Math 131B: Homework 3

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Problem 1. **Exercise 2.2.4**

Let $\epsilon > 0$ and choose $\delta = \epsilon$. If $d((x_1, y_1), (x_2, y_2)) < \delta$, then
$\Rightarrow d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} < \delta$
$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 < \delta^2$
Because the square of a number is non-negative, $|x_1 - x_2| < \delta = \epsilon$ and $|y_1 - y_2| < \delta = \epsilon$.
$\Rightarrow d(\pi_1(x_1, y_1), \pi_1(x_2, y_2)) < \epsilon$ and $d(\pi_2(x_1, y_1), \pi_2(x_2, y_2)) < \epsilon$
Hence, π_1 and π_2 are continuous.
Since $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ is continuous, and $f : \mathbb{R} \to X$ is continuous, their compositions $f \circ \pi_1 : \mathbb{R}^2 \to X$ and $f \circ \pi_2 : \mathbb{R}^2 \to X$ are also continuous. Because $g_1(x, y) := f(\pi_1(x, y)) = f(x)$ and $g_2(x, y) := f(\pi_2(x, y)) = f(y)$, $g_1(x, y)$ and $g_2(x, y)$ are continuous.

QED

Problem 2. **Exercise 2.2.10**

Let $g_y : \mathbb{R} \to \mathbb{R}^2$ $g_y(x) = (x, y)$ for some $y \in \mathbb{R}$ and let $g_x : \mathbb{R} \to \mathbb{R}^2$ $g_x(y) = (x, y)$ for some $x \in \mathbb{R}$.

We want to show $f_y(x) = f \circ g_y(x)$ and $f_x(y) = f \circ g_x(y)$ are continuous.

Let $\epsilon > 0$ and choose $\delta = \epsilon$.
If $d(x_1, x_2) < \delta$, then
$\Rightarrow |x_1 - x_2| < \delta$
$\Rightarrow \sqrt{(x_1 - x_2)^2 + (y - y)^2} < \delta = \epsilon$
$\Rightarrow d(g_y(x_1), g_y(x_2)) < \epsilon$
Hence g_y is continuous.

If $d(y_1, y_2) < \delta$, then
$\Rightarrow |y_1 - y_2| < \delta$
$\Rightarrow \sqrt{(x - x)^2 + (y_1 - y_2)^2} < \delta = \epsilon$
$\Rightarrow d(g_x(y_1), g_x(y_2)) < \epsilon$
Hence g_x is continuous.

Due to continuity preserved by composition, $f_x(y)$ and $f_y(x)$ are continuous. Since x and y are chosen arbitrarily, $y \rightarrowtail f(x, y)$ $\forall x \in \mathbb{R}$ and $x \rightarrowtail f(x, y)$ $\forall y \in \mathbb{R}$ are continuous separately.

Problem 3. **Exercise 2.2.11**

Let $g_y(x)$ and $g_x(y)$ be the functions defined in the previous problem.

We want to show $f(g_y(x))$ and $f(g_x(y))$ are continuous, but $f(x, y)$ is not continuous.

Let y and x_0 be arbitrary.
Let $\epsilon > 0$ and choose $\delta = \min\{|\frac{x_0}{2}|, \frac{\epsilon x_0^2}{2y}\}$
If $d(x, x_0) < \delta$ then
$d(f(g_y(x)), f(g_y(x_0))) = |\frac{xy}{x^2+y^2} - \frac{x_0 y}{x_0^2+y^2}|$
$= |\frac{(x-x_0)yxx_0}{(x^2+y^2)(x_0^2+y^2)}| \leq |\frac{(x-x_0)y}{(xx_0)}|$
$\leq |\frac{(x-x_0)2y}{x_0^2}| < |\frac{\delta 2y}{x_0^2}| \leq |\frac{\epsilon 2y(x_0^2)}{2y(x_0^2)}| = \epsilon$
Hence, $f(g_y(x))$ is continuous at x_0.

Let x and y_0 be arbitrary.
Let $\epsilon > 0$ and choose $\delta = \min\{|\frac{y_0}{2}|, \frac{\epsilon y_0^2}{2x}\}$
If $d(y, y_0) < \delta$ then
$d(f(g_x(y)), f(g_x(y_0))) = |\frac{xy}{x^2+y^2} - \frac{xy_0}{x^2+y_0^2}|$
$= |\frac{(y-y_0)xyy_0}{(x^2+y^2)(y_0^2+x^2)}| \leq |\frac{(y-y_0)x}{(yy_0)}|$
$\leq |\frac{(y-y_0)2x}{y_0^2}| < |\frac{\delta 2x}{y_0^2}| \leq |\frac{\epsilon 2x(y_0^2)}{2x(y_0^2)}| = \epsilon$
Hence, $f(g_x(y))$ is continuous at y_0.

Next, we will show $f(x, y)$ is not continuous at the origin.

Let $\epsilon = \frac{1}{2}$ and set $y = x$. If $(x, y) = (\frac{\sqrt{2}\delta}{4}, \frac{\sqrt{2}\delta}{4})$ then $d((x, y), (0, 0)) = \frac{\delta}{2} < \delta$. It follows $\frac{(\frac{\sqrt{2}\delta}{4})^2}{(\frac{\sqrt{2}\delta}{4})^2 + (\frac{\sqrt{2}\delta}{4})^2} = \frac{1}{2}$. Hence for all $\delta > 0$ there exists $d(f(x, y), f(0, 0)) \geq \frac{1}{2}$. Therefore, $f(x, y)$ is not continuous.
QED

Problem 4. **Exercise 2.2.12**

Let (x, y) be arbitrary. Let $\epsilon > 0$ and choose $\delta = \frac{\epsilon y}{x^2}$. If $|t| < \delta$ then $|f(xt, yt)| = |\frac{(xt)^2}{yt}| = |\frac{x^2 t}{y}| < \frac{\epsilon y x^2}{y x^2} = \epsilon$. Hence, $\lim_{t \to 0} f(xt, yt) = 0$.
f is continuous if $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$ for all paths. To show $f(x, y)$ is discontinuous at $(0, 0)$, we will show $\lim_{t\to 0} f(t, t^2) \neq 0$. Let $\epsilon = \frac{1}{2}$ and let $\delta > 0$. If $\sqrt{t^2 + t^4} \leq \delta$ then $|t| \leq \sqrt{\frac{-1+\sqrt{1+4\delta^2}}{2}}$. It follows $|f(\sqrt{\frac{-1+\sqrt{1+4\delta^2}}{2}}, \frac{-1+\sqrt{1+4\delta^2}}{2}) - f(0, 0)| = |\frac{\frac{-1+\sqrt{1+4\delta^2}}{2}}{\frac{-1+\sqrt{1+4\delta^2}}{2}} - 0| = 1 > \epsilon$.
Moreover, for every $\delta > 0$ there exists a (x', y') within δ of some (x, y) s.t $d((f(x'), f(y')), (f(x), f(y))) \geq \epsilon$.
QED

Problem 5. **Exercise 2.3.3**

Let $f : X \to Y$ be uniformly continuous. For every $\epsilon > 0$ there exists $\delta > 0$ s.t $d(f(x_1), f(x_2)) < \epsilon$ whenever $d(x_1, x_2) < \delta$. Let $x_0 \in X$. Then for every $\epsilon > 0$ $d(f(x), f(x_0)) < \epsilon$ whenever $d(x, x_0) < \delta$ by uniform continuity. Hence, f is continuous.
Let $g(x, y) = x^2 + y^2 + x - y$.
Let $\epsilon > 0$ and choose $\delta = \min\{1, \frac{\epsilon}{2(2+|x_0|+|y_0|)}\}$. If $d((x, y), (x_0, y_0)) < \delta$ then $|x - x_0| < \delta$ and $|y - y_0| < \delta$. It follows $|x+x_0+1||x-x_0| < |x+x_0+1|\delta \leq 2(|x_0|+1)\delta$ and $|y+y_0-1||y-y_0| < |y+y_0-1|\delta \leq 2(|y_0|+1)\delta$ by the triangle innequality. Thus, $d(g(x, y), g(x_0, y_0)) = |x^2 - x_0^2 + y^2 - y_0^2 + x - x_0 - y + y_0| \leq |x+x_0+1||x-x_0| + |y+y_0-1||y-y_0| \leq 2\delta(2+|x_0|+|y_0|) \leq \frac{2(2+|x_0|+|y_0|)\epsilon}{2(2+|x_0|+|y_0|)} < \epsilon$. Hence, g is continuous.
However $|(x + \frac{\sqrt{2\delta}}{2})^2 - (x)^2 + (y + \frac{\sqrt{2\delta}}{2})^2 - (y)^2 + (x + \frac{\sqrt{2\delta}}{2}) - x + y - (y + \frac{\sqrt{2\delta}}{2})| = |\frac{\sqrt{2\delta}}{2}(2x + \frac{\sqrt{2\delta}}{2}) + \frac{\sqrt{2\delta}}{2}(2y + \frac{\sqrt{2\delta}}{2})| = |\sqrt{2\delta}(x + y) + \delta| \geq \epsilon$ whenever $|x + y|$ is sufficiently large. Moreover, for every $\delta > 0$ there exists a (x', y') within δ of some (x, y) s.t $d((f(x'), f(y')), (f(x), f(y))) \geq \epsilon$. Hence, g is not uniformly continuous.
QED

Problem 6. **Exercise 2.3.4**

Let $\epsilon > 0$. By the uniform continuity of g, there exists a $\delta_1 > 0$ s.t $d(g(x), g(x')) < \epsilon$ whenever $d(x, x') < \delta_1$. By the uniform continuity of f, choose $\delta > 0$ to be small enough s.t $d(f(x), f(x')) < \delta_1$ whenever $d(x, x') < \delta$. Because $d(f(x), f(x')) < \delta_1$, we obtain $d(g(f(x)), g(f(x'))) < \epsilon$. Hence, $g \circ f$ is uniformly continuous.
QED

Problem 7. **Exercise 2.3.5**

Not a homework problem but used in the solution for 2.3.6

Let $\epsilon > 0$. By the uniform continuity of f and g, choose δ to be small enough so $d(f(x), f(x')) < \frac{\epsilon}{\sqrt{2}}$ and $d(g(x), g(x')) < \frac{\epsilon}{\sqrt{2}}$. It follows $d((f(x), g(x)), (f(x'), g(x'))) = \sqrt{(g(x) - g(x'))^2 + (f(x) - f(x'))^2} < \sqrt{2\frac{\epsilon^2}{2}} = \epsilon$ whenever $d(x, x') < \delta$.

Problem 8. **Exercise 2.3.6**

Let $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{2}$. If $d((x, y), (x', y')) < \delta$, this implies $|x - x'| < \delta$ and $|y - y'| < \delta$. It follows $|(x + y) - (x' + y')| = |(x - x') + (y - y')| \leq |x - x'| + |y - y'| < 2\delta = 2\frac{\epsilon}{2} = \epsilon$. Hence, addition is uniformly continuous.
Let $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{2}$. If $d((x, y), (x', y')) < \delta$, this implies $|x - x'| < \delta$ and $|y - y'| < \delta$. It follows $|(x - y) - (x' - y')| = |(x - x') - (y - y')| \leq |x - x'| + |y' - y| < 2\delta = 2\frac{\epsilon}{2} = \epsilon$. Hence, subtraction is uniformly continuous.
$|(\frac{\sqrt{2\delta}}{2} + x)|(\frac{\sqrt{2\delta}}{2} + y) - (xy)| = |\frac{\sqrt{2\delta}}{2}(x + y) + \frac{\delta}{2}| \geq \epsilon$ whenever $|x + y|$ is sufficiently large. Hence, multiplication is not uniformly continuous.
By exercise 2.3.5, we know that the direct sum preserves uniform continuity. Because the addition and

subtraction functions are uniformly continuous from $\mathbb{R}^2 \to \mathbb{R}$, $f+g$ and $f-g$ are uniformly continuous if f and g are uniformly continuous from $X \to \mathbb{R}$.
Let $f(x) = x+2$, $g(x) = \frac{x}{2}$. $|(x + \frac{\sqrt{2\delta}}{2} + 2)(\frac{x+\frac{\sqrt{2\delta}}{2}}{2}) - (x+2)(\frac{x}{2})| = |\frac{\sqrt{2\delta}}{2}(x+2+\frac{x}{2}) + \frac{\delta}{4}| \geq \epsilon$ whenever $|x+2+\frac{x}{2}|$ is sufficiently large.
$max(f,g)$, $min(f,g)$, and cf are uniformly continuous if f and g are uniformly continuous, but f/g is not nexessarily uniformly continuous. For max and min you can choose a δ small enough so that it works for both f and g. For cf you choose a δ small enough so that the difference in fs is less than $\frac{\epsilon}{c}$. If $f(x) = x$ and $g(x) = \frac{1}{1+x^2}$ f/g is not uniform continuous while f and g both are. QED

Problem 9. **Additional Problem**

a) f is uniformly continuous because for every $\epsilon > 0$, if $\delta = \epsilon$, $|f(x) - f(y)| < \epsilon$ whenever $|x-y| < \delta$.

b) f is not uniformly continuous because if $\epsilon < 1$ there exist $x \neq y$ for every $\delta > 0$ which implies there exists $d_{disc}(f(x), f(y)) = 1 > \epsilon$.

c) f is uniformly continuous because for every $\epsilon > 0$, if $\delta = 1$, $|f(x) - f(y)| = 0 < \epsilon$ whenever $d_{disc}(x,y) < \delta$.

d) f is uniformly continuous because for every $\epsilon > 0$, if $\delta = 1$, $d_{disc}(f(x), f(y)) = 0 < \epsilon$ whenever $d_{disc}(x,y) < \delta$.