Math 151A: Problem Set 1

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Problem 1: (T) Taylor's Theorem

Let $f(x) = e^{2x}$ for $x \in [0, 2]$.

a) Find Taylor's polynomial of degree-2, i.e. $P_2(x)$, around the point $x_0 = 0$ and use it to approximate the value of $f(1.5)$, i.e. $f(1.5) \approx P_2(1.5)$.

b) What is the error as a function of $\xi(x)$ when $x = 1.5$ (specify the domain for $\xi(1.5)$)?

c) What is the actual error (in magnitude)?

Solution:

a) $P_2(x) = 1 + 2x + 2x^2 \Rightarrow f(1.5) \approx P_2(1.5) = 8.5$

b) $R_2(x) = \frac{f'''(\xi(x))x^3}{6} = \frac{4e^{2\xi(x)}x^3}{3} \Rightarrow R_2(1.5) = \frac{4e^{2\xi(1.5)}1.5^3}{3}$ $\xi(1.5) \in (0, 1.5)$

c) $R_2(1.5) = e^3 - 8.5 \approx 11.5855$

Problem 2: (T) Bisection Method

Let $f(x) = \sqrt{\pi\, x} - \cos(\pi\, x)$ over the interval $[0, 1]$. We would like to find p such that $f(p) = 0$.

a) Show that the bisection method applied to this problem converges (apply the theorem from class).

b) How many iterations are needed to have a 10^{-q}-accurate approximation to the true root where $q > 1$? Write your answer in the form $n \geq Cq$ where C is an explicit constant that you need to provide.

Solution:

a) Applying the theorem from class, f is continuous over $[0, 1]$ and $f(0) \cdot f(1) < 0$. Thus, $\exists p \in (0, 1)$ s.t $f(p) = 0$. $f(x)$ over the interval $[0.1]$ is strictly monotonically increasing, so p is unique.

b) $|p_n - p| \leq 2^{-n} \leq 10^{-q} \Rightarrow n \geq \log_2(10)q$

Problem 3: (C) Bisection Method

Find a 10^{-5}-accurate approximation to $\sqrt[4]{25}$ using the Bisection Algorithm. To do so, you will need to define a function $f(x)$ whose root is $\sqrt[4]{25}$. The function $f(x)$ must only use simple operations: multiplication and addition/subtraction. Use the corollary from class to determine the number of steps required to achieve the given accuracy.

Solution:

Applying the theorem from class, $f(x) = x^4 - 25$ which is continuous over $[2,3]$ and $f(2) \cdot f(3) < 0 \Rightarrow \exists p \in (2,3)$ s.t $f(p) = 0$

$f'(x) > 0$ over the interval $[2,3]$, so f will only intersect with the $x - axis$ at 1 point. Thus, p is unique.

$a_0 = 2 \; b_0 = 3 \; p = \sqrt[4]{25}$

$|p_n - p| \leq \frac{3-2}{2^n}$

By the corollary from class, $\frac{1}{2^n} < 10^{-5} \Rightarrow n = 17 > log_2(\frac{1}{10^{-5}})$

$p_{17} = 2.236061096191406$ which is within 10^{-5} of $\sqrt[4]{25}$ ($|p_{17} - p| \approx 6.88 \cdot 10^{-6}$)

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% Bisection Method on function F(x)=x^4-25
clc;
clear all;

% Inputs: a, b, tol, N0
tol = 1e-5; % error tolerance
N0 = 20; % maximum number of iterations
a=2; % starting left point
b=3; % starting right point

% Start Iterating

n = 1;
Fa = a^4-25;
Fb = b^4-25;


while n <   N0

  p = a + (b-a)/2; % better way for writing p = (a+b)/2
  Fp = p^4-25; % evaluate the function at p

   if Fp==0 || (b-a)/2 < tol
       % close enough to actual root, stop iteration
       break;
   elseif sign(Fa)*sign(Fp) > 0
       % continue search in right half interval
       a = p;
       Fa = Fp;
   else
       % continue search in left half interval
       b = p;
       Fb = Fp;
   end
   n = n + 1;

end

fprintf('Iteration number = %d \n', n);
fprintf('p =  %.6f \n',p);
fprintf('F(p) = %.4f \n', p^4-25);
fprintf('Error= %.2e',abs(p-sqrt(5)));
```

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Name	Value
a	2.2361
b	2.2361
Fa	-6.4893e-04
Fb	3.3455e-05
Fp	-3.0774e-04
n	17
N0	20
p	2.2361
tol	1.0000e-05

Problem 4: (T) Bisection Method

You will show that the bisection method may not converge monotonically. Provide a continuous function $f(x)$ and an interval $[a, b]$ so that the error at the k-th step, denoted $E_k = |p_k - p|$, increases between some iterations although the sequence p_k converges to the unique root. To receive credit for this problem, you must justify your answer and prove that your example is convergent.

Solution:

$f(x) = x - 0.124$ on the interval $[-1, 1]$. Let $p_k = \frac{1}{2}(a_{k-1} + b_{k-1})$ and $a_0 = -1, b_0 = 1$.
We define $E_k = |f(p_k)| = |p_k - 0.124|$.
Given f is continuous over $[-1, 1]$ and $f(-1) \cdot f(1) < 0$ $\exists p \in (-1, 1)$ s.t $f(p) = 0$
$f'(x) > 0$ over the interval $[-1, 1]$, so f will only intersect with the $x - axis$ at 1 point.
Thus, p is unique.
$E_1 = 0.124, E_2 = 0.376, E_3 = 0.126, E_4 = 0.001...$
Since $E_2 > E_3 > E_1$ the sequence does not converge monotonically.

Problem 5: (T) Stopping Criteria for General Root-Finding Algorithms

Assume that we have a sequence p_n for $n = 1, 2, ...$ that is generated by an algorithm in order to find the root of a function $f(x)$. Let ϵ be the prescribed tolerance used to stop the iterative process.

You may use, without proof, that $\sum_{k=1}^{n} \frac{1}{k}$ diverges as $n \to \infty$.

a) Consider the stopping criterium $|p_n - p_{n-1}| < \epsilon$ for $n > 2$. Show that $p_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfies the criterium when $n \geq \frac{1}{\epsilon}$; however, $p_n \to \infty$ (thus the sequence does not converge to a finite value).

b) Consider the stopping criterium $\frac{|p_n - p_{n-1}|}{|p_n|} < \epsilon$ for $n > 2$ and $p_n \neq 0$. Show that $p_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfies the criterium; however, $p_n \to \infty$ (thus the sequence does not converge to a finite root).

c) Let $f(x) := (x-1)^{10}$, whose root is $p = 1$, and define the sequence $p_n = 1 + \frac{1}{n}$. Note that p_n goes to the root in the limit. Show that the stopping criterium $|f(p_n)| < 10^{-3}$ is achieved for all $n > 1$ but $|p - p_n| \leq 10^{-3}$ requires $n > 1000$.

Solution:

a) If $n > \frac{1}{\epsilon}$
$\Rightarrow \epsilon > \frac{1}{n} = |\frac{1}{n} + \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=1}^{n-1} \frac{1}{k}| = |\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n-1} \frac{1}{k}|$
Hence, the stopping criteria is met for $n > \frac{1}{\epsilon}$. However, since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, $\lim\limits_{k\to\infty} p_k = \infty$.

b) $|p_n| = |\sum_{k=1}^{n} \frac{1}{k}| \geq |\sum_{k=1}^{n} \frac{1}{n}| = 1 \Rightarrow \frac{1}{|p_n|} \leq 1$
It follows $\frac{|p_n - p_{n-1}|}{|p_n|} \leq \frac{|p_n - p_{n-1}|}{\sum_{k=1}^{n} \frac{1}{n}} = \frac{|p_n - p_{n-1}|}{\frac{n}{n}} = |p_n - p_{n-1}|$
By part a, $n > \frac{1}{\epsilon} \Rightarrow |p_n - p_{n-1}| < \epsilon$
Because $\frac{|p_n - p_{n-1}|}{|p_n|} \leq |p_n - p_{n-1}| \Rightarrow \frac{|p_n - p_{n-1}|}{|p_n|} < \epsilon$
Hence, the stopping criteria is met for $n > \frac{1}{\epsilon}$. However, since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, $\lim\limits_{k\to\infty} p_k = \infty$.

c) $|p_n - p| = |1 + \frac{1}{n} - 1| = \frac{1}{n} < 10^{-3} \Rightarrow n > 1000$
If $10^{\frac{3}{10}} < n \Rightarrow |f(p_n) - f(p)| = \frac{1}{n^{10}} < 10^{-3}$