Lecture 11: Completeness, UMVUEs, and the Lehmann-Scheffé Theorem

MATH 667-01
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- We first discuss some important theorems regarding unbiased estimators in Section 7.3 of Casella and Berger (2002)[1].
- We define complete statistics and state a result for completeness for exponential families as discussed in Section 6.2.
- Finally, we state a few results from Sections 7.3 and 7.5 closely related to work by Lehmann and Scheffé (1950)[2] showing that a complete sufficient statistic is the unique UMVUE of its mean.

[1] Casella, G. and Berger, R. (2002). *Statistical Inference, Second Edition*. Duxbury Press, Belmont, CA.

[2] Lehmann, E. L. and Scheffé, H. (1950). Completeness, similar regions, and unbiased estimation – part I. *Sankhya* **10**, 233–268.

Uniqueness of UMVUEs

- *Theorem L11.1* (Thm 7.3.19 on p.343): If there is a best unbiased estimator of $\tau(\theta)$, then it is unique.
- *Proof of Theorem L11.1*: Suppose W and W' are both best unbiased estimators of $\tau(\theta)$.
  Then $W^* = \frac{1}{2}(W + W')$ is an unbiased estimator of $\tau(\theta)$.
  Further, we have

$$
\begin{aligned}
\mathsf{Var}[W^*] &\overset{3.5}{=} \frac{1}{4}\mathsf{Var}[W + W'] \\
&\overset{3.15}{=} \frac{1}{4}\left(\mathsf{Var}[W] + \mathsf{Var}[W'] + 2\mathsf{Cov}[W, W']\right) \\
&\overset{9.5}{\leq} \frac{1}{4}\left(\mathsf{Var}[W] + \mathsf{Var}[W'] + 2\sqrt{\mathsf{Var}[W]\mathsf{Var}[W']}\right) \\
&= \frac{1}{4}\left(\mathsf{Var}[W] + \mathsf{Var}[W] + 2\sqrt{\mathsf{Var}[W]\mathsf{Var}[W]}\right) \\
&= \frac{1}{4}\left(4\ \mathsf{Var}[W]\right) = \mathsf{Var}[W].
\end{aligned}
$$

Uniqueness of UMVUEs

- *Proof of Theorem L11.1 continued*: Since W is a UMVUE, $\mathsf{Var}[W] \leq \mathsf{Var}[W^*]$ which implies that $\mathsf{Var}[W] = \mathsf{Var}[W^*]$.
- It follows that $\sqrt{\mathsf{Var}[W]\mathsf{Var}[W']} = \mathsf{Cov}[W, W']$, and consequently, *Theorem L9.1(b)* implies that $W' = a(\theta)W + b(\theta)$.
- Since $\mathsf{Var}[W] = \mathsf{Var}[W']$,

$$\begin{aligned} \mathsf{Var}[W] &= \mathsf{Cov}[W, W'] \\ &= \mathsf{Cov}[W, a(\theta)W + b(\theta)] \\ &\overset{3.15}{=} a(\theta)\mathsf{Var}[W] \end{aligned}$$

which implies that $a(\theta) = 1$.

- *Proof of Theorem L11.1 continued*: We also have

$$\tau(\theta) = \mathsf{E}[W'] = a(\theta)\mathsf{E}[W] + b(\theta) = a(\theta)\tau(\theta) + b(\theta).$$

- Since $a(\theta) = 1$, we obtain

$$\tau(\theta) = \tau(\theta) + b(\theta)$$

so that $b(\theta) = 0$.

- So,

$$W' = a(\theta)W + b(\theta) = 1 \cdot W + 0 = W$$

which proves that the UMVUE is unique.

Characterization of UMVUEs

- *Theorem L11.2* (Thm 7.3.20 on p.344): If $\mathsf{E}_\theta[W] = \tau(\theta)$, W is the best unbiased estimator of $\tau(\theta)$ if and only if W is uncorrelated with all unbiased estimators of zero.
- *Proof of Theorem L11.2*: Suppose W is the best unbiased estimator of $\tau(\theta)$ and let U be an unbiased estimator of 0. Then $W' = W + aU$ is an unbiased estimator of $\tau(\theta)$ for all a. Also, we have

$$\mathsf{Var}[W'] \overset{3.15}{=} \mathsf{Var}[W] + 2a\mathsf{Cov}[W, U] + a^2\mathsf{Var}[U].$$

The right side is minimized at $a^* = \dfrac{-\mathsf{Cov}[W, U]}{\mathsf{Var}[U]}$ since

$$\frac{d}{da}\mathsf{Var}[W + aU] = 2\mathsf{Cov}[W, U] + 2a\mathsf{Var}[U]$$

is positive when $a < a^*$ and negative when $a > a^*$.
So, $\mathsf{Var}[W + a^*U] \leq \mathsf{Var}[W]$ with equality only if $a^* = 0$.

- *Proof of Theorem L11.2 continued*: Conversely, suppose that W is uncorrelated with all unbiased estimators of 0, and W' is any other unbiased estimator of $\tau(\theta)$.
- Since $W' - W$ is an unbiased estimator of 0, W is uncorrelated with $W' - W$ which implies that $\text{Cov}[W, W' - W] = 0$.
- Then W is the UMVUE since

$$\begin{aligned}
\text{Var}[W'] &= \text{Var}[W + (W' - W)] \\
&\overset{3.15}{=} \text{Var}[W] + \text{Var}[W' - W] + 2\text{Cov}[W, W' - W] \\
&= \text{Var}[W] + \text{Var}[W' - W] \\
&\geq \text{Var}[W]
\end{aligned}$$

for any arbitrary W'.

- *Definition L11.1* (Def 6.2.21 on p.285): Let $f(t|\theta)$ be a family of pdfs or pmfs for a statistic $T(\boldsymbol{X})$. The family of probability distributions is called *complete* if

$$\mathsf{E}_\theta[g(T)] = 0 \text{ for all } \theta$$

implies

$$P_\theta(g(T) = 0) = 1 \text{ for all } \theta.$$

Equivalently, $T(\boldsymbol{X})$ is called a *complete statistic*.

- *Example L11.1*: Let $X_1, \ldots, X_n$ be iid Uniform$(0, \theta)$ random variables. Show that $T(X_1, \ldots, X_n) = X_{(n)}$ is a complete statistic.
- *Answer to Example L11.1*: Suppose $\mathsf{E}[g(T)] = 0$ for all $\theta > 0$. Then $\frac{d}{d\theta}\mathsf{E}[g(T)] = 0$. We can compute

$$\begin{aligned}
\frac{d}{d\theta}\mathsf{E}[g(T)] &\overset{9.21}{=} \frac{d}{d\theta}\int_0^\theta g(t)nt^{n-1}\theta^{-n}\ dt \\
&= \frac{d}{d\theta}\left[\theta^{-n}\int_0^\theta g(t)nt^{n-1}\ dt\right] \\
&= \frac{d}{d\theta}\left[\theta^{-n}\right]\int_0^\theta g(t)nt^{n-1}\ dt + \theta^{-n}\frac{d}{d\theta}\left[\int_0^\theta g(t)nt^{n-1}\ dt\right] \\
&= -n\theta^{-n-1}\int_0^\theta g(t)nt^{n-1}\ dt + \theta^{-n}g(\theta)n\theta^{n-1} \\
&= -n\theta^{-1}\mathsf{E}[g(T)] + g(\theta)n\theta^{-1} = g(\theta)n\theta^{-1}.
\end{aligned}$$

Since $n\theta^{-1} \neq 0$, we have $g(\theta) = 0$ for $\theta > 0$.

(Technically, this only justifies the completeness condition for Riemann-integrable functions.)

- *Theorem L11.3* (Thm 7.3.23 on p.347): Let T be a complete sufficient statistic for a parameter θ, and let $\phi(T)$ be any estimator based only on T. Then $\phi(T)$ is the unique UMVUE of its expected value.
- *Proof of Theorem L11.3*: Let $\tau(\theta) = \mathsf{E}[\phi(T)]$ and let W be any unbiased estimator of $\tau(\theta)$. *Theorem L10.5* implies that $\tilde{\phi}(T) = \mathsf{E}[W|T]$ is an unbiased estimator of $\tau(\theta)$ such that $\mathsf{Var}[\tilde{\phi}(T)] \leq \mathsf{Var}[W]$ for all θ. Since $\phi(T)$ and $\tilde{\phi}(T)$ are both unbiased, $\mathsf{E}[\phi(T) - \tilde{\phi}(T)] = 0$. Since T is complete, it follows that $P(\phi(T) - \tilde{\phi}(T) = 0) = 1$, or equivalently, $\phi(T) = \tilde{\phi}(T)$ with probability 1. Then, we have

$$\mathsf{Var}[\phi(T)] = \mathsf{Var}[\tilde{\phi}(T)] \leq \mathsf{Var}[W].$$

  Since W is any arbitrary unbiased estimator, $\phi(T)$ is a UMVUE of $\tau(\theta)$. By *Theorem L11.1*, it is unique.

- *Theorem L11.4* (Thm 7.5.1 on p.369): Unbiased estimators based on complete sufficient statistics are unique.
- *Proof of Theorem L11.4*: Suppose that T is a complete sufficient statistic for θ and $\mathrm{E}[\phi(T)] = \tau(\theta)$. By *Theorem L11.3*, $\phi(T)$ is the UMVUE of $\tau(\theta)$. By *Theorem L11.1*, UMVUEs are unique.
- So, since ϕ is arbitrary, $\phi(T)$ is the only function of T which is an unbiased estimator of $\tau(\theta)$.

- *Theorem L11.5* (p.347): If T is a complete sufficient statistic for a parameter θ and $h(X_1, \ldots, X_n)$ is any unbiased estimator of $\tau(\theta)$, then $\phi(T) = \mathsf{E}[h(X_1, \ldots, X_n)|T]$ is the UMVUE of $\tau(\theta)$.
- *Proof of Theorem L11.5*: Since $h(X_1, \ldots, X_n)$ is an unbiased etimator of $\tau(\theta)$ and T is sufficient for θ, $\phi(T)$ is an unbiased estimator of $\tau(\theta)$ by *Theorem L10.5*. Since T is complete and sufficient, *Theorem L11.3* implies that $\phi(T)$ is the (unique) UMVUE of $\tau(\theta)$.

Completeness and UMVUEs

- *Example L11.2*: Let $X_1, \ldots, X_n$ be iid Uniform$(0, \theta)$ random variables. Show that $\left(\frac{n+1}{n}\right) X_{(n)}$ is the UMVUE of θ.
- *Answer to Example L11.2*: We know that $\left(\frac{n+1}{n}\right) X_{(n)}$ is complete from *Example L11.1*. It is a sufficient statistic for θ by *Theorem L10.2* since the joint pdf can be expressed as $f(\boldsymbol{x}|\theta) = \frac{1}{\theta^n} I_{(0,\theta)}(x_{(n)})$.
- Let $\phi(T) = \frac{n+1}{n} T$. We know that $\mathsf{E}\left[\left(\frac{n+1}{n}\right) X_{(n)}\right] = \theta$ by *Example L9.4*.
- So *Theorem L11.3* implies that $\left(\frac{n+1}{n}\right) X_{(n)}$ is the unique UMVUE of θ.

- *Theorem L11.6* (Thm 6.2.25 on p.288): Let $X_1, \ldots, X_n$ be iid random variables with a pdf or pmf $f(x|\boldsymbol{\theta})$ that belongs to an exponential family given by

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta}) \exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta}) t_i(x)\right),$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$. Then $T(\boldsymbol{X}) = \left(\sum_{j=1}^{n} t_1(X_j), \ldots, \sum_{j=1}^{n} t_k(X_j)\right)$ is complete if the parameter space Θ contains an open set in $\mathbb{R}^k$.

- *Example L11.3*: Let X_1 and X_2 be independent identically distributed (iid) Poisson(θ) random variables.
  (a) Find a complete sufficient statistic for θ.
  (b) Find the UMVUE for $P(X_1 = 0) = e^{-\theta}$.
- *Answer to Example L11.3*: (a) We know $X_1 + X_2$ is sufficient for θ from *Example L10.7(a)*. Since the Poisson is an exponential family with pdf

$$f(x|\lambda) = \frac{1}{x!} I_{\mathbb{Z}^*(x)} e^{-\lambda} e^{x \ln \lambda}$$

  where $\lambda \in (0, \infty)$ which contains an open subset in $\mathbb{R}$, we know $\sum t(X_i) = \sum X_i = X_1 + X_2$ is complete by *Theorem L11.6*.

- *Answer to Example L11.3 continued*: We also know that $W = I_{\{0\}}(X_1)$ is an unbiased estimator of $e^{-\theta}$ from *Example L10.7*.
- So *Theorem L11.5* implies that

$$\begin{aligned} \phi(W|X_1+X_2) &= \mathsf{E}[W|X_1+X_2] \\ &\overset{10.22}{=} \left(\frac{1}{2}\right)^{X_1+X_2} \end{aligned}$$

is the UMVUE of $\tau(\theta)$.

- *Example L11.4*: Let $X_1, \ldots, X_n$ be iid Normal(μ, σ^2) random variables, where both μ and σ^2 are unknown. Show that $\bar{X}$ is the UMVUE of μ and S^2 is the UMVUE of σ^2.
- *Answer to Example L11.4 continued*: We know that $(\bar{X}, S^2)$ is sufficient for (μ, σ^2) from *Example L10.6*.
- Since this is a full exponential family as shown in *Example L6.5*, $(\bar{X}, S^2)$ is a complete statistic.
- Let $\phi_1(t_1, t_2) = t_1$. Then, by *Theorem L11.3*, $\phi_1(\bar{X}, S^2) = \bar{X}$ is the UMVUE of $\mathsf{E}[\phi_1(\bar{X}, S^2)] = \mathsf{E}[\bar{X}] \overset{3.19}{=} \mu$.
- Let $\phi_2(t_1, t_2) = t_2$. Then, by *Theorem L11.3*, $\phi_2(\bar{X}, S^2) = S^2$ is the UMVUE of $\mathsf{E}[\phi_2(\bar{X}, S^2)] = \mathsf{E}[S^2] \overset{3.22}{=} \sigma^2$.