D’Alembert Functional Equation

Lecture 7

February 2, 2017

D’Alembert Functional Equation

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January, 2017

Introduction

The well-known trigonometric identity

$$\cos(x+y) + \cos(x-y) = 2\cos(x)\,\cos(y)$$

implies the functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{DE}$$

for all $x, y \in \mathbb{R}$. In this lecture, we present the continuous solutions this functional equation.

• The above functional equation is known as the *d'Alembert functional equation*.

• It has a long history going back to d'Alembert (1769), Poisson (1804) and Picard (1922, 1928). The equation plays a central role in determining the sum of two vectors in Euclidean and non-Euclidean geometries.

• Cauchy (1821) determined the continuous solution of the d'Alembert functional equation.

Continuous Solution of d'Alembert Equation

Theorem 1 . *The continuous function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$f(x+y)+f(x-y)=2f(x)f(y), \quad \forall\, x, y \in \mathbb{R} \qquad \text{(DE)}$$

if and only if f *is one of the form*

$$f(x)=0, \tag{1}$$

$$f(x)=1, \tag{2}$$

$$f(x)=\cosh(\alpha x), \tag{3}$$

$$f(x)=\cos(\beta x), \tag{4}$$

where α, β *are real constants.*

Proof: Letting $x = 0 = y$ in $f(x+y)+f(x-y) = 2f(x)f(y)$, we obtain $f(0) = f(0)^2$. Hence $f(0) = 0$ or $f(0) = 1$.

If $f(0) = 0$, then letting $y = 0$ in the functional equation $f(x+y)+f(x-y) = 2f(x)f(y)$, we have $f(x) = f(x)\,f(0)$ which simplifies to $f(x) = 0$ for all $x \in \mathbb{R}$. This gives the solution (1).

Hence we assume from now on that f is not identically zero.

Next, we show that any solution of (DE) is an even function.

To see this, let $x = 0$ in $f(x+y) + f(x-y) = 2f(x)f(y)$.

Then we obtain $f(y) + f(-y) = 2f(0)f(y)$.

Since f is not identically zero, $f(0) \neq 0$ and $f(0) = 1$.

Hence the above equation yields $f(y) + f(-y) = 2f(y)$, which simplifies to $f(-y) = f(y)$ for all $y \in \mathbb{R}$. Thus f is an even function.

Since f is continuous on $\mathbb{R}$, f is also integrable on any finite interval. Hence, for $t > 0$, we have

$$\int_{-t}^{t} f(x+y)dy + \int_{-t}^{t} f(x-y)dy = 2f(x) \int_{-t}^{t} f(y)dy. \quad (5)$$

The first term in (5) can be written as

$$\int_{-t}^{t} f(x+y)dy = \int_{x-t}^{x+t} f(z)dz = \int_{x-t}^{x+t} f(y)dy.$$

Similarly, the second term in (5) can be written as

$$\int_{-t}^{t} f(x-y)dy = \int_{x+t}^{x-t} f(w)(-dw)$$

$$= \int_{x-t}^{x+t} f(w)dw$$

$$= \int_{x-t}^{x+t} f(y)dy.$$

Hence (5) becomes

$$\int_{x-t}^{x+t} f(y)dy + \int_{x-t}^{x+t} f(y)dy = 2f(x)\int_{-t}^{t} f(y)dy$$

which is

$$\int_{x-t}^{x+t} f(y)dy = f(x)\int_{-t}^{t} f(y)dy. \tag{6}$$

Since f is not identically zero, $f(0) = 1$.

Further, since f is continuous, there exists $t > 0$ such that (see Figure 1 on the next slide)

$$\int_{-t}^{t} f(y)\, dy > 0.$$

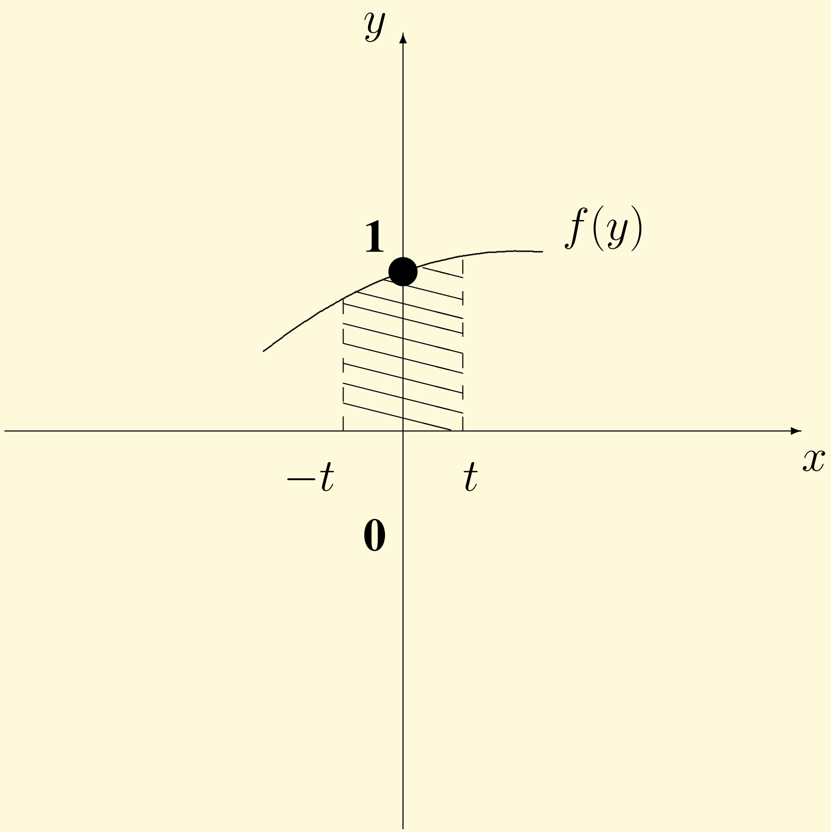

Figure 1. Illustration of the existence of $t > 0$.

- Recall that the fundamental theorem of calculus says that

if f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is differentiable on (a, b), and

$$g'(x) = f(x).$$

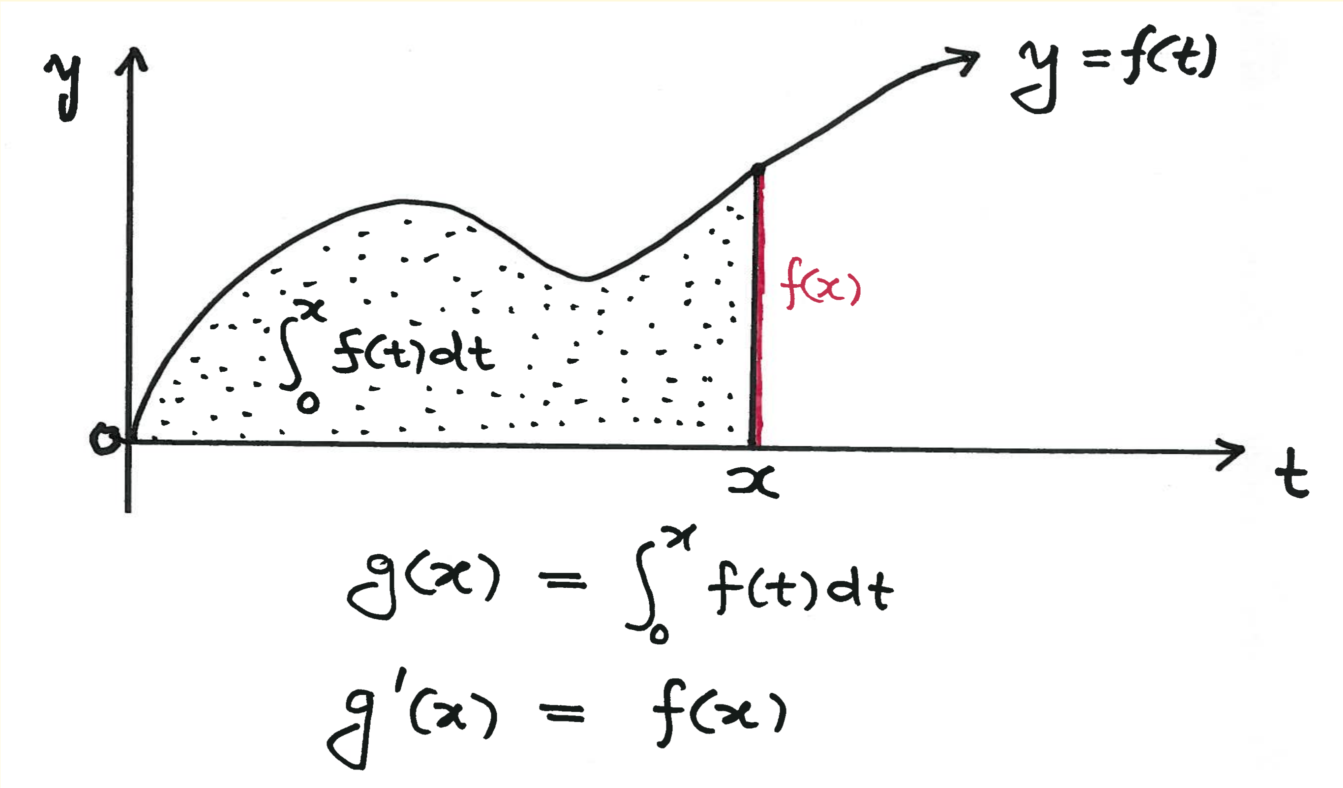

Illustration of Fundamental Theorem of Calculus

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By the fundamental theorem of calculus, the left-hand side of (6), that is of the following equality

$$\int_{x-t}^{x+t} f(y)dy = f(x) \int_{-t}^{t} f(y)dy$$

is differentiable with respect to x.

• **Hence the right-hand side of the above equality is also differentiable with respect to the variable x.**

Then differentiating (6) with respect to x, we get

$$\frac{d}{dx}\int_{x-t}^{x+t} f(y)dy = \frac{d}{dx}\left[f(x)\int_{-t}^{t} f(y)dy\right]$$

which is

$$f(x+t) - f(x-t) = f'(x)\int_{-t}^{t} f(y)dy. \tag{7}$$

This shows that f is twice differentiable and hence

$$f'(x+t) - f'(x-t) = f''(x) \int_{-t}^{t} f(y)dy.$$

Thus f is 3 times differentiable.

Proceeding step by step, we see that any continuous solution of (DE), that is of $f(x+y) + f(x-y) = 2f(x)f(y)$ is infinitely differentiable.

Letting $x = 0$ in (7), we obtain

$$f(t) - f(-t) = f'(0) \int_{-t}^{t} f(y)dy. \tag{8}$$

Since f is even, we have $f(t) = f(-t)$ and (8) yields

$$f'(0) \int_{-t}^{t} f(y)dy = 0. \tag{9}$$

Since $\int_{-t}^{t} f(y)dy > 0$, (9) gives

$$f'(0) = 0. \tag{10}$$

Since $f \in C^\infty(\mathbb{R})$, we differentiate (DE) with respect to y twice to get

$$f'(x+y) - f'(x-y) = 2f(x)f'(y)$$
$$f''(x+y) + f''(x-y) = 2f(x)f''(y)$$

for all $x, y \in \mathbb{R}$. Letting $y = 0$, we have

$$2f''(x) = 2f(x)f''(0).$$

Let $k = f''(0)$.

Then

$$f''(x) = kf(x)$$

which yields the following initial value problem (IVP)

$$\boxed{\begin{array}{c}\dfrac{d^2y}{dx^2} = ky,\ y(0) = 1,\ \ y'(0) = 0 \\ \text{(IVP)}\end{array}}$$

To solve this initial value problem we have to consider three cases: $k = 0$, $k > 0$ and $k < 0$.

Case 1. Suppose $k = 0$. Then IVP reduces to

$$\frac{d^2y}{dx^2} = 0.$$

Hence $y(x) = c_1\,x + c_2$.

Since $y(0) = 1$, $c_2 = 1$. Again since $y'(0) = 0$, we get $c_1 = 0$. Therefore $y(x) = 1$ is the solution in this case (which is $f(x) = 1$ for all $x \in \mathbb{R}$).

Case 2. Suppose $k > 0$. Letting $y = e^{mx}$ into

$$\frac{d^2y}{dx^2} = ky, \qquad \text{(DE}'\text{)}$$

we obtain $m^2 = k$ and hence $m = \pm\sqrt{k}$. Thus

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}, \quad \text{where } \alpha = \sqrt{k}.$$

Since

$$1 = y(0) = c_1 e^{\alpha \cdot 0} + c_2 e^{-\alpha \cdot 0} = c_1 + c_2,$$

hence $c_2 = (1 - c_1)$.

Thus

$$y(x) = c_1\, e^{\alpha\, x} + (1 - c_1)\, e^{-\alpha\, x}.$$

Now

$$\begin{aligned}
0 &= y'(0) \\
&= c_1\, \alpha e^{\alpha\, x} + (1 - c_1)\, (-\alpha)\, e^{-\alpha\, x}\Big|_{x=0} \\
&= c_1\, \alpha + (1 - c_1)\, (-\alpha) \\
&= c_1\, \alpha - \alpha + \, c_1 \alpha \\
&= 2\, c_1\, \alpha - \alpha.
\end{aligned}$$

Hence

$$2\,c_1\,\alpha = \alpha.$$

Since $\alpha \neq 0$, we have $c_1 = \frac{1}{2}$.

Therefore the solution of (DE′) is given by

$$y(x) = \frac{e^{\alpha\,x} + e^{-\alpha\,x}}{2} = \cosh(\alpha\,x).$$

Hence in this case we have $f(x) = \cosh(\alpha\,x)$ which is the solution listed in (3).

Case 3. Suppose $k < 0$. Letting $y = e^{mx}$ into

$$\frac{d^2y}{dx^2} = k\,y \qquad \text{(DE}'\text{)}$$

we obtain $m^2 = k$.

- Hence $m = \pm i\,\beta$, where $\beta = \sqrt{-k}$ and $i = \sqrt{-1}$.
- Thus the solution of (DE′) is given by

$$y(x) = c_1\,e^{i\beta\,x} + c_2\,e^{-i\beta\,x}.$$

Since

$$1 = y(0) = c_1 + c_2,$$

we have

$$c_2 = 1 - c_1.$$

Hence

$$y(x) = c_1\, e^{i\beta\, x} + (1 - c_1)\, e^{-i\beta\, x}.$$

Further, since

$$\begin{aligned} 0 &= y'(0) \\ &= i\,\beta\,c_1 - i\,\beta\,(1 - c_1) \\ &= 2\,i\,\beta\,c_1 - i\,\beta \end{aligned}$$

or

$$i\,\beta\,(2c_1 - 1) = 0,$$

we obtain

$$c_1 = \frac{1}{2}.$$

Hence, we have

$$y(x) = \frac{e^{i\beta\, x} + e^{-i\beta\, x}}{2} = \cos(\beta\, x).$$

Therefore the solution of the functional equation is given by

$$f(x) = \cos(\beta\, x)$$

which is (4).

This completes the proof of the theorem.

• Note that the continuity of f together with the functional equation gave us infinite differentiability of the solution f.

• Then by differentiating the functional equation, we obtained a differential equation.

• By solving this differential equation we found the solutions of the functional equation.

Remark 1 . This is one of the standard methods for solving functional equations when regularity properties like continuity are assumed.

Remark 2 . There is another standard method due to A. L. Cauchy. The Cauchy method consists of finding the solution of a functional equation on a dense set (like the set of rationals $\mathbb{Q}$) and then uses continuity to determine solutions in the set of real numbers $\mathbb{R}$.

General Solution of d'Alembert Equation

Definition 1 *A function* $E : \mathbb{R} \to \mathbb{C}$ *is said to be* exponential *if* E *satisfies the equation* $E(x+y) = E(x)E(y)$ *for all* $x, y \in \mathbb{R}$.

If E is a nonzero continuous function, then $E(x) = e^{\lambda x}$, where λ is an arbitrary complex constant.

If $E : \mathbb{R} \to \mathbb{C}$ is a nonzero exponential function, then we denote it by

$$E^*(y) = E(y)^{-1}. \tag{11}$$

Now we give some elementary properties of the exponential function.

Proposition 1 *If* $E : \mathbb{R} \to \mathbb{C}$ *is an exponential function and* $E(0)$ *is zero, then* $E(x) \equiv 0$ *for all* $x \in \mathbb{R}$.

Proof: Let $E : \mathbb{R} \to \mathbb{C}$ be an exponential function. Hence

$$E(x+y) = E(x)\,E(y) \tag{12}$$

for all $x, y \in \mathbb{R}$. Letting $y = 0$ in (12), we obtain

$$E(x) = E(x)\,E(0) \quad \text{for } x \in \mathbb{R}. \tag{13}$$

Since $E(0) = 0$, (13) yields

$$E(x) = 0 \quad \forall\, x \in \mathbb{R}. \tag{14}$$

Hence $E(x)$ is identically zero.

Proposition 2 *Let $E : \mathbb{R} \to \mathbb{C}$ be an exponential function. If $E(x) \not\equiv 0$, then $E(0) = 1$.*

Proof: Let $E : \mathbb{R} \to \mathbb{C}$ be an exponential function. Assume that $E(x)$ is not identically zero. Letting $x = 0 = y$ in (12), we get $E(0)\,[1 - E(0)] = 0$. Hence either $E(0) = 0$ or $E(0) = 1$.

We claim that $E(0) = 1$.

Suppose not. Then $E(0) = 0$. By Proposition 1, $E(x) \equiv 0$, is a contradiction to the fact that $E(x) \not\equiv 0$. Thus $E(0) = 1$.

This completes the proof of the proposition.

Proposition 3 *Let* $E : \mathbb{R} \to \mathbb{C}$ *be an exponential function. If* $E(x_0) = 0$ *for some* $x_0 \neq 0$, *then* $E(x) \equiv 0$ *for all* $x \in \mathbb{R}$.

Proof: Let $x\ (\neq x_0) \in \mathbb{R}$. Then, since $E(x_0) = 0$, we have

$$E(x) = E((x - x_0) + x_0) = E(x - x_0)\, E(x_0) = 0.$$

Hence $E(x) \equiv 0$. Thus E is nowhere zero or everywhere zero. This completes the proof.

Proposition 4 *Let* $E : \mathbb{R} \to \mathbb{C}$ *be an exponential function. If* $E(x)$ *is not identically zero, then*

$$E^*(-x) = E(x)$$

for all $x \in \mathbb{R}$.

Proof: Let $E : \mathbb{R} \to \mathbb{C}$ be exponential. Next, letting $y = -x$ in (12), we get

$$E(0) = E(x)\, E(-x). \tag{15}$$

Since $E(x)$ is not identically zero, by Proposition 2 we have $E(0) = 1$ and (15), that is $E(0) = E(x)\,E(-x)$ yields

$$E(-x) = \frac{1}{E(x)}.$$

Hence

$$E(-x) = E(x)^{-1}$$

or

$$E(-x) = E^*(x) \tag{16}$$

for all $x \in \mathbb{R}$.

Next replacing x by $-x$ in (16), that is in $E(-x) = E^*(x)$, we obtain

$$E^*(-x) = E(x) \tag{17}$$

and the proof of the proposition is now complete.

Proposition 5 *Let* $E : \mathbb{R} \to \mathbb{C}$ *be an exponential function. Suppose* $E(x)$ *is not identically zero. Then*

$$E^*(x+y) = E^*(x)E^*(y) \tag{18}$$

for all $x, y \in \mathbb{R}$.

Proof: Since $E(x)$ is not identically zero, $E(x)$ is never zero on $\mathbb{R}$ by Proposition 3. Now we consider

$$E^*(x+y) = \frac{1}{E(x+y)}$$
$$= \frac{1}{E(x)\,E(y)} = E(x)^{-1}\,E(y)^{-1} = E^*(x)\,E^*(y).$$

Hence

$$E^*(x+y) = E^*(x)\,E^*(y)$$

for all $x, y \in \mathbb{R}$.

• **Now we prove some elementary properties of the d'Alembert functional equation.**

Proposition 6 *Every nonzero solution $f : \mathbb{R} \to \mathbb{C}$ of the d'Alembert equation*

$$f(x+y) + f(x-y) = 2f(x)f(y) \qquad \text{(DE)}$$

is an even function.

Proof: Replacing y by $-y$ in the above equation (DE), we have

$$f(x+y)+f(x-y)=2f(x)f(-y). \tag{19}$$

Subtracting (19) from (DE), we obtain

$$f(y)=f(-y)$$

for all $y \in \mathbb{R}$. Hence f is an even function.

Let $G = (\mathbb{R}, +)$ be the additive group of reals and $\mathbb{C}$ be the set of complex numbers.

The continuous solution $g : G \longrightarrow \mathbb{C}$ of the exponential functional equation $g(x + y) = g(x)\, g(y)$ is of the form $g(x) = e^{\lambda x}$. The continuous periodic solution $f : G \longrightarrow \mathbb{C}$ of D'Alembert's is $f(x) = \cos(\alpha\, x)$.

• **How can we represent the solutions of D'Alembert's functional equation on abstract structures like group or semigroup?**

Notice that

$$\begin{aligned} f(x) &= \cos(\alpha x) \\ &= \frac{\left[e^{i\alpha\, x} + e^{-i\alpha\, x}\right]}{2} \\ &= \frac{\left[g(x) + g(-x)\right]}{2} \end{aligned}$$

where $g(x)$ is a solution of the exponential equation (i.e. a homomorphism from group $(G, +)$ into $(\mathbb{C}, \cdot)$.

Next we proceed to determine the nontrivial general solution of the functional equation (DE) **following Kannappan (1968).**

Theorem 2. *Every nontrivial solution* $f : \mathbb{R} \to \mathbb{C}$ *of the functional equation*

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{DE}$$

is of the form

$$f(x) = \frac{E(x) + E^*(x)}{2}, \tag{20}$$

where $E : \mathbb{R} \to \mathbb{C}^\star$ *(the set of nonzero complex numbers) is an exponential function.*

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Thank You