Math 114L: Problem Set 6

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Problem 1

(a) Let T be the set of $\mathcal{L}$-sentences that are true in all finite fields. By construction, T is satisfiable. Let $\psi_p : 0 \neq \underbrace{1+1+\cdots+1}_{\text{n many times}}$ be a sentence. Define a new theory $T^* = T \cup \{\psi_n : n \in \mathbb{N}\}$. By the compactness theorem, it suffices to show T^* is finitely satisfiable. Consider some finite $\Delta \subset T^*$. T is given to be satisfiable by any finite field. We want to show $\mathbb{F}_p$, the set of integers mod p, for some sufficiently large prime p, $\mathbb{F}_p \models \Delta$. $\mathbb{F}_p \models T$ because $\mathbb{F}_p$ is finite. If we choose p large enough such that p is greater than every n meantioned, $\mathbb{F}_p \models \{\psi_n : \psi_n \in \Delta\}$. This implies $\mathbb{F}_p \models \Delta$. It follows every finite $\Delta \subset T^*$ Δ is satisfiable, so T^* is satisfiable by the compactness theorem. Thus, T^* has a model K. K must be characteristic 0 because $0 \neq \underbrace{1+1+\cdots+1}_{\text{p many times}}$ for every prime p.

(b) A rational function f can be written as the ratio of two coprime polynomial functions $\frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$. $f(x)$ is surjective, so for every $y \in K$, there exists x s.t $f(x) = y$. We can write a polynomial as $P(x) = (\underbrace{1+\ldots+1}_{a_n \text{ many times}})\cdot(\underbrace{x\cdot\ldots\cdot x}_{n \text{ many times}})+\ldots+(\underbrace{1+\ldots+1}_{a_1 \text{ many times}})\cdot x+(\underbrace{1+\ldots+1}_{a_0 \text{ many times}})$. We can use the additive inverse to define negative coefficients. Consider the $\mathcal{L}$-sentence $\phi^* : \forall y \exists x(P(x) = y \cdot Q(x)) \rightarrow \forall x_1 \forall x_2 (P(x_1)\cdot Q(x_2) = P(x_2)\cdot Q(x_1) \rightarrow x_1 = x_2)$. ϕ^* is true in all finite fields because the pigeonhole principle tells us that every surjective function between finite sets of the same size must also be injective. By part (a) $F \models \phi$ for every finite field $\rightarrow K \models \phi$, so ϕ^* is true in K. Hence, every $f : K \rightarrow K$ rational function that is surjective is also injective.

(c) Consider the rational function $f(x) = x^2$. If we take ϕ^* with $P(x) = x \cdot x$ and $Q(x) = 1$, ϕ^* is true in K because in every finite field F either f is not surjective or f is injective by the pigeonhole principle. However, $f : \mathcal{C} \rightarrow \mathcal{C}$ is a surjective function but not injective.

Problem 2

(a) (i) $\exists x_1, \exists x_2, \dots \exists x_k (\bigwedge_{1 \le i < j \le k} x_i \neq x_j)$

(ii) $\forall x_1, \forall x_2, \dots \forall x_n \neg(R(x_1, x_2) \wedge R(x_2, x_3) \wedge \dots \wedge R(x_{n-1}, x_n))$

(b) For each $n \geq 3$, we can let G_n be C_{n+1}, a cycle graph consisting of $n+1$ vertices. For any vertex on the graph, you have to travel all the way around the ring to get back to the original vertex.

Problem 3

Let $\psi_n : \forall x_1 \forall x_2 \dots \forall x_n \neg(R(x_1, x_2) \wedge R(x_2, x_3) \wedge \dots \wedge R(x_{n-1}, x_n) \wedge R(x_n, x_1))$ be a sentence. Define a new theory $T = \phi \cup \{\psi_n : n \in \mathbb{N}\}$. To show T has a model G, we show that T is finitely satisfiable. For any finite subset $\Delta \subset T$ finitely many $\{\psi_n\}$ are mentioned. Let ψ_{n^*} be the largest n mentioned in Δ. We found previously C_{n^*+1} has a cycle of length n^*+1 and doesn't have any smaller cycles, so the graph satisfies ψ_{n^*} and every other $\{\psi_n : \psi_n \in \Delta\}$. Because every graph with a finite cycle satisfies ϕ and C_{n^*+1} doesn't have any cycles smaller than n^*+1, $C_{n^*+1} \models \Delta$. Since this holds for any arbitrary finite $\Delta \subset T$, T is satisfiable by the compactness theorem. Hence, there exists a graph that satisfies ϕ but has no finite cycles.

Problem 4

Let $\psi_n : \exists x_1 \exists x_2 \dots \exists x_n \bigwedge_{1 \le i < j \le n} x_i \neq x_j$ be a sentence. Define a new theory $T^* = T \cup \{\psi_n : n \in \mathbb{N}\}$. To show T has an infinite model, we show T^* is finitely satisfiable. For any finite $\Delta \subset T^*$, only finitely many $\{\psi_n\}$ are mentioned. Let ψ_{n^*} be the largest n mentioned in Δ. Any model that satisfies ψ_{n^*} satisfies $\{\psi_n : n \leq n^*\}$. Since T has an arbitrarily large model, there exists $M \models T$ and $M \models \psi_{n^*}$. Thus, $M \models \Delta$ for any finite $\Delta \subset T^*$. By compactness, T^* is satisfiable. Since there exists a model M s.t $M \models T$ and $M \models \{\psi_n : n \in \mathbb{N}\}$, T has an infinite model.

Problem 5

Suppose no such a p_0 exists. It follows for any p there is a field F of characteristic p that makes ϕ false. Our goal is to show $T = \neg\phi \cup \{\psi_n : n \in \mathbb{N}\}$ is consistent. Doing so will derive a contradiction because every field of characteristic 0 satisfies ϕ. We showed in an earlier problem there exists a field K of characteristic 0 that satisfies $\{\psi_n : n \in \mathbb{N}\}$. For any finite $\Delta \subset T$, only finitely many $\{\psi_n\}$ are mentioned. By assumption, there exists a field F of characteristic sufficiently large p that satisfies $\{\psi_n : \psi_n \in \Delta\}$ and makes ϕ false. Thus, Δ is satisfiable, so by compactness T is satisfiable. However, this is impossible because as mentioned before, every field of characteristic 0 satisfies ϕ. Thus, such a p_0 must exist.