MATH 668 Exam 2 Solutions

1. (a) The numerator of the F-statistic for $H_0 : \beta_0 - \beta_1 = 0$ can be written as

$$SSH/1 = (\mathbf{C}\hat{\boldsymbol{\beta}} - \mathbf{0})^\top \left(\mathbf{C}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{C}^\top\right)^{-1} (\mathbf{C}\hat{\boldsymbol{\beta}} - \mathbf{0}) = \left((1,-1)\begin{pmatrix}\hat{\beta}_0\\ \hat{\beta}_1\end{pmatrix}\right)^\top \left((1,-1)\begin{pmatrix}10 & 4\\ 4 & 2\end{pmatrix}^{-1}\begin{pmatrix}1\\ -1\end{pmatrix}\right)^{-1}\left((1,-1)\begin{pmatrix}\hat{\beta}_0\\ \hat{\beta}_1\end{pmatrix}\right)$$

$$= (\hat{\beta}_1 - \hat{\beta}_2)\left((1,-1)\tfrac{1}{20-16}\begin{pmatrix}2 & -4\\ -4 & 10\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}\right)^{-1}(\hat{\beta}_1 - \hat{\beta}_2)$$

$$= (\hat{\beta}_1 - \hat{\beta}_2)\left((1,-1)\begin{pmatrix}0.5 & -1\\ -1 & 2.5\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}\right)^{-1}(\hat{\beta}_1 - \hat{\beta}_2)$$

$$= (\hat{\beta}_1 - \hat{\beta}_2)5^{-1}(\hat{\beta}_1 - \hat{\beta}_2) = \tfrac{1}{5}(\hat{\beta}_1 - \hat{\beta}_2)^2.$$

The denominator can be written as

$$SSE/(n-k-1) = n\hat{\sigma}^2/(n-k-1) = 10\hat{\sigma}^2/(10-1-1) = \tfrac{10}{8}\hat{\sigma}^2.$$

So, the F-statistic can be expressed in the form $F = \dfrac{SSH/1}{SSE/(n-k-1)} = \dfrac{\frac{1}{5}(\hat{\beta}_1 - \hat{\beta}_2)^2}{\frac{10}{8}\hat{\sigma}^2} = .16\left(\dfrac{\hat{\beta}_0 - \hat{\beta}_1}{\hat{\sigma}}\right)^2.$

(b) $H_0 : \beta_0 = \beta_1$ should be rejected if $F > F_{.05,1,8} = 5.32$ which is equivalent to $\left|\dfrac{\hat{\beta}_0 - \hat{\beta}_1}{\hat{\sigma}}\right| > \sqrt{\dfrac{5.32}{.16}} \approx 5.767.$

2. Let $\boldsymbol{\beta}_1 = \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}$, $\hat{\boldsymbol{\beta}}_1^* = \begin{pmatrix}\hat{\boldsymbol{\beta}}_0^*\\ \hat{\boldsymbol{\beta}}_1^*\end{pmatrix}$, $\mathbf{X}_1 = \begin{pmatrix}1 & 1\\ 1 & 2\\ 1 & 3\end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix}1\\ 4\\ 9\end{pmatrix}$. Then we have

$E(\hat{\boldsymbol{\beta}}_1^*) = \boldsymbol{\beta}_1 + \mathbf{A}\boldsymbol{\beta}_2$ where

$$\mathbf{A} = (\mathbf{X}_1^\top\mathbf{X}_1)^{-1}\mathbf{X}_1^\top\mathbf{X}_2 = \begin{pmatrix}3 & 6\\ 6 & 14\end{pmatrix}^{-1}\begin{pmatrix}14\\ 36\end{pmatrix} = \tfrac{1}{42-36}\begin{pmatrix}14 & -6\\ -6 & 3\end{pmatrix}\begin{pmatrix}14\\ 36\end{pmatrix} = \begin{pmatrix}\frac{7}{3} & -1\\ -1 & 0.5\end{pmatrix}\begin{pmatrix}14\\ 36\end{pmatrix} = \begin{pmatrix}-\frac{10}{3}\\ 4\end{pmatrix}.$$

So, the bias for $\boldsymbol{\beta}_1$ is the second element of $E(\hat{\boldsymbol{\beta}}_1^*) - \boldsymbol{\beta}_1$ which is $E(\hat{\beta}_1) - \beta_1 = 4\beta_2$.

3. (a) The length of the confidence interval is $2t_{.025,6-2}s\sqrt{\boldsymbol{a}^\top(\mathbf{X}^\top\mathbf{X})^{-1}\boldsymbol{a}} = 4$ where $n = 6$, $\boldsymbol{a} = \begin{pmatrix}1\\ 0\end{pmatrix}$. Then $(\mathbf{X}^\top\mathbf{X})^{-1} = \frac{1}{6-4}\begin{pmatrix}1 & -2\\ -2 & 6\end{pmatrix} = \begin{pmatrix}.5 & -1\\ -1 & 3\end{pmatrix}$, so $\boldsymbol{a}^\top(\mathbf{X}^\top\mathbf{X})\boldsymbol{a} = .5$. Thus, $s = \dfrac{2}{t_{.025,6-2}\sqrt{.5}} = \dfrac{2}{2.776\sqrt{.5}} \approx 1.018886$. Then $\hat{\sigma}^2 = \dfrac{n-2}{n}s^2 \approx \dfrac{4}{6}(1.018886)^2 \approx 0.6920856$.

(b) Since $\hat{\varepsilon}_6 = 0$, $r_6 = 0$ which implies that $s_{(6)}^2 = \left(\frac{n-2-0}{n-3}\right)s^2 \approx \frac{4}{3}(1.018886)^2 \approx 1.384171$. Then $\hat{\sigma}_{(6)}^2 = \dfrac{5-2}{5}s_{(6)}^2 \approx \dfrac{3}{5}(1.384171) \approx 0.8305027$.

(c) Since $\mathbf{X}^\top\mathbf{X} - \begin{pmatrix}1\\ 0\end{pmatrix}(1,0) = \begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$, the length of the confidence interval without the 6th observation is

$$2t_{.025,5-2}s_{(6)}\sqrt{\boldsymbol{a}^\top\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}^{-1}\boldsymbol{a}} \approx 2t_{.025,3}\sqrt{1.384171}\sqrt{(1,0)\begin{pmatrix}1 & -2\\ -2 & 5\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}} \approx 2(3.182)\sqrt{1.384171}\sqrt{1} \approx 7.487297.$$

4. We have $\alpha_2 = -\alpha_1$ and $\gamma_2 = -\gamma_1$ so we can write
$y_{111} = \mu + \alpha_1 + \gamma_1 + \varepsilon_{111}$
$y_{112} = \mu + \alpha_1 + \gamma_1 + \varepsilon_{112}$
$y_{121} = \mu + \alpha_1 - \gamma_1 + \varepsilon_{121}$
$y_{122} = \mu + \alpha_1 - \gamma_1 + \varepsilon_{122}$
$y_{211} = \mu - \alpha_1 + \gamma_1 + \varepsilon_{211}$
$y_{212} = \mu - \alpha_1 + \gamma_1 + \varepsilon_{212}$

$y_{221} = \mu - \alpha_1 - \gamma_1 + \varepsilon_{221}$
$y_{222} = \mu - \alpha_1 - \gamma_1 + \varepsilon_{222}$.

Then we want to minimize $Q(\boldsymbol{\beta}) = \|\boldsymbol{y} - \mathbf{X}\boldsymbol{\beta}\|^2$ where $\boldsymbol{y} = \begin{pmatrix} 8 \\ 2 \\ 0 \\ 4 \\ 4 \\ 6 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$, $\boldsymbol{\beta} = \begin{pmatrix} \mu \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$ and we compute $\mathbf{X}^\top\mathbf{X} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = 8\mathbf{I}_3$ so that

$(\mathbf{X}^\top\mathbf{X})^{-1} = \dfrac{1}{8}\mathbf{I}$. Also, $\mathbf{X}^\top\boldsymbol{y} = \begin{pmatrix} 24 \\ 4 \\ 16 \end{pmatrix}$ so $\hat{\boldsymbol{\beta}} = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\boldsymbol{y} = \begin{pmatrix} 3 \\ .5 \\ 2 \end{pmatrix}$. If $H_0 : \alpha_1 = 0$ is true, then we can use the design matrix $\tilde{\mathbf{X}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$. We have $\tilde{\mathbf{X}}^\top\tilde{\mathbf{X}} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = 8\mathbf{I}_2$ and $\tilde{\mathbf{X}}^\top\boldsymbol{y} = \begin{pmatrix} 24 \\ 16 \end{pmatrix}$ so that $(\tilde{\mathbf{X}}^\top\tilde{\mathbf{X}})^{-1} = \frac{1}{8}\mathbf{I}$ and $(\tilde{\mathbf{X}}^\top\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}^\top\boldsymbol{y} = \frac{1}{8}\begin{pmatrix} 24 \\ 16 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Thus, it follows that $\hat{\boldsymbol{\beta}}_c = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$. Then $SSH = \|\mathbf{X}(\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}_c)\|^2 = \|\mathbf{X}\begin{pmatrix} 0 \\ .5 \\ 0 \end{pmatrix}\|^2 = \|\begin{pmatrix} .5 \\ .5 \\ .5 \\ .5 \\ -.5 \\ -.5 \\ -.5 \\ -.5 \end{pmatrix}\|^2 = 8(.25) = 2$ and $SSE = \boldsymbol{y}^\top\boldsymbol{y} - \hat{\boldsymbol{\beta}}^\top\mathbf{X}^\top\boldsymbol{y} = 136 - (3, .5, 2)\begin{pmatrix} 24 \\ 4 \\ 16 \end{pmatrix} = 136 - 106 = 30.$

So, $F = \dfrac{2/1}{30/(8-3)} = \dfrac{1}{3}$. The critical value for this test is $F_{.05,1,5} = 6.61$
so we fail to reject H_0 since $F < F_{.05,1,5}$.