My Solutions to Old Quals

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Let $(G, \mathring{u})$ be a group and $Z = Z(G)$ its center. Suppose that the group G/Z is cyclic. Show that G is Abelian.

My Solution

Let G/Z be cyclic with generator xZ. Every element in G/Z can be written as $x^k Z$ where $k \in \mathbb{Z}$ and $z \in Z$. Now let $g, h \in G$. Then

$$g = x^a z$$

and

$$h = x^b w$$

for $a, b \in \mathbb{Z}$ and $z, w \in Z$.

Thus

$$gh = x^a z x^b w = x^{a+b} zw = x^{b+a} wz = x^b w x^a z = hg$$

because $z, w \in Z$. THus $gh = hg$ which implies that G is Abelian. QED

Prove that every prime ideal is maximal in a PID.

My Solution

Let P be a prime ideal in the PID R. So $ab \in P$ iff a or $b \in P$. But R is a PID, so $P = (a)$ where $a \in R$. So we want to show P is maximal using this information, ie. if I is an ideal that contains P, then $I = P$ or $I = R$. So let I be an ideal containing P such that $P \subset I \subset R$. Let $I = (b)$ where $B \in R$ because R is a PID. Now, $a \in (a) \subset (b)$ which implies that there exists $c \in R$ such that $a = bc$. Since $a = bc \in P$, we know either b or c is in P because it's prime.

If $b \in P$, then $I = (b) \subset P$ which implies $P = I$. On the other hand, if $c \in P = (a)$, then there exists $d \in R$ such that $c = ad$. So

$$a = bc = bad$$

Because R is an integral domain (because it'e a PID), we can reduce the above formula to

$$1 = bd$$

Thus b is a unit which implies that $I = (b) = R$.

Thus either $P = I$ or $I = R$ for all I containing P, so P is maximal in R. QED