Math 167: Homework 3

Owen Jones

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Exercise 2.14 Row and column 1 dominate rows/columns $4 - n$, so the matrix below reduces to

P2

P 1

	1	2	3	4	...	n-1	n
1	0	-1	2	2	...	2	2
2	1	0	-1	2	...	2	2
3	-2	1	0	-1	...	2	2
4	-2	-2	1	0	..	2	2
...	...	...	...	...	...	...	...
n-1	-2	-2	-2	-2	...	0	-1
n	-2	-2	-2	-2	...	1	0

P2

P 1

	1	2	3
1	0	-1	2
2	1	0	-1
3	-2	1	0

Since the game is anti-symmetric, the value of the game is 0, and for $\mathbf{x}$ to be optimal, $\mathbf{x}^T A\mathbf{y} \geq V$ for all $\mathbf{y} \in \Delta_n$. Equalizing the payoffs of y_1, y_2, y_3, we obtain the optimal strategy for player I using the following system of equations,

$$\begin{aligned} 0x_1 + 1x_2 + (-2)x_3 &\geq 0 \\ (-1)x_1 + 0x_2 + x_3 &\geq 0 \\ 2x_1 + (-1)x_2 + 0x_3 &\geq 0 \\ x_1 + x_2 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Adding the first, twice the second, and the third equations yields, $0 \geq 0$, so if any of the first three equations are greater than 0, we would obtain a contradiction because $0 \not> 0$. Solving the above system equations we obtain $x_1 = x_3 = \frac{1}{4}, x_2 = \frac{1}{2}$. By symmetry, we obtain $x_1 = y_1, x_2 = y_2, x_3 = y_3$.

Exercise 2.15 Using symmetry we can reduce the 9×12 matrix to a more manageable 3×3 matrix. The $\frac{1}{4}$ results from there being 4 corners, midsides, corner-counterclockwise, and corner-clockwise selections with an equal probability of selecting 1 of the 4. E.g if Player I selects a corner at random and Player II places the submarine in a random corner-counterclockwise placement, there is a $\frac{1}{4}$ chance Player I hits Player II's submarine $(4 \times \frac{1}{4} \times \frac{1}{4})$. Since Player II can't place a submarine in a corner-clockwise location, we remove the column from the matrix. The midside row dominates the corner row. It follows the corner-counterclockwise column dominates the center column. Thus, the optimal strategy for Player II is to place a submarine in one of the corner-counterclockwise placements.

		P2	
		center	corner-counterclockwise
P 1	corner	0	1/4
	midside	1/4	1/4
	middle	1	0

Exercise 2.16 (a) The following table represents the payoff matrix for Player I

		P2 (Z)			
		a	b	c	d
P 1 (C)	a	1	1/2	0	0
	b	1/2	1	1/2	0
	c	0	1/2	1	1/2
	d	0	0	1/2	1

(b) We can reduce the 4×4 into a 2×2 by calling a and d outer where a and d are picked with equal probability and b and c inner where b and c are picked with equal probability. We can do this by symmetry.

		P2 (Z)	
		inner	outer
P 1 (C)	inner	3/4	1/4
	outer	1/4	1/2

(c) Equalizing payoffs for Player I we obtain $\frac{3}{4}I_n + \frac{1}{4}O_u = \frac{1}{4}I_n + \frac{1}{2}O_u, I_n + O_u = 1 \Rightarrow I_n = \frac{1}{3}, O_u = \frac{2}{3}$ for Player I with value $\frac{5}{12}$, and by symmetry Player II optimum strategy is $I_n = \frac{1}{3}, O_u = \frac{2}{3}$. It follows it's optimum for each player to choose $a = d = \frac{1}{3}$ and $b = c = \frac{1}{6}$.

Exercise 2.18 (Went to Prof's office hours and said we could leave it in recursive form) Γ_1 has a saddlepoint for the inspector to always inspect and Trumm to always be honest, so $\Gamma_1 = 0$. For $n \geq 2$, Γ_n doesn't have a pure Nash Equilibrium if $\Gamma_{n-1} > -1$, so we want to show $\Gamma_n > -1$ for all $n \geq 2$ by induction. This is trivial for $n = 2$ because we already found $\Gamma_1 = 0$. Assume for some $n \geq 2$ $\Gamma_{n-1} > -1$. It follows Γ_n doesn't have a pure Nash Equilibrium. Solving the following system of equations we obtain the optimal strategy for Player I.

$$x_1 - x_2 = \Gamma_{n-1}x_2 - x_1$$
$$x_1 + x_2 = 1$$

Thus, the optimal strategy for Player I is to inspect with probability $x_1 = \frac{\Gamma_{n-1}+1}{\Gamma_{n-1}+3}$ and wait with probability $x_2 = \frac{2}{\Gamma_{n-1}+3}$. By symmetry, Player II should cheat with probability $y_1 = \frac{\Gamma_{n-1}+1}{\Gamma_{n-1}+3}$ and be honest with probability $y_2 = \frac{2}{\Gamma_{n-1}+3}$, giving a game value of $\Gamma_n = \frac{\Gamma_{n-1}-1}{\Gamma_{n-1}+3}$. Since $\Gamma_{n-1} > -1$ then $\Gamma_n > \frac{-1-1}{-1+3} = -1$. Hence, Γ_{n+1} doesn't have a Nash Equilibrium. Thus, by induction, Γ_n doesn't have a Nash Equilibrium for all n, and each player should play the strategies discussed in the induction proof.

Exercise 2.22 Taking the original payoff matrix

P2

		1	2	3	4	5	6	...	n
P 1	1	0	-1	1	1	1	1	...	1
	2	1	0	-1	-1	1	1	...	1
	3	-1	1	0	-1	-1	-1	...	1
	4	-1	1	1	0	-1	-1	...	1
	5	-1	-1	1	1	0	-1	...	1
	6	-1	-1	1	1	1	0	...	1
	...	...	...	...	...	...	...	...	...
	n	-1	-1	-1	-1	-1	-1	...	0

Row/column 1 dominates all rows/columns greater than 4. Row/column 4 dominates row/column 3. Thus, we can reduce the original payoff matrix as follows:

P2

		1	2	4
P 1	1	0	-1	1
	2	1	0	-1
	4	-1	1	0

The matrix is skew-symmetric, therefore has a value of 0, and there exists no pure Nash Equilibrium. Thus, we use the following system of equations to find the optimun strategy for Player I.

$$
\begin{aligned}
x_2 - x_4 &\geq V = 0 \\
-x_1 + x_4 &\geq V = 0 \\
x_1 - x_2 &\geq V = 0 \\
x_1 + x_2 + x_4 &= 1
\end{aligned}
$$

Taking the sum of the LHS and RHS of the first three equations, we obtain $0 \geq 0$, so if the LHS of any of the first three equations is greater than 0, we obtain a contradition. Thus, we obtain $x_1 = x_2 = x_4 = \frac{1}{3}$ by simple algebra, and because the matrix is skew-symmetric, Player II has the same optimal strategy.

Exercise 2.23 Since each natural number has a successor, no matter high of a number a player chooses, there exists a strategy, the successor of their number, that is greater than their choice of number. Thus, regardless of either player's strategy, there is always a reason for both players to deviate, so there can't be a pure Nash Equilibrium.

P 2

		1	2	3	4	...	n	n+1
P 1	1	0	-1	-1	-1	...	-1	-1
	2	1	0	-1	-1	...	-1	-1
	3	1	1	0	-1	...	-1	-1
	4	1	1	1	0	...	-1	-1
	...	...	...	...	...	...	...	...
	n	1	1	1	1	...	0	-1
	n+1	1	1	1	1	...	1	0

For mixed strategies, for any n, row/column n dominates rows/columns $1-(n-1)$, so it follows that Player I and II will choose each natural number with probability 0. This is a contradiction because the sum of the probabilities will add up to 0 and not 1. Thus, there are no optimal mixed strategies and no mixed Nash Equilibrium. Since Player II can always choose the successor to Player I's choice, Player I can only guarantee to lose a dollar regardless of what they play i.e obtain a payoff of -1. This logic follows for Player II obtaining a payoff of 1.

Exercise 3.1 First we find the effective resistance between the beginning and end to find the value of the game. We will split the game into three Top (T), Middle (M), and Bottom (B). The effective resistance of $T = 1 + \frac{1}{1+1+\frac{1}{2}+1} = \frac{9}{7}$. The effective resistance of $M = \frac{1}{1+\frac{1}{1+\frac{1}{2}}} + \frac{1}{1+\frac{1}{2}} = \frac{19}{15}$. The effective resistance of $B = 1$. Thus, $V = \frac{1}{\frac{1}{T}+\frac{1}{M}+\frac{1}{B}} = \frac{171}{439}$.

$$\begin{aligned}
P(T_1) &= \frac{\frac{7}{9}}{\frac{7}{9}+\frac{15}{19}+1} \cdot \frac{1}{1+1+\frac{1}{2}+1} = \frac{38}{439} \\
P(T_2) &= \frac{\frac{7}{9}}{\frac{7}{9}+\frac{15}{19}+1} \cdot \frac{\frac{1}{2}}{1+1+\frac{1}{2}+1} = \frac{19}{439} \\
P(T_3) &= \frac{\frac{7}{9}}{\frac{7}{9}+\frac{15}{19}+1} \cdot \frac{1}{1+1+\frac{1}{2}+1} = \frac{38}{439} \\
P(T_4) &= \frac{\frac{7}{9}}{\frac{7}{9}+\frac{15}{19}+1} \cdot \frac{1}{1+1+\frac{1}{2}+1} = \frac{38}{439} \\
P(M_1) &= \frac{\frac{15}{19}}{\frac{7}{9}+\frac{15}{19}+1} \cdot \frac{1}{1+\frac{1}{1+\frac{1}{2}}} \cdot \frac{1}{1+\frac{1}{2}} = \frac{54}{439} \\
P(M_2) &= \frac{\frac{15}{19}}{\frac{7}{9}+\frac{15}{19}+1} \cdot \frac{1}{1+\frac{1}{1+\frac{1}{2}}} \cdot \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{27}{439} \\
P(M_3) &= \frac{\frac{15}{19}}{\frac{7}{9}+\frac{15}{19}+1} \cdot \frac{\frac{1}{1+\frac{1}{2}}}{1+\frac{1}{1+\frac{1}{2}}} \cdot \frac{1}{2} \cdot \frac{1}{1+\frac{1}{2}} = \frac{18}{439} \\
P(M_4) &= \frac{\frac{15}{19}}{\frac{7}{9}+\frac{15}{19}+1} \cdot \frac{\frac{1}{1+\frac{1}{2}}}{1+\frac{1}{1+\frac{1}{2}}} \cdot \frac{1}{2} \cdot \frac{1}{1+\frac{1}{2}} = \frac{18}{439} \\
P(M_5) &= \frac{\frac{15}{19}}{\frac{7}{9}+\frac{15}{19}+1} \cdot \frac{\frac{1}{1+\frac{1}{2}}}{1+\frac{1}{1+\frac{1}{2}}} \cdot \frac{1}{2} \cdot \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{9}{439} \\
P(M_6) &= \frac{\frac{15}{19}}{\frac{7}{9}+\frac{15}{19}+1} \cdot \frac{\frac{1}{1+\frac{1}{2}}}{1+\frac{1}{1+\frac{1}{2}}} \cdot \frac{1}{2} \cdot \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{9}{439} \\
P(B) &= \frac{171}{439}
\end{aligned}$$

By symmetry, the probabilities will be the same for the troll and traveller.

Exercise 3.2 Let n and k be arbitrary s.t $k \leq n$. We want to show for any subset $S \subseteq \{1, 2, \ldots, n\}, |S| \leq |f(S)|$ where $f(S)$ is the set of vetices S is connected to. Since each vertex in S has k edges incindent to it, we need to distribue $k \times |S|$ edges amongst $|f(S)|$ vertices. Since each vertex in $f(S)$ can have at most k edges incindent to it from vertices in S, there must be at least one vertex in $f(S)$ for every k edges. Thus, by the pigeonhole princple, $|f(S)|$ must be at least the size of S. Thus, by Hall's Marriage Theorem, any k-regular $n \times n$ graph must have perfect matching.

Exercise 3.5 Since this game is progressively bounded with $B(x) \leq 2n$ and can't end in a tie, one of the players must have a winning strategy. Let A_r be the set of actors, let A_s be the set of actresses, and let E be the set of edges that connect A_r to A_s. Suppose $G = (A_r, A_s, E)$ has a perfect matching. WLOG Player I picks actor $i \in A_r$. Let $A'_r = A_r \setminus \{i\}$. Since every subset of A'_r is a subset of A_r and there exists a perfect matching between A_r and A_s, A'_r has a matching with A_s of size $n-1$ by Hall's Marriage Theorem. We show (A'_r, A'_s, E') has a perfect matching of $n-1$ where $j = A_s \setminus A'_s \in f(i)$ where $f(i)$ is the set of vertices connected to i. Since there exists a perfect matching M between A_r and A_s, there must be some $e \in M \subseteq E$ that connects i and some $j \in f(i)$. If we take $M' = M \setminus e$, we will obtain a perfect matching between A_r' and $A_s' = A_s \setminus j$. Player II now selects actress j and can do this for every subsequent i' Player I selects.

Suppose $G = (A_r, A_s, E)$ does not have a perfect matching. Let M be the maximum matching between A_r and A_s. Choose $i \in A_r$ s.t no $e \in M$ is incindent on i. It follows that for every $j \in f(i)$ there exists a $e \in M$ incindent on j. If this were not the case, M could not be a maximum matching because we would be able to add an edge to connect i and j. We then play the subgame $G^* = (A_r', A_s', E')$ where A_r' and A_s' are the vertices in the maximum matching betwe A_r and A_s. Now we are playing a game with a perfect matching with Player II going first, so Player I has a winning strategy.