HW4 Solutions

1. (a) $Y \sim \text{Poisson}(n\lambda)$ as shown in HW3#3.

The posterior distribution of λ given $Y=y$ is $\text{Gamma}\left(y+1, \frac{\mu}{n\mu+1}\right)$ since

$$\pi(\lambda|y) \propto f(y|\lambda)\,\pi(\lambda) \propto (\lambda^y e^{-n\lambda})(e^{-\lambda/\mu})$$
$$= \lambda^y e^{-n\lambda-\lambda/\mu} = \lambda^{(y+1)-1} e^{-\frac{\lambda}{\frac{\mu}{n\mu+1}}}.$$

(b) The Bayes estimator is

$$E[\lambda|Y] = (Y+1)\left(\frac{\mu}{n\mu+1}\right) = (n\bar{X}+1)\left(\frac{\mu}{n\mu+1}\right) = \boxed{\left(\frac{n\mu}{n\mu+1}\right)\bar{X} + \left(\frac{1}{n\mu+1}\right)\mu}.$$

As $n\to\infty$, $\boxed{\frac{n\mu}{n\mu+1} \to 1}$ and $\boxed{\frac{1}{n\mu+1} \to 0}$.

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2. $$E[|X-a|] = \sum_{x=1}^{6} |x-a|\cdot f(x) = \sum_{x=1}^{6} \frac{|x-a|}{6}$$

If $a \le 1$, then $\sum_{x=1}^{6} \frac{|x-a|}{6} = \sum_{x=1}^{6} \frac{x-a}{6} = \frac{\sum_{x=1}^{6} x - 6a}{6} = \frac{21-6a}{6} = \frac{7}{2} - a.$

If $a > 6$, then $\sum_{x=1}^{6} \frac{|x-a|}{6} = \sum_{x=1}^{6} \frac{-(x-a)}{6} = \frac{-\sum_{x=1}^{6} x + 6a}{6} = \frac{-21+6a}{6} = -\frac{7}{2} + a.$

If $1 < a \le 6$, then

$$\sum_{x=1}^{6} \frac{|x-a|}{6} = \sum_{x=1}^{\lfloor a\rfloor} \frac{-(x-a)}{6} + \sum_{x=\lfloor a\rfloor+1}^{6} \frac{x-a}{6}$$
$$= -\sum_{x=1}^{\lfloor a\rfloor} \frac{x-a}{6} + \sum_{x=1}^{6} \frac{x-a}{6} - \sum_{x=1}^{\lfloor a\rfloor} \frac{x-a}{6}$$
$$= \sum_{x=1}^{6} \frac{x-a}{6} - 2\sum_{x=1}^{\lfloor a\rfloor} \frac{x-a}{6}$$
$$= \frac{7}{2} - a - 2\,\frac{\frac{\lfloor a\rfloor(\lfloor a\rfloor+1)}{2} - 2\lfloor a\rfloor a}{6}$$
$$= \frac{1}{6}\Big(21 - \lfloor a\rfloor(\lfloor a\rfloor+1) + (2\lfloor a\rfloor - 6)a\Big).$$

So
$$E[|X-a|] = \begin{cases} \frac{7}{2} - a & \text{if } a \le 1 \\ \frac{19}{6} - \frac{2}{3}a & \text{if } 1 < a \le 2 \\ \frac{5}{2} - \frac{1}{3}a & \text{if } 2 < a \le 3 \\ \frac{3}{2} & \text{if } 3 < a \le 4 \\ \frac{1}{6} + \frac{1}{3}a & \text{if } 4 < a \le 5 \\ -\frac{3}{2} + \frac{2}{3}a & \text{if } 5 < a \le 6 \\ -\frac{7}{2} + a & \text{if } a > 6 \end{cases}$$

is minimized when $a \in [3,4]$ since it is decreasing when $a < 3$, constant when $3 \le a \le 4$, and increasing when $a > 4$.

3. (a) Since $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} = \text{Gamma}(\alpha,\beta)$ with $\alpha = \frac{n-1}{2}$ and $\beta = 2$,

$$\text{Var}\left[\frac{(n-1)S^2}{\sigma^2}\right] = \alpha\beta^2 = \left(\frac{n-1}{2}\right)4 = 2(n-1).$$

Since $\text{Var}\left[\frac{(n-1)S^2}{\sigma^2}\right] = \frac{(n-1)^2}{\sigma^4}\text{Var}[S^2]$, we have

$$\text{Var}[S^2] = \frac{\sigma^4}{(n-1)^2}\text{Var}\left[\frac{(n-1)S^2}{\sigma^2}\right] = \frac{\sigma^4}{(n-1)^2}\cdot 2(n-1) = \frac{2\sigma^4}{n-1}.$$

(b)
$$\begin{aligned}
\text{MSE}[cS^2] &= \text{Var}[cS^2] + \left(E[cS^2]-\sigma^2\right)^2 \\
&= c^2\,\text{Var}[S^2] + (c\sigma^2 - \sigma^2)^2 \\
&= c^2\,\frac{2\sigma^4}{n-1} + \left((c-1)\sigma^2\right)^2 \\
&= \frac{\sigma^4}{n-1}\left[2c^2 + (c-1)^2(n-1)\right]
\end{aligned}$$

$$\frac{d}{dc}\text{MSE}[cS^2] = \frac{\sigma^4}{n-1}\left[4c + 2(c-1)(n-1)\right] = \frac{2\sigma^4}{n-1}\left[2c + (c-1)(n-1)\right]$$

The derivative equals 0 when
$$\begin{aligned}
2c + (c-1)(n-1) &= 0 \\
2c + nc - n - c + 1 &= 0 \\
(n+1)c - (n-1) &= 0 \\
c &= \frac{n-1}{n+1}.
\end{aligned}$$

The MSE is minimized here since $\text{MSE}[cS^2]$ is a convex function of c

because $$\frac{d^2}{dc^2}\text{MSE}[cS^2] = \frac{2\sigma^4}{n-1}\left[2 + (n-1)\right] = \frac{2\sigma^4(n+1)}{n-1} > 0.$$