Meeting 9:
Utility, part 2…

The general method of finding/assessing the utility function

As was described at Meeting 8, the utility function of a decision maker can be found by considering a gamble. Here, we formalise it a bit further.

Let

c_w = the least preferable (worst) consequence [for the decision maker]

c_b = the most preferable (best) consequence

in the decision problem.

Since the consequence is a function of both the action taken and the state of the world,

c_w corresponds to a pair $(a_w, \boldsymbol{\theta}_w)$

c_b corresponds to a pair $(a_b, \boldsymbol{\theta}_b)$.

Note! The action a_w is not a general "worst" action and $\boldsymbol{\theta}_w$ is not a general "worst" state of the world (correspondingly for a_b and $\boldsymbol{\theta}_b$). Only the pairs define the worst and best consequence respectively.

Without loss of generality set $U(a_w, \boldsymbol{\theta}_w) = 0$ and $U(a_b, \boldsymbol{\theta}_b) = 1$

$U(a_w, \boldsymbol{\theta}_w) = 0$
$U(a_b, \boldsymbol{\theta}_b) = 1$

For any pair of an action and a state of the world, $(a, \boldsymbol{\theta})$ the utility of the consequence $c(a, \boldsymbol{\theta})$ can be found by considering the gamble:

Lottery I: Obtain consequence $c(a, \boldsymbol{\theta})$ for certain
Lottery II: Obtain consequence $c(a_w, \boldsymbol{\theta}_w)$ with probability $1-p$ and consequence $c(a_b, \boldsymbol{\theta}_b)$ with probability p

The value of p for which the decision maker is indifferent in terms of expected utility between Lottery I and Lottery II is the utility of the consequence $c(a, \boldsymbol{\theta})$, i.e. $U(a, \boldsymbol{\theta})$.

From meeting 8:
This is so since the expected utility in Lottery I is $U(a, \boldsymbol{\theta})$, and the expected utility in Lottery II is $U(a_w, \boldsymbol{\theta}_w) \cdot (1-p) + U(a_b, \boldsymbol{\theta}_b) \cdot p = 0 \cdot (1-p) + 1 \cdot p = p$.

Example

A general practitioner (GP) is supposed to state the diagnosis of a patient, who has declared some symptoms.

The GP sees three possibilities for the symptoms declared:

1. The patient has disease A
2. The patient has disease B
3. The patient has no disease

These are the three possible states of nature and can be denoted θ_1, θ_2 and θ_3 respectively.

Now, the GP can choose to either…
…treat the patient for disease A (action a_1) …or…
…treat the patient for disease B (action a_2) …or…
…give no treatment (action a_3)

Note! This description of the decision problem is not realistic, but just to illustrate. A GP would do more than just select a treatment directly from the declared symptoms if there is more than one explanation to them.

Naturally, different combinations of action and state of the nature would lead to less or more preferable consequences.

Assume that

- the least preferable (worst) consequence is obtained when the patient has disease B and no treatment is given
- the most (and equally) preferable consequences are obtained when the patient has disease A and is treated for A, and when the patient has disease B and is treated for B respectively.

We could also obtain this most preferable consequence when the patient has no disease and is not treated. When would we not do that?

Hence,

$$U(a_3 , \boldsymbol{\theta_2}) = 0$$
$$U(a_1 , \boldsymbol{\theta}_1) = U(a_2 , \boldsymbol{\theta}_2) =1$$

a_1 = “Treat for disease A”
a_2 = “Treat for disease B”
a_3 = “Do not treat”

θ_1 = “The patient has disease A”
θ_2 = “The patient has disease B”
θ_3 =“The patient has no disease”

a_1 = “Treat for disease A”
a_2 = “Treat for disease B”
a_3 = “Do not treat”

θ_1 = “The patient has disease A”
θ_2 = “The patient has disease B”
θ_3 =“The patient has no disease”

Now, to find the utilities for all other combinations of action and state of the nature (treatment and presence/absence of disease) the GP should do the following:

Find $U(a_i, \theta_j)$ such that he is indifferent between

I. Obtaining for certain the consequence of taking action a_i when the state of nature is θ_j , i.e. $c(a_i, \theta_j)$
II. Obtain consequence $c(a_3, \theta_2)$ with probability $1–U(a_i, \theta_j)$ and consequence $c(a_1, \theta_1)$ (= $c(a_2, \theta_2)$) with probability $U(a_i, \theta_j)$

A possible table of utilities may then be

	θ_1	θ_2	θ_3
a_1	1	0.3	0.6
a_2	0.4	1	0.6
a_3	0.1	0	0.9

	θ_1	θ_2	θ_3
a_1	1	0.2	0.6
a_2	0.4	1	0.6
a_3	0.3	0	0.9

a_1 = "Treat for disease A"
a_2 = "Treat for disease B"
a_3 = "Do not treat"

θ_1 = "The patient has disease A"
θ_2 = "The patient has disease B"
θ_3 ="The patient has no disease"

Some rational explanations for the numbers in the table may be:

a) Treating when no disease is present may lead to inconveniences for the patient (undertaking some time-consuming activities and/or suffering from side-effects of prescribed drugs).
b) Treating for one disease while the other is present would have similar consequences like in a) with the addition that the other disease would possibly not be cured

a_1 = "Treat for disease A"
a_2 = "Treat for disease B"
a_3 = "Do not treat"

θ_1 = "The patient has disease A"
θ_2 = "The patient has disease B"
θ_3 = "The patient has no disease"

Now, the GP must assign probabilities to the three states of nature.

From epidemiological studies it might be known that

- the incidence rate of disease A is 5 in 100 persons per week
- the incidence rate of disease B is 1 in 100 persons per week.

Both these incidence rates are per week at this time of the year (think flus).

To assign prior probabilities from these epidemiological statistics we must assume that the GP is confident with restricting the set of possible states of nature to the three used here.

Calculating conditional incidence rates <u>may</u> then have given the following conditional prior probabilities (assuming a proportion of people with no disease among those visiting the GP):

$$P\left(\tilde{\theta} = \theta_1 \middle| \tilde{\theta} = \theta_1, \theta_2 \text{ or } \theta_3\right) \approx 0.20, P\left(\tilde{\theta} = \theta_2 \middle| \tilde{\theta} = \theta_1, \theta_2 \text{ or } \theta_3\right) \approx 0.01,$$
$$P\left(\tilde{\theta} = \theta_3 \middle| \tilde{\theta} = \theta_1, \theta_2 \text{ or } \theta_3\right) \approx 0.79$$

The much higher probability of θ_1 can be due to experience that people with disease A tend to visit the GP much more often than people with disease B.

	θ_1	θ_2	θ_3
a_1	1	0.2	0.6
a_2	0.4	1	0.6
a_3	0.3	0	0.9

a_1 = "Treat for disease A"
a_2 = "Treat for disease B"
a_3 = "Do not treat"

θ_1 = "The patient has disease A"
θ_2 = "The patient has disease B"
θ_3 = "The patient has no disease"

$$P\left(\tilde{\theta} = \theta_1 \middle| \tilde{\theta} = \theta_1, \theta_2 \text{ or } \theta_3\right) \approx 0.20, P\left(\tilde{\theta} = \theta_2 \middle| \tilde{\theta} = \theta_1, \theta_2 \text{ or } \theta_3\right) \approx 0.01,$$
$$P\left(\tilde{\theta} = \theta_3 \middle| \tilde{\theta} = \theta_1, \theta_2 \text{ or } \theta_3\right) \approx 0.79$$

Hence, the expected utilities of each action become

$$E\left(U(a_1, \tilde{\theta}) \middle| \tilde{\theta} = \theta_1, \theta_2 \text{ or } \theta_3\right) = EU_1 = 1 \cdot 0.20 + 0.2 \cdot 0.01 + 0.6 \cdot 0.79 \approx 0.68$$

$$E\left(U(a_2, \tilde{\theta}) \middle| \tilde{\theta} = \theta_1, \theta_2 \text{ or } \theta_3\right) = EU_2 = 0.4 \cdot 0.20 + 1 \cdot 0.01 + 0.6 \cdot 0.79 \approx 0.56$$

$$E\left(U(a_3, \tilde{\theta}) \middle| \tilde{\theta} = \theta_1, \theta_2 \text{ or } \theta_3\right) = EU_3 = 0.3 \cdot 0.20 + 0 \cdot 0.01 + 0.9 \cdot 0.79 \approx 0.77$$

…and the optimal action using the *EU*-criterion is a_3, i.e. "Do not treat"

Utility as a function of payoff

In some accounts for decision theory the utility function is assumed to be linear of payoff, i.e.

$$U(a,\boldsymbol{\theta}) = c + d \cdot R(a,\boldsymbol{\theta})$$

with $d > 0$.

Then, maximising the expected utility is equivalent to maximising the expected payoff.

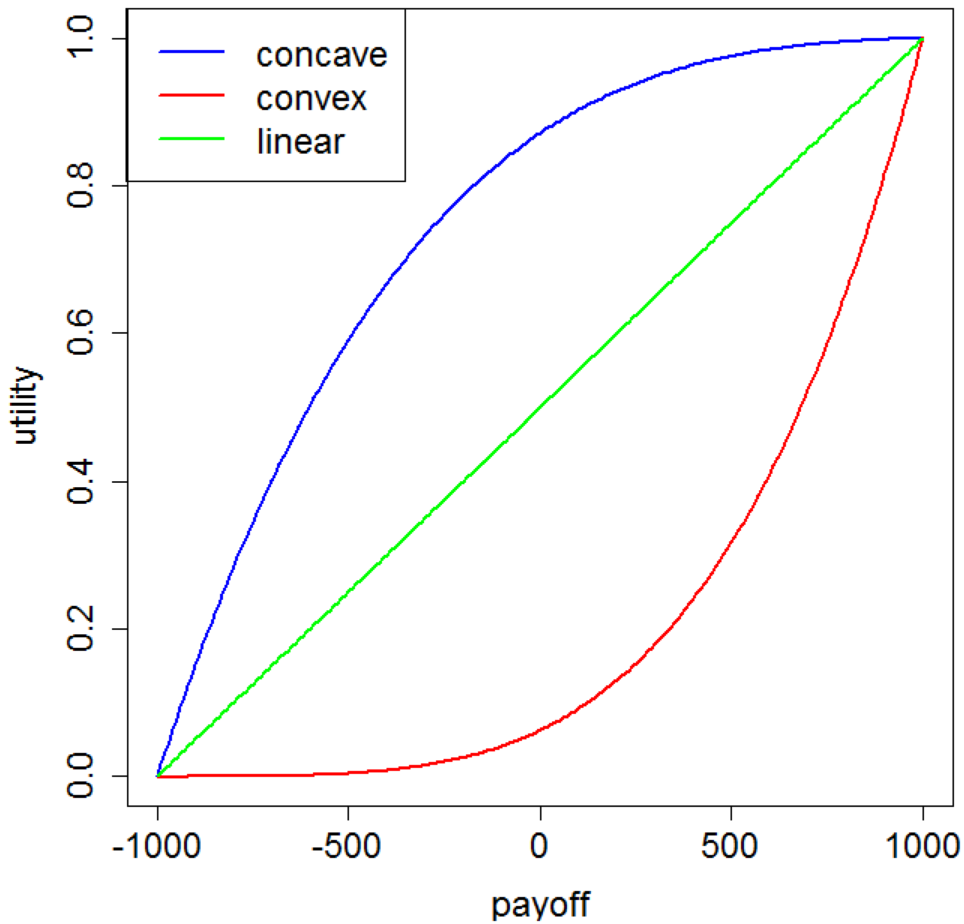
With such an assumption it is not possible to explain decisional behaviour among several individuals if they do not all adhere to the criterion of maximising the expected payoff.

Therefore, it is more general to assume that utility can be written as a function of payoff, but the function needs not to be linear:

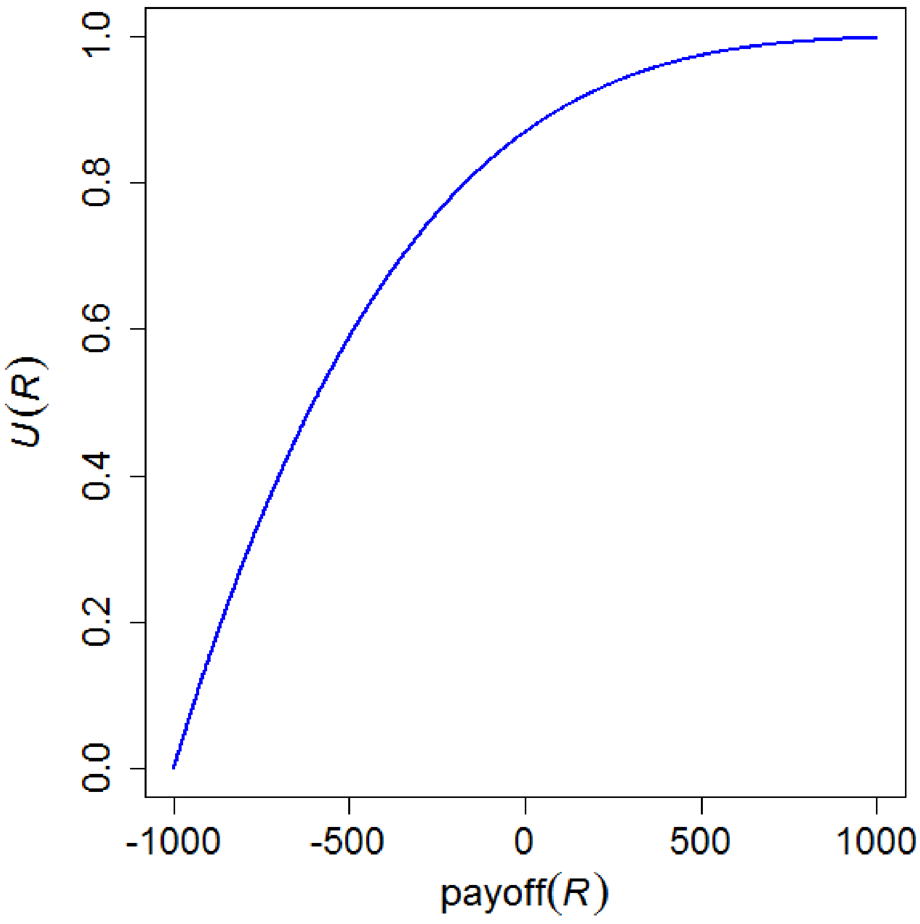
$$U(a,\boldsymbol{\theta}) = h\,(R(a,\boldsymbol{\theta}))$$

Besides the possibility that $U(a, \boldsymbol{\theta})$ can be linear in $R(a, \boldsymbol{\theta})$ the most common types of functions are

- $U(a, \boldsymbol{\theta})$ is a concave function of $R(a, \boldsymbol{\theta})$
- $U(a, \boldsymbol{\theta})$ is a convex function of $R(a, \boldsymbol{\theta})$



Concave utility functions



This functional form of the utility function characterizes a *risk avoider*.

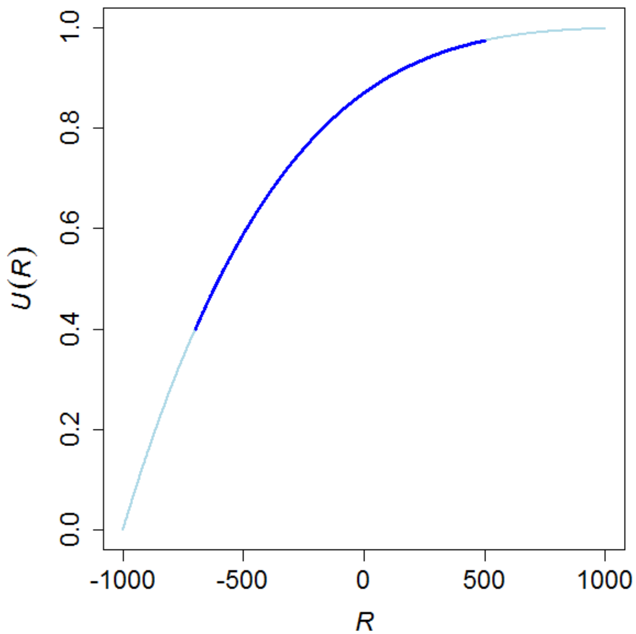
Why is it so?

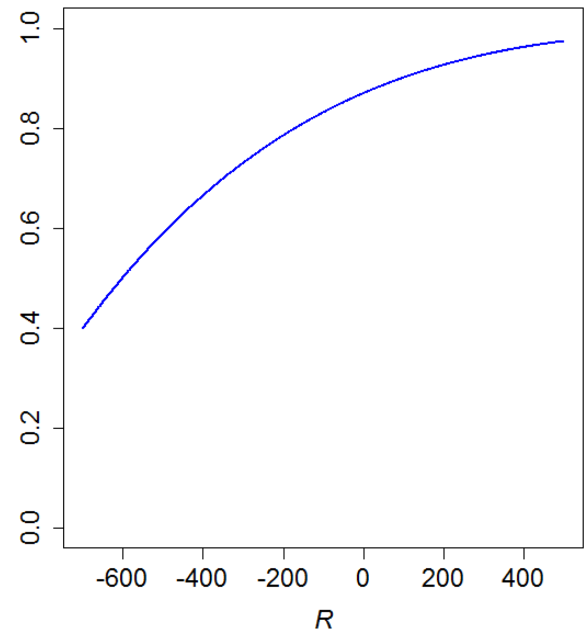
Consider the following bet:

Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

There are two actions: a_1 = "take the bet" and a_2 = "do not bet"

Focus on the range of money defined by the bet:





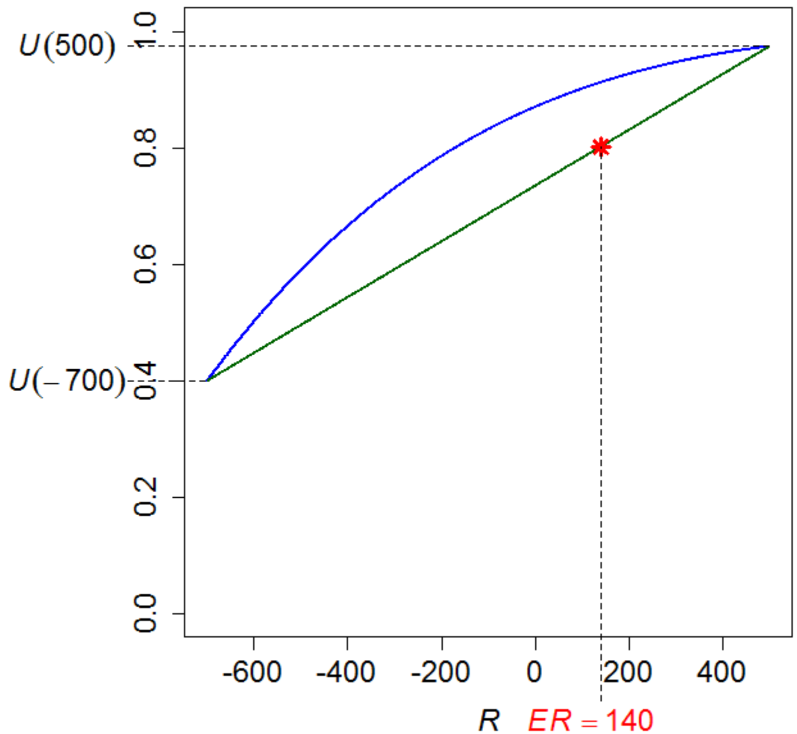
Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

Now, the expected payoff of the bet is

$$ER = 500 \cdot 0.7 - 700 \cdot 0.3 = 140$$

which is a convex combination of 500 and –700.

Since any convex combination of two points v_1 and v_2, i.e. any $v = p \cdot v_1 + (1-p) \cdot v_2$ where $0 < p < 1$, lies on the segment joining v_1 and v_2 (in one dimension this means $v_1 < v < v_2$) we can represent the expected payoff as a point on a straight line joining the points ($-700, U(-700)$) and ($500, U(500)$):



To clarify: a point in this sense is a point on the curve defining the utility function.

Hence, v_1 and v_2 both need to be on the curve, which then automatically defines the coordinates of the points.

The straight line corresponds with a utility function that is linear in payoff $\Rightarrow$ values along the line can be interpreted as payoff expressed in the same unit (scale) as $U(R)$.

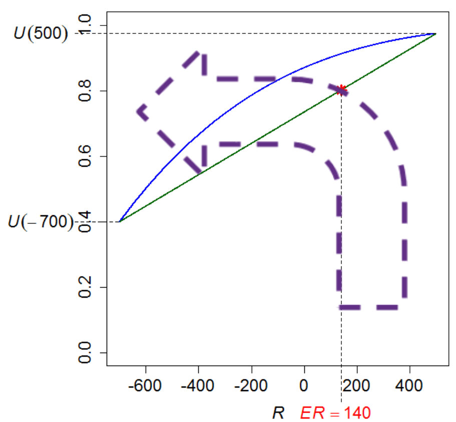
Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

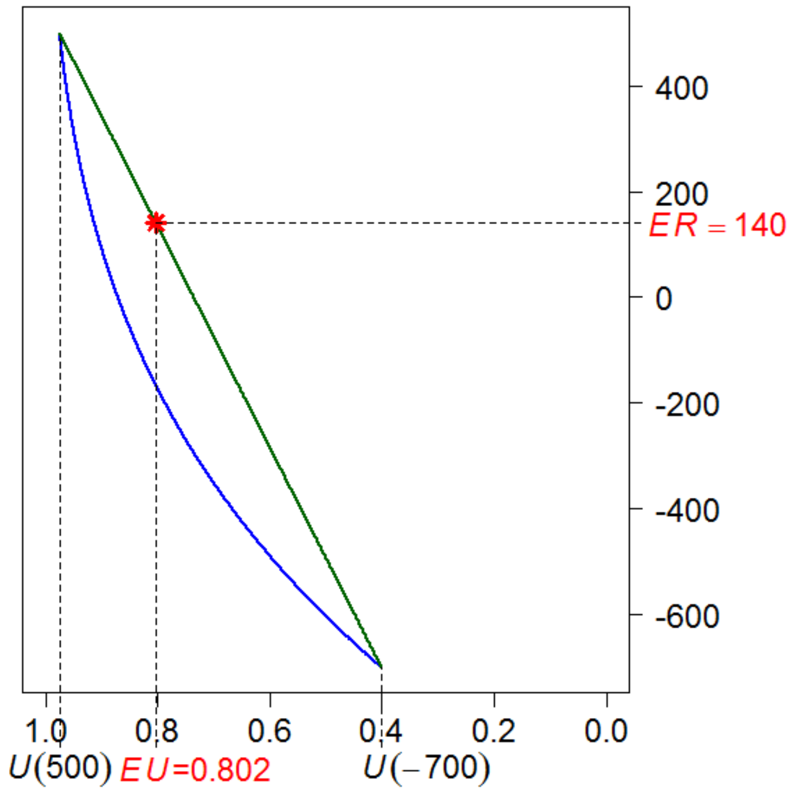
Now, it also holds that the expected utility (for money) of taking the bet is

$$EU(\text{bet}) = U(500) \cdot 0.7 + U(-700) \cdot 0.3$$

This is also a convex combination, but here of two utilities. This must be represented by the same point as was the corresponding convex combination of payoffs, but now we should view it from the "utility" perspective

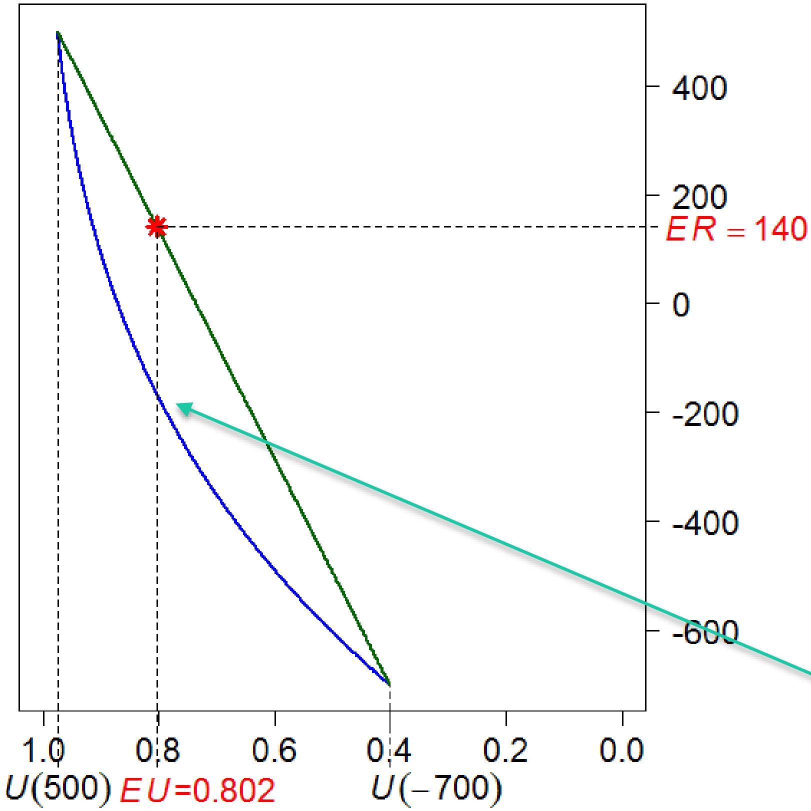
Using the mathematical function that was used to produce the curve we can calculate $EU \approx 0.802$





Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

Now, it becomes clearer why the utility function of a risk avoider is concave



Plotting R as a function of $U(R)$ we can see which cash equivalent (expressed on the same scale as $U(R)$) corresponds with which utility.

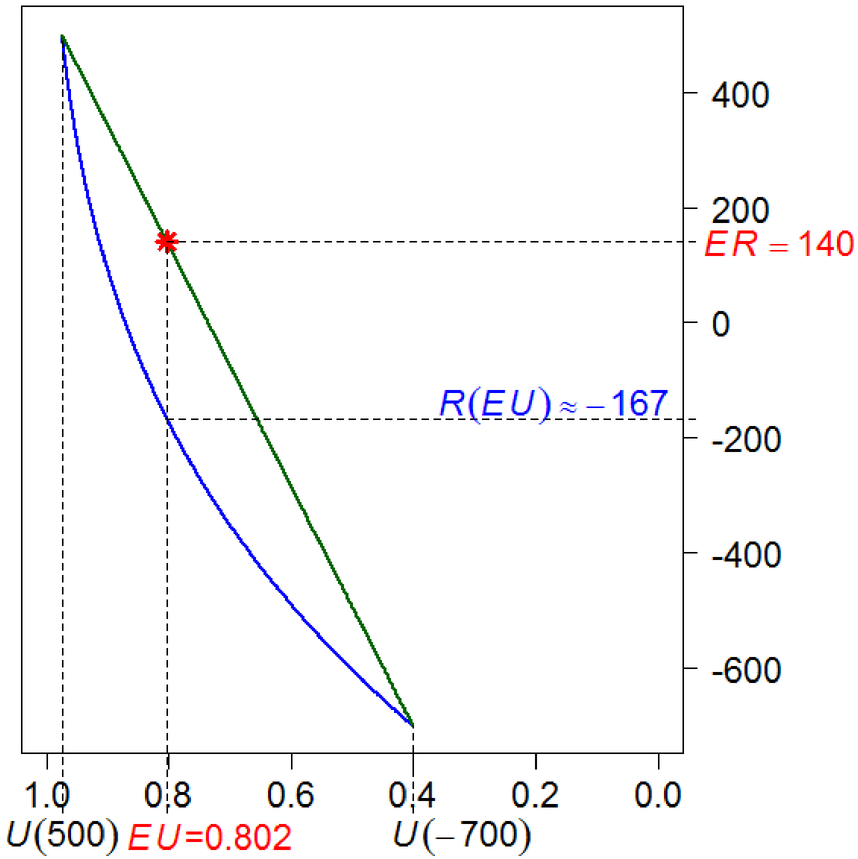
Here we can see that the cash equivalents for the utility of taking this bet are all lower than or equal to the payoff of taking the bet.

In particular, the value of EU is equal to the utility of a cash equivalent satisfying

$$R(EU) = U^{-1}(EU)$$

where U^{-1} is the inverse function of $U(R)$ restricted to $R \in (-700, 500)$

Here, again using the mathematical function "behind" the curve, we can calculate $R(EU) \approx -167$.



Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

Hence, the decision maker appreciates the expected utility of taking the bet to be equivalent to a payoff of SEK –167 …

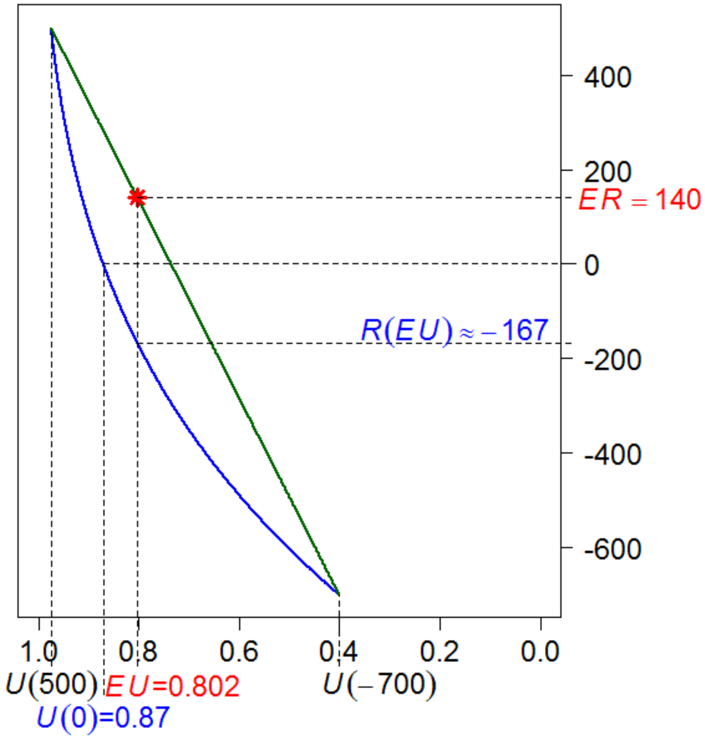
… while the expected payoff of taking the bet is SEK 140.

Now, the expected payoff of *not* taking the bet is (always) SEK 0.

The expected utility of not taking the bet must be equal to the utility when $R = 0$, i.e. $U(0)$.

This can again be calculated using the mathematical function behind $\Rightarrow U(0) \approx 0.87$

Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3



Now, the very decision problem is about taking the bet or not. The expected utility of taking the bet is 0.802 and the expected utility of not taking the bet is the utility corresponding with a payoff of SEK 0, which is 0.87.

Thus the optimal decision with the *EU*-criterion is to <u>not</u> take the bet.

R(*EU*) is called the *certainty equivalent* of the decision-maker. This is the lowest value of a certain payment that the decision-maker would prefer versus taking the bet. Apparently this value can be negative which would render a cost and not a return for the decision-maker.

Alternatively expressed, the decision-maker is in this case indifferent between

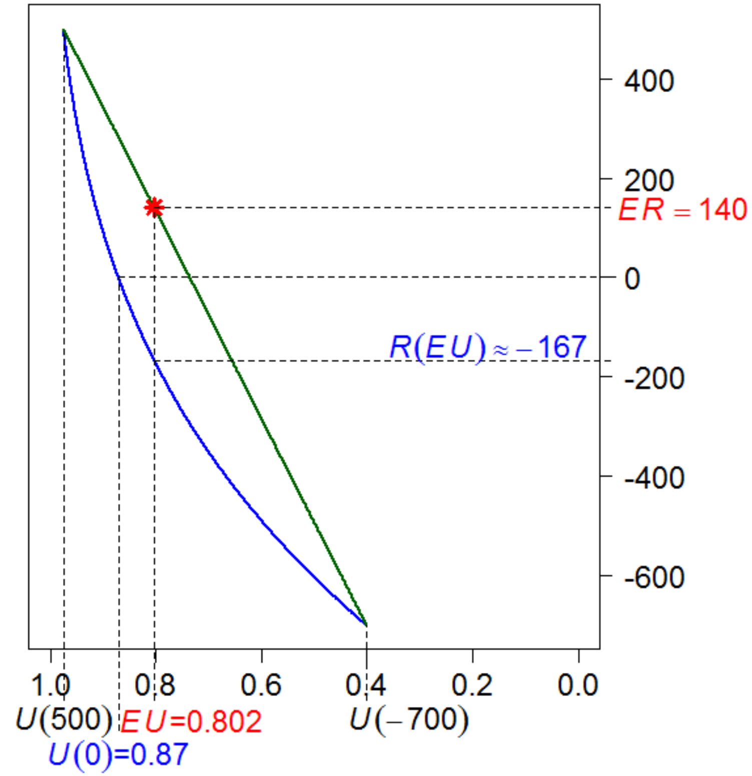
1. Obtaining SEK –167 for certain *and*
2. Obtain SEK 500 with probability 0.7 and losing SEK 700 with probability 0.3

Risk premium

Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

When a risk avoider is exposed to a “bet”, he or she will always have an expected utility of taking the bet that is lower than (or at highest equal to) the expected utility that is linear in payoff.

The payoff equivalent to the expected utility of this risk avoider is their certainty equivalent, *CE*, and the difference between the expected payoff, *ER*, and the certainty equivalent is called their *risk premium*, *RP*.



$$RP = ER - CE$$

In the above example the risk premium of the decision-maker is then approximately SEK 140 – (–167) = 307

Notice (again) that all functions and specific quantities are *<u>personal</u>* *to the decision-maker!*

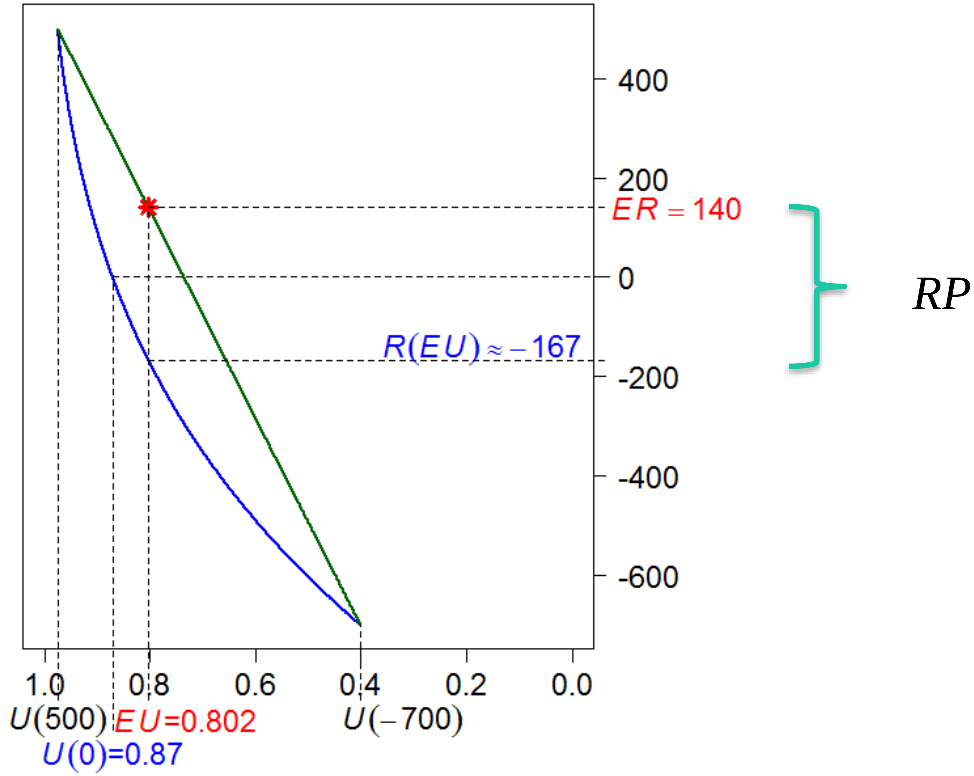
Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

What is then the difference between the certainty equivalent and the risk premium?

In the example above, the expected payoff was positive (ER = 140) while the certainty equivalent was negative (CE = –167).

The certainty equivalent is what the decision maker considers to be the expected utility in monetary terms of taking the bet. Hence they will never consider taking a bet with a negative certainty equivalent, but it would not generally suffice with a positive certain equivalent either.

The risk premium tells how much money must at least be *additionally paid* to the decision maker for making them take the bet. In the example above that amount was SEK 307.

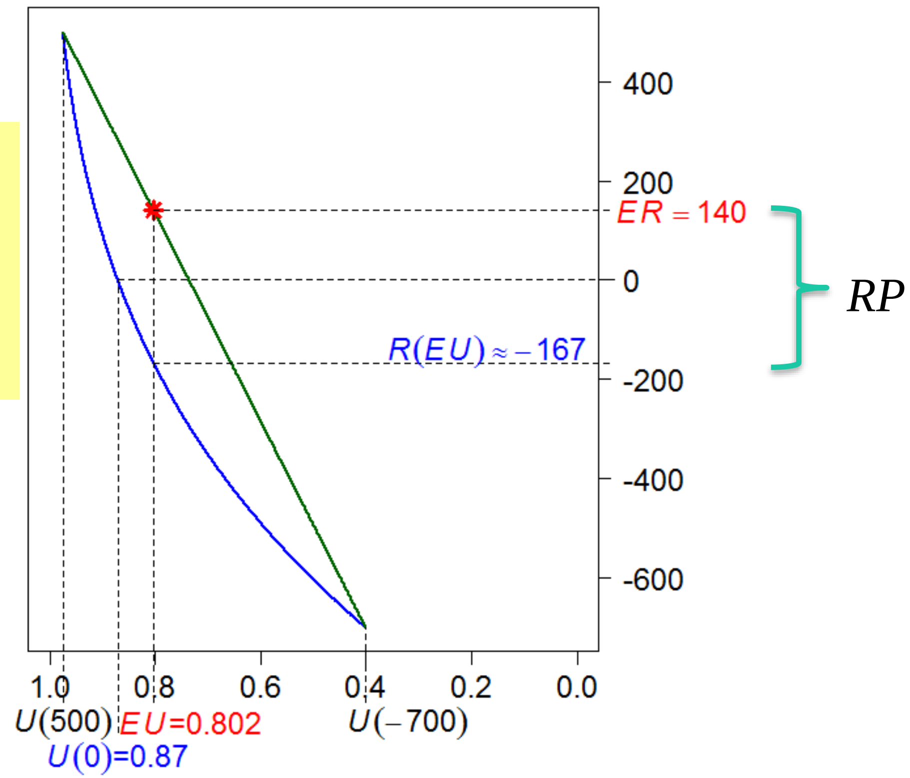


It is the shape of the utility function that implies that the decision-maker does not become indifferent between

Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

1. Obtaining SEK x for certain *and*
2. Obtain SEK 500 with probability 0.7 and losing SEK 700 with probability 0.3

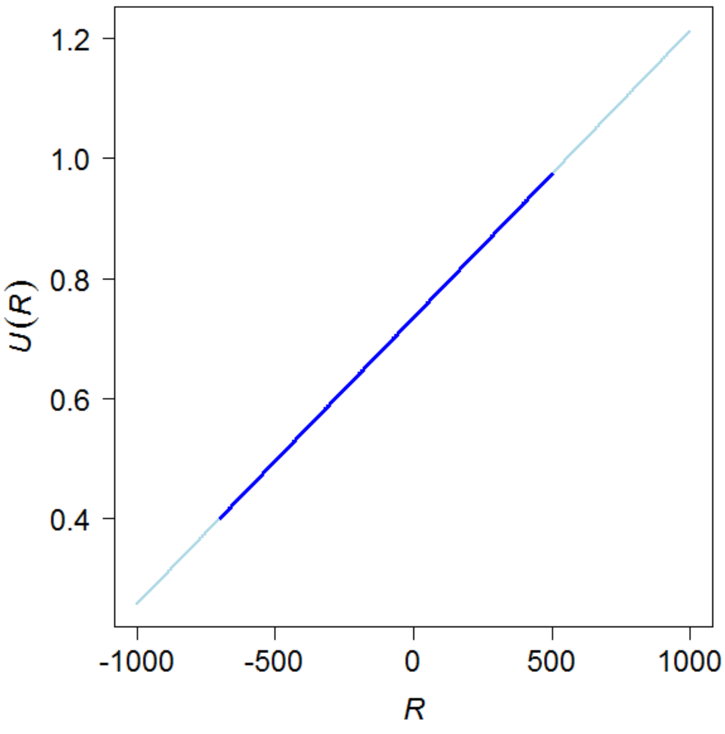
until $x = RP$.

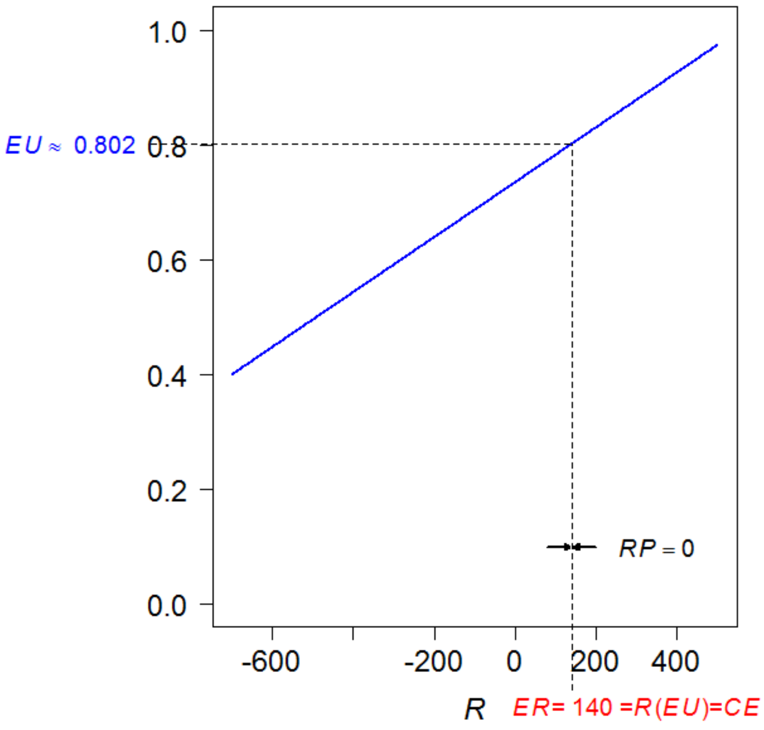


Hence, the higher the certainty equivalent the lower the risk premium.

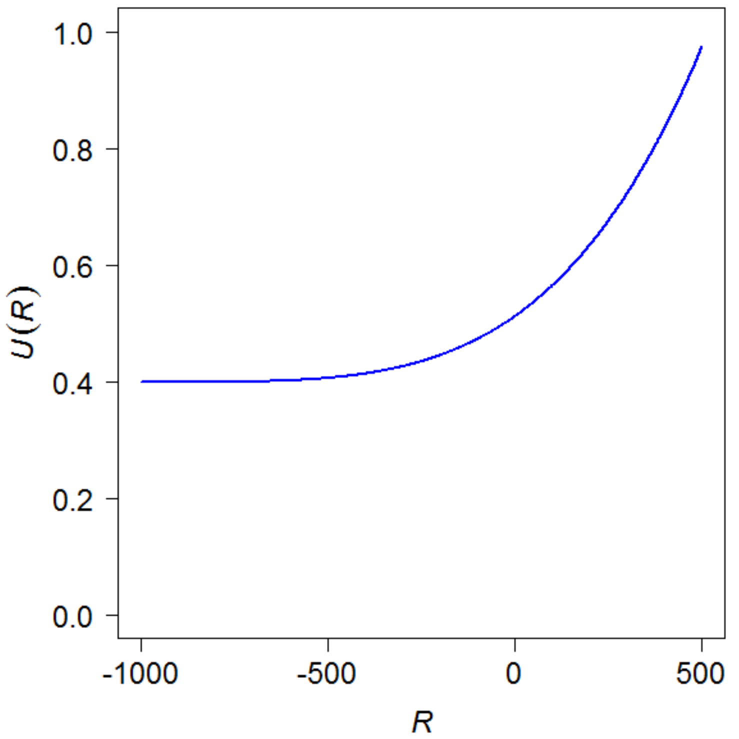
If the expected payoff of a bet is 0, the bet is said to be a *fair* bet. In that case the risk premium will be equal to the certainty equivalent in absolute sense since
$RP = 0 - CE$.

Now, if the decision-maker is *risk neutral* the expected utility for money of a bet coincides with the expected payoff (the utility is linear in payoff). This means that the certainty equivalent is always equal to *ER* and the risk premium is zero.

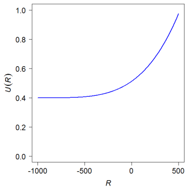




For a *risk taker* the situation is the opposite as for the risk avoider. The utility function of a risk avoider is convex, e.g.



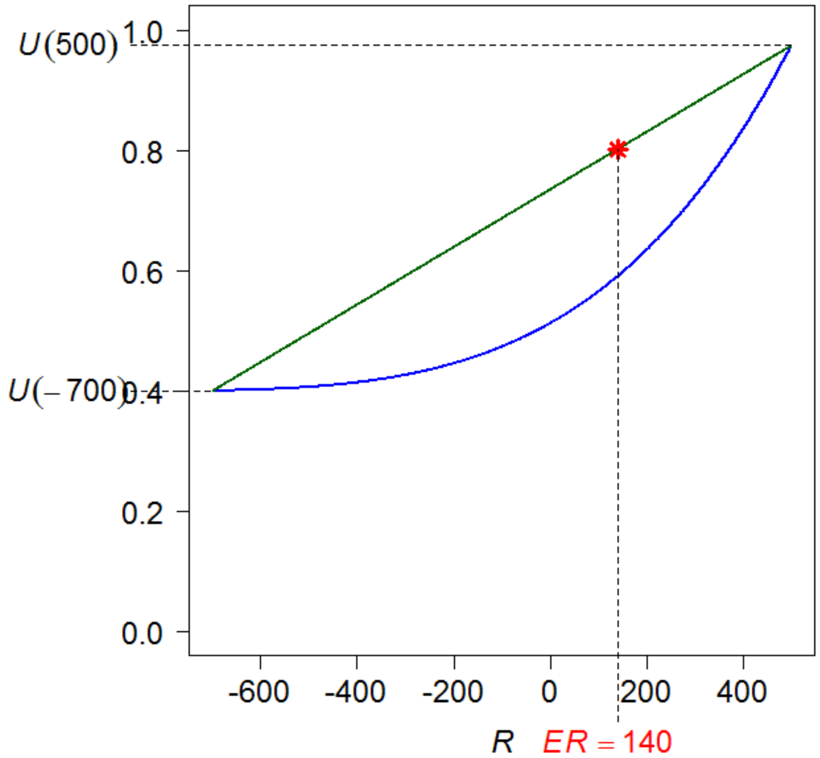
This function is here made such as the utilities for R = –700 and R = 500 are the same as with the previous utility function of a risk avoider.



Hence with the same bet as before, i.e.

Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

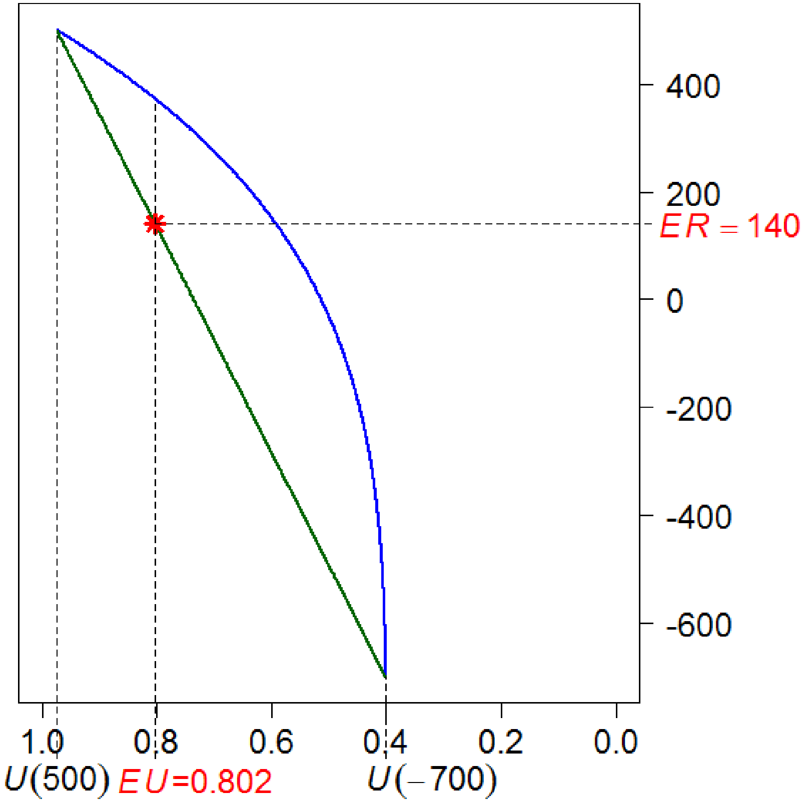
we can graphically illustrate this as



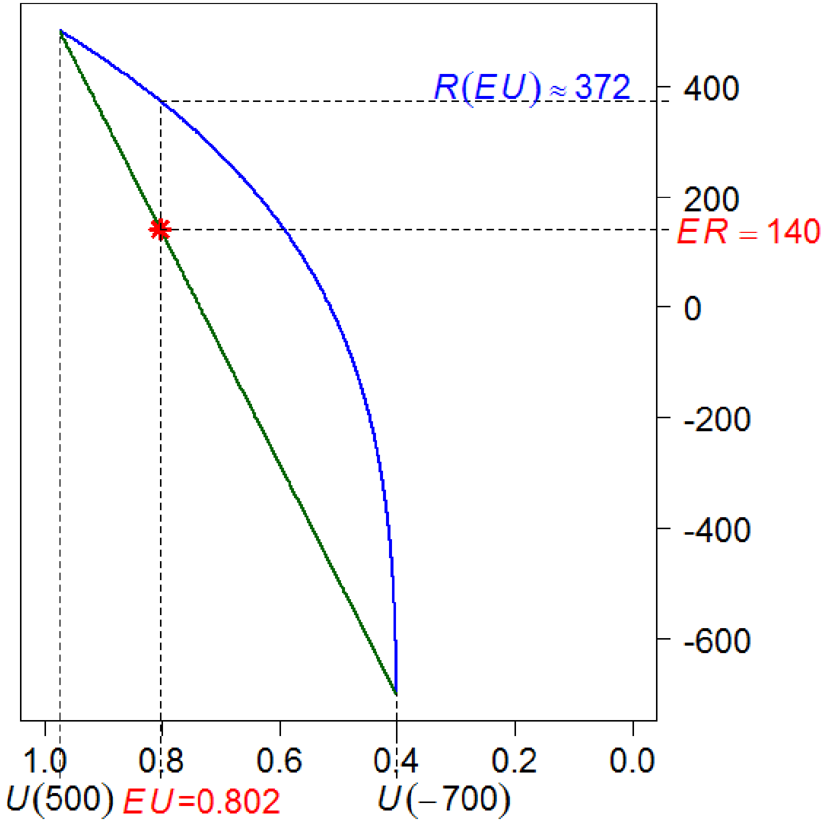
Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

The value of *EU* for taking the bet is the same here as before, i.e. 0.802

If we – as previously – plot *R* against *U*(*R*) we obtain:

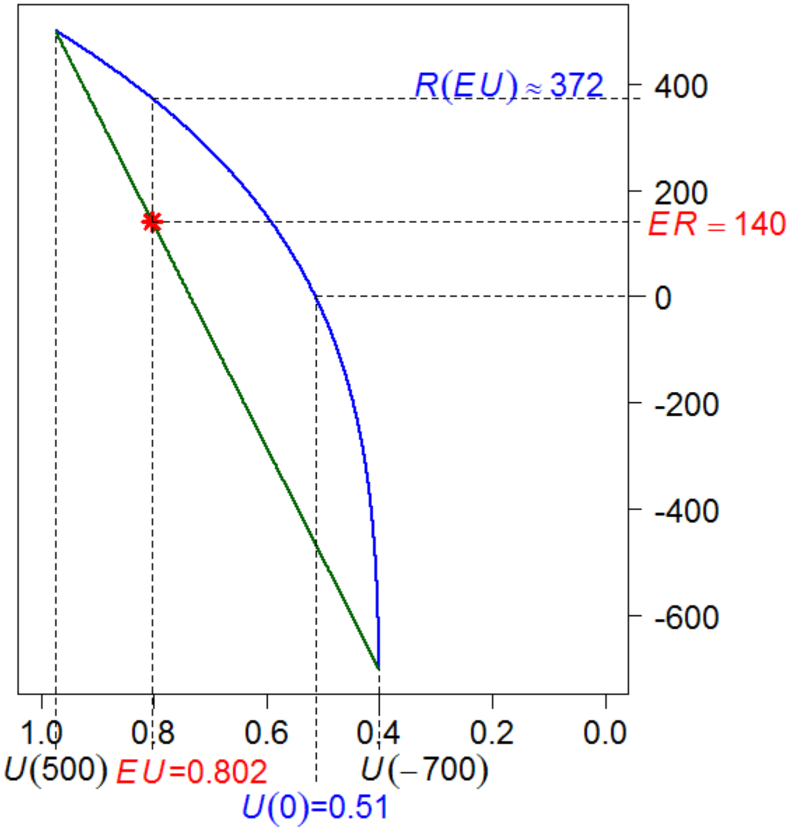


…and the certainty equivalent here becomes $CE = R(EU) \approx 372$



Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

The expected utility of not taking the bet is calculated as $U(0) \approx 0.51$



Thus the optimal decision with the *EU*-criterion is to take the bet, since $U(0) < EU$.

The risk premium with this utility function becomes

$$RP = ER - CE \approx 140 - 372 = -232 \text{ (SEK)}$$

…hence negative!

For a risk taker the certainty equivalent is always higher than (or at least equal to) the expected payoff.

Hence, the risk premium for taking a bet is always negative for a risk taker.

This means that a risk taker is willing to pay a certain amount for taking the bet.

Application to insurances

When it comes to deciding the premium for an insurance, which one (if any) of the insurance provider and the insurance taker is (closest to) being

- a risk avoider ?
- a risk taker ?