Math 131B: Homework 3

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Problem 1. **Exercise 1.5.4**

Let $f : \mathbb{R} \to \mathbb{R}$ $f(x) = 0$. $\mathbb{R}$ is open, but $\{0\}$ is not open because 0 is a boundary point.

QED

Problem 2. **Exercise 1.5.5**

Let $f : [1, \infty) \to (0, 1]$ $f(x) = \frac{1}{x}$. $[1, \infty)$ is closed because it contains all of its boundary points, 1, but $(0, 1]$ is not closed because $0 \notin (0, 1]$.

QED

Problem 3. **Exercise 1.5.10 (b)**

First we show that if (X, d) is a compact metric space, then (X, d) is complete. Let $(x^{(n)})_{n=m}^{\infty}$ be an arbitrary Cauchy sequence in (X, d). Since (X, d) is compact, there exists a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ that converges to some value $x_0 \in X$. It follows by Lemma 1.4.9 that $(x^{(n)})_{n=m}^{\infty}$ also converges to x_0. Because any arbitrary Cauchy sequence in (X, d) is convergent in (X, d), (X, d) is a complete metric space.

Next we show that (X, d) must also be totally bounded. Assume for the sake of contradiction that (X, d) is compact but not totally bounded. Therefore, $\exists \epsilon > 0$ s.t no finite number of balls of radius-ϵ will cover X. Because X requires infinitely many ϵ-balls to be covered entirely, $X \setminus (\bigcup_{i=1}^{n} B(x^{(i)}, \epsilon)) \neq \emptyset$ for all n. Thus, we can construct a sequence $(x^{(n)})_{n=1}^{\infty}$ in (X, d) where $x^{(n)} \notin \bigcup_{i=1}^{n-1} B(x^{(i)}, \epsilon)$. Since each term of the sequence is at least distance-ϵ from every other term of the sequence, $(x^{(n)})_{n=1}^{\infty}$ has no convergent subsequences. We obtain a contradiction because $(x^{(n)})_{n=1}^{\infty}$ is a sequence in (X, d) and (X, d) is compact, so $(x^{(n)})_{n=1}^{\infty}$ must have a convergent subsequence. Hence, (X, d) must be totally bounded.

QED

Problem 4. **Exercise 2.1.1** We will show a, b, and c are logically equivalent by showing $a \Rightarrow b$, $b \Rightarrow a$, $a \Rightarrow c$, and $c \Rightarrow a$.

$(a \Rightarrow b)$ Suppose f is continuous at x_0, and let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in X that converges to x_0. Because f is continuous at x_0, there exists $\delta > 0$ for every $\epsilon > 0$ s.t $d_Y(f(x^{(n)}), f(x_0)) < \epsilon$ whenever $d_X(x^{(n)}, x_0) < \delta$. Given $\epsilon > 0$ choose N to be sufficiently large s.t $n \geq N \Rightarrow d_X(x^{(n)}, x_0) < \delta$. Thus, $d_Y(f(x^{(n)}), f(x_0)) < \epsilon$ by the continuity of f. Hence, $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$.

$(b \Rightarrow a)$ Suppose the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y whenever a sequence $(x^{(n)})_{n=1}^{\infty}$ in X converges to x_0 with respect to the metric d_X, and assume for the sake of contradiction that f is not continuous at x_0. Because f is not continuous at x_0, $\exists \epsilon > 0$ s.t $\forall \delta > 0$ $d(x, x_0) < \delta$ yet $d(f(x), f(x_0)) \geq \epsilon$. Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in X s.t $d_X(x^{(n)}, x_0) < \frac{1}{n}$ while $d_Y(f(x^{(n)}), f(x_0)) \geq \epsilon$. We obtain a contradiction because we have a convergent sequence $(x^{(n)})_{n=1}^{\infty}$ where $(f(x^{(n)}))_{n=1}^{\infty}$ doesn't converge. Hence, f must be continuous at x_0.

$(a \Rightarrow c)$ Suppose f is continuous at x_0, and let $V \subset Y$ be an open set that contains $f(x_0)$. Because f is continuous at x_0, there exists $\delta > 0$ for every $\epsilon > 0$ s.t $d_Y(f(x), f(x_0)) < \epsilon$ whenever $d_X(x, x_0) < \delta$. By Proposition 1.2.15 (a), $\exists r_y > 0$ s.t $B_{(Y,d_Y)}(f(x_0), r_y) \subseteq V$. It follows $\exists r_x > 0$ s.t $f(x) \in B_{(Y,d_Y)}(f(x_0), r_y)$ whenever $x \in U = B_{(X,d_X)}(x_0, r_x)$. Hence, we have $U \subseteq X$ s.t $f(U) \subseteq V$.

$(c \Rightarrow a)$ Suppose for every open set $V \subset Y$ that contains $f(x_0)$, there exists an open set $U \subset X$ containing x_0 s.t $f(U) \subseteq V$, and assume for the sake of contradiction that f is not continuous at x_0. Because f is not continuous at x_0, $\exists \epsilon > 0$ s.t $\forall \delta > 0$ $d(x, x_0) < \delta$ yet $d(f(x), f(x_0)) \geq \epsilon$. Let $V = B_{(Y,d_Y)}(y_0, \epsilon)$. It follows there exists an open set $U \subset X$ s.t $f(U) \subseteq V$. By Proposition 1.2.15 (a), $\exists r > 0$ s.t $B_{(X,d_X)}(x_0, r) \subseteq U$. Because f is not continuous at x_0, there exists $x \in B_{(X,d_X)}(x_0, r)$ s.t $f(x) \notin B_{(Y,d_Y)}(y_0, \epsilon)$ which implies $f(U) \not\subseteq V$. Hence, we obtain a contradiction, so f must be continuous at x_0.

Thus, by transitivity, a,b, and c are logically equivalent.

Problem 5. **Exercise 2.1.4**

a) $f : \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ x+1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$, $g : \mathbb{R} \to \mathbb{R}$ $g(x) = 0$, $g \circ f(x) = 0$

b) $f : \mathbb{R} \to \mathbb{R}$ $f(x) = 0$, $g : \mathbb{R} \to \mathbb{R}$ $g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ x+1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$, $g \circ f(x) = 0$

c) $f : \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ x+1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$, $g : \mathbb{R} \to \mathbb{R}$ $g(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$, $g \circ f(x) = x+1$

Corrolary 2.1.7 is not an iff statement. Neither a) nor b) discuss when either $f(x)$ or $g(x)$ are discontinuous. Each of parts a),b), and c) of Exercise 2.1.7 include at least 1 discontinuous function for $f(x)$ or $g(x)$.