Math 151b: Problem Set 1

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Problem 1 $y(t) = e^{\lambda t} = e^{(a+bi)t} = e^{at}e^{bit} = e^{at}\cos(bt) + e^{at}i\sin(bt)$. The modulus $|y(t)| = \sqrt{(e^{at}\cos(bt))^2 + (e^{at}\sin(bt))^2} = \sqrt{e^{2at}(\cos^2(bt) + \sin^2(bt))} = \sqrt{e^{2at}} = |e^{at}| = e^{at}$

$$|y(t)| = \begin{cases} \infty & \text{if } a > 0 \quad \lim_{t\to\infty} e^{at} = \lim_{t\to\infty} \sum_{i=0}^{\infty} \frac{(at)^i}{i!} \geq \lim_{t\to\infty} t + 1 = +\infty \\ 0 & \text{if } a < 0 \quad \lim_{t\to\infty} e^{at} = \lim_{t\to\infty} \frac{1}{e^{|a|t}} = \frac{1}{\infty} = 0 \\ 1 & \text{if } a = 0 \quad \lim_{t\to\infty} e^{at} = \lim_{t\to\infty} 1 = 1 \end{cases}$$

Problem 2 (a) Because $f \in C^1(D)$, we can say that f_y is continuous on (c, d). It follows by the MVT that for any points $y_1, y_2 \in [c, d]$, there exists some ξ between y_1 and y_2 s.t $\frac{f(y_1,t)-f(y_2,t)}{y_1-y_2} = f_y(\xi)$. Because D is a closed region, f_y assumes a maximum and a minimum. Thus, there exists some L s.t $|f_y| \leq L$ for all $(y, t) \in D$. Moreover, $\frac{|f(y_1,t)-f(y_2,t)|}{|y_1-y_2|} \leq L \Rightarrow |f(y_1, t) - f(y_2, t)| \leq L|y_1 - y_2|$. Hence f is Lipschitz continuous in D.

(b) $f_y(y, t) = \frac{2yt^2}{t^2+1} \leq \frac{2\delta t_f^2}{t_f^2+1}$ assuming time to be positive. In the case where t is some other variable, we replace t_f with $\max(|t_0|, |t_f|)$. Thus, $|f(y_1, t) - f(y_2, t)| \leq \frac{2\delta t_f^2}{t_f^2+1}|y_1 - y_2|$. Hence f is Lipschitz continuous in D.

Problem 3 (a) $y_n = y_{n-1} + \lambda y_{n-1}h = y_{n-1}(1 - 10h)$ from Euler's method. Solving the characteric equation $x - (1 - 10h) = 0$ for our linear recurrence, we obtain $y_n = c(1 - 10h)^n$. Using our initial condition $y_0 = 1$, we find $c = 1$. Hence, $y_n = (1 - 10h)^n$.
https://artofproblemsolving.com/wiki/index.php/Characteristic_polynomial

(b) $y_1, y_2, y_3 = \begin{cases} -\frac{2}{3}, \frac{4}{9}, -\frac{296}{999} & \text{for } h = \frac{1}{6} \\ \frac{1}{6}, \frac{1}{36}, \frac{1}{216} & \text{for } h = \frac{1}{12} \end{cases}$. $h = \frac{1}{6}$ oscilates between positive and negative while $h = \frac{1}{12}$ stays positive because $1 - 10h > 0$.

(c) So long $h < \frac{1}{10}$ y_n will be positive for all $n \geq 1$.

Problem 4 (a) Let $u := y(t)$ and $v(t) := y'(t)$. We substitute in u and $v(t)$ to obtain $v'(t) + u \cdot v(t) + 4u = t^2$. Because $u' = v(t)$ we obtain the following system:

$$\begin{aligned} v' &= t^2 - u \cdot v(t) - 4u \\ u' &= v(t) \end{aligned}$$

(b) $u(t_{n+1}) = u(t_n + h) = u(t_n) + u'(t_n)h + O(h^2)$
$\approx u(t_n) + v(t_n)h$
$v(t_{n+1}) = v(t_n + h) = v(t_n) + v'(t_n)h + O(h^2)$
$\approx v(t_n) + (t_n^2 - u(t_n)v(t_n) - 4u(t_n))h$
$\Rightarrow \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} u'_n \\ v'_n \end{bmatrix} h$
$\Rightarrow \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} v_n \\ [nh]^2 - u_n v_n - 4u_n \end{bmatrix} h$
$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0.1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$
$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} + 0.1 \begin{bmatrix} 1 \\ 0.1^2 - 0.1 \cdot 1 - 4 \cdot 0.1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.951 \end{bmatrix}$
which gives us $u(0.2) \approx 0.2, v(0.2) \approx 0.951$ as approximations.