Math 131B: Homework 2

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Problem 1. **Exercise 1.3.1**

First, suppose E is relatively closed with respect to Y. Then $Y \setminus E$ is relatively open with respect to Y. By Proposition 1.3.4 a), $\exists V \subseteq X$ which is open in X s.t $Y \cap V = Y \setminus E$. Let $K = X \setminus V$ which is closed in X because V is open in X. It follows $Y \cap K = Y \cap (X \setminus V)$. *note: If $x \in Y \cap (X \setminus V)$, then $x \in Y$ and $x \in X \setminus V$. Because $x \in Y$ and $Y \subseteq X, x \in Y \setminus V$. Thus, $Y \cap (X \setminus V) \subseteq Y \setminus V$. If $x \in Y \setminus V$ then $x \in Y$ and $x \notin V$. Because $Y \subseteq X$ and $x \notin V$, $x \in X \setminus V$. Because $x \in Y$ and $x \in X \setminus V$, $x \in Y \cap (X \setminus V)$. Thus, $Y \setminus V \subseteq Y \cap (X \setminus V)$. Because the two sets are subsets of each other, they must be equal.* Because $Y \setminus V = Y \cap (X \setminus V)$ and $K = (X \setminus V)$, it follows $Y \cap K = Y \setminus V$, which is equivalent to $Y \setminus (Y \cap V)$ (set minus operator is the subset of Y that doesn't have any shared elements with V). Because $Y \cap V = Y \setminus E$, it follows $Y \setminus (Y \cap V) = Y \setminus (Y \setminus E)$. Because $E \subseteq Y$, $Y \setminus (Y \setminus E) = E$. Thus, if E is relatively closed with respect to Y, $\exists K \subseteq X$ which is closed in X s.t $Y \cap K = E$.

Next, suppose $E = Y \cap K$ for some $K \subseteq X$ which is closed in X. Then, $X \setminus K$ is open in X. In the previous part of the proof, we noted that $Y \cap (X \setminus K) = Y \setminus (Y \cap K)$, and because $(X \setminus K)$ is open, $Y \setminus (Y \cap K)$ is reletively open with respect to Y by proposition 1.3.4 a). As we've shown previously $Y \setminus (Y \cap K) = Y \setminus E$, so E must be relatively closed with respect to Y.

Hence, E is relatively closed with respect to Y if and only if $E = K \cap Y$ for some set $K \subseteq X$ which is closed in X.

QED

Problem 2. **Exercise 1.4.4**

Let $\epsilon > 0$. Because the sequence $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence in (X, d), there exists an $N \geq m$ such that $d(x^{(j)}, x^{(k)}) < \frac{\epsilon}{2}$ for all $j, k \geq N$. Because the subsequence $(x^{(n_i)})_{i=1}^{\infty}$ converges to x_0, there exists $n_l \geq N$ where $l \geq 1$ s.t $d(x^{(n_l)}, x_0) < \frac{\epsilon}{2}$. It follows that because $n_l \geq N$, $d(x^{(n_l)}, x^{(k)}) < \frac{\epsilon}{2}$ for any $k \geq N$. By the triangle innequality, if $n \geq N$, then $d(x^{(n)}, x_0) \leq d(x^{(n)}, x^{(n_l)}) + d(x_0, x^{(n_l)}) < 2 \cdot \frac{\epsilon}{2} = \epsilon$. Hence, the sequence $(x^{(n)})_{n=m}^{\infty}$ converges to x_0.

QED

Problem 3. **Exercise 1.4.7**

a) Let $x_0 \in \overline{Y}$. It follows there exists a sequence $(x^{(n)})_{n=m}^{\infty}$ in Y s.t $\lim_{n\to\infty} d_{|YxY}(x^{(n)}, x_0) = 0$. Because $(x^{(n)})_{n=m}^{\infty}$ is convergent, it must be Cauchy. *note: To prove Lemma 1.4.7, suppose $(x^{(n)})_{n=m}^{\infty}$ converges to x_0. For $\epsilon > 0$, choose N large enough s.t for any $j, k \geq N$, $d(x^{(j)}, x_0) < \frac{\epsilon}{2}$ and $d(x^{(k)}, x_0) < \frac{\epsilon}{2}$. By the triangle innequality, $d(x^{(k)}, x^{(j)}) \leq d(x^{(k)}, x_0) + d(x^{(j)}, x_0) < 2 \cdot \frac{\epsilon}{2} = \epsilon$. Hence, $(x^{(n)})_{n=m}^{\infty}$ is Cauchy.* Because Y is complete, any Cauchy sequence in Y must be convergent in Y. Thus, $x_0 \in Y \Rightarrow \overline{Y} \subseteq Y$. Because $Y \subseteq X \Rightarrow Y \subseteq \overline{Y}$. Thus, $Y = \overline{Y}$, so Y is closed in X.

QED

b) Let $(x^{(n)})_{n=m}^{\infty}$ be a Cauchy sequence in Y. Because (X, d) is a complete metric space and $Y \subseteq X$, the Cauchy sequence $(x^{(n)})_{n=m}^{\infty}$ must converge to some point we will call x_0 in X. Since there exists a sequence $(x^{(n)})_{n=m}^{\infty}$ in Y that converges to x_0, x_0 must be adherent to Y. Because Y is closed in X, x_0 is an element of Y. Since $(x^{(n)})_{n=m}^{\infty}$ is an arbitrary Cauchy sequence in Y that is convergent in Y, the subspace $(Y, d|YxY)$ is complete.

QED

Problem 4. **Exercise 1.5.12**

a) Let $(x^{(n)})_{n=m}^{\infty}$ be a Cauchy sequence in X. Thus, there exists $N \geq m$ s.t for all $j, k \geq N$, $d_{disc}(x^{(k)}, x^{(j)}) < 1$. Because $d_{disc}(x^{(k)}, x^{(j)}) < 1$, $x^{(k)}$ must be equal to $x^{(j)}$. Let $\epsilon > 0$. If $n \geq N$, $d_{disc}(x^{(n)}, x^{(N)}) = 0 < \epsilon$. Because $x^{(N)}$ is an element of X, $(x^{(n)})_{n=m}^{\infty}$ is a convergent sequence in X.

QED

b) X is compact iff X is finite.

($\Rightarrow$) X is compact if any open cover $X \subseteq \bigcup_{\alpha \in I} V_\alpha$ can be reduced to a finite collection of subsets that still covers X. Suppose $\bigcup_{\alpha \in I} V_\alpha$ is a collection of open balls of radius less than 1. Thus, $\bigcup_{\alpha \in I} V_\alpha$ is a collection of singleton subsets. It follows $\bigcup_{\alpha \in I} V_\alpha$ can only be reduced to a finite $F \subseteq I$ if there are finitely many $x \in X$. If there are infinitely many $x \in X$, you'd need infinitely many singleton subsets to cover X.

($\Leftarrow$) Suppose X is finite. If $X \subseteq \bigcup_{\alpha \in I} V_\alpha$ then for each $x \in X$ $\exists \alpha \in I$ s.t $x \in V_\alpha$. Labeling each element of X $\{x_1, x_2, ..., x_n\}$ with corresponding $\{\alpha_1, \alpha_2, ..., \alpha_n\}$, we can create a finite $F \subseteq I$ where $F = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $\bigcup_{\alpha \in F} V_\alpha$ still covers X. Thus, X is compact. QED

Problem 5. **Exercise 1.5.15**

Assume to the contrary $\cap_{\alpha \in I} K_\alpha = \emptyset$. Because $(K_\alpha)\alpha \in I$ is a collection of closed sets in X, $(K_\alpha^c)\alpha \in I$ is a collection of open sets in X. Because $\cap_{\alpha \in I} K_\alpha = \emptyset$, it follows that $\cup_{\alpha \in I} K_\alpha^c = X$. Thus, $\cup_{\alpha \in I} K_\alpha^c$ is an open cover for X. By the compactness of X, there exists a finite subset $F \subseteq I$ s.t $\cup_{\alpha \in F} K_\alpha^c = X$. Therefore $\cap_{\alpha \in F} K_\alpha = \emptyset$ which is a contradiction by the property that any finite subcollection of I has a non-empty intersection. Thus, $\cap_{\alpha \in I} K_\alpha \neq \emptyset$.

QED

Problem 6. **Additional problem**

a) Let $(x^{(n)})_{n=1}^{\infty}$ denote the sequence $x^{(n)} = \frac{1}{n}$ in E. Let $k \geq 1$ be arbitrary and $\epsilon > 0$. If ϵ is small enough s.t $\epsilon < \frac{1}{k}$, there exists $N > \frac{k}{1 - \epsilon k}$ s.t if $n \geq N$, $d(x^{(n)}, \frac{1}{k}) > \frac{1}{k} - \frac{1 - \epsilon k}{k} = \epsilon$. Thus, for any k there exists ϵ sufficiently small s.t the set $\{x^{(n)} : d(x^{(n)}, \frac{1}{k}) < \epsilon\}$ is finite. Hence, $(x^{(n)})_{n=1}^{\infty}$ doesn't have a subsequential limit.

QED

b) Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in $E \cup \{0\}$. If $(x^{(n)})_{n=m}^{\infty}$ converges to 0, then $(x^{(n)})_{n=m}^{\infty}$ has a subsequential limit. If not, for some $\epsilon > 0$, there exists $n \geq N$ for every $N \geq m$ s.t $d(x^{(n)}, 0) \geq \epsilon$. It follows there exist infinitely many n where $d(x^{(n)}, 0) \geq \epsilon$. However there exist only finitely many $k \in E \cup \{0\}$ s.t $\epsilon \leq k$, so for all but finitely many points in $E \cup \{0\}$ are contained in $B(0, \epsilon)$. Thus, we are distributing infinitely many terms of the sequence $(x^{(n)})_{n=m}^{\infty}$ among finitely many points in $(E \cup \{0\}) \setminus B(0, \epsilon)$. By the pigeonhole principle, there exists at least one $k \in (E \cup \{0\}) \setminus B(0, \epsilon)$ that is a subsequential limit. Moreover, for each $\epsilon > 0$ and $N \geq m$, there exists $n \geq N$ s.t $d(x^{(n)}, k) < \epsilon$. Hence, $(x^{(n)})_{n=m}^{\infty}$ must have a subsequential limit.

QED