Math 131B: Homework 8

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Problem 1. **Exercise 4.5.2**

(Lemma) If $n \geq 3$ then $(n+k)! > 2^k n!$ for all $k \in \mathbb{N}$
We will show this claim to be true by induction on k.

For the base case of $k = 1$ we have

$$(n+1)! = (n+1)n! > 3 \cdot n! > 2^1 n!$$

, so the claim holds for $k = 1$. We assume the claim to be true for some arbitrary k. Thus, it remains to show the claim holds for $k + 1$. Using the induction hypothesis, we obtain

$$(n+k+1)! = (n+k+1)(n+k)! > (n+k+1)2^k n! > 3 \cdot 2^k n! > 2^{k+1} n!$$

, so the claim holds for $k + 1$. Hence, by induction, the claim holds for all $k \in \mathbb{N}$
QED

We will show by induction on n that the claim $0 < \displaystyle\sum_{k=1}^{\infty} \frac{1}{(n+k)!} < \frac{1}{n!}$ for all $n \geq 3$

For the base case of $n = 3$ we have

$$0 < \sum_{k=1}^{\infty} \frac{1}{(3+k)!}$$

because each $\frac{1}{(3+k)!}$ is positive, and

$$\sum_{k=1}^{\infty} \frac{1}{(3+k)!} < \sum_{k=1}^{\infty} \frac{1}{2^k 3!} = \frac{\frac{1}{2 \cdot 3!}}{1 - \frac{1}{2}} = \frac{1}{3!}$$

by the sum of an infinite geometric series and the lemma proved above, so the claim holds for $n = 3$. We assume the claim to be true for some arbitrary n. Thus, it remains to show the claim holds for $n + 1$.

$$0 < \sum_{k=1}^{\infty} \frac{1}{(n+1+k)!}$$

because each $\frac{1}{(n+1+k)!}$ is positive, and

$$\sum_{k=1}^{\infty} \frac{1}{(n+1+k)!} < \sum_{k=1}^{\infty} \frac{1}{2^k (n+1)!} = \frac{\frac{1}{2 \cdot (n+1)!}}{1 - \frac{1}{2}} = \frac{1}{(n+1)!}$$

by the sum of an infinite geometric series and the lemma proved above, so the claim holds for $n + 1$. Hence, by induction, the claim holds for all $n \geq 3$.
QED

Next we show $n!e$ is not an integer for any $n \geq 3$ by contradiction. Assume for the sake of contradiction $n!e$ is an integer for some $n \geq 3$. By the definition of e, $n!e = n!\sum_{k=0}^{\infty} \frac{1}{k!} = \sum_{k=0}^{n} \frac{n!}{k!} + n!\sum_{k=1}^{\infty} \frac{1}{(n+k)!}$.
$0 < n!\sum_{k=1}^{\infty} \frac{1}{(n+k)!} < \frac{n!}{n!} = 1$ by Exercise 4.5.2, so it clearly is not an integer. $\sum_{k=0}^{n} \frac{n!}{k!} = \sum_{k=0}^{n} \prod_{i=k+1}^{n} i$ which is clearly an integer because each term is a product of integers which will also be an integer, and a finite sum of integers must be an integer. Thus, we have a contradiction because $\sum_{k=0}^{n} \frac{n!}{k!} < n!e < \sum_{k=0}^{n} \frac{n!}{k!} + 1$, so $n!e$ is between two consecutive integers for any $n \geq 3$. Hence, $n!e$ is not an integer for any $n \geq 3$. Because $n!e$ is not an integer for every $n \geq 3$, it follows that ne is not an integer for every $n \geq 3$. If there was an n s.t ne was an integer, $ne(n-1)!$ would also be an integer which contradicts $n!e$ is not an integer for any $n \geq 3$. Since $a = ne$ is not an integer for all n, it follows e cannot be expressed as a ratio of integers $\frac{a}{n}$, so e must be irrational.

Problem 2. **Exercise 4.5.4**
We will show $f : \mathbb{R} \to \mathbb{R}$ is infinetly differentiable using cases.

Case 1. ($x < 0$) f is clearly differentiable for $x < 0$ because the derivative of the zero function is the zero function, so by a simple induction $f^{(k)}(x) = 0$ for every integer $k \geq 0$ and $x < 0$.

Case 2. ($x > 0$) We show that f is of the form $f^{(k)}(x) = P_{2k}(\frac{1}{x})e^{-\frac{1}{x}}$ for every integer $k \geq 0$ and $x > 0$ by induction on k. For the base case of $k = 1$, we obtain $f^{(1)}(x) = \frac{1}{x^2}e^{-\frac{1}{x}}$ by the chain rule, so the claim holds for $k = 1$. Assume for some arbitrary k the claim holds, so it remains to show the claim holds for $k+1$. By the induction hypothesis, $f^{(k)}(x) = P_{2k}(\frac{1}{x})e^{-\frac{1}{x}}$ for some k, so we differentiate both sides to obtain $f^{(k+1)}(x)$. By the product rule and chain rule

$$\begin{aligned} f^{(k+1)}(x) &= (P_{2k-1}(\frac{1}{x})(-\frac{1}{x^2}))e^{-\frac{1}{x}} + P_{2k}(\frac{1}{x})(\frac{1}{x^2}e^{-\frac{1}{x}}) \\ &= (P_{2k+1}(\frac{1}{x}) + P_{2k+2}(\frac{1}{x}))e^{-\frac{1}{x}} \\ &= P_{2(k+1)}(\frac{1}{x})e^{-\frac{1}{x}} \end{aligned}$$

, so the claim holds for $k+1$. Hence, by induction the claim holds for all k.

Case 3. ($x = 0$) We will show f is differentiable at $x = 0$ and $f^{(k)}(0) = 0$ by using the limit definition of the derivative and induction on k. For the base case of $k = 1$, we obtain

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \frac{0}{0}$$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = 0 \text{ because } f(x) = 0 \text{ for all } x < 0$$

$$\text{Thus, it suffices to show } \lim_{x \to 0^+} \frac{e^{-\frac{1}{x}}}{x} = 0$$

$$\text{By the continuity of } y = \frac{1}{x} \text{ on } \mathbb{R}^+ \lim_{x \to 0^+} \frac{e^{-\frac{1}{x}}}{x} = \lim_{y \to \infty} \frac{e^{-y}}{\frac{1}{y}} = \lim_{y \to \infty} \frac{y}{e^y} = \lim_{y \to \infty} \frac{0}{e^y} = 0 \text{ by L'H twice}$$

, so the claim holds for $k = 1$. We assume the claim to be true for some arbitrary k. Thus, it remains to show the claim holds for $k+1$.

$$\lim_{x\to 0}\frac{f^{(k)}(x)-f^{(k)}(0)}{x}=\frac{0}{0}$$

$$\lim_{x\to 0^-}\frac{f^{(k)}(x)-f^{(k)}(0)}{x}=0 \text{ because } f^{(k)}(x)=0 \text{ for all } x<0$$

$$\text{Thus, it suffices to show } \lim_{x\to 0^+}\frac{P_{2k}(\frac{1}{x})e^{-\frac{1}{x}}}{x}=0$$

$$\text{By the continuity of } y=\frac{1}{x} \text{ on } \mathbb{R}^+ \lim_{x\to 0^+}\frac{P_{2k}(\frac{1}{x})e^{-\frac{1}{x}}}{x}=\lim_{y\to\infty}\frac{P_{2k}(y)e^{-y}}{\frac{1}{y}}$$

$$=\lim_{y\to\infty}\frac{P_{2k+1}(y)}{e^y}=\lim_{y\to\infty}\frac{0}{e^y}=0 \text{ by L'H 2(k+1) times.}$$

Thus, the claim holds for $k+1$, so by induction, the claim holds for all k.
If f is real analytic at $x=0$, it must have a Taylor series centered at 0 that converges to the function f. However, as we know, $f^{(k)}(0)=0$ for all k, so its Taylor series centered at 0 is the zero function, which is not the original function f. Hence, f is not real analytic at $x=0$

Problem 3. **Exercise 4.5.5**

(a) Because $\exp(x)$ and $\log(x)$ are inverses, $\exp(\log(x))=x$. Implicititely differentiating both sides, we obtain $(\exp(\log(x)))'=\exp(\log(x))\cdot\log'(x)=x\cdot\log'(x)=1$. Dividing both sides by x, we obtain our final solution $\log'(x)=\frac{1}{x}$.

(b) Let $x,y\in(0,\infty)$. Because $\exp(x)$ and $\log(x)$ are inverses, $\log(xy)=\log(\exp^{\log(x)}\cdot\exp^{\log(y)})$. Using Theorem 4.5.2($d$), $\log(\exp^{\log(x)}\cdot\exp^{\log(y)})=\log(\exp^{\log(x)+\log(y)})$, and because $\exp(x)$ and $\log(x)$ are inverses, $\log(\exp^{\log(x)+\log(y)})=\log(x)+\log(y)$ which is our desried result.

(c) By Theorem 4.5.2(e), $1=\exp(0)$, so $\log(1)=\log(\exp(0))=0$ because exp and log are inverses. $\frac{1}{x}=\frac{1}{exp(log(x))}$. By Theorem 4.5.2($e$) we obtain $\frac{1}{\exp(\log(x))}=\exp(-\log(x))$. Thus, $\log(\frac{1}{x})=\log(\exp(-\log(x)))=-\log(x)$.

(d) Because $\exp(x)$ and $\log(x)$ are inverses, $\log(x^y)=\log((\exp(\log(x))^y))$. By properties of exponents, $\log((\exp(\log(x))^y))=\log(\exp(y\cdot\log(x)))$, and $\log(\exp(y\cdot\log(x)))=y\cdot\log(x)$ because $\exp(x)$ and $\log(x)$ are inverses.

(e) $\dfrac{1}{1-t}=\sum_{n=0}^{\infty}t^n$ for $t\in(-1,1)$. For any $x\in(-1,1)$ the series $\sum_{n=0}^{\infty}t^n$ coverges uniformly to $\frac{1}{1-t}$ on $[-|x|,|x|]$. Thus, we can switch the order of integration and summation. It follows $\log(1-x)=\log(1-x)-\log(1-0)=-\int_0^x\frac{1}{1-t}=-\sum_{n=0}^{\infty}\int_0^x t^n=-\sum_{n=0}^{\infty}\frac{t^{n+1}}{n+1}=-\sum_{n=1}^{\infty}\frac{x^n}{n}$.
Substituting x for $1-x$, if $1-x\in(-1,1)$, then $\log(1-(1-x))=\log(x)=-\sum_{n=1}^{\infty}\frac{(1-x)^n}{n}=$ $-\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n}$, so $log(x)$ is analytic at $x=1$ with $R=1$

Problem 4. **Exercise 4.5.8**

Let $a\in(0,\infty)$. $\dfrac{1}{x}=\dfrac{\frac{1}{a}}{1-\frac{a-x}{a}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}}(x-a)^n$ for $x\in(0,2a)$ by the sum of a geometric series. For any $r<a$ the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}}(x-a)^n$ converges uniformly to $\frac{1}{x}$ on $[a-r,a+r]$, so $\log(x)=\log(a)+\int_a^x\frac{1}{t}=\log(a)+\sum_{n=0}^{\infty}\int_a^x\frac{(-1)^n}{a^{n+1}}(t-a)^n=\log(a)+\sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}(n+1)}(x-a)^{n+1}$ for

$x \in [a-r, a+r]$. Thus $\log(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ where $c_n = \begin{cases} \log(a) & \text{if } n = 0 \\ \frac{(-1)^{n+1}}{a^n n} & \text{if } n \neq 0 \end{cases}$ for $x \in (0, 2a)$. Hence $\log(x)$ is real analytic for all $a \in (0, \infty)$.

Problem 5. **Additional Problem**

We will show that any analytic function where $f(x) = 0$ for all $x < 0$ is the zero function for all $x \in \mathbb{R}$ by contradiction. Assume for the sake of contradiction $f(x) \neq 0$ for some $x \in \mathbb{R}$. Because f is analytic, it is continuous and infinitely differentiable. Let $a = \inf\{x \in \mathbb{R} : f(x) \neq 0\}$, so $f(x) = 0$ for all $x < a$. We will show by induction on k that $f^{(k)}(a) = 0$ for all $k \in \mathbb{N}$. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers less than a s.t $\lim_{n\to\infty} x_n = a$. For the base case $k = 0$, we know by the continuity of f that $\lim_{n\to\infty} x_n = a$ implies $\lim_{n\to\infty} f(x_n) = f(a)$. Since each $f(x_n) = 0$, $f(a) = 0$, so the base case holds. Let k be arbitrary and assume $f^{(k)}(a) = 0$. It remains to show $f^{(k+1)}(a) = 0$. For the case $k+1$, f is infinitely differentiable, so $f^{(k+1)}$ exists. The derivative of the zero function is the zero function, so $f^{(k+1)}(x) = 0$ for $x < a$. The continuity of $f^{(k+1)}$ tells us that $\lim_{n\to\infty} x_n = a$ implies $\lim_{n\to\infty} f^{(k+1)}(x_n) = f(a)$. Since each $f^{(k+1)}(x_n) = 0$, $f^{(k+1)}(a) = 0$, so the $k+1^{th}$ case holds. Hence, by induction, $f^{(k)}(a) = 0$ for all $k \geq 0$. By the uniqueness of a power series and Taylor's formula, if f is analytic at a, its power series centered at a is equal to its Taylor series. Since $f^{(k)}(a) = 0$ for each of its derivatives, its Taylor series converges to the zero function. However, we obtain a contradiction because for any $r > 0$ there exists $x \in (a, a+r)$ s.t $f(x) \neq 0$. This follows from the definition of a. If there exists $r > 0$ s.t $\forall x \in (a, a+r)$ $f(x) = 0$, there exists $b \in (a, a+r)$ s.t b is a lower bound for $\{x \in \mathbb{R} : f(x) \neq 0\}$ which contradicts our definition for a. Hence, f is not analytic on $\mathbb{R}$ if $f(x) \neq 0$ for some $x \in \mathbb{R}$.

Let g and h be analytic real valued functions where $g(x) = h(x)$ for all $x < 0$. Let f' be the difference of g and h. It follows $f'(x) = 0$ for all $x < 0$. By the previous parts of the problem, we showed that if f' is analytic and $f'(x) = 0$ for all $x < 0$, then $f'(x) = 0$ for all $x \in \mathbb{R}$. Thus, $g(x) - h(x) = 0 \Rightarrow g(x) = h(x)$ for all $x \in (x)$.