M621 Exam 1 review problems, set 1, Sept. 25

1. Draw a regular tetrahedron T Label the apex 1, and the base vertices $2, 3, 4$ clockwise. Let G be the group of symmetries of the tetrahedron. A tetrahedron has 4 vertices, 4 faces, and 6 edges.

   (a) Be able to explain –geometry, spatial sense– that G maps faces of T to faces of T, and that a symmetry α of G is determined by the image of the face $\{1, 2, 3\}$, and $\alpha(1)$. Then explain why $|T| \leq 12$.

   (b) List (in cycle notation) the 12 distinct elements of G.

   (c) The group D_{12} is also a group with 12 elements. Explain why G could not be isomorphic to D_{12}.

   (d) G acts on the four faces of T. Is the action transitive? Let $\sigma : G \to S_4$ be the homomorphism associated with the action. What is $ker(\sigma)$, the kernel of this action? Is σ onto?

2. Let G be a finite group, and let $H \leq G$. Assume that H is a proper subgroup. Let H act on G as follows: For all $h \in H, g \in G$, $h \cdot g = hg$.

   (a) Let $g \in G$. Describe the orbit of g (which is sometime denoted $\mathcal{O}(g)$).

   (b) Does H acts transitively on G?

   (c) Does H act faithfully on G?

   (d) As you showed, the orbits of the action of a group on A partition the set A. Explain how LaGrange's Theorem, that the order of H divides the order of G, follows from the above.

3. Provide a presentation of D_6.

4. Provide a presentation of $Z_2 \times Z_2$.

5. Suppose that $\Gamma : G \to K$ is an isomorphism. Prove that $\Gamma^{-1} : K \to G$ is an isomorphism.

6. Let $n \in \mathbb{N}$. A group G is *n-generated* if there exists a subset $A \subseteq G$, $n \geq |A|$, and $G =< A >$. Show that if K is a group, $\Gamma : G \to K$ is an onto homomorphism, and G is n-generated, then K is n-generated.

7. Suppose G is a cyclic group, $G =< g >$, where $g \in G$. Prove that every subgroup of G is cyclic.

8. Let G be a group, $g \in G$, and $|g| = n \in \mathbb{N}$. Let $k \in \mathbb{Z}$.

   (a) Prove that $g^k = e$ if and only if $n|k$.

   (b) Let $m \in \mathbb{N}$. Show that $|g^m| = \frac{n}{(n,m)}$. (You might want want to use that if x, y, z are integers, $x \neq 0$, then $x|yz$ and $(x, y) = 1$, then $x|z$.)

   (c) With m as above, prove that $< g^m >=< g^{(n,m)} >$.

9. Suppose $n \in \mathbb{N}$, $n \geq 2$, and $\{a, b\} \subseteq \{1, \ldots, n\}$ with $a \neq b$. Let $\beta \in S_n$. Prove that $\beta(ab)\beta^{-1} = (\beta(a)\beta(b))$.

10. How many elements of S_6 have order 2?

11. A fundamental set of group theory review problems (stuff from M521, stuff that isn't well-understood by M521 students, but must be completely understood by M621 students) now follows:

    (a) Let H be a subgroup of a group G, and let b, c be in G.

       i. $bH = H$ if and only if $b \in H$.

       ii. $bH = cH$ if and only if $bc^{-1} \in H$.

       iii. $bH \cap cH = \neq \emptyset$ implies that $bH = cH$. (From which we can conclude that the left cosets of H in G partition G.)

    (b) Another fundamental set of group theory review problems now follows: : Let H be a subgroup of a group G, and let b, c be in G. H is said to be a *normal* subgroup of G if for all $c \in G$, $cHc^{-1} \subseteq H$.

       i. Prove that if $\Gamma : G \to K$ is a homomorphism of G, then $ker(\Gamma)$ is a normal subgroup of G. (We've already proven that $ker(\Gamma)$ is a subgroup of G, so it suffices to show that if $c \in G$, then $c(ker(\Gamma))c^{-1} \subseteq ker(\Gamma)$.)

       ii. List the normal subgroups of S_3—the subgroups of S_3 are displayed in the text in the section on lattices of subgroups.

       iii. List the normal subgroups of D_8–the subgroups of D_8 are also displayed in the text.

       iv. Prove that H is normal in G if and only if for all $c \in G$, $cHc^{-1} = H$.

       v. Prove that H is normal in G if and only if for all $c \in G$, $cH = Hc$.

       vi. Prove that H is normal in G if and only if for all $b, c \in G$, $bHcH = bcH$.

12. Let $\Gamma : G \to K$ be an onto homomorphism, let $g \in G$, $k \in K$, with $\Gamma(g) = k$. Using that Γ is a homomorphism, prove that $\Gamma^{-1}(k) = gker(\Gamma)$.

13. Let G be a group. Let $A = Sub(G)$ be the set of all subsets of G. G acts on $Sub(G)$ "by conjugation": If $H \in Sub(G)$, $g \cdot H = gHg^{-1}$.

    (a) Let $G = S_4$. Are $H_1 =< (23) >$ and $H_2 =< (13) >$ in the same orbit under the action? Are $H_3 =< (12), (34) >$ and $H_4 =< (1234) >$ in the same orbit?

    (b) Determine G_{H_1}, the stabilizer of H_1, for the above action. Determine G_{H_3}. and G_{H_4}.

14. Let A and B be subgroups of a group G. Let $AB = \{ab : a \in A, b \in B\}$.

   (a) Show by a counterexample that AB is not always a subgroup of G.

   (b) Prove that if A is a normal subgroup, then AB is a subgroup of G.

   (c) Prove that if both A and B are normal subgroups, then AB is a normal subgroup.

15. Suppose $H \leq G$ and $[G : H] = 2$. Prove that H is normal in G.

16. True or false? If false, provide a specific counterexample.

   (a) If $H \leq G$, and a and b are in G, then $aH = bH$ implies $a = b$.

   (b) If N is a normal subgroup of G, and H is a subgroup of G, then $H \cap N$ is normal in H.

   (c) $C_G(a) = G$ if and only if $a \in Z(G)$.

   (d) $Z(G)$ is a normal subgroup of G.

   (e) For $n > 2$, $Z(S_3) = \{e\}$.

   (f) For a group G, $G/Z(G)$ is an Abelian group.

   (g) If G is cyclic, then $G \times G$ is cyclic.

   (h) If N is a normal subgroup of G, $\Gamma : G \to K$ is an onto homomorphism, then $\Gamma(N)$ is a normal subgroup of K.