Math 131B: Homework 8

Owen Jones

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Problem 1. **Exercise 4.7.3**

(a) The sum of angles identity gives us $\cos(x+\pi) = \cos(x)\cos(\pi) - \sin(x)\sin(\pi)$ and $\sin(x+\pi) = \sin(x)\cos(\pi) + \sin(\pi)\cos(x)$. Definition 4.7.4 tells us that $cos(\pi) = -1$ and $\sin(\pi) = 0$, so $\cos(x)\cos(\pi) - \sin(x)\sin(\pi)$ and $\sin(x+\pi) = \sin(x)\cos(\pi) + \sin(\pi)\cos(x)$ reduce to $\cos(x)(-1) - \sin(x)(0) = -\cos(x)$ and $\sin(x)(-1) + (0)\cos(x) = -\sin(x)$ respectively. It follows $\sin(x+2\pi) = -\sin(x+\pi) = \sin(x)$ and $\cos(x+2\pi) = -\cos(x+\pi) = \cos(x)$, so $\sin(x)$ and $\cos(x)$ are periodic with period 2π.

(b) By Theorem 4.7.5(a), $\sin(x) = -\sin(x+\pi)$, so if $sin(0) = 0$, then for all positive multiples of π $\sin(k\pi) = 0$. (This easily follows from a simple induction). Because $\sin(-x) = -\sin(x)$, the same holds for all negative multiples of π. Thus, $\frac{k\pi}{\pi} = k$ which is an integer. If $\frac{x}{\pi}$ is an integer, then x is a multiple of π, but we showed already that the sin of every multiple of π is 0, so $\sin(x) = 0$.

(c) The sum of angles identity gives us $\sin(\pi) = \sin(\frac{\pi}{2} + \frac{\pi}{2}) = 2\sin(\frac{\pi}{2})\cos(\frac{\pi}{2})$. Since $\sin(\pi) = 0$ and $\sin(\frac{\pi}{2}) \neq 0$ by Definition 4.7.4, $\cos(\frac{\pi}{2})$ must equal 0. We use Theorem 4.7.5(a) as we did in part (b) to show $\cos(x) = 0$ for $x \in \{k\pi - \frac{\pi}{2} : k \in \mathbb{N}\}$. We then use $\cos(-x) = \cos(x)$ to show $\cos(x) = 0$ for $x \in \{k\pi + \frac{\pi}{2} : k \in \mathbb{Z}\}$. Since $\frac{k\pi + \frac{\pi}{2}}{\pi} = k + \frac{1}{2}$, $\frac{x}{\pi}$ is an integer plus $\frac{1}{2}$. If $\frac{x}{\pi}$ is an integer plus $\frac{1}{2}$, then x is a multiple of π plus $\frac{\pi}{2}$, but we already showed that the cos of every multiple of π plus $\frac{\pi}{2}$ is 0, so $\sin(x) = 0$.

Lemma

WTS $\sin(\frac{\pi}{2}) = 1$ and $\sin(-\frac{\pi}{2}) = -1$. *Note: We showed* $\cos(\frac{\pi}{2}) = 0$ *in Exercise 4.7.3(c).*
$1 = \cos(0) = \cos(\frac{\pi}{2} + (-\frac{\pi}{2})) = \cos(\frac{\pi}{2})\cos(-\frac{\pi}{2}) - \sin(\frac{\pi}{2})\sin(-\frac{\pi}{2}) = 0 - \sin(\frac{\pi}{2})(-\sin(\frac{\pi}{2})) = \sin(\frac{\pi}{2})^2$ Since $\sin'(0) = \cos(0) = 1$ and $\frac{\pi}{2} < \pi$, the intermediate value theorem tells us $\sin(\frac{\pi}{2}) > 0$, so $\sin(\frac{\pi}{2})^2 = 1 \Rightarrow \sin(\frac{\pi}{2}) = 1$ and $-\sin(\frac{\pi}{2}) = \sin(-\frac{\pi}{2}) = -1$.

Problem 2. **Exercise 4.7.4**

Let x, y be real numbers such that $x^2 + y^2 = 1$. It follows that $x = \pm\sqrt{1-y^2}$. In particular, $0 \leq x^2$ and $0 \leq y^2$, so $0 \leq 1 - y^2 \leq 1 \Rightarrow |x| \leq 1$. Since $\sin(\theta)$ is a continuous function that assumes values of -1 and 1 on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the intermediate value theorem states there exists $\theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ s.t $\sin(\theta_1) = x$. In addition, $\sin(-\theta) = -\sin(\theta)$ and $\sin(\pi + \theta) = -\sin(\theta) \Rightarrow \sin(-(\pi \pm \theta)) = \sin(\theta)$. This tells there exists $\theta_2 = -(\theta_1 + \pi)$ for $\theta_1 \in [-\frac{\pi}{2}, 0]$ and $\theta_2 = -(\theta_1 - \pi)$ for $\theta_1 \in (0, \frac{\pi}{2}]$ s.t $\sin(\theta_2) = x$. It remains to show $y = \cos(\theta_1)$ or $y = \cos(\theta_2)$. Since $\theta_2 = -(\theta_1 \pm \pi) \Rightarrow \cos(\theta_2) = \cos(-(\theta_1 \pm \pi)) = \cos((\theta_1 \pm \pi)) = -\cos(\theta_1)$. Thus, $y = \pm\sqrt{1-x^2} = \pm\sqrt{1-\sin(\theta_1)^2}(\text{ or } \pm\sqrt{1-\sin(\theta_2)^2}) = \pm\sqrt{\cos(\theta_1)^2} = \pm\cos(\theta_1) = \cos(\theta_1)$ or $\cos(\theta_2)$

Problem 3. **Exercise 4.7.10**

(a) The Weierstrauss M-test tells us that if $\displaystyle\sum_{n=1}^{\infty} ||f^{(n)}||_\infty$ converges, then $\displaystyle\sum_{n=1}^{\infty} f^{(n)}(x)$ converges uniformly to $f(x)$. Since $|\cos(x)| \leq 1$, $\displaystyle\sum_{n=1}^{\infty} ||f^{(n)}||_\infty = \sum_{n=1}^{\infty} 4^{-n} = \frac{1}{3}$ by the sum of a geometric series, so $\sum_{n=1}^{\infty} 4^{-n}\cos(32^n \pi x)$ converges uniformly to $f(x)$.

(b) We will show $|f(\frac{j+1}{32^m}) - f(\frac{j}{32^m})| \geq 4^{-m}$ to be true by induction on m. For the base case choose $m = 1$ and let j be arbitrary.

$$|f(\frac{j+1}{32^1}) - f(\frac{j}{32^1})| = |\sum_{n=1}^{\infty} 4^{-n}(\cos(32^n\pi\frac{j+1}{32}) - \cos(32^n\pi\frac{j}{32}))|$$

Using the identity $\sum_{n=1}^{\infty} a_n = (\sum_{n=1}^{m-1} a_n) + a_m + \sum_{n=m+1}^{\infty} a_n$, the fact that cosine is periodic with period of 2π, and the fact that $\cos(x) = -\cos(x+\pi)$ we obtain

$$|f(\frac{j+1}{32^1}) - f(\frac{j}{32^1})| = |4^{-1}(2\cos(j\pi+\pi)) + \sum_{n=2}^{\infty} 4^{-n}(\cos(2\pi k_n) - \cos(2\pi l_k))|$$

$$\text{(where } k_n = \frac{32^{n-1}(j+1)}{2} \text{ and } l_n = \frac{32^{n-1}(j)}{2} \text{ which are clearly integers)}$$

$$= |4^{-1}(\pm 2) + \sum_{n=2}^{\infty} 4^{-n}(1-1)| = \frac{1}{2} \geq \frac{1}{4}$$

, so the claim holds for $m = 1$. Next, we assume for some arbitrary $m \geq 1$, the claim $|f(\frac{j+1}{32^m}) - f(\frac{j}{32^m})| \geq 4^{-m}$ holds. Thus, it remains to show the claim holds for $m+1$.

$$|f(\frac{j+1}{32^{m+1}}) - f(\frac{j}{32^{m+1}})| = |\sum_{n=1}^{\infty} 4^{-n}(\cos(32^n\pi\frac{j+1}{32^{m+1}}) - \cos(32^n\pi\frac{j}{32^{m+1}}))|$$

Using the identity $\sum_{n=1}^{\infty} a_n = (\sum_{n=1}^{m-1} a_n) + a_m + \sum_{n=m+1}^{\infty} a_n$, the fact that cosine is periodic with period of 2π, and the fact that $\cos(x) = -\cos(x+\pi)$ we obtain

$$\begin{aligned}|f(\frac{j+1}{32^{m+1}}) - f(\frac{j}{32^{m+1}})| = |\sum_{n=1}^{m} 4^{-n}(\cos(32^n\pi\frac{j+1}{32^{m+1}}) - \cos(32^n\pi\frac{j}{32^{m+1}}))\\ +4^{-m-1}(2\cos(j\pi+\pi))\\ + \sum_{n=m+2}^{\infty} 4^{-n}(\cos(2\pi k_n) - \cos(2\pi l_k))|\end{aligned}$$

$$\text{(where } k_n = \frac{32^{n-m-1}(j+1)}{2} \text{ and } l_n = \frac{32^{n-m-1}(j)}{2} \text{ which are clearly integers)}$$

$\sum_{n=m+2}^{\infty} 4^{-n}(\cos(2\pi k_n) - \cos(2\pi l_k)) = 0$ because cosine is periodic with period of 2π, so it suffices to show $|\sum_{n=1}^{m} 4^{-n}(\cos(32^n\pi\frac{j+1}{32^{m+1}}) - \cos(32^n\pi\frac{j}{32^{m+1}})) + 4^{-m-1}(2\cos(j\pi+\pi))| \geq 4^{-m-1}$.
Using the identity $|\cos(x) - \cos(y)| \leq |x-y|$, we obtain

$$\begin{aligned}&\sum_{n=1}^{m} 4^{-n}(\cos(32^n\pi\frac{j+1}{32^{m+1}}) - \cos(32^n\pi\frac{j}{32^{m+1}}))\\ &\leq \sum_{n=1}^{m} 4^{-n}|(\cos(32^n\pi\frac{j+1}{32^{m+1}}) - \cos(32^n\pi\frac{j}{32^{m+1}}))|\end{aligned}$$

$$\leq \sum_{n=1}^{m} 4^{-n}|\frac{32^n\pi}{32^{m+1}}| = \sum_{n=1}^{m} 8^n\frac{\pi}{32^{m+1}} = \frac{8^{m+1}-1}{7}\frac{\pi}{32^{m+1}} \leq \frac{\pi}{2^{2(m+1)}\cdot 7} \leq \frac{4^{-m-1}}{2}$$

Using the reverse triangle innequality, we obtain

$$\begin{aligned}
&|\sum_{n=1}^{m} 4^{-n}(\cos(32^n\pi\frac{j+1}{32^{m+1}}) - \cos(32^n\pi\frac{j}{32^{m+1}})) + 4^{-m-1}(2\cos(j\pi+\pi))| \\
\geq\ &||4^{-m-1}(2\cos(j\pi+\pi))| - |\sum_{n=1}^{m} 4^{-n}(\cos(32^n\pi\frac{j+1}{32^{m+1}}) - \cos(32^n\pi\frac{j}{32^{m+1}}))|| \\
&\geq |2\cdot 4^{-m-1} - \frac{4^{-m-1}}{2}| = \frac{3\cdot 4^{-m-1}}{2} \geq 4^{-m-1}
\end{aligned}$$

, so the claim holds for $m+1$. Hence, by induction, the claim holds for all $m \geq 1$ and j.

(c) Let x_0 be arbitrary and let $s_m = \frac{j_m}{32^m}$ and $t_m = \frac{j_m+1}{32^m}$ be sequences of rational numbers where j_m is an integer s.t $j_m \leq 32^m x_0 \leq j_m+1$. Assume for sake of contradiction that f is differentiable at x_0. If x_0 is a rational number whose denominator is a power of 2, then for all but finitely many m there exists j_m s.t $j_m = 32^m x_0$. Since t_m converges to x_0 and $t_m \neq x_0$ for all m, $\lim_{m\to\infty} \frac{f(t_m)-f(x_0)}{t_m - x_0}$ converges to $f'(x_0)$. Then by the definition of the derivative and the continuity of f

$$|f'(x_0)| = \lim_{m\to\infty} |\frac{f(t_m)-f(x_0)}{t_m-x_0}| = \lim_{m\to\infty} |\frac{f(t_m)-f(s_m)}{t_m-s_m}| \geq \lim_{m\to\infty} |\frac{32^m}{4^m}| = \infty$$

If not, then s_m and t_m converge to but never equal x_0, and $s_m < x_0 < t_0$. Thus, using the triangle innequality and the limit defintion of the derivative, we obtain

$$\begin{aligned}
|f'(x_0)| &= \lim_{m\to\infty} \frac{|\frac{f(x_0)-f(s_m)}{x_0-s_m}| + |\frac{f(x_0)-f(t_m)}{x_0-t_m}|}{2} \\
&\geq \lim_{m\to\infty} \frac{|\frac{f(x_0)-f(s_m)}{\frac{1}{32^m}}| + |\frac{f(x_0)-f(t_m)}{\frac{1}{32^m}}|}{2} \\
\text{because } \frac{1}{32^m} &= \frac{1}{t_m-s_m} \leq \frac{1}{t_m-x_0} \text{ and } \leq \frac{1}{x_0-s_m} \\
&\geq \lim_{m\to\infty} \frac{|\frac{f(s_m)-f(t_m)}{\frac{1}{32^m}}|}{2} \\
&\geq \lim_{m\to\infty} \frac{32^m}{2\cdot 4^m} \\
&= \infty
\end{aligned}$$

, so there exists at least one sequence that converges to x_0 where the limit definition of the derivative diverges. Thus, we have a contradiction, so f is not differentiable at x_0.

(d) $\sum_{n=1}^{\infty} ||f_n'||_\infty = \sum_{n=1}^{\infty} ||-8^n\pi\sin(32^n\pi x)||_\infty = \sum_{n=1}^{\infty} 8^n\pi$ which does not converge, so $\sum_{n=1}^{\infty} ||f_n'||_\infty$ is not absolutely convergent.

Problem 4. **Exercise 6.2.1**

If f is differentiable at x_0 and $f'(x_0) = L$ then by the limit definition of the derivative $\lim_{x\to x_0; x\in E\setminus\{x_0\}} \frac{f(x)-f(x_0)}{x-x_0} = L$. It follows $L = L\frac{x-x_0}{x-x_0}$ because $x - x_0 \neq 0$. Subtracting $L\frac{x-x_0}{x-x_0}$ from both sides, we obtain, $\lim_{x\to x_0; x\in E\setminus\{x_0\}} \frac{f(x)-f(x_0)-L(x-x_0)}{x-x_0} = \lim_{x\to x_0; x\in E\setminus\{x_0\}} \frac{f(x)-(f(x_0)+L(x-x_0))}{x-x_0} = 0.$

It follows from a proof from 131a that $\lim_{x\to x_0; x\in E\setminus\{x_0\}} \frac{f(x)-(f(x_0)+L(x-x_0))}{x-x_0} = 0 \Leftrightarrow \lim_{x\to x_0; x\in E\setminus\{x_0\}} |\frac{f(x)-(f(x_0)+L(x-x_0))}{x-x_0}| = 0$, so $(a) \Rightarrow (b)$.

If $\lim_{x\to x_0; x\in E\setminus\{x_0\}} |\frac{f(x)-(f(x_0)+L(x-x_0))}{x-x_0}| = 0$ then $\lim_{x\to x_0; x\in E\setminus\{x_0\}} \frac{f(x)-(f(x_0)+L(x-x_0))}{x-x_0} = 0.$

By algebra, $\lim\limits_{x\to x_0; x\in E\setminus\{x_0\}} \frac{f(x)-(f(x_0)+L(x-x_0))}{x-x_0} + L\frac{x-x_0}{x-x_0} = \frac{f(x)-f(x_0)}{x-x_0} = L\frac{x-x_0}{x-x_0} = L.$ This is the limit definition of the derivative, so of course, f is differentiable at x_0 with $f'(x_0) = L$. Thus, $(b) \Rightarrow (a)$.

Problem 5. **Exercise 6.2.2**

Suppose for the sake of contradiction that L_1 and L_2 are distinct linear transformations that satisfy $\lim\limits_{x\to x_0; x\in E\setminus\{x_0\}} \frac{|f(x)-(f(x_0)+L(x-x_0))|}{||x-x_0||} = 0$. It follows there exists at least one non-zero vector v s.t $L_1 v \neq L_2 v$, so we make the change of variables $x \to x_0 + vt$ where t is a scalar. It follows by the definition of a derivative

$$\lim_{t\to 0; t>0, x_0+vt\in E} \frac{||f(x_0+vt)-(f(x_0)+L_1 vt)||}{||vt||} = \lim_{t\to 0; t>0, x_0+vt\in E} \frac{||f(x_0+vt)-(f(x_0)+L_2 vt)||}{||vt||} = 0$$

Thus,

$$\lim_{t\to 0; t>0, x_0+vt\in E} \frac{||L_1 vt - L_2 vt||}{||vt||} \leq \lim_{t\to 0; t>0, x_0+vt\in E} \frac{||f(x_0+vt)-(f(x_0)+L_1 vt)||}{||vt||} + \frac{||f(x_0+vt)-(f(x_0)+L_2 vt)||}{||vt||}$$

by triangle innequality

so by the squeeze theorem

$$\lim_{t\to 0; t>0, x_0+vt\in E} \frac{||L_1 vt - L_2 vt||}{||vt||} = \lim_{t\to 0; t>0, x_0+vt\in E} \frac{||L_1 v - L_2 v||\cdot|t|}{||v||\cdot|t|} = \lim_{t\to 0; t\neq 0} \frac{||L_1 v - L_2 v||}{||v||} = 0$$

which is impossible because $L_1 v - L_2 v \neq 0$, so we obtain a contradiction and $L_1 = L_2$.