

Duality-of-Descriptions Framework: A Unified Theory for Quantum-Classical Electrical Systems

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Abstract

This paper presents a comprehensive mathematical framework for modeling electrical systems that exhibit both quantum and classical behaviors through multiple equivalent descriptions. The Duality-of-Descriptions framework introduces explicit mappings between continuous PDEs, lumped circuits, operator representations, and symbolic logic layers. We develop the theory to handle both homogeneous and heterogeneous data inputs, demonstrating practical applications to electrical engineering problems. The framework provides controlled approximations and duality metrics to evaluate mapping quality, enabling seamless transitions between wave-like and particle-like models while preserving observable predictions and conserved quantities.

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1 Introduction

Modern electrical systems increasingly operate at the boundary between classical and quantum regimes, requiring unified theoretical frameworks that can seamlessly transition between different mathematical descriptions. This paper introduces the Duality-of-Descriptions framework, which provides explicit mappings between multiple modeling formalisms while preserving physical observables.

1.1 Core Concept

A system S is described by multiple *models* M_i (continuous PDE, lumped circuit, stochastic, logical statement-layer). Duality is an explicit mapping $\mathcal{F}_{ij} : M_i \rightarrow M_j$ (and ideally a reverse mapping \mathcal{G}_{ji}) such that observable predictions and conserved quantities correspond under the mapping (up to controlled approximation).

2 Mathematical Framework

2.1 Model Spaces and Morphisms

Let \mathcal{M} be a category whose objects are modeling formalisms:

- **CONT**: Continuum PDEs for wave-like descriptions
- **LUMP**: State-space/circuit models for particle-like descriptions
- **OP**: Operator/Koopman representations
- **SYM**: Symbolic/logic descriptions

A morphism $f : M_i \rightarrow M_j$ is a concrete transformation (projection, modal decomposition, coarse-graining, aggregation, symbolic abstraction, etc.).

2.2 Hilbert Space Formulation

The total system state resides in the tensor product Hilbert space:

$$\mathcal{H} = \mathcal{H}_{\text{quantum}} \otimes \mathcal{H}_{\text{classical}} \quad (1)$$

We define joint states as superpositions:

$$|\Phi\rangle = \sum_{n,m} c_{nm} |\psi_n\rangle \otimes |E_m\rangle \quad (2)$$

where $|\psi_n\rangle \in \mathcal{H}_{\text{quantum}}$ and $|E_m\rangle \in \mathcal{H}_{\text{classical}}$.

3 Concrete Instantiations for Electrical Systems

3.1 Continuous (Wave-like) Model

For a transmission line or distributed circuit, the field variable $\phi(x, t)$ satisfies the wave PDE:

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi - \gamma \frac{\partial \phi}{\partial t} + s(x, t) \quad (3)$$

where c is the wave speed, γ is the damping coefficient, and $s(x, t)$ represents sources.

3.2 Lumped (Particle-like) Model

The state-space representation with circuit nodes $\mathbf{x}(t)$:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \quad \mathbf{y} = C\mathbf{x} \quad (4)$$

3.3 Koopman/Operator Lift

For nonlinear dynamics, we lift to a linear operator \mathcal{K} acting on observables g :

$$\mathcal{K}g = g \circ \Phi \quad (5)$$

where Φ is the flow map.

3.4 Symbolic/Statement Layer

A logical description L consists of temporal logic formulas, rules, and constraints describing permissible transitions and safety invariants.

4 Extended Theory for Heterogeneous Systems

4.1 Homogeneous and Heterogeneous Data Inputs

The framework is extended to handle two classes of data inputs:

4.1.1 Homogeneous Data

Similar data types that can be modeled uniformly (model-agnostic). For example, Cartesian coordinates (x, y, z) and polar coordinates (r, θ, ϕ) are isomorphic systems representing the same spatial information through different parametrizations.

4.1.2 Heterogeneous Data

Mixed data requiring transcriptional transformers with pattern recognition. The system must verify non-mutual exclusion of datasets to establish isomorphic relationships.

4.2 Wave Propagation and Decoherence

At the quantum-classical interface, system behavior emerges through:

- **Coherence:** Phase relations between wave components preserved
- **Decoherence:** Environmental coupling causes phase randomization
- **Interference:** Constructive and destructive wave interactions

5 Explicit Mapping Recipes

5.1 Modal Projection (CONT \rightarrow LUMP)

1. Solve eigenproblem: $\nabla^2 \psi_n + \lambda_n \psi_n = 0$
2. Project: $\phi(x, t) = \sum_n q_n(t) \psi_n(x)$
3. Truncate to first N modes \rightarrow state vector $q(t)$
4. Result matches lumped model parameters A, B, C

5.2 Aggregation/Homogenization

Use averaging and renormalization to produce effective parameters with controlled error estimates from multiscale analysis.

5.3 Koopman Embedding

Choose observable basis $\{g_k\}$ and approximate Koopman operator via EDMD/DMD for linear representation.

5.4 Symbolic Abstraction

Extract invariants (energy functionals, Lyapunov functions) and encode as logical rules.

6 Duality Metrics

We define measures to evaluate mapping quality:

- **Observable error:** $\epsilon_O = \|y_i(t) - \mathcal{F}_{ij}(y_j(t))\|$
- **Invariant preservation:** ΔE (difference in conserved quantities)
- **Information loss:** Relative entropy/KL divergence
- **Computational gain:** State dimension reduction vs fidelity loss

Duality holds when $\epsilon_O, \Delta E$ are below acceptable thresholds and the mapping has a stable pseudo-inverse.

7 Example: Virtual Circuit Control System

Consider monitoring electric current through an electromagnetic field using a coil-based system. The objective is to control electricity flow safely by modeling the system through multiple descriptions.

7.1 Classical Description (Particle Model)

Using Ohm's law and circuit theory:

$$V = IR \tag{6}$$

where current I represents particle flow through resistance R .

7.2 Quantum Description (Wave Model)

The system exhibits wave properties through:

$$\psi(x, t) = A(x, t)e^{iS(x, t)/\hbar} \tag{7}$$

allowing analysis of oscillations and interference patterns.

7.3 Dual Representation

The coherence operator bridges both descriptions, enabling prediction of wave-particle transitions based on measurement context.

8 Binding Condition and Theorem

[Modal Truncation Bound] Let M_{CONT} and M_{LUMP} be models with mapping \mathcal{P} by modal truncation. If the neglected spectral tail energy $E_{\text{tail}} < \varepsilon$ and truncation basis diagonalizes dominant operator, then for all bounded inputs \mathbf{u} :

$$\sup_{t \in [0, T]} \|y_{\text{CONT}}(t) - y_{\text{LUMP}}(t)\| \leq C(\varepsilon) \|u\| \quad (8)$$

where C depends on operator norms and damping.

9 Implementation Roadmap

1. **System Selection:** Choose canonical electrical system (e.g., transmission line with non-linear load)
2. **PDE Derivation:** Derive continuous model and compute mode shapes
3. **Projection:** Build lumped model via modal truncation
4. **Koopman Analysis:** Compute operator approximation for nonlinear dynamics
5. **Logic Extraction:** Encode invariants in temporal logic
6. **Validation:** Evaluate mappings with defined metrics
7. **Documentation:** Create category-theoretic framework repository

10 Tools and Methods

- Modal analysis (FEM/spectral methods)
- Model reduction (balanced truncation, POD)
- DMD/EDMD for Koopman approximation
- SMT/model-checkers for statement layer
- Python/MATLAB for numerical experiments

11 Applications and Extensions

11.1 Practical Applications

- Power grid stability analysis
- Quantum device modeling
- RF circuit design
- Control system synthesis

11.2 Future Extensions

- Multi-scale integration
- Non-Markovian environments
- Stochastic model incorporation
- Machine learning enhancement

12 Conclusion

The Duality-of-Descriptions framework provides a mathematically rigorous approach to modeling electrical systems across quantum and classical regimes. By establishing explicit mappings between different mathematical representations while preserving physical observables, this framework enables engineers to choose the most appropriate description for their specific application while maintaining consistency across scales.

The ability to handle both homogeneous and heterogeneous data inputs, combined with quantitative duality metrics, makes this framework particularly suitable for modern electrical systems that operate at the quantum-classical boundary. Future work will focus on experimental validation and extension to more complex multi-agent systems.

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