The College Admissions Problem Revisited

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Recall from last class

There are three kinds of market failure that impide efficient matching:

- Failure to provide thickness
- Failure to overcome congestion that thickness can bring
- Failure to make it safe for market participants to reliably reveal their information

In history, can we assume that educational institutions never ran into these problems in regards to their admissions?

Overcoming congestion that thickness brings

York University apologizes for accepting 500 students - by mistake

"'I hate university already,' student tweets after York mistakenly admits him

Taken from: https://www.cbc.ca/news/canada/toronto/york-university-1.3403066

Overcoming safety to state preferences: It’s not just a college admissions problem

CITY & STATE NEW YORK

Fiorello H. LaGuardia High School of Music & Art and Performing Arts.

How do students get into these high schools?

For eight of these schools, students must take the Specialized High School Admissions Test and then rank schools in order of preference. Students are then designated to a school based on their score on the SHSAT, their ranking of preferences and the amount of space at a school. For the ninth specialized high

Taken from: https://www.cityandstateny.com/articles/policy/education/new-york-city-specialized-high-schools-admissions-guide.html

Admissions policy algorithm

```
high_school_admissions(student_mark, preference_list):
    if (student_mark > threshold) AND (school is student's first preference):
        Accept student
    Otherwise:
        Reject student
```

Overview

- The Main Result
- The Formal Model
- Stability
- Relation to the Marriage Problem

The Main Result

Main Points

- At stable outcomes, colleges will NOT be indifferent between two groups of students enrolled
    - For every pair of stable outcomes, the college will prefer every student in one of the groups it is assigned over every student in the group of the other assignment
- Why?
    - Based on the preferences of 1 agent (college)

The Formal Model

Suppose...

There exists two sets of agents

Let C be a set of Colleges

$$C = \{C_1, C_2 ... C_n\}$$

Let S be a set of Students

$$S = \{s_1, s_2 ... s_m\}$$

The Rules

- Any student and college must mutually agree that the student will attend the college
- All colleges must have a quota; the number of positions a college must offer. This is denoted as

$$q_c$$

- Any college may choose to keep any of its positions unfilled
- Any student may remain unmatched if wished

Definition 1: Matching

The matching function is defined as

$$\mu : C \cup S \to C \cup S$$

with the properties...

Definition 1: Matching properties

i) For all students

$$|\mu(s)| = 1 \text{ and } \mu(s) \notin C \rightarrow \mu(s) = s$$

Translation: All students must be matched to only one thing. If a student is not matched to a college, then the student is matched to themselves.

Definition 1: Matching properties

ii) For all colleges

$$|\mu(C)| = q_c \text{ and if } |\mu(C) \cap S| = r < q_c$$

there are $q_c - r$ unfilled positions in college C

Translation: All colleges have a quota. If there are fewer students than the quota of the college, then the college contains (q_c - r) unfilled positions

Definition 1: Matching properties

iii) For all colleges and students

$$\mu(s) = C \leftrightarrow s \in \mu(C)$$

Translation: A student is matched with a college if and only if the college's set of matches contains the matched student.

Examples

1. $\mu(s_3) = C_{26}$
2. $\mu(C_3) = \{s_4, C_3, s_1, C_3\}$
3. $\mu(s_4) = s_4$

Preferences

P is an ordered list representing a college's or student's strict preferences

$$P(C) = s_1, s_2, C, s_3...$$

$$P(s) = C_2, C_1, C_3, s...$$

Other notations of preferences include

$$C_i >_s C_j$$

Denoting that a student s prefers college i over college j

And

$$s_i >_c s_j$$

Denoting college C prefers student i over student j

Introduction to preference relation (College assumption)

Suppose the following:

$$P(C_5) = s_1, s_2, s_3, s_4, C$$

$$\mu(C_5) = \{s_4, s_3\}$$

$$\mu'(C_5) = \{s_2, s_4\}$$

In this case, the assumption for colleges is that

$$\mu'(C_5) >_{c_5} \mu(C_5)$$

The Preference Relation (How colleges prefer matches)

The preference relation P#(C) over groups of students is responsive whenever

$$\mu'(C) = \mu(C) \cup \frac{\{s\}}{\{\sigma\}}, \sigma \in \mu(C) \text{ and } s \notin \mu(C)$$

Thus,

$$\mu'(C) >_c \mu(C)$$

In other words, if two assignments are the same, except for one student, college C will choose the match with the more preferred student

The Preference Relation Clarifications

- Different responsive preferences $P^{\#}(C)$ exist for any preference $P(C)$
- However, responsiveness does not specify whether a college with a quota of 2 prefers to be assigned its 1st and 4th choice students or its 2nd and 3rd choice students
- E.g. Would you rather have Albert Einstein and Elon Musk in your college vs. Bill Gates and Isaac Newton in your college?

Stability

Definition 3: Stability

A matching μ is *stable* if it is not blocked by any agent or any college-student pair.

μ is blocked by the student pair if:

$$\mu(s) \neq C$$

$$C >_s \mu(s)$$

$$\sigma \in \mu(C), s >_c \sigma$$

Definition 3: Stability Example

$$P(C_5) = s_1, s_2, s_3$$

$$P(C_3) = s_3, s_2, s_1$$

$$P(s_1) = C_5, C_3$$

$$P(s_3) = C_3, C_5$$

but

$$\mu(s_1) = C_3$$

$$\mu(s_3) = C_5$$

To make this stable...

$$\mu(s_1) = C_5$$

$$\mu(s_3) = C_3$$

Relation to Marriage Problem

Consider:

- College C has quota, q_c
- Now, suppose that colleges are the men in this model, and the students are women in the model

“How does a college obtain its quota then?”

- There are q_c duplicates of C, where C and C’ are the same college and P(C) = P(C’), each only having a quota of 1 (like how men are presented in the marriage problem)
- Each student’s preference list is modified by replacing C with C’s duplicates wherever C appears on the preference list

I.e. Comparing 1 student to 1 position in each college; rather than groups of students to 1 college

Agents now change to become students and college positions → 1-to-1 matching (Marriage Problem)

As an example

Suppose

$$P(C_1) = s_1, s_2, s_3 \text{ where } q_1 = 2$$

$$P(C_2) = s_2, s_1, s_3 \text{ where } q_2 = 1$$

$$P(C_3) = s_2, s_3, s_1 \text{ where } q_3 = 1$$

and

$$P(s_1) = C_1, C_2, C_3$$

$$P(s_2) = C_2, C_1, C_3$$

$$P(s_3) = C_3, C_2, C_1$$

The marriage-related problem equivalent:

$$P(C_{1_1}) = s_1, s_2, s_3$$

$$P(C_2) = s_2, s_1, s_3$$

$$P(C_3) = s_2, s_3, s_1$$

$$P(C_{1_2}) = s_1, s_2, s_3$$

and

$$P(s_1) = C_{1_1}, C_{1_2}, C_2, C_3$$

$$P(s_2) = C_2, C_{1_1}, C_{1_2}, C_3$$

$$P(s_3) = C_3, C_2, C_{1_1}, C_{1_2}$$

Theorem 0: The proposer deferred algorithm is always results in stable matches (near-identical to the marriage problem)

Mechanism in the college-admissions problem

Given theorem 0, we will equate μ as the proposed deferred algorithm to generate stable outcomes. This algorithm is a mechanism:

- Messages: The set of preferences for both colleges and students
- Allocation Rule: Every college/student will have a stable outcome

Theorem 1: When all preferences over individuals are strict, the set of students enrolled and positions filled in a college admissions problem is the same at every stable matching

Theorem 2: When all preferences over individuals are strict, any college that does not fill its quota at some stable matching is assigned precisely the same set of students at every stable matching

Example

Suppose each college has a quota of 1, where:

$$P(C_1) = s_1, s_2$$

$$P(C_2) = s_2, s_1$$

$$P(C_3) = s_1, s_2$$

$$P(s_1) = C_2, C_3, C_1$$

$$P(s_2) = C_1, C_3, C_2$$

College-proposed DA outcome:

$$\mu(C_1) = s_2,\ \mu(C_2) = s_1,\ \mu(C_3) = \{\}$$

Student-proposed DA outcome:

$$\mu(C_1) = s_2,\ \mu(C_2) = s_1,\ \mu(C_3) = \{\}$$

LEMMA 3: Suppose that colleges and students have strict individual preferences, and let μ and μ' be stable matchings for (S, C, P), such that $\mu(C) \neq \mu'(C)$ for some C. Let $\overline{\mu}$ and $\overline{\mu}'$ be the set of stable matchings corresponding to μ and μ' in the related marriage market.

Then if $\overline{\mu}(c_i) >_c \overline{\mu}'(c_i)$ for some position c_i of C, it follows that $\overline{\mu}(c_j) >_c \overline{\mu}'(c_j)$ for all positions of $c_j \in C$

Translation: the college will prefer every student in one stable matching group when compared to every student in the second stable matching group

Let's look at another example

Suppose the preferences are given by:

s_1: C_5, C_1	C_1: $s_1, s_2, s_3, s_4, s_5, s_6, s_7$
s_2: C_2, C_5, C_1	C_2: s_5, s_2
s_3: C_3, C_1	C_3: s_6, s_7, s_3
s_4: C_4, C_1	C_4: s_7, s_4
s_5: C_1, C_2	C_5: s_2, s_1
s_6: C_1, C_3	
s_7: C_1, C_3, C_4	

and let the quotas be

$$q_{C_1} = 3, \quad q_{C_j} = 1 \qquad (j = 2, \ldots, 5)$$

College-proposed DA outcome:

$$\mu(C_1) = \{s_1, s_3, s_4\}$$

$$\mu(C_2) = \{s_5\}$$

$$\mu(C_3) = \{s_6\}$$

$$\mu(C_4) = \{s_7\}$$

$$\mu(C_5) = \{s_2\}$$

$$>_c$$

Student-proposed DA outcome:

$$\mu(C_1) = \{s_5, s_6, s_7\}$$

$$\mu(C_2) = \{s_2\}$$

$$\mu(C_3) = \{s_3\}$$

$$\mu(C_4) = \{s_4\}$$

$$\mu(C_5) = \{s_1\}$$

Is the college admissions problem truly the same as the marriage problem?

The College Admissions Problem Is Not Equivalent to the Marriage Problem*

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Recall this slide from last week:

Theorem

Imposibility Theorem *(Roth). No stable matching mechanism exists for which stating the true preferences is a dominant strategy for every agent.*

However,

Theorem

(Dubins and Freedman; Roth) The men proposing deferred acceptance algorithm is strategy-proof for the men.

Suppose

True preferences are:

$P(C_1) = s_1, s_2, s_3, s_4$ with quota $q_{c_1} = 2$

$P(C_2) = s_1, s_2, s_3, s_4$ with quota $q_{c_2} = 1$

$P(C_3) = s_3, s_1, s_2, s_4$ with quota $q_{c_3} = 1$

"Strategic" preferences are:

$P(C_1) = s_4, s_2, s_3, s_1$ with quota $q_{c_1} = 2$

$P(C_2) = s_3, s_4, s_1, s_2$ with quota $q_{c_2} = 1$

$P(C_3) = s_4, s_3, s_1, s_2$ with quota $q_{c_3} = 1$

Student preferences are:

$P(s_1) = C_3, C_1, C_2$

$P(s_2) = C_2, C_1, C_3$

$P(s_3) = C_1, C_3, C_2$

$P(s_4) = C_1, C_2, C_3$

And the college deferrence algorithm is ran

To amplify: re-consider $P^{\#}(C)$

True preferences are:

$P(C_1) = s_1, s_2, s_3, s_4$ with quota $q_{c_1} = 2$

$P(C_2) = s_1, s_2, s_3, s_4$ with quota $q_{c_2} = 1$

$P(C_3) = s_3, s_1, s_2, s_4$ with quota $q_{c_3} = 1$

Alternatively:

$P(C_1) = s_4, s_2$

$P(C_2) = s_1$

$P(C_3) = s_3$

Stating truthful preferences lead to

$\mu(C_1) = \{s_3, s_4\}$

$\mu(C_2) = \{s_2\}$

$\mu(C_3) = \{s_1\}$

Strategic preferences lead to

$\mu(C_1) = \{s_2, s_4\}$

$\mu(C_2) = \{s_1\}$

$\mu(C_3) = \{s_3\}$

The main difference

In the marriage problem:

- It is proposer-optimal to always state true preferences
- There does not exist any outcome that every proposer prefers to the proposer-deferred algorithm outcome

However, in the college admissions problem:

- The points above are false
- As an implication, the proposed are inclined to strategically state preferences (stated in last class) and the proposers are inclined to strategically state preferences as well... making it still unsafe for all agents to state true preferences