

# Chapter II - Solutions

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## 1 Preface

These are solutions to exercises in *Analysis I* by Terence Tao. Most of them have been written by me, some with inputs from discussions on Math StackExchange.

## 2 Addition

**Exercise 2.2.1.** Prove Proposition 2.2.5 which states that addition is associative. (Hint: fix two of the variables and induct on the third.)

*Solution.* Let  $a, b$  and  $c$  be natural numbers and fix  $b$  and  $c$ . We will induct over  $a$ . For the base case, let  $a = 0$ . Hence from the definition of addition,

$$(0 + b) + c = b + c = 0 + (b + c)$$

Now let the hypothesis be true for some arbitrary non-zero  $a$ . Thus assume that  $(a + b) + c = a + (b + c)$ . To close the induction we need to prove associativity for  $a++$ . Since,  $(a++) + b = (a + b)++$ , we can write

$$((a++) + b) + c = ((a + b)++) = ((a + b) + c)++ \quad (1)$$

Also,

$$(a++) + (b + c) = (a + (b + c))++ \quad (2)$$

Since we assumed that  $a + (b + c) = (a + b) + c$ , we can conclude that  $(1) = (2)$  (by Axiom 4?). This closes the induction.  $\square$

**Exercise 2.2.2.** Prove Lemma 2.2.10 which states that for a positive number  $a$  there exists only one natural number  $b$  such that  $b++ = a$  (Hint: use induction.).

*Solution.* (by contradiction). Assume that there exists a natural number  $c$  such that  $c \neq b$  and  $b++ = c++ = a$ . However, this violates Axiom 4 which states that if  $m++ = n++$  then  $m = n$ . Thus our assumption is false.  $\square$

**Exercise 2.2.3.** Prove Proposition 2.2.12. (Hint: you will need many of the preceding propositions, corollaries, and lemmas.)

*Solution.*

(a)

- (b) From the definition we can write  $a = b + m, m \geq 0$  and  $b = c + n, n \geq 0$ . Thus  $a = c + m + n$  which, again from the definition, gives  $a \geq c$
- (c)  $a \geq b$  gives  $a = b + m, m \geq 0$  but  $b \geq a$  gives  $b = a + n, n \geq 0$ . Thus,  $a = a + m + n$  which, from Proposition 2.2.6, gives  $m + n = 0$ . Therefore  $m = n = 0$  and  $a = b$ .
- (d)  $a \geq b \implies a + c \geq b + c$  direction: we have  $a = b + m, m \geq 0 \implies a + c = (b + c) + m \geq b + c$  by definition.
- (e)  $a < b \implies b = a + m, m > 0$ . Note that there exists a predecessor of  $m$  thus  $m = n++$  and hence  $b = a + n++ = (a + n)++ \geq a++$
- (f)

**Exercise 2.2.5.**

**Exercise 2.2.6.**

□