Math 131B: Homework 1

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Problem 1: Exercise 1.1.5:

We will complete a proof by induction on n to show $(\sum_{i=1}^{n} a_i b_i)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = (\sum_{i=1}^{n} a_i)^2 (\sum_{i=1}^{n} b_i)^2$ is true for all n. We will denote the statement $P(n) : (\sum_{i=1}^{n} a_i b_i)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = (\sum_{i=1}^{n} a_i)^2 (\sum_{i=1}^{n} b_i)^2$.

Base case: The LHS of $P(1) : (\sum_{i=1}^{1} a_i b_i)^2 + \frac{1}{2}\sum_{i=1}^{1}\sum_{j=1}^{1}(a_i b_j - a_j b_i)^2$ and the RHS of $P(1) : (\sum_{i=1}^{1} a_i^2)(\sum_{i=1}^{1} b_i^2)$ are both equal to $a_1^2 b_1^2$. Since the left-hand-side and the right-hand-side are equal, the statement $P(1)$ holds.
Induction hypothesis: Assume for some arbitrary $n \geq 1$ the statement $P(n)$ holds.
Induction step: It remains to show the statement $P(n+1)$ holds by manipulating the LHS of $P(n+1)$ to look like the RHS.
First, foil $(\sum_{i=1}^{n+1} a_i b_i)^2 = (a_1 b_1 + a_2 b_2 + ... + a_{n+1} b_{n+1})(a_1 b_1 + a_2 b_2 + ... + a_{n+1} b_{n+1}) = \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_i b_i a_j b_j$

Second, foil $\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2$

$= \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_i^2 b_j^2 - 2a_i b_i a_j b_j + a_j^2 b_i^2$

$= \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_i^2 b_j^2 - \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_i b_i a_j b_j + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_j^2 b_i^2$

We can swap the index variables j and i, so $\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_j^2 b_i^2 = \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_i^2 b_j^2$

$\Rightarrow \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 = \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_i^2 b_j^2 - \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_i b_i a_j b_j$

Thus, the LHS of $P(n+1)$ is equivalent to $\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_i^2 b_j^2$

Factoring the above, we obtain $\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_i^2 b_j^2 = (a_1^2 + a_2^2 + ... + a_{n+1}^2)(b_1^2 + b_2^2 + ... + b_{n+1}^2) = (\sum_{i=1}^{n+1} a_i^2)(\sum_{i=1}^{n+1} b_i^2)$ which is exactly the RHS of $P(n+1)$. Thus, $P(n+1)$ holds.
Hence, by induction, $P(n)$ holds for all n.
QED

Because the square of a number is always non-negative

$(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i b_i)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$

Taking the square root of both sides

$|\sum_{i=1}^{n} a_i b_i| \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$

Thus, we obtain the Cauchy-Schwarz innequality.

QED

Foiling $(a_i + b_i)^2$, we obtain $\sqrt{\sum_{i=1}^{n}(a_i + b_i)^2} = \sqrt{\sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2}$

Using the Cauchy-Schwarz innequality, $\sqrt{\sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} + \sum_{i=1}^{n} b_i^2}$

Factoring $\sqrt{\sum_{i=1}^{n} a_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} + \sum_{i=1}^{n} b_i^2}$ we obtain $\sqrt{(\sqrt{\sum_{i=1}^{n} a_i} + \sqrt{\sum_{i=1}^{n} b_i})^2}$

Cancelling the square with the square root, we obtain $|\sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}| = \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$

$\Rightarrow \sqrt{\sum_{i=1}^{n}(a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$

Thus, we obtain the triangle innequality.

QED

Problem 2: Exercise 1.1.6:

a) For $x \in \mathbf{R}^n$ $d(x,x) = \sqrt{\sum_{i=1}^{n}(x_i - x_i)^2} = \sqrt{\sum_{i=1}^{n} 0} = 0$

b) For $x, y \in \mathbf{R}^n$ $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \geq 0$ because the square of a number is always non-negative.

c) For $x, y \in \mathbf{R}^n$ $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = d(y,x)$

d) For $x, y, z \in \mathbf{R}^n$ $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n}((x_i - z_i) + (z_i - y_i))^2} \leq \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2} = d(x,z) + d(y,z)$ by the Cauchy-Schwarz innequality.

Hence, $d(\mathbf{R}^n, d_{l1})$ is a metric space.
QED

Problem 3: Exercise 1.1.16

($\Rightarrow$)By the triangle innequality, $\lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty}(d(x_n, x) + d(y_n, x)) \leq \lim_{n\to\infty}(d(x_n, x) + d(y_n, y) + d(y, x))$
Because x_n converges to x and y_n converges to y, $\lim_{n\to\infty}(d(x_n, x)d(y_n, y)) = 0$
Thus, $\lim_{n\to\infty} d(x_n, y_n) \leq d(x, y)$
($\Leftarrow$) Similar to the forward direction, we can use the triangle innequality to show $d(x, y) \leq \lim_{n\to\infty} d(x, x_n) + d(y, x_n) \leq \lim_{n\to\infty}(d(x, x_n) + d(y, y_n) + d(x_n, y_n))$
Because x_n converges to x and y_n converges to y, $\lim_{n\to\infty}(d(x_n, x)d(y_n, y)) = 0$
Thus $d(x, y) \leq \lim_{n\to\infty} d(x_n, y_n)$
Because $d(x, y) \leq \lim_{n\to\infty} d(x_n, y_n)$ and $d(x, y) \geq \lim_{n\to\infty} d(x_n, y_n)$, this implies $d(x, y) = \lim_{n\to\infty} d(x_n, y_n)$
QED

Problem 4: Exercise 1.2.2

We WTS $a \Rightarrow b$ then $b \Rightarrow c$ then $c \Rightarrow a$

1) $a \Rightarrow b$
   If x_0 is an adherent point, $\forall r > 0, B(x_0, r) \cap E \neq \emptyset$
   The definition of an exterior point is the logical negation of an adherent point. This implies x_0 is not an exterior point of E.
   Since x_0 is not an exterior point of E, it must either be an interior or boundary point.

2) $b \Rightarrow c$
   If x_0 is an interior point of E $\exists r_1 > 0$ s.t $B(x_0, r) \subset E$
   If x_0 is a boundary point of E $\forall r > 0$ $\exists x \in B(x_0, r)$ s.t $x \in E$.
   In either case, we can construct a sequence x_n s.t $\forall n \in \mathbb{N}$ choose $x_n \in E$ s.t $d(x_n, x) < \frac{r_1}{n} \Rightarrow \lim_{n\to\infty} d(x_n, x) = 0$
   Thus, there exists a sequence $(x_n)_{n=1}^{\infty}$ in E that converges to x_0.

3) $c \Rightarrow a$
   We will prove this by contradiction.
   Assume to the contrary there exists a sequence $(x_n)_{n=1}^{\infty}$ in E that converges to x_0 and x_0 is an exterior point i.e $\exists r > 0$ s.t $B(x_0, r) \cap E = \emptyset$.

It follows if $d(x_0, x_n) < r$ $\exists n \in \mathbb{N}$ s.t $x_n \notin E$. This is a contradiction because $(x_n)_{n=1}^{\infty}$ is a sequence in E.
Thus, x_0 must be an adherent point.

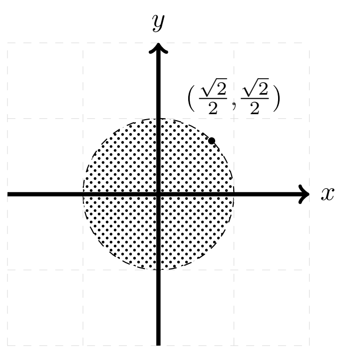
By transitivity $a \Leftrightarrow b \Leftrightarrow c$ (i.e if $a \Rightarrow b$ and $b \Rightarrow c$ then $a \Rightarrow c$). Thus, a,b, and c are equivalent.
QED

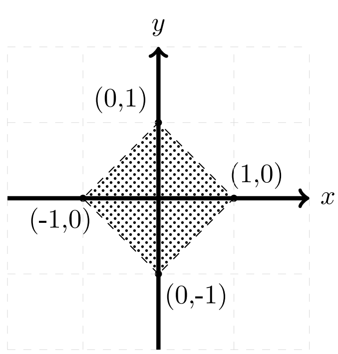
Problem 5: Exercise 1.2.4

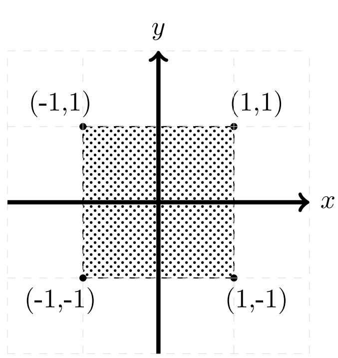
a) We will prove this by contradiction
Assume to the contrary $\exists x \in \overline{B}$ s.t $x \notin C \Rightarrow d(x, x_0) > r$.
By density, $\exists a$ s.t $d(x, x_0) - r > a > 0$.
By Prop 1.2.10, since x is an adherent point of B, there exists a sequence $(x_n)_{n=1}^{\infty}$ in B s.t $\lim_{n\to\infty} x_n = x$.
This is a contradiction because by the triangle innequality, $\lim_{n\to\infty} d(x, x_n) \geq \lim_{n\to\infty} d(x, x_0) - d(x_n, x_0) > r + a - r = a > 0 \Rightarrow \lim_{n\to\infty} d(x_n, x) \neq 0$.
Hence, $d(x, x_0) \leq r \Rightarrow x \in C$.

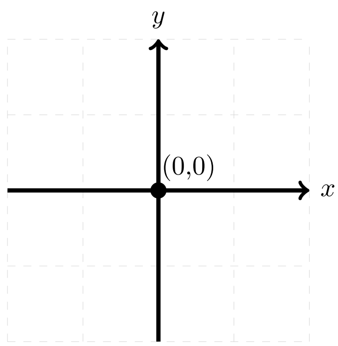
b) Let $(\mathbb{R}, d_{disc})$ be the metric space with $r = 1, x_0 = 0$,
$\overline{B} = 0, C = \mathbb{R} \Rightarrow \overline{B} \subset C$ and $\overline{B} \neq C$

Problem 6: Additional Problems

a) 

b) 

c) 



d)

Problem 7: Additional Problems

Let x_0 and r be arbitrary.
$\forall x \in V(x_0, r)$ choose $\epsilon(x) < d(x, x_0) - r$.
WTS, for any x, $B(x, \epsilon(x)) \subset V(x_0, r)$.
It suffices to show, for some arbitrary $b \in B(x, \epsilon(x))$, $d(b, x_0) > r \Rightarrow b \in V(x_0, r)$.
By the triangle innequality, $d(x_0, b) \geq d(x_0, x) - d(x, b)$.
$x \in V(x_0, r) \Rightarrow d(x_0, x) > r$ and $b \in B(x, \epsilon(x)) \Rightarrow d(x, b) < \epsilon(x) < d(x, x_0) - r$.
It follows $d(x_0, x) - d(x, b) > r \Rightarrow d(x_0, b) > r \Rightarrow b \in V(x_0, r)$.
Thus, $B(x, \epsilon(x)) \subset V(x_0, r)$ for any $x \in V(x_0, r) \Rightarrow x$ is an interior point of $V(x_0, r)$, $\forall x \in V(x_0, r)$.
Hence, $V(x_0, r)$ is an open set.