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DYNAMICS OF A FLOW-FORCED HEAT EXCHANGER

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The partial differential equation and boundary conditions describing the temperature response of a steam-water heat exchanger to an arbitrary variation in water velocity are solved. The solution is compared with the approximate linearized solution for a step change in velocity. It is found that significant error can result in the approximate solution for large magnitudes of step change or for high exchange to heat capacity ratios.

THE EQUATIONS DESCRIBING the dynamics of flow-forced heat exchangers have been solved only in their linearized form (7). The solutions obtained in this manner may be expected to give good accuracy only if the magnitude of the flow disturbance is small compared with the magnitude of the steady-state flow rate. It is pertinent to seek quantitative measures of the accuracy of these linearized solutions. To this end, a simple case of flow-forced heat exchange, for which an analytic solution may be found, is considered.

Description of System

Consider a heat exchanger in which one of the fluids remains at a constant, uniform temperature, independent of the other fluid. Such conditions are approximated in an exchanger with a condensing vapor as one of the fluids or in an exchanger where the flow rate or specific heat of one of the fluids is extremely high compared with that of the other. For convenience,

designate the constant temperature fluid as steam and the other as water. Assume that the water is in plug flow, has constant physical properties, and exhibits perfect radial mixing and negligible back-mixing. In addition, it may be assumed that the energies of compression and viscous dissipation are small compared with the other energies carried by the water stream.

Derivation of Equations

With these restrictions, the energy balance describing the temperature behavior of the water may be written:

$$\rho C_p \frac{\partial T}{\partial t'} + \rho(\bar{v} + \bar{v}') C_p \frac{\partial T}{\partial x'} = \frac{UA_h}{A_c} (W - T) \quad (1)$$

Define the dimensionless variables:

$$x = \frac{x'}{L}$$

$$t = \frac{\bar{v}t'}{L}$$

$$r = \frac{\bar{v}}{\bar{v}}$$

$$\theta = \frac{T - W}{T_0 - W}$$

$$P = \frac{UA_h L}{\rho \bar{v} C_p A_c} = \frac{UA}{w C_p}$$

In these variables, Equation 1 becomes:

$$\frac{\partial \theta}{\partial t} + (1 + r) \frac{\partial \theta}{\partial x} = -P\theta \quad (2)$$

It is well known that U will be a function of the velocity. If U_0 is the value of U when $r=0$ (i.e., at the steady state), then:

$$U = U_0(1 + r)^b \quad (3)$$

is the relationship suggested by standard heat transfer correlations. From Taylor's series:

$$(1 + r)^b = 1 + br + \frac{b(b-1)}{2!} r^2 + \frac{b(b-1)(b-2)}{3!} r^3 + \dots$$

it is evident that the approximation:

$$(1 + r)^b \approx 1 + br \quad (4)$$

is of reasonable accuracy. Thus, for a 100% increase in flow rate, with the worst value of b , 0.53, the error is only 8%, which is less than the error inherent in this form of correlation. Hence, for the present study, Equation 4 is accepted as the actual dependence of heat transfer coefficient on flow rate, and Equation 2 becomes:

$$\frac{\partial \theta}{\partial t} + (1 + r) \frac{\partial \theta}{\partial x} = -P_0(1 + br)\theta \quad (5)$$

where:

$$P_0 = \frac{U_0 A}{w C_p}$$

Equation 5 may be solved by the method of characteristic equations (2). According to this method, the equation is rewritten as two ordinary differential equations:

$$dt = \frac{dx}{1 + r} = - \frac{d\theta}{P_0\theta(1 + br)} \quad (6)$$

Equating the first and second members and solving:

$$x - t - \int_0^t r(\tau) d\tau = C_1$$

Similarly, using the first and third members:

$$\frac{\ln \theta}{P_0} + t + b \int_0^t r(\tau) d\tau = C_2$$

Here, C_1 and C_2 are arbitrary constants. The general solution to Equation 5 is obtained by putting $C_2 = f(C_1)$, where f is an arbitrary function. Thus:

$$\frac{\ln \theta}{P_0} + t + b \left[\int_0^t r(\tau) d\tau \right] = f \left\{ t + \int_0^t r(\tau) d\tau - x \right\} \quad (7)$$

The condition $\theta(0, t) = 1$ yields, when inserted into Equation 7:

$$t + b \int_0^t r(\tau) d\tau = f \left\{ t + \int_0^t r(\tau) d\tau \right\} \quad (8)$$

This is a functional equation for the unknown function f , in which t is a dummy variable. To solve it, set:

$$y = t + \int_0^t r(\tau) d\tau \quad (9)$$

Then:

$$t = y - \int_0^{\phi(y)} r(\tau) d\tau \quad (10)$$

where $\phi(y)$ is a function satisfying:

$$\phi(y) = y - \int_0^{\phi(y)} r(\tau) d\tau \quad (11)$$

Substitution of Equations 9 and 10 into Equation 8 yields the solution of Equation 8:

$$f(y) = y - (1 - b) \int_0^{\phi(y)} r(\tau) d\tau \quad (12)$$

Now that the function f is known, it may be used in Equation 7 to obtain the solution for θ :

$$\frac{\ln \theta}{P_0} = (1 - b) \int_0^t r(\tau) d\tau - x \\ \phi \left[t + \int_0^t r(\tau) d\tau - x \right] \quad (13)$$

The function ϕ may be eliminated from the solution, if desired. By the definition of ϕ , Equation 11:

$$\phi \left[t + \int_0^t r(\tau) d\tau - x \right] = t + \int_0^t r(\tau) d\tau - x \\ \phi \left[t + \int_0^t r(\tau) d\tau - x \right] \quad (14)$$

Furthermore, if $b = 0$, from Equation 13:

$$\frac{\ln \theta_0}{P_0} = \int_0^t r(\tau) d\tau - x \\ \phi \left[t + \int_0^t r(\tau) d\tau - x \right] \quad (15)$$

where θ_0 designates the solution for $b = 0$.

Comparison of Equations 14 and 15 indicates:

$$\phi \left[t + \int_0^t r(\tau) d\tau - x \right] = \frac{\ln \theta_0}{P_0} + t \quad (16)$$

so that Equation 13 becomes:

$$\frac{\ln \theta_0}{P_0} = \int_0^t r(\tau) d\tau - x \\ t + \frac{\ln \theta_0}{P_0} \quad (17)$$

which is an implicit equation for θ_0 , the temperature behavior when U is independent of r . It is clear from Equations 13 and 17 that:

$$\frac{\ln \theta}{P_0} + x = (1 - b) \left[\frac{\ln \theta_0}{P_0} + x \right] \quad (18)$$

The solution of Equation 18 is:

$$\theta = e^{-Pbx\theta_0^{(1-b)}} \quad (19)$$

Thus, Equations 17 and 19 form the solution to Equation 5, a result easily checked by differentiation.

Application. Consider a step change in the flow rate:

$$r(t) = \begin{cases} 0 & t < 0 \\ a & t > 0 \end{cases}$$

In this case, it is convenient to use Equation 11 to establish $\phi(y)$ and then use Equation 13 to obtain θ :

$$\phi(y) = y - a\phi(y) \\ \phi(y) = \frac{y}{1+a}$$

$$\phi \left[t + at - x \right] = t - \frac{x}{1+a}$$

$$\frac{\ln \theta}{P_0} = a(1-b)t - x; \quad t < \frac{x}{1+a}$$

$$\frac{\ln \theta}{P_0} = \frac{a(1-b)x}{1+a} - x = \frac{-(1+ab)x}{1+a}; \quad t > \frac{x}{1+a}$$

The solution is thus:

$$\theta = e^{-P_0[x - a(1-b)t]} \quad t < \frac{x}{1+a} \quad (20)$$

$$= e^{-P_0 \left[\frac{x(1+ab)}{1+a} \right]} \quad t > \frac{x}{1+a} \quad (21)$$

There are two time domains in the response. The first, described by Equation 20, operates until all of the fluid which was in the exchanger at time zero has passed the point x . The second, described by Equation 21, operates after that. In the second time domain, the temperature has reached a new steady-state value because the fluid now passing point x has traveled at the increased velocity during the entire heat transfer process.

Linearized Solution. Equation 5 may be linearized, to study the response when $r(t)$ is small compared to unity. The term "linearized" is a commonly used misnomer, in this case. Actually Equation 5 is linear. The dependence of r on the independent variable t in no way affects the property that any linear combinations of solution is also a solution. The linearity probably accounts for the ease of integrability of Equation 6.

Let:

$$\theta(x, t) = \bar{\theta}(x) + \tilde{\theta}(x, t) \quad (22)$$

where $\tilde{\theta}$ is small compared with $\bar{\theta}$, and $\bar{\theta}$ satisfies:

$$\frac{d\bar{\theta}}{dx} = -P_0\bar{\theta} \quad (23)$$

Substituting Equations 22 and 23 into Equation 5, subtracting Equation 23, and neglecting the term $r \frac{\partial \tilde{\theta}}{\partial x}$, yields:

$$\frac{\partial \tilde{\theta}}{\partial t} + \frac{\partial \tilde{\theta}}{\partial x} = P_0 r(1-b)e^{-P_0 x} - P_0 \tilde{\theta} \quad (24)$$

It is evident that the solution to Equation 23, $\bar{\theta} = e^{-P_0 x}$, represents the steady-state behavior of the exchanger. Since, for a step input, the exchanger is disturbed from steady state at time zero, we have $\tilde{\theta}(x, 0) = 0$.

The Laplace transform of Equation 24 is:

$$\frac{d\tilde{\theta}}{dx} + (s + P_0)\tilde{\theta} = RP_0(1-b)e^{-P_0 x} \quad (25)$$

where $R(s)$ is the Laplace transform of $r(t)$. The solution to Equation 25, with the boundary condition $\tilde{\theta}(0, s) = 0$ (constant entrance temperature) is:

$$\tilde{\theta} = \frac{P_0 R(1-b)e^{-P_0 x}}{s} [1 - e^{-sx}] \quad (26)$$

The inverse transform of Equation 26 is:

$$\tilde{\theta} = P_0 e^{-P_0 x} (1-b) \int_0^t r(\tau) [1 - S_x(t-\tau)] d\tau \quad (27)$$

where:

$$S_x(t) = \begin{cases} 0 & t < x \\ 1 & t > x \end{cases}$$

Clearly, Equation 27 is equivalent to:

$$\tilde{\theta} = P_0 e^{-P_0 x} (1-b) \int_{t-x}^t r(\tau) d\tau \quad (28)$$

Using a step input for $r(t)$, and adding $\bar{\theta}$ to $\tilde{\theta}$, results in:

$$\theta = e^{-P_0 x} [1 + P_0 a(1-b)t] \quad t < x$$

$$= e^{-P_0 x} [1 + P_0 a(1-b)x] \quad t > x \quad (29)$$

Discussion of Results

Comparison of the approximate solution, Equation 29, with the actual response to a step input Equations 20 and 21, shows that the exponential $e^{P_0 a(1-b)t}$ has been linearized. The accuracy of the approximate response during the first time domain is thus the same as the accuracy of $(1+y)$ for e^y and depends only on the parameter $P_0 a(1-b)$. This accuracy will be poor at high arguments, which means that the approximate solution should not be used for large step inputs or for exchangers with high heat exchange to heat capacity ratios, P_0 . Further, the point of separation of the two time domains is given by the approximate solution as $t = x$. For large step inputs, this will be considerably in error compared with the true time, $t = x/(1+a)$.

A graphic comparison of the true and approximate responses at the exchanger exit, $x = 1$, is given for the case $P_0 a(1-b) = 1$ in Figure 1. Curves are given for various values of a , in the form $e^{P_0 \theta}$ vs. t . It may be seen that for this case, the approximate response will be considerably in error, even for small values

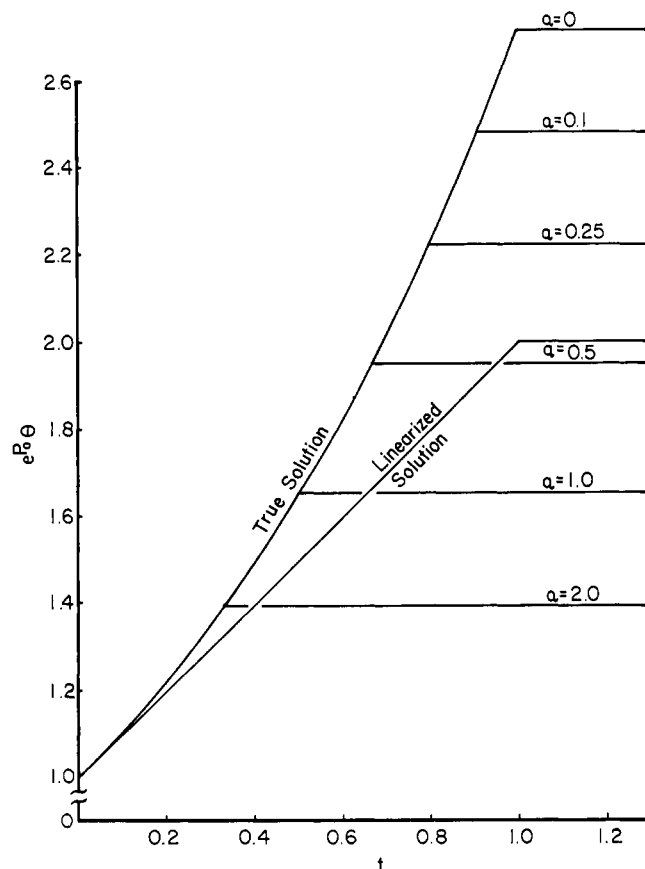


Figure 1. Comparison of exit temperature responses

$$P_0 a(1-b) = 1$$

Curve for $a = 0$ represents limiting condition as $a \rightarrow 0$ and $P \rightarrow \infty$

of a . This is because of the relatively high value of P_0 . Thus, if it is assumed that $b = 0.8$, then $P_0 a(1 - b) = 1$ and $a = 0.1$ require that $P_0 = 50$. In a more typical case, $P_0 a(1 - b) = 0.1$, $a = 0.1$, and the two solutions differ by less than 1% for all t .

Conclusions

A sufficiently high value of P_0 , the heat exchange to heat capacity ratio, will cause considerable error in the linearized solution, even for relatively small step inputs. For typical values ($P_0 \sim 1$, $a \sim 0.1$, $b \sim 0.5$) of the pertinent variables, the linearized solution is adequate to represent the response of the exchanger.

If $b = 1$, the exchanger cannot be disturbed from steady state by velocity changes, because the heat exchange and heat capacity effects offset each other. A high value of a , the magnitude of the step input, will cause considerable error in the linearized solution. This conclusion was expected.

The results of this work suggest that further study on the behavior of flow-forced heat exchangers is warranted. The case of distributed fluids on both sides of the exchanger, as well as other common configurations, should be studied to assess the error inherent in linearized solutions.

Nomenclature

A	= total heat transfer area
A_c	= water-side cross-sectional flow area
A_h	= heat transfer area per unit length of exchanger
a	= dimensionless magnitude of step change in velocity
b	= exponent for velocity dependence of heat transfer coefficient
C_p	= specific heat

C_1, C_2	= arbitrary constants used in solving Equation 5
f	= arbitrary function used in solving Equation 5
L	= length of heat exchanger
P	= heat exchange to heat capacity ratio, UA/wC_p
P_0	= value of P for $U = U_0$
$R(s)$	= Laplace transform of $r(t)$
r	= dimensionless velocity disturbance
S	= unit step function
s	= Laplace transform variable
T	= water temperature
T_0	= entering water temperature
t	= dimensionless time
t'	= time
U	= over-all heat transfer coefficient
U_0	= value for U for $r = 0$
\bar{v}	= steady-state velocity
$\bar{v}(t)$	= disturbance in water velocity
W	= steam temperature
w	= mass flow rate, mass/time
x	= dimensionless distance from exchanger inlet
x'	= distance from exchanger inlet
y	= dummy variable used in definition of ϕ
θ	= dimensionless water temperature
θ_0	= solution of Equation 5 when $b = 0$
$\bar{\theta}$	= steady-state solution of Equation 5
$\hat{\theta}$	= perturbation solution of Equation 5
ρ	= density
τ	= dummy variable of integration
$\phi(y)$	= function used in solving Equation 5

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THEORETICAL TIME-TEMPERATURE BEHAVIOR OF A SUPERCRITICAL MASS OF A SELF-HEATING CHEMICAL. *Simulation on a Digital Computer*

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By definition, the critical dimensions or mass (the size beyond which temperature increases without limit) of a self-heating chemical cannot be measured directly. Hence, experimental work must be done with supercritical masses whose allowable size depends upon the time available to the experimenter and upon the risk of eventual explosion which can be tolerated. As an aid to planning such experiments, an IBM 704 computer has been used to find the numerical solutions of Fourier's equation, including a power generation term and associated initial and boundary conditions, thus simulating the theoretical dynamic behavior of a mass of a self-heating chemical. The program has been used to simulate the behavior of spheres of a material similar to a mixture of NH_4NO_3 and fuel oil.

PRIOR TO 1939 there was no published definitive work on the concept of the critical state as applied to self-heating chemicals. In that year, Frank-Kamenetzky published a report (1) on the role of criticality in the thermal behavior of self-heating chemicals. He stated the general conditions that may cause a mass of self-heating chemicals to heat spontaneously to a point at which explosion may occur.

Accidents have often occurred involving mixtures of NH_4NO_3 and organic materials which might be better understood if there existed experimental and theoretical knowledge of how constitution and thermal history affect critical dimensions. Hainer made a complete report to the Commandant, U. S. Coast Guard Headquarters, in 1952 on a research project seeking to account for the Texas City, Tex., disaster (2).