S. K. FRIEDLANDER
The Johns Hopkins University, Baltimore, Md.

Theory of Aerosol Filtration

A method based on an analysis of the diffusion equation will help in filter design and correlation of experimental data

FIBROUS filters operated at low velocities are often used to separate particulate matter from gases. The principal criteria for measuring performance are pressure drop and efficiency of removal. Pressure drop can be predicted from correlations such as those proposed by Wong, Ranz, and Johnstone (*23*) and Chen (*4*). But in spite of careful experimental investigations of filtration efficiency (*4, 14, 18, 23*) and theoretical analyses (*5, 15*), no satisfactory general correlation for efficiency as a function of the several dimensionless groups describing the filtration process has been developed. In this report, the theory is reviewed and extended; a design correlation is derived and compared with experimental literature data.

The most common theoretical model of the fibrous filter is an array of cylinders set transverse to the aerosol flow; there has been some effort to study this model experimentally (*9*). Such a picture is an oversimplification, as it permits only a limited accounting for interaction among the fibers and the random nature of the system. In practice, the efficiency so defined is an effective value determined experimentally.

In an array of fibers with diameter d_F, and a fraction solids, α, at any point, the concentration of the aerosol passing through is N. For a single fiber, the removal efficiency is defined as

$$\eta = \frac{b}{d_F} \qquad (1)$$

where b is the width which corresponds to a region of flow completely cleared of all particles by the cylinder. In a differential distance, dh, in the flow direction, per unit width there are $\alpha dh/(\pi d_F^2/4)$ fibers; the removal over this distance by each fiber is:

$$-\frac{dN}{\frac{\alpha\, dh}{\pi d_F^2/4}} = bN = \eta\, d_F N \qquad (2)$$

Rearranging and integrating from $h = 0$ to $h = H$:

$$\eta = \frac{\pi d_F}{4H\alpha} \ln \frac{N_1}{N_2} \qquad (3)$$

In an experimental determination of η, the practice is to measure N_1 and N_2, the inlet and outlet concentrations of a homogeneous aerosol passed through the filter. An average fiber diameter, d_F, is determined by microscopic examination. The value of η so determined is a function of time, although little information on the relationship is available (*14*). Most of the data refer to initial efficiencies of fresh filters.

The central problem of filtration theory is prediction of η from the characteristics of the aerosol and filter. In the absence of gravitational and electrical influences, it is assumed that η depends on impaction or impingement, diffusion, and direct interception. Sieving, or removal by passages smaller than the particle diameter, is not important in fibrous filters, which are usually very porous, but may be more important for dense paper or cloth filters. Varying the porosity seems to affect, principally, velocity distribution in the neighborhood of the fibers. Wong, Ranz, and Johnstone (*23*) found little effect of fraction solids on η in the range $0.045 < \alpha < 0.098$; Chen (*4*) noted some variation with α and correlated his data by a linear relationship:

$$\eta_\alpha = \eta_0 (1 + 4.5\,\alpha) \qquad (4)$$

Up to the present, no effort has been made to take into account surface effects such as re-entrainment or rebound. Little is known about these phenomena (*10, 11*) and it is usually assumed that the concentration of particles at the fiber surface is 0. It might be possible to allow for these effects by postulating the existence of a "back-pressure." Here, however, it is assumed that surface concentration vanishes.

Impaction

"Impaction" describes deposition due to changes in direction of the gas. Heavy particles which cannot follow the motion of the carrier fluid collide with obstructing surfaces. The first quantitative studies of impaction were made by Sell (*20*), who assumed the applicability of Stokes' law and wrote a force balance on a particle in the form:

$$m\, d\vec{u}'/dt' = f(\vec{u'_F} - \vec{u'}) \qquad (5)$$

or in dimensionless terms:

$$2N_I\, d\vec{u}/dt + \vec{u} = \vec{u_F} \qquad (5a)$$

The group $N_I = mV/fd_F$, the "impaction parameter," represents the ratio of the distance moved by a particle injected into a stagnant gas with velocity V to the diameter of the cylinder. For a given fluid velocity distribution around the cylinder, it is possible (in principle) to solve for the path of a particle. The impaction efficiency can be obtained by determining the particle path just tangent to the cylinder at a given N_I. Several solutions (*1, 6, 16, 20*) for the

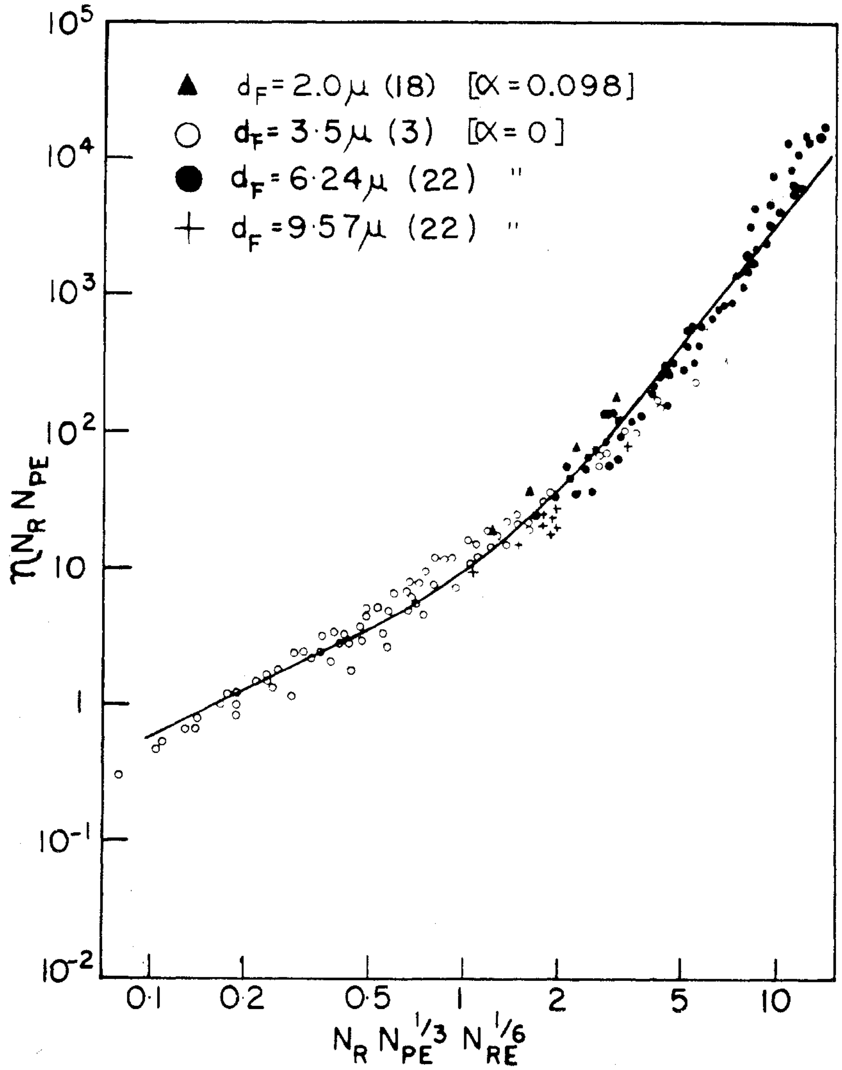

Figure 1. Recalculated data show a single-valued function with a slope of 3 at large values of $N_R N_{PE}^{1/3} N_{RE}^{1/6}$ and 1.2 for low values of parameter

case of potential flow around the cylinder are of particular value in calculating high velocity impaction, which occurs in the icing of airplane wings, and in predicting performance of viscous filters at high velocities. Davies (*5, 6*) has pointed out that, because dry fibrous filters are ordinarily operated at low velocities, the appropriate fluid velocity distributions are obtained from the laminar approximations of the equations of fluid motion.

In potential flow, the velocity near the stagnation point of the cylinder is proportional to the distance from the surface; Equation 5 when written for this region becomes linear, and physically meaningful solutions can be obtained only for values of $N_I > 1/8$ (*6, 16*). Thus for potential flow this value of N_I usually represents a minimum, below which no removal by impaction can be obtained. Davies (*6*) has indicated that minimum values for the impaction parameter also exist for viscous flow. However, a cutoff at a minimum N_I has not been experimentally verified, although impaction efficiency appears to decrease markedly in this region.

As the particles are not point masses but have a finite size, their path need take them only to within one radius of the surface for deposition to occur. This effect, "direct interception," is usually expressed in terms of the boundary conditions of the problem in the case of both impaction and diffusion, and is not reflected in the differential equations.

Diffusion

Particles smaller than about 1 micron exhibit Brownian movement, which in the neighborhood of a surface results in diffusion and deposition. Under certain limiting conditions (including absence of an external force field) distribution of the particles is determined by the diffusion equation:

$$\frac{dc'}{dt'} = D_{BM}\nabla^2 c' \qquad (6)$$

or in dimensionless terms:

$$\frac{dc}{dt} = \frac{2}{N_{PE}}\nabla^2 c \qquad (6a)$$

The group $N_{PE} = d_F V/D_{BM}$, the Peclet number, is a measure of the ratio of transport by convective forces to transport by molecular diffusion. Equation 6 is identical with the expressions describing diffusion or heat transfer in flowing fluids. Thus similar solutions can be expected for aerosol diffusion, except for the different boundary condition imposed by the finite radius of the particles, the direct interception effect. Solutions to Equation 6 depend on the velocity distribution in the neighborhood of the cylinder. There is, however, no counterpart to the lower cutoff point characterized by the critical N_I.

The importance of the diffusion mechanism in filtration was first emphasized by Langmuir (*16*), who attempted to predict filtration efficiencies by calculating rates of diffusion to single cylinders using a viscous flow velocity distribution, on the basis of an arbitrary assumption of contact time between aerosol and cylinder. A different approach (*8*) applied boundary layer theory to solve the flow diffusion equation, using the velocity distribution of Lamb for laminar flow. The theoretical results agreed well with experimental data for heat and mass transfer to single cylinders in liquids at low Reynolds numbers. (The characteristic parameters of the problem, Reynolds and Schmidt numbers, are similar for aerosol diffusion and transfer in liquids at low N_{RE}).

Modified Theory

Fundamental Equation. Mass transfer rates in packed beds depend on the Reynolds and Schmidt (or Peclet) numbers (*17*). For aerosol transfer, however, the situation is complicated by impaction and direct interception, which introduce parameters N_I and N_R. The principal difficulty in developing a filtration theory lies in establishing the nature of the interaction among the profusion of mechanisms.

To determine this interaction, we begin with the basic equation of motion for a particle undergoing Brownian movement in a gas (*2*):

$$m\frac{d\vec{u}'}{dt'} = f(\vec{u}_F' - \vec{u}') + \vec{K}(t') \qquad (7)$$

The net force acting on a particle is composed of the Stokes' resistance term, $f(\vec{u}_F' - \vec{u}')$, and the random time-dependent force, $\vec{K}(t')$, resulting from molecular bombardment. This expression is based on rather sweeping assumptions: The resistance term is derived for the nonaccelerating motion of a continuous, infinite fluid past a single sphere; the random acceleration depends on a kinetic picture of discrete impacts by what must be a discontinuous fluid. Nevertheless, results predicted from theory based on this approach have been confirmed experimentally. This constitutes the principal justification for its use.

Qualitatively, interesting facts emerge from a consideration of Equation 7. It differs from the usual force balance written on a particle (Equation 5), by the term $\vec{K}(t')$. This term becomes of increasing importance as the size of the particles decreases; mathematically, it represents the random force which produces the Brownian movement and diffusion. Thus the problem assumes a stochastic character and it is necessary to consider distributions of velocity and displacement among large numbers of particles. In the case of transfer near the stagnation point of a cylinder, there is some probability of finding a particle at the surface. Thus it appears that a minimum impaction parameter corresponding to zero efficiency does not exist. This conclusion is not in disagreement with experiment.

By subtracting $m d\vec{u}'_F/dt'$ from both sides of Equation 7, and letting the velocity of the particle relative to the fluid, $\vec{u}'_R = \vec{u}' - \vec{u}'_F$, the following expression is obtained:

$$m \frac{d\vec{u}'_R}{dt'} = -f\vec{u}'_R + \vec{K}(t') - m \frac{d\vec{u}'_F}{dt'} \quad (8)$$

This is the force balance written in a frame of reference moving with the velocity of the fluid in the neighborhood of the particle. The first term on the right represents the frictional resistance and the second the fluctuating force due to molecular bombardment. The third is a pseudoforce resulting from transformation to an accelerating reference frame. $d\vec{u}'_F/dt'$ represents the time rate of change of the velocity near the particle. As the particle does not follow the fluid motion exactly, $d\vec{u}'_F/dt'$ is not equal to the fluid acceleration; it approaches the fluid acceleration for very small particles. Under certain limiting conditions (*2, 21*) it is possible to pass from a force balance of this type to the following partial differential equation:

$$\frac{dc'}{dt'} = D_{\mathrm{BM}}\nabla^2 c' + \frac{1}{\beta}\nabla \cdot \frac{d\vec{u}'_F}{dt'} c' \quad (9)$$

often called the Smoluchowski equation. In dimensionless terms, it can be written:

$$\frac{dc}{dt} = \frac{2}{N_{\mathrm{PE}}}\nabla^2 c + 2N_I \nabla \cdot \frac{d\vec{u}_F}{dt} c \quad (10)$$

The principal limitation on passage from the force balance to the Smoluchowski equation is that $d\vec{u}'_F/dt'$ changes little during a time interval of the order of β^{-1}. This allows the particles to maintain a Gaussian (relative) velocity distribution and makes it possible to treat the problem with diffusion theory.

The time interval of interest (in which $d\vec{u}'_F/dt'$ changes) for flow past a cylinder is of the order d_F/V. Thus the requirement can be stated as $d_F/V >> \beta^{-1}$ or $1 >> V/\beta d_F = N_I$. In other words, the Smoluchowski equation can be applied to the fiber model, if the impaction parameter is much less than unity. This requirement is usually fulfilled for fibrous filters operated at low velocities to remove submicron particles. Equation 10 describes interaction among the diffusion, impaction, and (with appropriate boundary conditions) direct interception mechanisms. It incorporates the force balance on the individual particle.

New Method Of Correlation. Equation 10 is too complex for a general solution. Simplification is effected by assuming that at sufficiently low values of N_I the second term on the right can be neglected. How small N_I must be to allow this simplification will be best established by comparison with experimental data.

Without the impaction terms, Equation 10 reduces to Equation 6, the diffusion equation for a moving fluid. With particles of finite diameter, there is the complication of a boundary condition which depends on particle size:

$$\text{at} \quad r = 1 + N_R,\ c = 0 \quad (11)$$
$$r = \infty,\ c = 1$$

For the case of pure diffusion in a laminar flow field, an expression has been derived for efficiency of removal as a function of the Schmidt and Reynolds numbers (*8*) using the simplifying concept of the boundary layer (*19*). This approach is based on the existence of a narrow region near the surface, where the concentration drops from the main-stream value to that of the wall. In determining transfer rates, it is necessary to consider only what is occurring in the immediate neighborhood of the surface. Such an approach is usually valid when the Peclet number is large—i.e., when the rate of transfer by diffusion is small compared with convective transfer, as in transfer in flowing liquids and aerosols.

In cylindrical coordinates the steady-state flow-diffusion equation can be written:

$$v_r' \frac{\partial c'}{\partial r'} + v_\theta' \frac{\partial c'}{r'\partial\theta} = D\left(\frac{\partial^2 c'}{\partial r'^2} + \frac{1}{r'}\frac{\partial c'}{\partial r'} + \frac{\partial^2 c'}{r'^2\partial\theta^2}\right) \quad (12)$$

which simplifies to

$$v_r \frac{\partial c}{\partial r} + v_\theta \frac{\partial c}{\partial\theta} = \frac{2}{N_{\mathrm{PE}}}\frac{\partial^2 c}{\partial r^2} \quad (13)$$

written in the dimensionless boundary layer form (*7; 12*, p. 119).

To proceed, information on velocity distribution is necessary. Fortunately, we need consider flow only in the immediate neighborhood of the surface, as it is the boundary layer which is of importance. A reasonable approach, suggested by Langmuir (*15*), is to use Lamb's solution (*13*) for viscous flow; this is justifiable because the Reynolds number (based on fiber diameter) for flow through fibrous filters is usually small.

For $r \to 1$, the radial and angular velocities obtained from the Lamb solution are approximately:

$$v_r = -B(r-1)^2\cos\theta$$
$$v_\theta = 2B(r-1)\sin\theta \quad (14)$$

Coefficient B is less than unity and is a function of Reynolds number; for a bed of fibers, B probably also depends on porosity and fiber distribution. The boundary layer is assumed to be sufficiently thin to fall within the region where the approximation Equation 14 holds. This is a reasonable assumption because of the low particle diffusivities.

The form of the solution to Equation 13 to 14 for very small particles (pure diffusion, $N_R \to 0$) is given by (*8*):

$$\eta\, N_{\mathrm{PE}} \sim (BN_{\mathrm{PE}})^{1/3} \quad (15)$$

$$\text{or} \quad \eta N_R N_{\mathrm{PE}} \sim (BN_{\mathrm{PE}}N_R^3)^{1/3} \quad (16)$$

where the sign indicates proportionality.

At the other extreme, for pure direct interception ($N_{\mathrm{PE}} \to \infty$) the efficiency is given approximately by:

$$\eta = -2\int_0^{\pi/2} v_{r=1+N_R}\, d\theta$$
$$= 2BN_R^2\int_0^{\pi/2}\cos\theta\, d\theta = 2BN_R^2 \quad (17)$$

Multiplying both sides by $N_R N_{\mathrm{PE}}$:

$$\eta N_R N_{\mathrm{PE}} = 2BN_{\mathrm{PE}}N_R^3 \quad (18)$$

Thus on a log-log plot of $\eta N_R N_{\mathrm{PE}}$ *vs.* $(BN_{\mathrm{PE}}N_R^3)^{1/3}$, a slope of unity should be obtained as $(BN_{\mathrm{PE}}N_R^3)^{1/3}$ decreases (pure diffusion); a slope of 3 would be expected for large values of this parameter (direct interception). In the discussion which follows, parameter B is assumed to be proportional to $N_{\mathrm{RE}}^{1/2}$. This form correlates the data well and is in keeping with the results of McCune and Wilhelm (*17*) for mass transfer to liquids flowing through packed beds.

Application of Theory. Chen (*3, 4*) and Wong, Ranz, and Johnstone (*22, 23*) have made extensive, carefully controlled studies of the filtration of homogeneous liquid aerosols by beds of glass fibers. Chen used dioctyl phthalate aerosols ranging in diameter from 0.15 to 0.72 micron and filters composed of fibers with a mean diameter of about 3.5 microns. Wong used sulfuric acid aerosols ranging from about 0.45 to 1.3 microns and fibers of 3.51, 6.24, and 9.57 microns. The data of Chen were extrapolated to zero fraction solids while Wong found little variation of efficiency with porosity.

Most of these data have been recalculated and plotted (log-log) in Figure 1 with $\eta N_R N_{PE}$ *vs.* $N_R N_{PE}^{1/3} N_{RE}^{1/6}$. Included also are a few high porosity data of Ramskill and Anderson (*18*) taken from Chen's Table I (*4*). As predicted from the theory, a single-valued function is obtained with a slope of 3 at large values of $N_R N_{PE}^{1/3} N_{RE}^{1/6}$ and about 1.2 for low values of the parameter. There seems to be no influence of N_I which, for the data shown, ranges from 5 $(10)^{-4}$ to 1. At values of $N_I > 1$, the scatter became appreciable, indicating the increased importance of the impaction mechanism. We conclude, then, that neglecting of the impaction terms in Equation 10 and the boundary layer treatment of the diffusion equation are justified for N_{RE} and $N_I < 1$.

The excellent agreement among the data for different fiber mats is probably fortuitous. For any given type of fiber, aerosol, and porosity, a plot of this type should give a single line. Curves similar in form although not identical would be expected for different systems. The curve in Figure 1 can be approximated by the expression:

$$\eta N_R N_{PE} = 6(N_R N_{PE}^{1/3} N_{RE}^{1/6}) + 3(N_R N_{PE}^{1/3} N_{RE}^{1/6})^3 \quad (19)$$

Equation 19 (or Figure 1) might be used to estimate single fiber efficiencies for glass fiber beds of high porosity, if N_I and $N_{RE} < 1$. The penetration or over-all efficiency for particles of a given size in a bed of any thickness can then be calculated from Equation 3. For heterogeneous areosols, efficiency specifications can be stated as minimum removal for stipulated particle size. More rigorous calculations can be made if the inlet size distribution is known, by integrating over the entire particle size range.

The effect of porosity is difficult to estimate. In the direct interception region (high values of $N_R N_{PE}^{1/3} N_{RE}^{1/6}$), Wong found little variation with porosity over a rather limited range; Chen found a linear relationship (Equation 4). It is suggested that Equation 19 be used with the actual velocity, $V_s/(1 - \alpha)$, rather than the superficial velocity, V_s.

Equation 19 predicts a minimum in the curve of efficiency *vs.* particle size: As $D_{BM} = kT/3\pi\mu d_p = \gamma/d_p$, Equation 19 can be rewritten:

$$\eta = \frac{6\gamma^{2/3}}{\nu^{1/6} d_F^{1/2} d_p^{2/3} V^{1/2}} + \frac{3 d_p^2 V^{1/2}}{\nu^{1/2} d_F^{3/2}} \quad (20)$$

By differentiating with respect to d_p (all other variables constant) and equating to zero, the following expression is obtained for particle size at minimum filtration efficiency:

$$d_{pm} = 0.855 \times \frac{\gamma^{1/4} \nu^{1/8} d_F^{3/8}}{V_m^{3/8}} \quad (21)$$

The existence of the minimum is established, as $\partial^2\eta/\partial d_p^2$ is always positive.

Nomenclature

a_F = fiber radius, cm. or microns
a_p = particle radius, cm. or microns
b = region cleared of particles, cm.
B = function of N_{RE}
c = c'/c_M, dimensionless
c_M = mainstream particle concentration, particles per cc.
c' = particle concentration, particles per cc.
d_F = fiber diameter, cm. or microns
d_p = particle diameter, cm. or microns
d_{pm} = particle diameter at minimum efficiency, cm. or microns
D_{BM} = particle diffusivity, sq. cm. per second
f = $3\pi\mu d_p$, grams per second
h = distance from filter inlet, cm.
H = depth of filter bed, cm.
k = Boltzmann's constant
K = fluctuating force due to molecular bombardment, dynes
m = mass of particle, grams
N_1 = concentration of aerosol at inlet to filter bed, particles per cc.
N_2 = concentration of aerosol at exit of filter bed, particles per cc.
r = r'/a_F, dimensionless
r' = radial distance, cm.
t = $t'V/a_F$, dimensionless
t' = time, seconds
T = gas temperature, ° K.
u' = particle velocity, cm. per second
u'_F = gas velocity, cm. per second
u'_R = $u' - u'_F$, cm. per second
v'_r = velocity in radial direction, cm. per second
v'_θ = velocity in θ direction, cm. per second

Unprimed velocities in the text are obtained by dividing those listed above by V.

V = gas (and particle) velocity at $r = \infty$, cm. per second
V_m = gas velocity at minimum efficiency, cm. per second
V_s = superficial velocity, cm. per second

Dimensionless Groups

N_I = inertial parameter, mV/fd_F
N_{PE} = Peclet number, $d_F V/D_{BM}$
N_R = direct interception parameter, d_p/d_F
N_{RE} = Reynolds number, $d_F V/\nu$

Greek Symbols

α = fraction solids, dimensionless
β = f/m, sec.$^{-1}$
γ = $kT/3\pi\mu$
δ = boundary layer thickness, dimensionless as defined in text
η = efficiency, dimensionless
η_0 = efficiency for $\alpha = 0$, dimensionless
η_∞ = efficiency at any α, dimensionless
θ = angular coordinate, radians (measured from forward stagnation point)
μ = fluid viscosity, gram/cm./second
ν = μ/ρ
ρ = fluid density, grams per cc.

Literature Cited

(1) Albrecht, F., *Phys. Z.* **32,** 48 (1931).
(2) Chandrasekhar, S., *Revs. Mod. Phys.* **15,** 1 (1943).
(3) Chen, C. Y., Annual Report, Eng. Expt. Station, Univ. of Ill., Contract DA 18-108-CML-4789 (Jan 30, 1954).
(4) Chen, C. Y., *Chem. Revs.* **55,** 595 (1955).
(5) Davies, C. N., *Proc. Inst. Mech. Engrs.* (*London*) **B1,** 185 (1952).
(6) Davies, C. N., *Proc. Roy. Soc.* **A 234,** 269 (1956).
(7) Eckert, E. R. G., *VDI-Forschungsheft* **416,** 1942.
(8) Friedlander, S. K., *A.I.Ch.E. Journal* **3,** 43 (1957).
(9) Gallily, I., *J. Colloid Sci.* **10,** 558 (1955).
(10) Gillespie, T., *Ibid.*, **10,** 266 (1955).
(11) Gillespie, T., Rideal, E., *Ibid.*, **10,** 281 (1955).
(12) Goldstein, S., "Modern Developments in Fluid Dynamics," Vol. **I,** Oxford University Press, London, 1938.
(13) Lamb, H., "Hydrodynamics," 6th ed., p. 614, Cambridge University Press, London, 1932.
(14) LaMer, V. K., Columbia University Rept. **NYO-512** 1951.
(15) Langmuir, I., OSRD Rept. **865** (1942).
(16) Langmuir, I., Blodgett, K. B., General Electric Research Laboratory, Rept. **RL-225.**
(17) McCune, L. K., Wilhelm, R. H., Ind. Eng. Chem. **41,** 1124 (1949).
(18) Ramskill, E. A., Anderson, W. L., *J. Colloid Sci.* **6,** 416 (1951).
(19) Schlichting, H., "Boundary Layer Theory," McGraw-Hill, New York, 1955.
(20) Sell, W., *Forsch. Gebiete Ingenieurw.*, **2,** *Forschungsheft*, 347 (August 1931).
(21) Wax, N., ed., "Noise and Stochastic Processes," Dover, N. Y., 1954.
(22) Wong, J. B., Johnstone, H. F., Eng. Expt. Station Univ. of Ill., Tech. Rept. 11, Oct. 31, 1953.
(23) Wong, J. B., Ranz, W. E., Johnstone, H. F., *J. Appl. Phys.* **27,** 161 (1956).

Received for review October 5, 1957
Accepted May 7, 1958

Divisions of Industrial and Engineering Chemistry and Water, Sewage, and Sanitation Chemistry, Symposium on Air Pollution, 132nd Meeting, ACS, New York, N. Y., September 1957.