Final 2015

Owen Jones

12/8/2023

(Q-1) Base case: $(1+\frac{1}{1^2})(1+\frac{1}{2^2}) = 2 \cdot \frac{5}{4} = \frac{5}{2} \leq 5(1-\frac{1}{2}) = \frac{5}{2}$
Induction hypothesis: assume $\prod_{i=1}^{n}(1+\frac{1}{i^2}) \leq 5(1-\frac{1}{n})$
Induction step: suffices to show $5(1-\frac{1}{n})(1+\frac{1}{(n+1)^2}) \leq 5(1-\frac{1}{n+1})$.
$-2 \leq 0 \Rightarrow n^3+n^2-2 \leq n^3+n^2 \Rightarrow (n-1)((n+1)^2+1) \leq n^2(n+1) \Rightarrow n-1n\frac{(n+1)^2+1}{(n+1)^2} \leq \frac{n}{n+1} \Rightarrow 5(1-\frac{1}{n})(1+\frac{1}{(n+1)^2}) \leq 5(1-\frac{1}{n+1})$

(Q-2) given vector (a,b,c) and $(1,2,2)$, $(a+2b+2c)^2 \leq (a^2+b^2+c^2)(1^2+2^2+2^2) = 9 \Rightarrow |a+2b+2c| \leq 3$. Want $(a,b,c) \parallel (1,2,2)$ Thus, $(\frac{1}{3},\frac{2}{3},\frac{2}{3})$ will achieve the maximum value of 3.

(Q-3) $\gcd(4,3)=1$, so there exist x,y s,t $4x-3y=1$. By Euclidean algorithm, $x=1, y=1$ gives specific solution, and $4(x+1)-3(y+1)=1$ gives $x=1+3k, y=1+4k$ as our general solution.

(Q-4) $a+b+c=-2, ab+bc+ac=-9, abc=1$. $p_2 = -ab-bc-ac = 9, p_1 = ab^2c+a^2bc+abc^2 = 1\cdot -2 = -2, p_0 = 1, x^3+9x^2-2x-1$

(Q-5) $\frac{1}{F_{n-1}F_{n+1}} = \frac{F_{n+1}-F_{n-1}}{F_{n-1}F_{n+1}(F_{n+1}-F_{n-1})} = \frac{1}{F_{n-1}(F_{n+1}-F_{n-1})} - \frac{1}{F_{n+1}(F_{n+1}-F_{n-1})} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_{n+1}F_n}$
$\lim_{n\to\infty}\sum_{k=2}^{n}\frac{1}{F_{k-1}F_{k+1}} = \lim_{n\to\infty}\sum_{k=2}^{n}\frac{1}{F_{k-1}F_k} - \frac{1}{F_{k+1}F_k} = \frac{1}{F_1F_2} - \frac{1}{F_3F_2} + \frac{1}{F_2F_3} - \frac{1}{F_3F_2}\dots = \frac{1}{F_1F_2} - \frac{1}{F_{n+1}F_n} = 1$

(Q-6) $9r^2-6r+1 \Rightarrow r = \frac{1}{3} \Rightarrow a_n = \alpha\frac{1}{3^n} + \beta n\frac{1}{3^n} \Rightarrow a_n = \frac{6}{3^n} + \frac{9n}{3^n}$

(Q-7) $|PA|^2+|PB|^2+|PC|^2+|PD|^2$, $|PA|^2 = r^2+\frac{1}{2}d^2 - \vec{OP}\cdot\vec{OA}$. By symmetry $|PA|^2+|PB|^2+|PC|^2+|PD|^2 = 4r^2+2d^2$

(Q-8) Let $f(x) = ax+b\frac{x^2}{2}+c\frac{x^3}{3}-\sin(x)$. $f(\frac{\pi}{2}) = f(0) = 0$, so by rolle's theorem, $f'(x) = a+bx+cx^2-\cos(x) = 0$ for some $0 \leq x \leq \frac{\pi}{2}$.

(Q-9) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{4n+k} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{4+\frac{k}{n}} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(4+\frac{k}{n}) = \int_4^5\frac{1}{x}dx = \log(5)-\log(4) = \log(\frac{5}{4})$.

(Q-10) $1 \leq 2\sqrt{2}$
Suffices to show $2\sqrt{n}+\frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}$ If $2\sqrt{n}+\frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1} \Rightarrow$

$\frac{1}{\sqrt{n+1}} \leq 2(\sqrt{n+1}-\sqrt{n}) \Rightarrow \frac{1}{\sqrt{n+1}} \leq 2\frac{1}{\sqrt{n+1}+\sqrt{n}}$ which is true because $\frac{1}{\sqrt{n+1}} = \frac{2}{2\sqrt{n+1}} \leq \frac{2}{\sqrt{n}+\sqrt{n+1}}$

(Q-11) By AM-GM $\sqrt{ab} \cdot \sqrt{2bc} \cdot \sqrt{5cd} \cdot \sqrt{10ad} = 10|abcd| \leq (a+b)(b+2c)(c+5d)(a+10d) \Rightarrow 5000 \leq (a+b)(b+2c)(c+5d)(a+10d)$. Solve for when $a = b = 2c = 10d \Rightarrow 500 = 500d^4 \Rightarrow a = 10, b = 10, c = 5, d = 1$

(Q-12) By inspection, $(-2, 4)$ gives us a solution. It follows $(-2+5k, 4-9k)$ gives us the set of all integer solutions.

(Q-13) $P(x) = (x+1)(x+2)Q(x) + R(x)$ where the degree of $R(x)$ is less than $Q(x)$. It follows $P(-1) = R(-1) = 1 - a + b$ and $P(-2) = R(-2) = 2^{50} - 2a + b$. Setting $R(-1) = R(-2) = 0$, we find $a - 1 = b \Rightarrow a = 2^{50} - 1, b = 2^{50} - 2$.

(Q-14) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\frac{1}{2}((2k+1)-(2k-1))}{(2k-1)(2k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\frac{1}{2}}{(2k-1)} - \frac{\frac{1}{2}}{(2k+1)} = \lim_{n\to\infty} \frac{1}{2}\frac{1}{1} - \frac{1}{2}\frac{1}{2n+1} = \frac{1}{2}$

(Q-15) $\int_{y=0}^{y=10} \int_{x=0}^{x=\min(10,12-y)} dxdy = 100 - \int_{y=2}^{10} \int_{x=12-y}^{10} dxdy = 100 - \int_{y=2}^{y=10} y - 2dy = 100 - \frac{1}{2}y^2 - 2y|_{y=2}^{y=10} = 100 - 30 - 2 = 68 \Rightarrow P(x+y \leq 12) = \frac{17}{25}$

(Q-16) WLOG let G be the centroid of $\triangle ABC$. Let $\vec{a} = \vec{GA}, \vec{b} = \vec{GB}, \vec{c} = \vec{GC}$. We are given $\vec{d} = \frac{1}{3}(\vec{c}-\vec{b}), \vec{e} = \frac{1}{3}(\vec{a}-\vec{c}), \vec{f} = \frac{1}{3}(\vec{b}-\vec{a})$. We defined $\frac{\vec{a}+\vec{b}+\vec{c}}{3} = \vec{0}$. Clearly $\frac{\vec{d}+\vec{e}+\vec{f}}{3} = \frac{\frac{1}{3}(\vec{c}-\vec{b})+\frac{1}{3}(\vec{a}-\vec{c})+\frac{1}{3}(\vec{b}-\vec{a})}{3} = \vec{0}$

(Q-17) A simple induction shows $f(\sum_{i=1}^{n} x_i) = \sqrt{\sum_{i=1}^{n} f(x_i)^2}$. $f(0) = f(n \cdot 0) = \sqrt{nf(0)^2} = \sqrt{n}|f(0)| \Rightarrow f(0) = 0$ or $\sqrt{n} = \pm 1$. $\sqrt{n} \neq \pm 1$ unless $n = 1$. Thus, $f(0) = 0$. $f(0) = f(x + (-x)) = \sqrt{f(x)^2 + f(-x)^2} = 0 \Rightarrow f(x)^2 = -f(-x)^2$ which can only be true if $f(x) = 0$ because $f(x) \in \mathbb{R}$.

(Q-18) Let $I := \int_{x=0}^{x=1} \frac{e^{x+3}}{e^{x+3}+e^{4-x}} dx$. Let $y = 1 - x \Rightarrow dy = -dx \Rightarrow I = \int_{y=1}^{y=0} \frac{e^{4-y}}{e^{4-y}+e^{y+3}}(-dy) = \int_{y=0}^{y=1} \frac{e^{4-y}}{e^{4-y}+e^{y+3}} dy$. Because y is a dummy variable $I = \int_{x=0}^{x=1} \frac{e^{4-x}}{e^{4-x}+e^{x+3}} dx \Rightarrow I + I = \int_{x=0}^{x=1} \frac{e^{4-x}}{e^{4-x}+e^{x+3}} dx + \int_{x=0}^{x=1} \frac{e^{x+3}}{e^{4-x}+e^{x+3}} dx = \int_{x=0}^{x=1} \frac{e^{x+3}+e^{4-x}}{e^{4-x}+e^{x+3}} dx = 1 \Rightarrow I = \frac{1}{2}$

(Q-19) $1 \geq \sqrt{1}$. WTS $\frac{1}{\sqrt{n+1}} + \sqrt{n} \geq \sqrt{n+1}$. $\frac{1}{\sqrt{n+1}} \geq \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ which is clearly true.

(Q-20) Let $1, \omega, \omega^2$ be the roots of unity. $P(x) = (x^2 + x^1)Q(x) + R(x)$. $P(\omega) = R(\omega)$ and $P(\omega^2) = R(\omega^2)$ because ω, ω^2 are the roots of $x^2 + x^1$. It follows $P(\omega) = \omega^{3a} + \omega^{3b+1} + \omega^{3c+2} = 1 + \omega + \omega^2 = 0, P(\omega^2) = \omega^{6a} + \omega^{6b+2} + \omega^{6c+4} = 1 + \omega^2 + \omega = 0$

(Q-21) $\int_{y=0}^{y=1}\int_{x=\min(\frac{1}{4y},1)}^{x=1} dxdy = \int_{y=\frac{1}{4}}^{y=1}\int_{x=\frac{1}{4y}}^{x=1} dxdy = \int_{y=\frac{1}{4}}^{1} 1-\frac{1}{4y}dy = y-\frac{1}{4}\log(y)|_{\frac{1}{4}}^{1} =$
$1-(\frac{1}{4}-\frac{1}{4}\log(\frac{1}{4})) = \frac{3}{4}-\frac{1}{2}\log(2)$

(Q-22) Pf by induction:
$n=2$ there are 2 players, so there is only one game with a loser and a winner. Label the winner P_1 and the loser P_2.
By the induction hypothesis, there exists a way to order n many players s.t P_1 defeated P_2 etc. For the $n+1st$ player P^*, let W be the set of players defeated by P^*. Since this set is finite, it must have a minimum. Let P_i be the minimum of the set W. It follows, we can place P^* between P_{i-1} and P_i if $i>1$ or before P_1 if $i=1$.

(Q-23) $|\sin(x)| \leq \sin(x)$ trivially.
Assume $|\sin(nx)| \leq n\sin(x)$. $|\sin((n+1)x)| = |\sin(nx)\cos(x)+\cos(nx)\sin(x)| \leq |\sin(nx)||\cos(x)| + |\cos(nx)||\sin(x)| \leq |\sin(nx)| + |\sin(x)| \leq n\sin(x) + \sin(x) = (n+1)\sin(x)$

(Q-24) Divide S into 4 regions of equal size. By the pigeonhole principle, one of the 4 regions contains 3 points. The maximum area of a triangle inscribed inside a square is half the area of the square. WLOG let p_1, p_2, p_3 be on the region $[0,1]\times[0,1]$.
Let $b := |\vec{p_1p_2}|$ and let $h := \min(\sqrt{((p_{2x}-p_{1x})t+p_{1x}-p_{3x})^2+((p_{2y}-p_{1y})t+p_{1y}-p_{3y})^2})$.
If $b \leq \frac{\sqrt{2}}{2}$ and $A = \frac{1}{2}bh$

(Q-25) $\frac{a_1}{b_1}+\frac{a_2}{b_2}+\ldots+\frac{a_n}{b_n} \geq n\sqrt{\frac{a_1}{b_1}\frac{a_2}{b_2}\cdots\frac{a_n}{b_n}} = n$

(Q-26) $r^3-r^2-r+1 = (r+1)(r^2-2r+1) = (r+1)(r-1)^2 = 0$
$\alpha_1(-1)^n+\alpha_2+\alpha_3 n$
$\alpha_1+\alpha_2 = 1$
$\alpha_2+\alpha_3-\alpha_1 = 2$
$\alpha_1+\alpha_2+2\alpha_3 = 4$
$\alpha_1 = \frac{1}{4}, \alpha_2 = \frac{3}{4}, \alpha_3 = \frac{3}{2}$
$\frac{1}{4}(-1)^n+\frac{3}{4}+\frac{3}{2}n$

(Q-27) let $b_n := \sqrt{a_n}$
$r^2-r-2=0 \Rightarrow (r-1)(r+2)=0 \Rightarrow b_n = \alpha_1+\alpha_2(-2)^n$
$\Rightarrow a_n = (\alpha_1+\alpha_2(-2)^n)^2$
$1 = (\alpha_1+\alpha_2)^2 \Rightarrow 1 = \alpha_1^2+2\alpha_1\alpha_2+\alpha_2^2$
$1 = (\alpha_1-2\alpha_2)^2 \Rightarrow 1 = \alpha_1^2-4\alpha_1\alpha_2+4\alpha_2^2$
$a_n = (\frac{1}{3}+\frac{2}{3}(-2)^n)^2$

(Q-28) $1000-(\lfloor\frac{1000}{2}\rfloor+\lfloor\frac{1000}{3}\rfloor+\lfloor\frac{1000}{7}\rfloor-\lfloor\frac{1000}{6}\rfloor-\lfloor\frac{1000}{14}\rfloor-\lfloor\frac{1000}{21}\rfloor+\lfloor\frac{1000}{42}\rfloor)$
$1000-(500+333+142-166-71-47+23) = 286$

(Q-29) $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i} \Rightarrow \sum_{k=1}^{n}\sin((2k-1)\theta) = \sum_{k=1}^{n}\frac{e^{i((2k-1)\theta)}-e^{-i((2k-1)\theta)}}{2i} =$
$\sum_{k=1}^{n}\frac{e^{-i\theta}}{2i}(e^{2i\theta})^k-\frac{e^{i\theta}}{2i}(e^{-2i\theta})^k = \frac{e^{-i\theta}}{2i}\frac{e^{2in\theta}-1}{e^{2i\theta}-1}-\frac{e^{i\theta}}{2i}\frac{e^{-2in\theta}-1}{e^{-2i\theta}-1} = \frac{e^{2in\theta}-1}{2i(e^{i\theta}-e^{-i\theta})}+$

$\frac{e^{-2in\theta}-1}{2i(e^{i\theta}-e^{-i\theta})} = \frac{2i}{(e^{i\theta}-e^{-i\theta})}\frac{2-(e^{2in\theta}+e^{-2in\theta})}{4} = \frac{1-\cos(2n\theta)}{2\sin(\theta)} = \frac{\sin^2(n\theta)}{\sin(\theta)}$

(Q-30) $(1+\frac{1}{1^2})(1+\frac{1}{2^2}) = \frac{5}{2} \leq \frac{5}{2} = 5(1-\frac{1}{2})$
Assume $\prod_{k=1}^{n}(1+\frac{1}{k^2}) \leq 5(1-\frac{1}{n})$
Show $5(1-\frac{1}{n})(1+\frac{1}{(n+1)^2}) \leq 5(1-\frac{1}{n+1})$ $5(1-\frac{1}{n+1})-5(1-\frac{1}{n}) = 5(\frac{1}{n}-\frac{1}{n+1}) = \frac{5}{n(n+1)} \geq \frac{5}{(n+1)^2} \geq 5(1-\frac{1}{n})\frac{1}{(n+1)^2}$

(Q-31) $(a+2b+2c)^2 \leq (a^2+b^2+c^2)(1^2+2^2+2^2) = 3^2 \Rightarrow |a+2b+2c| \leq 3$. Choose $(a,b,c) \parallel (1,2,2)$. $(a,b,c) = (k,2k,2k) \Rightarrow 9k^2 = 1 \Rightarrow k = \frac{1}{3} \Rightarrow (a,b,c) = (\frac{1}{3},\frac{2}{3},\frac{2}{3})$

(Q-32) $x^3-(a+b+c)x^2+(ab+bc+ac)x-abc = x^3+2x^2-9x-1 = 0 \Rightarrow a+b+c = -2, ab+bc+ac = -9, abc = 1.$
$\Rightarrow x^3-(ab+bc+ac)x^2+(ab^2+a^2bc+abc^2)x-a^2b^2c^2 = x^3+9x^2-2x-1 = 0$

(Q-33) $\frac{1}{F_{n-1}F_{n+1}} = \frac{F_n}{F_{n-1}F_nF_{n+1}} = \frac{F_{n+1}-F_{n-1}}{F_{n-1}F_nF_{n+1}} = \frac{F_{n+1}}{F_{n-1}F_nF_{n+1}} - \frac{F_{n-1}}{F_{n-1}F_nF_{n+1}} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}$
$\sum_{n=2}^{infty}\frac{1}{F_{n-1}F_{n+1}} = \sum_{n=2}^{\infty}\frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}} = \frac{1}{F_1F_2} - \frac{1}{F_\infty F_{\infty+1}} = 1$ $F_n = F_{n-1}+F_{n-2} \Rightarrow r^2-r-1 = 0 \Rightarrow (r-\frac{1+\sqrt{5}}{2})(r-\frac{1-\sqrt{5}}{2})$
$F_n = \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$
$0 = \alpha_1+\alpha_2$
$1 = \alpha_1\frac{1+\sqrt{5}}{2} + \alpha_2\frac{1-\sqrt{5}}{2}$
$1 = \alpha_1(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}) \Rightarrow \alpha_1 = \frac{\sqrt{5}}{5}, \alpha_2 = -\frac{\sqrt{5}}{5}$

(Q-34) $9a_n = 6a_{n-1} - a_{n-2} \Rightarrow (r-\frac{1}{3}) = 0 \Rightarrow a_n\frac{\alpha_1}{3^n} + \frac{\alpha_2 n}{3^n} \Rightarrow 6 = \alpha_1 \Rightarrow 5 = 2+\frac{\alpha_2}{3} \Rightarrow \alpha_2 = 9 \Rightarrow a_n = \frac{2}{3^{n-1}} + \frac{n}{3^{n-2}}$

(Q-35) Let $P := re^{i\theta}$ and $A,B,C,D := \frac{d}{\sqrt{2}}, -\frac{d}{\sqrt{2}}, \frac{d}{\sqrt{2}}i, -\frac{d}{\sqrt{2}}i$.
$|PA|^2 = \frac{d^2}{2} - \sqrt{2}rd\cos(\theta) + r^2\cos^2(\theta) + r^2\sin^2(\theta)$
$|PB|^2 = \frac{d^2}{2} + \sqrt{2}rd\cos(\theta) + r^2\cos^2(\theta) + r^2\sin^2(\theta)$
$|PC|^2 = \frac{d^2}{2} - \sqrt{2}rd\sin(\theta) + r^2\cos^2(\theta) + r^2\sin^2(\theta)$
$|PD|^2 = \frac{d^2}{2} + \sqrt{2}rd\sin(\theta) + r^2\cos^2(\theta) + r^2\sin^2(\theta)$
$|PA|^2+|PB|^2+|PC|^2+|PD|^2 = 2d^2+4r^2$

(Q-36) Let $f(x) := ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 - \sin(x)$. $f(\frac{\pi}{2}) = f(0) = 0$, so there exists some $0 < \xi < \frac{\pi}{2}$ s.t $f'(\xi) = 0$. Thus, $0 = f'(\xi) = a + b\xi + c\xi^2 - \cos(\xi) \Rightarrow \cos(\xi) = a + b\xi + c\xi^2$ for some real value ξ.

(Q-37) $\sum_{k=0}^{r}\binom{r}{k}\binom{s}{n+k} = \binom{r}{0}\binom{s}{n}+\binom{r}{r}\binom{s}{n+r}+\sum_{k=1}^{r-1}(\binom{r-1}{k}+\binom{r-1}{k-1})\binom{s}{n+k}$
$= \binom{r}{0}\binom{s}{n}+\sum_{k=1}^{r-1}\binom{r-1}{k}\binom{s}{n+k}+\sum_{k=1}^{r-1}\binom{r-1}{k-1}\binom{s}{n+k}+\binom{r}{r}\binom{s}{n+r}$
$= \sum_{k=0}^{r-1}\binom{r-1}{k}\binom{s}{n+k} + \sum_{k=0}^{r-1}\binom{r-1}{k}\binom{s}{n+k+1}$

$\sum_{k=0}^{r-1} \binom{r-1}{k} \binom{s+1}{n+k+1}$. We can repeat this sequence of steps untill we obtain $\sum_{k=0}^{r-r} \binom{r-r}{k} \binom{s+r}{n+k+r} = \binom{s+r}{n+r} = \binom{s+r}{s+r-(n+r)} = \binom{s+r}{s-n}$