Selected solutions from Quiz 1 prep problems.

Comment. Let A and B be sets, and let $f : A \to B$ be a function. Recall that if U is a subset of B, then $f^{-1}(U) = \{x \in A : f(x) \in U\}$.

Note also that *if* f is a bijection, then f^{-1} means something somewhat different—-it's a function $f^{-1} : B \to A$. If f is not a bijection, f^{-1} (as defined above) can be regarded as a function, but a function with domain 2^B (the power set of B), and co-domain 2^A.

1. **Prep exercise.**: Suppose $\Gamma : G \to K$ is a homomorphism, and $A \leq K$ (that is, A is a subgroup of K). Then $\Gamma^{-1}(A) \leq G$.

   Proof. Since Γ is a homomorphism, $\Gamma(e_A) = e_B$, and since $A \leq K$, $e_B \in A$, from which it follows that $\Gamma^{-1}(A)$ contains e_A. So $\Gamma^{-1}(A)$ is non-empty. Let $u, v \in \Gamma^{-1}(A)$. In that case, there exists $y, z \in A$ such that $\Gamma(u) = y, \Gamma(v) = z$. Now, $\Gamma(uv^{-1}) = \Gamma(u)\Gamma(v)^{-1} = yz^{-1}$, an element contained in A since it is a subgroup of K. It follows that $uv^{-1} \in \Gamma^{-1}(A)$, so $\Gamma^{-1}(A)$ is a subgroup of G (by the One-Step Subgroup Test).

   Prep exercise. Find two partitions of $\mathbb{Z}$, each of which partitions $\mathbb{Z}$ into subsets, each partition separating 0 and 1, and so that each partition is compatible with the multiplication operation of $\mathbb{Z}$.

   Partition 1: Let $B = \{1, -1\}$, let $A = \mathbb{Z} - \{1, -1\}$. The partition $A|B$ of $\mathbb{Z}$ is compatible with multiplication.

   Partition 2: Let $C = 2\mathbb{Z}$, and let $D = 2\mathbb{Z} + 1$. The partition $C|D$ of $\mathbb{Z}$ is also compatible with multiplication.

2. Let $\Gamma : G \to K$ be a homomorphism. Assume you've shown that $\Gamma(e_G) = e_K$, and for all $g \in G$, all $n \in \mathbb{N}$, that $\Gamma(g^n) = \Gamma(g)^n$. The following is somewhat informal.

   (a) Show for all $g \in G$ that $\Gamma(g^{-1}) = \Gamma(g)^{-1}$: We have $e_K = \Gamma(gg^{-1}) = \Gamma(g)\Gamma(g^{-1})$, but $e_K = \Gamma(g)\Gamma(g^{-1})$ implies that $\Gamma(g^{-1}) = \Gamma(g)^{-1}$.

   (b) Show that if $|g| = n \in \mathbb{N}$, then $|\Gamma(g)| \leq n$: That $|g| = n$ implies that $g^n = e_G$. Thus, $\Gamma(g)^n = \Gamma(g^n) = \Gamma(e_G) = e_K$—-$|\Gamma(g)|$ is the least positive power of Γ that realizes the identity here, it follows that $n \geq |\Gamma(g)|$.

(c) Show that if Γ is an isomorphism of groups, then for all $g \in G$, $|g| = |\Gamma(g)|$: As we showed in class, $\Gamma^{-1} : K \to G$ is an isomorphism. If $|g| = n \in \mathbb{N}$, then $n \geq |\Gamma(g)|$, as we showed above, but $|\Gamma(g)| \geq |\Gamma^{-1}(\Gamma(g))| = |g|$ (using the same thing above), so $|g| = |\Gamma(g)|$, for this case. If $|g| = \infty$, the arguments above (using Γ^{-1}) show that $|\Gamma(g)|$ can't be finite, completing the exercise.

3. Let $G = D_8$. As we know, D_8 acts on the set $\{1, 2, 3, 4\}$. Let $B = \{\{1, 3\}, \{2, 4\}\}$.

   (a) Explain why for all $g \in G$, $g(B) = B$. (That is, show that $g(\{1, 3\} \in B$ and $g(\{2, 4\}) \in B$.)

   Any symmetry of the square moves $\{1, 3\}$ to $\{1, 3\}$ or to moves $\{1, 3\}$ to $\{2, 4\}$. A number of ways to justify that: The rotation r obviously moves $\{1, 3\}$ to $\{2, 4\}$, moves $\{2, 4\}$ to $\{1, 3\}$, and the reflection s fixes $\{1, 3\}$ and fixes $\{2, 4\}$...everything is generated by these two, from which the statement in the first sentence of this paragraph follow.

   Another justification: $\{1, 3\}$ and $\{2, 4\}$ are the two pairs that are distance two apart–since the symmetries are "rigid motions"—thus distance is preserved; hence, we do have an action on those two pairs.

   Let $\sigma : G \to S_2$, where $\sigma(g) = \sigma_g$, the action of g on B. As you proved above, $\sigma : G \to S_2$ is a homomorphism. **Determine** $ker(\sigma)$, listing the elements of D_8 in cycle notation: $ker(\sigma)$ consists of those symmetries α of the square that fix both $\{1, 3\}$ and $\{2, 4\}$, i.e., $\{\alpha(1), \alpha(3)\} = \{1, 3\}$ and $\{\alpha(2), \alpha(4)\} = \{2, 4\}$. (Note $\alpha \in ker(\sigma)$ does not require that α fix each element of $\{1, 2, 3, 4\}$ point wise.)

   $ker(\sigma) = \{e, (13), (24), (13)(24)\}$, as you can verify (by listing the remaining four elements of D_8, and of those four remaining β satisfy $\beta(\{1, 3\}) = \{2, 4\}$ and $\beta(\{2, 4\}) = \{1, 3\}$, so each such β is not in $ker(\sigma)$).

4. Let D_8 act on the set $C = \{\{i, j\} : 4 \geq j > i \geq 1\}$, the set of all six 2-element subsets of $\{1, 2, 3, 4\}$, under the following action: For all $g \in D_8$, and $\{i, j\} \in C$, $g \cdot \{i, j\} := \{g(i), g(j)\}$.

It is not difficult to see that the above is an action of D_8 on C—you are not asked to prove that. Answer the following.

(a) Determine the kernel of the above action (i.e., determine $ker(\sigma)$, where $\sigma : D_8 \to S_6$).

$ker(\sigma) = \{e\}$—it has a trivial kernel. In such cases, the action is said to be a *faithful action*: Indeed, if $\alpha \in D_8$, and $\alpha(\{1,2\} = \{1,2\})$ and $\alpha(\{2,3\}) = \{2,3\}$, since α is permutation of $\{1,2,3,4\}$, it follows readily that $\alpha(2) = 2$. The same argument type can be applied to $1,3,4$. So if α doesn't move any of the six two-element subsets, $\alpha = e$. (So the action here is *faithful.*)

(b) Explain why σ is **not** an onto map: *Explanation.* We're mapping from 8-element group (D_8) into an 6!-element group (S_6)—no way the function could be onto.