

# Lecture 9: UMVUEs and the Cramér-Rao Lower Bound

MATH 667-01  
Statistical Inference  
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- We discuss uniform minimum variance unbiased estimators as discussed in Section 7.3 of Casella and Berger (2002)<sup>1</sup>.
- We review correlation from Section 4.5.
- We discuss and prove the Cramér-Rao Inequality and some corollaries. The regularity conditions in these notes are from Section 7.3 of Casella and Berger (1990)<sup>2</sup>.
- We present several examples to illustrate the results.

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<sup>1</sup>Casella, G. and Berger, R. (2002). *Statistical Inference, Second Edition*. Duxbury Press, Belmont, CA.

<sup>2</sup>Casella, G. and Berger, R. (1990). *Statistical Inference*. Duxbury Press, Belmont, CA.

# Best Unbiased Estimator (UMVUE)

- In this lecture, we evaluate an estimator  $W$  of a parameter  $\theta$  based on the squared error loss function.
- If we consider only unbiased estimators, then  $E_{\theta}[(W - \theta)^2] = \text{Var}_{\theta}[W]$ .
- *Definition L9.1* (Def 7.3.7 on p.334): An estimator  $W^*$  is a *best unbiased estimator* of  $\tau(\theta)$  if it satisfies  $E_{\theta}[W^*] = \tau(\theta)$  for all  $\theta$  and, for any other unbiased estimator  $W$  with  $E_{\theta}[W] = \tau(\theta)$ , we have  $\text{Var}_{\theta}[W^*] \leq \text{Var}_{\theta}[W]$  for all  $\theta$ .
- $W^*$  is also called a *uniform minimum variance unbiased estimator* (UMVUE) of  $\tau(\theta)$ .

# Best Unbiased Estimator (UMVUE)

- *Example L9.1:* Let  $X_1, \dots, X_n$  be iid  $\text{Poisson}(\lambda)$ . Both  $\bar{X}$  and  $S^2$  are unbiased estimators of  $\lambda$  since  $E[X_1] = \text{Var}[X_1] = \lambda$  so that  $E[\bar{X}] = E[S^2] = \lambda$ . For what values of  $\lambda$  is the variance of  $\bar{X}$  smaller than the variance of  $S^2$ ?
- *Answer to Example L9.1:* We know  $\text{Var}[\bar{X}] \stackrel{3.19}{=} \frac{\text{Var}[X_1]}{n} = \frac{\lambda}{n}$ . It can be shown that

$$\begin{aligned}\text{Var}[S^2] &= \frac{1}{n} \left[ \lambda(1 + 3\lambda) - \frac{n-3}{n-1} \lambda^2 \right] \\ &= \frac{1}{n} \left[ \lambda + \frac{2n}{n-1} \lambda^2 \right]\end{aligned}$$

so  $\text{Var}[\bar{X}] < \text{Var}[S^2]$  for all  $\lambda$ .

- $E[X] = \mu_X$ ,  $E[Y] = \mu_Y$ ,  $\text{Var}[X] = \sigma_X^2$ ,  $\text{Var}[Y] = \sigma_Y^2$
- Assume  $0 < \sigma_X^2 < \infty$  and  $0 < \sigma_Y^2 < \infty$
- *Definition L9.2* (Def 4.5.2 on p.169): The *correlation of  $X$  and  $Y$*  is the number defined by

$$\rho_{XY} = \frac{\text{Cov}[X, Y]}{\sigma_X \sigma_Y}.$$

The value  $\rho_{XY}$  is also called the *correlation coefficient*.

- *Theorem L9.1* (Thm 4.5.7 on p.172): For any random variables  $X$  and  $Y$ ,
  - (a)  $-1 \leq \rho_{XY} \leq 1$ .
  - (b)  $|\rho_{XY}| = 1$  if and only if there exists numbers  $a \neq 0$  and  $b$  such that  $P(Y = aX + b) = 1$ . If  $\rho_{XY} = 1$  then  $a > 0$ , and if  $\rho_{XY} = -1$  then  $a < 0$ .

# Cramér-Rao Lower Bound

- *Theorem L9.2* (p.335): Let  $X_1, \dots, X_n$  be a sample with pdf  $f(\mathbf{x}|\theta)$ , and let  $W(\mathbf{X}) = W(X_1, \dots, X_n)$  be any estimator where  $E_\theta[W(\mathbf{X})]$  is a differentiable function of  $\theta$ . Suppose the joint pdf  $f(\mathbf{x}|\theta)$  satisfies

$$\frac{d}{d\theta} \int \cdots \int h(\mathbf{x}) f(\mathbf{x}|\theta) d\mathbf{x} = \int \cdots \int h(\mathbf{x}) \frac{\partial}{\partial \theta} f(\mathbf{x}|\theta) d\mathbf{x},$$

for any function  $h(\mathbf{x})$  with  $E_\theta[|h(\mathbf{X})|] < \infty$ . Then

$$\text{Var}_\theta[W(\mathbf{X})] \geq \frac{\left\{ \frac{d}{d\theta} E_\theta[W(\mathbf{X})] \right\}^2}{E_\theta \left[ \left( \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right)^2 \right]}.$$

- The inequality is referred to as the Cramér-Rao inequality.
- If  $W(\mathbf{X})$  is an unbiased estimator of  $\tau(\theta)$ , then the numerator becomes

$$\left( \frac{d}{d\theta} E_\theta[W(\mathbf{X})] \right)^2 = (\tau'(\theta))^2.$$

- *Proof of Theorem L9.2:* Since *Theorem L9.1(a)* implies

$$\left\{ \text{Cov} \left[ W(\mathbf{X}), \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right] \right\}^2 \leq \text{Var}[W(\mathbf{X})] \text{Var} \left[ \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right],$$

it follows that

$$\text{Var}[W(\mathbf{X})] \geq \frac{\left\{ \text{Cov} \left[ W(\mathbf{X}), \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right] \right\}^2}{\text{Var} \left[ \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right]}.$$

- *Proof of Theorem L9.2 continued:* Note that

$$\begin{aligned} \mathbb{E} \left[ \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right] &= \int_{\mathcal{X}} \frac{\partial}{\partial \theta} \ln f(\mathbf{x}|\theta) f(\mathbf{x}|\theta) d\mathbf{x} \\ &= \int_{\mathcal{X}} \frac{\frac{\partial}{\partial \theta} f(\mathbf{x}|\theta)}{f(\mathbf{x}|\theta)} f(\mathbf{x}|\theta) d\mathbf{x} \\ &= \int_{\mathcal{X}} \frac{\partial}{\partial \theta} f(\mathbf{x}|\theta) d\mathbf{x} \\ &= \frac{d}{d\theta} \int_{\mathcal{X}} f(\mathbf{x}|\theta) d\mathbf{x} = \frac{d}{d\theta} 1 = 0. \end{aligned}$$



- *Proof of Theorem L9.2 continued:* Then we have

$$\begin{aligned}\text{Cov} \left[ W(\mathbf{X}), \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right] &= \mathbb{E} \left[ W(\mathbf{X}) \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right] \\ &= \mathbb{E} \left[ W(\mathbf{X}) \frac{\frac{\partial}{\partial \theta} f(\mathbf{X}|\theta)}{f(\mathbf{X}|\theta)} \right] \\ &= \int_{\mathcal{X}} \frac{d}{d\theta} [W(\mathbf{x}) f(\mathbf{x}|\theta)] d\mathbf{x} \\ &= \frac{d}{d\theta} \mathbb{E}[W(\mathbf{X})]\end{aligned}$$

and

$$\text{Var} \left[ \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right] = \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right)^2 \right].$$

# Cramér-Rao Lower Bound (iid case)

- *Theorem L9.3* (p.337): Let  $X_1, \dots, X_n$  be iid with pdf  $f(x|\theta)$ , and let  $W(\mathbf{X}) = W(X_1, \dots, X_n)$  be any estimator where  $E_\theta[W(\mathbf{X})]$  is a differentiable function of  $\theta$ . If the joint pdf  $f(\mathbf{x}|\theta) = \prod f(x_i|\theta)$  satisfies

$$\frac{d}{d\theta} \int \cdots \int h(\mathbf{x}) f(\mathbf{x}|\theta) d\mathbf{x} = \int \cdots \int h(\mathbf{x}) \frac{\partial}{\partial \theta} f(\mathbf{x}|\theta) d\mathbf{x},$$

for any function  $h(\mathbf{x})$  with  $E_\theta[|h(\mathbf{X})|] < \infty$ , then

$$\text{Var}_\theta[W(\mathbf{X})] \geq \frac{\left(\frac{d}{d\theta} E_\theta[W(\mathbf{X})]\right)^2}{n E_\theta \left[ \left( \frac{\partial}{\partial \theta} \ln f(X|\theta) \right)^2 \right]}.$$

# Cramér-Rao Lower Bound (iid case)

- *Proof of Theorem L9.3 continued:* If we also assume that  $X_1, \dots, X_n$  is iid, then we have

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right)^2 \right] &= \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln \prod_{i=1}^n f(X_i|\theta) \right)^2 \right] \\ &= \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \sum_{i=1}^n \ln f(X_i|\theta) \right)^2 \right] \\ &= \sum_{i=1}^n \sum_{j=1}^n \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X_i|\theta) \right) \left( \frac{\partial}{\partial \theta} \ln f(X_j|\theta) \right) \right] \\ &= \sum_{i=1}^n \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X_i|\theta) \right)^2 \right] + \\ &\quad \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X_i|\theta) \right) \left( \frac{\partial}{\partial \theta} \ln f(X_j|\theta) \right) \right] \end{aligned}$$

# Cramér-Rao Lower Bound (iid case)

● *Proof of Theorem L9.3 continued:*

$$\begin{aligned} &= \sum_{i=1}^n \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X_i | \theta) \right)^2 \right] + \\ &\quad \sum_{i=1}^n \sum_{j \neq i} \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X_i | \theta) \right) \left( \frac{\partial}{\partial \theta} \ln f(X_j | \theta) \right) \right] \\ &= \sum_{i=1}^n \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X_i | \theta) \right)^2 \right] + \\ &\quad \sum_{i=1}^n \sum_{j \neq i} \mathbb{E} \left[ \frac{\partial}{\partial \theta} \ln f(X_i | \theta) \right] \mathbb{E} \left[ \frac{\partial}{\partial \theta} \ln f(X_j | \theta) \right] \\ &= \sum_{i=1}^n \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X_i | \theta) \right)^2 \right] \\ &= n \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X | \theta) \right)^2 \right]. \end{aligned}$$

# Cramér-Rao Lower Bound (iid case)

- The quantity  $E_{\theta} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X|\theta) \right)^2 \right]$  is called the *information number*, or *Fisher information* of the sample.
- *Theorem L9.4* (Lem 7.3.11 on p.338): If  $f(x|\theta)$  satisfies

$$\frac{d}{d\theta} E_{\theta} \left[ \frac{\partial}{\partial \theta} \ln f(X|\theta) \right] = \int \frac{\partial}{\partial \theta} \left[ \left( \frac{\partial}{\partial \theta} \ln f(x|\theta) \right) f(x|\theta) \right] dx$$

and  $\frac{d}{d\theta} \int f(x|\theta) dx = \int \frac{\partial}{\partial \theta} f(x|\theta) dx$ , then

$$E_{\theta} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X|\theta) \right)^2 \right] = -E_{\theta} \left[ \frac{\partial^2}{\partial \theta^2} \ln f(X|\theta) \right].$$

- The condition on  $f(x|\theta)$ , and consequently the result, is true for an exponential family.

# Cramér-Rao Lower Bound (iid case)

- *Proof of Theorem L9.4:* Note that

$$\begin{aligned}\frac{\partial^2}{\partial \theta^2} [\ln f(x|\theta)] &= \frac{\partial}{\partial \theta} \left\{ \frac{\frac{\partial}{\partial \theta} f(x|\theta)}{f(x|\theta)} \right\} \\ &= \frac{\frac{\partial^2}{\partial \theta^2} f(x|\theta)}{f(x|\theta)} - \frac{\left( \frac{\partial}{\partial \theta} f(x|\theta) \right)^2}{(f(x|\theta))^2}.\end{aligned}$$

- Then, we have

$$\begin{aligned}\mathbb{E} \left[ \frac{\frac{\partial^2}{\partial \theta^2} f(X|\theta)}{f(X|\theta)} \right] &= \int \frac{\partial^2}{\partial \theta^2} f(x|\theta) \, dx \\ &= \int \frac{\partial}{\partial \theta} \left\{ \frac{\partial}{\partial \theta} f(x|\theta) \right\} \, dx\end{aligned}$$

# Cramér-Rao Lower Bound (iid case)

• *Proof of Theorem L9.4 continued:*

$$\begin{aligned} &= \int \frac{\partial}{\partial \theta} \left\{ \frac{\partial}{\partial \theta} f(x|\theta) \right\} dx \\ &= \int \frac{\partial}{\partial \theta} \left\{ \frac{\frac{\partial}{\partial \theta} f(x|\theta)}{f(x|\theta)} f(x|\theta) \right\} dx \\ &= \int \frac{\partial}{\partial \theta} \left\{ \left( \frac{\partial}{\partial \theta} \ln f(x|\theta) \right) f(x|\theta) \right\} dx \\ &= \frac{d}{d\theta} \int \left\{ \left( \frac{\partial}{\partial \theta} \ln f(x|\theta) \right) f(x|\theta) \right\} dx \\ &= \frac{d}{d\theta} \int \left\{ \frac{\frac{\partial}{\partial \theta} f(x|\theta)}{f(x|\theta)} f(x|\theta) \right\} dx \\ &= \frac{d}{d\theta} \int \frac{\partial}{\partial \theta} f(x|\theta) dx = \frac{d}{d\theta} \frac{d}{d\theta} \int f(x|\theta) dx = \frac{d}{d\theta} [1] = 0 \end{aligned}$$

- *Proof of Theorem L9.4 continued:* So, it follows that

$$\begin{aligned} \mathbb{E} \left[ \frac{\partial^2}{\partial \theta^2} \ln f(X|\theta) \right] &= \mathbb{E} \left[ \frac{\frac{\partial^2}{\partial \theta^2} f(\mathbf{X}|\theta)}{f(\mathbf{X}|\theta)} \right] - \mathbb{E} \left[ \frac{\left( \frac{\partial}{\partial \theta} f(\mathbf{X}|\theta) \right)^2}{(f(\mathbf{X}|\theta))^2} \right] \\ &= 0 - \mathbb{E} \left[ \left( \frac{\frac{\partial}{\partial \theta} f(\mathbf{X}|\theta)}{f(\mathbf{X}|\theta)} \right)^2 \right] \\ &= -\mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right)^2 \right]. \end{aligned}$$



# Cramér-Rao Lower Bound (iid case)

- *Example L9.2:* Let  $X_1, \dots, X_n$  be iid  $\text{Poisson}(\lambda)$ . Find the Cramér-Rao lower bound on the variance of unbiased estimators of  $\lambda$ . Also, find the MLE and show that it is the UMVUE of  $\lambda$ .
- *Answer to Example L9.2:* Since  $\frac{\partial^2}{\partial \lambda^2} \ln f(x|\lambda) =$   
 $\frac{\partial^2}{\partial \lambda^2} [\ln \{ \lambda^x e^{-\lambda} (x!)^{-1} \}] = \frac{\partial^2}{\partial \lambda^2} [x \ln \lambda - \lambda - \ln(x!)] = -\frac{x}{\lambda^2},$   
we have

$$\mathbb{E} \left[ \frac{\partial^2}{\partial \lambda^2} \ln f(X|\lambda) \right] = \mathbb{E} \left[ -\frac{1}{\lambda^2} X \right] = -\frac{1}{\lambda^2} \mathbb{E}[X] = -\frac{1}{\lambda^2} \lambda = -\frac{1}{\lambda}.$$

By *Theorem L9.4*,

$$\mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X|\theta) \right)^2 \right] = -\mathbb{E} \left[ \frac{\partial^2}{\partial \lambda^2} \ln f(X|\lambda) \right] = \frac{1}{\lambda}.$$

# Cramér-Rao Lower Bound (iid case)

- *Answer to Example L9.2 continued:* So the Cramér-Rao lower bound for an unbiased estimator in the iid case is

$$\frac{\left(\frac{d}{d\theta}\mathbb{E}_{\theta}[W(\mathbf{X})]\right)^2}{n\mathbb{E}_{\theta}\left[\left(\frac{\partial}{\partial\theta}\ln f(X|\theta)\right)^2\right]} = \frac{1}{n\left(\frac{1}{\lambda}\right)} = \frac{\lambda}{n}.$$

The MLE of  $\lambda$  is  $\hat{\lambda} = \bar{X}$  and  $\text{Var}[\bar{X}] = \frac{\mathbb{E}[X_1]}{n} = \frac{\lambda}{n}$  so it attains the CRLB and is the UMVUE of  $\lambda$ .

# Cramér-Rao Lower Bound (iid case)

- *Example L9.3:* Let  $X_1, \dots, X_n$  be iid  $\text{Normal}(\mu, \sigma^2)$  random variables. Find the Cramér-Rao lower bound on unbiased estimators of  $\sigma^2$ . Does  $S^2$  satisfy the CRLB?
- *Answer to Example L9.3:* Since

$$\begin{aligned}\frac{\partial^2}{\partial(\sigma^2)^2} \ln f(x|\mu, \sigma^2) &= \frac{\partial^2}{\partial(\sigma^2)^2} \left[ -\frac{1}{2} \ln(2\pi) - \frac{1}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} (x - \mu)^2 \right] \\ &= \frac{1}{2\sigma^4} - \frac{(x - \mu)^2}{\sigma^6},\end{aligned}$$

*Theorem L9.4* implies that

$$\begin{aligned}\mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X|\mu, \sigma^2) \right)^2 \right] &= -\mathbb{E} \left[ \frac{\partial^2}{\partial \sigma^2} \ln f(X|\mu, \sigma^2) \right] \\ &= -\mathbb{E} \left[ \frac{1}{2\sigma^4} - \frac{(X - \mu)^2}{\sigma^6} \right]\end{aligned}$$

# Cramér-Rao Lower Bound (iid case)

- *Answer to Example L9.3 continued:*

$$\begin{aligned} &= -\mathbb{E} \left[ \frac{1}{2\sigma^4} - \frac{(X - \mu)^2}{\sigma^6} \right] \\ &= -\frac{1}{2\sigma^4} + \frac{\mathbb{E}[(X - \mu)^2]}{\sigma^6} \\ &= -\frac{1}{2\sigma^4} + \frac{\sigma^2}{\sigma^6} = \frac{1}{2\sigma^4}. \end{aligned}$$

Thus, the CRLB is  $\frac{1}{n\mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f(X|\theta) \right)^2 \right]} = \frac{2\sigma^4}{n}$ .

So,  $S^2$  does not satisfy the CRLB since

$$\text{Var}[S^2] = \frac{2\sigma^4}{n-1} = \frac{n}{n-1} \left( \frac{2\sigma^4}{n} \right) > \frac{2\sigma^4}{n} = \text{CRLB}.$$

# Cramér-Rao Lower Bound (iid case)

- *Example L9.4:* Let  $X_1, \dots, X_n$  be iid  $\text{Uniform}(0, \theta)$  random variables. Find the Cramér-Rao lower bound on the variance of unbiased estimators of  $\theta$ . Also, for  $Y = \max \{X_1, \dots, X_n\}$  show that  $\left(\frac{n+1}{n}\right) Y$  is an unbiased estimator which has a smaller variance than the Cramér-Rao lower bound.
- *Answer to Example L9.4:* Since  $\frac{\partial}{\partial \theta} \ln f(x|\theta) = \frac{\partial}{\partial \theta} \left[ \ln \frac{1}{\theta} \right] = -\frac{1}{\theta}$ , the CRLB is  $\frac{1}{n(-\theta^{-1})^2} = \frac{\theta^2}{n}$ .  
Since the CDF of  $Y$  is  $F(y) = P(Y \leq y) = \prod_{i=1}^n P(X_i \leq y) = \left(\frac{y}{\theta}\right)^n$  for  $0 < y < \theta$ ,  
the pdf of  $Y$  is  $f(y) = F'(y) = \frac{ny^{n-1}}{\theta^n} I_{(0,\theta)}(y)$ .

# Cramér-Rao Lower Bound (iid case)

- *Answer to Example L9.4 continued:*  $\left(\frac{n+1}{n}\right) Y$  is unbiased since
$$\begin{aligned} \mathbb{E} \left[ \left(\frac{n+1}{n}\right) Y \right] &= \frac{n+1}{n} \int_0^\theta y \frac{ny^{n-1}}{\theta^n} dy = \frac{n+1}{\theta^n} \int_0^\theta y^n dy = \\ &= \frac{n+1}{\theta^n} \left[ \frac{1}{n+1} y^{n+1} \right]_0^\theta = \frac{n+1}{\theta^n} \left[ \frac{1}{n+1} \theta^{n+1} \right] = \theta. \end{aligned}$$
- Similarly,  $\mathbb{E} \left[ \left(\frac{n+1}{n} Y\right)^2 \right] = \frac{(n+1)^2}{n\theta^n} \int_0^\theta y^{n+1} dy =$ 
$$\frac{(n+1)^2}{n\theta^n} \left[ \frac{1}{n+2} y^{n+2} \right]_0^\theta = \frac{(n+1)^2}{n\theta^n} \left[ \frac{1}{n+2} \theta^{n+2} \right] = \frac{(n+1)^2}{n(n+2)} \theta^2.$$
- So,  $\text{Var} \left[ \left(\frac{n+1}{n}\right) Y \right] = \frac{(n+1)^2}{n(n+2)} \theta^2 - \theta^2 = \frac{1}{n(n+2)} \theta^2.$
- It is now seen that  $\text{Var} \left[ \left(\frac{n+1}{n}\right) Y \right] = \frac{1}{n+2} \left( \frac{\theta^2}{n} \right) < \frac{\theta^2}{n} = \text{CRLB}.$

- *Theorem L9.5* (Cor 7.3.15 on p.341): Let  $X_1, \dots, X_n$  be iid  $f(x|\theta)$ , where  $f(x|\theta)$  satisfies the conditions of *Theorem L9.3*. Let  $L(\theta|\mathbf{x}) = \prod_{i=1}^n f(x_i|\theta)$  denote the likelihood function. If  $W(\mathbf{X}) = W(X_1, \dots, X_n)$  is any unbiased estimator of  $\tau(\theta)$ , then  $W(\mathbf{X})$  attains the Cramér-Rao Lower Bound if and only if

$$a(\theta)[W(\mathbf{x}) - \tau(\theta)] = \frac{\partial}{\partial \theta} \ln L(\theta|\mathbf{x})$$

for some function  $a(\theta)$ .

- *Proof of Theorem L9.5:* By Theorem L9.1(b),

$$\left\{ \text{Cov} \left[ W(\mathbf{X}), \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right] \right\}^2 = \text{Var}[W(\mathbf{X})] \text{Var} \left[ \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right]$$

if and only if there are functions  $b(\theta)$  and  $a(\theta)$  (where  $|a(\theta)| > 0$ ) such that

$$\frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) = a(\theta)W(\mathbf{X}) + b(\theta). \quad (1)$$

Since  $E[W(\mathbf{X})] = \tau(\theta)$  and  $E \left[ \frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right] \stackrel{9.8}{=} 0$ , taking the expected value of both sides of (1) yields  $0 = a(\theta)\tau(\theta) + b(\theta)$  so that

$$b(\theta) = -a(\theta)\tau(\theta). \quad (2)$$

Substituting (2) into (1), we have

$$\frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) = a(\theta) \{W(\mathbf{X}) - \tau(\theta)\}.$$



- *Example L9.5:* Let  $X_1, \dots, X_n$  be a random sample from a  $\text{Beta}(\theta, 1)$  population which has pdf

$$f(x) = \theta x^{\theta-1} I_{(0,1)}(x).$$

- (a) Compute  $\frac{\partial}{\partial \theta} \ln L(\theta|\mathbf{x})$  where  $L$  is the likelihood function.
- (b) Find the UMVUE for  $\frac{1}{\theta}$ .

- *Answer to Example L9.5:* We have

$$\begin{aligned} \frac{\partial}{\partial \theta} \ln L(\theta|\mathbf{x}) &= \frac{\partial}{\partial \theta} \sum_{i=1}^n \ln f(x_i|\theta) = \sum_{i=1}^n \frac{\partial}{\partial \theta} [(\theta - 1) \ln x_i + \ln \theta] \\ &= \sum_{i=1}^n \left\{ \ln x_i + \frac{1}{\theta} \right\} = \sum_{i=1}^n \ln x_i + \frac{n}{\theta}. \end{aligned}$$

# Cramér-Rao Lower Bound (iid case)

- *Answer to Example L9.5 continued:* Since  $f(x|\theta)$  is a member of an exponential family with  $c(\theta) = \theta$ ,  $w(\theta) = \theta - 1$ , and  $t(x) = \ln x$ , and  $\{w(\theta) : \theta \in (0, \infty)\} = (-1, \infty)$  contains an open subset of  $\mathbb{R}$

*Theorem L6.2* implies that  $\sum_{i=1}^n t(X_i) = \sum_{i=1}^n \ln X_i$  belongs to an exponential family with  $C(\theta) = [c(\theta)]^n = \theta^n$ ,  $w(\theta) = \theta - 1$ , and  $u_i = \sum_{i=1}^n t(x_i) = \sum_{i=1}^n \ln x_i$ .

By *Theorem L6.1*, we have

$$E\left[\sum_{i=1}^n \ln X_i\right] = -\frac{d}{d\theta} [\ln(\theta^n)] = -\frac{n}{\theta}.$$

- Hence,  $E\left[\frac{-1}{n} \sum_{i=1}^n \ln X_i\right] = \frac{1}{\theta}$ .
- *Theorem L9.5* shows that  $\frac{-1}{n} \sum_{i=1}^n \ln X_i$  is the UMVUE for  $\frac{1}{\theta}$  since

$$\frac{\partial}{\partial \theta} \ln L(\theta|\mathbf{x}) = -n \left( -\frac{1}{n} \sum_{i=1}^n \ln x_i - \frac{1}{\theta} \right).$$