Lecture 14: More Hypothesis Testing Examples

MATH 667-01
Statistical Inference
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- We give a couple more examples of the likelihood ratio tests (LRT).
- We compare these tests to other tests. One of these other tests includes a Bayesian test procedure discussed in Section 8.2.2 of Casella and Berger (2002)[1].
- We also prove a theorem conserning sufficient statistics and lieklihood ratio tests, and include an example comparing a likelihood ratio test based on a sufficient statistic with a test based on a different statistic.

[1] Casella, G. and Berger, R. (2002). *Statistical Inference, Second Edition*. Duxbury Press, Belmont, CA.

- *Example L14.1*: Suppose we toss a coin 5 times and count the total number of heads which occur. We assume each toss is independent and the probability of heads (denoted by p) is the same on each toss.
  Let $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5)$ where
  $x_i = \begin{cases} 1 & \text{if the } i\text{th toss is heads} \\ 0 & \text{if the } i\text{th toss is tails} \end{cases}$.
  We want to test $H_0 : p \leq .5$ versus $H_1 : p > .5$.
  (a) Show that the likelihood ratio test has a critical region of the form $\left\{\boldsymbol{x} : \sum_{i=1}^{5} x_i \geq K\right\}$.
  (b) Find the smallest integer K such that this is a level $.05$ test.

- *Answer to Example L14.1*: (a) Here $X_1, \ldots, X_5$ are iid Bernoulli(p) random variables so the likelihood function is

$$L(p; \boldsymbol{x}) = \prod_{i=1}^{5} p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^5 x_i}(1-p)^{5-\sum_{i=1}^5 x_i}.$$

Since the MLE of p is $\hat{p} \overset{7.13}{=} \bar{x}$,

$$\sup_{p\in\Theta} L(p; \boldsymbol{x}) = L(\bar{x}; \boldsymbol{x}) = \bar{x}^{5\bar{x}}(1-\bar{x})^{5-5\bar{x}}.$$

Since the maximizer of L over $\Theta_0 = \{p : p \in [0, 0.5]\}$ is $\tilde{p} = \begin{cases} \bar{x} & \text{if } \bar{x} \le .5 \\ .5 & \text{if } \bar{x} > .5 \end{cases}$,

$$\sup_{p\in\Theta_0} L(p; \boldsymbol{x}) = L(\tilde{p}; \boldsymbol{x}) = \begin{cases} \bar{x}^{5\bar{x}}(1-\bar{x})^{5-5\bar{x}} & \text{if } \bar{x} \le .5 \\ .5^{5\bar{x}}.5^{5-5\bar{x}} & \text{if } \bar{x} > .5 \end{cases}.$$

Likelihood Ratio Tests

- *Answer to Example L14.1 continued*: So, the likelihood ratio

$$\lambda(\boldsymbol{x}) = \begin{cases} \frac{.5^5}{\bar{x}^{5\bar{x}}(1-\bar{x})^{5(1-\bar{x})}} & \text{if } \bar{x} > .5 \\ 1 & \text{if } \bar{x} \leq .5 \end{cases}$$

  is a decreasing function of $\bar{x}$;
  thus, rejecting H_0 if $\lambda(\boldsymbol{x}) \leq c$ is equivalent to
  rejecting H_0 if $\bar{x} \geq \tilde{K} \Leftrightarrow \displaystyle\sum_{i=1}^{5} x_i \geq 5\tilde{K} = K$.

- *Answer to Example L14.1 continued*: (b) If $K = 5$, then the size of the test is

$$\sup_{p\in[0,.5]} P\left(\sum_{i=1}^{5} X_i = 5\right) = \sup_{p\in[0,.5]} \binom{5}{5} p^5(1-p)^0 = \sup_{p\in[0,.5]} p^5 = .03125.$$

If $K = 4$, then the size of the test is

$$\begin{aligned}
\sup_{p\in[0,.5]} P\left(\sum_{i=1}^{5} X_i \geq 4\right) &= \sup_{p\in[0,.5]} \left\{p^5 + \binom{5}{4} p^4(1-p)\right\} \\
&= \sup_{p\in[0,.5]} \left\{p^5 + 5p^4(1-p)\right\} \\
&= .5^5 + 5(.5)^4(.5) = .1875.
\end{aligned}$$

So, the LRT is a level $.05$ test when $K = 5$ but not when $K = 4$, and therefore, $K = 5$ is the smallest integer that makes the LRT a level $.05$ test.

- Hypothesis testing is much different from a Bayesian perspective where the parameter is considered random.
- From the Bayesian perspective, the natural approach is to compute

$$P(H_0 \text{ is true}|\boldsymbol{x}) = P(\theta \in \Theta_0|\boldsymbol{x}) = \int_{\Theta_0} \pi(\theta|\boldsymbol{x})d\theta$$

and

$$P(H_1 \text{ is true}|\boldsymbol{x}) = P(\theta \in \Theta_0^c|\boldsymbol{x}) = \int_{\Theta_0^c} \pi(\theta|\boldsymbol{x})d\theta$$

based on the posterior distribution $\pi(\theta|\boldsymbol{x})$.

- *Example L14.2*: Suppose we toss a coin 5 times and count the total number of heads which occur. We assume each toss is independent and the probability of heads (denoted by p) is the same on each toss.
  Consider a Bayesian model which assumes that p follows a Uniform$(0, 1)$ prior. What is the probability of the the null hypothesis $H_0 : p \leq .5$ if $\displaystyle\sum_{i=1}^{5} X_i = 5$?
- *Answer to Example L14.2*:
  Since $p|\boldsymbol{X} = \boldsymbol{x} \sim$ beta$(\sum_{i=1}^{5} x_i + 1, 5 - \sum_{i=1}^{5} x_i + 1)$ from slide 7.24, the probability is

$$P\left(p \leq .5 \middle| \sum_{i=1}^{5} x_i = 5\right) = \int_0^{.5} 6p^5 \; dp = 1/64 = .015625.$$

- *Theorem L14.1* (Thm 8.2.4 on p.377): If $T(\boldsymbol{X})$ is a sufficient statistic for θ and $\lambda^*(t)$ and $\lambda(\boldsymbol{x})$ are the LRT statistics based on T and $\boldsymbol{X}$, respectively, then $\lambda^*(T(\boldsymbol{x})) = \lambda(\boldsymbol{x})$ for every $\boldsymbol{x}$ in the sample space.
- *Proof of Theorem L14.1*: Let $g(t|\theta)$ be the pdf/pmf of $T(\boldsymbol{X})$ and let $L^*(\theta|t) = g(t|\theta)$ be the corresponding likelihood function. Then we have

$$\lambda(\boldsymbol{x}) \overset{13.3}{=} \frac{\sup\limits_{\Theta_0} L(\theta|\boldsymbol{x})}{\sup\limits_{\Theta} L(\theta|\boldsymbol{x})} \overset{1.11}{=} \frac{\sup\limits_{\Theta_0} f(\boldsymbol{x}|\theta)}{\sup\limits_{\Theta} f(\boldsymbol{x}|\theta)} \overset{10.7}{=} \frac{\sup\limits_{\Theta_0} g(T(\boldsymbol{x})|\theta)h(\boldsymbol{x})}{\sup\limits_{\Theta} g(T(\boldsymbol{x})|\theta)h(\boldsymbol{x})}$$

$$= \frac{\sup\limits_{\Theta_0} g(T(\boldsymbol{x})|\theta)}{\sup\limits_{\Theta} g(T(\boldsymbol{x})|\theta)} = \frac{\sup\limits_{\Theta_0} L^*(\theta|T(\boldsymbol{x}))}{\sup\limits_{\Theta} L^*(\theta|T(\boldsymbol{x}))} \overset{13.3}{=} \lambda^*(T(\boldsymbol{x})).$$

- *Example L14.3*: Suppose $X_1, \ldots, X_n$ is a random sample from a distribution with pdf

$$f(x|\theta) = e^{-(x-\theta)} I_{[\theta,\infty)}(x)$$

where $\theta \in \mathbb{R}$. Consider the one-sided test $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$.
(a) Show that the likelihood ratio test statistic has a critical region of the form $\left\{\boldsymbol{x} : x_{(1)} \geq K\right\}$.
(b) Compute the power function for this test.
(c) Find K so that the size of the test is $.05$.

- *Answer to Example L14.3*: (a) The likelihood function

$$L(\theta;\boldsymbol{x}) = \prod_{i=1}^{n} e^{-(x_i-\theta)} I_{[\theta,\infty)}(x_i) = e^{-\sum_{i=1}^{n} x_i} e^{n\theta} I_{[\theta,\infty)}(x_{(1)}).$$

is a positive, increasing function of θ on $(-\infty, x_{(1)}]$ and equal to 0 on $(x_{(1)}, \infty)$. So,

$$\sup_{\theta\in\Theta_0} L(\theta;\boldsymbol{x}) = \begin{cases} e^{-\sum_{i=1}^{n} x_i} e^{n x_{(1)}} & \text{if } x_{(1)} \le 0 \\ e^{-\sum_{i=1}^{n} x_i} & \text{if } x_{(1)} > 0 \end{cases} \quad \text{and}$$

$$\sup_{\theta\in\Theta} L(\theta;\boldsymbol{x}) = e^{-\sum_{i=1}^{n} x_i} e^{n x_{(1)}}.$$

The likelihood ratio $\lambda(\boldsymbol{x}) = \begin{cases} 1 & \text{if } x_{(1)} \le 0 \\ e^{-n x_{(1)}} & \text{if } x_{(1)} > 0 \end{cases}$ is a nonincreasing function of $x_{(1)}$ so, for $c < 1$, a LRT rejects H_0 when $\lambda(\boldsymbol{x}) \le c$, or equivalently, when $x_{(1)} \ge -\frac{1}{n}\ln c \equiv K$.

LRT and Sufficiency

- *Answer to Example L14.3*: (b) The power function for the LRT is

$$
\begin{aligned}
\beta(\theta) &= P_\theta(X_{(1)} \geq K) \\
&= P(X_1 \geq K \text{ and } X_2 \geq K \text{ and } \cdots \text{ and } X_n \geq K) \\
&= \prod_{i=1}^{n} P(X_i \geq K) \\
&= \prod_{i=1}^{n} \left(\int_K^\infty e^{-(x_i-\theta)}\ dx_i \right) \\
&= \prod_{i=1}^{n} \left(\left[-e^{-(x_i-\theta)} \right]_K^\infty \right) = \prod_{i=1}^{n} \left(e^{-(K-\theta)} \right) = \left(e^{-n(K-\theta)} \right).
\end{aligned}
$$

(c) Since $\sup_{\theta\in(-\infty,0]} \beta(\theta) = e^{-nK}$, we obtain the LRT with size $.05$ by choosing K so that $e^{-nK} = .05 \Leftrightarrow K = \frac{1}{n}\ln 20$.

LRT and Sufficiency

- What if we compare this LRT with a hypothesis test not based on a sufficient statistic?
- For example, consider a test with rejection region $\{\boldsymbol{x} : \bar{x} > C\}$. It can be shown that $\bar{X} - \theta \sim$ Gamma$(n, \frac{1}{n})$ so the power function for this test is

$$\beta'(\theta) = P_\theta(\bar{X} > C) = \int_{C-\theta}^{\infty} \frac{n^n}{(n-1)!} y^{n-1} e^{-yn}\, dy$$

  with $C \approx 1.57$ so that $\beta'(0) = \alpha$.
- Let $n = 10$. On the next slide, the power function $\beta(\theta)$ for the LRT with size $.05$ is shown in black and the power function for the other test based on $\bar{x}$ with size $.05$ is shown in red.
- We see that $\beta'(\theta) < \beta(\theta)$ when $\theta \in \Theta_0^c$ and $\beta'(\theta) \geq \beta(\theta)$ when $\theta \in \Theta_0$.

LRT and Sufficiency