Chapter II - Solutions

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1 Preface

These are solutions to exercises in *Analysis I* by Terence Tao. Most of them have been written by me, some with inputs from discussions on Math StackExchange.

2 Addition

Exercise 2.2.1. Prove Proposition 2.2.5 which states that addition is associative. (Hint: fix two of the variables and induct on the third.)

Solution. Let a, b and c be natural numbers and fix b and c. We will induct over a. For the base case, let $a = 0$. Hence from the definition of addition,

$$(0 + b) + c = b + c = 0 + (b + c)$$

Now let the hypothesis be true for some arbitrary non-zero a. Thus assume that $(a + b) + c = a + (b + c)$. To close the induction we need to prove associativity for $a{+}{+}$. Since, $(a{+}{+}) + b = (a + b){+}{+}$, we can write

$$((a{+}{+}) + b) + c = ((a + b){+}{+}) = ((a + b) + c){+}{+} \tag{1}$$

Also,

$$(a{+}{+}) + (b + c) = (a + (b + c)){+}{+} \tag{2}$$

Since we assumed that $a + (b + c) = (a + b) + c$, we can conclude that (1) = (2) (by Axiom 4?). This closes the induction.

□

Exercise 2.2.2. Prove Lemma 2.2.10 which states that for a positive number a there exists only one natural number b such that $b{+}{+} = a$ (Hint: use induction.).

Solution. (by contradiction). Assume that there exists a natural number c such that $c \neq b$ and $b{+}{+} = c{+}{+} = a$. However, this violates Axiom 4 which states that if $m{+}{+} = n{+}{+}$ then $m = n$. Thus our assumption is false. □

Exercise 2.2.3. Prove Proposition 2.2.12. (Hint: you will need many of the preceding propositions, corollaries, and lemmas.)

Solution.

(a) We know that if $a = b + m$, $m \geq 0$ then $a \geq b$. Let $a = a + 0$, then $m = 0$ and thus $a \geq a$.

(b) From the definition we can write $a = b + m$, $m \geq 0$ and $b = c + n$, $n \geq 0$. Thus $a = c + m + n$ which, again from the definition, gives $a \geq c$.

(c) $a \geq b$ gives $a = b + m$, $m \geq 0$ but $b \geq a$ gives $b = a + n$, $n \geq 0$. Thus, $a = a + m + n$ which, from Proposition 2.2.6, gives $m + n = 0$. Therefore $m = n = 0$ and $a = b$.

(d) $a \geq b \implies a + c \geq b + c$ direction: we have $a = b + m$, $m \geq 0 \implies a + c = (b + c) + m \geq b + c$ by definition.
Opposite dirction: $a + c \geq b + c \implies a + c = b + c$ **or** $a + c > b + c$. From the first case we get, $a = b$. From the second case, write $a + c = (b + c) + m$, $m > 0$. Cancelling c we get, $a = b + m$ which by definition is strictly greater than 0. Combining the two cases we get the desired inequality.

(e) $a < b \implies b = a + m, m > 0$. Note that there exists a predecessor of m thus $m = n{+}{+}$ and hence $b = a + n{+}{+} = (a + n){+}{+} \geq a{+}{+}$

(f) Forward direction: Let $b = a + m$, $m \geq 0$. But since by definition, $a \neq b$, we cannot invoke the cancellation law to conclude $m = 0$. Hence, $m > 0$.
Backward direction: By definition, $b \geq a$ but $d \neq 0 \implies b \neq a$. Hence, discarding the $a = b$ case we get $b > a$.

$\square$

Exercise 2.2.5.

Solution. $\square$

Exercise 2.2.6.

Solution. We proof this by induction and induct over n. For the base case, let $n = 0$, here m is also forced to be 0. So we need to prove that $P(0)$ is true.

$\square$