Set Theory - Solutions

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1 Preface

Nothing to put here for now.

2 Fundamentals

Exercise 3.1.1. Show that the definition of equality in Definition 3.1.4 (Definition 2.2 in notes) is reflexive, symmetric, and transitive.

Proof. Let A, B, C be sets. To prove reflexivity we have to prove that $\forall x(x \in A \iff x \in A)$. Since this statement is obviously true we have $A = A$.

To prove symmetry, consider A and B such that $A = B$. By definition we have $\forall x(x \in A \iff x \in B)$. We have to show that $\forall y(y \in B \iff y \in A)$. However, note that the result follows since the two statements are logically equivalent. Therefore, $B = A$.

To prove transitivity, note that for some $x \in A$ we have $x \in B$ since $A = B$ and because $B = C$ we have $x \in C$. Similarly for some $y \in C$ we have $y \in B$ and thus $y \in A$. Thus by definition we get $A = C$. $\square$

Exercise 3.1.2. Using only Definition 3.1.4 and Axiom 3.1, Axiom 3.2 and Axiom 3.3, prove that $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are all distinct.

Proof. If $\emptyset$ is equal to the other sets (say $\mathcal{S}$) we have to show that $\forall x(x \in \emptyset \iff x \in \mathcal{S})$. However, note that $\emptyset$ is empty while the others are non-empty. Hence, $\emptyset$ cannot be equalt to the other sets.

For the rest of the sets, Axiom 1 and Axiom 2 guarantee us to compare them with Definition 3.1.4. The sets $\{\emptyset\}$ (say A) and $\{\{\emptyset\}\}$ (say B) are not equal since $\emptyset \notin B$ and $\{\emptyset\} \notin A$. Hence, $\{\emptyset\} \neq \{\{\emptyset\}\}$. The other pairs can be proven similarly. $\square$