Actuariat IARD - ACT2040
Partie 8 - provisions pour sinistres
à payer, IBNR et triangles

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Provisions pour sinistres à payer

Références : de Jong & Heller (2008), section 1.5 et 8.1, and Wüthrich & Merz (2006), chapitres 1 à 3.

" *Les provisions techniques sont les provisions destinées à permettre le réglement intégral des engagements pris envers les assurés et bénéficaires de contrats. Elles sont liées à la technique même de l'assurance, et imposées par la réglementation.*"

"*It is hoped that more casualty actuaries will involve themselves in this important area. IBNR reserves deserve more than just a clerical or cursory treatment and we believe, as did Mr. Tarbell Chat 'the problem of incurred but not reported claim reserves is essentially actuarial or statistical'. Perhaps in today's environment the quotation would be even more relevant if it stated that the problem '...is more actuarial than statistical'.*" Bornhuetter & Ferguson (1972)

Le passif d'une compagnie d'assurance dommage

- les provisions techniques peuvent représenter 75% du bilan,
- le ratio de couverture (provision / chiffre d'affaire) peut dépasser 2,
- certaines branches sont à développement long, en montant

	n	$n+1$	$n+2$	$n+3$	$n+4$
habitation	55%	90%	94%	95%	96%
automobile	55%	80%	85%	88%	90%
dont corporels	15%	40%	50%	65%	70%
R.C.	10%	25%	35%	40%	45%

Introduction

Swiss Re

sigma

N° 2/2008

Provisionnement des sinistres en assurance de dommages :
progresser sur un défi stratégique

http ://media.swissre.com/documents/sigma2_2008_fr.pdf

Introduction

Les provisions pour sinistres n'ayant pas été encore réglés peuvent avoir de fortes conséquences sur le bilan et le compte de résultats des assureurs de dommages.

Les provisions pour sinistres sont des fonds constitués et inscrits au bilan d'un assureur, pour les sinistres n'ayant pas encore été réglés. Voici ce que l'on sait des provisions pour sinistres :

- Les provisions pour sinistres étant souvent supérieures au capital de l'assureur, toute modification de ces provisions peut avoir un impact important sur les bénéfices. Aux Etats-Unis, l'insuffisance des provisions pour sinistres a été la cause principale des cas de faillite de la période 1969 à 2002.

- Les provisions pour sinistres sont particulièrement importantes en assurance de responsabilité civile, où l'identification et le règlement des sinistres peuvent être longs. En RC, les provisions pour sinistres représentent en général de 300% à 450% des primes acquises, alors que les ratios de provisions techniques sont compris entre 60% et 100% en assurance de biens, et entre 10% et 30% seulement en assurance tous risques automobile.

- Les ratios de provisions techniques ont augmenté au fil du temps dans de nombreuses branches RC et accidents, et dans tout le secteur. Cette situation est due en partie à la part croissante des sinistres à développement lent.

- Ces derniers sont en effet particulièrement sujets aux ajustements de provisions. Aux Etats-Unis, par exemple, en assurance de RC médicale, les estimations initiales de sinistres pour l'année de survenance 2000 ont dû être augmentées de 25% (données 2006). Ce qui signifie qu'une activité semblant à l'origine plutôt lucrative peut finalement occasionner des pertes importantes.

- En réassurance les ratios de provisions pour sinistres sont plus élevés qu'en assurance directe, car les sinistres à développement lent sont lourdement réassurés.

http ://media.swissre.com/documents/sigma2_2008_fr.pdf

Introduction

Les méthodes de provisionnement ne permettent d'obtenir des résultats fiables que si les tendances et les cycles attendus sont intégrés dans les modèles.

Les assureurs utilisent plusieurs méthodes de détermination des provisions techniques. Les plus communes sont les méthodes Chain-Ladder, Bornhuetter-Ferguson, Benktander et Bootstrap. Les sinistres futurs étant soumis à des événements inattendus, les erreurs de prévision et de provisionnement sont inévitables, quelle que soit la méthode choisie. Mais lorsque des erreurs de prévision évitables se produisent, elles sont souvent dues à l'application de la méthode ou aux données utilisées dans les modèles. Il arrive aussi que des erreurs soient provoquées par le manque de connaissance de l'activité ou des tendances qui caractérisent les sinistres.

Les évolutions juridiques, les progrès de la médecine, l'allongement de la durée de vie et les attitudes sociales influent tous sur les sinistres à développement lent.

Mais les erreurs de provisionnement peuvent aussi être dues à des facteurs que ne peuvent contrôler les assureurs : des évolutions juridiques inattendues, l'allongement de l'espérance de vie, les progrès de la médecine et les changements dans les attitudes sociales. Les cycles de prix de l'assurance, qui sont la cause de révisions continuelles de provisions, posent cependant un autre défi.

http ://media.swissre.com/documents/sigma2_2008_fr.pdf

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Introduction

Graphique 1
Cas de faillite enregistrés par les
assureurs dommages, RC et accidents
américains, par cause

50   Cas de faillite

| | Croissance rapide | | Actifs surévalués | | Autres |
| | Provisions pour sinistres insuffisantes | | Allégation de fraude | | |
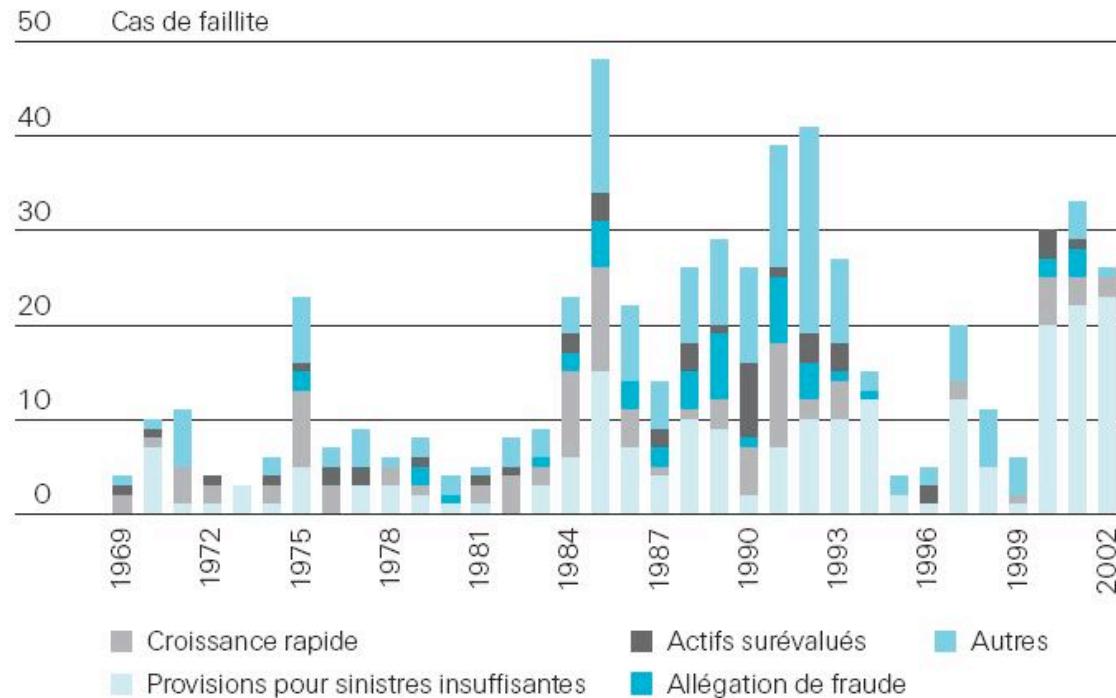
Source : A.M. Best : Best's Insolvency Study, Property/Casualty U.S. Insurers 1969–2002, mai 2004, p. 34

http ://media.swissre.com/documents/sigma2_2008_fr.pdf

Introduction

Graphique 3
Provisions pour sinistres, en % des primes acquises, 1980–2006
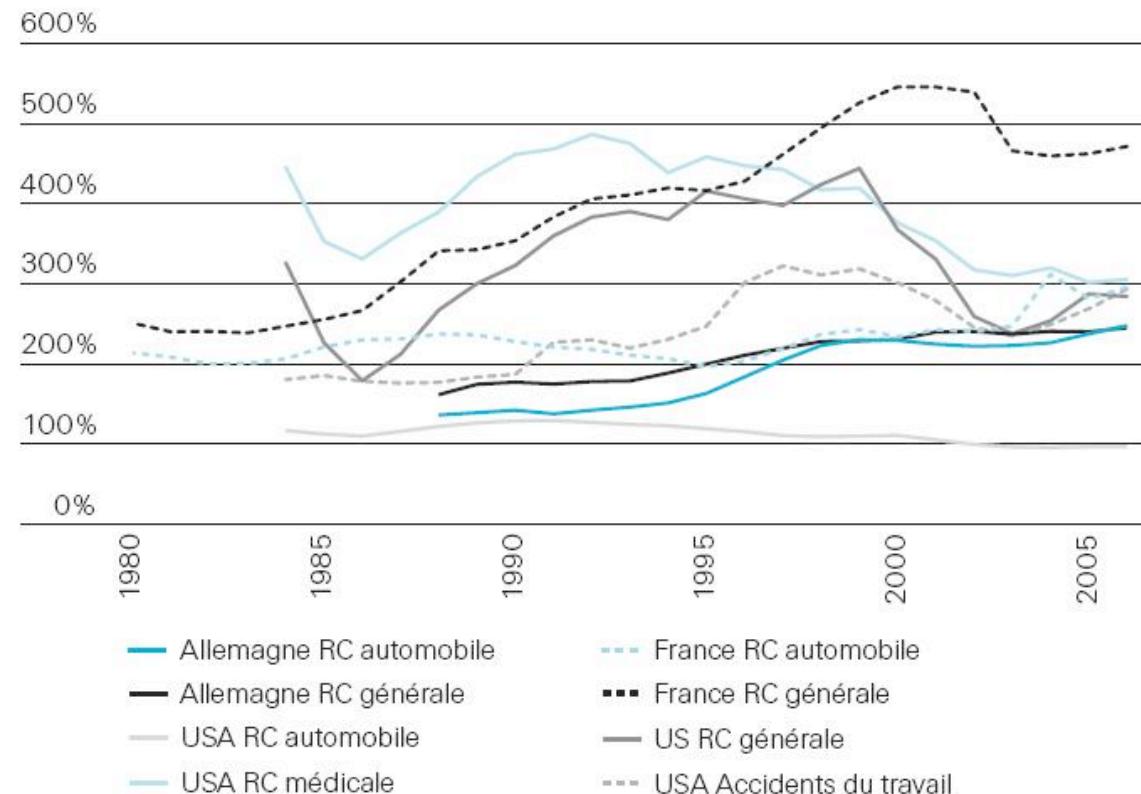
Sources : Autorités de surveillance des assurances, Best's Aggregates &Averages, Property/Casualty, édition 2007

http ://media.swissre.com/documents/sigma2_2008_fr.pdf

Introduction

|	ODP/Bootstrap	Mack	Bayesian/BF Method	Judgement	Scenarios	Regression/Curve Fitting
Description	Most common bootstrap model. Potential to use different distribution for the residuals	Calculation of standard error with and without tail factors.	Uses ODP model with a series of prior ULR estimates defined by a distribution	Based on professional experience	Can include any variation such as changing development patterns or single events	Fits Craighead curve to each origin year to derive initial estimate of ULR, then smoothes across origin years using regression
Data required	Cumulative claims triangles (paid or incurred)	Cumulative claim triangles (paid or incurred)	Cumulative claim triangles (paid or incurred)	Any	Any	Premium and claim amounts triangles
Is the method acceptable to the Profession?	Yes	Yes	Yes	Yes	Yes	Depends on purpose
Is the method easy to use and is it practical?	Yes	Yes	No	Yes	Yes	Yes
Can judgement or amendments be applied?	Yes	Amendments needed where gaps in published method	Requires prior distribution of ultimate position of each origin year	Yes - essential	Yes via choice of scenarios and manual adjustments or tweaks	Yes, perhaps too easily
Is the method easy to explain?	Principles easy to explain	No	Very difficult	Yes	Yes	Yes
When is method good? (Or not?)	Good if little negative development and residuals are iid and run-off pattern is same for all years.	Good only if run-off pattern is same for all years	Good if little negative development and residuals are iid and run-off pattern is same for all years.	Good if actuary has additional knowledge; bad if not experienced	Not good if volatile datasets or inexperienced actuary	Good if run-off pattern varies across origin years. Not good if there is much negative development

http ://www.actuaries.org.uk/system/files/documents/pdf/bhprizegibson.pdf

Introduction

	ODP/Bootstrap	Mack	Bayesian/BF Method	Judgement	Scenarios	Regression/Curve Fitting
Are extreme events included?	Only if in data	Only if in data	Yes, if in data and/or in prior distributions	Yes if desired	Yes if desired	Yes, if in data but can exclude if desired
Produce complete distribution of outcomes?	Yes if process error is simulated in addition to bootstrapping for parameter error	Produces mean and standard error only	Yes	Yes as any required percentile can be estimated using judgement	No – produces a few possible outcomes to which probabilities can be judgementally applied	No, just an approximate range
Type of uncertainty measured	Bootstrap method gives parameter uncertainty, process uncertainty can be simulated in addition	Process and parameter uncertainty	Process and parameter uncertainty	Potentially model error as well as parameter and process error	Usually just parameter uncertainty	Parameter uncertainty only (dependent variable in regression is expected ULR)
Time to program and complete	Easy to program in Excel though long time to run	Easy to program and quick to run	Specialist software required and very slow to run			Easy to do in Excel
Comparison of class results to aggregated	Automatic consistency between origin year and aggregate results	Automatic consistency between origin year and aggregate results	Automatic consistency between origin year and aggregate results	Should be consistent given enough care, but this not guaranteed	Does not produce separate assessment of aggregate uncertainty	Does not produce separate assessment of aggregate uncertainty

http ://www.actuaries.org.uk/system/files/documents/pdf/bhprizegibson.pdf

Introduction

http ://www.actuaries.org.uk/system/files/documents/pdf/bhprizegibson.pdf

Introduction

Commercial Property

http ://www.actuaries.org.uk/system/files/documents/pdf/bhprizegibson.pdf
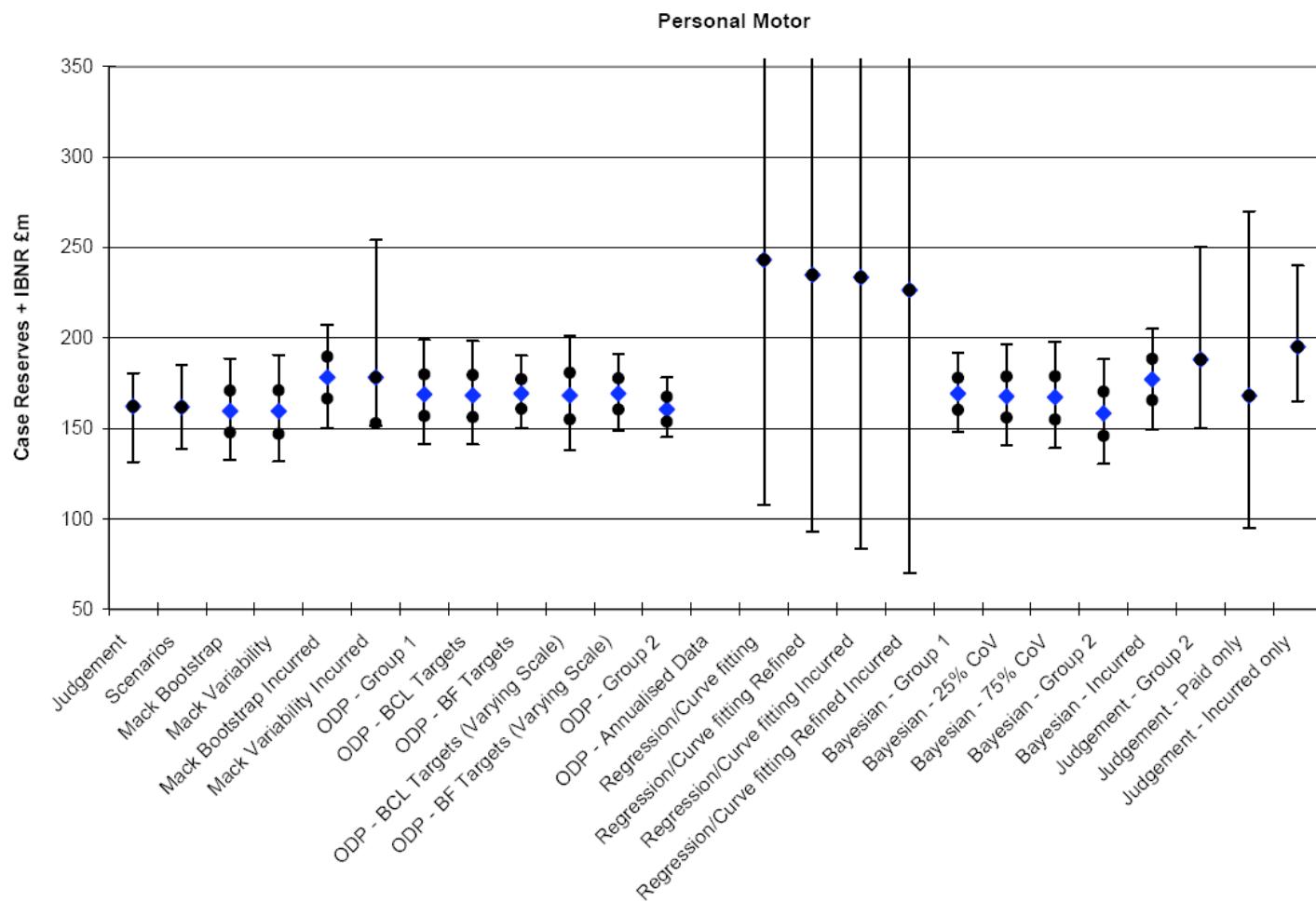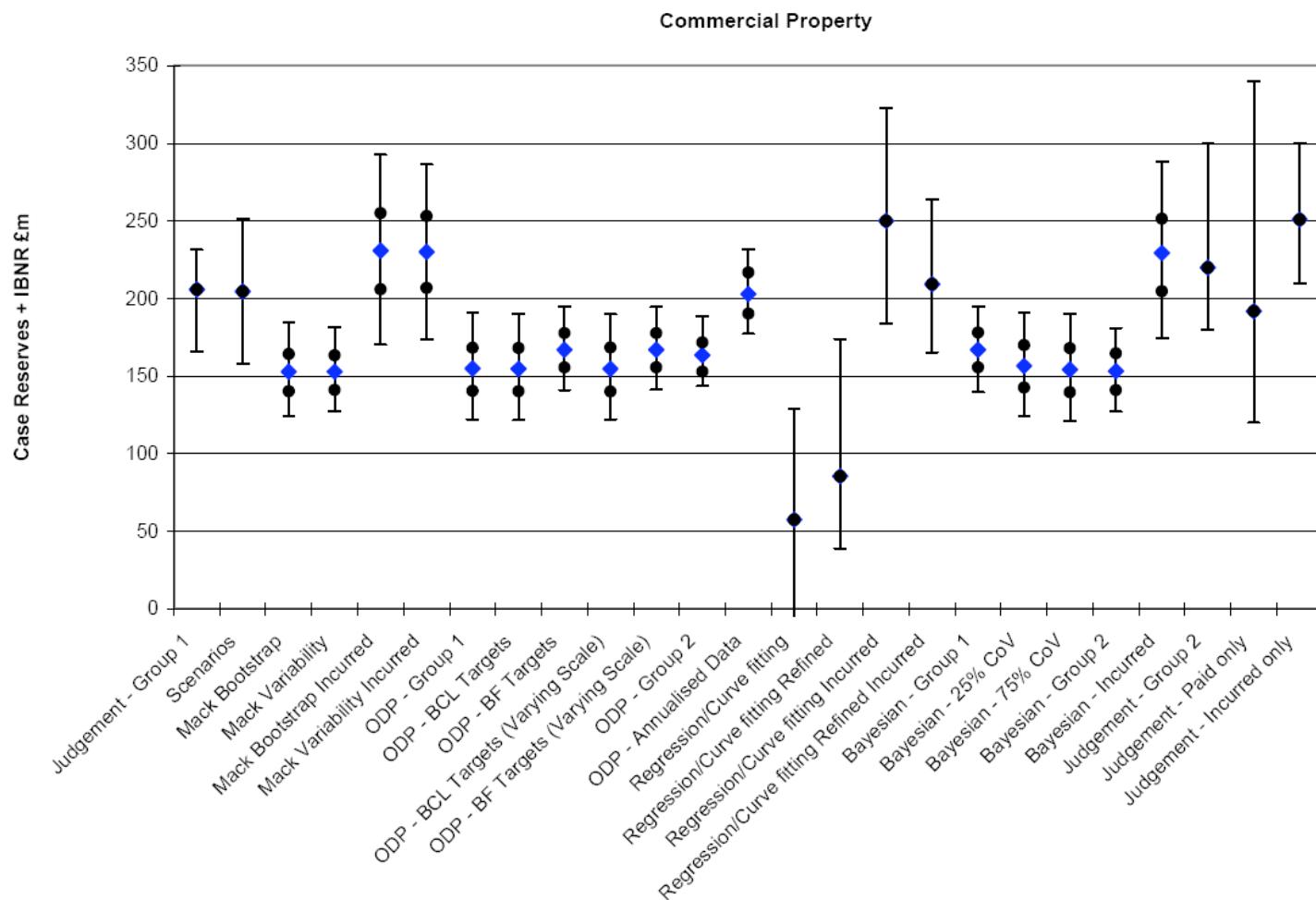
Assurance multirisques habitation
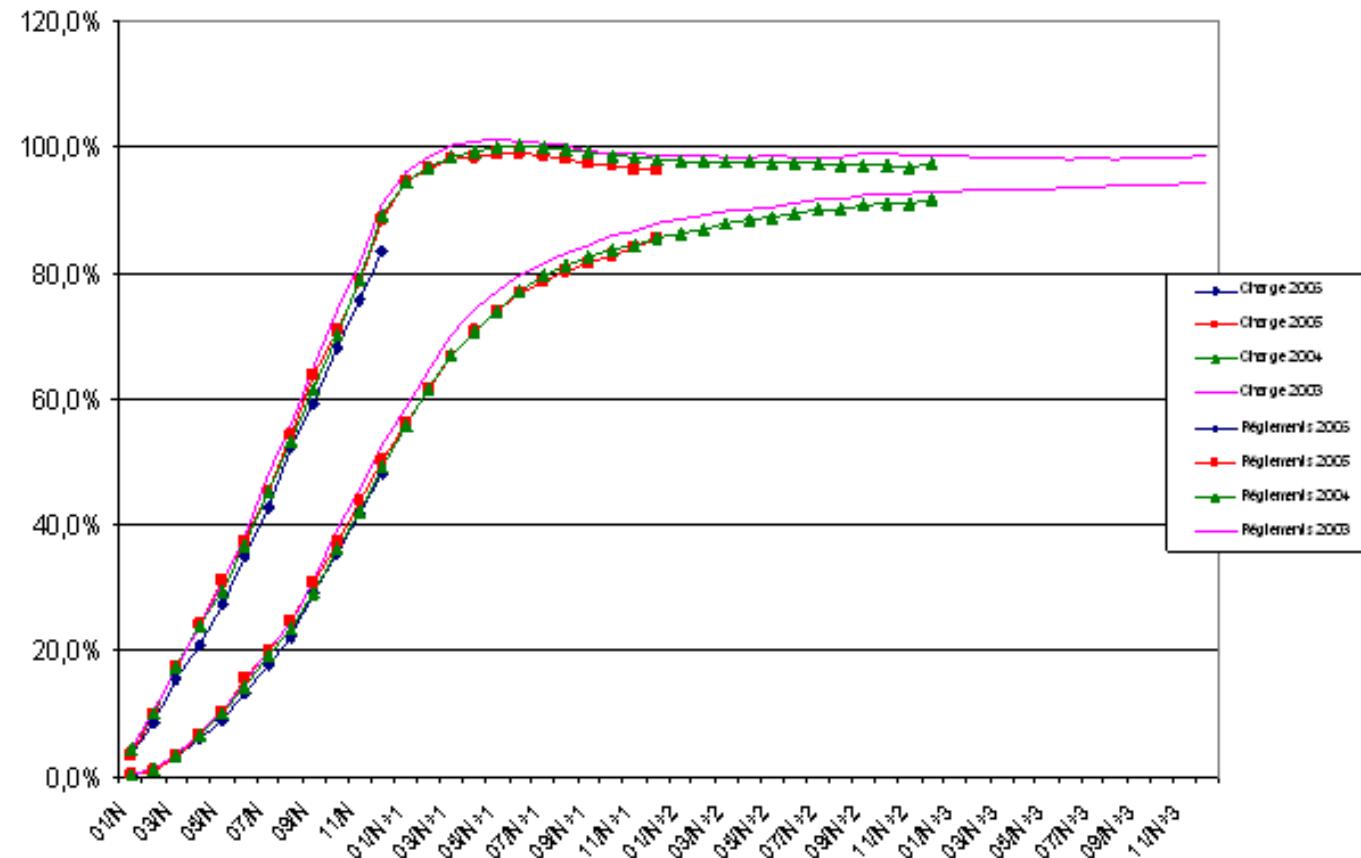
Assurance risque incendies entreprises
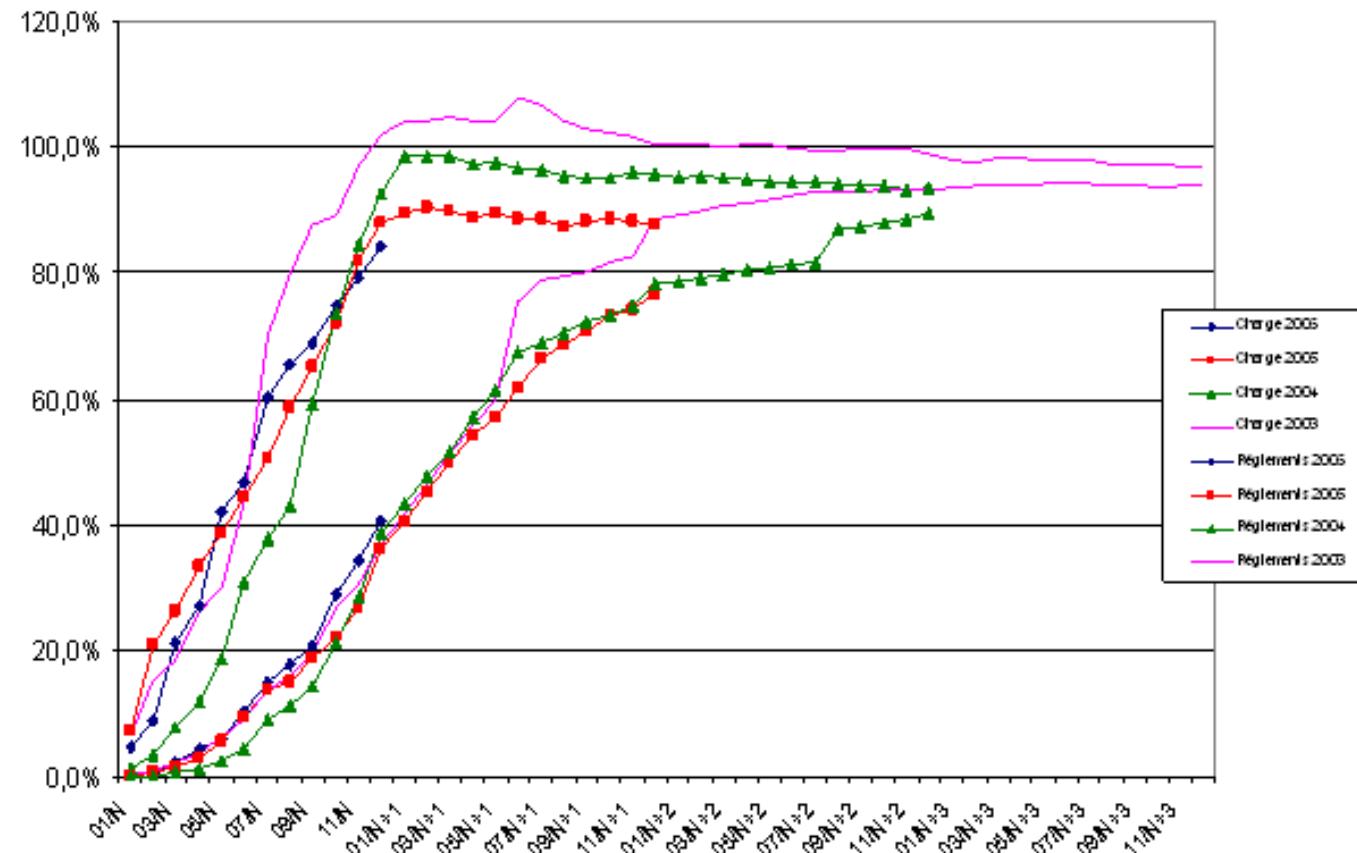
Assurance automobile (total)
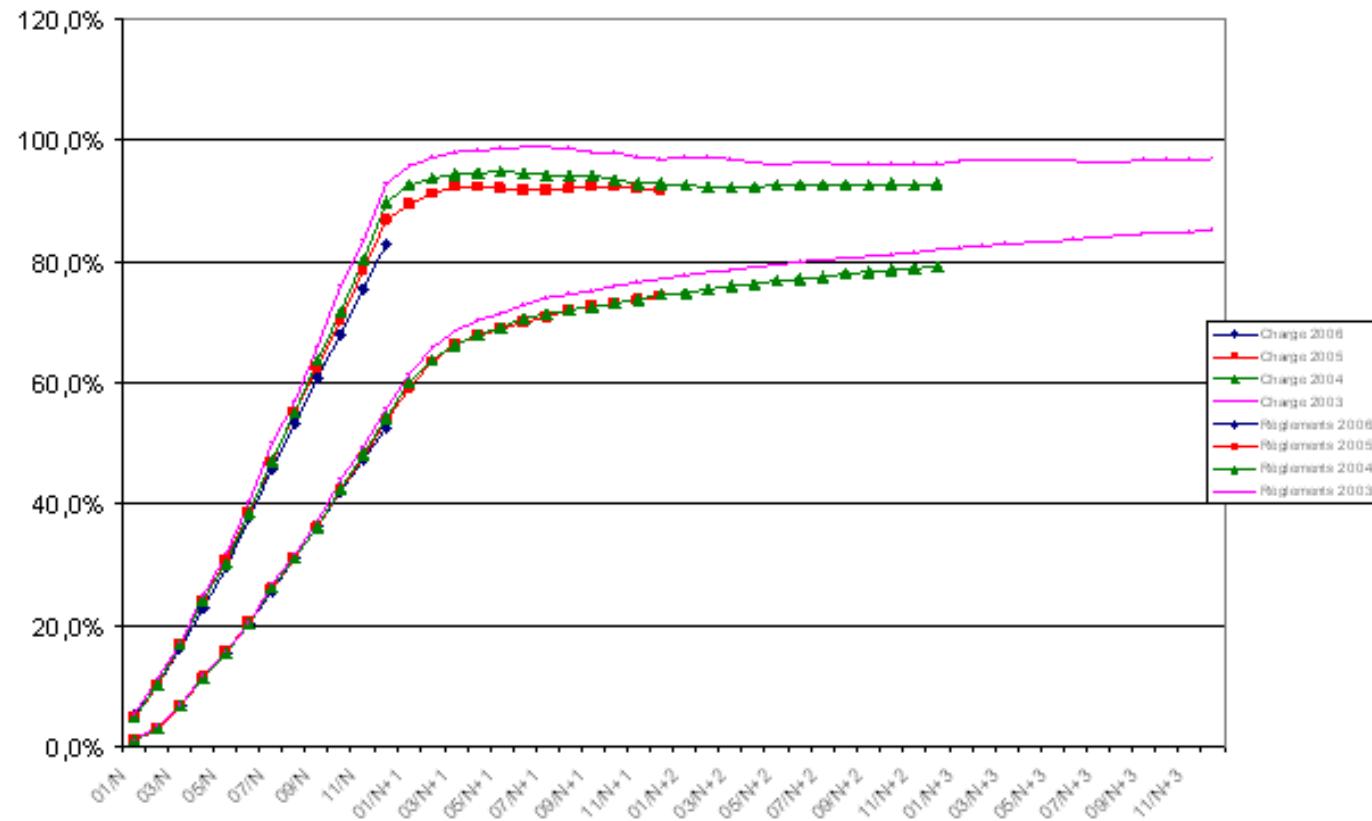
Assurance automobile matériel
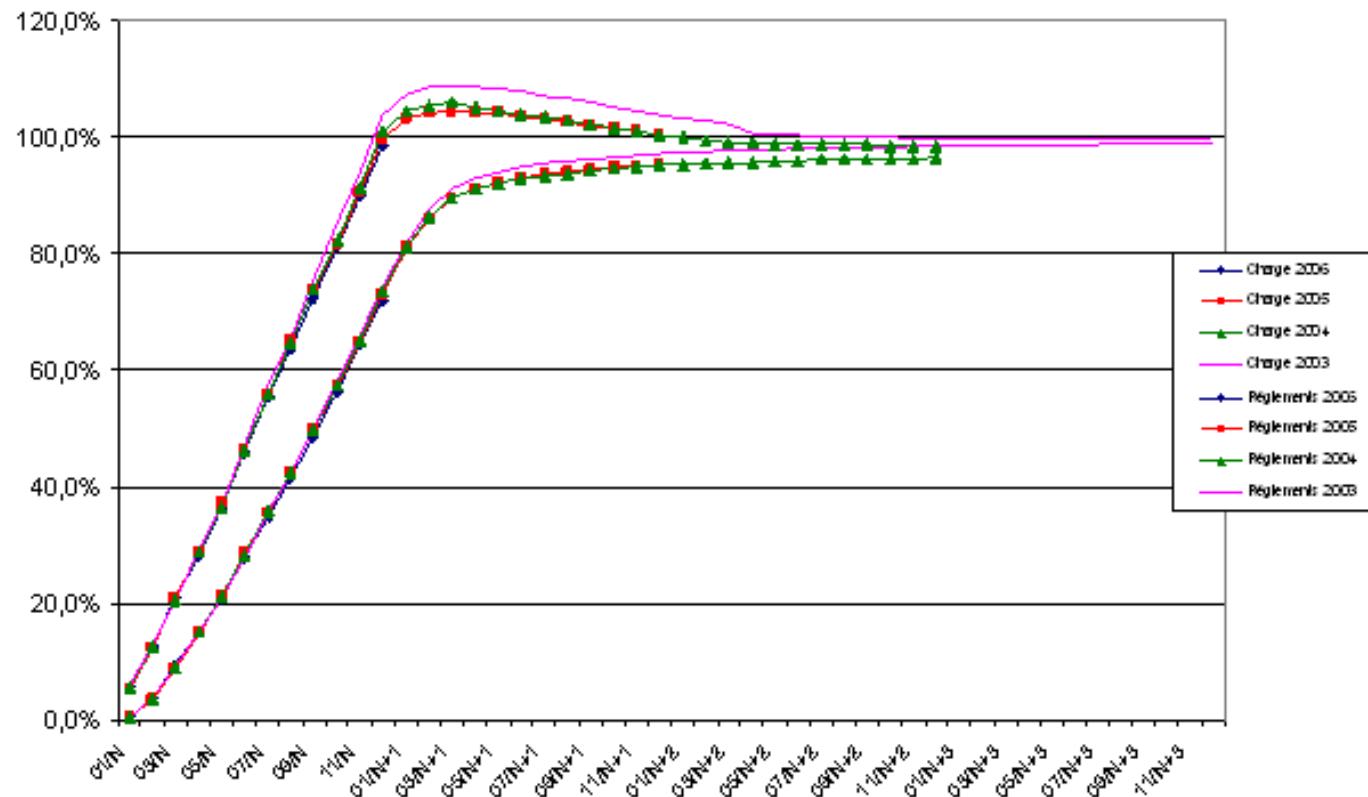
Assurance automobile corporel
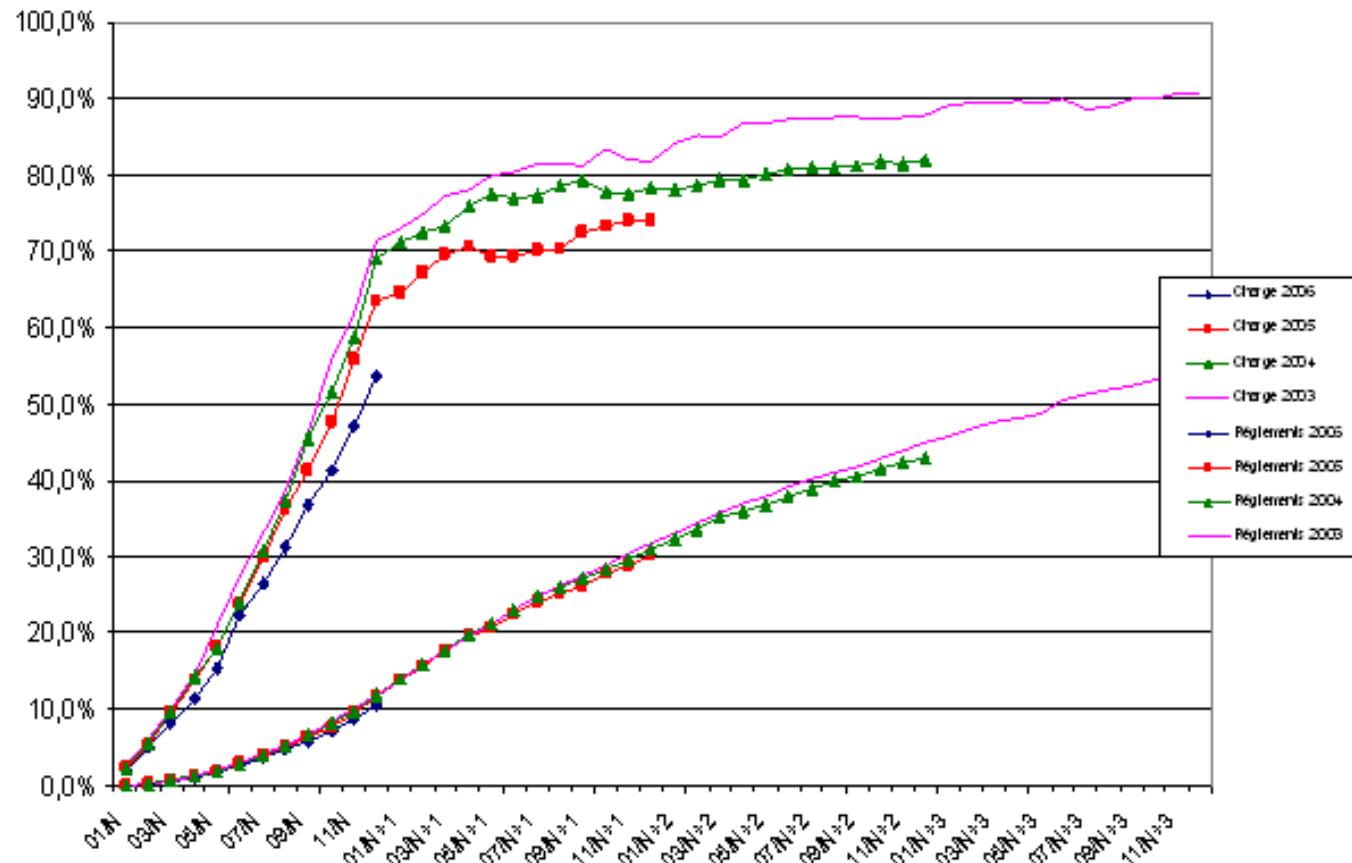
Assurance responsabilité civile entreprisee

Assurance responabilité civile médicale
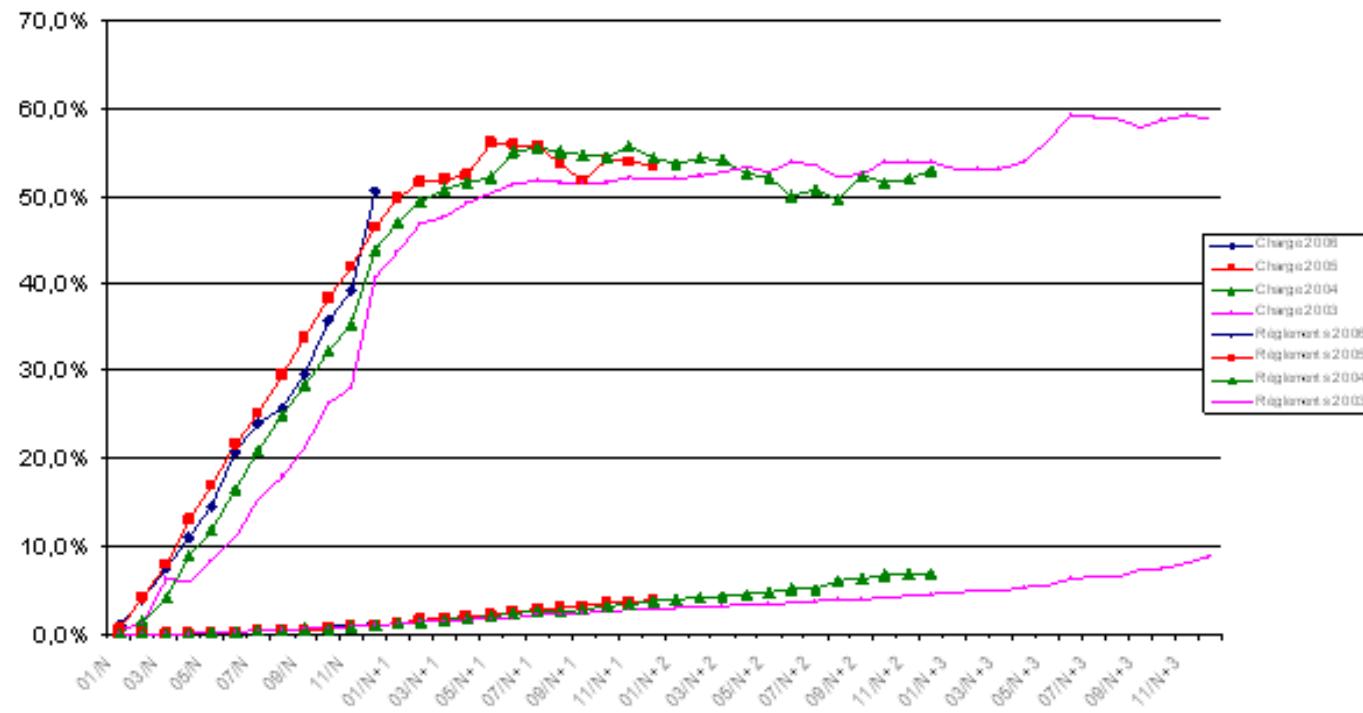
Assurance construction
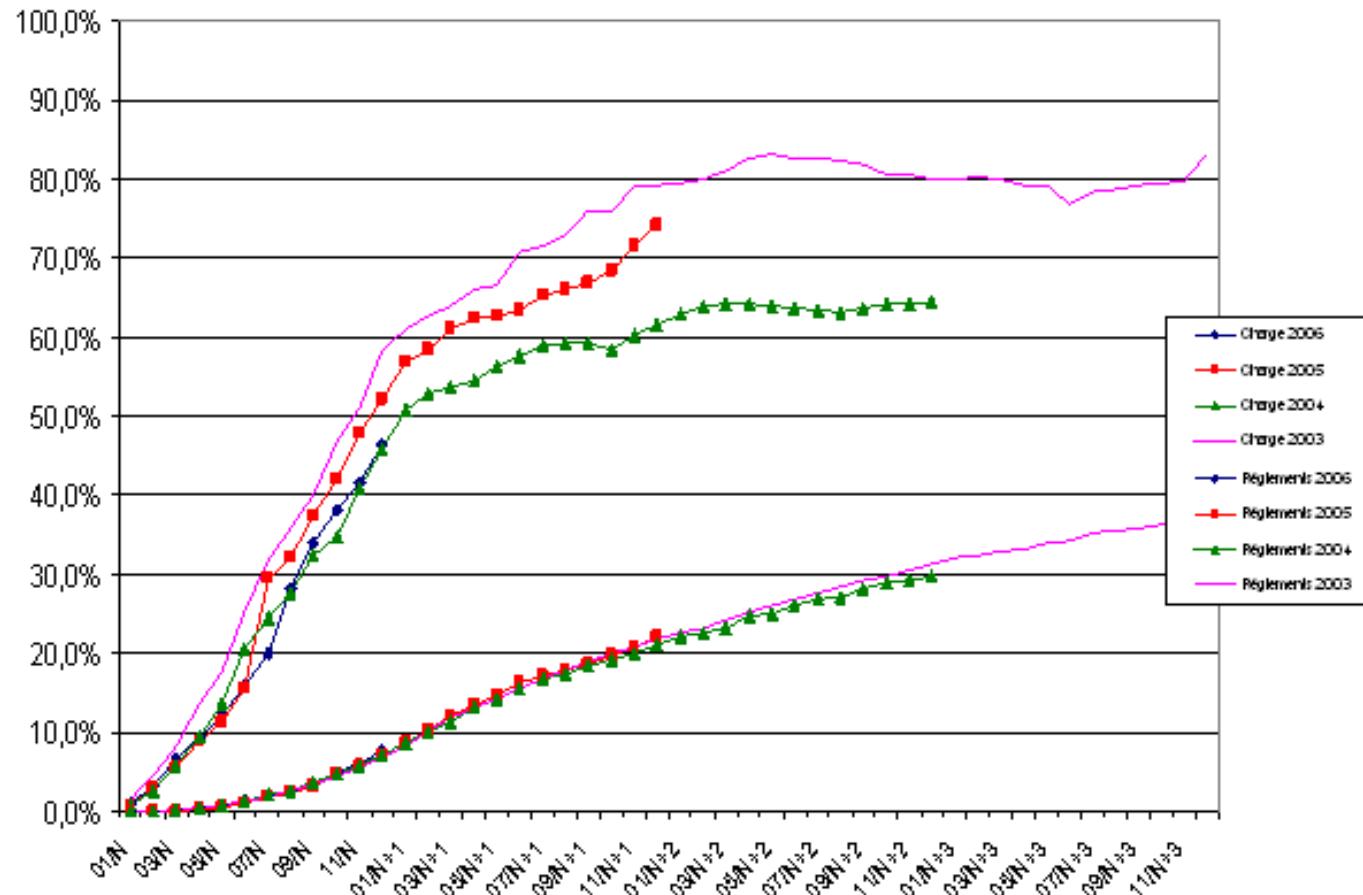
Les triangles : incréments de paiements

Noté $Y_{i,j}$, pour l'année de survenance i, et l'année de développement j,

	0	1	2	3	4	5
0	3209	1163	39	17	7	21
1	3367	1292	37	24	10	
2	3871	1474	53	22		
3	4239	1678	103			
4	4929	1865				
5	5217					

Les triangles : paiements cumulés

Noté $C_{i,j} = Y_{i,0} + Y_{i,1} + \cdots + Y_{i,j}$, pour l'année de survenance i, et l'année de développement j,

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420		
3	4239	5917	6020			
4	4929	6794				
5	5217					

Les triangles : nombres de sinistres

Noté $N_{i,j}$ sinistres survenus l'année i connus (déclarés) au bout de j années,

	0	1	2	3	4	5
0	1043.4	1045.5	1047.5	1047.7	1047.7	1047.7
1	1043.0	1027.1	1028.7	1028.9	1028.7	
2	965.1	967.9	967.8	970.1		
3	977.0	984.7	986.8			
4	1099.0	1118.5				
5	1076.3					

La prime acquise

Notée π_i, prime acquise pour l'année i

Year i	0	1	2	3	4	5
P_i	4591	4672	4863	5175	5673	6431
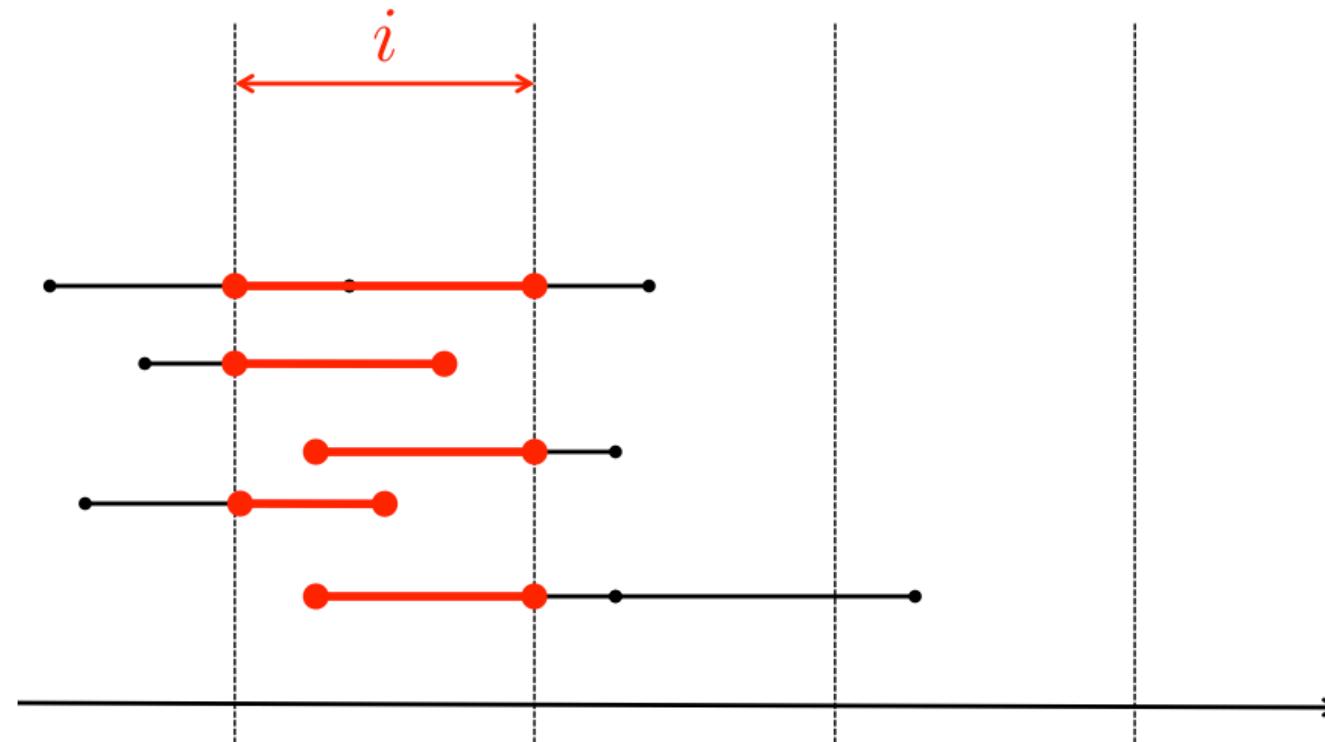
Diagramme de Lexis en assurance non-vie
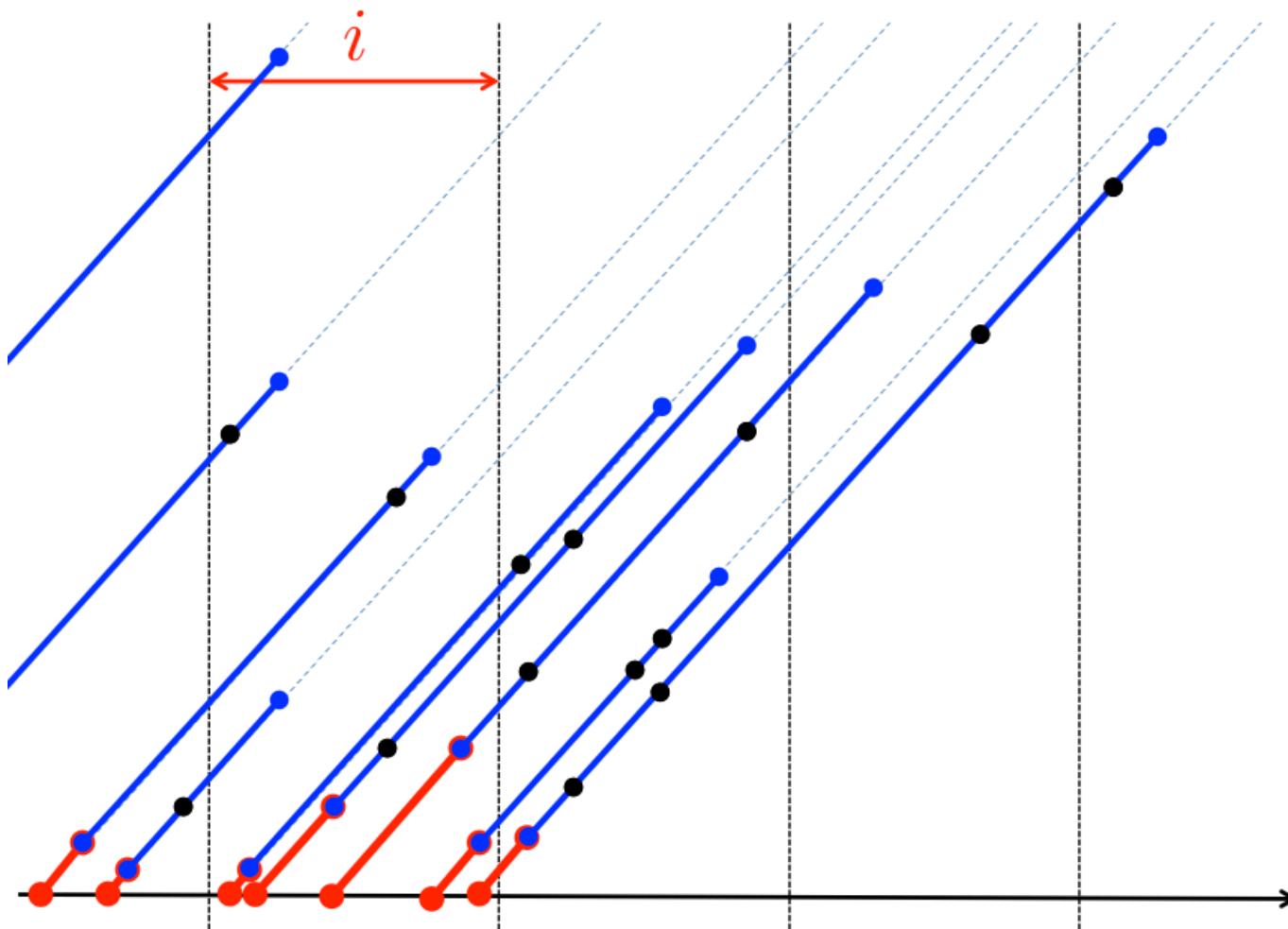
Triangles ?

Actually, there might be two different cases in practice, the first one being when initial data are missing

In that case it is mainly an index-issue in calculation.

Triangles ?

Actually, there might be two different cases in practice, the first one being when final data are missing, i.e. some tail factor should be included

	0	1	2	3	4	5
0	●	●	●	●	○	○
1	●	●	●	○	○	○
2	●	●	○	○	○	○
3	●	○	○	○	○	○
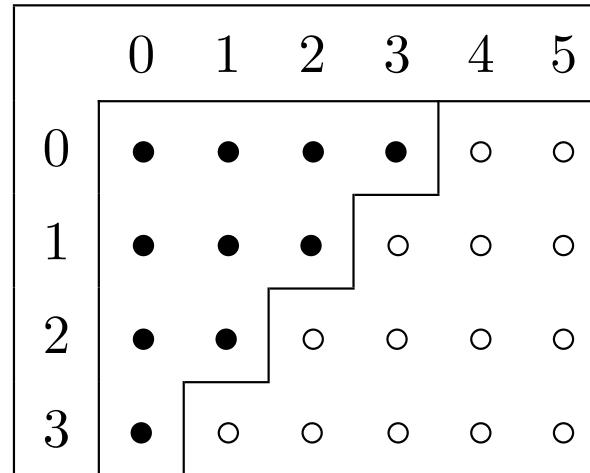
In that case it is necessary to extrapolate (with past information) the final loss

The Chain Ladder estimate

We assume here that

$$C_{i,j+1} = \lambda_j \cdot C_{i,j} \text{ for all } i, j = 1, \cdots, n.$$

A *natural* estimator for λ_j based on past history is

$$\widehat{\lambda}_j = \frac{\sum_{i=1}^{n-j} C_{i,j+1}}{\sum_{i=1}^{n-j} C_{i,j}} \text{ for all } j = 1, \cdots, n-1.$$

Hence, it becomes possible to estimate future payments using

$$\widehat{C}_{i,j} = \left[\widehat{\lambda}_{n+1-i}...\widehat{\lambda}_{j-1}\right] C_{i,n+1-i}.$$

La méthode Chain Ladder, en pratique

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420		
3	4239	5917	6020			
4	4929	6794				
5	5217					

La méthode Chain Ladder, en pratique

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420		
3	4239	5917	6020			
4	4929	6794				
5	5217					

$$\lambda_0 = \frac{4372 + \cdots + 6794}{3209 + \cdots + 4929} \sim 1.38093$$

La méthode Chain Ladder, en pratique

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420		
3	4239	5917	6020			
4	4929	6794				
5	5217	7204.3				

$$\lambda_0 = \frac{4372 + \cdots + 6794}{3209 + \cdots + 4929} \sim 1.38093$$

La méthode Chain Ladder, en pratique

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420		
3	4239	5917	6020			
4	4929	6794				
5	5217	7204.3				

$$\lambda_1 = \frac{4411 + \cdots + 6020}{4372 + \cdots + 5917} \sim 1.01143$$

La méthode Chain Ladder, en pratique

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420		
3	4239	5917	6020			
4	4929	6794	6871.7			
5	5217	7204.3	7286.7			

$$\lambda_1 = \frac{4411 + \cdots + 6020}{4372 + \cdots + 5917} \sim 1.01143$$

La méthode Chain Ladder, en pratique

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420		
3	4239	5917	6020			
4	4929	6794	6871.7			
5	5217	7204.3	7286.7			

$$\lambda_2 = \frac{4428 + \cdots + 5420}{4411 + \cdots + 5398} \sim 1.00434$$

La méthode Chain Ladder, en pratique

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420		
3	4239	5917	6020	6046.1		
4	4929	6794	6871.7	6901.5		
5	5217	7204.3	7286.7	7318.3		

$$\lambda_2 = \frac{4428 + \cdots + 5420}{4411 + \cdots + 5398} \sim 1.00434$$

La méthode Chain Ladder, en pratique

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420	5430.1	
3	4239	5917	6020	6046.1	6057.4	
4	4929	6794	6871.7	6901.5	6914.3	
5	5217	7204.3	7286.7	7318.3	7331.9	

$$\lambda_3 = \frac{4435 + 4730}{4428 + 4720} \sim 1.00186$$

La méthode Chain Ladder, en pratique

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	4752.4
2	3871	5345	5398	5420	5430.1	5455.8
3	4239	5917	6020	6046.1	6057.4	6086.1
4	4929	6794	6871.7	6901.5	6914.3	6947.1
5	5217	7204.3	7286.7	7318.3	7331.9	7366.7

$$\lambda_4 = \frac{4456}{4435} \sim 1.00474$$

La méthode Chain Ladder, en pratique

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	4752.4
2	3871	5345	5398	5420	5430.1	5455.8
3	4239	5917	6020	6046.15	6057.4	6086.1
4	4929	6794	6871.7	6901.5	6914.3	6947.1
5	5217	7204.3	7286.7	7318.3	7331.9	7366.7

One the triangle has been completed, we obtain the amount of reserves, with respectively 22, 36, 66, 153 and 2150 per accident year, i.e. the total is 2427.

Three ways to look at triangles

There are basically three kind of approaches to model development

- developments as percentages of total incured, i.e. consider $\varphi_0, \varphi_1, \cdots, \varphi_n$, with $\varphi_0 + \varphi_1 + \cdots + \varphi_n = 1$, such that

$$\mathbb{E}(Y_{i,j}) = \varphi_j \mathbb{E}(C_{i,n}), \text{ where } j = 0, 1, \cdots, n.$$

- developments as rates of total incured, i.e. consider $\gamma_0, \gamma_1, \cdots, \gamma_n$, such that

$$\mathbb{E}(C_{i,j}) = \gamma_j \mathbb{E}(C_{i,n}), \text{ where } j = 0, 1, \cdots, n.$$

- developments as factors of previous estimation, i.e. consider $\lambda_0, \lambda_1, \cdots, \lambda_n$, such that

$$\mathbb{E}(C_{i,j+1}) = \lambda_j \mathbb{E}(C_{i,j}), \text{ where } j = 0, 1, \cdots, n.$$

Three ways to look at triangles

From a mathematical point of view, it is strictly equivalent to study one of those. Hence,

$$\gamma_j = \varphi_0 + \varphi_1 + \cdots + \varphi_j = \frac{1}{\lambda_j} \frac{1}{\lambda_{j+1}} \cdots \frac{1}{\lambda_{n-1}},$$

$$\lambda_j = \frac{\gamma_{j+1}}{\gamma_j} = \frac{\varphi_0 + \varphi_1 + \cdots + \varphi_j + \varphi_{j+1}}{\varphi_0 + \varphi_1 + \cdots + \varphi_j}$$

$$\varphi_j = \begin{cases} \gamma_0 \text{ if } j = 0 \\ \gamma_j - \gamma_{j-1} \text{ if } j \geq 1 \end{cases} = \begin{cases} \dfrac{1}{\lambda_0} \dfrac{1}{\lambda_1} \cdots \dfrac{1}{\lambda_{n-1}}, \text{ if } j = 0 \\ \dfrac{1}{\lambda_{j+1}} \dfrac{1}{\lambda_{j+2}} \cdots \dfrac{1}{\lambda_{n-1}} - \dfrac{1}{\lambda_j} \dfrac{1}{\lambda_{j+1}} \cdots \dfrac{1}{\lambda_{n-1}}, \text{ if } j \geq 1 \end{cases}$$

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Three ways to look at triangles

On the previous triangle,

	0	1	2	3	4	n
λ_j	1,38093	1,01143	1,00434	1,00186	1,00474	1,0000
γ_j	70,819%	97,796%	98,914%	99,344%	99,529%	100,000%
φ_j	70,819%	26,977%	1,118%	0,430%	0,185%	0,000%

d-triangles

It is possible to define the d-triangles, with empirical λ's, i.e. $\lambda_{i,j}$

	0	1	2	3	4	5
0	1.362	1.009	1.004	1.002	1.005	
1	1.384	1.008	1.005	1.002		
2	1.381	1.010	1.001			
3	1.396	1.017				
4	1.378					
5						

The Chain-Ladder estimate

The Chain-Ladder estimate is probably the most popular technique to estimate claim reserves. Let $\mathcal{F}_t$ denote the information avalable at time t, or more formally the filtration generated by $\{C_{i,j}, i + j \leq t\}$ - or equivalently $\{X_{i,j}, i + j \leq t\}$

Assume that incremental payments are independent by occurence years, i.e. $C_{i_1,\cdot}$ $C_{i_2,\cdot}$ are independent for any i_1 and i_2.

Further, assume that $(C_{i,j})_{j \geq 0}$ is Markov, and more precisely, there exist λ_j's and σ_j^2's such that

$$\begin{cases} (C_{i,j+1}|\mathcal{F}_{i+j}) = (C_{i,j+1}|C_{i,j}) = \lambda_j \cdot C_{i,j} \\ \mathrm{Var}(C_{i,j+1}|\mathcal{F}_{i+j}) = \mathrm{Var}(C_{i,j+1}|C_{i,j}) = \sigma_j^2 \cdot C_{i,j} \end{cases}$$

Under those assumption, one gets

$$\mathbb{E}(C_{i,j+k}|\mathcal{F}_{i+j}) = (C_{i,j+k}|C_{i,j}) = \lambda_j \cdot \lambda_{j+1} \cdots \lambda_{j+k-1} C_{i,j}$$

Underlying assumptions in the Chain-Ladder estimate

Recall, see Mack (1993), properties of the Chain-Ladder estimate rely on the following assumptions

$$
\begin{cases}
H_1 & \mathbb{E}\left(C_{i,j+1}|C_{i,1},...,C_{i,j}\right) = \lambda_j.C_{ij} \text{ for all } i = 0,1,..,n \text{ and } j = 0,1,...,n-1 \\
H_2 & \left(C_{i,j}\right)_{j=1,...,n} \text{ and } \left(C_{i',j}\right)_{j=1,...,n} \text{ are independent for all } i \neq i'. \\
H_3 & \text{Var}\left(C_{i,j+1}|C_{i,1},...,C_{i,j}\right) = C_{i,j}\sigma_j^2 \text{ for all } i = 0,1,...,n \text{ and } j = 0,1,...,n-1
\end{cases}
$$

Testing assumptions

Assumption H_1 can be interpreted as a linear regression model, i.e.
$Y_i = \beta_0 + X_i \cdot \beta_1 + \varepsilon_i$, $i = 1, \cdots, n$, where ε is some error term, such that
$\mathbb{E}(\varepsilon) = 0$, where $\beta_0 = 0$, $Y_i = C_{i,j+1}$ for some j, $X_i = C_{i,j}$, and $\beta_1 = \lambda_j$.

Weighted least squares can be considered, i.e. $\min \left\{ \sum_{i=1}^{n-j} \omega_i \left(Y_i - \beta_0 - \beta_1 X_i \right)^2 \right\}$
where the ω_i's are proportional to $Var(Y_i)^{-1}$. This leads to

$$\min \left\{ \sum_{i=1}^{n-j} \frac{1}{C_{i,j}} \left(C_{i,j+1} - \lambda_j C_{i,j} \right)^2 \right\}.$$

As in any linear regression model, it is possible to test assumptions H_1 and H_2, the following graphs can be considered, given j

- plot $C_{i,j+1}$'s versus $C_{i,j}$'s. Points should be on the straight line with slope $\widehat{\lambda}_j$.

- plot (standardized) residuals $\epsilon_{i,j} = \dfrac{C_{i,j+1} - \widehat{\lambda}_j C_{i,j}}{\sqrt{C_{i,j}}}$ versus $C_{i,j}$'s.

Testing assumptions

H_2 is the accident year independent assumption. More precisely, we assume there is no calendar effect.

Define the diagonal $B_k = \{C_{k,0}, C_{k-1,1}, C_{k-2,2} \cdots, C_{2,k-2}, C_{1,k-1}, C_{0,k}\}$. If there is a calendar effect, it should affect adjacent factor lines,

$$A_k = \left\{ \frac{C_{k,1}}{C_{k,0}}, \frac{C_{k-1,2}}{C_{k-1,1}}, \frac{C_{k-2,3}}{C_{k-2,2}}, \cdots, \frac{C_{1,k}}{C_{1,k-1}}, \frac{C_{0,k+1}}{C_{0,k}} \right\} = {}_{''} \frac{\delta_{k+1}}{\delta_k} {}_{''},$$

and

$$A_{k-1} = \left\{ \frac{C_{k-1,1}}{C_{k-1,0}}, \frac{C_{k-2,2}}{C_{k-2,1}}, \frac{C_{k-3,3}}{C_{k-3,2}}, \cdots, \frac{C_{1,k-1}}{C_{1,k-2}}, \frac{C_{0,k}}{C_{0,k-1}} \right\} = {}_{''} \frac{\delta_k}{\delta_{k-1}} {}_{''}.$$

For each k, let N_k^+ denote the number of elements exceeding the median, and N_k^- the number of elements lower than the mean. The two years are independent, N_k^+ and N_k^- should be "closed", i.e. $N_k = \min\left(N_k^+, N_k^-\right)$ should be "closed" to $\left(N_k^+ + N_k^-\right)/2$.

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Since N_k^- and N_k^+ are two binomial distributions $\mathcal{B}\left(p = 1/2, n = N_k^- + N_k^+\right)$, then

$$\mathbb{E}\left(N_k\right) = \frac{n_k}{2} - \binom{n_k - 1}{m_k} \frac{n_k}{2^{n_k}} \text{ where } n_k = N_k^+ + N_k^- \text{ and } m_k = \left[\frac{n_k - 1}{2}\right]$$

and

$$V\left(N_k\right) = \frac{n_k\left(n_k - 1\right)}{2} - \binom{n_k - 1}{m_k} \frac{n_k\left(n_k - 1\right)}{2^{n_k}} + \mathbb{E}\left(N_k\right) - \mathbb{E}\left(N_k\right)^2.$$

Under some normality assumption on N, a 95% confidence interval can be derived, i.e. $\mathbb{E}\left(Z\right) \pm 1.96\sqrt{V\left(Z\right)}$.

	0	1	2	3	4	5
0	0.734	.0991	0.996	0.998	0.995	
1	0.723	0.992	0.995	0.998		
2	0.724	0.990	0.996			
3	0.716	0.983				
4	0.725					
5						
6	0.724	0.991	0.996	0.998	0.995	

	0	1	2	3	4	5
0	+	+	+	+	·	
1	−	+	−	−		
2	·	−	·			
3	−	−				
4	+					
5						
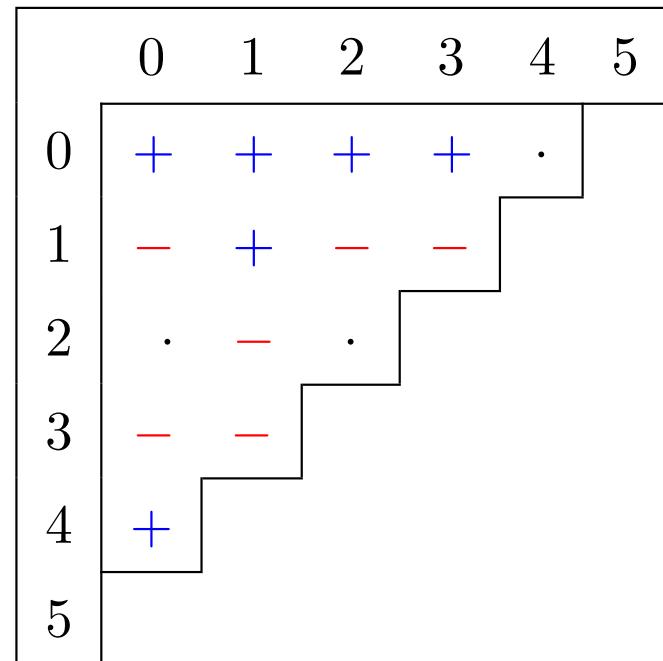
From Chain-Ladder to Grossing-Up

The idea of the Chain-Ladder technique was to estimate the λ_j's, so that we can derive estimates for $C_{i,n}$, since

$$\widehat{C}_{i,n} = \widehat{C}_{i,n-i} \cdot \prod_{k=n-i+1}^{n} \widehat{\lambda}_k$$

Based on the Chain-Ladder link ratios, $\widehat{\lambda}$, it is possible to define grossing-up coefficients

$$\widehat{\gamma}_j = \prod_{k=j}^{n} \frac{1}{\widehat{\lambda}_k}$$

and thus, the total loss incured for accident year i is then

$$\widehat{C}_{i,n} = \widehat{C}_{i,n-i} \cdot \frac{\widehat{\gamma}_n}{\widehat{\gamma}_{n-i}}$$

Variant of the Chain-Ladder Method (1)

Historically (see e.g.), the *natural* idea was to consider a (standard) average of individual link ratios.

Several techniques have been introduces to study individual link-ratios.

A first idea is to consider a simple linear model, $\lambda_{i,j} = a_j i + b_j$. Using OLS techniques, it is possible to estimate those coefficients simply. Then, we project those ratios using predicted one, $\widehat{\lambda}_{i,j} = \widehat{a}_j i + \widehat{b}_j$.

Variant of the Chain-Ladder Method (2)

A second idea is to assume that λ_j is the weighted sum of $\lambda_{...,j}$'s,

$$\widehat{\lambda}_j = \frac{\sum_{i=0}^{j-1} \omega_{i,j} \lambda_{i,j}}{\sum_{i=0}^{j-1} \omega_{i,j}}$$

If $\omega_{i,j} = C_{i,j}$ we obtain the chain ladder estimate. An alternative is to assume that $\omega_{i,j} = i + j + 1$ (in order to give more weight to recent years).

Variant of the Chain-Ladder Method (3)

Here, we assume that cumulated run-off triangles have an exponential trend, i.e.

$$C_{i,j} = \alpha_j \exp(i \cdot \beta_j).$$

In order to estimate the α_j's and β_j's is to consider a linear model on $\log C_{i,j}$,

$$\log C_{i,j} = \underbrace{a_j}_{\log(\alpha_j)} + \beta_j \cdot i + \varepsilon_{i,j}.$$

Once the β_j's have been estimated, set $\widehat{\gamma}_j = \exp(\widehat{\beta}_j)$, and define

$$\Gamma_{i,j} = \widehat{\gamma}_j^{\,n-i-j} \cdot C_{i,j}.$$

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The extended link ratio family of estimators

For convenient, link ratios are factors that give relation between cumulative payments of one development year (say j) and the next development year $(j+1)$. They are simply the ratios y_i/x_i, where x_i's are cumulative payments year j (i.e. $x_i = C_{i,j}$) and y_i's are cumulative payments year $j+1$ (i.e. $y_i = C_{i,j+1}$).

For example, the Chain Ladder estimate is obtained as

$$\widehat{\lambda}_j = \frac{\sum_{i=0}^{n-j} y_i}{\sum_{i=0}^{n-j} x_i} = \sum_{i=0}^{n-j} \frac{x_i}{\sum_{k=1}^{n-j} x_k} \cdot \frac{y_i}{x_i}.$$

But several other link ratio techniques can be considered, e.g.

$$\widehat{\lambda}_j = \frac{1}{n-j+1} \sum_{i=0}^{n-j} \frac{y_i}{x_i}, \ \text{ i.e. the simple arithmetic mean,}$$

$$\widehat{\lambda}_j = \left(\prod_{i=0}^{n-j} \frac{y_i}{x_i} \right)^{n-j+1}, \ \text{ i.e. the geometric mean,}$$

$$\widehat{\lambda}_j = \sum_{i=0}^{n-j} \frac{x_i^2}{\sum_{k=1}^{n-j} x_k^2} \cdot \frac{y_i}{x_i}, \text{ i.e. the weighted average "by volume squared",}$$

Hence, this technique can be related to wieghted least squares, i.e.

$$y_i = \beta x_i + \varepsilon_i, \text{ where } \varepsilon_i \sim \mathcal{N}(0, \sigma^2 x_i^{\delta}), \text{ for some } \delta > 0.$$

E.g. if $\delta = 0$, we obtain the arithmetic mean, if $\delta = 1$, we obtain the Chain Ladder estimate, and if $\delta = 2$, the weighted average "by volume squared".

The interest of this regression approach, is that standard error for predictions can be derived, under standard (and testable) assumptions. Hence

- standardized residuals $(\sigma x_i^{\delta/2})^{-1} \varepsilon_i$ are $\mathcal{N}(0, 1)$, i.e. QQ plot
- $\mathbb{E}(y_i | x_i) = \beta x_i$, i.e. graph of x_i versus y_i.

Properties of the Chain-Ladder estimate

Further

$$\widehat{\lambda}_j = \frac{\sum_{i=0}^{n-j-1} C_{i,j+1}}{\sum_{i=0}^{n-j-1} C_{i,j}}$$

is an unbiased estimator for λ_j, given $\mathcal{G}_j$, and $\widehat{\lambda}_j$ and $\widehat{\lambda}_j + h$ are non-correlated, given $\mathcal{F}_j$. Hence, an unbiased estimator for $\mathbb{E}(C_{i,j}|\mathcal{F}_i)$ is

$$\widehat{C}_{i,j} = \widehat{\lambda}_{n-i} \cdot \widehat{\lambda}_{n-i+1} \cdots \widehat{\lambda}_{j-2} \left(\widehat{\lambda}_{j-1} - 1 \right) \cdot C_{i,n-i}.$$

Recall that $\widehat{\lambda}_j$ is the estimator with minimal variance among all linear estimators obtained from $\lambda_{i,j} = C_{i,j+1}/C_{i,j}$'s. Finally, recall that

$$\widehat{\sigma}_j^2 = \frac{1}{n-j-1} \sum_{i=0}^{n-j-1} \left(\frac{C_{i,j+1}}{C_{i,j}} - \widehat{\lambda}_j \right)^2 \cdot X_{i,j}$$

is an unbiased estimator of σ_j^2, given $\mathcal{G}_j$ (see Mack (1993) or Denuit & Charpentier (2005)).

Prediction error of the Chain-Ladder estimate

We stress here that estimating reserves is a *prediction process* : based on past observations, we predict future amounts. Recall that prediction error can be explained as follows,

$$\underbrace{\mathbb{E}[(Y - \widehat{Y})^2]}_{\text{prediction variance}} = \mathbb{E}[\left((Y - \mathbb{E}Y) + (\mathbb{E}(Y) - \widehat{Y})\right)^2]$$

$$\approx \underbrace{\mathbb{E}[(Y - \mathbb{E}Y)^2]}_{\text{process variance}} + \underbrace{\mathbb{E}[(\mathbb{E}Y - \widehat{Y})^2]}_{\text{estimation variance}} .$$

- the *process variance* reflects randomness of the random variable
- the *estimation variance* reflects uncertainty of statistical estimation

Process variance of reserves per occurrence year

The amount of reserves for accident year i is simply

$$\widehat{R}_i = \left(\widehat{\lambda}_{n-i} \cdot \widehat{\lambda}_{n-i+1} \cdots \widehat{\lambda}_{n-2} \widehat{\lambda}_{n-1} - 1 \right) \cdot C_{i,n-i}.$$

Since

$$
\begin{aligned}
\text{Var}(\widehat{R}_i | \mathcal{F}_i) &= \text{Var}(C_{i,n} | \mathcal{F}_i) = \text{Var}(C_{i,n} | C_{i,i}) \\
&= \sum_{k=i+1}^{n} \prod_{l=k+1}^{n} \lambda_l^2 \sigma_k^2 \mathbb{E}[C_{i,k} | C_{i,i}]
\end{aligned}
$$

and a natural estimator for this variance is then

$$
\begin{aligned}
\widehat{\text{Var}}(\widehat{R}_i | \mathcal{F}_i) &= \sum_{k=i+1}^{n} \prod_{l=k+1}^{n} \widehat{\lambda}_l^2 \widehat{\sigma}_k^2 \widehat{C}_{i,k} \\
&= \widehat{C}_{i,n} \sum_{k=i+1}^{n} \frac{\widehat{\sigma}_k^2}{\lambda_k^2 \widehat{C}_{i,k}}.
\end{aligned}
$$

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Note that it is possible to get not only the variance of the ultimate cumulate payments, but also the variance of any increment. Hence

$$
\begin{aligned}
\mathrm{Var}(Y_{i,j}|\mathcal{F}_i) &= \mathrm{Var}(Y_{i,j}|C_{i,i}) \\
&= \mathbb{E}[\mathrm{Var}(Y_{i,j}|C_{i,j-1})|C_{i,i}] + \mathrm{Var}[\mathbb{E}(Y_{i,j}|C_{i,j-1})|C_{i,i}] \\
&= \mathbb{E}[\sigma_i^2 C_{i,j-1}|C_{i,i}] + \mathrm{Var}[(\lambda_{j-1} - 1)C_{i,j-1}|C_{i,i}]
\end{aligned}
$$

and a natural estimator for this variance is then

$$
\widehat{\mathrm{Var}}(Y_{i,j}|\mathcal{F}_i) = \widehat{\mathrm{Var}}(C_{i,j}|\mathcal{F}_i) + (1 - 2\widehat{\lambda}_{j-1})\widehat{\mathrm{Var}}(C_{i,j-1}|\mathcal{F}_i)
$$

where, from the expressions given above,

$$
\widehat{\mathrm{Var}}(C_{i,j}|\mathcal{F}_i) = \widehat{C}_{i,i} \sum_{k=i+1}^{j} -1 \frac{\sigma_k^2}{\lambda_k^2 \widehat{C}_{i,k}}.
$$

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Parameter variance when estimating reserves per occurrence year

So far, we have obtained an estimate for the process error of technical risks (increments or cumulated payments). But since parameters λ_j's and σ_j^2 are estimated from past information, there is an additional potential error, also called parameter error (or estimation error). Hence, we have to quantify $\mathbb{E}\left([R_i - \widehat{R}_i]^2\right)$. In order to quantify that error, Murphy (1994) assume the following underlying model,

$$C_{i,j} = \lambda_{j-1} C_{i,j-1} + \eta_{i,j} \tag{1}$$

with independent variables $\eta_{i,j}$. From the structure of the conditional variance,

$$\mathrm{Var}(C_{i,j+1}|\mathcal{F}_{i+j}) = \mathrm{Var}(C_{i,j+1}|C_{i,j}) = \sigma_j^2 \cdot C_{i,j},$$

it is natural to write Equation (**??**) as

$$C_{i,j} = \lambda_{j-1} C_{i,j-1} + \sigma_{j-1} \sqrt{C_{i,j-1}} \varepsilon_{i,j}, \tag{2}$$

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with independent *and centered variables with unit variance* $\varepsilon_{i,j}$. Then

$$\mathbb{E}\left([R_i - \widehat{R}_i]^2|\mathcal{F}_i\right) = \widehat{R}_i^2 \left(\sum_{k=0}^{n-i-1} \frac{\widehat{\sigma}_{i+k}^2}{\widehat{\lambda}_{i+k}^2 \sum C_{\cdot,i+k}} + \frac{\widehat{\sigma}_{n-1}^2}{[\widehat{\lambda}_{n-1} - 1]^2 \sum C_{\cdot,i+k}}\right)$$

Based on that estimator, it is possible to derive the following estimator for the conditional mean square error of reserve prediction for occurrence year i,

$$CMSE_i = \widehat{\mathrm{Var}}(\widehat{R}_i|\mathcal{F}_i) + \mathbb{E}\left([R_i - \widehat{R}_i]^2|\mathcal{F}_i\right).$$

Variance of global reserves (for all occurrence years)

The estimate total amount of reserves is $\widehat{\mathrm{Var}}(\widehat{R}) = \widehat{\mathrm{Var}}(\widehat{R}_1) + \cdots + \widehat{\mathrm{Var}}(\widehat{R}_n)$.

In order to derive the conditional mean square error of reserve prediction, define the covariance term, for $i < j$, as

$$CMSE_{i,j} = \widehat{R}_i \widehat{R}_j \left(\sum_{k=i}^{n} \frac{\widehat{\sigma}_{i+k}^2}{\widehat{\lambda}_{i+k}^2 \sum C_{\cdot,k}} + \frac{\widehat{\sigma}_j^2}{[\widehat{\lambda}_{j-1} - 1]\widehat{\lambda}_{j-1} \sum C_{\cdot,j+k}} \right),$$

then the conditional mean square error of overall reserves

$$CMSE = \sum_{i=1}^{n} CMSE_i + 2 \sum_{j>i} CMSE_{i,j}.$$

Statistical estimation if uncertainty

Under assumptions A_1 and A_2

$$\widehat{\lambda}_k = \frac{\sum_{i=1}^{n-k} C_{i,k+1}}{\sum_{i=1}^{n-k} C_{i,k}} \text{ for all } k = 1, \cdots, n-1.$$

is an unbiased estimator of λ_k.

Further, consider

$$\widehat{\sigma}_k^2 = \frac{1}{n-k-1} \sum_{i=1}^{n-k} C_{i,k} \left(\frac{C_{i,k+1}}{C_{i,k}} - \widehat{\lambda}_k \right)^2,$$

for all $k = 1, \cdots, n-2$. The value of $\widehat{\sigma}_{n-1}^2$ is simply extrapolated, so that

$$\frac{\widehat{\sigma}_{n-3}^2}{\widehat{\sigma}_{n-2}^2} = \frac{\widehat{\sigma}_{n-2}^2}{\widehat{\sigma}_{n-1}^2} \text{ , i.e. } \widehat{\sigma}_{n-1}^2 = \min \left\{ \frac{\sigma_{n-2}^4}{\sigma_{n-3}^2}, \min \left\{ \sigma_{n-3}^2, \sigma_{n-2}^2 \right\} \right\}.$$

Statistical estimation if uncertainty

Consider the initial triangles, then $\widehat{\lambda}_j$'s and $\widehat{\sigma}_j$'s are given by

k	0	1	2	3	4
$\widehat{\lambda}_k$	1.3809	1.0114	1.0043	1.0019	1.0047
$\widehat{\sigma}_k^2$	0.5254	0.1026	0.0021	0.0007	0.0002

A short word on Munich Chain Ladder

Munich chain ladder is an extension of Mack's technique based on paid (P) and incurred (I) losses.

Here we adjust the chain-ladder link-ratios λ_j's depending if the momentary (P/I) ratio is above or below average. It integrated correlation of residuals between P vs. I/P and I vs. P/I chain-ladder link-ratio to estimate the correction factor.

Use standard Chain Ladder technique on the two triangles,

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	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	4752.4
2	3871	5345	5398	5420	5430.1	5455.8
3	4239	5917	6020	6046.15	6057.4	6086.1
4	4929	6794	6871.7	6901.5	6914.3	6947.1
5	5217	7204.3	7286.7	7318.3	7331.9	7366.7

	0	1	2	3	4	5
0	4795	4629	4497	4470	4456	4456
1	5135	4949	4783	4760	4750	4750.0
2	5681	5631	5492	5470	5455.8	5455.8
3	6272	6198	6131	6101.1	6085.3	6085.3
4	7326	7087	6920.1	6886.4	6868.5	6868.5
5	7353	7129.1	6991.2	6927.3	6909.3	6909.3

Hence, we get the following figures

	latest P	latest I	latest P/I	ult. P	ult. I	ult. P/I
0	4456	4456	1.000	4456	4456	1.000
1	4730	4750	0.995	4752	4750	1.000
2	5420	5470	0.990	5455	5455	1.000
3	6020	6131	0.981	6086	6085	1.000
4	6794	7087	0.958	6947	6868	1.011
5	5217	7353	0.709	7366	6909	1.066
total	32637	35247	0.923	35064	34525	1.015

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Bornhuetter Ferguson

One of the difficulties with using the chain ladder method is that reserve forecasts can be quite unstable. The Bornhuetter-Ferguson (1972) method provides a procedure for stabilizing such estimates.

Recall that in the *standard* chain ladder model,

$$\widehat{C}_{i,n} = \widehat{F}_i C_{i,n-i}, \text{ where } \widehat{F}_i = \prod_{k=n-i}^{n-1} \widehat{\lambda}_k$$

Hence, a change of $\alpha\%$ in $C_{i,n-i}$ (due to sampling volatility) will generate a change in the forecast of $\alpha\%$. If $\widehat{R}_i$ denotes the estimated outstanding reserves,

$$\widehat{R}_i = \widehat{C}_{i,n} - C_{i,n-i} = \widehat{C}_{i,n} \cdot \frac{\widehat{F}_i - 1}{\widehat{F}_i}.$$

Bornhuetter Ferguson

Note that this model is a particular case of the family of estimators the the form

$$(1 - Z_i)\widehat{R}_i + Z_i R_i$$

which will be studied afterwards as using credibility theory.

For a bayesian interpretation of the Bornhutter-Ferguson model, England & Verrall (2002) considered the case where incremental paiments $X_{i,j}$ are i.i.d. overdispersed Poisson variables. Here

$$\mathbb{E}(X_{i,j}) = a_i b_j \text{ and } \text{Var}(X_{i,j}) = \varphi a_i b_j,$$

where we assume that $b_1 + \cdots + b_n = 1$. Parameter a_i is assumed to be a drawing of a random variable $A_i \sim \mathcal{G}(\alpha_i, \beta_i)$, so that $\mathbb{E}(A_i) = \alpha_i / \beta_i$, so that

$$\mathbb{E}(C_{i,n}) = \frac{\alpha_i}{\beta_i} = C_i^{\star} \text{ (say)},$$

which is simply a prior expectation of the final loss amount.

Bornhuetter Ferguson

The a posteriori distribution of $X_{i,j+1}$ is then

$$\mathbb{E}(X_{i,j+1}|\text{past observation}) = \left(Z_{i,j+1} C_{i,j} + [1 - Z_{i,j+1}] \frac{C_i^\star}{\widehat{F}_j} \right) \cdot (\lambda_j - 1)$$

where $Z_{i,j+1} = \dfrac{\widehat{F}_j^{-1}}{\beta \varphi + \widehat{F}_j}$, where $\widehat{F}_j = \lambda_{j+1} \cdots \lambda_n$.

Hence, Bornhutter-Ferguson technique can be interpreted as a Bayesian method, and a a credibility estimator (since bayesian with conjugated distributed leads to credibility).

- assume that accident years are independent
- assume that there exist parameters μ_i's and a pattern $\beta_1, \beta_2, \cdots, \beta_n$ with $\beta_n = 1$ such that

$$\mathbb{E}(C_{i,1}) = \beta_1 \mu_i$$

$$\mathbb{E}(C_{i,j+k}|C_{i,1}, \cdots, C_{i,j}) = C_{i,j} + [\beta_{j+k} - \beta_j] \mu_i$$

Hence, one gets that $\mathbb{E}(C_{i,j}) = \beta_j \mu_i$.

The sequence (β_j) denotes the claims development pattern. The Bornhuetter-Ferguson estimator for $\mathbb{E}(C_{i,n}|C_{i,1}, \cdots, C_{i,j})$ is

$$\widehat{C}_{i,n} = C_{i,j} + [1 - \widehat{\beta}_{j-i}]\widehat{\mu}_i$$

where $\widehat{\mu}_i$ is an estimator for $\mathbb{E}(C_{i,n})$.

If we want to relate that model to the classical Chain Ladder one,

$$\beta_j \text{ is } \prod_{k=j+1}^{n} \frac{1}{\lambda_k}$$

Consider the classical triangle. Assume that the estimator $\widehat{\mu}_i$ is a plan value (obtain from some business plan). For instance, consider a 105% loss ratio per accident year.

| i	0	1	2	3	4	5
premium	4591	4692	4863	5175	5673	6431
$\widehat{\mu}_i$	4821	4927	5106	5434	5957	6753
λ_i	$1,380$	$1,011$	$1,004$	$1,002$	$1,005$	
β_i	$0,708$	$0,978$	$0,989$	$0,993$	$0,995$	
$\widehat{C}_{i,n}$	4456	4753	5453	6079	6925	7187
$\widehat{R}_i$	0	23	33	59	131	1970

Boni-Mali

As point out earlier, the (conditional) mean square error of prediction (MSE) is

$$\text{mse}_t(\widehat{X}) \;=\; \mathbb{E}\left([X - \widehat{X}]^2 | \mathcal{F}_t\right)$$

$$=\; \underbrace{Var(X|\mathcal{F}_t)}_{\text{process variance}} \;+\; \underbrace{\left(\mathbb{E}\left(X|\mathcal{F}_t - \widehat{X}\right)\right)^2}_{\text{parameter estimation error}}$$

i.e $\widehat{X}$ is
$$\begin{cases} \text{a predictor for } X \\ \text{an estimator for } \mathbb{E}(X|\mathcal{F}_t). \end{cases}$$

But this is only a a long-term view, since we focus on the uncertainty *over the whole runoff period*. It is not a one-year solvency view, where we focus on changes *over the next accounting year.*

Boni-Mali
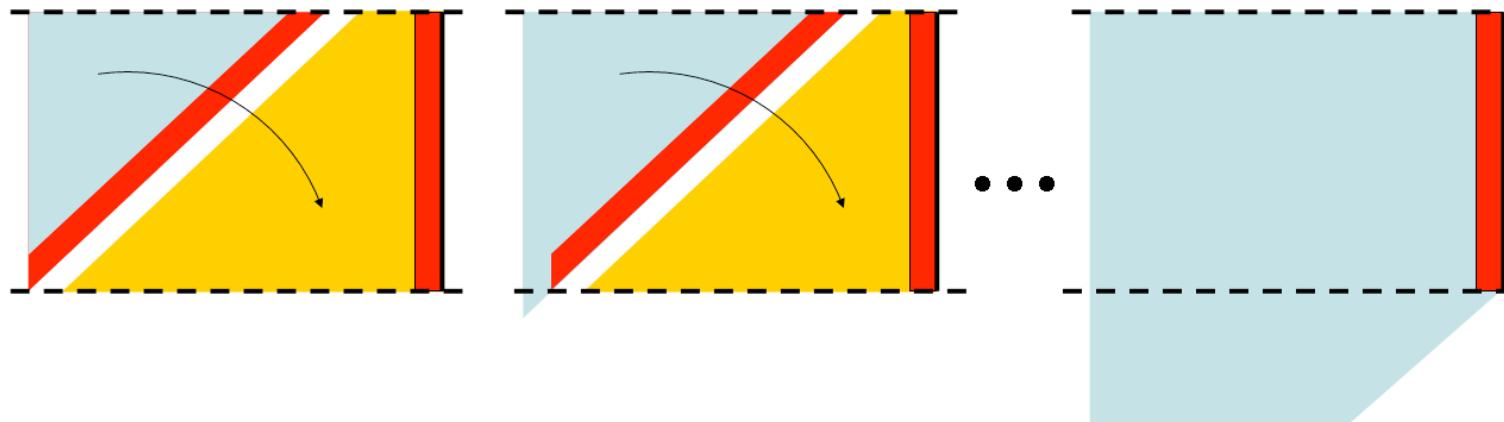
From time $t = n$ and time $t = n + 1$,

$$\widehat{\lambda_j}^{(n)} = \frac{\sum_{i=0}^{n-j-1} C_{i,j+1}}{\sum_{i=0}^{n-j-1} C_{i,j}} \text{ and } \widehat{\lambda_j}^{(n+1)} = \frac{\sum_{i=0}^{n-j} C_{i,j+1}}{\sum_{i=0}^{n-j} C_{i,j}}$$

and the ultimate loss predictions are then

$$\widehat{C}_i^{(n)} = C_{i,n-i} \cdot \prod_{j=n-i}^{n} \widehat{\lambda_j}^{(n)} \text{ and } \widehat{C}_i^{(n+1)} = C_{i,n-i+1} \cdot \prod_{j=n-i+1}^{n} \widehat{\lambda_j}^{(n+1)}$$

Boni-Mali

In order to study the one-year claims development, we have to focus on

$$\widehat{R}_i^{(n)} \text{ and } X_{i,n-i+1} + \widehat{R}_i^{(n+1)}$$

The boni-mali for accident year i, from year n to year $n+1$ is then

$$\widehat{BM}_i^{(n,n+1)} = \widehat{R}_i^{(n)} - \left[X_{i,n-i+1} + \widehat{R}_i^{(n+1)} \right] = \widehat{C}_i^{(n)} - \widehat{C}_i^{(n+1)}.$$

Thus, the conditional one-year runoff uncertainty is

$$\widehat{mse}(\widehat{BM}_i^{(n,n+1)}) = \mathbb{E}\left(\left[\widehat{C}_i^{(n)} - \widehat{C}_i^{(n+1)} \right]^2 | \mathcal{F}_n \right)$$

Boni-Mali

Hence,

$$
\widehat{mse}(\widehat{BM}_i^{(n,n+1)}) \;=\; [\widehat{C}_i^{(n)}]^2 \left[\frac{\widehat{\sigma}_{n-i}^2/[\widehat{\lambda}_{n-i}^{(n)}]^2}{C_{i,n-i}} + \frac{\widehat{\sigma}_{n-i}^2/[\widehat{\lambda}_{n-i}^{(n)}]^2}{\sum_{k=0}^{i-1} C_{k,n-i}} \right.
$$

$$
\left. + \sum_{j=n-i+1}^{n-1} \frac{C_{n-j,j}}{\sum_{k=0}^{n-j} C_{k,j}} \cdot \frac{\widehat{\sigma}_j^2/[\widehat{\lambda}_j^{(n)}]^2}{\sum_{k=0}^{n-j-1} C_{k,j}} \right]
$$

Further, it is possible to derive the MSEP for aggregated accident years (see Merz & Wüthrich (2008)).

From Chain Ladder to London Chain and London Pivot

La méthode dite London Chain a été introduite par Benjamin et Eagles, dans *Reserves in Lloyd's and the London Market* (1986). On suppose ici que la dynamique des $(C_{ij})_{j=1,..,n}$ est donnée par un modèle de type $AR(1)$ avec constante, de la forme

$$C_{i,k+1} = \lambda_k C_{ik} + \alpha_k \text{ pour tout } i, k = 1, .., n$$

De façon pratique, on peut noter que la méthode standard de Chain Ladder, reposant sur un modèle de la forme $C_{i,k+1} = \lambda_k C_{ik}$, ne pouvait être appliqué que lorsque les points $(C_{i,k}, C_{i,k+1})$ sont sensiblement alignés (*à k fixé*) sur une droite passant par l'origine. La méthode London Chain suppose elle aussi que les points soient alignés sur une même droite, mais on ne suppose plus qu'elle passe par 0.

Example :On obtient la droite passant au mieux par le nuage de points et par 0, et la droite passant au mieux par le nuage de points.

From Chain Ladder to London Chain and London Pivot

Dans ce modèle, on a alors $2n$ paramètres à identifier : λ_k et α_k pour $k = 1, ..., n$. La méthode la plus naturelle consiste à estimer ces paramètres à l'aide des moindres carrés, c'est à dire que l'on cherche, pour tout k,

$$\left(\widehat{\lambda}_k, \widehat{\alpha}_k\right) = \arg\min\left\{\sum_{i=1}^{n-k} \left(C_{i,k+1} - \alpha_k - \lambda_k C_{i,k}\right)^2\right\}$$

ce qui donne, finallement

$$\widehat{\lambda}_k = \frac{\frac{1}{n-k}\sum_{i=1}^{n-k} C_{i,k}C_{i,k+1} - \overline{C}_k^{(k)}\overline{C}_{k+1}^{(k)}}{\frac{1}{n-k}\sum_{i=1}^{n-k} C_{i,k}^2 - \overline{C}_k^{(k)2}}$$

où $\overline{C}_k^{(k)} = \dfrac{1}{n-k}\sum_{i=1}^{n-k} C_{i,k}$ et $\overline{C}_{k+1}^{(k)} = \dfrac{1}{n-k}\sum_{i=1}^{n-k} C_{i,k+1}$

et où la constante est donnée par $\widehat{\alpha}_k = \overline{C}_{k+1}^{(k)} - \widehat{\lambda}_k\overline{C}_k^{(k)}$.

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From Chain Ladder to London Chain and London Pivot

Dans le cas particulier du triangle que nous étudions, on obtient

k	0	1	2	3	4
$\widehat{\lambda}_k$	1.404	1.405	1.0036	1.0103	1.0047
$\widehat{\alpha}_k$	-90.311	-147.27	3.742	-38.493	0

From Chain Ladder to London Chain and London Pivot

The completed (cumulated) triangle is then

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420		
3	4239	5917	6020			
4	4929	6794				
5	5217					

From Chain Ladder to London Chain and London Pivot

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	4752
2	3871	5345	5398	5420	5437	5463
3	4239	5917	6020	6045	6069	6098
4	4929	6794	6922	6950	6983	7016
5	5217	7234	7380	7410	7447	7483

One the triangle has been completed, we obtain the amount of reserves, with respectively 22, 43, 78, 222 and 2266 per accident year, i.e. the total is 2631 (to be compared with 2427, obtained with the Chain Ladder technique).

From Chain Ladder to London Chain and London Pivot

La méthode dite London Pivot a été introduite par Straub, dans *Nonlife Insurance Mathematics* (1989). On suppose ici que $C_{i,k+1}$ et $C_{i,k}$ sont liés par une relation de la forme

$$C_{i,k+1} + \alpha = \lambda_k . (C_{i,k} + \alpha)$$

(de façon pratique, les points $(C_{i,k}, C_{i,k+1})$ doivent être sensiblement alignés (*à k fixé*) sur une droite passant par le point dit pivot $(-\alpha, -\alpha)$). Dans ce modèle, $(n+1)$ paramètres sont alors a estimer, et une estimation par moindres carrés ordinaires donne des estimateurs de façon itérative.

Introduction to factorial models : Taylor (1977)

This approach was studied in a paper entitled *Separation of inflation and other effects from the distribution of non-life insurance claim delays*

We assume the incremental payments are functions of two factors, one related to accident year i, and one to calendar year $i + j$. Hence, assume that

$$Y_{ij} = r_j \mu_{i+j-1} \text{ for all } i, j$$

$$
\begin{array}{cccccc}
r_1\mu_1 & r_2\mu_2 & \cdots & r_{n-1}\mu_{n-1} & r_n\mu_n \\
r_1\mu_2 & r_2\mu_3 & \cdots & r_{n-1}\mu_n \\
\vdots & \vdots \\
r_1\mu_{n-1} & r_2\mu_n \\
r_1\mu_n
\end{array}
\qquad
\begin{array}{cccccc}
r_1\mu_1 & r_2\mu_2 & \cdots & r_{n-1}\mu_{n-1} & r_n\mu_n \\
r_1\mu_2 & r_2\mu_3 & \cdots & r_{n-1}\mu_n \\
\vdots & \vdots \\
r_1\mu_{n-1} & r_2\mu_n \\
r_1\mu_n
\end{array}
$$
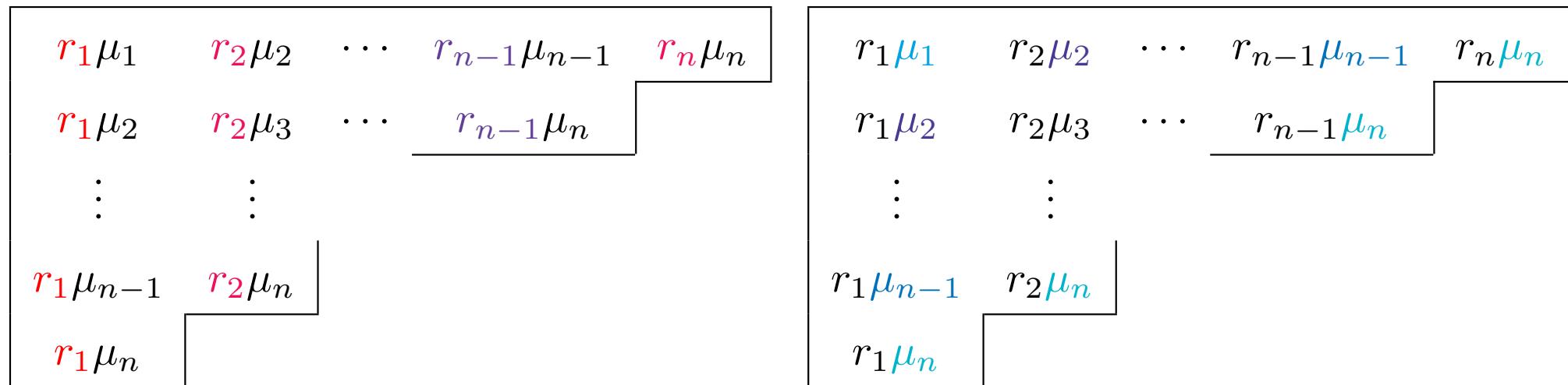
Hence, incremental payments are functions of development factors, r_j, and a calendar factor, μ_{i+j-1}, that might be related to some inflation index.

Introduction to factorial models : Taylor (1977)

In order to identify factors $r_1, r_2, .., r_n$ and $\mu_1, \mu_2, ..., \mu_n$, i.e. $2n$ coefficients, an additional constraint is necessary, e.g. on the r_j's, $r_1 + r_2 + ... + r_n = 1$ (this will be called arithmetic separation method). Hence, the sum on the latest diagonal is

$$d_n = Y_{1,n} + Y_{2,n-1} + ... + Y_{n,1} = \mu_n \left(r_1 + r_2 + ... + r_k \right) = \mu_n$$

On the first sur-diagonal

$$d_{n-1} = Y_{1,n-1} + Y_{2,n-2} + ... + Y_{n-1,1} = \mu_{n-1} \left(r_1 + r_2 + ... + r_{n-1} \right) = \mu_{n-1} \left(1 - r_n \right)$$

and using the nth column, we get $\gamma_n = Y_{1,n} = r_n \mu_n$, so that

$$r_n = \frac{\gamma_n}{\mu_n} \text{ and } \mu_{n-1} = \frac{d_{n-1}}{1 - r_n}$$

More generally, it is possible to iterate this idea, and on the ith sur-diagonal,

$$
\begin{aligned}
d_{n-i} &= Y_{1,n-i} + Y_{2,n-i-1} + ... + Y_{n-i,1} = \mu_{n-i} \left(r_1 + r_2 + ... + r_{n-i} \right) \\
&= \mu_{n-i} \left(1 - \left[r_n + r_{n-1} + ... + r_{n-i+1} \right] \right)
\end{aligned}
$$

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Introduction to factorial models : Taylor (1977)

and finally, based on the $n - i + 1$th column,

$$
\begin{aligned}
\gamma_{n-i+1} &= Y_{1,n-i+1} + Y_{2,n-i+1} + ... + Y_{i-1,n-i+1} \\
&= r_{n-i+1}\mu_{n-i+1} + ... + r_{n-i+1}\mu_{n-1} + r_{n-i+1}\mu_n
\end{aligned}
$$

$$
r_{n-i+1} = \frac{\gamma_{n-i+1}}{\mu_n + \mu_{n-1} + ... + \mu_{n-i+1}} \text{ and } \mu_{k-i} = \frac{d_{n-i}}{1 - [r_n + r_{n-1} + ... + r_{n-i+1}]}
$$

k	1	2	3	4	5	6
μ_k	4391	4606	5240	5791	6710	7238
r_k in %	73.08	25.25	0.93	0.32	0.12	0.29

Introduction to factorial models : Taylor (1977)

The challenge here is to forecast forecast values for the μ_k's. Either a linear model or an exponential model can be considered.
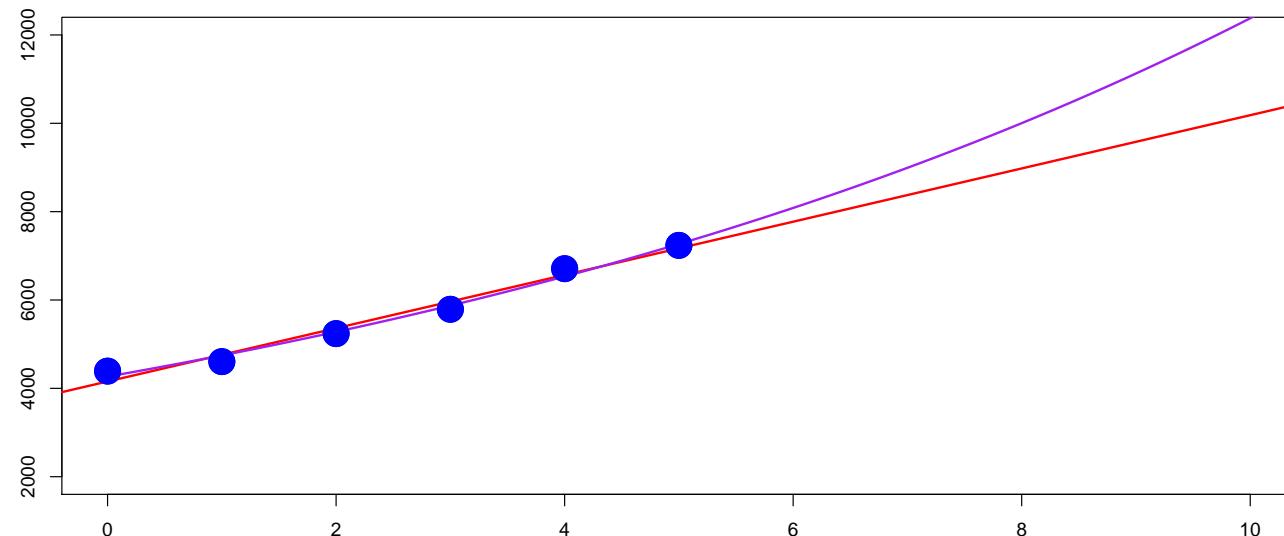
Lemaire (1982) and autoregressive models

Instead of a *simple* Markov process, it is possible to assume that the $C_{i,j}$'s can be modeled with an autorgressive model in two directions, rows and columns,

$$C_{i,j} = \alpha C_{i-1,j} + \beta C_{i,j-1} + \gamma.$$

Zehnwirth (1977)

Here, we consider the following model for the $C_{i,j}$'s

$$C_{i,j} = \exp\left(\alpha_i + \gamma_i \cdot j\right)\left(1 + j\right)^{\beta_i},$$

which can be seen as an extended Gamma model. α_i is a scale parameter, while β_i and γ_i are shape parameters. Note that

$$\log C_{i,j} = \alpha_i + \beta_i \log\left(1 + j\right) + \gamma_i \cdot j.$$

For convenience, we usually assume that $\beta_i = \beta$ et $\gamma_i = \gamma$.

Note that if $\log C_{i,j}$ is assume to be Gaussian, then $C_{i,j}$ will be lognormal. But then, estimators one the $C_{i,j}$'s while be overestimated.

Zehnwirth (1977)

Assume that $\log C_{i,j} \sim \mathcal{N}\left(X_{i,j}\beta, \sigma^2\right)$, then, if parameters were obtained using maximum likelihood techniques

$$
\begin{aligned}
\mathbb{E}\left(\widehat{C}_{i,j}\right) &= \mathbb{E}\left(\exp\left(X_{i,j}\widehat{\beta} + \frac{\widehat{\sigma}^2}{2}\right)\right) \\
&= C_{i,j}\exp\left(-\frac{n-1}{n}\frac{\sigma^2}{2}\right)\left(1 - \frac{\sigma^2}{n}\right)^{-\frac{n-1}{2}} > C_{i,j},
\end{aligned}
$$

Further, the homoscedastic assumption might not be relevant. Thus Zehnwirth suggested

$$
\sigma_{i,j}^2 = Var\left(\log C_{i,j}\right) = \sigma^2\left(1 + j\right)^h .
$$

Regression and reserving

De Vylder (1978) proposed a least squares factor method, i.e. we need to find $\boldsymbol{\alpha} = (\alpha_0, \cdots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_0, \cdots, \beta_n)$ such

$$(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}) = \operatorname{argmin} \sum_{i,j=0}^{n} (X_{i,j} - \alpha_i \times \beta_j)^2,$$

or equivalently, assume that

$$X_{i,j} \sim \mathcal{N}(\alpha_i \times \beta_j, \sigma^2), \text{ independent.}$$

A more general model proposed by De Vylder is the following

$$(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\gamma}}) = \operatorname{argmin} \sum_{i,j=0}^{n} (X_{i,j} - \alpha_i \times \beta_j \times \gamma_{i+j-1})^2.$$

In order to have an identifiable model, De Vylder suggested to assume $\gamma_k = \gamma^k$ (so that this coefficient will be related to some inflation index).