Linear Algebra

Riddhiman

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1 Preface

Self study Linear Algebra notes. Source is *Vector Calculus, Linear Algebra, and Differential Geometry* by John and Barbara Hubbard. These notes will start from Section 1.3. Notes for Sections 1.1 and 1.2 will be added later.

2 Matrix Multiplication as Linear Transformation

Definition 2.1 (Linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$)**:** A *linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping such that for all scalars a and for all $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^n$,

$$T(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = T(\vec{\mathbf{v}}) + T(\vec{\mathbf{w}}) \text{ and } T(a\vec{\mathbf{v}}) = aT(\vec{\mathbf{v}}) \tag{2.1}$$

This above was the definition of a linear transformation. The following theorem will relate matrices to linear transformation.

Theorem 2.1 (Matrices and linear transformation)

1. Any $m \times n$ matrix A defines a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by matrix multiplication:
$$T(\vec{\mathbf{v}}) = A\vec{\mathbf{v}}. \tag{2.2}$$

2. Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by multiplication by the $m \times n$ matrix $[T]$:
$$T(\vec{\mathbf{v}}) = [T]\vec{\mathbf{v}}, \tag{2.3}$$
where the i^{th} column of $[T]$ is $T(\vec{\mathbf{e}}_i)$.

Proof to part 2 of Theorem 2.1. Given that our domain is $\mathbb{R}^n$ we can write any vector $\vec{\mathbf{v}} \in \mathbb{R}^n$ as

$$\vec{\mathbf{v}} = u_1\vec{\mathbf{e}}_1 + u_2\vec{\mathbf{e}}_2 + \ldots + u_n\vec{\mathbf{e}}_n = \sum_{i=1}^{n} u_i\vec{\mathbf{e}}_i$$

Thus,

$$\begin{aligned} T(\vec{\mathbf{v}}) &= T\left(\sum_{i=1}^{n} u_i\vec{\mathbf{e}}_i\right) \\ &= \sum_{i=1}^{n} u_i T(\vec{\mathbf{e}}_i) \end{aligned}$$

Also note that since every i^{th} column of $[T]$ is $T(\vec{\mathbf{e}}_i)$. When $[T]$ is multiplied by $\vec{\mathbf{v}}$, the i^{th} of the resultant column vector can be denoted as $u_1T(\vec{\mathbf{e}}_1) + \ldots + u_nT(\vec{\mathbf{e}}_n)$ which is just $T(\vec{\mathbf{v}})$. $\square$