Exam 2 solutions

1. $$A_t - \hat{A}_t = -0.2\,(B_t - \hat{B}_t)$$

$$A_t - (100 - 6G_t) = -0.2\,(B_t - (60 - 5G_t))$$

$$A_t = 100 - 6G_t - 0.2B_t + 12 - G_t$$

predicted $A_t = \boxed{112 - 7G_t - 0.2B_t}$

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2. $$X = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \tilde{X} = \begin{bmatrix} 1 & -2 \\ \vdots & \vdots \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 4 & -1 & 7 \\ -1 & 7 & -7 \\ 7 & -7 & 19 \end{bmatrix}$$

model $\quad y = \beta_0 + \beta_1 x + \beta_2 x^2$

$$\tilde{\beta} = (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T y$$

$$\text{Bias}(\tilde{\beta}) = E[\tilde{\beta}] - \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T [\underset{4\times 2}{\tilde{X}} \;\; \underset{4\times 1}{x^2}]\,\beta - \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

$$= 0 + \begin{bmatrix} 4 & -1 \\ -1 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \beta_2$$

$$= \frac{1}{27} \begin{bmatrix} 7 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ -7 \end{bmatrix} \beta_2$$

$$= \frac{\beta_2}{27} \begin{bmatrix} 42 \\ -21 \end{bmatrix} = \begin{bmatrix} 14\beta_2/9 \\ -7\beta_2/9 \end{bmatrix}$$

$$\text{Bias}(\tilde{\beta}_0) = \boxed{\frac{14\beta_2}{9}}$$

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3. (a) $$\ell(\mu_1,\mu_2,\mu_3,\sigma^2) = \prod_{i=1}^{3}\prod_{j=1}^{6} \ln f(y_{ij}|\mu_1,\mu_2,\mu_3,\sigma^2)$$

$$= \prod_{i=1}^{3}\prod_{j=1}^{6} \ln\left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y_{ij}-\mu_i)^2}\right]$$

$$= -\frac{1}{2\sigma^2}\sum_{i=1}^{3}\sum_{j=1}^{6}(y_{ij}-\mu_i)^2 - 9\ln\sigma^2 - 9\ln(2\pi)$$

$$\frac{\partial \ell}{\partial \mu_{i'}} = -\frac{1}{2\sigma^2}\sum_{j=1}^{6}(y_{i'j}-\mu_{i'})(-2) = \frac{1}{\sigma^2}\sum_{j=1}^{6}(y_{i'j}-\mu_{i'}) = 0$$

$$\Updownarrow$$

$$\boxed{\hat{\mu}_{i'} = \bar{y}_{i'\cdot}} \quad \text{for } i'=1,2,3$$

$$\frac{\partial \ell}{\partial \sigma^2} = \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{3}\sum_{j=1}^{6}(y_{ij}-\mu_i)^2 - \frac{9}{\sigma^2} = 0$$

$$\Updownarrow$$

$$\boxed{\hat{\sigma}^2 = \frac{1}{18}\sum_{i=1}^{3}\sum_{j=1}^{6}(y_{ij}-\bar{y}_{i\cdot})^2}$$

(b) $$RSS_H = \sum_{i=1}^{3}\sum_{j=1}^{6}(y_{ij}-\bar{y}_{\cdot\cdot})^2$$

$$RSS = \sum_{i=1}^{3}\sum_{j=1}^{6}(y_{ij}-\bar{y}_{i\cdot})^2$$

$$F = \frac{(RSS_H - RSS)/2}{RSS/(18-3)} = \frac{15}{2}\cdot\frac{(RSS_H-RSS)}{RSS_H}\cdot\frac{RSS_H}{RSS}$$

$$= \frac{15}{2}\left(1-\frac{RSS}{RSS_H}\right)\cdot\frac{1}{\frac{RSS}{RSS_H}}$$

$$= \frac{15}{2} R^2\cdot\frac{1}{(1-R^2)}$$

Reject H_0 if $\quad F > \underset{3.68}{\underset{\shortparallel}{F(.05;2,15)}}$

$$\Updownarrow$$

$$7.5 \frac{R^2}{1-R^2} > 3.68$$

$$7.5R^2 > 3.68 - 3.68R^2$$

$$11.18R^2 > 3.68$$

$$\boxed{R^2 > \frac{3.68}{11.18}}$$

$$R^2 > 0.329$$

4.

$$y_{111} = \mu + \alpha_1 + \beta_1 + \gamma_{11} + e_{111}$$
$$y_{112} = \mu + \alpha_1 + \beta_1 + \gamma_{11} + e_{112}$$
$$y_{121} = \mu + \alpha_1 - \beta_1 - \gamma_{11} + e_{121}$$
$$y_{122} = \mu + \alpha_1 - \beta_1 - \gamma_{11} + e_{122}$$
$$y_{211} = \mu - \alpha_1 + \beta_1 - \gamma_{11} + e_{211}$$
$$y_{212} = \mu - \alpha_1 + \beta_1 - \gamma_{11} + e_{212}$$
$$y_{211} = \mu - \alpha_1 - \beta_1 + \gamma_{11} + e_{221}$$
$$y_{212} = \mu - \alpha_1 - \beta_1 + \gamma_{11} + e_{222}$$

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad y = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 5 \\ 1 \\ 3 \\ 1 \\ 5 \end{bmatrix}$$

$\hat{y}_{111} = \hat{y}_{112} = \bar{y}_{11\cdot} = 5, \quad \hat{y}_{121} = \hat{y}_{122} = \bar{y}_{12\cdot} = 6, \quad \hat{y}_{211} = \hat{y}_{212} = \bar{y}_{21\cdot} = 2, \quad \hat{y}_{221} = \hat{y}_{222} = \bar{y}_{22\cdot} = 3$

RSS =

$$\|y - \hat{y}\|^2 = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(y_{ijk} - \hat{y}_{ijk})^2 = 6 \cdot 1^2 + 2 \cdot 2^2 = 14$$

If $\alpha_1 = 0$, then use

$$\tilde{X} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow, \quad \beta^* = \begin{bmatrix} \mu \\ \beta_1 \\ \gamma_{11} \end{bmatrix}$$

$$\tilde{\beta} = (\tilde{X}^T\tilde{X})^{-1}\tilde{X}^T y = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 32 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 & 0 \\ 0 & 1/8 & 0 \\ 0 & 0 & 1/8 \end{bmatrix} \begin{bmatrix} 32 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -1/2 \\ 0 \end{bmatrix}$$

$$\tilde{y} = X\tilde{\beta} = \begin{bmatrix} 3.5 \\ 3.5 \\ 4.5 \\ 4.5 \\ 3.5 \\ 3.5 \\ 4.5 \\ 4.5 \end{bmatrix} \Rightarrow \quad RSS_H = \|y - \tilde{y}\|^2$$

$$= 2.5^2 + 0.5^2 + -2.5^2 + 0.5^2 + (-2.5)^2 + (-0.5)^2 + (-3.5)^2 + 0.5^2$$

$$= 32$$

$$F = \frac{(RSS_H - RSS)/1}{RSS/(8-4)} = \frac{18}{14/4} = \frac{36}{7} \approx 5.14$$

$F(.05; 1, 4) = 7.71$

So we fail to reject H_0 since $F < F(.05; 1, 4)$.

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5. (a) $E[y_{ij}] = E[\mu + a_i + e_{ij}] = \mu + E[a_i] + E[e_{ij}] = \mu + 0 + 0 = \boxed{\mu}$

(b) $var[y_{ij}] = var[\mu + a_i + e_{ij}] = var[a_i] + var[e_{ij}] = \boxed{\sigma_a^2 + \sigma_e^2}$

(c) $cov(y_{ij}, y_{ij'}) = cov(a_i + e_{ij}, a_i + e_{ij'})$

$= var(a_i) + cov(a_i, e_{ij'}) + cov(e_{ij}, a_i) + cov(e_{ij}, e_{ij'})$

$= \sigma_a^2 + 0 + 0 + 0 = \boxed{\sigma_a^2}$

(d) $\text{cov}(y_{ij}, y_{i'j}) = \text{cov}(a_i + e_{ij}, a_{i'} + e_{i'j}) = \boxed{0}$

since $a_i, a_{i'}, e_{ij}, e_{i'j}$ are independent

(e) $\text{cov}(y_{ij}, y_{i'j'}) = \text{cov}(a_i + e_{ij}, a_{i'} + e_{i'j'}) = \boxed{0}$

since $a_i, a_{i'}, e_{ij}, e_{i'j'}$ are independent