Quadratic functional equation

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Introduction

In this lecture, we examine

- biadditive functions
- quadratic functions
- quadratic functional equation.

First we show that every continuous biadditive function $f : \mathbb{R}^2 \to \mathbb{R}$ is of the form $f(x, y) = c\,xy$, where c is an arbitrary real constant.

- Second we give a general representation for the biadditive function in terms of a Hamel basis.
- Third, we determine the continuous solutions of the quadratic functional equation.
- Fourth, we present the representation of quadratic functions in terms of the diagonal of symmetric biadditive functions.

Additive Functions on $\mathbb{R}^2$

An additive function $f : \mathbb{R}^2 \to \mathbb{R}$ in two variables is defined as the solution of the equation

$$f(x+y, u+v) = f(x,u) + f(y,v) \tag{1}$$

for all $x, y, u, v \in \mathbb{R}$.

Decomposition of Additive Functions on $\mathbb{R}^2$

- Every additive function $f : \mathbb{R}^2 \to \mathbb{R}$ in two variable can be decomposed as the sum of two additive functions in one variable, that is,

$$f(x, v) = A_1(x) + A_2(v), \tag{2}$$

where $A_1, A_2 : \mathbb{R} \to \mathbb{R}$ are additive functions on $\mathbb{R}$.

To show (2) holds, one has to prove

(a) $f(x, v) = A_1(x) + A_2(v)$ for all $x, v \in \mathbb{R}$, and

(b) A_1 and A_2 are additive,

where $A_1(x) := f(x, 0)$ and $A_2(x) := f(0, x)$ for all $x \in \mathbb{R}$.

- Hence, the continuous solution of (1) is given by

$$f(x, v) = k_1 x + k_2 v, \tag{3}$$

where k_1, k_2 are arbitrary real constants.

Biadditive Functions on $\mathbb{R}^2$

Definition 1 *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be biadditive if and only if f is additive in each variable, that is,*

$$f(x+y,z) = f(x,z) + f(y,z),$$
$$f(x,y+z) = f(x,y) + f(x,z)$$

for all $x, y, z \in \mathbb{R}$.

An example of a biadditive function is the following

$$f(x, y) = c\,x\,y \quad \text{for } x, y \in \mathbb{R} \tag{4}$$

is biadditive, where c is a real constant.

To see this consider

$$\begin{aligned} f(x+y,z) &= c\,(x+y)\,z \\ &= cxz + cyz \\ &= f(x,z) + f(y,z). \end{aligned}$$

Similarly

$$\begin{aligned} f(x,y+z) &= c\,x\,(y+z) \\ &= cxy + cxz \\ &= f(x,y) + f(x,z). \end{aligned}$$

Hence $f(x) = c\,x\,y$ is an example of biadditive function.

The following theorem says that there are no other continuous biadditive functions besides $f(x, y) = c\,xy$.

Theorem 1 . *Every continuous biadditive map* $f : \mathbb{R}^2 \to \mathbb{R}$ *is of the form*

$$f(x, y) = cxy$$

for all $x, y \in \mathbb{R}$ *for some constant* c *in* $\mathbb{R}$.

Proof : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous biadditive map. Hence f satisfies

$$f(x+y, z) = f(x, z) + f(y, z) \tag{5}$$

for all $x, y, z \in \mathbb{R}$. Letting $x = 0 = y$ in (5), we obtain

$$f(0, z) = 0$$

for all $z \in \mathbb{R}$.

For fixed z, define

$$\phi(x) = f(x, z). \tag{6}$$

Then (5), that is $f(x + y, z) = f(x, z) + f(y, z)$, reduces to

$$\phi(x + y) = \phi(x) + \phi(y) \tag{7}$$

for all $x, y \in \mathbb{R}$. So ϕ is an additive function.

Since f is continuous in each variable, ϕ is also continuous on $\mathbb{R}$. Hence

$$\phi(x) = k\,x. \tag{8}$$

Since z is fixed, k depends on z. Hence we have

$$\phi(x) = k(z)\,x.$$

That is,

$$f(x, z) = x\,k(z). \tag{9}$$

Letting (9), that is, $f(x,z) = x\,k(z)$ into

$$f(x, y+z) = f(x,y) + f(x,z),$$

we get

$$x\,k(y+z) = x\,k(y) + x\,k(z)$$

for all $x, y, z \in \mathbb{R}$. Hence, we have

$$k(y+z) = k(y) + k(z).$$

Since f is continuous in each variable, k is also continuous and we obtain

$$k(y) = c\,y, \tag{10}$$

where c is a constant. Thus

$$f(x, y) = c\,x\,y \tag{11}$$

for all $x, y \in \mathbb{R}$.

Representation of Biadditive Functions

In the next theorem, we present a general representation of biadditive functions in terms of the elements of the Hamel basis of $\mathbb{R}$.

Theorem 2. *Every biadditive map $f : \mathbb{R}^2 \to \mathbb{R}$ can be represented as*

$$f(x, y) = \sum_{k=1}^{n} \sum_{j=1}^{m} \alpha_{kj} r_k s_j, \tag{12}$$

where

$$x = \sum_{k=1}^{n} r_k b_k, \qquad y = \sum_{j=1}^{m} s_j b_j,$$

the r_k, s_j being rational, while the b_j are elements of a Hamel basis B and $\alpha_{kj} = f(b_k, b_j)$.

Proof: Let B be a Hamel basis for the set of reals $\mathbb{R}$. Then $x \in \mathbb{R}$ can be represented as

$$x = \sum_{k=1}^{n} r_k\, b_k \tag{13}$$

with $b_k \in B$ and with rational coefficients r_k.

Similarly, $y \in \mathbb{R}$ can also be represented as

$$y = \sum_{j=1}^{m} s_j\, b_j \tag{14}$$

with $b_j \in B$ and with rational coefficients s_j.

Since f is biadditive, f satisfies

$$f(x_1 + x_2,\, y) = f(x_1,\, y) + f(x_2,\, y), \tag{15}$$

$$f(x,\, y_1 + y_2) = f(x,\, y_1) + f(x,\, y_2) \tag{16}$$

for all $x, x_1, x_2, y, y_1, y_2 \in \mathbb{R}$.

From (15) and (16), using induction, and the fact that f is additive in each variable, we have

$$f\left(\sum_{k=1}^{n} x_k,\, y\right) = \sum_{k=1}^{n} f(x_k,\, y), \tag{17}$$

$$f\left(x,\, \sum_{k=1}^{n} y_k\right) = \sum_{k=1}^{n} f(x,\, y_k). \tag{18}$$

Next we substitute

$$x_1 = x_2 = \cdots = x_n = x \quad \text{and} \quad y_1 = y_2 = \cdots = y_n = y$$

in (17) and (18) respectively, we get

$$f(nx,\, y) = n\, f(x,\, y) = f(x,\, ny). \tag{19}$$

From (19) with $t = \frac{m}{n} x$ (that is, $nt = mx$), we get

$$n\, f(t,\, y) = f(nt,\, y) = f(mx,\, y) = m\, f(x,\, y)$$

which is

$$f(t,\, y) = \frac{m}{n} f(x,\, y).$$

That is,

$$f\left(\frac{m}{n}x,\, y\right) = \frac{m}{n} f(x,\, y). \tag{20}$$

Since f is biadditive, we see that

$$f(x,\, 0) = 0 = f(0,\, y) \tag{21}$$

for all $x, y \in \mathbb{R}$. Next, substituting $x_2 = -x_1 = x$ in (15) and using (21), we obtain

$$f(-x,\, y) = -f(x,\, y). \tag{22}$$

From (22) and (20) we conclude that (19) is valid for all rational numbers. The same argument applies to the second variable, and so we have for all rational numbers r and all real x and y:

$$f(rx,\, y) = r\, f(x,\, y) = f(x,\, ry). \tag{23}$$

Hence by (13), (14), (17), (18) and (23), we obtain

$$\begin{aligned} f(x, y) &= f\left(\sum_{k=1}^{n} r_k b_k, \sum_{j=1}^{m} s_j b_j\right) \\ &= \sum_{k=1}^{n} r_k\, f\left(b_k, \sum_{j=1}^{m} s_j b_j\right) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{m} r_k\, s_j\, f\left(b_k,\, b_j\right) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{m} r_k\, s_j\, \alpha_{kj}, \end{aligned}$$

where $\alpha_{kj} = f(b_k,\, b_j)$. QED

Continuous Solution of Quadratic Equation

The following functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \quad \text{for all } x,y\in\mathbb{R} \tag{24}$$

is known as the *quadratic functional equation*. Next we determine its continuous solution.

Theorem 3 . *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in \mathbb{R}$. Then f is rationally homogeneous of degree two. Moreover on the set of rational numbers $\mathbb{Q}$, f has the form

$$f(r) = c\,r^2$$

for $r \in \mathbb{Q}$, where c is an arbitrary constant.

Proof: Letting $x = 0 = y$ in (24), we obtain

$$f(0) = 0. \tag{25}$$

Next, replacing y by $-y$ in (24), we see that

$$f(x - y) + f(x + y) = 2f(x) + 2f(-y). \tag{26}$$

Comparing (24) and (26), we have

$$f(y) = f(-y)$$

for all $y \in \mathbb{R}$. That is, f is an even function.

Next we show that f is a rationally homogeneous function of degree 2. We put $y = x$ in (24) to get

$$f(2x) = 4f(x)$$

or

$$f(2x) = 2^2 f(x) \quad \text{for } x \in \mathbb{R}.$$

Similarly

$$f(2x + x) + f(2x - x) = 2f(2x) + 2f(x)$$

or

$$f(3x) = 2f(2x) + f(x) = 8f(x) + f(x)$$

which is

$$f(3x) = 3^2 f(x) \quad \text{for } x \in \mathbb{R}.$$

Hence by induction, we get

$$f(nx) = n^2 f(x) \tag{27}$$

for all positive integers n. Next we show that (27) holds for all integers $n \in \mathbb{Z}$.

Suppose n is a negative integer. Then $-n$ is a positive integer. Hence

$$\begin{aligned} f(nx) &= f(-(-n)x) \\ &= f(-nx) \qquad \text{since } f \text{ is even} \\ &= (-n)^2 f(x) \\ &= n^2 f(x). \end{aligned}$$

Hence

$$f(nx) = n^2 f(x)$$

holds for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$.

Let r be an arbitrary rational number. Hence

$$r = \frac{k}{n}$$

for some integer $k \in \mathbb{Z}$ and some natural number $n \in \mathbb{N}$.

Therefore

$$k = r\,n.$$

We consider

$$\begin{aligned} k^2 f(x) &= f(kx) \\ &= f(rnx) \\ &= n^2 f(rx). \end{aligned}$$

Therefore

$$f(rx) = \frac{k^2}{n^2} f(x)$$

or

$$f(rx) = r^2 f(x). \tag{28}$$

That is, f is rationally homogeneous of degree 2.

Letting $x = 1$ in (28), we obtain

$$f(r) = c\,r^2 \quad \text{for } r \in \mathbb{Q}, \tag{29}$$

where $c := f(1)$.

Theorem 4 . *The general continuous solution of*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \qquad \text{(QE)}$$

for all $x, y \in \mathbb{R}$ *is given by*

$$f(x) = c\,x^2,$$

where c *is an arbitrary constant.*

Proof: Let f be the solution of (24) and suppose f to be continuous. For any real number $x \in \mathbb{R}$ there exists a sequence $\{r_n\}$ of rational numbers such that

$$\lim_{n \to \infty} r_n = x.$$

Since f satisfies (24), by previous theorem

$$f(r_n) = c\, r_n^2 \tag{30}$$

for all $n \in \mathbb{Z}$.

Using the continuity of f, we have

$$\begin{aligned}
f(x) &= f\left(\lim_{n\to\infty} r_n\right) \\
&= \lim_{n\to\infty} f(r_n) \\
&= \lim_{n\to\infty} (\, c\, r_n^2\,) \\
&= c \lim_{n\to\infty} r_n^2 \\
&= c\left(\lim_{n\to\infty} r_n\right)^2 \\
&= c\, x^2.
\end{aligned}$$

Hence $f(x) = cx^2$ for all $x \in \mathbb{R}$.

Definition 2 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called a* quadratic function *if* $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ *holds for all* $x, y \in \mathbb{R}$.

- **According to the previous theorem every continuous quadratic function** f **is of the form** $f(x) = c\,x^2$**, where** c **is an arbitrary real constant.**

Representation of Quadratic Functions

In the following theorem, we show that every real-valued quadratic function can be represented as the diagonal of a symmetric biadditive map.

Theorem 5 . *The function* $f : \mathbb{R} \to \mathbb{R}$ *is quadratic if and only if there exists a symmetric biadditive map* $B : \mathbb{R}^2 \to \mathbb{R}$ *such that* $f(x) = B(x, x)$.

Proof: Suppose $f(x) = B(x, x)$. Then

$$
\begin{aligned}
f(x+y) + f(x-y) \\
&= B(x+y, x+y) + B(x-y, x-y) \\
&= B(x, x+y) + B(y, x+y) + B(x, x-y) - B(y, x-y) \\
&= B(x,x) + B(x,y) + B(y,x) + B(y,y) \\
&\quad + B(x,x) - B(x,y) - B(y,x) + B(y,y) \\
&= 2B(x,x) + 2B(y,y) \\
&= 2f(x) + 2f(y).
\end{aligned}
$$

Thus f is a quadratic function.

Now we prove the converse. We suppose $f : \mathbb{R} \to \mathbb{R}$ is a quadratic function, and we define $B : \mathbb{R}^2 \to \mathbb{R}$ as

$$B(x, y) = \frac{1}{4}[f(x+y) - f(x-y)] \quad \text{for } x, y \in \mathbb{R}. \tag{31}$$

Letting $y = 0$ in

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \tag{32}$$

we have $f(0) = 0$ and $x = y$ gives

$$f(2x) = 4f(x). \tag{33}$$

Therefore

$$B(x,x) = \frac{[f(2x) - f(0)]}{4} = \frac{[4f(x)]}{4} = f(x).$$

Interchanging x with y in (32), we get

$$f(x+y) + f(y-x) = 2f(y) + 2f(x).$$

Comparing this equation with (32), we get

$$f(x-y) = f(y-x).$$

Hence f is an even function.

Next we obtain

$$B(x,y) = \frac{[f(x+y) - f(x-y)]}{4} = \frac{[f(y+x) - f(y-x)]}{4} = B(y,x).$$

Further,

$$\begin{aligned} B(-x,y) &= \frac{1}{4}[f(-x+y) - f(-x-y)] \\ &= \frac{1}{4}[f(x-y) - f(x+y)] \\ &= -\frac{1}{4}[f(x+y) - f(x-y)] \\ &= -B(x,y). \end{aligned}$$

Thus B is odd in the first variable. Similarly, one can show that B is odd in the second variable.

Next we show that B is additive in the first variable.

$$\begin{aligned}
4[B(x+y,z)+B(x-y,z)] \\
&= f(x+y+z)+f(x-y+z) \\
&\qquad - f(x+y-z)-f(x-y-z) \\
&= 2f(x+z)+2f(y)-2f(x-z)-2f(y) \\
&= 2f(x+z)-2f(x-z) \\
&= 8B(x,z).
\end{aligned}$$

Therefore we have shown

$$B(x+y,z)+B(x-y,z)=2B(x,z) \quad \text{for } x,z\in\mathbb{R}. \tag{34}$$

Interchanging x with y, we obtain

$$B(y+x,z)+B(y-x,z)=2B(y,z). \tag{35}$$

Subtracting (35) from (34), we have

$$B(x-y,z) - B(y-x,z) = 2B(x,z) - 2B(y,z). \tag{36}$$

Since B is odd in each variable, (36) yields

$$B(x-y,z) = B(x,z) - B(y,z).$$

Replacing $-y$ with y and using the fact that B is an odd function in the first variable, we get

$$B(x+y,z) = B(x,z) + B(y,z).$$

Therefore $B : \mathbb{R}^2 \to \mathbb{R}$ is additive in the first variable.

Since B is symmetric, B is also additive in the second variable.

Thus B is a biadditive function. The proof of the theorem is now complete.

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Thank You