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The physical basis of glacier volume-area scaling

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Abstract. Ice volumes are known for only a few of the roughly 160,000 glaciers worldwide but are important components of many climate and sea level studies which require water flux estimates. A scaling analysis of the mass and momentum conservation equations shows that glacier volumes can be related by a power law to more easily observed glacier surface areas. The relationship requires four closure choices for the scaling behavior of glacier widths, slopes, side drag and mass balance. Reasonable closures predict a volume-area scaling exponent which is consistent with observations, giving a physical and practical basis for estimating ice volumes. Glacier volume is insensitive to perturbations in the mass balance scaling, but changes in average accumulation area ratios reflect significant changes in the scaling of both mass balance and ice volume.

Introduction

Studies of the relation of melting glaciers to sea level rise and other aspects of glacier/climate interactions require regional and global estimates of ice volumes and surface areas. This necessitates either the impossible task of measuring the volume of all glaciers in the world, or using a theoretical technique which can scale up a few local observations to general statements about the global volume of ice. The following analysis solves a key part of many sea level and climate problems by giving a theoretical scaling basis for linking unknown ice volumes to easily observed ice surface quantities.

Historically, glaciers have been studied one at a time, with a fixed set of continuum physical rules and separately measured boundary conditions appropriate to each glacier [*Paterson*, 1994]. Glacier volumes, for example, are usually acquired by detailed radio echo sounding. Statistical scaling theories like the one presented in this paper, on the other hand, address the properties of many glaciers simultaneously. With this approach, glacier volumes are determined by statistically valid relationships to other known quantities, such as surface areas. In particular, in this paper the theoretically derived relationship between volume and surface area is confirmed by establishing empirically observed trends from over 100 different glaciers.

For most glaciers (excluding special cases such as surging) the accumulation and ablation of ice controls terminus advance and retreat, ice volume, and surface geometry. In particular, these quantities all vary with perturbations in the ice mass balance rate $\dot{b}$ (rate of ice input minus output, due to accumulation and ablation at the surface), and therefore global and regional trends in glacier geometry can be used as indicators of changing climate. In general, on any one glacier, there is an accumulation area at relatively high elevations ($\dot{b} > 0$ on a yearly average), and an ablation area at relatively low elevations ($\dot{b} < 0$ on a yearly average), but balance profiles as a function of elevation or horizontal distance are known in detail for only a few hundred of the roughly 160,000 glaciers worldwide [*Meier and Bahr*, 1996; *Haeberli et al.*, 1994]. The limited existing observations do suggest that the balance profiles versus elevation are nonlinear and perhaps quadratic [*Dyurgerov*, 1995; *Dyurgerov et al.*, 1995; *Haeberli et al.*, 1994; *Meier and Post*, 1962]. It is not known if quadratic profiles are characteristic of all or just a few glaciers, but there is some evidence that balance profiles are similar for many glaciers. By the end of the summer melt season, winter snow accumulated on the surface of a glacier has melted back to an easily observed elevation contour where $\dot{b} = 0$ (called the equilibrium line), and air photo observations of this contour show that the ra-

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tio of the accumulation area to the area of the entire glacier (accumulation area ratio (AAR)) falls within a narrow range of values (roughly 0.5 to 0.8) for noncalving glaciers in a near steady state condition [*Meier and Post*, 1962].

In the following analysis we demonstrate that the AAR is related to the mass balance profile and that both are intimately tied to a volume (V) and surface area (S) scaling relationship. If the mass balance profile changes (perhaps due to a change in climate), then the AAR and the exponent γ in $V \propto S^\gamma$ also change in a manner predicted by a rescaling of the mass and momentum conservation equations. The exponent γ is insensitive to small changes in the mass balance, but larger changes (easily measured with AARs) can have a dramatic effect on ice volumes.

The paper is organized as follows. The next section outlines common assumptions and their shortcomings for volume-area relationships. The following section uses a scaling analysis to relate glacier volumes to surface areas. The scaling analysis requires four closures describing the behavior of glacier width, slope, side drag, and mass balance. These closures are explored in a following section and summarized in a later section. The sensitivity of volume-area scaling to mass balance perturbations are then discussed, and the conclusions reiterate the primary volume-area results. For reference, the notation section defines variables used in the text.

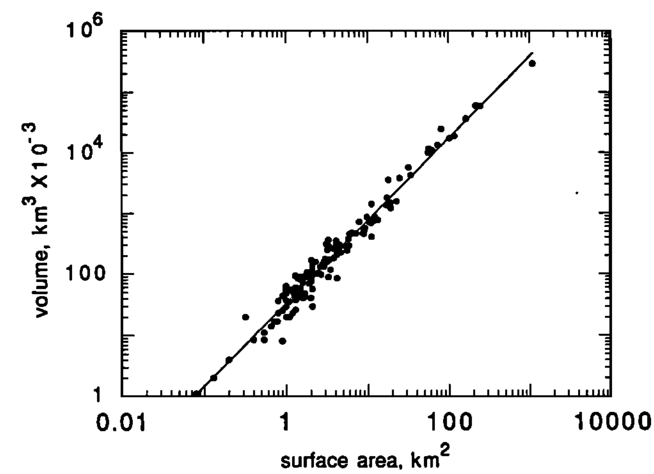

Figure 1. V versus S for 144 glaciers, not including ice caps, plotted on a log-log scale. Data were collected from Europe, North America, central Asia, and the Arctic (a list of 41 references is available from the authors upon request). Only those sources with recent reliable radio echo soundings were used for the volume estimates. As in a previous analysis with fewer glaciers [*Chen and Ohmura*, 1990], the straight line fit has a regression coefficient of 1.36. The squared correlation coefficient is $R^2 = 0.99$. The data are discussed in more detail by *Meier and Bahr* [1996] and in a paper in preparation by the authors. Exponents in the range from roughly 1.3 to 1.4 have been suggested by several Russian authors as well (e.g., for 103 glaciers between the Altai and Tien Shan mountains, *Macheret et al.* [1988] find an exponent of 1.379 ± 0.066 with an $R^2 = 0.95$).

Volume-Size Scaling

The volume of a glacier is proportional to the product of its length x, width w, and an average of its thickness h. In other words, $[V] \propto [x][w][h]$ where brackets indicate characteristic values. Similarly, the surface area is $[S] \propto [x][w]$. With some simple geometrical objects (e.g., spheres), the length, width, and height are expected to scale in the same manner, so that $[V] \propto [x]^3$ and $[S] \propto [x]^2$, and $[V] \propto [S]^{3/2}$. For glaciers, however, data show that $[V] \propto [S]^\gamma$ where $\gamma \approx 1.36$ (see Figure 1) [*Zhurovlyev*, 1985; *Macheret et al.*, 1988; *Chen and Ohmura*, 1990], and for ice sheets $\gamma \approx 1.25$ [*Paterson*, 1972]. The implication is that the length, width, and thickness of glaciers do not scale identically.

Different scaling behaviors for length, width, and thickness are usually attributed to a combination of ice dynamics and geometrical boundary conditions [e.g., *Paterson*, 1972]. For ice sheets, with no surrounding valley walls and essentially flat beds, it has long been argued that $[x] = [w]$, and therefore $[V] \propto [x]^2[h]$ and $[S] \propto [x]^2$. From arguments which assume perfectly plastic ice and/or uniform accumulation and ablation, it follows that $h \propto x^{1/2}$, $[h] \propto [x]^{1/2}$, and then $\gamma = 1.25$ [e.g., *Paterson*, 1972; *Nye*, 1951]. Similarly, for glaciers confined by mountain valleys it has been suggested that the width is roughly constant, and therefore $[V] \propto [x][h]$ while $[S] \propto [x]$ [e.g., *Paterson*, 1994]. For perfect plasticity and flat beds this suggests $\gamma = 3/2$ which is a poor match to the data (Figure 1).

However, these previous derivations make assumptions which are useful only as low-order approximations. Glacier beds are rarely flat, ice is not a perfectly plastic solid [*Glen*, 1958], accumulation and ablation are not uniform, and while intuitively appealing, valley glaciers are not one-dimensional surfaces with widths scaling as a constant. As a simple counterexample, consider a glacier which is completely contained within a mountain valley with a characteristic parabolic cross section. As the thickness of a glacier increases, the width also increases as $h \propto w^2$. Therefore $[w] \propto [h]^{1/2}$ and $[w]$ is not a constant. Also, data compiled for some large glaciers which are not contained within a single valley (e.g., the Barnard, Columbia, Harvard, Knik, Matanuska, Russell, and Tazlina, all in Alaska) show that $[x] \propto [S]^{0.5}$ where $[x]$ in this case is the average along-channel length [*Bahr and Peckham*, 1996]. With $[S] = [x][w]$, the observed relationship implies that for these very large branching glaciers $[w] \propto [x]$, which is the same as the width scaling commonly assumed for ice sheets. To include this and all other possibilities, in the following analyses we will assume that

$$[w] \propto [x]^q \tag{1}$$

where q is some constant.

With the width scaling in (1),

$$[V] \propto [x]^{1+q}[h] \tag{2a}$$

$$[S] \propto [x]^{1+q}. \tag{2b}$$

In the next section a scaling analysis of the mass and momentum equations will give a relationship between $[h]$ and $[x]$ which is used to relate $[V]$ to $[S]$ and then to predict γ.

Scaling Analysis

From observations, $[V] \propto [S]^\gamma$ with $\gamma \approx 1.36$. Therefore $[x][w][h] \propto [x]^\gamma[w]^\gamma$ and $[h] \propto [x]^{\gamma-1}[w]^{\gamma-1}$. It follows that $[h] \propto [x]^\theta$ where $\theta = (q+1)(\gamma-1)$. Our hypothesized relationship for width scaling (equation (1)) therefore implies that $[h]$ must be a power law in $[x]$. (From dimensional analysis this is not unexpected; the Buckingham Pi theorem often gives power law relationships between characteristic quantities [*Schmidt and Housen*, 1995].)

The constant θ is derived explicitly from a scaling analysis of the partial differential equations describing glacier dynamics. This scaling can be accomplished using many different techniques including the "shallow ice" nondimensionalizations of *Hutter* [1983], *Morland* [1984], *Fowler* [1992] and others, or a stretching symmetry analysis as detailed in the literature on similarity methods [e.g., *Logan*, 1987]. Both approaches give the same results, relating characteristic values for each glacier parameter (thickness, velocity, etc.) to the characteristic values of other glacier quantities. There is a large body of observational and analytical evidence suggesting that shear deformation is the principal mode of glacier flow with longitudinal compression and extension playing a secondary role, particularly for small valley glaciers and especially near the bed where the most significant deformation is occurring [*Paterson*, 1994]. By eliminating the typically small nonshear terms in the flow equations, both techniques show that characteristic quantities (indicated by square brackets) scale as follows.

Choose an $x$axis parallel and a $z$axis perpendicular (upwards) to the mean surface profile of a glacier. Let α be the angle the $x$axis makes with a horizontal which is perpendicular to gravity. Assume *Glen*'s [1958] nonlinear flow law. Then from mass conservation (continuity)

$$[\dot{b}] \propto \frac{[u_x][h]}{[x]} \tag{3a}$$

and from momentum conservation (ice flow)

$$[u_x] \propto [A][\rho]^n[g_x]^n[h]^{n+1}[F]^n \tag{3b}$$

where u_x is the down-glacier or x component of the velocity, $g_x = g\sin\alpha$ is the x component of gravity (g), and ρ is the ice density. F is a shape factor which accounts for drag on the valley sidewalls and glacier bed [*Nye*, 1965]. Variable n is a flow parameter, with $n \to \infty$ for perfectly plastic behavior [*Paterson*, 1994], $n = 3$ typical for ice, and $n = 1$ giving a Newtonian constitutive relationship with viscosity $(2A)^{-1}$. A is a temperature dependent parameter, although we will ignore this dependency in the simple scaling analysis considered here; isothermal conditions are reasonable for many of the world's mountain glaciers (considered in Figure 1), although isothermal conditions are inaccurate for most ice caps and ice sheets [*Paterson*, 1994]. Note that as in other scaling analyses [e.g., *Paterson*, 1972], equations (3a) and (3b) assume constants of proportionality which do not change significantly from glacier to glacier. This common assumption is partially justified with data in *Bahr* [1997a], *Chen and Ohmura* [1990], and in more detail by Peckham, in progress. (Also, the "constant" of proportionality may not strictly be constant, but may have a distribution about a nominal value, as discussed later for mass balance.)

The scaling relationships in (3) can be manipulated to express $[h]$ as a function of $[x]$ (necessary to express $[V]$ as a function of $[S]$). Combining equations (3a) and (3b) to eliminate $[u_x]$, $[\dot{b}][x] \propto [h]^{n+2}[A][\rho]^n[g_x]^n[F]^n$. The constants A, ρ, and g are largely independent of x (they do not vary with the size of the glacier), so

$$[h]^{n+2} \propto \frac{[\dot{b}][x]}{[\sin\alpha]^n[F]^n}. \tag{4}$$

To completely express $[h]$ as a function of $[x]$, closures for $[\sin\alpha]$, $[F]$, and $[\dot{b}]$ must be hypothesized. For the moment we propose that

$$[\sin\alpha] \propto [x]^{-r} \tag{5a}$$

$$[F] \propto [x]^{-f} \tag{5b}$$

$$\dot{b} \propto [x]^m \tag{5c}$$

for some constants r, f, and m. The power law forms are consistent with expectations from the Buckingham Pi theorem [*Schmidt and Housen*, 1995] and will be justified in the next section.

Substituting (5) into (4),

$$[h] \propto [x]^{\frac{1+m+n(f+r)}{n+2}}. \tag{6}$$

The volume-area relationship can now be constructed from (1), (2), and (6).

$$\begin{aligned}[V] &\propto [x][w][h]\\ [V] &\propto [x]^{1+q+\frac{1+m+n(f+r)}{n+2}}\\ [V] &\propto [S]^{1+\frac{1+m+n(f+r)}{(q+1)(n+2)}}\end{aligned} \tag{7}$$

Therefore γ is fixed for any choice of the scaling exponents q, r, f, m, and n. Appropriate values are suggested in the following section.

Choices for Closure

To complete the relationship between volume and surface area, four closure choices must be made, one for each of the scaling exponents related to glacier width (q), slope (r), side drag (f), and mass balance (m) (the exponent n is fixed by the choice of a constitutive law). These closures are each addressed separately.

Width Closure

As discussed above, $q = 0$ (width scales as a constant) is inappropriate, and $q = 1$ (width scales with length) is expected for large ice caps and ice sheets. A study of surface area, length, and width data from an inventory of over 24,000 Eurasian glaciers and 5400 Alps glaciers suggests that $[w] \propto [x]^{0.6}$ [*Bahr*, 1997b]. In other words, $q = 0.6$, agrees with available valley glacier data, so this is selected as the most likely value.

Slope Closure

If surface slopes are small (as with most very large glaciers), then as detailed in shallow-ice approximations,

$$[\sin\alpha] \propto \frac{[h]}{[x]} \tag{8}$$

(essentially "rise over run") [e.g., *Hutter*, 1983]. In this case, q is not independent of the other scaling constants, and from (5a), (6) and (8), $[x]^{-r} \propto [x]^{\frac{1+m+n(f+r)}{n+2}-1}$. Therefore

$$r = \frac{(1-m+n-nf)}{2\,(n+1)}. \tag{9}$$

This will be referred to as "shallow slope scaling."

On the other hand, for the large surface slopes typical of steep cirque glaciers, $[\sin\alpha]$ cannot be rescaled. In these cases the surface may follow the unpredictable bed topography, so that $r = 0$ seems most appropriate (this unscaled version will be referred to as "steep slope scaling"). However, longer glaciers typically have shallower slopes, so an inverse scaling with length, such as $r = 1$, may also be appropriate. Without some well-defined bed profile, choosing an exact value for slope closure is difficult.

Side Drag Closure

The shape factor can be approximated as

$$F = A_c/(hP)$$

where A_c is the cross-sectional area, and P is the perimeter length excluding the surface (i.e., perimeter of contact between ice and bed) [e.g., *Budd*, 1969]. Intuitively, the cross-sectional area divided by the basal perimeter is just a measure of the glacier width. This is shown rigorously by assuming a roughly polynomial shaped cross section, $h \propto w^{\beta}$ for some constant β (with $\beta = 2$ for parabolic channels). Then the perimeter is given by the arc length formula

$$P = 2\int_0^w \sqrt{1+\left(\frac{\partial h}{\partial w}\right)^2}\,dw \tag{10}$$

which scales as $[h]$ (because P is essentially the integral of a derivative of h). Similarly, the cross-sectional area is

$$A_c = 2\int_0^h h^{1/\beta}dh \propto w^{\beta+1} \tag{11}$$

which scales as $[w]^{\beta+1}$. Therefore the shape factor scales as $[F] \propto [A_c]/([h][P]) \propto [w]^{1-\beta}$, or

$$[F] \propto [w]/[h]. \tag{12}$$

In other words, the scaling exponent for the side drag is not independent of the other exponents which already describe the scaling of $[w]$ and $[h]$. From (6) and (12), $[x]^{-f} = [x]^{q-\frac{1+m+n(f+r)}{n+2}}$. Solving for f gives

$$f = 1/2\,(m + nr + 1 - q\,(n+2)). \tag{13}$$

The approximation $F = A_c/(hP)$ is valid when the half-width divided by the thickness is not too large, a good approximation for many valley glaciers [*Budd*, 1969]. When the halfwidth is large relative to the thickness, there is little side drag, and the shape factor tends to $F = 1$ [*Nye*, 1965]. Therefore for ice caps the side drag scaling exponent is expected to be $f = 0$.

Although our goal is to predict the volume-area scaling exponent γ, its observed value of $\gamma \approx 1.36$ for valley glaciers (Figure 1) can be used to suggest a reasonable value for f. Recall $[h] \propto [V]/[S] \propto [x]^{(q+1)(\gamma-1)}$, so $[w]/[h] \propto [x]^{q-(q+1)(\gamma-1)}$. For $q \approx 0.6$, as suggested above, $[w]/[h] \propto [x]^{0.024}$. In other words, the data show that $f \approx 0$ is expected. For $q = 0.6$ and $r = 0$ as suggested above, and for $m = 2$ as suggested below, (13) also predicts that $f = 0$ (when $n = 3$). As a reasonable closure value, therefore we will use $f = 0$.

Mass Balance Closure

The mass balance $\dot{b}$ is a function of position x because it is a climatically imposed boundary condition which can vary from point to point. As described elsewhere [e.g., *Jóhannesson et al.*, 1989], the characteristic value of the mass balance $[\dot{b}]$ is given as the magnitude of the mass balance at the terminus. However, if a glacier grows longer, then the terminus will move to a lower elevation and $[\dot{b}]$ will be correspondingly larger. Therefore $[\dot{b}]$ depends on the characteristic length. Rearranging (4) shows that the length dependence must be power law in form, as long as the slope and side drag also scale as power laws; recall $[h] \propto [V]/[S] \propto [x]^{(q+1)(\gamma-1)}$, so

$$\begin{aligned}[\dot{b}] &\propto \frac{[h]^{n+2}[\sin\alpha]^n[F]^n}{[x]} \\ [\dot{b}] &\propto [x]^{(n+2)(q+1)(\gamma-1)-nr-nf-1}. \end{aligned} \tag{14}$$

Again, a power law form is a reasonably expected consequence of the Buckingham Pi theorem.

Field data suggest that typical glacier mass balances have the approximate form

$$\dot{b} \approx -c_m x^m + c_0 \tag{15}$$

with $m \approx 2$ [*Dyurgerov*, 1995; *Dyurgerov et al.*, 1995; *Haeberli et al.*, 1994; *Meier and Post*, 1962] and positive constants c_m and c_0. For each individual glacier, this form can be thought of as the dominant terms in a power series expansion of the actual mass balance profile. In other words, the real mass balance profile on any particular glacier will have some complicated shape represented by the expansion

$$\dot{b} = \sum_{m=0}^{\infty} \pm c_m x^m. \tag{16}$$

The dominant terms which describe the overall shape are given in (15). Note that c_m and c_0 are constants for any one glacier but can have different values for different glaciers.

Assume for the moment that two different glaciers have similarly shaped but different mass balance profiles given by $\dot{b} = -c_m x^m + c_0$ (for the first glacier) and $\hat{\dot{b}} = -\hat{c}_m x^m + \hat{c}_0$ (for the second glacier). Let the glaciers have lengths L and $\hat{L}$ (see Figure 2). If both glaciers are in balance then

$$\int_0^L \dot{b}(x)dx = 0 = \int_0^{\hat{L}} \hat{\dot{b}}(x)dx \tag{17}$$

and by integration

$$c_m = \frac{c_0\,(m+1)}{L^m}. \tag{18}$$

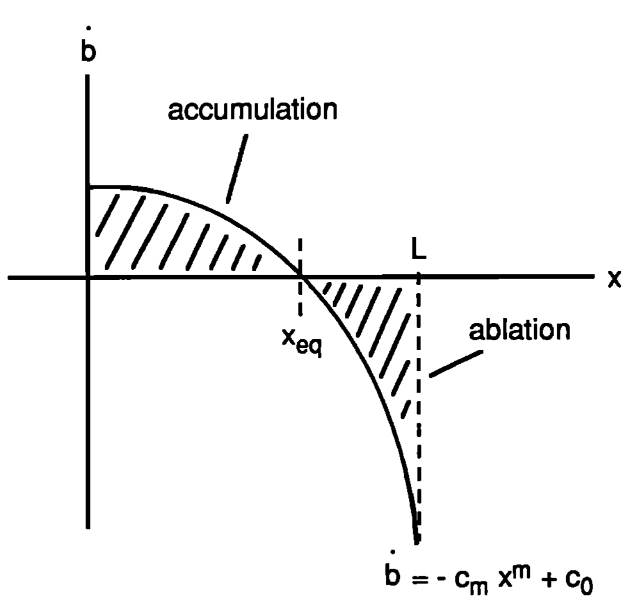

Figure 2. Diagram of the balance rate profile and accumulation and ablation areas (equal for steady state conditions)

Therefore, at the terminus, $\dot{b}(L) = -c_m L^m + c_0 = -m\,c_0$. Using (18) and an analogous equation for the second glacier, c_m and $\hat{c}_m$ can be eliminated, and

$$\frac{\dot{b}}{\hat{\dot{b}}} = \left(\frac{m\,c_0}{m\,\hat{c}_0}\right)\left(\frac{1-(m+1)(x/L)^m}{1-(m+1)(x/\hat{L})^m}\right). \tag{19}$$

If glacier lengths are selected as the characteristic lengths, then the second factor is $O(1)$, which suggests that the appropriate choice for the characteristic mass balances should be the magnitudes at the terminus $[\dot{b}] = -m\,[c_0]$ and $[\hat{\dot{b}}] = -m\,[\hat{c}_0]$. This choice agrees with previous analyses which also select the balance at the terminus as the characteristic mass balance [*Jóhannesson et al.*, 1989].

Equation (18), however, shows an alternative way to write the characteristic balance at the terminus.

$$[\dot{b}] = -m\,[c_0] = -[c_m][x]^m \frac{m}{m+1} \propto [x]^m \tag{20}$$

(where glacier length is selected as the characteristic length $[x]$). Both parts of (20) express the characteristic mass balance as the magnitude of the balance at the terminus. The right-hand side of equation (20) just makes the hidden dependence on characteristic length explicit, and this is the form we will use to help express the characteristic volume as a function of the characteristic surface area in (7).

As an important side note, recall that $\dot{b}$ is typically expressed as an increasing function of altitude z. With a change of coordinates that follow the surface of the glacier $z(x)$, $\dot{b}$ can be expressed just as easily as a function of length along a glacier, as done above. Note that when written as a function of altitude, no one would suggest that glaciers with a larger altitudinal range have larger mass balances. This is because the activity index $\partial\dot{b}/\partial z$ has a random distribution of possible values. In other words, the slope of $\dot{b}(z)$ can change from site to site depending on climate, so two glaciers with the same altitudinal range may span very different ranges of mass balance. Similarly, the slope of $\dot{b}(x)$ (roughly c_m) has a random distribution of possible values. We refer to this slope $\partial\dot{b}/\partial x$ as the "balance index." Two glaciers with the same length but different balance indices may span entirely different ranges of mass balance. Likewise, depending on the balance indices, a longer glacier does not always have a larger characteristic mass balance than a different shorter glacier. The characteristic mass balance must still scale with length (equation (20)), but the constants of proportionality $[c_m]$ can change (randomly with climate) for different glaciers.

Finally, to suggest an appropriate closure exponent m for the mass balance, we consider accumulation area ratios which can be expressed as a function of m. Suppose a glacier runs from $x = 0$ at its head to $x = L$ at its terminus, and suppose the equilibrium line is at $x = x_{\rm eq}$ (Figure 2). Assume a balance rate profile given by (15).

Note that the balance profile is concave downwards, as observed for real glaciers [*Dyurgerov*, 1995; *Dyurgerov et al.*, 1995; *Haeberli et al.*, 1994; *Meier and Post*, 1962]. By assuming approximately steady state conditions, the total mass balance over the length of the glacier is 0, and as derived in (18), $c_0/c_m = L^m/(m+1)$. Note that $\dot{b}(x_{eq}) = 0$, so from (15),

$$x_{eq} = \left(\frac{c_0}{c_m}\right)^{\frac{1}{m}}, \tag{21}$$

and then by eliminating c_0 and c_m,

$$x_{eq} = \frac{L}{(m+1)^{1/m}}. \tag{22}$$

Now recall that the accumulation area ratio is the accumulation area divided by the total glacier area. Therefore for a given characteristic or constant width,

$$\begin{aligned} \mathrm{AAR} &= \frac{[w]\, x_{eq}}{[w]\, L} \\ \mathrm{AAR} &= \left(\frac{1}{m+1}\right)^{\frac{1}{m}}. \end{aligned} \tag{23}$$

Accumulation area ratios are observed to fall approximately within the range 0.5 to 0.8 with typical values near two thirds for small mountain glaciers in a steady state balance [*Meier and Post*, 1962]. From (23), this spans $m = 1$ to $m = 11$. However, glacier inventory data for over 24,000 Eurasian glaciers gives an average AAR of 0.578, and inventory data for over 5400 glaciers in the Alps gives an average AAR of 0.580 [*Bahr*, 1997b]. These AARs both correspond to exponents of $m = 2.0$. This agrees with our intuition about the quadratic shape of balance profiles [*Dyurgerov*, 1995; *Dyurgerov et al.*, 1995; *Haeberli et al.*, 1994; *Meier and Post*, 1962], so $m = 2$ is selected as a reasonable closure value.

Closure Summary

Although many closures are possible, for valley glaciers $q \approx 0.6$ (width exponent) and $m \approx 2$ (mass balance exponent) are both suggested by data. The value $f \approx 0$ for side drag may also be appropriate. There is no supporting data for the slope closure, but of the reasonable values discussed, only the steep slope scaling $r = 0$ is consistent with the other closure choices for q, m, and f.

For ice caps and ice sheets, theory requires $q = 1$ and $f = 0$. Shallow slope scaling is reasonable, so from (9), $r = (4 - m)/8$. If we assume that ice caps and ice sheets follow a roughly "square root" shaped surface profile as suggested by data [e.g., *Paterson*, 1972], then $r \approx 1/2$. In that case $m = 0$, as frequently assumed elsewhere [e.g., *Paterson*, 1972].

With these closure values, volume-area scaling is predicted from (7) with $\gamma = 1 + [(1 + m + n(f + r))/(q+1)(n+2)]$. For valley glaciers the closure choices predict $\gamma = 1.375$, and for ice caps closure choices predict $\gamma = 1.25$. These are both in excellent agreement with all of the available data (Figure 1) [*Paterson*, 1972; *Chizhov and Kotlyakov*, 1982; *Zhurovlyev*, 1985; *Macheret et al.*, 1988; *Chen and Ohmura*, 1990; *Meier and Bahr*, 1996]. Other closure choices for q, m, f, and r will give the same values for γ. However, for valley glaciers, if we assume that $q = 0.6$ and $m = 2$ are roughly fixed by data, then f and r must have opposite signs and similar magnitudes to give a scaling exponent anywhere near the $\gamma = 1.36$ which is supported by the volume-area data. Both f and r are expected to be positive (scaling inversely with length as shown in (5)), so f and r must both be near 0, as was assumed above.

Although the accumulation area ratio data does not support the value of $m = 0$ for valley glaciers, this is a common assumption. In that case, for $q = 0.6$, the volume-area scaling exponent of 1.36 can only be reproduced if $f + r \approx 0.6$. As discussed above, f must be near 0 when $\gamma = 1.36$ and when $q = 0.6$. Therefore $r \approx 0.6$, which is a reasonable exponent for slope scaling. Again, however, $m = 0$ seems unreasonable; valley glaciers always have an accumulation area and an ablation area, which requires some kind of change in the mass balance with length (i.e., nonzero m; see (15)).

Although we have assumed throughout this paper that $n = 3$ in accordance with Glen's flow law, the scaling predictions can change with different n. If perfect plasticity is a reasonable assumption for ice sheets [*Nye*, 1951], then n approaches infinity, and the volume-area scaling exponent immediately reduces to $\gamma = 1.25$, as observed. Shallow slope scaling is still assumed, but because m is so small relative to the infinite magnitude of n, the value of m is irrelevant in this case.

Discussion

For most glaciers, mass balance is the boundary condition which must ultimately drive changes in glacier volumes. However, from experience and modeling experiments, we know that the volume is relatively insensitive to variations in the shape of the mass balance profile [e.g., *Paterson*, 1972]. Examining (7) shows why. The mass balance scaling exponent m changes the volume-area exponent by an amount $m/(q+1)(n+2)$. For $n = 3$ and the observed valley glacier value of $q = 0.6$, this means that mass balance contributes only $m/8$ to the total volume-area scaling exponent. For ice caps where $q = 1$, the contribution reduces to $m/10$. Therefore a 10% change in m will only alter the volume-area scaling exponent by roughly $\pm 1/100$. Small changes in m will have almost no effect on glacier volumes.

While $m/8$ and $m/10$ can be small, they are not insignificant. If $m \approx 2$ as suggested by valley glacier data, then the mass balance scaling contributes 0.25 to the total of $\gamma = 1.36$. This is a contribution of roughly 20% of the total volume-area scaling exponent. So small changes in m may be insignificant, but the magnitude of m determines a sizable fraction of the volume-area exponent. Hence large changes in m will cause noticeable changes in γ and glacier volumes.

Large changes in m can be tracked by AARs. In general, the AAR is not sensitive to small changes in m (equation (23)). However, any measured changes in the long-term average AAR must indicate larger changes in m. A reduction from an average AAR = 0.58 to AAR = 0.50, for example, indicates a change from $m = 2$ to $m = 1$. For valley glaciers, this could indicate a change from the current value of $\gamma = 1.36$ to $\gamma = 1.24$. For a typical valley glacier with a surface area of $10\,\mathrm{km}^2$, this decrease in the AAR would mean an almost 25% reduction in ice volume. Observed changes in average AARs therefore indicate changes in the volume-area exponent and may indicate significant changes in glacier volumes.

Conclusions

Characteristic glacier volumes depend on the product of a characteristic length $[x]$, width $[w]$, and thickness $[h]$. The preceding analysis shows that these dimensions can be rewritten to express the volume as a function of the characteristic surface area $[S]$, and four closure choices for the width, slope, side drag, and mass balance. Although many closures are possible, reasonable choices (supported by data) predict that $[V] \propto [S]^\gamma$, with $\gamma = 1.375$ for valley glaciers and $\gamma = 1.25$ for ice caps and ice sheets. These predicted scaling exponents are supported by volume-area data (Figure 1).

Changes in the volume-area scaling can be the result of changes in the specified width, slope, side drag, or mass balance scaling. Of these, the mass balance scaling behavior is most likely to be altered by changing climate. Glacier volumes are insensitive to small perturbations in the mass balance scaling exponent m, but measurable changes in average AARs are the result of much larger changes in m. These large changes can indicate significant changes in γ and glacier volumes.

AARs have the disadvantage of being noisy functions of time and space [*Meier and Post*, 1962; *Dugdale*, 1972; *Furbish and Andrews*, 1984], so multiple year or regional averages would be necessary to establish trends which might indicate significant changes in ice volume. However, even with the trouble of multiple year averages, measuring AARs and $[S]$ from satellite images (or aerial photographs) is significantly less costly than detailed radar measurements of ice volume. The volume-area scaling relationship gives both a practical and physically based method for estimating glacier volumes.

Notation

x	length along "horizontal" axis.
z	height along "vertical" axis.
w	width.
h	thickness.
V	volume.
S	surface area.
ρ	density.
g, g_x	gravity.
A	constitutive law parameter.
n	constitutive law parameter.
u_x	velocity (down-glacier).
F	shape factor (side drag).
A_c	cross-sectional area.
P	perimeter along bed.
$\dot{b}$	mass balance rate.
c_m, c_0	balance profile parameters.
α	x axis angle.
AAR	accumulation area ratio.
x_{eq}	x position of equilibrium line.
β	cross-sectional shape exponent.
γ	volume-surface area exponent.
θ	height-length exponent.
f	side drag exponent.
m	balance rate exponent.
q	width exponent.
r	slope exponent.

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