Math 106: Problem Set 3

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2/4/2024

3.2.1 Let s be a square number. It follows there exists some integer q s.t $s = q^2$.

$q \equiv 0 \pmod 4$ There exists some integer k s.t $q = 4k$. It follows $s = 16k^2$. $8 \mid 16k^2 \Rightarrow s \equiv 0 \pmod 8$.

$q \equiv 2 \pmod 4$ There exists some integer k s.t $q = 2+4k$. It follows $s = 16k^2+16k+4$. $8 \mid 16k^2 + 16k \Rightarrow s \equiv 4 \pmod 8$.

q is odd There exists some integer k s.t $q = \pm 1 + 4k$. It follows $s = 16k^2 \pm 8k + 1$. $8 \mid 16k^2 + 16k \Rightarrow s \equiv 1 \pmod 8$.

3.2.2 From **3.2.1**, we know any square leaves remainder $0, 1$, or 4 on division by 8. Thus, if we take 3 square numbers, their remainders added together (mod 8) will be some value. Some quick examples:
$0+0+0 = 0, 0+0+1 = 1, 0+1+1 = 2, 1+1+1 = 3, 4+0+0 = 4, 4+1+0 = 5, 4+1+1 = 6$
It remains to show it is impossible for the sum of 3 squares to leave 7 on division by 8.
Since 7 is odd, we must have either 1 or 3 odd squares. 3 odd squares added together leaves remainder 3 on division by 8, so we must have 1 odd square. Since the other two squares are even, they must leave remainder 0 or 4 on division by 8. Thus, the sum of the even squares leave remainder 0 or 4 on division by 8
Since neither 1 nor 5 is 7, it is impossible to leave remainder 7 on division by 8.

3.2.3 Let x_k by the k^{th} pentagonal number. From figure 3.1, we deduce $x_{k+1} = x_k + 3k + 1$.
Pf by induction:
Base case: $\frac{3 \cdot 1^2 - 1}{2} = 1$ which is the 1^{st} pentagonal number.
Induction hypothesis: Assume for some $k \geq 1$ $x_k = \frac{3k^2-k}{2}$.
Induction step: $x_{k+1} = \frac{3k^2-k}{2} + 3k + 1 = \frac{3k^2+5k+2}{2} = \frac{3(k+1)^2-(k+1)}{2}$.
Hence, by induction, the claim holds for all k.

3.2.4 Let t_k be the k^{th} triangular number. Thus $t_k = \sum_{i=0}^{k} i = \frac{k(k+1)}{2}$. We show $k^2 = t_{k-1} + t_k$.
$t_{k-1} + t_k = \frac{(k-1)k}{2} + \frac{k(k+1)}{2} = \frac{2k^2}{2} = k^2$

3.3.1 Let q be a prime divisor of $2^{n-1}p$.

Thus, either $\begin{cases} q = p & \text{if } q \mid p \\ q = 2 & \text{if } q \mid 2^{n-1} \end{cases}$ If $q = p$, then we can iterate through 2^{n-1} using the prime divisor property to show $1, 2, 2^2, \cdots 2^{n-1}$ are all proper divisors of $2^{n-1}p$. If $q = 2$, then we can iterate through $2^{n-2}p$ using the prime divisor property.

Thus, we obtain $\begin{cases} q = p \text{ and proceed to } 2^{n-2} \text{ case} & \text{if } q \mid p \\ q = 2 & \text{if } q \mid 2^{n-2} \end{cases}$ We can iterate through this case to show $p, 2p, 2^2p, \cdots 2^{n-2}p$ are all proper divisors of $2^{n-1}p$.

Thus, we only need to show that there are no other proper divisors of $2^{n-1}p$. Every number has a unique prime factorization. It follows that any proper divisor of $2^{n-1}p$ must must constructed from 2s and p. Moreover, any number $2^j p^k$ where $j > n-1$ or $p > 1$ can't be a divisor because $2^{j-n+1}p^{k-1} \nmid 1$.

3.3.2 If we divide a by b, we obtain a quotient q_1 and remainder r_2. It follows we can write r_2 as a linear combination of a and b i.e $r_2 = a - q_1 b$. Assume for some i, we can write r_i and r_{i+1} as a linear combination of a and b. We have $r_{i+2} = r_i - q_{i+1}r_i$ by division with remainder. Thus, $r_{i+2} = (am_i + bn_i) - q_{i+1}(am_{i+1} + bn_{i+1}) = a(m_i - q_{i+1}m_{i+1}) + b(n_i - q_{i+1}n_{i+1})$ which is a linear combination of a and b. Hence, $m_{i+2} = m_i - q_{i+1}m_{i+1}$ and $n_{i+2} = n_i - q_{i+1}n_{i+1}$ where $m_0 = 1, m_1 = 0, n_0 = 0, n_1 = 1$. When we terminate the Euclidean Algorithm after some k steps, we obtain $am_k + bn_k = \gcd(a, b)$.

3.3.3 If $\gcd(a, b) \mid c$ there exists an integer k s.t $\gcd(a, b)k = c$. It follows from **3.3.2** there exists m, n s.t $am + bn = \gcd(a, b)$. Thus, $a(mk) + b(nk) = c$ has an integer solution because mk, nk are both integers. Suppose $am + bn = c$ has an integer solution. $\gcd(a, b) \mid a$ and $\gcd(a, b) \mid b$, so $\gcd(a, b) \mid am + bn$. Thus, $\gcd(a, b) \mid c$ must also be true.

3.3.4 $\gcd(12, 15) = 3$. Thus, by **3.3.3**, if there exists a solution to $12x + 15y = 1$ then $3 \mid 1$. This is clearly false, so $12x + 15y = 1$ has no integer solutions.