# MATH 668-01 Homework 6

*Due: Friday, April 27, 2018 at 2pm*

**Instructions:** *Each of the following problems must be submitted to the instructor on or before the due date. Partial credit may be given for incorrect answers which make some positive progress. Late homework will not be accepted.*

1. (10 points) Let $\mathbf{I}_n$ be the $n \times n$ identity matrix and let $\boldsymbol{j}_n$ be the $n$-dimensional vector of 1's. Suppose $\boldsymbol{y} \sim N_n\left(\mu\boldsymbol{j}_n, \frac{1}{\tau}\mathbf{I}_n\right)$, $\mu|\tau \sim N\left(0, \frac{1}{\tau}\right)$, and $\tau \sim \text{Gamma}(\alpha, \delta)$. Show that

$$\frac{n}{n+1}\bar{y} \pm t_{\omega/2,n+2\alpha}\sqrt{\left(\left(\frac{n-1}{n+1}\right)s_y^2 + \left(\frac{n}{n+1}\right)\bar{y}^2 + \left(\frac{2}{n+1}\right)\delta\right)/(n+2\alpha)}$$

is a $100(1-\omega)\%$ credible set for $\mu$ where $s_y^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(y_i - \bar{y})^2$.

2. (10 points) A machine produces AAA batteries in batches of 3. Data on three batches are shown below.

| Batch | Weights (in grams) |
|---|---|
| 1st | 11.2, 11.4, 11.6 |
| 2nd | 11.9, 11.8, 11.3 |
| 3rd | 11.0, 11.2, 11.3 |

Assuming a one-way random effects model $y_{ij} = \mu + a_i + \varepsilon_{ij}$ where $\mu$ is the mean battery weight, $a_i \sim N(0, \sigma_1^2)$ is a random effect for each batch, and $\varepsilon_{ij} \sim N(0, \sigma^2)$ are independent random errors, compute the restricted maximum likelihood (REML) estimates of $E(\boldsymbol{y})$ and $\text{var}(\boldsymbol{y})$ where $\boldsymbol{y}^\top = (y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}, y_{31}, y_{32}, y_{33})$.

3. (10 points) Suppose that, for $i = 1, \ldots, n$, $\boldsymbol{y}_i$ are independent Poisson$(\lambda_i)$ where $\lambda_i = e^{\boldsymbol{x}_i^\top\boldsymbol{\beta}}$ where $\boldsymbol{\beta}$ is a fixed but unknown $p$-dimensional vector of model parameters.

(a – 2 pts) Show that the log-likelihood function has the form

$$l(\boldsymbol{\beta}) = \sum_{i=1}^{n}\left\{y_i\boldsymbol{x}_i^\top\boldsymbol{\beta} - e^{\boldsymbol{x}_i^\top\boldsymbol{\beta}} - \ln y_i!\right\}.$$

(b – 4 pts) Compute $\dfrac{\partial l}{\partial\boldsymbol{\beta}}$ and $\dfrac{\partial^2 l}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^\top}$.

(c – 4 pts) The dataset "us2003sars.txt" contains the number of cases of SARS in the United States where the response variable $y_i$ is the number of SARS cases on day $i$ and the explanatory variable $x_i$ is the number of days after December 31, 2002. Assume the Poisson regression model described above with $\boldsymbol{x}_i = \begin{pmatrix} 1 \\ x_i \end{pmatrix}$. Using three iterations of the IWLS algorithm with starting value $\hat{\beta}_0 = -5.000$ and $\hat{\beta}_1 = 0.070$, compute the maximum likelihood estimates of $\beta_0$ and $\beta_1$. Show at least three decimal places on each iteration.