M621, HW 8, due Tuesday Oct 18:SELECTED SOLNS 10.21

1. Let $n \in \mathbb{N}$ with $n > 1$.

   (a) Suppose $(a_1 \ldots a_k)$ is a k-cycle in S_n. Provide a *concise, clear* explanation (as if to a M521 student) why it is true that $(a_1 \ldots a_n) = (a_1 a_n) \ldots (a_1 a_2)$. (For example, $(123) = (13)(12)$.) With $\beta = (a_1 a_2) \ldots (a_1 a_n)$, you should explain why $\beta(a_1) = a_2, \beta(a_2) = a_3$, and so on, and that β fixes everything in $\{1, \ldots, n\} - \{a_1, \ldots, a_n\}$.

   (b) You just showed that every k-cycle is a product of 2-cycles. 2-cycles are often referred to as *transpositions*. Give a one or two sentence explanation of the following: S_n is generated by its transpositions. That is, $S_n =< \{(ij) : n \geq j > i \geq 1\} >$.

   (c) Show that $S_n =< \{(12), (23), \ldots, (n-1, n), (n1)\} >$. Suggestions: For $j \in \{1, \ldots, n\}$, $k \in \mathbb{N}$, you'll show that every transposition $(j, j+k)$ is generated by the set of n transpositions given above. When $k = 1$, there's nothing to prove. Proceed by induction on k. As usual, "j+k" is interpreted mod n. Of course, k need not be greater than $n-1$. (Continue proof on other side of sheet, if necessary.)

   In this problem, I would have liked to have seen a proof by induction.

   Proof. It suffices to show that every transposition can be represented as a product of transpositions from the set $X = \{(12), (23), \ldots, (n-1, n), (n1)\}$. A arbitrary transposition of S_n can be represented in the form $(j, j+k)$, where $n \geq j+k$. The proof is by induction on k. The base step of the induction, when $k = 1$, is trivial since transpositions of the form $(j, j+1)$, $n-1 \geq j$, are already contained in X. Let m be a positive integer, and suppose that whenever $m > r$, $r \in \{0\} \cup \mathbb{N}$, and $n \geq m + r$, all transpositions of the form $(j, j+r)$ are contained in $< X >$. (This is the induction hypothesis.) Now consider $(j, j+m)$, where $n \geq j+m$. Observe that $(j, m) = (j+m-1, j+m)(j, j+m-1)(j+m-1, j+m)$. By the induction hypothesis, $(j, m-1) \in < X >$. Also, observe that $(j+m-1, j+m)$ is in X. Hence, (j, m) is a product of elements of X, which means that $(j, m) \in < X >$. This completes the induction proof.

2. This is a longer proof, one that you will "sketch in" below. Prove the following proposition. **Proposition.** Suppose G is a group having normal subgroups H and K satisfying the following:

   (a) $H \cap K = \{e\}$.

   (b) $G = HK$.

   Then $G \cong H \times K$.

   Proof. The proof is by a series of claims (whose proofs you'll supply).

Claim 1 *For all $h \in H$ and all $k \in K$, $hk = kh$.*

Proof. Suppose $h \in H, k \in K$. We have $(hkh^{-1})k^{-1} = hkh^{-1}k^{-1} = h(kh^{-1}k^{-1})$. By normality of K, $hkh^{-1} \in K$ and by normality of H, $kh^{-1}k^{-1} \in H$. Thus, $hkh^{-1}k^{-1} \in H \cap K = \{e\}$. But $hkh^{-1}k^{-1} = e$ implies that $hk = kh$.

Claim 2 *Let h_1, h_2 be in H, and let k_1, k_2 be in K. Then $h_1k_1 = h_2k_2$ if and only if $h_1 = h_2$ and $k_1 = k_2$.*

Proof. If $h_1k_1 = h_2k_2$, $h_2^{-1}h_1 = k_2(k_1)^{-1}$. Of course $h^2h^{-1} \in H$ and $k_2(k_1)^{-1} \in K$. Since $H \cap K = \{e\}$, it follows that $h_2h^{-1} = e$ and $k_2(k_1)^{-1} = e$, from which it follows that $h_1 = h_2$ and $k_1 = k_2$. The other direction is obvious, completing the proof of the claim.

Since $G = HK$, it follows from Claim 2, that each element $g \in G$ has a *unique* representation as a product $g = hk$, where $h \in H$ and $k \in K$.

Claim 3 *The function $\Gamma : G \to H \times K$ given by $\Gamma(g) = (h, k)$, where $g = hk$ is the unique representation of g by an element of H times an element of K, is a homomorphism.*

Proof. By the comment above, every $g \in G$ has a unique representation as a product $g = hk$ of an element of H and an element of G. Thus the map Γ is a well-defined map.

Let g_1, g_2 be elements of G, and let $g_1 = h_1k_1$ and $g_2 = h_2k_2$ be the unique representations of g_1 and g_2. We have that $\Gamma(g_1g_2) = \Gamma(h_1k_1h_2k_2) = \Gamma(h_1h_2k_1k_2)$, the right-most equality by the first claim. Of course $h_1h_2 \in H$ and $k_1k_2 \in K$. Thus the unique representation of g_1g_2 is given by $(h_1h_2)(k_1k_2)$ is the unique representation of g_1g_2. Now by definition of Γ, we have $\Gamma(g_1g_2) = (h_1h_2, k_1k_2)$. Observe that $\Gamma(g_1g_2) = (h_1h_2, k_1k_2) = (h_1, k_1)(h_2, k_2) = \Gamma(g_1)\Gamma(g_2)$; thus, Γ is a homomorphism.

With the proof of the next claim, you've proved the proposition.

Claim 4 *Γ is a bijection.*

Proof. Let $(h, k) \in H \times K$. Let $g = hk$. Since elements of G are uniquely represented as described above, $\Gamma(g) = (h, k)$, completing the proof that Γ is onto.

Now suppose g_1, g_2 are in G, with $g_1 = h_1k_1$ and $g_2 = h_2k_2$ are the representations of g_1 and g_2. If $\Gamma(g_1) = \Gamma(g_2)$, then $(h_1, k_1) = \Gamma(g_1) =$

$\Gamma(g_2) = (h_2, k_2)$, then $h_1 = h_2$ and $k_1 = k_2$, from which it follows that $g_1 = g_2$. Thus Γ is one-to-one. $\square$

(EC: +.5) The converse to our proposition is true. What would the converse say?

Suppose G, H, K are groups, and $G = H \times K$. Then there exist normal subgroups H', K' of G such that $H' \cong H$ and $K' \cong K$, and $H' \cap K' = \{e\}$.

Slightly more general: Suppose G, H, K are groups, and $G \cong H \times K$. Then there exist normal subgroups H', K' of G such that $H' \cong H$ and $K' \cong K$, and $H' \cap K' = \{e\}$.