Mathematical Modeling Qualifier Exam

August 2016

Name:______________________________ ID:______________________________

Read every problem carefully. Do six problems out of the total eight problems.

To receive full credits, you MUST clearly state your reasoning statements, explicitly state all major theorems that you invoke, and demonstrate that all technical hypotheses of any theorem used are satisfied.

If you do more problems than required, you MUST indicate which problems you want to be graded. Otherwise, the problems will be chosen sequentially to grade. Organize all pages (with page number) you are going to turn in.

1. Consider the initial value problem in $\mathbf{R}$

$$\begin{cases} \dot{x} = x^2, \\ x(0) = 1. \end{cases} \tag{1}$$

(a) Find the first three successive approximations $u_1(t), u_2(t)$ and $u_3(t)$ by using integration.
(b) Use mathematical induction to show that for all $n \geq 1$,

$$u_n(t) = 1 + t + ... + t^n + \text{ higher order terms of } t.$$

(c) Solve the initial value problem (1) and show that the function $x(t) = \dfrac{1}{1-t}$ is a solution on the interval $(-\infty, 1)$.
(d) Show that the first $(n+1)$-terms in $u_n(t)$ agree with the first $(n+1)$-terms in Taylor series of the function $x(t) = \dfrac{1}{1-t}$ about $t = 0$.

2. Show that the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -2x - y - 3x^4 + y^2 \end{cases} \tag{2}$$

has no limit cycle in $\mathbf{R}^2$. (**Hint**: a function in the form $e^{\beta x}$ might be useful.)

3. Consider the following model defined in $\mathbf{D} = \Big\{(x, y) \in \mathbf{R}^2 : x \geq 0, y \geq 0\Big\}$:

$$\begin{cases} \dot{x} = 1 - x - \dfrac{2xy}{2+x} \\ \\ \dot{y} = y\Big(\dfrac{2x}{2+x} - 1\Big) \end{cases} \tag{3}$$

where $x(0) > 0$ and $y(0) > 0$.
(a) Show that the ω-limit set of any orbit is on the set

$$K = \Big\{(x, y) \in \mathbf{D} : x + y = 1\Big\}.$$

(b) Show that the system has a unique steady state in D and the unique equilibrium point is globally stable in D.

4. Show that the system

$$\begin{cases} \dot{x} = x - y - \left(x^2 + \frac{3}{2}y^2\right)x, \\ \\ \dot{y} = x + y - \left(x^2 + \dfrac{1}{2}y^2\right)y \end{cases} \tag{4}$$

has a limit cycle.

5. Determine the qualitative behavior near the non-hyperbolic critical point at the origin for the system

$$\begin{cases} \dot{x} = xy \\ \dot{y} = -y - x^2. \end{cases} \tag{5}$$

Sketch the phase portrait.

6. Consider the Lorenz model

$$\begin{cases} \dot{x} = \frac{3}{2}y - x, \\ \\ \dot{y} = \frac{1}{2}x - y - xz, \\ \\ \dot{z} = xy - z. \end{cases} \tag{6}$$

(a) Show that Lorenz model (6) has a unique equilibrium point.
(b) Show that the equilibrium point is globally stable. (**Hint**: Use a Liapunov function.)

7. Consider a spring-mass system that describes a motion with a force caused by friction or magnification or both:

$$\ddot{x} - \epsilon\dot{x} + x - x^3 + x^5 = 0, \qquad \epsilon \in (-2, 2). \tag{7}$$

(a) Without solving the equation, show that $\epsilon = 0$ is a bifurcation point. In more details, show that when $\epsilon = 0$ the system (7) is a Hamiltonian system and find an expression for the Hamiltonian. Then show that when $\epsilon < 0$ the origin is a stable equilibrium point; but when $\epsilon > 0$ the system (7) has a limit cycle.
(b) Classify the type of the bifurcation and then draw the bifurcation diagram. Label the diagram clearly.

8. Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{8}$$

where $q(\rho) = \rho u(\rho)$, and $u(\rho) \geq 0$ and $u(\rho) = u_m(1 - \dfrac{\rho}{\rho_m})$ with maximum vehicle speed $u_m > 0$ and maximum traffic density $\rho_m > 0$. Suppose that the initial traffic density is given by

$$\rho(x, 0) = \begin{cases} \dfrac{1}{4}\rho_m, & \text{if } x < -12, \\ \\ \dfrac{1}{2}\rho_m, & \text{if } -12 < x < 0, \\ \\ \dfrac{3}{4}\rho_m & \text{if } x > 0. \end{cases}$$

(a) Use the characteristic method to solve the equation.
(b) Show that initially the system has two shock waves, and determine the velocities of the two shock waves.
(c) Determine when the two shock waves meet.
(d) Determine the velocity of the new shock wave after the two shock waves meet, and sketch the graph of $\rho(x, t)$ in the (ρ, x)-plane for this case.

Mathematical Modeling Qualifier Exam

January 2016

Name:________________________________ ID:________________________________

Read every problem carefully. Do six problems out of the total eight problems.

You MUST show complete work to receive full credits. No work, no credit. If you do more problems than required, you MUST indicate which problems you want to be graded. Otherwise, the problems will be chosen sequentially to grade. Organize all pages (with page number) you are going to turn in.

1. Consider the predator-prey model

$$\dot{x} = x(1 - x - y), \quad \dot{y} = (4x - 1)y.$$

Let $x(0) > 0,\ y(0) > 0$. Determine $\lim_{t\to\infty}(x(t), y(t))$.

2. Consider the predator-prey model

$$\dot{x} = x(1 - y), \quad \dot{y} = (2x - 1)y.$$

Show that every non-constant positive solution is periodic.

3. Prove that the system

$$\dot{x} = x + y - 4x^3, \quad \dot{y} = y - x - 4y^3$$

has a limit cycle.

4. Consider

$$\dot{x} = y + x^2 - y^2, \quad \dot{y} = -x - 2xy.$$

(a) Find all the equilibria for this system and determine their stability.
(b) Find the Hamiltonian of the system.
(c) Find a solution curve that connects a saddle point.
(d) Does the curve obtained in (c) connect any other saddle points?

5. Consider the mass spring system subject to an external force $f(t)$:

$$\ddot{x} + \dot{x} + x = f(t).$$

Assume the initial data

$$x(0) = 0, \dot{x}(0) = 0.$$

Assume also that $f(t)$ is the force describing the striking effect on the mass of the mass spring system in a short time period $0 < T < 1/2$, given as below

$$f(t) = \begin{cases} \frac{\pi}{4T}\sin\frac{\pi t}{2T}, & 0 \le t < 2T \\ 0, & t \ge 2T. \end{cases}$$

(a) Solve the given initial value problem first for all $t \ge 0$.

(b) Compute the limits $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} \dot{x}(t)$.

(c) Compute the limits $\lim_{T\to 0} x(T)$ and $\lim_{T\to 0} \dot{x}(T)$, and discuss their physical meanings.

6. Suppose that the motion of a mass is described by the nonlinear differential equation

$$\frac{d^2x}{dt^2} + 2x + 3\left(\frac{dx}{dt}\right)^5 = 0.$$

Determine how solutions of this equation behave.

7. Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{1}$$

where $q(\rho) = \rho u(\rho)$, and $u(\rho) = u_{\text{max}}(1 - \frac{\rho}{\rho_{\text{max}}})$. Suppose that the initial traffic density is

$$\rho(x,0) = \begin{cases} \rho_{\text{max}}, & \text{if } x < 0, \\ \frac{\rho_{\text{max}}}{3}, & \text{if } 0 < x < 12, \\ 0, & \text{if } x > 12. \end{cases}$$

(a) Solve for $\rho(x,t)$ using the characteristic method.
(b) Sketch the graph of $\rho(x,t)$ in the (ρ, x) -plane for $t > 0$.
(c) Sketch how the speed of the vehicle originally at space $x = -12$ varies as $t > 0$.

8. Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0,$$

where $q(\rho) = \rho u(\rho)$, and

$$u(\rho) = u_{\text{max}}(1 - \frac{\rho}{\rho_{\text{max}}}).$$

Suppose that the initial traffic density is

$$\rho(x,0) = \begin{cases} \frac{\rho_{\text{max}}}{2}, & \text{if } x < -24 \\ \rho_{\text{max}}, & \text{if } -24 < x < 0 \\ 0, & \text{if } x > 0. \end{cases}$$

(a) Determine when the traffic line where $\rho = \rho_{\text{max}}$ completely dissipates.
(b) Describe how the traffic dynamics change in time after $\rho = \rho_{\text{max}}$ dissipates.
(c) Sketch how the speed of the vehicle originally at space $x = -48$ varies as $t > 0$.

Mathematical Modeling Qualifier Exam

August 2015

Name:______________________________ ID:______________________________

Read every problem carefully. Do six problems out of the total eight problems.

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1. Consider the Lotka-Volterra predator-prey model

$$\dot{x} = x(2-y), \quad \dot{y} = y(x-1).$$

Show that every non-constant positive solution is periodic.

2. Consider the system

$$\dot{x} = y + x\sqrt{x^2+y^2}\sin\frac{1}{\sqrt{x^2+y^2}}, \qquad \dot{y} = -x + y\sqrt{x^2+y^2}\sin\frac{1}{\sqrt{x^2+y^2}}.$$

Show that this system has infinitely many limit cycles, and discuss how the solutions behavior. (Hint: use polar coordinates.)

3. The Hamiltonian function of a Hamiltonian system is given by

$$H(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}.$$

(a) Find the equilibria of the system and discuss their stability

(b) Sketch the phase portrait for the system.

4. Consider the predator-prey model

$$\dot{x} = x(6 - x - \frac{3y}{1+x}), \ \dot{y} = y(x-2).$$

Assume that all positive solutions are bounded.
(a) Find all critical points and determine their local stability.
(b) Show that this system has a limit cycle in the first quadrant.

5. Suppose that the motion of a mass is described by the nonlinear differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x^5 = 0.$$

Determine how solutions of this equation behavior.

6. Suppose that the motion of a mass is described by the nonlinear differential equation

$$\frac{d^2x}{dt^2} + x + x^3 = 0.$$

Determine how solutions of this equation behavior.

7. Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{1}$$

where $q(\rho) = \rho u(\rho)$, and $u(\rho) = u_{\max}(1 - \frac{\rho}{\rho_{\max}})$. Suppose that the initial traffic density is

$$\rho(x,0) = \begin{cases} \rho_{\max}, & \text{if } x < 0, \\ \frac{\rho_{\max}}{2}, & \text{if } 0 < x < 10, \\ 0, & \text{if } x > 10. \end{cases}$$

(a) Solve for $\rho(x,t)$ using the characteristic method.
(b) Sketch the graph of $\rho(x,t)$ in the (ρ, x) -plane for $t > 0$.
(c) Sketch the speed of the vehicle initially at x=20 as time t goes on.

8. Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0,$$

where $q(\rho) = \rho u(\rho)$, and

$$u(\rho) = u_{\max}(1 - \frac{\rho}{\rho_{\max}}).$$

Suppose that the initial traffic density is

$$\rho(x,0) = \begin{cases} \frac{\rho_{\max}}{3}, & \text{if } x < -18 \\ \rho_{\max}, & \text{if } -18 < x < 0 \\ 0, & \text{if } x > 0. \end{cases}$$

(a) Determine when the traffic line where $\rho = \rho_{\max}$ completely dissipates.
(b) Describe how the traffic dynamics change in time after $\rho = \rho_{\max}$ dissipates.

Mathematical Modeling Qualifier Exam

August 2014

Name:______________________________ ID:______________________________

Read every problem carefully. Do six problems out of the total eight problems.

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1. Consider the Lotka-Volterra competition model

$$\dot{x} = x(3 - x - y), \;\; \dot{y} = y(4 - x - 2y).$$

Let $x(0) > 0, \;\; y(0) > 0$. Determine $\lim_{t\to\infty}(x(t), y(t))$.

2. Consider the predator-prey system

$$\dot{x} = x(5 - x - 2y), \quad \dot{y} = y(1 - y + x).$$

Assume that all positive solutions are bounded.
(a) Find all critical points and determine their local stability.
(b) Let $x(0) > 0, \;\; y(0) > 0$. Determine $\lim_{t\to\infty}(x(t), y(t))$.

3. Prove that the system

$$\dot{x} = -y + x(1 - 2x^2 - 3y^2), \quad \dot{y} = x + y(1 - 2x^2 - 3y^2)$$

has a limit cycle in an annulus.

4. Assume $\varepsilon \geq 0$. Consider

$$\dot{x} = -y - \varepsilon x, \quad \dot{y} = -x + x^3.$$

(a) Show that, as $\varepsilon = 0$, the system is Hamiltonian. Find the Hamiltonian of this system, and sketch a phase portrait.

(b) As $\varepsilon > 0$, what is the effect of ε on the stability of equilibrium points of the system?

5. Suppose that the motion of a mass m is described by the nonlinear differential equation

$$m\frac{d^2x}{dt^2} = -\alpha x^3 - \sigma(\frac{dx}{dt})^5$$

where $\alpha > 0, \;\; \sigma \geq 0$.

(a) Show that if $\sigma = 0$ then every nonzero solution $x(t)$ of the system is a periodic solution.

(b) Show that if $\sigma > 0$ then as $t \to \infty$, $x(t) \to 0$.

6. Suppose a spring system is on a table retarded by a Coulomb frictional force:

$$\ddot{x} + 4x = F_f,$$

where

$$F_f = \begin{cases} 1 & \text{if } \dot{x} < 0 \\ -1 & \text{if } \dot{x} > 0. \end{cases}$$

(a) If $\dot{x} > 0$, determine the energy equation. Sketch the resulting phase plane curves.

(b) If $\dot{x} < 0$, determine the energy equation. Sketch the resulting phase plane curves.

(c) Using the results of (a) and (b), sketch the solution in the phase plane. Show that the mass stops in a finite time.

(d) Consider a problem in which the mass is initially at $x = 0$ with velocity v_0. Determine how many times the mass passes $x = 0$ as a function of v_0.

7. Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0,$$

where $q(\rho) = \rho u(\rho)$, and $u(\rho) = 6[\ln \rho_{\text{max}} - \ln \rho]$.

Suppose that the initial traffic density is given by

$$\rho(x,0) = \begin{cases} \frac{1}{2}\rho_{\text{max}}, & \text{if } x < 0, \\ 0, & \text{if } x > 0. \end{cases}$$

For $t = 16$, where does $\rho = \frac{1}{4}\rho_{\text{max}}$?

8. Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0,$$

where $q(\rho) = \rho u(\rho)$, and

$$u(\rho) = u_{\text{max}}(1 - \frac{\rho}{\rho_{\text{max}}}).$$

Suppose that the initial traffic density is

$$\rho(x,0) = \begin{cases} \frac{\rho_{\text{max}}}{2}, & \text{if } x < -24 \\ \rho_{\text{max}}, & \text{if } -24 < x < 0 \\ 0, & \text{if } x > 0. \end{cases}$$

(a) Determine when the traffic line where $\rho = \rho_{\text{max}}$ completely dissipates.
(b) Describe how the traffic dynamics change in time after $\rho = \rho_{\text{max}}$ dissipates.

Mathematical Modeling Qualifier Exam

August 2013

Name:________________________________ ID:________________________________

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1. Consider the Lotka-Volterra competition model

$$\dot{x} = x(3 - x - y), \quad \dot{y} = y(4 - x - 2y).$$

Show that a solution $(x(t), y(t))$ with $x(0) > 0$ and $y(0) > 0$ will approach the equilibrium $(2, 1)$ as $t \to \infty$.

2. Consider the system

$$\dot{x} = y - x(\mu - \sqrt{x^2 + y^2}) \quad \dot{y} = -x - y(\mu - \sqrt{x^2 + y^2}).$$

(a) Show that this system undergoes a bifurcation at $\mu = 0$, and determine the stability of the bifurcated limit cycle.

(b) Sketch a bifurcation diagram.

(c) Obtain a Poincaré map for this system with $\mu = 1$ on the Poincaré section

$$\Sigma = \{(x, y) \in R^2 : 0 \leq x < \infty, y = 0.\}$$

(Hint: use polar coordinates.)

3. Consider

$$\dot{x} = y, \quad \dot{y} = -\sin x.$$

Find the Hamiltonian of this system, and sketch a phase portrait.

4. Consider the predator-prey model

$$\dot{x} = x(4 - x - \frac{2y}{1 + x}), \ \dot{y} = y(x - 1).$$

Assume that all positive solutions are bounded.
(a) Find all critical points and determine their local stability.
(b) Show that this system has a limit cycle in the first quadrant.

5. Suppose that a mass m is attached between two walls with a distance d apart via two springs of lengths l_1, l_2 with spring constants k_1, k_2 respectively. Assume that $d > l_1 + l_2$.

(a) What position of the mass would be called the equilibrium position of the mass? Compute the stretches of both springs at the equilibrium state.
(b) Show that the mass executes simple harmonic motion about its equilibrium position.
(c) What is the period of oscillation?
(d) Discuss the quantities in (a) and (c) when $\frac{k_1}{k_2} \to \infty$. What is the physical implication?

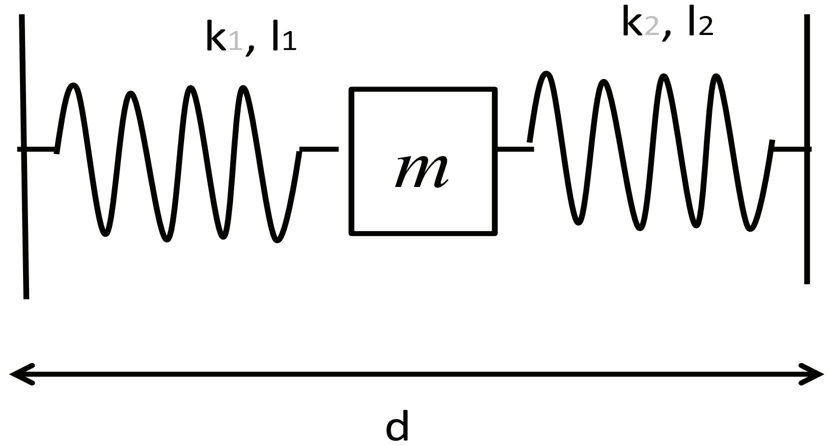

Figure 1: Mass is set on frictionless table, and its size is assumed to be zero.

6. Suppose that the motion of a mass m is described by the nonlinear differential equation

$$m\frac{d^2x}{dt^2} = -\alpha x^3 - \sigma(\frac{dx}{dt})^3$$

where $\alpha > 0,\ \sigma \geq 0$.

(a) Show that if $\sigma = 0$ then every nonzero solution $x(t)$ of the system is a periodic solution.

(b) Show that if $\sigma > 0$ then as $t \to \infty$, $x(t) \to 0$.

7. Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0, \qquad (1)$$

where $q(\rho) = \rho u(\rho)$, and $u(\rho) = u_{\max}(1 - \frac{\rho}{\rho_{\max}})$. Suppose that the initial traffic density is

$$\rho(x,0) = \begin{cases} \rho_{\max}, & \text{if } x < 0, \\ \rho_{\max}(L-x)/L, & \text{if } 0 < x < L, \\ 0, & \text{if } x > L. \end{cases}$$

(a) Solve for $\rho(x,t)$ using the characteristic method.
(b) Sketch the graph of $\rho(x,t)$ in the (ρ, x) -plane for $t > 0$.

8. Consider the same traffic model (1) in Problem 7. Suppose that the initial traffic density is given by

$$\rho(x,0) = \begin{cases} \frac{\rho_{\max}}{3}, & \text{if } x < 0, \\ \frac{\rho_{\max}}{2}, & \text{if } 0 < x < 1, \\ \rho_{\max}, & \text{if } x > 1. \end{cases}$$

(a) Show that initially the system has two shock waves, and determine the velocities of the two shock waves.
(b) Determine when the two shock waves meet.
(c) Determine the velocity of the new shock wave after the two shock waves meet, and sketch the graph of $\rho(x,t)$ in the (ρ, x) -plane for this case.
(d) When will the car initially at $x = -1$ pass the point $x = 1$?

Mathematical Modeling Qualifier Exam

January 2013

Name:________________________________ ID:________________________________

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1. Consider the Lotka-Volterra competition model

$$\dot{x} = x(3 - 2x - y), \quad \dot{y} = y(4 - x - 3y).$$

Show that a solution $(x(t), y(t))$ with $x(0) > 0$ and $y(0) > 0$ will approach the equilibrium $(1, 1)$ as $t \to \infty$.

2. Consider the system

$$\dot{x} = \mu x - y - x\sqrt{x^2 + y^2}, \quad \dot{y} = x + \mu y - y\sqrt{x^2 + y^2}.$$

(a) Show that this system undergoes a bifurcation at $\mu = 0$, determine the stability of the bifurcated solutions.

(b) Sketch a bifurcation diagram.

(c) Obtain a Poincaré map for this system with $\mu = 4$ on the Poincaré section

$$\Sigma = \{(x, y) \in R^2 : 0 \le x < \infty, y = 0.\}$$

(Hint: use polar coordinates.)

3. Consider

$$\dot{x} = y, \quad \dot{y} = x + x^2$$

Find the Hamiltonian of this system, sketch a phase portrait, and discuss the stability of all equilibria.

4. (a) Investigate the stability of the equilibrium $(0, 0)$ for the system

$$\dot{x} = -y - x^5, \quad \dot{y} = x - y^5.$$

(Hint: Consider the Lyapunov function $V(x, y) = x^2 + y^2$).

(b) Prove that the system

$$\dot{x} = x(1 - 4x + y), \quad \dot{y} = y(2 + 3x - 2y)$$

has no limit cycles in the first quadrant by applying Bendixson-Dulac's criteria with $\beta(x, y) = x^m y^n$.

5. Consider two masses each of mass m attached between two walls a distance d apart. Assume that all three springs have the same spring constants and un-stretched length l with $d = 3l$. See the picture below.

(a) Derive two second order ODEs to describe the motion of the two masses. What position of each mass would be called the equilibrium position of the system of masses?

(b) Show that the motion of the two masses are linear combinations of two simple harmonic motions. What are the frequencies of these two simple harmonic motions?

(c) Find out two energy functions such that they remain constant on the solution orbits.

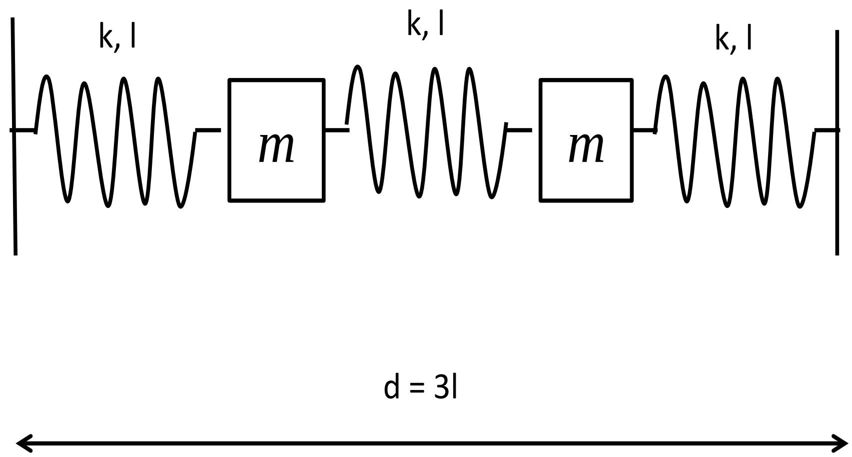

Figure 1: Masses are set on frictionless table, and their sizes are assumed to be zero.

6. Suppose that a mass spring system satisfies

$$\frac{d^2x}{dt^2} + (x^2 - 1)\frac{dx}{dt} + x = 0. \tag{1}$$

(a) Define the energy function

$$E(\dot{x}, x) = \frac{\dot{x}^2}{2} + x^2.$$

Compute $\frac{dE}{dt}$ along the solution orbits of (1).

(b) Based on the computation in (a), show that if there exists a periodic solution to (1), then the periodic solution can not be located in the domain $\{|x| < 1\}$ completely.

(c) Find all the equilibrium points, and discuss their stability.

7. Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{2}$$

where $q(\rho) = \rho \cdot u(\rho)$. Assume $u(\rho) = 40\left(1 - \frac{\rho}{16}\right)$ and the initial density is

$$\rho(x,0) = \begin{cases} 12 & x < 0 \\ 4 & x > 0 \end{cases}$$

(a) Find the equations of the characteristics, and sketch the characteristics in the (x,t)-plane.

(b) Solve (2) for $\rho(x)$ by using the characteristic method, and sketch the graph of ρ in (x,ρ)-plane.

(c) How long does it take the car originally at $x = -10$ to arrive at the point $x = 10$?

8. Consider the same traffic model (2) in Problem 7. The initial density is given by

$$\rho(x,0) = \begin{cases} 4 & x < 0 \\ 4 + x & 0 < x < 8 \\ 12 & x > 8 \end{cases}$$

Sketch the initial density, determine and sketch the density $\rho(t,x)$ at all later time.

Mathematical Modeling Qualifier Exam

August 2012

Name:______________________________ ID:______________________________

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1. Consider the model for two competing species

$$\begin{cases} x' = x(3 - x - y) \\ y' = y(4 - x - 2y). \end{cases} \tag{1}$$

(a) Find out all the equilibrium points in $\mathbf{R}^2$, and study their stability.

(b) Show that there is no closed orbit to the system in $\mathbf{R}^2_+$ by using the Dulac's theorem.

2. Consider the following vector field on $\mathbf{R}^2$:

$$\begin{cases} x' = 2\mu^2 x - y - x\sqrt{x^2+y^2}\left[3\mu - \sqrt{x^2+y^2}\right] \\ y' = x + 2\mu^2 y - y\sqrt{x^2+y^2}\left[3\mu - \sqrt{x^2+y^2}\right]. \end{cases} \tag{2}$$

(a) Show that this system undergoes a bifurcation at $\mu = 0$, and determine the stability of the bifurcated solutions.

(b) Sketch a bifurcation diagram.

3. Consider the planar system

$$\begin{cases} x' = -x^4 + 5\mu x^2 - 4\mu^2 \\ y' = -y, \end{cases} \tag{3}$$

where μ is a constant parameter.

(a) Find the equilibrium points of (3).

(b) Find the bifurcation value.

(c) Draw the phase portraits of (3) for various values of parameter μ.

(d) Draw the bifurcation diagram. Clearly label and explain the diagram.

4. Consider modified Rosenzweig-MacArthur model.

$$\begin{cases} u' = u(1-u)(0.4+u) - uv \\ v' = -\frac{1}{3}v(0.4+u) + uv. \end{cases} \tag{4}$$

(a) Find out all the equilibrium points and discuss their stability.

(b) Prove that there exists a limit cycle to the above system in $\mathbf{R}^2_+$.

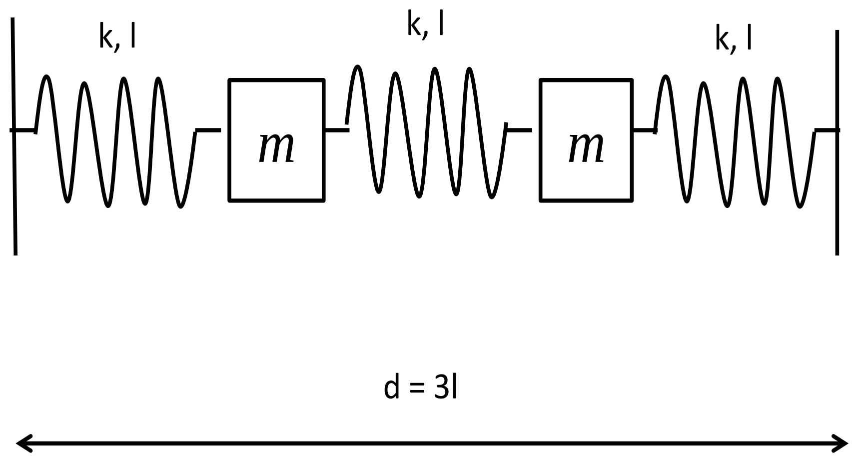


Figure 1: Masses are set on frictionless table, and their sizes are assumed to be zero.

5. Consider two masses with each of mass m attached between two walls a distance d apart. Assume that all three springs have the same spring constants and un-stretched length l with $d = 3l$. See the picture above.

   (a) Derive two second order ODEs to describe the motion of the two masses. What position of each mass would be called the equilibrium position of the system of masses?

   (b) What are the conditions on the initial data such that two masses remain equal distance to the walls.

   (c) What are the conditions on the initial data such that the distance between two masses remains constant.

   (d) Find out two energy functions such that they remain constant on the solution orbits.

6. Suppose that a mass spring system satisfies

$$\frac{d^2x}{dt^2} = -x + x^3. \tag{5}$$

   (a) Write the model (5) as an equivalent Hamiltonian system, then find all the equilibrium points, and discuss their stability.

   (b) Find the energy function for the model (5), and identify the corresponding potential energy.

   (c) Sketch the phase portrait in (x, v)-plane ($v = \frac{dx}{dt}$) by using the potential energy.

   (d) For the model (5) with friction as follows

$$\frac{d^2x}{dt^2} = -x + x^3 - \frac{dx}{dt}. \tag{6}$$

   Show that energy function obtained in (b) is a Lyapunov function for the system (6). What does it convey about the stability of the equilibrium points for the system (6)?

7. (Green light traffic problem) Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{7}$$

   where $q(\rho) = \rho \cdot u(\rho)$. Assume $u(\rho) = 36\left(1 - \frac{\rho}{18}\right)$ and the initial density is

$$\rho(x, 0) = \begin{cases} 18 & x < 0 \\ 0 & x > 0. \end{cases}$$

(a) Find the equations of the characteristics, and sketch the characteristics in the (x, t)-plane.

(b) Solve (7) for $\rho(x)$ by using the characteristic method, and sketch the graph of ρ in (x, ρ) plane.

(c) What is the car which will cross the traffic light (x=0) at $t = 5$?

8. Consider the same traffic model (7) in Problem 7. The initial density is given

$$\rho(x, 0) = \begin{cases} 3 & x < 0 \\ 6 & 0 < x < 18 \\ 12 & x > 18. \end{cases}$$

(a) Find the equations of the characteristics.

(b) Find the equations of all shock waves. Do the shock waves formed at $t = 0$ collide each other? If yes, find the time and location

(c) Sketch the characteristics and shock waves in the (x, t) plane before and after the shock collision.

(d) Solve (7) for $\rho(x)$ by using the characteristic method, and sketch the graph of ρ in (x, ρ) plane. Will the middle state $\rho = 6$ survive?

(e) Describe the terminal states of this traffic flow.

Mathematical Modeling Qualifier Exam
January 2011

Name ______________________________ ID ______________________________

Read every problem carefully. Do six problems out of the total eight problems.

You MUST show complete work to receive full credits. No work, no credit. If you do more problems than required, you MUST indicate which problems you want to be graded. Otherwise, the problems will be chosen sequentially to grade.

1. Consider the following system

$$\begin{cases} \dot{x} = x(1 - 2x - y) \\ \dot{y} = y(1 - x - 2y) \end{cases} \tag{1}$$

Show that the equilibrium point $(1/3, 1/3)$ is the global attractor of all positive solutions.

2. Consider following system

$$\begin{cases} \dot{x} = -4x + 2y + y^2 \\ \dot{y} = -16x + 6y - 27y^3 \end{cases} \tag{2}$$

By considering the trajectories across the square with coordinates $(1, 1)$, $(1, -1)$, $(-1, -1)$, $(-1, 1)$, centered at the origin, prove that the system has a limit cycle.

3. Consider following system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + x^3 \end{cases} \tag{3}$$

Find the Hamiltonian of this system and sketch a phase portrait.

4. Consider the model

$$\begin{cases} \dot{x} = -y + x(x^2 + y^2 - \mu) \\ \\ \dot{y} = x + y(x^2 + y^2 - \mu) \end{cases} \tag{4}$$

(a) Show that this system undergoes a Hopf bifurcation at $\mu = 0$.

(b) Sketch a bifurcation diagram.

(**Hint:** use polar coordinates.)

5. Consider the model

$$\begin{cases} \dot{x} = 4x - y - x\sqrt{x^2 + y^2} \\ \dot{y} = x + 4y - y\sqrt{x^2 + y^2} \end{cases} \tag{5}$$

(a) Obtain a Poincaré map for this system on the Poincaré section
$\Sigma = \{(x, y) \in R^2 : 0 \le x < \infty, y = 0\}$

(b) Find the fixed point of the Poincaré map and the limit cycle for the above system. Determine the stability of the limit cycle in (b)

6. Consider a spring-mass system with cubic friction

$$m\frac{d^2x}{dt^2} + \epsilon\Big(\frac{dx}{dt}\Big)^3 + x = 0, \qquad \epsilon > 0.$$

where $m > 0$ is the mass of the object.

(a) Find an engergy function and show that the engergy is dissiptive over the time.

(b) Show that there exists no periodic solution.

(c) Sketch the phase portrait in (x, v)-plane, where $v = \frac{dx}{dt}$.

7. (Green Light Problem.) Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{6}$$

with initial condition $\rho(x, 0) = f(x)$, where $\rho(x, t)$ is the density of the traffic, $q(\rho) = \rho u(\rho)$ is the traffic flow, and $u(\rho)$ is the velocity field. Assume that $u(\rho) = u_{max}(1 - \dfrac{\rho}{\rho_{max}})$ and the initial density is

$$f(x) = \begin{cases} \rho_{max} & x \le 0, \\ 0 & x > 0. \end{cases} \tag{7}$$

(a) Find the equations of the characteristics, and sketch the characteristics in the (x, t)-plane.

(b) Solve (6) for $\rho(x, t)$ using the characteristic method, and sketch the graph of ρ in (x, ρ)-plane.

(c) Find the path of the car starting at $x(0) = -L$, where $L > 0$, and then find the time elapsed when it passes the traffic light at $x = 0$. Draw the car path in the graph you sketched in (a).

8. (Red Light Problem.) Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{8}$$

with initial condition $\rho(x, 0) = f(x)$, where $\rho(x, t)$ is the density of the traffic $q(\rho) = \rho u(\rho)$ is the traffic flow, and $u(\rho)$ is the velocity field. Assume that $u(\rho) = 45(1 - \dfrac{\rho}{270})$ and the initial density is

$$f(x) = \begin{cases} 10 & x < 0, \\ 20 & 0 < x < L, \\ 40 & x > L. \end{cases} \tag{9}$$

where $L > 0$.

(a) Find the equations of the characteristics.

(b) Find the equations of all shockwaves, and the time and location that the shockwaves formed at $t = 0$ collide;

(c) Sketch the characteristics and shockwaves in the (x, t)-plane.

(d) Solve (8) for $\rho(x, t)$ using the characteristic method, and sketch the graph of ρ in (x, ρ)-plane.

Mathematical Modeling Qualifier Exam
August 2010

Name ______________________________ ID ______________________________

This Exam contains two parts and four pages.

Read every problem carefully. Do six problems out of the total eight problems.

You MUST show complete work to receive full credits. No work, no credit. If you do more problems than required, you MUST indicate which problems you want to be graded. Otherwise, the problems will be chosen sequentially to grade.

Part I:

1. Consider the following predator-prey model

$$\begin{cases} \dot{x} = x(7 - x - y) \\ \dot{y} = y(3 + x - y) \end{cases} \tag{1}$$

(a) Find all the equilibrium points and plot these points on (x, y)-plane.

(b) Find the linearized system of (1) at the equilibrium $E_0 = (0, 0)$ and study its stability.

(c) Show that the system does not have a closed orbit in the first quadrant.

(d) Let $E^* = (x^*, y^*)$ be the interior equilibrium point. Show that E^* is globally asymptotically stable in the first quadrant.

(e) Assume that x and y stand for prey and predator in a two species environment, respectively. Based on the mathematical results, describe ecological meaning in lay English.

(**Hint**: The functions $B(x, y) = \dfrac{1}{x^m y^n}$ for some specific integers m and n, and $V(x, y) = (x - x^* \ln x + y - y^* \ln y) - (x^* - x^* \ln x^* + y^* - y^* \ln y^*)$ might be helpful.)

2. Consider following system

$$\begin{cases} \dot{x} = a^2 x + y - x(x^2 + y^2) \\ \dot{y} = -x + a^2 y - y(x^2 + y^2). \end{cases} \tag{2}$$

where $a > 0$ is a constant.

(a) Obtain a Poincaré map for the system on $\sum = \{(x, y) \in R^2 : 0 \le x < \infty, y = 0\}$.

(b) Find the limit cycle in the system.

3. Consider the one-dimensional system

$$\dot{x} = -x^4 + 13\mu x^2 - 36\mu^2, \tag{3}$$

where μ is a constant parameter.

(a) Find the equilibrium points of the system (3).

(b) Find the bifurcation value(s) of the system and justify your answer.

(c) Draw the phase portraits of the system for various values of the paraemter μ.

(d) Draw the bifurcation diagram for the system. Clearly label and explain the diagram.

4. Consider the predator-prey model

$$\begin{cases} \dot{x} = x\left(1 - \dfrac{x}{30}\right) - \dfrac{xy}{x+10} \\ \\ \dot{y} = \dfrac{xy}{x+10} - \dfrac{y}{3} \end{cases} \tag{4}$$

It is known that the equilibrium points in the first quadrant are $E_0(0,0), E_1(30,0)$ and $E^*\left(5, \dfrac{25}{2}\right)$.

(a) Qualitatively analyze the model (4).

(b) Interpret the mathematical results to biological meanings.

(**Hint**: The following partial derivatives could be helpful.

$$\frac{\partial P}{\partial x} = 1 - \frac{1}{15}x - \frac{y}{x+10} + \frac{xy}{(x+10)^2}, \qquad \frac{\partial P}{\partial y} = -\frac{x}{x+10}$$

$$\frac{\partial Q}{\partial x} = \frac{y}{x+10} - \frac{xy}{(x+10)^2}, \qquad \frac{\partial Q}{\partial y} = \frac{x}{x+10} - \frac{1}{3}$$

)

Part II:

1. Consider the pendulum model

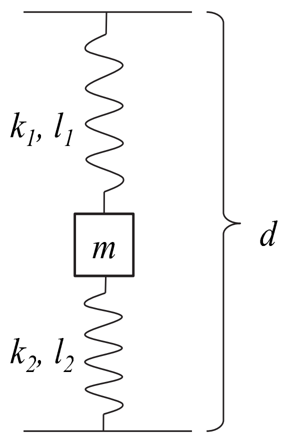
$$\ddot{x} + 2\sin x = 0. \tag{5}$$

(a) Show that the model (5) is a Hamiltonian system.

(b) Find a Hamiltonian or the energy function for the model (5).

(c) Sketch the phase portrait of the system in the (x, y)-plane, where $y = \dot{x}$.

2. Suppose that a mass m were attached vertically between two bars in a mechanical device as shown in the following figure. The distance between the two bars is d. The two springs have spring coefficients k_1 and k_2, lengths l_1 and l_2, respectively. Assume that the mass is at a point in the established one dimensional coordinate system (refer to the figure).



(a) Formulate a differential equation model regarding to the position x of the mass.

(b) Find the equilibrium position E^* of the mass, i.e., find the coordinate of E^*. If the two springs are identical (i.e., $k_1 = k_2$, and $l_1 = l_2$), is E^* right at the middle of the two bars? Why?

(c) Show that this model is Newtonian and find its energy function.

(d) Sketch the phase portrait of this system in the (x, v)-plane, where $v = \dfrac{dx}{dt}$. Justify your figure. (**Hint**: Notice that d might not be equal to $l_1 + l_2$.)

3. Consider the traffic flow model

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{6}$$

with initial condition $\rho(x, 0) = f(x)$, where $\rho(x, t)$ is the density of the traffic, $q(\rho) = \rho u(\rho)$ is the traffic flow, and $u(\rho)$ is the velocity field. Assume that f and q are smooth enough. Prove that $\rho = f(x - q'(\rho)t)$ is a solution of (6).

4. In the traffic flow model (6), assume that the velocity field $u(\rho) = 1 - \rho$ and the initial density is given by

$$\rho(x,0) = \begin{cases} \frac{1}{4} & x \le 0, \\ \frac{1}{3} & 0 < x \le a, \\ \frac{3}{4} & x > a. \end{cases} \tag{7}$$

(a) Sketch the initial density.

(b) Find the equations of the characteristics.

(c) Find the equations of all shockwaves, and the time and location that the shockwaves formed at $t = 0$ collide.

(d) Sketch the characteristics and shockwaves in the (x,t)-plane.

(e) Solve (6) for $\rho(x,t)$ using the characteristic method, and sketch the graph of ρ in the (x,ρ)-plane.

Mathematical Modeling Qualifier Exam

May, 2008

Name ______________________________________ **ID** _________________________

There are two parts in this Exam.

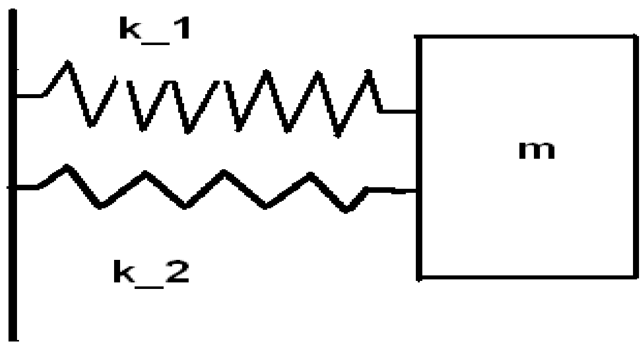
Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

Part I:

1. Suppose that a mass m were attached to two springs in parallel (see figure 1).

Figure 1:



Set up the model for the spring-mass system and answer the following questions.

(a) What position of the mass would be called the equilibrium position of the mass.

(b) Show that the mass executes simple harmonic motion about its equilibrium position.

(c) What is the period of oscillation?

2. Consider a nonlinear pendulum with Newtonian damping

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta - \beta\frac{d\theta}{dt}\left|\frac{d\theta}{dt}\right|,$$

where $\beta > 0$.

(a) By introducing the phase plane variable $v = \frac{d\theta}{dt}$, show that

$$\frac{dv}{d\theta} = \frac{-g\sin\theta - \beta v|v|}{Lv}.$$

(b) Instead of sketching the isoclines, show that

$$L\frac{d}{d\theta}(v^2) \pm \beta v^2 = -g\sin\theta.$$

(c) Under what conditions does the $+$ or $-$ sign apply?

(d) This is a linear differential equation for v^2. Solve this equation.

(e) Using this solution, roughly sketch the phase plane (trajectories on the $\theta - v$ plane).

3. Consider the traffic model

$$\frac{\partial\rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial\rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x,0) = \begin{cases} 2 & x \le 0 \\ 0 & 0 < x < 4 \\ 4 & x \ge 4. \end{cases}$$

Suppose that the traffic flux is given by the function $q(\rho) = 12\rho(1 - \frac{\rho}{4})$. Sketch the characteristics and the shocks. Find formulas for the density $\rho(x,t)$ and the shock $x_s(t)$.

4. The traffic flow in a highway with entrances and exits may be modeled as

$$\frac{\partial\rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial\rho}{\partial x} = \beta,$$

where

$$\beta(x,t) = \begin{cases} 0 & x \le 0 \\ 2 & x > 0. \end{cases}$$

Suppose that $q(\rho) = 4\rho(2 - \rho)$ and the initial density is

$$\rho(x,0) = \begin{cases} 2 & x \le 0 \\ 0 & x > 0. \end{cases}$$

(a) Sketch the graphs of characteristics and typical densities for different time t.

(b) Find formulas for the density $\rho(x,t)$ and the characteristics.

Part II:

1. Consider

$$x' = -\frac{1}{2}y(1+x) + x(1-4x^2-y^2), \;\; y' = 2x(1+x) + y(1-4x^2-y^2).$$

   (a) Prove that the origin is a unique critical point of the system and determine stability of the origin.

   (b) Use the Lyapunov function $V(x, y) = (1-4x^2-y^2)^2$ to show that $\Lambda^+(p) = \{(x, y)|4x^2+y^2 = 1\}$ for each $p \in \mathbf{R}^2$. (That is, $\{(x, y)|4x^2 + y^2 = 1\}$ is the global attractor of the system.)

2. (a) Prove that the system

$$x' = x(1-4x+y), \;\; y' = y(2+3x-2y)$$

   has no limit cycles by applying Bendixson's criteria with $\psi = x^m y^n$.

   (b) Consider the Hamiltonian system

$$\dot{x} = y,$$
$$\dot{y} = x - x^3$$

   Find the Hamiltonian of the system and sketch a phase portrait.

3. Consider the predator-prey system

$$x' = 2x - x^2 - xy,$$
$$y' = -y - y^2 + xy.$$

   (a) Find all critical points.

   (b) Discuss the linearized system in a neighborhood of each critical point. (Describe its type and stability.)

   (c) Use isoclines to sketch a phase portrait for the nonlinear system. (Note: $x \geq 0, y \geq 0$. There is no limit cycle.)

   (d) Interpret the solution in terms of species behavior.

4. Consider the one-parameter system of differential equations

$$x' = x^2 - x\mu^2,$$
$$y' = -y.$$

   Find the critical points, plot phase portraits, and sketch a bifurcation diagram.

Mathematical Modeling Qualifier Exam
Aug, 2008

Name __________________________________ **ID** ________________________

There are two parts in this Exam.

Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

Part I:

1. Consider

$$m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} + \alpha(e^{\beta x} - 1),$$

where α, β and c are positive constants.

(a) Find equilibrium point of the system, and determine its linear stability.

(b) Derive a first-order differential equation describing the phase plane ($\frac{dx}{dt}$ as a function of x).

(c) Find some typical isoclines for the case $\alpha = \beta = m = c = 1$. Use the isoclines to roughly sketch solutions in the phase plane based on the first-order differential equation. Show details.

2. Consider a nonlinear pendulum

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta.$$

(a) Derive the energy integral

$$\frac{L}{2}\left(\frac{d\theta}{dt}\right)^2 = g(\cos\theta - 1) + E.$$

(b) Use the energy integral to obtain an expression for the period of oscillation.

(c) Estimate the period if the energy is very large ($E >> 2g$). (Hint: $2g/E << 1$.)

3. (Green light problem) Assume that a traffic is stopped by an extremely long red light. Suppose that at $t = 0$ the traffic light turns from red to green. The traffic flow problem for the case may be modeled as following:

$$\frac{\partial\rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial\rho}{\partial x} = 0,$$

with the initail condition

$$\rho(x,0) = \begin{cases} \rho_{max} & x \le 0 \\ 0 & x > 0, \end{cases}$$

where ρ is the traffic density and q the traffic flux, $q = \rho u(\rho) = u_{max}\rho(1 - \rho/\rho_{max})$. If $u_{max} = 40$ and $\rho_{max} = 225$,

(1) Solve $\rho(x,t)$ for $x \in R$, $t > 0$.

(2) Find the car trajectory $x(t)$ for the car at $x(0) = -1$. When will the car pass the traffic light at $x = 0$?

(3) Sketch the graph of characteristics, the density at $t = 1$, and the car trajectory.

4. (Red light problem) Assume that a traffic is moving at a constant density ρ_0, and then stopped at a point $x = 0$ by a red light for a finite amount of time T_1. The traffic light turns green again at $t = T_1$. The traffic flow problem for the case may be modeled as following:

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x,0) = \begin{cases} \rho_0 & x < 0 \\ \rho_{max} & x = 0^- \\ 0 & x = 0^+ \\ \rho_0 & x > 0, \end{cases}$$

where ρ is the traffic density and q the traffic flux, $q = \rho u(\rho) = u_{max}\rho(1 - \rho/\rho_{max})$. If $u_{max} = 40$, $\rho_{max} = 200$, $\rho_0 = 100$, and $T_1 = 2$.

(1) Find the equations of the shock before the red light ('left' shock) and the shock after the red light ('right' shock).

(2) Solve $\rho(x,t)$ for $x \in R$, $0 < t < T_1$.

(3) Assume that the lead car catches up to the uniformly moving traffic at $t = T_u$, and the line of stopped cars completely dissipates at $t = T_d$. Solve $\rho(x,t)$ for $x \in R$, $T_1 < t < T_2$, where $T_2 = \min\{T_u, T_d\}$.

(4) Sketch the graph of characteristics, the shocks and fan-like solutions. Sketch the graphs of $\rho(x,t)$ for $t = 0$, $t = T_1$, and $t = T_2$.

Part II:

1. A certain species of fish can be divided into three age groups, each one year long. The Leslie matrix for the female portion of the population is given by

$$L = \begin{pmatrix} 0 & 4 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{pmatrix}$$

   (a) Find the long-term growth rate and the long-term distribution of age classes of the population. (**Hint**: The equation $-\lambda^3 + 2\lambda + 1 = 0$ has a root $\lambda = -1$.)

   (b) Determine a sustainable harvesting policy, if harvest the oldest age class only. (**Hint**: $(1 - d_1)[b_1 + b_2 c_1 (1 - d_2) + b_3 c_1 c_2 (1 - d_2)(1 - d_3)] = 1$.)

   (c) Describe how you determine the long-term distribution of the age classes of the population if the substainable harvesting policy in (b) is applied. (**Note**: You are not required to find the eigenvalues, give only formulas and describe the procedure.)

2. Consider the following one-parameter system of differential equations in polar form:

$$\dot{r} = r(\mu^2 - r^2), \quad \dot{\theta} = 1.$$

   Plot phase portraits for $\mu < 0$, $\mu = 0$ and $\mu > 0$. Sketch corresponding bifurcation diagrams.

3. Consider the Hamiltonian system

$$\dot{x} = y,$$
$$\dot{y} = x - x^3$$

   (a) Find all critical points and determine their local stability

   (b) Find the Hamiltonian of the system and sketch a phase portrait.

4. Solve the following differential equations

$$\dot{r} = r(4 - r^2), \quad \dot{\theta} = 1.$$

   By considering the line segment $\Sigma = \{(x, y) \in R^2 : 0 \le x < \infty\}$, find the Poincaré map for this system.

Mathematical Modeling Qualifier Exam

May, 2007

Name ______________________________ **ID** ____________________

There are two parts in this Exam.

Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

Part I:

1. (a) If the initial energy is sufficiently large, determine an expression for the time it takes a pendulum to go completely around.

   (b) Estimate the time if the energy is very large ($E >> 2g$).

2. Assume that a mass m satisfies $m\dfrac{d^2x}{dt^2} = -(x^2 - 1)$.

   (a) What are the equilibrium positions?

   (b) Derive an expression for conservation of energy.

   (c) Sketch trajectories in phase plane for the equation.

3. Consider the traffic model

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = 0,$$

   with the initail condition

$$\rho(x,0) = \begin{cases} \frac{1}{2} & x \le 0 \\ \frac{3}{2} & 0 < x < 2 \\ 1 & x \ge 2. \end{cases}$$

   Suppose that the traffic flux is given by the function $q(\rho) = 3\rho(2-\rho)$. Sketch the characteristics and shock. Find formulas for the density $\rho(x,t)$ and the shock $x_s(t)$.

4. Assume that $u = u_{max}(1 - \rho/\rho_{max})$.

   (a) Show that the time of intersection of neighboring characteristics starting at $x(0) = x_1$ is

$$t = \frac{\rho_{max}}{2u_{max}\frac{d\rho}{dx_1}}.$$

   (b) If at $t = 0$,

$$\rho(x,0) = \rho_{max} exp\left(\frac{-x^2}{L^2}\right).$$

   (1) Sketch the initial density. (2) Determine the time of the first shock. (3) Where does this shock first occur?

Part II:

1. Cosider a model of two competing species

$$x' = x(1 - x - y), \;\; y' = y(2 - x - \beta y),$$

where the parameter β satisfies the condition $0 < \beta < 2$.

(a) Find all critical points and their local stability.

(b) Show that the point $\left(0, \frac{2}{\beta}\right)$ is a global attractor for all positive solutions.

2. Consider the system

$$x' = x - y - x^3,$$
$$y' = x + y - y^3.$$

(a) This system has only one critical point. Determine its local stability.

(b) Show that this system has a limit cycle.

(c) Show that this system has at most one limit cycle.

3. Consider the following one-parameter system of differential equations in polar form:

$$\dot{r} = r(\mu - r)(\mu - 2r), \quad \dot{\theta} = -1.$$

Plot phase portraits for $\mu < 0$, $\mu = 0$ and $\mu > 0$. Sketch corresponding bifurcation diagrams.

4. (a) Solve the following differential equations

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1.$$

(b) By considering the line segment $\Sigma = \{(x, y) \in R^2 : 0 \leq x < \infty\}$, find the Poincaré map for this system.

Mathematical Modeling Qualifier Exam
Oct, 2007

Name ______________________________ **ID** ____________________

There are two parts in this Exam.

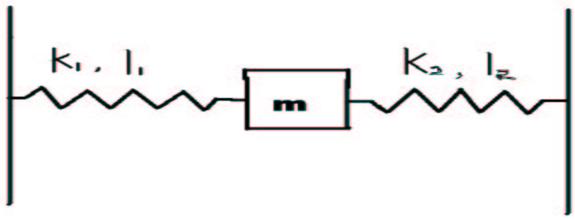
Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

Part I:

1. Suppose that a mass m were attached to two walls distance d apart. The spring coefficients and natural lengths of the springs are k_1, l_1 and k_2, l_2, respectively.

Figure 1:



(a) Find the equation describing the movement of the mass.

(b) Find the equilibrium position of the mass.

(c) Solve the equation.

2. The Van der Pol oscillator is described by the following nonlinear differential equation:

$$\frac{d^2x}{dt^2} - \epsilon\frac{dx}{dt}(1 - x^2) + \omega^2 x = 0,$$

where $\epsilon \geq 0$.

(a) Find the equilibrium position and determine its stability.

(b) If displacements are large, what do you expect happens? Give details of analysis.

(c) Sketch the trajectories in the phase plane if $\omega = 1$ and $\epsilon = \frac{1}{10}$. Describe any interesting features of the solution.

3. Consider the traffic model

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x,0) = \begin{cases} 1 & x \leq -2 \\ 3 & -2 < x < 0 \\ 0 & 0 < x < 4 \\ 1 & x \geq 4. \end{cases}$$

Suppose that the traffic flux is given by the function $q(\rho) = 2\rho(3-\rho)$. Sketch the characteristics and the shocks. Find formulas for the density $\rho(x,t)$ and the shock $x_s(t)$.

4. The traffic flow in a highway with entrances and exits may be modeled as

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = \beta,$$

where

$$\beta(x,t) = \begin{cases} 0 & x \leq 0 \\ \beta_0 & 0 < x < 2 \\ 0 & x \geq 2. \end{cases}$$

Suppose that $\beta_0 = 2$, $q(\rho) = 4\rho(2-\rho)$ and the initial density is $\rho(x,0) = 0$.

(a) Sketch the graphs of characteristics and typical densities for different time t.

(b) Find formulas for the density $\rho(x,t)$ and the characteristics.

Part II:

1. Cosider the specific Holling-Tanner model

$$x' = x\left(1 - \frac{x}{7}\right) - \frac{6xy}{(7+7x)}, \quad y' = 0.2y\left(1 - \frac{Ny}{x}\right).$$

where N is a constant with $x(t) \neq 0$ and $y(t)$ representing the populations of prey and predators, respectively. Sketch phase portraits when (a) $N = 2.5$, and (b) $N = 0.5$.

Hint: Construct a phase plane diagram in the usual way. Find the critical points, linearize around each one, determine the isoclines, and plot a phase plane portrait.

2. Show that the system

$$x' = -y + x(1 - 2x^2 - 3y^2), \quad y' = x + y(1 - 2x^2 - 3y^2).$$

has a unique limit cycle.

3. Consider the Hamiltonian system

$$\dot{x} = y,$$
$$\dot{y} = x - x^3$$

(a) Find all critical points and determine their local stability

(b) Find the Hamiltonian of the system and sketch a phase portrait.

4. Consider the Hénon map given by

$$x_{n+1} = 1 - \alpha x_n^2 + y_n,$$
$$y_{n+1} = \beta x_n,$$

where $\alpha > 0$ and $|\beta| < 1$.

(a) Find period-one critical points of the system, and describe how you determine their local stability.

(b) Show that when $\alpha = \frac{3(\beta-1)^2}{4}$ for fixed β, two period-two critical points bifurcate from a period-one critical point.

Mathematical Modeling Qualifier Exam

May, 2006

Name ____ **ID** ___________________________

There are two parts in this Exam.

Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

Part I:

1. If a spring-mass system has a friction force proportional to the cube of the velocity, then

$$m\frac{d^2x}{dt^2} + \sigma\left(\frac{dx}{dt}\right)^3 + kx = 0,$$

where $x = x(t)$ is the position of the mass at time t, m, σ and k are positive constants.

(a) Let $v = \frac{dx}{dt}$. Derive a first-order differential equation describing the phase plane. (That is, find a differential equation for v as a function of x.)

(b) Sketch the solution in the phase plane based on the first-order differential equation. Show the critical point and typical trajectories.

2. Consider the differential equation of a nonlinear pendulum

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta,$$

where θ is the angle from the vertical direction. ($\theta = 0$ is the vertical direction.)

(a) Let $\theta(0) = 0$, and $E = \frac{L}{2}\left[\frac{d\theta}{dt}(0)\right]^2$. ($E$ is the total energy when the potential energy is assumed to be 0 at $\theta = 0$.) Derive the energy integral formula for the system.

(b) Estimate the time it takes a pendulum to go completely around, if the energy E is very large ($E >> 2g$).

3. Consider the traffic model

$$\frac{\partial\rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial\rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x,0) = \begin{cases} 3 & x < 0 \\ 0 & 0 < x < 4 \\ 1 & x \geq 4. \end{cases}$$

Suppose that the traffic flux is given by the function $q(\rho) = 2\rho(3-\rho)$.

(a) Sketch the characteristics and the shocks.

(b)Find formulas for the density $\rho(x,t)$ and the shock $x_s(t)$.

4. The traffic flow in a highway with entrances and exits may be modeled as

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = \beta_0.$$

Suppose that $\beta_0 = 1$, $q(\rho) = 4\rho(2-\rho)$ and the initial density is

$$\rho(x,0) = \begin{cases} 2 & x \leq 0 \\ 0 & x > 0 \end{cases}$$

(a) Sketch the graphs of characteristics and typical densities for different time t.

(b) Find formulas for the density $\rho(x,t)$ and the characteristics.

Part II:

1. Consider the predator-prey system

$$x' = 4x - x^2 - xy,$$

$$y' = -y - y^2 + xy.$$

(a) Find all critical points and determine their local stability.

(b) Show that the coexistence critical point attracts all positive solutions.
(**Hint:** Use the Lyapunov function $V(x,y) = x - x^* \ln x + y - y^* \ln y - (x^* - x^* \ln x^* + y^* - y^* \ln y^*)$, where (x^*, y^*) denotes the coexistence critical point)

(c) Interpret positive solutions in terms of species behavior.

2. Consider the system

$$x' = x - y - x^3,$$
$$y' = x + y - y^3.$$

(a) This system has only one critical point $(x,y) = (0,0)$. Determine its local stability.

(b) Show that this system has one limit cycle.

(c) Show that this system has at most one limit cycle.

3. Consider the Hamiltonian system

$$\dot{x} = y,$$
$$\dot{y} = x - x^3$$

(a) Find all critical points and determine their local stability

(b) Find the Hamiltonian of the system and sketch a phase portrait.

4. Consider the Hénon map given by

$$x_{n+1} = 1 - \alpha x_n^2 + y_n,$$

$$y_{n+1} = \beta x_n,$$

where $\alpha > 0$ and $|\beta| < 1$.

(a) Find period-one critical points of the system, and describe how you determine their local stability.

(b) Show that when $\alpha = \frac{3(\beta-1)^2}{4}$ for fixed β, two period-two critical points bifurcate from a period-one critical point.

Mathematical Modeling Qualifier Exam

October, 2006

Name ______________________________ **ID** ____________________

There are two parts in this Exam.

Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

Part I:

1. Consider the equation of a nonlinear pendulum:

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta,$$

where θ is the angle from the vertical downward direction. ($\theta = 0$ is the vertical direction.)

(a) What are the two equilibrium positions?

(b) Derive an expression for conservation of energy.

(c) Derive an integral formula for the period of the nonlinear pendulum.

2. If a spring-mass system has a friction force proportional to the cube of the velocity, then

$$m\frac{d^2x}{dt^2} + \sigma\left(\frac{dx}{dt}\right)^3 + kx = 0,$$

where $x = x(t)$ is the position of the mass at time t, m, σ and k are positive constants.

(a) Let $v = \frac{dx}{dt}$ and sketch the solution in the phase plane.

(b) Let $v = \frac{dx}{dt}$ and solve the problem exactly.

(c) Use the results in (a) and (b) to predict how the system behave.

3. Consider the traffic model

$$\frac{\partial\rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial\rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x,0) = \begin{cases} 4 & x < 0 \\ 0 & 0 < x < 4 \\ 2 & x \geq 4. \end{cases}$$

Suppose that the traffic flux is given by the function $q(\rho) = 12\rho(1 - \frac{\rho}{4})$.

(a) Sketch the characteristics and the shocks.

(b)Find formulas for the density $\rho(x,t)$ and the shock $x_s(t)$.

4. Consider the nonlinear car-following model

$$\frac{d^2 x_n(t+T)}{dt^2} = -\lambda \frac{\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt}}{|x_n(t) - x_{n-1}(t)|}, \quad n = 0, 1, 2, ..., N-1,$$

where $x_n(t)$ is the location of the n^{th} car, N is the number of cars, $T > 0$ is a response time delay, and $\lambda > 0$ is a constant.

(a) Assume that $v_n(t) = \frac{dx_n(t)}{dt} = 0$, $(n = 0, 1, 2, ..., N-1)$, for $t < 0$, and $v_0(t)$ is known for $t > 0$. Formulate a general procedure to solve this nonlinear time-delay equation. **Do not evaluate the integrals.**

(b) Under some assumptions, the car-following model may be simplified to a linear model

$$\frac{d^2 x_n(t+T)}{dt^2} = -\lambda \left(\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt} \right).$$

Consider three cars only. Assume that $T = 1$, $\lambda = 2$. Let $v_n(t) = \frac{dx_n(t)}{dt}$. If $v_n(t) = 0$ for $t < 0$, n=0,1,2, and $v_0(t) = t + t^2/6$ for $t > 0$, compute $v_1(2.5)$ and $v_2(2.5)$.

Part II:

1. Consider the predator-prey system

$$x' = 2x - xy,$$
$$y' = -3y + xy.$$

(a) Find all critical points.

(b) Discuss the linearlized system in a neighborhood of each critical point. (Describe its type and stability.)

(3) Plot a phase portrait for the nonlinear system.

(4) Interpret the solution in terms of species behavior.

2. (a) Prove the system

$$x' = y + x\left(\frac{1}{2} - x^2 - y^2\right),$$
$$y' = -x + y\left(1 - x^2 - y^2\right)$$

has a stable limit cycle.

(b) Find the Hamiltonian of the system

$$\dot{x} = y,$$
$$\dot{y} = x - x^3$$

and sketch a phase portrait.

3. Consider the following one-parameter system of differential equations:

$$\dot{x} = -x^4 + 5\mu x^2 - 4\mu^2,$$
$$\dot{y} = -y.$$

Find the critical points, plot phase portraits, and sketch the corresponding bifurcation diagram.

4. (a) Obtain a Poincaré map for the system

$$\dot{x} = \mu x + y - x\sqrt{x^2 + y^2},$$

$$\dot{y} = x + \mu y - y\sqrt{x^2 + y^2}$$

on the Poincaré section $\Sigma = \{(x, y) \in R^2 : 0 \leq x < \infty, y = 0\}$.

(b) Use the characteristic multiplier to determine the stability of the limit cycle.

Mathematical Modeling Qualifier Exam

May, 2004

Name ID

You must show all work to receive full credit.

Do 5 problems out of 6 problems

1. The population dynamics of a species is governed by the discrete model

$$x_{n+1} = x_n[1 + r(1 - x_n)]$$

where $r > 0$.

(a) Find the nonnegative fixed points of period one and determine their stability.

(b) Find the value for r at which a period-doubling bifurcation occurs.

(**Hint:** you may want to use the fact $f(f(x)) - x = -rx(x-1)[r^2x^2 - (r^2+2r)x + r + 2]$ where $f(x) = x[1 + r(1 - x)]$.)

2. A certain species of fish can be divided into three age groups, each one year long. The Leslie matrix for the female portion of the population is given by

$$L = \begin{pmatrix} 0 & 3 & 5 \\ 0.3 & 0 & 0 \\ 0 & 0.8 & 0 \end{pmatrix}$$

(a) Find the long-term growth rate and the long-term distribution of the age classes of the population.

Hint: Use the following results produced by Maple:

A:=matrix([[0,3,5],[0.3,0,0],[0,0.8,0]]): eigenvals(A);

1.3400, -0.6700+0.6683I, -0.6700-0.6683I

eigenvects(A);

$[1.3400, 1, \{[3.5808, 0.8017, 0.4786]\}]$

$[-0.6700 + 0.6683I, 1, \{[4.3678 - 0.7635I, -1.1513 - 0.8065I, 0.2075 + 1.1700I]\}]$

$[-0.6700 - 0.6683I, 1, \{[4.3678 + 0.7635I, -1.1513 + 0.8065I, 0.2075 - 1.1700I]\}]$

(b) Determine a sustainable harvesting policy: harvest the oldest age class only.

(**Hint:** use the formula $(1-d_1)[b_1 + b_2c_1(1-d_2) + b_3c_1c_2(1-d_2)(1-d_3)] = 1$.)

(c) Describe how you determine the long-term distribution of the age classes of the population if the sustainable harvesting policy in (b) is applied.

3. Consider the Hénon map

$$x_{n+1} = \alpha + 0.3y_n - x_n^2, \; y_{n+1} = x_n.$$

This system has a chaotic attractor for $\alpha = \alpha_0 = 1.4$. Use the OGY method to control the chaos to a fixed point of period one:

(a) Find the matrix that is used to determne the chaos control;

(b) Describe how you use the matrix found in (a) to determine the parameters in the chaos control.

4. Consider the Lotka-Volterra competition model

$$\dot{x} = x(2 - \alpha x - y), \; \dot{y} = y(1 - x - y)$$

where the parameter α satisfies the condition $0 < \alpha < 2$.

(a) Find all critical points and their local stability.

(b) Show that the point $(\frac{2}{\alpha}, 0)$ is a global attractor for all positive solutions.

5. Consider the predator-prey system

$$\dot{x} = x(\beta - x) - \frac{xy}{1+x}, \; \dot{y} = y(\frac{x}{1+x} - \frac{1}{2})$$

where the parameter β satisfies the condition $\beta > 3$.

(a) Find all critical points and their local stability.

(b) Show that this system has a limit cycle in the first quadrant.

6. Consider the Duffing system given by

$$\dot{x} = y, \quad \dot{y} = x - x^3$$

(a) Find the Hamiltonian of the system.

(b) Find the equation that describe the homoclinic cycles in the system.

(c) Describe solution orbits inside the homoclinic cycles.