Exam 2 Solutions

1. (a) The joint density of $X_1, \dots, X_n$ is

$$f(\underline{x}|\alpha,\beta) = \prod_{i=1}^{n} \left\{ \frac{1}{\beta-\alpha} I_{(\alpha,\beta)}(x_i) \right\} = \frac{1}{(\beta-\alpha)^n} \prod_{i=1}^{n} I_{(\alpha,\beta)}(x_i)$$

$$= \frac{1}{(\beta-\alpha)} \prod_{i=1}^{n} I_{(\alpha,\infty)}(x_i) \prod_{i=1}^{n} I_{(-\infty,\beta)}(x_i) = \frac{1}{(\beta-\alpha)^n} I_{(\alpha,\infty)}(x_{(1)}) I_{(-\infty,\beta)}(x_{(n)})$$

$$= g(x_{(1)}, x_{(n)} | \alpha, \beta)\, h(\underline{x})$$

where $g(t_1, t_2|\alpha,\beta) = \frac{1}{(\beta-\alpha)^n} I_{(\alpha,\infty)}(t_1) I_{(-\infty,\beta)}(t_2)$ and $h(\underline{x}) = 1$. So $(x_{(1)}, x_{(n)})$ is sufficient for (α,β) by the Factorization Theorem.

(b) The method of moments estimator is the solution to $m_1 = \mu_1'(\alpha,\beta)$ and $m_2 = \mu_2'(\alpha,\beta)$ where $m_1 = \bar{X}$, $m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$, $\mu_1' = EX = \int_\alpha^\beta \frac{x}{\beta-\alpha}\,dx = \left[\frac{x^2}{2(\beta-\alpha)}\right]_\alpha^\beta = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)}$

and $\mu_2' = E[X^2] = \int_\alpha^\beta \frac{x^2}{\beta-\alpha}\,dx = \left[\frac{x^3}{3(\beta-\alpha)}\right]_\alpha^\beta = \frac{\beta^3-\alpha^3}{3(\beta-\alpha)}$.

So, solve the system $\left\{ \begin{array}{l} \bar{X} = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} \\ \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} \end{array} \right.$ to obtain the estimator.

To solve the system (not necessary for the exam), note that

$$\frac{\beta^2-\alpha^2}{2(\beta-\alpha)} = \frac{(\beta+\alpha)\cancel{(\beta-\alpha)}}{2\cancel{(\beta-\alpha)}} = \frac{\beta+\alpha}{2} \quad \text{and} \quad \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} = \frac{\beta^2+\alpha\beta+\alpha^2}{3}$$

$$\begin{array}{c|cccc} \alpha & 1 & 0 & 0 & -\alpha^3 \\ & & \alpha & \alpha^2 & \alpha^3 \\ \hline & 1 & \alpha & \alpha^2 & 0 \end{array}$$

Then $\left\{ \begin{array}{l} \bar{X} = \frac{\beta+\alpha}{2} \\ \frac{1}{n}\sum X_i^2 = \frac{\beta^2+\alpha\beta+\alpha^2}{3} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \beta+\alpha = 2\bar{X} \\ (\beta+\alpha)^2 - \alpha\beta = \frac{3}{n}\sum X_i^2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \alpha = 2\bar{X} - \beta \\ \alpha\beta = 4\bar{X}^2 - \frac{3}{n}\sum X_i^2 \end{array} \right\}$

$$\Rightarrow \left\{ \begin{array}{l} \alpha = 2\bar{X} - \beta \\ (2\bar{X}-\beta)\beta = 4\bar{X}^2 - \frac{3}{n}\Sigma X_i^2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \alpha = 2\bar{X} - \beta \\ \beta = \bar{X} + \sqrt{\frac{3(n-1)}{n}}\, S \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \alpha = \bar{X} - \sqrt{\frac{3(n-1)}{n}}\, S \\ \beta = \bar{X} + \sqrt{\frac{3(n-1)}{n}}\, S \end{array} \right.$$

$$0 = \beta^2 - 2\bar{X}\beta + \left(4\bar{X}^2 - \frac{3}{n}\Sigma X_i^2\right)$$

$$\beta = \frac{2\bar{X} + \sqrt{4\bar{X}^2 - 4\left(4\bar{X}^2 - \frac{3}{n}\Sigma X_i^2\right)}}{2} = \frac{2\bar{X} + \sqrt{\frac{12}{n}\Sigma X_i^2 - 12\bar{X}^2}}{2}$$

$$= \bar{X} + \sqrt{\frac{3}{n}\Sigma X_i^2 - 3\bar{X}^2} = \bar{X} + \sqrt{\frac{3}{n}\left(\Sigma X_i^2 - n\bar{X}^2\right)}$$

$$= \bar{X} + \sqrt{\frac{3(n-1)}{n} S^2} = \bar{X} + \sqrt{\frac{3(n-1)}{n}}\, S$$

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2. (a) If $W(X)$ is an unbiased estimator of β, then the numerator of the Cramér-Rao Lower Bound (CRLB) is $\left(\frac{d}{d\beta} E[W(X)]\right)^2 = \left(\frac{d}{d\beta}\beta\right)^2 = 1^2 = 1$. Since $X_1, \dots, X_n$ is iid, the denominator of the CRLB is $n E\left[\left(\frac{\partial}{\partial\beta}\log f(X|\beta)\right)^2\right]$.

We have $\frac{\partial}{\partial\beta}\log f(x|\beta) = \frac{\partial}{\partial\beta}\left[-\log\beta - \frac{x}{\beta}\right] = \frac{-1}{\beta} + \frac{x}{\beta^2}$

so that $E\left[\left(\frac{\partial}{\partial\beta}\log f(X|\beta)\right)^2\right] = \frac{1}{\beta^4}E\left[(X-\beta)^2\right] \underset{\substack{\uparrow \\ X \sim \text{Gamma}(1,\beta) \\ \Downarrow \\ EX = 1\cdot\beta = \beta}}{=} \frac{E[(X-EX)^2]}{\beta^4} = \frac{Var X}{\beta^4} \underset{\substack{\uparrow \\ X\sim \text{Gamma}(1,\beta) \\ Var X = 1\cdot\beta^2 = \beta^2}}{=} \frac{\beta^2}{\beta^4} = \frac{1}{\beta^2}$

and the CRLB is $\frac{1}{n/\beta^2} = \boxed{\frac{\beta^2}{n}}$.

(b) The derivative of the log-likelihood function $\ell(\beta|\underline{x}) = \Sigma \log f(x_i|\beta)$ is $\frac{\partial \ell}{\partial\beta} = \frac{-n}{\beta} + \frac{\Sigma x_i}{\beta^2}$.

We solve $\frac{\partial\ell}{\partial\beta} = 0$ to obtain $\frac{\Sigma x_i}{\beta^2} = \frac{n}{\beta} \Rightarrow \beta = \frac{\Sigma x_i}{n} = \bar{x}$.

This maximizes ℓ (and consequently $L(\beta|\underline{x})$) since

$$\left.\frac{\partial^2\ell}{\partial\beta^2}\right|_{\beta=\bar{x}} = \left.\frac{n}{\beta^2} - \frac{2\Sigma x_i}{\beta^3}\right|_{\beta=\bar{x}} = \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3} = -\frac{n}{\bar{x}^2} < 0.$$

(c) $E\bar{X} = \frac{1}{n}\sum EX_i = \frac{1}{n}\sum_{i=1}^{n}\beta = \frac{n\beta}{n} = \beta$ so $\text{Bias}\,\hat{\beta} = \beta - \beta = \boxed{0}$

$$\text{Var}\,\bar{X} = \frac{1}{n^2}\sum \text{Var}\,X_i = \frac{1}{n^2}\sum_{i=1}^{n}\beta^2 = \frac{n\beta^2}{n^2} = \boxed{\frac{\beta^2}{n}}.$$

(d) The MLE of $e^{-\beta}$ is $\widehat{e^{-\beta}} = e^{-\hat{\beta}} = \boxed{e^{-\bar{X}}}$.

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3. (a) The joint pmf of X_1 and X_2 is

$$f(x_1,x_2|\theta) = f(x_1|\theta)f(x_2|\theta) = \frac{\theta^{x_1}e^{-\theta}}{x_1!}\frac{\theta^{x_2}e^{-\theta}}{x_2!}\prod_{i=1}^{2} I_{\{0,1,2,\ldots\}}(x_i)$$

$$= \frac{\theta^{x_1+x_2}e^{-2\theta}}{x_1!\,x_2!}\prod_{i=1}^{2} I_{\{0,1,2,\ldots\}}(x_i) = g(x_1+x_2|\theta)\,h(x_1,x_2)$$

where $g(t|\theta) = \theta^{t}e^{-2\theta}$ and $h(\underline{x}) = \frac{1}{x_1!\,x_2!}\prod_{i=1}^{2} I_{\{0,1,2,\ldots\}}(x_i)$.

So X_1+X_2 is sufficient for θ.

(b) $E[T(X_1)] = P(T(X_1)=1) = P(X_1=0) = \frac{\theta^0 e^{-\theta}}{0!} = e^{-\theta}$

(c) Since $X_1+X_2 \sim \text{Poisson}(2\theta)$, we have

$$P(T(X_1)=1|X_1+X_2=y) = P(X_1=0|X_1+X_2=y) = \frac{P(X_1=0 \text{ and } X_1+X_2=y)}{P(X_1+X_2=y)}$$

$$= \frac{P(X_1=0 \text{ and } X_2=y)}{P(X_1+X_2=y)} = \frac{P(X_1=0)P(X_2=y)}{P(X_1+X_2=y)}$$

$$= \frac{\cancel{e^{-\theta}}\,\frac{\theta^y \cancel{e^{-\theta}}}{\cancel{y!}}}{\frac{(2\theta)^y \cancel{e^{-2\theta}}}{\cancel{y!}}} = \frac{\theta^y}{(2\theta)^y} = \boxed{\left(\frac{1}{2}\right)^y}.$$

(d) Note that $E[T(X_1)|X_1+X_2=y] = P(T(X_1)=1|X_1+X_2=y)$.

Since $T(X_1)$ is an unbiased estimator of $e^{-\theta}$ and X_1+X_2 is sufficient for θ (and consequently $e^{-\theta}$), the Rao-Blackwell Theorem implies that

$$\phi(X_1+X_2) = E[T(X_1)|X_1+X_2] = \boxed{\left(\frac{1}{2}\right)^{X_1+X_2}}$$

is a uniformly better unbiased estimator of $e^{-\theta}$.

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4. (a) The likelihood function for μ is

$$L(\mu|\underline{x}) = f(\underline{x}|\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\mu)^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\Sigma(x_i-\mu)^2}$$

and the log-likelihood $\ell(\mu|\underline{x}) = \log L(\mu|\underline{x}) = -\frac{n}{2}\log 2\pi - \frac{1}{2}\Sigma(x_i-\mu)^2$

is maximized at $\hat{\mu} = \bar{x}$ since

$$\frac{\partial \ell}{\partial \mu} = \Sigma(x_i - \mu) = 0 \Rightarrow \Sigma x_i - n\mu = 0 \Rightarrow \mu = \frac{\Sigma x_i}{n} = \bar{x} \quad \text{and}$$

$$\frac{\partial^2 \ell}{\partial \mu^2} = \Sigma(-1) = -n < 0.$$ So the likelihood ratio test statistic is

$$\lambda(\underline{x}) = \frac{\sup_{\mu \le 0} L(\mu|\underline{x})}{\sup_{\mu} L(\mu|\underline{x})} = \begin{cases} 1 & \text{if } \bar{x} \le 0 \\ \dfrac{\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\Sigma x_i^2}}{\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\Sigma(x_i-\bar{x})^2}} & \text{if } \bar{x} > 0 \end{cases} = \begin{cases} 1 & \text{if } \bar{x} < 0 \\ e^{\frac{n}{2}\bar{x}^2} & \text{if } \bar{x} > 0 \end{cases}$$

since

$$\frac{\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\Sigma x_i^2}}{\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(\Sigma x_i^2 - n\bar{x}^2)}} = e^{-\frac{n}{2}\bar{x}^2}.$$ So $\lambda(\underline{x})$ is

a decreasing function of $\bar{x}$. Thus, $\dfrac{\bar{x}}{1/\sqrt{n}}$ is large

if and only if $\lambda(\underline{x})$ is small so the critical region

has the form: $\left\{ \underline{x} : \dfrac{\bar{x}}{1/\sqrt{n}} \ge K \right\}$.

(b) When $\mu = 0$, $\bar{X} \sim N(0, \frac{1}{n})$ so $\dfrac{\bar{X}}{1/\sqrt{n}} \sim N(0,1)$ which implies

$P\left(\dfrac{\bar{X}}{1/\sqrt{n}} > 2.326\right) = .01$ (from the bottom row of the t-table).

So $K = 2.326$. (If $\mu < 0$, then $\dfrac{\bar{X}}{1/\sqrt{n}} \sim N(\mu, 1) \Rightarrow P\left(\dfrac{\bar{X}}{1/\sqrt{n}} > 2.326\right)$

$= P\left(\dfrac{\bar{X}-\mu}{1/\sqrt{n}} > 2.326 - \dfrac{\mu}{1/\sqrt{n}}\right) < .01$.)

(c) The power when $\mu = 1$ <u>and $n = 9$</u> is

$$P\left(\frac{\bar{X}}{1/\sqrt{9}} > 2.326\right) = P\left(\frac{\bar{X}-1}{1/\sqrt{9}} > 2.326 - \frac{1}{1/\sqrt{9}}\right) = P\left(\frac{\bar{X}-1}{1/\sqrt{9}} > -.674\right)$$

is $\boxed{\text{between } .7486 \text{ and } .7517}$.

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5. (a) The joint pmf of $X_1, \ldots, X_5$ is

$$f(x_1, \ldots, x_5 | \theta) = \theta^{\Sigma x_i}(1-\theta)^{5-\Sigma x_i} \prod_{i=1}^{n} I_{\{0,1\}}(x_i)$$

$T(\underline{X}) = \sum_{i=1}^{5} X_i$ is sufficient for θ.

By the Neyman-Pearson Lemma (corollary), the test with rejection region S which satisfies

$$t \in S \quad \text{if} \quad g(t|\theta_1) > k\, g(t|\theta_0)$$

$$\text{and} \quad t \in S^c \quad \text{if} \quad g(t|\theta_1) < k\, g(t|\theta_0)$$

for some $k \geq 0$ is a UMP level α test. Consider the likelihood ratios

$$\frac{g(t|\theta_1)}{g(t|\theta_0)} = \frac{\cancel{\binom{5}{t}} \left(\frac{3}{4}\right)^t \left(\frac{1}{4}\right)^{5-t}}{\cancel{\binom{5}{t}} \left(\frac{1}{2}\right)^t \left(\frac{1}{2}\right)^{5-t}} = \left(\frac{3}{2}\right)^t \left(\frac{1}{2}\right)^{-t} \left(\frac{1}{2}\right)^5 = 3^t \cdot \frac{1}{32}$$

which is increasing in t so $S = \{t \mid t \geq d\}$. If $\alpha = \frac{6}{32}$, then

$$\frac{6}{32} = P_{\theta=\frac{1}{2}}(T \geq d) \Rightarrow d = 4 \quad \text{since } P(T=5) = \left(\tfrac{1}{2}\right)^5 = \tfrac{1}{32}$$

$$P(T=4) = \binom{5}{4}\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^4 = \tfrac{5}{32}.$$

Equivalently, we reject H_0 when $\frac{g(t|\theta_1)}{g(t|\theta_0)} > k$ and fail to reject when $\frac{g(t|\theta_1)}{g(t|\theta_0)} < k$ for any $k \in \left[\frac{27}{32}, \frac{81}{32}\right)$.

(b) The probability of a Type II error is

$$P_{\theta=\frac{3}{4}}(T<4) = 1 - P_{\theta=\frac{3}{4}}(T \geq 4)$$

$$= 1 - P_{\theta=\frac{3}{4}}(T=4) - P_{\theta=\frac{3}{4}}(T=5)$$

$$= 1 - \binom{5}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 - \left(\frac{3}{4}\right)^5$$

$$= 1 - \frac{5 \cdot 3^4}{1024} - \frac{3^5}{1024} = 1 - \frac{405+243}{1024}$$

$$= 1 - \frac{648}{1024} = \frac{376}{1024} = \boxed{\frac{47}{128}} \approx .367.$$