Math 106: Group Project

Batman Begins

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d'Alembert's Lemma

If $p(z)$ is a nonconstant polynomial function and $p(z_0) \neq 0$, then any neighborhood of z_0 contains a point z_1 s.t $|p(z_1)| < |p(z_0)|$. Pf: Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be an n^{th} degree polynomial. Let $z_0 \in \mathbb{C}$ s.t $p(z_0) \neq 0$. Our goal is to find some Δz s.t $|p(z_0 + \Delta z)| < |p(z_0)|$.

$$p(z_0 + \Delta z) = a_n (z_0 + \Delta z)^n + a_{n-1}(z_0 + \Delta z)^{n-1} + \cdots + a_0$$

Let $0 \leq k \leq n$ be an integer. By the binomial formula,

$$(z + \Delta z)^k = \sum_{i=0}^{k} \binom{k}{i} z_0^{k-i} \Delta z^i = z_0^k + \sum_{i=1}^{k} \binom{k}{i} z_0^{k-i} \Delta z^i$$

Let $A_k := a_k \sum_{i=1}^{k} \binom{k}{i} z_0^{k-i}$. It follows

$$\begin{aligned}
&p(z_0 + \Delta z) = a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_0 + A_n \Delta z^n + A_{n-1} \Delta z^{n-1} + \cdots + A_1 \Delta z \\
&= p(z_0) + A_n \Delta z^n + A_{n-1} \Delta z^{n-1} + \cdots + A_1 \Delta z \\
&= p(z_0) + A_1 \Delta z + O(\Delta z^2) \\
&|\Rightarrow p(z_0 + \Delta z)| < |p(z_0)|
\end{aligned}$$

for sufficiently small Δz.

d'Alembert's Proof

Consider an arbitrary nonconstant polynomial function p. We observe that that $p(z)$ scales with $a_n z^n$ for large values of $|z|$. Thus, the continuous function $|p(z)|$ is increasing for $|z| > R$ for some sufficiently large R. The set of points $|z| \leq R$ is a closed ball, so the continuous function $|p(z)|$ assumes a maximum and minimum value by the extreme value theorem. Clearly, this minimum $0 \leq |p(z^*)|$ because $|\cdot|$ is a norm. This minimum value $0 = |p(z^*)|$ because any other minimum value would contradict d'Alembert's lemma.

Gauss's Proof

Consider an arbitrary nonconstant polynomial function p. We observe that that $p(z)$ scales with $a_n z^n$ for large values of $|z|$. Thus, the continuous function $|p(z)|$ is increasing for $|z| > R$ for some sufficiently large R. Inside the ball of radius R, we consider the curves $\Re[p(z)]$ and $\Im[p(z)]$.
Let $z = x + iy$

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$
$$= a_n (x+iy)^n + a_{n-1}(x+iy)^{n-1} + \cdots + a_0$$
$$(x+iy)^k = \sum_{j=1}^{k} \binom{k}{j} x^{k-j} (iy)^j$$

Observe for any term k, the odd values of j will contribute to the imaginary part of the polynomial $p(z)$ because $\binom{k}{j} x^{k-j}(iy)^j = ix^{k-j}y^j(i)^{j-1} = \pm ix^{k-j}y^j$. In addition, the even values of j will contribute to the real part because $\binom{k}{j} x^{k-j}(iy)^j = \binom{k}{j} x^{k-j}y^j(i)^j = \pm \binom{k}{j} x^{k-j}y^j$.
Collecting the real and imaginary terms, it follows $p_1(x,y) = \Re[p(z)]$ and $p_2(x,y) = \Im[p(z)]$ are polynomials of degrees $\leq n$.