Work for Example L5.2

Here is the partial fractions decomposition.

$$\frac{1}{(1+w^2)\left(1+\left(\frac{z-w}{m}\right)^2\right)} = \frac{Aw+B}{1+w^2} + \frac{C(z-w)+D}{1+\left(\frac{z-w}{m}\right)^2}$$

$$1 = (Aw+B)\left(1+\left(\frac{z-w}{m}\right)^2\right) + \left(C(z-w)+D\right)(1+w^2)$$

$$1 = (Aw+B)\left(\frac{1}{m^2}w^2 - \frac{2z}{m^2}w + 1 + \frac{z^2}{m^2}\right) + (-Cw+Cz+D)(1+w^2)$$

$$1 = \left(\frac{A}{m^2} - C\right)w^3 + \left(-\frac{2z}{m^2}A + \frac{1}{m^2}B + Cz + D\right)w^2 + \left(A\left(1+\frac{z^2}{m^2}\right) - \frac{2z}{m^2}B - C\right)w$$
$$+ B\left(1+\frac{z^2}{m^2}\right) + Cz + D$$

$$\frac{A}{m^2} - C = 0 \qquad ①$$

$$-\frac{2z}{m^2}A + \frac{1}{m^2}B + Cz + D = 0 \qquad ②$$

$$A\left(1+\frac{z^2}{m^2}\right) - \frac{2z}{m^2}B - C = 0 \qquad ③$$

$$B\left(1+\frac{z^2}{m^2}\right) + Cz + D = 1 \qquad ④$$

From ①, $C = \frac{A}{m^2}$. ⑤

Plugging ⑤ into ③, $A\left(1+\frac{z^2}{m^2} - \frac{1}{m^2}\right) = \frac{2z}{m^2}B$

$$A(m^2+z^2-1) = 2zB$$

$$A = \frac{2z}{m^2+z^2-1}B. \qquad ⑥$$

Plugging ⑥ into ⑤, $C = \frac{2z}{m^2(m^2+z^2-1)}B.$ ⑦

Plugging ⑥ and ⑦ into ②,

$$D = \left(\frac{2z}{m^2} \cdot \frac{2z}{m^2+z^2-1} - \frac{1}{m^2} - \frac{2z^2}{m^2(m^2+z^2-1)} \right) B$$

$$= \left(\frac{4z^2 - (m^2+z^2-1) - 2z^2}{m^2(m^2+z^2-1)} \right) B$$

$$= \frac{(z^2-m^2+1)B}{m^2(m^2+z^2-1)}. \quad ⑧$$

Plugging ⑦ and ⑧ into ④,

$$B\left(1 + \frac{z^2}{m^2} + \frac{2z^2}{m^2(m^2+z^2-1)} + \frac{z^2-m^2+1}{m^2(m^2+z^2-1)}\right) = 1$$

$$B = \frac{m^2(m^2+z^2-1)}{m^2(m^2+z^2-1) + z^2(m^2+z^2-1) + 2z^2 + z^2 - m^2 + 1}$$

$$B = \frac{m^2(m^2+z^2-1)}{z^4 + 2z^2m^2 + 2z^2 + m^4 - 2m^2 + 1} = \frac{m^2(m^2+z^2-1)}{z^4 + 2(m^2+1)z^2 + (m^2-1)^2}$$

$$= \frac{m^2(m^2+z^2-1)}{z^4 + \left[(m-1)^2 + (m+1)^2\right]z + \left[(m-1)(m+1)\right]^2}$$

$$= \frac{m^2(m^2+z^2-1)}{\left(z^2+(m-1)^2\right)\left(z^2+(m+1)^2\right)}$$

$$= \frac{m^2(m^2+z^2-1)}{a(z)}.$$

Then $A = \dfrac{2zm^2}{a(z)}$ from ⑥

$C = \dfrac{2z}{a(z)}$ from ⑤

$D = \dfrac{z^2-m^2+1}{a(z)}$ from ⑧.

Here is the computation of the integral on slide L6.13.

$$\int \left[\frac{2zm^2 w}{1+w^2} + \frac{m^2(m^2+z^2-1)}{1+w^2} + \frac{2z(z-w)}{1+\left(\frac{z-w}{m}\right)^2} + \frac{z^2-m^2+1}{1+\left(\frac{z-w}{m}\right)^2} \right] dw$$

$$\int \frac{2zm^2 w}{1+w^2}\,dw = zm^2\ln(1+w^2) + C_1$$

$$\int \frac{m^2(m^2+z^2-1)}{1+w^2}\,dw = m^2(m^2+z^2-1)\arctan w + C_2$$

$$\int \frac{2z(z-w)}{1+\left(\frac{z-w}{m}\right)^2}\,dw = \int \frac{-2zm^2u}{1+u^2}\,du = -zm^2\ln(1+u^2) + C_3 = -zm^2\ln\left(1+\left(\frac{z-w}{m}\right)^2\right) + C_3$$

Let $u = \frac{z-w}{m} \to z - w = mu$

$du = -\frac{1}{m}\,dw \to dw = -m\,du$

$$\int \frac{z^2-m^2+1}{1+\left(\frac{z-w}{m}\right)^2}\,dw = \int \frac{-m(z^2-m^2+1)}{1+u^2}\,du = -m(z^2-m^2+1)\arctan u + C_4$$
$$= -m(z^2-m^2+1)\arctan\left(\frac{z-w}{m}\right) + C_4$$

So, $$\int_0^N \left[\frac{2zm^2 w}{1+w^2} + \frac{m^2(m^2+z^2-1)}{1+w^2} + \frac{2z(z-w)}{\left(1+\left(\frac{z-w}{m}\right)^2\right)} + \frac{z^2-m^2+1}{1+\left(\frac{z-w}{m}\right)^2} \right] dw$$

$$= zm^2\ln(1+N^2) + m^2(m^2+z^2-1)\arctan N - zm^2\ln\left(1+\left(\frac{z-N}{m}\right)^2\right) - m(z^2-m^2+1)\arctan\left(\frac{z-N}{m}\right)$$
$$- 0 - 0 + zm^2\ln\left(1+\left(\frac{z}{m}\right)^2\right) + m(z^2-m^2+1)\arctan\left(\frac{z}{m}\right).$$

Thus $$\int_0^\infty [\quad]\,dw = zm^2\ln(m^2) + m^2(m^2+z^2-1)\frac{\pi}{2} - m(z^2-m^2+1)\left(-\frac{\pi}{2}\right)$$
$$+ zm^2\ln\left(1+\left(\frac{z}{m}\right)^2\right) + m(z^2-m^2+1)\arctan\left(\frac{z}{m}\right).$$

Similarly, $$\int_{-\infty}^0 [\quad]\,dw = -zm^2\ln\left(1+\left(\frac{z}{m}\right)^2\right) - m(z^2-m^2+1)\arctan\left(\frac{z}{m}\right)$$
$$- zm^2\ln(m^2) + m^2(m^2+z^2-1) - m(z^2-m^2+1)\left(-\frac{\pi}{2}\right).$$

So, $$\int_{-\infty}^\infty [\quad]\,dw = \pi\left[m^2(m^2+z^2-1) + m(z^2-m^2+1)\right]$$
$$= \pi m\left[m^3 + mz^2 - m + z^2 - m^2 + 1\right]$$
$$= \pi m\left[(m+1)(z^2+m^2-2m+1)\right]$$
$$= \pi m(m+1)\left(z^2+(m-1)^2\right).$$