Math 164: Problem Set 1

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2.6

$$x_1 + x_2 + 2x_3 + x_4 = 1$$
$$x_1 - 2x_2 - x_4 = -2$$

can be expressed in the form $Ax = b$ with matrix A and vectors x and b.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

We denote the columns of $A := [a_1, a_2, a_3, a_4]$. $a_i, b \in \mathbb{R}^2, i = 1 \ldots 4 \Rightarrow rankA, rank[A, b] \leq 2$. a_1, a_2 are linearly independent, so $rankA = rank[A, b] = 2$ It follows from theorem 2.1 there exists a solution x to the equation $Ax = b$.

Rewrite the equation $Ax = b$ as $x_1a_1 + x_2a_2 = b - x_3a_3 - x_4a_4$ and assign arbitrary values to x_3, x_4. Let $B := [a_1, a_2] \in \mathbb{R}^{2\times 2}$. Because B is an invertible matrix, we can solve for x_1, x_2 by computing $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B^{-1}(b - x_3a_3 - x_4a_4)$ to find the general solution for the system.

2.8 $\langle x, y\rangle_2 = 2x_1y_1 + 3x_2y_1 + 3x_1y_2 + 5x_2y_2 \Leftrightarrow x^T \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} y$

Note: for two vectors x, y of the same dimension, their Euclidean inner product is $x^T \cdot y = x \cdot y^T$. Moreover, $\langle x, x\rangle = \|x\|^2$

Positivity: Let $A := \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Observe $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$. $x^TA^T = (Ax)^T \Rightarrow x^TA^2x = (Ax)^T(Ax) = \|Ax\|^2 \geq 0$ because A is symmetric and the square of any number is non-negative. Also, $\langle x, x\rangle_2 = 0 \Leftrightarrow \|Ax\|^2 = 0 \Rightarrow Ax = 0 \Rightarrow x = A^{-1}0 = 0$ because A is invertible ($\det A = 1$).

Symmetry: $\langle x, y\rangle_2 = (Ax)^T(Ay) = (Ay)^T(Ax) = \langle y, x\rangle_2$ because $(Ax)^T(Ay) \in \mathbb{R}^{1\times 1}$, so $(Ay)^T(Ax) = ((Ay)^T(Ax))^T$.

Additivity: $\langle x + y, z\rangle_2 = (x + y)^TA^2z = x^TA^2z + y^TA^2z = \langle x, z\rangle_2 + \langle y, z\rangle_2$

Homogeneity: $\langle \alpha x, y\rangle_2 = (\alpha x)^TA^2y = \alpha x^TA^2y = \alpha\langle x, y\rangle_2$

2.9 $\mathbf{x} = (\mathbf{x} - \mathbf{y}) + \mathbf{y}$. By the triangle innequality $\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|$. Similarly $\mathbf{y} = (\mathbf{y} - \mathbf{x}) + \mathbf{x}$. By the triangle innequality $\|\mathbf{y}\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x}\|$. $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\|$. Observe $\|\mathbf{y}\| - \|\mathbf{x}\|, \|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{y} - \mathbf{x}\|$ by bringing $\|\mathbf{y}\|$ or $\|\mathbf{x}\|$ to the other side. $\max(\|\mathbf{y}\| - \|\mathbf{x}\|, \|\mathbf{x}\| - \|\mathbf{y}\|) = |\|\mathbf{y}\| - \|\mathbf{x}\||$. Thus, $|\|\mathbf{y}\| - \|\mathbf{x}\|| \leq \|\mathbf{y} - \mathbf{x}\|$.