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Neutron fibres: II. Some improving alternatives and analysis of bending losses

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Abstract. The possible confined propagation of slow (thermal) neutrons along cladded waveguides of small cross section (fibres) is studied, in order to improve a previously published analysis. Quite suitable possibilities for the core of the waveguide could be either Al or Si (or vacuum), so as to allow for confined propagation of neutrons along reasonable lengths, without large scattering and absorption losses. Tentative possibilities for the cladding could be either Ni or Fe. The bending losses are analysed for a curved neutron fibre in two spatial dimensions. Such bending losses turn out to be completely negligible, except in the case of a small radius of curvature R_c. To quote some typical values, for any fibre diameter between 0.1 and 1 μm and reasonable lengths for it (between 5 and 10 cm), bending losses can be neglected other than for $R_c \leqslant 10$ cm.

1. Introduction

Slow neutron beams have an increasing importance and usefulness regarding applied and basic research (see, for instance, White and Windsor 1984).

In a previous work (Alvarez-Estrada and Calvo 1984), we formulated several conjectures about confined propagation modes of thermal neutrons in fully solid (non-hollow) waveguides having small cross section (fibres), made up by materials, like Ti, with negative neutron–nucleus scattering amplitudes. We tried to exploit the possible analogy with optical fibres and pointed out some possible applications of neutron fibres to radiotherapy (for a lucid survey of slow neutron therapies, see Fowler 1981). There, we concentrated on uncladded Ti fibres, as the simplest theoretical (or ideal) possibility, even if their maximum length, z_0, is, in practice, rather restricted by neutron scattering and absorption by Ti nuclei. In fact, the flux decrease with z_0 of the coherent wave associated with the neutrons at the exit face of a fibre would follow the approximate law $\exp(-\mu_{tot}z_0)$, $\mu_{tot} = \rho(\sigma_{el} + \sigma_{abs})$, ρ being the number of nuclei per cm^3 and σ_{el} (σ_{abs})

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being the neutron elastic scattering (absorption) cross section for scattering by a nucleus in the material of the fibre. For Ti (see Bacon 1962, Maghabghab *et al* 1981)

$$\rho = 5.6 \times 10^{22} \text{ nuclei cm}^{-3} \qquad \sigma_{\text{el}} = 4.4 \times 10^{-24} \text{ cm}^2$$

$$\sigma_{\text{abs}} = 3.5 \times 10^{-24} \text{ cm}^2$$

and, hence, $\mu_{\text{tot}} = 0.45 \text{ cm}^{-1}$, so that for $z_0 = 5$ and 10 cm, one has $\exp(-\mu_{\text{tot}} z_0) \simeq 0.1$, 0.01, namely, 10% and 1% transmission, respectively. Thus, it would seem interesting to discuss other possibilities for neutron fibres of small cross section, namely, cladded (fully solid or hollow) ones, which could have smaller global losses by elastic scattering and absorption and, hence, could transmit an appreciable fraction of the neutron flux over a larger length, z_0. The price to be paid, eventually, is that their structure in the transverse cross section of the fibre would be more complicated.

Below, we shall discuss those improving alternatives (§ 2). We shall also study curved neutron fibres of small cross section and suitably large radius of curvature in two space dimensions, and present a theoretical study of the bending losses together with some numerical estimates.

2. Cladded neutron fibres: an improving alternative

2.1. Fibres with finite cladding and circular cross section

We shall follow the same conventions and notations as in Alvarez-Estrada and Calvo (1984). In particular, the description of the spin degrees of freedom for a neutron will be omitted (but its effect will be taken into account occasionally: see (2.2.2)).

We shall consider an infinite cladded fibre lying along the z axis. Both core and cladding have circular cross section with radii R_1, R_2 ($R_1 < R_2$), respectively (see figure 1). The approximate Schrödinger equation for the time-independent coherent

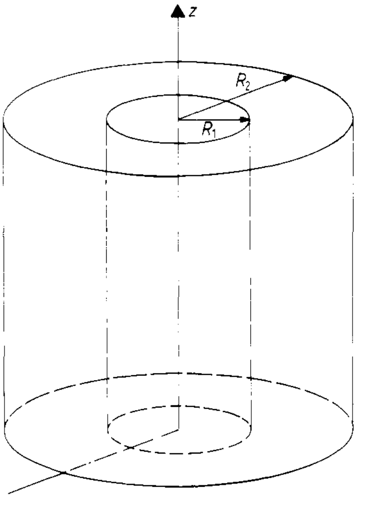

Figure 1. Fibre with finite cladding. The core (cladding) corresponds to $0 < |x| < R_1$ ($R_1 < |x| < R_2$).

wavefunction $\psi = \psi(\bar{x})$ of the neutron with total energy E is ($\bar{x} = x, z$)

$$\left(-\frac{\hbar^2}{2m}\Delta + V(\bar{x})\right)\psi(\bar{x}) = E\psi(\bar{x}) \tag{2.1.1}$$

$$V(\bar{x}) = \begin{cases} 2\pi\hbar^2\rho_1 b_1/m & |\underline{x}| < R_1 \quad \text{(core: } j = 1) & (2.1.2a) \\ 2\pi\hbar^2\rho_2 b_2/m & R_1 < |\underline{x}| < R_2 \quad \text{(clad: } j = 2) & (2.1.2b) \\ 2\pi\hbar^2\rho_3 b_3/m & R_2 < |\underline{x}| < +\infty \quad \text{(outer medium: } j = 3) & (2.1.2c) \end{cases}$$

where b_j is the (average coherent) amplitude for the scattering of a neutron by a nucleus in medium j, which has ρ_j nuclei cm^{-3} ($j = 1, 2, 3$). We assume:

$$\operatorname{Re} b_1 \geqslant 0 \qquad \operatorname{Re} b_2 > 0 \qquad \rho_1 \operatorname{Re} b_1 < \rho_2 \operatorname{Re} b_2. \tag{2.1.3}$$

In particular, either the core or the outer external medium or both could be a vacuum, so that $\rho_1 b_1 = 0$ or $\rho_3 b_3 = 0$.

In cylindrical coordinates ($|x|$, φ, z), equations (2.1.1) and (2.1.2) have the following factorised solutions (compare with § 3.2 in Alvarez-Estrada and Calvo 1984):

$$\psi(\bar{x}) = \exp \mathrm{i}\beta z \,\Phi_\alpha(x) \tag{2.1.4}$$

$$\Phi_\alpha(x) = \exp \mathrm{i}M\varphi \begin{cases} J_{|M|}(K_1|\underline{x}|) \quad |\underline{x}| < R_1 & (2.1.5a) \\ c_1 \exp K_2 R_1 H^{(1)}_{|M|}(\mathrm{i}K_2|\underline{x}|) + c_2 \exp(-K_2 R_1) & \\ \quad \times H^{(2)}_{|M|}(\mathrm{i}K_2|\underline{x}|) \quad R_1 < |\underline{x}| < R_2 & (2.1.5b) \\ c_3 H^{(1)}_{|M|}(K_3|\underline{x}|) + c_4 H^{(2)}_{|M|}(K_3|\underline{x}|) \quad R_2 < |\underline{x}| < +\infty & (2.1.5c) \end{cases}$$

$$K_1^2 + K^2 + 4\pi\rho_1 b_1 = 0 \qquad K_2^2 = 4\pi\rho_2 b_2 + K^2 \tag{2.1.6}$$

$$K_3^2 + K^2 + 4\pi\rho_3 b_3 = 0 \qquad E = (\hbar^2/2m)(\beta^2 - K^2) \tag{2.1.7}$$

where β is the propagation constant, $M = 0, \pm 1, \pm 2, \ldots$, α denotes collectively M and the Ks and $J_{|M|}$, $H^{(1)}_M$ and $H^{(2)}_{|M|}$ are the standard Bessel and Hankel functions (Abramowitz and Stegun 1965). The cs are suitable constants which satisfy from boundary conditions: both Φ_α and $\partial\Phi_\alpha/\partial|\underline{x}|$ have to be continuous at $|\underline{x}| = R_1, R_2$ for any $0 \leqslant \varphi \leqslant 2\pi$. The solutions of those equations for the cs are given in Appendix 1.

Strictly speaking, there can be no propagation modes in the situation under study, since the neutron could always escape by the tunnel effect across the cladding towards the outer region, so that $c_2 \neq 0$ and, consequently, $c_4 \neq 0$. However, for suitably large $R_2 - R_1$, such a tunnelling turns out to be negligible for a very large number of solutions of the form (2.1.4), (2.1.5).

In fact, in that case, the latter reduce approximately to equations (2.1.4) and (2.1.5a,b) with $c_2 \simeq 0$, while equation (2.1.5c) can be omitted since $c_3 \simeq 0$, $c_4 \simeq 0$, as we shall see in § 2.3. For that purpose, the following analysis in § 2.2 will be useful.

2.2. Fibres with infinite cladding and circular cross section

We consider the case of an infinite cladding ($R_2 = \infty$) in the absence of absorption, that is, $\operatorname{Im} b_1 = \operatorname{Im} b_2 = 0$, see figure 2. Then there exist propagation modes which are represented by Φ_α, as given by equations (2.1.4) and (2.1.5a,b) with

$$c_1 = \exp(-K_2 R_1) J_{|M|}(K_1 R_1)/H^{(1)}_{|M|}(\mathrm{i}K_2 R_1) \qquad c_2 = 0 \tag{2.2.1}$$

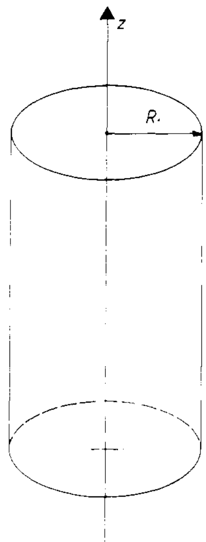

Figure 2. Fibre with infinite cladding. The core (infinite cladding) corresponds to $0 < |x| < R_1$ $(R_1 < |x| < +\infty)$.

(equation (2.1.5*c*) being disregarded). Notice that (i) K_1, K_2 verify equations (2.1.6), the second equation (2.1.7) and make the curly bracket in the right-hand side of equation (A2) vanish. In this case $(R_2 = +\infty)$, we shall give the quasiclassical approximation for the number of propagation modes with given K_1^2 in the range $0 \leqslant K_1^{(0)2} < K_1^2 < 4\pi(\rho_2 b_2 - \rho_1 b_1)$ which, *a priori*, can be expected to be reasonably valid when R_1 is not too small. Let Ω be the cross section of the core, with area $A(\Omega) = \pi R_1^2$. Then, the general quasiclassical formula (Landau and Lifshitz 1965, Martin 1972, Berry and Mount 1972) gives, by including a factor two in order to count both spin projections for the neutron:

$$\frac{2}{4\pi}\int_\Omega \mathrm{d}^2\underline{x}\,\frac{2m}{\hbar^2}\left(\frac{2\pi\hbar^2\rho_2 b_2}{m} - \frac{2\pi\hbar^2\rho_1 b_1}{m} - \frac{\hbar^2 K_1^{(0)2}}{2m}\right)$$

$$= 2A(\Omega)(\rho_2 b_2 - \rho_1 b_1)\left(1 - \frac{K_1^{(0)2}}{4\pi(\rho_2 b_2 - \rho_1 b_1)}\right) \qquad (2.2.2)$$

which extends equation (3.1.4) in Alvarez-Estrada and Calvo (1984). The total number of propagation modes is given by the right-hand side of (2.2.2), for $K_1^{(0)2} = 0$. Then, the fraction of propagation modes with any given K_1^2 in $0 \leqslant K_1^{(0)2} < K_1^2 < 4\pi(\rho_2 b_2 - \rho_1 b_1)$ over the total number of them is

$$1 - K_1^{(0)2}\ /[4\pi(\rho_2 b_2 - \rho_1 b_1)]. \qquad (2.2.3)$$

Use will be made of (2.2.3) in the next subsection.

2.3. *Approximate neglect of the tunnel effect for a suitably thick cladding*

Let us now assume the case of a finite but suitably large R_2, still with $\mathrm{Im}\, b_j = 0$, $j = 1, 2, 3$. From Appendix 1, it follows that for fixed R_1 and large R_2–R_1, c_1 and c_2/c_1 behave as a constant (consistently with the first equation (2.2.1)) and as

$$\exp[-2K_2(R_2 - R_1)] = \exp\{-2[4\pi(\rho_2 b_2 - \rho_1 b_1) - K_1^2]^{1/2}(R_2 - R_1)\} \tag{2.3.1}$$

respectively, while c_3, c_4 (as well as c_1', c_2', as given by equations (A6), (A7)) behave as $\exp[-K_2(R_2 - R_1)]$.

A comparative glance at equations (2.2.2), (2.2.3) and (2.3.1) leads to the approximate neglect of tunnelling across the cladding, since R_2 was, by assumption, suitably large. In fact, an appreciable tunnel effect towards the outer medium requires that $K_2(R_2 - R_1)$ be very small, that is, that K_1^2 be suitably close to (but smaller than) $4\pi(\rho_2 b_2 - \rho_1 b_1)$ and this can happen only for a very small fraction of the total number of solutions Φ_a.

Quantitatively, we consider typical values of ρ_j, b_j, R_j, $j = 1, 2$ such that

$$[4\pi(\rho_2 b_2 - \rho_1 b_1)]^{1/2} \simeq 10^{-2} \quad \text{to} \quad 10^{-3}\ \text{Å}^{-1} \qquad R_2 - R_1 \geqslant 5 \times 10^3\ \text{Å} \tag{2.3.2}$$

and $K_1^{(0)2}$ such that $\{1 - K_1^{(0)2}[4\pi(\rho_2 b_2 - \rho_1 b_1)]^{-1}\}^{1/2} \simeq 10^{-1}$ to 10^{-2}. Then, the waveguide with infinite cladding ($R_2 = +\infty$) would have only a tiny fraction about 10^{-2} to 10^{-4} of its propagation modes in the range $K_1^{(0)2} < K_1^2 < 4\pi(\rho_2 b_2 - \rho_1 b_1)$, by virtue of (2.2.3). Consequently, for the waveguide with finite cladding (R_2) fulfilling (2.3.2), tunnel effects would be, in principle, non-negligible for such a (very small) set of solutions Φ_a. Conversely, tunnelling would be fully negligible for the remainder of the Φ_as (almost all of them), which, then, constitute reliable approximations for propagation modes. This proves the statement at the end of § 2.1. Another (complementary) point of view is as follows. The time required for a thermal neutron (with kinetic energy about 0.02 eV) to travel along 1 m in the waveguide (z) axis is about 5×10^{-4} s. On the other hand, the time T_{tu} required for such a neutron to perform a tunnelling across the cladding towards the outer medium can be estimated by applying semiclassical methods similar to those used in the study of nuclear α-decay. Thus, T_{tu} turns out to be the inverse of a suitable probability for transmission across the cladding, which is determined by $|c_3|^2$. By extending directly to the present case the standard semiclassical approximation (for instance, see Eisberg and Resnick 1974) one finds:

$$T_{\mathrm{tu}} \simeq 2R_1[2(E - 2\pi\hbar^2\rho_1 b_1/m)/m]^{-1/2} \exp\{2[4\pi(\rho_2 b_2 - \rho_1 b_1)]^{1/2}(R_2 - R_1)\}. \tag{2.3.3}$$

The magnitude of T_{tu} is controlled by the exponential and for the values (2.3.2) it is several orders of magnitude larger than 5×10^{-4} s. In conclusion, for typical values like (2.3.2) tunnelling across the cladding is negligible for almost all Φ_as. Consequently, we are justified to consider the fibre with infinite cladding ($R_2 = +\infty$: see figure 2) in the remainder of this paper.

2.4. *Some improving alternatives for neutron fibres*

We shall discuss some possible materials for solid or hollow neutron fibres with a suitably thick cladding. Recall that in the cases to be considered below, the total amplitude for the nuclear scattering of thermal neutrons having energy E ($\sim$0.02 eV) is either positive (both in the cladding and, for solid waveguides, in the core) or vanishes (trivially, in the core of hollow waveguides). It seems physically natural that the following requirements should be fulfilled:

(i) Both the absorption and elastic cross sections (σ_{abs}, σ_{el}) and, hence, the total one (σ_{tot}) for the scattering of neutrons by the nucleus of an atom in the core have to be suitably small: they vanish, trivially, for the core of hollow fibres. Consequently, the important linear coefficient $\mu_{tot} = \rho\sigma_{tot} = \rho(\sigma_{abs} + \sigma_{el})$, which gives the decrease of the neutron (coherent) flux beyond a distance z_0 through $\exp(-\mu_{tot}z_0)$ (ρ being the number of nuclei per unit volume say, per cm^3) has to be suitably small. In the case of a compound, made up by molecules (each of them containing n_A atoms of type A and n_B atoms of type B), the relevant linear coefficient μ_{tot} (which determines the decrease of neutron coherent flux through the same law, $\exp(-\mu_{tot}z_0)$) also has to be small. We recall that if the compound has density d(g cm^{-3}) and ρ molecules cm^{-3}, then

$$\begin{aligned}\mu_{tot} &= d(n_A(A) + n_B(B))^{-1}(\mu_{tot,A}\, d_A^{-1} n_A(A) + \mu_{tot,B}\, d_B^{-1} n_B(B)) \\ &= \rho(n_A\sigma_{tot,A} + n_B\sigma_{tot,B}) \qquad (2.4.1)\end{aligned}$$

where (A) and (B) are respectively, the atomic weight of atoms of type A and B and d_A and d_B are the densities of atoms A and B for a homogeneous medium of atoms, while $\sigma_{tot,x}$ and $\mu_{tot,x}$ are the total neutron cross section and the linear coefficient, as defined above. We also remember that $(n_A(A) + n_B(B))\rho$ equals d times Avogadro's number. Clearly, the linear absorption coefficient μ_{abs} for a compound made up by molecules is given by formulae similar to (2.4.1), provided that $\sigma_{tot,x}$ is replaced by $\sigma_{abs,x}$ (see Bacon 1962, page 63).

(ii) The absorption cross section for neutrons by the nuclei of the cladding should be small.

(iii) The positive quantity $\rho_2b_2 - \rho_1b_1$ ($j = 2$: cladding; $j = 1$: core) should be relatively large. Recall that it controls:

(a) the height of the potential well in the core where the neutrons will be confined,

(b) the total number of propagation modes in quasiclassical approximation (see (2.2.2) with $K_1^{(0)} = 0$),

(c) the smallness of the tunnelling effect across the cladding, when the latter is not strictly infinite (equations (2.3.1), (2.3.3)),

(d) the magnitude of the acceptance angle i_c for neutrons entering into the core from vacuum (see Alvarez-Estrada and Calvo 1984, subsections 2.2.2–3): recall that $\sin i_c \simeq \hbar(2mE)^{-1/2}, [\rho_2b_2 - \rho_1b_1]^{1/2}$. We remark that if the core is a compound, say, a homogeneous aggregate of molecules, then, the quantity ρ_1b_1 should be understood as $\rho(n_Ab_A + n_Bb_B)$, where b_x is the (positive) scattering amplitude for a neutron by a nuclei of type x = A, B while ρ, n_A, n_B have the same meaning as before.

(iv) They should be relatively abundant, easy to produce or prepare and share some good mechanical and chemical properties. At this point, it may be interesting to recall the high degree of development reached by the technologies of glass and ceramic fibres, by virtue of their wide usefulness in optical communications and space sciences (see, for instance McCreight *et al* 1965).

In table 1, we collect values for σ_{abs} and σ_{el} for certain (pure) elements, together with ρ, ρb, μ_{abs} and μ_{tot} for those elements and some glass and ceramic compounds. Table 1 also includes some (rather rough) estimates for air (having considered only its contents in nitrogen and oxygen).

Graphite, Si, Al, SiO_2 and Al_2O_3 could be possible candidates for cores of solid fibres, while Ni and Fe would be tentative possibilities for claddings. Clearly, the smallest values of μ_{tot} are obtained for pure Al and Si. Consequently a cladded fibre having Ni or Fe as cladding and Si or Al (or vacuum) in the core would represent a considerable

Table 1.

Material	$\rho\,(10^{22}$ atoms, molecules $\mathrm{cm}^{-3})$	$\rho b\,(10^{10}\,\mathrm{cm}^{-2})$	$\sigma_{el}\,(10^{-24}\,\mathrm{cm}^2)$	$\sigma_{abs}\,(10^{-24}\,\mathrm{cm}^2)$	$\mu_{abs}\,(\mathrm{cm}^{-1})$	$\mu_{tot}\,(\mathrm{cm}^{-1})$
Ni	9.13	9.4	18.0	2.7	0.25	1.89
Fe	8.48	8.14	11.8	1.4	0.12	1.12
Graphite (C)	11.3	7.47	5.51	0.003	0.0005	0.62
Si	5.19	2.18	2.2	0.06	0.004	0.11
SiO_2	2.2	3.46	—	—	0.001	0.236
Al	6.03	2.11	1.5	0.13	0.008	0.098
Al_2O_3	2.34	5.69	—	—	0.006	0.374
Air	2.7×10^{-3}	4.6×10^{-3}	—	—	4.6×10^{-5}	5.7×10^{-4}

improvement with respect to the simpler Ti fibre discussed previously in (Alvarez-Estrada and Calvo 1984), as the former would allow for neutrons to be guided through longer distances. In particular, for Si or Al cores, confined propagation of neutrons along, say, $z_0 = 10$ cm could occur with a flux decrease factor $\exp(-\mu_{tot} z_0) \simeq e^{-1} \simeq 0.37$.

2.5. Fibre with infinite cladding and circular cross section: approximate formulae for propagation modes

We shall give some approximate formulae for the propagation modes for infinite cladding ($R_2 = \infty$) and $\operatorname{Im} b_1 = \operatorname{Im} b_2 = 0$ which fulfil (2.1.4), (2.1.5*a*,*b*), (2.2.1), (2.16), the second (2.17) and make the curly bracket in equation (A2) vanish. The approximate expressions presented below will be the counterparts of similar ones for optical fibres and, accordingly, details of the calculations will be omitted: they constitute a direct extension of those given by Snyder (1969, see also section 8.6 in Marcuse 1972). In analogy with optical waveguides, it is useful to introduce the following dimensionless parameter for neutron fibres

$$v = R_1[4\pi(\rho_2 b_2 - \rho_1 b_1)]^{1/2}. \tag{2.5.1}$$

For $[4\pi(\rho_2 b_2 - \rho_1 b_1)]^{1/2} \simeq 10^{-2}$ to 10^{-3} Å^{-1} and $R_1 \simeq 10^{-4}$ cm, one has $v \simeq 10$ to 10^2. Clearly, (2.16) yields the following general relationship

$$v^2 = (K_1 R_1)^2 + (K_2 R_1)^2. \tag{2.5.2}$$

First, we consider $M = 0$ and the (too idealised) situation where R_1 is so small that only one propagation mode would be allowed. Then, $K_2 R_1$ is very small ($K_2 R_1 \ll K_1 R_1$) with ($\Gamma = 1.7817$)

$$\ln(\Gamma K_2 R_1/2) \simeq J_0(v)/vJ_1(v). \tag{2.5.3}$$

Next, let R_1 be such that just one propagation mode with $M = 1$ exists. Then $K_2 = 0$ and $v = v_c = K_1 R_1$ fulfils $J_0(v_c) = 0$ and (2.5.1) (cut-off condition). For slightly larger values of R_1, K_2 ($K_2 R_1 \ll K_1 R_1$) so that no more than one propagation mode with $M = 1$ is allowed, one has

$$(R_1 K_2)^2 \ln(\Gamma K_2 R_1/2) \simeq vJ_0(v)/J_1(v). \tag{2.5.4}$$

Let $M = 2, 3, 4, \ldots$. Then, one propagation mode with such a value for M occurs when $K_2 = 0$ and the radius R_1 satisfies (2.5.1), $v = v_c = K_1 R_1$ and $J_{M-1}(v_c) = 0$ (cut-off condition). Again, for slightly larger values of $R_1 K_2$ ($K_2 R_1 \ll K_1 R_1$, $K_1 R_1 \simeq v - (K_2 R_1)^2/2v_c$) such that no further propagation modes with that value of M could exist, the following approximation holds:

$$(R_1 K_2)^2 \simeq \frac{2vJ_{|M|-1}(v)}{(\mathrm{d}J_{|M|-1}(v)/\mathrm{d}v)_{v_c} - (|M| - 1)^{-1} J_{|M|}(v_c)}. \tag{2.5.5}$$

We now consider the opposite case: $v \simeq K_2 R_1 \gg K_1 R_1$ ('far from cut-off'). First, let $M = 0$. Then, one gets:

$$K_1 R_1 \simeq (K_1 R_1)_\infty \exp\left(-\frac{1}{v}\right) \quad \text{with} \quad J_0((K_1 R_1)_\infty) = 0. \tag{2.5.6}$$

Finally, for $M = 1, 2, 3, \ldots$, the result is

$$K_1 R_1 \simeq (K_1 R_1)_\infty \left(\frac{v - 2M}{v + 2M}\right)^{1/2M} \quad \text{with} \quad J_M((K_1 R_1)_\infty) = 0. \tag{2.5.7}$$

3. Bending losses in neutron fibres: a two-dimensional analysis

3.1. Introduction

Bending losses are among the most important phenomena giving rise to losses for a confined beam in a waveguide. In order to estimate them in the case of optical waveguides, several approximate treatments, based essentially upon two-dimensional analyses, have been put forward by Marcatili (1969) and Marcuse (1972, § 9.6). Bending losses in curved (hollow) neutron guides of large cross section have been considered by Jacrot (1970) and Schaerpf and Eichler (1973). Also, bending losses for neutron microguides (sandwiches of alternate nickel and aluminium layers, with total thickness about 0.037 mm) have been studied by Marx (1971).

We shall outline here a quantum-mechanical analysis of bending losses in neutron fibres of small cross section and large radius of curvature (R_c). We shall simplify our three-dimensional problem by assuming that it is replaced by a two-dimensional one, with $\underline{x} = (x, y)$ being substituted by x, that is, we suppose either that the fibre is one-dimensional in its transverse cross section or, equivalently, that all relevant quantities (Schrödinger equation V, propagation modes, etc) are y-independent. Our treatment will constitute a Green function variant of a method used in § 9.6 of Marcuse (1972) in order to study the curvature losses of electromagnetic radiation in an optical fibre in two spatial dimensions. We refer to Marcuse (1972) for a lucid exposition of the basic philosophy behind the approximations, while we believe that our Green function approach helps to clarify several aspects in his method. Like in his (optical fibre) problem, we hope that the results of our two-dimensional calculation will yield an acceptable order of magnitude estimate of the true bending losses for the three-dimensional neutron fibre.

First, let us assume temporarily the infinite fibre to be strictly straight (with $R_c = \infty$, without curvature losses), its transverse section lying in $-R_1 < x < +R_1$ (see figure 3).

The actual (lossless) propagation modes for the neutron, namely, $\psi(x, z) = \exp i\beta z \Phi(x)$ are assumed to satisfy the two-dimensional analogue of (2.1.1) with a potential

$$V(x) = \begin{cases} 2\pi\hbar^2 \rho_2 b_2/m \equiv V_0 & |x| > R_1 \quad (3.1.1a) \\ V_1(x) & |x| < R_1. \quad (3.1.1b) \end{cases}$$

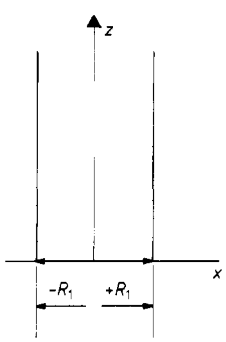

Figure 3. Fibre with infinite cladding in two spatial dimensions (x, z). The core (infinite cladding) corresponds to $-R_1 < x < R_1$ $(R_1 < |x| < \infty)$.

That is, for the sake of generality in the formulation of the two-dimensional bending losses problem, we allow for a smooth variation in the potential for $|x| < R_1$, although at the end we shall set $V_1(x) = 2\pi\hbar^2\rho_1 b_1/m$. A standard Green function argument yields directly for $-\infty < x < +\infty$:

$$\Phi(x) = \frac{2m}{\hbar^2}\frac{1}{2K_2}\int_{-R_1}^{+R_1} \mathrm{d}x' \exp[-K_2|x-x'|][V_0 - V_1(x')]\Phi(x'). \qquad (3.1.2)$$

In fact, by using (2.1.6) and (2.1.7) one can show that the resulting $\psi(x, z)$ satisfies the two-dimensional Schrödinger equation similar to (2.1.1), V being given in (3.1.1a,b).

3.2. Integral equation approach to bending losses

We shall assume that the two-dimensional fibre is perfectly straight from $z = -\infty$ up to some $z = z_0$ and that it has a suitably large radius of curvature R_c ($<+\infty$) for $z > z_0$ (see figure 4). We assume an incoming thermal neutron which has a confined propagation

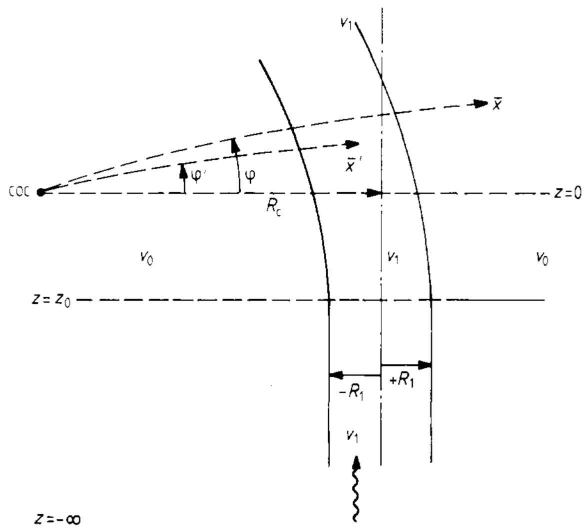

Figure 4. Curved fibre with infinite cladding in two spatial dimensions. The part of the fibre in $-\infty < z < z_0$ (lower part) is strictly straight. The part above $z = z_0$ (upper part) is curved. The centre of curvature is COC. The incoming propagation mode $\exp(\mathrm{i}\beta z\Phi(x))$, represented by one wiggly arrow, proceeds from $z = -\infty$.

initially in $-\infty < z < z_0$ and so is represented by the propagation mode $\psi(x, z) = \exp(\mathrm{i}\beta z)\Phi(x)$, where $\Phi(x)$ satisfies (3.1.2). It is natural to treat the propagation of the total (coherent) wavefunction for the neutron throughout all two-dimensional space (even for $z > z_0$, where bending losses will occur) by using an integral equation and to accept that the latter has to be a homogeneous one (by virtue of the actual incoming wave condition and of the fact that the potential is non-vanishing at $z \to -\infty$ along the fibre):

$$\psi(\bar{x}) = \int \mathrm{d}^2\bar{x}'(-1/4\mathrm{i})H_0^{(1)}\{[(2m/\hbar^2)(E - V_0)]^{1/2}|\bar{x} - \bar{x}'|\}$$
$$\times (2m/\hbar^2)(V_0 - V_1(\bar{x}'))\psi(\bar{x}') \qquad (3.2.1)$$

where $\bar{x} = (x_1, z)$, $\bar{x}' = (x_1', z')$, the integration in (3.2.1) extends over the whole two-dimensional domain occupied by the fibre (see figure 4). We are using the standard Green function for the two-dimensional analogue of (2.1.1):

$$(-1/4\mathrm{i})H_0^{(1)}\{[(2m/\hbar^2)(E-V_0)]^{1/2}|\bar{x}-\bar{x}'|\} = \int \frac{\mathrm{d}^2\bar{l}}{(2\pi)^2} \frac{\exp \mathrm{i}\bar{l}(\bar{x}-\bar{x}')}{\bar{l}^2 - (2m/\hbar^2)(E-V_0) - \mathrm{i}\varepsilon}$$

$$= -\frac{1}{4\mathrm{i}} \sum_{M=-\infty}^{+\infty} \exp \mathrm{i}M(\varphi - \varphi')H_{|M|}^{(1)}\{[(2m/\hbar^2)(E-V_0)]^{1/2} r_>\}$$

$$\times J_{|M|}\{[(2m/\hbar^2)(E-V_0)]^{1/2} r_<\} \tag{3.2.2}$$

where $\bar{l} = (l_x, l_z)$, $\varepsilon \to 0^+$, $x_1 = |\bar{x}| \cos \varphi$, $z = |\bar{x}| \sin \varphi$ and similarly for x_1', z' while $r_<$, $r_>$ are the smaller and the larger of $|\bar{x}|$, $|\bar{x}'|$, respectively. It is easy to see that the right-hand side of (3.2.1) fulfils the two-dimensional analogue of (2.1.1).

In the limit $R_c \to \infty$, the neutron should continue to have a permanently confined propagation along the straight fibre without bending losses in $-\infty < z < +\infty$. Consequently, the solution $\psi(\bar{x})$ of (3.2.1) should reduce exactly to the incoming wave $\exp \mathrm{i}\beta z \Phi(x)$, where $\Phi(x)$ satisfies (3.1.2), and (2.1.6), (2.1.7) hold. More precisely, such a consistency condition can indeed be proved provided that (see figure 4)

(i) one sets (for $z > z_0$)

$$x_1' = R_c + x' \qquad z' = R_c\varphi' \qquad \mathrm{d}^2\bar{x}' = \mathrm{d}x_1'\mathrm{d}z' = \mathrm{d}x' R_c \mathrm{d}\varphi'$$

$$0 \leqslant \varphi' < 2\pi \qquad z_0 \to -\infty$$

(ii) one replaces $V_1(\bar{x}')$ by $V_1(x')$ for $z > z_0$

(iii) one accepts that the main contribution to the series in (3.2.2) comes from the terms with $M = \beta R_c$

(iv) one sets $x_1 = R_c + x$, $z = R_c\varphi$, and assumes that $\bar{x}$ is kept fixed and lies outside the fibre with $x > R_1$

(v) the Hankel function $H_{|M|}^{(1)}\{[(2m/\hbar^2)(E-V_0)]^{1/2}|\bar{x}|\}$ with $M = \beta R_c$ is replaced by the following asymptotic formula (see (9.1.3) and (9.3.2) in Abramowitz and Stegun 1965) which is valid when both order and argument are large and the former is somewhat larger than the latter (as one may approximate $|\bar{x}| \sim x + R_c$)

$$(-\mathrm{i}) \frac{\exp |M|(\alpha - \tanh \alpha)}{[(\pi |M| \tanh \alpha)/2]^{1/2}} \tag{3.2.3}$$

where

$$\cosh \alpha = \frac{|M|}{[(2m/\hbar^2)(E-V_0)]^{1/2}|\bar{x}|}. \tag{3.2.4}$$

In turn, one approximates (3.2.3) by

$$(-\mathrm{i}) \frac{\exp[\beta R_c \tanh^{-1}(K_2/\beta) - K_2 R_c] \exp(-K_2 x)}{[(\pi K_2 R_c)/2]^{1/2}}. \tag{3.2.5}$$

For brevity, we shall omit details of these approximations and refer to §9.6 in Marcuse 1972 for them,

(vi) similar approximations are carried out for the Bessel function, successively, with

$M = \beta R_c$, $|\bar{x}'| \simeq x' + R_c$ (see (9.3.2) in Abromawitz and Stegun 1965):

$$J_{|M|}\{[(2m/\hbar^2)(E - V_0)]^{1/2}|\bar{x}'|\} \simeq \frac{\exp|M|(\tanh\alpha' - \alpha')}{[2\pi|M|\tanh\alpha']^{1/2}}$$

$$\simeq \frac{\exp[K_2R_c - \beta R_c \tanh^{-1}(K_2/\beta)]\exp K_2x'}{2[(\pi K_2R_c)/2]^{1/2}} \tag{3.2.6}$$

with

$$\cosh\alpha' = \frac{|M|}{[(2m/\hbar^2)(E - V_0)]^{1/2}|\bar{x}'|}$$

in this case.

It is now entirely trivial to check that when all these approximations are performed, (3.2.1) coincides with (3.1.2). A remarkable feature is that all R_c-dependences cancel out. For brevity, we omit unnecessary details.

We now come back to the case of a suitably large but finite R_c, with z_0 very large in magnitude and negative. As the neutron propagates along the slightly curved fibre, the incoming propagation mode $\exp i\beta z\Phi(x)$ becomes $\psi(\bar{x})$, which satisfies (3.2.1) and describes bending losses as we shall see. For any $\bar{x}$, fixed outside the fibre and in the side which is opposite to the one where the centre of curvature lies (see figure 4) we are allowed to make use of all previous conditions and approximations (i)–(iii) plus (vi) except (iv)–(v), in (3.2.1). Then, the latter becomes the following explicit representation for $\psi(\bar{x})$ in terms of the incoming propagation mode:

$$\psi(\bar{x}) \simeq \exp i\beta R_c\varphi H^{(1)}_{\beta R_c}\{[(2m/\hbar^2)(E - V_0)]^{1/2}|\bar{x}|\}R_c^{1/2}B \tag{3.2.7}$$

$$B = \frac{1}{4i}\left(\frac{2\pi}{K_2}\right)^{1/2}\exp[K_2R_c - \beta R_c\tanh^{-1}(K_2/\beta)]\int_{R_1}^{+R_1} dx'\,\frac{2m}{\hbar^2}(V_1(x') - V_0)$$
$$\times \exp(K_2x')\Phi(x'). \tag{3.2.8}$$

3.3. *Probability fluxes and a general formula for the bending loss coefficient*

In order to analyse the bending losses, we take the limit $|\bar{x}| \to \infty$ ($|\bar{x}|/R_c \to \infty$) for fixed φ. Then, (3.2.7) becomes the outgoing cylindrical wave

$$\psi(\bar{x}) \sim \exp i\beta R_c\varphi\left(\frac{2}{\pi[(2m/\hbar^2)(E - V_0)]^{1/2}|\bar{x}|}\right)^{1/2}R_c^{1/2}$$
$$\times \exp i\left[\left(\frac{2m}{\hbar^2}(E - V_0)\right)^{1/2}|\bar{x}| - \pi/4 - \beta R_c\pi/2\right]B. \tag{3.3.1}$$

Since $\psi(\bar{x})$ has an outgoing wave behaviour, the neutron represented by the former no longer has a confined propagation along the fibre and has a non-vanishing probability to escape far from the waveguide. In order to evaluate that probability, we recall that the quantum-mechanical current $\bar{j}$ associated with a generic neutron wavefunction ψ is

$$\bar{j} = \frac{\hbar}{m}\mathrm{Re}[\psi^*(-i\bar{\nabla})\psi]. \tag{3.3.2}$$

Then, the probability current associated to the incoming propagation mode $\exp i\beta z\Phi(x)$

for $-\infty < z < z_0$, z_0 being large in magnitude and negative, is

$$\bar{j}_{\text{in}} = \left(0, \frac{\hbar\beta}{m}|\Phi(x)|^2\right) \tag{3.3.3}$$

and the total incoming probability flux across the whole transverse cross section of the fibre in $-\infty < z < z_0$ is

$$F_{\text{in}} = \frac{\hbar\beta}{m}\int_{-\infty}^{+\infty} \mathrm{d}x\,|\Phi(x)|^2. \tag{3.3.4}$$

The probability current associated with the wavefunction (3.3.1) is ($|\bar{x}| \to \infty$, $|\bar{x}|/R_c \to \infty$)

$$\bar{j}_{\text{out}} \sim \frac{2\hbar}{\pi m} R_c |B|^2 \frac{\bar{x}}{|\bar{x}|^2}. \tag{3.3.5}$$

We consider an interval of unit length in the fibre, determined by two radii, and the corresponding arc $c(1)$ of length $L(\to\infty)$ determined by those radii at $|\bar{x}| \to \infty$ shown in figure 5. Clearly, $L = |\bar{x}|/R_c$.

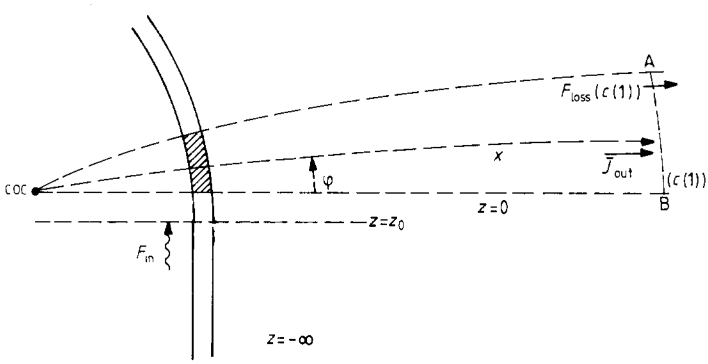

Figure 5. Curved fibre in two dimensions (see the caption to figure 4). The total incoming probability flux (wiggly arrow) is F_{in}. The shaded sector in the curved part, determined by two radii (namely, coc A and coc B) has unit length along the fibre. The arc $c(1)$ determined by AB has length L ($\to\infty$). The outgoing probability current is j_{out} and the associated probability flux across $c(1)$ is $F_{\text{loss}}(c(1))$.

The probability flux associated to $\bar{j}_{\text{out}}$ across the arc $c(1)$ is

$$F_{\text{loss}}(c(1)) \equiv \int_{c(1)} \bar{j}_{\text{out}} \frac{\bar{x}}{|\bar{x}|}\,\mathrm{d}l = \frac{2\hbar}{\pi m}|B|^2 \frac{R_c}{|\bar{x}|} L = \frac{2\hbar}{\pi m}|B|^2. \tag{3.3.6}$$

The bending loss coefficient is defined as $\tau \equiv F_{\text{loss}}(c(1))/F_{\text{in}}$. The use of (3.3.4) and (3.3.6) yields

$$\tau = \frac{1}{4\beta K_2}\exp 2R_c[K_2 - \beta\tanh^{-1}(K_2/\beta)] \times \left|\int_{-R_1}^{+R_1} \mathrm{d}x'(2m/\hbar^2)(V_0 - V_1(x'))\exp K_2 x' \Phi(x')\right|^2 \Big/ \int_{-\infty}^{+\infty} \mathrm{d}x'|\Phi(x')|^2. \tag{3.3.7}$$

Next, we consider the closed line formed by the two radii which intersect the fibre at $z\,(z > z_0)$ and $z + \Delta z$ (Δz being small) and the arc $c(\Delta z)$ at $|\bar{x}| \to \infty$, $|\bar{x}|/R_c \to \infty$ (which has length $(|\bar{x}|/R_c)\Delta z$): see figure 6. Let $F(z)$ be the probability flux of $\bar{j}$ along the fibre at z, that is, across the transverse cross section of the fibre which coincides with the radius which crosses it at z. Since the current associated to the total stationary wave ψ is conserved ($\nabla \cdot \bar{j} = 0$), the corresponding property of flux conservation yields (see figure 6)

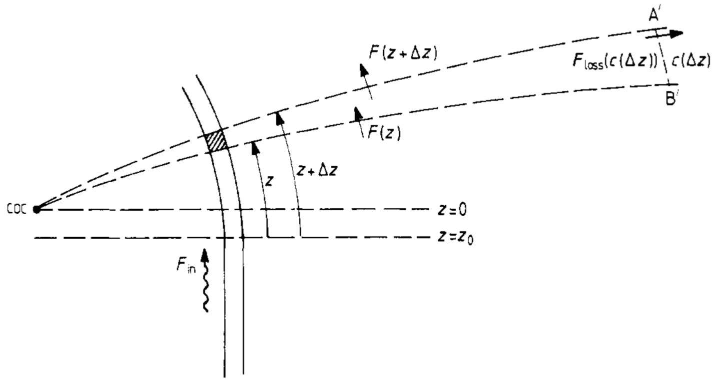

Figure 6. Curved fibre in two dimensions (see also captions to figures 4 and 5). The shaded sector in the curved part, determined by the two radii COC A′ and COC B′, has length Δz along the fibre. The arc $c(\Delta z)$ has length $L(\Delta z/1)$. The associated outgoing probability flux across $c(\Delta z)$ is $F_{\text{loss}}\,(c(\Delta z))$. The probability fluxes across COC B′ and COC A′ are $F(z)$ and $F(z + \Delta z)$, respectively. The wiggly arrow corresponds to the total incoming probability flux.

$$F(z + \Delta z) - F(z) = -F_{\text{loss}}(c(\Delta z)) \simeq F_{\text{loss}}(c(1))\Delta z = -\tau F_{\text{in}}\Delta z \tag{3.3.8}$$

as Δz is small†. By integrating (3.3.8) in $z \geqslant z_0$ with the condition $F(z_0) = F_{\text{in}}$, we get the approximate formula

$$F(z) \simeq F_{\text{in}}[1 - \tau(z - z_0)] \tag{3.3.9}$$

provided that $z - z_0 < \tau^{-1}$. If $0 \leqslant z - z_0 \ll \tau^{-1}$, (3.3.9) can be replaced by

$$F(z) \simeq F_{\text{in}} \exp[-\tau(z - z_0)] \tag{3.3.10}$$

which can also be obtained from (3.3.9) as, then, $\tau F_{\text{in}}\Delta z \simeq \tau F(z)\Delta z$.

3.4. Bending losses for homogeneous neutron fibre: some numerical estimates

We shall outline the application to a homogeneous neutron fibre, with

$$V_1(x) = 2\pi\hbar^2\rho_1 b_1/m \qquad \text{for} \qquad -R_1 < x < R_1. \tag{3.4.1}$$

† The lengths of $c(1)$ and $c(\Delta Z)$ are measured relative to an interval of unit length along the fibre. The following remarks, which are consistent with (3.3.7), (3.3.8) may be useful. The probability flux associated with $\bar{j}_{\text{out}}$ across an arc $c(2\pi R_c)$ corresponding to $0 \leqslant \varphi < 2\pi$ for $|\bar{x}| \to \infty$ is $F_{\text{loss}}\,(c(2\pi R_c)) = 2\pi R_c F_{\text{loss}}(c(1))$. Then, τ may also be defined as $F_{\text{loss}}(c(2\pi R_c))/(2\pi R_c F_{\text{in}})$.

Then, one has (Φ_0 being a constant)

$$\Phi(x) = \begin{cases} \Phi_0 \cos K_1 R_1 \exp[-K_2(|x| - R_1)] & |x| > R_1 \qquad (3.4.2a) \\ \Phi_0 \cos K_1 x & |x| < R_1 \qquad (3.4.2b) \end{cases}$$

$$\tan K_1 R_1 = K_2/K_1 \qquad K_1^2 + K_2^2 = 4\pi(\rho_2 b_2 - \rho_1 b_1) \tag{3.4.3}$$

K_1, K_2 being related to E, β through (2.1.6) and the second (2.1.7). Notice that the second (3.4.3) is the counterpart for the slab of (2.5.2) for the homogeneous cylindrical fibre. On the other hand:

$$\int_{-\infty}^{+\infty} \mathrm{d}x' |\Phi(x')|^2 = |\Phi_0|^2 R_1 \left[\left(1 + \frac{\sin 2K_1R_1}{2K_1R_1}\right) + \frac{\cos^2 K_1R_1}{K_2R_1}\right] \tag{3.4.4}$$

$$\int_{-R_1}^{+R_1} \mathrm{d}x' \frac{2m}{\hbar^2}[V_0 - V_1(x')] \exp K_2 x' \Phi(x') = 2K_2 \Phi_0 \cos K_1 R_1 \exp K_2 R_1. \tag{3.4.5}$$

Then, (3.3.7) and (3.4.4), (3.4.5) yield the following expression for the bending loss:

$$\tau = \frac{K_2^2}{\beta} \exp[2R_c(K_2 - \beta \tanh^{-1}(K_2/\beta))] \exp 2K_2R_1$$
$$\times K_1^2/(1 + K_2R_1) 4\pi(\rho_2 b_2 - \rho_1 b_1) \tag{3.4.6}$$

we consider typical values

$$[4\pi(\rho_2 b_2 - \rho_1 b_1)]^{1/2} \simeq 10^{-3} \text{ to } 10^{-2}\ \text{Å}^{-1} \qquad E \simeq 0.02\ \text{eV}$$
$$R_1 \simeq 10^3 \text{ to } 10^4\ \text{Å} \qquad R_c \simeq 1\ \text{cm to } 1\ \text{m}. \tag{3.4.7}$$

There are a whole set of propagation modes for the straight fibre, all with $\beta \simeq 1$ Å^{-1}. Thus, K_1 may be very close to (but smaller than) $[4\pi(\rho_2 b_2 - \rho_1 b_1)]^{1/2}$ so that K_2 is positive and very small (say, with $K_2 R_1 \ll 1$): case (i). The other extreme situation (case (ii)) corresponds to a very small positive K_1, so that K_2 is very close to (but smaller than) $[4\pi(\rho_2 b_2 - \rho_1 b_1)]^{1/2}$, with $K_2 R_1 \gg 1$. Notice that $0 \leqslant K_2 \leqslant [4\pi(\rho_2 b_2 - \rho_1 b_1)]^{1/2}$ always holds, due to (3.4.3). Then, the inequality $K_2/\beta \ll 1$ holds in all cases and one may use the following approximation (see § 9.6 in Marcuse 1972)

$$\beta \tanh^{-1}(K_2/\beta) - K_2 \simeq \frac{1}{3}\frac{K_2^3}{\beta^2}. \tag{3.4.8}$$

In case (i), by using $K_2 R_1 \ll 1$, $K_1^2 \simeq 4\pi(\rho_2 b_2 - \rho_1 b_1)$, (3.4.6) yields:

$$\tau \simeq \frac{K_2^2}{\beta} \exp\left(-\frac{2}{3}\frac{K_2^3}{\beta^2} R_c\right). \tag{3.4.9}$$

In case (ii), by letting $K_2 R_1 \gg 1$, $K_2^2 \simeq 4\pi(\rho_2 b_2 - \rho_1 b_1)$, (3.4.6) becomes:

$$\tau \simeq \frac{1}{\beta}\frac{K_1^2}{K_2 R_1} \exp 2K_2\left[R_1 - \frac{1}{3}\left(\frac{K_2}{\beta}\right)^2 R_c\right]. \tag{3.4.10}$$

We shall be interested in fibres where the largest value of $z - z_0$ varies between 5 and 10 cm.

In case (i), let us allow for K_2 any value less than (or equal to) 10^{-5} Å^{-1}. Then, τ is less than $10^{-10}/\exp(10^6)$ Å^{-1} for $R_c = 10$ cm and, so, the largest value of $\tau(z - z_0)$ is less than $1/10\exp(10^6)$ for $z = 10$ cm.

In case (ii), there is a sensible (exponential) dependence of τ, as given by (3.4.10), on $K_2[R_1 - \frac{1}{3}(K_2/\beta)^2 R_c]$.

If the latter is not positive, then for $K_1 \leqslant 10^{-5}$ Å^{-1} and $K_2 R_1 \sim 10$, τ is less than about 10^{-11} Å^{-1} and, so, for $z - z_0 \sim 10$ cm, then $\tau(z - z_0)$ is less than 10^{-2}. However, for the same K_1, $R_1 \sim 10^4$ Å, 10^{-3} Å$^{-1} \leqslant K_2 \leqslant 5 \times 10^{-3}$ Å^{-1} and $R_c \sim 10$ cm, $K_2[R_1 - \frac{1}{3}(K_2/\beta)^2 R_c]$ may change from negative to positive values and, so, τ and $\tau(z - z_0)$ could be large. Thus, we conclude that the bending losses are fully negligible in all cases, except in case (ii) for suitably small values of R_c such that $K_2[R_1 - \frac{1}{3}(K_2/\beta)^2 R_c]$ becomes positive. In the latter case, τ changes quite rapidly and the curvature losses may be large.

4. Conclusions

We have proposed and analysed some improving alternatives for neutron fibres of small cross section, which could allow for confined propagation of neutrons along lengths which are longer than that for an uncladded Ti fibre (studied in earlier work). Essentially, the actual improved fibres are cladded, the core being either Al or Si (or vacuum).

We have studied the bending losses for a curved neutron fibre in two spatial dimensions. In general, the bending losses are fully negligible except for propagation modes with K_2 close to $[4\pi(\rho_2 b_2 - \rho_1 b_1)]^{1/2}$ and very small K_1, *provided that the curvature radius decreases suitably*.

Acknowledgment

We are grateful to Dr J W Fowler, from Gray Laboratory, Mount Vernon Hospital, for an interesting correspondance containing useful comments.

Appendix 1

The boundary conditions on Φ_α and $\partial\Phi_\alpha/\partial|x|$ (see (5.1.3)), namely, their continuity, at $|x| = R_1, R_2$ yield the following expressions for the cs:

$$c_1 = \Delta_1^{-1} \exp(-K_2 R_1)\left\{J_{|M|}(K_1 R_1)\left[\frac{\mathrm{d}H^{(2)}_{|M|}(\mathrm{i}K_2|x|)}{\mathrm{d}|x|}\right]_{R_1} - H^{(2)}_{|M|}(\mathrm{i}K_2 R_1)\left[\frac{\mathrm{d}J_{|M|}(K_1|x|)}{\mathrm{d}|x|}\right]_{R_1}\right\} \tag{A1}$$

$$c_2 = \Delta_1^{-1} \exp K_2 R_1\left\{H^{(1)}_{|M|}(\mathrm{i}K_2 R_1)\left[\frac{\mathrm{d}J_{|M|}(K_1|x|)}{\mathrm{d}|x|}\right]_{R_1} - J_{|M|}(K_1 R_1)\left[\frac{\mathrm{d}H^{(1)}_{|M|}(\mathrm{i}K_2|x|)}{\mathrm{d}|x|}\right]_{R_1}\right\} \tag{A2}$$

$$\Delta_1 = H^{(1)}_{|M|}(\mathrm{i}K_2 R_1)\left[\frac{\mathrm{d}H^{(2)}_{|M|}(\mathrm{i}K_2|x|)}{\mathrm{d}|x|}\right]_{R_1} - H^{(2)}{}_{|M|}(\mathrm{i}K_2 R_1)\left[\frac{\mathrm{d}H^{(1)}_{|M|}(\mathrm{i}K_2|x|)}{\mathrm{d}|x|}\right]_{R_1} \tag{A3}$$

$$c_3 = \Delta_2^{-1}\left\{c_1'\left[\frac{\mathrm{d}H^{(2)}_{|M|}(K_3|x|)}{\mathrm{d}|x|}\right]_{R_2} - c_2' H^{(2)}_{|M|}(K_3R_2)\right\} \tag{A4}$$

$$c_4 = \Delta_2^{-1}\left\{c_1'\left[\frac{\mathrm{d}H^{(1)}_{|M|}(K_3|x|)}{\mathrm{d}|x|}\right]_{R_2} - c_2' H^{(1)}_{|M|}(K_3R_2)\right\} \tag{A5}$$

$$c_1' = c_1 \exp K_2R_1 H^{(1)}_{|M|}(\mathrm{i}K_2R_2) + c_2 \exp(-K_2R_1) H^{(2)}_{|M|}(\mathrm{i}K_2R_2) \tag{A6}$$

$$c_2' = c_1 \exp K_2R_1\left[\frac{\mathrm{d}H^{(1)}_{|M|}(\mathrm{i}K_2|x|)}{\mathrm{d}|x|}\right]_{R_2} + c_2 \exp(-K_2R_1)\left[\frac{\mathrm{d}H^{(2)}_{|M|}(\mathrm{i}K_2|x|)}{\mathrm{d}|x|}\right]_{R_2} \tag{A7}$$

$$\Delta_2 = H^{(1)}_{|M|}(K_3R_2)\left[\frac{\mathrm{d}H^{(2)}_{|M|}(K_3|x|)}{\mathrm{d}|x|}\right]_{R_2} - H^{(2)}_{|M|}(K_3R_2)\left[\frac{\mathrm{d}H^{(1)}_{|M|}(K_3|x|}{\mathrm{d}|x|}\right]_{R_2}. \tag{A8}$$

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