office hours

T Th 10:45-11:45

Dawn of a New Day

602 

Ch 11 Product Measure -more than 1 dimension .. Oh God...

• Product σ-algebra

Suppose we have 2 measure spaces:
$(X, \mathcal{A}, \mu)$, & $(Y, \mathcal{B}, \nu)$. Define a measurable rectangle as a set in the form

$A \times B$, $A \in \mathcal{A}$ & $B \in \mathcal{B}$.

Now define $\mathcal{C}_0$ as the collection of finite union of disjoint measurable rectangles

i.e. $\mathcal{C}_0 = \{ \bigcup_{i=1}^{n} A_i \times B_i \mid A_i \in \mathcal{A}, B_i \in \mathcal{B}, A_i \times B_i \cap A_j \times B_j \neq \emptyset \text{ if } i \neq j \}$

We can check $\mathcal{C}_0$ is an algebra by

i) $\emptyset \in \mathcal{C}_0$ ii) $E \in \mathcal{C}_0 \Rightarrow E^c \in \mathcal{C}_0$

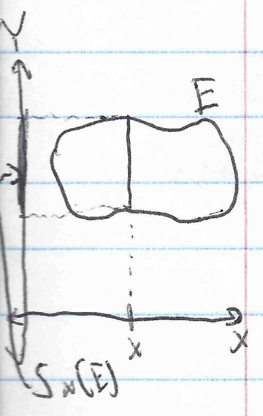
iii) $E_i \in \mathcal{C}_0$, $1 \le i \le n$, $\Rightarrow \bigcup_{i=1}^{n} E_i \in \mathcal{C}_0$, We're taking his word that $\mathcal{C}_0$ is an algebra.

Now we can define the product σ-algebra as

$\mathcal{A} \times \mathcal{B} = \sigma(\mathcal{C}_0)$ So we have a space: $X \times Y$

algebra: $\mathcal{A} \times \mathcal{B}$

measure: $\mu \times \nu$



$\forall E \subset X \times Y$, define the x-section of E by

$s_x(E) = \{ y \in Y \mid (x,y) \in E \}$ for a fixed $x \in X$.

Similarly, we define the y section of E by

$t_y(E) = \{ x \in X \mid (x,y) \in E \}$ for a fixed $y \in Y$.

Cont.

This is capital S, the previous was small

For a given function $f: X \times Y \to \mathbb{R}$, we define
$S_x f : Y \to \mathbb{R} \ \forall x$ &
$T_y f : X \to \mathbb{R} \ \forall y$ by $S_x f(y) = f(x,y)$
& $T_y f(x) = f(x,y)$

Lemma: (1) If $E \in \mathcal{A} \times \mathcal{B}$, then $s_x(E) \in \mathcal{B} \ \forall x$ & $t_y(E) \in \mathcal{A} \ \forall y$.

(2) If f is $\mathcal{A} \times \mathcal{B}$ measurable, then $S_x f$ is $\mathcal{B}$ measurable $\forall x$ & $T_y f$ is $\mathcal{A}$ measurable $\forall y$.

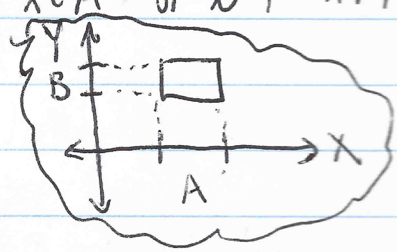
Proof:

(1) WTS $E \in \mathcal{A} \times \mathcal{B} \Rightarrow s_x(E) \in \mathcal{B} \ \forall x$ (the 2nd case is identical to prove)

Define $\mathcal{C} = \{ E \in \mathcal{A} \times \mathcal{B} \mid s_x(E) \in \mathcal{B} \ \forall x \}$
$\subset \mathcal{A} \times \mathcal{B}$

WTS $\mathcal{C}$ is a σ-algebra containing all measurable rectangles.

i) If $E = A \times B$, $A \in \mathcal{A}$ & $B \in \mathcal{B}$,
then $s_x(E) = s_x(A \times B) = B$ if $x \in A$ or $\emptyset$ if $x \notin A$
so $s_x(E) \in \mathcal{B}$
$\Rightarrow E \in \mathcal{C}$.



Furthermore,
In particular $\emptyset = \emptyset \times \emptyset \in \mathcal{C}$.

ii) WTS $E \in \mathcal{C} \Rightarrow E^c \in \mathcal{C}$
$y \in s_x(E^c) \Leftrightarrow (x,y) \in E^c \Leftrightarrow y \notin s_x(E) \Leftrightarrow y \in (s_x(E))^c$
$\Rightarrow s_x(E^c) = (s_x(E))^c$ ☆

But $E \in \mathcal{C} \Rightarrow s_x(E) \in \mathcal{B} \Rightarrow (s_x(E))^c \in \mathcal{B} \Rightarrow$ ☆
$\Rightarrow s_x(E^c) \in \mathcal{B}$. $\Rightarrow E^c \in \mathcal{C}$

Proof cont.

Proof cont.

iii) WTS $E_i \in \mathcal{C}$, $i=1,2,\dots$ $\Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{C}$.

So we need to check $S_x\left(\bigcup_{i=1}^{\infty} E_i\right) \in \mathcal{B}$.

$$
\begin{aligned}
y \in S_x\left(\bigcup_{i=1}^{\infty} E_i\right) &\iff (x,y) \in \bigcup_{i=1}^{\infty} E_i \\
&\iff (x,y) \in E_i \text{ for some } i \\
&\iff y \in S_x(E_i) \text{ for some } i \\
&\iff y \in \bigcup_{i=1}^{\infty} S_x E_i
\end{aligned}
$$

$\Rightarrow S_x\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} \underbrace{S_x(E_i)}_{\in \mathcal{B}} \in \mathcal{B}$ so it's closed under countable union.

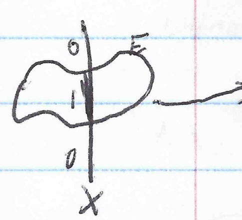
So by i) ii) & iii)

$\mathcal{C}$ is a σ algebra containing all the measurable rectangles

So by def of $\mathcal{A} \times \mathcal{B}$, $\mathcal{C} = \mathcal{A} \times \mathcal{B}$.

2) WTS f is $\mathcal{A}\times\mathcal{B}$ measurable $\Rightarrow S_x f$ is $\mathcal{B}$ measurable $\forall x$.

the other case is identical proofs



i) If $f = \chi_E$ where $E \in \mathcal{A}\times\mathcal{B}$. Then

$$
\begin{aligned}
S_x f(y) &= S_x \chi_E(y) \\
&= \chi_{S_x(E)}(y)
\end{aligned}
$$

By (1) $(S_x(E) \in \mathcal{B}) \Rightarrow S_x f$ is $\mathcal{B}$ measurable.

ii) If $f = \sum_{i=1}^{n} \chi_{E_i}$, $E_i \in \mathcal{A}\times\mathcal{B}$. Then

f is $\mathcal{B}$ measurable by i) & linearity.

Proof cont.

these $\int$'s are capital

iii) Consider if f: non-negative & $\mathcal{A} \times \mathcal{B}$ measurable.
we can take $\mathcal{A} \times \mathcal{B}$ measurable simple functions $r_n \ni r_n \uparrow f$.
Then (i) $S_x r_n \to S_x f \quad \forall x$

~~(ii)~~ $S_x r_n$ is $\mathcal{B}$ measurable $\forall n$ by ii)

$\Rightarrow S_x f$ is $\mathcal{B}$ measurable

iv) f is $\mathcal{A} \times \mathcal{B}$ measurable. $f = f^+ - f^-$

$\Rightarrow$ by linearity $S_x f$ is measurable by iii)

QED

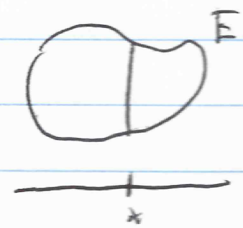
Consider metric spaces $(X, \mathcal{A}, \mu)$ & $(Y, \mathcal{B}, \nu)$

- For $E \in \mathcal{A} \times \mathcal{B}$, define $h(x) = \nu(S_x(E))$, similarly
  $''\sigma(e_0)$ $\quad k(y) = \mu(t_y(E))$

all together

- Proposition: Suppose μ & ν are σ-finite. Then
  (1) h is $\mathcal{A}$ measurable & k is $\mathcal{B}$ measurable.
  (2) $\int h(x)\, \mu(dx) = \int k(y)\, \nu(dy)$

What does this mean?



(2) $\forall x, y, \quad \chi_{S_x(E)}(y) = S_x \chi_E(y)$

So $\int h(x)\, \mu(dx)$

$= \int \nu(s_x(E))\, \mu(dx)$

$= \int \left(\int \chi_{s_x(E)}(y)\, \nu(dy) \right) \mu(dx)$

$= \int \left(\int S_x \chi_E(y)\, \nu(dy) \right) \mu(dx)$

$= \int\int \chi_E(x,y)\, \nu(dy)\, \mu(dx)$

☆ more or less, it's like how we define double integrals. ☆

Proof next time

- Define a product measure (on metric space $(X \times Y, \mathcal{A} \times \mathcal{B})$) $\mu \times \nu$ by

$$\mu \times \nu(E) = \int h(x)\, \mu(dx) = \int k(y) \nu(dy)$$

Moreover, in the case where $E = A \times B$ (a measurable rectangle) $A \in \mathcal{A}$ & $B \in \mathcal{B}$, $\mu \times \nu(A \times B) = \mu(A)\, \nu(B)$ b/c $h(x) = \nu(S_x(A \times B)) = \chi_A \nu(B)$.

• Proof of last time's proposition

(-) First consider the case when μ & ν are finite measures. Define $\mathcal{C} \subset \mathcal{A} \times \mathcal{B}$ for which (1) & (2) of the proposition hold true. So h is $\mathcal{C}$ measurable & (2) holds true $\forall\, E \in \mathcal{C}$. We will show $\mathcal{C}$ is a monotone class containing $\mathcal{C}_0$. So, WTS

(i) $A_i \uparrow A$ & $A_i \in \mathcal{C} \Rightarrow A \in \mathcal{C}$

& (ii) $A_i \downarrow A$ & $A_i \in \mathcal{C} \Rightarrow A \in \mathcal{C}$. So first, ...

i) WTS measurable rectangles belong to $\mathcal{C}$.

If $E = A \times B$, $A \in \mathcal{A}$, $B \in \mathcal{B}$. Well, $h(x) = \nu(S_x(E))$ $= \nu(S_x(A \times B)) = \chi_A(x) \nu(B)$ which we know is $\mathcal{A}$ measurable, b/c $\nu(B)$ is finite & measurable, & $\chi_A(x)$ is obviously measurable (1 or 0) & $\int h(x) \mu(dx) = \int \chi_A(x)\, \nu(B)\, \mu(dx)$ $= \mu(A) \nu(B)$. Similarly,

$k(y) = \mu(t_y(A \times B)) = \mu(A) \chi_B(y)$ is $\mathcal{B}$ measurable for same reasons &

$\int k(y)\, \nu(dy) = \int \mu(A)\, \chi_B(y)\, \nu(dy) = \mu(A)\, \nu B$.

So we've proven h & k are $\mathcal{A}$ & $\mathcal{B}$ measurable respectively, &

$\int h(x)\, \mu\, dx = \int k(y)\, \nu(dy)$. $\Rightarrow A \times B \in \mathcal{C}$.

ii) WTS $\mathcal{C}_0 \subset \mathcal{C}$.

Let $E = \bigcup_{i=1}^{n} E_i \in \mathcal{C}_0$ where E_i are disjoint measurable rectangles.

$S_x(E) = S_x\left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} s_x(E_i)$

So $h(x) = \nu(S_x(E)) = \nu\left(\bigcup_{i=1}^{n} S_x(E_i)\right) = \sum_{i=1}^{n} \nu(s_x(E_i))$ which is $\mathcal{A}$ measurable by i). Setting $h_i(x) = \nu(s_x(E_i))$, then we can say $h(x) = \sum_{i=1}^{n} h_i(x)$. Similarly, $k(y) = \mu(t_y(E)) = \mu\left(\bigcup_{i=1}^{n} t_y(E_i)\right)$

$= \sum_{i=1}^{n} \mu(t_y(E_i))$ which is $\mathcal{B}$ measurable by i) (proof is exact same).

$= \sum_{i=1}^{n} k_i(y)$, $k_i(y) = \mu(t_y(E_i))$.

Now, $\int h(x)\,\mu(dx) = \int \sum_{i=1}^{n} h_i(x)\,\mu(dx) = \sum_{i=1}^{n} \int h_i(x)\,\mu(dx)$

$= \sum_{i=1}^{n} \int k_i(y)\,\nu(dy) = \int \sum k_i(y)\,\nu(dy) = \int k(y)\,\nu(dy)$. Thus $\mathcal{C}_0 \subset \mathcal{C}$.

iii) WTS $\mathcal{C}$ is a monotone class.

a) Let $E_n \uparrow E$ & $E_n \in \mathcal{C}$ $\forall n$. Notice, $E_n \in \mathcal{C} \iff E \in \mathcal{A} \times \mathcal{B}$ & (1) & (2) hold true. So we need to show (1) & (2) hold true for E. Set $h_n(x) = \nu(s_x(E_n))$ & $k_n(x) = \mu(t_x(E_n))$ $\forall n$. So b/c $E_n \uparrow E$, $h_n \uparrow h = \nu(s_x(E))$ & $k_n \uparrow k = \mu(t_x(E))$.

$\Rightarrow h$ is $\mathcal{A}$ measurable & k is $\mathcal{B}$ measurable b/c lim sup & lim inf of a measurable function is also measurable.

We notice that $\int h_n(x)\,\mu(dx) = \int k_n(y)\,\nu(dy)$ $\forall n$. We know $h_n \uparrow h$ & $k_n \uparrow k$, so, by Monotone Convergence Thm, $\int h(x)\,\mu(dx) = \int k(y)\,\nu(dy)$ (take limit of both sides) $\Rightarrow E \in \mathcal{C}$.

b) on next page.

b) Let $E_n \downarrow E$ & $E_n \in \mathcal{C}$ $\forall n$. WTS $E \in \mathcal{C}$.

Setting $h_n, k_n, h,$ & k as the same as in a), then can we say $h_n \downarrow h$ & $k_n \downarrow k$? Well, μ & ν are finite as we've defined before, so yes! Thus b/c they converge pointwise, h & k must be $\mathcal{A}$ & $\mathcal{B}$ measurable respectively.

Moreover, $\int h_n(x)\, \mu(dx) = \int k_n(y)\, \nu(dy)$ $\forall n$.

By the Dominated Convergence Thm. w/ a dominant function $\chi_{X \times Y}$, we have $\int h(x) \mu(dx) = \int k(y)\, \nu(dy)$, thus $E \in \mathcal{C}$.

Thus by i) ii) & iii), $\mathcal{C}$ is a monotone class containing $\mathcal{C}_0$.

What can we claim from this?

Recall the Thm: If $\mathcal{C}_0$ is an algebra, then $M(\mathcal{C}_0) = \sigma(\mathcal{C}_0)$ where $M(\mathcal{C}_0)$ is the smallest monotone class containing $\mathcal{C}_0$.

Well, we've shown $\mathcal{A} \times \mathcal{B} = \sigma(\mathcal{C}_0) = M(\mathcal{C}_0) \subset \mathcal{C}$. By the definition of $\mathcal{C}$, $\mathcal{C} \subset \mathcal{A} \times \mathcal{B} \Rightarrow \mathcal{C} = \mathcal{A} \times \mathcal{B}$.

(--) Now we want to consider the case where μ & ν are σ-finite.

$\Rightarrow F_i \uparrow X$ & $\mu(F_i) < \infty$ $\forall i$,

& $G_i \uparrow Y$ & $\nu(G_i) < \infty$ $\forall i$. Set $\forall i$,

$\mu_i(A) = \mu(A \cap F_i)$ $\forall A \in \mathcal{A}$ &

$\nu_i(B) = \nu(B \cap G_i)$ $\forall B \in \mathcal{B}$.

Set $h_i(x) = \nu_i(s_x(E)) = \nu(s_x(E) \cap G_i)$ &

$k_i(y) = \mu_i(t_y(E)) = \mu(t_y(E) \cap F_i)$ $\forall i$.

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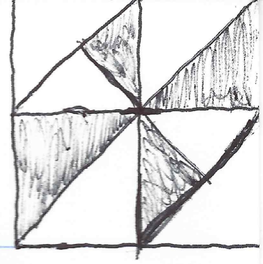


Then by (–) (step 1), h_i is $\mathcal{A}$ measurable $\forall i$ & k_i is $\mathcal{B}$ meas. $\forall i$.
Also $h_i \uparrow h$ & $k_i \uparrow k$. Therefore h & k are $\mathcal{A}$ & $\mathcal{B}$ meas. respectively.

the final part (proving the integration is equal) is left as an exercise that he couldn't do himself.

Now $\int h_i(x)\, \mu_i(dx) = \int h_i(x)\, \chi_{F_i}\, \mu(dx)$.

$\int \chi_E\, \mu_i = \mu_i(E) = \mu(E \cap F_i) = \chi_{F_i}\, \mu_i(E) = \int \chi_E\, \chi_{F_i}\, \mu(dx)$

Prove it for characteristic, then simple, then regular functions.
he doesn't understand how to prove it though, so good luck. In the end,
$\int h(x)\, \mu(dx) = \int k(y)\, \nu(dy)$ QED

Go kill yourself Gio.

- To show $\mu \times \nu$ is a measure, we must show
  i) $\mu \times \nu(\emptyset) = 0$ ii) $\mu \times \nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu \times \nu(E_i)$
  where E_i is a disjoint measure $\forall i$.

- Recall $A \in \mathcal{A}$ & $B \in \mathcal{B}$, $\mu \times \nu(A \times B) = \mu(A)\,\nu(B)$

<u>Fubini's Thm</u>
Suppose $f: X \times Y \to \mathbb{R}$ is $\mathcal{A} \times \mathcal{B}$ measurable. Suppose μ & ν are σ-finite. If either a) f is non-negative, or
b) $\int |f(x,y)|\, d(\mu \times \nu)(x,y) < \infty$, then

$S_x f$ (1) $y \mapsto f(x,y)$ is $\mathcal{B}$ measurable $\forall x$
$T_y f$ (2) $x \mapsto f(x,y)$ is $\mathcal{A}$ measurable $\forall y$
(3) $h(x) = \int f(x,y)\, \nu(dy)$ is $\mathcal{A}$ measurable
(4) $k(y) = \int f(x,y)\, \mu(dx)$ is $\mathcal{B}$ measurable
(5) we have $\int f(x,y)\, d(\mu \times \nu)(x,y)$
$= \int \left(\int f(x,y)\, d\mu(x) \right) d\nu(y)$
$= \int \left(\int f(x,y)\, d\nu(y) \right) d\mu(x)$

Proof follows

Proof: i) $f = \chi_E$, $E \in \mathcal{A} \times \mathcal{B}$

then we're done for (1) - (5) from our previous proposition.

ii) $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ $\quad a_i > 0 \quad E_i \in \mathcal{A} \times \mathcal{B}$.

True by i) & linearity

iii) f: non-negative simple function.

Consider simple functions $f_n \uparrow f$. Then

(1) - (5) follow by ii) by applying the monotone convergence thm.

iv) For $f \ni \int |f(x,y)| \, d(\mu \times \nu)(x,y) < \infty$,

write $f = f^+ - f^-$, then by linearity, we can see that (1) - (5) follow. $\quad$ QED.

- If $\iint |f(x,y)| \, \mu(dx) \, \nu(dy) < \infty$, $|f|$: nonnegative

$\Rightarrow \int |f|(x,y) \, d(\mu \times \nu)(x,y)$

$= \int \left(\int |f|(x,y) \, d\mu(x) \right) d\nu(y) < \infty$

$\Rightarrow$ by the Fubini Thm for f, (1) - (5) hold true.

Ch 11 Hmwk ($2, 4(i)$, ~~(8)~~ $8, 10, 11, 12$)

due Feb 2nd.

Ch 12

• Positive & negative sets.

- Def - Let $\mathcal{A}$ be a σ-algebra. A signed measure is a function $\mu : \mathcal{A} \to (-\infty, \infty]$ ∋

  i) $\mu(\emptyset) = 0$

  ii) $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ whenever $A_i \in \mathcal{A}$ & pairwise disjoint.

  (Here "=" in ii) means that the series converges absolutely if the left hand side is finite).

- Def - Let μ be a signed measure on $\mathcal{A}$.

  i) $A \in \mathcal{A}$ is called a positive set for μ if $\mu(B) \geq 0$ whenever $B \subset A$, $B \in \mathcal{A}$.

  ii) $A \in \mathcal{A}$ is called a negative set for μ if $\mu(B) \leq 0$ whenever $B \subset A$, $B \in \mathcal{A}$

  iii) $A \in \mathcal{A}$ is called a null set for μ if ~~$\mu(A)$~~ $\mu(B) = 0$ whenever $B \subset A$, $B \in \mathcal{A}$.

  Note: our usual measure that we use is called positive measure instead of signed.

- $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n \to \infty} \mu(\bigcup_{i=1}^{n} A_i)$ (proof is identical for signed as well as positive).

Ex | Let m be a Lebesgue measure on $(\mathbb{R}, \underbrace{\mathcal{L}}_{\text{Lebesgue } \sigma\text{-algebra}})$. Let f be integrable.

$\mu(A) = \int_A f \, dm$, $A \in \mathcal{L}$.

  i) Let μ be a signed measure.

  ii) Let $P = \{t \mid f(t) \geq 0\}$, $N = \{t \mid f(t) < 0\}$.

  Notice P is a positive set & N is a negative set.

This suggests a decomposition $\mathbb{R} = P \underbrace{\cup}_{\text{union}} N$ ✩(Hahn decomposition)

$C = \{t \mid f(t) = 0\}$ null set ✩ unique up to C.

iii) μ as in i)

$f = f^+ - f^-$, $\mu^+(A) = \int_A f^+$ $\quad \mu = \mu^+ - \mu^-$ ← (Jordan decomposition)

$\mu^-(A) = \int_A f^-$

Proposition: Let μ be a function $\ni \mu : \mathcal{A} \to [-\infty, \infty]$ where $\mathcal{A}$ is a σ-algebra. Let E be a measurable set $E \in \mathcal{A} \ni \mu(E) < 0$. Then $\exists$ a measurable subset F of E that is a negative set & $\mu(F) < 0$.

Proof: If E is a negative set then we're done. If not, $\exists$ a measurable subset w/ positive measure. Let n_1 be the smallest # in $\mathbb{N} \ni \exists E_1 \subset E$ w/ $\mu(E_1) \geq \frac{1}{n_1}$. We now construct a pairwise disjoint E_k, $k \geq 2$ inductively below. Suppose $E_1, E_2, \dots, E_{k-1}$ are pairwise disjoint measurable subsets of E w/ $\mu(E_i) > 0$ where $i = 1, \dots, k-1$. Define $F_k = E \setminus \left(\bigcup_{i=1}^{k-1} E_i\right)$

$\mu(F_k) = \mu(E) - \mu\left(\bigcup_{i=1}^{k-1} E_i\right) = \mu(E) - \sum_{i=1}^{k-1} \mu(E_i) \leq \mu(E) < 0$.

Hence if F_k is a negative set, then we are done by setting $F = F_k$. If F_k is not a negative set choose n_k as the smallest natural # $\ni \exists E_k \subset F_k$ is measurable & $\mu(E_k) \geq \frac{1}{n_k}$. We stop the process if $\exists k \ni F_k$ is a negative set & $\mu(F_k) < 0$.

If not, set $F = \bigcap_{k=1}^{\infty} F_k = E \setminus \left(\bigcup_{k=1}^{\infty} E_k\right)$ where

$\mu(E) = \mu\left(F \cup \left(\bigcup_{k=1}^{\infty} E_k\right)\right) = \lim_{n\to\infty} \mu\left(F \cup \left(\bigcup_{k=1}^{n} E_k\right)\right)$

$= \mu(F) + \lim_{n\to\infty} \sum_{k=1}^{n} \mu(E_k) = \mu(F) + \sum_{k=1}^{\infty} \mu(E_k)$,

$-\infty < \mu(E) < 0$, $\mu(E_k) > 0$.

$\Rightarrow \mu(F) \leq \mu(E) < 0$. Now we need to show that F is a negative set. Suppose not, then $\exists G \subset F$ in $\mathcal{A} \ni \mu(G) > 0$. Cont.

we could write $\mu(G) > \frac{1}{N}$ for some $N \in \mathbb{N}$. But then from our construction of F_k, $\exists\, n_k \ni n_{k'} > N$ for some k' b/c $\frac{1}{n_k} \to 0$. This means we could choose G instead of $E_{n_{k'}}$
$\Rightarrow$ $G \not\subset F$ which contradicts our construction of $G \subset F$, thus F must be a negative set. QED.

<u>Def:</u> $A \Delta B = (A \setminus B) \cup (B \setminus A)$ for two sets A & B.

• <u>Thm: Hahn decomposition</u>

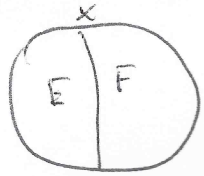
(1) Let μ be a function $\ni \mu : \mathcal{A} \to (-\infty, \infty]$ (signed measure), Then $\Rightarrow \exists$ disjoint $E, F \in \mathcal{A} \ni E \cup F = X$, and E is a negative set & F is a positive set.

(2) If E' & F' are another pair that satisfies (1), then $E \Delta E' = F \Delta F'$ is a null set wrt μ.

<u>Proof:</u> (1)

If there is no negative set (other than $\emptyset$), then we're done b/c $X = X$. If not, define $L = \inf\{\mu(A_i) \mid A_i$: negative sets$\}$. Choose negative sets $A_n \ni \mu(A_n) \to L$. Set $B_1 = A_1$, $B_2 = A_2 \setminus B_1$, $B_3 = A_3 \setminus (B_1 \cup B_2)$, $\ldots$, $B_n = A_n \setminus (\bigcup_{i=1}^{n-1} B_i)$ $\forall n$, $\Rightarrow B_n$ are disjoint negative sets. $(A \subset B_n \subset A_n)$. We also know $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$. Define $E = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$.
$\forall\, C \subset E$, we write $\mu(C) = \mu(C \cap \bigcup_{n=1}^{\infty} B_n)$
$= \lim_{n\to\infty} \mu(C \cap \bigcup_{i=1}^{n} B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(C \cap B_i) \le 0 \Rightarrow E$ is a negative set. Moreover, $\mu(E) = \mu(A_n \cup (E \setminus A_n))$
$= \mu(A_n) + \mu(E \setminus A_n)$ b/c they're disjoint.
$\le \mu(A_n)$ b/c $\mu(E \setminus A_n)$ is negative.
As $n \to \infty$, $\mu(E) \le L$. By the definition of L,
$\mu(E) \ge L \Rightarrow \mu(E) = L > -\infty$

Cont.



Now take $F = E^c$. WTS F is a positive set. Suppose not.
$\exists\, B \subset F \ni \mu(B) < 0 \Rightarrow \exists\, C \subset B$ a negative set $\ni \mu(C) < 0$.
$\Rightarrow E \cup C$ is a negative set $\ni \mu(E \cup C) < \mu(E) = L$
So we have a contradiction to the definition of L, Thus
F is a positive set.

(2)

Let E' & F' be another pair satisfying the decomposition of (1).
Then consider $E \setminus E'$. Let A be a subset $\ni$
$A \subset E \setminus E' = F' \setminus F \overset{b/c}{=} (E')^c \setminus E^c$. It reduces to prove this.
$F' \setminus F \subset F' \Rightarrow \mu(A) \geq 0$ where $A \subset E \setminus E' \subset E$
$\Rightarrow \mu(A) \leq 0$. Thus $\mu(A) = 0$. Therefore any subset
of $E \setminus E'$ measure zero. $\Rightarrow E \setminus E'$ is a null set.
~~By symmetry of the proof~~ By the same proof, one can show that
$E' \setminus E$ is the null set. Then $E' \Delta E$ is the null set.
Similarly one can prove $F' \Delta F$ is a null set. Thus the
decomposition is unique up to null set. QED.

Defn μ & ν are called mutually singular denoted by
$\mu \perp \nu$ if $\exists$ disjoint $E, F \in \mathcal{A} \ni$
i) $E \cup F = X$ &
ii) $\mu(E) = \nu(F) = 0$. Where μ is on F & ν is on E.

Ex) Suppose m is a Lebesgue measure on $[0,1]$. Define
$\mu(A) = m(A \cap [0, \frac{1}{2}])$ & $\nu(A) = m(A \cap [\frac{1}{2}, 1])$.
Then setting $E = (\frac{1}{2}, 1]$, $F = [0, \frac{1}{2}]$, $X = E \cup F$, $E \cap F = \emptyset$.
$\mu(E) = 0$, $\nu(F) = 0$. $\Rightarrow \mu \perp \nu$.

I wish

- <u>Jordan decomposition Thm:</u>

If μ is a signed measure on $(X, \mathcal{A})$, $\exists$ positive measures μ^+, μ^- $\ni$ $\mu = \mu^+ - \mu^-$ & $\mu^+ \perp \mu^-$. The decomposition is unique.

<u>Proof:</u>

By the Hahn decomposition, $\exists$ E that is a negative set & F that is a positive set wrt μ $\ni$ $X = E \cup F$ & $E \cap F = \emptyset$

Define $\forall A \in \mathcal{A}$, $\mu^+(A) = \mu(A \cap F) \geq 0$ & $\mu^- = -\mu(A \cap E) \geq 0$.

This gives a desired decomposition. To show the uniqueness, assume that $\mu = \nu^+ - \nu^-$ is another decomposition. Choose E' $\ni$ $\nu^+(E') = 0$ & $\nu^-((E'^c) : F') = 0$ $\ni$ $X = E' \cup F' = X$, $E' \cap F' = \emptyset$.

If $A \subset F'$, $\nu^-(A) \leq \nu^-(F') = 0 \Rightarrow \nu^-(A) = 0$, thus $\mu(A) = \nu^+(A) - \nu^-(A) = \nu^+(A) \geq 0$
$\Rightarrow$ F' is a positive set for μ. Similarly, by a symmetric proof, E' is a negative set, $\Rightarrow$ F' & E' gives another Hahn decomposition of X, $\Rightarrow$ $E \Delta E'$ & $F \Delta F'$ are null sets.

Now to show $\mu^+ = \nu^+$ $\forall A \in \mathcal{A}$, we write
$\nu^+(A) = \nu^+(A \cap F') - \underbrace{\nu^-(A \cap F')}_{=0} = \mu(A \cap F')$
$= \mu(A \cap F) = \mu^+(A \cap F) = \mu^+(A)$.

Thus $\nu^+(A) = \mu^+(A) \Rightarrow \nu^-(A) = \mu^-(A)$ by symmetry,
$\Rightarrow$ the decomposition is unique. QED

- Let's define $|\mu| = \mu^+ + \mu^-$. This is called the <u>total variation measure</u> of the signed measure μ, &
$|\mu|(X)$ is called the <u>total variation</u> of μ.

Ch. 13

- <u>The Radon - Nikodym Thm.</u>

Suppose f is non-negative & integrable w.r.t. μ. Define

(*)...... $\nu(A) = \int_A f \, d\mu$, $A \in \mathcal{A}$ where $\mathcal{A}$ is σ-algebra. Then ν is a measure (by a seperate proof) for a fixed f. Notice $\nu(A) = 0$ if $\mu(A) = 0$.

- <u>Question:</u> Given 2 measures μ & ν, can we find f to make (*) hold?

- <u>Def:</u> A measure ν is ~~called~~ said to be <u>absolutely continuous</u> wrt a measure μ if $\nu(A) = 0$ whenever $\mu(A) = 0$. In this case we write <u>$\nu << \mu$</u>.

- <u>Proposition:</u> Suppose ν is a finite measure. Then we have $\nu << \mu$ iff $\forall \epsilon > 0$ $\exists$ $\delta > 0$ $\ni$ $\mu(A) < \delta$ $\Rightarrow \nu(A) < \epsilon$ where $A \in \mathcal{A}$. (note $\mathcal{A}$ is the σ-algebra).

<u>Proof:</u>

($\Leftarrow$) For a given A & $\forall \epsilon > 0$, $\exists \delta > 0 \ni \mu(A) < \delta \Rightarrow \nu(A) < \epsilon$. To show $\nu << \mu$, we want to show if $\mu(A) = 0$ then $\nu(A) = 0$. Well, if $\mu(A) = 0$, then $\nu(A) < \epsilon$ $\forall \epsilon > 0$ by our assumption, $\Rightarrow \nu(A) = 0 \Rightarrow \nu << \mu$.

($\Rightarrow$) Assume that $\nu << \mu$. By way of contradiction, suppose that $\exists$ $\epsilon > 0 \ni$ no corresponding δ exists.

$\Rightarrow \exists$ $E_k \ni \mu(E_k) < 2^{-k}$ but $\nu(E_k) \geq \epsilon$.

Set $F = \bigcap_{n=1}^{\infty} F_n$ where $F_n = \bigcup_{k=n}^{\infty} E_k$.

(Notice $F_1 \supset F_2 \supset \dots$) Then $\mu(F) = \mu(\bigcap_{n=1}^{\infty} F_n)$

Cont.

So $\mu(F) = \mu(\bigcap_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu(\bigcup_{k=n}^{\infty} E_k)$
$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) < \lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{2^k} = 0 \Rightarrow \mu(F) = 0.$
Also know $\mu(F_1) = \mu(\bigcup_{k=n}^{\infty} E_k) \leq \sum \frac{1}{2^k} < \infty$, so F_1 is finite, so we can take the limit.
But $\nu(F) = \lim_{n\to\infty} \nu(\bigcup_{k=n}^{\infty} E_k) \geq \lim_{n\to\infty} \nu(E_n) > \varepsilon$, therefore even though $\mu = 0$, $\nu > 0$, thus the proposition should be true & we've proven the iff statement. QED.

- Lemma: Let μ & ν be finite measures on $(X, \mathcal{A})$. Then either $\mu \perp \nu$ or $\exists\, \varepsilon > 0$ & $G \in \mathcal{A} \ni \mu(G) > 0$ & G is a positive set for $\nu - \varepsilon\mu$.

Proof:

For a signed measure, $\nu - \frac{1}{n}\mu$ where $n \in \mathbb{N}$. Consider the Hahn decomposition $X = E_n \cup F_n$ where E_n is a negative set, F_n is a positive set. Set $F = \bigcup^{\infty} F_n$, $E = \bigcap_{n=1}^{\infty} E_n$ $(E^c = F)$. $\forall n$, $E \subset E_n$, $\Rightarrow \nu(E) \leq \nu(E_n) \leq \frac{1}{n}\mu(E_n) \leq \frac{1}{n}\mu(X)\ \forall n$
$\Rightarrow \nu(E) = 0$. If $\mu(E^c) = 0 \Rightarrow \nu \perp \mu \Rightarrow$ i). If not, $\mu(E^c) > 0$ so $\mu(F) = \mu(\bigcup_{n=1}^{\infty} F_n) > 0$.
$\Rightarrow \mu(F_n) \geq 0$ ... rest of proof in book.

- R.N. Thm. - Let μ be σ-finite measure on $(X, \mathcal{A})$ & ν be a finite measure on $(X, \mathcal{A}) \ni \nu \ll \mu$.
$\Rightarrow \exists$ a μ-integrable non-negative function f which is $\mathcal{A}$-meas. $\ni \nu(A) = \int_A f\, d\mu\ \forall A \in \mathcal{A}$. This is unique up to a set of μ-measure 0.

- Rmk on Thm: f is called the Radon-Nikodym derivative of γ wrt. μ (density). We write $f = \frac{d\gamma}{d\mu} \Rightarrow d\gamma = f\, d\mu$.

proof of the uniqueness:

Assume there are two such functions f & g $\ni$

$\gamma(A) = \int_A f\, d\mu = \int_A g\, d\mu \quad \forall A \in \mathcal{A}$.

$\Rightarrow 0 = \int_A (f-g)\, d\mu \quad \forall A \in \mathcal{A}. \Rightarrow f-g=0$ a.e. wrt. μ.

proof of existence:

Step 1) Assume μ is finite. We construct f below:

Define $\mathcal{E} = \{ g : \begin{matrix}\text{non-negative}\\ \text{measurable}\end{matrix} \mid \begin{matrix}\int_A g\, d\mu \le \mu(A) \\ \forall A \in \mathcal{A}\end{matrix} \}$. Notice $\mathcal{E} \ni 0 \Rightarrow \mathcal{E} \neq \emptyset$. Set $L = \sup_{g \in \mathcal{E}} \int_X g\, d\mu$, so that $L \le \gamma(X) < \infty$. So the sup. is finite.

Take a sequence $g_n \in \mathcal{E} \ni \int g_n\, d\mu \to L$ & we set $h_n = \max(g_1, g_2, \dots, g_n)$. To show $h_n \in \mathcal{E}$, we first consider $h_2 = \max(g_1, g_2)$. Set $B = \{x \in X \mid g_1(x) \ge g_2(x)\}$

Note $B \in \mathcal{A}$.

we write $\int_A h_2\, d\mu = \int_{A \cap B} h_2\, d\mu + \int_{A \cap B^c} h_2\, d\mu$.

$= \int_{A\cap B} g_1\, d\mu + \int_{A \cap B^c} g_2\, d\mu \le \gamma(A \cap B) + \gamma(A \cap B^c)$

$= \gamma(A) \quad \forall A \in \mathcal{A} \Rightarrow h_2 \in \mathcal{E}$. Then we use induction to prove $h_n \in \mathcal{E}$. Note that $h_n \uparrow \lim_{n\to\infty} h_n = f$. By the monotone convergent thm applied to

$\int_A h_n\, d\mu \le \gamma(A) \Rightarrow \int f\, d\mu \le \gamma(A)\ \forall A \in \mathcal{A}$.

Proof cont. here

So $f \in \mathcal{E}$.

Notice as well that $\int_X f\, d\mu \ge \int_X h_n\, d\mu \ge \int_X g_n\, d\mu \quad \forall n$

$\Rightarrow \int_X f\, d\mu \ge L$, But $f \in \mathcal{E}$ so $\Rightarrow \int_X f\, d\mu = L$.

Step 2) WTS $\int_A f\,d\mu = \nu(A)\ \forall A \in \mathcal{A}$. Define λ by $\lambda(A) = \nu(A) - \int_A f\,d\mu\ \forall A \in \mathcal{A}$. Is λ a measure? Yes, b/c $\nu(A) \geq \int_A f\,d\mu\ \forall A$ by Step (1).

To show $\lambda \perp \mu$, let's suppose not (by way of contr.).

Then by a lemma, $\exists\ \varepsilon > 0$ & $G \in \mathcal{A} \ni \mu(G) > 0$ & G is a positive set for $\lambda - \varepsilon\mu$ b/c λ & μ are finite.

For a set $A \in \mathcal{A}$, $\nu(A) - \int_A f\,d\mu = \lambda(A) \geq \lambda(A \cap G)$

$\geq \varepsilon\mu(A \cap G) = \int_A \varepsilon\chi_G\,d\mu$

$\Rightarrow \nu(A) \geq \int_A f + \varepsilon\chi_G\,d\mu\ \ \forall A \in \mathcal{A}$.

$\Rightarrow f + \varepsilon\chi_G \in \tilde{E}$, but $\int_X (f + \varepsilon\chi_G)\,d\mu$

$= \int_X f\,d\mu + \varepsilon\mu(G) = L + c$ where $c > 0$,

So $\int_X f\,d\mu + \varepsilon\mu(G) > L$, but this is a contradiction to the def. of L, Thus $\lambda \perp \mu$.

$\Rightarrow \exists\ H \in \mathcal{A} \ni \mu(H) = 0$ & $\lambda(H^c) = 0$. Then $\nu \ll \mu \Rightarrow \nu(H) = 0$. Hence $\lambda(H) = \nu(H) - \int_H f\,d\mu = 0$.

$\Rightarrow \lambda = 0 \Rightarrow \nu(A) = \int_A f\,d\mu\ \ \forall A \in \mathcal{A}$.

Step 3) Assume μ is σ-finite. $\Rightarrow \exists\ F_i \uparrow X$ where $\mu(F_i) < \infty\ \forall i$. Set $\mu_i = \mu|_{F_i}$ by $\mu_i(A) = \mu(A \cap F_i)\ \forall A \in \mathcal{A}$ & $\nu_i = \nu|_{F_i}$ by $\nu_i(A) = \nu(A \cap F_i)\ \forall A \in \mathcal{A}$.

$\Rightarrow \nu_i \ll \mu_i$ b/c $\mu_i(A) = 0 \Rightarrow \mu(A \cap F_i) = 0 \Rightarrow \nu(A \cap F_i) = 0$

$\Rightarrow \nu_i(A) = 0$. By the result in Step 2, $\exists\ f_i \ni$ $d\nu_i = f_i\,d\mu_i$ ($\Leftrightarrow \nu_i(A) = \int_A f_i\,d\mu_i\ \forall A \in \mathcal{A}$). Let $j \geq i$, then $\mu_j|_{F_i} = \mu_i$ by definition. $\Rightarrow f_j|_{F_i} = f_i$ by the uniqueness of f a.e. Define $f(x) = f_i(x)$ where $x \in F_i$. $\forall A \in \mathcal{A}$, $\nu(A \cap F_i) = \nu_i(A) = \int_A f_i\,d\mu_i = \int_{A \cap F_i} f\,d\mu\ \ \forall i$. Letting $i \to \infty$, we find that $\nu(A) = \int_A f\,d\mu$.

QED.

• **Lebesgue decomposition Thm:** Let μ be σ-finite & ν be finite on $(X, \mathcal{A})$. $\Rightarrow \exists$ measures λ & $\rho \ni \nu = \lambda + \rho$, $\rho \ll \mu$, & $\lambda \perp \mu$.

Proof:

Consider μ to be finite. Let $\mathcal{F} = \{ g$ that are non-negative measurable $\mid \int_A g \, d\mu \le \nu(A) \ \forall A \in \mathcal{A} \}$,

$\infty > L = \sup_{g \in \mathcal{F}} \int_X g \, d\mu \mid g_i \in \mathcal{F} \ni \int_X g_i \, d\mu \to L$, $h_i = \max_{1 \le j \le i} (g_j)$, $f = \lim_{i \to \infty} h_i$;

&

$\int_A f \, d\mu \le \nu(A) \ \forall A \in \mathcal{A}$. Define a measure ρ by

$\rho(A) = \int_A f \, d\mu \ \forall A \in \mathcal{A}$, Moreover $\mu(A) = 0 \Rightarrow \rho(A) = 0$

$\Rightarrow \rho \ll \mu$. Define $\lambda = \nu - \rho$ (so that $\nu = \lambda + \rho$), so λ is a measure.

Thus $\nu = \lambda + \rho$ & $\rho \ll \mu$. So we only need to show $\lambda \perp \mu$.

Suppose not, $\Rightarrow \exists \ \varepsilon > 0$ & $F \in \mathcal{A} \ni \mu(F) > 0$ & F is a positive set for $\lambda - \varepsilon\mu$ b/c λ & μ are finite.

Now $\nu(A) - \int_A f \, d\mu = (\nu - \rho)(A) = \lambda(A) \ge \lambda(A \cap F)$

$\ge \varepsilon \mu(A \cap F) = \int_A \varepsilon \chi_F \, d\mu \Rightarrow \nu(A) \ge \int_A (f + \varepsilon \chi_F) \, d\mu \ \forall A \in \mathcal{A}$,

$\Rightarrow f + \varepsilon \chi_F \in \mathcal{F}$. But $\int_X (f + \varepsilon \chi_F) \, d\mu = L + \varepsilon \mu(F) > L$.

Thus we have a contradiction by the definition of L, so $\lambda \perp \mu$.

If μ is σ-finite, we can argue the same thing using F_i's (like last proof) QED.

Homework
Ch. 12 (4)
Ch 13 (6, 8, 9, 12)

→ tip! $\frac{d\mu}{d\rho} = f \iff d\mu = f\, d\rho$

due 2/16

Ch 14 Differentiation

- For $f: \mathbb{R} \to \mathbb{R}$, we'll show:
(1) The derivative of $\int_a^x f(y)\,dy = f$ a.e. if f is integrable.
(2) Functions of bounded variation are differentiable
(3) $\int_a^b f'(y)\,dy = f(b) - f(a)$ if f is absolutely continuous.

Notation. • Let f be a function from $\mathbb{R}^n \to \mathbb{R}$. Let $B(x,r)$ be the open ball centered at x w/ radius r. Let m be a Lebesgue measure on $\mathbb{R}^n$.

- Covering Lemma: $E \subset \mathbb{R}^n$ is covered by a collection of balls $\{B_\alpha\}_{\alpha \in I}$ & $\exists\, R > 0 \ni$ diameter of $B_\alpha \le R\ \forall \alpha$.
$\Rightarrow \exists$ a disjoint sequence $B_1, B_2, \ldots$ of $\{B_\alpha\}_{\alpha \in I} \ni$
$m(E) \le 5^n \sum_{k=1}^{\infty} m(B_k)$.

Proof:
Define $d(B_\alpha)$ = diameter of B_α $\forall \alpha$. Choose $B_1 \ni$
$d(B_1) \ge \frac{1}{2} \sup d(B_\alpha)$. Now given $B_1, \ldots, B_k$, we inductively choose B_{k+1}, which is disjoint from $B_1, \ldots, B_k \ni$
☆ $d(B_{k+1}) \ge \frac{1}{2} \sup \{ d(B_\alpha) \mid B_\alpha \text{ disjoint from } B_1, \ldots, B_k \}$. We stop if $\nexists$ such B_{k+1}. Otherwise we continue to construct B_k, $k = 1, 2, \ldots$ If $\sum m(B_k) = \infty$, we're done. If not, $\sum_{k=1}^{\infty} m(B_k) < \infty$, we have
means ball gets smaller as $k \to \infty$ → $m(B_k) \to 0$. Define B_k^* as the ball centered at the center of B_k but $d(B_k^*) = 5\, d(B_k)$. Now we WTS
$E \subset \bigcup_{k=1}^{\infty} B_k^*$. It's enough to show that $B_\alpha \subset \bigcup_{k=1}^{\infty} B_k^*$ $\forall \alpha$, b/c $\{B_\alpha\}$ is a cover of E.

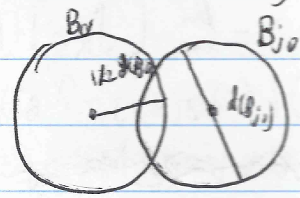
cont.

Fix $\alpha \in I$. If $B_\alpha = B_k$ for some k, then we're done ~~b/c same region~~ b/c B_α would be in the union.
If not, $B_\alpha \neq B_k \ \forall k$. Since $m(B_k) \to 0$, choose the smallest $k \ni d(B_{k+1}) < \frac{1}{2} d(B_\alpha)$. $\Rightarrow$ B_α must intersect at least one of $B_1, \dots, B_k$. $\Rightarrow B_\alpha \cap B_{j_0} \neq \emptyset$ for some $j_0 \le k$.

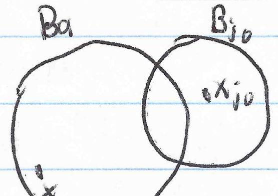
Proof cont. We know that $\frac{1}{2} d(B_\alpha) \le d(B_{j_0})$
$\boxed{\text{Choose the smallest } j_0 \ni}$. This inequality follows from our definition ☆. Now we observe $B_\alpha \subset B_{j_0}^*$.

Explanation: $d(B_{j_0}^*) = 5(B_{j_0})$

$\frac{1}{2} d(B_\alpha) \le d(B_{j_0})$



In any case, $5d(B_{j_0})$ completely overtakes whatever B_α is. With a better drawing, we could understand better.



Setting x_{j_0} to be the center of B_{j_0}, & x to be any point in B_α, we write
$|x_{j_0} - x| \le |x_{j_0} - y| + |y - x|$ by triangle ineq.
$\forall \ y \in B_\alpha \cap B_{j_0}$

$$|x_{j_0} - y| + |y - x| \le \frac{1}{2} d(B_{j_0}) + d(B_\alpha) \le \left(\frac{1}{2} + 2\right) d(B_{j_0}) = \frac{5}{2} d(B_{j_0})$$

$$\Rightarrow x \in B_{j_0}^* \Rightarrow B_\alpha \subset B_{j_0}^* \Rightarrow B_\alpha \subset \bigcup_{k=1}^{\infty} B_k^* \Rightarrow E \subset \bigcup_{k=1}^{\infty} B_k^*.$$

So,

$$m(E) \le m\left(\bigcup_k B_k^*\right) \le \sum_{k=1}^{\infty} m(B_k^*) = 5^n \sum_{k=1}^{\infty} m(B_k)$$

so we're done. QED

- **Def:** f is <u>locally integrable</u> when $f:\mathbb{R}^n \to \mathbb{R}$ if $\int_K |f| < \infty$ for every compact set K.
- **Def:** For a locally integrable function $f:\mathbb{R}^n \to \mathbb{R}$, we ~~defined~~ define the <u>maximal function</u>, MF, of f by
$$M f(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy.$$

– Q: Is MF measurable? Yes

We need to show that $\{x \mid MF(x) > a\}$ is open $\forall a$. This would prove Borel measurable $\Rightarrow$ Lebesgue measurable. First we show $F(x) = \int_{B(x,r)} |f(y)|\, dy$ is continuous $\forall r$. It suffices to show that $0 = \lim_{x \to x_0} F(x) - F(x_0) = \lim_{x\to x_0} \left(\int_{\mathbb{R}^n} |f(y)| (\chi_{B(x,r)} - \chi_{B(x_0,r)})\, dy \right)$ by DCT, $= \int_{\mathbb{R}^n} \lim_{x \to x_0} |f(y)| (\chi_{B(x,r)} - \chi_{B(x_0,r)})\, dy = 0.$

$\Rightarrow F(x)$ is continuous $\Rightarrow$ (*) $\{x \mid \int_{B(x,r)} |f(y)|\, dy > A\}$ is open $\forall A$. Now we notice that $Mf(x) > a$ iff $\exists r > 0 \ni \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy > a \iff \exists r > 0 \ni \int_{B(x,r)} f(y)\, dy > a\, m(B(x,r))$.
Hence we write $\{x \mid Mf(x) > a\} = \bigcup_{r>0} \underbrace{\{x \mid \int_{B(x,r)} |f(y)|\, dy > a\, m(B(x,r))\}}_{\text{Open by } (*)}$

$\Rightarrow MF$ is continuous. $\Rightarrow$ Borel measurable $\Rightarrow$ Lebesgue measurable.

- Q: f is an integrable function. Does this $\Rightarrow$ Mf is integrable? No

Let, $Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy$. Take $f = \chi_{B(0,\frac{1}{2})}$.

~~[illegible]~~ • Thm: Suppose f is integrable. $\Rightarrow m(\{x \mid Mf(x) > \beta\}) \leq \frac{5^n}{\beta} \int |f(x)|\, dx \;\; \forall \beta > 0$.

Proof: Define $E_\beta = \{x \mid Mf(x) > \beta\}$ for a fixed $\beta > 0$. If $x \in E_\beta \Rightarrow \exists\, B_x$ where B_x is a ball centered at $x \ni$ ⊛
$\frac{1}{m(B_x)} \int_{B_x} |f(y)|\, dy > \beta$. $\Rightarrow m(B_x) < \frac{1}{\beta} \int_{B_x} |f(y)|\, dy \leq \frac{1}{\beta} \int_x |f(y)|\, dy$.
So $\{B_x\}_{x \in E_\beta}$ is a cover of E_β & ~~[illegible]~~
$d(B_x) \leq c\left(\frac{1}{\beta} \int_x |f(y)|\, dy\right)^{1/n}$ independent of x, for some constant c. By the covering Lemma, $\exists$ disjoint sequence B_k where $k = 1, \dots,$ of $\{B_x\}_{x \in E_\beta} \ni m(E_\beta) \leq 5^n \sum_{k=1}^{\infty} m(B_k)$.

$$\leq 5^n \sum_{k=1} \frac{1}{\beta} \int_{B_k} |f(y)|\, dy \text{ by ⊛ } = \frac{5^n}{\beta} \int_{\bigcup_{k=1} B_k} |f(y)|\, dy$$
$$\leq \frac{5^n}{\beta} \int |f(y)|\, dy. \qquad \text{QED}$$

he likes this proof, pray it's not on the test.

• Thm: Let $f_r(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\, dy$. If f is locally integrable, then $f_r(x) \to f(x)$ a.e. as $r \to 0$.

Proof:

To show $f_r \to f$ a.e. x, It suffices to show that $f_r \to f$ a.e. $x \in B(0,N)$ for a sufficiently large N. WLOG, assume $f = 0$ on $B^c(0, 2N)$. Hence f is integrable.

Since f is integrable, $\exists$ a compactly supported continuous function $g \ni \int |f - g| < \varepsilon$ for any fixed $\varepsilon > 0$. Define ~~[illegible]~~
$g_r(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} g(y)\, dy$. Then we notice that
$|g_r(x) - g(x)| \leq \frac{1}{m(B(x,r))} \int_{B(x,r)} |g(y) - g(x)|\, dy$.

cont. on next page.

Proof: cont.

$\forall \varepsilon > 0, \exists \delta > 0 \ni |g(y) - g(x)| < \varepsilon$ if $|y - x| < \delta$ for a fixed x.

Take $r \approx \delta$. So $\frac{1}{m(B(x,r))} \int_{B(x,r)} |g(y) - g(x)| \, dy \le \frac{1}{m(B(x,r))} \int_{B(x,r)} \varepsilon \, dy$

$= \varepsilon$ So $g_r(x) \to g(x)$ as $r \to 0 \; \forall x$.

Now we write $\limsup_{r \to 0} |f_r(x) - f(x)| \le \limsup_{r \to 0} |f_r(x) - g_r(x)|$

$+ \limsup_{r \to 0} |g_r(x) - g(x)| + \limsup_{r \to 0} |g(x) - f(x)|$. (B)

(A) $\{x \mid \limsup_{r \to 0} |f_r(x) - f(x)| > \beta\} \subset \{x \mid \limsup_{r \to 0} |f_r(x) - g_r(x)| + |g(x) - f(x)| > \beta\}$

$\subset \{x \mid \limsup_{r \to 0} |f_r(x) - g_r(x)| > \frac{\beta}{2}\} \cup \{x \mid |g(x) - f(x)| > \frac{\beta}{2}\}$ (C).

Note as well $\{x \mid \limsup_{r \to 0} |f_r - g_r| \le \frac{\beta}{2}\} \cap \{x \mid |g - f| \le \frac{\beta}{2}\} \subset$

$\{x \mid \limsup_{r \to 0} |f_r - g_r| + |g - f| \le \beta\}$. So $m(A) \le m(B) \le m(C)$.

So $|f_r| \le$ $\le \frac{1}{m(B)} \int_B |f| \le Mf = \sup \frac{1}{m(B)} \int |f| \; \forall r$.

$|f_r - g_r| \le m(f - g) \; \forall r > 0$

$\le m(\{x \mid M(f-g)(x) > \frac{\beta}{2}\}) + m(\{x \mid |f - g| > \frac{\beta}{2}\})$.

$\le 2 \frac{5^n}{\beta} \int |f - g| + 2 \frac{\int |f-g|}{\beta}$ by Chebyshev's inequality

(*) $\le 2 \frac{(5^n + 1)}{\beta} \varepsilon \implies m(\{x \mid \limsup_{r \to 0} |f_r(x) - f(x)| > \beta\}) = 0, \; \forall \beta > 0$.

well $m(\{x \mid \limsup_{r \to 0} |f_r(x) - f(x)| > 0\}) = m(\bigcup_{j \ge 0} \{x \mid \limsup_{r \to 0} |f_r(x) - f(x)| > \frac{1}{j}\})$

$\le \sum_{j=1}^{\infty} m(\{x \mid \limsup_{r \to 0} |f_r(x) - f(x)| > \frac{1}{j}\}) = 0$ by (*)

$\implies f_r(x) \to f(x)$ a.e. as $r \to 0$.

QED

Ch 11 – Thm 14.2 on p. 120

End of Exam 1

Assume f is locally integrable.

• Thm: $\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)-f(x)|\,dy \to 0$ as $r\to 0$ a.e. in X.

proof: $\forall c\in\mathbb{Q}$, $\exists$ a set N_c of measure 0 $\ni$
$\frac{1}{m(B(x,r))}\int|f(y)-c|\,dy \to |f(x)-c|$ $\forall x\notin N_c$ by our previous thm w/ f replaced by $|f-c|$.

Set $N=\bigcup_{c\in\mathbb{Q}} N_c$ & $m(N)=0$. Now suppose $\forall x\notin N$ & $\forall \varepsilon>0$, we can choose $c\in\mathbb{Q}\ni |f(x)-c|<\varepsilon$. Then,

$\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)-f(x)|\,dy \le \frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)-c|\,dy + |f(x)-c|$

take lim sup... So $\limsup_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)-f(x)|\,dy$

$\le 2|f(x)-c| < 2\varepsilon$ $\forall \varepsilon>0$ $\Rightarrow$ our thm. QED

– Antiderivatives

Let. $f:\mathbb{R}\to\mathbb{R}$, $B(x,h)=(x-h,x+h)$ & m be 1-d Lebesgue measure.
For an integrable function f, define the antiderivative F of f by
$F(x)=\int_a^x f(t)\,dt$ where $a\in\mathbb{R}$. Note F is continuous.

• Thm: Let $f:\mathbb{R}\to\mathbb{R}$ to be integrable, & $a\in\mathbb{R}$. Define $F(x)=\int_a^x f(y)\,dy$.
F is differentiable & $F'=f$ a.e.

Proof: WTS $\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=f(x)$ a.e. in x. It is sufficient to show $\lim_{h\to 0}\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|=0$.

For $h>0$, $\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|=\left|\frac{1}{h}\int_x^{x+h}(f(y)-f(x))\,dy\right|$

$\le\frac{1}{h}\int_{x-h}^{x+h}|f(y)-f(x)|\,dy=\frac{2}{2h}\int_{x-h}^{x+h}|f(y)-f(x)|\,dy=\frac{2}{m(B(x,h))}\int_{B(x,h)}|f(y)-f(x)|\,dy$

$\to 0$ a.e. as $h\to 0$, w/ similar proof, $h<0$.

QED.

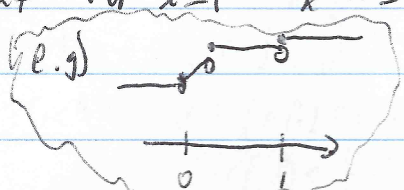
~~Prop: If $f: \mathbb{R} \to \mathbb{R}$ is increasing & right continuous, then f is differentiable a.e.~~

Never mind I guess? This class is trash.

Lemma: $H: \mathbb{R} \to \mathbb{R}$ where H is increasing & right continuous & constant for $x \geq 1$ & $x \leq 0$.

for this proof, recall def. of L-Stieltjes measure.

(e.g.)



Let λ be the Lebesgue Stieltjes measure defined w/ H. Then, if $\lambda \perp m$, then

$$\lim_{r \to 0} \frac{\lambda(B(x,r))}{m(B(x,r))} = 0 \quad m\text{- a.e. } x.$$

Proof:

$\forall\, r > 0$, $\lambda(B(x,r))$ is Borel-meas. function in x. He won't prove this though b/c he probably doesn't know how. Set $r_j = \frac{1}{2^j}$. If $r_{j+1} \leq r < r_j$, then

$$\frac{\lambda(B(x,r))}{m(B(x,r))} \leq \frac{\lambda(B(x,r_j))}{m(B(x,r_{j+1}))} \leq \frac{1}{2} r_j = r_{j+1}$$

$$\hookrightarrow = \frac{\lambda(B(x,r_j))}{m(B(x,\frac{1}{2}r_j))} = \frac{\lambda(B(x,r_j))}{\frac{1}{2} m(B(x,r_j))}.$$

Hence it is sufficient to show

$$(*) \quad \lim_{j \to \infty} \frac{\lambda(B(x,r_j))}{m(B(x,r_j))} = 0 \quad \text{a.e. on } X.$$

$\lambda \perp m \Rightarrow \exists\ E\ \&\ F = E^c \ni \lambda(F) = 0,\ m(E) = 0$. Fix $\varepsilon > 0$.

We first want to find a bounded open set $G \ni$ $F \cap [0,1] \subset G$ & $\lambda(G) < \varepsilon$.

For F, $\exists$ open set $G' \ni G' \supset F$ & $\lambda(G' \setminus F) < \varepsilon$. We can do this b/c $\lambda(\mathbb{R}) < \infty$. Then $\lambda(G' \setminus F) + \lambda(F) = \lambda(G')$. but $\lambda(F) = 0$. $\lambda(G' \setminus F) < \varepsilon \Rightarrow \lambda(G') < \varepsilon$. Take $G = G' \cap (-1,2)$.

Then i) $F \cap [0,1] \subset G' \cap (-1,2) = G$

ii) $\lambda(G) = \lambda(G' \cap (-1,2)) = \lambda(G') < \varepsilon$.

cont.

Lemma proof cont.

Define $A_\beta = \{x \in F \cap [0,1] \mid \limsup_{j\to\infty} \frac{\lambda(B(x,r_j))}{m(B(x,r_j))} > \beta\}$ $\forall$ fixed $\beta > 0$ & show below that $m(A_\beta) = 0$.

$x \in A_\beta \Rightarrow x \in F \cap [0,1] \subset G$, & $\exists$ $B_x = B(x, r_j)$ for some j $\ni$ $\frac{\lambda(B_x)}{m(B_x)} > \beta$ as well as $B_x \subset G$. Then

$A_\beta \subset \bigcup_{x \in A_\beta} B_x$, $m(B_x) \le m(G) < \infty$, so by the covering lemma, disjoint $\{B^i\}_{i=1}^\infty \subset \{B_x\}_{x\in A_\beta}$ $\ni$

$$m(A_\beta) \le 5 \sum_{i=1}^\infty m(B_i) < \frac{5}{\beta} \sum_{i=1}^\infty \lambda(B_i) = \frac{5}{\beta} \lambda\left(\bigcup_{i=1}^\infty B^i\right)$$
$$\le \frac{5}{\beta}\lambda(G) < \frac{5}{\beta}\varepsilon \Rightarrow m(A_\beta) = 0 \quad \forall \beta > 0.$$

Then $m\left(\{x \mid \limsup_{j\to\infty} \frac{\lambda(B(x,r_j))}{m(B(x,r_j))} > 0\}\right)$

$$= m\left(\bigcup_{N=1}^\infty \{x \mid \limsup_{j\to\infty} \frac{\lambda(B(x,r_j))}{m(B(x,r_j))} > \frac{1}{N}\}\right) = \lim_{N\to\infty} m\left(\{x \mid \limsup_{j\to\infty} \frac{\lambda(B(x,r_j))}{m(B(x,r_j))} > \frac{1}{N}\}\right)$$

$= 0$. QED.

• __Proposition:__ $F: \mathbb{R} \to \mathbb{R}$ is an increasing & right continuous function, then F' exists a.e. & $\int_a^b F'(x)\,dx \le F(b) - F(a)$ $\forall a, b \in \mathbb{R}$ (i.e. F is locally integrable).

__Proof:__

It is sufficient to show that F' $\exists$ a.e. on $[0,1]$.

Why? i) Same proof works on $[-N, N]$, $\forall N$

ii) To show differentiability at x, we can consider F on $[-N, N]$ including x.

↳ so we will show this.

rest of proof on next page.

Proof cont.

Re defined $F(x) = \begin{cases} \lim_{y \to 0^+} F(y), & x \leq 0 \\ F(x) & 0 < x \leq 1 \\ F(1) & x > 1 \end{cases}$ $\Rightarrow$ i) F is right cont. & increasing ii) The differentiability of this modified F is the same as that of the original F on $(0,1]$.

Define γ as the Lebesgue Stieltjes measure from F. By the Lebesgue decomposition, $\gamma = \lambda + \rho \ni \lambda \perp m$ & $\rho \ll m$. Note that $\rho([0,1]) \leq \gamma([0,1]) = F(1) - F(0) \geq 0$. By R-N thm, $\exists$ nonnegative integrable function $f \ni \rho(A) = \int_A f\,dm$ $\forall$ measurable A. Define $H(x) = \lambda([0,x]) = \gamma((0,x]) - \rho((0,x]) = F(x) - F(0) - \int_0^x f(y)\,m(dy)$.

$\Rightarrow$ i) H is right continuous & increasing

ii) H is constant $x \leq 0$, $x \geq 1$

iii) $\lambda \perp m$, iv) λ is the L-S measure defined by H.

Then $\lim_{r \to 0} \frac{\lambda(B(x,r))}{m(B(x,r))} = 0$ a.e. x by the previous lemma. $B(x,r) = (x-r, x+r)$

$\Rightarrow \lim_{r \to 0} \frac{H(x+r) - H(x-r)}{2r} = 0$. Then we find that

$\limsup_{h \to 0^+} \frac{H(x+h) - H(x)}{h} \leq \lim_{h \to 0^+} \frac{H(x+h) - H(x-h)}{h} = 0$

$\Rightarrow \lim_{h \to 0^+} \frac{H(x+h) - H(x)}{h} = 0$ a.e. x. The proof for the left hand derivative is similar. Then we can deduce that H' exists & $H'(x) = 0$ a.e.

We write $F(x) = -H(x) + F(0) + \int_0^x f(y)\,dy$. By the fund. thm of calculus, $F'(x) = 0 + 0 + f(x)$ a.e. x $\Rightarrow F'(x) = f(x)$ a.e. x $\Rightarrow F'$ exists a.e.

Moreover $\int_a^b F'(x)\,m(dx) = \int_a^b f(x)\,dx = \rho((a,b]) \leq \gamma((a,b]) = F(b) - F(a) \Rightarrow \int_a^b F'(x)\,m(dx) \leq \gamma((a,b]) = F(b) - F(a)$. $\forall a < b$.

QED.

Proposition: $F: \mathbb{R} \to \mathbb{R}$ is an increasing function.
$\Rightarrow F'$ exists a.e. & $\int_a^b F'(x)\,dx \le F(b) - F(a)$ $\forall a<b$.

Proof: Define $G(x) = \lim_{y \to x^+} F(y)$. Then:
i) $G(x)$ is defined $\forall x \in \mathbb{R}$.
ii) $G = F$ a.e. b/c F has at most countably many discontinuities
iii) G is increasing & right continuous.
$\Rightarrow G'$ exists a.e. We will show that $G'(x)$ exists &
$G(x) = F(x) \Rightarrow F'(x) = G'(x)$. $\Rightarrow F' = G'$ a.e.
Fix x where $G'(x)$ exists & $G(x) = F(x)$. Fix $\varepsilon > 0$.
$\forall h > 0$, $\exists x_h \in (x+h, x+h+\varepsilon h) \ni F(x_h) = G(x_h)$.
Then $F(x+h) \le F(x_h) = G(x_h) \le G(x + (1+\varepsilon)h)$ b/c F & G are increasing. $\frac{F(x+h)-F(x)}{h} \le \frac{G(x+(1+\varepsilon)h) - G(x)}{h}$
$\Rightarrow \limsup_{h \to 0^+} \frac{F(x+h)-F(x)}{h} \le \limsup_{h\to 0^+} \frac{G(x+(1+\varepsilon)h)-G(x)}{h}$
$= (1+\varepsilon) \limsup_{h \to 0^+} \frac{G(x+(1+\varepsilon)h) - G(x)}{(1+\varepsilon)h} = (1+\varepsilon) G'(x)$. Similarly one can show that $\liminf_{h\to 0^+} \frac{F(x+h)-F(x)}{h} \ge (1-\varepsilon) G'(x)$ w/ a symmetric proof. Thus F' exists a.e.

Now wts. that $F' = G'$ a.e. (F' is locally integrable).
Fix $a \le b$. Chose $a_n \downarrow a$ & $b_n \uparrow b \ni F(a_n) = G(a_n)$ $\forall n$
& $F(b_n) = G(b_n)$ $\forall n$ b/c $F = G$ a.e.
$\Rightarrow F(b) - F(a) \ge F(b_n) - F(a_n) = G(b_n) - G(a_n)$
$\ge \int_{a_n}^{b_n} G'(x)\,dx$ by previous prop.
$= \int_{a_n}^{b_n} F'(x)\,dx$. $\Rightarrow F(b) - F(a) \ge \int_{a_n}^{b_n} F'(x)\,dx$
$= \int F'(x) \chi_{(a_n, b_n)}\,dx$
i) $F'(x)\chi_{(a_n,b_n)} \to F'(x)\chi_{(a,b)}$ as $n \to \infty$
ii) $|F'(x)\chi_{(a_n,b_n)}| \le |F'(x)|\chi_{(a,b)}$ & is integrable, by the DCT, $F(b) - F(a) \ge \int F'(x)\chi_{(a,b)}\,dx = \int_a^b F'(x)\,dx$.

QED

• **Def:** A function f is of <u>bounded variation</u> on $[a,b]$ if $V_f[a,b] = \sup\{\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|\} < \infty$ where sup is taken over all partitions $a = x_0 < x_1 < x_2 < \dots < x_k = b$ of $[a,b]$.

1) If f is increasing on $[a,b]$, $\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{k} (f(x_i) - f(x_{i-1}))$ $= f(b) - f(a) < \infty$.

2) f & g are of bounded variation on $[a,b]$, $f + cg$ is of bounded variation on $[a,b]$.

• **Def:** f is Lipschitz continuous if $\exists\ c_1 > 0 \ni |f(y) - f(x)| \le c_1 |y - x| \ \forall x, y$.

3) f is Lipschitz continuous, $\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{k} c_1 |x_i - x_{i-1}|$ $= c_1(b - a) < \infty$.

• **Lemma:** If f is a bounded variation on $[a,b]$, Then

(*) $f = f_1 - f_2$ where f_i is increasing on $[a,b]$ for $i = 1, 2$. (hence f is differentiable).

proof:

Define $\tilde{f}_1(y) = \sup\{\sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+\}$ &

$\tilde{f}_2(y) = \sup\{\sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-\}$ where the sup is taken over all ~~partitioning~~ partitions of $[a,y]$. ($x_0 = a < x_1 < x_2 < \dots < x_k = y$, $y \in [a,b]$)

$y_1 < y_2 \Rightarrow \tilde{f}_1(y_1) \le \tilde{f}_1(y_2)$. Hence $\tilde{f}_1$ & $\tilde{f}_2$ are increasing functions & measurable. $f(y) - f(a) = \sum_{i=1}^{k} f(x_i) - f(x_{i-1}) = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+$ $- \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^- \Rightarrow \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+ = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^- + f(y) - f(a)$.

Taking the sup, $\tilde{f}_1(y) = \tilde{f}_2(y) + f(y) - f(a) \Rightarrow f(y) = \tilde{f}_1(y) + f(a) - \tilde{f}_2(y)$

Therefore (*) holds true. QED.

Lemma: If f is of bounded variation on $[a,b]$ (so by previous lemma, then $f = f_1 - f_2$, & f_i is increasing for $i=1,2$) where $[c,d] \subset [a,b]$, then $f_i(d) - f_i(c) \leq V_f[c,d]$ for $i = 1,2$.

Proof:

Let $P = \{x_0 = a < x_1 < \ldots < x_n = d\}$ be a partition of $[a,d]$. Define $P_0 = P$ w/ c in it ~~as an endpoint~~. Let $P' =$ subset of P_0 w/ the points less than or equal to c. Let $P'' =$ subset of P_0 w/ the points greater than or equal to c. $\sum_{P} [f(x_i) - f(x_{i-1})]^+$
$\leq \sum_{P_0} [f(x_i) - f(x_{i-1})]^+ = \sum_{P'} [f(x_i) - f(x_{i-1})]^+ + \sum_{P''} [f(x_i) - f(x_{i-1})]^+$.
On the other hand, b/c $[g+h]^+ \leq [g]^+ + [h]^+$, $[f(x_i) - f(x_{i-1})]^+$
$= [f_1(x_i) - f_1(x_{i-1}) + (-\frac{f_2(x_i)}{f_2(x_{i-1})})]^+ \leq [f_1(x_i) - f_1(x_{i-1})]^+ + [f_2(x_i) + f_2(x_{i-1})]^+$
$= f_1(x_i) - f_1(x_{i-1}) + 0$.
Then $\sum_{P} [f(x_i) - f(x_{i-1})]^+ \leq \sum_{P'} (f_1(x_i) - f_1(x_{i-1})) + \sum_{P''} |f(x_i) - f(x_{i-1})|$
$\leq f_1(c) - f_1(a) + V_f[c,d]$.
$\Rightarrow \sum_{P} [f(x_i - f(x_{i-1})] + f_1(a) \leq f_1(c) + V_f[c,d]$. Take sup over all such P's. So, $\tilde{f}_1(d) + f_1(a) \leq (c) + V_f[c,d]$ &
$f_1(d) = \tilde{f}_1(d) + f_1(a)$. $\Rightarrow f_1(d) - f_1(c) \leq V_f[c,d]$. Proof for f_2 is similar, so we are done. QED.

Remark: If we write f of bounded variation on $[a,b]$ as $f = f_1 - f_2$ where f_1 & f_2 are increasing. Then the total variation of f on $[a,b]$ is defined by $f_1(b) + f_2(b) - (f_1(a) + f_2(a))$.

• Recall: f is absolutely continuous if $\forall \varepsilon > 0$, $\exists \delta > 0 \ni \{(a_i, b_i)\}_{i=1}^{k}$ is a finite collection of disjoint intervals w/ $\sum_{i=1}^{k} |b_i - a_i| < \delta$ $\Rightarrow \sum_{i=1}^{k} |f(b_i) - f(a_i)| < \varepsilon$.

• Lemma: Let f be an absolutely continuous function on $[a,b]$. Then ① f is of bounded variation on $[a,b]$. ($\Rightarrow f$ is differentiable a.e. & $\int_a^b f'(x)\,dx \leq f(b) - f(a)$.) Moreover $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Proof:

For $\varepsilon > 0$, $\exists \delta > 0 \ni \sum_{i=1}^{k} |f(b_i) - f(a_i)| < 1$ wherever $\{(a_i, b_i)\}_{i=1}^{k}$ is a set of finitely many disjoint intervals w/ $\sum_{i=1}^{k} |b_i - a_i| < \delta$. $\forall$ fixed j, then $V_f[a + j\delta, a + (1+j)\delta] \leq 1$. $\Rightarrow V_f[a,b] < \infty$. $\Rightarrow f$ is a function of bounded variation. QED

Thus f is diff. a.e. by previous lemma.

• Lemma: If f is a function of bounded variation, we can write $f = f_1 - f_2$ where f_i is increasing for $i = 1, 2$. In addition, if f is absolutely continuous, then f_i is absolutely continuous for $i = 1, 2$.

Proof:

$\forall \varepsilon > 0$, $\exists \delta > 0 \ni \sum_{\ell=1}^{m} |f(B_\ell) - f(A_\ell)| < \varepsilon$ ⊛ wherever $\{(A_\ell, B_\ell)\}$ is a set of finitely many disjoint intervals w/ $\sum_{\ell=1}^{m} |B_\ell - A_\ell| < \delta$. Now to show f_1 is absolutely continuous, we'll consider $\{(a_i, b_i)\}_{i=1}^{k}$ as a set of disjoint k many intervals with $\sum_{i=1}^{k} |b_i - a_i| < \delta$ & show that $\sum_{i=1}^{k} |f_1(b_i) - f_1(a_i)| < \varepsilon$. Consider a partition of (a_i, b_i) for a fixed i as $a_i = s_{i,0} < s_{i,1} < \dots < s_{i,j_i - 1} < s_{i,j_i} = b_i$. $\sum_{i=1}^{k} \sum_{j=0}^{j_i - 1} (s_{i,j+1} - s_{i,j}) = \sum_{i=1}^{k} (b_i - a_i) < \delta$. By ⊛ w/ $\{(s_{i,j}, s_{i,j+1}) \mid {}^{1 \leq i \leq k}_{0 \leq j \leq j_i - 1}\}$

$\sum \sum^{j_i - 1} |f(s_{i,j+1}) - f(s_{i,j})| < \varepsilon$. cont.

proof cont.

Take the sup over all partitions on (a_i, b_i).

$\sum_{i=1}^{k} V_f[a_i, b_i] \leq \varepsilon$. Using a previous lemma,

$\Rightarrow f_1(b_i) - f_1(a_i) \leq V_f[a_i, b_i]$

$\Rightarrow \sum_{i=1}^{k} |f_1(b_i) - f_1(a_i)| \leq \sum_{i=1}^{k} V_f[a_i, b_i] \leq \varepsilon$. QED

Case for f_2 is similar to this ↑

Also notice $\leq \varepsilon$. We need to use $\frac{\varepsilon}{2}$ so we can say $< \varepsilon$.

He's an idiot. Should have pointed that out sooner.

Ch 14 Hmwrk:

Due 3/30

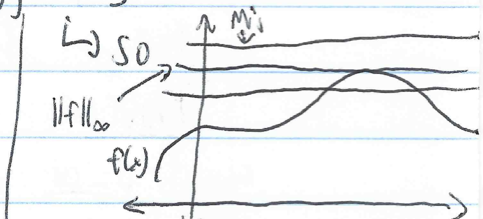
(1, 2, 5, 7, 8)

Ch. 15 L^p Spaces

- **Norms:**

— We have $(X, \mathcal{A}, \mu)$, a σ-finite measure space ~~measurable space~~. Define
<u>L^p norm</u> of f by $\|f\|_p = \left(\int_X |f(x)|^p \, d\mu\right)^{1/p}$ where $1 \le p < \infty$,
$\|f\|_\infty = \inf\{M \ge 0 \mid \mu(\{x \mid |f(x)| \ge M\}) = 0\}$

↳ we define this as essential sup, or ess sup. We'll get to this later?
Go kill yourself Gie,



Sort of like an upper bound up to set of measure zero

— The space $L^p = L^p(X) = L^p(\mu) = L^p(X, \mu)$
is the set $\{f \mid \|f\|_p < \infty\}$.

— Q: Is $\|\cdot\|_p$ a norm on L^p?
We need to check:
i) $\|af\|_p = |a| \, \|f\|_p, \ \forall a \in \mathbb{R}$
ii) $\|f+g\|_p \le \|f\|_p + \|g\|_p$
iii) $\|f\|_p = 0 \Rightarrow f = 0$ a.e. $\longrightarrow$ & $f = 0$ in L^p

} b/c definition of <u>norm</u>.

Note: i) is true by definition of $\|\cdot\|_p$
ii) is the triangle inequality or Minkowski's inequality
~~and~~ iii) is by previous proofs involving integration, & b/c $f = g$ in L^p if $f = g$ a.e.

— For any $1 \le p \le \infty$, we define the <u>conjugate</u> q of p by $\frac{1}{p} + \frac{1}{q} = 1$ if $1 < p < \infty$, $q = 1$ if $p = \infty$, or $q = \infty$ if $p = 1$.

Proposition) (Hölder's inequality). For any $1 \le p \le \infty$ & its conjugate q, & measurable function f & g, we have

(*) $\int |fg| \, d\mu \le \|f\|_p \|g\|_q$. In particular, when $p=q=2$, (*) is called Cauchy-Schwarz inequality.

Notice $\int |fg| \, d\mu = \|fg\|_1$.

Proof:

case 1) $p=\infty$. If $\|f\|_\infty = M$, then $|f| \le M$ a.e. & hence $\int |fg| \le M \int |g| = \|f\|_\infty \|g\|_1$

Thus we see that (*) holds true.

case 2) $p=1$. Then $q=\infty$. Hence by case 1, (*) holds true.

case 3) $1<p, q<\infty$.

i) if $\|f\|_p = 0$, $f=0$ a.e. So $\int |fg| \, d\mu = \|fg\|_1 = \|0\|_1 = 0$ & $\|f\|_p \|g\|_q = 0$.

Hence (*) holds

ii) if $\|g\|_q = 0$, (*) holds true as in i).

iii) $\|f\|_p = \infty$ or $\|g\|_q = \infty$, then (*) holds true b/c LHS $<\infty$ by definition, so RHS $=\infty$ is bigger than LHS.

iv) $0 < \|f\|_p < \infty$, $0 < \|g\|_q < \infty$. Define $F(x) = \frac{|f(x)|}{\|f\|_p}$, & $G(x) = \frac{|g(x)|}{\|g\|_q}$. Then $\|F(x)\|_p = \frac{1}{\|f\|_p} \|f\|_p = 1$ & similarly, $\|G(x)\|_q = 1$.

$\int |FG| = \int FG = \int \frac{|fg|}{\|f\|_p \|g\|_q} \le 1$. Thus it is sufficient to show that $\int FG \le 1$.

Let $y = e^x$, so $y'' = e^x > 0$ so y is convex (concave up). Hence $e^{\lambda a + (1-\lambda) b} \le \lambda e^a + (1-\lambda) e^b$ $\forall$ $a<b$, $0 \le \lambda \le 1$. If $F(x) \ne 0$, & $G(x) \ne 0$, let $a = p \log F(x)$ & $b = q \log G(x)$, $\lambda = \frac{1}{p}$ & $1-\lambda = \frac{1}{q}$. Cont. on next page.

Then write $e^{\frac{1}{p} p \log F(x) + \frac{1}{q} q \log G(x)} \leq \frac{1}{p} e^{p \log F(x)} + \frac{1}{q} e^{q \log G(x)}$

$\Rightarrow e^{\log F(x) G(x)} \leq \frac{1}{p} e^{\log F(x)^p} + \frac{1}{q} e^{\log G(x)^q}$

$\Rightarrow F(x) G(x) \leq \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q$

By integrating over X, we find that

$\Rightarrow \|FG\|_1 \leq \frac{1}{p} \|F\|_p^p + \frac{1}{q} \|G\|_q^q = \frac{1}{p} + \frac{1}{q} = 1$ so we're done.

QED

gives us $a^p + b^p \leq (a+b)^p \leq c_p(a^p + b^p)$

- <u>Lemma</u>: If $a, b \geq 0$ & $1 \leq p < \infty$, then $(a+b)^p \leq 2^{p-1} a^p + 2^{p-1} b^p$.

  <u>Proof:</u>

  If $a = 0$, then $b^p \leq 2^{p-1} b^p$. Duh. b/c $p \geq 1$. Same for $b = 0$.

  Consider the remaining case where $a, b > 0$. Whatever. He's an idiot.

  This would be an easy induction proof.

- <u>Minkowski's inequality</u>: If f & g are measurable, then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ where $1 \leq p \leq \infty$.

<u>Proof:</u>

- Case 1) Let $p = 1$ or $p = \infty$. Then $|f+g| \leq |f| + |g|$ by the triangle inequality.

  Then $\int |f+g| \leq \int (|f| + |g|) = \int |f| + \int |g|$

  $\Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p$ ---- call this $(*)$ w/ $p = 1$.

  ess sup $\Rightarrow (*)$ w/ $p = \infty$.

- Case 2) Let ~~[illegible]~~ $1 < p < \infty$. If $\|f_p\| = \infty$, $\|g\|_p = \infty$, or $\|f+g\|_p = 0$, then $(*)$ is obvious. So consider the case where $\|f\|_p, \|g\|_p < \infty$ & $\|f+g\|_p > 0$. Now we write $(a+b)^p \leq 2^{p-1} a^p + 2^{p-1} b^p$ w/ $a = |f|$ & $b = |g|$. So $|f+g|^p \leq (|f| + |g|)^p \leq (|f|^p + |g|^p) 2^{p-1}$

  Integrating this inequality, we find

  $\int |f+g|^p = \|f+g\|_p^p \leq 2^{p-1} \|f_p\|^p + 2^{p-1} \|g\|_p^p < \infty \Rightarrow f + g \in L^p$.

cont.

Case 2 cont) $\Rightarrow \|f+g\|_p < \infty$. Now we write

$|f+g|^p = |f+g|\,|f+g|^{p-1} \leq (|f|+|g|)\,|f+g|^{p-1}$

$= |f|\,|f+g|^{p-1} + |g|\,|f+g|^{p-1}$. Integrating this we find that

$\|f+g\|_p^p \leq \int |f|\,|f+g|^{p-1} + \int |g|\,|f+g|^{p-1} = \| |f|\,|f+g|^{p-1} \|_1 + \| |g|\,|f+g|^{p-1} \|_1$

using Hölder's inequality w/ p & $q = \frac{p}{p-1}$, then

b/c $q = \frac{p}{p-1}$ & $\frac{1}{q} = \frac{p-1}{p}$

$\leq \|f\|_p \left(\int |f+g|^{(p-1)\frac{p}{p-1}} \right)^{\frac{p-1}{p}} + \|g\|_p \left(\int |f+g|^{(p-1)\frac{p}{p-1}} \right)^{\frac{p-1}{p}}$

$\leq \|f_p\|\,\|f+g\|_p^{p-1} + \|g\|_p \|f+g\|_p^{p-1}$

$\Rightarrow \frac{\|f+g\|_p^p}{\|f+g\|_p^{p-1}} \leq \|f\|_p + \|g\|_p \Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p$.

QED.

- Now we can say L^p is a linear space equipped w/ the p norm, so $g = f$ in L^p if $g = f$ a.e.
- $f_n \to f$ in $L^p(X, \mu)$ if $\int_X |f_n - f|^p \, d\mu \to 0$ as $n \to \infty$, iff $\|f_n - f\|_p^p \to 0$ iff $\|f_n - f\|_p \to 0$.
- recall ess sup f. We also have ess inf f now.

sup of function up to measure 0 set   inf of function up to measure 0 set

• **Completeness of L^p**

- note: A space $(X, \|\cdot\|_{norm})$ is complete if any Cauchy sequence $\{a_n\} \subset X$ is convergent, $\exists\, A \in X \ni \lim_{n\to\infty} a_n = A$ in X where $\|a_n - A\|_{norm} \to 0$ as $n \to \infty$.

- $\{f_n\}_{n=1}^{\infty} \subset L^p(X, \mu)$ is Cauchy if $\|f_n - f_m\|_p \to 0$ as $n, m \to \infty$.

- **Thm:** If $1 \leq p \leq \infty$, then L^p is complete.

Proof:

~~WTS~~ We pick a Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ & will show that it is convergent. B/c $\{f_n\}$ is Cauchy, $\forall \varepsilon > 0$, $\exists\, N \in \mathbb{N} \ni \|f_m - f_n\|_p < \varepsilon$ if $m, n \geq N$.

Step 1) For $\varepsilon = \frac{1}{2^{j+1}}$ (j is fixed where $j \geq 1$), $\exists\, n_j > 0 \ni$ $\|f_n - f_m\|_p \leq \frac{1}{2^{j+1}}$ if $n, m \geq n_j$. (note $n_{j+1} \geq n_j$). We consider $\{f_{n_j}\}_{j=0}^{\infty}$, $f_{n_0} = 0$ w/ $n_0 = 0$.

Our candidate for the limit of f_n is $\sum_{m=1}^{\infty} (f_{n_m} - f_{n_{m-1}})$. $(**)$

Step 2) WTS $(**)$ is absolutely convergent. Define $g_j(x) = \sum_{m=1}^{j} |f_{n_m}(x) - f_{n_{m-1}}(x)|$

So $g_j(x)$ is an increasing function in j. Define $g(x) = \lim_{j\to\infty} g_j(x)$ (could be ∞). By Minkowski's inequality,

$\|g_j\|_p \leq \sum_{m=1}^{j} \|f_{n_m} - f_{n_{m-1}}\|_p \quad \forall j$,

$\hookrightarrow \leq \|f_{n_1} - f_{n_0}\|_p + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^j} \leq \|f_{n_1}\|_p + \frac{1}{2}$.

$\Rightarrow \|g_j\|_p \leq \|f_{n_1}\|_p + \frac{1}{2} \quad \forall j \geq 1$.

Notice $|g_j|^p$ is nonnegative & measurable. Hence by Fatou's Lemma,

$\int |g|^p \leq \lim_{j\to\infty} \int |g_j|^p \leq \lim_{j\to\infty} \|g_j\|_p^p \leq \lim_{j\to\infty} (\|f_{n_1}\|_p + \frac{1}{2})^p < \infty$.

$\Rightarrow g \in L^p$. Moreover g is finite a.e, & $g_j \to g$ a.e.

To be continued next time.

Thm cont:

This also tells us $f = \sum_{m=1}^{\infty} (f_{n_m} - f_{n_{m-1}})$ is finite a.e. & $f \in L^p$ where $f \le g \Rightarrow \|f\|_p \le \|g\|_p < \infty$.

To show $f_{n_j} \to f$ in L^p as $j \to \infty$, Let's set $f(x) = 0$ $\forall x$ where the absolute convergence does not hold. Now, we write
$f(x) = \lim_{k\to\infty} \sum_{m=1}^{k} (f_{n_m}(x) - f_{n_{m-1}}(x)) = \lim_{k\to\infty} f_{n_k}(x)$.

$\|f - f_{n_j}\|_p^p = \int |f - f_{n_j}|^p \le \liminf_{k\to\infty} \int |f_{n_k} - f_{n_j}|^p$ (by Fatou's w/ $\{|f_{n_k} - f_{n_j}|^p\}_{k=1}^{\infty}$) $\le \liminf_{k\to\infty} \|f_{n_k} - f_{n_j}\|_p^p \le \liminf_{k\to\infty} \frac{1}{2^{j+1}} = \frac{1}{2^{j+1}}$ $\forall j$

$\Rightarrow \|f - f_{n_j}\| \to 0$ as $j \to \infty$ Hence $f_{n_j} \to f$ in L^p as $j \to \infty$.

Step 3) To show $f_m \to f$ in L^p as $m \to \infty$, $\forall \varepsilon > 0$, choose $N \in \mathbb{N}$ $\ni \|f_{n_j} - f\|_p < \frac{\varepsilon}{2}$ where $j \ge N$ & $\|f_m - f_{n_j}\|_p < \frac{1}{2^{j+1}} < \frac{\varepsilon}{2}$ where $j \ge N$ & $m \ge n_N$.

Then we write $\|f_m - f\|_p \le \|f_m - f_{n_j}\| + \|f_{n_j} - f\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

QED.

• **Proposition:** The set $C_c(\mathbb{R})$ of continuous functions w/ compact support is dense in $L^p(\mathbb{R})$ $\forall 1 \le p < \infty$.

this means $\forall$ function in $L^p(\Omega)$, $\exists$ a sequence $\{f_n\} \in C_c(\mathbb{R}) \ni f_n \to f$ in L^p as $n \to \infty$.

- **Corollary:** $C([a,b])$ is dense in $L^2([a,b])$.
  Note $\|f\|_2 = \left(\int_a^b |f|^2\right)^{1/2}$.

- **Convolution**
  f is a function $\ni f:\mathbb{R}^n \to \mathbb{R}$ where m is the Lebesgue measure in $\mathbb{R}^n$.

def — The convolution of measurable f & g is defined by
$f*g(x) = \int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy = \int (f\circ K)(x,y)\,g(y)\,dy$ where
$K(x,y) = x-y$.
Question: Is $f*g$ measurable? yes
Question: $f*g = g*f$? yes. didn't see that coming.

- **Proposition** — (1) If $f,g \in L^1 \Rightarrow \|f*g\|_1 \le \|f\|_1\|g\|_1$ & hence $f*g \in L^1$.

$x\in\mathbb{R}^n$

(2) If $f\in L^1$ & $g \in L^p$ $\forall p \ni 1 \le p \le \infty$
$\Rightarrow \|f*g\|_p \le \|f\|_1 \|g\|_p$ & hence $f*g \in L^p$.

Proof)

(1) $\|f*g\|_1 = \int |f*g|\,dx = \int \left|\int f(x-y)g(y)\,dy\right| dx$
$\le \iint |f(x-y)||g(y)|\,dy\,dx \le \iint |f(x-y)||g(y)|\,dx\,dy$
$\le \int |g(y)| \int |f(x-y)|\,dx\,dy \le \int |f(x)|dx \int |g(y)|\,dy \le \|f\|_1\|g\|_1$.

(2) $\|f*g\|_p^p = \|g*f\|_p^p = \int \left|\int g(x-y) f(y)\,dy\right|^p dx$ $(**)$
Note $\left|\int g(x-y)f(y)\,dy\right| \le \int |g(x-y)||f(y)|\,dy$
$\le \int |f(y)|^{1/q} |f(y)|^{1-1/q} |g(x-y)|\,dy$ where $\frac{1}{p}+\frac{1}{q}=1$
$\le \left(\int |f(y)|^{1/q\cdot q}\,dy\right)^{1/q} \left(\int |f(y)|^{p/p} |g(x-y)|^p\,dy\right)^{1/p}$
$\le \|f\|_1^{1/q} \left(\int |f(y)||g(x-y)|^p\,dy\right)^{1/p}$ $(*)$. By $(**)$ & $(*)$
$\|f*g\|_p^p \le \int \|f\|_1^{p/q} \int |f(y)||g(x-y)|^p\,dy\,dx$
$= \|f\|_1^{p/q} \iint |f(y)||g(x-y)|^p\,dy\,dx \le \|f\|_1^{p/q} \iint |f(y)||g(x-y)|^p\,dx\,dy$ by Fubini

Prop. cont.

$\leq \|f\|_1^{p/q} \int |f(y)| \int |g(x-y)|^p \, dx \, dy = \|f\|_1^{p/q} \|g\|_p^p \int |f(y)| dy$

$= \|f\|_1^{1+p/q} \|g\|_p^p$. So far we have $\|f*g\|_p^p \leq \|f\|_1^{1+p/q} \|g\|_p^p$.

$\Rightarrow \|f*g\|_p \leq \|f\|_1^{1/p+1/q} \|g\|_p = \|f\|_1 \|g\|_p$.

Proof for when $p=\infty$ is left for us.

QED.

- Note: If $\Omega \subset \mathbb{R}^n$ is bounded (i.e. $m(\Omega)<\infty$), then $L^p(\Omega) \subset L^q(\Omega)$ $\forall\ 1 \leq q < p \leq \infty$.

Proof: WTS $\|f\|_q < \infty$ if $\|f\|_p < \infty$.

$\|f\|_q^q = \int_\Omega |f|^q \cdot 1 \leq \left(\int_\Omega (|f|^q)^{p/q}\right)^{q/p} \left(\int_\Omega 1^r\right)^{1/r}$

where r is the conjugate of p/q. $1 \leq p/q < \infty$

so $\leq \|f\|_p^q (m(\Omega))^{1/r} < \infty$. So we're done. QED.

In general, $L^p \not\subset L^q$, $q \leq p$.

(Counter example) In $\mathbb{R}$, $L^2(\mathbb{R}) \not\subset L^1(\mathbb{R})$.

$f(x) = \begin{cases} 1/x & x \geq 1 \\ 0 & x < 1 \end{cases}$. $\|f\|_1 = \int_1^\infty \frac{1}{x} = \ln x \,\big]_1^\infty = \infty$ s.t. $f \notin L^1$.

but $\|f\|_2^2 = \int_1^\infty \frac{1}{x^2} < \infty$ so $f \in L^2$.

QED.

- One application of convolution is the process called "mollification" to approximate an L^p function by smooth functions. We introduce a smooth function called "mollifier", $\varphi: \mathbb{R}^n \to \mathbb{R}$ $\ni$ i) φ is compactly supported

ii) $\int \varphi = 1$

For example, $\varphi(x) = \begin{cases} c_1 e^{\frac{-1}{1-|x|^2}}, & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$ w/ a proper $c_1 \ni \int \varphi = 1$.

• We define $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$ so that

i) φ_ε is supported in ~~[illegible]~~ $[-\varepsilon, \varepsilon]$

ii) $\int \varphi_\varepsilon = 1 \quad \forall \varepsilon$.

So $\varphi_\varepsilon \to \delta_0$ (point mass at $x=0$) as $\varepsilon \to 0$ in "some sense".

• <u>Thm</u>: If $f \in L^p$ where $1 \le p \le \infty$, then

(1) $f * \varphi_\varepsilon$ is infinitely differentiable & moreover,

$$\frac{\partial^{\alpha_1+\alpha_2+\dots+\alpha_n}(f*\varphi_\varepsilon)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}} = f * \frac{\partial^{\alpha_1+\dots+\alpha_n}\varphi_\varepsilon}{\partial x_1^{\alpha_1}\cdots\partial^{\alpha_n}x_n}$$

$\forall \varepsilon > 0$ & $\forall \alpha_i \in \mathbb{N} \cup \{0\}$, $1 \le i \le n$.

(2) $f * \varphi_\varepsilon \to f$ a.e. as $\varepsilon \to 0$.

(3) If f is continuous, $f * \varphi_\varepsilon \to f$ uniformly on any compact set as $\varepsilon \to 0$.

(4) If $1 \le p < \infty$, then $f * \varphi_\varepsilon \to f$ in L^p.

<u>Proof 1)</u> ISTS that $\frac{\partial(f*\varphi_\varepsilon)}{\partial x_i} = f * \frac{\partial \varphi_\varepsilon}{\partial x_i} \quad \forall\, 1 \le i \le n$.

Then inductively we're done.

Let i) φ has support in $B(0,R)$ (φ_ε has support in $B(0,\varepsilon R)$)

ii) e_i is the unit vector in the ith direction, $1 \le i \le n$.

So $\frac{f*\varphi_\varepsilon(x+he_i) - f*\varphi_\varepsilon(x)}{h} = \frac{\int f(y)\varphi_\varepsilon(x+he_i-y) - \int f(y)\varphi_\varepsilon(x-y)dy}{h}$

$= \frac{1}{h}\int f(y)\left(\varphi_\varepsilon(x+he_i-y) - \varphi_\varepsilon(x-y)\right)dy$

i) $\frac{\varphi_\varepsilon(x+he_i-y) - \varphi_\varepsilon(x-y)}{h} \to \frac{\partial \varphi_\varepsilon}{\partial x_i}$ pointwise

ii) on the other hand since φ_ε is differentiable many times,

Comes from Taylor expansion

$\left|\frac{\varphi_\varepsilon(x-y+he_i) - \varphi_\varepsilon(x-y)}{h}\right| \le c|h|$ ~~[illegible]~~ $\le C$ for some constant c by the mean value thm.

(cont.)

Thm w/no end cont.

In addition, $\varphi_\varepsilon(x-y) \neq 0$ if $|x-y| \le \varepsilon R$ &

$\varphi_\varepsilon(x-y+he_i) \neq 0$ if $|x-y+he_i| < \varepsilon R$

$\underset{|x-y|-he_i}{\overset{\vee\!\mid}{|x-y+he_i|}}$ so $|x-y| \le \varepsilon R + h$

$\Rightarrow \varphi_\varepsilon(x-y+he_i) - \varphi_\varepsilon(x-y) \neq 0$ if $|x-y| < \varepsilon R + h$.

$=$ $\ge \varepsilon R + 1$.

Hence we found that $\left|\frac{\varphi_\varepsilon(x-y+he_i) - \varphi_\varepsilon(x-y)}{h}\right| \le C_* \chi_{B}(x, \varepsilon R+1)(y)$.

$\int |f(y)\, C_* \chi_{B(x,\varepsilon R+1)}|\, dy \le C_* \int_{B(x,\varepsilon R+1)} |f(y)|\, dy$

$\le C_* \left(\int_{B(x,\varepsilon R+1)} |f(y)|^p\, dy\right)^{1/p} \left(\int_{B(x,\varepsilon R+1)} 1^q\, dy\right)^{1/q} \le C_* \|f\|_p\, m(B(0,\varepsilon R+1))^{1/q}$

$< \infty$

Hence by DCT, we find that $\lim_{h\to 0} \frac{f*\varphi_\varepsilon(x+he_i) - f*\varphi_\varepsilon(x)}{h}$

$= \int f(y) \frac{\partial \varphi_\varepsilon}{\partial x_i}(x-y)\, dy = \left(f * \frac{\partial \varphi_\varepsilon}{\partial x_i}\right)(x) = \frac{\partial (f*\varphi_\varepsilon)}{\partial x_i}$. QED

Proof of 2)

$f*\varphi_\varepsilon \to f$ a.e. as $\varepsilon \to 0$, $|f*\varphi_\varepsilon(x) - f(x)|$

$= \left|\int f(y)\varphi_\varepsilon(x-y)\, dy - f\int \varphi(y)\, dy\right| = \left|\int (f(y)-f(x))\varphi_\varepsilon(x-y)\, dy\right|$

$\le \left|\frac{1}{\varepsilon^n}\int (f(y)-f(x))\varphi\left(\frac{x-y}{\varepsilon}\right) dy\right| \le \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon R)} |f(y)-f(x)|\varphi\left(\frac{x-y}{\varepsilon}\right) dy$.

Btw, ~~$\varepsilon B(0,R) = B(0,\varepsilon R)$~~ $\varepsilon^n m(B(0,R)) = m(B(0,\varepsilon R))$

$\le \|\varphi\|_\infty \frac{m(B(0,R))}{m(B(x,\varepsilon R))} \int_{B(x,\varepsilon R)} |f(y)-f(x)|\, dy \to 0$ as $\varepsilon \to 0$.

Proof of 3) f is continuous, $f*\varphi_\varepsilon \to f$ uniformly on any compact set ~~[-N,N]~~ ~~then~~ $B(0,N)$ $\forall N$. Let $N \in \mathbb{N}$ be fixed.

WTS $\sup_{|x|\le N} |f*\varphi_\varepsilon(x) - f(x)| \to 0$ as $\varepsilon \to 0$.

$\le \sup_{|x|\le N} \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon R)} |f(y)-f(x)|\varphi\left(\frac{x-y}{\varepsilon}\right) dy$

$\le \sup_{|x|\le N} \frac{1}{\varepsilon^n} \sup_{y\in B(x,\varepsilon R)} |f(y)-f(x)| \int_{B(x,\varepsilon R)} \varphi\left(\frac{x-y}{\varepsilon}\right) dy$.

$\le \sup_{|x|\le N} \frac{1}{\varepsilon^n} \sup_{y\in B(x,\varepsilon R)} |f(y)-f(x)|\, \|\varphi\|_\infty\, m(B(0,\varepsilon R))$

$\le \sup_{|x|\le N} \sup_{y\in B(x,\varepsilon R)} |f(y)-f(x)|\, m(B(0,R))\, \|\varphi\|_\infty \to 0$ as $\varepsilon \to 0$.

Proof of 4) WTS if $f \in L^p$ where $1 \le p < \infty$, then $f * \varphi_\varepsilon \to f$ in L^p.

- Step 1) WTS $\forall\, \varepsilon > 0$, $\exists$ a compactly supported step function $g \ni$ $\|f - g\|_p < \varepsilon$.

  ↑ simple function.

  i) Set $f_N = f\chi_{B(0,N)}$
  - $\Rightarrow f - f_N \to 0$ a.e. as $N \to \infty$
  - $\|(f - f_N)^p\|_1 \le \int |f|^p < \infty$.

  Hence by DCT, $\lim_{N\to\infty} \|f - f_N\|_p^p = \lim_{N\to\infty} \int |f - f_N|^p = 0$.

  $\Rightarrow f_N \to f$ in L^p

  ii) Choose sufficiently large $N \ni \|f - f_N\|_p < \frac{\varepsilon}{2}$

  iii) For f_N, choose a simple function $g \ni g \le f_N$ & $\|g - f_N\|_p < \frac{\varepsilon}{2}$.

  By i) & ii), $\|f - g\|_p \le \|f - f_N\|_p + \|f_N - g\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

  Note that g is supported in $B(0,N)$.

- Step 2) WTS $g * \varphi_\varepsilon \to g$ in L^p as $\varepsilon \to 0$.

  Note that : i) $g \in L^\infty \Rightarrow \|g * \varphi_\varepsilon\|_\infty \le \|g\|_\infty \|\varphi_\varepsilon\|_1 = \|g\|_\infty < \infty$.

  ii) By 2 in Thm, $g * \varphi_\varepsilon \to g$ a.e. as $\varepsilon \to 0$.

  iii) g is supported in $B(0,N)$. φ_ε is supported in $B(0, \varepsilon R)$

→ $\Rightarrow g * \varphi_\varepsilon(x) = \int g(y)\varphi_\varepsilon(x-y)\,dy$ is supported in $B(0, N + \varepsilon R)$.

  From i) & iii), $|g * \varphi_\varepsilon| = |(g * \varphi_\varepsilon)\chi_{B(0,N+\varepsilon R)}|$ & $\int |(g * \varphi_\varepsilon)\chi_{B(0,N+\varepsilon R)}| = \int_{B(0,N+\varepsilon R)} |g * \varphi_\varepsilon| \le \|g\|_\infty m(B(0, N+\varepsilon R))$

  By DCT, we write $\lim_{\varepsilon\to 0} \|g - g * \varphi_\varepsilon\|_p^p$

  $= \lim_{\varepsilon\to 0} \int |g - g * \varphi_\varepsilon|^p = \int \lim_{\varepsilon\to 0} |g - g * \varphi_\varepsilon|^p = 0$

  $\Rightarrow g * \varphi_\varepsilon \to g$ in L^p.

HW Due 4/18

(1, 4, 6, 8, 12, 13, 14)

- Step 3) WTS $f*\varphi_\varepsilon \to f$ in L^p as $\varepsilon \to 0$. We write

$\|f*\varphi_\varepsilon - f\|_p \le \|f*\varphi_\varepsilon - g*\varphi_\varepsilon\| + \|g*\varphi_\varepsilon - g\|_p + \|g-f\|_p$.

By steps 1) & 2), $\forall \varepsilon>0$, $\|f*\varphi_\varepsilon - f\|_p \le 2\varepsilon + \|f*\varphi_\varepsilon - g*\varphi_\varepsilon\|_p$

So $\|f*\varphi_\varepsilon - g*\varphi_\varepsilon\|_p = \|(f-g)*\varphi_\varepsilon\|_p \le \|f-g\|_p\|\varphi_\varepsilon\|_1$

$\le \|f-g\|_p < \varepsilon$ by step 1. So $\|f*\varphi_\varepsilon - f\|_p < 3\varepsilon$. QED.

- Bounded Linear Functionals

$L^p(X,\mathcal{M}) = \{f: \text{measurable} \mid \|f\|_p < \infty\}$ $(X, \mathcal{A}, \mu)$ where X is σ-finite.

$\|f\|_p = \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{1/p} & 1 \le p < \infty \\ \operatorname{ess\,sup}_X |f| & p=\infty \end{cases}$

A linear functional on L^p is a map $H: L^p \to \mathbb{R}$ satisfying

i) $H(f+g) = H(f) + H(g)$

ii) $H(af) = aH(f)$ $\forall f \in L^p$ & $\forall a \in \mathbb{R}$.

H is a bounded linear function if $\|H\| = \sup\{|Hf| \mid \|f\|_p \le 1\}$ is finite.

- The space of bounded linear functionals is denoted by $\mathcal{L}(L^p(X,\mathcal{M}), \mathbb{R})$.
- The dual space of L^p, denoted by $(L^p)' = (L^p)^*$, is the set of all bounded linear functionals.

Our goal here is to identify $(L^p)'$ w/ L^q, where $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$.

- Define $\operatorname{sgn}(x) = \begin{cases} -1, & x<0 \\ 0, & x=0 \\ 1, & x>0 \end{cases}$.

• Thm: $1 \le p \le \infty$, q is the conjugate of p. Let $f \in L^p$. Then

$*$ $\quad \|f\|_p = \sup_{\|g\|_q \le 1} \int_X fg \, dm$.

Proof:

$\int fg \le \|fg\|_1 \le \|f\|_p \|g\|_q \le \|f\|_p \; \forall g$ where $\|g\|_q \le 1$.

$\sup_{\|g\|_q \le 1} \int fg \le \|f\|_p$.

So now we just need to show RHS of $* \ge$ LHS $*$.

Case 1) $p = 1$

Take $g(x) = \operatorname{sgn} f(x) \Rightarrow$ i) $|g| \le 1 \Rightarrow \|g\|_\infty \le 1$ ii) $fg = f\operatorname{sgn}(f) = |f|$

$\|f\|_1 = \int |f| = \int fg \le \sup_{\|g\|_\infty \le 1} \int fg \Rightarrow$ RHS $* \ge$ LHS $*$

Case 2) $p = \infty$

If $\|f\|_\infty = 0$, there is nothing to prove.

Let $\|f\|_\infty = M > 0$. X is σ finite, so $\exists F_n \uparrow X$ &

$m(F_n) < \infty \; \forall n$. For a fixed $0 < a < M$, define

$A_n = \{x \in F_n \mid |f(x)| > a\}$. Then, since $\|f\|_\infty = M$,

$m(A_n) > 0$ for sufficiently large n.

Define $g_n(x) = \frac{(\operatorname{sgn} f)(x) \chi_{A_n}(x)}{m(A_n)} \Rightarrow \|g_n\| = \frac{1}{m(A_n)} \int_{A_n} |\operatorname{sgn} f| = 1$

$\forall n$. Also $\int f g_n = \frac{1}{m(A_n)} \int_{A_n} |f| \ge \frac{1}{m(A_n)} a\, m(A_n) \ge a \; \forall n$,

Hence $a \le \int f g_n \; \forall n$. So $a \le \sup_n \int f g_n \le \sup_{\|g\|_1 \le 1} \int fg$

$\Rightarrow M \le \sup_{\|g\|_1 \le 1} \int fg \Rightarrow$ where $M = \|f\|_\infty$. So

RHS $*$ = LHS $*$.

Case 3) $1 < p < \infty$

Let $\|f\|_p > 0$. $\exists F_n \uparrow X \ni m(F_n) < \infty \; \forall n$.

Writing $f = f^+ - f^-$, we introduce sequences of non-negative simple functions q_n & $r_n \ni q_n \uparrow f^+$ & $r_n \uparrow f^-$ as

$n \to \infty$. Define $s_n(x) = (q_n - r_n)\chi_{F_n}$ (simple function).

We know $s_n(x) \to f(x)$, $\|s_n\|_p < \infty \; \forall n$.

cont.

Case 3 (cont.)) $|f-s_n|^p = |f^+ - q_n \chi_{F_n}|^p + |f^- - r_n \chi_{F_n}|^p$ which is monotone, Hence, by the Monotone convergence theorem,
$\lim_{n\to\infty} \|f-s_n\|_p^p = \lim_{n\to\infty} \int |f-s_n|^p = 0 \Rightarrow s_n \to f$ in L^p as $n\to\infty$.
Then $\|s_n\|_p > 0$ for sufficiently large n. Define $g_n(x)$ by
$g_n(x) = (\text{sgn} f)(x) \frac{|s_n(x)|^{p-1}}{\|s_n\|_p^{p/q}}$, This g_n is a simple func.
$\|g_n\|_q = \frac{1}{\|s_n\|_q^{p/q}} \left(\int |(\text{sgn} f)|^q |s_n|^{pq-q}\right)^{1/q}$
$= \frac{1}{\|s_n\|_p^{p/q}} \left(\int |s_n|^{pq-q}\right)^{1/q}$, but $\frac{1}{p} + \frac{1}{q} = 1$, so
$q + p = pq \Rightarrow pq - q = p$
$\|g_n\|_q = \frac{1}{\|s_n\|_p^{p/q}} \left(\int |s_n|^p\right)^{1/q} = 1 \Rightarrow \|g_n\|_q = 1 \ \forall n$. On the other hand, $|f| \geq |s_n|$. So
$\int f g_n = \frac{1}{\|s_n\|_p^{p/q}} \int |f|\, |s_n|^{p-1} \geq \frac{1}{\|s_n\|_p^{p/q}} \int |s_n|^p = \frac{\|s_n\|_p^p}{\|s_n\|_p^{p/q}}$
$= \|s_n\|_p^{p-p/q} = \|s_n\|_p \ \forall n$.
$\Rightarrow \int f g_n \geq \|s_n\|_p \Rightarrow \sup_n \int f g_n \geq \|f\|_p$
$\Rightarrow \sup_{\|g\|_q \leq 1} fg \geq \|f\|_p$. QED

- Corollary: Let f be measurable, $\int fg < \infty$, $\forall$ simple functions g. Then $\|f\|_p = \sup \left\{ \int fg \;\middle|\; \begin{matrix} g \text{ is a simple function} \\ \|g\|_q \leq 1 \end{matrix} \right\} \ \forall\, 1 < p < \infty$ where q is the conjugate of p.

• <u>Proposition:</u> $1<p<\infty$ & q is the conjugate of p. let $g \in L^q$.
Define $H(f) = \int fg \quad \forall f \in L^p$, ($H: L^p \to \mathbb{R}$). Then H is a bounded linear functional on L^p.

<u>Proof:</u>

i) $H(f_1 + cf_2) = \int (f_1 + cf_2) g = \int f_1 g + c \int f_2 g$
$= H(f_1) + cH(f_2)$ So H is a linear functional.

ii) To show the boundedness of H on L^p, WTS $\|H\| = \sup_{\|f\|_p \le 1} |H(f)|$
$\le \sup_{\|f\|_p \le 1} \left| \int fg \right| \le \sup_{\|f\|_p \le 1} \|fg\|_1 \le \sup_{\|f\|_p \le 1} \|f\|_p \|g\|_q$
$\le \|g\|_q$

Moreover, $\|H\| = \sup_{\|f\|_p \le 1} \left| \int fg \right| \ge \sup_{\|f\|_p \le 1} \int fg = \|g\|_q$
$\Rightarrow \|H\| = \|g\|_q$

QED

$\forall g \in L^q$, $H(f) = \int fg$, $f \in L^p$
then $H(f) \in \mathcal{L}(L^p; \mathbb{R}) = (L^p)'$

• <u>Thm:</u> If $1<p<\infty$, let $H \in \underbrace{\mathcal{L}(L^p, \mathbb{R})}_{=(L^p)'}$. Then $\exists\, g \in L^q$ $\ni H(f) = \int fg$ & $\|g\|_q = \|H\|$.

Exam 2 Ends Here