MATH 114L: Homework 1

Questions

Question 1: Show that the connectives $\{\vee, \wedge\}$ can be expressed in terms of the connectives $\{\rightarrow, \neg\}$.

Question 2: For which natural numbers n are there elements of PL_0 of length n?

Question 3: (Unique readability) Show that a sequence ϕ is an element of PL_0 if and only if there is a finite sequence of sequences $(\phi_1, \ldots, \phi_n)$ such that $\phi_n = \phi$ and for each $i \leq n$ either there exists m such that $\phi_i = (A_m)$ or there exists $j < i$ such that $\phi_i = (\neg\phi_j)$ or there exist j_1, j_2 both less than i such that ϕ_i is equal to $(\phi_{j_1} \rightarrow \phi_{j_2})$.

Definition 2: For $\phi \in PL_0$ define:

- $C(\phi)$ = number of instances of logical connectives in ϕ,
- $S(\phi)$ = total number of symbols occurring in ϕ (i.e., $S(\phi)$ is just the length of ϕ),
- $D(\phi)$ = total number of instances of binary connectives which occur in ϕ,
- $E(\phi)$ = total number of instances of atomic propositions A_i which occur in ϕ.

Prove by induction that:

- $E(\phi) = D(\phi) + 1$,
- $S(\phi) \geq 3C(\phi)$.

Question 4, 5 and 6: Please complete the following exercises from Yannis lecture notes (page 15, Section 4, Propositional logic): 1.1, 1.6, 1.7.

Solutions

Solution to Question 2:

Proof. We show that the set of natural numbers n which are the lengths of formulas in L_0 is $\mathbb{N} \setminus \{0, 2, 3, 6\}$.

First note that we have length-1 formulas $\langle A_i \rangle$. Observe that the shortest formulas involving a connective $\neg$ or $\rightarrow$ must have the form $(\neg A_i)$, which has length 4. So it's clear that there are no formulas of length 2 or 3. The shortest formulas involving just one occurrence of $\rightarrow$ have the form $(A_i \rightarrow A_j)$, which has length 5. One can check that any formula involving at least two connectives must have length at least 7, so the number 6 is ruled out as a possible length. Now note that if ϕ is a formula of length m, then $(\neg\phi)$ is a formula of length $m+3$. Since we have formula of length 4 and of length 5, we have formulas of length $4+3k$ and $5+3k$ for all $k \geq 0$. Moreover, we have the following formula of length 9, $((A_1 \rightarrow A_1) \rightarrow A_1)$, so we get formulas of length $9+3k$ for all $k \geq 0$. Putting everything together shows that $\mathbb{N} \setminus \{0, 2, 3, 6\}$ is indeed the set of possible lengths. □

Solution to Question 3:

Proof. Recall the definition of L_0 from class: L_0 is the smallest set of (finite) sequences of symbols $(,), \neg, \rightarrow, A_1, A_2, \ldots$ such that:

1. $A_i \in L_0$, for each $i = 1, 2, \ldots$;
2. if $\phi \in L_0$, then $(\neg\phi) \in L_0$;
3. if $\phi, \psi \in L_0$, then $(\phi \rightarrow \psi) \in L_0$.

We can call this definition of L_0 the "top-down" definition. The right-hand side of the iff in this exercise gives a "bottom-up" definition.

For the purposes of the proof, let L_1 denote the set of (finite) sequences ϕ of symbols $(,), \neg, \rightarrow, A_1, A_2, \ldots$, which satisfy the "bottom-up" characterization on the right-hand side of the iff. We show $L_0 = L_1$ by the method of double inclusion.

Step 1: Show that $L_0 \subseteq L_1$. Since L_0 is the smallest set satisfying (1), (2), (3), it suffices to show that L_1 also satisfies (1), (2), (3). We check each of these conditions for L_1 in turn. For (1), note that the sequence $\langle \phi_1, \ldots, \phi_n \rangle = \langle \phi_1 \rangle = \langle A_i \rangle$ of length 1 witnesses that the length 1 formulas A_i are in L_1. For (2), suppose that $\phi \in L_0$, in other words $\phi = \phi_n$ for some sequence $\langle \phi_1, \ldots, \phi_n \rangle$ as in the bottom-up characterization. But then the sequence $\langle \phi_1, \ldots, \phi_n, (\neg\phi) \rangle$ shows that $(\neg\phi) \in L_1$. Finally, for (3), suppose that ϕ, ψ are both in L_0, as witnessed by the sequences $\langle \phi_1, \ldots, \phi_n \rangle$ and $\langle \psi_1, \ldots, \psi_m \rangle$ where $\phi = \phi_n$ and $\psi = \psi_m$. Then the sequence $\langle \phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_m, (\phi \rightarrow \psi) \rangle$ shows that $(\phi \rightarrow \psi) \in L_1$.

Step 2: Show that $L_1 \subseteq L_0$. For this step, we prove by induction on n that any finite sequence $\phi \in L_1$ whose membership in L_1 is witnessed by a sequence $\langle \phi_1, \ldots, \phi_n \rangle$ of length n must be in L_0. The base case is $n = 1$, in which case we can only have the finite sequences $\langle \phi_1, \ldots, \phi_n \rangle = \langle A_i \rangle$. But by condition (1), $A_i \in L_0$, so the base case holds. Now assume for the inductive hypothesis that for all $\phi \in L_1$ whose membership in L_1 is witnessed by a sequence of length $< n$, we have $\phi \in L_0$. Suppose that $\langle \phi_1, \ldots, \phi_n \rangle$ is a sequence of length n witnessing the membership of ϕ_n in L_1. Then either $\phi_n = (\neg\phi_i)$ for some $i < n$

or $\phi_n = (\phi_i \to \phi_j)$ for some $i, j < n$. In the first case $\phi_i \in L_0$ by the inductive hypothesis, and so by condition

(2) in the definition of L_0, we must have $\phi_n \in L_0$. In the second case, both ϕ_i and ϕ_j are in L_0 by the inductive hypothesis, and so by condition (3) in the definition of L_0, we must have $(\phi_i \to \phi_j) \in L_0$. This completes the proof. $\square$

Solution to Definition 2:

Proof. We use a very common method of proof in logic for this exercise, namely induction on the length of the formula ϕ.

(a) We prove by induction on the length n of a formula ϕ that $E(\phi) = D(\phi)+1$. The base case is $n = 1$, in which case ϕ is a length-1 formula A_i. But then $E(\phi) = 1$ and $D(\phi) = 0$ so $E(\phi) = D(\phi) + 1$. Assume for induction that $E(\phi) = D(\phi) + 1$ for all formulas of length less than n. Suppose that ϕ has length n. By the readability theorem, there exist formulas ψ_1 and ψ_2 of length less than n such that ϕ has one of the following forms:

$$(\neg\psi_1), (\psi_1 \to \psi_2), (\psi_1 \wedge \psi_2), (\psi_1 \vee \psi_2).$$

If $\phi = (\neg\psi_1)$, then $E(\phi) = E(\psi_1) = D(\psi_1) + 1 = D(\phi) + 1$, where we applied the inductive hypothesis to ψ_1. In the other three cases, we have $E(\phi) = E(\psi_1) + E(\psi_2) = (D(\psi_1) + 1) + (D(\psi_2) + 1) = (D(\psi_1) + D(\psi_2) + 1) + 1 = D(\phi) + 1$. This completes the inductive step.

(b) We prove by induction on the length n of a formula ϕ that $S(\phi) \geq 3C(\phi)$. The base case is $n = 1$, in which case ϕ is a length-1 formula A_i. But then $S(\phi) = 1$ and $C(\phi) = 0$ so $S(\phi) \geq 3C(\phi)$. Assume for induction that $S(\phi) \geq 3C(\phi)$ for all formulas of length less than n. Suppose that ϕ has length n. By the readability theorem, there exist formulas ψ_1 and ψ_2 of length less than n such that ϕ has one of the following forms:

$$(\neg\psi_1), (\psi_1 \to \psi_2), (\psi_1 \wedge \psi_2), (\psi_1 \vee \psi_2).$$

If $\phi = (\neg\psi_1)$, then $S(\phi) = S(\psi_1)+3 \geq 3C(\psi_1)+3 = 3(C(\psi_1)+1) = 3C(\phi)$, where we applied the inductive hypothesis to ψ_1. In the other three cases, we have $S(\phi) = S(\psi_1)+S(\psi_2)+3 \geq (3C(\psi_1))+(3C(\psi_2))+3 = 3(C(\psi_1)+C(\psi_2) + 1) = 3C(\phi)$. This completes the inductive step.

$\square$