MATH 667 EXAM 1 (September 28, 2017: 2:30pm–3:45pm) NAME KEY

The exam is closed book; students are permitted to prepare one 8.5×11 page of formulas, notes, etc. that can be used during the exam. A calculator is permitted but not necessary for the exam. Do 4 out of the 5 problems (10 points each, 40 points total). Clearly indicate the problem that you are omitting; if it is not clear, then the first 4 problems will be graded.

Problem 1. (10 points) Let $X_1,\ldots,X_n$ be independent random variables such that $X_j \sim \text{Normal}(\mu, j)$ has probability density function

$$f_j(x|\mu) = \frac{1}{\sqrt{2\pi j}}e^{-\frac{1}{2j}(x-\mu)^2}$$

for $j = 1,\ldots,n$ where $\mu \in \mathbb{R}$ is unknown. (Here we are assuming that the variance for X_j is known to be j.)

(a - 2 pts) What is the joint probability density function of $X_1,\ldots,X_n$?

(b - 7 pts) Find the maximum likelihood estimator of μ.

(c - 1 pt) What is the maximum likelihood estimator of $\sin(\mu)$?

(a) $$f(x_1,\ldots,x_n|\mu) = \prod_{j=1}^{n} f_j(x_j|\mu) = \prod_{j=1}^{n}\frac{1}{\sqrt{2\pi j}}e^{-\frac{1}{2j}(x_j-\mu)^2} = \boxed{\frac{1}{(2\pi)^{n/2}}\cdot\frac{1}{\sqrt{n!}}\,e^{-\frac{1}{2}\sum_{j=1}^{n}\frac{(x_j-\mu)^2}{j}}}$$

(b) The log-likelihood function is

$$\ell(\mu) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln n! - \frac{1}{2}\sum_{j=1}^{n}\frac{(x_j-\mu)^2}{j}$$

Differentiating this function of μ, we obtain

$$\ell'(\mu) = -\frac{1}{2}\cdot\sum_{j=1}^{n}\frac{2(x_j-\mu)}{j}(-1) = \sum_{j=1}^{n}\frac{x_j}{j} - \sum_{j=1}^{n}\frac{\mu}{j} = \sum_{j=1}^{n}\frac{x_j}{j} - \mu\sum_{j=1}^{n}\frac{1}{j}.$$

Solving $\ell'(\mu)=0$, we obtain

$$\boxed{\hat{\mu} = \frac{\sum_{j=1}^{n}\frac{x_j}{j}}{\sum_{j=1}^{n}\frac{1}{j}}}$$

sign of ℓ'

$+ \quad -$

$\hat{\mu}$

since ℓ' is a decreasing linear function

which maximizes ℓ since $\ell(\mu)$ is increasing if $\mu<\hat{\mu}$ and decreasing if $\mu>\hat{\mu}$.

(c) By the invariance property of the MLE,

$$\widehat{\sin\mu} = \boxed{\sin\hat{\mu}}.$$

Problem 2. (10 points) Let $Y_1, \dots, Y_5$ be a random sample from a normal population with mean 3 and variance 2.

(a - 2 pts) Compute $\mathrm{E}\left[\sum_{i=1}^{5} Y_i\right]$.

(b - 2 pts) Compute $\mathrm{Var}\left[\sum_{i=1}^{5} Y_i\right]$.

(c - 3 pts) Compute $P\left(S^2 \le 2.5\right)$ where $S^2 = \frac{1}{4}\sum_{i=1}^{5}(Y_i - \bar{Y})^2$ and $\bar{Y} = \frac{1}{5}\sum_{i=1}^{5} Y_i$.

(d - 3 pts) Compute $P\left(V^2 \le 10\right)$ where $V^2 = \sum_{i=1}^{5}(Y_i - 3)^2$.

If needed, use the standard normal and/or χ^2 table attached to this exam and/or use the normal pdf $f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ and/or the chi-squared pdf $f(x|p) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2} I_{(0,\infty)}(x)$ where p is the degrees of freedom.

(a) $E\left[\sum_{i=1}^{5} Y_i\right] = \sum_{i=1}^{5} E[Y_i] = \sum_{i=1}^{5} 3 = 5 \cdot 3 = \boxed{15}$

(b) Since $Y_1, \dots, Y_5$ are independent,

$$\mathrm{Var}\left[\sum_{i=1}^{5} Y_i\right] = \sum_{i=1}^{5} \mathrm{Var}[Y_i] = \sum_{i=1}^{5} 2 = 5 \cdot 2 = \boxed{10}$$

(c) $P(S^2 \le 2.5) = P\left(\frac{4S^2}{2} \le 5\right)$

Let $Q = \frac{(n-1)S^2}{\sigma^2} = \frac{4S^2}{2}$, We know $Q \sim \chi^2_4$ so

$$P(Q \le 5) = 1 - P(Q > 5) = 1 - .29 = \boxed{.71}$$

from χ^2-table with df=4

(d) $P(V^2 \le 10) = P\left(\sum_{i=1}^{5}\left(\frac{Y_i - 3}{\sqrt{2}}\right)^2 \le \frac{10}{2}\right)$

Since $Z_i = \frac{Y_i - 3}{\sqrt{2}}$, $i = 1, 2, 3, 4, 5$ are independent Normal(0,1),

$\sum_{i=1}^{5} Z_i^2 = \sum_{i=1}^{5}\left(\frac{Y_i - 3}{\sqrt{2}}\right)^2 \sim \chi^2_5$ so

$$P\left(\sum_{i=1}^{5}\left(\frac{Y_i - 3}{\sqrt{2}}\right)^2 \le 5\right) = 1 - .42 = \boxed{.58}$$

from χ^2-table with df=5

Problem 3. (10 points) Suppose that Z_1 and Z_2 are independent identically distributed normal$(0, 1)$ random variables with probability density function

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

Let $\bar{Z} = \dfrac{Z_1 + Z_2}{2}$ and $S_Z^2 = \sum_{i=1}^{2} (Z_i - \bar{Z})^2$.

(a - 5 pts) Find the joint probability density function of $U_1 = \bar{Z}$ and $U_2 = Z_2 - \bar{Z}$.

(b - 5 pts) Prove that $\bar{Z}$ and S_Z^2 are independent.

(a) $\left.\begin{array}{l} u_1 = \frac{z_1+z_2}{2} \\ u_2 = z_2 - \frac{z_1+z_2}{2} = \frac{z_2 - z_1}{2} \end{array}\right\} \Rightarrow \begin{array}{l} z_1 = u_1 - u_2 \\ z_2 = u_1 + u_2 \end{array}$

$$f_{U_1,U_2}(u_1,u_2) = f_{Z_1,Z_2}(u_1-u_2, u_1+u_2)\,|J| \quad \text{where } J = \begin{vmatrix} \frac{\partial z_1}{\partial u_1} & \frac{\partial z_1}{\partial u_2} \\ \frac{\partial z_2}{\partial u_1} & \frac{\partial z_2}{\partial u_2} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1-(-1) = 2$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u_1-u_2)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u_1+u_2)^2} \cdot 2$$

$$= \frac{1}{\pi} e^{-\frac{1}{2}u_1^2 + u_1u_2 - \frac{1}{2}u_2^2 - \frac{1}{2}u_1^2 - u_1u_2 - \frac{1}{2}u_2^2}$$

$$= \boxed{\frac{1}{\pi} e^{-u_1^2} e^{-u_2^2}} = \left(\frac{1}{\sqrt{\pi}} e^{-u_1^2}\right)\left(\frac{1}{\sqrt{\pi}} e^{-u_2^2}\right).$$

(b) Since the joint pdf is the product of a function of u_1 and of a function of u_2, U_1 and U_2, U_1 and U_2 are independent. Consequently,

since $\bar{Z} = U_1$ (function of U_1) and

$$S_Z^2 = (Z_1 - \bar{Z})^2 + (Z_2 - \bar{Z})^2$$
$$= \left(\frac{Z_1 - Z_2}{2}\right)^2 + \left(\frac{Z_2 - Z_1}{2}\right)^2$$
$$= (-U_2)^2 + U_2^2 = 2U_2^2 \text{ is a function of } U_2,$$

$\bar{Z}$ and S_Z^2 are independent.

Problem 4. (10 points) Let X be a random variable with probability mass function

$$f_X(x|p) = P(X = x) = \begin{cases} p(1-p)^x & \text{if } x \text{ is a nonnegative integer} \\ 0 & \text{otherwise} \end{cases}.$$

(a - 7 pts) A family of probability density functions is called an exponential family if it can be expressed as

$$f(x|\theta) = h(x)c(\theta)\exp\{w(\theta)t(x)\}.$$

Is $\{f_X(x|p)\}$ an exponential family? If yes, define θ and find $h(x)$, $c(\theta)$, $w(\theta)$, and $t(x)$. If not, justify your answer.

(b - 3 pts) What is E[X]?

(a) $f_X(x|p) = I_{\mathbb{Z}^+}(x)\, p\, e^{\ln(1-p)^x} = I_{\mathbb{Z}^+}(x)\, p\, e^{x\ln(1-p)}$ where $\mathbb{Z}^+$ is the set of nonnegative integers

Yes, $\{f_X(x|p)\}$ is an exponential family where $\theta = p$, $h(x) = I_{\mathbb{Z}^+}(x)$, $c(\theta) = \theta$, $w(\theta) = \ln(1-\theta)$, and $t(x) = x$.

(b) $E[w(\theta)t(X)] = -\frac{d}{d\theta}[\ln c(\theta)]$

$\Downarrow$

$E\left[\frac{-1}{1-\theta} X\right] = -\frac{1}{\theta}$

$\Downarrow$

$-\frac{1}{1-\theta} E[X] = -\frac{1}{\theta}$

$\Downarrow$

$E[X] = \frac{1-\theta}{\theta} = \boxed{\frac{1-p}{p}}$

Problem 5. (10 points) Suppose $X_1, \ldots, X_n$ are independent identically distributed random variables each with probability density function

$$f(x|\mu,\sigma) = \frac{3}{2\pi\left(1+\left(\frac{x-\mu}{\sigma}\right)^6\right)}, -\infty < x < \infty$$

where the parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown. Find the method of moments estimator of (μ, σ). It can be shown that $\mathrm{E}[X]$, $\mathrm{E}[X^2]$, and $\mathrm{Var}[X]$ exist; you can use this fact without proof.

This is a location-scale family so there is a random variable Z such that $X = \mu + \sigma Z$ where Z has pdf $f(z|0,1)$.

$$\int_{-\infty}^{0} \frac{z}{1+z^6}\,dz \overset{u=-z}{=} \int_{\infty}^{0} \frac{-u}{1+u^6}(-du) = -\int_0^{\infty} \frac{u}{1+u^6}\,du$$

We have
$$E[Z] = \int_{-\infty}^{\infty} z \cdot f(z|0,1)\,dz$$
$$= \frac{3}{2\pi}\left[\int_{-\infty}^{0} \frac{z}{1+z^6}\,dz + \int_0^{\infty} \frac{z}{1+z^6}\,dz\right] = 0$$ since $f(z|0,1)$ is symmetric about 0 and $E[Z^2]$ exists,

and
$$E[Z^2] = \int_{-\infty}^{\infty} z^2 f(z|0,1)\,dz$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{3z^2}{1+(z^3)^2}\,dz$$
$$= \frac{1}{2\pi}\Big[\arctan(z^3)\Big]_{-\infty}^{\infty} = \frac{1}{2\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{1}{2\pi}\pi = \frac{1}{2}$$

so $E[X] = \mu + \sigma \cdot 0 = \mu$ and $\mathrm{Var}[X] = \sigma^2 \mathrm{Var}[Z] = \frac{\sigma^2}{2}$

$$\Downarrow$$

$$E[X^2] = \mathrm{Var}[X] + (E[X])^2 = \frac{\sigma^2}{2} + \mu^2.$$

The method of moments estimator is the solution to the system

$$\tilde{\mu} = \bar{X}$$
$$\frac{\tilde{\sigma}^2}{2} + \tilde{\mu}^2 = \frac{\sum X_i^2}{n}$$

so

$$\tilde{\mu} = \bar{X}$$
$$\tilde{\sigma}^2 = 2\left(\frac{\sum_{i=1}^n X_i^2}{n} - \tilde{\mu}^2\right) = 2\left(\frac{\sum X_i^2 - n\bar{X}^2}{n}\right) = \frac{2}{n}\sum_{i=1}^n (X_i - \bar{X})^2.$$