1 Robustness analysis

An important aspect to analyze when doing controller design, is the robustness of the strategy against variations of the parameters that determine the model. In the case of a directional drilling system, there are two main parameters that can be considered to be uncertain, the active weight on bit Π and the bit walk angle ϖ. In the case of Π, this uncertainty is related to different factors. In particular, changes in the applied hook-load, the variation of the interaction of the rock with the drillsting and the bit, and the decrease of sharpness of the bit as the trajectory evolves, are some of the factors that may affect the value of this parameter. It has to be mentioned, that all of this changes are usually considered to influence the fluctuation of Π slowly with respect to overall evolution of the borehole. Regarding the bit walk angle ϖ, this parameter is present in the process due to its 3D nature, being one of the main factors that causes undesired behaviors such as borehole spiraling. The effect of this parameter will be analyzed and dealt with in subsequent sections, since the terms related to this parameter add nonlinear effects and coupling to the already complex dynamics of the system.

Considering that the previous controller design was based on a nominal value of weight on bit Π, to test if the designed strategy is able to cope with the uncertainty of this parameter, it will be considered as $\Pi = \bar{\Pi} + \delta\Pi$, where $\bar{\Pi}$ represents the nominal value of weight on bit.

2 Error dynamics for robust stability analysis and controller design

In this section, the error dynamics of the system are derived once again, since the change of parameter Π affects the structure of the system. Recalling Equation (??), the states of the system are given by:

$$\begin{aligned}\begin{bmatrix} x'_\Theta \\ x'_\Phi \end{bmatrix} &= \begin{bmatrix} A_0 & 0 \\ 0 & A_0 \end{bmatrix} \begin{bmatrix} x_\Theta(\xi) \\ x_\Phi(\xi) \end{bmatrix} + \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} x_\Theta(\xi_1) \\ x_\Phi(\xi_1) \end{bmatrix} + \begin{bmatrix} A_2 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_\Theta(\xi_2) \\ x_\Phi(\xi_2) \end{bmatrix} \\ &+ \begin{bmatrix} B_{0\Theta} & 0 \\ 0 & B_{0\Phi}(\Theta, \check{\Theta}, \Theta', \check{\Theta}') \end{bmatrix} \begin{bmatrix} \Gamma^*_\Theta \\ \Gamma^*_\Phi \end{bmatrix} + \begin{bmatrix} B_{1\Theta} & 0 \\ 0 & B_{1\Phi}(\Theta, \check{\Theta}) \end{bmatrix} \begin{bmatrix} \Gamma^{*\prime}_\Theta \\ \Gamma^{*\prime}_\Phi \end{bmatrix} + \begin{bmatrix} BW \\ 0 \end{bmatrix},\end{aligned} \tag{1}$$

and with output equations given by:

$$y_\Theta = C_\Theta x_\Theta + D_\Theta \Gamma^*_\Theta + EW_y, \tag{2}$$

$$y_\Phi = C_\Phi x_\Phi + D_\Phi \Gamma^*_\Phi \frac{\sin\check{\Theta}}{\sin\Theta}. \tag{3}$$

Herein matrices A_0, A_1, A_2, B_{0i}, B_{1i}, C_i and D_i for $i = \Theta, \Phi$ defined the same as in (??), evaluated at $\Pi = \bar{\Pi} + \delta\Pi$. It has to be note that matrices $B_{0\Phi}$ and $B_{1\Phi}$ are kept (i.e. not introducing the α term) and have a dependency with respect to Θ, $\check{\Theta}$, Θ' and $\check{\Theta}'$. From here on the dependency is not made explicit.

As before, an input filter is introduced as:

$$\Gamma^{*\prime}_i = -\bar{b}_0 \Gamma^*_i - \bar{b}_1 u_i, \tag{4}$$

where $\bar{b}_0$ and $\bar{b}_0$ are defined as in (??) and evaluated at $\bar{\Pi}$. An important remark is, that this input filter will not be able to get rid of the Γ^*_i-related terms (not even in the case of the inclination dynamics), due to the difference between nominal matrices $\bar{B}_{0i}$ and $\bar{B}_{1i}$ and their real versions. The system dynamics of the system can be derived (using the actual versions of the state matrices), applying the input filter and including it as a state of the system, as following:

$$\begin{bmatrix} x'_\Theta \\ \Gamma^{*'}_\Theta \\ x'_\Phi \\ \Gamma^{*'}_\Phi \end{bmatrix} = \begin{bmatrix} A_0 & (B_{0\Theta} - B_{1\Theta}\bar{b}_0) & 0 & 0 \\ 0 & -\bar{b}_0 & 0 & 0 \\ 0 & 0 & A_0 & (B_{0\Phi} - B_{1\Phi}\bar{b}_0) \\ 0 & 0 & 0 & -\bar{b}_0 \end{bmatrix} \begin{bmatrix} x_\Theta(\xi) \\ \Gamma^*_\Theta(\xi) \\ x_\Phi(\xi) \\ \Gamma^*_\Phi(\xi) \end{bmatrix} + \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_\Theta(\xi_1) \\ \Gamma^*_\Theta(\xi_1) \\ x_\Phi(\xi_1) \\ \Gamma^*_\Phi(\xi_1) \end{bmatrix}$$
$$+ \begin{bmatrix} A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_\Theta(\xi_2) \\ \Gamma^*_\Theta(\xi_2) \\ x_\Phi(\xi_2) \\ \Gamma^*_\Phi(\xi_2) \end{bmatrix} + \begin{bmatrix} -B_{1\Theta}\bar{b}_1 & 0 \\ -\bar{b}_1 & 0 \\ 0 & -B_{1\Phi}\bar{b}_1 \\ 0 & -\bar{b}_1 \end{bmatrix} \begin{bmatrix} u_\Theta \\ u_\Phi \end{bmatrix} + \begin{bmatrix} BW \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5}$$

The feedforward input can be now defined as:

$$u_{ri} = B^T(x'_{ri}(\xi) - \bar{A}_0 x_{ri}(\xi) - \bar{A}_1 x_{ri}(\xi_1) - \bar{A}_2 x_{ri}(\xi_2)). \tag{6}$$

This feedforward input can be utilized to define once again a desired input Γ^*_{id} as in Equation (**??**) and define an error coordinate $\Delta\Gamma^*_i$. Then the error dynamics can be defined as in (7), using the fact that $x_i = e_i + x_{ri}$.

$$\begin{bmatrix} e'_\Theta \\ \Delta\Gamma^{*'}_\Theta \\ e'_\Phi \\ \Delta\Gamma^{*'}_\Phi \end{bmatrix} = \begin{bmatrix} A_0 & (B_{0\Theta} - B_{1\Theta}\bar{b}_1) & 0 & 0 \\ 0 & -b_0 & 0 & 0 \\ 0 & 0 & A_0 & (B_{0\Phi} - B_{1\Phi}\bar{b}_1) \\ 0 & 0 & 0 & -b_0 \end{bmatrix} \begin{bmatrix} e_\Theta(\xi) \\ \Delta\Gamma^*_\Theta(\xi) \\ e_\Phi(\xi) \\ \Delta\Gamma^*_\Phi(\xi) \end{bmatrix} + \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_\Theta(\xi_1) \\ \Delta\Gamma^*_\Theta(\xi_1) \\ e_\Phi(\xi_1) \\ \Delta\Gamma^*_\Phi(\xi_1) \end{bmatrix}$$
$$+ \begin{bmatrix} A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_\Theta(\xi_2) \\ \Delta\Gamma^*_\Theta(\xi_2) \\ e_\Phi(\xi_2) \\ \Delta\Gamma^*_\Phi(\xi_2) \end{bmatrix} + \begin{bmatrix} -B_{1\Theta}\bar{b}_1 & 0 \\ -\bar{b}_1 & 0 \\ 0 & -B_{1\Phi}\bar{b}_1 \\ 0 & -\bar{b}_1 \end{bmatrix} \begin{bmatrix} v_\Theta \\ v_\Phi \end{bmatrix} + \begin{bmatrix} (-B - B_{1\Theta}\bar{b}_1) \\ 0 \\ (-B - B_{1\Phi}\bar{b}_1) \\ 0 \end{bmatrix} \begin{bmatrix} u_{r\Theta} \\ u_{r\Phi} \end{bmatrix} + \begin{bmatrix} F_{p\Theta} \\ 0 \\ F_{p\Phi} \\ 0 \end{bmatrix}, \tag{7}$$

where the perturbation terms $F_{p\Theta}$ and $F_{p\Phi}$ are defined as:

$$F_{p\Theta} = (B_{0\Theta} - B_{1\Theta}\bar{b}_0)\Gamma^*_{\Theta d} + dA_0 x_{r\Theta}(\xi) + dA_1 x_{r\Theta}(\xi_1) + dA_2 x_{r\Theta}(\xi_2) + BW, \tag{8}$$
$$F_{p\Phi} = (B_{0\Phi} - B_{1\Phi}\bar{b}_0)\Gamma^*_{\Phi d} + dA_0 x_{r\Theta}(\xi) + dA_1 x_{r\Theta}(\xi_1) + dA_2 x_{r\Theta}(\xi_2), \tag{9}$$

where $\Delta A_t = A_t - \bar{A}_t$ for $t = 0, 1, 2$. Then, the state feedback controller v_i is implemented in the same way as in (**??**), (**??**) and (**??**) i.e.

$$z'_{1i} = \zeta \begin{bmatrix} k_{1i} & 0 & 0 \end{bmatrix} (\check{x}_i - x_{ri}), \tag{10}$$
$$z'_{2i} = -\gamma z_{2i} + \gamma(z_{1i} + K_i(\check{x}_i - x_{ri})), \tag{11}$$
$$v_i = z_{2i}. \tag{12}$$

Accounting for the observer design, the same integral action will be included and defined as in (**??**), namely,

$$q'_i = \zeta[l_{1i}, l_{2i}](y_i - \check{y}_i).$$

Then the observer dynamics can be defined as:

$$\begin{bmatrix} \check{x}'_\Theta \\ q'_\Theta \\ \check{x}'_\Phi \\ q'_\Phi \end{bmatrix} = \begin{bmatrix} \bar{A}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{A}_0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \check{x}_\Theta(\xi) \\ q_\Theta(\xi) \\ \check{x}_\Phi(\xi) \\ q_\Phi(\xi) \end{bmatrix} + \begin{bmatrix} \bar{A}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{A}_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \check{x}_\Theta(\xi_1) \\ q_\Theta(\xi_1) \\ \check{x}_\Phi(\xi_1) \\ q_\Phi(\xi_1) \end{bmatrix} + \begin{bmatrix} \bar{A}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{A}_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \check{x}_\Theta(\xi_2) \\ q_\Theta(\xi_2) \\ \check{x}_\Phi(\xi_2) \\ q_\Phi(\xi_2) \end{bmatrix}$$
$$+ \begin{bmatrix} L_\Theta(y_\Theta - \check{y}_\Theta) \\ \zeta[l_{1\Theta}, l_{2\Theta}](y_\Theta - \check{y}_\Theta) \\ L_\Phi(y_\Phi - \check{y}_\Phi) \\ \zeta[l_{1\Phi}, l_{2\Phi}](y_\Phi - \check{y}_\Phi) \end{bmatrix} + \begin{bmatrix} Bq_\Theta \\ 0 \\ Bq_\Phi \\ 0 \end{bmatrix} + \begin{bmatrix} B(u_{r\Theta} + v_\Theta) \\ 0 \\ B(u_{r\Phi} + v_\Phi) \\ 0 \end{bmatrix}, \tag{13}$$

and with corresponding output equations

$$\check{y}_\Theta = \bar{C}_\Theta \check{x}_\Theta + \bar{D}_\Theta \Gamma^*_\Theta, \tag{14}$$
$$\check{y}_\Phi = \bar{C}_\Phi \check{x}_\Phi + \bar{D}_\Phi \Gamma^*_\Phi. \tag{15}$$

Since matrices C_i and D_i differ from their nominal versions, the description for the derivatives of the states of the integral action for both inclination and azimuth are defined by (after substituting the output equations and considering the fact that $\check{x}_i = e_i + x_{ri} - \delta_i$):

$$q'_\Theta = \zeta[l_{1\Theta}, l_{2\Theta}](\Delta C_\Theta e_\Theta + \Delta C_\Theta x_{r\Theta} + \Delta D_\Theta \Delta\Gamma^*_\Theta + \Delta D_\Theta \Gamma^*_{\Theta d} + \bar{C}_\Theta \delta_\Theta + EW_y), \tag{16}$$
$$q'_\Phi = \zeta[l_{1\Phi}, l_{2\Phi}](\Delta C_\Phi e_\Phi + \Delta C_\Phi x_{r\Phi} + D_\Phi(\Delta\Gamma_\Phi + \Gamma^*_{\Phi d})^* \frac{\sin\check{\Theta}}{\sin\Theta} - \bar{D}_\Phi(\Delta\Gamma^*_\Phi + \Gamma^*_{\Phi d}) + \bar{C}_\Phi \delta_\Phi), \tag{17}$$

where $\Delta D_i = D_i - \bar{D}_i$ and $\Delta C_i = C_i - \bar{C}_i$. Afterwards, the observer error dynamics can be obtained. The main difference is that in this case, due to model uncertainty, the observer dynamics are not decoupled from the system dynamics (not even for the inclination). The complete system closed-loop error dynamics (for state vector $X(\xi)$ defined as in (**??**)) are given by

$$\begin{aligned} X'(\xi) =& A_{0cl}X(\xi) + A_{1cl}X(\xi_1) + A_{2cl}X(\xi_2) + \\ & P_{cl}(u_{r\Theta}, u_{r\Phi}, \Gamma^*_{\Theta d}, \Gamma^*_{\Phi d}, x_{r\Theta}(\xi), x_{r\Theta}(\xi_1), x_{r\Theta}(\xi_2), x_{r\Phi}(\xi), x_{r\Phi}(\xi_1), x_{r\Phi}(\xi_2), \Theta, \check{\Theta}, W, W_y), \end{aligned} \tag{18}$$

where:

$$A_{0cl} = \begin{bmatrix} A_{0\Theta} & 0 \\ 0 & A_{0\Phi} \end{bmatrix}, \qquad A_{1cl} = \begin{bmatrix} A_{1\Theta} & 0 \\ 0 & A_{1\Phi} \end{bmatrix}, \qquad A_{2cl} = \begin{bmatrix} A_{2\Theta} & 0 \\ 0 & A_{2\Phi} \end{bmatrix}, \tag{19}$$

where the system matrices in (19) and the vector $P_{cl}(\xi, \xi_1, \xi_2, \Theta, \check{\Theta}, W, W_y)$ (where all the dependencies on terms related to trajectory have been substituted by ξ, ξ_1 and ξ_2) are given by:

$$A_{0\Theta} = \left[\begin{array}{cccc|cc}
A_0 & (B_{0\Theta} - B_{1\Theta}\bar{b}_0) & 0 & -B_{1\Theta}\bar{b}_1 & 0 & 0 \\
0 & -\bar{b}_0 & 0 & -\bar{b}_1 & 0 & 0 \\
\zeta\left[k_{1\Theta}, 0, 0\right] & 0 & 0 & 0 & -\zeta\left[k_{1\Theta}, 0, 0\right] & 0 \\
\gamma K_\Theta & 0 & \gamma & -\gamma & -\gamma K_\Theta & 0 \\
\hline
(\Delta A_0 - L_\Theta \Delta C_\Theta) & (B_{0\Theta} - B_{1\Theta}\bar{b}_0 - L_\Theta \Delta D_\Theta) & 0 & (-B - B_{1\Theta}\bar{b}_1) & \bar{A}_0 - L_\Theta \bar{C}_\Theta & -B \\
\zeta\left[l_{1\Theta}, l_{2\Theta}\right]\Delta C_\Theta & \zeta\left[l_{1\Theta}, l_{2\Theta}\right]\Delta D_\Theta & 0 & 0 & \zeta\left[l_{1\Theta}, l_{2\Theta}\right]\bar{C}_\Theta & 0
\end{array}\right],$$

$$A_{0\Phi} = \left[\begin{array}{cccc|cc}
A_0 & (B_{0\Phi} - B_{1\Phi}\bar{b}_0) & 0 & -B_{1\Phi}\bar{b}_1 & & \\
0 & -\bar{b}_0 & 0 & -\bar{b}_1 & 0 & 0 \\
\zeta\left[k_{1\Phi}, 0, 0\right] & 0 & 0 & 0 & -\zeta\left[k_{1\Phi}, 0, 0\right] & 0 \\
\gamma K_\Phi & 0 & \gamma & -\gamma & -\gamma K_\Phi & 0 \\
\hline
(\Delta A_0 - L_\Phi \Delta C_\Phi) & (B_{0\Phi} - B_{1\Phi}\bar{b}_0 - L_\Phi(D_\Phi \frac{\sin\check{\Theta}}{\sin\Theta} - \bar{D}_\Phi)) & 0 & (-B - B_{1\Phi}\bar{b}_1) & \bar{A}_0 - L_\Phi \bar{C}_\Phi & -B \\
\zeta\left[l_{1\Phi}, l_{2\Phi}\right]\Delta C_\Phi & \zeta\left[l_{1\Phi}, l_{2\Phi}\right](D_\Phi \frac{\sin\check{\Theta}}{\sin\Theta} - \bar{D}_\Phi) & 0 & 0 & \zeta\left[l_{1\Phi}, l_{2\Phi}\right]\bar{C}_\Phi & 0
\end{array}\right],$$

$$A_{1i} = \left[\begin{array}{cccc|cc}
A_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\Delta A_1 & 0 & 0 & 0 & \bar{A}_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \quad A_{2i} = \left[\begin{array}{cccc|cc}
A_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\Delta A_2 & 0 & 0 & 0 & \bar{A}_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],$$

$$P_{cl}(\xi, \xi_1, \xi_2, \Theta, \check{\Theta}, W, W_y) = \left[\begin{array}{c}
F_{p\Theta} + (-B - B_{1\Theta}\bar{b}_1)u_{r\Theta} \\
0 \\
0 \\
0 \\
\hline
F_{r\Theta} + (-B - B_{1\Theta}\bar{b}_1)u_{r\Theta} \\
\zeta\left[l_{1\Theta}, l_{2\Theta}\right](\Delta C_\Theta x_{r\Theta} + \Delta D_\Theta \Gamma^*_{\Theta d} + E W_y) \\
\hline
F_{p\Phi} + (-B - B_{1\Phi}\bar{b}_1)u_{r\Phi} \\
0 \\
0 \\
0 \\
\hline
F_{r\Phi} + (-B - B_{1\Phi}\bar{b}_1)u_{r\Phi} \\
\zeta\left[l_{1\Phi}, l_{2\Phi}\right](\Delta C_\Phi x_{r\Phi} + (D_\Phi \frac{\sin\check{\Theta}}{\sin\Theta} - \bar{D}_\Phi)\Gamma^*_{\Phi d})
\end{array}\right].$$

The non-defined terms $F_{r\Theta}$ and $F_{r\Phi}$ are given by:

$$F_{r\Theta} = (B_{0\Theta} - B_{1\Theta}\bar{b}_0 - L_\Theta \Delta D_\Theta)\Gamma^*_{\Theta d} + (\Delta A_0 - L_\Theta \Delta C_\Theta)x_{r\Theta}(\xi) + \Delta A_1 x_{r\Theta}(\xi_1) + \Delta A_2 x_{r\Theta}(\xi_2) + BW - L_\Theta E W_y, \tag{20}$$

$$F_{r\Phi} = \left(B_{0\Phi} - B_{1\Phi}\bar{b}_0 - L_\Phi\left(D_\Phi \frac{\sin\check{\Theta}}{\sin\Theta} - \bar{D}_\Phi\right)\right)\Gamma^*_{\Phi d} + (\Delta A_0 - L_\Phi \Delta C_\Phi)x_{r\Phi}(\xi) + \Delta A_1 x_{r\Phi}(\xi_1) + \Delta A_2 x_{r\Phi}(\xi_2), \tag{21}$$

where $\Delta A_0 = A_0 - \bar{A}_0$, $\Delta A_1 = A_1 - \bar{A}_1$ and $\Delta A_1 = A_1 - \bar{A}_1$.

3 Linearization of the system with uncertain weight on bit

Since there is also dependency on Θ and $\check{\Theta}$ (due to the $B_{0\Phi}$ and $B_{1\Phi}$ vectors), the same substitutions as in (**??**) and (**??**) are used to write down the equations in terms of e_i, δ_i and the desired trajectory terms (ξ-dependent).

Prior to performing the linearization, it is important to recall the dependencies with respect to the states of the terms in the system matrices, to have a clear view of where coupling terms may appear. In the case of the inclination, the state matrices and vectors are linear and dependent on constant (uncertain) terms, meaning that these terms will remain the same. Contrariwise, vectors $B_{0\Phi}$ and $B_{1\Phi}$ are dependent on states of the inclination dynamics. Let us first analyze $B_{0\Phi}$, which contains elements in terms of Θ, $\check{\Theta}$, Θ' and $\check{\Theta}'$. As in (**??**) and (**??**), Θ and $\check{\Theta}$ can be expressed in terms of e_Θ and δ_Θ. In the case of the derivatives, this need to be substituted from the system dynamics in Equation (**??**), for the case with uncertain Π. The dependencies of vectors $B_{0\Phi}$ and $B_{1\Phi}$ can then be expressed in terms of the states of the system as:

$$\begin{aligned} B_{0\Phi} &= B_{0\Phi}(e_\Theta(\xi), e_\Theta(\xi_1), e_\Theta(\xi_2), \delta_\Theta(\xi), \delta_\Theta(\xi_1), \delta_\Theta(\xi_2), \Delta\Gamma^*_\Theta(\xi), z_{2\Theta}(\xi)) \\ B_{1\Phi} &= B_{1\Phi}(e_\Theta(\xi), \delta_\Theta(\xi)) \end{aligned} \tag{22}$$

As before, the linearization is performed according to Equation (**??**) (in this case using the actual value of Π) and defining the perturbation vector as in (**??**). The linearized system is given by:

$$\bar{X}'(\xi) = \bar{A}_{0cl}\bar{X}(\xi) + \bar{A}_{1cl}\bar{X}(\xi_1) + \bar{A}_{2cl}\bar{X}(\xi_2), \tag{23}$$

where the linearized matrices $\bar{A}_{0cl}$, $\bar{A}_{1cl}$ and $\bar{A}_{2cl}$, are given by:

$$\bar{A}_{0cl} = \begin{bmatrix} A_{0\Theta} & 0 \\ A_{0c} & A_{0\Phi} \end{bmatrix}, \qquad \bar{A}_{1cl} = \begin{bmatrix} A_{1\Theta} & 0 \\ A_{1c} & A_{1\Phi} \end{bmatrix}, \qquad \bar{A}_{2cl} = \begin{bmatrix} A_{2\Theta} & 0 \\ A_{2c} & A_{2\Phi} \end{bmatrix}, \tag{24}$$

and sub-matrices defined as:

$$A_{0i} = \left[\begin{array}{cccc|cc} A_0 & (B_{0i} - B_{1i}\bar{b}_0) & 0 & -B_{1i}\bar{b}_1 & 0 & 0 \\ 0 & -b_0 & 0 & -b_1 & 0 & 0 \\ \zeta\left[k_{1i}, 0, 0\right] & 0 & 0 & 0 & -\zeta\left[k_{1i}, 0, 0\right] & 0 \\ \gamma K_i & 0 & \gamma & -\gamma & -\gamma K_i & 0 \\ \hline (\Delta A_0 - L_i \Delta C_i) & (B_{0i} - B_{1i}\bar{b}_0 - L_i \Delta D_i) & 0 & (-B - B_{1i}\bar{b}_1) & A_0 - L_i \bar{C}_i & -B \\ \zeta\left[l_{1i}, l_{2i}\right]\Delta C_i & \zeta\left[l_{1i}, l_{2i}\right]\Delta D_i & 0 & 0 & \zeta\left[l_{1i}, l_{2i}\right]\bar{C}_i & 0 \end{array}\right],$$

$$A_{0c} = \left[\begin{array}{cccc|cc} p_{0e1}(\xi) & p_{0e2}(\xi) & 0 & p_{0e3}(\xi) & p_{0e4}(\xi) & p_{0e4}(\xi) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline p_{0\delta 1}(\xi) & p_{0\delta 2}(\xi) & 0 & p_{0\delta 3}(\xi) & p_{0\delta 4}(\xi) & p_{0\delta 5}(\xi) \\ 0 & 0 & 0 & 0 & p_{0q}(\xi) & 0 \end{array}\right]$$

$$A_{1i} = \left[\begin{array}{cccc|cc} A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \Delta A_1 & 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \qquad A_{1c} = \left[\begin{array}{cccc|cc} p_{1e1}(\xi) & 0 & 0 & 0 & p_{1e2}(\xi) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline p_{1\delta 1}(\xi) & 0 & 0 & 0 & p_{1\delta 2}(\xi) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$A_{2i} = \left[\begin{array}{cccc|cc} A_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \Delta A_2 & 0 & 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \qquad A_{2c} = \left[\begin{array}{cccc|cc} p_{2e1}(\xi) & 0 & 0 & 0 & p_{2e2}(\xi) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline p_{2\delta 1}(\xi) & 0 & 0 & 0 & p_{2\delta 2}(\xi) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The definition of the p coefficients inside the coupling matrix will depend on the chosen linearization point. It is important to know, is that these coefficients are all dependent on ξ and the actual value of Π, but as in the previous case, they are all multiplied by the states corresponding to the inclination, knowing that these are asymptotically stable.

There are two key differences with respect to the nominal case. First, looking at the structure of matrices A_{0i}, it can be concluded that the separation principle does not longer hold, since there is mutual coupling between the error dynamics and the observer error dynamics in both the inclination and azimuth. This means, that it is not possible to design controller and observer gains separately as in the nominal case. Secondly, in the specific case of $A_{0\Phi}$, it has to be pointed out that this matrix is ξ-dependent. This is because vector $B_{0\Phi}$ depends on Θ' and $\check{\Theta}'$, which are not equal due to the difference between Π and $\bar{\Pi}$, i.e.:

$$\begin{aligned} \Theta' - \check{\Theta}' =& (\beta_1 - \bar{\beta}_1)\Theta_d + (\beta_2 - \bar{\beta}_2)\langle\Theta\rangle_{1d} + (\beta_3 - \bar{\beta}_3)\langle\Theta\rangle_{2d} + (\beta_4 - \bar{\beta}_4)\Theta_1 d + (\beta_5 - \bar{\beta}_5)\Theta_2 d \\ &+ B^T(B_{0\Theta} - B_{1\Theta}\bar{b}_0)\Delta\Gamma^*_{\Theta} + B^T(-B_{1\Theta}\bar{b}_1 - B)u_{r\Theta} + B^T(-B_{1\Theta}\bar{b}_1 - B)z_{2\Theta} \\ &+ B^T(B_{0\Theta} - B_{1\Theta}\bar{b}_0)\Delta\Gamma^*_{\Theta d} \end{aligned} \tag{25}$$

where the β and $\bar{\beta}$ terms are defined as in (**??**) for the real and nominal values of Π, respectively. In the nominal case, this difference became equal to zero at the equilibrium where $e_i = 0$ and $\delta_i = 0$, which is not the case when taking into account uncertainty on Π. This two differences render the robust controller synthesis much more difficult, as it will be explained in detail in further sections.

In order to perform the linearization, an equilibrium point for the system given by Equation (18) needs to be found. In the case for $\Pi = \bar{\Pi}$, the error dynamics $e_i(\xi)$ and the observer error dynamics $\delta_i(\xi)$ and their delayed versions, were chosen to be equal to zero. In this case it was possible to find a unique solution for the determined system of equations given in (??). Defining all errors and derivatives of system (18) equal to zero results in:

$$
\begin{aligned}
0 =&(B_{0\Theta} - B_{1\Theta}\bar{b}_0)\Delta\Gamma^*_\Theta - (B_{1\Theta}\bar{b}_1)z_{2\Theta} + F_{p\Theta} + (-B - B_{1\Theta}\bar{b}_1)u_{r\Theta},\\
0 =& -\bar{b}_0\Delta\Gamma^*_\Theta - \bar{b}_1 z_{2\Theta},\\
0 =&\gamma z_{1\Theta} - \gamma z_{2\Theta},\\
0 =&(B_{0\Theta} - B_{1\Theta}\bar{b}_0 - L_\Theta\Delta D_\Theta)\Delta\Gamma^*_\Theta + (-B - B_{1\Theta}\bar{b}_1)z_{2\Theta} - B_{q\Theta} + F_{r\Theta} + (-B - B_{1\Theta}\bar{b}_1)u_{r\Theta},\\
0 =&\zeta \begin{bmatrix} l_{1\Theta} & l_{2\Theta}\end{bmatrix} \Delta D_\Theta\Delta\Gamma^*_\Theta + \zeta \begin{bmatrix} l_{1\Theta} & l_{2\Theta}\end{bmatrix} (\Delta C_\Theta x_{r\Theta} + \Delta D_\Theta\Gamma^*_{\Theta d} + EW_y),\\
0 =&(B_{0\Phi} - B_{1\Phi}\bar{b}_0)\Delta\Gamma^*_\Phi - (B_{1\Phi}\bar{b}_1)z_{2\Phi} + F_{p\Phi} + (-B - B_{1\Phi}\bar{b}_1)u_{r\Phi},\\
0 =& -\bar{b}_0\Delta\Gamma^*_\Phi - \bar{b}_1 z_{2\Phi},\\
0 =&\gamma z_{1\Phi} - \gamma z_{2\Phi},\\
0 =&(B_{0\Phi} - B_{1\Phi}\bar{b}_0 - L_\Phi\Delta D_\Phi)\Delta\Gamma^*_\Phi + (-B - B_{1\Phi}\bar{b}_1)z_{2\Phi} - B_{q\Phi} + F_{r\Phi} + (-B - B_{1\Phi}\bar{b}_1)u_{r\Phi},\\
0 =&\zeta \begin{bmatrix} l_{1\Phi} & l_{2\Phi}\end{bmatrix} \Delta D_\Phi\Delta\Gamma^*_\Phi + \zeta \begin{bmatrix} l_{1\Phi} & l_{2\Phi}\end{bmatrix} (\Delta C_\Phi x_{r\Phi} + \Delta D_\Phi\Gamma^*_{\Phi d}).
\end{aligned}
\tag{26}
$$

In this equation it has to be noticed that, since $\Theta = \check{\Theta}$, then $B_{1\Theta} = B_{1\Phi}$ and $\Delta D_\Phi = D_\Phi - \bar{D}_\Phi$. Nevertheless, vector $B_{0\Phi}$ is still different from $B_{0\Theta}$, because of its dependency on the derivatives of the inclination and its estimate.

It can be seen as well that the system is overdetermined, since it possesses less unknowns than equations, and in general this type of systems have no solution. Two approaches can be taken to find an equilibrium around which the system could be linearized. Considering that the variation of Π is slow and not of a significantly big value along the trajectory, an initial approach is to consider the same linearization point as in the nominal case given in (??). The problem of this approach is that this linearization would only be valid for variations of Π along the trajectory that do not make the system diverge to much from its linearized version. The second approach is, that instead of taking $e_i(\xi) = e_i(\xi_1) = e_i(\xi) = 0$ and $\delta_i(\xi) = \delta_i(\xi_1) = \delta_i(\xi) = 0$, retain the errors at the current value of ξ as part of the equations to obtain a determined system of equations (since $e_i(\xi)$ and $\delta_i(\xi)$ would add up to ten unknowns). The problem of this second approach is that the error at current ξ for the equilibrium point could result different to zero, which would mean that the system's equilibrium would not be at the reference. Another problem of this approach is that the equilibrium point may result in ξ-dependent terms for certain coordinates, rendering the controller design more complicated. Both approaches will be considered.

Starting to analyze the latter option, first the equations for the equilibrium point (without writing the explicit dependency on ξ) would be expressed as:

$$
\begin{aligned}
0 =&A_0 e_\Theta + (B_{0\Theta} - B_{1\Theta}\bar{b}_0)\Delta\Gamma^*_\Theta - (B_{1\Theta}\bar{b}_1)z_{2\Theta} + F_{p\Theta} + (-B - B_{1\Theta}\bar{b}_1)u_{r\Theta},\\
0 =& -\bar{b}_0\Delta\Gamma^*_\Theta - \bar{b}_1 z_{2\Theta},\\
0 =&\zeta \begin{bmatrix} k_{1\Theta} & 0 & 0\end{bmatrix} e_\Theta - \zeta \begin{bmatrix} k_{1\Theta} & 0 & 0\end{bmatrix} \delta_\Theta,\\
0 =&\gamma K_\Theta e_\Theta + \gamma z_{1\Theta} - \gamma z_{2\Theta} - \gamma K_\Theta \delta_\Theta,\\
0 =&(\Delta A_0 - L_\Theta\Delta C_\Theta)e_\Theta + (B_{0\Theta} - B_{1\Theta}\bar{b}_0 - L_\Theta\Delta D_\Theta)\Delta\Gamma^*_\Theta + (-B - B_{1\Theta}\bar{b}_1)z_{2\Theta}\\
&+ (\bar{A}_0 - L_\Theta\bar{C}_\Theta)\delta_\Theta - B_{q\Theta} + F_{r\Theta} + (-B - B_{1\Theta}\bar{b}_1)u_{r\Theta},\\
0 =&\zeta \begin{bmatrix} l_{1\Theta} & l_{2\Theta}\end{bmatrix} \Delta C_\Theta e_\Theta + \zeta \begin{bmatrix} l_{1\Theta} & l_{2\Theta}\end{bmatrix} \Delta D_\Theta\Delta\Gamma^*_\Theta + \zeta \begin{bmatrix} l_{1\Theta} & l_{2\Theta}\end{bmatrix} \bar{C}_\Theta\delta_\Theta + \zeta \begin{bmatrix} l_{1\Theta} & l_{2\Theta}\end{bmatrix} (\Delta C_\Theta x_{r\Theta} + \Delta D_\Theta\Gamma^*_{\Theta d} + EW_y),\\
0 =&A_0 e_\Phi + (B_{0\Phi} - B_{1\Phi}\bar{b}_0)\Delta\Gamma^*_\Phi - (B_{1\Phi}\bar{b}_1)z_{2\Phi} + F_{p\Phi} + (-B - B_{1\Phi}\bar{b}_1)u_{r\Phi},\\
0 =& -\bar{b}_0\Delta\Gamma^*_\Phi - \bar{b}_1 z_{2\Phi},\\
0 =&\zeta \begin{bmatrix} k_{1\Phi} & 0 & 0\end{bmatrix} e_\Phi - \zeta \begin{bmatrix} k_{1\Phi} & 0 & 0\end{bmatrix} \delta_\Phi,\\
0 =&\gamma K_\Phi e_\Phi + \gamma z_{1\Phi} - \gamma z_{2\Phi} - \gamma K_\Phi \delta_\Phi,\\
0 =&(\Delta A_0 - L_\Phi\Delta C_\Phi)e_\Phi + (B_{0\Phi} - B_{1\Phi}\bar{b}_0 - L_\Phi\Delta D_\Phi)\Delta\Gamma^*_\Phi + (-B - B_{1\Phi}\bar{b}_1)z_{2\Phi}\\
&+ (\bar{A}_0 - L_\Phi\bar{C}_\Phi)\delta_\Phi - B_{q\Phi} + F_{r\Phi} + (-B - B_{1\Phi}\bar{b}_1)u_{r\Phi},\\
0 =&\zeta \begin{bmatrix} l_{1\Phi} & l_{2\Phi}\end{bmatrix} \Delta C_\Phi e_\Phi + \zeta \begin{bmatrix} l_{1\Phi} & l_{2\Phi}\end{bmatrix} \Delta D_\Phi\Delta\Gamma^*_\Phi + \zeta \begin{bmatrix} l_{1\Phi} & l_{2\Phi}\end{bmatrix} \bar{C}_\Phi\delta_\Phi + \zeta \begin{bmatrix} l_{1\Phi} & l_{2\Phi}\end{bmatrix} (\Delta C_\Phi x_{r\Phi} + \Delta D_\Phi\Gamma^*_{\Phi d}).
\end{aligned}
\tag{27}
$$

This system of equations is determined, having a unique solution. The values of the equilibrium points for this system are given by:

$$
\begin{aligned}
e_{\Theta} &= \begin{bmatrix} 0 \\ d_{\langle e_{\Theta}\rangle_1}(\xi) \\ d_{\langle e_{\Theta}\rangle_2}(\xi) \end{bmatrix}, \\
\Delta\Gamma^*_{\Theta} &= 0, \\
z_{1\Theta} &= d_{z_{1\Theta}}(\xi), \\
z_{2\Theta} &= 0, \\
\delta_{\Theta} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \\
q_{\Theta} &= d_{q_{\Theta}}(\xi), \\
e_{\Phi} &= \begin{bmatrix} 0 \\ d_{\langle e_{\Phi}\rangle_1}(\xi) \\ 0 \end{bmatrix}, \\
\Delta\Gamma^*_{\Phi} &= 0, \\
z_{1\Phi} &= d_{z_{1\Phi}}(\xi), \\
z_{2\Phi} &= 0, \\
\delta_{\Phi} &= \begin{bmatrix} 0 \\ d_{\langle \delta_{\Phi}\rangle_1}(\xi) \\ 0 \end{bmatrix}, \\
q_{\Phi} &= d_{q_{\Phi}}(\xi),
\end{aligned}
\tag{28}
$$

where the d terms are dependent on the desired trajectory. It is important to notice, that for both the inclination and azimuth, the equilibrium of the system is at $e_{i0} = 0$ and $\delta_{i0} = 0$.

In order to check the robustness of the controller against uncertainty on the active weight on bit, the rightmost-pole of the closed-loop system is computed for a grid of values of $\Pi \in [-0.5\bar{\Pi}, 0.5\bar{\Pi}]$

- Explain that the same change for the dependency on Θ will be made
- Workout the equations to get the equilibrium points
- Explain why to use the previous equilibrium point
- Write linear system and values of coefficients