Math 164: Problem Set 9

Owen Jones

March 3, 2024

12.18 The problem is equivalent to minimizing $\frac{1}{2}\|\mathbf{x}-\mathbf{b}\|^2$. If $\mathbf{x} \in \mathcal{R}(\mathbf{A})$, $\exists \mathbf{y} \in \mathbb{R}^n$ s.t $\mathbf{Ay} = \mathbf{x}$
It follows $\mathbf{x}^* = \mathbf{Ay}^* = \mathbf{A}(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{b}$ using the least squares method with with variable $\mathbf{y}$.

12.20 Let $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$, so our problem becomes
$\min\limits_{\mathbf{Ay}=\mathbf{b}-\mathbf{Ax}_0} \|\mathbf{y}\|$. Using Theorem 12.2, $\mathbf{y}^* = \mathbf{A}^\top(\mathbf{AA}^\top)^{-1}(\mathbf{b} - \mathbf{Ax}_0)$ is a unique minimizer to our constrained minimization problem. Since $\mathbf{x}^* = \mathbf{y}^* + \mathbf{x}_0$,
$\mathbf{x}^* = \mathbf{A}^\top(\mathbf{AA}^\top)^{-1}(\mathbf{b} - \mathbf{Ax}_0) + \mathbf{x}_0$
$= \mathbf{A}^\top(\mathbf{AA}^\top)^{-1}\mathbf{b} - \mathbf{A}^\top(\mathbf{AA}^\top)^{-1}\mathbf{Ax}_0 + \mathbf{I}_n\mathbf{x}_0$
$= \mathbf{A}^\top(\mathbf{AA}^\top)^{-1}\mathbf{b} + (\mathbf{I}_n - \mathbf{A}^\top(\mathbf{AA}^\top)^{-1}\mathbf{A})\mathbf{x}_0$

12.23 Let $\mathbf{y} \in \mathcal{R}(\mathbf{A}^\top)$ satisfy $\mathbf{Ay} = \mathbf{b}$. It follows $\exists \mathbf{z} \in \mathbb{R}^m$ s.t $\mathbf{A}^\top\mathbf{z} = \mathbf{y}$.
Since $\mathbf{Ay}$ and $\mathbf{Ax}^*$ both equal $\mathbf{b}$
$\mathbf{Ay} - \mathbf{Ax}^* = \mathbf{AA}^\top\mathbf{z} - \mathbf{A}(\mathbf{A}^\top(\mathbf{AA}^\top)^{-1}\mathbf{b}) = 0$
$\Rightarrow \mathbf{AA}^\top(\mathbf{z} - (\mathbf{AA}^\top)^{-1}\mathbf{b}) = 0$
$\Rightarrow \mathbf{z} = (\mathbf{AA}^\top)^{-1}\mathbf{b}$ because $\mathbf{AA}^\top$ is invertible.
Thus, $\mathbf{y} = \mathbf{A}^\top\mathbf{z} = \mathbf{A}^\top(\mathbf{AA}^\top)^{-1}\mathbf{b} = \mathbf{x}^*$

20.2 (a) Let $f(\mathbf{x}) = x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3$,
$h(\mathbf{x}) = [x_1 + 2x_2 - 3, 4x_1 + 5x_3 - 6]^\top$,
and $l(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda^\top h(\mathbf{x})$.
We first want to find an $(\mathbf{x}^*, \lambda^*)$ that satisfies the Lagrange condition:

$$Dl(\mathbf{x}^*, \lambda^*) = [D_x l(\mathbf{x}^*, \lambda^*), D_\lambda(\mathbf{x}^*, \lambda^*)] = \mathbf{0}^\top$$

It follows

$$Dl(\mathbf{x},\lambda) = \begin{bmatrix} 2 & 2 & 0 & 1 & 4 \\ 2 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 2 & 0 & 0 & 0 \\ 4 & 0 & 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} -4 \\ -5 \\ -6 \\ 3 \\ 6 \end{bmatrix} = \mathbf{0}^\top$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & 0 & 1 & 4 \\ 2 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 2 & 0 & 0 & 0 \\ 4 & 0 & 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ -6 \\ 3 \\ 6 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{16}{5} \\ -\frac{1}{10} \\ -\frac{34}{25} \\ -\frac{27}{5} \\ -\frac{6}{5} \end{bmatrix}$$

by Gaussian elimination.
Next, we want to show this point satisfies the SONC

$$L(\mathbf{x}^*,\lambda^*) = \mathbf{F}(\mathbf{x}^*) + \lambda^*\mathbf{H}(\mathbf{x}^*) = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

because $H_k(\mathbf{x}^*) = 0, k = 1,2$

$$T(\mathbf{x}^*) := \{\mathbf{v} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 5 \end{bmatrix} \mathbf{v} = 0\} = span\{[\frac{5}{4}, -\frac{5}{8}, 1]^\top\}$$

$$\forall \mathbf{v} \in T(\mathbf{x}^*), \mathbf{v}^\top L(\mathbf{x}^*,\lambda^*)\mathbf{v} = \alpha^2[\frac{5}{4}, -\frac{5}{8}, 1]^\top \begin{bmatrix} 2 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{5}{4} \\ -\frac{5}{8} \\ 1 \end{bmatrix}$$

$$= \alpha^2[\frac{5}{4}, -\frac{5}{8}, 1]^\top \begin{bmatrix} \frac{5}{4} \\ -\frac{5}{4} \\ 0 \end{bmatrix} = \frac{75}{32}\alpha^2 > 0, \forall \alpha \neq 0$$

, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{16}{5} \\ -\frac{1}{10} \\ -\frac{34}{25} \\ -\frac{27}{5} \\ -\frac{6}{5} \end{bmatrix}$$

is a strict local minimizer.

(b) Let

$$
\begin{aligned}
&f(x) = 4x_1 + x_2^2\\
&h(x) = x_1^2 + x_2^2 - 9\\
&l(x,\lambda) = f(x) + \lambda h(x)\\
&\Rightarrow \nabla l(x,\lambda) = \begin{bmatrix} 4+2\lambda x_1 \\ 2x_2 + 2\lambda x_2 \\ x_1^2 + x_2^2 - 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We first observe $\lambda \neq 0$. If so, we contradict the first innequality. Thus, $x_1 = -\frac{2}{\lambda}$.
We have two cases for x_2. $x_2 = 0$ or $x_2 \neq 0$
The first case gives the first two candidates $[3,0]^\top \lambda = -\frac{2}{3}, [-3,0]^\top \lambda = \frac{2}{3}$.
The second case gives the second two candidates $[2,\sqrt{5}]^\top \lambda = -1, [2,-\sqrt{5}]^\top \lambda = -1$.
All 4 candidates are regular.

$$
\begin{aligned}
&L(\mathbf{x},\lambda) = \mathbf{F}(\mathbf{x}) + \lambda \mathbf{H}(\mathbf{x}) = \begin{bmatrix} 2\lambda & 0 \\ 0 & 2+2\lambda \end{bmatrix}\\
&T(\mathbf{x}) = \{\mathbf{v} \in \mathbb{R}^2 : [2x_1, 2x_2]\mathbf{v} = 0\}
\end{aligned}
$$

$$
\begin{aligned}
&L([3,0]^\top, -\frac{2}{3}) = \begin{bmatrix} -\frac{4}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}\\
&T([3,0]^\top) = span\{[0,1]^\top\}\\
&\forall \mathbf{v} \in T([3,0]^\top) \neq \mathbf{0}, \mathbf{v}^\top \begin{bmatrix} -\frac{4}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \mathbf{v} = \alpha^2 \frac{4}{9} > 0\\
&L([-3,0]^\top, \frac{2}{3}) = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{10}{3} \end{bmatrix} > 0\\
&L([2,\sqrt{5}]^\top, -1) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\\
&T([2,\sqrt{5}]^\top) = span\{[-\sqrt{5}, 2]^\top\}\\
&\forall \mathbf{v} \in T([2,\sqrt{5}]^\top) \neq \mathbf{0}, \mathbf{v}^\top \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{v} = -\alpha^2 10 < 0\\
&L([2,\sqrt{5}]^\top, -1) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\\
&T([2,-\sqrt{5}]^\top) = span\{[\sqrt{5}, 2]^\top\}\\
&\forall \mathbf{v} \in T([2,\sqrt{5}]^\top) \neq \mathbf{0}, \mathbf{v}^\top \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{v} = -\alpha^2 10 < 0
\end{aligned}
$$

Thus, $[3,0]^\top \lambda = -\frac{2}{3}, [-3,0]^\top \lambda = \frac{2}{3}$ are strict local minimizers and $[2,\sqrt{5}]^\top \lambda = -1, [2,-\sqrt{5}]^\top \lambda = -1$ are strict local maximizers.

(c)

$$
\begin{aligned}
f(x) &= x_1 x_2 \\
h(x) &= x_1^2 + 4x_2^2 - 1 \\
l(x,\lambda) &= -x_1 x_2 + \lambda x_1^2 + 4\lambda x_2^2 \\
\nabla l(x,\lambda) &= \begin{bmatrix} x_2 + 2\lambda x_1 \\ x_1 + 8\lambda x_2 \\ x_1^2 + 4x_2^2 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$x_1 = -8\lambda x_2 \Rightarrow x_2(1 - 16\lambda^2) = 0$. $x_2 \neq 0$ by inspection.
$\lambda = \frac{1}{4} \Rightarrow [\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}]^\top, [-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}]^\top$ are solutions.
$\lambda = -\frac{1}{4} \Rightarrow [\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}]^\top, [-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}]^\top$ are solutions.

$$
\begin{aligned}
L(\mathbf{x}, \frac{1}{4}) &= \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix} \\
L(\mathbf{x}, -\frac{1}{4}) &= \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & -2 \end{bmatrix} \\
T(\mathbf{x}) &= \{\mathbf{v} : [2x_1, 8x_2]\mathbf{v} = 0\} \\
T([\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}]^\top) &= T([-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}]^\top) = span\{[1,1]^\top\} \\
T([\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}]^\top) &= T([-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}]^\top) = span\{[1,-1]^\top\}
\end{aligned}
$$

For $\mathbf{v} \in span\{[1,-1]^\top\}, \mathbf{v}^\top L(\mathbf{x}, -\frac{1}{4})\mathbf{v} < 0$
For $\mathbf{v} \in span\{[1,1]^\top\}, \mathbf{v}^\top L(\mathbf{x}, \frac{1}{4})\mathbf{v} > 0$
Thus, $[\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}]^\top, [-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}]^\top$ are strict local maximizers and $[\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}]^\top, [-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}]^\top$ are strict local minimizers.

20.3 $l(\mathbf{x},\lambda) = (\mathbf{a}^\top\mathbf{x})(\mathbf{b}^\top\mathbf{x}) + \lambda_1(x_1 + x_2) + \lambda_2(x_2 + x_3) = 0$

$$\nabla l = \begin{bmatrix} \nabla_{\mathbf{x}} l \\ h(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} (\mathbf{a}\mathbf{b}^\top + \mathbf{b}\mathbf{a}^\top)\mathbf{x} + [\nabla h_1, \nabla h_2]\lambda \\ h(\mathbf{x}) \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \lambda \\ h(\mathbf{x}) \end{bmatrix}$$

$$= \begin{bmatrix} x_2 + \lambda_1 \\ x_1 + x_2 + \lambda_1 + \lambda_2 \\ x_2 + \lambda_2 \\ x_1 + x_2 \\ x_2 + x_3 \end{bmatrix} = \mathbf{0}$$

By inspection, $\mathbf{x}^* = \mathbf{0}, \lambda^* = \mathbf{0}$ solves the system uniquely.

$L(\mathbf{x}^*, \lambda^*) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$T(\mathbf{x}^*) = span\{[1, -1, 1]^\top\}$

$\forall \mathbf{v} \in T(\mathbf{x}^*), \mathbf{v}^\top L(\mathbf{x}^*, \lambda^*)\mathbf{v} = -4\alpha^2 < 0$, so $\mathbf{0}$ is a strict local maximizer.

20.4 $\nabla l(\mathbf{x}^\top, \lambda^\top) = \begin{bmatrix} x_1 + \lambda \\ x_1 + 4 + 4\lambda \\ h(\mathbf{x}^*) \end{bmatrix} = \mathbf{0}$

$\Rightarrow x_1 = \frac{4}{3} \Rightarrow \nabla f(\mathbf{x}^*) = [\frac{4}{3}, \frac{16}{3}]^\top$

20.9 $f(\mathbf{x}) = \dfrac{\mathbf{x}^\top \begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix} \mathbf{x}}{\mathbf{x}^\top \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}}$

Observe that if $\mathbf{x}^*$ is a maximizer, $\alpha\mathbf{x}$ for some scalar α is also a maximizer because

$$f(\alpha\mathbf{x}^*) = \frac{(\alpha\mathbf{x}^*)^\top \begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix} (\alpha\mathbf{x}^*)}{(\alpha\mathbf{x}^*)^\top \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} (\alpha\mathbf{x}^*)}$$

$$= \frac{\alpha^2}{\alpha^2} \frac{\mathbf{x}^\top \begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix} \mathbf{x}}{\mathbf{x}^\top \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}} = f(\mathbf{x}^*)$$

It follows we can solve an equivalent problem to maximize $\mathbf{x}^\top \begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix} \mathbf{x}$ constrained to $\mathbf{x}^\top \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x} = 1$

$l(\mathbf{x}, \lambda) = \mathbf{x}^\top \begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix} \mathbf{x} + \lambda(1 - \mathbf{x}^\top \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x})$

$\nabla l_x = 2(\begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix} - \lambda \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix})\mathbf{x} = \mathbf{0}$
$\Leftrightarrow (\lambda \mathbf{I}_2 - \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix})\mathbf{x} = 0$
which is equivalent to solving an eigenvalue eigenvector problem.
$\det(\lambda \mathbf{I}_2 - \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}) = (\lambda - 10)(\lambda - 5)$ $\lambda = 10$ will give us a much larger value.
$\mathbf{x}^* = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ gives us a solution that satisfies $h(\mathbf{x}^*) = 0$.
Thus, any scalar multiple of $\mathbf{x}^*$ will give us a maximizer.

20.10 Let $\mathbf{Q}_0 = \frac{1}{2}(\begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 4 & 3 \end{bmatrix})$
$\nabla l(\mathbf{x}^*, \lambda^*) = 2\mathbf{Q}_0\mathbf{x}^* - 2\lambda^*\mathbf{x} = 0$
which is equivalent to solving for $\mathbf{Q}_0$ eigenvalues and corresponding eigenvectors.
$\det \begin{bmatrix} \lambda - 3 & -2 \\ -2 & \lambda - 3 \end{bmatrix} = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$
Thus, the maximizers are the eigenvectors for $\lambda = 5$ constrained to $\|x\| = 1$
$[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}], [-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]$

20.14 Suppose $\mathbf{x}^* = [1, 1]$ is a solution to $\nabla l(\mathbf{x}, \lambda) = \begin{bmatrix} a + 2\lambda x_1 \\ b + 2\lambda x_2 \\ x_1^2 + x_2^2 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
It follows $a = -2\lambda$ and $b = -2\lambda$, so $a = b$.