Final Test MATH 635 Fall 2017

(1) This test should be done independently, any collaboration and discussion with your classmates are forbidden.
(2) You must show complete (necessary) work to receive full credits. No work, no credit.
(3) You must write your solution in a neat and concise fashion. Also organize your final solution according to the order of index appearing below.
(4) Test is due at 11:00 on Thursday, Dec. 7. Any late submission will not be accepted unless an extension is given by the instructor.

1. (10pts) Determine the center manifold and describe the dynamics on the center manifold. Describe the stability and instability of the origin.

$$\begin{cases} x' = -xy - x^6 \\ y' = -y + x^2 \end{cases}$$

2. (10pts) Reduce the quadratic system

$$\begin{cases} x' = y + x^2 + y^2 \\ y' = xy, \end{cases}$$

to its normal form.

3. (10pts) Consider the model for two competing species

$$\begin{cases} x' = x(2 - x - y) \\ y' = y(5 - 2x - 3y) \end{cases}$$

(a) Find out all the equilibrium points in $\mathbf{R}_+^2$, and study their stability.

(b) Determine the $\omega(\Gamma_{(2,3)}), \omega(\Gamma_{(1,0.5)})$ and $\alpha(\Gamma_{(1,0.5)})$ by its phase portrait, where $\Gamma_{(a,b)}$ denotes the orbit passing through the point (a, b).

4. (15pts) Let M be closed, bounded, a positively invariant region for the planar system

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

and $p \in M$.

(a) Show that $\omega(\Gamma_p)$ is connected.

(b) Assume there are three steady states $p_1, p_2, p_3 \in \omega(\Gamma_p)$, and two trajectories Γ_1 and Γ_2, such that $\alpha(\Gamma_1) = p_1$, $\omega(\Gamma_1) = p_2$, $\alpha(\Gamma_2) = p_2$, $\omega(\Gamma_2) = p_3$. Show that there must be a trajectory Γ_3, such that

$$\alpha(\Gamma_3) = p_3, \omega(\Gamma_3) = p_1.$$

(Hint: Read relevant material in the textbook, and write in your own words.)

5. (15pts) Show that the nonlinear system

$$\begin{cases} x' = -y + xz^2 - xw^2 \\ y' = x + yz^2 - yw^2 \\ z' = -z(x^2 + y^2) \\ w' = w(x^2 + y^2) \end{cases}$$

has a periodic orbit $\gamma(t) = (\cos t, \sin t, 0, 0)^t$. Find the linearization of this system about $\gamma(t)$. The fundamental matrix $\Phi(t)$ for this system, and the characteristic exponents and multipliers of $\gamma(t)$. What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$?

6. (10pts) Consider the following vector field on $\mathbf{R}^2$:

$$\begin{cases} x' = \mu x - y - x(x^2 + y^2) \\ y' = x - \mu y - y(x^2 + y^2) \end{cases}$$

where $\mu \in \mathbf{R}$. Define the cross section Σ to the vector field by

$$\Sigma = \{(r, \theta) \in \mathbf{R} \times [0, 2\pi] \quad | \quad r > 0, \theta = 0\}$$

Compute the Poincaré map P and find out the fixed point of P, and determine their stability. (Hint: divide it into cases $\mu < 0, = 0, > 0$)

7. (15pts) Consider the planar system

$$\begin{cases} x' = x - y - x^5 \\ y' = x + y - y^5 \end{cases}$$

(a) Show that $(0, 0)$ is the only equilibrium point and study its stability.

(b) Use the Poincaré-Bendixson Theorem to show there exists a periodic orbit to the above system in certain annular region, determine the inner and outer radius of this annular region as accurate as possible.

8. (15pts) Consider the system

$$\begin{cases} x' = y \\ y' = -x(x - a)(1 - x) \end{cases},$$

where $0 < a < 1$. Prove that this system is Hamiltonian system and find its Hamiltonian function.

(a) Find all its equilibrium points and study their stability.

(b) Draw some typical solutions orbits. What is the range for a so that there is homoclinic orbit from $(0, 0)$ to itself in the right half plane $x > 0$.