Introduction to functional equations

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Abstract: In this introductory talk, we present an overview of the field of functional equations and then determine the solutions two simple functional equations, namely Sincov functional equation and Cauchy functional equation. This talk will be accessible to undergraduate and graduate students.

What are functional equations?

- Functional equations are equations in which the unknowns are functions.

Some examples of functional equations are:

$$f(x+y) = f(x)+f(y)$$

$$f(x+y) = f(x)f(y)$$

$$f(xy) = f(x)f(y)$$

$$f(xy) = f(x)+f(y)$$

$$
\begin{aligned}
f(x+y) &= f(x)g(y) + g(x)f(y) \\
f(x+y) &= f(x) + f(y) + f(x)f(y) \\
f(x+y) &= 2f(x)f(y) - f(x-y) \\
f(x+y) &= g(xy) + h(x-y)
\end{aligned}
$$

$$
\begin{aligned}
f(x+y)+f(x-y) &= 2f(x)+2f(y)\\
f(pr,qs)+f(ps,qr) &= 2f(p,q)+2f(r,s)\\
f(x,y)+f(x+y,z) &= f(y,z)+f(x,y+z)\\
f(x,y)+f(y,z) &= f(x,z)
\end{aligned}
$$

- **The field of functional equations includes differential equations, difference equations, integral equations, and iterations.**

- To solve a functional equation means to find the unknown function (or functions).

- **In order to find solutions, the unknown functions must often be restricted to a specific nature (such as bounded, continuous, differentiable).**

- Functional equations is a field of mathematics which is over 270 years old.

- More than 15,500 papers have been published in this area (according to MathSciNet).

History of functional equations

- Functional equations appeared in the literature around the same time as the modern theory of functions.

- In 1747 and 1750, d'Alembert published three papers and these three papers were the first on functional equations.

Jean d’Alembert (1717-1783)

Functional equations were studied by

- D'Alembert (1747)
- Euler (1768)
- Poisson (1804)
- Cauchy (1821)
- Abel (1823)
- Darboux (1875)

and many others.

Jean d'Alembert

(1717-1783)

Leonard Euler

(1707-1783)

Simeon Poisson

(1781-1840)

Augustin Cauchy

(1789-1857)

Niels Abel

(1802-1829)

Gaston Darboux

(1842-1917)

- David Hilbert (1900) pointed out that while the theory of differential equations provides elegant and powerful techniques for solving functional equations, the differentiability assumptions are not inherently required.

David Hilbert (1862-1943)

- Motivated by Hilbert's suggestions many researchers in functional equations have treated various functional equations without any (or mild) regularity assumptions.

- This effort has given rise to the modern theory of functional equations. The theory of functional equations forms a modern mathematical discipline.

- This discipline has developed very rapidly in the last six decades largely due to the effort of Janos Aczél.

Professor Janos Aczél

Applications

Functional Equations have found applications in many areas of mathematics, sciences and engineering.

Fo example, they are used in

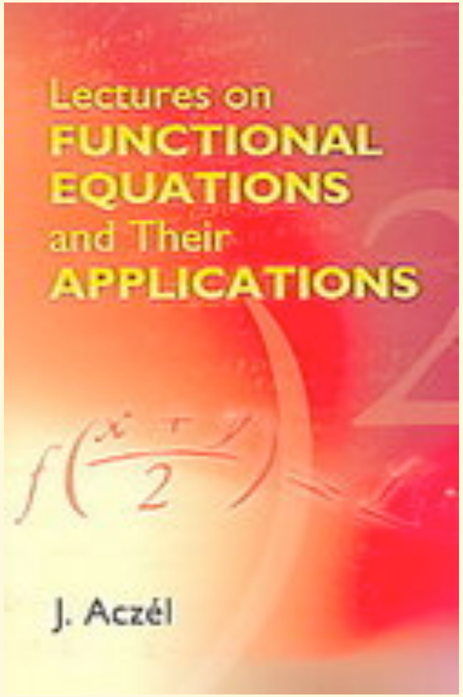
- Algebra
- Combinatorics
- Economics
- Psychology
- Statistics
- Computer Science
- Dynamical Systems
- Dynamical Programing
- Information Theory
- Probability Theory

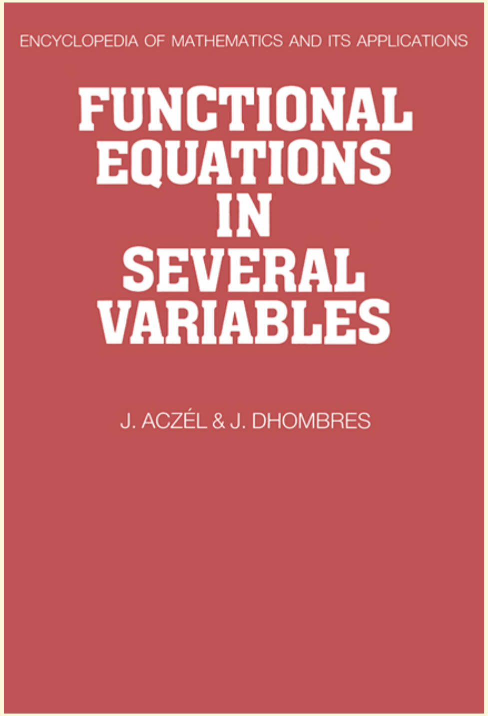
International Conferences

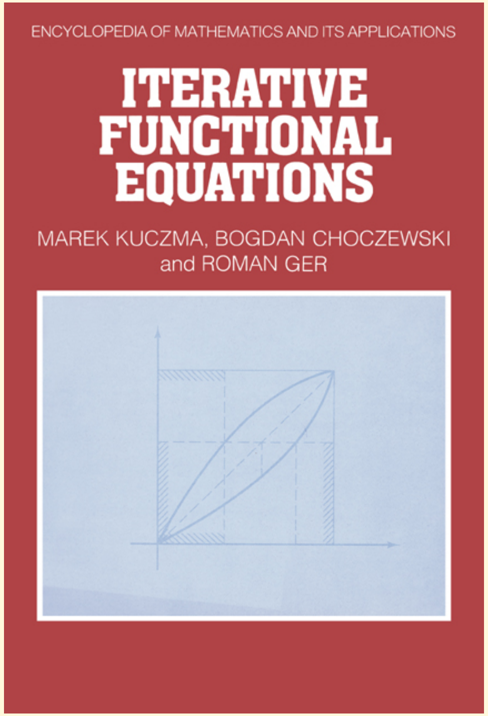
Each year, there are several conferences and seminars on functional equations. Among these the ISFE (started in 1962) and ICFEI (started in 1984) are premier conferences in this field.

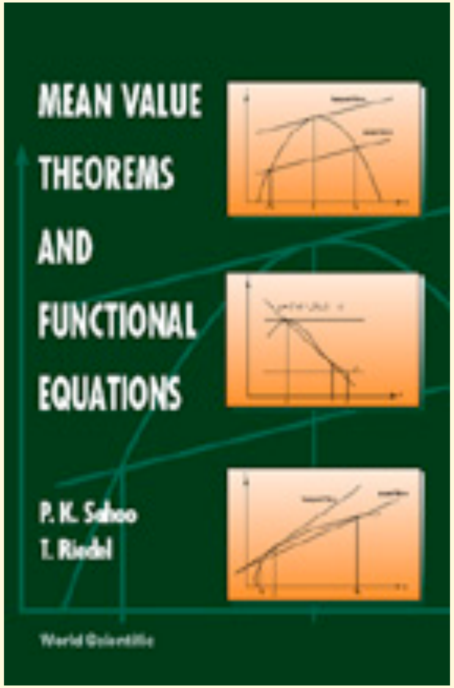
44th ISFE in Louisville, Kentucky, USA, June 14-21, 2006

Few books on functional equations





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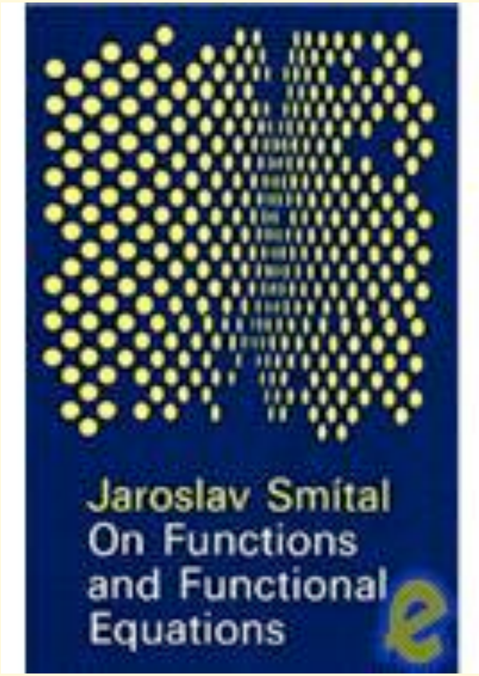
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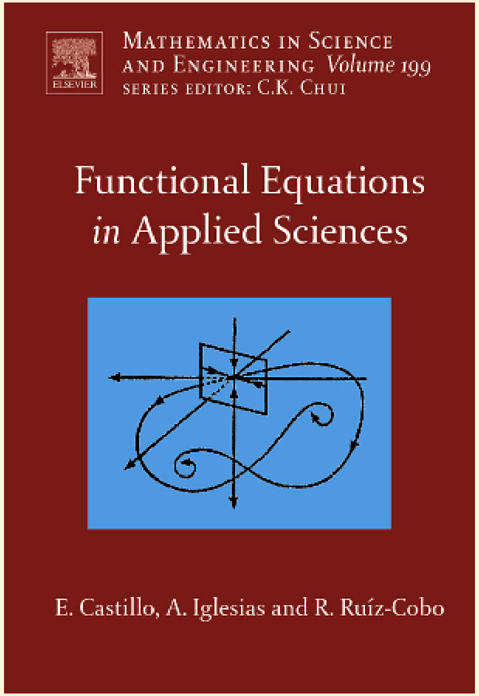
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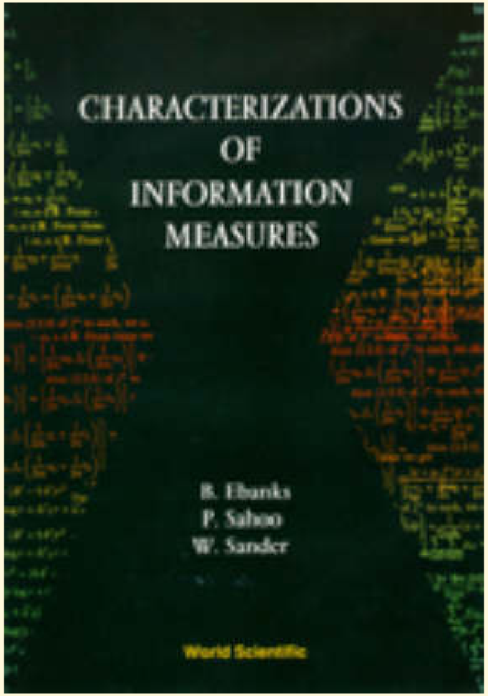
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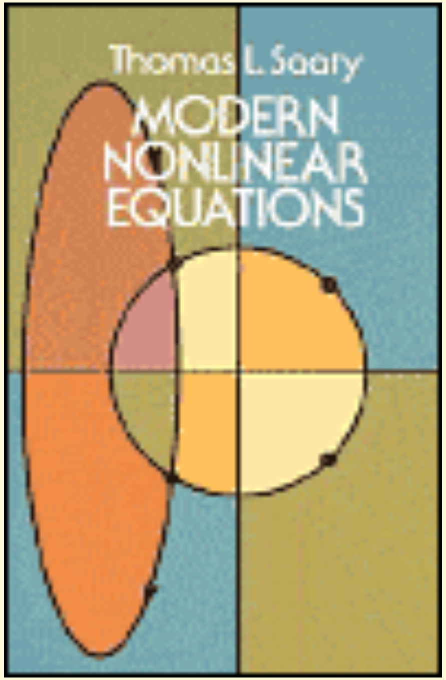
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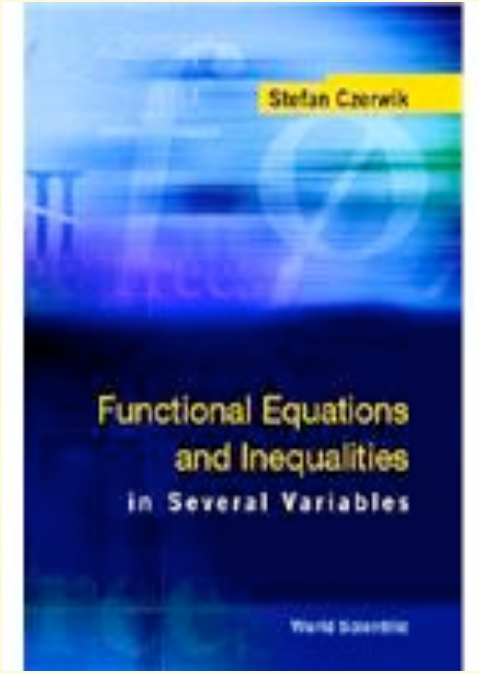
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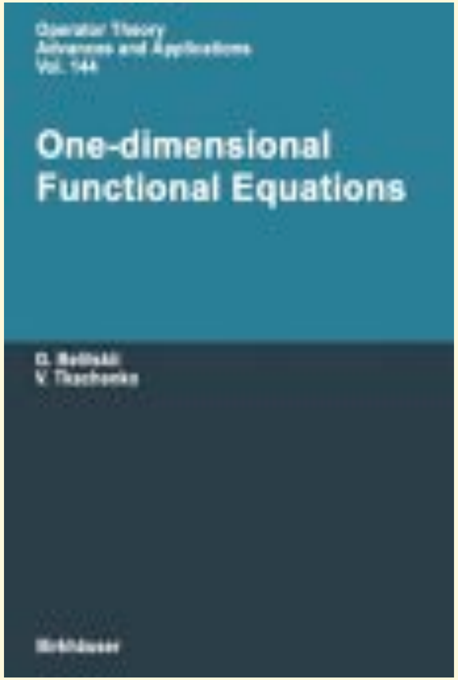
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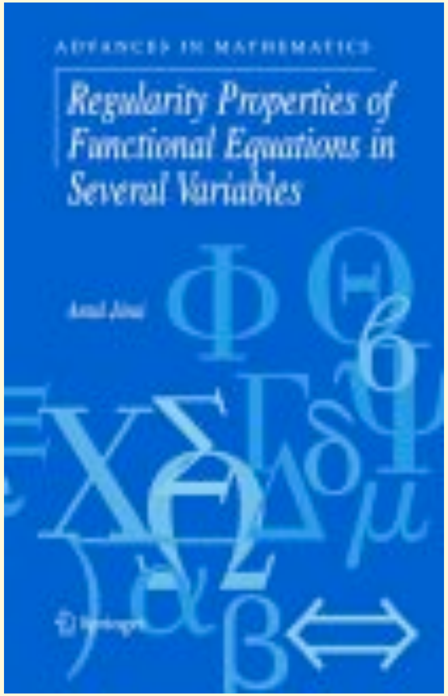
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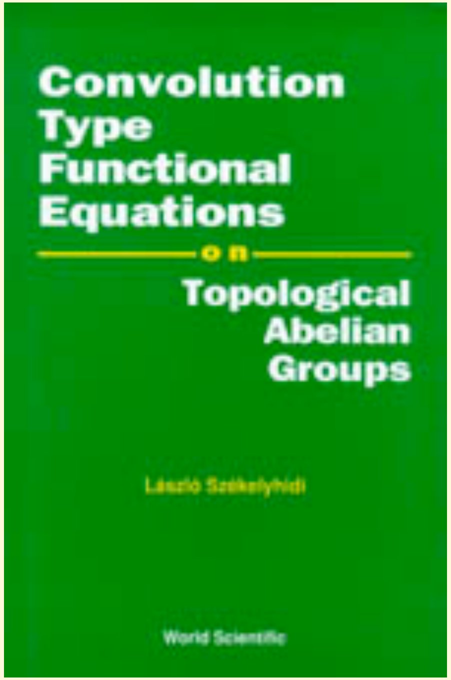
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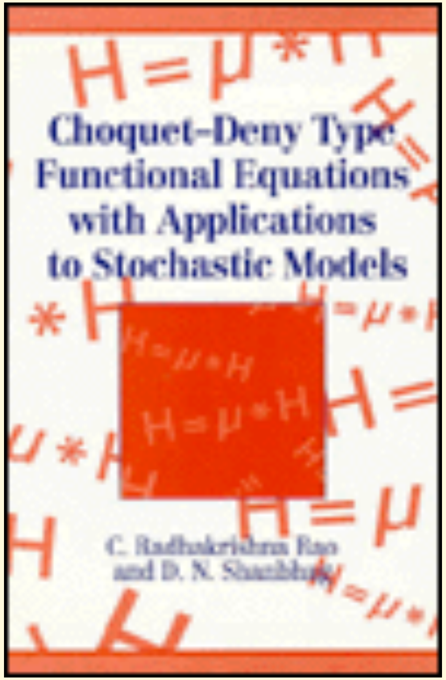
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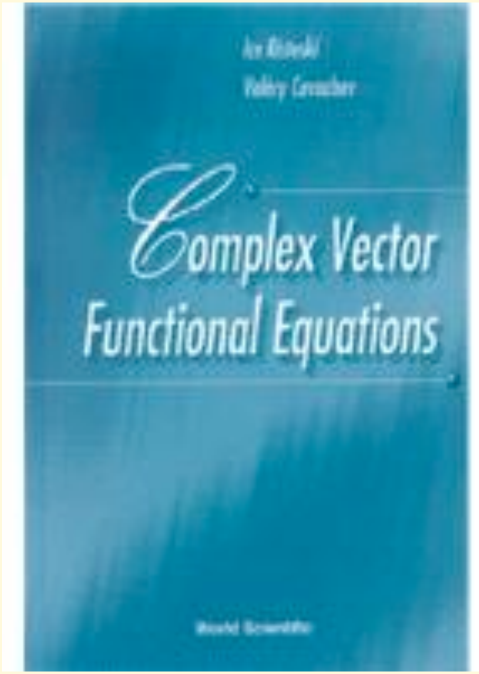
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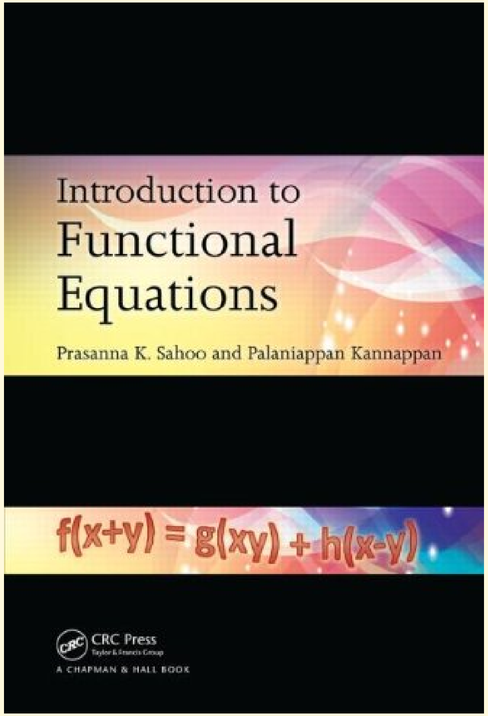
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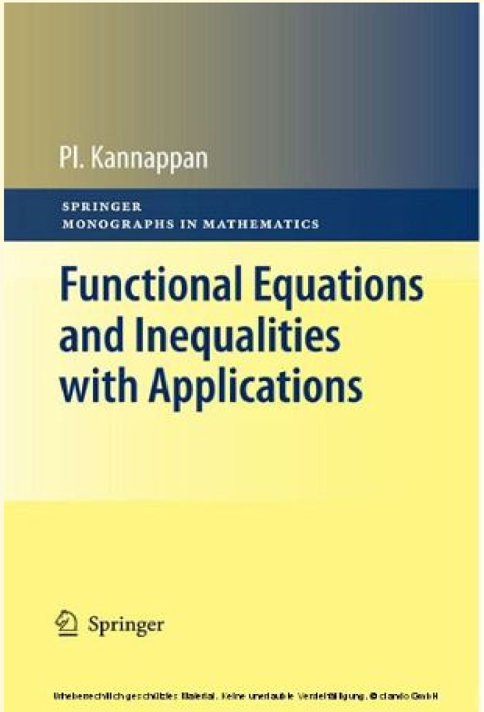
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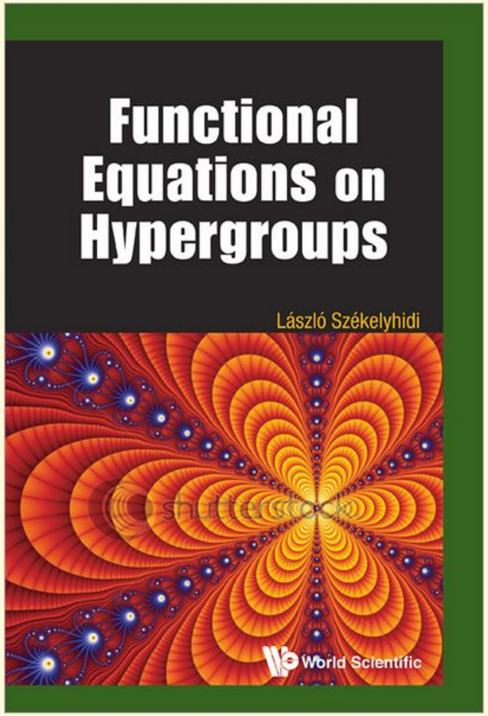
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Sincov Functional Equation

We will examine a very simple functional equation which has many applications.

This functional equation is the **Sincov functional equation**.

• In 1903, the Russian mathematician D.M. Sincov studied the following functional equation

$$f(x, y) + f(y, z) = f(x, z) \tag{1}$$

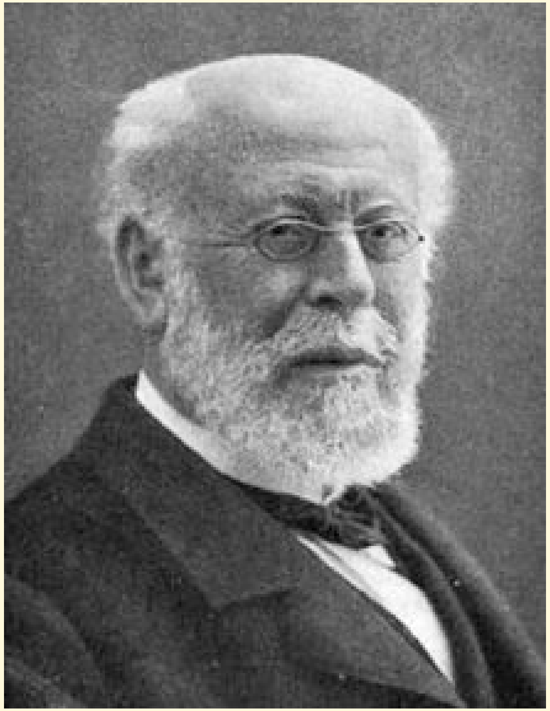
for all $x, y, z \in \mathbb{R}$ (the set of real numbers).

Dmitrii Sintsov (1867-1946)

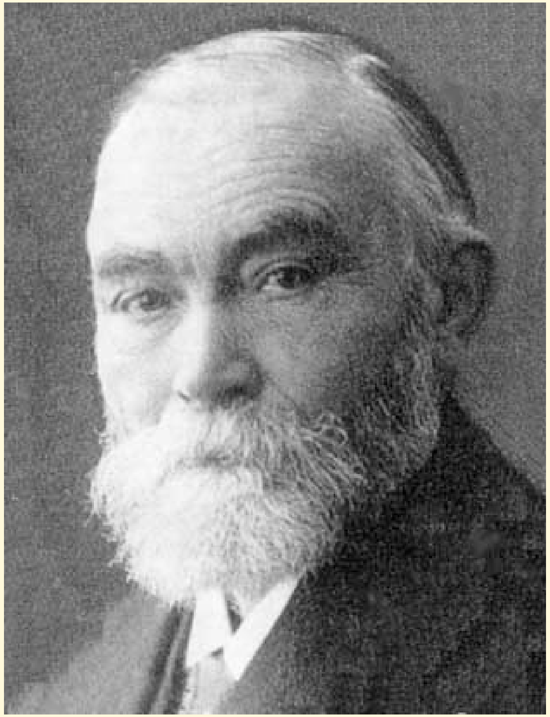
• Others like Moritz Cantor (1896) and Gottlob Frege (1874) treated this functional equation

$$f(x, y) + f(y, z) = f(x, z)$$

before Sincov.



Moritz Cantor (1829-1920)



Gottlob Frege (1848-1925)

The functional equation (1), that is

$$f(x,y) + f(y,z) = f(x,z)$$

is known as the Sincov functional equation.

- **The most general solution of the Sincov functional equation is $f(x,y) = \phi(x) - \phi(y)$, where ϕ is an arbitrary function.**

The multiplicative version of (1) is the following:

$$f(x,y)\,f(y,z) = f(x,z) \tag{2}$$

for all $x, y, z \in \mathbb{R}$.

- The most general solution of (2) are $f(x,y) = 0$ and $f(x,y) = \frac{\phi(x)}{\phi(y)}$, where ϕ is a nonzero arbitrary function.

- The additive Sincov functional equation

$$f(x,y) + f(y,z) = f(x,z)$$

reduces to the well-known Cauchy functional equation

$$g(x+y) = g(x) + g(y)$$

if we take $f(x,y) = g(y-x)$.

To see this just substitute $u = y - x$ and $v = z - y$ so that $u + v = z - x$.

Applications of Sincov Equation

- The Sincov functional equation arises in the axiomatic derivation of the maximum entropy principle (see Shore and Johnson (1980)).

- The Sincov functional equation is used in the characterizations of information measures on open domains (see Ebanks, Sahoo and Sander (1998)).

• It also arises in the study of subgroup consistent poverty indices in economics (see Foster and Shorrocks (1991)).

• Further, the Sincov functional equation has found applications in actuarial mathematics (see Shiu (1988)).

In their book, Dodson and Poston (1977) define an affine space as follows:

• *An affine space is a triple* (S, V, f) *where* S *is a set,* V *a vector space and* $f : S^2 \to V$ *such that*

$$f(x,y) + f(y,z) = f(x,z)$$

for $x, y, z \in S$*, and for all* $x \in S$*, the map* $f_x(y) = f(x,y)$ *is a bijection.*

- In the definition of affine space, we again encounter the Sincov functional equation.

Cauchy Functional Equation

We will examine another very simple functional equation which has many applications.

This functional equation is the Cauchy equation and it it of the form

$$f(x+y) = f(x) + f(y), \quad \text{for all } x, y \in \mathbb{R}.$$

This theorem was first proved by A. L. Cauchy in 1821.

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Cauchy equation*

$$f(x+y) = f(x) + f(y), \ \textit{for all } x, y \in \mathbb{R}.$$

Then f is linear, that is $f(x) = m\,x$, where m is a real constant.

Proof. The Fundamental Theorem of Calculus says that if f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$

is continuous and differentiable on (a, b), and $g'(x) = f(x)$.

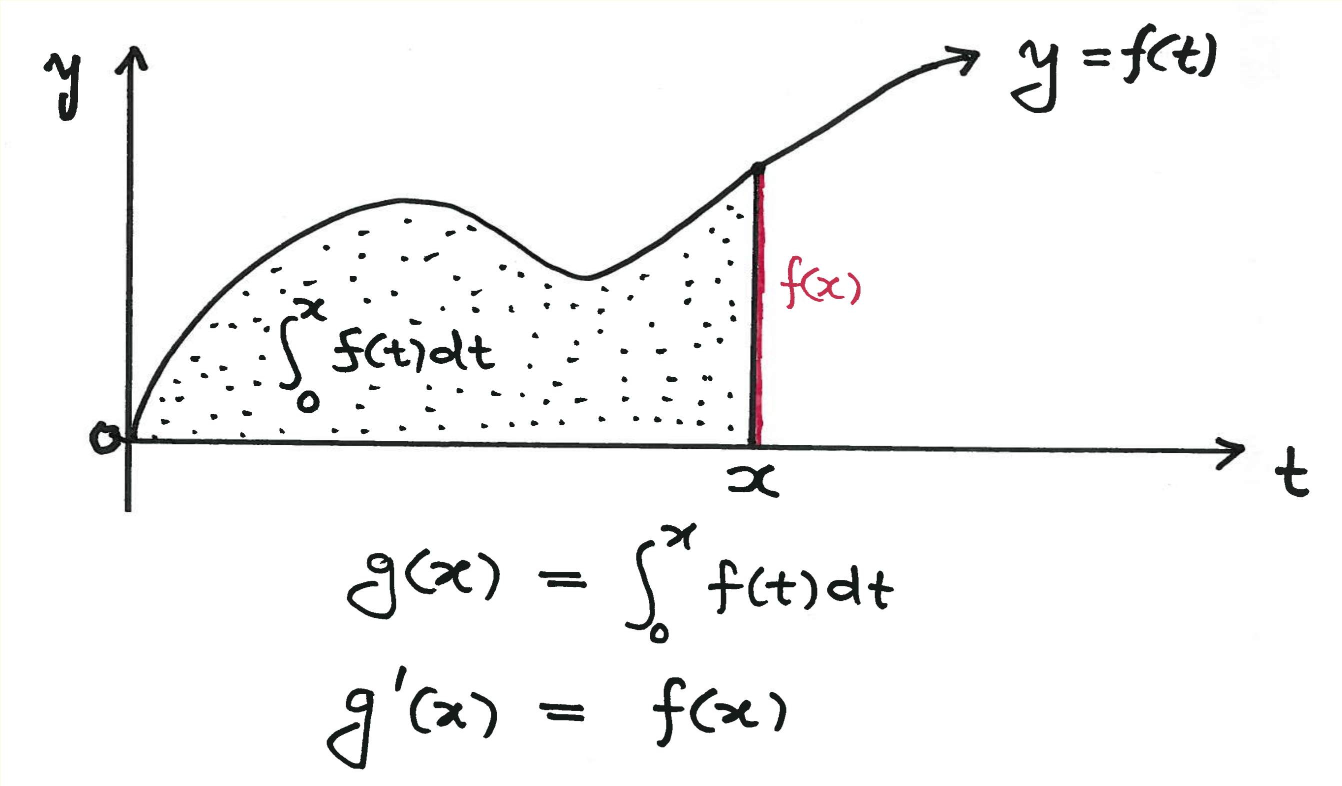


Illustration of Fundamental Theorem of Calculus

First, let us fix x and then integrate both side of the Cauchy equation with respect to y from 0 to 1 to get

$$
\begin{aligned}
f(x) &= \int_0^1 f(x)\,dy \\
&= \int_0^1 \left[f(x+y) - f(y)\right] dy \\
&= \int_x^{1+x} f(u)\,du - \int_0^1 f(y)\,dy \\
&= \int_d^{1+x} f(u)\,du - \int_d^x f(u)\,du - \int_0^1 f(y)\,dy.
\end{aligned}
$$

Hence

$$f(x) = \int_d^{1+x} f(u)\,du - \int_d^{x} f(u)\,du - \int_0^1 f(y)\,dy.$$

Since f is continuous, by the FTC, we get

$$\begin{aligned} f'(x) &= f(1+x) - f(x) \\ &= f(1) + f(x) - f(x) \\ &= f(1) \\ &= m \text{ (say)}. \end{aligned}$$

Therefore solving the differential equation,

$$\frac{d}{dx}f(x) = m$$

we get

$$f(x) = m\,x + b,$$

where d is a constant of integration. It can be easily shown that $b = 0$ and thus $f(x) = mx$.

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Thank You

Appendix

Jean d'Alembert

d'Alembert was a French mathematician who was a pioneer in the study of differential equations and their use of in physics. He studied the equilibrium and motion of fluids.

Simeon Poisson

Poisson's most important works were a series of papers on definite integrals and his advances in Fourier series. This work was the foundation of later work in this area by Dirichlet and Riemann.

Niels Abel

In 1824 Abel proved the impossibility of solving algebraically the general equation of the fifth degree.

Augustin Cauchy

Cauchy pioneered the study of analysis, both real and complex, and the theory of permutation groups. He also researched in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics.

Leonard Euler

Leonhard Euler was a Swiss mathematician who made enormous contibutions to a wide range of mathematics and physics including analytic geometry, trigonometry, geometry, calculus and number theory.

Gaston Darboux

Darboux made important contributions to differential geometry and analysis and the Darboux integral is named after him.

David Hilbert

Hilbert's work in geometry had the greatest influence in that area after Euclid. A systematic study of the axioms of Euclidean geometry led Hilbert to propose 21 axioms and he analysed their significance. He made contributions to mathematics and physics.

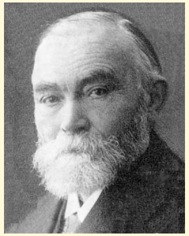
Janós Aczél

Janós Aczél will be best remembered for his pioneering work on the theory of functional equations, with applications in many fields, such as information measures, index numbers, group decision making, aggregation, production functions, laws of science, theory of measurement and utility theory.

Dmitrii Sintsov

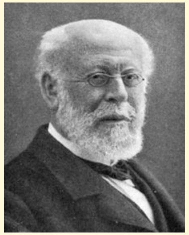
Sintsov's main areas of interest were in the theory of conics and applications of this geometrical theory to the solution of differential equations and to the theory of nonholonomic differential geometry.

Gottlob Frege



Gottlob Frege was one of the founders of modern symbolic logic putting forward the view that mathematics is reducible to logic. He is widely regarded as a logician on a par with Aristotle, Gödel, and Taraski.

Moritz Cantor



Moritz Cantor is best remembered for the four volume work *Vorlesungen ber Geschichte der Mathematik* which traces the history of mathematics up to 1799. The first volume was published in 1880 and the last volume appeared in 1908.

Thank You