MATH 667-01 Homework 5

Due: Tuesday, October 24, 2017

Instructions: *Each of the following problems must be submitted to the instructor on or before the due date. Partial credit may be given for incorrect answers which make some positive progress. Late homework will not be accepted.*

1. (10 points) Let $X_1, \ldots, X_n$ be independent identically distributed random variables with probability density function

$$f(x|\theta) = \theta e^{-\theta x} I_{(0,\infty)}(x).$$

(a - 5 pts) Compute the Cramér-Rao Lower Bound (CRLB) on the variance of unbiased estimators of $\dfrac{1}{\theta}$.

(b - 5 pts) Find the maximum likelihood estimator of $\dfrac{1}{\theta}$ and show that it satisfies the CRLB.

2. (10 points)

(a - 1 pt) If $f(\boldsymbol{x}|\mu,\sigma^2) = (2\pi)^{-n/2}\exp\left\{-\dfrac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$ and $f(\boldsymbol{y}|\mu,\sigma^2) = (2\pi)^{-n/2}\exp\left\{-\dfrac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right\}$, show that

$$\frac{f(\boldsymbol{x}|\mu,\sigma^2)}{f(\boldsymbol{y}|\mu,\sigma^2)} = \exp\left\{\frac{\mu}{\sigma^2}\left(\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}y_i\right) - \frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}x_i^2 - \sum_{i=1}^{n}y_i^2\right)\right\}.$$

(b - 3 pts) If $X_1, \ldots, X_n$ is a random sample from a Normal distribution with mean μ and variance μ, show that $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$ is sufficient but not minimal sufficient for μ.

(c - 3 pts) If $X_1, \ldots, X_n$ is a random sample from a Normal distribution with mean μ and variance μ, show that $\sum_{i=1}^{n} X_i^2$ is minimial sufficient for μ.

(d - 3 pts) If $X_1, \ldots, X_n$ is a random sample from a Normal distribution with mean μ and variance μ^2, show that $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$ is minimal sufficient for μ.

3. (10 points) Suppose that $X_1, \ldots, X_n$ are independent identically distributed random variables with probability mass function

$$P(X=x) = \begin{cases} p(1-p)^x & \text{if } x \text{ is a nonnegative integer} \\ 0 & \text{otherwise} \end{cases}.$$

(a - 1 pt) Show that $G_1 = \begin{cases} 1 & \text{if } X_1 > 1 \\ 0 & \text{if } X_1 \le 1 \end{cases}$ is an unbiased estimator of $(1-p)^2$.

(b - 4 pts) Show that $\sum_{i=1}^{n} X_i$ is a sufficient statistic for p.

(c - 4 pts) Compute $P\left(X_1 = x \,\middle|\, \sum_{i=1}^{n} X_i = t\right)$.

(d - 1 pt) Find an unbiased estimator of $(1-p)^2$ which is uniformly better than G_1. Justify your answer.