M621, HW2, due Sept. 8

1. For the regular n-gon: let r be the $\frac{360}{n}$ rotation in the clockwise direction, let u be a reflection of the regular n-gon, let s be the reflection through the line of symmetry of the regular n-gon that passes through 1 and the center of the regular n-gon, and let e be the identity element of D_{2n}, the identity function.

   (a) For any $w \in D_{2n}$, let $Fix(w) = \{k \in \{1, \ldots, n\} : w(k) = k\}$. Three examples: $Fix(r) = \emptyset$, $Fix(e) = \{1, \ldots, n\}$, and if $n = 4$, and s is the reflection through the line of symmetry of the square that passes through 1, then $Fix(s) = \{1, 3\}$.

      i. **Brief explanation.** If n is odd, and u is a reflection, what is $|Fix(u)|$?

      ii. **Brief explanation.** If n is even, and u is a reflection, what possibilities are there for $|Fix(u)|$?

      iii. **Brief explanation.** If $n > k > 0$, what is $|Fix(r^k)|$?

      iv. It is clear that if $n > k \geq 0$, the rotation r^k is completely determined by $r(1)$. In fact, for any $j \in \{1, \ldots, n\}$, r^k is completely determind by $r^k(j)$. For $j \in \{1, \ldots, n\}$, what is $r^k(j)$? (Your answer will probably involve "mod n".)

      v. **Brief explanation.** Suppose u is a reflection, explain why if u is a reflection, then $u \neq r^k$ for any $n > k \geq 0$. (If n is odd, each reflection has one fixed point, but if n is even, there are reflections that have no fixed points. You'll want to make sure you deal with both of those cases.)

      vi. **Brief explanation.** Determine the following, briefly explaining for your answer.

         A. $rs(1)$

         B. $rs(2)$

         C. $sr^{-1}(1)$

D. $sr^{-1}(2)$

vii. Comment briefly on the following: "Since rs and sr^{-1} agree on the vertices that make up an edge (namely 1 and 2), it follows that $rs = sr^{-1}$".

viii. For $n > 2$, show that D_{2n} is not Abelian (by producing a pair of elements $y, z \in D_{2n}$ such that $yz \neq zy$).

2. Suppose G is a group, and for all $g \in G$, $g^2 = e$. Prove that G is Abelian.

3. **Short answer, but provide specifics.** Find a group G such that for all $g \in G$, $g^3 = e$, but G is not Abelian.

4. Suppose that G is a group, x, y are elements of G, $n \in \mathbb{N}$, and $(xy)^n = e$. Prove that $(yx)^n = e$. Suggestion: Find a power of yx "inside" $(xy)^n$.

5. Using exercise 4 directly above, prove that if G is a group, $n \in \mathbb{N}$, then $|xy| = |yx|$. (Make sure you consider consider the possibility that one or both of xy, yx have infinite order.)

6. (Suggesed, but voluntary, +1 EC—perhaps you'll have a chance to write it on the board next Thursday) Let $n \in \mathbb{N}$. Recall that S_n is the group of permutations of $\{1, \ldots, n\}$. Let α, β be elements of S_n. Prove that $Fix(\beta\alpha\beta^{-1}) = \beta(Fix(\alpha)\}$.