Math 151A: Midterm

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Problem 1: Solution

a) $g(x) = x^{\frac{1}{5}}$. $g \in C[\frac{1}{5^{\frac{5}{4}}}, \frac{3}{2}]$. g assumes a minimum of $5^{-\frac{1}{4}}$ at the endpoint $x = \frac{1}{5^{\frac{5}{4}}}$ and a maximum of $(\frac{3}{2})^{\frac{1}{5}}$ at the endpoint $x = \frac{3}{2}$, so $g(x) \in [\frac{1}{5^{\frac{5}{4}}}, \frac{3}{2}]$ for all $x \in [\frac{1}{5^{\frac{5}{4}}}, \frac{3}{2}]$. $|g'(x)|$ exists $\forall x \in (\frac{1}{5^{\frac{5}{4}}}, \frac{3}{2})$, and $|g'(x)| < 1$ $\forall x \in (\frac{1}{5^{\frac{5}{4}}}, \frac{3}{2})$ because $|g'(x)|$ assumes a maximum of 1 at the endpoint $x = \frac{1}{5^{\frac{5}{4}}}$, and $|g'(x)|$ is strictly decreasing over the interval $[\frac{1}{5^{\frac{5}{4}}}, \frac{3}{2}]$. It follows by density of the real numbers, for each $x \in (\frac{1}{5^{\frac{5}{4}}}, \frac{3}{2})$, there exists k s.t $|g'(x)| \leq k < 1$. By Theorem 2.3, there exists a unique fixed point in $[\frac{1}{5^{\frac{5}{4}}}, \frac{3}{2}]$. In addition, Theorem 2.4 states that for any $p_0 \in [\frac{1}{5^{\frac{5}{4}}}, \frac{3}{2}]$, the sequence $p_n = g(p_{n-1})$, $n \geq 1$, converges to the unique fixed point in $[\frac{1}{5^{\frac{5}{4}}}, \frac{3}{2}]$.

b) $\lim_{n\to\infty} p_n = \lim_{n\to\infty} p_n^{\frac{1}{5}} = 1^{\frac{1}{5}} = 1$.

p_n converges linearly to the point 1 if $\lim_{n\to\infty} |\frac{g(p_n) - g(1)}{p_n - 1}| = \lambda$ for some $\lambda < 1$.

It follows $\lim_{n\to\infty} |\frac{g(p_n) - g(1)}{p_n - 1}| = \lim_{n\to\infty} |\frac{p_n^{\frac{1}{5}} - 1}{p_n - 1}|$

$= \lim_{n\to\infty} |\frac{p_n^{\frac{1}{5}} - 1}{(p_n^{\frac{1}{5}} - 1)(p_n^{\frac{4}{5}} + p_n^{\frac{3}{5}} + p_n^{\frac{2}{5}} + p_n^{\frac{1}{5}} + 1)}|$

$= \lim_{n\to\infty} |\frac{1}{p_n^{\frac{4}{5}} + p_n^{\frac{3}{5}} + p_n^{\frac{2}{5}} + p_n^{\frac{1}{5}} + 1}|$

$= |\frac{1}{1^{\frac{4}{5}} + 1^{\frac{3}{5}} + 1^{\frac{2}{5}} + 1^{\frac{1}{5}} + 1}| = \frac{1}{5}$

Thus, there exists a $\lambda = \frac{1}{5} < 1$

p_n only converges linearly if $\lim_{n\to\infty} \frac{|g(p_n) - g(1)|}{|p_n - 1|^\alpha}$ diverges for all $\alpha > 1$.

$\lim_{n\to\infty} \frac{|g(p_n) - g(1)|}{|p_n - 1|^\alpha} = \lim_{n\to\infty} \frac{\frac{|g(p_n) - g(1)|}{|p_n - 1|}}{|p_n - 1|^{\alpha-1}}$ where $\alpha > 1$.

$= \lim_{n\to\infty} \frac{\frac{1}{5}}{|p_n - 1|^{\alpha-1}} = \infty$ because $|p_n - 1|^{\alpha-1}$ converges to 0.

Problem 2: Solution

a) $\lim_{n\to\infty} \frac{|g(p_n)-1|}{|p_n-1|} = \lim_{n\to\infty} \frac{|\frac{1}{2}e^{-p_n+1}(\cos(\pi p_n)^2+1)-1|}{|p_n-1|}$
$= \frac{|\frac{1}{2}e^{0}(\cos(\pi)^2+1)-1|}{|1-1|} = \frac{1}{|1-1|} \Rightarrow$ zero over zero.
By L'H $= \lim_{x\to 1} \frac{|-\frac{1}{2}e^{1-x}(\cos(\pi x)^2+1)-\pi e^{1-x}\cos(\pi x)\sin(\pi x)|}{|1|}$
$= |-\frac{1}{2}e^{0}(\cos(\pi)^2+1)-\pi e^{0}\cos(\pi)\sin(\pi)| = 1$ (sublinearly)

b) $g(x) = x - \frac{e^{1-x}(\cos(\pi x)^2+1)-2x}{-e^{1-x}(\cos(\pi x)^2+2\pi\sin(\pi x)\cos(\pi x)+1)+2}$ (at least quadratically by Newton's method)

c) $\lim_{n\to\infty} \frac{|g(p_n)-1|}{|p_n-1|} = \lim_{n\to\infty} \frac{|\frac{19}{20}p_n+\frac{1}{40}e^{-p_n+1}(\cos(\pi p_n)^2+1)-1|}{|p_n-1|}$ (zero over zero)
By L'H $= \lim_{x\to 1} |\frac{19}{20}-\frac{1}{40}e^{-x+1}(\cos(\pi x)^2+2\pi\sin(\pi x)\cos(\pi x)+1)| = \frac{18}{20}$ (linearly)

d) $\lim_{n\to\infty} \frac{|g(p_n)-1|}{|p_n-1|} = \lim_{n\to\infty} \frac{|3p_n-e^{-p_n+1}(\cos(\pi p_n)^2+1)-1|}{|p_n-1|}$ (zero over zero)
By L'H $= \lim_{x\to 1} |3+e^{-x+1}(\cos(\pi x)^2+2\pi\sin(\pi x)\cos(\pi x)+1)| = 5$ (diverges)

b) fastest, c) second, a) third, d) slowest.

Problem 3: Solution

a) table

	0	1	2	3	4
-1	33				
0	$\frac{51}{20}$	$-\frac{609}{20}x+\frac{51}{20}$			
3	-3	$-\frac{37}{20}x+\frac{51}{20}$	$\frac{143}{20}x^2-\frac{466}{20}x+\frac{51}{20}$		
2	$-\frac{21}{20}$	$-\frac{39}{20}x+\frac{57}{20}$	$-\frac{1}{20}x^2-\frac{34}{20}x+\frac{51}{20}$	$-\frac{48}{20}x^3+\frac{239}{20}x^2-\frac{322}{20}x+\frac{51}{20}$	
5	q	$-\frac{21+20q}{60}x+\frac{21+8q}{12}$	$\frac{69+10q}{60}x^2-\frac{462-50q}{60}x+\frac{585+60q}{60}$	$\frac{72+10q}{300}x^3-\frac{375-50q}{300}x^2+\frac{-78+60}{300}x+\frac{51}{20}$	

$$\frac{q}{180}x^4-\frac{q}{45}x^3+\frac{11}{25}x^4+\frac{q}{180}x^2-\frac{104}{25}x^3+\frac{q}{30}x+\frac{1239}{100}x^2-\frac{673}{50}x+\frac{51}{20}$$

$q=504/5$

$P_4(x)=x^4-\frac{32}{5}x^3+\frac{259}{20}x^2-\frac{101}{10}x+\frac{51}{20}$.

b) $g(x)=x-\frac{P_4(x)}{P_4'(x)}$. Starting at $x=1.211$ we get $p_{100}=0.500$.

Problem 4: Solution

a) Let $g(x) = \frac{1}{2}(x + \frac{\alpha}{x})$. $g \in C[\min\{p_0, \frac{\sqrt{3\alpha}}{3}\}, \max\{p_0, g(p_0), \frac{2\sqrt{3\alpha}}{3}\}]$. g assumes a minimum of $\sqrt{\alpha}$ at $x = \sqrt{\alpha}$ when $g'(x) = 0$, and g assumes a maximum of $\frac{2\sqrt{3\alpha}}{3}$ when $\frac{\sqrt{3\alpha}}{3} \leq p_0 \leq \sqrt{3a}$ and a maximum of $g(p_0)$ otherwise. Thus, g maps onto itself. Want to show $|p_{n+1} - \sqrt{\alpha}| \leq |p_n - \sqrt{\alpha}|$ for all $n > 1$. $|p_{n+1} - \sqrt{\alpha}| = |\frac{1}{2}(p_n + \frac{\alpha}{p_n}) - \sqrt{\alpha}| = |\frac{p_n - \sqrt{\alpha}}{2p_n}||p_n - \sqrt{\alpha}|$. This implies $|p_{n+1} - \sqrt{\alpha}| \leq |p_n - \sqrt{\alpha}|$ if $|\frac{p_n - \sqrt{\alpha}}{2p_n}| \Rightarrow p_n > \frac{\sqrt{a}}{3}$. If $p_0 \leq \frac{\sqrt{a}}{3}$, then $p_1 = \frac{1}{2}(p_0 + \frac{\alpha}{p_0}) > \frac{\alpha}{2\frac{\sqrt{a}}{3}} = \frac{3\sqrt{a}}{2} > \frac{\sqrt{a}}{3}$.

By induction $|p_n - \sqrt{\alpha}| = |\frac{p_{n-1} - \sqrt{\alpha}}{2p_{n-1}}||p_{n-1} - \sqrt{\alpha}| = ... = \prod_{i=1}^{n-1} |\frac{p_i - \sqrt{\alpha}}{2p_i}||p_1 - \sqrt{\alpha}|$. $|\frac{p_i - \sqrt{\alpha}}{2p_i}| = |\frac{p_{i-1} - \sqrt{\alpha}}{2p_{i-1}}||\frac{p_{i-1} - \sqrt{\alpha}}{p_{i-1} + \frac{\alpha}{p_{i-1}}}| \leq |\frac{p_{i-1} - \sqrt{\alpha}}{2p_{i-1}}|$ because $|\frac{p_{i-1} - \sqrt{\alpha}}{p_{i-1} + \frac{\alpha}{p_{i-1}}}| < 1$ ($-\sqrt{\alpha} < \frac{\alpha}{p_{i-1}}, p_{i-1} > \sqrt{\alpha}$ and $\sqrt{\alpha} - p_{i-1} < p_{i-1} + \frac{\alpha}{p_{i-1}}, p_{i-1} < \sqrt{\alpha}$).

It follows $|\frac{p_1 - \sqrt{\alpha}}{2p_1}|$ is an upper bound for $|\frac{p_n - \sqrt{\alpha}}{2p_n}|$

Thus, $\lim_{n \to \infty} |p_n - \sqrt{\alpha}| \leq |\dfrac{p_1 - \sqrt{\alpha}}{2p_1}|^{n-1} |p_1 - \sqrt{\alpha}| = 0$ by the squeeze theorem. Hence p_n converges for all $p_0 > 0$.

b) If $p_0 < 0$, p_n converges to $-\sqrt{\alpha}$. The proof is similar to how we show p_n converges to $\sqrt{\alpha}$ for $p_0 > 0$. $g \in C[\min\{p_0, g(p_0), -\frac{2\sqrt{3\alpha}}{3}, \max\{p_0, -\frac{\sqrt{3\alpha}}{3}\}\}]$. g assumes a maximum of $\sqrt{\alpha}$ and minimum of $g(p_0)$ or $-\frac{2\sqrt{3\alpha}}{3}$. Then, we show $|\frac{p_1 + \sqrt{\alpha}}{2p_1}| < 1$ and an upper bound for $|\frac{p_n + \sqrt{\alpha}}{2p_n}|$ for $n > 1$. Then, we use the squeze theorem as we did previously to show $\lim_{n \to \infty} |p_n + \sqrt{\alpha}|$ converges to 0.