Math 170S: Homework 1

Owen Jones

10/20/2023

Problem 1. $f(x) = \frac{\lambda^x}{x!}e^{-\lambda}$ $P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P(X_1 = x_1)P(X_2 = x_2)\ldots P(X_n = x_n)$ by independence. Thus, $L(x_1, x_2, \ldots, x_n, \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!}e^{-\lambda}$.

log preserves maximization, so $\arg\max_{\lambda} L(x_1, x_2, \ldots, x_n, \lambda) = \arg\max_{\lambda} \log(L(x_1, x_2, \ldots, x_n, \lambda))$.

$\arg\max_{\lambda} L(x_1, x_2, \ldots, x_n, \lambda) = \sum_{i=1}^{n} \log(\frac{\lambda^{x_i}}{x_i!}e^{-\lambda}) = \sum_{i=1}^{n} x_i \log(\lambda) - \log(x_i!) - \lambda$ $\frac{\partial L}{\partial \lambda} = 0 = \sum_{i=1}^{n} \frac{x_i}{\lambda} - 1 \Rightarrow$

$\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n} = \overline{x}$

We know $L(\hat{\lambda})$ is a maximum by the second derivative test $\frac{\partial^2 L}{\partial \lambda^2} = \sum_{i=1}^{n} -\frac{x_i}{\lambda^2} < 0$ because the $x_i s$ are positive and any number squared is positive.

Problem 2. For 5 heads and 25 tails $L(X_1 = x_1, X_2 = x_2, \ldots X_{30} = x_{30}, p) = \prod_{i=1}^{30} p^{x_i}(1-p)^{1-x_i}$ for $x_i \in \{0, 1\}$.

$\arg\max_{p} L(p) = \sum_{i=1}^{30} x_i \log(p) + (1 - x_i)\log(1-p)$ $\frac{\partial L}{\partial p} = 0 = \sum_{i=1}^{30} \frac{x_i}{p} - \frac{1-x_i}{1-p} = \frac{5}{p} - \frac{25}{1-p} \Rightarrow 5(1-p) = 25p \Rightarrow \hat{p} = \frac{1}{6}$

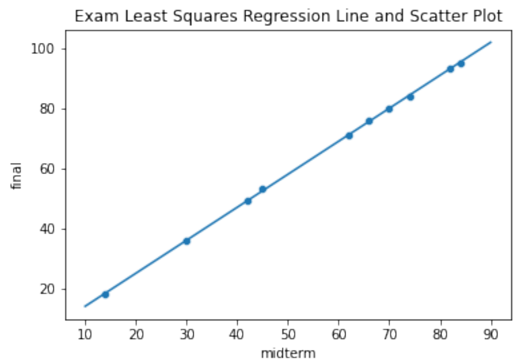
We know $L(\hat{\lambda})$ is a maximum by the second derivative test $\frac{\partial^2 L}{\partial p^2} = -\frac{5}{p^2} - \frac{25}{(1-p)^2} = -216 < 0$

Problem 3.

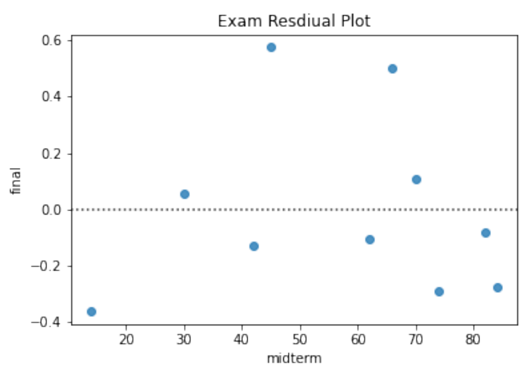
1. WTS $f(x|y, \theta) \geq 0$ for all x and $\int_y^\infty f(x|y,\theta)dx = 1$ (using $x \geq y$.)
   $f(x|y,\theta) = \theta y^\theta x^{-\theta-1}$. θ and y are both positive, so $\theta y^\theta > 0$. $x \geq y > 0$, so $x > 0$. A positive number raised to any power is positive, so $x^{-\theta-1} > 0$. Hence, $\theta y^\theta x^{-\theta-1}$ for all x.
   $\int_y^\infty \theta y^\theta x^{-\theta-1}dx = -y^\theta x^{-\theta}|_y^\infty = -y \lim_{x\to\infty} x^{-\theta} - (-y^{\theta-\theta}) = 1$ $(\theta > 1)$

2. $L(\theta) = \prod_{i=1}^{n} \theta y^\theta x_i^{-\theta-1}$ $\arg\max_{\theta} L(\theta) = \arg\max_{\theta} \log(L(\theta)) = \sum_{i=1}^{n} \log(\theta) + \theta\log(y) - (\theta+1)\log(x_i)$.
   $\frac{\partial L}{\partial \theta} = 0 = \sum_{i=1}^{n} \frac{1}{\theta} + \log(y) - \log(x_i) \Rightarrow \frac{n}{\theta} = \sum_{i=1}^{n} \log(\frac{x_i}{y}) \Rightarrow \hat{\theta} = \frac{n}{\sum_{i=1}^{n} \log(\frac{x_i}{y})}$
   We know $L(\hat{\theta})$ is a maximum by the second derivative test $\frac{\partial^2 L}{\partial \theta^2} = -\frac{n}{\theta^2} < 0$ because the square of any number is positive and a negative times a positive is negative.

Problem 4. 1. $\beta_1 = \frac{cov(x,y)}{s_x^2} = \dfrac{\sum(y-\overline{y})(x-\overline{x})}{\sum(x-\overline{x})^2} = 1.09878929,\ \beta_0 = \overline{y} - b\overline{x} = 2.9788896$



2.



3. The residuals from the Midterm vs. Final plot don't follow a trend (points are randomly distributed above and below the regression line). This implies a linear regression fits the data well.

4. $L(\beta_0, \beta_1, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(y_i - (\beta_0 + \beta_1 x_i))^2}{2\sigma^2})$. $\arg\max_{\sigma^2} L(\sigma^2) = \log(\arg\max_{\sigma^2} L(\sigma^2)) = \sum_{i=1}^{n} -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{(y_i - (\beta_0 + \beta_1 x_i))^2}{2\sigma^2}\ \frac{\partial L}{\partial \sigma^2} = 0 = \sum_{i=1}^{n} -\frac{1}{2\sigma^2} + \frac{(y_i - (\beta_0 + \beta_1 x_i))^2}{2(\sigma^2)^2} \Rightarrow$

$$\hat{\sigma^2} = \frac{\sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i))^2}{n}$$

We know $L(\hat{\sigma^2})$ is a maximum by the second derivative test $\frac{\partial^2 L}{\partial(\sigma^2)^2} = \frac{n}{2(\sigma^2)^2} - \frac{\sum_{i=1}^{n}(y_i - (\beta_0 + \beta_1 x_i))^2}{(\sigma^2)^3} = \frac{n}{2(\sum_{i=1}^{n}(y_i - (\beta_0 + \beta_1 x_i))^2)^2} - \frac{n^3}{(\sum_{i=1}^{n}(y_i - (\beta_0 + \beta_1 x_i))^2)^2} < 0$ because $n^3 > \frac{n}{2}$ for all natural numbers and variance is positive.

Problem 5. We need two equations because we have two unknowns.

$$E[X] = \int_0^1 x f(x|\alpha,\beta)dx = \frac{\alpha}{\alpha+\beta}\int_0^1 \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta-1}dx = \frac{\alpha}{\alpha+\beta}\int_0^1 beta(\alpha+1,\beta)dx = \frac{\alpha}{\alpha+\beta}$$

$$E[X^2] = \int_0^1 x^2 f(x|\alpha,\beta)dx = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}\int_0^1 beta(\alpha+2,\beta)dx = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$\beta = \frac{(1-E[X])^2}{E[X](E[X^2]-E[X]^2)}$ $\alpha = E[X^2]\frac{1-E[X]}{E[X^2]-E[X]^2}$