Introduction

Prior to the introduction of complex numbers mathematicians have been dealing with real numbers; therefore, the introduction of complex numbers was considered to be "impossible numbers" since it involves the square root of a negative number which could not be observed in the physical world hence why it was coined the term "imaginary".

For example, if we were asked to find the solutions of the equation

$$x^2 - 4 = 0 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = 2, -2$$

We say that the equation has a solution of x = 2 and -2.

What about the solution for

$$x^2 + 1 = 0 \quad \Rightarrow \quad x^2 = -1 \quad \Rightarrow \quad x = \sqrt{-1}$$

Mathematicians then could not comprehend the idea of $\sqrt{-1}$ so they deemed it to be an "impossible number" since it could not be observed in the physical world. They concluded that anytime there is a square root of a negative number that the equation simply has no solution. And thus, the general solution of the quadratic was agreed to be of the form:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The most important reason as to why the concept of complex numbers arises is that complex numbers simplify many mathematical problems, more specifically, problems that involve two dimensions as complex numbers are in the form a + b*i*, where a and b are real numbers and *i* is the imaginary unit with the special property $i^2 = -1$ (Stillwell 276).

For this project, our group will tackle problems related to complex numbers, including the quadratic and cubic problem, which cannot be explained without the incorporation of complex numbers; angle division, which features the multiplication of complex numbers; cotes theorem, which helps us understanding the relationship between complex exponentials and trigonometric functions; and finally, the fundamental theorem of algebra, which explains the solution of polynomial equations through complex numbers.

New Concepts

We first begin by exploring the recurrence of complex numbers. Mathematicians first encountered complex numbers when trying to find solutions to quadratic equations. However, the occurrence of complex numbers was not too prevalent in quadratic equations. After mathematicians had found the general solution to the quadratic equation they set their sights on the next big step, finding a general solution to cubic equations. Thanks to the contributions of Scipione del Ferro, Niccolò Fontana Tartaglia, and Gerolamo Cardano the general solution of the cubic was discovered in the form of

$$y^3 = py + q$$

$$y = \sqrt[3]{\frac{q}{2} + \sqrt{(\frac{q}{2})^2 - (\frac{p}{3})^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{(\frac{q}{2})^2 - (\frac{p}{3})^3}}$$ (Stillwell 277).

Though, even this general solution has a problem. For example,

$$x^3 = 15x + 4$$

Applying the general solution of the cubic equation we get

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$$

When $(\frac{q}{2})^2 - (\frac{p}{3})^3 < 0$ we have a complex number. Earlier we discussed that mathematicians deemed the equations with such square roots to have no solution. However, we learned that a cubic always has at least one real root so that can't be the case. Let $x = 4$.

$$x^3 = 15x + 4 \Rightarrow (4)^3 = 15(4) + 4 \Rightarrow 64 = 64$$

We see here that a solution does indeed exist for the cubic equation. The notion that an equation does not have a solution when there is a complex number is now being questioned on its validity. The recurrence of complex numbers becomes unavoidable when solving cubics making it difficult for mathematicians to simply ignore like they once did with the quadratic. During this time the idea of complex numbers was best captured by Cardano when he said, they are "as subtle as they are useless". This begs the question on what the meaning or purpose of complex numbers is?

Unlike many other mathematicians Bombelli took the idea of complex numbers seriously. From the previous example, Bombelli sought to reduce the expression $\sqrt[3]{a + b\sqrt{-1}}$ to the form $c + d\sqrt{-1}$ (Stillwell 278). He was able to reduce it to that form by cubing the expressions to get

$$\sqrt[3]{2 + 11\sqrt{-1}} = 2 + \sqrt{-1} \quad , \quad \sqrt[3]{2 - 11\sqrt{-1}} = 2 - \sqrt{-1}$$

Therefore x can be rewritten as

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1} \Rightarrow x = 4$$

which gives us the same solution as before when we directly plugged in $x = 4$. Through his approach he was able to use complex numbers to arrive at a solution for the cubic. This revolutionized the way mathematicians viewed complex numbers. They now recognize that they could use complex numbers as an intermediate step to help them arrive at a solution.

New Theorems:

Angle Division

The story of complex numbers and angle division begins with Viete, who showed that solving the cubic is equivalent to trisecting an arbitrary angle. He found that trisecting an angle with cosine c is equivalent to solving the cubic equation:

$$4y^3 - 3y = c$$

Continuing on this path, Viete attempted to find polynomial expressions for cos(nθ) and sin(nθ).

Inspired by this work, Newton came up with an equation that relates y = sin(nθ) and x = sin(θ).

$$y = nx - \frac{n(n^2-1)}{3!}x^3 + \frac{n(n^2-1)(n^2-3^2)}{5!}x^5 + \ldots$$

A solution to this equation was found by de Moivre for n values of the form 4m+1.

$$x = \frac{1}{2}\sqrt[n]{y + \sqrt{y^2 - 1}} + \frac{1}{2}\sqrt[n]{y - \sqrt{y^2 - 1}}$$

This solution strongly resembles the solution for a cubic found by Cardano, yet again showing the link between the cubic and complex. It is from this solution we derive our version of de Moivre's formula that works for all values of n, a powerful method for angle division in the complex plane.

$$(cos\theta + isin\theta)^n = cos(n\theta) + isin(n\theta)$$

Cotes Theorem

Around this time, mathematicians tended to avoid results about $\sqrt{-1}$, although they used it in calculations to deduce results about the reals. One such example is Cotes theorem, which relates the regular n-gon to the polynomial $x^n - 1$. Proving this theorem requires solving the equation:

$$PA_k^2 = 1 - 2x\,cos(\frac{2k\pi}{n}) + x^2$$

This is easily done by splitting it into complex linear factors and applying de Moivre's theorem, showing us how valuable complex numbers are for the factorization of polynomials.

Fundamental Theorem of Algebra

Simply put, the fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root (Stillwell 285). This theorem includes polynomials that have real coefficients, since every real number is a special type of complex number, a complex number whose imaginary part equal to zero. Next, we will discuss the three points of significance of FTA.

Firstly, FTA demonstrates that the field of complex numbers is algebraically closed, which entails that every polynomial equation has a solution within the complex numbers. However, we cannot make the same argument for the field of real numbers, due to the fact that there exists polynomial equations with real coefficients that do not have real solutions. For example, $x^2 + 4 = 0$ is a polynomial equation with real coefficients; however, we can observe that it does not have real solutions.

Next, FTA guarantees that there exists roots for polynomial equations that reside beyond real number roots, which is essential for solving many problems in calculus. For example, it is crucial to be able to find the roots of a derivative of a polynomial, or the critical points, which helps us determine the local maxima and local minima of the functions. This is a connection to differential calculus. It is also connected to multiple theorems of analysis, which includes the argument principle and so on.

Last but not least, FTA ensures that a polynomial with degree *n* will have exactly *n* roots according to the complex number system, including multiplicity with respect to each root.

Gauss's Proof

Gauss starts with a polynomial with real coefficients

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

All real polynomials can be reduced into a product of linear and irreducible quadratic factors. For appropriate values of r and ϕ, a linear factor can be expressed as $z \pm r, r \geq 0$ and a quadratic factor can be expressed as $z^2 + 2r\cos\phi z + r^2, r > 0$ with complex roots $r(\cos\phi \pm i\sin\phi)$. Substituting $z = r(\cos\phi + i\sin\phi)$ we define 2 new polynomials

$$U(r,\phi) = a_0 + a_1\cos(\phi)r + a_2\cos(2\phi)r^2 + \cdots + a_n\cos(n\phi)r^n$$
$$T(r,\phi) = a_1\sin(\phi)r + a_2\sin(2\phi)r^2 + \cdots + a_n\sin(n\phi)r^n$$

where $U(r,\phi) = Re(P(z))$ and $T(r,\phi) = Im(P(z))$.
Consider the curves $U = 0$ and $T = 0$. Our goal is find the intersection of these two curves within a sufficiently large circle of radius R. For if we can determine (r,ϕ) such that $U = 0$ and $T = 0$ simultaneously, then $z \pm r \mid P(z)$ or $z^2 + 2r\cos\phi z + r^2 \mid P(z)$.
Gauss observes the behavior of the two curves inside the circle. By Bezout's lemma, $U = 0$ and $T = 0$ should each intersect the circle $2n$ times for a total of $4n$ intersections with the circle. Gauss divides the circumference of circle into

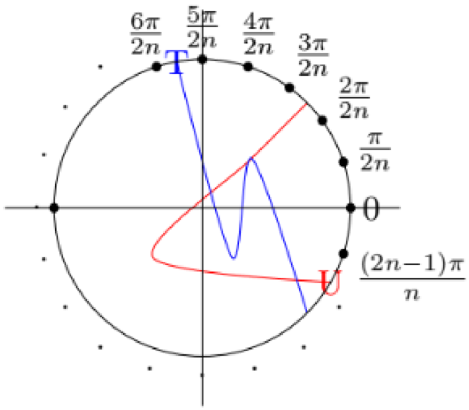

Figure 1: Division of circle into sectors each with an alternating intersection with T or U.

$4n$ arcs for each of the intersections. Gauss observes the intersections alternate between U and T labeling U's the odd intersections and T's the even. Since the curves do not stop abruptly, all branches of the curves must intersect the circle at two distinct points. Figure 1 shows how the circle is divided into arcs and an intersecting branch of $U = 0$ and $T = 0$. He uses the fact that the intersections alternate and that the branches are connected at two points to construct a geometric proof by contradiction to show that T and U must intersect inside the circle.

Conclusion

In conclusion, many of the theorems and ideas discussed showcase the connections complex numbers make between the different fields of math. For example, the angle division theorem discussed earlier was a tremendous discovery, as it related algebra and trigonometry, creating another bridge between the realms of algebra and geometry. Furthermore, the Fundamental Theorem of Algebra sheds light on the approach to differential calculus, as it provides a foundation for root finding of derivatives of a polynomial function.

Contribution Statement

Victor Shi:

1. **Introduction Part I: the importance of Complex Number**
    a. Briefly introduces how mathematicians in the past come about the problem of "impossible numbers", which leads to the discussion of complex numbers
2. **New Theorems Part III: Fundamental Theorem of Algebra**
    a. Describes the definition and importance of Gauss's Fundamental Theorem of Algebra, connecting it to differential equations
3. **Conclusion**
    a. Provided an ending paragraph on the connection between complex numbers and different fields of mathematics

Nam Truong:

2. **Introduction Part II**
    a. Briefly introduces how finding the general solution of the quadratic equation has led mathematicians to focus on finding the general solution of the cubic equation.
3. **New Concepts**
    a. Introduced Bombelli's approach to using Cardano's equation and explained how his approach gave purpose/made sense of complex numbers.

Owen Jones:

1. **Intuition and Explanation of Proof through LaTeX**

Nitin Veeraperumal:

1. **New Theorems Part I and II**
    a. Highlights the motivation and development behind the angle division theorem and describes how it connects algebra and geometry.
    b. Notes early mathematicians' reservations about the imaginary number and provides an example of complex numbers being used in an intermediary step of the Cotes Theorem proof.
2. **Conclusion**
    a. Reiterates the significance of how the angle division theorem connects algebra and geometry, and how complex numbers are vital in describing the connection between various areas of mathematics.

Works Cited

"C. F. Gauss's Proofs of the Fundamental Theorem of Algebra", Harel Cain

"The Fundamental Theorem of Algebra: A Visual Approach", Daniel J. Velleman

"d'Alembert's Lemma", France Dacar

"Mathematics and Its History, 3rd Edition", John Stillwell