The exam is closed book; students are permitted to prepare one 8.5×11 page of formulas, notes, etc. that can be used during the exam. A calculator is permitted but not necessary for the exam.

Do all 4 problems (the problems with the three highest scores are worth 30% each, the problem with the lowest score is worth 10%).

Problem 1. (10 points) Suppose $\begin{pmatrix} y_1 \\ x_1 \end{pmatrix}, \ldots, \begin{pmatrix} y_n \\ x_n \end{pmatrix}$ are independent $N_2\left(\begin{pmatrix} \mu_y \\ \mu_x \end{pmatrix}, \begin{pmatrix} \sigma_{yy} & \sigma_{yx} \\ \sigma_{yx} & \sigma_{xx} \end{pmatrix}\right)$ random variables. Define $\beta_0 = \mu_y - \sigma_{yx}\sigma_{xx}^{-1}\mu_x$ and $\beta_1 = \sigma_{xx}^{-1}\sigma_{yx}$ so that $E(y_i|x_i) = \beta_0 + \beta_1 x_i$ and $\sigma^2 = \text{var}(y_i|x_i) = \sigma_{yy} - \sigma_{yx}\sigma_{xx}^{-1}\sigma_{yx}$.

(a - 7 pts) If $n = 10, \sum_{i=1}^{10} x_i = 4$ and $\sum_{i=1}^{10} x_i^2 = 2$, find a constant C such that the F-statistic for testing $H_0 : \beta_0 = \beta_1$ versus $H_A : \beta_0 \neq \beta_1$ can be expressed in the form

$$F = C\left(\frac{\hat{\beta}_0 - \hat{\beta}_1}{\hat{\sigma}}\right)^2$$

where $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\sigma}^2$ are the maximum likelihood estimators of β_0, β_1, and σ^2, respectively.

(b - 3 pts) For what values of $\frac{\hat{\beta}_0 - \hat{\beta}_1}{\hat{\sigma}}$ should H_0 be rejected at level .05?

Problem 2. (10 points) Suppose $y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i$ for $i = 1, \ldots, 3$ where $x_i = i$ are known values, the regression parameters β_0, β_1, and β_2 are unknown, and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are independent and identically distributed random variables with mean 0 and unknown variance σ^2. Now, suppose that the x^2 term is excluded from the model, and least squares estimation is used to model the y's based on an intercept term and the x's (that is, suppose that we incorrectly use a linear model and find values $\hat{\beta}_0^*$ and $\hat{\beta}_1^*$, which minimize $\sum_{i=1}^{3}(y_i - \beta_0^* - \beta_1^* x_i)^2$).

What is the bias of using $\hat{\beta}_1^*$ to estimate β_1? Write your answer as a function of the true (but unknown) parameter values β_0, β_1, and β_2.

Problem 3. (10 points) Suppose $\boldsymbol{y} \sim N_6(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$ where $\mathbf{X}$ is an 6×2 matrix and $\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$ is a vector of fixed but unknown parameters. Also, suppose that

- $\mathbf{X}^\top\mathbf{X} = \begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix}$,
- the length of the 95% confidence interval for β_0 is 4,
- the 6th row of $\mathbf{X}$ is $(1, 0)$, and
- the residual for the 6th observation is 0.

(a - 3 pts) Compute the maximum likelihood estimate of σ^2.
(b - 4 pts) Compute the maximum likelihood estimate of σ^2 with the 6th observation removed from the data set.
(c - 3 pts) Compute the length of the 95% confidence interval for β_0 based only on the first 5 observations in the data set (that is, with the 6th observation removed from the data set).

Problem 4. (10 points) Suppose that

$$y_{ijk} = \mu + \alpha_i + \gamma_j + \varepsilon_{ijk} \text{ for } i, j, k = 1, 2$$

where $\alpha_1 + \alpha_2 = \gamma_1 + \gamma_2 = 0$ and ε_{ijk} are independent Normal(0,σ^2) random variables.
Given the data

		i	
	y_{ijk}	1	2
j	1	8,2	4,6
	2	0,4	0,0

test the hypothesis $H_0 : \alpha_1 = \alpha_2 = 0$ at level 0.05.

Here are some formulas which might be helpful:

$$\hat{\boldsymbol{y}} = \mathbf{X}_1\hat{\boldsymbol{\beta}}_1 + \mathbf{X}_2\hat{\boldsymbol{\beta}}_2 \text{ where } \hat{\boldsymbol{\beta}}_1^* = (\mathbf{X}_1^\top\mathbf{X}_1)^{-1}\mathbf{X}_1^\top\boldsymbol{y},\ \mathbf{A} = (\mathbf{X}_1^\top\mathbf{X}_1)^{-1}\mathbf{X}_1^\top\mathbf{X}_2, \text{ and } \hat{\boldsymbol{\beta}}_1 = \hat{\boldsymbol{\beta}}_1^* - \mathbf{A}\hat{\boldsymbol{\beta}}_2$$

$$E(\hat{\boldsymbol{\beta}}_1^*) = \boldsymbol{\beta}_1 + \mathbf{A}\boldsymbol{\beta}_2 \text{ where } \mathbf{A} = (\mathbf{X}_1^\top\mathbf{X}_1)^{-1}\mathbf{X}_1^\top\mathbf{X}_2$$

$$E(s_1^2) = \sigma^2 + \frac{\boldsymbol{\beta}_2^\top\mathbf{X}_2^\top(\mathbf{I} - \mathbf{X}_1(\mathbf{X}_1^\top\mathbf{X}_1)^{-1}\mathbf{X}_1^\top)\mathbf{X}_2\boldsymbol{\beta}_2}{n-p-1}$$

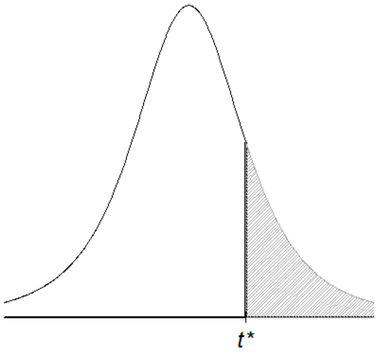
$$F = \frac{SSH/q}{SSE/(n-k-1)} \text{ where}$$

$$SSH = (\mathbf{C}\hat{\boldsymbol{\beta}} - \boldsymbol{t})^\top\left(\mathbf{C}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{C}^\top\right)^{-1}(\mathbf{C}\hat{\boldsymbol{\beta}} - \boldsymbol{t}) \text{ and } SSE = \boldsymbol{y}^\top(\mathbf{I} - \mathbf{H})\boldsymbol{y} = \boldsymbol{y}^\top\boldsymbol{y} - \hat{\boldsymbol{\beta}}^\top\mathbf{X}^\top\boldsymbol{y}$$

$$\boldsymbol{a}^\top\hat{\boldsymbol{\beta}} \pm t_{\alpha/2,n-k-1}s\sqrt{\boldsymbol{a}^\top(\mathbf{X}^\top\mathbf{X})^{-1}\boldsymbol{a}} \text{ is a } 100(1-\alpha)\% \text{ confidence interval for } \boldsymbol{a}^\top\boldsymbol{\beta}$$

$$\hat{\boldsymbol{\beta}}_c = \hat{\boldsymbol{\beta}} - (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{C}^\top(\mathbf{C}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{C}^\top)^{-1}(\mathbf{C}\hat{\boldsymbol{\beta}} - \boldsymbol{t}) \text{ and } \hat{\sigma}_c^2 = \tfrac{Q(\hat{\boldsymbol{\beta}}_c)}{n}$$

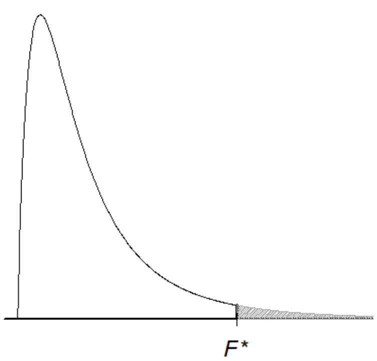
$$\hat{\varepsilon}_i = y_i - \mathcal{X}_i^\top\hat{\boldsymbol{\beta}}$$
$$\hat{y}_{(i)} = \mathcal{X}_i^\top\hat{\boldsymbol{\beta}}_{(i)}$$
$$SSE_{(i)} = \boldsymbol{y}_{(i)}^\top\boldsymbol{y}_{(i)} - \hat{\boldsymbol{\beta}}_{(i)}^\top\mathbf{X}_{(i)}^\top\boldsymbol{y}_{(i)}$$
$$r_i = \frac{\hat{\varepsilon}_i}{s\sqrt{1-h_{ii}}}$$
$$\hat{\boldsymbol{\beta}}_{(i)} = \hat{\boldsymbol{\beta}} - \frac{\hat{\varepsilon}_i}{1-h_{ii}}(\mathbf{X}^\top\mathbf{X})^{-1}\mathcal{X}_i$$
$$s_{(i)}^2 = \left(\frac{n-k-1-r_i^2}{n-k-2}\right)s^2$$



If T is a random variable with a t distribution having df degrees of freedom, then the critical value t^* in the table is the value such that the shaded area is $p = P(T > t^*)$.

t distribution critical values

	Upper-tail probability p				
df	.10	.05	.025	.01	.005
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.25
10	1.372	1.812	2.228	2.764	3.169
∞	1.282	1.645	1.960	2.326	2.576



If f is a random variable with an F distribution having $df1$ and $df2$ degrees of freedom, then the critical value F^* in the table is the value such that the shaded area is $P(f > F^*) = p$.

F distribution critical values

		$p = .05$				
		$df1$				
		1	2	3	4	5
	1	161.45	199.5	215.71	224.58	230.16
	2	18.51	19.00	19.16	19.25	19.30
	3	10.13	9.55	9.28	9.12	9.01
	4	7.71	6.94	6.59	6.39	6.26
$df2$	5	6.61	5.79	5.41	5.19	5.05
	6	5.99	5.14	4.76	4.53	4.39
	7	5.59	4.74	4.35	4.12	3.97
	8	5.32	4.46	4.07	3.84	3.69
	9	5.12	4.26	3.86	3.63	3.48
	10	4.96	4.10	3.71	3.48	3.33

	$p = .025$				
	$df1$				
	1	2	3	4	5
1	647.79	799.50	864.16	899.58	921.85
2	38.51	39.00	39.17	39.25	39.30
3	17.44	16.04	15.44	15.10	14.88
4	12.22	10.65	9.98	9.60	9.36
5	10.01	8.43	7.76	7.39	7.15
6	8.81	7.26	6.60	6.23	5.99
7	8.07	6.54	5.89	5.52	5.29
8	7.57	6.06	5.42	5.05	4.82
9	7.21	5.71	5.08	4.72	4.48
10	6.94	5.46	4.83	4.47	4.24