Math 106: Problem Set 1

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1.2.3 WLOG let $m \in \mathbb{Z}$. If m is even, then $m \equiv 0 \mod (2) \Rightarrow m = 2k$ for some $k \in \mathbb{Z}$. Otherwise, m is odd, so $m \equiv 1 \mod (2) \Rightarrow m = 2k+1$ for some $k \in \mathbb{Z}$. Thus, because m is arbitrary, m^2 can be any perfect square. Either $m^2 = (2k+1)^2 = 4k^2 + 4k + 1 \equiv 1 \mod (4)$ because $4k^2$ and $4k$ are clearly divisible by 4, or $m^2 = 4k^2 \equiv 0 \mod (4)$ because $4k^2$ is clearly divisible by 4. Hence, every perfect square leaves remainder 0 or 1 on division by 4.

1.2.4 WLOG let a be odd. Suppose for the sake of contradiction b is also odd. Since a^2 and b^2 are odd, then c^2 must be even. It follows $c^2 \equiv 0 \mod (4)$ and $b^2 \equiv 1 \mod (4)$ by **1.2.3**. Thus, $c^2 - b^2 \equiv 3 \mod (4)$. However, as we showed in **1.2.3**, $a^2 \equiv 1 \mod (4)$, so we obtain a contradiction. Thus, b cannot be odd. Hence, both a and b cannot both be odd.

1.3.1 If (a, b, c) is a pythagorean triple, then $\frac{a}{c} = \frac{1-t^2}{1+t^2}, \frac{b}{c} = \frac{2t}{1+t^2}$ where $t = \frac{q}{p}$ for some integers p, q. Substituting $\frac{q}{p}$ we obtain $\frac{a}{c} = \frac{1-\frac{q^2}{p^2}}{1+\frac{q^2}{p^2}}, \frac{b}{c} = \frac{2\frac{q}{p}}{1+\frac{q^2}{p^2}}$. Multiplying by $\frac{p^2}{p^2}$, $\frac{a}{c} = \frac{p^2-q^2}{p^2+q^2}, \frac{b}{c} = \frac{2pq}{p^2+q^2}$.

1.3.2 If $\frac{a}{c} = \frac{p^2-q^2}{p^2+q^2}, \frac{b}{c} = \frac{2pq}{p^2+q^2}$ from **1.3.1**, then if we set $c := r(p^2+q^2)$ then $a = r(p^2-q^2)$ and $b = 2rpq$

1.3.4 $\cos(\theta) = \frac{x}{1} = \frac{1-t^2}{1+t^2}, \sin(\theta) = \frac{y}{1} = \frac{2t}{1+t^2}$ by the solution pair $(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$. $\Rightarrow \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{\frac{2t}{1+t^2}}{\frac{1+t^2}{1+t^2}+\frac{1-t^2}{1+t^2}} = \frac{\frac{2t}{1+t^2}}{\frac{2}{1+t^2}} = t$

1.4.2 Let h be the shared side of $\triangle ac_1h$ and $\triangle bc_2h$. Let α_1 be the angle opposite c_1 and β_1 be the angle opposite c_2. Let α_2 and β_2 be the angle opposite h for $\triangle ac_1h$ and $\triangle bc_2h$ respectively. $\alpha_1 + \beta_1 = 90^\circ \Rightarrow \alpha_1 = \beta_2, \alpha_2 = \beta_1$. Thus, $\triangle ac_1h, \triangle bc_2h, \triangle ab(c_1+c_2)$ are all similar. It follows $\frac{a}{c_1+c_2} = \frac{c_1}{a}$ and $\frac{b}{c_1+c_2} = \frac{c_2}{b} \Rightarrow a^2 = c_1(c_1+c_2)$ and $b^2 = c_2(c_1+c_2) \Rightarrow a^2+b^2 = (c_1+c_2)^2$

1.5.1 For some arbitrary odd integer m, there exists some integer q such that $m = 2q+1$. Squaring both sides we obtain $m^2 = (2q+1)^2 = 4q^2+4q+1 = 2r+1$ where $r = 2q^2+2q$. Since m^2 can be written in the form $m^2 = 2r+1$ for some integer r, m^2 is also odd.

1.5.2 Squaring $2q+1$ we obtain $4q^2+4q+1=4s+1$ where $s=q^2+q$. It follows $4 \mid ((2q+1)^2-1) \Rightarrow (2q+1)^2 \equiv 1 \pmod 4$. Any odd integer $m \equiv 1 \pmod 4$ or $m \equiv 3 \pmod 4$. Any even integer $m \equiv 0 \pmod 4$ or $m \equiv 2 \pmod 4$. Thus, $m^2 \equiv 1^2 \pmod 4, 3^2 \pmod 4 \Rightarrow m^2 \equiv 1 \pmod 4$ if m is odd, and $m^2 \equiv 0^2 \pmod 4, 2^2 \pmod 4 \Rightarrow m^2 \equiv 0 \pmod 4$ if m is even.

1.6.1 Let x_1, x_2 be real numbers with the same sign or 0, and let y_1, y_2 be real numbers that satisfy $x_1y_2 = x_2y_1$. WLOG by translation let $A := (-x_1, -y_1), B := (0,0), C := (x_2, y_2)$. Let $t \in [0,1]$ and $((x_2+x_1)t - x_1, (y_2+y_1)t - y_1)$ be the set of points between A and C. If $x_1 = 0$ or $x_2 = 0$ then either A or C must be at the origin or both A and C must be on the line $x=0$. In either case, A, B, C are colinear and $AB + BC = AC \Leftrightarrow \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2} = \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2}$. Otherwise, set $t = \frac{x_1}{x_1+x_2}$.
$(y_2+y_1)t - y_1 = \frac{(y_2+y_1)x_1}{x_1+x_2} - \frac{y_1(x_1+x_2)}{x_1+x_2} = \frac{y_2x_1 - y_1x_2}{x_1+x_2} = 0$. Thus, A, B, C are all colinear. Moreover, $AB + BC = AC \Leftrightarrow \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2} = \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2}$.

1.6.2 Let $L = \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$
$\Rightarrow x_1^2+y_1^2 = L^2 - 2L\sqrt{x_2^2+y_2^2} + x_2^2 + y_2^2$
$\Rightarrow 2L\sqrt{x_2^2+y_2^2} = L^2 + x_2^2 + y_2^2 - x_1^2 - y_1^2$
$\Rightarrow 4L^2(x_2^2+y_2^2) = L^4 + 2L^2(x_2^2+y_2^2-x_1^2-y_1^2) + (x_2^2+y_2^2-x_1^2-y_1^2)^2$
$\Rightarrow 0 = L^4 - 2L^2(x_1^2+x_2^2+y_1^2+y_2^2) + (x_2^2+y_2^2-x_1^2-y_1^2)^2$
$(x_2^2+y_2^2-x_1^2-y_1^2)^2$
$= x_1^4 + x_2^4 + y_1^4 + y_2^4 + 2x_2^2y_2^2 - 2x_1^2x_2^2 - 2x_2^2y_1^2 - 2y_2^2x_1^2 - 2y_1^2y_2^2 + 2x_1^2y_1^2$
$(x_1^2+x_2^2+y_1^2+y_2^2)^2$
$= x_1^4 + x_2^4 + y_1^4 + y_2^4 + 2x_1^2x_2^2 + 2x_1^2y_1^2 + 2x_2^2y_1^2 + 2x_2y_2^2 + 2y_1^2y_2^2 + 2x_1^2y_2^2$
$\Rightarrow L^2 = (x_2^2+y_2^2+x_1^2+y_1^2) \pm \sqrt{4x_1^2x_2^2 + 4x_2^2y_1^2 + 4y_1^2y_2^2 + 4x_1^2y_2^2}$
$\Rightarrow L^2 = (x_2^2+y_2^2+x_1^2+y_1^2) \pm \sqrt{4x_1^2x_2^2 + 4y_1^2y_2^2 + 8x_1x_2y_1y_2}$
$\Rightarrow L^2 = (x_2^2+y_2^2+x_1^2+y_1^2) \pm 2\sqrt{(x_1x_2+y_1y_2)^2}$
$\Rightarrow L^2 = (x_1 \pm x_2)^2 + (y_1 \pm y_2)^2$ will only consider $+$
$\Rightarrow L = \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2}$ which is equal to the RHS