Math 100: Problem Set 6

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(Q-1) Suppose we express n as a sum of n many 1s. Between any two 1s, we can choose whether or not to split the sum and start a new number. For example, we place a single split between the $k-th$ and the $k+1-st$ 1s. Thus, we express n as $k, n-k$. Since we can place any number of splits between 0 and $n-1$, the number of ways to express n as a sum of positive integers is $\sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$ by the binomial formula.

(Q-2) We can equivalently express this problem as we have 14 1s (because we can have groups of 0) and we have to make 4 splits. Thus, the number of ways can be written as $\binom{14}{4} = 1001$

(Q-3) Consider a set of n objects with ordered subset of size k with elements $x_1, x_2, x_3, \ldots, x_{k-1}, x_k$. Let $x_1^*, x_2^*, x_3^*, \ldots, x_{k-1}^*$ be the successors of the first $k-1$ elements of the subset. The subset is unfriendly if $x_i^* \neq x_{i+1}$ for all $1 \leq i < k$. Thus, to create a subset of size k, we can only choose from a pool of $n-(k-1)$ objects because none of $x_1^*, x_2^*, x_3^*, \ldots, x_{k-1}^*$ can be chosen. It follows the number of unfriendly subsets is given by $\binom{n-k+1}{k}$.

(Q-4) ${}_1P_1 = 1! = 1$. The only permutation of S_1 is $\{1\}$ where $a_1 = 1$, so $g_1 = 0$. ${}_2P_2 = 2! = 2$. The two permutations of S_2 are $\{1,2\}$ and $\{2,1\}$, so the only derangement is $\{2,1\}$ where $a_1 = 2$, and $a_2 = 1$. Thus, $g_2 = 1$.

We use the PIE to show $g_n = \sum_{k=0}^{n} (-1)^k {}_nP_{n-k} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

We know there are ${}_nP_n$ permutations of S_n, so we subtract ${}_nP_{n-1}$, the number of permutations with at least one fixed point, and then alternate adding and subtracting $(-1)^k_n P_{n-k}$ to avoid double counting. ${}_nP_{n-k}$ is the number of permutations with k fixed points because we fix k out of n elements of S_n and find the number of permutations for any k elements fixed.

WTS $n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = (n-1)((n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + (n-2)! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!})$

$$= (n-1)\frac{(-1)^{n-1}}{(n-1)!}(n-1)! + (n-1)((n-1)! + (n-2)!)\sum_{k=0}^{n-2}\frac{(-1)^k}{k!}$$

$$= (n-1)(-1)^{n-1} + n!\sum_{k=0}^{n-2}\frac{(-1)^k}{k!} = \frac{n!}{(n-1)!}(-1)^{n-1} + \frac{n!}{n!}(-1)^n + n!\sum_{k=0}^{n-2}\frac{(-1)^k}{k!}$$

$$= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

(Q-5) $\sum_{k=0}^{n}(-1)^k(1)^{n-k}\binom{n}{k} = (1+(-1))^n = 0$ by the binomial formula.

(Q-6) $(k-1)\times k\binom{n}{k} = \frac{n!}{(n-k)!(k-2)!} = n(n-1)\binom{n-2}{k-2}$

$$\Rightarrow n(n-1)\sum_{k=2}^{n}\binom{n-2}{k-2} = n(n-1)2^{n-2}$$

(Q-7) $(1+x)^n = \sum_{k=0}^{n}x^k\binom{n}{k}$

$$\Rightarrow \frac{d}{dx}(1+x)^n = n(1+x)^{n-1} = \sum_{k=1}^{n}kx^{k-1}\binom{n}{k}$$

$$\Rightarrow nx(x+1)^{n-1} = \sum_{k=1}^{n}kx^k\binom{n}{k}$$

$$\Rightarrow \frac{d}{dx}nx(x+1)^{n-1} = n(x+1)^{n-1} + n(n-1)x(x+1)^{n-2} = \sum_{k=1}^{n}k^2x^{k-1}\binom{n}{k}$$

$$\Rightarrow n(n+1)2^{n-2} = \sum_{k=1}^{n}k^2\binom{n}{k}$$

(Q-8) $n(x+1)^{n-1} + n(n-1)x(x+1)^{n-2} = \sum_{k=1}^{n}k^2x^{k-1}\binom{n}{k}$

$$\Rightarrow n(-1+1)^{n-1} + n(n-1)(-1)(-1+1)^n n - 2 = \sum_{k=1}^{n}(-1)^{k-1}k^2\binom{n}{k}$$

$$= \begin{cases} 1 & \text{for } n=1 \\ -2 & \text{for } n=2 \\ 0 & \text{for } n\geq 3 \end{cases}$$

(Q-9) Lemma: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!}$

$= \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} + \frac{(n-1)!}{k!((n-1)-k)!}$

$= \binom{n-1}{k} + \binom{n-1}{k-1}$

WTS by induction on k that $\binom{s+r}{s-n} = \sum_{i=0}^{k} \binom{s+r-k}{n+r-i}\binom{k}{i}$

Base case: $P(1)$: $\binom{s+r}{s-n} = \binom{s+r}{s+r-(n+r)} = \binom{s+r}{n+r}$

$= \binom{s+r-1}{n+r} + \binom{s+r-1}{n+r-1} = \binom{1}{0}\binom{s+r-1}{n+r} + \binom{1}{1}\binom{s+r-1}{n+r-1}$

$\sum_{i=0}^{1} \binom{s+r-1}{n+r-i}\binom{1}{i}$

Induction hypothesis: Let $0 \le k < r$ be arbitrary. Assume $\sum_{i=0}^{k} \binom{s+r-k}{n+r-i}\binom{k}{i}$ holds for k.

Induction step: Expand each term of the series:

$\binom{s+r-k}{n+r-i}\binom{k}{i} = \binom{s+r-k-1}{n+r-i}\binom{k}{i} + \binom{s+r-k-1}{n+r-i-1}\binom{k}{i}$

Separate into two different series and reindex

$\sum_{i=0}^{k} \binom{s+r-k-1}{n+r-i-1}\binom{k}{i} \to \sum_{i=1}^{k+1} \binom{s+r-k-1}{n+r-i}\binom{k}{i-1}$

This implies $\sum_{i=0}^{k} \binom{s+r-k-1}{n+r-i}\binom{k}{i} + \sum_{i=1}^{k+1} \binom{s+r-k-1}{n+r-i}\binom{k}{i-1}$

$= \binom{s+r-k-1}{n+r-0}\binom{k}{0} + \binom{s+r-k-1}{n+r-k}\binom{k}{k} + \sum_{i=1}^{k} \binom{s+r-k-1}{n+r-i}\binom{k+1}{i}$

$= \sum_{i=0}^{k+1} \binom{s+r-k-1}{n+r-i}\binom{k+1}{i}$

because $\binom{k}{0} = \binom{k}{k} = \binom{k+1}{0} = \binom{k+1}{k+1} = 1$.

Since our claim holds for any $0 \le k \le r$ by induction set $k = r$.

It follows $\binom{s+r}{s-n}$

$= \sum_{i=0}^{r} \binom{s}{n+r-i}\binom{r}{i}$

$= \sum_{i=0}^{r} \binom{s}{n+r-i}\binom{r}{r-i}$

$= \sum_{i=0}^{r} \binom{s}{n+i}\binom{r}{i}$ by reindexing $i \to r-i$

(Q-10) We take $r = s = n$ and $n = 0$. Thus, $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$ by what we found in (Q-9).