Math 151b: Problem Set 7

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Problem 1

(a) Case 1: $\mathbf{v} = \mathbf{0}$
Trivially $\|A\mathbf{v}\| = \|\mathbf{0}\| = 0 \leq 0 = \|\mathbf{A}\|0 = \|A\| \cdot \|\mathbf{v}\|$
Case 2: $\mathbf{v} \neq \mathbf{0}$
Let $\mathbf{w} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \Rightarrow \|\mathbf{w}\| = 1$.
$\|A\mathbf{w}\| \leq \max\limits_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \|A\| \Rightarrow \|A\mathbf{w}\| \leq \|A\|$.
Thus, $\|A\mathbf{w}\| = \|A\frac{\mathbf{v}}{\|\mathbf{v}\|}\| = \frac{1}{\|\mathbf{v}\|}\|A\mathbf{v}\| \leq \|A\| \Rightarrow \|A\mathbf{v}\| \leq \|A\| \cdot \|\mathbf{v}\|$.

(b) Suppose $\|\mathbf{x}\| = 1$.
Observe $\mathit{Range}(\mathbf{B}) \subset \mathbb{R}^n$.
$\|AB\mathbf{x}\| \leq \|A\| \cdot \|B\mathbf{x}\| \leq \|A\| \cdot \|B\| \cdot \|\mathbf{x}\| = \|A\| \cdot \|B\|$ by part (a).
Since this innequality holds for any $\|\mathbf{x}\| = 1$, the innequality must hold for $\|AB\| = \max\limits_{\|\mathbf{x}\|=1} \|AB\mathbf{x}\| \leq \|A\| \cdot \|B\|$.
Thus, $\|AB\| \leq \|A\| \cdot \|B\|$.

Problem 2

We will prove $\rho(\cdot)$ is not a matrix norm by contrapositive. Let $\mathbf{A} \in \mathbb{R}^{n \times n} \neq \mathbf{0}$ be a strictly triangular matrix. Because the eigenvalues of a triangular matrix are the elements along the main diagonal, $\lambda_i = 0$, $i = 1, \ldots, n$. It follows $\rho(\mathbf{A}) = \max\limits_{1 \leq i \leq n} |\lambda_i| = 0$. Because $\rho(\mathbf{A}) = 0 \not\Rightarrow \mathbf{A} = \mathbf{0}$, $\rho(\cdot)$ is not a matrix norm.

Problem 3

$G_j = D^{-1}(L+U), G_g = (D-L)^{-1}U$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_j = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$G_g = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

(a) For G_j $\det(\lambda I - G_j) = \lambda^3 + \frac{5}{4}\lambda$, $\lambda = 0, \frac{\sqrt{5}i}{2}, -\frac{\sqrt{5}i}{2}$
$\Rightarrow \rho(G_j) = |\frac{\sqrt{5}i}{2}| = \frac{\sqrt{5}}{2} < 1$

(b) For G_g eigenvalues are elements along diagonal, $\lambda = 0, -\frac{1}{2}$
$\Rightarrow \rho(G_g) = |-\frac{1}{2}| = \frac{1}{2} < 1$

Problem 4

(a) $G_j = D^{-1}(L+U) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}$
For G_j $\det(\lambda I - G_j) = \lambda(\lambda^2 - 2) - 2(\lambda - 2) + 4(\lambda + 1) = \lambda^3$
$\Rightarrow \rho(G_j) = |0| = 0 < 1$

(b) $(D-L)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$
$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$
$G_g = (D-L)^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}$
For G_g eigenvalues are elements along diagonal, $\lambda = 0, 2$
$\Rightarrow \rho(G_g) = |2| = 2 > 1$

Problem 5

(a) Proof by induction:
Base case: $\|\mathbf{x}^{(1)} - \mathbf{x}\| = \|(G\mathbf{x}^{(0)} + c) - (G\mathbf{x} + c)\|$
$= \|G(\mathbf{x}^{(0)} - \mathbf{x})\| \le \|G\|\|\mathbf{x}^{(0)} - \mathbf{x}\|$ by Problem 1.
Induction hypothesis: Assume for some k $\|\mathbf{x}^{(k)} - \mathbf{x}\| \le \|G\|^k\|\mathbf{x}^{(0)} - \mathbf{x}\|$.
Induction step: $\|\mathbf{x}^{(k+1)} - \mathbf{x}\| = \|(G\mathbf{x}^{(k)} + c) - (G\mathbf{x} + c)\|$
$= \|G(\mathbf{x}^{(k)} - \mathbf{x})\| \le \|G\|\|\mathbf{x}^{(k)} - \mathbf{x}\|$
$\le \|G\|^{k+1}\|\mathbf{x}^{(0)} - \mathbf{x}\|$ by the induction hypothesis.
Hence, by induction, the claim holds for all k.

(b) (i) Proof by induction:
Base case: $\|\mathbf{x}^{(2)} - \mathbf{x}^{(1)}\| = \|(G\mathbf{x}^{(1)} + c) - (G\mathbf{x}^{(0)} + c)\|$

$= \|G(\mathbf{x}^{(1)} - \mathbf{x}^{(0)})\| \leq \|G\|\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$ by Problem 1.
Induction hypothesis: Assume for some k $\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\| \leq \|G\|^{k-1}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$.
Induction step: $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| = \|(G\mathbf{x}^{(k)} + c) - (G\mathbf{x}^{(k-1)} + c)\|$
$= \|G(\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)})\| \leq \|G\|\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|$
$\leq \|G\|^k\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$ by the induction hypothesis.
Hence, by induction, the claim holds for all k.

(ii) Proof by induction:
Base case: the base case is proved in part (a). We know $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| \leq \|G\|^k\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$
Induction hypothesis: Assume for some $m > k$

$$\|\mathbf{x}^{(m)} - \mathbf{x}^{(k)}\| \leq \|G\|^k\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \sum_{i=0}^{m-k-1} \|G\|^i$$

Induction step: $\|\mathbf{x}^{(m+1)} - \mathbf{x}^{(m)}\| \leq \|G\|^m\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$ by part (a).
$\|\mathbf{x}^{(m+1)} - \mathbf{x}^{(m)}\| + \|\mathbf{x}^{(m)} - \mathbf{x}^{(k)}\|$
$\leq \|G\|^{m-k}\|G\|^k\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| + \|G\|^k\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \sum_{i=0}^{m-k-1} \|G\|^i$
$= \|G\|^k\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \sum_{i=0}^{m-k} \|G\|^i$.
By the triangle innequality $\|\mathbf{x}^{(m+1)} - \mathbf{x}^{(k)}\| \leq \|\mathbf{x}^{(m+1)} - \mathbf{x}^{(m)}\| + \|\mathbf{x}^{(m)} - \mathbf{x}^{(k)}\|$

Thus, $\|\mathbf{x}^{(m+1)} - \mathbf{x}^{(k)}\| \leq \|G\|^k\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \displaystyle\sum_{i=0}^{m-k} \|G\|^i$.
Hence, by induction, the claim holds for all m.

(iii) Suppose $\|G\| < 1$.
$\|\mathbf{x}^{(m)} - \mathbf{x}\| \leq \|G\|^m\|\mathbf{x}^{(0)} - \mathbf{x}\|$ from part (a).
It follows $\lim_{m\to\infty} \|\mathbf{x}^{(m)} - \mathbf{x}\| = 0 \Rightarrow \mathbf{x}^{(m)} \to \mathbf{x}$ as $m \to \infty$.
Thus, $\|\mathbf{x}^{(k)} - \mathbf{x}\| = \lim_{m\to\infty} \|\mathbf{x}^{(k)} - \mathbf{x}^{(m)}\|$.

$\|\mathbf{x}^{(m)} - \mathbf{x}^{(k)}\| \leq \|G\|^k\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \displaystyle\sum_{i=0}^{m-k-1} \|G\|^i$ from part (b)(ii).

$\|G\|^k\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \displaystyle\sum_{i=0}^{m-k-1} \|G\|^i = \frac{\|G\|^k - \|G\|^m}{1 - \|G\|}$ for all $m > k$ by the sum of a geometric series.

Using $\|G\| < 1$, $\displaystyle\lim_{m\to\infty} \frac{\|G\|^k - \|G\|^m}{1 - \|G\|} = \frac{\|G\|^k}{1 - \|G\|}$.

Hence, by the squeeze theorem, $\|\mathbf{x}^{(k)} - \mathbf{x}^{(m)}\| \leq \frac{\|G\|^k - \|G\|^m}{1-\|G\|}$ for all $m > k$ implies $\|\mathbf{x}^{(k)} - \mathbf{x}\| \leq \frac{\|G\|^k}{1-\|G\|}$.
(Note: I know the squeeze theorem requires the innequality to hold for every element of the sequence, but we can easily get around that by defining 2 sequences $A_n := \|\mathbf{x}^{(k)} - \mathbf{x}^{(k+1+n)}\|$ and $B_n := \frac{\|G\|^k - \|G\|^{k+1+n}}{1-\|G\|}$ which satisfy the condition $A_n \leq B_n$ for all $n \geq 0$).

Problem 6

(a) $D = \begin{bmatrix} 10 & & & \\ & 10 & & \\ & & 8 & \\ & & & 5 \end{bmatrix}, L = \begin{bmatrix} & & & \\ -5 & & & \\ & 4 & & \\ & & 1 & \end{bmatrix}, U = \begin{bmatrix} & -5 & & \\ & & 4 & \\ & & & 1 \\ & & & \end{bmatrix}, b = \begin{bmatrix} 6 \\ 25 \\ -11 \\ -11 \end{bmatrix}$

$$G_j = \begin{bmatrix} \frac{1}{10} & & & \\ & \frac{1}{10} & & \\ & & \frac{1}{8} & \\ & & & \frac{1}{5} \end{bmatrix} \begin{bmatrix} & -5 & & \\ -5 & & 4 & \\ & 4 & & 1 \\ & & 1 & \end{bmatrix} = \begin{bmatrix} & -\frac{1}{2} & & \\ -\frac{1}{2} & & \frac{2}{5} & \\ & \frac{1}{2} & & \frac{1}{8} \\ & & \frac{1}{5} & \end{bmatrix}$$

$$c_j = \begin{bmatrix} \frac{1}{10} & & & \\ & \frac{1}{10} & & \\ & & \frac{1}{8} & \\ & & & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 6 \\ 25 \\ -11 \\ -11 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{5}{2} \\ -\frac{11}{8} \\ -\frac{11}{5} \end{bmatrix}$$

$$\mathbf{x}_j^{(1)} = G_j \mathbf{x}_j^{(0)} + c_j = c_j = \begin{bmatrix} \frac{3}{5} \\ \frac{5}{2} \\ -\frac{11}{8} \\ -\frac{11}{5} \end{bmatrix}$$

$$\mathbf{x}_j^{(2)} = G_j \mathbf{x}_j^{(1)} + c_j = (G_j + I) c_j = \begin{bmatrix} 1 & -\frac{1}{2} & & \\ -\frac{1}{2} & 1 & \frac{2}{5} & \\ & \frac{1}{2} & 1 & \frac{1}{8} \\ & & \frac{1}{5} & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{5} \\ \frac{5}{2} \\ -\frac{11}{8} \\ -\frac{11}{5} \end{bmatrix} = \begin{bmatrix} -\frac{13}{20} \\ \frac{33}{20} \\ -\frac{2}{5} \\ -\frac{99}{40} \end{bmatrix}$$

$$(D - L)^{-1} = \begin{bmatrix} 10 & & & \\ 5 & 10 & & \\ & -4 & 8 & \\ & & -1 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \frac{1}{8} & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & 4 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \frac{1}{10} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -5 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{10} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & & & \\ -\frac{1}{20} & \frac{1}{10} & & \\ -\frac{1}{40} & \frac{1}{20} & \frac{1}{8} & \\ -\frac{1}{200} & \frac{1}{100} & \frac{1}{40} & \frac{1}{5} \end{bmatrix}$$

$$G_g = \begin{bmatrix} \frac{1}{10} & & & \\ -\frac{1}{20} & \frac{1}{10} & & \\ -\frac{1}{40} & \frac{1}{20} & \frac{1}{8} & \\ -\frac{1}{200} & \frac{1}{100} & \frac{1}{40} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} & -5 & & \\ & & 4 & \\ & & & 1 \\ & & & \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & & \\ 0 & \frac{1}{4} & \frac{2}{5} & \\ 0 & \frac{1}{8} & \frac{1}{5} & \frac{1}{8} \\ 0 & \frac{1}{40} & \frac{1}{25} & \frac{1}{40} \end{bmatrix}$$

$$c_g = \begin{bmatrix} \frac{1}{10} & & & \\ -\frac{1}{20} & \frac{1}{10} & & \\ -\frac{1}{40} & \frac{1}{20} & \frac{1}{8} & \\ -\frac{1}{200} & \frac{1}{100} & \frac{1}{40} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 6 \\ 25 \\ -11 \\ -11 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{11}{5} \\ -\frac{11}{40} \\ -\frac{451}{200} \end{bmatrix}$$

$$\mathbf{x}_g^{(1)} = G_g\mathbf{x}_g^{(0)} + c_g = c_g = \begin{bmatrix} \frac{3}{5} \\ \frac{11}{5} \\ -\frac{11}{40} \\ -\frac{451}{200} \end{bmatrix}$$

$$\mathbf{x}_g^{(2)} = G_g\mathbf{x}_g^{(1)} + c_g = (G_g + I)c_j = \begin{bmatrix} 1 & -\frac{1}{2} & & \\ 0 & \frac{5}{4} & \frac{2}{5} & \\ 0 & \frac{1}{8} & \frac{6}{5} & \frac{1}{8} \\ 0 & \frac{1}{40} & \frac{1}{25} & \frac{41}{40} \end{bmatrix} \begin{bmatrix} \frac{3}{5} \\ \frac{11}{5} \\ -\frac{11}{40} \\ -\frac{451}{200} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{66}{25} \\ -\frac{539}{1600} \\ -\frac{18139}{8000} \end{bmatrix}$$

Answers!!

$\mathbf{x}_j^{(1)} = \begin{bmatrix} \frac{3}{5} & \frac{5}{2} & -\frac{11}{8} & -\frac{11}{5} \end{bmatrix}^\top$

$\mathbf{x}_j^{(2)} = \begin{bmatrix} -\frac{13}{20} & \frac{33}{20} & -\frac{2}{5} & -\frac{99}{40} \end{bmatrix}^\top$

$\mathbf{x}_g^{(1)} = \begin{bmatrix} \frac{3}{5} & \frac{11}{5} & -\frac{11}{40} & -\frac{451}{200} \end{bmatrix}^\top$

$\mathbf{x}_g^{(2)} = \begin{bmatrix} -\frac{1}{2} & \frac{66}{25} & -\frac{539}{1600} & -\frac{18139}{8000} \end{bmatrix}^\top$

(b) See Python code

(c) The Gauss-Seidel method took half as many iterations to converge within the prescribed tolerance of 10^{-6}.
Jacobian method: 4115 iterations
GS method: 1983 iterations

In [1]:
```
import numpy as np
from numpy import linalg
from matplotlib import pyplot as plt
from math import sqrt
```

In [2]:
```
def LDU(A):
    D=np.diag(np.diagonal(A))
    L=-1*(np.tril(A)-D)
    U=-1*(np.triu(A)-D)
    return D,L,U
```

In [3]:
```
def Jacobi(A,b,x_0,tol=1e-6,n_max=10000):
    D,L,U=LDU(A)
    n=0
    D_inv=linalg.inv(D)
    G_j=np.matmul(D_inv,L+U)
    c_j=D_inv.dot(b)
    while linalg.norm(A.dot(x_0)-b,np.inf)>=tol and n<n_max:

        x_0=G_j.dot(x_0)+c_j
        n+=1
    return x_0,linalg.norm(A.dot(x_0)-b,np.inf),n
```

In [4]:
```
def GS(A,b,x_0,tol=1e-6,n_max=10000):
    D,L,U=LDU(A)
    n=0
    D_L_inv=linalg.inv(D-L)
    G_g=np.matmul(D_L_inv,U)
    c_g=D_L_inv.dot(b)
    while linalg.norm(A.dot(x_0)-b,np.inf)>=tol and n<n_max:
        x_0=G_g.dot(x_0)+c_g
        n+=1
    return x_0,linalg.norm(A.dot(x_0)-b,np.inf),n
```

In [5]:
```
def BVP(f,a,b,c,alpha,beta,numpts):
    xvec=np.linspace(a,b,numpts+1)
    h=xvec[1]-xvec[0]
    bvec=f(xvec[1:-1])
    bvec[0]=bvec[0]-alpha/h**2
    bvec[-1]=bvec[-1]-beta/h**2
    A=-1*(2/h**2+c)*np.identity(numpts-1)+np.diag((1/h**2)*np.ones(numpts-2),k=1
    return A,bvec
```

In [6]:
```
f= lambda x:x*0
```

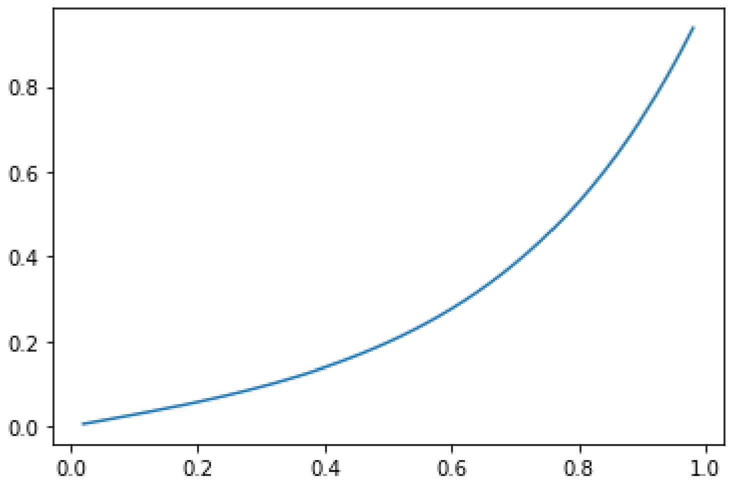
In [14]:
```
A,b=BVP(f,0,1,10,0,1,50)
```

In [15]:
```
u_j,res_j,n_j=Jacobi(A,b,np.zeros(np.shape(A)[0]))
```

In [16]:
```
u_g,res_g,n_g=GS(A,b,np.zeros(np.shape(A)[0]))
```

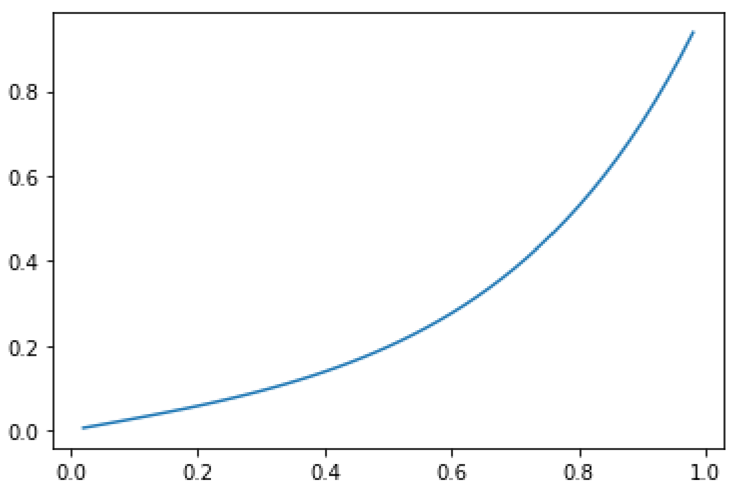
In [17]:
```
plt.plot(np.linspace(0,1,len(u_j)+2)[1:-1],u_j)
```

Out[17]:
```
[<matplotlib.lines.Line2D at 0x7fdbe83d2c40>]
```



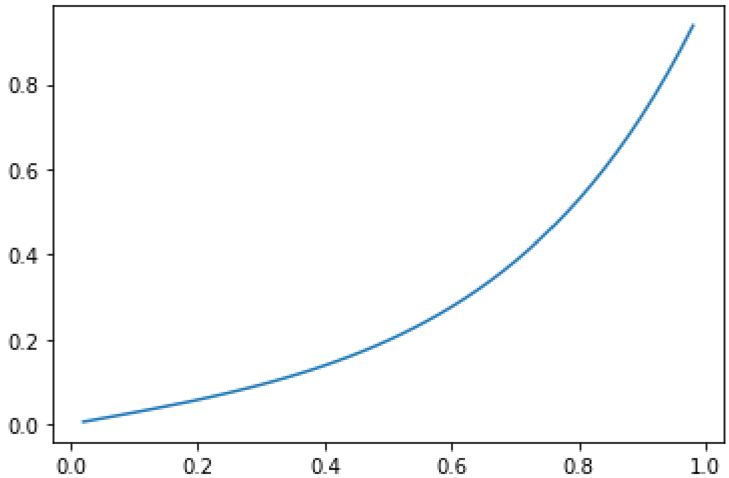
In [18]:
```
plt.plot(np.linspace(0,1,len(u_g)+2)[1:-1],u_g)
```

Out[18]:
```
[<matplotlib.lines.Line2D at 0x7fdc487adeb0>]
```



In [19]:
```
plt.plot(np.linspace(0,1,np.shape(A)[0]+2)[1:-1],linalg.inv(A).dot(b))
```

Out[19]:
```
[<matplotlib.lines.Line2D at 0x7fdc4888e310>]
```



In [20]:
```
n_j
```

Out[20]: 4115

In [21]:
```
n_g
```

Out[21]: 1983

In [ ]: