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Equivalence of Physically Based Statistical Fracture Theories for Reliability Analysis of Ceramics in Multiaxial Loading

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Two weakest-link fracture statistics formulations for multiaxial loading, Batdorf's flaw density and orientation distribution approach and Evans' elemental strength approach, are compared for identical fracture criteria and distribution functions. It is demonstrated that despite some fundamental differences in the methodology used in calculating fracture probabilities for multiaxial loading the two approaches give identical predictions. A recent contradictory conclusion reported in the literature is shown to be incorrect. [Key words: mechanical properties, modeling, flaws, statistics, fracture.]

I. Introduction

THREE different statistical formulations are commonly used to predict fracture probabilities of ceramics in general multiaxial loading: (a) Weibull's approach, (b) Batdorf's flaw density and orientation approach, and (c) Evans' elemental strength approach.

For multiaxial loading, Weibull[1] proposed the following equation for the probability of failure, F:

$$F = 1 - \exp\left[-\int_A\left\{\int_\omega\left(\frac{\sigma_n}{\sigma_0}\right)^m d\omega\right\}dA\right] \tag{1}$$

where A is the surface area stressed in tension, σ_n is the normal tensile stress in an arbitrary direction relative to the principal stresses, σ_0 and m are the two Weibull parameters called scale and shape parameter, respectively, and $d\omega$ is an elemental solid angle on a unit sphere in the direction of σ_n. The integration over the solid angle is carried out for all orientations for which σ_n is tensile. Weibull assumed that the probability of failure was zero for orientations for which σ_n was compressive.

Although Weibull's fracture statistics formulation is used frequently in the analysis of multiaxial fracture of ceramics,[2–4] the method suffers from several intrinsic deficiencies. Physical characteristics of the strength-controlling flaws are not explicitly considered in the analysis. Thus, for example, in the case of cracklike flaws the orientation relationship between the cracks and the applied stresses via a suitable fracture criterion and the cumulative effect on fracture probability is not taken into account. In other words, a formal link between statistical fracture theory and fracture mechanics theory is not established in Weibull's formulation.

Batdorf's flaw density and orientation approach[5,6] and Evans' elemental strength approach[7] are two physically based, weakest-link formulations of fracture statistics for multiaxial loading. They explicitly consider cracklike flaws, their size and orientation distributions and specific fracture criteria based on linear elastic fracture mechanics theory. There are, however, some fundamental differences between the two approaches. Batdorf assumed an intrinsic (i.e., material characteristic) flaw-size distribution that is independent of the applied stress state and expressed it in terms of a critical stress normal to the plane of the crack. Strength distributions in different stress states are assessed by calculating the contributions to the cumulative fracture probability of all the flaws for which an effective stress (defined by a pertinent fracture criterion) exceeds the critical stress. On the other hand, Evans' formulation, following the methodology of Matthews *et al.*,[8] does not make an a priori assumption regarding the flaw-size distribution. An elemental strength distribution for a reference stress state (for example, uniaxial) is first established from test data. This base strength distribution is correlated to strength distributions in other stress states through a consideration of flaw orientations, normal and shear stress components acting on flaws, and a suitable fracture criterion. Despite these important differences, the equivalence of the two approaches was suggested when both Batdorf[6] and Evans[7] obtained good agreements between the biaxial strengths reported by Giovan and Sines[9] for an alumina ceramic and their theoretical predictions based on uniaxial strength.

Recently, Lamon[10] employed Weibull, Batdorf, and Evans approaches to analyze the uniaxial and biaxial strength data for an alumina ceramic reported by Shetty *et al.*[11] The Weibull and Batdorf analyses underestimated the inert biaxial strengths based on the uniaxial strengths, while Evans' elemental strength approach gave satisfactory correlation. This result is surprising because of the previously demonstrated equivalence of the two approaches. Lamon,[10] however, did use a critical noncoplanar strain energy release rate fracture criterion as opposed to the coplanar strain energy release rate criterion originally used by Batdorf[6] and Evans[7] in their analyses. This study was, therefore, undertaken to critically compare the predictions of the two approaches for identical flaw-size and elemental strength distributions and fracture criteria. The objective was to see if the equivalence of the two approaches was valid for different fracture criteria or if it was true only for specific fracture criteria as suggested by Lamon's recent results.

II. Theoretical Formulations and Calculations

In Batdorf's approach,[5,6] the cumulative probability of fracture, F, is given by the following equation:

$$F = 1 - \exp\left[-\int_A\int_0^{\sigma_h}\frac{\Omega}{4\pi}\frac{dN(\sigma_c)}{d\sigma_c}\,d\sigma_c\,dA\right] \tag{2}$$

where σ_c is the critical normal stress of a particular crack; and σ_h, depending on the fracture criterion, is the highest value that σ_c can achieve. It can be greater than the applied principal stress, σ, for some fracture criteria with strong shear sensitivity. Ω is a solid angle in the principal stress space enclosing all the normals to crack planes so that an effective stress σ_e, which is a function of the principal stress, σ, and crack orientation, will satisfy the fracture criterion. $N(\sigma_c)$ is the crack-size distribution function representative of the test material. It may characterize a surface flaw population in

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which case the stress integration is carried out over the stressed area, as shown in Eq. (2). For volume flaws, the integration is over the stressed volume. One can also have a situation of concurrent surface and volume flaws. For analytical simplicity, the flaw-size distribution function is assumed here to be the following form:

$$N(\sigma_c) = \bar{k}(\sigma_c)^m \tag{3}$$

where $\bar{k}$ and m are the scale and shape parameters, respectively, in analogy with the Weibull parameters.

Note that the first integral, which integrates the critical stress, involves two factors: flaw-size distribution and orientation distribution. For a specified infinitesimal critical stress range, the solid angle, Ω, that represents the flaw orientations that contribute to fracture is first determined. For equitriaxial stress state, Ω is 4π. For other cases, Ω is a function of the applied stress state, the particular fracture criterion used, and the orientation distribution of the flaws. Figure 1 shows the geometric representation of Ω in the three-dimensional stress space for uniaxial and biaxial stress states for random distribution of flaw orientations. Note in Fig. 1 that θ_c is now determined by the specific fracture criterion employed and is expressed as a function of σ and σ_c. Once Ω is expressed in terms of σ and σ_c, the two integrations, one for σ_c over the range 0 to σ_h and the other over the stressed area (or volume), are carried out either analytically or by numerical methods.

The elemental strength approach for surface flaws takes the following form:

$$F = 1 - \exp\left[-\int_A dA \int_0^S g(S)\, dS\right] \tag{4}$$

where S is the fracture strength, and $g(S)\, dS$ is the number of flaws per unit area with a strength between S and $S + dS$. In this approach, size and orientation distributions of flaws are still considered separately, but the physical bases for their consideration are different from Batdorf's. Assuming that the lower bound strength pertinent to the flaw population of concern is negligible compared with the applied stress, McClintock's strength density function,[12] similar to Weibull's two-parameter function, is used to describe the equitriaxial strength density function $g_T(S_T)$.

$$g_T(S_T) = \frac{mS_T^{m-1}}{S_0^m} \tag{5}$$

Strength density functions for equibiaxial (or uniaxial) stress states are, then, derived from this base strength density function using relationships between single-flaw strength in equitriaxial stress state, S_T, and equibiaxial stress state, S_B (or uniaxial stress state, S_U) based on an appropriate fracture criterion and flaw orientation. Two equivalent sectorial elements oriented at an angle θ, with respect to the applied stress, are integrated from zero to $\pi/2$ in the principal stress space to get the distribution function for all flaw orientations. Figure 2 shows the orientation relations used in developing

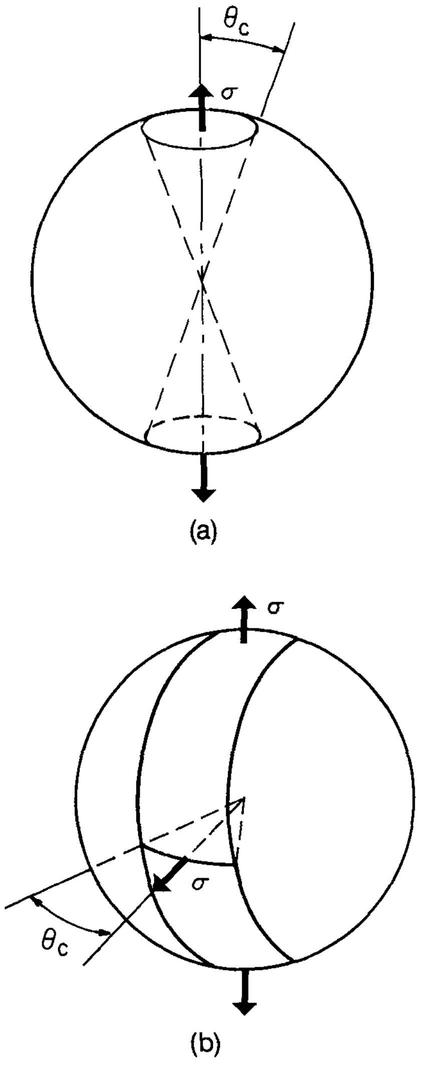

Fig. 1. Solid angles (Ω) in the three-dimensional stress space for uniaxial (a) and equibiaxial (b) stress states used in the flaw density and orientation approach (after Batdorf and Crose).[5]

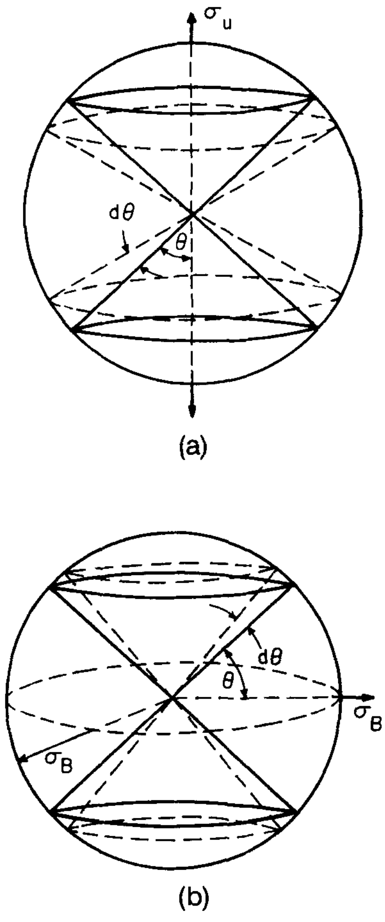

Fig. 2. Orientation relations in uniaxial (a) and equibiaxial (b) stress states developed in the elemental strength approach (after Evans).[7]

the strength distribution functions for uniaxial and equibiaxial stress states.

It should be evident from the above discussion that there is a fundamental difference between the two approaches in extending the fracture statistics from a reference stress state, say equitriaxial, to other stress states such as biaxial or uniaxial. In Batdorf's approach, fracture statistics in different stress states are linked to a material-characteristic flaw-size distribution function through the orientation factor Ω which varies with stress state. Evans, on the other hand, converts strength distribution functions from one stress state to another without making any specific assumptions regarding flaw-size distributions. Only for the equitriaxial stress state the equivalence of the two approaches becomes apparent if $dN(\sigma_c)/d\sigma_c$ and $g(S_T)$ are selected to have the same functional forms and Ω takes the value of 4π. For other stress states, the equivalence is not obvious and one must carry out the respective integrations over the orientations to demonstrate the equivalence.

In this paper, the critical normal stress (shear insensitive) and a noncoplanar strain energy release rate (shear sensitive) criteria are used to demonstrate the calculations for uniaxial and equibiaxial stress states. In fracture mechanics terms, the critical normal stress fracture criterion takes the form

$$K_{\rm I} = K_{\rm IC} \tag{6}$$

i.e., a crack will extend when the mode I stress intensity, $K_{\rm I}$, equals or exceeds $K_{\rm IC}$, the critical stress intensity or fracture toughness of the material. The noncoplanar strain energy release rate criterion for combined mode I and mode II loading can be adequately represented by the following empirical equation:[13]

$$\left[\frac{K_{\rm I}}{K_{\rm IC}}\right] + \left[\frac{K_{\rm II}}{CK_{\rm IC}}\right]^2 = 1 \tag{7}$$

Equation (7) was originally suggested by Palaniswamy and Knauss[13] with a constant $C = (2/3)^{1/2} = 0.82$ to fit their numerical results of the maximization of strain energy release rate for noncoplanar crack extension under mode I and mode II loading. Singh and Shetty[14] have shown that fracture toughness envelopes for polycrystalline ceramics in combined mode I and mode II loading can also be described by Eq. (7) with the constant C taking values in the range 1.5 to 2. The higher values of C have been attributed to increased resistance of polycrystalline ceramics in mode II loading due to the interlocking surface asperities on crack faces.

The integrals within the exponential brackets of both formulations (Eqs. (2) and (4)) were calculated for the above two fracture criteria assuming randomly oriented penny-shaped cracks. The stressed areas are expressed as A_{eff}, effective areas stressed uniformly by the maximum stress. A_{eff} is dependent on the specimen or component geometry and stress gradients. The final expressions are summarized as follows.

(1) Uniaxial Stress State

(A) Critical Normal Stress Criterion

(1) Flaw density and orientation (Batdorf) formulation:

$$F = 1 - \exp\left[-\frac{\bar{k}\sigma_u^m}{2m+1}A_{effu}\right] \tag{8}$$

(2) Elemental strength (Evans) formulation:

$$F = 1 - \exp\left[-{\rm I}_u\left(\frac{S_u}{S_0}\right)^m A_{effu}\right] \tag{9}$$

where I_u is given by

$$\mathrm{I}_u = \int_0^{\pi/2} [\cos^2\theta]^m \sin\theta\, d\theta \tag{10}$$

where θ is the orientation angle as shown in Fig. 2.

(B) Noncoplanar Strain Energy Release Rate Criterion

(1) Flaw density and orientation (Batdorf) formulation:

$$F = 1 - \exp[-\bar{k}m\sigma_u^m(\mathrm{I}_h + \mathrm{I}_h')A_{effu}] \tag{11}$$

where I_h and I_h' are given by

$$\mathrm{I}_h = \int_0^1 (1 - \cos\theta_C)\left(\frac{\sigma_C}{\sigma}\right)^{m-1} d\left(\frac{\sigma_C}{\sigma}\right) \tag{12a}$$

$$\mathrm{I}_h' = \int_1^{\sigma_h/\sigma} (\cos\theta_{C1} - \cos\theta_{C2})\left(\frac{\sigma_C}{\sigma}\right)^{m-1} d\left(\frac{\sigma_C}{\sigma}\right) \tag{12b}$$

θ_C or $(\theta_{C1}, \theta_{C2})$ are the root or roots of Eq. (7) given as

$$\sin\theta_C = \sin\theta_{C2} = \left[1 - \frac{(1 + C^2X) - \sqrt{1 + 2C^2X - (4C^2 - C^4)X^2}}{2}\right]^{1/2} \tag{13}$$

$$\sin\theta_{C1} = \left[1 - \frac{(1 + C^2X) + \sqrt{1 + 2C^2X - (4C^2 - C^4)X^2}}{2}\right]^{1/2} \tag{14}$$

$$X = \left(\frac{\sigma_C}{\sigma}\right)$$

(2) Elemental strength (Evans) formulation: The same form as Eq. (9) would be obtained except I_u is now given by

$$\mathrm{I}_u = \int_0^{\pi/2}\left[\frac{\cos^2\theta + \cos\theta\sqrt{\cos^2\theta + \dfrac{4}{C^2}\sin^2\theta}}{2}\right]^m \sin\theta\, d\theta \tag{15}$$

(2) Equibiaxial Stress State

(A) Critical Normal Stress Criterion

(1) Flaw density and orientation (Batdorf) formulation:[9]

$$F = 1 - \exp\left[-\frac{\sqrt{\pi}\,\Gamma(m)\bar{k}m\sigma_b^m}{(2m+1)\Gamma\left(m + \frac{1}{2}\right)}A_{effb}\right] \tag{16}$$

(2) Elemental strength (Evans) formulation:

$$F = 1 - \exp\left[-\mathrm{I}_b\left(\frac{S_b}{S_0}\right)^m A_{effb}\right] \tag{17}$$

where I_b is given by

$$\mathrm{I}_b = \int_0^{\pi/2} [\cos^2\theta]^m \cos\theta\, d\theta \tag{18}$$

(B) Noncoplanar Strain Energy Release Rate Criterion

(1) Flaw density and orientation (Batdorf) formulation:

$$F = 1 - \exp[-\bar{k}m(\mathrm{I}_a + \mathrm{I}_a')\sigma_b^m A_{effb}] \tag{19}$$

where I_a and I_a' are

$$\mathrm{I}_a = \int_0^1 \sin\theta_C\left(\frac{\sigma_C}{\sigma}\right)^{m-1} d\left(\frac{\sigma_C}{\sigma}\right) \tag{20a}$$

$$\mathrm{I}_a' = \int_1^{\sigma_h/\sigma} (\sin\theta_{C2} - \sin\theta_{C1})\left(\frac{\sigma_C}{\sigma}\right)^{m-1} d\left(\frac{\sigma_C}{\sigma}\right) \tag{20b}$$

where θ_C or $(\theta_{C1}, \theta_{C2})$ have the same expressions as in the uniaxial case.

(2) Elemental strength (Evans) formulation: The same form as Eq. (17) would be obtained except I_b is now given by

$$\mathrm{I}_b = \int_0^{\pi/2}\left[\frac{\cos^2\theta + \cos\theta\sqrt{\cos^2\theta + \dfrac{4}{C^2}\sin^2\theta}}{2}\right]^m \cos\theta\, d\theta \tag{21}$$

Note that in the above equations $\sigma_u(S_u)$ and $\sigma_b(S_b)$ denote the maximum tensile stresses in the specimens or components subjected to uniaxial and equibiaxial stress states, respectively.

The equivalence of the two formulations can now be examined by comparing the final expressions. The most convenient way to compare the two formulations is to write expressions for the ratios of the maximum stresses in equibiaxial and uniaxial stress states at the same fracture probability and effective stressed areas, $A_{effb} = A_{effu}$. These expressions for σ_b/σ_u and S_b/S_u corresponding to the Batdorf and Evans formulations, respectively, are listed in Table I for the two fracture criteria. Note that these stress ratios are functions of only the shape parameter, m. If these stress ratios are equal for different fracture criteria, then it can be concluded that Batdorf's and Evans' formulations are exactly equivalent. It is interesting to note from Table I that the expressions for the stress ratios look sufficiently different that one cannot conclude a priori without quantitative calculations that the two formulations are equivalent.

The expressions for the stress ratios listed in Table I were evaluated for different values of m. The integrals listed in the table were evaluated numerically using Simpson's rule.[15] The accuracy of the numerical integrations were ensured by checking convergence with increasing sensitivity. For both fracture criteria, the ratios of the stresses in equibiaxial and uniaxial stress states were equal in the two formulations; i.e., $\sigma_b/\sigma_u = S_b/S_u$. Thus, the calculations indicated that Batdorf and Evans formulations are equivalent for both normal and shear-sensitive fracture criteria. Figure 3 shows a plot of t stress ratio as a function of the shape parameter, m. T stress ratios are less than 1 for all values of m, which impl biaxial weakening relative to uniaxial stress state.

III. Discussion

The results of this study have demonstrated that the t physically based fracture statistics formulations of Batd and Evans are exactly equivalent in their predictions of fr ture probabilities in multiaxial loading. This equivalence easily recognized for the equitriaxial stress state, but it is r readily apparent a priori for biaxial and uniaxial stress stat This is due to the difference between the two approaches the manner fracture statistics in the equitriaxial stress st are correlated with the corresponding fracture statistics equibiaxial and uniaxial stress states. The calculations in t study were performed for two limiting cases of the shear se sitivity of the flaws: the normal stress criterion which sumes complete shear insensitivity and an empirical fractu criterion that closely simulates the shear sensitivity of t noncoplanar strain energy release rate criterion. Although t calculations were restricted to the equibiaxial stress state, i believed that the equivalence would be true for unbalanc biaxial stress states as well.

The initial motivation for this study was the unexpect result reported by Lamon,[10] which showed different theore cal predictions based on the two theories for fracture prol bilities of alumina disks tested in flexure. Lamon[10] used t

Table I. Ratios of Maximum Principal Stresses in Equibiaxial and Uniaxial Stress States for Identical Fracture Probabil Formulated According to Batdorf's (σ_b/σ_u) and Evans' (S_b/S_u) Approaches

Fracture criterion	σ_b/σ_u	S_b/S_u
Critical normal stress (Eq. (6))	$\left[\dfrac{\Gamma(m+1/2)}{\sqrt{\pi}\Gamma(m)m}\right]^{1/m}$	$\left[\dfrac{I_u(\text{Eq. (10)})}{I_b(\text{Eq. (18)})}\right]^{1/m}$
Noncoplanar strain energy release rate (Eq. (7))	$\left[\dfrac{I_h + I_h'}{I_a + I_a'}\right]^{1/m}$	$\left[\dfrac{I_u(\text{Eq. (15)})}{I_b(\text{Eq. (21)})}\right]^{1/m}$

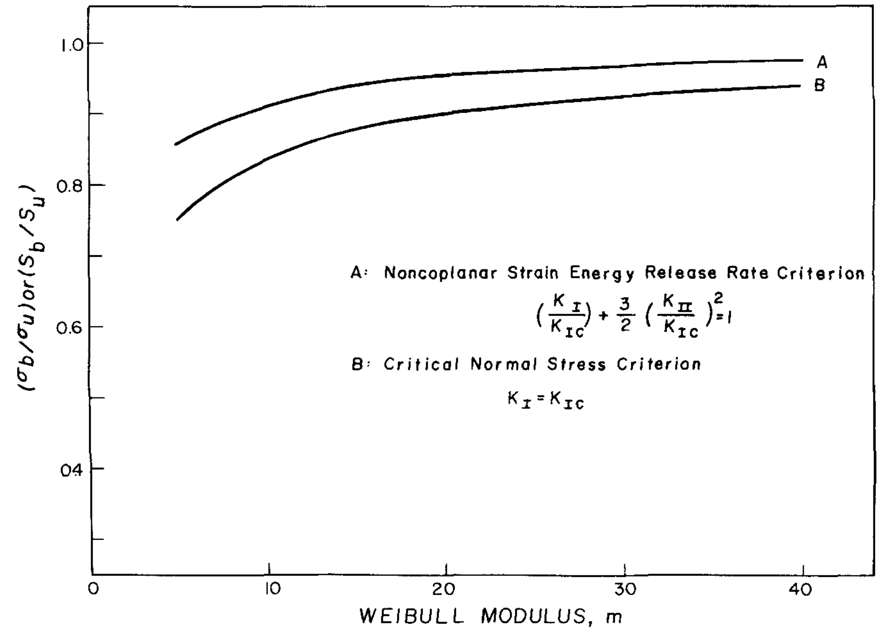

Fig. 3. Ratios of equibiaxial and uniaxial fracture stresses for equal fracture probabilities and identical uniformly stressed areas based on a noncoplanar strain energy release rate and critical normal stress fracture criteria.

following noncoplanar strain energy release rate criterion proposed by Hellen and Blackburn:[16]

$$G_{max} = \frac{(1+\nu)(1+\chi)}{4E}[K_{\rm I}^4 + 6K_{\rm I}^2K_{\rm II}^2 + K_{\rm II}^4]^{1/2} \qquad (22)$$

where E is Young's modulus, ν is Poisson's ratio, $\chi = (3 - 4\nu)$ under plane strain conditions, and $\chi = (3 - \nu)/(1 + \nu)$ under plane stress conditions. This fracture criterion represents an intermediate case of shear sensitivity of fracture and corresponds very closely to Eq. (7) with $C = 1.0$. Equation (22) was also used in the Batdorf and Evans formulations and found to give identical results. Thus, it is safe to conclude that the two formulations lead to same predictions irrespective of the fracture criteria. The results of this study do not support the conclusion reached by Lamon.[10]

IV. Conclusions

(1) The flaw-size and orientation approach developed by Batdorf and the elemental strength approach of Evans are shown to be equivalent in their predictions of fracture probability in multiaxial loading.

(2) The equivalence of the two approaches is readily apparent in equitriaxial stress state, but it is not so obvious for equibiaxial and uniaxial stress states which depend on specific fracture criteria.

(3) Recent conclusions in the literature regarding intrinsic superiority of one or the other approach are incorrect.

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