Teaching session 3 : Hand In assignment.

Assignment 1:

Prove the Kalman filtering recursion for the following state space model with initial prior on state $f(z_1) = N(z; m_0, P_0)$

where: $e_t \sim N(0, Q_t)$

$v_t \sim N(0, R_t)$

$$z_t = A_{t-1} z_{t-1} + e_t \quad ①$$

$$x_t = C_t z_t + v_t \quad ②$$

Property 1: $f(y_1) f(y_2 | y_1) = f(y_1, y_2)$

$$N(y_1; u_1, \Sigma) \; N(y_2; By_1, R) = N\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}; \begin{bmatrix} u \\ Bu \end{bmatrix} \begin{bmatrix} \Sigma & \Sigma B^T \\ B\Sigma & B\Sigma B^T + R \end{bmatrix}\right)$$

Property 2: Marginalisation and conditioning

$$f(y_1, y_2) = N\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}; \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$$

then:
$$f(y_1) = N(y_1; u_1, \Sigma_{11})$$
$$f(y_2) = N(y_2; u_2, \Sigma_{22})$$
$$f(y_1 | y_2) = N(y_1; u_1 + \Sigma_{12} \Sigma_{22}^{-1} (y_2 - u_2), \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21})$$

$$f(y_2 | y_1) = N(y_2; u_2 + \Sigma_{21} \Sigma_{11}^{-1} (y_1 - u_1), \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12})$$

Assume that the posterior distribution of previous step is gaussian.

$$p(z_{t-1} | x_{1:t-1}) = N(z_{t-1} | m_{t-1}, P_{t-1})$$

Chapman - Kolmogorov eq:

$$p(z_t | x_{1:t-1}) = \int p(z_t | z_{t-1}) \, p(z_{t-1} | x_{1:t-1}) \, dx_{t-1}$$

$$= \int N(z_t | A_{t-1} z_{t-1}, Q_{t-1}) \, N(z_{t-1} | m_{t-1}, P_{t-1})$$

Using gaussian distribution computation rules from previous slides, we get the prediction step.

$$p(z_{t+1} | x_{1:t}) = N(z_{t} | A_{t-1} m_{t-1}, A_{t-1} P_{t-1} A_{t-1}^T + Q_{t})$$

$$= N(z_{t} | m_{t|t-1}, P_{t|t-1})$$

The joint distribution is given by x_t and z_t.

$$p(z_{t}, x_{t} | x_{1:t-1}) = p(x_{t} | z_{t})\, p(z_t | x_{1:t-1})$$

$$= N\left(\begin{bmatrix} z_{t} \\ x_{t} \end{bmatrix} \middle| \begin{pmatrix} m_{t|t-1} \\ C m_{t|t-1} \end{pmatrix}, \begin{pmatrix} P_{t|t-1} & P_{t|t-1} C^T \\ C P_{t|t-1} & C P_{t|t-1} C^T + R \end{pmatrix} \right)$$

The conditional distribution of z_t and x_t is given as

$$p(z_t | x_t, y_{1:t-1}) = p(z_t | x_{1:t})$$

$$= N(z_t | m_{t|t}, P_{t|t})$$

where

$$m_{t|t} = m_{t|t-1} + K_t (x_t - C_t m_{t|t-1})$$

$$P_{t|t} = (I - K_t C_t) P_{t|t-1}$$

$$K_t = P_{t|t-1} C_t^T (C_t P_{t|t-1} C_t^T + R)^{-1}$$