MATH 668-01 Homework 4

Due: Thursday, March 8, 2018

Instructions: *Each of the following problems must be submitted to the instructor on or before the due date. Partial credit may be given for incorrect answers which make some positive progress. Late homework will not be accepted.*

1. (10 points) Consider the regression model $\boldsymbol{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \beta_3\boldsymbol{x} + \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon} \sim N_{10}(\mathbf{0}, \sigma^2\mathbf{I})$, $\mathbf{X}_1$ is a 10×3 full rank matrix, and $\boldsymbol{x}$ is a 10-dimensional column vector with unit length (i.e., $\|\boldsymbol{x}\|^2 = \boldsymbol{x}^\top\boldsymbol{x} = 1$). Suppose $\boldsymbol{x}^\top\boldsymbol{y} = \dfrac{\boldsymbol{x}^\top\boldsymbol{y}}{\boldsymbol{x}^\top\boldsymbol{x}} = 0.70$, $\boldsymbol{u} = \mathbf{X}_1^\top\boldsymbol{x} = \dfrac{\mathbf{X}_1^\top\boldsymbol{x}}{\boldsymbol{x}^\top\boldsymbol{x}} = \begin{pmatrix} 0.65 \\ 3.10 \\ 1.78 \end{pmatrix}$, and

$$\left((\mathbf{X}_1 - \boldsymbol{x}\boldsymbol{u}^\top)^\top(\mathbf{X}_1 - \boldsymbol{x}\boldsymbol{u}^\top)\right)^{-1}(\mathbf{X}_1 - \boldsymbol{x}\boldsymbol{u}^\top)^\top(\boldsymbol{y} - 0.70\boldsymbol{x}) = \begin{pmatrix} 0.07 \\ -0.05 \\ 0.02 \end{pmatrix}.$$

(a - 5 pts) Compute the least squares estimate of $\boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta}_1 \\ \beta_3 \end{pmatrix}$.

(b - 5 pts) Also suppose $\mathbf{X}_1^\top\boldsymbol{y} = \begin{pmatrix} 1.16 \\ 1.35 \\ 2.16 \end{pmatrix}$ and $\boldsymbol{y}^\top\boldsymbol{y} = 0.80$. Test $H_0 : \boldsymbol{\beta}_1 = \mathbf{0}$ versus $H_1 : \boldsymbol{\beta}_1 \neq \mathbf{0}$ at level .05.

2. (10 points) To find the maximum yield y for a chemical reaction, an experimenter chose values of the following variables:

$$\begin{aligned} x_1 &= \text{temperature in Fahrenheit} \\ x_2 &= \text{percentage concentration of reagent} \\ x_3 &= \text{reaction time in hours.} \end{aligned}$$

The data is stored in the file "YieldData.txt". Consider the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i1}^2 + \beta_5 x_{i2}^2 + \beta_6 x_{i3}^2 + \beta_7 x_{i1}x_{i2} + \beta_8 x_{i1}x_{i3} + \beta_9 x_{i2}x_{i3} + \varepsilon_i$$

for $i = 1, \ldots, n$ where $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^\top \sim N(\mathbf{0}, \sigma^2\mathbf{I})$.
(a - 5 pts) Compute the maximum likelihood estimates of $\beta_0, \beta_1, \ldots, \beta_9$ and σ^2.
(b - 5 pts) Perform an F-test of $H_0 : \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = \beta_9 = 0$ at level .05.

3. (10 points) Let $\mathbf{X} = (\boldsymbol{j}, \mathbf{X}_1)$ be a matrix of rank $p+1$ where $\mathbf{X}_1$ is an $n \times p$ matrix, and let $\mathbf{C} = (\mathbf{0}_p, \mathbf{I}_p)$.

(a - 4 pts) Show that $(\mathbf{C}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{C}^\top)^{-1} = \mathbf{X}_1^\top(\mathbf{I} - \frac{1}{n}\mathbf{J})\mathbf{X}_1$ where $\mathbf{J} = \boldsymbol{j}\boldsymbol{j}^\top$.

(b - 4 pts) Show that $\mathbf{X}^\top(\mathbf{I} - \frac{1}{n}\mathbf{J})\mathbf{X} = \begin{pmatrix} 0 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{X}_1^\top(\mathbf{I} - \frac{1}{n}\mathbf{J})\mathbf{X}_1 \end{pmatrix}$.

(c - 2 pts) Let $\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\boldsymbol{\beta}}_1 \end{pmatrix}$ where $\hat{\boldsymbol{\beta}}_1$ is a p-dimensional vector. Show that

$$(\mathbf{C}\hat{\boldsymbol{\beta}})^\top(\mathbf{C}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{C}^\top)^{-1}\mathbf{C}\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}^\top\mathbf{X}^\top(\mathbf{I} - \frac{1}{n}\mathbf{J})\mathbf{X}\hat{\boldsymbol{\beta}}.$$