0$ (rad/s) according to the guideline given by eq 46 is used for simplicity to recover the corrupted limit cycle for measurement and relay feedback. The corresponding results listed in Table 2 demonstrate that improved accuracy for the measured parameters and model identification can thus be obtained. Even for the case that a random noise causing NSR = 30% is added, the proposed filter together with the averaging for 10 oscillation periods can ensure the identification robustness, as indicated by the resulting models listed in Table 2.

Example 2. Consider the first-order unstable process studied in the recent literature:[31]

$$G = \frac{e^{-0.4s}}{s - 1}$$

By using two different unbiased relay tests, Marchetti et al.[31] derived the process model, $G_m = 0.928e^{-0.392s}/(0.757s - 1)$, indicating enhanced identification accuracy when compared to Shen et al.[26] Using the proposed algorithms respectively based on a biased ($u_+ = 1.2$ and $u_- = -0.9$) or unbiased ($u_+ = -u_- = 1.0$) relay test with $\epsilon_+ = -\epsilon_- = 0.1$ and $\alpha = 0.1$, the resulting

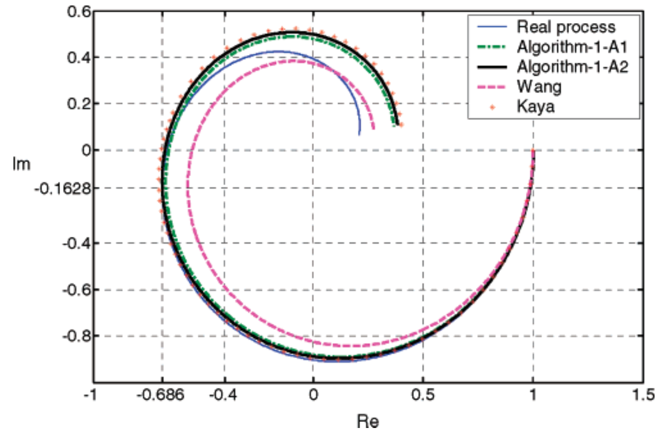

Figure 3. Nyquist fitting of FOPDT models for example 3.

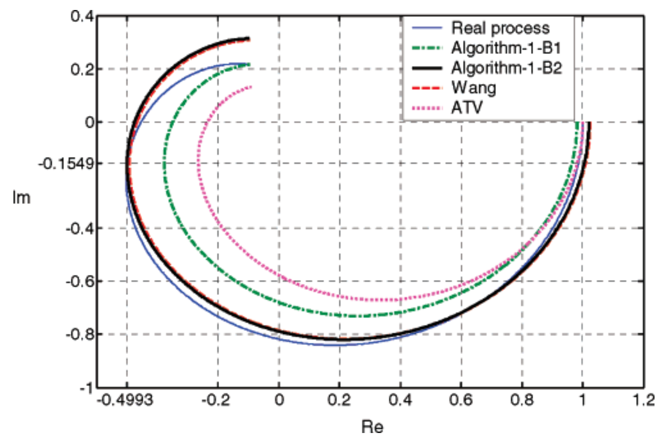

Figure 4. Nyquist fitting of FOPDT models for example 4.

models are listed in Table 1. It is again seen that the proposed algorithms result in good accuracy.

Example 3. Consider the fifth-order process studied in the recent literature:[27,28]

$$G = \frac{(-s + 1)e^{-s}}{(s + 1)^5}$$

By using a biased relay test, Wang et al.[27] derived a FOPDT model, $G_m = 1.00e^{-4.24s}/(2.99s + 1)$, and Kaya and Atherton[28] gave a FOPDT model, $G_m = 1.00e^{-5.082s}/(2.292s + 1)$, both of which had shown the superiority over other relay identification methods. For comparison, the proposed Algorithm-1-A1 and Algorithm-1-A2 are used by performing a biased relay test as in example 1, resulting in the FOPDT models listed in Table 1. The Nyquist plots of these FOPDT models are shown in Figure 3. It can be seen that the proposed Algorithm-1-A2 yields the

Table 2. Denoising Effects for Identification against Measurement Noises

		% error in the intermediate values				identified model	
NSR	denoising	A_+	A_-	A_u	φ_u	Algorithm-1-A1	Algorithm-1-A2
5%	averaging	1.5	−0.37	−3.89	−2.08	$\frac{0.9939e^{-1.994s}}{9.5672s + 1}$	$\frac{0.9939e^{-2.0092s}}{9.9402s + 1}$
10%	averaging	2.74	−2.36	−9.28	−5.32	$\frac{1.0236e^{-1.928s}}{9.3637s + 1}$	$\frac{1.0236e^{-1.9971s}}{10.2331s + 1}$
10%	filtering	1.72	−0.96	4.19	−2.05	$\frac{1.0108e^{-2.134s}}{10.4172s + 1}$	$\frac{1.0108e^{-2.2417s}}{10.1769s + 1}$
30%	filtering	−0.91	2.78	2.71	−2.68	$\frac{1.0292e^{-2.222s}}{10.8582s + 1}$	$\frac{1.0292e^{-2.2211s}}{10.3754s + 1}$

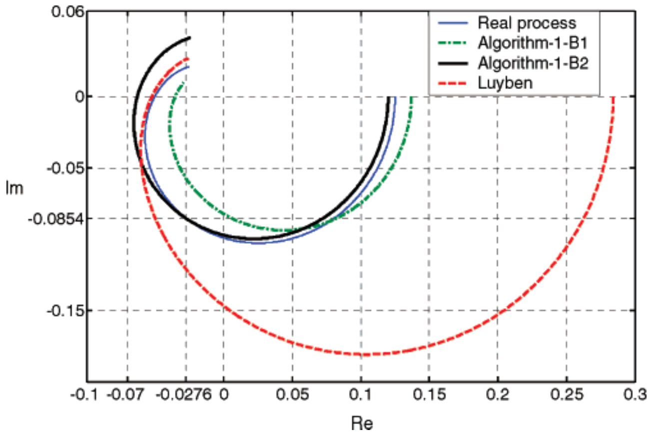

Figure 5. Nyquist fitting of FOPDT models for example 5.

best fitting effect, owing to using the precise fitting condition of the process response at the oscillation frequency, i.e., (−0.686, −j0.1628), as shown in Figure 3. The widely used fitting error of frequency response, $E = \max_{\omega\in[0,\omega_c]} \{|[G_m(j\omega) - G(j\omega)]/G(j\omega)|\}$, may be employed to quantitatively evaluate the fitting effect, where ω_c is the cutoff angular frequency corresponding to $\angle G(j\omega_c) = -\pi$. It can be verified that the FOPDT model obtained from the proposed Algorithm-1-A2 results in $E = 2.71\%$, which is much smaller than those of the other methods. The proposed Algorithm-1-A1 yields slightly inferior fitting, compared to Algorithm-1-A2, but with less computation effort.

Example 4. Consider the high-order process studied in the recent literature:[38]

$$G = \frac{2.4(s+5)e^{-s}}{(s+1)^3(s+3)(s+4)}$$

On the basis of a step test, Wang et al.[38] derived a FOPDT model, $G_m = 1.0239e^{-2.3602s}/(2.2548s + 1)$, which had demonstrated its superiority over the widely recognized area-based or graphical methods.[39] With the knowledge of the process gain obtained exactly from step tests, the well-known ATV method[2] may give a FOPDT model, $G_m = 1.0e^{-s}/(2.2892s + 1)$. By performing an unbiased relay test as in example 1, the proposed Algorithm-1-B1 and Algorithm-1-B2 result in the FOPDT models listed in Table 1. The Nyquist plots of these FOPDT models are shown in Figure 4. It can be seen that the proposed Algorithm-1-B2 yields the best fitting owing to using the precise fitting condition at the oscillation frequency, i.e., (−0.4993, −j0.1549), as shown in Figure 4.

Example 5. Consider the third-order process studied in the recent literature:[20]

$$G = \frac{e^{-s}}{8(s+1)^3}$$

On the basis of an unbiased relay test, Luyben[20] presented a FOPDT model, $G_m = 0.284e^{-1.96s}/(5.97s + 1)$. Correspondingly, the proposed Algorithm-1-B1 and Algorithm-1-B2 are used to obtain a FOPDT process model from an unbiased relay test as in example 2. The identification results are shown in Table 1. The Nyquist plots of these FOPDT models are shown in Figure 5. It is once again seen that improved fitting is captured by the proposed algorithms. The proposed Algorithm-1-B2 yields the best fitting, in particular for the referred low-frequency range, owing to that the model response coincides with the process at the oscillation frequency, i.e., (−0.0276, −j0.0854).

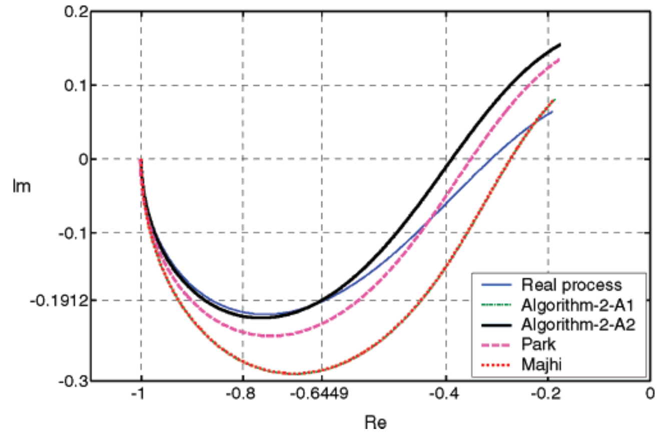

Figure 6. Nyquist fitting of FOPDT models for example 6.

Example 6. Consider the second-order unstable process studied in the recent literature:[22,25,30]

$$G = \frac{e^{-0.5s}}{(2s-1)(0.5s+1)}$$

From an unbiased relay test, Vivek and Chidambaram[22] derived a FOPDT model, $G_m = 0.7534e^{-1.0412s}/(2.1642s - 1)$. From a biased relay test, Park et al.[30] gave a FOPDT model, $G_m = 1.002e^{-1.067s}/(2.347s - 1)$. By using a modified relay structure, Majhi and Atherton[25] obtained a FOPDT model, $G_m = 1.0e^{-1.061s}/(2.875s - 1)$, with a prior knowledge of the process gain. For comparison, the proposed algorithms (2-A1, 2-A2, 2-B1, and 2-B2) are respectively used to obtain the FOPDT models listed in Table 1, from a single biased/unbiased relay test as in example 2. The Nyquist plots of the FOPDT models obtained from the proposed algorithms (2-A1 and 2-A2) and the work of Park et al.[30] and Majhi and Atherton[25] are shown in Figure 6. It can be seen that apparently improved fitting is captured by the proposed Algorithm-2-A2 over the referred low-frequency range, owing to using the precise fitting condition at the oscillation frequency, i.e., (−0.6449, −j0.1912). The proposed Algorithm-2-A1 results in inferior Nyquist fitting compared to Algorithm-2-A2 but is similar to the work of Majhi and Atherton.[25] For using unbiased relay test, the proposed Algorithm-2-B1 and Algorithm-2-B2 give evidently improved identification of the process gain compared to the work of Vivek and Chidambaram,[22] and it can be easily verified through the Nyquist plot that the proposed Algorithm-2-B2 gives the best fitting for the process frequency response.

6. Conclusions

Four alternative identification algorithms have been proposed for obtaining a FOPDT model for a stable process, while four alternative FOPDT identification algorithms for unstable processes. The general relay response expressions of a FOPDT stable/unstable process have been derived, which may be used for assessing the fluctuation range of process response under relay feedback. Through applications to the first-order and higher-order processes studied in the recent literature, it has been demonstrated that all the proposed algorithms can give good accuracy in the sense that the model structure matches well with the process to be identified. For higher-order processes, the proposed algorithms (1-A2, 1-B2, 2-A2, and 2-B2) with more computation effort can obtain enhanced identification accuracy compared to the other algorithms for implementation simplicity. Identification robustness of the proposed algorithms have been illustrated through measurement noise tests, and the proposed denoising method based on a low-pass Butterworth filter has demonstrated good performance for recovering the corrupted limit cycle from measurement noises causing high NSR.

Acknowledgment

This work is supported in part by the Hong Kong Research Grants Council under Project No. 612906.

Appendix. Proof of Proposition 1

The initial step response of a FOPDT stable process arising from the relay output $\Delta\mu - \mu_0$ can be derived as

$$y_0(t) = k_p(\Delta\mu - \mu_0)(1 - e^{-(t-\theta)/\tau_p}) \quad (A1)$$

When it comes to the first relay switch point denoted by t_0 as shown in Figure 2, the relay output changes to $\Delta\mu + \mu_0$, indicating that a step change of $2\mu_0$ is added to the process input. Using the linear superposition principle, the process output response can be derived as

$$y_1(t) = y_0(t + t_0) + 2k_p\mu_0(1 - e^{-(t-\theta)/\tau_p}) \quad (A2)$$

By using a time shift of $t_0 + \theta$, eq A2 can be rewritten as

$$y_1|_{\text{shift}}(t) = y_0(t + t_0 + \theta) + 2k_p\mu_0 - 2k_p\mu_0 e^{-t/\tau_p} \quad (A3)$$

When it comes to the second relay switch point, the relay output changes to $\Delta\mu - \mu_0$, indicating that a step change of $-2\mu_0$ is added to the process input. Using the linear superposition principle, the process output response can be derived as

$$y_2(t) = y_1(t + P_+) - 2k_p\mu_0(1 - e^{-(t-\theta)/\tau_p}) \quad (A4)$$

By using a time shift of $t_0 + \theta + P_+$, eq A4 can be rewritten as

$$y_2|_{\text{shift}}(t) = y_0(t + t_0 + \theta + P_+) + 2k_p\mu_0(1 - 1) - 2k_p\mu_0 e^{-t/\tau_p}(e^{-P_+/\tau_p} - 1) \quad (A5)$$

At the third relay switch point, the relay output changes again to $\Delta\mu + \mu_0$, indicating that a step change of $2\mu_0$ is once again added to the process input. The process output response with a time shift of $t_0 + \theta + P_u$ can be derived as

$$y_3|_{\text{shift}}(t) = y_0(t + t_0 + \theta + P_u) + 2k_p\mu_0(1 - 1 + 1) - 2k_p\mu_0 e^{-t/\tau_p}(e^{-P_u/\tau_p} - e^{-P_-/\tau_p} + 1) \quad (A6)$$

The process output response following the fourth relay switch point is the result of four interlaced step changes respectively with a magnitude of $2\mu_0$. The process output response with a time shift of $t_0 + \theta + P_u + P_+$ can thus be derived as

$$y_4|_{\text{shift}}(t) = y_0(t + t_0 + \theta + P_u + P_+) + 2k_p\mu_0(1 - 1 + 1 - 1) - 2k_p\mu_0 e^{-t/\tau_p}(e^{-(P_u+P_+)/\tau_p} - e^{-P_u/\tau_p} + e^{-P_+/\tau_p} - 1) \quad (A7)$$

Hence, the time shifted process output response after each relay switch points can be summarized as

$$y_{2n+1}|_{\text{shift}}(t) = y_0(t + t_0 + \theta + nP_u) + 2k_p\mu_0 - 2k_p\mu_0 E e^{-t/\tau_p} \quad (A8)$$

$$y_{2n+2}|_{\text{shift}}(t) = y_0(t + t_0 + \theta + nP_u + P_+) - 2k_p\mu_0 F e^{-t/\tau_p} \quad (A9)$$

where $n = 0, 1, 2,...$ and

$$E = 1 + \sum_{k=1}^{n}(e^{-kP_u/\tau_p} - e^{-[(k-1)P_u+P_-]/\tau_p}) \quad (A10)$$

$$F = \sum_{k=0}^{n}(e^{-(kP_u+P_+)/\tau_p} - e^{-kP_u/\tau_p}) \quad (A11)$$

Note that $0 < e^{-P_u/\tau_p} < 1$. It follows as $n \to \infty$ that

$$\sum_{k=0}^{n} e^{-kP_u/\tau_p} = \frac{1}{1 - e^{-P_u/\tau_p}} \quad (A12)$$

Substituting eq A12 into eqs A10 and A11, respectively, we can obtain

$$E = \frac{1}{1 - e^{-p_u/\tau_p}} - \frac{e^{-p_-/\tau_p}}{1 - e^{-p_u/\tau_p}} = \frac{1 - e^{-p_-/\tau_p}}{1 - e^{-p_u/\tau_p}} \quad (A13)$$

$$F = (e^{-p_+/\tau_p} - 1)\frac{1}{1 - e^{-p_u/\tau_p}} = -\frac{1 - e^{-p_+/\tau_p}}{1 - e^{-p_u/\tau_p}} \quad (A14)$$

It can be seen from eq A1 that $y_0(t + t_0 + \theta + nP_u) = y_0(t + t_0 + \theta + nP_u + P_+) = k_p(\Delta\mu - \mu_0)$ as $n \to \infty$. Hence, in the limit cycle, it follows that

$$y_+(t) = \lim_{n\to\infty} y_{2n+1}|_{\text{shift}}(t) = k_p(\Delta\mu + \mu_0) - 2k_p\mu_0 E e^{-t/\tau_p}, \quad t \in [0, P_+] \quad (A15)$$

$$y_-(t) = \lim_{n\to\infty} y_{2n+2}|_{\text{shift}}(t) = k_p(\Delta\mu - \mu_0) - 2k_p\mu_0 F e^{-t/\tau_p}, \quad t \in (P_+, P_u] \quad (A16)$$

In view of the fact that $E > 0$, $F < 0$, and e^{-t/τ_p} decreases monotonously with respect to t, we can conclude from eqs A15 and A16 that $y_+(t)$ increases monotonously for $t \in [0, P_+]$, and $y_-(t)$ decreases monotonously for $t \in (P_+, P_u]$.

From the above derivation for eqs A15 and A16, it can be seen that the limit cycle can be definitely formed for a FOPDT stable process. It should be noted that before the process output response moves into the limit cycle, the time intervals between each relay switch points may not equal the corresponding half periods in the limit cycle. Nevertheless, we can equalize them as done in eqs A4−A7 to derive the exact expression of the limit cycle, in view of the fact that the limit cycle does not include these initial response differences. In other words, the process response in steady oscillation has the same limit cycle with that of the corresponding ideal oscillation which has identical time intervals between the sequential relay switch points from beginning to end.

Literature Cited

(1) Åström, K. J.; Hägglund, T. Automatic tuning of simple regulators with specification on phase angle and amplitude margins. *Automatica* **1984**, *20*, 645−651.

(2) Luyben, W. L. Derivation of transfer functions for highly nonlinear distillation columns. *Ind. Eng. Chem. Res.* **1987**, *26*, 2490−2495.

(3) Atherton, D. P. Relay Autotuning: An overview and alternative approach. *Ind. Eng. Chem. Res.* **2006**, *45*, 4075−4080.

(4) Hang, C. C.; Åström, K. J.; Wang, Q. G. Relay feedback auto-tuning of process controllers−A tutorial review. *J. Process Control* **2002**, *12*, 143−163.

(5) Chang, R. C.; Shen, S. H.; Yu, C. C. Derivation of transfer function from relay feedback systems. *Ind. Eng. Chem. Res.* **1992**, *31*, 855−860.

(6) Tan, K. K.; Lee, T. H.; Wang, Q. G. Enhanced automatic tuning procedures for process control of PI/PID controllers. *AIChE J.* **1996**, *42*, 2555−2562.

(7) Wang, Y. G.; Shao, H. H. PID autotuner based on gain- and phase-margin specifications. *Ind. Eng. Chem. Res.* **1999**, *38* (8), 3007−3012.

(8) Yu, C. C. *Autotuning of PID Controllers: Relay Feedback Approach*, 2nd ed.; Springer-Verlag: London, 2006.

(9) Wang, P.; Gu, D. Y.; Zhang, W. D. Modified relay feedback identification based on describing function analysis. *Ind. Eng. Chem. Res.* **2007**, *46* (5), 1538−1546.

(10) Sung, S. W.; Lee, I. B. Enhanced relay feedback method. *Ind. Eng. Chem. Res.* **1997**, *36* (12), 5526−5530.

(11) Wang, Q. G.; Hang, C. C.; Bi, Q. A technique for frequency response identification from relay feedback. *IEEE Trans. Control Syst. Technol.* **1999**, *7* (1), 122−128.

(12) Scali, C.; Marchetti, G.; Semino, D. Relay with additional delay for identification and autotuning of completely unknown processes. *Ind. Eng. Chem. Res.* **1999**, *38*, 1987−1997.

(13) Arruda, G. H. M.; Barros, P. R. Relay-based closed loop transfer function frequency points estimation. *Automatica* **2003**, *39*, 309−315.

(14) Loh, A. P.; Cai, X.; Tan, W. W. Auto-tuning of phase lead lag compensators. *Automatica* **2004**, *40*, 423−429.

(15) Sung, S. W.; Lee, I. B; Lee, B. K. On-line process identification and automatic tuning method for PID controllers. *Chem. Eng. Sci.* **1998**, *53* (10), 1847−1859.

(16) Tan, K. K.; Lee, T. H.; Jiang, X. On-line relay identification, assessment and tuning of PID controller. *J. Process Control* **2001**, *11*, 483−496.

(17) Kaya, I. IMC based automatic tuning method for PID controllers in a Smith predictor configuration. *Comput. Chem. Eng.* **2004**, *28*, 281−290.

(18) Leva, A. Model-based proportional-integral-derivative autotuning improved with relay feedback identification. *IEE Control Theory Appl.* **2005**, *152* (2), 247−256.

(19) Liu, T.; Zhang, W. D.; Gu, D. Y. New IMC-based control strategy for open-loop unstable cascade processes. *Ind. Eng. Chem. Res.* **2005**, *44* (4), 900−909.

(20) Luyben, W. L. Getting more information from relay feedback tests. *Ind. Eng. Chem. Res.* **2001**, *40*, 4391−4402.

(21) Vivek, S.; Chidambaram, M. Identification using single symmetrical relay feedback test. *Comput. Chem. Eng.* **2005**, *29*, 1625−1630.

(22) Vivek, S.; Chidambaram, M. An improved relay auto tuning of PID controllers for unstable FOPTD systems. *Comput. Chem. Eng.* **2005**, *29*, 2060−2068.

(23) Panda, R. C.; Yu, C. C. Shape factor of relay response curves and its use in autotuning. *J. Process Control* **2005**, *15*, 893−906.

(24) Huang, H. P.; Jeng, J. C.; Luo, K. Y. Auto-tune system using single-run relay feedback test and model-based controller design. *J. Process Control* **2005**, *15*, 713−727.

(25) Majhi, S.; Atherton, D. P. Online tuning of controllers for an unstable FOPDT process. *IEE Control Theory Appl.* **2000**, *147* (4), 421−427.

(26) Shen, S. H.; Wu, J. S.; Yu, C. C. Use of biased-relay feedback for system identification. *AIChE J.* **1996**, *42*, 1174−1180.

(27) Wang, Q. G.; Hang, C. C.; Zou, B. Low-order modelling from relay feedback. *Ind. Eng. Chem. Res.* **1997**, *36*, 375−381.

(28) Kaya, I.; Atherton, D. P. Parameter estimation from relay autotuning with asymmetric limit cycle data. *J. Process Control* **2001**, *11*, 429−439.

(29) Srinivasan, K.; Chidambaram, M. Modified relay feedback method for improved system identification. *Comput. Chem. Eng.* **2003**, *27*, 727−732.

(30) Park, J. H.; Sung, S. W.; Lee, I. B. An enhanced PID control strategy for unstable processes. *Automatica* **1998**, *34* (6), 751−756.

(31) Marchetti, G.; Scali, C.; Lewin, D. R. Identification and control of open-loop unstable processes by relay methods. *Automatica* **2001**, *37*, 2049−2055.

(32) Ramakrishnan, V.; Chidambaram, M. Estimation of a SOPTD transfer function model using a single asymmetrical relay feedback test. *Comput. Chem. Eng.* **2003**, *27*, 1779−1784.

(33) Thyagarajan, T.; Yu, C. C. Improved autotuning using the shape factor from feedback. *Ind. Eng. Chem. Res.* **2003**, *42*, 4425−4440.

(34) Wang, Q. G.; Lee, T. H.; Lin, C. *Relay Feedback: Analysis, Identification and Control*; Springer-Verlag: London, 2003.

(35) Thyagarajan, T.; Yu, C. C.; Huang, H. P. Assessment of controller performance: A relay feedback approach. *Chem. Eng. Sci.* **2003**, *58*, 497−512.

(36) Panda, R. C.; Yu, C. C. Analytical expressions for relay feed back responses. *J. Process Control* **2003**, *13*, 489−501.

(37) Lin, C.; Wang, Q. G.; Lee, T. H. Relay feedback: A complete analysis for first-order systems. *Ind. Eng. Chem. Res.* **2004**, *43*, 8400−8402.

(38) Wang, Q. G.; Guo, X.; Zhang, Y. Direct identification of continuous time delay systems from step responses. *J. Process Control* **2001**, *11*, 531−542.

(39) Åström, K. J.; Hägglund, T. *PID Controllers: Theory, Design, and Tuning*, 2nd ed.; Instrument Society of America: Research Triangle Park, NC, 1995.

Received for review June 22, 2007
Revised manuscript received November 2, 2007
Accepted November 2, 2007

IE070856D