$\vec{\mu} = (\mu_1, \dots, \mu_n)$

$Y_i \sim N(\mu_i, \sigma^2 = 0.2)$, $i = 1, \dots, n$

$P(\mu_1 = 1)$

$P(\mu_{i+1} | \mu_i) = N(\mu_i, 0.2)$, $i = 1, \dots, n-1$

Lab 4

Q2)

Gibbs sampler:-

$$X_{t+1,i} \sim f(\cdot | X_{t+1,1}, \dots, X_{t+1,i-1}, X_{t,i+1}, \dots, X_{t,d})$$

We need to sample (to get a value for a particular X)

$$f(\underbrace{\cdot}_{\text{we are interested in}} | \underbrace{X_{t+1,1}, \dots, X_{t,d}}_{\text{will be constant since we will integrate over this.}}) \leftarrow \text{conditional distr.}$$

We are interested in the conditional distr., $\therefore$ from Baye's theorem,

$$\underbrace{P(\theta|D)}_{\text{Posterior}} \propto \underbrace{P(D|\theta)}_{\text{Likelihood}} \underbrace{P(\theta)}_{\text{Prior}}$$

We are given,

$\vec{\mu} = (\mu_1, \dots, \mu_n) \leftarrow$ unknown parameters

$Y_i \sim N(\mu_i$, variance $= 0.2)$, $i = 1, \dots, n$

where prior is,

$P(\mu_1) = 1$

$P(\mu_{i+1}, \mu_i) = N(\mu_i, 0.2)$, $i = 1, \dots, n-1$

$\therefore$ $$P(\vec{\mu}\mid\vec{y}) = \frac{P(\vec{y}\mid\vec{\mu})\,P(\vec{\mu})}{\int P(\vec{y}\mid\vec{\mu})\,P(\vec{\mu})\,d\vec{\mu}}$$

$$P(\vec{\mu}\mid\vec{y}) \propto P(\vec{y}\mid\vec{\mu})\,P(\vec{\mu})$$

$\therefore$ w.k.t. $N(\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$\therefore$ Likelihood,

$$L(\vec{y}\mid\vec{\mu}) = \frac{1}{(\sqrt{2\pi\sigma^2})^n}\prod_{i=1}^{n} e^{-\frac{1}{2\sigma^2}(y_i-\mu_i)^2}$$

Prior,

$$P(\mu_{i+1}\mid\mu_i) = N(\mu_i, 0.2) = \underbrace{P(\mu_1)}_{1}\,P(\mu_2\mid\mu_1)\cdots P(\mu_n\mid\mu_{n-1})$$

$[i=1,\ldots,n-1]$

$\therefore$ $$P(\vec{\mu}) = \frac{1}{(\sqrt{2\pi\sigma^2})^{n-1}}\prod_{i=1}^{n-1} e^{-\frac{1}{2\sigma^2}(\mu_{i+1}-\mu_i)^2}$$

$P(\vec{\mu},\vec{y}) = L(\vec{y}\mid\vec{\mu})\cdot P(\mu_{i+1}\mid\mu_i)$

Now, the posterior (conditional distr.) we are interested in:

$$P(\vec{\mu}\mid\vec{y}) \propto e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n-1}(\mu_{i+1}-\mu_i)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu_i)^2}$$

$$\hookrightarrow \propto \boxed{e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n-1}(\mu_{i+1}-\mu_i)^2+(y_i-\mu_i)^2\right]-\frac{1}{2\sigma^2}(y_n-\mu_n)^2}}$$

Now,

~~considering~~ $P(\mu_i\mid\vec{\mu}_{-i},\vec{y}) \rightarrow$ ~~$P(\mu_i$~~

$\odot$ Considering, $P(\mu_1\mid\vec{\mu}_{-1},\vec{y}) \rightarrow P(\mu_1\mid\mu_2,\mu_3,\ldots\mu_n,\vec{y})$

$\hookrightarrow$ we are interested only in μ_1

$$P(\mu_1\mid\vec{\mu}_{-1},\vec{y}) \propto \exp\left[-\frac{1}{2\sigma^2}(\mu_2-\mu_1)^2+(y_1-\mu_1)^2\right]$$

$$= \exp\left[-\frac{1}{2\sigma^2}(\mu_1-\mu_2)^2+(\mu_1-y_1)^2\right]$$

On comparing with hint,

$$\exp\left[\frac{-1}{d}\left((x-a)^2+(x-b)^2\right)\right] \propto \exp\left[-\frac{(x-(a+b)/2)^2}{d/2}\right]$$

$$\Rightarrow \exp\left[-\frac{(\mu_1-(\mu_2+y_1)/2)^2}{\cancel{2}\sigma^2/\cancel{2}}\right]$$

$$\therefore P(\mu_1 \mid \vec{\mu}_{-1}, \vec{y}) \sim N\left(\frac{\mu_2+y_1}{2}, \frac{\sigma^2}{2}\right) \longrightarrow (1)$$

(2) Now,

Considering $P(\mu_n \mid \vec{\mu}_{-n}, \vec{y})$,

$$\propto \exp\left[-\frac{1}{2\sigma^2}\left[(\mu_n-\mu_{n-1})^2+(\mu_n-y_n)^2\right]\right]$$

$$\propto \exp\left[-\frac{(\mu_n-(\mu_{n-1}+y_n)/2)^2}{\sigma^2}\right]$$

$$\sim N\left(\frac{\mu_{n-1}+y_n}{2}, \frac{\sigma^2}{2}\right) \longrightarrow (2)$$

(3) Now,

~~Considering~~ Taking $i=3$,

$$\sum_{i=2}^{3}(\mu_{i+1}-\mu_i)^2+(y_i-\mu_i)^2$$

$$= \underbrace{(\mu_3-\mu_2)^2+(y_2-\mu_2)^2}_{i=2}+\underbrace{(\mu_4-\mu_3)^2+(y_3-\mu_3)^2}_{i=3}$$

$$= \exp\left[\frac{-1}{2\sigma^2}\left((\mu_i-\mu_{i-1})^2+(\mu_{i+1}-\mu_i)^2+(y_i-\mu_i)^2\right)\right]$$

$$\propto \exp\left(-\frac{(\mu_i-(\mu_{i-1}+\mu_{i+1}+y_i)/3)^2}{2\sigma^2/3}\right)$$

$$\therefore P(\mu_i \mid \vec{\mu}_{-i}, \vec{y}) \sim N\left(\frac{\mu_{i-1}+\mu_{i+1}+y_i}{3}, \frac{2\sigma^2}{3}\right) \longrightarrow (3)$$