Math 151A: Problem Set 4

Prof. Schaeffer

Instructions:

- Due on Friday, May 12th by 1pm
- Late HW will not be accepted (but this assignment will be accepted until Monday, May 15th by 1pm)
- Write down all of the details and attach your code to the end of the assignment for full credit (as a PDF).
- If you LaTeX your solutions, you will get 5% extra credit.
- (T) are "pencil-and-paper" problems and (C) means that the problem includes a computational/programming component.

Problem 1: (T) Lagrange Polynomials and Neville's Method

Use Neville's method to obtain approximations to $f(0.1)$ using the Lagrange interpolating polynomials of degrees one, two, and three if
$f(0) = 1$, $f(0.25) = 1.65$, $f(0.5) = 2.72$, and $f(0.75) = 4.48$. You should explicitly compute the table associated with Neville's method:

x_i	$x - x_i$	$Q_{i,0}$	$Q_{i,1}$	$Q_{i,2}$	$Q_{i,3}$
0	0.1	1	-	-	-
0.25	$0.1 - 0.25$	1.65	$\frac{f(0.25)0.1 - f(0)(0.1-0.25)}{0.25-0}$	-	-
0.5	$0.1 - 0.5$	2.72	$\frac{f(0.5)(0.1-0.25) - f(0.25)(0.1-0.5)}{0.5-0.25}$	$\frac{\frac{1.07\cdot 0.1+0.145}{0.25}(0.1) - \frac{0.65\cdot 0.1+0.25}{0.25}(0.1-0.5)}{0.5-0}$	-
0.75	$0.1 - 0.75$	4.48	$\frac{f(0.75)(0.1-0.5) - f(0.5)(0.1-0.75)}{0.75-0.5}$	$\frac{\frac{1.76\cdot 0.1-0.2}{0.25}(0.1-0.25) - \frac{1.07\cdot 0.1+0.145}{0.25}(0.1-0.75)}{0.75-0.25}$	$\frac{\frac{0.69(0.1)^2+0.0175\cdot 0.1+0.15875}{0.125}(0.1) - \frac{0.42(0.1)^2+0.22\cdot 0.1+0.125}{0.125}(0.1-0.75)}{0.75-0}$

$P_3(x) = 2.88x^3 + 1.2x^2 + 2.12x + 1$, $P_3(0.1) = 1.22688$
(The true value is $f(0.1) = 1.2214$.)

Problem 2: (T) Lagrange Polynomials and Neville's Method

Suppose $x_j = j$, for $j = 0, 1, 2, 3$, and it is known that:

$$P_{0,1}(x) = 2x + 1,\ P_{0,2}(x) = x + 1,\ P_{1,2,3}(2.5) = 3$$

Determine $P_{0,1,2,3}(2.5)$.

Solution:

$P_{0,1,2}(2.5) = \frac{P_{0,2}(2.5)(2.5-1) - P_{0,1}(2.5)(2.5-2)}{2-1}$

$P_{0,1,2,3}(2.5) = \frac{P_{1,2,3}(2.5)(2.5) - P_{0,1,2}(2.5)(2.5-3)}{3-0} = \frac{(3)(2.5) - ((3.5)(1.5) - (6)(0.5))}{3} = 2.875$

Problem 3: (T) Lagrange Polynomials and Neville's Method

Suppose $x_j = 2\,j$, for $j = 0, 1, 2, 3, 4$ and it is known that:

$$P_{1,2}(1) = 2, \qquad P_{1,2,3}(1) = 1, \quad P_{1,4}(1) = 6.$$

Determine $P_{1,2,3,4}(1)$.

Solution:

$P_{1,2,4}(1) = \frac{P_{1,4}(1)(1-2)-P_{1,2}(1)(1-4)}{4-2}$

$P_{1,2,3,4}(1) = \frac{P_{1,2,4}(1)(1-3)-P_{1,2,3}(1)(1-4)}{4-3} = \frac{(6)(-1)-(2)(-3)}{2}(-2) - (1)(-3) = 3$

Problem 4: (T) Newton's Divided Differences

a) Find the degree-2 interpolating polynomial via Newton's divided difference for $f(x) = \frac{x}{1+x}$ using nodes $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$.

b) What are the degree-2 interpolating polynomials associated with Lagrange's construction and Neville's construction? Compare them to the solution of Part (a).

Solution:

a) Using nodes $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$, $P_2(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x - x_0) + \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}(x - x_0)(x - x_1)$
$= 0 + \frac{\frac{1}{2}-0}{1-0}(x - 0) + \frac{\frac{\frac{2}{3}-\frac{1}{2}}{2-1} - \frac{\frac{1}{2}-0}{1-0}}{2-0}(x - 0)(x - 1)$
$= \frac{2}{3}x - \frac{1}{6}x^2$

b) Nevilles's:$P_2(x) = \frac{2}{3}x - \frac{1}{6}x^2$
Lagrange:$P_2(x) = \frac{2}{3}x - \frac{1}{6}x^2$
The degree-2 interpolating polynomials associated with Lagrange's and Neville's construction will be the same by uniqueness.

Problem 5: (T) Numerical Differentiation/Finite Difference

Show that the following finite difference formula is first-order accurate:

$$f^{(2)}(x) = \frac{f(x+2\,h) - 2f(x+h) + f(x)}{h^2} + O(h).$$

For full credit, you must state any assumptions on $f(x)$ and justify each step in your solution. Hint: Apply Taylor's theorem to each of the terms on the right-hand side.

Solution:

Suppose $x \in (a,b)$ where $f \in C^3[a,b]$ and $h \neq 0$ is sufficiently small to ensure $x+h, x+2h \in (a,b)$

Taylor expansions of $f(x+2h)$ and $f(x+h)$ about x:

$f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2} \cdot f''(x) + \frac{h^3}{6} \cdot f'''(\xi_1(x))$

$f(x+2h) = f(x) + 2h \cdot f'(x) + \frac{4h^2}{2} \cdot f''(x) + \frac{8h^3}{6} \cdot f'''(\xi_2(x))$

$\Rightarrow f(x+2h) - 2f(x+h) + f(x) = h^2 \cdot f''(x) + h^3 \cdot (\frac{4}{3} f'''(\xi_2(x)) - \frac{1}{3} f'''(\xi_1(x)))$

$\Rightarrow f''(x) = \frac{f(x+2h)-2f(x+h)+f(x)}{h^2} - h \cdot (\frac{4}{3} f'''(\xi_2(x)) - \frac{1}{3} f'''(\xi_1(x)))$ by algebra.

Thus, $O(h) = -h \cdot (\frac{4}{3} f'''(\xi_2(x)) - \frac{1}{3} f'''(\xi_1(x)))$ which is linear.

Hence $f''(x) = \frac{f(x+2h)-2f(x+h)+f(x)}{h^2} + O(h)$ is first order accurate.