MATH 668 Homework 2 Solutions

1. (a) Since $\begin{pmatrix} 4x_1 - 3x_2 + 1 \\ x_2 - x_3 + 2 \end{pmatrix} = \begin{pmatrix} 4 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

$$E\left[\begin{pmatrix} 4 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} 4 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix} E\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

(b) $\text{cov}(4x_1-3x_2+1, x_2-x_3+2) = \text{cov}(4x_1-3x_2, x_2-x_3) = \text{cov}\left(\begin{pmatrix} 4 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right)$

$$= \begin{pmatrix} 4 & -3 & 0 \end{pmatrix} \text{cov}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 & -3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = -2$$

(c) Since $\text{cov}\left[\begin{pmatrix} 4 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] = \text{cov}\left[\begin{pmatrix} 4 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right] = \begin{pmatrix} 4 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix} \text{cov}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ -3 & 1 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 4 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ -3 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 10 & -2 \\ -2 & 5 \end{pmatrix}$, the correlation matrix for $\begin{pmatrix} 4x_1 - 3x_2 + 1 \\ x_2 - x_3 + 2 \end{pmatrix}$ is

$$\mathbf{P}_\rho = \begin{pmatrix} 1 & \frac{-2}{\sqrt{10}\sqrt{5}} \\ \frac{-2}{\sqrt{10}\sqrt{5}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-\sqrt{2}}{5} \\ \frac{-\sqrt{2}}{5} & 1 \end{pmatrix}.$$

2. (a) Since $y_3 - y_2 + 1 = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + 1$,

$$E\left[\begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + 1\right] = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} E\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + 1 = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + 1 = -2 + 1 = -1$$

and $\text{var}\left[\begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + 1\right] = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \text{cov}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 13 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = 10$,

$y_3 - y_2 + 1 \sim N(-1, 10)$.

(b) Since $\begin{pmatrix} y_2 \\ y_1 \\ y_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ 1 & 5 & 2 \\ 2 & 2 & 13 \end{pmatrix}\right)$,

$E\left(\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} \Big| y_2 = 0\right) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} 1^{-1}(0-1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$ and

$\text{var}\left(\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} \Big| y_2 = 0\right) = \begin{pmatrix} 5 & 2 \\ 2 & 13 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} 1^{-1} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 13 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$,

$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} \Big| y_2 = 0 \sim N_2\left(\begin{pmatrix} -1 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}\right)$ so $y_1|y_2 = 0$ and $y_3|y_2 = 0$ are independent since $\text{cov}(y_1, y_3|y_2 = 0) = 0$ and

$P(y_1 > 0 \text{ and } y_3 > 0|y_2 = 0) = P(y_1 > 0|y_2 = 0)P(y_3 > 0|y_2 = 0) = P\left(\frac{y_1-(-1)}{\sqrt{4}} > \frac{1}{2}|y_2 = 0\right) P\left(\frac{y_3-(-3)}{3} > \frac{3}{3}|y_2 = 0\right) = P\left(\frac{y_1-(-1)}{\sqrt{4}} > 0.5|y_2 = 0\right) P\left(\frac{y_3-(-3)}{3} > 1|y_2 = 0\right) = 0.0489511$ as shown below.

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(1-pnorm(.5))*(1-pnorm(1))
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## [1] 0.0489511
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(c) Since $\begin{pmatrix} y_3 - y_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$,

$E\begin{pmatrix} y_3 - y_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} E\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ and

$\text{var}\begin{pmatrix} y_3 - y_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{cov}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 1 \\ 1 & 5 \end{pmatrix}$,

so $\begin{pmatrix} y_3 - y_2 \\ y_1 \end{pmatrix} \sim N_2\left(\begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 & 1 \\ 1 & 5 \end{pmatrix}\right)$.

Then $E(y_1|y_3 - y_2 = 1) = 0 + 1(10)^{-1}(1 - (-2)) = 0 + 0.3 = 0.3$ and

$\text{var}(y_1|y_3 - y_2 = 1) = 5 - 1(10)^{-1}(1) = 5 - 0.1 = 4.9$

so $y_1|y_3 - y_2 = 1 \sim N(0.3, 4.9)$.

Then $P(y_1 > 0|y_3 - y_2 = 1) = P\left(\frac{y_1-(-0.3)}{\sqrt{4.9}} > \frac{0.3}{\sqrt{4.9}}\middle| y_3 - y_2 = 1\right) = P\left(\frac{y_1-(-0.3)}{\sqrt{4.9}} > 0.1355262\middle| y_3 - y_2 = 1\right) = 0.5539021$ as shown below.

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1-pnorm(-.3/sqrt(4.9))
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## [1] 0.5539021
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3. There are many answers. Here are two methods to find an answer.

Method 1: Find the eigenvectors of $\mathbf{\Sigma} = \begin{pmatrix} 1+\alpha & 1 \\ 1 & 1 \end{pmatrix}$. The characteristic equation is $0 = \det(\mathbf{\Sigma} - \lambda\mathbf{I}) = \begin{pmatrix} 1+\alpha-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1+\alpha-\lambda)(1-\lambda) - 1 = \lambda^2 - (2+\alpha)\lambda + \alpha$. Using the quadratic formula, the solutions are $\lambda = \dfrac{(2+\alpha) \pm \sqrt{(2+\alpha)^2 - 4\alpha}}{2} = 1 + \dfrac{\alpha}{2} \pm \sqrt{1 + \dfrac{\alpha}{2}}$. Solving the system $\begin{pmatrix} 1+\alpha-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, an eigenvector corresponding to λ is the vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ where $(1+\alpha-\lambda)x_1 + x_2 = 0$ so $x_2 = -x_1(1+\alpha-\lambda) = x_1\left(-\dfrac{\alpha}{2} \pm \sqrt{1+\dfrac{\alpha}{2}}\right)$. So any multiple of $\begin{pmatrix} 1 \\ -\frac{\alpha}{2} \pm \sqrt{1+\frac{\alpha}{2}} \end{pmatrix}$ is an eigenvector corresponding to λ. Letting $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -\frac{\alpha}{2} + \sqrt{1+\frac{\alpha}{2}} & -\frac{\alpha}{2} - \sqrt{1+\frac{\alpha}{2}} \end{pmatrix}^\top = \begin{pmatrix} 1 & -\frac{\alpha}{2} + \sqrt{1+\frac{\alpha}{2}} \\ 1 & -\frac{\alpha}{2} - \sqrt{1+\frac{\alpha}{2}} \end{pmatrix}$, we see that $\mathbf{A}\boldsymbol{x} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 + \frac{\alpha^2}{2} + (2-\alpha)\sqrt{1+\frac{\alpha^2}{4}} & 0 \\ 0 & 2 + \frac{\alpha^2}{2} - (2-\alpha)\sqrt{1+\frac{\alpha^2}{4}} \end{pmatrix}\right)$.

Method 2: Consider a general matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and compute $\text{cov}(\mathbf{A}\boldsymbol{x}) = \mathbf{A}\ \text{cov}(\boldsymbol{x})\ \mathbf{A}^\top = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1+\alpha & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} (1+\alpha)a^2 + 2ab + b^2 & (1+\alpha)ac + bc + ad + bd \\ (1+\alpha)ac + ad + bc + bd & (1+\alpha)c^2 + 2cd + d^2 \end{pmatrix}$. If the off-diagonal element $(1+\alpha)ac + bc + ad + bd$ equals 0, then x_1 and x_2 are independent. If $c = 0$, then we can find a solution that does not depend on α; we only need $ad + bd = 0 \Rightarrow (a + b)d = 0$. In addition, if we let $d = 1$, then we get $a + b = 0 \Rightarrow a = -b$. Say we then choose $a = 1$ and $b = -1$ so that $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Then we see that $\mathbf{A}\boldsymbol{x} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right)$.