MATH 668 EXAM 1 (February 22, 2018: 2:30pm–3:45pm) NAME __________________________

The exam is closed book; students are permitted to prepare one 8.5×11 page of formulas, notes, etc. that can be used during the exam. A calculator is permitted but not necessary for the exam.

Do all 4 problems (the problems with the three highest scores are worth 30% each, the problem with the lowest score is worth 10%).

Problem 1. (10 points) Suppose $\boldsymbol{x} \sim N_n(\mathbf{0}, \mathbf{\Sigma})$ where $\mathbf{\Sigma}$ is an $n \times n$ positive definite matrix such that $\mathbf{\Sigma} = \mathbf{C}\mathbf{C}^\top$ where $\mathbf{C}$ is a lower triangular $n \times n$ matrix with positive elements on its diagonal.

(a - 5 pts) Find the distribution of $\mathbf{C}^{-1}\boldsymbol{x}$.

(b - 5 pts) Let $\mathbf{B}$ be an $n \times n$ symmetric matrix with rank r. If $\mathbf{C}^\top\mathbf{B}\mathbf{C}$ is idempotent, then show that $\boldsymbol{x}^\top\mathbf{B}\boldsymbol{x} \sim \chi^2(r)$.

(Note: If $\boldsymbol{y} \sim N(\boldsymbol{\mu}, \mathbf{\Sigma})$, the mgf of $\boldsymbol{y}$ is $M_{\boldsymbol{y}}(\boldsymbol{t}) = e^{\boldsymbol{t}'\boldsymbol{\mu}+\boldsymbol{t}'\mathbf{\Sigma}\boldsymbol{t}/2}$ and the mgf of $\boldsymbol{y}^\top\mathbf{A}\boldsymbol{y}$ is
$M_{\boldsymbol{y}^\top\mathbf{A}\boldsymbol{y}}(t) = \det(\mathbf{I} - 2t\mathbf{A}\mathbf{\Sigma})^{-1/2}e^{-\boldsymbol{\mu}^\top(\mathbf{I}-(\mathbf{I}-2t\mathbf{A}\mathbf{\Sigma})^{-1})\mathbf{\Sigma}^{-1}\boldsymbol{\mu}/2}$.
If $v \sim \chi^2(p, \lambda)$, then the mgf of v is $M_v(t) = (1-2t)^{-p/2}e^{-\lambda[1-1/(1-2t)]}$. You may use these if they are helpful, but you are also free to use any theorems that we have used in class.)

Problem 2. (10 points) Suppose that $\boldsymbol{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_{16} \end{pmatrix} \sim N_{16}(\mathbf{0}, \mathbf{I})$, $\boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_9 \end{pmatrix} \sim N_9(\mathbf{0}, 4\mathbf{I})$, and that $\boldsymbol{z}$ and $\boldsymbol{y}$ are independent random vectors. Also, let $\boldsymbol{j}$ be a 16-dimensional vector of ones.

(a - 5 pts) Compute $P\left(\boldsymbol{j}^\top \boldsymbol{z} > \sqrt{\boldsymbol{y}^\top \boldsymbol{y}}\right)$ using the attached tables.

(b - 5 pts) Compute $P\left(\boldsymbol{j}^\top \boldsymbol{z} < 1 \text{ and } \boldsymbol{y}^\top \boldsymbol{y} < 20\right)$ using the attached tables.

Problem 3. (10 points) For fixed $\lambda > 0$ and observed values of $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, let

$$\tilde{Q}(b) = \sum_{i=1}^{n} (y_i - bx_i)^2 + \lambda b^2.$$

(a - 4 pts) Find the value of b which minimizes $\tilde{Q}$ (as a function of λ, $x_1, \ldots, x_n$, and $y_1, \ldots, y_n$).

(b - 2 pts) Denote the minimizer in part (a) as $\hat{\beta}_\lambda$. Show that $\hat{\beta}_\lambda$ is the minimizer of $\tilde{Q}$.

(c - 3 pts) Suppose that $y_1, \ldots, y_n$ are independent random variables such that $E(y_i) = \beta x_i$ for $i = 1, \ldots, n$, and $x_1, \ldots, x_n$ are values of a fixed non-random explanatory variable. Compute $E(\hat{\beta}_\lambda)$.

(d - 1 pt) What happens to $E(\hat{\beta}_\lambda)$ as $\lambda \to 0$?

Problem 4. (10 points) Consider the linear regression model

$$\boldsymbol{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

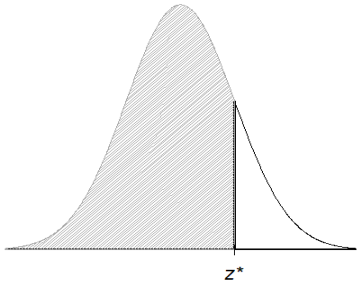
where $\boldsymbol{\varepsilon}$ is a random variable such that $E(\boldsymbol{\varepsilon}) = \mathbf{0}_n$, $\text{cov}(\boldsymbol{\varepsilon}) = \sigma^2 \mathbf{I}_n$. Also, $\mathbf{X} = (\boldsymbol{j}, \boldsymbol{x})$ is an $n \times 2$ design matrix, $\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$ is a 2-dimensional vector of fixed but unknown coefficients, σ^2 is the fixed but unknown variance, $\boldsymbol{x}$ is a known non-random n-dimensional column vector, and $\boldsymbol{j}$ is an n-dimensional column vector of ones.

Suppose we have a data set with the following summary statistics:

$$n = \boldsymbol{j}^\top \boldsymbol{j} = 7,\ \boldsymbol{j}^\top \boldsymbol{x} = -1,\ \boldsymbol{x}^\top \boldsymbol{x} = 3,\ \boldsymbol{j}^\top \boldsymbol{y} = -5, \boldsymbol{x}^\top \boldsymbol{y} = 4,\ \boldsymbol{y}^\top \boldsymbol{y} = 9.$$

(a - 6 pts) Compute the least squares estimate of $\boldsymbol{\beta}$ based on the data set.

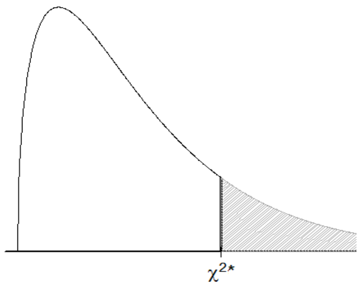
(b - 4 pts) Denote your answer to (a) as $\hat{\boldsymbol{\beta}}$ and let $Q(\boldsymbol{b}) = (\boldsymbol{y} - \mathbf{X}\boldsymbol{b})^\top (\boldsymbol{y} - \mathbf{X}\boldsymbol{b})$.
Compute $Q(\hat{\boldsymbol{\beta}})$ and divide it by an appropriate constant to obtain an unbiased estimate of σ^2 based on the data set.



If Z is a standard normal random variable, then the shaded area is $P(Z \leq z^*)$.

Standard Normal Cumulative Probabilities

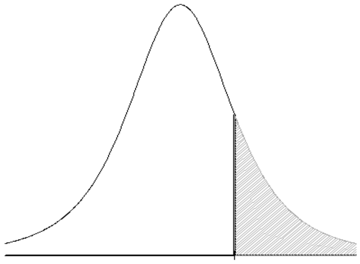
z	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998



If Q is a random variable with a χ^2 distribution having *df* degrees of freedom, then the critical value χ^{2*} in the table is the value such that the shaded area is
$p = P(Q > \chi^{2*})$.

Critical Values for χ^2 distribution

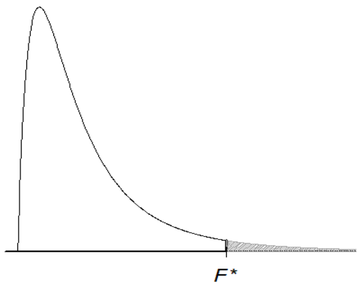
	p											
df	**.995**	**.98**	**.88**	**.83**	**.75**	**.47**	**.35**	**.24**	**.16**	**.05**	**.02**	**.01**
1	0	0	0	0	0	1	1	1	2	4	5	7
2	0	0	0	0	1	2	2	3	4	6	8	9
3	0	0	1	1	1	3	3	4	5	8	10	11
4	0	0	1	1	2	4	4	5	7	9	12	13
5	0	1	2	2	3	5	6	7	8	11	13	15
6	1	1	2	3	3	6	7	8	9	13	15	17
7	1	2	3	4	4	7	8	9	11	14	17	18
8	1	2	4	4	5	8	9	10	12	16	18	20
9	2	3	4	5	6	9	10	12	13	17	20	22
10	2	3	5	6	7	10	11	13	14	18	21	23
11	3	4	6	7	8	11	12	14	16	20	23	25
12	3	4	7	7	8	12	13	15	17	21	24	26
13	4	5	7	8	9	13	14	16	18	22	25	28
14	4	5	8	9	10	14	15	17	19	24	27	29
15	5	6	9	10	11	15	16	18	20	25	28	31
16	5	7	10	11	12	16	18	20	22	26	30	32
17	6	7	11	11	13	17	19	21	23	28	31	33
18	6	8	11	12	14	18	20	22	24	29	32	35
19	7	9	12	13	15	19	21	23	25	30	34	36
20	7	9	13	14	15	20	22	24	26	31	35	38
21	8	10	14	15	16	21	23	25	27	33	36	39
22	9	11	15	16	17	22	24	26	28	34	38	40
23	9	11	15	17	18	23	25	27	30	35	39	42
24	10	12	16	17	19	24	26	28	31	36	40	43
25	11	13	17	18	20	25	27	30	32	38	42	44



If T is a random variable with a t distribution having *df* degrees of freedom, then the critical value t^* in the table is the value such that the shaded area is
$p = P(T > t^*)$.

Critical Values for t distribution

	p									
df	**.36**	**.24**	**.16**	**.08**	**.05**	**.02**	**.01**	**.001**	**.0001**	**.00001**
1	0.5	1.1	1.8	3.9	6.3	15.9	31.8	318	3183	31831
2	0.4	0.9	1.3	2.2	2.9	4.8	7.0	22.3	70.7	224
3	0.4	0.8	1.2	1.9	2.4	3.5	4.5	10.2	22.2	47.9
4	0.4	0.8	1.1	1.7	2.1	3.0	3.7	7.2	13.0	23.3
5	0.4	0.8	1.1	1.6	2.0	2.8	3.4	5.9	9.7	15.5
6	0.4	0.8	1.1	1.6	1.9	2.6	3.1	5.2	8.0	12.0
7	0.4	0.7	1.1	1.6	1.9	2.5	3.0	4.8	7.1	10.1
8	0.4	0.7	1.1	1.5	1.9	2.4	2.9	4.5	6.4	8.9
9	0.4	0.7	1.1	1.5	1.8	2.4	2.8	4.3	6.0	8.1
10	0.4	0.7	1.0	1.5	1.8	2.4	2.8	4.1	5.7	7.5
11	0.4	0.7	1.0	1.5	1.8	2.3	2.7	4.0	5.5	7.1
12	0.4	0.7	1.0	1.5	1.8	2.3	2.7	3.9	5.3	6.8
13	0.4	0.7	1.0	1.5	1.8	2.3	2.7	3.9	5.1	6.5
14	0.4	0.7	1.0	1.5	1.8	2.3	2.6	3.8	5.0	6.3
15	0.4	0.7	1.0	1.5	1.8	2.2	2.6	3.7	4.9	6.1
16	0.4	0.7	1.0	1.5	1.7	2.2	2.6	3.7	4.8	6.0
17	0.4	0.7	1.0	1.5	1.7	2.2	2.6	3.6	4.7	5.8
18	0.4	0.7	1.0	1.5	1.7	2.2	2.6	3.6	4.6	5.7
19	0.4	0.7	1.0	1.5	1.7	2.2	2.5	3.6	4.6	5.6
20	0.4	0.7	1.0	1.5	1.7	2.2	2.5	3.6	4.5	5.5
21	0.4	0.7	1.0	1.5	1.7	2.2	2.5	3.5	4.5	5.5
22	0.4	0.7	1.0	1.5	1.7	2.2	2.5	3.5	4.5	5.4
23	0.4	0.7	1.0	1.5	1.7	2.2	2.5	3.5	4.4	5.3
24	0.4	0.7	1.0	1.5	1.7	2.2	2.5	3.5	4.4	5.3
25	0.4	0.7	1.0	1.4	1.7	2.2	2.5	3.5	4.4	5.2



If f is a random variable with an F distribution having $df1$ and $df2$ degrees of freedom, then the critical value F^* in the table is the value such that the shaded area is $P(f > F^*) = .05$.

Level .05 critical values for F distribution

$df2$ = degrees of freedom in the denominator	$df1$ = degrees of freedom in the numerator									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	161	200	216	225	230	234	237	239	241	242
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24