Math 164: Problem Set 2

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3.2 **a.** $\begin{bmatrix}\mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3}\end{bmatrix}\begin{bmatrix}1 & 2 & 4\\ 3 & -1 & 5\\ -4 & 5 & 3\end{bmatrix} = \begin{bmatrix}\mathbf{e'_1} & \mathbf{e'_2} & \mathbf{e'_3}\end{bmatrix}$

$$\Rightarrow \boldsymbol{T} = \begin{bmatrix}1 & 2 & 4\\ 3 & -1 & 5\\ -4 & 5 & 3\end{bmatrix}^{-1} = \frac{1}{42}\begin{bmatrix}28 & -14 & -14\\ 29 & -19 & -7\\ -11 & 13 & 7\end{bmatrix}$$

b. $\begin{bmatrix}\mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3}\end{bmatrix} = \begin{bmatrix}\mathbf{e'_1} & \mathbf{e'_2} & \mathbf{e'_3}\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 1 & -1 & 0\\ 3 & 4 & 5\end{bmatrix}$

$$\Rightarrow \boldsymbol{T} = \begin{bmatrix}1 & 2 & 3\\ 1 & -1 & 0\\ 3 & 4 & 5\end{bmatrix}$$

3.3 $\begin{bmatrix}\mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3}\end{bmatrix} = \begin{bmatrix}\mathbf{e'_1} & \mathbf{e'_2} & \mathbf{e'_3}\end{bmatrix}\begin{bmatrix}2 & 2 & 3\\ 1 & -1 & 0\\ -1 & 2 & 1\end{bmatrix}$

$$\Rightarrow \boldsymbol{T} = \begin{bmatrix}2 & 2 & 3\\ 1 & -1 & 0\\ -1 & 2 & 1\end{bmatrix}$$

$$\Rightarrow \boldsymbol{T}^{-1} = \begin{bmatrix}1 & -4 & 3\\ 1 & -5 & -3\\ -1 & 6 & 4\end{bmatrix}$$

$$\Rightarrow \boldsymbol{B} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}$$

$$\Rightarrow \boldsymbol{B} = \begin{bmatrix}3 & -10 & -8\\ -1 & 8 & 4\\ 2 & -13 & -7\end{bmatrix}$$

3.5 $\det(\lambda I_4 - \boldsymbol{A}) = (\lambda+1)\det(\lambda I_3 - \begin{bmatrix}1 & 0 & 0\\ 5 & 2 & 1\\ 1 & 0 & 3\end{bmatrix})$

$$= (\lambda+1)(\lambda-1)\det(\lambda I_2 - \begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}) = (\lambda+1)(\lambda-1)(\lambda-2)(\lambda-3)$$

$$\begin{bmatrix} 3+1 & 0 & 0 & 0 \\ -1 & 3-1 & 0 & 0 \\ -2 & -5 & 3-2 & -1 \\ 1 & -1 & 0 & 3-3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+1 & 0 & 0 & 0 \\ -1 & 2-1 & 0 & 0 \\ -2 & -5 & 2-2 & -1 \\ 1 & -1 & 0 & 2-3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$x_1 + 2x_2 = 0$
$2x_1 + 5x_2 + 3x_3 + x_4 = 0$
$x_1 - x_2 - 4x_4 = 0$
$\Rightarrow x_2 + 3x_3 + x_4 = 0$
$-3x_2 - 4x_4 = 0$
$\Rightarrow 3x_3 - \frac{1}{3}x_4$
$$\begin{bmatrix} -1+1 & 0 & 0 & 0 \\ -1 & -1-1 & 0 & 0 \\ -2 & -5 & -1-2 & -1 \\ 1 & -1 & 0 & -1-3 \end{bmatrix} \begin{bmatrix} 24 \\ -12 \\ 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$x_1 = 0$
$2x_1 + 5x_2 + x_3 + x_4 = 0$
$x_1 - x_2 - x_4 = 0$
$\Rightarrow 5x_2 + x_3 + x_4 = 0$
$x_2 + 2x_4 = 0$
$9x_4 = x_3$
$$\begin{bmatrix} 1+1 & 0 & 0 & 0 \\ -1 & 1-1 & 0 & 0 \\ -2 & -5 & 1-2 & -1 \\ 1 & -1 & 0 & 1-3 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
Giving us eigenvectors $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 24 \\ -12 \\ 1 \\ 9 \end{bmatrix}, v_4 \begin{bmatrix} 0 \\ -2 \\ 9 \\ 1 \end{bmatrix}$ with corresponding eigenvalues $3, 2, -1, 1$ for $\boldsymbol{A}$.
Let $[v_1, v_2, v_3, v_4]$ be our new basis for $\mathbb{R}^4$. $\boldsymbol{T} = [v_1, v_2, v_3, v_4]^{-1}[e_1, e_2, e_3, e_4] = [v_1, v_2, v_3, v_4]^{-1}$ If $\boldsymbol{B}$ is our linear transformation with respect to basis $[v_1, v_2, v_3, v_4]$, then $\boldsymbol{B} = \boldsymbol{TAT}^{-1}$
$= [v_1, v_2, v_3, v_4]^{-1}\boldsymbol{A}[v_1, v_2, v_3, v_4] = [v_1, v_2, v_3, v_4]^{-1}[\lambda_1 v_1, \lambda_2 v_2, \lambda_3 v_3, \lambda_4 v_4] =$
$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.6 We can rewrite the characteristic polynomial for $\boldsymbol{I_n} - \boldsymbol{A}$, $\det(\lambda' \boldsymbol{I_n} - (\boldsymbol{I_n} - \boldsymbol{A}))$, as $\det((\lambda' - 1)\boldsymbol{I_n} + \boldsymbol{A})$.
It follows $\det((\lambda' - 1)\boldsymbol{I_n} + \boldsymbol{A}) = (-1)^n \det((1 - \lambda')\boldsymbol{I_n} - \boldsymbol{A})$. If λ_i $i = 1 \ldots n$ are the eigenvalues of $\boldsymbol{A}$, then $\det(\lambda \boldsymbol{I_n} - \boldsymbol{A}) = 0$ if $\lambda = \lambda_i$ for $i = 1 \ldots n$.

Take $\lambda_i' = 1 - \lambda_i \Rightarrow (-1)^n \det((1 - \lambda_i')\boldsymbol{I_n} - \boldsymbol{A})$
$= (-1)^n \det((1 - (1 - \lambda_i)')\boldsymbol{I_n} - \boldsymbol{A})$
$= (-1)^n \det(\lambda_i \boldsymbol{I_n} - \boldsymbol{A}) = 0.$
Hence, $\det(\lambda' \boldsymbol{I_n} - (\boldsymbol{I_n} - \boldsymbol{A})) = 0$ if $\lambda' = 1 - \lambda_i \Leftrightarrow 1 - \lambda_i$ are the eigenvalues of $\boldsymbol{I_n} - \boldsymbol{A}$.

3.8 $\mathcal{N}(A) \triangleq \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{Ax} = 0\}$. $\det(0\boldsymbol{I_3} - \mathbf{A}) = 4(-2) - (-2)(4) = 0$ Row-reduce $\mathbf{A}$ to reduced row-echelon form:

$$\begin{bmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ 0 & 0 & 0 \\ 2 & -3 & 1 \end{bmatrix} R_2 = R_2 + R_3 - R_1,$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -3 & 1 \end{bmatrix} R_1 = \tfrac{1}{2} R_1,$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} R_3 = R_3 - R_1,$$

$\Rightarrow 2x_1 - x_2 = 0, -2x_2 + x_3 = 0$. Let $x_2 = s \Rightarrow x_1 = \frac{s}{2}, x_3 = 2s$.

Thus, $\mathcal{N}(A) = span(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix})$.

3.16 All principal minors are nonnegative:
$\det(2) = 2$,
$\det \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 0$,
$\det(\boldsymbol{A}) = 2\det \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} - 2\det \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} + 2\det \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 0.$
$\det(\lambda I_3 - \boldsymbol{A}) = (\lambda - 2)\det \begin{bmatrix} \lambda - 2 & 2 \\ -2 & \lambda \end{bmatrix} + 2\det \begin{bmatrix} -2 & -2 \\ -2 & \lambda \end{bmatrix} - 2\det \begin{bmatrix} -2 & \lambda - 2 \\ -2 & 2 \end{bmatrix}$
$= (\lambda - 2)(\lambda^2 - 2\lambda + 4) + 2(-2\lambda - 4) - 2(-8 + 2\lambda)$
$= \lambda^3 - 2\lambda^2 + 4\lambda - 2\lambda^2 - 4\lambda - 8 - 4\lambda - 8 + 16 - 4\lambda$
$= \lambda^3 - 4\lambda^2 - 8\lambda.$
$\det((-1)I_3 - \boldsymbol{A}) = 3, \det((-2)I_3 - \boldsymbol{A}) = -8$, so by IVT, $\boldsymbol{A}$ must have a negative eigenvalue. Thus, $\boldsymbol{A}$ cannot be positive semidefinite.

3.17 (a) $\boldsymbol{Q}$ is said to be indefinite if the matrix takes on both positive and negative eigenvalues. $\det(\lambda I_3 - \boldsymbol{Q}) = \lambda \det \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} + 1\det \begin{bmatrix} -1 & -1 \\ -1 & \lambda \end{bmatrix} - 1\det \begin{bmatrix} -1 & \lambda \\ -1 & -1 \end{bmatrix} = \lambda(\lambda - 1)(\lambda + 1) - 2(\lambda + 1) = (\lambda^2 - \lambda - 2)(\lambda + 1) = (\lambda + 1)^2(\lambda - 2) \Rightarrow \boldsymbol{Q}$ has both positive and negative eigenvalues. Thus, $\boldsymbol{Q}$ is indefinite.

(b) $\mathbf{x}^\top \boldsymbol{Q}\mathbf{x} = \langle \mathbf{x}, \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \end{bmatrix} \rangle = \langle \mathbf{x}, \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix} \rangle = \langle \mathbf{x}, -\mathbf{x} \rangle = -\|\mathbf{x}\|^2 < 0$ for all $\mathbf{x} \neq 0$, so $\boldsymbol{Q}$ is negative definite.

3.18 (a) $f(x_1,x_2,x_3)=x_2^2$. For $x\in\mathbb{R}, x^2\geq 0 \Rightarrow x_2^2\geq 0$ for all $\mathbf{x}\in\mathbb{R}^3$. Thus, f is positive semidefinite.

(b) $f(x_1,x_2,x_3) = x_1^2+2x_2^2-x_1x_3 = \mathbf{x}^\top \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \mathbf{x}$. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$.

$\det(\lambda I_3 - \boldsymbol{A}) = \det \begin{bmatrix} \lambda-1 & 0 & \frac{1}{2} \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-\frac{1}{4(\lambda-1)} \end{bmatrix} = (\lambda-2)(\lambda^2-\lambda-\frac{1}{4}) = (\lambda-2)(\lambda-\frac{1}{2}(1-\sqrt{2}))(\lambda-\frac{1}{2}(1+\sqrt{2}))$. Because $\boldsymbol{A}$ has positive and negative eigenvalues, $\boldsymbol{A}$ is indefinite.

(c) $f(x_1,x_2,x_3) = x_1^2+x_3^2+2x_1x_2+2x_1x_3+2x_2x_3 = \mathbf{x}^\top \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x}$.

Let $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. $\det(\lambda I_3 - \boldsymbol{A}) = (\lambda-1)\det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda-1 \end{bmatrix} + \det\begin{bmatrix} -1 & -1 \\ -1 & \lambda-1 \end{bmatrix} - \det\begin{bmatrix} -1 & \lambda \\ -1 & -1 \end{bmatrix} = \lambda^3-2\lambda^2-2\lambda = \lambda(\lambda-1+\sqrt{3})(\lambda-1-\sqrt{3})$. Because $\boldsymbol{A}$ has positive and negative eigenvalues, $\boldsymbol{A}$ is indefinite.

3.20 $f(x_1,x_2,x_3) = x_1^2+x_2^2+5x_3^2+2\xi x_1x_2-2x_1x_3+4x_2x_3 = \mathbf{x}^\top \begin{bmatrix} 1 & \xi & -1 \\ \xi & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix} \mathbf{x}$

Suffices to find values for ξ s.t $\det\begin{bmatrix} 1 & \xi \\ \xi & 1 \end{bmatrix} > 0$ and $\det\begin{bmatrix} 1 & \xi & -1 \\ \xi & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix} > 0$

$\det\begin{bmatrix} 1 & \xi \\ \xi & 1 \end{bmatrix} = 1-\xi^2 \Rightarrow -1<\xi<1$

$\det\begin{bmatrix} 1 & \xi & -1 \\ \xi & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix} = 1\det\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} - \xi\det\begin{bmatrix} \xi & 2 \\ -1 & 5 \end{bmatrix} - 1\begin{bmatrix} \xi & 1 \\ -1 & 2 \end{bmatrix}$

$= 1-\xi(5\xi+2)-(2\xi+1) = 1-5\xi^2-2\xi-2\xi-1 = -5\xi^2-4\xi \Rightarrow \xi(5\xi+4) < 0 \Rightarrow -\frac{4}{5}<\xi<0$.

Thus, $-\frac{4}{5}<\xi<0$ for the quadratic form to be positive definite.

3.21 We are given $\boldsymbol{Q}$ is a symmetric positive definite matrix.

Positivity: Given that $\boldsymbol{Q}$ is positive definite $\mathbf{x}^\top\boldsymbol{Q}\mathbf{x}>0$ for all $\mathbf{x}\neq 0$, and $\mathbf{x}^\top\boldsymbol{Q}\mathbf{x}=0$ if $\mathbf{x}=0$.

Symmetry: $\langle\mathbf{x},\mathbf{y}\rangle_{\boldsymbol{Q}} = \mathbf{x}^\top\boldsymbol{Q}\mathbf{y} = (\boldsymbol{Q}^\top\mathbf{x})^\top\mathbf{y} = (\boldsymbol{Q}\mathbf{x})^\top\mathbf{y} = \langle\boldsymbol{Q}\mathbf{x},\mathbf{y}\rangle = \langle\mathbf{y},\boldsymbol{Q}\mathbf{x}\rangle = \mathbf{y}^\top\boldsymbol{Q}\mathbf{x} = \langle\mathbf{y},\mathbf{x}\rangle_{\boldsymbol{Q}}$

$$
\begin{aligned}
\text{Additivity:}\quad & \langle \mathbf{x}+\mathbf{y},\mathbf{z}\rangle_{Q} = \langle \mathbf{z},\mathbf{x}+\mathbf{y}\rangle_{Q} = \mathbf{z}^{\top}\boldsymbol{Q}(\mathbf{x}+\mathbf{y}) = (\boldsymbol{Q}\mathbf{z})^{\top}(\mathbf{x}+\mathbf{y}) \\
& = \langle \boldsymbol{Q}\mathbf{z},\mathbf{x}+\mathbf{y}\rangle = \langle \boldsymbol{Q}\mathbf{z},\mathbf{x}\rangle + \langle \boldsymbol{Q}\mathbf{z},\mathbf{y}\rangle = \mathbf{z}^{\top}\boldsymbol{Q}\mathbf{x} + \mathbf{z}^{\top}\boldsymbol{Q}\mathbf{y} \\
& = \langle \mathbf{z},\mathbf{x}\rangle_{Q} + \langle \mathbf{z},\mathbf{y}\rangle_{Q} = \langle \mathbf{x},\mathbf{z}\rangle_{Q} + \langle \mathbf{y},\mathbf{z}\rangle_{Q} \\
\text{Homogeneity:}\quad & \langle \mathbf{r}\mathbf{x},\mathbf{y}\rangle_{Q} = (\mathbf{r}\mathbf{x})^{\top}\boldsymbol{Q}\mathbf{y} = \mathbf{r}(\mathbf{x})^{\top}\boldsymbol{Q}\mathbf{y} = \mathbf{r}\langle \mathbf{x},\mathbf{y}\rangle_{Q}
\end{aligned}
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