Math 114L: Problem Set 1

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Question 1

Suppose for the sake of contradiction that T is not satisfiable. It follows by The Completeness Theorem for PL that T is not consistent. This implies there exists some ϕ s.t $T \vdash \phi$ and $T \vdash \neg\phi$. Because a deduction from a set of formulas T requires a finite sequence of steps, there exists a finite subset $T_0 \subseteq T$ that contains all the required formulas to prove ϕ and $(\neg\phi)$. It follows $T_0 \vdash \phi$ and $T_0 \vdash \neg\phi$, and by soundness, $T_0 \models \phi$ and $T_0 \models (\neg\phi)$. T_0 is satisfiable by assumption, so there exists v s.t $v \models T_0$. However, $v(\phi) =$T and $v((\neg\phi)) =$T cannot both be true. Thus, $T_0 \models \phi$ and $T_0 \models (\neg\phi)$ cannot both be true. Moreover, $T \vdash \phi$ and $T \vdash (\neg\phi)$ cannot both be true, so T is consistent. Because every consistent set of formulas is satisfiable, T must also be satisfiable.

Question 2

Suppose $\Gamma \cup \{\phi\}$ logically implies ψ. Consider some assignment v s.t $v \models \Gamma$. $v \models (\phi \to \psi)$ iff $v \not\models \phi$ or $v \models \psi$. If $v \models \phi$, then $v \models \Gamma \cup \{\phi\}$. Because $v \models \Gamma \cup \{\phi\}$, $v \models \psi$ by assumption. If $v \not\models \phi$ then $(\phi \to \psi)$ is vacuously true.
Suppose Γ logically implies $(\phi \to \psi)$. Consider some assignment v s.t $v \models \Gamma \cup \{\phi\}$. Then $v \models \phi$, and because $v \models \Gamma$, $v \models (\phi \to \psi)$. $v \models (\phi \to \psi)$ iff $v \not\models \phi$ or $v \models \psi$. Because $v \models \phi$, $v \models \psi$ must be true.

Question 3

A can see the hats of B and C. If A saw 2 white hats, A would deduce they are wearing a black hat because not all the hats are white. A's answer signals to B that B or C is wearing a black hat. If B saw C wearing a white hat, B would deduce they are wearing a black hat because otherwise A would have known the color of their (A's) own hat. Since neither A nor B knew the color of their hats, C can confidently conclude they are wearing a black hat.

Question 4

- Trivially, if $\Gamma_0 \models \phi$ then $\Gamma \models \phi$ for any $\Gamma_0 \subset \Gamma$.
  Suppose Γ logically implies ϕ. It follows $\Gamma \cup \{(\neg\phi)\}$ is not satisfiable. By compactness, there exists some $\Gamma' \subset \Gamma \cup \{(\neg\phi)\}$ that is not satisfiable. In particular, we want to choose a Γ' with the least number of elements. Let $\Gamma_0 = \Gamma' \setminus \{(\neg\phi)\}$. Thus, Γ_0 logically implies ϕ because $\Gamma_0 \cup \{(\neg\phi)\}$ is not satisfiable. Moreover, because of the way we chose Γ', for any proper $\Gamma_1 \subset \Gamma_0$, $\Gamma_1 \cup \{(\neg\phi)\}$ is satisfiable. In other words, Γ_0 is independent.

- Consider the infinite set $\Gamma = \{A_1, \neg(A_1 \to (\neg A_2)), \neg(A_1 \to (A_2 \to (\neg A_3))), \ldots\}$ for propositional variables $A_1, A_2, A_3, \ldots \in PL_0$.
  Case 1: Γ_0 is empty
  $\Gamma_0 \not\models A_1$
  Case 2: $\Gamma_0 = \{\psi_k\}$ contains one formula
  Let ψ_k be the k-th formula in the sequence. $\Gamma \models \psi_{k+1}$ but $\Gamma_0 \not\models \psi_{k+1}$ where $\psi_{k+1} \equiv \neg(\psi_k \to (\neg A_{k+1}))$.
  Case 3: WLOG let $m > k$ $\Gamma_0 = \{\psi_k, \psi_m, \ldots\}$ contains at least two formulas
  Γ_0 is logically equivalent to $\Gamma_0 \setminus \psi_k$, so Γ_0 is not independent.
  Γ_0 cannot be both logically equivalent to Γ and independent.

- If Γ is finite, we showed earlier in the problem we can find a logically equivalent and independent subset Γ_0. We set $\Delta = \Gamma_0$. If Γ is infinite, we showed that some sets have no logically equivalent and independent subsets. If that is the case, let $\Delta = \Gamma$. Otherwise, we choose Δ to be a logically equivalent and independent subset in a similar manner to a finite set.

Question 5

(a) $\Gamma = \{A_1, (\neg A_1)\}$

(b) $\Gamma = \{A_1, A_2, (A_1 \to (\neg A_2))\}$

(c) $\Gamma = \{A_1, A_2, A_3, (\neg(A_1 \to (\neg A_2)) \to (\neg A_3))\}$

Question 6

(a) $\forall i, j \in \{1, 2, \ldots, n\}, (A_{i,j} \to A_{j,i}) \land (\neg A_{i,i})$

(b) $(\forall i, j \in \{1, 2, \ldots, n\}, (A_{i,j} \to A_{j,i}) \land (\neg A_{i,i})) \land (\exists i \in \{1, 2, \ldots, n\}, \forall j \in \{1, 2, \ldots, n\} \to (\neg A_{i,j}))$

(c) $(\forall i, j \in \{1, 2, \ldots, n\}, (A_{i,j} \to A_{j,i}) \land (\neg A_{i,i})) \land (\forall i \in \{1, 2, \ldots, n\}, \exists j, k \in \{1, 2, \ldots, n\} j \neq k \land A_{i,k} \land A_{i,k})$

Question 7

If there are n propositional variables and each p_i can be assigned T or F, there are a total of 2^n ways to assign $\vec{p} = \{p_1, p_2, \ldots, p_n\}$. For each inequivalent formula χ_i $F_{\vec{p}}^{\chi_i}(\vec{x})$ is either T or F, giving us $\sum_{i=0}^{2^n} \binom{2^n}{i} = 2^{2^n}$ potential inequivalent formulas.
We'll show each one of those inequivalent formulas are achievable by induction.
Base case: $n = 1$

p_1	$\neg p_1$	$p_1 \wedge \neg p_1$	$p_1 \vee \neg p_1$
T	F	F	T
F	T	F	T

giving us the four possible inequivalent formulas.
Induction hypothesis: Assume for some n each of the 2^{2^n} possible inequivalent formulas are achievable.
Induction step:
We define each of the possible inequivalent functions as follows:
$(p_{n+1} \wedge \chi_i) \vee (\neg p_{n+1} \wedge \chi_j)$ $\forall i, j \in \{1, \ldots, N\}$ giving us a total of $N^2 = \left(2^{2^n}\right)^2 = 2^{2^{n+1}}$ possible inequivalent formulas. To show each is logically inequivalent, we consider $(p_{n+1} \wedge \chi_i) \vee (\neg p_{n+1} \wedge \chi_j)$ and $(p_{n+1} \wedge \chi_k) \vee (\neg p_{n+1} \wedge \chi_l)$. When p_{n+1} is assigned to be true, χ_i is equivalent to χ_k iff $i = k$, and when p_{n+1} is assigned to be false, χ_j is equivalent to χ_l iff $j = l$ by the induction hypothesis. Thus, $(p_{n+1} \wedge \chi_i) \vee (\neg p_{n+1} \wedge \chi_j)$ is equivalent to $(p_{n+1} \wedge \chi_k) \vee (\neg p_{n+1} \wedge \chi_l)$ iff $i = k$ and $j = l$.
Hence, by induction, there are a total of 2^{2^n} inequivalent formulas for n propositional variables.