Exam 1 Solutions

1. (a) $f(x|\theta) = (\theta-1)\,x^{-\theta}\, I_{(1,\infty)}(x)$

$$= I_{(1,\infty)}(x)\ (\theta-1)\ e^{\theta(-\log x)} = h(x)c(\theta)e^{w_1(\theta)\cdot t_1(x)}$$

where $h(x) = I_{(1,\infty)}(x)$, $c(\theta) = \theta-1$, $w_1(\theta) = \theta$, and $t_1(x) = -\log x$

So this family of pdfs is a one-dimensional exponential family.

(b) $E\left[\dfrac{dw_1(\theta)}{d\theta}\, t_1(X)\right] = -\dfrac{d}{d\theta}\log c(\theta)$

Now, $\dfrac{dw_1(\theta)}{d\theta} = \dfrac{d}{d\theta}[\theta] = 1$ and $\dfrac{d}{d\theta}[\log c(\theta)] = \dfrac{d}{d\theta}[\log(\theta-1)] = \dfrac{1}{\theta-1}$

So that $E[1\cdot(-\log X)] = -\dfrac{1}{\theta-1}$

$$\Downarrow$$

$$E[\log X] = \boxed{\frac{1}{\theta-1}}.$$

---

2. (a) $P(X=0) = \int_0^{1/2} P(X=0, U=u)\,du$

$$= \int_0^{1/2} P(X=0|U=u)\cdot P(U=u)\,du$$

$$= \int_0^{1/2} (1-u)\cdot 2\,du = \left[-(1-u)^2\right]_0^{1/2} = -\tfrac{1}{4}+1 = \tfrac{3}{4}$$

$$P(X=1) = \int_0^{1/2} P(X=1|U=u)\cdot P(U=u)\,du$$

$$= \int_0^{1/2} u\cdot 2\,du = \left[u^2\right]_0^{1/2} = \tfrac{1}{4} - 0 = \tfrac{1}{4}$$

So $X \sim \text{Bernoulli}(\tfrac{1}{4})$.

(b) $EX = \tfrac{1}{4}$, $EU = \int_0^{1/2} u\cdot f_U(u)\,du = \int_0^{1/2} 2u\,du = \left[u^2\right]_0^{1/2} = \tfrac{1}{4}$

$$E[XU] = E\big[E[XU|U]\big] = E\big[U\,E[X|U]\big] = E[U\cdot U]$$

since $X|U=u \sim \text{Bernoulli}(u)$

So $E[XU] = E[U^2] = \int_0^{1/2} u^2 \cdot 2\,du = \left[\frac{2u^3}{3}\right]_0^{1/2}$

$$= \frac{2(\frac{1}{8})}{3} = \frac{1}{12}$$

So $\mathrm{Cov}(X,U) = E[XU] - E[X]E[U]$

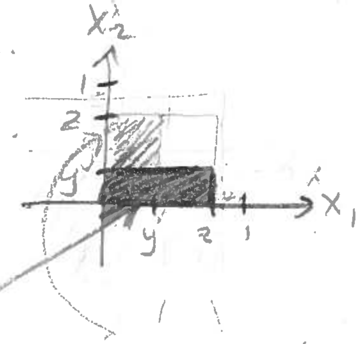
$$= \frac{1}{12} - \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{12} - \frac{1}{16} = \frac{4-3}{48} = \boxed{\frac{1}{48}}$$

---

3. (a) $F_{X_{(1)},X_{(2)}}(y,z) = P(X_{(1)} \le y \text{ and } X_{(2)} \le z)$

If $\max\{X_1, X_2\} \le z$, then $X_1 \le z$ and $X_2 \le z$.

If $\min\{X_1, X_2\} \le y$, then $X_1 \le y$ or $X_2 \le y$.



So $\{X_{(1)} \le y \text{ and } X_2 \le z\}$ if and only if $\{X_1 \le z \text{ and } X_2 \le y\}$ or $\{X_1 \le y \text{ and } y < X_2 \le z\}$

If $0 < y < z \le 1$, then

$$F_{X_{(1)},X_{(2)}}(y,z) = P(X_1 \le z \text{ and } X_2 \le y) + P(X_1 \le y \text{ and } y < X_2 \le z)$$

$$= P(X_1 \le z)\,P(X_2 \le y) + P(X_1 \le y)\,P(y < X_2 \le z)$$

$$= z \cdot y + y \cdot (z-y)$$

$$= y(2z-y)$$

Then $\frac{\partial^2}{\partial y \partial z}[y(2z-y)] = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial z}[y(2z-y)]\right\}$

$$= \frac{\partial}{\partial y}[y \cdot 2] = 2$$

so $\boxed{f_{X_{(1)},X_{(2)}}(y,z) = 2 \cdot I_{(0,z)}(y)\, I_{(0,1)}(z)}$

(b)

$$\left.\begin{array}{l} v = \frac{y}{z} \\ w = z \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} y = vz = vw \\ z = w \end{array}\right.$$

so this is a one-to-one transformation.

Since $J = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} w & v \\ 0 & 1 \end{vmatrix} = w$,

the joint density of V and W is

$$\begin{aligned} f_{V,W}(v,w) &= f_{X_{(1)},X_{(2)}}(vw, w) \cdot |J| \\ &= 2 \cdot I_{(0,w)}(vw) \cdot I_{(0,1)}(w) \cdot w \\ &= 2w\, I_{(0,1)}(w)\, I_{(0,w)}(vw) \\ &= 2w\, I_{(0,1)}(w) \cdot I_{(0,1)}(v) \\ &= f_W(w) \cdot f_V(v) \end{aligned}$$

so V and W are independent.

---

4. (a) $Y_1 + Y_2 \sim \text{Normal}(0,4)$ since $E[Y_1+Y_2] = E[Y_1] + E[Y_2] = 0 + 0 = 0$

and $Var[Y_1+Y_2] \overset{\text{indep.}}{=} Var[Y_1] + Var[Y_2] = 2 + 2 = 4$

$$P(Y_1+Y_2 > 1) = P\left(\frac{(Y_1+Y_2)-0}{\sqrt{4}} > \frac{1-0}{2}\right) = P\left(Z > \tfrac{1}{2}\right) \overset{\text{Normal table}}{=} 1 - .6915 = .3085$$

Also, $\left(\frac{Y_3}{\sqrt{2}}\right), \left(\frac{Y_4}{\sqrt{2}}\right), \left(\frac{Y_5}{\sqrt{2}}\right)$ are iid Normal(0,1)

so $\frac{Y_3^2}{2} + \frac{Y_4^2}{2} + \frac{Y_5^2}{2} \sim \chi^2_3$

and

$$P(Y_3^2+Y_4^2+Y_5^2>5) = P\left(\frac{Y_3^2+Y_4^2+Y_5^2}{2} > \frac{5}{2}\right) \overset{\chi^2 \text{ table}}{=} .48$$

Since Y_1+Y_2 is independent of $Y_3^2+Y_4^2+Y_5^2$,

$$P(Y_1+Y_2>1 \text{ and } Y_3^2+Y_4^2+Y_5^2>5) = P(Y_1+Y_2>1)\,P(Y_3^2+Y_4^2+Y_5^2>5)$$

$$= \boxed{(.3085)(.48)}$$

(b) $$\frac{\bar{Y}-0}{S/\sqrt{5}} = \frac{\bar{Y}}{\sqrt{\frac{1}{4}\Sigma(Y_i-\bar{Y})^2}/\sqrt{5}} = \frac{2\sqrt{5}\,\bar{Y}}{\Sigma(Y_i-\bar{Y})^2} \sim t_4$$

So $$P\left(\sqrt{5}\,\bar{Y} < \sqrt{\sum_{i=1}^{5}(Y_i-\bar{Y})^2}\right) = P\left(\frac{2\sqrt{5}\,\bar{Y}}{\sqrt{\Sigma(Y_i-\bar{Y})^2}} < 2\right)$$

$$= P(T<2) \overset{t\text{-table}}{=} 1-.06 = \boxed{.94}$$

---

5. (a) $$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)\frac{1}{x!}4^x e^{-4}$$

$$= \sum_{x=2}^{\infty} x(x-1)\frac{1}{x!}4^x e^{-4}$$

$$= \sum_{x=2}^{\infty} \frac{1}{(x-2)!}4^{x-2}4^2 e^{-4}$$

$$= 4^2 \sum_{x=2}^{\infty} \frac{1}{(x-2)!}4^{x-2}e^{-4} = 4^2 \cdot 1 = \boxed{16}$$

(b) $a = EX = \boxed{4}$, $b = \sqrt{\operatorname{Var} X} = \sqrt{EX^2-(EX)^2} = \sqrt{20-4^2} = \sqrt{4} = \boxed{2}$

since $EX^2 - EX = 16 \Rightarrow EX^2 = 16+4 = 20$

(c) $$P(3.99 < \bar{X} < 4.01) = P\left(\frac{\sqrt{10000}\,(3.99-4)}{2} < \frac{\sqrt{10000}\,(\bar{X}-4)}{2} < \frac{\sqrt{10000}\,(4.01-4)}{2}\right)$$

$$\overset{\text{CLT}}{\approx} P\left(\frac{-1}{2} < Z < \frac{1}{2}\right)$$

$$= P\left(Z<\frac{1}{2}\right) - P\left(Z<-\frac{1}{2}\right)$$

$$= .6915 - (1-.6915) = \boxed{.3830}$$