Math 100: Problem Set 5

Owen Jones

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(Q-1) If $a^{-1}ba = b^{-1}$ and $b^{-1}ab = a^{-1}$ then $(b^{-1}ab)ba = b^{-1}$. It follows $ab^2a = 1$. We also observe that $b^{-1}ab = a^{-1} = (a^{-1}ba)ab = a^{-1} \Rightarrow ba^2b = 1$. $b^{-1}ab = a^{-1} \Rightarrow b = aba$. Thus $ab^2a = a^2ba^2ba^2 = 1$. Since $ba^2b = 1$ $ab^2a = a^2ba^2ba^2 = a^2(ba^2b)a^2 = a^2 \cdot a^2 = a^4 = 1$. By similar logic, $a^{-1}ba = b^{-1} \Rightarrow a = bab$. Thus, $ba^2b = b^2ab^2ab^2 = 1 \Rightarrow b^2ab^2ab^2 = b^2(ab^2a)b^2 = b^2 \cdot b^2 = b^4 = 1$.

(Q-2) Since a' is a unique element s.t $aa' = 1$, it follows for all $x \in R$ where $x \neq 0_R$, $ax \neq 0_R$. If such an x existed $aa' + ax = a(a' + x) = 1 \Rightarrow a'$ is not unique. $a(a'a) = (aa')a = a$ by associativity. It follows $a(a'a - 1_R) = 0$. Therefore, $a'a - 1_R = 0$ because $ax \neq 0_R$ for all $x \neq 0_R$. Hence, $a'a = 1$.

(Q-3)

- Fermat's Little Theorem states that for any x, $x^p - x \equiv 0 \mod (p)$. It follows that for each $x \in \mathbb{Z}_p$, x is a root of $x^p - x$. Thus, because $\mathbb{Z}_p$ is a field, we can write $x^p - x$ as the product of its factors. $x^p - x = Q(x)\displaystyle\prod_{i=0}^{p-1}(x - i)$. $Q(x) = 1$ because $x^p - x$ can have at most p factors.
- Given $\displaystyle\prod_{i=0}^{p-1}(x - i) \equiv x^p - x \mod (p)$, we know for $x \neq 0$ $\displaystyle\prod_{i=1}^{p-1}(x - i) \equiv x^{p-1} - 1 \mod (p)$. Take $x = p$. Thus, $\prod_{i=1}^{p-1}(p - i) = (p-1)! \equiv p^{p-1} - 1 \mod (p)$. Because p^{p-1} is a multiple of p, $p^{p-1} - 1 \equiv -1 \mod (p)$. Hence, $(p-1)! \equiv -1 \mod (p)$.

(Q-4) Since $2^p - 1$ and 2 are primes, we can write the sum of n's factors excluding n as $\displaystyle\sum_{i=0}^{p-1} 2^i + (2^p - 1)\sum_{i=0}^{p-2} 2^i = \frac{2^p - 1}{2 - 1} + (2^p - 1)\frac{2^{p-1} - 1}{2 - 1} = 2^{p-1}(2^p - 1)$

(Q-5) $\displaystyle\sum_{i=1}^{n} i \sum_{j=0}^{i-1} 10^j = \sum_{i=1}^{n} i\frac{10^i - 1}{10 - 1} = -\frac{n(n+1)}{18} + \frac{1}{9}\sum_{i=1}^{n} i10^i$

$\displaystyle= \frac{n10^{n+1}}{81} - \frac{10^{n+1} - 10}{729} - \frac{n(n+1)}{18}$

(Q-6) Solving explicitely for $a_n = \frac{5\cdot 3^{n-1}-1}{2}$.

It follows $\displaystyle\sum_{i=1}^{n} a_n = \frac{5}{2}\sum_{i=1}^{n} 3^{n-1} - \frac{n}{2} = \frac{5(3^n-1)}{4} - \frac{n}{2}$

(Q-7) $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. It follows $\displaystyle\sum_{k=1}^{n} \sin((2k-1)\theta) = \sum_{k=1}^{n} \frac{e^{i(2k-1)\theta} - e^{-i(2k-1)\theta}}{2i}$

$$= \frac{1}{2i}\left(e^{i\theta}\frac{e^{2i\theta\cdot n}-1}{e^{2i\theta}-1} - e^{-i\theta}\frac{e^{-2i\theta\cdot n}-1}{e^{-2i\theta}-1}\right) = \frac{1}{2i}\left(\frac{e^{2i\theta\cdot n}-1}{e^{i\theta}-e^{-i\theta}} + \frac{e^{-2i\theta\cdot n}-1}{e^{i\theta}-e^{-i\theta}}\right) = \frac{i(2-(e^{2i\theta\cdot n}+e^{-2i\theta\cdot n}))}{2(e^{i\theta}-e^{-i\theta})} = \frac{1-\cos(2n\theta)}{2\sin(\theta)} = \frac{\sin^2(n\theta)}{\sin(\theta)}$$

(Q-8) (a) $\frac{k-1}{k!} = \frac{1}{(k-1)!} - \frac{1}{k!} \Rightarrow \sum_{k=2}^{n} \frac{k-1}{k!} = \sum_{k=2}^{n}(\frac{1}{(k-1)!} - \frac{1}{k!}) = 1 - \frac{1}{n!}$

(b) $k \times k! = (k+1)! - k! \Rightarrow \sum_{k=1}^{n} k \times k! = (n+1)! - 1$

(c) $\frac{2k}{k(k+1)(k+2)} = \frac{2}{(k+1)(k+2)} = \frac{2}{k+1} - \frac{2}{k+2} \Rightarrow \sum_{k=1}^{n} \frac{2k}{k(k+1)(k+2)} = 1 - \frac{2}{n+2}$

(Q-9) $1 - \frac{1}{k^2} = \frac{k^2-1}{k^2} = \frac{(k-1)(k+1)}{k^2} \Rightarrow \prod_{k=2}^{n} \frac{(k-1)(k+1)}{k^2}$
$= \frac{(1)(3)}{(2)(2)}\frac{(2)(4)}{(3)(3)}\frac{(3)(5)}{(4)(4)} \cdots \frac{(n-2)(n-1)}{(n-2)(n-2)}\frac{(n-2)(n)}{(n-1)(n-1)}\frac{(n-1)(n+1)}{(n)(n)}$
$= \frac{1(n+1)}{2n} \Rightarrow \lim_{n\to\infty} \prod_{k=2}^{n} \frac{(k-1)(k+1)}{k^2} = \frac{1}{2}$

(Q-10) We know $F_k = F_{k+1} - F_{k-1} \Rightarrow \sum_{k=1}^{n} F_{2k-1} = \sum_{k=1}^{n} F_{2k} - F_{2k-2} = F_{2n} - F_0 = F_{2n}$