M 622 HW, due Monday, April 21

1. Let p and q be prime numbers, and let $t(x) = x^p - q$.

   (a) Explain why $t(x)$ is irreducible over $\mathbb{Q}$.

   (b) Describe the roots of $t(x)$ (they are contained in $\mathbb{C}$).

   (c) Describe the splitting K of $t(x)$, and determine $[K : \mathbb{Q}]$.

   (d) Let $G = Gal(K/\mathbb{Q})$. Use the appropriate part of the Sylow Theorem to explain why G has just one Sylow-p subgroup, which we'll call N. Use another part of the Sylow Theorem to explain why N is normal.

   (e) By the Fundamental Theorem of Galois Theory (FTGT), the fixed field J of N is a splitting field of some polynomial $a(x) \in \mathbb{Q}[x]$. Since $|N| = p$, use FTGT (be explicit about which part) to determine $[J : \mathbb{Q}]$, and then determine J explicitly and $a(x)$ explicitly.

   (f) Use FTGT to explain why G can't be Abelian.

   (g) Is G solvable? Explain—there are several ways to do so.

2. Suppose F is a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{R}$, $t(x) \in F[x]$, and S is the splitting field of $t(x) \in F[x]$. Show that if $[S : F]$ is odd, then $S \subset \mathbb{R}$.

3. An old exercise from M622 states that if p is a prime number, then if α is a transposition of S_p and β is a p-cycle of S_p , then $S_p = <\alpha, \beta>$, prove that if $p(x) \in \mathbb{Q}[x]$ is a degree 5 irreducible polynomial having exactly two non-real roots, then the Galois group of $p(x)$ is S_5.