Math 164: Midterm

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1. $x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})} = x^{(k)} - \frac{x^{(k)3}}{3x^{(k)2}} = \frac{2}{3}x^{(k)2}$

2. $\mathbf{x}^\top(aa^\top)\mathbf{x} = (a^\top\mathbf{x})^\top(a^\top\mathbf{x}) = \langle a, \mathbf{x}\rangle^2 \geq 0$

3. $\phi''(\alpha) = \mathbf{d}^\top F(\mathbf{x} + \alpha\mathbf{d})\mathbf{d}$

4. $f(x) = \frac{1}{2}\mathbf{x}^\top(ab^\top)\mathbf{x}$ Let $\mathbf{Q} = \frac{1}{2}(ab^\top + ba^\top)$. $\nabla f(x) = \frac{1}{2}\mathbf{Q}\mathbf{x} = \frac{1}{2}(ab^\top + ba^\top)\mathbf{x}$

5. $F(x) = \frac{1}{2}(ab^\top + ba^\top)$.

6. (a) $f(x_1, x_2) = {}^\top\mathbf{x} \Rightarrow \nabla f(x_1, x_2) = [c_1, c_2]^\top$. Since $\nabla f(x_1, x_2) \neq 0\ \forall \mathbf{x} \in \Omega$, we check $\mathbf{d}^\top\nabla f(x_1, x_2) \leq 0$ for all feasible directions. Any point on the line $x_1 + x_2 = 1$, take $\mathbf{d} = [1, -1]^\top \Rightarrow \mathbf{d}^\top\nabla f = c_1 - c_2 > 0$. Any point on the line $x_1 = 0$ except $[1, 0]^\top$, take $\mathbf{d} = [1, 0]^\top \Rightarrow \mathbf{d}^\top\nabla f = c_1 > 0$. Any point on the line $x_2 = 0$ except $[1, 0]^\top$, take $\mathbf{d} = [1, 0]^\top \Rightarrow \mathbf{d}^\top\nabla f = c_1 > 0$. At the point $[1, 0]^\top$, $d_1 < 0$ and $d_2 > d_1$ because $\mathbf{d}$ needs to be below the line $x_1 + x_2 = 1$. Thus, $d_1c_2 + d_2c_2 \leq 0$

   (b) Suppose x^* is a minimizer. It follows x^* must be a critical point. Thus, $\nabla f(x^*) = 0$. Since, $\nabla f(x^*) = 0$, x^* satisifies the FONC. Suppose x^* satisifies the FONC. Thus, $\nabla f(x^*) = 0$. Because $Q > 0$, f is convex and x^* is a global minimizer.

   (c) Suppose x^* is a minimizer. Let $\phi(\alpha) = f(x^* + \alpha d)$. It follows there exists some $\alpha_0 > 0$ s.t $\phi(0) \leq \phi(\alpha)\ \forall \alpha \in [0, \alpha_0]$. Taylor expanding $\phi(\alpha)$ at $\alpha = 0$ we get $0 \leq \alpha d^\top\nabla f(x^*) + \frac{\alpha^2}{2}d^\top Qd$. Because $-\frac{\alpha}{2}d^\top Qd \leq d^\top\nabla f(x^*)$ for all α, for sufficiently small α $0 \leq \nabla d^\top f(x^*)$.

7. (a) $f(x) = \frac{1}{2}(A(x-a))^\top(A(x-a)) = \frac{1}{2}(x-a)^\top A^2(x-a)$, so $\nabla f = A^2(x-a) = 0$ when $x = a$

   (b) $\alpha_k = \frac{\|A^2(x^{(k)}-a)\|^2}{\|A^3(x^{(k)}-a)\|^2}$

   (c) $\alpha_k = \frac{1}{c^2}$. $x^{(k+1)} = x^{(k)} - \frac{c^2(x^k - a)}{c^2} = a$ Thus, the steepest descent converges in 1 iteration.