Math 114L: Problem Set 5

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Problem 1

(1) To show T_R is consistent, it suffices to show there exists a structure G s.t $G \models T_R$. Let G_0 be some countable graph. We want to construct some countable graph $G_1 \supset G_0$ s.t if X and Y are disjoint finite subsets of G_0, then there is a vertex $z \in G_1$ s.t $R(x, z)$ for $x \in X$ and $\neg R(y, z)$ for $y \in Y$. We construct G_1 by adding a new vertex z_X for each finite $X \subseteq G_0$. For each $X \subseteq G_0$ and $x \in X$, we add new edges between x and z_X s.t $R(x, z_X)$ holds. It follows for each finite disjoint subsets X and Y of G_0, we can find $z \in G_1$ s.t $R(x, z)$ for each $x \in X$ and $\neg R(y, z)$ for each $y \in Y$.

By adding new points to G_{n+1} in a similar manner as before, we can iterate the construction of a sequence of countable graphs. For each $n \in \mathbb{N}$, we can construct a countable $G_{n+1} \supset G_n$ s.t for each disjoint subsets X and Y of G_n, we can find $z \in G_{n+1}$ s.t $R(x, z)$ for each $x \in X$ and $\neg R(y, z)$ for each $y \in Y$. Let $G = \bigcup\limits_{n \in \mathbb{N}} G_n$ which by compactness is a countable model for T_R.

(2) Let $M \models T_R$. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be in M s.t $\{a_1, \ldots, a_n\} \cap \{b_1, \ldots, b_m\} = \emptyset$. It follows there exists d_1 s.t $\bigwedge_{i=1}^n R(a_i, d_1) \bigwedge_{j=1}^m \neg R(b_j, d_1)$. We can iterate this process for each $k \in \mathbb{N}$ to find $\{d_1, \ldots, d_k\}$ points such that for $l \leq k$ $\bigwedge_{i=1}^n R(a_i, d_l) \bigwedge_{j=1}^m \neg R(b_j, d_l)$. We can do this because for the finite subset $\{a_1, \ldots, a_n, d_1, \ldots, d_{k-1}\}$ there exists a d_k s.t $\bigwedge_{i=1}^n R(a_i, d_k) \bigwedge_{l=1}^{k-1} R(d_l, d_k) \bigwedge_{j=1}^m \neg R(b_j, d_k)$. It follows we can construct a countably infinite set $D \subset M$ s.t for each $d \in D$ $\bigwedge_{i=1}^n R(a_i, d) \bigwedge_{j=1}^m \neg R(b_j, d)$.

Problem 2

(1) Consider the sentence $\varphi = \exists x \forall y (y \leq x)$. Because $\mathcal{M}$ has a last element, $\mathcal{M} \models \varphi$. By elementary equivalence, $\mathcal{N} \models \varphi$. Thus, $\mathcal{N}$ must have a last element.

(2) Extend L_M to a new language $L' = L_M \cup \{a\}$. Construct a new theory $T' = T_{ord} \cup \{a > m | m \in M\}$ in the language L' by extending the original

theory T_{ord}. Any finite subset of T' mentions only finitely many elements of M. Because M doesn't have a last element, for any finite set of elements $\{m_1, m_2, \ldots, m_k\}$ we can find an element $a \in M$ s.t the set of sentences $\{a > m_i | i = 1, \ldots, k\}$ is consistent with T_{ord}. By compactness, T' is satisfiable. Thus, T' has a model $\mathcal{N}$ s.t $M \subseteq N$, $M \equiv N$, and there exists $a \in N$ s.t a is larger than every element in M.

(3) L_M to a new language $L' = L_M \cup \{c_i\}$ by adding infinitely many constants. Construct a new theory $T'' = T_{ord} \cup \{a < c_i < b | a, b \in M, a < b\}$ in the language L' by extending the original theory T_{ord}. Any finite subset of T'' mentions only finitely many pairs (a, b) of M and each sentence is satisfiable by an appropriate choice of c_i because M is given to be dense. By compactness, T'' is satisfiable. Thus, T'' has a models $\mathcal{N}$ s.t $M \subseteq N$, $M \equiv N$, and there exists $c \in N \setminus M$ s.t $a < c < b$ for each pair $a, b \in M$ where $a < b$.

Problem 3

Assume to the contrary no such natural number n exists such that φ is true in all finite models of T with size at least n. Construct a sequence of models $M_1, M_2, \ldots$ where $|M_n| \geq n$ and $M_n \models T$ but $M_n \not\models \varphi$. Consider the set of sentences $T \cup \{\neg\varphi\} \cup \{\exists x_1, \ldots x_n \bigwedge_{i \neq j} (x_i \neq x_j) | n \in \mathbb{N}\}$. By assumption, there is a model $|M_n| \geq n$ for each $n \in \mathbb{N}$. By compactness, $T \cup \{\neg\varphi\}$ has an infinite model. Let M be an infinite model of $T \cup \{\neg\varphi\}$. Thus, $M \models T$ and $M \models \neg\varphi$ which is clearly a contradiction. Hence, a finite n must exist.

Problem 4

Assume the theory T is complete. Let M and N be models of T. Our goal is to show $M \equiv N$ i.e for every $\mathcal{L}$-sentence ϕ $M \models \phi$ iff $N \models \phi$. Since T is complete, for every $\mathcal{L}$-sentence ϕ either $T \vdash \phi$ or $T \vdash \neg\phi$. By soundedness and the completeness of first order logic $T \vdash \phi \leftrightarrow T \models \phi$. If $T \vdash \phi$, then ϕ is true in every models of T, including M and N, so $M \models \phi$ and $N \models \phi$. If $T \vdash \neg\phi$, then $\neg\phi$ is true in every models of T, including M and N, so $M \not\models \phi$ and $N \not\models \phi$. Thus, M and N agree on the truth values for every $\mathcal{L}$-sentence ϕ.
Assume for every two models M and N of T, $M \equiv N$. Our goal is to show T is complete i.e for every $\mathcal{L}$-sentence ϕ either $T \vdash \phi$ or $T \vdash \neg\phi$. Assume to the contrary T is not complete. Thus, there exists a sentence ϕ s.t neither $T \vdash \phi$ nor $T \vdash \neg\phi$. Because T doesn't prove ϕ, we can find a model M s.t $M \not\models \phi$. Because T doesn't prove $\neg\phi$, we can find a model N s.t $N \models \phi$. Thus, we obtain a contradiction because we assumed $M \models \phi$ iff $N \models \phi$ for every $\mathcal{L}$-sentence ϕ. Hence, T must be complete.