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Introduction

• We have seen that a continuous solution of the additive Cauchy equation $f(x+y) = f(x) + f(y)$ is linear.

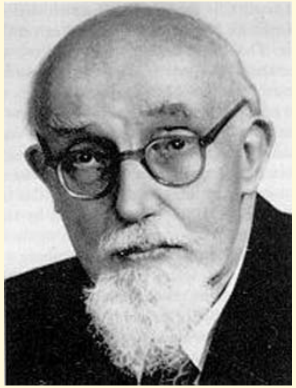
• In other words continuous additive functions are linear.

• Even if we relax the continuity condition to continuity at a point, still additive functions are linear.

It is known that every regular additive function is always linear. Regular means one of the following: measurable, differentiable, continuous, locally integrable, integrable, bounded, monotonic, etc.

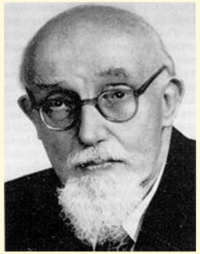
- For many years the existence of discontinuous additive functions was an open problem.

- Mathematicians could neither prove that every additive function is continuous nor exhibit an example of a discontinuous additive function.



It was the German mathematician Georg Hamel in 1905 who first succeeded in proving that there exist discontinuous additive functions.

Georg Hamel

 Georg Hamel (1877-1954) worked in function theory, mechanics and the foundations of mathematics. He is perhaps best known for the *Hamel basis*, published in 1905, when he made an early and explicit use of the Axiom of Choice to construct a basis for the real numbers as a vector space over the rational numbers.

• First, we show that the non-linear solution of the additive Cauchy equation displays a very strange behavior.

We begin with the following definition.

Definition 1 *The graph of a function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is the set*

$$G = \{(x, y) \,|\, x \in \mathbb{R}, \quad y = f(x)\}.$$

It is easy to note that the graph G of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is subset of the plane $\mathbb{R}^2$.

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{Q}$ be a subset of $\mathbb{R}$. The set $\mathbb{Q}$ is said to be dense in $\mathbb{R}$ if, x is any point in $\mathbb{R}$, then there is a point ρ in $\mathbb{Q}$ "infinitely close" to x.

- **This means x can never be finite distance away from points in $\mathbb{Q}$.**

Let $\mathbb{Q}$ be a subset of $\mathbb{R}$. The set $\mathbb{Q}$ is dense in $\mathbb{R}$ if, for any $x \in \mathbb{R}$ and any $\epsilon > 0$, there exists a $\rho \in \mathbb{Q}$ such that

$$|x - \rho| \leq \epsilon.$$

- The set of rational numbers are dense in reals.
- The set of integers are not dense in reals.

Theorem 1. *The graph of every non-linear solution* $f : \mathbb{R} \to \mathbb{R}$ *of the additive Cauchy equation is everywhere dense in the plane* $\mathbb{R}^2$.

Proof: The graph G of f is given by

$$G = \{(x, y) \mid x \in \mathbb{R},\ y = f(x)\}.$$

Choose a nonzero x_1 in $\mathbb{R}$. Since f is a non-linear solution of the additive Cauchy equation

$$f(x) \neq mx$$

for any real constant m.

Hence there exists a nonzero real number x_2 such that

$$\frac{f(x_1)}{x_1} \neq \frac{f(x_2)}{x_2};$$

otherwise writing $m = \frac{f(x_1)}{x_1}$ and letting $x_2 = x$, we will have $f(x) = mx$ for all $x \neq 0$, and since $f(0) = 0$ this implies that f is linear contrary to our assumption that f is non-linear.

This implies that

$$\begin{vmatrix} x_1 & f(x_1) \\ x_2 & f(x_2) \end{vmatrix} \neq 0,$$

so that the vectors $\mathbf{v}_1 = (x_1, f(x_1))$ and $\mathbf{v}_2 = (x_2, f(x_2))$ are linearly independent and thus span the whole plane $\mathbb{R}^2$. This means that for any vector $\mathbf{v} = (x, f(x))$ there exist real numbers r_1 and r_2 such that

$$\mathbf{v} = r_1 \mathbf{v}_1 + r_2 \mathbf{v}_2.$$

• If we permit only rational numbers ρ_1, ρ_2, then by their appropriate choice, we can get with $\rho_1\mathbf{v}_1 + \rho_2\mathbf{v}_2$ arbitrarily close to any given plane vector **v** (since the rational numbers $\mathbb{Q}$ are dense in reals $\mathbb{R}$ and hence $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$). Now,

$$\begin{aligned}\rho_1\mathbf{v}_1 + \rho_2\mathbf{v}_2 &= \rho_1(x_1, f(x_1)) + \rho_2(x_2, f(x_2)) \\ &= (\rho_1 x_1 + \rho_2 x_2,\ \rho_1 f(x_1) + \rho_2 f(x_2)) \\ &= (\rho_1 x_1 + \rho_2 x_2,\ f(\rho_1 x_1 + \rho_2 x_2)).\end{aligned}$$

Thus, the set $\widehat{G}$ given by

$$\left\{(x,y)\middle|x=\rho_1x_1+\rho_2x_2,\ y=f(\rho_1x_1+\rho_2x_2),\ \rho_1,\rho_2\in\mathbb{Q}\right\}$$

is everywhere dense in $\mathbb{R}^2$. Since

$$\widehat{G}\subset G,$$

the graph G of our non-linear additive function f is also dense in $\mathbb{R}^2$. The proof of the theorem is now complete.

- The graph of f forms a dense subset of $\mathbb{R}^2$ yet it intersects every horizontal line at precisely one point.

- The graph of f forms a dense subset of $\mathbb{R}^2$ yet it intersects every vertical line at precisely one point.

- What is more amazing that we can choose f so that every rational line intersects the graph at most one point!

A line is called a rational line if the equation of the line can be written as with rational numbers, that is, if it has an equation

$$a\,x + b\,y + c = 0$$

with a, b, c rational numbers.

The graph of an additive continuous function is a straight line that passes through the origin.

The graph of a non-linear additive function is dense in the plane.

Next, we introduce the concept of Hamel basis to construct a discontinuous additive function.

Let us consider the set

$$S = \{s \in \mathbb{R} \,|\, s = u + v\sqrt{2} + w\sqrt{3},\ \ u, v, w \in \mathbb{Q}\}$$

whose elements are the rational linear combination of $1, \sqrt{2}, \sqrt{3}$.

Further, this rational combination is unique.

That is, if an element $s \in S$ has two different rational linear combinations, for instance,

$$s = u + v\sqrt{2} + w\sqrt{3} = u' + v'\sqrt{2} + w'\sqrt{3},$$

then $u = u'$, $v = v'$ and $w = w'$. To prove this we note that this assumption implies that

$$(u - u') + (v - v')\sqrt{2} + (w - w')\sqrt{3} = 0.$$

Letting $a = (u - u')$, $b = (v - v')$ and $c = (w - w')$, we see that the above expression reduces to

$$a + b\sqrt{2} + c\sqrt{3} = 0.$$

Next, we show that $a = 0 = b = c$. The above expression yields

$$b\sqrt{2} + c\sqrt{3} = -a,$$

and squaring both sides, we have $2bc\sqrt{6} = a^2 - 2b^2 - 3c^2$.

Hence

$$2bc\sqrt{6} = a^2 - 2b^2 - 3c^2$$

implies that b or c is zero; otherwise, we may divide both sides by $2bc$ and get

$$\sqrt{6} = \frac{a^2 - 2b^2 - 3c^2}{2bc}$$

contradicting the fact that $\sqrt{6}$ is an irrational number.

If $b = 0$, then from

$$2bc\sqrt{6} = a^2 - 2b^2 - 3c^2.$$

we have $a + c\sqrt{3} = 0$. Hence $c = 0$ (else $\sqrt{3} = -\frac{a}{c}$ is a rational number contrary to the fact that $\sqrt{3}$ is an irrational number).

Similarly if $c = 0$, we obtain that $b = 0$. Thus both b and c are zero. Hence it follows immediately that $a = 0$.

If we call

$$B = \left\{ 1,\ \sqrt{2},\ \sqrt{3} \right\},$$

then every element of S is a *unique* rational linear combination of the elements of B. This set B is called a Hamel basis for the set S. Formally, a Hamel basis is defined as follows.

Definition 2 *Let S be a set of real numbers and let B be a subset of S. Then B is called a Hamel basis for S if every member of S is a unique (finite) rational linear combination of B.*

If the set S is the set of reals $\mathbb{R}$, then using the axiom of choice it can be shown that a Hamel basis B for $\mathbb{R}$ exists.

There is a close connection between additive functions and Hamel bases.

To exhibit an additive function it is sufficient to give its values on a Hamel basis, and these values can be assigned arbitrarily.

Theorem 2. *Let B be a Hamel basis for $\mathbb{R}$. If two additive functions have the same value at each member of B, then they are equal.*

Proof: Let f_1 and f_2 be two additive functions having the same value at each member of B. Then $f_1 - f_2$ is additive. Let us write $f = f_1 - f_2$.

Let x be any real number. Then there are numbers b_1, b_2, ..., b_n in B and rational numbers r_1, r_2, ..., r_n such that

$$x = r_1 b_1 + r_2 b_2 + \cdots + r_n b_n.$$

Hence

$$\begin{aligned}
f_1(x) - f_2(x) &= f(x) \\
&= f(r_1 b_1 + r_2 b_2 + \cdots + r_n b_n) \\
&= f(r_1 b_1) + f(r_2 b_2) + \cdots + f(r_n b_n) \\
&= r_1 f(b_1) + r_2 f(b_2) + \cdots + r_n f(b_n) \\
&= r_1[f_1(b_1) - f_2(b_1)] + r_2[f_1(b_2) - f_2(b_2)] \\
&\qquad + \cdots + r_n[f_1(b_n) - f_2(b_n)] \\
&= 0.
\end{aligned}$$

Thus, we have $f_1 = f_2$ and the proof is complete.

Theorem 3 . *Let B be a Hamel basis for $\mathbb{R}$. Let $g : B \to \mathbb{R}$ be an arbitrary function defined on B. Then there exists an additive function $f : \mathbb{R} \to \mathbb{R}$ such that $f(b) = g(b)$ for each $b \in B$.*

Proof: For each real number x there can be found $b_1, b_2, ..., b_n$ in B and rational numbers $r_1, r_2, ..., r_n$ with

$$x = r_1 b_1 + r_2 b_2 + \cdots + r_n b_n.$$

We define $f(x)$ to be

$$r_1 g(b_1) + r_2 g(b_2) + \cdots + r_n g(b_n).$$

This defines $f(x)$ for all x. This definition is unambiguous since for each x, the choice of $b_1, b_2, ..., b_n, r_1, r_2, ..., r_n$ is unique, except for the order in which b_i and r_i are selected. For each b in B, we have $f(b) = g(b)$ by definition of f. Next, we show that f is additive on the reals. Let x and y be any two real numbers.

Then

$$x = r_1 a_1 + r_2 a_2 + \cdots + r_n a_n,$$

$$y = s_1 b_1 + s_2 b_2 + \cdots + s_m b_m,$$

where $r_1, r_2, \ldots, r_n$, $s_1, s_2, \ldots, s_m$ are rational numbers and $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_m$ are members of the Hamel basis B.

The two sets $\{a_1, a_2, ..., a_n\}$ and $\{b_1, b_2, ..., b_m\}$ may have some members in common. Let the union of these two sets be $\{c_1, c_2, ..., c_l\}$. Then $\ell \leq m + n$, and

$$x = u_1 c_1 + u_2 c_2 + \cdots + u_\ell c_\ell$$

$$y = v_1 c_1 + v_2 c_2 + \cdots + v_\ell c_\ell,$$

where $u_1, u_2, ..., u_\ell$, $v_1, v_2, ..., v_\ell$ are rational numbers, several of which may be zero.

Hence

$$x + y = (u_1 + v_1)c_1 + (u_2 + v_2)c_2 + \cdots + (u_\ell + v_\ell)c_\ell$$

and

$$\begin{aligned} f(x+y) & \\ &= f(\,(u_1 + v_1)c_1 + (u_2 + v_2)c_2 + \cdots + (u_\ell + v_\ell)c_\ell\,) \\ &= (u_1 + v_1)\,g(c_1) + (u_2 + v_2)\,g(c_2) + \cdots + (u_\ell + v_\ell)\,g(c_\ell) \\ &= [\,(u_1 g(c_1) + u_2 g(c_2) + \cdots + u_\ell g(c_\ell)\,] \\ &\qquad + [\,(v_1 g(c_1) + v_2 g(c_2) + \cdots + v_\ell g(c_\ell)\,] \\ &= f(x) + f(y). \end{aligned}$$

Hence f is additive on the set of real numbers $\mathbb{R}$. QED.

With the help of a Hamel basis, next we construct a non-linear additive function. Let B be a Hamel basis for the set of real numbers $\mathbb{R}$. Let $b \in B$ be any element of B. Define

$$g(x) = \begin{cases} 0 & \text{if } x \in B \setminus \{b\} \\ 1 & \text{if } x = b. \end{cases}$$

By Theorem 3, there exists an additive function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = g(x)$ for each $x \in B$. Note that this f cannot be linear since for $x \in B$ and $x \neq b$, we have

$$0 = \frac{f(x)}{x} \neq \frac{f(b)}{b}.$$

Therefore f is a non-linear additive function.

We end this section with the following remark.

Remark 1 *No concrete example of a Hamel basis for* $\mathbb{R}$ *is known; we only know that it exists.*

The graph of a discontinuous additive function on $\mathbb{R}$ *is not easy to draw as the set* $\{\, f(x) \,|x \in \mathbb{R}\,\}$ *is dense in* $\mathbb{R}$.

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Thank You