Math 100: Problem Set 2

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(Q-1)

- Using AM.GM innequality we obtain $(a+b) \geq 2\sqrt{ab}, (a+c) \geq 2\sqrt{ac}, (b+c) \geq 2\sqrt{bc}$. It follows, $(a+b)(b+c)(a+c) \geq 2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ac} = 8abc$.
- $a^2b^2 + b^2c^2 + c^2a^2 = \frac{a^2b^2+b^2c^2}{2} + \frac{b^2c^2+c^2a^2}{2} + \frac{c^2a^2+a^2b^2}{2}$. By AM.GM $\frac{a^2b^2+b^2c^2}{2} + \frac{b^2c^2+c^2a^2}{2} + \frac{c^2a^2+a^2b^2}{2} \geq ab^2c + abc^2 + a^2bc = abc(a+b+c)$
- If $a+b+c = 1 \Rightarrow (a+b+c)^2 = 1$. It follows $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ac = \frac{a^2+b^2}{2} + \frac{b^2+c^2}{2} + \frac{c^2+a^2}{2} + 2ab + 2bc + 2ac$. By AM.GM $\frac{a^2+b^2}{2} + \frac{b^2+c^2}{2} + \frac{c^2+a^2}{2} + 2ab + 2bc + 2ac \geq 3ab + 3bc + 3ac$. Because $3ab + 3bc + 3ac \leq (a+b+c)^2 \leq 1 \Rightarrow ab + bc + ac \leq \frac{1}{3}$

(Q-2) $b^{n+1} - a^{n+1} = (b-a)\displaystyle\sum_{k=0}^{n} a^k b^{n-k}$. We rewrite $(n+1)(b-a)a^n = (b-a)\displaystyle\sum_{k=0}^{n} a^n$ and $(n+1)(b-a)b^n = (b-a)\displaystyle\sum_{k=0}^{n} b^n$. Observe for each k $a^n \leq a^k b^{n-k} \leq b^n$. It follows that because $0 < a < b$ there exists at least one k s.t $a^n < a^k b^{n-k} < b^n$. Hence, $(b-a)(n+1)a^n < b^{n+1} - a^{n+1} < b^{n+1}$

(Q-3) By AM.GM $\frac{a^2b+b^2c+c^2a}{3} \geq \sqrt[3]{a^2b \cdot b^2c \cdot c^2a} = \sqrt[3]{a^3b^3c^3} = abc$
and $\frac{a^2c+b^2a+c^2b}{3} \geq \sqrt[3]{a^2c \cdot b^2a \cdot c^2b} = \sqrt[3]{a^3b^3c^3} = abc$.
Hence, $9 \cdot \frac{a^2b+b^2c+c^2a}{3} \frac{a^2c+b^2a+c^2b}{3}$
$= (a^2b + b^2c + c^2a)(a^2c + b^2a + c^2b) \geq 9a^2b^2c^2$

(Q-4) By AM.GM $\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} = n\frac{\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} =}{n} \geq n\sqrt[n]{\frac{a_1}{b_1}\frac{a_2}{b_2}\cdots\frac{a_n}{b_n}} = n\sqrt[n]{\frac{a_1a_2\ldots a_n}{b_1b_2\ldots b_n}} = n\sqrt[n]{\frac{a_1a_2\ldots a_n}{a_1a_2\ldots a_n}}$
(because $b_1, b_2, \ldots b_n$ is a rearrangement of $a_1, a_2 \ldots a_n$)
$= n$
Hence, $\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n$.

(Q-5)

- WTS by induction $P(n) : n! < (\frac{n+1}{2})^n$ is true for all integers $n > 2$. $P(3) : 3! = 6 < (\frac{3+1}{2})^3 = 8$. Assume for some $n > 2$ the statement $P(n)$ holds. First we show that $2 \leq (1 + \frac{1}{n+1})^{n+1}$. Using binomial expansion $(1+\dfrac{1}{n+1})^{n+1} = \displaystyle\sum_{k=0}^{n+1}\binom{n+1}{k}(\frac{1}{n+1})^{k} = 1\cdot 1 + \frac{n+1}{n+1} + \cdots > 2$ for $n > 2$. It follows $(1+\frac{1}{n+1})^{n+1} = (\frac{n+2}{n+1})^{n+1} > 2 \Rightarrow \frac{(n+1+1)^{n+1}}{2^{n+1}} > \frac{(n+1)^{n+1}}{2^{n}}$. $(n+1)! = (n+1)n! < (n+1)(\frac{n+1}{2})^{n}$ by the induction hypothesis. Thus, $\frac{(n+1+1)^{n+1}}{2^{n+1}} > (n+1)\frac{(n+1)^{n}}{2^{n}} > (n+1)!$. Therefore, the claim holds for $P(n+1)$. Hence, by induction, the claim holds for all n.

- WTS by induction $P(n): 1\times 3\times 5\times\cdots\times(2n-1) < n^{n}$ for $n > 2$. $P(3): 1\times 3\times 5 = 15 < 3^{3} = 27$ Assume for some $n > 2$ the statement $P(n)$ holds. $1\times 3\times 5\times\cdots\times(2n-1)\times(2n+1) < 2n^{n+1}+n^{n}$ by the induction hypothesis. Thus, it suffices to show $2n^{n+1}+n^{n} < (n+1)^{n+1}$ It follows by binomial expansion $(n+1)^{n+1} = \displaystyle\sum_{k=0}^{n}\binom{n+1}{k}n^{k} = n^{n+1}+(n+1)n^{n}+\cdots+1 > 2n^{n+1}+n^{n}$ for $n > 2$. Thus, the claim holds for $P(n+1)$. Hence, by induction, the claim holds for all $n > 2$.

(Q-6)

- For each $i \in 1\cdots n$ consider $p_i x_i$ as the sum of p_i many $x_i's$. Thus, we can consider $\frac{p_1x_1+p_2x_2+\cdots+p_nx_n}{p_1+p_2+\cdots+p_n}$ to be the arithmetic mean of $p_1+p_2+\cdots+p_n$ many positive numbers. It follows that the geometric mean of $p_1+p_2+\cdots+p_n$ many positive numbers is $\sqrt[p_1+p_2+\cdots+p_n]{\displaystyle\prod_{i=1}^{n}\prod_{k=1}^{p_i}x_i} = \sqrt[p_1+p_2+\cdots+p_n]{x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}}$. By AM.GM we obtain $\sqrt[p_1+p_2+\cdots+p_n]{x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}} \le \frac{p_1x_1+p_2x_2+\cdots+p_nx_n}{p_1+p_2+\cdots+p_n}$
- If each p_i is a positive rational number, then the denominators must have a least commmon multiple which we will call y. Then, for each i, yp_i is an integer. Then, we can consider yp_ix_i as the sum of yp_i many $x_i's$. It follows by the previous proof that
$\frac{p_1x_1+p_2x_2+\cdots+p_nx_n}{p_1+p_2+\cdots+p_n} = \frac{yp_1x_1+yp_2x_2+\cdots+yp_nx_n}{yp_1+yp_2+\cdots+yp_n}$
$\ge \sqrt[yp_1+yp_2+\cdots+yp_n]{x_1^{yp_1}x_2^{yp_2}\cdots x_n^{yp_n}}$
$= \sqrt[y(p_1+p_2+\cdots+p_n)]{(x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n})^{y}} = \sqrt[p_1+p_2+\cdots+p_n]{x_1^{p_1}x_2^{p_2}\cdots + x_n^{p_n}}$

(Q-7)

- For vectors $\vec{u}$ and $\vec{v}$, set $u_i = \sqrt{p_i}$ and $v_i = \sqrt{p_i}x_i$. It follows by the Cauchy-Schwarz innequality that $|\vec{u}\cdot\vec{v}| \le ||\vec{u}||||\vec{v}||$.
Thus, $|\sqrt{p_1}^2x_1+\cdots+\sqrt{p_n}^2x_n| \le \sqrt{\sqrt{p_1}^2+\cdots+\sqrt{p_n}^2}\sqrt{\sqrt{p_1}^2x_1^2+\cdots\sqrt{p_n}^2x_n^2}$.
Since both sides of the innequality are positive, we obtain
$(p_1x_1+\cdots+p_nx_n)^2 \le (p_1+\cdots+p_n)(p_1x_1^2+\cdots+p_nx_n^2)$ by squaring both sides.
- For vectors $\vec{u}$ and $\vec{v}$, set $\vec{u} = (a\sqrt{b}, b\sqrt{c}, c\sqrt{a})$ and $\vec{v} = (c\sqrt{b}, a\sqrt{c}, b\sqrt{a})$
It follows by the Cauchy-Schwarz innequality that $|\vec{u}\cdot\vec{v}| \le ||\vec{u}||||\vec{v}||$.
Thus, using the fact $a, b, c > 0$, $3abc \le \sqrt{a^2b+b^2c+c^2a}\sqrt{c^2b+a^2c+b^2a}$.
Squaring both sides, we obtain $9a^2b^2c^2 \le (a^2b+b^2c+c^2a)(c^2b+a^2c+b^2a)$.

(Q-8) Let $f(x) = (x+1)^a$ where $x \neq 0$. For the case where $x = 0$, $1 + a \cdot 0 = (1+0)^a$.

It follows by MVT there exists some c between 0 and x s.t $f'(c) = \frac{(x+1)^a - 1^a}{x}$. It follows $(x+1)^a = a(c+1)^{a-1}x + 1$. If $0 < a < 1$ then it suffices to show $a(c+1)^{a-1}x + 1 \leq 1 + ax$.

$x < 0$ $0 < c+1 < 1$, so $1 < \frac{1}{c+1}$. Since $0 < 1-a < 1$, $1 < (\frac{1}{c+1})^{1-a}$. Thus, $a(c+1)^{a-1}x + 1 \leq ax + 1$.

$x > 0$ $1 < c+1 < 2$, so $0 < \frac{1}{c+1} < 1$. Since $0 < 1-a < 1$, $0 < (\frac{1}{c+1})^{1-a} < 1$. Thus, $a(c+1)^{a-1}x + 1 \leq ax + 1$.

If $a > 1$ then $0 < a - 1$. It follows that for $x < 0$, $0 < c+1 < 1$, so $0 < (c+1)^{a-1} < 1$. Thus, $a(c+1)^{a-1}x + 1 \geq ax + 1$. By similar argument, we have $(c+1)^{a-1} > 1$ for $x > 0$, so $a(c+1)^{a-1}x + 1 \geq ax + 1$. If $a < 0$ then $a - 1 < -1$. It follows that for $x < 0$, $0 < c+1 < 1$, so $1 < (c+1)^{a-1}$. Thus, $a(c+1)^{a-1}x + 1 \geq ax + 1$ (using $ax > 0$ because both are negative). By similar argument, we have $0 < (c+1)^{a-1} < 1$ for $x > 0$, so $a(c+1)^{a-1}x + 1 \geq ax + 1$ (using $ax < 0$ because a is negative and x is positive).

(Q-9) Let $f(t) = \log(t+1)$. It follows by MVT that there exists c between 0 and x s.t $f'(c) = \frac{\log(x+1)}{x}$. Since $0 < c < x$, it follows that $f'(c) = \frac{1}{1+c} > \frac{1}{1+x}$. Thus, $\frac{x}{x+1} < \log(x+1)$ for $x > 0$. Moreover, there exists c between $\frac{x^2+3x}{2}$ and $\frac{x}{2}$ s.t $f'(c) = \frac{2\log(x+1)}{x^2+2x}$. It follows $\log(x+1) = \frac{x(x+2)}{2(c+1)} < \frac{x(x+2)}{2(x+1)}$

(Q-10) $\frac{\sin(a)-0}{a-0} = \cos(c_1)$ for some $0 < c_1 < a$ and $\frac{\sin(b)-0}{b-0} = \cos(c_2)$ for some $0 < c_2 < b$. $\frac{\cos(c_1)-\cos(c_2)}{c_1-c_2} = -sin(c_3)$ for some c_3 between c_1 and c_2, so because $\cos(x)$ is decreasing and $-\sin(c_3)$ $c_1 > c_2$. Thus, $\cos(c_2) > \cos(c_1) \Rightarrow \frac{\sin(a)}{\sin(b)} < \frac{a}{b}$ because $\frac{\sin(a)}{a} < \frac{\sin(b)}{b}$. $\frac{\tan(a)-0}{a-0} = \frac{1}{\cos(c_4)^2}$ for some $0 < c_4 < a$ and $\frac{\tan(b)-0}{b-0} = \frac{1}{\cos(c_5)^2}$ for some $0 < c_5 < b$. $\frac{\cos(c_4)-\cos(c_5)}{c_4-c_5} = -\sin(c_6)$ for some c_6 between c_4 and c_5, so $c_4 > c_5$. Because $\cos(c_5) > \cos(c_4) > 0 \Rightarrow \frac{1}{\cos(c_5)^2} < \frac{1}{\cos(c_4)^2} \Rightarrow \frac{\tan(b)}{b} < \frac{\tan(a)}{a} \Rightarrow \frac{a}{b} < \frac{\tan(a)}{\tan(b)}$, so $\frac{\sin(a)}{\sin(b)} < \frac{a}{b} < \frac{\tan(a)}{\tan(b)}$