M621 HW, due April 6 I will return these to your mailbox Thursday night April 6.

1. Let p be a prime, let $t(x) = x^4 - p \in \mathbb{Q}[x]$, and let $S \subseteq \mathbb{C}$ be a splitting field of $t(x)$ over $\mathbb{Q}$.

   (a) Show that $t(x)$ is irreducible.

   (b) Determine the roots of $t(x)$.

   (c) Determine $[S : \mathbb{Q}]$, with explanation.

   (d) Since $S/\mathbb{Q}$ is Galois, $|Aut(S/\mathbb{Q})| = [S : \mathbb{Q}]$. Moreover, as we showed, since $\deg(t(x)) = 4$, $Aut(S/\mathbb{Q})$ is isomorphic to a subgroup of S_4. Based on your knowledge of S_4 and its Sylow subgroups, explain why $Aut(S/\mathbb{Q})$ is isomorphic to a Sylow-2 subgroup of $Aut(S/\mathbb{Q})$. [Note each Sylow-2 subgroup of S_4 is isomorphic to D_8. So $Aut(S/\mathbb{Q}) \cong D_8$.]

2. Let $\gamma = \sqrt{2+\sqrt{2}}$.

   (a) Find $m_{\gamma,\mathbb{Q}(\sqrt{2})}(x)$. (It has degree two.)

   (b) Find the minimal polynomial $m_{\gamma,\mathbb{Q}}(x)$. (It has degree 4.) Show some work.

   (c) Find the roots of $m_{\gamma,\mathbb{Q}}(x)$.

   (d) Show that $\mathbb{Q}(\gamma)$ is the splitting field of $m_{\gamma,\mathbb{Q}}(x)$ (over $\mathbb{Q}$).

   (e) +1 EC: Determine $Aut(\mathbb{Q}(\gamma)/\mathbb{Q})$ up to isomorphism, providing a brief well-reasoned argument.