Math 131B: Homework 7

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Problem 1. **Exercise 4.2.2**

Let $x_0 \in \mathbb{R} \setminus 1$. We want to show $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(1-x_0)^{n+1}}(x-x_0)^n = \frac{1}{1-x}$ for x between 1 and $2x_0 - 1$. To show this we use the formula for the sum of a geometric series $\displaystyle\sum_{n=0}^{\infty} a \cdot r^n = \frac{a}{1-r}$ for $r \in (-1, 1)$. It follows $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(1-x_0)^{n+1}}(x-x_0)^n = \sum_{n=0}^{\infty} \frac{1}{(1-x_0)}(\frac{x-x_0}{1-x_0})^n = \frac{\frac{1}{(1-x_0)}}{1-\frac{x-x_0}{1-x_0}} = \frac{1}{(1-x_0)-(x-x_0)} = \frac{1}{1-x}$ for x between 1 and $2x_0 - 1$. Hence $f(x)$ is analytic for all $x \in \mathbb{R} \setminus 1$.

Problem 2. **Exercise 4.2.3**

We want to show by induction on k that function $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ which is real-analytic at a has a k^{th} derivative given by $f^{(k)}(x) = \displaystyle\sum_{n=0}^{\infty} c_{n+k}\frac{(n+k)!}{n!}(x-a)^n$. For the base case $k = 0$ we obtain $f^{(0)}(x) = \displaystyle\sum_{n=0}^{\infty} c_{n+0}\frac{(n+0)!}{n!}(x-a)^n = f(x)$, so the claim holds for $k = 0$. Now assume for some arbitrary $k \geq 0$ the claim $f^{(k)}(x) = \displaystyle\sum_{n=0}^{\infty} c_{n+k}\frac{(n+k)!}{n!}(x-a)^n$ holds. To show the $k+1^{st}$ case, we differentiate both sides giving us $f^{(k+1)}(x) = \displaystyle\sum_{n=1}^{\infty} c_{n+k}\frac{(n+k)!}{n!} \cdot n \cdot (x-a)^{n-1} = \sum_{n=1}^{\infty} c_{n+k}\frac{(n+k)!}{(n-1)!}(x-a)^{n-1}$. Reindexing the variable n s.t each $n = n-1$ we obtain $\displaystyle\sum_{n=1}^{\infty} c_{n+k}\frac{(n+k)!}{(n-1)!}(x-a)^{n-1} = \sum_{n=0}^{\infty} c_{n+k+1}\frac{(n+k+1)!}{n!}(x-a)^n$. Thus, the claim holds for $k+1$ Hence, by induction, the claim holds for all k.

Problem 3. **Exercise 4.2.5**

Let a, b be real numbers and let $n \geq 0$ be an integer. It follows $(x-a)^n = ((x-b)+(b-a))$. Using the binomial formula we can expand $(x-a)^n = ((x-b)+(b-a))^n = \displaystyle\sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(x-b)^m \cdot (b-a)^{n-m}$. Hence, $(x-a)^n = \displaystyle\sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(x-b)^m \cdot (b-a)^{n-m}$, so we obtain our desired solution.

Writing a Taylor's expansion of the function $f(x) = (x-a)^n$ centered at $x = b$, we obtain $f(x) = \displaystyle\sum_{m=0}^{\infty} \frac{f^{(m)}(b)}{m!}(x-b)^m = \sum_{m=0}^{\infty} \frac{(b-a)^{n-m}n!}{m!(n-m)!}(x-b)^m$ because $f^{(n)}(b) = \frac{(b-a)^{n-m}n!}{(n-m)!}$ by a simple induction (or taking for granted Exercise 4.2.1 is true). Thus, Exercise 4.2.5 is consistent with Taylor's Theorem and Exercise 4.2.1.

Problem 4. **Exercise 4.2.6**

Let $P_n(x)$ be a polynomial of degree n and let a be a real number. We can express $P_n(x) = \sum_{k=0}^{n} b_k x^k$ as a sum of monomials. Using Exercise 4.2.5, we can express $P_n(x) = \sum_{k=0}^{n} b_k x^k = \sum_{k=0}^{n} b_k \sum_{m=0}^{k} \frac{k!}{m!(k-m)!}(x-a)^m \cdot (a)^{k-m}$. It follows, we can express $P_n(x) = \sum_{m=0}^{\infty} c_m (x-a)^m$ where $c_m = \sum_{k=m}^{n} b_k \frac{k!}{m!(k-m)!}(a)^{k-m}$. Hence, $P_n(x)$, an arbitrary polynomial of degree n, is analytic at an arbitrary real number a, so any polynomial is analytic on $\mathbb{R}$.