IDCT Derivation

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1 Discrete Cosine Transform

$C(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) cos\left[\frac{\pi(2x+1)u}{2N}\right]$

2 Inverse Discrete Cosine Transform

2.1 Definition:

$C(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) cos\left[\frac{\pi(2x+1)u}{2N}\right]$

$$\alpha(u) = \begin{cases} \sqrt{\frac{1}{N}} & \text{if } u = 0 \\ \sqrt{\frac{2}{N}} & \text{if } u \neq 0 \end{cases}$$

2.2 The Core Lemma:

$\sum_{u=0}^{N-1} \alpha^2(u) cos\left[\frac{\pi(2x+1)u}{2N}\right] cos\left[\frac{\pi(2k+1)u}{2N}\right] = \delta_{x,k}$

where $\delta_{x,k} = 1$ if $x = k$ and $\delta_{x,k} = 0$ otherwise.

2.3 Derivation

We multiply both sides of the forward equation by the basis function corresponding to index k, weighted by $\alpha(u)$, and sum over the frequency domain $u = 0$ to $N - 1$.

$\sum_{u=0}^{N-1} \alpha(u) C(u) \cos\left[\frac{\pi(2k+1)u}{2N}\right]$

$= \sum_{u=0}^{N-1} \alpha(u) \left(\alpha(u) \sum_{x=0}^{N-1} f(x) \cos\left[\frac{\pi(2x+1)u}{2N}\right]\right) \cos\left[\frac{\pi(2k+1)u}{2N}\right]$

Now rearranging.

RHS $= \sum_{x=0}^{N-1} f(x) \left[\sum_{u=0}^{N-1} \alpha^2(u) \cos\left[\frac{\pi(2x+1)u}{2N}\right] \cos\left[\frac{\pi(2k+1)u}{2N}\right]\right]$

The inside term bracket is exactly the $\delta_{x,k}$ from *Definition*:

$\left[\sum_{u=0}^{N-1} \alpha^2(u) \cos\left[\frac{\pi(2x+1)u}{2N}\right] \cos\left[\frac{\pi(2k+1)u}{2N}\right]\right] = \delta_{x,k}$

Therefore, the equation simplifies to:

RHS $= \sum_{x=0}^{N-1} f(x) \cdot \delta_{x,k}$

The summation over x contains N terms. However, due to the definition of the Kronecker Delta $\delta_{x,k}$:

- For all $x \neq k$, the term is $f(x) \cdot 0 = 0$.
- For the single case $x = k$, the term is $f(k) \cdot 1 = f(k)$.

Thus, the sum collapses to a single term:

$$\text{RHS} = f(k)$$

Thus,
$f(k) = \sum_{u=0}^{N-1} \alpha(u) C(u) \cos\left[\frac{\pi(2k+1)u}{2N}\right]$