M621: Some quiz 2 (Nov. 3) review problems

1. Suppose G is a group with subgroups H and K.
   (a) Suppose that $(|H|, |K|) = 1$. Show that $H \cap K = \{e\}$.
   (b) Suppose that $H \leq N_G(K)$. Show that $HK \leq G$.
   (c) Provide an example of an H, K, both subgroups, such that HK is not a subgroup of G.
   (d) Suppose $|H|, |K|$ are both finite. Prove that $|HK| = \frac{|H||K|}{|H \cap K|}$. This is an interesting fact, and has an interesting proof, one that you should master.
2. Suppose G is finite and Abelian, and $p||G|$, where p is prime. Without using the Sylow Theorem, prove that there exists an element $g \in G$ having order p. The standard proof is by induction on $|G|$, a good proof to master.
3. Suppose H is a subgroup of G. Prove H is normal in G if and only if H is a union of conjugacy classes of G.
4. Explain, as if to a 521 student, why it is true that if G is a finite group and K is a homomorphic image of G, then $|K|||G|$.
5. Suppose G is a group, N is a normal subgroup of G, and $K \leq G$ with $K \geq N$. Prove that K is normal in G if and only if K/N is normal in G/N.
6. List the conjugacy classes of A_4.
7. Let $n \in \mathbb{N}$ be greater than 2. Prove that there exists an isomorphic copy of A_{n-2} in S_n.
8. Suppose that $H \leq G$.
   (a) Prove that $[G : H] = 2$ implies H is normal in G.
   (b) Show, by an example, that $[G : H] = 3$ does not imply H is normal in G.
   (c) Harder, but a very good exercise: Prove that if p is the least prime dividing $|G|$, and $[G : H] = p$, then H is normal in G.
9. Show that A_4 has no subgroup of order 6. (Thus the converse of Lagrange's Theorem is not in general true.)
10. Prove that if G is a group and $G/Z(G)$ is cyclic, then G is Abelian. (This is an important exercise.)
11. State and prove the Orbit-Stabilizer Proposition. (This is a very important problem.)

12. State the Class Equation. Prove that it is valid. (This a very important problem.)

13. Use the Orbit-Stabilizer result and the Class Equation to prove that if G is a p-group, then $Z(G)$ is non-trivial.

14. Prove that if $|G| = p^2$, where p is a prime, then G is Abelian.

15. Let G be a group. Prove that $Aut(G)$, the automorphisms of G, forms a group (with operation composition $\circ$).

16. Let G be a group, with $g \in G$, and let $c_g : G \to G$ be the function given $c_g(h) = ghg^{-1}$, for all $h \in G$.

    (a) Show that c_g is an automorphism of G.

    (b) Let $Inn(G) = \{c_g : g \in G\}$. Show that $Inn(G)$ is trivial if and only if G is Abelian.

    (c) Show that $Inn(G)$ is a normal subgroup of $Aut(G)$.

    (d) Consider the map $\Gamma : G \to Inn(G)$ given by $\Gamma(g) = c_g$ for all $g \in G$. Show that Γ is an onto homomorphism, and that $ker(\Gamma) = Z(G)$.

17. Show that $Aut(Z_n) \cong Z_n^\times$, the latter the group of units of Z_n with operation multiplication mod n. This is a good problem (one we did in the undergrad course), one that'll help your general understanding of things.

18. Suppose G is a finite group with $|G| = p^\alpha m$, where $(p, m) = 1$, and p is a prime. Suppose $P \leq G$ and $|P| = p^\alpha$. Let Q be any p-subgroup of G. Prove that $Q \cap N_G(P) = Q \cap P$. (This is a good problem, the lemma that leads into the proof of the Sylow Theorem. It's on page 140, and we'll do it in class on Tuesday, Oct. 25.)