Lipschitz Continuous

Let $D = [t_0, t_f] \times [-M, M]$
$f : D \to \mathbb{R}$ is L.C if $\exists L > 0$ s.t $|f(t, y_1) - f(t, y_2)\| < L|y_1 - y_2\|$ for $(t, y_1), (t, y_2) \in D$

One step methods

LTE: $\tau_{n+1} = y(t_{n+1}) - y_{n+1}$ $\tau_{n+1} = O(h^{p+1}) \Rightarrow p^{th}$ order accurate/consistent
GE: $e_N = |y_(t_N) - y_N| \leq \frac{\theta}{s}(\exp(Ns) - 1)$ $e_N = O(h^p) \Rightarrow p^{th}$ order convergent.

RK Methods

RK: $k_i = f(t_n + c_i h, y_n + h \sum_{j=1}^{s} a_{ij} k_j)$ $y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i$
$e^\top b = 1$, $Ae = c$

LMM

AB: $y_{n+1} - y_n = h[\frac{3}{2} f(t_n, y_n) - \frac{1}{2} f(t_{n-1}, y_{n-1})]$,
AM: $y_{n+1} - y_n = \frac{h}{2}(f(t_{n+1}, y_{n+1}) + f(t_n, y_n))$,
General 2-step: $y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h(\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n)$
$\mathscr{L}_h y(t) = y(t+2h) + \alpha_1 y(t+h) + \alpha_0 y(t) - h[\beta_2 y'(t+2h) + \beta_1 y'(t+h) + \beta_0 y'(t)]$
$\rho(r) = r^2 + \alpha_1 r + \alpha_0$ $\sigma(r) = \beta_2 r^2 + \beta_1 r + \beta_0$ consistent if $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$
K-step method: $y_{n+k} + \alpha_{k-1} y_{n+k-1} + \cdots + \alpha_0 y_n = h[\beta_k y_{n+k} + \beta_{k-1} y_{n+k-1} + \cdots + \beta_0 y_0]$

1. Explicit: $\beta_k = 0$
2. Consistent: $\rho(1) = 0$ & $\rho'(1) = \sigma(1)$
3. Zero-Stable: $\rho(r)$ satisfies Root Condition $|r_i| \leq 1$ and $|r_i| = 1$ is simple.
4. Convergent: (2) + (3)

Absolute Stability: A method with time step h is absolutely stable to the test problem $y' = \lambda y$ if $\lim_{n\to\infty} y_n = 0$
$\mathscr{R} = \{z \in \mathcal{C} :$ the method is absolutely stable with $z = \lambda h\}$
LMM $\rho(r, z) = \rho(r) - z\sigma(r)$
$\mathscr{R} = \{z \in \mathcal{C} : \rho(r, z) = \rho(r) - z\sigma(r)$ satisfies the strict root condition$\}$
$interior(\mathscr{R})$: set of z that satisfies strict root condition.
$exterior(\mathscr{R})$: set of z that fails strict root condition.
$\partial\mathscr{R}$: set of z s.t $\rho(r, z)$ has $|r_{\max}| = 1$.
Boundary locus: $\{z = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}\}\theta \in [0, 2\pi]$ Way of determining stability region from boundary.
Stiff Problems

- Two widely varying time scales
- Use small step size due to stability (rather than accuracy)
- Need methods with large stability regions

BVP

Dirichlet: $u(a) = \alpha$, $u(b) = \beta$ Neumann: $u'(a) = \alpha$, $u'(b) = \beta$
$u''(x) = f(x)$ for $a < x < b$ with $u(a) = \alpha$, $u(b) = \beta$
Finite difference approx $\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = f(x_i)$

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ f_{N-1} \end{bmatrix} - \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ \beta \end{bmatrix}$$

$\tau_i = \frac{u_{i+1}-2u_i+u_{i-1}}{h^2} - f(x_i)$ $\tau_i = O(h^p) \to p^{th}$ order accurate.
$Ae = \tau$ where e is global error vector and τ is LTE.
For symmetric matrices $\rho(A) = \max|\lambda_i\|$
$\|e^h\|_2 = O(h^{\frac{3}{2}})$

Matrix Splitting

Jacobi Method: $x^{(k+1)} = G_j x^{(k)} + c_j$, $G_j = D^{-1}(L+U)$, $c_j = D^{-1}b$
Gauss-Seidel: $x^{(k+1)} = G_g x^{(k)} + c_g$, $G_g = (D-L)^{-1}U$, $c_g = (D-L)^{-1}b$
If $\rho(G) < 1$ $\lim_{k\to\infty} G^k = 0$ & $\sum_{k=0}^{\infty} G^k = (I-G)^{-1}$ exists.

Overdetermined System

Diagonally dominant: $\sum_{j\neq i}^{n}|a_{ij}| < |a_{ii}|$
Exact Solution:$Ax_0 = b$, Approximation: $Ax = b - r$, Residual: $r = b - Ax$, error: $e = x_0 - x$, $\|e\| \leq \|A^{-1}\|\|r\|$,
Relative error: $\frac{\|e\|}{\|x_0\|}$ (unknown), Relative residual: $\frac{\|r\|}{\|b\|}$ (known),
$\frac{\|e\|}{\|x_0\|} \leq cond(A)\frac{\|r\|}{\|b\|}$

$$\underbrace{\begin{bmatrix} 1 & x_1 & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^n \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_m & \cdots & x_m^n \end{bmatrix}}_{A^{m\times n+1}} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}_{b}$$

Normal Equation: $A^\top Ax = A^\top b$
QR Decomposition: $A = QR$ Q is an orthonormal spanning set of $range(A)$
$Rx_0 = Q^\top b$