HW1 solutions

1. Let X be the number of red marbles selected in the 4 draws.

Then $X \sim \text{Binomial}(n=4, p=\frac{6}{10}=.6)$ and

$$P(X=1) = \binom{4}{1}(.6)^1(.4)^{4-1} = 4(.6)(.4)^3 = \boxed{.1536}.$$

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2. The loglikelihood function for estimating M is

$$\ell(M) = \ln L(M; X=1) = \ln\left\{\binom{4}{1}\left(\tfrac{M}{10}\right)^1\left(1-\tfrac{M}{10}\right)^3\right\}$$
$$= \ln 4 + \ln\left(\tfrac{M}{10}\right) + 3\ln\left(\tfrac{10-M}{10}\right) = \ln M + 3\ln(10-M) + \underbrace{(\ln 4 - 4\ln 10)}_{C}.$$

where $M \in \{1, 2, \ldots, 9\}$.

Then $\ell'(M) = \frac{1}{M} - \frac{3}{10-M} = \frac{10-M-3M}{M(10-M)} = \frac{10-4M}{M(10-M)}$

so $\ell'(M) > 0$ if $M < 2.5$ and $\ell'(M) < 0$ if $M > 2.5$.

Thus, $\ell(M)$ is maximized at either $M=2$ or $M=3$; we check both.

$$\ell(2) = \ln 2 + 3\ln 8 + C = \ln 1024 + C$$
$$\ell(3) = \ln 3 + 3\ln 7 + C = \ln 1029 + C$$

So $\boxed{\hat{M} = 3}$ is the MLE.

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3. First, we find the MLE of M for each possible value of x.

If $x=0$, $\ell(M; X=0) = \ln\left(1-\frac{M}{10}\right)^4 = 4\ln(10-M) - 4\ln 10$, is decreasing $M \in \{0, 1, 2, \ldots, 9\}$

so it is maximized at $M=0$.

If $x=1$, then ℓ is maximized at $M=2$ (from part 2).

If $x=2$, then $\ell(M; X=2) = \ln\left\{\binom{4}{2}\left(\frac{M}{10}\right)^2\left(1-\frac{M}{10}\right)^2\right\}$
$$= 2\ln M + 2\ln(10-M) + \ln 6 - 4\ln 10, \quad M \in \{1, \ldots, 9\}$$

and $\ell'(M) = \frac{2}{M} - \frac{2}{10-M} = \frac{20-2M-2M}{M(10-M)} = \frac{20-4M}{M(10-M)}$.

So ℓ is maximized at $M=5$.

ℓ': $+$ | $-$ at 5

If $x=3$, then $\ell(M;X=3) = \ln\{\binom{4}{3}(\frac{M}{10})^3(1-\frac{M}{10})^1\}$

$$= 3\ln M + \ln(10-M) + \underbrace{\ln 4 - 4\ln 10}_{C}, \quad M\in\{1,\dots,9\}.$$

and $\ell'(M) = \dfrac{3}{M} - \dfrac{1}{10-M} = \dfrac{3(10-M)-M}{M(10-M)} = \dfrac{30-4M}{M(10-M)}$.

ℓ': $+$ before 7.5, $-$ after 7.5

So ℓ is maximized at $M=7$

because $\ell(7) = 3\ln 7 + \ln 3 + C = \ln 1029 + C$

and $\ell(8) = 3\ln 8 + \ln 2 + C = \ln 1024 + C$

If $x=4$, then $\ell(M;X=4) = \ln M^4 = 4\ln M$, $M\in\{1,2,\dots,10\}$

is increasing so it is maximized at $M=10$.

So, the rule based on the likelihood principle is

$$\hat{M}(x) = \begin{cases} 0 & \text{if } x=0 \\ 3 & \text{if } x=1 \\ 5 & \text{if } x=2 \\ 7 & \text{if } x=3 \\ 10 & \text{if } x=4 \end{cases}.$$

If $M=6$, then the pmf of $\hat{M}(X)$ is

$P(\hat{M}(X)=0) = P(X=0) = \binom{4}{0}(.6)^0(.4)^4 = .0256$

$P(\hat{M}(X)=3) = P(X=1) = \binom{4}{1}(.6)^1(.4)^3 = .1536$

$P(\hat{M}(X)=5) = P(X=2) = \binom{4}{2}(.6)^2(.4)^2 = .3456$

$P(\hat{M}(X)=7) = P(X=3) = \binom{4}{3}(.6)^3(.4)^1 = .3456$

$P(\hat{M}(X)=10) = P(X=4) = \binom{4}{4}(.6)^4 = .1296$.