Formulas of PL

Formulas: We define them recursively by the 3 clauses

a) Each propositional variable A_i is a formula.

b) $(\neg\phi)$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, and $(\phi \rightarrow \psi)$ are also formulas if ϕ and ψ are.

c) No string that is not built by a) or b) is a formula.

Definition: A set of strings is propositionally closed if it contains all the propositional variables and is closed under sentential conncetives. A string is a formula if it belongs to every propositionally closed set S.
Induction on formulas:

1) Check that each propositional variable has property P.

2) If ϕ has property P then so does $(\neg\phi)$

3) If ϕ ans ψ have property P then so does $(\phi \bullet \psi)$ for any binary conncetive $\vee, \wedge, \rightarrow$.

Unique readability:
For every formula exactly one of the following is true:

1) ϕ is a prop. variable

2) There is a unique formula ψ s.t $\phi = (\neg\psi)$.

3) There are unique formulas ψ, χ s.t $\phi = (\psi \vee \chi)$, $(\psi \wedge \chi)$, or $(\psi \rightarrow \chi)$

Semantics of PL

We assign a function $\upsilon : \{A_i | i \in \mathbb{N}\} \rightarrow \{T, F\}$ to each propositional variable and extend to all formulas by induction on formulas.
Theorem: For any truth assigment $\upsilon : \{A_i | i \in \mathbb{N}\} \rightarrow \{T, F\}$, there is a unique extension $\upsilon : PL \rightarrow \{T, F\}$.
Definition: A formula $\varphi \in PL$ is satisfiable if there is an assigment $\upsilon : \{A_i | i \in \mathbb{N}\} \rightarrow \{T, F\}$ s.t $\upsilon(\varphi) = T$
Tarski conditions:

- $\upsilon \models A_i \leftrightarrow \upsilon(A_i) = T$
- $\upsilon \models \neg\varphi \leftrightarrow \upsilon \not\models \varphi$
- $\upsilon \models (\varphi \wedge \psi) \leftrightarrow \upsilon \models \varphi$ and $\upsilon \models \psi$
- $\upsilon \models (\varphi \vee \psi) \leftrightarrow \upsilon \models \varphi$ or $\upsilon \models \psi$
- $\upsilon \models (\varphi \rightarrow \psi) \leftrightarrow \upsilon \not\models \varphi$ or $\upsilon \models \psi$

Tautologies:

- $\models \varphi \to (\psi \to \phi)$
- $\models (\varphi \to \psi) \to ((\varphi \to (\psi \to \chi)) \to (\varphi \to \chi))$
- $\models (\varphi \to \psi) \to ((\varphi \to (\neg\psi)) \to (\neg\varphi))$
- $\models (\neg(\neg\varphi)) \to \varphi$
- $\models \varphi \to (\psi \to (\varphi \wedge \psi))$
- $\models \varphi \to (\varphi \vee \psi)$
- $\models (\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$

Logical Consequence and Soundness

$T \models \varphi$ means "for every assigment that satisfies all formulas in T then v also satisfies φ"
Modus Ponens: For any two formulas φ, ψ $\varphi, (\varphi \to \psi) \models \psi$.
Definition: A deduction or proof from a set of formulas T is a finite sequence of formulas $\chi_0, \chi_1, \ldots, \chi_k$ such that for each $n \leq k$ one of the following holds:

(D1) $\chi_n \in T$ (assumption)

(D2) $\chi_n \equiv \chi_i$ for some $i < n$ (repetition)

(D3) χ_n is an axiom

(D4) χ_n can be inferred with MP for some χ_i, χ_j with $i, j < n$

$T \vdash \chi$ iff there is a proof $\chi_0, \chi_1, \ldots, \chi_k$ from T s.t $\chi \equiv \chi_k$. Then χ is a theorem of T.
Definition: A set of formulas is deductively closed if it contains all the axioms and it is closed under MP.
Lemma: For every T and φ $T \vdash \varphi$ iff $\varphi \in S$ for every deductively closed $S \supseteq T$.
Theorem: (Soundness) For any set of formulas Γ and formula φ if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Consistency and Completeness

Definition: A set of formulas Γ is consistent if there is no formula φ s.t $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.
Definition: A set of formulas is strongly complete if $\forall\varphi \in PL$ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.
Step 1: A set of formulas S is consistent and strongly complete iff there is an assigment v to the propositional variables s.t for every formula φ $v \models \varphi \leftrightarrow \varphi \in S$.

The Deduction Theorem: For any set of formulas T and all φ, ψ if $\Gamma, \varphi \vdash \psi$ then $\Gamma \vdash (\varphi \to \psi)$
Step 2: If Γ is consistent then for every formula φ either $\Gamma \cup \varphi$ or $\Gamma \cup \varphi$ is consistent.
Step 3: (The Completeness Theorem for PL)

(1) Every consistent set of formulas is satisfiable

(2) If $\Gamma \models \chi$ then $\Gamma \vdash \chi$

Structures

Language: is a set of constants, function symbols, and relation symbols. $\mathcal{L} = \{c_j\}_{j \in J} \cup \{R_i\}_{i \in I} \cup \{R_k\}_{k \in K}$
Terms and formulas:
Let $\mathcal{L}$ be a language we consider $S \supseteq \mathcal{L}$ of symbols where we add:

(1) The logical symbols $\neg, \vee, \wedge, \to, \forall, \exists, =$

(2) Parentheses ()

(3) The individual variables $v_0, v_1, \ldots$

Definition: An $\mathcal{L}$-term is defined by recursion where:

a) Each variable v_i is a term

b) Each constant symbol is a term

c) If $\tau_1, \ldots, \tau_n$ are terms and f is an $n{-}ary$ function symbol then $f(\tau_1, \ldots, \tau_n)$ is a term

Definition: An $\mathcal{L}$-formula is defined by recursion where:

a) If s, t are terms then $s = t$ is a formula

b) If $\tau_1, \ldots, \tau_n$ are terms and R is an $n{-}ary$ relation symbol then $R(\tau_1, \ldots, \tau_n)$ is a formula

c) If φ, ψ are formulas and v is a variable $(\neg\varphi), (\varphi \bullet \psi), \exists v \varphi, \forall v \varphi$ are formulas.

Let φ be an $\mathcal{L}$-formula then

(1) φ is quantifier free if $\exists$ and $\forall$ do not occur in φ

(2) The variable x_i if free in φ if it is not quantified

Semantics

A structure in the language $\mathcal{L} = \{c_j\}_{j\in J} \cup \{R_i\}_{i\in I} \cup \{R_k\}_{k\in K}$ is a pair $A = (A, I)$ where:

- A is a non-empty set (we call A) to be the universe
- I is an interpretation that assigns:
  - $I(c) = c^A$ an element of A.
  - $I(R) = R^A$ is an $n - ary$ relation $R^A \subset A^n$ to each $n - ary$ relation R.
  - $I(f) = f^A : A^n \to A$ an $n - ary$ function for each function symbol.

Definition: An isomorphism $\sigma : A \to B$ where both are $\mathcal{L}$-structures is a bijection such that:

(1) For each constant symbol c_A $\sigma(c_A) = c_B$

(2) For each $n{-}ary$ relation symbol R and $a_1, a_2, \ldots, a_n \in A$, $(a_1, a_2, \ldots, a_n) \in R^A \leftrightarrow (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_n)) \in R^B$

(3) For eahc $n{-}ary$ function symbol f and $a_1, a_2, \ldots, a_n \in A, = \sigma(f^A(a_1, a_2, \ldots, a_n)) = f^B(\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_n))$

If A and B are isomorphic then we denote this by $A \cong B$
Theorem: Let M and N be $\mathcal{L}$-structures and $f : M \to N$ an isomorphism. If υ is an assigment $\upsilon : Var \to M$ then $f \circ \upsilon : Var \to N$ is an assignment on N and for any formula φ and tuple $\overline{m} \in M$ $(M, \upsilon) \models \varphi(\overline{m})$ iff $(M, f \circ \upsilon) \models \varphi(f(\overline{m}))$.
Definition: An automorphism σ of M is an isomorphism of M onto M. Definition: (Substructure) Let $A = (A, I)$ $B = (B, J)$ be $\mathcal{L}$-structures. A is a substructure of B if

- $A \subseteq B$
- For every constant symbol $c^B = c^A \in A$
- For any $n - ary$ function and $a_1, a_2, \ldots, a_n \in A$, $f^A(a_1, a_2, \ldots, a_n) = f^B(a_1, a_2, \ldots, a_n)$
- For any $n - ary$ relation and $a_1, a_2, \ldots, a_n \in A$ $R^B(a_1, a_2, \ldots, a_n) \leftrightarrow R^A(a_1, a_2, \ldots, a_n)$

Definition: An assigment into a structure A is any association of objects in A with variables $\pi : Variables \to A$.
The value of an assigment is proven by induction on formulas.
Satisfaction: Let A be and $\mathcal{L}$-structure and π an assigment in A and φ an $\mathcal{L}$-formula
We say $A, \pi \models \phi \leftrightarrow value(\varphi, \pi) = 1 \leftrightarrow$ the assigment π satisfies the formula in the structure A.
Tarski conditions:

- $A, \pi \models s = t \leftrightarrow value^A(s, \pi) = value^A(t, \pi)$
- $A, \pi \models \neg\varphi \leftrightarrow A, \pi \not\models \varphi$
- $A, \pi \models \varphi \wedge \psi \leftrightarrow A, \pi \models \varphi$ and $A, \pi \models \psi$
- $A, \pi \models \varphi \vee \psi \leftrightarrow A, \pi \models \varphi$ or $A, \pi \models \psi$
- $A, \pi \models \varphi \rightarrow \psi \leftrightarrow A, \pi \not\models \varphi$ or $A, \pi \models \psi$
- $A, \pi \models \exists v\varphi \leftrightarrow$ there is $a \in A$ such that $(A, \pi(v := a)) \models \varphi$
- $A, \pi \models \forall v\varphi \leftrightarrow$ for all $a \in A$ $(A, \pi(v := a)) \models \varphi$

Definition: A proposition Φ in ordinary English about an $\mathcal{L}$-structure A is expressed or formalized by a sentence if Φ and φ mean the same thing (ϕ is a statement)$\rightarrow$ φ is a sentence if all variables are quantified.

Definable Sets

Let $R \subseteq A^n$ where A is an $\mathcal{L}$-structure and R is a relation in the universe of A^n. We say that R is definable if there is (1) an $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$ (2) a tuple $\overline{a}_0 \in A^{|\overline{y}|}$ such that $A \models \varphi(\overline{b}, \overline{a}_0)$ iff $\overline{b} \in A$ and we say that it is definable over a_0

If R is definable over a_0 then there is an $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$ such that:

$(m_1, m_2, \ldots, m_n) \in R$ iff $(M, v) \models \varphi(\overline{m}, a_0) \leftrightarrow (M, \sigma \circ v) \models \varphi(\sigma(\overline{m}), a_0)$

Theories

Definition: Let $\mathcal{L}$ be a fixed language on $\mathcal{L}$-theory T is any (possibly infinite) set of sentences T. The members of T are called axioms.

Definition 2: An $\mathcal{L}$-structure M is a model of an $\mathcal{L}$-theory T if for any sentence $\varphi \in T$ $M \models \phi$ i.e $M \models T$.

Definition 3: The models of a theory T T is the class of $\mathcal{L}$-structures $Mod(T) = \{M | M$ is an $\mathcal{L}$-structure and $M \models T\}$

Definition: Let $\mathcal{L}$ be a fixed language and let Φ be a property of $\mathcal{L}$-structures we say that it is elementary if there is an $\mathcal{L}$-theory T such that:

For M and $\mathcal{L}$-structure M has property $\Phi \leftrightarrow M \models T$ Definition: Let M and N be $\mathcal{L}$-structures then they are elementary equivalent if $\forall\varphi$ $\mathcal{L}$-sentence $M \models \varphi$ iff $N \models \varphi$ we denote this by $M \equiv N$,

Logical Consequence and Proofs

Definition: Let T be an $\mathcal{L}$-Theory and φ an $\mathcal{L}$-sentence we say that φ is a logical consequence $T \models \varphi$ iff every $\mathcal{L}$-structure $M \models T$ then $M \models \varphi$.

Hilbert's axiom schemes

(a) Propositional Axiom Schemes: The set of logical tautologies
(b) Predicate Axiom Schemes

- $\forall \upsilon \varphi(\upsilon, \vec{u}) \rightarrow \varphi(\tau, \vec{u})$ τ free for υ in $\varphi(\upsilon, \vec{u})$
- $\forall \upsilon(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall \upsilon \psi)$ υ not free in φ
- $\varphi(\tau, \vec{u}) \rightarrow \exists \upsilon \varphi(\upsilon, \vec{u})$ τ free for υ in $\varphi(\upsilon, \vec{u})$

(c) Rules of Inference

- From φ and $\varphi \rightarrow \psi$ infer ψ MP
- From φ infer $\forall \upsilon \varphi$ (Generalization)
- From $\varphi \rightarrow \psi$ infer $\exists \upsilon \varphi \rightarrow \psi$ provided υ is not free in ψ (Exists Elimination)

(d) Identity Axioms

- $\upsilon = \upsilon$ $\upsilon = \upsilon' \rightarrow \upsilon' = \upsilon$ $\upsilon = \upsilon' \rightarrow ((\upsilon' \rightarrow \upsilon'') \rightarrow (\upsilon \rightarrow \upsilon''))$
- $(v_1 = w_1 \wedge \ldots \wedge v_n = w_n) \rightarrow (R(v_1, \ldots, v_n) \rightarrow R(w_1, \ldots, w_n))$ any $n - ary$ relation symbol
- $(v_1 = w_1 \wedge \ldots \wedge v_n = w_n) \rightarrow (f(v_1, \ldots, v_n) = f(w_1, \ldots, w_n))$ any $n - ary$ function symbol

Definition: A deduction from a theory T is any sequence of formulas $\varphi_0, \ldots \varphi_n$ where each φ_i is either:

- an axiom $\varphi_i \in$ Hilbert's Axioms
- a hypothesis $\varphi_i \in T$
- repetition $\varphi_i = \varphi_j$ for $j < i$
- or follows from MP, Generalization, or Exists Elimination

Definition: (Soundness for first order logic) Let T be a theory and χ a sentence if $T \vdash \chi$ then $T \models \chi$
Lemma: Let M be an $\mathcal{L}$-structure and χ an axiom in the Hilbert's list (not inference) then $M \models \chi$

Completeness Theorem

Definition 1: Γ a theory and φ a sentence if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$
Definition 2: If Γ is consistent then it is satisfiable
Consistency: A theory T is consistent if there is no sentence φ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$
An $\mathcal{L}$-theory is satisfiable if there is an $\mathcal{L}$-structure M such that $M \models T$
An $\mathcal{L}$-theory is complete if for every sentence φ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$
Definition: A theory T has the Henkin property if:

(a) it is consistent

(b) it is complete

(c) if $\exists v \varphi \in H$ then there is a constant c such that $\varphi(c) \in H$

Idea: add a bunch of constants to witness the existentials
Definition: Fix a Henkin set we say that it is deductively closed if for every sentence χ if $H \vdash \chi$ then $\chi \in H$ For all sentences $\varphi, \psi, \exists v \varphi(v), \forall v \varphi(v)$

(a) $\neg\varphi \leftrightarrow \varphi \notin H$

(b) $\varphi \wedge \psi \in H \leftrightarrow \varphi \in H$ and $\psi \in H$

(c) $\varphi \vee \psi \in H \leftrightarrow \varphi \in H$ or $\psi \in H$

(d) $\varphi \rightarrow \psi \in H \leftrightarrow \varphi \notin H$ or $\psi \in H$

(e) $\exists v \varphi(v) \in H \leftrightarrow$ there is some constant c such that $\varphi(c) \in H$

(f) $\forall v \varphi(v) \in H \leftrightarrow$ for all constants c $\varphi(c) \in H$

Every Theory can be extended to a Henkin Set
Every Henkin set is satisfiable

Compactness Theorem*

(PL Logic) Suppose T is an infinite set of formulas. Prove that if every finite subset $T_0 \subset T$ is satisfiable, then T is satisfiable.
(First Order) If every finite subset of a theory T in a finite vocabulary has a model, then T has a model.