Math 100: Problem Set 8

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(Q-1) (a) $P = (\frac{0+a+b}{3}, \frac{0+0+c}{3}) = (\frac{a+b}{3}, \frac{c}{3})$.

$O = (X, Y)$ where (X, Y) is the solution to $\sqrt{(X-0)^2 + (Y-0)^2} = \sqrt{(X-a)^2 + (Y-0)^2} = \sqrt{(X-b)^2 + (Y-c)^2}$

$\Rightarrow X = \frac{a}{2} \Rightarrow 0 = -2b(\frac{a}{2}) + b^2 - 2cY + c^2 \Rightarrow Y = \frac{b^2+c^2-ab}{2c} \Rightarrow O = (\frac{a}{2}, \frac{b^2+c^2-ab}{2c})$

(b) $\frac{AP}{PD} + \frac{BP}{PE} + \frac{CP}{PF} = \frac{AP^2}{AP \cdot PD} + \frac{BP^2}{BP \cdot PE} + \frac{CP^2}{CP \cdot PF}$. WTS $AP \cdot PD = R^2 - x^2$.

Observe $\triangle AOD$ is isosceles with $AO = DO = R$. Let $\alpha = \angle OAD$.

By Law of Cosines $R^2 = R^2 + AD^2 - 2R \cdot AD \cos(\alpha) \Rightarrow \cos(\alpha) = \frac{AD}{2R}$

Consider $\triangle AOP$

$x^2 = R^2 + AP^2 - 2AP \cdot R\cos(\alpha)$

$= R^2 + AP^2 - AP \cdot AD = R^2 - AP \cdot DP$

$\Rightarrow AP \cdot DP = R^2 - x^2$

By the same reasoning, we can show $BP \cdot EP = CP \cdot FP = R^2 - x^2$.

Thus, $\frac{AP}{PD} + \frac{BP}{PE} + \frac{CP}{PF} = \frac{AP^2+BP^2+CP^2}{R^2-x^2}$

$AP^2 + BP^2 + CP^2 = AO^2 + BO^2 + CO^2 + 3PO^2 - 2(\vec{AO} + \vec{BO} + \vec{CO}) \cdot \vec{OP} = 3R^2 + 3x^2 - 6\vec{OP} \cdot \vec{OP} = 3R^2 - 3x^2$

(Q-2) Let a and b be given.

Choose c s.t $x^2 + y^2 = c^2$ intersects with $y = -\frac{b}{a}x + b$.

$\frac{a^2+b^2}{a^2}x^2 - 2\frac{b^2}{a}x + b^2 = c^2 \Rightarrow c = \sqrt{\frac{a^2+b^2}{a^2}x^2 - 2\frac{b^2}{a}x + b^2}$ for some x (c is squared so $\pm$ doesn't matter).

For positive b, we want $\frac{-b}{a} = \frac{-x}{|\frac{b}{a}x-b|} = \frac{x}{\frac{b}{a}x-b}$ and for negative b we want $\frac{-b}{a} = \frac{x}{|\frac{b}{a}x-b|} = \frac{x}{\frac{b}{a}x-b}$.

It follows $x = \frac{ab^2}{a^2+b^2}$.

Thus $c = \sqrt{\frac{b^4}{a^2+b^2} - 2\frac{b^4}{a^2+b^2} + \frac{b^4+a^2b^2}{a^2+b^2}} = \frac{ab}{\sqrt{a^2+b^2}}$

(Q-3) Want to find the intersections of $y^2 = ax$ and $(x-h)^2 + (y-k)^2 = r^2$. It follows $(\frac{y^2}{a} - h)^2 + (y-k)^2 = r^2 \Rightarrow \frac{y^4}{a^2} - 2\frac{y^2}{ah} + h^2 + y^2 - 2ky + k^2 - r^2 = 0 \Rightarrow y_1y_2y_3y_4 = a^2(h^2 + k^2 - r^2)$ by Vieta's formulas.

(Q-4) For each side of the quadrilateral $ABCD$ let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be a vector from the origin to the respective vertix. Let $x \in (0, 1)$. For each of the points E, F, G, H let $\vec{e}, \vec{f}, \vec{g}, \vec{h}$ be a vector from the origin to the respective point. $\vec{e} = \vec{a}x + \vec{d}(1-x), \vec{f} = \vec{a}x + \vec{b}(1-x), \vec{g} = \vec{c}x + \vec{b}(1-x), \vec{h} = \vec{c}x + \vec{d}(1-x)$. WTS $\vec{EF} \parallel \vec{GH}$ and $|\vec{EF}| = |\vec{GH}|$. $\vec{EF} = \vec{f} - \vec{e} = \vec{b}(1-x) - \vec{d}(1-x)$, $\vec{GH} = \vec{h} - \vec{g} = \vec{d}(1-x) - \vec{b}(1-x)$. Since $\vec{GH}$ is just $\vec{EF}$ rotated $180\,\text{deg}$, the two sides are parallel and of equal length. We us similar logic to show $\vec{HE} \parallel \vec{FG}$ and $|\vec{HE}| = |\vec{FG}|$. Thus, $EFGH$ is a quadrilateral with two set of parallel lines.

(Q-5) WLOG let the vertices of $ABCD$ be $(0,0), (2a,0), (2b,2c), (2a+2b,2c)$. The centers of the squares are located at $M_1 = (a, -a), M_2 = (b-c, b+c), M_3 = (a+2b, 2c+a), M_4 = (2a+b+c, c-b)$. WTS $\vec{M_1M_3} = \vec{M_2M_4}$ and $\vec{M_1M_3} \perp \vec{M_2M_4}$. $|\vec{M_1M_3}| = \sqrt{4b^2+4a^2+4ac+4c^2}$ and $|\vec{M_2M_4}| = \sqrt{4b^2+4a^2+4ac+4c^2}$ $\vec{M_1M_3} \cdot \vec{M_2M_4} = 2b \cdot (2a+2c) + (2a+2c) \cdot (-2b) = 0$. Because the diagonals of the quadrilateral $M_1M_2M_3M_4$ are equal in length and perpendicular, it must be a square.

(Q-6) For each side of the quadrilateral $ABCD$ let $\vec{a} = \vec{OA}, \vec{b} = \vec{OB}, \vec{c} = \vec{OC}, \vec{d} = \vec{OD}$ in the complex plane.
Let $\omega = e^{\frac{2\pi i}{3}}$
$\vec{M_1} = \omega(\vec{a} - \vec{b}) + \vec{a}$
$\vec{M_2} = \omega^2(\vec{b} - \vec{c}) + \vec{b}$
$\vec{M_3} = \omega(\vec{c} - \vec{d}) + \vec{c}$
$\vec{M_4} = \omega^2(\vec{d} - \vec{a}) + \vec{d}$
WTS $\vec{M_1M_2} + \vec{M_3M_4} = 0$
$\vec{M_1M_2} + \vec{M_3M_4} = 0\vec{b} + 0\vec{d} - \omega^2\vec{c} - (1+\omega)\vec{a} - \omega^2\vec{a} - (1+\omega)\vec{c} = 0$ using the identity $1 + \omega + \omega^2 = 0$
The same strategy is used to show $\vec{M_2M_3} + \vec{M_4M_1} = 0$
Since $M_1M_2M_3M_4$ is a quadrilateral with two sets of parallel lines, it must be a parallelogram.

(Q-7) Let $ABCD$ be a tetrahedron where $AB \perp CD$ and $AC \perp BD$. We want to show $AD \perp CD$.
$\vec{AB} = \vec{AC} + \vec{CB}$ and $\vec{CD} = \vec{CB} + \vec{BD}$. $\vec{AB} \cdot \vec{CD}$
$= \vec{AC} \cdot \vec{CB} + \vec{AC} \cdot \vec{BD} + \vec{CB} \cdot \vec{CB} + \vec{CB} \cdot \vec{BD}$
$= \vec{AC} \cdot \vec{CB} + \vec{CB} \cdot (\vec{CB} + \vec{AC}) = \vec{AC} \cdot \vec{CB} + \vec{AB}\vec{CD}$
$= \vec{AC} \cdot \vec{CB} = 0$, so $\vec{AC} \perp \vec{CB}$

(Q-8) Let A_k correspond to $z_k = e^{\frac{2\pi ik}{n}}$ for $i = 1 \dots n$ and fix P to be z.
$\sum_{k=1}^{n} PA_k^4 = \sum_{k=1}^{n} |z_k - z|^4 = \sum_{k=1}^{n} (z - z_k)(z - z_k)(\overline{z} - \overline{z_k})(\overline{z} - \overline{z_k})$
$z\overline{z} = 1, z_k\overline{z_k} = 1, z_1 + z_2 \dots + z_n = 0, \overline{z_1} + \overline{z_2} \dots + \overline{z_n} = 0$ because conjugate is just a reflection about the x-a
$\sum_{i=1}^{n} z^2\overline{z}^2 = n$

$$\sum_{i=1}^{n} z_k^2 \overline{z_k}^2 = n$$
odd and even orders sum to 0
$$\sum_{i=1}^{n} z_k z \overline{z_k z} = n$$
other even orders sum to 0.

Thus, $\displaystyle\sum_{k=1}^{n} PA_k^4 = 6n$

(Q-9) WLOG let $G = (0,0)$ be the centroid of $\triangle ABC$ with vertices $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$. It follows $(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}) = (0,0)$. Observe $(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = 0$ and $(y_1 + y_2 + y_3)^2 = y_1^2 + y_2^2 + y_3^2 + 2y_1y_2 + 2y_1y_3 + 2y_2y_3 = 0$. It follows $3(GA^2 + GB^2 + GC^2) = 3(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2)$
$= 2(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2) - (2x_1x_2 + 2x_1x_3 + 2x_2x_3 + 2y_1y_2 + 2y_1y_3 + 2y_2y_3)$
$= (x_1^2 - 2x_1x_2 + x_2^2) + (x_1^2 - 2x_1x_3 + x_3^2) + (x_2^2 - 2x_2x_3 + x_3^2) + (y_1^2 - 2y_1y_2 + y_2^2) + (y_1^2 - 2y_1y_3 + y_3^2) + (y_2^2 - 2y_2y_3 + y_3^2)$
$= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_1 - x_3)^2 + (y_1 - y_3)^2 + (x_2 - x_3)^2 + (y_2 - y_3)^2$
$= AB^2 + BC^2 + AC^2$

(Q-10) Let a, b, c, d, e, f be the complex positions of the vertices of the $ABCDEF$ hexagon. Let $x = e^{\frac{\pi i}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ because a, b c, d and e, f are r apart. (We construct an equilateral triangle with side length r with one of the vertices at the origin. Each angle is $\frac{\pi}{3}$, so arc length is $r\frac{\pi}{3}$). Let $b = ax, d = cx, f = ex$. WTS $z_1 = \frac{ax+c}{2}, z_2 = \frac{cx+e}{2}, z_3 = \frac{ex+a}{2}$ are equidistant. This is true iff $z_3 = z_2x^{\pm 1} + z_1(1 - x^{\pm 1})$. $z_2x^1 + z_1(1 - x^1) = \frac{cx^2+ex}{2} + \frac{ax+c}{2} - \frac{ax^2+cx}{2} = \frac{c(x^2-x+1)+ex+a(x-x^2)}{2} = \frac{c(-\frac{2}{2}+1-\frac{\sqrt{3}}{2}i+\frac{\sqrt{3}}{2}i)+ex+a(\frac{1}{2}+\frac{1}{2}+\frac{\sqrt{3}}{2}i-\frac{\sqrt{3}}{2}i)}{2} = \frac{a+ex}{2} = z_3$. By symmetry, this equality will hold for all z_1, z_2, z_3