MATH 667-01 Homework 3

Due: Thursday, September 21, 2017

Instructions: *Each of the following problems must be submitted to the instructor on or before the due date. Partial credit may be given for incorrect answers which make some positive progress. Late homework will not be accepted.*

1. (10 points) A random variable is said to have a Cauchy(μ, σ) pdf if its pdf has the form

$$f(x|\mu,\sigma) = \frac{1}{\sigma\pi\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)}.$$

The mean and variance do not exist for the Cauchy distribution. So the parameters μ and σ^2 are not the mean and variance. But they do have important meanings. Show that if X is a random variable with a Cauchy(μ, σ) distribution, then:

(a - 5 pts) μ is the median of the distribution of X, that is, $P(X \geq \mu) = P(X \leq \mu) = \frac{1}{2}$.

(b - 5 pts) $\mu + \sigma$ and $\mu - \sigma$ are the quartiles of the distribution of X, that is,
$P(X \geq \mu + \sigma) = P(X \leq \mu - \sigma) = \frac{1}{4}$.
(Hint: Prove this first for $\mu = 0$ and $\sigma = 1$ and then use properties of location-scale families.)

2. (10 points) Let X be a Beta$(\alpha, 1)$ random variable with pdf

$$f_X(x|\alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} I_{(0,\infty)}(x)$$

where $\alpha > 0$.

(a - 5 pts) A family of probability density functions is called an exponential family if it can be expressed as $f(x|\theta) = h(x)c(\theta)\exp\{w(\theta)t(x)\}$. Is $\{f_X(x|\alpha)\}$ an exponential family? If yes, define θ and find $h(x)$, $c(\theta)$, $w(\theta)$, and $t(x)$. If not, justify your answer.

(b - 5 pts) Express E$[\ln X]$ in terms of the function Γ and its derivatives.

3. (10 points) Suppose that $X_1, \ldots, X_n$ are independent Poisson(λ) random variables with pmf

$$P(X_i = x) = \begin{cases} \frac{1}{x!}\lambda^x e^{-\lambda} & \text{if } x \text{ is a nonnegative integer} \\ 0 & \text{otherwise} \end{cases}.$$

(a - 5 pts) Does the pmf of $Y = \sum_{i=1}^{n} X_i$ belong to an exponential family? Justify your answer.

(b - 5 pts) Let $n = 3$ and $\lambda = 1.2$. Compute $P(\bar{X} \geq \lambda)$.