Cauchy functional equations

January 12, 2017

Cauchy functional equations

Ron Sahoo

Department of Mathematics

University of Louisville, Louisville, Kentucky 40292 USA

January, 2017

Cauchy functional equations

The functional equations

$$f(x+y) = f(x) + f(y)$$

$$f(x+y) = f(x)\,f(y)$$

$$f(xy) = f(x) + f(y)$$

$$f(xy) = f(x)\,f(y)$$

for all $x, y \in \mathbb{R}$ are called Cauchy functional equations. Here $f : \mathbb{R} \to \mathbb{R}$ is an unknown function.

The first functional equation, namely

$$f(x+y) = f(x) + f(y) \tag{1}$$

for all x, $y \in \mathbb{R}$ is known as the additive Cauchy functional equation.

The functional equation (1) was first studied by A.M. Legendre (1791) and C.F. Gauss (1809) but A.L. Cauchy (1821) first found its general continuous solution.

What are functional equations?

- Functional equations are equations in which the unknowns are functions.
- Functional equations is a field of mathematics which is over 270 years old.
- Functional equations appeared in the literature around the same time as the modern theory of functions.

- **The field of functional equations includes differential equations, difference equations, integral equations, and iterations.**

- To solve a functional equation means to find the unknown function (or functions).

- **In order to find solutions, the unknown functions must often be restricted to a specific nature (such as bounded, continuous, differentiable).**

Definition 1 . *A function* $f : \mathbb{R} \to \mathbb{R}$ *is said to be an additive function if it satisfies the additive Cauchy functional equation* $f(x+y) = f(x) + f(y)$ *for all* $x, y \in \mathbb{R}$.

Definition 2 . *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called a linear function if and only if it is of the form* $f(x) = c\,x$ *where* c *is any real constant.*

- The graph of a linear function $f(x) = c\,x$ is a non-vertical line that passes through the origin.

- The linear functions satisfy the additive Cauchy equations.

- Are there any other functions that satisfy the additive Cauchy equation?

- We begin by showing that the only continuous solutions of the additive Cauchy equation are those which are linear.

This theorem was first proved by A. L. Cauchy in 1821.

Theorem 1 . *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Cauchy equation*

$$f(x+y) = f(x) + f(y), \quad \textit{for all } x, y \in \mathbb{R}.$$

Then f is linear, that is $f(x) = m\,x$, where m is a real constant.

Proof. The Fundamental Theorem of Calculus says that if f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is continuous and differentiable on (a, b), and $g'(x) = f(x)$.

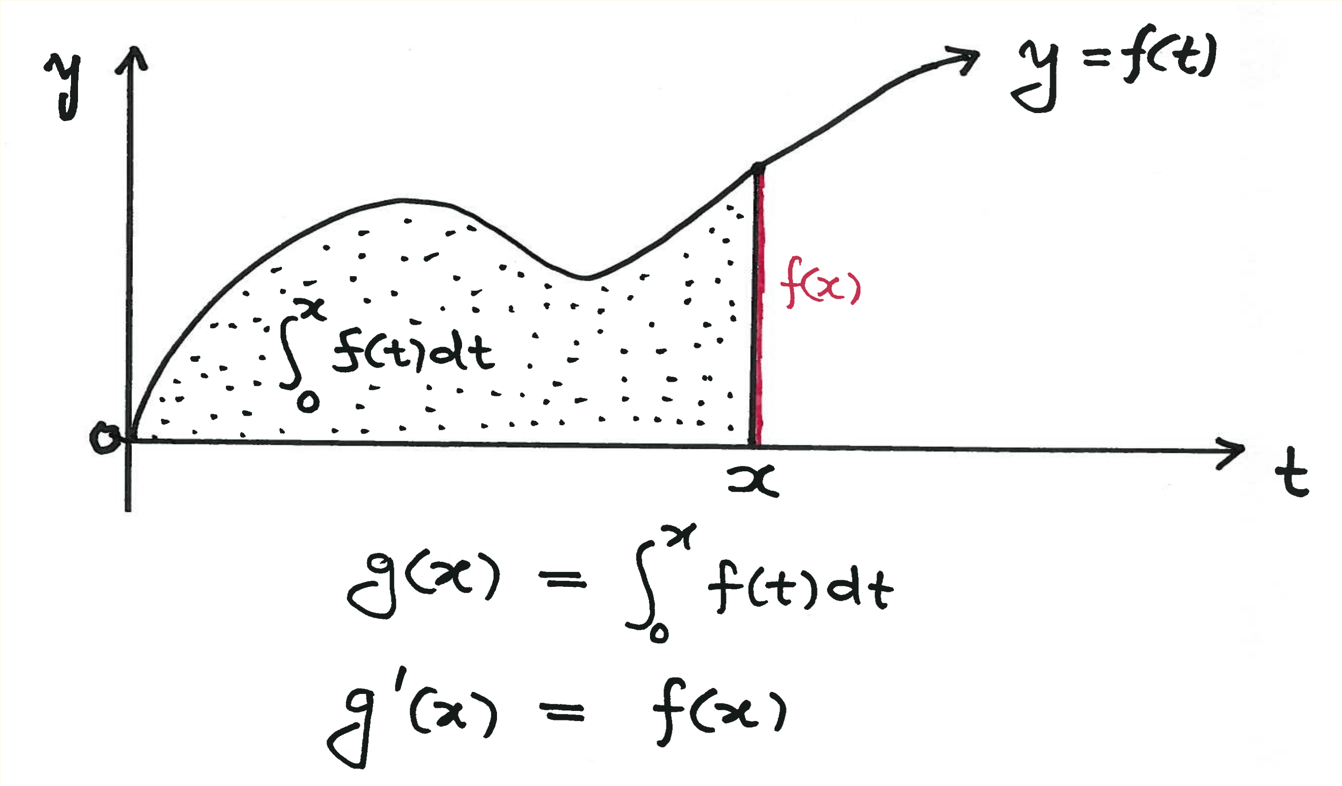

Illustration of Fundamental Theorem of Calculus

First, let us fix x and then integrate both side of the Cauchy equation with respect to y from 0 to 1 to get

$$
\begin{aligned}
f(x) &= \int_0^1 f(x)\,dy \\
&= \int_0^1 \left[f(x+y) - f(y)\right] dy \\
&= \int_x^{1+x} f(u)\,du - \int_0^1 f(y)\,dy \\
&= \int_d^{1+x} f(u)\,du - \int_d^x f(u)\,du - \int_0^1 f(y)\,dy.
\end{aligned}
$$

Hence

$$f(x) = \int_d^{1+x} f(u)\,du - \int_d^x f(u)\,du - \int_0^1 f(y)\,dy.$$

Since f is continuous, by the FTC, we get

$$\begin{aligned} f'(x) &= f(1+x) - f(x) \\ &= f(1) + f(x) - f(x) \\ &= f(1) \\ &= m \text{ (say)}. \end{aligned}$$

Therefore solving the differential equation,

$$\frac{d}{dx}f(x) = m$$

we get

$$f(x) = m\,x + b,$$

where d is a constant of integration. It can be easily shown that $b = 0$ and thus $f(x) = mx$. This completes the proof of the theorem.

• Notice that in Theorem 1, we use the continuity of f to conclude that f is also integrable.

• The integrability of f forced the solution f of the additive Cauchy equation to be linear.

• Thus every integrable solution of the additive Cauchy equation is also linear.

Definition 3. **A function $f : \mathbb{R} \to \mathbb{R}$ is said to be locally integrable if and only if it is integrable over every finite interval.**

Finite interval means interval of finite length. It is known that every locally integrable solution of the additive Cauchy equation is also linear.

We give as a short proof of this using an argument provided by Shapiro (1973).

Assume f is a locally integrable solution of the additive Cauchy equation. Hence $f(x + y) = f(x) + f(y)$ holds for all x and y in $\mathbb{R}$.

Since f is additive and locally integrable, we get

$$\begin{aligned} y\,f(x) &= \int_0^y f(x)dz \\ &= \int_0^y [f(x+z) - f(z)]\,dz \\ &= \int_x^{x+y} f(u)du - \int_0^y f(z)dz \\ &= \int_0^{x+y} f(u)du - \int_0^x f(u)du - \int_0^y f(u)du. \end{aligned}$$

The right side of the above equality is invariant under the interchange of x and y. Hence it follows that

$$y\,f(x) = x\,f(y)$$

for all $x, y \in \mathbb{R}$. Therefore, for $x \neq 0$, we obtain

$$\frac{f(x)}{x} = c,$$

where c is an arbitrary constant.

The last equality implies that $f(x) = cx$ for all $x \in \mathbb{R} \setminus \{0\}$.

Letting $x = 0$ and $y = 0$ in (1), we get $f(0) = 0$.

Together with this and the above, we conclude that f is a linear function in $\mathbb{R}$.

This completes the proof.

Although the proof of Theorem 1 is brief and involves only calculus, this proof is not very instructive.

We will present now a different proof which will help us to understand the behavior of the solution of the additive Cauchy equation a bit more.

We begin with the following definition.

Definition 4 . *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be rationally homogeneous if and only if*

$$f(rx) = r\,f(x) \tag{2}$$

for all $x \in \mathbb{R}$ and all rational numbers r.

The following theorem shows that any solution of the additive Cauchy equation is rationally homogeneous.

Theorem 2 . *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a solution of the additive Cauchy equation. Then* f *is rationally homogeneous. Moreover,* f *is linear on the set of rational numbers* $\mathbb{Q}$.

Proof: Letting $x = 0 = y$ in (1) see that $f(0) = f(0) + f(0)$ and hence

$$f(0) = 0. \tag{3}$$

Substituting $y = -x$ in (1) and then using (3), we see that f is an odd function in $\mathbb{R}$, that is

$$f(-x) = -f(x) \tag{4}$$

for all $x \in \mathbb{R}$. Thus, so far, we have shown that a solution of the additive Cauchy equation is zero at the origin and it is an odd function.

Next, we will show that a solution of the additive Cauchy equation is rationally homogeneous. For any x,

$$f(2x) = f(x + x) = f(x) + f(x) = 2\,f(x).$$

Hence

$$f(3x) = f(2x+x) = f(2x)+f(x) = 2f(x)+f(x) = 3\,f(x);$$

so in general (using induction)

$$f(nx) = n\,f(x) \tag{5}$$

for all positive integers n.

If n is a negative integer, then $-n$ is a positive integer and by (5) and (4), we get

$$\begin{aligned} f(nx) &= f(-(-n)x) \\ &= -f(-nx) \\ &= -(-n)\, f(x) \\ &= n\, f(x). \end{aligned}$$

Thus, we have shown $f(nx) = n\, f(x)$ for all integers n and all $x \in \mathbb{R}$.

Next, let r be an arbitrary rational number. Hence, we have

$$r = \frac{k}{\ell}$$

where k is an integer and ℓ is a natural number. Further, $kx = \ell(rx)$. Using the integer homogeneity of f, we obtain

$$k\, f(x) = f(kx) = f(\ell(rx)) = \ell\, f(rx);$$

that is,

$$f(rx) = \frac{k}{\ell} f(x) = r\, f(x).$$

Thus, f is rationally homogeneous.

Further, letting $x = 1$ in the above equation and defining $c = f(1)$, we see that

$$f(r) = c\,r$$

for all rational numbers $r \in \mathbb{Q}$.

Hence, f is linear on the set of rational numbers and the proof is now complete.

Now we present the second proof of Theorem 1.

Proof: Let f be a continuous solution of the additive Cauchy equation. For any real number x there exists a sequence $\{r_n\}$ of rational numbers with $r_n \to x$.

Since f satisfies the additive Cauchy equation, by Theorem 2, f is linear on the set of rational numbers. That is,

$$f(r_n) = c\, r_n$$

for all n.

Now using the continuity of f, we get

$$\begin{aligned} f(x) &= f\left(\lim_{n\to\infty} r_n\right) \\ &= \lim_{n\to\infty} f(r_n) \\ &= \lim_{n\to\infty} c\, r_n \\ &= c\, x \end{aligned}$$

and the proof is now complete.

The following theorem is due to Darboux (1875).

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a solution of the additive Cauchy equation* (1). *If f is continuous at a point, then it is continuous everywhere.*

Proof: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at a point t in $\mathbb{R}$ and let x be any arbitrary point. Hence, we have

$$\lim_{y \longrightarrow t} f(y) = f(t).$$

Next, we show that f is continuous at x.

Consider

$$\begin{aligned}
\lim_{y\to x} f(y) &= \lim_{y\to x} f(y-x+x-t+t)\\
&= \lim_{y\to x} [f(y-x+t)+f(x-t)]\\
&= \lim_{y\to x} f(y-x+t)+\lim_{y\to x} f(x-t)\\
&= f(t)+f(x-t)\\
&= f(t)+f(x)-f(t)\\
&= f(x).
\end{aligned}$$

Thus we proved that f is continuous at the arbitrary point x and the arbitrariness of x implies that f is continuous everywhere in $\mathbb{R}$.

Now the proof of the theorem is complete.

The next theorem follows from Theorem 1 and Theorem 3.

Theorem 4 . *Let f be a solution of the additive Cauchy functional equation (1). If f is continuous at a point, then f is linear; that is, $f(x) = c\,x$ for all $x \in \mathbb{R}$.*

References

[1] B. Ebanks, P.K. Sahoo and W. Sander, Characterizations of Information Measures, World Scientific, Singapore, 1998.

[2] P. K. Sahoo and Pl. Kannappan, Introduction to Functional Equations, CRC Press, Boca Raton, 2011.

[3] P. K. Sahoo and T. Riedel, Mean Value Theorems and Functional Equations, World Scientific, Singapore, 1998.

Thank You