Math 621: Final exam review problems 2

1. A warm-up problem: True or false? If false, provide a specific counterexample.

   (a) If R is an integral domain, then every non-trivial homomorphic image of R is integral domain

   (b) If F is a field, then, up to isomorphism, F has only one non-trivial homomorphic image.

   (c) If R is a commutative ring, then every homomorphic image of R is a commutative ring.

   (d) If G is a cyclic group, every homomorphic image of G is cyclic.

   (e) If G is a cyclic group, then $G \times G$ is a cyclic group.

   (f) If G is a cyclic group, then every subgroup of G is cyclic.

   (g) If R is a commutative ring with 1, a, b, c are elements of R, with $a \neq 0$, then $ab = ac$ implies that $b = c$.

   (h) If R is a ring with 1, the product of any two units of R is also a unit of R.

   (i) If R is a ring with 1, the sum of any two units of R is also a unit of R.

   (j) If G is a group, N is a normal subgroup of G, and A is a subgroup of G, then $NA = AN$.

   (k) If G is a group, N is a normal subgroup of G, and A is a subgroup of G, then for all $a \in A, n \in N$, $an = na$.

   (l) If G is a group, N is a normal subgroup of G, and A is a subgroup of G, then for all $a \in A, n \in N$, there exists $n' \in N$ such that $an = n'a$.

2. It is important to review the Class Equation. Suppose G is a finite group. Assume there are exactly k conjugacy classes of G that have more than one element. Then $|G| = |Z(G)| + \sum_{i=1}^{i=k}[G : C_G(a_i)]$, where $a_1, \ldots, a_k$ are representatives of the k distinct conjugacy classes of G that have more than one element.

   You should be able to explain why it is true that

   (a) $g \in Z(G)$ if and only if the congugacy class of g is $\{g\}$ if and only if $C_G(g) = G$ if and only if $g \in Z(G)$,

   (b) $[G : C_G(a_i)]$ is the number of elements in the conjugacy class of a_i, and

   (c) be able to use the above two facts to explain the Class Equation.

3. Use the Class Equation to prove that if G is a p-group ($|G| = p^\alpha$, where p is prime, and $\alpha \in \mathbb{N}$), then G has a non-trivial center (that is, $|Z(G)| > 1$).

4. Describe all ring homomorphisms from Z_{30} to Z_{40}, and in each case, describe the image and the kernel of the homomorphism. How do you know these are the only such homomorphisms? (This problem is one of the evils of life as we know it, to be endured by all who choose to pass this way.)

5. Let G be a group, $H \leq G$ a subgroup of G. Prove that $\cap\{gHg^{-1} : g \in G\}$ is a normal subgroup of G. Then prove that $\cap\{gHg^{-1} : g \in G\}$ is the largest normal subgroup of G that is contained in H.

6. Suppose G is a finite group, $H \leq G$, with H a proper subgroup of G, and $[G : H] = k$. Let G act on the left cosets of H in G. So G is acting on a k-element set.

   (a) Show that the kernal of this action is $\cap\{gHg^{-1} : g \in G\}$.

   (b) Suppose G is simple. Explain why G embeds in S_k.

   (c) Show that if G is simple, and $|G| > m!$ (where $m \in \mathbb{N}$), then G has no subgroup of index m.

7. Home-brewed problem: Suppose G is a group, A, B are subgroups of G, N is a normal subgroup of G, and A is a normal subgroup of B. Recall that since N is normal, NA and NB are both subgroups of G.

   (a) Prove that NA is normal in NB.

   (b) Then find a surjective homomorphism $\Gamma : B/A \to NB/NA$. Explain clearly and concisely why your map Γ is indeed a surjective homomorphism.

   (c) Is your map Γ above necessarily an isomorphism? If not, provide a specific counterexample.

8. Prove that A_4 has no subgroup of order 6. (Suggestion: Use that if G is a group with a subgroup H, then $[G : H] = 2$ implies that H is normal in G.)

9. Let $n \in \mathbb{N}$. Provide an embedding $\iota : S_n \to A_{n+2}$. Prove that ι is indeed an injective homomorphism.

   Is it true that for all $n \in \mathbb{N}$ that S_n can always be embedded in A_{n+1}? If so, prove it; if not, provide a specific counterexample.