Let p be a nonconstant polynomial function of a complex variable.
If $p(z_0) \neq 0$, then every neighborhood of the point z_0 contains a point z such that $|p(z)| < |p(z_0)|$

Intuition

$p(z) = p(z_0 + \Delta z)$

$= a_0 + a_1(z_0 + \Delta z) + a_2(z_0 + \Delta z)^2 + \cdots + a_n(z_0 + \Delta z)^n$

Consider!! $a_m(z_0 + \Delta z)^m = a_m z_0^m + a_m \sum_{k=1}^{m} \binom{m}{k} z_0^{m-k} \cdot \Delta z^k$

we assume Δz to be small $\Rightarrow (\Delta z)^2, (\Delta z)^3 \ldots (\Delta z)^n \ll \Delta z$

$\Rightarrow p(z) = p(z_0) + A\Delta z + \epsilon$ for some constant A where $|\epsilon| \ll |A\Delta z|$

choose Δz to be in the direction of the origin. Think of the picture!!

$\Rightarrow |p(z)| < |p(z_0)|$

The Goal

Polynomials can be broken up into linear and irreducible quadratic factors over the real line.
Our goal is to show that eiter $(x \pm r), r \geq 0$ or $x^2 + 2r\cos\phi x + r^2, r > 0$ is a factor of polynomial $p(x)$ for an appropriate choice of r and θ. Moreover, we want to show $r(\cos\phi \pm i\sin\phi)$ is a root of $p(x)$.

Clever Substitution

Let $x = r(\cos\phi + i\sin\phi) \Rightarrow x^k = r^k(\cos k\phi + i\sin k\phi)$
Split $p(x)$ into

$$U(r,\phi) = a_0 + a_1\cos(\phi)r + a_2\cos(2\phi)r^2 + \cdots + a_n\cos(n\phi)r^n$$
$$T(r,\phi) = a_1\sin(\phi)r + a_2\sin(2\phi)r^2 + \cdots + a_n\sin(n\phi)r^n$$

Consider the curves $U(r,\phi) = 0$ and $T(r,\phi) = 0$