Math 151A: Final

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Problem 1: (a) $g(x)$ is a continuous function whose derivative $g'(x) = (\sqrt{9-x})' = \frac{-1}{2\sqrt{9-x}} < 0$ on the open interal $(0, 9)$, in other words decreasing, so $g(x)$ assumes a maximum of $g(0) = 3$ at one endpoint and a minimum of $g(9) = 0$ at the other. Thus, $g(x) \in [0, 9]$ for all $x \in [0, 9]$, so $p_k = g(p_{k-1}) \in [0, 9]$ for all $k \geq 0$.

(b)

$$g(x) \leq g(0) = 3 \text{ if } 0 \leq x \leq 9 \text{ because } g(x) \text{ is a decreasing function.}$$
$$\text{Thus, } |g'(g(x))| = |\frac{-1}{2\sqrt{9-\sqrt{9-x}}}| \leq |g'(3)| < 1$$
$$\Rightarrow |g'(p_k)| = |g'(g(p_{k-1}))| \leq |g'(3)| < 1 \text{ for all } k \geq 1$$

Thus, $|g(p_k) - g(p)| \leq |g'(3)||p_k - p| \Rightarrow |g(p_k) - g(p)| \leq |g'(3)|^2|p_{k-1} - p| \leq ... \leq |g'(3)|^{k-1}|g(p_0) - p|$, so p_k converges to the unique fixed point p for $p_0 \in [0, 9]$ by the squeeze theorem and the mean value theorem. Moreover, p_k converges to the unique fixed point p by a slight variation of the fixed point theorem because $g(p_k) \in [0, 9]$ for $p_k \in [0, 9]$ and for all but finitely many $|g'(p_k)|$ are bounded above by some $L < 1$.

Problem 2: $f(x+h,y) = f(x,y) + \frac{\partial f(x,y)}{\partial x}h + \frac{\partial f(x,y)}{\partial y}\cdot 0 + \frac{\partial^2 f(x,y)}{\partial x^2}\frac{h^2}{2} + \frac{\partial^2 f(x,y)}{\partial y^2}\cdot 0 + 2\frac{\partial^2 f(x,y)}{\partial x \partial y}\cdot 0 + \frac{\partial^3 f(x,y)}{\partial x^3}\frac{h^3}{6} + 3\frac{\partial^3 f(x,y)}{\partial x^2 \partial y}\cdot 0 + 3\frac{\partial^3 f(x,y)}{\partial x \partial y^2}\cdot 0 + \frac{\partial^3 f(x,y)}{\partial y^3}\cdot 0 + O(h^4)$

$f(x-h,y) = f(x,y) + \frac{\partial f(x,y)}{\partial x}(-h) + \frac{\partial f(x,y)}{\partial y}\cdot 0 + \frac{\partial^2 f(x,y)}{\partial x^2}\frac{h^2}{2} + \frac{\partial^2 f(x,y)}{\partial y^2}\cdot 0 + 2\frac{\partial^2 f(x,y)}{\partial x \partial y}\cdot 0 + \frac{\partial^3 f(x,y)}{\partial x^3}\frac{-h^3}{6} + 3\frac{\partial^3 f(x,y)}{\partial x^2 \partial y}\cdot 0 + 3\frac{\partial^3 f(x,y)}{\partial x \partial y^2}\cdot 0 + \frac{\partial^3 f(x,y)}{\partial y^3}\cdot 0 + O(h^4)$

$f(x,y+h) = f(x,y) + \frac{\partial f(x,y)}{\partial x}\cdot 0 + \frac{\partial f(x,y)}{\partial y}h + \frac{\partial^2 f(x,y)}{\partial x^2}\cdot 0 + \frac{\partial^2 f(x,y)}{\partial y^2}\frac{h^2}{2} + 2\frac{\partial^2 f(x,y)}{\partial x \partial y}\cdot 0 + \frac{\partial^3 f(x,y)}{\partial x^3}\cdot 0 + 3\frac{\partial^3 f(x,y)}{\partial x^2 \partial y}\cdot 0 + 3\frac{\partial^3 f(x,y)}{\partial x \partial y^2}\cdot 0 + \frac{\partial^3 f(x,y)}{\partial y^3}\frac{h^3}{6} + O(h^4)$

$f(x,y-h) = f(x,y) + \frac{\partial f(x,y)}{\partial x}\cdot 0 + \frac{\partial f(x,y)}{\partial y}(-h) + \frac{\partial^2 f(x,y)}{\partial x^2}\cdot 0 + \frac{\partial^2 f(x,y)}{\partial y^2}\frac{h^2}{2} + 2\frac{\partial^2 f(x,y)}{\partial x \partial y}\cdot 0 + \frac{\partial^3 f(x,y)}{\partial x^3}\cdot 0 + 3\frac{\partial^3 f(x,y)}{\partial x^2 \partial y}\cdot 0 + 3\frac{\partial^3 f(x,y)}{\partial x \partial y^2}\cdot 0 + \frac{\partial^3 f(x,y)}{\partial y^3}\frac{-h^3}{6} + O(h^4)$

$f(x+h,y+h) = f(x,y) + \frac{\partial f(x,y)}{\partial x}h + \frac{\partial f(x,y)}{\partial y}h + \frac{\partial^2 f(x,y)}{\partial x^2}\frac{h^2}{2} + \frac{\partial^2 f(x,y)}{\partial y^2}\frac{h^2}{2} + 2\frac{\partial^2 f(x,y)}{\partial x \partial y}\frac{h^2}{2} + \frac{\partial^3 f(x,y)}{\partial x^3}\frac{h^3}{6} + 3\frac{\partial^3 f(x,y)}{\partial x^2 \partial y}\frac{h^3}{6} + 3\frac{\partial^3 f(x,y)}{\partial x \partial y^2}\frac{h^3}{6} + \frac{\partial^3 f(x,y)}{\partial y^3}\frac{h^3}{6} + O(h^4)$

$f(x-h,y-h) = f(x,y) + \frac{\partial f(x,y)}{\partial x}(-h) + \frac{\partial f(x,y)}{\partial y}(-h) + \frac{\partial^2 f(x,y)}{\partial x^2}\frac{h^2}{2} + \frac{\partial^2 f(x,y)}{\partial y^2}\frac{h^2}{2} + 2\frac{\partial^2 f(x,y)}{\partial x \partial y}\frac{h^2}{2} + \frac{\partial^3 f(x,y)}{\partial x^3}\frac{-h^3}{6} + 3\frac{\partial^3 f(x,y)}{\partial x^2 \partial y}\frac{-h^3}{6} + 3\frac{\partial^3 f(x,y)}{\partial x \partial y^2}\frac{-h^3}{6} + \frac{\partial^3 f(x,y)}{\partial y^3}\frac{-h^3}{6} + O(h^4)$

$\Rightarrow$

$0 = f(x,y) + \frac{\partial f(x,y)}{\partial x}h + \frac{\partial^2 f(x,y)}{\partial x^2}\frac{h^2}{2} + \frac{\partial^3 f(x,y)}{\partial x^3}\frac{h^3}{6} - f(x+h,y) + O(h^4)$

$0 = f(x,y) + \frac{\partial f(x,y)}{\partial x}(-h) + \frac{\partial^2 f(x,y)}{\partial x^2}\frac{h^2}{2} + \frac{\partial^3 f(x,y)}{\partial x^3}\frac{-h^3}{6} - f(x-h,y) + O(h^4)$

$0 = f(x,y) + \frac{\partial f(x,y)}{\partial y}h + \frac{\partial^2 f(x,y)}{\partial y^2}\frac{h^2}{2} + \frac{\partial^3 f(x,y)}{\partial y^3}\frac{h^3}{6} - f(x,y+h) + O(h^4)$

$0 = f(x,y) + \frac{\partial f(x,y)}{\partial y}(-h) + \frac{\partial^2 f(x,y)}{\partial y^2}\frac{h^2}{2} + \frac{\partial^3 f(x,y)}{\partial y^3}\frac{-h^3}{6} - f(x,y-h) + O(h^4)$

$2\frac{\partial^2 f(x,y)}{\partial x \partial y}\frac{h^2}{2} = -f(x,y) - \frac{\partial f(x,y)}{\partial x}h - \frac{\partial f(x,y)}{\partial y}h - \frac{\partial^2 f(x,y)}{\partial x^2}\frac{h^2}{2} - \frac{\partial^2 f(x,y)}{\partial y^2}\frac{h^2}{2} - \frac{\partial^3 f(x,y)}{\partial x^3}\frac{h^3}{6} - 3\frac{\partial^3 f(x,y)}{\partial x^2 \partial y}\frac{h^3}{6} - 3\frac{\partial^3 f(x,y)}{\partial x \partial y^2}\frac{h^3}{6} - \frac{\partial^3 f(x,y)}{\partial y^3}\frac{h^3}{6} + f(x+h,y+h) + O(h^4)$

$2\frac{\partial^2 f(x,y)}{\partial x \partial y}\frac{h^2}{2} = -f(x,y) - \frac{\partial f(x,y)}{\partial x}(-h) - \frac{\partial f(x,y)}{\partial y}(-h) - \frac{\partial^2 f(x,y)}{\partial x^2}\frac{h^2}{2} - \frac{\partial^3 f(x,y)}{\partial x^3}\frac{-h^3}{6} - 3\frac{\partial^3 f(x,y)}{\partial x^2 \partial y}\frac{-h^3}{6} - 3\frac{\partial^3 f(x,y)}{\partial x \partial y^2}\frac{-h^3}{6} - \frac{\partial^3 f(x,y)}{\partial y^3}\frac{-h^3}{6} + f(x-h,y-h) + O(h^4)$

$\Rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2h^2} \\ \frac{1}{2h^2} \\ \frac{1}{2h^2} \\ \frac{1}{2h^2} \\ \frac{1}{2h^2} \\ \frac{1}{2h^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{h^2} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{-1}{h^2} \end{bmatrix}$$

$\Rightarrow$

$\frac{\partial^3 f(x,y)}{\partial x^3}\frac{h^3}{6}\frac{1}{2h^2} + \frac{\partial^3 f(x,y)}{\partial x^3}\frac{-h^3}{6}\frac{1}{2h^2} = 0$

$\frac{\partial^3 f(x,y)}{\partial y^3}\frac{h^3}{6}\frac{1}{2h^2} + \frac{\partial^3 f(x,y)}{\partial y^3}\frac{-h^3}{6}\frac{1}{2h^2} = 0$

$(-\frac{\partial^3 f(x,y)}{\partial x^3}\frac{h^3}{6} - 3\frac{\partial^3 f(x,y)}{\partial x^2 \partial y}\frac{h^3}{6} - 3\frac{\partial^3 f(x,y)}{\partial x \partial y^2}\frac{h^3}{6} - \frac{\partial^3 f(x,y)}{\partial y^3}\frac{h^3}{6})\frac{1}{2h^2} + (-\frac{\partial^3 f(x,y)}{\partial x^3}\frac{-h^3}{6} - 3\frac{\partial^3 f(x,y)}{\partial x^2 \partial y}\frac{-h^3}{6} -$

$3\frac{\partial^3 f(x,y)}{\partial x \partial y^2}\frac{-h^3}{6} - \frac{\partial^3 f(x,y)}{\partial y^3}\frac{-h^3}{6})\frac{1}{2h^2} = 0,$
so the $O(h^3)$ terms cancel out by symmetry.

$\Rightarrow$

$$\frac{2\frac{\partial^2 f(x,y)}{\partial x \partial y}\frac{h^2}{2} + 2\frac{\partial^2 f(x,y)}{\partial x \partial y}\frac{h^2}{2}}{2h^4} = \frac{h^2(2f(x,y) - f(x+h,y) - f(x-h,y) - f(x,y+h) - f(x,y-h) + f(x+h,y+h) + f(x-h,y-h)) + O(h^4)}{2h^4}$$

$\Rightarrow$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{2f(x,y) - f(x+h,y) - f(x-h,y) - f(x,y+h) - f(x,y-h) + f(x+h,y+h) + f(x-h,y-h)}{2h^2} + O(h^2)$$

Hence, we obtain the desired result.

Problem 3: We have six variables, so we'll need six equations and should expect exactness to at least degree-5.

$$\int_{-1}^{1} 1dx = 2 = c_1 \cdot 1 + c_2 \cdot 1 + c_3 \cdot 1$$
$$\int_{-1}^{1} xdx = 0 = c_1x_1 + x_2x_2 + c_3x_3$$
$$\int_{-1}^{1} x^2dx = \frac{2}{3} = c_1x_1^2 + c_2x_2^2 + c_3x_3^2$$
$$\int_{-1}^{1} x^3dx = 0 = c_1x_1^3 + c_2x_2^3 + c_3x_3^3$$
$$\int_{-1}^{1} x^4dx = \frac{2}{5} = c_1x_1^4 + c_2x_2^4 + c_3x_3^4$$
$$\int_{-1}^{1} x^5dx = 0 = c_1x_1^5 + c_2x_2^5 + c_3x_3^5$$

Using Wolfram Alpha system of equations solver we obtain $\begin{matrix} c_1 = \frac{5}{9} & c_2 = \frac{8}{9} & c_3 = \frac{5}{9} \\ x_1 = -\sqrt{\frac{3}{5}} & x_2 = 0 & x_3 = \sqrt{\frac{3}{5}} \end{matrix}$

The formula is not exact to degree-6 because $\int_{-1}^{1} x^6dx = \frac{2}{7} \neq c_1x_1^6 + c_2x_2^6 + c_3x_3^6 = \frac{6}{25}$, so our formula is exact to at most degree-5.

Problem 4: (a)

$$A = \begin{bmatrix} 10 & -1 & 6 \\ -1 & 8 & 9 \\ 6 & 9 & 1 \end{bmatrix} \vec{v}^{(0)} = \begin{bmatrix} 0.5 \\ 0.6 \\ 0.6 \end{bmatrix}$$

$$\vec{w}^{(1)} = A\vec{v}^{(0)} = \begin{bmatrix} 8 \\ \frac{97}{10} \\ 9 \end{bmatrix}, \vec{v}^{(1)} = \frac{\vec{w}^{(1)}}{||\vec{w}^{(1)}||_2} = \begin{bmatrix} \frac{80}{\sqrt{23909}} \\ \frac{97}{\sqrt{23909}} \\ \frac{90}{\sqrt{23909}} \end{bmatrix}$$

$$\vec{w}^{(2)} = A\vec{v}^{(1)} = \begin{bmatrix} \frac{1243}{\sqrt{23909}} \\ \frac{1506}{\sqrt{23909}} \\ \frac{1443}{\sqrt{23909}} \end{bmatrix}, \vec{v}^{(2)} = \frac{\vec{w}^{(2)}}{||\vec{w}^{(2)}||_2} = \begin{bmatrix} \frac{1243}{\sqrt{5895334}} \\ \frac{753\sqrt{2}}{\sqrt{2947667}} \\ \frac{1443}{\sqrt{5895334}} \end{bmatrix}$$

$$\lambda^{(2)} = r(\vec{v}^{(2)}) = (\vec{v}^{(2)})^T A\vec{v}^{(2)} = \frac{92573743}{5895334} \approx 15.7$$

(b)

$$\frac{||A\vec{v}^{(2)} - \lambda^{(2)}\vec{v}^{(2)}||_2}{||\lambda^{(2)}\vec{v}^{(2)}||_2} = 0.00675$$

by symbolic solver.

(c)

$$\begin{bmatrix} 10 & -1 & 6 \\ -1 & 8 & 9 \\ 6 & 9 & 1 \end{bmatrix} - \frac{92573743}{5895334} \begin{bmatrix} \frac{1243}{\sqrt{5895334}} \\ \frac{753\sqrt{2}}{\sqrt{2947667}} \\ \frac{1443}{\sqrt{5895334}} \end{bmatrix} \begin{bmatrix} \frac{1243}{\sqrt{5895334}} & \frac{753\sqrt{2}}{\sqrt{2947667}} & \frac{1443}{\sqrt{5895334}} \end{bmatrix}$$
$$= \begin{bmatrix} 5.88470 & -5.98605 & 1.22255 \\ -5.98605 & 1.95894 & 3.21168 \\ 1.22255 & 3.21168 & -4.54613 \end{bmatrix}$$

(d) $\vec{v}^{(40)} = \begin{bmatrix} 0.7982 \\ -0.5990 \\ -0.0639 \end{bmatrix}, \lambda^{(40)} = 10.2788$ with $tol = 10^-5$

$\vec{v}^{(40)}$ and $\lambda^{(40)}$ is an approximation for one of the other two eigenvalue-eigenvector pairs of matrix A.

$\frac{A\vec{v}^{(40)}}{\lambda^{(40)}} = \begin{bmatrix} 0.79752 \\ -0.59980 \\ -0.06476 \end{bmatrix}$

```matlab
% Fixed-Point Method for function A
clc;
clear all;

% Inputs: p0, tol, N0
tol = 1e-5;  % error tolerance
N0 = 5000;    % maximum number of iteration
v0= [0.5;0.6;0.6];     % starting point
l= 0;

% Start Iterating
j = 1;
w = v0;
v = w;
A = [5.8847 -5.98605 1.22255;-5.98605 1.95894 3.21168;1.22255 3.21168 -4.54613];

while j < N0

  w = A*v0;
  v = w/norm(w);
  l= transpose(v)*A*v;
  if norm(A*v-l*v)/norm(l*v)<tol
      % close enough to actual root, stop
      break;
  else
      v0=v;
      j = j + 1;
  end

end

fprintf('Iteration number = %d \n', j);
fprintf('v = %.4f \n',v);
fprintf('l = %.4f \n',l);
fprintf('Error = %.4f \n', norm(A*v-l*v)/norm(l*v));
```



```
Iteration number = 40
v = 0.7982
v = -0.5990
v = -0.0639
l = 10.2788
Error = 0.0000
>>
```