

Lecture 13: Likelihood Ratio Tests for a Random Sample from a Normal Population

MATH 667-01
Statistical Inference
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- We introduce the likelihood ratio test (LRT) discussed in Section 8.2.1 of Casella and Berger (2002)¹.
- First, we illustrate it for testing a hypothesis concerning the mean for normal populations when the variance is assumed to be known.
- Next, we consider when the variance is unknown. In this case, the test statistic is related to a t distribution rather than a normal distribution, so we give some general background on the t distribution from Section 5.3.2 before deriving the LRT.

¹Casella, G. and Berger, R. (2002). *Statistical Inference, Second Edition*. Duxbury Press, Belmont, CA.

- *Definition L13.1* (Def 8.2.1 on p.375): The *likelihood ratio test statistic* for testing $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_0^c$ is

$$\lambda(\mathbf{x}) = \frac{\sup_{\Theta_0} L(\theta; \mathbf{x})}{\sup_{\Theta} L(\theta; \mathbf{x})}.$$

A *likelihood ratio test* (LRT) is any test that has a rejection region of the form $\{\mathbf{x} : \lambda(\mathbf{x}) \leq c\}$, where c is any number satisfying $0 \leq c \leq 1$.

Test for Normal Population with Variance Known

- *Example L13.1:* Suppose X_1, \dots, X_n is a random sample from a $\text{Normal}(\mu, \sigma^2)$ population with μ unknown but σ^2 known and suppose that the experimenter is interested in testing

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu \neq \mu_0.$$

- (a) Show that the likelihood ratio test has a critical region of the form $\left\{x : \frac{|\bar{x} - \mu_0|}{\sigma/\sqrt{n}} \geq K\right\}$.
- (b) Find K so that the size of the test is 0.01.
- (c) What is the probability of a Type II error for the test in part (b) if $\mu_0 = 0$, $\sigma^2 = 4$, $\mu = 1$, and $n = 25$?

Test for Normal Population with Variance Known

- *Answer to Example L13.1:* (a) The likelihood function is

$$\begin{aligned}L(\mu; \mathbf{x}) &= (2\pi\sigma^2)^{-n/2} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right\} \\&= (2\pi\sigma^2)^{-n/2} \exp \left\{ -\frac{1}{2\sigma^2} \left(\sum_{i=1}^n x_i^2 - 2\mu \sum_{i=1}^n x_i + n\mu^2 \right) \right\} \\&= (2\pi\sigma^2)^{-n/2} \exp \left\{ -\frac{1}{2\sigma^2} \left(\sum_{i=1}^n x_i^2 - 2\mu n\bar{x} + n\mu^2 \right) \right\}.\end{aligned}$$

Since $\Theta_0 = \{\mu_0\}$,

$$\sup_{\mu \in \Theta_0} L(\mu; \mathbf{x}) = L(\mu_0; \mathbf{x}) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} (\sum_{i=1}^n x_i^2 - 2n\mu_0\bar{x} + n\mu_0^2)}.$$

Since the MLE of μ is $\hat{\mu} = \bar{x}$,

$$\sup_{\mu \in \Theta} L(\mu; \mathbf{x}) = L(\bar{x}; \mathbf{x}) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} (\sum_{i=1}^n x_i^2 - n\bar{x}^2)}.$$

Test for Normal Population with Variance Known

- *Answer to Example L13.1 continued:* The likelihood ratio is

$$\begin{aligned}\lambda(\mathbf{x}) &= \frac{(2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2} \left(\sum_{i=1}^n x_i^2 - 2n\mu_0\bar{x} + n\mu_0^2\right)\right\}}{(2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2} \left(\sum_{i=1}^n x_i^2 - n\bar{x}^2\right)\right\}} \\&= \exp\left\{-\frac{1}{2\sigma^2} \left(n\bar{x}^2 - 2n\mu_0\bar{x} + n\mu_0^2\right)\right\} \\&= \exp\left\{-\frac{n}{2\sigma^2} (\bar{x} - \mu_0)^2\right\} = \exp\left\{-\frac{1}{2} \left(\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right)^2\right\},\end{aligned}$$

and we reject H_0 if $\lambda(\mathbf{x}) \leq c$.

Since $h(t) = \sqrt{-2 \ln t}$ is a decreasing function of t , $\lambda(\mathbf{x}) \leq c$ if and only if

$$\frac{|\bar{x} - \mu_0|}{\sigma/\sqrt{n}} = h(\lambda(\mathbf{x})) \geq h(c) = \sqrt{-2 \ln c} = K.$$

Test for Normal Population with Variance Known

- *Answer to Example L13.1 continued:* (b) If H_0 is true, then $\bar{X} \sim \text{Normal}(\mu_0, \frac{\sigma^2}{n})$ so that $Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim \text{Normal}(0, 1)$.
The value of K such that

$$P(|Z| \geq K) = .01 \Leftrightarrow P(Z \geq K) = .005$$

is $K = 2.576$. (This can be obtained by looking up the cumulative probability .995 on the normal table or using the R command `qnorm(.995)`.)

Test for Normal Population with Variance Known

- *Answer to Example L13.1 continued:* (c) For testing $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$ when the true value of the parameter is $\mu = 1$, the probability of a Type II error is

$$\begin{aligned} P\left(\frac{|\bar{X} - 0|}{2/\sqrt{25}} \geq 2.576\right) &= P(|\bar{X}| \geq 1.03) \\ &= P(\bar{X} \geq 1.03) + P(\bar{X} \leq -1.03) \\ &= P\left(\frac{\bar{X} - 1}{2/\sqrt{25}} \geq .08\right) + P\left(\frac{\bar{X} - 1}{2/\sqrt{25}} \leq -5.08\right) \\ &\approx .4681 + .0000 = .4681. \end{aligned}$$

- A more precise answer can be obtain using R:

```
> K=qnorm(.995)
> (1-pnorm(K-2.5))+pnorm(-K-2.5)
[1] 0.4697776
```


- So, if X_1, \dots, X_n is a random sample from a Normal population, then we know that $\bar{X} \sim \text{Normal}(\mu, \sigma^2/n)$ so that $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0, 1)$.
- Unfortunately, the expression $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ involves two (unknown) parameters μ and σ , and in situations where we want to make inferences about μ , we would prefer an expression in which μ is the only unknown parameter.
- What if we replace σ by its estimate S ? Then we get

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}{\frac{S/\sqrt{n}}{\sigma/\sqrt{n}}} = \frac{Z}{\sqrt{S^2/\sigma^2}} = \frac{Z}{\sqrt{V/p}}$$

where $Z \sim \text{Normal}(0, 1)$, $V \sim \chi_p^2$, Z and V are independent, and $p = n - 1$.

- *Definition L13.2* (Def 5.3.4 on p.223): If $Z \sim \text{Normal}(0, 1)$, $V \sim \chi_p^2$, and Z and V are independent, then we say random variable $T = Z/\sqrt{V/p}$ has *Student's t distribution with p degrees of freedom*, and we write $T \sim t_p$.
- Let X_1, \dots, X_n be a random sample from a $\text{Normal}(\mu, \sigma^2)$ distribution. The quantity $(\bar{X} - \mu)/(S/\sqrt{n})$ has *Student's t distribution with $n - 1$ degrees of freedom*.
- *Theorem L13.1*: (a) If $T \sim t_p$, then the pdf of T is

$$f_T(t) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1 + t^2/p)^{(p+1)/2}}, -\infty < t < \infty.$$

(b) If $T \sim t_p$ with $p > 1$, then $E[T] = 0$.

(c) If $T \sim t_p$ with $p > 2$, then $\text{Var}[T] = \frac{p}{p-2}$.

- *Proof of Theorem L13.1(a):* Since $Z \sim \text{Normal}(0, 1)$ and $V \sim \chi_p^2$ are independent, their joint pdf is

$$\begin{aligned} f_{Z,V}(z, v) &\stackrel{2.4}{=} f_Z(z)f_V(v) \\ &= \frac{1}{\sqrt{2\pi}}e^{-z^2/2} \frac{1}{\Gamma(p/2)2^{p/2}}v^{(p/2)-1}e^{-v/2}I_{(0,\infty)}(v). \end{aligned}$$

- Now, we make the bivariate transformation $T = Z/\sqrt{V/p}$ and $U = V$.

Proof of Theorem L13.1(a) continued:

- This transformation can be inverted as follows.

$$\left. \begin{array}{l} t = \frac{z}{\sqrt{v/p}} \\ u = v \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} z = t\sqrt{u/p} \\ v = u \end{array} \right.$$

- Since $J = \left| \frac{\partial(z, v)}{\partial(t, u)} \right| = \left| \begin{array}{cc} \sqrt{u/p} & 1/(2\sqrt{up}) \\ 0 & 1 \end{array} \right| = \sqrt{\frac{u}{p}}$, the joint pdf of T and U is

$$\begin{aligned} f_{T,U}(t, u) &\stackrel{4.18}{=} f_{Z,V} \left(t\sqrt{\frac{u}{p}}, u \right) |J| \\ &= \frac{e^{-t^2 u / (2p)} u^{(p/2)-1} e^{-u/2} \sqrt{u/p}}{\sqrt{2\pi} \Gamma(p/2) 2^{p/2}} I_{(0,\infty)}(u). \end{aligned}$$

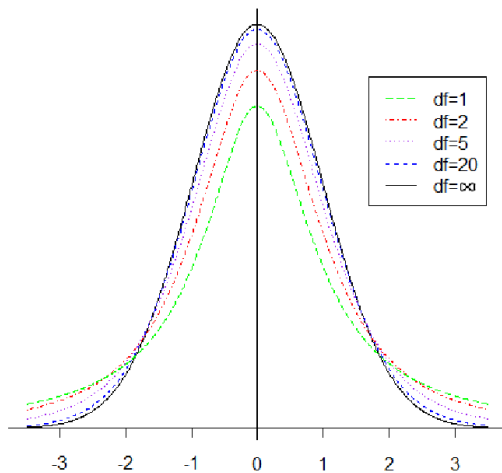
Proof of Theorem L13.1(a) continued:

- Then, to get the marginal distribution of T , we integrate out U and obtain

$$\begin{aligned}
 f_T(t) &= \int_0^\infty f_{T,U}(t, u) \, du \\
 &= \frac{1}{\sqrt{2\pi p} \Gamma(p/2) 2^{p/2}} \int_0^\infty u^{(p-1)/2} e^{-u(1+t^2/p)/2} \, du \\
 &= \frac{1}{\sqrt{2\pi p} \Gamma(p/2) 2^{p/2}} \int_0^\infty u^{(p+1)/2-1} \exp \left\{ -\frac{u}{2 \left(\frac{1}{1+t^2/p} \right)} \right\} \, du \\
 &= \frac{\Gamma\left(\frac{p+1}{2}\right) \left(2 \left(\frac{1}{1+t^2/p}\right)\right)^{(p+1)/2}}{\sqrt{2\pi p} \Gamma(p/2) 2^{p/2}} \int_0^\infty \frac{u^{(p+1)/2-1} \exp \left\{ -\frac{u}{2 \left(\frac{1}{1+t^2/p} \right)} \right\}}{\Gamma\left(\frac{p+1}{2}\right) \left(2 \left(\frac{1}{1+t^2/p}\right)\right)^{(p+1)/2}} \, du \\
 &= \frac{\Gamma\left(\frac{p+1}{2}\right) \left(2 \left(\frac{1}{1+t^2/p}\right)\right)^{(p+1)/2}}{\sqrt{2\pi p} \Gamma(p/2) 2^{p/2}} = \frac{\Gamma\left(\frac{p+1}{2}\right) \left(\frac{1}{1+t^2/p}\right)^{(p+1)/2}}{\sqrt{\pi p} \Gamma(p/2)}.
 \end{aligned}$$

t Distribution

t distribution



t Distribution

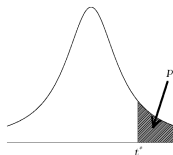


Table for t distribution

t distribution critical values

df	Upper-tail probability p								
	.25	.20	.15	.10	.05	.025	.02	.01	.005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.90	31.82	63.66
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841
\vdots									
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581
∞	0.674	0.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576
	50%	60%	70%	80%	90%	95%	96%	98%	99%
Confidence level C									

Test for Normal Population with Variance Unknown

- *Example L13.2:* Suppose X_1, \dots, X_n is a random sample from a $\text{Normal}(\mu, \sigma^2)$ population with μ and σ^2 unknown and suppose that the experimenter is interested in testing

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu \neq \mu_0$$

where $\Theta = \{(\mu, \sigma^2) : \mu \in (-\infty, \infty) \text{ and } \sigma \in (0, \infty)\}$.

- (a) Show that the likelihood ratio test statistic has a critical region of the form $\left\{ \mathbf{x} : \frac{|\bar{x} - \mu_0|}{s/\sqrt{n}} \geq K \right\}$.
- (b) Find K so that the size of the test is 0.01.

Test for Normal Population with Variance Unknown

- *Answer to Example L13.2:* (a) The likelihood function is

$$L(\mu, \sigma^2; \mathbf{x}) = (2\pi\sigma^2)^{-n/2} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right\}.$$

Since the **maximizer** of L over $\Theta_0 = \{(\mu_0, \sigma^2) : \sigma^2 > 0\}$ is

$(\mu_0, \tilde{\sigma}^2)$ where $\tilde{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu_0)^2$, we have

$$\begin{aligned} \sup_{\theta \in \Theta_0} L(\theta; \mathbf{x}) &= (2\pi\tilde{\sigma}^2)^{-n/2} e^{-\frac{1}{2\tilde{\sigma}^2} \sum_{i=1}^n (x_i - \mu_0)^2} \\ &= (2\pi\tilde{\sigma}^2)^{-n/2} e^{-\frac{1}{2\tilde{\sigma}^2} n\tilde{\sigma}^2} = (2\pi e\tilde{\sigma}^2)^{-n/2}. \end{aligned}$$

Since the MLE of (μ, σ^2) is $(\bar{x}, \hat{\sigma}^2)$ with $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$,

$$\begin{aligned} \sup_{\theta \in \Theta} L(\theta; \mathbf{x}) &= (2\pi\hat{\sigma}^2)^{-n/2} e^{-\frac{1}{2\hat{\sigma}^2} \sum_{i=1}^n (x_i - \bar{x})^2} \\ &= (2\pi\hat{\sigma}^2)^{-n/2} e^{-\frac{1}{2\hat{\sigma}^2} n\hat{\sigma}^2} = (2\pi e\hat{\sigma}^2)^{-n/2}. \end{aligned}$$

Test for Normal Population with Variance Unknown

- *Answer to Example L13.2 continued:* The likelihood ratio is

$$\begin{aligned}\lambda(\mathbf{x}) &= \frac{(2\pi e\tilde{\sigma}^2)^{-n/2}}{(2\pi e\hat{\sigma}^2)^{-n/2}} = (\tilde{\sigma}^2/\hat{\sigma}^2)^{-n/2} \\&= \left(\frac{\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right)^{-n/2} \\&= \left(1 + \frac{n(\bar{x} - \mu_0)^2}{(n-1)s^2} \right)^{-n/2} = \left(1 + \frac{1}{n-1} \left(\frac{\bar{x} - \mu_0}{s/\sqrt{n}} \right)^2 \right)^{-n/2}\end{aligned}$$

and we reject H_0 if $\lambda(\mathbf{x}) \leq c$.

Since $h(t) = (n-1)(t^{-2/n} - 1)$ is a decreasing function of t ,
 $\lambda(\mathbf{x}) \leq c$ if and only if

$$\frac{|\bar{x} - \mu_0|}{s/\sqrt{n}} = h(\lambda(\mathbf{x})) \geq h(c) = (n-1)(c^{-2/n} - 1) = K.$$

Test for Normal Population with Variance Unknown

- *Answer to Example L13.2 continued:*

(b) If H_0 is true, then $T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim t_{24}$.

The value of K such that

$$P(|T| \geq K) = .01 \Leftrightarrow P(T \geq K) = .005$$

is $K = 2.797$. (This can be obtained by looking up the upper-tail probability .005 on the t-table or using the R command `qt(.995,df=24)`.)

Test for Normal Population with Variance Unknown

- *Example L13.3:* Data were collected on pollution in a river around a chemical plant. A government regulation requires that no more than 10 parts per million (ppm) of chemical Z should be present in the river. A scientist collects 16 independent observations from the river and computes the sample mean and sample standard deviation to be $\bar{x} = 10.5$ ppm and $s = 1.6$ ppm, respectively.
 - (a) What are appropriate null and alternative hypotheses?
 - (b) Assuming a normal population and iid observations, perform a hypothesis test at level .05 to determine if there is statistically significant evidence based on the scientist's data that the chemical plant has violated the regulation.

Test for Normal Population with Variance Unknown

- *Answer to Example L13.3:* (a) Let μ be the amount of chemical Z in the river in ppm. The null hypothesis is $H_0 : \mu \leq 10$ versus $H_a : \mu > 10$.

(b) So this is a one-sided test and we need to find the critical region. The work is similar to *Example L13.2* except that the maximizer of L over $\Theta_0 = \{(\mu, \sigma^2) : \sigma^2 > 0\}$ is $(\tilde{\mu}, \tilde{\sigma}^2)$ where

$$\tilde{\mu} = \begin{cases} \bar{x} & \text{if } \bar{x} \leq 10 \\ 10 & \text{if } \bar{x} > 10 \end{cases} \quad \text{and} \quad \tilde{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \tilde{\mu})^2 \quad (\text{see the}$$

method of successive maximization in *Example L2.2*).

So, the likelihood ratio is

$$\lambda(\mathbf{x}) = \begin{cases} \left(1 + \frac{1}{n-1} \left(\frac{\bar{x} - \mu_0}{s/\sqrt{n}} \right)^2 \right)^{-n/2} & \text{if } \bar{x} > 10 \\ 1 & \text{if } \bar{x} \leq 10 \end{cases} .$$

Test for Normal Population with Variance Unknown

- *Answer to Example L13.3 continued:* For $c < 1$, $\lambda(\mathbf{x}) \leq c$ if and only if $\frac{|\bar{x} - 10|}{s/\sqrt{n}} \geq K$ and $\bar{x} > 10$. This implies that the critical region has the form $\left\{ \mathbf{x} : \frac{\bar{x} - 10}{s/\sqrt{n}} \geq K \right\}$.

If H_0 is true, then $T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim t_{15}$.

The value of K such that

$$P(T \geq K) = .05$$

is $K = 1.753$.

So, we fail to reject H_0 since $t = \frac{10.5 - 10}{1.6/\sqrt{16}} = 1.25$ is not in the critical region.

Test for Normal Population with Variance Unknown

- An alternate way to make the decision for *Example L13.3* is to compute the p -value for the test. Since we reject H_0 when $T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$ is large, the p -value $P(T > 1.25)$ is the probability that we will observe a test statistic at least as large as the observed value. If this p -value is smaller than the size of the test .05, then we reject H_0 . In this case, we can see from the t -table that the p -value is between .10 and .15 since $t = 1.25$ is between 1.341 and 1.074. So, we fail to reject H_0 . The exact p -value can be computed in R as follows:

```
> 1-pt(1.25,df=15)  
[1] 0.1152253
```