MATH 667 EXAM 3 (December 14, 2015: 8:00am–10:30am) NAME ________________________

The exam is closed book; students are permitted to prepare one 4x6 notecard of formulas, notes, etc. that can be used during the exam. A calculator is permitted but not necessary for the exam.

Do 4 out of the 5 problems (20 points each, 80 points total). Clearly indicate the problem that you are omitting; if it is not clear, then the first 4 problems will be graded.

Problem 1. (20 points) Suppose that a sample of size 20 is collected from a population which follows a normal distribution with probability density function $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ where μ and σ^2 are unknown, and suppose that the only two summary statistics recorded are the sum of the first 10 observations and the sample variance of the last 10 observations.

(a - 10 pts) Show that $\dfrac{\frac{1}{10}\sum_{i=1}^{10} X_i - \mu}{\sqrt{\frac{1}{10}\cdot\frac{1}{9}\sum_{i=11}^{20}\left(X_i - \frac{1}{10}\sum_{j=11}^{20} X_j\right)^2}}$ follows a t-distribution, and find the degrees of freedom for its distribution?

(b - 10 pts) For the observed data, suppose that the sum of the first 10 observations is 60 and that the sample variance of the last 10 observations is 40. Find a 95% confidence interval for μ that is centered at the sample mean of the first 10 observations.

Use the t table attached to this exam.

Problem 2. (20 points) Suppose that $X_1, \ldots, X_n$ is a random sample from a normal(θ,θ^2) population where $\theta > 0$ each with probability density function $f(x) = \dfrac{1}{\sqrt{2\pi}\theta} e^{-\frac{1}{2\theta^2}(x-\theta)^2}$.

(a - 6 pts) Find a two-dimensional sufficient statistic for θ.

(b - 6 pts) Find the method of moments estimator of θ.

(c - 8 pts) Find the maximum likelihood estimator of θ. For this problem, it is not necessary to show that the estimator maximizes the likelihood.

Problem 3. (20 points)

(a - 5 pts) Let X be a Poisson random variable having probability mass function

$$P(X = x) = \begin{cases} \dfrac{1}{x!}\lambda^x e^{-\lambda} & \text{if } x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathrm{E}[X^2 - X]$ (or equivalently $\mathrm{E}[X(X-1)]$).

(b - 5 pts) Show that $\bar{X}$ is an unbiased estimator of λ.

(c - 5 pts) What is the variance of $\bar{X}$?

(d - 5 pts) Show that $\bar{X}$ satisfies the Cramér-Rao Lower Bound for an unbiased estimator of λ.

Problem 4. (20 points) Let X be a random variable with probability mass function

$$f(x|\theta) = \begin{cases} \dfrac{1}{5}\left(\dfrac{5^{|x|+1}}{61}\right)^{\theta-1} & \text{for } x = -2, -1, 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}.$$

(a - 7 pts) Consider a hypothesis test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ based on a single observation x which rejects H_0 if $x \geq 1$ and fails to reject H_0 otherwise. What is the size of the test?
(b - 7 pts) Compute the probability of a Type II error for the test in part (a).
(c - 6 pts) Let α denote the size of the test in part (a). Is the test in part (a) a uniformly most powerful (UMP) level α test for $H_0 : \theta = 1$ versus $H_1 : \theta = 2$? If so, prove it? If not, find a UMP level α test if one exists.

Problem 5. (20 points) Suppose that $X_1, \ldots, X_n$ is independent, identically distributed random variables from a distribution with probability density function $f(x) = \dfrac{2x}{\theta^2} I_{[0,\theta]}(x)$ where $\theta \in (0, 1]$ is unknown, and suppose that the experimenter is interested in testing

$$H_0 : \theta = 1 \text{ versus } H_1 : \theta < 1.$$

(a - 7 pts) Show that the likelihood ratio test statistic has a critical region of the form

$$\{(x_1, \ldots, x_n) : \max\{x_1, \ldots, x_n\} \leq K\}.$$

(b - 6 pts) Find the value of K such that $P_\theta(\max\{x_1, \ldots, x_n\} \leq K) = .01$.
(c - 7 pts) Find the 99% confidence interval for θ obtained by inverting the likelihood ratio test in parts (a) and (b).