Math 114L: Problem Set 1

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(Note: The Problems are numbered strangely in the hw assignment. I tried to make it clear which question I was answering, but let me know if there is anything unclear.)

Problem 1:

$(p \vee q)$ $\neg p \rightarrow q$

p	q	$(p \vee q)$	$(\neg p)$	$(\neg p) \rightarrow q$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	F

$(p \wedge q)$ $\neg(p \rightarrow \neg q)$

p	q	$(p \wedge q)$	$(p \rightarrow \neg q)$	$\neg(p \rightarrow \neg q)$
T	T	T	F	T
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

Problem 2:

By induction:
First, we show that we can construct a string of length $3k + 1$ for some non-negative integer k.
Base Case: We are given $A_i \in PL_0$. It follows we can construct a string of length 1.
Induction hypothesis: Assume for some k, we can construct a string $\phi \in PL_0$

of length $3k+1$.
Induction step: By (a) we can construct $(\neg\phi) \in PL_0$ of length $3(k+1)+1$.
Thus, any string of length $3k+1$ can be constructed. Next, we show that we can construct a string of length $3k+2$ for any $k \geq 1$
Given $\phi, \psi \in PL_0$ of lengths $3(k-1)+1$ and 1 with $k \geq 1$, we can construct $(\phi \to \psi)$ of length $3k+2$.
Once, again, we can use a similar argument to show we can construct a string of length $3k$ for $k \geq 3$ $(3(k-2)+2)+3+1 = 3k$ where $k \geq 3$ using strings ϕ, ψ of length $3(k-2)+2$ and 1 $(\phi \to \psi)$.
Hence, we can construct a string of length n for every natural number except $2, 3$, and 6.

Problem 3:

Suppose $\phi \in PL_0$. Then, it can be defined recursively by the 3 clauses. It follows ϕ must either be a propositional variable, there exists another variable $\psi \in PL_0$ s.t $\phi = (\neg\psi)$, or there exist $\psi, \chi \in PL_0$ s.t $\phi = (\psi \to \chi)$ because all other strings are not formulas. Hence, we can construct a finite sequence of sequences s.t each element satisifies one of the three conditions.
Suppose there is a finite sequence of sequences $(\phi_1, \ldots, \phi_n)$ such that $\phi_n = \phi$ and for each $i \leq n$ either there exists m s.t $\phi_i = A_m$ or there exists $j < i$ such that $\phi_i = (\neg\phi_j)$ or there exist j_1, j_2 both less than i such that $\phi_i = (\phi_{j_1} \to \phi_{j_2})$.
We will show by strong induction, $\phi_i \in PL_0$ for all $i \leq n$.
Base case: For $i = 1$, neither the second nor the third property can hold because ϕ_1 is the first element of the sequence. Thus, $\phi_1 = A_m \in PL_0$ for some m.
Induction hypothesis: Assume for $1 \leq i < n$ $\phi_1, \ldots, \phi_i \in PL_0$.
Induction step: Suppose $\phi_{i+1} = A_m$ for some m, then $\phi_{i+1} \in PL_0$ because $A_m \in PL_0$. Suppose there exists some $j < i+1$ s.t $\phi_{i+1} = (\neg\phi_j)$. Because $\phi_j \in PL_0$ and PL_0 is closed under the connective $\neg$, $\phi_{i+1} \in PL_0$. Suppose there exist j_1, j_2 both less than $i+1$ such that $\phi_{i+1} = (\phi_{j_1} \to \phi_{j_2})$. Because $\phi_{j_1}, \phi_{j_2} \in PL_0$ and PL_0 is closed under the connective $\to$, $\phi_{i+1} \in PL_0$.
Hence, by induction, $\phi_i \in PL_0$ for all $i \leq n$. Moreover, $\phi = \phi_n \in PL_0$

Definition 2 Proofs:

(i) Proof by induction on E
Base case: $\phi = A_i$. It follows $E(\phi) = 1$ and $D(\phi) = 0$, so $E(\phi) = D(\phi)+1$
Induction hypothesis: Assume for some n, m we have some $\phi, \psi \in PL_0$ s.t $E(\phi) = n$, $E(\psi) = m$ and $E(\phi) = D(\phi)+1, E(\psi) = D(\psi)+1$.
Induction step: Any element of PL_0 can be constructed recursively from logical connectives and atomic propositions. Consider the statements $\phi' = (\neg\phi)$ and $\chi = (\phi \to \psi)$.
$E(\phi') = E(\phi) = n$ and $D(\phi) = D(\phi) = n-1$ because we neither add a binary connective nor a atomic proposition, so $E(\phi') = D(\phi')+1$.

$E(\chi) = E(\phi \to \psi) = E(\phi) + E(\psi) = n + m$ and $D(\chi) = D(\phi \to \psi) = D(\phi) + D(\psi) + 1 = n + m - 1$, so $E(\chi) = D(\chi) + 1$
Hence, $E(\phi) = D(\phi) + 1$ for any $\phi, \psi \in PL_0$

(ii) Proof by induction on S
Base case: $\phi = A_i$. $S(\phi) = 1 \geq 3 \cdot 0 = 3C(\phi)$
Induction hypothesis: Assume for some $n, m, p \neq 2, 3, 6$ we have some $\phi, \psi, \chi \in PL_0$ s.t $S(\phi) = n, S(\psi) = m, S(\chi) = p$ and $S(\phi) \geq 3C(\phi), S(\psi) \geq 3C(\psi), S(\chi) \geq 3C(\chi)$.
Induction step: Let $\phi' = (\neg\phi), \psi' = (\psi \to \chi)$.
$S(\phi') = S(\phi) + 3 \geq 3C(\phi) + 3 = 3(C(\phi) + 1) = 3(C(\phi'))$.
$S(\psi') = S(\psi) + S(\chi) + 3 \geq 3C(\psi) + 3C(\chi) + 3 = 3(C(\psi) + C(\chi) + 1) = 3(C(\psi'))$. In problem 2, we showed that we can construct strings in PL_0 for any length $\neq 2, 3, 6$, so we can choose values of n, m, p to construct new strings of any length.

Problem 1.1

Let L be the set of all formulas. Suppose $\alpha, \beta \in L$ are some formulas in L. It follows by the definition of a formula $\alpha, \beta \in S$ for every propositionally closed set S. Thus, $L \subseteq S$. If $\alpha \in S$ then $(\neg\alpha) \in S$, and if $\alpha, \beta \in S$ and $\bullet$ is a binary connective then $(\alpha \bullet \beta) \in S$. Since this holds for every propositionally closed set $(\neg\alpha)$ and $(\alpha \bullet \beta)$ are formulas. Hence, L is propositionally closed. Since $L \subseteq S$ for every propositionally closed set S, L is also the smallest propositionally closed set.

Problem 1.6

p	q	$(p \to q)$	$(q \to p)$	$\neg(q \to p)$	$(p \to q) \land \neg(q \to p)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	F	F

Problem 1.7

$(\phi \downarrow \phi) \quad (\neg\phi)$

ϕ	$(\phi \downarrow \phi)$	$\neg\phi$
T	F	F
F	T	T

$(\phi \downarrow \phi) \downarrow (\psi \downarrow \psi) \quad (\phi \wedge \psi)$

ϕ	ψ	$(\phi \downarrow \phi) \downarrow (\psi \downarrow \psi)$	$(\phi \wedge \psi)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$(\phi \downarrow \psi) \downarrow (\phi \downarrow \psi) \quad (\phi \vee \psi)$

ϕ	ψ	$(\phi \downarrow \psi) \downarrow (\phi \downarrow \psi)$	$(\phi \vee \psi)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

$((\phi \downarrow \phi) \downarrow \psi) \downarrow ((\phi \downarrow \phi) \downarrow \psi) \quad (\phi \to \psi)$

ϕ	ψ	$((\phi \downarrow \phi) \downarrow \psi) \downarrow ((\phi \downarrow \phi) \downarrow \psi)$	$(\phi \to \psi)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

(a) Every propositional variable A_i is a formula.
(b) If ϕ and ψ are formulas then
$(\phi \downarrow \phi) \quad (\neg\phi)$
$(\phi \downarrow \phi) \downarrow (\psi \downarrow \psi) \quad (\phi \wedge \psi)$
$(\phi \downarrow \psi) \downarrow (\phi \downarrow \psi) \quad (\phi \vee \psi)$
$((\phi \downarrow \phi) \downarrow \psi) \downarrow ((\phi \downarrow \phi) \downarrow \psi) \quad (\phi \to \psi)$
are formulas
(c) No string is a formula except by virtue of (a) and (b).
Proof by Induction on n
Base case: $n = 1$ There are only four unary bit functions, and each of them is written below, relative to the variable p:
$f_1(x) = 1 \quad (p \downarrow (p \downarrow p)) \downarrow (p \downarrow (p \downarrow p))$
$f_2(x) = 0 \quad (p \downarrow p) \downarrow ((p \downarrow p) \downarrow (p \downarrow p))$
$f_3(x) = x \quad p$
$f_4(x) = 1 - x \quad (p \downarrow p)$
Induction hypothesis: Assume every n-ary bit function can be defined by a $\downarrow$-formula with n propositional variables.
Induction step: Suppose f is (n+1)-ary. Consider the two functions obtained by

fixing the last variable of f to be 0 or 1 and choose by the induction hypothesis formulas which define them relative to the variables $p_1, \dots, p_n$:
$f_1(x_1, \dots, x_n) = f(x_1, \dots, x_n, 1)$ defined by ϕ_1
$f_0(x_1, \dots, x_n) = f(x_1, \dots, x_n, 0)$ defined by ϕ_0
$\alpha = ((p_{n+1} \downarrow p_{n+1}) \downarrow (\phi_1 \downarrow \phi_1))$
$\beta = (((p_{n+1} \downarrow p_{n+1}) \downarrow (p_{n+1} \downarrow p_{n+1})) \downarrow (\phi_0 \downarrow \phi_0))$
Using lemma $2A.1$ to check that if p_{n+1} is a new propositional variable, then the formula
$((\alpha \downarrow \beta) \downarrow (\alpha \downarrow \beta))$
defines f relative to the list $p_1, \dots, p_n, p_{n+1}$.