$$\frac{\partial f}{\partial x} = -1 + 3x^2\frac{dx}{dt} \qquad \left.\frac{\partial f}{\partial x}\right|_{4,0} = -1 = -k$$

$$\frac{\partial f}{\partial(dx/dt)} = x^3 \qquad \left.\frac{\partial f}{\partial(dx/dt)}\right|_{4,0} = 64 = -c.$$

From the table or the diagram above, we see $x = 4$ is an unstable equilibrium position.

This determines what happens near the equilibrium position. The nonlinear terms in the neighborhood of the equilibrium position have been neglected. Are we justified in doing so? A complete answer to that question is postponed, but will be analyzed in later sections on population dynamics. For the moment, let us just say that in "most" cases the results of a linearized stability analysis explains the behavior of the solution in the immediate vicinity of the equilibrium position.

In the case in which the linearized stability analysis predicts the equilibrium solution is unstable, the displacement grows (usually exponentially). Eventually the solution is perturbed so far from the equilibrium that neglecting the nonlinear terms is no longer a valid approximation. When this occurs we can not rely on the results of a linear stability analysis. The solution may or may not continue to depart from the equilibrium position. To analyze this situation, the solution can be discussed in the phase plane.

EXERCISES

27.1. Suppose that

$$\frac{d^2z}{dt^2} + c\frac{dz}{dt} + kz = 0,$$

where c and k are parameters which can be negative, positive or zero.

(a) Under what circumstances does z oscillate with an amplitude staying constant? growing? decaying?

(b) For what values of c and k do there exist initial conditions such that z exponentially grows? decays?

27.2. Assume that for a spring-mass system the restoring force does not depend on the velocity and the friction force does not depend on the displacement. Thus

$$m\frac{d^2x}{dt^2} = -f(x) - g\left(\frac{dx}{dt}\right).$$

If $x = 0$ is an equilibrium point, then what can be said about $f(x)$ and $g(dx/dt)$? Analyze the linear stability of the equilibrium solution $x = 0$.

27.3. Where in the phase plane is it possible for trajectories to cross?

27.4. Analyze equation 27.4 if $c = 0$ with $k > 0$, $k = 0$, and $k < 0$.

28. Nonlinear Pendulum with Damping

As an example of an autonomous system, let us consider a nonlinear pendulum with a damping force. If the frictional force is proportional to the velocity of the mass with frictional coefficient c, then

$$\boxed{L\frac{d^2\theta}{dt^2} = -g\sin\theta - k\frac{d\theta}{dt},} \tag{28.1}$$

where $k = cL/m$ is positive, $k > 0$. Can you envision a situation in which damping occurs in this manner?

We recall the phase plane for the nonlinear pendulum without friction sketched in Fig. 24-4. In particular we are now interested in determining effects due to friction. Before this problem is mathematically solved, can you describe what you expect to occur

1. If a small angle with small velocity is initially prescribed?
2. If an extremely large initial velocity is prescribed?

Hopefully your intuition is good and the mathematics will verify your predictions.

Since an energy integral does not exist for equation 28.1, we must again introduce the phase plane variable, the angular velocity,

$$\boxed{v = d\theta/dt.}$$

$d^2\theta/dt^2 = dv/dt = (d\theta/dt)(dv/d\theta) = v\,dv/d\theta$ and hence equation 28.1 becomes a first-order differential equation,

$$Lv\frac{dv}{d\theta} = -g\sin\theta - kv,$$

or equivalently

$$\boxed{\frac{dv}{d\theta} = \frac{-g\sin\theta - kv}{Lv}.} \tag{28.2}$$

Unlike the frictionless pendulum, we cannot integrate this immediately to obtain energy curves. Instead, the phase plane is sketched. Along $v = 0$, $dv/d\theta = \infty$. There the direction field is vertical. We also recall, for example, that in the upper half plane θ increases since $v = d\theta/dt > 0$ (and arrows thus point to the right as in Fig. 28-1):

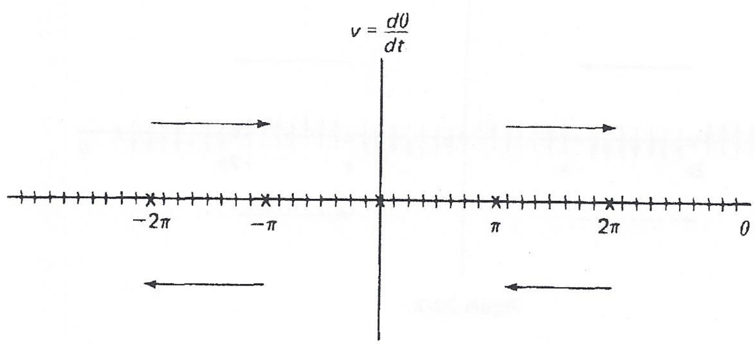

Figure 28-1.

The equilibrium positions are marked with an 'x'. They occur where $dv/d\theta = 0/0$ (see Sec. 27). Note again that $\theta = 0$ is expected to be stable, and that $\theta = \pi$ is expected to be an unstable equilibrium position. This can be verified in a straightforward manner by doing a linearized stability analysis in the neighborhood of the equilibrium positions, as suggested in Sec. 27. For example, near $\theta = 0$, $\sin\theta$ is approximated by θ and hence

$$L\frac{d^2\theta}{dt^2} = -g\theta - k\frac{d\theta}{dt}. \tag{28.3}$$

This equation is mathematically analagous to the equation describing a linear spring-mass system with friction (see Secs. 10–13). $\theta = 0$ is a stable equilibrium position. The angle θ is damped; it is underdamped if $k^2 < 4Lg$ (sufficiently small friction) and overdamped if $k^2 > 4Lg$.

Along the isocline $v = 0$, the solutions in the phase plane must have vertical tangents. However, at $v = 0$ is v increasing or decreasing? In other words, should arrows be introduced on the vertical slashes pointing upwards or downwards? The sign of dv/dt at $v = 0$ determines whether v is increasing or decreasing there. It cannot be determined from the phase plane differential equation, equation 28.2. Instead the time-dependent equation must be analyzed. From equation 28.1, $dv/dt = -g\sin\theta$ at $v = 0$. Thus at $v = 0$, dv/dt is positive where $\sin\theta$ is negative (for example, $-\pi < \theta < 0$) and vice versa. (Can you give a physical interpretation of this result?) Consequently we have Fig. 28-2.

Other than the isocline along which $dv/d\theta = \infty$ (namely $v = 0$), the next most important isocline (and also usually an easy one to determine) is the one along which $dv/d\theta = 0$. From equation 28.2, the curve along which $dv/d\theta = 0$ is

$$v = -\frac{g}{k}\sin\theta.$$

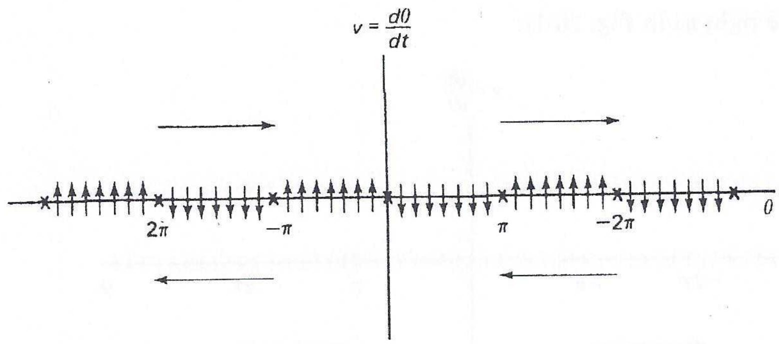

Figure 28-2.

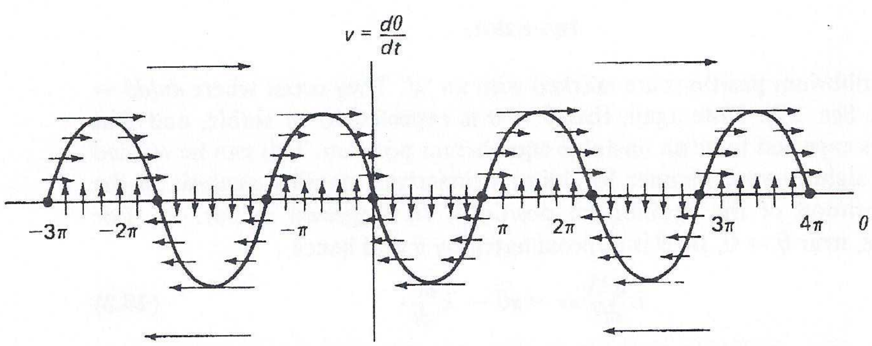

Figure 28-3 Nonlinear pendulum with damping: direction field corresponding to $dv/d\theta$ equal 0 and ∞.

Sketching this curve, the direction field, and the corresponding arrows yields Fig. 28-3.

Before we attempt to make sketches of the solution, the sign of dv/dt should be calculated from

$$\frac{dv}{dt} = -g\frac{\sin\theta}{L} - \frac{kv}{L}.$$

If $dv/dt = h(v, \theta)$, then we know that the sign of dv/dt usually changes at $h(v, \theta) = 0$, in this example the sinuous isocline that has been drawn in Fig. 28-3. As this curve is crossed, the sign of dv/dt changes (if the zero is a simple zero, which it is in this case). On one side of this curve dv/dt is positive, and on the other side dv/dt is negative. For example if $v > -g/k \sin\theta$, then $dv/dt < 0$. Thus trajectories go downward (as indicated by $\downarrow$) above the sinuous sketched curve and vice versa. An alternate method to calculate the sign of dv/dt is to analyze the sign for very large v (both positive and negative). Then, since

$$\frac{dv}{dt} \approx -\frac{k}{L}v,$$

v must decrease if v is sufficiently large and positive. (How large is sufficiently large?) This yields the same result.

In every region of the phase plane, it has been determined whether both v and θ are increasing or decreasing with time. Let us use arrows in the following way to suggest the general direction of the trajectories: for example, if θ decreases in time $\longleftarrow$ and v increases in time $\uparrow$, then the symbol $\nwarrow$ is introduced in the corresponding region of the phase plane. Our results for the nonlinear pendulum with friction are shown in Fig. 28-4.

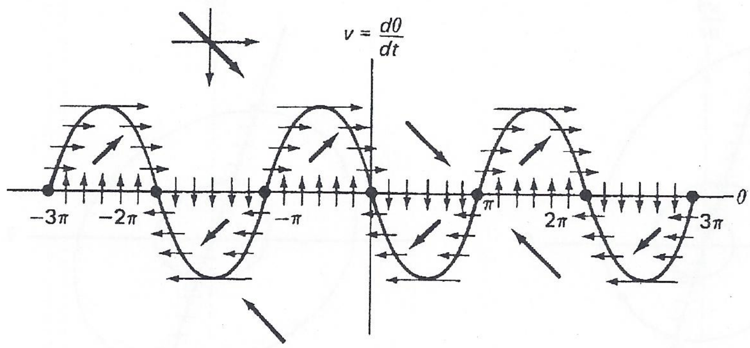

Figure 28-4 General direction of the trajectories.

To determine the accurate behavior of the trajectories, we may further use the method of isoclines. Before doing so, however, we note that often the qualitative behavior can be more easily obtained by first analyzing the phase plane in the neighborhood of the equilibrium positions. Eventually in the context of population dynamics (see Sec. 47), we will discuss a wide range of possible phase planes in the vicinity of equilibrium points. In the meantime, let us discuss the two specific equilibrium positions that occur for a nonlinear pendulum with friction.

Near the "natural" position of the pendulum, $\theta = 0$, the phase plane is as shown in Fig. 28-5. The curve along which $dv/d\theta = 0$, $v = -g/k \sin\theta$, is

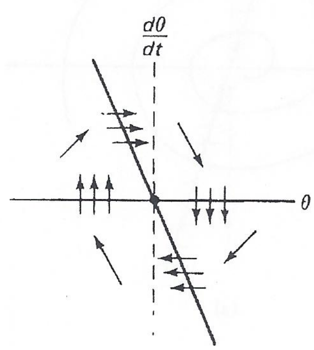

Figure 28-5 Damped pendulum: qualitative behavior of the trajectories in the neighborhood of the natural equilibrium position.

approximated near $\theta = 0$ by the straight line $v = -(g/k)\theta$. The linearized stability analysis in the vicinity of $\theta = 0$ (see equation 28.3) helps to determine the approximate behavior in the neighborhood of $\theta = 0$ of the phase plane. Without the linearized stability analysis, the above diagram suggests the motion may be one of the four types sketched in Fig. 28-6. Note that all four sketches satisfy the qualitative behavior suggested by the arrows in Fig. 28.5. In the vicinity of the equilibrium position, the trajectories (a) spiral

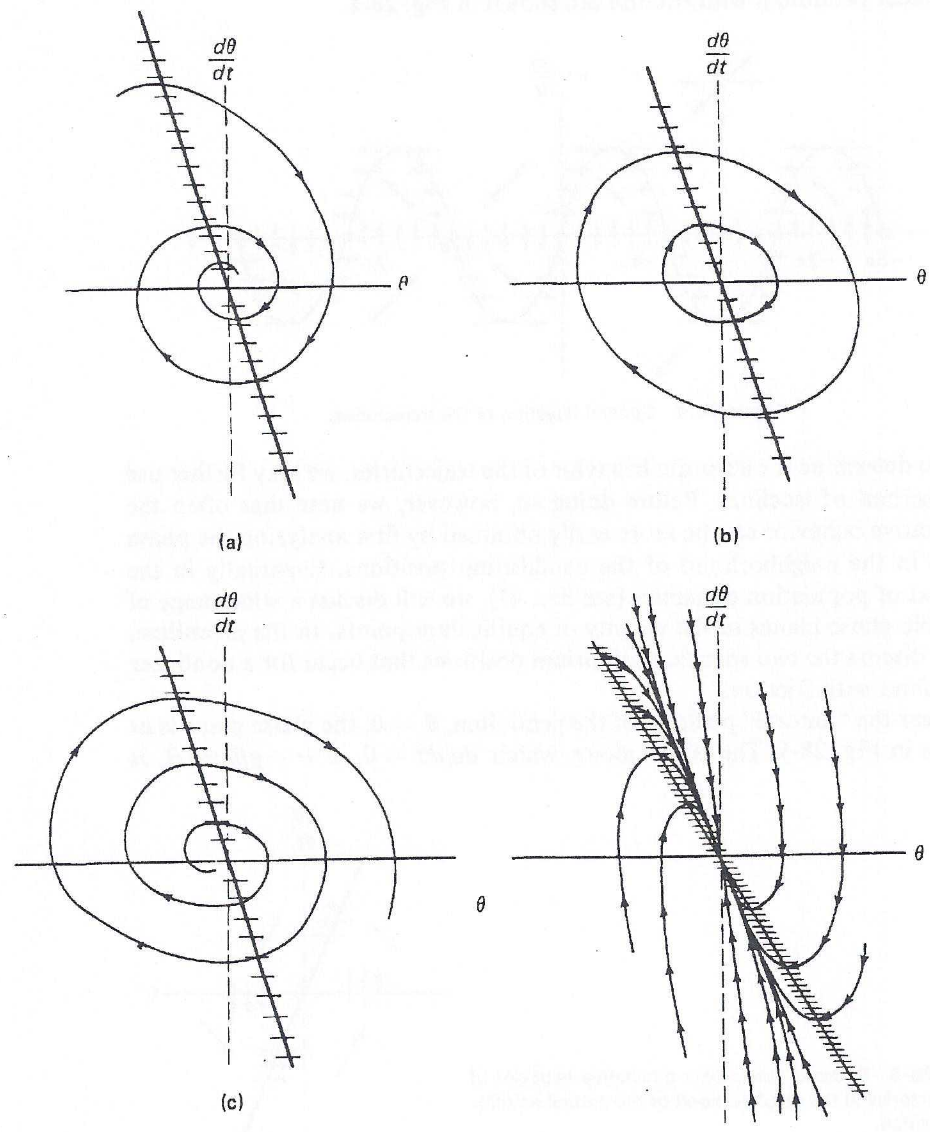

Figure 28-6 Some different types of trajectories in the neighborhood of an equilibrium position (not necessarily for a pendulum).

in towards the equilibrium position, in which case we would say the equilibrium position is stable; or (b) the trajectories could "circle" around the equilibrium position (that is, the trajectories would be closed curves), in which case we say the equilibrium position is neutrally stable; or (c) the trajectories could spiral out from the equilibrium position, in which case we would say the equilibrium position is unstable; or (d) the trajectories tend directly towards the equilibrium position without oscillating around it, a different kind of equilibrium position (details of this last case, called a stable node, are given in Sec. 47).

As we have suggested, the type of equilibrium position is easily determined from the linearized analysis valid in the vicinity of the equilibrium position. From equation 28.3, we see that if there is sufficiently small friction (i.e., *if* $k^2 < 4Lg$), since the solution is an underdamped oscillation, the trajectories must be as illustrated in (a) and spiral inwards. Alternatively, the decay of the energy implies that either (a) or (d) is the correct case (see Fig. 26.1). If $k^2 < 4Lg$, the effect of friction (even the slightest amount of friction, $k > 0$, no matter how small) is to transform the trajectories from closed curves ($k = 0$) to spirals. This is not surprising because we observed in exercise 26.2d the same phenomena when comparing a damped linear oscillator to an undamped one. If $k^2 > 4Lg$, the trajectories will be different. Exercise 28.2 discusses this latter case.

Can a similar analysis be done near $\theta = \pi$? See Fig. 28-7. The trajectories seem to tend towards the equilibrium position if they are in certain regions in

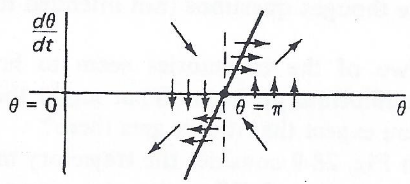

Figure 28-7 Phase plane of a damped pendulum: qualitative behavior near the inverted equilibrium position.

the phase plane, while in other regions the trajectories move away. Clearly this indicates that $\theta = \pi$ is an unstable equilibrium position of the pendulum (as we already suspect on physical grounds and can verify using the linearized analysis of Sec. 27). Let us roughly sketch in Fig. 28-8 some trajectories in the neighborhood of $\theta = \pi$:

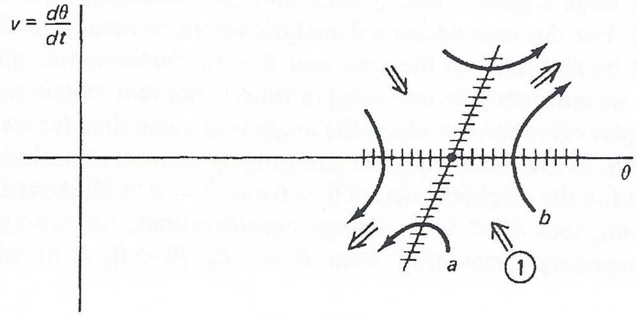

Figure 28-8.

Consider trajectories near the area marked ① in the figure. Some trajectories on the left must cross the isocline along which $dv/d\theta = 0$ and then curve downward as illustrated by curve *a*. Others, more to the right, must turn towards the right as illustrated by curve *b*. Thus there must be a trajectory (in between) which "enters" the unstable equilibrium position. In a similar manner we can easily see that there are four trajectories which enter this unstable equilibrium position (two enter backwards in time) as illustrated in Fig. 28-9. In Sec. 47B we will call such an equilibrium position a **saddle point.** In the neighborhood of the equilibrium, the trajectories that enter a saddle point can be shown to be approximated by *two* straight lines as sketched (see exercise 26.2 and the further developments of exercise 28.1).

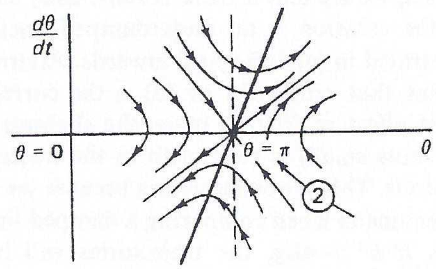

Figure 28-9 Trajectories for a damped pendulum in the vicinity of the unstable equilibrium position.

Some thought questions (not intended to be difficult) follow:

1. Two of the trajectories seem to be tending towards the unstable equilibrium position. What might this correspond to physically? Do you expect that it ever gets there?
2. In Fig. 28-9 consider the trajectory marked ②. Can you explain what is happening? What do you expect will eventually happen to that solution?

Determining the phase plane in the neighborhood of the equilibrium position is not sufficient to completely understand the behavior of the nonlinear pendulum with damping. We can easily imagine an initial condition near $\theta = 0$ with a large angular velocity such that the pendulum does not remain near $\theta = 0$. For this case additional analysis would be necessary. The motion would not be restricted to the area near $\theta = 0$. Furthermore, since $\theta = \pi$ is unstable, we will rarely be interested in trajectories that remain near $\theta = \pi$. To investigate solutions for which the angle is at some time far away from an equilibrium, in the phase plane we can roughly connect the solution curves that are valid in the neighborhood of $\theta = 0$ and $\theta = \pi$ as illustrated in Fig. 28-10 (assuming that $k^2 < 4Lg$). Energy considerations, for example, show that the trajectory emanating from $\theta = -\theta_0$ $(0 < \theta_0 < \pi)$ with

$v = d\theta/dt = 0$ must fall short of $\theta = \theta_0$, when again $v = 0$. The maximum displacement of a pendulum diminishes after each oscillation due to the small friction (as illustrated in Fig. 28-10).

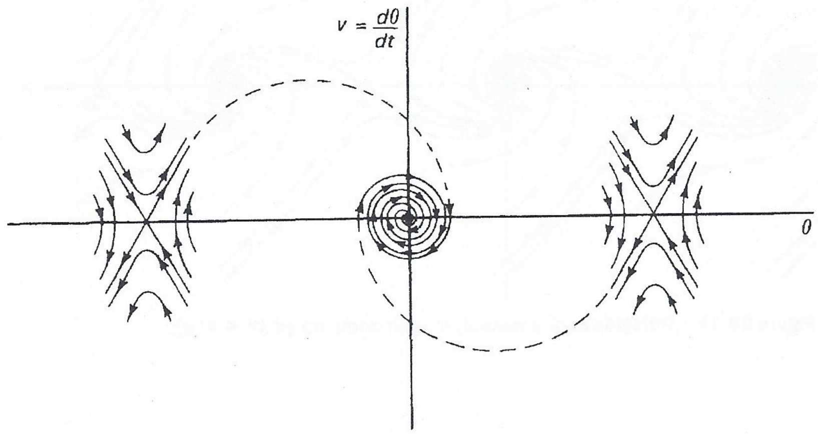

Figure 28-10 Phase plane if $k^2 < 4Lg$: sketch illustrating trajectories in the neighborhood of both equilibrium positions.

To improve this rough sketch, and, in particular, to sketch the phase plane for large velocities, the method of isoclines should be systematically employed. Along the curve

$$v = g\frac{\sin\theta}{k}, \qquad \frac{dv}{d\theta} = \frac{-2k}{L}.$$

More generally along the curves

$$v = \frac{\lambda g \sin\theta}{k}, \qquad \frac{dv}{d\theta} = \frac{-(1+\lambda)k}{\lambda L}.$$

Using these isoclines we obtain an improved sketch of the phase plane in Fig. 28-11 (if $k^2 < 4Lg$). The sketch of the trajectories of the slightly damped nonlinear pendulum shows that we can understand the solution of a complicated mathematical problem without obtaining an explicit solution. Under certain determinable circumstances, the pendulum oscillates with smaller and smaller amplitude. If a sufficiently large initial angular velocity is given, the pendulum will go around a finite number of times (this contrasts with the frictionless nonlinear pendulum in which the pendulum continually goes around and around). The pendulum will be continually slowing down and eventually it will not be able to go completely around. Then the pendulum will oscillate around its natural position with decreasing amplitude (except in the very special cases in which the pendulum approaches the inverted position with zero velocity, but never gets there).

Exercise 28.2 modifies the above discussion when $k^2 > 4Lg$.

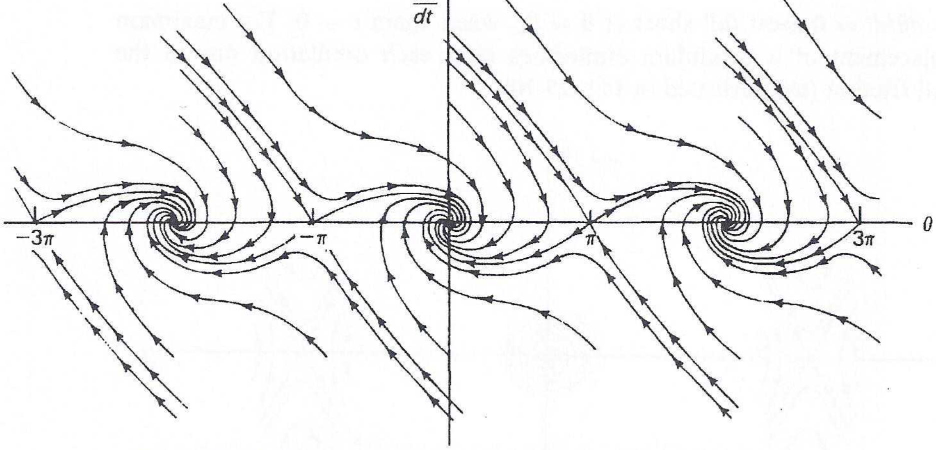

Figure 28-11 Trajectories of a pendulum with damping (if $k^2 < 4Lg$).

EXERCISES

28.1. Consider equation 28.1.

(a) Approximate $\sin\theta$ in the neighborhood of the unstable equilibrium position.

(b) Show that the resulting approximation of the phase plane equation can be put into the form

$$\frac{dv}{d\theta^*} = \frac{c\theta^* + dv}{a\theta^* + bv},$$

where $\theta^* = \theta - \pi$.

(c) Show that two solution curves are straight lines going through the origin ($v = 0, \theta^* = 0$). [Hint: See exercise 26.2]. Show that one straight line has positive slope and the other negative.

28.2. For the nonlinear pendulum with friction, equation 28.1, sketch the solution in the phase plane if friction is sufficiently large, $k^2 > 4Lg$. Pay special attention to the phase plane in the neighborhood of $\theta = 0$ and $\theta = \pi$. Show that there are straight line solutions in the neighborhood of both $\theta = 0$ and $\theta = \pi$. For sketching purposes, you may assume that $L = 1$, $g = 1$, $k = 3$. [Hint: See exercises 26.2 and 28.1.]

28.3. If a spring-mass system has a friction force proportional to the cube of the velocity, then

$$m\frac{d^2x}{dt^2} + \sigma\left(\frac{dx}{dt}\right)^3 + kx = 0.$$

(a) Derive a first-order differential equation describing the phase plane (dx/dt as a function of x).

(b) Sketch the solution in the phase plane.

28.4. Consider the spring-mass system of exercise 28.3 without a restoring force (i.e., $k = 0$).
(a) How do you expect the solution to behave?
(b) Let $v = dx/dt$ and sketch the solution in the phase plane.
(c) Let $v = dx/dt$ and solve the problem exactly.
(d) Show that the solutions of parts (b) and (c) verify part (a).

28.5. Consider a linear pendulum with linearized friction

$$L\frac{d^2\theta}{dt^2} = -g\theta - k\frac{d\theta}{dt}.$$

Under what condition does the pendulum continually oscillate back and forth with decreasing amplitude?

28.6. Consider a nonlinear pendulum with Newtonian damping (see exercise 10.6):

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta - \beta\frac{d\theta}{dt}\left|\frac{d\theta}{dt}\right|,$$

where $\beta > 0$.
(a) Show that an energy integral does not exist.
(b) By introducing the phase plane variable $v = d\theta/dt$, show that

$$\frac{dv}{d\theta} = \frac{-g\sin\theta - \beta v\,|v|}{Lv}.$$

(c) Instead of sketching the isoclines, show that

$$L\frac{d}{d\theta}(v^2) \pm \beta v^2 = -g\sin\theta.$$

(d) Under what conditions does the + or − sign apply?
(e) This is a linear differential equation for v^2. Solve this equation.
(f) Using this solution, roughly sketch the phase plane.
(g) What qualitative differences do you expect to occur between this problem and the one discussed in Sec. 28?

28.7. The Van der Pol oscillator is described by the following nonlinear differential equation:

$$\frac{d^2x}{dt^2} - \epsilon\frac{dx}{dt}(1 - x^2) + \omega^2 x = 0,$$

where $\epsilon \geq 0$.
(a) Briefly describe the physical effect of each term.
(b) If $\epsilon = 0$, what happens?
(c) Is the equilibrium position $x = 0$, linearly stable or unstable?
(d) If displacements are large, what do you expect happens?
(e) Sketch the trajectories in the phase plane if $\omega = 1$ and $\epsilon = \frac{1}{10}$. Describe any interesting features of the solution.

28.8. Reconsider the Van der Pol oscillator of exercise 28.7.

(a) Numerically integrate the differential equation with $\omega = 1$ and $\epsilon = \frac{1}{10}$.

(b) Can the scaling of time justify letting $\omega = 1$ always?

(c) Compare your numerical results to the sketch in the phase plane (exercise 28.7e).

28.9. Rescale equation 28.1 to determine the important dimensionless parameters.

28.10. Consider equation 28.1. Assume $k^2 < 4Lg$. (Let $k = 1, L = 1, g = 1$). Using a computer, solve the initial value problem:

$$\theta(0) = 0, \qquad \frac{d\theta}{dt}(0) = \Omega_0.$$

Determine how many times the pendulum goes completely around as a function of Ω_0.

28.11. Using a linearized stability analysis, show that $\theta = \pi$ is an unstable equilibrium position of a nonlinear pendulum with friction.

29. Further Readings in Mechanical Vibrations

In the preceding sections, only some of the simplest models of mechanical vibrations have been introduced. We progressed from linear undamped oscillators to nonlinear ones with frictional forces. The behavior of the nonlinear systems we have analyzed seem qualitatively similar to linear ones. However, we have *not* made a complete mathematical analysis of all possible problems. By including other types of nonlinear and frictional forces (for example, as occur in certain electrical devices), we could find behavior not suggested by linear problems especially if external periodic forces are included. However, our goal is only to *introduce* the concepts of applied mathematics, and thus we may end our preliminary investigation of mechanical vibrations at this point. For further studies, I refer the interested reader to the following excellent books:

ANDRONOW, A. A., CHAIKIN, C. E., and WITT, A. A., *Theory of Oscillations.* Princeton, N.J.: Princeton University Press, 1949. (This includes a good discussion of the theory of the clock!)

STOKER, J. J., *Nonlinear Vibrations.* New York: Interscience, 1950.

Other interesting problems involving the coupling of two or more spring-mass systems and/or pendulums as well as problems involving rigid bodies in two and three dimensions can be found in a wide variety of texts, whose titles often contain "mechanics" or "dynamics." In particular, two well-known ones are:

GOLDSTEIN, H., *Classical Mechanics*. Reading, Mass.: Addison-Wesley, 1950.

LANDAU, L. D. and LIFSHITZ, E. M., *Mechanics*. Readings Mass.: Addison-Wesley, 1960.