Math 100: Problem Set 7

Owen Jones

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(Q-1) $\frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$

$$\frac{1}{x+2} = \sum_{i=0}^{\infty} \frac{1}{2}(-\frac{x}{2})^i$$

$$\frac{1}{x+3} = \sum_{i=0}^{\infty} \frac{1}{3}(-\frac{x}{3})^i$$

$$\frac{1}{x^2+5x+6} = \sum_{i=0}^{\infty} \frac{1}{2}(-\frac{x}{2})^i - \frac{1}{3}(-\frac{x}{3})^i = \sum_{i=0}^{\infty} x^i(-1)^i(\frac{1}{2^{i+1}} - \frac{1}{3^{i+1}})$$

(Q-2) Let the characteristic polynomial for $a_n = 4a_{n-1} - 5a_{n-2} + 3a_{n-3}$ be $z^3 - 4z^2 + 5z - 2 = 0$. We use the rational roots theorem to guess roots $z = 1$ and $z = 2$. It follows $(z-1)^2(z-2) = z^3 - 4z^2 + 5z - 2$. If $a_n = (p+qn) + (r)2^n$, then solving the system of equations:

$$\begin{aligned} 1 &= p + r \\ 0 = p + q + 2r \Leftrightarrow -1 &= q + r \\ -5 = p + 2q + 4r \Leftrightarrow -6 = 2q + 3r \Leftrightarrow -4 &= r \\ p = 5, q = 3, r &= -4 \end{aligned}$$

gives us $a_n = (5+3n) + (-4)2^n$.

(Q-3) Let the characteristic polynomial for $a_n = 7a_{n-1} - 12a_{n-2}$ be $z^2 - 7z + 12 = (z-3)(z-4)$. If $a_n = p3^n + q4^n$, then solving the system of equations:

$$\begin{aligned} 2 &= p + q \\ 17 &= 3p + 4q \\ p &= -9, q = 11 \end{aligned}$$

gives us $a_n = 11 \cdot 4^n - 9 \cdot 3^n$. Thus, $\sum_{i=0}^{n} 11 \cdot 4^n - 9 \cdot 3^n = 11\frac{4^{n+1}-1}{3} - 9\frac{3^{n+1}-1}{2}$.

(Q-4) Let $b_n = \sqrt{a_n}$. It follows $b_n = b_{n-1} + 2b_{n-2}$. Let the characteristic polynomial for $b_n = b_{n-1} + 2b_{n-2}$ be $z^2 - z - 2 = (z-2)(z+1) = 0$. If

$b_n = p(-1)^n + q2^n$, then solving the system of equations:

$$
\begin{aligned}
1 &= p + q \\
1 &= -p + 2q \\
p &= \frac{1}{3}, q = \frac{2}{3}
\end{aligned}
$$

gives us $b_n = \frac{1}{3}(-1)^n + \frac{2}{3}2^n \Rightarrow a_n = (\frac{1}{3}(-1)^n + \frac{2}{3}2^n)^2$.

(Q-5) Let $b_n = \ln(a_n)$. It follows $b_n = \frac{1}{2}(b_{n-1} - b_{n-2})$. Let the characteristic polynomial for $b_n = \frac{1}{2}(b_{n-2} - b_{n-1})$ be $z^2 + \frac{1}{2}z - \frac{1}{2} = (z+1)(z-\frac{1}{2})$ If $b_n = p(-1)^n + q(\frac{1}{2})^n$ then solving the system of equations:

$$
\begin{aligned}
\log(8) &= p + q \\
-\frac{1}{2}\log(8) &= -p + \frac{q}{2} \\
p = \log(4), q &= \log(2)
\end{aligned}
$$

gives us $a_n = 4^{(-1)^n} 2^{(\frac{1}{2})^n}$

(Q-6) Soolving the characteristic polynomial we obtain $y_n = p \cdot a^n + c(n)$ where $c(n)$ is a function of n

$$
\begin{aligned}
1 &= p + c(0) \\
a + b &= p \cdot a + c(1) \\
p = \frac{a}{a-b}, c(n) &= \frac{b^{n+1}}{b-a}
\end{aligned}
$$

which gives us $y_n = \frac{a^{n+1} - b^{n+1}}{a-b}$

(Q-7) We will prove the inclusion-exclusion principle by induction on n.
Base case: The case for $n = 1$ is trivial, and we prove the case for $n = 2$ because it will be useful in the induction step. $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.
Induction hypothesis: assume for some arbitrary n the inclusion-exclusion principle holds.
Induction step: $|A_1 \cup \cdots \cup A_n \cup A_{n+1}| = |(A_1 \cup \cdots \cup A_n) \cup A_{n+1}| = |A_1 \cup \cdots \cup A_n| + |A_{n+1}| - |(A_1 \cup \cdots \cup A_n) \cap A_{n+1}|$ by the inclusion-exclusion principle for $n = 2$.
$|A_1 \cup \ldots \cup A_n| = \sum\limits_i |A_i| - \sum\limits_{i<j} |A_i \cap A_j| + \sum\limits_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1}|A_1 \cap \ldots \cap A_n|$ by the induction hypothesis.
Let $B_i = A_i \cap A_{n+1}$. By the distibutive property $|(A_1 \cup \cdots \cup A_n) \cap A_{n+1}| = |B_1 \cup \cdots \cup B_n|$
It follows $|B_1 \cup \cdots \cup B_n| = \sum\limits_i |B_i| - \sum\limits_{i<j} |B_i \cap B_j| + \sum\limits_{i<j<k} |B_i \cap B_j \cap B_k| -$

$\ldots + (-1)^{n-1}|B_1 \cap \ldots \cap B_n|$ by the induction hypothesis.
$= \sum_i |A_i \cap A_{n+1}| - \sum_{i<j} |A_i \cap A_j \cap A_{n+1}| + \sum_{i<j<k} |A_i \cap A_j \cap A_k \cap A_{n+1}| -$
$\ldots + (-1)^{n-1}|A_1 \cap \ldots \cap A_n \cap A_{n+1}|$
Thus, $|A_1 \cup \cdots \cup A_n \cup A_{n+1}| = |A_1 \cup \cdots \cup A_n| + |A_{n+1}| - |(A_1 \cup \cdots \cup A_n) \cap A_{n+1}|$
$= \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots + (-1)^n |A_1 \cap \ldots \cap A_n \cap A_{n+1}|$

(Q-8) $1000 - \lfloor \frac{1000}{2} \rfloor - \lfloor \frac{1000}{3} \rfloor - \lfloor \frac{1000}{7} \rfloor + \lfloor \frac{1000}{6} \rfloor + \lfloor \frac{1000}{14} \rfloor + \lfloor \frac{1000}{21} \rfloor - \lfloor \frac{1000}{42} \rfloor$
$= 1000 - 500 - 333 - 142 + 166 + 71 + 47 - 23 = 286$

(Q-9) $P((x + Y < 1) \cap (XY < \frac{2}{9}))$
$= \int_0^1 \int_0^{\min(1, 1-x, \frac{2}{9x})} dydx = \int_0^{\frac{1}{3}} \int_0^{1-x} dydx + \int_{\frac{1}{3}}^{\frac{2}{3}} \int_0^{\frac{2}{9x}} dydx + \int_{\frac{2}{3}}^1 \int_0^{1-x} dydx =$
$\frac{1}{3} + \frac{2}{9}\log(2)$

(Q-10) $P(|X - Y| \geq \alpha) = 1 - P(|X - Y| < \alpha)$
$= 1 - \int_0^1 \int_{\max(0, x-\alpha)}^{\min(x+\alpha, 1)} dydx$
Case 1: $\alpha < \frac{1}{2}$
$1 - \int_0^1 \int_{\max(0,x-\alpha)}^{\min(x+\alpha,1)} dydx = 1 - \int_0^{\alpha} \int_0^{x+\alpha} dydx - \int_{\alpha}^{1-\alpha} \int_{x-\alpha}^{x+\alpha} dydx - \int_{1-\alpha}^1 \int_{x-\alpha}^1 dydx$
$= (1 - a)^2$
Case 2: $\alpha \geq \frac{1}{2}$
$1 - \int_0^1 \int_{\max(0,x-\alpha)}^{\min(x+\alpha,1)} dydx = 1 - \int_0^{1-\alpha} \int_0^{x+\alpha} dydx - \int_{1-\alpha}^{\alpha} \int_0^1 dydx - \int_{\alpha}^1 \int_{x-\alpha}^1 dydx$
$= (1 - a)^2$