Classroom discussion details Thurs. Sept 15

0.1 Conjugation in S_n

Let k, n be positive integers, with $n \geq k$, let $\alpha = (a_1 \ldots a_k) \in S_n$, a k-cycle in S_n, let $\beta \in S_n$, and let $\gamma = \beta(a_1 \ldots a_k)\beta^{-1}$. We proved Lemma 0.1 in class using fixed elements and "move" elements of a permutation: Recall that if $\phi \in S_n$, then $Fix(\phi) = \{y \in \{1, \ldots, n\} : \phi(y) = y\}$—so $Fix(\phi)$ is the set of elements (of $\{1, \ldots, n\}$) that are fixed by ϕ. Let $Move(\phi) = \{1, \ldots, n\} - Fix(\phi)$, the set of elements that are "moved" by ϕ, i.e., $Move(\phi) = \{z \in \{1, \ldots, n\} : \phi(z) \neq z\}$. Notice that $Fix(\phi) \cup Move(\phi) = \{1, \ldots, n\}$.

Lemma 0.1 $\beta(a_1 \ldots a_k)\beta^{-1} = (\beta(a_1) \ldots \beta(a_k))$.

Proof.

The explanation begins with a description of $Move(\gamma)$ and $Fix(\gamma)$.

Claim 1 $Move(\gamma) = \beta^{-1}(\{a_1, \ldots, a_k\}) = \{\beta(a_1), \ldots, \beta(a_k)\}$.

Suppose $y \in \beta^{-1}(\{a_1, \ldots, a_k\})$. So for some i, $k \geq i \geq 1$, there exists $a_i \in \{a_1, \ldots, a_k\}$ such that $\beta(a_i) = y$. Let's compute: $\beta(a_1 \ldots a_n)\beta^{-1}(y) = \beta(a_1 \ldots a_k)\beta^{-1}(\beta(a_i)) = (\beta(a_1 \ldots a_k)(\beta^{-1}\beta(a_i)) = \beta(a_1 \ldots a_k)(a_i) = \beta(a_{i+1})$ (where $i+1$ is taken mod k). So $\gamma(\beta(a_i)) = \beta(a_{i+1})$. Note that $a_i \neq a_{i+1}$, and that β is a permutation implies that $y = \beta(a_i)$ is in $Move(\gamma)$. Thus, $\beta^{-1}(\{a_1, \ldots, a_k\}) \subseteq Move(\gamma)$. For the reverse inclusion, suppose $z \notin \beta^{-1}(\{a_1, \ldots, a_k\})$. In that case, $\beta(z) \notin \{a_1 \ldots a_k\}$, $\gamma(z) = \beta(a_1 \ldots a_k)\beta^{-1}(z)$, with $\beta^{-1}(z) \notin \{a_1, \ldots, a_k\}$. But then $(a_1 \ldots a_k)(\beta^{-1}(z)) = \beta^{-1}(z)$ (since $Move(a_1 \ldots a_k) = \{a_1, \ldots, a_k\}$), which implies that that $\beta^{-1}(z)$ is fixed by $(a_1 \ldots a_k)$). So $\beta(a_1 \ldots a_k)\beta^{-1}(z) = \beta\beta^{-1}(z) = z$—that is, $z \notin \beta^{-1}(\{a_1, \ldots, a_k\})$. The claim is proved.

From the proof of the Claim, we have a bit more. If $y \in Move(\gamma)$, we have shown there exists i such that $y = \beta(a_i)$, and $\gamma(y) = \gamma(\beta(a_i)) = \beta(a_{i+1})$. Since the only elements that are moved by γ are in the set $\{\beta(a_1), \ldots, \beta(a_k)\}$, the only elements that will occur in the cycle notation representation of γ will be $\{\beta(a_1), \ldots, \beta(a_k)\}$. But we know that under γ, $\beta(a_1) \to \beta(a_2) \to \beta(a_3) \ldots \beta(a_k) \to \beta(a_1)$. So the cycle notation representation of γ must be

$$(\beta(a_1)\ldots\beta(a_k)),$$

completing the proof of the lemma. $\square$

Uses of the above lemma:

1. $\beta = (456)(12)$. What is $\beta(2,3,5,9,1)\beta^{-1}$? Answer. $(\beta(2)\beta(3)\beta(5)\beta(9)\beta(1)) = (1,3,6,9,2)$.

2. Notice that if $(a_1 \ldots a_k)$ is a k-cycle, then any conjugate $\beta(a_1 \ldots a_k)\beta^{-1}$ of $(a_1 \ldots a_k)$ is also a k-cycle. This will turn out to be a usable fact, one that will be generalized below.

Proposition 0.2 *Let $\beta \in S_n$, and let $(a_1, \ldots, a_{k_1}), \ldots (b_1, \ldots, b_{k_2}), \ldots (z_1, \ldots, z_{k_j})$ be a set of j non-trivial cycles of lengths $k_1, \ldots, k_j$ respectively. Then $\rho = \beta(a_1 \ldots a_k) \ldots (z_1 \ldots z_{k_j})\beta^{-1} = (\beta(a_1) \ldots \beta(a_{k_1}) \ldots (\beta(z_1) \ldots \beta(z_{k_j})$.*

Proof. Observe that $\beta(a_1 \ldots a_k) \ldots (z_1 \ldots z_{k_j})\beta^{-1} = (\beta(a_1 \ldots a_{k_1})\beta^{-1}) \ldots (\beta(z_1 \ldots z_{k_j})\beta^{-1}$. Now apply Lemma 0.1. $\square$

Uses of the proposition.

1. Let $\beta = (452)(29)$. Then $\beta(143)(35)(29)\beta^{-1} = (153)(32)(29)$.

2. **Definition.** Let $\phi \in S_n$. It has a unique-up-to-ordering representation as a product of *disjoint* cycles. Suppose ϕ is given by $\phi = \alpha_1 \ldots \alpha_j$, where α_1 is a n_1-cycle, α_2 is an n_2-cycle, and so on, with α_j is an n_j cycle. Then ϕ is said to have **shape** $n_1, \ldots, n_j$. For example, $\phi = (12)(57)(368)$ has shape $2, 2, 3$. The shape of a permutation is usually given as a sequence of positive integers that increase from left to right (as does $2, 2, 3$). By Proposition 0.2, "conjugation preserves shape" in S_n.