

Vectors, matrices and derivatives - Solutions

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July 2023

1 Preface

Solutions to some problems from *Vector Calculus, Linear Algebra, and Differential Geometry* by John and Barbara Hubbard. Starting from Section 1.3.

2 Matrix Multiplication as Linear Transformation

Exercise 1.3.15. Prove part 1 of Theorem 1.3.4: show that the mapping from $\mathbb{R}^n \rightarrow \mathbb{R}^m$ described by the product $A\vec{v}$ is indeed linear.

Solution. For this we will need to invoke the definition of matrix multiplication:

Definition 2.1: Matrix multiplication

If A is an $m \times n$ matrix whose (i, j) th entry is $a_{i,j}$ and B is an $n \times p$ matrix whose (i, j) th entry is $b_{i,j}$, then $C = AB$ is the $m \times p$ matrix whose (i, j) th entry is:

$$c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}$$

Now, since the product of A and \vec{v} by Definition 2.1 is a vector with dimensions $m \times 1$, we can say that $A\vec{v} \in \mathbb{R}^m$. Hence, let A denote a transformation $\mathcal{T} : \mathbb{R}^n \rightarrow \mathbb{R}^m$. We wish to prove that \mathcal{T} is a linear transformation, in other words we need to prove that

$$\mathcal{T}(\vec{v} + \vec{w}) = \mathcal{T}(\vec{v}) + \mathcal{T}(\vec{w}) \text{ and } \mathcal{T}(a\vec{v}) = a\mathcal{T}(\vec{v})$$

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Then the i th element of $A\vec{v}$ looks like $c_{i,1} = \sum_{k=1}^n a_{i,k} v_{k,1}$. Sim-

ilarly, consider $\vec{w} \in \mathbb{R}^n$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$. Then i th element of $A\vec{w}$ will look like

$d_{i,1} = \sum_{k=1}^n a_{i,k} w_{k,1}$. Now consider the i th element of $A(\vec{v} + \vec{w})$,

$$\begin{aligned} f_i &= \sum_{k=1}^n a_{i,k} (v_k + w_k) \\ &= \sum_{k=1}^n a_{i,k} v_k + \sum_{k=1}^n a_{i,k} w_k \\ &= c_i + d_i \end{aligned}$$

From this we can see that $A(\vec{v} + \vec{w}) = A(\vec{v}) + A(\vec{w})$. But remember that $A(\vec{u}) = \mathcal{T}(\vec{u})$ for some $\vec{u} \in \mathbb{R}^n$, thus we $\mathcal{T}(\vec{v} + \vec{w}) = \mathcal{T}(\vec{v}) + \mathcal{T}(\vec{w})$ which is what we wanted to prove. The second statement $\mathcal{T}(a\vec{v}) = a\mathcal{T}(\vec{v})$ can be proven similarly. Thus \mathcal{T} must be a linear transformation. □

This is a pretty long-winded proof. Some lines are repetitive and the scaling and addition of the linear transformation could have been handled together. But I can't be bothered to improve it now. The idea is clear.