M622 HW 2 due Jan 26

1. If R is an integral domain with a, b, c non-0 elements and $a = bc$, then $(a) = (b)$ if and only if c is a unit of R.

2. If R is an integral domain with a, b, c non-0 elements and $a = bc$ with neither b nor c are units, then (a) is properly contained in (b). (Use exercise1—very short proof.)

3. (Don't turn in—Instead YoYo and Christen will present.) Using exercises if 1 and 2, and the definition of a maximal ideal, show that R is a PID, and b is irreducible in R, then (b) is a maximal ideal of R.

4. If R is a commutative ring with 1 with ideals A and B, with $AB = \{a_1b_1 + \ldots a_kb_k : \{a_1, \ldots, a_k\} \subseteq A,\ \{b_1, \ldots, b_k\} \subseteq B,\ k \in \mathbb{N}\}$, finite sums of instances of ab, where $a \in A, b \in B$. Show AB is an ideal.

5. Let R be a commutative ring with 1 with ideals I and J satisfying $I+J = R$.

   (a) Assume that $I \cap J = \{0\}$. Prove that $R \cong (I \times J)$. (Suggestion: Show that every element $r \in R$ can be uniquely represented $r = i + j$, for some $i \in I, j \in J$, and define a map $\Gamma : R \to I \times J$, and show Γ is an isomorphism of rings.)

   (b) You will continue to assume that $I + J = R$, but no longer assume that $I \cap J = \{0\}$. Use the first part to show that $R/(I \cap J) \cong (R/I) \times (R/J)$. (Suggestion: Work in $R/(I \cap J)$ with two appropriate ideals of $R/(I \cap J)$, and use (a) above.

6. Number 6 (a), (b), page 293. Everyone will turn in 6(a) and 6(b). For 6(c)—don't turn in: Instead Adam and Israel will present—I suggest to theythem that use 5(a), properties that guarantee factorizations of rings into direct products.