MATH 668-01 Homework 3

Due: Thursday, February 15, 2018

Instructions: *Each of the following problems must be submitted to the instructor on or before the due date. Partial credit may be given for incorrect answers which make some positive progress. Late homework will not be accepted.*

1. (10 points) Suppose that $\boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}$ is a random vector with n-dimensional mean vector $\boldsymbol{\mu}$ and $n \times n$ covariance matrix $\boldsymbol{\Sigma}$.

Let $\boldsymbol{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \\ 0 \end{pmatrix}$ be an n-dimensional constant vector for which the last element is 0 (the result is true for any n-dimensional vector $\boldsymbol{a}$, so you don't need this assumption but you can use it if it is helpful). Show that

$$\begin{aligned} E\left(\boldsymbol{a}^\top \boldsymbol{y} y_n\right) &= \mathrm{tr}(\mathrm{cov}(y_n, \boldsymbol{y})\boldsymbol{a}) + \boldsymbol{a}^\top \boldsymbol{\mu} E(y_n) \\ &= \mathrm{cov}(y_n, \boldsymbol{y})\boldsymbol{a} + \boldsymbol{a}^\top \boldsymbol{\mu} E(y_n). \end{aligned}$$

2. (10 points) Suppose $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{\Sigma} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

(a - 2pts) Let $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. Show that $\mathbf{B}^2 = 5\boldsymbol{\Sigma}$.

(b - 3pts) Let $\mathbf{A} = \begin{pmatrix} .5 & -.5 \\ -.5 & .5 \end{pmatrix}$ and $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Find a scalar c such that $c\boldsymbol{y}^\top \mathbf{B}^{-1}\mathbf{A}\mathbf{B}^{-1}\boldsymbol{y} \sim \chi^2(1)$.

(c - 2pts) Show that $\boldsymbol{j}^\top \mathbf{B}^{-1}\boldsymbol{y}$ and $\boldsymbol{y}^\top \mathbf{B}^{-1}\mathbf{A}\mathbf{B}^{-1}\boldsymbol{y}$ are independent, where $\boldsymbol{j} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(d - 3pts) Find a scalar d such that $\dfrac{d\boldsymbol{j}^\top \mathbf{B}^{-1}\boldsymbol{y}}{\boldsymbol{y}^\top \mathbf{B}^{-1}\mathbf{A}\mathbf{B}^{-1}\boldsymbol{y}} \sim t(1)$.

3. (10 points) For fixed $\lambda \geq 0$ and observed values of $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, let

$$\tilde{Q}(b_0, b_1) = \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i)^2 + \lambda b_1^2.$$

(a - 7 pts) Find the values of b_0 and b_1 which minimize $\tilde{Q}$ (as a function of λ, $x_1, \ldots, x_n$, and $y_1, \ldots, y_n$).
(b - 3 pts) Denote the minimizers in part (a) as $\hat{\beta}_{0,\lambda}$ and $\hat{\beta}_{1,\lambda}$. What happens to the $\hat{\beta}_{1,\lambda}$ if λ is very large? What happens to $\hat{\beta}_{0,\lambda}$ if λ is very large?