3 THE ALGEBRA OF REVOLUTION AND THE REVOLUTION IN ALGEBRA

3.1 BOOLEAN VARIATIONS ON THE DIALECTIC

IN THE PREVIOUS CHAPTER, we have seen how the classical image of thought against which Hegel struggles in *The Science of Logic* finds its most concentrated expression, in that text, in Hegel's critique of the then-embryonic project of *mathematizing logic*. We argued that the there remains something in that image which Hegel's dialectic never fully metabolizes, a blind spot in the dialectic whose symptoms are, to a large extent, crystallized in the problematic status of the notion of *calculation* in the *Logic*, and the constellation of notions that surround it: notions of exteriority and mechanicity, for instance, and their relation to logic, a relation whose still unsatisfactory treatment by Hegel appeared to us to indicate a threshold that the discipline of dialectical logic must ultimately cross, if it is not to relapse into a predicament like that of the 'beautiful soul', shying away from the defiles of calculation in which the idea of logic must realize itself in exteriority without losing itself.[1] If the time for a direct engagement of dialectics with mathematical logic was, perhaps, not yet ripe in 1831, the situation would change in the coming decades, with George Boole's publication of *The Mathematical Analysis of Logic* in 1847 and *The Laws of Thought* in 1853, which, following a long period of incubation, at last set the discipline of logic on the unsure path of mathematical science, a path which, in the century and a half that followed, saw the science undergo technical developments and transformations that remain, today, a source of profound philosophical challenges. With respect to the dialectic, Boole's own work has the singular virtue of concentrating the classical points of resistance to the dialectic with unprecedented clarity and concision, providing for us something of a control specimen, a degree-zero of dialectical thought. There are few divisions in the history of logic as stark and dramatic as that between Hegel and Boole.

1 See, for instance, the Remarks appended to Chapter 2 of Section 2 of Volume 1 of *The Science of Logic*, (Hegel 2010, 170-182) where philosophy, which 'must know how to distinguish what is by nature a self-external material' and 'must know that, so far as this material goes, the concept can make its way forward only externally,' i.e. non-philosophically, so as 'to prevent ideas from interfering with the peculiar nature of externality and accidentality, and the ideas themselves, because of the disproportionateness of the material, from being distorted and reduced to a formalism' (Hegel 2010, 177), cringes away from a thorough engagement with calculation and its bearing on logic as logic. The Remarks in Chapter 1 of Section 1 of Volume II of the *Logic* (Hegel 2010, 540-546, 604-608) further continue with this line of attitude, more anxious to quarantine mathematics from logic than to undertake a genuinely philosophical and dialetical engagement with the former, an engagement that should not shy away from an exteriorization and realization of the idea of logic in the mathematical medium.

This situation, as maximally antithetical as history has been able to provide, provides us with a starting point for the current chapter. Here, we will attempt to bring the tension between the dialecticization and mathematization of logic into clearer focus by examining two attempted formalizations of dialectical logic that deviate as little as possible from the formal structure of Boolean algebra: the philosopher and Catholic priest Dominique Dubarle's endeavour to inscribe the logic of the Hegelian concept into the theory of Boolean rings, and the Marxist theoretician and engineer Juan Grompone's effort to embed the structure of the materialist dialectic (after Engels) in the context of lattice theory.

Rings and lattices are abstract algebraic structures that began to enter mathematics in the last decades of the nineteenth century, and which became the objects of relatively autonomous mathematical subdisciplines within the first decades of the twentieth. Though their historical roots are manifold and applications extraordinarily diverse, both can be seen to generalize, among other things, Boole's algebra of logic, responding in different ways to the central *epistemological obstacle* encountered in Boole's logic – the appearance of 'logically uninterpretable' formulas in logical calculations, which arise as a consequence of the algebra's lack of operational closure (under $+$) – and both can be seen to issue from the essential *epistemological break* that Boole's work heralds: a liberation of algebra from the province of quantity, and a movement towards viewing mathematical theories as autonomous, abstract structures that would no longer need to appeal to any privileged domain of objects or experiences to be assured of their validity.

The nature of this chapter's subject matter demands a certain interweaving of philosophical and mathematical discourse. No prior mathematical, or logico-mathematical knowledge has been presupposed, however. The mathematics encountered in this chapter are relatively simple – this is, in fact, a second motive for treating Dubarle and Grompone's work early on in this study, before proceeding to material that is more mathematically demanding – and my intention is to move through it slowly and didactically, in part to prepare the reader with the mathematical background and sensitivity necessary for subsequent chapters.

3.2 DUBARLE'S DIALECTICAL RINGS

DOMINIQUE DUBARLE'S contributions to philosophy are marked by an extraordinarily subtle intellect, combining scrupulous attention to detail with an inventiveness completely unburdened by dogmatism. Since his encounter with the work of Cavaillès and Hegel around the time of the Second World War,[2] the relation between logico-mathematical and dialec-

2 According to Charles Journet, in a 1954 letter to Jacques Maritain, Dominique Dubarle discovered Hegel by reading Jean Hyppolite's 1938 translation of and commentary upon *The Phenomenology of Spirit* during his internment in a Nazi prison camp between 1940 and 1943, and again, in a second camp, in 1944. Journet came to greet Dubarle when he left the camp in 1944; in the conversation that followed, Dubarle is said to have told his fellow theologian 'that we are decisively lacking in audacity, and that we must do with Hegel what

tical thought has remained a constant concern in his work, from his early articles from the late 1940s until his death in 1987 at the age of eighty.

In the section that follows, our focus will be on a text from 1972, co-authored with André Doz[3] but largely prefigured in Dubarle's 1968 presentation for Jean Hyppolite's seminar, 'Hegel et la pensée moderne'.[4] It is in that text that we find Dubarle's subtle and far ranging mediations on dialectics and mathematical logic brought to an abrupt point, in a highly experimental *formalization of the dialectic*.

Before examining that text, however, it is worth taking the time to orient ourselves in Dubarle's philosophical understanding of logico-mathematical formalization, so as to discern more clearly what is at stake in *Logique et dialectique*. Dubarle's 1955 article, 'Remarques sur la Philosophie de la formalisation logico-mathématique' provides us with as good a starting point as any.

3.2.1 *Dubarle's understanding of the meaning of logico-mathematical formalization*

The phenomenon of logico-mathematical formalization is, before all else, for Dubarle, the movement by logical and epistemological reflection upon mathematics becomes, itself, a mathematical discipline and object – it is that by which mathematics makes its own reflection immanent to itself. This is a drama staged in two interwoven acts: (1) the mathematization of logic, anticipated by Leibniz and brought to fruition by Boole, Frege and their successors, and (2) the logical formalization of mathematical theories, whose

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Saint Thomas did with Aristotle' (Journet and Maritain 2005, 402). It was during the same years as Dubarle's internment that Jean Cavaillès wrote his masterpiece in the dialectical epistemology of mathematics, *Sur la logique et la théorie de la science*, itself written in a prison camp in 1942, and published post-humously – after Cavaillès execution in 1943 at the hands of the Nazis for his work in the French Resistance – in 1947 (see Cavaillès 1994 (1932-47, 474-560). Dubarle was among the first to engage with Cavaillès' work in detail, and published an article in the *Revue de Métaphysique et de Morale* in 1948 (vol. LIII), with the title 'Le dernier écrit philosophique de Jean Cavaillès'. In 1967-68, he would attend Jean Hyppolite's seminar on Hegel, with the title 'Hegel et la pensée moderne', along with Jacques Derrida, Louis Althusser, Alain Badiou and Dominique Janicaud, among others. It is there that he would present the first version of his attempt to inscribe the Hegelian dialectic in the abstract algebra of Boolean rings (see Dubarle 1970).

3 André Doz is an accomplished scholar and translator of Hegel's work in his own right, author of Doz (1987) and translator of Hegel (1970). The core of Dubarle and Doz (1972), however, is evidently Dubarle's own work, in view of its similarity to Dubarle (1970).

4 It is perhaps of some interest to note that the mathematics underpinning Dubarle (1970) and Dubarle and Doz (1972) were already, for the most part, worked out in a 1963 manuscript, without any direct reference to Hegel. The manuscript was published posthumously in 1989, as **?**. It would be contrary to the spirit and general historical tendency of mathematical science, however, to see the genetic independence of Dubarle's algebra (specifically, the system $\mathbb{B}^2_\Delta$, which we will soon examine in detail) from his analysis of Hegel's doctrine of the concept as cause to suspect that the connection between the two of being merely accidental or external. (This may be the case, of course, but *this* is not grounds for believing so.) The deepest harmonies in mathematics are often discovered through what is, from a historical and genetic perspective, sheer accident and coincidence.

principal historical protagonist we encounter in David Hilbert's metamathematical research programme. Through this double movement, epistemological theses concerning mathematics – pertaining to questions concerning the consistency and completeness of theories, the power of certain forms of calculation, and so on – are themselves transformed into mathematical conjectures, bearing immanently mathematical consequences.

The first global consequence of the discipline's logico-mathematical formalization is a mathematization of mathematical subjectivity. The material given to mathematical thought by the work of formal mathematics and metamathematics, since the first quarter of the twentieth century, is no longer compelled to take the shape of mathematical *objects*, standing over and against unthematised mathematical thought, but the syntactico-noetic structure of the thought itself: 'The act of formal mathematics is therefore no longer, like the act of intuitive mathematics, the act of thinking an *object* of thought, but the act of thinking a *project* of thought' (Dubarle 1955, 370). Though the intuitive and cognitive apprehension of traditional mathematical objectivities – spaces, structures, numbers, and so on – remains possible, and even prevalent in mathematical practice, this practice is now essentially informed by the possibility of 'stepping back' and directing its attention to the noetic and syntactic 'projects' that seize upon those objectivities. This reflexivity is not just a matter of calling old forms by new names, but a genuine change in the orientation of mathematical thought, one which produces dramatic practical effects by animating the discipline with unprecedented mobility, allowing it, for instance, to grasp formal, syntactico-noetic unities that transcend their various objective realizations (theories can now be contemplated independently of the various domains that furnish them with models, abstract algebraic structures – such as rings and lattices, among others – can be reflected upon independently of their various 'concrete' instantiations in arithmetic, topology and even logic). Correlatively, these project-structures can be taken in turn as objects, or models, of other project-structures or theories. This change in the essential orientation of mathematics has, by the present day, affected the discipline so thoroughly that to characterize mathematics as 'the science of quantity', or the science of *any* particular kind of objectivity – a characterization that only two hundred years ago appeared a common-sensical platitude (we find it frequently in Hegel's texts, for instance), would, today, seem almost self-evidently absurd.

The second global consequence of this drama is a *reciprocal inclusion* of logic in mathematics and mathematics in logic, a situation which appears, at first, to tend towards a cohesive and monolithic unity of the two in a seamless immanent block. *Extensionally*, Dubarle observes, logic and mathematics appear to have become equal and virtually identical: every mathematical theory can be transposed into the register of pure logic, whether in a variant of set theory (which can legitimately be understood as a pure logical theory, as Husserl argues in *Formal and Transcendental Logic* (Husserl 1969)) or in higher order logic (Dubarle 1955, 366), but any logical framework in which mathematics can be wholly immersed is *itself* a mathematical theory, subject to forms of algebraic, geometrical, topological and arithmetical analysis and manipulation. The novelty and subtlety of Dubarle's analysis, as worked out in Dubarle (1955), lies in his argument that this reciprocal inclusion, and extensional equality, of mathematics and logic *does not*, itself, imply a strict *identity* between the two modalities of thought.

Dubarle's analysis begins on a local register. What the logico-mathematical formalization of a theory delivers to thought is, first and foremost, the syntactical 'materiality' of the theory – the theory transformed into what, playfully and polemically, the formalists used to characterize as 'games played with marks on paper', whose rules and procedures are, themselves, written out in unambiguous and fully thematised literal assemblages. This is the initial form by which the logical envelopment of a mathematical theory manifests itself. But this 'exteriorization' of mathematics in a logical formalism has a virtually antithetical correlate, evincing a logical dimension that is locally, but *essentially*, unthematised:

> Over these procedures, logic arbitrates, but this is no longer a logic of schemata that are represented and destined for discursive combination [*destinés à s'amalgamer au discours*], but a logic of effective, imperative, regulative schemata. Here logic remains, as if in the margin of the materiality of the posited discourse, and, by this very fact, it appears as if it can only be formulated in the proposition of a "metalanguage", afferent to the formalized theory, the metalanguage that the mathematicians describing the formal mathematics are usually content to draw from ordinary language with its natural advantages. (Dubarle 1955, 373-4)

This *local* distinction between the formal mathematics materialized in logico-syntactical schemata the imperative logical forms that reflection thematises in a metalanguage is, however, inevitably sublated: the metalanguage is, in every case, itself subject to thematization and formalization. But this sublation is sublated in turn, as this formalization simply shifts the bump in the rug. The scene being played out here is a familiar one, resembling in detail the many stagings of 'bad infinity' in the Hegelian doctrine of being – subject to being swept down the channel of blind progression without concept, such as Jean-Yves Girard sees in the Tarskian endeavour to locate the *meaning* of logical propositions in their 'truth conditions', an endeavour that leads to explaining conjunction by meta-conjunction, itself explained by meta-meta-conjunction, and so on.[5]

Dubarle does not stake any claims on the perpetually sublated *local* distinction between the logically formalized mathematics being described and normatively charged logic that governs that description, but, in an essentially Hegelian gesture, it is the structure underlying this dynamic he attempts to grasp. In the flickering movement between object language and

5 We will return to Girard's critique of Tarskian semantics, and its implicit 'bad infinity', in Chapter ?? on page ??.

metalanguage, he seeks to pinpoint logic in its pure state – logical normativity as abstracted from the materiality of the object language, logic *without* mathematics.

This, however, is a nothingness, a '*a virtuality without efficacy and totally illegible to thought*' (Dubarle 1955, 377). What is essential to logic, according to Dubarle, is that it *needs* a materiality other to itself in order to operate at all, without which it 'would remain entirely virtual and silent' (Dubarle 1955, 377). In itself, logic is *without force*: 'Logic has no force of its own; all of its energy is derived from an act that is not its own' (Dubarle 1955, 377). This is a conclusion which, at first, appears antithetical to the logicisms of either Frege or Hegel, one which seems to answer Frege's question, 'How do the empty forms of logic come to disgorge so rich a content?' (Frege 1960, § 16: 22) – a question which Uwe Petersen calls the 'central problem of speculative philosophy' and identifies with Hegel's – with the answer that they *cannot*, not in and of themselves, for even if logic is capable of enveloping and metabolizing the whole of mathematics (to say nothing of metaphysics), it cannot do so by its own power.

This thesis is a subtle one. It is not simply a question of distinguishing logical axioms and rules of inference from non-logical axioms – axioms for which a countermodel can be produced, for instance (see, for instance Badiou 2007, 27-8). Even a 'purely' logical theory, such as a second order logic akin to Frege's,[6] after all, is capable of 'disgorging so rich a content' as classical mathematics in its entirety, without the aid of any 'axioms' (as opposed to rules of inference). What is necessary, to begin, is not an axiom but the *marking of a difference* that is not, itself, logical, a difference which is not commanded by a norm or logical principle. With Hilbert, for instance, we see that this may be nothing but the difference between letters on a page (Hilbert 1967). What is necessary is that this beginning be, with respect to logic, contingent.

Whether this thesis finds itself in conflict with the Hegelian understanding of logic is a still more delicate question. Everything would seem to hinge on how the problem of the *beginnning*, in the *Logic*, is understood. Without attempting to untangle this problem in all its complexity, here, three observations will suffice:

(1) It is not quite correct to say that Hegel's *Logic* begins *ex nihilo*, as it appears to do. Even the 'indeterminate immediate' with which, after a long introductory procedure, it appears to commence is explicitly understood as a *result*, the result of the long process of mediations that compose the entirety of *The Phenomenology of Spirit*, and the point of departure of those mediations is the contingent and irrational stimulus of sense-certainty. The point here is not at all to *ground* the logic in the given of sensation – the mediations carried out in the *Phenomenology*, and prolonged in the introductions to the *Science of Logic*, serve to erase and sublate contingency and givenness, not to enshrine it as a foundation. The point is that what Dubarle calls the 'energy' on which logic draws, on which it draws even in the work

6 Or Petersen's system $L^i D^Z_\lambda$, which we will examine in Chapter ?? on page ??.

of erasing its trace, is contingent and non-logical. The *Science of Logic* does not begin from nothing, but from the result of a protracted labour of *nihilation*, on whose momentum it draws.

(2) The very task of *beginning* can be seen as the trace of this act, a trace that is both absolutely minimal – as minimal as possible – and ineffaceable: 'this beginning remains as the underlying ground of all that follows without vanishing from it [...] remaining everywhere immanent in its further determinations' (Hegel 2010, 49).[7] Even the first great scene of the *Logic* – the staging of *being* and *nothing* – is not, itself, presented as a logical operation. There is no sublation at work between the being and nothing – to draw on terminology we will introduce in section**??**, their difference is neither limitation, nor reference, nor opposition, and the 'passage' between the two (to the extent that they are two) is neither transition, nor reflection, nor development. What we find there is a *repetition without concept*, a repetition which is then thematised in the *first* logical concept, which is that of *becoming*.[8]

(3) It would be careless to neglect, moreover, the rich profusion of differences in which Hegel's *beginning* is immersed, like a fish in water: the element of language, which *appears* to supply not only the beginning but the very movement of the logic with such constant external stimulus that Hegel's correspondent, J.W.A. Pfaff, complained that every transition or inference in the text seemed to him to constitute a new act or postulate (Hegel 1962, 356, 360). Hegel, no doubt, intends the impulses lent by language to the logic to be, themselves, internalized and sublated in its course, but even if they – or at least what in them is contingent or accidental – are in some fashion nihilated, to begin from the trace of such nihilations is not to begin *ex nihilo*.

The non-logical nature of the difference – the energy – with which logic begins its work is not an impurity that can be done away with. Without a MacGuffin, there is no plot.[9].

3.2.2 *The particular difficulties that Dubarle takes Hegel's thought to pose for formalization*

Logic, true logic, cannot be mathematized; what *can* be mathematized, in logic, can only be its inessential chaff, whose transformation into a calculus could serve only to husk it away from its essential kernel, which, for its

7 In this interpretation of the problem of *beginning* in Hegel, I am heavily indebted to Mladen Dolar's reading of *The Science of Logic*.

8 The initiating role of repetition without concept in the *Logic* finds a syntactical icon in Hegel's text: '*Sein, reiner sein* [...] *Nichts,* das *reiner Nichts*' (Hegel 2010, 59): it is to a mere repetition of the indeterminate that we owe the fading of being into nothing and the ontologization of nothing into being (a shift indexed by the unexpected intrusion of the definite article 'das'). Again, I am grateful to Mladen Dolar for having brought this to my attention in his 2012 seminars on Hegel's *Logic* at the Jan van Eyck Academie.

9 To play the role of what Hitchcock calls a MacGuffin – the intrinsically meaningless trigger of any storyline – is, in fact, the function that Hans Blumenberg assigns to the question of being in western thought (see Blumenberg 1991)

part, remains untouched.[10] This is the stance that Hegel appears to take in *The Science of Logic*, and it is one from which (to the best of my knowledge), he never strays. It is in full consciousness of Hegel's profound opposition to the mathematization of logic *tout court* that Dominique Dubarle undertakes the attempt to mathematize, and then formalize, the Hegelian logic of the concept. His motives are profoundly dialectical: in an relentless oscillation that spans over two decades of research,[11] Dubarle uses the question of the dialectic's formalization to clarify and problematize, in a concentrated fashion, *both* the rational coherence of dialectical logic *and* the limits of formalization – subjecting the dialectic itself to the drama of its exteriorization and mediation by the signs of mathematics, in order to discern the shifting frontiers that demarcate the understanding from reason, and restlessly fold the one into the other.

To set the stage, Dubarle, with his co-author André Doz, quote Hegel's most explicit and extensive indictments of the project – still embryonic in his day – of mathematizing logic. The text is one that will be of recurring interest to us, and so we will reproduce it here at length:

> The great Euler, infinitely fertile and sharp of mind in detecting and arranging the deep relations of algebraic quantities, the dry, prosaic Lambert in particular, and others, have attempted to construct a *notation* for this class of relations between determinations of the concept based on lines, figures, and the like, the general intention being to elevate – or in fact rather to debase – the logical modes of relation to the status of a *calculus*.

10 The tension between this stance and the broader tendencies of Hegel's thought – as concern the relation between the essential and the inessential, for example – should be apparent, and this is an issue that I will deal with elsewhere (in Chapter I, for instance).

11 The first outlines of the problem are already legible in a text published in *Revue de Métaphysique et de Morale* in 1955, under the title, "Remarques sur la Philosophie de la formalisation logico-mathématique" (Dubarle 1955), where Dubarle undertakes a subtle and sophisticated reflection on the meaning of formalization, and the inherently dialectical – even Hegelian – stakes of David Hilbert's formalism and meta-mathematical research programme. In an essay written in 1963, but not published until 1987 (?), we find Dubarle experimenting with generalizations and modifications of the Boolean logical algebra (retracing, as we will see, terrain already well known to abstract algebraists since the first decades of the twentieth century – the theory of Boolean rings and lattices), which he will later put to work in a presentation given to Jean Hyppolite's 1967-68 seminar, *Hegel et la Pensée Moderne*, titled "Logique formalisante et logique hégelienne" (Dubarle 1970). This material would be further refined and expanded upon, with the aid of Hegel scholar André Doz, in the 1972 monograph, *Logique et dialectique* (Dubarle 1970). 1977 would find Dubarle returning to the concept of formalization itself, with *Logos et formalisation du langage* (Dubarle 1977), in an effort to clarify the concept by reflecting upon its technical use in mathematics, and upon the tension attested to, by both Hegel and Heraclitus, between the articulation of λόγος ἐνδιάθετος, the *logos in interiority*, in the exteriority proper to λόγος προφορικός. In the same year, Dubarle conducted his celebrated seminar on the ontology of Thomas Aquinas, to which he brought the instruments of logical formalization and geometrical schematism to bear in an effort to distill, from Aquinas, an "ontology of the subject", unfolding, along the way, the motif of projection at work in Dubarle (1955) and the projective-geometrical schematism by which he sought to rescue the infinity and interiority of the Hegelian concept from the "regime of exteriority" imposed upon it by his own ultra-Boolean algebra in Dubarle and Doz (1972).

> One need only compare the nature of a sign with what the sign ought to indicate immediately to see that even the project of a logical notation is unworkable. [. . .]
>
> It is characteristic of objects of this kind [*the signs of mathematics*] as contrasted with the determinations of the concept, that they are mutually *external*, that they have a *fixed* determination. Now when concepts are made to conform to such signs, they cease to be concepts. Their determinations are not inert things, like numbers and lines whose connections lie outside them; they are living movements; the distinguished determinateness of the one side is immediately also internal to the other side; what would be a complete contradiction for numbers and lines is essential to the nature of the concept.
>
> Since the human being has in language a means of designation that is appropriate to reason, it is otiose to look for a less perfect means of representation to bother oneself with. It is essentially only spirit that can grasp concept as concept, for the latter is not just the property of spirit but its pure self. It is futile to want to fix it by means of spatial figures and algebraic signs for the sake of the *outer eye* and a *non-conceptual, mechanical manipulation*, such as a *calculus*. (Hegel 2010, 544)

The text is enigmatic. The most obvious question concerns the peculiar right of passage that Hegel gives to language, but withholds from mathematical symbolism. The raw material of both media, in the last instance, appears to be the same: discretely articulated and mutually external signs, susceptible to combination and repetition. What is it, then, that absolves language from the exteriority of its symbolic articulation, and what is it that 'fixes' mathematical thought determinations, constraining them to the manifest exteriority of their syntactic expression? The answer Hegel points to in this passage is mediated by an obscure pair of metaphors: the signs of language are animated by sort of spiritual 'life', while those of mathematics are moved only by a sort of 'mechanical manipulation'. The understanding of the living and the mechanical that underwrites these metaphors appears to return us to the dichotomy of the internal and the external, in a classical (vaguely Aristotelean) fashion: the living, as natural, contains within itself its own motive power, while the mechanical, as artificial, is moved only from without. There is little to be gained by reiterating the problem of interiority and exteriority in this fashion, shrouding it in what appears to be a dogmatic and impoverished dichotomy of the living and the mechanical.[12] It seems that Hegel is making a double gesture here: the signs of mathematics are fixed in virtue of a *determination* that, at least in their immediacy, lies outside of them: lines and numbers, for example, between which Hegel

12 Discrediting this dichotomy – or at least complicating and enriching it – is one of the great merits of Simondon's mechanology, as developed in Simondon (1989 (1958).

observes no *internal* connections, no 'organic' necessity.[13] I will leave aside, for now, the question as to whether or not this is an accurate description of the relations between mathematical determinations. In any case, if these determinations are 'fixed' in the way Hegel says they are, then this fixity appears to have been thrusted upon the symbolic apparatus itself, and intervenes not at the level of the signs themselves, but on their regimentation by means of axioms and rules – the axioms and rules of arithmetic or geometry, for instance. Fixity flows back, from the determinations the symbols refer to, into the symbols themselves *as* symbols, or at least into the general symbolic practice of mathematical thought. This seems to rest upon a generalization that, perhaps, would not have appeared as spurious in Hegel's day as it does to us now, a generalization that passes from the character of arithmetic and geometry to *mathematical thinking in general*, a mode of thinking which has long since left the nest of numbers and lines in which the history, before Boole, had reared it. From this point of view, it appears that the constraints that inhibit a mathematical apprehension of logic are constraints that the practice of *formalization* will not tighten but dissolve.

In short, if, in the determinations of mathematical syntax, there is a measure of '*fixity*' or '*exteriority*' that does not arrest the flow of language this can only be the result of controls that are imposed *mathematically*, and which should therefore be mathematically intelligible. But the great lesson of modern mathematics – which still loomed on the horizon in Hegel's day – is that what is mathematically intelligible is mathematically variable. It should therefore fall within the power of mathematical logic to think that which, in its syntax, resists and obstructs dialectical reasoning, and to rigorously experiment with these obstructions and their suspension. It is with this in mind that the problem of the fixity of mathematical determinations should be staged.

With regards to Dubarle's enterprise, the form that this problem takes is that of finding a mathematical – and, specifically, an algebraic – schema that would exhibit the *structure* that differentiates the form of the concept from that of number, a form according to which, for instance, 'the distinguished determinateness of the one side [of the concept] is immediately also internal to the other side'. In a moment, we will have the opportunity to gauge the success of this undertaking in characterizing such formations.

The converse problem – what absolves language from the fixity of its symbolic articulation? – is one which continued to occupy Dubarle in his 1977 book, *Logos et formalisation du langage*. There, Dubarle approaches the problem in terms of the ancient dualism of λόγος προφορικός and λόγος ἐνδιάθετος. [EXPLAIN, GIVE HISTORICAL CONTEXT]

To illustrate how this dualism functions, Dubarle begins with what is perhaps its simplest and purest historical instance, which we find with Parmenides.

13 This is consonant with Hegel's peculiar account of calculation and geometrical proof, in the Remarks following his discussion of Quantity and in the Preface to *The Phenomenology of Spirit*. [ADD CITATIONS]

The same scheme – this tripartite articulation of the discourse into (1) a logos subsisting in interiority, (2) a logos articulated in exteriority which serves as commentary and explanation, and (3) a single 'capital signifier' which serves to suture the two – is characteristic of Hegel's style, Dubarle argues, but with a crucial difference:

> with the Greek philosopher, the proposition of the trace specifically expressive of the *Logos*, to which the prolix language of reflection is attached, is reduced once and for all to a single, fixed word [namely, 'ἔστιν'/'is']; with Hegel, by contrast, the proposition of the trace involves diversification and transformation. The language of the Concept is a chain of principle signifiers, producing and undoing themselves in a mobile, rhythmic fashion, in a sort of perpetually liquifying concatenation, with its modes of rest, marked by the nominations, and transitions to various modalities from nomination to nomination. This is a great philosophical novelty.

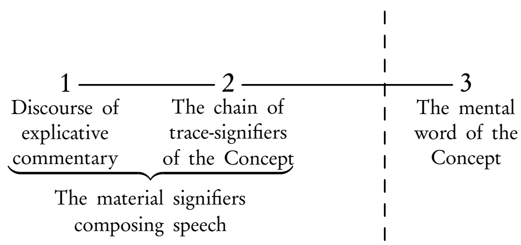


Figure 1: The triple articulation of philosophical λόγος, after Dubarle (1977, 236)

The articulation between regimes of exteriority and regimes of interiority in Hegel's discourse thus appears to mirror, to some extent, the articulation between object language and meta-language in the practice of formal mathematics – and, like a mirror, inverts it. [SEE LFL 233-4, QUOTED IN NOTES, DISCUSS]

Recall that any formalization of mathematical *knowledge*, according to Dubarle, remains subject to a unformalized regime of logical *acts* that will put it to work,[14] a regime that is not ineffable, but essentially formalizable in a 'metalanguage' – which, once formalized, is similarly subject to the acts that put it in motion.

Hegel's manifest discourse, Dubarle suggests, may be understood as something of a 'metalanguage', but a metalanguage of a peculiar sort: one whose 'object language' remains essentially latent, a silent and unwritten λόγος ἐνδιαθετος.

14 This is, in fact, one of the lessons that can be drawn from Lewis Carroll's fable of Achilles and the Tortoise.

15

As an exegetical strategy, this sort of approach can find much to recommend it in the Introduction to the *Phenomenology of Spirit*, with its famous staging of the text that follows as the observations of a 'spectator' whose only task is to behold the activity of spirit, and describe what it sees. [FIND QUOTE] It is less clear, however, that the same holds true of the *Logic*, where everything seems to take place as if the distinction between the actor and the spectator had been collapsed.

'The logical thought of Hegel himself, like that of those who have tried to follow in his footsteps,' Dubarle writes,

> is expressed in a spoken (or written) human discourse. Now, in its brute being and as an expressive system giving the thought its being-there (*Dasein*) in the universe of human communication, this discourse can and should be the object of human **understanding**, accountable to all that the understanding is now capable of [*justiciable de tout ce que l'entendement peut maintenant que ce discours a été produit*]. Being the discourse of a *philosophical* thought, it does not escape the condition common

15 Dubarle writes:

> Tout d'abord, à suivre le discours de l'exposé hégélien de la pensée spéculative, telle qu'il a été rédigé, il apparaît aisément que le principal de celui-ci, semblable d'ailleurs en cela ùne bonne partie du discours philosophique et aussi du parler courant, a pour caractéristique d'être « discours de commentaire », présupposant une composante première et préalable du dire, au sujet de laquelle il s'impose de fournir des explications, dans un dire de réfléchissement sur ce premier dire. On a déjà noté, dans l'analyse faite de la production du langage formalisé, ce caractère très naturel du langage usuel qui consiste à s'accompagner de façon quasi-incessante de son auto-commentaire, parlant mais aussi disant, et comme d'une même jet « ce que parler veut dire » (cf. chap. IV, p. 157-158). De même on a, dés le début de la réflexion sur le langage formalisé une fois celui-ci présenté, insisté sur la présence indistincte, au sein du langage naturel de ce que le passage au langage formalisé oblige de distinguer en parlant d'un côte du « langage-objet » constitué par le langage formalisé lui-même, et de l'autre du « métalangage » expressif de tout ce qu'il y a encore à dire concernant le langage-objet et, s'appuyant à celui-ci, concernant l'intelligibilité dont il est le véhicule.
>
> Si l'on revient alors au discours hégélien avec de telles notions en tête, on a comme l'évidence que ce discours a une allure très proche d'un métalangage, mais d'un métalangage singulier. Car, à première vue tout au moins, il ne se rapporte plus à un « langage-object » s'effectuant au monde des langages extérieurs comme le sont les langages formalisés, mais à ce dire immanent à l'esprit qui est précisément le Concept, lui et le jeu de ses déterminations vivantes, vécues par l'intelligence à même l'intelligibilité vivante. Ainsi le discours dialectique de Hegel est-il, pour autant qu'il nous est accessible, l'auto-commentaire extérieur du Concept, quasi-métalangage descriptif de ce qui se [234] dit, modes et contenu, au-dedans de l'esprit. C'est à sa façon un langage de réfléchissement, langage d'annonce pour ainsi dire kérygmatique d'un autre dire sans matérialité extérieure et qui n'existe plus qu'à l'état de parole vécue en même temps que dite au seul lieu de la vie mentale. (Dubarle 1977, 233-4)

> to *every* human discourse (which is to say, to be effectuated through the use of an articulated language). Not only can its materiality and the syntax of that materiality be studied; its expressive functioning and the how [*le comment*] of its semantics can likewise be explored. And if, after this reconsideration that the understanding itself performs, some regularity in the discursive dispositions that govern the communication are uncovered in a more or less manifest fashion, then, by its very nature, this regularity will present itself as a schema subject to mathematization. ((Dubarle 1970, 117))

[THE SCOPE AND AIM OF DUBARLE'S PROJECT: CONTROL, RIGOUR, CONTINUATION OF DIALECTICS AND DISPLACEMENT OF EXCESS, BUT ALSO A KANTWARDS DRIFT...]

3.2.3 *Review of Hegel's logic of the concept, in* The Science of Logic*, Vol. II, Section I, Chapter I*

Dubarle presents his dialectical algebra as a mathematization of certain structural aspects of Hegel's logic of the concept, as developed in Chapter I of Section I of Volume II of *The Science of Logic*, and in Subsection (a) of Section A of Chapter IX of the *Enclyclopædia Logic*. While it is not feasible to attempt a fully adequate examination of these texts within the bounds of the present study, it would be helpful to have at least a rudimentary analysis of these texts in view, in order to better come to grips with what is at stake in Dubarle's formalism.

Hegel's logic of the concept is concerned, first of all, with the *internal structure* of the concept as such, rather than with relations between concepts. On a first approximation, the concept is analysed into three *moments*: the universal, the particular and the singular. What interests us is the relation or operation that articulates these moments, and organizes them into a structure. The name of this operation is '*development*'.

DOCTRINE	NEGATIVITY	OPERATION
being	limit	transition/passing into
essence	reference	reflection/shining in
concept	position/opposition	development/implying

Table 1: Modes of difference and passage in the logic

As concept is to essence and being, so development is to transition (passing into) and reflection (shining in): the mode by which the moments of the thought determinations relate to (or transcend) the modes of negativity or difference that determine them.[16]

16 Hegel makes these distinctions explicitly in his discussion of *particularity*, in part B of Chapter I of the Doctrine of the Concept:

The moments of an ontological category,[17] for instance, are articulated by *limitation*; each moment relates to its others through the mediation of a limit, whose transgression is indissociable from its articulation. This transgression, or 'ought', takes the form of a *transition* of one ontological moment into its other. A transition operates like a derivation that exhausts its premise in the act of deriving its conclusion, and it is for this reason that – contrary to certain 'paraconsistent' interpretations of Hegel – it is possible to transit or 'pass over' from a moment to its contrary without engendering inconsistency or *actual* contradiction. In general, however, the transitions are not irreversible, and so we frequently find ourselves in a process of oscillation between two contrary moments – without every arriving at their simultaneous conjunction.

a moment	α		
is determined w.r.t. another	α	$\rightarrow$	β
into which it passes	$\cancel{\alpha}$		β
to find itself determined w.r.t. the first	α	$\leftarrow$	β
into which it passes	α		$\cancel{\beta}$
and so on		$\vdots$	

The next phase is, typically, the *thematization* – in the sense in which Husserl and Cavaillès use this term: the apprehension of an operation as an 'object' – of the oscillation that the transitions compose, their 'synthesis'.

[SKETCH, REFINE]

[DO THE SAME FOR ESSENCE]

'The onward movement of the concept,' however

> is no longer either a transition into, or a reflection on something else, but DEVELOPMENT. For in the notion, the elements distinguished are without more ado at the same time declared to be identical with one another and with the whole, and

> Difference, as it presents itself here, is in its concept and therefore in its truth. All previous difference has this unity in the concept. As it is present immediately IN BEING, DIFFERENCE IS THE *limit* OF AN *other*; as present IN REFLECTION, IT IS RELATIVE, POSITED AS REFERRING ESSENTIALLY TO ITS OTHER; here is where the unity of the concept thus begins to be *posited*; at first, however, the unity is only *reflective shine* in an other. – The true significance of the transitoriness and the dissolution of these determinations is just this, that they attain to their concept, to their truth; being, existence, something, or whole and part, and so on, substance and accidents, cause and effect, are thought determinations on their own; AS DETERMINATE *concepts*, HOWEVER, THEY ARE GRASPED IN SO FAR AS EACH IS COGNIZED IN UNITY WITH ITS OTHERS OR IN OPPOSITION TO THEM. (Hegel 2010, 535)

17 By 'ontological category', I mean a category of the doctrine of being, *with the exception of* being *and* nothing, which, as we will later see, have a different structure. Being and nothing are not articulated by any limit whatsoever, since the two (to the extent that they are two) are *pure and indeterminate immediacy*. They 'pass into' one another not in virtue of a limit which they transgress, but by virtue of a pure repetition that shows us their lack of any real difference.

> the specific character of each is a free being of the whole notion. (Hegel 1892, ¶ 161, 288-9)[18]

Development is not simply another mode of passage or sublation alongside transition and reflection, but a complex mode that incorporates the dynamics of its predecessors. When α *develops into*, or 'implies', β, this event does not exhaust α, but preserves it, insofar as this passage also mobilizes a 'reflection' of the transition back into α *as a constitutive part of* α.

On the face of it, what most dramatically sets Hegel's theory of the concept apart from its predecessors – and the Kantian theory of concepts in particular – is precisely such an articulation between universality and particularity in terms of development, rather than in terms of magnitude (relations of magnitude remaining, for Hegel, strictly *external* relations). The relation between the universal and the particular is not, in any sense, the relation between a subsuming and a subsumed concept – such as the relation between *animal* and *mammal*, for instance. 'α *develops into* β' (which I will abbreviate, here, as $\alpha \rightsquigarrow \beta$) differs from the relation of subsumption[19] in at least two crucial ways, one which concerns the relation, and one which concerns the relata:

1. Concerning the relation itself, $\alpha \rightsquigarrow \beta$ is not an ordering relation, as $\alpha \subseteq \beta$ is (reading $\subseteq$ as 'is subsumed by') is. To be more precise, $\rightsquigarrow$ is not antisymmetrical,[20] as witnessed by the fact that the universal develops into the particular *and* the particular develops back into the universal (and also into the singular), without the two (or three) being *immediately* identical with one another.
2. Concerning the relata, the α and β articulated by a development are not separate *concepts*, but *moments within a concept*.

The universal constitutes itself in positing the particular as its other, as its 'negation'; if the universal and the particular were simply ontological moments, this negation would take the form of a limit that the universal

18 In all of the quotations from (Hegel 1892) in this text, Wallace's translation has been modified by substituting 'concept' for 'notion', when translating '*Begriff*' (and so on for 'conceptual'/'notional', etc.), to bring the terminology more into line with Giovanni's 2010 translation of *The Science of Logic* (Hegel 2010). This saves me from having to fall back on awkward circumlocutions when referring to Hegel's logic of the concept, etc.

19 The relation of subsumption is traditionally understood as a relation between two concepts A and B – with B subsuming A – which obtains if either (i) the totality of properties or 'notes' that determines B is included in the totality of notes that determines A (the is the relation that the ensemble of notes {*animal, biped*} stands in relation to the ensemble {*rational, animal, biped*} (intensional subsumption) or (ii) the totality of objects falling under A is included in the totality of objects falling under B (extensional subsumption). (Note the inversion of the relation, as we pass between the intensional and extensional version of the notion.)

20 A relation R is said to be 'antisymmetrical' if and only if aRb and bRa implies $a = b$, or, the only situation in which both aRb and bRa could both be true is one in which a and b are one and the same – a provision which amounts only to saying that antisymmetrical relations may nevertheless be reflexive, like 'less than or equal to' ($\leq$).
An *ordering relation* is, typically, one which is (a) antisymmetrical, (b) transitive (if aRb and bRc then aRc), and (c) reflexive (aRa).

passes over, *transitioning* into the particular, which would no longer be the universal (though it would, perhaps, go on to pass back into the universal in a transition symmetrical to the first, giving a second beat to the oscillation). This is not what takes place in the doctrine of the concept, however, for

> In so far as the universal possesses determinateness, this determinateness is not only the *first* negation but also the reflection of this negation into itself. According to that first negation, taken by itself, the universal is *particular*, and in this guise we shall consider it in a moment. In the other determinateness however, the universal is still essentially universal, and this side we have here still to consider. – For this determinateness, as it is in the concept, is the total reflection – a *doubly reflective shine*, both *outwards*, as reflection into the other, and *inwards*, as reflection into itself. The outward shining establishes a distinction with respect to an *other*; the universal accordingly takes on a particularity which is resolved in a higher universality. Inasmuch as it now is also only a relative universal, it does not lose its character of universality; it preserves itself in its determinateness, not just because it remains indifferent to it – for then it would be only *posited together* with it – but because of what has just been called the *inward shining*. There determinateness, as determinate *concept, is bent back into itself* . . . (Hegel 2010, 532)

Development compounds transition with a doubled form of reflection, and it is this that preserves the relative stability of the terms it relates, even as it has them pass into one another. The result of this is that the oscillations characteristic of the ontological sphere [[and the oppositions characteristic of the eidetic sphere?]] give way to *totalizations*, or unifications: 'Each distinction is confounded in the course of the very reflection that should isolate it and hold it fixed. Only a way of thinking that is *merely representational*, for which abstraction has isolated them,' Hegel goes on to argue, 'is capable of holding the universal, the particular, and the singular rigidly apart. Then they can be counted. . . ' (Hegel 2010, 548).

This abstraction, however, is, itself, not altogether alien to the dynamics of the concept – and this is a point that will become crucial for Dubarle's analysis, as we will see in a moment – on the contrary, it is 'the *soul* of singularity' Between the speculative totalization of the concept that confounds its distinctions, and the abstraction that sunders them into countable parts, works the *act* or *operation* of abstraction that must, itself, be incorporated into the concept, as '*posited abstraction*' (Hegel 2010, 548).[21]

21 Likewise, the singular is, like the most abstract determinations of qualitative being, a 'this' – but since a *this* is a *this* 'only in so far as it is *pointed at*', the abstractive action of ostentation must, itself, be incorporated into the 'this' that the singular is: 'the singular surely is also a *this*, as an immediate which is the result of mediation, but does not have this mediation outside it; it is itself repelling separation, *posited abstraction*, yet is, precisely in this separation, a positive connection.

The universal refers to the particular, as its negation, but in such a way – mediated by double reflection – that it remains itself and does not get lost in transition, as the qualitative determinations of being tend to do. The particular, symmetrically [?] refers to the universal, without losing itself in it. This reference without loss is, itself, what shapes the concept as a totality, as a *singularity*, which is not a third moment separate from the first two (universality and particularity), but precisely the thematization of their mutual reference, their *development* into one another. 'Singularity is essentially the negativity of the determinations of the concept, but not merely as if it stood as a third something distinct from them'; what is posited in the moment of the singular is 'that each of the distinct determinations' – the universal and the particular – 'is the *totality*', and the distinctions are thus 'confounded'.

But a fourth moment now comes to intervene, the old maid of *abstraction* that the point of the game until now had been to discard from the hand of the concept. Abstraction – posited or thematized abstraction – now stands to singularity as universality stood to particularity: the holding of the moments of particularity and universality apart, while allowing this distinction to be seized as the very principle of their unity. Abstraction, the enemy of the speculative apprehension of the concept, is now exposed as its 'soul' (Hegel 2010, 548), its animation and principle of totality.

As a purely mnemonic, and admittedly inadequate, device – but one which helps to anticipate Dubarle's schematism – we can record the main moments of Hegel's analysis of the concept, here, with the following figure:

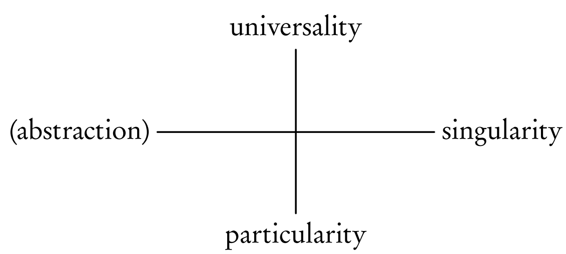


Figure 2: moments of the concept

3.2.4 *The quadruplicity of the concept and reinstitution of the void, as opening the dialectic to a formal and classical reconstruction*

Dubarle stakes a great deal on the double thesis that (a) a quadruple analysis of the Hegelian concept – in contrast to the more familiar threefold analysis of the concept, along the lines of the well-worn but somewhat dubious interpretation of Hegel through the prism of *thesis–antithesis–synthesis* – is possible, and provides for a more univocal and rigorous rational 'control'[22] of the logic of conceptual development, and (b) that such an analysis, while it can find certain motivating suggestions in Hegel's text, bears conse-

22

quences that go dramatically against the grain of Hegel's '*absolute idealism*' (or '*absolute realism*').

There are, at the very least, three occasions at which Hegel's text appears to motivate a fourfold analysis of the concept.

(I) The first is the one I have sketched out in 3.2.3 above, whereby Hegel, perhaps only implicitly, shows how the interpenetration of the universal and the particular should not simply lead us towards a simple unity of the two – a singularity in which all distinctions are confounded – but to a singularity which is in constitutive tension with the very act of division, *abstraction*, which must be recognized as its 'soul' – just as the universal is the soul of the particular.

Here, a fourfold explication of the concept proceeds on the basis of the *duality between singularity and abstraction* (see Figure 2 on the preceding page).

(II) The second concerns a possible ambiguity in Hegel's account of the *particular*, which Dubarle sees as implicitly *split*. The most dramatic staging of this scission, at least in the chapter on the concept, is Hegel's aside on the diversity of nature:

> We can *wonder* at nature, at the manifoldness of its genera and species, in the infinite diversity of its shapes, for wonder is *without concept* and its object is the irrational. It is allowed to nature, since nature is the self-externality of the concept, to indulge in this diversity, just as spirit, even though it possesses the concept in the shape of the concept, lets itself go into pictorial representation and runs wild in the infinite manifoldness of the latter. The manifold genera and species of nature must not be esteemed to be anything more than arbitrary notions of spirit engaged in pictorial representations. Both indeed show traces and intimations of the concept, but they do not exhibit it in trustworthy copy, for they are the sides of its free self-externality; the concept is the absolute power precisely because it can let its difference go free in the shape of self-subsistent diversity, external necessity, accidentality, arbitrariness, opinion – all of which, however, must not be taken as anything more than the abstract side of *nothingness*. (Hegel 2010, 536)[23]

23 It is difficult not to be reminded of Herodotus:

> But no sooner had the strait been bridged than a great storm came on and cut apart and scattered all their work. Xerxes flew into a rage at this, and he commanded that the Hellespont be struck with three hundred strokes of the whip and that a pair of foot-chains be thrown into the sea. It's even been said that he sent off a rank of branders along with the rest to the Hellespont! He also commanded the scourgers to speak outlandish and arrogant words: "You hateful water, our master lays his judgement on you thus, for you have unjustly punished him even though he's done you no wrong! Xerxes the king will pass over you, whether you wish it or not! It is fitting that no man offer you sacrifices, for you're a muddy and salty river!" In these ways he commanded that the sea

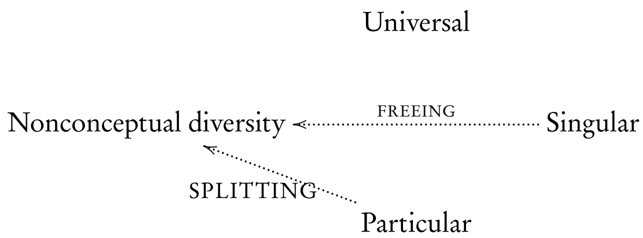


Figure 3: The scission of the particular

Hegel at once marks the difference between this, familiar notion of particularity and the particularity that is a moment of the concept, while in the same breath grounding the possibility of this *particularity without concept* in a dimension of the concept itself – its 'freeing of difference' or externality. The concept is structured in such a way as to open *constitutively* onto the non-conceptual, and it may be possible to count this opening itself as one of its moments (as the obverse or dual of the concept's totality, perhaps).

Here, the quadruplicity of the concept is premised on the *scission of the particular* (see Figure 3).[24]

(III) Thirdly, there is the famously enigmatic passage from the Chapter on the Absolute Idea where Hegel explicitly countenances an analysis of the concept that would unfold it in four moments – while at the same time making light of the importance that *any* quantitative determinations might have

be punished and also that the heads be severed from all those who directed the bridging of the Hellespont.

24 Commenting on a passage just prior to the one we have quoted above, where Hegel specifies that the particular comprises only two species: the universal and the particular itself, and that 'the impotence of nature' is that it proliferates irrationally beyond these two necessary species, 'and loses itself in a blind manifoldness void of concept' (Hegel 2010, 536), Dubarle writes:

> The moment of particularity gives rise, as if by nature, to a sort of internal fault, following the intervention of the principle of difference, a fault which presents itself in a more or less accentuated fashion in the form of a notional duality of opposed determinations. The concept thus divides itself (*sich urteilt*) in displaying the split [*scindée*] actuality of what, within it, is the particularization of the real. The scission in two (*Entzweiung*) of the moment of particularity can be presented in a more or less brute and material fashion depending on the conceptual domains in which it intervenes. But one can seize its conceptual scheme in its pure state in what Hegel says of the genus of the universal and its rational dissociation into two species. ((Dubarle and Doz 1972, 146))

$$
\begin{tikzcd}
 & \text{Universal} \arrow[dl] \arrow[dd] & \\
\text{Absolute negative} \arrow[rr] & & \text{Singular} \\
 & \text{Particular} \arrow[ur] &
\end{tikzcd}
$$

Figure 4: Dubarle's quadruplicity schema

for the concept.[25] This extraordinarily dense and pregnant passage runs as follows:

> In this turning point of the method, the course of cognition returns at the same time back into itself. This negativity is as self-sublating contradiction the *restoration* of the *first immediacy*, of simple universality; for the other of the other, the negative of the negative, is immediately the *positive*, the *identical*, the *universal*. In the whole course, if one at all cares *to count*, this *second* immediate is *third* to the first immediate and the mediated. But it is also *third* to the first or formal negative and to the absolute negativity or second negative; now in so far as that first negative is already the second term, the term counted as *third* can also be counted as *fourth*, and instead of a *triplicity*, the abstract form may also be taken to be a *quadruplicity*; in this way the negative or the *difference* is counted as a *duality*. (Hegel 2010, 746)

Dubarle plots the quadruplicity gestured at here as a forking of the developments that lead from the universal (the first immediate) to the singular (the second immediate) into two simultaneous paths: one which passes from universal, to particular, to singular, and one which passes from universal (first or formal negative), to absolute negativity (second negative), to singular, to obtain the schema in Figure 4.

Can these figures, rough and approximate as they are, be consistently superimposed? This depends on whether or not there is a way of unifying, theoretically, (1) abstraction, as the dual of singularity/totality, (2) non-conceptual difference (or the non-conceptual residue of particularity, freed from the concept), and (3) the absolute negative. Do these themes converge in any meaningful way?

25 The same note is struck earlier, in the discussion of singularity:

> Only a way of thinking that is *merely representational*, for which abstraction has isolated them, is capable of holding the universal, the particular, and the singular rigidly apart. Then they can be counted; and for a further distinction this representation relies on one which is *entirely external to being*, on their *quantity*, and nowhere is such a distinction as inappropriate as here. (?, 548)

Between (1) and (2) there does appear to be a discernible correlation, whether noetico-noematic or intensional-extensional. The 'abstraction' that opposes the unity of the concept finds its counterpart in a disunity without concept; both are, however, expressions of the 'freeing power' of the total concept: abstraction is the soul of the singular, the way that the universal is the soul of the particular while the multiplied residue of the particular is, on Hegel's analysis, [FINISH AND DEVELOP THIS THOUGHT. CORRECT, I THINK, BUT SKETCHY.

3.2.4.1 *The conceptual void*

Given the concept's relative indifference to such abstract figurations, this hairsplitting may seem relatively harmless, if not tedious. The stakes that Dubarle places on the quadruple articulation of the concept of nevertheless extremely high. The first and most important philosophical reason for this has to do with his interpretation of the obscure moment, the leftmost point in Figures 2, 3, and 4, as the 'void'.[26] 'The void in question,' he writes

> is not just any void whatsoever. It is, if one may express it in this way, the *conceptual* void, not the *merely gnoseological* void where the word 'nothingness' comes to designate something that the intellect aims at as something positive and effective all the while testing it and declaring it recalcitrant to knowledge. This conceptual void, moreover, is not at all the 'absolute nothingness' of the *ontological* void, since the conceptually void term can be relativized and, while nevertheless exercising at that moment the logical function of the void term, conventionally that which remains outside every effective conceptualization and implementation in the thought whose development is at issue. The void term, the word 'nothingness', if it is to serve as a name for that term, is thus something like the logical mask something whose determinations remain excluded from discourse. (Dubarle and Doz 1972, 137)

To assert that the obscure, fourth vertex of the concept – which can be recognized in the various dissections laid out above – functions as a void term has a number of consequences, Dubarle argues.

The first is that it introduces, into the Hegelian development of the concept, the place of the *false*. The void is summoned to play a role 'analogous to that of the zero in Boolean logic', that to which nothing real – nothing in the universe with which the discourse is concerned – corresponds.[27]

26 'Nothing remains to stop us,' Dubarle asserts, 'from substituting, in the logico-mathematical formalism of control, the logically void term for this "second negative" or this "absolute negativity" ' (Dubarle and Doz 1972, 148).

27 Cf. Raymond (1977, 90), where Pierre Raymond remarks that at the point of the Boolean zero, 'logical reasoning and reality no longer coincide; there is no adequation between them: in effect, to the real unity of the world corresponds, in thought, the couple true/false, such that the false can be made to correspond to [ontological] nothingness by deforming it; the necessity of the universe has nothing to do with necessity for man, since the former has no

Like the Eleatic Stranger of the *Sophist*, Dubarle goes on to indicate the parricidal character of the adjunction of the void to the concept. He argues that Hegel's refusal to recognize the logical status of the conceptual void is moved by

> the wish to maintain a simple and direct concordance between conceptual comprehension and the very reality it comprehends. One should here speak of Hegel's 'absolute conceptualism'. The absolute idealism that it demands is at the same time an absolute realism. Absolute knowledge should fully verify the coincidence that Parmenides has already said must exist between the true act of thought and that which is truly thought, the full coincidence of the conceptual and the real, since, according to the Hegelian formula: 'all that is rational is actually real (*wirklich*) and all that is effectively real is rational. This obliges us to cast outside the sphere of this knowledge the concept of nothingness, at least in the ordinary sense of the word, because it is admitted that that concept would play the role of a sufficient counter-example, as something 'conceptual' which is not 'actually real', and would thereby introduce an initial fault between the thesis of absolute idealism and the commitment to absolute realism.(Dubarle and Doz 1972, 144-5)

The insistence on the *conceptuality* of the void, Dubarle argues, is as destructive to Hegel's absolutism as Plato's insistence on the *reality* of non-being is to that of Parmenides. The essential possibility of mismatch between the conceptual and the real that the void inscribes into the concept, as Dubarle understands it, would a subordination of conceptual development to 'a *judgment* concerning the reality or non-reality [...] of the concept proposed' (Hegel 2010, 145).

It may not, at first, be obvious that Hegel has not already prepared a place for the conceptual void, given the ambiguous role played by what, in his discussion of the singular in the first chapter of the subjective logic, he calls 'abstraction' – grasped as 'the *soul* of singularity' (Hegel 2010, 548), and as that which lets 'difference go free in the shape of self-subsistent diversity, external necessity, accidentality, arbitrariness, opinion – all of which, however, must not be taken as anything more than the abstract side of *nothingness*' (Hegel 2010, 536). But this is where the entire problem now seems to be condensed: the moment of *abstraction*, whether it is counted as a fourth moment or blended inexplicitly into the texture of the particular and the singular, is what *impells the concept towards judgment from within* – 'the concept is itself this act of abstracting; the positing of its determinations over against each other is its own determining,' Hegel writes, and '*Judgment* is this positing of the determinate concepts through the concept itself' (Hegel 2010, 550). The *conceptual void*, by contrast, marks a lack in the concept

degrees of error, of contradiction – we could, here, employ Boole's reasoning backwards: the possible contradictions of the universe can be explained by a coherent theory'.

– not its negative power for the 'realization' of its other, but a symptom that calls for its external supplementation through subjective judgment. The judging subject, for Dubarle, enters as a bridge between the conceptual and the real that the former is now seen to be incapable of constructing on its own – and this is so insofar as the 'power' of abstraction has been replaced by an inert void. The void is the stump that remains after the amputation of *abstraction* – or, to put it in a way that is both accurate and somewhat convoluted, the void is what remains after the abstraction of abstraction from *abstraction*, and, consequently, from the other moments as well. It is this abstraction of abstraction that leaves us with moments that are, strictly speaking, *abstract*: products of abstraction systematically divorced from the act that engenders them.

3.2.4.2 *Excursus on the Griss-Heyting-Brouwer debate on negation in mathematics*

The problem Dubarle is confronting here is one which had already erupted at the heart of mathematical logic, among the Dutch intuitionists in the middle of the twentieth century. Beginning in 1908, in the wake of the 'foundational crisis' in mathematics – when the discipline found itself confronted by profound antinomies the moment it sought to its most fundamental concepts explicit, the concepts of number, set, property and the like – a revision of mathematical method was proposed by the idealist mathematician L.E.J. Brouwer, initiating a programme to which he would later give the name of 'intuitionism'. Like Hegel, Brouwer began by rejecting the classical, Kantian image of truth. Mathematical truth, reasoned Brouwer, is never a truth concerning objects outside itself, to which it may relate only contingently and externally. Mathematics is a discipline which produces its own objects, and the truth of its propositions is radically immanent to the constructions which demonstrate these truths to be so. It therefore makes no sense to insist upon the validity of certain classical 'laws of thought' in mathematics. In particular, once we see that mathematical truth is wholly immanent to the movement of its concepts, to the work of mathematical construction, we can no longer sustain the LAW OF THE EXCLUDED MIDDLE, which says of any proposition p that either p or its negation must be true, with no third option (*tertium non datur*). Of a piece with and implied by this rejection of the law of the excluded middle was a rejection of the equivalence classically asserted between a proposition and the negation of its negation: we may no longer take $\neg\neg p$ to be equivalent to p – this, for the intuitionist, would amount to asserting that (a) a proof that *any proof of the contradictoriness of p can be logically transformed into the proof of a contradiction* and (b) a direct proof of p can be logically transformed into one another, something which is not always possible. (To put it another way, what the intuitionist denies is that (a) demonstrating that a contradiction follows from the assumption that such and such an object does not exist, is not equivalent to (b) the actual construction of that object in thought.)

In the 1940s, another Dutch mathematician, G.F.C. Griss, an idealist philosopher drawing on the thought on Hegel and, perhaps to a greater extent, Bergson (in particular, on the critique of negativity the latter developed in *Creative Evolution*) and disciple of Brouwer, began to argue that Brouwer had not gone far enough in his immanent and procedural revision of the idea of mathematical truth.[28]It is not enough to simply *weaken* the classical notion of negation – to break its symmetry and refuse it the capacity to exhaust the possible – *we must expunge it from mathematics entirely*. Since the absurd, the false, the *conceptual void*, cannot itself be constructed in intuition, we have no right to speak of constructions which can be transformed into constructions of the absurd. We can no longer consider mathematically valid, for instance, the proposition that $3 \neq 5$. We must instead assert that 3 and 5 are *separate*, and that the interval which separates them – the 2 that may be added to 3 to reach 5 – can be positively constructed. No place is left in mathematics, asserts Griss, for abstract negation; every negation must be replaced by a constructible *determination*, or rejected as empty and void of concept.

Griss' heretical 'affirmationism' was opposed by Brouwer and his disciple, Arend Heyting (who was the first to successfully formalize an intuitionistic logical calculus). Heyting raised a crucial problem concerning the possibility of elaborating a formal logic for a negationless mathematics such as Griss had envisioned (and which he had already begun to construct), and proceeded to outline the following difficulties:

> According to Brouwer, and also in this point Griss agrees with him, logic is not presupposed in mathematics, but yet in mathematical reasonings we can detect certain regularities the study of which belongs to logic. One might suppose that the logic of negationless mathematics is obtained by omitting from the intuitionistic logic all those rules which contain a negation, but this simple method does not work because it is necessary to impose a restriction upon the conjunction of two propositions. "*a* is a square" is a proposition; "*a* is a circle" as well, but "*a* is a square and *a* is a circle" is not a proposition. Thus the conjunction of two propositions *p* and *q* can only be formed if an instance is known which makes this conjunction true. In other words, the notion of a well-formed formula in this logic will depend upon the interpretation; A PURELY FORMAL LOGIC IS IMPOSSIBLE (Heyting (1954, 94-5)).

Griss' idealism and logic are not exactly Hegel's (which he does, however, draw upon, and with which engages with in some detail in *Idealistische filosofie*, before shifting towards something closer to an empiricist idealism (see Griss 1946)), but the common ground is clear enough for the point to be made. A logic can be genuinely *formal*, as Kant and Dubarle require,

28 See Griss (1948-49) for Griss' own presentation of his position. A brief discussion of Griss' negationless mathematics can also be found in Deleuze's *Difference and Repetition*. [cite]

only to the extent that it can mark the *conceptual void*, the false. (This is not necessarily also a condition for the *formalization* of logic, which I take to be distinct.) If Hegel purges the conceptual void from logic – a thesis that, itself, remains uncertain, since it may be possibly to locate the void in the moments of purging, as when such figures as the indifferent diversity of nature[29], or the conceptless machinations of arithmetic[30] are cast aside as debris in the onward movement of the logic itself – this is to the extent that he envisions truth as *nothing apart* from the movement of the concept. There is no content which would be both alien to the formal procession of the concept and on which its forms would depend for their truth. Since the truth of the forms deployed by Hegel's logic is not contingent upon anything external, but immanent to their very movement, there is no need to develop the logic in such a way that it remains neutral with respect to the

29 See, for instance, Hegel's aside on the diversity of nature, in his discussion of the moment of particularity – a passage which, as discussed in 3.2.4 on page 46, exemplifies the 'scission of the particular' that Dubarle takes to be implicit, even if insufficiently emphasised, in Hegel's logic:

> We can *wonder* at nature, at the manifoldness of its genera and species, in the infinite diversity of its shapes, for wonder is *without concept* and its object is the irrational. It is allowed to nature, since nature is the self-externality of the concept, to indulge in this diversity, just as spirit, even though it possesses the concept in the shape of the concept, lets itself go into pictorial representation and runs wild in the infinite manifoldness of the latter. The manifold genera and species of nature must not be esteemed to be anything more than arbitrary notions of spirit engaged in pictorial representations. Both indeed show traces and intimations of the concept, but they do not exhibit it in trustworthy copy, for they are the sides of its free self-externality; the concept is the absolute power precisely because it can let its difference go free in the shape of self-subsistent diversity, external necessity, accidentality, arbitrariness, opinion – all of which, however, must not be taken as anything more than the abstract side of *nothingness*. (Hegel 2010, 536)

30 See, for instance, Hegel's second Remark on the subject of arithmetic in the chapter on quantum:

> Not only does arithmetic not contain the concept and the intellectual task of conceptualization that goes with it: it is the very opposite of the concept. Here, because of the indifference of the combined to the combining – a combining that lacks necessity – thought finds itself engaged in an activity which is at the same time the utter externalization of itself, a *tour de force* in which it *moves in an element void of thought*, drawing relations where there is no capacity for necessary relations. The subject matter is the abstract thought of *externality* itself. (Hegel 2010, 178)

For reasons such as these, Hegel argues, the forms of number, calculation and arithmetic 'contain no demand for speculative thought' (177), and he makes it clear that his own discussion of arithmetical calculation – contained in a few Remarks and not in the body of the *Logic* – 'cannot be said to be a philosophical treatment of them' (177), which he considers neither necessary nor possible.

In view of the similarity of the language Hegel uses in this passage and in the passage quoted in the previous footnote, it is tempting to listen here for an echo of Galileo's famous aphorism: the book of nature is written the language of mathematics, which, in Hegel's idiom, is the language of exteriority. Ultimately, however, I do not expect that this would sit well with Hegel's own (anti-Newtonian?) philosophy of nature, but further research would be needed to see how this tension plays out.

truth or falsity of its determinations – no need, and indeed no possibility, for the logic to remain 'formal'.[31]

For Hegel's logic of the concept to be *made* formal, *the moment of abstraction must be seized as void*: not as that which, in singularity, effects its 'primal division' ('*Ur-teil*') in judgment, but as that which signals the need for judgment to *intervene from without*, insofar as, following Dubarle, it indicates the lack of a principled coincidence of the real and the conceptual.

3.2.4.3 *Classical logic as an "inalienable acquisition of the understanding", according to Dubarle*

Already in his contribution to Hyppolite's 'Hegel et la pensée moderne' seminar, Dubarle insists that any formalization of dialectical logic should show it to be 'an *extension* of the classical formalism, considered as an inalienable acquisition of rationality. [. . .] Into it, one must be able to plunge, without distortion, the formalism by which the mathematics of traditional logic has come to specify its constitution' (Dubarle 1970, 118). As shown by a posthumously published manuscript from 1963, in which Dubarle develops the algebra that he would later apply to the Hegelian concept (and which we will examine in detail in a moment), Dubarle had arrived at this conviction well before taking up the challenge of formalizing dialectical logic. It is, in fact, on their failure to function as *conservative extensions* of classical logic that Dubarle blames what he takes to be the lack of success of both intuitionistic logic and certain many-valued logics.[32] This commitment is unwaveringly maintained through the project carried out in Dubarle and Doz (1972).

The possibility of adhering to this commitment, even in the face of the Hegelian dialectic, is precisely what Dubarle takes his quadrisection of the concept to afford, that it grants, in the symbolic traces of reason, all of the necessary footholds for the understanding to reassert its 'inalienable acquisition'. The four nodes of the universal, the singular, the particular and the void – if arranged in a certain fashion – suffice to provide an algebraic model

31 This point demands some clarification: the 'formality' of a logic is not the same thing as its *formalization*. The notion of formality that is at issue here is closely linked to a very specific image of truth. A formalization serves to identify a certain structure, one which is perhaps invariant across numerous transformations, and identify that structure with a certain series of gestures, with certain movements of thought considered in themselves and as immanent to a syntax. If the transformations which preserve this invariance are transformations of something which is conceived as 'content', and especially as true or false content, then the formalization may also be understood as an extraction of the 'formal' as opposed to the contentual, but the two concepts, formality and formalization, are nevertheless conceptually distinct. What continues to historically bind the formalized to the notion of formality, today, is coupling of syntax with a certain notion of 'semantics'. This distinction, between formality and formalization, will have been discussed at length in Chapter I.

32 Whether or not intuitionistic logic can be seen as a conservative extension of classical logic is, in fact, open to debate. On the one hand, as Gödel has shown, it is a relatively simple matter to transpose *all* of the theorems of classical logic into intuitionistic logic, provided we append a double negation to every formula and its subformulas. As Girard has more recently pointed out, however, this transposition does *not* preserve the computational behaviour of classical logic, or the dynamics of its cut-elimination procedure.

for the classical 'laws of thought'. This consequence is not immediate, however, and some measure of force or cunning must be applied in order to wrangle (or lure) the Hegelian concept into the cage of a Boolean ring. There is a great deal at stake – and there are sacrifices made – in this taming of reason, and so a detailed examination of Dubarle's manoeuvres is called for.

3.2.5 *Dubarle's algebraic schematization and formalization of the logic of the concept*

At bottom, Dubarle's algebra is simply $\mathbb{B}^2$, the product of two two-element Boolean rings. A ring is a general, abstract algebraic structure – only slightly less general than a group – commonly defined[33] as a set R together with two operations, denoted $+$ and $\cdot$, which are such that

1. R is closed under $+$ and $\cdot$: if $x \in R$ and $y \in R$ then $x + y \in R$ and $x \cdot y \in R$;
2. $+$ and $\cdot$ are both *associative*: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;
3. there exists, in R, an element that serves as an *identity* for $+$, commonly written as 0, which is such that $a + 0 = a$ for all a;
4. there exists, in R, an element that serves as an *identity* for $\cdot$, commonly written as 1, which is such that $a \cdot 1 = a$ for all a;
5. there exists, in R, an *additive inverse* for every a, which is to say some element b such that $a + b = 0$;
6. $+$ is commutative: $a + b = b + a$;
7. $+$ and $\cdot$ distribute over one another:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

$$a + (b \cdot c) = (a + b) \cdot (a + c)$$

The most familiar example of a ring is the algebra of integers, with $+$ and $\cdot$ receiving their standard interpretation as addition and multiplication, respectively. A ring is further determined as *Boolean* if and only if its elements are universally subject to Boole's 'fundamental law':

$$x \cdot x = x$$

or $x^2 = x$, which is to say that its elements are universally *idempotent under multiplication*. I have already indicated this law's conceptual bearing upon logic, and once we have worked through the details of Dubarle's system, I will return to the effects of its introduction into the dialectic.

A two-element Boolean ring differs from George Boole's own logical algebra only in being closed under $+$. In Boole's system, recall, the lack of $+$'s logical closure, expressed by the fact that $1 + 1 = 2$, where 2, to the

33 The concept of ring was first introduced by Richard Dedekind in the late nineteenth century, and the term itself is due to David Hilbert. [DIG UP CITATIONS]

extent that it is not idempotent under $\cdot$, is not a 'logical value'. This non-closure lead to the incidence of *logically uninterpretable* moments in Boole's logical calculations: between the equations taken as logical premises, and the equation expressing the argument's logical conclusion, we may witness the intercession of equations that *cannot* be interpreted as logical assertions, equations employing non-idempotent values. Insofar as a Boolean ring is, by definition, both *closed under* $+$ and *idempotent under* $\cdot$, there is no chance of calculations straying from the fold; there are no more burrows into the underworld of number, and everything happens in the broad daylight of logical interpretability.

A Boolean ring is therefore no longer arithmetical in nature. Its conception, and in fact the concept of ring in general, became possible only in light of an essential shift in the history of mathematics, a shift which, in fact, has one of its many anticipations in Boole. As late as 1831, there was nothing unusual about Hegel's insistence that mathematics is, essentially, the science of magnitude – an assertion that, today, would be not only false but incredible. Though Boole was among the first to envision and effectively produce a mathematical theory which was no longer a theory of quantity, he was nevertheless obliged to *embed* his theory in the quantitative – in arithmetic – for it to be recognizably mathematical, and it was this exigency, this 'epistemological obstacle', that is responsible for the problematic status of $+$ as a logical operation, for so far as logic is concerned – or classical logic in any case – it is a contingent and accidental matter that, with the exception of zero, arithmetic fails to supply us with a number idempotent under $+$.

The concept of ring belongs essentially to the age of *modern* mathematics, of mathematics understood as the science of abstract structures, autonomous unto themselves.[34] The transposition of Boolean algebra into the theory of idempotent rings has no need to navigating around the appearance of 'non-logical values', since it is now permitted to stipulate, abruptly, that

$$1 + 1 = 0$$

or, more generally, that every element is its own additive inverse. The standard logical reading of $+$ is to interpret it as *exclusive disjunction*, or, less equivocally, as *symmetric difference*, which gives a fairly natural meaning to the equation $a + a = 0$, which is often written as $a \not\equiv a = 0$: to say 'a or a, but not both' is to say something equal to simple falsity, or, alternatively: 'the difference between a and a is nothing at all (i.e., 0)'.

34 The word 'modern' has a meaning and temporal extension similar to what it has when we speak of 'modern art'. Since the late nineteenth century, the axiomatic absolves itself from the notion of self-evident truths describing a familiar domain (space, number, etc.) just as the sculpture absolves itself from monument and painting absolves itself from picture. As with art, mathematical 'modernity' reached its climax around the time of the Second World War, leading to a phase that has been called its 'transmodernity' or 'postmodernity': see, for instance, Fernando Zalamea's brilliant account of this transition in [CITE, ADD TO BIBDESK]. For a succinct and precise account of sculptural modernity and postmodernity, I have found nothing that surpasses Rosalind Krauss's article, 'Sculpture in the Expanded Field' [CITE].

A two-element Boolean ring – which we will denote $\mathbb{B}$ – can be described by a pair of 'truth tables' in a straightforward manner, as seen in Table 2.

$\cdot$	0	1
0	0	0
1	0	1

$+$	0	1
0	0	1
1	1	0

Table 2: Tables for the Boolean ring of order 2, $\mathbb{B}$

There is no need, however, to restrict ourselves to two-element rings. Consider, for instance, the ring-product $\mathbb{B} \times \mathbb{B}$ where the underlying set is the cartesian product of B and B, the set of ordered pairs

$$\{\langle x, y\rangle \mid x \in B \text{ and } y \in B\}$$

and where the operations of product and sum are defined as follows:

$$\langle a_1, b_1\rangle \cdot \langle a_2, b_2\rangle = \langle (a_1 \cdot a_2), (b_1 \cdot b_2)\rangle$$

$$\langle a_1, b_1\rangle + \langle a_2, b_2\rangle = \langle (a_1 + a_2), (b_1 + b_2)\rangle$$

which yields the tables in Table 3.[35]

$\cdot$	$\langle 0,0\rangle$	$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$
$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$
$\langle 0,1\rangle$	$\langle 0,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$	$\langle 0,1\rangle$
$\langle 1,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$
$\langle 1,1\rangle$	$\langle 0,0\rangle$	$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$

$+$	$\langle 0,0\rangle$	$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$
$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$
$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$	$\langle 0,0\rangle$	$\langle 0,1\rangle$
$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$

Table 3: Tables for Boolean ring of order 4, $\mathbb{B}^2$

This, too, is a Boolean ring, and conforms to all of the axioms of the theory (criteria 1-7, above, and Boole's fundamental law). Moreover, we can see that these tables validate every theorem of Boolean logic, when $+$ is interpreted as exclusive disjunction (or symmetric difference) and $\cdot$ is interpreted as conjunction.

The negation of a, written a', will be the element x such that $a + x = \langle 1,1\rangle$, and such that $a \cdot x = \langle 0,0\rangle$ – a definition which, in fact, incorporates the classical laws of non-contradiction and excluded middle.

Because $\cdot$ is *associative, commutative* and *idempotent*, it is possible to use it to define an ordering relation in the algebra – a fact that will take on greater importance when we turn to the algebra of lattices. To be precise, we can define $a \subseteq b$ to hold when $a \cdot b = a$. The proof of this is as follows:[36]

35 As the table for $+$ shows, $(R, +)$ also describes the Klein group, the smallest non-cyclic group, as Dubarle observes in **?**.

36 Dubarle does not spend much time reflecting on the order structure of the algebra, which in view of the developments in Section 3.3 are, perhaps, of greater importance in this context

1. R is an ordering relation if R is *reflexive, transitive and antisymmetric*. If these properties can be shown to hold of $\subseteq$, as defined above, then $\subseteq$ is an ordering relation.
2. REFLEXIVITY: A relation R is reflexive when, for all a, R relates a to itself (aRa). That this holds for $\subseteq$ follows immediately from idempotency, for if $a \cdot a = a$ then $a \subseteq a$ by definition of $\subseteq$.
3. TRANSITIVITY: A relation R is transitive when, for all a, b, c, aRb and bRc imply aRc. To show that this holds for $\subseteq$ means showing that

   (i) $a = a \cdot b$ and

   (ii) $b = b \cdot c$ together imply

   (iii) $a = a \cdot c$

   which can be shown as follows:

   $$\begin{array}{lcll} a & = & a \cdot (b \cdot c) & \text{substituting } (b \cdot c) \text{ for } b, \text{ in virtue of (ii)} \\ a & = & (a \cdot b) \cdot c & \text{by the associativity of } (\cdot) \\ a & = & a \cdot c & \text{substituting } (a \cdot b) \text{ for } a, \text{ in virtue of (i).} \end{array}$$

4. ANTISYMMETRY: A relation R is antisymmetric when aRb and bRa only if $a = b$. Showing that $\subseteq$ is antisymmetric means showing that

   (i) $a = a \cdot b$ and

   (ii) $b = b \cdot a$ together imply

   (iii) $a = b$

   but this follows immediately from the *commutativity* of $\cdot$ (from the fact that $a \cdot b = b \cdot a$) and from the transitivity of identity.[37] In other words, $a = a \cdot b = b \cdot a = b$.
5. Having shown that $\subseteq$, understood as $a \subseteq b =_{\text{df}} a = a \cdot b$, is reflexive, transitive, and antisymmetric, we have shown that it is an ordering relation. This completes the proof.

There is some danger of confusion here, which should be cleared up right away. The ordering relation that $\subseteq$ expresses is *not* a necessarily a linear order. An order is *linear*, or *total*, when it serves to compare any two elements of the set over which it is defined, which is to say that for all a, b, *either* $a \subseteq b$ or $b \subseteq a$ is true. This is indeed the case with $\mathbb{B}$ – since every element is either 0 or 1 – but it is no longer true of $\mathbb{B}^2$. The order structure $\mathbb{B}^2$ exhibits is that of a *partial order* – and, more precisely, it is a *lattice*, a structure we will explore in more detail in Section 3.3. Figure 5 serves to

than in the scope of his own project. The proof which follows is mine, but is by no means anything special – it is the sort of thing that is left as an exercise in introductory textbooks on abstract algebra, such as, for instance, Grätzer (1999), which I have relied on for much of what concerns the theory of lattices in the current chapter.

37 Not to be confused with the transitivity of $\subseteq$, which was the subject of point 3, above.

illustrate this structure,[38] reading $a \subseteq b$ when a is either identical with or above and connected to b.

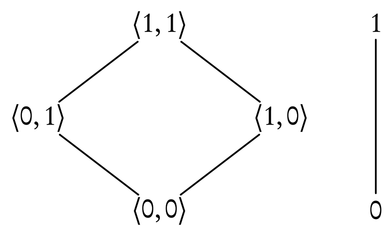


Figure 5: The order structure of $\mathbb{B}^2$ and $\mathbb{B}$, respectively

To read this figure:

1. we take x *being identical to or beneath and connected to* y *in the diagram* to mean that $x \subseteq y$;
2. we take *the highest point identical with or beneath and connected to both* x *and* y to indicate $x \cap y$ (sometimes called the 'greatest lower bound' of x and y, but which, here, we are calling x and y's conjunction, intersection or product);
3. we take *the lowest point identical with or above and connected to both* x *and* y to indicate $x \cup y$ (the 'least upper bound' of x and y, or, alternatively, their (inclusive) disjunction, or union);
4. a point x' which, with respect to some x, is such that the highest point below and connected to both x and x' is the bottom element (here: Λ), and such that the lowest point above and connected to both x and x' is the top element (here: S), is called the *complement* or (classical) negation of x.

This ordering relation is of interest because it replicates the relation of logical *entailment*, when the algebra is interpreted as a logic of propositions, and the relation of *subsumption* when it is interpreted as logic of terms (as a syllogistic logic, for instance). How, precisely, the relation should be understood when the algebra is interpreted as the *internal logic of the concept*, in the Hegelian sense, remains problematic, as we will see in 3.2.5.1.

In any case, this ordering opens the algebra to a logical interpretation, and in what follows we will substitute a logical symbolism for the ring theoretic notation used so for, following Dubarle's own (more or less standard) choice of signs, shown in 4.

It is this algebra $\mathbb{B} \times \mathbb{B}$, or $\mathbb{B}^2$, which can be grasped as either a ring or a lattice (since it is both), that Dubarle elects as an algebraic representative of the structure of the Hegelian concept.

38 Readers who notice that the order structure describing $\mathbb{B}$ has the shape of a line, and that that of $\mathbb{B}^2$ has the shape of a square, may well predict that the order structure (and lattice) characteristic of $\mathbb{B}^3$ must have the form of a *cube*. They are correct.

RING-THEORETIC			LOGICAL
product:	$a \cdot b$	conjunction:	$a \cap b$
sum:	$a + b$	symmetric difference:	$a \not\equiv b$
	$a + b + (a \cdot b)$	(inclusive) disjunction:	$a \cup b$
multiplicative identity:	1	true/universe:	V
additive identity:	0	false/void:	Λ

Table 4: Dubarle's notation

3.2.5.1 *Dubarle's operational completion of the* $\mathbb{B}^2$ *algebra, in preparation for a formalization of conceptual* aufheben

Equipped only with the classical connectives of conjunction, exclusive disjunction (symmetric difference) and negation – and those which may be constructed on this basis, such as inclusive disjunction ($a \cup b =_{\mathrm{df}} a \not\equiv b \not\equiv (a \cap b)$) or the conditional ($a \rightarrow b =_{\mathrm{df}} a' \cup b$) – $\mathbb{B}^2$ lacks the logico-algebraic property of *functional completeness*. A set of operations for an algebra $\mathbb{A}$ is 'functionally complete' when it is possible to obtain from it, by concatenating or composing the operations, all of the possible functions mapping $\mathbb{A} \times \mathbb{A}$ to $\mathbb{A}$ – all the binary operations that the algebra is *capable* of supporting.[39] To achieve functional completeness, Dubarle supplements product, sum and complement (or conjunction, symmetric difference and negation) with a new batch of operations in Hegelian reglia. Of new, 'dialectical' operations, only one needs to be introduced axiomatically – the operation that Dubarle calls 'abstraction' – and out of it the rest can be easily reconstructed. Dubarle writes the 'abstraction' operator as ΔA, and for it the following two axioms:[40]

Δ1. $\Delta A \subseteq A$ (i.e. $\Delta A = \Delta A \cap A$, recalling that $\cap$ is synonymous with $\cdot$);[41]
Δ2. $\Delta(A \cup B) = (\Delta A \cup \Delta B)$

Semantically, Δ is a function of one variable mapping $\mathbb{B}^2$ to $\mathbb{B}^2$. The intuitive idea that Dubarle suggests here – which, as we will see, is either misleading or suspect – is that ΔA 'abstracts', from a total concept A, that which, in it, is 'universal'. In fact, Dubarle suggests an alternative, equivalent approach to the notions of universality and abstraction that begins by introducing a

39 Note that in the present case, this means considering the functions from $\mathbb{B}^2 \times \mathbb{B}^2$ to $\mathbb{B}^2$, which is to say, from $(\mathbb{B} \times \mathbb{B}) \times (\mathbb{B} \times \mathbb{B})$ to $\mathbb{B} \times \mathbb{B}$, and not from $\mathbb{B} \times \mathbb{B}$ to $\mathbb{B}$. (We are slightly abusing notation here, to let the same sign $\times$ stand for both the cartesian product of the underlying sets of algebras and the structure product of the algebras themselves.)

40 In Dubarle and Doz (1972, 164), the axiom we have labelled Δ2 is written $(A \cup B) = (\Delta A \cup \Delta B)$, which seems to be a misprint, in view of Dubarle's latter assertion that we can identify ΔA with $A \cap U$. As printed in Dubarle and Doz (1972), the second axiom is, in fact, false: putting S for both A and B, the axiom yields $S = U$, from which $P = \Lambda$ follows, reducing the $\mathbb{B}^2$ algebra to $\mathbb{B}$. This suspicion is borne out by the earlier version of the algebra presented in Dubarle (1970, 144), where we find $\nabla(A \cup B) = (\nabla A \cup \nabla B)$ presented as an axiom, which can be shown to be equivalent to $\Delta(A \cup B) = (\Delta A \cup \Delta B)$.

41 See the demonstration in 3.2.5 on page 57.

constant, instead of an operation, into the algebra, which would be written U and which would serve to denote the element $\langle 0, 1\rangle$ of $\mathbb{B}^2$. 'Abstraction' would then be defined

$$\Delta A =_{\mathrm{df}} A \cap U$$

It can be seen right away that this definition yields an operation that satisfies the two axioms written above.[42]

1. $A \cap U \subseteq A$ is satisfied, given that
   (i) we have defined $x \subseteq y$ as $x \cap y = x$, so
   (ii) according to this definition, the axiom can be rewritten as $(A \cap U) \cap A = A \cap U$, now
   (iii) $\cap$ (which is just another sign for $\cdot$) is *associative* and *commutative*, which lets us again rewrite the axiom as $(A \cap A) \cap U = A \cap U$. But,
   (iv) $\cap$ is also *idempotent*, which means that $A \cap A = A$. Substituting equals for equals, axiom 1 then becomes $A \cap U = A \cap U$, which is indisputable.
2. This axiom now takes the form $(A \cup B) \cap U = (A \cap U) \cup (B \cap U)$. This can be proven as follows:
   (i) First, we will prove a LEMMA that will remain useful to us throughout this chapter: that $\cap$ is *distributive* over $\cup$.
   (ii) Recall that we have defined $x \cup y$ as $x \not\equiv y \not\equiv (x \cap y)$.[43] To show that $\cap$ distributes over $\cup$ therefore amounts to showing that

   $$(x \not\equiv (y \not\equiv (x \cap y))) \cap d = ((x \cap d) \not\equiv (y \cap d)) \not\equiv ((x \cap d) \cap (y \cap d))$$

   where the left side of the equation is just another way of writing $(x \cap d) \cup (y \cap d)$, according to our definition of $\cup$. This is shown as follows

   $$\begin{aligned} & (x \not\equiv (y \not\equiv (x \cap y))) \cap d \\ = \quad & (x \cap d) \not\equiv ((y \not\equiv (x \cap y)) \cap d) && \text{distributing } \cap \text{ over } \not\equiv \\ = \quad & (x \cap d) \not\equiv ((y \cap d) \not\equiv (x \cap y) \cap d) && \text{and doing so again} \\ = \quad & (x \cap d) \not\equiv ((y \cap d) \not\equiv (x \cap y) \cap (d \cap d)) && \text{by idempotence of } \cap \\ = \quad & (x \cap d) \not\equiv ((y \cap d) \not\equiv (x \cap d) \cap (y \cap d)) && \text{assoc. and comm. of } \cap \end{aligned}$$

42 Like the proof in 3.2.5, above, the demonstration that follows is fairly elementary. I include it here for three reasons: (1) to give the reader unfamiliar with algebraic logic a sense of how proofs, even elementary proofs, are carried out, (2) to provide some basic, didactic foundations for the more advanced mathematics and logic that will concern us in subsequent chapters, and (3) to record the roles played by the essential algebraic properties of associativity, idempotence and commutativity in the establishment of certain, basic laws of logic, a fact that will concern us later when we come to study a rarefied form of these properties in our inquiry into the *structural rules* of logic. It will be in the structural rules that we will uncover some of the most subtle and profound points of resistance concerning the relation of dialectics to mathematical logic. (To anticipate this, we may say, without yet making any attempt at explanation, that associativity has much to do with *cut*, idempotence with *contraction* and *weakening*, and commutativity with *exchange*.)

43 Remember the shift from ring-theoretic to logical notation, summarized in Table 4 on page 59.

and so, $(x \cup y) \cap d = (x \cap d) \cup (y \cap d)$ by definition of $\cup$

$\cap$ is therefore distributive over $\cup$, in virtue of its distributivity of $\not\equiv$ and the algebraic properties of associativity, commutativity and idempotence that characterise a Boolean ring. This gives us our lemma.[44]

(iii) The theorem we wish to prove is now just a special case of our lemma:

$$(A \cup B) \cap U = (A \cap U) \cup (B \cap U)$$

.

The universal constant, similarly, can be defined by means of 'abstraction', since it is just the result of taking an abstraction of the whole:

$$U =_{\mathrm{df}} \Delta V$$

Where V is 'the whole', or the maximal element in the algebra, which takes the form $\langle 1, 1\rangle$ above. (Dubarle will later seek to determine it as the singular.) Product, or conjunction (which, here, is written as either $\cap$ or $\cdot$), is here understood as something like a *set intersection* – an interpretation which is technically unimpeachable, since set intersection (defined over the set of subsets of a given set), by itself determines a Boolean algebra, in which it functions as product or conjunction (with set union taking the role of inclusive disjunction). This characterization of abstraction as *taking what A has in common with the universal*, or as winnowing out what is universal in A, nevertheless harbours philosophical and interpretational problems that will become increasingly pronounced as we progress.

Dubarle goes on to define the operation of '*deposition*', written ∇A, as the counterpart of 'abstraction':

$$\nabla A = A \cap P = A \cap (\Delta V)'$$

which selects from A what, in it, 'opposes the *posited* determinations of actuality to the sublimations of the universal' (Dubarle and Doz 1972, 153): A's intersection, conjunction, or product with the particular. Since Dubarle takes the particular to be the algebraic complement, or classical *negation* of the universal (recall that $\langle 0, 1\rangle \cdot \langle 1, 0\rangle = \langle 0, 0\rangle$)

He proceeds, then, to construct an operation of *concrescence*, which 'ties A's own moment of universality to *to the systematic totality* of the Concept's particularities' (Dubarle and Doz 1972, 153). Written $\blacktriangledown A$, it is formally defined as follows:

$$\blacktriangledown A =_{\mathrm{df}} (\Delta A')'$$

which is to say, as the negation of the abstraction of the negation of A – or, to put it another way, $\blacktriangledown A$ expresses the union, or inclusive disjunction, of A with the particular, P ($\blacktriangledown A = A \cup P$).

44 Incidentally, it turns out that $\cup$ also distributes over $\cap$ – notwithstanding $\not\equiv$'s failure to do so. The pair $\cap, \cup$ thus exhibits a greater operational harmony than does the pair $\cap, \not\equiv$, a fact that recommends a shift in perspective, when it comes to Boolean algebras, from rings to lattices, where $\cup$ is the primary operation and $\not\equiv$ is derived. Proving the distributivity of $\cup$ over $\cap$ is left as an exercise for the reader.

'Relief' (*rélèvement*) comes next. It is the counterpart of concrescence just as abstraction is the counterpart of deposition, and is defined

$$\blacktriangle A =_{\text{df}} (\nabla A')'$$

the negation of the deposition of the negation of A, or, what amounts to the same, the 'assumption of A into the totality of the universal' (Dubarle and Doz 1972, 153), the union $A \cup U$.

$$\begin{array}{llllll}
\text{'Abstraction':} & \Delta A & = & (\blacktriangledown A')' & = & A \cap U \\
\text{'Deposition':} & \nabla A & = & (\blacktriangle A')' & = & A \cap P \\
\text{'Relief':} & \blacktriangle A & = & (\nabla A')' & = & A \cup U \\
\text{'Concrescence':} & \blacktriangledown A & = & (\Delta A')' & = & A \cup P
\end{array}$$

Table 5: Supplementary operations in Dubarle's algebra

These four supplementary operations – abstraction, deposition, concrescence, and relief – obtained either by introducing, into the language, a constant that picks out one of the two middle terms of 5 (here, U, semantically interpretable as a name for $\langle 0, 1 \rangle$), or by introducing a pair of axioms for the operation he calls 'abstraction' (Δ), serve to render the $\mathbb{B}^2$ algebra functionally complete. We will give the name $\mathbb{B}^2_\Delta$ to this enriched algebra. As the terminology selected by Dubarle suggests, it will be in $\mathbb{B}^2_\Delta$ that he attempts to formalize the Hegelian logic of the concept. A consideration of some basic equations will help us better orient ourselves in this algebra, and anticipate certain points at which it will rest uneasily with Hegel's logic.

Let us first consider the ways in which these operations compose with one another. Composing each white sign with its black mate yields a constant function sending any A to the universal or the particular, respectively:

$$\begin{array}{lllll}
\Delta \blacktriangle A & = & \blacktriangle \Delta A & = & U \\
\nabla \blacktriangledown A & = & \blacktriangledown \nabla A & = & P
\end{array}$$

Further, in virtue of the absorption identities,[45] signs of opposite colour and inverted orientation are functional inverses to one another:

$$\begin{array}{lllll}
\Delta \blacktriangledown A & = & \blacktriangledown \Delta A & = & A \\
\nabla \blacktriangle A & = & \blacktriangle \nabla A & = & A
\end{array}$$

Since Dubarle has positioned U and P as *complements* to each other – i.e. $U' = P$ and $P' = U$ – the following theorems trivially hold:

$$\begin{array}{lllll}
\Delta \nabla A & = & \nabla \Delta A & = & \Lambda, \text{ for all } A \\
\blacktriangle \blacktriangledown A & = & \blacktriangledown \blacktriangle A & = & V, \text{ for all } A
\end{array}$$

45 The absorption identities are an expression of the idempotence of $\cap$ and $\cup$, and state $a \cup (a \cap b) = a = a \cap (a \cup b)$.

The reason why $\Delta\nabla A = \Lambda$ is that

$$\Delta\nabla A = (A \cap P) \cap U = A \cap (P \cap U)$$

and since $P = U'$, $P \cap U = \Lambda$ (and for all x, $x \cap \Lambda = \Lambda$). Read through the lens of its spontaneous, set-theoretic interpretation, '$P \cap U = \Lambda$' states, simply, that *the particular and the universal have nothing – no elements – in common*, but this seems absolutely incompatible with Hegel's text, which never tires of telling us that the universal 'is itself while reaching out to its other' – the particular – 'and embracing it' (Hegel 2010, 532); the particular, in turn, is 'taken up into the universality and [...] pervaded by it, just as it pervades it in turn, EQUAL IN EXTENSION AND IDENTIAL WITH IT' (Hegel 2010, 533), such that the relation of the universal to the particular is, ultimately, 'not an outwardly directed *limitation*,' such as would characterize *ontological* relations (see subsection 3.2.3, above), 'but is *positive*, for by virtue of the universality it stands in FREE SELF-REFERENCE' (Hegel 2010, 533). How is this to be reconciled with the algebraic formula that states that the intersection, conjunction, product, or, in order-theoretic terms, the greatest lower bound of universality and particularity is the abstract *void*?

Two observations are called for here. The first concerns what I have called the spontaneous, set-theoretic interpretation of the formalism – an interpretation to which Dubarle's language, at times, perhaps gives too much license. It is important, first of all, to remember that we are not operating inside set theory, even on the formal level, despite the fact that everything written above can be *translated* or transposed into the language of set theory. The susceptibility of the $\mathbb{B}^2_\Delta$ formalism to a set-theoretic semantics must remain here a purely abstract possibility, one which, even if it were to be carried out, would by no means oblige us to adopt a set-theoretically informed metaphysics on the level of the system's philosophical interpretation. This is to say that nothing obliges us to, first, gloss $U \cap P = \Lambda$ as

$$\neg(\exists x)(x \in U \wedge x \in P)$$

and *then* interpret the '$\in$' as the relation of conceptual determination or containment. This symbol, after all, occurs nowhere in the object-language of $\mathbb{B}^2_\Delta$, and it is to $\mathbb{B}^2_\Delta$ that Dubarle entrusts the task of describing the operations of conceptual 'development' described by Hegel, operations which are meant to include, among other things, the relation of U to its 'content', to its particular determinations. It is possible, for instance, to follow the *letter* of Hegel's text and take the function of *negation* as what links U to the particular determinations P that furnish its content:

> even the *abstract universal* entails this much, that in order to obtain it there is required the *leaving aside* of other determinations of the concrete. As determinations in general, these determinations are *negations*, and *leaving them aside* is a further *negating*. Even in the abstract universal, therefore, the negation of negation is already present. (Hegel 2010, 531)

'But', Hegel goes on to write,

> this double negation comes to be represented as if it were *external* to it, both as if the properties of the concrete that are left out were different from the ones that are retained as the content of the abstraction, and as if this operation of leaving some aside while retaining the rest went on outside them. With respect to this movement, the universal has not yet acquired the determination of *externality*; it is still in itself that absolute negation which is, precisely, the negation of negation or ABSOLUTE NEGATIVITY. (Hegel 2010, 531)

'Absolute negativity', recall,[46] is, with 'abstraction', a name for the moment of Hegel's concept that Dubarle seeks to substitute for what he calls 'the conceptual void,' the abstract Λ – in an operation that, he fully admits, does some violence to the original Hegelian schema. It seems appropriate, therefore, to expect that the *product* of U and P is what unites the reciprocally-determining negative action of both moments – their abstracting and deposing tendencies – thereby distilling the 'absolute negativity' that, at this phase of the logic, appears as their identity: Λ.

The reader familiar with Hegel's text will, of course, have noticed while the immediate objections to the $U \cap P = \Lambda$ formula, on page 63, drew on Hegel's description of the relations between the *concrete* universal and particular, it was on his descriptions of the *abstract universal* that we have tried to build a charitable defence of the formula. The tension between these two levels of description is the subject of our second observation. But this is, in fact, the plane on which Dubarle's algebraization of the logic appears to intervene: it serves only to slice, from the Hegelian totality of the concept, the abstract phase of its determinations, in order to prepare it for algebraic analysis. What mediates, in Hegel's text, between the concrete structure of universality, particularity and singularity that is his primary object of attention, and the abstract namesakes of these moments, from which he seeks to differentiate the true conception of the concept, is the obscure moment of *abstraction* (Figure 2 on page 45). The neutralization of the moment of abstraction, which replaces it with the inert void, Λ, serves to efface the *action* of abstraction from the logic of the concept, and makes the other moments – U, P, S – appear as products of abstraction, abstracted *from* the process of abstraction that yields them.

The stage is already set, but if this algebra is to house the Hegelian dialectic of the concept, then certain features of it, already apparent here, demand attention. For instance, we know that Dubarle intends to plot the moments of universality, particularity, singularity and void onto the four points of

46 Referring to the passage quoted on page 47, Dubarle remarks that 'Nothing remains to stop us, from substituting, in the logico-mathematical formalism of control, the logically void term for this "second negative" or this "absolute negativity" ' (Dubarle and Doz 1972, 148). See section 3.2.4.1 on page 48, above, for a discussion of this point.

the $\mathbb{B}^2_\Delta$ algebra. Let us consider this structure more closely (it is reproduced in Figure **??**).[47]

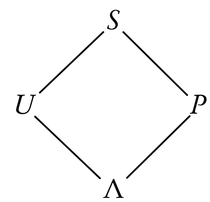


Figure 6: $\mathbb{B}^2_\Delta$, with nodes as moments

The information contained in Figure 6 is recapitulated in 'truth table' form in 6.

The first characteristic that should strike us is its perfect *symmetry*, a symmetry, in fact, that takes the form of a *structural invariance under complementation*, which is to say, under Boolean *negation*. There is nothing, in effect, to distinguish the left side of the graph from its right side, except for their mere distinction, and likewise there is nothing to distinguish the top from the bottom. Formally, what this amounts to is a strong principle of *duality* in the algebra. Every theorem in $\mathbb{B}^2_\Delta$ can be written as an order relation, of the form

$$X \subseteq Y$$

where X and Y are formulas. What the duality of the algebra entails is that if we substitute every X with its complement X' and do the same for Y, and invert $\subseteq$ into $\supseteq$, we again obtain a theorem of the algebra.

$$X' \supseteq Y'$$

The question, then, is this: what remains of the Hegelian concept if we take only what remains invariant under this symmetry? What is the *quotient* of the concept and classical negation?

[ATTEMPT AN ANSWER]

Finally, before moving on, it may help to underscore the essentially *conservative* nature of $\mathbb{B}^2_\Delta$ with respect to Boole's bivalent, classical algebra.

As can immediately be seen from Table 6, or from the an inspection of Figure 6, all of the canonical classical 'laws of thought' are conserved in $\mathbb{B}^2_\Delta$. We have, for instance

1. The law of the excluded middle, since $x \cup x' = S$ for all x.
2. The law of non-contradiction, since $x \cap x' = \Lambda$ for all x.
3. The law of distribution, as we have already seen in 2 on page 61
4. The absorption identities $x \cup (x \cap y) = x$ and $x \cap (x \cup y) = x$.
5. The law of double negation, whereby $x'' = x$.
6. The principle '*ex falso sequitur quodlibet*', given that $\Lambda \subseteq x$ for all x.

47 Instructions for reading lattice diagrams can be found above, following Figure 5 on page 58.

$\cap$	Λ	U	P	S
Λ	Λ	Λ	Λ	Λ
U	Λ	U	Λ	U
P	Λ	Λ	P	P
S	Λ	U	P	S

$\cup$	Λ	U	P	S
Λ	Λ	U	P	S
U	U	U	S	S
P	P	S	P	S
S	S	S	S	S

$\not\equiv$	Λ	U	P	S
Λ	Λ	U	P	S
U	U	Λ	S	P
P	P	S	Λ	U
S	S	P	U	Λ

negation	*abstraction*	*relief*	*deposition*	*concrescence*
$(\Lambda)' = S$	$\Delta(\Lambda) = \Lambda$	$\blacktriangle(\Lambda) = U$	$\nabla(\Lambda) = \Lambda$	$\blacktriangledown(\Lambda) = P$
$(U)' = P$	$\Delta(U) = U$	$\blacktriangle(U) = U$	$\nabla(U) = \Lambda$	$\blacktriangledown(U) = S$
$(P)' = U$	$\Delta(P) = \Lambda$	$\blacktriangle(P) = S$	$\nabla(P) = P$	$\blacktriangledown(P) = P$
$(S)' = \Lambda$	$\Delta(S) = U$	$\blacktriangle(S) = S$	$\nabla(S) = P$	$\blacktriangle(S) = S$

Table 6: Tables for logical operations in $\mathbb{B}^2_\Delta$, with Dubarle's constants

Certain of these laws, indeed, appear to inhere at deeper levels of the logic than do others – and several seem to flow inevitably from the decision to establish ourselves in an idempotent ring, or, what amounts to the same, a distributive lattice. Dubarle's conservative strategy, in fact, can be seen as an intention to remain, staunchly, in the *space* of classical logic, and to endeavour to reconstruct, as far as possible, the Hegelian variations on logic as movements and rearrangements *within* that space – varying the size of the logic, for instance, by taking successive powers of the original $\mathbb{B}$ algebra. Dubarle, however, ties this mathematical conservativism to a subtle philosophical imagination, and the interest of his work largely lies in the dexterity with which he folds the delicate fabric of the dialectic into the austere architecture of Boolean logic – leaving this architecture almost entirely as he found it, with almost no trace of its Hegelian tenant. His work is thus an exemplar of the approach van Benthem calls for in his highly critical 1979 review of Richard Routley's attempt to formalize the dialectic in a paraconsistent relevance logic (an approach we will examine in Chapter **??**), where he writes that

> Instead of taking dialectical ideas like the admissibility of contradictions, or the non-equivalence of $\neg\neg A$ and A, at face value (in which case they, naturally, cannot be fitted into classical logic), one should try a little harder to find an interpretation which does fit into the classical framework. Instantaneous formalization of deviant principles does not point at liberality, but at lack of logical phantasy. (van Benthem 1975, 345)

It is for this reason that we are beginning this study of formalizations of the dialectic with Dubarle – the closest of all the case studies to classical logic – the logic that has been with us, with minor variations, since antiquity (with the work of Aristotle and Chrysippus), and which was the first to be fully mathematized, with Boole. Whether, to use a favourite expression of Hegel's, this 'excessive tenderness'[48] for classical logic is ultimately compatible with the dialectic – be it Hegel's or the dialectical pressures internal to mathematical logic – is a question to which we shall soon return.

3.2.5.2 *Dubarle's algebraic schematization of the Hegelian concept, and the* Aufhebung *operation*

The task that Dubarle now sets himself is to analyse, through the medium of his extended Boolean algebra $\mathbb{B}^2_\Delta$, the sequence of operations that take place when, as Hegel describes it,

> As negativity in general, that is, according to the *first immediate* negation, the universal has determinateness *in it* above all *as particularity*; as *a second universal*, as the negation of negation, it is *absolute determinateness*, that is, *singularity* and *concreteness*. (Hegel 2010, 532)

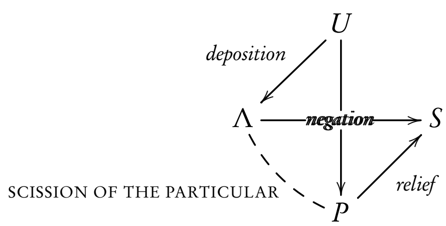
or, following a description that makes clearer the quadruple structure of the concept – a structure that Dubarle insists upon – the dynamic under investigation is what takes place when

> the course of cognition returns at the same time back into itself. This negativity is as self-sublating contradiction the *restoration* of the *first immediacy*, of simple universality; for the other of the other, the negative of the negative, is immediately the *positive*, the *identical*, the *universal*. In the whole course, if one at all cares *to count*, this *second* immediate is *third* to the first immediate and the mediated. But it is also *third* to the first or formal negative and to the absolute negativity or second negative; now in so far as that first negative is already the second term, the term counted as *third* can also be counted as *fourth*, and instead of a *triplicity*, the abstract form may also be taken to be a *quadruplicity*; in this way the negative or the *difference* is counted as a *duality*. (Hegel 2010, 746)

48 'It is an excessive tenderness for the world to keep contradiction away from it, to transfer it to spirit instead, to reason, and to leave it there unresolved. In fact, spirit is the one which is strong enough that it can endure contradiction, but it is spirit again which knows how to resolve it' (Hegel 2010, 201). Cf. also (Hegel 2010, 367): 'The ordinary tenderness for things, the overriding worry of which is that they do not contradict themselves, forgets instead, here as elsewhere, that contradiction is not thereby dissolved but is rather shoved elsewhere, into subjective or external reflection; forgets that the two moments of this reflection, of which it speaks as assumed facts in its effort at removing them or displacing them, are in fact contained in it as sublated, and each referring to the other in one unity.' Elsewhere, in his lectures on the history of philosophy, Hegel makes the same remark again, in almost identical language: 'Nevertheless Kant shows here too much tenderness for things: it would be a pity, he thinks, if they contradicted themselves. But that mind, which is far higher, should be a contradiction – that is not a pity at all. The contradiction is therefore by no means solved by Kant; and since mind takes it upon itself, and contradiction is self-destructive, mind is in itself all derangement and disorder.' [FIND CITATION REF.]

It is clear that the operation that Hegel calls 'negation' cannot be transposed immediately onto the structure of Boolean complementation, the algebraic representative of classical, 'Aristotelean' negation, given the classical identity of A and A''. For Dubarle, in effect, the desired transposition will pass through an analysis mediated by his supplementary operations of *abstraction, deposition* and *relief*, by means of which, he will argue, the Hegelian *Aufhebung* – or an abstract adumbration of the same – can be reassembled.

I have already given some indication as to the *shape* that Dubarle understands this description to trace, with Figure 4 on page 48. The point now is to see, at once, how this figure expresses an *algebraic* sequence of operations, how this sequence fits with the standard *logical* interpretation of Boolean rings, and how – if at all – it serves to capture and clarify some aspect of the Hegelian logic of the concept, or at least that trail of the logic that the concept imprints in the exteriority of its linguistic expression. Let us now look more closely at how Dubarle proceeds.



$$(\Lambda, P) \longleftarrow \left\{ \begin{matrix} \textit{deposition} \\ \textit{negation} \end{matrix} U \right|$$

$$\left| (\Lambda, P)^{\textit{negation}}_{\textit{relief}} \right\} \longrightarrow S$$

Figure 7: Dubarle's schematization of the Hegelian concept

As seen in Figure 7, Dubarle begins by splitting what Hegel calls 'negation' into two pairs of composite operations. The 'first negation', which Dubarle writes $\left\{ {}^{\alpha}_{\beta} U \right|$ decomposes into a forking pair of functions:

- α which sends U, which can equally be read as 'the first or formal negative', since it is equal to ΔS, to 'the absolute negativity or second negative', whose symbolic trace, here, is the conceptual void Λ. What realizes this operation in $\mathbb{B}^2_\Delta$ is the function of *deposition*: $\nabla(\Delta S) = \Lambda$.
- β which sends U, as 'the first immediate' to the 'the mediated' P, and which is realized in $\mathbb{B}^2_\Delta$ by the operation of *negation* or complementation: $U' = P$;

The 'second negation', written $\left| (\Lambda, P)^{\gamma}_{\delta} \right\}$, serves to recompose what the first negation has split apart, and is further analysed into another, merging functional pair:

- γ which sends 'the absolute negativity or second negative' Λ to the 'second immediate', or singular concept S, a function which, like α is realized complementation or negation in $\mathbb{B}^2_\Delta$: $\Lambda' = S$;

δ which sends 'the mediated' P to 'the second immediate' S, and which $\mathbb{B}^2_\Delta$ realizes with the operation of *relief*: $\blacktriangle P = S$.

The result of these operations is a figure that resembles the one Hegel describes above (Hegel 2010, 746): two overlapping triads, as in Figure ??.

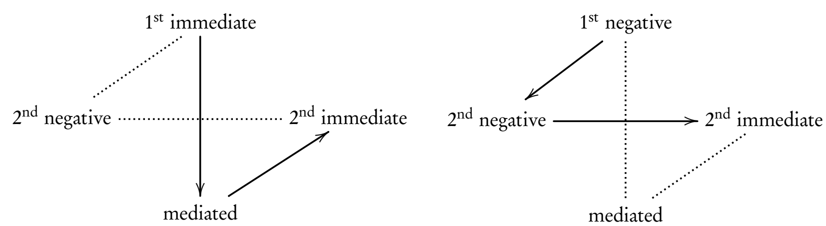


Figure 8: Quadruplicity in two overlapping triads

The question that now confronts us concerns the relation between this *figuration* of the *Aufhebung* and the logical material on which it is drawn. It seems that there are two distinct, but interwoven layers of description in Dubarle's algebraicization of the Hegelian dialectic. If the schematization presented above is to count as a logico-algebraic analysis of the concept's dialectical developments, then we must be able to ascertain the logical status of the developments $\left\{ {}^{\alpha}_{\beta} U \mid \text{and} \mid (\Lambda, P)^{\gamma}_{\delta} \right\}$. Is it possible to count them as rules of inference, axiomatic conditionals, or, in Hegelian language, as forms by which the concept *develops*? That is to say, given U, does this schema entitle us to *derive* the moments Λ and P? Given (Λ, P), does the schema work to imply S?

The problem that arises here is this: if it is possible to infer one moment from another, according to the arrows in Figure 7, then we are faced with a difficulty. Suppose that X and Y are any two moments connected by an arrow in the schema above – an arrow that is now to be read as something like an implication (a word Hegel himself uses for the relations of development), so that we have, in effect, $X \to Y$. The difficulty arises when we ask whether, after inferring Y from X – as guided by $X \to Y$, we are still confronted with X. Given that we are operating in what remains an essentially Boolean logic – indeed, insofar as we remain within the category of lattices (which is a much broader category) – there is no way for the logical passage, the inference, from X to Y to absolve us of X. X therefore remains, alongside Y, 'after' the development or inference has taken place. But the schema in question is woven of functions leading from moments to their *negations*, or to the abstract Λ itself. If even one of these implications is followed through, as an inference, then the logic collapses into triviality: it plunges into the void, from which everything follows.[49]

49 When I first noticed this difficulty, the problem seemed to lie in what Boole named the 'fundamental law of thought', and to which $\mathbb{B}^2_\Delta$, as a Boolean algebra, and even as a lattice, is necessarily subject. It is this law, the identity $A = A \cap A$, that, as a structural consequence,

For this problem to be avoided, either implication or contradiction, or both, must be freed from their classical constraints and laid open to an analysis that Boolean algebra does not appear to have the resources to carry out. The cluster of tasks and problems that this problem points towards will be the work of subsequent chapters.

A chemist can prepare a batch of pigments, and use them to paint a picture of Saturn's rings, but despite the wealth of chemical analysis that went into preparing the pigments, this should not count as a *chemical analysis* of the rings of Saturn. The situation with Dubarle's logic is analogous. It allows for a *figure* of the dialectic to be constructed, but it is a figure that is logically inanimate. 'Beneath' the schema of the *Aufhebung* drawn on its surface, there lives a logic that is, indeed, elegant and operational, but the channels of this logic do not flow in the fourfold direction of what Hegel called the development of the concept.

3.2.6 *The Fano configuration as* matheme *of conceptual infinity and interiority*

The fundamental tension animating both Dubarle's general reflections on formalization, and his algebraicization of the Hegelian concept in particular, is, as we have seen in section 3.2.2, the tension between the 'regime of interiority' proper to the concept in and for itself, the λόγος ἐνδιάθετος, on the one hand, and the 'regime of exteriority' that Dubarle follows Hegel in taking to be inherent in mathematical formalization, and to language as λόγος προφορικός, on the other.

In a schema reproduced in Figure 1 on page 39, we find Dubarle's attempt (in Dubarle (1977)) to describe a certain organization of discourse that, with both Parmenides and Hegel, serves relate and subordinate the philosopher's λόγος προφορικός to the absolute λόγος ἐνδιάθετος at which it aims. In the fifth chapter of *Logique et dialectique*, Dubarle attempts to grasp this configuration, or one similar to it, in quasi-mathematical terms – drawing from it a rudimentary structure from finite projective geometry.

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serves to 'idealize' the formulas put to work in logical implication. Without it – as we will see in Chapter **??** and **??** – implication functions in such a way as to 'consume' its antecedent in delivering its consequence. If this law did not hold, it seemed, then the arrows of 7 could indeed operate as forms of implication without engendering antinomies. The problem, today, appears more complex than I had initially thought, insofar as Dubarle aims to articulate the structure of conceptual *development*, and not that of ontological *transition* or essential *reflection*. Though the latter two modes of passage do, to varying degrees, involve a movement out of the antecedent towards the consequent, development no longer operates in this fashion. Hegel is explicit on this point: a moment remains in itself while developing towards its posited other. However, this does not seem to be a purely and simply classical form of implication: the 'imperfective' (to use Girard's term) quality of development is *constituted* by an intricate weave of ontological transitions and essential reflections, and the endurance of the conceptual moment in its identity appears to be the result of a tight knot of more emphemeral forms of negativity and passage. An adequately detailed analysis of the difference between the three forms of passage in Hegel's *Logic* now appears to be an essential task for any attempt to articulate the relation between dialectical logic and existing logico-mathematical systems, but remains to be fully carried out.

The structure in question is PG[1,2], a projective line comprising of three points: two 'finite' points, written $\frac{0}{1}$ and $\frac{1}{1}$, meant to correspond to the concepts of 'non-being' and 'being' in Parmenidean discourse, respectively – or that side of the discourse which is actually articulated and made manifest. These two points can be taken as the values of a minimal Boolean algebra $\mathbb{B}$, anchoring the austere logic of Parmenides' teachings. There is some difficulty, however, in understanding the 'being' that is articulated by the finite point $\frac{1}{1}$ as adequate to the supreme *concept* of Being, as One and as All, insofar as it must articulated against non-being, in a logic that straddles that of which one may speak and think, and that of which neither speech nor thought is possible – according to the famous Eleatic interdiction. At the level of $\frac{0}{1}$ and $\frac{1}{1}$, Parmenidean thought thus remains vulnerable to the criticism Plato articulated in *The Sophist*: that in order to demarcate the true – being – from the false – non-being – one is forced to submit to a λόγος that straddles and mixes the two (and this is the role assigned to the Platonic logic of the five supreme genera). In order for the way of truth is not to be contaminated by nothingness, division and exteriority, the Eleatic is forced to retreat, and insist upon the difference between the supreme, ineffable BEING and its projections into exteriority, the 'seeming' mixture of being and non-being.[50]

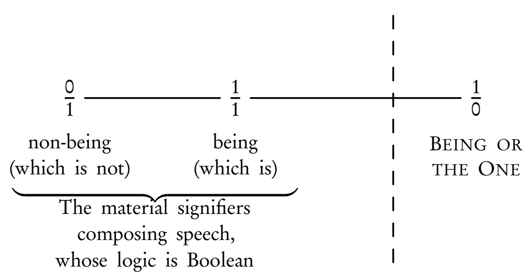


Figure 9: The projective scheme of Parmenidean λόγος, after Dubarle (1977, 236) and Dubarle and Doz (1972, 172)

Dubarle suggests an analogous configuration for the Hegelian *concept*, which amounts to an embedding of the $\mathbb{B}^2_\Delta$ algebra in the finite projective plane PG[2,2], known as the *Fano configuration* (see Figure 10 on the facing page).

At this point, the *figurative* tendencies of Dubarle's enterprise become increasingly apparent.[51] The geometrical rigging intended to show how the

50 It is unclear how appropriate to Parmenides' thought is the profoundly Christianized notion of the infinite that Dubarle employs here. The perfection of the One, for Parmenides, implies its strict *finitude*, with infinity, limitlessness (απεριον), being a mark of imperfection and non-being.

51 Hegel, it is worth noting, takes a special interest in those 'impasses of formalization' at which mathematical thought finds itself thrown back on figurative, pictorial representation, and

exteriorized abstractions of $\mathbb{B}^2_\Delta$ are nonetheless suspended from the infinite and interiorizing regime of the concept itself is almost altogether extrinsic to the logic. At first sight, everything appears as if we are facing an extension of the logical algebra itself: we have seen that the system $\mathbb{B}^2_\Delta$ can be specified as the set of ordered pairs describing the structure-product $\mathbb{B} \times \mathbb{B}$ (see Figures 5 on page 58 and 3 on page 57), and Dubarle's embedding of $\mathbb{B}^2_\Delta$ in the Fano configuration begins by appending to each of the ordered pairs corresponding to the elements of $\mathbb{B}^2_\Delta$ ($\langle 0,0\rangle = \Lambda, \langle 0,1\rangle = U, \langle 1,0\rangle = P, \langle 1,1\rangle = S$) an additional 1, prefixing the pair. This additional 1 is meant as an index of the finitude of the Λ, U, P, S so adjusted, and it is suggested that it be interpreted as a divisor – the idea being that it leaves the element unchanged.

A second set of elements is then introduced, elements Dubarle calls infinite. Like U, P, S, these are designated by ordered triples, but triples beginning with 0. Again we are invited to imagine 0 as that by which the ordered pair corresponding to U, P or S is 'divided', suggesting the infinitary character of these additional elements. The elements are arranged as shown in Figure 10.

What is peculiar about these newly added elements is that they are not made subject to the operations $\cap$, $\cup$ and $(\ )'$ that determine the structure of $\mathbb{B}^2_\Delta$ – nor, therefore, are they admissible values for the operations of deposition, abstraction, concrescence or relief. Only one operation is suggested for them, which Dubarle calls *transference*, and which is defined as follows:

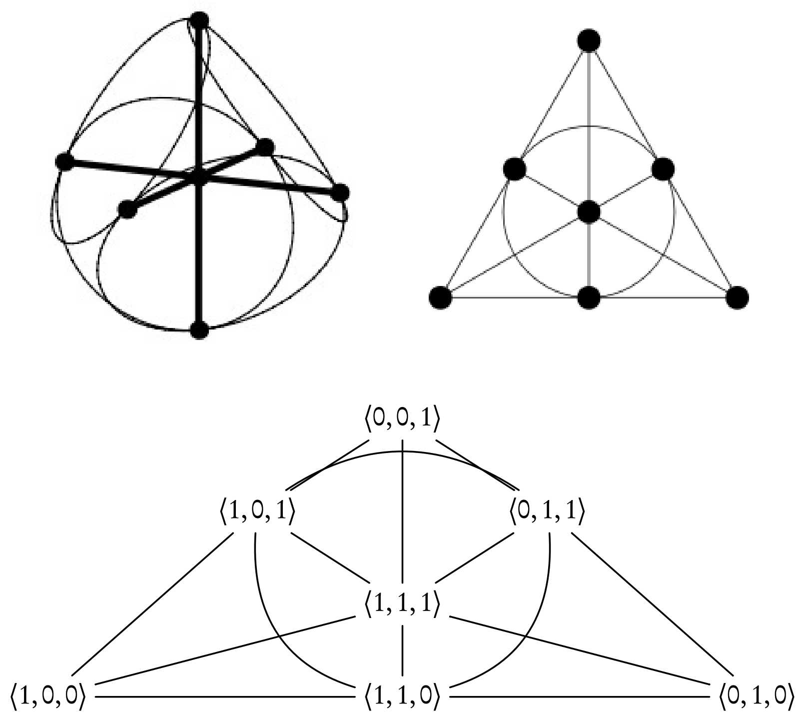
$$\langle a_1, b_1, c_1\rangle + \langle a_2, b_2, c_2\rangle =_{\mathrm{df}} \langle (a_1 + a_2), (b_1 + b_2), (c_1 + c_2)\rangle$$

This definition of the sum of ordered triples is strictly analogous to our earlier extension of $+$ for ordered pairs in 3 on page 57.[52] Recall that the sums on the right are sums of values of the Boolean ring $\mathbb{B}$, and so $1 + 0 = 0 + 1 = 1$, $0 + 0 = 0$ and, in particular, $1 + 1 = 0$.

understands such impasses symptoms indicating that mathematics has, somehow, touched on the real, the concept that outstrips it. This is how he diagnoses the deadlocks of the infinitesimal calculus, in a Remark appended to his discussion of the particular concept:

> The higher mathematics, which also proceeds to the infinite and allows itself contradictions, can no longer employ its customary signs for representing such determinations. In order to indicate the still conceptually uncomprehended representation of the *infinite approximation* of two ordinates, or when it equates a curve to an infinite number of infinitely small straight lines, all it does is to design two straight lines *outside each other* or to draw straight lines inside but still *distinct* from a curve; for the infinite, which is the point at issue here, higher mathematics falls back on *pictorial representations*. (Hegel 2010, 545)

52 Dubarle mixed together both '$\not\equiv$' and '$+$' in his own presentation of this operation, a notational complication which seemed unnecessary, since the definition is so strictly analogous to the definition of $+$ for ordered pairs, used earlier, where we used '$+$' for both the sum of pairs *and* the sum of single values.



Finite Universal	$\frac{\langle 0,1\rangle}{1} = \langle 1,0,1\rangle$	*Infinite Universal*	$\frac{\langle 0,1\rangle}{0} = \langle 0,0,1\rangle$
Finite Particular	$\frac{\langle 1,0\rangle}{1} = \langle 1,1,0\rangle$	*Infinite Particular*	$\frac{\langle 0,1\rangle}{0} = \langle 0,0,1\rangle$
Finite Singular	$\frac{\langle 1,1\rangle}{0} = \langle 1,1,1\rangle$	*Infinite Singular*	$\frac{\langle 0,1\rangle}{0} = \langle 0,0,1\rangle$

Finite Void $\frac{\langle 0,0\rangle}{1} = \langle 1,0,0\rangle$

Figure 10: Fano configuration as matheme of the concept

Dubarle then defines a triadic relation of *colinearity*: Three triples $\langle m_1, n_1, p_1\rangle, \langle m_2, n_2, p_2\rangle$, and $\langle m_3, n_3, p_3\rangle$, are said to be *colinear* if and only if

$$\begin{aligned}
\langle m_1, n_1, p_1\rangle + \langle m_2, n_2, p_2\rangle &= \langle m_3, n_3, p_3\rangle \\
\langle m_2, n_2, p_2\rangle + \langle m_3, n_3, p_3\rangle &= \langle m_1, n_1, p_1\rangle \\
\langle m_3, n_3, p_3\rangle + \langle m_1, n_1, p_1\rangle &= \langle m_2, n_2, p_2\rangle
\end{aligned}$$

all hold. The results of such calculations match the diagram in Figure 10 perfectly: all and only those points lying on a single line – including the circle inscribed in the triangle, which counts as a line – stand in such a relation to one another.

The peculiarity of this construction can, at least in part, be seen in the *ad hoc* behaviour of $+$. On the one hand, it behaves exactly as does $+$ or $\not\equiv$ in the theory of Boolean rings, and is defined for the Fano configuration in a manner strictly analogous to its definition for ring-products. It is, however, set up in such a way as to be no longer interdefinable with $\cup$ by way of $\cap$, and this is because the infinite values are strangely preserved from what Dubarle calls the 'generic' operations governing $\mathbb{B}^2_\Delta$ (Dubarle and Doz 1972, 184). It is unclear, moreover, what *logical work* the relation of colinearity or operation of transference is intended to perform with respect to $\mathbb{B}^2_\Delta$ – if there are any interesting logical facts we can learn about $\mathbb{B}^2_\Delta$ by immersing it in the Fano configuration, apart from seemingly inert statements informing us of relations of colinearity. What is meant to follow, and what is meant to be expressed, by statements like 'the finite universal and the finite particular are colinear with infinite singularity' or 'the void and the finite universal are colinear with infinite universality'?

There may be something of interest here that is, at least in this light, hidden from view, but if so Dubarle does not indicate what this might be. Ultimately, the Fano configuration presented in the fifth chapter of *Logique et dialectique* appears to operate as an object for contemplation rather than a mathematical instrument, an abstract figuration of certain metaphors of projection and transcendence, or a poem in the shape of a formalism. It is less an extension of mathematical formalization into what, previously, had been concealed in the obscurity of λόγος ἐνδιάθετος than it is an icon serving to illustrate an impasse of formalization, such as Hegel himself claimed to detect in the pictorial representations that the practitioners of the infinitesimal calculus fell back upon in an effort to grapple mathematically with the infinite. This service this structure performs, in Dubarle's text, is neither that of a mathematical instrument nor a mathematical object, but that of a *matheme*, in the sense that Lacan and Badiou have given to this term: a *fragment* of mathematics enlisted as a poetic apparatus for tracing the deadlocks of mathematical thought.[53]

53 This is precisely how the 'matheme for the event' works in Badiou's *L'être et l'événement*, for instance, where the formula $e_x = \{x \in X, e_x\}$ is introduced to indicate a localised rupture in being qua being, while maintaining that the formal structure of the latter is characterised by the Zermelo-Fraenkel axioms of set theory, axioms which include the *axiom of foundation*, which is incompatible with the existence of such sets as e_x. This is not taken to be a flaw in the theory, however, which could be easily modified in any number of ways to allow for such non-well-founded sets. Rather, the role of the 'matheme of the event' is to sit *essentially* outside the ontological regime of ZF. It is written in the language of ZF, but as a poetic fragment, a piece of formalism that *cannot be used* in actual mathematical calculations, or which is not intended to be used in that way (see Badiou 1988). The profound counter-systematicity of Lacan's many 'mathemes' can be understood in the same fashion: we are not supposed to mathematically derive *anything* from the equation of $\sqrt{-1}$ with 'the erectile organ', for instance, at least not in a systematic fashion. What bits of actual mathematics enter into this reverie are themselves essentially *demathematized*, used as poetic devices for tracing the deadlocks of the symbolic order.

3.3 GROMPONE'S DIALECTICAL LATTICES

During the American-backed military dictatorship that ruled Uruguay between 1973 and 1985, the Marxist philosopher and engineer Juan Grompone, 'nourished by the need for an intellectual and ideological resistance to all that the dictatorship represented,' undertook the project of formalizing the Hegelian and Marxist dialectic, a project that owed its survival to its esotericism and abstraction. In a time during which it was extraordinarily dangerous to be found carrying a copy of *Capital*, it was quite safe to be found reading Birkhoff's *Lattice Theory* (?, 19-20). The product of Grompone's studies came to light only after the dictatorship had fallen, with the 1985 publication of *Estudios sobre la lógica dialéctica*.

3.3.1 *Engel's formulation of the 'laws of the dialectic'*

The philosophical source of Grompone's conception of the dialectic is not Hegel but Engels, and it is Engel's (already somewhat schematic) 'Laws of the Dialectic' that gives Grompone his point of departure. Of these three laws, which Engels enumerates as

1. the conversion of quantity into quality
2. the interpenetration of contraries
3. the negation of the negation

Grompone seeks to formalize only the last two, considering the first to be a 'material' rather than 'formal' principle – though he will often insist that, in the last instance, all three principles, and the entire dialectical logic they adumbrate, have their source and ground in Nature, and not in the Concept. The means for mathematizing the two remaining laws – that contraries interpenetrate, and that the doubling of negation does not return us to our initial assertion – will be the mathematical theory of lattices.

3.3.2 *Grompone's logical naturalism, and the scope and aim of his attempt to formalize dialectical logic*

3.3.3 *Grompone's lattice-theoretic generalization of the concept of dialectical logic, and of the algebra presented by Dubarle*

Grompone puts the machinery of lattice theory to a far more complex and sophisticated use than what we find in Dubarle (who appeals, for his part, to the theory of Boolean rings – itself congruent with the theory of distributive (or 'Boolean') lattices).

Interestingly, Grompone makes no effort to argue for the reinsertion of the abstract (Boolean) zero into the structure of the dialectic, but instead sees the strictly *dialectical* moments of logic as suspended *between* truth and falsity, between 1 and 0. His definition of a 'simple dialectical lattice' runs as follows:

> DEFINITION: We will call a *simple dialectical* lattice of order n, $\mathbb{D}_n$, the lattice formed by the elements $\bot$, $\top$, and n mutually contrary elements.

It is worth noting that, without any apparent awareness of Dubarle's work, Grompone is thus able to subsume Dubarle's algebra under the category of 'dialectical lattices' that he himself defines. He calls it the 'yin-yang lattice', $\mathbb{D}_2$. Here it is again, to refresh our memories.

$$\begin{tikzcd} & S = \top \arrow[dl, no head] \arrow[dr, no head] & \\ P = yin \arrow[dr, no head] & & U = yang \arrow[dl, no head] \\ & \Lambda = \bot & \end{tikzcd} \tag{3.1}$$

It can be seen straightaway that it satisfies the definition of a simple dialectical lattice: between $\top$ and $\bot$, here interpreted as the absolutely true and the absolutely false (rather than as the Singular and the Void), two other mutually contrary (complementary) elements are suspended, which Dubarle calls 'the Particular' and 'the Universal', but which Grompone names '*yin*' and '*yang*'. The idea here is that a (non-trivial) dialectical lattice is one in which we encounter pairs of negated terms, neither of which is completely true or completely false.

This lattice can be equipped with various negations. The most straightforward is classical complementation: $yin' = yang$, $yang' = yin$, $\top' = \bot$, and $\bot' = \top$.

Though the structure of the 'yin-yang lattice', or *simple dialectical lattice of the second order*, is indeed identical to that of the 'ultra-Boolean ring' proposed by Dubarle, its theoretical interpretation is quite different. The 'dialectical' component of this lattice is, for Grompone, confined to a single stratum, the 'dialectical values' that fall into the 'faults and fissures' (**?**, 16) separating the non-dialectical, Boolean values of abstract truth and falsity (which correspond formally to Dubarle's Singular and conceptual Void). For Dubarle, by contrast, the dialectical spans the entire lattice, and incorporates the $\top$ and the $\bot$ into itself as essential moments.

The difference becomes more pronounced when we turn to Grompone's interpretation of the structure of Hegelian negation. Recall the somewhat formulaic passage cited earlier, where Hegel writes:

> As negativity in general, that is, according to the *first immediate* negation, the universal has determinateness *in it* above all *as particularity*; as *a second universal*, as the negation of negation, it is *absolute determinateness*, that is, *singularity* and *concreteness*. (Hegel 2010, 532)

To schematize this operation, Grompone does not, like Dubarle, split negation into pairs of composite operations, but rather installs it within a non-Boolean lattice, the 'simple dialectical lattice of order 3', $\mathbb{D}_3$:

$$\begin{tikzcd} & \top \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \textit{thesis (Universal)} \arrow[dr, dash] & \textit{antithesis (Particular)} \arrow[d, dash] & \textit{synthesis (Singular)} \arrow[dl, dash] \\ & \bot & \end{tikzcd} \tag{3.2}$$

Negation on this lattice is defined as follows. When applied to the 'classical' values of abstract truth or abstract falsity, it functions just like classical, Boolean negation: $\top' = \bot$, $\bot' = \top$. But when applied to the 'dialectical' values, we have the familiar 'Hegelian' scheme: the negation of the *thesis* is the *antithesis*, the negation of the *antithesis* is the *synthesis*, and the negation of the *synthesis* brings about a return to the *thesis*, completing the Hegelian circle with a return to the 'first immediate' – albeit without a place for the subtle and crucial difference that Hegel insists upon between the first immediate and its post-dialectical return. The result is that Grompone's 'Hegel Lattice' spins like a top, and that the *negation of the negation of the negation* returns us to where we started, without any difference whatsoever.

Like Dubarle's algebraicization of the dialectic, Grompone's is profoundly *symmetrical*. [FOLLOW UP THIS THOUGHT]

[NON-DISTRIBUTIVITY] This lattice is the kernel of non-distributivity, and the basis for what Von Neumann and Birkhoff have dubbed *quantum logic*.

The simple and schematic nature of Grompone's dialectical lattices allows them to be generalized and complexified in a straightforward manner. It is possible, for instance, to increase the number of dialectical values, and to introduce distinct 'levels' of dialectical values between the abstract $\bot$ and $\top$ (the 'rank' of the lattice), so as to obtain more complex and nuanced structures like this one, a complex dialectical lattice of rank 3 and order 5:

3.3.4 *The capacities and resources of dialectical lattice theory*

3.3.5 *Critique of Grompone's attempt*

3.4 THE BOOLEAN ALGEBRAIZATION OF LOGIC, AS CONCENTRATION OF CERTAIN "ANTI-DIALECTICAL" THEMES

3.4.1 *Boole's 'Fundamental Law of Thought' as a crystallization of formality and fixity*

George Boole's great merit was not only to be the first to at last set logic on the unsure path of science, but to grasp the possibility of embedding logic in, while demarcating it from, the algebra of quantities by means of a single law – a law he called 'THE FUNDAMENTAL LAW OF THOUGHT' – which would serve to capture in a single but articulate crystalline structure the most essential obstacles to dialectical logic: the *formality* and the *fixity* of conceptual determinations.

What, in Boole's eyes, justifies the 'fundamental law of thought' – the idempotence of conjunction, or the identity of A *and* $A = A$ – is precisely the thought that when our signs have a 'fixed interpretation' – when they refer univocally to some *thing* outside thought, a thing whose unconditional exteriority of thought is already presupposed in Boole's putatively non-metaphysical claim that the laws of logic are 'laws of thought, and not laws of things' – then the repetition of the thought does not affect the identity of the thing thought. 'Suppose,' for instance

> that by a definite act of conception the attention has been fixed upon men, and that by another exercise of the same faculty we limit it to those of the race who are white. Then any further repetition of the latter mental act, by which the attention is limited to white objects, does not in any way modify the conception arrived at, viz., that of white men. (Boole 2005/1853, 32).

Considerations of the same nature give Boole his justification for the law of *commutativity*, the identity $AB = BA$.

> Difference in the order of the qualities or attributes of an object, apart from all questions of causation, is a difference in conception merely. The law [of commutativity] expresses as a general truth, that the same thing may be conceived in different ways, and states the nature of that difference; and it does no more than this. (Boole 2005/1853, 21)

In sum, we can say that, for Boole, logic's operations – like the freedoms of modern democracy – are licensed to the extent that *they leave everything in its place*. They express the radical exteriority of thought to its objects, and the pure 'formality' of logic. The freedoms they grant to language – both to logical reasoning and, Boole observes, to poetic diction, understood as pure ornament – is underwritten by the indifference of the thing to which discourse is tethered.

Boole's genius is that, even as he instates this law as an unyielding precondition of logical thought, he leaves perfectly legible the conditions under which it would have be challenged. What justifies the nullity of repetition (or commutation) in logic, Boole seems to argue, is the *stability* and *exteriority* of what thought thinks with respect to the thought that think it. So long as thought and its object remain mutually exterior, it does indeed appear unthinkable that the *course of reasoning* itself would change the constitution of its object. It is on this presupposition that the laws of Boolean algebra are justified.

$M1$	$\vdash p \Rightarrow [q \Rightarrow p]$
$M2$	$\vdash [p \Rightarrow [q \Rightarrow r]] \Rightarrow [[p \Rightarrow q] \Rightarrow [p \Rightarrow r]]$
$M3$	$\vdash [\neg p \Rightarrow \neg q] \Rightarrow [q \Rightarrow p]$
$B1$	$\vdash A \subseteq (A \cap A)$
$B2$	$\vdash (A \cap B) \subseteq A$
$B3$	$\vdash [(A \cap B') \subseteq (A \cap A')] \Rightarrow (A \subseteq B)$
$B4$	$\vdash [A \subseteq B] \Rightarrow [(A \cap B') \subseteq (C \cap C')]$
$B5$	$\vdash [A \subseteq B] \Rightarrow [(A \cap C) \subseteq (C \cap B)]$
$B6$	$\vdash [A \subseteq B] \Rightarrow [[B \subseteq C] \Rightarrow [A \subseteq C]]$
$\Delta 1$	$\vdash \Delta A \subseteq A$
$\Delta 2$	$\vdash \Delta(A \cup B) = (\Delta A \cup \Delta B)$
Modus Ponens	from $\vdash p \Rightarrow q$ and $\vdash p$ infer $\vdash q$, for any propositions p, q
Substitution 1	for any M axiom, all instances of any letter may be replaced with any propositic
Substitution 2	for any B axiom, all instances of any letter may be replaced with any constant

Table 7: Axioms for $\mathbb{B}^2_\Delta$

4

GO BACK TO AN-FANG

4.1 THE PROBLEM OF BEGINNING AND THE DEAD-END OF CALCULATION

In 1812, the Nuremberg professor of mathematics, J.W.A. Pfaff, exchanged a series of letters with Hegel on the subject of the recently published *Wissenschaft der Logik*. From what we can gather from Pfaff's remaining letters – Hegel's replies being lost – the exchange hinged on a series of deadlocks in the relation between speculative and mathematical thought.

What Pfaff asks Hegel – repeatedly, it seems, and without satisfaction – is *how* one proceeds in speculative logic, *how* one step succeeds another, given that everything must proceed from an utterly empty beginning, rejecting every appeal to laws or axioms. As Pfaff sees it, however, the axiomaticity Hegel rejects at the outset takes its revenge by obscurely and inexplicitly infecting every step of the logic with arbitrariness: hidden postulates blindly absorbed from the medium of language in which the speculative logician swims (and which, if made explicit, can be justified only in a retroactive and circular fashion, as Pfaff observes).[1] 'There was some mathematical audacity on my part,' Pfaff writes, in a passage that's worth quoting at length,

> to assert that [in your *Logic*] I could not see any *proofs*, that in every case all I could see were *new postulates*, pretensions to have the new *spring* from what already exists. [...] This is why I went a bit further, and said that there was something *arbitrary* there (a lack of necessary connections between asserted propositions). [...]
>
> After an attentive reading of your letter, my persisting rebellion now has to do simply with the fact – of which I am now wholly convinced – that not only do I fail to perceive the difference between philosophical and mathematical thought, but that, furthermore, I am unable to practice the former, because each of its acts appears to me as a new postulate. The enigma is always this: HOW does the thinking subject develop a new proposition? HOW does the new, which was not yet there *in the thought*, spring from the old? HOW is synthesis possible? HOW

1 In Pfaff's second letter, we read: 'It seems to me that you are always compelled to move in a circle... Starting from a given point, you posit, in advance, these operations, etc., that are only arrived at later on; and so, in order for it all to be justified, you are compelled to return to the point from which you started. If, in your circle, you are not compelled to move in a straight line (like a mathematician), nor even, like a comet, in a parabola, but in a closed curve like the planets, like the sojourn of the gods, *this is because you have need of language, while the mathematician is mute*' (Hegel 1962, 359).

> does thought progress? HOW does one unite freedom and necessity, creation and construction, invention and demonstration? [...] This is the only question that can truly interest a man who thinks... the 'how'...; as for the results, well, there they are, *collected* in mathematics (the collections of logarithmic tables, of integral formulas, etc.). Their formation, their necessary formation, *that* is what is truly of interest. (Hegel 1962, 360)

What's peculiar about Pfaff's criticism is the extent to which, at least on the surface, it echoes Hegel's own criticism of mathematical demonstration. In the Preface to *The Phenomenology of Spirit* and in several Remarks in *The Science of Logic*† he argues that it is the *mathematical* mode of demonstration that fails to establish a necessary connection between its several steps. 'We see that 7 and 5 make 12,' for example,

> by adding five more ones to the seven, on our finger tips or in some other way; and the result is then imprinted in memory *by rote*, FOR THERE IS NO INTERNAL CONSTRAINT TO THE PROCEDURE. Similarly, we know that $7 \times 5 = 35$ by counting off on finger tips, etc. (by adding to one seven another seven and by repeating the operation five times), and the result is equally memorized. The labor of this counting, the ascertaining of the sums or the products, is relieved by ready-made addition or multiplication tables which one has only to learn by heart.' (Hegel 2010, 172)

What breaks the symmetry between Pfaff and Hegel's reciprocal criticisms of method is the Hegelian distinction between *internal* and *external* necessity. If Dominque Dubarle is correct in detecting an indirect reply to Pfaff's '*how*'s in a passage inserted into the *Logic*'s second edition, then this is, indeed, how Hegel himself marks his conflict with Pfaff: 'The question of the *how*,' he writes,

> means: in which way and manner? in what relation? and so forth, and requires the application of a particular category; but there can be no question here of a 'way' or 'manner', of the categories of the understanding. The question of the *how* is itself one that belongs to the bad practices of reflection, which demands comprehensibility, but for that it presupposes its fixed categories and is thereby assured from the start to be forearmed against the answer to what it asks. (Hegel 2010, 72)

Calculation is, for Hegel, an arbitrary sequencing of facts because, as he understands it, it relies on a separation of its necessitating principles – embalmed as axioms or 'fixed categories' – from the subject matter (whatever that might be) whose development those principles guide. This separation or abstraction leaves the subject matter of mathematics 'void of concept',

† See specimen texts # 5 and # 6.

an empty husk that can be adequately represented by a meaningless stock of letters manipulated by the calculator from without. This mode of operation, argues Hegel, is so utterly foreign to genuine, logical reasoning – which refuses any *a priori* separation of method and subject matter[2] – that any attempt to mathematise logic, genuine logic, is doomed to failure. 'It is characteristic' of the signs of mathematics, 'as contrasted with the determinations of the concept', he writes, 'that they are mutually *external*, that they have a *fixed* determination. Now when concepts are made to conform to such signs, they cease to be concepts', and so 'It is futile to want to fix [them] by means of spatial figures and algebraic signs for the sake of the *outer eye* and a *non-conceptual, mechanical manipulation*, such as a *calculus*' (Hegel 2010, 544).

The abstraction to which calculative thought subjects its material seems, at times, more than Hegel's dialectic can bear: when it appears in that process – in the Remarks interjected into his treatment of quantity, for instance – it occupies the peculiar and uncommon position of an *unsublatable* exteriority. As a *speculative* category, of course, 'Quantity' is not immune to sublation, but the calculative practice associated with it, Hegel freely admits, *cannot* be given a philosophical treatment, cannot be immanently and conceptually developed.[†] The concept can make its way past calculation only by sidestepping it, as an undialectical by-product of the dialectic.[‡]

2 As Hegel writes at the very beginning of his Introduction to *The Science of Logic*, 'In no science is the need to begin with the fact [*Sache*] itself, without preliminary reflections, felt more strongly than in the science of logic. In every other science, the matter that it treats, and the scientific method, are distinguished from each other; the content, moreover, does not make an absolute beginning but is dependent on other concepts and is connected on all sides with other material. It is therefore permitted to these sciences to speak of their ground and its context, as well of their method, in the form of lemmas; to apply presupposed forms of definitions and the like without further ado, as known and accepted; and to make use of customary ways of argumentation in order to establish their general concepts and fundamental determinations.

Logic, on the contrary, cannot presuppose any of these forms of reflection, these rules and laws of thinking, for they are part of its content and they first have to be established within it' (Hegel 2010, 23).

† See specimen text # 6.

‡ Consider, for example, the following remark, reprinted in specimen text # 6: 'The step-by-step determination of the species of calculation that we have just given cannot be said to be a philosophical treatment of them, not an exposition of their inner meaning as it were, for it is not in fact an immanent development of the concept. However, philosophy must know how to distinguish what is by nature a self-external material; it must know that, so far as this material goes, the concept can make its way forward only externally, and its moments also can be only in the form peculiar to their externality, as here equality and inequality. To distinguish the spheres to which a specific form of the concept belongs, that is, in which the concept is present in concrete existence, is an essential requirement when philosophizing about real objects. This is to prevent ideas from interfering with the peculiar nature of externality and accidentality, and the ideas themselves, because of the disproportionateness of the material, from being distorted and reduced to a formalism. Here, however, the externality in which the moments of the concepts appear in this external matter, number, is the appropriate form; since these moments display the subject matter in the conceptual form of the understanding which is appropriate to it, and also, since they contain no demand for speculative thought and therefore have the semblance of being easy, they deserve to be employed in elementary textbooks' (Hegel 2010, 177).

This is not, however, the first time the notion of unsublatable exteriority, devoid of concept and devoid of inner negativity, surfaces in Hegel's logic. What is surprising is that what appears here as a dead branch of the dialectic – pure exteriority without concept – is not treated in an even remotely similar fashion when it first makes an entrance in the *Logic*, in the scene of the *Logic*'s *beginning*, which, unsublated, 'is the ever present and self-preserving foundation of all subsequent developments, remaining everywhere immanent in its further determinations' (Hegel 2010, 49), as Hegel himself puts it. There is, indeed, no inner negativity animating the empty repetition of 'being, pure being' or 'nothing, the pure nothing' (Hegel 2010, 59), which is mediated by neither limitation, reflection or development. In the beginning, they are nothing but empty *loci*, names purged of all content, all 'determinations'. The dialectic nevertheless proceeds, and must proceed, from this not-yet negative moment. How then does the artificial separation of necessity from the bare letters of a calculus result in a dead end, something speculation can only side-step, rather than return to and retroactively *ground*? What is the *obstacle* that blocks a genuinely dialectical apprehension of calculation, an obstacle not present in the no less empty exteriority of the *beginning*?

THE BAR OF FORMALITY. The bar that separates the dead letters of calculation from the empty beginning of Hegel's logic, the bar that reflects every mathematician's 'how' *away* from the dialectic rather than into it, and which makes the exteriority of the calculative a dead end rather than an unsublatable beginning for dialectical thought, can – it seems – be summed up in a single word: it is the *formality* of mathematical thought that prevents it from being able to apprehend that logic of *absolute form* that is the dialectic.

4.2 FORMALITY AND FORMALISATION

Hegel does not waste any time distancing the project to be undertaken in the *Science of Logic* from the 'formal' conception of the discipline that we find in Kant's characterisation of what he calls 'general logic' – the logic of the Aristotelean and Stoic tradition, which he considers a finished and unshakable science – and which persists, in an attenuated and enriched fashion, in the Kantian conception of '*transcendental logic*'. This is the conception of logic which, Hegel writes,

> has hitherto rested on a separation, presupposed once and for all in ordinary consciousness, of the *content* of knowledge and its *form*, or of *truth* and *certainty*. Presupposed *from the start* is that the material of knowledge is present in and for itself as a ready-made world outside thinking; that thinking is by itself empty, that it comes to this material as a form from outside, fills itself with it, and only then gains a content, thereby becoming real knowledge.

> Further, these two component parts [...] are said to stand to each other in this order: the object is complete and finished all by itself and, for its actuality, can fully dispense with thought; thought, for its part, is something deficient and in need of a material in order to complete itself, and also, as a pliable indeterminate form, must adapt itself to its matter. Truth is the agreement of thought with the subject matter, and in order to produce this agreement – for it is not there on its own account – thought is expected to be subservient and responsive to the subject matter. (Hegel 2010, 24)[3]

Hegel's various criticisms of mathematics all pivot on the conviction that mathematical thought remains 'formal' in this sense, and it is for this reason that he takes a mathematisation of *logic* to be a way of detaining the latter in a tradition whose subversion is long overdue.[4]

This thesis, which, at the time Hegel was writing, certainly seemed reasonable enough, is nevertheless *false*. The protagonists of logic's early mathematisation, in the nineteenth century, indeed *meant* to codify a vision of logic that was, undisputably, *formal* in orientation. But the mathematical language in which they carried out this task was more truthful than their intentions, and the project of logic's mathematisation gave way to logico-mathematical *formalisation* – taking a direction very different from the steady path planned for it.

The formalisation of logic is not its submission to formality, but the process of formality's *sublation*. This is the thesis I want to defend here, today. It is *through logic's formalisation* that the science grasps a deeper and more concrete concept of logical form, one that opposes and outstrips the notion of formality dismissed in the first pages of Hegel's *Logic*, destroying the grip of this notion on logic while at the same time preserving and grounding it as the *use* of a λόγος that subtends the distinction of content and form. Since vague talk of sublation comes cheap, I'll try to make this more precise: logic's formalisation culminates in a vision of logical form that is so profoundly analogous to Hegel's own dialectic of *essence, ground, form and matter* that its products deserve to be called *formalisations of dialectical logic* – at least up to a certain point.[5]

3 As Hegel makes clear, this conception of logic is left unaffected by the 'Copernican hypothesis' by which Kant adjusts the programme of metaphysics. A methodological hiatus remains between Kant's transcendental theory of objects – according to which objects must conform to the *a priori* structures of subjectivity in order to be objects of experience in the first place – and the *formal* theory of logic, and even of 'transcendental' logic, which is just formal logic enriched with rules and axioms obtained through the transcendental study of the *a priori* conditions of experience. (See CPR A56-7/B80-1, e.g.)

4 This is how the Preface to the first edition of *The Science of Logic* begins: 'The complete transformation that the ways of philosophical thought have undergone among us in the past twenty-five odd years, the higher standpoint in self-awareness that spirit has attained in this period of time, has so far had little influence on the shape of the *logic*' (Hegel 2010, 7).

5 We could go further and say that, today, logic is in the process of answering to the Hegelian demand to become a '*formal* science, yet the science of *absolute form* [... which] has in it

4.2.1 *The process of formalisation*

4.2.1.1 *Abstract algebra and axiomatic method*

Note: if time is tight, skip to 4.2.1.3 on page 88.

In the classical era, sculpture was essentially an art of monument. The artwork joined real or mythical history at two points: through its shape, as a figurative representation of a historical figure or moment, and through the pedestal that connected it to the site that grounded its meaning. The movement of sculptural modernity, as Rosalind Krauss argues in 'Sculpture in the Expanded Field' (Krauss 1979), consisted in a gradual abstraction from these two bonds – absorbing the pedestal into the figure, absolving it of its site, and abstracting its figure from its historical or mythical referents, absolving it of representational duties. An analogous movement characterises mathematical modernity. Where geometry, in the classical era, served to represent and idealise supposedly given or intuited space, to whose evidence its axioms served to rivet it like a pedestal, with the dawn of mathematical modernity this would change dramatically. As with the emergence of abstract algebra, geometrical (and other mathematical) constructions were absolved of their representational function to become mathematics for mathematics' sake – if they retained representational usefulness, this was in some sense accidental to their validity as mathematical. And the axiom, like the modern pedestal, no longer served to attach these abstract constructions to an evidential *base* whose reality or truth could not be doubted; the positing of axioms, instead, served to mark a point of rupture with the lifeworld, their arbitrariness was made a virtue that usurped that of evidence, and they were fully integrated into the activity of mathematical construction as that which guaranteed the independence and freedom of the latter. This allowed, among other things, for the free proliferation of non-Euclidean geometries. On the algebraic front, the same attitude (often working with different methods) opened onto the study of *operations* satisfying certain formal properties (like the distributivity of one operation over another, invertibility, commutativity, associativity, etc.) without limiting *what* those operations are operating on to the narrow channels of number – or, indeed, to any pre-given domain.

These two gestures profoundly transformed the very idea of mathematics, which, when Hegel was writing, could still be described as *the science of magnitudes* without straining anyone's credulity. The epistemological break represented by abstract algebra and axiomatic method has made such an attitude impossible to maintain. The 'objects' of mathematics would, from the late nineteenth century onwards, be nothing but the correlates of an

a content or reality of its own [...] and the content is only these determinations of the absolute form and nothing else' (Hegel 2010, 523).

autonomous movement of decision and construction, constrained only by the modes of consistency to which it commits itself. [6]

4.2.1.2 *Mathematisation of logic*

I do not think that it is an accident that the first genuinely abstract algebras, the first mathematical practices to fully understand their freedom from any objective given, were those which struggled to apprehend rational thought *itself* as an object: these were the algebras of logic, that came with Boole and his successors. It is as if the objectification of the subject was a sort of threshold for the subject to cross in order to be freed of any *particular* object.

Boole's work presents a particularly interesting case, on the threshold of the practices of logical formalisation and mathematical logic in such a way that it condenses the historical contradiction between the epistemological break currently in progress and the epistemological obstacles that still hampered it. Boole's algebra of logic is the first non-numerical algebraic system, an algebra whose signs are not to be understood as referring to *quantities*. His work, however, precedes – by mere decades – the fully modern era of universal algebra, which gave the algebraist the right to construct abstract systems in complete autonomy from the domain of quantity.

Boole begins by observing that the algebraic laws that govern the addition and multiplication of quantities – the *commutativity* of the operations (that $a + b = b + a$ and $a \times b = b \times a$), and the *distributivity* of $\times$ over $+$ – also govern the *logical* operations of *disjunction* and *conjunction*, 'or' and 'and'. Logical deduction, he saw, can be mapped onto algebraic calculation, embedded in the algebra of quantities, with the variables being interpreted as ranging over classes (predicates and subjects of judgments) or propositions, rather than numbers – provided, that is, that we only interpret as logical values (classes or propositions) those values that satisfy an additional law, which Boole called *the fundamental law of thought*, which is simply

$$x \times x = x$$

or *idempotency under multiplication*, understood as the identity of 'x and x' with 'x', itself.

What, in Boole's eyes, justifies the 'fundamental law of thought' – the idempotence of conjunction, or the identity of A *and* $A = A$ – is precisely the thought that when our signs have a 'fixed interpretation' – when they refer univocally to some *thing* outside thought. 'Suppose,' for instance

> that by a definite act of conception the attention has been fixed upon men, and that by another exercise of the same faculty we limit it to those of the race who are white. Then any further repetition of the latter mental act, by which the attention is

6 To adopt another analogy, what took place in mathematics in the late nineteenth century is roughly comparable to what took place in political theory in the seventeenth, with Hobbes – who, by the way, had a few prescient insights into the future of mathematical formalisation.

> limited to white objects, does not in any way modify the conception arrived at, viz., that of white men. (Boole 2005/1853, 32).[7]

This is how Boole generally proceeds, guided by the notion that logic's operations are licensed to the extent that *they leave everything in its place*. His algebra encodes thought's exteriority to its objects, and logic's austere 'formality'. The freedoms his laws grant to language – opening the way for logical reasoning and, Boole observes, allowing poetic diction to be understood as pure ornament – is underwritten by the indifference of the thing to which discourse is tethered. 'The object,' to make Hegel speak through Boole, 'is complete and finished all by itself and, for its actuality, can fully dispense with thought' (Hegel 2010, 24). What emerges from this vision of logic is a set of algebraic laws that serve to neutralise anything we could call *syntactic time*, imposing explicitly the very structure of 'fixity' that Hegel took to be inherent in mathematical symbolism, a fixity in virtue of which 'concepts made to conform to such signs cease to be concepts'.[†] By making explicit the laws of fixity, and formulating them mathematically, Boole nevertheless succeeds in opening a gap between the condition of fixity and mathematical symbolism as such – the fundamental law of thought, after all, did not come for free, but had to be explicitly *imposed* on the movements of syntax.[8]

My reasons for emphasising this point, which for the time being seems tangential, will be clear in a moment, when we see the extent to which Girard's work hinges on a suspension of Boole's law.

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7 Considerations of the same nature give Boole his justification for the law of *commutativity*, the identity $xy = yx$. As Boole puts it, 'Difference in the order of the qualities or attributes of an object, apart from all questions of causation, is a difference in conception merely. The law [of commutativity] expresses as a general truth, that the same thing may be conceived in different ways, and states the nature of that difference; and it does no more than this. (Boole 2005/1853, 21)

† See specimen text # 6.

8 By the same token, Boole leaves perfectly legible certain conditions under which the fundamental might be challenged. What justifies the nullity of repetition (or commutation) in logic, Boole seems to argue, is the *stability* and *exteriority* of what thought thinks with respect to the thought that think it. So long as thought and its object remain external to one another, it does indeed appear unthinkable that the *course of reasoning* itself would change the constitution of its object. For this reason, though Boole's logic is indeed a mathematisation of mathematical thought, it is not yet the mathematics of *thought thinking itself*.
Boole's own interpretation of 'the fundamental law of thought' gives us every reason to anticipate the failure of every attempt to formalise dialectical logic that remains within its scope: the ultraboolean algebra proposed by Dominique Dubarle, where the relations between the universal, the particular, the singular and the void – which Dubarle sees as an essential moment of the Hegelian dialectic, misrecognised by Hegel himself – are mapped onto the algebraic relations obtaining in a $\mathbb{B}^2$ Boolean ring or lattice, and the more sophisticated lattice-theoretic formalisation of the dialectic proposed by Juan Grompone. This law serves to enforce the *fixity* of logical determinations by appealing to the *formality* of their manipulation: to the notion that they leave their object untouched, and that there is no such thing as *logical time*.

4.2.1.3 *Formal languages and logico-mathematical axiomatisation*

By the beginning of the twentieth century, the movement of mathematics' abstraction – or 'modernisation' – had come full circle. So long as axiomatisation continued to be carried out in the medium of natural language, its practitioners could not be assured of the integrity and independence of their abstractions – there is great difficulty in making it absolutely clear that hidden presuppositions and appeals to intuition did not leak into the axiomatic vessel so long as it was constructed from the dark and noisy elements of language. This is, of course, the threat that Pfaff took to unravel Hegel's claims to have made a *presuppositionless beginning* in his *Logic* – and there is some suggestion in one of Pfaff's letters that Hegel, himself, observed that even mathematics is not immune to this threat (see Hegel 1962, 361). In the element of language, even mathematics cannot be as 'mute' as Pfaff desired.[†]

The sophisticated mathematisation of logical quantification and relations that Frege and Russell developed around the turn of the century provided a means to solve the problem of *mute axiomatisation*. The extremely basic mathematical structures of *functions* and *sets* could now be expressed and specified without any explicit use of natural language (at least on the strata concerned, a complete departure from natural language being impossible, as the most serious formalists never failed to point out[9]). By this route, the axioms of a system could be stated *in a language that was itself formalised*. Inferences from the axioms to the theorems could, likewise, be encoded entirely in the formal language of the underlying logic. And the logic, itself, proved to be amenable to axiomatic treatment.

The result of this was that, under the seal of the axiomatic method, a complete and mutual embrace of mathematics and logic became possible.

The greatest protagonist of this new axiomatic method was undoubtedly David Hilbert, with the movement of *formalism* that he inaugurated, and the *metamathematical* research programme that he advanced. We have already learned something of this programme, and its dialectical significance, from John Bova and Emmanuel Barot's presentations, yesterday, so I won't spend too much time on it. For now, it can be summed up along three lines:

FIRST, Hilbertian formalism is a means for mathematising epistemological reflection on mathematics, transforming, for example, epistemological theses concerning the consistency and completeness of a mathematical system into mathematical conjectures *bearing mathematical, and not only philosophical, consequences*. The most important objective of 'the Hilbert Pro-

† See specimen text # 7.

9 As Haskell B. Curry, who we'll meet in a moment, remarks, 'The construction of a formal system has to be explained in a communicative language understood by both the speaker and the hearer. We call this language the *U-language* (the language being used). It is a language in the habitual sense of the word. It is well determined but not rigidly fixed; new locutions may be introduced in it by way of definition, old locutions may be made more precise, etc. *Everything we do depends on the U-language; we can never transcend it; whatever we study we study by means of it*. Of course, there is always vagueness inherent in the U-language; but we can, by skillful use, obtain any degree of precision by a process of successive approximation' (Curry and Feys 1958, 25).

gramme', as it came to be called, was to *prove* that the axioms of arithmetic and analysis *are not contradictory*, and to do so by means internal to arithmetic itself – means that are essentially *finitary*.

SECOND, the precondition for this reflexivity was, for Hilbert, a prior reduction of a mathematical system to 'a game of marks on paper', as some of the formalists polemically put it: a simple set of rules for rewriting formulas, beginning with a set of *axioms*, so as to transform them into *theorems*. In order to make mathematical reason an *object* for itself, the formalists saw, it is first necessary to identify it, as far as possible, with its mode of expression – seeking, in the play of inscriptions, both a materialisation of mathematical ideality and an idealisation of its material support. This gesture presses towards an increasing closure of the gap between the rational *necessity* of mathematics and the apparent *accidentality* of its expression, and it is only by seeing how and where this gulf is sealed that we can understand the legitimacy, for example, of incorporating calculations over the number of characters occurring in a given formula into demonstrations bearing on the mathematical structures those formula express – to indicate just one practice common in formal mathematics.

THIRD, formalism should thus be seen as an effort to achieve a *finite but nevertheless adequate apprehension of the infinite*. Any effective syntactic *technique*, any game of inscriptions that can actually by *played*, must ultimately be finite, even if indefinitely extendible. To be something that can actually be *written*, for instance, a proposition can only contain a finite number of symbols, and a proof only a finite number of steps. The *theme* of these demonstrations is nevertheless, in general, mathematical structures of infinite complexity and size: the multitude of sets, numbers, spaces, etc. whose *concepts* formalisation transubstantiates into techniques of inscription.

The logical formalism Hilbert used differed only slightly, and inessentially, from Russell's. Its propositional fragment contains only one rule, *modus ponens*. To this, the quantificational fragment adds the rule of universal generalisation: to infer from $A[x]$, when x is free (i.e. left completely indeterminate), $(\forall x)A[x]$. A formal system, in Hilbert's sense, would also

$$\frac{A \to B \qquad A}{B}$$

Figure 11: Modus ponens

contain a series of *axioms*: a set of logical axioms, being a more or less arbitrary selection of tautologies (true on any interpretation of their variables), and a set of axioms specific to the theory in question (specifying the properties of sets, groups, numbers, or whatever domain is being axiomatised).

The resulting logical syntax is a useful object for analysis, so far as metalogical proofs of the system's completeness (or incompleteness) and consistency (or inconsistency) were concerned, but a bit of a mess to read and painful to work with. Formalised proofs, with Hilbert, began to become mathematical objects in their own right, but through a glass darkly – only their most general features could be discerned (typically, whether or not a proof *exists* for certain mathematical formulas, such as a contradiction, or the set of statements true of some structure).[10]

THE SEQUENT CALCULUS. Everything changes with Gerhard Gentzen's invention of the sequent calculus, constructed in the dusk of the 'Hilbert programme' and the wake of its death at the hands of Gödel's incompleteness theorems, in an effort to prove that, even if we cannot demonstrate the consistency of arithmetic by finite means, we may do so by *transfinite* means – which means proving the consistency of arithmetic by means of super-arithmetic. The tactic worked, but apart from giving the Hilbert Programme the funeral procession it deserved, it was of little independent interest. The tools he constructed to prove the result, however, were revolutionary.

The sequent calculus differs from axiomatic formalizations of logic, such as Hilbert's, Frege's or Russell's, for instance, in its absence of 'axioms', in the strict sense. Or, rather, in its reduction of all axiomaticity to assertions of the form

$$A \vdash A$$

read 'A implies A', taken as a primitive principle of identity. For the rest of the logic, what the sequent calculus offers in place of axioms is an elegant collection of *rules of inference*, a pair of which attends to each *connective* in the logic in question. (I say 'the logic in question' because the sequent calculus can, in fact, be used to formalize a great number of *different* logics, as we will see.)

A sequent calculus proof takes the form of a 'tree'. At its root is placed the 'sequent' that is to be proved, which has the form

$$\Gamma \vdash \Delta$$

where Γ and Δ are (possibly empty) series of formulas. The series on the left – Γ – contains the 'premises' of the sequent; the series on the right – Δ – its 'consequences'. The left series is read (informally) as the 'conjunction' of its formulas; the right aside, as the 'disjunction' of the formulas in Δ. This difference reflects an essential logical duality, or symmetry, between conjunction and disjunction, such as is expressed – in classical logic – by the De Morgan equivalences, for instance:

$$A \wedge B \leftrightarrow \neg(\neg A \vee \neg B)$$

10 As Girard remarks, 'Before Gentzen, logic was formulated in 'Hilbert-style' formal systems; basically a lot of axioms and a couple of rules, essentially *Modus Ponens*: from A and $A \Rightarrow B$ deduce B, and *generalisation*: from $A[x]$, deduce $\forall x A[x]$. These systems have no good structural property, for instance, it is impossible to try the slightest automated deduction in the presence of *Modus Ponens*: indeed, in order to prove B (supposedly obtained by *Modus Ponens*), one must guess the premise A, which can be any formula, including B. The only setting in which Hilbert-style systems are justified is the narrow context of the incompleteness theorem: indeed, one does not seek positive properties of formalism, but rather its limits. In this very negative perspective, Hilbert-style systems do no worse than other systems and some of them have the advantage of being immediately understandable' (Girard 2011a, 41).

$$[L\neg]\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, \vdash \neg A, \Delta}\ [R\neg]$$

Figure 12: Negation in the sequent calculus

The formulas in Γ or Δ may, of course, themselves contain connectives, and the purpose of the calculus' *rules of inference* is to show how these complex formula can be systematically decomposed into their simplest constituents, branching the sequent-tree accordingly, in order to arrive at its 'leaves': axiomatic assertions of identity, of the form $A \vdash A$ (see figure 17 on page 115 for details). The tree can also be read from the top down, with each downward step representing a deductive inference.

As observed by the Belgian dialectician, Leo Apostel, Gentzen's sequent calculus (and systems of natural deduction)

> aim at the most complete solidarity possible between the definition of logical constants and the demonstrations in which those constants appear: to demonstrate a thesis is now equivalent to constructing it on the basis of the simplest theses. The proof thereby becomes a genesis, and it is no longer possible to assert that the operations are applied to concepts from the outside. (Apostel 1979, 113)

The complaint Hegel makes against mathematical demonstration in the Preface to the *Phenomenology of Spirit*[†] loses its grip on mathematical logic at this point, where its capacity to outstrip the old notion of formality is becoming increasingly apparent.

The convergence of 'form' and 'content', or the apprehension of logic's form *as* its content, is pressed further than this by Gentzen's calculus. The form and formalism of the system exposes certain deep geometrical properties underlying the 'pure play of symbols' articulated by the formalists.[11] In its syntactic structure, for instance, it *shows* how the fundamental concept of *implication* can be understood as an axis of symmetry, which, at bottom, is nothing other than logical *negativity* itself. This can be seen in the rules for the negation operator (figure 12). Moreover, as Girard points out, the very rules of the calculus 'are unchanged if one swaps left and right, provided we perform the permutations $\vee/\wedge$ and $\exists/\forall$. One knows that negation exchanges these connectives: one concludes that negation is nothing but the exchange left/right. [...] This is an obvious change of paradigm,' he goes on to argue, for 'instead of concentrating on the intended meaning of negation, one works on its geometry, which does not care about our intentions' (Girard 2011a, 47).

† See specimen text # 5.

11 'The current formalist ideology says that mathematics is a pure play on symbols. [...] The mistake might lie in a subtle shift from pure play to 'meaningless': who told us that a play on symbols is meaningless, has not its own geometry? By the way, Gentzen – Hilbert's most conspicuous follower – disclosed a structure (sequent calculus) underlying the 'play on symbols' ... and sequent calculus has its own geometrical structure-this is precisely what ludics is about' (Girard 2003, 132).

The deepest aperture that Gentzen's formalism opens onto the life of logical form is focused through his reflections of the *cut* rule of the calculus (figure 13). This rule, essentially, is *modus ponens*. It expresses our freedom

$$\frac{\Gamma \vdash C, \Delta \qquad \Gamma', C \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \textsc{cut}$$

Figure 13: The cut rule

to make use of *lemmas* in a proof, or to combine proofs together into more complex demonstrations.

Now, Gentzen's most famous and significant theorem – his *Hauptsatz* – is that CUT is *eliminable* from the calculus. Every proof using CUT can be transformed, algorithmically, into one where CUT is absent: an 'explicit' proof, which makes no appeal to lemmas, and in which every subformula decorating the 'leaves' of the proof appears as well in the proven 'root' sequent. *Cut-elimination* is of crucial importance in logic for a number of reasons. Consider, for instance, the light it throws on the problem of consistency. *Inconsistency* in a sequent calculus formalization of a logic is equivalent to the derivability of the empty sequent

$$\vdash$$

But the only possible way of deriving such a sequent is through CUT. If a system enjoys *cut-elimination*, then no sequent derivable without CUT is derivable *with* CUT, and the system is therefore consistent.

As we will see in a moment, moreover, the algorithm of cut-elimination is even more revealing than its results: its study opens onto an unsuspectedly rich and complex field of investigation, where the *dynamics* of logical form come to light.

The convergence of 'form' and 'content' brought on by the sequent calculus, or the progressive sublation of formality, which Apostel notes, encounters a revealing obstacle in what are called the *structural rules* of the calculus. These concern the global structure of proofs, the format of the logical horizon that is in place before a single logical constant is written, and determine the very nature of what it is to be a logical *proposition*. Collectively, they endow the formula with a certain *ideality*, and open a gap between the type of logical information it contains and its particular occurrences in the syntactic structure of the proof. In effect, the structural rules can be seen as a sophisticated refinement of Boole's fundamental insights as to what constitutes the logical. The names of the rules are WEAKENING, EXCHANGE and CONTRACTION.

EXCHANGE is the rule that lets us disregard the *order* of information in a given series of formulas. It is equivalent to Boole's law of *commutativity*, the collapse of syntactic succession into simulated simultaneity.

WEAKENING is the rule that lets us forget or discard 'unneeded' information, as we move upwards in the proof, or add potentially 'irrelevant' information as we move downwards.

CONTRACTION, together with weakening, transposes Boole's *fundamental law of thought* into the calculus, and works to 'idealise' formulas in the proof, so as to make them eternally reusable and available. Without contraction formula behave less like ideal propositions and more like exhaustible 'resources'.

The collective effect of these rules is to reopen a gap between the punctual location of formulas and the global geometry of the proof: it lets logical information 'wander' through the proof tree, unbinding, in a sense, logical content from the syntax that inscribes it. By and large, these rules are fundamentally antagonistic to the smooth operation of cut-elimination: so long as contraction reigns free, in particular, cut-elimination does not converge towards a unique result, a unique cut-free kernel, and retains a degree of 'algorithmic inconsistency', which we will look at more closely in a moment.

4.2.1.4 *Formal systems*

The investigation into the *dynamics* of proofs, a manner of studying logical forms that owes nothing to the form-content dichotomy, was made possible when another of Hilbert's greatest students, and perhaps the most radical of the formalists, Haskell B. Curry, pushed the movement of formalisation to its limits, and developed the concept of a *completely formalised system*.

A formalised system, in Curry's sense, is specified by (1) an initial stock of primitive entities – generally represented by letters, and (2) rules for constructing new entities by combining the primitives – new entities that are essentially *identical* to the process of their construction (Curry and Feys 1958, 15). It is in this last respect that the concept of a formal system differs from that of an abstract algebra, where, typically, 'the same element may be obtained by the operations in many different ways' (Curry and Feys 1958, 17), or, to put it another way, where the *equality* of various constructions is treated as a more or less *immediate fact*. In a formal system, a proposition expressing an equality is *itself* an entity that must be constructed by a determinate process, on the basis of syntactical differences that it does not simply neutralise.[12]

A *completely* formalised system is one that is utterly univocal, one that admits no pre-given division of its objects into *types*. Types, in logic and computation theory, are classifications imposed on terms so as to restrict their scope of application, usually for the sake of avoiding paradox, incoherence, or the 'crashing' of computational processes. It was by stratifying propositional functions through a hierarchy of types, for instance, that Russell avoided the antinomy he discovered at the heart of Frege's logic. In an untyped, or univocal system, by contrast, every entity operates on every other, and 'any construct formed from the primitive entities by means of

12 The spirit of a formal system is, in this way, suprisingly close to that of Hegel's own reflections on quasi-algebraic identities – as when he says, for example, that the identity expressed by $A = A$ conceals a difference and negative movement inscribed in the *propositional form* itself (see Hegel 2010, 360).

the allowed operations must be significant in the sense that it is admissible as an entity' (Curry and Feys 1958, 5).

For a system to be considered 'formal', it must, at least, traffic in schematic expressions whose terms can be freely substituted for other terms – whose terms play the role of *variables*, or empty slots. So long as the work of substitution is guided by an independent *substitution rule*, then the formality of the system is in some sense 'abstract', or ungrounded by the milieu in which it operates. Now, the implementation of a substitution rule, as Curry has shown, generally presupposes that a *typing system* has been put in place, usually in an 'intuitive' or implicit fashion (Curry and Feys 1958, 33). The typing of entities is thus a precondition for grasping the expressions of a formal system as *formal*, in the sense of being *schemata* that are essentially indifferent to their content, which can be interchanged at will – provided that the right *type* of content is inserted into their empty slots. It is this notion of formality – a notion close to, and in any case necessary for, the one that Hegel rejects at the outset of his *Logic* – that collapses under the force of *complete formalisation*, which demands a univocity of logic incompatible with the presupposition of types.

Apostel sums the situation up well in his Appendix to *Logique et dialectique*, where he recounts how

> After mathematical logic's period of construction, which culminates with Russell, followed a period of formalisation, which transformed the systems of the preceding epoch into systems speaking of signs and combinations of signs. But, in a characteristically dialectical transformation, with the extreme formalist Haskell Curry, those systems, which in Carnap speak of signs, are transformed into systems speaking of indeterminate [*quelconques*] structures, of objects in the most general sense of the word. After having sought to exhaust the richest content, the development of logic is restrained to the narrowest form, only to once again overflow into the most complete generality.

He then goes on to articulate a suggestion that I will try to develop in some detail in the last section of this presentation, writing that

> Nothing more clearly illustrates the dialectic of form and ground that Hegel recognised as essential than this process, which he gave short shrift in criticising, at certain places in his 'Logic', the diminution to which the pure concept is subjected when the Euler circles reduce it to merely sensory. It is the inevitable and constant transmutation that he tells us of in his dialectic of essence, a dialectic at work before his very eyes, and which he willfully, without reason, discounts.
>
> We conclude,

he writes,

> that if we examine the profound affinity between form and matter in the Hegelian system, then formal and dialectical logic should be two faces of the same unity. (Apostel 1979, 117)

To unfold this suggestion, we must look at things more closely.

The completely formalised systems at the heart of Curry's research programme (in the mid-twentieth century) were what he called *combinatory logic* and the related system (which, strictly speaking, is still 'less' formalised than combinatory logic, but still essentially untyped) called the *lambda calculus*, as developed by Alonzo Church. Without going into too much detail, we can say that the main purpose of these systems is to analyse the idea of an *effectively calculable function*, in a struggle to give exact meaning to the notion of determinate, finite procedure at the heart of the formalist enterprise. The concept that emerged was, in effect, that of a *computer programme* in its most basic and abstract form (Alan Turing's universal abstract machine was, in fact, a contemporary, parallel and provably equivalent prong in this campaign).

With W.A. Howard, Curry went on to discover an isomorphism that would link together formalisations two rebellions against the formal: the lambda calculus developed by Curry and Church and the sequent calculus developed by Gentzen. The Curry-Howard isomorphism, as it came to be called, is a strict correspondence between *proofs* in the sequent calculus (or natural deduction) of intuitionistic logic and functions, or *programmes* in the lambda calculus. Through its lens, the procedure of CUT-ELIMINATION – which justifies the combination of proofs and the use of theorems as *lemmas* in proofs of greater complexity, a practice indispensable to mathematical practice – fully reveals its dynamic and computational significance: Each CUT, or combination of proofs – as lemmas in a more complex proof – becomes a combination of programmes whose *execution* is the process of cut-elimination itself.

This correspondence, however, demands a prior bridling of the lambda calculus with an apparatus of *types* – it is, strictly speaking, an isomorphism of logic with the *typed* lambda calculi – and so still remains at the *threshold* of logic's sublation of its own formality, a threshold which it, here, outstrips only in part. Types, here, are imposed to prevent certain paradoxical and self-referential formations from arising in the calculus, so as to ensures that *every programme terminates, or yields a result rather than 'crashes'*. The Curry-Howard isomorphism, which correlates logical *proofs* with *programmes*, also correlates programme *types* with *propositions*. A proposition, from this point of view, is perceived as a set of proofs, or, more precisely, as a *classification* determining which proofs can be combined, or CUT, with which. The proposition, on this view, finds its meaning not by referring to a model, or pre-given state of affairs, but according to the formal dynamics obtaining between the ways it can be proven and the ways those proofs can be used.

4.2.1.5 *Syntax and semantics, and the relapse into formality*

Even if these developments herald a radical and thoroughgoing sublation of the notion of logic's formality through its formalisation, and a conception of the discipline in terms of the action of form on form, an action that develops its own content immanently and dynamically, without any need of extraneous hypostases, the image of formality would nevertheless continue to detain logic's historical development through the notion of *semantics*, or through a certain use of that notion, which, in *The Concept of Model*, Badiou has diagnosed as ideological (Badiou 2007).

The great bastion of abstract formality in modern mathematical logic is what Alfred Tarski called the *semantic conception of truth*.[13] In non-technical terms, this is the conception of truth that proceeds according to definitions of the form

'snow is white' is true IF AND ONLY IF snow is white.

When it comes to defining the truth conditions of formal, compound propositions, this conception operates by equivalences such as

$$\begin{aligned} A \wedge B \ \texttt{is true} \ &\Leftrightarrow \ A \ \texttt{is true} \ \text{AND} \ B \ \texttt{is true} \\ A \vee B \ \texttt{is true} \ &\Leftrightarrow \ A \ \texttt{is true} \ \text{OR} \ B \ \texttt{is true} \\ \neg A \ \texttt{is true} \ &\Leftrightarrow \ A \ \texttt{is} \ \text{NOT} \ \texttt{true} \\ (\forall x)A[x] \ \texttt{is true} \ &\Leftrightarrow \ \text{FOR ALL} \ \texttt{objects} \ \mathfrak{x} \ \texttt{in any model,} \ A[\mathfrak{x}] \ \texttt{is true} \end{aligned}$$

and so on. These are not *quite* the empty tautologies that they appear to be at first sight, since there *is* a difference between the left and right sides of the equivalences.[14] The expressions which are *named* and whose truth is asserted on the left are expressions in the 'object language', the syntax under investigation. But as Tarski and, before him, Gödel, showed, a non-contradictory syntax *cannot* express its own truth predicate. If it could, it would be exposed to the famous 'Liar Paradox' – through a simple act of diagonalisation it could construct statements that amount to saying '$A \Leftrightarrow \neg(A \ \texttt{is true})$' – statements which are at once true and false (or

13 For a clear overview of Tarski's theory of truth, see Tarski (1969).

14 Regarding the 'snow is white' example, Tarski offers the following clarification: 'It might seem at first sight that [this example], when regarded as a definition, exhibits an essential flaw widely discussed in traditional logic as a vicious circle. The reason is that certain words, for example 'snow', occur in both the definiens and the definiendum. Actually, however, these occurrences have an entirely different character. The word 'snow' is a syntactical, or organic, part of the definiens; in fact the definiens is a sentence, and the word 'snow' is its subject. The definiendum is also a sentence; it expresses the fact that the definiens is a true sentence. Its subject is a name of the definiens formed by putting the definiens in quotes. [...] Hence the word 'snow', which undoubtedly occurs in the definiendum as a part, does not occur there as a syntactical part. [...] However, words which are not syntactical parts of the definiendum cannot create a vicious circle, and the danger of a vicious circle vanishes' (Tarski 1969, 64). To further dispel fears of a vicious circle, Tarski points out that the definition, ' "snow is white' is true if and only if snow is white" ' could just as well have been written: "The string of three words, the first of which is the string of the letters Es, En, O and Double-U, the second is the string of letters I and Es, and the third is the string of the letters Double-U, Aitch, I, Te, and E, is a true sentence if and only if snow is white" (ibid.).

which, if Boole's fundamental law is suspended, *oscillate* between truth and falsehood). The truth predicate for a syntax $\mathscr{S}$ must therefore be expressed in *another syntax*, which would be a *metalanguage* with respect to $\mathscr{S}$ (call it $\mathscr{S}'$). The connectives on the right hand side of the equivalences just mentioned must therefore be connectives in the *metalanguage*. When we define the truth of $A \wedge B$ in $\mathscr{S}'$, we *use* the metalinguistic 'AND', but only mention or name the object-linguistic *and* (denoted by the wedge symbol, $\wedge$). Of course, the metalanguage, so long as it is non-contradictory, is subject to the same limitations. *It* must call on yet another metalanguage to express *its* truth predicate, and so on, in a badly infinite progression that, as Hegel would say, *expresses* the contradiction at the heart of the semantic conception of truth by the very flight that avoids it *without resolving it*.[15]

Equivalences of this sort, shuttling across the badly infinite hierarchy of metalanguages, are neither false nor useless. They perform the invaluable service of mediating between various syntactic constructions, between explicit formalisms and informal, historically given mathematical structures through which the formalisms can be 'interpreted': they open the channels of communication on which the theory of *models* depends. What *is* both false and useless is the notion that such a procedure in any way serves to explicate the logical forms themselves.[†] Semantics is a good way of *using* logic, of *applying it* to material that comes from elsewhere (even if it is simply furnished by a neighbouring domain of logic itself). But its efficiency is bought at the price of logic's *transparency*, and to attempt to *explain* a syntax through its semantics is to get lost in a hall of mirrors.

The way in which the syntax/semantics opposition is often handled bears further traces of the formal image of logic, which Hegel attributes to Kant, in framing it as a relation between a *subjective technique of representation*, on the one hand – finitary syntax – and a *represented object*, on the other – a (usually infinite) structure, interpreted as the semantic content or model of the syntax. This notion of *syntax as subject* is one which neither Hegel nor Girard will reject entirely. The question for both, rather, will be how to radicalise it so as to release it from the image of formality, over-awed with semantics. What syntax will retain of the subject will be, moreover, precisely what is occluded by the formal image of logic that charges it with little more than representing the specular image of its own semantics: its temporal, procedural aspect (the dimension of syntactic time), its autonomy or self-legislation, and its 'existential' negativity.

4.3 THE IDEA OF TRANSCENDENTAL SYNTAX

The same *modus operandi*, and the same formal image of logic found in the semantic conception of truth animates the canonical versions of *transcen-*

15 'The process to infinity is in general the expression of contradiction [...] *not the resolution of it*' (Hegel 2010, 191).

† This is the butt of Girard's 'Broccoli logic' joke. See specimen text # 15.

dental logic, which tout themselves as an inquiry into logic's conditions of possibility or grounds.

Hegel's double criticism of the Kantian programme of transcendental logic is well-known: his transcendental logic preserves unquestioned the *form* of logic received from the scholastic tradition, and goes on to bar any explication of those forms *in and for themselves* by keeping transcendental logic an essentially *formal* inquiry – abiding by the prejudice that the forms of thought are incapable of determining their own truth and internal content:

> Just as the Kantian philosophy did not consider the categories in and for themselves, but declared them to be finite determinations unfit to hold the truth, on the only inappropriate ground that they are subjective forms of self-consciousness, still less did it subject to criticism the forms of the concepts that make up the content of ordinary logic. What it did, rather, is to pick a portion of them, namely the functions of judgments, for the determination of categories, and simply accepted them as valid presuppositions. Even if there were nothing more to the forms of logic than these formal functions of judgment, for that reason alone they would already be worthwhile investigating to see how far, by themselves, they correspond to the *truth*. A logic that does not perform this task can at most claim the value of a natural description of the phenomena of thought as they simply occur. (Hegel 2010, 525)

For Hegel, it will be through a reflective subversion of logic's *formality* that the *forms*, themselves, can be scientifically grasped – which is to say, rationally transformed and not merely repeated. On this point, Girard's approach is infinitely closer to Hegel's than to Kant's, as we will see.[16]

Husserl is no better, where the question of actually *grounding* the syntactical structures of logic are concerned. We can consider, for example, his attempt to explicate the *transcendental* grounds of the law of non-contradiction

$$\vdash \neg(A \wedge \neg A)$$

by pointing out that

> whoever thinks of two judgments as judged by someone, and, on making them distinct, recognises that one *contradicts* the other, *cannot do otherwise* than *deny* the *conjunctive* judgment formed out of them. (Husserl 1969, § 75, 190)

It is clear that this game of mirrors leads nowhere, other than on a badly infinite flight away from the syntax itself, whose blind repetition in the (meta-) language of subjective consciousness does nothing to explicate any grounds whatsoever. It's true that the quest for logic's grounds never ceases

16 See also Hegel's remarks on Kant's transcendental logic at Hegel (2010, 40-42).

to concern Husserl, but the effect of this quest on his work is consistently centrifugal. Where the forms of logic should be explicated *in and for themselves* remains the empty centre of his entire corpus, with every attempt to penetrate logic's depths winds up being flung outwards into the living context of logic's *use*. This centrifuge has been quite productive for transcendental phenomenology in general, but it must be admitted that nothing like a transcendental *logic* has actually been carried out in those quarters.

It is from this point of view that we must recognise Girard's project of *transcendental syntax* as a radical departure from the Kantian and Husserlian traditions, traditions with which, I think, he is too quick to associate himself. The project was first articulated under that name in 2011 (in Girard (2011b) and again in Girard (2012)), but retrospectively embraces the whole of his research since the mid-1980s.

Girard's own methodological categories help to explicate the distance he takes from the Kantian and Husserlian methods of transcendental logic. In 'Syntaxe transcendentale, manifeste', he delineates four strata – or 'infernos' – of logical investigation.

-1 THE ALETHIC INFERNO: this is the realm of 'truth', semantically conceived. Within limited contexts – those where the infinite or the numerical is not at stake, and where indirect modes of self-reference cannot take shape – it serves as a baggy garment for *provability*. Propositions are alethically equal when they are equi-demonstrable; semantic 'truth', where it works best, is effectively the *quotient* we obtain from the syntax when we identify all provable statements with *truth*, in the abstract, and all refutable statements with the equally abstract *false*. What disappears in this collapse of *proofs* into *provabilities* is the *action of proof* itself. Alethic or semantic interpretations cannot discern between two syntactically different proofs of the same theorem, and so quite a bit falls into the blind spots of this particular perspective. What is retained is a bleak and impoverished *repetition* of the initial syntax – or the quotient obtained when one identifies all equiprovable formula.[17] This approach to logic is best suited to its CLASSICAL variant: the traditional 'laws of thought': the law of the excluded middle, non-contradiciton, Boole's fundamental law of thought, etc. all find a fairly straightforward justification from this point of view. Kantian and Husserlian transcendental logic plays itself out entirely on this plane, hence their permanent indenture to the 'formal' image of logic, which they might supplement with additional content (gleaned from various critical or phenomenological reflections on the *a priori* conditions of experience), but whose horizon they perpetually fail to outstrip.[18]

17 As in a Lindenbaum algebra, which is a lattice structure in which we take the equivalence classes of each formula as the elements of the lattice, defined as $|\alpha| = \{x : x \Leftrightarrow \alpha\}$, and the ordering relation $|\alpha| \leq |\beta|$ to hold whenever $(\exists x \in |\alpha|)(\exists y \in |\beta|)(x \Rightarrow y)$. A Boolean lattice can be seen as the Lindenbaum algebra of classical propositional logic.

18 Things only get worse for classical logic, from the point of view of the commited proof theorist, with Gödel's incompleteness theorems. There have traditionally been two kinds

-2 THE FUNCTIONAL INFERNO: this inferno goes somewhat deeper than the preceding by attending to the *action* of proving, interpreted in terms of the functional transits linking one proof to another. Rather than interpret $A \to B$ as IF A IS TRUE THEN B IS TRUE, for instance, it interprets it as a *function* or map sending a proof of A to a proof of B. Category theory and the lambda calculi, mediated by the Curry-Howard isomorphism, make invaluable contributions to the exploration of this inferno, whose native logic tends to be INTUITIONISTIC (i.e. rejects the law of the excluded middle, and favours a somewhat asymmetric relation between premises and conclusions).[19]

-3 THE INTERACTIVE INFERNO: the dynamics of *cut-elimination* are what fuel the third inferno. Gentzen and Curry's advances furnish crucial prerequisites for its exploration, but since cut-elimination is essentially a phenomenon of *logical symmetry* – the symmetry of negation, of $\neg\neg A$ with A – its adequate investigation is quite limited by the intuitionistic framework, which falls short of this symmetry. It's no use retreating to classical logic at this point, since classical logic maintains the symmetry of negation at the price of a cut-elimination process that is algorithmically incoherent (non-convergent) – it is not in general possible to reduce a given classical proof to a unique cut-free kernel. Though alethically non-contradictory, classical logic has for this reason been said to be *computationally inconsistent*. What Girard showed in 1987 was that a far deeper and more systematic critique of classical logic than intuitionism offers is needed in order to grasp the phenomenon of cut-elimination adequately. In particular, he showed that where classical logic fails, where it 'leaks', is in its collapse of the subtle differences of logical time – a collapse enshrined in Boole's fundamental law, which identifies A with its repetitions. [20] In order to grasp cut-elimination in a way that is both computationally sound –

of reaction to the incompleteness results: (1) the transcendent, semantics-oriented reaction, which, when seeing that there are 'unprovable truths', cries, 'So much the worse for provability! Truth flies free!' (2) the immanent, syntax-oriented reaction, which responds, 'So much the worse for truth! It's too clumsy to capture even *provability* without stumbling and overstepping its bounds!' From the latter point of view, the existence of undecidable statements makes a plea against one of classical logic's emblematic laws: the law of the excluded middle, since it does not survive the shift from semantic truth to syntactic provability in any sufficiently powerful system, where '$\vdash A$ OR $\vdash \neg A$' is refutable. For the intuitionist, who *identifies* A with the set of A's proofs, $A \vee \neg A$ is not a valid law.

19 I.e., in the sequent calculus, intuitionistic logic is distinguished from classical logic only in this: one may have at most *one* formula on the right side of the sequent, while the left may contain any finite number of formula. This is enough to prevent the law of the excluded middle, or the identity of A with $\neg\neg A$ from being provable in the intuitionistic calculus, since proving either of these propositions requires the use of *contraction* on the right side of the sequent (see 18 on page 115 for an illustration of the proof of the law of the excluded middle in the classical sequent calculus, and notice that this proof could not be written without placing more than one formula on the right hand side).

20 The reason intuitionistic logic has a convergent cut-elimination procedure (an algorithmically coherent dynamics of proofs) is that it effectively prohibits the use of contraction and weakening on *one* side of the sequent. This, in fact, amounts to an asymmetrical notion of logical time, as Kripke models of intuitionistic logic help to illustrate: whereas in classical

traces a convergent dynamic process – *and* fully symmetrical, it is necessary to *fully suspend the structural rules of the sequent calculus* – the structural rules which, in embryonic form, in Boole, were the very buttresses of the *formality* and *fixity* of logical determinations. This leads to *linear logic*, a logic where attention is paid to the *occurrences* of formulas in a proof rather than the mere 'types' that occur, where something like logical time once again becomes visible, and where negation is no longer an operation that inverts 'truth values' (swaps the true for the false, and vice versa), but which expresses something like the relation between an action and *its* reaction – its 'determinate negation' as a Hegelian reading might have it. Implication in linear logic, moreover, is no longer – as in classical or intuitionistic logic – a relation expressing the entailment of one timeless proposition by another, but the action of one logical determination *passing into* another. If we combine A and $A \multimap B$, for example, we obtain B, but in doing so, we *lose* A – A is 'consumed' in the process of producing B, so to speak.

Negation in linear logic lends itself well to an interpretation in terms of the alternation of agency between two players in a 'game', or two adversaries in an argument or struggle. For this reason, the linear-logical study of the dynamics of cut-elimination lends itself quite naturally to certain varieties of *game semantics*.

-4 THE DEONTIC INFERNO: The concrete problem that compels the descent to the fourth inferno, in a nutshell, is this: I've mentioned that the second, 'functional' inferno surpasses the explanatory power of the first, by opening up the study of *proofs*, rather than just the *provability* of propositions, by interpreting the connectives (and, or, etc.) as functional or algorithmic modes of combination of proofs. The deepest aperture into this field is provided by the dynamics of *cut-elimination*, but cut-elimination seems to pivot on a logical symmetry – the symmetry of logical negativity – that the *intuitionistic* structure of the second inferno stymies. A deeper grasp on logical symmetry is provided by shifting to the third inferno, the interactive, where *linear logic* prevails. But something about negation still escapes this perspective. According to the functional interpretation, the negation of A is understood as $A \to \bot$, as a function sending a proof of A to the proof of a contradiction or falsehood. But what could the 'proof' of a contradiction be, *how* could it be? The notion seems to be void of constructive content, and it was for this reason that the radical intuitionist – and, in fact, the proponent of an idealism that would attempt to synthesise Bergson and Hegel (Griss 1946)

logic, propositions are always either *eternally true* or *eternally false*, in intuitionistic logic they are either *eternally proven* or *currently unproven*. Linear logic symmetrises logical time, by distinguishing statements that are *currently active* in a proof from those which are *currently inactive*. I discuss the intuitionistic image of logical time to some extent in Fraser (2006).

– G.F.C. Griss made a motion for the complete rejection of negation from logic (Griss 1948-49). But as the moderate intuitionist Arend Heyting pointed out, this would not only destroy logic's *formality*, but forbid any means of reconstituting it.[21] The notion of a 'proof of the contradictory' remains, in Girard's words, the 'empty pivot' of the constructive understanding of logic (Girard 2003, 131,145).

To grasp the concrete, procedural content of logical negativity, we must allow proofs to interact with something like 'false proofs' – but false proofs cannot be produced if we forearm the activity of 'proving' against contradiction in advance, by an apparatus of rules and types. Nor is it enough to tamper with those rules so as to allow 'true contradictions', as Priest and the other paraconsistentists try to do, since this would be, as Emmanuel has already pointed out, to neuter the element of negativity whose deep logical structure we are trying to encounter. The grail of the fourth inferno, therefore, would be a form of contradiction or opposition that is *prior* to, and, indeed, constitutive of logical rules – prior, in fact, to the very idea of *truth* or *provability*, and hinging on a notion of negation that would not mean *refuation* but *recusal* or *challenge*.

This is what leads us to the fourth inferno, which Girard calls 'deontic', and it is on level that we begin to sense the intention of his downward flight. The study of 'interaction' in the third inferno is indeed a study of form acting on form, form disgorging its content as the dynamics of that action. It is nevertheless constrained by a normative apparatus of *types* – fixed determinations arming the logician in advance against the question of how form, left to itself, will perform its operations. The 'how' the logician asks the third inferno is still, to some extent, the 'how' which, in Hegel's words, 'means: in which way and manner? in what relation? and so forth, and requires the application of a particular category,' and so 'presupposes its fixed categories and is thereby assured from the start to be forearmed against the answer to what it asks' (Hegel 2010, 72). But here, in the fourth inferno, whose exploration is animated by a desire to see what is *behind* the 'fixed categories' of the typing apparatus constraining logic,'there can be no question here of a 'way' or 'manner', of the categories of the understanding' (Hegel 2010, 72).

21 'According to Brouwer, and also in this point Griss agrees with him, logic is not presupposed in mathematics, but yet in mathematical reasonings we can detect certain regularities the study of which belongs to logic. One might suppose that the logic of negationless mathematics is obtained by omitting from the intuitionistic logic all those rules which contain a negation, but this simple method does not work because it is necessary to impose a restriction upon the conjunction of two propositions. "*a* is a square" is a proposition; "*a* is a circle" as well, but "*a* is a square and *a* is a circle" is not a proposition. Thus the conjunction of two propositions p and q can only be formed if an instance is known which makes this conjunction true. In other words, the notion of a well-formed formula in this logic will depend upon the interpretation; a purely formal logic is impossible' (Heyting 1954, 94-5).

The motive of Girard's descent is the motive of Platonic dialectic: to seek the unconditioned conditions for the possibility of λόγος, through a meticulous destruction of its presuppositions, a careful 'oblivion of syntax' as he calls it (in Girard 2012). ('My dream,' he told me when I met him in February, 'would be to have no axioms at all'.) This is, ultimately, I think, an endless task, but one which can nevertheless serve to orient logical research. The apprehension of what is at stake in this task, for transcendental syntax, thus demands a return to the question of *beginning*.

To strip logic of presupposed *types* means to move, as close as possible, to the dimension of bare literality that formalisation progressively lays bare – to the grounds of symbolic manipulation, no longer made subject to what Hegel called the 'fixed' and 'formal' determinations of the understanding. It will be from this perspective, to the extent to which is is actually obtainable in mathematical practice, that transcendental syntax will seek to reconstitute logical formality, to show how it emerges, immanently, from the empty exteriority of the beginning it sights.

4.4 AN-FANG: INTRODUCTION TO LUDICS

It's a well-known custom in the Platonic dialogues to fall back on myth and allegory to guide thought's movement towards the the obscure foundations of λόγος. A similar practice is found in Girard's writings, executed in a somewhat more fragmentary fashion, drawing on snatches of poetry or fiction with little explanation before proceeding with (often daunting) technical elaborations.[22] The text of 2001's 'Locus Solum: From the rules of logic to the logic of rules' – the first, and, for all its labyrinthine complexity, most elementary text to broach the 'deontic inferno' – is exemplary of this practice. There is, first of all, of course, the title's allusion to Roussel's novel, *Locus Solus*, or 'Solitary Place', from which Girard draws the pun of *Locus Solum*, 'Solely the Place', or 'only the place matters'. Now, neither Girard nor I have actually gotten around to reading Roussel's book, so I won't spend much time reconstructing this particular allusion.

More to the point is the text's epigraph, which repeats the first lines of Cordwainer Smith's short story, 'The Dead Lady of Clown Town'. It is an extraordinarily strange text to find at the head of an essay in mathematical logic, an essay to which the journal *Mathematical Structures in Computer Science* devoted an entire issue. It is also what marked my first impression of Girard's work, when I encountered it entirely by accident about five years ago, before I could understand a word of it.

This is it:

> Go back to An-fang, the Peace Square at An-Fang, the Beginning Place at An-Fang, where all things start (...) An-Fang was near a city, the only living city with a pre-atomic name (...)

22 For example, before introducing *linear logic*, in 1987, as 'a logic behind logic [...] a continuation of the constructivization that began with intuitionistic logic', we are reminded of the surrealist Audibert's invocation of '*La secrète noirceur du lait*' (Girard 1987, 2).

> The headquarters of the People Programmer was at An-Fang, and there the mistake happened: A ruby trembled. Two tourmaline nets failed to rectify the laser beam. A diamond noted the error. Both the error and the correction went into the general computer. (Epigraph to Girard 2001)[†]

It's all here, the entire ludic formalism, in allusion, pun and code, and I can't think of a better way to introduce it than to go through this allegory line by line.

Go back to An-Fang, the Peace Square at An-Fang, the Beginning Place at An-Fang, where all things start... When Goethe's Faust translated the Gospel of John, he chose to subvert the Lutheran translation, '*Im Anfang war das Wort*' with the subversively romantic '*Im Anfang war die Tat*', in the Beginning was the Act. If there is a contradiction between the Gospels of Faust and John, it is effectively sublated in the formalist Hilbert's famous pronouncement, that

> The solid philosophical attitude that I think is required for the grounding of pure mathematics – as well as for all scientific thought, understanding, and communication – is this: *In the beginning was the sign* [*Am Anfang ist das Zeichen*]. (Hilbert 1998, 202)[23]

or the word as act, as technique. What we have here is a bit different: in the beginning is *the beginning*, or rather the *signs*, "the beginning" – but these are not the type of signs that, in formalism, are preserved from ambiguity and ensure certainty. They are the material of a multi-lingual pun! *Im Anfang ist "An-Fang"*': broken Chinese for 'Peace Square'! ('*An*' = 'peace', '*Fang*' = 'square', but in the sense of an equilateral rectangle rather than a plaza or place).[24] What's the meaning of this slip of logic into *lalangue*, at the very moment when its *beginning* is at stake?

Ludics approaches the Hilbertian sign from the angle of its *locativity*, present but scarcely visible in the formalist framework. 'Locativity' designates the quasi-material trace of the sign's occurrence, subtracted from the type of sign that occurs (from its codified uses). It concerns the place or *locus* where the sign is inscribed – the *letter* rather than the *signifier*, to use slightly similar Lacanian distinction. Insofar as their can be ambiguity in syntax, or interference between syntactic expressions – interference normally avoided, for instance, by renaming variables before combining two syntactic expressions in a third, so that there is no confusion as to what binds them – this is due to the *locative* aspects of the sign. The condition of possibility of a pun – and a bilingual pun above all – is precisely the *locative* aspect of signifiers: the

† The excerpt is reproduced as specimen text # 16.

23 Michael Detlefsen suggests that, as an allusion to Goethe's 'In the beginning was the act', Hilbert's remark might have been meant as a jab at Brouwer's (Faustian?) maxim that 'Mathematics is created by a free *action* independent of experience'. See Detlefsen's helpful lecture notes on formalism at http://philosophy.nd.edu/people/all/profiles/detlefsen-michael/documents/InSearchofFormalism.L3f.HO_000.pdf.

24 Thanks to Tzuchien Tho for pointing this out to me.

accidental but always possible circumstance that the phonemes the German language uses to express *beginning*, for example, might interfere with those used to designate 'Peace Square' in mock-Chinese. Ordinarily, the use of formal languages abstracts from this aspect of the sign, according to techniques that Girard calls 'spiritual', as opposed to 'locative' – techniques concerned with *structure* alone, whereby everything is identified with its isomorphic images. Ludics takes the locative aspects of syntax seriously, it *begins* with the locative, and seeks to study *how* what has traditionally been taken as accidental dross can contribute to the very structuration of logic's 'spiritual', trans-locative dimension. This, by the way, is the meaning of the monograph's title: nothing takes place but the place...

An-Fang was near a city, the only living city with a pre-atomic name... Since Girard makes free use of ellipses in this epigraph, we should take seriously every line he makes a point of including, even if this means stretching the pun at some points. The stretch is worth making here, exploiting this reference to pre-atomic history to anticipate ludics's subversion of logical *atomicity*.

Logical *atoms* are unanalysable propositions, normally indicated by naked letters, without any connective symbols or grammatical structure. In a sequent calculus proof, one works by decomposing the sequent to be proved – following the logical rules assigned to each connective – until one reaches atomic propositions, and the proof or analysis terminates. If, at this point, every 'leaf' of the proof-tree contains a sequent of the form

$$A \vdash A$$

then the sequent at the root is *proved*, it is a theorem. The notion that every meaningful proposition can ultimately be decomposed into *atoms* has a long history. It is one of the fundamental theses of Leibniz, one with which Hegel broke emphatically, and one reasserted against Hegel by Russell, who took the practical need for atoms in mathematical logic to show that Hegel was mistaken in rejecting semantic and logical atomism. If we do not presuppose the existence of *atomic propositions*, Russell argued, the acquisition of knowledge has *nowhere to begin*.[25]

Of atoms, however, ludics has none. At bottom, it does not even operate on *propositions*, and retains only the *loci* of propositions as its base. The

25 As A. Ushenko explains, 'the question at issue,' in much of Russell's dispute with Hegel, 'is the principle of atomicity at the basis of analysis. According to the principle of atomicity, a single statement can be a complete unit of meaning and therefore completely true: if we know the logical structure, or syntax, of the statement together with the dictionary meanings of its connotative constituents, we have the necessary and sufficient equipment for the understanding of the statement as a whole without the aid of a wider context [...]. If our concern is knowledge, or true propositions alone, the principle of atomicity is summed up by the dictum that a single true statement is completely true even though it can never be the complete truth. Another way to state the dictum is to observe that the oath which binds a witness to tell 'the whole truth' forces him into perjury, although he can, if he wants to, tell the court 'nothing but the truth'. Russell proceeds to argue that Hegel and the enemies of analysis cannot account for the *beginning* of knowledge, since knowledge must begin with a single sentence, and continue through a sequence of single sentences. (Ushenko 1949, 108-9)

basic entities of ludics are what Girard calls *designs*, or strategies. They resemble proof-trees in the sequent calculus, except that while a proof-tree has a sequent at its root, such as

$$\Gamma \vdash \Delta$$

where Γ and Δ are sequences of propositions, the root of a design is a sequent or '*pitchfork*', of the form

$$\xi_n \vdash \xi_{I=\{m_1 \dots m_i\}}$$

which is a relation between a *locus* (which we can think of as a certain *action* or gesture) and collection of several other *loci* (which we can think of as anticipations of certain *re*-actions).

In a sequent proof, we would proceed by isolating the main connectives on each side of the sequent – they might be conjunction, disjunction, or whatever – and determining the structure of the immediately upper nodes of the proof tree in accordance with the rules assigned to those connectives. A design has no set of rules to follow. Instead, it is allowed to branch out in any fashion we like (so long as a certain rhythm between 'positive' and 'negative' moves is preserved – but this technicality isn't altogether crucial here). Instead of having each proposition branch out into sub-propositions – for example, having

$$\Gamma, \Delta \vdash A \wedge B$$

branch into

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}$$

– we have each locus branch into its *subloci*. But since we began with bare loci, without any propositional structure imposed on them in advance, there is nothing written at the root of a design that preordains *how* the design will decompose or ramify. The process of ramification may indeed carry on indefinitely, in an infinite analysis of the initial loci – actions and anticipations – that never encounters a single *atom*.

The headquarters of the People Programmer[26] *was at An-Fang and there the mistake happened. A ruby trembled...* Since there are no atoms to be found in designs, there are no *axioms* either. A kernel of what we could call *axiomaticity* nevertheless persists. It persists in the form of a pure, abstract arbitrariness, as pure contingency, under the name of the *daimon*.

The *daimon* is the rule of ludics that allows any locus whatsoever to be inscribed without justification, without being subject to further inquiries from the 'opponent'. It is the only action by which the branching of a design may be brought to a halt. Formally, it is written

$$\frac{}{\vdash \xi_n} \maltese$$

26 The reference to 'the People Programmer' isn't at all irrelevant here – it hints, in fact, at a certain concern Girard has with the transcendental conditions for, and 'formatting' of, logical subjectivity – but it would take me too far afield to go down this path today, so we'll cut straight to the trembling ruby and get on with it.

It can be seen as the bare form of the axiom, purified of all content and every appeal to evidence. It is an interruption of thought, or a brute decision.

Allowing designs to have recourse to the *daimon* opens the field of logic considerably. Its intervention populates the arena not only with the locative skeletons of *proofs*, but also with the bones of 'paraproofs' – unfinished proofs and sophisms. These are what occupy the 'pivot' left vacant by functional proof theory (see Girard 2003, 131, 139-145), and open logic to a complete duality between proofs and tests (every design has a counter-design). To give some illustration of the expressive capacities this affords the formalism, we could turn to Myriam Quatrini recent work, where she shows that most of the sophistical stratagems Schopenhauer catalogues in *The Art of Always Being Right* – stratagems which this arch-enemy of Hegel liked to call *dialectical* – can be transcribed into ludics with the *daimon*'s aid.

Two tourmaline nets failed to rectify the laser beam. A diamond noted the error. Both the error and the correction went into the general computer… The 'error' in question, of course, is the *daimon*, which we have just invoked. The 'general computer', on the same allegorical reading, is nothing other than ludics itself, or 'dialectics', as Girard suggests the formalism could have also been called. What's won from this deal with the devil is a complete arena in which designs can be 'tested' against one another, eliminating the practical need to shuttle between syntactic proofs, on the one hand, and semantic counter-models on the other. With the return of the sophist to the arena of logic, the logician no longer needs to hypostatise a transcendent world – an external model – against which to test syntactical operations.

Testing, here, is what Girard calls the *interaction* between designs. Interaction is a process analogous to *cut-elimination* in the sequent calculus. 'Cut' does not exist as a *rule* in ludics, but as a phenomenon built into its very geometry. There is a 'cut' between two designs when they share the same locus at their roots, on opposite sides of the pitchfork, as in figure ??. *Inter-*

$$\dfrac{\dfrac{\vdash \xi_{0.1} \qquad \dfrac{\dfrac{}{\vdash \xi_{0.3.1}}\maltese \qquad \dfrac{}{\vdash \xi_{0.3.2}}\maltese}{\xi_{0.3} \vdash}}{\vdash \xi_0}}{\xi \vdash} \qquad\qquad \dfrac{\dfrac{\dfrac{}{\vdash \xi_{0.1}, \xi_{0.2}}\maltese \qquad \dfrac{\xi_{0.3.2} \vdash \xi_{0.1}}{\vdash \xi_{0.1}, \xi_{0.3}}}{\xi_0 \vdash}}{\vdash \xi}$$

$$\mathfrak{D} \qquad\qquad\qquad \mathfrak{E}$$

Figure 14: Poised for combat

action proceeds by a progressive elimination of 'cuts' – the mutual deletion of shared loci on opposite sides of the pitchfork. The interaction poised to take place in figure?? results in the arrangement seen in figure?? after one round.

The process continues so long as shared loci, oppositely positioned, link the two designs. The first design to invoke the *daimon* 'loses' – this is the price of axiomaticity in ludics. The victory, however, is not necessarily eternal, since the winning strategy must engage with further 'tests', against

$$\dfrac{\vdash \xi_{0.1} \qquad \dfrac{\dfrac{}{\vdash \xi_{0.3.1}}\maltese \qquad \dfrac{}{\vdash \xi_{0.3.2}}\maltese}{\xi_{0.3} \vdash}}{\vdash \xi_0} \qquad\qquad \dfrac{\dfrac{}{\vdash \xi_{0.1}, \xi_{0.2}}\maltese \qquad \dfrac{\xi_{0.3.2} \vdash \xi_{0.1}}{\vdash \xi_{0.1}, \xi_{0.3}}}{\xi_0 \vdash}$$

$$\mathfrak{D} \qquad\qquad\qquad\qquad \mathfrak{E}$$

Figure 15: After the first round

which it too would be forced to summon the daimon and admit defeat – the judges must also be judged. In a moment, we will see how this provides for a completely syntactical notion of falsity, and therefore truth, to be defined in ludics.

$$\vdash \xi_{0.1} \qquad \dfrac{\dfrac{}{\vdash \xi_{0.3.1}}\maltese \qquad \dfrac{}{\vdash \xi_{0.3.2}}\maltese}{\xi_{0.3} \vdash} \qquad\qquad \dfrac{}{\vdash \xi_{0.1}, \xi_{0.2}}\maltese \qquad \dfrac{\xi_{0.3.2} \vdash \xi_{0.1}}{\vdash \xi_{0.1}, \xi_{0.3}}$$

$$\mathfrak{D} \qquad\qquad\qquad\qquad \mathfrak{E}$$

Figure 16: After the second round: $\mathfrak{E}$ invokes the *daimon*, and $\mathfrak{D}$ wins

4.5 LUDICS AND THE DIALECTIC OF FORM

It is with 'Locus Solum', and the ludic formalism it advances, that Girard first makes explicit his project's proximity to dialectics – though with some hesitance. On the first page of the text, right after the epigraph, Girard directs the reader's attention to a lengthy ($>$ 100 pp!) 'dictionary' appended to the monograph, which he calls 'a sort of final introduction, since one can only be introduced to known material. For instance,' he adds, 'if you go to DIALECTICS you will learn about the word *ludics*' (Girard 2001, 303). The entry itself reads as follows:

> DIALECTICS: Illustrated by Hegel, Marx, not to mention comrade Stalin. . . , this is a word with a history. For this reason, even if ludics has a lot to do with (the early) dialectics, I decided to create a new expression rather than having to assume the (mostly negative) consequences of the reuse of such a notorious word. By the way, this proximity of ludics to dialectics, which it would be unfair to deny, is not the result of, say, an attempt in the style of Engels' *Dialektik der Natur*, but only the result of a long familiarity with logic, and the contemplation of the symmetries of proof-theory. (Girard 2001, 418)

Girard's hesitancy has two motives: (1) a distrust of the philosophical tradition of dialectical thought, and (2) an even greater distrust of what is called, today, *philosophical logic* – a field he routinely caricatures for its *ad hoc* and superficial attempts to subordinate and mould mathematical logic to the shape of a externally supplied subject matter, usually coming from metaphysics or common sense, inventing 'non-classical logics' tailored to fit the customer. For Girard, what demands formalisation is not something external to mathematical logic, but the leaks and blind spots already immanent to historically existing mathematical logic. It isn't difficult to sense an effectively Hegelian attitude at work in this, or to see how this attitude leads to an antipathy towards many attempts to 'formalise the dialectic' – such as those of Priest or Dubarle, for instance – an antipathy whose motives are perfectly congruent with those of the strictest Hegelian: a hatred of the *ad hoc*.

Nevertheless, the next ten years would dissolve Girard's resistance towards inscribing his research programme, and ludics in particular, in the space of the PFD, without giving way on the principles of his initial reluctance. In 2012's 'Transcendental Syntax 2.0', Girard, without backpedalling, asserts that ludics is indeed enough to

> vindicate one of the greatest philosophers, Hegel, and his *contradictory foundations*. From a fregean viewpoint, founding A on its negation is stupid, since A and $\neg A$ cannot coexist by consistency! The fact is that [...] ludics rests upon contradictory foundations without the slightest problem. The mistake about Hegel [*a mistake Girard takes Priest and Da Costa to have made*] is to take negation in the *alethic* (i.e., -1) sense, for which there cannot be any honest contradictory foundations; there are dishonest ones, e.g., paraconsistent "logic", making Hegel a sibling of Bernard Madoff... a real insult for such a great philosopher! (Girard 2012, 6)

The argument remains aphoristic, and ends here. What I want to do now is to show that it can be developed in detail, that there is indeed, in ludics, a concept of the (sub-alethic) contradictory foundations, and of the contradictory foundations of formality, in particular. This concept of formality founded through contradiction – and not imposed from above, in order to avoid contradiction – is, in its contours, essentially Hegelian, even as it is developed without any explicit effort to mimic Hegel's thought (with which Girard seems only passingly familiar).

Though the motif of contradictory foundations runs throughout Hegel's speculative and dialectical thought, its most explicit thematisation is to be found in Book II of *The Science of Logic*, under the heading of *The Doctrine of Essence*. There, the concept of contradiction serves as a bridge between the study of the determinations of reflection (or 'essentialities'), and that of the category of *ground*, in which the concept of *form*, itself, finds its place. An analysis of ludics will show that this concept of form can indeed be related back to the notion of form which, earlier, I said is sublated through the historical process of *formalisation* – a notion of form that the radical formalist Haskell Curry showed is contingent on a confinement of the calculative by means of *types*. Recall that the *a priori* imposition of typing apparatus on the calculative represents the traditionally *Kantian* response to the discovery of mathematical antinomies, as alethic contradictions. What we will find in ludics is an essentially dialectical approach to the immanent genesis of types – or *forms* – through a free reflection on *non*-alethic contradiction.

The semantic, or, more precisely, *alethic* vision of logic submits fully to the image of formality that Hegel decries from the outset of the *Logic* onwards – even if, as we have seen, it never ceases to provide the material required for a sublation of this image[27] The ideological arrangement of this image in terms of a syntactical 'appearance' finding purchase in a semantic 'reality', may, however, lead us to expect the alethic vision to be susceptible to the same dialectic of appearance and reality that drives the movement of the Hegelian Doctrine of Essence – unveiling its deeper rational structure not in the badly infinite ascent of metalanguages piled on metalanguages in pursuit of a fugitive phantom of truth, but in a purely negative game of mirrors that immanently unfolds on a single and univocal plane.

This is what takes place in Girard's descent to the 'fourth inferno' of transcendental syntax – an inferno that, in fact, is to be found in the 'well-hidden geometry' of the syntactic surface itself (Girard 1998, 1). A proof – grasped, at first, as the tree-like object that the sequent calculus formalisations of logic make of it – is stripped of all imaginary decoration, all the impenetrable atoms that the realist (Russell, for example) takes to promise a metaphysically atomistic interpretation in the real, and the entire apparatus of pre-formulated rules that refer the question of *how* we must proceed to the authority of presupposed forms. What results is what Girard calls a *design*, or a strategy, determined in a purely locative fashion. The strategy is not tested against a *model*, a (typically infinite) construction brought into a more or less external relation with the syntactic proof-structure and susceptible to being (ideologically) understood as the *given* reality against which the proof, or a collection of proofs, is measured, but against *other strategies*, without any their correctness or erroneousness being determined, beforehand, in any fashion. The basic unit of these strategies is not, any longer, a formula or proposition, but a positive or negative *locus*, an empty position answering to – or making a demand upon – its polar counterpart.

The relation between a positive locus and its negative counterparts (or vice versa) is not a static or extrinsic relation, but, immediately, a process of interaction that proceeds like a generalised form of the *cut-elimination algorithm* discovered by Gentzen. This process is an effective and concrete process of *contradiction*, but a form of contradiction that is deeper than and prior to any determination of truth-values – anterior to any fixed support for alethic reference to reality. When this takes place, one of three things may happen:

27 As Badiou, for example, shows in *The Concept of Model* (Badiou 2007)

1. The conflict between the strategies goes on endlessly, in the sort of badly infinite progression that Hegel took to be the symptom of an unresolved contradiction. In this case, Hegel's diagnosis is apt: the situations in which interaction *diverges* are those in which the two designs are locked in a form of abstract 'dissensus'. It is a form of antagonism on which no propositions can be grounded, a logical phenomenon that has no counterpart on the alethic plane. The name that Girard gives to this outcome, or lack of an outcome, is $\mathfrak{Fid}$ or *fidelity*.

2. The conflict *converges* on a particular design, usually different from the two that entered the interaction. Interaction can therefore be used as a way of transforming one design into another, in a concrete and effective fashion. The most methodologically important of these transformations is a procedure called *faxing*, which serves to delocate or transpose a design onto a different locative base – performing the material labour underlying abstract or 'spiritual' conceptions of proof structures, mediated by a particular class of designs called $\mathfrak{Fax}$. If, for example, we want to transpose a design $\mathfrak{D}$ with base $\vdash \xi$ onto base $\vdash \rho$, we have it interact with a $\mathfrak{Fax}$ design of base $\xi \vdash \rho$, a process which converges on a design isomorphic (structurally identical) to $\mathfrak{D}$ but rooted in $\vdash \rho$.

3. The third possible outcome is that, step by step, the opposed loci cancel each other out entirely, collapsing the two trees into a design called $\mathfrak{Dai}$

$$\frac{}{\vdash}\maltese$$

The empty ground into which the contradiction founders is, here, a pure void, the empty pitchfork (sequent) summoned by the *daimon*. When this takes place, the two strategies – call them $\mathfrak{X}$ and $\mathfrak{Y}$ – are said to be *orthogonal* to one another (written $\mathfrak{X} \perp \mathfrak{Y}$).

Understood as structural invariance, form owes its existence to the mobility of $\mathfrak{Fax}$.[28] But it is from the third sort of outcome – the foundering of contradiction into the void, $\mathfrak{Dai}$ – that *form*, as typing, is reconstituted. To get this, we first define the *orthogonal* of a set $\mathscr{S}$ of designs as

$$\mathscr{S}^{\perp} := \{\mathfrak{D} \mid \exists \mathfrak{X} \in \mathscr{S} . \mathfrak{D} \perp \mathfrak{X}\}$$

A form, as type, proposition, or, in the terminology of ludics, as 'behaviour', is a set of designs that is *equal to its biorthogonal*:

$$\mathscr{S} \text{ is a type, iff } \mathscr{S} = (\mathscr{S}^{\perp})^{\perp}$$

28 We could parrot Hegel here to describe this process as one of 'adding to abstract identity the extra factor of that movement' whose 'hidden necessity' is inscribed in the form of the proposition (Hegel 2010, 360), to recall Hegel's reflections on the law of identity in the Doctrine of Essence.

If the reflective play of negativity between designs describes a logic of reflection, or essence, then by taking the pivot of form's constitution as $\mathfrak{Dai}$, we are taking as the *ground* of form the moment in which, in Hegel's words, 'the positive and the negative constitute the essential determination in which essence is lost in its negation,' such that 'these self-subsisting determinations of reflection' – the designs – 'sublate themselves,' cancel each other out through a process of interaction or elimination, 'and the determination that has foundered to the ground is the true determination of essence' (Hegel 2010, 386). In contrast to $\mathfrak{Fid}$, the bad infinity of unresolved contradiction, $\mathfrak{Dai}$ is the 'resolved contradiction' into which finite determinations – designs orthogonal to one another, and therefore punctuated by the mark of finitude, which is the *daimon* – 'founder', or fall to ground, to use Hegel's own pun. $\mathfrak{Dai}$ is, relative to essence, an 'absolute' which *is*, 'because the finite is the immanently self-contradictory opposition, because it *is not*' – it receives its logical determination only in the case of its orthogonality with what destroys it.

Formality, to put it another way, now appears as the interaction of bare negativities – the opposition of polarised designs, which stripped of all logical decoration, presuppose no distinction between form and content – under the quotient of their collapse into ground ($\mathfrak{Dai}$) or as invariant under bi-orthogonality.

This is where we find the buttresses of the alethic inferno in the deontic, the supports on which *propositional forms*, capable of bearing *truth*, will rest. A behaviour, or interactive type, is, in effect, the ludic embodiment of the logical *proposition*. Its equality with its biorthogonal captures a notion of completeness that owes nothing to an external or semantic reference to a model. To understand this, we can think of a *complete* set of proofs $\mathscr{D}$ as one that is turns out to be valid when measured against all potential counter-examples, or counter-models. Ludics fulfills this office with the set of counter-proofs or 'tests', $\mathscr{D}^{\perp}$.[29] But $\mathscr{D}$ measures up to these tests if it is orthogonal to them, if it meets them head-on, and hence if it is equal to the orthogonal of its orthogonal, $\mathscr{D} = \mathscr{D}^{\perp\perp}$. A propositional form, type or behaviour is 'true' when it contains a design that 'wins' against all challengers – all of its orthogonal counter-designs. The technical conditions for truth, in ludics, are somewhat subtle, but broadly speaking, they boil down to two essential properties: to be victorious, a design must be (1) 'exact and parsimonious' – these are delicate conditions, but effectively they mean that the victorious design must move in a way that closely anticipates its opponents, without too many unnecessary moves, and (2) 'stubborn'; the winning design must not give up, must not invoke the *daimon*, which is nevertheless

29 A technical aside: if we have a purported proof of a sequent $\vdash \Delta$ in the sequent calculus, then its counter-model, if one exists, can be obtained by attempting to prove $\Delta \vdash$ or, equivalently, $\vdash \neg\Delta = \{\neg A : A \in \Delta\}$, and then building a model out of the subformula unfolded in that 'counter-proof'. Now, the locative infrastructure of these two structures – the attempted proofs of $\vdash \Delta$ and $\Delta \vdash$ – will be orthogonal, in the sense described above. The pairing of a design with its orthogonal design is thus a way of bringing the duality between syntactic proofs and semantic counterexampels onto a single, homogeneous plane.

the quilting point of all logical determinacy. If neither party in a dispute plays the *daimon*, no logical form will crystalise: the cut-net will diverge, and be lost to $\mathfrak{Fid}$. Truth owes its existence to the errors of its opponents, without which it is nothing more than faith.

Contradiction, on the deontic plane – the plane of recusals rather than refutations – is orthogonality, and the foundering of the contradictory to ground is the convergence of orthogonal designs on $\mathfrak{Dai}$. This is, in any case, how I am attempting to map the dialectic of essence onto the dynamics of ludics, in a way that works out and satisfies, I think, suggestions which, in Girard's writings, remain on the level of aphorism. $\mathfrak{Dai}$ is the ground into which essence founders, the *result* of essential (deontic and locative) contradiction. *Logical form*, or *types*, emerge in ludics on the basis of the symmetry that this empty ground induces between designs: it is identity in biorthogonality, and nothing more. In dramatic contrast to the way in which types are *applied*, transcendently, to the pure lambda calculus – 'such that,' as Jean-Baptiste Joinet writes, 'typing would appear as an apparatus of control, a disciplinary action taken on an otherwise vagabond dynamics' (Joinet 2009, 44), here, in ludics, typing answers to Hegel's own description of *form* as

> absolute negativity itself or the negative absolute self-identity by virtue of which essence is indeed not being but essence. [...] One cannot therefore ask, *how form comes to essence*, for form is only the internal reflective shining of essence, its own reflection inhabiting it. (Hegel 2010, 391)[30]

30 Jean-Baptiste Joinet sums the situation up well when he recounts how 'The vision, which could be called 'cybernetic', of logic as external, *transcendent* control, imposing particular properties on a pre-existing interaction (*anterior* to that control), that was methodologically useful insofar as it came to disengage (in virtue of having 'enlarged, displaced and refined' the point of view on the dynamics, and in virtue of having recovered symmetries, isomorphisms and quotients in the space of proofs) a fruitful and general frameowrk for taking up those processes *in abstracto*, later found itself replaced or at least ompleted by a point of view that may be contrasted with it by being qualified as *immanentist*, a point of view according to which the regulative norm of the dynamics, *logic*, emerges directly from the interaction as a solution to the problem of *internal completeness*' (Joinet 2009, 50).

AXIOM AND CUT

$$[\text{AXIOM}]\ \frac{}{A \vdash A} \qquad \frac{\Gamma_1 \vdash C, \Delta_1 \qquad \Gamma_2, C \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ [\text{CUT}]$$

OPERATIONAL RULES

$$[\text{L}\neg]\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, \vdash \neg A, \Delta}\ [\text{R}\neg]$$

$$[\text{L}\wedge]\ \frac{\Gamma, A_{i\in(1,2)} \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta} \qquad \frac{\Gamma_1 \vdash A_1, \Delta \qquad \Gamma_2 \vdash A_2, \Delta}{\Gamma_1, \Gamma_2 \vdash A_1 \wedge A_2, \Delta}\ [\text{R}\wedge]$$

$$[\text{L}\vee]\ \frac{\Gamma_1, A_1 \vdash \Delta \qquad \Gamma_2, A_2 \vdash \Delta}{\Gamma_1, \Gamma_2, A_1 \vee A_2 \vdash \Delta} \qquad \frac{\Gamma \vdash A_{i\in(1,2)}}{\Gamma \vdash A_1 \vee A_2, \Delta}\ [\text{R}\vee]$$

$$[\text{L}\rightarrow]\ \frac{\Gamma_1 \vdash A_1, \Delta \qquad \Gamma_2, A_2 \vdash \Delta}{\Gamma_1, \Gamma_2, A_1 \rightarrow A_2 \vdash \Delta} \qquad \frac{\Gamma, A_1 \vdash A_2, \Delta}{\Gamma \vdash A_1 \rightarrow A_2}\ [\text{R}\rightarrow]$$

$$[\text{L}\forall]\ \frac{\Gamma, A[x/t] \vdash \Delta}{\Gamma, \forall x A[x] \vdash \Delta} \qquad (y \text{ not free in } \Gamma, \Delta)\ \frac{\Gamma \vdash A[x/y], \Delta}{\Gamma \vdash \forall x A[x], \Delta}\ [\text{R}\forall]$$

$$[\text{L}\exists]\ \frac{\Gamma, A[x/y] \vdash \Delta}{\Gamma, \exists x A[x] \vdash \Delta}\ (y \text{ not free in } \Gamma, \Delta) \qquad \frac{\Gamma \vdash A[x/t], \Delta}{\Gamma \vdash \exists x A[x], \Delta}\ [\text{R}\exists]$$

STRUCTURAL RULES

$$\text{LEFT WEAKENING [LW]}\ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \text{RIGHT WEAKENING [RW]}$$

$$\text{LEFT CONTRACTION [LC]}\ \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \text{RIGHT CONTRACTION [RC]}$$

$$\text{LEFT EXCHANGE [LE]}\ \frac{\Gamma \vdash \Delta}{\sigma\Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \sigma\Delta}\ \text{RIGHT EXCHANGE [LE]}$$

$\sigma\Phi$ is a permutation of Φ

Figure 17: The rules of the classical sequent calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ [\text{Identity}]}{A \vdash A \vee \neg A}\ [\text{R}\vee]}{\vdash (A \vee \neg A)\, \neg A}\ [\text{R}\neg]}{\vdash (A \vee \neg A)\, (A \vee \neg A)}\ [\text{R}\vee]}{\vdash A \vee \neg A}\ [\text{Right Contraction}]$$

Figure 18: Classical sequent proof of the law of the excluded middle

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