Math 621 HW 1, due Thursday, Sept. 1

Please work out on scrap paper, download these pages, and after editing your work, write yours proofs, answers, on them. Put your name somewhere. Thanks.

1. Let S be a non-empty set, and let $*$ be a binary operation on S. Suppose $*$ is associative, and suppose that $e \in S$ is an identity element of $(S, *)$, an element e such that for all $s \in S$ $e * s = s = s * e$.

   (a) (2 points) Prove that the identity of $(S, *)$ is unique. That is,show that if f is also an identity element of $(S, *)$, then $e = f$.

   (b) (2 points) Suppose that s has an inverse v in $(S, *)$. (So $s * v = e = v * s$.) Prove the inverse of an element of $(S, *)$, if it exists, is unique.

   (c) (2 points) If s has an inverse, denote it by s^{-1}. Suppose that u and v are both contained in S, and both u and v have inverses in $(S, *)$. Show that $u * v$ has an inverse in $(S, *)$.

(d) (2 points) Suppose that every element of $(S, *)$ has an inverse. Show that if s, t, u are in S, and $s * u = t * u$, then $s = t$.

(e) (6 points) Suppose that *every* element s in $(S, *)$ has an inverse s^{-1}.

Let b be an element of S. Define a binary operation $\circ$ on S as follows: For all $s, t \in S$, let $s \circ t = s * b * t$. (There's no need to parenthesize "$s * b * t$" since $*$ is associative.)

i. (2 out of 6 points) Show that $\circ$ is associative.

ii. **Short answer.** (2 out of 6 points) $(S, \circ)$ has an identity element. What is it?

iii. **Short answer.** (2 out of 6 points) Every element $s \in S$ has an inverse $\overline{s}$ in $(S, \circ)$. What is $\overline{s}$, the inverse of s, in $(S, \circ)$?

iv. **Optional**: +1 point, and perhaps the chance for you to write your solution on the board.

Find a bijection $\Gamma : S \to S$ such that for all $s, t \in S$, $\Gamma(s * t) = \Gamma(s) \circ \Gamma(t)$. Verify that Γ satisfies the equation of the previous sentence, and that Γ is a bijection.

(f) **Optional**, +1 point, and perhaps the chance for you to write your solution on the board.

Recall that if $\mathbb{Z}$ is partitioned into a collection of non-empty subsets $S_1, \ldots, S_k$, the partition is said to be *compatible with addition* if whenever a, b, c are in $\mathbb{Z}$, and a and b are same subset, then $a + c$ and $b + c$ are in the same subset. More formally, if there exists $i \in \{1, \ldots, k\}$ such that $\{a, b\} \subseteq S_i$, then there exists $j \in \{1, \ldots, k\}$ such that $\{a + c, b + c\} \subseteq S_j$.

Show that if $\mathbb{Z}$ is partitioned into two sets S and T, with $0 \in S$, and with $1 \in T$, and the partition is compatible with addition in $\mathbb{Z}$, then $S = 2\mathbb{Z}$ (the even integers) and $T = 1 + 2\mathbb{Z}$ (the odd integers). Suggestion: You could begin by showing that $0 \in S$ and the partition is compatible with addition implies that S is closed under addition and that $\{s + 1 : s \in S\} = S + 1 \subseteq T$.