HW7 solutions

1. (a) Here are the likelihood ratios

x	1	2	3	4	5	6
$\frac{f(x\mid H_1)}{f(x\mid H_0)}$	12	$\frac{4}{3}$	$\frac{6}{5}$	$\frac{50}{7}$	$\frac{1}{2}$	$\frac{6}{19}$

If we only reject when $x=1$, then $P_{H_0}(\text{reject } H_0) = .01$.

Next, we add $x=4$ to the rejection region. Then $P_{H_0}(X\in\{1,4\}) = .01+.07=.08$.

Next, we add $x=2$ to the rejection region, but only reject H_0 with probability $\phi(2)$.

Then

$$\begin{aligned} P_{H_0}(\text{reject } H_0) &= P_{H_0}(X=1) + P_{H_0}(X=4) + P_{H_0}(X=2 \text{ and reject } H_0) \\ &= .01 + .07 + P_{H_0}(X=2)\, P_{H_0}(\text{reject } H_0 \mid X=2) \\ &= .08 + .03\,\phi(2) = .10. \end{aligned}$$

So $.08 + .03\,\phi(2) = .10 \Rightarrow .03\,\phi(2) = .02 \Rightarrow \phi(2) = \frac{.02}{.03} = \frac{2}{3}$.

By the Neymen-Pearson Lemma (Thm L15.1),

$$\phi(x) = \begin{cases} 1 & \text{if } x=1,4 \\ 2/3 & \text{if } x=2 \\ 0 & \text{if } x=3,5,6 \end{cases}$$

is the test function for a UMP test, and we see that its size is .10.

(b) $$\pi(H_0\mid 1) = \frac{\pi(H_0)f(1\mid H_0)}{\pi(H_0)f(1\mid H_0)+\pi(H_1)f(1\mid H_1)} = \frac{.8(.01)}{.8(.01)+.2(.12)} = .25$$

$$\pi(H_0\mid 2) = \frac{.8(.03)}{.8(.03)+.2(.04)} = .75$$

$$\pi(H_0\mid 3) = \frac{.8(.05)}{.8(.05)+.2(.06)} = \frac{10}{13} \approx .769$$

$$\pi(H_0\mid 4) = \frac{.8(.07)}{.8(.07)+.2(.50)} = \frac{14}{39} \approx .359$$

$$\pi(H_0\mid 5) = \frac{.8(.08)}{.8(.08)+.2(.04)} = \frac{8}{9} \approx .889$$

$$\pi(H_0\mid 6) = \frac{.8(.76)}{.8(.76)+.2(.24)} = \frac{38}{41} \approx .927$$

2. (a) The joint pdf of X_1 and X_2 is

$$f(x_1,x_2) = f(x_1)f(x_2) = \frac{1}{\beta}e^{-x_1/\beta}\cdot\frac{1}{\beta}e^{-x_2/\beta} = \frac{1}{\beta^2}e^{-\frac{x_1+x_2}{\beta}}\prod_{i=1}^{2} I_{(0,\infty)}(x_i)$$

so we see that X_1+X_2 is a sufficient statistic for β by the Factorization Theorem.

Now, we show $T = X_1+X_2$ has a nondecreasing MLR.

Since X_1 and X_2 are iid exponential with mean β, $T \sim \text{Gamma}(2,\beta)$.

(Note: The MGF of a $\text{Gamma}(\alpha,\beta)$ is $M(t) = (1-\frac{t}{\beta})^{-\alpha}$ and $X_i \sim \text{Gamma}(1,\beta)$ so MGF of X_1+X_2 is $M_{X_1+X_2}(t) = E[e^{t(X_1+X_2)}] = E[e^{tX_1}e^{tX_2}]$
$= E[e^{tX_1}]E[e^{tX_2}] = (1-\frac{t}{\beta})^{-1}(1-\frac{t}{\beta})^{-1} = (1-\frac{t}{\beta})^{-2}$ which is the MGF of a $\text{Gamma}(2,\beta)$ random variable.) The pdf of T is $g(t) = \frac{1}{\Gamma(2)\beta^2}te^{-t/\beta}I_{(0,\infty)}(t)$

If $\theta_2 > \theta_1$, then $\dfrac{g(t|\theta_2)}{g(t|\theta_1)} = \left(\frac{\theta_1}{\theta_2}\right)^2 e^{-\frac{t}{\theta_2}+\frac{t}{\theta_1}} = \left(\frac{\theta_1}{\theta_2}\right)^2 e^{\frac{(\theta_2-\theta_1)t}{\theta_1\theta_2}}$ is an increasing function of t (since $\theta_1>0$, $\theta_2>0$, and $\theta_2-\theta_1>0$.).

By the Karlin-Rubin Theorem (Thm L15.3), the test which rejects $H_0: \beta<\beta_0$ iff $T>t_0$ is a UMP level α test. To attain level .05, we need

$$.05 = P_{H_0}(T>t_0) = \int_{t_0}^{\infty}\frac{1}{\beta_0^2}te^{-t/\beta_0}dt = \frac{1}{\beta_0^2}\left[-\beta_0 te^{-t/\beta_0} - \beta_0^2e^{-t/\beta_0}\right]_{t_0}^{\infty}$$

D	I
t	e^{-t/β_0}
1	$-\beta_0 e^{-t/\beta_0}$
0	$\beta_0^2 e^{-t/\beta_0}$

$$= \left[-\left(\frac{t}{\beta_0}+1\right)e^{-t/\beta_0}\right]_{t_0}^{\infty}$$

$$= \left(\frac{t_0}{\beta_0}+1\right)e^{-t_0/\beta_0}.$$

Solving $(1+\frac{t_0}{\beta_0})e^{-t_0/\beta} = .05$, we obtain $\frac{t_0}{\beta_0} = m \Rightarrow t_0 = m\beta_0$

So the critical region for the UMP level .05 test is $\{(x_1,x_2) : x_1+x_2 > m\beta_0\}$.

(b) The corresponding 95% confidence interval for β is $\left[\frac{X_1+X_2}{m}, \infty\right)$

since
$$P(X_1+X_2 \leq m\beta_0) = .95$$
$$\Downarrow$$
$$P\left(\frac{X_1+X_2}{m} \leq \beta\right) = .95.$$

3. (a) Let $Y = \frac{\max\{X_1, \dots, X_n\}}{\theta} = \max\{\frac{X_1}{\theta}, \dots, \frac{X_n}{\theta}\}$.

The cdf of Y is $F(y) = P(Y \le y)$ for $y \in (0,1)$

$$= P\left(\frac{X_1}{\theta} \le y, \dots, \frac{X_n}{\theta} \le y\right)$$
$$= P(X_1 \le \theta y, \dots, X_n \le \theta y)$$
$$= P(X_1 \le \theta y) \cdots P(X_n \le \theta y)$$
$$= \left(P(X_1 \le \theta y)\right)^n$$
$$= \left(\int_0^{\theta y} \frac{1}{\theta}\,dx\right)^n = \left(\left[\frac{x}{\theta}\right]_0^{\theta y}\right)^n = \begin{cases} 0 & \text{if } y < 0 \\ y^n & \text{if } 0 < y \le 1 \\ 1 & \text{if } y \ge 1 \end{cases}.$$

Then the pdf of Y is $f(y) = F'(y) = ny^{n-1} I_{(0,1)}(y)$

so we see that $Y = \frac{\max\{X_1, \dots, X_n\}}{\theta} \sim \underbrace{\text{Beta}(n,1)}_{\text{does not depend on } \theta}$ so it is a pivot.

(b) To obtain a $100(1-\alpha)\%$ confidence interval of the form $[L(\underline{X}), U(\underline{X})]$, we need to find θ_L and θ_u such that

$$P\left(\theta_L \le \frac{\max\{X_1, \dots, X_n\}}{\theta} \le \theta_u\right) = P\left(\frac{\max\{X_1, \dots, X_n\}}{\theta_u} \le \theta \le \frac{\max\{X_1, \dots, X_n\}}{\theta_L}\right) = 1 - \alpha.$$

Since $P(\theta_L \le Y \le \theta_u) = 1 - P(Y < \theta_L \text{ or } Y > \theta_u)$

$$= 1 - P(Y < \theta_L) - P(Y > \theta_u)$$
$$= 1 - \theta_L^n - (1 - \theta_u^n) = \theta_u^n - \theta_L^n,$$

we can choose any, positive θ_L and θ_u such that $\theta_u^n - \theta_L^n = 1 - \alpha$.

In particular, we can pick $\alpha_L > 0$ and $\alpha_u > 0$ such that $\alpha_L + \alpha_u = \alpha$ and

$$\theta_u^n = 1 - \alpha_u \quad \text{and} \quad \theta_L^n = \alpha_L$$
$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$
$$\theta_u = \sqrt[n]{1 - \alpha_u} \qquad \theta_L = \sqrt[n]{\alpha_L}$$

so $\left[\frac{\max\{X_1, \dots, X_n\}}{\sqrt[n]{1-\alpha_u}}, \frac{\max\{X_1, \dots, X_n\}}{\sqrt[n]{\alpha_L}}\right]$ is a $100(1-\alpha)\%$ confidence interval for θ.

4. (a) Since $Var[X_i] = \lambda < \infty$, $\bar{X}_n \to E[X_i] = \lambda$ in probability by the Weak Law of Large Numbers (Thm L17.4). Thus, $P(|\bar{X}_n - \mu| < \varepsilon) \to 1$ as $n \to \infty$ by Def. L17.2 which implies $\bar{X}_n$ is a consistent estimator of λ.

(b) The MGF of X_i exists ($E[e^{tX_i}] = e^{\lambda(e^t - 1)}$ in a nbhd of $t=0$) so the Central Limit Theorem (Thm L18.2) implies $\frac{\sqrt{n}(\bar{X}_n - \lambda)}{\sqrt{\lambda}} \to N(0,1)$ in distribution; equivalently, $\sqrt{n}(\bar{X}_n - \lambda) \to N(0, \lambda)$ in distribution.

Letting $g(\lambda) = \sqrt{\lambda}$, the Delta Method (Thm L18.4) implies that

$$\sqrt{n}\left(g(\bar{X}_n) - g(\lambda)\right) = \sqrt{n}\left(\sqrt{\bar{X}_n} - \sqrt{\lambda}\right) \to N(0, \tfrac{1}{4}) \text{ in distribution}$$

since $\sigma^2 (g'(\lambda))^2 = \lambda \left(\frac{1}{2\sqrt{\lambda}}\right)^2 = \lambda\left(\frac{1}{4\lambda}\right) = \frac{1}{4}$. Now we check if this is the nCRLB.

$$\frac{d}{d\lambda}[\ln f(x|\lambda)] = \frac{d}{d\lambda}\left[\ln \frac{\lambda^x e^{-\lambda}}{x!}\right] = \frac{d}{d\lambda}\left[x \ln\lambda - \lambda - \ln x!\right] = \frac{x}{\lambda} - 1$$

$$\Rightarrow E\left[\left(\frac{d}{d\lambda} \ln f(X|\lambda)\right)^2\right] = E\left[\left(\frac{X}{\lambda} - 1\right)^2\right] = \frac{1}{\lambda^2} E[(X-\lambda)^2] = \frac{1}{\lambda^2} Var[X] = \frac{1}{\lambda^2}\lambda = \frac{1}{\lambda}$$

$$nCRLB = \frac{\left[\frac{d}{d\lambda}\sqrt{\lambda}\right]^2}{E\left[\left(\frac{d}{d\lambda} \ln f(X|\lambda)\right)^2\right]} = \frac{\left[\frac{1}{2\sqrt{\lambda}}\right]^2}{\frac{1}{\lambda}} = \frac{\frac{1}{4\lambda}}{\frac{1}{\lambda}} = \frac{1}{4}.$$

So, $\sqrt{n}\left(\sqrt{\bar{X}_n} - \sqrt{\lambda}\right) \to N(0, nCRLB)$ which is what we need to write that $\sqrt{\bar{X}_n}$ is asymptotically efficient for $\sqrt{\lambda}$.

(c) Pick $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $\alpha_1 + \alpha_2 = .05$ so that

$$.95 = 1 - \alpha = P\left(-z_{\alpha_1} \le \frac{\sqrt{n}\left(\sqrt{\bar{X}_n} - \sqrt{\lambda}\right)}{\frac{1}{2}} \le z_{\alpha_2}\right) = P\left(\sqrt{\bar{X}_n} - \frac{1}{2\sqrt{n}} z_{\alpha_1} \le \sqrt{\lambda} \le \sqrt{\bar{X}_n} + \frac{1}{2\sqrt{n}} z_{\alpha_2}\right)$$

which gives us an approximate 95% confidence interval $\left[\sqrt{\bar{X}_n} - \frac{1}{2\sqrt{n}} z_{\alpha_1}, \sqrt{\bar{X}_n} + \frac{1}{2\sqrt{n}} z_{\alpha_2}\right]$ for $\sqrt{\lambda}$.

If $\alpha_1 = \alpha_2 = .025$, $n = 10000$, and $\bar{x} = 4.4$, then the approximate 95% CI for $\sqrt{\lambda}$ is

$$\sqrt{4.4} \pm \frac{1}{2(100)} 1.960 \implies [2.0878, 2.1074].$$