D'Alembert Functional Equation Continued

Lecture 8

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D’Alembert Functional Equation Continued

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Introduction

The well-known trigonometric identity

$$\cos(x+y) + \cos(x-y) = 2\cos(x)\,\cos(y)$$

implies the functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y) \qquad \text{(DE)}$$

for all $x, y \in \mathbb{R}$. In this lecture, we present the continuous solutions this functional equation.

General Solution of d'Alembert Equation

Definition 1 *A function* $E : \mathbb{R} \to \mathbb{C}$ *is said to be* exponential *if* E *satisfies the equation* $E(x+y) = E(x)E(y)$ *for all* $x, y \in \mathbb{R}$.

If E is a nonzero continuous function, then $E(x) = e^{\lambda x}$, where λ is an arbitrary complex constant.

If $E : \mathbb{R} \to \mathbb{C}$ is a nonzero exponential function, then we denote it by

$$E^*(y) = E(y)^{-1}. \tag{1}$$

Now we give some elementary properties of the exponential function.

Proposition 1 *If* $E : \mathbb{R} \to \mathbb{C}$ *is an exponential function and* $E(0)$ *is zero, then* $E(x) \equiv 0$ *for all* $x \in \mathbb{R}$.

Proof: Let $E : \mathbb{R} \to \mathbb{C}$ be an exponential function. Hence

$$E(x+y) = E(x)\,E(y) \tag{2}$$

for all $x, y \in \mathbb{R}$. Letting $y = 0$ in (2), we obtain

$$E(x) = E(x)\,E(0) \quad \text{for } x \in \mathbb{R}. \tag{3}$$

Since $E(0) = 0$, (3) yields

$$E(x) = 0 \quad \forall\, x \in \mathbb{R}. \tag{4}$$

Hence $E(x)$ is identically zero.

Proposition 2 *Let* $E : \mathbb{R} \to \mathbb{C}$ *be an exponential function. If* $E(x) \not\equiv 0$, *then* $E(0) = 1$.

Proof: Let $E : \mathbb{R} \to \mathbb{C}$ be an exponential function. Assume that $E(x)$ is not identically zero. Letting $x = 0 = y$ in (2), we get $E(0)\,[1 - E(0)] = 0$. Hence either $E(0) = 0$ or $E(0) = 1$.

We claim that $E(0) = 1$.

Suppose not. Then $E(0) = 0$. By Proposition 1, $E(x) \equiv 0$, is a contradiction to the fact that $E(x) \not\equiv 0$. Thus $E(0) = 1$.

This completes the proof of the proposition.

Proposition 3 *Let* $E : \mathbb{R} \to \mathbb{C}$ *be an exponential function. If* $E(x_0) = 0$ *for some* $x_0 \neq 0$*, then* $E(x) \equiv 0$ *for all* $x \in \mathbb{R}$*.*

Proof: Let $x\ (\neq x_0) \in \mathbb{R}$. Then, since $E(x_0) = 0$, we have

$$E(x) = E((x - x_0) + x_0) = E(x - x_0)\, E(x_0) = 0.$$

Hence $E(x) \equiv 0$. Thus E is nowhere zero or everywhere zero. This completes the proof.

Proposition 4 *Let $E : \mathbb{R} \to \mathbb{C}$ be an exponential function. If $E(x)$ is not identically zero, then*

$$E^*(-x) = E(x)$$

for all $x \in \mathbb{R}$.

Proof: Let $E : \mathbb{R} \to \mathbb{C}$ be exponential. Next, letting $y = -x$ in (2), we get

$$E(0) = E(x)\,E(-x). \tag{5}$$

Since $E(x)$ is not identically zero, by Proposition 2 we have $E(0) = 1$ and (5), that is $E(0) = E(x)\,E(-x)$ yields

$$E(-x) = \frac{1}{E(x)}.$$

Hence

$$E(-x) = E(x)^{-1}$$

or

$$E(-x) = E^*(x) \tag{6}$$

for all $x \in \mathbb{R}$.

Next replacing x by $-x$ in (6), that is in $E(-x) = E^*(x)$, we obtain

$$E^*(-x) = E(x) \tag{7}$$

and the proof of the proposition is now complete.

Proposition 5 *Let* $E : \mathbb{R} \to \mathbb{C}$ *be an exponential function. Suppose* $E(x)$ *is not identically zero. Then*

$$E^*(x+y) = E^*(x)E^*(y) \tag{8}$$

for all $x, y \in \mathbb{R}$.

Proof: Since $E(x)$ is not identically zero, $E(x)$ is never zero on $\mathbb{R}$ by Proposition 3. Now we consider

$$\begin{aligned} E^*(x+y) &= \frac{1}{E(x+y)} \\ &= \frac{1}{E(x)\,E(y)} = E(x)^{-1}\,E(y)^{-1} = E^*(x)\,E^*(y). \end{aligned}$$

Hence

$$E^*(x+y) = E^*(x)\,E^*(y)$$

for all $x, y \in \mathbb{R}$.

• **Now we prove some elementary properties of the d'Alembert functional equation.**

Proposition 6 *Every nonzero solution* $f : \mathbb{R} \to \mathbb{C}$ *of the d'Alembert equation*

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{DE}$$

is an even function.

Proof: Replacing y by $-y$ in the above equation (DE), we have

$$f(x+y)+f(x-y)=2f(x)f(-y). \tag{9}$$

Subtracting (9) from (DE), we obtain

$$f(y)=f(-y)$$

for all $y \in \mathbb{R}$. Hence f is an even function.

Let $G = (\mathbb{R}, +)$ be the additive group of reals and $\mathbb{C}$ be the set of complex numbers.

- **The nonzero continuous solution** $g : G \to \mathbb{C}$ **of the exponential functional equation** $g(x+y) = g(x)\,g(y)$ **is of the form** $g(x) = e^{\lambda x}$**.**

- **The continuous periodic solution** $f : G \to \mathbb{C}$ **of D'Alembert's is** $f(x) = \cos(\alpha\, x)$**.**

• **How can we represent the solutions of D'Alembert's functional equation on abstract structures like group or semigroup?**

Notice that

$$
\begin{aligned}
f(x) &= \cos(\alpha x) \\
&= \frac{\left[e^{i\alpha\, x} + e^{-i\alpha\, x}\right]}{2} \\
&= \frac{\left[g(x) + g(-x)\right]}{2}
\end{aligned}
$$

where $g(x)$ is a solution of the exponential equation (i.e. a homomorphism from group $(G, +)$ into $(\mathbb{C}, \cdot)$.

• In 1968, Pl. Kannappan determine the general nonzero solution $f : G \to \mathbb{C}$ of d'Alembert functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)\,f(y) \quad \forall x, y \in G$$

when f satisfies $f(xyz) = f(xzy)$ for all $x, y, z \in G$.

Theorem 1 . *Every nonzero solution $f : \mathbb{R} \to \mathbb{C}$ of the functional equation*

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{DE}$$

is of the form

$$f(x) = \frac{E(x) + E^*(x)}{2}, \tag{10}$$

where $E : \mathbb{R} \to \mathbb{C}^\star$ (the set of nonzero complex numbers) is an exponential function.

Let $D(\mathbb{R},\mathbb{C})$ be the set of all nonzero solutions of D'Alembert equation. To prove the theorem one has to establish the following 13 steps.

Step 1. Since $f \in D(\mathbb{R},\mathbb{C})$, thus $f(0) = 1$.

Step 2. Since $f \in D(\mathbb{R},\mathbb{C})$, $f(2x) = 2f(x)^2 - 1$.

Step 3. Since $f \in D(\mathbb{R},\mathbb{C})$, f satisfies

$$f(2x) + f(2y) = 2f(x+y)\, f(x-y)$$

Step 4. Since $f \in D(\mathbb{R}, \mathbb{C})$, f satisfies

$$\left[\, f(x+y) - f(x-y)\,\right]^2 = 4\left[f(x)^2 - 1\right]\left[f(y)^2 - 1\right].$$

Step 5. Show f satisfies

$$\left[f(x+y) - f(x)\, f(y)\,\right]^2 = \left[f(x)^2 - 1\right]\left[f(y)^2 - 1\right].$$

Step 6. Assume $f \in D(\mathbb{R}, \mathbb{C})$ and $f(x) \in \{-1, 1\}$. Then demonstrate that $f(x) = \frac{f(x)+f^*(x)}{2}$ is a solution of the equation $f(x+y) = f(x)\, f(y)$.

Step 7. Assume $f(x_o) \notin \{-1, 1\}$ for some $x_o \in \mathbb{R}$. Let $\alpha := f(x_o)$ and $\beta^2 := \alpha^2 - 1 \neq 0$. Then show

$$E(x) := f(x) + \frac{1}{\beta}\left[f(x + x_o) - f(x)f(x_o)\right]$$

is well defined.

Step 8. For this case, show $E(x)^2 - 2\,E(x)\,f(x) + 1 = 0$.

Step 9. If $E(x)$ is nowhere zero, and $f(x) = \frac{E(x)+E^*(x)}{2}$.

Step 10. Show

$$f(x_o + x)f(y) + f(x_o + y)f(x)$$
$$= f(x_o + x + y) + \alpha\big[2f(x)f(y) - f(x + y)\big]$$

Step 11. Show

$$f(x_o + x)\, f(x_o + y)$$
$$= f(x)f(y) + \alpha\, f(x_o + x + y) - f(x + y)$$

Step 12. Using Steps 10-11, show $E(x+y) = E(x)\,E(y)$.

Step 13. Show $f(x) = \frac{E(x)+E^*(x)}{2}$ satisfies (DE).

Proof: Let f be a nonzero solution of (DE); that is, f is not an identically zero function. Letting $x = 0 = y$ in (DE), we obtain $f(0)[1 - f(0)] = 0$. Hence either $f(0) = 0$ or $f(0) = 1$. Since $f(x)$ is not identically zero,

$$f(0) = 1. \tag{11}$$

Letting $y = x$ in (DE) and using (11), we get

$$f(2x) = 2f(x)^2 - 1. \tag{12}$$

Replacing x by $x + y$ and y by $x - y$ in (DE), we get

$$f(x + y + x - y) + f(x + y - x + y) = 2f(x + y)f(x - y).$$

Hence

$$f(2x) + f(2y) = 2f(x + y)f(x - y) \tag{13}$$

for all $x, y \in \mathbb{R}$.

Next, we compute

$$
\begin{aligned}
\big[\, f(x+y) - f(x-y)\,\big]^2
&= \big[f(x+y) + f(x-y)\,\big]^2 - 4f(x+y)f(x-y) \\
&= \big[\,2f(x)f(y)\,\big]^2 - 4f(x+y)f(x-y) \\
&= 4f(x)^2 f(y)^2 - 2\big[\,f(2x) + f(2y)\,\big] \\
&= 4f(x)^2 f(y)^2 - 2\big[\,2f(x)^2 - 1 + 2f(y)^2 - 1\,\big] \\
&= 4f(x)^2 f(y)^2 - 4f(x)^2 - 4f(y)^2 + 4 \\
&= 4\big[\,f(x)^2 - 1\,\big]\big[\,f(y)^2 - 1\,\big].
\end{aligned}
$$

Therefore

$$f(x+y) - f(x-y) = \pm 2\sqrt{[f(x)^2 - 1][f(y)^2 - 1]}.$$

Adding this to (DE), we get

$$f(x+y) = f(x)f(y) \pm \sqrt{[f(x)^2 - 1][f(y)^2 - 1]}.$$

Hence

$$[f(x+y) - f(x)f(y)]^2 = [f(x)^2 - 1][f(y)^2 - 1]. \qquad (14)$$

Now we consider two cases based on whether

(1) $f(x) \in \{1, -1\}$ **for all** $x \in \mathbb{R}$ **or** **(2)** $f(x) \notin \{1, -1\}$ **for some** $x \in \mathbb{R}$.

Case 1. Suppose $f(x) \in \{1, -1\}$. Hence by (14), we get

$$f(x+y) = f(x)f(y) \tag{15}$$

for all $x, y \in \mathbb{R}$. Since $f(x)$ is either 1 or -1, we have

$$f^*(x) = f(x).$$

Hence

$$f(x) = \frac{f(x) + f^*(x)}{2}$$

is a solution of (DE). Note that

$$f(x) = \frac{E(x) + E^*(x)}{2}$$

with $E(x) \in \{1, -1\}$.

Case 2. Suppose $f(x) \notin \{1, -1\}$ for some x. Hence

$$f(x_0)^2 \neq 1$$

for some $x_0 \in \mathbb{R}$. Let $\alpha = f(x_0)$. Hence $\alpha^2 - 1 \neq 0$. Let us call

$$\beta^2 = \alpha^2 - 1. \tag{16}$$

Now we define

$$E(x) = f(x) + \frac{1}{\beta}[f(x+x_0) - f(x)f(x_0)]$$
$$= \frac{1}{\beta}[f(x+x_0) + (\beta-\alpha)f(x)] \qquad (17)$$

for all $x \in \mathbb{R}$. Clearly E is well defined. To see this, let $x_1 = x_2$ and consider

$$E(x_1) = \frac{1}{\beta}[f(x_1+x_0) + (\beta-\alpha)f(x_1)]$$
$$= \frac{1}{\beta}[f(x_2+x_0) + (\beta-\alpha)f(x_2)]$$
$$= E(x_2).$$

Hence E is well defined. Next, we compute

$$\begin{aligned}
[E(x) - f(x)]^2 &= \frac{1}{\beta^2}[f(x + x_0) - f(x)f(x_0)]^2 \\
&= \frac{1}{\beta^2}[f(x)^2 - 1][f(x_0)^2 - 1] \qquad \text{(by (14))} \\
&= \frac{\alpha^2 - 1}{\beta^2}[f(x)^2 - 1] \\
&= f(x)^2 - 1, \qquad (18)
\end{aligned}$$

since $\beta^2 = \alpha^2 - 1$.

Hence (18) yields

$$E(x)^2 - 2E(x)f(x) + f(x)^2 = f(x)^2 - 1$$

which is

$$E(x)^2 - 2E(x)f(x) + 1 = 0.$$

$E(x) = 0$ leads to the contradiction $1 = 0$.

Therefore, $E(x) \neq 0$, and then

$$f(x) = \frac{E(x)^2 + 1}{2E(x)}$$
$$= \frac{E(x) + E^*(x)}{2}.$$

Next we show that $E(x)$ satisfies

$$E(x+y) = E(x)\,E(y).$$

To show this we need the following:

$$
\begin{aligned}
&2[f(x_0+x)f(y)+f(x_0+y)f(x)] \\
&\quad = f(x_0+x+y)+f(x_0+x-y)+f(x_0+y+x) \\
&\qquad\quad + f(x_0+y-x) \quad \text{(by (DE))} \\
&\quad = 2f(x_0+x+y)+f(x_0+x-y)+f(x_0+y-x) \\
&\quad = 2f(x_0+x+y)+f(x_0+(x-y))+f(x_0-(x-y)) \\
&\quad = 2f(x_0+x+y)+2f(x_0)f(x-y) \\
&\quad = 2[f(x_0+x+y)+f(x_0)\{2f(x)f(y)-f(x+y)\}] \\
&\quad = 2[f(x_0+x+y)+\alpha\{2f(x)f(y)-f(x+y)\}] \qquad (19)
\end{aligned}
$$

and

$$
\begin{aligned}
&2f(x_0+x)f(x_0+y)\\
&= f(x_0+x+x_0+y)+f(x_0+x-x_0-y) \quad \text{(by (DE))}\\
&= f(x_0+(x_0+x+y))+f(x-y)\\
&= [2f(x_0)f(x_0+x+y)-f(x_0+x+y-x_0)]\\
&\qquad +[2f(x)f(y)-f(x+y)] \quad \text{(by (DE))}\\
&= [2f(x_0)f(x_0+x+y)-f(x+y)]+[2f(x)f(y)-f(x+y)]\\
&= 2[f(x)f(y)+\alpha f(x_0+x+y)-f(x+y)]. \qquad (20)
\end{aligned}
$$

Next, we consider

$$
\begin{aligned}
&E(x)\,E(y)\\
&=\frac{1}{\beta^2}[f(x+x_0)+(\beta-\alpha)f(x)][f(y+x_0)+(\beta-\alpha)f(y)]\\
&=\frac{1}{\beta^2}[f(x+x_0)f(y+x_0)+(\beta-\alpha)\{f(x)f(x_0+y)\\
&\qquad+f(y)f(x_0+x)\}+(\beta-\alpha)^2f(x)f(y)]\\
&=\frac{1}{\beta^2}[f(x)f(y)+\alpha f(x_0+x+y)-f(x+y)\\
&\qquad+(\beta-\alpha)\{f(x_0+x+y)+2\alpha f(x)f(y)-\alpha f(x+y)\}\\
&\qquad+(\beta-\alpha)^2f(x)f(y)]\quad \text{(by (20) and (19))}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\beta^2}[\{(\beta-\alpha)^2 + 2\alpha(\beta-\alpha)+1\}f(x)f(y) + \beta f(x_0+x+y) \\
&\qquad\qquad - \{1+(\beta-\alpha)\alpha\}f(x+y)] \\
&= \frac{1}{\beta^2}[(\beta^2-\alpha^2+1)f(x)f(y) \\
&\qquad\qquad + \beta f(x_0+x+y) - (\beta\alpha-\beta^2)f(x+y)] \\
&= \frac{1}{\beta^2}[\beta f(x_0+x+y) + \beta(\beta-\alpha)f(x+y)] \\
&= \frac{1}{\beta}[f(x_0+x+y) + (\beta-\alpha)f(x+y)] \\
&= E(x+y).
\end{aligned}
$$

Hence $E : \mathbb{R} \to \mathbb{C}^{\star}$ is an exponential function. This completes the "only if" part.

The "if" part can be shown by direct verification. Consider

$$
\begin{aligned}
&f(x+y)+f(x-y)\\
&=\frac{E(x+y)+E^*(x+y)}{2}+\frac{E(x-y)+E^*(x-y)}{2}\\
&=\frac{E(x)E(y)+E^*(x)E^*(y)+E(x)E(-y)+E^*(x)E^*(-y)}{2}\\
&=\frac{E(x)E(y)+E^*(x)E^*(y)+E(x)E^*(y)+E^*(x)E(y)}{2}\\
&=\frac{E(x)[E(y)+E^*(y)]+E^*(x)[E^*(y)+E(y)]}{2}\\
&=\frac{[E(x)+E^*(x)][E(y)+E^*(y)]}{2}\\
&=2f(x)f(y).
\end{aligned}
$$

This completes the proof.

Remark 1 *In Theorem 1, the function* $E : \mathbb{R} \to \mathbb{C}^\star$ *is a homomorphism from the additive group of reals,* $\mathbb{R}$*, to the multiplicative group of nonzero complex numbers,* $\mathbb{C}^\star$.

The last theorem can be generalized to the following theorem which was originally proved by Kannappan (1968a).

Theorem 2 *Let G be an arbitrary abelian group and $\mathbb{C}^\star$ be the multiplicative group of nonzero complex numbers. Then every nontrivial solution $f : G \to \mathbb{C}$ of the functional equation (DE), that is,*

$$f(x+y) + f(x-y) = 2f(x)f(y),$$

is of the form

$$f(x) = \frac{g(x) + g^*(x)}{2},$$

where $g : G \to \mathbb{C}^\star$ is a homomorphism of the group G into $\mathbb{C}^\star$.

The proof is almost identical to the proof of the last theorem.

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Thank You