M621, HW 5, Solution, 09.27

1. About 1/2 to 2/3 of the class got problem 1...some confusion here. Therefore, it's done in some detail below.

   We begin with a a fixed element x of G. Its orbit under this particular action is called $\mathcal{O}$ with $\mathcal{O} = \{\langle \S : \langle \in \mathcal{H}\}$. We want to show that $|\mathcal{O}| = |\mathcal{H}|$. To do that we construct a bijection f from H to $\mathcal{O}$: For all $h \in H$, let $f(h) = hx$. Observe that f is a well-defined function—we'll show it's a bijection. Suppose $h_1, h_2 \in H$, and $f(h_1) = f(h_2)$. Then $h_1 x = f(h_1) = f(h_2) = h_2 x$. Since x, h_1, h_2 are all in the group G, we can use cancel on the right, and we have $h_1 = h_2$. Hence, f is one-to-one. Now let $y \in \mathcal{O}$. By definition of $\mathcal{O}$, there exists $h \in H$ such that $y = hx$. But then $f(h) = y$, so y is in the image of f, and f has been shown to be onto.

   By the paragraph above, each orbit of our action has the same number of elements, namely $|H|$. As proven in exercise 18 (which was done for HW 3), the orbits of an action partition the set A (the set on which the group acts—in this case, $A = G$).

   Each orbit has $|H|$ elements, so $|G| = k|H|$, where k is the number of orbits. Thus, $|H|$ divides $|G|$, which is Lagrange's Theorem.

   Comments. 1. It helps if you name your functions (such as "f" or "F" or "α", or whatever you like), and specify the domain and co-domain (e.g. $F : H \to \mathcal{O}$ or $f : H \to \mathcal{O}$ or..). Also, make sure your function makes sense, and is doing what you want it to do.

   2. "x" was a fixed but arbitrary element of G, and that seemed caused some people some problems. The authors could have called it "$\mathcal{O}_\S$", the orbit of x under the action—not that how they wrote it is any way wrong. This goes back to the first comment—make sure you're aimed in the right direction before you take off.

   3. If you didn't get all of it (or even if you did) , it's a very good problem to go over, get right, and understand well.

2. Problem 2, pg. 52 6(a) see https://crazyproject.wordpress.com/2010/01/24/every-subgroup-is-contained-in-its-normalizer/

3. Problem 3: $\mathbb{N}$ is a subset of $\mathbb{Z}$ that is closed under $+$, but $\mathbb{N}$ is not a subgroup of $(\mathbb{Z}, +)$.

4. Problem 4: See https://crazyproject.wordpress.com/2010/01/27/there-is-a-unique-group-homomorphism-from-zz-to-any-group-where-the-image-of-1-is-fixed/