M622 HW, due Mar. 21. Write explanations in sentences.

1. Let K/F be a splitting field of a separable polynomial $t(x) \in F[x]$ of degree n, and let $\{r_1, \dots, r_n\} \subseteq K$ be the roots of $t(x)$. You have proven that if $\sigma \in Aut(K/F)$, and r is one of the roots of $t(x)$, then $\sigma(r)$ is also a root of $t(x)$. Prove it again below.

2. The splitting field K is generated by K and $\{r_1, \dots, r_n\}$, i.el, $K = F(r_1, \dots, r_n)$. As we've shown, every element in $k \in K$ can be expressed as $q(r_1, \dots, r_n)$, where q is a multivariate polynomial $q(x_1, \dots, x_n)$ with coefficients in F. Briefly explain why it is that if $\sigma \in Aut(K/F)$ fixes each of the roots $\{r_1, \dots, r_n\}$ of $t(x)$, then $\sigma = id_K$. (You will have shown that $Aut(K/F)$ acts **faithfully** on the roots $\{r_1, \dots, r_n\}$ of $t(x)$, and, therefore, that $Aut(K/F)$ is isomorphic to a subgroup of S_n.)

3. Let $t(x) = x^4 - 3 \in \mathbb{Q}[x]$. Determine the roots of $t(x)$ in $\mathbb{C}$, determine the splitting field K for $t(x) = x^4 - 3 \in \mathbb{Q}[x]$, and determine $[K : \mathbb{Q}]$.

   (a) List the roots:

   (b) Describe the splitting field–briefly explain.

   (c) What is $[K : \mathbb{Q}]$?

   (d) We showed in class that if S is a splitting field of a separable polynomial $t(x) \in \mathbb{F}[x]$, then $[S : F] = |Aut(S/F)|$. With K the splitting field of $x^4 - 2 \in \mathbb{Q}[x]$, you will have shown that $[K : \mathbb{Q}] = Aut(K/\mathbb{Q})$ has fewer than $4! = 24$ elements. And you also showed each element of $Aut(K/Q)$ is completely determined by the permutation it induces on the four roots of $x^4 - 3$. Fnd a permutation of those four roots that (you listed above) could not possibly be induced by any element of $Aut(K/Q)$—briefly **explain**.

4. We showed in class that the splitting field of $\Phi_8(x)$ is $K := \mathbb{Q}(\psi)$, where $\psi = cis(2\pi i/8) := \psi$, a primitive 8-th root of unity, and that $[K : \mathbb{Q}] = \phi(8) = 4$, the number of positive integers k, $8 > k \geq 1$, with $(8, k) = 1$. Note that the four primitive 8th roots of unity are $X = \{\psi, \psi^3, \psi^5, \psi^7\}$. Let $\sigma \in Aut(K/Q)$. As you showed in earlier exercises, σ permutes X, and is determined completely by the permutation it induces on X.

   (a) Show that if $\sigma(\psi) = \psi^k$, where $k \in \{1, 3, 5, 7\}$, then σ is completely determined.

   (b) Provide an isomorphism $\Gamma : Aut(K/Q) \to U(8)$, the latter the group of units of Z_8. (The group of units is also denoted $Z_8^{\times}$ or Z_8^{*}.) Briefly explain why your map Γ is a homomorphism.