M621 Short HW 5 due Sept 29.: **Solutions, 10.1** G is a group, $H \leq G$, and $b \in G$.

1. Show that $bHb^{-1} \leq G$, and the map $F : H \to bHb^{-1}$ given by $F(h) = bhb^{-1}$, all $h \in H$, is an isomorphism. **Do this—don't turn in.**

   Proof. Let $H \leq G$, and let $b \in G$. We'll show $F : H \to bHb^{-1}$, given by $F(h) = bhb^{-1}$ for all $h \in H$, is an isomorphism of groups. With $y, z \in H$, we have $F(yz) = byzb^{-1} = byb^{-1}bzb^{-1} = (byb^{-1})(bzb^{-1}) = F(y)F(z)$. So F is compatible with the group operations. Suppose h, k are in H, and $F(h) = F(k)$. In that case, $bhb^{-1} = bkb^{-1}$, and it follows from the cancellativity properties that $h = k$. Lastly, if $g \in bHb^{-1}$, then there exists $h \in H$ such that $g = bhb^{-1}$, which means that $F(h) = g$, showing that g is in the image of F, and proving that F is an onto map. $\square$

   (I probably should have asked to prove that $H \leq G$ and $b \in G$ implies that bHb^{-1} is a subgroup of G: Of course $e = beb^{-1}$, so bHb^{-1} is non-empty. If $\{u, v\} \subseteq bHb^{-1}$, then there exist $h_u, h_v \in H$ such that $u = bh_ub^{-1}$ and $v = bh_vb^{-1}$. So $uv^{-1} = bh_ub^{-1}(bh_vb^{-1})^{-1} = bh_ub^{-1}b(h_v)^{-1}b^{-1} = bh_uh_v^{-1}b^{-1}$. Since $H \leq G$, $h_uh_v^{-1} \in H$. By the 1-Step Subgroup Test, bHb^{-1} is a subgroup of G.)

2. Let A be the set of left cosets of H in G. So $A = \{bH : b \in G\}$. Let G act on A as follows: For all $g \in G$, and all $bH \in A$, let $g \cdot bH = gbH$. This does indeed define a group action—you don't have to prove it. As you know, a group action determines a homomorphism $\sigma : G \to S_A$, where $g \to \sigma_g$, and $\sigma_g : A \to A$ is the permutation of A given by $a \to g \cdot a$, all $a \in A$. **Prove** $ker(\sigma) = \cap\{bHb^{-1} : b \in G\}$.

   Proof. We have $g \in ker(\sigma)$ if and only if for all $b \in G$, $gbH = bH$ if and only if $b^{-1}gb \in H$ if and only if for all $b \in G$, $g \in bHb^{-1}$ if and only if $g \in \cap\{bHb^{-1} : b \in G\}$. $\square$

3. Let $A = \{bHb^{-1} : b \in G\}$, the set of all *conjugates* of H in G. Define an action of G on A as follows: For all $g \in G$, all $bHb^{-1} \in A$, we let $g \cdot bHb^{-1} = g(bHb^{-1})g^{-1}$ (which is $(gb)H(gb)^{-1}$, another conjugate of H). This defines an action of G on A. Show G_H, the stabilizer of H in G under the action, is $N_G(H)$, the normalizer of H in G.

   Proof. This is an exercise in important definitions.

Under the action given in the exercise, G acts on the conjugates of a given subgroup H of G by $g \cdot bHb^{-1} = (gb)H(gb)^{-1}$, for all $g \in G$, and each conjugate bHb^{-1} of H, we are asked what is the stabilizer of H under the action. That would be $\{g \in G : g \cdot H = H\}$, which is $\{g \in G : gHg^{-1} = H\}$; the right-most set in the last equation is, by definition, $N_G(H)$. So $G_H = N_G(H)$. $\square$

Comment. A classic lemma: Suppose G acts transitively on a set A. Let a and b in A, and suppose that $g \cdot a = b$. Then $G_b = gG_ag^{-1}$. See if you can prove it.

Some questions—Exan 1 like questions. Let S_4 act on all 2-element subgroups of S_4 by conjugation, i.e., if $H \leq S_4$ has two elements, and $g \in S_4$, then $g \cdot H = gHg^{-1}$. So A is the set of all two-element subgroups of S_4. Questions: 1. Does S_4 act transitively on A? 2. Does S_4 act faithfully on A?