My Solutions to Old Modeling Quals

Jacob S Townson

April-August 2018

Prelude

This document is written in reference to qualifying exams given at the University of Louisville in past years. These solutions are not given from the University, but of my work alone as a way to study for my own qualifying exam. If any tips or recommendations come up and you feel you should share, feel free to raise an issue on GitHub where I have this document saved and open to the public here. To see the qualifying exams for yourself, visit this link. When referencing the Perko book, this is in reference to the *Differential Equations and Dynamical Systems, 3rd Edition.* Thank you for reading, and for all advice.

Jacob Townson

What to Expect:

- Mechanical Vibrations: Newton's Laws, spring-mass systems, two-mass oscillators, friction, damping, pendulum, linear stability and equilibria, energy analysis, phase place analysis, nonlinear oscillations, control oscillations, inverse problem
- Traffic Flow: Welocities and velocity fields, traffic flow and density, conservation laws, linear and nonlinear car-following models, steady state, first order partial differential equations, green light models and rarefaction solution, shock waves, highway with entrance, traffic wave propagation, optimization problem. (**NOTE: WE DID NOT COVER TRAFFIC FLOW IN CLASS, THUS IT IS LIKELY TO NOT BE ON THE QUAL THIS SUMMER [2018]**)
- Dynamical Systems: Nonlinear systems in the plane, interacting species, limit cycles, Hamiltonian systems, Liapunov functions and stability, bifurcation theory, three-dimensional autonomous system and chaos, Poincare maps and nonautonomous systems in the plane, linear discrete synamical systems

My Solutions from Quals

Summer 2016

Consider the initial value problem in $\mathbb{R}$ where $\dot{x} = x^2$ and $x(0) = 1$. (a) Find the first three successive approximations $u_1(t), u_2(t)$, and $u_3(t)$ by using integration. (b) Use mathematical induction to show that for all $n \geq 1$, $u_n(t) = 1 + t + ... + t^n + O(t)$. (c) Solve the IVP and show that the function $x(t) = \frac{1}{1-t}$ is a solution on the interval $(-\infty, 1)$. (d) Show that the first $(n+1)$-terms in $u_n(t)$ agree with the first $(n+1)$-terms in Taylor series of the function $x(t) = \frac{1}{1-t}$ about $t = 0$.

My Solution:

Show that the system with $\dot{x} = y$ and $\dot{y} = -2x - y - 3x^4 + y^2$ has no limit cycle in $\mathbb{R}^2$. (HINT: a function in the form $e^{\beta x}$ might be useful)

My Solution:

Consider the following model defined in $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$: $\dot{x} = 1 - x - \frac{2xy}{2+x}$, $\dot{y} = y\left(\frac{2x}{2+x} - 1\right)$ where $x(0) > 0$ and $y(0) > 0$.

(a) Show that the ω-limit set of any orbit is on the set $K = \{(x, y) \in \mathbb{D} : x + y = 1\}$. (b) Show that the system has a unique steady state in $\mathbb{D}$ and the unique equilibrium point is globally stable in $\mathbb{D}$.

My Solution:

Show that the system with $\dot{x} = x - y - (x^2 + \frac{3}{2}y^2)x$ and $\dot{y} = x + y - (x^2 + \frac{1}{2}y^2)y$ has a limit cycle.

My Solution:

Determine the qualitative behavior near the non-hyperbolic critical point at the origin for the system with $\dot{x} = xy$ and $\dot{y} = -y - x^2$. Sketch the phase portrait.

My Solution:

Consider the Lorenz model with $\dot{x} = \frac{3}{2}y - x$, $\dot{y} = \frac{1}{2}x - y - xz$, and $\dot{z} = xy - z$.

(a) Show that the Lorenz model has a unique equilibrium point. (b) Show that the equilibrium point is globally stable. (HINT: Use a Liapunov function.)

My Solution:

Consider a spring-mass system that describes a motion with a force caused by friction or magnification or both: $\ddot{x} - \epsilon\dot{x} + x - x^3 + x^5 = 0$ where $\epsilon \in (-2, 2)$.

(a) Without solving the equation, show that $\epsilon = 0$ is a bifurcation point. In more details, show that when $\epsilon = 0$ the system is a Hamiltonian system and find an expression for the Hamiltonian. Then sow that when $\epsilon < 0$ the origin is a stable equilibrium point; but when $\epsilon > 0$ the system has a limit cycle. (b) Classify the type of the bifurcation and then draw the bifurcation diagram. Label the diagram clearly.

My Solution:

Winter 2016

Consider the predator-prey model with $\dot{x} = x(1 - x - y)$ and $\dot{y} = (4x - 1)y$. Let $x(0) > 0$ and $y(0) > 0$. Determine $\lim_{t\to\infty}(x(t), y(t))$.

My Solution:

Consider the predator-prey model with $\dot{x} = x(1 - y)$ and $\dot{y} = (2x - 1)y$. Show that every non-constant positive solution is periodic.

My Solution:

Prove that the system with $\dot{x} = x + y - 4x^3$ and $\dot{y} = y - x - 4y^3$ has a limit cycle.

My Solution:

Consider $\dot{x} = y + x^2 - y^2$ and $\dot{y} = -x - 2xy$. (a) Find all equilibria for this system and determine their stability. (b) Find the Hamiltonian of the system. (c) Find a solution curve that connects a saddle point. (d) Does the curve obtained in (c) connect any other saddle points?

My Solution:

Consider the mass spring system subject to an external force $f(t)$:

$$\ddot{x} + \dot{x} + x = f(t)$$

Assume $x(0) = 0$ and $\dot{x}(0) = 0$. Assume also that $f(t)$ is the force describing the striking effect on the mass of the mass spring system in a short time period $0 < T < \frac{1}{2}$, given as $f(t) = \frac{\pi}{4T} \sin \frac{\pi t}{2T}$ when $0 \leq t < 2T$ and $f(t) = 0$ when $t \geq 2T$.

(a) Solve the given IVP first for all $t \geq 0$. (b) Compute the limits $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} \dot{x}(t)$. (c) Compute the limits $\lim_{T\to 0} x(T)$ and $\lim_{T\to 0} \dot{x}(T)$, and discuss their physical meanings.

My Solution:

Suppose that the motion of a mass is described by the nonlinear differential equation $\frac{d^2x}{dt^2} + 2x + 3(\frac{dx}{dt})^5 = 0$. Determine how solutions of this equation behave.

My Solution:

Summer 2015

Consider the Lotka-Volterra predator-prey model with $\dot{x} = x(2 - y)$ and $\dot{y} = y(x - 1)$. Show that every non-constant positive solution is periodic.

My Solution:

Consider the system

$$\dot{x} = y + x\sqrt{x^2 + y^2} \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right), \dot{y} = -x + y\sqrt{x^2 + y^2} \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right)$$

Show that this system has infinitely many limit cycles, and discuss how the solutions behavior. (Hint: use polar coordinates.)

My Solution:

The Hamiltonian function of a Hamiltonian system is given by $H(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$. (a) Find the equilibria of the system and discuss their stability. (b) Sketch the phase portrait for the system.

My Solution:

Consider the predator-prey model

$$\dot{x} = x\left(6 - x - \frac{3y}{1+x}\right), \dot{y} = y(x-2)$$

Assume that all positive solutions are bounded. (a) Find all critical points and determine their local stability. (b) Show that this system has a limit cycle in the first quadrant.

My Solution:

Suppose that the motion of a mass is described by the nonlinear differential equation $\frac{d^2x}{dt^2} + \frac{dx}{dt} + x^5 = 0$. Determine how solutions of this equation behave.

My Solution:

Suppose that the motion of a mass is described by the nonlinear differential equation $\frac{d^2x}{dt^2} + x + x^3 = 0$. Determine how solutions of this equation behave.

My Solution:

Summer 2014

Important Notes