HW2 solutions

1. (a) $\ell(\sigma) = \ln \prod_{i=1}^{n} f(x_i|\sigma) = \sum_{i=1}^{n} \ln f(x_i|\sigma)$

$$= \sum_{i=1}^{n} \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2\sigma^2}(x_i - i)^2}\right)$$

$$= \sum_{i=1}^{n} \left\{ -\ln\sigma - \tfrac{1}{2}\ln(2\pi) - \frac{1}{2\sigma^2}(x_i - i)^2 \right\}$$

$$= -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - i)^2$$

$$\frac{d\ell}{d\sigma} = -\frac{n}{\sigma} + \frac{\cancel{2}}{\cancel{2}\sigma^3}\sum_{i=1}^{n}(x_i - i)^2$$

$$\frac{d\ell}{d\sigma} = 0 \iff \frac{n}{\hat\sigma} = \frac{1}{\hat\sigma^3}\sum_{i=1}^{n}(x_i - i)^2 \Rightarrow \hat\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - i)^2 \Rightarrow \hat\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - i)^2}$$

Then $\boxed{\hat\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - i)^2}}$ maximizes ℓ

since $\frac{d\ell}{d\sigma} = -\frac{n\sigma^2}{\sigma^3} + \frac{1}{\sigma^3}(n\hat\sigma^2) = -\frac{n}{\sigma^3}(\sigma^2 - \hat\sigma^2)$ is positive if $\sigma < \hat\sigma$, negative if $\sigma > \hat\sigma$.

(b) $Z_i = \frac{X_i - i}{\sigma} \sim N(0,1)$

$\Rightarrow Z_i^2 \sim \chi_1^2$ by Theorem L4.3(a)

Also, $Z_1^2, \ldots, Z_n^2$ are independent by Thm. L3.5

$\Rightarrow Z_1^2 + \cdots + Z_n^2 \sim \chi_n^2$ by Thm L4.3(b).

Since $\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n}\left(\frac{X_i - i}{\sigma}\right)^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - i)^2 = \frac{n\hat\sigma^2}{\sigma^2} = Q$,

we have that $\boxed{Q \sim \chi_n^2}$.

2. $E[S] = E[\sqrt{S^2}] = \frac{\sigma}{\sqrt{10-1}} E\left[\sqrt{\frac{(10-1)S^2}{\sigma^2}}\right]$

Since $Y = \frac{(10-1)S^2}{\sigma^2} \sim \chi^2_{\underbrace{10-1}_{9}}$, we need to compute $E[\sqrt{Y}]$.

$$E[\sqrt{Y}] = \int_0^\infty \sqrt{y} \frac{1}{\Gamma(\frac{9}{2}) 2^{9/2}} y^{\frac{9}{2}-1} e^{-\frac{y}{2}} dy$$

$$= \frac{1}{\Gamma(\frac{9}{2}) 2^{9/2}} \int_0^\infty y^4 e^{-\frac{y}{2}} dy$$

$$= \frac{\Gamma(5)\, 2^5}{\Gamma(\frac{9}{2}) 2^{9/2}} \int_0^\infty \frac{1}{\Gamma(5)\, 2^5} y^{5-1} e^{-\frac{y}{2}} dy$$

$\Gamma(5) = 4! = 24$

$\Gamma(\frac{9}{2}) = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \underbrace{\Gamma(\tfrac{1}{2})}_{=\sqrt{\pi}}$

$$= \frac{24\sqrt{2}}{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}} \cdot 1$$

$$= \frac{128\sqrt{2}}{35\sqrt{\pi}}.$$

So $E[S] = \frac{\sigma}{3} \cdot \frac{128\sqrt{2}}{35\sqrt{\pi}} = \boxed{\frac{128\sqrt{2}}{105\sqrt{\pi}} \sigma}$

3. $P\left(\sum_{i=1}^{9} X_i > 12 \text{ or } \sum_{i=1}^{9}(X_i-\bar{X})^2 > 64\right) = 1 - P\left(\sum_{i=1}^{9} X_i \le 12 \text{ and } \sum_{i=1}^{9}(X_i-\bar{X})^2 \le 64\right)$

$$= 1 - P\left(\sum_{i=1}^{9} X_i \le 12\right) P\left(\sum_{i=1}^{9}(X_i-\bar{X})^2 \le 64\right)$$

Since $\sum_{i=1}^{9} X_i \sim N(9\cdot 1 = 9, 9\cdot 16 = 144)$, $\dfrac{\sum_{i=1}^{9} X_i - 9}{12} \sim N(0,1)$ and

$$P\left(\sum_{i=1}^{9} X_i \le 12\right) = P\left(\frac{\sum_{i=1}^{9} X_i - 9}{12} \le 0.25\right) \approx 0.5987063.$$

from Normal table or pnorm(.25)

Next, $\dfrac{\sum_{i=1}^{9}(X_i-\bar{X})^2}{16} \sim \chi^2_8$ so

$$P\left(\sum_{i=1}^{9}(X_i-\bar{X})^2 \le 64\right) = P\left(\frac{\sum_{i=1}^{9}(X_i-\bar{X})^2}{16} \le 4\right) \approx 0.1428765.$$

from pchisq(4, df $= 8$)

or

$$\int_0^4 \frac{1}{\Gamma(4)2^4} y^3 e^{-y/2}\, dy = \frac{1}{96}\left[-(2y^3+12y^2+48y+96)e^{-y/2}\right]_0^4 = 1 - \frac{19}{3e^2}$$

So, we have

$$P\left(\sum_{i=1}^{9} X_i > 12 \text{ or } \sum_{i=1}^{9}(X_i-\bar{X})^2 > 64\right)$$

$$\approx 1 - (.5987063)(.1428765)$$

$$\approx \boxed{.9144589}$$