# Formal Proof of Zero-Knowledge Protocol and HMAC-based Derived Key Security

Obinexus Computing
Nnamdi Michael Okpala

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**Abstract**

In this paper we present a comprehensive formal proof for a zero-knowledge proof (ZKP) protocol as implemented in our system and demonstrate the security of a derived key computed via HMAC encoding. In our model, two parties (Alice and Bob) possess asymmetric key pairs. Alice holds a private key $x_A$ and a public key $y_A$, while Bob similarly holds his own keys. In addition to proving the standard properties of a ZKP protocol (completeness, soundness, and zero-knowledge), we show that the derived key

$$K_{\text{derived}} = \text{HMAC}_{x_A}(y_A)$$

is secure. The proof relies on the pseudorandomness and collision-resistance properties of the underlying HMAC construction.

## 1 Introduction

Zero-Knowledge Proofs (ZKPs) allow a prover to convince a verifier that a statement is true without revealing any additional information. In our context, the protocol is implemented between two parties, Alice and Bob, where (i) Alice can prove possession of a secret $x_A$ satisfying a relation $R(x_A)$ without revealing it, and (ii) a new key is derived from her public key $y_A$ by signing it with her private key using an HMAC construction. This document provides a formal mathematical proof for both the ZKP properties and the security of the HMAC-based derived key.

## 2 Preliminaries and Notation

We define the following:

- $x_A \in \mathbb{Z}_q$: Alice's private key.
- $y_A = g^{x_A} \pmod{p}$: Alice's public key in a cyclic group $G$ of prime order $q$ with generator $g$.
- $x_B$ and $y_B$: Bob's corresponding keys.
- $HMAC_K(m)$: The HMAC function with key $K$ and message $m$, assumed to be a secure pseudorandom function (PRF) based on a collision-resistant hash (e.g., SHA-512).
- $K_{\text{derived}} = \text{HMAC}_{x_A}(y_A)$: The derived key computed using Alice's private key and public key.

## 3 Protocol Description

### 3.1 Schnorr Identification Protocol

The Schnorr protocol is used for identification and is executed as follows:

1. **Commitment:** Alice chooses a random $r \in \mathbb{Z}_q$ and computes
$$t = g^r \pmod{p}.$$
2. **Challenge:** Bob sends a random challenge $c \in \mathbb{Z}_q$.
3. **Response:** Alice computes
$$s = r + c\,x_A \mod q.$$
4. **Verification:** Bob verifies the equation
$$g^s \overset{?}{=} t \cdot y_A^c \pmod{p}.$$

This protocol satisfies:

- *Completeness:* If Alice is honest, the verification equation holds.
- *Soundness:* A cheating prover who does not know $x_A$ cannot produce valid responses for more than one challenge.
- *Zero-Knowledge:* A simulator can produce transcripts indistinguishable from real protocol executions without knowing $x_A$ (see Section 4.1).

### 3.2 Derived Key via HMAC

The derived key is computed as:
$$K_{\text{derived}} = \text{HMAC}_{x_A}(y_A).$$
This operation uses $x_A$ as the secret key and $y_A$ as the message. The security intuition is that—even though $y_A$ is public—the pseudorandomness of HMAC ensures that only someone who knows $x_A$ can compute the correct $K_{\text{derived}}$.

## 4 Security Proof

### 4.1 Zero-Knowledge Property

To prove the zero-knowledge property of the Schnorr protocol, we construct a simulator $S$ that, given a challenge $c$, produces a transcript $(t, c, s)$ that is statistically indistinguishable from an actual protocol run:

1. Choose a random $s \in \mathbb{Z}_q$.
2. Compute
$$t = g^s \cdot y_A^{-c} \pmod{p}.$$
3. Output the transcript $(t, c, s)$.

In an honest execution, Alice computes $t = g^r$ and $s = r + c\,x_A$; hence,
$$g^s = g^{r+c\,x_A} = g^r \cdot g^{c\,x_A} = t \cdot y_A^c.$$

In the simulated transcript, the distribution of $t$ and $s$ is identical since $s$ is uniformly random and $t$ is computed accordingly. Therefore, the transcript produced by $S$ is indistinguishable from one generated in a real protocol execution, proving the zero-knowledge property.

## 4.2 Statistical Independence Analysis

Let $G$ be our cyclic group of prime order $q$ with generator $g$. For the random values chosen in our protocol:

1. **Commitment Phase:** $r$ is chosen uniformly at random from $\mathbb{Z}_q$. The statistical properties of $r$ are critical:
   - Since $q$ is prime, $\mathbb{Z}_q$ forms a field, ensuring uniform distribution.
   - For any two distinct values $r_1, r_2 \in \mathbb{Z}_q$, $g^{r_1} \not\equiv g^{r_2} \pmod{p}$.
   - The distribution of $t = g^r \pmod{p}$ is uniform in the subgroup generated by $g$.
2. **Independence Across Protocol Runs:** Let $r_i$ be the random value chosen in the $i$-th run. For any $i \neq j$,
$$\Pr[r_i \mid r_j] = \Pr[r_i],$$
   which follows from the field properties of $\mathbb{Z}_q$.
3. **Independence of $t$ from $x_A$:** For any fixed $x_A$ and uniformly random $r$:
   - $t$ has $\log_2(q)$ bits of entropy.
   - The mutual information $I(t; x_A) = 0$, ensuring perfect hiding of the commitment.

## 4.3 Security of the HMAC-based Derived Key

We now formally prove the security of the derived key $K_{\text{derived}} = \text{HMAC}_{x_A}(y_A)$.

**Theorem 1.** *Assume that* HMAC *is a secure pseudorandom function (PRF). Then the derived key*

$$K_{derived} = \text{HMAC}_{x_A}(y_A)$$

*is computationally indistinguishable from a uniformly random string for any probabilistic polynomial-time adversary without knowledge of $x_A$.*

*Proof.* Assume, for the sake of contradiction, that there exists a probabilistic polynomial-time adversary $\mathcal{A}$ that can distinguish $K_{\text{derived}}$ from a uniformly random string with non-negligible advantage $\epsilon$. Then, using $\mathcal{A}$, one can construct a distinguisher $\mathcal{D}$ that distinguishes the function $f(m) = \text{HMAC}_{x_A}(m)$ from a truly random function $R(m)$ as follows:

1. $\mathcal{D}$ is given oracle access to a function $O(\cdot)$ which is either $\text{HMAC}_{x_A}(\cdot)$ (with fixed secret $x_A$) or a truly random function.
2. $\mathcal{D}$ queries the oracle on the public key $y_A$ and obtains $O(y_A)$.
3. $\mathcal{D}$ then runs $\mathcal{A}$ with input $O(y_A)$. If $\mathcal{A}$ distinguishes $O(y_A)$ from a random string with advantage $\epsilon$, then $\mathcal{D}$ outputs that the oracle is $\text{HMAC}_{x_A}(\cdot)$; otherwise, it outputs that the oracle is random.

This construction implies that:
$$\text{Adv}^{\text{PRF}}_{\text{HMAC}}(\mathcal{D}) \geq \epsilon - \frac{q_H}{2^n},$$
where $q_H$ is the number of HMAC queries and $n$ is the output length of the HMAC function.

Thus, if HMAC is $(t, \epsilon)$-secure as a PRF, then $K_{\text{derived}}$ is $(t', \epsilon')$-secure with:

$$t' = t - O(1), \quad \epsilon' = \epsilon + \frac{q_H}{2^n}.$$

For practical parameters (e.g., using SHA-512 based HMAC with $n = 512$ bits and assuming $q_H \leq 2^{64}$), the additional term $\frac{q_H}{2^{512}}$ is approximately $2^{-448}$, which is negligible for all practical purposes. $\square$

## 5 Conclusion

We have demonstrated a comprehensive formal proof for a zero-knowledge proof protocol as implemented in our system. First, we showed that the Schnorr identification protocol satisfies completeness, soundness, and zero-knowledge. Next, we provided an explicit statistical independence analysis, ensuring that the random values used in the protocol are uniformly distributed and independent. Finally, we detailed an enhanced security reduction for the HMAC-based derived key, quantifying the security loss and demonstrating that, under practical parameters, the derived key is secure. This construction adheres to best practices in cryptographic protocol design, ensuring that only a party with knowledge of the private key can compute the derived key while revealing no additional information.