Formulas of PL

Formulas: We define them recursively by the 3 clauses

a) Each propositional variable A_i is a formula.

b) $(\neg\phi)$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, and $(\phi \to \psi)$ are also formulas if ϕ and ψ are.

c) No string that is not built by a) or b) is a formula.

Definition: A set of strings is propositionally closed if it contains all the propositional variables and is closed under sentential conncetives. A string is a formula if it belongs to every propositionally closed set S.
Induction on formulas:

1) Check that each propositional variable has property P.

2) If ϕ has property P then so does $(\neg\phi)$

3) If ϕ ans ψ have property P then so does $(\phi \bullet \psi)$ for any binary conncetive $\vee, \wedge, \to$.

Unique readability:
For every formula exactly one of the following is true:

1) ϕ is a prop. variable

2) There is a unique formula ψ s.t $\phi = (\neg\psi)$.

3) There are unique formulas ψ, χ s.t $\phi = (\psi \vee \chi)$, $(\psi \wedge \chi)$, or $(\psi \to \chi)$

Semantics of PL

We assign a function $\upsilon : \{A_i | i \in \mathbb{N}\} \to \{T, F\}$ to each propositional variable and extend to all formulas by induction on formulas.
Theorem: For any truth assigment $\upsilon : \{A_i | i \in \mathbb{N}\} \to \{T, F\}$, there is a unique extension $\upsilon : PL \to \{T, F\}$.
Definition: A formula $\varphi \in PL$ is satisfiable if there is an assigment $\upsilon : \{A_i | i \in \mathbb{N}\} \to \{T, F\}$ s.t $\upsilon(\varphi) = T$
Tarski conditions:

- $\upsilon \models A_i \leftrightarrow \upsilon(A_i) = T$
- $\upsilon \models \neg\varphi \leftrightarrow \upsilon \not\models \varphi$
- $\upsilon \models (\varphi \wedge \psi) \leftrightarrow \upsilon \models \varphi$ and $\upsilon \models \psi$
- $\upsilon \models (\varphi \vee \psi) \leftrightarrow \upsilon \models \varphi$ or $\upsilon \models \psi$
- $\upsilon \models (\varphi \to \psi) \leftrightarrow \upsilon \not\models \varphi$ or $\upsilon \models \psi$

Tautologies:

- $\models \varphi \to (\psi \to \phi)$
- $\models (\varphi \to \psi) \to ((\varphi \to (\psi \to \chi)) \to (\varphi \to \chi))$
- $\models (\varphi \to \psi) \to ((\varphi \to (\neg\psi)) \to (\neg\varphi))$
- $\models (\neg(\neg\varphi)) \to \varphi$
- $\models \varphi \to (\psi \to (\varphi \wedge \psi))$
- $\models \varphi \to (\varphi \vee \psi)$
- $\models (\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$

Logical Consequence and Soundness

$T \models \varphi$ means "for every assigment that satisfies all formulas in T then v also satisfies φ"
Modus Ponens: For any two formulas φ, ψ $\varphi, (\varphi \to \psi) \models \psi$.
Definition: A deduction or proof from a set of formulas T is a finite sequence of formulas $\chi_0, \chi_1, \ldots, \chi_k$ such that for each $n \leq k$ one of the following holds:

(D1) $\chi_n \in T$ (assumption)

(D2) $\chi_n \equiv \chi_i$ for some $i < n$ (repetition)

(D3) χ_n is an axiom

(D4) χ_n can be inferred with MP for some χ_i, χ_j with $i, j < n$

$T \vdash \chi$ iff there is a proof $\chi_0, \chi_1, \ldots, \chi_k$ from T s.t $\chi \equiv \chi_k$. Then χ is a theorem of T.
Definition: A set of formulas is deductively closed if it contains all the axioms and it is closed under MP.
Lemma: For every T and φ $T \vdash \varphi$ iff $\varphi \in S$ for every deductively closed $S \supseteq T$.
Theorem: (Soundness) For any set of formulas Γ and formula φ if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Consistency and Completeness

Definition: A set of formulas Γ is consistent if there is no formula φ s.t $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.
Definition: A set of formulas is strongly complete if $\forall \varphi \in PL$ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.
Step 1: A set of formulas S is consistent and strongly complete iff there is an assigment v to the propositional variables s.t for every formula φ $v \models \varphi \leftrightarrow \varphi \in S$.

The Deduction Theorem: For any set of formulas T and all φ, ψ if $\Gamma, \varphi \vdash \psi$ then $\Gamma \vdash (\varphi \to \psi)$
Step 2: If Γ is consistent then for every formula φ either $\Gamma \cup \varphi$ or $\Gamma \cup \varphi$ is consistent.
Step 3: (The Completeness Theorem for PL)

(1) Every consistent set of formulas is satisfiable

(2) If $\Gamma \models \chi$ then $\Gamma \vdash \chi$

Structures

Language: is a set of constants, function symbols, and relation symbols. $\mathcal{L} = \{c_j\}_{j \in J} \cup \{R_i\}_{i \in I} \cup \{R_k\}_{k \in K}$
Terms and formulas:
Let $\mathcal{L}$ be a language we consider $S \supseteq \mathcal{L}$ of symbols where we add:

(1) The logical symbols $\neg, \vee, \wedge, \to, \forall, \exists, =$

(2) Parentheses ()

(3) The individual variables $v_0, v_1, \ldots$

Definition: An $\mathcal{L}$-term is defined by recursion where:

a) Each variable v_i is a term

b) Each constant symbol is a term

c) If $\tau_1, \ldots, \tau_n$ are terms and f is an $n{-}ary$ function symbol then $f(\tau_1, \ldots, \tau_n)$ is a term

Definition: An $\mathcal{L}$-formula is defined by recursion where:

a) If s, t are terms then $s = t$ is a formula

b) If $\tau_1, \ldots, \tau_n$ are terms and R is an $n{-}ary$ relation symbol then $R(\tau_1, \ldots, \tau_n)$ is a formula

c) If φ, ψ are formulas and v is a variable $(\neg\varphi), (\varphi \bullet \psi), \exists v \varphi, \forall v \varphi$ are formulas.

Let φ be an $\mathcal{L}$-formula then

(1) φ is quantifier free if $\exists$ and $\forall$ do not occur in φ

(2) The variable x_i if free in φ if it is not quantified

Semantics

A structure in the language $\mathcal{L} = \{c_j\}_{j\in J} \cup \{R_i\}_{i\in I} \cup \{R_k\}_{k\in K}$ is a pair $A = (A, I)$ where:

- A is a non-empty set (we call A) to be the universe
- I is an interpretation that assigns:
  - $I(c) = c^A$ an element of A.
  - $I(R) = R^A$ is an $n - ary$ relation $R^A \subset A^n$ to each $n - ary$ relation R.
  - $I(f) = f^A : A^n \to A$ an $n - ary$ function for each function symbol.

Definition: An isomorphism $\sigma : A \to B$ where both are $\mathcal{L}$-structures is a bijection such that:

(1) For each constant symbol c_A $\sigma(c_A) = c_B$

(2) For each $n-ary$ relation symbol R and $a_1, a_2, \ldots, a_n \in A$, $(a_1, a_2, \ldots, a_n) \in R^A \leftrightarrow (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_n)) \in R^B$

(3) For eahc $n-ary$ function symbol f and $a_1, a_2, \ldots, a_n \in A, = \sigma(f^A(a_1, a_2, \ldots, a_n)) = f^B(\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_n))$

If A and B are isomorphic then we denote this by $A \cong B$
Theorem: Let M and N be $\mathcal{L}$-structures and $f : M \to N$ an isomorphism. If υ is an assigment $\upsilon : Var \to M$ then $f \circ \upsilon : Var \to N$ is an assignment on N and for any formula φ and tuple $\overline{m} \in M$ $(M, \upsilon) \models \varphi(\overline{m})$ iff $(M, f \circ \upsilon) \models \varphi(f(\overline{m}))$.
Definition: An automorphism σ of M is an isomorphism of M onto M. Definition: (Substructure) Let $A = (A, I)$ $B = (B, J)$ be $\mathcal{L}$-structures. A is a substructure of B if

- $A \subseteq B$
- For every constant symbol $c^B = c^A \in A$
- For any $n - ary$ function and $a_1, a_2, \ldots, a_n \in A$, $f^A(a_1, a_2, \ldots, a_n) = f^B(a_1, a_2, \ldots, a_n)$
- For any $n - ary$ relation and $a_1, a_2, \ldots, a_n \in A$ $R^B(a_1, a_2, \ldots, a_n) \leftrightarrow R^A(a_1, a_2, \ldots, a_n)$

Definition: An assigment into a structure A is any association of objects in A with variables $\pi : Variables \to A$.
The value of an assigment is proven by induction on formulas.
Satisfaction: Let A be and $\mathcal{L}$-structure and π an assigment in A and φ an $\mathcal{L}$-formula
We say $A, \pi \models \phi \leftrightarrow value(\varphi, \pi) = 1 \leftrightarrow$ the assigment π satisfies the formula in the structure A.
Tarski conditions:

- $A, \pi \models s = t \leftrightarrow value^A(s, \pi) = value^A(t, \pi)$
- $A, \pi \models \neg\varphi \leftrightarrow A, \pi \not\models \varphi$
- $A, \pi \models \varphi \wedge \psi \leftrightarrow A, \pi \models \varphi$ and $A, \pi \models \psi$
- $A, \pi \models \varphi \vee \psi \leftrightarrow A, \pi \models \varphi$ or $A, \pi \models \psi$
- $A, \pi \models \varphi \to \psi \leftrightarrow A, \pi \not\models \varphi$ or $A, \pi \models \psi$
- $A, \pi \models \exists v\varphi \leftrightarrow$ there is $a \in A$ such that $(A, \pi(v := a)) \models \varphi$
- $A, \pi \models \forall v\varphi \leftrightarrow$ for all $a \in A$ $(A, \pi(v := a)) \models \varphi$

Definition: A proposition Φ in ordinary English about an $\mathcal{L}$-structure A is expressed or formalized by a sentence if Φ and φ mean the same thing (ϕ is a statement)$\to$ φ is a sentence if all variables are quantified.

Definable Sets

Let $R \subseteq A^n$ where A is an $\mathcal{L}$-structure and R is a relation in the universe of A^n. We say that R is definable if there is (1) an $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$ (2) a tuple $\overline{a}_0 \in A^{|\overline{y}|}$ such that $A \models \varphi(\overline{b}, \overline{a}_0)$ iff $\overline{b} \in A$ and we say that it is definable over a_0

If R is definable over a_0 then there is an $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$ such that:
$(m_1, m_2, \ldots, m_n) \in R$ iff $(M, v) \models \varphi(\overline{m}, a_0) \leftrightarrow (M, \sigma \circ v) \models \varphi(\sigma(\overline{m}), a_0)$

Theories

Definition: Let $\mathcal{L}$ be a fixed language on $\mathcal{L}$-theory T is any (possibly infinite) set of sentences T. The members of T are called axioms.
Definition 2: An $\mathcal{L}$-structure M is a model of an $\mathcal{L}$-theory T if for any sentence $\varphi \in T$ $M \models \phi$ i.e $M \models T$.
Definition 3: The models of a theory T T is the class of $\mathcal{L}$-structures $Mod(T) = \{M | M \text{ is an } \mathcal{L}\text{-structure and } M \models T\}$
Definition: Let $\mathcal{L}$ be a fixed language and let Φ be a property of $\mathcal{L}$-structures we say that it is elementary if there is an $\mathcal{L}$-theory T such that:
For M and $\mathcal{L}$-structure M has property $\Phi \leftrightarrow M \models T$ Definition: Let M and N be $\mathcal{L}$-structures then they are elementary equivalent if $\forall\varphi$ $\mathcal{L}$-sentence $M \models \varphi$ iff $N \models \varphi$ we denote this by $M \equiv N$,

Logical Consequence and Proofs

Definition: Let T be an $\mathcal{L}$-Theory and φ an $\mathcal{L}$-sentence we say that φ is a logical consequence $T \models \varphi$ iff every $\mathcal{L}$-structure $M \models T$ then $M \models \varphi$.
Hilbert's axiom schemes

(a) Propositional Axiom Schemes: The set of logical tautologies
(b) Predicate Axiom Schemes

- $\forall \upsilon \varphi(\upsilon, \vec{u}) \rightarrow \varphi(\tau, \vec{u})$ τ free for υ in $\varphi(\upsilon, \vec{u})$
- $\forall \upsilon(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall \upsilon \psi)$ υ not free in φ
- $\varphi(\tau, \vec{u}) \rightarrow \exists \upsilon \varphi(\upsilon, \vec{u})$ τ free for υ in $\varphi(\upsilon, \vec{u})$

(c) Rules of Inference

- From φ and $\varphi \rightarrow \psi$ infer ψ MP
- From φ infer $\forall \upsilon \varphi$ (Generalization)
- From $\varphi \rightarrow \psi$ infer $\exists \upsilon \varphi \rightarrow \psi$ provided υ is not free in ψ (Exists Elimination)

(d) Identity Axioms

- $\upsilon = \upsilon$ $\upsilon = \upsilon' \rightarrow \upsilon' = \upsilon$ $\upsilon = \upsilon' \rightarrow ((\upsilon' \rightarrow \upsilon'') \rightarrow (\upsilon \rightarrow \upsilon''))$
- $(v_1 = w_1 \wedge \ldots \wedge v_n = w_n) \rightarrow (R(v_1, \ldots, v_n) \rightarrow R(w_1, \ldots, w_n))$ any $n-ary$ relation symbol
- $(v_1 = w_1 \wedge \ldots \wedge v_n = w_n) \rightarrow (f(v_1, \ldots, v_n) = f(w_1, \ldots, w_n))$ any $n-ary$ function symbol

Definition: A deduction from a theory T is any sequence of formulas $\varphi_0, \ldots \varphi_n$ where each φ_i is either:

- an axiom $\varphi_i \in$ Hilbert's Axioms
- a hypothesis $\varphi_i \in T$
- repetition $\varphi_i = \varphi_j$ for $j < i$
- or follows from MP, Generalization, or Exists Elimination

Definition: (Soundness for first order logic) Let T be a theory and χ a sentence if $T \vdash \chi$ then $T \models \chi$
Lemma: Let M be an $\mathcal{L}$-structure and χ an axiom in the Hilbert's list (not inference) then $M \models \chi$

Completeness Theorem

Definition 1: Γ a theory and φ a sentence if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$
Definition 2: If Γ is consistent then it is satisfiable
Consistency: A theory T is consistent if there is no sentence φ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$
An $\mathcal{L}$-theory is satisfiable if there is an $\mathcal{L}$-structure M such that $M \models T$
An $\mathcal{L}$-theory is complete if for every sentence φ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$
Definition: A theory T has the Henkin property if:

(a) it is consistent

(b) it is complete

(c) if $\exists v\varphi \in H$ then there is a constant c such that $\varphi(c) \in H$

Idea: add a bunch of constants to witness the existentials
Definition: Fix a Henkin set we say that it is deductively closed if for every sentence χ if $H \vdash \chi$ then $\chi \in H$ For all sentences $\varphi, \psi, \exists v\varphi(v), \forall v\varphi(v)$

(a) $\neg\varphi \leftrightarrow \varphi \notin H$

(b) $\varphi \wedge \psi \in H \leftrightarrow \varphi \in H$ and $\psi \in H$

(c) $\varphi \vee \psi \in H \leftrightarrow \varphi \in H$ or $\psi \in H$

(d) $\varphi \to \psi \in H \leftrightarrow \varphi \notin H$ or $\psi \in H$

(e) $\exists v\varphi(v) \in H \leftrightarrow$ there is some constant c such that $\varphi(c) \in H$

(f) $\forall v\varphi(v) \in H \leftrightarrow$ for all constants c $\varphi(c) \in H$

Every Theory can be extended to a Henkin Set
Every Henkin set is satisfiable

Compactness Theorem

(PL Logic) Suppose T is an infinite set of formulas. Prove that if every finite subset $T_0 \subset T$ is satisfiable, then T is satisfiable.
(First Order) If every finite subset of a theory T in a finite vocabulary has a model, then T has a model.
(Compactness-Completeness) Let $\mathcal{L}$ be a language and T be a first order theory. The following are equivalent:

1. T is consistent
2. T is finitely satisfiable
3. T is satisfiable
4. T is finitely satisfiable

Applications

1. Any set A can be lineraly ordered
2. Any torsion free group can be lineraly ordered

3. There is a non-archimedian field F such that $F \equiv (\mathbb{R}, +, \cdot, 0, 1)$

4. Ramsey's theorem: For every natural number k there is a natural number n such that for every coloring $c : [n]^2 \to \{R, B\}$ there is $S \subset \{1, \ldots, n\}$ of size k that is monocromatic.

5. Let $\mathcal{L}$ be the language of fields. For every ϕ, a $\mathcal{L}$ sentence, there is some $N \subset \mathbb{N}$ such that for every prime number $p \geq N$ such that $(\mathbb{C}, +, \cdot, 0, 1) \models \phi$ iff $(F_p^{alg}, +, \cdot, 0, 1) \models \phi$.

Elementary Substructure

Let M be a structure, and let A be a subset of M. Then A is an elementary substructure of M if for every formula $\phi(v_1, \ldots, v_n)$ and every tuple $\bar{a} \in A^n$, if $M \models \phi(\bar{a})$, then $A \models \phi(\bar{a})$.

Tarski Vaught criterion

Let M be a structure, and let $A, B \subseteq M$. Then A is an elementary substructure of M with respect to B, denoted by $A \prec_B M$, if for every formula $\phi(v_1, \ldots, v_n)$ and every tuple $\bar{a} \in A^n$, if $M \models \phi(\bar{a}, \bar{b})$ for some $\bar{b} \in B^n$, then $A \models \phi(\bar{a})$.

Tarski Gödel

0.1 Arithmetization

The document begins by explaining the concept of arithmetization, which is the process of encoding mathematical statements and proofs as numerical sequences. This is a crucial technique used in the proofs of both Tarski's and Gödel's theorems. It introduces the coding of sequences and terms, including the coding of the language of Peano Arithmetic (PA) and how terms and formulas are represented arithmetically.

0.2 The Theorems

- **Tarski's Undefinability Theorem**: This theorem asserts that the set of all true arithmetic statements (Truth(N)) is not arithmetically definable. In other words, there is no single arithmetic formula that can capture all and only the truths about natural numbers.

- **Gödel's First Incompleteness Theorem**: This theorem states that any consistent formal system that is capable of expressing basic arithmetic is incomplete. There are true statements within the system that cannot be proven using the system's axioms and rules.

- **Gödel's Second Incompleteness Theorem**: This theorem extends the first by showing that such a system cannot prove its own consistency.

0.3 Problems

The document concludes with a set of problems designed to reinforce the concepts discussed. These problems involve proving various properties of the coding functions and the relations used in the proofs of the theorems.

1 Key Concepts and Definitions

- **Arithmetization**: The technique of representing statements, sequences, and proofs in arithmetic form. This involves encoding symbols, terms, and formulas as natural numbers.
- **Sequence Coding**: A method for encoding finite sequences of natural numbers. The document details specific functions and properties related to this coding.
- **Formulas and Proofs**: The document explains how formulas and proofs are encoded, and how these encodings are used to state and prove the theorems of Tarski and Gödel.

1.1 Theorem Statements

- **Tarski's Theorem**: The set of arithmetical truths (Truth(N)) is not arithmetical.
- **Gödel's First Incompleteness Theorem**: No arithmetical theory can be both sound (all its theorems are true) and complete (it proves all true statements).
- **Gödel's Second Incompleteness Theorem**: No consistent arithmetical theory can prove its own consistency.

2 Proof Techniques

- **Self-Reference and Diagonalization**: Techniques used in the proofs involve creating self-referential statements and using diagonalization to show that certain sets cannot be captured by arithmetic formulas.
- **Arithmetical Functions and Relations**: Various arithmetical functions and relations are defined and used to construct the necessary statements for the proofs.

Midterm Solutions

Exercise 1 (Propositional logic)

Let $L = \{A_i : i \in \mathbb{N}\}$ be a propositional language. Let φ and θ be L-sentences and let Σ be an L-theory.

(1) State the definition of a Σ-proof of φ.

(2) Suppose that $\Sigma \vdash (\varphi \to \theta)$. Prove that $\Sigma \vdash ((\neg\theta) \to (\neg\varphi))$.

Solution:

(1) See course notes.

(2) The easiest way to do this problem is to use the Completeness Theorem for propositional logic. This says that if Σ is an L-theory, and φ is a propositional formula, then $\Sigma \vdash \varphi$ iff $\Sigma \models \varphi$. Suppose that $\Sigma \vdash (\varphi \to \theta)$. Then $\Sigma \models (\varphi \to \theta)$. We want to show that $\Sigma \models ((\neg\theta) \to (\neg\varphi))$, which will show $\Sigma \vdash ((\neg\theta) \to (\neg\varphi))$. Suppose that v is a truth assignment that satisfies Σ. By assumption, $\Sigma \models (\varphi \to \theta)$, so v satisfies $(\varphi \to \theta)$. But $(\varphi \to \theta)$ is logically equivalent to $((\neg\theta) \to (\neg\varphi))$, so we conclude that v also satisfies $((\neg\theta) \to (\neg\varphi))$. This completes the proof.

Exercise 2 (Definable sets)

(1) Let $M = (A; |, 1)$, where $A = \mathbb{N}\backslash\{0\} = \{1, 2, 3, \ldots\}$, $|$ is the binary relation on A defined by $n|m$ if and only if n divides m, and 1 is a distinguished point. Show that $P = \{p \in \mathbb{N} : p \text{ is a prime number}\}$ is a definable set.

(2) Consider the structure $M_1 = (\mathbb{Z}; +)$ with the standard interpretation. Prove that $\mathbb{N}$ is not a definable subset of $\mathbb{Z}$ in the language $L = \{+\}$.

Solution:

(1) Since a positive integer x is prime iff $x \neq 1$ and every positive integer divisor of x is either 1 or x, the set P of primes is defined by the formula

$$\varphi(x) = \text{“}x \neq 1 \wedge \forall y(y|x \to (y = 1 \vee y = x))\text{”}.$$

(Note that the number 1 is not prime, by definition.) Alternatively, since a positive integer x is prime iff x has exactly two positive integer divisors, the set P of primes is also defined by the formula

$$\psi(x) = \text{“}\exists y \exists z(y \neq z \wedge y|x \wedge z|x \wedge \forall w(w|x \to (w = y \vee w = z)))\text{”}.$$

(2) The only rigorous way we know of to show that a subset is not definable is to use the following Fact: if $D \subseteq A$ is definable and $G : A \to A$ is an automorphism, then whenever $a \in D$ we have $G(a) \in D$. Thus, to show

that a set is not definable, we need to use a nontrivial automorphism of the structure. There is only one nontrivial automorphism of $(\mathbb{Z}; +)$, namely the map $G : \mathbb{Z} \to \mathbb{Z}$ given by $G(n) = -n$ for all $n \in \mathbb{Z}$. Now just observe that $1 \in \mathbb{N}$ but $G(1) = -1 \notin \mathbb{N}$. Hence, the Fact implies that $\mathbb{N}$ is not a definable subset of $(\mathbb{Z}; +)$.

Exercise 3 (Sentences and isomorphism)

Let $L = \{\leq\}$. Consider

(a) $M_1 = (\mathbb{N}; \leq)$, standard interpretation.

(b) $M_2 = (\mathbb{Q}; \leq)$, standard interpretation.

(1) Find an L-sentence φ such that $M_1 \models \varphi$ and $M_2 \not\models \varphi$.

(2) Conclude that M_1 and M_2 are not isomorphic, i.e. $M_1 \ncong M_2$.

(3) Provide an example of an automorphism of M_2.

Solution:

(1) There are many possibilities. One example of such a sentence φ is

$$\exists x \forall y (x \leq y).$$

Another possibility is

$$\exists x \exists y (x \neq y \wedge x \leq y \wedge \neg \exists z (z \neq x \wedge z \neq y \wedge x \leq z \wedge z \leq y)),$$

which says that there exist consecutive elements x and y with no element in between them.

(2) We proved in class that isomorphic models are always elementarily equivalent. Since M_1 and M_2 are not elementarily equivalent by part (1), we conclude that $M_1 \ncong M_2$.

(3) There are many examples of automorphisms of M_2 besides the identity map. For example, $G(q) = q + 1$ and $H(q) = q/2$ are automorphisms.

Exercise 4 (True or False)

(1) Let L be a finite language containing only relational symbols and constants. Let M and N be L-structures, then $M \equiv N$ if and only if $M \cong N$.

(2) Consider the structures $M_1 = (\mathbb{Z}; +, 0)$ and $M_2 = (2\mathbb{Z}; +, 0)$. Then $M_1 \equiv M_2$.

(3) Let L be a fixed language, and let T be a first-order theory. If there is an L-structure $M \models T$, then T is consistent.

(4) $(\mathbb{N}, +)$ is a substructure of $(\mathbb{Z}, +)$.

Solution:

(1) False. We only know this is true for finite structures M and N. We will see soon that $(\mathbb{R}; \leq) \equiv (\mathbb{Q}; \leq)$ but these structures are not isomorphic.

(2) True. The structures M_1 and M_2 are even isomorphic via $G : \mathbb{Z} \to 2\mathbb{Z}$ given by $G(n) = 2n$. Hence they are elementarily equivalent. (Given the tools we have at our disposal at this point in the course, the only way we know how to show that two structures are elementarily equivalent is to show they are isomorphic.)

(3) True. This follows from the Soundness Theorem for first-order logic.

(4) True. One just has to check that $\mathbb{N}$ is closed under $+$.