Gauss's Proof

Gauss starts with a polynomial with real coefficients

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

All real polynomials can be reduced into a product of linear and irreducible quadratic factors. For appropriate values of r and ϕ, a linear factor can be expressed as $z \pm r, r \geq 0$ and a quadratic factor can be expressed as $z^2 + 2r\cos\phi z + r^2, r > 0$ with complex roots $r(\cos\phi \pm i\sin\phi)$. Substituting $z = r(\cos\phi + i\sin\phi)$ we define 2 new polynomials

$$U(r,\phi) = a_0 + a_1\cos(\phi)r + a_2\cos(2\phi)r^2 + \cdots + a_n\cos(n\phi)r^n$$
$$T(r,\phi) = a_1\sin(\phi)r + a_2\sin(2\phi)r^2 + \cdots + a_n\sin(n\phi)r^n$$

where $U(r,\phi) = Re(P(z))$ and $T(r,\phi) = Im(P(z))$.
Consider the curves $U = 0$ and $T = 0$. Our goal is find the intersection of these two curves within a sufficiently large circle of radius R. For if we can determine (r,ϕ) such that $U = 0$ and $T = 0$ simultaneously, then $z \pm r \mid P(z)$ or $z^2 + 2r\cos\phi z + r^2 \mid P(z)$.
Gauss observes the behavior of the two curves inside the circle. By Bezout's lemma, $U = 0$ and $T = 0$ should each intersect the circle $2n$ times for a total of $4n$ intersections with the circle. Gauss divides the circumference of circle into

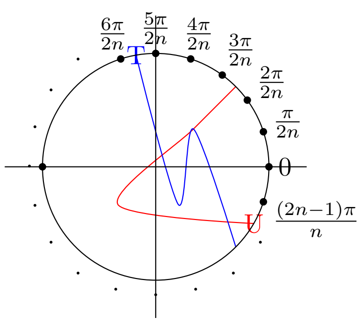

Figure 1: Division of circle into sectors each with an alternating intersection with T or U.

$4n$ arcs for each of the intersections. Gauss observes the intersections alternate between U and T labeling U's the odd intersections and T's the even. Since the curves do not stop abruptly, all branches of the curves must intersect the circle at two distinct points. Figure 1 shows how the circle is divided into arcs and an intersecting branch of $U = 0$ and $T = 0$. He uses the fact that the intersections alternate and that the branches are connected at two points to construct a geometric proof by contradiction to show that T and U must intersect inside the circle.