

# Feedback Particle Filter

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**Abstract**—The feedback particle filter introduced in this paper is a new approach to approximate nonlinear filtering, motivated by techniques from mean-field game theory. The filter is defined by an ensemble of controlled stochastic systems (the particles). Each particle evolves under feedback control based on its own state, and features of the empirical distribution of the ensemble. The feedback control law is obtained as the solution to an optimal control problem, in which the optimization criterion is the Kullback–Leibler divergence between the actual posterior, and the common posterior of any particle. The following conclusions are obtained for diffusions with continuous observations: 1) The optimal control solution is exact: The two posteriors match exactly, provided they are initialized with identical priors. 2) The optimal filter admits an innovation error-based gain feedback structure. 3) The optimal feedback gain is obtained via a solution of an Euler–Lagrange boundary value problem; the feedback gain equals the Kalman gain in the linear Gaussian case. Numerical algorithms are introduced and implemented in two general examples, and a neuroscience application involving coupled oscillators. In some cases it is found that the filter exhibits significantly lower variance when compared to the bootstrap particle filter.

**Index Terms**—Mean-field games, nonlinear filtering, optimal transportation, particle filtering.

## I. INTRODUCTION

We consider a scalar filtering problem

$$dX_t = a(X_t)dt + \sigma_B dB_t, \quad (1a)$$

$$dZ_t = h(X_t)dt + \sigma_W dW_t \quad (1b)$$

where  $X_t \in \mathbb{R}$  is the state at time  $t$ ,  $Z_t \in \mathbb{R}$  is the observation,  $a(\cdot), h(\cdot)$  are  $C^1$  functions, and  $\{B_t\}, \{W_t\}$  are mutually independent standard Wiener processes. Unless otherwise noted, the stochastic differential equations (SDEs) are expressed in Itô form.

The objective of the filtering problem is to compute or approximate the posterior distribution of  $X_t$  given the filtration (history of observations)  $\mathcal{Z}_t := \sigma(Z_s : s \leq t)$ . The posterior  $p^*$  is defined so that, for any measurable set  $A \subset \mathbb{R}$

$$\int_{x \in A} p^*(x, t) dx = \mathbb{P}\{X_t \in A | \mathcal{Z}_t\}. \quad (2)$$

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The filter is infinite-dimensional since it defines the evolution, in the space of probability measures, of  $\{p^*(\cdot, t) : t \geq 0\}$ . If  $a(\cdot), h(\cdot)$  are linear functions, the solution is given by the finite-dimensional Kalman filter. The theory of nonlinear filtering is described in the classic monograph [3].

The article [4] surveys numerical methods to approximate the nonlinear filter. One approach described in this survey is particle filtering.

The particle filter is a simulation-based algorithm to approximate the filtering task [5]–[7]. The key step is the construction of  $N$  stochastic processes  $\{X_t^i : 1 \leq i \leq N\}$ . The value  $X_t^i \in \mathbb{R}$  is the state for the  $i$ th particle at time  $t$ . For each time  $t$ , the empirical distribution formed by the “particle population” is used to approximate the conditional distribution. Recall that this is defined for any measurable set  $A \subset \mathbb{R}$  by

$$p^{(N)}(A, t) = \frac{1}{N} \sum_{i=1}^N \mathbb{1}\{X_t^i \in A\}. \quad (3)$$

A common approach in particle filtering is called *sequential importance sampling*, where particles are generated according to their importance weight at every time stage [4], [7]. By choosing the sampling mechanism properly, particle filtering can approximately propagate the posterior distribution, with the accuracy improving as  $N$  increases [8].

The objective of this paper is to introduce an alternative approach to the construction of a particle filter for (1a), (1b) inspired by mean-field optimal control techniques; cf., [9], [10]. In this approach, the model for the  $i$ th particle is defined by a controlled system

$$dX_t^i = a(X_t^i) dt + \sigma_B dB_t^i + dU_t^i \quad (4)$$

where  $X_t^i \in \mathbb{R}$  is the state for the  $i$ th particle at time  $t$ ,  $U_t^i$  is its control input, and  $\{B_t^i\}$  are mutually independent standard Wiener processes. Certain additional assumptions are made regarding admissible forms of control input.

Throughout the paper, we denote conditional distribution of a particle  $X_t^i$  given  $\mathcal{Z}_t$  by  $p$  where, just as in the definition of  $p^*$

$$\int_{x \in A} p(x, t) dx = \mathbb{P}\{X_t^i \in A | \mathcal{Z}_t\}. \quad (5)$$

The initial conditions  $\{X_0^i\}_{i=1}^N$  are assumed to be independent and identically distributed (i.i.d.), and drawn from initial distribution  $p^*(x, 0)$  of  $X_0$  (i.e.,  $p(x, 0) = p^*(x, 0)$ ).

The control problem is to choose the control input  $U_t^i$  so that  $p$  approximates  $p^*$ , and consequently  $p^{(N)}$  [defined in (3)] approximates  $p^*$  for large  $N$ . The synthesis of the control input is cast as an optimal control problem, with the Kullback–Leibler metric serving as the cost function. The optimal control input is obtained via analysis of the first variation.

The main result of this paper is to derive an explicit formula for the optimal control input, and demonstrate that under general conditions we obtain an exact match:  $p = p^*$  under optimal control. The optimally controlled dynamics of the  $i$ th particle have the following Itô form:

$$dX_t^i = a(X_t^i)dt + \sigma_B dB_t^i + \underbrace{K(X_t^i, t)dI_t^i + \Omega(X_t^i, t)dt}_{\text{optimal control, } dU_t^{i*}} \quad (6)$$

in which  $\Omega(x, t) := (1/2)\sigma_W^2 K(x, t)K'(x, t)$ ,  $K'(x, t) = (\partial K / \partial x)(x, t)$ , and  $I_t^i$  is a certain modified form of the *innovation process* that appears in the nonlinear filter

$$dI_t^i := dZ_t - \frac{1}{2} \left( h(X_t^i) + \hat{h}_t \right) dt \quad (7)$$

where  $\hat{h}_t := E[h(X_t^i) | \mathcal{Z}_t] = \int h(x)p(x, t)dx$ . In a numerical implementation, we approximate

$$\hat{h}_t \approx \hat{h}_t^{(N)} := \frac{1}{N} \sum_{i=1}^N h(X_t^i). \quad (8)$$

The gain function  $K$  is shown to be the solution to the following Euler–Lagrange boundary value problem (E-L BVP)

$$-\frac{\partial}{\partial x} \left( \frac{1}{p(x, t)} \frac{\partial}{\partial x} \{p(x, t)K(x, t)\} \right) = \frac{1}{\sigma_W^2} h'(x) \quad (9)$$

with boundary conditions  $\lim_{x \rightarrow \pm\infty} p(x, t)K(x, t) = 0$ , where  $h'(x) = (d/dx)h(x)$ .

Note that the gain function needs to be obtained for each value of time  $t$ . If the right-hand side of (9) is nonnegative valued, it then follows from the minimum principle for elliptic BVPs that the gain function  $K$  is nonnegative valued [11].

The contributions of this paper are as follows:

- **Variational Problem.** The construction of the feedback particle filter is based on a variational problem, where the cost function is the Kullback–Leibler (K-L) divergence between  $p^*(x, t)$  and  $p(x, t)$ . The feedback particle filter (6)–(9), including the formula (7) for the innovation error and the E-L BVP (9), is obtained via analysis of the first variation.
- **Consistency.** The particle filter model (6) is consistent with nonlinear filter in the following sense: Suppose the gain function  $K(x, t)$  is obtained as the solution to (9), and the priors are consistent,  $p(x, 0) = p^*(x, 0)$ . Then, for all  $t \geq 0$  and all  $x$

$$p(x, t) = p^*(x, t).$$

- **Algorithms.** Numerical techniques are proposed for synthesis of the gain function  $K(x, t)$ . If  $a(\cdot)$  and  $h(\cdot)$  are linear and the density  $p^*$  is Gaussian, then the gain function is simply the Kalman gain. At time  $t$ , it is a constant given in terms of variance alone. The variance is approximated empirically as a sample covariance.

In the nonlinear case, numerical approximation techniques are described. Other approaches using Galerkin methods or the

sum of Gaussian approximation also exist but are omitted on account of space. Details for these can be found in [2], [12].

In recent decades, there have been many important advances in importance sampling based approaches for particle filtering; cf., [4], [7], and [13]. A crucial distinction here is that there is no resampling of particles.

We believe that the introduction of control in the feedback particle filter has several useful features/advantages:

- **Does not require sampling.** There is no re-sampling required as in the conventional particle filter. This property allows the feedback particle filter to be flexible with regards to implementation and does not suffer from sampling-related issues.
- **Innovation error.** The innovation error-based feedback structure is a key feature of the feedback particle filter (6). The innovation error in (6) is based on the average value of the prediction  $h(X_t^i)$  of the  $i$ th particle and the prediction  $\hat{h}_t$  due to the entire population.

The feedback structure is easier to see when the filter is expressed in its Stratonovich form:

$$dX_t^i = a(X_t^i)dt + dB_t^i + K(X_t^i, t) \circ \left( dZ_t - \frac{1}{2} \left( h(X_t^i) + \hat{h}_t \right) dt \right). \quad (10)$$

Given that the Stratonovich form provides a mathematical interpretation of the (formal) ODE model [14, Sec. 3.3], we also obtain the ODE model of the filter. Denoting  $Y_t \doteq (dZ_t/dt)$  and white noise process  $\dot{B}_t^i \doteq (dB_t^i/dt)$ , the ODE model of the filter is given by

$$\frac{dX_t^i}{dt} = a(X_t^i) + \dot{B}_t^i + K(X_t^i, t) \cdot \left( Y_t - \frac{1}{2} \left( h(X_t^i) + \hat{h}_t \right) \right).$$

The “innovation” of the feedback particle filter thus lies in the (modified) definition of innovation error for a particle filter [see (7)]. Moreover, the feedback control structure that existed thus far only for Kalman filter now also exists for particle filters (compare parts (a) and (b) of Fig. 1). For the linear case, the optimal gain function is the Kalman gain. For the nonlinear case, the Kalman gain is replaced by a nonlinear function of the state.

- **Feedback structure.** Feedback is important on account of the issue of *robustness*. A filter is based on an idealized model of the underlying dynamic process that is often nonlinear, uncertain and time-varying. The self-correcting property of the feedback provides robustness, allowing one to tolerate a degree of uncertainty inherent in any model.

In contrast, a conventional particle filter is based upon importance sampling. Although the innovation error is central to the Kushner–Stratonovich’s stochastic partial differential equation (SPDE) of nonlinear filtering, it is conspicuous by its absence in a conventional particle filter.

Arguably, the structural aspects of the Kalman filter have been as important as the algorithm itself in design, integration, testing, and operation of the overall system. Without such structural features, it is a challenge to create scalable cost-effective solutions.

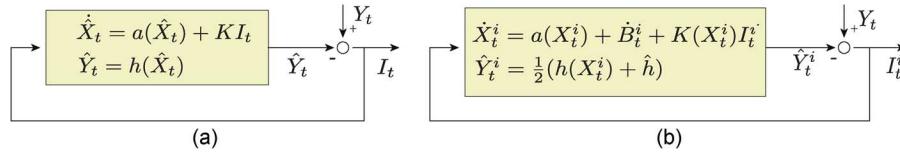


Fig. 1. Innovation error-based feedback structure for (a) Kalman filter and (b) feedback particle filter.

- **Variance reduction.** Feedback can help reduce the high variance that is sometimes observed in the conventional particle filter. Numerical results in Section V support this claim—see Fig. 3 for a comparison of the feedback particle filter and the bootstrap particle filter.
- **Ease of design, testing and operation.** On account of structural features, feedback particle filter-based solutions are expected to be more robust, cost-effective, and easier to debug and implement.
- **Applications.** Bayesian inference is an important paradigm used to model functions of certain neural circuits in the brain [15], [16]. Compared to techniques that rely on importance sampling, a feedback particle filter may provide a more neuro-biologically plausible model to implement filtering and inference functions. This is illustrated here with the aid of a filtering problem involving nonlinear oscillators. Another application appears in [17] and [18].

#### A. Comparison With Relevant Literature

Our work is motivated by recent development in mean-field games, but the focus there has been primarily on optimal control [9], [10].

In nonlinear filtering, there are two directly related works: Crisan and Xiong [19], and Mitter and Newton [20]. In each of these papers, a controlled system is introduced, of the form

$$dX_t^i = (a(X_t^i) + u(X_t^i, t)) dt + \sigma_B dB_t^i.$$

The objective is to choose the control input to obtain a solution of the nonlinear filtering problem.

The approach in [20] is based on consideration of a finite-horizon optimal control problem. It leads to an HJB equation whose solution yields the optimal control input.

The work of Crisan and Xiong is closer to our paper in terms of both goals and approaches. Although we were not aware of their work prior to submission of our original conference papers [1], [2], Crisan and Xiong provide an explicit expression for a control law that is similar to the feedback particle filter, with some important differences. One, the considerations of Crisan and Xiong (and also of Newton and Mitter) require introduction of a *smooth approximation* of the process “ $(dZ/dt) - \hat{h}_t$ ,” which we avoid with our formulation. Two, the filter derived in Crisan and Xiong has a structure based on a gain feedback with respect to the smooth approximation, while the feedback particle filter is based on a modified formula for the innovation error  $I_t^i$  as given in (7). This formula is fundamental to construction of particle filters in continuous time settings. We clarify here that the formula for innovation error is *not* assumed, rather it comes about as a result of the analysis of the variational problem.

Remarkably, both the feedback particle filter and Crisan and Xiong’s filter require solution of the same boundary value

problem, and as such have the same computational complexity. The BVP is solved to obtain the gain function. However, the particular solution described in Crisan and Xiong for the BVP may not work in all cases, including the linear Gaussian case. Additional discussion appears in Section III-E.

Apart from these works, Daum and Huang have introduced the *information flow filter* for the continuous-discrete time filtering problem [21]. Although an explicit formula for the filter is difficult to obtain, a closely related form of the boundary value problem appears in their work. There is also an important discussion of both the limitations of the conventional particle filter, and the need to incorporate feedback to ameliorate these issues. Several numerical experiments are presented that describe high variance and robustness issues, especially where signal models are unstable. These results provide significant motivation to the work described here.

#### B. Outline

The variational setup is described in Section II: It begins with a discussion of the continuous-discrete filtering problem: the equation for dynamics is defined by (1a), but the observations are made only at discrete times. The continuous-time filtering problem [for (1a), (1b)] is obtained as a limiting case of the continuous-discrete problem.

The feedback particle filter is introduced in Section III. Extension to the multivariable case is briefly described in Section III-D, followed by a comparison with Crisan and Xiong’s filter in Section III-E.

Algorithms are discussed in Section IV, and numerical examples are described in Section V, including the neuroscience application involving coupled oscillator models. These models (also considered in our earlier mean-field control paper [10]) provided some of the initial motivation for the present work.

## II. VARIATIONAL PROBLEM

The precise formulation of the optimal control problem begins with the continuous time model, with sampled observations. The equation for dynamics is given by (1a), and the observations are made only at discrete times  $\{t_n\}$

$$Y_{t_n} = h(X_{t_n}) + W_{t_n}^\Delta \quad (11)$$

where  $\Delta := t_{n+1} - t_n$  and  $\{W_{t_n}^\Delta\}$  is i.i.d. and drawn from  $\mathcal{N}(0, (\sigma_W^2 / \Delta))$ .

The particle model in this case is a hybrid dynamical system: For  $t \in [t_{n-1}, t_n]$ , the  $i$ th particle evolves according to the stochastic differential equation

$$dX_t^i = a(X_t^i) dt + \sigma_B dB_t^i, \quad t_{n-1} \leq t < t_n \quad (12)$$

where the initial condition  $X_{t_{n-1}}^i$  is given. At time  $t = t_n$  there is a potential jump that is determined by the input  $U_{t_n}^i$

$$X_{t_n}^i = X_{t_n^-}^i + U_{t_n}^i \quad (13)$$

where  $X_{t_n^-}^i$  denotes the right limit of  $\{X_t^i : t_{n-1} \leq t < t_n\}$ . The specification (13) defines the initial condition for the process on the next interval  $[t_n, t_{n+1}]$ .

The filtering problem is to construct a control law that defines  $\{U_{t_n}^i : n \geq 1\}$  such that  $p(\cdot, t_n)$  approximates  $p^*(\cdot, t_n)$  for each  $n \geq 1$ . To solve this problem we first define “belief maps” that propagate the conditional distributions of  $X_t$  and  $X_t^i$ .

#### A. Belief Maps

The observation history is denoted  $\mathcal{Y}_n := \sigma\{Y_{t_k} : k \leq n, k \in \mathbb{N}\}$ . For each  $n$ , various conditional distributions are considered:

- 1)  $p_n^*$  and  $p_n^{*-}$ : The conditional distribution of  $X_{t_n}$  given  $\mathcal{Y}_n$  and  $\mathcal{Y}_{n-1}$ , respectively.
- 2)  $p_n$  and  $p_n^-$ : The conditional distribution of  $X_{t_n}^i$  given  $\mathcal{Y}_n$  and  $\mathcal{Y}_{n-1}$ , respectively.

These densities evolve according to recursions of the form

$$p_n^* = \mathcal{P}^*(p_{n-1}^*, Y_{t_n}), \quad p_n = \mathcal{P}(p_{n-1}, Y_{t_n}). \quad (14)$$

The mappings  $\mathcal{P}^*$  and  $\mathcal{P}$  can be decomposed into two parts. The first part is identical for each of these mappings: the transformation that takes  $p_{n-1}$  to  $p_n^-$  coincides with the mapping from  $p_{n-1}^*$  to  $p_n^{*-}$ . In each case it is defined by the Kolmogorov forward equation associated with the diffusion on  $[t_{n-1}, t_n]$ .

The second part of the mapping is the transformation that takes  $p_n^{*-}$  to  $p_n^*$ , which is obtained from Bayes’ rule: Given the observation  $Y_{t_n}$  made at time  $t = t_n$

$$p_n^*(s) = \frac{p_n^{*-}(s) \cdot p_{Y|X}(Y_{t_n}|s)}{p_Y(Y_{t_n})}, \quad s \in \mathbb{R} \quad (15)$$

where  $p_Y$  denotes the probability density function (pdf) for  $Y_{t_n}$ , and  $p_{Y|X}(\cdot|s)$  denotes the conditional distribution of  $Y_{t_n}$  given  $X_{t_n} = s$ . Applying (11) gives

$$p_{Y|X}(Y_{t_n}|s) = \frac{1}{\sqrt{\frac{2\pi\sigma_w^2}{\Delta}}} \exp\left(-\frac{(Y_{t_n} - h(s))^2}{\frac{2\sigma_w^2}{\Delta}}\right).$$

Combining (15) with the forward equation defines  $\mathcal{P}^*$ .

The transformation that takes  $p_n^-$  to  $p_n$  depends upon the choice of control  $U_{t_n}^i$  in (13). At time  $t = t_n$ , we seek a control input  $U_{t_n}^i$  that is *admissible*.

*Definition 1 (Admissible Input):* The control sequence  $\{U_{t_n}^i : n \geq 0\}$  is *admissible* if there is a sequence of maps  $\{v_n(x; y_0^n)\}$  such that  $U_{t_n}^i = v_n(X_{t_n^-}^i, Y_{t_0}, \dots, Y_{t_n})$  for each  $n$ , and moreover:

- i)  $E[|U_{t_n}^i|] < \infty$ , and with probability one

$$\lim_{x \rightarrow \pm\infty} v_n(x, Y_{t_0}, \dots, Y_{t_n}) p_n^-(x) = 0.$$

- ii)  $v_n$  is twice continuously differentiable as a function of  $x$ .
- iii)  $1 + v'_n(x)$  is nonzero for all  $x$ , where  $v'_n(x) = (d/dx)v_n(x)$ . ■

We will suppress the dependency of  $v_n$  on the observations (and often the time-index  $n$ ), writing  $U_{t_n}^i = v(x)$  when  $X_{t_n^-}^i =$

$x$ . Under the assumption that  $1 + v'(x)$  is nonzero for all  $x$ , we can write

$$p_n(x^+) = \frac{p_n^-(x)}{|1 + v'(x)|}, \quad \text{where } x^+ = x + v(x). \quad (16)$$

#### B. Variational Problem

Our goal is to choose an admissible input so that the mapping  $\mathcal{P}$  approximates the mapping  $\mathcal{P}^*$  in (14). More specifically, given the pdf  $p_{n-1}$  we have already defined the mapping  $\mathcal{P}$  so that  $p_n = \mathcal{P}(p_{n-1}, Y_{t_n})$ . We denote  $\hat{p}_n^* = \mathcal{P}^*(p_{n-1}, Y_{t_n})$ , and choose  $v_n$  so that these pdfs are as close as possible. We approach this goal through the formulation of an optimization problem with respect to the K-L divergence metric. That is, at time  $t = t_n$ , the function  $v_n$  is the solution to the following optimization problem:

$$v_n(x) = \arg \min_v D(p_n \| \hat{p}_n^*). \quad (17)$$

Based on the definitions, for any  $v$  the K-L divergence can be expressed as

$$D(p_n \| \hat{p}_n^*) = - \int_{\mathbb{R}} p_n^-(x) \{ \ln |1 + v'(x)| + \ln (p_n^-(x + v(x)) p_{Y|X}(Y_{t_n}|x + v(x))) \} dx + C \quad (18)$$

where  $C = \int_{\mathbb{R}} p_n^-(x) \ln(p_n^-(x) p_Y(Y_{t_n})) dx$  is a constant that does not depend on  $v$ ; cf., Appendix A for the calculation.

The solution to (17) is described in the following proposition, whose proof appears in Appendix B.

*Proposition 2.1:* Suppose that the admissible input is obtained as the solution to the sequence of optimization problems (17). Then for each  $n$ , the function  $v = v_n$  is a solution of the following Euler-Lagrange (E-L) BVP:

$$\left( \frac{d}{dx} \right) \left( \frac{p_n^-(x)}{|1 + v'(x)|} \right) = p_n^-(x) \frac{\partial}{\partial v} (\ln (p_n^-(x + v) p_{Y|X}(Y_{t_n}|x + v))) \quad (19)$$

with boundary condition  $\lim_{x \rightarrow \pm\infty} v(x) p_n^-(x) = 0$ . ■

We refer to the minimizer as the *optimal control function*. Additional details on the continuous-discrete time filter appear in our conference paper [1].

### III. FEEDBACK PARTICLE FILTER

We now consider the continuous time filtering problem (1a), (1b) introduced in Section I.

#### A. Belief State Dynamics and Control Architecture

The model for the particle filter is given by the Itô diffusion

$$dX_t^i = a(X_t^i) dt + \sigma_B dB_t^i + \underbrace{u(X_t^i, t) dt + K(X_t^i, t) dZ_t}_{dU_t^i} \quad (20)$$

where  $X_t^i \in \mathbb{R}$  is the state for the  $i$ th particle at time  $t$ , and  $\{B_t^i\}$  are mutually independent standard Wiener processes. We assume the initial conditions  $\{X_0^i\}_{i=1}^N$  are i.i.d., independent of  $\{B_t^i\}$ , and drawn from the initial distribution  $p^*(x, 0)$  of  $X_0$ .

Both  $\{B_t^i\}$  and  $\{X_0^i\}$  are also assumed to be independent of  $X_t, Z_t$ .

As in Section II, we impose admissibility requirements on the control input  $U_t^i$  in (20):

**Definition 2 (Admissible Input):** The control input  $U_t^i$  is *admissible* if the random variables  $u(x, t)$  and  $K(x, t)$  are  $\mathcal{Z}_t = \sigma(Z_s : s \leq t)$  measurable for each  $t$ . Moreover, each  $t$ ,

- i)  $E[|u(X_t^i, t)| + |K(X_t^i, t)|^2] < \infty$ , and with probability one

$$\lim_{x \rightarrow \pm\infty} u(x, t)p(x, t) = 0 \quad (21a)$$

$$\lim_{x \rightarrow \pm\infty} K(x, t)p(x, t) = 0. \quad (21b)$$

where  $p$  is the posterior distribution of  $X_t^i$  given  $\mathcal{Z}_t$ , defined in (5).

- ii)  $u : \mathbb{R}^2 \rightarrow \mathbb{R}, K : \mathbb{R}^2 \rightarrow \mathbb{R}$  are twice continuously differentiable in their first arguments. ■

The functions  $\{u(x, t), K(x, t)\}$  represent the continuous-time counterparts of the optimal control function  $v_n(x)$  [see (17)]. We say that these functions are *optimal* if  $p \equiv p^*$ , where recall  $p^*$  is the posterior distribution of  $X_t$  given  $\mathcal{Z}_t$  as defined in (2). Given  $p^*(\cdot, 0) = p(\cdot, 0)$ , our goal is to choose  $\{u, K\}$  in the feedback particle filter so that the evolution equations of these conditional distributions coincide.

The evolution of  $p^*(x, t)$  is described by the Kushner–Stratonovich (K-S) equation:

$$dp^* = \mathcal{L}^\dagger p^* dt + \frac{1}{\sigma_W^2} (h - \hat{h}_t)(dZ_t - \hat{h}_t dt)p^* \quad (22)$$

where  $\hat{h}_t = \int h(x)p^*(x, t)dx$ , and  $\mathcal{L}^\dagger p^* = -(\partial(p^*a)/\partial x) + (\sigma_B^2/2)(\partial^2 p^*/\partial x^2)$ .

The evolution equation of  $p(x, t)$  is described next. The proof appears in Appendix C.

**Proposition 3.1:** Consider the process  $X_t^i$  that evolves according to the particle filter model (20). The conditional distribution of  $X_t^i$  given the filtration  $\mathcal{Z}_t, p(x, t)$ , satisfies the forward equation

$$dp = \mathcal{L}^\dagger pdt - \frac{\partial}{\partial x}(Kp)dZ_t - \frac{\partial}{\partial x}(up)dt + \sigma_W^2 \frac{1}{2} \frac{\partial^2}{\partial x^2}(pK^2)dt. \quad (23)$$

### B. Consistency With the Nonlinear Filter

The main result of this section is the construction of an optimal pair  $\{u, K\}$  under the following assumption:

**Assumption A1:** The conditional distributions  $(p^*, p)$  are  $C^2$ , with  $p^*(x, t) > 0$  and  $p(x, t) > 0$ , for all  $x \in \mathbb{R}, t > 0$ . ■

We henceforth choose  $\{u, K\}$  as the solution to a certain E-L BVP based on  $p$ : the function  $K$  is the solution to

$$-\frac{\partial}{\partial x} \left( \frac{1}{p(x, t)} \frac{\partial}{\partial x} \{p(x, t)K(x, t)\} \right) = \frac{1}{\sigma_W^2} h'(x) \quad (24)$$

with boundary condition (21b). The function  $u(\cdot, t) : \mathbb{R} \rightarrow \mathbb{R}$  is obtained as

$$u(x, t) = K(x, t) \left( -\frac{1}{2} (h(x) + \hat{h}_t) + \frac{1}{2} \sigma_W^2 K'(x, t) \right) \quad (25)$$

where  $\hat{h}_t = \int h(x)p(x, t)dx$ . We assume moreover that the control input obtained using  $\{u, K\}$  is admissible. The particular form of  $u$  given in (25) and the BVP (24) is motivated by considering the continuous-time limit of (19), obtained on letting  $\Delta := t_{n+1} - t_n$  go to zero; the calculations appear in Appendix D.

Existence and uniqueness of  $\{u, K\}$  is obtained in the following proposition 4. Its proof is given in Appendix E.

**Proposition 3.2:** Consider the BVP (24), subject to Assumption A1. Then,

- 1) There exists a unique solution  $K$ , subject to the boundary condition (21b).
- 2) The solution satisfies  $K(x, t) \geq 0$  for all  $x, t$ , provided  $h'(x) \geq 0$  for all  $x$ . ■

The following theorem shows that the two evolution (22) and (23) are identical. The proof appears in Appendix F.

**Theorem 3.3:** Consider the two evolution equations for  $p$  and  $p^*$ , defined according to the solution of the forward (23) and the K-S (22), respectively. Suppose that the control functions  $u(x, t)$  and  $K(x, t)$  are obtained according to (24) and (25), respectively. Then, provided  $p(x, 0) = p^*(x, 0)$ , we have for all  $t \geq 0$ ,

$$p(x, t) = p^*(x, t).$$

**Remark 1:** Theorem 3.3 is based on the *ideal setting* in which the gain  $K(X_t^i, t)$  is obtained as a function of the posterior  $p = p^*$  for  $X_t^i$ . In practice the algorithm is applied with  $p$  replaced by the empirical distribution of the  $N$  particles.

In this ideal setting, the empirical distribution of the particle system will approximate the posterior distribution  $p^*(x, t)$  as  $N \rightarrow \infty$ . The convergence is in the weak sense in general. To obtain almost sure convergence, it is necessary to obtain sample path representations of the solution to the stochastic differential equation for each  $i$  (see, e.g., [22]). Under these conditions the solution to the SDE (4) for each  $i$  has a functional representation

$$X_t^i = F \left( X_0^i, B_{[0,t]}^i; Z_{[0,t]} \right)$$

where the notation  $Z_{[0,t]}$  signifies the entire sample path  $\{Z_s : 0 \leq s \leq t\}$  for a stochastic process  $Z$ ;  $F$  is a continuous functional (in the uniform topology) of the sample paths  $\{B_{[0,t]}^i, Z_{[0,t]}\}$  along with the initial condition  $X_0^i$ . It follows that the empirical distribution has a functional representation:

$$p^{(N)}(A, t) = \frac{1}{N} \sum_{i=1}^N \mathbb{1} \left\{ F \left( X_0^i, B_{[0,t]}^i; Z_{[0,t]} \right) \in A \right\}.$$

The sequence  $\{(X_0^i, B_{[0,t]}^i) : i = 1, \dots\}$  is i.i.d. and independent of  $Z$ . It follows that the summand  $\{\mathbb{1}\{F(X_0^i, B_{[0,t]}^i; Z_{[0,t]}) : i = 1, \dots\}$  is also i.i.d. given  $Z_{[0,t]}$ . Almost sure convergence follows from the Law of Large Numbers for scalar i.i.d. sequences.

In current research we are considering the more difficult problem of performance bounds for the approximate implementations described in Section IV.

*Remark 2:* On integrating (24) once, we obtain an *equivalent* characterization of the E-L BVP:

$$\frac{\partial}{\partial x}(pK) = -\frac{1}{\sigma_W^2}(h - \hat{h}_t)p \quad (26)$$

now with a single boundary condition  $\lim_{x \rightarrow -\infty} pK(x, t) = 0$ . The resulting gain function can be readily shown to yield admissible control input under certain additional technical assumptions on density  $p$  and the function  $h$ .

Given the scope of this paper, and the fact that the same *a priori* bounds apply also to the multivariable case, we defer additional discussion to a future publication; cf. [12].

*Remark 3:* Although the methodology and the filter is presented for Gaussian process and observation noise, the case of non-Gaussian process noise is easily handled—simply replace the noise model in the filter with the appropriate model of the process noise.

For other types of observation noise, one would modify the conditional distribution  $p_{Y|X}$  in the optimization problem (17). The derivation of filter would then proceed by consideration of the first variation (see Appendix D).

### C. Example: Linear Model

It is helpful to consider the feedback particle filter in the following simple linear setting:

$$dX_t = \alpha X_t dt + \sigma_B dB_t \quad (27a)$$

$$dZ_t = \gamma X_t dt + \sigma_W dW_t \quad (27b)$$

where  $\alpha, \gamma$  are real numbers. The initial distribution  $p^*(x, 0)$  is assumed to be Gaussian with mean  $\mu_0$  and variance  $\Sigma_0$ .

The following lemma provides the solution of the gain function  $K(x, t)$  in the linear Gaussian case.

**Lemma 3.4:** Consider the linear observation (27b). If  $p(x, t)$  is assumed to be Gaussian with mean  $\mu_t$  and variance  $\Sigma_t$ , then the solution of E-L BVP (9) is given by

$$K(x, t) = \frac{\Sigma_t \gamma}{\sigma_W^2}. \quad (28)$$

The formula (28) is verified by direct substitution in the ODE (9) where the distribution  $p$  is Gaussian.

The optimal control yields the following form for the particle filter in this linear Gaussian model:

$$dX_t^i = \alpha X_t^i dt + \sigma_B dB_t^i + \frac{\Sigma_t \gamma}{\sigma_W^2} \left( dZ_t - \gamma \frac{X_t^i + \mu_t}{2} dt \right). \quad (29)$$

Now we show that  $p = p^*$  in this case. That is, the conditional distributions of  $X_t$  and  $X_t^i$  coincide, and are defined by the well-known dynamic equations that characterize the mean and the variance of the continuous-time Kalman filter. The proof appears in Appendix G.

**Theorem 3.5:** Consider the linear Gaussian filtering problem defined by the state-observation equations (27a), (27b). In this case the posterior distributions of  $X_t$  and  $X_t^i$  are Gaussian, whose conditional mean and covariance are given by the respective SDE and the ODE:

$$d\mu_t = \alpha \mu_t dt + \frac{\Sigma_t \gamma}{\sigma_W^2} (dZ_t - \gamma \mu_t dt) \quad (30)$$

$$\frac{d}{dt} \Sigma_t = 2\alpha \Sigma_t + \sigma_B^2 - \frac{(\gamma)^2 \Sigma_t^2}{\sigma_W^2}. \quad (31)$$

Notice that the particle system (29) is not practical since it requires computation of the conditional mean and variance  $\{\mu_t, \Sigma_t\}$ . If we are to compute these quantities, then there is no reason to run a particle filter!

In practice  $\{\mu_t, \Sigma_t\}$  are approximated as sample means and sample covariances from the ensemble  $\{X_t^i\}_{i=1}^N$ :

$$\begin{aligned} \mu_t &\approx \mu_t^{(N)} := \frac{1}{N} \sum_{i=1}^N X_t^i \\ \Sigma_t &\approx \Sigma_t^{(N)} := \frac{1}{N-1} \sum_{i=1}^N \left( X_t^i - \mu_t^{(N)} \right)^2. \end{aligned} \quad (32)$$

The resulting (29) for the  $i$ th particle is given by

$$dX_t^i = \alpha X_t^i dt + \sigma_B dB_t^i + \frac{\Sigma_t^{(N)} \gamma}{\sigma_W^2} \left( dZ_t - \gamma \frac{X_t^i + \mu_t^{(N)}}{2} dt \right). \quad (33)$$

It is very similar to the mean-field “synchronization-type” control laws and oblivious equilibria constructions as in [9], [10]. The model (29) represents the mean-field approximation obtained by letting  $N \rightarrow \infty$ .

The control term in the linear feedback particle filter (32), (33) appears also in the work of Reich [23], where it is referred to as the *ensemble square root filter*.

### D. Feedback Particle Filter for the Multivariable Model

Consider the model (1a), (1b) in which the state  $X_t$  is  $d$ -dimensional, with  $d \geq 2$ , so that  $a(\cdot)$  is a vector-field on  $\mathbb{R}^d$ . For ease of presentation  $\sigma_B$  is assumed to be scalar, and the observation process  $Z_t \in \mathbb{R}$  real-valued.

To aid comparison with Crisan and Xiong’s work, we express the feedback particle filter in its Stratonovich form

$$dX_t^i = a(X_t^i) dt + \sigma_B dB_t^i + K(X_t^i, t) \circ dI_t^i \quad (34)$$

where the innovation error is as before:

$$dI_t^i := dZ_t - \frac{1}{2} \left( h(X_t^i) + \hat{h}_t \right) dt \quad (35)$$

and the gain function  $K(x, t) = (K_1, K_2, \dots, K_d)^T$  is now vector-valued. It is given by the solution of a BVP, the multivariable counterpart of (26):

$$\nabla \cdot (pK) = -\frac{1}{\sigma_W^2} (h - \hat{h}_t) p \quad (36)$$

where  $\nabla \cdot$  denotes the divergence operator.

It is straightforward to prove consistency by repeating the steps in the proof of Theorem 3.3, now with the Kolmogorov forward operator:

$$\begin{aligned} dp &= \mathcal{L}^\dagger p dt - \nabla \cdot (Kp) dZ_t - \nabla \cdot (up) dt \\ &\quad + \frac{1}{2} \sigma_W^2 \sum_{i,j=1}^d \frac{\partial^2 \left[ (KK^T)_{ij} p \right]}{\partial x_i \partial x_j} dt \end{aligned} \quad (37)$$

where  $\mathcal{L}^\dagger p = -\nabla \cdot (pa) + (1/2)\sigma_B^2 \Delta p$ ,  $\Delta$  is the Laplacian, and  $u$  is the multivariable counterpart of (25)

$$u = -K(x, t) \frac{h(x) + \hat{h}_t}{2} + \Omega(x, t)$$

where  $\Omega = (\Omega_1, \Omega_2, \dots, \Omega_d)^T$  is the Wong–Zakai correction term

$$\Omega_l(x, t) := \frac{1}{2}\sigma_W^2 \sum_{k=1}^d K_k(x, t) \frac{\partial K_l}{\partial x_k}(x, t).$$

As with the scalar case, the multivariable feedback particle filter requires solution of a BVP (36) at each time step.

Following the work of Crisan and Xiong [19], we might assume the following representation in an attempt to solve (36):

$$pK = \nabla \phi. \quad (38)$$

where  $\phi$  is assumed to be sufficiently smooth. Substituting (38) in (36) yields the Poisson equation

$$\Delta \phi = -\frac{1}{\sigma_W^2}(h - \hat{h}_t)p. \quad (39)$$

A solution to Poisson's equation with  $d \geq 2$  can be expressed in terms of Green's function:

$$G(r) = \begin{cases} \frac{1}{2\pi} \ln(r), & \text{for } d = 2; \\ \frac{1}{d(2-d)\omega_d} r^{2-d}, & \text{for } d > 2 \end{cases}$$

where  $\omega_d$  is the volume of the unit ball in  $\mathbb{R}^d$ . A solution to (39) is then given by

$$\phi(x) = -\frac{1}{\sigma_W^2} \int_{\mathbb{R}^d} G(|y - x|) (h(y) - \hat{h}_t) p(y, t) dy$$

where  $|y - x| := (\sum_{j=1}^d (y_j - x_j)^2)^{1/2}$  is the Euclidean distance.

On taking the gradient and using (38), one obtains an explicit formula for the gain function

$$K(x, t) = \frac{1}{\sigma_W^2} \frac{\Gamma(p, h)(x)}{p(x, t)} =: K_g(x, t) \quad (40)$$

where  $\Gamma(p, g)(x) := (1/d\omega_d) \int ((y - x)/y - x^d) (g(y) - \hat{g}) p(y) dy$ .

While this leads to a solution to (36), it may not lead to an admissible control law. This difficulty arises in the prior work of Crisan and Xiong.

### E. Comparison With Crisan and Xiong's Filter

In [19] and in [13, Sec. 4], a particle filter of the following form is presented:

$$dX_t^i = a(X_t^i) dt + \sigma_B dB_t^i + K_g(X_t^i, t) \left( \frac{d}{dt} \tilde{I}_t \right) dt \quad (41)$$

where  $\tilde{I}_t$  is a certain smooth approximation obtained from the standard form of the innovation error  $I_t := Z_t - \int_0^t \hat{h}_t dt$ . A consistency result is described for this filter.

We make the following comparisons:

- 1) Without taking a smooth approximation, the filter (41) is formally equivalent to the following SDE expressed here in its Stratonovich form:

$$dX_t^i = a(X_t^i) dt + \sigma_B dB_t^i + K_g(X_t^i, t) \circ (dZ_t - \hat{h}_t dt). \quad (42)$$

In this case, using (37), it is straightforward to show that the consistency result *does not hold*. In particular, there is an extra second-order term that is not present in the K-S equation for evolution of the true posterior  $p^*(x, t)$ .

- 2) The feedback particle filter introduced in this paper does not require a smooth approximation and yet achieves consistency. The key breakthrough is the modified definition of the innovation error [compare (42) with (34)]. Note that the innovation error (35) is not assumed *a priori* but comes about via analysis of the variational problem. This is one utility of introducing the variational formulation. Once the feedback particle filter has been derived, it is straightforward to prove consistency (see the Proof of Theorem 3.3).
- 3) The computational overhead for the feedback particle filter and the filter of Crisan and Xiong are equal. Both require the approximation of the integral (40), and division by (a suitable regularized approximation of)  $p(x, t)$ . Numerically, the Poisson equation formulation (39) of the E-L BVP (36) is convenient. There exist efficient numerical algorithms to approximate the integral solution (40) for a system of  $N$  particles in arbitrary dimension; cf. [24].

However, while appealing, the function  $K_g$  is *not* the correct form of the gain function in the multivariable case, even for the linear Gaussian model. It is straightforward to verify that the Kalman gain is a solution of the boundary value problem (36). Using the Kalman gain for the gain function in (34) yields the feedback particle filter for the multivariable linear Gaussian case. The filter is a straightforward extension of (33) in Section III-C.

However, the Kalman gain solution is not of the form (38). Thus, the integral solution (40) does *not* equal the Kalman gain in the linear Gaussian case (for  $d \geq 2$ ).

Moreover, the gradient form solution is unbounded:  $|K_g(x, t)| \rightarrow \infty$  as  $|x| \rightarrow \infty$ , and  $E[|K_g|] = E[|K_g|^2] = \infty$ . A proof is given in Appendix H.

It follows that the control input obtained using  $K_g$  is not admissible, and hence the Kolmogorov forward operator is no longer valid. Filter implementations using  $K_g$  suffer from numerical issues on account of large unbounded gains. In contrast, the feedback particle filter using Kalman gain works both in theory and in practice.

The choice of gain function in the multivariable case requires careful consideration of the uniqueness of the solutions of the BVP: The solution of (36) is not unique, even though uniqueness holds when  $pK$  is assumed to be of a gradient-form.

Before closing this section, we note that [19, Prop. 2.4] concerns another filter that does not rely on smooth approximation:

$$dX_t^i = a(X_t^i) dt + \sigma_B dB_t^i + K_g(X_t^i, t) \circ dZ_t - \frac{1}{\sigma_W^2} \frac{\Gamma(p, \frac{1}{2}|h|^2)(X_t^i)}{p(X_t^i, t)} dt.$$

Our calculations indicate that consistency is also an issue for this filter. The issue with  $K_g$  also applies to this filter. A more complete comparison needs further investigation.

#### IV. SYNTHESIS OF THE GAIN FUNCTION

Implementation of the nonlinear filter (6) requires solution of the E-L BVP (9) to obtain the gain function  $K(x, t)$  for each fixed  $t$ .

##### A. Direct Numerical Approximation of the BVP Solution

The explicit closed-form formula (76) for the solution of the BVP (9) can be used to construct a direct numerical approximation of the solution. Using (76), we have

$$K(x, t) = \frac{1}{p(x, t)} \frac{1}{\sigma_W^2} \int_{-\infty}^x (\hat{h}_t - h(y)) p(y, t) dy.$$

The approximation involves three steps:

- 1) Approximation of  $\hat{h}_t$  by using a sample mean:

$$\hat{h}_t \approx \frac{1}{N} \sum_{j=1}^N h(X_t^j) =: \hat{h}_t^{(N)}.$$

- 2) Approximation of the integrand:

$$(\hat{h}_t - h(y)) p(y, t) \approx \frac{1}{N} \sum_{j=1}^N (\hat{h}_t^{(N)} - h(X_t^j)) \delta(y - X_t^j)$$

where  $\delta(\cdot)$  is the Dirac delta function.

- 3) Approximation of the density  $p(x, t)$  in the denominator, e.g., as a sum of Gaussian:

$$p(x, t) \approx \frac{1}{N} \sum_{j=1}^N q_t^j(x) =: \tilde{p}(x, t) \quad (43)$$

where  $q_t^j(x) = q(x; X_t^j, \varepsilon) = (1/\sqrt{2\pi\varepsilon}) \exp(-(1/2\varepsilon)(x - X_t^j)^2)$ . The appropriate value of  $\varepsilon$  depends upon the problem. As a function of  $N$ ,  $\varepsilon$  can be made smaller as  $N$  grows. As  $N \rightarrow \infty$ ,  $\varepsilon \rightarrow 0$ .

This yields the following numerical approximation of the gain function:

$$K(x, t) = \frac{1}{\tilde{p}(x, t)} \frac{1}{\sigma_W^2} \frac{1}{N} \sum_{j=1}^N (\hat{h}_t^{(N)} - h(X_t^j)) H(x - X_t^j) \quad (44)$$

where  $H(\cdot)$  is the Heaviside function.

Note that the gain function needs to be evaluated only at the particle locations  $X_t^i$ . An efficient  $O(N^2)$  algorithm is easily constructed to do the same:

$$\begin{aligned} K(X_t^i, t) &= \frac{1}{\tilde{p}(X_t^i, t)} \frac{1}{\sigma_W^2} \frac{1}{N} \\ &\times \left( \sum_{j: X_t^j < X_t^i} (\hat{h}_t^{(N)} - h(X_t^j)) \right. \\ &\quad \left. + \frac{1}{2} (\hat{h}_t^{(N)} - h(X_t^i)) \right) \\ K'(X_t^i, t) &= \frac{1}{\sigma_W^2} (\hat{h}_t^{(N)} - h(X_t^i)) - \tilde{b}(X_t^i) K(X_t^i, t) \end{aligned} \quad (45)$$

where  $\tilde{b}(x) := (\partial/\partial x)(\ln \tilde{p})(x, t)$ . For  $\tilde{p}$  defined using the sum of Gaussian approximation (43), a closed-form formula for  $\tilde{b}(x)$  is easily obtained.

##### B. Algorithm

For implementation purposes, we use the Stratonovich form of the filter [see (10)] together with an Euler discretization. The resulting discrete-time algorithm appears in Algorithm 1. At each time step, the algorithm requires approximation of the gain function. A direct numerical solution-based algorithm for the same is summarized in Algorithm 2.

In practice, one can use a less computationally intensive algorithm to approximate the gain function. An algorithm based on sum-of-Gaussian approximation of density appears in our conference paper [2]. In the application example presented in Section V-C, the gain function is approximated by using Fourier series.

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##### Algorithm 1 Implementation of feedback particle filter

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```

1: Initialization
2: for  $i = 1$  to  $N$  do
3:   Sample  $X_0^i$  from  $p(x, 0)$ 
4: end for
5: Assign value  $t = 0$ 
1: Iteration [from  $t$  to  $t + \Delta t$ ]
2: Calculate  $\hat{h}_t^{(N)} = (1/N) \sum_{i=1}^N h(X_t^i)$ 
3: for  $i = 1$  to  $N$  do
4:   Generate a sample,  $\Delta V$ , from  $N(0, 1)$ 
5:   Calculate  $\Delta I_t^i = \Delta Z_t - (1/2)(h(X_t^i) + \hat{h}_t^{(N)})\Delta t$ 
6:   Calculate the gain function  $K(X_t^i, t)$  (e.g., by using
Algorithm 2)
7:    $X_{t+\Delta t}^i = X_t^i + a(X_t^i)\Delta t + \sigma_B \sqrt{\Delta t} \Delta V + K(X_t^i, t)\Delta I_t^i$ 
8: end for
9:  $t = t + \Delta t$ 

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##### Algorithm 2 Synthesis of gain function $K(x, t)$

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- 1: Calculate  $\hat{h}_t \approx \hat{h}_t^{(N)}$ ;
- 2: Approximate  $p(x, t)$  as a sum of Gaussian:

$$p(x, t) \approx \tilde{p}(x, t) = \frac{1}{N} \sum_{j=1}^N q_t^j(x)$$

where  $q_t^j(x) = (1/\sqrt{2\pi\varepsilon}) \exp(-((x - X_t^j)^2/2\varepsilon))$ .

- 3: Calculate the gain function

$$K(x, t) = \frac{1}{\tilde{p}(x, t)} \frac{1}{\sigma_W^2} \frac{1}{N} \sum_{j=1}^N (\hat{h}_t^{(N)} - h(X_t^j)) H(x - X_t^j)$$

where  $H(\cdot)$  is the Heaviside function.

---

##### C. Further Remarks on the BVP

Recall that the solution of the nonlinear filtering problem is given by the Kushner–Stratonovich nonlinear evolution PDE. The feedback particle filter instead requires, at each time  $t$ , a solution of the linear BVP (36) to obtain the gain function  $K$ :

$$\nabla \cdot (pK) = -\frac{1}{\sigma_W^2} (h - \hat{h}_t)p.$$

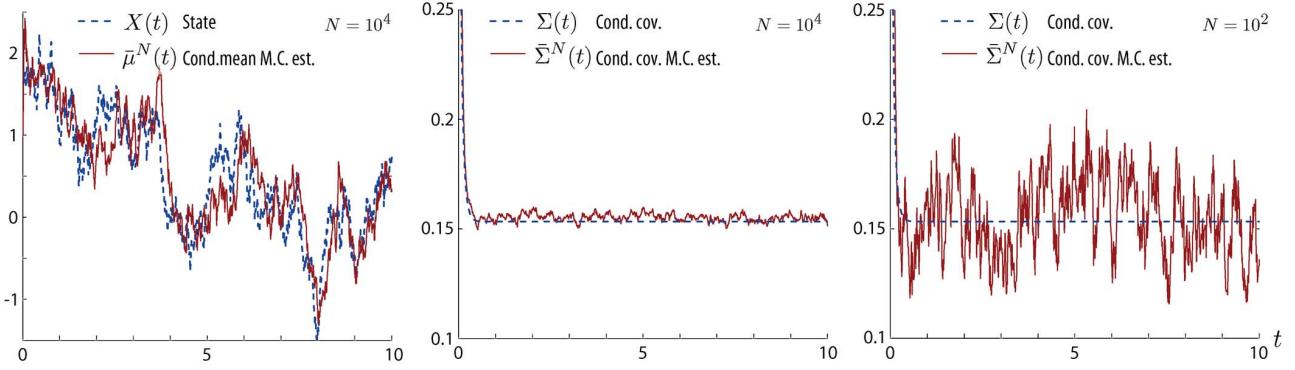


Fig. 2. (a) Comparison of the true state  $\{X_t\}$  and the conditional mean  $\{\bar{\mu}_t^{(N)}\}$ . (b) and (c) plots of estimated conditional covariance with  $N = 10000$  and  $N = 100$  particles, respectively. For comparison, the true conditional covariance obtained using Kalman filtering equations is also shown.

We make the following remarks:

- 1) There are close parallels between the proposed algorithm and the vortex element method (VEM) developed by Chorin and others for solution of the Navier–Stokes evolution equation; cf., [25], [26]. In VEM, as in the feedback particle filter, one obtains the solution of a Navier–Stokes equation by flowing a large number of particles. The vector-field for the particles is obtained by solving a linear BVP at each time.
- Algorithms based on VEM are popular in the large Reynolds number regime when the domain is not too complicated. The latter requirement is necessary to obtain solution of the linear BVP in tractable fashion [24].
- 2) One may ask what is the benefit, in terms of accuracy and computational cost, of the feedback particle filter-based solution when compared to a direct solution of the nonlinear evolution equation (Kushner–Stratonovich PDE), or even its linear counterpart (Zakai equation)?
- The key point, we believe, is robustness on account of the feedback control structure. Specifically, the self-correcting property of the feedback provides robustness, allowing one to tolerate a degree of uncertainty inherent in any model or approximation scheme. This is expected to yield accurate solutions in a computationally efficient manner. A complete answer will require further analysis, and as such reflects an important future direction.
- 3) The biggest computational cost of our approach is the need to solve the BVP at each time-step, that additionally requires one to approximate the density. We are encouraged however by the extensive set of tools in feedback control: after all, one rarely needs to solve the HJB equations in closed-form to obtain a reasonable feedback control law. Moreover, there are many approaches in nonlinear and adaptive control to both approximate control laws as well as learn/adapt these in online fashion; cf., [27].

## V. NUMERICS

### A. Linear Gaussian Case

Consider the linear system:

$$dX_t = \alpha X_t dt + dB_t \quad (46a)$$

$$dZ_t = \gamma X_t dt + \sigma_W dW_t, \quad X_0 \sim N(1, 1) \quad (46b)$$

where  $\{B_t\}, \{W_t\}$  are mutually independent standard Wiener process, and parameters  $\alpha = -0.5$ ,  $\gamma = 3$ , and  $\sigma_W = 0.5$ .

Each of the  $N$  particles is described by the linear SDE

$$dX_t^i = \alpha X_t^i dt + dB_t^i + \frac{\gamma \bar{\Sigma}_t^{(N)}}{\sigma_W^2} \left[ dZ_t - \gamma \frac{X_t^i + \bar{\mu}_t^{(N)}}{2} dt \right] \quad (47)$$

where  $\{B_t^i\}$  are mutually independent standard Wiener process; the particle system is initialized by drawing initial conditions  $\{X_0^i\}_{i=1}^N$  from the distribution  $N(1, 1)$ , and the parameter values are chosen according to the model.

In the simulation discussed next, the mean  $\bar{\mu}_t^{(N)}$  and the variance  $\bar{\Sigma}_t^{(N)}$  are obtained from the ensemble  $\{X_t^i\}_{i=1}^N$  according to (32).

Fig. 2 summarizes some of the results of the numerical experiments: Part (a) depicts a sample path of the state  $\{X_t\}$  and the mean  $\{\bar{\mu}_t^{(N)}\}$  obtained using a particle filter with  $N = 10000$  particles. Part (b) provides a comparison between the estimated variance  $\bar{\Sigma}_t^{(N)}$  and the true error variance  $\Sigma_t$  that one would obtain by using the Kalman filtering equations. The accuracy of the results is sensitive to the number of particles. For example, part (c) of the figure provides a comparison of the variance with  $N = 100$  particles.

*Comparison with the Bootstrap Filter:* We next provide a performance comparison between the feedback particle filter and the bootstrap particle filter for the linear problem (46a), (46b) in regard to both error and running time.

For the linear filtering problem, the optimal solution is given by the Kalman filter. We use this solution to define the relative mean-squared error:

$$mse = \frac{1}{T} \int_0^T \left( \frac{\Sigma_t^{(N)} - \Sigma_t}{\Sigma_t} \right)^2 dt \quad (48)$$

where  $\Sigma_t$  is the error covariance using the Kalman filter, and  $\Sigma_t^{(N)}$  is its approximation using the particle filter.

Fig. 3(a) depicts a comparison between  $mse$  obtained using the feedback particle filter (47) and the bootstrap filter. The latter implementation is based on an algorithm taken from [28, Ch. 9]. For simulation purposes, we used a range of values of  $\alpha \in \{-0.5, 0, 0.5\}$ ,  $\gamma = 3$ ,  $\sigma_B = 1$ ,  $\sigma_W = 0.5$ ,  $\Delta t = 0.01$ , and  $T = 50$ . The plot is generated using simulations with  $N = 20, 50, 100, 200, 500, 1000$  particles.

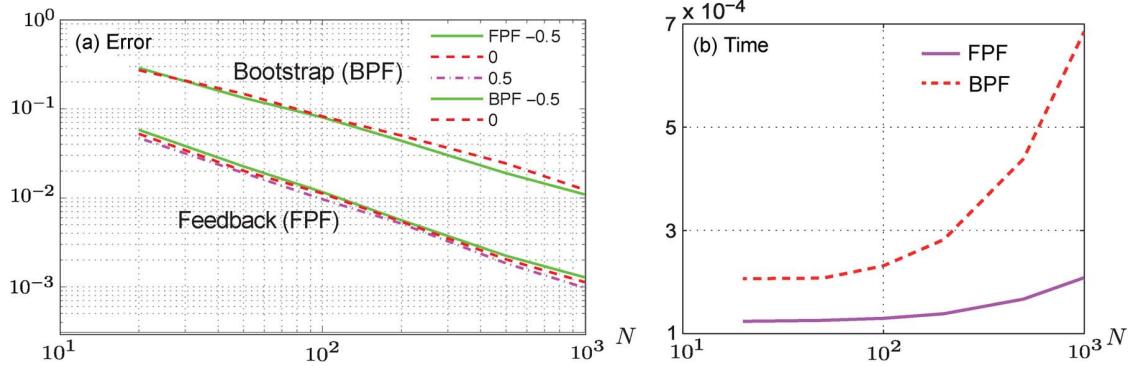


Fig. 3. Comparison of (a) the  $mse$ , (b) the computational time using feedback particle filter and the bootstrap particle filter.

These numerical results suggest that feedback can help reduce the high variance that is sometimes observed with the conventional particle filter. The variance issue can be especially severe if the signal process (1a) is unstable [e.g.,  $\alpha > 0$  in (46a)]. In this case, individual particles can exhibit numerical instabilities due to time-discretization, floating point representation, etc. With  $\alpha > 0$ , our numerical simulations with the bootstrap filter “blew-up” (similar conclusions were also arrived independently in [21]) while the feedback particle filter is provably stable based on observability of the model (46a), (46b) (see Fig. 3 mse plot with  $\alpha = 0.5$ ).

Fig. 3(b) depicts a comparison between computational time for the two filtering algorithms on the same problem. The time is given in terms of computation time per iteration cycle (see Algorithm 1 in Section IV-B) averaged over 100 trials. For simulation purpose, we use MATLAB R2011b (7.13.0.564) on a 2.66-GHz iMac as our test platform.

These numerical results suggest that, for a linear Gaussian implementation, feedback particle filter has a lower computational cost compared to the conventional bootstrap particle filter. The main reason is that the feedback particle filter avoids the computationally expensive resampling procedure.

We also carried out simulations where the gain function is approximated using Algorithm 2. In this case, the mse of the filter is comparable to the mse depicted in Fig. 3(a). However, the computation time is larger than the bootstrap particle filter. This is primarily on account of the evaluation of the exponentials in computing  $\tilde{p}(x, t)$ . Detailed comparisons between the feedback particle filter and the bootstrap particle filter will appear elsewhere.

In general, the main computational burden of the feedback particle filter is to obtain gain function which can be made efficient by using various approximation approaches; cf. [12].

### B. Nonlinear Example

This nonlinear SDE is chosen to illustrate the tracking capability of the filter in highly nonlinear settings

$$dX_t = X_t(1 - X_t^2)dt + \sigma_B dB_t \quad (49a)$$

$$dZ_t = X_t dt + \sigma_W dW_t. \quad (49b)$$

When  $\sigma_B = 0$ , the ODE (49a) has two stable equilibria at  $\pm 1$ . With  $\sigma_B > 0$ , the state of the SDE “transitions” between these two “equilibria.”

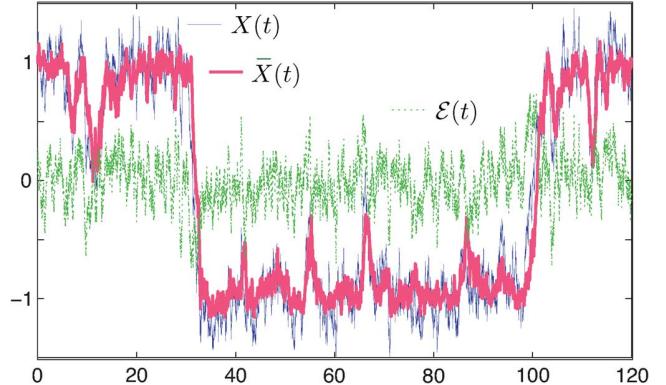


Fig. 4. Comparison of the true state  $X(t)$  and the conditional mean  $\bar{X}(t)$  by using feedback particle filter. The error  $\mathcal{E}(t) = X(t) - \bar{X}(t)$  remains small even during a transition of the state.

Fig. 4 depicts the simulation results obtained using the nonlinear feedback particle filter (6), with  $\sigma_B = 0.4$ ,  $\sigma_W = 0.2$ . The implementation is based on an algorithm described in [2, Sec. 4], and the details are omitted here on account of space. We initialize the simulation with two Gaussian clusters. After a brief period of transients, these clusters merge into a single cluster, which adequately tracks the true state including the transition events.

### C. Application: Nonlinear Oscillators

We consider the filtering problem for a nonlinear oscillator:

$$d\theta_t = \omega dt + \sigma_B dB_t \mod 2\pi \quad (50)$$

$$dZ_t = h(\theta_t)dt + \sigma_W dW_t \quad (51)$$

where  $\omega$  is the frequency,  $h(\theta) = (1/2)[1 + \cos(\theta)]$ , and  $\{B_t\}$ , and  $\{W_t\}$  are mutually independent standard Wiener processes. For numerical simulations, we pick  $\omega = 1$  and the standard deviation parameters  $\sigma_B = 0.5$  and  $\sigma_W = 0.4$ . We consider oscillator models because of their significance to applications including neuroscience; cf., [10].

The feedback particle filter is given by

$$\begin{aligned} d\theta_t^i &= \omega dt + \sigma_B dB_t^i + K(\theta_t^i, t) \\ &\circ \left[ dZ_t - \frac{1}{2} \left( h(\theta_t^i) + \hat{h}_t \right) dt \right] \mod 2\pi \end{aligned} \quad (52)$$

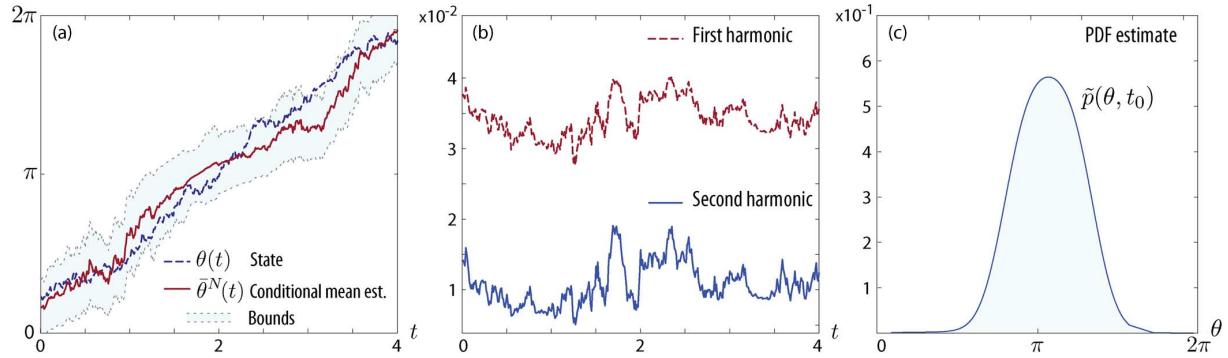


Fig. 5. Summary of the numerical experiments with the nonlinear oscillator filter: (a) Comparison of the true state  $\{\theta_t\}$  and the conditional mean  $\{\bar{\theta}_t^N\}$ . (b) The mean-squared estimate of the first and second harmonics of the density  $p(\theta, t)$  and (c) a plot of a typical empirical distribution.

$i = 1, \dots, N$ , where the function  $K(\theta, t)$  is obtained via the solution of the E-L equation:

$$-\frac{\partial}{\partial \theta} \left( \frac{1}{p(\theta, t)} \frac{\partial}{\partial \theta} \{p(\theta, t)K(\theta, t)\} \right) = -\frac{\sin \theta}{2\sigma_W^2}. \quad (53)$$

Although the (53) can be solved numerically to obtain the optimal control function  $K(\theta, t)$ , here we investigate a solution based on perturbation method. Suppose, at some time  $t$ ,  $p(\theta, t) = (1/2\pi) =: p_0$ , the uniform density. In this case, the E-L equation is given by:

$$\partial_{\theta\theta} K = \frac{\sin \theta}{2\sigma_W^2}.$$

A straightforward calculation shows that the solution in this case is given by

$$K(\theta, t) = -\frac{\sin \theta}{2\sigma_W^2} =: K_0(\theta). \quad (54)$$

To obtain the solution of the E-L (53), we assume that the density  $p(\theta, t)$  is a small harmonic perturbation of the uniform density. In particular, we express  $p(\theta, t)$  as

$$p(\theta, t) = p_0 + \epsilon \tilde{p}(\theta, t) \quad (55)$$

where  $\epsilon$  is a small perturbation parameter. Since  $p(\theta, t)$  is a density,  $\int_0^{2\pi} \tilde{p}(\theta, t) d\theta = 0$ .

We are interested in obtaining a solution of the form

$$K(\theta, t) = K_0(\theta) + \epsilon \tilde{K}(\theta, t). \quad (56)$$

On substituting the ansatz (55) and (56) in (53), and retaining only  $O(\epsilon)$  term, we obtain the following linearized equation:

$$\partial_{\theta\theta} \tilde{K} = -2\pi \partial_\theta [(\partial_\theta \tilde{p}) K_0]. \quad (57)$$

The linearized E-L (57) can be solved easily by considering a Fourier series expansion of  $\epsilon \tilde{p}(\theta, t)$

$$\epsilon \tilde{p}(\theta, t) = P_c(t) \cos \theta + P_s(t) \sin \theta + \text{h.o.h} \quad (58)$$

where “h.o.h” denotes the terms due to higher order harmonics. The Fourier coefficients are given by

$$P_c(t) = \frac{1}{\pi} \int_0^{2\pi} p(\theta, t) \cos \theta d\theta, \quad P_s(t) = \frac{1}{\pi} \int_0^{2\pi} p(\theta, t) \sin \theta d\theta.$$

For a harmonic perturbation, the solution of the linearized E-L (57) is given by

$$\begin{aligned} \epsilon \tilde{K}(\theta, t) &= \frac{\pi}{4\sigma_W^2} (P_c(t) \sin 2\theta - P_s(t) \cos 2\theta) \\ &=: K_1(\theta; P_c(t), P_s(t)). \end{aligned} \quad (59)$$

For “h.o.h” terms in the Fourier series expansion (58) of the density in  $p(\theta, t)$ , the linearized E-L (57) can be solved in a similar manner. In numerical simulation provided here, we ignore the higher order harmonics, and use a control input as summarized in the following proposition:

*Proposition 5.1:* Consider the E-L equation (53) where the density  $p(\theta, t)$  is assumed to be a small harmonic perturbation of the uniform density  $1/2\pi$ , as defined by (55) and (58). As  $\epsilon \rightarrow 0$ , the gain function is given by the following asymptotic formula:

$$K(\theta, t) = K_0(\theta) + K_1(\theta; P_c(t), P_s(t)) + o(\epsilon) \quad (60)$$

where  $P_c(t), P_s(t)$  denote the harmonic coefficients of density  $p(\theta, t)$ . For large  $N$ , these are approximated by using the formulae

$$P_c(t) \approx \frac{1}{\pi N} \sum_{j=1}^N \cos \theta_j(t), \quad P_s(t) \approx \frac{1}{\pi N} \sum_{j=1}^N \sin \theta_j(t). \quad (61)$$

We next discuss the result of numerical experiments. The particle filter model is given by (52) with gain function  $K(\theta_t^i, t)$ , obtained using formula (60). The number of particles  $N = 10\,000$  and their initial condition  $\{\theta_0^i\}_{i=1}^N$  was sampled from a uniform distribution on circle  $[0, 2\pi]$ .

Fig. 5 summarizes some of the results of the numerical simulation. For illustration purposes, we depict only a single cycle from a time-window after transients due to initial condition have converged. Part (a) of the figure compares the sample path of the actual state  $\{\theta_t\}$  (as a dashed line) with the estimated mean  $\{\bar{\theta}_t^N\}$  (as a solid line). The shaded area indicates  $\pm$  one standard deviation bounds. Part (b) of the figure provides a comparison of the magnitude of the first and the second harmonics (as dashed and solid lines, respectively) of the density  $p(\theta, t)$ . The density at any time instant during the time-window is approximately harmonic (see also part (c) where the density at one typical time instant is shown).

Note that at each time instant  $t$ , the estimated mean, the bounds and the density  $p(\theta, t)$  shown here are all approximated from the ensemble  $\{\theta_t^i\}_{i=1}^N$ . For the sake of illustration, we have used a Gaussian mixture approximation to construct a smooth approximation of the density.

The reader is referred to our papers [17] and [18] for additional applications involving nonlinear filtering with coupled oscillator models. In these papers, Galerkin approaches are described for the approximation of the gain function.

## VI. CONCLUSION

In this paper, we introduced a new formulation of the nonlinear filter, referred to as the *feedback particle filter*. The feedback particle filter provides for a generalization of the Kalman filter to a general class of nonlinear non-Gaussian problems. Feedback particle filter inherits many of the properties that has made the Kalman filter so widely applicable over the past five decades, including innovation error and the feedback structure (see Fig. 1).

Feedback is important on account of the issue of *robustness*. In particular, feedback can help reduce the high variance that is sometimes observed in the conventional particle filter. Numerical results are presented to support this claim (see Fig. 3).

Even more significantly, the structural aspects of the Kalman filter have been as important as the algorithm itself in design, integration, testing and operation of a larger system involving filtering problems (e.g., navigation systems). We expect feedback particle filter to similarly provide for an integrated framework, now for nonlinear non-Gaussian problems. We refer the reader to our paper [29] where feedback particle filter-based algorithms for nonlinear filtering with data association uncertainty are described.

## APPENDIX

### A. Calculation of K-L Divergence

Recall the definition of K-L divergence for densities:

$$D(p_n \parallel \hat{p}_n^*) = \int_{\mathbb{R}} p_n(s) \ln \left( \frac{p_n(s)}{\hat{p}_n^*(s)} \right) ds.$$

We make a coordinate transformation  $s = x + v(x)$  and use (16) to express the K-L divergence as

$$\begin{aligned} D(p_n \parallel \hat{p}_n^*) &= \int_{\mathbb{R}} \frac{p_n^-(x)}{|1+v'(x)|} \ln \left( \frac{p_n^-(x)}{|1+v'(x)|\hat{p}_n^*(x+v(x))} \right) |1+v'(x)| dx \\ &= \int_{\mathbb{R}} \frac{p_n^-(x)}{|1+v'(x)|} \ln \left( \frac{p_n^-(x)}{|1+v'(x)|\hat{p}_n^*(x+v(x))} \right) |1+v'(x)| dx \end{aligned}$$

The expression for K-L divergence given in (18) follows on using (15).  $\blacksquare$

### B. Solution of the Optimization Problem

Denote

$$\begin{aligned} \mathcal{L}(x, v, v') &= -p_n^-(x) \\ &\times (\ln |1+v'| + \ln (p_n^-(x+v)p_{Y|X}(Y_{t_n}|x+v))) . \end{aligned} \quad (62)$$

The optimization problem (17) is a calculus of variation problem

$$\min_v \int \mathcal{L}(x, v, v') dx.$$

The minimizer is obtained via the analysis of first variation given by the well-known Euler–Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial v} = \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial v'} \right).$$

Explicitly substituting the expression (62) for  $\mathcal{L}$ , we obtain (19).

### C. Derivation of the Forward Equation

We denote the filtration  $\mathcal{B}_t = \sigma(X_0^i, B_s^i : s \leq t)$ , and we recall that  $\mathcal{Z}_t = \sigma(Z_s : s \leq t)$  for  $t \geq 0$ . These two filtrations are *independent* by construction.

On denoting  $\tilde{a}(x, t) = a(x) + u(x, t)$ , the particle evolution (20) is expressed as

$$X_t^i = X_0^i + \int_0^t \tilde{a}(X_s^i, s) ds + \int_0^t \mathsf{K}(X_s^i, s) dZ(s) + \sigma_B B_t^i. \quad (63)$$

By assumption on Lipschitz continuity of  $\tilde{a}$  and  $\mathsf{K}$ , there exists a unique solution that is adapted to the larger filtration  $\mathcal{B}_t \vee \mathcal{Z}_t = \sigma(X_0^i, B_s^i, Z_s : s \leq t)$ . In fact, there is a functional  $F_t$  such that

$$X_t^i = F_t(X_0^i, B_t^i, Z^t) \quad (64)$$

where  $Z^t := \{Z_s : 0 \leq s \leq t\}$  denotes the trajectory.

The conditional distribution of  $X_t^i$  given  $\mathcal{Z}_t = \sigma(Z_s : s \leq t)$  was introduced in Section II-A: Its density is denoted  $p(x, t)$ , defined by any bounded and measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  via

$$\mathbb{E}[f(X_t^i) | \mathcal{Z}_t] = \int_{\mathbb{R}} p(x, t) f(x) dx =: \langle p_t, f \rangle.$$

We begin with a result that is the key to proving Prop. 3.1. The proof of the Lemma 7.1 below appears in [2]. It is omitted here on account of space.

*Lemma 7.1:* Suppose that  $f$  is an  $\mathcal{B}_t \vee \mathcal{Z}_t$ -adapted process satisfying  $\mathbb{E} \int_0^t |f(s)|^2 ds < \infty$ . Then

$$\mathbb{E} \left[ \int_0^t f(s) ds | \mathcal{Z}_t \right] = \int_0^t \mathbb{E}[f(s) | \mathcal{Z}_s] ds \quad (65)$$

$$\mathbb{E} \left[ \int_0^t f(s) dZ_s | \mathcal{Z}_t \right] = \int_0^t \mathbb{E}[f(s) | \mathcal{Z}_s] dZ_s. \quad (66)$$

We now provide a proof of the Prop. 3.1.  $\blacksquare$

*Proof of Proposition 3.1:* Applying Itô's formula to (20) gives, for any smooth and bounded function  $f$

$$\begin{aligned} df(X_t^i) &= \mathcal{L}f(X_t^i) dt + \sigma_B \frac{\partial f}{\partial x}(X_t^i) dB_t^i \\ &\quad + \mathsf{K}(X_t^i, t) \frac{\partial f}{\partial x}(X_t^i) dZ_t \end{aligned}$$

where  $\mathcal{L}f := (a+u)(\partial f/\partial x) + (1/2)(\sigma_W^2 K^2 + \sigma_B^2)(\partial^2 f/\partial x^2)$ . Therefore,

$$\begin{aligned} f(X_t^i) &= f(X_0^i) + \int_0^t \mathcal{L}f(X_s^i) ds + \sigma_B \int_0^t \frac{\partial f}{\partial x}(X_s^i) dB_s^i \\ &\quad + \int_0^t K(X_s^i, s) \frac{\partial f}{\partial x}(X_s^i) dZ_s. \end{aligned}$$

Taking conditional expectations on both sides:

$$\begin{aligned} \langle p_t, f \rangle &= \mathbb{E}[f(X_0^i) | \mathcal{Z}_t] + \mathbb{E}\left[\int_0^t \mathcal{L}f(X_s^i) ds | \mathcal{Z}_t\right] \\ &\quad + \sigma_B \mathbb{E}\left[\int_0^t \frac{\partial f}{\partial x}(X_s^i) dB_s^i | \mathcal{Z}_t\right] \\ &\quad + \mathbb{E}\left[\int_0^t K(X_s^i, s) \frac{\partial f}{\partial x}(X_s^i) dZ_s | \mathcal{Z}_t\right]. \end{aligned}$$

On applying Lemma 7.1, and the fact that  $B_t^i$  is a Wiener process, we conclude that

$$\langle p_t, f \rangle = \langle p_0, f \rangle + \int_0^t \langle p_s, Lf \rangle ds + \int_0^t \left\langle p_s, K \frac{\partial f}{\partial x} \right\rangle dZ_s.$$

The forward (23) follows using integration by parts. ■

#### D. Euler-Lagrange Equation for the Continuous-Time Filter

In this section we describe, formally, the continuous-time limit of the discrete-time E-L BVP (19). In the continuous-time case, the control and the observation models are of the form [see (20) and (1b)]:

$$\begin{aligned} dU_t^i &= u(X_t^i, t) dt + K(X_t^i, t) dZ_t, \\ dZ_t &= h(X_t) dt + \sigma_W dW_t. \end{aligned}$$

In discrete-time, these are approximated as

$$\begin{aligned} \Delta U_t^i &= u(X_t^i, t) \Delta t + K(X_t^i, t) \Delta Z_t, \\ \Delta Z_t &= h(X_t) \Delta t + \sigma_W \Delta W_t, \end{aligned} \quad (67)$$

where  $\Delta t$  is the small time-increment at  $t$ . It follows that the conditional distribution of  $Y_t \doteq (\Delta Z_t/\Delta t)$  given  $X_t$  is the density

$$p_{Y|X}(Y_t | \cdot) = \frac{1}{\sqrt{\frac{2\pi\sigma_W^2}{\Delta t}}} \exp\left(-\frac{(\Delta Z_t - h(\cdot)\Delta t)^2}{2\sigma_W^2 \Delta t}\right). \quad (68)$$

Substituting (67), (68) in the E-L BVP (19) for the continuous-discrete time case, we arrive at the formal equation:

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{p(x, t)}{1 + u' \Delta t + K' \Delta Z_t} \right) &= p(x, t) \\ &= \frac{\partial}{\partial v} (\ln p(x+v, t) + \ln p_{Y|X}(Y_t | x+v)) \Big|_{v=u\Delta t+K\Delta Z_t}. \end{aligned} \quad (69)$$

For notational ease, we use primes to denote partial derivatives with respect to  $x$ :  $p$  is used to denote  $p(x, t)$ ,  $p' := (\partial p/\partial x)(x, t)$ ,  $p'' := (\partial^2 p/\partial x^2)(x, t)$ ,  $u' := (\partial u/\partial x)(x, t)$ ,  $K' := (\partial K/\partial x)(x, t)$ , etc. Note that the time  $t$  is fixed.

A sketch of calculations to obtain (24) and (25) starting from (69) appears in the following three steps:

Step 1) The three terms in (69) are simplified as

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{p}{1 + u' \Delta t + K' \Delta Z_t} \right) &= p' - f_1 \Delta t - (p' K' + p K'') \Delta Z_t \\ p \frac{\partial}{\partial v} \ln p(x+v) \Big|_{v=u\Delta t+K\Delta Z_t} &= p' + f_2 \Delta t + \left( p'' K - \frac{p'^2 K}{p} \right) \Delta Z_t \\ p \frac{\partial}{\partial v} \ln p_{Y|X}(Y_t | x+v) \Big|_{v=u\Delta t+K\Delta Z_t} &= \frac{p}{\sigma_W^2} (h' \Delta Z_t - h h' \Delta t) + p h'' K \Delta t \end{aligned}$$

where we have used Itô's rules  $(\Delta Z_t)^2 = \sigma_W^2 \Delta t$ ,  $\Delta Z_t \Delta t = 0$ , etc., and where

$$\begin{aligned} f_1 &= (p' u' + p u'') - \sigma_W^2 (p' K'^2 + 2p K' K''), \\ f_2 &= \left( p'' u - \frac{p'^2 u}{p} \right) + \sigma_W^2 K^2 \left( \frac{1}{2} p''' - \frac{3p' p''}{2p} + \frac{p'^3}{p^2} \right). \end{aligned}$$

Collecting terms in  $O(\Delta Z_t)$  and  $O(\Delta t)$ , after some simplification, leads to the following ODEs:

$$\mathcal{E}(K) = \frac{1}{\sigma_W^2} h'(x) \quad (70)$$

$$\mathcal{E}(u) = -\frac{1}{\sigma_W^2} h(x) h'(x) + h''(x) K + \sigma_W^2 G(x, t) \quad (71)$$

where  $\mathcal{E}(K) = -(\partial/\partial x)((1/p(x, t))(\partial/\partial x)\{p(x, t)K(x, t)\})$ , and  $G = -2K' K'' - (K')^2 (\ln p)' + (1/2)K^2 (\ln p)'''$ .

Step 2) Suppose  $(u, K)$  are admissible solutions of the E-L BVP (70), (71). Then it is claimed that

$$-(pK)' = \frac{h - \hat{h}}{\sigma_W^2} p \quad (72)$$

$$-(pu)' = -\frac{(h - \hat{h})\hat{h}}{\sigma_W^2} p - \frac{1}{2} \sigma_W^2 (pK^2)'' \quad (73)$$

Recall that admissible here means

$$\lim_{x \rightarrow \pm\infty} p(x, t)u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} p(x, t)K(x, t) = 0. \quad (74)$$

To show (72), integrate (70) once to obtain

$$-(pK)' = \frac{1}{\sigma_W^2} hp + Cp$$

where the constant of integration  $C = -(\hat{h}/\sigma_W^2)$  is obtained by integrating once again between  $-\infty$  to  $\infty$  and using the

boundary conditions for  $K$  (74). This gives (72). To show (73), we denote its right hand side as  $\mathcal{R}$  and claim

$$\left(\frac{\mathcal{R}}{p}\right)' = -\frac{hh'}{\sigma_W^2} + h''K + \sigma_W^2 G. \quad (75)$$

The equation (73) then follows by using the ODE (71) together with the boundary conditions for  $u$  (74). The verification of the claim involves a straightforward calculation, where we use (70) to obtain expressions for  $h$  and  $K''$ . The details of this calculation are omitted on account of space.

**Step 3)** The E-L equation for  $K$  is given by (70) which is the same as (24). The proof of (25) involves a short calculation starting from (73), which is simplified to the form (25) by using (72).

**Remark 4:** The derivation of Euler–Lagrange equation, as presented above, is a heuristic on account of Step 1. A similar heuristic also appears in the original paper of Kushner [30]. There, the Kushner–Stratonovich PDE (22) is derived by considering a continuous-time limit of the Bayes formula (15). The Itô’s rules are used to obtain the limit. Rigorous justification of the calculation in Step 1, or its replacement by an alternate argument is the subject of future work.

The calculation in Steps 2 and 3 require additional regularity assumptions on density  $p$  and function  $h : p$  is  $C^3$  and  $h$  is  $C^2$ .

#### E. Proof of Proposition 3.2.

Consider the ODE (24). It is a linear ODE whose unique solution is given by

$$K(x, t) = \frac{1}{p(x, t)} \times \left( C_1 + C_2 \int_{-\infty}^x p(y, t) dy - \frac{1}{\sigma_W^2} \int_{-\infty}^x h(y) p(y, t) dy \right) \quad (76)$$

where the constant of integrations  $C_1 = 0$  and  $C_2 = (\hat{h}_t / \sigma_W^2)$  because of the boundary conditions for  $K$ . Part 2 is an easy consequence of the minimum principle for elliptic PDEs [11].

#### F. Proof of Thm. 3.3

It is only necessary to show that with this choice of  $\{u, K\}$ , we have  $dp(x, t) = dp^*(x, t)$ , for all  $x$  and  $t$ , in the sense that they are defined by identical stochastic differential equations. Recall  $dp^*$  is defined according to the K-S (22), and  $dp$  according to the forward (23).

If  $K$  solves the E-L BVP (24) then using (76)

$$\frac{\partial}{\partial x}(pK) = -\frac{1}{\sigma_W^2}(h - \hat{h})p. \quad (77)$$

On multiplying both sides of (25) by  $-p$ , we have

$$\begin{aligned} -up &= \frac{1}{2}(h - \hat{h})pK - \frac{1}{2}\sigma_W^2(pK)\frac{\partial K}{\partial x} + \hat{h}pK \\ &= -\frac{1}{2}\sigma_W^2\frac{\partial(pK)}{\partial x}K - \frac{1}{2}\sigma_W^2(pK)\frac{\partial K}{\partial x} + \hat{h}pK \\ &= -\frac{1}{2}\sigma_W^2\frac{\partial}{\partial x}(pK^2) + \hat{h}pK \end{aligned}$$

where we have used (77) to obtain the second equality. Differentiating once with respect to  $x$  and using (77) once again,

$$-\frac{\partial}{\partial x}(up) + \frac{1}{2}\sigma_W^2\frac{\partial^2}{\partial x^2}(pK^2) = -\frac{\hat{h}}{\sigma_W^2}(h - \hat{h})p. \quad (78)$$

Using (77), (78) in the forward (23), we have

$$dp = \mathcal{L}^\dagger p + \frac{1}{\sigma_W^2}(h - \hat{h})(dZ_t - \hat{h}dt)p.$$

This is precisely the SDE (22), as desired.

#### G. Proof of Thm. 3.5

The Gaussian density is given by

$$p(x, t) = \frac{1}{\sqrt{2\pi\Sigma_t}} \exp\left(-\frac{(x - \mu_t)^2}{2\Sigma_t}\right). \quad (79)$$

The density (79) is a function of the stochastic process  $\mu_t$ . Using Itô’s formula

$$dp(x, t) = \frac{\partial p}{\partial \mu} d\mu_t + \frac{\partial p}{\partial \Sigma} d\Sigma_t + \frac{1}{2} \frac{\partial^2 p}{\partial \mu^2} d\mu_t^2$$

where  $(\partial p / \partial \mu) = ((x - \mu_t) / \Sigma_t) p$ ,  $(\partial p / \partial \Sigma) = (1/2\Sigma_t)((((x - \mu_t)^2 / \Sigma_t) - 1)p)$ , and  $(\partial^2 p / \partial \mu^2) = (1/\Sigma_t)((((x - \mu_t)^2 / \Sigma_t) - 1)p)$ . Substituting these into the forward (23), we obtain a quadratic equation  $Ax^2 + Bx = 0$ , where

$$\begin{aligned} A &= d\Sigma_t - \left( 2\alpha\Sigma_t + \sigma_B^2 - \frac{\gamma^2\Sigma_t^2}{\sigma_W^2} \right) dt, \\ B &= d\mu_t - \left( \alpha\mu_t dt + \frac{\gamma\Sigma_t}{\sigma_W^2}(dZ_t - \gamma\mu_t dt) \right). \end{aligned}$$

This leads to the model (30) and (31). ■

#### H. BVP for Multivariable Feedback Particle Filter

Consider the multivariable linear system

$$dX_t = \alpha X_t dt + \sigma_B dB_t \quad (80a)$$

$$dZ_t = \gamma^T X_t dt + \sigma_W dW_t \quad (80b)$$

where  $X_t \in \mathbb{R}^d$ ,  $Z_t \in \mathbb{R}^1$ ,  $\alpha$  is an  $d \times d$  matrix,  $\gamma$  is an  $d \times 1$  vector,  $\{B_t\}$  is a  $d$ -dimensional Wiener process,  $\{W_t\}$  is a scalar Wiener process, and  $\{B_t\}, \{W_t\}$  are assumed to be mutually independent. We assume the initial distribution  $p^*(x, 0)$  is Gaussian with mean vector  $\mu_0$  and variance matrix  $\Sigma_0$ .

The following proposition shows that the Kalman gain is a solution of the multivariable BVP (36), the Kalman gain solution does not equal the solution  $K_g$  [see (40)], and that the solution given by  $K_g$  is not integrable with respect to  $p$ .

**Proposition 7.2:** Consider the  $d$ -dimensional linear system (80a), (80b), where  $d \geq 2$ . Suppose  $p(x, t)$  is assumed to be Gaussian:  $p(x, t) = (1/(2\pi)^{d/2}|\Sigma_t|^{1/2}) \exp(-(1/2)(x - \mu_t)^T \Sigma_t^{-1}(x - \mu_t))$ , where  $x = (x_1, x_2, \dots, x_d)^T$ ,  $\mu_t$  is the mean,  $\Sigma_t$  is the covariance matrix, and  $|\Sigma_t| > 0$  denotes the determinant.

1) One solution of the BVP (36) is given by the Kalman gain

$$K(x, t) = \frac{1}{\sigma_W^2} \Sigma_t \gamma \quad (81)$$

- 2) Suppose that the Kalman gain  $K$  given in (81) is nonzero. For this solution to (36), there does not exist a function  $\phi$  such that  $pK = \nabla\phi$ .
- 3) Consider the solution  $K_g(x, t)$  of the BVP (36) as given by (40). This gain function is unbounded:  $|K_g(x, t)| \rightarrow \infty$  as  $|x| \rightarrow \infty$ , and moreover

$$\int_{\mathbb{R}^d} |K_g(x, t)| p(x, t) dx = \infty, \quad \int_{\mathbb{R}^d} |K_g(x, t)|^2 p(x, t) dx = \infty.$$

■

*Proof:* The Kalman gain solution (81) is verified by direct substitution in the BVP (36) where the distribution  $p$  is Gaussian.

The proof of claim 2 follows by contradiction. Suppose a function  $\phi$  exists such that  $pK = \nabla\phi$ , then we have

$$\frac{\partial(Kp)_i}{\partial x_j} = \frac{\partial^2\phi}{\partial x_j \partial x_i} = \frac{\partial^2\phi}{\partial x_i \partial x_j} = \frac{\partial(Kp)_j}{\partial x_i}, \quad \forall i, j \quad (82)$$

where  $(Kp)_i$  is the  $i$ th entry of vector  $Kp$ . By direct evaluation, we have

$$\frac{\partial(Kp)_i}{\partial x_j} = 2K_i \cdot (\Sigma_t^{-1}(x - \mu_t))_j p.$$

Using (82), we obtain

$$K_i (\Sigma_t^{-1})_{jk} = K_j (\Sigma_t^{-1})_{ik}, \quad \forall i, j, k \quad (83)$$

Setting  $k = i$ , summing over the index  $i$  and using (81), we arrive at

$$\text{tr}(\Sigma_t^{-1}) K = \Sigma_t^{-1} K$$

where  $\text{tr}(\Sigma_t^{-1})$  denotes the trace of the matrix  $\Sigma_t^{-1}$ . This provides a contradiction because  $K \neq 0$  and  $\Sigma_t$  is a positive definite symmetric matrix with  $|\Sigma_t| > 0$ .

We now establish claim 3. For the solution  $K_g$  as given by (40):

$$pK_g(x, t) = \frac{1}{\sigma_W^2} \frac{1}{d\omega_d} \int \frac{y-x}{|y-x|^d} (h(y) - \hat{h}) p(y, t) dy$$

For this integral, with  $h(y) \equiv \gamma^T y$ , we have the following asymptotic formula for  $|x| \sim \infty$ :

$$pK_g(x, t) \sim C \frac{1}{|x|^d} + o\left(\frac{1}{|x|^d}\right)$$

where  $C$  does not vary as a function of  $|x|$  (its value depends only upon the angular coordinates). For example, in dimension  $d = 2$ ,  $C$  is given by

$$\begin{aligned} C(x_1, x_2) &= C(|x| \cos(\theta), |x| \sin(\theta)) \\ &= -\frac{1}{d\omega_d} \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \frac{\Sigma_t \gamma}{\sigma_W^2} \end{aligned}$$

where  $\Sigma_t \gamma / \sigma_W^2$  is the Kalman gain vector.

The result follows because  $(1/p|x|^d) \rightarrow \infty$  and  $1/|x|^d$  is not integrable in  $\mathbb{R}^d$ . Using the Cauchy–Schwarz inequality

$$\int_{\mathbb{R}^d} |K_g(x, t)| p(x, t) dx \leq \left( \int_{\mathbb{R}^d} |K_g(x, t)|^2 p(x, t) dx \right)^{\frac{1}{2}}$$

which shows that  $K_g$  is not square-integrable. ■

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