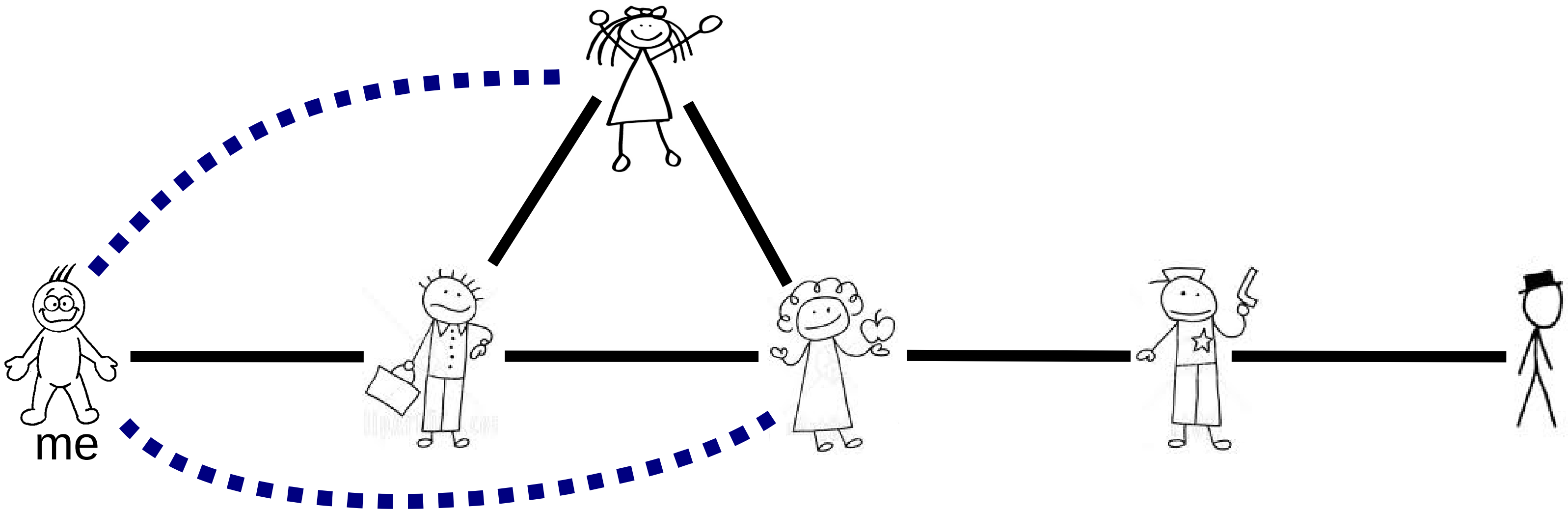
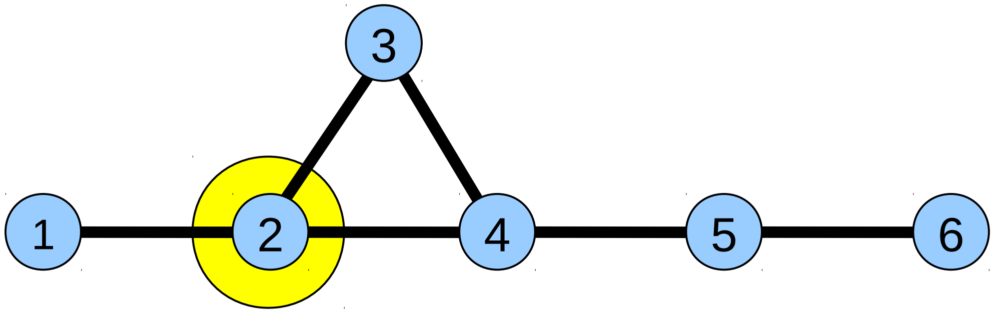


Predict who I will add as friend next

Facebook's algorithm: find friends-of-friends

Adjacency Matrix



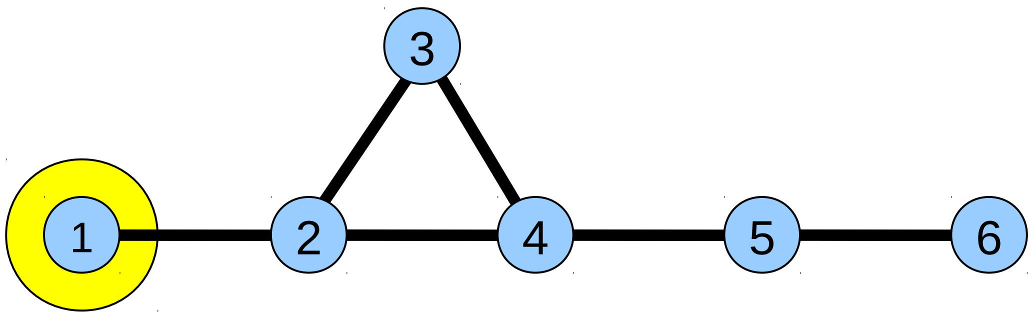
$\mathbf{A}_{uv} = 1$ when u and v are connected

$\mathbf{A}_{uv} = 0$ when u and v are not connected

$\mathbf{A}$ is square and symmetric

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left| \begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right| \end{array}$$

Square of Adjacency Matrix



$$(\mathbf{AA})_{uv} = \mathbf{A}_{u:}\mathbf{A}_{:v}$$

equals the number of common friends of u and v

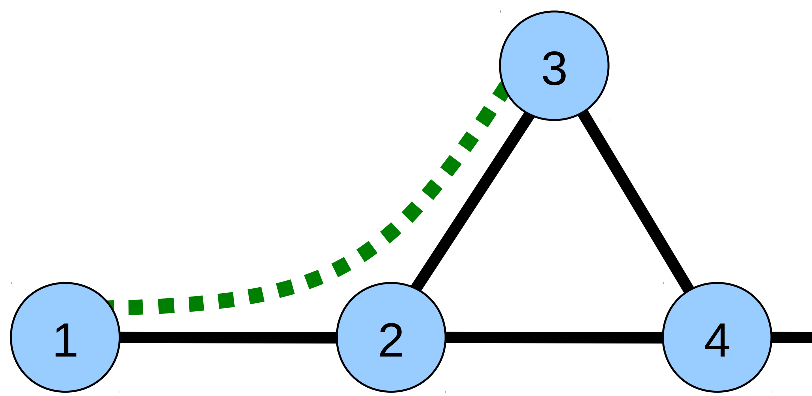
$$\mathbf{A}^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left| \begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right| \end{array}$$

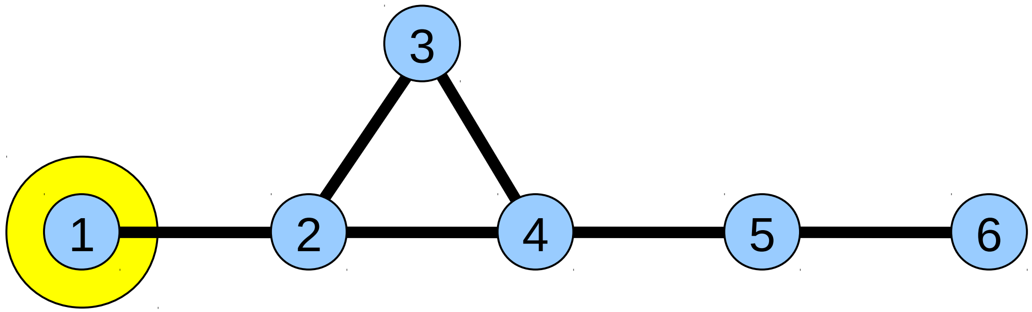
Square of Adjacency Matrix: Example

Count the number of ways a person can be found as the friend of a friend

Matrix product $\mathbf{A}\mathbf{A} = \mathbf{A}^2$

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$





Compute the number of friends-of-friends-of-friends:

$$\mathbf{A}^3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^3 = \begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} 0 & 3 & 1 & 1 & 1 & 0 \\ 3 & 2 & 4 & 5 & 1 & 1 \\ 1 & 4 & 2 & 4 & 1 & 1 \\ 1 & 5 & 4 & 2 & 4 & 0 \\ 1 & 1 & 1 & 4 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Problem: $\mathbf{A}^3$ is not sparse!

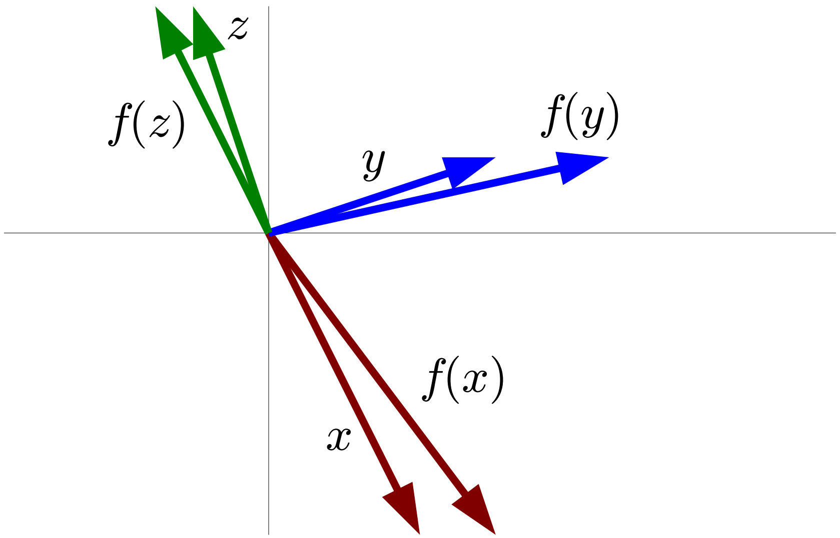
Hard to compute for very large matrices

Geometrical Interpretation of Matrices as Vector-Functions

Interpret a real $n \times n$ matrix $\mathbf{A}$ as a function f from n-vectors to n-vectors:

$$f(x) = \mathbf{A}x$$

Example with $\mathbf{A} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}$



Diagonal Matrices

Definition: A matrix $\mathbf{X}$ is diagonal if $\mathbf{X}_{uv} = 0$ whenever $u \neq v$

Diagonal matrices are easy to take powers of:

If $\mathbf{X}$ is diagonal, then $\mathbf{X}^k$ is diagonal and given by

$$(\mathbf{X}^k)_{uu} = (\mathbf{X}_{uu})^k$$

Example:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}^3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Geometrical interpretation: stretch/mirror along the axes

Definition: a matrix $\mathbf{U}$ is orthogonal when $\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}$

Interpret a matrix $\mathbf{U}$ as a vector-function: $f(x) = \mathbf{U}x$

This means:

(a) $\mathbf{U}_{:u}^{\mathrm{T}}\mathbf{U}_{:v} = 1$ when $u = v$

(b) $\mathbf{U}_{:u}^{\mathrm{T}}\mathbf{U}_{:v} = 0$ when $u \neq v$

In other words:

(a) The columns of $\mathbf{U}$ have unit length

(b) The columns of $\mathbf{U}$ are orthogonal to each other

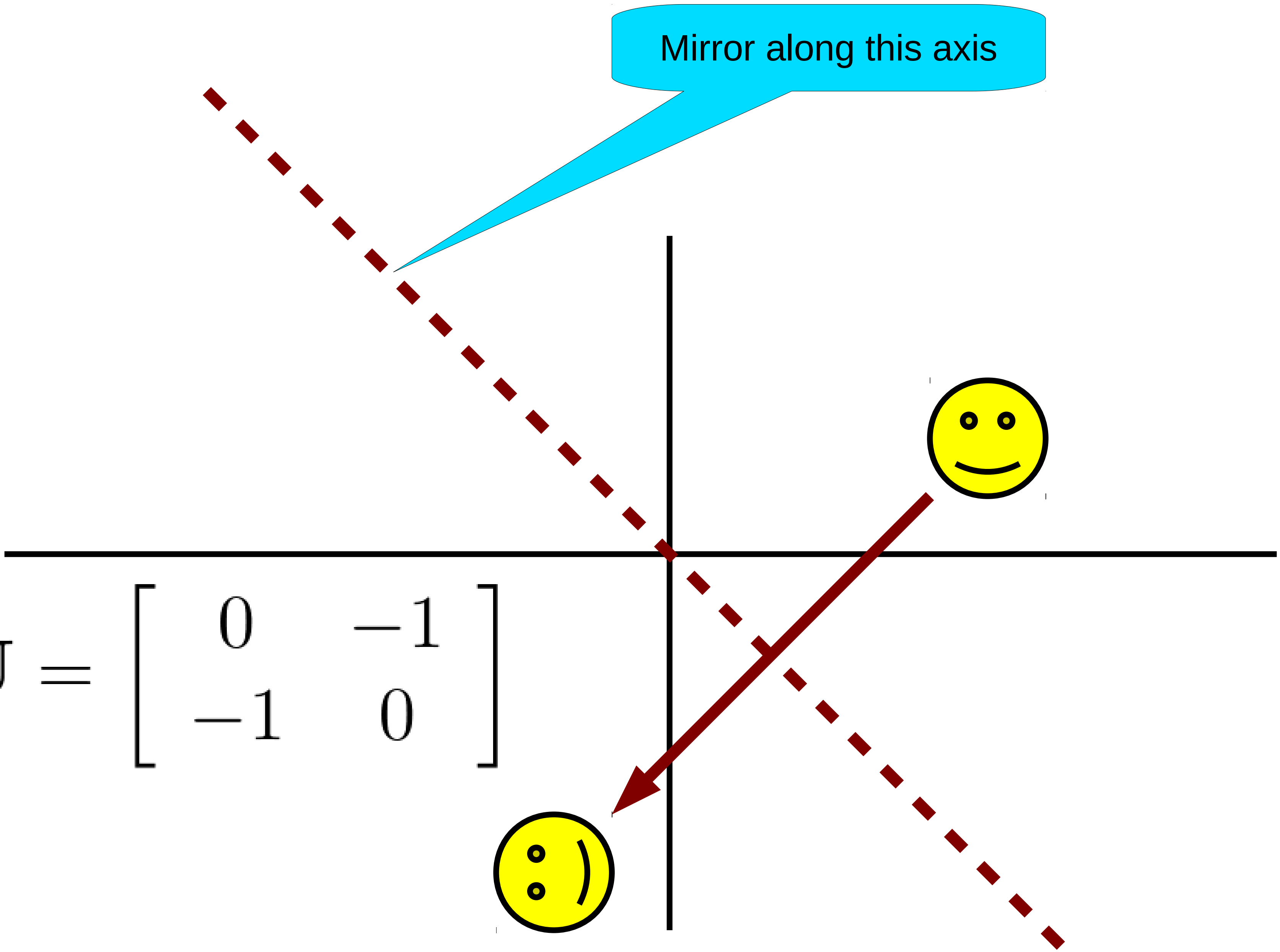
It follows:

(a) f preserves vector length

(b) f preserves angles

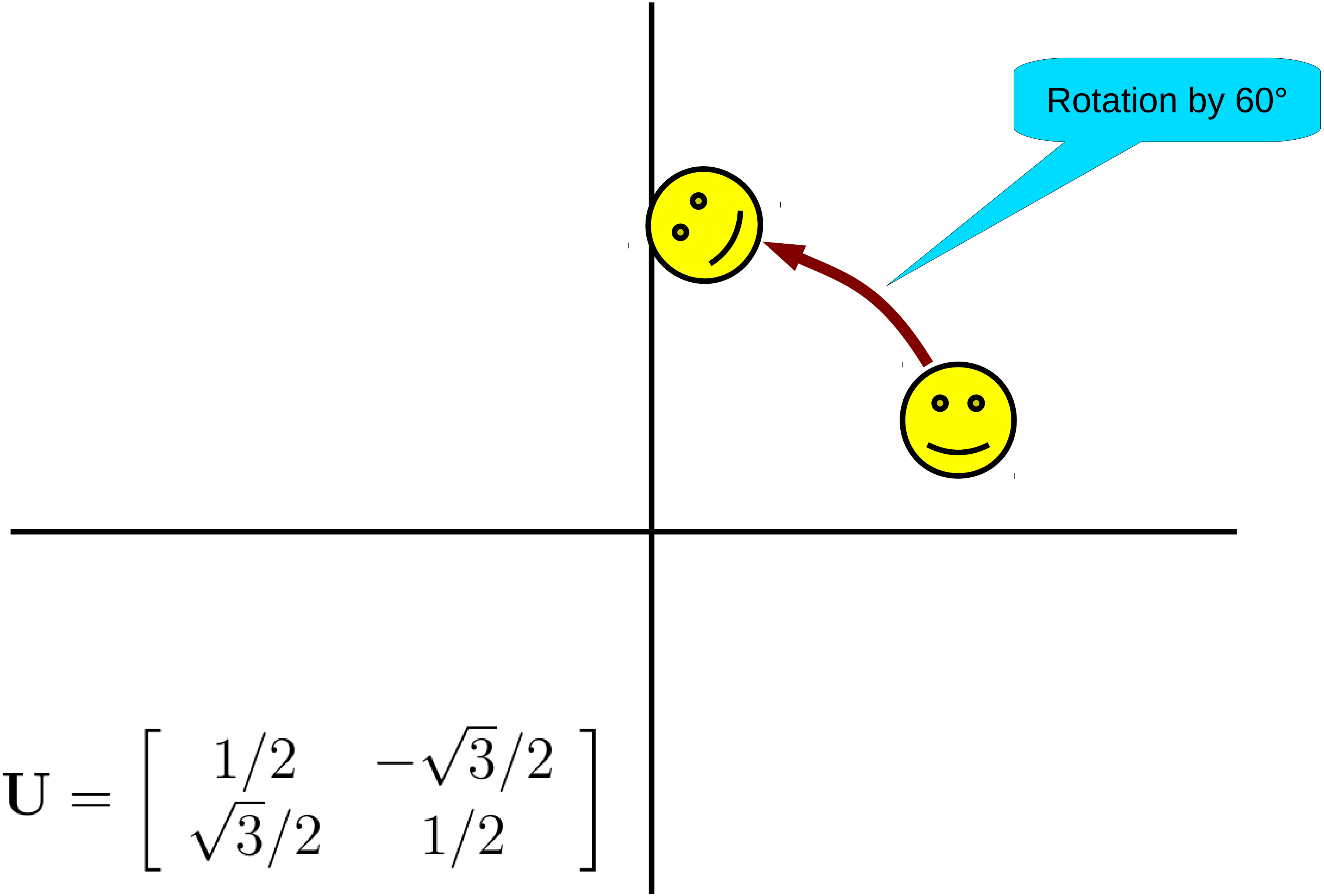
Geometrical interpretation: f is a rotation, reflection or a combination of both

Orthogonal Matrices: Example 1



$$\mathbf{U} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Orthogonal Matrices: Example 2



$$\mathbf{U} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

Eigenvalue Decomposition

For a real, symmetric matrix $\mathbf{A}$, the eigenvalue decomposition of $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1}$$

where

$\mathbf{U}$ is orthogonal (the eigenvectors)

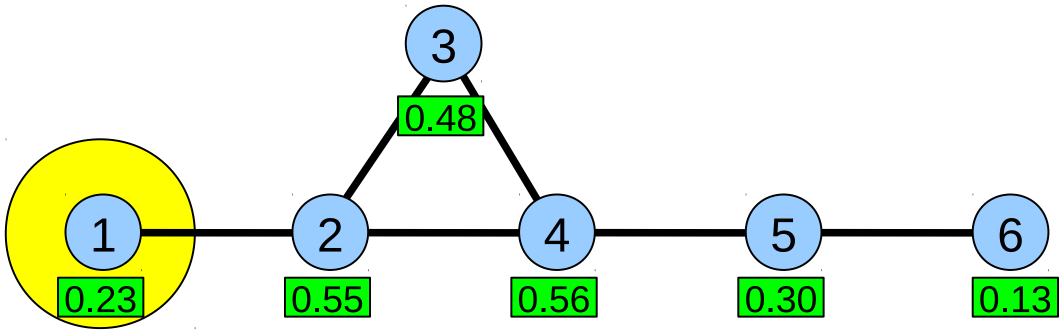
$\mathbf{\Lambda}$ is diagonal (the eigenvalues)

Interpretation as a vector-function $f(x) = \mathbf{A}x$:

- Rotate, stretch/mirror, rotate back
- I.e.: stretch in arbitrary direction

Every real symmetric matrix is a stretch/mirror in arbitrary direction!

Eigenvalue Decomposition: Example



$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{A} =$$

$$\underbrace{\begin{bmatrix} 0.25 & -0.43 & -0.17 & 0.76 & 0.30 & 0.23 \\ -0.44 & 0.59 & 0.10 & 0.21 & 0.33 & 0.55 \\ -0.10 & -0.56 & 0.51 & -0.39 & 0.18 & 0.48 \\ 0.61 & 0.18 & -0.41 & -0.32 & -0.12 & 0.56 \\ -0.52 & -0.28 & -0.37 & 0.09 & -0.64 & 0.30 \\ 0.30 & 0.20 & 0.63 & 0.34 & -0.59 & 0.13 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} -1.74 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.37 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.59 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.27 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.33 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 0.25 & -0.43 & -0.17 & 0.76 & 0.30 & 0.23 \\ -0.44 & 0.59 & 0.10 & 0.21 & 0.33 & 0.55 \\ -0.10 & -0.56 & 0.51 & -0.39 & 0.18 & 0.48 \\ 0.61 & 0.18 & -0.41 & -0.32 & -0.12 & 0.56 \\ -0.52 & -0.28 & -0.37 & 0.09 & -0.64 & 0.30 \\ 0.30 & 0.20 & 0.63 & 0.34 & -0.59 & 0.13 \end{bmatrix}^T}_{U}$$

Eigenvector

Eigenvalue

$\mathbf{U}$ contains eigenvectors, $\mathbf{\Lambda}$ contains eigenvalues

Computing A³

Use the eigenvalue decomposition $\mathbf{A} = \mathbf{U\Lambda U}^{\mathrm{T}}$

$$\mathbf{A}^3 = \mathbf{U\Lambda}\boxed{\mathbf{U}^{\mathrm{T}}\ \mathbf{U}}\mathbf{\Lambda}\boxed{\mathbf{U}^{\mathrm{T}}\ \mathbf{U}}\mathbf{\Lambda U}^{\mathrm{T}} = \mathbf{U}\boxed{\mathbf{\Lambda}^3}\mathbf{U}^{\mathrm{T}}$$

Exploit $\mathbf{U}$ and $\mathbf{\Lambda}$:

- $\boxed{\mathbf{U}^{\mathrm{T}}\mathbf{U}} = \mathbf{I}$ because $\mathbf{U}$ is orthogonal
- $(\boxed{\mathbf{\Lambda}^k})_{ii} = \mathbf{\Lambda}_{ii}^{\ k}$ because $\mathbf{\Lambda}$ is diagonal

Result: Just cube all eigenvalues

Requirements for a Link Prediction Function

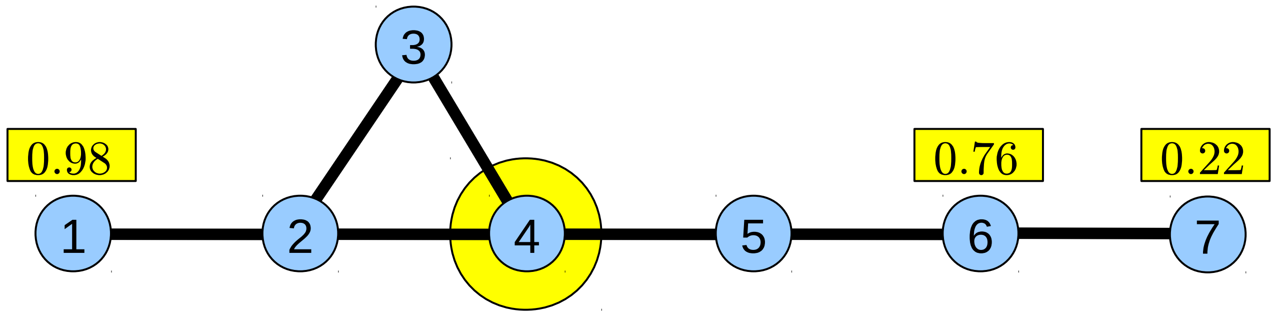
A good link prediction function should:
- Give a higher score when there are more paths connecting two nodes
- Give a higher score when paths are shorter

I.e.,

$$F(\mathbf{A}) = a\mathbf{A}^2 + b\mathbf{A}^3 + c\mathbf{A}^4 + \ldots$$

with $a > b > c > \ldots > 0$

Matrix Exponential



$$\exp(\mathbf{A}) = \mathbf{I} + \mathbf{A} + 1/2\ \mathbf{A}^2 + 1/6\ \mathbf{A}^3 + \ldots$$

$$\exp \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix} = \begin{array}{cc} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} & \\ \begin{vmatrix} 1.66 & 1.72 & 0.93 & 0.98 & 0.28 & 0.06 & 0.01 \\ 1.72 & 3.57 & 2.70 & 2.93 & 1.04 & 0.29 & 0.06 \\ 0.93 & 2.70 & 2.86 & 2.71 & 0.99 & 0.28 & 0.06 \\ \boxed{0.98} & \boxed{2.93} & \boxed{2.71} & \boxed{3.63} & \boxed{1.95} & \boxed{0.76} & \boxed{0.22} \\ 0.28 & 1.04 & 0.99 & 1.95 & 2.35 & 1.59 & 0.64 \\ 0.06 & 0.29 & 0.28 & 0.76 & 1.59 & 2.23 & 1.38 \\ 0.01 & 0.06 & 0.06 & 0.22 & 0.64 & 1.38 & 1.59 \end{vmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \end{array}$$

Link Prediction Functions as Spectral Transformations

$$\mathbf{A}^2 = \mathbf{U}\boxed{\mathbf{\Lambda}^2}\mathbf{U}^\mathrm{T}$$ Friend of a friend

$$\mathbf{A}^3 = \mathbf{U}\boxed{\mathbf{\Lambda}^3}\mathbf{U}^\mathrm{T}$$ Friend of a friend of a friend

$$\exp(\mathrm{a}\mathbf{A}) = \mathbf{U}\boxed{\exp(\mathrm{a}\mathbf{\Lambda})}\mathbf{U}^\mathrm{T}$$ Matrix exponential

$$(\mathbf{I} - \mathrm{a}\mathbf{A})^{-1}$$ Neumann kernel

Link prediction functions change the eigenvalues, but do not change the eigenvectors. They are spectral transformations.