Complex Numbers and the Fundamental Theorem of Algebra

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Reasoning Behind Choice

Why did we choose this topic?

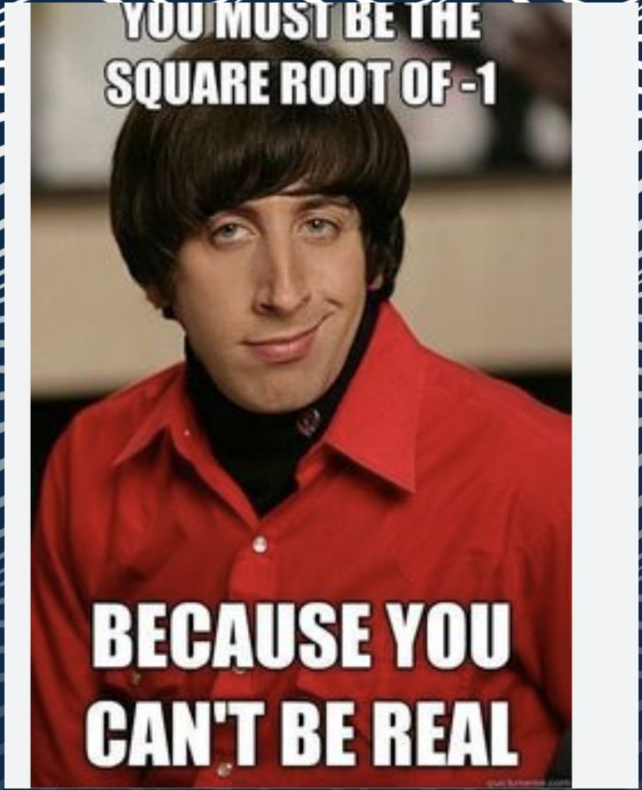
Unique

We chose this topic because it’s unique and no other group has presented it.

Introduction

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = 2, -2$$

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = \sqrt{-1}$$



How we are going to approach this topic

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Background Information

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Open Problem

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Mathematical Connections

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Open Problem

What’s next after the quadratic equation?

What is the general solution of the cubic equation?

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New Concepts

Featuring Ferro, Tartaglia, Cardano, and Bombelli

General Solution of the Cubic

The del Ferro–Tartaglia–Cardano solution of the cubic equation

$$y^3 = py + q$$

is

$$y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

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$x^3 = 15x + 4$

$$x^3 = \overset{\overset{p}{\downarrow}}{15}x + \overset{\overset{q}{\downarrow}}{4}$$

$$x = \sqrt[3]{\frac{4}{2} + \sqrt{\left(\frac{4}{2}\right)^2 - \left(\frac{15}{3}\right)^2}} + \sqrt[3]{\frac{4}{2} - \sqrt{\left(\frac{4}{2}\right)^2 - \left(\frac{15}{3}\right)^2}}$$

$$x = \sqrt[3]{2 + \sqrt{(2)^2 - (5)^2}} + \sqrt[3]{2 - \sqrt{(2)^2 - (5)^2}}$$

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$$

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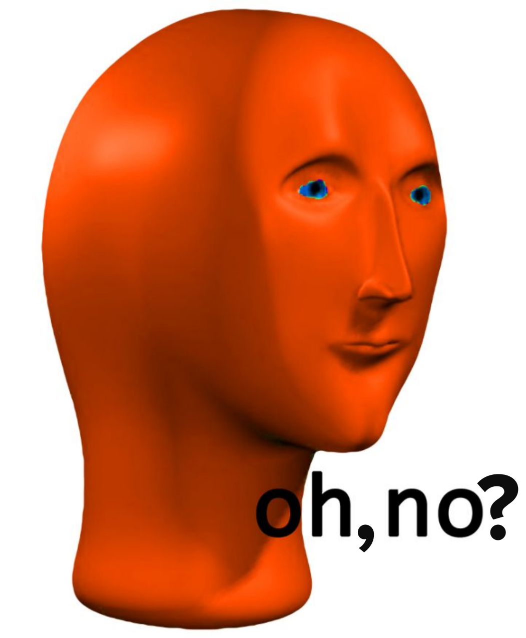

$$x^3 = 15x + 4$$

Let $x = 4$

$$(4)^3 = 15(4) + 4$$

$$64 = 60 + 4$$

$$64 = 64$$

A solution exists!

Introducing Bombelli

The del Ferro–Tartaglia–Cardano solution of the cubic equation

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$$x^3 = \overset{p}{15}x + \overset{q}{4}$$

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$$

$$\sqrt[3]{a + b\sqrt{-1}} = c + d\sqrt{-1}$$

$$\sqrt[3]{2 + 11\sqrt{-1}} = 2 + \sqrt{-1}$$

$$\sqrt[3]{2 - 11\sqrt{-1}} = 2 - \sqrt{-1}$$

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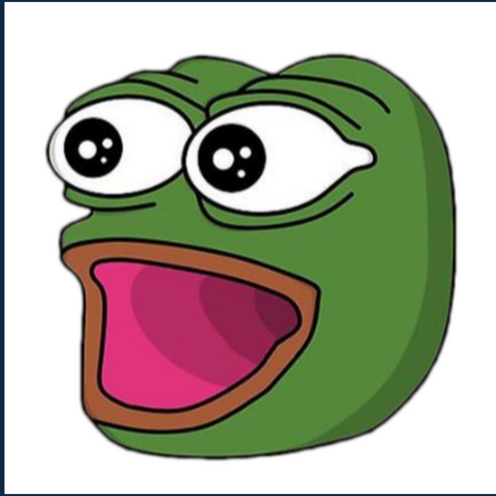
$$x^3 = \overset{p}{\overset{\downarrow}{15}}x + \overset{q}{\overset{\downarrow}{4}}$$

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$$

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$$

$$x = 2 \cancel{+\sqrt{-1}} + 2 \cancel{-\sqrt{-1}}$$

$$\boxed{x = 4}$$



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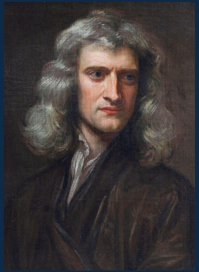
New Theorems

Angle Division, Cote’s, and Fundamental Theorem of Algebra

Angle Division

$$4y^3 - 3y = c$$

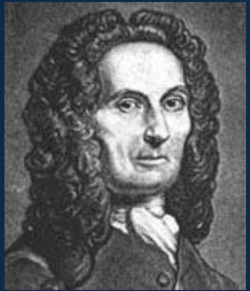
Viète's cubic equation



$$y = nx - \frac{n(n^2-1)}{3!}x^3 + \frac{n(n^2-1)(n^2-3^2)}{5!}x^5 + \cdots$$

$$y = \sin n\theta \text{ and } x = \sin\theta$$

Newton's equation



$$x = \frac{1}{2}\sqrt[n]{y + \sqrt{y^2-1}} + \frac{1}{2}\sqrt[n]{y - \sqrt{y^2-1}},$$

de Moivre's solution to Newton's

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

de Moivre's formula, our version

Cote’s Theorem

If $A_0, \ldots, A_{n-1}$ are equally spaced points on the unit circle with center O, and if P is a point on OA_0 such that $OP = x$, then (Figure 14.4)

$$PA_0 \cdot PA_1 \cdots PA_{n-1} = 1 - x^n.$$

$$PA_k^2 = 1 - 2x\cos\frac{2k\pi}{n} + x^2.$$

What???

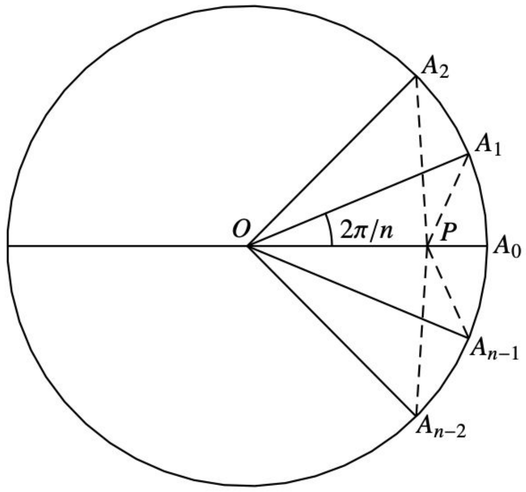
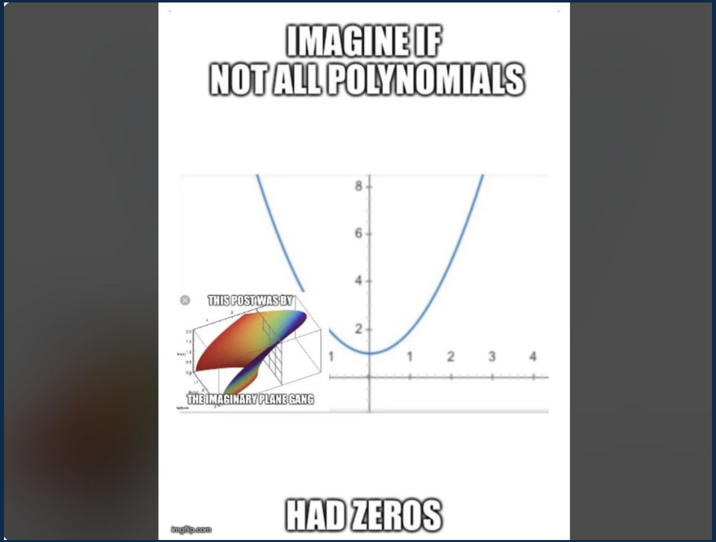
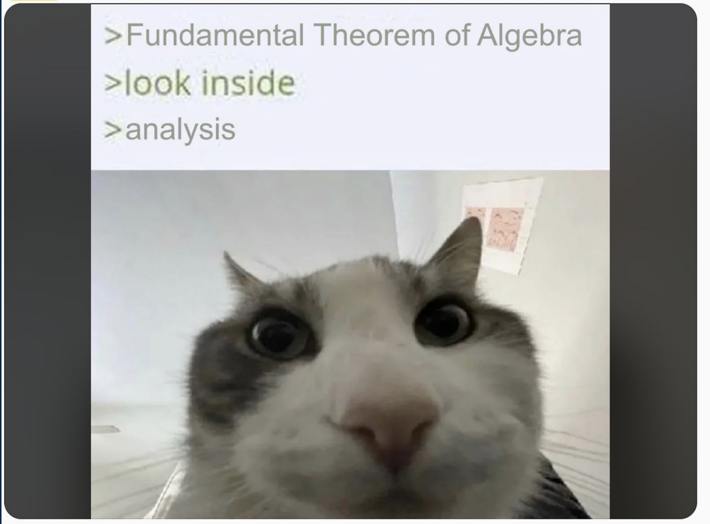


Figure 14.4: Cotes’s theorem

Fundamental Theorem of Algebra





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Mathematical Proofs

- d’Alembert’s Lemma
- d’Alembert’s Proof
- Gauss’s first proof

Jean le Rond d’Alembert

- First proof (1746)
- Revised by Jean-Robert Argand in (1806)

Carl Friedrich Gauss

- First proof (1799)
- Proposes two other proofs later on in career
- Alexander Ostrowski (1920) shows Gauss's proof can be made rigorous

Gauss's "Proof"

The Goal

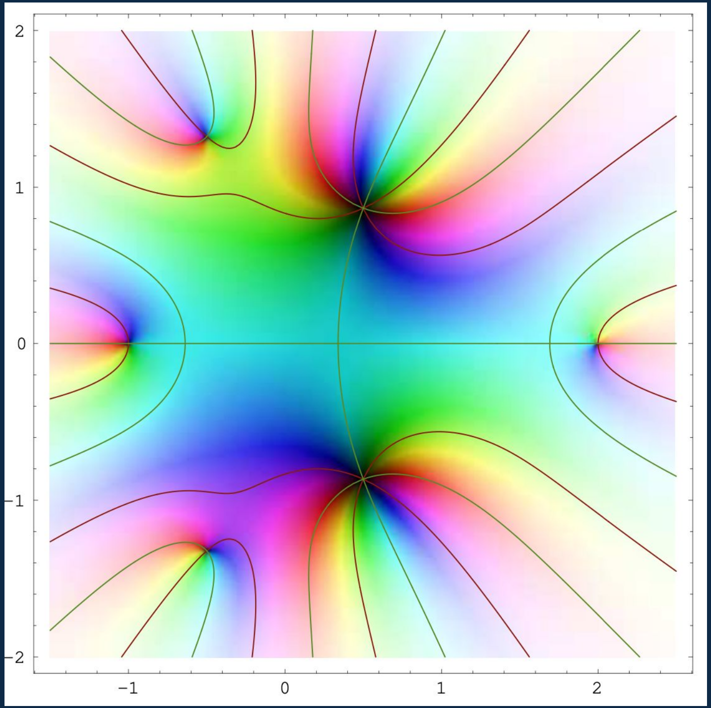
Polynomials can be broken up into linear and irreducible quadratic factors over the real line.
Our goal is to show that eiter $(x \pm r), r \geq 0$ or $x^2 + 2r\cos\phi x + r^2, r > 0$ is a factor of polynomial $p(x)$ for an appropriate choice of r and θ. Moreover, we want to show $r(\cos\phi \pm i\sin\phi)$ is a root of $p(x)$.

Clever Substitution

Let $x = r(\cos\phi + i\sin\phi) \Rightarrow x^k = r^k(\cos k\phi + i\sin k\phi)$
Split $p(x)$ into

$$U(r,\phi) = a_0 + a_1\cos(\phi)r + a_2\cos(2\phi)r^2 + \cdots + a_n\cos(n\phi)r^n$$
$$T(r,\phi) = a_1\sin(\phi)r + a_2\sin(2\phi)r^2 + \cdots + a_n\sin(n\phi)r^n$$

Consider the curves $U(r,\phi) = 0$ and $T(r,\phi) = 0$



Gauss’s “Proof”

- Why do the intersections alternate?
- Curves entering and exiting the circle
- Proof by Contradiction
- Pigeonhole principle

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Mathematical Connections

What are the connections between arithmetic, geometry, and calculus?

Works Cited

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