Partial Differential Equations in Finance

Jordan Stout

Abstract

In this paper, I will introduce how partial differential equations are used in finance. First I will be applying the concepts we have learned in class to a portfolio to derive the Black-Scholes equation and explain about the concept of continuously revised delta hedging. I will then give an example of how the Black-Scholes equation is solved analytically by converting it into the heat equation.

1 Derivation

The goal is to derive a formula that find the value of a European style stock option. To do so we make five assumptions. First, that there is no change in the number of shares outstanding. Second, the underlying stock pays no dividends within the lifetime of the contract. Third, the price of the stock one period ahead has a log-normal distribution with mean μ and volatility q, which are both constant over the lifetime of the contract. Fourth, there is a risk-free interest rate which remains constant. Finally, individuals can borrow and lend at the risk-free interest rate.

Suppose we wish to price a European style stock option V, with the underlying stock value of S, which is set to mature at T. We shall assume that S follows a geometric Brownian motion with mean growth rate of μ and volatility σ. The first step of the derivation is to use Ito's Lemma to give us the SDE (1).

$$dV = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 \tag{1}$$

We must also recall that our model is based off of geometric Brownian motion, which is our case is represented by (2)

$$dS = \mu S dt + \sigma S dX \tag{2}$$

Plugging (2) into (1) and simplifying we arrive at

$$dV = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})dt + \sigma S\frac{\partial V}{\partial S}dX \tag{3}$$

Our proofs defining statements is that a fully hedged portfolio will grow at the risk free rate. Therefore, we must see how our portfolio changes in time.

Specifically, we are interested in the infinitesimal change of a mixture of a call option and a quantity of stock. The quantity will be represented by Δ

$$d(V+\Delta S) = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \Delta\mu S)dt + \Delta S(\frac{\partial V}{\partial S} + \Delta)dX \quad (4)$$

This leads us to a choice for Δ which will eliminate the term associated with the randomness. If we set $\Delta = -\frac{\partial V}{\partial S}$ we receive (5).

$$d(C+\Delta S) = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})dt \quad (5)$$

This technique is known as Delta-Hedging and provides us with a portfolio that is free of randomness. The key idea is to buy and sell the underlying asset in just the right way to eliminate risk, or as I like to think of it, flipping a coin and calling heads and tails. This elimination of randomness is how we can say that it should grow at the risk free rate. Hence the growth of our portfolio must be equal the risk free rate compounding.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = r(V - S\frac{\partial V}{\partial S}) \quad (6)$$

Rearranging, we arrive at the second order, linear PDE known as the Black-Scholes equation.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad (7)$$

2 Introduction

The Black-Scholes equation describes the price of a European style call options from time of purchase to its maturity. "European style" refers to the caveat that the option cannot be exercised before its expiration date. The unsimplified Black-Scholes equation is shown in Equation 7. Where σ^2 is the variance of the return on the option also known as its volatility, V is the value of the option, S is the underlying stock's price ($0 \leq S$), r is the risk-free interest rate, t is the time until expiration and T is the expiration date ($0 \leq t \leq T$).

Fischer Black and Myron Scholes won the 1997 Nobel Price for Economic Science for their work with the Black-Scholes Model. From here, we wish to now be able to find an analytical solution the Black-Scholes equation. Before finding an analytical solution, a mathematician would be crazy to attempt to solve a PDE without trying to simplify it first. Conviniently, it is possible, through a series of substitutions, to transform Black-Scholes into the form of the Heat Equation.

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} \quad (8)$$

3 Transformation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 (S\frac{\partial}{\partial S})^2 V + (r - \frac{1}{2}\sigma^2) S\frac{\partial V}{\partial S} - rV = 0 \tag{9}$$

After rewriting Equation 1 as Equation 3, we will transform Black-Scholes in terms of forward time. Time in the heat equation refers to time since t=0, while t in Black-Scholes refers to time until expiration. To solve this inconsistency we make the substitution $t = T - \tau$. For convenience we also make the substitution $S = e^y$.

$$S\frac{\partial}{\partial S} \to \frac{\partial}{\partial y} \tag{10}$$

$$\frac{\partial}{\partial t} \to -\frac{\partial}{\partial \tau} \tag{11}$$

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial y^2} - (r - \frac{1}{2}\sigma)\frac{\partial V}{\partial y} + rV = 0 \tag{12}$$

Next our goal is to omit the rV term. To do this we will make the substitution $u = e^{r\tau}V$. Using the product rule we end at the PDE

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\frac{\partial^2 u}{\partial y^2} - (r - \frac{1}{2}\sigma^2)\frac{\partial u}{\partial y} = 0 \tag{13}$$

The first partial derivative with respect to y does not cancel unless $r = \frac{1}{2}\sigma^2$ because we have not taken into account the drift of the Brownian motion. To cancel the drift (which is linear in time), we make the substitution.

$$x = y + (r - \frac{\sigma^2}{2})\tau \tag{14}$$

Under our new coordinate system (x, τ), we have the following relationships

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} \tag{15}$$

$$\frac{\partial}{\partial \tau} = -(r - \frac{1}{2}\sigma^2)\frac{\partial}{\partial y} \tag{16}$$

Applying (8), (9) and (10) to (7) we arrive at (11) below

$$-\frac{\partial u}{\partial \tau} - (r - \frac{1}{2}\sigma^2)\frac{\partial u}{\partial x} + (r - \frac{1}{2}\sigma^2)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{17}$$

This simplifies to a more recognizable...

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} \tag{18}$$

4 Conclusion and Further Work

The purpose of this paper was to learn something new. I had only ever seen the Black-Scholes equation, and it was intriguing because it insinuated that there was a profound world of mathematical finance much more complex than previously imagined. There are three pieces of further work I would like to pursue. First, solving the derived heat equation in terms of the financial variables. Second I would like to solve Black-Scholes numerically. Finally I would like to research more on SDEs because I only learned enough to understand Ito's Lemma, but as far as mathematical finance goes I imagine SDEs are a force to be reckoned with.

5 Bibliography

1. Cheng, Steve. "Analytic Solution of Black-Scholes PDE." *Analytic Solution of Black-Scholes PDE*, 22 Mar. 2013, planetmath.org/AnalyticSolutionOfBlackScholesPDE.

2. Stecher. "Converting the Black-Scholes PDE to The Heat Equation." *math.tamu.edu*, Texas A&M , Apr. 2012.

3. Kwan, Clarence C.Y. 2019, *Solving the Black-Scholes Partial Differential Equation via the Solution Method for a One-Dimensional Heat Equation.*

4. "Brownian Motion and Random Walks." *MIT*, web.mit.edu/8.334/www/grades/projects/projects17/OscarMickelin/brownian.html.

5. Holton, Glyn. "Brownian Motion (Wiener Process)." *GlynHolton.com*, 10 Oct. 2016, www.glynholton.com/notes/brownian_motion/.

6. Stephanie. "Lognormal Distribution: Definition, Examples." *Statistics How To*, 17 Sept. 2020, www.statisticshowto.com/lognormal-distribution/.

7. "Ito's Lemma." *QuantStart*, www.quantstart.com/articles/Itos-Lemma/.