M621, HW 8, due Tuesday Oct 18

1. Let $n \in \mathbb{N}$ with $n > 1$.

   (a) Suppose $(a_1 \ldots a_k)$ is a k-cycle in S_n. Provide a *concise, clear* explanation (as if to a M521 student) why it is true that $(a_1 \ldots a_n) = (a_1 a_n) \ldots (a_1 a_2)$. (For example, $(123) = (13)(12)$.) With $\beta = (a_1 a_2) \ldots (a_1 a_n)$, you should explain why $\beta(a_1) = a_2, \beta(a_2) = a_3$, and so on, and that β fixes everything in $\{1, \ldots, n\} - \{a_1, \ldots, a_n\}$.

   (b) You just showed that every k-cycle is a product of 2-cycles. 2-cycles are often referred to as *transpositions.* Give a one or two sentence explanation of the following: S_n is generated by its transpositions. That is, $S_n =< \{(ij) : n \geq j > i \geq 1\} >$.

   (c) Show that $S_n =< \{(12), (23), \ldots, (n-1, n), (n1)\} >$. Suggestions: For $j \in \{1, \ldots, n\}$, $k \in \mathbb{N}$, you'll show that every transposition $(j, j+k)$ is generated by the set of n transpositions given above. When $k = 1$, there's nothing to prove. Proceed by induction on k. As usual, "j+k" is interpreted mod n. Of course, k need not be greater than $n - 1$. (Continue proof on other side of sheet, if necessary.)

2. This is a longer proof, one that you will "sketch in" below. Prove the following proposition. **Proposition.** Suppose G is a group having normal subgroups H and K satisfying the following:

   (a) $H \cap K = \{e\}$.

   (b) $G = HK$.

   Then $G \cong H \times K$.

   Proof. The proof is by a series of claims (whose proofs you'll supply).

   Claim 1 *For all $h \in H$ and all $k \in K$, $hk = kh$.*

   Proof.

   Claim 2 *Let h_1, h_2 be in H, and let k_1, k_2 be in K. Then $h_1 k_1 = h_2 k_2$ if and only if $h_1 = h_2$ and $k_1 = k_2$.*

   Proof.

Since $G = HK$, it follows from Claim 2, that each element $g \in G$ has a *unique* representation as a product $g = hk$, where $h \in H$ and $k \in K$.

Claim 3 *The function $\Gamma : G \to H \times K$ given by $\Gamma(g) = (h, k)$, where $g = hk$ is the unique representation of g by an element of H times an element of K, is a homomorphism.*

Proof.

With the proof of the next claim, you've proved the proposition.

Claim 4 *Γ is a bijection.*

Proof.

□

(EC: +.5) The converse to our proposition is true. What would the converse say?