Remaining Cauchy functional equations

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Introduction

In Chapter 5 of his book *Cours d'Analyse*, A. L. Cauchy (1821) also studied three other functional equations, namely,

$$f(x+y) = f(x)f(y), \tag{1}$$

$$f(xy) = f(x) + f(y) \tag{2}$$

and

$$f(xy) = f(x)f(y) \tag{3}$$

besides the additive Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{4}$$

for all $x, y \in \mathbb{R}$.

In this lecture, we solve the remaining Cauchy functional equations.

The general solution of each of these functional equations is determined in terms of the additive function.

Finally, using the general solution, the continuous solution is provided for each of these functional equations.

Solution of Exponential Cauchy Equation

First, we determine the general solution of the exponential Cauchy functional equation (1) without assuming any regularity condition such as continuity, boundedness or differentiability on the unknown function f.

Theorem 1 . *If the functional equation* (1)*, that is,*

$$f(x+y) = f(x)f(y),$$

holds for all real numbers x and y, then the general solutions of (1) *are given by*

$$f(x) = e^{A(x)} \quad \text{and} \quad f(x) = 0 \quad \forall\, x \in \mathbb{R}, \tag{5}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and e is the Napierian base of logarithm.

Proof: It is easy to see that $f(x) = 0$ for all $x \in \mathbb{R}$ is a solution of (1), that is

$$f(x+y) = f(x)\,f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Hence from now on we suppose that $f(x)$ is not identically zero.

To complete the rest of the proof of the theorem, we need to show

(a) f is nowhere zero,

(b) f is strictly positive,

(c) $A := \ln f$ is additive.

We claim that $f(x)$ is nowhere zero. Suppose not. Then there exists a y_o such that $f(y_o) = 0$. From (1), we get

$$f(y) = f((y - y_o) + y_o) = f(y - y_o)\, f(y_o) = 0$$

for all $y \in \mathbb{R}$.

This is a contradiction to our assumption that $f(x)$ is not identically zero. Hence $f(x)$ is nowhere zero.

Letting $x = \frac{t}{2} = y$ in $f(x+y) = f(x)\,f(y)$, we see that

$$f(t) = f\left(\frac{t}{2}\right)^2$$

for all $t \in \mathbb{R}$. Hence $f(x)$ is a strictly positive. Now taking natural logarithm of both sides of $f(x+y) = f(x)\,f(y)$, we obtain

$$\ln f(x+y) = \ln f(x) + \ln f(y).$$

Defining $A : \mathbb{R} \to \mathbb{R}$ by $A(x) = \ln f(x)$, we have

$$A(x+y) = A(x) + A(y). \tag{6}$$

Hence we have the asserted solution $f(x) = e^{A(x)}$ and the proof is now complete.

The following corollary is obvious from the above theorem.

Corollary 1 *If the equation* (1)*, that is,* $f(x+y) = f(x)f(y)$*, holds for all real numbers* x *and* y*, then the general continuous solutions of* (1) *are given by*

$$f(x) = e^{c\,x} \quad \text{and} \quad f(x) = 0 \quad \forall\, x \in \mathbb{R}, \tag{7}$$

where c *is an arbitrary real constant.*

Next we give the definition of exponential functions.

Definition 1 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called a (real-valued) real exponential function if it satisfies* $f(x+y) = f(x)\,f(y)$ *for all* $x, y \in \mathbb{R}$.

Let n be a positive integer. Suppose the functional equation

$$f(x+y+nxy) = f(x)\,f(y) \tag{8}$$

holds for all reals $x > -\frac{1}{n}$ and all $y > -\frac{1}{n}$.

- When $n \to 0$, the functional equation (8) reduces to the exponential Cauchy functional equation.
- This equation was studied by Thielman (1949).

Theorem 2 . *Every solution f of the functional equation*

$$f(x + y + nxy) = f(x)\, f(y)$$

for all reals $x > -\frac{1}{n}$ and all $y > -\frac{1}{n}$ is of the form

$$f(x) = 0 \qquad \text{or} \qquad f(x) = e^{A(\,\ln(1+nx)\,)}, \tag{9}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function.

Proof: We write the functional equation (8) as

$$f\left(\frac{(1+nx)(1+ny)-1}{n}\right) = f(x)\, f(y). \tag{10}$$

Next we define $0 < 1 + nx = e^u$ and $0 < 1 + ny = e^v$ so that $u = \ln(1 + nx)$ and $v = \ln(1 + ny)$.

Now rewriting (10), we obtain

$$f\left(\frac{e^{u+v}-1}{n}\right) = f\left(\frac{e^{u}-1}{n}\right) f\left(\frac{e^{v}-1}{n}\right) \tag{11}$$

for all $u, v \in \mathbb{R}$.

Letting

$$\phi(u) = f\left(\frac{e^u - 1}{n}\right) \tag{12}$$

in (11), we have

$$\phi(u+v) = \phi(u)\,\phi(v) \tag{13}$$

for all $u, v \in \mathbb{R}$.

Hence by Theorem 1, we have

$$\phi(x) = e^{A(x)} \quad \text{or} \quad \phi(x) = 0 \quad \forall\, x \in \mathbb{R}, \tag{14}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and e is the Napierian base of logarithm.

Therefore from

$$\phi(u) = f\left(\frac{e^u - 1}{n}\right)$$

and $\phi(x) = e^{A(x)}$ or $\phi(x) = 0$, we obtain

$$f(x) = 0 \qquad or \qquad f(x) = e^{A(\,\ln(1+nx)\,)},$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function. The proof of the theorem is now complete.

The following corollary is obvious.

Corollary 2 *Every continuous solution f of the functional equation (8) holding for all reals $x > -\frac{1}{n}$ and all $y > -\frac{1}{n}$ is of the form*

$$f(x) = 0 \qquad \text{or} \qquad f(x) = (1 + nx)^k, \tag{15}$$

where k is an arbitrary constant.

Solution of Logarithmic Cauchy Equation

Now we consider the second Cauchy functional equation (2), that is

$$f(xy) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R} \setminus \{0\}.$$

This functional equation is known as the logarithmic Cauchy equation.

Theorem 3 . *If the functional equation (2), that is,*

$$f(xy) = f(x) + f(y)$$

holds for all $x, y \in \mathbb{R} \setminus \{0\}$*, then the general solution of (2) is given by*

$$f(x) = A(\ln |x|) \quad \forall x \in \mathbb{R} \setminus \{0\}, \tag{16}$$

where A *is an additive function.*

Proof: To prove this theorem, we have to show

(a) f is even on $\mathbb{R} \setminus \{0\}$, and

(b) $A(s) := \ln(e^s)$ is additive on $\mathbb{R}$.

Substitute $x = t$ and $y = t$ in $f(xy) = f(x) + f(y)$ to get

$$f(t^2) = 2f(t).$$

Letting $x = -t$ and $y = -t$ in $f(xy) = f(x)+f(y)$, we have

$$f(t^2) = 2f(-t).$$

Hence we see that

$$f(t) = f(-t) \quad \forall t \in \mathbb{R} \setminus \{0\}. \tag{17}$$

Next, suppose the functional equation $f(xy) = f(x) + f(y)$ holds for all $x > 0$ and $y > 0$. Let

$$x = e^s \qquad \text{and} \qquad y = e^t \tag{18}$$

so that

$$s = \ln x \qquad \text{and} \qquad t = \ln y. \tag{19}$$

Note that $s, t \in \mathbb{R}$ since $x, y \in \mathbb{R}_+$ where

$$\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x > 0\}.$$

Substituting $x = e^s$ and $y = e^t$ in $f(xy) = f(x) + f(y)$, we obtain

$$f(e^{s+t}) = f(e^s) + f(e^t).$$

Defining

$$A(s) = f(e^s) \tag{20}$$

and using the last equation we have

$$A(s+t) = A(s) + A(t)$$

for all $s, t \in \mathbb{R}$.

Hence from the definition $A(s) = f(e^s)$, we have

$$f(x) = A(\ln x) \qquad \forall x \in \mathbb{R}_+. \tag{21}$$

Since $f(t) = f(-t)$, we see that the general solution of $f(xy) = f(x) + f(y)$ is

$$f(x) = A(\ln |x|) \qquad \forall x \in \mathbb{R} \setminus \{0\}$$

and the proof is now complete.

The following corollary is a consequence of the last theorem.

Corollary 3 *The general solution of the functional equation $f(xy) = f(x) + f(y)$ for all $x, y \in \mathbb{R}_+$ is given by*

$$f(x) = A(\ln x), \tag{22}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function.

The following result is also trivial.

Corollary 4 *The general solution of the functional equation $f(xy) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ is given by*

$$f(x) = 0 \qquad \forall x \in \mathbb{R}. \tag{23}$$

Proof: Substitute $y = 0$ in (2) to get $f(0) = f(x) + f(0)$ and hence we have the asserted solution. QED

Corollary 5 *The general continuous solution of the functional equation* $f(xy) = f(x) + f(y)$ *for all* x, y *in* $\mathbb{R} \setminus \{0\}$ *is given by*

$$f(x) = c \ln |x| \qquad \forall x \in \mathbb{R} \setminus \{0\}, \tag{24}$$

where c *is an arbitrary real constant.*

Definition 2 *A function* $f : \mathbb{R}_+ \to \mathbb{R}$ *is called a logarithmic function if it satisfies* $f(xy) = f(x) + f(y)$ *for all* $x, y \in \mathbb{R}_+$.

Solution of Multiplicative Cauchy Equation

Now we treat the last Cauchy equation (3), that is

$$f(xy) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

This equation is the most complicated of the three equations considered in this chapter.

In the following theorem we need the notion of the signum function. The signum function is denoted by $\operatorname{sgn}(x)$ and defined as

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases} \tag{25}$$

It is easy to see that

$$|\mathrm{sgn}\,(x)| = \begin{cases} 1 & \text{if } x \in \mathbb{R} \setminus \{0\} \\ 0 & \text{if } x = 0. \end{cases}$$

Hence the following function

$$f(x) = \begin{cases} e^{A(\ln|x|)} & \text{if } x \in \mathbb{R} \setminus \{0\} \\ 0 & \text{if } x = 0. \end{cases}$$

can be compactly written as $f(x) = e^{A(\ln|x|)}\ |sgn(x)|$.

Theorem 4. *The general solutions of $f(xy) = f(x)f(y)$ holding for all $x, y \in \mathbb{R}$ are given by*

$$f(x) = 0, \tag{26}$$

$$f(x) = 1, \tag{27}$$

$$f(x) = e^{A(\ln|x|)}\,|sgn(x)|, \tag{28}$$

and

$$f(x) = e^{A(\ln|x|)}\,sgn(x), \tag{29}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and e is the Napierian base of logarithm.

Proof: The steps to prove this theorems are the following:

Step 1. Show $f(0) = 0$, $f(0) = 1$, $f(1) = 0$ and $f(1) = 1$

Step 2. $f(x) \geq 0$ for all $x \in \mathbb{R}_+$

Step 3. If $f(x_o) = 0$, the $f(x) = 0$ for all $x \in \mathbb{R}$

Step 4. If $f(x) \neq 0$ for all $x \in \mathbb{R} \setminus \{0\}$, then $f(x) = 1$

Step 5. If $f(0) = 0$, then $f(x)$ is nowhere zero on $\mathbb{R} \setminus \{0\}$

Step 6. If $f(x)$ is nowhere zero on $\mathbb{R} \setminus \{0\}$, then the map $A(s) := \ln f(e^s)$ is additive on $\mathbb{R}$

Step 7. Show $f(1) = 1$ since $f(1) = 0$ yields a cotradiction

Step 8. Show $f(1) = 1$ implies $f(-1) = 1$ or $f(-1) = -1$

Step 9. If $f(1) = 1$, then

$$f(x) = e^{A}(\ln(|x|))\,|sgn(x)|$$

for all $x \in \mathbb{R} \setminus \{0\}$

Step 10. If $f(-1) = -1$, then show

$$f(x) = e^{A}(\ln(|x|))\, sgn(x)$$

for all $x \in \mathbb{R} \setminus \{0\}$

Letting $x = 0 = y$ in (3), we obtain $f(0)[1 - f(0)] = 0$ and hence either

$$f(0) = 0 \qquad \text{or} \qquad f(0) = 1. \tag{30}$$

Substituting $x = 1 = y$ in (3), we have $f(1)[1 - f(1)] = 0$ and hence either

$$f(1) = 0 \qquad \text{or} \qquad f(1) = 1. \tag{31}$$

Let x be a positive real number, that is $x > 0$. Then (3), that is, $f(xy) = f(x)f(y)$ implies

$$f(x) = f(\sqrt{x})^2 \geq 0. \tag{32}$$

Suppose there exists an $x_0 \in \mathbb{R}$, $x_0 \neq 0$ such that $f(x_0) = 0$. Let $x \in \mathbb{R}$ be an arbitrary real number. Then from (3) we have

$$f(x) = f\left(x_0 \frac{x}{x_0}\right) = f(x_0)\, f\left(\frac{x}{x_0}\right) = 0$$

for all $x \in \mathbb{R}$ and we obtain the solution (26).

From now on we suppose that $f(x) \neq 0$ for all $x \in \mathbb{R} \setminus \{0\}$. From (30) we have either $f(0) = 0$ or $f(0) = 1$. If $f(0) = 1$, then letting $y = 0$ in (3), we obtain

$$f(0) = f(x)f(0)$$

and hence

$$f(x) = 1.$$

for all $x \in \mathbb{R}$. Thus we have the asserted solution (27).

Next we consider the case $f(0) = 0$. In this case we claim that f is nowhere zero in $\mathbb{R} \setminus \{0\}$. Suppose not. Then there exists a y_o in $\mathbb{R} \setminus \{0\}$ such that $f(y_o) = 0$. Letting $y = y_o$ in (3), we have

$$f(xy_o) = f(x)f(y_o) = 0.$$

Hence

$$f(x) = 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$$

which is a contradiction to our assumption that f is not identically zero. Thus f is nowhere zero in $\mathbb{R} \setminus \{0\}$.

From the fact that f is nowhere zero in $\mathbb{R} \setminus \{0\}$ and (32), we have

$$f(x) > 0 \qquad \text{for} \qquad x > 0. \tag{33}$$

Let

$$x = e^s \qquad \text{and} \qquad y = e^t \tag{34}$$

so that

$$s = \ln x \qquad \text{and} \qquad t = \ln y. \tag{35}$$

Note that $s, t \in \mathbb{R}$ since $x, y \in \mathbb{R}_+$.

Substituting (34) into (3), we obtain

$$f(e^{s+t}) = f(e^s)f(e^t).$$

Since $f(t) > 0$ for all $t > 0$, taking the natural logarithm of both sides of the last equation, we have

$$A(s+t) = A(s) + A(t),$$

where

$$A(s) = \ln f(e^s) \quad \forall s \in \mathbb{R}. \tag{36}$$

Thus A is an additive function. From (36)and (35), we obtain

$$f(x) = e^{A(\ln|x|)} \quad \forall x \in \mathbb{R}_+. \tag{37}$$

From (31) we see that either $f(1) = 0$ or $f(1) = 1$. If $f(1) = 0$, then letting $y = 1$ in (3), we obtain

$$f(x) = 0 \qquad \forall x \in \mathbb{R} \setminus \{0\}$$

contrary to our assumption that f is not identically zero on $\mathbb{R} \setminus \{0\}$. Hence $f(1) = 1$. Now letting $x = -1 = y$ in (3), we get $f(1) = f(-1)^2$ and hence

$$f(-1) = 1 \qquad \text{or} \qquad f(-1) = -1. \tag{38}$$

If $f(-1) = 1$, then letting $y = -1$ in (3), we have

$$f(-x) = f(x)f(-1) = f(x)$$

for all $x \in \mathbb{R} \setminus \{0\}$. Thus (37) yields $f(x) = e^{A(\ln |x|)}$ for all $x \in \mathbb{R} \setminus \{0\}$. Since $f(0) = 0$, we have

$$f(x) = \begin{cases} e^{A(\ln |x|)} & \text{if } x \in \mathbb{R} \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases}$$

which is the asserted solution (28).

If $f(-1) = -1$, then letting $y = -1$ in (3), we have

$$f(-x) = f(x)f(-1) = -f(x)$$

for all $x \in \mathbb{R} \setminus \{0\}$. Hence (37) yields

$$f(x) = \begin{cases} e^{A(\ln|x|)} & \text{if } x > 0 \\ -e^{A(\ln|x|)} & \text{if } x < 0 \end{cases}$$

for all $x \in \mathbb{R} \setminus \{0\}$.

Together with the fact that $f(0) = 0$, we have

$$f(x) = \begin{cases} e^{A(\ln|x|)} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -e^{A(\ln|x|)} & \text{if } x < 0 \end{cases}$$

which is the asserted solution (29). Now the proof of the theorem is complete.

Corollary 6 *The general continuous solution of the equation $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ is given by*

$$f(x) = 0, \tag{39}$$

$$f(x) = 1, \tag{40}$$

$$f(x) = |x|^\alpha, \tag{41}$$

and

$$f(x) = |x|^\alpha \, sgn(x), \tag{42}$$

where α is an arbitrary positive real constant.

Proof: By Theorem 4 either $f = 0$, or $f = 1$, or f has the form (28) or (29), where $A : \mathbb{R} \to \mathbb{R}$ is an additive function. Since f is continuous and $A(t) = \ln f(e^t)$, A is also continuous on $\mathbb{R}$. Therefore $A(t) = \alpha\, t$, where $\alpha \in \mathbb{R}$ is an arbitrary constant. Hence from (28) and (29), we get

$$f(x) = |x|^{\alpha} \quad \text{and} \quad f(x) = |x|^{\alpha}\, \mathrm{sgn}\,(\mathrm{x}),$$

respectively.

The only thing remaining to be shown is $\alpha > 0$. If we had $\alpha = 0$, then (41) will yield $f(x) = 1$ for $x \neq 0$, and by continuity of f we must have $f(0) = 1$. Hence we will have $f = 1$, already listed in (40). Formula (42) with $\alpha = 0$ yields

$$f(x) = 1 \qquad \text{for} \qquad x > 0$$

and

$$f(x) = -1 \qquad \text{for} \qquad x < 0$$

and thus f cannot be continuous.

Similarly if $\alpha < 0$, then f given by (41) and (42) satisfies

$$\lim_{x \to 0^+} f(x) = \infty$$

and hence cannot be continuous at 0. Now the proof of the corollary is complete.

Definition 3 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called a multiplicative function if it satisfies* $f(xy) = f(x)\,f(y)$ *for all* $x, y \in \mathbb{R}$.

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Thank You