# Math 151b: Problem Set 4

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**Problem 1**

(a)

$$L_{n-1}(t) = \frac{(t-t_n)(t-t_{n+1})}{(t_{n-1}-t_n)(t_{n-1}-t_{n+1})} = \frac{(t-t_n)(t-t_{n+1})}{2h^2}$$

$$L_n(t) = \frac{(t-t_{n-1})(t-t_{n+1})}{(t_n-t_{n-1})(t_n-t_{n+1})} = -\frac{(t-t_{n-1})(t-t_{n+1})}{h^2}$$

$$L_{n+1}(t) = \frac{(t-t_{n-1})(t-t_n)}{(t_{n+1}-t_n)(t_{n+1}-t_{n-1})} = \frac{(t-t_{n-1})(t-t_n)}{2h^2}$$

$$p(t) = \sum_{i=n-1}^{n+1} y'(t_i)L_i(t)$$

$$p(t) = \frac{y'(t_{n-1})}{2h^2}(t-t_n)(t-t_{n+1}) - \frac{y'(t_n)}{h^2}(t-t_{n-1})(t-t_{n+1})$$

$$+ \frac{y'(t_{n+1})}{2h^2}(t-t_{n-1})(t-t_n)$$

(b)

$$s = \frac{t-t_n}{h} \Rightarrow ds = \frac{1}{h}dt, \frac{t_n-t_n}{h} = 0, \frac{t_{n+1}-t_n}{h} = 1$$

$$\int_{t_n}^{t_{n+1}} (t-t_n)(t-t_{n+1})dt = \int_0^1 hs(hs-h)hds = h^3\int_0^1 s(s-1)ds$$

$$\int_{t_n}^{t_{n+1}} (t-t_{n-1})(t-t_{n+1})dt = \int_0^1 (hs+h)(hs-h)hds = h^3\int_0^1 (s^2-1)ds$$

$$\int_{t_n}^{t_{n+1}} (t-t_{n-1})(t-t_n)dt = \int_0^1 (hs+h)hshds = h^3\int_0^1 s(s+1)ds$$

(c)

$$\int_{t_n}^{t_{n+1}} p(t)dt$$
$$= \int_{t_n}^{t_{n+1}} \frac{y'(t_{n-1})}{2h^2}(t-t_n)(t-t_{n+1}) - \frac{y'(t_n)}{h^2}(t-t_{n-1})(t-t_{n+1})$$
$$+ \frac{y'(t_{n+1})}{2h^2}(t-t_{n-1})(t-t_n)dt$$
$$= \frac{y'(t_{n-1})h}{2}\int_0^1 s(s-1)ds - y'(t_n)h\int_0^1 (s^2-1)ds + \frac{y'(t_{n+1})h}{2}\int_0^1 s(s+1)ds$$
$$= -\frac{y'(t_{n-1})h}{12} + \frac{8y'(t_n)h}{12} + \frac{5y'(t_{n+1})h}{12}$$
$$= h[\frac{5}{12}f(t_{n+1}, y_{n+1}) + \frac{8}{12}f(t_n, y_n) - \frac{1}{12}f(t_{n-1}, y_{n-1})]$$

(d) assume $y(t_n) = y_n$ where $y'(t) = f(t, y(t))$

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} y'(t)dt$$
$$\int_{t_n}^{t_{n+1}} y'(t)dt \approx \int_{t_n}^{t_{n+1}} p(t)dt$$
$$= h[\frac{5}{12}f(t_{n+1}, y_{n+1}) + \frac{8}{12}f(t_n, y_n) - \frac{1}{12}f(t_{n-1}, y_{n-1})]$$
$$\Rightarrow y_{n+1} - y_n = h[\frac{5}{12}f(t_{n+1}, y_{n+1}) + \frac{8}{12}f(t_n, y_n) - \frac{1}{12}f(t_{n-1}, y_{n-1})]$$

(e)

$$\mathcal{L}_h y(t) = y(t+2h) - y(t+h) - h[\frac{5}{12}y'(t+2h) + \frac{8}{12}y'(t+h) - \frac{1}{12}y'(t)]$$
$$y(t+2h) = y + 2hy' + 2h^2y'' + \frac{4h^3}{3}y''' + O(h^4)$$
$$y(t+h) = y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + O(h^4)$$
$$y(t+2h) - y(t+h) = hy' + \frac{3h^2}{2}y'' + \frac{7h^3}{6}y''' + O(h^4)$$
$$y'(t+2h) = y' + 2hy'' + 2h^2y''' + O(h^3)$$
$$y'(t+h) = y' + hy'' + \frac{h^2}{2}y''' + O(h^3)$$
$$h[\frac{5}{12}y'(t+2h) + \frac{8}{12}y'(t+h) - \frac{1}{12}y'(t)] = hy' + \frac{3h^2}{2}y'' + \frac{7h^3}{6} + O(h^4)$$
$$\Rightarrow y(t+2h) - y(t+h) - h[\frac{5}{12}y'(t+2h) + \frac{8}{12}y'(t+h) - \frac{1}{12}y'(t)] = O(h^4)$$
$$\Rightarrow \mathcal{L}_h y(t) = O(h^4)$$

Thus, (2) is third order accurate.

**Problem 2**
A LMM is consistent if $\rho(1) = 0$ $\rho'(1) = \sigma(1)$.
$\rho(r) = r^2 - 1$ and $\sigma(r) = \frac{3}{4}r - \frac{1}{4}$
$\rho(1) = 0$ and $\rho'(r) = 2r \Rightarrow \rho'(1) = 2 \neq \frac{1}{2} = \sigma(1)$. Since the LMM does not satisfy $\rho'(1) = \sigma(1)$, the LMM is not consistent.

**Problem 3**
$\rho(r) = r + \alpha$ and $\sigma(r) = \beta_1 r + \beta_0$.
We are given the LMM is consistent, so $\rho(1) = 0 \Rightarrow \alpha = -1$.
It follows the only root $r$ of our characteristic generating polynomial is 1. Since all roots $|r_i| \leq 1$ and simple, the one step method is strongly zero-stable. Hence, it is convergent.

**Problem 4**
The characteristic polynomials for this LMM are $\rho(r) = r^3 + r^2 - r - 1, \sigma(r) = r^3 + r^2 + r + 1$.
$\rho(1) = 0, \rho'(1) = 4, \sigma(1) = 4$, so the LMM is consistent.
$\rho(r) = r^2(r+1) - (r+1) = (r-1)(r+1)^2 \Rightarrow y_n = c_1 + (c_2 + c_3 n)(-1)^n$. Because the root $r = -1$ has multiplicity 2, there exists an $|r_i| = 1$ that is not a simple root, so the LMM is not zero stable. Hence, the LMM is not convergent.