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The Convex Conjugate of L_1 -norm is

$$f^*(y) = \begin{cases} 0 & |y| \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

$$\therefore f(x) = |x| = f^*(y) = \max_{y \in \mathbb{R}^n} x^T y - f^*(y) = \max_{|y| \leq 1} x^T y$$

Adding $-\sum_{i=1}^n 1 - \sqrt{1 - y_i^2}$

$$h(x) = \max_{|y| \leq 1} \sum x_i y_i - \sum 1 - \sqrt{1 - y_i^2}$$

$$\frac{\partial}{\partial y_i} \sum x_i y_i - \sum 1 - \sqrt{1 - y_i^2} = x_i - \frac{y_i}{\sqrt{1 - y_i^2}}$$

$$\frac{\partial}{\partial y_i} = 0 \rightarrow x_i = \frac{y_i}{\sqrt{1 - y_i^2}} \rightarrow y_i = \frac{x_i}{\sqrt{1 + x_i^2}}$$

$$\therefore h(x) = \sum \frac{x_i^2}{\sqrt{1 + x_i^2}} + \sqrt{1 - \frac{x_i^2}{1 + x_i^2}} - 1$$

$$= \sum \frac{x_i^2}{\sqrt{x_i^2 + 1}} + \frac{1 - \frac{x_i^2}{x_i^2 + 1}}{\sqrt{1 - \frac{x_i^2}{x_i^2 + 1}}} - 1$$

$$= \sum \frac{x_i^2}{\sqrt{x_i^2 + 1}} + \frac{1}{\frac{(x_i^2 + 1) \sqrt{1 - \frac{x_i^2}{x_i^2 + 1}}}{x_i^2 + 1}} - 1$$

$$= \sum \frac{x_i^2}{\sqrt{x_i^2 + 1}} + \frac{1}{\sqrt{x_i^2 + 1}} - 1$$

$$\begin{aligned} & \rightarrow \sqrt{(x_i^2 + 1)^2 - x_i^2(x_i^2 + 1)} \\ & = \sqrt{x_i^4 + 2x_i^2 + 1 - x_i^4 - x_i^2} \\ & = \sqrt{x_i^2 + 1} \end{aligned}$$

$$\therefore h(x) = \sum_{i=1}^n \sqrt{x_i^2 + 1} - 1$$