Math 106: Problem Set 8

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10.4.1 The Taylor series of $\frac{\sin\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!}\ldots$ is a polynomial with roots at π^2k^2 for $k \in \mathbb{N}$. Using Descartes's factor theorem, we can write $\frac{\sin\sqrt{x}}{\sqrt{x}} = (1-\frac{x}{\pi^2})(1-\frac{x}{4\pi^2})(1-\frac{x}{9\pi^2})\ldots$ as a product of its roots. Using the composition of functions, $\frac{\sin\sqrt{x^2}}{\sqrt{x^2}} = (1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2})(1-\frac{x^2}{9\pi^2})\ldots$. $\frac{\sin|x|}{|x|} = \frac{\sin x}{x}$ by properties of odd and even functions (Consider casewise $x<0$ and $x \geq 0$). Multiplying by x, we obtain $\sin x = x\displaystyle\prod_{k\in\mathbb{N}}(1-\frac{x^2}{\pi^2k^2})$.

10.4.2 $\sin\frac{\pi}{2} = 1$. Thus, $1 = \dfrac{\pi}{2}\displaystyle\prod_{k\in\mathbb{N}}(\frac{4k^2-1}{4k^2}) \Rightarrow \frac{2}{\pi} = \prod_{k\in\mathbb{N}}\frac{2k-1}{2k}\frac{2k+1}{2k}$. Taking the reciprocal and dividing by two, we obtain $\frac{\pi}{4} = \frac{1}{2}\prod_{k\in\mathbb{N}}\frac{2k}{2k-1}\frac{2k}{2k+1}$

10.6.1 $\displaystyle\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{5}}(\phi^{n+1}-(1-\phi)^{n+1})}{\frac{1}{\sqrt{5}}(\phi^n-(1-\phi)^n)} = \lim_{n\to\infty}\frac{\phi^{n+1}-(1-\phi)^{n+1}}{\phi^n-(1-\phi)^n}$. Observe $|1-\phi|<1 \Rightarrow \displaystyle\lim_{n\to\infty}(1-\phi)^n = 0$. Thus, $\displaystyle\lim_{n\to\infty}\frac{\phi^{n+1}-(1-\phi)^{n+1}}{\phi^n-(1-\phi)^n} = \lim_{n\to\infty}\frac{\phi^{n+1}}{\phi^n} = \phi = \frac{1+\sqrt{5}}{2}$

10.7.1 Suppose there are finitely many primes $p_1, p_2, \ldots, p_n$. The Euler product is defined to be a generating formula for $\zeta(s)$.

$$\zeta(1) = \prod_{k=1}^{n}\frac{1}{1-\frac{1}{p_k}} = \prod_{k=1}^{n}\sum_{m=0}^{\infty}\frac{1}{p_k^m} = 1 + \sum\frac{1}{p_1^{m_1}p_2^{m_2}\ldots p_n^{m_n}} = \sum_{n=1}^{\infty}\frac{1}{n}$$

To derive a contradiction, we show the two series are not equal.
Trivially, every term of the series $1+\sum\dfrac{1}{p_1^{m_1}p_2^{m_2}\ldots p_n^{m_n}}$ is the reciprocal of a natural number. Thus, it suffices to show $1+\sum\dfrac{1}{p_1^{m_1}p_2^{m_2}\ldots p_n^{m_n}}$ is missing terms of the harmonic series. Consider $\frac{1}{p_1p_2\ldots p_n+1}$. $p_1p_2\ldots p_n+1$ is a natural number, so clearly, $\frac{1}{p_1p_2\ldots p_n+1}$ is a term of the harmonic series.

However, $p_1p_2\ldots p_n + 1$ is not divisible by any of $p_1, p_2, \ldots, p_n$. Thus, $1 + \sum \dfrac{1}{p_1^{m_1}p_2^{m_2}\ldots p_n^{m_n}}$ is missing the term $\frac{1}{p_1p_2\ldots p_n+1}$. Hence, we obtain a contradiction because the Euler product cannot be a generating function for the $\zeta(s)$ if we have finitely many primes.