1. Introduction to Mathematical Models in the Physical Sciences

Science attempts to establish an understanding of all types of phenomena. Many different explanations can sometimes be given that agree qualitatively with experiments or observations. However, when theory and experiment quantitatively agree, then we can usually be more confident in the validity of the theory. In this manner mathematics becomes an integral part of the scientific method.

Applied mathematics can be said to involve three steps:

1. the formulation of a problem—the approximations and assumptions, based on experiments or observations, that are necessary to develop, simplify, and understand the mathematical model;
2. the solving of realistic problems (including relevant computations);
3. the interpretation of the mathematical results in the context of the nonmathematical problem.

In this text, we will attempt to give equal emphasis to all three aspects.

One cannot underestimate the importance of good experiments in developing mathematical models. However, mathematical models are important in their own right, aside from an attempt to mimic nature. This occurs because the real world consists of many interacting processes. It may be impossible in an experiment to entirely eliminate certain undesirable effects. Furthermore one is never sure which effects may be negligible in nature. A mathematical model has an advantage in that we are able to consider only certain effects, the object being to see which effects account for given observations and which effects are immaterial.

The process of applying mathematics never ends. As new experiments or observations are made, the mathematical model is continually revised and improved. To illustrate this we first study some problems from physics involving mechanical vibrations.

A spring-mass system is analyzed, simplified by many approximations including linearization (Secs. 2–9). Experimental observations necessitate the consideration of frictional forces (Secs. 10–13). A pendulum is then analyzed (Secs. 14–16) since its properties are similar to those of a spring-mass system. The nonlinear frictionless pendulum and spring-mass systems are briefly studied, stressing the concepts of equilibrium and stability (Secs. 17–18).

before energy principles and phase plane analysis are used (Secs. 19–20). Examples of nonlinear frictionless oscillators are worked out in detail (Secs. 21–25). Nonlinear systems which are damped are then discussed (Secs. 26–28). Mathematical models of increasing difficulty are formulated; we proceed in the following manner:

1. linear systems (frictionless).
2. linear systems with friction.
3. nonlinear systems (frictionless).
4. nonlinear systems with friction.

2. Newton's Law

To begin our investigations of mathematical models, a problem with which most of you are somewhat familiar will be considered. We will discuss the motion of a mass attached to a spring as shown in Fig. 2-1:

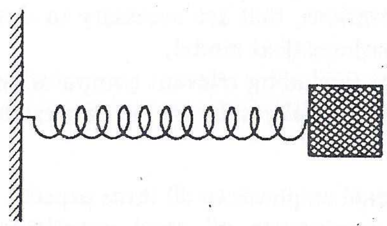

Figure 2-1 Spring-mass system.

Observations of this kind of apparatus show that the mass, once set in motion, moves back and forth (oscillates). Although few people today have any intrinsic interest in such a spring-mass system, historically this problem played an important part in the development of physics. Furthermore, this simple spring-mass system exhibits behavior of more complex systems. For example, the oscillations of a spring-mass system resemble the motions of clock-like mechanisms and, in a sense, also aid in the understanding of the up-and-down motion of the ocean surface.

Physical problems cannot be analyzed by mathematics alone. This should be the first fundamental principle of an applied mathematician (although apparently some mathematicians would frequently wish it were not so). A spring-mass system cannot be solved without formulating an equation which describes its motion. Fortunately many experimental observations culminated in **Newton's second law of motion** describing how a particle reacts to a force. Newton discovered that the motion of a point mass is well described by the now famous formula

$$\vec{F} = \frac{d}{dt}(m\vec{v}), \qquad (2.1)$$

where $\vec{F}$ is the vector sum of all forces applied to a point mass of mass m. The forces $\vec{F}$ equal the rate of change of the **momentum** $m\vec{v}$, where $\vec{v}$ is the velocity of the mass and $\vec{x}$ its position:

$$\vec{v} = \frac{d\vec{x}}{dt}. \tag{2.2}$$

If the mass is constant (which we assume throughout this text), then

$$\boxed{\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a},} \tag{2.3}$$

where $\vec{a}$ is the vector acceleration of the mass

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2}. \tag{2.4}$$

Newton's second law of motion (often referred to as just **Newton's law**), equation 2.3, states that the force on a particle equals its mass times its acceleration, easily remembered as "*F* equals *ma*." The resulting acceleration of a point mass is proportional to the total force acting on the mass.

At least two assumptions are necessary for the validity of Newton's law. There are no point masses in nature. Thus, this formula is valid only to the extent in which the finite size of a mass can be ignored.* For our purposes, we will be satisfied with discussing only point masses. A second approximation has its origins in work by twentieth century physicists in which Newton's law is shown to be invalid as the velocities involved approach the speed of light. However, as long as the velocity of a mass is significantly less than the speed of light, Newton's law remains a good *approximation*. We emphasize the word approximation, for although mathematics is frequently treated as a science of exactness, mathematics is applied to models which only approximate the real world.

EXERCISES

2.1. Consider Fig. 2-2, which shows two masses (m_1 and m_2) attached to the opposite ends of a rigid (and massless) bar:

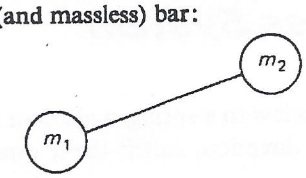

Figure 2-2.

*Newton's second law can be applied to finite sized rigid bodies if $\vec{x}$, the position of the point mass, is replaced by $\vec{x}_{cm}$, the position of the center of mass (see exercise 2.1).

m_1 is located at $\vec{x}_1$ and m_2 is located at $\vec{x}_2$. The bar is free to move and rotate due to imposed forces. The bar applies a force $\vec{F}_1$ to mass m_1 and also a force $\vec{F}_2$ to m_2 as seen in Fig. 2-3:

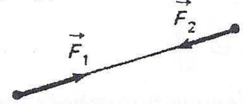

Figure 2-3.

Newton's third law of motion, stating that the forces of action and reaction are equal and opposite, implies that $\vec{F}_2 = -\vec{F}_1$.

(a) Suppose that an external force $\vec{G}_1$ is applied to m_1, and $\vec{G}_2$ to m_2. By applying Newton's second law to each mass, show the law can be applied to the rigid body consisting of both masses, if $\vec{x}$ is replaced by the center of mass $\vec{x}_{cm}$ [i.e., show $m(d^2\vec{x}_{cm}/dt^2) = \vec{F}$, where m is the total mass, $m = m_1 + m_2$, $\vec{x}_{cm}$ is the center of mass, $\vec{x}_{cm} = (m_1\vec{x}_1 + m_2\vec{x}_2)/(m_1 + m_2)$, and $\vec{F}$ is the sum of forces applied, $\vec{F} = \vec{G}_1 + \vec{G}_2$]. The motion of the center of mass of the rigid body is thus determined. However, its rotation remains unknown.

(b) Show that $\vec{x}_{cm}$ lies at a point on the rigid bar connecting m_1 to m_2.

2.2. Generalize the result of exercise 2.1 to a rigid body consisting of N masses.

2.3. Figure 2-4 shows a rigid bar of length L:

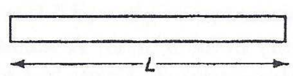

Figure 2-4.

(a) If the mass density $\rho(x)$ (mass per unit length) depends on the position along the bar, then what is the total mass m?

(b) Using the result of exercise 2.2, where is the center of mass $\vec{x}_{cm}$? [Hint: Divide the bar up into N equal pieces and take the limit as $N \longrightarrow \infty$.]

(c) If the total force on the mass is $\vec{F}$, show that

$$m\frac{d^2\vec{x}_{cm}}{dt^2} = \vec{F}.$$

3. Newton's Law as Applied to a Spring-Mass System

We will attempt to apply Newton's law to a spring-mass system. It is assumed that the mass moves only in one direction, call it the x direction, in which case the mass is governed by

$$\boxed{m\frac{d^2x}{dt^2} = F} \qquad (3.1)$$

If there were no forces F, the mass could move only at a constant velocity. (This statement, known as Newton's first law, is easily verifiable—see exercise 3.1.) Thus the observed variability of the velocity must be due to forces probably exerted by the spring. To develop an appropriate model of the spring force, one should study the motions of spring-mass systems under different circumstances. Let us suppose a series of experiments were run in an attempt to measure the spring force. At some position the mass could be placed and it would not move; there the spring exerts no force on the mass. This place at which we center our coordinate axis, as we see in Fig. 3-1, $x = 0$, is called the **equilibrium** or unstretched position of the spring*:

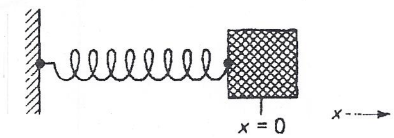

Figure 3-1 Equilibrium: no force exerted by the spring.

The distance x is then referred to as the displacement from equilibrium or the amount of stretching of the spring. If we stretch the spring (that is let $x > 0$), then the spring exerts a force pulling the mass back towards the equilibrium position (that is $F < 0$). Similarly, if the spring is contracted ($x < 0$), then the spring pushes the mass again towards the equilibrium position ($F > 0$). Such a force is called a **restoring** force. Furthermore, we would observe that as we increase the stretching of the spring, the force exerted by the spring would increase. Thus we might obtain the results shown in Fig. 3-2, where a curve is smoothly drawn connecting the experimental data points marked with an "x":

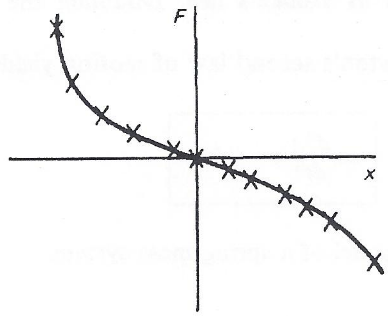

Figure 3-2 Experimental spring force.

We have assumed that the force only depends on the amount of stretching of the spring; the force does not depend on any other quantities. Thus, for example, the force is assumed to be the same no matter what speed the mass is moving at.

*Throughout this text, we assume that the width of the mass is negligible.

A careful examination of the experimental data shows that the force depends, in a complex manner, on the stretching. However, for stretching of the spring which is not too large (corresponding to at most a moderate force), Fig. 3-3 shows that this curve can be approximated by a straight line:

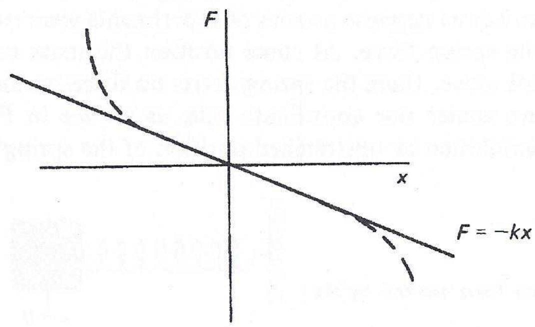

Figure 3-3 Hooke's Law: approximation of experimental spring force.

Thus

$$\boxed{F = -kx} \tag{3.2}$$

is a good approximation for the spring-force as long as the mass is not very far from its equilibrium position. k is called the spring constant. It depends on the elasticity of the spring. This linear relationship between the force and the position of the mass was discovered by the seventeenth century physicist Hooke and is thus known as **Hooke's law**. Doubling the displacement, doubles the force.

Using Hooke's law, Newton's second law of motion yields

$$\boxed{m\frac{d^2x}{dt^2} = -kx,} \tag{3.3}$$

the simplest mathematical model of a spring-mass system.

EXERCISES

3.1. Newton's first law states that with no external forces a mass will move along a straight line at constant velocity.

(a) If the motion is only in the x direction, then using Newton's second law ($m(d^2x/dt^2) = F$), prove Newton's first law.

(b) If the motion is in three dimensions, then using Newton's second law ($m(d^2\vec{x}/dt^2) = \vec{F}$), prove Newton's first law. (Why is the motion in a straight line?)

3.2. If a spring is permanently deformed due to a large force, then do our assumptions fail?

4. Gravity

In Sec. 3, we showed that the differential equation

$$\boxed{m\frac{d^2x}{dt^2} = -kx} \tag{4.1}$$

describes the motion of a spring-mass system. Some of you may object to this model, since you may find it difficult to imagine a horizontally oscillating spring-mass system such as that shown in Fig. 4-1:

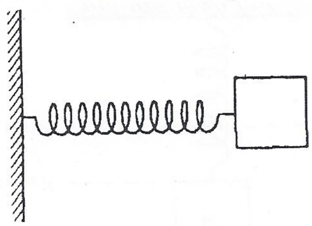

Figure 4-1.

It may seem more reasonable to consider a vertical spring-mass system as illustrated in Fig. 4-2:

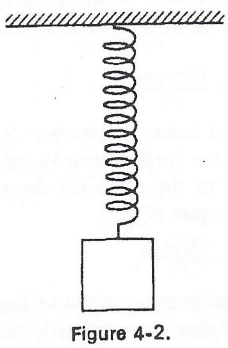

Figure 4-2.

The derivation of the equation governing a horizontal spring-mass system does not apply to the vertical system. There is another force—gravity. We approximate the gravitational force as a constant* $-mg$, the mass m times the acceleration due to gravity $-g$. The two forces add vectorially and hence Newton's law becomes

$$m\frac{d^2y}{dt^2} = -ky - mg, \tag{4.2}$$

where y is the vertical coordinate. $y = 0$ is the position at which the spring exerts no force.

Is there a position at which we could place the mass and it would not move, what we have called an *equilibrium position*? If there is, then it follows that $dy/dt = d^2y/dt^2 = 0$, and the two forces must balance:

$$0 = -ky - mg.$$

Thus we see

$$y = -\frac{m}{k}g$$

is the equilibrium position of this spring-mass-gravity system (represented by Fig. 4-3), not $y = 0$:

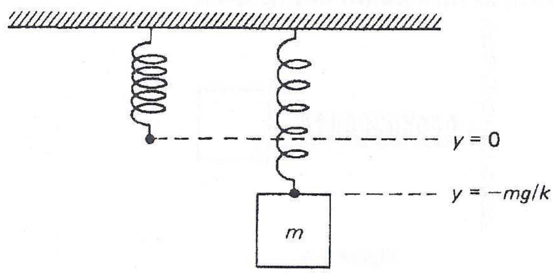

Figure 4-3 Gravitational effect on spring-mass equilibrium.

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*Actually, a gravitational force of attraction $\vec{F}$ exists directed between any two point masses m_1 and m_2. Its magnitude is inversely proportional to the square of the distance between them, r,

$$|\vec{F}| = \frac{Gm_1m_2}{r^2},$$

the so-called **inverse-square law**, where G is a universal constant determined experimentally. If the earth is spherically symmetric, then the force due to the earth's mass acting on any point mass is directed towards the center of the earth (or downwards). Thus the radial component of the gravitational force on a mass m is

$$F = \frac{-GmM}{r^2},$$

where M is the mass of the earth. If the displacement of the spring is small as compared to the radius of the earth r_0 (not a very restrictive assumption!), then the gravitational force

Only at that position will the force due to gravity balance the upward force of the spring. The spring sags downwards a distance mg/k when the mass is added, a result that should not be surprising. For a larger mass, the spring sags more. The stiffer the spring (k larger), the smaller the sag of the spring (also quite reasonable).

It is frequently advantageous to translate coordinate systems from one with an origin at $y = 0$ (the position of the unstretched spring) to one with an origin at $y = -mg/k$ (the equilibrium position with the mass). Let Z equal the **displacement** from this equilibrium position:

$$Z = y - \left(-\frac{mg}{k}\right) = y + \frac{mg}{k}.$$

Upon this substitution, equation 4.2 becomes

$$m\frac{d^2Z}{dt^2} = -kZ.$$

This is the same as equation 4.1. Thus the mass will move vertically around the new vertical equilibrium position in the same manner as the mass would move horizontally around its horizontal equilibrium position. For this reason we may continue to study the horizontal spring-mass system even though vertical systems are more commonplace.

EXERCISES

4.1. A mass m is thrown upward with initial speed v_0. Assume that gravity is constant. How high does the mass go before it begins to fall? Does this height depend in a reasonable way on m, v_0, and g?

4.2. A mass m is rolled off a table (at height h above the floor) with horizontal speed v_0. Where does the mass land? What trajectory did the mass take?

4.3. A mass m is thrown with initial speed v_0 at an angle θ with respect to the horizon. Where does the mass land? What trajectory did the mass take? For what angle does the mass land the farthest away from where it was thrown (assuming the same initial speed)?

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can be approximated by $-GmM/r_0^2$, a constant. Thus the universal constant G is related to g by

$$g = \frac{GM}{r_0^2}.$$

The rotation of the earth only causes very small modifications of this result. In addition, since the earth is not spherically symmetric, there are local variations to this formula. Furthermore, inhomogeneities in the earth's internal structure cause measurable variations, (which are useful in mineral and oil exploration).

5. Oscillation of a Spring-Mass System

We now proceed to analyze the differential equation describing a spring-mass system,

$$\boxed{m\frac{d^2x}{dt^2} = -kx.} \tag{5.1}$$

The restoring force is proportional to the stretching of the spring. Although this equation has been derived using many approximations and assumptions, it is hoped that the understanding of its solution will aid in more exact investigations (some of which we will pursue). Equation 5.1 is a second-order linear differential equation with constant coefficients. As you should recall from a course in differential equations, the general solution of this differential equation is

$$x = c_1 \cos \omega t + c_2 \sin \omega t, \tag{5.2}$$

where

$$\omega^2 = \frac{k}{m}$$

and where c_1 and c_2 are arbitrary constants. However, for those readers who did not recognize that equation 5.2 is the general solution of equation 5.1, a brief review of the standard technique to solve constant coefficient linear differential equations is given. The general solution of a second-order linear homogeneous differential equation is a linear combination of two homogeneous solutions. For constant coefficient differential equations, the homogeneous solutions are usually in the form of simple exponentials, e^{rt}. The specific exponential(s) are obtained by directly substituting the assumed form e^{rt} into the differential equation.

If e^{rt} is substituted into equation 5.1, then a quadratic equation for r results,

$$mr^2 = -k.$$

The two roots are imaginary,

$$r = \pm i\omega$$

where $\omega = \sqrt{k/m}$. Thus the general solution is a linear combination of $e^{i\omega t}$ and $e^{-i\omega t}$,

$$x = ae^{i\omega t} + be^{-i\omega t}, \tag{5.3}$$

where a and b are arbitrary constants. However, the above solution involves the exponential function of an imaginary argument. The displacement x must be real. To show how equation 5.3 can be expressed in terms of real functions, we must recall that

$$\boxed{e^{i\omega t} = \cos \omega t + i \sin \omega t} \tag{5.4a}$$

as is derived in exercise 5.6 using the Taylor series of sines, cosines, and exponentials. A similar expression for $e^{-i\omega t}$, can be derived from equation 5.4a by replacing ω by $-\omega$. This results in

$$\boxed{e^{-i\omega t} = \cos \omega t - i \sin \omega t} \tag{5.4b}$$

where the evenness of the cosine function [$\cos(-y) = \cos y$] and the oddness of the sine function [$\sin(-y) = -\sin y$] has been used. Equations 5.4a and 5.4b are called *Euler's formulas*, which when applied to equation 5.3 yield

$$x = (a + b) \cos \omega t + i(a - b) \sin \omega t.$$

The desired result

$$x = c_1 \cos \omega t + c_2 \sin \omega t$$

follows, if the constants c_1 and c_2 are defined by

$$c_1 = a + b$$
$$c_2 = i(a - b).$$

The constants c_1 and c_2 are arbitrary since given any value of c_1 and c_2, there exists values of a and b, namely

$$a = \tfrac{1}{2}(c_1 - ic_2)$$
$$b = \tfrac{1}{2}(c_1 + ic_2).$$

Since the algebra is a bit involved, it is useful to *memorize* the result we have just derived:

> An arbitrary linear combination of $e^{i\omega t}$ and $e^{-i\omega t}$,
>
> $$x = ae^{i\omega t} + be^{-i\omega t},$$
>
> is equivalent to an arbitrary linear combination of $\cos \omega t$ and $\sin \omega t$,
>
> $$x = c_1 \cos \omega t + c_2 \sin \omega t.$$

In the above manner you should now be able to state without any hesitation that the general solution of

$$m\frac{d^2x}{dt^2} = -kx$$

is

$$x = c_1 \cos \omega t + c_2 \sin \omega t,$$

where $\omega = \sqrt{k/m}$ and c_1 and c_2 are arbitrary constants. The general solution is a linear combination of two oscillatory functions, a cosine and a sine. An *equivalent* expression for the solution is

$$\boxed{x = A \sin (\omega t + \phi_0).} \tag{5.5}$$

This is shown by noting

$$\sin (\omega t + \phi_0) = \sin \omega t \cos \phi_0 + \cos \omega t \sin \phi_0,$$

in which case

$$c_1 = A \sin \phi_0$$
$$c_2 = A \cos \phi_0.$$

If you are given c_1 and c_2, it is seen that both A and ϕ_0 can be determined. Dividing the two equations yields an expression for $\tan \phi_0$, and using $\sin^2 \phi_0 + \cos^2 \phi_0 = 1$ results in an equation for A^2:

$$A = (c_1^2 + c_2^2)^{1/2}$$

$$\phi_0 = \tan^{-1} \frac{c_1}{c_2}.$$

The expression, $x = A \sin (\omega t + \phi_0)$, is especially convenient for sketching the displacement as a function of time. It shows that the sum of any multiple of $\cos \omega t$ plus any multiple of $\sin \omega t$ is itself a sinusoidal function as sketched in Fig. 5-1:

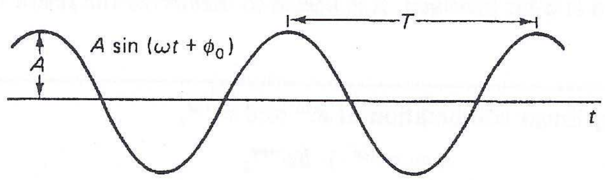

Figure 5-1 Period and amplitude of oscillation.

A is called the **amplitude** of the oscillation; it is easily computed from the above equation if c_1 and c_2 are known. The **phase** of oscillation is $\omega t + \phi_0$,

ϕ_0 being the phase at $t = 0$. In many situations, this agrees with the observed motion of a spring-mass system.

This motion is referred to as **simple harmonic motion.** The mass oscillates sinusoidally around the equilibrium position $x = 0$. The solution is periodic in time. As illustrated in Fig. 5-1, the mass after reaching its maximum displacement (x largest), again returns to the same position T units of time later. The entire oscillation repeats itself every T units of time, called the **period** of oscillation. Mathematically a function $f(t)$ is said to be periodic with period T if

$$f(t + T) = f(t).$$

To determine the period T, we recall that the trigonometric functions are periodic with period 2π. Thus for a complete oscillation, as t increases to $t + T$, from equation 5.5 $\omega t + \phi_0$ must change by 2π:

$$\omega(t + T) + \phi_0 - \omega t - \phi_0 = 2\pi.$$

Consequently the period T is

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.} \tag{5.6}$$

ω, called the **circular frequency** (as is explained in exercise 5.7), is the number of periods in 2π units of time:

$$\boxed{\omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}.} \tag{5.7}$$

The number of oscillations in one unit of time is the **frequency** f,

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}},$$

measured in cycles per second (sometimes known as a Hertz). Since a spring-mass system normally oscillates with frequency $1/2\pi\sqrt{k/m}$, this value is referred to as the *natural frequency* of a spring-mass system of mass m and spring constant k. Other physical systems have natural frequencies of oscillation. Perhaps in a subsequent course you will determine the natural frequencies of oscillation of a vibrating string or a vibrating drum head!

EXERCISES

5.1. Sketch $x = 2 \sin (3t - \pi/2)$.

5.2. If $x = -\cos t + 3 \sin t$, what is the amplitude and phase of the oscillation? Sketch this function.

5.3. If $x = -\cos t + 3 \sin (t - \pi/6)$, what is the amplitude of the oscillation?

5.4. If $x = -\sin 2t$, what is the frequency, circular frequency, period, and amplitude of the oscillation? Sketch this function.

5.5. It was shown that $x = ae^{i\omega t} + be^{-i\omega t}$ is equivalent to $x = c_1 \cos \omega t + c_2 \sin \omega t$. Show that if c_1 and c_2 are real (that is if x is real), then b is the complex conjugate of a.

5.6. The Taylor series for $\sin x$, $\cos x$, and e^x are well known for real x:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

The above Taylor expansions are also valid for complex x.
(a) Show that $e^{i\omega t} = \cos \omega t + i \sin \omega t$, for ω real.
(b) Show that $e^{-i\omega t} = \cos \omega t - i \sin \omega t$, for ω real.

5.7. Consider a particle moving around a circle, with its position designated by the polar angle θ. Assume its **angular velocity** $d\theta/dt$ is constant, $d\theta/dt = \omega$. Show that the x component of the particle's position executes simple harmonic motion (and also the y component). ω is measured in radians per unit of time or revolutions per 2π units of time, the **circular frequency.**

5.8. (a) Show that $x = c_1 \cos \omega t + c_2 \sin \omega t$ is the general solution of $m(d^2x/dt^2) = -kx$. What is the value of ω?
(b) Show that an equivalent expression for the general solution is $x = B \cos (\omega t + \theta_0)$. How do B and θ_0 depend on c_1 and c_2?

6. Dimensions and Units

In the previous section, the formula for the circular frequency of a simple spring-mass system was derived,

$$\omega = \sqrt{\frac{k}{m}}.$$

As a check on our calculations we claim that the dimensions of both sides of this equation agree. Checking formulas by dimensional analysis is an important general procedure you should follow. Frequently this type of check will detect embarrassing algebraic errors.

In dimensional analysis, brackets indicate the dimension of a quantity. For example, the notation [x] designates the dimension of x, which is a length

L, i.e., $[x] = L$, measured in units of feet, inches, miles, meters, or smoots.* In any calculation to eliminate possible confusion only one unit of length should be used. In this text we will use metric units in the *m–k–s* system, i.e., *m*eters for length, *k*ilogram for mass, and *s*econds for time. However, as an aid in conversion to those familiar with the British–American system the equivalent length in feet or miles and the equivalent mass in pounds will appear afterwards in parentheses.

What is the dimension of dx/dt, the velocity? Clearly,

$$\left[\frac{dx}{dt}\right] = \frac{L}{\tau},$$

a length L divided by a time τ. Mathematically we note that

$$\left[\frac{dx}{dt}\right] = [x]/[t].$$

The dimensions of a derivative of any quantity is always the ratio of the dimension of that quantity divided by the dimension of the variable which we are differentiating with respect to. This is shown in general directly from the definition of a derivative,

$$\frac{dy}{dz} = \lim_{\Delta z \to 0} \frac{y(z + \Delta z) - y(z)}{\Delta z}.$$

The dimensions of the two sides must agree. The right-hand side is the difference between two values of y (hence having the dimension of y) divided by a small value of z (having the dimension of z). Thus,

$$\left[\frac{dy}{dz}\right] = [y]/[z].$$

This result can be used to determine the dimensions of an acceleration:

$$\left[\frac{d^2x}{dt^2}\right] = \frac{[x]}{[t]^2} = \frac{L}{\tau^2}.$$

Note that

$$\left[\frac{d^2x}{dt^2}\right] \neq \frac{L^2}{\tau^2}.$$

This is obvious from a physical point of view. However, this result is shown below (since sometimes the dimension of a quantity might not be as obvious

---

*Units are chosen to facilitate communication and understanding of the magnitudes of quantities. Meters (and the other metric units) are familiar to all of the world except the general public in the United States, who seem reluctant to change. At the other extreme, a smoot is a unit of length only used to measure the distance across the Charles River in Boston on the frequently walked Harvard Bridge. Local folklore says that this unit was the length of a slightly inebriated student as he was rolled across the bridge by some "friends".

as it is in this problem):

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right).$$

Thus,

$$\left[\frac{d^2x}{dt^2}\right] = \frac{\left[\frac{dx}{dt}\right]}{[t]} = \frac{L/\tau}{\tau} = \frac{L}{\tau^2}.$$

Since g is an acceleration, the units will be a length per unit of time squared. As discussed in Sec. 4, g is only an approximation, the value we use is $g = 9.8$ meters/sec^2 (32 feet/sec^2). Everywhere at the surface of the earth, the gravitational acceleration is within 1 percent of this value.

What is the dimension of k? Since $F = -kx$, then

$$[k] = \frac{[F]}{[x]} = \frac{\left[m\frac{d^2x}{dt^2}\right]}{[x]} = \frac{ML/\tau^2}{L} = \frac{M}{\tau^2},$$

where M is a unit of mass. In this way ω can be shown to have the same dimension as $\sqrt{k/m}$ (see exercise 6.1).

EXERCISES

6.1. (a) What is the dimension of ω?
(b) Show that ω has the same dimensions as $\sqrt{k/m}$.
You will probably need to note that a radian has no dimension. The formula $(d/d\theta) \sin\theta = \cos\theta$ shows this to be true.

6.2 Suppose a quantity y having dimensions of time is *only* a function of k and m.
(a) Give an example of a possible dependence of y on k and m.
(b) Can you describe the most general dependence that y can have on k and m?

7. Qualitative and Quantitative Behavior of a Spring-Mass System

To understand the predictions of the mathematical model of a spring-mass system, the effect of varying the different parameters is investigated. An important formula is the one derived for the period of oscillation,

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{7.1}$$

Suppose that we use a firmer spring, that is one whose spring constant k is larger, with the same mass. Without relying on the mathematical formula, what differences in the motion should occur? Let us compare two different springs represented in Fig. 7-1:

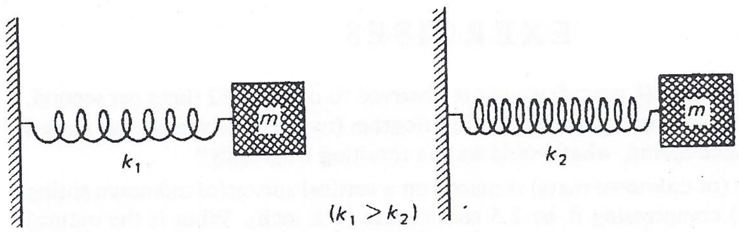

Figure 7-1.

The one which is firmer has a larger restoring force and hence it returns more quickly to its equilibrium position. Thus we suspect that the larger k is, the shorter the period. Equation 7.1 also predicts this qualitative feature. On the other hand, if the mass is increased using the same spring, then the formula shows that the period increases. The system oscillates more slowly (is this reasonable?).

In any problem we should compare as much as possible our intuition about what should happen with what the formula predicts. If the two agree, then we expect that our formula gives us the quantitative effects for the given problem—one of the major purposes for using mathematics. In particular for a spring-mass system, we might have suspected without using any mathematics that increasing the mass increases the period, but it is doubtful that we could have known that quadrupling the weight results in an increase in the period by a factor of two!

In mathematical models, usually the qualitative effects are at least partially understood. Quantitative results are often unknown. When quantitative results are known (perhaps due to precise experiments), then mathematical models are desirable in order to discover which mechanisms best account for the known data, i.e., which quantities are important and which can be ignored. In complex problems sometimes two or more effects interact. Although each by itself is qualitatively and quantitatively understood, their interaction may need mathematical analysis in order to be understood even qualitatively.

If our intuition about a problem does not correspond to what a mathematical formula predicts, then further investigations of the problem are necessary. Perhaps the intuition is incorrect, in which case the mathematical formulation and solution has aided in directly improving one's qualitative understanding. On the other hand, it may occur that the intuition is correct and consequently that either there was a mathematical error in the derivation of the formula or the model upon which the analysis is based may need improvement.

EXERCISES

7.1. A .227 kilogram ($\frac{1}{2}$ pound) weight is observed to oscillate 12 times per second. What is the spring constant? If a .91 kilogram (two pound) weight was placed on the same spring, what would be the resulting frequency?

7.2. A weight (of unknown mass) is placed on a vertical spring (of unknown spring constant) compressing it by 2.5 centimeters (one inch). What is the natural frequency of oscillation of this spring-mass system?

8. *Initial Value Problem*

In the previous sections, we have shown that

$$x = c_1 \cos \omega t + c_2 \sin \omega t \tag{8.1}$$

is the general solution of the differential equation describing a spring-mass system,

$$m \frac{d^2x}{dt^2} = -kx, \tag{8.2}$$

where c_1 and c_2 are arbitrary constants and $\omega = \sqrt{k/m}$. The constants c_1 and c_2 will be determined from the initial conditions of the spring-mass system.

One way to initiate motion in a spring-mass system is to strike the mass. A simpler method is to pull (or push) the mass to some position (say x_0) and then let go. Mathematically we wish to determine the solution of equation 8.2 which satisfies the initial conditions that the mass is at x_0 at $t = 0$,

$$x(0) = x_0,$$

and at $t = 0$, the velocity of the mass, dx/dt, is zero,

$$\frac{dx}{dt}(0) = 0.$$

In the above example of an initial value problem the mass is *initially at rest.* Two initial conditions are necessary since the differential equation involves the second derivative in time. To solve this initial value problem, the arbitrary constants c_1 and c_2 are determined so that equation 8.1 satisfies the initial conditions. For example, at $t = 0$, $x = x_0$. Thus

$$c_1 = x_0.$$

In general the velocity is obtained by differentiating the formula for displacement:

$$\frac{dx}{dt} = -c_1\omega \sin \omega t + c_2\omega \cos \omega t.$$

In this problem the mass is at rest at $t = 0$. Setting the initial velocity equal to zero yields

$$c_2 = 0.$$

In this manner

$$x = x_0 \cos \omega t$$

satisfies this initial value problem. The mass executes **simple harmonic motion,** as shown in Fig. 8-1. The spring is initially stretched and hence the spring pulls the mass to the left. This process continues indefinitely.

Other initial value problems are posed in the exercises. It should be noted that the form of the general solution

$$x = c_1 \cos \omega t + c_2 \sin \omega t$$

is more suited for the initial value problem than

$$x = A \sin (\omega t + \phi_0),$$

although the latter, as previously mentioned, is often best for simple sketches.

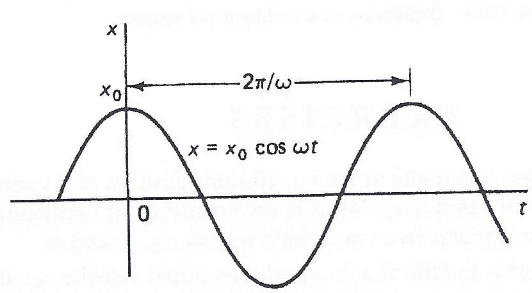

Figure 8-1a Oscillation of a spring-mass system.

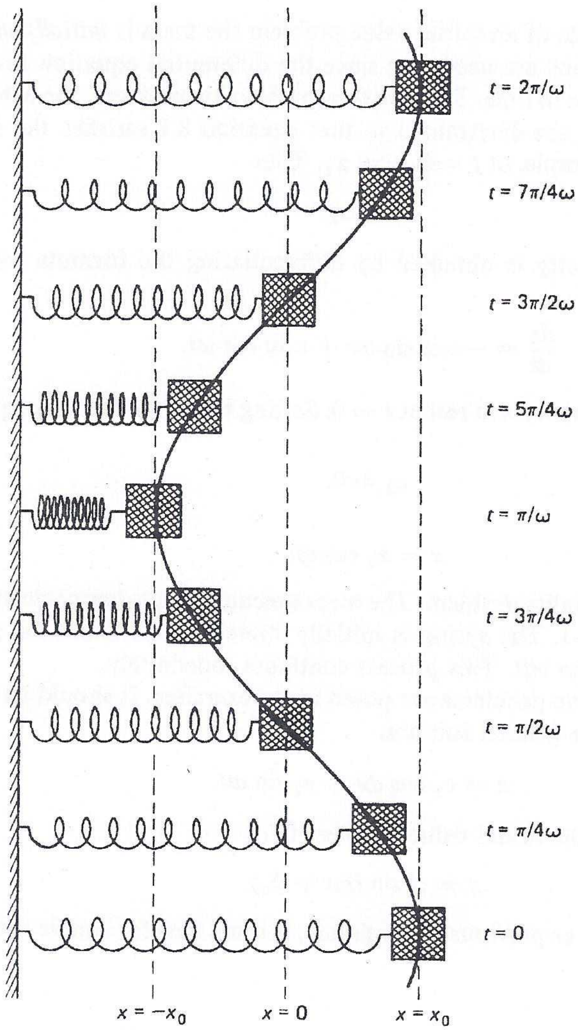

Figure 8-1b Oscillation of a spring-mass system.

EXERCISES

8.1. Suppose that a mass is initially at the equilibrium position of a spring, but is initially moving with velocity v_0. What is the amplitude of oscillation? Show that the amplitude depends in a reasonable way on v_0, k, and m.

8.2. Suppose that a mass is intially at $x = x_0$ with an initial velocity v_0. Show that the resulting motion is the sum of two oscillations, one corresponding to the mass initially at rest at $x = x_0$ and the other corresponding to the mass initially at the equilibrium position with velocity v_0. What is the amplitude of the total oscillation?

8.3. Consider a spring-mass system initially at rest with initial displacement x_0. Show that the maximum and minimum displacements occur halfway between times at which the mass passes its equilibrium position.

9. A Two-Mass Oscillator

In the previous sections we carefully developed a mathematical model of a spring-mass system. We analyzed the resulting oscillations which occur when the force can be approximated as being simply proportional to the stretching of the spring. Before discussing modifications to this model, as an example let us consider a more complicated spring-mass system. Suppose instead of attaching a spring and a mass to a rigid wall, we attach a spring and a mass to another mass which is also free to move. Let us assume that the two masses are m_1 and m_2, while the connecting spring is known to have an unstretched length l and spring constant k. We insist that the system is constrained to move only horizontally (as might occur if both masses slide along a table) as shown in Fig. 9-1:

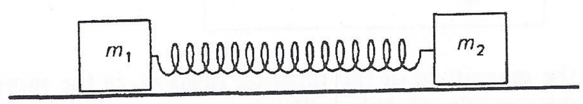

Figure 9-1.

We wish to know the manner in which the two masses, m_1 and m_2, move.

To analyze that question, we must return to fundamental principles; we must formulate Newton's law of motion for *each* mass. The force on each mass equals its mass times its acceleration. In order to obtain expressions for the accelerations, we introduce the position of each mass (for example, in Fig. 9-2, x_1 and x_2 are the distances each mass is from a fixed origin):

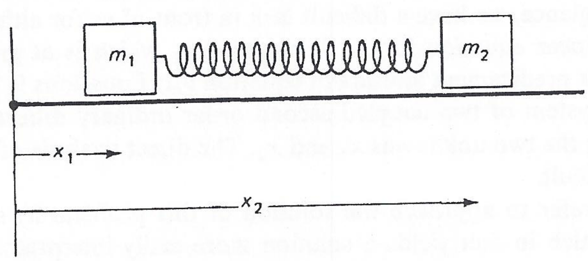

Figure 9-2.

Although the unstretched length of the spring is l, it is *not* necessary that $x_2 - x_1 = l$, for in many circumstances the spring may be stretched or compressed. Certainly, for example, we may impose initial conditions such that initially $x_2 - x_1 \neq l$. Now from Newton's law of motion, it follows that if F_1 is the force on mass m_1 and if F_2 is the force on mass m_2, then

$$m_1 \frac{d^2x_1}{dt^2} = F_1 \quad \text{and} \quad m_2 \frac{d^2x_2}{dt^2} = F_2.$$

To complete the derivation of the equations of motion, we must determine the two forces, F_1 and F_2. The only force on each mass is due to the spring. Each force is an application of Hooke's law; the force is proportional to the *stretching* of the spring (it is *not* proportional to the length of the spring). The stretching of the spring is the length of the spring $x_2 - x_1$ minus the unstretched length l: $x_2 - x_1 - l$. The magnitude of the force is just the spring constant k times the stretching, but the direction of the force is quite important. An error may be made if we are not careful. If, for example, the spring is stretched ($x_2 - x_1 - l > 0$), then the mass m_1 is being pulled to the right. The force is in the positive x_1 direction, $k(x_2 - x_1 - l)$. In any situation the force on mass m_1 is $k(x_2 - x_1 - l)$ and thus

$$\boxed{m_1 \frac{d^2x_1}{dt^2} = k(x_2 - x_1 - l).} \tag{9.1}$$

However, although the magnitude of the force on mass m_2 is the same, the spring force acts in the opposite direction. Thus

$$\boxed{m_2 \frac{d^2x_2}{dt^2} = -k(x_2 - x_1 - l).} \tag{9.2}$$

Having derived the equations describing the motion of this system of two masses coupled by a spring, we have only begun the process of applying mathematics to this physical situation. We must now proceed to solve these equations. At first glance, we have a difficult task in front of us for although equation 9.1 is a linear equation for x_1, it contains x_2 which is at present unknown! A similar predicament appears in equation 9.2. Equations 9.1 and 9.2 form a linear system of two coupled second order ordinary differential equations involving the two unknowns x_1 and x_2. The direct analysis of such a system is not difficult.

However, we prefer to approach the solution of this problem in a different way; one which in fact yields a solution more easily interpreted. By inspection we note that the equations simplify if they are added together. In that manner

$$m_1 \frac{d^2x_1}{dt^2} + m_2 \frac{d^2x_2}{dt^2} = 0. \tag{9.3}$$

The sum of the two forces vanishes (since one force is minus the other). Equation 9.3 can be re-expressed as

$$\frac{d^2}{dt^2}(m_1x_1 + m_2x_2) = 0. \tag{9.4}$$

Thus the center of mass of the system, $(m_1x_1 + m_2x_2)/(m_1 + m_2)$, (as discussed in exercise 2.1a), does not accelerate, but moves at a constant velocity (determined from initial conditions). If this system of two masses attached to a spring is viewed as a single entity, then since there are no external forces on it, the system obeys Newton's first law and will not accelerate. By saying there is no acceleration, we have shown that we mean that the center of mass of the system of two masses moves at a constant velocity.

The simple expression for the motion of the center of mass is quite interesting, but hardly aids in understanding the possibly complex behavior of each individual mass. Try subtracting equation 9.2 from equation 9.1; you will soon discover that although adding the two equations results in a significant simplification, subtracting the two is not helpful. Instead we note that the force only depends on the stretching of the spring, $x_2 - x_1 - l$. Perhaps we can directly determine a differential equation for the stretching of the spring. We see this can be accomplished by dividing each equation by the mass (i.e., equation 9.1 by m_1 and equation 9.2 by m_2) and then subtracting rather than by subtracting right away. Through this subtle trick* we discover that

$$\frac{d^2}{dt^2}(x_2 - x_1) = -\frac{k}{m_2}(x_2 - x_1 - l) - \frac{k}{m_1}(x_2 - x_1 - l).$$

Letting z be the stretching of the spring

$$z = x_2 - x_1 - l, \tag{9.5}$$

we see that

$$\boxed{\frac{d^2z}{dt^2} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)z.} \tag{9.6}$$

By this rather lucky manipulation, we see the remarkable result that the stretching of the spring executes simple harmonic motion. The circular frequency is $\sqrt{k(1/m_1 + 1/m_2)}$.

*You should note that we did not use any mathematical motivation for these steps. Instead as often occurs in applied mathematics, the solution of the mathematical equation of interest simplifies if we use insight gained from the physical problem.

We also note that this is the same type of oscillation that would result if a certain mass m were placed on the *same* spring, that is one with spring constant k but fixed at the other end. The mass m necessary for this analogy is such that

$$\boxed{\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}.} \tag{9.7}$$

This mass m is less than either m_1 or m_2 (since $1/m > 1/m_1$ and $1/m > 1/m_2$); it is thus called the **reduced mass**:

$$\boxed{m = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}} = \frac{m_1 m_2}{m_1 + m_2}.} \tag{9.8}$$

Attaching a spring-mass system to a movable mass reduces the effective mass; the stretching of the spring executes simple harmonic motion as though a smaller mass was attached. But don't forget that the entire system may move, i.e., the center of mass moves at a constant velocity!

We could discuss more complex spring-mass systems. Instead, we will return in the next sections to the study of a single mass attached to a spring.

EXERCISES

9.1. Suppose that the initial positions and initial velocities were given for a spring-mass system of the type discussed in this section:

$$x_1(0) = \alpha \qquad \frac{dx_1}{dt}(0) = \gamma$$

$$x_2(0) = \beta \qquad \frac{dx_2}{dt}(0) = \delta.$$

Determine the position of each mass at future times. [Hint: See equations 9.4, 9.5, and 9.6.]

9.2. Consider two masses (of mass m_1 and m_2) attached to a spring (of unstretched length l and spring constant k) in a manner similar to that discussed in this section. However, suppose the system is aligned vertically (rather than horizontally) and that consequently a constant gravitational force is present. Analyze this system and compare the results to those of this section.

9.3.–9.6.

In the following exercises consider two springs, each obeying Hooke's law; one spring with spring constant k_1 (and unstretched length l_1) and the other with spring constant k_2 (and unstretched length l_2) as shown in Fig. 9-3:

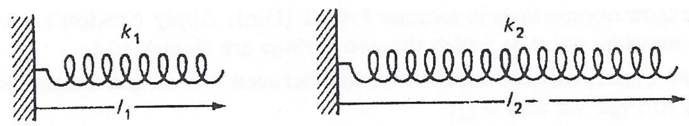

Figure 9-3.

9.3. Suppose that a mass m were attached between two walls a distance d apart (refer to Figures 9-3 and 9-4):

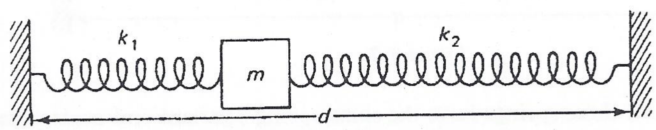

Figure 9-4.

(a) Briefly explain why it is *not* necessary for $d = l_1 + l_2$.
(b) What position of the mass would be called the equilibrium position of the mass? If both springs are identical, where should the equilibrium position be? Show that your formula is in agreement.
(c) Show that the mass executes simple harmonic motion about its equilibrium position.
(d) What is the period of oscillation?
(e) How does the period of oscillation depend on d?

9.4. Suppose that a mass m were attached to two springs in **parallel** (refer to Figs. 9-3 and 9-5):

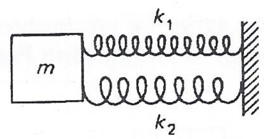

Figure 9-5.

(a) What position of the mass would be called the equilibrium position of the mass?
(b) Show that the mass executes simple harmonic motion about its equilibrium position.
(c) What is the period of oscillation?
(d) If the two springs were to be replaced by one spring, what would be the unstretched length and spring constant of the new spring such that the motion would be *equivalent*?

9.5. Suppose a mass m were attached to two springs in **series** (refer to Figs. 9-3 and 9-6):

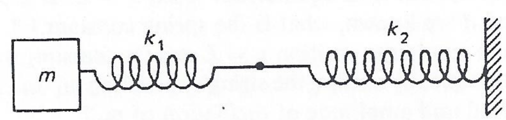

Figure 9-6.

Answer the same questions as in exercise 9.4a–d. [Hint: Apply Newton's law also to the massless point at which the two springs are connected.]

9.6. Consider two masses each of mass m attached between two walls a distance d apart (refer to Figs. 9-3 and 9-7):

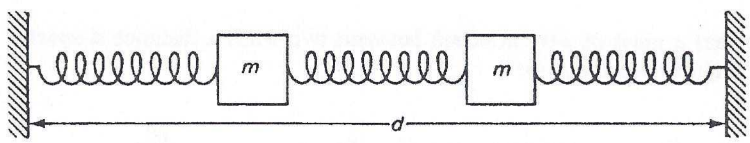

Figure 9-7.

Assume that all three springs have the same spring constant and unstretched length.

(a) Suppose that the left mass is a distance x from the left wall and the right mass a distance y from the right wall. What position of each mass would be called the equilibrium position of the system of masses?

(b) Show that the distance between the masses oscillates. What is the period of that oscillation?

(c) Show that x–y executes simple harmonic motion with a period of oscillation different from (b).

(d) If the distance between the two masses remained constant, describe the motion that could take place both qualitatively and quantitatively.

(e) If $x = y$, describe the motion that could take place both qualitatively and quantitatively.

9.7. Consider a mass m_1 attached to a spring (of unstretched length d) and pulled by a constant force F_2, $F_2 = m_2 g$, as illustrated in Fig. 9-8.

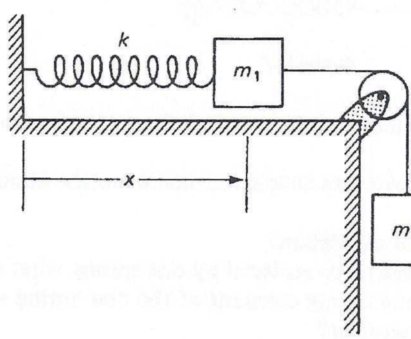

Figure 9-8.

(a) Suppose that the system is in equilibrium when $x = L$. Is $L > d$ or is $L < d$? If L and d are known, what is the spring constant k?

(b) If the system is at rest in the position $x = L$ and the mass m_2 is suddenly removed (for example, by cutting the string connecting m_1 and m_2), then what is the period and amplitude of oscillation of m_1?

10. Friction

Our mathematical model shows that the displacement of a simple spring-mass system continues to oscillate for all time. The amplitude of oscillation remains constant; the mass never stops completely nor does the amplitude even decay! Does this correspond to our experience? If we displaced the mass to the right, as shown in Fig. 10-1, then we would probably expect the mass to oscillate in the manner sketched in Fig. 10-2. We suspect that the mass oscillates around its equilibrium position with smaller and smaller magnitude until it stops.

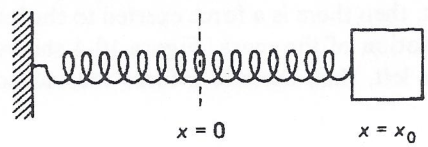

Figure 10-1.

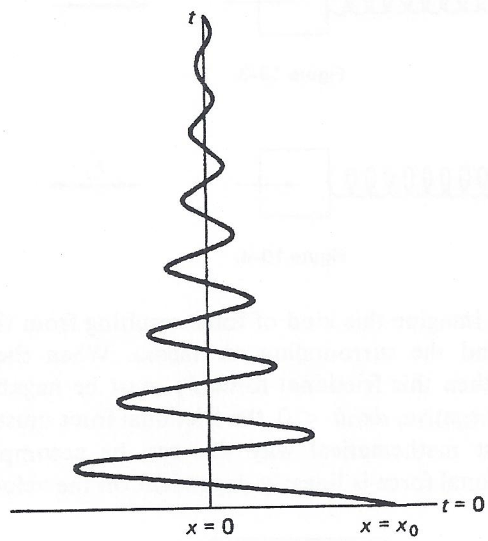

Figure 10-2.

Must we reject a mathematical model that yields perfect periodic motion? Absolutely not, for we can at least imagine a spring-mass system that exhibits *many* oscillations before it finally appears to significantly decay. In this case the mathematical model of a spring-mass system,

$$m\frac{d^2x}{dt^2} = -kx,$$

is a good approximation *for times that are not particularly long*. Furthermore, the importance of simple harmonic motion is in its aid in understanding more complicated periodic motion.

How can we improve our model to account for the experimental observation that the amplitude of the mass decays? Perhaps when the restoring force was approximated by Hooke's law, the possibility of decay was eliminated. However, in later sections we will show that this is not the case as the equation $m(d^2x/dt^2) = -f(x)$, representing any restoring force, never has oscillatory solutions that decay in time.

In order to account for the observed decay, we must include other forces. What causes the amplitude of the oscillation to diminish? Let us conjecture that there is a **resistive** force, that is, a force preventing motion. When the spring is moving to the right, then there is a force exerted to the left, as shown in Fig. 10-3, resisting the motion of the mass. Figure 10-4 shows that when the spring is moving to the left, then there is a force exerted to the right.

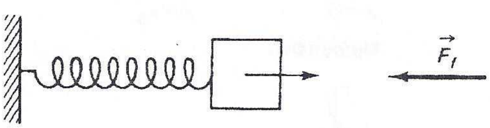

Figure 10-3.

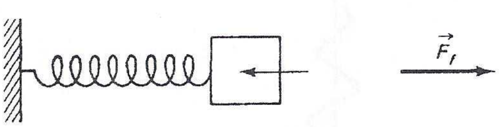

Figure 10-4.

For example, we can imagine this kind of force resulting from the "friction" between the mass and the surrounding air media. When the velocity is positive, $dx/dt > 0$, then this frictional force F_f must be negative, $F_f < 0$. When the velocity is negative, $dx/dt < 0$, the frictional force must be positive, $F_f > 0$. The simplest mathematical way this can be accomplished is to assume that the frictional force is linearly dependent on the velocity:

$$\boxed{F_f = -c\frac{dx}{dt}} \qquad \textbf{(10.1)}$$

where c is a *positive* constant ($c > 0$) referred to as the friction coefficient. This force-velocity relationship is called a **linear damping force**; a damped oscillation meaning the same as an oscillation which decays. The accuracy of this assumed form of the force-velocity relationship should be verified experimentally. Here we claim that in some situations (but certainly not all) this is a good approximate expression for the force resulting from the resistance between an object and its surrounding fluid (liquid or gas) media (especially if the velocities are not too large). Independent forces add vectorially; hence, the differential equation describing a spring-mass system with a linear damping force and a linear restoring force is

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

or equivalently

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.} \qquad \textbf{(10.2)}$$

The force corresponding to the friction between a spring-mass system and a table, illustrated in Fig. 10-5, does not act in the way previously described, $F_f \neq -c(dx/dt)$. Instead, experiments indicate that once the mass is moving

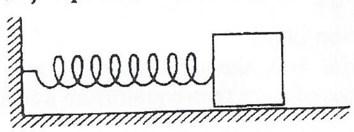

Figure 10-5.

the friction force is resistive but has a magnitude which is approximately constant independent of the velocity. We model this experimental result by stating

$$F_f = \begin{cases} \gamma & \text{for } dx/dt < 0 \\ -\gamma & \text{for } dx/dt > 0, \end{cases} \qquad \textbf{(10.3)}$$

as sketched in Fig. 10-6 (see exercise 10.4). γ depends on the roughness of the surface (and the weight of the mass). In later sections exercises will discuss the mathematical solution of problems involving this type of friction, called

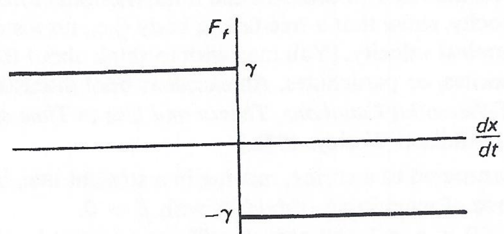

Figure 10-6 Coulomb friction.

Conlomb friction. However, in this text for the most part, we will limit our discussion to linear damping,

$$F_f = -c\frac{dx}{dt}.$$

EXERCISES

10.1. Suppose that an experimentally observed frictional force is approximated by $F_f = \alpha(dx/dt)^3$, where α is a constant.
(a) What is the sign of α?
(b) What is the dimension of α?
(c) Show that the resulting differential equation is nonlinear.

10.2. From the differential equation for a spring-mass system with linear damping, show that $x = 0$ is the only equilibrium position of the mass.

10.3. What is the dimension of the constant c defined for linear damping?

10.4. Consider Coulomb friction, equation 10.3.
(a) What is the sign of γ?
(b) What is the dimension of γ?
(c) Assume that if $dx/dt = 0$, then F_f could be any value such that $|F_f| \leq \gamma$. What values of x are then equilibrium positions of the spring-mass system?

10.5. Assume the same form of friction as in exercise 10.4. If initially $x = x_0 > 0$ and the velocity is $v_0 > 0$, then at what time does the mass of a spring-mass system first stop moving to the right? Will the mass continue to move after that time?

10.6. In some problems *both* linear damping and Coulomb friction occur. In this case, sketch the total frictional force as a function of the velocity.

10.7. In certain physical situations the damping force is proportional to the velocity squared, known as **Newtonian damping.** In this case show that

$$m\frac{d^2x}{dt^2} = -kx - \alpha\frac{dx}{dt}\left|\frac{dx}{dt}\right|,$$

where $\alpha > 0$.

10.8. If gravity is approximated by a constant and if the frictional force is proportional to the velocity, show that a free-falling body (i.e., no restoring force) approaches a **terminal** velocity. [You may wish to think about this effect for raindrops, meteorites, or parachutes. An excellent brief discussion is given by Dickinson, *Differential Equations, Theory and Use in Time and Motion*, Reading, Mass.: Addison-Wesley, 1972.]

10.9. A particle not connected to a spring, moving in a straight line, is subject to a *retardation* force of magnitude $\beta(dx/dt)^n$, with $\beta > 0$.
(a) Show that if $0 < n < 1$, the particle will come to rest in a finite time. How far will the particle travel, and when will it stop?

(b) What happens if $n = 1$?
(c) What happens if $1 < n < 2$?
(d) What happens if $n > 2$?

10.10. Consider Fig. 10-7, which shows a glass of mass m starting at $x = 0$ and sliding on a table of length H:

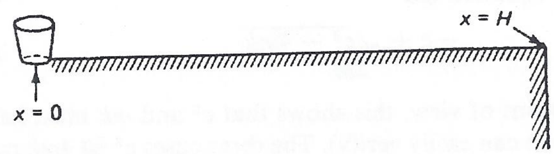

Figure 10-7.

(a) For what initial velocities v_0 will the glass fall off the table on the right if the only force is Coulomb friction, equation 10.3?
(b) Suppose that the frictional force instead is

$$F_f = \begin{cases} -\gamma - c\dfrac{dx}{dt} & \text{if } \dfrac{dx}{dt} > 0 \\ \gamma - c\dfrac{dx}{dt} & \text{if } \dfrac{dx}{dt} < 0. \end{cases}$$

Describe physically what F_f represents. Answer the same question as in part (a).
(c) Compare the results of parts (a) and (b).

11. Oscillations of a Damped System

A spring-mass system with no forces other than a spring force and friction is governed by

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.} \tag{11.1}$$

It must be verified that solutions to this equation behave in a manner consistent with our observations. If this is not true then perhaps a spring-mass system decays as a result of forces other than a linear damping force.

In order to solve a constant coefficient homogeneous ordinary differential equation recall that the two linearly independent solutions "almost always" can be written in the form of exponentials e^{rt}, where r satisfies the **characteris-**

tic equation obtained by direct substitution,

$$\boxed{mr^2 + cr + k = 0.} \tag{11.2}$$

The two roots of this equation are

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}. \tag{11.3}$$

From a dimensional point of view, this shows that c^2 and mk must have the same dimensions (as you can easily verify). The three cases $c^2 \gtreqless 4mk$ must be distinguished. A different form of the general solution corresponds to each case, since the roots are respectively real and unequal, real and equal, and complex.

EXERCISES

11.1. Show that c^2 has the same dimension as mk.

11.2. What dimension should the roots of the characteristic equation have? Verify that the roots have this dimension.

12. *Underdamped Oscillations*

If $c^2 < 4mk$, then the coefficient of friction is small; the damping force is not particularly large. We call this the **underdamped** case. In this case, the roots of the characteristic equation are complex conjugates of each other (see equation 11.3)

$$r = -\frac{c}{2m} \pm i\omega,$$

where

$$\boxed{\omega = \frac{\sqrt{4mk - c^2}}{2m} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}.}$$

Thus the general solution of the differential equation (11.1) is

$$x = ae^{[(-c/2m)+i\omega]t} + be^{[(-c/2m)-i\omega]t}.$$

By factoring $e^{-ct/2m}$, we see

$$x = e^{-ct/2m}(ae^{i\omega t} + be^{-i\omega t}).$$

Recalling that an arbitrary linear combination of $e^{i\omega t}$ and $e^{-i\omega t}$ is equivalent to an arbitrary linear combination of $\cos \omega t$ and $\sin \omega t$, we observe that the motion of a linearly damped spring-mass system in the underdamped case ($c^2 < 4mk$) is described by

$$x = e^{-ct/2m}(c_1 \cos \omega t + c_2 \sin \omega t)$$

or

$$\boxed{x = Ae^{-ct/2m} \sin (\omega t + \phi_0).} \qquad (12.1)$$

A sketch of the solution is most easily accomplished using the latter form. The solution is the product of an exponential and a sinusoidal function (each sketched in Fig. 12-1). At the maximum value of the sinusoidal function,

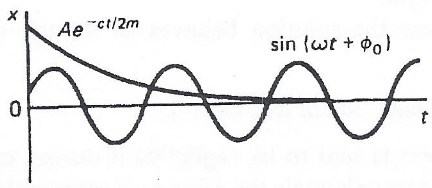

Figure 12-1.

x equals the exponential alone, while at the minimum value of the sinusoidal function, x equals minus the exponential. Thus we first sketch the exponential $Ae^{-ct/2m}$ and its negative $-Ae^{-ct/2m}$ in dashed lines. Periodically at the "x's" the solution lies on the two exponential curves drawn in Fig. 12-2 (exactly where depends on the phase ϕ_0):

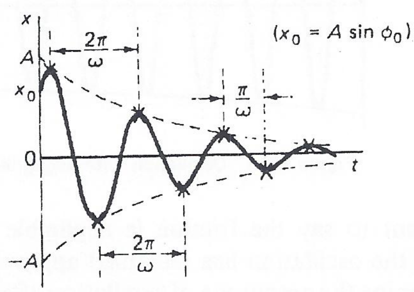

Figure 12-2 Underdamped oscillation.

Halfway between the marks, the function is zero. It varies smoothly throughout. Thus in the figure we sketch in a solid line the motion of a spring-mass system in the slightly damped case*. The exponential $Ae^{-ct/2m}$ is called the amplitude of the oscillation. Thus the amplitude exponentially decays. Indeed this seems quite similar to what we expect to observe in the case of a

*Although the zeroes are easy to locate, the maxima and minima are not halfway between the appropriate zeroes.

spring-mass system with **sufficiently small friction**. However, the mass never absolutely stops.

As long as there is friction ($c > 0$), no matter how small, it cannot be ignored as the amplitude of the oscillation will diminish in time only with friction. In this case though the solution is not exactly periodic, we can speak of an approximate circular frequency

$$\boxed{\omega = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}.} \tag{12.2}$$

Notice this expression for the frequency reduces to the frictionless value if there is no friction, $c = 0$, $\omega = \sqrt{k/m}$.

We will show that for some time the solution behaves as though there were no friction if

$$c^2 \ll 4mk \qquad \text{(read "much less than").}$$

The friction of a spring-mass system is said to be **negligible** if during many oscillations the amplitude remains approximately the same as is represented in Fig. 12-3.

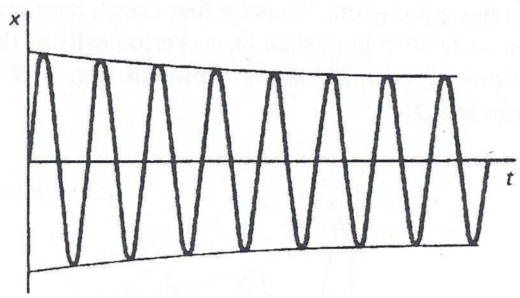

Figure 12-3 Oscillation with negligible damping.

It is equivalent to say the friction is negligible if after one "period" the amplitude of the oscillation has remained approximately constant. Thus we wish to determine the amplitude of oscillation after one period. The period of oscillation follows from equations 12.1 and 12.2 and is

$$T = \frac{2\pi}{\sqrt{k/m - c^2/4m^2}} = \frac{2\pi}{\sqrt{k/m[1 - (c^2/4mk)]}}.$$

However if $c^2 \ll 4mk$, we can approximate the period by its frictionless value:

$$T \approx \frac{2\pi}{\sqrt{k/m}} \qquad \text{(read "approximately equals").}$$

This is the first term in a Taylor series expansion of the period.* In this manner we can roughly estimate the amplitude of oscillation after one period,

$$Ae^{-ct/2m} \approx Ae^{-2\pi[c/2m\sqrt{(k/m)}]} = Ae^{-2\pi(c^2/4mk)^{1/2}}.$$

Since e^{-x} approximately equals 1 if x is small, it follows that if $c^2 \ll 4mk$, then the exponential has not decayed much in one "period."

Using a numerical criteria, the damping might be said to be negligible if after one "period" the mass returns to at least 95 percent of its original position, that is if

$$e^{-2\pi(c^2/4mk)^{1/2}} \geq .95.$$

Taking the natural logarithm of both sides, yields

$$-2\pi\left(\frac{c^2}{4mk}\right)^{1/2} \geq \log(.95)$$

or

$$\frac{c^2}{4mk} \leq \left[\frac{-\log(.95)}{2\pi}\right]^2.$$

The natural logarithm of .95 can be obtained from a mathematical table. However, the natural logarithm near 1 may be accurately approximated using the Taylor series formula

$$\log(1 - x) \approx -x.$$

Using this formula, damping is negligible (with a 95 percent criteria) if

$$\frac{c^2}{4mk} \leq \left(\frac{.05}{2\pi}\right)^2 = \frac{.0025}{4\pi^2} \approx .00006.$$

This calculation has been simplified using the rough numerical approximation $\pi^2 \approx 10$, since $\pi^2 = 9.8696\ldots$.

EXERCISES

12.1. If friction is sufficiently small ($c^2 < 4mk$), it has been shown that

$$x = Ae^{-ct/2m}\sin(\omega t + \phi_0),$$

where

$$\omega = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}.$$

(a) Show that $\omega = \sqrt{\frac{k}{m}}\sqrt{1 - \frac{c^2}{4mk}}$.

(b) Determine the first few terms of the Taylor series for small z of $(1 - z)^{1/2}$. [Hint: Use the binomial expansion (see page 89).]

*Improvements to this approximation of the period are suggested in the exercises.

(c) Using the result of (b), improve the approximation

$$\omega \approx \sqrt{\frac{k}{m}} \qquad \text{valid if } c^2 \ll 4mk.$$

12.2. At what time has the amplitude of oscillation of a spring-mass system with negligible friction decayed to $1/e$ of its original value? Does this time depend in a reasonable way on k, m, and c?

12.3. Show that the ratio of two consecutive local maximum displacements is a constant.

12.4. If $c < 0$, the force is called a negative friction force.

(a) In this case, show that $x \longrightarrow \infty$ as $t \longrightarrow \infty$.

(b) If in addition $c^2 < 4mk$, then roughly sketch the solution.

12.5. Determine the motion of an underdamped spring-mass system which is initially at its equilibrium position ($x = 0$) with velocity v_0. What is the amplitude of the oscillation?

12.6. Show that the local maximum or minimum for the displacement of an under-damped oscillation does *not* occur halfway between the times at which the mass passes its equilibrium position. However, show that the time period between successive local maxima (or minima) is constant. What is that period?

12.7. Consider a spring-mass system with linear friction (but without gravity). Suppose that there is an additional force, $B \cos \omega_0 t$, a periodic forcing function. Assume that B and ω_0 are known.

(a) If the coefficient of friction is zero (i.e., $c = 0$), then determine the general solution of the differential equation. Show that the solution is oscillatory if $\omega_0 \neq \sqrt{k/m}$. Show that the solution algebraically grows in time if $\omega_0 = \sqrt{k/m}$ (this is called **resonance** and occurs if the forcing frequency ω_0 is the same as the natural frequency $\sqrt{k/m}$).

(b) If $c^2 < 4mk$, show that the general solution consists of the sum of oscillatory terms (of frequency ω_0) and terms which exponentially decay in time. Thus for sufficiently large times the motion is accurately approximated by an oscillation of constant amplitude. What is that amplitude? Note it is independent of initial conditions.

(c) If $c^2 \ll 4mk$, show that for large times the oscillation approximately obeys the following statements: If the forcing frequency is less than the natural frequency ($\omega_0 < \sqrt{k/m}$), then the mass oscillates "in phase" with the forcing function (i.e., when the forcing function is a maximum, the stretching of the spring is a maximum, and vice versa). On the other hand, if the forcing frequency is greater than the natural frequency ($\omega_0 > \sqrt{k/m}$), then the mass oscillates "180 degrees out of phase" with the forcing function (i.e., when the forcing function is a maximum, the compression of the spring is a maximum and vice versa).

(d) In part (b) the ratio of the amplitude of oscillation of the mass to the amplitude of the forcing function is called the **response**. If $c^2 \ll 4mk$, then at what frequency is the response largest?

12.8. Consider the effect of striking the mass of a spring-mass system that is initially at rest. The problem is to solve

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t),$$

subject to the initial conditions

$$x(0) = x_0$$
$$\frac{dx}{dt}(0) = 0,$$

where $f(t)$ is the force due to the striking. Assume that $f(t)$ is approximately constant for some short length of time Δt, and then zero thereafter:

$$f(t) = \begin{cases} f_0 & 0 \le t \le \Delta t \\ 0 & t > \Delta t. \end{cases}$$

(a) Show that if $c^2 < 4mk$, then

$$x = \frac{f_0}{k} + \left(x_0 - \frac{f_0}{k}\right)e^{-ct/2m}\left(\cos \omega_0 t + \frac{c}{2m\omega_0}\sin \omega_0 t\right)$$

for $0 \le t \le \Delta t$.

(b) Calculate the position and velocity of the mass at $t = \Delta t$.

(c) Assume that the force is large for the short length of time Δt, i.e., as $\Delta t \longrightarrow 0$, $f_0 \longrightarrow \infty$. Further assume that as $\Delta t \longrightarrow 0$, $f_0 \Delta t = I$ (called the **impulse**). Calculate the limit of part (b) as $\Delta t \longrightarrow 0$, and show that as $\Delta t \longrightarrow 0$,

$$x(\Delta t) \longrightarrow x_0$$
$$\frac{dx}{dt}(\Delta t) \longrightarrow \frac{I}{m}.$$

(d) Briefly explain the following conclusion: the effect of an impulsive force is only to instantaneously increase the velocity by I/m. This explains a method by which a nonzero initial velocity occurs.

12.9. Reconsider exercise 12.8 for an alternate derivation.

(a) Show that for $0 \le t \le \Delta t$,

$$m\frac{dx}{dt} + cx - cx_0 + k\int_0^t x\, d\bar{t} = f_0 t.$$

(b) If Δt is small, show that

$$m\frac{dx}{dt}(\Delta t) \approx f_0\, \Delta t.$$

[Hint: Use Taylor expansions.]

(c) If (as before) $\Delta t \longrightarrow 0$ and $f_0 \longrightarrow \infty$ such that $f_0\, \Delta t \longrightarrow I$, show that the new initial conditions after the impulse are $x(0) = x_0$ and $(dx/dt)(0) = I/m$.

13. Overdamped and Critically Damped Oscillations

On the other hand, if the **friction is sufficiently large**, then

$$c^2 > 4mk,$$

and we call the system **overdamped**. The motion of the mass is no longer a decaying oscillation. The solution of equation 11.1 is

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \tag{13.1}$$

where r_1 and r_2 are real and both negative,

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}$$

$$r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}.$$

If the friction is sufficiently large, we should expect that the mass decays to its equilibrium position quite quickly. Exercise 13.1 shows that it does not oscillate. Instead, the mass either decays to its equilibrium position as seen in Fig. 13-1(a) or (b), or it shoots past the equilibrium position exactly once before returning monotonically towards the equilibrium position as seen in Fig. 13-1(c):

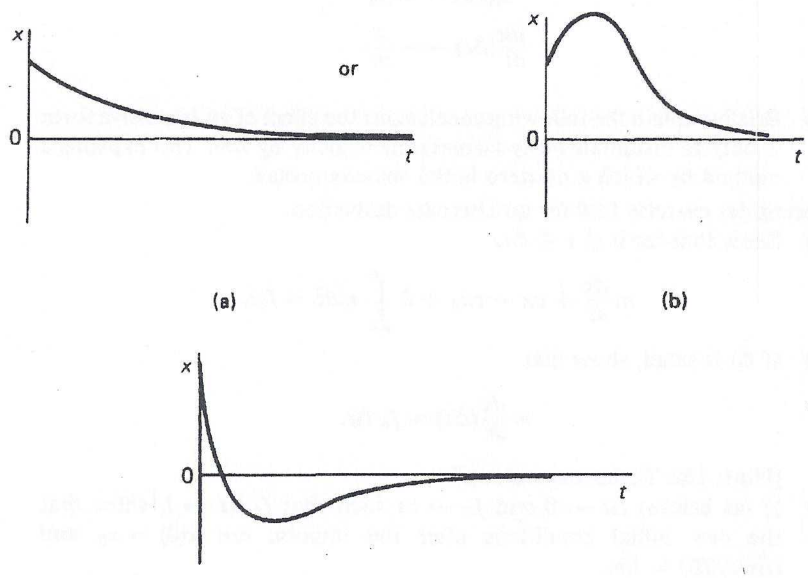

Figure 13-1 Overdamped oscillations.

It cannot cross the equilibrium position more than once. The mass crosses its equilibrium position only if the initial velocity is sufficiently negative (assuming that the initial position is positive).

When $c^2 = 4mk$, the spring-mass system is said to be **critically damped.** Mathematically $c^2 = 4mk$ requires a separate discussion since both exponential solutions become the same. However, from a physical point of view this case is *insignificant.* This is because the quantities c, m, and k are all experimentally measured quantities—there is no possibility that these measurements could be such that $c^2 = 4mk$ exactly. Any small deviation from equality will result in either of the previous two cases. For mathematical completeness, the solution in this case is

$$x = e^{-ct/2m}(At + B). \tag{13.2}$$

Although there is an algebraic growth t, the solution still returns to its equilibrium position as $t \longrightarrow \infty$ since the exponential decay is much stronger than an algebraic growth. A sketch would indicate no qualitative difference between this case and the overdamped case. In particular, the solution may go through the equilibrium position once at most.

EXERCISES

13.1. Assume that friction is sufficiently large ($c^2 > 4mk$).

(a) Show that the mass either decays to its equilibrium position (without passing through it), or that the mass shoots past its equilibrium position exactly once before returning monotonically towards its equilibrium position.

(b) If the initial position x_0 of the mass is positive, then show that the mass crosses its equilibrium position only if the initial velocity is sufficiently negative. What is the value of this critical velocity? Does it depend in a reasonable way on x_0, c, m, and k?

13.2. Do exercise 13.1 for the critically damped case $c^2 = 4mk$.

13.3. Assume that friction is extremely large ($c^2 \gg 4mk$).

(a) If the mass is initially at $x = 0$ with a positive velocity, then roughly sketch the solution you expect by physical reasoning.

(b) Estimate the characteristic roots if $c^2 \gg 4mk$.

(c) Based on part (b), sketch the approximate solution. What is the approximate maximum amplitude?

13.4. Assume that friction is extremely large ($c^2 \gg 4mk$).

(a) If the mass is initially at $x = x_0$ with velocity v_0, then approximate the solution. [Hint: Use the result of exercise 13.3b.]

(b) Solve the differential equation governing the spring-mass system (with friction) if the mass term can be neglected. Show that the two initial conditions cannot be satisfied. Why not? Applying which one of the two initial conditions yields a result consistent with (a)? Show that the solutions obtained in (a) and (b) are quite similar except for small times.

(c) Solve the differential equation governing the spring-mass system (with friction) if the restoring force term can be neglected. Show that the two initial conditions can be satisfied. Show that this solution does not approximate very well the solution to (a) for all time.

13.5. Assume that $c^2 > 4mk$.

(a) Solve the initial value problem, that is, at $t = 0$, $x = x_0$ and $dx/dt = v_0$.

(b) Take the limit of the solution obtained in (a) as $c^2 \longrightarrow 4mk$. Show that the limiting solution is the same as for the case $c^2 = 4mk$. [Hint: Let $r_1 \longrightarrow r_2$.]

13.6. (a) Using a computer, determine the motion of an overdamped spring-mass system (let $m = 1$, $k = 1$, $c = 3$), satisfying the initial conditions $x(0) = 1$ and $(dx/dt)(0) = v_0$ for various *negative* initial velocities.

(b) Show that the mass crosses its equilibrium position only if the initial velocity is sufficiently negative.

(c) Estimate this critical velocity.

(d) Compare your result to exercise 13.1b.

13.7. Consider a vertical spring-mass system with linear friction. Suppose two additional forces are present, gravity and any other force $f_1(t)$ only depending on time. Show that the motion is the same as that which would occur without gravity, but with a spring of greater length with the same spring constant.

13.8. Assume

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$$

and also assume a particular solution is known. Show that the difference between the exact solution and the particular solution tends towards zero as $t \longrightarrow \infty$ (if $c > 0$) independent of the initial conditions.

14. A Pendulum

We have investigated in some depth a spring-mass system which is governed by a linear differential equation. We may now be wondering what effect the neglected nonlinear terms may have. To give us additional motivation to analyze nonlinear problems, we now discuss a common physical system whose mathematical formulation results in a specific nonlinear equation.

Consider a pendulum of length L, shown on the next page in Fig. 14-1. At one end the pendulum is attached to a fixed point and is free to rotate about it. A mass m is attached at the other end as illustrated below. We know from observations that a pendulum oscillates in a manner at least qualitatively

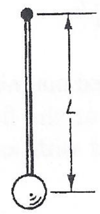

Figure 14-1 A pendulum.

similar to a spring-mass system. To make the problem easier, we assume the mass m is large enough so that, as an approximation, we state that all the mass is contained at the bob of the pendulum (that is the mass of the rigid shaft of the pendulum is assumed negligible). Again we apply Newton's second law of motion,

$$\vec{F} = m\vec{a}.$$

The pendulum moves in two dimensions (unlike the spring-mass system which was constrained to move in one dimension). However, a pendulum also involves only one degree of freedom as it is constrained to move along the circumference of a circle of radius L. This is represented in Fig. 14-2.

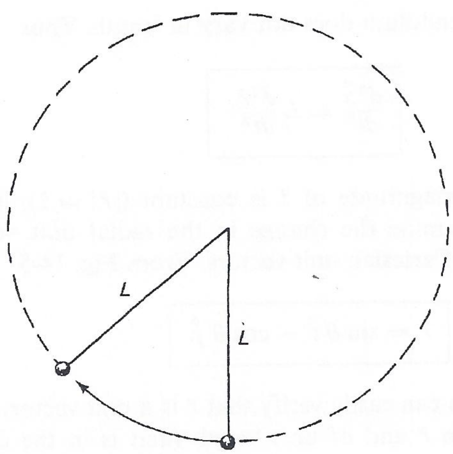

Figure 14-2.

Consequently, we will now develop the form Newton's law takes in polar coordinates. In two or three dimensions, Newton's law for a mass m is

$$m\frac{d^2\vec{x}}{dt^2} = \vec{F},$$

where $\vec{x}$ is the position vector of the mass (the vector from the origin to the mass). (In 3-dimensional rectangular coordinates, $\vec{x} = x\hat{i} + y\hat{j} + z\hat{k}$, the acceleration is given by

$$\frac{d^2\vec{x}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k},$$

since $\hat{i}, \hat{j}, \hat{k}$ are unit vectors which not only have fixed magnitude but also have fixed directions.) In polar coordinates (centered at the fixed vertex of the pendulum), the position vector is pointed outward with length L,

$$\vec{x} = L\hat{r}, \tag{14.1}$$

where $\hat{r}$ is the radial unit vector. The polar angle θ is introduced such that $\theta = 0$ corresponds to the pendulum in its "natural" position.* (See Fig. 14-3.)

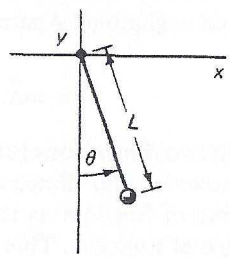

Figure 14-3.

L is constant since the pendulum does not vary in length. Thus

$$\boxed{\frac{d^2\vec{x}}{dt^2} = L\frac{d^2\hat{r}}{dt^2}.} \tag{14.2}$$

However, although the magnitude of $\hat{r}$ is constant ($|\hat{r}| = 1$), its direction varies in space. To determine the change in the radial unit vector $\hat{r}$, we express it in terms of the Cartesian unit vectors. From Fig. 14-5

$$\boxed{\hat{r} = \sin\theta\,\hat{i} - \cos\theta\,\hat{j}.} \tag{14.3a}$$

Note the minus sign. You can easily verify that $\hat{r}$ is a unit vector. The θ-unit vector is perpendicular to $\hat{r}$ and of unit length (and is in the direction of

*Be careful—this definition of the polar angle differs from the standard one shown in Fig. 14-4.

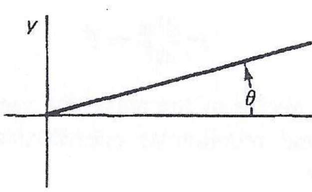

Figure 14-4.

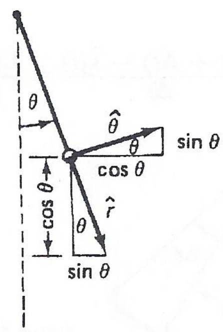

Figure 14-5 Radial and angular unit vectors.

increasing θ). Thus also from Fig. 14-5:

$$\boxed{\hat{\theta} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}.} \tag{14.3b}$$

In order to calculate the acceleration vector $d^2\vec{x}/dt^2$, the velocity vector $d\vec{x}/dt$ must first be calculated:

$$\frac{d\vec{x}}{dt} = L\frac{d\hat{r}}{dt} + \frac{dL}{dt}\hat{r}.$$

Since L is a constant for a pendulum, $dL/dt = 0$, and hence

$$\frac{d\vec{x}}{dt} = L\frac{d\hat{r}}{dt}.$$

From equation 14.3a,

$$\frac{d\hat{r}}{dt} = \frac{d\theta}{dt}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}),$$

which we note in general is more simply written as

$$\boxed{\frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{\theta}.} \tag{14.4a}$$

Similarly,

$$\boxed{\frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt}\hat{r}.} \tag{14.4b}$$

Thus the velocity vector is in the direction of $\hat{\theta}$,

$$\frac{d\vec{x}}{dt} = L\frac{d\theta}{dt}\hat{\theta}.$$

The magnitude of the velocity is $L(d\theta/dt)$, if motion lies along the circumference of a circle. Why is it obvious that if L is constant, then the velocity is in the θ direction? [Answer: If L is constant, then in a short length of time the position vector has changed only a little, but the change must be in the θ direction (see Fig. 14-6). In fact we see geometrically that

$$\frac{d\vec{x}}{dt} \approx \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t} \approx \frac{L\Delta\theta\,\hat{\theta}}{\Delta t} \approx L\frac{d\theta}{dt}\hat{\theta}.\Big]$$

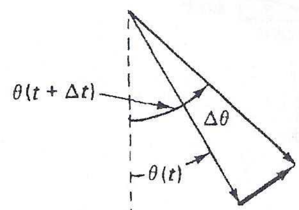

Figure 14-6 Rate of change of position vector.

The θ component of the velocity is the distance L times the **angular velocity** $d\theta/dt$. The acceleration vector is obtained as the derivative of the velocity vector:

$$\frac{d^2\vec{x}}{dt^2} = L\frac{d}{dt}\left(\frac{d\theta}{dt}\hat{\theta}\right) = L\left[\frac{d^2\theta}{dt^2}\hat{\theta} - \left(\frac{d\theta}{dt}\right)^2\hat{r}\right].$$

The angular component of the acceleration is $L(d^2\theta/dt^2)$. It exists only if the angle is accelerating. If L is constant, the radial component of the acceleration, $-L(d\theta/dt)^2$, is always directed inwards. It is called the centripetal acceleration and will occur even if the angle is only steadily increasing (i.e., even if the angular velocity $d\theta/dt$ is constant).

For *any* forces $\vec{F}$, when a mass is constrained to move in a circle, Newton's law implies

$$\boxed{mL\frac{d^2\theta}{dt^2}\hat{\theta} - mL\left(\frac{d\theta}{dt}\right)^2\hat{r} = \vec{F}.} \qquad \textbf{(14.5)}$$

For a pendulum, what are the forces? Clearly, there is a gravitational force $-mg\hat{j}$, which should be expressed in terms of polar coordinates. From the definitions of $\hat{r}$ and $\hat{\theta}$, equation 14.3,

$$\boxed{\begin{aligned}\hat{i} &= \hat{r}\sin\theta + \hat{\theta}\cos\theta\\ \hat{j} &= -\hat{r}\cos\theta + \hat{\theta}\sin\theta.\end{aligned}} \qquad \textbf{(14.6)}$$

Thus the gravitional force,

$$m\vec{g} = -mg\hat{j} = mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta}.$$

Perhaps this is more readily seen by breaking the direction $-\hat{j}$ into its polar components, as is done in Fig. 14-7.

Are there any other forces on the mass? If there were no other forces, then the mass would not move along the circle. The mass is held by the rigid shaft of the pendulum, which exerts a force $-T\hat{r}$ towards the origin of as yet

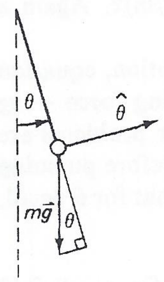

Figure 14-7.

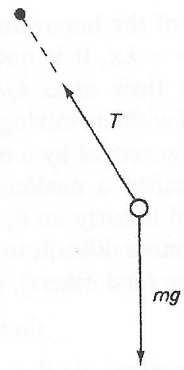

Figure 14-8 Forces on pendulum.

unknown magnitude T (and, as will be shown, of nonconstant magnitude). The forces on the bob of the pendulum are illustrated in Fig. 14-8. This results in motion along the circle. Thus

$$mL\frac{d^2\theta}{dt^2}\hat{\theta} - mL\left(\frac{d\theta}{dt}\right)^2\hat{r} = mg\cos\theta\hat{r} - mg\sin\theta\,\hat{\theta} - T\hat{r}.$$

Each component of this vector force equation yields an ordinary differential equation:

$$\boxed{\begin{aligned} mL\frac{d^2\theta}{dt^2} &= -mg\sin\theta \\ -mL\left(\frac{d\theta}{dt}\right)^2 &= mg\cos\theta - T. \end{aligned}} \qquad \begin{aligned} &\textbf{(14.7a)} \\ &\textbf{(14.7b)} \end{aligned}$$

(A two-dimensional vector equation is equivalent to two scalar equations.) T could be obtained from the second equation (if desired, which it frequently isn't), after determining θ from the first equation. Equation 14.7a implies that the mass times the θ component of the acceleration must balance the θ component of the gravitational force.

The mass m can be cancelled from both sides of equation 14.7a. Thus, the motion of the pendulum does not depend on the magnitude of the mass m attached to the pendulum. Only varying the length L (or g) will affect the motion. This qualitative fact has been determined even though we have not as yet solved the differential equation. Furthermore, only the ratio g/L is important as from equation 14.7a

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta.} \qquad \textbf{(14.8)}$$

(This is an advantageous procedure whenever possible in applied mathematics

—the determination of the important parameters. In fact for a spring-mass system, $m(d^2x/dt^2) = -kx$, it is not the two parameters k and m that are important, but only their ratio k/m, since $d^2x/dt^2 = -(k/m)x$. Again a conclusion is reached without solving the equation.)

The pendulum is governed by a nonlinear differential equation, equation 14.8, and hence is called a **nonlinear pendulum**. The restoring force $-mg \sin\theta$ does not depend linearly on θ, the unknown! Nonlinear problems are usually considerably more difficult to solve than linear ones. Before pursuing this nonlinear problem (and others), we recall from calculus that for θ small,

$$\sin\theta \approx \theta.$$

Geometrically the functions $\sin\theta$ and θ are nearly identical for small θ (θ being the linearization of $\sin\theta$ around the origin as seen in Fig. 14-9):

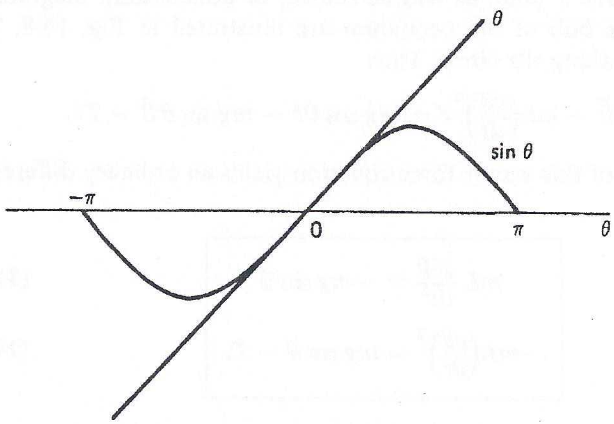

Figure 14-9 For θ small, θ approximates $\sin\theta$.

Using that approximation, the differential equation becomes

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta} \tag{14.9}$$

called the equation of a **linearized pendulum**. This is the same type of differential equation as the one governing a linearized spring-mass system without friction. Hence a linearized pendulum also executes simple harmonic motion. The pendulum oscillates with circular frequency

$$\boxed{\omega = \sqrt{\frac{g}{L}}} \tag{14.10}$$

and period $T = 2\pi\sqrt{L/g}$, *as long as θ is small.* This result can be checked as to its dimensional consistency. The effect of changing the length of the pendulum (or changing the magnitude of gravity) can be qualitatively and quantitatively determined immediately. Again the period of oscillation is independent of amplitude (as an approximation for small amplitude oscillations). Apparently this was first realized by Galileo, who observed the swinging of lamps suspended from long cords (i.e., a pendulum) in churches. You must remember that these observations were made before accurate clocks existed, and thus Galileo used his pulse to measure time!

In later sections we will investigate what happens if θ is *not* small.

EXERCISES

14.1. Consider the differential equation of a pendulum. Show that the period of *small amplitude* oscillations (around its natural position) is $T = 2\pi\sqrt{L/g}$. Briefly discuss the dependence of the period on L, g, and m.

14.2. Consider a mass m located at $\vec{x} = x\hat{\imath} + y\hat{\jmath}$, where x and y are unknown functions of time. The mass is free to move in the x-y plane without gravity (i.e., it is *not* connected to the origin via the shaft of a pendulum) and hence the distance L from the origin may vary with time.

(a) Using polar coordinates as introduced in Sec. 14, what is the velocity vector?

(b) Show that the acceleration vector $\vec{a} = (d^2/dt^2)\vec{x}$ is

$$\vec{a} = \left(L\frac{d^2\theta}{dt^2} + 2\frac{dL}{dt}\frac{d\theta}{dt}\right)\hat{\theta} + \left(\frac{d^2L}{dt^2} - L\left(\frac{d\theta}{dt}\right)^2\right)\hat{r}.$$

(c) If L is independent of t and $d\theta/dt$ is constant, sketch the trajectories. What direction is $\vec{a}$. Is this reasonable?

(d) If θ is independent of t, sketch possible trajectories. From part (b), show that the acceleration is in the correct direction.

14.3. Consider a mass m located at $\vec{x} = x\hat{\imath} + y\hat{\jmath}$ and only acted upon by a force in the direction of $\hat{r}$ with magnitude $-g(L)$ depending only on $L \equiv |\vec{x}|$, called a **central force.**

(a) Derive the differential equations governing the angle θ and the distance L [Hint: Use the equation for $\vec{a}$ determined in exercise 14.2b].

(b) Show that $L^2(d\theta/dt)$ is constant [Hint: Differentiate $L^2(d\theta/dt)$ with respect to t]. This law is called **Conservation of Angular Momentum.**

14.4. A particle's angular momentum around a fixed point is defined as a vector, the cross-product of the position vector and the momentum:

$$\vec{x} \times m\frac{d\vec{x}}{dt}$$

where $\vec{x}$ is the position vector relative to the fixed point. If all forces are in the direction of the fixed point (called **central forces**, see exericse 14.3), then show that angular momentum is conserved, i.e., show

$$\frac{d}{dt}\left(\vec{x} \times m\frac{d\vec{x}}{dt}\right) = 0.$$

14.5. Find an approximate expression for the radial force exerted by the shaft of the pendulum in the case of small oscillations. Show that the tension $T \approx mg$. Is this reasonable? Improve that approximation, and show that the radial force is *not* constant in time.

14.6. Consider a *swinging spring* of unstretched length L_0 and of spring constant k attached to a mass m as illustrated in Fig. 14-10.

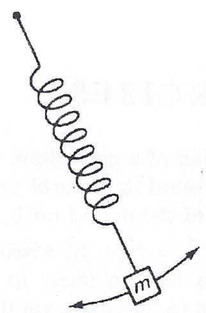

Figure 14-10 A swinging spring.

Show that the only force in addition to gravity acting on the mass is $-k(L - L_0)\hat{r}$. Derive the differential equations governing the motion of the mass. [Hint: Use the result of exercise 14.2].

14.7. One of **Kepler's laws of planetary motion** states that the radius vector drawn from the sun to a planet describes equal areas in equal times, that is, the rate of change of the area is a constant. Prove this using the result of exercise 14.3. [Hint: The differential area subtended is $dA = \frac{1}{2}L^2\,d\theta$ (Why?), and thus what is dA/dt?].

14.8. Newton knew an experimentally determined value of g, the radius of the earth, and the distance the moon is from the center of the earth. Using this information and assuming the moon moves in a circular orbit, estimate how long it takes the moon to go around the earth. [Hint: Recall $g = GM/r_0^2$]. Newton in the seventeenth century essentially used the preceeding ideas to help verify the inverse-square universal law of gravitation.

14.9. Consider an object resting on the equator. It obviously moves in a circle with a radius of approximately 4000 miles.

(a) Estimate the velocity of the object.

(b) There are two forces holding the object from moving radially: the force exerted by the surface of the earth and the gravitational force. Show that the radial acceleration of the object is much less than the gravitational acceleration.

14.10. Refer to exercise 14.9. Since most of you do not live at the equator, approximately calculate your present acceleration. In what direction is it?

14.11. From equation 14.3, show that $\hat{r}$ is perpendicular to $\hat{\theta}$.

14.12. Consider any nonconstant vector $\hat{r}$ of constant length. Show that $d\hat{r}/dt$ is perpendicular to $\hat{r}$.

14.13. Consider a pendulum whose length is varied in a prescribed manner, as shown in Fig. 14-11, i.e., $L = L(t)$ is known:

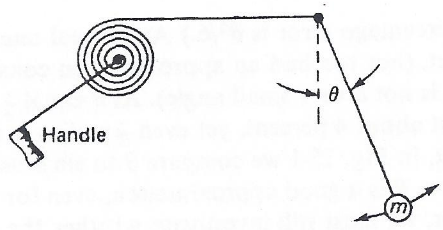

Figure 14-11 A variable length pendulum.

Derive the differential equation governing the angle θ. [Hint: Use the result of exercise 14.2.]

14.14. Show that $d\hat{\theta}/dt = -\hat{r}(d\theta/dt)$.

15. How Small is Small?

The nonlinear ordinary differential equation describing the motion of a pendulum was simplified to a linear one by using the approximation

$$\sin\theta \approx \theta.$$

Is this a good approximation? The Taylor series of $\sin\theta$,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots,$$

is valid for all θ. What error is introduced by neglecting all the nonlinear terms? An application of an extension of the mean value theorem (more easily remembered as the Taylor series with remainder*), yields

$$\sin\theta = \theta - \frac{\theta^3}{3!}\cos\bar{\theta},$$

*The formula for the Taylor series (with remainder) of $f(x)$ around $x = x_0$ is

$$f(x) = \underline{\sum_{n=0}^{N-1} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n} + f^{(N)}(\bar{x})\frac{(x - x_0)^N}{N!},$$

where $\bar{x}$ is an intermediate point ($x_0 < \bar{x} < x$ if $x > x_0$), but $\bar{x}$ is otherwise unknown. The underlined expression is the first N terms of the Taylor series. The remainder appears like the $N + 1$st term except that the Nth derivative is evaluated at the unknown intermediate point $\bar{x}$ rather than at the point x_0. The above formula is valid as long as $f(x)$ is continuous in the interval and the derivatives of $f(x)$ through the Nth derivative are also continuous.

where $\bar{\theta}$ is such that $0 < \bar{\theta} < \theta$, but $\bar{\theta}$ is otherwise unknown. The error E in our approximation, $E = -(\theta^3/6) \cos \bar{\theta}$, is bounded by $\theta^3/6$,

$$|E| < \frac{\theta^3}{6},$$

since $|\cos \bar{\theta}| < 1$. (The percentage error is $\theta^2/6$.) At θ equal one radian, the error is at most 16 percent, (not too bad an approximation considering one radian is about 57° which is not a very small angle). At θ equal $\frac{1}{2}$ radian, the error is reduced to at most about 4 percent, yet even $\frac{1}{2}$ radian is not a particularly small angle. In fact, in Fig. 15-1 we compare θ to $\sin \theta$ using a set of tables. It is seen that $\sin \theta \approx \theta$ is a good approximation, even for angles that are not too small. However, we must still investigate whether the solution to the linear equation, $d^2\theta/dt^2 = -(g/L)\theta$, is a good approximation to the solution to the more difficult nonlinear equation, $d^2\theta/dt^2 = -(g/L) \sin \theta$! Approximate equations do not *always* have solutions that are a good approximation to the solution of the exact equation. As an oversimplified example we know that $(1000.5)\pi$ in some sense approximately equals 1000π, but $\cos (1000\pi)$ is not a good approximation to $\cos (1000.5\pi)$, (since $\cos (1000\pi) = 1$ and $\cos (1000.5\pi) = 0$).

θ (degrees)	θ (radians)	$\sin \theta$
1	0.0174533	0.0174524
5	0.08726	0.08716
10	0.17453	0.17365
15	0.26179	0.25882
20	0.34907	0.34202
25	0.43633	0.42262
30	0.52360	0.50000
35	0.61087	0.57358
40	0.69813	0.64279

Figure 15-1.

EXERCISES

15.1. For what angles is $\sin \theta \approx \theta$ a valid approximation with an error guaranteed to be less than 10 percent?

15.2. (a) Using the Taylor series with remainder, what is the maximum percent error that occurs when $\sin \theta$ is approximated by θ for $\theta = 30°$?
(b) For $\theta = 30°$, what is the actual percent error?
(c) Compare part (a) to part (b).

16. A Dimensionless Time Variable

Returning to the equation of a nonlinear pendulum,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta,$$

let us suppose that we need to compute solutions to this equation numerically. Initial conditions are needed to solve a differential equation on the computer. In general,

$$\theta(0) = \theta_0$$
$$\frac{d\theta}{dt}(0) = \Omega_0.$$

This suggests three parameters of significance in the calculation, θ_0, Ω_0, and g/L. To determine all solutions to the differential equation, it appears we must vary these *three* parameters. However, if time is measured in a certain way, then we will show that only two parameters are important. Let us **scale** the time t by any constant time Q, by which we mean let

$$t = Q\tau.$$

Using the chain rule

$$\frac{d}{dt} = \frac{1}{Q}\frac{d}{d\tau},$$

yields the differential equation

$$\frac{d^2\theta}{d\tau^2} = -\frac{g}{L}Q^2\sin\theta,$$

to be solved with the modified initial conditions:

$$\theta(0) = \theta_0$$
$$\frac{d\theta}{d\tau}(0) = Q\Omega_0.$$

Q is chosen such that there are less parameters necessary in the problem. For example, let $(g/L)Q^2 = 1$ or $Q = \sqrt{L/g}$, in which case

$$\boxed{\begin{aligned} &\frac{d^2\theta}{d\tau^2} = -\sin\theta \\ &\theta(0) = \theta_0 \\ &\frac{d\theta}{d\tau}(0) = \Omega_0\sqrt{\frac{L}{g}}. \end{aligned}} \qquad (16.1)$$

This resulting problem has only two parameters of significance, θ_0 and $\Omega_0\sqrt{L/g}$. To obtain numerical solutions, we only have to vary these two parameters.

Let us describe what this scaling of time represents physically. The circular frequency of small amplitude oscillations is $\omega_0 = \sqrt{g/L}$ and thus

$$\boxed{\tau = \sqrt{\frac{g}{L}}\,t = \omega_0 t.} \tag{16.2}$$

The variable τ has no dimensions, and is thus called a **dimensionless** time variable! The only important parameters in the nonlinear problem (if we measure time based on the frequency of a small oscillation) is θ_0, the initial amplitude of the pendulum, and

$$\Omega_0\sqrt{\frac{L}{g}} = \frac{\Omega_0}{\omega_0},$$

the ratio of the initial angular velocity Ω_0 to the circular frequency of small oscillations!

EXERCISES

16.1. Using an appropriate dimensionless time variable, what equation governs a frictionless spring-mass system with a linear restoring force?

16.2. Consider a linearized pendulum. Show, by independently scaling both time and the angle θ, that the three parameters θ_0, Ω_0, and g/L reduce to one parameter. Why doesn't this work for the nonlinear pendulum?

16.3. Show that Ω_0/ω_0 is a dimensionless parameter.

17. Nonlinear Frictionless Systems

For a spring-mass system without friction, we have shown

$$\boxed{m\frac{d^2x}{dt^2} = -f(x).} \tag{17.1}$$

Here the force, $-f(x)$, depends only on the position of the mass. If the equilibrium position is $x = 0$, then the spring exerts no force there, $f(0) = 0$.

We assume that $f(x)$ is such that it can be expanded in a Taylor series,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots .$$

Since $f(0) = 0$,

$$m\frac{d^2x}{dt^2} = -kx - \frac{f''(0)}{2}x^2 - \cdots,$$

where $k = f'(0)$. If the force is a restoring force, then $f(x)$ is positive for positive x (and vice versa), and hence k is positive. Thus the result *for small amplitudes of oscillation* in which the nonlinear terms can be neglected, is Hooke's law (the linearized spring-mass equation).

We will now consider motions of a spring-mass system such that the amplitudes are not necessarily small. Then equation 17.1 is appropriate. The nonlinear pendulum also satisfies a differential equation of that form, since

$$L\frac{d^2\theta}{dt^2} = -g \sin \theta. \qquad \textbf{(17.2)}$$

Before solving these nonlinear ordinary differential equations, what properties do we expect the solution to have? For small amplitudes the solution most likely oscillates periodically. For larger amplitudes oscillations are still expected, at least for the nonlinear pendulum. Futhermore, there are certain equilibrium positions for the nonlinear pendulum, that is, if the pendulum is in that position and at rest it will stay there. For $\theta = \theta_E$ to be an equilibrium position, $\theta = \theta_E$ must solve the differential equation 17.1. Since θ_E is a constant $d\theta_E/dt = d^2\theta_E/dt^2 = 0$, and thus equation 17.2 implies

$$\sin \theta_E = 0.$$

Consequently, $\theta_E = 0, \pi$. (Other mathematical solutions to this equilibrium problem are physically equivalent.) $\theta_E = 0$ is the "natural" position of a pendulum, as shown in Fig. 17-1, while $\theta_E = \pi$ as demonstrated in Fig. 17-2 is the "inverted" position of a pendulum:

Figure 17-1 Natural equilibrium position of a pendulum.

Figure 17-2 Inverted equilibrium position of a pendulum.

It is only at these two positions that the forces will balance, yielding no motion. However, there is a fundamental difference between these solutions that is immediately noticeable. Although both are equilibrium positions,

$\theta_E = 0$ is stable and $\theta_E = \pi$ is very unstable! Any mathematical solution must illustrate this striking difference between these two equilibrium positions.

In general if equation 17.1 is valid then there is an equilibrium position at any value of x such that $f(x) = 0$.

18. Linearized Stability Analysis of an Equilibrium Solution

The concept of the stability of solutions is considered to be one of the fundamental aspects of applied mathematics. Now we are pursuing this subject with respect to the stability of the equilibrium positions of a pendulum. In the discussion of population dynamics later in the text, we will investigate the stability of equilibrium populations. Other areas in which stability questions are important include, for example, economics, chemistry, and widely diverse fields of engineering and physics.

As illustrated by the two equilibrium positions for a nonlinear pendulum, the concept of stability is not a difficult one. Basically, an equilibrium solution of a time-dependent equation is said to be **stable** if the (usually time-dependent) solution stays "near" the equilibrium solution for *all* initial conditions "near" the equilibrium. More precise mathematical definitions of different kinds of stability can be given. (See, for example, W. Boyce and R. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, New York: John Wiley & Sons, 1969.) In a precise discussion, what is meant by "near" is carefully defined. However, for our purposes the abstractness of the rigorous definitions of stability is unnecessary. When an equilibrium solution is not stable, it is said to be **unstable**. For example, even if only one initial condition exists for which the solution tends "away" from the equilibrium, then the equilibrium is unstable. On the other hand an equilibrium is not stable just because there exists one initial condition such that the solution stays near the equilibrium. We repeat, to be stable it must stay near for *all* initial conditions. In summary, an equilibrium solution is unstable if solutions tend "away" from the equilibrium and stable if solutions either tend "toward" the equilibrium or stay the same "distance away" (for example, for the natural position of the linearized pendulum the amplitude of oscillation remains the same; in some sense the solution does not tend towards the equilibrium, but stays the same "distance away".)

In this section, we give a mathematical method to distinguish between stable and unstable equilibrium solutions. If a mass m is acted upon by a force $-f(x)$, then

$$\boxed{m\frac{d^2x}{dt^2} = -f(x).} \tag{18.1}$$

$x = x_E$ is an equilibrium position if

$$f(x_E) = 0. \tag{18.2}$$

To analyze the stability of this equilibrium position, we investigate positions x of the mass near its equilibrium position. For x near to x_E, the function $f(x)$ can be approximated using the first few terms of its Taylor series around $x = x_E$:

$$f(x) = f(x_E) + (x - x_E)f'(x_E) + \frac{(x - x_E)^2}{2!}f''(x_E) + \cdots$$

Thus

$$m\frac{d^2x}{dt^2} = -f(x_E) - (x - x_E)f'(x_E) - \frac{(x - x_E)^2}{2!}f''(x_E) - \cdots$$

Since $x = x_E$ is an equilibrium position, $f(x_E) = 0$ and thus

$$m\frac{d^2x}{dt^2} = -(x - x_E)f'(x_E) - \frac{(x - x_E)^2}{2!}f''(x_E) - \cdots$$

For x sufficiently near to x_E, $(x - x_E)^2$ is much smaller than $x - x_E$ and consequently the quadratic term $[(x - x_E)^2 f''(x_E)/2!]$ can be ignored, as well as the higher-order terms of the Taylor series (if $f'(x_E) \neq 0$). As an approximation

$$m\frac{d^2x}{dt^2} = -(x - x_E)f'(x_E).$$

Although this equation can be explicitly solved, it is more convenient to introduce the **displacement from equilibrium** y:

$$y = x - x_E.$$

Using y as the new dependent variable

$$\boxed{m\frac{d^2y}{dt^2} = -f'(x_E)y.} \tag{18.3}$$

The coefficient $f'(x_E)$ is a *constant*; the displacement from equilibrium y approximately satisfies the above linear differential equation with constant coefficients.

We say the equilibrium solution is **stable** if for initial conditions sufficiently near the equilibrium solution, the solution stays close to the equilibrium solution. Otherwise the equilibrium solution is said to be **unstable**. Thus the stability of the equilibrium solution is determined by the time dependence of the displacement from equilibrium. A simple analysis of equation 18.3, known as a **linearized stability analysis**, shows that:

1. If $f'(x_E) > 0$, then the mass executes simple harmonic motion (about its equilibrium position). In this case we say the equilibrium position is **stable.*** This shows the importance of simple harmonic motion as it will describe motion near a stable equilibrium position. For slight departures from equilibrium, in this case the force tends to restore the mass.
2. If $f'(x_E) < 0$, then the system has some exponential decay and some exponential growth. Since the solution consists of a combination of these two effects, the displacement from equilibrium will exponentially grow for most initial conditions. Thus in this case the equilibrium point is said to be **unstable.** If displaced from equilibrium, the force will push it further away.
3. If $f'(x_E) = 0$, then this linearized stability analysis is inconclusive as additional terms of the Taylor series need to be calculated.

As an example, let us investigate the linearized stability of the equilibrium positions of a nonlinear pendulum:

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta.$$

The equilibrium positions are $\theta_E = 0$ and $\theta_E = \pi$. $f(\theta) = g\sin\theta$, and therefore $f'(\theta) = g\cos\theta$. We note that

$$\begin{aligned} f'(0) &= g > 0 \\ f'(\pi) &= -g < 0. \end{aligned}$$

Thus, as we know by our experience with pendulums, $\theta_E = 0$ is a stable equilibrium position of a nonlinear pendulum, while $\theta_E = \pi$ is an unstable equilibrium position.

An *equivalent* method known as a **perturbation method** is sometimes used to investigate the linearized stability of an equilibrium solution. To facilitate remembering that x is near to x_E, a small parameter ϵ is introduced, $0 < \epsilon \ll 1$, such that

$$x(t) = x_E + \epsilon x_1(t).$$

$\epsilon x_1(t)$ is now the displacement from equilibrium (also known as the amount the position is perturbed or the perturbation). If this expression is substituted into the differential equation

$$m\frac{d^2x}{dt^2} = -f(x),$$

---

*The term **neutrally stable** is sometimes used, indicating that the solution does not tend to equilibrium as $t \longrightarrow \infty$.

then

$$\epsilon m \frac{d^2x_1}{dt^2} = -f(x_E + \epsilon x_1)$$

results, since x_E is a constant. Again using the Taylor series it is seen that

$$\epsilon m \frac{d^2x_1}{dt^2} = -f(x_E) - \epsilon x_1 f'(x_E) - \frac{\epsilon^2 x_1^2}{2!} f''(x_E) - \cdots$$

By neglecting the $O(\epsilon^2)$* terms, this reduces to equation 18.3. Using either method, the difficult to solve nonlinear differential equation is approximated by an easily analyzed linear differential equation.

EXERCISES

18.1. In this problem we will investigate the stability of circular planetary orbits. In exercises 14.2 and 14.3 it was shown that

$$m\left[\frac{d^2L}{dt^2} - L\left(\frac{d\theta}{dt}\right)^2\right] = -g(L)$$

$$L^2 \frac{d\theta}{dt} = \text{constant} \equiv H_0.$$

(a) Show that $m[(d^2L/dt^2) - L^{-3}H_0^2] = -g(L)$, a second-order differential equation. In parts (b) and (c) assume the radial force is an inverse-power law, i.e.,

$$g = \frac{c}{L^n} \quad \text{with } c > 0.$$

(b) What is the radius L_0 of an allowable *circular* orbit? [Hint: $L_0 = \{c/[m(d\theta/dt)^2]\}^{1/(n+1)}$.]

(c) Using a linearized stability analysis, show that this circular orbit is stable only if $n < 3$ (i.e., for an inverse-square law, a circular orbit is stable).

18.2. Assume that $x = x_0$ is a *stable* equilibrium point of equation 18.1. What is the period of small oscillations around that equilibrium point?

18.3. Suppose

$$\frac{d^2x}{dt^2} = x - x^2$$

(a) Determine all possible equilibrium solutions. [Hint: The answer is $x = 0$ and $x = 1$.]

*The symbol $O(\epsilon^2)$ is read as "order ϵ^2." It indicates that the order of magnitude of the most important neglected term is ϵ^2. A more careful treatment of this symbol can be developed. The author believes this is unnecessary in this text.

(b) Is $x = 0$ a stable equilibrium solution?
(c) Is $x = 1$ a stable equilibrium solution?

18.4. Consider a spring-mass system with a nonlinear restoring force satisfying

$$m\frac{d^2x}{dt^2} = -kx - \alpha x^3,$$

where $\alpha > 0$. Which positions are equilibrium positions? Are they stable?

18.5. Consider a system which satisfies

$$m\frac{d^2x}{dt^2} = -kx + \alpha x^3,$$

where $\alpha > 0$.
(a) Show that the force does not always restore the mass towards $x = 0$.
(b) Which positions are equilibrium positions? Are they stable?

18.6. Consider a stranded moonship of mass m_s somewhere directly between the moon (of mass m_m) and the earth (of mass m_e). The gravitational force between any two masses is an *attractive* force of magnitude $G(m_1m_2/r^2)$, (where G is a universal gravitational constant, m_1 and m_2 are the two masses, and r is the distance between the masses).
(a) If the moonship is located at a distance y from the center of the earth, show that

$$m_s\frac{d^2y}{dt^2} = -G\frac{m_em_s}{y^2} + G\frac{m_mm_s}{(r_0 - y)^2},$$

where r_0 is the constant distance between the earth and moon. (Assume that the earth has no effect on the moon, i.e., assume that both are fixed in space.)
(b) Calculate the equilibrium position of the moonship.
(c) Is the equilibrium position stable? Is your conclusion reasonable?
(d) Compare this problem to exercise 18.7.

18.7. Consider an isolated positive electrically charged particle of charge q_e (and mass m_e) located directly between two fixed positive charged particles of charge q_{p_1} and q_{p_2} respectively. The electrical force between two charged particles is $-F(q_1q_2/r^2)$ (alike charges repel and different charges attract), where F is a universal electrical constant, q_1 and q_2 are the two charges, and r is the distance between the two charges.
(a) If the middle particle is located at a distance y from one of the particles, show that

$$m_e\frac{d^2y}{dt^2} = F\frac{q_eq_{p_1}}{y^2} - F\frac{q_eq_{p_2}}{(r_0 - y)^2},$$

where r_0 is the approximately constant distance between the two positively charged particles.
(b) Calculate the equilibrium position of the positively charged particle.
(c) Is the equilibrium position stable? Is your conclusion reasonable?
(d) Compare this problem to exercise 18.6.

18.8. Consider a mass m attached to the exact middle of a stretched string of length l.

(a) Suppose that the mass when attached forms in equilibrium a 30°–30°–120° triangle (due to gravity g) as in Fig. 18-1.

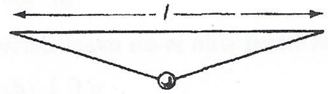

Figure 18.1 Vertically vibrating mass.

Calculate the *tension* T in the string. (The tension is the force exerted by the string.)

(b) Assume that the tension remains the same when the mass is displaced vertically a *small* distance y. Calculate the period of oscillation of the mass.

18.9. Consider a nonlinear pendulum. Using a linearized stability analysis, show that the inverted position is unstable. What is the exponential behavior of the angle in the neighborhood of this unstable equilibrium position?

18.10. Suppose

$$m\frac{d^2x}{dt^2} = \alpha(e^{\beta x} - 1) \quad \text{with } \alpha > 0 \text{ and } \beta > 0.$$

(a) What are the dimensions of α and β?
(b) Determine all equilibrium positions.
(c) Describe the motion in the neighborhood of the equilibrium position, $x = 0$.

18.11. Consider equation 18.3. If $f'(x_E) > 0$, show that the amplitude of the oscillation is small if the initial displacement from equilibrium *and* the initial velocity are small.

19. Conservation of Energy

In the previous section, we were able to analyze the solution of

$$\boxed{m\frac{d^2x}{dt^2} = -f(x)} \tag{19.1}$$

in the neighborhood of an equilibrium position. Here we continue the investigation of this nonlinear equation representing a spring-mass system without friction. We are especially interested now in determining the behavior of solutions to equation 19.1 valid far away from an equilibrium position.

The general solution to a second-order differential equation (even if nonlinear) contains two arbitrary constants. One of these constants can be obtained in a manner to be described. First multiply both sides of equation

19.1 by the velocity dx/dt,

$$m\frac{dx}{dt}\frac{d^2x}{dt^2} = -f(x)\frac{dx}{dt}.$$

The left-hand side is an exact derivative, since

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right] = \frac{dx}{dt}\frac{d^2x}{dt^2}.$$

Thus,

$$m\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right] = -f(x)\frac{dx}{dt}.$$

After multiplying by dt,

$$md\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right] = -f(x)\,dx.$$

Both sides of this equation can now be integrated. If there is a function $F(x)$ such that $dF/dx = f$ (i.e., $F(x) = \int^x f(\bar{x})\,d\bar{x}$), then *indefinite* integration yields

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = -F(x) + E$$

or

$$\boxed{\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + F(x) = E,} \qquad \textbf{(19.2a)}$$

where E is a constant of integration. The quantity $\frac{1}{2}m(dx/dt)^2 + F(x)$, which we will show is the total energy, remains the same throughout the motion; it is said to be **conserved.**

Especially in cases in which $f(x)$ does not have a simple integral, it is often more advantageous to do a *definite* integration from the initial position x_0 with initial velocity v_0, i.e.,

$$x(t_0) = x_0$$

$$\frac{dx}{dt}(t_0) = v_0.$$

Thus

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - \frac{1}{2}mv_0^2 = -\int_{x_0}^{x} f(\bar{x})\,d\bar{x}.$$

This expression corresponds to the one obtained by indefinite integration if $E = \frac{1}{2}mv_0^2$. An alternate expression, more easily interpreted, can be derived by noting

$$\int_{x_0}^{x} f(\bar{x})\,d\bar{x} = \int_{x_0}^{x_1} f(\bar{x})\,d\bar{x} + \int_{x_1}^{x} f(\bar{x})\,d\bar{x},$$

where x_1 is any fixed position. Then

$$\boxed{\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \int_{x_1}^{x} f(\bar{x})\,d\bar{x} = \frac{1}{2}mv_0^2 + \int_{x_1}^{x_0} f(\bar{x})\,d\bar{x} \equiv E.} \tag{19.2b}$$

The quantity

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \int_{x_1}^{x} f(\bar{x})\,d\bar{x}$$

is again said to be **conserved**, since it is constant throughout the motion, being initially equal to

$$\frac{1}{2}mv_0^2 + \int_{x_1}^{x_0} f(\bar{x})\,d\bar{x}.$$

This constant of the motion is called the **total energy**. Part of it

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}mv^2$$

is called the **kinetic energy**; it is that portion of energy due to the *motion* of the mass (hence the term kinetic). $\int_{x_1}^{x} f(\bar{x})\,d\bar{x}$ is the work* necessary to raise the mass from x_1 to x. The force necessary to raise the mass is minus the external force [for example, with gravity $-mg\hat{j}$, the force necessary to raise a mass is $+mg\hat{j}$]. Thus if the external force is $-f(x)$, i.e., $m(d^2x/dt^2) = -f(x)$, then the work necessary to raise the mass from x_1 to x is $\int_{x_1}^{x} f(x)\,dx$. It is thus in essence the work or energy that is stored in the system for "potential" usage, and hence is called the **potential energy** (relative to the position $x = x_1$). Equation 19.2 is called the equation of **conservation of energy** or the **energy equation**. The total energy is shared between kinetic energy and potential energy. For example, as a mass speeds up it must gain its kinetic energy from the potential energy already stored in the system. Often (but not always) it is convenient to measure potential energy relative to an equilibrium position (i.e., let $x_1 = x_E$).

The principle of **conservation of energy** is frequently quite useful (and always important). As an example, consider an object being thrown vertically subject only to the force of gravity. Newton's law implies

$$m\frac{d^2y}{dt^2} = -mg,$$

*The definition of work W is the force times the distance (i.e., for a variable force g, $w = \int_{x_1}^{x} g\,ds$). In two or three spatial dimensions if the force is $\vec{g}$, the work done by the force is $W = \int_{\vec{x}_1}^{\vec{x}} \vec{g}\cdot d\vec{s}$.

where y is the height above the ground. Suppose the object is initially at $y = 0$, but thrown upward with velocity v_0 (the initial velocity $dy/dt = v_0$). A reasonable question is: What is the highest point the object reaches before it falls back to the ground. Although it is not difficult to directly solve this differential equation (integrate twice!) and then determine the highest point (as was done in exercise 4.1), conservation of energy more immediately yields the result. The potential energy relative to the ground level (a convenient position since there is no equilibrium for this problem) is

$$F(y) = \int_0^y mg\, d\bar{y} = mgy.$$

Conservation of energy implies that the sum of the kinetic energy and the potential energy does not vary in time,

$$\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + mgy = E.$$

The constant E equals the initial kinetic energy,

$$E = \frac{1}{2}mv_0^2,$$

because initially the potential energy is zero since initially $y = 0$. Furthermore at the highest point $y = y_{\max}$, the velocity must equal zero, $dy/dt = 0$ at $y = y_{\max}$. Hence at the highest point the kinetic energy must be zero. At that point all of the energy must be potential energy. Thus conservation of energy implies

$$mgy_{\max} = E = \frac{1}{2}mv_0^2.$$

Consequently the highest point is

$$y_{\max} = \frac{v_0^2}{2g}.$$

Other applications are considered in the exercises.

Suppose the potential energy relative to a position $x = x_1$ is known

$$F(x) = \int_{x_1}^{x} f(\bar{x})\, d\bar{x}$$

and is sketched, yielding (for example) Fig. 19-1. The derivative of the potential energy is $f(x)$,

$$\frac{dF(x)}{dx} = f(x).$$

The applied force is $-f(x)$. Thus *the derivative of the potential energy is minus the force.* Since $f(x_E) = 0$, potential energy at an equilibrium position has an extremum point $dF/dx = 0$ (a relative minimum if $f'(x_E) > 0$, a relative

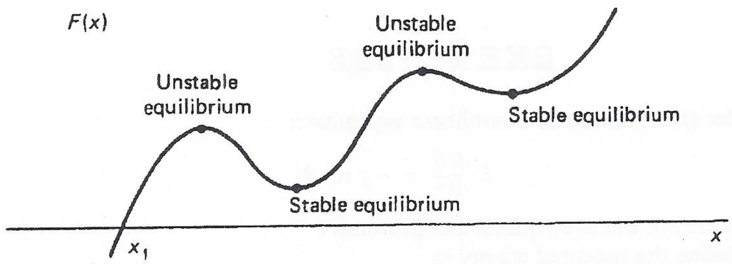

Figure 19-1 Potential energy $F(x)$ illustrating equilibrium positions.

maximum if $f'(x_E) < 0$, and if $f'(x_E) = 0$ either a relative minimum, a relative maximum, or a saddle point). Recall in studying the linearized stability of an equilibrium position, it was determined that an equilibrium position is stable if $f'(x_E) > 0$ and unstable if $f'(x_E) < 0$. Thus the potential energy has a relative minimum at a stable equilibrium position. Similarly at an unstable equilibrium position, the potential energy has a relative maximum. It is easy to remember these results as they are the same as would occur if a ball were placed on a mountain shaped like the potential energy curve. These facts are noted in Fig. 19-1.

From conservation of energy, equation 19.2, an expression for the velocity, dx/dt, can be obtained,

$$\frac{dx}{dt} = \pm\sqrt{\frac{2E}{m} - \frac{2}{m}\int_{x_1}^{x} f(\bar{x})\, d\bar{x}}.$$

The sign of the square root must be chosen appropriately. It is positive if the velocity is positive and vice versa. Furthermore, this first order differential equation is separable:

$$\frac{dx}{\pm\sqrt{\frac{2E}{m} - \frac{2}{m}\int_{x_1}^{x} f(\bar{x})\, d\bar{x}}} = dt.$$

Integrating from $t = t_0$ (where $x = x_0$), yields an implicit solution

$$t = t_0 + \int_{x_0}^{x} \frac{d\bar{x}}{\pm\sqrt{\frac{2E}{m} - \frac{2}{m}\int_{x_1}^{\bar{x}} f(\bar{x})\, d\bar{x}}},$$

where the appropriate sign again must be chosen in the integrand depending on whether the velocity is positive or negative. However, this equation solves for t as a function of x. We usually are more interested in the position x as a function of t. Furthermore, this formula is not particularly helpful in understanding the qualitative behavior of the solution.

EXERCISES

19.1. Consider the equation of a nonlinear pendulum:

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta.$$

(a) What are the two equilibrium positions?

(b) Define the potential energy as

$$F(\theta) \equiv \int_0^\theta g\sin\bar{\theta}\,d\bar{\theta}.$$

Evaluate this potential and formulate conservation of energy.

(d) Show that this potential energy (as a function of θ) has a relative minimum at the stable equilibrium position and a relative maximum at the unstable equilibrium position.

19.2. The equation of a linearized pendulum is $L(d^2\theta/dt^2) = -g\theta$. By multiplying by $d\theta/dt$ and integrating, determine a quantity which is a constant of motion.

19.3. In general show that the potential energy when a mass is at rest equals the total energy.

19.4. A linear spring-mass system (without friction) satisfies $m(d^2x/dt^2) = -kx$.

(a) Derive that

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2 = \text{constant} \equiv E.$$

(b) Consider the initial value problem such that at $t = 0$, $x = x_0$ and $dx/dt = v_0$. Evaluate E.

(c) Using the expression for conservation of energy, evaluate the maximum displacement of the mass from its equilibrium position. Compare this to the result obtained from the exact explicit solution.

(d) What is the velocity of the mass when it passes its equilibrium position?

19.5. Derive an expression for the potential energy if the only force is gravity.

19.6. Suppose a mass m located at (x, y) is acted upon by a force field, $\vec{F}$ (i.e., $m(d^2\vec{x}/dt^2) = \vec{F}$). The kinetic energy is defined as

$$\frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right];$$

it again equals $\frac{1}{2}mv^2$. The potential energy is defined as

$$-\int_{\vec{x}_1}^{\vec{x}} \vec{F}\cdot d\vec{s},$$

where $\vec{x}_1$ is any fixed position. If there exists a function $\phi(x, y)$ such that

$$\vec{F} = -\nabla\phi,$$

then show that the total energy (kinetic energy plus potential energy) is conserved. Such a force field $\vec{F}$ is called a **conservative force field.** In this case show that the potential energy equals $\phi(\vec{x}) - \phi(\vec{x}_1)$.

19.7. A mass m is thrown upwards at velocity v_0 against the inverse-square gravitational force ($F = -GmM/y^2$).
(a) How high does the mass go?
(b) Determine the velocity at which the mass does not return to earth, the so-called **escape velocity.**
(c) Estimate this value in kilometers per hour (miles per hour).

19.8. Using a computer, numerically integrate any initial value problem for a frictionless spring-mass system with a linear restoring force. Is the energy constant?

20. Energy Curves

In the previous section a complex expression was derived for the motion of a spring-mass system. A better understanding of the solution can be obtained by analyzing the energy equation 19.2,

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \int_{x_1}^{x} f(\bar{x})\,d\bar{x} = \frac{1}{2}mv_0^2 + \int_{x_1}^{x_0} f(\bar{x})\,d\bar{x},$$

representing conservation of energy.

Let us consider x and dx/dt as variables rather than x and t. In this manner conservation of energy yields a relationship between x and dx/dt, namely

$$\boxed{\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + F(x) = E,} \tag{20.1}$$

where the potential energy only depends on the position,

$$\boxed{F(x) = \int_{x_1}^{x} f(\bar{x})\,d\bar{x},} \tag{20.2}$$

and the total energy is constant,

$$\boxed{E = \frac{1}{2}mv_0^2 + \int_{x_1}^{x_0} f(\bar{x})\,d\bar{x} = \text{constant.}} \tag{20.3}$$

Graphing equation 20.1 as in Fig. 20-1 will yield some curve in the dx/dt vs. x space, for each value of E:

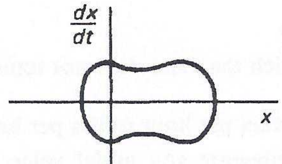

Figure 20-1 Typical energy curve.

For each time, the solution $x(t)$ corresponds to one point on this curve since if $x(t)$ is known so is dx/dt. As time changes, the point corresponding to the solution changes, sketching a curve in the dx/dt vs. x space. Along this curve energy is conserved. This coordinate system is called the **phase plane**, since we have expressed the equation in terms of the two variables x and dx/dt, referred to as the two phases of the system (position and velocity). The curve sketching the path of the solution is called the **trajectory** in the phase plane.

The actual graph of the curves of constant energy (corresponding to different constant values of E) depends on the particular potential energy function, equation 20.2. For example, if $f(x) = kx$ (let $x_1 = 0$), then

$$F(x) = \frac{kx^2}{2}.$$

Sketching the potential energy and the total constant energy yields Fig. 20-2. Since the kinetic energy is positive ($\frac{1}{2}m(dx/dt)^2 \geq 0$), the total energy is

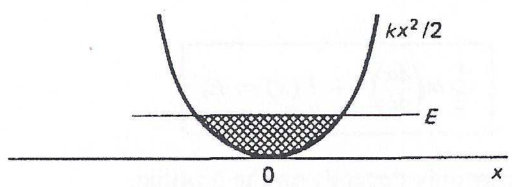

Figure 20-2.

greater than the potential energy, $E \geq kx^2/2$ as sketched in the hatched region. The values of x are restricted. For these values of x there will be two possible values of dx/dt, determined from conservation of energy,

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + k\frac{x^2}{2} = E.$$

In the next sections we will illustrate how to use this information to sketch the phase plane.

Suppose *part* of one such energy curve in the phase plane relating x and dx/dt is known, and looks as sketched in Fig. 20-3. Although x and dx/dt are as yet unknown functions of t, they satisfy a relation indicated by the curve in

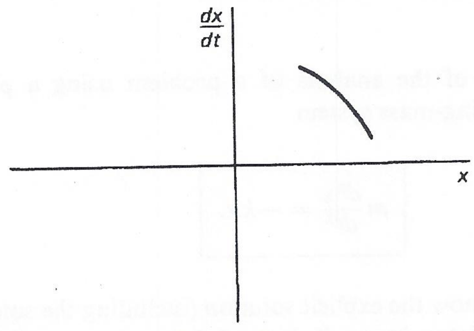

Figure 20-3.

the phase plane. This curve is quite significant because we can determine certain qualitative features of the solution directly from it. For example for the curve in Fig. 20-3, since the solution is in the *upper half plane*, $dx/dt > 0$, it follows that x increases as t increases. Arrows are added to the phase plane diagram to indicate the direction the solution changes with time. In the phase plane shown in Fig. 20-4, since x increases, the solution $x(t)$ moves to the right as time increases. As another example, suppose that the curve shown in Fig. 20-5 corresponds to the solution in the phase plane. Again in the upper half plane $dx/dt > 0$ (and hence x increases). However, in the lower half plane, x decreases.

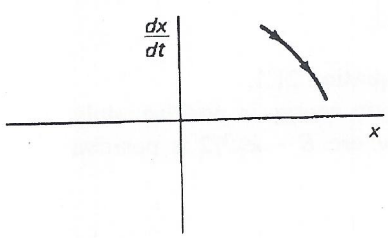

Figure 20-4.

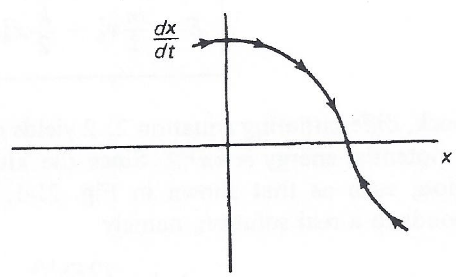

Figure 20-5.

Although the explicit solution of nonlinear equations only occasionally is easily interpreted, the solution in the phase plane often quickly suggests the qualitative behavior.

21. Phase Plane of a Linear Oscillator

As a simple example of the analysis of a problem using a phase plane, consider the linear spring-mass system

$$\boxed{m\frac{d^2x}{dt^2} = -kx.} \tag{21.1}$$

Although we already know the explicit solution (including the solution of the initial value problem), let us ignore it. Instead, let us suppose that we do not know the solution nor any of its properties. We will show how the energy integral determines the qualitative features of the solution.

The energy integral is formed by multiplying the above equation by dx/dt and then integrating:

$$\boxed{\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + k\frac{x^2}{2} = E,} \tag{21.2}$$

where the constant E can be determined by the initial conditions of the mass. Thus,

$$\boxed{E = \frac{m}{2}v_0^2 + \frac{k}{2}x_0^2.} \tag{21.3}$$

As a check, differentiating equation 21.2 yields equation 21.1.

The potential energy is $kx^2/2$. Since the kinetic energy is positive, only the region, such as that shown in Fig. 21-1, where $E - kx^2/2$ is positive corresponds to a real solution, namely

$$|x| < \left(\frac{2E}{k}\right)^{1/2}.$$

The mass cannot have potential energy greater than the total energy.

The energy equation relates x and dx/dt. A typical curve, defined by equation 21.2 corresponding to one value of E, $E = E_0$, is an ellipse in the phase plane shown in Fig. 21-2 with intercepts at $x = \pm\sqrt{2E_0/k}$ and at $dx/dt = \pm\sqrt{2E_0/m}$.

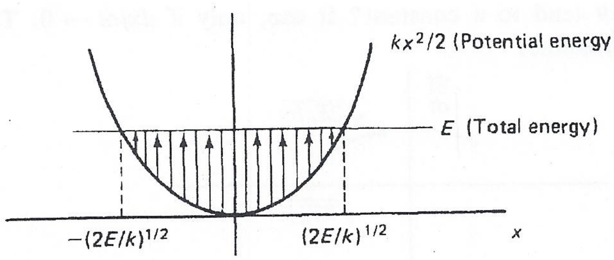

Figure 21-1 Potential energy of a spring-mass system.

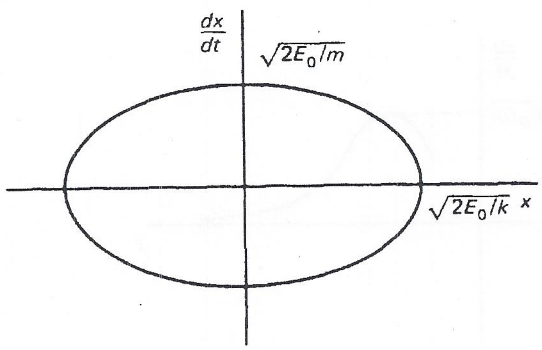

Figure 21-2 Elliptical trajectory in the phase plane.

How does this solution behave in time? Recall, in the upper half plane x increases, while in the lower half plane x decreases. Thus we have Fig. 21-3. The solution goes around and around (clockwise) in the phase plane. After one circuit in the phase plane, no matter where it starts, the solution returns to the same position with the same velocity. It then repeats the same trajectory in the same length of time. This process continues and thus the solution is periodic. Or is it? How do we know the solution in the phase plane doesn't continually move in the direction of the arrow but never reaches a certain point? Suppose that, as in Fig. 21-4, the solution only approaches a point. As illustrated, dx/dt and x tend to constants as $t \longrightarrow \infty$. Can x steadily tend to a

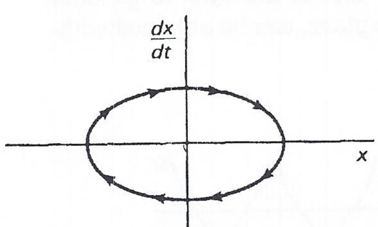

Figure 21-3 Oscillation in the phase plane.

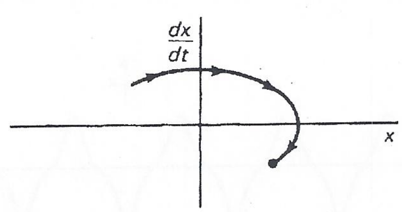

Figure 21-4.

constant, and dx/dt tend to a constant? It can, only if $dx/dt \longrightarrow 0$. Thus Fig. 21-5 appears possible:

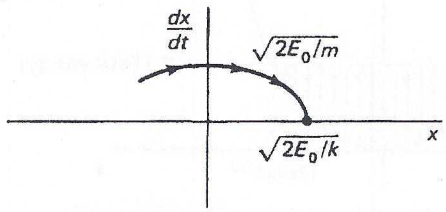

Figure 21-5.

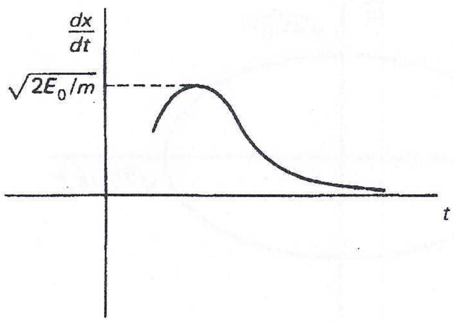

Figure 21-6.

However, if $x \longrightarrow \sqrt{2E_0/k}$ as $t \longrightarrow \infty$, then the above trajectory implies that dx/dt must depend on time as shown in Fig. 21-6. As illustrated dx/dt approaches 0 but does *not* reach that point in a *finite* amount of time. Clearly, $d^2x/dt^2 \longrightarrow 0$. The differential equation, $d^2x/dt^2 = -(k/m)x$, then implies that $x \longrightarrow 0$, i.e., $x \not\rightarrow \sqrt{2E_0/k}$ as $t \longrightarrow \infty$. Consequently, the solution never "stops." It oscillates between a maximum value of x, $x = +\sqrt{2E_0/k}$, and a minimum value, $x = -\sqrt{2E_0/k}$. (Note that at the maximum and minimum values, the velocity, dx/dt equals zero.) Since the solution oscillates, perhaps as shown in Fig. 21-7, the solution is periodic in time. Many of the qualitative features of the solution in the phase plane agree with the exact results.

Furthermore an expression for the period, that is the time to go once completely around the closed curve in the phase plane, can be obtained with-

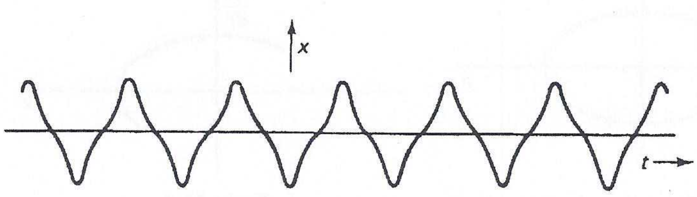

Figure 21-7.

out the explicit solution. The velocity v is determined from equation 21.2 as a function of x,

$$\boxed{v = \pm\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}.} \tag{21.4}$$

Since $v = dx/dt$, it follows that

$$\boxed{dt = \frac{dx}{v},} \tag{21.5}$$

which can be used to determine the period. If we integrate equation 21.5 over an entire period T, then the result is

$$\boxed{T = \oint \frac{dx}{v},} \tag{21.6}$$

where $\oint$ represents the integral of dx/v as the displacement x traverses a complete cycle (the plus sign in equation 21.4 must be used in equation 21.6 if v is positive and vice versa). This calculation is rather awkward. Instead, the time it takes the moving spring-mass system to go from the equilibrium position ($x = 0$) to the maximum displacement ($x = \sqrt{2E_0/k}$) is, by symmetry, exactly one quarter of the period, as indicated by Fig. 21-8. Integrating equation 21.5 in this manner yields

$$\boxed{\frac{T}{4} = \int_0^{\sqrt{2E_0/k}} \frac{dx}{\sqrt{\frac{2E_0}{m} - \frac{k}{m}x^2}},} \tag{21.7}$$

since in this case the sign of v is always positive. The integral in equation 21.7 can be calculated (by trigonometric substitution or by using a table of integrals), yielding the result

$$T = 2\pi\sqrt{\frac{m}{k}}$$

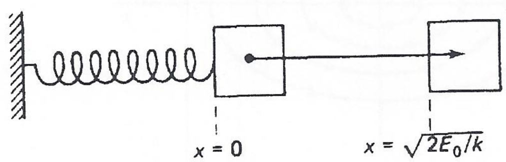

Figure 21-8 Maximum displacement of a spring-mass system.

which agrees with the result obtained from the explicit solution. Interestingly enough the period does not depend on the energy (i.e., the period does not depend on the amplitude of the oscillation). This is a general property of linear systems. Can you give a simple mathematical explanation of why for linear problems the period cannot depend on the amplitude? [Answer: For a linear (homogeneous) differential equation, if $x(t)$ is a solution, then $Ax(t)$ is also a solution for any constant A. Thus if $x(t)$ is periodic with period T, $Ax(t)$ is also periodic with the same period. The period does not depend on the amplitude parameter A.] However, it will be shown that for a nonlinear system the period can depend on the amplitude.

The only information not determined from the phase plane curve is that the solution is exactly sinusoidal (rather than some other periodic function). Appropriate integration of equation 21.5 shows the solution to be sinusoidal. Instead let us briefly show that the ellipses in the phase plane also follow from the knowledge that the solution is sinusoidal. Since $x(t) = A \sin(\omega t + \phi_0)$, where $\omega = \sqrt{k/m}$, it follows that $dx/dt = A\omega \cos(\omega t + \phi_0)$. The phase plane is a curve relating x and dx/dt, but not depending on t. Thus, t must be eliminated from these two equations. This is accomplished as follows:

$$\left(\frac{x}{A}\right)^2 + \left(\frac{\frac{dx}{dt}}{A\omega}\right)^2 = 1,$$

or equivalently

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2 = \frac{k}{2}A^2.$$

In exercise 21.5, it is verified that

$$\frac{k}{2}A^2 = E = \frac{m}{2}v_0^2 + \frac{k}{2}x_0^2.$$

Actually only one closed curve in the phase plane has been illustrated. A few more curves corresponding to other values of the energy E are sketched in Fig. 21-9 to indicate the phase plane for the linear oscillator:

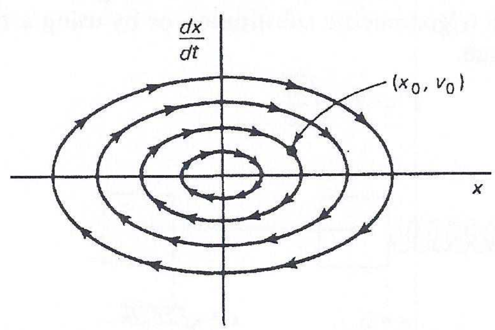

Figure 21-9 Linear oscillator: trajectories in the phase plane.

The initial value problem,

$$x(t_0) = x_0$$
$$\frac{dx}{dt}(t_0) = v_0,$$

is satisfied as follows. First the initial values (x_0, v_0) are located in the phase plane. Then the curve (see Fig. 21-9) which goes through it is determined (note that $E = (m/2)v_0^2 + (k/2)x_0^2$), representing how x changes periodically in time.

In summary, for equation 21.1 the energy curves determined the trajectories in the phase plane. Those energy curves are closed curves implying that the solution oscillates periodically. The amplitude of oscillation is obtained from the initial conditions. Thus the entire qualitative behavior of the solution can be determined by analyzing the phase plane.

EXERCISES

21.1. Suppose the motion of a mass m was described by the nonlinear differential equation $m(d^2x/dt^2) = -\beta x^3$, where $\beta > 0$.
(a) What is the dimensions of the constant β?
(b) What are the equilibrium positions?
(c) Derive an expression for conservation of energy.
(d) Using a phase plane analysis, show that the position x oscillates around its equilibrium position.
(e) If at $t = t_0$, $x = x_0$ and $dx/dt = v_0$, then what is the maximum displacement from equilibrium? Also, what velocity is the mass moving at when it is at $x = 0$?

21.2. Suppose that the motion of a mass m were governed by $m\, d^2x/dt^2 = kx$, where $k > 0$. Show that the phase plane indicates the equilibrium position $(x = 0)$ is unstable.

21.3. Assume that a mass m satisfies $m\,(d^2x/dt^2) = -x^2$.
(a) What are the equilibrium positions?
(b) Derive an expression for conservation of energy.
(c) Using a phase plane analysis, show that for most initial conditions the mass eventually tends towards $-\infty$. Is that reasonable? However, show that for certain initial conditions the mass tends towards its equilibrium position.
(d) How long does it take that solution to approach the equilibrium position?
(e) Would you say the equilibrium solution is stable or unstable?

21.4. Show that the test for the stability of an equilibrium solution by the linearized stability analysis of Sec. 18 is inconclusive for exercises 21.1 and 21.3. Can you suggest a generalization to the criteria developed in Sec. 18?

21.5. The motion of a frictionless spring-mass system with a linear restoring force is described by

$$x(t) = A \sin(\omega t + \phi_0),$$

where $\omega = \sqrt{k/m}$. Show that the total energy E satisfies

$$E = \frac{k}{2}A^2 = \frac{m}{2}v_0^2 + \frac{k}{2}x_0^2,$$

where x_0 is the initial position and v_0 the initial velocity of the mass.

21.6. The phase plane equation for a linear oscillator can be used to directly obtain the solution.

(a) Show that

$$dt = \frac{dx}{\pm\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}}.$$

(b) Assume $x = x_0$ at $t = 0$. (Why is $E \geq (k/2)x_0^2$?) Integrate the above expression to obtain t as a function of x. Now solve for x as a function of t. Is your answer reasonable?

21.7. Evaluate the following integral to determine the period T:

$$\frac{T}{4} = \int_0^{\sqrt{2E/k}} \frac{dx}{\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}}.$$

Show that the period does *not* depend on the amplitude of oscillation.

21.8. Consider the linear spring-mass system, equation 21.1. Show that the average value of the kinetic energy equals the average value of the potential energy, and both equal one-half of the total energy.

22. Phase Plane of a Nonlinear Pendulum

The energy integral sketched in the phase plane can be used to determine the qualitative behavior of nonlinear oscillators. As an example consider the differential equation of a nonlinear pendulum,

$$\boxed{L\frac{d^2\theta}{dt^2} = -g \sin\theta.} \tag{22.1}$$

Multiplying each side of equation 22.1 by $d\theta/dt$ and integrating, yields

$$\frac{L}{2}\left(\frac{d\theta}{dt}\right)^2 = g(\cos\theta - 1) + E. \tag{22.2}$$

Here the potential energy has been calculated relative to the natural position of the pendulum $\theta = 0$:

$$\int_0^\theta g \sin \bar{\theta}\, d\bar{\theta} = g(1 - \cos \theta).$$

At $\theta = 0$, the "energy"* E consists only of kinetic energy. Again as a check, differentiate equation 22.2 with respect to t yielding equation 22.1. The energy E is constant and determined from the initial conditions,

$$E = \frac{L}{2}\Omega_0^2 + g(1 - \cos \theta_0),$$

where $\Omega_0 = d\theta/dt(t_0)$ and $\theta_0 = \theta(t_0)$.

To sketch the trajectories in the phase plane ($d\theta/dt$ as a function of θ), equation 22.2 must be analyzed. Unfortunately equation 22.2 does not represent an easily recognizable type of curve (for the linear oscillator in Sec. 21, we immediately noticed equation 21.2 implied that the trajectories were ellipses). Instead, the energy integral equation 22.2 is used as the basis for sketching the trajectories. First, we sketch in Fig. 22-1 the potential energy $g(1 - \cos \theta)$ as a function of θ:

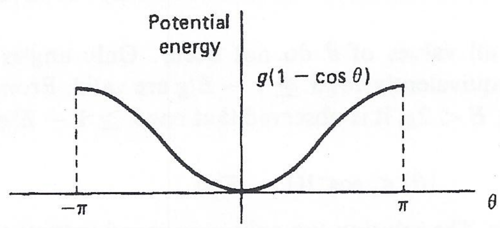

Figure 22-1 Nonlinear pendulum: potential energy.

Drawing vertical lines whose length equals the difference between the total energy E and the potential energy, as in Fig. 22-2 yields an expression for $L/2(d\theta/dt)^2$, the kinetic energy, which must be positive. Figure 22-2 gives a graphical representation of $L/2(d\theta/dt)^2$ dependence on θ, which is easily related to $d\theta/dt$ dependence on θ.

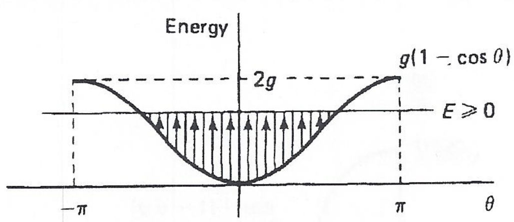

Figure 22-2 Nonlinear pendulum: kinetic energy.

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*E here does not have the units of energy. Instead multiplying equation 22.2 by mL yields an expression for energy. Thus mLE is actually the constant energy.

The solution curve corresponding to $E = 0$ is trivial (see Fig. 22-2). $d\theta/dt = 0$ and hence $\cos\theta = 1$, as marked on Fig. 22-3. These isolated points are the only stable equilibrium positions of a pendulum. If the initial energy is zero, then the pendulum must be at its stable equilibrium position. The pendulum will not move from that position.

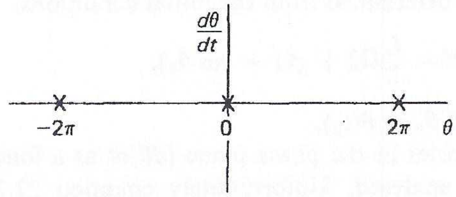

Figure 22-3 Positions of zero energy.

To sketch the remaining curves, it helps to notice that the curves must be even in $d\theta/dt$ (i.e., replacing $d\theta/dt$ by $-d\theta/dt$ does not change eq. 22.2). Also the curves are even in θ. Furthermore, curves in the phase plane are periodic in θ, with period 2π. Thus the curves are sketched only for $d\theta/dt > 0$ and $0 < \theta < \pi$.

For $2g > E > 0$, all values of θ do not occur. Only angles such that $E \geq g(1 - \cos\theta)$ or equivalently $\cos\theta \geq 1 - E/g$ are valid. From Fig. 22-2, where as sketched $0 < E < 2g$, it is observed that $\cos\theta \geq 1 - E/g$ is equivalent to

$$|\theta| \leq \cos^{-1}(1 - E/g)$$

as long as $-\pi \leq \theta \leq \pi$. The solution can only correspond to these angles. Our sketch of the phase plane is improved by noting that the magnitude of $d\theta/dt$ is larger in regions where the difference between the total energy E and the potential energy is the greatest. Thus, for each fixed energy level such that $0 < E < 2g$, $L/2(d\theta/dt)^2 = g(\cos\theta - 1) + E$ yields Fig. 22-4, where we have used the fact that $(d\theta/dt)^2 = 2E/L$ when $\theta = 0$. The calculation of the slope of the curve sketched in Fig. 22-4 is outlined in exercise 22.2. We have included an arrow to indicate changes in the solution as t increases. The evenness in θ

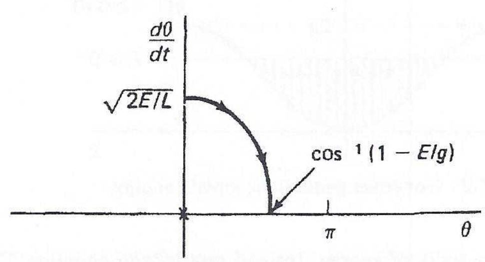

Figure 22-4 Trajectory in the phase plane.

and $d\theta/dt$ yields solutions which as before must be periodic in time, as shown in Fig. 22-5. The periodic solution oscillates around the stable equilibrium position. For each fixed E in this range, the largest angle is called θ_{max} (see Fig. 22-6):

$$\boxed{\theta_{max} = \cos^{-1}(1 - E/g).}$$

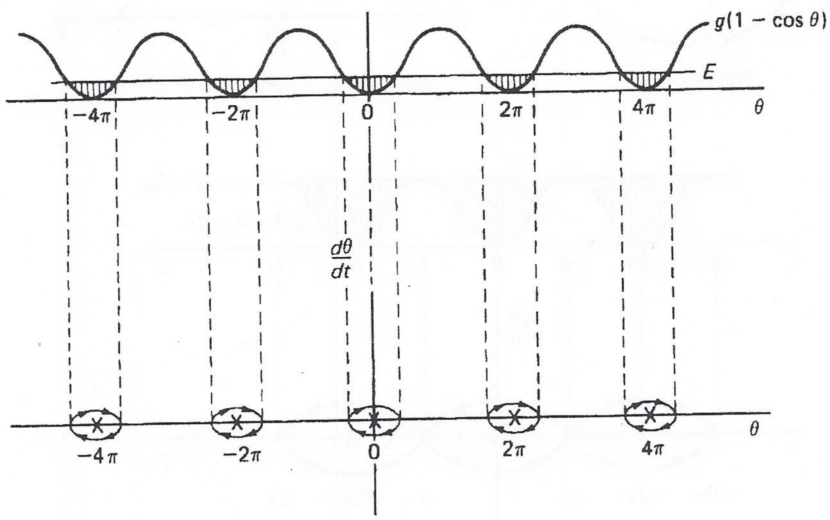

Figure 22-5 Potential energy, kinetic energy, and energy curves.

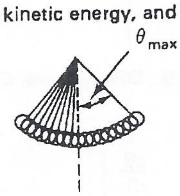

Figure 22-6 Oscillation of a pendulum.

For small energy, the solution is nearly the periodic solution of the linearized pendulum. As E increases away from zero, the motion represents a periodic solution (though not sinusoidal) with larger and larger amplitudes. Sketching the phase plane for other values of E such that $2g > E > 0$, yields Fig. 22-7.

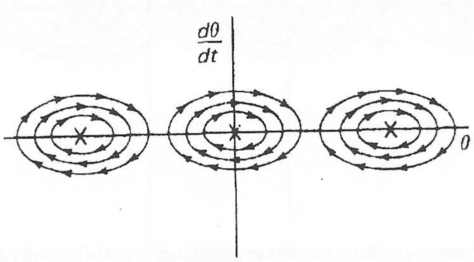

Figure 22-7 Nonlinear pendulum: trajectories in the phase plane (for sufficiently small energy).

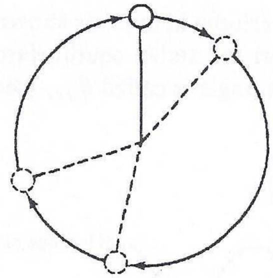

Figure 22-8.

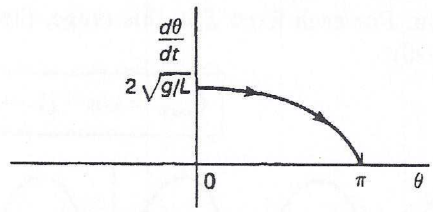

Figure 22-9 Phase plane if $E = 2g$.

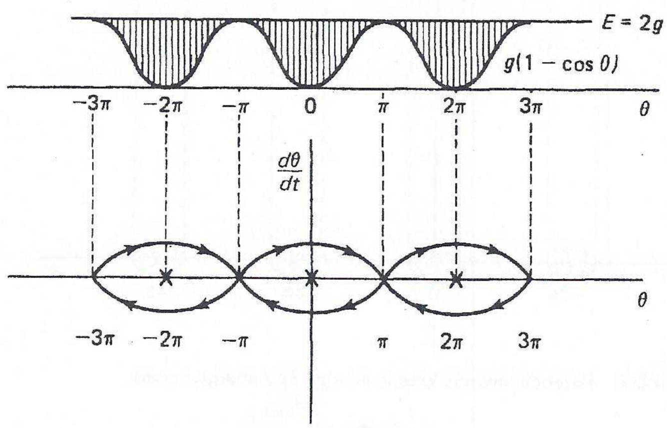

Figure 22-10 Energy curve, $E = 2g$.

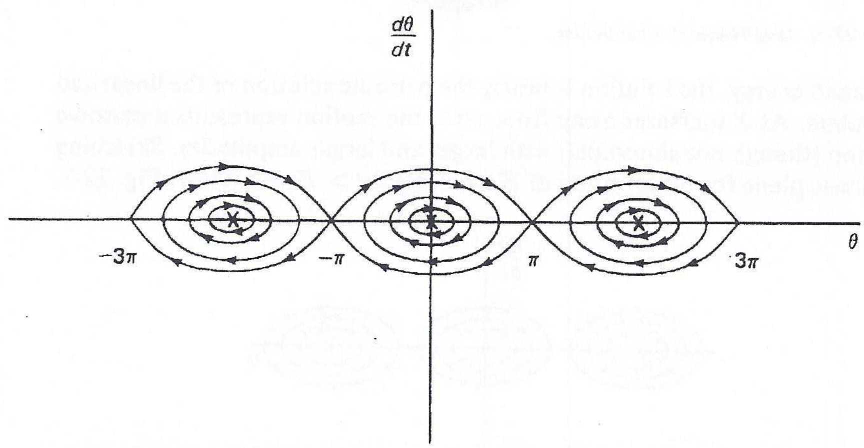

Figure 22-11 Phase plane for a nonlinear pendulum (for sufficiently small energies including critical energy).

If $E = 2g$, the energy is at the level necessary for all angles to be possible as illustrated in Fig. 22-8. In the phase plane, the curve corresponding to $E = 2g$ is that shown in Fig. 22-9. Thus, we have Fig. 22-10. Using this last result, the still incomplete phase plane is sketched in Fig. 22-11.

The energy integral enables us to sketch the trajectories in the phase plane. Note the key steps:

1. Sketch the potential energy as a function of θ.
2. For a representative value of the total energy E, diagram the kinetic energy (the difference between the total energy and the potential energy).
3. From the kinetic energy, sketch the angular velocity $d\theta/dt$ as a function of θ.

EXERCISES

22.1. Suppose a spring-mass system is on a table retarded by a Coulomb frictional force (equation 10.3):

$$m\frac{d^2x}{dt^2} + kx = F_f, \quad \text{where } F_f = \begin{cases} \gamma & \text{if } \dfrac{dx}{dt} < 0 \\ -\gamma & \text{if } \dfrac{dx}{dt} > 0. \end{cases}$$

(a) If $dx/dt > 0$, determine the energy equation. Sketch the resulting phase plane curves. [Hint: By completing the square show that the phase plane curves are ellipses centered at $v = 0$, $x = -\gamma/k$; *not* centered at $x = 0$.]

(b) If $dx/dt < 0$, repeat the calculation of part (a). [Hint: The ellipses now are centered at $v = 0$, $x = \gamma/k$.]

(c) Using the results of (a) and (b), sketch the solution in the phase plane. Show that the mass stops in a finite time!!!

(d) Consider a problem in which the mass is initially at $x = 0$ with velocity v_0. Determine how many times the mass passes $x = 0$ as a function of v_0.

22.2. Consider the phase plane determined by equation 22.2.

(a) Show that $d/d\theta\,(d\theta/dt) = 0$ at $\theta = 0$ (if $E \neq 0$), which has been used in the figures of this section.

(b) Verify $d/d\theta\,(d\theta/dt) = \infty$ at $d\theta/dt = 0$ (if $E \neq 0$ and if $E \neq 2g$) as also assumed in the figures.

(c) If $E = 2g$, calculate $(d/d\theta)\,(d\theta/dt)$, and briefly explain how that information is used in the last three sketches of Sec. 22.

22.3. Consider a nonlinear pendulum. Show that the sum of the potential energy (mgy, where y is the *vertical* distance of the pendulum above its natural position) and the kinetic energy ($\frac{1}{2}mv^2$, where v is the speed of the mass) is a constant. [Hint: See exercise 19.6.]

23. Can a Pendulum Stop?

The phase plane, Fig. 22-10, for the limiting energy curve $E = 2g$ shows that the pendulum tends towards the inverted position (either $\theta = -\pi$ or $\theta = \pi$). For example, $E = 2g$ corresponds to initially starting a pendulum at $\theta = \pi/2$ with just the right velocity such that the pendulum approaches the top with zero velocity, as shown in Fig. 23-1. It appears to reach that position with zero

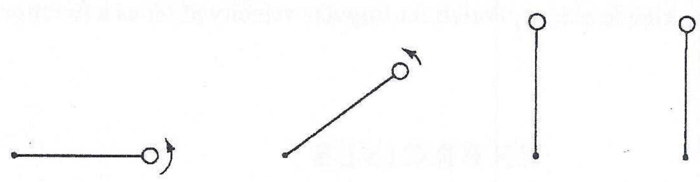

Figure 23-1 Pendulum approaching unstable inverted equilibrium.

angular velocity ($d\theta/dt = 0$). There is some theoretical difficulty with this solution as the uniqueness theorem† for ordinary differential equations implies that if the pendulum ever reaches the top with exactly zero velocity, then it would have to stay there (both in positive and negative time), since that point is an equilibrium position. How do we remedy this nonuniqueness difficulty‡? It will be shown that *the pendulum never reaches the top*; instead it only approaches the top, taking an infinite amount of time to reach the top. We will show this in two ways. Thus our phase plane picture is correct (but slightly misleading).

The first technique will involve an approximation. We would like to know what happens as $\theta \longrightarrow \pi$. How long does it take the pendulum to get there if it is initially close to $\theta = \pi$ with exactly the critical energy? From equation 22.2, since $E = 2g$,

$$L\left(\frac{d\theta}{dt}\right)^2 = 2g(\cos\theta + 1). \tag{23.1}$$

Expanding the above energy equation in a Taylor series around $\theta = \pi$, yields

$$L\left(\frac{d\theta}{dt}\right)^2 = 2g\left[1 + \cos\pi - (\theta - \pi)\sin\pi - \frac{(\theta - \pi)^2}{2!}\cos\pi + \cdots\right].$$

†The theorem states that for equation 22.1 there is a unique solution (for all time) satisfying any given initial conditions.

‡The difficulty is that there *appears* to be more than one solution corresponding to the initial condition at $t = t^*$ that $\theta = \pi$ and $d\theta/dt = 0$. One solution is $\theta = \pi$ for all time and another solution is one in which $\theta \neq \pi$ (at least for $t < t^*$).

We neglect all terms of the Taylor series beyond the first nonzero one. In that manner the following is a reasonable *approximation*:

$$L\left(\frac{d\theta}{dt}\right)^2 = g(\theta - \pi)^2 \quad \text{or equivalently} \quad \frac{d\theta}{dt} = \pm(\theta - \pi)\sqrt{\frac{g}{L}}.$$

The $\pm$ sign indicates a pendulum can be swinging clockwise or counterclockwise either towards or away from the equilibrium position, $\theta = \pi$. For the case we are investigating in which the pendulum swings towards the equilibrium position,

$$\frac{d\theta}{dt} = -(\theta - \pi)\sqrt{\frac{g}{L}} \tag{23.2}$$

(if $\theta < \pi$, then $d\theta/dt > 0$ and if $\theta > \pi$, then $d\theta/dt < 0$). This first-order constant coefficient differential equation is easily solved (especially if $\theta - \pi$, rather than θ, is considered as the dependent variable). Thus $\theta - \pi = Ae^{-(t-t_0)\sqrt{g/L}}$. This equation implies that if θ is initially (at $t = t_0$) near π, then it takes an infinite amount of time for θ to reach π, (i.e., it never reaches π). Furthermore, equation 23.2 shows that the trajectories (for $E = 2g$) can be approximated near $\theta = \pi$ by two straight lines with slopes in the phase plane equal to $\pm\sqrt{g/L}$ (as sketched in Figs. 22-10 and 22-11).

A more rigorous derivation of this result is now briefly discussed. If $E = 2g$, then from equation 23.1

$$\frac{d\theta}{dt} = +\sqrt{\frac{2g}{L}}\sqrt{\cos\theta + 1},$$

where the positive sign is again chosen to ensure that θ increases (assuming $\theta < \pi$ initially). Since

$$\frac{d\theta}{\sqrt{\frac{2g}{L}}\sqrt{\cos\theta + 1}} = dt,$$

if $\theta = \theta_0$ at $t = 0$, then the time t^* at which $\theta = \pi$ is given by

$$\frac{1}{\sqrt{\frac{2g}{L}}}\int_{\theta_0}^{\pi}\frac{d\theta}{\sqrt{\cos\theta + 1}} = t^*.$$

This integral is divergent as $\theta \longrightarrow \pi$. Hence $t^* = \infty$.

EXERCISES

23.1. Assume that a nonlinear pendulum is initially at its stable equilibrium position.

(a) How large an initial angular velocity is necessary for the pendulum to go completely around?

(b) At what initial angular velocity will the pendulum never pass its equilibrium position again?

23.2. If a pendulum is initially at its unstable equilibrium position, then how large an initial angular velocity is necessary for the pendulum to go completely around?

23.3. Consider $d^2x/dt^2 = -f(x)$, with $x = x_0$ a linearly unstable equilibrium position.

(a) Show that

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \int_{x_0}^{x} f(\bar{x})\,d\bar{x} = E = \text{constant}.$$

(b) For what value of the constant E is it possible that the solution approaches this unstable equilibrium point, but approaches it in a manner such that the velocity approaches zero.

(c) Show that if E is the value obtained in part (b), and if x is near x_0, then it will take an infinite amount of time for x to reach $x = x_0$.

23.4. Using a computer, numerically integrate differential equation 22.1 with initial conditions $\theta(0) = 0$, $d\theta/dt(0) = \Omega_0$. [Hint: Recall Sec. 16]. Estimate the value of Ω_0 for which the pendulum first goes all away around. Compare your answer to the exact answer.

24. What Happens If a Pendulum Is Pushed Too Hard?

If $E > 2g$, then there is more initial energy than is needed for the pendulum to almost go around. We thus expect the pendulum to complete one cycle around (see for example Fig. 24-1) and (since there is no friction) continue revolving indefinitely. If $E > 2g$, then Fig. 24-2 shows that all angles occur. Thus in the phase plane we have Fig. 24-3, since $E = L/2(d\theta/dt)^2 + g(1 - \cos\theta)$. Using this result, the sketch of the phase plane for the pendulum is completed in Fig. 24-4. For $E > 2g$, $|\theta|$ keeps increasing (and $\to \infty$). As expected, the pendulum rotates around and around (clockwise or counterclockwise in real space depending on whether θ is increasing or decreasing). The entire qualita-

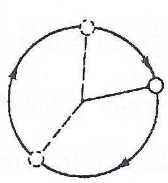

Figure 24-1 Rotating pendulum.

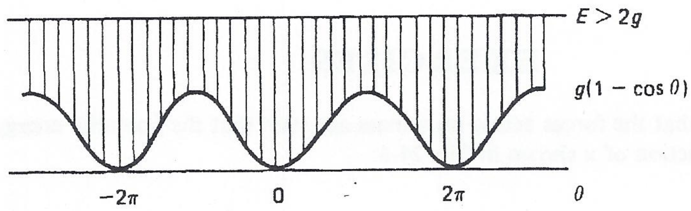

Figure 24-2.

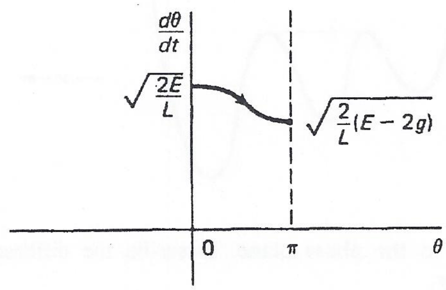

Figure 24-3 Energy curve (large energy).

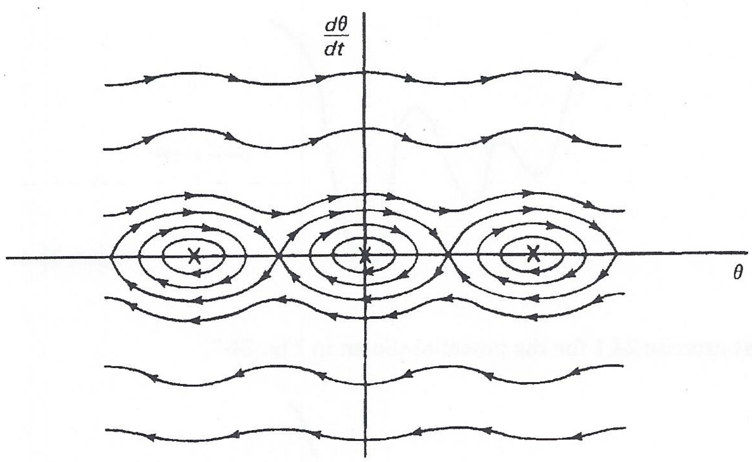

Figure 24-4 Nonlinear pendulum: trajectories in the phase plane.

tive behavior of the nonlinear pendulum has been determined using the energy curves in the phase plane.

Note that the curve in the phase plane for which $E = 2g$ separates the phase plane into two distinct regions of entirely different qualitative behavior. Such a curve is called a **separatrix**, in this case

$$L(d\theta/dt)^2 = 2g(1 + \cos\theta).$$

EXERCISES

24.1. Assume that the forces acting on a mass are such that the potential energy is the function of x shown in Fig. 24-5:

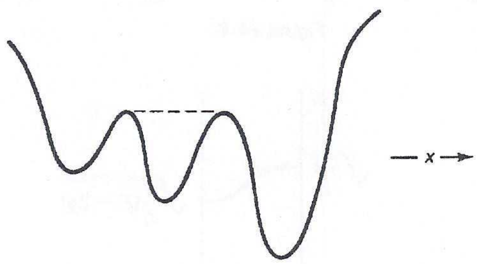

Figure 24-5.

Sketch the solutions in the phase plane. Describe the different kinds of motion that can occur.

24.2. Repeat exercise 24.1 for the potential shown in Fig. 24-6.

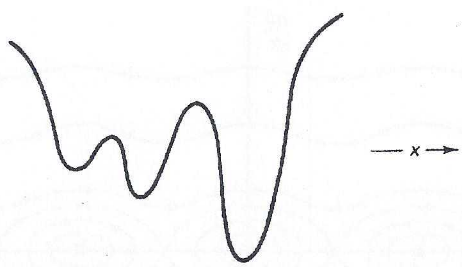

Figure 24-6.

24.3. Repeat exercise 24.1 for the potential shown in Fig. 24-7.

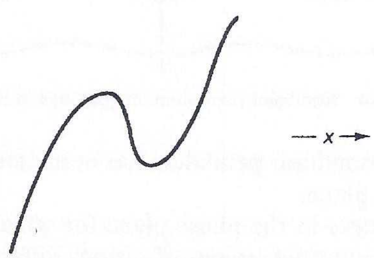

Figure 24-7.

24.4. Assume that the following equation describes a spring-mass system:

$$m\frac{d^2x}{dt^2} = -kx - \alpha x^3,$$

where $\alpha > 0$. Sketch the solution in the phase plane. Interpret the solution (see exercise 18.4).

24.5. Suppose that a spring-mass system satisfies $m(d^2x/dt^2) = -kx + \alpha x^3$, where $\alpha > 0$. Sketch the solution in the phase plane. Interpret the solution (see exercise 18.5).

24.6. Suppose that the potential energy is known. Referring to Fig. 24-8:

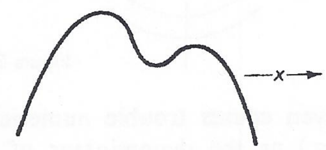

Figure 24-8.

(a) Locate all equilibrium positions.
(b) Sketch the force as a function of x [Hint: Your answer should be consistent with part (a).]
(c) Sketch the solutions in the phase plane.
(d) Explain how part (c) illustrates which of the equilibrium positions are stable and which unstable.

24.7. Suppose $m\,(d^2x/dt^2) = -ke^{2\alpha x}$ where $\alpha > 0$ and $k > 0$.
(a) Determine all (if any) equilibrium positions.
(b) Formulate conservation of energy.
(c) Sketch the solution in the phase plane.
(d) Suppose that a mass starts at $x = -1$. For what initial velocities will the mass reach $x = 0$?

25. Period of a Nonlinear Pendulum

Using the energy integral,

$$\frac{L}{2}\left(\frac{d\theta}{dt}\right)^2 = g(\cos\theta - 1) + E,$$

the qualitative behavior of the nonlinear pendulum,

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta,$$

was obtained in Secs. 22–24. For infinitesimally small amplitudes the period of oscillation is $2\pi\sqrt{L/g}$.

Is the period unaltered as the amplitude becomes larger? Using the energy integral, an expression for the period is obtained (see Sec. 21),

$$T = 4\int_0^{\theta_{\max}} \frac{d\theta}{\sqrt{\frac{2}{L}}\sqrt{g(\cos\theta - 1) + E}}, \qquad (25.1)$$

where $g(\cos\theta_{max} - 1) + E = 0$. (Recall, θ_{max} is the largest angle of oscillation.) This expression is valid when the pendulum oscillates back and forth as in Fig. 25-1 (that is, if $E < 2g$):

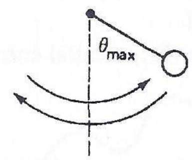

Figure 25-1.

This integral is not easily evaluated. It even causes trouble numerically (rectangles, trapezoids, Simpson's rule, etc.) as the denominator of the integrand $\longrightarrow 0$ at the endpoint $\theta = \theta_{max}$.

In order to determine the manner in which the period T depends on the energy E (or θ_{max}), a change of variables is employed. Let

$$u = 1 - \frac{g}{E}(1 - \cos\theta).$$

Under this transformation, the limits of the integral do not depend on E,

$$\boxed{T(E) = 2\sqrt{\frac{L}{g}}\int_0^1 \frac{du}{u^{1/2}(1-u)^{1/2}\left[1 - \frac{E(1-u)}{2g}\right]^{1/2}}.} \tag{25.2}$$

Although this is still difficult to evaluate analytically or numerically, it can be approximated for small values of E. For $E = 0$,

$$\boxed{T(0) = 2\sqrt{\frac{L}{g}}\int_0^1 \frac{du}{u^{1/2}(1-u)^{1/2}},}$$

corresponding to the period of an infinitesimal amplitude. Since $1 - u \geq 0$, it is seen that for $E > 0$ the denominator of the integrand in (25.2) is smaller than that which occurs when $E = 0$:

$$1 - \frac{E(1-u)}{2g} < 1 \quad \text{for } E > 0 \text{ and } 0 \leq u < 1.$$

Thus

$$T(E) > T(0), \quad \text{for } E > 0,$$

showing that the *period of a nonlinear pendulum is larger than that corresponding to an infinitesimal amplitude.* Evaluating $T(0)$ involves methods of integration. Let

$$w = u^{1/2} \quad \left(dw = \frac{du}{2u^{1/2}}\right), \tag{25.3}$$

in which case, as expected, $T(0)$ equals the period of a linearized pendulum,

$$T(0) = 4\sqrt{\frac{L}{g}}\int_0^1 \frac{dw}{(1-w^2)^{1/2}} = 4\sqrt{\frac{L}{g}}\sin^{-1} w\Big|_0^1 = 2\pi\sqrt{\frac{L}{g}}.$$

Let us attempt to calculate the first effects of the nonlinearity. The period is given by equation 25.2. E is small and thus

$$\frac{E(1-u)}{2g} < 1.$$

For E very small, this quantity is much smaller than 1. Consequently, the first few terms of a Taylor expansion of the integrand (around $E = 0$) will yield a good approximation

$$\left[1 - \frac{E(1-u)}{2g}\right]^{-1/2} = 1 + \frac{E}{4g}(1-u) + \cdots$$

Only the desire to keep the amount of calculations to a minimum, prevents us from developing additional terms in this approximation. The use of the binomial expansion facilitates the above calculation.* Thus

$$T(E) = 2\sqrt{\frac{L}{g}}\int_0^1 \frac{1}{u^{1/2}(1-u)^{1/2}}\left[1 + \frac{E}{4g}(1-u) + \cdots\right] du$$

or, equivalently,

$$T(E) = T(0) + \frac{E}{2g}\sqrt{\frac{L}{g}}\int_0^1 \left(\frac{1-u}{u}\right)^{1/2} du + \cdots$$

To evaluate this additional term, the transformation given by equation 25.3 is again made. Using it yields

$$\int_0^1 \left(\frac{1-u}{u}\right)^{1/2} du = 2\int_0^1 (1-w^2)^{1/2}\, dw.$$

This last integral can be evaluated using trigonometric substitutions, integration-by-parts, or integral tables (for the lazy ones among us). In that manner

$$2\int_0^1 (1-w^2)^{1/2}\, dw = w(1-w^2)^{1/2} + \sin^{-1} w\Big|_0^1 = \frac{\pi}{2}.$$

---

*The binomial expansion

$$(1+a)^n = 1 + na + \frac{n(n-1)}{2}a^2 + \frac{n(n-1)(n-2)}{3!}a^3 + \cdots$$

is an example of a Taylor series. It is valid for all n (including negative and noninteger n) as long as $|a| < 1$. Many approximations requiring a Taylor series need only an application of a binomial expansion, saving the tedious effort of actually calculating a Taylor series via its definition.

Thus, for E small, an expression for the increased period is obtained,

$$T(E) = \sqrt{\frac{L}{g}}\left(2\pi + \frac{\pi}{4}\frac{E}{g} + \cdots\right).$$

The dependence of the period on the energy has been determined for small energies. For larger values of E (corresponding to a larger maximum angle), the period may be obtained by numerically evaluating either equation 25.1 or equation 25.2.

EXERCISES

25.1. (a) If the initial energy is sufficiently large, determine an expression for the time it takes a pendulum to go completely around.

(b) Estimate this time if the energy is very large ($E \gg 2g$). Give a physical interpretation of this answer.

25.2. Consider formula 25.1.

(a) Using a computer, numerically evaluate the period of a nonlinear pendulum as a function of the energy E.

(b) Also determine the period as a function of $\theta_{\max}$.

25.3. Consider the differential equation of a nonlinear pendulum, equation 22.1. Numerically integrate (using a computer) this equation. Evaluate the period as a function of the energy E and of $\theta_{\max}$. [Hint: Assume initially $\theta = \theta_0$ and $d\theta/dt = 0$].

25.4. Compare the results of exercise 25.3 to exercise 25.2.

25.5. (a) From equation 25.2, show that a nonlinear pendulum has a longer period than the linearized pendulum.

(b) Show that $dT/dE > 0$. Briefly describe a physical interpretation of this result.

25.6. If $\theta_{\max} = 5°$, approximately what percentage has the period of oscillation increased from that corresponding to a linearized pendulum?

25.7. The transformation $w = u^{1/2}$ has been used in Sec. 25.

(a) Relate the new variable w directly to θ. [Hint: You will need the trigonometric identity $\cos 2\theta = 1 - 2\sin^2\theta$.]

(b) The integrations performed in Sec. 25 can be analyzed by the trigonometric substitution suggested by the triangle in Fig. 25-2.

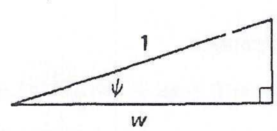

Figure 25-2.

Show that, as defined in Fig. 25-2,

$$\sin\psi = \frac{\sin\theta/2}{\sin k/2},$$

where $\cos k = 1 - E/g$.

(c) Using ψ as the new integration variable, directly transform equation 25.2. Approximate the period for E small, i.e., for $\sin k/2$ small.

26. Nonlinear Oscillations with Damping

In the last few sections, we have analyzed the behavior of nonlinear oscillators neglecting frictional forces. We have found that the properties of nonlinear oscillators are quite similar to those of linear oscillators with the major differences being:

1. For nonlinear oscillators the period (of a periodic solution) depends on the amplitude of oscillation.
2. More than one equilibrium solution is possible.

Since even in linear problems we know that friction cannot be completely neglected, we proceed to investigate systems in which the frictional and restoring-type forces interact in a rather arbitrary way,

$$\boxed{m\frac{d^2x}{dt^2} = h\left(x, \frac{dx}{dt}\right).} \tag{26.1}$$

The forces depend only on the position and velocity of the mass.

In order to understand how to analyze this type of equation, recall that with no friction (but allowing a nonlinear restoring force) a significant amount of information was obtained by considering the energy integral as it related to the solution in the phase plane. However, with a frictional force, energy is not expected to be conserved.

As an example, reconsider the linear oscillator with linear damping,

$$\boxed{m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}.} \tag{26.2}$$

Let us attempt to form an energy integral by multiplying both sides of this equation by dx/dt and then integrating:

$$\boxed{\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2 = E_0 - c\int_0^t \left(\frac{dx}{dt}\right)^2 dt,} \tag{26.3}$$

where E_0 is the initial energy, i.e., the energy at $t = 0$,

$$\boxed{E_0 = \frac{m}{2}v_0^2 + \frac{k}{2}x_0^2.}$$

Explicitly we see that the energy depends on time,

$$\boxed{E(t) = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2;}$$

it is not constant. It decreases in time (see equation 26.3) and is said to dissipate. Thus, if we sketched ellipses in the phase plane (corresponding to the conservation of energy), then the solutions would continually remain inside smaller and smaller ellipses, *perhaps* as illustrated in Fig. 26-1. Thus the solution cannot be periodic. However, the energy equation is not convenient for a more detailed understanding of the solution. We are unable to sketch the phase plane from this equation, as in equation 26.3 dx/dt does not depend only on x. We will return to this example in a later exercise.

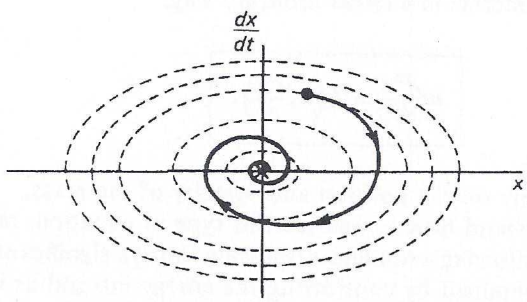

Figure 26-1 Energy dissipation in the phase plane.

A slightly different technique can be used to obtain information about the solution. Reconsider the *general* form of Newton's law allowing restoring and frictional forces,

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right).$$

Here we have incorporated the mass m into the function f. This is *not* the most general form of a second-order differential equation. The most general form is

$$\frac{d^2x}{dt^2} = g\left(x, \frac{dx}{dt}, t\right).$$

In the equation representing Newton's law, there is no explicit dependence on t. Such an equation is called **autonomous.** The simplest property of an autonomous equation is that translating the time origin does not change the equation (see exercise 26.10). Autonomous equations are quite important since in many physical phenomena translation in time is insignificant, and in this text we restrict attention to such equations.

Qualitative features of the solution of autonomous systems can be obtained by considering the equation in the phase plane. An energy integral will not always exist, but it will be shown that an autonomous equation can be interpreted as a relationship between dx/dt and x. Let

$$\boxed{v = \frac{dx}{dt}.}$$

A simple use of the chain rule shows that

$$\boxed{\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.} \qquad \textbf{(26.4)}$$

In this way, the general autonomous equation,

$$\boxed{\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right),} \quad \text{becomes} \quad \boxed{v\frac{dv}{dx} = f(x, v).}$$

Using v and x as variables, a first order differential equation is derived,

$$\boxed{\frac{dv}{dx} = \frac{f(x, v)}{v}.} \qquad \textbf{(26.5)}$$

If an energy integral had existed, then this first order differential equation could be directly integrated yielding solution curves as before. Here we are not necessarily as fortunate since first order differential equations cannot always be solved explicitly. However, the solution of first-order differential equations can always be sketched in the following way. Suppose that

$$\frac{dv}{dx} = g(x, v).$$

At each value of x, the differential equation prescribes the slope of the solution dv/dx (if v is known). Through each point in a v–x plane, a short straight line is drawn with slope equal to $g(x, v)$ as illustrated in Fig. 26-2.

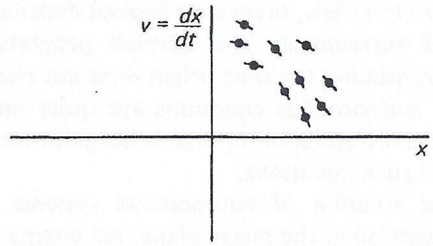

Figure 26-2 Direction field.

This graph is called the **direction field** of the differential equation. At each point the solution must be parallel to these "dashes." By roughly connecting these line segments, the solution in the phase plane (v as a function of x) can be sketched (given x and v initially). To facilitate sketching of the direction field, *curves along which the slope of the solution is a constant* sometimes can be calculated. These curves are called **isoclines*** (Be careful to distinguish between isoclines and solution curves. Sometimes to confuse matters they are the same, but usually they are quite distinct from each other.) From equation 26.5, isoclines are curves along which $f(x, v)/v$ is constant. One isocline is immediately obtainable, namely when $v = 0$, $dv/dx = \infty$. As a review of the method of isoclines, the solution to equation 26.5 is sketched by noting that along the x-axis ($v = 0$) any solution must have an infinite slope. To indicate this small dashes are drawn with infinite slope on Fig. 26-3. $v = 0$ is an isocline. Solution curves which cross the x-axis must be parallel to these dashes.

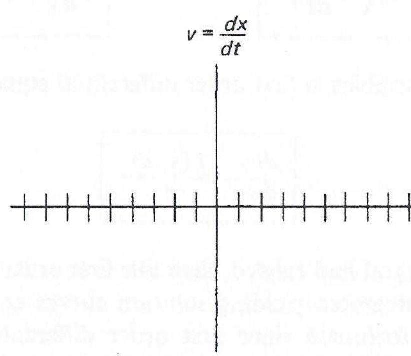

Figure 26-3.

Let us consider a specific example. For a spring-mass system with a linear restoring force but without friction,

$$m\frac{d^2x}{dt^2} = -kx.$$

**Iso* meaning equal as in *iso*tope or *iso*bar; *cline* meaning slope as in in*cline*.

Although we can solve this problem explicitly or solve it by sketching the solution in the phase plane using the energy integral, we will illustrate the method of isoclines to sketch the solution in the phase plane. Letting $v = dx/dt$ and using the chain rule $(d^2x/dt^2) = v(dv/dx)$ yields the first order differential equation

$$\frac{dv}{dx} = -\frac{kx}{mv}. \tag{26.6}$$

(Explicitly integrating this equation is equivalent to the energy integral.) We sketch the solution by first drawing the isoclines, curves along which the slope of the solution is constant. As above along $v = 0$, $dv/dx = \infty$. The isocline is the straight line $v = 0$, along which the slope of the solution is always infinite, indicated by the vertical slashes. Furthermore $x = 0$ is another isocline, as along $x = 0$, $dv/dx = 0$ (a constant). Other isoclines, for example, are $v = (k/m)x$ along which $dv/dx = -1$, and $v = -(k/m)x$ along which $dv/dx = +1$. These isoclines are sketched in Fig. 26-4. To locate the

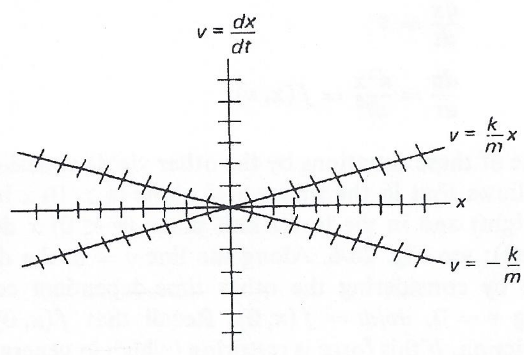

Figure 26-4 Isoclines for $md^2x/dt^2 = -kx$.

most general isocline, we look for the curve along which the slope of the solution is the constant λ, $dv/dx = \lambda$. From equation 26.6,

$$-\frac{k}{m}\frac{x}{v} = \lambda.$$

Solving for v, we see these isoclines are $v = -(k/m\lambda)x$. *For this example* all the isoclines are straight lines (in general isoclines need *not* be straight lines). We indicate in Fig. 26-5 some of these other isoclines. Any solution must be tangent to the slashes. In this manner a solution in the phase plane can be sketched (as indicated by the darkened curve in Fig. 26-5). In particular, note the difference between the solution curves (ellipses) and the isoclines (straight lines). Actually a *rough* sketch might not guarantee solutions are the ellipses which our previous analysis tells us they must be.

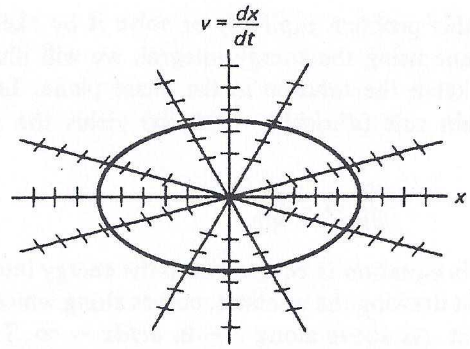

Figure 26-5 Direction field sketched by the method of isoclines.

What happens as time increases? As before when considering energy curves in the phase plane, arrows are used to indicate the direction of changes in time. We note (returning to the general time-dependent equations),

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = f(x, v).$$

(Note that dividing one of these equations by the other yields equation 26.5.) From $dx/dt = v$, it follows that in the upper half plane ($v > 0$) x increases (arrows point to the right) and in the lower half plane ($v < 0$) x decreases (arrows point to the left); see Fig. 26-6. Along the line $v = 0$, the direction in time is determined by considering the other time-dependent equation, $dv/dt = f(x, v)$. Along $v = 0$, $dv/dt = f(x, 0)$. Recall that $f(x, 0)$ is the "force" assuming no friction. *If* this force is restoring (which in general it does *not* have to be),

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right),$$

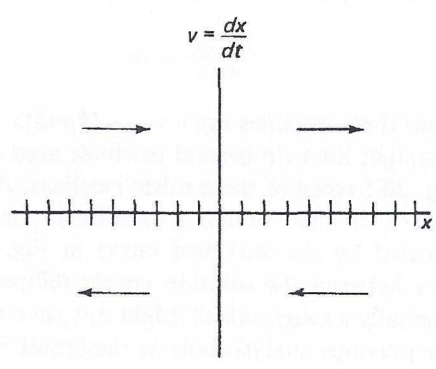

Figure 26-6.

then it is concluded that for $x > 0$, $f(x, 0) < 0$ (and vice versa). Hence $dv/dt > 0$ for $x < 0$ at $v = 0$, and thus v increases in the left half plane. Similarly for $x > 0$ at $v = 0$, v decreases in the right half plane. These results yield (if the force is always restoring) Fig. 26-7.

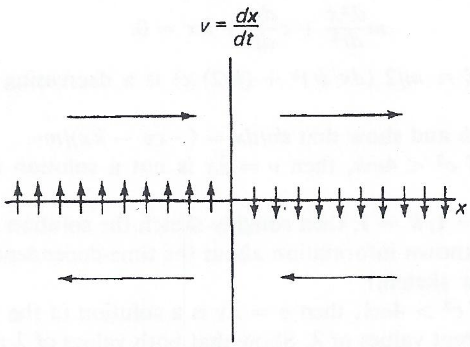

Figure 26-7.

For the example $m(d^2x/dt^2) = -kx$, arrows can be added to the phase plane yielding Fig. 26-8, the same result as previously sketched in Sec. 21.

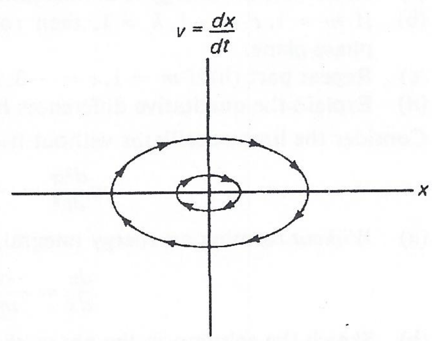

Figure 26-8 Phase plane for a linear spring-mass system.

EXERCISES

26.1. Consider $dv/dx = (cx + dv)/(ax + bv)$. Curves along which the slope (dv/dx) is constant are called **isoclines**. Thus the method of isoclines implies

$$\frac{dv}{dx} = \lambda \quad \text{when } \lambda = \frac{cx + dv}{ax + bv}.$$

(a) For this problem, show that the isoclines are all straight lines.

(b) What is the slope of each isocline? (Be careful to distinguish between the slope of the solution and the slope of an isocline.)
(c) Is it possible for the slope of an isocline to be the same as the slope of the solution? In this case show that the isocline itself is a solution curve.

26.2. Consider a linear oscillator with linear friction:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

(a) Show that $E \equiv m/2\,(dx/dt)^2 + (k/2)\,x^2$ is a decreasing function of time.
(b) Let $v = dx/dt$ and show that $dv/dx = (-cv - kx)/mv$.
(c) Show that if $c^2 < 4mk$, then $v = \lambda x$ is not a solution in the phase plane.
(d) If $m = 1, c = 1, k = 1$, then roughly sketch the solution in the phase plane. (Use known information about the time-dependent solution to improve your sketch.)
(e) Show that if $c^2 > 4mk$, then $v = \lambda x$ is a solution in the phase plane for two different values of λ. Show that both values of λ are negative.
(f) If $m = 1, c = 3, k = 1$, then roughly sketch the solution in the phase plane. [Hint: Use the results of part (e)].
(g) Explain the qualitative differences between parts (d) and (f).

26.3. Reconsider exercise 26.2 if friction is negative, $c < 0$.
(a) Show that the energy is an increasing function of time.
(b) If $m = 1, c = -1, k = 1$, then roughly sketch the solution in the phase plane.
(c) Repeat part (b) if $m = 1, c = -3, k = 1$.
(d) Explain the qualitative differences between parts (b) and (c).

26.4. Consider the linear oscillator without friction:

$$m\frac{d^2x}{dt^2} = -kx.$$

(a) *Without* forming an energy integral, let $v = dx/dt$ and show that

$$\frac{dv}{dx} = \frac{-kx}{mv}.$$

(b) Sketch the solution in the phase plane.
(c) Interpret the solution.

26.5. Briefly explain why only one solution curve goes through each point in the phase plane except for an equilibrium point in which case there may be more than one.

26.6. Suppose that $dv/dx = v^2 - x$.
(a) Show that the isoclines are not straight lines.
(b) Sketch the solution.

26.7. Consider a spring-mass system with cubic friction

$$m\frac{d^2x}{dt^2} + \sigma\left(\frac{dx}{dt}\right)^3 + kx = 0, \qquad \sigma > 0.$$

Show that $E \equiv m/2(dx/dt)^2 + k/2\,x^2$ is a decreasing function of time.

26.8. Consider

$$m\frac{d^2x}{dt^2} = -kx - \alpha x^3 - \sigma\left(\frac{dx}{dt}\right)^3.$$

Assume $k > 0$, $m > 0$, $\alpha > 0$, and $\sigma > 0$. Show that as $t \longrightarrow \infty$, $x \longrightarrow 0$. [Hint: Form an energy integral and show $dE/dt \leq 0$. Since $E \geq 0$ (why?), show that Fig. 26-9 implies that $E \longrightarrow 0$:

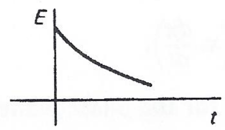

Figure 26.9 Energy decay.

Why can't $E \longrightarrow E_0 > 0$ as $t \longrightarrow \infty$?]

26.9. Reconsider exercise 26.8 if $\alpha < 0$. Show that the conclusion of the problem may no longer be valid.

26.10. Consider the general second-order autonomous equation:

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right).$$

Show that if $x = g(t)$ is a solution, then $x = g(t - t_0)$ is another solution for any t_0.

26.11. Assume

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + f(x) = 0,$$

with $c > 0$ and $f(0) = 0$.

(a) Give a physical interpretation of this problem.

(b) By considering the energy, show that there are no periodic solutions (other than $x(t) \equiv 0$).

26.12. Show that if a solution in the phase plane is a straight line, it corresponds to x growing or decaying exponentially in time.

26.13. Consider

$$m\frac{d^2x}{dt^2} = \alpha(e^{\beta x} - 1) - c\frac{dx}{dt}.$$

(a) What first-order differential equation determines the solution curves in the phase plane?

(b) What curves are isoclines?

26.14. Suppose that $m\, d^2x/dt^2 = kx$ with $m > 0$ and $k > 0$.

(a) Briefly explain why $x = 0$ is an unstable equilibrium position.

(b) Using the method of isoclines sketch the solution in the phase plane (for ease of computation, let $m = 1$ and $k = 1$). [Hint: At least sketch the isoclines corresponding to the slope of the solution being 0, ∞, ± 1.]

(c) Explain how part (b) illustrates part (a).

27. Equilibrium Positions and Linearized Stability

The general autonomous system,

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right),$$

yields a first-order differential equation for the phase plane

$$\boxed{\frac{dv}{dx} = \frac{f(x, v)}{v}} \tag{27.1}$$

where $v = dx/dt$ (see Sec. 26). The slope of the solution in the phase plane is uniquely determined everywhere. Well, not quite: at any point where both the numerator and denominator is zero, dv/dx is not uniquely determined since $dv/dx = 0/0$ (it depends on how you approach that point). Such points are called singular points of the phase plane equation 27.1. Singular points occur whenever

$$\boxed{\begin{aligned} v &= 0 \\ f(x, 0) &= 0. \end{aligned}} \tag{27.2}$$

In other words, the velocity is zero, $v = 0$, and there are no forces at any such singular point. These singular points thus represent **equilibrium positions**, values of x for which the forces cancel if there is no motion. For example, such points were encountered in the discussion of a nonlinear pendulum without friction. As in that problem we are quite interested in determining which such equilibrium points are stable.

As has been shown in Sec. 18, stability can be investigated most easily by considering a **linearized stability analysis**. The analysis here differs from the previous one only by certain mathematical details now necessitated by the possible velocity dependence of the force. Suppose that $x = x_E$ is an equilibrium time-independent solution of the equation of motion,

$$\boxed{\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right).} \tag{27.3}$$

If x is initially near x_E with a small velocity, then it is reasonable to expand

$f(x, dx/dt)$ in a Taylor series of a function of two variables*:

$$f\left(x, \frac{dx}{dt}\right) = f(x_E, 0) + (x - x_E)\frac{\partial f}{\partial x}\bigg|_{x_E, 0} + \frac{dx}{dt}\frac{\partial f}{\partial\left(\frac{dx}{dt}\right)}\Bigg|_{x_E, 0} + \cdots$$

Since $f(x_E, 0) = 0$ (x_E is an equilibrium solution),

$$\frac{d^2x}{dt^2} = (x - x_E)\frac{\partial f}{\partial x}\bigg|_{x_E, 0} + \frac{dx}{dt}\frac{\partial f}{\partial\left(\frac{dx}{dt}\right)}\Bigg|_{x_E, 0},$$

where higher-order terms in the Taylor series have been neglected since x is near x_E and dx/dt is small. Again introducing the displacement from equilibrium, z,

$$z = x - x_E,$$

it follows that

$$\boxed{\frac{d^2z}{dt^2} = -kz - c\frac{dz}{dt},} \tag{27.4}$$

where

$$\boxed{-k = \frac{\partial f}{\partial x}\bigg|_{x_E, 0}}$$

$$\boxed{-c = \frac{\partial f}{\partial\left(\frac{dx}{dt}\right)}\Bigg|_{x_E, 0}}$$

The notation used has taken advantage of the analogy of equation 27.4 to a spring-mass system with friction. However, here it is *not* necessary that k and c be positive!

This equation is a constant coefficient second-order homogeneous ordinary differential equation; exactly the kind analyzed earlier. Solutions are exponentials e^{rt}, where

$$\boxed{r = \frac{-c \pm \sqrt{c^2 - 4k}}{2}}$$

*The formula for the Taylor series of a function of two variables is

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + h\frac{\partial f}{\partial x}\bigg|_{x_0, y_0} + k\frac{\partial f}{\partial y}\bigg|_{x_0, y_0} + \cdots$$

or equivalently

$$f(x, y) = f(x_0, y_0) + (x - x_0)\frac{\partial f}{\partial x}\bigg|_{x_0, y_0} + (y - y_0)\frac{\partial f}{\partial y}\bigg|_{x_0, y_0} + \cdots$$

(except if $c^2 = 4k$, in which case the solutions are $e^{-ct/2}$ and $te^{-ct/2}$). The equilibrium solution is said to be linearly stable if, for all initial conditions near $x = x_E$ and $v = 0$, the displacement from equilibrium does not grow. The following table indicates the behavior of the equilibrium solution:

$$c^2 - 4k > 0 \quad \begin{cases} \text{Unstable if } c \leq 0. \\ \text{Also unstable if } c > 0 \text{ but } k < 0. \\ \text{Otherwise stable (i.e., } c > 0 \text{ and } k \geq 0). \end{cases}$$

$$c^2 - 4k = 0 \quad \begin{cases} \text{Stable if } c > 0. \\ \text{Unstable if } c \leq 0. \end{cases}$$

$$c^2 - 4k < 0 \quad \begin{cases} \text{Unstable if } c < 0. \\ \text{Stable if } c \geq 0 \text{ (sometimes said to be} \\ \textit{neutrally stable}^* \text{ if } c = 0 \text{ since the solu-} \\ \text{tion purely oscillates if } c = 0). \end{cases}$$

This information can also be communicated using a stability diagram in c–k parameter space, Fig. 27-1. The equilibrium position is stable only if the linearized displacement z satisfies a differential equation corresponding to a linear spring-mass system $k \geq 0$ with damping $c \geq 0$ (except if $c = k = 0$).

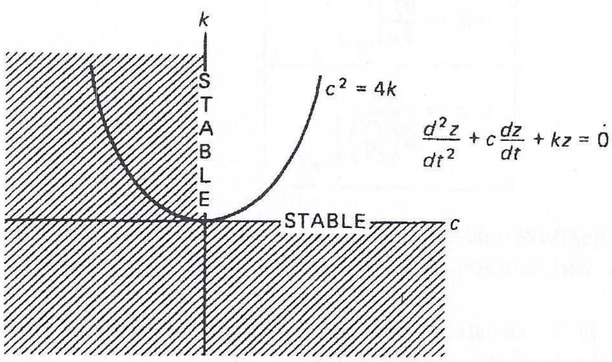

Figure 27-1 Stability diagram (the hatched region is *unstable*).

As an example, suppose that

$$\frac{d^2x}{dt^2} = -(x - 4) + x^3\frac{dx}{dt}.$$

We see that $x = 4$ is the only equilibrium position ($dx/dt = 0$ and $d^2x/dt^2 = 0$). Letting $f(x, dx/dt) = -(x - 4) + x^3(dx/dt)$, we can determine the stability of $x = 4$ by simply calculating the partial derivatives of $f(x, dx/dt)$:

*See page 58.