Dawn of the First Day

textbook - Real Analysis for Graduate Students by Richard Bass. ver. 3.1

Exams:
10/18
12/01

Ch1) notation stuff

- most of it - duh

$A \triangle B = (A - B) \cup (B - A)$ (symmetric difference)

$A_i \uparrow$ means $A_1 \subset A_2 \subset A_3 \subset \dots$

$A_i \uparrow A$ means $A_1 \subset A_2 \subset A_3 \subset \cdots$ & $\bigcup_{i=1}^{\infty} A_i = A$

and vice versa for decreasing

$x \vee y = \max(x, y)$ $\quad x \wedge y = \min(x, y)$

$x^+ = x \vee 0$ $\quad x^- = (-x) \vee 0$

Chapter 2 - Families of Sets

2.1 Algebras & σ-algebras

Let X be a set.

Def: An algebra is a collection $\mathcal{A}$ of subsets of X $\ni$

1) $\emptyset \in \mathcal{A}$, $X \in \mathcal{A}$
2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
3) $A_1, A_2, \dots, A_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{A}$, $\bigcap_{i=1}^{n} A_i \in \mathcal{A}$

Def: $\mathcal{A}$ is a σ-algebra if in addition from this

4) $A_1, A_2, A_3, \dots, A_n, \dots \in \mathcal{A}$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A} \ \& \ \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$$

- Suppose X is any set, $\mathcal{A} = \{\emptyset, X\}$ is the most simple σ-algebra we could have

Can also have $A \subset X$, $A \neq \emptyset$ & $A \neq X$, $\mathcal{A} = \{\emptyset, A, A^c, X\}$

Continued

Prove: $\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c = \bigcap_{i=1}^{\infty} A_i$

proof: step 1) $\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \subset \bigcap_{i=1}^{\infty} A_i$

Let $x \in \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c$

or equivalently $x \notin \bigcup_{i=1}^{\infty} A_i^c \Rightarrow x \notin \bigcup_{i=1}^{m} A_i^c, \forall m$

$\Rightarrow x \in \left(\bigcup_{i=1}^{m} A_i^c\right)^c, \forall m \Rightarrow x \in \bigcap_{i=1}^{m} A_i \ \forall m$

$\Rightarrow x \in \bigcap_{i=1}^{\infty} A_i$

step 2) $\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \supset \bigcap_{i=1}^{\infty} A_i$

Let $x \in \bigcap_{i=1}^{\infty} A_i$. Then $x \in A_i, \forall i$

$x \in \bigcap_{i=1}^{m} A_i, \forall m$

$x \notin \left(\bigcap_{i=1}^{m} A_i\right)^c \ \forall m$ so $x \notin \left(\bigcup_{i=1}^{m} A_i^c\right), \forall m$

so $x \notin \bigcup_{i=1}^{\infty} A_i^c$, thus $x \in \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c$

QED

hmwrk – reproduce this proof

★ countable

$\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c \in \mathcal{A} \Rightarrow \bigcap_{i=1}^{\infty} A_i^c \in \mathcal{A}$ (connects step 4 in σ algebra)

Ex X is a set, $\mathcal{A} = \{\emptyset, X\}$ then this is a σ algebra

ex X $\mathcal{A}$ = set of all subsets of X – this is also a σ algebra.

Ex $X = \mathbb{R}$, $\mathcal{A} = \{A \subset \mathbb{R} \mid A \text{ is countable or } A^c \text{ is countable}\}$

$A \subset \mathbb{R}$ A is not countable $\not\Rightarrow A^c$ is countable

ex – $A = [0, 1]$ Prove it's a σ algebra

Proof: 1) $\emptyset$ (countable) $\in \mathcal{A}$

$X^c = \mathbb{R}^c = \emptyset \Rightarrow X \in \mathcal{A}$.

2) $A \in \mathcal{A} \Rightarrow A$ is countable or A^c is countable.

$A = (A^c)^c \Rightarrow A^c \in \mathcal{A}$

3) $A_1, A_2, \ldots \in \mathcal{A}$

cont.

3) cont, $A_1, A_2, \ldots \in \mathcal{A}$

- Case 1) A_i is countable for $\forall i$
- Case 2) $\exists\ i_0$, $A_{i_0}^c$ is countable

-1) $\bigcup_{i=1}^{\infty} A_i \rightarrow$ countable union of countable sets, so it's countable, hence $\in \mathcal{A}$
thus our intersection is in it as well.

-2) $\bigcup_{i=1}^{\infty} A_i$, so we know we can also see if $(\bigcup_{i=1}^{\infty} A_i)^c = \bigcap_{i=1}^{\infty} A_i^c \subset A_{i_0}^c$
$\Rightarrow$ b/c $A_{i_0}^c$ is countable, then b/c $\bigcap_{i=1}^{\infty} A_i^c$ is smaller, then it's countable. $\Rightarrow (\bigcup_{i=1}^{\infty} A_i)^c$: countable $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

Thus this is a σ-algebra. QED.

Ex) $X = [0,1]$, $\mathcal{A} = \{\emptyset, X, [0,\frac{1}{2}], (\frac{1}{2},1]\}$
Prove $\mathcal{A}$ is a σ-algebra.

Proof: step 1 & 2 are obvious.
step 3) Consider $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$ (each i can be one of these 4 subsets)

$$\bigcup_{i=1}^{\infty} A_i = \begin{cases} \emptyset & \text{if } A_i = \emptyset\ \forall i \\ [0,\frac{1}{2}] & \text{if } A_i = [0,\frac{1}{2}] \text{ for some } i \ \&\ (\frac{1}{2},1], X \notin \{A_i\} \\ (\frac{1}{2},1] & \text{if } A_i = (\frac{1}{2},1] \text{ for some } i \ \&\ [0,\frac{1}{2}], X \notin \{A_i\} \\ X & \text{if } A_i = X \text{ for some } i, \text{ or if } [0,\frac{1}{2}] \ \&\ (\frac{1}{2},1] \in \{A_i\} \end{cases}$$

$\forall$ cases $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ thus $\mathcal{A}$ is a σ-algebra.

Lemma 2.7 - If $\mathcal{A}_\alpha$ is a σ-algebra $\forall\ \alpha$ in some ~~non zero~~ non-empty index set I,
then $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ-algebra.
or X

example of this $\bigcap_{i=1}^{\infty} A_i = \bigcap_{\alpha \in I} A_\alpha$ where $I = \mathbb{N}$
but could have $\bigcap_{i=1}^{N} A_i = \bigcap_{\alpha \in J} A_\alpha$ where $J = \{1, \ldots, N\}$ - not countable

Cont.

Think about the proof of $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$

Prove $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ-algebra on X

1) $\emptyset \in \mathcal{A}_\alpha \ \forall \alpha \Rightarrow \emptyset \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$

$X \in \mathcal{A}_\alpha \ \forall \alpha \Rightarrow X \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$

2) Assume $A \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha \Rightarrow A \in \mathcal{A}_\alpha \ \forall \alpha$

we know each $\mathcal{A}_\alpha$ is σ-algebra, thus

$\Rightarrow A^c \in \mathcal{A}_\alpha \ \forall \alpha \Rightarrow A^c \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$

3) Assume $A_1, A_2, \ldots, A_n, \ldots \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$

$\Rightarrow A_1, A_2, \ldots \in \mathcal{A}_\alpha \ \forall \alpha \Rightarrow \bigcup_{i=1}^{\infty} A_i, \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}_\alpha \ \forall \alpha$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i, \bigcap_{i=1}^{\infty} A_i \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$ thus we have proved the lemma.

QED.

X: a set, ~~$\mathcal{C}$~~ $\mathcal{C}$: a collection of subsets of X

Define: $\sigma(\mathcal{C}) = \bigcap_\alpha \mathcal{A}_\alpha$, $\ni \mathcal{A}_\alpha$ is a σ-algebra & $\mathcal{C} \subset \mathcal{A}_\alpha$.

ⓐ $\sigma(\mathcal{C})$ is not empty!

$\mathcal{C} \subset \mathcal{A}^* = \sigma$-algebra of all subsets of X

ⓑ $\sigma(\mathcal{C})$ is a σ-algebra.

- $\sigma(\mathcal{C})$ is called the σ-algebra generated by $\mathcal{C}$. (i.e. $\sigma(\mathcal{C})$ is the smallest σ-algebra containing $\mathcal{C}$)

- 1) if $\mathcal{C}$ is a σ-algebra, $\sigma(\mathcal{C}) = \mathcal{C}$

2) $\mathcal{C}_1 \subset \mathcal{C}_2$, $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$

3) $\sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C})$

look up def. of metric space.

- X: a metric space ~~a metric space~~
  ex $\Rightarrow (\mathbb{R}, |\cdot|)$, open sets are defined ex) $(0,1)$

Def: If $\mathcal{G}$ is the collection of open subsets of X, $\sigma(\mathcal{G})$ is called the Borel σ-algebra on X. often denoted by $\mathcal{B}$.
- Elements in $\mathcal{B}$ are called Borel sets & are said to be Borel measurable.

Proposition $\to X = \mathbb{R}$. The Borel σ-algebra $\mathcal{B}$ is generated by each of:

(1) $\mathcal{C}_1 = \{(a,b) \mid a,b \in \mathbb{R}\}$,
(2) $\mathcal{C}_2 = \{[a,b] \mid a,b \in \mathbb{R}\}$,
(3) $\mathcal{C}_3 = \{(a,b] \mid a,b \in \mathbb{R}\}$,
(4) $\mathcal{C}_4 = \{(a,\infty) \mid a,b \in \mathbb{R}\}$

Proof: (1) WTS $\sigma(\mathcal{C}_1) = \mathcal{B}$ so first let's show $\sigma(\mathcal{C}_1) \subset \mathcal{B}$.
Let $\mathcal{G}$: set of all open subsets of $X = \mathbb{R}$. so $\mathcal{C}_1 \subset \mathcal{G}$
$\Rightarrow \sigma(\mathcal{C}_1) \subset \sigma(\mathcal{G}) = \mathcal{B}$
Now we need to show $\sigma(\mathcal{C}_1) \supset \mathcal{B}$.
$\forall G \in \mathcal{G} \Rightarrow G = \bigcup_{i=1}^{\infty} G_i$, $G_i \in \mathcal{C}_1$. If G is bdd., then
$G = \bigcup_{i=1}^{N} G_i$, $G_i \in \mathcal{C}_1$. If (a,∞), $\forall a$, $(a,\infty) = \bigcup_{n=1}^{\infty} (a,n)$
$\mathcal{B} \subset \sigma(\mathcal{C}_1) \Leftarrow \mathcal{G} \subset \sigma(\mathcal{C}_1) \Leftarrow G \in \sigma(\mathcal{C}_1) \Leftarrow \sigma(\mathcal{C}_1)$
Thus $\sigma(\mathcal{C}_1) = \mathcal{B}$.

Proof cont.

Proof cont: (2) - first WTS $\sigma(\mathcal{E}_2) \subset \mathcal{B}$.

Take any $[a,b] \in \mathcal{E}_2$. $[a,b] = \bigcap_{n \geq k}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n})$ for sufficiently large k.

$[a,b] \in \sigma(\mathcal{E}_1) = \mathcal{B} \Rightarrow \mathcal{E}_2 \subset \mathcal{B} \Rightarrow \sigma(\mathcal{E}_2) \subset \mathcal{B}$.

- next WTS $\sigma(\mathcal{E}_2) \supset \mathcal{B}$.

here it's good enough to show $\sigma(\mathcal{E}_2) \supset \sigma(\mathcal{E}_1) \overset{b/c}{=} \mathcal{B}$.

Take $(a,b) \in \mathcal{E}_1$. $(a,b) = \bigcup_{n \geq k}^{\infty} [a+\frac{1}{n}, b-\frac{1}{n}]$ for a sufficiently large k.

$\therefore \mathcal{E}_1 \subset \sigma(\mathcal{E}_2)$

$\mathcal{B} = \sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$. Thus $\sigma(\mathcal{E}_2) = \mathcal{B}$.

(3) & (4) are done similarly

due 1 week from today → HW: Prove that $\mathcal{B}$ on $\mathbb{R}$ is generated by (3),(4)

hint use that $\mathcal{B} = \sigma(\mathcal{E}_1)$

Def: A monotone class is a collection of subsets, $\mathcal{M}$, of $X \ni$

$A_1 \subset A_2 \subset \ldots$ ← (1) $A_i \uparrow A$, $A_i \in \mathcal{M}$, $\forall i \Rightarrow A \in \mathcal{M}$.

$A_1 \supset A_2 \supset A_3 \ldots$ ← (2) $A_i \downarrow A$, $A_i \in \mathcal{M}$, $\forall i \Rightarrow A \in \mathcal{M}$.

properties

- $\mathcal{M}_1$ & $\mathcal{M}_2$ are collections of subsets of X. $\mathcal{M}_1 \cap \mathcal{M}_2$ is monotone if $\mathcal{M}_1$ & $\mathcal{M}_2$ are monotone.
- Consider $\bigcap_{i \in I} \mathcal{M}_i \ni \mathcal{M}_i \supset \mathcal{N}$ & $\forall i$ $\mathcal{M}_i$ are monotone,
  ↳ this is the smallest monotone class containing $\mathcal{N}$.

Theorem: (Monotone Class Theorem)

If $\mathcal{A}_0$ is algebra, $\mathcal{A}$ is smallest σ-algebra containing $\mathcal{A}_0$, & $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}_0$, then $\mathcal{A} = \mathcal{M}$.

1

Proof of monotone class theorem:

WTS $\mathcal{A} \subset \mathcal{M}$ & $\mathcal{M} \subset \mathcal{A}$. First let's do $(\mathcal{M} \subset \mathcal{A})$.

$\mathcal{A}$ is monotone b/c any σ-algebra is monotone. & $\mathcal{A}_0 \subset \mathcal{A}$.

$\mathcal{M} \subset \mathcal{A}$ by definition.

WTS $\mathcal{A} \in \mathcal{M}$.

X: a set, $\mathcal{A}_0$ algebra, $\mathcal{A}$: smallest σ-algebra containing $\mathcal{A}_0$,

$\mathcal{M}$ see previous page

2nd part:

$\mathcal{M} \subset \mathcal{A}$: done, now prove $\mathcal{M} \supset \mathcal{A}$

We'll show $\mathcal{M}$ is a σ-algebra below step by step:

1) Define $\mathcal{N}_1 = \{A \in \mathcal{M} \mid A^c \in \mathcal{M}\}$

$\mathcal{A}_0 \subset \mathcal{N}_1 \subset \mathcal{M}$

b/c $A \in \mathcal{A}_0 \Rightarrow A^c \in \mathcal{A}_0 \subset \mathcal{M}$

$\subset \mathcal{M}$

Now let's prove $\mathcal{N}_1$ is monotone

Pick $A_i \in \mathcal{N}_1, \forall i \ni A_i \uparrow A$. need to verify $A \in \mathcal{N}_1$

$A_i \in \mathcal{N}_1 \ \forall i \Rightarrow A_i, A_i^c \in \mathcal{M} \ \forall i$

$A_i^c \downarrow A^c$ b/c $\begin{cases} A_1 \subset A_2 \subset \cdots \\ \Rightarrow A_1^c \supset A_2^c \supset \cdots \\ \bigcup_{i=1}^{\infty} A_i = A \end{cases}$ thus $A^c = \left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$

Since $\mathcal{M}$ is monotone, $A, A^c \in \mathcal{M} \Rightarrow A \in \mathcal{N}_1$

similarly, one can verify that $A_i \in \mathcal{N}_1 \ \forall i$ & $A_i \downarrow A$ $\Rightarrow A \in \mathcal{N}_1$. Therefore $\mathcal{N}_1$ is a monotone class containing $\mathcal{A}_0 \Rightarrow \mathcal{N}_1 = \mathcal{M}$ b/c it's saying $\mathcal{N}_1$ is smaller than $\mathcal{M}$, but $\mathcal{M}$ is the smallest monotone class.

Thus we've proved $\mathcal{M}$ is closed under the compliment

HW: Show N_3 is a monotone class

Proof is similar to 2) & 3)

2) Define $N_2 = \{A \in M \mid A \cap B \in M \ \ \forall B \in A_0\}$

$A_0 \subset N_2 \subset M$

We now show that N_2 is a monotone class.

Pick $A_i \uparrow A$, $A_i \in N_2 \ \forall i$, & need to verify $A \in N_2$

$\forall B \in A_0$, $A \cap B = (\bigcup_{i=1}^{\infty} A_i) \cap B = \bigcup_{i=1}^{\infty} (A_i \cap B)$

$A_i \cap B \in M$, $\forall i$ by $A_i \in N_2 \ \forall i$

$A_1 \cap B \subset A_2 \cap B \subset A_3 \cap B \subset \ldots$ by $A_1 \subset A_2 \subset A_3 \subset \ldots$

$\Rightarrow A \cap B \in M$ b/c M is a monotone class.

$\Rightarrow A \in N_2$

With the similar proof for $A_i \downarrow A$, we claim N_2 is monotone.

Thus $A \cap B \in M$, $\forall A \in M$, $B \in A_0$.

3) Define $N_3 = \{A \in M \mid A \cap B \in M, \ \forall B \in M\}$

$A_0 \subset N_3 \subset M$

Now let's show N_3 is a monotone class.

choose $A_i \in N_3 \ \forall i$, $A_i \uparrow A$, want to verify $A \in N_3$

if N_3 is a monotone class (prove for HW),

$\Rightarrow N_3 = M \Rightarrow M$ is closed under an intersection. $\Rightarrow$ M is closed under finitely many intersections.

$\longrightarrow$ more HW (2.1, 2.2, 2.3, 2.4) from book by 9/13

~~Final step~~: WTS $\mathcal{M}$ is a σ-algebra.

1) $\emptyset, X \in \mathcal{M}$, true b/c $\emptyset, X \in \mathcal{A}_0 \subset \mathcal{M}$.

2) $A^c \in \mathcal{M}$ $\forall A \in \mathcal{M}$ by 1) from previous.

3) $A_i \in \mathcal{M}$ $\forall i \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$, $\bigcap_{i=1}^{\infty} A_i \in \mathcal{M}$

For $A_i \in \mathcal{M}$, define $B_n = \bigcap_{i=1}^{n} A_i$ $\forall n = 1, 2, \ldots$

$B_1 \supset B_2 \supset \ldots \rightarrow \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{\infty} A_i \Rightarrow B_n \downarrow \bigcap_{i=1}^{\infty} A_i$

$\Rightarrow \bigcap_{n=1}^{\infty} A_i \in \mathcal{M}$

Thus $\mathcal{M}$ is a σ-algebra $\Rightarrow \mathcal{M} \supset \mathcal{A} \supset \mathcal{A}_0$ by def. of $\mathcal{A}$

One more hint for HW:

$X, \mathcal{A}_i$ on X, $\forall i$

$A_j \in \bigcup_{i=1}^{\infty} \mathcal{A}_i$ $\forall j \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \bigcup_{i=1}^{\infty} \mathcal{A}_i \leftarrow$ NO! bad

Ch 3 Measures

3.1) Def: X is a set, $\mathcal{A}$: σ-algebra on X. Now we can define (and examples) a measure on $(X, \mathcal{A})$ is a function $\mu: \mathcal{A} \to [0, \infty] \ni$

1) $\mu(\emptyset) = 0$,

2) $\forall$ pairwise disjoint $A_i \in \mathcal{A}$ $\forall i$,

$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

The triple $(X, \mathcal{A}, \mu)$ is called a measure space.

Ex) X: a set

$\mathcal{A}$: Collection of all subsets of X

$\mu(A) = \#$ of elements in A $\forall A \in \mathcal{A}$,

Ex) X: a set, $\mathcal{A}$: a σ-algebra on X, Define $\delta_{x_0}(A) = \begin{cases} 1 & \text{if } x_0 \in A, \\ 0 & \text{otherwise} \end{cases}$ for $x_0 \in X$

point mass at x_0

Ch. 3 cont.

- **Proposition 3.5**

The following hold true: $(X, \mathcal{A}, \mu)$

(1) $A, B \in \mathcal{A}$, $A \subset B \Rightarrow \mu(A) \le \mu(B)$,

(2) $A_i \in \mathcal{A}$, $\forall i$, $A = \bigcup_{i=1}^{\infty} A_i$
$$\Rightarrow \mu(A) \le \sum_{i=1}^{\infty} \mu(A_i)$$

(3) $A_i \in \mathcal{A}$, $\forall i$, $A_i \uparrow A \Rightarrow \mu(A) = \lim_{n\to\infty} \mu(A_n)$

(4) $A_i \in \mathcal{A}$, $\forall i$, $A_i \downarrow A$, $\mu(A_1) < \infty$
$$\Rightarrow \mu(A) = \lim_{n\to\infty} \mu(A_n)$$

- Proofs:

(1) $A, B \in \mathcal{A}$, $A \subset B \overset{\text{WTS}}{\Rightarrow} \mu(A) \le \mu(B)$

$B = B \setminus A \cup A$

$\mu(B) = \mu((B \setminus A) \cup A) = \mu(B \setminus A) + \mu(A) \ge \mu(A)$

QED

(2) WTS $\Rightarrow \mu(A) \le \sum_{i=1}^{\infty} \mu(A_i)$

$B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $B_3 = A_3 \setminus (A_1 \cup A_2), \dots,$

$B_i = A_i \setminus (\bigcup_{j=1}^{i-1} A_j)$, $\forall i$.

B_i's are pairwise disjoint. $B_i \subset A_i$ $\forall i$.

$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$, thus $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$

$\mu(A) = \mu(\bigcup_{i=1}^{\infty} A_i) = \mu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i)$ b/c μ is a measure & B_i's are pairwise disjoint.

$\le \sum_{i=1}^{\infty} \mu(A_i)$ b/c $B_i \subset A_i$.

thus $\mu(A) \le \sum_{i=1}^{\infty} \mu(A_i)$

(3) & (4) on next page

(3) WTS $\mu(A) = \lim_{n\to\infty} \mu(A_n)$

use same B_i as from proof of (2), ie. $B_i = A_i \setminus \left(\bigcup_{j=1}^{i-1} A_j\right)\ \forall i$

$B_i \subset A_i$, $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$.

$$\mu(A) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} \mu(A_n)$$

QED

(4) WTS $\mu(A) = \lim_{n\to\infty} \mu(A_n)$

Define $B_1 = A_1 \setminus A_1 = \emptyset$ $\quad B_3 = A_1 \setminus A_3$ $\quad B_1 \subset B_2 \subset B_3$

$B_2 = A_1 \setminus A_2$

$B_i = A_1 \setminus A_i \ \forall i \in \mathbb{N} \Rightarrow B_1 \subset B_2 \subset B_3 \subset \dots \subset B_n \subset \dots$

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = \bigcup_{i=1}^{\infty} (A_1 \cap A_i^c)$$

$$= A_1 \cap \left(\bigcup_{i=1}^{\infty} A_i^c\right) = A_1 \cap \left(\bigcap_{i=1}^{\infty} A_i\right)^c$$

by def of A_i's

$$= A_1 \cap A^c = A_1 \setminus A$$

$$\Rightarrow B_i = A_1 \setminus A_i \uparrow A_1 \setminus A$$

Then by (3) $\Rightarrow \lim_{n\to\infty} \mu(B_n) = \mu(A_1 \setminus A)$

$$\Rightarrow \lim_{n\to\infty} \mu(A_1 \setminus A_n) = \mu(A_1 \setminus A)$$

by the way $A_1 \supset A \Rightarrow A_1 = (A_1 \setminus A) \cup A$

$$\Rightarrow \mu(A_1) = \mu(A_1 \setminus A) + \mu(A)$$

$\Rightarrow \mu(A_1 \setminus A) = \mu(A_1) - \mu(A)$ similarly b/c $A_1 \supset A_n$

$$\mu(A_1 \setminus A_n) = \mu(A_1) - \mu(A_n)$$

applying what we've found:

$$\lim_{n\to\infty} (\mu(A_1) - \mu(A_n)) = \mu(A_1) - \mu(A)$$

$$\mu(A_1) - \lim_{n\to\infty} \mu(A_n) = \mu(A_1) - \mu(A) \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(A)$$

QED

(30)

Q) What if we drop $\mu(A_1) < \infty$ in (4)?

$X = \mathbb{N}$, μ = counting measure.

Define $A_i = \{ j \in \mathbb{N} \mid j \geq i \}$ $\forall i$.

$A_1 \supset A_2 \supset A_3 \supset \dots$

$A_i \downarrow \emptyset$ $\quad \mu(\emptyset) = 0$

$\lim_{i\to\infty} \mu(A_i) = \lim_{i\to\infty} \infty = \infty$

$(X, \mathcal{A}, \mu)$

<u>Def:</u> A measure μ is <u>finite measure</u> if $\mu(X) < \infty$.

A measure μ is <u>σ-finite</u> if $\exists$ $E_i \in \mathcal{A}_j$ $i \ni$ $\mu(E_i) < \infty$ $\forall i$, & $X = \bigcup_{i=1}^{\infty} E_i$.

- Suppose we have $(X, \mathcal{A}, \mu)$ is a σ-finite measure space.

$\exists$ $E_i \in \mathcal{A}$ $\forall i \ni \mu(E_i) < \infty$ $\forall i$, & $X = \bigcup_{i=1}^{\infty} E_i$.

- <u>Define:</u> $F_n = \bigcup_{i=1}^{n} E_i$ $\forall n$, Then $F_1 \subset F_2 \subset \dots$, $\bigcup_{i=1}^{\infty} F_i = X$.

(b/c $F_1 = E_1$, $F_2 = E_1 \cup E_2$, ...)

$\mu(F_n) = \mu(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} \mu(E_i) < \infty$ $\forall n$.

$\Rightarrow$ Hence, WLOG, we can assume that E_i's in the def of σ-finite measure is increasing.

- <u>Define:</u> $(X, \mathcal{A}, \mu)$: measure space. $A \subset X$ is called a "<u>null set</u>" if $\exists B \in \mathcal{A} \ni A \subset B$ & $\mu(B) = 0$.

$A \in \mathcal{A}$?, not always: $A \notin \mathcal{A}$.

☆ empty set is a null set, but not the only one ☆

Ch3 Hwork: (3.1, 3.2, 3.3, 3.4, 3.6)

Due 9/15

- $(X, \mathcal{A}, \mu)$: measure space

Def: If $\mathcal{A}$ contains all the null sets, $(X, \mathcal{A}, \mu)$ is called <u>complete</u>.

Def: The completion of $\mathcal{A}$ is the smallest σ-algebra $\bar{\mathcal{A}}$ containing $\mathcal{A}$ $\ni$ $(X, \bar{\mathcal{A}}, \bar{\mu})$ is complete where $\bar{\mu}$ is an extension of μ $\ni$ $\bar{\mu}(B) = \mu(B)$ $\forall B \in \mathcal{A}$.

- What does this mean?

to determine if something is the null set, we must have μ first. null set doesn't have to be in the σ-algebra though, so we want to complete the measure space. we need to make the σ-algebra bigger to fit the null sets.

<u>side note</u>: $C \in \bar{\mathcal{A}}$ iff $\exists$ unique A & B $\ni$ $C = A \cup B$ & $A \in \mathcal{A}$ & B is a null set. <u>Example problem 3.8</u>

- $(X, \mathcal{A}, \mu)$: finite measure space w/ $\mu(X) = 1$. Then we consider this measure space as probability measure space & write $(\Omega, \mathcal{F}, \mathbb{P})$ by setting $\Omega = X$, $\mathcal{F} = \mathcal{A}$, & $\mathbb{P} = \mu$.

σ field

prob. space

Ch 4 Construction of Measures

4.1 Outer measure

Def: Let X be a set. An <u>outer measure</u> is a function μ^* defined on the collection of all subsets of X satisfying:

(1) $\mu^*(\emptyset)=0$

(2) $A\subset B \Rightarrow \mu^*(A)\le \mu^*(B)$

(3) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu^*(A_i)$, $A_i\subset X$ $\forall i$.

ie: $(X, \mathcal{A}$: σ-algebra containing all subsets of X $, \mu)$

- a set is a null set w/ respect to μ^* if $\mu^*(N)=0$

Proposition: Suppose $\mathcal{C}$ is a collection of subsets of $X \ni \emptyset\in\mathcal{C}$ & $\exists D_1, D_2, \ldots \in \mathcal{C} \ni X=\bigcup_{i=1}^{\infty} D_i$.

Suppose $\ell: \mathcal{C}\to[0,\infty)$ w/ $\ell(\emptyset)=0$.

Define: $\mu^*(E)=\inf\left\{\sum_{i=1}^{\infty} \ell(A_i) \,\middle|\, A_i\in\mathcal{C}\ \forall i,\ E\subset\bigcup_{i=1}^{\infty} A_i\right\}$

Then μ^* is an outer measure.

Proof: $\mu^*(\emptyset)=0$

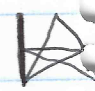
(1) $0\le \mu^*(\emptyset)\le \ell(\emptyset)=0 \Rightarrow \mu^*(\emptyset)=0$

(2) $A\subset B \Rightarrow \mu^*(A)\le\mu^*(B)$

$\forall$ choice of $A_i\in\mathcal{C}$ $\forall i \ni B\subset\bigcup_{i=1}^{\infty} A_i$, we also have

$\cancel{A\subset\bigcup_{i=1}^{\infty} A_i \Rightarrow \sum_{i=1}^{\infty} \ell(A_i) \in \{\ell(A_i)\} \le \mu^*(B)}$

$\Rightarrow \mu^*(A)\le\sum_{i=1}^{\infty} \ell(A_i) \Rightarrow \mu^*(A)\le\mu^*(B)$.



proof of prop. cont.

(3) For $A_1, A_2, \ldots \subset X$, $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu^*(A_i)$.
Let $A_i \subset X$, $\forall i = 1, 2, \ldots$, we want to show that $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu^*(A_i) + \varepsilon$ $\forall \varepsilon > 0$. This is equivalent to b/c of the arbitrarily small ε.

$\varepsilon > 0$ given. $\forall A_i$, choose $C_{i1}, C_{i2}, \ldots$ from $\mathcal{C}$ $\ni$
$A_i \subset \bigcup_{j=1}^{\infty} C_{ij}$ & $\sum_{j=1}^{\infty} \ell(C_{ij}) \le \mu^*(A_i) + \frac{\varepsilon}{2^i}$

Then since $\bigcup_{i=1}^{\infty} A_i \subset \bigcup_i \bigcup_j C_{ij}$, so $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_i \left(\sum_j \ell(C_{ij})\right)$

$$\le \sum_i \left(\mu^*(A_i) + \frac{\varepsilon}{2^i}\right)$$

$$= \sum_i \mu^*(A_i) + \sum_i \frac{\varepsilon}{2^i} = \sum_i \mu^*(A_i) + \varepsilon$$

→ geometric series.

Ex] $X = \mathbb{R}$, $\mathcal{C}\{(a,b] \mid a,b \in \mathbb{R}\}$. $\ell((a,b]) = b-a$ $\forall b \ge a$ $\ni a, b \in \mathbb{R}$.

Define μ^* by $\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \ell(A_i) \,\middle|\, \begin{array}{l} A_i \in \mathcal{C}\ \forall i, \\ E \subset \bigcup_{i=1}^{\infty} A_i \end{array}\right\}$

Then μ^* is an outer measure.

But μ^* is not a measure on the collection of all subsets of $\mathbb{R}$.

If we restrict μ^* to a σ-algebra $\mathcal{L}$ (which is strictly smaller than the collection of all subsets of $\mathbb{R}$). Then μ^* is measure on $\mathcal{L}$.
(Lebesgue Measure) (Lebesgue σ-algebra)

Ex] p. 23 has another good one. Honestly, just read the book.

Def: μ^*: outer measure. $A \subset X$ is μ^*-measurable if
$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \forall E \subset X.$$

proposition based on previous def.

a) call $(*)$: $(\mu^*(E) = \mu^*(E\cap A) + \mu^*(E\cap A^c)$ $\forall E\subset X$.

$E = (E\cap A)\cup(E\cap A^c) \Rightarrow \mu^*(E) = \mu^*((E\cap A)\cup(E\cap A^c))$

$\leq \mu^*(E\cap A) + \mu^*(E\cap A^c)$. now want to show $\geq$.

$\left\{ \begin{array}{l} \text{so for } (*), \text{ it is enough to show that} \\ \mu^*(E) \geq \mu^*(E\cap A) + \mu^*(E\cap A^c) \quad \forall E\subset X \end{array}\right.$

b) A is μ^*-measurable iff A^c is μ^*-measurable

c) a) holds true automatically if $\mu^*(E) = \infty$.

Call this B

Theorem: i) μ^* is an outer measure on X. Then the collection of $\mathcal{A}$ of all μ^*-measurable sets is a σ-algebra.

ii) If we set μ as the restriction of μ^* to $\mathcal{A}$. Then μ is measure. Moreover, $\mathcal{A}$ contains

iii) all the null sets.

Proof: i) $\mathcal{A}$ is a σ-algebra : a) $\emptyset, X\in\mathcal{A}$, b) $A\in\mathcal{A}\Rightarrow A^c\in\mathcal{A}$.

c) $A_i\in\mathcal{A}$ $\forall i \Rightarrow \bigcup_{i=1}^{\infty} A_i\in\mathcal{A}$

-a: $\mu^*(E\cap\emptyset) + \mu^*(E\cap\emptyset^c) = \mu^*(\emptyset) + \mu^*(E) = \mu^*(E)$ $\forall E\subset X$.

$\Rightarrow \emptyset\in\mathcal{A}$. Then by B, $X\in\mathcal{A}$; so b follows from

-b: B.

-c: As a first step, let's show that $A_1\cup A_2 \in \mathcal{A}$ if $A_1, A_2\in\mathcal{A}$.

We write $\forall E$, $\mu^*(E) = \mu^*(E\cap A_1) + \mu^*(E\cap A_1^c)$
by $(A_1\in\mathcal{A})$

so $= \mu^*(E\cap A_1\cap A_2) + \mu^*(E\cap A_1\cap A_2^c) + \mu^*(E\cap A_1^c\cap A_2) + \mu^*(E\cap A_1^c\cap A_2^c)$

(by $A_1\cap E\subset X$, $A_2\in\mathcal{A}$)

On the other hand,

$(A_1\cap A_2)\cup(A_1\cap A_2^c)\cup(A_1^c\cap A_2) \supset (A_1\cup A_2)$

$\mu^*(E\cap(A_1\cap A_2)\cup E\cap(A_1\cap A_2^c)\cup E\cap(A_1^c\cap A_2)) \geq \mu^*(E\cap(A_1\cup A_2))$

cont

$$\Rightarrow \mu^*(E) \geq \mu^*(E\cap(A_1\cup A_2)) + \mu^*(E\cap A_1^c \cap A_2^c)$$
$$= \mu^*(E\cap(A_1\cup A_2)) + \mu^*(E\cap(A_1\cup A_2)^c) \Rightarrow A_1\cup A_2 \in \mathcal{A}.$$

---

b/c we've proven if $A\in\mathcal{A}$, $A^c\in\mathcal{A}$ & $\forall A_1, A_2\in\mathcal{A} \Rightarrow A_1\cup A_2 \Rightarrow \mathcal{A}$, then we know $A_1\cap A_2\in\mathcal{A}$.

Now all we need to show for i) is that if $A_i\in\mathcal{A}\ \forall i \Rightarrow \bigcup_{i=1}^{\infty} A_i\in\mathcal{A}$.

- take $A_i\in\mathcal{A}\ \forall j$, let $E\subset X$. Setting $B=\bigcup_{i=1}^{\infty} A_i$ $(B_n = \bigcup_{i=1}^{n} A_i)$

Need to show that $\mu^*(E)\geq \mu^*(E\cap B)+\mu^*(E\cap B^c)$.

Define a pairwise disjoint C_i, $\ni C_1 = A_1$, $C_2 = A_2\setminus A_1$, $C_i = A_i \setminus \left(\bigcup_{j=1}^{i-1} A_j\right)\ \forall i, j$: $\bigcup_{i=1}^{n} C_i = B_n$, $\bigcup_{i=1}^{\infty} C_i = B$ b/c of limits & pairwise disjointness.

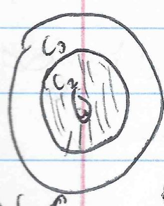
Note that $C_i\in\mathcal{A}\ \forall i$.

Let's look at $\mu^*(E\cap B_n) = \mu^*(E\cap B_n\cap C_n)+\mu^*(E\cap B_n\cap C_n^c)$

$= \mu^*(E\cap C_n)+\mu^*(E\cap B_{n-1})$

$= \mu^*(E\cap C_n) + \mu^*(E\cap B_{n-1}\cap C_{n-1}) + \mu^*(E\cap B_{n-1}\cap C_{n-1}^c)$

$= \mu^*(E\cap C_n)+\mu^*(E\cap C_{n-1}) + \mu^*(E\cap B_{n-2})$



$B\cap C_2^c$

By induction, we obtain $\mu^*(E\cap B_n) = \sum_{i=1}^{n} \mu(E\cap C_i)$

Now we write $\mu^*(E) = \mu^*(E\cap B_n)+\mu^*(E\cap B_n^c)\quad \forall n$

$= \sum_{i=1}^{n} \mu^*(E\cap C_i)+\mu^*(E\cap B_n^c)\quad \forall n$

Hence at the limit, $\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E\cap C_i)+\mu^*(E\cap B^c)$

From this, $\mu^*(E)\geq \mu^*\Big(\underbrace{\bigcup_{i=1}^{\infty} (E\cap C_i)}_{\text{by DeMorgan law } = E\cap(\bigcup_{i=1}^{\infty} C_i)}\Big) + \mu^*(E\cap B^c)$

$= \mu^*(E\cap B)+\mu^*(E\cap B^c)$. This implies $B\subset\mathcal{A}$. Hence w/ similar proof for $\bigcap A_i$'s, $\mathcal{A}$ is a σ-algebra.

Proof continued

which we have known is a σ-algebra.

iii) Take C_i as the disjoint sets in $\mathcal{A}$. Need to show that $\mu^*(\bigcup_{i=1}^{\infty} C_i) = \sum_{i=1}^{\infty} \mu^*(C_i)$. Then b/c $\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E \cap C_i) + \mu^*(E \cap B^c)$,

$\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E \cap C_i) + \mu^*(E \cap (\bigcup_{i=1}^{\infty} C_i)^c)$ take $E = \bigcup_{i=1}^{\infty} C_i$.

$\mu^*(\bigcup_{i=1}^{\infty} C_i) = \sum_{i=1}^{\infty} \mu^*(C_i)$ ✓ QED

iiii) Let $A \subset X$, $\mu^*(A) = 0$. WTS $A \in \mathcal{A}$.

$\forall E \subset X$,

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \le \underbrace{\mu^*(A)}_{0} + \mu^*(E \cap A^c)$$
$$\le \mu^*(E)$$
$$\Rightarrow A \in \mathcal{A}.$$

- Key points we can come up with σ-algebra to make the outer measure a measure.

• 4.2 Lebesgue - Stieltjes measures

$X = \mathbb{R}$, $\mathcal{C}$ = collection of $(a, b]$, $a, b \in \mathbb{R} \cup \{\pm\infty\}$.

$\alpha(\cdot)$: increasing & right-continuous function.

$\begin{cases} \text{i) } x < y \Rightarrow \alpha(x) \le \alpha(y) \\ \text{ii) } \lim_{x \to x_0} \alpha(x) = \alpha(x_0) \end{cases}$

Ex) $\alpha(x) = x$

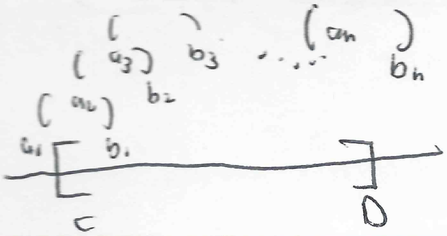
Define: $m^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \ell(A_i) \;\middle|\; \begin{array}{l} A_i \in \mathcal{C} \ \forall i \\ E \subset \bigcup_{i=1}^{\infty} A_i \end{array} \right\}$

where $\ell((a, b]) = \alpha(b) - \alpha(a)$

m^* is an outer measure on $\mathbb{R}$, & is a measure on the collection of all m^*-measurable sets.

- Note that if K & L are adjacent, i.e., $K = (a, b]$ & $L = (b, c]$, then $K \cup L = (a, c]$ & $\ell(K) + \ell(L) = \alpha(b) - \alpha(a) + \alpha(c) - \alpha(b)$
$= \alpha(c) - \alpha(a) = \ell(K \cup L)$

drawing of it



Lemma: Let $\{J_k=(a_k,b_k),\ a_k,b_k\in\mathbb{R}\}_{k=1}^{n}$ be a finite collection of finite open intervals covering a finite closed interval $[C,D]$. Then $\sum_{k=1}^{n}(\alpha(b_k)-\alpha(a_k))\geq\alpha(D)-\alpha(C)$

Proof: WLOG $a_1<C$, $D<b_n$ (as seen in pic.).

$a_{i+1}<b_i<b_{i+1}$, $i=1,\dots,n-1$

We write $\alpha(D)-\alpha(C)\leq\alpha(b_n)-\alpha(a_1)$

$$\leq\alpha(b_n)-\alpha(b_{n-1})+\alpha(b_{n-1})-\alpha(b_{n-2})\cdots+\alpha(b_2)-\alpha(b_1)+\alpha(b_1)-\alpha(a_1)$$

$$\leq\alpha(b_n)-\alpha(a_n)$$
$$+\alpha(b_{n-1})-\alpha(a_{n-1})$$
$$\vdots$$
$$+\alpha(b_1)-\alpha(a_1)=\sum_{k=1}^{n}(\alpha(b_k)-\alpha(a_k))$$ QED

Proposition: If $I=[e,f]$, $e,f\in\mathbb{R}$, then $m^*(I)=\ell(I)$.

Proof: i) WTS $m^*(I)\leq\ell(I)$

Take $A_1=I$, $A_i=\emptyset$ $\forall i\geq 2$. $m^*(I)\leq\sum_{i=1}^{\infty}\ell(A_i)$
$=\ell(I)$

ii) WTS $m^*(I)\geq\ell(I)$

Let $A_i=(c_i,d_i]$ $\forall i$ $\ni$ $I\subset\bigcup_{i=1}^{\infty}A_i$. We aim to show that $\forall\varepsilon>0$, $\ell(I)\leq\sum_{i=1}^{\infty}\ell(A_i)+\varepsilon$.

Homework Stuff

Must say $A_i\in\mathcal{A}$ $\forall i$ disjoint.

Recall Proposition 4.8

$I = (e,f]$, $e,f \in \mathbb{R} \Rightarrow m^*(I) = \ell(I)$

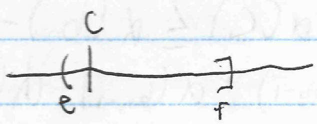
Proof) Last time we proved i) $m^*(I) \leq \ell(I)$

Today: ii) $m^*(I) \geq \ell(I)$

Choose $A_i = (c_i, d_i)$, $c_i, d_i \in \mathbb{R}$ $\forall i \ni I \subset \bigcup_{i=1}^{\infty} A_i$

☆ → $\forall \varepsilon > 0$, choose $C \in (e,f) \ni \underbrace{\alpha(C) - \alpha(e) < \frac{\varepsilon}{2}}_{☆}$

Why?

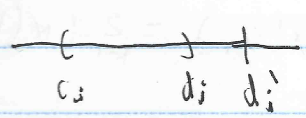


b/c α is right continuous ~~from the right~~

Summary of this Proof

set $D = f$. $\forall i$ choose $d_i' > d_i \ni \alpha(d_i') - \alpha(d_i) < \frac{\varepsilon}{2^{i+1}}$

by the right continuity of α.



Let $\{B_i = (c_i, d_i')\}_{i=1}^{\infty}$ is an open cover of a compact set $[C, D]$ ↓

$\Rightarrow \exists \{B_{i_j}\}_{j=1}^{n}$ covers $[C, D]$ b/c compactness

$$
\begin{aligned}
\ell(I) &= \alpha(f) - \alpha(e) \quad \text{by definition} \\
&\leq \alpha(f) - \alpha(C) + \frac{\varepsilon}{2} \quad \text{by ☆} \\
&\leq \underbrace{\alpha(D) - \alpha(C)}_{\text{over closed interval, finite set, and covers the set, so}} + \frac{\varepsilon}{2} \\
&\leq \sum_{j=1}^{n} \left(\alpha(d_{i_j}') - \alpha(c_{i_j})\right) + \frac{\varepsilon}{2} \quad \text{by the lemma we proved in the last lecture} \\
&\leq \sum_{j=1}^{n} \left(\alpha(d_{i_j}) - \alpha(c_{i_j})\right) + \sum_{j=1}^{n} \frac{\varepsilon}{2^{i+1}} + \frac{\varepsilon}{2} \\
&\leq \sum_{j=1}^{n} \ell(A_{i_j}) + \frac{\varepsilon}{2} \sum_{j=1}^{n} \frac{1}{2^{i_j}} + \frac{\varepsilon}{2}
\end{aligned}
$$

$\Rightarrow \ell(I) \leq \sum_{i=1}^{\infty} \ell(A_i) + \varepsilon$. By taking inf. over all $\{A_i\}_{i=1}^{\infty}$,

$\ell(I) \leq m^*(I) + \varepsilon$

$\Rightarrow \ell(I) \leq m^*(I)$ QED

• Proposition 4.9

Every set in the Borel σ-algebra on $\mathbb{R}$ is m^*-measurable.

Proof: It is sufficient to show that an interval $(c,d]$ is m^*-measurable b/c $\sigma(\{(c,d] \mid c,d \in \mathbb{R}\})$ = Borel σ-algebra. & the set of all m^* measurable sets is a σ-algebra.

$\forall$ set E, WTS $m^*(E) = m^*(E \cap J) + m^*(E \cap J^c)$ where $J = (c,d]$. Note $m^*(E) < \infty$.

Choose $I_i = (a_i, b_i]$, $a_i, b_i \in \mathbb{R}$ $\forall i$ $\ni$ $E \subset \bigcup_{i=1}^{\infty} I_i$ &

$\star$ $\star$ $m^*(E) \geq \sum_{i=1}^{\infty} l(I_i) - \varepsilon$

BTW, since $E \subset \bigcup_i I_i$, $m^*(E \cap J) \leq \sum_i l(I_i \cap J)$

$\leq \sum_i m^*(I_i \cap J)$ ~~by proposition proved before~~

by $I_i \cap J$ is an interval in the form $(\alpha, \beta]$ for some α, β, and by proposition 4.8. Similarly,

$m^*(E \cap J^c) \leq \sum_i m^*(I_i \cap J^c)$

$\Rightarrow m^*(E \cap J) + m^*(E \cap J^c) \leq \sum_i (m^*(I_i \cap J) + m^*(I_i \cap J^c))$.

Setting $K_1 = (-\infty, c]$, $K_2 = (d, \infty)$ $\quad K_1 \; J \; K_2$

$m^*(I_i \cap J + m^*(\underbrace{I_i \cap J^c}_{K_1 \cup K_2})$

$= m^*(I_i \cap J) + m^*((I_i \cap K_1) \cup (I_i \cap K_2))$

$= m^*(I_i \cap J) + m^*(I_i \cap K_1) + m^*(I_i \cap K_2)$

$= l(I_i \cap J) + l(I_i \cap K_1) + l(I_i \cap K_2) = \sum_{i=1}^{\infty} l(I_i)$

so $m^*(E \cap J) + m^*(E \cap J^c) = \sum_{i=1}^{\infty} l(I_i) \leq m^*(E) + \varepsilon$ by $\star$

$\Rightarrow m^*(E) \geq m^*(E \cap J) + m^*(E \cap J^c)$ QED

So what does this mean?. Who f**kn knows!?.

• drop * & call m as the Lebesgue-Stieltjes measure. As a special case when $\alpha(x) = x$, the collection of all m^*-measurable sets is called the Lebesgue σ-algebra.

A set is Lebesgue measurable if it is in the lebesgue σ-algebra.

what are open & closed sets

Ex) Let m be a Lebesgue measure on $\mathbb{R}$.

i) $x \in \mathbb{R}$, $\{x\}$ is closed (so it's in the Borel σ-algebra which is contained in ...) so $\{x\}$ is a Lebesgue measurable

ii) $m(\{x\}) = \lim_{n\to\infty} m((x-\frac{1}{n}, x])$ (by proof in HW)

$= \lim_{n\to\infty}(x-(x-\frac{1}{n})) = \lim_{n\to\infty} \frac{1}{n} = 0$

using this: $m([a,b]) = m(\{a\} \cup (a,b])$

$= m(\{a\}) + m((a,b]) = m((a,b]) = b-a$.

Similarly, $m((a,b)) = m([a,b)) = b-a$.

- $m(\text{countable set}) = 0$

google it · Cantor set $[\quad]$ it's in the homework, num

ex 4.11 in book ← HW read it

Ex) Cantor set

$F_0 := [0, 1]$ $\quad F_1 = [0, \frac{1}{3}] \; [\frac{2}{3}, 1]$

$F_2 = [0, \frac{1}{9}][\frac{2}{9}, \frac{3}{9}] \; [\frac{6}{9}, \frac{7}{9}][\frac{8}{9}, 1]$ $\quad F = \bigcap_{i=0}^{\infty} F_i$ ~~$[0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}}$~~

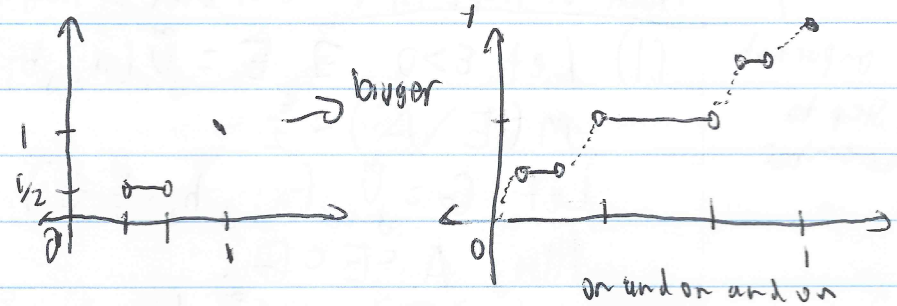
$= \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$

so:

every point is limit of sequence.

F is closed, uncountable, every point is a limit point, contains no intervals, $m(F) = 0$ (lesbegue measure)

- Cantor Function



the special thing about this function is that it's continuous.

Ex) $\{q_i\}_{i=1}^{\infty}$: enumeration of the rational #'s (collection of rational #'s (make a list of them))

$\forall \varepsilon > 0$, $I_i = (q_i - \frac{\varepsilon}{2^i}, q_i + \frac{\varepsilon}{2^i})$

so that $m(I_i) = \frac{\varepsilon}{2^{i-1}}$ $\forall i \in \mathbb{N}$.

$m(\bigcup_{i=1}^{\infty} I_i) \leq \sum_{i=1}^{\infty} m(I_i) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i-1}} = 2\varepsilon$.

Define $A = [0,1] \setminus \bigcup_{i=1}^{\infty} I_i$

$m(A) = ?$

well, $m(A) = m([0,1]) - m(\bigcup_{i=1}^{\infty} I_i) \geq 1 - 2\varepsilon > 0$

just showed the measure of all irrational #'s between 0 & 1 is close to 1. Neat.

Proposition 4.14 : m : Lebesgue measure on $\mathbb{R}$, $A \subseteq [0,1]$ lebesgue measurable set

Then (1) $\forall \varepsilon > 0$, $\exists G$: open set $\ni A \subset G$ & $m(G \setminus A) < \varepsilon$

(2) $\forall \varepsilon > 0$, $\exists F$: closed set $\ni F \subset A$ & $m(A \setminus F) < \varepsilon$

(3) $\exists H$: countable intersection of decreasing sequence of open sets $\ni A \subset H$ & $m(H \setminus A) = 0$.

(4) $\exists F$: countable union of an increasing sequence of closed sets $\ni F \subset A$ & $m(A \setminus F) = 0$

Homework due by 9/29 — Prove (4) prop. 4.14

$\frac{\varepsilon}{2}\left(\frac{1}{2^j}\right)$

important step to remember

Proof of prop. 4.14 – just when I thought things were getting interesting...

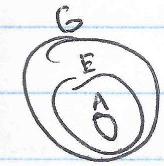
(1) Let $\varepsilon > 0$, $\exists\ E = \bigcup_{j=1}^{\infty} (a_j, b_j] \ni A \subset E$ &
$m(E \setminus A) < \frac{\varepsilon}{2}$

Let $G = \bigcup_{j=1}^{\infty} (a_j, b_j + \frac{\varepsilon}{2}\frac{1}{2^j})$: open set

Also, $A \subset E \subset G$.

$m(G \setminus E) = m(\bigcup_{j=1}^{\infty} (b_j, b_j + \frac{\varepsilon}{2}\frac{1}{2^j}))$

$\le \sum_{j=1}^{\infty} m((b_j, b_j + \frac{\varepsilon}{2}\frac{1}{2^j})) = \sum_{j=1}^{\infty} \frac{\varepsilon}{2}\frac{1}{2^j} = \frac{\varepsilon}{2}$



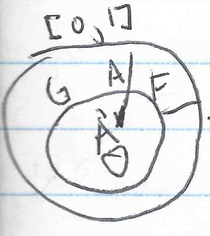
Hence we find $m(G \setminus A) = m((G \setminus E) \cup (E \setminus A))$

$\le m(G \setminus E) + m(E \setminus A) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ QED

(2) Set $A' = [0,1] \setminus A$ (sort of like the compliment)

$\subset [0,1]$, by (1), $\exists$ G : open set $\ni$

$A' \subset G$ & $m(G \setminus A') < \varepsilon$



Now we set $F = [0,1] \setminus G$: closed : & $F \subset A$. On the other hand, $A \setminus F = A \cap F^c = A \cap G$

$= G \cap (A')^c = G \setminus A'$

Then $m(A \setminus F) = m(G \setminus A') < \varepsilon$ QED

(3) by (1), $\exists\ G_i$: open $\ni G_i \supset A$ & $m(G_i \setminus A) < \frac{1}{2^i}\ \forall i$.

Set $H_i = \bigcap_{j=1}^{i} G_j$: open $\forall i$ $H_1 \supset H_2 \supset \ldots$

$m(H_i) \le m(G_i)\ \forall i$

Now set $H = \bigcap_{i=1}^{\infty} H_i$ (we don't know if this is open b/c it's infinite)

$\downarrow$ in Borel σ-algebra thus $\rightarrow$ in Lebesgue σ-algebra

$\hookrightarrow$ Lebesgue measurable

$\Rightarrow A \subset H$ & $m(H \setminus A) \le m(H_i \setminus A)\ \forall i$

$\le m(G_i \setminus A)\ \forall i$

$\le \frac{1}{2^i}\ \forall i \Rightarrow m(H \setminus A) = 0.$ QED

Note: Prop 4.14 – could also work for $[a, b]$, not just $[0, 1]$. So why the bulls did we use $[0,1]$? IDK man.

• Another remark: The countable intersection of open sets are called G_δ sets.

seems important. why's it a remark... idk man.

The countable union of closed sets are called F_σ sets

• New property:

Let, $A \subset [0, 1]$: Lebesgue measurable

$$A = V \setminus N_1 \quad V: G_\delta \text{ set}, \; N_1: \text{Null set}$$
$$= F \cup N_2 \quad F: F_\sigma \text{ set}, \; N_2: \text{Null set}$$

Everything we've done today is true as well, if we use Lebesgue - Stieltjes measure instead of Lebesgue. I thought that was intuitive.

• Recall: $\mu^* = \inf\{\sum l(A_i) \mid A_i\, (a_i, b_i]\}$

$$l((a_i, b_i]) = b_i - a_i$$

we're gonna show it can't be a measure on all $\mathbb{R}$.

proof of this is in section 4.4.

Oct 18 Exam 1

4.5 Famous Thm

- The Carathéodory extension theorem.

Theorem: $\mathcal{A}_0$ is an algebra. $\ell : \mathcal{A}_0 \to [0,\infty]$ is a measure,
Define, for $\forall E \subset X$,

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \ell(A_i) \;\middle|\; \begin{array}{l} A_i \in \mathcal{A}_0 \ \forall i \\ E \subset \bigcup_{i=1}^{\infty} A_i \end{array} \right\}$$

Then:
(1) μ^* is an outer measure, (2) $\mu^*(A) = \ell(A)$, $\forall A \in \mathcal{A}_0$,
(3) every set in $\mathcal{A}_0$ & every μ^*-null set is μ^*-measurable,
(4) if ℓ is σ-finite, $\exists$ a unique extension to $\sigma(\mathcal{A}_0)$

Notes: - What does it mean for ℓ to be a measure on $\mathcal{A}_0$?
$\ell(\emptyset) = 0$. $\forall A_i \in \mathcal{A}_0$ $\forall i \ni \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_0$, we have
$\ell(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \ell(A_i)$
- $\mathcal{A}_0$ is an algebra on X, thus $X \subset \mathcal{A}_0$.

Proof: (1) $\ell(\emptyset) = 0$, $X \subset X \in \mathcal{A}_0 \Rightarrow \mu^*$ is an outer measure
(2) Let $E \subset \mathcal{A}_0$.
~~Assume $\ell(E)$~~ - By choosing $A_1 = E$, $A_i = \emptyset$, $\forall i \geq 2$, ($\in \mathcal{A}_0$, $\in \mathcal{A}_0$)
by def. of μ^*, we find that $\underline{\mu^*(E) \leq \ell(E)}$.
- Now we want to prove it's true for the opposite inequality. $\forall \{A_i\}_{i=1}^{\infty}$, $A_i \in \mathcal{A}_0$ $\forall i$, $E \subset \bigcup_{i=1}^{\infty} A_i$,
define $B_i = E \cap (A_i \setminus \bigcup_{j=1}^{i-1} A_j)$, $\forall i$. So the B_i's are disjoint. B/c all parts of B_i are in $\mathcal{A}_0$, then $B_i \in \mathcal{A}_0$ $\forall i$.
$\bigcup_{i=1}^{\infty} B_i \supset E$ b/c $E \subset \bigcup_{i=1}^{\infty} A_i$, so $B_i \subset A_i$ $\forall i$.
We can further our first statement to $\bigcup_{i=1}^{\infty} B_i = E$.

$\exists E \subset \bigcup B_i$
$\bigcup B_i = \bigcup E \cap (A_i \setminus A_j)$
$= E \cap (\bigcup A_i)$
$= E$

So $\ell(E) = \ell(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \ell(B_i) \leq \sum_{i=1}^{\infty} \ell(A_i)$
$\Rightarrow \ell(E) \leq \mu^*(E)$ b/c μ^* is inf of all covers, & this is an arbitrary cover

Proof cont. F my A

(3) a) WTS $A \in \mathcal{A}_0 \Rightarrow A$ is μ^*-measurable.

so we WTS $\forall E \subset X$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$

well, $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ is given by def.

we only need to show $\geq$ case.

Choose $B_i \in \mathcal{A}_0$ $\forall i \ni A \subset \bigcup_{i=1}^{\infty} B_i$ & $\sum_{i=1}^{\infty} l(B_i) \leq \mu^*(E) + \varepsilon$

for a fixed & arbitrary $\varepsilon > 0$.

$\mu^*(E) + \varepsilon \geq \sum_{i=1}^{\infty} l(B_i) = \sum_{i=1}^{\infty} (l(B_i \cap A) + l(B_i \cap A^c))$

$= \sum_{i=1}^{\infty} l(B_i \cap A) + \sum_{i=1}^{\infty} l(B_i \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$

$\cup B_i \supset E \Rightarrow \cup(B_i \cap A) \supset E \cap A$. $\{B_i \cap A\}$ is a cover of $E \cap A$.

so $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$

b) WTS ~~A~~ if A is μ^*-null set $\Rightarrow A$ is μ^*-measurable.

we did this before. Do it yourself. Go kill yourself Gie.

(4) l is a measure & σ-finite. $\exists A_i \in \mathcal{A}_0 \ni l(A_i) < \infty$ $\forall i$ & $X = \bigcup_{i=1}^{\infty} A_i$.

recall, $l(X) < \infty \Rightarrow l$ is σ-finite. not the other way.

WTS μ^* is the unique extension of l. Here μ^* is defined as in the thm WTS if $\exists$ 2 extensions μ^* & γ, then $\mu^* = \gamma$ ie.

$\mu^*(E) = \gamma(E)$ $\forall E \in \sigma(\mathcal{A}_0)$

case 1: l is finite. well, $\mu^*(X) = l(X) < \infty$ so μ^* is finite as well.

The collection of all μ^*-measurable set forms a σ-algebra. $\supset \mathcal{A}_0$.

$\Rightarrow$ the σ-algebra $\supset \sigma(\mathcal{A}_0)$. $\Rightarrow$ If $E \in \sigma(\mathcal{A}_0)$ then E is μ^*-measurable. & $\mu^*(E) = \inf\{\sum_{i=1}^{\infty} l(A_i) \mid {A_i \in \mathcal{A}_0 \atop E \subset \cup A_i} \forall i\}$. $l = \gamma$ on $\mathcal{A}_0$

$\Rightarrow \sum l(A_i) = \sum \gamma(A_i)$. Hence if $E \subset \bigcup_{i=1}^{\infty} A_i$, $A_i \in \mathcal{A}_0$, then $\gamma(E)$

$\leq \sum \gamma(A_i)$ by def. of measure $= \sum l(A_i) \Rightarrow \underline{\gamma(E) \leq \mu^*(E)}$

- Now we need to show $\gamma(E) \geq \mu^*(E)$.

$\forall \varepsilon > 0$, choose $A_i \in \mathcal{A}_0$ $\forall i \ni E \subset \bigcup_{i=1}^{\infty} A_i$ & $\mu^*(E) + \varepsilon \geq \sum_{i=1}^{\infty} l(A_i)$

cont.

HW (4.2, 4.3, 4.4, 4.5, 4.8)

Due Oct. 6th

Proof of (4) cont.

$\mu^*(E)+\varepsilon \ge \sum_{i=1}^{\infty} \ell(A_i) = \sum_{i=1}^{\infty} \mu^*(A_i) \ge \mu^*(\bigcup_{i=1}^{\infty} A_i)$. Hence setting

$A=\bigcup_{i=1}^{\infty} A_i$, $\mu^*(E)+\varepsilon \ge \mu^*(A)$

$\Rightarrow \mu^*(A)-\mu^*(E)\le\varepsilon \Rightarrow \mu^*(A\setminus E)\le \varepsilon \quad (E\subset A)$

Set $B_i := \bigcup_{j=1}^{i} A_j$, $(B_1\subset B_2\subset\dots)\uparrow A$. $B_i\in A_0\ \forall i$.

$\mu^*(A)=\lim_{i\to\infty}\mu^*(B_i)=\lim_{i\to\infty}\nu(B_i)$ b/c they're both extensions from ℓ.

$=\nu(A) \Rightarrow \mu^*(E)\le\mu^*(A)=\nu(A)=\nu(E)+\nu(A\setminus E)$

$\le \nu(E)+\mu^*(A\setminus E)\le \nu(E)+\varepsilon$

$\Rightarrow \mu^*(E)\le\nu(E)$ QED so $\mu^*(E)=\nu(E)$

Case 2:

Need to show μ^* is the unique extension of ℓ when ℓ is σ-finite, so WTS $\mu^*(E)=\nu(E)$ $E\in\sigma(A_0)$.

We write $X=\bigcup_{i=1}^{\infty} K_i \ni \ell(K_i)<\infty\ \forall i$. $K_1\subset K_2\subset\dots$

Then by the previous result on the finite measure ~~case~~ case, we have a unique extension of ℓ_i defined by $\ell_i(A)=\ell(A\cap K_i)$ $\forall A\in\sigma(A_0)$

$\mu^*(A)=\lim_{i\to\infty} \mu(A\cap K_i)=\lim_{i\to\infty} \ell_i(A)=\lim_{i\to\infty}\nu(A\cap K_i)$

$=\nu(A)$ so $\mu^*(A)=\nu(A)$. QED

Ch 5 kill me Measureable functions

Measurability –

Consider $(X, \mathcal{A})$ as a measurable space.

set, σ-algebra

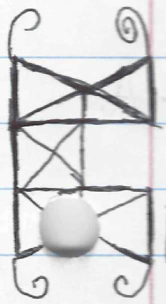
Def: A function $f : X \to \mathbb{R}$ is measurable (or $\mathcal{A}$ measurable) if

$x \in X \longrightarrow \{x \mid f(x) > a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$.

$\boxed{\text{Ex}}$ f : a constant function, i.e. $f(x) = c \quad c \in \mathbb{R}$.

$\{x \mid f(x) > a\} = \emptyset$ if $c \le a$, or $X \quad c > a$

so $$\{x \mid f(x) > a\} = \begin{cases} \emptyset & c \le a \\ X, & c > a \end{cases}$$



$\emptyset, X \in \mathcal{A}$, Hence f is measurable

$\boxed{\text{Ex}}$ $$f(x) = \begin{cases} 1 & x \in A \ (\subset X) \\ 0 & x \notin A \end{cases}$$

$$\{x \in X \mid f(x) > a = \begin{cases} \emptyset, & a \ge 1 \\ A, & 0 \le a < 1 \\ X, & a < 0 \end{cases}$$

Hence f is measurable iff $A \in \mathcal{A}$.

to see why, lookup Borel σ-algebra

$\underline{\text{Ex}}$ $X = \mathbb{R}$, $\mathcal{A}$ = Borel σ-algebra on $\mathbb{R}$.

$f(x) = x \qquad \{x \in \mathbb{R} \mid f(x) > a\} = (a, \infty) \in \mathcal{A} \quad \forall a \in \mathbb{R}$

$\Rightarrow f$ is measurable.

$(X, \mathcal{A}) \quad F : X \to \mathbb{R}$.

Proposition: Suppose f is a real valued function. Then the following conditions are equivalent:

(1) $\{x \in X \mid f(x) > a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$

(2) $\{x \in X \mid f(x) \le a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$

(3) $\{x \in X \mid f(x) < a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$

(4) $\{x \in X \mid f(x) \ge a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$

$x \in X$

previous WTS
Proof of previous proposition: (1) $\Leftrightarrow$ 2

$\{x \in X \mid f(x) > a\}^c = \{x \mid f(x) \le a\} \in \mathcal{A}$, so they are both $\in \mathcal{A}$

- By the same proof, we know (3) $\Leftrightarrow$ (4)

WTS now (1) $\Leftrightarrow$ (4), this would show (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4)

Assume (1) holds true

$\{x \mid f(x) \ge a\} = \bigcap_{n=1}^{\infty} \underbrace{\{x \mid f(x) > a - \tfrac{1}{n}\}}_{\in \mathcal{A}} \in \mathcal{A}$ so $\forall a$.

So (1) $\Rightarrow$ (4)

Now assume (4) holds true,

we write $\{x \mid f(x) > a\} = \bigcup_{n=1}^{\infty} \underbrace{\{x \mid f(x) \ge a + \tfrac{1}{n}\}}_{\in \mathcal{A}} \in \mathcal{A}$

So (4) $\Rightarrow$ (1) thus (4) $\Leftrightarrow$ (1)

thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) QED

-Proposition: X is a metric space, $\mathcal{A}$ is a σ-algebra on X containing all open sets, & $f: X \to \mathbb{R}$ and is continuous, then f is measurable.

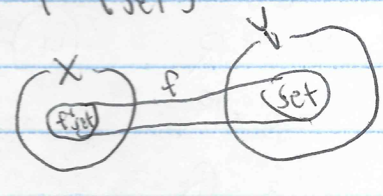
notes: basically saying a continuous function is measurable.

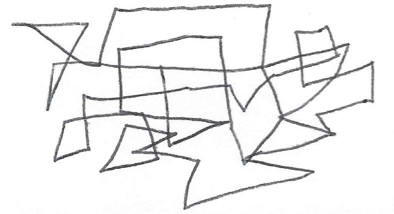
-Definition of continuity of a function (topological)

$f: X \to Y$. f is continuous if $f^{-1}(V)$ is open in X whenever V is open in Y

inverse image

note $f^{-1}(\text{set})$





Proof of ~~def~~ prop: $\{x \mid f(x) > a\} = f^{-1}((a, \infty))$: open based on previous def $\in A$.

Hence f is measurable by definition. QED

ignore this for now: Recall Borel σ-algebra is σ-algebra of all open sets. So it fits our definition.

• Proposition – Let $c \in \mathbb{R}$. Suppose f, g are measurable real valued functions. Then $f+g$, cf, $-f$, fg, & $\max(f,g)$, $\min(f,g)$ are also measurable. note $\max(f,g)(x) = \max(f(x), g(x))$

to see proof, look in book probably.

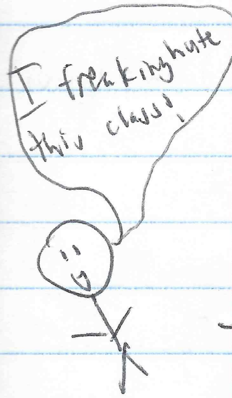
• Proposition – If f_i is a measurable function $\forall i$, then so are $\sup_i(f_i)$, $\inf_i(f_i)$, $\lim_{i\to\infty} \sup(f_i)$, & $\lim_{i\to\infty} \inf(f_i)$ provided that they are finite.

added homework problem: Suppose $E \subset \mathbb{R}$, $m(E) > 0$ where m is a Lebesgue measure.

Does this imply E contains an interval?

Prove this or give a counter example

proof on next page



Proof of previous proposition: - To show $\sup_i f_i$ is measurable, we observe that $\{x \mid \sup_i f_i(x) > a\} = \bigcup_{i=1}^{\infty} \underbrace{\{x \mid f_i(x) > a\}}_{\in \mathcal{A}} \in \mathcal{A}$ $\forall a \in \mathbb{R}$

so the union has to be in $\mathcal{A}$ as well, thus QED.

- Concerning $\limsup_{i \to \infty} f_i$, we have $\limsup_i f_i = \inf_n \left(\sup_{k \geq n} f_k \right)$ which is measurable

inf's are similar to this QED

- **Def:** - We say $f = g$ almost everywhere written $f = g$ a.e. if $\{x \mid f(x) \neq g(x)\}$ has measure zero.
  - We say $f_i \to f$ a.e., if $\{x \mid f_i(x) \not\to f(x)\}$ has measure zero.

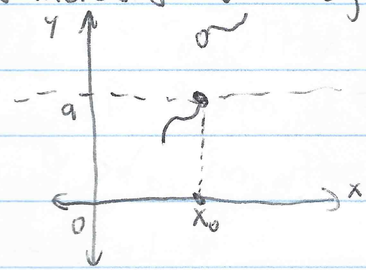
- **Def** - Suppose X is a metric space, $\mathcal{B}$ is a Borel σ-algebra on X. We say $f: X \to \mathbb{R}$ is Borel measurable if f is measurable w.r.t. $\mathcal{B}$.

- **Def** - If $f: \mathbb{R} \to \mathbb{R}$ is measurable w.r.t. Lebesgue σ-algebra ~~measure~~, then we say f is Lebesgue measurable.

- X is a metric space, f is continuous, then f is Borel measurable. But if f is Borel measurable, it's not necessarily continuous.
- $X = \mathbb{R}$, f is continuous, then f is Borel measurable, thus f is Lebesgue measurable. But if f is Lebesgue measurable, it doesn't necessarily mean it's Borel measurable.

• **Proposition:** $f:\mathbb{R}\to\mathbb{R}$ & f is monotone. $\Rightarrow$ f is Borel measurable.

proof: WLOG, assume f is increasing. $\forall\, a>0$, let $x_0 = \sup\{x \mid f(x)\le a\}$



If $f(x_0)\le a$, $\{x \mid f(x)>a\} = (x_0,\infty)$

If $f(x_0)>a$, $\{x \mid f(x)>a\} = [x_0,\infty)$

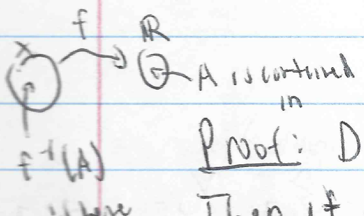
Both of these sets are Borel, which implies f is Borel measurable.

QED.

• **Proposition:** $(X, \mathcal{A})$ measurable space.

$f: X\to\mathbb{R}$ measurable function.

If A is in the Borel σ-algebra on $\mathbb{R}$, then $f^{-1}(A)\in\mathcal{A}$.



Proof: Define $\mathcal{C} = \{A\subset\mathbb{R} \mid f^{-1}(A)\in\mathcal{A}\}$

Then it is enough to show that the Borel-σ-algebra on $\mathbb{R}$: $\mathcal{B}\subset\mathcal{C}$.

Step 1) - Show $\mathcal{C}$ is a σ-algebra. If $A_i\in\mathcal{C}$ $\forall\, i$, then

$$f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} \underbrace{f^{-1}(A_i)}_{\in\mathcal{A}} \in \mathcal{A} \quad \text{so}$$

Hence $\bigcup A_i\in\mathcal{C}$. That is, $\mathcal{C}$ is closed under countable unions.

If $A\in\mathcal{C}$, $f^{-1}(A^c) = (\underbrace{f^{-1}(A)}_{\in\mathcal{A}})^c \in\mathcal{A}$ so $\Rightarrow$ $\mathcal{C}$ is closed under compliment. $\Rightarrow A^c\in\mathcal{C}$.

$f^{-1}(\mathbb{R}) = X\in\mathcal{A} \Rightarrow \mathbb{R}\in\mathcal{C}$, thus $\mathcal{C}$ is a σ-algebra.

step 2) Now we need to show $\mathcal{B} \subset \mathcal{C}$.

We know f is $\mathcal{A}$ measurable.

$\{x \mid f(x) > a\} = f^{-1}((a,\infty)) \in \mathcal{A} \ \forall a$ b/c f is $\mathcal{A}$-measurable.

$\Rightarrow (a,\infty) \in \mathcal{C} \ \forall a$.

$\Rightarrow \underbrace{\sigma(\{(a,\infty)\}_{a\in\mathbb{R}})}_{= \mathcal{B} \text{ by former proposition}} \subset \mathcal{C}$ b/c $\mathcal{C}$ is a σ-algebra.

Hence $\mathcal{B} \subset \mathcal{C}$. QED.

Exam based on homeworks turned in & things studied in class.

5 problems. I hate you Gie. Go die in a burn fire.

Slit your throat and shove eels inside.

5.2 Approximation of functions

Def: $(X, \mathcal{A})$ is a measurable space. $\forall E \subset \mathcal{A}$, define the characteristic function of E, χ_E,

$$\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases} \quad \text{(measurable)}.$$

Def: A simple function, s, is $s(x) = \sum_{i=1}^{n} a_i \chi_{E_i}$ where $a_i \in \mathbb{R}$, $E_i \in \mathcal{A}$, $\forall i$. (measurable).

Proposition: Suppose f is a non-negative measurable function.

$\Rightarrow \exists$ a sequence of non-negative measurable simple functions $\{s_n\}_{n=1}^{\infty}$ increasing to f.

Note: $\{s_n\}_{n=1}^{\infty}$ increasing to f is to get us pointwise convergence to f.

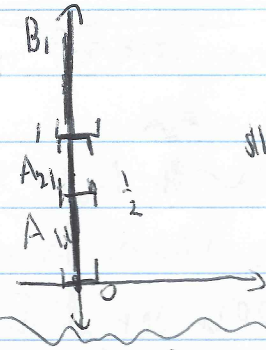
Proof: We want to construct an $s_n \ni$ it increases to f.

Define $A_{in} = \{ x \mid \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n} \}$

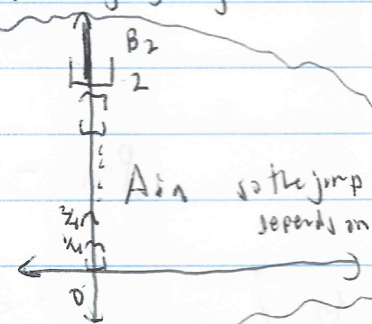
$B_n = \{ x \mid f(x) \ge n \}$ $\quad n = \{1, 2, 3, \dots\}$

$i = \{1, 2, \dots, n2^n\}$

Explanation: $n=1$



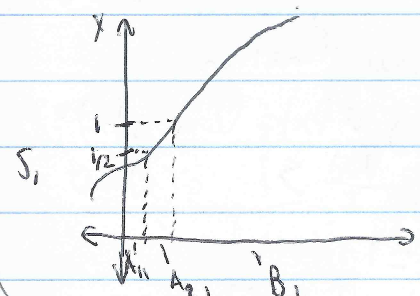
showing how they converge

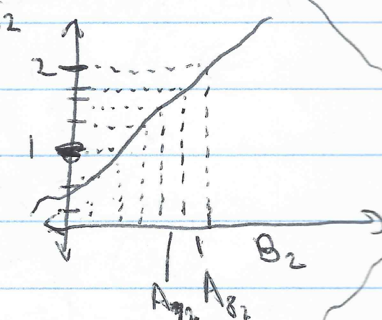


so the jump depends on n

$$s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{A_{in}} + n \chi_{B_n} \quad \forall n$$

Further explanation





$s_n \le f(x) \ \forall x.$

$s_n \to f$ pointwise.

Failure

$\forall x, \forall \varepsilon > 0, \exists N \in \mathbb{N} \ni |f(x) - s_n(x)| < \varepsilon$ if $n \ge N$

QED?

Def:

If f is a function. (support of f) $= \operatorname{supp} f = \{ x \mid f(x) \ne 0 \}$

• Lusin's theorem.

$f:[0,1]\to\mathbb{R}$ is Lebesgue measurable. $\forall\ \varepsilon>0$, $\exists$ a closed $F\subset[0,1] \ni m([0,1]\setminus F)<\varepsilon$

& $f|_F$ = (restriction of f to F)

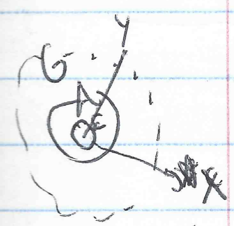
$f|_F$ is continuous ~~...~~ on F where

Proof: step 1) ~~let $f=\chi_A$ ... A be Lebesgue measurable~~

Let $f=\chi_A$ where A is Lebesgue measurable in $[0,1]$

$\forall\ \varepsilon>0\ \exists$ G: open set & E (a closed set)

$\ni E\subset A\subset G$, & $m(G\setminus A)=m(A\setminus E)<\frac{\varepsilon}{2}$ by former prope



E is compact b/c it's closed & bdd.

Let $\delta=\inf\{|x-y| \mid x\in E,\ y\in G^c\}>0$

Define $g(x)=\max\left(1-\frac{d(x)}{\delta},0\right)$ where $d(x)=\inf\{|x-y| \mid y\in E\}$

$\Rightarrow$ i) g is continuous function

ii) If $x\in E$, $d(x)=0$, $g(x)=1$

iii) If $x\in G^c$, $\frac{d(x)}{\delta}\geq 1$ $g(x)=0$

iv) $0\leq g(x)\leq 1$

Take $F=(E\cup G^c)\cap[0,1]$, so f is closed

& $m([0,1]\setminus F)=m([0,1]\cap(E\cup G^c)^c)$

$=m(G\cap E^c)=m(G\setminus E)<\varepsilon$ $\quad \overset{\text{"}}{E^c\cap G}$

by our above definitions.

$f=\chi_A$

WTS $f|_F=g$

well $f|_F=\begin{cases}1 & \text{in } E\cap[0,1]\\ 0 & \text{in } G^c\cap[0,1]\end{cases}$

Hence $f|_F=g$ & hence $f|_F$ is continuous.

QED?.

- step 2) $f = \sum_{i=1}^{M} a_i \chi_{A_i}$, A_i is Lebesgue measurable, $a_i > 0$.

$\forall \varepsilon > 0$, choose F_i as a closed set in $[0,1] \ni$

$m([0,1] \setminus F_i) < \frac{\varepsilon}{M}$ & $\chi_{A_i}|_{F_i}$ is continuous

Set $F = \bigcap_{i=1}^{M} F_i$ closed in $[0,1]$.

$m([0,1] \setminus F) = m([0,1] \cap (\bigcap_{i=1}^{M} F_i)^c) = m([0,1] \cap (\bigcup_{i=1}^{M} F_i^c))$

$= m(\bigcup_{i=1}^{M} [0,1] \cap F_i^c) \le \sum_{i=1}^{M} m([0,1] \setminus F_i) < \varepsilon$

$f|_F = \sum_{i=1}^{M} a_i \chi_A |_{F_i}$ Hence this is continuous b/c we know this is continuous by our choice of F_i.

QED?

- step 3) Let f be non-negative, Lebesgue measurable, & bdd. by $K > 0$.

Set $A_{in} = \{x \mid \frac{i-1}{2^n} < f(x) < \frac{i}{2^n}\}$

$n = 1, 2, \dots$, $i = 1, \dots, n2^n$

recall $f(x) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{A_{in}} + n \chi_{B_n}$ but K is bdd, so for some reason we can get rid of the B_n.

so $f_n(x) = \sum_{i=1}^{k2^n+1} \frac{i-1}{2^n} \chi_{A_{in}}$ & now $i = 1, \dots, k2^n+1$

simple function & measurable b/c blow me

$f_n \to f$ pointwise

Define $h_n(x) = f_{n+1}(x) - f_n(x)$: simple function - simple function

$\Rightarrow$ simple function & measurable

Moreover $|h_n(x)| = |f_{n+1}(x) - f_n(x)|$

$\le \left|\frac{1}{2^{n+1}} - \frac{1}{2^n}\right| = \frac{1}{2^{n+1}} < \frac{1}{2^n} \quad \forall n$

Step 3 cont. on next page.

For $f_0 =$ characteristic function. S_0, choose F_0 that's closed $\ni$
$m([0,1]\setminus F_0) < \frac{\varepsilon}{2}$ & $f_0|_{F_0}$ is continuous.
For $n \geq 1$, choose F_n that's closed $\ni$ $m([0,1]\setminus F_n) < \frac{\varepsilon}{2^{n+1}}$
& $f_n|_{F_n}$ is continuous.
Now set $F = \bigcap_{n=0}^{\infty} F_n$ since each F_n is closed, F is also closed.

$$m([0,1]\setminus F) = m\left([0,1] \cap \left(\bigcup_{n=0}^{\infty} F_n^c\right)\right) = m\left(\bigcup_{n=0}^{\infty}\left([0,1]\cap F_n^c\right)\right)$$
$$= m\left(\cup\left([0,1]\setminus F_n\right)\right) \leq \sum_{n=0}^{\infty} m([0,1]\setminus F_n) < \varepsilon.$$

Now we need to show $f_n|_{F_n}$ is continuous.
On F, $f_0(x) + \sum_{n=1}^{\infty} h_n(x) = \lim_{N\to\infty} F_N(x) \to f(x)$ pointwise
based on our construction
Note $|h_n(x)| < \frac{1}{2^n}$ $\forall n \geq 1$.
Also $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

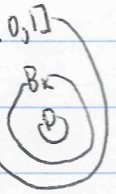
$$\Rightarrow f_0 + \sum_{n=1}^{\infty} h_n(x) \to f \text{ uniformly}$$

by the Weierstrass M test.
Then $f(x)$ is continuous on F, by some math 501 stuff.

chopped the [illegible] part).
- Step 4) f is non-negative function & Lebesgue measurable on $[0,1]$.
Let $\varepsilon > 0$. We need to find a closed set $F \ni m([0,1]\setminus F) < \varepsilon$
& $f|_F$ is continuous.
Set $B_K = \{x \mid f(x) \leq K\}$. $B_K \uparrow [0,1]$ as $K \to \infty$. By choosing $K > 0$ sufficiently large, $\cancel{m([0,1]\setminus B_K) < \frac{\varepsilon}{3}}$
$m(B_K) > 1 - \frac{\varepsilon}{3}$ (can be made arbitrarily close to $[0,1]$.
We can choose D as a closed set in $[0,1] \ni D \subset B_K$ &
$m(B_K \setminus D) < \frac{\varepsilon}{3}$.
Choose E as a closed set in $[0,1] \ni \underbrace{f \cdot \chi_D}_{\text{measurable \& bounded.}}|_E$ is continuous,
$m([0,1]\setminus E) < \frac{\varepsilon}{3}$

cont on next page

Now take $F = E \cap D$, and this is closed b/c $\cap$ of 2 closed sets.



$m([0,1]\setminus(E\cap D)) = m([0,1]\cap(E^c \cup D^c)) = m(([0,1]\cap E^c)\cup([0,1]\cap D^c))$

$\le m([0,1]\setminus E) + m([0,1]\setminus D) = ([0,1]\setminus B_k)\cup(B_k\setminus D)$

$\le m([0,1]\setminus E) + m([0,1]\setminus B_k) + m(B_k\setminus D) < \varepsilon.$

Now we just need to show it's continuous.

Note $f\cdot\chi_D|_E = f|_F$ so it's continuous? QED.

Step 5) Let f be Lebesgue measurable.

Recall $f^+ = \max(f,0)$ & $f^- = \max(-f,0)$

$f = f^+ - f^-$. Keep this def. in mind.

Choose F^+ & F^- closed in $[0,1]$ $\ni$ $m([0,1]\setminus F^+) = m([0,1]\setminus F^-) < \frac{\varepsilon}{2}$

$f^+|_{F^+}$, $f^-|_{F^-}$ are continuous.

Set $F = F^+ \cap F^-$ to be closed, so

$m([0,1]\setminus F) < \varepsilon$ by very similar computations as before.

Moreover $f|_F = \underbrace{f^+|_{F^+}}_{\text{Cont.}} - \underbrace{f^-|_{F^-}}_{\text{Cont.}}$ So the whole thing is continuous. Thus we're done.

Ch 6 - The Lebesgue Integral

• <u>Def:</u> Let $(X, \mathcal{A}, \mu)$ be a measure space.
If $s = \sum_{i=1}^{n} a_i \chi_{E_i}$ is non-negative & a measurable simple function
(note s is measurable iff $\forall i$ E_i is measurable).
The <u>Lebesgue Integral</u> of s is $\int s \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i)$.

- Remark: In the case where a_i & $\mu(E_i) = \infty$ for some i,
we compute $a_i \mu(E_i) = 0$

• If $f \geq 0$ & is a measurable function, ~~[illegible]~~
$\int f \, d\mu = \sup\{ \int s \, d\mu \mid \substack{0 \leq s \leq f \\ s: \text{simple function}} \}$

• If f is a measurable function,
$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$, this is provided that $\int f^+ d\mu$ & $\int f^- d\mu$ are not both infinite.
(positive part) (negative part)

• <u>Remarks:</u> - i) A simple function can be written in more than one way; i.e.
$s = a \chi_{A \cup B}$ where $A \cap B = \emptyset$ $\Rightarrow$
$s = a\chi_A + a\chi_B$ as well.
- ii) If $s = \sum_{i=1}^{n} a_i \chi_{A_i} = \sum_{i=1}^{m} b_i \chi_{B_i}$, then $\int s \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i) = \sum_{i=1}^{m} b_i \mu(B_i)$
- iii) If s is a simple function & $s \geq 0$, then $\int s \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i)$
$= \sup\{ \int s' \, d\mu \mid \substack{0 \leq s' \leq s \\ s' \text{ is simple}} \}$

• <u>Def:</u> f is <u>integrable</u> if f is measurable & $\int |f| \, d\mu < \infty$.

• <u>Proposition:</u> Suppose we have a measure space $(X, \mathcal{A}, \mu)$.
(1) $f: X \to \mathbb{R}$ is measurable $\ni$ $a \leq f(x) \leq b$ $\forall x \in X$ & $\mu(X) < \infty$,
then ~~[illegible]~~ $a\mu(X) \leq \int f \, d\mu \leq b\,\mu(X)$
(2) $f, g: X \to \mathbb{R}$ are measurable functions, integrable, & $f(x) \leq g(x)$ $\forall x \in X$
$\Rightarrow \int f \, d\mu \leq \int g \, d\mu$.

Homework: 5.3, 5.5, 1st question in 5.9, 6.2, 6.3, 6.

Due next Thursday

prop cont.

(3) $f: X \to \mathbb{R}$ is integrable, & $c \in \mathbb{R} \Rightarrow \int cf \, d\mu = c \int f \, d\mu$.

(4) $\mu(A) = 0$ & f is integrable $\Rightarrow \int f \chi_A \, d\mu = 0$.

used def. given to prove these results.

Understandings (1) $\Rightarrow$ bounded $\Rightarrow$ bounded

(2) duh (3) duh

(4) characteristic of $A \cdot f$ is integrable & thus $\int f \chi_A \, d\mu =$

• Notation: • $\int f \chi_A \, d\mu = {}_A\int f \, d\mu$ χ_A means we're integrating on A

has to be measurable to integrate.

• $\int f \, d\mu = \int f(x) \, d\mu(x) = \int f(x) \, \mu(dx)$

$= \int f$

this is why we write it this.

• Suppose $f: \mathbb{R} \to \mathbb{R}$, and m is a Lebesgue measure

$\int f \, dm = \int f(x) \, dm(x) = \int f(x) \, m(dx)$

& ${}_a^b\int f(x) \, dx = \int_{[a,b]} f(x) \, m(dx)$

Ch 7 - Limit Theorems.

- **Monotone Convergence Theorem**

Suppose f_n is a non-negative & measurable function $\forall n$. Consider $\{f_n : \begin{smallmatrix} f_n \geq 0 \\ \text{measurable} \end{smallmatrix}\}_{n=1}^{\infty} \ni$

$f_1(x) \leq f_2(x) \leq \dots \; \forall x$ & $\lim_{n\to\infty} f_n(x) = f(x) \; \forall x$, $\Rightarrow$

$\int f_n \, d\mu \to \int f \, d\mu$.

- understanding: Have a sequence of measurable ≥ 0 functions, pointwise, the function is increasing, $f_n \to f$ pointwise, then $\int f_n \to \int f$.

- Proof: - 1) WTS f is measurable.

Set $\ell = [0, a]$ for $a \in \mathbb{R}$ & $a > 0$. Then if $f(x) \in \ell$, then $f_n(x) \in \ell \; \forall n$ b/c $0 \leq f_n(x) \leq f(x) \; \forall n$.

Similarly, if $f_n(x) \in \ell$, then $f(x) \in \ell$ b/c $\lim_{n\to\infty} f_n(x) = f(x) \in \ell$ b/c ℓ is closed.

$\Rightarrow f(x) \in \ell$ iff $f_n(x) \in \ell \; \forall n$.

$\Rightarrow \{x \mid f(x) \in \ell\} = \bigcap_{n=1}^{\infty} \{x \mid f_n(x) \in \ell\} \in \mathcal{A}$

$\Rightarrow f$ is measurable.

- 2) WTS $\int f_n \, d\mu \to \int f \, d\mu$.

Consider the sequence $\{\int f_n \, d\mu\}_{n=1}^{\infty}$; but we know this is an increasing sequence b/c $f_n(x) \leq f_{n+1}(x) \; \forall n \; \forall x$.

Define $0 \leq L = \lim_{n\to\infty} \int f_n \, d\mu$ (could possibly be ∞). Since $f_n(x) \leq f(x) \; \forall n, x$

$\int f_n \, d\mu \leq \int f \, d\mu \; \forall n$.

$\Rightarrow L \leq \int f \, d\mu$. Next wts $L \geq \int f \, d\mu$. Let s be any non-negative measurable simple function $\ni s = \sum_{i=1}^{m} a_i \chi_{E_i} \leq f$.

$\forall c \in (0,1) \ni A_n = \{x \mid f_n \geq c\, s(x)\}$, so $A_n \uparrow X$ b/c $f_n(x) \to f(x)$ as $n \to \infty$ $\forall x \in X$. $\int f_n \geq \int f_n \chi_{A_n} = \int_{A_n} f_n \geq \int_{A_n} c\, s = c \sum_{i=1}^{m} a_i \, m(E_i \cap A_n)$

$\sum a_i \chi_{E_i}$ in integrating A_n, $= \sum a_i \, m(E_i \cap A_n)$

Proof cont. Note $E_i \cap A_n \uparrow E_i$ as $n \to \infty$ for each i.

So $c \sum_{i=1}^{m} a_i \mu(E_i \cap A_n) \to c \sum_{i=1}^{m} a_i \mu(E_i)$ as $n \to \infty$

$$c \sum_{i=1}^{m} a_i \mu(E_i) = c\int s$$

Then $\int f_n \geq c \sum_{i=1}^{m} a_i \mu(E_i \cap A_n)$

$$\int f_n \to L, \qquad c \sum_{i=1}^{m} a_i \mu(E_i \cap A_n) \to c\sum_{i=1}^{m} a_i \mu(E_i) = c\int s$$

$\Rightarrow L \geq c\int s \quad \forall\, c \in (0,1) \Rightarrow L \geq \int s.$

By taking sup over all s, $L \geq \int f$ QED

Thm: If $f, g \geq 0$ & measurable, or f, g are integrable, then
$\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu.$

Note, in book he expands this to the complex #'s, we will stick to the reals

Proof:

Step 1) f, g: non-negative measurable simple functions.

$f = \sum_{i=1}^{m} a_i \chi_{A_i}$, $g = \sum_{i=1}^{n} b_i \chi_{B_i}$ where $a_i, b_i \geq 0$ &
A_i, B_i are measurable.

Set $c_i \begin{cases} a_i, & 1 \leq i \leq m \\ b_i & m+1 \leq i \leq m+n \end{cases}$ & $C_i = \begin{cases} A_i & 1 \leq i \leq m \\ B_i & m+1 \leq i \leq m+n \end{cases}$

So

$$f+g = \sum_{i=1}^{m+n} c_i \chi_{C_i} \Rightarrow \int f+g = \int \sum_{i=1}^{m+n} c_i \chi_{C_i}$$
$$= \sum_{i=1}^{m+n} c_i \mu(C_i) = \sum_{i=1}^{m} c_i \mu(C_i) + \sum_{i=m+1}^{m+n} c_i \mu(C_i)$$
$$= \sum_{i=1}^{m} a_i \mu(A_i) + \sum_{i=1}^{n} b_i \mu(B_i) = \int f\,d\mu + \int g\,d\mu.$$

Step 2) f, g are non negative measurable functions

We have proven in this case $\exists \{s_n\}_{n=1}^{\infty}$ (sequence of non negative measurable simple functions) $\ni s_1 \leq s_2 \leq \dots$, $s_n \to f$ pointwise.

So similarly let $\{t_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable simple functions $\ni t_1 \leq t_2 \leq \dots$ & $t_n \to g$ pointwise thm proof continued:

$\Rightarrow \{s_n + t_n\}_{n=1}^{\infty}$ is a sequence of non-negative measurable simple functions,

so $s_1+t_1 \leq s_2+t_2 \leq \ldots \ni s_n+t_n \to f+g$ pointwise.

so $\int (f+g)\,d\mu = \int \lim_{n\to\infty} (s_n+t_n)\,d\mu = \lim_{n\to\infty} \int (s_n+t_n)\,d\mu$ by the monotone convergence thm. so $= \lim_{n\to\infty} \left(\int s_n\,d\mu + \int t_n\,d\mu \right)$ by step 1,

so $= \lim_{n\to\infty} \int s_n\,d\mu + \lim_{n\to\infty} \int t_n\,d\mu = \int f\,d\mu + \int g\,d\mu$ by the monotone convergence thm. again.

Step 3) Assume f, g are integrable. $\Rightarrow f$ & g are measurable & $\int |f|\,d\mu < \infty$ & $\int |g|\,d\mu < \infty$. Is $f+g$ measurable? Yes, we know this from previous stuff. So $\int |f+g|\,d\mu \leq \int (|f|+|g|)\,d\mu$ by the triangle inequality. $= \int |f|\,d\mu + \int |g|\,d\mu$ by step 2, $< \infty$ b/c both are finite. $\Rightarrow f+g$ is integrable.

$$(f+g) = (f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$$

$$\Rightarrow (f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

$$\star \Rightarrow \int (f+g)^+ + \int f^- + \int g^- = \int (f+g)^- + \int f^+ + \int g^+ \quad \text{by step 2.}$$

Notice $\int (f+g) = \int (f+g)^+ - \int (f+g)^-$ by definition.

$$= \int f^+ - \int f^- + \int g^+ - \int g^- \quad \text{by } \star$$

$= \int f + \int g$ by definition of f and g, b/c they're integrable.

QED.

Taking this as truth as well as $\int cf = c\int f$, then we have linearity

ie $\int$ linear combination $=$ linear combination broken into multiple integrals.

- Let's look at a thing:

$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu$ by monotone convergence thm.

this changing the limit is a big deal & very useful for later work.

Recall liminf: $\{x_n\} \subset \mathbb{R}$

$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} (\inf_{i\geq n} x_i)$

$= \sup_{n\geq 0} (\inf_{i\geq n} x_i)$

- **Thm: Fatou's Lemma:**

Suppose we have a sequence of functions $\{f_n \geq 0\}_{n=1}^{\infty}$ where $\forall n$ f_n is measurable.

$\Rightarrow \int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$

Proof:

Set $g_n = \inf_{i\geq n} f_i$ (we do this b/c $\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n$)

$$\Rightarrow \begin{cases} g_n \geq 0 \quad \forall n \\ g_1(x) \leq g_2(x) \leq \dots, \forall x \\ g_n(x) \to \liminf_{n\to\infty} f_n(x) \end{cases}$$

Note $g_n \leq f_n$ $i \geq n$ b/c g_n is the inf. so

$g_n(x) \leq f_i(x)$ $i \geq n, \forall x$. So $\int g_n \leq \int f_i$ $i \geq n$.

$\Rightarrow \int g_n \leq \inf_{i\geq n} \int f_i \Rightarrow \lim_{n\to\infty} \int g_n \leq \lim_{n\to\infty} \inf_{i\geq n} \int f_i = \liminf_{n\to\infty} \int f_n$

well, by the monotone convergence thm,

$\lim_{n\to\infty} \int g_n = \int \lim_{n\to\infty} g_n = \int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$

QED

Thm: Dominated Convergence Theorem

Suppose we have f_n as a measurable real valued func. Sowe have the sequence $\{f_n\}_{n=1}^{\infty} \ni f_n(x) \to f(x)$ $\forall x$. If $\exists$ a non-negative integrable function $g \ni |f_n(x)| \leq g(x)$ $\forall x$, then $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$.

Proof on next page.

② inequalities for reference in proof. ①

Proof: WTS $\limsup_{n\to\infty} \int f_n \le \int f \le \liminf_{n\to\infty} \int f_n$ which $\Rightarrow$ the $\lim_{n\to\infty} \int f_n = \int f$.

①: We know $|f_n(x)| \le g(x) \Rightarrow -g(x) \le f_n(x) \le g(x)$

$\Rightarrow f_n(x) + g(x) \ge 0 \quad \forall x \,\&\, n$, and $f_n(x)+g(x)$ is measurable. This satisfies condition for Fatou's Lemma, so by Fatou's Lemma,

$$\int \liminf_{n\to\infty} (f_n+g) \le \liminf_{n\to\infty} \int (f_n+g).$$

$= \int (\liminf_{n\to\infty} f_n + g) = \int (f+g)$

$= \liminf_{n\to\infty} \left(\int f_n + \int g \right)$ b/c linearity & g integ.

$= \liminf_{n\to\infty} \int f_n + \int g$

$\Rightarrow \int f + \int g \le \liminf_{n\to\infty} \int f_n + \int g \Rightarrow \int f \le \liminf_{n\to\infty} \int f_n$. So we have ①.

②: $|f_n| \le g(x) \Rightarrow -g(x) \le f_n(x) \le g(x) \Rightarrow g - f_n \ge 0$ & is measurable $\forall n$.

Then by Fatou's Lemma,

$$\int \liminf_{n\to\infty} (g-f_n) \le \liminf_{n\to\infty} \int (g-f_n)$$

$= \int (g-f) = \int g - \int f$

$\liminf_{n\to\infty} \left(\int g - \int f_n \right) = \int g + \liminf_{n\to\infty} \left(-\int f_n \right) = \int g - \limsup_{n\to\infty} \int f_n$.

$\int g - \int f \le \int g - \limsup_{n\to\infty} \int f_n$

$\Rightarrow -\int f \le -\limsup_{n\to\infty} \int f_n \Rightarrow \int f \ge \limsup_{n\to\infty} \int f_n$ so we have ②,

thus

$\lim_{n\to\infty} \int f_n = \int f$. QED.

almost everywhere.

Def: If $f = g$ a.e. in X ~~this says~~ this means that $\exists E \subset X \ni \mu(E) = 0$ & $f(x) = g(x) \; \forall x \in X \setminus E$.

This means $f(x) = 1$ on $x \in \mathbb{R}$.

$g(x) = \begin{cases} 1 & \text{on } x \in \mathbb{R}\setminus\mathbb{Q} \\ 0 & \text{on } x \in \mathbb{Q} \end{cases} \Rightarrow f = g$ a.e. on $\mathbb{R}$.

Ch. 7 homework: (7.3, 7.5, 7.6, 7.7, 7.8, 7.14)

Due 11/15 Tuesday

Ex| Let f_n be a measureable non-negative sequence of functions $\ni f_n \uparrow$ a.e. We'll set $A = \{x \mid f_n(x) \uparrow f(x)\}$ so $\mu(A^c) = 0$.

In this case:

b/c $\chi_A + \chi_{A^c} = 1$

$$\lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_n (\chi_A + \chi_{A^c}) = \lim_{n\to\infty} \left(\int f_n \chi_A + \int f_n \chi_{A^c} \right)$$

$$= \lim_{n\to\infty} \int f_n \chi_A$$

We'll come back to this after 8.1 b/c Gié's an idiot :)

Ch 8 Properties of Lebesgue Integrals

What do we know? This is what we've got so far.

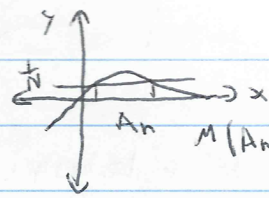
1) $\int \sum_{i=1}^{n} a_i \chi_{A_i} = \sum_{i=1}^{\infty} a_i m(A_i)$
2) $\int f = \sup \{ \int \text{simple functions} \mid \text{simple function} \leq f \}$ $f \geq 0$ & measurable.
3) f is measurable, $f = f^+ - f^-$.
4) $\int cf + g = c\int f + \int g$

8.1 – Criteria for a function to be zero a.e.

• **Proposition:** When f is measurable & non-negative $\ni \int f \, dm = 0$,
$\implies f = 0$ a.e.

Proof: By way of contradiction, suppose $f \neq 0$ a.e. Then since f is non-negative, $\exists N > 0 \ni m(A_N) > 0$ where $A_N = \{x \mid f(x) > \frac{1}{N}\}$.

(aside: What does this mean?



$m(A_n) > 0$

if this is not possible, $f=0$ a.e., which is what we're assuming is false)

Proof cont.

$$0 = \int f \geq \int f \chi_{A_N}$$

χ_{A_N} is measurable b/c A_N is measurable by definition

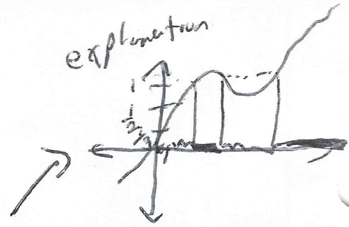
$$= \int_{A_N} f \geq \int_{A_N} \frac{1}{N} = \frac{1}{N} m(A_N) > 0$$

but $0 = \int f > 0$, which is a contradiction, thus $f = 0$ a.e. QED.

• **Prop:** Let f be integrable $\ni \int_A f = 0$ $\forall$ measurable set A.
$\implies f = 0$ a.e.

Proof on next page.

explanation



$A_1 \supseteq$
$\cap$
$A_2 \supseteq$
$\cap$
$A_3 \supseteq$

Proof: – Step 1) WTS $\mu(\{x \mid f(x) > 0\}) = 0$.

– Step 2) WTS $\mu(\{x \mid f(x) < 0\}) = 0$.

– Step 1): Define $A_n = \{x \mid f(x) > \frac{1}{n}\}$ $\forall n = 1, 2, \ldots$

So $A_1 \subset A_2 \subset A_3 \subset \ldots$, $\bigcup_{n=1}^{\infty} A_n = \{x \mid f(x) > 0\}$.

$\forall n \quad 0 = \int_{A_n} f \geq \int_{A_n} \frac{1}{n} = \frac{1}{n}\mu(A_n) \geq 0$

but $0 \not> 0$ so

$\Rightarrow \mu(A_n) = 0 \ \forall n. \Rightarrow \mu(\{x \mid f(x) > 0\}) = \mu(\bigcup_{n=1}^{\infty} A_n)$

$\leq \sum_{n=1}^{\infty} \mu(A_n) = 0$

$\Rightarrow \mu(\{x \mid f(x) > 0\}) = 0$.

– Step 2): Almost identical to step 1, except we let

$B_n = \{x \mid f(x) < -\frac{1}{n}\}$ $\ni$

$\bigcup_{n=1}^{\infty} B_n = \{x \mid f(x) < 0\}$, then we're good.

⋮

QED.

• Corollary: Let m be a Lebesgue measure & $a \in \mathbb{R}$. If

$f: \mathbb{R} \to \mathbb{R}$ & integrable, then

$\int_a^x f(y)\,dy = 0 \ \forall x \in \mathbb{R} \Rightarrow f = 0$ a.e.

Proof of this Corollary:

By earlier propositions, it's sufficient to show that

$\int_E f = 0 \ \forall$ Lebesgue measurable sets E.

That is, we will show $\int_a^x f(y)\,dy = 0 \Rightarrow \int_E f = 0$.

More on this proof later.

1) $\forall$ interval (c, d), $\int_c^d f(y)\,dy = \int_{(c,d)} f = \int f \chi_{(c,d)}\,dm$

$= \int f \chi_{(c,a)\cup(a,d)} = \int f \chi_{(c,a)} + \int f \chi_{(a,d)}$

$= \int_c^a f(y)\,dy + \int_a^d f(y)\,dy = -\int_a^c f(y)\,dy + \int_a^d f(y)\,dy$

but $\int_a^x f(y)\,dy = 0 \ \forall x$, so $\Rightarrow \int_c^d f(y)\,dy = 0$. QED.

2) Let G be the finite union of open intervals.

$\int_G f = 0$ by 1) & linearity of integration. (there are more steps, but it's ok)

3) Let G be an open set in $\mathbb{R}$. So

$G = \bigcup_{i=1}^{\infty} G_i$ where G_i are disjoint open intervals.

$\int_G f = \int f\chi_G = \int f\chi_{\bigcup_{i=1}^{\infty} G_i} = \int \sum_{i=1}^{\infty} f\chi_{G_i}$.

We define $F_n = \sum_{i=1}^{n} f\chi_{G_i}$ $\forall n$

i) $F_n \to f\chi_G$ as $n\to\infty$ pointwise,

ii) $F_n \le |f|$ & $\int |f| < \infty$ b/c f is integrable so

$\lim_{n\to\infty} \int F_n = \int f\chi_G = \int_G f$

On the other hand, $\int F_n = \int \sum_{i=1}^{n} f\chi_{G_i} = \sum_{i=1}^{n} \int f\chi_{G_i} = \sum_{i=1}^{n} \int_{G_i} f = 0$ $\forall n$

by step 2)

$\Rightarrow 0 = \int_G f$.

4) Consider a sequence of open sets $G_n \ni G_n \downarrow H$ $\forall n$. So $H = \bigcap_{i=1}^{\infty} G_i$

$\int_H f = \int f\chi_H = \int f\chi_{\bigcap_{i=1}^{\infty} G_i}$ Here $G_1 \supset G_2 \supset \dots$

$\hookrightarrow = \int \lim_{n\to\infty} f\chi_{G_n}$

Set $F_n = f\chi_{G_n}$, we observe that

i) $f\chi_{G_n} \to f\chi_H$ pointwise

ii) $f\chi_{G_n} \le |f|$, & $\int |f| < \infty$

$\Rightarrow$ by Dominant Conv. theorem, $\int_H f = \lim_{n\to\infty} \int f\chi_{G_n} = 0$ by step 3).

5) E: Lebesgue measurable set

So consider an E that's a Lebesgue measurable set.

So $\exists G_n$ that's an open set $\ni \forall n$ $G_n \downarrow H$ & $m(E\setminus H) = 0$.

$\int_E f = \int f\chi_E = \int f\chi_{H\cup(E\setminus H)} = \int f\chi_H + f\chi_{E\setminus H}$

these are disjoint

$= \int f\chi_H + \cancel{\int f\chi_{E\setminus H}}^{0}$

$= \int f\chi_H = 0$ by step 4)

So we've shown $\int_E f = 0$ $\forall$ Lebesgue measurable sets E, so by a prop., $f = 0$ a.e. QED.

Note: support: $\overline{\{x \mid f(x) \neq 0\}}$ = smallest closed set $A \ni A^c \supset \{x \mid f(x) = 0\}$.

Approximation Result

Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ where f is Lebesgue measurable & integrable, $\forall \varepsilon > 0$ $\exists$ a continuous g w/ compact support $\ni \int |f-g| < \varepsilon$.

Side note: Observe that $f = f^+ - f^-$, if $\exists$ continuous g_1 & g_2 w/ compact supports $\ni$ $\int |f^+ - g_1| < \frac{\varepsilon}{2}$, $\int |f^- - g_2| < \frac{\varepsilon}{2}$

Then $\int |f - (g_1 + g_2)|$

$= \int |(f^+ - g_1) - (f^- - g_2)| \leq \int |f^+ - g_1| + |f^- - g_2|$

$= \int |f^+ - g_1| + \int |f^- - g_2| < \varepsilon$

This says if we can approximate the positive and negative part separately, we're done. So it's sufficient to prove this for a non-negative function f.

Proof: From side note we can assume $f \geq 0$, setting $f_n = f \chi_{[-n,n]}$ $\forall n$, we notice that $f_n \to f$ pointwise & $f_n \leq |f|$, & $\int |f| < \infty$. Hence by Dominant Conv. Thm, $\int f = \lim_{n \to \infty} \int f_n$. Hence $(\forall \varepsilon > 0)$ $\exists N \ni$ $|\int f - \int f_N| < \frac{\varepsilon}{2}$. Hence, from now on, we'll find a continuous g w/ compact support $\ni$

(*) $\int |f_N - g| < \frac{\varepsilon}{2}$ b/c

$\int |f-g| \leq \int |f - f_n + f_N - g| \leq \int |f - f_N| + \int |f_N - g| < \varepsilon$.

So all we have to do is prove (*).

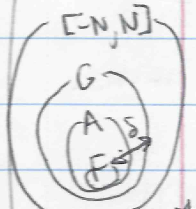
In short, we need to show $\exists$ a continuous $g \ni$ $\int |f_N - g| < \frac{\varepsilon}{2}$ where $f_N = f \chi_{[-N,N]}$ & $f \geq 0$, N is fixed.

Exam 2 12/1 → It's in your calendar.

Final 12/9

- Approximation Result cont.

Proof cont.



case 1) If $f_N = \chi_A$, where A is Lebesgue measurable in $[-N,N]$, then

$\exists$ G that's open & F that's closed $\ni F \subset A \subset G \subset [-N,N]$ & $m(G \setminus F) < \frac{\varepsilon}{2}$.

Since F is compact, $\exists\, \delta > 0 \ni \delta = \min(\text{dist}(F, G^c))$

*(Note F is compact b/c $[-N,N]$ is bdd., & F is closed)

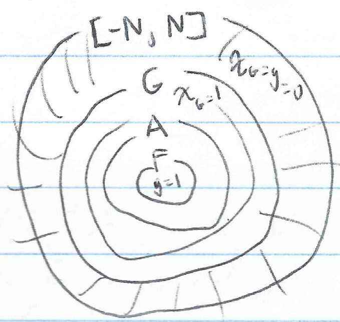
*(Also note, $(\min(F, G^c))$ exists b/c F is compact & G^c is closed. If F wasn't compact, we couldn't do this)

Define $g(x) = \left(1 - \frac{\text{dist}(x,F)}{\delta}\right)^+$, Then g is continuous & $0 \le g \le 1$, $g = 1$ if $x \in F$, $0 = g$ if $x \in G^c$

hard to see why, but makes sense.

ie $g = \begin{cases} 1 & x \in F \\ 0 & x \in G^c \end{cases}$ w/ compact support in $[-N,N]$

We know g & χ_A, χ_G are all positive b/c each is between 0 & 1. $g \le \chi_G$ b/c



Similarly $\chi_A \ge \chi_F$

$\Rightarrow \int |g - \chi_A| = \int g - \chi_A \le \int \chi_G - \chi_F = \int \chi_{(G \setminus F)}$

$= m(G \setminus F) < \frac{\varepsilon}{2}$ So we're done w/ case 1, b/c $\Rightarrow \int |g - \chi_A| < \frac{\varepsilon}{2} \Rightarrow$ ⊛ when $f_N = \chi_A$.

Case 2 on next page.

Ch8 hwrk: (1, 3, 5, 6)

Due 11/22

case 2) WTS ⊛⊛ when $f_N = \sum_{i=1}^{p} a_i \chi_{A_i}$ where $\forall i$, A_i are Lebesgue measurable in $[-N,N]$ & $a_i > 0$ $\forall i$ b/c we only consider the non-negative functions.

By case 1), we know that $\exists\, g_i$ that are continuous w/ compact support in $[-N,N]$ $\forall i \ni \int |\chi_{A_i} - g_i| < \frac{\varepsilon}{2a_i p}$.

Define $g = \sum_{i=1}^{p} g_i \rightarrow$ sum of continuous functions, so it's continuous, & it also has a compact support in $[-N,N]$.

$$\int |f_N - g| = \int \left| \sum_{i=1}^{p} a_i \chi_{A_i} - \sum_{i=1}^{p} a_i g_i \right| = \int \left| \sum_{i=1}^{p} a_i (\chi_{A_i} - g_i) \right|$$

$$\le \int \sum_{i=1}^{p} a_i |\chi_{A_i} - g_i| = \sum_{i=1}^{p} a_i \int |\chi_{A_i} - g_i|$$

$$< \sum_{i=1}^{p} a_i \frac{\varepsilon}{2a_i p} = \sum_{i=1}^{p} \frac{\varepsilon}{2p} = \frac{\varepsilon}{2} \Rightarrow \text{⊛⊛ is true when}$$

$f_N = \sum_{i=1}^{p} a_i \chi_{A_i}$.

case 3) Consider when $f_N = f \chi_{[-N,N]}$ where f is non-negative, Lebesgue measurable & integrable.

With this f, $\Rightarrow \exists \{S_n\}_{n=1}^{\infty}$ simple functions supported in $[-N,N]$ b/c we have $f\chi_{[-N,N]}$, $\ni$

$$\underset{\text{pointwise}}{S_n \uparrow f_N} \quad \& \quad \int S_n \nearrow \int f_N \leftarrow \text{(monotone convergence increasing)}$$

$$\Rightarrow \exists M \in \mathbb{N} \ni \underbrace{\int f_N - \int S_M}_{= \int |f_N - S_M| \text{ b/c they're always positive.}} < \frac{\varepsilon}{4}$$

For S_M, choose a continuous $g \ni \int |S_M - g| < \frac{\varepsilon}{4}$ by case 2).

$$\int |f_N - g| = \int |f_N - S_M + S_M - g| \le \int |f_N - S_M| + |S_M + -g|$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

QED

Ch 9 Riemann Integrals.

Notation

- Consider a function f on $[a,b] \to \mathbb{R}$ to be a bounded function.

  $\int f$ is a Lebesgue integral

  $R(f)$ is a Riemann integral.

- To define $R(f)$, introduce a partition $P = \{x_i\}_{i=0}^{n} \ni$

  $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ of $[a,b]$,

  Define $U(P,f) = \sum_{i=1}^{n} \sup_{x_{i-1} \le x \le x_i} f(x)(x_i - x_{i-1})$ (upper sum), &

  $L(P,f) = \sum_{i=1}^{n} \inf_{x_{i-1} \le x \le x_i} f(x)(x_i - x_{i-1})$

  Set $\overline{R}(f) = \inf\{U(P,f) \mid P: \text{partition of } [a,b]\}$ (over all partitions.

  $\underline{R}(f) = \sup\{L(P,f) \mid P: \text{partition of } [a,b]\}$

  If $\overline{R}(f) = \underline{R}(f)$, then the Riemann integral is defined

  $R(f) = \overline{R}(f) = \underline{R}(f)$.

- Thm: Let f be a function defined on $[a,b] \to \mathbb{R}$ & be bdd.

f is Riemann integrable iff the set of points where f is discontinuous has Lebesgue measure 0. In this case, f is Lebesgue measurable &

$R(f) = \int f$.

Proof:

$[\Rightarrow \ \& \ \int f = R(f)]$: $\forall$ partitions $P = \{x_i\}_{i=0}^{n}$ of $[a,b]$, define a simple function $T_p(x) = \sum_{i=1}^{n} \sup_{x_{i-1} \le y \le x_i} f(y) \chi_{[x_{i-1},x_i]}(x)$. We can define another simple function $S_p(x) = \sum_{i=1}^{n} \inf_{x_{i-1} \le y \le x_i} f(y) \chi_{[x_{i-1},x_i]}(x)$.

$\Rightarrow \int T_p = \sum_{i=1}^{n} \sup_{x_{i-1} \le y \le x_i} f(y)(x_i - x_{i-1}) = U(P,f)$ similarly,

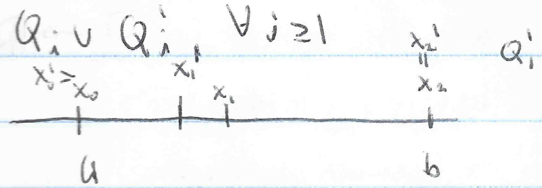
$\Rightarrow \int S_p = L(P,f)$.

Since f is Riemann integrable, $\exists$ a sequence of partitions $\{Q_i\}_{i=1}^{\infty} \ni$ $Q_1 \subset Q_2 \subset \dots$ & $U(Q_i,f) \downarrow R(f)$. Here we've used the fact that $Q_i \subset Q_j \Rightarrow U(Q_i,f) \ge U(Q_j,f)$ $\forall$ i & $j \ni i < j$. We can also find a $\{Q'_i\}_{i=1}^{\infty} \ni Q'_1 \subset Q'_2 \subset \dots$ & $L(Q'_i,f) \uparrow R(f)$ for the same reasons.

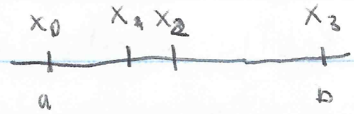
Proof cont.

Now define $P_i = Q_i \cup Q_i^{i}$, $\forall i \geq 1$

b/c,



So $P_1 = Q_1 \cup Q^2$



So $P_1 \subset P_2 \subset P_3 \subset \ldots$ & $U(P_i, f) \downarrow R(f)$ as $i \to \infty$

& $L(P_i, f) \uparrow R(f)$ as $i \to \infty$,

$\forall x$, $T_{P_i}(x)$ is a sequence of simple functions, ie,

$T_{P_1}(x) \geq T_{P_2}(x) \geq \ldots$ for any fixed point x.

ie) $\{T_{P_i}(x)\}_{i=1}^{\infty}$ is a decreasing sequence & bdd.

Hence by the Monotone convergence thm (bdd. & monotone) $\exists$

$T(x) \ni T_{P_i}(x) \to T(x)$ $\forall x \in [a,b]$.

Similarly, $\exists$ $S(x) \ni S_{P_i}(x) \to S(x)$ as $i \to \infty$ $\forall x \in [a,b]$

Moreover, $T_{P_i}(x) \geq f(x) \geq S_{P_i}(x)$ $\forall i$ & $x \in [a,b]$,

& $T(x) \geq f(x) \geq S(x)$ $\forall x \in [a,b]$.

But since f is bdd., the pointwise convergence of T_{P_i} & S_{P_i} is uniform (pointwise convergence to bdd. means uniform). $\forall \varepsilon > 0$, $\exists N$

$\ni |T_{P_i}(x) - T(x)| < \varepsilon$ if $i \geq N$ $\forall x \in [a,b]$. Similar for $S_{P_i}(x)$.

Now we notice that

i) $T_{P_i}(x) - S_{P_i}(x) \to T(x) - S(x)$ as $i \to \infty$ $\forall x \in [a,b]$.

ii) $|T_{P_i}(x) - S_{P_i}(x)| \leq |T_{P_i}(x)| + |S_{P_i}(x)| \leq 2 \sup_{a \leq x \leq b} |f| \chi_{[a,b]}$

& $\int (2 \sup_{a \leq x \leq b} |f| \chi_{[a,b]}) < \infty$,

so it's integrable.

By applying Dominant convergence Thm,

$\int \lim_{i \to \infty} (T_{P_i}(x) - S_{P_i}(x)) = \lim_{i \to \infty} \int (T_{P_i} - S_{P_i})$

$0 \leq \int (T - S)$ $\quad$ $\lim_{i \to \infty} (U(P_i, f) - L(P_i, f)) = 0$

$\Rightarrow \int (T - S) = 0$

cont.

$\Rightarrow$ by a Thm we proved before, $T - S = 0$ a.e.

$\Rightarrow f = T = S$ a.e. We know T is Lebesgue measurable b/c its the limit of sequence of simple functions, & we know $T = f$ a.e.

$\Rightarrow$ <u>f is Lebesgue measurable.</u>

Now to show f is continuous a.e., define $\mathcal{C} = \{a \le x \le b \mid T(x) = S(x) = f(x)\}$, & we will show that T is continuous on $\mathcal{C}$. (b/c T a.e.)

We will show this using undergrad def. of continuity from analysis.

Let $x_0 \in \mathcal{C}$, $\varepsilon > 0$,

i) choose $N \ni |T(y) - T_{P_i}(y)| < \frac{\varepsilon}{3}$ $\forall y \in [a,b]$ if $i \ge N$ by the uniform convergence of $T_{P_i} \to T$ on $[a,b]$.

ii) Choose $\delta > 0$ small enough $\ni (x_0 - \delta, x_0 + \delta) \subseteq [x_{i_0-1}, x_{i_0}]$ for some $x_{i_0-1}, x_{i_0} \in P_N$.

Then we write
$$|T(x) - T(x_0)| \le |T(x) - T_{P_N}(x)| + |T_{P_N}(x) - T_{P_N}(x_0)| + |T_{P_N}(x_0) - T(x_0)| < \frac{\varepsilon}{3} + 0 + \frac{\varepsilon}{3} < \varepsilon$$

So <u>f is cont. a.e.</u>

Now we notice that

i) $T_{P_i}(x) \to T(x) \Rightarrow T_{P_i} \to T = f$ a.e.

ii) $|T_{P_i}| \le \sup_{a \le x \le b} |f| \, \chi_{[a,b]}$: integrable, so by dominant convergence thm,

$$\int f = \int T = \int \lim_{i\to\infty} T_{P_i} = \lim_{i\to\infty} \int T_{P_i} = \lim_{i\to\infty} U(P_i, f) = R(f)$$

thus $\underline{\int f = R(f)}$

This is one direction... now we have $\Leftarrow$.

($\Leftarrow$ & $R(f) = \int f$)...

Assume f is measurable & cont. a.e. Let P_i be the partition of $[a,b]$ into 2^i many sub intervals of the same size. Then we can construct our $T_{P_i} \downarrow f$ & $S_{P_i} \uparrow f$ as before.

we will finish next time

$[\Leftarrow$ & $R(f)=\int f]$ restarting

Assume f is cont. a.e. $\Rightarrow \exists$ a continuous function $h \ni f=h$ a.e.

$\ni P_i = \{x_j\}_{j=0}^{2^i}$ Define a partition $P_i \; \forall i$ of $[a,b]$ including 2^i subintervals of the same size. Define $T_{P_i}(x) = \sum \sup_{x_{i-1}\le y\le x_i} h(y)\,\chi_{[x_{i-1},x_i]}$

$$S_{P_i}(x) = \sum \inf_{x_{i-1}\le y\le x_i} h(y)\,\chi_{[x_{i-1},x_i]}$$

$\Rightarrow T_{P_i} \downarrow h$ & $S_{P_i} \uparrow h$ uniformly in $a \le x \le b$ as $n\to\infty$

$\Rightarrow T_{P_i} \downarrow f$ a.e. & $S_{P_i} \uparrow f$ a.e. uniformly in X as $n \to \infty$.

$U(P_i,f) = \int T_{P_i}$, since $T_{P_i} \to f$ a.e. & $|T_{P_i}| \le \sup_{a\le h\le b} |h|\,\chi_{[a,b]}$ & $\int \sup_{a\le h\le b} |h|\,\chi_{[a,b]} < \infty$. Hence by the Dominant Convergence Thm, $\lim_{i\to\infty} U(P_i,f) = \lim_{i\to\infty}\int T_{P_i} = \int f$.

Similarly, $\lim_{i\to\infty} L(P_i,f) = \int f \Rightarrow R(f) = \int f$.

QED.

- Summarizing what we've got -

- Fatou's Lemma: very powerful, but not most convenient b/c of inequality.
- Monotone Convergence Thm: $0 \le f_1 \le f_2 \le \dots$ has to be monotone & positive
- dominant convergence thm DCT: have to find dominant function
- Generalized DCT: easier to use this in application than regular DCT.

Ch 10 - Types of Convergence

• Definitions:

~~Let M be a measure, we say a sequence of functions f_n~~ Go kill yourself

- Suppose we have a sequence of functions $\{f_n : \mathbb{R} \to \mathbb{R}\}_{n=1}^{\infty}$. We say

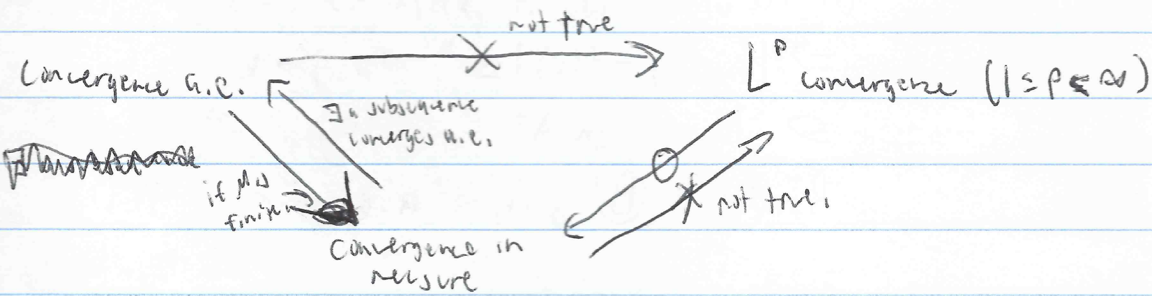
i) f_n converges almost everywhere to f, $f_n \to f$ a.e., if $\exists$ a set of measure 0 $\ni$ for x not in this set, $f_n(x) \to f(x)$.

ii) Suppose M is a measure, we say f_n converges in measure M to f if $\forall\, \varepsilon > 0$, $M(\{x \mid |f_n(x) - f(x)| > \varepsilon\}) \to 0$ as $n \to \infty$.

P should be integrable.

iii) $\forall\, p \ni 1 \le p < \infty$, we say $f_n \to f$ in L^p if $\int |f - f_n|^p \to 0$ as $n \to \infty$

⊖ Remark: $L^p(\mathbb{R}) = \{f \text{ measurable} \ni \int_{\mathbb{R}} |f|^p < \infty\}$



• Proof of these implications:

- Proposition: If M is finite, $f_n \to f$ a.e. $\Rightarrow$ $f_n \to f$ in measure. where our measure space is $(X, \mathcal{A}, M)$

Proof: $\forall\, \varepsilon > 0$, define $A_n = \{x \mid |f_n(x) - f(x)| > \varepsilon\}$. Then we need to show $M(A_n) \to 0$ as $n \to \infty$. Since $f_n \to f$ a.e., consider χ_{A_n} $\Rightarrow$ $\chi_{A_n}(x) \to 0$ a.e. as $n \to \infty$. Consider $|\chi_{A_n}| \le 1$, $\int 1 \, dM < M(X) < \infty$

Hence by the DCT, $\lim_{n\to\infty} \int \chi_{A_n} \, dM = \int \lim_{n\to\infty} \chi_{A_n} \, dM = \int 0 = 0$

$\lim_{n\to\infty} M(A_n) \Rightarrow \lim_{n\to\infty} M(A_n) = 0$ So we're done

More on next page.

- **Proposition:** Assume $f_n \to f$ in measure μ. Show that this $\Rightarrow \exists\, n_j \ni f_{n_j} \to f$ a.e.

Proof: $\forall \varepsilon > 0$, $\mu(\{x \mid |f_n(x) - f(x)| > \varepsilon\}) \to 0$ as $n \to \infty$.

Let's define $A_j = \{x \mid |f_{n_j}(x) - f(x)| > \frac{1}{j}\}$ $\forall j$. Choose

when $k=1$ $\bigcup_{j=1}^{\infty} A_j$

$n_j \ni \mu(\{x \mid |f_{n_j}(x) - f(x)| > \frac{1}{j}\}) < \frac{1}{2^j}$

$= \mu(A_j) < \frac{1}{2^j}$. Now define $A = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j$

$k=2$ $\bigcup_{j=2}^{\infty} A_j$ $\to$ so we have a decreasing sequence b/c

$\bigcup_{j=1}^{\infty} A_j \supset \bigcup_{j=2}^{\infty} A_j \supset \bigcup_{j=3}^{\infty} A_j \supset \dots$

$\mu(A) = \mu(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j) = \lim_{k\to\infty} \mu(\bigcup_{j=k}^{\infty} A_j) \le \lim_{k\to\infty} \sum_{j=k}^{\infty} \mu(A_j)$

$< \lim_{k\to\infty} \sum_{j=k}^{\infty} \frac{1}{2^j} = \lim_{k\to\infty} \frac{1/2^k}{1-\frac{1}{2}} = \lim_{k\to\infty} \frac{1}{2^{k-1}} = 0$

$\Rightarrow \mu(A) = 0$ ---- (A)

On the other hand, if $x \notin A \Rightarrow x \notin \bigcup_{j=k}^{\infty} A_j$ for some k.

$\Rightarrow |f_{n_j}(x) - f(x)| \le \frac{1}{j}$ for $j \ge k$

$\Rightarrow f_{n_j} \to f$ on A^c ---- (B)

By (A) & (B), $f_{n_j} \to f$ a.e. QED.

- In case μ is not finite, convergence a.e. $\Rightarrow$ convergence in measure?

No, counter ex.:

$X = \mathbb{R}$, $\mu = m$: Lebesgue measure. Consider a sequence $f_n = \chi_{(n,n+1)}$

$\forall n$, $f_n(x) \to 0$ as $n \to \infty$ $\forall x$.

For $\varepsilon = \frac{1}{2}$, $\mu(\{x \mid |f_n(x) - f(x)| > \frac{1}{2}\}) = m((n, n+1)) = 1 \not\to 0$.

QED.

Chebyshev's Inequality

Lemma: $1 \le p < \infty$,
$\mu(\{x \mid |f(x)| \ge a\}) \le \frac{\int |f|^p d\mu}{a^p}$

~~Proposition~~: Chebyshev's Inequality

Lemma: $1 \le p < \infty$. $\mu(\{x \mid |f(x)| \ge a\}) \le \frac{\int |f|^p d\mu}{a^p}$

Proof: Define $A = \{x \mid |f(x)| \ge a\}$.

$\chi_A \le \frac{|f|^p}{a^p} \chi_A$

Then we have $\mu(A) \le \int_A \frac{|f|^p}{a^p} d\mu \le \frac{1}{a^p} \int |f|^p d\mu$.

Proposition: $f_n \to f$ in L^p $\Rightarrow$ $f_n \to f$ in measure.

Proof: $\forall \varepsilon > 0$, $\mu(\{x \mid |f_n(x) - f(x)| > \varepsilon\})$

$\le \frac{\int |f_n - f|^p}{\varepsilon^p}$ by Chebyshev's inequality.

but $\frac{\int |f_n - f|^p}{\varepsilon^p} \to 0$ as $n \to \infty$ because $f_n \to f$ in L^p.

$\Rightarrow f_n \to f$ in measure.

Ex) $f_n \to f$ a.e. $\overset{?}{\not\Rightarrow}$ $f_n \to f$ in L^p

$f_n \to f$ in measure $\not\Rightarrow$

Counterexamples: Define $f_n = n^2 \chi_{(0, \frac{1}{n})}$

$\mu = m$: Lebesgue measure on $\mathbb{R}$. We notice that

i) $f_n \to 0$ pointwise a.e.

ii) $\forall \varepsilon > 0$, $m(\{x \mid |f_n - 0| > \varepsilon\}) = m((0, \frac{1}{n})) = \frac{1}{n}$

$\Rightarrow f_n \to 0$ in measure.

iii) $\forall 1 \le p < \infty$, $\int |f_n - 0|^p = \int n^{2p} \chi_{(0, \frac{1}{n})} = n^{2p} \frac{1}{n} = n^{2p-1}$

$\to \infty$ as $n \to \infty$

Thus $f_n \not\to 0$ in L^p. Thus we have found our counterex. to prove this false.

• Egorov's Thm: (Eggroll's thm)

Suppose μ is a finite measure on $(X, \mathcal{A})$, $f_n \to f$ a.e., then $\forall\ \varepsilon > 0$, $\exists\ A \in \mathcal{A} \ni \mu(A) < \varepsilon$ &

$f_n \to f$ uniformly on A^c.

Proof:

Define $A_{n,k} = \bigcup_{m=n}^{\infty} \{x \mid |f_m(x) - f(x)| > \frac{1}{k}\}$

$A_{1,k} \supset A_{2,k} \supset \dots \ \forall k$.

Since $f_n \to f$ a.e. $\forall$ fixed x & k, $\exists$

$N = N(x,k) \ni |f_n(x) - f(x)| < \frac{1}{k}$ if $m \geq N$.

$\Rightarrow \mu(\bigcap_{n=1}^{\infty} A_{n,k}) = 0$

$\Rightarrow \lim_{n \to \infty} \mu(A_{n,k}) = 0$

$\exists \{n_k\}$ of $\{n\} \ni \mu(A_{n_k,k}) < \frac{\varepsilon}{2^k}$

Set $A = \bigcup_{k=1}^{\infty} A_{n_k,k}$

Then $\mu(A) = \mu(\bigcup_{k=1}^{\infty} A_{n_k,k}) \leq \sum_{k=1}^{\infty} \mu(A_{n_k,k})$

$< \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$ Thus $\mu(A) < \varepsilon$. Now we just need to show the uniform convergence part.

If $x \in A^c$ $(x \notin A)$, then $x \notin A_{n_k,k}\ \forall k$.

$\Rightarrow |f_m(x) - f(x)| \leq \frac{1}{k}$ by definition of $A_{n,k}$ if $m \geq n_k$

$\Rightarrow f_n \to f$ uniformly on A^c by definition of uniform convergence.

QED