Exam 3 Solutions

1. (a) Since $\dfrac{\sum_{i=1}^{10} X_i/10 - \mu}{\sigma/\sqrt{10}} \sim \text{Normal}(0,1)$ and

$\dfrac{\sum_{i=11}^{20} \left(X_i - \sum_{j=11}^{20} X_j/10\right)^2}{\sigma^2} \sim \chi^2_{10-1}$ are independent, we know that

$$\frac{\dfrac{\frac{1}{10}\sum_{i=1}^{10} X_i - \mu}{\sigma/\sqrt{10}}}{\sqrt{\dfrac{\sum_{i=11}^{20}\left(X_i - \frac{1}{10}\sum_{j=11}^{20} X_j\right)^2}{\sigma^2}\Big/9}} = \frac{\frac{1}{10}\sum_{i=1}^{10} X_i - \mu}{\sqrt{\frac{1}{10}\cdot\frac{1}{9}\sum_{i=11}^{20}\left(X_i - \frac{1}{10}\sum_{j=11}^{20} X_j\right)^2}} \sim t_9 .$$

(b) $$P\left(-t_{9,.025} \le \frac{\frac{1}{10}\sum_{i=1}^{10} X_i - \mu}{\sqrt{\frac{1}{10}\cdot\frac{1}{9}\sum_{i=11}^{20}\left(X_i - \frac{1}{10}\sum_{j=11}^{20} X_j\right)^2}} \le t_{9,.025}\right) = .95$$

$$\Rightarrow P\left(\frac{1}{10}\sum_{i=1}^{10} X_i - t_{9,.025}\sqrt{\frac{1}{10}\cdot\frac{1}{9}\sum_{i=11}^{20}\left(X_i - \frac{1}{10}\sum_{j=11}^{20} X_j\right)^2} \le \mu \le \frac{1}{10}\sum_{i=1}^{10} X_i + t_{9,.025}\sqrt{\frac{1}{10}\cdot\frac{1}{9}\sum_{i=11}^{20}\left(X_i - \frac{1}{10}\sum_{j=11}^{20} X_j\right)^2}\right) = .95$$

So $$\frac{1}{10}\sum_{i=1}^{10} x_i \pm t_{9,.025}\sqrt{\frac{1}{10}\cdot\frac{1}{9}\sum_{i=11}^{20}\left(x_i - \frac{1}{10}\sum_{j=11}^{20} x_j\right)^2}$$

$$\Rightarrow \frac{60}{10} \pm 2.262\sqrt{\frac{1}{10}\cdot 40}$$

$$\Rightarrow 6 \pm 2.262 \cdot 2$$

$$\Rightarrow 6 \pm 4.524$$

$$\Rightarrow (6 - 4.524,\ 6 + 4524)$$

$\Rightarrow \boxed{(1.476,\ 10.524)}$ is a 95% CI for μ.

2. (a) The joint pdf of $X_1, \dots, X_n$ is

$$f(\underline{x}|\theta) = \frac{1}{(2\pi)^{n/2}\theta^n} e^{-\frac{1}{2\theta^2}\sum_{i=1}^n (x_i-\theta)^2} = \frac{1}{(2\pi)^{n/2}\theta^n} e^{-\frac{1}{2}\sum_{i=1}^n \left(\frac{x_i}{\theta}-1\right)^2}$$

$$= \frac{1}{(2\pi)^{n/2}\theta^n} e^{-\frac{1}{2}\left[\frac{1}{\theta^2}\sum x_i^2 - \frac{2}{\theta}\sum x_i + n\right]}$$

$$= \frac{1}{(2\pi)^{n/2}e^{n/2}} \cdot \frac{1}{\theta^n} e^{-\frac{1}{2\theta^2}\sum_{i=1}^n x_i^2 + \frac{1}{\theta}\sum_{i=1}^n x_i} = g\left(\sum_{i=1}^n x_i^2, \sum_{i=1}^n x_i \,\Big|\, \theta\right) h(\underline{x})$$

where $g(t_1, t_2|\theta) = \frac{1}{\theta^n} e^{-\frac{1}{2\theta^2}t_1 + \frac{1}{\theta}t_2}$ and $h(\underline{x}) = \frac{1}{(2\pi e)^{n/2}}$.

So, $\boxed{\left(\sum_{i=1}^n X_i^2, \sum_{i=1}^n X_i\right)}$ is sufficient for θ by the Factorization Theorem.

(b) The method of moments estimator is the solution to $m_1 = \mu_1'(\theta)$ which is $\boxed{\tilde{\theta} = \bar{X}}$.

(c) The log-likelihood function for θ is

$$\ell(\theta|\underline{x}) = -\frac{n}{2}\log 2\pi - \frac{n}{2} - n\log\theta - \frac{1}{2\theta^2}\sum x_i^2 + \frac{1}{\theta}\sum x_i .$$

Its derivative is

$$\frac{\partial \ell}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^3}\sum x_i^2 - \frac{1}{\theta^2}\sum x_i = -\frac{n}{\theta^3}\left[\theta^2 + \bar{x}\theta - \frac{\sum x_i^2}{n}\right].$$

Setting $\frac{\partial \ell}{\partial \theta} = 0$ and solving for θ, we obtain

$$\theta^2 + \bar{x}\theta - \frac{\sum x_i^2}{n} = 0 \implies \theta = \frac{-\bar{x} + \sqrt{\bar{x}^2 + 4\frac{\sum x_i^2}{n}}}{2}$$

$$= -\frac{\bar{x}}{2} + \sqrt{\frac{\bar{x}^2}{4} + \frac{\sum x_i^2}{n}}$$

This is not necessary for the exam, but to show θ

$\hat{\theta} = \frac{-\bar{x}}{2} + \sqrt{\frac{\bar{x}^2}{4} + \frac{\sum x_i^2}{n}}$ maximizes ℓ, we show that

$\left.\frac{\partial^2 \ell}{\partial \theta^2}\right|_{\theta=\hat{\theta}} < 0$. Now, we have

$$\frac{\partial^2 \ell}{\partial \theta^2} = \frac{n}{\theta^2} + \frac{2}{\theta^3}\Sigma x_i - \frac{3}{\theta^4}\Sigma x_i^2 = \frac{n}{\theta^4}\left[\theta^2 + 2\bar{x}\theta - 3\frac{\Sigma x_i^2}{n}\right].$$

Evaluating $\frac{\partial^2 \ell}{\partial \theta^2}$ at $\theta = \hat{\theta}$, we obtain

$$\left.\frac{\partial^2 \ell}{\partial \theta^2}\right|_{\theta=\hat{\theta}} = \frac{n}{\hat{\theta}^4}\left[\hat{\theta}^2 + 2\bar{x}\hat{\theta} - \frac{3\Sigma x_i^2}{n}\right]$$

$$= \frac{n}{\hat{\theta}^4}\left[\underbrace{\left(\hat{\theta}^2 + \bar{x}\hat{\theta} - \frac{\Sigma x_i^2}{n}\right)}_{=0} + \bar{x}\hat{\theta} - \frac{2\Sigma x_i^2}{n}\right]$$

$$= \frac{n}{\hat{\theta}^4}\left[\bar{x}\hat{\theta} - 2\frac{\Sigma x_i^2}{n}\right]$$

$$= \frac{n}{\hat{\theta}^4}\left[\bar{x}\left(\frac{-\bar{x}}{2} + \sqrt{\frac{\bar{x}^2}{4} + \frac{\Sigma x_i^2}{n}}\right) - 2\frac{\Sigma x_i^2}{n}\right]$$

$$= \frac{n}{\hat{\theta}^4}\left[\bar{x}\sqrt{\frac{\bar{x}^2}{4} + \frac{\Sigma x_i^2}{n}} - \frac{\bar{x}^2}{2} - 2\frac{\Sigma x_i^2}{n}\right]$$

$$= \frac{n}{\hat{\theta}^4}\left[\bar{x}\sqrt{\frac{\bar{x}^2}{4} + \frac{\Sigma x_i^2}{n}} - 2\left(\frac{\bar{x}^2}{4} + \frac{\Sigma x_i^2}{n}\right)\right]$$

$$= \frac{n}{\hat{\theta}^4}\sqrt{\frac{\bar{x}^2}{4} + \frac{\Sigma x_i^2}{n}}\ \underbrace{\left(\bar{x} - 2\sqrt{\frac{\bar{x}^2}{4} + \frac{\Sigma x_i^2}{n}}\right)}_{= -2\hat{\theta}}$$

$$= \frac{-2n}{\hat{\theta}^3}\sqrt{\frac{\bar{x}^2}{4} + \frac{\Sigma x_i^2}{n}} < 0$$

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3. (a) $E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)\frac{1}{x!}\lambda^x e^{-\lambda} = \sum_{x=2}^{\infty} \frac{x(x-1)}{x!}\lambda^x e^{-\lambda}$

$$= \lambda^2 \sum_{x=2}^{\infty} \frac{1}{(x-2)!}\lambda^{x-2} e^{-\lambda} = \lambda^2 \sum_{y=0}^{\infty} \frac{1}{y!}\lambda^y e^{-\lambda}$$

$$= \lambda^2(1) = \boxed{\lambda^2}.$$

(b) $E[X] = \sum_{x=0}^{\infty} x \frac{1}{x!}\lambda^x e^{-\lambda} = \sum_{x=1}^{\infty} \frac{1}{(x-1)!}\lambda^x e^{-\lambda} = \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} = \lambda$

So $E[\bar{X}] = E\left[\frac{1}{n}\sum X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n}\lambda = \frac{n\lambda}{n} = \lambda.$

(c) $Var[\bar{X}] = Var\left[\frac{1}{n}\sum X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} Var\, X_i = \frac{1}{n^2}\sum_{i=1}^{n}\lambda = \frac{n\lambda}{n^2} = \boxed{\frac{\lambda}{n}}$

since $Var\, X = E[X^2] - (EX)^2 = E[X^2 - X] + EX - (EX)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$

(d) If $W(X)$ is an unbiased estimator of λ, then the numerator of the Cramér-Rao Lower Bound (CRLB) is $\left(\frac{d}{d\lambda}E[W(X)]\right)^2 = \left(\frac{d}{d\lambda}\lambda\right)^2 = 1^2 = 1$. Since $X_1, \dots, X_n$ is iid, the denominator of the CRLB is $n\,E\left[\left(\frac{\partial}{\partial\lambda}\log f(X|\theta)\right)^2\right]$.

We have $\frac{\partial}{\partial\lambda}\log f(x|\lambda) = \frac{\partial}{\partial\lambda}\left[-\log x! + x\log\lambda - \lambda\right] = \frac{x}{\lambda} - 1.$

So $E\left[\left(\frac{X}{\lambda}-1\right)^2\right] = E\left[\frac{1}{\lambda^2}X^2 - \frac{2}{\lambda}X + 1\right] = \frac{1}{\lambda^2}(\lambda^2+\lambda) - \frac{2}{\lambda}\cdot\lambda + 1 = 1 + \frac{1}{\lambda} - 2 + 1 = \frac{1}{\lambda}.$

So the CRLB is $\frac{1}{n\cdot\frac{1}{\lambda}} = \boxed{\frac{\lambda}{n}}.$

4. (a) When H_0 is true, $\theta = 1$ so $f(x|\theta) = \frac{1}{5} I_{\{-2,-1,0,1,2\}}(x)$

and $P_{\theta=1}(X \geq 1) = P_{\theta=1}(X=1) + P_{\theta=2}(X=1) = \boxed{\frac{2}{5}}$.

(b) The probability of a Type II error is

$$P_{\theta=2}(X<1) = 1 - P_{\theta=2}(X \geq 1)$$

$$= 1 - P_{\theta=2}(X=1) - P_{\theta=2}(X=2)$$

$$= 1 - \frac{1}{5} \cdot \frac{25}{61} - \frac{1}{5} \cdot \frac{125}{61} = 1 - \frac{5+25}{61} = \boxed{\frac{31}{61}}$$

(c) No, this is not a UMP level $\alpha = .4$ test.

The following table gives the ratios $\frac{f(x|\theta=2)}{f(x|\theta=1)}$ for each x

x	-2	-1	0	1	2
$\frac{f(x\|\theta=2)}{f(x\|\theta=1)}$	$\frac{125}{61}$	$\frac{25}{61}$	$\frac{1}{61}$	$\frac{25}{61}$	$\frac{125}{61}$

.

By the Neyman-Pearson Lemma, the UMP level $\alpha = .4$ test has the form $x \in R$ when $\frac{f(x|\theta=2)}{f(x|\theta=1)} > c$ and $x \in R^c$ when $\frac{f(x|\theta=2)}{f(x|\theta=1)} < c$ if it satisfies $\alpha = P_{\theta=1}(X \in R)$.

If $\alpha = .4$, then $R = \{-2, 2\}$ and $c \in \left(\frac{25}{61}, \frac{125}{61}\right)$.

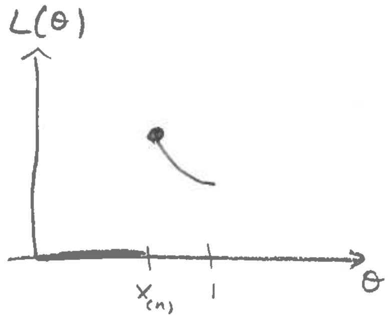
(For the UMP test, the probability of a Type I error is

$P_{\theta=2}(X \in \{-1,0,1\}) = \frac{11}{61}$.)

5. (a) The likelihood function for θ is

$$L(\theta|\underline{x}) = \prod_{i=1}^{n} f(x_i|\theta) = \frac{2^n \prod_{i=1}^{n} x_i}{\theta^{2n}} \prod_{i=1}^{n} I_{[0,\theta]}(x_i)$$

$$= \frac{2^n \prod_{i=1}^{n} x_i}{\theta^{2n}} \cdot I_{[x_{(n)},\infty)}(\theta)$$



$L(\theta|\underline{x})$ is maximized at $\hat{\theta} = X_{(n)}$.

So the likelihood ratio test statistic is

$$\lambda(\underline{x}) = \frac{L(1|\underline{x})}{L(\hat{\theta}|\underline{x})} = \frac{2^n \prod_{i=1}^{n} x_i}{\frac{2^n \prod_{i=1}^{n} x_i}{x_{(n)}^{2n}}} = X_{(n)}^{2n}.$$

which is an increasing function of $X_{(n)}$.

So, $\lambda(\underline{x})$ is small if and only if $X_{(n)}$ is small, and thus, the critical region has the form $\{\underline{x} : \max x_i \le K\}$.

(b) $P_\theta(X_{(n)} \le K) = P_\theta(X_1 \le K, \dots, X_n \le K) = (P_\theta(X_1 \le K))^n = \left(\frac{K^2}{\theta^2}\right)^n$

$\Rightarrow \left(\frac{K}{\theta}\right)^{2n} = .01 \Rightarrow \frac{K}{\theta} = .1^{1/n} \Rightarrow K = \boxed{\theta \sqrt[n]{.1}}$

(c) $P_\theta(X_{(n)} > \theta \sqrt[n]{.1}) = .99$

$\Rightarrow P_\theta(X_{(n)} \sqrt[n]{10} > \theta) = .99$

So $\boxed{[0, X_{(n)} \sqrt[n]{10})}$ is a 99% CI for θ.