Exam 1 Solutions

1. (a) $Q(\beta) = (y - X\beta)^T (y - X\beta) = y^T y - 2\beta^T X^T y + \beta^T X^T X \beta$

$$\frac{\partial Q(\beta)}{\partial \beta} = \boxed{-2X^T y + 2X^T X \beta}$$

(b) (i) $E[a^T\hat\beta] = a^T E[\hat\beta] = a^T\beta$

$\operatorname{var}[a^T\hat\beta] = a^T \operatorname{var}[\hat\beta]\, a = a^T(\sigma^2 (X^T X)^{-1})\, a = \sigma^2 a^T (X^T X)^{-1} a$

Since $\hat\beta$ is a linear transformation of y,

$$\boxed{a^T\hat\beta \sim N(a^T\beta, \sigma^2 a^T (X^T X)^{-1} a)}.$$

(ii) $$\frac{(n-2)s^2}{\sigma^2} = \frac{RSS}{\sigma^2} \sim \boxed{\chi^2(n-2)}$$

(iii) Since $\hat\beta = (X^T X)^{-1} X^T y$ and $r = (I - X(X^T X)^{-1} X^T) y$ are linear transformations of y, they are both normally distributed.

Also, $\operatorname{cov}(\hat\beta, r) = \operatorname{cov}((X^T X)^{-1} X^T y,\ I - X(X^T X)^{-1} X^T y)$

$$= (X^T X)^{-1} X^T \operatorname{var}(y)\, (I - X(X^T X)^{-1} X^T)$$

$$= \sigma^2 (X^T X)^{-1} X^T (I - X(X^T X)^{-1} X^T)$$

$$= \sigma^2 ((X^T X)^{-1} X^T - \cancel{(X^T X)^{-1} X^T X}(X^T X)^{-1} X^T) = \underline{0}.$$

If two normal random variables are uncorrelated, then they are independent; so $\hat\beta$ and r are independent. Since $r^T r$ is a function of r, $\hat\beta$ and $r^T r$ are also independent.

(iv) $$\frac{\dfrac{a^T\hat\beta - a^T\beta}{\sqrt{\sigma^2 a^T (X^T X)^{-1} a}}}{\sqrt{\left\{\dfrac{(n-2)s^2}{\sigma^2}\right\} / (n-2)}} \sim t(n-2)$$ since

$$z = \frac{a^T\hat\beta - a^T\beta}{\sqrt{\sigma^2 (a^T (X^T X)^{-1} a)}} \sim N(0,1),$$

$$q = \frac{(n-2)s^2}{\sigma^2} \sim \chi^2(n-2),$$

and z and q are independent since z is a function of $\hat\beta$ and q is a function of r.

2. Test $H_0: \beta_1 = 2$ vs. $H_A: \beta_1 < 2$.

Test statistic: $t = \dfrac{\hat\beta_1 - 2}{s/\sqrt{S_{xx}}}$,

$$\left.\begin{aligned} S_{xy} &= \Sigma x_i y_i - n\bar{x}\bar{y} = 6 - 0 = 6 \\ S_{xx} &= \Sigma x_i^2 - n\bar{x}^2 = 6 - 0 = 6 \end{aligned}\right\} \Rightarrow \hat\beta_1 = \frac{S_{xy}}{S_{xx}} = \frac{6}{6} = 1$$

$$\hat\beta_0 = \bar{y} - \hat\beta_1 \bar{x} = 0 - 0 = 0$$

$$S_{yy} = \Sigma y_i^2 - n\bar{y}^2 = 16 - 0 = 16$$

$$R = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}} = \frac{6}{\sqrt{6 \cdot 16}} = \frac{3}{2\sqrt{6}} \Rightarrow R^2 = \frac{9}{4 \cdot 6} = \frac{3}{8}$$

$$s^2 = \frac{(1 - R^2) \cdot S_{yy}}{n-2} = \frac{\frac{5}{8} \cdot 16}{8} = \frac{5}{4}$$

observed $t = \dfrac{1-2}{\sqrt{\frac{5}{4}}/\sqrt{6}} = \dfrac{-1}{\sqrt{5/24}} = -\sqrt{\dfrac{24}{5}} \approx -2.191 < -2$

critical value $t^* = t(.05\,;8) = 1.86$.

So we reject H_0 since $t < -t^*$.

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3. (a) The MGF of q is

$$M_q(t) = \prod_{i=1}^{k} M_{q_i}(t) = \prod_{i=1}^{k} (1-2t)^{-\nu_i/2} = (1-2t)^{-\frac{1}{2}\sum_{i=1}^{k} \nu_i}.$$

This is the MGF of a $\chi^2\left(\sum_{i=1}^{k} \nu_i\right)$ distribution.

(b) Note that $V^{-1} = (T^T)^{-1} \Lambda^{-1} T^{-1} = T \Lambda^{-1} T^T$.

So $y^T V^{-1} y = y^T T \Lambda^{-1} T^T y = (y^T T \Lambda^{-1/2})(\Lambda^{-1/2} T^T y)$.

Next, $z = \Lambda^{-1/2} T^T y \sim N(0, I)$ since

$$\begin{aligned} \text{var}(z) = \Lambda^{-1/2} T^T \, V \, T \Lambda^{-1/2} &= \Lambda^{-1/2} T^T (T \Lambda T^T) T \Lambda^{-1/2} \\ &= \Lambda^{-1/2} (T^T T) \Lambda (T^T T) \Lambda^{-1/2} \\ &= \Lambda^{-1/2} (\Lambda^{1/2} \Lambda^{1/2}) \Lambda^{-1/2} = I. \end{aligned}$$

So $z_1, \ldots, z_m$ are iid $N(0,1)$.

$\Rightarrow z_1^2, \ldots, z_m^2$ are iid $\chi^2(1)$.

From part (a) $z^T z = z_1^2 + \ldots + z_m^2 \sim \chi^2(m)$

The result follows since $z^T z = y^T V^{-1} y$.

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4. $$\|X(\hat\beta - \beta)\|^2 < \underbrace{2 s^2 F(.05; 2, n-2)}_{\shortparallel}$$

$$2 \frac{n}{n-2} \hat\sigma^2 F(.05; 2, n-2) \overset{n=5}{=} 2 \cdot \frac{5}{3} \hat\sigma^2 \underbrace{F(.05;2,3)}_{} \quad \overset{9.55}{\shortparallel}$$

$$= \frac{95.5}{3} \hat\sigma^2$$

is a 95% confidence ellipse for β

$$\|X(\hat\beta - \beta)\|^2 = (\hat\beta - \beta)^T X^T X (\hat\beta - \beta)$$

$$= \begin{bmatrix} \hat\beta_0 - \beta_0 & \hat\beta_1 - \beta_1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} \hat\beta_0 - \beta_0 \\ \hat\beta_1 - \beta_1 \end{bmatrix}$$

$$= 5(\hat\beta_0 - \beta_0)^2 + 2(3)(\hat\beta_0 - \beta_0)(\hat\beta_1 - \beta_1) + 2(\hat\beta_1 - \beta_1)^2$$

So, the 95% confidence ellipse is

$$5(\hat\beta_0 - \beta_0)^2 + 6(\hat\beta_0 - \beta_0)(\hat\beta_1 - \beta_1) + 2(\hat\beta_1 - \beta_1)^2 \le \frac{95.5}{3} \hat\sigma^2$$

$$\Downarrow$$

$$\frac{15}{95.5}\left(\frac{\hat\beta_0 - \beta_0}{\hat\sigma}\right)^2 + \frac{18}{95.5}\left(\frac{\hat\beta_0 - \beta_0}{\hat\sigma}\right)\left(\frac{\hat\beta_1 - \beta_1}{\hat\sigma}\right) + \frac{6}{95.5}\left(\frac{\hat\beta_1 - \beta_1}{\hat\sigma}\right)^2 \le 1$$

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5. (a) $$(X^T X)^{-1} x_n = \frac{1}{S_{xx}} \begin{bmatrix} \Sigma x_i^2/n & -\bar x \\ -\bar x & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_n \end{bmatrix}$$

$$= \frac{1}{S_{xx}} \begin{bmatrix} \frac{\Sigma x_i^2}{n} - \bar x x_n \\ -\bar x + x_n \end{bmatrix} = \frac{1}{S_{xx}} \begin{bmatrix} \frac{1}{n}(\Sigma x_i^2 - n\bar x^2) + \bar x^2 - \bar x x_n \\ x_n - \bar x \end{bmatrix}$$

$$= \frac{1}{S_{xx}} \begin{bmatrix} \frac{S_{xx}}{n} - \bar{x}(x_n - \bar{x}) \\ x_n - \bar{x} \end{bmatrix} = \begin{bmatrix} \frac{1}{n} - \frac{\bar{x}(x_n-\bar{x})}{S_{xx}} \\ \frac{x_n - \bar{x}}{S_{xx}} \end{bmatrix}$$

(b) $h_{nn} = x_n^T (X^TX)^{-1} x_n$

$$= \begin{bmatrix} 1 & x_n \end{bmatrix} \begin{bmatrix} \frac{1}{n} - \frac{\bar{x}(x_n-\bar{x})}{S_{xx}} \\ \frac{x_n - \bar{x}}{S_{xx}} \end{bmatrix}$$

$$= \frac{1}{n} - \frac{\bar{x}(x_n - \bar{x})}{S_{xx}} + \frac{x_n(x_n - \bar{x})}{S_{xx}}$$

$$= \frac{1}{n} + \frac{(x_n - \bar{x})(x_n - \bar{x})}{S_{xx}} = \boxed{\frac{1}{n} + \frac{(x_n - \bar{x})^2}{S_{xx}}}$$

(c) $\hat{\beta}_{(n)} = \hat{\beta} - \frac{r_{nn}}{1 - h_{nn}} (X^TX)^{-1} x_n$

$$\frac{r_{nn}}{1 - h_{nn}} = \frac{y_n - \hat{\beta}_0 - \hat{\beta}_1 x_n}{1 - \frac{1}{n} - \frac{(x_n - \bar{x})^2}{S_{xx}}}$$

Let $a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so that $\hat{\beta}_{1,(n)} = a^T \hat{\beta}_{(n)}$.

Then $\hat{\beta}_{1,(n)} = \hat{\beta}_1 - \frac{y_n - \hat{\beta}_0 - \hat{\beta}_1 x_n}{1 - \frac{1}{n} - \frac{(x_n-\bar{x})^2}{S_{xx}}} \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \\ \frac{x_n - \bar{x}}{S_{xx}} \end{bmatrix}$

$$= \hat{\beta}_1 - \frac{(y_n - \hat{\beta}_0 - \hat{\beta}_1 x_n)}{\left(1 - \frac{1}{n} - \frac{(x_n - \bar{x})^2}{S_{xx}}\right)} \cdot \frac{(x_n - \bar{x})}{S_{xx}}$$

$$= \hat{\beta}_1 - \frac{(y_n - \hat{\beta}_0 - \hat{\beta}_1 x_n)(x_n - \bar{x})}{(1 - \frac{1}{n}) S_{xx} - (x_n - \bar{x})^2}$$