MATH 562-01 MATHEMATICAL STATISTICS

Exam 2 (10/24/16, Monday) Name:________________________

1. (10 points) Let $X_1, X_2, \cdots, X_n$ be a random sample from the discrete distribution

$$P[X = x] = \begin{cases} \dfrac{\theta^{2x} e^{-\theta^2}}{x!} & x = 0,1,2,\cdots \\ 0 & otherwise \end{cases}$$

where $\theta > 0$. Are the MME $\hat{\theta}$ and MLE $\tilde{\theta}$ unbiased? Carefully justify your answers.

2. (10 points each) Let X and Y be independent random variables with $E(X) = 1$, $E(Y) = 2$ and $\text{var}(X) = \text{var}(Y) = \sigma^2$. Find the constant k such that $T = k\left(X^2 - Y^2\right) + Y^2$ is an unbiased estimator for σ^2.

3. (10 points) A sequence of independent Bernoulli trials with probability of success p is performed. Let X denote the number of trials until the first success. Four independent realizations of X are obtained: $x_1 = 1, x_2 = 1, x_3 = 3$, and $x_4 = 1$. If, a priori, p has probability density function $h(p) = 2p$, for $p \in (0,1)$, and squared errors loss $L(t, p) = (t - p)^2$ is used, find the Bayesian estimate for p.

4. (20 points) A random sample of size n is drawn from a normal population $N(\mu_1, 2)$, and another random sample of the same size is drawn independently from another normal population $N(\mu_2, 5)$.

   (1). Find the MLE $\tilde{\theta}$ for $\theta = \mu_1 - \mu_2$. (NOTE: Specify the distribution you use, and justify every step carefully.)

   (2). Show that $\tilde{\theta}$ is a UMVUE, so it is an efficient estimator.

5. (20 points) Let $X_1, X_2, \cdots, X_n$ be a random sample from $UNIF(0, \theta)$. We know that the MLE $\tilde{\theta} = \max\{X_1, X_2, \cdots, X_n\}$ is biased. Also, the MLE $\tilde{\theta}$ has probability density function $f_{\tilde{\theta}}(x) = \dfrac{nx^{n-1}}{\theta^n}$, for $x \in (0, \theta)$.

   (1). Find the constant k such that $T = k\tilde{\theta}$ is an unbiased estimator. Is $\tilde{\theta}$ itself asymptotically unbiased?

   (2). Find $\text{var}(T)$, the variance of T, and $\dfrac{1}{E\left[\left(\dfrac{\partial}{\partial \theta} \ln L(\theta)\right)^2\right]}$, the lower bound given in the Cramer-Rao Theorem. Compare them and explain why the *CRLB* does not hold in this case.

6. (10 points) Let $X_1, X_2, \cdots, X_n$ be a random sample from $BIN(m, p)$. Show that the MLE $\tilde{p} = \frac{\overline{X}}{m}$ for p is efficient, i.e., a UMVUE. (Hint: You may use known results for $BIN(m, p)$.)

7. (20 points) Let $X_1, X_2, \cdots, X_n$ be a random sample from and exponential population $EXP(\theta)$, where $\theta \in (0, \infty)$, and $T = u(X_1) = \begin{cases} 0, & 0 < X_1 < 1 \\ 1, & X_1 \geq 1 \end{cases}$.

   (1). Show that T is an unbiased estimator for $P[X > 1] = e^{-\frac{1}{\theta}}$.

   (2). By the Rao-Blackwell Theorem, $S = E\left[T \middle| \sum_{i=1}^{n} X_i \right]$ is an efficient estimator of $P[X > 1] = e^{-\frac{1}{\theta}}$. Find S. (Hint: Use $f_{X_1 \left| \sum_{i=1}^{n} X_i = s \right.}(x_1) = \frac{n-1}{s^{n-1}}(s - x_1)^{n-2},\ x_1 \in (1, s)$.)

   NOTE: We just apply the Rao-Blackwell Theorem here, without proof. Rao-Blackwell provides a way to find efficient estimator by iteration, a quite effective method.