Math 114L: Problem Set 4

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Problem 1

Part 1:
Let $P^{\mathcal{M}} := \{n \in \mathbb{N} : n \text{ is prime or even}\}$.
Let $Q^{\mathcal{M}} := \{n \in \mathbb{N} : n \text{ is odd}\}$.
$\mathcal{M} \models T$ since (1) every number is odd or even, (2) there are infinitely many $x \in P^{\mathcal{M}} \setminus Q^{\mathcal{M}}$ (even numbers), (3) there are infinitely many $x \in Q^{\mathcal{M}} \setminus P^{\mathcal{M}}$ (odd composite numbers), (4) there are infinitely many $x \in P^{\mathcal{M}} \cap Q^{\mathcal{M}}$ (odd primes).
Part 2:
Let $\mathcal{M}, \mathcal{N} \models T$. (2) says $P^{\mathcal{M}} \setminus Q^{\mathcal{M}}$ and $P^{\mathcal{N}} \setminus Q^{\mathcal{N}}$ are countably infinite, so we construct a bijection $f_1 : P^{\mathcal{M}} \setminus Q^{\mathcal{M}} \to P^{\mathcal{N}} \setminus Q^{\mathcal{N}}$. Using similar arguments with (3) and (4) we construct bijections $f_2 : Q^{\mathcal{M}} \setminus P^{\mathcal{M}} \to Q^{\mathcal{N}} \setminus P^{\mathcal{N}}$ and $f_3 : P^{\mathcal{M}} \cap Q^{\mathcal{M}} \to P^{\mathcal{N}} \cap Q^{\mathcal{N}}$. (1) says $M = P^{\mathcal{M}} \cup Q^{\mathcal{M}} = (P^{\mathcal{M}} \setminus Q^{\mathcal{M}}) \cup (Q^{\mathcal{M}} \setminus P^{\mathcal{M}}) \cup (P^{\mathcal{M}} \cap Q^{\mathcal{M}})$, so $f := f_1 \sqcup f_2 \sqcup f_3$ is a bijection $f : \mathcal{M} \to \mathcal{N}$ that preserves P and Q.

Problem 2

(1) Let $G_n = ([n+2], \{(1,2), (2,1), \ldots, (n+1, n+2), (n+2, n+1)\})$. The shortest path connecting 1 and $n+2$ is length $n+1$ $\{(1,2), (2,3), \ldots, (n+1, n+2)\}$, so G_n is not n connected. For $1 \leq a < b \leq n+2$ $\{(a, a+1), (a+1, a+2), \ldots (b-1, b)\}$ is a path of length $b - a \leq n+1$ from a to b, and by symmetry, we can find a path from b to a in a similar manner. Thus, G_n is $n+1$ connected.

(2) Consider the sentence $\varphi_n := \forall a, b(\neg(a = b) \to \bigvee_{k \in [n]} \exists c_1, \ldots c_{k-1} E(a, c_1) \wedge E(c_1, c_2) \wedge \ldots \wedge E(c_{k-1}, b))$. $G \models \varphi_n$ iff for every vertex $a \neq b$ $\exists 1 \leq k \leq n$ s.t a, b are k connected.

(3) Suppose for the sake of contradiction the set of all connected graphs form an elementary class. It follows there exists a theory T s.t $M \models T$ iff M is connected. Let $T^* = T \cup \{\neg\phi_n : n \in \mathbb{N}\}$. If T^* is satisfiable, then any model G s.t $G \models T^*$ is not n connected for any $n \in \mathbb{N}$. Let $\Delta \subset T^*$ be finite, and let n be the largest s.t $\neg\varphi_n \in \Delta$. The graph we found in (1),

G_n, satisfies T since G_n is $n+1$ connected, and $G_n \models \neg\varphi_n$ because G_n is not k connected for any $1 \leq k \leq n$. Since $G_n \models \Delta$, this implies T^* is satisfiable by compactness.

Problem 3

$M \cong N \to M \equiv N$, so it suffices to show the reverse direction.
Suppose $M \equiv N$. Since M is finite, we can enumerate its underlying set $M = \{m_1, m_2, \ldots, m_n\}$.
Case 1: $\mathcal{L}$ is a finite language.
Consider the sentence

$$\varphi = \exists x_1, x_2, \ldots, x_n (\bigwedge_{\substack{i,j\in[n] \\ \text{s.t } i\neq j}} \neg(x_i = x_j)) \wedge (\forall x \bigvee_{i\in[n]} x = x_i)$$

$$\wedge (\bigwedge_{c\in\mathcal{L}} \bigwedge_{\substack{i \text{ s.t} \\ m_i = c^M}} x_i = c)$$

$$\wedge (\bigwedge_{f\in\mathcal{L}} \bigwedge_{\substack{i,i_1,\ldots,i_k\in[n] \\ \text{s.t } m_i = f^M(m_{i_1},\ldots,m_{i_k})}} x_i = f(x_{i_1}, \ldots, x_{i_k}))$$

$$\wedge (\bigwedge_{R\in\mathcal{L}} \bigwedge_{\substack{i_1,\ldots,i_k\in[n] \\ \text{s.t } (m_{i_1},\ldots,m_{i_k})\in R^M}} R(x_{i_1}, \ldots, x_{i_k}))$$

$$\wedge (\bigwedge_{R\in\mathcal{L}} \bigwedge_{\substack{i_1,\ldots,i_k\in[n] \\ \text{s.t } (m_{i_1},\ldots,m_{i_k})\notin R^M}} \neg R(x_{i_1}, \ldots, x_{i_k}))$$

Clearly, $M \models \varphi$ because $m_1 \ldots m_n$ witness $x_1, \ldots x_n$. By elementary equivalence, $N \models \varphi$. Let $n_1, \ldots n_n$ be the witness of $x_1, \ldots x_n$. Define embedding $\eta : M \to N$ by sending each m_i to n_i. By first line, domain and codomain are the same size, and η is surjective. Thus, η is a bijection. By the second line, $m_i = c^M$ implies $n_i = \eta(m_i) = c^N$ for all $m_i \in M$, so η preserves constants. By the third line, $m_i = f^M(m_{i_1}, \ldots, m_{i_k})$ implies $n_i = \eta(m_i) = f^N(\eta(m_{i_1}), \ldots \eta(m_{i_k}))$ for all $m_i, m_{i_1}, \ldots, m_{i_k} \in M$, so η preserves functions. By the fourth and fifth line we have $(m_{i_1}, \ldots, m_{i_k}) \in R^M$ iff $(\eta(m_{i_1}), \ldots \eta(m_{i_k})) \in R^N$, so η preserves relation symbols.
Hence, η preserves constant, function, and relation symbols, and therefore, $M \cong N$.

Problem 4

Let I be a set s.t $|I| > |M|$. Let $\mathcal{L}' = \mathcal{L} \cup \{c_i : i \in I\}$ be an extension of $\mathcal{L}$ and $T = Th_{\mathcal{L}}(M) \cup \{c_i \neq c_j : i \neq j \in I\}$. We want to show T is satisfiable. Let $\Delta \subset T$ be finite. Pick a finite $J \subset I$ s.t every c_i that that occur in Δ have an index in J. Since M is infinite, we can interpret each $\{c_i : i \in J\}$ as distinct elements of M. Thus, $M \models \Delta$. By compactness, T is satisfiable. Let N be an $\mathcal{L}^*$ structure s.t $N \models T$ $N \models Th_{\mathcal{L}}(N) \Rightarrow Th_{\mathcal{L}}(N) \supset Th_{\mathcal{L}}(M) \Rightarrow Th_{\mathcal{L}}(N) = Th_{\mathcal{L}}(M)$, so $N \equiv M$. However, $|N| \leq |I| > |M|$, so $|N| > |M|$. Hence, N and M cannot be isomorphic.

Problem 5

(1) Extend the language $\mathcal{L}$ to a new language $\mathcal{L}^* = \mathcal{L} \cup \{a\}$. Define a theory $T^* = Th(M) \cup \{D(p, a) : p \in \mathbb{P}\}$ in the new language $\mathcal{L}^*$. It suffices to show T^* is finitely satisfiable. Let $T_0 \subset T^*$ be finite. It follows $T_0 = \{\varphi_1, \ldots, \varphi_n\} \cup \{D(p, a) : p \in P\}$ for finitely many $\varphi_i \in Th(M)$ sentences and some finite subset $P \subset \mathbb{P}$. $M \models T_0$ because $M \models \{D(p, a) : p \in P\}$ if we for example let $a = \prod_{p_i \in P} p_i$ and $M \models \varphi_i$ for any $\varphi_i \in Th(M)$. Since T^* is finitely satisfiable, then T^* must be satisfiable by compactness. Thus, there exists some structure N s.t $N \models T^*$ and because $N \models Th(M)$ $N \equiv_{\mathcal{L}} M$ in the language $\mathcal{L}$.

(2) Assume the twin prime conjecture is true. Extend the language $\mathcal{L}$ to a new language $\mathcal{L}^* = \mathcal{L} \cup \{p_1, p_2\}$. Define a theory $T^* = Th(M) \cup \{T(p_1) \wedge T(p_2)\} \cup \{n < p_1 : n \in \mathbb{N}\} \cup \{p_2 = p_1 + 2\}$ in the new language $\mathcal{L}^*$. It suffices to show T^* is finitely satisfiable. Let $T_0 \subset T^*$ be finite. It follows $T_0 = \{\varphi_1, \ldots, \varphi_n\} \cup \{T(p_1) \wedge T(p_2)\} \cup \{n < p_1 : n \in N_0\} \cup \{p_2 = p_1 + 2\}$ for finitely many $\varphi_i \in Th(M)$ sentences and some finite subset $N_0 \subset \mathbb{N}$. $M \models \{T(p_1) \wedge T(p_2)\} \cup \{n < p_1 : n \in N_0\} \cup \{p_2 = p_1 + 2\}$ because the twin prime conjecture states there exists infinitely many pairs of primes (p_1, p_2) s.t $p_2 = p_1 + 2$ allowing us to find some p_1, p_2 larger than the largest element of N_0. $M \models \varphi_i$ for any $\varphi_i \in Th(M)$. Thus, $M \models T_0$. Since T^* is finitely satisfiable, then T^* must be satisfiable by compactness. Thus, there exists some structure N s.t $N \models T^*$ and because $N \models Th(M)$ $N \equiv_{\mathcal{L}} M$ in the language $\mathcal{L}$.

Problem 6

Assume to the contrary such a k does not exist. It follows for each $n \in \mathbb{N}$ there exists a finite model M_n with n or more elements that makes φ false. Let $M' = \bigcup_{n \in \mathbb{N}} M_n$ be the union of all M_n. Because M' is a model with infinitely many elements, it must make φ true. However, each M_n makes φ false, so their union must make φ false. Hence, we obtain a contradiction, and there must exist some k s.t all models with k or more elements must make φ true.