Lecture 9: UMVUEs and the Cramér-Rao Lower Bound

MATH 667-01
Statistical Inference
University of Louisville

October 5, 2017

Last modified: 10/17/2017

Introduction

- We discuss uniform minimum variance unbiased estimators as discussed in Section 7.3 of Casella and Berger (2002)[1].
- We review correlation from Section 4.5.
- We discuss and prove the Cramér-Rao Inequality and some corollaries. The regularity conditions in these notes are from Section 7.3 of Casella and Berger (1990)[2].
- We present several examples to illustrate the results.

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[1] Casella, G. and Berger, R. (2002). *Statistical Inference, Second Edition*. Duxbury Press, Belmont, CA.

[2] Casella, G. and Berger, R. (1990). *Statistical Inference*. Duxbury Press, Belmont, CA.

Best Unbiased Estimator (UMVUE)

- In this lecture, we evaluate an estimator W of a parameter θ based on the squared error loss function.
- If we consider only unbiased estimators, then $\mathsf{E}_\theta[(W-\theta)^2] = \mathsf{Var}_\theta[W]$.
- *Definition L9.1* (Def 7.3.7 on p.334): An estimator W^* is a *best unbiased estimator* of $\tau(\theta)$ if it satisfies $\mathsf{E}_\theta[W^*] = \tau(\theta)$ for all θ and, for any other unbiased estimator W with $\mathsf{E}_\theta[W] = \tau(\theta)$, we have $\mathsf{Var}_\theta[W^*] \leq \mathsf{Var}_\theta[W]$ for all θ.
- W^* is also called a *uniform minimum variance unbiased estimator* (UMVUE) of $\tau(\theta)$.

- *Example L9.1*: Let $X_1, \ldots, X_n$ be iid Poisson(λ). Both $\bar{X}$ and S^2 are unbiased estimators of λ since $\mathsf{E}[X_1] = \mathsf{Var}[X_1] = \lambda$ so that $\mathsf{E}[\bar{X}] = \mathsf{E}[S^2] = \lambda$. For what values of λ is the variance of $\bar{X}$ smaller than the variance of S^2?
- *Answer to Example L9.1*: We know $\mathsf{Var}[\bar{X}] \stackrel{3.19}{=} \dfrac{\mathsf{Var}[X_1]}{n} = \dfrac{\lambda}{n}$. It can be shown that

$$\begin{aligned} \mathsf{Var}[S^2] &= \frac{1}{n}\left[\lambda(1+3\lambda) - \frac{n-3}{n-1}\lambda^2\right] \\ &= \frac{1}{n}\left[\lambda + \frac{2n}{n-1}\lambda^2\right] \end{aligned}$$

so $\mathsf{Var}[\bar{X}] < \mathsf{Var}[S^2]$ for all λ.

Review: Correlation

- $\mathsf{E}[X] = \mu_X$, $\mathsf{E}[Y] = \mu_Y$, $\mathsf{Var}[X] = \sigma_X^2$, $\mathsf{Var}[Y] = \sigma_Y^2$
- Assume $0 < \sigma_X^2 < \infty$ and $0 < \sigma_Y^2 < \infty$
- *Definition L9.2* (Def 4.5.2 on p.169): The *correlation of X and Y* is the number defined by

$$\rho_{XY} = \frac{\mathsf{Cov}[X, Y]}{\sigma_X \sigma_Y}.$$

  The value ρ_{XY} is also called the *correlation coefficient*.
- *Theorem L9.1* (Thm 4.5.7 on p.172): For any random variables X and Y,
  - (a) $-1 \le \rho_{XY} \le 1$.
  - (b) $|\rho_{XY}| = 1$ if and only if there exists numbers $a \neq 0$ and b such that $P(Y = aX + b) = 1$. If $\rho_{XY} = 1$ then $a > 0$, and if $\rho_{XY} = -1$ then $a < 0$.

Cramér-Rao Lower Bound

- *Theorem L9.2* (p.335): Let $X_1, \ldots, X_n$ be a sample with pdf $f(\boldsymbol{x}|\theta)$, and let $W(\boldsymbol{X}) = W(X_1, \ldots, X_n)$ be any estimator where $\mathsf{E}_\theta[W(\boldsymbol{X})]$ is a differentiable function of θ. Suppose the joint pdf $f(\boldsymbol{x}|\theta)$ satisfies

$$\frac{d}{d\theta} \int \cdots \int h(\boldsymbol{x}) f(\boldsymbol{x}|\theta)\ d\boldsymbol{x} = \int \cdots \int h(\boldsymbol{x}) \frac{\partial}{\partial\theta} f(\boldsymbol{x}|\theta)\ d\boldsymbol{x},$$

  for any function $h(\boldsymbol{x})$ with $\mathsf{E}_\theta[\ |h(\boldsymbol{X})|\ ] < \infty$. Then

$$\mathsf{Var}_\theta[W(\boldsymbol{X})] \geq \frac{\left\{\frac{d}{d\theta}\mathsf{E}_\theta[W(\boldsymbol{X})]\right\}^2}{\mathsf{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right)^2\right]}.$$

- The inequality is referred to as the Cramér-Rao inequality.
- If $W(\boldsymbol{X})$ is an unbiased estimator of $\tau(\theta)$, then the numerator becomes

$$\left(\frac{d}{d\theta}\mathsf{E}_\theta[W(\boldsymbol{X})]\right)^2 = \left(\tau'(\theta)\right)^2.$$

- *Proof of Theorem L9.2*: Since *Theorem L9.1*(a) implies

$$\left\{\mathsf{Cov}\left[W(\boldsymbol{X}), \frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right]\right\}^2 \leq \mathsf{Var}[W(\boldsymbol{X})]\mathsf{Var}\left[\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right],$$

it follows that

$$\mathsf{Var}[W(\boldsymbol{X})] \geq \frac{\left\{\mathsf{Cov}\left[W(\boldsymbol{X}), \frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right]\right\}^2}{\mathsf{Var}\left[\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right]}.$$

- *Proof of Theorem L9.2 continued*: Note that

$$
\begin{aligned}
\mathsf{E}\left[\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right] &= \int_{\mathcal{X}} \frac{\partial}{\partial\theta}\ln f(\boldsymbol{x}|\theta) f(\boldsymbol{x}|\theta)\ d\boldsymbol{x} \\
&= \int_{\mathcal{X}} \frac{\frac{\partial}{\partial\theta} f(\boldsymbol{x}|\theta)}{f(\boldsymbol{x}|\theta)} f(\boldsymbol{x}|\theta)\ d\boldsymbol{x} \\
&= \int_{\mathcal{X}} \frac{\partial}{\partial\theta} f(\boldsymbol{x}|\theta)\ d\boldsymbol{x} \\
&= \frac{d}{d\theta}\int_{\mathcal{X}} f(\boldsymbol{x}|\theta)\ d\boldsymbol{x} = \frac{d}{d\theta} 1 = 0.
\end{aligned}
$$

- *Proof of Theorem L9.2 continued*: Then we have

$$
\begin{aligned}
\mathsf{Cov}\left[W(\boldsymbol{X}), \frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right] &= \mathsf{E}\left[W(\boldsymbol{X})\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right] \\
&= \mathsf{E}\left[W(\boldsymbol{X})\frac{\frac{\partial}{\partial\theta}f(\boldsymbol{X}|\theta)}{f(\boldsymbol{X}|\theta)}\right] \\
&= \int_{\mathcal{X}} \frac{d}{d\theta}[W(\boldsymbol{x})f(\boldsymbol{x}|\theta)]\ d\boldsymbol{x} \\
&= \frac{d}{d\theta}\mathsf{E}[W(\boldsymbol{X})]
\end{aligned}
$$

and

$$
\mathsf{Var}\left[\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right] = \mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right)^2\right].
$$

- *Theorem L9.3* (p.337): Let $X_1, \dots, X_n$ be iid with pdf $f(x|\theta)$, and let $W(\boldsymbol{X}) = W(X_1, \dots, X_n)$ be any estimator where $\mathsf{E}_\theta[W(\boldsymbol{X})]$ is a differentiable function of θ. If the joint pdf $f(\boldsymbol{x}|\theta) = \prod f(x_i|\theta)$ satisfies

$$\frac{d}{d\theta} \int \cdots \int h(\boldsymbol{x}) f(\boldsymbol{x}|\theta)\ d\boldsymbol{x} = \int \cdots \int h(\boldsymbol{x}) \frac{\partial}{\partial\theta} f(\boldsymbol{x}|\theta)\ d\boldsymbol{x},$$

for any function $h(\boldsymbol{x})$ with $\mathsf{E}_\theta[\ |h(\boldsymbol{X})|\ ] < \infty$, then

$$\mathsf{Var}_\theta[W(\boldsymbol{X})] \geq \frac{\left(\frac{d}{d\theta}\mathsf{E}_\theta[W(\boldsymbol{X})]\right)^2}{n\mathsf{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\ln f(X|\theta)\right)^2\right]}.$$

Cramér-Rao Lower Bound (iid case)

- *Proof of Theorem L9.3 continued*: If we also assume that $X_1, \ldots, X_n$ is iid, then we have

$$\mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right)^2\right] = \mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln\prod_{i=1}^{n} f(X_i|\theta)\right)^2\right]$$

$$= \mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\sum_{i=1}^{n}\ln f(X_i|\theta)\right)^2\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X_i|\theta)\right)\left(\frac{\partial}{\partial\theta}\ln f(X_j|\theta)\right)\right]$$

$$= \sum_{i=1}^{n}\mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X_i|\theta)\right)^2\right] +$$

$$\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X_i|\theta)\right)\left(\frac{\partial}{\partial\theta}\ln f(X_j|\theta)\right)\right]$$

Cramér-Rao Lower Bound (iid case)

- *Proof of Theorem L9.3 continued*:

$$
\begin{aligned}
&= \sum_{i=1}^{n} \mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X_i|\theta)\right)^2\right] + \\
&\quad \sum_{i=1}^{n}\sum_{j\neq i} \mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X_i|\theta)\right)\left(\frac{\partial}{\partial\theta}\ln f(X_j|\theta)\right)\right] \\
&= \sum_{i=1}^{n} \mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X_i|\theta)\right)^2\right] + \\
&\quad \sum_{i=1}^{n}\sum_{j\neq i} \mathsf{E}\left[\frac{\partial}{\partial\theta}\ln f(X_i|\theta)\right]\mathsf{E}\left[\frac{\partial}{\partial\theta}\ln f(X_j|\theta)\right] \\
&= \sum_{i=1}^{n} \mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X_i|\theta)\right)^2\right] \\
&= n\mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X|\theta)\right)^2\right].
\end{aligned}
$$

Cramér-Rao Lower Bound (iid case)

- The quantity $\mathsf{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\ln f(X|\theta)\right)^2\right]$ is called the *information number*, or *Fisher information* of the sample.
- *Theorem L9.4* (Lem 7.3.11 on p.338): If $f(x|\theta)$ satisfies

$$\frac{d}{d\theta}\mathsf{E}_\theta\left[\frac{\partial}{\partial\theta}\ln f(X|\theta)\right] = \int \frac{\partial}{\partial\theta}\left[\left(\frac{\partial}{\partial\theta}\ln f(x|\theta)\right) f(x|\theta)\right]\, dx$$

and $\frac{d}{d\theta}\int f(x|\theta)\, dx = \int \frac{\partial}{\partial\theta} f(x|\theta)\, dx$, then

$$\mathsf{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\ln f(X|\theta)\right)^2\right] = -\mathsf{E}_\theta\left[\frac{\partial^2}{\partial\theta^2}\ln f(X|\theta)\right].$$

- The condition on $f(x|\theta)$, and consequently the result, is true for an exponential family.

- *Proof of Theorem L9.4*: Note that

$$
\begin{aligned}
\frac{\partial^2}{\partial\theta^2}\left[\ln f(x|\theta)\right] &= \frac{\partial}{\partial\theta}\left\{\frac{\frac{\partial}{\partial\theta} f(x|\theta)}{f(x|\theta)}\right\} \\
&= \frac{\frac{\partial^2}{\partial\theta^2} f(x|\theta)}{f(x|\theta)} - \frac{\left(\frac{\partial}{\partial\theta} f(x|\theta)\right)^2}{(f(x|\theta))^2}.
\end{aligned}
$$

- Then, we have

$$
\begin{aligned}
\mathsf{E}\left[\frac{\frac{\partial^2}{\partial\theta^2} f(X|\theta)}{f(X|\theta)}\right] &= \int \frac{\partial^2}{\partial\theta^2} f(x|\theta)\ dx \\
&= \int \frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial\theta} f(x|\theta)\right\}\ dx
\end{aligned}
$$

- *Proof of Theorem L9.4 continued*:

$$
\begin{aligned}
&= \int \frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial\theta} f(x|\theta)\right\}\ dx \\
&= \int \frac{\partial}{\partial\theta}\left\{\frac{\frac{\partial}{\partial\theta} f(x|\theta)}{f(x|\theta)} f(x|\theta)\right\}\ dx \\
&= \int \frac{\partial}{\partial\theta}\left\{\left(\frac{\partial}{\partial\theta}\ln f(x|\theta)\right) f(x|\theta)\right\}\ dx \\
&= \frac{d}{d\theta}\int \left\{\left(\frac{\partial}{\partial\theta}\ln f(x|\theta)\right) f(x|\theta)\right\}\ dx \\
&= \frac{d}{d\theta}\int \left\{\frac{\frac{\partial}{\partial\theta} f(x|\theta)}{f(x|\theta)} f(x|\theta)\right\}\ dx \\
&= \frac{d}{d\theta}\int \frac{\partial}{\partial\theta} f(x|\theta)\ dx = \frac{d}{d\theta}\frac{d}{d\theta}\int f(x|\theta)\ dx = \frac{d}{d\theta}[1] = 0
\end{aligned}
$$

- *Proof of Theorem L9.4 continued*: So, it follows that

$$
\begin{aligned}
\mathsf{E}\left[\frac{\partial^2}{\partial\theta^2}\ln f(X|\theta)\right] &= \mathsf{E}\left[\frac{\frac{\partial^2}{\partial\theta^2} f(X|\theta)}{f(X|\theta)}\right] - \mathsf{E}\left[\frac{\left(\frac{\partial}{\partial\theta} f(X|\theta)\right)^2}{(f(X|\theta))^2}\right] \\
&= 0 - \mathsf{E}\left[\left(\frac{\frac{\partial}{\partial\theta} f(X|\theta)}{f(X|\theta)}\right)^2\right] \\
&= -\mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X|\theta)\right)^2\right].
\end{aligned}
$$

Cramér-Rao Lower Bound (iid case)

- *Example L9.2*: Let $X_1, \ldots, X_n$ be iid Poisson(λ). Find the Cramér-Rao lower bound on the variance of unbiased estimators of λ. Also, find the MLE and show that it is the UMVUE of λ.
- *Answer to Example L9.2*: Since $\frac{\partial^2}{\partial\lambda^2} \ln f(x|\lambda) = \frac{\partial^2}{\partial\lambda^2} \left[\ln\left\{\lambda^x e^{-\lambda}(x!)^{-1}\right\}\right] = \frac{\partial^2}{\partial\lambda^2} \left[x \ln\lambda - \lambda - \ln(x!)\right] = -\frac{x}{\lambda^2}$, we have

$$\mathsf{E}\left[\frac{\partial^2}{\partial\lambda^2} \ln f(X|\lambda)\right] = \mathsf{E}\left[-\frac{1}{\lambda^2} X\right] = -\frac{1}{\lambda^2}\mathsf{E}[X] = -\frac{1}{\lambda^2}\lambda = -\frac{1}{\lambda}.$$

By *Theorem L9.4*,

$$\mathsf{E}\left[\left(\frac{\partial}{\partial\theta} \ln f(X|\theta)\right)^2\right] = -\mathsf{E}\left[\frac{\partial^2}{\partial\lambda^2} \ln f(X|\lambda)\right] = \frac{1}{\lambda}.$$

- *Answer to Example L9.2 continued*: So the Cramér-Rao lower bound for an unbiased estimator in the iid case is

$$\frac{\left(\frac{d}{d\theta}\mathsf{E}_\theta[W(\boldsymbol{X})]\right)^2}{n\mathsf{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\ln f(X|\theta)\right)^2\right]} = \frac{1}{n\left(\frac{1}{\lambda}\right)} = \frac{\lambda}{n}.$$

The MLE of λ is $\hat{\lambda} = \bar{X}$ and $\mathsf{Var}[\bar{X}] = \frac{\mathsf{Var}[X_1]}{n} = \frac{\lambda}{n}$ so it attains the CRLB and is the UMVUE of λ .

- *Example L9.3*: Let $X_1, \ldots, X_n$ be iid Normal(μ, σ^2) random variables. Find the Cramér-Rao lower bound on unbiased estimators of σ^2. Does S^2 satisfy the CRLB?
- *Answer to Example L9.3*: Since

$$\begin{aligned} \frac{\partial^2}{\partial(\sigma^2)^2} \ln f(x|\mu, \sigma^2) &= \frac{\partial^2}{\partial(\sigma^2)^2} \left[-\frac{1}{2} \ln(2\pi) - \frac{1}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2 \right] \\ &= \frac{1}{2\sigma^4} - \frac{(x-\mu)^2}{\sigma^6}, \end{aligned}$$

Theorem L9.4 implies that

$$\begin{aligned} \mathsf{E}\left[\left(\frac{\partial}{\partial\theta} \ln f(X|\mu, \sigma^2) \right)^2 \right] &= -\mathsf{E}\left[\frac{\partial^2}{\partial\sigma^2} \ln f(X|\mu, \sigma^2) \right] \\ &= -\mathsf{E}\left[\frac{1}{2\sigma^4} - \frac{(X-\mu)^2}{\sigma^6} \right] \end{aligned}$$

- *Answer to Example L9.3 continued*:

$$
\begin{aligned}
&= -\mathsf{E}\left[\frac{1}{2\sigma^4} - \frac{(X-\mu)^2}{\sigma^6}\right] \\
&= -\frac{1}{2\sigma^4} + \frac{\mathsf{E}[(X-\mu)^2]}{\sigma^6}] \\
&= -\frac{1}{2\sigma^4} + \frac{\sigma^2}{\sigma^6} = \frac{1}{2\sigma^4}.
\end{aligned}
$$

Thus, the CRLB is $\dfrac{1}{n\mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X|\theta)\right)^2\right]} = \dfrac{2\sigma^4}{n}$.

So, S^2 does not satisfy the CRLB since
$\mathsf{Var}[S^2] = \frac{2\sigma^4}{n-1} = \frac{n}{n-1}\left(\frac{2\sigma^4}{n}\right) > \frac{2\sigma^4}{n} = CRLB$.

- *Example L9.4*: Let $X_1, \ldots, X_n$ be iid Uniform$(0, \theta)$ random variables. Find the Cramér-Rao lower bound on the variance of unbiased estimators of θ. Also, for $Y = \max\{X_1, \ldots, X_n\}$ show that $\left(\frac{n+1}{n}\right) Y$ is an unbiased estimator which has a smaller variance than the Cramér-Rao lower bound.
- *Answer to Example L9.4*: Since $\frac{\partial}{\partial\theta} \ln f(x|\theta) = \frac{\partial}{\partial\theta}\left[\ln \frac{1}{\theta}\right] = -\frac{1}{\theta}$, the CRLB is $\frac{1}{n(-\theta^{-1})^2} = \frac{\theta^2}{n}$.
  Since the CDF of Y is
  $F(y) = P(Y \leq y) = \prod_{i=1}^{n} P(X_i \leq y) = \left(\frac{y}{\theta}\right)^n$ for $0 < y < \theta$,
  the pdf of Y is $f(y) = F'(y) = \dfrac{ny^{n-1}}{\theta^n} I_{(0,\theta)}(y)$.

- *Answer to Example L9.4 continued*: $\left(\frac{n+1}{n}\right)Y$ is unbiased since $\mathsf{E}\left[\left(\frac{n+1}{n}\right)Y\right] = \frac{n+1}{n}\int_0^\theta y\frac{ny^{n-1}}{\theta^n}\,dy = \frac{n+1}{\theta^n}\int_0^\theta y^n\,dy = \frac{n+1}{\theta^n}\left[\frac{1}{n+1}y^{n+1}\right]_0^\theta = \frac{n+1}{\theta^n}\left[\frac{1}{n+1}\theta^{n+1}\right] = \theta$.
- Similarly, $\mathsf{E}\left[\left(\frac{n+1}{n}Y\right)^2\right] = \frac{(n+1)^2}{n\theta^n}\int_0^\theta y^{n+1}\,dy = \frac{(n+1)^2}{n\theta^n}\left[\frac{1}{n+2}y^{n+2}\right]_0^\theta = \frac{(n+1)^2}{n\theta^n}\left[\frac{1}{n+2}\theta^{n+2}\right] = \frac{(n+1)^2}{n(n+2)}\theta^2$.
- So, $\mathsf{Var}\left[\left(\frac{n+1}{n}\right)Y\right] = \frac{(n+1)^2}{n(n+2)}\theta^2 - \theta^2 = \frac{1}{n(n+2)}\theta^2$.
- It is now seen that $\mathsf{Var}\left[\left(\frac{n+1}{n}\right)Y\right] = \frac{1}{n+2}\left(\frac{\theta^2}{n}\right) < \frac{\theta^2}{n} = \text{CRLB}$.

Attainment

- *Theorem L9.5* (Cor 7.3.15 on p.341): Let $X_1, \ldots, X_n$ be iid $f(x|\theta)$, where $f(x|\theta)$ satisfies the conditions of *Theorem L9.3*. Let $L(\theta|\boldsymbol{x}) = \prod_{i=1}^{n} f(x_i|\theta)$ denote the likelihood function. If $W(\boldsymbol{X}) = W(X_1, \ldots, X_n)$ is any unbiased estimator of $\tau(\theta)$, then $W(\boldsymbol{X})$ attains the Cramér-Rao Lower Bound if and only if

  $$a(\theta)[W(\boldsymbol{x}) - \tau(\theta)] = \frac{\partial}{\partial\theta} \ln L(\theta|\boldsymbol{x})$$

  for some function $a(\theta)$.

Attainment

- *Proof of Theorem L9.5*: By *Theorem L9.1*(b),

$$\left\{\mathsf{Cov}\left[W(\boldsymbol{X}), \frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right]\right\}^2 = \mathsf{Var}[W(\boldsymbol{X})]\mathsf{Var}\left[\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right]$$

if and only if there are functions $b(\theta)$ and $a(\theta)$ (where $|a(\theta)| > 0$) such that

$$\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta) = a(\theta)W(\boldsymbol{X}) + b(\theta). \tag{1}$$

Since $\mathsf{E}[W(X)] = \tau(\theta)$ and $\mathsf{E}\left[\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta)\right] \overset{9.8}{=} 0$, taking the expected value of both sides of (1) yields $0 = a(\theta)\tau(\theta) + b(\theta)$ so that

$$b(\theta) = -a(\theta)\tau(\theta). \tag{2}$$

Substituting (2) into (1), we have

$$\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}|\theta) = a(\theta)\left\{W(\boldsymbol{X}) - \tau(\theta)\right\}.$$

- *Example L9.5*: Let $X_1, \ldots, X_n$ be a random sample from a Beta$(\theta, 1)$ population which has pdf

$$f(x) = \theta x^{\theta-1} I_{(0,1)}(x).$$

(a) Compute $\frac{\partial}{\partial\theta} \ln L(\theta|\boldsymbol{x})$ where L is the likelihood function.
(b) Find the UMVUE for $\frac{1}{\theta}$.

- *Answer to Example L9.5*: We have

$$\begin{aligned} \frac{\partial}{\partial\theta} \ln L(\theta|\boldsymbol{x}) &= \frac{\partial}{\partial\theta} \sum_{i=1}^{n} \ln f(x_i|\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \left[(\theta-1)\ln x_i + \ln\theta\right] \\ &= \sum_{i=1}^{n} \left\{ \ln x_i + \frac{1}{\theta} \right\} = \sum_{i=1}^{n} \ln x_i + \frac{n}{\theta}. \end{aligned}$$

Cramér-Rao Lower Bound (iid case)

- *Answer to Example L9.5 continued*: Since $f(x|\theta)$ is a member of an exponential family with $c(\theta) = \theta$, $w(\theta) = \theta - 1$, and $t(x) = \ln x$, and
  $\{w(\theta) : \theta \in (0, \infty)\} = (-1, \infty)$ contains an open subset of $\mathbb{R}$
  Theorem L6.2 implies that $\displaystyle\sum_{i=1}^{n} t(X_i) = \sum_{i=1}^{n} \ln X_i$ belongs to an exponential family with $C(\theta) = [c(\theta)]^n = \theta^n$, $w(\theta) = \theta - 1$, and $u_i = \sum_{i=1}^{n} t(x_i) = \sum_{i=1}^{n} \ln x_i$.
  By *Theorem L6.1*, we have
  $\mathsf{E}\left[\sum_{i=1}^{n} \ln X_i\right] = -\frac{d}{d\theta}\left[\ln(\theta^n)\right] = -\frac{n}{\theta}$.
- Hence, $\mathsf{E}\left[\frac{-1}{n}\sum_{i=1}^{n} \ln X_i\right] = \frac{1}{\theta}$.
- *Theorem L9.5* shows that $\frac{-1}{n}\sum_{i=1}^{n} \ln X_i$ is the UMVUE for $\frac{1}{\theta}$ since

$$\frac{\partial}{\partial\theta}\ln L(\theta|\boldsymbol{x}) = -n\left(-\frac{1}{n}\sum_{i=1}^{n}\ln x_i - \frac{1}{\theta}\right).$$