Math 114L: Problem Set 3

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Question 1

induction on terms
Base case: $s_1(x) = s_2(x)$ and $s_1(c) = s_2(c)$ because s_1 is the same assignment to variable x and constant c.
Induction hypothesis: Let $\tau_1, \ldots, \tau_n$ be terms where $(s_1, \tau_i) = (s_2, \tau_i)$.
Induction step: Let f be a function. $(s_1, f(\tau_1, \tau_2, \ldots, \tau_n)) = (f((s_1, \tau_1), (s_1, \tau_2), \ldots, (s_1, \tau_n)))$.
BY IH $(f((s_1, \tau_1), (s_1, \tau_2), \ldots, (s_1, \tau_n))) = (f((s_2, \tau_1), (s_2, \tau_2), \ldots, (s_2, \tau_n))) = (s_1, f(\tau_1, \tau_2, \ldots, \tau_n))$.
Induction on formulas
Assume for formulas ϕ and ψ $\phi(s_1) = \phi(s_2)$ and $\psi(s_1) = \psi(s_2)$.
$M \models \neg\phi(s_1)$ iff $M \not\models \phi(s_1) \Rightarrow M \not\models \phi(s_2)$ iff $M \models \neg\phi(s_2)$ by IH.
$M \models \phi(s_1) \rightarrow \psi(s_1)$ if $M \models \psi(s_1)$ or $M \not\models \phi(s_1)$ thus $M \models \psi(s_1) \rightarrow M \models \psi(s_2)$ or $M \not\models \phi(s_1) rightarrow M \not\models \phi(s_2)$ thus $M \models \psi(s_2)$ or $M \not\models \phi(s_2)$ so $M \models \phi(s_2) \rightarrow \psi(s_2)$. $M \models \forall x \phi(x, s_1)$ iff $\phi(x, s_1)$ is true for every assignment x. $\phi(x, s_1) = \phi(x, s_2)$ iff $M \models \forall x \phi(x, s_2)$.

Question 2

(1) $\varphi \equiv \neg(\forall y \exists x(x + x = y))$

(2) $\varphi \equiv (\forall y \exists x(x + y = 0))$

(3) $\varphi \equiv (\forall y \exists x(x + x = y))$

(4) $\varphi \equiv (\forall y \exists x(x + x = y))$

(5) let $\psi(z) \equiv (\exists x(x + x = z))$ (z is even).
$\varphi \equiv (\exists x_1, x_2 \forall y(\psi(x_1 + y) \vee \psi(x_2 + y)))$

Question 3

(1) $\varphi \equiv (\exists y(\neg(y \leq x) \rightarrow (\forall z(z \leq x) \vee (y \leq z))))$

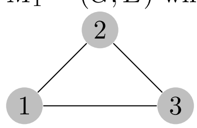
(2) $\varphi \equiv (\exists y(\neg(x \leq y)))$

(3) $\varphi \equiv (\exists y(\neg(x \leq y)))$

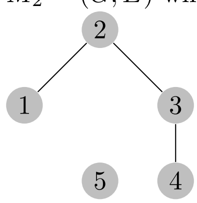
Question 4

(1) $(\forall x \neg E(x,x)) \wedge (\forall y \forall z E(y,z) \to E(z,y))$

(2) $M_1 = (G, E)$ where G is given by the graph shown below,



$M_2 = (G, E)$ where G is given by the graph shown below,



$M_3 = (G, E)$ where G is an undirected tree.

Question 5

(a) $\{n : \forall x \forall y((n = x \times y) \to ((x = 1) \vee (y = 1)))\}$

(b) $\{(n, m) : \exists x(n = m \times x)\}$

(c) $\varphi(m, k) \equiv \exists x(m = k \times x)$
$\psi(n) \equiv \forall x \forall y((n = x \times y) \to ((x = 1) \vee (y = 1)))$
$\{n : \exists x, \exists y, \exists z(\neg((x = y) \vee (y = z) \vee (x = z)) \wedge (\varphi(n, x \times y \times z)) \to (\psi(x) \wedge \psi(y) \wedge \psi(z)))\}$

Question 6

(a) Consider the automorphism $G(x) = -x$. If $\mathbb{N} \subset \mathbb{Z}$ is definable, then $a \in \mathbb{N}$ iff $G(a) \in \mathbb{N}$. This is clearly impossible because the negative numbers are not natural numbers.

(b) Consider the automorphism $G(x) = x/2$. If $\mathbb{N} \subset \mathbb{Q}$ is definable, then $a \in \mathbb{N}$ iff $G(a) \in \mathbb{N}$. This is clearly impossible because fractions are not natural numbers.

Question 7

(a) Let $A = \mathbb{N}$ and let $f^M(x) = x$. $\{a\} = \{\exists x(f^M(x) = a)\}$ with $\overline{a}_0 = \{a\}$.

(b) Let $B = \{-1, 1\}$ and let $f^N(x) = x$. $g(x) = -x$ is an automorphism, so $b \in \{b\}$ iff $g(b) \in \{b\}$ which makes $\{1\}, \{-1\}$ not definable.

Question 8

(a) Suppose $|A| \neq |B|$. Because $|A|$ is finite, WLOG let $|A| = n$. Consider two L-sentences. $\phi_n := \forall y(y = x_1 \lor y = x_2 \lor \ldots \lor y = x_n)$ and $\psi_n := \exists x_1, \exists x_2, \ldots, \exists x_n(\neg(x_1 = x_2 \lor x_1 = x_3 \lor \ldots \lor x_1 = x_n \lor x_2 = x_3 \lor \ldots \lor x_2 = x_n \ldots \lor x_{n-1} = x_n) \land \phi_n)$. If $|B| > |A|$, then $\forall x_1, \forall x_2, \ldots \forall x_n \exists y \neg(y = x_1 \lor y = x_2 \lor \ldots \lor y = x_n)$ and if $|B| < |A|$ by the pigeonhole principle $\forall x_1, \forall x_2, \ldots \forall x_n(x_1 = x_2 \lor x_1 = x_3 \lor \ldots \lor x_1 = x_n \lor x_2 = x_3 \lor \ldots \lor x_2 = x_n \ldots \lor x_{n-1} = x_n)$. Hence, we have found a ψ_n s.t $M \models \psi_n$ and $N \not\models \psi_n$.

(b) (i) Suppose V and W are isomorphic to each other. It follows there exists some map $T : V \to W$. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. For any $w \in W$, there exists $v \in V$ s.t $w = Tv$. $v = \alpha_1 v_1 + \ldots \alpha_n v_n \Rightarrow w = Tv = T(\alpha_1 v_1 + \ldots \alpha_n v_n) = \alpha_1 T v_1 + \ldots + \alpha_n T v_n$ It follows $\{Tv_1, Tv_2, \ldots, Tv_n\}$ forms a basis for W. Thus, W and V have the same dimension.

Suppose V and W have the same dimension. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V and $\{w_1, w_2, \ldots, w_n\}$ be a basis for W. We can define an isomorphism $T = [w_1, w_2, \ldots, w_n]^{-1}[v_1, v_2, \ldots, v_n]$ which is a bijection from $V \to W$. By basic linear algebra, the transformation is linear and preserves 0. Thus, V is isomorphic to W.

(ii) From (i) we know that V is isomorphic to W if, V and W have the same dimension. By fact 5.4, this implies $V \equiv W$.

Question 9

(a) Let $\sigma : \mathbb{R} \to (0, 1)$ be the logistic sigmoid function. The domain and codomain are infinite and the logistic sigmoid function is strictly increasing, so σ has an inverse. Moreover, $x \leq y \to \sigma(x) \leq \sigma(y)$ and vice versa. Since we have found a bijection for $\mathbb{R} \to (0, 1)$ that preserves $\leq$, $M_1 \cong M_2$. Because $M_1 \cong M_2$, it follows $M_1 \equiv M_2$ by fact 5.4.

(b) Consider the sentence $\varphi := \forall y \exists x \neg(x \leq y)$. $M_1 \models \varphi$ but $M_3 \not\models \varphi$ because $(0, 1]$ has a maximum. Thus, $M_1 \not\equiv M_3$.