Math 170E: Homework 5

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Problem 1. We expect the number of drop that fall on a 5 square inch region to follow a Poisson distribution because we are counting the number of events that occur during a fixed interval of time. $f(0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$ where $\lambda = 5in^2 \cdot \frac{1}{10}min \cdot \frac{\frac{20drops}{1in^2}}{1min} = 10drops$ is the expected number of drops for a given interval, so $f(0) = e^{-10}$

Problem 2. (1) $E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^3 \frac{2}{9}x^2dx = \frac{2}{27}x^3|_0^3 = 2$

(2) $E[X^2] = \int_{-\infty}^{\infty} x^2f(x)dx = \int_0^3 \frac{2}{9}x^3 = \frac{2}{36}x^4|_0^3 = \frac{9}{2}$

(3) $f(y) = \begin{cases} \frac{1}{9} & \text{for } y = 1 \\ \frac{1}{3} & \text{for } y = 2 \\ \frac{5}{9} & \text{for } y = 3 \end{cases}$

(4) $E[Y] = \displaystyle\sum_{y \in Y} yf(y) = \frac{1}{9} + \frac{2}{3} + \frac{15}{9} = \frac{22}{9}$

Problem 3. (1) $G(w) = \begin{cases} 0 & \text{for } w \leq a \\ \frac{w-a}{b-a} & \text{for } a < w < b \\ 1 & \text{for } b \leq w \end{cases}$

(2) W has a uniform distribution $U(a, b)$

(3) $E[YW] = aE[Y] + (b-a)E[Y^2] = \frac{a}{2} + \frac{b-a}{3} = \frac{2b+a}{6}$

Problem 4. (1) $f(x) = \frac{1}{\theta}e^{\frac{-x}{\theta}} = \frac{2}{3}e^{\frac{-2x}{3}}$

(2) $P(X > 2) = \int_2^{\infty} f(x)dx = e^{\frac{-4}{3}}$

Problem 5. (1) $\mu = M'(0) = \frac{3}{(1-3\cdot 0)^2} = 3$, $E[X^2] = M''(0) = \frac{18}{(1-3\cdot 0)^3} = 18$, $Var(X) = E[X^2] - \mu^2 = 9$

(2) $M_Y(t) = E[e^{(2X+1)t}] = e^tE[e^{2Xt}] = e^tM_X(2t) = \frac{e^t}{1-6t}$

Problem 6. (1) $f(w) = \frac{(16w)^2e^{-16w}}{\Gamma(2)}$

(2) $P(W < 0.5) = \int_0^{0.5} \frac{(16w)^2e^{-16w}}{\Gamma(2)}dw = -\frac{1}{8}e^{-16w}(128w^2 + 16w + 1)|_0^{\frac{1}{2}} = \frac{1}{8} - \frac{41}{8}e^{-8}$

(3) $E[\frac{1}{W}] = \int_0^{\infty} 256we^{-16w}dw = -16xe^{-16x} - 16e^{-16x}|_0^{\infty} = 1$
$(0 < xe^{-16x} = \dfrac{x}{\sum_{n=0}^{\infty} \frac{(16x)^n}{n!}} = \dfrac{x}{1 + 16x + 128x^2 + \cdots} < \dfrac{x}{128x^2}$ for all positive x because each element of the series is positive, and a smaller denominator implies a larger value. Thus, $\displaystyle\lim_{x\to\infty} \frac{x}{128x^2} = 0 \Rightarrow \lim_{x\to\infty} xe^{-16x} = 0$ by the squeeze theorem.)