

---

**PROBABILITY  
AND  
MATHEMATICAL STATISTICS**

---

**Prasanna Sahoo  
Department of Mathematics  
University of Louisville  
Louisville, KY 40292 USA**

---

Copyright ©2007. All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the author.

---

---

# PREFACE

---

This book is both a tutorial and a textbook. This book presents an introduction to probability and mathematical statistics and it is intended for students already having some elementary mathematical background. It is intended for a one-year junior or senior level undergraduate or beginning graduate level course in probability theory and mathematical statistics. The book contains more material than normally would be taught in a one-year course. This should give the teacher flexibility with respect to the selection of the content and level at which the book is to be used. This book is based on over 15 years of lectures in senior level calculus based courses in probability theory and mathematical statistics at the University of Louisville.

Probability theory and mathematical statistics are difficult subjects both for students to comprehend and teachers to explain. Despite the publication of a great many textbooks in this field, each one intended to provide an improvement over the previous textbooks, this subject is still difficult to comprehend. A good set of examples makes these subjects easy to understand. For this reason alone I have included more than 350 completely worked out examples and over 165 illustrations. I give a rigorous treatment of the fundamentals of probability and statistics using mostly calculus. I have given great attention to the clarity of the presentation of the materials. In the text, theoretical results are presented as theorems, propositions or lemmas, of which as a rule rigorous proofs are given. For the few exceptions to this rule references are given to indicate where details can be found. This book contains over 450 problems of varying degrees of difficulty to help students master their problem solving skill.

In many existing textbooks, the examples following the explanation of a topic are too few in number or too simple to obtain a through grasp of the principles involved. Often, in many books, examples are presented in abbreviated form that leaves out much material between steps, and requires that students derive the omitted materials themselves. As a result, students find examples difficult to understand. Moreover, in some textbooks, examples

are often worded in a confusing manner. They do not state the problem and then present the solution. Instead, they pass through a general discussion, never revealing what is to be solved for. In this book, I give many examples to illustrate each topic. Often we provide illustrations to promote a better understanding of the topic. All examples in this book are formulated as questions and clear and concise answers are provided in step-by-step detail.

There are several good books on these subjects and perhaps there is no need to bring a new one to the market. So for several years, this was circulated as a series of typeset lecture notes among my students who were preparing for the examination 110 of the Actuarial Society of America. Many of my students encouraged me to formally write it as a book. Actuarial students will benefit greatly from this book. The book is written in simple English; this might be an advantage to students whose native language is not English.

I cannot claim that all the materials I have written in this book are mine. I have learned the subject from many excellent books, such as *Introduction to Mathematical Statistics* by Hogg and Craig, and *An Introduction to Probability Theory and Its Applications* by Feller. In fact, these books have had a profound impact on me, and my explanations are influenced greatly by these textbooks. If there are some similarities, then it is due to the fact that I could not make improvements on the original explanations. I am very thankful to the authors of these great textbooks. I am also thankful to the Actuarial Society of America for letting me use their test problems. I thank all my students in my probability theory and mathematical statistics courses from 1988 to 2003 who helped me in many ways to make this book possible in the present form. Lastly, if it weren't for the infinite patience of my wife, Sadhna, this book would never get out of the hard drive of my computer.

The author on a Macintosh computer using  $\text{\TeX}$ , the typesetting system designed by Donald Knuth, typeset the entire book. The figures were generated by the author using MATHEMATICA, a system for doing mathematics designed by Wolfram Research, and MAPLE, a system for doing mathematics designed by Maplesoft. The author is very thankful to the University of Louisville for providing many internal financial grants while this book was under preparation.

Prasanna Sahoo, *Louisville*

---

# TABLE OF CONTENTS

---

<b>1. Probability of Events</b>	<b>1</b>
1.1. Introduction	
1.2. Counting Techniques	
1.3. Probability Measure	
1.4. Some Properties of the Probability Measure	
1.5. Review Exercises	
<b>2. Conditional Probability and Bayes' Theorem</b>	<b>27</b>
2.1. Conditional Probability	
2.2. Bayes' Theorem	
2.3. Review Exercises	
<b>3. Random Variables and Distribution Functions</b>	<b>45</b>
3.1. Introduction	
3.2. Distribution Functions of Discrete Variables	
3.3. Distribution Functions of Continuous Variables	
3.4. Percentile for Continuous Random Variables	
3.5. Review Exercises	
<b>4. Moments of Random Variables and Chebychev Inequality</b>	<b>73</b>
4.1. Moments of Random Variables	
4.2. Expected Value of Random Variables	
4.3. Variance of Random Variables	
4.4. Chebychev Inequality	
4.5. Moment Generating Functions	
4.6. Review Exercises	

<b>5. Some Special Discrete Distributions . . . . .</b>	<b>107</b>
5.1. Bernoulli Distribution	
5.2. Binomial Distribution	
5.3. Geometric Distribution	
5.4. Negative Binomial Distribution	
5.5. Hypergeometric Distribution	
5.6. Poisson Distribution	
5.7. Riemann Zeta Distribution	
5.8. Review Exercises	
<b>6. Some Special Continuous Distributions . . . . .</b>	<b>141</b>
6.1. Uniform Distribution	
6.2. Gamma Distribution	
6.3. Beta Distribution	
6.4. Normal Distribution	
6.5. Lognormal Distribution	
6.6. Inverse Gaussian Distribution	
6.7. Logistic Distribution	
6.8. Review Exercises	
<b>7. Two Random Variables . . . . .</b>	<b>185</b>
7.1. Bivariate Discrete Random Variables	
7.2. Bivariate Continuous Random Variables	
7.3. Conditional Distributions	
7.4. Independence of Random Variables	
7.5. Review Exercises	
<b>8. Product Moments of Bivariate Random Variables . . . . .</b>	<b>213</b>
8.1. Covariance of Bivariate Random Variables	
8.2. Independence of Random Variables	
8.3. Variance of the Linear Combination of Random Variables	
8.4. Correlation and Independence	
8.5. Moment Generating Functions	
8.6. Review Exercises	

<b>9. Conditional Expectations of Bivariate Random Variables</b>	<b>237</b>
9.1. Conditional Expected Values	
9.2. Conditional Variance	
9.3. Regression Curve and Scedastic Curves	
9.4. Review Exercises	
<b>10. Functions of Random Variables and Their Distribution</b>	<b>257</b>
10.1. Distribution Function Method	
10.2. Transformation Method for Univariate Case	
10.3. Transformation Method for Bivariate Case	
10.4. Convolution Method for Sums of Random Variables	
10.5. Moment Method for Sums of Random Variables	
10.6. Review Exercises	
<b>11. Some Special Discrete Bivariate Distributions</b>	<b>289</b>
11.1. Bivariate Bernoulli Distribution	
11.2. Bivariate Binomial Distribution	
11.3. Bivariate Geometric Distribution	
11.4. Bivariate Negative Binomial Distribution	
11.5. Bivariate Hypergeometric Distribution	
11.6. Bivariate Poisson Distribution	
11.7. Review Exercises	
<b>12. Some Special Continuous Bivariate Distributions</b>	<b>317</b>
12.1. Bivariate Uniform Distribution	
12.2. Bivariate Cauchy Distribution	
12.3. Bivariate Gamma Distribution	
12.4. Bivariate Beta Distribution	
12.5. Bivariate Normal Distribution	
12.6. Bivariate Logistic Distribution	
12.7. Review Exercises	

<b>13. Sequences of Random Variables and Order Statistics . .</b>	<b>351</b>
13.1. Distribution of Sample Mean and Variance	
13.2. Laws of Large Numbers	
13.3. The Central Limit Theorem	
13.4. Order Statistics	
13.5. Sample Percentiles	
13.6. Review Exercises	
 <b>14. Sampling Distributions Associated with</b>	
<b>    the Normal Population . . . . .</b>	<b>391</b>
14.1. Chi-square distribution	
14.2. Student's $t$ -distribution	
14.3. Snedecor's $F$ -distribution	
14.4. Review Exercises	
 <b>15. Some Techniques for Finding Point</b>	
<b>    Estimators of Parameters . . . . .</b>	<b>409</b>
15.1. Moment Method	
15.2. Maximum Likelihood Method	
15.3. Bayesian Method	
15.3. Review Exercises	
 <b>16. Criteria for Evaluating the Goodness</b>	
<b>    of Estimators . . . . .</b>	<b>449</b>
16.1. The Unbiased Estimator	
16.2. The Relatively Efficient Estimator	
16.3. The Minimum Variance Unbiased Estimator	
16.4. Sufficient Estimator	
16.5. Consistent Estimator	
16.6. Review Exercises	



<b>17. Some Techniques for Finding Interval</b>	
<b>Estimators of Parameters</b> . . . . .	489
17.1. Interval Estimators and Confidence Intervals for Parameters	
17.2. Pivotal Quantity Method	
17.3. Confidence Interval for Population Mean	
17.4. Confidence Interval for Population Variance	
17.5. Confidence Interval for Parameter of some Distributions not belonging to the Location-Scale Family	
17.6. Approximate Confidence Interval for Parameter with MLE	
17.7. The Statistical or General Method	
17.8. Criteria for Evaluating Confidence Intervals	
17.9. Review Exercises	
<b>18. Test of Statistical Hypotheses</b> . . . . .	533
18.1. Introduction	
18.2. A Method of Finding Tests	
18.3. Methods of Evaluating Tests	
18.4. Some Examples of Likelihood Ratio Tests	
18.5. Review Exercises	
<b>19. Simple Linear Regression and Correlation Analysis</b> . . .	577
19.1. Least Squared Method	
19.2. Normal Regression Analysis	
19.3. The Correlation Analysis	
19.4. Review Exercises	
<b>20. Analysis of Variance</b> . . . . .	613
20.1. One-way Analysis of Variance with Equal Sample Sizes	
20.2. One-way Analysis of Variance with Unequal Sample Sizes	
20.3. Pair wise Comparisons	
20.4. Tests for the Homogeneity of Variances	
20.5. Review Exercises	

<b>21. Goodness of Fits Tests</b> . . . . .	645
21.1. Chi-Squared test	
21.2. Kolmogorov-Smirnov test	
21.3. Review Exercises	
<b>References</b> . . . . .	663
<b>Answers to Selected Review Exercises</b> . . . . .	669

---

## REFERENCES

---

- [1] Aitken, A. C. (1944). *Statistical Mathematics*. 3rd edn. Edinburgh and London: Oliver and Boyd,
- [2] Arbous, A. G. and Kerrich, J. E. (1951). Accident statistics and the concept of accident-proneness. *Biometrics*, 7, 340-432.
- [3] Arnold, S. (1984). Pivotal quantities and invariant confidence regions. *Statistics and Decisions* 2, 257-280.
- [4] Bain, L. J. and Engelhardt. M. (1992). *Introduction to Probability and Mathematical Statistics*. Belmont: Duxbury Press.
- [5] Bartlett, M. S. (1937). Properties of sufficiency and statistical tests. *Proceedings of the Royal Society*, London, Ser. A, 160, 268-282.
- [6] Bartlett, M. S. (1937). Some examples of statistical methods of research in agriculture and applied biology. *J. R. Stat. Soc., Suppl.*, 4, 137-183.
- [7] Brown, L. D. (1988). *Lecture Notes*, Department of Mathematics, Cornell University. Ithaca, New York.
- [8] Brown, M. B. and Forsythe, A. B. (1974). Robust tests for equality of variances. *Journal of American Statistical Association*, 69, 364-367.
- [9] Campbell, J. T. (1934). The Poisson correlation function. *Proc. Edin. Math. Soc.*, Series 2, 4, 18-26.
- [10] Casella, G. and Berger, R. L. (1990). *Statistical Inference*. Belmont: Wadsworth.
- [11] Castillo, E. (1988). *Extreme Value Theory in Engineering*. San Diego: Academic Press.
- [12] Cherian, K. C. (1941). A bivariate correlated gamma-type distribution function. *J. Indian Math. Soc.*, 5, 133-144.

- [13] Dahiya, R., and Guttman, I. (1982). Shortest confidence and prediction intervals for the log-normal. *The canadian Journal of Statistics* 10, 777-891.
- [14] David, F.N. and Fix, E. (1961). Rank correlation and regression in a non-normal surface. *Proc. 4th Berkeley Symp. Math. Statist. & Prob.*, 1, 177-197.
- [15] Desu, M. (1971). Optimal confidence intervals of fixed width. *The American Statistician* 25, 27-29.
- [16] Dynkin, E. B. (1951). Necessary and sufficient statistics for a family of probability distributions. English translation in *Selected Translations in Mathematical Statistics and Probability*, 1 (1961), 23-41.
- [17] Eisenhart, C., Hastay, M. W. and Wallis, W. A. (1947). *Selected Techniques of Statistical Analysis*, New York: McGraw-Hill.
- [18] Feller, W. (1968). *An Introduction to Probability Theory and Its Applications, Volume I*. New York: Wiley.
- [19] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications, Volume II*. New York: Wiley.
- [20] Ferentinos, K. K. (1988). On shortest confidence intervals and their relation uniformly minimum variance unbiased estimators. *Statistical Papers* 29, 59-75.
- [21] Freund, J. E. and Walpole, R. E. (1987). *Mathematical Statistics*. Englewood Cliffs: Prentice-Hall.
- [22] Galton, F. (1879). The geometric mean in vital and social statistics. *Proc. Roy. Soc.*, 29, 365-367.
- [23] Galton, F. (1886). Family likeness in stature. With an appendix by J.D.H. Dickson. *Proc. Roy. Soc.*, 40, 42-73.
- [24] Ghahramani, S. (2000). *Fundamentals of Probability*. Upper Saddle River, New Jersey: Prentice Hall.
- [25] Graybill, F. A. (1961). *An Introduction to Linear Statistical Models*, Vol. 1. New York: McGraw-Hill.

- [26] Guenther, W. (1969). Shortest confidence intervals. *The American Statistician* 23, 51-53.
- [27] Guldberg, A. (1934). On discontinuous frequency functions of two variables. *Skand. Aktuar.*, 17, 89-117.
- [28] Gumbel, E. J. (1960). Bivariate exponential distributions. *J. Amer. Statist. Ass.*, 55, 698-707.
- [29] Hamedani, G. G. (1992). Bivariate and multivariate normal characterizations: a brief survey. *Comm. Statist. Theory Methods*, 21, 2665-2688.
- [30] Hamming, R. W. (1991). *The Art of Probability for Scientists and Engineers* New York: Addison-Wesley.
- [31] Hogg, R. V. and Craig, A. T. (1978). *Introduction to Mathematical Statistics*. New York: Macmillan.
- [32] Hogg, R. V. and Tanis, E. A. (1993). *Probability and Statistical Inference*. New York: Macmillan.
- [33] Holgate, P. (1964). Estimation for the bivariate Poisson distribution. *Biometrika*, 51, 241-245.
- [34] Kapteyn, J. C. (1903). *Skew Frequency Curves in Biology and Statistics*. Astronomical Laboratory, Noordhoff, Groningen.
- [35] Kibble, W. F. (1941). A two-variate gamma type distribution. *Sankhya*, 5, 137-150.
- [36] Kolmogorov, A. N. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Erg. Math., Vol 2, Berlin: Springer-Verlag.
- [37] Kolmogorov, A. N. (1956). *Foundations of the Theory of Probability*. New York: Chelsea Publishing Company.
- [38] Kotlarski, I. I. (1960). On random variables whose quotient follows the Cauchy law. *Colloquium Mathematicum*. 7, 277-284.
- [39] Isserlis, L. (1914). The application of solid hypergeometrical series to frequency distributions in space. *Phil. Mag.*, 28, 379-403.
- [40] Laha, G. (1959). On a class of distribution functions where the quotient follows the Cauchy law. *Trans. Amer. Math. Soc.* 93, 205-215.

- [41] Levene, H. (1960). In Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling. I. Olkin *et. al.* eds., Stanford University Press, 278-292.
- [42] Lundberg, O. (1934). *On Random Processes and their Applications to Sickness and Accident Statistics*. Uppsala: Almqvist and Wiksell.
- [43] Mardia, K. V. (1970). *Families of Bivariate Distributions*. London: Charles Griffin & Co Ltd.
- [44] Marshall, A. W. and Olkin, I. (1967). A multivariate exponential distribution. *J. Amer. Statist. Ass.*, 62, 30-44.
- [45] McAlister, D. (1879). The law of the geometric mean. *Proc. Roy. Soc.*, 29, 367-375.
- [46] McKay, A. T. (1934). Sampling from batches. *J. Roy. Statist. Soc.*, Supplement, 1, 207-216.
- [47] Meyer, P. L. (1970). *Introductory Probability and Statistical Applications*. Reading: Addison-Wesley.
- [48] Mood, A., Graybill, G. and Boes, D. (1974). *Introduction to the Theory of Statistics* (3rd Ed.). New York: McGraw-Hill.
- [49] Moran, P. A. P. (1967). Testing for correlation between non-negative variates. *Biometrika*, 54, 385-394.
- [50] Morgenstern, D. (1956). Einfache Beispiele zweidimensionaler Verteilungen. *Mitt. Math. Statist.*, 8, 234-235.
- [51] Papoulis, A. (1990). *Probability and Statistics*. Englewood Cliffs: Prentice-Hall.
- [52] Pearson, K. (1924). On the moments of the hypergeometrical series. *Biometrika*, 16, 157-160.
- [53] Pestman, W. R. (1998). *Mathematical Statistics: An Introduction* New York: Walter de Gruyter.
- [54] Pitman, J. (1993). *Probability*. New York: Springer-Verlag.
- [55] Plackett, R. L. (1965). A class of bivariate distributions. *J. Amer. Statist. Ass.*, 60, 516-522.

- [56] Rice, S. O. (1944). Mathematical analysis of random noise. *Bell. Syst. Tech. J.*, 23, 282-332.
- [57] Rice, S. O. (1945). Mathematical analysis of random noise. *Bell. Syst. Tech. J.*, 24, 46-156.
- [58] Rinaman, W. C. (1993). *Foundations of Probability and Statistics*. New York: Saunders College Publishing.
- [59] Rosenthal, J. S. (2000). *A First Look at Rigorous Probability Theory*. Singapore: World Scientific.
- [60] Ross, S. (1988). *A First Course in Probability*. New York: Macmillan.
- [61] Ross, S. M. (2000). *Introduction to Probability and Statistics for Engineers and Scientists*. San Diego: Harcourt Academic Press.
- [62] Roussas, G. (2003). *An Introduction to Probability and Statistical Inference*. San Diego: Academic Press.
- [63] Sahai, H. and Ageel, M. I. (2000). *The Analysis of Variance*. Boston: Birkhauser.
- [64] Seshadri, V. and Patil, G. P. (1964). A characterization of a bivariate distribution by the marginal and the conditional distributions of the same component. *Ann. Inst. Statist. Math.*, 15, 215-221.
- [65] H. Scheffé (1959). *The Analysis of Variance*. New York: Wiley.
- [66] Snedecor, G. W. and Cochran, W. G. (1983). *Statistical Methods*. 6th eds. Iowa State University Press, Ames, Iowa.
- [67] Sveshnikov, A. A. (1978). *Problems in Probability Theory, Mathematical Statistics and Theory of Random Functions*. New York: Dover.
- [68] Tardiff, R. M. (1980). L'Hospital rule and the central limit theorem. *American Statistician*, 35, 43-44.
- [69] Taylor, L. D. (1974). *Probability and Mathematical Statistics*. New York: Harper & Row.
- [70] Tweedie, M. C. K. (1945). Inverse statistical variates. *Nature*, 155, 453.
- [71] Waissi, G. R. (1993). A unifying probability density function. *Appl. Math. Lett.* 6, 25-26.

- [72] Waissi, G. R. (1994). An improved unifying density function. Appl. Math. Lett. 7, 71-73.
- [73] Waissi, G. R. (1998). Transformation of the unifying density to the normal distribution. Appl. Math. Lett. 11, 45-28.
- [74] Wicksell, S. D. (1933). On correlation functions of Type III. *Biometrika*, 25, 121-133.
- [75] Zehna, P. W. (1966). Invariance of maximum likelihood estimators. *Annals of Mathematical Statistics*, 37, 744.



**CHAPTER 13**

3. 0.115.

4. 1.0.

5.  $\frac{7}{16}$ .

6. 0.352.

7.  $\frac{6}{5}$ .

8. 100.64.

9.  $\frac{1+\ln(2)}{2}$ .

10.  $[1 - F(x_6)]^5$ .

11.  $\theta + \frac{1}{5}$ .

12.  $2e^{-w} [1 - e^{-w}]$ .

13.  $6 \frac{w^2}{\theta^3} \left(1 - \frac{w^3}{\theta^3}\right)$ .

14.  $N(0, 1)$ .

15. 25.

16.  $X$  has a degenerate distribution with MGF  $M(t) = e^{\frac{1}{2}t}$ .

17.  $POI(1995\lambda)$ .

18.  $\left(\frac{1}{2}\right)^n (n+1)$ .

19.  $\frac{8^8}{11^9} 35$ .

20.  $f(x) = \frac{60}{\theta} \left(1 - e^{-\frac{x}{\theta}}\right)^3 e^{-\frac{3x}{\theta}}$  for  $0 < x < \infty$ .

21.  $X_{(n+1)} \sim Beta(n+1, n+1)$ .

**CHAPTER 14**

1.  $N(0, 32)$ .
2.  $\chi^2(3)$ ; the MGF of  $X_1^2 - X_2^2$  is  $M(t) = \frac{1}{\sqrt{1-4t^2}}$ .
3.  $t(3)$ .
4.  $f(x_1, x_2, x_3) = \frac{1}{\theta^3} e^{-\frac{(x_1+x_2+x_3)}{\theta}}$ .
5.  $\sigma^2$
6.  $t(2)$ .
7.  $M(t) = \frac{1}{\sqrt{(1-2t)(1-4t)(1-6t)(1-8t)}}$ .
8. 0.625.
9.  $\frac{\sigma^4}{n^2} 2(n-1)$ .
10. 0.
11. 27.
12.  $\chi^2(2n)$ .
13.  $t(n+p)$ .
14.  $\chi^2(n)$ .
15.  $(1, 2)$ .
16. 0.84.
17.  $\frac{2\sigma^2}{n^2}$ .
18. 11.07.
19.  $\chi^2(2n-2)$ .
20. 2.25.
21. 6.37.

**CHAPTER 15**

$$1. \sqrt{\frac{3}{n} \sum_{i=1}^n X_i^2}.$$

$$2. \frac{1}{\bar{X}-1}.$$

$$3. \frac{2}{\bar{X}}.$$

$$4. -\frac{n}{\sum_{i=1}^n \ln X_i}.$$

$$5. \frac{n}{\sum_{i=1}^n \ln X_i} - 1.$$

$$6. \frac{2}{\bar{X}}.$$

$$7. 4.2$$

$$8. \frac{19}{26}.$$

$$9. \frac{15}{4}.$$

$$10. 2.$$

$$11. \hat{\alpha} = 3.534 \text{ and } \hat{\beta} = 3.409.$$

$$12. 1.$$

$$13. \frac{1}{3} \max\{x_1, x_2, \dots, x_n\}.$$

$$14. \sqrt{1 - \frac{1}{\max\{x_1, x_2, \dots, x_n\}}}.$$

$$15. 0.6207.$$

$$18. 0.75.$$

$$19. -1 + \frac{5}{\ln(2)}.$$

$$20. \frac{\bar{X}}{1+\bar{X}}.$$

$$21. \frac{\bar{X}}{4}.$$

$$22. 8.$$

$$23. \frac{n}{\sum_{i=1}^n |X_i - \mu|}.$$

24.  $\frac{1}{N}$ .

25.  $\sqrt{\bar{X}}$ .

26.  $\hat{\lambda} = \frac{n\bar{X}}{(n-1)S^2}$  and  $\hat{\alpha} = \frac{n\bar{X}^2}{(n-1)S^2}$ .

27.  $\frac{10n}{p(1-p)}$ .

28.  $\frac{2n}{\theta^2}$ .

29.  $\begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}$ .

30.  $\begin{pmatrix} \frac{n\lambda}{\mu^3} & 0 \\ 0 & \frac{n}{2\lambda^2} \end{pmatrix}$ .

31.  $\hat{\alpha} = \frac{\bar{X}}{\beta}$ ,  $\hat{\beta} = \frac{1}{\bar{X}} \left[ \frac{1}{n} \sum_{i=1}^n X_i^2 - \bar{X} \right]$ .

32.  $\hat{\theta}$  is obtained by solving numerically the equation  $\sum_{i=1}^n \frac{2(x_i - \theta)}{1 + (x_i - \theta)^2} = 0$ .

33.  $\hat{\theta}$  is the median of the sample.

34.  $\frac{n}{\lambda}$ .

35.  $\frac{n}{(1-p)p^x}$ .

**CHAPTER 16**

1.  $b = \frac{\sigma_2^2 - \text{cov}(T_1, T_2)}{\sigma_1^2 + \sigma_2^2 - 2\text{cov}(T_1, T_2)}.$
2.  $\hat{\theta} = \overline{|X|}, E(\overline{|X|}) = \theta$ , unbiased.
4.  $n = 20.$
5.  $k = \frac{1}{2}.$
6.  $a = \frac{25}{61}, b = \frac{36}{61}, \hat{c} = 12.47.$
7.  $\sum_{i=1}^n X_i^3.$
8.  $\sum_{i=1}^n X_i^2$ , no.
9.  $k = \frac{4}{\pi}.$
10.  $k = 2.$
11.  $k = 2.$
13.  $\ln \prod_{i=1}^n (1 + X_i).$
14.  $\sum_{i=1}^n X_i^2.$
15.  $X_{(1)}$ , and sufficient.
16.  $X_{(1)}$  is biased and  $\overline{X} - 1$  is unbiased.  $X_{(1)}$  is efficient then  $\overline{X} - 1.$
17.  $\sum_{i=1}^n \ln X_i.$
18.  $\sum_{i=1}^n X_i.$
19.  $\sum_{i=1}^n \ln X_i.$
22. Yes.
23. Yes.

**24.** Yes.

**25.** Yes.

**26.**  $\hat{\theta} = 3 \overline{X}.$

**27.**  $\hat{\theta} = \frac{50}{30} X.$

**CHAPTER 17**

7. The pdf of  $Q$  is  $g(q) = \begin{cases} n e^{-nq} & \text{if } 0 < q < \infty \\ 0 & \text{otherwise.} \end{cases}$

The confidence interval is  $\left[ X_{(1)} - \frac{1}{n} \ln \left( \frac{2}{\alpha} \right), X_{(1)} - \frac{1}{n} \ln \left( \frac{2}{2-\alpha} \right) \right]$ .

8. The pdf of  $Q$  is  $g(q) = \begin{cases} \frac{1}{2} e^{-\frac{1}{2}q} & \text{if } 0 < q < \infty \\ 0 & \text{otherwise.} \end{cases}$

The confidence interval is  $\left[ X_{(1)} - \frac{1}{n} \ln \left( \frac{2}{\alpha} \right), X_{(1)} - \frac{1}{n} \ln \left( \frac{2}{2-\alpha} \right) \right]$ .

9. The pdf of  $Q$  is  $g(q) = \begin{cases} n q^{n-1} & \text{if } 0 < q < 1 \\ 0 & \text{otherwise.} \end{cases}$

The confidence interval is  $\left[ X_{(1)} - \frac{1}{n} \ln \left( \frac{2}{\alpha} \right), X_{(1)} - \frac{1}{n} \ln \left( \frac{2}{2-\alpha} \right) \right]$ .

10. The pdf  $g(q)$  of  $Q$  is given by  $g(q) = \begin{cases} n q^{n-1} & \text{if } 0 \leq q \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

The confidence interval is  $\left[ \left( \frac{2}{\alpha} \right)^{\frac{1}{n}} X_{(n)}, \left( \frac{2}{2-\alpha} \right)^{\frac{1}{n}} X_{(n)} \right]$ .

11. The pdf of  $Q$  is given by  $g(q) = \begin{cases} n(n-1)q^{n-2}(1-q) & \text{if } 0 \leq q \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

12.  $\left[ X_{(1)} - z_{\frac{\alpha}{2}} \frac{1}{\sqrt{n}}, X_{(1)} + z_{\frac{\alpha}{2}} \frac{1}{\sqrt{n}} \right]$ .

13.  $\left[ \hat{\theta} - z_{\frac{\alpha}{2}} \frac{\hat{\theta}+1}{\sqrt{n}}, \hat{\theta} + z_{\frac{\alpha}{2}} \frac{\hat{\theta}+1}{\sqrt{n}} \right]$ , where  $\hat{\theta} = -1 + \frac{n}{\sum_{i=1}^n \ln x_i}$ .

14.  $\left[ \frac{2}{\bar{X}} - z_{\frac{\alpha}{2}} \sqrt{\frac{2}{n\bar{X}^2}}, \frac{2}{\bar{X}} + z_{\frac{\alpha}{2}} \sqrt{\frac{2}{n\bar{X}^2}} \right]$ .

15.  $\left[ \bar{X} - 4 - z_{\frac{\alpha}{2}} \frac{\bar{X}-4}{\sqrt{n}}, \bar{X} - 4 + z_{\frac{\alpha}{2}} \frac{\bar{X}-4}{\sqrt{n}} \right]$ .

16.  $\left[ X_{(n)} - z_{\frac{\alpha}{2}} \frac{X_{(n)}}{(n+1)\sqrt{n+2}}, X_{(n)} + z_{\frac{\alpha}{2}} \frac{X_{(n)}}{(n+1)\sqrt{n+2}} \right]$ .

17.  $\left[ \frac{1}{4} \bar{X} - z_{\frac{\alpha}{2}} \frac{\bar{X}}{8\sqrt{n}}, \frac{1}{4} \bar{X} + z_{\frac{\alpha}{2}} \frac{\bar{X}}{8\sqrt{n}} \right]$ .

**CHAPTER 18**

1.  $\alpha = 0.03125$  and  $\beta = 0.763$ .
2. Do not reject  $H_o$ .
3.  $\alpha = 0.0511$  and  $\beta(\lambda) = 1 - \sum_{x=0}^7 \frac{(8\lambda)^x e^{-8\lambda}}{x!}$ ,  $\lambda \neq 0.5$ .
4.  $\alpha = 0.08$  and  $\beta = 0.46$ .
5.  $\alpha = 0.19$ .
6.  $\alpha = 0.0109$ .
7.  $\alpha = 0.0668$  and  $\beta = 0.0062$ .
8.  $C = \{(x_1, x_2) \mid \bar{x}^2 \geq 3.9395\}$ .
9.  $C = \{(x_1, \dots, x_{10}) \mid \bar{x} \geq 0.3\}$ .
10.  $C = \{x \in [0, 1] \mid x \geq 0.829\}$ .
11.  $C = \{(x_1, x_2) \mid x_1 + x_2 \geq 5\}$ .
12.  $C = \{(x_1, \dots, x_8) \mid \bar{x} - \bar{x} \ln \bar{x} \leq a\}$ .
13.  $C = \{(x_1, \dots, x_n) \mid 35 \ln \bar{x} - \bar{x} \leq a\}$ .
14.  $C = \left\{ (x_1, \dots, x_5) \mid \left( \frac{\bar{x}}{2\bar{x}-2} \right)^{5\bar{x}-5} \bar{x}^5 \leq a \right\}$ .
15.  $C = \{(x_1, x_2, x_3) \mid |\bar{x} - 3| \geq 1.96\}$ .
16.  $C = \left\{ (x_1, x_2, x_3) \mid \bar{x} e^{-\frac{1}{3}\bar{x}} \leq a \right\}$ .
17.  $C = \left\{ (x_1, x_2, \dots, x_n) \mid \left( \frac{e}{10\bar{x}} \right)^{3\bar{x}} \leq a \right\}$ .
18.  $\frac{1}{3}$ .
19.  $C = \{(x_1, x_2, x_3) \mid x_{(3)} \leq \sqrt[3]{117}\}$ .
20.  $C = \{(x_1, x_2, x_3) \mid \bar{x} \geq 12.04\}$ .
21.  $\alpha = \frac{1}{16}$  and  $\beta = \frac{255}{256}$ .
22.  $\alpha = 0.05$ .