(Heine-Borel Theorem) Let (R^n, d) be a Euclidean space with either the Euclidean metric, the taxicab metric, or the sup norm metric. Let E be a subset of R^n. Then E is compact if and only if it is closed and bounded.

(Continuous maps preserve compactness). Let $f : X \to Y$ be a continuous map from one metric space (X, dX) to an- other (Y, dY). Let $K \subseteq X$ be any compact subset of X. Then the image $f(K) := f(x) : x \in K$ of K is also compact.

(Maximum principle). Let (X, d) be a compact metric space, and let $f : X \to R$ be a continuous function. Then f is bounded. Furthermore, f attains its maximum at some point $x_{max} \in X$, and also attains its minimum at some point $x_{min} \in X$.
Continuous functions on compact sets are automatically uniformly continuous.

(Connected spaces). Let (X, d) be a metric space. We say that X is disconnected iff there exist disjoint non-empty open sets V and W in X such that $V \cup W = X$. We say that X is connected iff it is non-empty and not disconnected. (The connectedness of X is equivalent to for any $x, y \in X$ where $x < y$ $[x, y] \subseteq X$)
Continuity preserves connectedness.

Let (X, dX) and (Y, dY) be metric spaces, let E be a subset of X, and let $f : X \to Y$ be a function. Let $x_0 \in X$ be an adherent point of E and $L \in Y$. Then the following four statements are logically equivalent: (a) $\lim_{x \to x_0; x \in E} f(x) = L$, (b) For every sequence $(x^{(n)})_{n=1}^{\infty}$ in E which converges to x_0 with respect to the metric dX, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to L with respect to the metric dY, (c) For every open set $V \subset Y$ which contains L, there exists an open set $U \subset X$ containing x_0 such that $f(U \cap E) \subseteq V$, (d) If one defines the function $g : E \cup x_0 \to Y$ by defining $g(x_0) := L$, and $g(x) := f(x)$ for $x \in E \setminus x_0$, then g is continuous at x_0. Furthermore, if $x_0 \in E$, then $f(x_0) = L$.

(Pointwise convergence). For every x and every $\epsilon > 0$ there exists $N > 0$ such that $dY\ (f(n)(x), f(x)) < \epsilon$ for every $n > N$. We call the function f the pointwise limit of the functions $f^{(n)}$.

(Uniform convergence). For every $\epsilon > 0$ there exists $N > 0$ such that $dY\ (f(n)(x), f(x)) < \epsilon$ for every $n > N$ and every x. We call the function f the uniform limit of the functions $f^{(n)}$.

Uniform limits preserve Continuity. We can exchange the order of limits and uniform convergence in complete metric spaces.

(Bounded functions). A function $f : X \to Y$ from one metric space (X, dX) to another (Y, dY) is bounded if $f(X)$ is a bounded set, i.e., there exists a ball $B(Y, dY)(y_0, R)$ in Y such that $f(x) \in B(Y, dY)(y_0, R)$ for all $x \in X$. (Uniform limits preserve boundedness)

(Metric space of bounded functions). $B(X \to Y) := \{f | f : X \to Y \text{is a bounded function}\}$ with notion of distance $d_\infty(f, g) := \sup\{dY(f(x), g(x)) : x \in X\}$ (The space of continuous functions is complete.)

(Sup norm). $||f||_\infty = \sup\{|f(x)| : x \in X\}$ (Weierstrauss M-test). $\displaystyle\sum_{n=1}^{\infty} f^{(n)}$ converges uniformly if $\displaystyle\sum_{n=1}^{\infty} ||f^{(n)}||_\infty$ converges.

Let $[a, b]$ be an interval, and for each integer $n \geq 1$, let $f(n) : [a, b] \to R$ be a Riemann-integrable function. Suppose $f(n)$ converges uniformly on $[a, b]$ to a function $f : [a, b] \to R$. Then f is also Riemann integrable, and $\displaystyle\lim_{n\to\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} f$.

If f'_n converges uniformly, and $f_n(x_0)$ converges for some x_0, then f_n also converges uniformly, and $\frac{d}{dx} \lim_{n\to\infty} f^{(n)}(x) = \lim_{n\to\infty} \frac{d}{dx} f^{(n)}(x)$.

Exchanging the order of series and integration/differentiation uses the same logic as exchanging limits with integration/differentiation.