M621 HW 7, due Oct 7: **Solns. Oct. 9**

1. **Pg. 85, number 5**. Be sure to use the definition of "order of an element" in your explanation.

   Solution. Let $g \in G$, and let N be a normal subgroup of G. Note that N is the identity of G/N. Using that N is normal, it follows that if $k \in \mathbb{N}$, then $(gN)^k = g^k N$ if and only if $g^k \in N$. By the definition of order of an element, it follows that $|gN| =$

   $$\begin{cases} \infty, & \text{if } \{k \in \mathbb{N} : g^k \in N\} = \emptyset \\ \min\{k \in \mathbb{N} : g^k \in N\}, & \text{if } \{k \in \mathbb{N} : g^k \in N\} \neq \emptyset \end{cases}$$

   page 89, 41.

   Solution. As observed in the statement of the problem, the inverse of a commutator is also a commutator, from which it follows that the commutator subgroup of G consists of the set of products of commutators of G.

   Let g, x, y be elements of G. Observe that $g(xyx^{-1}y^{-1})g^{-1} = (gxg^{-1})(gyg^{-1})(gx^{-1}g^{-1})^{-1}((gy^{-1}g^{-1})^{-1}$. Thus, the conjugate of a commutator is a commutator, from which it follows readily that the conjugate of a product of commutators is a product of commutators. Since the commutator subgroup consists of products of commutators, it follows that the commutator subgroup of G is a normal subgroup of G.

   To show that G/N is Abelian, it suffices to show that for any a, b in G, $aNbN = bNaN$. Using the normality of N, we have $aNbN = abN$. Observe that $(ab)^{-1}ba = b^{-1}a^{-1}ba$ is a commutator, and therefore contained in N; thus, $abN = baN$. That N is normal implies that $aNbN = abN = baN = bNaN$.

   The extra-credit part is not much more difficult: Suppose A is a normal subgroup of G, and G/A is Abelian. We show that the commutator subgroup N is contained in A. To do so, it suffices to show that for any$a, b \in G$, we have $aba^{-1}b^{-1} \in A$. Since G/A is Abelian, $b^{-1}Aa^{-1}A = a^{-1}Ab^{-1}A$. Since A is normal, we have $(b^{-1}a^{-1})^{-1}a^{-1}b^{-1} \in A$, so $aba^{-1}b^{-1} \in A$.