Math 151b: Problem Set 8

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Problem 1:

(a) Taking the log of both sides $\log(y) = \log(\alpha) - \beta x$.

Let $\mathbf{A} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -\beta \\ \log(\alpha) \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} \log(y_1) \\ \log(y_2) \\ \vdots \\ \log(y_m) \end{bmatrix}$

(b) Let $\overline{X} = \frac{1}{m}\sum_{i=1}^m x_i$, $\overline{X^2} = \frac{1}{m}\sum_{i=1}^m x_i^2$, $\overline{\log(Y)} = \frac{1}{m}\sum_{i=1}^m \log(y_i)$, $\overline{X\log(Y)} = \frac{1}{m}\sum_{i=1}^m x_i \log(y_i)$.

$$
\begin{aligned}
\mathbf{x}^* &= (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{b} \\
&= \begin{bmatrix} m\overline{X^2} & m\overline{X} \\ m\overline{X} & m \end{bmatrix}^{-1} \begin{bmatrix} m\overline{X\log(Y)} \\ m\overline{\log(Y)} \end{bmatrix} = \begin{bmatrix} \overline{X^2} & \overline{X} \\ \overline{X} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \overline{X\log(Y)} \\ \overline{\log(Y)} \end{bmatrix} \\
&= \frac{1}{\overline{X^2} - \overline{X}^2} \begin{bmatrix} 1 & -\overline{X} \\ -\overline{X} & \overline{X^2} \end{bmatrix} \begin{bmatrix} \overline{X\log(Y)} \\ \overline{\log(Y)} \end{bmatrix} \\
&= \frac{1}{\overline{X^2} - \overline{X}^2} \begin{bmatrix} \overline{X\log(Y)} - \overline{X}\cdot\overline{\log(Y)} \\ \overline{X^2}\cdot\overline{\log(Y)} - \overline{X}\cdot\overline{X\log(Y)} \end{bmatrix} \\
&\Rightarrow \beta = \frac{\overline{X}\cdot\overline{\log(Y)} - \overline{X\log(Y)}}{\overline{X^2} - \overline{X}^2}, \alpha = \exp\left(\frac{\overline{X^2}\cdot\overline{\log(Y)} - \overline{X}\cdot\overline{X\log(Y)}}{\overline{X^2} - \overline{X}^2}\right)
\end{aligned}
$$

Problem 2:

(a) $(P_\mathbf{q})^2 = (\mathbf{q}\mathbf{q}^\top)(\mathbf{q}\mathbf{q}^\top) = \mathbf{q}(\mathbf{q}^\top\mathbf{q})\mathbf{q}^\top = \mathbf{q}\|\mathbf{q}\|^2\mathbf{q}^\top = \mathbf{q}\mathbf{q}^\top = P_\mathbf{q}$ because $\mathbf{q}$ is a unit vector. $P_\mathbf{q}^\top = (\mathbf{q}\mathbf{q}^\top)^\top = (\mathbf{q}^\top)^\top(\mathbf{q})^\top = \mathbf{q}\mathbf{q}^\top = P_\mathbf{q}$. By transitivity, $P_\mathbf{q}^\top = (P_\mathbf{q})^2$

(b) $(I-P)^2 = I^2 - 2IP + P^2 = I - 2P + P = I - P$ because if P is an orthogonal projector, then $P^2 = P$. $(I-P)^\top = I^\top - P^\top = I - P$ because if P is an orthogonal projector, then $P^\top = P$. Thus, $I - P$ is an orthogonal projector.

Since P is an orthogonal projector onto W, $P\mathbf{v} \in W$ and $\mathbf{v} - P\mathbf{v} \in W^\perp$ for all $\mathbf{v} \in \mathbb{R}^m$. Thus, $(I - P)\mathbf{v} = \mathbf{v} - P\mathbf{v} \in W^\perp$, so $(I - P)$ is an orthogonal projector onto $W^\perp$. We showed in part (a) that $P_\mathbf{q}$ is an orthogonal projector onto $span\{\mathbf{q}\}$, so by what we showed earlier in part (b), $I - P_\mathbf{q}$ must be an orthogonal projector onto $span\{\mathbf{q}\}^\perp$.

Problem 3:

(a) Proof by induction:
Base Case:

$$
\begin{aligned}
&P_{\perp\mathbf{q}_2} P_{\perp\mathbf{q}_1} = (I - \mathbf{q}_2\mathbf{q}_2^\top)(I - \mathbf{q}_1\mathbf{q}_1^\top) \\
&= I^2 - \mathbf{q}_2\mathbf{q}_2^\top - \mathbf{q}_1\mathbf{q}_1^\top + \mathbf{q}_2(\mathbf{q}_2^\top\mathbf{q}_1)\mathbf{q}_1^\top \\
&= I - \mathbf{q}_2\mathbf{q}_2^\top - \mathbf{q}_1\mathbf{q}_1^\top
\end{aligned}
$$

$\mathbf{q}_2^\top\mathbf{q}_1 = 0$ because $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ is an orthonormal set.
Induction hypothesis: Assume for some $2 \leq k < n$

$$
P_{\perp\mathbf{q}_k} \ldots P_{\perp\mathbf{q}_2} P_{\perp\mathbf{q}_1} = I - \mathbf{q}_k\mathbf{q}_k^\top - \cdots - \mathbf{q}_2\mathbf{q}_2^\top - \mathbf{q}_1\mathbf{q}_1^\top
$$

Induction step:

$$
\begin{aligned}
&P_{\perp\mathbf{q}_{k+1}} P_{\perp\mathbf{q}_k} \ldots P_{\perp\mathbf{q}_2} P_{\perp\mathbf{q}_1} = (I - \mathbf{q}_{k+1}\mathbf{q}_{k+1}^\top)(I - \mathbf{q}_k\mathbf{q}_k^\top - \cdots - \mathbf{q}_2\mathbf{q}_2^\top - \mathbf{q}_1\mathbf{q}_1^\top) \\
&= I - \mathbf{q}_k\mathbf{q}_k^\top - \cdots - \mathbf{q}_2\mathbf{q}_2^\top - \mathbf{q}_1\mathbf{q}_1^\top - \mathbf{q}_{k+1}\mathbf{q}_{k+1}^\top \\
&+ \mathbf{q}_{k+1}(\mathbf{q}_{k+1}^\top\mathbf{q}_k)\mathbf{q}_k^\top + \cdots + \mathbf{q}_{k+1}(\mathbf{q}_{k+1}^\top\mathbf{q}_2)\mathbf{q}_2^\top + \mathbf{q}_{k+1}(\mathbf{q}_{k+1}^\top\mathbf{q}_1)\mathbf{q}_1^\top \\
&= I - \mathbf{q}_{k+1}\mathbf{q}_{k+1}^\top - \mathbf{q}_k\mathbf{q}_k^\top - \cdots - \mathbf{q}_2\mathbf{q}_2^\top - \mathbf{q}_1\mathbf{q}_1^\top
\end{aligned}
$$

$\mathbf{q}_{k+1}^\top\mathbf{q}_i = 0$ for all $i = 1 \ldots k$ because $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ is an orthonormal set.
Hence, by induction, $P_{\perp\mathbf{q}_k} \ldots P_{\perp\mathbf{q}_2} P_{\perp\mathbf{q}_1} = I - \mathbf{q}_k\mathbf{q}_k^\top - \cdots - \mathbf{q}_2\mathbf{q}_2^\top - \mathbf{q}_1\mathbf{q}_1^\top$ for all $2 \leq k \leq n$.

(b) If we change the order we apply the orthogonal projectors, we simply change the order of the linear combination of $\mathbf{q}_i\mathbf{q}_i^\top$s. However, addition and subtraction are commutative, so this would not change $P_{\perp\mathbf{q}_k} \ldots P_{\perp\mathbf{q}_2} P_{\perp\mathbf{q}_1}$. Thus, $P_{\perp W} = P_{\perp\mathbf{q}_k} \ldots P_{\perp\mathbf{q}_2} P_{\perp\mathbf{q}_1}$ for any permutation of $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$.

Problem 4:

(a) Suppose we have a collection of data points in a vertical line i.e, they all have the same x coordinate but different y coordinates $y_1, y_2 \ldots y_m$.
Let $\mathbf{A} = \begin{bmatrix} 1 & x \\ 1 & x \\ \vdots & \vdots \\ 1 & x \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$. $\mathbf{b} \notin span\{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\} \Rightarrow \mathbf{b} \notin$

$\mathcal{R}(\mathbf{A})$.

$\mathbf{A}^\top\mathbf{A} = \begin{bmatrix} m & mx \\ mx & mx^2 \end{bmatrix}$ is not invertible because the columns are linearly dependent, so there are multiple least square solutions.

$\begin{bmatrix} m & mx \\ mx & mx^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m y_i \\ x\sum_{i=1}^m y_i \end{bmatrix}$

$\begin{bmatrix} 1 \\ x \end{bmatrix} (c_0 m + c_1 mx) = \begin{bmatrix} 1 \\ x \end{bmatrix} \sum_{i=1}^m y_i$

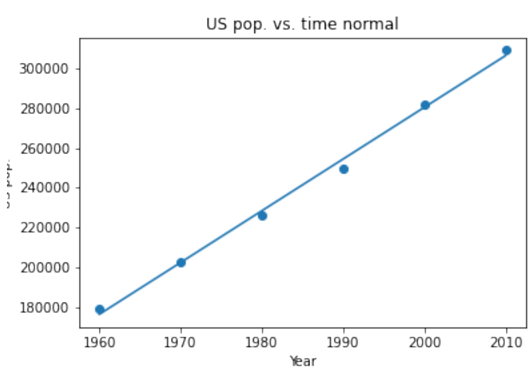
so any c_0, c_1 that satisfies $c_0 + c_1 x = \frac{1}{m}\sum_{i=1}^m y_i$ will be a least square solution.

(b) Let $a_1, a_2, \dots, a_n$ denote the columns of A. $a_i = Qr_i$ where r_i is the i^{th} column of R. It follows $\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = Q(\alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n)$. Q is an orthogonal matrix, so it has rank n. Thus, $Q(\alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n) = 0$ iff $\alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n = 0$.
If A has full rank, then $\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = 0$ iff $\alpha_i = 0$ for $i = 1 \dots n$. This implies $\alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n = 0$ iff $\alpha_i = 0$ for $i = 1 \dots n$. Thus, the columns of R are linearly independent. Since R is upper triangular, this is only true when the product of the diagonal is nonzero. Thus, all the entries of the diagonal are nonzero.
If all the diagonal entries of R are nonzero, the its columns are linearly independent. Thus, $Q(\alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n) = 0$ iff $\alpha_i = 0$ for $i = 1 \dots n$. This implies $\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = 0$ iff $\alpha_i = 0$ for $i = 1 \dots n$ because $a_i = Qr_i$. Since the columns of A are linearly independent, A has full rank.

Problem 5:

(a) (i) Condition number for overdetermined system: 230733.08869696865
Condition number for normal equation: 53237758117.525375
$cond(A) = \sqrt{cond(A^\top A)}$

(ii) $c_0 = -4.88868679 \times 10^6$, $c_1 = 2.58447429 \times 10^3$



In [2]:
```
import numpy as np
from numpy import linalg
from matplotlib import pyplot as plt
```

In [3]:
```
xdata=np.arange(1960,2020,10)
```

In [4]:
```
ydata=np.array([179323, 203302, 226542, 249633, 281422, 308746])
```

In [5]:
```
Amat=np.array([np.ones(6),xdata]).transpose()
```

In [6]:
```
bvec=ydata
```

In [7]:
```
Anormal=np.matmul(Amat.transpose(),Amat)
```

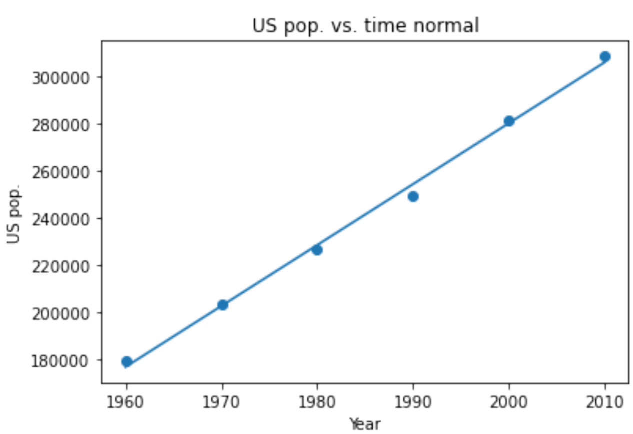
In [8]:
```
bnormal=Amat.transpose().dot(bvec)
```

In [9]:
```
xvec=linalg.solve(Anormal,bnormal)
```

In [10]:
```
xvec
```

Out[10]:
```
array([-4.88868679e+06,  2.58447429e+03])
```

In [24]:
```
c_0_A,c_1_A=xvec
plt.scatter(xdata,ydata)
plt.plot(xdata,xdata*c_1_A+c_0_A)
plt.xlabel('Year')
plt.ylabel('US pop.')
plt.title('US pop. vs. time normal')
plt.savefig('US pop vs time normal')
```



In [26]:
```
linalg.cond(Amat)
```

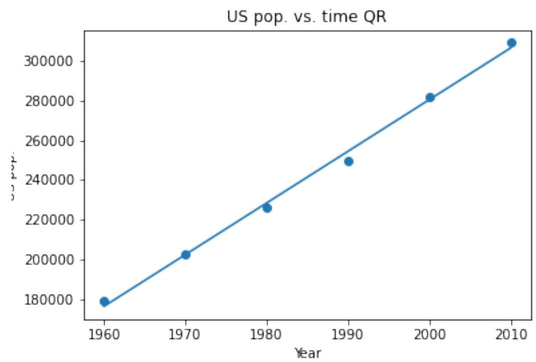
Out[26]:
```
230733.08869696865
```

In [19]:
```
linalg.cond(Anormal)
```

Out[19]:
```
53237758117.525375
```

In [ ]:

(b) Condition number for QR: 230733.08869696865



In [2]:
```
import numpy as np
from numpy import linalg
from matplotlib import pyplot as plt
```

In [3]:
```
xdata=np.arange(1960,2020,10)
```

In [4]:
```
ydata=np.array([179323, 203302, 226542, 249633, 281422, 308746])
```

In [18]:
```
Amat=np.array([np.ones(6),xdata]).transpose()
```

In [6]:
```
bvec=ydata
```

In [7]:
```
Q,R=linalg.qr(Amat)
```

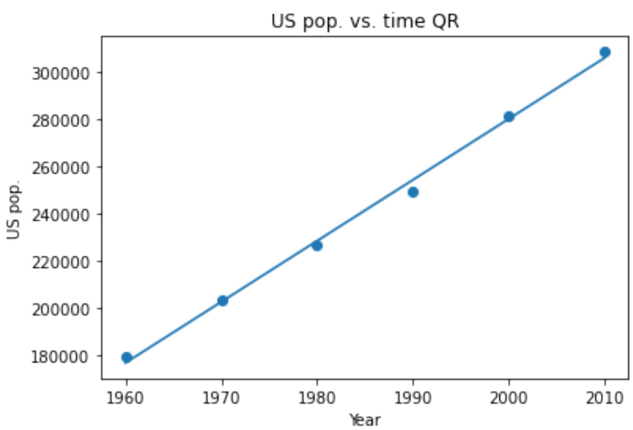
In [8]:
```
bqr=Q.transpose().dot(bvec)
```

In [9]:
```
xvec=linalg.solve(R,bqr)
```

In [10]:
```
xvec
```

Out[10]:
```
array([-4.88868679e+06,  2.58447429e+03])
```

In [17]:
```
c_0,c_1=xvec
plt.scatter(xdata,ydata)
plt.plot(xdata,xdata*c_1+c_0)
plt.xlabel('Year')
plt.ylabel('US pop.')
plt.title('US pop. vs. time QR')
plt.savefig('US pop vs time QR')
```



In [12]:
```
linalg.cond(Amat)
```

Out[12]:
```
230733.08869696865
```

In [13]:
```
linalg.cond(R)
```

Out[13]:
```
230733.08869696865
```

(c)
  (a) Condition number for overdetermined system: 62250894343.79537
      Condition number for normal equation: $8.00160109303428 \times 10^{21}$

  (b) Condition number for QR: 62250894343.79537

  (c) Condition number for recentered data QR method: 2843.8719526832056