Lecture 5: Location-Scale Families

MATH 667-01
Statistical Inference
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Introduction

- We define and prove a few general properties of location and scale families of distributions described in Section 3.5 of Casella and Berger (2001)[1].
- We will also prove a result discussed in Section 5.2 for the sampling distribution of $\bar{X}$ for a random sample from a location-scale family.

[1] Casella, G. and Berger, R. (2001). Statistical Inference, second edition. Duxbury Press.

- *Theorem L5.1* (Thm 3.5.1 on p.116): Let $f(x)$ be any pdf and let μ and $\sigma > 0$ be any given constants. Then the function

$$g(x|\mu, \sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right).$$

  is a pdf.

- *Proof of Theorem L5.1*: A function is a pdf if it is nonnegative and if it integrates to 1.
  1. Clearly, $g(x|\mu, \sigma) \geq 0$ since f is nonnegative and $\sigma > 0$.
  2. Letting $z = \dfrac{x-\mu}{\sigma}$ so that $dz = \dfrac{dx}{\sigma} \Rightarrow dx = \sigma dz$. Then

$$\int g(x|\mu, \sigma)\ dx = \int \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)\ dz = \int \frac{1}{\sigma} f(z) \sigma dz = \int f(z) dz = 1.$$

Location and Scale Families

- *Definition L5.1* (Def 3.5.2 on p.116): Let $f(x)$ be any pdf. Then the family of pdfs $f(x-\mu)$, indexed by the parameter μ, $-\infty < \mu < \infty$, is called the *location family with standard pdf* $f(x)$ and μ is called the *location parameter* for the family.
- *Definition L5.2* (Def 3.5.4 on p.119): Let $f(x)$ be any pdf. Then for any $\sigma > 0$, the family of pdfs $(1/\sigma)f(x/\sigma)$, indexed by the parameter σ, is called the *scale family with standard pdf* $f(x)$ and σ is called the *scale parameter* of the family.
- *Definition L5.3* (Def 3.5.5 on p.119): Let $f(x)$ be any pdf. Then for any μ, $-\infty < \mu < \infty$, and any $\sigma > 0$, the family of pdfs $(1/\sigma)f((x-\mu)/\sigma)$, indexed by the parameter (μ, σ), is called the *location-scale family with standard pdf* $f(x)$; μ is called the *location parameter* and σ is called the *scale parameter*.

- Let $Z \sim$ Normal$(0, 1)$; its pdf is $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$. Then the Normal$(\mu, \sigma)$ family of pdfs

$$f(x|\mu, \sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

  is a location-scale family with standard pdf $f(z)$.

- An exponential family of pdfs with scale parameter β (see p.101) has the form

$$f(x|\beta) = \frac{1}{\beta} e^{-x/\beta} I_{(0,\infty)}(x).$$

  It is a scale family with standard pdf $f(z) = e^{-z} I_{(0,\infty)}(z)$.

- *Theorem L5.2* (Thm 3.5.6 on p.120): Let $f(\cdot)$ be any pdf. Let μ be any real number and let σ be any positive real number. Then X is a random variable with pdf $(1/\sigma)f((x-\mu)/\sigma)$ if and only if there exists a random variable Z with pdf $f(z)$ and $X = \sigma Z + \mu$.
- *Proof of Theorem L5.2*:

  Suppose there exists a random variable Z with pdf $f(z)$ and $X = \sigma Z + \mu$. Let $g(z) = \sigma z + \mu$. Then $g^{-1}(x) = \frac{x-\mu}{\sigma}$ and the pdf of X is

  $$f_X(x) = f_Z(g^{-1}(x)) \left| \frac{d}{dx} \left[g^{-1}(x) \right] \right| \quad = f\left(\frac{x-\mu}{\sigma} \right) \frac{1}{\sigma}.$$

  Conversely, suppose X has pdf $f((x-\mu)/\sigma)/\sigma$. Let $g(x) = \frac{x-\mu}{\sigma}$ and $Z = g(X)$ so that $X = \sigma Z + \mu$. Then $g^{-1}(z) = \sigma z + \mu$ and the pdf of Z is

$$f_Z(z) = f_X(g^{-1}(z)) \left| \frac{d}{dz} \left[g^{-1}(z) \right] \right| \quad = f\left(\frac{(\sigma z + \mu) - \mu}{\sigma} \right) \frac{1}{\sigma} \sigma \ = f(z).$$

- *Theorem L5.3* (Thm 3.5.7 on p.121): Let Z be a random variable with pdf $f(z)$. Suppose $\mathsf{E}[Z]$ and $\mathsf{Var}[Z]$ exist. If X is a random variable with pdf $(1/\sigma)f((x-\mu)/\sigma)$, then

$$\mathsf{E}[X] = \sigma\mathsf{E}[Z] + \mu \text{ and } \mathsf{Var}[X] = \sigma^2\mathsf{Var}[Z].$$

- *Proof of Theorem L5.3*:
  By *Theorem L5.2*, there exists a random variable Z^* with pdf $f(z)$ and $X = \sigma Z^* + \mu$.
  By *Theorem L3.1(a)*, $\mathsf{E}[X] = \sigma\mathsf{E}[Z^*] + \mu \; = \sigma\mathsf{E}[Z] + \mu$.
  By *Theorem L3.2*, $\mathsf{Var}[X] = \sigma^2\mathsf{Var}[Z^*] \; = \sigma^2\mathsf{Var}[Z]$.

- *Example L5.1*: Suppose X is a random variable with pdf

$$f(x|\alpha,\beta,\mu) = \frac{1}{\Gamma(\alpha)\beta}\left(\frac{x-\mu}{\beta}\right)^{\alpha-1} e^{-(x-\mu)/\beta} I_{(\mu,\infty)}.$$

(a) If $\mu = 0$ and $\beta = 1$, find $\mathsf{E}[X]$ and $\mathsf{Var}[X]$.
(b) Find $\mathsf{E}[X]$ and $\mathsf{Var}[X]$.

- *Answer to Example L5.1*: (a) For any positive integer n, we have

$$\begin{aligned} \mathsf{E}[X^n] &= \int_0^\infty x^n f(x|\alpha)\, dx \\ &= \int_0^\infty x^n \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}\, dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+n-1} e^{-x}\, dx \end{aligned}$$

- *Answer to Example L5.1 continued*:

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+n-1} e^{-x} \, dx$$
$$= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \frac{(\alpha+n-1)\cdots\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = (\alpha+n-1)\cdots\alpha.$$

So, $\mathsf{E}[X] = \alpha$ and
$\mathsf{Var}[X] = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = (\alpha+1)\alpha - \alpha^2 = \alpha.$
(b) Applying *Theorem L5.3* to the answer for part (a), we have

$$\mathsf{E}[X] = \beta\alpha + \mu$$
$$\mathsf{Var}[X] = \beta^2\alpha.$$

Sampling Distribution of $\bar{X}$ for Location-Scale Families

- (p.216): Suppose that $X_1, \ldots, X_n$ is a random sample from a distribution with pdf $\dfrac{1}{\sigma} f\left(\dfrac{x-\mu}{\sigma}\right)$.
- Then there exist independent random variables $Z_1, \ldots, Z_n$ where $X_i = \sigma Z_i + \mu$ and the pdf of Z_i is $f(z)$.
- So $Z_1, \ldots, Z_n$ is a random sample from a distribution with pdf $f(z)$.
- $$\begin{aligned} \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}\sum_{i=1}^{n}(\sigma Z_i + \mu) \ &= \frac{\sigma}{n}\sum_{i=1}^{n} Z_i + \frac{1}{n} n\mu \\ &= \sigma \bar{Z} + \mu \end{aligned}$$
- If $g(z)$ is the pdf of $\bar{Z}$, then $\dfrac{1}{\sigma} g\left((x-\mu)/\sigma\right)$ is the pdf of $\bar{X}$.

- *Theorem L5.4* (Thm 5.2.9 on p.215): If X and Y are independent continuous random variables with pdfs $f_X(x)$ and $f_Y(y)$, then the pdf of $Z = X + Y$ is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z - w)\ dw.$$

- *Proof of Theorem L5.4*: Let $W = X$ and consider the transformation from (X, Y) to (Z, W). The inverse of this transformation is $X = W$ and $Y = Z - W$.

  The Jacobian is $J = \left|\frac{\partial(x,y)}{\partial(w,z)}\right| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$

  The joint density of (Z, W) is
  $f_{Z,W}(z, w) = f_{X,Y}(w, z - w)|J| = f_X(w) f_Y(z - w).$
  Integrating out w, the pdf of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z - w)\ dw.$$

- *Example L5.2*: A random variable is said to have a Cauchy(μ, σ) pdf if its pdf has the form

$$f(x|\mu,\sigma) = \frac{1}{\sigma\pi\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)}.$$

(a) If $X \sim$ Cauchy$(0,1)$ and $Y \sim$ Cauchy$(0,m)$ are independent random variables, then use the partial fractions decomposition

$$\frac{1}{(1+w^2)\left(1+(\frac{z-w}{m})^2\right)} = \frac{1}{a(z)}\left(\frac{b(w,z)}{1+w^2} + \frac{c(w,z)}{1+(\frac{z-w}{m})^2}\right)$$

where $a(z) = \left(z^2+(m+1)^2\right)\left(z^2+(m-1)^2\right)$,
$b(w,z) = m^2(2zw+m^2+z^2-1)$, and
$c(w,z) = 2z(z-w)+z^2-m^2+1$ to show that
$X+Y \sim$ Cauchy$(0,m+1)$.

- *Example L5.2 continued:*
  (b) If $Z_1, \ldots, Z_n$ are independent Cauchy$(0,1)$ random variables, then show that $\bar{Z} \sim$ Cauchy$(0,1)$.
  (c) If $X_1, \ldots, X_n$ are independent Cauchy(μ, σ) random variables, find the pdf of $\bar{X}$.
- *Answer to Example L5.2*:
  (a) The pdf of $Z = X + Y$ is

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)\, dw \\
&= \frac{1}{m\pi a(z)} \int_{-\infty}^{\infty} \left(\frac{b(w,z)}{1+w^2} + \frac{c(w,z)}{1+(\frac{z-w}{m})^2} \right)\, dw \\
&= \frac{1}{m\pi^2 a(z)} \pi(m^2+m)(z^2+(m-1)^2) \\
&= \frac{m+1}{\pi((m+1)^2+z^2)} = \frac{1}{(m+1)\pi \left(1 + \left(\frac{z}{m+1}\right)^2\right)}.
\end{aligned}
$$

Sampling Distribution of $\bar{X}$ for Location-Scale Families

- *Answer to Example L5.2 continued*:
  (b) It can be shown that $\sum_{i=1}^{n} Z_i \sim \text{Cauchy}(0, n)$ by induction.
  - *Basis step*: $Z_1 \sim \text{Cauchy}(0, 1)$
  - *Inductive step*: Assume $\sum_{i=1}^{k} Z_i \sim \text{Cauchy}(0, k)$. Then, $\sum_{i=1}^{k+1} Z_i = \sum_{i=1}^{k} Z_i + Z_{k+1} \sim \text{Cauchy}(0, k+1)$ by part (a).

  Since $\bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$, Theorem L5.2 implies that $\bar{Z}$ has pdf

$$\frac{1}{1/n} f\left(\frac{z}{1/n}\right) = nf(nz) \ = n\frac{1}{n\pi\left(1 + (nz/n)^2\right)} \ = \frac{1}{\pi\left(1 + z^2\right)}.$$

- (c) Since $g(z) = \dfrac{1}{\pi(1+z^2)}$ is the pdf of $\bar{Z}$, the work on slide 5.10 implies that the pdf of $\bar{X}$ is

$$\frac{1}{\sigma} g\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\pi\left(1 + (\frac{x-\mu}{\sigma})^2\right)}$$

  so $\bar{X} \sim \text{Cauchy}(\mu, \sigma)$.