M621 Some Test 2 type problems Below G is a group, b, c are elements of G, H, K are subgroups of G, p is a prime, and n is a positive integer.

1. Suppose $s \in \mathbb{N}$. If $|b| = ps$, then $|b^s| = p$.

2. Let G be a group. A subgroup H of G is a *maximal subgroup of* G if (COMPLETE THE DEFINITION). A subgroup N of G is a *maximal normal subgroup of* G if COMPLETE THE DEFINiTION).

3. Let G be a group. The group G is *simple* if COMPLETE THE DEFINITION.

4. Prove that an Abelian group G is simple if and only if there exists a prime p such that $G \cong Z_p$.

5. Let N be a normal subgroup of G. Prove that G/N is simple if and only if N is a maximal normal subgroup of G. (You could use the Fourth Isomorphism Theorem, the part that states every normal subgroup A of G/N is of form K/N, where K is a normal subgroup of G and K contains N.)

6. A *composition series* of a group G is a sequence (COMPLETE THE DEFINITION). The *factors* of a composition series are (COMPLETE THE DEFINITION).

7. Suppose $n > 2$. Show that $\{e\} \trianglelefteq < r^2 > \trianglelefteq < r > \trianglelefteq D_{2n}$ is a composition series for D_{2n}, Then show that $\{e\} \trianglelefteq < s > \trianglelefteq < s, r^2 > \trianglelefteq D_{2n}$ is also a composition series for D_{2n}.

8. A group G is *solvable* if (COMPLETE THE DEFINITION).

   (a) Using one of the above composition series for D_{2n} to show that D_{2n} is solvable.

   (b) A_5 is simple, as we proved. Explain why A_5 has just one composition series, and then explain why A_5 is not solvable.

9. The Third isomorphism Theorem states that if G is a group, H, K are normal subgroups of G, and H is contained in K, then $G/K \cong G/H/K/H$. Prove the Third Isomorphism Theorem—you'll use the First Isomorphism Theorem.

10. Suppose $n > 1$, and $\phi : Z_n \to Z_n$ is a homomorphism.

    (a) Show that for all $x \in Z_n$, $\phi(x) = \phi(1)x$. (FYI: Homomorphisms of the form $f : G \to G$ are called *endomorphisms of* G. Note that the bijective endomorphisms of a group are its automorphisms. The endomorphisms of a group form a monoid, a binary associative structure $(X, *)$ having an identity element. The endomorphisms of G form a monoid with operation composition of endomorphisms.)

    (b) Show that ϕ above is an automorphism if and only if $(n, \phi(1)) = 1$.

    (c) Suppose ϕ, γ are both automorphisms of Z_n, with $\phi(1) = j, \gamma(1) = k$, where $n > j \geq k \geq 1$ and $(n, j) = 1 = (n, k)$. Show that $\phi \circ \gamma(x) = kjx$, for all $x \in Z_n$ (where kj is, as usual, taken mod n).

    (d) Recall that $Z_n^{\times}$ is the group with underlying $\{j \in Z_n : (n, j) = 1\}$ with operation **multiplication** (mod n of course). Show that $Aut(Z_n) \cong Z_n^{\times}$. Provide a specific isomorphism $f : Z_n^{\times} \to Aut(Z_n)$, and show f is an isomorphism.

11. Show that S_n can be embedded in A_{n+2}. Provide a specific isomorphism $\iota : S_n \hookrightarrow A_{n+2}$, and show ι is an isomorphism.

12. State the Sylow Theorem.

13. Cauchy's Theorem states if G is a finite group, and $p||G|$, then G has an element b such that $p = |b|$. Prove Cauchy's Theorem as a corollary of the the Sylow Theorem (the part of the Sylow Theorem that states that there exist p-Sylow subgroups).

14. Suppose H is a subgroup of a finite group G, and $p||H|$. Let B be a p-Sylow subgroup of H. Show that there exists a p- Sylow subgroup P of G such that $B = H \cap P$.

15. A finite group G acts on its p-Sylow subgroups by conjugation. One aspect of the Sylow Theorem indicates that this action is transitive. Using these comments, along with the Orbit-Stablizer stuff, explain why if P is a Sylow p-subgroup of G, then $[G : N_G(P)] = n_p$.

16. Suppose that $|G| = pq$, where p, q are primes, and $p < q$.

    (a) Show that G has a normal q-Sylow subgroup.

    (b) Suppose that in addition q is not equal to $tp + 1$ for any $t \in \mathbb{N}$. Show that G has a normal p-Sylow subgroup, and that $G \cong Z_p \times Z_q$ (using the result of an earlier homework assignment).

17. Suppose G is a finite group with a Sylow p-subgroup P, and Q is a subgroup of G with $|Q| = p^{\beta}$ (that is, Q is a p-subgroup of G).

18. This is a good exercise, certainly too long for an exam, but it would be very useful to review. We did it in detail in class: Suppose G is a group, $|G| = 12$, and G has more than one 3-Sylow subgroup.

   (a) How many 3-Sylow subgroups does G have?

   (b) Show that G acts on the set A of 3-Sylow subgroups of G.

   (c) Show that if P is a 3-Sylow subgroup of G, then $[G : N_G(P)] = 4$; use this to explain why this implies that $N_G(P) = P$, and show that the action above is also faithful (that is, that the map $\sigma : G \to S_A$ is one-to-one).

   (d) Since the action is faithful, G is embedded in S_4, i.e. $\sigma(G) \cong G$. Thus, there's an isomorphic copy of G in S_4. Explain why that isomorphic copy ($\sigma(G)$) must be isomorphic to A_4. [Suggestions: Count 3-cycles. Use Lagrange.] **You've proven that a group having 12 elements that has more than one 3-Sylow subgroup is isomorphic to A_4.**

   (e) As we know, D_{12} is a non-Abelian group of order 12. Show that D_{12} is not isomorphic to A_4—check on the number of elements of given orders. From the parts of the exercise above, it must be that D_{12} has a normal Sylow-3 subgroup. Check to see if that's true.

19. Prove that $N_Q(P) = P \cap Q$. (This was proved and discussed in class.)

20. True or false? If "false", provide a specific counterexample.

   (a) If A, B, C are subgroups of G, $A \trianglelefteq B$ and $B \trianglelefteq C$, then $A \trianglelefteq C$.

   (b) If N is a normal subgroup of G, then G acts on the elements of N by conjugation. (So $A = N$.) The kernel of σ, the homomorphism $\sigma : G \to S_A$, determined by the action is $C_G(N)$.

   (c) Let H be a subgroup of G. Then $\cap\{gHg^{-1} : g \in G\}$ is a normal subgroup of G.