Real Analysis Qualifying Exam
Spring 2017

Complete **three** problems from Part A and **three** problems from Part B. Throughout the test the Lebesgue measure on $\mathbb{R}$ is denoted by m and the Lebesgue outer measure on $\mathbb{R}$ is denoted by m^*.

Part A:

Problem 1. Prove that the series $\sum_{k=0}^{\infty} \sin^k t$ converges uniformly for $t \in [-\pi/4, \pi/4]$ and then evaulate the series

$$\sum_{k=0}^{\infty} \int_{-\pi/4}^{\pi/4} \sin^k t\, dt.$$

Problem 2. Prove that if $A \subset \mathbb{R}$ has Lebesgue measure 0, then $m\left(\{e^x : x \in A\}\right) = 0$.

Problem 3. Let $\{E_n\} \subset \mathcal{M}$ be a sequence of Lebesgue measurable subsets of $[0,1]$. Prove:

(a) If $\sum m(E_n) < \infty$ then $m(\limsup E_n) = 0$

(b) If $m(E_n) \to 0$ it may not be true that $m(\limsup E_n) = 0$.

Problem 4. Prove that if $f : [0,1] \to (0,\infty)$ is absolutely continuous, then so is $1/f$.

Problem 5. Prove that if $f \in L^p([0,\infty))$, $1 \le p \le \infty$, then

$$\lim_{n\to\infty} \int_0^{\infty} f(x)e^{-nx}\, dx = 0.$$

Part B:

Problem 6. Define the function $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ if x is irrational, and by $f(x) = \dfrac{1}{q}$ if x is rational and $x = \dfrac{p}{q}$ when written in least terms. Decide whether or not f is Riemann integrable on $[0,1]$ and if so, evaulate its integral.

Problem 7. Let $\{p_n\}$ be a sequence of polynomials. Suppose that for every point $x \in [0, 1]$ there exists an index n satisfying $p_n(x) = 0$. Prove at least one of the polynomials is identically zero.

Problem 8. Let $A \subset \mathbb{R}$. Prove that the following are equivalent to each other:

(a) A is not Lebesgue measurable.

(b) There is an $\varepsilon > 0$ such that whenever B is measurable and $A \subset B$, then $m^*(B \setminus A) \geq \varepsilon$.

Problem 9. Prove that if f is absolutely continuous on $[0, 1]$ and there is a $g \in C([0, 1])$ such that $f' = g$ a.e., then f is differentiable on $[0, 1]$ and $f' = g$.

Problem 10. Let $h \in L^\infty(\mathbb{R})$. Define a functional $T : L^1(\mathbb{R}) \to \mathbb{R}$ by

$$Tf = \int_{\mathbb{R}} fh \, dm$$

Prove $\sup_{\|f\|_1 \leq 1} Tf = \|h\|_\infty$.

Spring 2017 A #1

Prove that the series $\sum_{n=1}^{\infty} \sin^n(t)$ converges uniformly for $t \in [-\pi/4, \pi/4]$. Then evaluate the series $\sum_{n=1}^{\infty} \int_{-\pi/4}^{\pi/4} \sin^n(t)\,dt$

Note: <u>Weierstrass M test</u>

Let $\sum_{n=1}^{\infty} f_n$ be a series of real valued functions on a subset $A \subset \mathbb{R}$.

Suppose $\exists$ convergent series $\sum_{n=1}^{\infty} M_n$, $M_n \geq 0$

s.t. $\forall n \in \mathbb{N}$ & $x \in A$ $|f_n(x)| \leq M_n$

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly

- for $t \in [-\pi/4, \pi/4]$

$$\sin^n(t) \leq \left(\frac{1}{\sqrt{2}}\right)^n \quad \forall n$$

with $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$ converges by geometric series

Thus by weirstrass M test the seq converges uniformly

- Since the seq converges uniformly on a compact set

$$\sum_{n=1}^{\infty} \int_{-\pi/4}^{\pi/4} \sin^n(t)\,dt = \int_{-\pi/4}^{\pi/4} \sum_{n=1}^{\infty} \sin^n(t)\,dt = \int_{-\pi/4}^{\pi/4} \frac{1}{1-\sin(t)}\,dt$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1+\sin(t)}{\cos^2(t)}\,dt = \int_{-\pi/4}^{\pi/4} \sec^2(t)\,dt + \int_{-\pi/4}^{\pi/4} \tan(t)\sec(t)\,dt = \tan(t)\Big|_{-\pi/4}^{\pi/4} + \sec(t)\Big|_{-\pi/4}^{\pi/4}$$

$$= (1) - (-1) + \sqrt{2} - \sqrt{2} = 2$$

Spring 2077 A #2

Prove that if $A \subset \mathbb{R}$ has Lebesgue measure zero then $m(\{e^x : x \in A\}) = 0$

We will show $m(e^A \cap [n, n+1]) = 0 \quad \forall\, n \in \mathbb{Z}$

$$\Rightarrow \quad m(e^A) = m\left(\bigcup_{n \in \mathbb{Z}} (e^A \cap [n, n+1])\right) \leq \sum_{n \in \mathbb{Z}} m(e^A \cap [n, n+1]) = 0$$

We know that e^x is increasing

& is ABS Cont on $[n, n+1]$

Let $\{(a_i, b_i)\}$ be a collection of disjoint intervals covering $A \cap [n, n+1]$ s.t. $\sum |b_i - a_i| < \delta$

Let $f_n = e^x \cdot \chi_{[n,n+1]}$ for a fixed n.

Hence $m(e^A \cap [n, n+1]) \leq m\left(f_n\left(\bigcup (a_i, b_i)\right)\right)$ since $\bigcup (a_i, b_i)$ covers $A \cap [n, n+1]$

$\leq \sum |f_n(b_i) - f_n(a_i)|$ since e^x is increasing

$< \varepsilon$ By uniform continuity.

Thus $m(e^A \cap [n, n+1]) = 0$

Spring 2017 A #3

Let $\{E_n\} \subset \mathcal{M}$ be a sequence of Lebesgue measurable subsets of $[0,1]$ Prove:

a) If $\sum m(E_n) < \infty$ then $m(\limsup E_n) = 0$

b) If $m(E_n) \to 0$ it may not be true that $m(\limsup E_n) = 0$

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a) Let $\varepsilon > 0$, since $\sum m(E_n) < \infty$

$$\exists N \text{ s.t. } \sum_{k=N} m(E_n) < \varepsilon$$

$$\limsup(E_n) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subset \bigcup_{k=N}^{\infty} E_k$$

Hence $m(\limsup(E_n)) \leq m\left(\bigcup_{k=N}^{\infty} E_k\right) \leq \sum_{k=N}^{\infty} m(E_k) < \varepsilon$

$$\Longrightarrow m(\limsup E_n) = 0$$

b) Let $E_1 = [0,1]$

$E_2 = [0, 1/2]$, $E_3 = [1/2, 1]$

$E_4 = [0, 1/4]$, $E_5 = [1/4, 1/2]$, $E_6 = [1/2, 3/4]$, $E_7 = [3/4, 1]$

$E_8 = [0, 1/8]$, . . . , - - - -

& so on. $\lim_{n\to\infty} m(E_n) = 0$

But $\forall n \in \mathbb{N}$ $\bigcup_{k=n}^{\infty} E_k = [0,1]$ thus $\limsup(E_k) = [0,1]$

Spring 2017 A #4

Prove if $f: [0,1] \to (0,\infty)$ is ABS cont then so is $1/f$

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given f is ABS cont.

$\Rightarrow$ for $\varepsilon > 0$ $\exists\, \delta > 0$ s.t.

if $\{(a_i, b_i)\}_{i=1}^{n}$ is a finite collection of disjoint intervals of $[0,1]$

w/ $\sum_{i=1}^{n} |b_i - a_i| < \delta \Rightarrow \sum_{i=1}^{n} |f(a_i) - f(b_i)| < \varepsilon$

Since f is ABS cont $\Rightarrow$ f is continuous & Hence it is Bounded on $[0,1]$

So let $m = \min(f)$

& choose δ s.t. $\sum_{i=1}^{n} |b_i - a_i| < \delta \Rightarrow \sum_{i=1}^{n} |f(a_i) - f(b_i)| < m^2 \varepsilon$

$$\sum_{i=1}^{n} \left|\frac{1}{f}(a_i) - \frac{1}{f}(b_i)\right| = \sum_{i=1}^{n} \left|\frac{1}{f(a_i)} - \frac{1}{f(b_i)}\right|$$

$$= \sum_{i=1}^{n} \left|\frac{f(b_i) - f(a_i)}{f(a_i) \cdot f(b_i)}\right|$$

$$\leq \frac{1}{m^2} \cdot \sum_{i=1}^{n} |f(b_i) - f(a_i)| < \frac{m^2 \varepsilon}{m^2} = \varepsilon$$

Hence $\frac{1}{f}$ is ABS cont.

Spring 2017 A # S

Prove if $f \in L^p([0,\infty))$, $1 \le p \le \infty$ then

$$\lim_{n\to\infty} \int_0^\infty f(x) e^{-nx} dx = 0$$

Let $A = \{x \mid f(x) < 1\}$ $B = \{x \mid f(x) \ge 1\}$

Note: Since $f \in L^p$ $m(B) < \infty$

$$\int_0^\infty f(x)e^{-nx} dx = \int f(x)e^{-nx}\chi_A + f(x)e^{-nx}\chi_B dx$$

where $\int_0^\infty f(x)e^{-nx}\chi_A dx < \int_0^\infty e^{-x} = 1$

& $\int f(x)e^{-nx}\chi_B dx \le \int |f|^p \chi_B < \infty$
Since $f \in L^p$

Thus By LDCT

$$\lim_{n\to\infty} \int_0^\infty f(x)e^{-nx} dx = \int_0^\infty \lim_{n\to\infty} f(x)e^{-nx} dx$$
$$= \int 0 \, dx$$
$$= 0.$$

Spring 2017 B #6

Define $f:[0,1] \to \mathbb{R}$ by $f(x)=0$ if x is irrational and by $f(x)=\frac{1}{q}$ if x is Rational & $x=\frac{p}{q}$ when written in least terms.
Decide if f is Riemann Integrable on $[0,1]$ & if so evaluate the integral.

Let $A = \{ x \mid \lim_{x \to a} f(x) \neq f(a) \}$

then $A = \{ x \mid x \in \mathbb{Q} \}$

Thus $m(A) = 0$.

Also f is clearly measurable

$$\{ x \mid f(x) \geq a \} = \begin{cases} (0,\infty) \\ \end{cases}$$

hence f is Lebesgue measurable

AND $\int f = R f$

w/ $\int f = 0$ since $f = 0$ a.e.

Spring 2017 B #7

let $\{P_n\}$ be a sequence of polynomials
suppose that for every pt $x \in [0,1]$
$\exists$ index n satisfying $P_n(x) = 0$
Prove at least one of the polynomials
is identically zero.

Suppose $\nexists$ an n s.t. $P_n \equiv 0$

for each $n \in \mathbb{N}$

define $S_n = \{x \in [0,1] \mid P_n(x) = 0\}$

Note, S_n has to be finite since each poly can only have a finite number of zeros.

now consider $\bigcup_{n=1}^{\infty} S_n \supseteq [0,1]$ since for each $x \in [0,1]$
$\exists$ a polynomial s.t. $P_n(x) = 0$

But a countable collection of finite sets is countable. But $[0,1]$ is uncountable

contradiction.

Spring 2017 B #8

Let $A \subset \mathbb{R}$ Prove TFAE

a) A is not Lebesgue measurable.

b) There is an $\varepsilon > 0$ s.t. whenever B is measurable and $A \subset B$ then $m^*(B \setminus A) \geq \varepsilon$

(a) $\Rightarrow$ (b) : ~~suppose~~ Let A be Not Lebesgue measurable

& suppose $\forall \varepsilon > 0$ $\exists$ B s.t.

$A \subset B$ & $m^*(B \setminus A) < \varepsilon$

So for every $\frac{1}{n}$ let $B_n \supset A$

& $m^*(B_n \setminus A) < \frac{1}{n}$

$A \subset \bigcap B_n$ & $m^*((\bigcap B_n) \setminus A) = 0$

Since $m^*((\bigcap B_n) \setminus A) = 0$

$\Longrightarrow (\bigcap B_n) \setminus A$ is measurable

Lebesgue measure is complete

Hence $A = \left[(\bigcup (B_n^c)) \cup (\bigcap (B_n) \setminus A)\right]^c$ is measurable, contradiction

Spring 2017 B #9

prove if f is ABS cont on $[0,1]$ and there is a $g \in C([0,1])$ s.t. $f' = g$ a.e. then f is diffable on $[0,1]$ and

$f' = g$

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given f ABS CONT on $[0,1]$ thus f is of Bounded variation & Diffable a.e.

Furthermore $f(x) = f(0) + \int_0^x f'(t)\,dt$

$= f(0) + \int_0^x g(t)\,dt$ since $f' = g$ a.e.

$$f'(x) = \lim_{h \to 0} \frac{\int_0^{x+h} g(t)dt - \int_0^x g(t)dt}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} g(t)\,dt$$

$= g(x)$ $\forall x \in [0,1]$ since $g \in C([0,1])$

thus f is Diffable on $[0,1]$

& $f' = g$ $\forall x \in [0,1]$

Spring 2017 B #10

Let $h \in L^\infty(\mathbb{R})$. Define a functional

$T : L^1(\mathbb{R}) \to \mathbb{R}$ by $Tf = \int_{\mathbb{R}} fh \, d\mu$

Prove $\sup_{\|f\|_1 \le 1} Tf = \|h\|_\infty$

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will show $\le$:

$$\sup_{\|f\|_1 \le 1} Tf = \sup_{\|f\|_1 \le 1} \int fh \le \sup_{\|f\|_1 \le 1} \|f\|_1 \|h\|_\infty$$ By Hölder's since $1, \infty$ conj

$$\le 1 \cdot \|h\|_\infty$$

$$= \|h\|_\infty$$

will show $\ge$: we may assume $\|h\|_\infty = M > 0$

$\mathbb{R}$ is σ-finite so $\exists\, F_n \uparrow X$ s.t. $m(F_n) < \infty$

Define $A_n = \{x \in F_n \mid |h(x)| > a\}$ for $0 < a < M$, fixed.

Hence $m(A_n) > 0$

Define $g_n(x) = \dfrac{\operatorname{sgn}(h) \cdot \chi_{A_n}}{m(A_n)} \Rightarrow \|g_n\|_1 = 1 \; \forall n$

& $\int h g_n = a \; \forall n$

Thus $a < \int h g_n \; \forall n$

$$\Rightarrow \sup_{0 < a < M} a < \sup_{\|g_n\|_1 \le 1} \int h g_n$$

$$\|f\|_\infty = M < \sup_{\|f\|_1 \le 1} \int fh = \sup_{\|f\|_1 \le 1} Tf$$

Analysis Qualifying Examination

Department of Mathematics
University of Louisville
August 11, 2016

This test has two sections. Do four problems from each section. If you do more than four problems, indicate the solutions to be graded.

Section A

1. Let $C \subset [0,1]$ be a closed set. Prove that χ_C is Riemann integrable if and only if ∂C has Lebesgue measure zero.

2. Let $S \subset \mathbb{R}$. Prove the following statements are equivalent to each other:

   (a) S is Lebesgue measurable.

   (b) There is a G_δ set G and a set N of measure zero such that $S = G \setminus N$.

3. If f is nonnegative and integrable on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 \sqrt[n]{f} = \lambda\{x : f(x) > 0\}.$$

4. Let $f \in L^1(\mathbb{R})$. If $\int_a^b f = 0$ for all rational numbers a and b with $a < b$, then $f = 0$ a.e..

5. Suppose that $f \in L^2([0,1])$ and $\|f\|_2 = 1$. Define $g(x) = xf(x)$. Prove that $g \in L^1([0,1])$ and that $\|g\|_1 \le \frac{1}{\sqrt{3}}$.

Section B

6. Let $\{a_k\}$ be a sequence of real numbers with the property that $|a_k| \leq 1$ for all k. Prove that both series

$$f(x) = \sum_{k=1}^{\infty} a_k x^k, \quad g(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

converge uniformly on every compact subinterval of $(-1, 1)$ and that $f'(x) = g(x)$ for all $x \in (-1, 1)$.

7. Give an example of a continuous function $f : [0, 1] \to \mathbb{R}$ with the property that $f(0) = 0$, $f(1) = 1$, yet $f'(x) \leq -1$ for almost every $x \in [0, 1]$.

8. Let $E_1, E_2, E_3, \ldots$ be a sequence of measurable subsets of $\mathbb{R}$ with the property that $\sum_{n=1}^{\infty} \lambda(E_n) < \infty$. Show almost every $x \in \mathbb{R}$ is contained in only finitely many of the E_n.

9. Let $f : [0, 1] \to \mathbb{R}$ be Lebesgue measurable. Prove that if

$$p \leq f(x) \leq q$$

for all $x \in [0, 1]$, then $\int_{[0,1]} f$ exists and

$$p \leq \int_{[0,1]} f \leq q.$$

10. Define a sequence of functions $f_n \in L^1[0, 1]$ by

$$f_n(x) = \begin{cases} n, & x \leq 1/n \\ 0, & x > 1/n \end{cases}.$$

Does f_n converge in $L^1[0, 1]$? If so, to what function?

11. Let H be a real Hilbert space. Prove that $\langle x, y \rangle = 0$ if and only if $\|x\| \leq \|x + \lambda y\|$ for every $\lambda \in \mathbb{R}$.

(Aug 2016)

Aug 2016 A #1

Let $C \subset [0,1]$ be a closed set.
Prove that χ_C is Riemann Integrable
IFF ∂C has Lebesgue measure Zero.

$\Leftarrow$: ∂C has Lebesgue measure Zero

hense $m(\{x \mid \lim_{x \to a} f(x) \neq f(a)\}) = 0$

Thus Rieman Integral exists and agrees with its Lebesgue Integral

$\Rightarrow$: χ_C is Riemann Integral
IFF the Lebesgue integral $\exists$
and $m(\{x \mid \lim_{x \to a} f(x) \neq f(a)\}) = 0$
χ_C is discontinuous at its Boundary Pts hence $m(\partial C) = 0$

Aug 2016 A #2

Let $S \subset \mathbb{R}$. Prove TFAE

a) S is Lebesgue measurable.

b) There is a G_δ set G and a set N of measure zero s.t. $S = G \setminus N$

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b $\Rightarrow$ a: Given G is a G_δ set

G is Borel measurable Hence Lebesgue measurable

N is Lebesgue measurable since Lebesgue σ-Alg is complete

Thus $S = G \setminus N$ is Lebesgue measurable

a $\Rightarrow$ b: (Prop 4.14)

Aug 2016 A #3

It f is non-negative & Integrable on $[0,1]$ then

$$\lim_{n\to\infty}\int_0^1 \sqrt[n]{f} = \lambda\{x ; f(x)>0\}$$

$$\int_0^1 \sqrt[n]{f} = \int_0^1 \sqrt[n]{f}\cdot\chi_{\{f=0\}} + \int_0^1 \sqrt[n]{f}\cdot\chi_{\{f>0\}}$$

where $\int_0^1 \sqrt[n]{f}\cdot\chi_{\{f=0\}} = \int_0^1 \sqrt[n]{0} = 0$

thus $\int_0^1 \sqrt[n]{f} = \int_0^1 \sqrt[n]{f}\cdot\chi_{\{f>0\}}$

since $\int_0^1 \sqrt[n]{f} \cdot < \int_0^1 f < \infty$ (f int)

$$\lim_{n\to\infty}\int_0^1 \sqrt[n]{f}\cdot\chi_{\{f>0\}} \overset{D.C.T.}{=} \int_0^1 \lim_{n\to\infty}\sqrt[n]{f}\cdot\chi_{\{f>0\}}$$

$$= \int_0^1 1\cdot\chi_{\{f>0\}}$$

$$= m(\{x \mid f(x)>0\})$$

Note: forgot

$$\int \sqrt[n]{f}\,\chi_{\{0<f<1\}}$$

Bound this shit by 1 which is Integrable on $[0,1]$

Aug 2016 A #4 Let $f \in L^1(\mathbb{R})$ If $\int_a^b f = 0$ $\forall$ Rational numbers a & b w/ $a<b$ then $f=0$ a.e.

CLAIM: f integrates to 0 over arbitrary open sets.

Thus for $\varepsilon>0$ Choose B: open s.t. $\{f>0\} \subset B$ & $m(B \setminus \{f>0\}) < \delta$

Hence $\left|\int_{\{f>0\}} f\right| \leq \left|\int_B f - \int_{\{f>0\}} f\right| = \left|\int_{B \setminus \{f>0\}} f\right| \leq \int_{B\setminus\{f>0\}} |f| < \varepsilon$

since $m(B\setminus\{f>0\})<\delta$

$= 0$ since true $\forall \varepsilon > 0$

Thus $m(\{x \mid f(x) > 0\}) = 0$

Pf of Claim

for $(a,b) \in \mathbb{R}\times\mathbb{R}$ $\exists$ $\{a_n\},\{b_n\} \in \mathbb{Q}$ s.t. $a_n \downarrow a$, $b_n \uparrow b$

Thus $\int_{(a,b)} f = \int_{\cup(a_n,b_n)} f \overset{DCT}{=\!=} \lim_{n\to\infty} \int_{(a_n,b_n)} f$ w/ $|f|$ as majorant since $f \in L^1$

$= 0$

Now for any B: open, write $B = \bigcup_{n=1}^{\infty} (a_n, b_n)$ w/ (a_n,b_n) arbitrary disjoint intervals

then $\int_B f = \int \sum_{n=1}^{\infty} f \cdot \chi_{(a_n,b_n)} \overset{DCT}{=\!=} \sum_{n=1}^{\infty} \int f \cdot \chi_{(a_n,b_n)}$ w/ $|f|$ as majorant since $f \in L^1$

$= \sum 0 = 0$

Thus $\int_B f = 0$ for any open B.

Similar Pf holds for

$m(\{x \mid f(x) < 0\}) = 0$

Hence $f = 0$ a.e.

Aug 2016 A #5

Suppose $f \in L^2([0,1])$ & $\|f\|_2 = 1$
Define $g(x) = x \cdot f(x)$ prove
that $g \in L^1([0,1])$ & $\|g\|_1 \leq \frac{1}{\sqrt{3}}$

Claim: $x \in L^2([0,1])$

$$\left(\int_0^1 |x|^2\right)^{1/2} = \left(\frac{1}{3}x^3\Big|_0^1\right)^{1/2} < \infty$$

So $x \in L^2$

given 2, 2 are conjugate

$$\|x \cdot f(x)\|_1 \leq \|x\|_2 \|f\|_2 \text{ by Holders}$$
$$\leq \left(\frac{1}{3}x^3\Big|_0^1\right)^{1/2} \cdot 1$$
$$\leq \frac{1}{\sqrt{3}}$$

Aug 2016 B #6

Let $\{a_n\}$ be a sequence of $\mathbb{R}$ s.t. $|a_n| \leq 1$ $\forall n$ Prove that both

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \; ; \; g(x) = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1}$$

converge uniformly on every compact subinterval of $(-1,1)$ & that

$f'(x) = g(x)$ $\forall$ $x \in (-1,1)$

Show f is uniformly convergent on $(-1,1)$

- Since $|a_n| \leq 1$ $\sum_{n=1}^{\infty} a_n x^n \leq \sum_{n=1}^{\infty} x^n$ which converges uniformly since $|x|<1$ By weierstrass m test And geometric series.

~~Let [a,b] be compact compact subinterval of (-1,1)~~

Let C $\subset [0,1]$ be a compact Interval of $(-1,1)$

and let $a = \max\{|x| : x \in C\}$

then

$\sum |x|^n \leq \sum a^n$ which converges by geo seq.

Now APPLY weierstrass m-test.

Aug 2016 B #7

give an example of a cont function $f:[0,1]\to\mathbb{R}$ w/ the property $f(0)=0$ $f(1)=1$ yet $f'(x)\le -1$ for almost every $x\in[0,1]$.

modified cantor function

Aug 2016 B #8

Let $E_1, E_2, \dots$ be a sequence of measurable subsets of $\mathbb{R}$ w/ the property $\sum_{n=1}^{\infty} \lambda(E_n) < \infty$. Show almost every $x \in \mathbb{R}$ is contained in only finitely many of the E_n.

Bwoc suppose

$\exists\ I \subset \mathbb{R}$ s.t. $m(I) = c > 0$

& $I \subset \tilde{E}_n \quad \forall n$

where $\{\tilde{E}_n\}$ is a subsequence

But $\sum_{n=1}^{\infty} m(\tilde{E}_n) \geq \sum_{n=1}^{\infty} m(I) = \sum_{n=1}^{\infty} c$ which diverges since $c > 0$

Aug 2016 B #9

Let $f : [0,1] \to \mathbb{R}$ be Lebesgue measurable. Prove that if $p \leq f(x) \leq q$ $\forall\, x \in [0,1]$

then $\int_{[0,1]} f$ $\exists$ and

$$p \leq \int_{[0,1]} f \leq q$$

$$\int_{[0,1]} 1 = 1$$

Since $p \leq f \leq q$

$$p \leq \int_{[0,1]} f \leq q$$

Let $a = \max\{|p|, |q|\}$

then $|f| < a$

$$\Rightarrow \int_{[0,1]} |f| < a$$

Thus f is integrable.

Aug 2016 #10

Define sequence of functions $f_n \in L^1([0,1])$ by $f_n(x) = \begin{cases} n & x \in 1/n \\ 0 & x > 1/n \end{cases}$

Does f_n converge in $L^1([0,1])$?
If so what is the function?

Suppose $f_n \xrightarrow{L^p} f$

then $\|f_n\|_1 \to \|f\|_1$

But $\int |f_n| = \frac{1}{n} \cdot n = 1 \quad \forall n$

Hence $\|f\| = 1$

Contradiction since

$\lim_{n \to \infty} f_n \to 0$

Analysis Qualifying Examination

Department of Mathematics
University of Louisville
January 2016, 9:00am–12:30pm

This test has two sections. Do three problems from each section. If you do more than three problems, indicate the solutions to be graded.

Section A

1. Let S be dense in $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. Prove or give a counterexample: f is measurable if and only if $\{x : f(x) \geq s\}$ is measurable for all $\cancel{y}\ s \in S$.

2. Suppose $\lambda(S)$ denotes the Lebesgue measure of the set $S \subset \mathbb{R}$. Let $g : [0, 1] \to \mathbb{R}$ be absolutely continuous and $E \subset [0, 1]$ be such that $\lambda(E) = 0$. Prove that $\lambda(g(E)) = 0$.

3. Let $f_n(x) = x^n$ for each $n \geq 1$. Prove that the sequence $\{f_n\}$ converges uniformly on $[-\delta, \delta]$ for each $0 < \delta < 1$, and converges non-uniformly on $(-1, 1)$.

4. Let $\lambda(G)$ denote the Lebesgue measure of the set G. Find an open set G which is dense in $[0, 1]$ such that $\lambda(G) < 1$ and $\lambda(G \cap I) > 0$ for any interval $I \subset [0, 1]$.

Section B

5. Is $L^p([a, b])$ separable, where $1 < p < \infty$?

6. Suppose that $1 < p, q < \infty$ and that $\frac{1}{p} + \frac{1}{q} = 1$. Prove that if $f_n \to f$ in $L^p(\mathbb{R})$ and $g_n \to g$ in $L^q(\mathbb{R})$, then $f_n g_n \to fg$ in $L^1(\mathbb{R})$.

7. Let $f \in L^1([0, 1])$. Prove that

Hilbert shit

$$\lim_{n \to \infty} \int_0^1 f(x) \cos nx \, dx = 0.$$

8. Assume that $f \in L^\infty([0, 1])$. Prove that $f \in L^p([0, 1])$ for each $1 \leq p < \infty$ and that $\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$.

Jan 2016

Jan 2016 A #1

Let S be dense in $\mathbb{R}$, $f:\mathbb{R}\to\mathbb{R}$. Prove or Counter Ex
f is measurable IFF $\{x : f(x) \geq s\}$ is measurable $\forall s \in S$

$\Rightarrow$: f measurable

$\Rightarrow \{x \mid f(x) \geq a\}$ is $\mathcal{L}$-measurable $\forall a \in \mathbb{R}$

hence $\{x \mid f(x) \geq s\}$ is $\mathcal{L}$ meas $\forall s \in \mathbb{R}$

$\Leftarrow$: $\{x : f(x) \geq s\}$ measurable $\forall s \in S$

ISTS $\{x : f(x) > s\}$ is measurable $\forall s \in S^c$

let $s \in S^c$

given S is dense in $\mathbb{R}$

Let $\{s_n\} \subset S$ s.t. $s_n \downarrow s$

thus $\{x : f(x) > s\} = \bigcup_{n=1}^{\infty} \{x \mid f(x) \geq s_n\}$

is the union of measurable sets

Hence is measurable.

Jan 2016 A #2

Suppose $\lambda(S)$ denotes Lebesgue measure of the set $S \subset \mathbb{R}$. Let $g:[0,1] \to \mathbb{R}$ be Abs. cont. & $E \subset [0,1]$ be s.t. $\lambda(E) = 0$. prove that $\lambda(g(E)) = 0$

Given $E \subset [0,1]$ w/ $\lambda(E) = 0$

Let $\{(a_i, b_i)\}$ be a collection of disjoint intervals covering E w/ $\sum |b_i - a_i| < \delta$.

g ABS cont $\Rightarrow$ g is continuous.

thus for each (a_i, b_i)

Let $(c_i, d_i) \subseteq (a_i, b_i)$ s.t. $\{f(c_i), f(d_i)\} \in \{\max(f|_{(a_i, b_i)}), \min(f|_{(a_i, b_i)})\}$

Now consider that

$\sum |c_i - d_i| < \sum |b_i - a_i| < \delta$ Hence

$$\begin{aligned}
\lambda(g(E)) &\le \lambda\left(g\left(\bigcup_{i=1}^{\infty} (a_i, b_i)\right)\right) = \lambda\left(\bigcup_{i=1}^{\infty} g(a_i, b_i)\right) \\
&\le \sum \lambda(g(a_i, b_i)) \\
&\le \sum |f(c_i) - f(d_i)| \quad \text{w/ } \sum |c_i - d_i| < \delta \\
&< \varepsilon \quad \text{for } \varepsilon > 0 \\
&= 0
\end{aligned}$$

Jan 2016 A #3

Let $f_n(x) = x^n$ for each $n \geq 1$. Prove that $\{f_n\}$ converges uniformly on $[-\delta, \delta]$ for each $0 < \delta < 1$ & converges non-uniformly on $(-1, 1)$

Let $\delta \in (0,1)$, Let $\varepsilon > 0$

Let $N = \dfrac{\log(\varepsilon)}{\log(\delta)}$

hence for $n \geq N = \dfrac{\log(\varepsilon)}{\log(\delta)}$

$n \log(\delta) \leq \log(\varepsilon)$

$\log(\delta^n) \leq \log(\varepsilon)$

Since $\log(\delta)$ & $\log(\varepsilon) < 0$ $\quad \delta^n \leq \varepsilon$

thus $x^n < \delta^n < \varepsilon$

$\forall x \in (-\delta, \delta)$

w.t.S. $|x|^n < \varepsilon$

$\iff \log(|x|^n) < \log(\varepsilon)$

$n \log(|x|) < \log(\varepsilon)$

$n < \dfrac{\log(\varepsilon)}{\log(|x|)}$

~~Let $x \in (-1,1)$~~

Fix $x \in (-1, 1)$

$\Rightarrow x^n < \varepsilon$

By previous pf,

But not uniform since N depends on x

Jan 2016 A # 4

Let $\lambda(G)$ denote Lebesgue measure of the set G.
Find an open set G which is dense in $[0,1]$
such that $\lambda(G) < 1$ & $\lambda(G \cap I) > 0$
for any interval $I \subset [0,1]$

Let $\{q_i\}_{i=1}^{\infty}$ enumerate the Rationals in $\mathbb{Q} \cap [0,1]$
for each n Define I_n as an Interval
containing q_n, $m(I_n) < \frac{1}{4 \cdot 2^n}$ & $I_n \subset [0,1]$
Let $G = \bigcup_{i=1}^{\infty} I_n$ w/ $m(G) = m(\bigcup_{i=1}^{\infty} I_n) \leq \sum_{i=1}^{\infty} m(I_n)$

$$< \sum_{i=1}^{\infty} \frac{1}{4 \cdot 2^n}$$

$$= \frac{1}{4} \frac{1}{1 - 1/2} = \frac{1}{2}$$

Since G contains the Rationals on $[0,1]$
it is dense on $[0,1]$

G is open since its the coutable union of open Intervals

Let $I \subset [0,1]$ thus it contains an rational
Hence It intersects I_n nontrivially.
Thus $m(I \cap G) > 0$

Jan 2016 B #5

Is $L^p([a,b])$ Seperable, where $1 < p < \infty$?

Yes (Seperable $\Rightarrow$ contains countable dense subset)

Let $S[a,b] \subset L^p([a,b])$
be step functions on $[a,b]$

AND let $S_{\mathbb{Q}}[a,b] \subset S[a,b]$
be step functions of $[a,b]$
with rational endpts.

Since $\mathbb{Q}$ is Dense in $\mathbb{R}$
$S_{\mathbb{Q}}[a,b]$ is Dense in $S[a,b]$
thus Dense in $L^p([a,b])$

Jan 2016 B #5 pt 2

Show $L^\infty([a,b])$ is Not Separable.
what about L^1?

Let $A = \{ \chi_{(a,t)} ; a \le t \le b \}$

<u>CLAIM</u> for any $\chi_{(a,t_1)}, \chi_{(a,t_2)} \in A$
s.t. $t_1 \ne t_2 \Rightarrow \|\chi_{(a,t_1)} - \chi_{(a,t_2)}\| = 1$

<u>Pf</u> wlog assume $t_1 < t_2$
Consider on the interval (a, t_1) $|\chi_{(a,t_1)} - \chi_{(a,t_2)}| = 0$
on the Interval (t_1, t_2) $|\chi_{(a,t_1)} - \chi_{(a,t_2)}| = 1$
& on the interval (t_2, b) $|\chi_{(a,t_1)} - \chi_{(a,t_2)}| = 0$
thus $\|\chi_{(a,t_1)} - \chi_{(a,t_2)}\|_\infty = 1$

Consider $\{ B(\chi_{(a,t)}, 1/3) \}$ is an uncountable collection of disjoint Balls

Thus given Any Dense set $S \subset L^\infty([a,b])$ has elements in each Ball (By Def of density)
thus S is uncountable.

2016 Jan B #6

Suppose $1 < p, q < \infty$. $\frac{1}{p} + \frac{1}{q} = 1$
prove if $f_n \to f$ in $L^p(\mathbb{R})$ and $g_n \to g$ in $L^q(\mathbb{R})$ then $f_n g_n \to fg$ in $L^1(\mathbb{R})$

I STS $\lim_{n\to\infty} \|f_n g_n - fg\| = 0$

$$\lim_{n\to\infty} \|f_n g_n - fg\|_1 = \|f_n g_n - f_n g + f_n g - fg\|_1$$

$$= \lim_{n\to\infty} \|f_n g_n - f_n g\|_1 + \|f_n g - fg\|_1$$

$$= \lim_{n\to\infty} \|f_n\|_p \cdot \|g_n - g\|_q + \|f_n - f\|_p \cdot \|g\|_q$$

where $\lim_{n\to\infty} \|f_n\|_p \to \|f\|_p < \infty$

$\|g_n - g\|_q \to 0$ & $\|f_n - f\|_p \to 0$

$= 0$

Jan 2016 B #7

Let $f \in L^1([0,1])$ Prove that

$$\lim_{n\to\infty} \int_0^1 f(x) \cos nx \, dx = 0$$

HILBERT SHIT Disguised.

Jan 2016 B #8

Assume $f \in L^\infty([0,1])$. Prove that $f \in L^p([0,1])$ for each $1 \leq p < \infty$ & that

$$\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$$

Claim 1 $f \in L^\infty([0,1]) \Rightarrow f \in L^1([0,1])$

Pf $f \in L^\infty \Rightarrow f \leq M = \text{esssup}(f)$ a.e. $\Rightarrow \int_{[0,1]} |f| \leq \int_{[0,1]} M = M < \infty$

Thus $f \in L^1$

Claim 2 $f \in L^p([0,1])$ for $1 \leq p \leq \infty$

Pf consider $\int |f|^p = \int_{\{|f|<1\}} |f|^p + \int_{\{|f| \geq 1\}} |f|^p$

- where for $p > 1$: $\int_{\{|f|<1\}} |f|^p < \int_{\{|f|<1\}} |f| < \infty$ since $f \in L^1$
- where $\int_{\{|f| \geq 1\}} |f|^p \leq \int_{\{|f| \geq 1\}} M^p = M^p \cdot m(\{|f| \geq 1\}) < \infty$

Thus $\int |f|^p < \infty \Rightarrow f \in L^p$

Claim 3 $\limsup \|f\| \leq \|f\|_\infty$

Pf $|f| \leq \|f\|_\infty$ a.e. $\Rightarrow |f|^p \leq \|f\|_\infty^p \Rightarrow \left(\int_{[0,1]} |f|^p\right)^{1/p} \leq \left(\int_{[0,1]} \|f\|_\infty^p\right)^{1/p} = \|f\|_\infty < \infty$

Hence $\limsup \|f\|_p \leq \|f\|_\infty$

Claim 4 $\|f\|_\infty \leq \liminf \|f\|_p$

Pf Let $t \in [0, M = \|f\|_\infty)$ $\int |f| = \int_{\{|f|<t\}} |f| + \int_{\{|f| \geq t\}} |f|$

$\Rightarrow \int |f|^p \geq \int_{\{|f|<t\}} |f|^p \geq \int_{\{|f|<t\}} t^p$

Hence $\left(\int |f|^p\right)^{1/p} \geq \left(\int_{\{|f|<t\}} t^p\right)^{1/p} = t \cdot m(\{|f|<t\})^{1/p}$

$\Rightarrow \liminf \|f\|_p \geq \liminf t \cdot m(\{|f|<t\})^{1/p} = t \cdot 1$ since $m(\{|f|<t\}) < \infty$ & $\lim_{p \to \infty} M^{1/p} = 1$

Thus $\liminf \|f\|_p \geq t \quad \forall t \in [0, \|f\|_\infty)$

$\Rightarrow \liminf \|f\|_p \geq \|f\|_\infty$

It follows that By Claim 3 & 4 $\limsup \|f\|_p \leq \|f\|_\infty \leq \liminf \|f\|_p \Rightarrow \lim_{p \to \infty} \|f\|_p = \|f\|_\infty$

Analysis Qualifying Examination

Department of Mathematics
University of Louisville
August 13, 2015, 9:00am–12:30pm

This test has two sections. Do three problems from each section. If you do more than three problems, indicate the solutions to be graded.

Section A

1. Let $C \subset \mathbb{R}$ denote the Cantor set. Let $\chi_C(x) = 1$ if $x \in C$ and 0 otherwise. Explain why χ_C is Riemann integrable and compute $\int_0^1 \chi_C(x)\,dx$.

2. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $m(E) = 1$. Prove there exists a Lebesgue measurable set $F \subset E$ with $m(F) = \frac{1}{2}$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space. If $f_n : X \to \mathbb{R}$ is a sequence of functions such that $\sum_{n=1}^{\infty} \int_X |f_n|\,d\mu$ converges, then prove that $f_n \to 0$ almost everywhere.

4. Prove that

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ x^2 \cos\left(\frac{1}{x^2}\right) & \text{if} \quad x \neq 0 \end{cases}$$

is continuous but not absolutely continuous on $[-1, 1]$.

5. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. If f is μ-measurable and

$$p \leq f(x) \leq q$$

for all $x \in X$, then prove that $\int_X f\,d\mu$ exists and

$$p\,\mu(X) \leq \int_X f\,d\mu \leq q\,\mu(X).$$

Section B

6. Suppose that $1 < p, q < \infty$ and that $\frac{1}{p} + \frac{1}{q} = 1$. Prove that if $f_n \to f$ in $L^p(\mathbb{R})$ and $g_n \to g$ in $L^q(\mathbb{R})$, then $f_n g_n \to fg$ in $L^1(\mathbb{R})$.

7. Evaluate $\displaystyle \frac{d}{dt} \int_0^1 \frac{\sin(xt)}{x}\, dx$. Justify your computations.

8. Let H be a Hilbert space. Prove that if $\{x_\alpha\}_{\alpha \in A}$ is an orthonormal set in H then

$$\sum_{\alpha \in A} |\langle x, x_\alpha \rangle|^2 \leq \|x\|^2$$

for all $x \in H$.

9. If $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $p \geq 1$, then prove that $f \in L^p(\mathbb{R})$.

10. If $f : \mathbb{R} \to [0, \infty)$ is measurable, then $\displaystyle \lim_{n \to \infty} \int_{-n}^{n} f = \int_{\mathbb{R}} f$.

Aug 2015

Aug 2015 A #1

Let $C \subset \mathbb{R}$ denote the Cantor Set.
Let $\chi_C(x) = 1$ if $x \in C$, 0 o.w.
Explain why χ_C is Riemann Integrable
& compute $\int_0^1 \chi_C(x)\,dx$.

given that $m(C)=0$

$\chi_C(x) = 0$ a.e. $\Rightarrow \int \chi_C = 0$

& since $\chi_C(x)$ is Bounded
with $m(\{x \mid \lim_{x \to a} f(x) \neq f(a)\}) = m(C) = 0$

$R\int f = \int f = 0$

Aug 2015 A #2

Let $E \subset \mathbb{R}$ be Lebesgue measurable set w/ $m(E) = 1$. prove $\exists$ Lebesgue measurable set $F \subset E$ w/ $m(F) = 1/2$

$$E = \bigcup_{n=1}^{\infty} E \cap [n, n+1] \implies m(E) = \lim_{n \to \infty} m(E \cap [-n, n])$$

$$= 1$$

Pick $n \in \mathbb{N}$ sufficiently large s.t. $m(E \cap [-n, n]) > 1/2$

consider $f(x) = \int_{-n}^{x} \chi_E$ for $x \in [-n, n]$

since f is continuous w/ $f(-n) = 0$ and $f(n) > 1/2$, $\exists$ $c \in [-n, n]$ s.t. $f(c) = 1/2$ by Intermediate value thrm.

Thus $(-n, c) \cap E$ is a Lebesgue measurable set with $m((-n, c) \cap E) = 1/2$

Aug 2015 A # 3

Let $(\mathbb{X}, \mathcal{A}, \mu)$ be a measure space. If $f_n : \mathbb{X} \to \mathbb{R}$ is a seq of functions s.t. $\sum_{n=1}^{\infty} \int_{\mathbb{X}} |f_n| \, d\mu$ converges then prove that $f_n \to 0$ A.E.

$$\sum_{n=1}^{\infty} \int_{\mathbb{X}} |f_n| \, d\mu = \lim_{H \to \infty} \sum_{n=1}^{H} \int_{\mathbb{X}} |f_n| \, d\mu$$

$$= \lim_{H \to \infty} \int_{\mathbb{X}} \sum_{n=1}^{H} |f_n| \, d\mu$$

$$= \int_{X} \lim_{H \to \infty} \sum_{n=1}^{H} |f_n| \, d\mu \quad \text{By M.C.T.}$$

Since $\sum_{i=1}^{H} |f_n|$ is increasing

[illegible] converges.

$$\lim_{H \to \infty} \sum_{i=1}^{H} |f_n| = \sum_{i=1}^{\infty} |f_n| \text{ [illegible]}$$

$$= \int \sum_{n=1}^{\infty} |f_n| \, d\mu$$

$$< \infty$$

$$\Longrightarrow |f_n| \to 0 \quad \text{a.e. } n \to \infty$$

Hence $f_n \to 0$ a.e.

prove that $\underline{\text{Aug 2018 A \#4}}$

$$f(x) = \begin{cases} 0 & \text{if } x=0 \\ x^2\cos(\frac{1}{x^2}) & \text{, if } x\neq 0 \end{cases}$$

is continuous But not ABS Cont on $[-1,1]$

Consider ~~$\cos(\frac{1}{x^2})$~~

$x^2\cos(\frac{1}{x^2})$ is Constructed of functions that are continuous everywhere on its Domain

So it is Continuous on $\mathbb{R}\setminus\{0\}$

to show continuous at $x=0$

consider that $-1 \leq \cos(\frac{1}{x^2}) \leq 1$

$$\Rightarrow -x^2 \leq x^2\cos(\tfrac{1}{x^2}) \leq x^2$$

w/ $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$ hence by Squeeze thm

$$\lim_{x\to 0} x^2\cos(\tfrac{1}{x^2}) = f(0) = 0$$

Suppose f is ABS Continuous. on $[-1,1]$
Hence it is of Bounded-variation on $[-1,1]$
thus $V_f[0,1] < \infty$

Consider the partion with endpoints

$$P = \{-1\} \cup \left[\left\{ \pm\sqrt{\tfrac{1}{n\pi}} : n \in \mathbb{Z} \right\} \cap [-1,1]\right] \cup \{1\}$$

Hence for $x_i, x_{i+1} \in P$

$$\sum |f(x_i) - f(x_{i+1})| = \sum \left| f\left(\sqrt{\tfrac{1}{n\pi}}\right) - f\left(\sqrt{\tfrac{1}{(n+1)\pi}}\right) \right|$$

$$= \sum \left| \left[\left(\sqrt{\tfrac{1}{n\pi}}\right)^2 \cdot \cos\left(\frac{1}{\sqrt{\tfrac{1}{n\pi}}^{\,2}}\right)\right] - \left[\left(\sqrt{\tfrac{1}{(n+1)\pi}}\right)^2 \cdot \cos\left(\frac{1}{\sqrt{\tfrac{1}{(n+1)\pi}}^{\,2}}\right)\right] \right|$$

$$= \sum \left| \left(\frac{1}{n\pi}\cos(n\pi)\right) - \left(\frac{1}{(n+1)\pi} \cdot \cos((n+1)\pi)\right) \right|$$

$$= \sum \left| \frac{1}{n\pi} - \frac{-1}{(n+1)\pi} \right|$$ since n is even & $n+1$ is odd

$$= \frac{1}{\pi} \sum \left| \frac{2n+1}{n^2+n} \right|$$ which diverges

By Harmonic series

thus $V_f[0,1] \not< \infty$

Contradiction to assumption of f being ABS cont.

Aug 2015 A #5

Let $(X, \mathcal{A}, \mu)$ be a finite measure space.

If ~~[illegible]~~ f is μ-measurable

& $p \leq f(x) \leq q \quad \forall\, x \in X$

then prove $\int_X f\, d\mu$ $\exists$ and

$p \cdot \mu(X) \leq \int_X f\, d\mu \leq q\, \mu(X)$

Since X is finite, & $p \leq f \leq q$

$$\mu(X) = \int_X 1\, d\mu$$

$\Rightarrow$

$$p \cdot \mu(X) \leq \int_X f\, d\mu \leq q\, \mu(X)$$

now to show $\int_X f\, d\mu$ is integrable.

Consider

$$|f| \leq \max\{|p|, |q|\} = M$$

Hence ~~[illegible]~~

$$|f| \leq M$$

$$\Rightarrow \int_X |f|\, d\mu \leq \mu(X) \cdot M < \infty$$

Hence Integrable.

Aug 2015 B #6

Suppose $1<p,\ q<\infty$ & that $\frac{1}{p}+\frac{1}{q}=1$
Prove that if $f_n \to f$ in $L^p(\mathbb{R})$ & $g_n \to g$
in $L^q(\mathbb{R})$ then $f_n \cdot g_n \to f \cdot g$ in $L^1(\mathbb{R})$

ISTS $\lim_{n\to\infty} \|f_n g_n \to fg\|_1 = 0$

$$\lim_{n\to\infty} \|f_n g_n - fg\|_1 = \lim_{n\to\infty} \|f_n g_n - f_n g\|_1 + \|f_n g - fg\|_1$$

$$\leq \lim_{n\to\infty} \|f_n\|_p \|g_n - g\|_q + \|g\|_q \|f_n - f\|_p$$

where $\lim_{n\to\infty} \|f_n\|_p \to \|f\|_p < \infty$

$\|g_n - g\|_q \to 0$

& $\|f_n - f\|_p \to 0$

$= 0$

Aug 2015 B # 7

Evaluate $\frac{d}{dt}\int_0^1 \frac{\sin(xt)}{x}\,dx$.. Justify computation

Let $u = xt \Rightarrow du = \frac{dx}{t}$

when $x = 0 \Rightarrow u = 0$, $x = 1 \Rightarrow u = t$

$$\frac{d}{dt}\int_0^1 \frac{\sin(xt)}{x}\,dx = \frac{d}{dt}\int_0^t \frac{\sin(u)}{u}\,du$$

$$= \frac{\sin(t)}{t} \quad \text{By F.T.C.}$$

Aug 2015 B #9

If $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ $p \geq 1$
prove that $f \in L^p(\mathbb{R})$

given $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$
$\Rightarrow M = \text{ess sup}(f) < \infty$ & $\int |f| < \infty$

$$\int |f|^p = \int_{\{x \mid |f| < 1\}} |f|^p + \int_{\{x \mid |x| > 1\}} |f|^p$$

for $p > 0$
$$\int_{\{x \mid |f(x)| < 1\}} |f|^p < \int_{\{x \mid |f| < 1\}} |f|$$

$$\leq \underbrace{\int_{\{\}} |f|}_{< \infty} + \underbrace{\int_{\{\}} M^p}_{= M^p \cdot m(\{x \mid |f(x)| > 1\}) < \infty}$$

$< \infty$

so $\left(\int |f|^p\right)^{1/p} < \infty$ $\forall p$

$\Rightarrow$ $f \in L^p$

Aug 2015 B # 10

If $f: \mathbb{R} \to [0, \infty]$ is measurable,

then $\lim_{n\to\infty} \int_{-n}^{n} f = \int_{\mathbb{R}} f$

Since f is positive

$f \cdot \chi_{[-n,n]}$ is a sequence of Increasing Positive functions

~~Thus $\lim_{n\to\infty}$~~

thus $\lim_{n\to\infty} \int f \cdot \chi_{[-n,n]} \overset{MCT}{=} \int \lim_{n\to\infty} f \cdot \chi_{[-n,n]}$

$= \int_{\mathbb{R}} f$

ANALYSIS QUALIFIER EXAM

JANUARY 2013

The Lebesgue measurable subsets of $\mathbb{R}$ are denoted by $\mathcal{L}$ and Lebesgue measure is denoted by λ.

This test has two sections and two pages with five problems in each section. You are expected to do three problems from each section. Solutions to at most three problems from each section will be graded. If you do more than three problems, indicate the solutions to be graded.

Section A

Problem 1. Show that every dense subset of $L^\infty([0,1])$ is uncountable.

Problem 2. Let f be a Lebesgue measurable function on $\mathbb{R}$ with the property that

$$\sup_{\{g\in L^2(\mathbb{R}):\|g\|_2\le 1\}} \int_{\mathbb{R}} |fg|\, d\lambda \le 1.$$

Prove that $f \in L^2(\mathbb{R})$ and $\|f\|_2 \le 1$.

Problem 3. Let $f \ge 0$ and $f \in L^p[0,1]$ for all $p \in [1,\infty)$. If $\|f\|_p^p = \|f\|_1$ for all $p \in [1,\infty)$, then there is a set S such that $f = \chi_S$ a.e.

Problem 4. If E is a measurable subset of $\mathbb{R}$, then there is an interval I such that $\lambda(E\cap I) > \frac{9}{10}\lambda(I)$ or $\lambda(E^c \cap I) > \frac{9}{10}\lambda(I)$.

Problem 5. A measure space (X,μ) is σ-finite iff there is an $f : X \to (0,\infty)$ such that $f \in L^1(X,\mu)$.

Section B

Problem 6. (a) Find a sequence $f_n : [0,1] \to \mathbb{R}$ such that $\int_0^1 |f_n(x)| = 2$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} f_n(x) = 1, \ \forall x \in [0,1].$$

(b) If the f_n are as in part (a), then prove

$$\lim_{n\to\infty} \int_0^1 |f_n(x) - 1|\, dx = 1.$$

Problem 7. Show that $\mathcal{G} = \{f \in C[0,1] : \int_0^1 f^2 > 1\}$ is open in $C[0,1]$. (Assume $C[0,1]$ has the uniform metric.)

Problem 8. Let (X, ρ) be a metric space and suppose K and F are nonempty disjoint subsets of X with K compact and F closed.

(a) Prove there is a $\delta > 0$ such that $\rho(x, y) \geq \delta$ for all $x \in K$ and $y \in F$.

(b) Show that part (a) may fail if K is closed, but not compact.

Problem 9. The limit superior of a sequence of sets $\{E_k\}$ is defined as

$$\limsup E_k = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k.$$

Let $\{E_k : k \in \mathbb{N}\}$ be a sequence of sets in $\mathcal{L}$.

(a) Prove that if $\sum_{k \in \mathbb{N}} \lambda(E_k) < \infty$, then $\lambda(\limsup E_k) = 0$.

(b) Is it true in general that $\lambda(\limsup E_k) = \limsup \lambda(E_k)$?

Problem 10. Show that

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is in $BV[-1, 1]$, but

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is not.

Jan 2013 A #1

Show every Dense subset of $L^\infty([0,1])$ is uncountable
(Equivalent to showing L^∞ is NOT ~~complete~~ separable)

Let $A = \{\chi_{(0,t)} ; t \in (0,1)\}$

then for any $\chi_{(0,t_1)}, \chi_{(0,t_2)} \in A$ s.t. $t_1 \neq t_2$

$\|\chi_{(0,t_1)} - \chi_{(0,t_2)}\|_\infty = 1$

Since wlog assume $t_1 < t_2$

then

$$\chi_{(0,t_1)} - \chi_{(0,t_2)} = \begin{cases} 0 & \text{on } (0,t_1) \\ 1 & \text{on } (t_1,t_2) \\ 0 & \text{on } (t_2,1) \end{cases}$$

Consider $\{B(\chi_{(0,t)}, 1/3)\}$ is an uncountable collection of disjoint Balls.

& given ANY Dense set $S \subset L^\infty[0,1]$ has elements in each Ball By Def of Density

Thus S is uncountable.

January 2013 A #2

Let f be lebesgue measurable on $\mathbb{R}$ s.t.

$$\sup_{\{g \in L^2(\mathbb{R}) : \|g\|_2 \leq 1\}} \int_{\mathbb{R}} |fg| \, d\lambda \leq 1$$

Prove that $f \in L^2(\mathbb{R})$ & $\|f\|_2 \leq 1$

Let $s_n \uparrow |f|$ be a sequence of simple functions. Let $s_n' = s_n \chi_{[-n,n]}$

then

Jan 2013 A#3

Let $f \geq 0$ & $f \in L^p[0,1]$ $\forall p \in [1,\infty)$

If $\|f\|_p^p = \|f\|_1$ $\forall p \in [1,\infty)$ then there is a set S s.t. $f = \chi_S$ a.e.

$\|f\|_1 = \int |f|^p \quad \forall p \in [1,\infty)$

~~Claim~~ suppose $m(\{x \mid f > 1\}) \neq 0$

Pick $\varepsilon \in (1,\infty)$ s.t.

$1 < \varepsilon \leq f < \infty$ $\quad 1 < \varepsilon \leq f < \infty$

then $\|f\|_1 = \int |f|^p \geq \int_{\{1<|f|\}} |f|^p \geq \int \varepsilon^p \to \infty$ since $\varepsilon > 1$

thus

$m(\{x \mid f > 1\}) = m(\bigcup_{k:1<\varepsilon<\infty}$

Claim $m(\{x \mid 0 < f < 1\}) = 0$

Pick $0 < |f| \leq \varepsilon < 1$

$\|f\|_1 = \int |f|^p \leq \int_{\{0<|f|\leq\varepsilon<1\}} \varepsilon^p \to 0$ as $p \to \infty$

Hence $f = \begin{cases} 0 \\ 1 \end{cases}$ a.e.

thus defines $S = \{x \mid f(x) = 1\}$

then $f = \chi_S$ a.e.

Jan 2013 A #4

If E is measurable subset of $\mathbb{R}$, then there is an Interval I s.t. $\lambda(E\cap I) > \frac{9}{10}\lambda(I)$ OR $\lambda(E^c \cap I) > \frac{9}{10}\lambda(I)$

Suppose Not

$\Rightarrow \forall I,\ \lambda(E\cap I) \le \frac{9}{10}\lambda(I)$ & $m(E^c \cap I) \le \frac{9}{10}\lambda(I)$

Suppose E has finite measure & let $E \subset \bigcup I_n$

Then ~~m(E) = m(E ∩ ⋃ I_n)~~ $m(E) = m(E \cap \bigcup I_n) = m(\bigcup E\cap I_n)$

$\le m(E\cap I_n) \le \frac{9}{10}\sum m(I_n)$

Thus $\forall$ covers of E $\quad m(E) \le \frac{9}{10}\sum m(I_n)$

But By Def $\quad m(E) = \mathrm{INF}\{m(I_n) : E \subset \bigcup I_n\}$

Hence $\quad m(E) \le \frac{9}{10} m(E) \Rightarrow m(E) = 0$

Now for E any measure

$m(E\cap(-n,n)\cap I) \le m(E\cap I) \le \frac{9}{n} m(I)$

Hence $\quad m(E\cap(-n,n)) = 0 \ \forall n$

where $m(E) = m(\bigcup_{n=1}^{\infty}(E\cap(-n,n))) \le \sum m(E\cap(-n,n)) = 0$

Note Same proof works for $m(E^c) = 0$

But ~~m(E)~~ that's fucked

Jun 2013 A #5

~~Claim~~ A measure space (X, M) is σ-finite IFF there is an $f : X \to (0, \infty)$ s.t. $f \in L^1(X, M)$

$\Leftarrow$: $f : X \to (0,\infty)$ s.t. $f \in L^1(X, M)$

$$X = \{f > 0\} = \bigcup_{n=1}^{\infty} \{f > \tfrac{1}{n}\}$$

where $\mu(\{f > \frac{1}{n}\}) \leq \int \frac{|f|}{\left(\frac{1}{n}\right)} d\mu$

Chebyshev's Ineq

$\leq n \cdot \int |f| < \infty$ since $f \in L^1$

$\Rightarrow$: Since X is σ-finite we have $X = \bigcup_{i=1}^{\infty} A_n$ disjoint w/ $\mu(A_n) < \infty$

we can construct

Define $f = \sum_{n=1}^{\infty} a_n$

where $a_n = \begin{cases} \frac{1}{\mu(A_n) \cdot n^2} \cdot \chi_{A_n} & \text{if } \mu(A_n) > 0 \\ \frac{1}{n^2} \chi_{A_n} & \text{if } \mu(A_n) = 0 \end{cases}$

then $\int f \overset{MCT}{=} \sum_{n=1}^{\infty} \int \frac{1}{\mu(A_n) n^2} \chi_{A_n} = \sum_{n=1}^{\infty} \frac{1}{\mu(A_n) n^2} \int \chi_{A_n} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

hence $f \in L^1$

Jan 2013 B #6

a) Find a sequence $f_n : [0,1] \to \mathbb{R}$ s.t.

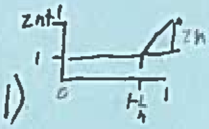
$$\int_0^1 |f_n(x)| = 2 \quad \forall n \in \mathbb{N} \ \& \ \lim_{n\to\infty} f_n(x) = 1 \quad \forall x \in [0,1]$$

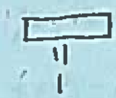
b) If f_n are as in part (a), then prove

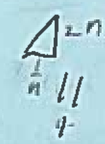
$$\lim_{n\to\infty} \int_0^1 |f_n(x) - 1|\, dx = 1$$

---

a) Let $f_n(x) = \begin{cases} 1 & x \in [0, 1-\frac{1}{n}] \\ -2n^2x - 2n^2 + 2n + 1 & x \in [1-\frac{1}{n}, 1) \\ 1 & x = 1 \end{cases}$





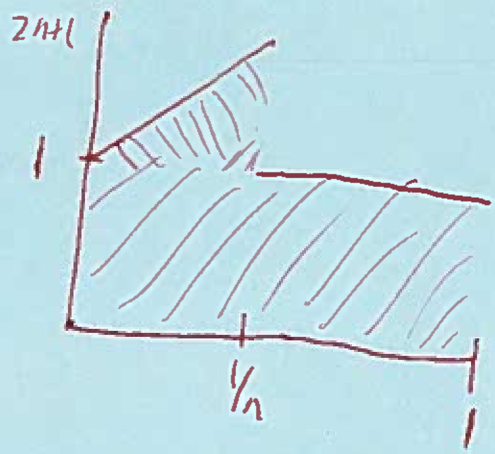


a)

~~$f_n(x) =$~~

$$f_n(x) = \begin{cases} \frac{2n^2}{1-n} + 1 & 0 \le x \le \frac{1}{n} \\ 1 & \text{o.w} \end{cases}$$

a)



Jan 2013 B #7

Show that $G = \{f \in C[0,1] : \int_0^1 f^2 > 1\}$ is open in $C[0,1]$ (Assume $C[0,1]$ has uniform metric)

Jan 2013 B #8

Let (X, ρ) be a metric space & suppose K, F are nonempty disjoint subsets of X w/ K compact and F closed

a) Prove there is a $\delta > 0$ s.t. $\rho(x, y) \geq \delta$ $\forall$ $x \in K$ & $y \in F$

b) Show that part (a) may fail if K is closed but Not compact.

Jan 2013 B #9 Limit superior of a sequence of sets $\{E_k\}$ is defined as

$$\limsup E_k = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k$$

Let $\{E_k : k \in \mathbb{N}\}$ be a sequence of sets in $\mathcal{L}$

a) Prove that if $\sum_{k\in\mathbb{N}} \lambda(E_k) < \infty \Rightarrow \lambda(\limsup E_k) = 0$

b) Is it true in general that

$$\lambda(\limsup E_k) = \limsup \lambda(E_k)?$$

---

a) Let $\varepsilon > 0$, given $\sum_{k=1}^{\infty} m(E_k) < \infty$

$$\exists \ N \text{ s.t. } \sum_{k=N}^{\infty} m(E_k) < \varepsilon$$

$$\lim\sup(E_k) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subset \bigcup_{k=N}^{\infty} E_k$$

Hence $m(\limsup(E_k)) \leq m(\bigcup_{k=N}^{\infty} E_k) \leq \sum_{k=N}^{\infty} m(E_k) < \varepsilon$

$$\Rightarrow m(\limsup(E_k)) = 0$$

b) Consider the sequence of function

$E_1 = [0,1]$

$E_2 = [0, 1/2]$, $E_3 = [1/2, 1]$

$E_4 = [0, 1/4]$, $E_5 = [1/4, 1/2]$ $E_6 = [1/2, 3/4]$ $E_7 = [3/4, 1]$

$E_8 = [0, 1/8]$ $E_9 = [1/8, 2/8]$ . . . .

$E_{\_} = [0, 1/16]$ . . . . . .

$\lim_{n\to\infty} m(E_k) = 0$ But $\forall n \in \mathbb{N}$ $\bigcup_{k=n}^{\infty} E_k = [0,1]$

Problem 10 Show that

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is in $BV[-1,1]$

But

$$g(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is NOT.

SEE Aug 2015

#4