Math 100: Problem Set 9

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(Q-1) Show $f(2^{\frac{n}{2}}) = [f(1)]^{2^n}$ by induction.
Base case: $f(\sqrt{1^2+1^2}) = f(2^{\frac{1}{2}}) = f(1)f(1) = [f(1)]^{2^1}$
Induction hypothesis: Assume for some n $f(2^{\frac{n}{2}}) = [f(1)]^{2^n}$
Induction step: WTS claim holds for $n \pm 1$.
$(n-1) : [f(1)]^{2^{n-1}} = ([f(1)]^{2^n})^{\frac{1}{2}} = (f(2^{\frac{n}{2}}))^{\frac{1}{2}} = (f(\sqrt{2^{n-1}+2^{n-1}}))^{\frac{1}{2}} = (f(2^{\frac{n-1}{2}})f(2^{\frac{n-1}{2}}))^{\frac{1}{2}} = f(2^{\frac{n-1}{2}})$
$(n+1) : f(2^{\frac{n+1}{2}}) = f(\sqrt{2^n+2^n}) = f(2^{\frac{n}{2}})^2 = [f(1)]^{2^n 2} = [f(1)]^{2^{n+1}}$
Lemma: $f(\sqrt{x_1^2+x_2^2+\cdots+x_n^2}) = f(x_1)f(x_2)\cdots f(x_n)$
Base case: $f(\sqrt{x^2+y^2}) = f(x)f(y)$ is given.
Induction hypothesis: Assume $f(\sqrt{x_1^2+x_2^2+\cdots+x_n^2}) = f(x_1)f(x_2)\cdots f(x_n)$
Induction step: $f(\sqrt{x_1^2+x_2^2+\cdots+x_n^2+x_{n+1}^2}) = f((\sqrt{x_1^2+x_2^2+\cdots+x_n^2})^2 + x_{n+1}^2) = f(\sqrt{x_1^2+x_2^2+\cdots+x_n^2})f(x_{n+1}) = f(x_1)f(x_2)\cdots f(x_n)f(x_{n+1})$
Thus, the claim holds for all n.
WTS $f(\sqrt{\frac{m}{2^n}}) = [f(1)]^{\frac{m}{2^n}}$ by induction on m
Base case: $f(\sqrt{2^{-n}}) = [f(1)]^{2^{-n}}$ by previous proof.
Induction hypothesis: Assume for some m $f(\sqrt{\frac{m}{2^n}}) = [f(1)]^{\frac{m}{2^n}}$ is true.
Induction step: WTS claim holds for $m \pm 1$
$(m+1) : f(\sqrt{\frac{m+1}{2^n}}) = f(\sqrt{\frac{m}{2^n}+\frac{1}{2^n}}) = [f(1)]^{\frac{m}{2^n}}[f(1)]^{\frac{1}{2^n}} = [f(1)]^{\frac{m+1}{2^n}}$
$(m-1) : [f(1)]^{\frac{m-1}{2^n}} = ([f(1)]^{\frac{m}{2^n}})^{\frac{m-1}{m}} = (f(\sqrt{\frac{m}{2^n}}))^{\frac{m-1}{m}} = ((f(\sqrt{\frac{1}{2^n}}))^m)^{\frac{m-1}{m}} = ([f(1)]^{\frac{1}{2^n}})^{m-1} = [f(1)]^{\frac{m-1}{2^n}}$
For any $x \in \mathbb{R}$ we can can construct a sequence S_k where each s_k is of the form $\sqrt{\frac{m}{2^n}}$ that converges to x. By the continuity of f, $\lim_{k\to\infty} S_k = x \Rightarrow \lim_{k\to\infty} f(S_k) = f(x)$.

(Q-2) (a) Let $x_0 \in [0,1] \cap \mathbb{Q}$ be arbitrary, and let $\epsilon = f(x_0)$. Suppose $\delta > 0$. By the denseness of real numbers, there exists $x \in \mathbb{I}$ s.t $|x - x_0| < \delta$. It follows $|f(x) - f(x_0)| = f(x_0) \geq \epsilon$. Hence, $f(x)$ is discontinuous for all rationals in $[0,1]$.

(b) Let $x_0 \in [0,1] \cap \mathbb{I}$ be arbitrary and let $\epsilon > 0$.

On the interval $(0,1)$, there are $q-1$ values of x such that $f(x) = \frac{1}{q}$. Since there are finitely many values of x s.t $f(x) = \frac{1}{q}$ for all $q \in \mathbb{N}$, we can pick a $\delta > 0$ small enough s.t for all $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \frac{1}{q}$ for all $q \in \mathbb{N}$. Thus, we can make $|f(x) - f(x_0)| < \epsilon$. Hence, $f(x)$ is continuous for $x \in [0,1] \cap \mathbb{I}$.

(Q-3) We assume $f(0) > 0$ and $f(1) < 1$. If either $f(0) = 0$ or $f(1) = 1$ are true, we are done. Let $g(x) := f(x) - x$. It follows $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$, so $g(1) < 0 < g(0)$. Note $x \in [0,1]$. By the continuity of g, there exists $\delta > 0$ s.t $|g(x) - g(0)| < |g(0)| \Rightarrow g(x) < 0$ for all $0 < x < \delta$. Moreover, there exists $\delta > 0$ s.t $|g(x) - g(1)| < g(1) \Rightarrow 0 < g(x)$ for all $0 < 1 - x < \delta$. Let $S := \{x \in [0,1] : g(x) > 0\}$, and let $c := \sup(S)$. By the definition of c, there exists some $x_0 \in (c - \delta, c)$ for $\delta > 0$ s.t $g(x_0) > g(c) \geq 0$. If there wasn't, $\sup(S) < c$. In addition, for all $x \in (c, c+\delta)$ $g(x) \leq 0$. It follows $g(c+\delta) \leq 0 \leq g(c) < g(x_0)$. By continuity, there exists $\delta > 0$ s.t $0 \leq g(c) \leq |g(c+\delta) - g(c)| < \epsilon \Rightarrow g(c) = 0 \Rightarrow f(c) = c$.

(Q-4) Let $f : [7am, 5pm] \times [0,1]$ and $g : [7am, 5pm] \times [0,1]$ be continuous functions for the path the hiker takes from the bottom to the top and top to bottom. WTS there is a point $c \in [7am, 5pm]$ where $f(c) = g(c)$. Let $h(x) := g(x) - f(x) \Rightarrow h(7am) = 1, h(5pm) = -1$. By the IVT, there exists $c \in (7am, 5pm)$ s.t $h(c) = 0 \Rightarrow f(c) = g(c)$.

(Q-5) (a) $g'(0) = \lim_{h\to 0} \frac{g(h)-g(0)}{h} = \lim_{h\to 0} \frac{h + 2h^2\sin(\frac{1}{h}) - 0}{h} = \lim_{h\to 0} 1 + 2h\sin(\frac{1}{h})$. Since we are taking the limit as h goes to 0, we can assume h is small, i.e $|h| < \frac{1}{2}$. $\sin(x)$ is bounded between -1 and 1, so $|2h\sin(\frac{1}{h})| < 1 \Rightarrow g'(0) > 1$.

(b) WTS for $\delta > 0$ there exists $0 < |c| < \delta$ s.t $g'(c) < 0$. $g'(c) = c + 4c\sin(\frac{1}{c}) - 2\cos(\frac{1}{c})$. Let $c = \frac{1}{2\pi k}$ for $k \in \mathbb{N}$ sufficiently large s.t $c < \delta$. Thus, $g(c) = \frac{1}{2\pi k} - 2 < 0$.

(Q-6) (a) Let $f(x) := 5x^4 - 4x + 1$. It follows $f(\frac{1}{2}) = 5(\frac{1}{2})^4 - 4(\frac{1}{2}) + 1 = \frac{5}{16} - 2 + 1 = \frac{21}{16} - 2 < 0$ and $f(1) = 2$. It follows by the IVT, there exists a root between $[\frac{1}{2}, 1]$.

(b) Let $f(x) := a_0 + a_1 x + \cdots + a_n x^n$ where $a_0 + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$. Let $g(x) : \frac{a_0}{1} + \frac{a_1}{2}x + \cdots + \frac{a_n x^{n+1}}{n+1}$ be an antiderivative of $f(x)$. Observe $g(0) = g(1) = 0$. It follows by Rolle's Theorem that there exists a point on the interval $[0,1]$ s.t $g'(x) = f(x) = 0$

(Q-7) $\lim_{n\to\infty} 4^n(1 - \cos(\frac{\theta}{2^n}))$

$= \lim_{n\to\infty} 4^n(1 - \sum_{k=0}^{\infty} \frac{(\frac{\theta}{2^n})^{2k}(-1)^k}{(2k)!})$

$= \lim_{n\to\infty} 4^n(\sum_{k=1}^{infty} \frac{(\frac{\theta}{2^n})^{2k}(-1)^{k-1}}{(2k)!})$

$= \lim_{n\to\infty} \sum_{k=1}^{infty} \frac{\theta^{2k}(-1)^{k-1}}{4^{n\,k-1}(2k)!}$

$= \lim_{n\to\infty} \frac{\theta^2}{2} + \sum_{k=2}^{infty} \frac{\theta^{2k}(-1)^{k-1}}{4^{n\,k-1}(2k)!} = \frac{\theta^2}{2} + 0 = \frac{\theta^2}{2}$

(Q-8) $L = \lim_{x\to\infty} x\int_0^x e^{t^2-x^2}dt = \infty\cdot 0$ because $\lim_{x\to\infty} x\int_0^x e^{t^2}dt = \infty$ and $\lim_{x\to\infty} e^{-x^2} = 0$. It follows by LH $\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}dt = \lim_{x\to\infty} \frac{xe^{x^2}+\int_0^x e^{t^2}dt}{2xe^{x^2}} = \frac{\infty}{\infty}$. By L'H $\lim_{x\to\infty} \frac{xe^{x^2}+\int_0^x e^{t^2}dt}{2xe^{x^2}} = \lim_{x\to\infty} \frac{2e^{x^2}+2x^2e^{x^2}}{2e^{x^2}+4x^2e^{x^2}} = \lim_{x\to\infty} \frac{1+x^2}{1+2x^2} = \frac{1}{2}$.

(Q-9) (a) $\lim_{n\to\infty}\sum_{k=1}^n \frac{n}{k^2+n^2}$. Observe $\frac{n}{k^2+n^2} \geq \frac{n}{2n^2} = \frac{1}{2n}$. The sum $\lim_{n\to\infty}\sum_{k=1}^n \frac{1}{2k}$ diverges, so $\lim_{n\to\infty}\sum_{k=1}^n \frac{n}{k^2+n^2}$ diverges.

(b) Let $L = \lim_{n\to\infty}(\prod_{k=1}^n(1+\frac{k}{n}))^{\frac{1}{n}} \Rightarrow \log(L) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \log(1+\frac{k}{n}) = \int_1^2 \log(x)dx = \log(\frac{4}{e}) \Rightarrow L = \frac{4}{e}$

(Q-10) Let $f(x) := 2x - \int_0^x f(t)dt - 1$. WTS there is one root on the interval $[0,1]$. It suffices to show where $f'(x) = 0$. $f'(x) = 2 - f(x) \Rightarrow e^x f'(x) + e^x f(x) = 2e^x \Rightarrow e^x f(x) = 2e^x + c \Rightarrow f(x) = 2 + ce^{-x}$. To find c we use $f(0) = -1 \Rightarrow c = -3 \Rightarrow f(x) = 2 - 3e^{-x}$. Because $f'(x) = 3e^{-x} > 0$, f is strictly increasing. Solving for $f(x) = 2 - 3e^{-x} = 0$, $2 = 3e^{-x} \Rightarrow x = \log(3/2)$.