

The state space model is given by,

$$z_t = A z_{t-1} + e_t \quad ; e_t \sim N(0, R)$$

$$x_t = C z_t + v_t \quad ; v_t \sim N(0, Q)$$

To find the state space representation for  $ARIMA(p, d, q) \times (P, D, Q)$ ,

$$\phi^p(B^d) \phi^p(B) (1-B^d)^d (1-B)^d x_t = \theta^q(B^d) \theta^q(B) \omega_t$$

$$\phi^p(B^d) \phi^p(B) (1-B^d)^d (1-B)^d \underbrace{[\theta^q(B^d) \theta^q(B)]^{-1} x_t}_{z_t} = \omega_t$$

$$\therefore \phi^p(B^d) \phi^p(B) (1-B^d)^d (1-B)^d z_t = \omega_t \quad \leftarrow \text{AR}$$

Matrix A will be,

$$A = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \dots & \phi_r \\ 1 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 1 \end{bmatrix}$$

where  $r = \max(P + p + d + D, Q + q + 1)$

The AR equation can be written as,

$$\begin{bmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-r+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_r \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} z_{t-1} \\ \vdots \\ z_{t-r} \end{bmatrix} + \begin{bmatrix} \omega_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Matrix C will be,

$$C = [1 \quad \theta_1 \quad \theta_2 \quad \dots \quad \theta_r]$$

where  $r = \max(P + p + d + D, Q + q + 1)$

The MA equation can be written as,

$$x_t = [1 \quad \theta_1 \quad \theta_2 \quad \dots \quad \theta_r] \begin{bmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-r+1} \end{bmatrix}$$

On substituting,

$$p=3, d=2, q=1, P=2, D=1, Q=1, s=5$$

$$r = \max(3+2+2+5, 5+2+1+1) \\ = \max(13, 9) \\ = 13$$

$$\begin{bmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-14} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_{13} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{bmatrix} \begin{bmatrix} z_{t-1} \\ z_{t-2} \\ \vdots \\ z_{t-13} \end{bmatrix} + \begin{bmatrix} \omega_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

And,

$$x_t = \begin{bmatrix} 1 & \theta_1 & \theta_2 & \dots & \theta_{12} \end{bmatrix} \begin{bmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-14} \end{bmatrix}$$