Math 106: Problem Set 7

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9.2.1 $\sum_{i=1}^{n}(i+1)^2 - i^2 = 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \Rightarrow \sum_{i=1}^{n} i = \frac{1}{2}(\sum_{i=1}^{n}(i+1)^2 - i^2 - \sum_{i=1}^{n} 1)$
$= \frac{1}{2}((n+1)^2 - 1 - n) = \frac{n(n+1)}{2}$ by sum of telescoping series.
$\sum_{i=1}^{n}(i+1)^3 - i^3 = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \Rightarrow \sum_{i=1}^{n} i^2 = \frac{1}{3}(\sum_{i=1}^{n}(i+1)^3 - i^3 - 3\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1)$
$= \frac{1}{3}((n+1)^3 - 1 - 3\frac{n(n+1)}{2} - n) = \frac{n(n+1)(2n+1)}{6}$ by sum of telescoping series.

9.2.2 $\frac{1}{n}\sum_{i=1}^{n} f(\frac{i}{n}) = \frac{1}{n}\sum_{i=1}^{n}\frac{i^2}{n^2} = \frac{n(n+1)(2n+1)}{6n^3}$
$\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty}\frac{n^3}{n^3}(1+\frac{1}{n})(\frac{1}{3}+\frac{1}{6n})$. Limit of the product is the product of the limits, so $\lim_{n\to\infty}\frac{n^3}{n^3}(1+\frac{1}{n})(\frac{1}{3}+\frac{1}{6n}) = \frac{1}{3}$.

9.3.1 $(\frac{3x}{2}+1)^2 = \frac{9x^2}{4}+3x+1 = x^3-3x^2+3x+1 \Rightarrow 0 = x^3 - \frac{21}{4}x^2$ by subtracting $\frac{9x^2}{4}+3x+1$ from both sides. The geometric interpretation of the double root is that there are two lines tangent to the curve at $x=0$. We find the tangent lines using the method in 3.5.1 by drawing a line through rational points $(0,1)$ and $(\frac{21}{4}, \frac{71}{8})$ or through $(0,-1)$ and $(\frac{21}{4}, -\frac{71}{8})$.

9.3.2 By implicit differentiation $2y\frac{dy}{dx} = 3x^2 - 6x + 5 \Rightarrow \frac{dy}{dx} = \frac{5}{2}$ at $(0,1)$. Thus, we should substitute $y = \frac{5}{2}x + 1$.

9.5.1

$$x = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \dots$$
$$x^2 = (a_0 + a_1 y + a_2 y^2 + \dots)^2 = a_0^2 + 2a_0 a_1 y + (2a_0 a_2 + a_1^2)y^2 + (2a_0 a_3 + 2a_1 a_2)y^3 + \dots$$

We substitute $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ into either equation

$$x = a_0 + a_1(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots) + a_2(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots)^2 + a_3(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots)^3 + \ldots$$
$$\Rightarrow x = a_0 + a_1 x + (a_2 - \frac{a_1}{2})x^2 + (\frac{a_1}{3} - a_2 + a_3)x^3 + \ldots$$

Comparing coefficients, $a_0 = 0$, $a_1 = 1$, $a_2 - \frac{a_1}{2} = 0$, and $\frac{a_1}{3} - a_2 + a_3 = 0$.
Thus, $a_0 = 0, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}$