Kuznetsov's book
4 on Syllabus

Chapter 4

Planar ODE

In this chapter we prove the famous Poincaré-Bendixson Theorem that classifies the asymptotic behaviour of bounded orbits in smooth planar ODEs. We shall see that in a generic planar system any such orbit tends either to an equilibrium or to a cycle (periodic orbit). Then we give some conditions ensuring existence or nonexistence of cycles in planar systems, in particular, when the system is close to a special (Hamiltonian) system. We apply many results and techniques to the analysis of several prey-predator ecological models. Basic facts about multidimensional Hamiltonian systems are summarized in an appendix.

4.1 Limit sets

Before focusing on the special case $n = 2$, we introduce some notions that are applicable in any dimension. So, consider

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{4.1}$$

where $f \in C^1$. Let $\varphi(t,x) = \varphi^t(x)$, where φ^t is the (local) flow generated by (4.1). Theorem 1.5 implies that the map $(t,x) \mapsto \varphi(t,x)$ is surely continue in any point (t,x), where it is defined.

Definition 4.1 *A point $y \in \mathbb{R}^n$ is an ω-**limit point** of $x \in \mathbb{R}^n$, if $\varphi(t,x)$ is defined for all $t \geq 0$ and there exists an increasing sequence $\{t_i\} \to \infty$, such that $\varphi(t_i,x) \to y$ for $i \to \infty$. The set of all ω-limit points of x is called the ω-**limit set** of x and denoted by $\omega(x)$.*

Remarks:

(1) We use the expression $\{t_i\} \to \infty$ to indicate that the sequence $\{t_i\}$ diverges. Similarly, $\{p_i\} \to p$ means that the sequence of points $p_i \in \mathbb{R}^n$ converges to $p \in \mathbb{R}^n$.

(2) Considering decreasing unbounded sequences $\{t_i\} \to -\infty$, we obtain the α-limit points of $x \in \mathbb{R}^n$ and the set $\alpha(x)$. (Recall that the Greek alphabet begins with α and ends with ω.)

(3) Since $\omega(\varphi^t(x)) = \omega(x)$, one can use the notation $\omega(\Gamma)$, where $\Gamma = \Gamma(x)$ is the orbit through x, or $\omega(\Gamma^+)$, where $\Gamma^+ = \Gamma^+(x)$ is the *positive half-orbit* through x:

$$\Gamma^+(x) = \{y \in \mathbb{R}^n : y = \varphi^t(x),\ t \geq 0\}.$$

(4) If $x^0 \in \mathbb{R}^n$ is an equilibrium point of (4.1), then $\omega(x^0) = x^0$. If $x \in \Gamma_0$, where Γ_0 is a periodic orbit of (4.1), then $\omega(x) = \Gamma_0$.

For any two subsets $A, B \subset \mathbb{R}^n$ define

$$\rho(A, B) = \inf_{x \in A, y \in B} \|x - y\| .$$

If A consists of one point $x \in \mathbb{R}^n$ and B is fixed, the function $x \mapsto \rho(x, B)$ is continuous. The following four lemmas are valid for all $n \geq 1$.

Lemma 4.2 (Basic properties) *Let $x \in \mathbb{R}^n$ and assume that $\Gamma^+(x)$ is bounded. Then the set $\omega(x)$ is* (i) *nonempty,* (ii) *bounded,* (iii) *closed, and* (iv) *connected.*

Proof: Since $\Gamma^+(x)$ is bounded, $\varphi(t, x)$ is defined for all $t \geq 0$.

(i) Consider $x_i = \varphi(i, x)$, $i = 1, 2, \ldots$. The sequence $\{x_i\}$ is bounded and infinite. So, there is a convergent subsequence: $\{x_{I(k)}\} \to p$. This means $p \in \omega(x)$, so $\omega(x) \neq \emptyset$.

(ii) If $\omega(x)$ is unbounded, then $\Gamma^+(x)$ is also unbounded, a contradiction.

(iii) Let $\{p_i\}$ be a sequence of points $p_i \in \omega(x)$ and $\{p_i\} \to p \in \mathbb{R}^n$. Let us prove that $p \in \omega(x)$.

Indeed, for each p_i there is an increasing sequence of times $\{t_k^{(i)}\} \to \infty$ as $k \to \infty$, such that the corresponding sequence $\{x_k^{(i)}\}$ defined by

$$x_k^{(i)} = \varphi(t_k^{(i)}, x)$$

converges to p_i, i.e. $\|x_k^{(i)} - p_i\| \to 0$ for $k \to \infty$. For each i, take the first $K = K(i)$ such that

$$\|x_{K(i)}^{(i)} - p_i\| \leq \frac{1}{i}.$$

Then

$$\|x_{K(i)}^{(i)} - p\| = \|(x_{K(i)}^{(i)} - p_i) + (p_i - p)\| \leq \|x_{K(i)}^{(i)} - p_i\| + \|p_i - p\| \leq \frac{1}{i} + \|p_i - p\| \to 0$$

as $i \to \infty$. This means that $p \in \omega(x)$, so $\omega(x)$ is closed.

(iv) Suppose $\omega(x) = A \cup B$, where $A, B \subset \mathbb{R}^n$ are nonempty, bounded, and disjoint, i.e. $A \cap B = \emptyset$. Then, since $\omega(x)$ is a closed subset of $\mathbb{R}^n$, so are A, B and accordingly

$$\rho(A, B) = \delta > 0.$$

Since A and B are composed of ω-limit points, there exist increasing and unbounded sequences of times $\{t_i^A\}$ and $\{t_i^B\}$ such that

$$\rho(A, \varphi(t_i^A, x)) < \frac{\delta}{2} \quad \text{and} \quad \rho(B, \varphi(t_i^B, x)) < \frac{\delta}{2}.$$

This implies

$$\rho(A, \varphi(t_i^B, x)) > \frac{\delta}{2}$$

and, by continuity of ρ and φ, there exists a sequence $\{\tau_j\}$, such that

$$\rho(A, \varphi(\tau_j, x)) = \frac{\delta}{2}.$$

The sequence $\{\varphi(\tau_j, x)\}$ is infinite and bounded. Thus, may be for a subsequence $\{\tau_{J(k)}\}$, we have

$$\{\varphi(\tau_{J(k)}, x)\} \to z, \quad k \to \infty,$$

with $z \notin A, B$. But clearly $\in \omega(x)$. Contradiction. Therefore, $\omega(x)$ is connected. □

Lemma 4.3 (Convergence) *Let $\Gamma^+(x)$ be bounded. Then $\rho(\varphi(t, x), \omega(x)) \to 0$ as $t \to +\infty$.*

Proof: Suppose the claim is false, then there is an $\varepsilon > 0$ and a sequence $\{t_i\} \to \infty$ of times such that

$$\rho(\varphi(t_i, x), \omega(x)) \geq \varepsilon > 0 \tag{4.2}$$

for all $i = 1, 2, \ldots$. As in the proof of Lemma 4.2(i), the sequence $\{x_i\}$ with $x_i = \varphi(t_i, x)$ is bounded and infinite. So, it has a convergent subsequence:

$$\{x_{I(k)}\} \to p \in \mathbb{R}^n.$$

Using the continuiy of ρ, we get from (4.2)

$$\rho(p, \omega(x)) > 0,$$

which is a contradiction, since by definition $p \in \omega(x)$. □

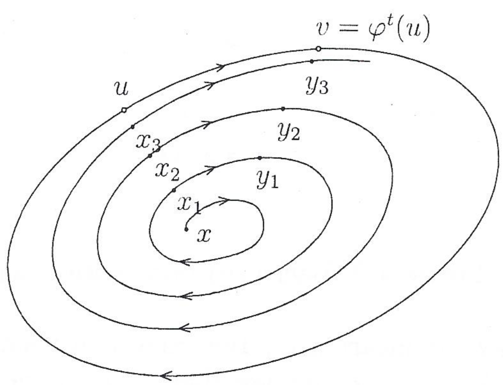


Figure 4.1: Forward invariance of $\omega(x)$.

Lemma 4.4 (Invariance) *Let $\Gamma^+(x)$ be bounded. Then $\omega(x)$ is an invariant set of the flow φ^t, i.e. if $y \in \omega(x)$ then $\varphi^t(y) \in \omega(x)$ for all $t \in \mathbb{R}$.*

Proof: First consider forward invariance. Let $u \in \omega(x)$. We want to prove that $\varphi^t(u)$ is defined and $v = \varphi^t(u) \in \omega(x)$ for $t > 0$. By definition, there is an increasing sequence of times $\{t_i\} \to \infty$ such that

$$x_i = \varphi(t_i, x) \to u,$$

as $i \to \infty$. Consider the sequence

$$y_i = \varphi^t(x_i) = \varphi(t, \varphi(t_i, x))$$

(see Figure 4.1). Since φ is a continuous function of both of its arguments, $\varphi(t, u)$ exists and

$$\{y_i\} \to \varphi(t, u) = v.$$

On the other hand, by the semigroup property,

$$y_i = \varphi(t + t_i, x),$$

so

$$\varphi(\tau_i, x) \to v$$

as $i \to \infty$ for $\tau_i = t + t_i$ and $\{\tau_i\} \to \infty$. Therefore, $v \in \omega(x)$.

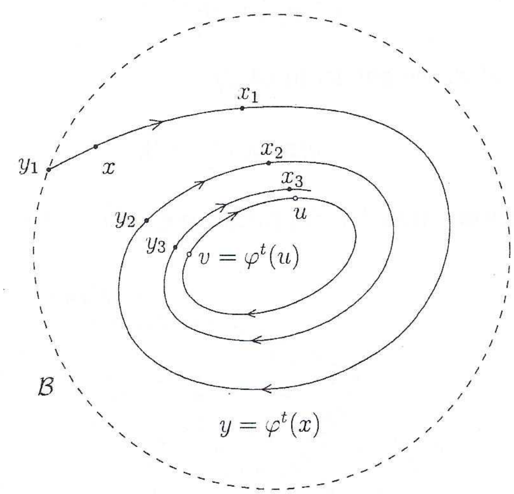


Figure 4.2: Backward invariance of $\omega(x)$.

For $t < 0$ (backward invariance), the same construction works with a slight modification. Namely, it could happen that $\varphi^t(x_i)$ is not defined for $i \le N$ for some integer $N > 0$, but $\varphi^t(x_i)$ is well defined for $i > N$ (see Figure 4.2, where $\mathcal{B}$ represents infinitely-remote points). In this case, start from τ_{N+1}, i.e. take $\tilde{\tau}_i = \tau_{N+i}$. $\square$

Lemma 4.5 (Transitivity) *Let $x, y, z \in \mathbb{R}^n$. If $z \in \omega(y)$ and $y \in \omega(x)$, then $z \in \omega(x)$.*

Proof: Let $\{x_i\} \to y$, where $x_i = \varphi(t_i, x)$ for some increasing sequence $\{t_i\} \to +\infty$, while $\{y_i\} \to z$, where $y_i = \varphi(\tau_i, y)$ for some increasing sequence $\{\tau_j\} \to +\infty$. Define

$$z_i = \varphi^{\tau_i}(x_i) = \varphi(\tau_i, \varphi(t_i, x)) = \varphi(\tau_i + t_i, x).$$

Then, by continuity of $\varphi(\cdot, \cdot)$,

$$z_i = \varphi(\tau_i, \varphi(t_i, x)) \to z.$$

Therefore, for an increasing sequence of times $\{\tau_i + t_i\} \to +\infty$,

$$z_i = \varphi(\tau_i + t_i, x) \to z. \quad \square$$

4.2 The Poincaré-Bendixson Theorem

For the rest of this section, fix $n = 2$, *i.e.* consider a planar system

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2)\,, \\ \dot{x}_2 &= f_2(x_1, x_2)\,, \end{cases} \tag{4.3}$$

where $f_{1,2}$ are C^1-functions of (x_1, x_2). Natural candidates for ω-limit sets in such systems are equilibria and cycles. Recall that a point x^0 is an equilibrium if $f(x^0) = 0$, while a cycle is a closed orbit Γ_0 that corresponds to a nonequilibrium solution for which $\varphi(t, x) = \varphi(t + T, x)$ with some $T > 0$. The minimal T is called the *period* of Γ_0.

Definition 4.6 *A closed line segment L is called* **transverse** *for* (4.3) *if the vector field f neither vanishes in L, nor is anywhere tangent to L.*

A transverse segment can obviously be constructed through any point $x \in \mathbb{R}^2$ where $f(x) \neq 0$ (for example, take a sufficiently small segment orthogonal to $f(x)$). Consider such a segment L, then there are no equilibria in L and any orbit crosses L at a nonzero angle. Introduce a coordinate s along L.

Definition 4.7 *A continuous image of a circle without self-intersections is called a* ***Jordan curve.***

According to a famous theorem by Jordan, the complement to a Jordan curve $\Gamma \subset \mathbb{R}^2$ is the union of two disjoint connected open sets, the interior $\mathrm{Int}(\Gamma)$ and the exterior $\mathrm{Ext}(\Gamma)$, the first one of which is bounded while the second is unbounded.

Lemma 4.8 (Monotonicity) *If an orbit of* (4.3) *intersects L for an increasing sequence of times $\{t_i\}$, then the corresponding sequence of intersection points $\{p_i\}$ is either constant or strictly monotone.*

Proof: Consider an orbit that crosses a transverse segment L at two *distinct* points

$$p_0 = \varphi(t_0, x), \;\; p_1 = \varphi(t_1, x),$$

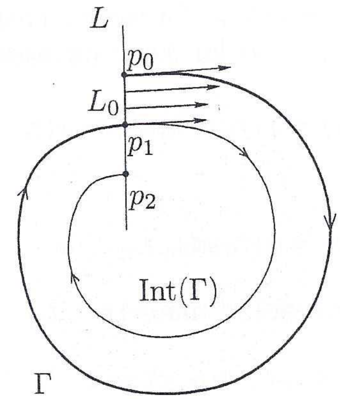


Figure 4.3: Monotonicity of the intersections.

with $t_0 < t_1$, and has no other intersections with L for $t \in [t_0, t_1]$. Denote by Γ the closed (piecewise-smooth) curve composed of the orbit part connecting p_0 and p_1, and the subsegment $L_0 \subset L$ between these same points. Γ is a Jordan curve. Suppose that orbits starting at points in L_0 enter $\mathrm{Int}(\Gamma)$ for small positive times (see Figure 4.3). By continuity, the same is true for the orbit starting at p_1. Therefore, $\mathrm{Int}(\Gamma)$ is a forward invariant open set for (4.3), since no orbit can cross its boundary Γ outwards.

This implies that the orbit starting at p_1 will belong to $\mathrm{Int}(\Gamma)$ for all positive times. Thus, all its possible intersections with L, including the next one p_2, must be located in $\mathrm{Int}(\Gamma)$. This implies that the corresponding sequence of intersection points $\{p_i\}$ is monotone.

If $p_0 = p_1$, the orbit is periodic and all its further intersections with L occur at the same point, giving a constant sequence $p_i = p_0$.

The case, when orbits starting at points in L_0 enter $\mathrm{Int}(\Gamma)$ for small negative times reduces to the case considered above by reversing time. □

The map $p_i \mapsto p_{i+1}$ or, in terms of the coordinate s along L, $s_i \mapsto s_{i+1}$ is obviously a Poincaré map as introduced in Chapter 3.

Lemma 4.9 *Let $p \in \omega(x)$ and let L be a transverse segment through p. Then there is an increasing sequence $\{t_i\} \to \infty$ such that for $q_i = \varphi(t_i, x)$ we have $\{q_i\} \to p$ and $q_i \in L$.*

Proof: By definition, there is a sequence of points $\{p_i\} \to p$, such that

$$p_i = \varphi(t_i, x)$$

for an increasing sequence of times $\{t_i\} \to +\infty$. All these points belong to the same orbit $\Gamma(x)$ but, in general, $p_i \notin L$. For each point p_i near p, take $q_i = \varphi(\tau(p_i), p_i)$, where $\tau(x)$ is the minimal (in absolute value) time needed to move along $\Gamma(x)$ from p_i to L (see Figure 4.4). When $\{p_i\} \to p$, we have $\tau(p_i) \to 0$ so that $\rho(q_i, p_i) \to 0$ and therefore $\{q_i\} \to p$ as $i \to \infty$.

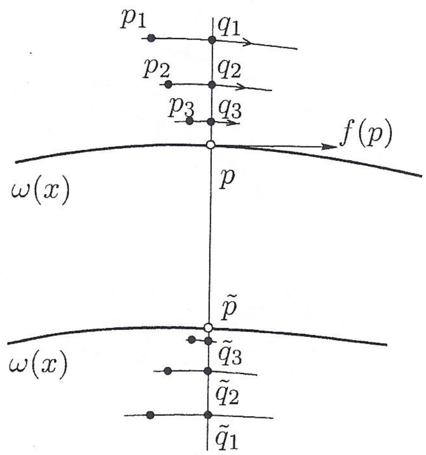


Figure 4.4: $L \cap \omega(x)$ consists of only one point p.

The existence of $\tau(x)$ can be established as in Lemma 3.11 of Chapter 3. Indeed, let $a \in \mathbb{R}^2$ be a vector orthogonal to L, so that for any point $q \in L$ we have

$$\langle a, q - p \rangle = 0 \ .$$

The condition that the orbit starting at x arrives at $t = \tau$ to a point $q \in L$ can be written as

$$g(x, \tau) = \langle a, \varphi(\tau, x) - p \rangle = 0 \ .$$

The scalar function g is defined near $(p, 0)$ and smooth. It obviously satisfies $g(p, 0) = 0$. Moreover, since L is a transverse segment,

$$g_\tau(p, 0) = \langle a, f(p) \rangle \neq 0 \ .$$

Therefore, the Implicit Function Theorem guarantees the existence of a function $x \mapsto \tau(x)$ having the property $\tau(p) = 0$, defined and smooth in a neighbourhood of p, and satisfying $g(x, \tau(x)) = 0$ in this neighbourhood. □

Lemma 4.10 *The set $\omega(x)$ can intersect the transverse segment L in at most one point.*

Proof: Let $p \in \omega(x) \cap L$. By Lemma 4.9, there is a sequence of points $\{q_i\} \to p$ with $q_i \in L$ and belonging to $\Gamma(x)$ (see Figure 4.4 again).

According to Lemma 4.8, the sequence $\{q_i\}$ is monotone. Any monotone sequence along a line has at most one limit point. □

The following lemma is often used to prove the existence of a periodic orbit.

Lemma 4.11 *If $\omega(x)$ is nonempty and does not contain equilibria, then it contains a periodic orbit Γ_0.*

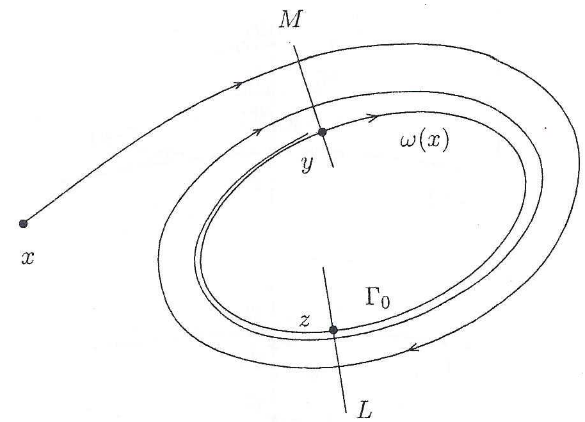


Figure 4.5: Lemmas 4.11 and 4.12.

Proof: Take $y \in \omega(x)$ and consider $z \in \omega(y)$ (Figure 4.5). By Lemma 4.5, $z \in \omega(x)$ and, thus, z is not an equilibrium. Take a transverse segment L through z, and consider a sequence $\{y_i\} \to z$ with

$$y_i = \varphi(t_i, y) \in L \cap \Gamma(y)$$

(the existence of such a sequence is guaranteed by Lemma 4.9). Notice that by Lemma 4.4 all $y_i \in \omega(x)$. By Lemma 4.8, the sequence $\{y_i\}$ is either strictly monotone or constant. If $\{y_i\}$ is monotone, $\omega(x)$ has more than one intersection with L, a contradiction with Lemma 4.10. Thus, it is constant, i.e.

$$y_i = z,$$

implying that $\Gamma_0 = \Gamma(y)$ is a periodic orbit. □

Lemma 4.12 *If $\omega(x)$ contains a periodic orbit Γ_0, then $\omega(x) = \Gamma_0$.*

Proof: There is a transverse segment M through any point $y \in \Gamma_0$ (see Figure 4.5). By Lemma 4.10, the intersection $\omega(x) \cap M = y$. Therefore, there is an open annulus in which Γ_0 is the only subset of $\omega(x)$. Since $\omega(x)$ is connected, $\omega(x) \setminus \Gamma_0$ is empty. □

We are ready now to prove the central result of this section.

Theorem 4.13 (Poincaré-Bendixson) *Let $x \in \mathbb{R}^2$ and assume that $\Gamma^+(x)$ belongs to a bounded closed subset of $\mathbb{R}^2$ containing only a finite number of equilibria. Then one of the following possibilities holds:*

(i) *$\omega(x)$ is an equilibrium;*
(ii) *$\omega(x)$ is a periodic orbit;*
(iii) *$\omega(x)$ consists of equilibria and orbits having these equilibria as their α- and ω-limit sets.*

Proof:

(i) If $\omega(x)$ contains only equilibria, it is a union of a finite number of points. However, according to Lemma 4.2, $\omega(x)$ is connected. Therefore, in this case $\omega(x)$ is just one equilibrium.

(ii) If $\omega(x)$ contains no equilibria, there is a periodic orbit $\Gamma_0 \subset \omega(x)$ (Lemma 4.11). However, by Lemma 4.12 $\omega(x) = \Gamma_0$.

(iii) Suppose $\omega(x)$ contains both equilibrium and nonequilibrium points. Let $q \in \omega(x)$ be a nonequilibrium point. Then, Lemma 4.5 (and its reformulation for α-limit sets) implies $\omega(q), \alpha(q) \subset \omega(x)$.

If $\omega(q)$ does not contain an equilibrium, then $\omega(q)$ (and hence $\omega(x)$) contains a periodic orbit (Lemma 4.11). Therefore, by Lemma 4.12, $\omega(x)$ coincides with this periodic orbit. However, $\omega(x)$ contains equilibria, a contradiction.

Thus, $\omega(q)$ contains an equilibrium. Assume that $\omega(q)$ also contains a nonequilibrium point y. Take a transverse segment L through y. Due to Lemma 4.4, all points of $\Gamma(q)$ belong to $\omega(x)$. By Lemma 4.9, the orbit $\Gamma(q)$ must intersect L infinitely-many times near y. Moreover, all the intersection points must be different (otherwise, $\Gamma(q)$ is a periodic orbit, which is already excluded). Thus, $\omega(x)$ intersects with L more than once, a contradiction.

Therefore, $\omega(q)$ contains only equilibria and, as in step (i), is just one equilibuim. Essentially the same arguments can be applied to show that $\alpha(q)$ is just one equilibrium. $\square$

Remarks:

(1) If f is an analytic vector field, in particular a polynomial vector field, then (4.3) automatically has a finite number of equilibria in any closed bounded subset of the plane.

(2) With Lemma 4.3, option (iii) of the Poincaré-Bendixson Theorem implies that $\omega(x)$ actually consists of a contour formed by equilibria and their connecting (homoclinic or heteroclinic) orbits, i.e. orbits tending to these equilibria as $t \to \pm\infty$. Clearly then, case (iii) is exceptional.

It can be proved that there can be at most one heteroclinic orbit $\Gamma \subset \omega(x)$ connecting two specific equilibria $x^{1,2} \in \omega(x)$ such that $x^1 \neq x^2$. Moreover, any equilibrium $x^0 \in \omega(x)$ of an analytical system (4.3) can have only a finite number of homoclinic orbits $\Gamma_i \in \omega(x)$.

(3) A periodic orbit Γ_0 is called a *limit cycle* if there is an annulus around Γ_0 in which there are no other periodic orbits. A limit cycle can be stable, unstable, or semi-stable (meaning stable from only one side). $\diamondsuit$

4.3 Stability, existence, and uniqueness of planar cycles

As we have already mentioned in Section 3.2 of Chapter 3, for planar systems one can express the only nontrivial multiplier λ_2 of the monodromy matrix corresponding to a periodic solution $\varphi(t)$ of (4.3) as

$$\lambda_2 = \exp\left(\int_0^T \operatorname{div} f(\varphi(t))\, dt\right), \tag{4.4}$$

The cycle is (exponentially) stable if $\lambda_2 < 1$ and unstable if $\lambda_2 > 1$.

Linear stability of a T-periodic cycle of a smooth planar system (4.3) can also be studied explicitly using the so called *normal coordinates* (cf. Section 3.2). Suppose that (4.3) has a periodic orbit (cycle) Γ_0 and

$$\varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} \tag{4.5}$$

is the corresponding periodic solution, $\varphi(t+T) = \varphi(t)$ for all $t \in \mathbb{R}$. Introduce new coordinates (τ, ξ) in a neighbourhood of the cycle by the relations:

$$\left\{ \begin{array}{rcl} x_1 & = & \varphi_1(\tau) + \xi\dot{\varphi}_2(\tau), \\ x_2 & = & \varphi_2(\tau) - \xi\dot{\varphi}_1(\tau), \end{array} \right. \tag{4.6}$$

where $\tau \in [0, T]$ and $\xi \in \mathbb{R}$ has small absolute value (see Figure 4.6). Note that $\xi = 0$ corresponds to the cycle. The Jacobian matrix of (4.6) is

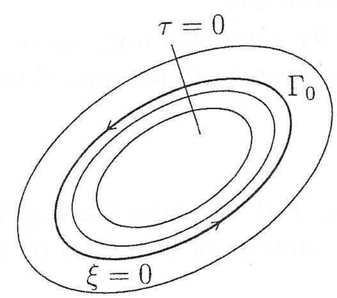


Figure 4.6: Normal coordinates near the cycle Γ_0.

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial \tau} & \frac{\partial x_1}{\partial \xi} \\ \frac{\partial x_2}{\partial \tau} & \frac{\partial x_2}{\partial \xi} \end{pmatrix} = \begin{pmatrix} \dot{\varphi}_1 + \xi\ddot{\varphi}_2 & \dot{\varphi}_2 \\ \dot{\varphi}_2 - \xi\ddot{\varphi}_1 & -\dot{\varphi}_1 \end{pmatrix},$$

so that

$$\det(J) = -(\dot{\varphi}_1^2 + \dot{\varphi}_2^2) - \xi(\dot{\varphi}_1\ddot{\varphi}_2 - \dot{\varphi}_2\ddot{\varphi}_1).$$

Since $\dot{\varphi}(\tau) = f(\varphi(\tau)) \neq 0$ along the cycle,

$$\det(J)|_{\xi=0} = -(\dot{\varphi}_1^2 + \dot{\varphi}_2^2) = -\|\dot{\varphi}\|^2 \neq 0.$$

This implies that the coordinate transformation(4.6) is regular in a neighbourhood of Γ_0. Now write (4.3) in the (τ, ξ)-coordinates, i.e.

$$\begin{pmatrix} \dot{\tau} \\ \dot{\xi} \end{pmatrix} = J^{-1} f(x),$$

where x is given by (4.6). First, compute the inverse of J:

$$J^{-1} = \frac{1}{\det(J)} \begin{pmatrix} -\dot{\varphi}_1 & -\dot{\varphi}_2 \\ -\dot{\varphi}_2 + \xi\ddot{\varphi}_1 & \dot{\varphi}_1 + \xi\ddot{\varphi}_2 \end{pmatrix}.$$

Since φ satisfies (4.3), we have

$$\begin{cases} \dot\varphi_1 &= f_1(\varphi_1, \varphi_2), \\ \dot\varphi_2 &= f_2(\varphi_1, \varphi_2), \end{cases} \tag{4.7}$$

so that

$$f(x) = \begin{pmatrix} \dot\varphi_1 + (a_{11}\dot\varphi_2 - a_{12}\dot\varphi_1)\xi \\ \dot\varphi_2 + (a_{21}\dot\varphi_2 - a_{22}\dot\varphi_1)\xi \end{pmatrix} + O(\xi^2),$$

where

$$a_{ij} = \frac{\partial f_i}{\partial x_i}, \quad i,j = 1,2.$$

Thus

$$\dot\tau = -\frac{1}{\det(J)}\left[\dot\varphi_1^2 + \dot\varphi_2^2 + (-a_{12}\dot\varphi_1^2 + (a_{11} - a_{22})\dot\varphi_1\dot\varphi_2 + a_{21}\dot\varphi_2^2)\xi + O(\xi^2)\right]$$

and

$$\dot\xi = -\frac{1}{\det(J)}\left[-(\dot\varphi_1\ddot\varphi_1 + \dot\varphi_2\ddot\varphi_2)\xi + (a_{11}\dot\varphi_2^2 - (a_{12} + a_{21})\dot\varphi_1\dot\varphi_2 + a_{22}\dot\varphi_1^2)\xi + O(\xi^2)\right],$$

which yields

$$\frac{d\xi}{d\tau} = \left[-\frac{\dot\varphi_1\ddot\varphi_1 + \dot\varphi_2\ddot\varphi_2}{\dot\varphi_1^2 + \dot\varphi_2^2} + \frac{a_{11}\dot\varphi_2^2 - (a_{12} + a_{21})\dot\varphi_1\dot\varphi_2 + a_{22}\dot\varphi_1^2}{\dot\varphi_1^2 + \dot\varphi_2^2}\right]\xi + O(\xi^2).$$

Differentiating both equations of (4.7) with respect to time, multiplying the first by $\dot\varphi_1$ and the second by $\dot\varphi_2$, and adding the resulting expressions, we find

$$(a_{12} + a_{21})\dot\varphi_1\dot\varphi_2 = \dot\varphi_1\ddot\varphi_1 + \dot\varphi_2\ddot\varphi_2 - a_{11}\dot\varphi_1^2 - a_{22}\dot\varphi_2^2.$$

Using this formula, we obtain

$$\frac{d\xi}{d\tau} = \left[(a_{11} + a_{22}) - \frac{2(\dot\varphi_1\ddot\varphi_1 + \dot\varphi_2\ddot\varphi_2)}{\dot\varphi_1^2 + \dot\varphi_2^2}\right]\xi + O(\xi^2).$$

Therefore

$$\frac{d\xi}{d\tau} = A_1(\tau)\xi + O(\xi^2), \quad A_1(\tau) = \frac{\partial f_1}{\partial x_1}(\varphi(\tau)) + \frac{\partial f_2}{\partial x_2}(\varphi(\tau)) - \frac{d}{d\tau}\ln\|\dot\varphi(\tau)\|^2. \tag{4.8}$$

The solution $\xi(\tau, \xi_0)$ to (4.8) with $\xi(0, \xi_0) = \xi_0$ can be expanded as

$$\xi(\tau, \xi_0) = a_1(\tau)\xi_0 + O(\xi_0^2),$$

where $a_1(0) = 1$. From (4.8) we get the linear equation

$$\dot a_1 = A_1(\tau)a_1,$$

which gives, using the periodicity of $\varphi(\tau)$,

$$\begin{aligned} a_1(T) &= \exp\left(\int_0^T A_1(\tau)\, d\tau\right) \\ &= \exp\left(\int_0^T \left[\frac{\partial f_1}{\partial x_1}(\varphi(\tau)) + \frac{\partial f_2}{\partial x_2}(\varphi(\tau))\right] d\tau - \ln\|f(\varphi(\tau))\|^2\Big|_0^T\right) \\ &= \exp\left(\int_0^T \operatorname{div} f(\varphi(\tau))\, d\tau\right). \end{aligned}$$

So the Poincaré map $\xi_0 \mapsto \xi_1 = \xi(T, \xi_0)$ on the local transverse segment $\tau = 0$ has the expansion

$$\xi_1 = e^{\int_0^T \operatorname{div} f(\varphi(\tau))\, d\tau}\xi_0 + O(\xi_0^2).$$

Therefore, the cycle is linearly stable if

$$\int_0^T \text{div } f(\varphi(\tau))\, d\tau < 0$$

and unstable if the opposite inequality is true, in accordance with Section 3.2 of Chapter 3.

Using (4.4), one can derive several important results on existence and uniqueness of periodic orbits in planar systems. Let Γ_0 be a periodic orbit of (4.3) and let $\varphi(t)$ be the corresponding T-periodic solution. Recall that Γ_0 is naturally oriented by the advance of time. Denote by L_0 the same closed orbit but always oriented counterclockwise. Then, if Γ_0 and L_0 have the same (i.e., counterclockwise) orientation, we have

$$\begin{aligned}\oint_{L_0} f_1 dx_2 - f_2 dx_1 &= \int_0^T [f_1(\varphi(t))\dot{\varphi}_2(t) - f_2(\varphi(t))\dot{\varphi}_1(t)]dt \\ &= \int_0^T [f_1(\varphi(t))f_2(\varphi(t)) - f_2(\varphi(t))f_1(\varphi(t)]dt \;=\; 0.\end{aligned}$$

The result remains valid if Γ_0 has opposite (i.e., clockwise) orientation. In this case

$$\begin{aligned}\oint_{L_0} f_1 dx_2 - f_2 dx_1 &= \int_0^T [-f_1(\varphi(t))\dot{\varphi}_2(t) + f_2(\varphi(t))\dot{\varphi}_1(t)]dt \\ &= \int_0^T [-f_1(\varphi(t))f_2(\varphi(t)) + f_2(\varphi(t))f_1(\varphi(t)]dt \;=\; 0.\end{aligned}$$

Now recall that a domain Ω is *connected* if and only if any two points from Ω can be linked by a continuous curve lying entirely in the domain. The domain is called *simply connected* if in addition any closed curve in this domain can be continuously shrunk within the domain to a point. In simply-connected domains Green's Theorem holds, which asserts that

$$\oint_{L_0} f_1 dx_2 - f_2 dx_1 = \int_{\text{Int}(L_0)} \text{div} f(x)\, dx, \tag{4.9}$$

provided that L_0 is a piecewise-smooth counterclockwise-oriented Jordan curve. Combining these two observations we can prove (by contradiction) the following result.

Theorem 4.14 (Bendixson's Criterion) *Let Ω be a simply-connected domain in $\mathbb{R}^2$. If*

$$\text{div} f(x) = \frac{\partial f_1(x)}{\partial x_1} + \frac{\partial f_2(x)}{\partial x_2}$$

is not identically zero on any open subset of Ω and does not change sign in Ω, then (4.3) has no periodic orbits lying entirely in Ω $\square$.

Since the vector fields $f(x)$ and $\mu(x)f(x)$, where the function $\mu : \mathbb{R}^2 \to \mathbb{R}$ is smooth and everywhere positive, define orbitally equivalent systems of differential equations, one also has the following variant of the above theorem.

Theorem 4.15 (Dulac's Criterion) *Let Ω be a simply-connected domain in $\mathbb{R}^2$ and suppose μ is a scalar positive C^1-function on Ω. If*

$$\operatorname{div}(\mu(x)f(x)) = \frac{\partial}{\partial x_1}(\mu(x)f_1(x)) + \frac{\partial}{\partial x_2}(\mu(x)f_2(x))$$

is not identically zero on any open subset of Ω and does not change sign in Ω, then (4.3) *has no periodic orbits lying entirely in Ω.* □

An annulus is an example of a domain that is not simply-connected. Indeed, closed curves that encircle the "hole" in the domain cannot be shrunk within the domain to a point. Two Jordan curves are called equivalent if they can be deformed into each other, such that each intermediate curve is located within the domain and remains a Jordan curve. In *doubly-connected* domains there are exactly two equivalence classes of Jordan curves: Those that can be shrunk to a point and those that cannot.

Theorem 4.16 *Let Ω be a doubly-connected domain in $\mathbb{R}^2$ and suppose that*

$$\operatorname{div} f(x) = \frac{\partial f_1(x)}{\partial x_1} + \frac{\partial f_2(x)}{\partial x_2}$$

is either positive or negative in Ω.

Then (4.3) *has at most one periodic orbit lying entirely in Ω.*

Proof: According to Bendixson's Criterion, there cannot be periodic orbits that can be shrunk within Ω to a point. Thus, if there are periodic orbits of (4.3) in Ω, they must be nested around the "hole" in the domain. Each orbit can be either *isolated* (i.e., has an annular neigbourhood in which there are no other periodic orbits) or *nonisolated.*

Any nonisolated periodic orbit is nonhyperbolic (has $\mu_1 = 1$), so that

$$\int_0^T \operatorname{div}\ f(\varphi(\tau))\ d\tau = 0.$$

However, this is excluded by the assumption. Thus, any periodic orbit in Ω must be isolated.

Assume that there are two or more isolated periodic orbits in Ω and consider a pair of them, such that there are no other periodic orbits *in between.* Then either

(i) one of them is nonhyperbolic with $\mu_2 = 1$, or

(ii) both periodic orbits are hyperbolic and have opposite stability: One is (exponentially) stable with $\mu_2 < 1$, while the other one is unstable with $\mu_2 > 1$.

In both cases we get a contradiction with the assumption. □

As with Dulac's Criterion, applying Theorem 4.16 to an orbitally equivalent system, we get a stronger result.

Theorem 4.17 *Let Ω be a doubly-connected domain in $\mathbb{R}^2$ and suppose that μ is a scalar positive C^1-function in Ω. If*

$$\operatorname{div}(\mu(x)f(x)) = \frac{\partial}{\partial x_1}(\mu(x)f_1(x)) + \frac{\partial}{\partial x_2}(\mu(x)f_2(x))$$

is either positive or negative in Ω, then (4.3) has at most one periodic orbit lying entirely in Ω. □

4.4 Phase plane analysis of prey-predator models

In this section we apply the Poincaré-Bendixson theory to analyse planar ODEs appearing in ecological modelling. In addition, we introduce the important notion of zero-isoclines. These divide the phase plane into regions with different "slopes" of the vector field. By regarding qualitative information about these slopes in various regions, one often can obtain strong conclusions about ω-limit sets of the systems. This methodology is called "phase plane analysis".

4.4.1 Lotka-Volterra Model

Consider a simple aquatic ecosystem composed of two fish species, one of which (called the *predator*) has the other (called the *prey*) as its main source of food. Assume that the prey population grows exponentially in the absence of predators, while the predator population decreases exponentially in the absence of prey. Encounters between prey and predator occur according to the Law of Mass Action, i.e. proportional to the densities of both species. Upon encounter, the prey is swallowed with some probablity and then instantaneously converted into predator offspring. These assumptions translate into the differential equations (*Lotka-Volterra model*):

$$\begin{cases} \dot{v} &= av - bvp, \\ \dot{p} &= -cp + dvp, \end{cases} \tag{4.10}$$

where v stands for prey (victim) density and p for the predator density. Our first task is to derive qualitative and quantitative information about the behaviour of v and p as functions of time t for various nonnegative initial values and, in particular, how this behaviour depends on the four positive parameters a, b, c and d. As a follow-up we shall consider several modifications of (4.10).

4.4.2 Zero-isoclines and equilibria

It is helpful to start the analysis of (4.10) by drawing some auxiliary curves, the so called *isoclines* (more precisely, *zero-isoclines*). The v-isoclines are the curves on which $\dot{v} = 0$. Likewise, the p-isoclines are the curves where $\dot{p} = 0$. Clearly, an equilibrium of (4.10) is to be found at the intersection of two such isoclines. For the Lotka-Volterra system, the v-isoclines are the straight lines $v = 0$ and

$$p = \frac{a}{b},$$

while the p-isoclines are the straight lines $p = 0$ and

$$v = \frac{c}{d}.$$

Accordingly, there are two equilibrium points: $(0,0)$ which is called *trivial*, and

$$(\bar{v}, \bar{p}) = \left(\frac{c}{d}, \frac{a}{b}\right), \tag{4.11}$$

which is called *nontrivial*, *internal*, or *positive*.

Before proceeding, let us address the effect of fishing on the positive equilibrium. Suppose, fishing produces for both prey and predator a per capita probability μ to be caught per unit of time. This means that a should be replaced by $a - \mu$ and $(-c)$ by $(-c - \mu)$ in (4.10). The corresponding equilibrium will become

$$\left(\frac{c+\mu}{d}, \frac{a-\mu}{b}\right).$$

This shows that fishing *increases* $\bar{v}$ while it *decreases* $\bar{p}$. Therefore, in a period in which fishing effort is much reduced, we may expect the $\bar{v}$ to $\bar{p}$ ratio to decrease[1].

The isoclines divide the nonnegative quarter plane into four regions (see Figure 4.7). To indicate the direction of the flow, we put vertical arrows on the v-isoclines

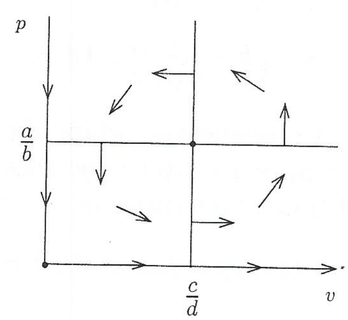


Figure 4.7: Zero-isoclines of the Lotka-Volterra system (4.10)

and horizontal arrows on the p-isoclines. Note that the direction of the arrows flips when passing the nontrivial equilibrium. Placing directional arrows inside the regions by continuity, we see at a glance that orbits of (4.10) have a tendency to spin around the nontrivial equilibrium. However, we can not see whether they spiral inward or outward.

To determine the local behaviour of orbits of (4.10) near equilibria, one may attempt to apply the Grobman-Hartman Theorem. One might suspect that the trivial equilibrium is a saddle point from the fact that both coordinate axes are invariant. The Jacobian matrix of (4.10) evaluated at the trivial equilibrium,

$$\begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix},$$

[1]This was Volterra's explanation for an increase of sharks observed by fishermen in the Adriatic Sea when they resumed fishing after World War I. During the war fishing had stopped.

obviously has two real eigenvalues: $\lambda_1 = a > 0$, $\lambda_2 = -c < 0$. Thus, the origin is a hyperbolic *saddle*, as expected. The Jacobi matrix of (4.10) evaluated at the nontrivial equilibrium (4.11),

$$\begin{pmatrix} a - b\bar{p} & -b\bar{v} \\ d\bar{p} & d\bar{v} - c \end{pmatrix} = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{pmatrix},$$

has purely imaginary eigenvalues $\lambda_{1,2} = \pm i\sqrt{ac}$. Therefore, this equilibrium is non-hyperbolic and the Grobman-Hartman Theorem is not applicable. To analyse the phase portrait of (4.10) near the nontrivial equilibrium and in the whole positive quarter plane (*quadrant*), one has to apply some other techniques.

Fortunately, the Lotka-Volterra system has a *constant of motion*

$$L(v,p) = dv - c\ln v + bp - a\ln p. \tag{4.12}$$

Indeed, along the orbits

$$\begin{aligned} \frac{d}{dt}L(v(t),p(t)) &= \frac{\partial L}{\partial v}(v(t),p(t))\dot{v}(t) + \frac{\partial L}{\partial p}(v(t),p(t))\dot{p}(t) \\ &= \left(d - \frac{c}{v(t)}\right) v(t)(a - bp(t)) \\ &+ \left(b - \frac{a}{p(t)}\right) p(t)(-c + dv(t)) \;=\; 0, \end{aligned}$$

meaning that L is constant. Consequently, orbits of (4.10) in the positive quadrant correspond to level sets $L(v,p) = L_0$. Actually, these level sets are closed curves. To see this, notice that (4.12) can be written as

$$L(v,p) = L_1(v) + L_2(p)$$

with functions $L_{1,2}$ that have similar properties. Indeed, $L_1(v) = dv - c\ln v$ has one global quadratic minimum at $v = \frac{c}{d}$, while $L_2(p) = bp - a\ln p$ has one global quadratic minimum at $p = \frac{a}{b}$. In fact, $L(v,p)$ as a function of two variables (v,p) has a unique global quadratic minimum at the nontrivial equilibrium (4.11). Therefore, its level curves are closed and form a nested family surrounding the nontrivial equilibrium. This completes the construction of the phase portrait of the Lotka-Volterra system in the positive quadrant (see Figure 4.8). The nontrivial equilibrium is a *center*: It is Lyapunov stable but not asymptotically stable.

It is interesting to calculate the average values of v and p over one period. To this end, we write the first equation in (4.10) as

$$\frac{d}{dt}\ln v(t) = \frac{1}{v(t)}\frac{dv(t)}{dt} = a - bp(t)$$

and integrate it to obtain

$$\ln\left(\frac{v(t)}{v(0)}\right) = at - b\int_0^t p(\tau)\, d\tau.$$

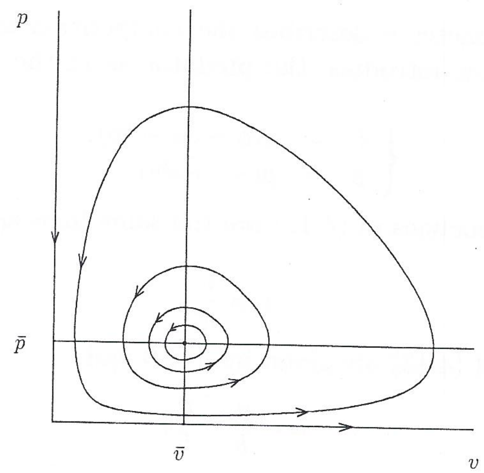


Figure 4.8: Phase portrait of the Lotka-Volterra system.

When $t = T$ (the period), we have $v(T) = v(0)$, so the left-hand side vanishes and consequently

$$\frac{1}{T}\int_0^T p(\tau)\,d\tau = \frac{a}{b} = \bar{p}.$$

Similarly,

$$\frac{1}{T}\int_0^T v(\tau)\,d\tau = \frac{c}{d} = \bar{v}.$$

In other words, the average of $v(t)$ and $p(t)$ over one period is equal to the equilibrium values $\bar{v}$ and $\bar{p}$, respectively. Thus, the equilibrium values are representative characteristics of the nonequilibrium dynamics of (4.10).

4.4.3 Stabilization by competition

The Lotka-Volterra system (4.10) gives an oversimplified description of prey-predator interaction. Yet it is a convenient reference point for the incorporation of additional mechanisms. In particular, the neutral stability of the nontrivial equilibrium allows us to classify such mechanisms as either *stabilizing* or *destabilizing*, depending on whether the corresponding equilibrium in a modified model is locally asymptotically stable or unstable.

We begin with modifying the equation for the isolated prey population. The so-called *logistic equation*

$$\dot{v} = av - ev^2 = v(a - ev)$$

describes in a phenomenological manner that, even in the absence of a predator, prey growth may be limited, with the prey population size settling down at the carrying capacity

$$K = \frac{a}{e}.$$

The nonnegative parameter e describes the competition among prey for limited external resources. Now introduce the predator as in the original Lotka-Volterra model, i.e. consider

$$\begin{cases} \dot{v} &= v(a - ev - bp), \\ \dot{p} &= p(-c + dv). \end{cases} \tag{4.13}$$

It is clear that the p-isoclines of (4.13) are the same lines as for the Lotka-Volterra system, i.e. $p = 0$ and

$$v = \frac{c}{d},$$

while the v-isoclines of (4.13) are given by $v = 0$ and

$$p = \frac{a}{b} - \frac{e}{b}v.$$

There are two typical configurations of the isoclines. When

$$\frac{a}{e} < \frac{c}{d},$$

the system (4.13) has two equilibria on the v-axis,

$$(0,0), \quad \left(\frac{a}{e},\, 0\right) = (K, 0),$$

the first of which is a saddle, while the second is globally asymptotically stable. This can be seen from Figure 4.9, where the isoclines and the equilibria are shown

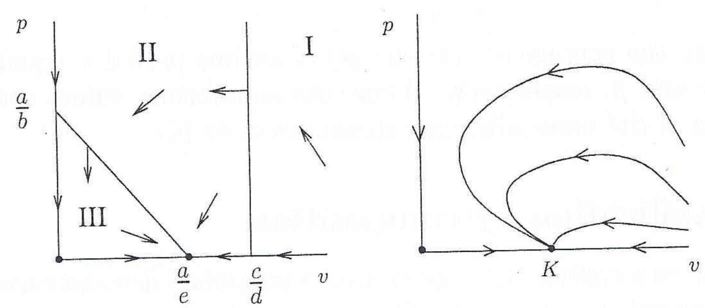


Figure 4.9: Zero-isoclines and the phase portrait of (4.13): $ad - ce < 0$.

together with the flow directions. As one can prove, all orbits starting in region I enter region II, while all orbits starting in II eventually either enter region III or converge to $(0, K)$. The triangular region III is *positively invariant*, i.e. the orbits can only enter it, not leave it. Using monotonicity arguments, we can show that all orbits starting inside this triangle tend to the point $(K, 0)$ as $t \to \infty$.

In case of the opposite inequality

$$\frac{a}{e} > \frac{c}{d},$$

the system (4.13) still has two equilibria on the v-axis,

$$(0,0), \quad (K, 0),$$

which are both hyperbolic *saddles*, as well as a nontrivial equilibrium

$$(\tilde{v}, \tilde{p}) = \left(\frac{c}{d}, \frac{ad - ec}{bd}\right), \tag{4.14}$$

which is a locally asymptotically *stable node* or *focus*. This follows from the analysis of the eigenvalues of this equilibrium and the Grobman-Hartman Theorem, since the nontrivial equilibrium is hyperbolic. Indeed, the Jacobian matrix of (4.13) evaluated at the nontrivial equilibrium (4.14) is

$$\begin{pmatrix} a - 2e\tilde{v} - b\tilde{p} & -b\tilde{v} \\ d\tilde{p} & d\tilde{v} - c \end{pmatrix} = \begin{pmatrix} -\frac{ec}{d} & -\frac{bc}{d} \\ \frac{ad - ec}{b} & 0 \end{pmatrix}.$$

Its eigenvalues $\lambda_{1,2}$ satisfy the characteristic equation

$$\lambda^2 - \sigma\lambda + \Delta = 0,$$

where

$$\sigma = \lambda_1 + \lambda_2 = -\frac{ec}{d} < 0, \quad \Delta = \lambda_1\lambda_2 = \frac{c(ad - ec)}{d} > 0.$$

Therefore, Re $\lambda_{1,2} < 0$, since all parameters are positive and $ad - ec > 0$.

Figure 4.10 suggests that the orbits of (4.13) tend to spiral around the nontrivial equilibrium.

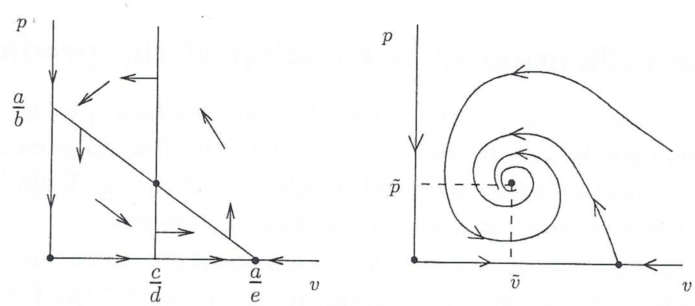


Figure 4.10: Zero-isoclines and the phase portrait of (4.13): $ad - ce > 0$.

In fact, the nontrivial equilibrium (4.14) of (4.13) is not only locally but *globally* asymptotically stable, i.e., all orbits starting in the positive quarter plane tend to this equilibrium. To prove this, notice first that the constant of motion L for the Lotka-Volterra system (4.10) can be written as

$$L(v, p) = d(v - \bar{v} \ln v) + b(p - \bar{p} \ln p),$$

where $(\bar{v}, \bar{p})$ are the coordinates of the nontrivial equilibrium (4.11). Now introduce a similar function

$$V(v, p) = d(v - \tilde{v} \ln v) + b(p - \tilde{p} \ln p) \tag{4.15}$$

where $(\tilde{v}, \tilde{p})$ are the coordinates of the nontrivial equilibrium (4.14). The function (4.15) is nonincreasing along the orbits of the modified system (4.13):

$$\begin{aligned}\frac{d}{dt}V(v(t),p(t)) &= d\left(1-\frac{d}{dv(t)}\right)v(t)(a-ev(t)-bp(t)) + \\ &\quad b\left(1-\frac{ad-ec}{bdp(t)}\right)p(t)(-c+dv(t)) \\ &= -de\left(v(t)-\frac{c}{d}\right)^2 \leq 0,\end{aligned}$$

with equality only on the p-isocline

$$v = \frac{c}{d}.$$

The function V is therefore a *Lyapunov function* for (4.13). It has a unique extremum (minimum) at the nontrivial equilibrium $(\tilde{v}, \tilde{p})$ and its closed level curves $V(v,p) = V_0$ surround this point. Moreover, the above formula shows that any orbit of (4.13), starting in the positive quadrant of the (v,p)-plane, passes any level $V(v,p) = V_0$ only once. At such a passage, the orbit goes from the *outer* into the *inner* region delimited by the corresponding level curve. The passage is transversal everywhere except on the p-isocline, where it is tangential. These facts imply that the nontrivial equilibrium in (4.13) is globally asymptotically stable. Thus, the competition among the prey is a stabilizing factor.

4.4.4 Destabilization by saturation of the predator

The *functional response* is by definition the per predator per unit of time eaten number of prey, usually as a function of prey density. The *numerical response* is the per predator per unit of time produced number of offspring. Usually, one takes the numerical response proportional to the functional response.

In the Lotka-Volterra model the functional response increases linearly with the prey density, which is another oversimplification. In reality, the predator functional response should approach some constant level at high prey densities. The so called Holling Type II functional response

$$\frac{bv}{1+\beta bv}$$

takes this saturation effect into account. For large v it approaches the maximal digestion/handling capacity $\frac{1}{\beta}$. Taken alone, this is a destabilizing effect.

Indeed, one can show that the following modification of the Lotka-Volterra system

$$\begin{cases} \dot{v} = v\left(a - \dfrac{bp}{1+\beta bv}\right), \\ \dot{p} = p\left(-c + \dfrac{dv}{1+\beta bv}\right), \end{cases} \tag{4.16}$$

has a unique nontrivial equilibrium

$$(v_1, p_1) = \left(\frac{c}{d - \beta bc}, \frac{ad}{b(d - \beta bc)} \right)$$

when $d - \beta bc > 0$. The Jacobian matrix of (4.16) evaluated at the equilibrium (v_1, p_1) is reduced to

$$\begin{pmatrix} \frac{\beta abc}{d} & -\frac{bc}{d} \\ \frac{a(d - \beta bc)}{b} & 0 \end{pmatrix}.$$

Therefore, its eigenvalues $\lambda_{1,2}$ satisfy

$$\sigma = \lambda_1 + \lambda_2 = \frac{\beta abc}{d} > 0, \ \Delta = \lambda_1 \lambda_2 = \frac{ac(d - \beta bc)}{d} > 0,$$

and thus the equilibrium (v_1, p_1) is hyperbolic and locally unstable. It can be either an unstable *node* or a *focus*.

Moreover, any orbit of (4.16) starting in the positive quadrant tends to this equilibrium as $t \to -\infty$ (see Figure 4.11). To prove this global result, we first notice

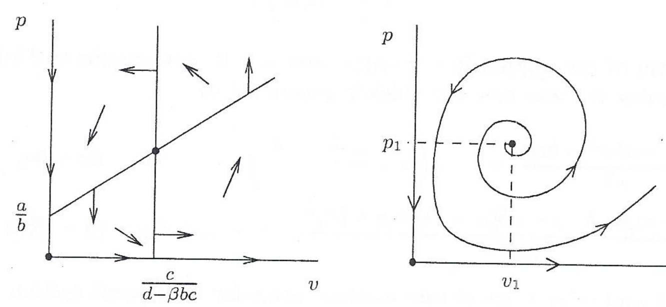


Figure 4.11: Zero-isoclines and the phase portrait of the system (4.16).

that orbits of (4.16) in the positive quadrant coincide with those of the polynomial system

$$\begin{cases} \overset{\frac{dv}{d\tau}}{\dot{v}} = v(a + \beta abv - bp), \\ \frac{dp}{d\tau}\ \dot{p} = p(-c + (d - \beta bc)v), \end{cases} \qquad (4.17)$$

✓ We're rescaled the time here, we can't use the same dots

which is obtained by multiplying the right-hand sides of (4.16) by

$$(1 + \beta bv).$$

This factor is obviously positive in the positive quadrant of the (v, p)-plane, so the direction of the motion along the orbits is preserved. According to Chapter 1, the systems (4.17) and (4.16) are *orbitally equivalent* in the positive quadrant.

Introduce new variables in the positive quadrant, namely:

$$\begin{cases} \xi = \ln v, \\ \eta = \ln p. \end{cases}$$

In these variables, (4.17) takes the form

$$\begin{cases} \dot{\xi} &= a + \beta ab\, e^{\xi} - b\, e^{\eta} \equiv P(\xi, \eta), \\ \dot{\eta} &= -c + (d - \beta bc)e^{\xi} \equiv Q(\xi, \eta). \end{cases} \tag{4.18}$$

Since

$$\frac{\partial P}{\partial \xi} + \frac{\partial Q}{\partial \eta} = \beta ab\, e^{\xi} > 0$$

for all ξ, Bendixson's Criterion guarantees the absence of cycles of (4.18) in the whole (ξ, η)-plane. Thus, (4.17) and hence (4.16) have no periodic orbits in the positive quadrant. Moreover, as we will show below, any negative half-orbit of the system is bounded. Together these facts imply that for $d - \beta bc > 0$ any orbit of (4.16) starting in the positive quadrant tends to the nontrivial equilibrium (v_1, p_1) as $t \to -\infty$. Otherwise, according to the Poincaré-Bendixson Theorem for α-limit sets, it must tend to a cycle at $t \to -\infty$, which is impossible. Similar arguments prove that any nonequilibrium orbit of (4.16) is unbounded as $t \to +\infty$.

To prove that all negative half-orbits of (4.17) are bounded, introduce new variables

$$z = \frac{1}{v}, \quad u = \frac{p}{v},$$

so that u is the ratio of the population densities and $z = 0$ corresponds to "infinitely large" prey density. The dynamics of these new variables is generated by

$$\begin{aligned} \dot{z} &= -\frac{av + \beta abv^2 - bvp}{v^2} = -z\left(a + \frac{\beta ab}{z} - \frac{bu}{z}\right) = -\beta ab - az + bu, \\ \dot{u} &= \frac{-(a+c)pv + (d - \beta b(a+c))v^2 p + bvp^2}{v^2} = -(a+c)u + (d - \beta b(a+c))\frac{u}{z} + \frac{bu^2}{z}. \end{aligned}$$

Multiplying both $\dot{z}$ and $\dot{u}$ by z, we obtain another orbitally equivalent system:

$$\begin{cases} \dot{z} &= -\beta abz - az^2 + bzu, \\ \dot{u} &= (d - \beta b(a+c))u - (a+c)zu + bu^2, \end{cases} \tag{4.19}$$

that should be studied near the u-axis, i.e., for small $z \geq 0$ and any $u \geq 0$. The results of such an analysis are illustrated in Figure 4.12. Clearly, both the z-axis and the u-axis are invariant. The

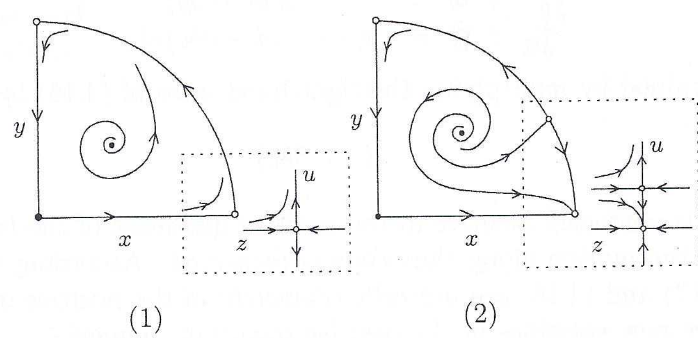


Figure 4.12: Behaviour of the system (4.16) at the infinity: (1) $(d - \beta bc) - \beta ab \geq 0$; (2) $(d - \beta bc) - \beta ab < 0$.

origin $(z, u) = (0, 0)$ is always an equilibrium of (4.19). It is a *saddle* if

$$d - \beta b(a + c) = (d - \beta bc) - \beta ab > 0$$

and a *stable node* if

$$d - \beta b(a + c) = (d - \beta bc) - \beta ab < 0.$$

In the latter case, there is another equilibrium point on the positive u-axis at

$$u = \frac{\beta b(a + c) - d}{b} > 0,$$

which is a *saddle*.

What does this mean for the system (4.17)? It is useful to consider the image of the (v, p)-portrait of (4.17) in the (x, y)-plane under the map

$$(x, y) = \left(\frac{v}{\sqrt{1 + v^2 + p^2}}, \frac{p}{\sqrt{1 + v^2 + p^2}} \right),$$

which is called the *Poincaré projection*. This map projects the whole (v, p)-plane into the unit circle in the (x, y)-plane, so that the ratio $u = p/v$ is preserved and points on the circle correspond to "infinity". The images of the positive quadrant of the (v, p)-plane are also shown in Figure 4.12 with the corresponding phase portraits. There are at most three infinitely-remote equilibria: Two studied above and one more at the "end" of the p-axis, which is a *saddle* for all $(d - \beta bc) > 0$. From neither of these equilibria emanates an orbit into the positive quadrant: *"Infinity is always stable"*. This implies that all negative half-orbits of (4.17) with positive initial data are bounded.

There is an interesting difference between the two mentioned cases. When $(d - \beta bc) - \beta ab > 0$ (and also when the equality holds), all orbits spiral indefinitely as $t \to +\infty$ while going to infinity. This means that predators gain and lose control over the prey again and again. In contrast, when $(d - \beta bc) - \beta ab < 0$, almost all orbits starting in the positive quadrant spiral for a while around the nontrivial equilibrium but eventually tend to infinity "along the v-axis", i.e. $u(t) = v(t)/p(t) \to 0$ as $t \to +\infty$. There is only one exceptional orbit that connects the unstable positive equilibrium and the saddle at infinity; along this orbit

$$\frac{v(t)}{p(t)} \to \frac{\beta b(a + c) - d}{b}$$

as $t \to +\infty$. So the predator may gain and lose control over the prey a number of times, but eventually the control will be lost and the prey density will grow exponentially, while the predator density also grows exponentially but, generically, at strictly lower rate.

4.4.5 Rosenzweig-MacArthur model

By combining the stabilizing effect of the competition among prey and the destabilizing effect of the predator saturation, one arrives at the model

$$\begin{cases} \dot{v} &= v\left(a - ev - \dfrac{bp}{1 + \beta bv}\right), \\ \dot{p} &= p\left(-c + \dfrac{dv}{1 + \beta bv}\right), \end{cases} \tag{4.20}$$

which is a variant of a more general *Rosenzweig-MacArthur model*:

$$\begin{cases} \dot{v} &= v(h(v) - p\varphi(v)), \\ \dot{p} &= p(-c + \psi(v)). \end{cases} \tag{4.21}$$

The prey isocline in (4.20) is the parabola

$$p(v) = \frac{1}{b}(a - ev)(1 + \beta bv),$$

which has its top for

$$v_0 = \frac{1}{2}\left(\frac{a}{e} - \frac{1}{\beta b}\right).$$

Figure 4.13 illustrates possible isocline configurations. It is not difficult to prove that all positive half-orbits of (4.20) are bounded.

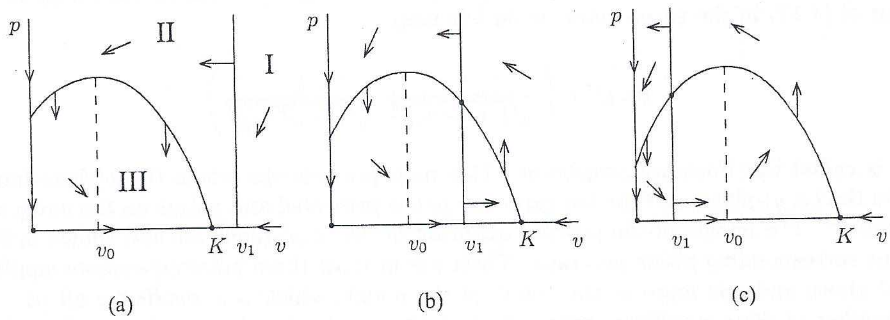


Figure 4.13: Zero-isoclines of (4.20).

When the predator isocline

$$v = v_1 = \frac{c}{d - \beta bc}$$

lies to the right of the trivial equilibrium

$$(K, 0) = \left(\frac{a}{e}, 0\right)$$

(see Figure 4.13(a)), all orbits of (4.20) with positive initial data will tend to this

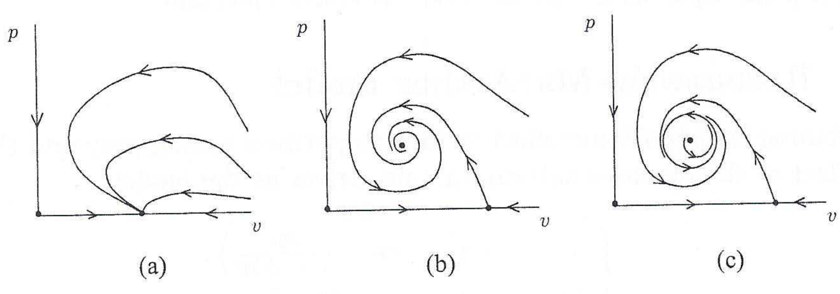


Figure 4.14: Generic phase portraits of the Rosenzweig-MacArthur system (4.20).

equilibrium as $t \to +\infty$ (see Fig. 4.14(a)). This fact can be proved just like the corresponding statement in Section 4.4.3. The equilibrium remains globally asymptotically stable also when the predator isocline goes through $(K, 0)$.

When the predator isocline goes through the top of the parabola or lies to the right of the top but to the left of the equilibrium $(K, 0)$ (see Figure 4.13(b)), the system (4.20) has a unique nontrivial equilibrium $(v_1, p(v_1))$ which is either a *stable node* or a *stable focus*. Moreover, all orbits of (4.20) with positive initial data tend to this equilibrium as $t \to +\infty$ (see Fig. 4.14(b)). This can be proved with some effort using Dulac's Theorem 4.15.

To simplify further computations, reduce the number of parameters in (4.20) by the linear scaling:

$$v = \frac{a}{e}\,x, \quad p = \frac{ad}{be}\,y, \quad t = \frac{\beta b}{d}\,\tau.$$

This brings (4.20) to the form

$$\begin{cases} \dot{x} &= rx(1-x) - \dfrac{xy}{m+x} \equiv R(x,y), \\ \dot{y} &= -\gamma y + \dfrac{xy}{m+x} \equiv S(x,y), \end{cases} \tag{4.22}$$

where the dot means now the derivative with respect to τ and

$$r = \frac{\beta ab}{d}, \quad \gamma = \frac{\beta bc}{d}, \quad m = \frac{e}{\beta ab}.$$

After the scaling, $x = 1$ corresponds to the equilibrium of the prey in the absence of the predator, while the coordinates of the nontrivial equilibrium are

$$(x_1, y_1) = \left(\frac{\gamma m}{1-\gamma}, \frac{rm(1-\gamma-\gamma m)}{(1-\gamma)^2}\right). \tag{4.23}$$

This equilibrium is indeed in the positive quadrant $\mathbb{R}^2_+$ if $\gamma < 1$ and $1 - \gamma - \gamma m > 0$. The top of the parabola $\dot{x} = 0$ is now located at

$$x_0 = \frac{1-m}{2}. \tag{4.24}$$

Therefore, the isocline configuration shown in Figure 4.13(b) appears when

$$\frac{1-m}{1+m} < \gamma < \frac{1}{1+m}. \tag{4.25}$$

We now look for a function $\mu : \mathbb{R}^2_+ \to \mathbb{R}$ (to use in Dulac's Theorem 4.15) in the form

$$\mu(x,y) = \left(\frac{x}{m+x}\right)^{\alpha} y^{\delta},$$

with some $\alpha, \delta \in \mathbb{R}$. According to Theorem 4.15, system (4.22) will have no cycles in $\mathbb{R}^2_+$ if the expression

$$Q(x,y) = \frac{\partial}{\partial x}(\mu(x,y)R(x,y)) + \frac{\partial}{\partial y}(\mu(x,y)S(x,y))$$

is not identically equal to zero and does not change sign. This expression can be computed as

$$Q(x,y) = \left[-\frac{m(\alpha+1)y}{(m+x)^2} + \frac{r}{m+x}P_{\alpha,\omega}(x)\right]\mu(x,y),$$

where

$$\omega = \frac{\delta+1}{r} \tag{4.26}$$

and

$$P_{\alpha,\omega}(x) = -2x^2 - [(\alpha+2)m + (\gamma-1)\omega - 1]x + (\alpha+1-\gamma\omega)m.$$

Therefore, $Q(x, y)$ will be nonpositive in $\mathbb{R}^2$ if we can select values of α and ω such that $\alpha \geq -1$ and $P_{\alpha,\omega}(x) \leq 0$ for all $x > 0$. The discriminant[2] of the second-degree polynomial $P_{\alpha,\omega}(x)$ is

$$D_\alpha(\omega) = (1-\gamma)^2\omega^2 - 2[(\alpha m - 1)(1-\gamma) + 2m(1+\gamma)]\omega + (\alpha+2)^2m^2 + 2(3\alpha+2)m + 1$$

and in turn the discriminant of the polynomial $D_\alpha(\omega)$ is given by

$$D(\alpha) = -32m[(1-\gamma)(1-\gamma-\gamma m)\alpha + 1 - \gamma - 2\gamma m].$$

If there is an α^* such that $D(\alpha^*) > 0$, then $D_{\alpha^*}(\omega) = 0$ has two real solutions $\omega_1 < \omega_2$. If now ω^* is any real number such that $\omega_1 < \omega^* < \omega_2$, then $D_{\alpha^*}(\omega^*) < 0$ and $P_{\alpha^*,\omega^*}(x)$ has no real roots. Since the coefficient of x^2 is negative, $P_{\alpha^*,\omega^*}(x) < 0$ for all x. Choosing $\delta^* = r\omega^* - 1$ according to (4.26) completes the argument.

Consider now three cases when (4.25) holds:

(1) If

$$\frac{1-m}{1+m} < \gamma < \frac{1}{1+2m},$$

choose α^* such that

$$-1 < \alpha^* < -\frac{1-\gamma-2\gamma m}{(1-\gamma)(1-\gamma-\gamma m)} < 0$$

and it follows that $D(\alpha^*) > 0$.

(2) If

$$\gamma = \frac{1}{1+2m},$$

for any $-1 \leq \alpha^* < 0$ we have

$$D(\alpha^*) = -\frac{64m^3\alpha^*}{(1+2m)^2} > 0.$$

(3) If

$$\frac{1}{1+2m} < \gamma < \frac{1}{1+m},$$

set $\alpha^* = 0$. Then, clearly, $D(0) > 0$.

If $\gamma = \frac{1-m}{1+m}$, i.e. when the predator isocline $\dot{y} = 0$ goes through the top of the parabola $\dot{x} = 0$, we can take $\alpha^* = -1$. Then $D(-1) = 0$, implying that

$$D_{-1}(\omega) = \frac{(m^2 + 2\omega m - 1)^2}{(m+1)^2}$$

has the double root

$$\omega^* = \frac{1-m^2}{2m}.$$

With these values of α^* and ω^*, the polynomial P_{α^*,ω^*} takes the form

$$P_{\alpha^*,\omega^*}(x) = -\tfrac{1}{2}(2x + m - 1)^2 \leq 0.$$

Since it vanishes only along the predator isocline $x = x_0$, Theorem 4.15 guarantees the absence of cycles also in this case.

Finally, when the predator isocline lies to the left of the top (see Figure 4.13(c)), the nontrivial equilibrium is an *unstable focus*. Since positive half-orbits of (4.20) could not approach the nontrivial equilibrium due to its repellor character, the Poincaré-Bendixson Theorem implies that there must be at least one *periodic orbit*

[2]Recall that for a polynomial $Ax^2 + Bx + C$ the *discriminant* is defined as $D = B^2 - 4AC$.

in the positive quadrant. Since any such orbit must encircle an equilibrium, we conclude that there exists at least one periodic orbit around the nontrivial equilibrium in (4.20). The corresponding solutions $(v(t), p(t))$ are periodic. Thus, the model (4.20) predicts in this case periodic oscillations in the population densities.

With much more effort, one can prove that the system (4.20) has *at most one* periodic orbit. Therefore, when it exists, the nontrivial equilibrium is either globally asymptotically stable, or unstable and is surrounded by the unique stable periodic orbit (*limit cycle*) (see Figure 4.14(c)). The proof is sketched below.

Recall that the considered case occurs when $0 < x_1 < x_0$, i.e. when

$$0 < \frac{\gamma m}{1-\gamma} < \frac{1-m}{2}$$

(see (4.24) and (4.23)). The proof of uniqueness is heavily based on the *mirror symmetry* of the prey isocline $y = r(1-x)(m+x) \equiv g(x)$, with respect to the vertical line $x = x_0$.

Consider in $\mathbb{R}^2_+$ the system

$$\begin{cases} \dot{x} &=& x[g(x) - y] &\equiv& R_1(x, y), \\ \dot{y} &=& y[-\gamma m + (1-\gamma)x] &\equiv& S_1(x, y), \end{cases} \tag{4.27}$$

that is obtained by the multiplication of the right-hand side of (4.22) by $(m + x)$ and hence is orbitally equivalent to (4.22). Its advantage is that it is polynomial. The divergence of the vector field $F = (R_1, S_1)^T$ is given by

$$\text{div } F = \frac{\partial R_1}{\partial x} + \frac{\partial S_1}{\partial x} = g(x) - y + xg'(x) - \gamma m + (1-\gamma)x = \frac{1}{x}\frac{dx}{dt} + xg'(x) + \frac{1}{y}\frac{dy}{dt}.$$

As we have seen, in the considered case (4.22) (and, therefore, (4.27)) has at least one periodic orbit around the nontrivial equilibrium. If

$$I_1 = \int_0^{T_0} \text{div } F(\xi(t), \eta(t))\, dt < 0$$

for *any* periodic solution $(\xi(t), \eta(t))$ with period T_0 of (4.27), then all cycles are hyperbolic and exponentially stable (see equation (4.4)). Arguments similar to those used to prove Theorem 4.16, show that there may exist at most one such cycle, i.e., when the cycle exists, it is unique. Notice that I_1 can be computed as

$$I_1 = \int_0^{T_0} \xi(t)g'(\xi(t))\, dt = r\int_0^{T_0} \xi(t)(1 - m - 2\xi(t))\, dt = 2r\int_0^{T_0} \xi(t)(x_0 - \xi(t))\, dt,$$

since, due to periodicity,

$$\oint_\Gamma \frac{dx}{x} = \oint_\Gamma \frac{dy}{y} = 0,$$

where Γ is the cycle. Note that Γ has counterclockwise orientation. Define

$$I_2 = \int_0^{T_0} \xi(t)(x_0 - \xi(t))\, dt, \tag{4.28}$$

so that $I_1 = 2rI_2$.

Lemma 4.18 *The mirror image Γ' of any cycle Γ of (4.27) with respect to the vertical line $x = x_0$ intersects Γ in precisely six points: A, B, P, Q, P', Q', such that P' and Q' are the mirror images of P and Q, respectively. Moreover,*

(*i*) $x_A = x_B = x_0,\ y_A < g(x_0) < y_B$;
(*ii*) $x_{P'} < x_1 < x_0 < x_P,\ \ x_{Q'} < x_1 < x_0 < x_Q$;
(*iii*) $y_{P'} = y_P < g(x_P),\ \ y_{Q'} = y_Q > g(x_Q)$.

(*see* Figure 4.15(a)).

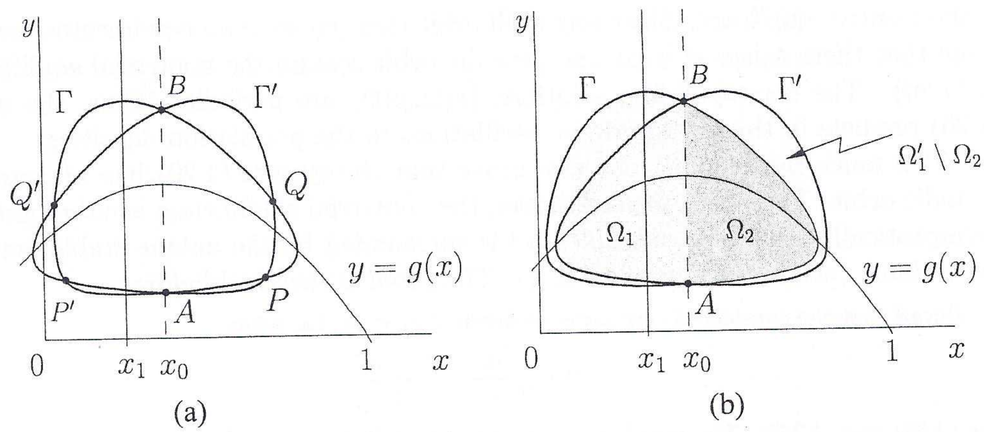


Figure 4.15: Geometry of the limit cycle: (a) real; (b) hypothetical.

Proof: Condition (i) is merely the definition of A and B.

For the rest, it is sufficient to prove that the mirror image of that part of Γ, which is located to the left of the line $x = x_0$, intersects Γ in two points: P and Q. Note that simple comparison and monotonicity arguments show that there can exist no more than two such points, one of which (say, P) is located below the prey isocline $y = g(x)$, while the other (Q) is above this isocline.

Consider the following auxiliary function

$$V(x, y) = (1-\gamma)\int_1^x \frac{(\xi - x_1)\, d\xi}{\xi} + y.$$

The derivative of V along solutions of (4.27) satisfies

$$\dot{V} = (1-\gamma)\frac{x - x_1}{x}\dot{x} + \dot{y} = (1-\gamma)g(x)(x - x_1) = \frac{g(x)}{y}\dot{y}.$$

Thus, by periodicity,

$$\oint_\Gamma \frac{g(x)\, dy}{y} = \int_0^{T_0} \dot{V}\, dt = 0.$$

Introduce two open domains Ω_1 and Ω_2, such that Ω_1 is the sub-domain inside Γ located strictly to the left from $x = x_0$, while Ω_2 is the sub-domain inside Γ located strictly to the right from $x = x_0$. Clearly, $\Omega_1 \cap \Omega_2 = \emptyset$.

Now assume that the right part of Γ' does not intersect Γ, i.e., that Ω_2 is located strictly inside the mirror image Ω_1' of Ω_1 (see Figure 4.15(b)). Thus, $\Omega_1' \setminus \Omega_2$ is a nonempty simply-connected domain.

By Green's Theorem,

$$0 = \oint_\Gamma \frac{g(x)\, dy}{y} = \int_\Omega \frac{g'(x)}{y}\, dxdy = 2r\int_\Omega \frac{x_0 - x}{y}\, dxdy,$$

where Ω is the domain inside Γ. Since

$$\begin{aligned}
\int_\Omega \frac{x_0 - x}{y}\, dxdy &= \int_{\Omega_1} \frac{x_0 - x}{y}\, dxdy + \int_{\Omega_2} \frac{x_0 - x}{y}\, dxdy \\
&= -\int_{\Omega_1'} \frac{x_0 - x}{y}\, dxdy + \int_{\Omega_2} \frac{x_0 - x}{y}\, dxdy \\
&= -\int_{\Omega_1' \setminus \Omega_2} \frac{x_0 - x}{y}\, dxdy,
\end{aligned}$$

and $x_0 < x$ in $\Omega_1' \setminus \Omega_2$, we have

$$0 = -2r \int_{\Omega_1' \setminus \Omega_2} \frac{x_0 - x}{y} \, dx dy > 0,$$

which is a contradiction. Therefore, Ω_2 cannot be strictly inside the mirror image Ω_1' of Ω_1, meaning that the right part of Γ' must intersect Γ. It is easy to verify that the above arguments also exclude the possibility for Γ' and Γ to have a single common point to the right from $x = x_0$.

Thus, Γ' and Γ intersect at two points to the right of $x = x_0$: P and Q, which, together with their mirror images P' and Q', satisfy conditions (ii) and (iii). □

Lemma 4.19 *Let* $p(\xi) = 2\xi^2 - 4x_0\xi + 2x_0x_1$. *Then*

(i) $p(x_{Q'}) > 0$;
(ii) $p(x_{P'}) > 0$.

Proof: Let $y(x)$ represent the upper part of Γ. Then

$$\frac{dy}{dx} = (1 - \gamma)\frac{(x - x_1)y}{(g(x) - y)x}.$$

Lemma 4.18 implies

$$\left.\frac{dy}{dx}\right|_Q > -\left.\frac{dy}{dx}\right|_{Q'}$$

(see Figure 4.15(a)) or

$$\frac{(x_Q - x_1)y_Q}{(g(x_Q) - y_Q)x_Q} > -\frac{(x_{Q'} - x_1)y_{Q'}}{(g(x_{Q'}) - y_{Q'})x_{Q'}}.$$

Taking into account $y_Q = y_{Q'}$, $g(x_Q) = g(x_{Q'})$, and $g(x_Q) < y_Q$, we obtain

$$\frac{x_{Q'}}{x_1 - x_{Q'}} < \frac{x_Q}{x_Q - x_1}.$$

Substituting $x_Q = 2x_0 - x_{Q'}$ into this inequality, we immediately get $p(x_{Q'}) > 0$. The proof of (ii) is similar. □

The integral (4.28) can be written as the sum of six integrals (see Figure 4.15(a)):

$$I_2 = I_{AP} + I_{PQ} + I_{QB} + I_{BQ'} + I_{Q'P'} + I_{P'A},$$

where each integral is computed over the corresponding time interval. Let

$$0 < t_P < t_Q < t_B < t_{Q'} < t_{P'} < t_A = T_0$$

be the respective times of passage through the points P, Q, B, Q', P', A, while tracing over one period the periodic solution $(\xi(t), \eta(t))$, starting at point A and returning to A.

Let $y = \eta_1(x)$ represent Γ on $[t_{P'}, t_A]$, and let $y = \eta_2(x)$ represent Γ on $[0, t_P]$ (see Figure 4.16). We can rewrite $I_{P'A} + I_{AP}$ using the mirror symmetry of the prey isocline $y = g(x)$ with respect to the vertical line $x = x_0$, i.e. $g(2x_0 - x) = g(x)$. We get

$$\begin{aligned}
I_{P'A} + I_{AP} &= \left(\int_{t_{P'}}^{t_A} + \int_0^{t_P}\right) \xi(t)(x_0 - \xi(t)) \, dt \\
&= \int_{P'A} \frac{(x_0 - x)}{g(x) - \eta_1(x)} \, dx + \int_{AP} \frac{(x_0 - x)}{g(x) - \eta_2(x)} \, dx \\
&= \int_{P'A} \frac{(x_0 - x)}{g(2x_0 - x) - \eta_1(x)} \, dx + \int_{AP} \frac{(x_0 - x)}{g(x) - \eta_2(x)} \, dx \\
&= -\int_{AP} \frac{(x_0 - x)}{g(x) - \eta_1(2x_0 - x)} \, dx + \int_{AP} \frac{(x_0 - x) \, dx}{g(x) - \eta_2(x)} \\
&= \int_{x_A}^{x_P} \frac{(x_0 - x)(\eta_2(x) - \eta_1(2x_0 - x))}{(g(x) - \eta_1(2x_0 - x))(g(x) - \eta_2(x))} \, dx \; < \; 0,
\end{aligned}$$

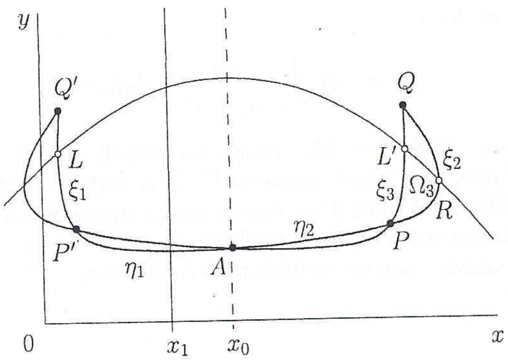


Figure 4.16: Geometrical constructions for the proof of uniqueness.

since from Lemma 4.18 it follows that for all x with $x_A = x_0 < x < x_P$

$$\eta_2(x) - \eta_1(2x_0 - x) > 0, \quad g(x) - \eta_2(x) > 0, \quad g(x) - \eta_1(2x_0 - x) > 0,$$

holds.

Similar arguments imply that

$$I_{QB} + I_{BQ'} = \left(\int_{t_Q}^{t_B} + \int_{t_B}^{t_{Q'}} \right) \xi(t)(x_0 - \xi(t))\, dt < 0.$$

Let now $x = \xi_1(y)$ represent Γ on $[t_{Q'}, t_{P'}]$ and let $x = \xi_2(y)$ represent Γ on $[t_P, t_Q]$ (Figure 4.16). Then

$$I_{Q'P'} = \int_{t_{Q'}}^{t_{P'}} \xi(t)(x_0 - \xi(t))\, dt = -\frac{1}{1-\gamma} \int_{y_P}^{y_Q} \frac{\xi_1(y)(x_0 - \xi_1(y))}{(\xi_1(y) - x_1)y}\, dy$$

and

$$I_{PQ} = \int_{t_P}^{t_Q} \xi(t)(x_0 - \xi(t))\, dt = \frac{1}{1-\gamma} \int_{y_P}^{y_Q} \frac{\xi_2(y)(x_0 - \xi_2(y))}{(\xi_2(y) - x_1)y}\, dy.$$

The last integral can be re-written as follows

$$\begin{aligned} I_{PQ} &= \frac{1}{1-\gamma} \left[\int_{y_P}^{y_Q} \frac{\xi_2(y)(x_0 - \xi_2(y))}{(\xi_2(y) - x_1)y}\, dy + \int_{y_Q}^{y_P} \frac{\xi_3(y)(x_0 - \xi_3(y))}{(\xi_3(y) - x_1)y}\, dy \right. \\ &\quad \left. - \int_{y_Q}^{y_P} \frac{\xi_3(y)(x_0 - \xi_3(y))}{(\xi_3(y) - x_1)y}\, dy \right] \\ &= \frac{1}{1-\gamma} \left[\oint_{PRQL'P} \frac{x(x_0 - x)}{(x - x_1)y}\, dy + \int_{y_P}^{y_Q} \frac{\xi_3(y)(x_0 - \xi_3(y))}{(\xi_3(y) - x_1)y}\, dy \right], \end{aligned}$$

where $x = \xi_3(y) = 2x_0 - \xi_1(y)$ represents the mirror image of $x = \xi_1(y)$, and the first integral is taken over the closed contour $PRQL'P$ (see Figure 4.16). Denote the compact domain bounded by this contour by Ω_3. Green's Theorem implies

$$\oint_{PRQL'P} \frac{x(x_0 - x)}{(x - x_1)y}\, dy = \int_{\Omega_3} \frac{(-x^2 + 2x_1 x - x_0 x_1)}{(x - x_1)^2 y}\, dy.$$

However, since $x_0 > x_1$ and

$$-x^2 + 2x_1 x - x_0 x_1 = -[(x - x_1)^2 + x_1(x_0 - x_1)] < 0,$$

we have

$$\oint_{PRQL'P} \frac{x(x_0 - x)}{(x - x_1)y}\, dy < 0.$$

Therefore

$$I_{PQ} < \frac{1}{1-\gamma} \int_{y_P}^{y_Q} \frac{\xi_3(y)(x_0 - \xi_3(y))}{(\xi_3(y) - x_1)y}\, dy = \frac{1}{1-\gamma} \int_{y_P}^{y_Q} \frac{(2x_0 - \xi_1(y))(\xi_1(y) - x_0)}{(2x_0 - \xi_1(y) - x_1)y}\, dy.$$

Now we can estimate

$$\begin{aligned} I_{Q'P'} + I_{PQ} &< \frac{1}{1-\gamma}\left[-\int_{y_P}^{y_Q} \frac{\xi_1(y)(x_0 - \xi_1(y))}{(\xi_1(y) - x_1)y}\, dy + \int_{y_P}^{y_Q} \frac{(2x_0 - \xi_1(y))(\xi_1(y) - x_0)}{(2x_0 - \xi_1(y) - x_1)y}\, dy \right] \\ &= \frac{1}{1-\gamma} \int_{y_P}^{y_Q} \frac{(x_0 - \xi_1(y))(2\xi_1^2(y) - 4x_0\xi_1(y) + 2x_0x_1)}{(\xi_1(y) - x_1)(2x_0 - \xi_1(y) - x_1)y}\, dy \\ &= \frac{1}{1-\gamma} \int_{y_P}^{y_Q} \frac{(x_0 - \xi_1(y))p(\xi_1(y))}{(\xi_1(y) - x_1)(2x_0 - \xi_1(y) - x_1)y}\, dy, \end{aligned}$$

where

$$p(\xi) = 2\xi^2 - 4x_0\xi + 2x_0x_1$$

is the polynomial already introduced in Lemma 4.19. This polynomial has two roots:

$$\xi_\pm = x_0 \pm \sqrt{x_0(x_0 - x_1)} > 0,$$

and is positive for $\xi < \xi_-$. If we prove that $\xi_1(y)$ satisfies $\xi_1(y) < \xi_-$ for all $y \in [y_P, y_Q]$, we can conclude that $p(\xi_1(y)) > 0$, and, since by Lemma 4.18,

$$\xi_1(y) - x_1 < 0, \quad 2x_0 - \xi_1(y) - x_1 > 0,$$

we would get

$$I_{Q'P'} + I_{PQ} < 0.$$

To complete the proof, notice that by Lemma 4.19 we have $p(x_{Q'}) > 0$ and $p(x_{P'}) > 0$. Moreover, $x_{Q'} < x_0$ and $x_{P'} < x_0$ while $\xi_+ > x_0$. Hence, both $x_{P'}, x_{Q'} < \xi_-$. Since, obviously,

$$\xi_1(y) \leq \max\{x_{P'}, x_{Q'}\},$$

we obtain that $\xi_1(y) < \xi_-$ for all $y \in [y_P, y_Q]$.

Thus, $I_2 < 0$ and therefore $I_1 < 0$ along any periodic orbit Γ.

4.5 Planar Hamiltonian and related systems

There is a special class of planar systems, for which a complete characterization of phase portraits is possible and which appear in Classical Mechanics. Moreover, it is possible to draw some conclusions about generic small perturbations of such systems.

4.5.1 Hamiltonian systems with one degree of freedom

Let $H = H(q,p)$ be a smooth (at least C^2) function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Define

$$H_q(q,p) = \frac{\partial H(q,p)}{\partial q}, \quad H_p(q,p) = \frac{\partial H(q,p)}{\partial p}.$$

Definition 4.20 *A planar system*

$$\begin{cases} \dot{q} &= H_p(q,p), \\ \dot{p} &= -H_q(q,p), \end{cases} \tag{4.29}$$

is called a ***Hamiltonian system*** *with one degree of freedom and* ***Hamiltonian*** *(or* ***energy function****)* H*.*

Example 4.21 (Lotka-Volterra system revisited)

Consider once again the Lotka-Volterra system from Section 4.4.1:

$$\begin{cases} \dot{x} &= ax - bxy, \\ \dot{y} &= -cy + dxy. \end{cases} \tag{4.30}$$

The linear scaling

$$x = \xi_0 \xi,\ y = \eta_0 \eta,\ t = \tau_0 \tau$$

with

$$\xi_0 = \frac{a}{d},\ \eta_0 = \frac{a}{b},\ \ \tau_0 = \frac{1}{a}$$

reduces (4.30) to

$$\begin{cases} \dot{\xi} &= \xi - \xi\eta, \\ \dot{\eta} &= -\gamma\eta + \xi\eta, \end{cases} \tag{4.31}$$

where $\dot{\xi}$ and $\dot{\eta}$ are the derivatives of ξ and η with respect to the new time τ, and

$$\gamma = \frac{c}{a}$$

is a new parameter. Introduce new coordinates in the positive quadrant $\mathbb{R}^2_+$ by the

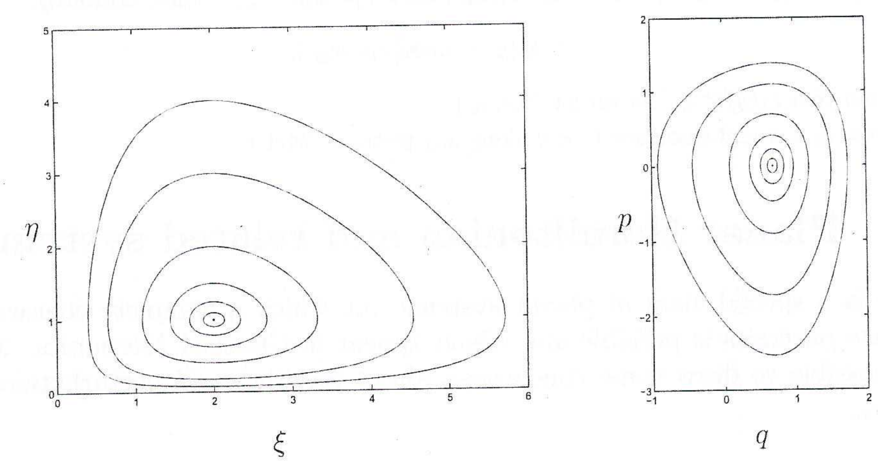


Figure 4.17: Phase portraits of the Lotka-Volterra system in the (ξ,η)- and (q,p)-coordinates.

formulas:

$$\begin{cases} q &= \ln \xi, \\ p &= \ln \eta. \end{cases}$$

One has

$$\begin{aligned} \dot{q} &= \frac{\dot{\xi}}{\xi} = 1 - \eta = 1 - e^p, \\ \dot{p} &= \frac{\dot{\eta}}{\eta} = -\gamma + \xi = -\gamma + e^q. \end{aligned}$$

Thus, (4.31) is smoothly equivalent in $\mathbb{R}^2_+$ to the Hamiltonian system (4.29) with

$$H(q,p) = \gamma q - e^q + p - e^p.$$

The transformation $(\xi, \eta) \mapsto (q, p)$ maps closed orbits of (4.31) in $\mathbb{R}^2_+$ onto closed orbits of (4.29) in $\mathbb{R}^2$, preserving their time parameterization (see Figure 4.17). $\diamond$

The following result explains why (4.29) is also called a *conservative system* (see also Exercise 4.7.12).

Theorem 4.22 (Conservation of energy) *The Hamiltonian* $H(q,p)$ *is a constant of motion for the Hamiltonian system, i.e.* $H(q(t), p(t)) = \text{const}$ *along solutions.*

Proof: $\dot{H} = H_q \dot{q} + H_p \dot{p} = H_q H_p - H_p H_q = 0$. $\square$

This theorem has as an immediate implication that any orbit of (4.29) belongs to a level set $\{(q,p) \in \mathbb{R}^2 : H(q,p) = h\}$ of the Hamiltonian function H. Next we can deduce further implications from the geometry of such level sets. As illustrated in the lower half of Figure 4.18, a connected component of a level set is typically either

- a single point (corresponding to a maximum or a minimum of H);
- a closed curve (these occur in families parametrized by the value of H; a strict maximum or minimum is always surrounded by such a family);
- a number of points connected by curves (more on this case below).

Another direct consequence of (4.29) is that equilibria of a Hamiltonian system correspond exactly to critical points of H. So on the one hand we can use the classification of nondegenerate critical points into extrema (maxima and minima) and saddle points, and on the other hand we can use the classification of equilibria according to their stability properties. How do these two ways of classification correspond to each other ?

A first observation is that (4.29) cannot have asymptotically stable equilibria (nor asymptotically stable cycles). Indeed, if all orbits starting in a neighbourhood of an equilibrium converge towards this equilibrium, then H must be constant on this neighbourhood (since H is continuous and constant along orbits). So the entire neighbourhood must consist of equilibria, a contradiction.

As H is assumed to be C^2, the Jacobian matrix at some equilibrium exists and is given by

$$A = \begin{pmatrix} H^0_{qp} & H^0_{pp} \\ -H^0_{qq} & -H^0_{qp} \end{pmatrix},$$

where the superscript indicates the evauation of all quantities at the equilibrium. Thus Tr $A = H^0_{qp} - H^0_{qp} = 0$ and the eigenvalues of A are either real and of opposite sign or purely imaginary, depending on the sign of

$$\det A = H^0_{qq} H^0_{pp} - (H^0_{qp})^2 \neq 0.$$

If we consider the equilibrium as a critical point of H, we are led to consider the Hessian matrix

$$B = \begin{pmatrix} H^0_{qq} & H^0_{qp} \\ H^0_{qp} & H^0_{pp} \end{pmatrix},$$

which has exactly the same determinant as A. When the determinant is positive, H has an extremum (a maximum if $H^0_{pp} < 0$ and a minimum if $H^0_{pp} > 0$). Thus, when A has purely imaginary eigenvalues, the corresponding equilibrium is a (nonlinear) center surrounded by a family of periodic orbits. When the determinant is negative, H has a saddle point and A has one positive and one negative real eigenvalue. Thus, the equilibrium is also a saddle point in the sense of planar dynamical systems (so when locally a level set consists of two intersecting curves, these are exactly the local stable and the local unstable manifolds of the point of intersection, which is a saddle equilibrium point).

In order to formulate our conclusions as a lemma, it is useful to introduce a new term.

Definition 4.23 *An equilibrium is called* ***simple****, if it has no zero eigenvalue.*

Lemma 4.24 *A simple equilibrium of a Hamiltonian system* (4.29) *with one degree of freedom is either a saddle or a center. The first case applies when*

$$H^0_{qq} H^0_{pp} - (H^0_{qp})^2 > 0$$

and the second when the opposite inequality holds. □

Remarks:

(1) With any smooth function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ one can also associate the *gradient flow* generated by the system

$$\left\{ \begin{array}{rcl} \dot{q} & = & -H_q(q,p), \\ \dot{p} & = & -H_p(q,p). \end{array} \right. \tag{4.32}$$

For this flow $\dot{H} = -[(H_q)^2 + (H_p)^2] \leq 0$, so orbits converge towards minima of H following steepest descent and crossing level curves of H orthogonally.

(2) Returning to (4.29), we observe that one might wonder how the period varies as a function of the value of H when there is a family of closed orbits ? Is the period a monotone function ? This is a notoriously difficult problem. With considerable effort one can prove monotonicity for some specific systems. But as far as we know, there is no general result and, in fact, the period may not be monotone for other specific systems.

(3) Finally note that the divergence of the vector field $(H_p, -H_q)^T$ is zero. As a consequence, planar Hamiltonian flows preserve area (see Theorem 4.35 in the Appendix to this chapter for a general result). ◇

4.5.2 Potential systems with one degree of freedom

Definition 4.25 *A planar system* (4.29) *with*

$$H(q,p) = \frac{1}{2}p^2 + U(q),$$

where U is a smooth scalar function, is called a **potential system.**

Here the term $\frac{1}{2}p^2$ is called the *kinetic energy,* while $U(q)$ is called the *potential energy.*

The corresponding Hamiltonian system (4.29) has the form:

$$\begin{cases} \dot{q} &= p, \\ \dot{p} &= -U'(q). \end{cases} \tag{4.33}$$

Eliminating p we get Newton's 2nd Law:

$$\ddot{q} = F(q),$$

with the *potential force* $F(q) = -U'(q)$. (Side remark for physicists: Notice that the "mass" is put equal to one, which can always be achieved by scaling.)

The phase portrait of (4.33) in the (q,p)-plane is completely determined by the potential energy $U(q)$ (see Figure 4.18). The portrait is symmetric with respect

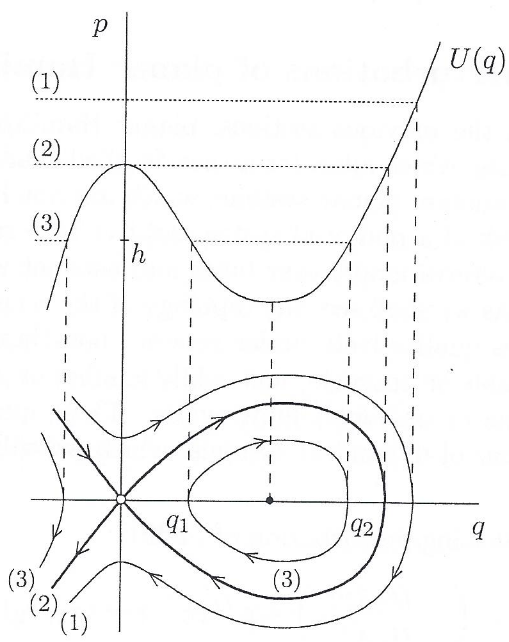


Figure 4.18: Newton dynamics with one degree of freedom.

to the reflection in the q-axis and time reversal. Equilibrium points of (4.33) have the form $(q^0, 0)$, where q^0 is a critical point of $U(q)$, i.e. $U'(q^0) = 0$. If q^0 is a local maximum of U, the equilibrium is a saddle; if q^0 is a local minimum of U, the equilibrium is a center. A connected subset of any level set $H(q,p) = h$, where h is a regular value, is diffeomorphic to either a circle (closed curve) or a line. Any closed orbit defines a periodic solution with period (see Exercise 4.7.11)

$$T = \int_{q_1}^{q_2} \sqrt{\frac{2}{h - U(q)}}\, dq, \tag{4.34}$$

where $q_1 < q_2$ are the coordinates of the intersections of the orbit with the q-axis and, therefore, $U(q_1) = U(q_2) = h$. Critical level sets contain equilibria and homo- or heteroclinic orbits, which are asymptotic to them.

Example 4.26 (Famous potential systems with $m = 1$)

(1) *Harmonic oscillator*:
$$H = \frac{1}{2}(p^2 + q^2).$$

(2) *Ideal pendulum*:
$$H = \frac{1}{2}p^2 - \cos q.$$

(3) *Duffing oscillator*:
$$H = \frac{1}{2}(p^2 - q^2) + \frac{q^4}{4}.$$

You are invited to draw their phase portraits yourself. $\diamond$

4.5.3 Small perturbations of planar Hamiltonian systems

As we have seen in the previous sections, planar Hamiltonian systems have very special phase portraits, which allow for rather detailed characterization. Can we use this information to analyse planar systems which are not Hamiltonian but close to them? One can think of a potential system subject to a small friction, or about a generalized Lotka-Volterra model that takes into account weak competition among prey or predators. As we shall see, the topology of the phase portrait of a Hamiltonian system changes qualitatively under generic (non-Hamiltonian) perturbations: Centers become (stable or unstable) foci, while families of periodic orbits disappear, giving rise to (stable or unstable) limit cycles. These qualitative changes are examples of *bifurcations* of dynamical systems, which we will study systematically in Chapters 5 and 6.

Consider the following perturbation of (4.29):

$$\dot{x} = \begin{pmatrix} H_{x_2}(x) \\ -H_{x_1}(x) \end{pmatrix} + \varepsilon f(x), \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \tag{4.35}$$

where $\varepsilon \in \mathbb{R}$ is a small parameter, and $H : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ are smooth functions. Since we are interested in non-Hamiltonian perturbations, $\operatorname{div} f$ does not vanish.

We begin with a simple result concerning equilibria.

Theorem 4.27 *Consider the system* (4.35) *and assume that* $f(0) = 0$.

(i) *If* $x = 0$ *is a simple saddle at* $\varepsilon = 0$, *then* $x = 0$ *is a saddle of* (4.35) *for all* ε *with sufficiently small* $|\varepsilon|$.

(ii) *If* $x = 0$ *is a simple center at* $\varepsilon = 0$, *then* $x = 0$ *is a focus of* (4.35) *for all* ε *with sufficiently small* $|\varepsilon| > 0$. *Moreover, this focus is stable for* $\varepsilon \operatorname{div} f(0) < 0$ *and unstable for* $\varepsilon \operatorname{div} f(0) > 0$.

Proof: Write the Jacobian matrix of (4.35) at the equilibrium $x = 0$ as

$$A(\varepsilon) = \begin{pmatrix} H^0_{x_1x_2} & H^0_{x_2x_2} \\ -H^0_{x_1x_1} & -H^0_{x_2x_1} \end{pmatrix} + \varepsilon f^0_x.$$

Its eigenvalues $\lambda_{1,2}(\varepsilon)$ depend smoothly on ε.

Part (i) is then obvious, since if $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ have opposite sign at $\varepsilon = 0$, then this property will hold for all sufficiently small ε. Thus, by the Grobman-Hartman Theorem, $x = 0$ is a saddle for such parameter values.

When $x = 0$ is a center at $\varepsilon = 0$, the matrix $A(\varepsilon)$ has a pair of nonreal eigenvalues $\lambda_1 = \bar{\lambda}_2(\varepsilon)$ for all sufficiently small ε and

$$2\mathrm{Re}\ \lambda_{1,2}(\varepsilon) = \mathrm{Tr}\ A(\varepsilon) = \varepsilon \mathrm{Tr}\ f^0_x = \varepsilon \operatorname{div} f(0).$$

Applying the Grobman-Hartman Theorem for $\varepsilon \neq 0$, we obtain Part (ii) of the theorem. □

Example 4.28 (Perturbed Lotka-Volterra system)

Consider the following small perturbation of the (scaled) Lotka-Volterra system (4.31):

$$\begin{cases} \dot{\xi} &= \xi - \xi\eta - \varepsilon\xi^2, \\ \dot{\eta} &= -\gamma\eta + \xi\eta, \end{cases} \tag{4.36}$$

where $0 < \varepsilon \ll 1, \gamma > 0$ and the $\varepsilon\xi^2$-term describes weak competition among prey. Introducing the same variables as in Example 4.21,

$$\begin{cases} q &= \ln \xi, \\ p &= \ln \eta, \end{cases}$$

we obtain

$$\begin{cases} \dot{q} &= 1 - e^p - \varepsilon e^q, \\ \dot{p} &= -\gamma + e^q. \end{cases}$$

This system has for $(1 - \gamma\varepsilon) > 0$ an equilibrium

$$(q^0(\varepsilon), p^0(\varepsilon)) = (\ln \gamma, \ln(1 - \gamma\varepsilon)),$$

which is a center if $\varepsilon = 0$. Translating the origin of the coordinate system to this equilibrium by the transformation

$$\begin{cases} x_1 &= q - \ln \gamma, \\ x_2 &= p - \ln(1 - \varepsilon\gamma), \end{cases}$$

we obtain the system

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & 1 - e^{x_2} + \varepsilon\gamma(e^{x_2} - e^{x_1}), \\ \dot{x}_2 & = & -\gamma(1 - e^{x_1}), \end{array} \right. \tag{4.37}$$

which has the form (4.35) with $H(x) = x_2 - e^{x_2} + \gamma(x_1 - e^{x_1})$ and

$$f(x) = \begin{pmatrix} \gamma(e^{x_2} - e^{x_1}) \\ 0 \end{pmatrix}, \quad f(0) = 0.$$

The system (4.37) satisfies the conditions of Theorem 4.27. Since

$$\varepsilon \operatorname{div} f(0) = -\varepsilon\gamma < 0,$$

the equilibrium $(q^0(\varepsilon), p^0(\varepsilon))$ is a stable focus for sufficiently small $\varepsilon > 0$.

Note that this result perfectly agrees with our analysis in Section 4.4.3, where it was shown that system (4.13) has a unique positive globally asymptotically stable equilibrium when $ad - ce > 0$. Indeed, system (4.13) coincides with system (4.36) if we set $a = b = d = 1, c = \gamma$, and $e = \varepsilon$, so that the above condition turns into $(1 - \varepsilon\gamma) > 0$, which is obviously true for small ε. $\diamond$

Remark: Consider a slightly more general system

$$\dot{x} = \begin{pmatrix} H_{x_2}(x) \\ -H_{x_1}(x) \end{pmatrix} + \varepsilon f(x, \varepsilon), \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \varepsilon \in \mathbb{R}, \tag{4.38}$$

where H is smooth and has a nondegenerate critical point $x = 0$, while f is a smooth function of (x, ε). The Implicit Function Theorem assures that (4.38) has a smooth family $x^0(\varepsilon)$ of equilibria for small $\varepsilon \neq 0$, such that $x^0(0) = 0$. Translating the origin to $x^0(\varepsilon)$, we can assume without loss of generality that $f(0, \varepsilon) = 0$ for all ε with small $|\varepsilon|$. Then, the equilibrium $x = 0$ is either a saddle or, when $\operatorname{div} f(0,0) \neq 0$, a focus. The stability of the focus is determined by the sign of $\operatorname{div} f(0,0)$. $\diamond$

Now we want to study limit cycles of perturbed planar Hamiltonian systems.

Theorem 4.29 (Pontryagin, 1934) *Let L_0 be a clockwise-oriented cycle of (4.35) for $\varepsilon = 0$ corresponding to a periodic solution $\varphi(t)$ with the (minimal) period T_0 and let $\Omega_0 \subset \mathbb{R}^2$ denote the domain inside the cycle L_0. If*

$$M_0 = \int_{\Omega_0} \operatorname{div} f(x)\, dx = 0,$$

while

$$M_1 = \int_0^{T_0} \operatorname{div} f(\varphi(t))\, dt \neq 0,$$

then

(i) *there exists an annulus around L_0 in which the system (4.35) has, for all ε with sufficiently small $|\varepsilon|$, a unique cycle L_ε, such that $L_\varepsilon \to L_0$ as $\varepsilon \to 0$;*

(ii) *this cycle L_ε is stable for $\varepsilon M_1 < 0$ and unstable for $\varepsilon M_1 > 0$.*

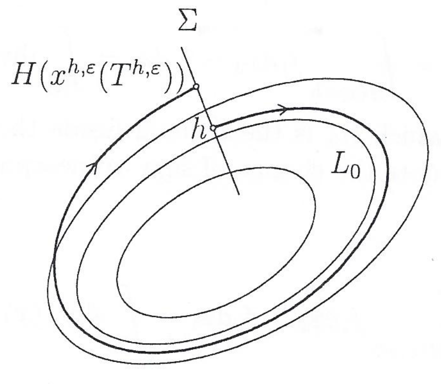


Figure 4.19: A segment Σ orthogonal to L_0, together with unperturbed and perturbed orbits.

Proof:

(i) Consider a cycle L_0 for which the assumptions of the theorem are valid. Without loss of generality, suppose that $H(x) = 0$ for $x \in L_0$. Fix a segment Σ orthogonal to L_0 and consider the value h of the Hamilton function H as a local coordinate along Σ (see Figure 4.19). Denote by $x^{h,\varepsilon} = x^{h,\varepsilon}(t)$ the solution of (4.35) starting at a point in Σ with a small coordinate h, and let $T^{h,\varepsilon}$ be the minimal time needed by the solution to return back to Σ. Due to the smooth dependence of solutions on the initial point and the parameter, $T^{h,\varepsilon}$ is a smooth function of (h, ε) in a neighbourhood of $(0,0)$ with $T^{0,0} = T_0$, since $x^{0,0} = \varphi$.

Introduce another smooth function

$$\Delta(h, \varepsilon) = H(x^{h,\varepsilon}(T^{h,\varepsilon})) - H(x^{h,\varepsilon}(0)) = H(x^{h,\varepsilon}(T^{h,\varepsilon})) - h,$$

which one calls for obvious reasons the *displacement function*. In general, $\Delta(h, \varepsilon) \neq 0$. However, $\Delta(h, 0) = 0$, since H is constant along orbits of the unperturbed system. Moreover, if $\Delta(h, \varepsilon) = 0$, the perturbed system (4.35) has a closed orbit for the corresponding value of ε passing through the point in Σ with the coordinate h. We will analyse the equation $\Delta(h, \varepsilon) = 0$ with the help of the Implicit Function Theorem.

Along the solutions of (4.35), one has

$$\begin{aligned} \Delta(h, \varepsilon) &= \int_0^{T^{h,\varepsilon}} dH(x^{h,\varepsilon}(t)) = \int_0^{T^{h,\varepsilon}} [H_{x_1}(x^{h,\varepsilon}(t))\dot{x}_1^{h,\varepsilon}(t) + H_{x_2}(x^{h,\varepsilon}(t))\dot{x}_2^{h,\varepsilon}(t)]dt \\ &= \varepsilon \int_0^{T^{h,\varepsilon}} [H_{x_1}(x^{h,\varepsilon}(t))f_1(x^{h,\varepsilon}(t)) + H_{x_2}(x^{h,\varepsilon}(t))f_2(x^{h,\varepsilon}(t))]\, dt \\ &= \varepsilon \int_0^{T^{h,0}} [-\dot{x}_2^{h,0}(t)f_1(x^{h,0}(t)) + \dot{x}_1^{h,0}(t)f_2(x^{h,0}(t))]\, dt + O(\varepsilon^2), \end{aligned}$$

where the last integral is computed along solutions of (4.35) with $\varepsilon = 0$. Since these solutions have clockwise orientation, we can write

$$\Delta(h, \varepsilon) = \varepsilon M(h) + O(\varepsilon^2), \tag{4.39}$$

where

$$M(h) = \oint_{H(x)=h} f_1 dx_2 - f_2 dx_1 = \int_{\Omega_h} \operatorname{div} f(x)\, dx.$$

(The last equality, in which Ω_h is the domain inside the level curve $H(x) = h$, is Green's formula (4.9); note the change of sign to incorporate the *clockwise* orientation of L_0.)

By assumption,

$$M(0) = \oint_{H(x)=0} f_1 dx_2 - f_2 dx_1 = \int_{\Omega_0} \operatorname{div} f(x)\, dx = M_0 = 0.$$

Thus we have

$$M(h) = \int_0^h \left(\int_0^{T^{s,0}} \operatorname{div} f(x(\tau, s)) |\det(J_1(\tau, s))|\, d\tau \right) ds,$$

where J_1 is the Jacobian matrix of the map $(\tau, h) \mapsto x(\tau, h) = x^{h,0}(\tau)$. Hence

$$M'(0) = \int_0^{T^{0,0}} \operatorname{div} f(\varphi(\tau))\ |\det(J_1(\tau, 0))|\ d\tau.$$

Differentiating the identity

$$H(x_{h,0}(\tau)) = h$$

with respect to h, we obtain

$$H_{x_1}(x^{h,0}(\tau)) \frac{\partial x_1^{h,0}(\tau)}{\partial h} + H_{x_2}(x^{h,0}(\tau)) \frac{\partial x_2^{h,0}(\tau)}{\partial h} = 1.$$

Therefore,

$$\begin{aligned} \det(J_1(\tau, h)) &= \det \begin{pmatrix} \dfrac{\partial x_1^{h,0}(\tau)}{\partial \tau} & \dfrac{\partial x_1^{h,0}(\tau)}{\partial h} \\ \dfrac{\partial x_2^{h,0}(\tau)}{\partial \tau} & \dfrac{\partial x_2^{h,0}(\tau)}{\partial h} \end{pmatrix} \\ &= \det \begin{pmatrix} H_{x_2}(x^{h,0}(\tau)) & \dfrac{\partial x_1^{h,0}(\tau)}{\partial h} \\ -H_{x_2}(x^{h,0}(\tau)) & \dfrac{\partial x_2^{h,0}(\tau)}{\partial h} \end{pmatrix} = 1\,. \end{aligned}$$

Hence $|\det(J_1(\tau, 0))| = 1$ and

$$M'(0) = \int_0^{T_0} \operatorname{div} f(\varphi(t))\, dt = M_1 \neq 0$$

by the second assumption.

Thus, we have

$$\Delta(h, \varepsilon) = \varepsilon(M_0 + hM_1 + O(h^2)) + O(\varepsilon^2) = \varepsilon F(h, \varepsilon)$$

for some smooth function F. If $\varepsilon = 0$, $\Delta = 0$ for all small $|h|$ and all orbits are closed (Hamiltonian case). If $\varepsilon \neq 0$, the equation

$$F(h,\varepsilon) = 0$$

is such that we can apply the Implicit Function Theorem. Indeed,

$$F(0,0) = M_0 = 0, \quad F_h(0,0) = M_1 \neq 0,$$

by the assumptions we made. Thus, there is a unique smooth function $h = h(\varepsilon)$, $h(0) = 0$, such that

$$F(h(\varepsilon),\varepsilon) = 0$$

for all small $|\varepsilon|$. This implies that $\Delta(h(\varepsilon),\varepsilon) = 0$ for all ε with sufficiently small $|\varepsilon| \neq 0$. Therefore, there exists a unique cycle L_ε through $h(\varepsilon)$.

(ii) Since the map $h \mapsto h + \Delta(h,\varepsilon)$ has derivative

$$1 + \frac{\partial \Delta(h(\varepsilon),\varepsilon)}{\partial h}$$

in the fixed point $h(\varepsilon)$ and the sign of

$$\frac{\partial \Delta(h(\varepsilon),\varepsilon)}{\partial h}$$

coincides with the sign of εM_1, the stability assertions follow at once. □

Remarks:

(1) If L_0 has counter-clockwise orientation, the expression (4.39) will take the form

$$\Delta(h,\varepsilon) = -\varepsilon M(h) + O(\varepsilon^2).$$

The subsequent analysis can then be carried out with obvious modifications. It shows that statement (i) is still valid, but the cycle L_ε is stable when $\varepsilon M_1 > 0$ and unstable when $\varepsilon M_1 < 0$.

(2) When L_0 is oriented counter-clockwise,

$$\oint_{L_0} f_1(x)dx_2 - f_2(x)dx_1 = \int_0^{T_0} f(\varphi(t)) \wedge \dot{\varphi}(t)\, dt,$$

where the *wedge product* of two vectors $u, v \in \mathbb{R}^2$ is defined by $u \wedge v = u_1 v_2 - u_2 v_1$. ◇

Example 4.30 (Van der Pol equation)

The second-order equation,

$$\ddot{x} + x = \varepsilon \dot{x}(1 - x^2), \tag{4.40}$$

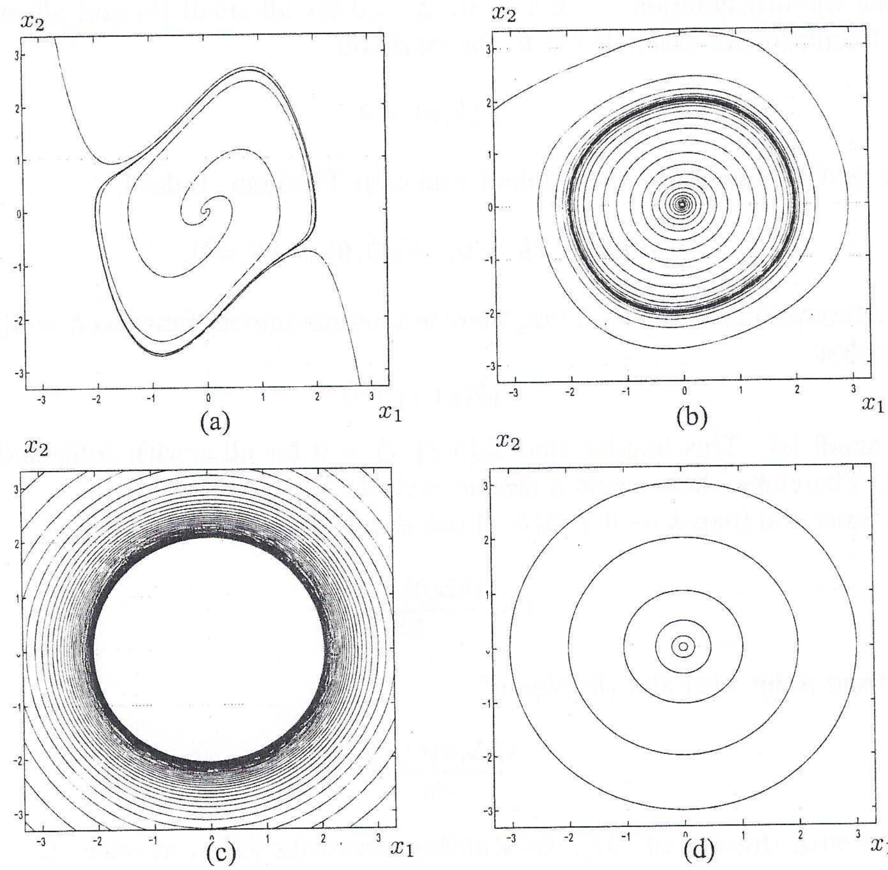


Figure 4.20: Phase portraits of Van der Pol equation: (a) $\varepsilon = 1$; (b) $\varepsilon = 0.1$; (c) $\varepsilon = 0.01$; (d) $\varepsilon = 0$.

is called the *van der Pol equation*[3]. It can be rewritten as the equivalent planar system

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & -x_1 + \varepsilon x_2(1 - x_1^2). \end{array} \right. \tag{4.41}$$

The system with $\varepsilon = 0$ is Hamiltonian (*harmonic oscillator*) with $H(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, which has a family of 2π-periodic solutions

$$\varphi(t) = \begin{pmatrix} r \sin t \\ r \cos t \end{pmatrix}, \quad r > 0.$$

Since $f_1 = 0,\ f_2 = x_2(1 - x_1^2)$, and the periodic orbits are oriented clockwise, we get

$$M(r) = -\oint_{x_1^2+x_2^2=r^2} f_1(x)dx_2 - f_2(x)dx_1 = \int_0^{2\pi} r^2 \cos^2 t(1 - r^2 \sin^2 t)\, dt = \frac{\pi}{4} r^2(4 - r^2).$$

[3]van der Pol, B. 'Forced oscillations in a circuit with nonlinear resistance (receptance with reactive triode)', *London, Edinburgh and Dublin Phil. Mag.* **3** (1927), 65-80

Therefore, $M(r) = 0$ for $r = 2$. Along this solution,

$$M_1 = \int_0^{2\pi} \operatorname{div} f(\varphi(t))\, dt = \int_0^{2\pi} (1 - 4\sin^2 t) dt = -2\pi < 0.$$

Thus, by Theorem 4.29, a unique and stable limit cycle bifurcates from the circle $r = 2$ for small $\varepsilon > 0$ (see Figure 4.20). One can prove that (4.41) has exactly one limit cycle for all $\varepsilon > 0$ (see Exercise 4.7.6). $\diamond$

4.6 References

The best references for the qualitative theory of autonomous planar ODEs and the theory of their bifurcations are still the classical books [Andronov, Leontovich, Gordon & Maier 1971, Andronov, Leontovich, Gordon & Maier 1973]. Pontryagin's method to locate limit cycles by perturbing planar Hamiltonian systems is also presented in [Andronov et al. 1973]. Further results on nonlinear planar ODEs, e.g., the index theory and blow-up techniques to study degenerate equilibrium points, can be found in [Arnol'd 1973, Arnol'd 1983, Arnol'd & Il'yashenko 1988, Perko 2001] and, in particular, in [Dumortier, Llibre & Artés 2006].

The theory of Hamiltonian systems is a classical and highly developed topic, see [Verhulst 1996] for a brief introduction and [Arnol'd 1989, Marsden & Ratiu 1999] for advanced presentations.

Phase portraits of various prey-predator models and their bifurcations are studied in [Bazykin 1998].

4.7 Exercises

E 4.7.1 (Bautin's example)

Prove that the planar quadratic system

$$\begin{cases} \dot{x} &=& y, \\ \dot{y} &=& ax + by + \alpha x^2 + \beta y^2, \end{cases}$$

has no periodic orbits, provided $b \neq 0$. *Hint*: Use Dulac's Criterion with factor $\mu(x, y) = e^{\gamma x}$, where γ is chosen suitably.

E 4.7.2 (Normal form for Andronov-Hopf bifurcation)

Consider the following planar system depending on parameter α:

$$\begin{cases} \dot{x} &=& \alpha x - y - x(x^2 + y^2) + R(x, y, \alpha), \\ \dot{y} &=& x + \alpha y - y(x^2 + y^2) + S(x, y, \alpha), \end{cases} \tag{4.42}$$

where R and S are smooth functions of (x, y, α), such that $R, S = \mathcal{O}(\rho^4)$ as $\rho \to 0$ with $\rho^2 = x^2 + y^2$. In Chapter 5, we will show that a generic planar system near the *supercritical Andronov-Hopf bifurcation* is locally smoothly orbitally equivalent to (4.42).

(a) Prove that there exists $\varepsilon > 0$ such that for all $\alpha \leq 0$ with sufficiently small $|\alpha|$ all positive half-orbits of (4.42) starting in the disk

$$D(\varepsilon) = \{(x, y) \in \mathbb{R} : x^2 + y^2 \leq \varepsilon^2\}$$

remain in $D(\varepsilon)$ and converge to $O = (0,0)$ as $t \to \infty$. *Hint*: Use polar coordinates.

(b) Prove that for all sufficiently small $\alpha > 0$ all positive half-orbits of (4.42) starting in the annulus

$$A(\alpha) = \left\{ (x,y) \in \mathbb{R}^2 : \frac{2\alpha}{3} \le x^2 + y^2 \le \frac{3\alpha}{2} \right\}$$

remain in $A(\alpha)$. Use the Poincaré-Bendixson Theorem to conclude that (4.42) has at least one periodic orbit in $A(\alpha)$ for all sufficiently small $\alpha > 0$.

(c) Prove that for all sufficiently small $\alpha > 0$ holds: $(\text{div } F)(x, y, \alpha) < 0$ for all $(x, y) \in A(\alpha)$, where F is the vector field defining (4.42). Conclude using Bendixson's Criterion that (4.42) has exactly one periodic orbit in $A(\alpha)$ for all sufficiently small $\alpha > 0$.

(d) Prove that there exists $\varepsilon > 0$ such that for all sufficiently small $\alpha > 0$ simultaneously hold:

- $A(\alpha) \subset D(\varepsilon)$;
- all positive half-orbits of (4.42) starting in $D(\varepsilon)$ remain in $D(\varepsilon)$;
- all positive orbits starting in $D(\varepsilon) \setminus (A(\alpha) \cup O)$ enter $A(\alpha)$.

(e) Combine the results obtained to sketch and describe all possible phase portraits of (4.42) in a fixed small neighbourhood of the origin for all α with sufficiently small $|\alpha|$.

E 4.7.3 (Chemostat dynamics)

A *chemostat* is a laboratory device for growing microorganisms continuously (see Figure 4.21). Let

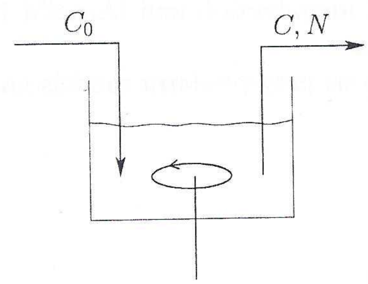


Figure 4.21: A chemostat.

C_0 denote the concentration of a limiting resource (*substrate*) in the inflow to the chemostat, while C and $N = (N_1, N_2, \ldots)$ denote the concentrations of different types of microorganisms in the chemostat (and in the outflow).

(a) If only one microorganism is present, then, according to some model and after a scaling, the dynamics of (C, N) are described by

$$\begin{cases} \dot{C} &= C_0 - C - K(C)N, \\ \dot{N} &= -N + \alpha K(C)N, \end{cases}$$

where K is a positive and increasing smooth function with $K(0) = 0$ and $K(C) \to 1$ as $C \to \infty$, e.g.,

$$K(C) = \frac{C}{1+C}.$$

Assume that $\alpha > 1$ (otherwise the microorganisms are washed out, no matter how large is C_0). Prove that

(i) $C(t) + \frac{1}{\alpha}N(t) \to C_0$ for $t \to \infty$;

(ii) If $\alpha K(C_0) < 1$, then the equilibrium $(C_0, 0)$ is globally asymptotically stable;

(iii) If $\alpha K(C_0) > 1$, then the equilibrium $(\overline{C}, \overline{N})$ is asymptotically stable and all orbits with $N(0) > 0$ tend to this equilibrium as $t \to \infty$. Here $\overline{C}$ is characterized by $\alpha K(\overline{C}) = 1$ and

$$\overline{N} = \frac{C_0 - \overline{C}}{K(\overline{C})} = \alpha(C_0 - \overline{C}).$$

(b) If we consider the competition of two different microorganisms for the same substrate, we have to analyse the 3-dimensional system

$$\begin{cases} \dot{C} &= C_0 - C - K_1(C)N_1 - K_2(C)N_2, \\ \dot{N}_1 &= -N_1 + \alpha_1 K_1(C)N_1, \\ \dot{N}_2 &= -N_2 + \alpha_2 K_2(C)N_2 \end{cases}$$

where $\alpha_i > 1$, and K_i have the same properties as K in part (a).
(i) Show that $C(t) + \frac{1}{\alpha_1}N_1(t) + \frac{1}{\alpha_2}N_2(t) \to C_0$ as $t \to \infty$;
(ii) Motivated by (i), consider the planar system

$$\begin{cases} \dot{N}_1 &= -N_1 + \alpha_1 K_1\left(C_0 - \frac{1}{\alpha_1}N_1 - \frac{1}{\alpha_2}N_2\right)N_1, \\ \dot{N}_2 &= -N_2 + \alpha_2 K_2\left(C_0 - \frac{1}{\alpha_1}N_1 - \frac{1}{\alpha_2}N_2\right)N_2. \end{cases} \tag{4.43}$$

We assume that $\alpha_i K_i(C_0) > 1$ for $i = 1, 2$ and define $\overline{C}_i$ by $\alpha_i K_i(\overline{C}_i) = 1$.
Prove that $(\alpha_1(C_0 - \overline{C}_1), 0)$ attracts all orbits with $N_i(0) > 0$ if $\overline{C}_1 < \overline{C}_2$, while $(0, \alpha_2(C_0 - \overline{C}_2))$ attracts all such orbits if $\overline{C}_1 > \overline{C}_2$. (*Hint*: What can you say about the slope of the isoclines?)

E 4.7.4 (Infectious diseases)

(a) The system

$$\begin{cases} \dot{S} &= b - \mu S - \beta SI, \\ \dot{I} &= -\mu I + \beta SI - \alpha I, . \end{cases} \tag{4.44}$$

arises in the context of modelling the spread of an infectious disease. Here S stands for "susceptibles" and I for "infecteds", which are also infectious; b is the population birth rate and all newborns are susceptible; μ is the per capita death rate and α is the per capita rate at which infecteds lose their infectiousness.
(i) Show that the equilibrium

$$\left(\frac{b}{\mu}, 0\right)$$

is globally asymptotically stable when

$$R_0 = \frac{\beta b}{\mu(\mu + \alpha)} < 1.$$

(ii) Show that the equilibrium

$$(\overline{S}, \overline{I}) = \left(\frac{\mu + \alpha}{\beta}, \frac{b}{\mu + \alpha} - \frac{\mu}{\beta}\right)$$

attracts all orbits with $S(0), I(0) > 0$ when $R_0 > 1$. (*Hint*: Try to find a Lyapunov function in an already familiar form.)

(b) The system

$$\begin{cases} \dot{S} &= bS + bR - \mu S - \gamma\frac{SI}{N}, \\ \dot{I} &= -\mu I + \gamma\frac{SI}{N} - \alpha I, \\ \dot{R} &= -\mu R + f\alpha I, \end{cases}$$

where $N = S+I+R$, describes the spread of an infectious disease in a population that grows with rate $(b-\mu)$ in the absence of the disease. The variables S and I have the same meaning as in part (a), while R denotes the "removed" individuals, which are immune for the rest of their life. After an exponentially distributed (with parameter α) infectious period, individuals die with probability $1-f$ and become "removed" with probability f. Infected individuals have zero per capita birth rate. All parameters are positive and we assume that $b > \mu$. In the transmission term $\gamma SI/N$, the N in the denominator makes sure that the contact rate of an individual is independent of the population size (think, for example, of a sexually transmitted disease). In part (a) we did not need to worry about this point, since the population size did stabilize anyhow at b/μ.
(i) Verify that the system is *first-order homogeneous*, i.e., $F(\lambda x) = \lambda F(x)$, where $x = (S, I, R)^T$ and F is the vector field at the right-hand side. As a consequence, exponential solutions are feasible even though the system is nonlinear.
(ii) Rewrite the system in terms of N, y and z, where $y = I/N$, $z = R/N$. (*Hint*: You should obtain a partially decoupled system: The equations for y and z do not depend on N.)
(iii) Study phase portraits of the (y,z)-subsystem.

Hints: First prove that the triangle $\{(y,z) : y \geq 0, z \geq 0, 1-y-z \geq 0\}$ is forward-invariant. Then study the cases $\gamma \leq \alpha + b$ and $\gamma > \alpha + b$ separatly. In the former case, prove that $y(t) \to 0$ (and, consequently, $z(t) \to 0$) as $t \to \infty$ for all $y(0), z(0) \geq 0$. In the latter case, use Dulac's Criterion with factor $(yz)^{-1}$ to exclude cycles.
(iv) Use the information obtained in (iii) to derive the (asymptotic) growth rate of N. Could the infectious agent slow down, or even stop, the host population growth?

E 4.7.5 (Slow-fast oscillations)

Periodic solutions of planar *slow-fast systems*

$$\begin{cases} \dot{\xi} &= P(\xi, \eta), \\ \varepsilon\dot{\eta} &= Q(\xi, \eta), \end{cases}$$

where $\varepsilon \ll 1$, can be constructed by concatenation of slow motions near the isocline $Q(\xi, \eta) = 0$ with fast "jumps" towards this curve.

(a) Consider the following variant of the *van der Pol oscillator* (cf. Example 4.30):

$$\begin{cases} \dot{x} &= -y, \\ \varepsilon\dot{y} &= x + y - y^3, \end{cases} \tag{4.45}$$

where $\varepsilon > 0$ is a parameter. Analyse the dynamics of this system for $\varepsilon \ll 1$ (*Hint*: See Figure 4.22. *Warning*: We realize that "Analyse" is a somehow ambiguous assignment. What we meant is "Make plausible that the phase portraits are as shown in the figure", while leaving it to the reader's own consciousness whether or not to provide rigorous proofs. These proofs require a serious effort.)

(b) Consider the planar system

$$\begin{cases} \varepsilon\dot{x} &= x\left(1 - \dfrac{x}{K} - \dfrac{y}{1+x}\right), \\ \dot{y} &= y\left(-1 + \theta\dfrac{x}{1+x}\right). \end{cases} \tag{4.46}$$

(i) Show that (4.46) can be obtained from the Rosenzweig-MacArthur model (4.20) by the scaling

$$\tau = ct, \quad x = \beta b v, \quad y = \frac{b}{a}p,$$

and introduction of the new parameters:

$$\varepsilon = \frac{c}{a}, \quad \theta = \frac{d}{\beta bc}, \quad K = \frac{\beta ab}{e}.$$

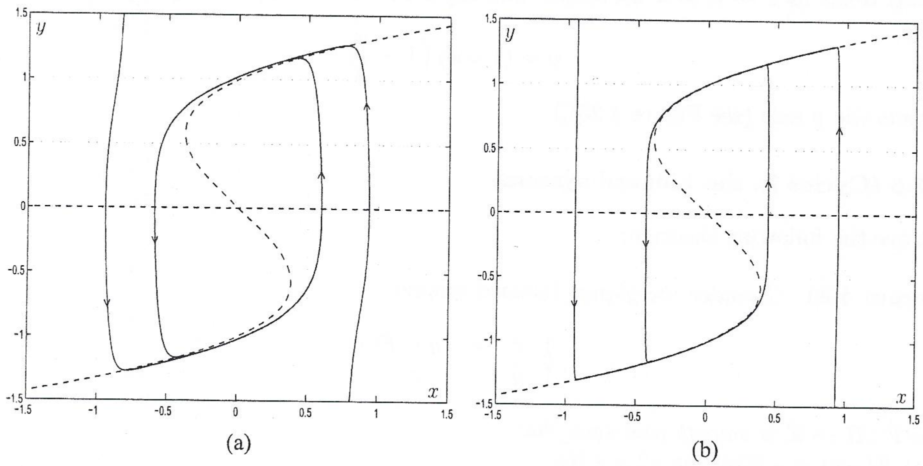


Figure 4.22: Phase portraits of (4.45): (a) $\varepsilon = 0.1$; (b) $\varepsilon = 0.01$.

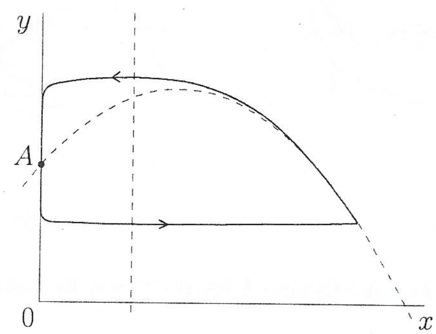


Figure 4.23: A slow-fast limit cycle in (4.46).

(ii) Assume that

$$\theta > 1, \quad K > 1 + \frac{2}{\theta - 1}.$$

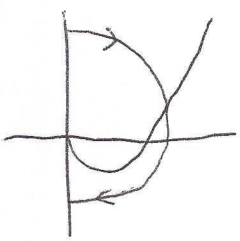
Interpret this assumption in biological terms. Sketch the phase portrait of (4.46) for $\varepsilon \ll 1$.

(*Warning*: A careful and nontrivial analysis shows that the low part of the slow-fast cycle in (4.46) tends as $\varepsilon \to 0$ to a horizontal line segment located *below* the point A, where the prey isocline

$$y = (1+x)\left(1 - \frac{x}{K}\right)$$

intersects the y-axis (see Figure 4.23).)

Proof that took forever in class w/ this drawing!



E 4.7.6 (Cycles in the Liénard system)

(a) Prove the following theorem:

Theorem 4.31 *Consider the planar* Lienard system

$$\begin{cases} \dot{x} &= y - F(x), \\ \dot{y} &= -x, \end{cases} \tag{4.47}$$

where $F : \mathbb{R} \to \mathbb{R}$ is smooth and such that

(i) $F(-x) = -F(x)$ for all $x \in \mathbb{R}$;

(ii) $F(x) \to +\infty$ as $x \to +\infty$ and there exists a constant $\beta > 0$ such that $F(x) > 0$ and $F'(x) > 0$ for $x > \beta$;

(iii) $F'(0) < 0$ so that there exists a constant $\alpha > 0$ such that $F(x) < 0$ for all $0 < x < \alpha$.

Then the system (4.47) has at least one periodic orbit. If moreover $\alpha = \beta$, the periodic orbit is unique and all orbits of (4.47) with $(x(0), y(0)) \neq (0,0)$ tend to this periodic orbit as $t \to +\infty$.

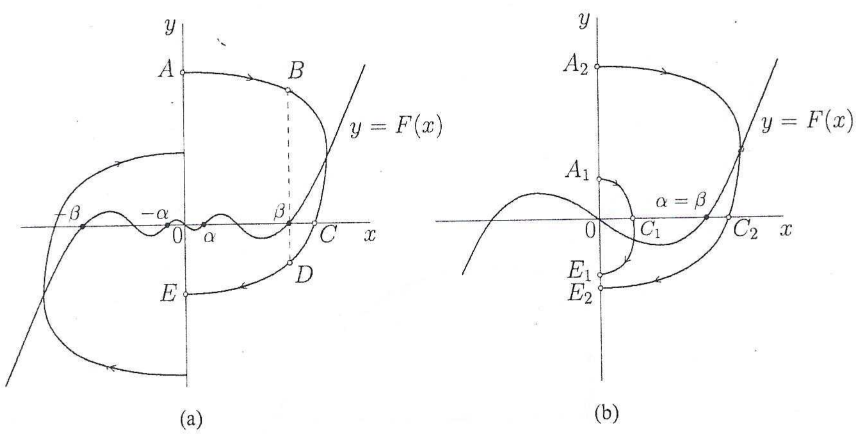


Figure 4.24: Analysis of limit cycles in Liénard system.

Hints:

(1) Prove that the origin is an unstable node or focus;

(2) Introduce the function $R(x,y) = \frac{1}{2}(x^2 + y^2)$. Show that $\dot{R} = -xF(x)$.

(3) Consider an orbit of (4.47) starting at a point $A = (0, y_0)$ with sufficiently large $y_0 > 0$ and show that it comes back to the y-axis at a point $E = (0, y_1)$ with $y_1 < 0$ and $|y_1| < y_0$ (see Figure 4.24(a)). For this, introduce

$$I(y_0) = R(0, y_1) - R(0, y_0)$$

and write

$$I(y_0) = \int_{ABCDE} dR = \left(\int_{AB} + \int_{DE}\right) \frac{xF(x)dx}{F(x)-y} + \int_{BCD} F(x)dy$$

and notice that both integrals are negative, implying $I(y_0) < 0$, when $y_0 \to +\infty$.

(4) Then use the symmetry of the system with respect to $(x, y) \mapsto (-x, -y)$ to construct a forward-invariant set composed of the orbit arc connecting A and E, its reflection, and two segments of the y-axis: $[-y_1, y_0]$ and $[-y_0, y_1]$. Apply the Poincaré-Bendixson Theorem.

(5) When $\alpha = \beta$, prove first that for any orbit crossing the interval $0 < x < \alpha$ we have $|y_1| > y_0$ (see Figure 4.24(b)). Use the fact that $F(x) < 0$ and , therefore,

$$I(y_0) = \int_{BCD} F(x)dy > 0.$$

(6) Use (3) and the monotonicity of F for $x \geq \beta$ to prove that $I(y_0)$ has only one zero. This implies the uniqueness of the limit cycle and its stability.

(b) Using Theorem 4.31, prove that the cycle in the van der Pol equation (4.40) from Example 4.30 is unique. (*Hint*: To write (4.40) in the form (4.47), introduce a new variable $y = \dot{x} + F(x)$, where

$$F(x) = \int_0^x f(s)\, ds$$

and $f(s) = \varepsilon(s^2 - 1)$. Then the conditions of Theorem 4.31 are satisfied with $\alpha = \beta = \sqrt{3}$.)

E 4.7.7 (A nontrivial ω-limit set)

Consider the system[4]

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} H_y(x,y) \\ -H_x(x,y) \end{pmatrix} + \mu H(x,y) \begin{pmatrix} H_x(x,y) \\ H_y(x,y) \end{pmatrix}, \tag{4.48}$$

where μ is a parameter,

$$H(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$$

is a scalar function, and

$$\begin{pmatrix} H_x(x,y) \\ H_y(x,y) \end{pmatrix} = \begin{pmatrix} -x + x^3 \\ y \end{pmatrix}$$

is its gradient.

(a) Plot the phase portrait of (4.48) at $\mu = 0$ with special attention to equilibria and critical level-sets. (*Answer*: Figure 4.25(a)).

(b) Prove that for $\mu < 0$ all nonequilibrium orbits of (4.48) tend to the level-set $H(x, y) = 0$. (*Hint*: $\dot{H} = \mu\,(H_x^2 + H_y^2)H$.)

(c) Using (b), prove that the ω-limit set of all points with $H(x, y) > 0$ consists of the saddle point $(x, y) = (0, 0)$ and two orbits homoclinic to this saddle (see Figure 4.25(b)).

E 4.7.8 (A planar Hamiltonian system)

Consider the system

$$\begin{cases} \dot{x} &= -\mu y + x^2 - y^2, \\ \dot{y} &= \mu x - 2xy, \end{cases} \tag{4.49}$$

appearing in the study of bifurcations of a periodic orbit near 1:3 resonance in Hamiltonian systems with two degrees of freedom [Arnol'd 1989].

[4]Batalova, Z.S. and Neimark, Yu.I. 'A certain dynamical system with homoclinic structure', In: *Theory of Oscillations, Applied Mathematics and Cybernetics*, v.1, Gorkii State University, 1973, pp. 131–147. In Russian.

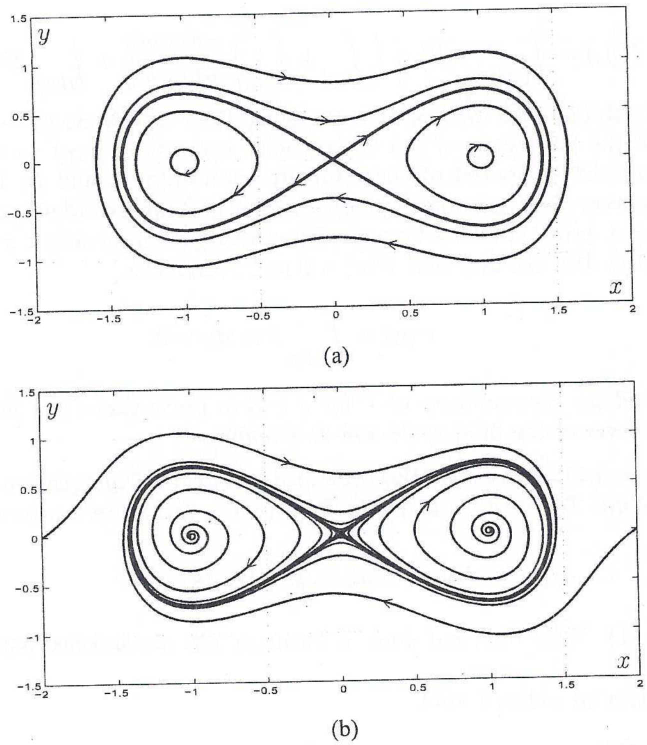


Figure 4.25: Phase orbits of (4.48): (a) $\mu = 0$; (b) $\mu = -0.5$.

(a) Prove that this system is Hamiltonian for all values of the parameter μ and find its Hamiltonian function H.

(b) Draw the phase portrait of (4.49) for $\mu < 0$, $\mu = 0$, and $\mu > 0$. (*Hint*: The level set $H(x, y) = h$ passing through nontrivial equilibria of (4.49) when $\mu \neq 0$ is the union of three straight lines.)

E 4.7.9 (Rock-scissors-paper dynamics)

Consider a planar system

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & (x_1 + \frac{1}{3})\,[x_1 + 2x_2 - \varepsilon(x_2 + \Psi(x)), \\ \dot{x}_2 & = & -\,(x_1 + \frac{1}{3})\,[2x_1 + x_2 - \varepsilon(x_1 + x_2 - \Psi(x)), \end{array} \right. \tag{4.50}$$

where $\Psi(x) = x_1^2 + x_1x_2 + x_2^2$.

(a) Prove that (4.50) is $\mathbb{Z}_3$-symmetric, i.e. it does not change under the transformation $x \mapsto Tx$, where

$$T = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},\quad T^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\quad T^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) Find all equilibria of (4.50).

(c) Show that (4.50) is Hamiltonian for $\varepsilon = 0$, find its Hamiltonian function $H = H(x)$ and verify that $H(Tx) = H(x)$. Sketch the phase portrait of the system for $\varepsilon = 0$. (*Hint*: Use the trick of Exercise 4.7.8.)

(d) Prove that $\dot{H} = -3\varepsilon\Phi H$ along orbits of (4.50) and conclude that it has no periodic orbits. Sketch the phase portrait of the system for $\varepsilon \neq 0$.

E 4.7.10 (Effective Kepler potential)

Draw the phase portrait of a potential system with one degree of freedom with

$$U(q) = -\frac{k}{q} + \frac{M}{q^2}.$$

Classify all orbits.

E 4.7.11 (Planar potential systems)

(a) Show that

$$T(h) = \frac{dS(h)}{dh},$$

where $S(h)$ is the area of the domain bounded by a noncritical closed level curve

$$\frac{p^2}{2} + U(q) = h$$

and where $T(h)$ is the corresponding period.
(b) Using (a), prove formula (4.34) for the period $T(h)$ of oscillations of a potential mechanical system.
(c) Find the period of small oscillations near a quadratic minimum of $U(q)$ at q_0 .
(*Answer*: $T_0 = 2\pi/\sqrt{U''(q_0)}$.)
(d) Find linear approximations to the orbits tending to a simple saddle $(q_0, 0)$ as $t \to \pm\infty$.
(*Answer*: $p = \pm\sqrt{U''(q_0)}(q - q_0)$.)

E 4.7.12 (Conservative systems)

Definition 4.32 *A smooth system*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2m}, \tag{4.51}$$

is called ***conservative*** *if it is orbitally equivalent to a Hamiltonian system (see Appendix to this chapter):*

$$\dot{x} = J_{2m} H_x(x), \quad x \in \mathbb{R}^{2m},$$

i.e., there is a positive smooth function $\rho = \rho(x)$ and a smooth function $H = H(x)$, such that $\rho(x)f(x) = J_{2m}H_x(x)$ for all $x \in \mathbb{R}^{2m}$.

Prove the following result.

Theorem 4.33 *If* (4.51) *is a conservative system and φ^t is the corresponding flow, then*

$$W(t) = \int_{\varphi^t(D_0)} \rho(x)\, dx = \text{const}$$

for any domain $D_0 \subset \mathbb{R}^{2m}$, for which $W(0)$ exists.

(*Hint*: Mimic the proof of Lemma 4.36 in the Appendix and take into account that $\operatorname{div} f + \langle \rho_x, f \rangle = \operatorname{div}(\rho f)$.)

E 4.7.13 (Conservative ecosystem)

(a) Prove that the Lotka-Volterra system (4.31) is conservative in any open region of $\mathbb{R}^2_+$ and find the corresponding Hamiltonian function.

(*Answer*: $\rho(\xi,\eta) = (\xi\eta)^{-1}$ and $H(\xi,\eta) = \gamma \ln \xi - \xi + \ln \eta - \eta$.)

(b) Consider the following modification of Lotka-Volterra system (4.31) with $\gamma = 1$:

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & \xi - \dfrac{\xi\eta}{(1+\alpha\xi)(1+\beta\eta)} \equiv f_1(\xi,\eta), \\ \dot{\eta} & = & -\eta + \dfrac{\xi\eta}{(1+\alpha\xi)(1+\beta\eta)} \equiv f_2(\xi,\eta), \end{array} \right. \tag{4.52}$$

where $\alpha, \beta > 0$ are parameters[5].

(i) Prove that (4.52) is conservative (see Exercise 4.7.12) in $\mathbb{R}^2_+$ if $\alpha = \beta$, and has no closed orbits if $\alpha \neq \beta$. (*Hint*: Look for ρ in the form

$$\rho(\xi,\eta) = \xi^a \eta^b (1+\alpha\xi)(1+\beta\eta),$$

where a and b are unknown coefficients. By a proper selection of a and b, one can achieve that $\operatorname{div}(\rho f) = (\alpha - \beta)\xi\rho$.)

(ii) Sketch all topologically different phase portraits of (4.52) that exist for different (α, β).

(ii) Can you prove that (4.52) has a family of closed orbits for $\alpha = \beta$, provided that

$$0 < \alpha = \beta < \frac{1}{4},$$

using the symmetry of the system? (*Hint*: The transformation $(\xi, \eta, t) \mapsto (\eta, \xi, -t)$ does not change (4.52), i.e. the system is *reversible*.).

E 4.7.14 (Another reversible system)

Prove that the phase portrait of the planar system

$$\left\{ \begin{array}{rcl} \dot{x} & = & y, \\ \dot{y} & = & x + xy - x^3, \end{array} \right.$$

is as presented in Figure 4.26.

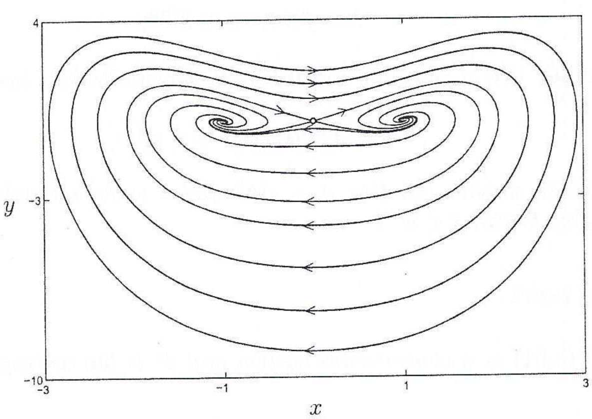


Figure 4.26: Phase portrait of a reversible system.

(*Hint*: Use the symmetry of the system under a reflection and time reversal: $(x, t) \mapsto (-x, -t)$ or prove that it is conservative in (a subset of) $\mathbb{R}^2$.)

[5] Bazykin, A.D., Berezovskaya, F.S., Denisov, G.A., and Kuznetsov, Yu.A. 'The influence of predator saturation effect and competition among predators on predator-prey system dynamics', *Ecol. Modelling* **14** (1981), 39-57.

E 4.7.15 (Properties of Poisson brackets)

(a) Prove properties (P.1)–(P.5) of the Poisson bracket introduced in the Appendix (see Definition 4.38).

(b) Compute $\{q_i, q_j\}$, $\{q_i, p_j\}$, and $\{p_i, p_j\}$.

E 4.7.16 (An integrable Hamiltonian system with two degrees of freedom)

Read Appendix to this chapter and consider the system

$$\dot{x} = J_4 H_x, \quad x \in \mathbb{R}^4, \tag{4.53}$$

with

$$H(q,p) = q_1^2 q_2 + 2q_1 p_1 p_2 - p_1^2 q_2.$$

(a) Prove that not all level sets of H are compact, i.e., that some of them are unbounded. (*Hint*: Take $q_2 = 1, p_2 = 0$ and draw the level sets of H in the resulting 2-dimensional (q_1, p_1)-space. You will see (unbounded) hyperbolas.)

(b) Find a quadratic constant of motion F of (4.53) independent of H. (*Hint*: Try

$$F(q,p) = ap_1^2 + bq_1^2 + cp_2^2 + q_2^2$$

with unknown coefficients a, b, c. The condition $\{F, H\} = 0$ is equivalent to a linear system with solution $2a = 2b = c = 1$.)

(c) Prove that, although the level sets of H are not bounded, the orbits of (4.53) are bounded. (*Hint*: F is definite, i.e., its level sets are compact.)

(d) Show that most solutions of (4.53) lie on invariant tori (*Hint*: The system is integrable, with only a few degeneracies. *Warning*: Computing on which levels these degeneracies occur is less trivial.)

4.8 Appendix: General properties of Hamiltonian systems

Consider now a smooth function $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $H = H(q,p)$, called the *Hamiltonian* or *energy function*. Let

$$H_p(q,p) = \left(\frac{\partial H(q,p)}{\partial p_1}, \ldots, \frac{\partial H(q,p)}{\partial p_m}\right)^T, \quad H_q(q,p) = \left(\frac{\partial H(q,p)}{\partial q_1}, \ldots, \frac{\partial H(q,p)}{\partial q_m}\right)^T.$$

Definition 4.34 *A system*

$$\begin{cases} \dot{q} &= H_p(q,p), \\ \dot{p} &= -H_q(q,p), \end{cases} \tag{4.54}$$

is called a ***Hamiltonian system*** *with m degrees of freedom.*

The Hamiltonian function is defined modulo an additive constant. The system (4.54) can be written as

$$\dot{x} = J_{2m} H_x(x), \tag{4.55}$$

where $x = (q,p)^T \in \mathbb{R}^{2m}$, and

$$J_{2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

Similar to Definition 4.25, (4.54) with

$$H(q,p) = \frac{1}{2}\langle p, p\rangle + U(q),$$

where $q, p \in \mathbb{R}^m$ and U is a smooth scalar function, is called a potential system with m degrees of freedom. Here the term $\frac{1}{2}\langle p, p\rangle$ is called the *kinetic energy*, while $U(q)$ is called the *potential energy* of the system.

The corresponding $2m$-dimensional Hamiltonian system has the form:

$$\begin{cases} \dot{q} &= p, \\ \dot{p} &= -U_q(q). \end{cases} \tag{4.56}$$

Although details of the dynamics generated by (4.54) or (4.56) with $m > 1$ are not known, some general results have been established. The most basic and elementary of these are presented below.

Theorem 4.22, saying that the Hamiltonian H is a constant of the motion, is valid for all $m \geq 1$, since

$$\dot{H} = \langle H_q, \dot{q}\rangle + \langle H_p, \dot{p}\rangle = \langle H_q, H_p\rangle - \langle H_p, H_q\rangle = 0.$$

Therefore, the phase space of a Hamiltonian system is foliated by $(2m-1)$-dimensional (codim 1) invariant level sets $H(q,p) = h$ (*energy levels*). As in the planar case ($m = 1$), this immediately excludes asymptotically stable equilibria and cycles in multidimensional Hamiltonian systems.

Theorem 4.35 (Liouville) *The flow generated by a Hamiltonian system* (4.54) *is volume preserving.*

The proof of this theorem is based on the following lemma valid for general autonomous ODEs.

Lemma 4.36 *Let φ^t be the flow generated by a smooth system*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{4.57}$$

and let $V(t)$ denote the volume of the t-shift $D_t = \varphi^t(D_0)$ of a measurable domain $D_0 \subset \mathbb{R}^n$. Then

$$\left.\frac{dV(t)}{dt}\right|_{t=0} = \int_{D_0} \mathrm{div} f(x)\, dx,$$

where the divergence of the vector field $f(x)$ is defined by

$$\mathrm{div} f(x) = \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i}.$$

Proof: For any diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $y = \Phi(x)$, we have

$$\int_{\Phi(D_0)} dy = \int_{D_0} \left|\det\left(\frac{\partial \Phi}{\partial x}\right)\right| dx.$$

Thus, for $\Phi = \varphi^t$,

$$V(t) = \int_{\varphi^t(D_0)} dy = \int_{D_0} \det\left(\frac{\partial \varphi^t}{\partial x}\right) dx.$$

From the definition of solutions to (4.57), we have

$$\varphi^t(x) = x + tf(x) + o(t).$$

Since

$$\frac{\partial \varphi^t}{\partial x} = I_n + tf_x + o(t),$$

we get using the definition of the determinant

$$\begin{aligned} \det\left(\frac{\partial \varphi^t}{\partial x}\right) &= \det(I_n + tf_x) + o(t) = \prod_{i=1}^{n}(1 + t\lambda_i) + o(t) \\ &= 1 + t\sum_{i=1}^{n}\lambda_i + o(t) = 1 + t\,\mathrm{Tr}(f_x) + o(t) \\ &= 1 + t\,\mathrm{div} f + o(t), \end{aligned}$$

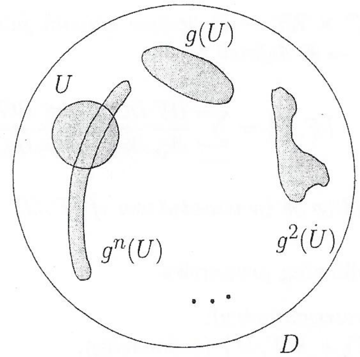


Figure 4.27: The recurrence phenomenon

where λ_i stand for the eigenvalues of the Jacobian matrix. Therefore,

$$V(t) = \int_{D_0} (1 + t \operatorname{div} f + o(t))\, dx = V(0) + t \int_{D_0} \operatorname{div} f\, dx + o(t)$$

and

$$\left.\frac{dV}{dt}\right|_{t=0} = \lim_{t\to 0} \frac{V(t) - V(0)}{t} = \int_{D_0} \operatorname{div} f\, dx. \quad \square$$

Proof of Theorem 4.35: For a Hamiltonian vector field

$$f(x) = J_{2m} H_x(x)$$

and so one has

$$\operatorname{div} f = \sum_{k=1}^{m} \frac{\partial^2 H}{\partial q_k \partial p_k} - \frac{\partial^2 H}{\partial p_k \partial q_k} = 0.$$

Thus $\dot V(0) = 0$ and, since for an autonomous system we can put the origin of the time axis wherever we want, $\dot V(t) = 0$ for any t, implying $V(t) = \text{const}$. $\square$

Suppose now that the invariant domain

$$D = \{(q,p) \in \mathbb{R}^{2m} : H(q,p) \le h\}$$

is bounded. Consider the orbits of (4.54) in this domain. The Liouville Theorem 4.35 implies that after some time almost any such orbit will return to an arbitrary small neighbourhood of its starting point. This is a consequence of the following topological theorem, applied to the unit time-shift $g = \varphi^1$ along the orbits of (4.54).

Theorem 4.37 (Poincaré Recurrence) *Let $D \in \mathbb{R}^n$ be a bounded domain and let $g : D \to D$ be an invertible, continuous, and volume-preserving mapping. Then in any neighbourhood U of any point in D there is a point x which returns to U after repeated application of the map, i.e. $g^j(x) \in U$ for some integer j.*

Proof: Take a neighbourhood U of an arbitrary point in D and consider its images:

$$U,\ g(U),\ g^2(U), \dots, g^j(U), \dots$$

(see Figure 4.27). All these sets have the same positive volume. Since the volume of D is finite, these images cannot be all disjunct. Therefore, for some $k \le 0,\ l \le 0,\ k > l$,

$$g^k(U) \cap g^l(U) \ne \emptyset.$$

Thus, $g^{k-l}(U) \cap U \ne \emptyset$. Therefore, we can find a point $x \in U$ such that $g^j(x) \in U$ with $j = k - l$. $\square$.

Definition 4.38 *Let $F, G : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be two smooth functions. The* ***Poisson bracket*** *is the function $\{F, G\} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ defined by*

$$\{F, G\} = \sum_{i=1}^{m} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}.$$

The functions F and G are said to be in ***involution*** *if $\{F, G\} = 0$.*

The Poisson bracket has the following properties:

(P.1) $\{F, G\} = -\{G, F\}$ (*antisymmetry*);
(P.2) $\{F, G + \lambda H\} = \{F, G\} + \lambda\{F, H\}$ (*bilinearity*);
(P.3) $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$ (*Jacobi's identity*);
(P.4) $\{FG, H\} = F\{G, H\} + G\{F, H\}$ (*Leibnitz' rule*),

where F, G, H are smooth functions and $\lambda \in \mathbb{R}$. The first three properties mean that the set of all smooth functions on $\mathbb{R}^{2m}$ with the usual addition and multiplication by constants and the above defined Poisson bracket is a *Lie algebra*. Using notations introduced in (4.55), we can write

$$\{F, G\} = \langle F_x, J_{2m} G_x \rangle.$$

Theorem 4.39 (Poisson) *A smooth function F is a constant of the motion of* (4.55) *if and only if it is in involution with the Hamiltonian H : $\{F, H\} = 0$.*

Proof: Along solutions of (4.55),

$$\dot{F} = \langle F_x, \dot{x} \rangle = \langle F_x, J_{2m} H_x \rangle = \{F, H\}. \ \square$$

Lemma 4.40 *The Poisson bracket $\{F, G\}$ of two constants of the motion F and G of* (4.55) *is also a constant of motion for* (4.55).

Proof: Indeed, using the properties of $\{\cdot, \cdot\}$ and Lemma 4.39, we get

$$\{\{F, G\}, H\} = \{F, \{G, H\}\} + \{G, \{H, F\}\} = 0 + 0 = 0. \ \square$$

Thus, all constants of motion of a Hamiltonian system also form a Lie algebra that is a *subalgebra* of the Lie algebra of all smooth functions.

There is a special class of Hamiltonian systems, called *integrable*, which have m functionally independent constants of the motion in involution (one of them, of course, is the Hamiltonian H). Solutions of such systems can be found analytically by integration. The dynamics of an integrable system is relatively simple: Its phase space is almost completely foliated by m-dimensional invariant tori on which all orbits are either periodic or quasi-periodic. This follows from a fundamental theorem, which we formulate without proof.

Theorem 4.41 (Liouville-Arnold) *Consider m smooth functions*

$$F^{(1)}, F^{(2)}, \ldots, F^{(m)} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

and introduce the following set:

$$M_f = \{x \in \mathbb{R}^m \times \mathbb{R}^m : F^{(i)}(x) = f_i, \ i = 1, 2, \ldots, m\},$$

where $f = (f_1, f_2, \ldots, f_m)^T \in \mathbb{R}^m$. Suppose that

(L.1) *The functions $F^{(i)}$ are in involution: $\{F^{(i)}, F^{(j)}\} = 0, \ i, j = 1, 2, \ldots, m$.*
(L.2) *The gradients $F_x^{(i)}(x), \ i = 1, 2, \ldots, m$, are linearly independent in any point $x \in M_f$.*

Then

(i) M_f *is a smooth* m*-dimensional invariant manifold for the Hamiltonian system* $\dot{x} = J_{2m}H_x$ *with* $H = F^{(1)}$.

(ii) *If* M_f *is bounded and connected, then it is diffeomorphic to the* m*-dimensional torus*

$$\mathbb{T}^m = \{(\varphi_1, \ldots, \varphi_m) : \varphi_i \bmod 2\pi\}.$$

(iii) *The motion on this torus is governed by the equation*

$$\dot{\varphi} = \omega_f, \ \varphi \in \mathbb{T}^m$$

for some $\omega_f \in \mathbb{R}^m$.

(iv) *There is a neighbourhood of* M_f *in which one can introduce new coordinates* $(I, \varphi) \in \mathbb{R}^m \times \mathbb{T}^m$*, such that the Hamiltonian system* $\dot{x} = J_{2m}H_x(x)$ *will take the form:*

$$\begin{cases} \dot{\varphi} &= \omega(I), \\ \dot{I} &= 0, \end{cases} \tag{4.58}$$

where $\omega(0) = \omega_f$. $\square$

Remark:
The system (4.58) is Hamiltonian with $H = H(I)$ such that

$$\omega(I) = \frac{\partial H(I)}{\partial I}.$$

The coordinates (I, φ) are called the *action-angle variables.* $\diamond$

Obviously, any Hamiltonian system with one degree of freedom ($m = 1$) is integrable. Although several important Hamiltonian systems with several degrees of freedom, for instance Kepler's two-body problem, are integrable, integrability is exceptional when $m \geq 2$. In general, a nonintegrable Hamiltonian system (4.55) behaves very different from an integrable one: Invariant tori may exist but are separated by domains of "chaotic motions".

Example 4.42 (Elastic Pendulum)

Consider a small ball attached to a massless spring that can both oscillate in the radial direction and swing like a pendulum in the plane (see Figure 4.28). This is called the *elastic pendulum.* Let

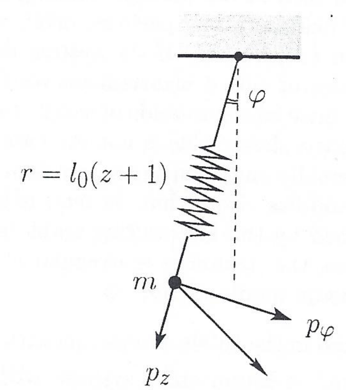


Figure 4.28: Elastic pendulum.

m be the mass of the ball and let l_0 and l be, respectively, the length of the spring in the vertical position in the absence of load and under load. Denote by g the acceleration of gravity, by s the spring constant, and by r the distance between the ball and the suspension point.

Introduce

$$\omega_z = \sqrt{\frac{s}{m}}, \quad \omega_\varphi = \sqrt{\frac{g}{l}}, \quad \sigma = ml^2.$$

Then the Hamiltonian (energy) of the system is given by

$$H = \frac{1}{2\sigma}\left(p_z^2 + \frac{p_\varphi^2}{(z+1)^2}\right) + \frac{\sigma}{2}\omega_z^2\left(z + \frac{\omega_\varphi^2}{\omega_z^2}\right)^2 - \sigma\omega_\varphi^2(z+1)\cos\varphi,$$

where φ is the angular displacement of the pendulum and

$$z = \frac{r - l_0}{l_0}$$

measures its radial displacement.

Obviously, the corresponding system (4.55) with

$$q = \begin{pmatrix} z \\ \varphi \end{pmatrix}, \quad p = \begin{pmatrix} p_z \\ p_\varphi \end{pmatrix}, \quad x = \begin{pmatrix} q \\ p \end{pmatrix},$$

is Hamiltonian with two degrees of freedom. The energy levels $H = h$ are 3-dimensional submanifolds in the 4-dimensional phase space. To analyse the dynamics in a level set, introduce the plane

$$z = 0$$

and parametrize it by (φ, p_φ). Given the energy h, consider a map

$$P : \begin{pmatrix} \varphi \\ p_\varphi \end{pmatrix} \mapsto \begin{pmatrix} \varphi' \\ p'_\varphi \end{pmatrix},$$

where (φ', p'_φ) is the point of the *second* intersection with the plane of the orbit starting at the point (φ, p_φ). One can show that P is an area-preserving map. This map can be approximated by numerical integration with high accuracy[6]. Orbits of P computed in this manner are depicted in Figure 4.29(a) for the following numerical values of the parameters:

$$h = 0.6125, \quad \sigma = 1, \quad \omega_z = 4, \quad \omega_\varphi = 1.$$

Note that the eigenfrequencies ω_z and ω_φ are in $4:1$ resonance.

Several fixed points can be seen in the picture. Among them we can clearly distinguish four saddle points. Actually, they belong to one periodic orbit crossing the plane $z = 0$ four times, and so forming two period-two cycles of P. If the system describing the elastic pendulum were integrable, we would see families of closed invariant curves filling the whole plane, separated by coinciding stable and unstable invariant manifolds of saddle fixed points. As Figure 4.29(b), giving a zoom-in of the right upper part, shows, this is not the case. The stable and unstable manifolds of the saddles intersect transversally and form the *heteroclinic structure* for P (see Chapter 3) that is most pronounced near the saddles. Note that, in fact, it is just a *homoclinic structure* for one cycle in the phase space, formed by the intersecting stable and unstable manifolds of this cycle. Inside the homoclinic structure, the dynamics is irregular ("chaotic") and consists of permanent meandering between long-periodic saddle cycles. $\diamondsuit$

Thus, as Poincaré discovered in the 1890s, homoclinic structures prevent integrability. It should be noted that "chaotic domains" in Hamiltonian systems with two degrees of freedom are confined between invariant tori, so that orbits starting inside them cannot leave these domains. Note that this is not true for Hamiltonian systems with $m > 2$, where the chaotic domains can overlap. This phenomenon is called *Arnold diffusion.*

[6]Tuwankotta, J.M. and Quispel, G. R. W. 'Geometric numerical integration applied to the elastic pendulum at higher-order resonance', *J. Comput. Appl. Math.* **154** (2003), 229-242.

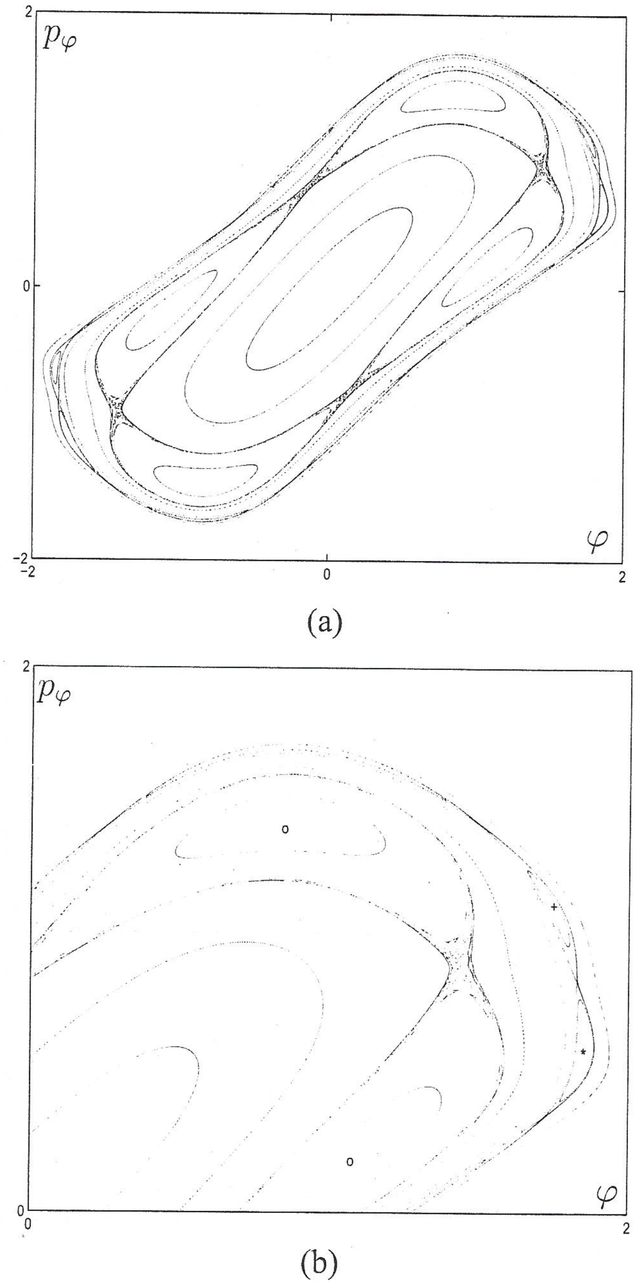


Figure 4.29: Poincare map of the elastic pendulum.