622 Homework due Feb. 16

Friendly reminder: Do not consult with the internet when doing homework problems. Feel free to ask me if you have a question (email, office hours, class), or discuss problems with your classmates. But that's it.

1. Recall that if K is any field, either an isomorphic copy of $\mathbb{Q}$ is contained in K (in which case $char(K) = 0$), or there exist a unique prime number p such that K contains an isomorphic copy of $\mathbb{Z}_p$, the p-element field (in which $char(K) = p$).

   Using the above, prove that if K is a finite field, then there exists a prime number p and a positive integer n such that $|K| = p^n$.

   Suggestion: Explain why K can be regarded as a vector space over some $\mathbb{Z}_p$, and go from there.

2. Let $K = \mathbb{Q}(2^{1/3})$, the smallest subfield of $\mathbb{C}$ that contains $2^{1/3}$. Let $2 + (5)2^{1/3} \in K$. Use the Euclidean algorithm to find the inverse of $2 + (5)2^{1/3}$, representing $(2 + (5)2^{1/3})^{-1}$ as an element of the form

   $$(*) \;\; a_2(2^{2/3}) + a_1 2^{1/3} + a_0,$$

   where a_2, a_1, a_0 are contained in $\mathbb{Q}$. So $a_2(2^{2/3}) + a_1 2^{1/3} + a_0$ is an element in $\mathbb{Q}[2^{1/3}]$).

That the Euclidean algorithm can be used here is based on the fact that $x^3 - 2 \in \mathbb{Q}$ is irreducible; thus, $b(x) = 5x + 2$ and $x^3 - 2$ are relatively prime polynomials, which means that there exist $s(x), t(x) \in \mathbb{Q}[x]$ such that $1 = s(x)b(x) + t(x)(x^3 - 2)$. Find $s(x), t(x)$ using the Euclidean algorithm and then "backtracking" to determine a_2, a_1, a_0 in (*).

[One "take-away" from this: You might be able to explain why $\mathbb{Q}(2^{1/3}) = \mathbb{Q}[2^{1/3}]$, an important and useful fact, one which generalizes to extensions formed adding a root of an irreducible to a field.]