PROBABILITY
AND
MATHEMATICAL STATISTICS

Prasanna Sahoo
Department of Mathematics
University of Louisville
Louisville, KY 40292 USA

PREFACE

This book is both a tutorial and a textbook. This book presents an introduction to probability and mathematical statistics and it is intended for students already having some elementary mathematical background. It is intended for a one-year junior or senior level undergraduate or beginning graduate level course in probability theory and mathematical statistics. The book contains more material than normally would be taught in a one-year course. This should give the teacher flexibility with respect to the selection of the content and level at which the book is to be used. This book is based on over 15 years of lectures in senior level calculus based courses in probability theory and mathematical statistics at the University of Louisville.

Probability theory and mathematical statistics are difficult subjects both for students to comprehend and teachers to explain. Despite the publication of a great many textbooks in this field, each one intended to provide an improvement over the previous textbooks, this subject is still difficult to comprehend. A good set of examples makes these subjects easy to understand. For this reason alone I have included more than 350 completely worked out examples and over 165 illustrations. I give a rigorous treatment of the fundamentals of probability and statistics using mostly calculus. I have given great attention to the clarity of the presentation of the materials. In the text, theoretical results are presented as theorems, propositions or lemmas, of which as a rule rigorous proofs are given. For the few exceptions to this rule references are given to indicate where details can be found. This book contains over 450 problems of varying degrees of difficulty to help students master their problem solving skill.

In many existing textbooks, the examples following the explanation of a topic are too few in number or too simple to obtain a through grasp of the principles involved. Often, in many books, examples are presented in abbreviated form that leaves out much material between steps, and requires that students derive the omitted materials themselves. As a result, students find examples difficult to understand. Moreover, in some textbooks, examples are often worded in a confusing manner. They do not state the problem and then present the solution. Instead, they pass through a general discussion, never revealing what is to be solved for. In this book, I give many examples to illustrate each topic. Often we provide illustrations to promote a better understanding of the topic. All examples in this book are formulated as questions and clear and concise answers are provided in step-by-step detail.

There are several good books on these subjects and perhaps there is no need to bring a new one to the market. So for several years, this was circulated as a series of typeset lecture notes among my students who were preparing for the examination 110 of the Actuarial Society of America. Many of my students encouraged me to formally write it as a book. Actuarial students will benefit greatly from this book. The book is written in simple English; this might be an advantage to students whose native language is not English.

I cannot claim that all the materials I have written in this book are mine. I have learned the subject from many excellent books, such as *Introduction to Mathematical Statistics* by Hogg and Craig, and *An Introduction to Probability Theory and Its Applications* by Feller. In fact, these books have had a profound impact on me, and my explanations are influenced greatly by these textbooks. If there are some similarities, then it is due to the fact that I could not make improvements on the original explanations. I am very thankful to the authors of these great textbooks. I am also thankful to the Actuarial Society of America for letting me use their test problems. I thank all my students in my probability theory and mathematical statistics courses from 1988 to 2003 who helped me in many ways to make this book possible in the present form. Lastly, if it weren't for the infinite patience of my wife, Sadhna, this book would never get out of the hard drive of my computer.

The author on a Macintosh computer using TEX, the typesetting system designed by Donald Knuth, typeset the entire book. The figures were generated by the author using MATHEMATICA, a system for doing mathematics designed by Wolfram Research, and MAPLE, a system for doing mathematics designed by Maplesoft. The author is very thankful to the University of Louisville for providing many internal financial grants while this book was under preparation.

Prasanna Sahoo, *Louisville*

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CHAPTER 13

3. 0.115.

4. 1.0.

5. $\frac{7}{16}$.

6. 0.352.

7. $\frac{6}{5}$.

8. 100.64.

9. $\frac{1+\ln(2)}{2}$.

10. $[1-F(x_6)]^5$.

11. $\theta + \frac{1}{5}$.

12. $2\,e^{-w}\,[1-e^{-w}]$.

13. $6\,\frac{w^2}{\theta^3}\left(1-\frac{w^3}{\theta^3}\right)$.

14. $N(0,1)$.

15. 25.

16. X has a degenerate distribution with MGF $M(t) = e^{\frac{1}{2}t}$.

17. $POI(1995\lambda)$.

18. $\left(\frac{1}{2}\right)^n (n+1)$.

19. $\frac{8^8}{11^9}\,35$.

20. $f(x) = \frac{60}{\theta}\left(1-e^{-\frac{x}{\theta}}\right)^3 e^{-\frac{3x}{\theta}}$ for $0 < x < \infty$.

21. $X_{(n+1)} \sim Beta(n+1, n+1)$.

CHAPTER 14

1. $N(0,32)$.

2. $\chi^2(3)$; the MGF of $X_1^2 - X_2^2$ is $M(t) = \frac{1}{\sqrt{1-4t^2}}$.

3. $t(3)$.

4. $f(x_1, x_2, x_3) = \frac{1}{\theta^3} e^{-\frac{(x_1+x_2+x_3)}{\theta}}$.

5. σ^2

6. $t(2)$.

7. $M(t) = \frac{1}{\sqrt{(1-2t)(1-4t)(1-6t)(1-8t)}}$.

8. 0.625.

9. $\frac{\sigma^4}{n^2} 2(n-1)$.

10. 0.

11. 27.

12. $\chi^2(2n)$.

13. $t(n+p)$.

14. $\chi^2(n)$.

15. $(1,2)$.

16. 0.84.

17. $\frac{2\sigma^2}{n^2}$.

18. 11.07.

19. $\chi^2(2n-2)$.

20. 2.25.

21. 6.37.

CHAPTER 15

1. $\sqrt{\frac{3}{n}\sum_{i=1}^{n} X_i^2}$.

2. $\frac{1}{\bar{X}-1}$.

3. $\frac{2}{\bar{X}}$.

4. $-\frac{n}{\sum_{i=1}^{n} \ln X_i}$.

5. $\frac{n}{\sum_{i=1}^{n} \ln X_i} - 1$.

6. $\frac{2}{\bar{X}}$.

7. 4.2

8. $\frac{19}{26}$.

9. $\frac{15}{4}$.

10. 2.

11. $\hat{\alpha} = 3.534$ and $\hat{\beta} = 3.409$.

12. 1.

13. $\frac{1}{3}\max\{x_1, x_2, ..., x_n\}$.

14. $\sqrt{1 - \frac{1}{\max\{x_1, x_2, ..., x_n\}}}$.

15. 0.6207.

18. 0.75.

19. $-1 + \frac{5}{\ln(2)}$.

20. $\frac{\bar{X}}{1+\bar{X}}$.

21. $\frac{\bar{X}}{4}$.

22. 8.

23. $\frac{n}{\sum_{i=1}^{n} |X_i - \mu|}$.

24. $\frac{1}{N}$.

25. $\sqrt{\overline{X}}$.

26. $\hat{\lambda} = \frac{n\bar{X}}{(n-1)S^2}$ and $\hat{\alpha} = \frac{n\bar{X}^2}{(n-1)S^2}$.

27. $\frac{10\,n}{p\,(1-p)}$.

28. $\frac{2n}{\theta^2}$.

29. $\begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}$.

30. $\begin{pmatrix} \frac{n\lambda}{\mu^3} & 0 \\ 0 & \frac{n}{2\lambda^2} \end{pmatrix}$.

31. $\widehat{\alpha} = \frac{\overline{X}}{\widehat{\beta}}$, $\quad \widehat{\beta} = \frac{1}{\overline{X}} \left[\frac{1}{n} \sum_{i=1}^{n} X_i^2 - \overline{X} \right]$.

32. $\widehat{\theta}$ is obtained by solving numerically the equation $\sum_{i=1}^{n} \frac{2(x_i-\theta)}{1+(x_i-\theta)^2} = 0$.

33. $\widehat{\theta}$ is the median of the sample.

34. $\frac{n}{\lambda}$.

35. $\frac{n}{(1-p)\,p^2}$.

CHAPTER 16

1. $b = \frac{\sigma_2^2 - cov(T_1, T_2)}{\sigma_1^2 + \sigma_2^2 - 2cov(T_1, T_2}$.

2. $\widehat{\theta} = \overline{|X|},\ E(\overline{|X|}) = \theta$, unbiased.

4. $n = 20$.

5. $k = \frac{1}{2}$.

6. $a = \frac{25}{61},\ b = \frac{36}{61},\ \widehat{c} = 12.47$.

7. $\displaystyle\sum_{i=1}^{n} X_i^3$.

8. $\displaystyle\sum_{i=1}^{n} X_i^2$, no.

9. $k = \frac{4}{\pi}$.

10. $k = 2$.

11. $k = 2$.

13. $\displaystyle\ln \prod_{i=1}^{n} (1 + X_i)$.

14. $\displaystyle\sum_{i=1}^{n} X_i^2$.

15. $X_{(1)}$, and sufficient.

16. $X_{(1)}$ is biased and $\overline{X} - 1$ is unbiased. $X_{(1)}$ is efficient then $\overline{X} - 1$.

17. $\displaystyle\sum_{i=1}^{n} \ln X_i$.

18. $\displaystyle\sum_{i=1}^{n} X_i$.

19. $\displaystyle\sum_{i=1}^{n} \ln X_i$.

22. Yes.

23. Yes.

24. Yes.

25. Yes.

26. $\widehat{\theta} = 3\,\overline{X}$.

27. $\widehat{\theta} = \frac{50}{30}\,X$.

CHAPTER 17

7. The pdf of Q is $g(q) = \begin{cases} n\,e^{-n\,q} & \text{if } 0 < q < \infty \\ 0 & \text{otherwise.} \end{cases}$

The confidence interval is $\left[X_{(1)} - \frac{1}{n}\,\ln\left(\frac{2}{\alpha}\right),\;\; X_{(1)} - \frac{1}{n}\,\ln\left(\frac{2}{2-\alpha}\right)\right]$.

8. The pdf of Q is $g(q) = \begin{cases} \frac{1}{2}\,e^{-\frac{1}{2}\,q} & \text{if } 0 < q < \infty \\ 0 & \text{otherwise.} \end{cases}$

The confidence interval is $\left[X_{(1)} - \frac{1}{n}\,\ln\left(\frac{2}{\alpha}\right),\;\; X_{(1)} - \frac{1}{n}\,\ln\left(\frac{2}{2-\alpha}\right)\right]$.

9. The pdf of Q is $g(q) = \begin{cases} n\,q^{n-1} & \text{if } 0 < q < 1 \\ 0 & \text{otherwise.} \end{cases}$

The confidence interval is $\left[X_{(1)} - \frac{1}{n}\,\ln\left(\frac{2}{\alpha}\right),\;\; X_{(1)} - \frac{1}{n}\,\ln\left(\frac{2}{2-\alpha}\right)\right]$.

10. The pdf $g(q)$ of Q is given by $g(q) = \begin{cases} n\,q^{n-1} & \text{if } 0 \leq q \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

The confidence interval is $\left[\left(\frac{2}{\alpha}\right)^{\frac{1}{n}}\,X_{(n)},\;\; \left(\frac{2}{2-\alpha}\right)^{\frac{1}{n}}\,X_{(n)}\right]$.

11. The pdf of Q is given by $g(q) = \begin{cases} n\,(n-1)\,q^{n-2}\,(1-q) & \text{if } 0 \leq q \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

12. $\left[X_{(1)} - z_{\frac{\alpha}{2}}\,\frac{1}{\sqrt{n}},\; X_{(1)} + z_{\frac{\alpha}{2}}\,\frac{1}{\sqrt{n}}\right]$.

13. $\left[\widehat{\theta} - z_{\frac{\alpha}{2}}\,\frac{\widehat{\theta}+1}{\sqrt{n}},\; \widehat{\theta} + z_{\frac{\alpha}{2}}\,\frac{\widehat{\theta}+1}{\sqrt{n}}\right]$, where $\widehat{\theta} = -1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$.

14. $\left[\frac{2}{\overline{X}} - z_{\frac{\alpha}{2}}\sqrt{\frac{2}{n\,\overline{X}^2}},\; \frac{2}{\overline{X}} + z_{\frac{\alpha}{2}}\sqrt{\frac{2}{n\,\overline{X}^2}}\right]$.

15. $\left[\overline{X} - 4 - z_{\frac{\alpha}{2}}\,\frac{\overline{X}-4}{\sqrt{n}},\; \overline{X} - 4 + z_{\frac{\alpha}{2}}\,\frac{\overline{X}-4}{\sqrt{n}}\right]$.

16. $\left[X_{(n)} - z_{\frac{\alpha}{2}}\,\frac{X_{(n)}}{(n+1)\,\sqrt{n+2}},\; X_{(n)} + z_{\frac{\alpha}{2}}\,\frac{X_{(n)}}{(n+1)\,\sqrt{n+2}}\right]$.

17. $\left[\frac{1}{4}\,\overline{X} - z_{\frac{\alpha}{2}}\,\frac{\overline{X}}{8\,\sqrt{n}},\; \frac{1}{4}\,\overline{X} + z_{\frac{\alpha}{2}}\,\frac{\overline{X}}{8\,\sqrt{n}}\right]$.

CHAPTER 18

1. $\alpha = 0.03125$ and $\beta = 0.763$.

2. Do not reject H_o.

3. $\alpha = 0.0511$ and $\beta(\lambda) = 1 - \sum_{x=0}^{7} \frac{(8\lambda)^x e^{-8\lambda}}{x!}, \;\lambda \neq 0.5.$

4. $\alpha = 0.08$ and $\beta = 0.46$.

5. $\alpha = 0.19$.

6. $\alpha = 0.0109$.

7. $\alpha = 0.0668$ and $\beta = 0.0062$.

8. $C = \{(x_1, x_2) \,|\, \overline{x}^2 \geq 3.9395\}$.

9. $C = \{(x_1, ..., x_{10}) \,|\, \overline{x} \geq 0.3\}$.

10. $C = \{x \in [0,1] \,|\, x \geq 0.829\}$.

11. $C = \{(x_1, x_2) \,|\, x_1 + x_2 \geq 5\}$.

12. $C = \{(x_1, ..., x_8) \,|\, \overline{x} - \overline{x} \ln \overline{x} \leq a\}$.

13. $C = \{(x_1, ..., x_n) \,|\, 35 \ln \overline{x} - \overline{x} \leq a\}$.

14. $C = \left\{ (x_1, ..., x_5) \,|\, \left(\frac{\overline{x}}{2\overline{x}-2}\right)^{5\overline{x}-5} \overline{x}^5 \leq a \right\}$.

15. $C = \{(x_1, x_2, x_3) \;|\; |\overline{x} - 3| \geq 1.96\}$.

16. $C = \left\{ (x_1, x_2, x_3) \;|\; \overline{x}\, e^{-\frac{1}{3}\overline{x}} \leq a \right\}$.

17. $C = \left\{ (x_1, x_2, ..., x_n) \;|\; \left(\frac{e}{10\,\overline{x}}\right)^{3\overline{x}} \leq a \right\}$.

18. $\frac{1}{3}$.

19. $C = \left\{ (x_1, x_2, x_3) \;|\; x_{(3)} \leq \sqrt[3]{117} \right\}$.

20. $C = \{(x_1, x_2, x_3) \;|\; \overline{x} \geq 12.04\}$.

21. $\alpha = \frac{1}{16}$ and $\beta = \frac{255}{256}$.

22. $\alpha = 0.05$.