MATH 667 EXAM 2 (November 24, 2015: 9:30am–10:45am) NAME ________________________

The exam is closed book; students are permitted to prepare one 4x6 notecard of formulas, notes, etc. that can be used during the exam. A calculator is permitted but not necessary for the exam.

Do 4 out of the 5 problems (20 points each, 80 points total). Clearly indicate the problem that you are omitting; if it is not clear, then the first 4 problems will be graded.

Problem 1. (20 points) Suppose that $X_1, \ldots, X_n$ are independent identically distributed (iid) uniform random variables each with probability density function $f(x) = \dfrac{1}{\beta - \alpha} I_{(\alpha,\beta)}(x)$.
where α and β are unknown, and $\alpha < \beta$.
(a - 10 pts) Find a two-dimensional sufficient statistic for (α, β).
(b - 10 pts) Find a system of equations that could be used to obtain the method of moments estimators of α and β. It is not necessary to solve the equations, but any integrals involved in the equations should be evaluated.

Problem 2. (20 points) Suppose that $X_1, \ldots, X_n$ are independent identically distributed (iid) exponential random variables each with probability density function $f(x) = \dfrac{1}{\beta} e^{-x/\beta} I_{(0,\infty)}(x)$.
(a - 6 pts) Calculate the Cramér-Rao Lower Bound for an unbiased estimator of β.
(b - 6 pts) Find the maximum likelihood estimator of β. Justify that it maximizes the likelihood function.
(c - 6 pts) Find the bias and variance of the maximum likelihood estimator of β.
(Hint: If Y is a gamma(α, β) random variable, its probability density function is $f(y|\alpha,\beta) = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} I_{(0,\infty)}(y)$, its mean is $\mathrm{E}Y = \alpha\beta$, and its variance is $\mathrm{Var}\ Y = \alpha\beta^2$, and its moment generating function is $M_Y(t) = \left(\dfrac{1}{1-\beta t}\right)^\alpha$ when $t < \dfrac{1}{\beta}$.)
(d - 2 pts) Find the maximum likelihood estimator of $e^{-\beta}$.

Problem 3. (20 points) Let X_1 and X_2 be independent identically distributed (iid) Poisson(θ) random variables each with probability mass function $f(x) = \dfrac{\theta^x e^{-\theta}}{x!}, x = 0, 1, 2, \ldots$.
(a - 5 pts) Find a sufficient statistic for θ.
(b - 5 pts) Show that $T(X_1) = \begin{cases} 1 & \text{if } X_1 = 0 \\ 0 & \text{otherwise} \end{cases}$ is an unbiased estimator of $\tau(\theta) = e^{-\theta}$.
(c - 5 pts) Using the fact that the sum of independent Poisson random variables with means $\mu_1, \ldots, \mu_n$ is a Poisson random variable with mean $\mu_1 + \ldots + \mu_n$, compute $P(T(X_1) = 1 | X_1 + X_2 = y)$.
(d - 5 pts) For the estimator $T(X_1)$ in part (b), find a uniformly better unbiased estimator of $e^{-\theta}$. Justify your answer with an appropriate theorem or calculations.

Problem 4. (20 points) Suppose that $X_1, \ldots, X_n$ is a random sample from a normal$(\mu, 1)$ population each with probability density function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}$ where μ is unknown, and suppose that the experimenter is interested in testing

$$H_0 : \mu \leq 0 \text{ versus } H_1 : \mu > 0.$$

(a - 8 pts) Show that the likelihood ratio test statistic has a critical region of the form $\left\{ (x_1, \ldots, x_n) : \frac{\bar{x}}{1/\sqrt{n}} \geq K \right\}$.
(b - 6 pts) Find the value of K (to at least 2 decimal places) such that the test in part (a) has size .01.
(c - 6 pts) What is the power of the test in part (a) when $\mu = 1$?

Use the standard normal and/or t tables attached to this exam.

Problem 5. (20 points) Let $X_1, \ldots, X_5$ be independent identically distributed (iid) Bernoulli(θ) random variables each with probability mass function $f(x|\theta) = \theta^x (1-\theta)^{1-x} I_{\{0,1\}}(x)$.

(a - 12 pts) Find the uniformly most powerful (UMP) test level $\alpha = \frac{6}{32} = .1875$ test for $H_0 : \theta = \frac{1}{2}$ versus $H_1 : \theta = \frac{3}{4}$. Justify your answer with an appropriate theorem or calculations.

(b - 8 pts) Compute the probability of a Type II error for the test in part (a).