Math 131B: Homework 6

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Problem 1. **Exercise 3.6.1**

We want to show $\int_a^b \sum_{n=1}^{\infty} f^{(n)} = \sum_{n=1}^{\infty} \int_a^b f^{(n)}$. Let $(s^{(n)})_{n=1}^{\infty}$ be the sequence of partial sums for the sequence $(f^{(n)})_{n=1}^{\infty}$ Given the series $\sum_{n=1}^{\infty} f^{(n)}$ converges uniformly, it follows $(s^{(n)})_{n=1}^{\infty}$ converges to f. By theorem 3.6.1, $\int_a^b \lim_{n\to\infty} s^{(n)} = \lim_{n\to\infty} \int_a^b s^{(n)}$. We use the linearity of the integral to switch the order of integration and summation for finitely many terms: $\lim_{n\to\infty} \int_a^b s^{(n)} = \lim_{n\to\infty} \sum_{i=1}^{n} \int_a^b f^{(i)}$. It follows $\lim_{n\to\infty} \sum_{i=1}^{n} \int_a^b f^{(i)}$ is equivalent to $\sum_{n=1}^{\infty} \int_a^b f^{(n)}$, so we obtain our desired result.

Problem 2. **Exercise 3.7.1**

We use the beginning of the proof given in the textbook. It remains to show the sequence of functions $(f_n)_{n=1}^{\infty}$ converges uniformly to the function $f : [a,b] \to \mathbb{R}$ $f(x) := L - \int_a^{x_0} g + \int_a^{x} g$ for all $x \in [a,b]$ and that f is differentiable with derivative g.

Let $\epsilon > 0$. By the uniform convergence of f_n' to g, we can use theorem 3.6.1 to choose an N large enough s.t $d(\int_a^{x_0} f_n', \int_a^{x_0} g) < \frac{\epsilon}{3}$, $d(\int_a^{x} f_n', \int_a^{x} g) < \frac{\epsilon}{3}$, and $d(f_n(x_0), L) < \frac{\epsilon}{3}$ whenever $n > N$ and $x \in [a,b]$. It follows $d(f(x), f_n(x_0) - \int_a^{x_0} f_n' + \int_a^{x} f_n') < \epsilon$ by the triangle innequality. The fundamental theorem of calculus gives us $f_n(x_0) - \int_a^{x_0} f_n' + \int_a^{x} f_n' = f_n(x_0) - (f_n(x_0) - f_n(a)) + (f_n(x) - f_n(a)) = f_n(x)$, so $d(f(x), f_n(x)) < \epsilon$ for any arbitrary $x \in [a,b]$. Hence, $(f_n)_{n=1}^{\infty}$ converges uniformly to f.

$\int_a^{x} g - \int_a^{x_0} g = \int_{x_0}^{x} g = f(x) - f(x_0)$ by algebra and because $f_n'(x_0)$ converges to $f(x_0)$. g is integrable on $[a,b]$, so by the fundamental theorem of calculus, g must be the derivative of f because f is the antiderivative of g. Hence, f is differentiable with derivative g. $f_n'(x) = \frac{x}{\sqrt{\frac{1}{n^2}+x^2}}$ diverges at $x = 0$, so $f_n'(x)$ does not converge uniformly. Hence, theorem 3.7.1 doesn't apply.

Problem 3. **Exercise 3.7.3**

Let $(s^{(n)})_{n=1}^{\infty}$ be the sequence of partial sums for the sequence $(f^{(n)})_{n=1}^{\infty}$. Because there exists $x_0 \in [a,b]$ s.t $s^{(n)}(x_0)$ is convergent and $s'^{(n)} = \sum_{i=1}^{n} f_i'$ is uniformly convergent by the Weierstrass M-test, we can use Theorem 3.7.1 to exchange the order of limits and differentiation. It follows $\frac{d}{dx} \lim_{n\to\infty} s^{(n)} = \lim_{n\to\infty} \frac{d}{dx} s^{(n)}$. Since each $f^{(n)}$ is differentiable, we can exchange the order of summation and differentiation for finitely many n to obtain $\frac{d}{dx} \lim_{n\to\infty} s^{(n)} = \lim_{n\to\infty} \frac{d}{dx} s^{(n)} = \lim_{n\to\infty} \frac{d}{dx} \sum_{i=1}^{n} f^{(i)} =$

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{d}{dx} f^{(i)} = \sum_{n=1}^{\infty} \frac{d}{dx} f^{(n)}$ which is our desired result.

Problem 4. **Exercise 4.1.1**

(a) Suppose $x \in \mathbb{R}$ s.t $|x-a| > R$. The Root test states $\sum_{n=1}^{\infty} c_n(x-a)^n$ diverges if $\limsup |c_n(x-a)^n|^{\frac{1}{n}} > 1$. $\limsup |c_n(x-a)^n|^{\frac{1}{n}} > \limsup |c_n \cdot R^n|^{\frac{1}{n}} = \limsup |c_n|^{\frac{1}{n}}|R| = 1$, so $\sum_{n=1}^{\infty} c_n(x-a)^n$ diverges if $|x-a| > R$.

(b) Suppose $x \in \mathbb{R}$ s.t $|x-a| < R$. The Root test states $\sum_{n=1}^{\infty} c_n(x-a)^n$ converges if $\limsup |c_n(x-a)^n|^{\frac{1}{n}} < 1$. $\limsup |c_n(x-a)^n|^{\frac{1}{n}} < \limsup |c_n \cdot R^n|^{\frac{1}{n}} = \limsup |c_n|^{\frac{1}{n}}|R| = 1$, so $\sum_{n=1}^{\infty} c_n(x-a)^n$ converges if $|x-a| < R$.

(c) $\sum_{n=1}^{\infty} c_n(x-a)^n$ converges uniformly on $[a-r, a+r]$ for $0 < r < R$ if $\sum_{n=1}^{\infty} ||c_n(x-a)^n||_\infty$ is convergent by Weierstrass M-test. It follows $\limsup |c_n(r)^n|^{\frac{1}{n}} = \frac{r}{R} < \frac{R}{R} = 1$, so $\sum_{n=1}^{\infty} c_n(r)^n$ converges by the Root test. Thus, $\sum_{n=1}^{\infty} c_n(x-a)^n$ converges uniformly on $[a-r, a+r]$ for $0 < r < R$. Let $x_0 \in (a-R, a+R)$. It follows there exists r_1 between 0 and R s.t $x_0 \in [a-r_1, a+r_1]$. Since $\sum_{n=1}^{\infty} c_n(x-a)^n$ converges uniformly on $[a-r_1, a+r_1]$ and $c_n(x-a)^n$ is continuous at x_0 for each n, the limiting function f must also be continuous at x_0. Hence, f is continuous for $x \in (a-R, a+R)$

(d) For any $0 < r < R$, $\limsup |nc_n(r)^{n-1}|^{\frac{1}{n}} < \limsup |n|^{\frac{1}{n}}|c_n|^{\frac{1}{n}}|(R)^{\frac{n-1}{n}}| = 1$, so $\sum_{n=1}^{\infty} ||nc_n(x-a)^{n-1}||_\infty$ converges. By the Weierstrass M-test, $nc_n(x-a)^{n-1}$ converges uniformly to some function f' on $[a-r, a+r]$. Pick $x_0 \in (a-R, a+R)$. It follows there exists $0 < r_x < R$ s.t $x_0 \in [a-r_x, a+x]$. Because each $c_n(x-a)^n$ is differentiable, $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ converges uniformly to some function f' on $[a-r, a+r]$, and $\sum_{n=1}^{\infty} c_n(x_0-a)^n$ converges to some value L, Theorem 3.7.1 states $\sum_{n=1}^{\infty} c_n(x-a)^n$ converges uniformly to some differentiable function f whose derivative is f'. Because this holds for any $x_0 \in (a-R, a+R)$, f is differentiable over $(a-R, a+R)$.

(e) $\sum_{n=1}^{\infty} c_n(x-a)^n$ converges uniformly on $[y, z]$ to f by (c) because $[y, z]$ is a compact set. Each $c_n(x-a)^n$ is integrable on $[y, z]$, so by Corollary 3.6.2, we can switch the order of summation and integration. Thus, by the fundamental theorem of calculus, $\int_y^z f = \sum_{n=1}^{\infty} c_n \frac{(z-a)^{n+1} - (y-a)^{n+1}}{n+1}$.

Problem 5. **Exercise 4.1.2**

(a) $\sum_{n=1}^{\infty} x^n$

(b) $\sum_{n=1}^{\infty} \frac{1}{n} x^n$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$

(d) $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$

(e) $\sum_{n=1}^{\infty} x^n$

Problem 6. **Additional Problem**

Let $x_0 \in (-1,1)$. The power series $\sum_{k=1}^{\infty}(-1)^k(x)^{2k}$ can be rewritten as $\sum_{k=1}^{\infty}(-1 \cdot x^2)^k$. If $x_0 \in (-1,1)$, then $-x_0^2 \in (-1,0)$. Because we know the series $\sum_{k=0}^{\infty} x^k$ converges pointwise to $\frac{1}{1-x}$ for $x \in (-1,1)$, $\sum_{k=0}^{\infty}(-1 \cdot x_0^2)^k$ converges to $\frac{1}{1-(-x_0^2)} = \frac{1}{1+x_0^2}$. Hence $\sum_{k=0}^{\infty}(-1)^k(x)^{2k}$ converges pointwise to $\frac{1}{1+x^2}$. Moreover, we can use the Weierstrass M-test to show $\sum_{k=0}^{\infty}(-1 \cdot x_0^2)^k$ converges uniformly for any subinterval $[-r,r]$. $\sum_{k=0}^{\infty} ||(-1 \cdot x^2)^k||_{\infty} = \sup\{\frac{1}{1+x^2} : x \in [-r,r]\} = 1$, so $\sum_{k=1}^{\infty}(-1)^k(x)^{2k}$ converges uniformly on $[-r,r]$. For any $x \in (-1,1)$ there exists $0 < r < 1$ s.t $x \in [-r,r]$. Since $\sum_{k=0}^{\infty}(-1)^k(t)^{2k}$ converges uniformly on $[-r,r]$ and each $\int_0^x (-1)^k(t)^{2k}dt = \frac{(-1)^k}{2k+1}x^{2k+1}$ we can use Corollary 3.6.2 to show $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}x^{2k+1} = \sum_{k=0}^{\infty} \int_0^x (-1)^k(t)^{2k} = \int_0^x \sum_{k=0}^{\infty}(-1)^k(t)^{2k} = \int_0^x \frac{1}{t^2+1} = \arctan(x) - \arctan(0) = \arctan(x)$ Hence, we obtain our desired result.