Math 151A: Problem Set 5

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Instructions:

- Due on Friday, May 19th by 1pm.
- Late HW will not be accepted.
- Write down all of the details and attach your code to the end of the assignment for full credit (as a PDF).
- If you LaTeX your solutions, you will get 5% extra credit.
- (T) are "pencil-and-paper" problems and (C) means that the problem includes a computational/programming component.

Problem 1: (T) Numerical Differentiation

Using the Lagrange polynomial approximation, we can show that:

$$f^{(4)}(x_0) = \frac{f(x_0+2h) - 4f(x_0+h) + 6f(x_0) - 4f(x_0-h) + f(x_0-2h)}{h^4} + O(h^2).$$

In this problem, you will derive this formula using the method of undetermined coefficients. Start with the following expression:

$$Af(x_0+2h) + Bf(x_0+h) + Cf(x_0) + Df(x_0-h) + Ef(x_0-2h).$$

Expand each term using Taylor polynomials of degree 5 (where the sixth order term $O(h^6)$ is the error) and choose the coefficients in order to get an approximation to the fourth derivative.

Hint: You will need to solve a 5-by-5 linear system. Once you have the coefficients, check that the sixth equation (corresponding to the terms involving "$h^5 f^{(5)}(x_0)$") is zero.

Solution:

$f(x_0 \pm h) = f(x_0) \pm h \cdot f'(x_0) + \frac{h^2 \cdot f^{(2)}(x_0)}{2} \pm \frac{h^3 \cdot f^{(3)}(x_0)}{6} + \frac{h^4 \cdot f^{(4)}(x_0)}{24} \pm \frac{h^5 \cdot f^{(5)}(x_0)}{120} + \frac{h^6 \cdot f^{(6)}(\xi_1)}{720}$

$f(x_0 \pm 2h) = f(x_0) \pm 2h \cdot f'(x_0) + \frac{4h^2 \cdot f^{(2)}(x_0)}{2} \pm \frac{8h^3 \cdot f^{(3)}(x_0)}{6} + \frac{16h^4 \cdot f^{(4)}(x_0)}{24} \pm \frac{32h^5 \cdot f^{(5)}(x_0)}{120} + \frac{64h^6 \cdot f^{(6)}(\xi_2)}{720}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & -1 & -2 \\ 4 & 1 & 0 & 1 & 4 \\ 8 & 1 & 0 & -1 & -8 \\ 16 & 1 & 0 & 1 & 16 \end{bmatrix} \times \begin{bmatrix} \frac{1}{h^4} \\ -\frac{4}{h^4} \\ \frac{6}{h^4} \\ -\frac{4}{h^4} \\ \frac{1}{h^4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{h^4} \end{bmatrix}$$

Checking to see if the coefficients for $f^{(5)}(x_0)$ sum to zero. $\frac{h \cdot f^{(5)}(x_0)}{120}(1 \cdot 32 - 4 \cdot 1 + 6 \cdot 0 + 4 \cdot 1 - 1 \cdot 32) = 0$

Since they do, our error term comes from the sixth order term,

$\Rightarrow f^{(4)}(x_0) = \frac{f(x_0+2h)-4f(x_0+h)+6f(x_0)-4f(x_0-h)+f(x_0-2h)}{h^4} + (O)(h^2)$

Problem 2: (T) Richardson's Extrapolation

We can use lower-order formulae to generate approximations with higher accuracy. Consider the finite difference formula:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f^{(2)}(x_0) - \frac{h^2}{6} f^{(3)}(x_0) + O(h^3) \tag{1}$$

which includes more truncation terms than we used in the class.

a) Replacing h with $2h$ in equation (1) yields the following formula:

$$f'(x_0) = \frac{f(x_0+2h) - f(x_0)}{2h} - h f^{(2)}(x_0) - \frac{2h^2}{3} f^{(3)}(x_0) + O(h^3). \tag{2}$$

Using equations (1) and (2), show that :

$$f'(x_0) = \frac{-f(x_0+2h) + 4f(x_0+h) - 3f(x_0)}{2h} + \frac{h^2}{3} f^{(3)}(x_0) + O(h^3)$$

b) Repeat the process from Part (a) with:

$$f'(x_0) = \frac{-f(x_0+2h) + 4f(x_0+h) - 3f(x_0)}{2h} + \frac{h^2}{3} f^{(3)}(x_0) + O(h^3)$$

to show:

$$f'(x_0) = \frac{f(x_0+4h) - 12f(x_0+2h) + 32f(x_0+h) - 21f(x_0)}{12h} + O(h^3).$$

Solution:

a) Equation (2) subtracted from $2\times$ equation (1) yields:
$2(\frac{f(x_0+h)-f(x_0)}{h} - \frac{h}{2} f^{(2)}(x_0) - \frac{h^2}{6} f^{(3)}(x_0) + O(h^3)) - (\frac{f(x_0+2h)-f(x_0)}{2h} - h f^{(2)}(x_0) - \frac{2h^2}{3} f^{(3)}(x_0) + O(h^3))$
$= \frac{4f(x_0+h)-4f(x_0)}{2h} - 2\frac{h}{2} f^{(2)}(x_0) - 2\frac{h^2}{6} 2 f^{(3)}(x_0) + 2O(h^3) - (\frac{f(x_0+2h)-f(x_0)}{2h} - h f^{(2)}(x_0) - \frac{2h^2}{3} f^{(3)}(x_0) + O(h^3))$
$= \frac{4f(x_0+h)-3f(x_0)-f(x_0+2h)}{2h} + \frac{h^2}{3} f^{(3)}(x_0) + O(h^3)$ by adding like terms.

b) Replacing h with $2h$ for the function

$$f'(x_0) = \frac{-f(x_0+2h) + 4f(x_0+h) - 3f(x_0)}{2h} + \frac{h^2}{3} f^{(3)}(x_0) + O(h^3) \tag{3}$$

we obtain

$$f'(x_0) = \frac{-f(x_0+4h) + 4f(x_0+2h) - 3f(x_0)}{4h} + \frac{4h^2}{3} f^{(3)}(x_0) + O(h^3) \tag{4}$$

$$\begin{bmatrix} -18 & -9 \\ 24 & 0 \\ -6 & 12 \\ 0 & -3 \end{bmatrix} \times \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -21 \\ 32 \\ -12 \\ 1 \end{bmatrix}$$

Subtracting $\frac{1}{3}$ equation (4) from $\frac{4}{3}$ equation (3) we obtain
$f'(x_0) = \frac{f(x_0+4h)-12f(x_0+2h)+32f(x_0+h)-21f(x_0)}{12h} + O(h^3)$.

Problem 3: (T) An Application to Parameter Estimation, Population Data

Consider the logistic growth model commonly used in biology, demography, probability, sociology, etc.:

$$\frac{d}{dt}f(t) = r\left(f(t) - f(t)^2\right)$$

where $r > 0$ is the constant growth rate parameter. Suppose we are given data on a population (in millions) over the last few years:

t	2011	2012	2013	2014	2015
$f(t)$	0.33000	0.33443	0.33890	0.34340	0.34792

and would like to fit the model to the data. Using the ordinary differential equation at $t = 2013$ and the forward difference (2-point right-sided approximation), the central difference (3-point centered approximation), and a 5-point approximation to the derivative (check the textbook), approximate the value of r (total of 3 approximations).

Solution:

Forward $f'(t) = \frac{f(t+h)-f(t)}{h} + O(h) \Rightarrow r = \frac{f(2014)-f(2013)}{f(2013)+f(2013)^2} = \frac{0.34340-0.33890}{0.33890+0.11485321} = 0.009917$

Central $f'(t) = \frac{f(t+h)-f(t-h)}{2t} + O(h^2) \Rightarrow r = \frac{f(2014)-f(2012)}{2(f(2013)+f(2013)^2)} = \frac{0.34340-0.33433}{2(0.33890+0.11485321)} = 0.009994$

5-Point $f'(t) = \frac{1}{12h}[f(t-2h) - 8f(t-h) + 8f(t+h) - f(t+2h) + O(h^4)] \Rightarrow r = \frac{f(2011)-8f(2012)+8f(2014)-f(2015)}{12(f(2013)+f(2013)^2)} = \frac{0.33000-8\cdot 0.33443+8\cdot 0.34340-0.34792}{12(0.33890+0.11485321)} = 0.009888$

Problem 4: (C) Finite Differences

Write a program that computes an approximation to the first derivatives of:

a) $f(x) = (x-1)^3$

b) $f(x) = (x-1)^2 \sin(5x)$

over the interval $x \in [-1, 1]$ using the forward, backward, and central differences with $h = 0.01$ (i.e. the grid starts at left endpoint -1 with spacing h up to the right endpoint 1). **Provide 6 plots** (i.e. your derivative approximation vs x), corresponding to each of the combinations. For credit, you must label the plots.

Solution:

5/18/23 5:15 PM /Users/theelusiveg.../finite_differences.m 1 of 1

```
% Find approximation to the first derivatives of the functions f(x)=(x-1)^3
% and g(x)=sin(5x)(x-1)^3 using the forward, backward and central
% differences.

clc;
clear all;

F_1 = @(x) sin(5*x).*power(x-1,2);
G_1 = @(x) power(x-1,3);
x=linspace(-1,1,200);
F=F_1(x);
G=G_1(x);

h=x(2)-x(1);
xCentral=x(2:end-1);
dFCentral=(F(3:end)-F(1:end-2))/(2*h);
dGCentral=(G(3:end)-G(1:end-2))/(2*h);
xForward=x(1:end-1);
dFForward=(F(2:end)-F(1:end-1))/h;
dGForward=(G(2:end)-G(1:end-1))/h;
xBackward=x(2:end);
dFBackward=(F(2:end)-F(1:end-1))/h;
dGBackward=(G(2:end)-G(1:end-1))/h;
hold on
tiledlayout(3,2);
nexttile
plot(xCentral,dFCentral,'r')
legend('F-Central')
nexttile
plot(xForward,dFForward,'k');
legend('F-Forward')
nexttile
plot(xBackward,dFBackward,'g');
legend('F-Backward')
nexttile
plot(xCentral,dGCentral,'b')
legend('G-Central')
nexttile
plot(xForward,dGForward,'c');
legend('G-Forward')
nexttile
plot(xBackward,dGBackward,'m');
legend('G-Backward')
```