Math 151b: Problem Set 5

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Problem 1

(a)

$$y_{n+1} - y_n = h[\theta\lambda y_{n+1} + (1-\theta)\lambda y_n]$$
$$y_{n+1} - h\lambda\theta y_{n+1} = y_n + h\lambda(1-\theta)y_n$$
$$y_{n+1}(1 - z\theta) = (1 + z(1-\theta))y_n$$
$$y_{n+1} = \frac{1 + (1-\theta)z}{1 - z\theta} y_n$$

(b)

$$|w|^2 = w\overline{w} = \frac{1 + (1-\theta)z}{1 - z\theta}\frac{1 + (1-\theta)\overline{z}}{1 - \overline{z}\theta}$$
$$= \frac{1 + (1-\theta)(z+\overline{z}) + (1-\theta)^2|z|^2}{1 - \theta(z+\overline{z}) + \theta^2|z|^2}$$
$$\Rightarrow |w|^2 - 1 = \frac{1 + (1-\theta)(z+\overline{z}) + (1-\theta)^2|z|^2}{1 - \theta(z+\overline{z}) + \theta^2|z|^2} - \frac{1 - \theta(z+\overline{z}) + \theta^2|z|^2}{1 - \theta(z+\overline{z}) + \theta^2|z|^2}$$
$$= \frac{(1-2\theta)|z|^2 + (z+\overline{z})}{1 - \theta(z+\overline{z}) + \theta^2|z|^2} = \frac{(1-2\theta)|z|^2 + (z+\overline{z})}{|1-\theta z|^2}$$

Because $\theta, 1 \in \mathbb{R} \Rightarrow \overline{1 - \theta z} = 1 - \theta\overline{z}$

$$\Rightarrow 1 - \theta(z+\overline{z}) + \theta^2|z|^2 = \overline{1-\theta z}(1-\theta z) = |1-\theta z|^2$$

If $|w| < 1$ then $|w|^2 - 1 < 0 \Leftrightarrow \frac{(1-2\theta)|z|^2 + (z+\overline{z})}{|1-\theta z|^2} < 0$
$\Leftrightarrow (1-2\theta)|z|^2 + (z+\overline{z}) < 0.$

(c) $(1-2\theta)|z|^2 + (z+\overline{z}) = (1-2\theta)z\overline{z} + (z+\overline{z})$
$= (1-2\theta)z\overline{z} + (z+\overline{z}) + \frac{1}{1-2\theta} - \frac{1}{1-2\theta} = (1-2\theta)(z + \frac{1}{1-2\theta})(\overline{z} + \frac{1}{1-2\theta}) - \frac{1}{1-2\theta}$
$= (1-2\theta)|z + \frac{1}{1-2\theta}|^2 - \frac{1}{1-2\theta}$

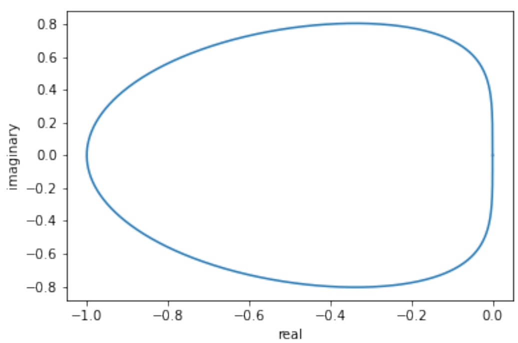
(d) (i) Combining (b) and (c) we get
$|w| < 1 \Leftrightarrow (1-2\theta)|z + \frac{1}{1-2\theta}|^2 - \frac{1}{1-2\theta} < 0$
Because $1 - 2\theta < 0 \Rightarrow |z + \frac{1}{1-2\theta}|^2 > \frac{1}{1-2\theta}^2$
Taking the square root of both sides $|z - \frac{1}{2\theta-1}| > \frac{1}{2\theta-1}$

(ii) Because $1 - 2\theta > 0 \Rightarrow |z + \frac{1}{1-2\theta}|^2 < \frac{1}{1-2\theta}^2$
Taking the square root of both sides $|z + \frac{1}{1-2\theta}| < \frac{1}{1-2\theta}$

(iii) For $\theta = \frac{1}{2}$ we need $\Re(z) < 0$ from the trapezoidal stability region lecture note.

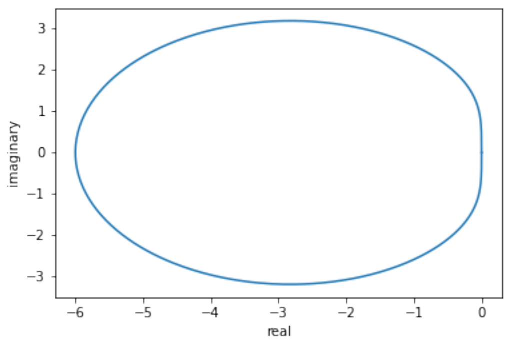
Problem 2

(a) Boundary Region



$y_{n+2} - y_{n+1} = h(\frac{3}{2}\lambda y_{n+1} - \frac{1}{2}\lambda y_n)$
$\Rightarrow y_{n+2} - (1 + \frac{3}{2}z)y_{n+1} + \frac{1}{2}zy_n = 0$
$\Rightarrow \rho(r; z) = r^2 - (1 + \frac{3}{2}z)r + \frac{1}{2}z$
Boundary $z = \frac{e^{2\theta i} - e^{\theta i}}{\frac{3}{2}e^{\theta i} - \frac{1}{2}}$ for $\theta \in [0, 2\pi]$
$r = \frac{1+\frac{3}{2}z+\sqrt{\frac{9}{4}z^2+z+1}}{2}, \frac{1+\frac{3}{2}z-\sqrt{\frac{9}{4}z^2+z+1}}{2}$
stability region inside boundary because $z = -0.5 + 0i$ satisfies root condition.

(b) Boundary Region



$y_{n+2} - y_{n+1} = h(\frac{5}{12}\lambda y_{n+2} + \frac{8}{12}\lambda y_{n+1} - \frac{1}{12}\lambda y_n)$
$\Rightarrow (1 - \frac{5}{12}z)y_{n+2} - (1 + \frac{8}{12}z)y_{n+1} + \frac{1}{12}zy_n = 0$
$\Rightarrow \rho(r; z) = (1 - \frac{5}{12}z)r^2 - (1 + \frac{8}{12}z)r + \frac{1}{12}z$
$r = \frac{1+\frac{2}{3}z+\sqrt{\frac{23}{48}z^2+z+1}}{2-\frac{5}{6}z}, \frac{1+\frac{2}{3}z-\sqrt{\frac{23}{48}z^2+z+1}}{2-\frac{5}{6}z}$
stability region inside boundary because $z = -3 + 0i$ satisfies root condition.

Problem 3

(a)

$$\begin{aligned} y_{n+1} &= y_n + hk_2 \\ &= y_n + h\lambda(y_n + hk_1/2) \\ &= y_n + h\lambda(y_n + h\lambda y_n/2) \\ &= y_n + z(y_n + zy_n/2) \\ &= (1 + z + \frac{z^2}{2})y_n \end{aligned}$$

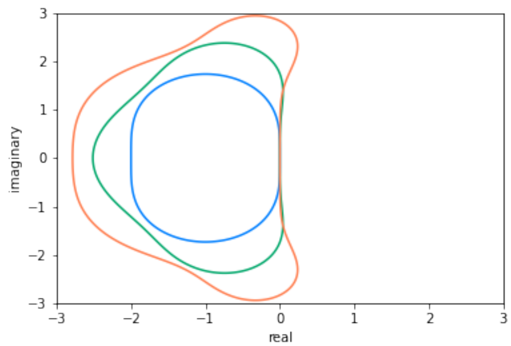
(b)

$$
\begin{aligned}
y_{n+1} &= y_n + h(\frac{1}{6}k_1 + \frac{4}{6}k_2 + \frac{1}{6}k_3) \\
&= y_n + h(\frac{1}{6}\lambda y_n + \frac{4}{6}\lambda(y_n + h\lambda y_n/2) + \frac{1}{6}\lambda(y_n - h\lambda y_n + 2h\lambda(y_n + h\lambda y_n/2))) \\
&= y_n + (\frac{1}{6}zy_n + \frac{4}{6}z(y_n + zy_n/2) + \frac{1}{6}z(y_n - zy_n + 2z(y_n + zy_n/2))) \\
&= (1 + \frac{1}{6}z + \frac{4}{6}z(1 + z/2) + \frac{1}{6}z(1 - z + 2z(1 + z/2)))y_n \\
&= (1 + \frac{1}{6}z + \frac{4}{6}(z + z^2/2) + \frac{1}{6}(z + z^2 + z^3))y_n \\
&= (1 + z + \frac{z^2}{2} + \frac{z^3}{6})y_n
\end{aligned}
$$

(c)

$$
\begin{aligned}
&y_{n+1} = y_n + h(\frac{1}{6}k_1 + \frac{2}{6}k_2 + \frac{2}{6}k_3 + \frac{1}{6}k_4) \\
&h\frac{1}{6}k_1 = \frac{z}{6}y_n \\
&h\frac{2}{6}k_2 = (\frac{2z}{6} + \frac{z^2}{6})y_n \\
&h\frac{2}{6}k_3 = (\frac{2z}{6} + \frac{z^2}{6} + \frac{z^3}{12})y_n \\
&h\frac{1}{6}k_4 = (\frac{z}{6} + \frac{z^2}{6} + \frac{z^3}{12} + \frac{z^4}{24})y_n \\
&\Rightarrow y_{n+1} = y_n + \frac{z}{6}y_n + (\frac{2z}{6} + \frac{z^2}{6})y_n + (\frac{2z}{6} + \frac{z^2}{6} + \frac{z^3}{12})y_n + (\frac{z}{6} + \frac{z^2}{6} + \frac{z^3}{12} + \frac{z^4}{24})y_n \\
&= (1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24})y_n
\end{aligned}
$$

(d) The area of the stability regions increases when we use higher degree polynomials.



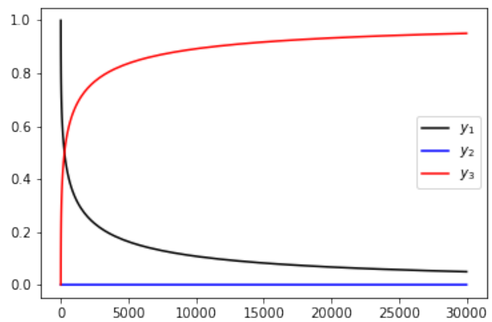
Problem 4

(a) If $y(t) = (0,0,1)^\top$ then
$y'(t) = (-\alpha \cdot 0 + \beta \cdot 0 \cdot 1, \alpha \cdot 0 - \beta \cdot 0 \cdot 1 - \gamma \cdot 0^2, \gamma \cdot 0^2)^\top = (0,0,0)^\top$
$\frac{d}{dt}[\sum_{i=1}^{3} y(t)] = \sum_{i=1}^{3} y'(t) = (-\alpha y_1 + \beta y_2 y_3) + (\alpha y_1 - \beta y_2 y_3 - \gamma y_2^2) + (\gamma y_2^2) = 0$
Thus, by the FTC $\sum_{i=1}^{3} y(t) - \sum_{i=1}^{3} y(0) = \int_0^t \sum_{i=1}^{3} y'(t) dt = 0 \Rightarrow \sum_{i=1}^{3} y(t) = \sum_{i=1}^{3} y(0) = 1$ for all $t > 0$.
Since $\sum_{i=1}^{3} \mathbf{y}_e = 1$ the reaction model admits the unique steady state $\mathbf{y}_e = (0,0,1)^\top$.

(b) Solver uses 30001 time steps. The numerical solution doesn't appear to be anywhere close to the steady state solution. Used a max step size of 10^{-4}

(c) Solver uses 27 time steps. Requires significantly fewer time steps used then for part (b). Also, doesn't appear to be closer to the steady state.

(d) I used a final time of $T = 3 \times 10^4$ to obtain a $y_3(T) \approx 0.95$. The solver only used 95 time steps.



In [2]:
```
import math as m
import numpy as np
import cmath as cm
from matplotlib import pyplot as plt
from scipy import integrate
```

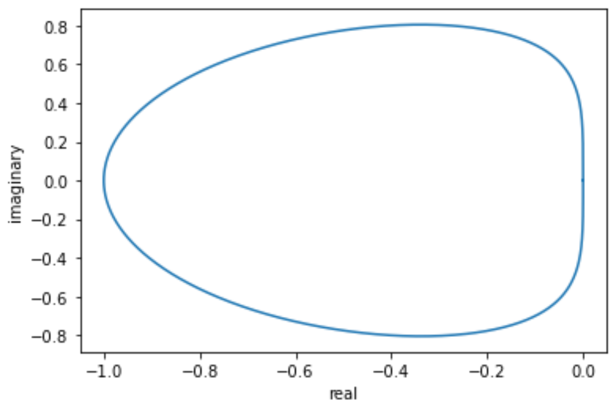
In [13]:
```
rho_poly = lambda r: r**2-r
sigma_poly = lambda r: 3/2*r-1/2
thvec=np.linspace(0,2*m.pi,1000)
root_con = lambda z:max(abs((1+3/2*z+cm.sqrt(9/4*z**2+z+1)/2)),abs((1+3/2*z-cm.s
```

In [26]:
```
z=np.array([rho_poly(cm.exp(t*1j))/sigma_poly(cm.exp(t*1j)) for t in thvec])
```

In [15]:
```
root_con(-0.5)
```

Out[15]:
```
0.7653882032022076
```

In [27]:
```
plt.plot(z.real,z.imag)
plt.xlabel('real')
plt.ylabel('imaginary')
plt.savefig('AB_2_boundary')
```



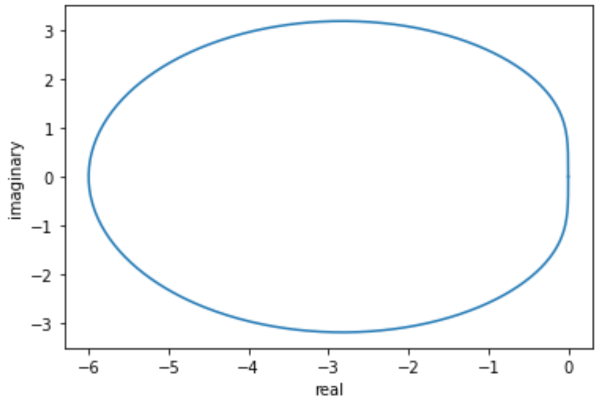
In [17]:
```
rho_poly_2 = lambda r: r**2-r
sigma_poly_2 = lambda r: 5/12*r**2+8/12*r-1/12
root_con_2 = lambda z: max(abs((1+2/3*z+cm.sqrt(23/48*z**2+z+1))/(2-5/6*z)),abs(
```

In [24]:
```
z_2=np.array([rho_poly_2(cm.exp(t*1j))/sigma_poly_2(cm.exp(t*1j)) for t in thvec
```

In [19]:
```
root_con_2(-3)
```

Out[19]:
```
0.5601534739054566
```

In [25]:
```
plt.plot(z_2.real,z_2.imag)
plt.xlabel('real')
plt.ylabel('imaginary')
plt.savefig('AM_2_boundary')
```



In [21]:

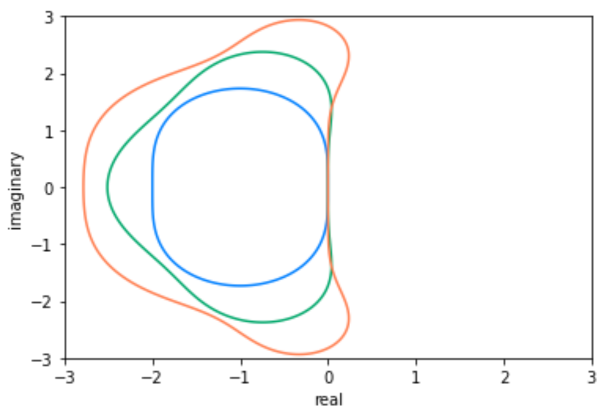
```
RK_2_T=lambda z:abs(1+z+1/2*z**2)
RK_3_T=lambda z:abs(1+z+1/2*z**2+1/6*z**3)
RK_4_T=lambda z:abs(1+z+1/2*z**2+1/6*z**3+1/24*z**4)
```

In [22]:

```
azure = [(0, 128/255, 1.0)]
jade = [(0, 168/255, 107/255)]
coral = [(1.0, 127/255, 80/255)]
```

In [23]:

```
xv = np.linspace(-3, 3, 301)
yv = np.linspace(-3, 3, 301)
xx, yy = np.meshgrid(xv, yv)
zz=xx+yy*1j
plt.contour(xx,yy,RK_2_T(zz),[0,1],colors=azure)
plt.contour(xx,yy,RK_3_T(zz),[0,1],colors=jade)
plt.contour(xx,yy,RK_4_T(zz),[0,1],colors=coral)
plt.xlabel('real')
plt.ylabel('imaginary')
plt.savefig('RK_boundaries')
```



In [4]:

```
concentration_f= lambda t,y: np.array([-4e-2*y[0]+1e4*y[1]*y[2],4e-2*y[0]-1e4*y[
```

In [34]:

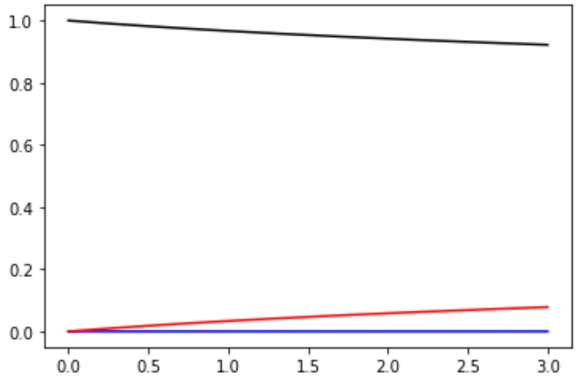
```
c_RK45=integrate.solve_ivp(concentration_f,[0,3],np.array([1,0,0]),method='RK45'
```

In [35]:

```
plt.plot(c_RK45.t,c_RK45.y[0],'k-')
plt.plot(c_RK45.t,c_RK45.y[1],'b-')
plt.plot(c_RK45.t,c_RK45.y[2],'r-')
```

Out[35]:

```
[<matplotlib.lines.Line2D at 0x7f95281fa400>]
```



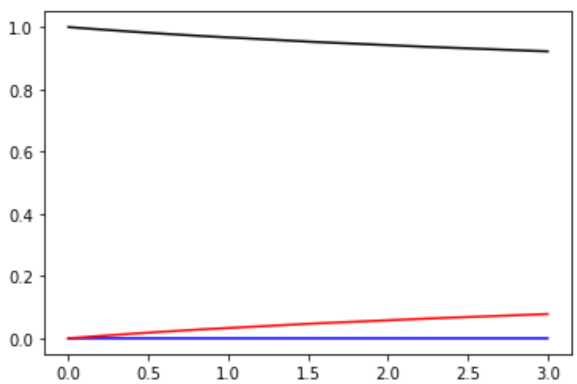
In [36]:
```
len(c_RK45.t)
```

Out[36]: 30001

In [43]:
```
c_BDF=integrate.solve_ivp(concentration_f,[0,3],np.array([1,0,0]),method='BDF')
```

In [53]:
```
plt.plot(c_BDF.t,c_BDF.y[0],'k-')
plt.plot(c_BDF.t,c_BDF.y[1],'b-')
plt.plot(c_BDF.t,c_BDF.y[2],'r-')
```

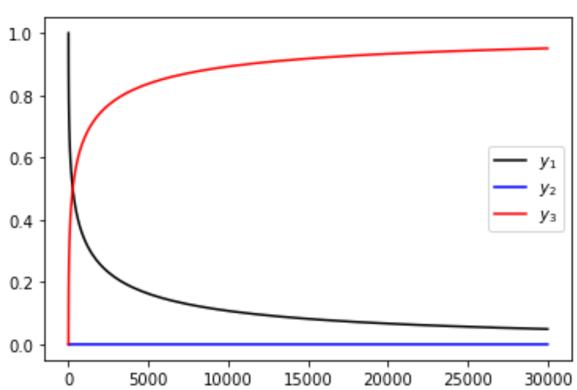
Out[53]: [<matplotlib.lines.Line2D at 0x7f8af9fd4730>]



In [46]:
```
len(c_BDF.t)
```

Out[46]: 27

In [8]:
```
c_BDF_2=integrate.solve_ivp(concentration_f,[0,3e4],np.array([1,0,0]),method='BD
plt.plot(c_BDF_2.t,c_BDF_2.y[0],'k-',label='$y_1$')
plt.plot(c_BDF_2.t,c_BDF_2.y[1],'b-',label='$y_2$')
plt.plot(c_BDF_2.t,c_BDF_2.y[2],'r-',label='$y_3$')
plt.legend()
c_BDF_2.y[2][-1]
plt.savefig('stiff_solver_chemical_reaction_rate')
```



In [12]:
```
len(c_BDF_2.t)
```

Out[12]: 95

In [11]:
```
c_BDF_2.y[2][-1]
```

Out[11]: 0.9509191008000204

In [ ]:
```

```