*MATH 667 EXAM 1 (October 22, 2015: 9:30am–10:45am)* NAME ________________________

**The exam is closed book; students are permitted to prepare one 4x6 notecard of formulas, notes, etc. that can be used during the exam. A calculator is permitted but not necessary for the exam.**

**Do 4 out of the 5 problems (20 points each, 80 points total). Clearly indicate the problem that you are omitting; if it is not clear, then the first 4 problems will be graded.**

**Problem 1.** (20 points) Suppose $X$ is a random variable with probability density function (pdf)

$$f(x|\theta) = \begin{cases} \dfrac{\theta - 1}{x^\theta} & \text{for } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 1$.

(a - 10 pts) Show that this family of pdfs is an exponential family.

(b - 10 pts) Find E[$\log X$].

**Problem 2.** (20 points) Suppose that the distribution of $X$, conditional on $U = u$, is Bernoulli($u$) with probability mass function

$$P(X = x|U = u) = \begin{cases} u^x(1-u)^{1-x} & \text{if } x \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases},$$

and that the marginal distribution of $U$ is uniform on the interval $(0, \frac{1}{2})$.

(a - 10 pts) Find the marginal distribution of $X$.

(b - 10 pts) Compute Cov($X, U$), the covariance of $X$ and $U$.

**Problem 3.** (20 points) Let $X_1$ and $X_2$ be independent uniform random variables on the interval $(0, 1)$.
(a - 10 pts) Find the joint probability density function (pdf) of $X_{(1)}$ and $X_{(2)}$ where $X_{(1)} = \min\{X_1, X_2\}$ and $X_{(2)} = \max\{X_1, X_2\}$.
(b - 10 pts) Let $V = \dfrac{X_{(1)}}{X_{(2)}}$ and $W = X_{(2)}$. Show that $V$ and $W$ are independent.

**Problem 4.** (20 points) Suppose that $Y_i$, $i = 1, \ldots, 5$ are independent Normal random variables with mean 0 and variance 2.
(a - 10 pts) Compute $P\Big(Y_1 + Y_2 > 1 \text{ and } Y_3^2 + Y_4^2 + Y_5^2 > 5\Big)$.

(b - 10 pts) Let $\bar{Y} = \dfrac{Y_1 + Y_2 + Y_3 + Y_4 + Y_5}{5}$. Compute $P\left(\sqrt{5}\,\bar{Y} < \sqrt{\sum_{i=1}^{5}(Y_i - \bar{Y})^2}\right)$.

Use the standard normal, $\chi^2$, and/or $t$ tables attached to this exam.

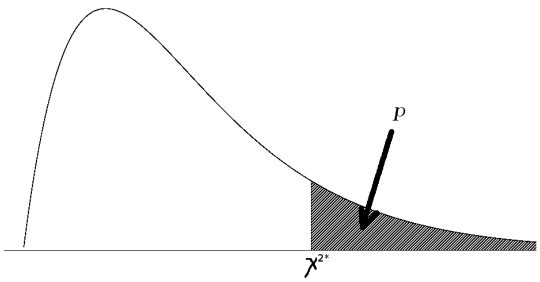
**Problem 5.** (20 points)
(a - 6 pts) Let $X$ be a Poisson random variable with mean $\lambda = 4$ having probability mass function

$$P(X = x) = \begin{cases} \dfrac{1}{x!}4^x e^{-4} & \text{if } x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathrm{E}[X^2 - X]$ (or equivalently $\mathrm{E}[X(X-1)]$).

(b - 6 pts) Suppose that $X_1, \ldots, X_n$ are independent Poisson random variables with mean 4, and let $\bar{X}_n$ denote the sample mean of this random sample of size $n$. For what real numbers $a$ and $b$ does $\dfrac{\sqrt{n}(\bar{X} - a)}{b}$ converge in distribution to a standard normal random variable?

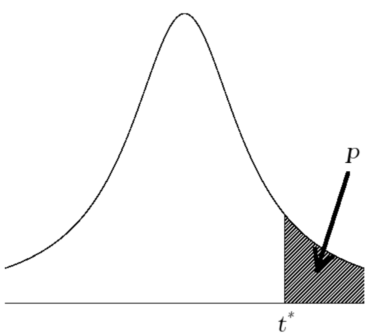
(b - 8 pts) Suppose that $X_1, \ldots, X_{10000}$ are independent Poisson random variables with mean 4 and let $\bar{X}$ denote its sample mean. Use the central limit theorem to approximate $P\Big(3.99 < \bar{X} < 4.01\Big)$.
Use the standard normal table attached to this exam.




The critical value $\chi^{2*}$ is the value such that the area under the density curve of a $\chi^2$ distribution with *df* degrees of freedom to the right of $\chi^{2*}$ is equal to $p$.

**$\chi^2$ distribution critical values**

| | Upper-tail probability $p$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| df | .64 | .55 | .48 | .42 | .35 | .29 | .24 | .17 | .12 | .08 | .04 | .02 |
| 1 | 0.2 | 0.4 | 0.5 | 0.7 | 0.9 | 1.1 | 1.4 | 1.9 | 2.4 | 3 | 4 | 5 |
| 2 | 0.9 | 1.2 | 1.5 | 1.7 | 2.1 | 2.5 | 2.9 | 3.5 | 4.2 | 5 | 6 | 8 |
| 3 | 1.7 | 2.1 | 2.5 | 2.8 | 3.3 | 3.7 | 4.2 | 5.0 | 5.8 | 7 | 8 | 10 |
| 4 | 2.5 | 3.0 | 3.5 | 3.9 | 4.4 | 5.0 | 5.5 | 6.4 | 7.3 | 8 | 10 | 12 |
| 5 | 3.4 | 4.0 | 4.5 | 5.0 | 5.6 | 6.2 | 6.7 | 7.8 | 8.7 | 10 | 12 | 13 |




The critical value $t^*$ is the value such that the area under the density curve of a $t$ distribution with *df* degrees of freedom to the right of $t^*$ is equal to $p$. It is also the value such that the area under the curve between $-t^*$ and $t^*$ is equal to $C$.

**$t$ distribution critical values**

| | Upper-tail probability $p$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| df | .20 | .18 | .15 | .12 | .10 | .08 | .07 | .06 | .05 | .02 |
| 1 | 1.4 | 1.6 | 2.0 | 2.5 | 3.1 | 3.9 | 4.5 | 5.2 | 6.3 | 15.9 |
| 2 | 1.1 | 1.2 | 1.4 | 1.7 | 1.9 | 2.2 | 2.4 | 2.6 | 2.9 | 4.8 |
| 3 | 1.0 | 1.1 | 1.2 | 1.5 | 1.6 | 1.9 | 2.0 | 2.2 | 2.4 | 3.5 |
| 4 | 0.9 | 1.0 | 1.2 | 1.4 | 1.5 | 1.7 | 1.8 | 2.0 | 2.1 | 3.0 |
| 5 | 0.9 | 1.0 | 1.2 | 1.3 | 1.5 | 1.6 | 1.8 | 1.9 | 2.0 | 2.8 |
| $\infty$ | 0.84 | 0.92 | 1.04 | 1.17 | 1.28 | 1.41 | 1.48 | 1.55 | 1.64 | 2.05 |
| | 60% | 64% | 70% | 76% | 80% | 84% | 86% | 88% | 90% | 96% |
| | Confidence level $C$ | | | | | | | | | |