Chapter II - Solutions

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1 Preface

These are solutions to exercises in *Analysis I* by Terrence Tao. Most of them have been written by me, some with inputs from discussions on Math StackExchange.

2 Addition

Exercise 2.2.1. Prove Proposition 2.2.5 which states that addition is associative. (Hint: fix two of the variables and induct on the third.)

Solution. Let a, b and c be natural numbers and fix b and c. We will induct over a. For the base case, let $a = 0$. Hence from the definition of addition,

$$(0 + b) + c = b + c = 0 + (b + c)$$

Now let the hypothesis be true for some arbitrary non-zero a. Thus assume that $(a + b) + c = a + (b + c)$. To close the induction we need to prove associativity for $a{++}$. Since, $(a{++}) + b = (a + b){++}$, we can write

$$((a{++}) + b) + c = ((a + b){++}) = ((a + b) + c){++} \tag{1}$$

Also,

$$(a{++}) + (b + c) = (a + (b + c)){++} \tag{2}$$

Since we assumed that $a + (b + c) = (a + b) + c$, we can conclude that (1) = (2) (by Axiom 4?). This closes the induction.

□

Exercise 2.2.2. Prove Lemma 2.2.10 which states that for a positive number a there exists only one natural number b such that $b{++} = a$ (Hint: use induction.).

Solution. (by contradiction). Assume that there exists a natural number c such that $c \neq b$ and $b{++} = c{++} = a$. However, this violates Axiom 4 which states that if $m{++} = n{++}$ then $m = n$. Thus our assumption is false. □

Exercise 2.2.3. Prove Proposition 2.2.12. (Hint: you will need many of the preceding propositions, corollaries, and lemmas.)

Solution.

(a)

(b) From the definition we can write $a = b + m, m \geq 0$ and $b = c + n, n \geq 0$. Thus $a = c + m + n$ which, again from the definition, gives $a \geq c$

(c) $a \geq b$ gives $a = b + m, m \geq 0$ but $b \geq a$ gives $b = a + n, n \geq 0$. Thus, $a = a + m + n$ which, from Proposition 2.2.6, gives $m + n = 0$. Therefore $m = n = 0$ and $a = b$.

(d) $a \geq b \implies a + c \geq b + c$ direction: we have $a = b + m, m \geq 0 \implies a + c = (b + c) + m \geq b + c$ by definition.

(e) $a < b \implies b = a + m, m > 0$. Note that there exists a predecessor of m thus $m = n{+}{+}$ and hence $b = a + n{+}{+} = (a + n){+}{+} \geq a{+}{+}$

(f)

Exercise 2.2.5.
Exercise 2.2.6. $\square$