Analysis Exam 1 Study Guide.

2 problems from notes

2 problems from hmwrk.

- **Proof Techniques:** - Proving equality ex: $(\bigcup_{i=1}^{\infty} A_i^c)^c = \bigcap_{i=1}^{\infty} A_i$
  Must show they are subsets of each other ie. $A \subset B$, $B \subset A$.
- Prove it's an algebra: if it's a collection $\mathcal{A}$ of subsets of X $\ni$
  1) $\emptyset \in \mathcal{A}$, $X \in \mathcal{A}$ 2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ where $A \subset X$.
  3) $A_1, A_2, \dots, A_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{A}$ or $\bigcap_{i=1}^{n} A_i \in \mathcal{A}$ (if one is true, so is the other.
- Prove it's a σ-algebra:
  In adition to being an algebra, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ & $\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$.
- Prove a collection $\mathcal{M}$ is a monotone class of X:
  (1) $A_i \uparrow A$, where $A_i \in \mathcal{M}$ $\forall i$ then $A \in \mathcal{M}$
  (2) $A_i \downarrow A$, where $A_i \in \mathcal{M}$ $\forall i$ then $A \in \mathcal{M}$ } if $A \in \mathcal{M}$, then $\mathcal{M}$ is a monotone class.
- Pay attention → Prove μ is a measure on a metric space $(X, \mathcal{A})$ where $\mu : \mathcal{A} \to [0, \infty)$ $\ni$
  1) $\mu(\emptyset) = 0$ 2) $\forall$ pairwise disjoint $A_i \in \mathcal{A}$ $\forall i$, $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$
- when comparing sets, and we want to manipulate our set to have the properties we want, we use a set $B \in \mathcal{A}$; or $B_n = \bigcap_{i=1}^{n} A_i$; or $B_1 = A_1$ - $B_2 = A_2 \setminus A_1$ - $B_3 = A_3 \setminus (A_1 \cup A_2)$ $\ni$ $B_i = A_i \setminus (\bigcup_{j=1}^{i-1} A_j)$ $\forall i$
- when constructing measures to prove measurability, we need ϵ, or $\frac{1}{2^i}$, something that converges.

- ~~Homework~~ **Homework Problems**
- Prove $(\bigcup_{i=1}^{\infty} A_i^c)^c = \bigcap_{i=1}^{\infty} A_i$

  Answer: Let $x \in \bigcap_{i=1}^{\infty} A_i$, so $x \in A_i$ $\forall i$. Clearly then $x \in \bigcap_{i=1}^{m} A_i$ $\forall m$, and from this we know $x \notin (\bigcap_{i=1}^{m} A_i)^c$ $\forall m$. By De Morgan's Law, we have $x \notin (\bigcup_{i=1}^{m} A_i^c)$ $\forall m$. Thus $x \in (\bigcup_{i=1}^{m} A_i^c)^c$ $\forall m$. Because this is true $\forall m$, it's also true in the infinite case, so $x \in (\bigcup_{i=1}^{\infty} A_i^c)^c$.
  Thus $\bigcap_{i=1}^{\infty} A_i \subseteq (\bigcup_{i=1}^{\infty} A_i^c)^c$. The other direction can be proved similarly.

- 2.1: Find an example of a set X & a monotone class M consisting of subsets of X $\ni$ $\emptyset \in M$, $X \in M$, but M is not a σ-algebra.

  Answer: consider $X = \{1, 2, 3\}$, & $A = \{1\}$. Then $M = \{\emptyset, A, X\}$ is monotone b/c $\emptyset \subset A \subset X$ & M is closed under unions.
  However $A^c = \{2, 3\} \notin M$, so M is not a σ-algebra.

2.2: Find an example of a set X & two σ-algebras $\mathcal{A}_1$ & $\mathcal{A}_2$ each consisting of subsets of $X \ni \mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra.

Answer: Let $X = \{1,2,3\}$, $\mathcal{A}_1 = \{\{1\}, \{2,3\}, X, \emptyset\}$ & $\mathcal{A}_2 = \{\{2\}, \{1,3\}, X, \emptyset\}$.
So $\mathcal{A}_1 \cup \mathcal{A}_2 = \{\{1\}, \{2\}, \{2,3\}, \{1,3\}, X, \emptyset\}$.
Notice $\{1\} \cup \{2\} = \{1,2\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$, so it's not a σ-algebra.

2.3 Suppose $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ are σ-algebras consisting of subsets of X. Is $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ necessarily a σ-algebra? If not, give a counterexample.

Answer: We run into trouble when trying to show $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ is closed under countable unions ie. $\forall A_1, A_2, \dots \in \bigcup_{i=1}^{\infty} \mathcal{A}_i$, that $\bigcup_{j=1}^{\infty} A_j \in \bigcup_{i=1}^{\infty} \mathcal{A}_i$, ie. $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}_i$ for some i.
- Consider $X = \mathbb{N}$ & $\mathcal{A}_n =$ all subsets of $\{1, \dots, n\}$ & it's complements.
Then $\mathcal{A}_1 = \{\emptyset, \{1\}, \{1\}^c, X\}$
$\subset \mathcal{A}_2 = \{\emptyset, \{1\}, \{1\}^c, \{2\}, \{2\}^c, \{1,2\}, \{1,2\}^c, X\} \subset \mathcal{A}_3 = \dots$
Let $A_j = \{2j\}$. So $A_1 = \{2\}$, $A_2 = \{4\}$ and so on. Thus $\forall j$, $A_j \in \cup A_i$, but this union isn't contained in any of the $\mathcal{A}_k$, thus it isn't in $\bigcup_{i=1}^{\infty} \mathcal{A}_i$. So no $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ is not necessarily a σ-algebra. QED

2.4 Suppose $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$ are monotone classes. Let $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$. Suppose $A_j \uparrow A$ & $\forall j$ $A_j \in \mathcal{M}$. Is A necessarily in $\mathcal{M}$? If not, give a counter example.

Answer: Let $A_n = \{2k \in \mathbb{N} \mid k \leq n\}$ & $\mathcal{M}_n = \{\emptyset, \mathbb{N}\} \cup \{E_k \mid k \leq n\}$ $\forall n$ where $\mathcal{M}_n$ is a monotone class on $\mathbb{N}$ & E_k are the even numbers up to k.
Notice $\mathcal{M}_n \subset \mathcal{M}_{n+1}$ $\forall n$. $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n = \{\emptyset, \mathbb{N}\} \cup \{E_k \mid k \in \mathbb{N}\}$. Thus $\mathcal{M}$ is not a monotone class b/c $\{E_k \mid k \in \mathbb{N}\}$ is an increasing sequence of members of $\mathcal{M}$, but $\bigcup_{k \in \mathbb{N}} E_k$ is the set of even numbers which isn't in $\mathcal{M}$. So no, $A \in \mathcal{M}$ is not necessary. QED.

3.1: Suppose $(X, \mathcal{A})$ is a measurable space & μ is a non-negative set function that is finitely additive & $\ni \mu(\emptyset) = 0$ & $\mu(B)$ is finite for some non-empty $B \in \mathcal{A}$. Suppose that whenever A_i is an increasing sequence of sets in $\mathcal{A}$, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i)$. Show that μ is a measure.

Answer: It's given that $\mu(\emptyset) = 0$ ✓

We know finite additivity only occurs for pairwise disjoint sets, so $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$ for some pairwise disjoint $A_i \in \mathcal{A}$. WTS $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{j=1}^{\infty} \mu(A_i)$.
So, let $B_1 = A_1$, $B_2 = A_1 \cup A_2, \ldots, B_n = \bigcup_{i=1}^{n} A_i$. Then we know $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} \forall i$ & $B_1 \subset B_2 \subset \ldots \subset B_n \subset \ldots$. So by assumption, we have $\mu(\bigcup_{i=1}^{\infty} B_i) = \lim_{i \to \infty} \mu(B_i)$.
Hence, $\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(\bigcup_{i=1}^{\infty} B_i) = \lim_{i \to \infty} \mu(B_i) = \lim_{i \to \infty} \mu(\bigcup_{j=1}^{i} A_j) = \lim_{i \to \infty} \sum_{j=1}^{i} \mu(A_j) = \sum_{j=1}^{\infty} \mu(A_j)$

Thus μ is a measure. QED

3.2 Suppose $(X, \mathcal{A})$ is a measurable space & μ is a non-negative set function that is finitely additive & $\ni \mu(\emptyset) = 0$ & $\mu(X) < \infty$. Suppose that A_i is a sequence of sets in $\mathcal{A}$ that decrease to $\emptyset$, then $\lim_{i \to \infty} \mu(A_i) = 0$. Show μ is a measure.

Answer: It's given $\mu(\emptyset) = 0$ ✓

Since finite additivity is given, we need to show countable additivity as in 3.1.
So for some pairwise disjoint $A_i \in \mathcal{A}$, we have $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$. WTS $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. Let $B_1 = \bigcup_{i=1}^{\infty} A_i$, $B_2 = \bigcup_{i=2}^{\infty} A_i, \ldots, B_n = \bigcup_{i=n}^{\infty} A_i, \ldots$.
Thus we have $B_1 \supset B_2 \supset \ldots \supset B_n \supset \ldots$ & $B_i \downarrow \emptyset$. Thus by assumption $\lim_{i \to \infty} \mu(B_i) = 0$.
So we have $\bigcup_{i=1}^{\infty} A_i = (\bigcup_{i=1}^{n-1} A_i) \cup B_n$. Note that since the A_i's are pairwise disjoint, $\bigcup_{i=1}^{n-1} A_i$ & B_n are disjoint. Thus

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \mu((\bigcup_{j=1}^{n-1} A_i) \cup B_n) = \mu(\bigcup_{i=1}^{n-1} A_i) + \mu(B_n) = \sum_{i=1}^{n-1} \mu(A_i) + \mu(B_n)$$

Hence $\lim_{n \to \infty} \mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \to \infty} (\sum_{i=1}^{n-1} \mu(A_i) + \mu(B_n)) = \lim_{i \to \infty} \sum_{i=1}^{n-1} \mu(A_i) + 0 = \sum_{i=1}^{\infty} \mu(A_i)$

So $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. QED

3.3: Let X be an uncountable set & let $\mathcal{A}$ be the collection of subsets A of $X \ni$ either A or A^c is countable. Define $\mu(A)=0$ if A is countable & $\mu(A)=1$ if A is uncountable. Prove μ is a measure.

Answer: First we need to show $\mathcal{A}$ is a σ-algebra, then we can prove μ is a measure. Suppose A_i is countable. Then the union of all of these will be countable, thus $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Suppose A_k is uncountable for some k, thus A is uncountable. Then A^c is countable & $(\bigcup_{i=1}^{\infty} A_i)^c \subset A_k^c$, thus the compliment of the union is inside of $\mathcal{A}$.
Thus $\mathcal{A}$ is a σ-algebra.
Now we can begin to show μ is a measure.
1) $\mu(\emptyset)=0$ b/c $\emptyset$ is countable & contained in $\mathcal{A}$.
2) If A_i is countable $\forall i$, then $\mu(\bigcup A_i)=0=\Sigma\mu(A_i)$ by properties of countable sets. Suppose $\exists\, k \ni A_k$ is uncountable. Then A_k^c is countable. Thus $(\bigcup A_i)^c \subset A_k^c$, so $\mu(\bigcup A_i)=1$. As for $\Sigma\mu(A_i)$, we can see it's equal to 1 since there is only one nonzero term (ie. $\mu(A_k)$). Thus μ is a measure on $\mathcal{A}$. QED.

3.4: $(X, \mathcal{A}, \mu)$ is a measure space, & $A, B \in \mathcal{A}$. Prove that $\mu(A)+\mu(B)=\mu(A\cup B)+\mu(A\cap B)$.

Answer: We know $A\cup B = (A/B)+(A\cap B)+(B/A)$.
So $\mu(A\cup B)=\mu(A/B)+\mu(A\cap B)+\mu(B/A)$ b/c they're disjoint.
Note $A/B = A/(A\cap B)$. So $\mu(A\cup B)=\mu(A)-\mu(A\cap B)+\mu(A\cap B)+\mu(B)-\mu(A\cap B)$
$\Rightarrow \mu(A\cup B)=\mu(A)+\mu(B)-\mu(A\cap B)$. Thus
$\mu(A)+\mu(B)=\mu(A\cup B)+\mu(A\cap B)$ QED.

3.6: Prove that if $(X, \mathcal{A}, \mu)$ is a measure space, $B\in\mathcal{A}$ & we define $\nu(A)=\mu(A\cap B)$ for $A\in\mathcal{A}$, then ν is a measure.

Answer: $\nu(\emptyset)=\mu(\emptyset\cap B)=\mu(\emptyset)=0$ ✓ WTS for pairwise disjoint $A_i \in \mathcal{A}$ that $\nu(\bigcup_{i=1}^{\infty} A_i)=\sum_{i=1}^{\infty}\nu(A_i)$
So let $A_i\in\mathcal{A}$ $\forall i$ be pairwise disjoint. Then
$\nu(\bigcup_{i=1}^{\infty} A_i)=\mu(\bigcup_{i=1}^{\infty} A_i\cap B)=\mu((A_1\cap B)\cup(A_2\cap B)\cup\dots)=\mu(A_1\cap B)+\mu(A_2\cap B)+\dots$
$=\nu(A_1)+\nu(A_2)+\dots$
$=\sum_{i=1}^{\infty}\nu(A_i)$ QED.

$\mathcal{C}_1 = \{(a,b) \mid a,b \in \mathbb{R}\}$

$\mathcal{C}_2 = \{[a,b] \mid a,b \in \mathbb{R}\}$

$\mathcal{C}_3 = \{(a,b] \mid a,b \in \mathbb{R}\}$

$\mathcal{C}_4 = \{(a,\infty) \mid a,b \in \mathbb{R}\}$

- Let $\mathcal{C}_3 = \{(a,b] \mid a,b \in \mathbb{R}\}$ & $\mathcal{B}$ is the Borel-σ-algebra. Show $\sigma(\mathcal{C}_3) = \mathcal{B}$

<u>Answer:</u> We have $\sigma(\mathcal{C}_1) = \mathcal{B}$ where $\mathcal{C}_1 = \{(a,b) \mid a,b \in \mathbb{R}\}$ so it is enough to show $\sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_3)$, ie $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_3)$ & $\sigma(\mathcal{C}_3) \subset \sigma(\mathcal{C}_1)$.

(1) $\forall (a,b]$ in $\mathcal{C}_3$, $(a,b] = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n})$. From this we can see that $\mathcal{C}_3 \subset \sigma(\mathcal{C}_1) \Rightarrow \sigma(\mathcal{C}_3) \subset \sigma(\mathcal{C}_1) = \mathcal{B}$.

(2) $\forall (a,b)$ in $\mathcal{C}_3$, $(a,b) = \bigcup_{n=1}^{\infty} (a, b-\frac{1}{n})$ for sufficiently large k. Thus $\mathcal{C}_1 \subset \sigma(\mathcal{C}_3)$
$\Rightarrow \mathcal{B} = \sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_3)$.

Thus $\sigma(\mathcal{C}_3) = \mathcal{B}$. QED

- Show $\sigma(\mathcal{C}_4) = \mathcal{B}$. Similar to the previous problem, we can show this if $\sigma(\mathcal{C}_4) = \sigma(\mathcal{C}_3)$.

<u>Answer:</u> $(a,b] = (a,\infty) - (b,\infty)$, so $\mathcal{C}_3 \subset \sigma(\mathcal{C}_4) \Rightarrow \sigma(\mathcal{C}_3) \in \sigma(\mathcal{C}_4)$.

For the reverse situation, $(a,\infty) = \bigcup_{n=1}^{\infty} (a, a+n]$, then $\mathcal{C}_4 \subset \sigma(\mathcal{C}_3)$
$\Rightarrow \sigma(\mathcal{C}_4) \subset \sigma(\mathcal{C}_3) = \mathcal{B}$. Thus $\mathcal{B} = \sigma(\mathcal{C}_3) = \sigma(\mathcal{C}_4)$. QED.

- <u>Monotone Class Thm</u>: Let $\mathcal{A}_0$ be an algebra. $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{A}_0$. Also let $\mathcal{M}$ be the smallest monotone class containing $\mathcal{A}_0$.

-Define $N_3 = \{A \in \mathcal{M} \mid A \cap B \in \mathcal{M}.\ \forall B \in \mathcal{M}\}$. Show N_3 is a monotone class.

<u>Answer:</u> Recall $\mathcal{A}_0 \subset N_3 \subset \mathcal{M}$

Now pick $A_i \in N_3 \ni A_i \uparrow A\ \forall i$. WTS $A \in N_3$. $\forall B \in \mathcal{M}$,
$A \cap B = (\bigcup_{i=1}^{\infty} A_i) \cap B = \bigcup_{i=1}^{\infty} (A_i \cap B) \in \mathcal{M}$. Hence $A \in N_3$.

Similarly we can show if given $A_i \in N_3$ where $A_i \downarrow A\ \forall i$ that $A \in N_3$.

Thus N_3 is a monotone class. But since $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}_0$, we have $N_3 = \mathcal{M}$. QED.

4.2 Let m be a Lebesgue measure & A be a Lebesgue measurable set of $\mathbb{R}$ w/ $m(A)<\infty$. Let $\varepsilon>0$. Show $\exists$ G open & F closed $\ni F \subseteq A \subseteq G$ & $m(G-F)<\varepsilon$.

Answer: A is either bounded or unbounded.

-Case 1) A is bounded. Then $\exists\, a,b \in \mathbb{R} \ni A \subset [a,b]$. So by prop. 4.14 we know $\forall \varepsilon>0\ \exists$ G that's an open set $\ni A \subset G$ & $m(G \setminus A)<\varepsilon$; Also $\forall \varepsilon>0\ \exists$ F that's a closed set $\ni F \subset A$ & $m(A \setminus F)<\varepsilon$. Choose an open set $G \ni A \subset G$ & $m(G\setminus A)<\frac{\varepsilon}{2}$ & closed set $F \subset A \ni m(A\setminus F<\frac{\varepsilon}{2}$. $m(G-F)=m(G\setminus A)+m(A\setminus F)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$ ✓.

-Case 2) A is unbounded. We can't use prop 4.14 here b/c it's unbounded. However, we can use prop 4.14 to prove $\exists$ G that's an open set $\ni A \subset G$ & $m(G \setminus A)<\varepsilon$. Now we need to find our F. Let $A_k=\{x \in A \mid |x| \leq k\}$. Note $A_1 \subset A_2 \subset \ldots$, $A_k \uparrow A$, $\cup A_k = A$. By prop 3.5, we know $m(A)=\lim_{k\to\infty} m(A_k)$, $\forall \varepsilon>0$, $\exists N \ni m(A) \leq m(A_n)+\frac{\varepsilon}{2}$ where A_n is bounded. Because of this, $\exists$ F that's a closed set $\ni F \subset A_n$ & $m(A_n \setminus F)<\frac{\varepsilon}{2}$ by prop 4.14. Thus $m(A \setminus F)<\varepsilon$. Thus we know $\forall \varepsilon>0\ \exists$ a closed set $F \subset A \ni m(A\setminus F)<\varepsilon$. Choose an open set $G \ni A \subset G$ & $m(G\setminus A)<\varepsilon/2$ & closed set $F \subset A \ni m(A\setminus F)<\varepsilon/2$. Thus $m(G-F)=m(G\setminus A)+m(A\setminus F)<\varepsilon/2+\frac{\varepsilon}{2}=\varepsilon$. QED.

4.4 Let m be Lebesgue-Stieltjes measure corresponding to a right continuous increasing function α. Show that $\forall x$, $m(\{x\})=\alpha(x)-\alpha(x-)$.

Answer: Recall if $I=(e,k]$, $e,k \in \mathbb{R}$, then $m(I)=\ell(I)$ & $\alpha(x-)=\lim_{y\to x^-}\alpha(y)$.

Let $\{x_n\}$ be a sequence $\ni x_n \to x$ & $x_j<x\ \forall j$. Then write $\{x\}=\bigcap_n (x_n,x]$. By continuity from above, $m(\{x\})=\lim_{n\to\infty} m((x_n,x])$
$=\lim_{n\to\infty}\alpha(x)-\alpha(x_n)=\alpha(x)-\alpha(x-)$. QED.

4.3: If $(X, \mathcal{A}, \mu)$ is a measure space, define $\mu^*(A) = \inf\{\mu(B) \mid A \subset B \text{ \& } B \in \mathcal{A}\}\ \forall A \subset X$. Show μ^* is an outer measure. Show that each set in $\mathcal{A}$ is μ^* measurable & μ^* agrees w/ μ on $\mathcal{A}$.

Answer: – Show μ^* is an outer measure

Well, $\mu^*(\emptyset) = \inf\{\mu(B) \mid \emptyset \subset B, B \in \mathcal{A}\} = 0$ ✓

– WTS $A \subset B \Rightarrow \mu^*(A) \le \mu^*(B)$.

Note for every $C \in \mathcal{A} \ni B \subset C$, we know $A \subset C$ since $A \subset B$, so $\{\mu(C) \mid B \subset C, C \in \mathcal{A}\} \subset \{\mu(D) \mid A \subset D, D \in \mathcal{A}\}$. Hence, $\inf\{\mu(C) \mid B \subset C, C \in \mathcal{A}\} \ge \inf\{\mu(D) \mid A \subset D, D \in \mathcal{A}\}$ so $\mu^*(A) \le \mu^*(B)$.

– WTS $\mu^*(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu^*(A_i)$, $A_i \subset X\ \forall i$.

We will show $\mu^*(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu^*(A_i) + \varepsilon\ \forall \varepsilon > 0$. Let $A_1, A_2, \ldots$ be a sequence of sets $\ni A_i \subset X\ \forall i$. $\forall i$ let B_i be $\ni A_i \subset B_i$, $B_i \in \mathcal{A}$. $\forall \varepsilon > 0\ \exists B_i \ni \mu(B_i) < \mu^*(A_i) + \frac{\varepsilon}{2^i}$ (Clearly $\exists M \in \mathcal{A} \ni A \subset M$ & $\mu^*(A) = \mu(M)$ so we know we can find such a B_i). Clearly $\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} B_i$ since $A_i \subset B_i\ \forall i$. So,

$$\mu^*\Big(\bigcup_{i=1}^{\infty} A_i\Big) \le \mu^*\Big(\bigcup_{i=1}^{\infty} B_i\Big) \underset{\uparrow}{=} \mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) \le \sum_{i=1}^{\infty} \mu(B_i) < \sum_{i=1}^{\infty}\Big(\mu^*(A_i) + \frac{\varepsilon}{2^i}\Big) = \sum_{i=1}^{\infty} \mu^*(A_i) + \varepsilon.$$

proven at (J*) Thus $\mu^*(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu^*(A_i)$.

(J*) – Show μ^* agrees w/ μ on $\mathcal{A}$. Let $A \in \mathcal{A}$. So $\mu^*(A) = \inf\{\mu(B) \mid A \subset B \in \mathcal{A}\}$. Since we are looking at the inf, the smallest $B \ni A \subset B \in \mathcal{A}$ is $A \in \mathcal{A}$. So $\mu^*(A) = \inf\{\mu(A) \mid A \subset A \in \mathcal{A}\}$ thus $\mu^*(A) = \mu(A)$.

– Show each set in $\mathcal{A}$ is μ^* measurable. I.e. $\forall A \in \mathcal{A}$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)\ \forall E \subset X$ (so it's sufficient to show $\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \cap A^c)$). Consider the set of elts in $\mathcal{A} \ni E \subset C \ni C \in \mathcal{A}$, ie. $\mathcal{C} = \{C \mid E \subset C \text{ \& } A \in \mathcal{A}\}$. Clearly $\inf\{\mu(C) \mid C \in \mathcal{C}\} = \mu^*(E)$. Now let's arrange the elts of $\mathcal{C} \ni C_1 \supset C_2 \supset \ldots$ & $\overline{C} = \bigcap_{i=1}^{\infty} C_i$ so $C_i \downarrow \overline{C}$. So $\overline{C} \subset C_i\ \forall i \Rightarrow \mu(\overline{C}) \le \mu(C_i)\ \forall i$. So $\inf\{\mu(C) \mid C \in \mathcal{C}\} = \mu(\overline{C}) = \mu^*(E)$. Let $A \in \mathcal{A}$. So

$$\mu^*(E) = \mu(\overline{C}) = \mu(\overline{C} \cap A) + \mu(\overline{C} \cap A^c)$$
$$\ge \inf\{\mu(C \cap A) \mid E \subset C \text{ \& } C \in \mathcal{A}\} + \inf\{\mu(C \cap A^c) \mid E \subset C \in \mathcal{A}\}$$
$$= \mu^*(E \cap A) + \mu^*(E \cap A^c). \quad \text{QED.}$$

4.5: Suppose m is a Lebesgue ~~[illegible]~~ measure. Define $x+A=\{x+y \mid y\in A\}$ & $cA=\{cy \mid y\in A\}$ $\forall$ $x\in\mathbb{R}$ & $c\in\mathbb{R}$. Show if A is a Lebesgue measurable set, then $m(x+A)=m(A)$ & $m(cA)=|c|\,m(A)$.

Answer: - Observe that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of sets, then $\{I_k\}_{k=1}^{\infty}$ covers A iff $\{I_k+y\}_{k=1}^{\infty}$ covers $A+y$. Moreover, if each I_k is an open interval, then each I_k+y is an open interval of the same length & so $\sum_{i=1}^{\infty} l(I_k)=\sum_{k=1}^{\infty} l(I_k+y)$. Thus $m(x+A)=m(A)$.

- Similarly, consider $m(cA)=\inf\{\sum_{i=1}^{\infty} l(R_i) \mid R_i\in\mathcal{C}_2\ \forall i,\ cA\subset\bigcup_{i=1}^{\infty}R_i\}$.

- Let $c>0$. Notice $\forall R_i$ we can write it in terms of $R_i=cM_i$, so $R_i=(cm_{i_1}, cm_{i_2}]$ & $l(R_i)=l(cM_i)=cm_{i_2}-cm_{i_1}=c(m_{i_2}-m_{i_1})$ ✓

Now let $c<0$. Then $\forall R_i$ we can write in terms of $R_i=c(m_{i_1}, m_{i_2}]=[cm_{i_2}, cm_{i_1})$ so $l(R_i)=l([cm_{i_2}, cm_{i_1}))=cm_{i_1}-cm_{i_2}=c(m_{i_1}-m_{i_2})=|c|(m_{i_2}-m_{i_1})$

So $m(cA)=|c|mA$ ✓ QED.

4.8

- If X is a metric space, $\mathcal{B}$ is the Borel σ-algebra & μ is a measure on $(X,\mathcal{B})$ then the support of μ is the smallest closed set $F\ni\mu(F^c)=0$. Show that if F is a closed subset of $[0,1]$, then $\exists$ a finite measure whose support is F.

Answer: B/c F is closed & $F\subset[0,1]$ is separable, we know $\exists A\subset F \ni A$ is countable & F is the closure of A. Let's consider 2 cases for A.

- Case 1) $|A|<\infty$. Let $\mu_1(E)=\frac{1}{|A|}\cdot|E\cap A|$ for $E\subset\mathcal{B}$. Verify μ_1 is a measure.
$\mu(\emptyset)=\frac{1}{|A|}\cdot|\emptyset\cap A|=\frac{1}{|A|}\cdot 0=0$ ✓. Let $E_i\cap E_j=\emptyset$ for $i\neq j$ & $E_i\in\mathcal{B}\ \forall i$.
$\mu(\bigcup_{i=1}^{\infty}E_i)=\frac{1}{|A|}\cdot|\bigcup_{i=1}^{\infty}E_i\cap A|=\frac{1}{|A|}|(E_1\cap A)\cup(E_2\cap A)\cup\ldots|=\frac{1}{|A|}|E_1\cap A|+\ldots$
$=\sum_{i=1}^{\infty}\mu(E_i)$ ✓. Now we need to verify for $\mu(F^c)=0$. $\mu(F^c)=\frac{1}{|A|}|F^c\cap A|=\frac{1}{|A|}|\emptyset|=0$ ✓

- Case 2) $|A|=\infty$. Let $\mu_2(E)=\sum_{i=1}^{\infty}\frac{1}{2^n}f(E)$ where $f(E)=\begin{cases}1 & \text{if } x\in E\cap A\\ 0 & \text{if } x\notin E\cap A\end{cases}$ for some $x\in A$.
Verify μ_2 is a measure. $\mu(\emptyset)=\sum_{i=1}^{\infty}\frac{1}{2^n}f(\emptyset)=0$ ✓. Let $E_i\cap E_j=\emptyset$ for $i\neq j$ & $E_i\in\mathcal{B}\ \forall i$.
$\mu(\bigcup_{i=1}^{\infty}E_i)=\begin{cases}\frac{1}{2^n} & \text{if } x\in\bigcup_{i=1}^{\infty}E_i\\ 0 & \text{if } x\notin\bigcup_{i=1}^{\infty}E_i\end{cases}=\sum_{i=1}^{\infty}\mu(E_i)$ ✓. Verify $\mu_2(F^c)=0$. Note $x\notin F^c\ \forall x\in A$, so $\mu_2(F^c)=\sum_{i=1}^{\infty}\frac{1}{2^n}\cdot 0=0$ ✓

- Claim: F is the smallest closed set $\ni\mu(F^c)=0$. Any set $\ni E\cap A=\emptyset$ will make $\mu_i(E)=0$. So, since F is the closure of A, F is the smallest closed set $\ni F^c\cap A=\emptyset$. Thus F is the support for μ_1 & μ_2. QED

- Suppose $E \subset \mathbb{R}$, & $m(E) > 0$ is a Lebesgue measure. Does that imply E contains an interval? Prove this or give a counter example.

<u>Answer</u>: Let U be an open set containing all the rationals $\ni m(U) < (b-a)$. Thus $m([a,b] \setminus U) \geq (b-a) - m(U) > 0$. The set $[a,b] \setminus U$ is compact, & a subset of $[a,b]$ but it contains no rationals, thus it has no interior. So $\exists E \subset \mathbb{R} \ni E$ contains no intervals. So it's not true. QED.

- <u>Important Definitions & Theorems</u> $\longrightarrow$ Note, these are all proven in notes / book. Must read them for the rest. There are also more of them. Good Luck! :)

- An <u>algebra</u> (see how to prove it's an algebra)
- A <u>σ-algebra</u> (" " " " " ")
- A <u>monotone class</u> (" " " " (monotone class))
- <u>Monotone class Thm</u>: Suppose $\mathcal{A}_0$ is a σ-algebra, $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{A}_0$, & $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}_0$. Then $\mathcal{M} = \mathcal{A}$.
- A <u>measure</u> (see how to prove a function on a metric space is a measure)
- $(X, \mathcal{A}, \mu)$ is a measure space where X is a set, $\mathcal{A}$ is a σ-algebra, & μ is a measure.
- A measure μ is a <u>finite measure</u> if $\mu(X) < \infty$.
- A measure is <u>σ-finite</u> if $\exists$ sets $E_i \in \mathcal{A}$ for $i = 1, 2, \ldots \ni \mu(E_i) < \infty \ \forall i$ & $X = \bigcup_{i=1}^{\infty} E_i$.
- An <u>outer measure</u> is a function μ^* defined on the collection of all subsets of $X \ni$ (1) $\mu^*(\emptyset) = 0$; (2) if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$; (3) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ whenever $A_1, A_2, \ldots$ are subsets of X.
- Proof in notes — <u>Prop. 4.2</u>, Suppose $\mathcal{C}$ is a collection of subsets of $X \ni \emptyset \in \mathcal{C}$ & $\exists D_1, D_2, \ldots$ in $\mathcal{C} \ni X = \bigcup_{i=1}^{\infty} D_i$. Suppose $\ell : \mathcal{C} \to [0, \infty]$ with $\ell(\emptyset) = 0$. Define:
$\mu^*(E) = \inf\left\{ \sum_{i=1}^{\infty} \ell(A_i) \mid A_i \in \mathcal{C} \ \forall i \ \& \ E \subset \bigcup_{i=1}^{\infty} A_i \right\}$, then <u>$\mu^*$ is an outer measure.</u>
- Proof in notes — <u>Carathéodory Thm</u>: If μ^* is an outer measure on X, then the collection $\mathcal{A}$ of μ^*-measurable sets is a σ-algebra. If μ is the restriction of μ^* to $\mathcal{A}$, then μ is a measure. Moreover, $\mathcal{A}$ contains all the null sets
- Let μ^* be an outer measure. A set $A \subset X$ is <u>μ^*-measurable</u> if
$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \ \forall E \subset X$. (Can see this used in this proof) ☆
- $\mu^*(E) = \inf\left\{ \sum_{i=1}^{\infty} \ell(A_i) \mid A_i \in \mathcal{C} \ \forall i \ \& \ E \subset \bigcup_{i=1}^{\infty} A_i \right\}$
- Every set in the Borel σ-algebra on $\mathbb{R}$ is m^* measurable (proof in notes & book)

- dropping the $*$, we call m a Lebesgue-Stieltjes measure. In the special case where $\alpha(x)=x$, m is a Lebesgue measure. In the special case of this, the collection of m^*-measurable sets is called the Lebesgue σ-algebra. A set is Lebesgue measurable if it's in the L-σ-algebra.
- Carathéodory extension thm: Suppose $\mathcal{A}_0$ is an algebra & $\ell: \mathcal{A}_0 \to [0,\infty]$ is a measure on $\mathcal{A}_0$. Define $\mu^*(E) = \inf\{\sum_{i=1}^{\infty} \ell(A_i) \mid \forall A_i \in \mathcal{A}_0, E \subset \bigcup_{i=1}^{\infty} A_i\}$ for $E \subset X$. Then,
(1) μ^* is an outer measure; (2) $\mu^*(A) = \ell(A)$ if $A \in \mathcal{A}_0$;
(3) Every set in $\mathcal{A}_0$ & every μ^* null set is μ^*-measurable (Proof in notes / book)
(4) if ℓ is σ-finite, then $\exists$ a unique extension to $\sigma(\mathcal{A}_0)$.
- A function $f: X \to \mathbb{R}$ is measurable or $\mathcal{A}$ measurable if $\{x \mid f(x) > a\} \in \mathcal{A}$ $\forall a \in \mathbb{R}$. A complex-valued function is measurable if both its real & imaginary parts are measurable.

proof in notes / book

- Prop 5.6: If X is a metric space, $\mathcal{A}$ contains all the open sets, & $f: X \to \mathbb{R}$ is cont., then f is measurable.
- Prop 5.7: Let $c \in \mathbb{R}$. If f & g are measurable real-valued functions, then so are $f+g$, $-f$, cf, fg, $\max(f,g)$, & $\min(f,g)$
- We say $f=g$ almost everywhere, written $f=g$ a.e., if $\{x \mid f(x) \neq g(x)\}$ has measure zero. Similarly, we say $f_i \to f$ a.e. if the set of x where $f_i(x)$ does not converge to $f(x)$ has measure zero.
- If $f: \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.
  ↳ X is a metric space, $\mathcal{B}$ is Borel σ-algebra on X. We say $f: X \to \mathbb{R}$ is Borel measurable if f is measurable wrt $\mathcal{B}$.
- Let $(X, \mathcal{A})$ be a measurable space & let $f: X \to \mathbb{R}$ be a measurable function. If A is in the Borel σ-algebra on $\mathbb{R}$, then $f^{-1}(A) \in \mathcal{A}$.
- Let $(X, \mathcal{A})$ be a measurable space. If $E \in \mathcal{A}$, define the characteristic function of E by $\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$. Note this is measurable.
- A simple function s is a function of the form $s(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ where $a_i \in \mathbb{R}$, $E_i \in \mathcal{A}$ $\forall i$.
- Suppose f is a non-negative function. Then $\exists$ a sequence of non-negative measurable simple functions $\{s_n\}_{n=1}^{\infty}$ increasing to f.
- If f is a function, the support of f = $\text{supp } f = \{x \mid f(x) \neq 0\}$
- Lusin's Thm: Suppose $f: [0,1] \to \mathbb{R}$ is Lebesgue measurable, m is a Lebesgue measure, & $\varepsilon > 0$. $\exists$ a closed set $F \subset [0,1] \ni m([0,1] - F) < \varepsilon$ & the restriction of f to F (ie. $f|_F$) is a continuous function on F.