Math 106: Problem Set 5

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Quadratic Formula

- $x^2 + px + c = 0$
  $x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} - c$
  $(x + \frac{p}{2})^2 = \frac{p^2}{4} - c$
  $x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - c}$
  $x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - c}$
  $x = \frac{-p\pm\sqrt{p^2-4c}}{2}$

- $ax^2 + bx + c = 0$
  $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
  Let $p = \frac{b}{a}, c = \frac{c}{a}$
  $x = \frac{-p\pm\sqrt{p^2-4c}}{2} \Rightarrow x = \frac{-\frac{b}{a}\pm\sqrt{\frac{b^2}{a^2}-4\frac{c}{a}}}{2}$
  $x = \frac{-\frac{b}{a}\pm\frac{1}{a}\sqrt{b^2-4ac}}{2}$
  $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

6.4.4 Suppose $\sqrt[3]{2} = a+b\sqrt{c}$. Cubing both sides, $2 = a^3+3a^2b\sqrt{c}+3ab^2c+b^3c\sqrt{c}$. $(a^3 + 3ab^2c - 2) + (3a^2b + b^3c)\sqrt{c} = 0 \Leftrightarrow a^3 + 3ab^2c - 2 = 3a^2b + b^3c = 0$ from **6.4.3**. Thus, $a^3 + 3ab^2c = 2, 3a^2b + b^3c = 0$.

6.4.5 Because $3a^2b+b^3c = 0 \Rightarrow -3a^2b-b^3c = 0$. Thus, $2 = a^3-3a^2b\sqrt{c}+3ab^2c-b^3c\sqrt{c}$. Taking the cube root of both sides $\sqrt[3]{2} = \sqrt[3]{a^3 - 3a^2b\sqrt{c} + 3ab^2c - b^3c\sqrt{c}} = a-b\sqrt{c}$. This is a contradiction because $2\sqrt[3]{2} = a+b\sqrt{c}+a-b\sqrt{c} = 2a \Rightarrow \sqrt[3]{2} = a$ which is impossible because $\sqrt[3]{2} \notin F_k$ but $a \in F_k$.

6.5.2 $y^3 = 2 \Rightarrow p = 0, q = 2$
$y = \sqrt[3]{\frac{2}{2} + \sqrt{(\frac{2}{2})^2 - (\frac{0}{3})^3}} + \sqrt[3]{\frac{2}{2} - \sqrt{(\frac{2}{2})^2 - (\frac{0}{3})^3}}$
$= \sqrt[3]{1 + \sqrt{1}} + \sqrt[3]{1 - \sqrt{1}} = \sqrt[3]{2}$.

6.5.3 $y = \sqrt[3]{\frac{6}{2} + \sqrt{(\frac{6}{2})^2 - (\frac{6}{3})^3}} + \sqrt[3]{\frac{6}{2} - \sqrt{(\frac{6}{2})^2 - (\frac{6}{3})^3}}$
$= \sqrt[3]{3+1} + \sqrt[3]{3-1} = \sqrt[3]{4} + \sqrt[3]{2}$
$\Rightarrow (\sqrt[3]{4} + \sqrt[3]{2})^3 = 4 + 3\sqrt[3]{32} + 3\sqrt[3]{16} + 2 = 6 + 6(\sqrt[3]{4} + \sqrt[3]{2})$

6.7.1 $x^n - a^n = x^n - a^n + \sum_{i=1}^{n-1} a^i x^{n-i} - a^i x^{n-i}$
$= x^n - ax^{n-1} + ax^{n-1} - a^2x^{n-2} + a^2x^{n-2} + \cdots + a^{n-1}x - a^n$
$= (x-a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-1})$
$\frac{x^n - a^n}{x-a}$ is the sum of a geometric series with $a_0 = a^{n-1}$ and common ratio $r = \frac{x}{a}$.

6.7.2 $p(x) - p(a) = \sum_{i=0}^{k} a_i(x^i - a^i)$. We showed that $(x-a) \mid (x^i - a^i)$ for $i \in \mathbb{N}$ in **6.7.1**, so $(x-a)$ divides a linear combination of $(x^i - a^i)$. Thus, $(x-a) \mid \sum_{i=0}^{k} a_i(x^i - a^i) \Rightarrow (x-a) \mid p(x) - p(a)$.

6.7.3 Suppose $p(a) = 0$. By **6.7.2** we have $(x-a) \mid p(x) - p(a)$. However, $p(x) - p(a) = p(x) - 0 = p(x)$, so $(x-a) \mid p(x)$.