# MATH 668 Homework 6 Solutions

1. Using Theorem 11.2.2 with $\boldsymbol{a}=1$, $\boldsymbol{\beta}=\mu$, $\mathbf{X}=\boldsymbol{j}$, and $\boldsymbol{\phi}=0$, we have $\left.\dfrac{\mu-\boldsymbol{\phi}_*}{\sqrt{\mathbf{W}_*}}\right|\boldsymbol{y}\sim t(n+2\alpha)$ where

$\mathbf{V}_*=(1+\boldsymbol{j}^\top\boldsymbol{j})^{-1}=\dfrac{1}{1+n}$, $\boldsymbol{\phi}_*=\dfrac{1}{1+n}(0+\boldsymbol{j}^\top\boldsymbol{y})=\dfrac{n\bar{y}}{n+1}$ and $\mathbf{W}_*=\left[\dfrac{(\boldsymbol{y}-\mathbf{X}\boldsymbol{\phi})^\top(\mathbf{I}+\mathbf{X}\mathbf{V}\mathbf{X}^\top)^{-1}(\boldsymbol{y}-\mathbf{X}\boldsymbol{\phi})+2\delta}{n+2\alpha}\right]\dfrac{1}{1+n}$

$$=\left[\frac{\boldsymbol{y}^\top\boldsymbol{y}+\boldsymbol{\phi}^\top\mathbf{V}^{-1}\boldsymbol{\phi}-\boldsymbol{\phi}_*^\top\mathbf{V}_*^{-1}\boldsymbol{\phi}_*+2\delta}{n+2\alpha}\right]\frac{1}{1+n}=\left[\frac{\sum_{i=1}^n y_i^2+0-\frac{n\bar{y}}{1+n}\left(\frac{1}{1+n}\right)^{-1}\frac{n\bar{y}}{1+n}+2\delta}{n+2\alpha}\right]\frac{1}{1+n}=\frac{\sum_{i=1}^n y_i^2-\frac{n^2\bar{y}^2}{1+n}+2\delta}{(1+n)(n+2\alpha)}$$

$$=\frac{(\sum_{i=1}^n y_i^2-n\bar{y}^2)+n\bar{y}^2-\frac{n^2\bar{y}^2}{1+n}+2\delta}{(1+n)(n+2\alpha)}=\frac{(n-1)s_y^2+\left(n-\frac{n^2}{1+n}\right)\bar{y}^2+2\delta}{(1+n)(n+2\alpha)}=\frac{(n-1)s_y^2+\frac{n(1+n)-n^2}{1+n}\bar{y}^2+2\delta}{(1+n)(n+2\alpha)}$$

$=\dfrac{(n-1)s_y^2+\frac{n}{n+1}\bar{y}^2+2\delta}{(n+1)(n+2\alpha)}$. So, it follows that

$$1-\omega=P\left(\left|\frac{\mu-\frac{n\bar{y}}{n+1}}{\sqrt{\frac{(n-1)s_y^2+\frac{n}{n+1}\bar{y}^2+2\delta}{(n+1)(n+2\alpha)}}}\right|\leq t_{\omega/2,n+2\alpha}\right)=P\left(\left|\mu-\frac{n\bar{y}}{n+1}\right|\leq t_{\omega/2,n+2\alpha}\sqrt{\frac{(n-1)s_y^2+\frac{n}{n+1}\bar{y}^2+2\delta}{(n+1)(n+2\alpha)}}\right)$$

$$=P\left(\frac{n\bar{y}}{n+1}-t_{\omega/2,n+2\alpha}\sqrt{\frac{(n-1)s_y^2+\frac{n}{n+1}\bar{y}^2+2\delta}{(n+1)(n+2\alpha)}}\leq\mu\leq\frac{n\bar{y}}{n+1}+t_{\omega/2,n+2\alpha}\sqrt{\frac{(n-1)s_y^2+\frac{n}{n+1}\bar{y}^2+2\delta}{(n+1)(n+2\alpha)}}\right).$$

2. We can compute the REML estimate of $\boldsymbol{\Sigma}$ with the following iterative method.

```
y=c(11.2,11.4,11.6,11.9,11.8,11.3,11.0,11.2,11.3)
C=cbind(diag(8),0)
K=C%*%(diag(9)-1/9*matrix(1,9,9))
sigma=c(1,1)
M=matrix(0,2,2)
q=c(0,0)
Z=diag(3)%x%rep(1,3)
ZZt=Z%*%t(Z)
Sigma=sigma[1]*diag(9)+sigma[2]*ZZt
P=t(K)%*%solve(K%*%Sigma%*%t(K))%*%K
M[1,1]=sum(diag(P%*%P))
M[1,2]=sum(diag(P%*%P%*%ZZt))
M[2,1]=sum(diag(P%*%ZZt%*%P))
M[2,2]=sum(diag(P%*%ZZt%*%P%*%ZZt))
q[1]=sum((P%*%y)^2)
q[2]=sum((t(Z)%*%P%*%y)^2)
sigma=solve(M)%*%q
sigma
```

```
##             [,1]
## [1,] 0.05555556
## [2,] 0.04407407
```

```
Sigma=sigma[1]*diag(9)+sigma[2]*ZZt
P=t(K)%*%solve(K%*%Sigma%*%t(K))%*%K
M[1,1]=sum(diag(P%*%P))
M[1,2]=sum(diag(P%*%P%*%ZZt))
```

```r
M[2,1]=sum(diag(P%*%ZZt%*%P))
M[2,2]=sum(diag(P%*%ZZt%*%P%*%ZZt))
q[1]=sum((P%*%y)^2)
q[2]=sum((t(Z)%*%P%*%y)^2)
sigma=solve(M)%*%q
sigma
```

```
##            [,1]
## [1,] 0.05555556
## [2,] 0.04407407
```

So, $\hat{\sigma}^2 = 0.05555556$ and $\hat{\sigma}_1^2 = 0.04407407$ and the estimate of $\text{var}(\boldsymbol{y}) = \boldsymbol{\Sigma}$ is

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 0.09962963 & 0.04407407 & 0.04407407 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.04407407 & 0.09962963 & 0.04407407 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.04407407 & 0.04407407 & 0.09962963 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.09962963 & 0.04407407 & 0.04407407 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.04407407 & 0.09962963 & 0.04407407 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.04407407 & 0.04407407 & 0.09962963 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.09962963 & 0.04407407 & 0.04407407 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.04407407 & 0.09962963 & 0.04407407 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.04407407 & 0.04407407 & 0.09962963 \end{pmatrix}$$

as shown below.

```r
Sigma
```

```
##             [,1]       [,2]       [,3]       [,4]       [,5]       [,6]
##  [1,] 0.09962963 0.04407407 0.04407407 0.00000000 0.00000000 0.00000000
##  [2,] 0.04407407 0.09962963 0.04407407 0.00000000 0.00000000 0.00000000
##  [3,] 0.04407407 0.04407407 0.09962963 0.00000000 0.00000000 0.00000000
##  [4,] 0.00000000 0.00000000 0.00000000 0.09962963 0.04407407 0.04407407
##  [5,] 0.00000000 0.00000000 0.00000000 0.04407407 0.09962963 0.04407407
##  [6,] 0.00000000 0.00000000 0.00000000 0.04407407 0.04407407 0.09962963
##  [7,] 0.00000000 0.00000000 0.00000000 0.00000000 0.00000000 0.00000000
##  [8,] 0.00000000 0.00000000 0.00000000 0.00000000 0.00000000 0.00000000
##  [9,] 0.00000000 0.00000000 0.00000000 0.00000000 0.00000000 0.00000000
##             [,7]       [,8]       [,9]
##  [1,] 0.00000000 0.00000000 0.00000000
##  [2,] 0.00000000 0.00000000 0.00000000
##  [3,] 0.00000000 0.00000000 0.00000000
##  [4,] 0.00000000 0.00000000 0.00000000
##  [5,] 0.00000000 0.00000000 0.00000000
##  [6,] 0.00000000 0.00000000 0.00000000
##  [7,] 0.09962963 0.04407407 0.04407407
##  [8,] 0.04407407 0.09962963 0.04407407
##  [9,] 0.04407407 0.04407407 0.09962963
```

Then we can compute the EGLS estimate of $\boldsymbol{\beta}$ based on $\hat{\boldsymbol{\Sigma}}$ using the formula $\hat{\boldsymbol{\beta}} = (\mathbf{X}^\top \hat{\boldsymbol{\Sigma}} \mathbf{X})^{-1} \mathbf{X}^\top \hat{\boldsymbol{\Sigma}} \boldsymbol{y} = 11.41111$ as shown below.

```r
X=rep(1,9)
solve(t(X)%*%solve(Sigma)%*%X)%*%t(X)%*%solve(Sigma)%*%y
```

```
##          [,1]
## [1,] 11.41111
```

Then the estimate of $E(\boldsymbol{y})$ is $X\hat{\boldsymbol{\beta}} = 11.41111\boldsymbol{j}$.

3. (a) The log-likelihood function can be expressed as

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n} \ln f(y_i) = \sum_{i=1}^{n} \ln \frac{\lambda_i^{y_i} e^{-\lambda_i}}{x_i!} = \sum_{i=1}^{n} (y_i \ln \lambda_i - \lambda_i - \ln x_i!) = \sum_{i=1}^{n} \left( y_i \ln e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} - e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} - \ln x_i! \right)$$

$$= \sum_{i=1}^{n} \left( y_i \boldsymbol{x}_i^\top \boldsymbol{\beta} - e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} - \ln x_i! \right).$$

(b) The vector of partial derivatives of $\ell$ is

$$\frac{\partial \ell}{\partial \boldsymbol{\beta}} = \sum_{i=1}^{n} \left( y_i \frac{\partial}{\partial \boldsymbol{\beta}} \left[ \boldsymbol{x}_i^\top \boldsymbol{\beta} \right] - \frac{\partial}{\partial \boldsymbol{\beta}} \left[ e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} \right] - 0 \right) = \sum_{i=1}^{n} \left( y_i \frac{\partial}{\partial \boldsymbol{\beta}} \left[ \boldsymbol{x}_i^\top \boldsymbol{\beta} \right] - e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} \frac{\partial}{\partial \boldsymbol{\beta}} \left[ \boldsymbol{x}_i^\top \boldsymbol{\beta} \right] \right) = \sum_{i=1}^{n} \left( y_i \boldsymbol{x}_i - e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} \boldsymbol{x}_i \right)$$

$$= \sum_{i=1}^{n} \left( y_i - e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} \right) \boldsymbol{x}_i$$

The martix of second partial derivatives of $\ell$ is

$$\frac{\partial^2 \ell}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^\top} = \frac{\partial}{\partial \boldsymbol{\beta}} \left[ \frac{\partial \ell}{\partial \boldsymbol{\beta}^\top} \right] = \frac{\partial}{\partial \boldsymbol{\beta}} \left[ \sum_{i=1}^{n} \left( y_i - e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} \right) \boldsymbol{x}_i^\top \right] = \sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{\beta}} \left[ y_i - e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} \right] \boldsymbol{x}_i^\top = \sum_{i=1}^{n} \left( 0 - \frac{\partial}{\partial \boldsymbol{\beta}} \left[ e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} \right] \right) \boldsymbol{x}_i^\top$$

$$= -\sum_{i=1}^{n} e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} \frac{\partial}{\partial \boldsymbol{\beta}} \left[ \boldsymbol{x}_i^\top \boldsymbol{\beta} \right] \boldsymbol{x}_i^\top = -\sum_{i=1}^{n} e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} \boldsymbol{x}_i \boldsymbol{x}_i^\top = -\sum_{i=1}^{n} \boldsymbol{x}_i \left( e^{\boldsymbol{x}_i^\top \boldsymbol{\beta}} \right) \boldsymbol{x}_i^\top.$$

(c) The following code performs 3 steps of the IWLS procedure to obtain the MLE

$\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} -5.253984 \\ 0.06794086 \end{pmatrix}.$

```r
setwd("C:/Users/ryan/Desktop/S18/668/data")
SARS=read.table("us2003sars.txt",header=TRUE)
x=SARS$date
y=SARS$cases

X=cbind(1,x)
beta=c(-5,0.07)
for (t in 1:3){
 eta=c(X%*%beta)
 lambda=exp(eta)
 m=lambda
 W=diag(lambda)
 beta=beta+solve(t(X)%*%W%*%X)%*%t(X)%*%(y-m)
 cat(paste("beta_(",t,")= ",sep=""),beta,"\n")
}
```

```
## beta_(1)=  -5.22976 0.06859278
## beta_(2)=  -5.254418 0.06798139
## beta_(3)=  -5.253984 0.06794086
```