Math 100: Problem Set 3

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(Q-1) (a) WTS by induction on the number of the number of steps required by the Euclidean algorithm to produce the greatest common divisor of F_n and F_{n+1} that the $\gcd(F_n, F_{n+1}) = F_1 = 1$. $P(1) : F_2 = q \cdot F_1 + r = F_1 + F_0 = F_1$, so we compute $\gcd(F_2, F_1) = F_1 = 1$ in $n = 1$ steps. $P(n)$: Assume we can compute $\gcd(F_{n+1}, F_n) = F_1 = 1$ for some $n > 0$ in n steps. $P(n+1)$: We compute the first step $F_{n+2} = q \cdot F_{n+1} + r$ where $q = 1$ and $r = F_n$ by the definition of the Fibonacci sequence. It follows $\gcd(F_{n+2}, F_{n+1}) = \gcd(F_{n+1}, F_n)$. Since we know we can compute $\gcd(F_{n+1}, F_n) = F_1 = 1$ in n steps, we can compute $\gcd(F_{n+2}, F_{n+1}) = F_1 = 1$ in $n+1$ steps. Hence, by induction, $\gcd(F_{n+1}, F_n) = F_1 = 1$ in n steps for all n.

(b) Want to show by induction that T_n and T_{n+k} are relatively prime for all k.
$P(1) : T_{n+1} = q \cdot T_n + r$ where $q = (T_n - 1)$ and $r = 1$. $T_n \in \mathbb{N}$ for all n is provable by a simple induction. Thus, $\gcd(T_{n+1}, T_n) = \gcd(T_n, 1) = 1$.
$P(k)$: Assume for some $k \geq 1$ $\gcd(T_{n+k}, T_n) = \gcd(T_n, 1) = 1$.
$P(k+1) : T_{n+k} = q \cdot T_n + 1$ for some q by the induction hypothesis. This implies $q(q \cdot T_n + 1) \in \mathbb{N}$ It follows $T_{n+k+1} = (T_{n+k} - 1)T_{n+k} + 1 = (q \cdot T_n + 1 - 1)(q \cdot T_n + 1) + 1 = q(q \cdot T_n + 1)T_n + 1$. Thus, $\gcd(T_{n+k+1}, T_n) = \gcd(T_n, 1) = 1$. Hence, by induction, $\gcd(T_{n+k}, T_n) = \gcd(T_n, 1) = 1$ for all n and k. Setting $m = k + n$, we obtain T_m and T_n are relatively prime.

(Q-2) It suffices to show $\gcd(a+b, c+d) = 1$. We know this is true iff there exist integers s, t s.t $s(a+b)+t(c+d) = 1$. Using $d(a+b)-b(c+d) = ad-bc = 1$ set $s = d, t = -b$. Thus, $\gcd(a+b, c+d) = 1 \Rightarrow \frac{a+b}{c+d}$ is irreducible.

(Q-3) Let $\gcd(a_1, \ldots, a_m) = s$ and $\gcd(b_1, \ldots, b_n) = t$. Since s and t divide each a_i and b_j respectively, st divides each $a_i b_j$. Thus, $\gcd(a_1 b_1, \ldots, a_m b_n)$ is a multiple of st. Suppose for the sake of contradiction, there exists some prime number p s.t pst divides $\gcd(a_1 b_1, \ldots, a_m b_n)$. It follows ps must divide each a_i or pt must divide each b_j. We know this is false because $\gcd(a_1, \ldots, a_m) = s$ and $\gcd(b_1, \ldots, b_n) = t$. It follows there exists some i, j s.t ps does not divide a_i and pt does not divide b_j. Thus, pst cannot divide $a_i b_j$. Hence, $st = \gcd(a_1 b_1, \ldots, a_m b_n)$.

(Q-4) $100y+x = 200x+2y+2 \Rightarrow 98y-199x = 2$. $98y-199x \equiv -3x \mod (98) \equiv 2 \mod (98) \Rightarrow -3x = -96 \mod (98) \Rightarrow x \equiv 32 \mod (98)$ Thus, $x = 32+98k \Rightarrow 98y-199(32+98k) = 2 \Rightarrow y = 65+199k$. We obtain the pair $(x, y) = (32+98k, 65+199k)$. Original check is \$32.65

(Q-5) For the set $\{1, 2, \dots, 100\} \mod (10)$ has 10 congruence classes each containing 10 numbers.
mod (12) has 12 congruence classes 8 of which contain 8 numbers and 4 of which contain 9 numbers.
mod (13) has 13 congruence classes 9 of which contain 8 numbers and 4 of which contain 7 numbers.
mod (11) has 11 congruence classes 1 of which contains 10 numbers and 10 of which contain 9.
If we consider for example congruence class 1 for $\mod (10)$ we can pick at most 5 numbers from the set $\{1, 11, 21, \dots, 91\}$ without selecting 2 numbers that differ by 10.
It follows by the pigeonhole principle we can pick at most 50 numbers without selecting 2 that differ by 10, 52 numbers without 2 that differ by 12, 52 numbers without 2 that differ by 13, and 55 without 2 that differ by 11.

(Q-6) We will show by induction on n for that $4^{3n+1} + 2^{3n+1} + 1$ is divisible by 7 for $n \geq 0$.
$P(0) : 4^{3\cdot0+1} + 2^{3\cdot0+1} + 1 = 7$ which is clearly divisible by 7.
$P(n)$: Assume for some $n \geq 0$ $4^{3n+1} + 2^{3n+1} + 1$ is divisible by 7.
$P(n+1) : 4^{3(n+1)+1} + 2^{3(n+1)+1} + 1$
$= 64 \cdot 4^{3n+1} + 8 \cdot 2^{3n+1} + 1$
$= 56 \cdot 4^{3n+1} + 8(4^{3n+1} + 2^{3n+1} + 1) - 7$ which is divisible by 7 because 7 divides 56, $4^{3n+1} + 2^{3n+1} + 1$, and 7.
Hence, by induction, 7 divides $4^{3n+1} + 2^{3n+1} + 1$ for all n.

(Q-7) (a) Because the square of an even number is even, if there exists a perfect square in the sequence $\{11, 111, \dots\}$, it must the square of an odd number. Let $x = 2k+1$ for some integer k. WTS $x^2 \notin \{11, 111, \dots\}$ for all k. $x^2 = (2k+1)^2 = 4(k^2+k)+1 \equiv 1 \mod (4)$. WTS each element in the sequence $\{11, 111, \dots\} \equiv 3 \mod (4)$ by induction.
$P(1) : 11 = 2 \cdot 4 + 3$
$P(n)$: Assume for some $_n$ in the sequence $\{11, 111, \dots\}$ $s_n \equiv 3 \mod (4)$.
$P(n+1)$: $s_{n+1} = 10s_n + 1$, so $s_n \equiv 3 \mod (4) \Rightarrow 10s_n \equiv 30 \mod (4) \Rightarrow 10s_n \equiv 2 \mod (4) \Rightarrow s_{n+1} \equiv 3 \mod (4)$.
Thus, by induction, $\{11, 111, \dots\} \equiv 3 \mod (4)$, but since all odd perfect squares are $\equiv 1 \mod (4)$, $\{11, 111, \dots\}$ contains no perfect squares.

(b) Let k and m be integers, so $(2k+1)^2$ and $(2m+1)^2$ are odd squares. Their difference is $(2k+1)^2 - (2m+1)^2 = ((2k+1)-(2m+1))((2k+$

$1) + (2m + 1)) = 4(k - m)(k + m + 1)$. Either $k - m$ or $k + m + 1$ must be even. If k, m have same parity then $k - m$ is even otherwise $k + m + 1$ is even. Thus, $(k - m)(k + m + 1)$ is divisible by 2, so $4(k - m)(k + m + 1)$ is divisible by 8.

(Q-8) Suppose for the sake of contradiction $\frac{21n-3}{4}$ and $\frac{15n+2}{4}$ are both integers. Thus, $21n - 3 \equiv 15n + 2 \mod (4) \Rightarrow 6n \equiv 5 \mod (4) \Rightarrow 6n \equiv 1 \mod (4)$. This is impossible because $6n$ must be even, and $4k + 1$ is clearly odd. Hence, they can't be both integers.

(Q-9) Let n be arbitary and arrange the first $(2n + 1)^2$ prime numbers $p_1, p_2, \ldots, p_{(2n+1)^2-1}, p_{(2n+1)^2}$ in an $2n + 1 \times 2n + 1$ array we will call $M_{2n+1\times 2n+1}$. Let R_i be the product of the elements of row i and C_i be the product of the elements of column i. Since each R_i and R_j for $i \neq j$ are relatively prime, it follows by the Chinese Remainder Theorem there exists a unique solution a to

$$\begin{aligned}
a &\equiv -n \mod (R_1) \\
a &\equiv -(n-1) \mod (R_2) \\
&\ldots \\
a &\equiv 0 \mod (R_{n+1}) \\
&\ldots \\
a &\equiv (n-1) \mod (R_{2n}) \\
a &\equiv n \mod (R_{2n+1})
\end{aligned}$$

and because each C_i and C_j for $i \neq j$ are relatively prime, there exists a unique solution b to

$$\begin{aligned}
b &\equiv -n \mod (C_1) \\
b &\equiv -(n-1) \mod (C_2) \\
&\ldots \\
b &\equiv 0 \mod (C_{n+1}) \\
&\ldots \\
b &\equiv (n-1) \mod (C_{2n}) \\
b &\equiv n \mod (C_{2n+1})
\end{aligned}$$

Take any point (x, y) in the $2n + 1 \times 2n + 1$ square centered at (a, b). It follows $x \equiv 0 \mod (R_i)$ for some i and $y \equiv 0 \mod (C_j)$ for some j. In other words, x and y are multiples of R_i and C_j respectively. However, R_i and C_j are not relatively prime because they have a common factor of $p_{(2n+1)i+j}$. Thus, $\gcd(x, y) \neq 1$. Since the $2n+1 \times 2n+1$ square centered at (a, b) contains every latice point within n of (a, b) and all of them are invisible, we have found a point that is at least n away from any visible lattice point.

(Q-10) Let a be an integer with decimal representation $a = \sum_{i=0}^{n} 10^i a_i$. Let a^* be the result of moving the initial digit a_n to the end. It follows $a^* = a_n + \sum_{i=0}^{n-1} 10^{i+1} a_i$.

Multiplying a by 10 and adding a_n we obtain $10a + a_n = a_n + \sum_{i=0}^{n} 10^{i+1} a_i = 10^{n+1} a_n + a^*$. Thus, $a^* = 2a$ iff $8a = a_n(10^{n+1} - 1)$. This is impossible because $a_n = 8$ and $a = 10^{n+1} - 1$ cannot be true simultaneously, and they must both be true for $a^* = 2a$ because $10^{n+1} - 1$ and 8 are relatively prime.