A functional equation associated with elementary school arithmetic

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Abstract: In this talk, I will present an interesting functional equation known as the cocycle equation. This equation has a long history. I will derive this equation using the traditional method of manual addition of two multi-digit numbers that students learn in elementary school arithmetic. The solutions of this equation will be presented on free abelian groups. This talk will assume only a basic knowledge of abstract algebra and therefore will be be accessible to undergraduate and graduate students.

- This talk is about a functional equation that arises from the elementary school arithmetic.
- We will see how this equation can be solved on a free abelian group.

Suppose we ask an elementary school student to add the number 25 to the number 18. The student will add the ones and will get 13. So he will write 3 and carry the digit 1. Then he will add the tens and the carry digit to get a 4. Therefore the sum he will have is 43.

$$\begin{array}{ccc} & 1 & \\ & 1 & 8 \\ + & 2 & 5 \\ \hline & 4 & 3 \end{array}$$

Illustration of right to left addition of two-digit numbers

Let us create a formal group theoretic framework for the study of arithmetic.

Let us consider addition in the finite abelian group $\mathbb{Z}_{100}$.

This will serve as a model for addition of two-digit numbers.

$$\begin{array}{cccc} & & 1 & \\ & & 1 & 8 \\ + & & 2 & 5 \\ \hline & & 4 & 3 \end{array}$$

Illustration of right to left addition of two-digit numbers

$\mathbb{Z}_{10}$ is a subgroup of $\mathbb{Z}_{100}$ consisting of the multiples of 10.

Let us call this subgroup H. It represents subgroup of “tens”.

$$\begin{array}{ccc} & 1 & \\ & \color{red}{1} & 8 \\ + & \color{red}{2} & 5 \\ \hline & \color{red}{4} & 3 \end{array}$$

Illustration of right to left addition of two-digit numbers

Also, $\mathbb{Z}_{10}$ is a quotient group of $\mathbb{Z}_{100}$, namely, $\mathbb{Z}_{100}/H$. We call this group G. This group G is a group of "ones".

For any $x \in \mathbb{Z}_{100}$, let us write

$$x = [a]\,[b],$$

where $a \in H$ and $b \in G$. This means we are thinking of a as the tens digit of x and b as the ones digit of x.

For example, we write the number 48 as $[4][8]$.

For any two elements $[a_1][b_1]$ and $[a_2][b_2]$ of $\mathbb{Z}_{100}$, their sum

$$[a_1][b_1] + [a_2][b_2]$$

in $\mathbb{Z}_{100}$ can be written uniquely in the form $[a][b]$, with a and b are functions of a_1, a_2, b_1, b_2.

The element b of G depends only on the elements b_1 and b_2 of G.

The ones digit of a sum depends only on the ones digits of the summands. This is the way one works from right to left when adding multi-digit numbers by hand.

The element a of H depends on the elements a_1, a_2, b_1, b_2.

The tens digit of a sum depends on all four values a_1, a_2, b_1, b_2 because adding two multi-digit numbers is not merely adding columns.

The formula for a is

$$a = a_1 + a_2 + z(b_1, b_2) \tag{1}$$

where $z : G \times G \to H$ is a function expressing how the ones digits of the summands affects the tens digit of the sum.

The formula for b is

$$b = b_1 + b_2. \tag{2}$$

Here the addition symbol "+" is the group operation of G.

$$
\begin{array}{cccc}
 & & z(b_1, b_2) & \\
 & & [a_1] & [b_1] \\
+ & & [a_2] & [b_2] \\
\hline
 & & [a_1 + a_2 + z(b_1, b_2)] & [b_1 + b_2]
\end{array}
$$

Illustration of right to left addition of two-digit numbers

$$
\begin{array}{lcc}
 & z(b_1+b_2,b_3) & \\
 & [a_1+a_2+z(b_1,b_2)] & [b_1+b_2] \\
+ & [a_3] & [b_3] \\
\hline
 & [a_1+a_2+a_3+z(b_1,b_2)+z(b_1+b_2,b_3)] & [b_1+b_2+b_3]
\end{array}
$$

Illustration of right to left addition of two-digit numbers

Since the group operation of the group $(\mathbb{Z}_{100}, +)$ is associative, thus we have

$$([a_1][b_1] + [a_2][b_2]) + [a_3][b_3] = [a_1][b_1] + ([a_2][b_2] + [a_3][b_3])$$

for any $a_1, a_2, a_3 \in H$ and $b_1, b_2, b_3 \in G$.

Using (1) and (2), we compute

$$\big([a_1][b_1] + [a_2][b_2]\big) + [a_3][b_3]$$

$$= \big[a_1 + a_2 + a_3 + z(b_1, b_2) + z(b_1 + b_2, b_3)\big]\,[b_1 + b_2 + b_3].$$

Similarly, we also have

$$[a_1][b_1] + \big([a_2][b_2] + [a_3][b_3]\big)$$

$$= \big[a_1 + a_2 + a_3 + z(b_2, b_3) + z(b_1, b_2 + b_3)\big]\,[b_1 + b_2 + b_3].$$

Hence the consideration of the tens digit of the sum yields the functional equation

$$z(b_1, b_2) + z(b_1 + b_2, b_3) = z(b_2, b_3) + z(b_1, b_2 + b_3)$$

for all $b_1, b_2, b_3 \in G$.

The last functional equation can be rewritten as

$$F(a,b)+F(a+b,c)=F(b,c)+F(a,b+c) \qquad \text{(CE)}$$

for all $a,b,c\in G$ and $F : G\times G\to H$ is an unknown function to be determined.

The functional equation (CE) is known as the 2-cocycle functional equation. This equation has a long history, with connection to

- Factor Systems and Group Extension
- Cohomology Theory
- Information Theory
- Formal Groups
- Algebra of Polyhedra

• In 1956, S. Kurepa determined the differentiable solution of (CE) when $G = H = \mathbb{R}$ and proved that F is of the form

$$F(a, b) = f(a + b) - f(a) - f(b) \quad \forall a, b \in \mathbb{R} \tag{3}$$

where $f : G \longrightarrow H$ is an arbitrary function.

• In 1959, J. Erdös proved that every symmetric solution $F : G^2 \longrightarrow H$ of (CE) is of the form (3), where G is an abelian group and H is a divisible abelian group.

• In 1968, Jessen, Karpf and Thorup proved J. Erdös' result in a different way.

• Using some ideas from Jessen, Karpf and Thorup (1968), we will present the solution of the functional equation (CE) when G is a free abelian group.

• If H is a divisible abelian group and G is an abelian, then we also get the same solution.

Free Abelian Groups

First, we present the definition and some examples of free abelian group.

A free abelian group is an abelian group that has a basis in the sense that every element of the group can be written in one and only one way as a finite linear combination of elements of the basis with integer coefficients.

Not all abelian groups have basis, the ones they do are called free abelian groups.

Free abelian groups are the free groups that are abelian. In fact the only free groups that are abelian are the trivial group (free group of rank 0) and the cyclic group (free group of rank 1).

Example 1. Let $G = \mathbb{Z} \oplus \mathbb{Z} = \{\, (a, b) \,|\, a, b \in \mathbb{Z} \,\}$. Then $(G, +)$ is a free abelian group with basis $\{(1, 0),\ (0, 1)\}$.

Example 2. $(\mathbb{Q}, +)$ is not a free abelian group.

Example 3. $(\mathbb{Z}_n, +)$ is not a free abelian group.

If G is a free abelian group with a basis X and $card(X) = n$, then G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$.

• For every nonempty set B, there exists a free abelian group with basis B.

• Subgroup of a free abelian group is a free abelian group.

• All free groups having B as basis are isomorphic.

• If G is a free abelian group with basis B, then for every arbitrary $f : B \longrightarrow H$ (some abelian group) there exists unique homomorphism $F : G \longrightarrow H$ which extends f.

Theorem 1 (J-K-T, 1968) *Let H be an abelian group and G be a free abelian group. Then every symmetric solution $F : G^2 \to H$ of*

$$F(a,b) + F(a+b,c) = F(b,c) + F(a,b+c) \tag{CE}$$

for all $a, b, c \in G$ is of the form

$$F(a,b) = f(a+b) - f(a) - f(b), \tag{S}$$

where $f : G \to H$ is an arbitrary function.

Proof: It is easy to see that F given by (S) is symmetric and satisfies the 2-cocycle functional equation (CE).

Now we prove that F given by (S) is the only solution of the 2-cocycle functional equation (CE).

Letting $b = 0$ in (CE), we get

$$F(a, 0) = F(0, c).$$

Letting $a = 0 = b$ in (CE), we get

$$F(0, c) = F(0, 0).$$

Hence

$$F(a, 0) = F(0, c) = F(0, 0). \tag{4}$$

for all $a, c \in G$.

Define a set W by

$$W = G \times H.$$

Let us define a binary operation "$+$" in W as

$$(a,\ x) + (b,\ y) = (\, a + b,\ \ x + y + F(a, b)\,) \tag{5}$$

for all $a, b \in G$ and all $x, y \in H$.

We show that $(W, +)$ is an abelian group. That is

(1) $+$ is commutative;

(2) W has an identity element;

(3) each element in W has an inverse;

(4) $+$ is associative.

Step 1. For each $a, b \in G$ and $x, y \in H$,

$$
\begin{aligned}
(a,\ x) + (b,\ y) & \\
&= (\,a+b,\ \ x+y+F(a,b)\,) && \text{(by definition)} \\
&= (\,b+a,\ \ y+x+F(a,b)\,) && (G \text{ and } H \text{ abelian}) \\
&= (\,b+a,\ \ y+x+F(b,a)\,) && (F \text{ symmetric}) \\
&= (b,\ y) + (a,\ x) && \text{(by definition)}
\end{aligned}
$$

Hence the binary operation "$+$" is commutative in W.

Step 2. For each $a \in G$ and $x \in H$,

$$
\begin{aligned}
(a,\ x) + (0,\ -F(0,0)) \\
&= (\,a+0,\ \ x - F(0,0) + F(a,0)\,) \\
&= (\,a,\ \ x - F(0,0) + F(0,0)\,) \qquad (\text{ by (CE) }) \\
&= (\,a,\ \ x\,).
\end{aligned}
$$

Hence $(0,\ -F(0,0))$ is the identity element in W. We denote this identity element by 0_W.

Step 3. For each $a \in G$ and $x \in H$,

$$
\begin{aligned}
(a,\ x) + (-a,\ \ -x - F(a,-a) - F(0,0)) \\
&= (\, a - a,\ \ x - x - F(a,-a) - F(0,0) + F(a,-a)\,) \\
&= (\, 0,\ \ -F(0,0)\,) \\
&= 0_W.
\end{aligned}
$$

Therefore, every element (a, x) in W has an inverse $(-a,\ \ -x - F(a,-a) - F(0,0))$ in W.

Step 4. For any $(a, x), (b, y), (c, z) \in W$, we have

$$\begin{aligned}
&\big((a,x) + (b, y)\big) + (c, z) \\
&\quad = \big(a + b, x + y + F(a, b)\big) + (c, z) \\
&\quad = \big(a + b + c, x + y + z + F(a, b) + F(a + b, c)\big) \\
&\quad = \big(a + b + c, x + y + z + F(b, c) + F(a, b + c)\big) \\
&\quad = (a, x) + \big(b + c, y + z + F(b, c)\big) \\
&\quad = (a, x) + \big((b, y) + (c, z) \big).
\end{aligned}$$

Hence "+" is associative.

Consider the projection $\phi : W \to G$ defined by $\phi(a,x) = a$.

Then

$$\begin{aligned}\phi(\,(a,x)+(b,y)\,) &= \phi(\,a+b, x+y+F(a,b)\,)\\ &= a+b\\ &= \phi(a,x)+\phi(b,y).\end{aligned}$$

Hence ϕ is a homomorphism from W onto G.

Next, we show a function $f : G \to H$ satisfies

$$F(a, b) = f(a + b) - f(a) - f(b) \tag{6}$$

for all $a, b \in G$ if and only if

$$K := \{(a, f(a)) \mid a \in G\} \tag{7}$$

is a subgroup of the abelian group W.

Suppose $F(a,b) = f(a+b) - f(a) - f(b)$ for all $a, b \in G$.

We want to show K is a subgroup of W. From (6), we have

$$F(a,-b) = f(a-b) - f(a) - f(-b)$$

$$F(b,-b) = f(0) - f(b) - f(-b)$$

$$F(0,0) = f(0) - f(0) - f(0).$$

Hence

$$F(a,-b) - F(0,0) - F(b,-b)$$
$$= f(a-b) - f(a) + f(b). \qquad (8)$$

Let $(a, f(a))$ and $(b, f(b))$ be any two elements in K. We want to show that

$$\big(\, a - b, \;\; f(a - b) \,\big) \in K.$$

Using (8), we compute

$$
\begin{aligned}
&(a, f(a)) - (b, f(b)) \\
&\quad = (a, f(a)) + \big(-b, \; -f(b) - F(b,-b) - F(0,0) \big) \\
&\quad = \big(a - b, f(a) - f(b) - F(b,-b) - F(0,0) + F(a,-b)\big) \\
&\quad = \big(a - b, \; f(a) - f(b) + f(a-b) - f(a) + f(b) \big) \\
&\quad = \big(a - b, \;\; f(a-b) \big) \in K.
\end{aligned}
$$

Hence K is a subgroup of W.

Next, we assume that K is a subgroup of W. We want to show that (6) holds.

Since $K < W$, therefore, whenever $(a, f(a))$ and $(b, f(b))$ are elements of K, we have

$$
\begin{aligned}
& (a, f(a)) + (b, f(b)) \in K \\
\Rightarrow & \big(a + b,\ f(a) + f(b) + F(a, b) \big) \in K.
\end{aligned}
$$

Hence from the definition of K, we have

$$f(a+b) = f(a) + f(b) + F(a,b)$$

which is

$$F(a,b) = f(a+b) - f(a) - f(b). \tag{6}$$

Thus, the existence of a function $f : G \to H$ satisfying (6) is equivalent to the existence of a subgroup S of W which by ϕ is mapped bijectively onto G.

Since G is a free abelian group, G has a basis $\{a_i \,|\, i \in I\}$ in the sense that every element of G can be written in one and only one way as a finite linear combination of elements of the basis with integer coefficients.

Let x_i for $i \in I$ be arbitrary elements of the group H. Let

$$S = \langle \{(a_i, x_i) \,|\, i \in I\} \rangle$$

be the subgroup generated by the set $\{(a_i, x_i) \,|\, i \in I\}$. Then S is a subgroup of W. It is easy to see that the projection map ϕ maps S bijectively onto G.

Hence f is determined by its values

$$f(a_i) = x_i, \quad i \in I,$$

and these value can be arbitrarily chosen. This completes the proof of the theorem.

Theorem 2 (Hosszú, 1965) *Let H and G be abelian groups. Suppose H is uniquely 2-divisible. The most general solution $F : G^2 \to H$ of*

$$F(a,b) + F(a+b,c) = F(b,c) + F(a,b+c) \qquad \text{(CE)}$$

is of the form

$$F(a,b) = S(a,b) + A(a,b)$$

where S is a symmetric solution of (CE) *and A is an anti-symmetric one, additive in both variables.*

Proof: Define

$$S(a,b) = \frac{1}{2}\left[F(a,b) + F(b,a)\right]$$

and

$$A(a,b) = \frac{1}{2}\left[F(a,b) - F(b,a)\right].$$

Then $S(a,b) = S(b,a)$ and $A(a,b) = -A(b,a)$.

Also $S(a,b)$ and $A(a,b)$ are solutions of (CE). To see $A(a,b)$ is a solution of (CE), consider

$$\begin{aligned}
&2\big[A(a,b)+A(a+b,c)\big]\\
&\qquad\stackrel{def}{=} F(a,b)-F(b,a)+F(a+b,c)-F(c,a+b)\\
&\qquad= F(a,b)+F(a+b,c)-F(b,a)-F(c,a+b)\\
&\qquad\stackrel{(CE)}{=} F(b,c)+F(a,b+c)-F(c,b)-F(c+b,a)\\
&\qquad= F(b,c)-F(c,b)+F(a,b+c)-F(b+c,a)\\
&\qquad= 2\big[A(b,c)+A(a,b+c)\big].
\end{aligned}$$

Similarly, it can be shown that $S(a, b)$ is a solution of (CE).

• Since A is antisymmetric, it is enough to show that A is additive in the first variable.

$$
\begin{aligned}
2\,A(a+b,c)
&\overset{def}{=} F(a+b,c) - F(c,a+b) \\
&\overset{(\mathrm{CE})}{=} F(a,b+c) + F(b,c) - F(a,b) \\
&\qquad - \left[F(c+a,b) + F(c,a) - F(a,b)\right] \\
&\overset{(\mathrm{CE})}{=} F(a,b+c) + F(b,c) \\
&\qquad - \left[F(a,b+c) + F(c,b) - F(a,c) + F(c,a)\right] \\
&\overset{def}{=} 2\,A(b,c) + 2\,A(a,c).
\end{aligned}
$$

This completes the proof of the theorem.

In 1995, Szekelyhidi found the bounded solution of the functional equation (CE) on a large class of groups known as amenable group.

To present his result, I will first remind you the definition of amenable group.

Amenable Groups

Let G be a finite group with $G = \{x_1, x_2, .x_3, .., x_n\}$. Let $f : G \to \mathbb{C}$. Define

$$M(f) = \frac{1}{n} \sum_{k=1}^{n} f(x_k).$$

Then M satisfies

$$M(\alpha f + \beta g) = \alpha M(f) + \beta M(g).$$

Hence M is a linear functional on the space of all complex valued functions f defined on G. When f is real-valued, it is easy to see that

$$\min\{f(x_1), ..., f(x_n)\} \leq M(f) \leq \max\{f(x_1), ..., f(x_n)\}$$

Moreover, it is also easy to check that

$$M(\overline{f}) = \overline{M(f)}$$

for all $f : G \to \mathbb{C}$.

Define

$$f_j(x_i) = f(x_i x_j) \quad \text{and} \quad {}_jf(x_i) = f(x_j x_i).$$

Then

$$\begin{aligned} M(f_j) &= \frac{1}{n}\sum_{i=1}^{n} f(x_i x_j) \\ &= \frac{1}{n}\sum_{i=1}^{n} f(x_i) = M(f). \end{aligned}$$

- Hence $M(f_j) = M(f)$ and M is right invariant.
- Similarly, $M({}_jf) = M(f)$ and M is left invariant.

Therefore $M(f)$ is an invariant mean for the finite group G.

This leads to the following definition of invariant mean for an arbitrary group.

Definition 1 *Let* G *be a group (or a semigroup). Let* $B(G, \mathbb{C})$ *be a space of all bounded complex valued functions defined on* G *with norm*

$$||f|| = \sup\{f(x) \mid x \in G\}.$$

A linear functional M *in* $B(G, \mathbb{C})$ *is said to be a right invariant mean if the following conditions hold:*

$$M(\overline{f}) = \overline{M(f)} \quad \forall\, f \in B(G, \mathbb{C}); \tag{9}$$

$$\inf\{f(x) \mid x \in G\} \leq M(f) \leq \sup\{f(x) \mid x \in G\} \tag{10}$$

for all real valued $f \in B(G, \mathbb{C})$;

$$M(f_x) = M(f) \tag{11}$$

for all $x \in G$ *and* $f \in B(G, \mathbb{C})$ *where* $f_x(t) = f(tx)$.

A mean which is both left and right invariant is said to be two-sided invariant or just invariant.

A semigroup S that admits a left (right) invariant mean on $B(S, \mathbb{C})$ is called a left (right) amenable. If the semigroup S admits a two-sided invariant mean on $B(S, \mathbb{C})$, then S is said to be two-sided amenable or simply amenable.

- The group of integers is amenable.
- Every finite group (semigroup) is amenable.
- Every commutative group (semigroup) is amenable.
- Every subgroup of an amenable group is amenable.
- If a group is left (right) amenable, then it is amenable.
- The free group on two generators is non-amenable.

Theorem 3 (Szekelyhidi, 1995) *Let G be a right amenable semigroup and let $F : G \times G \to \mathbb{C}$ be a bounded function satisfying*

$$F(x,y) + F(xy,z) = F(x,yz) + F(y,z)$$

for all $x, y, z \in G$. Then there exists a unique bounded function $f : G \to \mathbb{C}$ with

$$F(x,y) = f(x) + f(y) - f(xy) \quad \forall\, x, y \in G.$$

Proof: Let M denote any right invariant mean defined on the set of all bounded complex valued functions on G.

We write M_x to indicate that M is applied on the argument as a function of x.

We apply M on both sides of

$$F(x,y) + F(xy,z) = F(x,yz) + F(y,z)$$

as functions of x, for any fixed $y, z \in G$.

Thus

$$M_x[F(x,y)] + M_x[F(xy,z)] = M_x[F(x,yz)] + M_x[F(y,z)].$$

By right invariance of M and the fact that $M(1) = 1$, we obtain

$$M_x[F(x,y)] + M_x[F(x,z)] = M_x[F(x,yz)] + F(y,z).$$

Therefore

$$F(y,z) = M_x[F(x,y)] + M_x[F(x,z)] - M_x[F(x,yz)].$$

So defining

$$f(y) := M_x[F(x,y)] \quad \forall\, y \in G, \tag{12}$$

we have

$$F(y,z) = f(y) + f(z) - f(yz). \tag{13}$$

As F is bounded, it follows from (12) that f is bounded. Next we prove that f is unique. Suppose not. Then there exists g such that

$$F(y,z) = g(y) + g(z) - g(yz). \tag{14}$$

From (13) and (14), we have

$$(f-g)(yz) = (f-g)(y) + (f-g)(z) \quad \forall\, y,z \in G.$$

Since $f - g$ is a complex homomorphism of G, therefore it cannot be bounded unless it is zero. To see this, let $\phi = f - g$. Since ϕ is a complex homomorphism of group G, therefore $\phi(x^n) = n\phi(x)$ for all $x \in G$. Hence

$$\phi(x) = \lim_{n \to \infty} \phi(x) = \lim_{n \to \infty} \frac{\phi(x^n)}{n} = 0$$

for each $x \in G$. Thus $\phi = 0$. Hence $f = g$. So f is unique. This completes the proof of the theorem.

Generalized Cocycle Equation: In 1994, motivated by Szekelyhidi's result, Páles considered the following functional equation

$$F(x,y)+\frac{1}{n}\sum_{i=1}^{n}F(x\phi_i(y),z)=\frac{1}{n}\sum_{i=1}^{n}F(x,y\phi_i(z))+F(y,z)$$

where $\phi_1,\phi_2,\ldots,\phi_n : S \to S$ are pairwise distinct homomorphisms of the semigroup S and the set $\{\phi_1,\phi_2,\ldots,\phi_n\}$ is a group under function composition.

If $n = 1$ and $\phi_1(x) = x$, then the last functional equation reduces to 2-cocycle functional equation.

If $n = 2$, $\phi_1(x) = x$ and $\phi_2(x) = x^{-1}$, then the equation reduces to another interesting functional equation

$$\begin{aligned} F(x,y) + \frac{1}{2}\left[F(xy,z) + F(xy^{-1},z)\right] \\ = \frac{1}{2}\left[F(xy,z) + F(x,yz^{-1})\right] + F(y,z). \end{aligned}$$

Theorem 4 (Páles, 1994) *Suppose S is a semigroup and $f : S \to \mathbb{C}$ an arbitrary function. Then the function $F : S \times S \to \mathbb{C}$ defined by*

$$F(x, y) = f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(x\phi_i(y)) \tag{15}$$

is a solution of the functional equation

$$F(x, y) + \frac{1}{n}\sum_{i=1}^{n} F(x\phi_i(y), z) = \frac{1}{n}\sum_{i=1}^{n} F(x, y\phi_i(z)) + F(y, z).$$

Proof. Computing the left side of the functional equation, we get

$$
\begin{aligned}
F(x,y) &+ \frac{1}{n}\sum_{i=1}^{n} F(x\phi_i(y), z) \\
&\overset{(15)}{=} f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(x\phi_i(y)) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\left[f(x\phi_i(y)) + f(z) - \frac{1}{n}\sum_{j=1}^{n} f(x\phi_i(y)\phi_j(z))\right]
\end{aligned}
$$

$$
\begin{aligned}
&= f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(x\phi_i(y)) + \frac{1}{n}\sum_{i=1}^{n} f(x\phi_i(y)) \\
&\quad + f(z) - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(x\phi_i(y)\phi_j(z)) \\
&= f(x) + f(y) + f(z) - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(x\phi_i(y)\phi_j(z)).
\end{aligned}
$$

Similarly, we have

$$\frac{1}{n}\sum_{i=1}^{n} F(x, y\phi_i(z)) + F(y, z)$$

$$= f(x) + f(y) + f(z) - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(x\phi_j(y\phi_i(z))).$$

Hence

$$F(x,y)+\frac{1}{n}\sum_{i=1}^{n}F(x\phi_i(y),z)-\frac{1}{n}\sum_{i=1}^{n}F(x,y\phi_i(z))-F(y,z).$$

$$=f(x)+f(y)+f(z)-\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}f(x\phi_i(y)\phi_j(z))$$

$$-f(x)-f(y)-f(z)+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}f(x\phi_j(y\phi_i(z)))$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[f(x\phi_j(y\phi_i(z))) - f(x\phi_i(y)\phi_j(z)) \right]$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[f(x\phi_j(y)\phi_j(\phi_i(z)))) - f(x\phi_i(y)\phi_j(z)) \right]$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[f(x\phi_j(y)\phi_i(z))) - f(x\phi_i(y)\phi_j(z)) \right] = 0$$

where we have used the fact that ϕ_i are homomorphisms and $\{\phi_1, \phi_2, ..., \phi_n\}$ is a group under composition. This proof is now complete.

In the next theorem, we present the bounded solutions of the generalized cocycle equation on right-amenable semigroups. The proof of this theorem is due to Páles but it is based on Szekelyhidi's proof.

Theorem 5 (Páles, 1994) *Let S be a right-amenable semigroup and $F : S \times S \to \mathbb{C}$ be a bounded function. Then F satisfies*

$$F(x,y)+\frac{1}{n}\sum_{i=1}^{n}F(x\phi_i(y),z)=\frac{1}{n}\sum_{i=1}^{n}F(x,y\phi_i(z))+F(y,z)$$

for all $x,y,z \in S$ if and only if there exists a bounded function $f : S \to \mathbb{C}$ such that

$$F(x,y) = f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(x\phi_i(y))$$

for all $x, y \in S$.

Moreover, the representing function $f : S \to \mathbb{C}$ *of* F *is unique.*

Proof. Suppose F can be represented by (15), that is

$$F(x,y) = f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(x\phi_i(y))$$

with some bounded f. Then from (15), we see that F is bounded. By Theorem 4, $F(x,y)$ given by

$$F(x,y) = f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(x\phi_i(y))$$

is a solution of generalized cocycle equation.

To prove the converse, let M be a right-invariant mean on the space of bounded complex-valued functions on S. We write M_x to indicate that M is applied on the argument as function of x.

Assume now that a bounded function F satisfies the generalized cocycle equation. Fix $y, z \in S$ and apply M_x on both sides of the generalized cocycle equation.

Then using the right-invariance and $M(1) = 1$, we obtain

$$
\begin{aligned}
M_x\Big[F(x,y) + \frac{1}{n}\sum_{i=1}^{n} F(x\phi_i(y), z)\Big] \\
= M_x\Big[\frac{1}{n}\sum_{i=1}^{n} F(x, y\phi_i(z)) + F(y,z)\Big]
\end{aligned}
$$

Hence

$$M_x\big[F(x,y)\big] + \frac{1}{n}\sum_{i=1}^{n} M_x\big[F(x\phi_i(y),z)\big]$$
$$= \frac{1}{n}\sum_{i=1}^{n} M_x\big[F(x,y\phi_i(z))\big] + F(y,z)$$

Since $M_x\big[F(x\phi_i(y),z)\big] = M_x\big[F(x,z)\big]$, we have

$$M_x\big[F(x,y)\big] + M_x\big[F(x,z)\big]$$
$$= \frac{1}{n}\sum_{i=1}^{n} M_x\big[F(x,y\phi_i(z))\big] + F(y,z)$$

Define a function $f : S \to \mathbb{C}$ by

$$f(u) = M_x\big[F(x,u)\big]. \tag{16}$$

Then from the last equality, we obtain

$$F(x,y) = f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(x\phi_i(y)).$$

which is (15).

It remains to prove that f is uniquely determined. Suppose not. Then there exist $f_1, f_2 : S \to \mathbb{C}$ such that

$$F(x,y) = f_1(x) + f_1(y) - \frac{1}{n}\sum_{i=1}^{n} f_1(x\phi_i(y)) \tag{17}$$

and

$$F(x,y) = f_2(x) + f_2(y) - \frac{1}{n}\sum_{i=1}^{n} f_2(x\phi_i(y)). \tag{18}$$

Let $g := f_1 - f_2$. Then (17) and (18) imply

$$g(x) + g(y) = \frac{1}{n}\sum_{i=1}^{n} g(x\phi_i(y)). \tag{19}$$

Applying M_x to both sides of (19), we get

$$M_x\big[g(x)\big] + g(y) = \frac{1}{n}\sum_{i=1}^{n} M_x\big[g(x\phi_i(y))\big].$$

which is

$$M_x\big[g(x)\big] + g(y) = M_x\big[g(x)\big].$$

Therefore $g(y) = 0$ for all $y \in S$. Hence

$$f_1(y) = f_2(y) \qquad \forall\, y \in S.$$

This shows that f is unique and the proof of the theorem is now complete.

• Based on the functional equation (CE), we make several remarks.

Remark 1 *The function F satisfying* (CE) *and the conditions $F(x,e)=0=F(e,x)$ is called a cocycle.*

Remark 2 *The set of all cocycles from* $G \times G$ *to* H *is denoted by* $\mathbb{Z}^2(G, H)$. *Hence*

$$\mathbb{Z}^2(G, H) = \{F : G \times G \to H \mid$$
$$F \in S(G, H) \;\&\; F(a, e) = 0 = F(e, b)\},$$

where $S(G, H)$ *is the set of solution of 2-cocycle functional equation* (CE).

Recall that the real-valued solution of 2-cocycle equation on free abelian groups is of the form

$$F(a,b) = f(ab) - f(a) - f(b) \qquad \forall\, a, b \in G.$$

Remark 3 $\mathbb{Z}^2(G,H)$ *is an abelian group under point-wise addition of cocycles.*

Remark 4 *Given a function $f : G \to H$ the map*

$$(a, b) \mapsto f(a + b) - f(a) - f(b)$$

is called the coboundary of f if $f(0) = 0$.

• Note that this maps measures how much f deviates being a homomorphism of G into H.

Remark 5 *The set of all coboundaries from $G \times G$ to H is denoted by $\mathbb{B}^2(G, H)$.*

Remark 6 *$\mathbb{B}^2(G, H)$ is an abelian group under point-wise addition of coboundaries.*

Remark 7 *$\mathbb{B}^2(G, H)$ is a normal subgroup of $\mathbb{Z}^2(G, H)$.*

Remark 8 *The quotient group*

$$\mathbb{H}^2(G, H) = \mathbb{Z}^2(G, H)/\mathbb{B}^2(G, H)$$

is called the second cohomolohy group of G.

Remark 9 *Szekelyhidi's theorem can be rephrases as: For any discrete amenable group G, the second bounded cohomology group $\mathbb{H}_b^2(G, \mathbb{C}) = 0$.*

Remark 10 *Ramsey (2005) proved the following result: For any discrete amenable group* G*, the bounded cohomology group* $\mathbb{H}_b^n(G, \mathbb{R}) = 0$ *for any* $n \in \mathbb{N}$.

Remark 11 *We have studied (see Faiziev and Sahoo (2005)) the second bounded cohomology of amalgamated product of groups.*

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Thank You

A group G is said to be divisible if for every $x \in G$ and for every positive integer n there exists a $y \in G$ such that $ny = x$.

Example 1. $(\mathbb{Z}, +)$ is not a divisible group.

Example 2. $(\mathbb{Q}, +)$ is a divisible group.

Example 3. $(\mathbb{R}, +)$ is a divisible group.