Vectors, matrices and derivates - Solutions

Riddhiman

July 2023

1 Preface

Solutions to some problems from *Vector Calculus, Linear Algebra, and Differential Geometry* by John and Barbara Hubbard. Starting from Section 1.3.

2 Matrix Multiplication as Linear Transformation

> **Exercise 1.3.15.** Prove part 1 of Theorem 1.3.4: show that the mapping from $\mathbb{R}^n \to \mathbb{R}^m$ described by the product $A\vec{\mathbf{v}}$ is indeed linear.

Solution. For this we will need to invoke the definition of matrix multiplication:

Definition 2.1: Matrix multiplication

If A is an $m \times n$ matrix whose (i,j)th entry is $a_{i,j}$ and B is an $n \times p$ matrix whose (i,j)th entry is $b_{i,j}$, then $C = AB$ is the $m \times p$ matrix whose (i,j)th entry is:

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

.

Now, since the product of A and $\vec{\mathbf{v}}$ by Definition 2.1 is a vector with dimensions $m \times 1$, we can say that $A\vec{\mathbf{v}} \in \mathbb{R}^m$. Hence, let A denote a transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$. We wish to prove that $\mathcal{T}$ is a linear transformation, in other words we need to prove that

$$\mathcal{T}(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \mathcal{T}(\vec{\mathbf{v}}) + \mathcal{T}(\vec{\mathbf{w}}) \text{ and } \mathcal{T}(a\vec{\mathbf{v}}) = a\mathcal{T}(\vec{\mathbf{v}})$$

Let $\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Then the ith element of $A\vec{\mathbf{v}}$ looks like $c_{i,1} = \sum_{k=1}^{n} a_{i,k} v_{k,1}$. Similarly, consider $\vec{\mathbf{w}} \in \mathbb{R}^n$ and $\vec{\mathbf{w}} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$. Then ith element of $A\vec{\mathbf{w}}$ will look like

$d_{i,1} = \sum_{k=1}^{n} a_{i,k} w_{k,1}$. Now consider the ith element of $A(\vec{\mathbf{v}} + \vec{\mathbf{w}})$,

$$\begin{aligned} f_i &= \sum_{k=1}^{n} a_{i,k}(v_k + w_k) \\ &= \sum_{k=1}^{n} a_{i,k} v_k + \sum_{k=1}^{n} a_{i,k} w_k \\ &= c_i + d_i \end{aligned}$$

From this we can see that $A(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = A(\vec{\mathbf{v}}) + A(\vec{\mathbf{w}})$. But remember that $A(\vec{\mathbf{u}}) = \mathcal{T}(\vec{\mathbf{u}})$ for some $\vec{\mathbf{u}} \in \mathbb{R}^n$, thus we $\mathcal{T}(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \mathcal{T}(\vec{\mathbf{v}}) + \mathcal{T}(\vec{\mathbf{w}})$ which is what we wanted to prove. The second statement $\mathcal{T}(a\vec{\mathbf{v}}) = a\mathcal{T}(\vec{\mathbf{v}})$ can be proven similarly. Thus $\mathcal{T}$ must be a linear transformation.

$\square$

This is a pretty long-winded proof. Some lines are repetitive and the scaling and addition of the linear transformation could have been handled together. But I can't be bothered to improve it now. The idea is clear.