Math 170S: Homework 7

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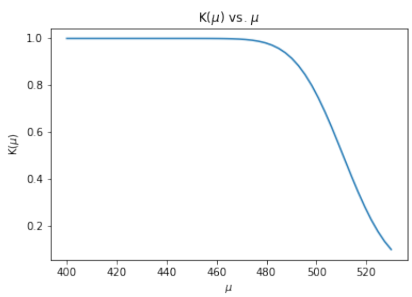
Problem 1.

1. $K(\mu) = P(\overline{X} \in C|\mu)$. $X \sim \mathcal{N}(\mu, 8100) \Rightarrow \overline{X} \sim \mathcal{N}(\mu, 225)$. Let $Z := \frac{\overline{X}-\mu}{15} \sim \mathcal{N}(0,1)$. Let $z_\alpha := \frac{510.77-\mu}{15}$ $P(\overline{X} \in C|\mu) \Leftrightarrow P(Z \leq z_\alpha) = \phi(\frac{510.77-\mu}{15})$. Thus, $K(\mu) = \phi(\frac{510.77-\mu}{15})$.
2. $\alpha = K(530) = \phi(\frac{510.77-530}{15}) = \phi(-1.282) = 0.0999$
3. $K(510.77) = \phi(\frac{510.77-510.77}{15}) = \phi(0) = 0.5$
4. Graph below shows $K(\mu)$ vs. μ



5. (i) $P(\overline{X} \leq 507.35) = \phi(\frac{507.35-530}{15}) = 0.0655$
   (ii) $P(\overline{X} \leq 497.45) = \phi(\frac{497.45-530}{15}) = 0.0150$

Problem 2.

1. By Neyman-Pearson $\forall \mu_1 > 0.5, \exists k$ s.t $\Lambda(x) = \frac{L(0.5)}{L(\mu_1)} \leq k$ for $x \in C$ and $\Lambda(x) = \frac{L(0.5)}{L(\mu_1)} \geq k$ for $x \in C'$ where $P(\Lambda(X) \leq k|H_0) = \alpha$.

$$\Lambda(x) = \frac{L(0.5)}{L(\mu_1)} = \frac{\prod_{i=1}^{10} \frac{(0.5)^{x_i} e^{-0.5}}{x_i!}}{\prod_{i=1}^{10} \frac{(\mu_1)^{x_i} e^{-\mu_1}}{x_i!}} = \prod_{i=1}^{10} \frac{(0.5)^{x_i} e^{-0.5}}{(\mu_1)^{x_i} e^{-\mu_1}} = (\frac{0.5}{\mu_1})^{\sum_{i=1}^{10} x_i} e^{10(\mu_1 - 0.5)} \leq k$$

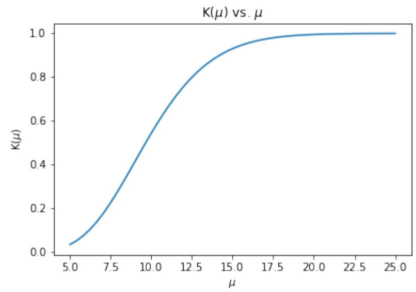
Taking the log of both sides $\displaystyle\sum_{i=1}^{10} x_i \log(\frac{0.5}{\mu_1}) + 10(\mu_1 - 0.5) \leq \log(k)$.

Using the simple alternative hypothesis $H_1 : \mu = \mu_1$ where $\mu_1 > 0.5$, it follows $\log(\frac{0.5}{\mu_1}) < 0$. Thus, $\displaystyle\sum_{i=1}^{10} x_i \geq \frac{\log(k) - 10(\mu_1 - 0.5)}{\log(\frac{0.5}{\mu_1})}$. Let $c := \frac{\log(k)-10(\mu_1-0.5)}{\log(\frac{0.5}{\mu_1})}$

By Neyman-Pearson, we can define a best region $C := \{(x_1, \ldots, x_{10}) : \displaystyle\sum_{i=1}^{10} x_i \geq c\}$

2. Want to find c s.t $P(\displaystyle\sum_{i=1}^{10} x_i \geq c|H_0) = 0.068$. $P(\displaystyle\sum_{i=1}^{10} x_i \geq c|H_0) = 1 - P(\displaystyle\sum_{i=1}^{10} x_i < c|H_0)$, so $P(\displaystyle\sum_{i=1}^{10} x_i \geq c|H_0) = 0.068 \Leftrightarrow P(\displaystyle\sum_{i=1}^{10} x_i < c|H_0) = 0.932$ Using Python, $c = 9$ gives us $P(\sum_{i=1}^{10} x_i < c|H_0) = 0.932 \Rightarrow P(\sum_{i=1}^{10} x_i \geq 9|H_0) = 0.068$

3. Graph below shows $K(\mu)$ vs. μ



Problem 3.

1. By Neyman-Pearson $\exists k$ s.t $\Lambda(x) = \frac{L(3)}{L(5)} \leq k$ for $x \in C$ and $\Lambda(x) = \frac{L(3)}{L(5)} \geq k$ for $x \in C'$ where $P(\Lambda(X) \leq k | H_0) = \alpha$.

   $$\Lambda(x) = \frac{L(3)}{L(5)} = \frac{\prod_{i=1}^{n} (\frac{1}{3}) e^{-\frac{x_i}{3}}}{\prod_{i=1}^{n} (\frac{1}{5}) e^{-\frac{x_i}{5}}} = \frac{(\frac{1}{3})^n e^{-\frac{n\overline{x}}{3}}}{(\frac{1}{5})^n e^{-\frac{n\overline{x}}{5}}} = (\tfrac{5}{3})^n e^{n\overline{x}(\frac{1}{5}-\frac{1}{3})} \leq k$$

   Taking the log of both sides and solving for $\sum_{i=1}^{n} x_i$, we obtain $\sum_{i=1}^{n} x_i \geq \frac{\log(k) - n\log(\frac{5}{3})}{\frac{1}{5}-\frac{1}{3}}$. Let $c := \frac{\log(k) - n\log(\frac{5}{3})}{\frac{1}{5}-\frac{1}{3}}$.

   By Neyman-Pearson, we can define a best region $C := \{(x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i \geq c\}$

2. $P(\sum_{i=1}^{12} x_i \geq c) = 0.1 \Leftrightarrow 1 - P(\sum_{i=1}^{12} x_i < c) = 0.9$. Using Python and $2\theta \sum_{i=1}^{12} x_i \sim \chi^2(24)$, $c = 5.533$ gives us $P(\sum_{i=1}^{12} x_i \geq c) = P(\chi^2(24) \geq 6 \cdot 5.533) = 0.1 \Rightarrow C := \{(x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i \geq 5.533\}$

3. We choose the same critical region $C := \{(x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i \geq 5.533\}$.

Problem 4.

1. $$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{L(230)}{L(\overline{x})} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{20\pi}} e^{-\frac{(x_i-230)^2}{20}}}{\prod_{i=1}^{n} \frac{1}{\sqrt{20\pi}} e^{-\frac{(x_i-\overline{x})^2}{20}}} = \frac{e^{-\sum_{i=1}^{n} \frac{(x_i-230)^2}{20}}}{e^{-\sum_{i=1}^{n} \frac{(x_i-\overline{x})^2}{20}}}$$

   $= \exp(-\frac{1}{20}(\sum_{i=1}^{n} (x_i - 230)^2 - (x_i - \overline{x})^2)) = \exp(-\frac{1}{20}((n\overline{x}^2 - 460n\overline{x} + n230^2)))$

   $= \exp(-\frac{n}{20}(\overline{x} - 230)^2) \leq k \Rightarrow -\frac{n}{10}(\overline{x} - 230)^2 \leq \log(k) \Rightarrow (\overline{x} - 230)^2 \geq -\frac{20}{n}\log(k) \Rightarrow \frac{|\overline{x} - 230|}{\frac{10}{\sqrt{n}}} \geq \sqrt{2\log(\frac{1}{k})} = c.$

2. Because $Z = \frac{\overline{x}-230}{\frac{10}{\sqrt{n}}} \sim \mathcal{N}(0,1)$ set $c = z_0.1 = 1.282$. If $\overline{x} = 232.6$ and $n = 16$, then we should choose to accept H_0.

3. $P(Z > \frac{232.6-230}{\frac{10}{\sqrt{16}}}) = 1 - \phi(\frac{232.6-230}{\frac{10}{\sqrt{16}}}) = 0.1492$