M621, final prep problems I: Here are some good problems to work on to help you prepare for the final. Don't hesitate to email me if you have a question, or come by my office (assuming I'm there of course).

1. Let G be a group. Consider the commutator subgroup $[G;G]$ of G, the subgroup generated by the set $\{aba^{-1}b^{-1} : a, b \in G\}$. (An element of the form $aba^{-1}b^{-1}$ is known as a commutator.) It is known that $[G, G]$ is a normal subgroup of G.

   (a) Prove that $G/[G, G]$ is Abelian.

   (b) Prove that if $\Gamma : G \to K$ is a surjective homomorphism, and K is Abelian, then $[G, G] \subseteq ker(\Gamma)$.

2. Suppose G is a group and $Z = Z(G)$ is its center. Prove that $G/Z(G)$ is cyclic, then G is Abelian.

3. Prove that any finite group is isomorphic to a subgroup of A_n for some $n \in \mathbb{N}$, where A_n is the alternating group acting on $\{1, \ldots, n\}$.

4. Let G be an arbitrary group with a center $Z(G)$. Prove that the inner auto- morphism group $Inn(G)$ is isomorphic to $G/Z(G)$.

5. Suppose R is a commutative ring with 1. Prove that R is a field if and only if the only ideals of R are the trivial ideal $\{0\}$ and R itself.

6. Prove that a group of order 56 has at least one normal Sylow p-subgroup.

7. Prove that if $f : A \to B$ is a surjective ring homomorphism, then whenever I is an ideal of A, then $f(I)$ is an ideal of B. Give an example to show that $f(A)$ need not be an ideal of B if f is not surjective.

   Suppose J is an ideal of B and f is surjective. Show that $f^{-1}(J)$ is an ideal of R. Can the subjectivity hypothesis be dropped?

8. A ring R is called Boolean if every element $a \in R$ is idempotent; that is $a^2 = a$. Prove that every Boolean ring is commutative.

9. Let R be a commutative ring with 1, and let I be an ideal of R. Prove the following.

   (a) I is a maximal ideal if and only if R/I is a field.

   (b) I is a prime ideal if and only if R/I is an integral domain.

10. State and prove e the Third Isomorphism Theorem

    (a) for groups

    (b) for rings

11. Consider the cyclic group $(Z_{10}, +)$.

    (a) If f is an automorphism of this group, what are all possible values of $f(1)$?

    (b) Determine $Aut(Z_10)$ up to isomorphism.

    (c) A subgroup H of a group G is said to be characteristic if for each $f \in Aut(G)$, $f(H) = H$. Show that every subgroup of Z_{10} is characteristic.

12. Prove that a group of order 300 can't be simple.