Homework Week 4 CS146

Prove that the inverse-gamma distribution and scaled-inverse-chi-squared distribution are equivalent, except that the parameters of the distributions are defined differently. Show that the two distributions have the same algebraic form in terms of the random variable. What is the relationship between the α and β parameters of the inverse-gamma and the ν and τ parameters of the scaled-inverse-chi-squared distribution?

Inverse-Gamma Dist:

$$f(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right)$$

Scaled-Inverse-Chi-Squared:

$$\frac{(\tau^2\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \frac{\exp\left[\frac{-\nu\tau^2}{2x}\right]}{x^{1+\nu/2}}$$

If we subsitute α with $\nu/2$ and β with $\tau^2\nu/2$ we get:

$$\frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\exp\left[\frac{-\beta}{x}\right]}{x^{1+\alpha}}$$

Which is equivalent to:

$$\frac{\beta^\alpha}{\Gamma(\alpha)} \exp\left[\frac{-\beta}{x}\right] x^{-1-\alpha}$$

Calculate the derivatives of the normal-inverse-gamma pdf with respect to its random variables, x and σ² and determine the values of x and σ² (in terms of the distribution parameters μ, λ, α, β) that maximize the pdf.

The PDF:

$$\frac{\sqrt{\lambda}}{\sigma\sqrt{2\pi}} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} e^{-\frac{2\beta+\lambda(x-\mu)^2}{2\sigma^2}}$$

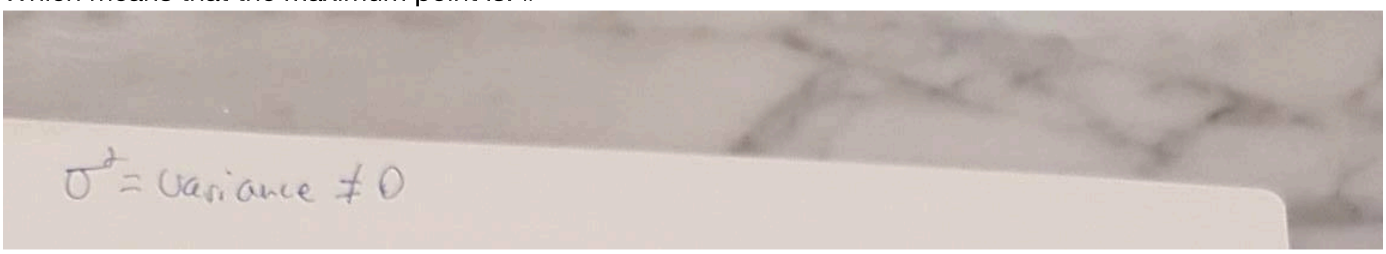
The derivative with respect to σ^2 (see Apendix for further explanation):

$$C_1 \cdot \sigma^{2^{(-\alpha-1.5)}} \cdot e^{-\frac{2\beta+\lambda(x-\mu)^2}{2\sigma^2}} \cdot [(-\alpha-1.5)(\sigma^{2)^{-1}}) + \frac{2\beta+\lambda(x-mu)^2}{2(\sigma^2)^2}]$$

where

$$C_1 = \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \frac{\beta^\alpha}{\Gamma(\alpha)} [\frac{1}{\alpha^2}]^{(\alpha+1)}$$

Which means that the maximum point is: #



$$\left[(-\alpha - 1\tfrac{1}{2})\,(\sigma^2)^{-1} + \frac{2\beta + 1(x-\mu)^2}{2(\sigma^2)^2}\right] = 0$$

$$\frac{\alpha + 1\tfrac{1}{2}}{\sigma^2} = \frac{2\beta + 1(x-\mu)^2}{2(\sigma^2)^2} \quad / \cdot \sigma^2$$

$$\alpha + 1\tfrac{1}{2} = \frac{2\beta + 1(x-\mu)^2}{2\sigma^2} \quad / \cdot \sigma^2$$

$$\sigma^2 = \frac{2\beta + 1(x-\mu)^2}{2\alpha + 3}$$

The derivative with respect to x: #

$f(x)$:

$$\underbrace{\frac{\sqrt{\lambda}}{\sigma\sqrt{2\pi}} \; \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1}} \; e^{\frac{-2\beta-\lambda(x-\mu)^2}{2\alpha^2}}$$

$$= C_1 \; e^{\frac{-2\beta-\lambda(x-\mu)^2}{2\alpha^2}}$$

$$= \underbrace{C_1 \cdot e^{\frac{-2\beta}{2\alpha^2}}} \cdot e^{\frac{-\lambda(x-\mu)^2}{2\alpha^2}}$$

$$= C_2 \cdot e^{\frac{-\lambda(x-\mu)^2}{2\alpha^2}}$$

$$= C_2 \cdot e^{C_3 \cdot (x-\mu)^2} \; , C_3 = \frac{-\lambda}{2\alpha^2}$$

$$f'(x) = C_2 \cdot C_3 \; (x-\mu) \cdot e^{C_3 (x-\mu)^2} = 0$$

$$x - \mu = 0$$

$$\boxed{x = \mu}$$

Appendix: Derivation: #

$\sigma^2 = \text{variance} \neq 0$

$$\left[(-\alpha - 1\tfrac{1}{2}) (\sigma^2)^{-1} + \frac{2\beta + 1(x-\mu)^2}{2(\sigma^2)^2} \right] = 0$$

$$\frac{\alpha + 1\tfrac{1}{2}}{\sigma^2} = \frac{2\beta + 1(x-\mu)^2}{2(\sigma^2)^2} \quad / \cdot (\sigma^2)$$

$$\alpha + 1\tfrac{1}{2} = \frac{2\beta + 1(x-\mu)^2}{2\sigma^2} \quad / \cdot \sigma^2 \quad : \alpha + 1\tfrac{1}{2}$$

$$\sigma^2 = \frac{2\beta + 1(x-\mu)^2}{2\alpha + 3}$$

#

$$f(x) = \frac{\sqrt{\lambda}}{\sigma\sqrt{2\pi}} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} e^{-\frac{2\beta+\lambda(x-\mu)^2}{2\sigma^2}}$$

$$= \underbrace{\frac{\sqrt{\lambda}}{\sqrt{2\pi}} \frac{\beta^{\alpha}}{\Gamma(\alpha)}}_{C_1} \left(\frac{1}{(\sigma^2)^{1/2}}\right) \left(\frac{1}{\sigma^2}\right)^{\alpha+1} e^{-\frac{2\beta+\lambda(x-\mu)^2}{2\sigma^2}}$$

$$= C_1 \frac{1}{(\sigma^2)^{1/2}} \left(\frac{1}{(\sigma^2)^{\alpha+1}}\right) e^{\frac{-2\beta+\lambda(x-\mu)^2}{2\sigma^2}}$$

$$= C_1 \frac{1}{(\sigma^2)^{\alpha+1\frac{1}{2}}} e^{\frac{-2\beta+\lambda(x-\mu)^2}{2\sigma^2}}$$

$$f'(x) = C_1 \Big[(-\alpha - 1\tfrac{1}{2})(\sigma^2)^{(-\alpha-2.5)} \cdot e^{\frac{-2\beta+\lambda(x-\mu)^2}{2\sigma^2}}$$

$$+ \sigma^{2^{-\alpha-1\frac{1}{2}}} \cdot \frac{2(-2\beta+\lambda(x-\mu)^2)}{4(\sigma^2)^2} \cdot e^{-\frac{2\beta+\lambda(x-\mu)^2}{2\sigma^2}} \Big]$$

$$= C_1 \sigma^{2^{-\alpha-1\frac{1}{2}}} e^{-\frac{2\beta+\lambda(x-\mu)^2}{2\sigma^2}} \Big[(-\alpha - 1\tfrac{1}{2})(\sigma^2)^{-1}$$

$$+ \frac{2\beta+\lambda(x-\mu)^2}{2(\sigma^2)^2} \Big]$$