Lecture 13: Likelihood Ratio Tests for a Random Sample from a Normal Population

MATH 667-01
Statistical Inference
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Introduction

- We introduce the likelihood ratio test (LRT) discussed in Section 8.2.1 of Casella and Berger (2002)[1].
- First, we illustrate it for testing a hypothesis concerning the mean for normal populations when the variance is assumed to be known.
- Next, we consider when the variance is unknown. In this case, the test statistic is related to a t distribution rather than a normal distribution, so we give some general background on the t distribution from Section 5.3.2 before deriving the LRT.

[1] Casella, G. and Berger, R. (2002). *Statistical Inference, Second Edition*. Duxbury Press, Belmont, CA.

- *Definition L13.1* (Def 8.2.1 on p.375): The *likelihood ratio test statistic* for testing $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_0^c$ is

$$\lambda(\boldsymbol{x}) = \frac{\sup\limits_{\Theta_0} L(\theta; \boldsymbol{x})}{\sup\limits_{\Theta} L(\theta; \boldsymbol{x})}.$$

A *likelihood ratio test* (LRT) is any test that has a rejection region of the form $\{\boldsymbol{x} : \lambda(\boldsymbol{x}) \leq c\}$, where c is any number satisfying $0 \leq c \leq 1$.

Test for Normal Population with Variance Known

- *Example L13.1*: Suppose $X_1, \ldots, X_n$ is a random sample from a Normal(μ, σ^2) population with μ unknown but σ^2 known and suppose that the experimenter is interested in testing

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu \neq \mu_0.$$

(a) Show that the likelihood ratio test has a critical region of the form $\left\{\boldsymbol{x} : \frac{|\bar{x}-\mu_0|}{\sigma/\sqrt{n}} \geq K\right\}$.
(b) Find K so that the size of the test is 0.01.
(c) What is the probability of a Type II error for the test in part (b) if $\mu_0 = 0$, $\sigma^2 = 4$, $\mu = 1$, and $n = 25$?

- *Answer to Example L13.1*: (a) The likelihood function is

$$\begin{aligned} L(\mu;\boldsymbol{x}) &= (2\pi\sigma^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\} \\ &= (2\pi\sigma^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}x_i^2-2\mu\sum_{i=1}^{n}x_i+n\mu^2\right)\right\} \\ &= (2\pi\sigma^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}x_i^2-2\mu n\bar{x}+n\mu^2\right)\right\}. \end{aligned}$$

Since $\Theta_0=\{\mu_0\}$,

$$\sup_{\mu\in\Theta_0}L(\mu;\boldsymbol{x})=L(\mu_0;\boldsymbol{x})=(2\pi\sigma^2)^{-n/2}e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}x_i^2-2n\mu_0\bar{x}+n\mu_0^2\right)}.$$

Since the MLE of μ is $\hat{\mu}=\bar{x}$,

$$\sup_{\mu\in\Theta}L(\mu;\boldsymbol{x})=L(\bar{x};\boldsymbol{x})=(2\pi\sigma^2)^{-n/2}e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}x_i^2-n\bar{x}^2\right)}.$$

Test for Normal Population with Variance Known

- *Answer to Example L13.1 continued*: The likelihood ratio is

$$
\begin{aligned}
\lambda(\boldsymbol{x}) &= \frac{(2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^n x_i^2 - 2n\mu_0\bar{x} + n\mu_0^2\right)\right\}}{(2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^n x_i^2 - n\bar{x}^2\right)\right\}} \\
&= \exp\left\{-\frac{1}{2\sigma^2}\left(n\bar{x}^2 - 2n\mu_0\bar{x} + n\mu_0^2\right)\right\} \\
&= \exp\left\{-\frac{n}{2\sigma^2}\left(\bar{x} - \mu_0\right)^2\right\} = \exp\left\{-\frac{1}{2}\left(\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right)^2\right\},
\end{aligned}
$$

and we reject H_0 if $\lambda(\boldsymbol{x}) \le c$.
Since $h(t) = \sqrt{-2\ln t}$ is a decreasing function of t, $\lambda(\boldsymbol{x}) \le c$ if and only if

$$
\frac{|\bar{x} - \mu_0|}{\sigma/\sqrt{n}} = h(\lambda(\boldsymbol{x})) \ge h(c) = \sqrt{-2\ln c} = K.
$$

- *Answer to Example L13.1 continued*: (b) If H_0 is true, then $\bar{X} \sim \text{Normal}(\mu_0, \frac{\sigma^2}{n})$ so that $Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim \text{Normal}(0, 1)$. The value of K such that

$$P\left(|Z| \geq K\right) = .01 \Leftrightarrow P\left(Z \geq K\right) = .005$$

is $K = 2.576$. (This can be obtained by looking up the cumulative probability $.995$ on the normal table or using the R command `qnorm(.995)`.)

- *Answer to Example L13.1 continued*: (c) For testing $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$ when the true value of the parameter is $\mu = 1$, the probability of a Type II error is

$$\begin{aligned}
P\left(\frac{|\bar{X}-0|}{2/\sqrt{25}} < 2.576\right) &= P\left(|\bar{X}| < 1.03\right) \\
&= P\left(-1.03 < \bar{X} < 1.03\right) \\
&= P\left(-5.08 < \frac{\bar{X}-1}{2/\sqrt{25}} < .08\right) \\
&\approx .5319 - .0000 = .5319.
\end{aligned}$$

- A more precise answer can be obtain using R:

```
> K=qnorm(.995)
> pnorm(K-2.5)-pnorm(-K-2.5)
[1] 0.5302224
```

- So, if $X_1, \dots, X_n$ is a random sample from a Normal population, then we know that $\bar{X} \sim \text{Normal}(\mu, \sigma^2/n)$ so that $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0, 1)$.
- Unfortunately, the expression $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ involves two (unknown) parameters μ and σ, and in situations where we want to make inferences about μ, we would prefer an expression in which μ is the only unknown parameter.
- What if we replace σ by its estimate S? Then we get

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}}{\dfrac{\sqrt{S^2/n}}{\sigma/\sqrt{n}}} = \frac{Z}{\sqrt{S^2/\sigma^2}}. = \frac{Z}{\sqrt{V/p}}$$

  where $Z \sim \text{Normal}(0, 1)$, $V \sim \chi^2_p$, Z and V are independent, and $p = n - 1$.

t Distribution

- *Definition L13.2* (Def 5.3.4 on p.223): If $Z \sim$ Normal$(0, 1)$, $V \sim \chi^2_p$, and Z and V are independent, then we say random variable $T = Z/\sqrt{V/p}$ has *Student's t distribution with p degrees of freedom*, and we write $T \sim t_p$.
- Let $X_1, \ldots, X_n$ be a random sample from a Normal(μ, σ^2) distribution. The quantity $(\bar{X} - \mu)/(S/\sqrt{n})$ has *Student's t distribution with $n - 1$ degrees of freedom*.
- *Theorem L13.1*: (a) If $T \sim t_p$, then the pdf of T is

$$f_T(t) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1 + t^2/p)^{(p+1)/2}}, -\infty < t < \infty.$$

(b) If $T \sim t_p$ with $p > 1$, then $\mathsf{E}[T] = 0$.
(c) If $T \sim t_p$ with $p > 2$, then $\mathsf{Var}[T] = \dfrac{p}{p-2}$.

- *Proof of Theorem L13.1(a)*: Since $Z \sim$ Normal$(0,1)$ and $V \sim \chi^2_p$ are independent, their joint pdf is

$$\begin{aligned} f_{Z,V}(z,v) &\overset{2.4}{=} f_Z(z) f_V(v) \\ &= \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \frac{1}{\Gamma(p/2) 2^{p/2}} v^{(p/2)-1} e^{-v/2} I_{(0,\infty)}(v). \end{aligned}$$

- Now, we make the bivariate transformation $T = Z/\sqrt{V/p}$ and $U = V$.

Proof of Theorem L13.1(a) continued:

- This transformation can be inverted as follows.

$$\left.\begin{array}{l} t = \frac{z}{\sqrt{v/p}} \\ u = v \end{array}\right\} \Leftrightarrow \left\{ \begin{array}{l} z = t\sqrt{u/p} \\ v = u \end{array}\right.$$

- Since $J = \left|\frac{\partial(z,v)}{\partial(t,u)}\right| = \left|\begin{array}{cc} \sqrt{u/p} & 1/(2\sqrt{up}) \\ 0 & 1 \end{array}\right| = \sqrt{\frac{u}{p}}$, the joint pdf of T and U is

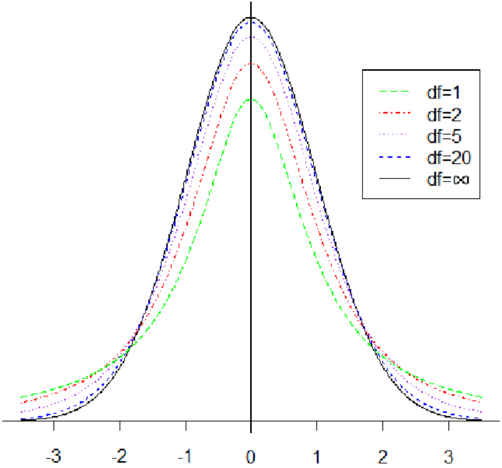
$$\begin{aligned} f_{T,U}(t,u) &\overset{4.18}{=} f_{Z,V}\left(t\sqrt{\frac{u}{p}}, u\right)|J| \\ &= \frac{e^{-t^2u/(2p)}u^{(p/2)-1}e^{-u/2}\sqrt{u/p}}{\sqrt{2\pi}\Gamma(p/2)2^{p/2}} I_{(0,\infty)}(u). \end{aligned}$$

Proof of Theorem L13.1(a) continued:

- Then, to get the marginal distribution of T, we integrate out U and obtain

$$\begin{aligned}
f_T(t) &= \int_0^\infty f_{T,U}(t,u)\,du \\
&= \frac{1}{\sqrt{2\pi p}\,\Gamma(p/2)2^{p/2}} \int_0^\infty u^{(p-1)/2} e^{-u(1+t^2/p)/2}\,du \\
&= \frac{1}{\sqrt{2\pi p}\,\Gamma(p/2)2^{p/2}} \int_0^\infty u^{(p+1)/2-1} \exp\left\{-\frac{u}{2\left(\frac{1}{1+t^2/p}\right)}\right\} du \\
&= \frac{\Gamma\left(\frac{p+1}{2}\right)\left(2\left(\frac{1}{1+t^2/p}\right)\right)^{(p+1)/2}}{\sqrt{2\pi p}\,\Gamma(p/2)2^{p/2}} \int_0^\infty \frac{u^{(p+1)/2-1}\exp\left\{-\frac{u}{2\left(1+t^2/p\right)^{-1}}\right\}}{\Gamma\left(\frac{p+1}{2}\right)\left(2\left(\frac{1}{1+t^2/p}\right)\right)^{(p+1)/2}}\,du \\
&= \frac{\Gamma\left(\frac{p+1}{2}\right)\left(2\left(\frac{1}{1+t^2/p}\right)\right)^{(p+1)/2}}{\sqrt{2\pi p}\,\Gamma(p/2)2^{p/2}} = \frac{\Gamma\left(\frac{p+1}{2}\right)\left(\frac{1}{1+t^2/p}\right)^{(p+1)/2}}{\sqrt{\pi p}\,\Gamma(p/2)}.
\end{aligned}$$

t distribution



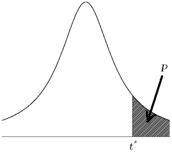


Table for t distribution

t distribution critical values

	Upper-tail probability p								
df	.25	.20	.15	.10	.05	.025	.02	.01	.005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.90	31.82	63.66
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841
					$\vdots$				
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581
∞	0.674	0.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576
	50%	60%	70%	80%	90%	95%	96%	98%	99%
	Confidence level C								

- *Example L13.2*: Suppose $X_1, \ldots, X_n$ is a random sample from a Normal(μ, σ^2) population with μ and σ^2 unknown and suppose that the experimenter is interested in testing

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu \neq \mu_0$$

where $\Theta = \left\{(\mu, \sigma^2) : \mu \in (-\infty, \infty) \text{ and } \sigma \in (0, \infty)\right\}$.
(a) Show that the likelihood ratio test statistic has a critical region of the form $\left\{\boldsymbol{x} : \frac{|\bar{x}-\mu_0|}{s/\sqrt{n}} \geq K\right\}$.
(b) Find K so that the size of the test is 0.01.

- *Answer to Example L13.2*: (a) The likelihood function is

$$L(\mu, \sigma^2; \boldsymbol{x}) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\}.$$

Since the maximizer of L over $\Theta_0 = \left\{(\mu_0, \sigma^2) : \sigma^2 > 0\right\}$ is $(\mu_0, \tilde{\sigma}^2)$ where $\tilde{\sigma}^2 = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(X_i - \mu_0)^2$, we have

$$\begin{aligned}\sup_{\theta\in\Theta_0} L(\theta; \boldsymbol{x}) &= (2\pi\tilde{\sigma}^2)^{-n/2} e^{-\frac{1}{2\tilde{\sigma}^2}\sum_{i=1}^{n}(x_i-\mu_0)^2} \\ &= (2\pi\tilde{\sigma}^2)^{-n/2} e^{-\frac{1}{2\tilde{\sigma}^2}n\tilde{\sigma}^2} = (2\pi e\tilde{\sigma}^2)^{-n/2}.\end{aligned}$$

Since the MLE of (μ, σ^2) is $(\bar{x}, \hat{\sigma}^2)$ with $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$,

$$\begin{aligned}\sup_{\theta\in\Theta} L(\theta; \boldsymbol{x}) &= (2\pi\hat{\sigma}^2)^{-n/2} e^{-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{n}(x_i-\bar{x})^2} \\ &= (2\pi\hat{\sigma}^2)^{-n/2} e^{-\frac{1}{2\hat{\sigma}^2}n\hat{\sigma}^2} = (2\pi e\hat{\sigma}^2)^{-n/2}.\end{aligned}$$

Test for Normal Population with Variance Unknown

- *Answer to Example L13.2 continued*: The likelihood ratio is

$$
\begin{aligned}
\lambda(\boldsymbol{x}) &= \frac{(2\pi e\tilde{\sigma}^2)^{-n/2}}{(2\pi e\hat{\sigma}^2)^{-n/2}} = \left(\tilde{\sigma}^2/\hat{\sigma}^2\right)^{-n/2} \\
&= \left(\frac{\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right)^{-n/2} \\
&= \left(1 + \frac{n(\bar{x} - \mu_0)^2}{(n-1)s^2}\right)^{-n/2} = \left(1 + \frac{1}{n-1}\left(\frac{\bar{x} - \mu_0}{s/\sqrt{n}}\right)^2\right)^{-n/2}
\end{aligned}
$$

and we reject H_0 if $\lambda(\boldsymbol{x}) \leq c$.
Since $h(t) = (n-1)(t^{-2/n} - 1)$ is a decreasing function of t, $\lambda(\boldsymbol{x}) \leq c$ if and only if

$$
\frac{|\bar{x} - \mu_0|}{s/\sqrt{n}} = h(\lambda(\boldsymbol{x})) \geq h(c) = (n-1)(c^{-2/n} - 1) = K.
$$

Test for Normal Population with Variance Unknown

- *Answer to Example L13.2 continued*:

  (b) If H_0 is true, then $T = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim t_{24}$.

  The value of K such that

  $$P\left(|T| \geq K\right) = .01 \Leftrightarrow P\left(T \geq K\right) = .005$$

  is $K = 2.797$. (This can be obtained by looking up the upper-tail probability $.005$ on the t-table or using the R command `qt(.995,df=24)`.)

- *Example L13.3*: Data were collected on pollution in a river around a chemical plant. A government regulation requires that no more than 10 parts per million (ppm) of chemical Z should be present in the river. A scientist collects 16 independent observations from the river and computes the sample mean and sample standard deviation to be $\bar{x} = 10.5$ ppm and $s = 1.6$ ppm, respectively.

  (a) What are appropriate null and alternative hypotheses?

  (b) Assuming a normal population and iid observations, perform a hypothesis test at level $.05$ to determine if there is statistically significant evidence based on the scientist's data that the chemical plant has violated the regulation.

- *Answer to Example L13.3*: (a) Let μ be the amount of chemical Z in the river in ppm. The null hypothesis is $H_0 : \mu \leq 10$ versus $H_a : \mu > 10$.
  (b) So this is a one-sided test and we need to find the critical region. The work is similar to *Example L13.2* except that the maximizer of L over $\Theta_0 = \left\{(\mu, \sigma^2) : \sigma^2 > 0\right\}$ is $(\tilde{\mu}, \tilde{\sigma}^2)$ where $\tilde{\mu} = \begin{cases} \bar{x} & \text{if } \bar{x} \leq 10 \\ 10 & \text{if } \bar{x} > 10 \end{cases}$ and $\tilde{\sigma}^2 = \dfrac{1}{n}\sum_{i=1}^{n}(x_i - \tilde{\mu})^2$ (see the method of successive maximization in *Example L2.2*).
  So, the likelihood ratio is

$$\lambda(\boldsymbol{x}) = \begin{cases} \left(1 + \frac{1}{n-1}\left(\frac{\bar{x}-\mu_0}{s/\sqrt{n}}\right)^2\right)^{-n/2} & \text{if } \bar{x} > 10 \\ 1 & \text{if } \bar{x} \leq 10 \end{cases}.$$

- *Answer to Example L13.3 continued*: For $c < 1$, $\lambda(\boldsymbol{x}) \leq c$ if and only if $\dfrac{|\bar{x} - 10|}{s/\sqrt{n}} \geq K$ and $\bar{x} > 10$. This implies that the critical region has the form $\left\{\boldsymbol{x} : \frac{\bar{x}-10}{s/\sqrt{n}} \geq K\right\}$.

  If H_0 is true, then $T = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim t_{15}$.

  The value of K such that

  $$P\left(T \geq K\right) = .05$$

  is $K = 1.753$.

  So, we fail to reject H_0 since $t = \frac{10.5-10}{1.6/\sqrt{16}} = 1.25$ is not in the critical region.

- An alternate way to make the decision for *Example L13.3* is to compute the p-value for the test. Since we reject H_0 when $T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$ is large, the p-value $P(T > 1.25)$ is the probability that we will observe a test statistic at least as large as the observed value. If this p-value is smaller than the size of the test .05, then we reject H_0. In this case, we can see from the t-table that the p-value is between $.10$ and $.15$ since $t = 1.25$ is between 1.341 and 1.074. So, we fail to reject H_0. The exact p-value can be computed in R as follows:

```
> 1-pt(1.25,df=15)
[1] 0.1152253
```