Identity: $(\sum_{i=1}^{n} a_i b_i)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$ Triangle Innequality: $(\sum_{i=1}^{n}(a_i + b_i)^2)^{\frac{1}{2}} \leq (\sum_{i=1}^{n} a_i^2)^{\frac{1}{2}} + (\sum_{i=1}^{n} b_i^2)^{\frac{1}{2}}$

Open and Closed Sets: Open contains none of the boundary points. Closed contains all of the boundary points.
A subset E of X is only open if there exists a ball centered at each point in E that fits entirely inside E. A subset E is only closed if every convergent sequence converges in E.
Open balls are open sets and closed balls are closed sets. Any singleton set is closed. A set is open iff the complement is closed.
Finite collections of open (intersection) and closed (union) sets are open and closed respectively.
Infinite/Finite collections of open (union) and closed (intersection) sets are open and closed respectively.
$Int(E)$ is the largest open set in E, $\overline{E}$ is the smallest closed set which contains E.

Relative topology: Let (X, d)be a metric space, let Y be a subset of X, and let E be a subset of Y. We say that E is relatively open with respect to Y if it is open in the metric subspace $(Y, d_{|Y \times Y})$. Similarly, we say that E is relatively closed with respect to Y if it is closed in the metric space $(Y, d_{|Y \times Y})$.

a) E is relatively open with respect to Y iff $E = V \cap Y$ for some set $V \subset X$ which is open in X.
b) E is relatively closed with respect to Y iff $E = K \cap Y$ for some set $K \subset X$ which is closed in X.

Complete metric spaces: A metric space (X, d) is said to be complete iff every Cauchy sequence in (X, d) is in fact convergent in (X, d). If a subspace is complete, it must be closed in the metric space. If a metric space (X, d) is complete, every subspace must be closed in (X, d).

Compact metric spaces: A metric space (X, d) is said to be compact iff every sequence in (X, d) has at least one convergent subsequence. A subset Y of a metric space X is said to be compact if the subspace $(Y, d_{Y \times Y})$ is compact. Alternatively: Any collection of open sets that cover X can be reduced to a finite subcover.

Bounded sets: Let (X, d) be a metric space Y is bounded iff it can fit entirely inside a ball. Compact sets are complete and bounded.

Heine-Borel: Compact subsets are closed and bounded. For Euclidean spaces with Euclidean metric, a metric space is compact iff it is closed and bounded.

Every nested sequence of compact subsets of X is non-empty.

Let (X, d) be a metric space.

a) If Y is a compact subset of X, and $Z \subset Y$, then Z is compact iff Z is closed.
b) If $Y_1, Y_2, ..., Y_n$ are a finite collection of compact subsets of X, then their union is also compact.
c) Every finite subset of X (including the empty set) is compact.

Continuity preserves convergence

a) f is continuous at x_0.
b) Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.
c) For every open set $V \subset Y$ that contains $f(x_0)$, there exists an open set $U \subset X$ containing x_0 such that $f(U) \subset V$.

In addition

c) Whenever V is an open set in Y, the set $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is an open set in X.
d) Whenever F is an closed set in Y, the set $f^{-1}(F) = \{x \in X : f(x) \in F\}$ is a closed set in X.

Continuity and product spaces: Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions, and let $f \oplus g : X \to \mathbb{R}^2$ be their direct sum. We give $\mathbb{R}^2$ the Euclidean metric.

a) If $x_0 \in X$, then f and g are both continuous at x_0 iff $f \oplus g$ is continuous at x_0.
b) f and g are both continuous iff $f \oplus g$ is continuous.