MATH 668 Exam 1 Solutions

1. (a) $\mathbf{C}^{-1}\boldsymbol{x} \sim N_n(\mathbf{0}, \mathbf{I})$ since $E(\mathbf{C}^{-1}\boldsymbol{x}) = \mathbf{C}^{-1}E(\boldsymbol{x}) = \mathbf{C}^{-1}\mathbf{0} = \mathbf{0}$ and $\text{cov}(\mathbf{C}^{-1}\boldsymbol{x}) = \mathbf{C}^{-1}\text{cov}(\boldsymbol{x})(\mathbf{C}^{-1})^\top = \mathbf{C}^{-1}\boldsymbol{\Sigma}(\mathbf{C}^{-1})^\top = \mathbf{C}^{-1}\mathbf{C}\mathbf{C}^\top(\mathbf{C}^\top)^{-1} = \mathbf{I}$

(b) Let $\boldsymbol{z} = \mathbf{C}^{-1}\boldsymbol{x}$. Since $\boldsymbol{z} \sim N_n(\mathbf{0}, \mathbf{I})$ and $\mathbf{B}$ has rank r, we know $\boldsymbol{x}^\top\mathbf{B}\boldsymbol{x} = \boldsymbol{x}^\top\mathbf{C}^{-\top}(\mathbf{C}^\top\mathbf{B}\mathbf{C})\mathbf{C}^{-1}\boldsymbol{x} = \boldsymbol{z}^\top(\mathbf{C}^\top\mathbf{B}\mathbf{C})\boldsymbol{z} \sim \chi^2(r)$ if and only if $(\mathbf{C}^\top\mathbf{B}\mathbf{C})(\mathbf{I}) = \mathbf{C}^\top\mathbf{B}\mathbf{C}$ is idempotent.

2. $\boldsymbol{j}^\top\boldsymbol{z} \sim N(0, \boldsymbol{j}^\top\boldsymbol{j} = 16) \implies \frac{1}{4}\boldsymbol{j}^\top\boldsymbol{z} \sim N(0,1)$; $\boldsymbol{y}^\top\left(\frac{1}{4}\mathbf{I}_9\right)\boldsymbol{y} \sim \chi^2(9)$ since $\left(\frac{1}{4}\mathbf{I}_9\right)(4\mathbf{I}_9) = \mathbf{I}_9$ is idempotent and $\left(\frac{1}{4}\mathbf{I}_9\right)$ has rank 9 $\boldsymbol{j}^\top\boldsymbol{z}$ and $\boldsymbol{y}^\top\boldsymbol{y}$ are independent since $\boldsymbol{z}$ and $\boldsymbol{y}$ are independent

(a) $P\left(\boldsymbol{j}^\top\boldsymbol{z} > \sqrt{\boldsymbol{y}^\top\boldsymbol{y}}\right) = P\left(\dfrac{\boldsymbol{j}^\top\boldsymbol{z}/4}{\sqrt{\boldsymbol{y}^\top(\frac{1}{4}\mathbf{I})\boldsymbol{y}/9}} > \dfrac{1/4}{\sqrt{(1/4)/9}}\right) = P\left(\dfrac{\boldsymbol{j}^\top\boldsymbol{z}/4}{\sqrt{\boldsymbol{y}^\top(\frac{1}{4}\mathbf{I})\boldsymbol{y}/9}} > 1.5\right) \approx .08$ since $\dfrac{\boldsymbol{j}^\top\boldsymbol{z}/4}{\sqrt{\boldsymbol{y}^\top(\frac{1}{4}\mathbf{I})\boldsymbol{y}/9}} \sim t(9)$

(b) $P\left(\boldsymbol{j}^\top\boldsymbol{z} < 1 \text{ and } \boldsymbol{y}^\top\boldsymbol{y} < 20\right) = P\left(\boldsymbol{j}^\top\boldsymbol{z} < 1\right)P\left(\boldsymbol{y}^\top\boldsymbol{y} < 20\right) = P\left(\boldsymbol{j}^\top\boldsymbol{z}/4 < 1/4\right)P\left(\boldsymbol{y}^\top(\frac{1}{4}\mathbf{I})\boldsymbol{y} < 20/4\right) = P\left(\boldsymbol{j}^\top\boldsymbol{z}/4 < 0.25\right)P\left(\boldsymbol{y}^\top(\frac{1}{4}\mathbf{I})\boldsymbol{y} < 5\right) \approx (0.5987)(1 - 0.83) \approx 0.1018.$

3. (a) Differentiating $\tilde{Q}$ with respect to b, we obtain $\dfrac{d\tilde{Q}}{db} = -2\sum_{i=1}^{n} x_i(y_i - bx_i) + 2\lambda b$. Setting this to 0, we denote the solution as $\hat{\beta}_\lambda$. Then $-2\sum_{i=1}^n x_i(y_i - \hat{\beta}_\lambda x_i) + 2\lambda\hat{\beta}_\lambda = 0 \implies$ $\sum_{i=1}^n x_iy_i - \hat{\beta}_\lambda\sum_{i=1}^n x_i^2 - \lambda\hat{\beta}_\lambda = 0 \implies \sum_{i=1}^n x_iy_i = \hat{\beta}_\lambda(\sum_{i=1}^n x_i^2 + \lambda) \implies \hat{\beta}_\lambda = \dfrac{\sum_{i=1}^n x_iy_i}{\sum_{i=1}^n x_i^2 + \lambda}$.

(b) $\tilde{Q}$ is a strictly convex function since $\dfrac{d^2\tilde{Q}}{db^2} = 2\sum_{i=1}^{n} x_i^2 + 2\lambda > 0$ for all b. Thus, it is minimized at its critical value $\hat{\beta}_\lambda$.

(c) $E(\hat{\beta}_\lambda) = E\left(\dfrac{\sum_{i=1}^n x_iy_i}{\sum_{j=1}^n x_j^2 + \lambda}\right) = \dfrac{1}{\sum_{j=1}^n x_j^2 + \lambda}E\left(\sum_{i=1}^{n} x_iy_i\right) = \dfrac{1}{\sum_{j=1}^n x_j^2 + \lambda}\sum_{i=1}^{n} x_iE(y_i) = \dfrac{1}{\sum_{j=1}^n x_j^2 + \lambda}\sum_{i=1}^{n} x_i^2\beta = \beta\dfrac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2 + \lambda}$

(d) $\lim_{\lambda\to 0} E(\hat{\beta}_\lambda) = \lim_{\lambda\to 0}\dfrac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2 + \lambda} = \beta\dfrac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2 + 0} = \beta$

4. $\mathbf{X}^\top\mathbf{X} = \begin{pmatrix} \boldsymbol{j}^\top\boldsymbol{j} & \boldsymbol{j}^\top\boldsymbol{x} \\ \boldsymbol{x}^\top\boldsymbol{j} & \boldsymbol{x}^\top\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{X}^\top\boldsymbol{y} = \begin{pmatrix} \boldsymbol{j}^\top\boldsymbol{y} \\ \boldsymbol{x}^\top\boldsymbol{y} \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

(a) $\hat{\boldsymbol{\beta}} = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\boldsymbol{y} = \left(\frac{1}{21-1}\begin{pmatrix} 3 & 1 \\ 1 & 7 \end{pmatrix}\right)\begin{pmatrix} -5 \\ 4 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} -11 \\ 23 \end{pmatrix} = \begin{pmatrix} -0.55 \\ 1.15 \end{pmatrix}$

(b) $s^2 = \dfrac{Q(\hat{\boldsymbol{\beta}})}{7-1-1} = \dfrac{\boldsymbol{y}^\top\boldsymbol{y} - \boldsymbol{y}^\top\mathbf{X}\hat{\boldsymbol{\beta}}}{5} = \dfrac{9 - (-5,4)\frac{1}{20}\begin{pmatrix} -11 \\ 23 \end{pmatrix}}{5} = \dfrac{9 - \frac{1}{20}(55+92)}{5} = \dfrac{180-147}{20(5)} = \dfrac{33}{100} = 0.33$