HW3 solutions

1. (a) If $X \sim \text{Cauchy}(\mu, \sigma)$, Theorem L5.2 implies that there is a $Z \sim \text{Cauchy}(0,1)$ such that $X = \sigma Z + \mu$.

So,
$$P(X \geq \mu) = P(\sigma Z + \mu \geq \mu) = P(Z \geq 0)$$
$$= \int_0^\infty \frac{1}{\pi(1+z^2)}\,dz = \frac{1}{\pi}\Big[\arctan z\Big]_0^\infty = \frac{1}{\pi}\left(\frac{\pi}{2} - 0\right) = \frac{1}{2}.$$

Also, $P(X \leq \mu) = P(X < \mu) = 1 - P(X \geq \mu) = \frac{1}{2}$.

(b)
$$P(X \geq \mu + \sigma) = P(\sigma Z + \mu \geq \mu + \sigma) = P(Z \geq 1)$$
$$= \int_1^\infty \frac{1}{\pi(1+z^2)}\,dz = \frac{1}{\pi}\Big[\arctan z\Big]_1^\infty = \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{4}.$$

$$P(X \leq \mu - \sigma) = P(Z \leq -1) = \int_{-\infty}^{-1} \frac{1}{\pi(1+z^2)}\,dz = \frac{1}{\pi}\Big[\arctan z\Big]_{-\infty}^{-1} = \frac{1}{\pi}\left(-\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{4}.$$

---

2. (a) Yes, we can write

$$f_X(x|\alpha) = \underbrace{e^{-x} I_{(0,\infty)}(x)}_{h(x)} \underbrace{\frac{1}{\Gamma(\alpha)}}_{c(\alpha) = \frac{1}{\Gamma(\alpha)}} e^{\overbrace{(\alpha-1)}^{w(\alpha)} \overbrace{\ln x}^{t(x)}}.$$

(b) Since $w'(\alpha) = 1$,

$$E[\ln X] = E[w'(\alpha)\, t(X)] = -\frac{d}{d\alpha}[\ln c(\alpha)]$$
$$= \frac{d}{d\alpha}\left[-\ln \frac{1}{\Gamma(\alpha)}\right] = \frac{d}{d\alpha}[\ln \Gamma(\alpha)] = \boxed{\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}}.$$

This is called the digamma function.

---

3. (a) The pmf of X_i belongs to an exponential family

$$f(x|\lambda) = \frac{1}{x!} I_{\mathbb{Z}^*}(x) e^{-\lambda} e^{(\ln \lambda)x}, \quad \lambda > 0$$

where $\mathbb{Z}^*$ is the set of all nonnegative integers

with $c(\lambda) e^{-\lambda}$, $w(\lambda) = \ln \lambda$, and $t(x) = x$.

Since $\{w(\lambda) \mid \lambda \in (0,\infty)\} = \mathbb{R}$ contains an open subset of $\mathbb{R}$,

Theorem L 6.2 implies that

$\sum_{j=1}^{n} t(X_j) = \sum_{j=1}^{n} X_j$ is an exponential family with pmf

$$f_{\Sigma X_j}(u) = H(u)\,(c(\lambda))^n e^{(\ln \lambda)u} = H(u) e^{-n\lambda} \lambda^u.$$

(b) It can be seen that $\sum_{j=1}^{n} X_j \sim \text{Poisson}(n\lambda)$ since

$f_{\Sigma X_j}(u)$ has the form of a Poisson pmf with $H(u) = \frac{n^u}{u!} I_{\mathbb{Z}^*}(u)$

so that $f_{\Sigma X_j}(u) = e^{-n\lambda} \frac{(n\lambda)^u}{u!} I_{\mathbb{Z}^*}(u)$.

Then $P(\bar{X} \geq \lambda) = P(\frac{1}{n} \Sigma X_i \geq \lambda) = P(\Sigma X_i \geq n\lambda)$

$$= 1 - P(\Sigma X_i < n\lambda) = 1 - \sum_{u=0}^{\lfloor n\lambda \rfloor} e^{-n\lambda} \frac{(n\lambda)^u}{u!}.$$

with $n = 3$ and $\lambda = 1.2$,

$$P(\bar{X} \geq 1.2) = 1 - \sum_{u=0}^{\lfloor 3.6 \rfloor} e^{-3.6} \frac{(3.6)^u}{u!}$$

$$= 1 - e^{-3.6}\left(1 + 3.6 + \frac{(3.6)^2}{2} + \frac{(3.6)^3}{6}\right) \approx \boxed{0.484784}$$

<u>Alternately</u>: We can show that $\sum_{i=1}^{n} X_i \sim \text{Poisson}(n\lambda)$ as follows using the fact that the MGF of a Poisson(λ) random variable is $M(t) = e^{\lambda(e^t - 1)}$.

$$M_{\Sigma X_i}(t) = E[e^{t\Sigma X_i}] = E[\Pi e^{tX_i}] = \Pi E[e^{tX_i}] = \prod_{i=1}^{n} e^{\lambda(e^t-1)}$$

$$= (e^{\lambda(e^t - 1)})^n = e^{n\lambda(e^t - 1)}.$$

This is the MGF of a Poisson random variable with mean $n\lambda$ so

$\Sigma X_i \sim \text{Poisson}(n\lambda)$.