Math 131B: Homework 5

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Problem 1. **Exercise 2.4.2**

$f : X \to Y$ is clearly continuous if f is the constant function, so we only need to show the other direction. Let $f : X \to Y$ be a continuous function. Suppose $x_0 \in X$. Let $V = f^{-1}(B_{d_Y}(f(x_0), 1))$ which by the continuity of f is open in X. Suppose for the sake of contradiction, $X \setminus V \neq \emptyset$. Let $U = \bigcup_{x \in X \setminus V} f^{-1}(B_{d_Y}(f(x), 1))$ which by the continuity of f is an open set equal to $X \setminus V$. Since X is connected and $U \cup V = X$, $U \cap V \neq \emptyset$. However, this is clearly impossible because any $x \in U \cap V$ cannot simultaneously have the property $f(x) = f(x_0)$, and $f(x) \neq f(x_0)$. Hence, $X \setminus V = \emptyset$ and f is constant.

QED

Problem 2. **Exercise 2.4.7**

Suppose E is a path connected subset of X, and suppose for the sake of contradiction E is disconnected. Let U and V be disjoint non-empty relatively open subsets s.t $U \cup V = E$. Let $x \in U$ and $y \in V$. It follows there exists a continuous function $\gamma : [0, 1] \to E$ s.t $\gamma(0) = x$ and $\gamma(1) = y$. Let $a = sup\{z \in [0, 1] : \gamma(z) \in U\}$ where $\gamma(a) \in E$. Because U is relatively open in E, $\gamma(a) \in U$ implies $\gamma(a) \neq y$ and there exists $\epsilon > 0$ s.t $B_{(E,d_{E\times E})}(\gamma(a), \epsilon) \subseteq U$, but this is a contradiction because then a cannot be an upper bound for $\{z \in [0, 1] : \gamma(z) \in U\}$. Thus, $\gamma(a) \in V$. Because V is relatively open in E, $\gamma(a) \neq x$ and $\epsilon > 0$ s.t $B_{(E,d_{E\times E})}(\gamma(a), \epsilon) \subseteq V$, but this is a contradiction because then a cannot be the least upper bound for $\{z \in [0, 1] : \gamma(z) \in U\}$. Thus $\gamma(a) \notin U \cup V$ and $\gamma(a) \in E$, so $U \cup V \neq E$. Hence E is not disconnected.

QED

Problem 3. **Exercise 3.2.4**

Let $\epsilon = 1$. Because f_n converges uniformly to f, there exists $N > 0$ s.t $d_Y(f_n(x), f(x)) < 1$ for every $n > N$ and $x \in X$. Because f is bounded, there exists a ball $B_{(Y,d_Y)}(y_0, R_f)$ in Y s.t $f(x) \in B_{(Y,d_Y)}(y_0, R_f)$ for all $x \in X$. Because each f_n for $n \in \{1...N\}$ is bounded, there exists a ball $B_{(Y,d_Y)}(y_n, R_n)$ in Y s.t $f_n(x) \in B_{(Y,d_Y)}(y_n, R_n)$ for all $x \in X$. It follows by the triangle innequality, $d_Y(y_0, f_n) \leq d_Y(y_0, f) + d_Y(f, f_n) < R_f + 1$ for every $x \in X$ and $n > N$, and $d_Y(y_0, f_n) \leq d_Y(y_0, y_n) + d_Y(y_n, f_n) < R_n + d_Y(y_0, y_n)$ for every $x \in X$ and $n \leq N$. Let $R = max\{R_f + 1, d_Y(y_0, y_1) + R_1, ..., d_Y(y_0, y_n) + R_n\}$. Thus, $f_n(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$ and all positive integers n.

QED

Problem 4. **Exercise 3.3.6**

Suppose $(f^{(n)})_{n=1}^{\infty}$ is a sequence of bounded functions from one metric space (X, d_X) to another (Y, d_Y), and suppose this sequence converges uniformly to another function $f : X- \to Y$. Since each $f^{(n)}(x)$ is bounded, there exists $B_{(Y,d_Y)}(y_n, R_n)$ in Y s.t $f^{(n)}(x) \in B_{(Y,d_Y)}(y_n, R_n)$ for each $x \in X$. Because $(f^{(n)})_{n=1}^{\infty}$ converges uniformly to $f(x)$, for all but finitely many n $d_Y(f^{(n)}(x), f(x)) < 1$ for each $x \in X$. It follows for any sufficiently large n, $d_Y(f(x), y_n) \leq d_Y(f^{(n)}(x), f(x)) + d_Y(f^{(n)}(x), y_n) < R_n + 1$ for each $x \in X$. Thus, for any sufficiently large n, $f(x) \in B_{(Y,d_Y)}(y_n, R_n + 1)$ for each $x \in X$, so $f(x)$ is bounded.

QED

3.2.4 assumes f is bounded and shows that every function in the sequence is contained in a single ball, and 3.3.6 proves that f is bounded if it is the limit of a sequence of bounded functions.

Problem 5. **Exercise 3.3.7**

Let $f_n : (0,1) \to (\frac{n}{n+1}, n)$ $f_n(x) = \frac{1}{x+\frac{1}{n}}$ which converges to $f : (0,1) \to (1, \infty)$ $f(x) = \frac{1}{x}$. For every $x \in X$ and $\epsilon > 0$, there exists $N = \frac{1}{x^2\epsilon}$ s.t if $n > N$ $d_Y(f_n(x), f(x)) = |\frac{x+\frac{1}{n}}{x(x+\frac{1}{n})} - \frac{x}{x(x+\frac{1}{n})}| = |\frac{1}{x^2 n + x}| < |\frac{1}{x^2 n}| < \epsilon$. Hence, f_n converges pointwise to f, where each f_n is bounded, but f is unbounded.

QED

Problem 6. **Exercise 3.3.8**

Because each $f_n(x)$ and $g_n(x)$ are uniformly bounded, Excercise 3.2.4 states that there also exists some $M > 0$ s.t $|f(x)| < M$ and $|g(x)| < M$. Let $\epsilon > 0$ and choose N to be sufficiently large s.t $d(f_n(x), f(x)) < \frac{\epsilon}{2M}$ and $d(f_n(x), f(x)) < \frac{\epsilon}{2M}$ by the uniform convergence of f_n and g_n.

If $n > N$, $d(f_n(x)g_n(x), f(x)g(x)) = |f_n(x)g_n(x) - f(x)g(x)|$

$= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)|$

$\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| < M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon$.

Hence $f_n(x)g_n(x)$ converges uniformly to $f(x)g(x)$.

QED

Problem 7. **Exercise 3.4.1**

($d_\infty(f, f) = 0$) This is clearly true because $d_Y(f(x), f(x)) = 0$ for all $x \in X$, so $\sup\{d_Y(f(x), f(x)) : x \in X\} = 0$. Thus, $d_\infty(f, f) = 0$.

(Positivity) $d_Y(f(x), g(x)) \geq 0$ for every $x \in X$, so $\sup\{d_Y(f(x), g(x)) : x \in X\} \geq 0$. If f and g are distinct, there exists at least one point where $g(x) \neq f(x)$, so $\sup\{d_Y(f(x), g(x)) : x \in X\} > 0$
Thus, $d_\infty(f, g) > 0$ for distinct functions f and g.

(Symmetry) $d_Y(f(x), g(x)) = d_Y(g(x), f(x))$ for every $x \in X$, so
$\sup\{d_Y(f(x), g(x)) : x \in X\} = \sup\{d_Y(g(x), f(x)) : x \in X\}$.
Thus, $d_\infty(f, g) = d_\infty(g, f)$.

(Triangle innequality) $d_Y(f(x), g(x)) \leq d_Y(f(x), h(x)) + d_Y(h(x), g(x))$ for all $x \in X$,
so $\sup\{d_Y(f(x), g(x)) : x \in X\} \leq \sup\{d_Y(f(x), h(x)) + d_Y(g(x), h(x)) : x \in X\}$.
Because the sum of the supremums of two functions is greater than or equal to the supremum of the sum of two functions,
$\sup\{d_Y(f(x), g(x)) : x \in X\} \leq \sup\{d_Y(f(x), h(x)) : x \in X\} + \sup\{d_Y(g(x), h(x)) : x \in X\}$
Thus, $d_\infty(f, g) \leq d_\infty(f, h) + d_\infty(g, h)$.

Hence, $B(X \to Y)$ is a metric space.

Problem 8. **Exercise 3.4.2**

Let $(f^{(n)})_{n=1}^\infty$ be a sequence of functions in the space $B(X \to Y)$ with metric d_∞ and let f be another function in $B(X \to Y)$. First we show that if $(f^{(n)}(x))_{n=1}^\infty$ converges to f in the metric d_∞, then $(f^{(n)})_{n=1}^\infty$ converges uniformly to f. If $\lim_{n\infty} d_\infty(f^{(n)}, f) = 0$, then for every $\epsilon > 0$ there exists $N > 0$ s.t $\sup\{d_Y(f^{(n)}(x), f(x)) : x \in X\} < \epsilon$ whenever $n > N$. Since $\sup\{d_Y(f^{(n)}(x), f(x)) : x \in X\} < \epsilon$ and $d_Y(f^{(n)}(x), f(x)) \leq \sup\{d_Y(f^{(n)}(x), f(x)) : x \in X\}$ for each $x \in X$, $d_Y(f^{(n)}(x), f(x)) < \epsilon$ for each $x \in X$. Hence, $f^{(n)}$ converges uniformly to f.

Next we show that if $(f^{(n)})_{n=1}^\infty$ converges uniformly to f, then $(f^{(n)})_{n=1}^\infty$ converges to f in the metric d_∞. If for every $\epsilon > 0$, there exists $N > 0$ s.t $d_Y(f^{(n)}(x), f(x)) < \frac{\epsilon}{2}$ for every $n > N$ and $x \in X$, then $\sup\{d_Y(f^{(n)}(x), f(x)) : x \in X\} \leq \frac{\epsilon}{2} < \epsilon$. Thus, $d_\infty(f^{(n)}, f) < \epsilon$. Hence $(f^{(n)})_{n=1}^\infty$ converges to f with respect to the metric d_∞.

Problem 9. **Additional Problem**

If E is disconnected, there exist two open, non-empty, disjoint sets A and B s.t $A \cup B = E$. Using the fact that A and B are complements of each other in E, $B = \overline{B} \cap E$ and $A = \overline{A} \cap E$ are closed in E. Because A and B are subsets of E, any point in $\overline{A}$ not in E will not be in B and vice-versa, so because A and B are disjoint, $A \cap \overline{B} = B \cap \overline{A} = \emptyset$.

Suppose there exist sets A and B s.t $A \cup B = E$, $A \cap \overline{B} = \emptyset$, and $\overline{A} \cap B = \emptyset$. $A \subseteq \overline{A}$ and $\overline{A} \cap B = \emptyset$, so $A \cap B = \emptyset$. B is open in E because $\overline{A} \cap B = \emptyset$, $\overline{A} \cap E$ is closed in E, and $E \setminus \overline{A} = B$. The same logic holds to show A is also open in E. Because A and B are disjoint, nonempty, open, and $A \cup B = E$, E is disconnected.