Jensen functional equation

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Introduction

In this lecture, we present

- a brief introduction to convex functions
- the solution of Jensen equation on $\mathbb{R}$
- the continuous solution of Jensen equation on $[a, b]$.

Convex Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be convex if and only if it satisfies the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1}$$

for all $x, y \in \mathbb{R}$

Convex functions were first introduced by J.L.W.V. Jensen in 1905, although functions satisfying the condition (1) had been treated by Hadamard (1893) and Hölder (1889).

In 1905, Jensen wrote

> It seems to me that the notion of convex functions is just as fundamental as positive or increasing functions. If I am not mistaken in this, the notion ought to find its place in elementary expositions of the theory of real functions.

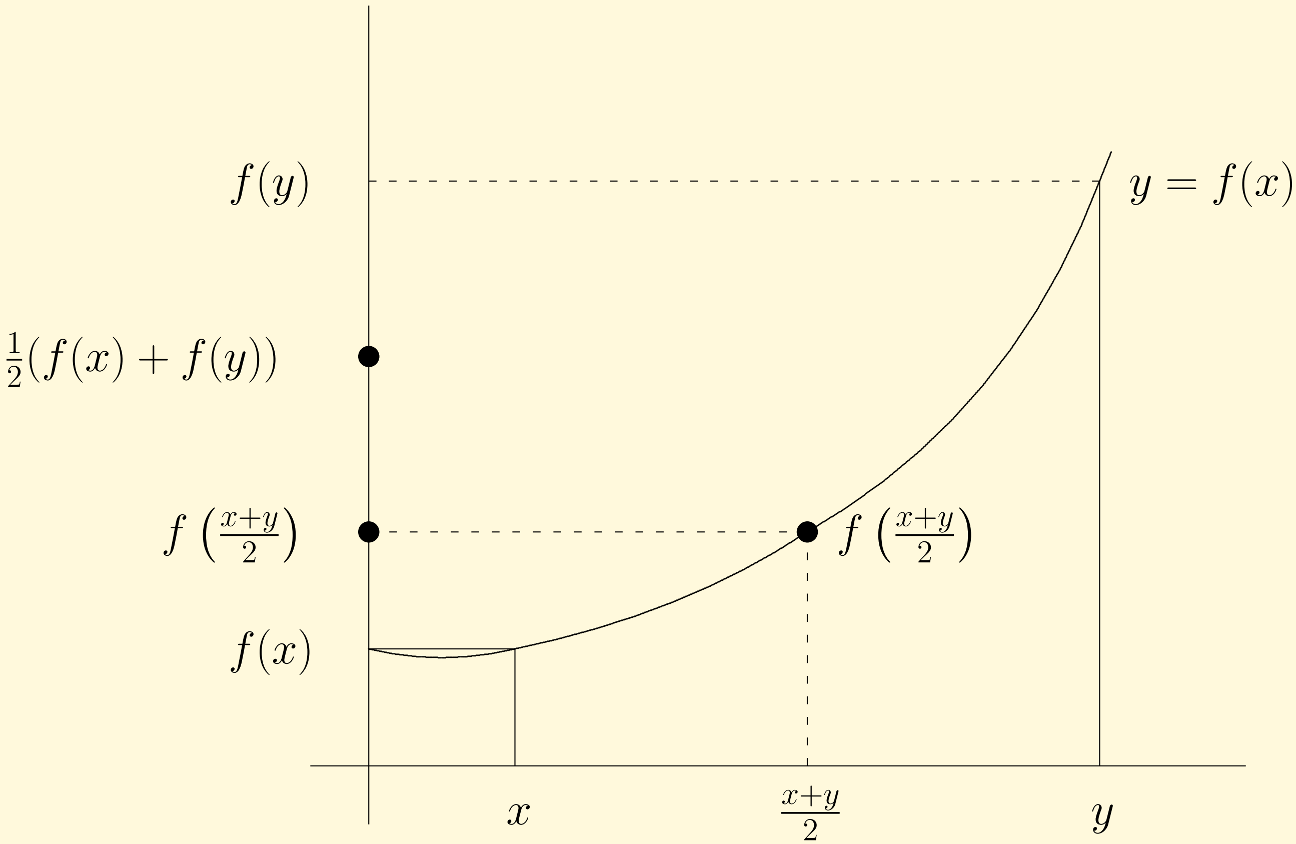

Figure 1. Geometrical interpretation of convexity.

Example. The followings are examples of convex functions

(a) $f(x) = mx + c$ on $\mathbb{R}$ for any $m, c \in \mathbb{R}$

(b) $f(x) = x^2$ on $\mathbb{R}$

(c) $f(x) = e^{\alpha x}$ on $\mathbb{R}$ for any $\alpha \geq 1$ or $\alpha \leq 0$

(d) $f(x) = |x|^\alpha$ on $\mathbb{R}$ for any $\alpha \geq 1$

(e) $f(x) = x \log x$ on $\mathbb{R}_+$

(f) $f(x) = \tan x$ on $\left[0,\ \frac{\pi}{2}\right]$

A finite sum of convex functions is also a convex function. However, the product of convex functions is not necessarily convex. For example,

$$f(x) = x^2 \quad \text{and} \quad g(x) = e^x$$

are convex functions on $\mathbb{R}$ but their product

$$h(x) = x^2 e^x$$

is not a convex function on $\mathbb{R}$.

If $A : \mathbb{R} \to \mathbb{R}$ is an additive function, then A is also a convex function. Since

$$A\left(\frac{x+y}{2}\right) = \frac{1}{2}A(x+y) = \frac{1}{2}(A(x)+A(y)),$$

A satisfies

$$A\left(\frac{x+y}{2}\right) \leq \frac{A(x)+A(y)}{2}.$$

Therefore A is a convex function.

If $A : \mathbb{R} \to \mathbb{R}$ is an additive function and $f : \mathbb{R} \to \mathbb{R}$ is a convex function, then their composition $f(A(x))$ is a convex function.

Jensen Functional Equation

The following functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}(f(x) + f(y))$$

for all $x, y \in \mathbb{R}$ is called the *Jensen functional equation*.

Definition 1 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is said to be Jensen if it satisfies*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \quad \forall\, x, y \in \mathbb{R}.$$

Definition 2 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is said to be affine if it is of the form*

$$f(x) = cx + a,$$

where c, a *are arbitrary constants.*

We want to show that every continuous Jensen function on $\mathbb{R}$ is affine.

Theorem 1 . *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies Jensen equation*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \tag{JE}$$

for all $x, y \in \mathbb{R}$ if and only if

$$f(x) = A(x) + a, \tag{2}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and a is a real arbitrary constant.

Proof: It is easy to verify that (2) satisfies the Jensen equation (JE).

Letting $y = 0$ in (JE), we get

$$f\left(\frac{x}{2}\right) = \frac{f(x)}{2} + \frac{a}{2}, \tag{3}$$

where $a = f(0)$. Putting (3) in (JE) we see that

$$\frac{f(x+y)+a}{2} = \frac{f(x)+f(y)}{2}$$

which is

$$f(x+y) + a = f(x) + f(y). \tag{4}$$

Define $A : \mathbb{R} \to \mathbb{R}$ by

$$A(x) = f(x) - a. \tag{5}$$

Then from (4), we see that

$$A(x + y) = A(x) + A(y).$$

Hence we have the asserted solution

$$f(x) = A(x) + a,$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function.

The following theorem is obvious from the last theorem.

Theorem 2 . *Every continuous Jensen function is affine.*

The proof of Theorem 1 does not extend to functions defined on a closed and bounded interval.

Next we determine the general continuous solution of (JE) on a closed and bounded interval $[a, b]$ for some a, b in $\mathbb{R}$.

First we need the following definition.

Definition 3 *Let m and n be two positive integers. A rational number of the form*

$$\frac{m}{2^n}$$

is called a dyadic rational number.

Theorem 3 . *The continuous solution of*

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2} \tag{JE}$$

for all $x, y \in [a,b]$ *is given by*

$$f(x)=\alpha+\beta\,x, \tag{6}$$

where α *and* β *are arbitrary constants.*

Proof: Define a new function $F : [0,1] \to \mathbb{R}$ as

$$F(y) = f((b-a)y + a) \quad \text{for } y \in [0,1]. \tag{7}$$

Since

$$(b-a)\,y + a \in [a, b]$$

$$(b-a)\,y \in [a-a,\ b-a]$$

$$(b-a)\,y \in [0,\ (b-a)]$$

$$y \in [0,1],$$

therefore the domain of the function F is $[0,1]$. Next we show that F satisfies (JE).

For this, using (7), we compute $F\left(\frac{x+y}{2}\right)$ as

$$\begin{aligned} F\left(\frac{x+y}{2}\right) &= f\left((b-a)\left(\frac{x+y}{2}\right)+a\right) \\ &= f\left(\frac{[(b-a)x+a]+[(b-a)y+a]}{2}\right) \\ &= \frac{f((b-a)x+a)+f((b-a)y+a)}{2} \\ &= \frac{F(x)+F(y)}{2}, \quad \forall\, x, y \in [0,1]. \end{aligned}$$

Thus F satisfies the Jensen functional equation on $[0,1]$.

Letting $x = 0$ and $y = 1$ in $F(\frac{x+y}{2}) = \frac{F(x)+F(y)}{2}$, we get

$$F\left(\frac{1}{2}\right) = \frac{F(0)+F(1)}{2} = \frac{c+d}{2} = c + \frac{1}{2}(d-c),$$

where $c = F(0)$ and $d = F(1)$.

Similarly, letting $x = 0$ and $y = \frac{1}{2}$ in $F(\frac{x+y}{2}) = \frac{F(x)+F(y)}{2}$,

we have

$$F\left(\frac{1}{4}\right) = \frac{F(0) + F\left(\frac{1}{2}\right)}{2} = \frac{c + c + \frac{1}{2}(d-c)}{2} = c + \frac{1}{4}(d-c).$$

Now letting $x = \frac{1}{2}$ and $y = 1$ in $F(\frac{x+y}{2}) = \frac{F(x)+F(y)}{2}$, we obtain

$$F\left(\frac{3}{4}\right) = \frac{F\left(\frac{1}{2}\right) + F(1)}{2} = c + \frac{3}{4}(d-c).$$

Next we will show that if x is any real number of the form $\frac{m}{2^k}$ where m and k are positive integers with $0 \le m \le 2^k$, then

$$F(x) = c + x(d-c). \tag{8}$$

We proceed by induction on k. We have already shown that the assertion is true for $k = 1, 2$.

Assume that (8) holds for $k = n$ and consider two cases:

$$
\begin{tikzcd}
 & \text{case (a)} \quad x = \frac{2m}{2^{n+1}} \\
\text{cases} \arrow[ur] \arrow[dr] & \\
 & \text{case (b)} \quad x = \frac{2m+1}{2^{n+1}}
\end{tikzcd}
$$

In case (a) we have

$$\begin{aligned} F\left(\frac{2m}{2^{n+1}}\right) &= F\left(\frac{m}{2^n}\right) \\ &= c + \frac{m}{2^n}\Big(d - c\Big) \\ &= c + \frac{2m}{2\,2^n}\Big(d - c\Big) \\ &= c + \frac{2m}{2^{n+1}}\Big(d - c\Big). \end{aligned}$$

In the case (b)

$$
\begin{aligned}
F\left(\frac{2m+1}{2^{n+1}}\right) &= F\left(\frac{1}{2}\left[\frac{m}{2^n}+\frac{m+1}{2^n}\right]\right) \\
&= \frac{F\left(\frac{m}{2^n}\right)+F\left(\frac{m+1}{2^n}\right)}{2} \\
&= \frac{1}{2}\left[c+\frac{m}{2^n}(d-c)+c+\frac{m+1}{2^n}(d-c)\right] \\
&= c+\frac{2m+1}{2^{n+1}}(d-c).
\end{aligned}
$$

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Hence the Jensen equation is satisfied for all dyadic rationals x in $[0, 1]$.

Since F is continuous and the subset of all dyadic rationals in $[0, 1]$ is dense in $[0, 1]$, we have $F(x) = c + x(d - c)$ for all $x \in [0, 1]$.

This yields $f(x) = \alpha + \beta x$, where α, β are real constants.

The proof of the theorem is now complete.

Remark 1 *We have seen in the proof of the above theorem that the function F defined by $F(x) = f((b-a)x + a)$ satisfies the Jensen functional equation on the interval $[0, 1]$. Following the proof of Theorem 1, one can easily show that $F(x) = A(x) + \alpha$, where $A : [0, 1] \to \mathbb{R}$ is an additive function and α is an arbitrary constant.*

Remark 2 *Using an extension theorem, the additive function can be extended from* $[0,1]$ *to* $\mathbb{R}$. *Thus the general solution* $f : [a,b] \to \mathbb{R}$ *of the Jensen equation can be given by*

$$f(x) = A\left(\frac{x-a}{b-a}\right) + \alpha,$$

where $A : \mathbb{R} \to \mathbb{R}$ *is an additive function.*

Hence we have the following theorem.

Theorem 4 . *The general solution of*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \tag{JE}$$

for all $x, y \in [a, b]$ *is given by*

$$f(x) = A\left(\frac{x-a}{b-a}\right) + \alpha, \tag{9}$$

where α *is an arbitrary constant and* $A : \mathbb{R} \to \mathbb{R}$ *is an additive function.*

A Related Functional Equation

Popoviciu (1965) demonstrated that if I is a nonempty interval and $f : I \to \mathbb{R}$ is a convex function, then f satisfies the inequality

$$3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) \geq 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]$$

for all $x, y, z \in I$.

If we change the inequality sign to an equality sign in the above inequality, then we have a functional equation of Jensen type.

In this section, our goal is to determine the general solution of this Jensen type functional equation, namely,

$$3f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z) \\ = 2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right] \qquad (10)$$

for all $x, y, z \in \mathbb{R}$.

Theorem 5 . *The function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the functional equation* (10) *for all* $x, y, z \in \mathbb{R}$ *if and only if*

$$f(x) = A(x) + b \tag{11}$$

for all $x \in \mathbb{R}$, *where* $A : \mathbb{R} \to \mathbb{R}$ *is an additive function and* b *is an arbitrary real constant.*

Proof: It is easy to see that if f is of the form (11), then f is a solution of the functional equation (10).

Now we prove the converse. That is, every solution of (10) is of the form (11). First, we define a function $A : \mathbb{R} \to \mathbb{R}$ by

$$A(x) = f(x) - b \tag{12}$$

for all $x \in \mathbb{R}$, where $b = f(0)$.

Then $A(0) = 0$ and the function A satisfies

$$3A\left(\frac{x+y+z}{3}\right) + A(x) + A(y) + A(z) = 2\left[A\left(\frac{x+y}{2}\right) + A\left(\frac{y+z}{2}\right) + A\left(\frac{z+x}{2}\right)\right] \tag{13}$$

for all $x, y, z \in \mathbb{R}$. Substitute $y = x$ and $z = -2x$ in (10) to obtain

$$A(-2x) = 4\,A\left(-\frac{x}{2}\right) \tag{14}$$

for all $x \in \mathbb{R}$.

Replacing x by $-x$ in (14), we have

$$A(2x) = 4\,A\left(\frac{x}{2}\right) \tag{15}$$

for all $x \in \mathbb{R}$. Again replacing x by $2x$ in (15), we have

$$A(4x) = 4\,A\,(x) \tag{16}$$

for all $x \in \mathbb{R}$.

Putting $y = z = 0$ in (13) and taking account of (15), we obtain

$$3A\left(\frac{x}{3}\right) = A\,(2\,x) - A\,(x) \tag{17}$$

for all $x \in \mathbb{R}$. Substituting $y = x$ and $z = 0$ in (13) and taking account of (17), we obtain

$$A\,(4\,x) = A\,(2\,x) - 4\,A\left(\frac{x}{2}\right) \tag{18}$$

for all $x \in \mathbb{R}$.

From (15), (16) and (18) it follows that

$$A(2x) = 2A(x) \tag{19}$$

for all $x \in \mathbb{R}$. Putting $y = x$ and $z = -x$ in (13) and taking account of (17) and (18), we obtain

$$A(-x) = -A(x) \tag{20}$$

for all $x \in \mathbb{R}$.

Finally substituting $z = -x - y$ in (13) and taking account of (18) and (19), we obtain

$$A(x + y) = A(x) + A(y)$$

for all $x, y \in \mathbb{R}$. Therefore $A : \mathbb{R} \to \mathbb{R}$ is an additive function and hence from (12) we obtain the asserted solution (11). This completes the proof of the theorem.

Remark 3 *If we let $x = u + v$ and $y = u - v$ in (JE), that is in $f((x + y)/2) = [f(x) + f(y)]/2$, then we have*

$$f(u) = \frac{1}{2}[f(u + v) + f(u - v)]$$

for all $u, v \in \mathbb{R}$. Hence the Jensen functional equation can also be written as $f(x + y) + f(x - y) = 2\, f(x)$.

This representation has some advantages over (JE) while studying the Jensen equation on algebraic structures.

Remark 4 *For an arbitrary group G, we denote $\cdot$ as its group operation and e as the identity element. To simplify our writing, we write xy, instead of $x \cdot y$.*

If G is abelian, the group operation and the identity element are denoted by $+$ and 0, respectively. In this case we write xy as $x + y$. Similar notations will be adapted for semigroups.

Theorem 6 . *[Sinopoulos (2000)] Let* $(S,+)$ *be a commutative semigroup,* G *a 2-cancellative abelian group, and* σ *an endomorphism of* S *such that* $\sigma(\sigma x) = x$ *for* $x \in S$*. Then the general solution* $f : S \to G$ *of the Jensen functional equation*

$$f(x+y) + f(x+\sigma y) = 2f(x) \qquad \forall\, x, y \in S \tag{21}$$

is given by $f(x) = A(x) + a$ *for all* $x \in S$*, where* $a \in G$ *is an arbitrary constant and* $A : S \to G$ *is an arbitrary additive function with* $A(\sigma x) = -A(x)$ *for all* $x \in S$*.*

Proof: We replace y by $y + \sigma y$ in $f(x+y) + f(x+\sigma y) = 2f(x)$ to obtain

$$f(x + y + \sigma y) = f(x). \tag{22}$$

Replacing x by $x + z$ in $f(x+y) + f(x+\sigma y) = 2f(x)$, we have

$$f(x + z + y) + f(x + z + \sigma y) = 2f(x + z). \tag{23}$$

Interchanging y with z in (23), we obtain

$$f(x + y + z) + f(x + y + \sigma z) = 2f(x + y). \tag{24}$$

Adding the equations (23) and (24) and using (21), we have

$$
\begin{aligned}
f(x+z+y) + f(x+z+\sigma y) \\
+ f(x+y+z) + f(x+y+\sigma z) \\
= 2f(x+z) + 2f(x+y)
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
2f(x+y+z) + f(x+(z+\sigma y)) + f(x+\sigma(z+\sigma y)) \\
= 2f(x+z) + 2f(x+y).
\end{aligned}
$$

Using $f(x+y)+f(x+\sigma y)=2f(x)$, we obtain

$$2f(x+y+z)+2f(x)=2f(x+z)+2f(x+y). \tag{25}$$

Setting $z=\sigma x$ in (25) and using (22), we get

$$f(y)+f(x)=f(x+\sigma x)+f(x+y). \tag{26}$$

Interchanging x with y, we see that $f(x+\sigma x)=f(y+\sigma y)$; that is, $f(x+\sigma x)$ is a constant, say, a.

So (26) yields

$$[f(x+y)-a]=[f(x)-a]+[f(y)-a] \tag{27}$$

which leads to with $A(x)=f(x)-a$.

Substituting $f(x)=A(x)+a$ back into

$$f(x+y)+f(x+\sigma y)=2f(x),$$

we see that $A(\sigma y)=-A(y)$ and this completes the proof.

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Thank You