Proof of theorem 1

We will use the following definitions:

- $\mathcal{V}$, the set of cache-enabled routers/nodes in the network with cardinality V.
- $\mathcal{M}$, a set of M equally unit sized content items.
- ζ, the new users' connection rate.
- ϕ, the connented user's disconnection rate.
- z, the Zipf distribution exponent of the content items popularity.
- ϑ_m, $m \in \mathcal{M}$ the popularity of item m in the Zipf distribution.
- r_m, the aggregate incoming request rate for an information item $m \in \mathcal{M}$.

where r_m is given by:

$$r_m = \zeta \vartheta_m = \zeta \frac{1/k^z}{\sum_{i=1}^{M} 1/i^z}, \tag{1}$$

assuming that the particular item is ranked k-th out of the M information items within the Zipf distribution.

Probability of absorption into state 0

The probability of retrieving a requested item $m \in \mathcal{M}$ at time $t > 0$, assuming that at time $t = 0$ the network fragments and the content origin for that particular item is not reachable, depends only on the probability that another user has already retrieved that item in the past and is still connected to the network (*i.e.,* assuming zero router cache capacity).

We define as $\{X_m(t), 0 \leq t < \infty\}$ the Markov process with stationary transition probabilities (where the possible values of $X_m(t)$ are non-negative integers), that depicts the number of users which have already retrieved item m and are connected to the network at time t. Clearly, if at any time instance $t' \geq 0$, $X_m(t') = 0$, the requested item can no longer be retrieved, since *(i)* it is not cached in the network, *(ii)* the content origin is not reachable and *(iii)* there are no connected users who have previously retrieved the item and can assist in content retrieval.

According to the stochastic modelling theory $X_m(t)$ is a *birth and death* process with one absorbing state. We define as the zero state, the state at which $X_m(t) = 0$. This is an absorbing state (no user with a cached copy of item m is attached at a router of the network), since after that state the requested item cannot be retrieved. The probability of absorption into state 0 is not, a priori, a certain event, since the states of the process may wander forever among the states $(1, 2, \ldots)$ or possibly drift to infinity.

We define as $\lambda_m(n)$ the birth rate of the process when the process is at state n (n connected users in the network who have the content item m) and as $\mu_m(n)$ the death rate of the same process. Clearly $\lambda_m(0) = 0$. In our case we have for $\lambda_m(n)$:

$$\lambda_m(n) = \begin{cases} 0 & \text{if } n = 0, \\ r_m & \text{if } n > 0. \end{cases} \tag{2}$$

Note that the birth rate of the process is independent of its actual state when $n > 0$ and it depends only on the popularity of the corresponding item.

For the death rate of the process we have:

$$\mu_m(n) = n\phi. \tag{3}$$

Let $u_m(n)$ $n = (1, 2, \ldots)$ denote the probability of absorption into state 0 from the initial state n. We can write a recursion formula for $u_m(n)$ by considering the the possible states after the first transition. We know that the first transitions entails the movements:

$$n \to n+1 \text{ with probability } \frac{\lambda_m(n)}{\mu_m(n) + \lambda_m(n)}$$

$$n \to n-1 \text{ with probability } \frac{\mu_m(n)}{\mu_m(n) + \lambda_m(n)}$$

Invoking a step analysis we obtain:

$$u_m(n) = \frac{\lambda_m(n)}{\mu_m(n) + \lambda_m(n)} u_m(n+1) + \frac{\mu_m(n)}{\mu_m(n) + \lambda_m(n)} u_m(n-1) \text{ when } n \geq 1, \tag{4}$$

where $u_m(0) = 1$.

In order to solve (4) subject to the conditions $u_m(0) = 1$ and $0 \leq u_m(n) \leq 1$ $(n \geq 1)$, we rewrite it and we have:

$$(u_m(n+1) - u_m(n)) = \frac{\mu_m(n)}{\lambda_m(n)} (u_m(n) - u_m(n-1)), \qquad n \geq 1.$$

Defining $v_m(n) = u_m(n+1) - u_m(n)$ we get

$$v_m(n) = \frac{\mu_m(n)}{\lambda_m(n)} v_m(n-1), \qquad n \geq 1.$$

Iterating the last equation yields the formula $v_m(n) = \rho_m(n) v_m(0)$, where

$$\rho_m(n) = \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{\mu_m(1) \cdots \mu_m(n)}{\lambda_m(1) \cdots \lambda_m(n)} = \dfrac{\phi \cdot 2\phi \cdots i\phi}{\lambda_m \lambda \cdots \lambda_m} = \left(\dfrac{\phi}{r_m}\right)^n \cdot n! & \text{if } i > 0. \end{cases} \tag{5}$$

and with $v_m(n) = u_m(n+1) - u_m(n)$,

$$u_m(n+1) - u_m(n) = v_m(n) = \rho_m(n) = \rho_m(n)(u_m(1) - u_m(0)) = \rho_m(n)(u_m(1) - 1)$$

Summing the last equations from $n = 1$ to $n = s - 1$, we have

$$u_m(s) - u_m(1) = (u_m(1) - 1) \sum_{n=1}^{s-1} \rho_m(n), \quad m > 1. \tag{6}$$

since $u_m(s)$ by its very meaning is bounded by 1, we observe that if

$$\sum_{n=1}^{\infty} \rho_m(n) = \infty, \tag{7}$$

then necessarily $u_m(1) = 1$ and $u_m(s) = 1$ $\forall m \geq 2$. Differently, if (7) holds, then ultimately absorption into state 0 is certain from any initial state.

Suppose now that $0 < u_m(1) < 0$, then we have

$$\sum_{n=1}^{\infty} \rho_m(n) < \infty.$$

It is straightforward to observe that $u_m(s)$ is decreasing in s, since passing from state s to state 0 requires entering the intermediate states in the intervening time. Moreover, it can shown that $u_m(s) \to 0$ as $s \to \infty$. Now assuming $s \to \infty$ in (6) allows us to solve for $u_m(1)$, thus

$$u_m(1) = \frac{\sum_{n=1}^{\infty} \rho_m(n)}{1 + \sum_{n=1}^{\infty} \rho_m(n)}$$

and then from (6) we get:

$$u_m(s) = \frac{\sum_{n=s}^{\infty} \rho_m(n)}{1 + \sum_{n=1}^{\infty} \rho_m(n)}, \qquad m \geq 1.$$

Concluding we have for the probability of absorption into state 0 from the initial state s

$$u_m(s) = \begin{cases} 1 & \text{if } \displaystyle\sum_{n=1}^{\infty} \rho_m(n) = \infty, \\ \dfrac{\sum_{n=s}^{\infty} \rho_m(n)}{1 + \sum_{n=1}^{\infty} \rho_m(n)} & \text{if } \displaystyle\sum_{n=1}^{\infty} \rho_m(n) < \infty. \end{cases} \tag{8}$$

Mean time until absorption

Here we will compute the mean time to absorption starting form state s. We assume that condition (7) holds, and that absorption is certain. Let $T_m(n)$ be the mean absorption time starting from state n (*i.e.,* this could be an infinite state). Considering the possible states following the first transition, instituting a first step analysis, and recalling the fact that the mean waiting time in state n is $(\lambda_m(n) + \mu_m(n))^{-1}$ (it is actually exponentially distributed with parameter $\lambda_m(n) + \mu_m(n)$), we deduce the recursion relation

$$T_m(n) = \frac{1}{\lambda_m(n) + \mu_m(n)} + \frac{\lambda_m(n)}{\lambda_m(n) + \mu_m(n)} T_m(n+1) + \frac{\mu_m(n)}{\lambda_m(n) + \mu_m(n)} T_m(n-1), \; n \geq 1, \tag{9}$$

where $T_m(0) = 0$. Letting $y_m(n) = T_m(n) - T_m(n+1)$ and rewriting (9) leads to

$$y_m(n) = \frac{1}{\lambda_m(n)} + \frac{\mu_m(n)}{\lambda_m(n)} y_m(n-1), \qquad n \geq 1. \tag{10}$$

Iterating (10) we get

$$y_m(1) = \frac{1}{\lambda_m(1)} + \frac{\mu_m(1)}{\lambda_m(1)} y_m(0)$$
$$y_m(2) = \frac{1}{\lambda_m(2)} + \frac{\mu_m(2)}{\lambda_m(2)} y_m(1) = \frac{1}{\lambda_m(2)} + \frac{\mu_m(2)}{\lambda_m(1)\lambda_m(2)} + \frac{\mu_m(1)\mu_m(2)}{\lambda_m(1)\lambda_m(2)} y_m(0)$$
$$y_m(3) = \frac{1}{\lambda_m(3)} + \frac{\mu_m(3)}{\lambda_m(2)\lambda_m(3)} + \frac{\mu_m(2)\mu_m(3)}{\lambda_m(1)\lambda_m(2)\lambda_m(3)} + \frac{\mu_m(1)\mu_m(2)\mu_m(3)}{\lambda_m(1)\lambda_m(2)\lambda_m(3)} y_m(0)$$

and finally

$$y_m(s) = \sum_{n=1}^{s} \frac{1}{\lambda_m(n)} \prod_{j=n+1}^{s} \frac{\mu_m(j)}{\lambda_m(j)} + \left(\prod_{j=1}^{s} \frac{\mu_m(j)}{\lambda_m(j)} \right) y_m(0)$$

Assuming the product $\prod_{s+1}^{s} \frac{\mu_m(j)}{\lambda_m(j)} = 1$ and using the notation in (5) the expression for $y_m(s)$ becomes

$$y_m(s) = \sum_{n=1}^{s} \frac{1}{\lambda_m(n)} \frac{\rho_m(s)}{\rho_m(n)} + \rho_m(s) y_m(0),$$

and since $y_m(s) = T_m(s) - Tm(s+1)$ and $y_m(0) = T_m(0) - T_m(1) = -T_m(1)$, then:

$$\frac{1}{\rho_m(s)} \left(T_(s) - T_m(s+1) \right) = \sum_{n=1}^{s} \frac{1}{\lambda_m(n)\rho_m(n)} - T_m(1). \qquad (11)$$

If $\sum_{n=1}^{\infty} \frac{1}{\lambda_m(n)\rho_m(n)} = \infty$, then inspection of (11) reveals that necessarily $T_m(1) = \infty$. Indeed, it is probabilistically evident that $T_m(s) < T_m(s+1)$ $\forall s$, and this property would be violated for large s if we assume to the contrary that $T_m(1)$ is finite.

Now assume that $\sum_{n=1}^{\infty} \frac{1}{\lambda_m(n)\rho_m(n)} < \infty$, then letting $s \to \infty$ in (11) gives

$$T_m(1) = \sum_{n=1}^{\infty} \frac{1}{\lambda_m(n)\rho_m(n)} - \lim_{s\to\infty} \frac{1}{\rho_m(s)} \left(T_m(s) - T_m(s+1) \right).$$

It is more involved but still possible to prove that

$$\lim_{s\to\infty} \frac{1}{\rho_m(s)} \left(T_m(s) - T_m(s+1) \right) = 0,$$

and then

$$T_m(1) = \sum_{n=1}^{\infty} \frac{1}{\lambda_m(n)\rho_m(n)}.$$

From the above analysis the corresponding time to absorption for the initial state s is:

$$T_m(s) = \begin{cases} \infty & \text{if } \displaystyle\sum_{n=1}^{\infty} \frac{1}{\lambda_m(n)\rho_m(n)} = \infty, \\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{\lambda_m(n)\rho_m(n)} + & \\ \displaystyle + \sum_{k=1}^{s-1} \rho_m(k) \sum_{j=k+1}^{\infty} \frac{1}{\lambda_m(j)\rho_m(j)} & \text{if } \displaystyle\sum_{n=1}^{\infty} \frac{1}{\lambda_m(n)\rho_m(n)} < \infty, \end{cases} \tag{12}$$