# Lecture 15: Uniformly Most Powerful Tests, the Neyman-Pearson Lemma, and the Karlin-Rubin Theorem

MATH 667-01
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# Introduction

- We give the definition of a uniformly most powerful test in Section 8.3.2 of Casella and Berger (2002)[1].
- Then the proof for the Neyman-Pearson Lemma is given.
- In Section 8.3.2, pdfs/pmfs with a monotone likelihood ratio (MLR) are also discussed and the Karlin-Rubin Theorem is proven.

[1] Casella, G. and Berger, R. (2002). *Statistical Inference, Second Edition*. Duxbury Press, Belmont, CA.

# Uniformly Most Powerful (UMP) Tests

- *Definition L15.1* (p.389): Let $\mathcal{C}$ be a class of tests for testing $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_0^c$. A test in class $\mathcal{C}$, with power function $\beta(\theta)$, is a *uniformly most powerful* (UMP) *class $\mathcal{C}$ test* if $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Theta_0^c$ and every $\beta'(\theta)$ that is a power function of a test in class $\mathcal{C}$.
- Often, $\mathcal{C}$ will be the class of all level $\alpha$ tests.
- *Definition L15.2* (Def 8.3.11 on p.388): A *test function*, $\phi(\boldsymbol{x})$, for a hypothesis testing procedure is a function on the sample space whose value is one if $\boldsymbol{x}$ is in the rejection region and zero if $\boldsymbol{x}$ is in the acceptance region. That is, $\phi(\boldsymbol{x})$ is the indicator function of the rejection region.

- *Theorem L15.1* (Thm 8.3.12 on p.388): Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where the pdf or pmf corresponding to $\theta_i$ is $f(\boldsymbol{x}|\theta_i), i = 0, 1$, using a test with rejection region $R$ that satisfies

$$(*) \quad \boldsymbol{x} \in R \text{ if } f(\boldsymbol{x}|\theta_1) > kf(\boldsymbol{x}|\theta_0) \text{ and } \boldsymbol{x} \in R^c \text{ if } f(\boldsymbol{x}|\theta_1) < kf(\boldsymbol{x}|\theta_0),$$

for some $k \geq 0$, and

$$(**) \qquad \alpha = P_{\theta_0}(\boldsymbol{X} \in R).$$

Then

a. (Sufficiency) Any test that satisfies (*) and (**) is a UMP level $\alpha$ test.
b. (Necessity) If there exists a test satisfying (*) and (**) with $k > 0$, then every UMP level $\alpha$ test is a size $\alpha$ test and every UMP level $\alpha$ test satisfies (*) except perhaps on a set $A$ satisfying $P_{\theta_0}(\boldsymbol{X} \in A) = P_{\theta_1}(\boldsymbol{X} \in A) = 0$.

- *Proof of Theorem L15.1*: Let $\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in R \\ 0 & \text{if } \boldsymbol{x} \in R^c \end{cases}$ and let $\phi'(\boldsymbol{x})$ be a test function of any other level $\alpha$ test. Since $0 \leq \phi'(\boldsymbol{x}) \leq 1$, we have $1 - \phi'(\boldsymbol{x}) \geq 0$ and $f(\boldsymbol{x}|\theta_1) - kf(\boldsymbol{x}|\theta_0) > 0$ if $\boldsymbol{x} \in R$. If $x \in R^c$, we have $0 - \phi'(\boldsymbol{x}) \leq 0$ and $f(\boldsymbol{x}|\theta_1) - kf(\boldsymbol{x}|\theta_0) < 0$. So, for all $\boldsymbol{x}$, we have
$$0 \leq \Big(\phi(\boldsymbol{x}) - \phi'(\boldsymbol{x})\Big)\Big(f(\boldsymbol{x}|\theta_1) - kf(\boldsymbol{x}|\theta_0)\Big). \tag{1}$$
The test functions are related to the power functions as follows:
$$\beta(\theta) = P_\theta(X \in R) = \mathsf{E}[\phi(X)] = \int \phi(\boldsymbol{x})f(\boldsymbol{x}|\theta)\ d\boldsymbol{x}$$
and
$$\beta'(\theta) = \int \phi'(\boldsymbol{x})f(\boldsymbol{x}|\theta)\ d\boldsymbol{x}.$$

- *Proof of Theorem L15.1 continued*: Since (1) holds for all $\boldsymbol{x}$,

$$
\begin{aligned}
0 &\leq \int \Big(\phi(\boldsymbol{x}) - \phi'(\boldsymbol{x})\Big)\Big(f(\boldsymbol{x}|\theta_1) - kf(\boldsymbol{x}|\theta_0)\Big)\, d\boldsymbol{x} \qquad (2) \\
&= \int \phi(\boldsymbol{x}) f(\boldsymbol{x}|\theta_1) d\boldsymbol{x} - \int \phi'(\boldsymbol{x}) f(\boldsymbol{x}|\theta_1) d\boldsymbol{x} - k \int \phi(\boldsymbol{x}) f(\boldsymbol{x}|\theta_0) d\boldsymbol{x} + k \int \phi'(\boldsymbol{x}) f(\boldsymbol{x}|\theta_0) d\boldsymbol{x} \\
&= \beta(\theta_1) - \beta'(\theta_1) - k\beta(\theta_0) + k\beta'(\theta_0) \\
&= \beta(\theta_1) - \beta'(\theta_1) - k\Big(\beta(\theta_0) - \beta'(\theta_0)\Big).
\end{aligned}
$$

Now, $\beta'(\theta_0) \leq \alpha$ ($\phi'$ is a level $\alpha$ test) and $\beta(\theta_0) = \alpha$ ($\phi$ is a size $\alpha$ test) so it follows that

$$\beta(\theta_0) - \beta'(\theta_0) \geq 0$$

which implies that $\beta(\theta_1) - \beta'(\theta_1) \geq 0$. Hence, $\beta(\theta) \geq \beta'(\theta)$ for every $\theta$ in $\Theta_0^c = \{\theta_1\}$ which shows that $\phi$ is a UMP level $\alpha$ test.

- *Proof of Theorem L15.1 continued*: Suppose $\phi'$ is the test function for a UMP level $\alpha$ test.
  Since $\phi$ is also a UMP level $\alpha$ test, $\beta(\theta_1) = \beta'(\theta_1)$.
  Thus, it follows that $0 \leq -k\Big(\beta(\theta_0) - \beta'(\theta_0)\Big)$ so that

$$\beta(\theta_0) - \beta'(\theta_0) \leq 0 \Rightarrow \alpha - \beta'(\theta_0) \leq 0 \Rightarrow \alpha \leq \beta'(\theta_0).$$

  But since $\phi'$ is a level $\alpha$ test, $\beta'(\theta_0) \leq \alpha$ so we know $\beta'(\theta_0) = \alpha$; that is, $\phi'$ is a size $\alpha$ test. This also implies that the integral in (2) equals 0, and consequently, $\phi'$ must satisfy (*) except possibly on a set of measure 0.

- *Example L15.1*: Suppose the pmf of $X$ under $H_0$ and $H_1$ are given in the following table.

| | $x$ | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| $f(x\|H_0)$ | .01 | .02 | .02 | .03 | .92 |
| $f(x\|H_1)$ | .02 | .02 | .10 | .63 | .23 |

  (a) Find a UMP test with size $.05$ for testing $H_0$ versus $H_1$.
  (b) Find the probability of a Type II error of the test in part (a).

- *Answer to Example L15.1*: By the Neyman-Pearson Lemma, we look at rejection regions satisfying

$$x \in R \text{ if } f(x|H_1) > kf(x|H_0) \text{ and } x \in R^c \text{ if } f(x|H_1) < kf(x|H_0)$$

  for some $k$.

- *Answer to Example L15.1 continued*: The following table computes the ratios $\frac{f(x|H_1)}{f(x|H_0)}$.

| | $x$ | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| $f(x\|H_0)$ | .01 | .02 | .02 | .03 | .92 |
| $f(x\|H_1)$ | .02 | .02 | .10 | .63 | .23 |
| $f(x\|H_1)/f(x\|H_0)$ | 2 | 1 | 5 | 21 | .25 |

  If $k \in (21, \infty)$, $R = \emptyset$ and $P(X \in R|H_0) = 0$.
  If $k \in (5, 21)$, $R = \{4\}$ and $P(X \in R|H_0) = .03$.
  If $k \in (2, 5)$, $R = \{3, 4\}$ and $P(X \in R|H_0) = .05$.
  So rejecting $H_0$ if and only if $X \in \{3, 4\}$ is the UMP test with size $.05$.

- The probability of a Type II error is

$$\begin{aligned} P(X \notin R|H_1) &= P(X \in \{1, 2, 5\} |H_1) \\ &= f(1|H_1) + f(2|H_1) + f(5|H_1) = .27. \end{aligned}$$

- *Example L15.2*: Let $X_1, \ldots, X_n$ be iid Normal$(\mu, 1)$ random variables. Show that there is no UMP test for testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$.
- *Answer to Example L15.2*: For a specified value of $\alpha$, a level $\alpha$ test in this problem is a test which satisfies $P_{\theta_0}(\text{reject } H_0) \leq \alpha$.
  Let $\mu_1$ be a value less than $\mu_0$ and consider testing $H_0' : \mu = \mu_0$ versus $H_1' : \mu = \mu_1$.
  Let $z_\alpha = \Phi^{-1}(1 - \alpha)$ with $\Phi$ being the cdf of a standard normal random variable. It can be shown that the test that rejects $H_0'$ if and only if $\bar{X} < \mu_0 - z_\alpha \sigma / \sqrt{n}$ is the UMP test of $H_0'$ versus $H_1'$ by *Theorem L15.1(a)*; name ths test "Test 1". By *Theorem L15.1(b)*, any other test that has as high a power as Test 1 must have the same rejection region except possibly on a set of measure 0. That is, if there is a UMP test for $H_0$ versus $H_1$, it must be Test 1 because no other test has as high a power at $\mu_1$.

- *Answer to Example L15.2 continued*: Now consider a test which rejects $H_0$ if $\bar{X} > \mu_0 + z_\alpha \sigma/\sqrt{n}$, which is a level $\alpha$ test; call this "Test 2".
  Let $\beta_i(\mu)$ be the power function for Test $i$. For any $\mu_2 > \mu_0$,

$$\begin{aligned}
\beta_2(\mu_2) &= P_{\mu_2}\left(\bar{X} > \mu_0 + \frac{\sigma z_\alpha}{\sqrt{n}}\right) \\
&= P_{\mu_2}\left(\frac{\bar{X} - \mu_2}{\sigma/\sqrt{n}} > \frac{\mu_0 - \mu_2}{\sigma/\sqrt{n}} + z_\alpha\right) \\
&> P\left(Z > z_\alpha\right) \\
&= P\left(Z < -z_\alpha\right) \\
&= P_{\mu_2}\left(\frac{\bar{X} - \mu_2}{\sigma/\sqrt{n}} < \frac{\mu_0 - \mu_2}{\sigma/\sqrt{n}} - z_\alpha\right) \\
&= P_{\mu_2}\left(\bar{X} < \mu_0 - \frac{\sigma z_\alpha}{\sqrt{n}}\right) = \beta_1(\mu_2).
\end{aligned}$$

  Thus, Test 1 is not a UMP test since Test 2 as higher power than Test 1 at $\theta_2$. Thus, no UMP level $\alpha$ exists for testing $H_0$ versus $H_1$.

- *Theorem L15.2* (Cor 8.3.13 on p.389): Consider the hypothesis testing problem posed in the Neyman-Pearson Lemma. Suppose $T(\boldsymbol{X})$ is a sufficient statistic for $\theta$ and $g(t|\theta_i)$ is the pdf or pmf of $T$ corresponding to $\theta_i, i = 0, 1$. Then any test based on $T$ with rejection region $S$ (a subset of the sample space of $T$) is a UMP level $\alpha$ test if it satisfies

$$t \in S \text{ if } g(t|\theta_1) > kg(t|\theta_0)$$

and

$$t \in S^c \text{ if } g(t|\theta_1) < kg(t|\theta_0),$$

for some $k \geq 0$ where $\alpha = P_{\theta_0}(T \in S)$.

- *Proof of Theorem L15.2*: The result follows immediately from part (a) of the Neyman-Pearson Lemma since

$$\boldsymbol{x} \in R \text{ if } f(\boldsymbol{x}|\theta_1) \overset{10.7}{=} g(T(\boldsymbol{x})|\theta_1)h(\boldsymbol{x}) > kg(T(\boldsymbol{x})|\theta_0)h(\boldsymbol{x}) \overset{10.7}{=} kf(\boldsymbol{x}|\theta_0)$$

and

$$\boldsymbol{x} \in R^c \text{ if } f(\boldsymbol{x}|\theta_1) \overset{10.7}{=} g(T(\boldsymbol{x})|\theta_1)h(\boldsymbol{x}) < kg(T(\boldsymbol{x})|\theta_0)h(\boldsymbol{x}) \overset{10.7}{=} kf(\boldsymbol{x}|\theta_0).$$

- *Definition L15.3* (Def 8.3.16 on p.391): A family of pdfs or pmfs $\{g(t|\theta) : \theta \in \Theta\}$ for a univariate random variable $T$ with real-valued parameter $\theta$ has a *monotone likelihood ratio* (MLR) if, for every $\theta_2 > \theta_1$, $g(t|\theta_2)/g(t|\theta_1)$ is a monotone (nonincreasing or nondecreasing) function of $t$ on $\{t : g(t|\theta_1) > 0$ or $g(t|\theta_2) > 0\}$.
  Note that $c/0$ is defined as $\infty$ if $0 < c$.
- The following result is referred to as the Karlin-Rubin Theorem.
- *Theorem L15.3* (Thm 8.3.17 on p.391): Consider testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Suppose that $T$ is a sufficient statistic for $\theta$ and the family of pdfs of pmfs $\{g(t|\theta) : \theta \in \Theta\}$ of $T$ has a nondecreasing MLR. Then for any $t_0$, the test that rejects $H_0$ if and only if $T > t_0$ is a UMP level $\alpha$ test where $\alpha = P_{\theta_0}(T > t_0)$.

- *Proof of Theorem L15.3*: Let $\theta'$ be any value in $\Theta$ greater than $\theta_0$. First, we consider the test $H_0' : \theta = \theta_0$ versus $H_1' : \theta = \theta'$. For $\mathcal{T}' = \{t : t > t_0,$ and $g(t|\theta') > 0$ or $g(t|\theta_0) > 0\}$, define $k' = \inf_{t \in \mathcal{T}'} \frac{g(t|\theta')}{g(t|\theta_0)}$. Since $T$ has a nondecreasing MLR, $\frac{g(t|\theta')}{g(t|\theta_0)} > k'$ if and only if $t > t_0$. By *Theorem L15.2*, this test is UMP for testing $H_0'$ versus $H_1'$.

  Now, we show that $\beta(\theta)$ is a nondecreasing function of $\theta$. Let $\theta_1 < \theta_2$. If $t_1 < t_2$, then

$$\frac{g(t_2|\theta_2)}{g(t_2|\theta_1)} \geq \frac{g(t_1|\theta_2)}{g(t_1|\theta_1)} \Leftrightarrow g(t_1|\theta_1)g(t_2|\theta_2) \geq g(t_1|\theta_2)g(t_2|\theta_1).$$

- *Proof of Theorem L15.3 continued*: Summing/integrating $t_1$ on $(-\infty, t_2)$, we see that

$$P_{\theta_1}(T \le t_2) g(t_2|\theta_2) \ge P_{\theta_2}(T \le t_2) g(t_2|\theta_1) \Rightarrow \frac{g(t_2|\theta_2)}{g(t_2|\theta_1)} \ge \frac{P_{\theta_2}(T \le t_2)}{P_{\theta_1}(T \le t_2)}.$$

Summing/integrating $t_2$ on $[t_1, \infty)$, we see that

$$g(t_1|\theta_1) P_{\theta_2}(T > t_1) \ge g(t_1|\theta_2) P_{\theta_1}(T > t_1) \Rightarrow \frac{P_{\theta_2}(T > t_1)}{P_{\theta_1}(T > t_1)} \ge \frac{g(t_1|\theta_2)}{g(t_1|\theta_1)}.$$

Thus, for any $t$, we have

$$\frac{P_{\theta_2}(T > t)}{P_{\theta_1}(T > t)} \ge \frac{P_{\theta_2}(T \le t)}{P_{\theta_1}(T \le t)} \Rightarrow \frac{1 - P_{\theta_2}(T \le t)}{1 - P_{\theta_1}(T \le t)} \ge \frac{P_{\theta_2}(T \le t)}{P_{\theta_1}(T \le t)}$$

$$\Rightarrow \frac{P_{\theta_1}(T \le t)}{1 - P_{\theta_1}(T \le t)} \ge \frac{P_{\theta_2}(T \le t)}{1 - P_{\theta_2}(T \le t)}.$$

- *Proof of Theorem L15.3 continued*: Since $\frac{x}{1-x}$ is a nondecreasing function for $x \in (0,1)$, this implies that $P_{\theta_2}(T > t) \geq P_{\theta_1}(T > t)$.
  Now, for testing $H_0$, the size of the test is

$$\sup_{\theta \in (-\infty, \theta_0]} \beta(\theta) = \beta(\theta_0) = \alpha$$

  since $\beta(\theta)$ is nondecreasing. Let $\beta^*$ be the power function for any other level $\alpha$ test for $H_0$. This is also a level $\alpha$ test for $H_0'$ so $\beta(\theta) \geq \beta^*(\theta')$ for any $\theta' > \theta_0$. So the test for $H_0$ versus $H_1$ is a UMP level $\alpha$ test.

- *Example L15.3*: Suppose $X_1, \ldots, X_n$ is a random sample from a distribution with pdf

$$f(x|\theta) = e^{-(x-\theta)} I_{[\theta,\infty)}(x)$$

and let $\theta_0$ be a fixed number in $\Theta = (-\infty, \infty)$. Consider the one-sided test $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$. Show that the likelihood ratio test which rejects $H_0$ if $X_{(1)} \geq K$ is a UMP level $\alpha$ test where $\alpha = P_{\theta=0}(X_{(1)} \geq K)$.

- *Answer to Example L15.3*: Note that $X_{(1)}$ is sufficient for $\theta$ (see slide 14.11). It can be shown that $X_{(1)} - \theta$ is exponentially distributed with mean $\frac{1}{n}$ so $X_{(1)}$ has pdf $g(t|\theta) = ne^{-n(t-\theta)} I_{[\theta,\infty)}(t)$ (similar to work on slide 14.12). This family of pdfs has a MLR since, for $\theta_1 < \theta_2$,
$$\frac{g(t|\theta_2)}{g(t|\theta_1)} = \begin{cases} 0 & \text{if } \theta_1 \leq t < \theta_2 \\ e^{n(\theta_1-\theta_2)} & \text{if } \theta_2 \leq t \end{cases}$$
is a nondecreasing function of $t$. So, by the Karlin-Rubin Theorem, the LRT is a UMP level $\alpha$ test.