Math 142: Modeling	**Name:** ______________________
Homework # 1	**Due: Friday Oct6**

Directions Complete the exercises. Your solutions to the exercises should be submitted to Gradescope before the indicated due date above. Please follow rules regarding Gradescope submission as described in the syllabus.

References Except for the help of the instructor or TAs and the class textbooks and notes, if you use any resources, for example, a book, a website, or you discussed with your friends, please acknowledge them in this References section.

- I discussed Problem ?? with STUDENT A, STUDENT B, ...
- I used BOOK/WEBSITE to help me do Problem ??.

Exercises

1. Review of linear algebra and calculus

   (a) Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$$

   Solution: We find λ_1, λ_2 s.t $\det(A - \lambda I) = 0$. Solving the quadratic $(1-\lambda)^2 - 9 = 0$, we obtain $\lambda_1 = -2, \lambda_2 = 4$.

   (b) It will often be useful to know whether a particular matrix has an eigenvalue bigger than 1. Without explicitly calculating its eigenvalues, show that the matrix

$$B = \begin{pmatrix} 1 & 3 & 1 \\ -3 & 0 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

   has one eigenvalue that is larger than 1 . Hint: start by calculating $\det(B - I)$. Then explain why $\det(B - \lambda I)$ is negative for sufficiently large values of λ. Finally explain then why $\det(B - \lambda I)$ must be go through zero for some $\lambda > 1$.

   Solution: WTS by the intermediate value theorem, there exists a $\lambda > 1$ s.t $\det(B - \lambda I) = 0$.
   $\det(B - I) = 0(-6) - (-3)(14) + 0(4) = 42$
   Let $f(t) := \det(B - tI)$ which is a cubic polynomial, a continuous function. By Cramer's rule, the only contribution to the cubic term comes from $(1-t)\begin{vmatrix} -t & 1 \\ 1 & 6-t \end{vmatrix}$. It follows that the constant coefficient of the cubic term of the function $f(t)$ is -1, and because the rate of growth of the cubic term dominates the lower order terms, $f(t) < 0$ for sufficiently large enough values of t. Thus, because $f(t) < 0 < f(1)$ for sufficiently large t, the IVT states there exists some $1 < \lambda < t$ s.t $f(\lambda) = \det(B - \lambda I) = 0$ Hence, B has an eigenvalue greater than 1.

(c) The Michaelis-Menten formula is often used as a model for how the rate of a chemical reaction depends on the amount of chemical, x that is reacting. According to the formula the rate of the reaction is:

$$r(x) = \frac{kx}{x+a}$$

where k and a are both positive constants, and the amount of chemical, $x \geq 0$. We will use calculus tools to draw the graph of the function $r(x)$.

(a) Show that $r'(x) > 0$. That is, r is an increasing function.

Solution:

$$r'(x) = \frac{d}{dx} r(x) = k \frac{(x+a) - x}{(x+a)^2} = k \frac{a}{(x+a)^2} > 0$$

for all $x \geq 0$ because the square of a number is always positive and k and a are both positive constants.

(b) Carefully explaining your reasoning, show that $r(x) \to k$ as $x \to \infty$.

Solution:

$$\lim_{x\to\infty} r(x) = k \frac{x}{x+a} = k \lim_{x\to\infty} \frac{x}{x+a} = k \frac{1}{1} = k$$

by infinity over infinity L'H rule.

(c) Show that $r''(x) < 0$. What does this mean for the shape of the graph of $r(x)$?

Solution:

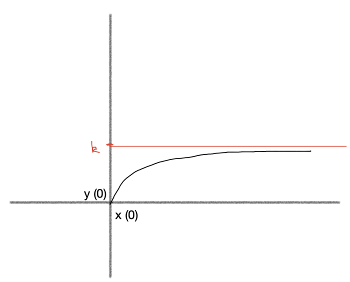
$$r''(x) = \frac{d}{dx} r'(x) = k \frac{-2a}{(x+a)^3} < 0$$

because $(x+a)^3 > 0$ for $x \geq 0$ (multiplying positive numbers yields a positive product) and k and a are both positive constants, so $k \frac{-2a}{(x+a)^3} < 0$.

Thus, we can expect the graph of $r(x)$ to assymptotically approach k from below with an x and y intercept of 0. We can also say that the function is always increasing, but the rate at which it is increasing is decreasing (concave down).

(d) Putting all of your information together, draw a sketch showing the shape of the graph of $r(x)$ as a function of x.

Solution:



(e) We will meet compartment models, which show up as models for how matter or energy moves through systems. As part of the modeling, we will derive a formula for the concentration of a solute in a tank, that has inflows and outflows. We will find that the concentration of solute, $c(t)$, evolves according to an equation:

$$\frac{dc}{dt} = c_\infty - c$$

where c_∞ will be the constant concentration of solute in the tank inflow. (You don't need to derive this equation). Solve this differential equation, assuming that $c(0) = c_0$ (the unknown constants c_0 and c_∞ will appear in your answer). Then, by analyzing your solution, show that, no matter what the value of $c_0, c(t) \to c_\infty$ as $t \to \infty$

Solution:

$$\frac{dc}{dt} = c_\infty - c \Rightarrow \frac{dc}{dt} + c = c_\infty \Rightarrow c(t) = \frac{\int e^{\int 1dt} c_\infty dt + c}{e^{\int 1dt}} = c_\infty + ce^{-t} = c_\infty + (c_0 - c_\infty)e^{-t}$$

given the initial condition of $c(0) = c_0$

Since $\lim_{t\to\infty} e^{-t} = 0$,

$\Rightarrow \lim\limits_{t\to\infty} c(t) = \lim\limits_{t\to\infty} c_\infty + (c_0 - c_\infty)e^{-t} = c_\infty + (c_0 - c_\infty) \lim\limits_{t\to\infty} e^{-t} = c_\infty + 0 = c_\infty$

2. Our model for population growth started from a word equation for how population size changes from time k to time $k + 1$; this question will give you some practice on how to write down these word equations:, both for population growth and for other systems that can be modeled using recurrence equations.

   You are building a mathematical model for the number of students attending UCLA. Write a word equation relating to the population N_k in one quarter to the population N_{k+1} in the next quarter. Your **word** equation should include the following terms.

   - # students admitted to the university that quarter
   - # students that completed their degree (graduated)
   - # students that returned from a break
   - # students that are taking the quarter off

   Solution:

   $$\begin{pmatrix} \text{\# students} \\ \text{@ quarter } k+1 \end{pmatrix} = \begin{pmatrix} \text{\# students} \\ \text{@ quarter } k \end{pmatrix} + \begin{pmatrix} \text{\# returned from break} \\ \text{@ quarter } k \end{pmatrix} + \begin{pmatrix} \text{\# admitted} \\ \text{@ quarter } k \end{pmatrix} - \begin{pmatrix} \text{\# quarter off} \\ \text{@ quarter } k \end{pmatrix} - \begin{pmatrix} \text{\# completed degree} \\ \text{@ quarter } k \end{pmatrix}$$

3. You are trying to build a mathematical model for the population of red-wolves in North Carolina. Red wolves are a critically endangered species, once found throughout the southeastern United States but now found mainly in a single North Carolina wildlife reserve. The data in this question come from a report published by the U.S. Fish and Wildlife Service in 2007: "Red Wolf (Canis rufus) 5-Year Status Review: Summary and Evaluation".

   (a) If the population size t-years after the start of a study on red wolf numbers is N_t, start by writing down a word equation so that N_{t+1} can be predicted from N_t :

   $$N_{t+1} = N_t + (\text{ number of pups born }) - (\text{ number of wolves that die})$$

   We will start by deriving expressions for the birth and death rates.

(a) It is possible to count the number of pups born in given year. A subset of these data (taken from the US Fish and Wildlife Service report) is given in the following table:

Year	N_t	Number of pups born
1990	18	3
1993	44	16
1996	70	16
1999	126	37
2002	123	33
2005	115	41

Plot the number of pups against the population size N_t. Show that the data are consistent with the following formula for the number of pups born in one year: Number of pups born in one year $= 0.28N_t$

Solution:

In [1]:
```
import pandas as pd
import numpy as np
import seaborn as sns
import matplotlib.pyplot as plt
```

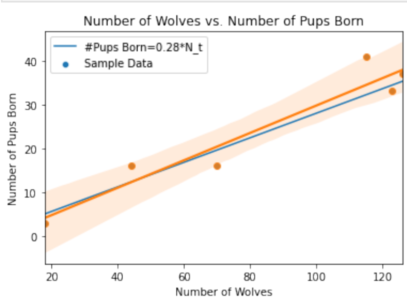
In [2]:
```
df = pd.DataFrame(
{
    "year": [1990,1993,1996,1999,2002,2005],
    "N_t": [18,44,70,126,123,115],
    "pups_born": [3,16,16,37,33,41]})
```

In [3]:
```
df.set_index("year")
```

Out[3]:

	N_t	pups_born
year		
1990	18	3
1993	44	16
1996	70	16
1999	126	37
2002	123	33
2005	115	41

In [4]:
```
sns.scatterplot(data=df,x="N_t",y="pups_born",legend="auto", label="Sample Data")
sns.lineplot(data=df, x="N_t",y=df["N_t"].multiply(0.28),legend="auto",label="#Pups Born=0.28*N_t")
sns.regplot(data=df,x="N_t",y="pups_born")
plt.title("Number of Wolves vs. Number of Pups Born")
plt.xlabel("Number of Wolves")
plt.ylabel("Number of Pups Born")
plt.savefig("Q_2_Hw_1_142.png")
```



In [ ]:

The sample regression line doesn't differ significantly from the proposed linear relationship between the number of wolves and number of pups born, so it would not be unreasonable to assume that the sample data comes from a population that is consistent with the formula Number of pups born in one year$= 0.28N_t$.

(b) Conservationists also measure the number of red wolf deaths in each year: the main causes of death are being killed by hunters and ranchers, or by being struck by vehicles. They estimate that in one year 22% of red wolves are killed. Write down recurrence relation for the population size N_t including both birth and death rates.

Solution:

$N_{k+1} = N_k + b\Delta t N_k - m\Delta t N_k$

$\Delta t = 1 \text{ year}, b = 0.28, m = 0.22$

$N_{t+1} = N_t + 0.28N_t - 0.22N_t = (1.06)N_t$

(c) Assuming that the current population size is $N_0 = 130$ wolves, use your formula from (a) to predict the population size N_t for the next 5 years (that is, calculate $N_1, N_2, \ldots, N_5$).

Solution:

$N_1 = (1.06)N_0 = 137.8$

$N_2 = (1.06)N_1 = 146.1$

$N_3 = (1.06)N_2 = 154.8$

$N_4 = (1.06)N_3 = 164.1$

$N_5 = (1.06)N_4 = 174.0$

(d) The current conservation goal for the wild red wolf population is to reach 220 individuals. When, according to your model, will that population size be reached?

Solution:

$N_t = (1.06)^t N_0$

$\log(N_t) = t\log(1.06) + \log(N_0)$

$\Rightarrow t = \frac{\log(\frac{N_t}{N_0})}{\log(1.06)}$

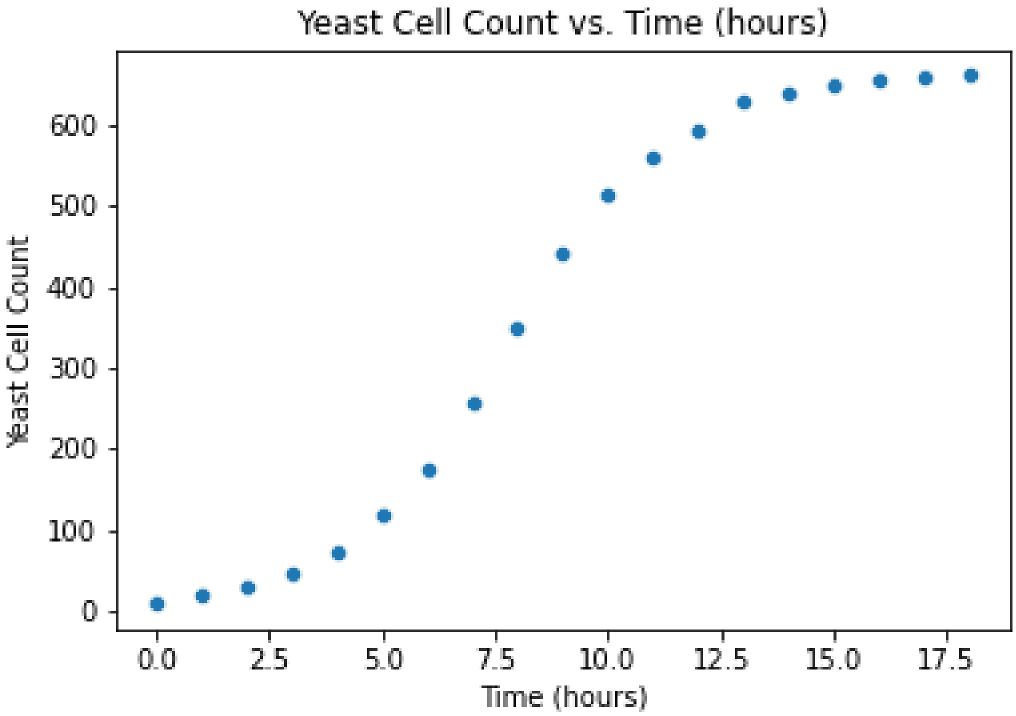
$N_t = 220, N_0 = 130$

$\Rightarrow t = \frac{\log(\frac{220}{130})}{\log(1.06)} = 9.0287$

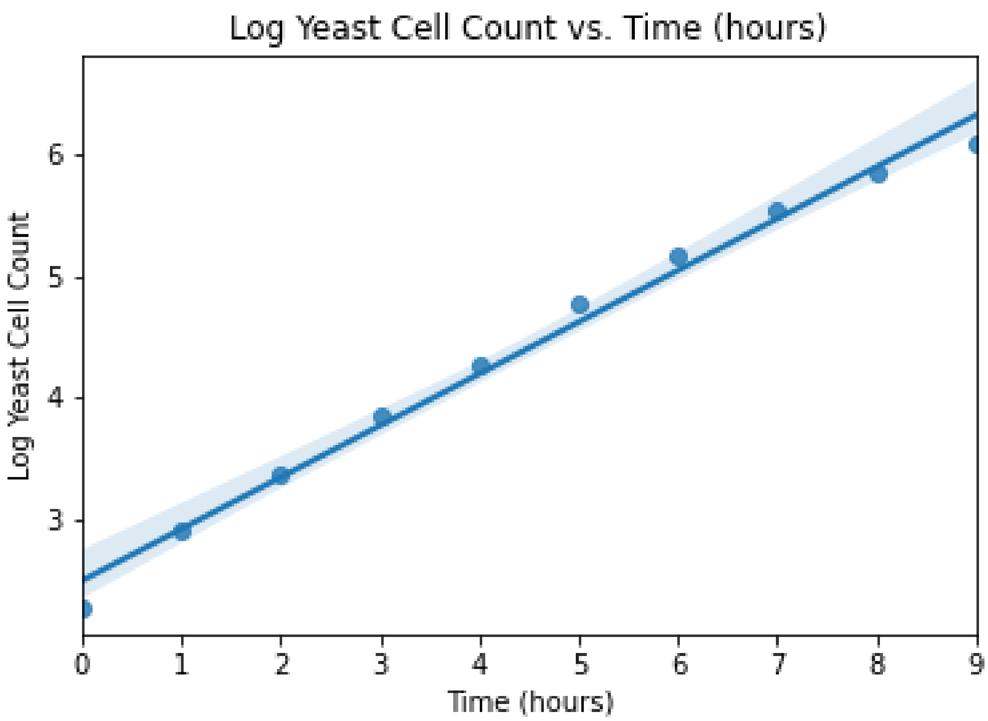
The population will have reached 220 individuals after 10 (*9.0287*) years.

4. You are trying to use your population growth model from class to model the growth of a population of yeast cells, and to compare to real data. The real data we will use is from a paper by Carlson. You should start by downloading the data from the BioQuest website: Bioquest Data The data consists of a count of yeast cells (the count is measured in terms of millions of cells per ml of the growth medium), as a function of time.

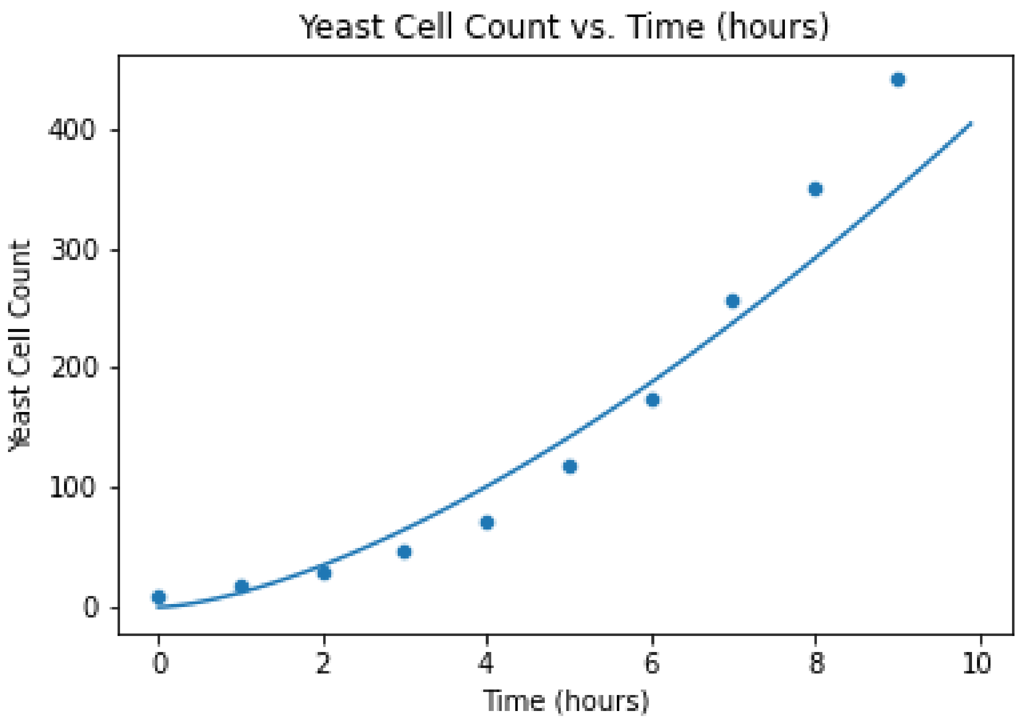
(a) Show by plotting the data for population size against time that the population size in Carlson's experiment that the population does not grow exponentially (or at least, does not grow exponentially over the entire time covered by the experiment). We discussed, in class, that populations can not grow exponentially indefinitely because cells compete for resources.



(b) In fact, in the first few hours of the experiment, the population growth is approximately exponential. Explain how to plot your data in a way that shows that for early times there is exponential growth, and estimate the reproductive rate associated with that exponential growth.



We can plot the log of the Yeast Cell Count against Time (hours). If there exists a linear relationship between the transformed variables for the first few hours ($t < 10$), then the population growth is approximately exponential.



$N_{k+1} \approx (1.5328)N_k \Rightarrow R_0 \approx 0.5328$

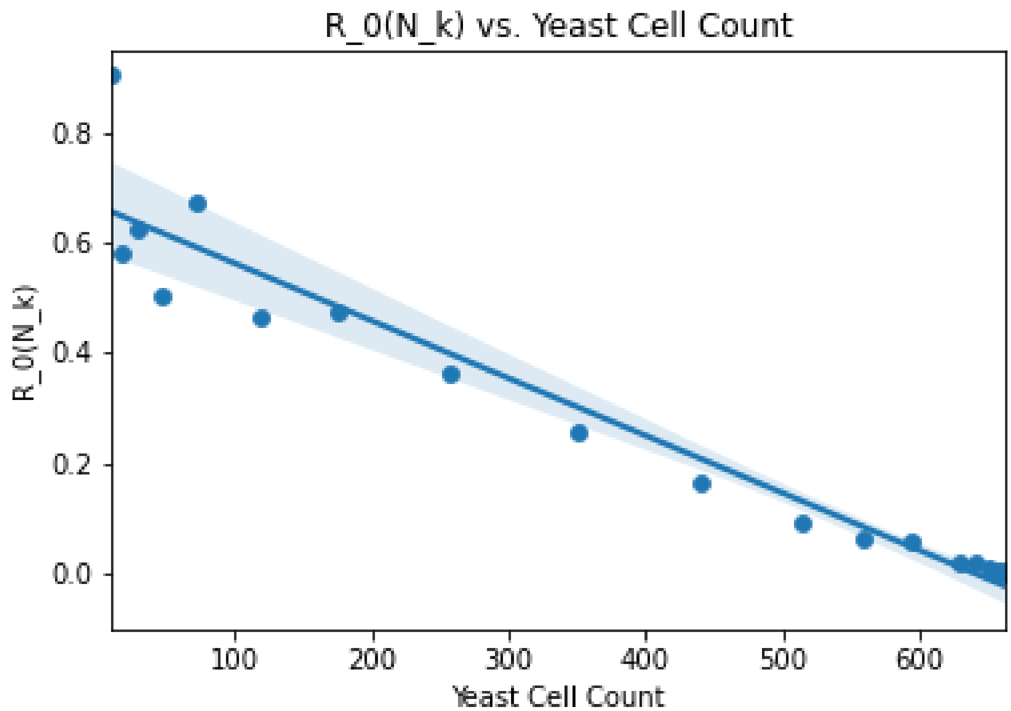
(c) Let's consider how we might modify our model to account for the fact that the reproductive rate is not a constant. Suppose that the reproductive rate R_0 depends on the population size. That is $R_0 \equiv R_0(N_k)$. So our model for population growth is:

$$N_{k+1} = (1 + R_0(N_k)\,\Delta t)\,N_k$$

How can we find the function $R_0(N_k)$? Rearrange our equation in the form:

$$\frac{N_{k+1} - N_k}{N_k \Delta t} = R_0(N_k)$$

the left hand side of the equation is the relative increase in population size between censuses (divided by time). For the Carlson data plot this quantity against N_k.



(d) Some researchers have argued that the correct model for the growing population is

$$R_0\left(N_k\right) = r\left(1 - \frac{N_k}{K}\right),$$

where r and K are positive constants. This model is known as the logistic growth model. Do you think this model is consistent with your graph from (c)? Briefly justify your answer.
The graph in part (c) follows the same inverse relationship as the proposed formula, and there is a strong correlation between the two variables.

(e) The positive constant K is usually called the carrying capacity of the population. Thinking about the new model, can you briefly describe what K represents?
The carrying capacity represents the max number of organisms that an environment/ecosystem can support. This can occur when environmental pressures caause the net reproduction rate to go to 0.

In [1]:

```python
import pandas as pd
import numpy as np
import seaborn as sns
import matplotlib.pyplot as plt
```

In [2]:

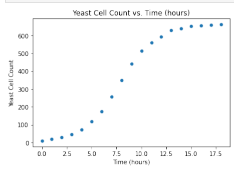
```python
bacteria_data=pd.read_csv("data/M142_Homework1_Data.csv")
bacteria_data
```

Out[2]:

	Time (hours)	Yeast Cell Count
0	0	9.6
1	1	18.3
2	2	29.0
3	3	47.2
4	4	71.1
5	5	119.1
6	6	174.6
7	7	257.3
8	8	350.7
9	9	441.0
10	10	513.3
11	11	559.7
12	12	594.8
13	13	629.4
14	14	640.8
15	15	651.1
16	16	655.9
17	17	659.6
18	18	661.8

In [48]:

```python
sns.scatterplot(data=bacteria_data,x="Time (hours)",y="Yeast Cell Count")
plt.title("Yeast Cell Count vs. Time (hours)")
plt.savefig("Q_4_Hw_1_142_a.png")
```

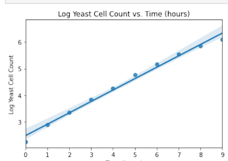


In [10]:

```python
bacteria_exp=bacteria_data[bacteria_data["Time (hours)"]<10]
```

In [50]:

```python
sns.regplot(data=bacteria_exp,x=bacteria_exp["Time (hours)"],y=np.log(bacteria_exp["Yeast Cell Count"]))
plt.ylabel("Log Yeast Cell Count")
plt.title("Log Yeast Cell Count vs. Time (hours)")
plt.savefig("Q_4_Hw_1_142_b.png")
```



In [21]:

```python
fit=np.exp(np.polyfit(bacteria_exp["Time (hours)"],np.log(bacteria_exp["Yeast Cell Count"]),1))
fit
```

Out[21]: array([1.5328075 , 12.03305581])

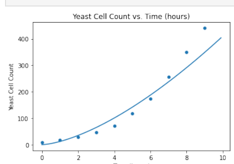
In [40]:

```python
corr=np.corrcoef(bacteria_exp["Time (hours)"], np.log(bacteria_exp["Yeast Cell Count"]))
corr[1][0]
```

Out[40]: 0.9944986908084287

In [49]:

```python
x_0=np.arange(0,10,0.1)
y_0=fit[1]*np.power(x_0,fit[0])
sns.lineplot(x=x_0,y=y_0)
sns.scatterplot(data=bacteria_exp,x=bacteria_exp["Time (hours)"],y=bacteria_exp["Yeast Cell Count"])
plt.title("Yeast Cell Count vs. Time (hours)")
plt.savefig("Q_4_Hw_1_142_c.png")
```



In [25]:

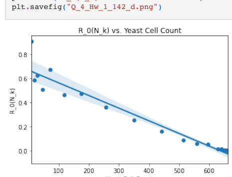
```python
r_0=np.array([])
for x in np.arange(len(bacteria_data)):
    if x==18:
        r_0=np.append(r_0,0)
    else:
        r_0=np.append(r_0,(bacteria_data["Yeast Cell Count"][x+1]-bacteria_data["Yeast Cell Count"][x])/bacteria_data["Yeast Cell Count"][x])
```

In [26]:

```python
bacteria_data["R_0(N_k)"]=r_0
```

In [51]:

```python
sns.scatterplot(data=bacteria_data,x="Yeast Cell Count",y="R_0(N_k)")
sns.regplot(data=bacteria_data,x="Yeast Cell Count",y="R_0(N_k)")
plt.title("R_0(N_k) vs. Yeast Cell Count")
plt.savefig("Q_4_Hw_1_142_d.png")
```



In [41]:

```python
corr_2=np.corrcoef(bacteria_data["Yeast Cell Count"], bacteria_data["R_0(N_k)"])
corr_2[1][0]
```

Out[41]: -0.9652771067429227

In [38]:

```python
fit_2=np.polyfit(bacteria_data["Yeast Cell Count"],bacteria_data["R_0(N_k)"],1)
fit_2
```

Out[38]: array([-0.00104662, 0.66844655])

In [43]:

```python
k=-1*1/fit_2[0]
k
```

Out[43]: 955.4535128353328