

Mathematical Calculations

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<2025-12-04 Thu>

Contents

1 Discrete Cosine Transform 2

2 Inverse Discrete Cosine Transform 2

2.1 Definition: 2

2.2 The Core Lemma: 2

2.3 Derivation 2

#LATEX:

1 Discrete Cosine Transform

$$C(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) \cos \left[\frac{\pi(2x+1)u}{2N} \right]$$

2 Inverse Discrete Cosine Transform

2.1 Definition:

$$C(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) \cos \left[\frac{\pi(2x+1)u}{2N} \right]$$

$$\alpha(u) = \begin{cases} \sqrt{\frac{1}{N}} & \text{if } u = 0 \\ \sqrt{\frac{2}{N}} & \text{if } u \neq 0 \end{cases}$$

2.2 The Core Lemma:

$$\sum_{u=0}^{N-1} \alpha^2(u) \cos \left[\frac{\pi(2x+1)u}{2N} \right] \cos \left[\frac{\pi(2k+1)u}{2N} \right] = \delta_{x,k}$$

where $\delta_{x,k} = 1$ if $x = k$ and $\delta_{x,k} = 0$ otherwise.

2.3 Derivation

We multiply both sides of the forward equation by the basis function corresponding to index k , weighted by $\alpha(u)$, and sum over the frequency domain $u = 0$ to $N - 1$.

$$\begin{aligned} & \sum_{u=0}^{N-1} \alpha(u) C(u) \cos \left[\frac{\pi(2k+1)u}{2N} \right] \\ &= \sum_{u=0}^{N-1} \alpha(u) \left(\alpha(u) \sum_{x=0}^{N-1} f(x) \cos \left[\frac{\pi(2x+1)u}{2N} \right] \right) \cos \left[\frac{\pi(2k+1)u}{2N} \right] \end{aligned}$$

Now rearranging.

$$\text{RHS} = \sum_{x=0}^{N-1} f(x) \left[\sum_{u=0}^{N-1} \alpha^2(u) \cos \left[\frac{\pi(2x+1)u}{2N} \right] \cos \left[\frac{\pi(2k+1)u}{2N} \right] \right]$$

The inside term bracket is exactly the $\delta_{x,k}$ from *Definition*:

$$\left[\sum_{u=0}^{N-1} \alpha^2(u) \cos \left[\frac{\pi(2x+1)u}{2N} \right] \cos \left[\frac{\pi(2k+1)u}{2N} \right] \right] = \delta_{x,k}$$

Therefore, the equation simplifies to:

$$\text{RHS} = \sum_{x=0}^{N-1} f(x) \cdot \delta_{x,k}$$

The summation over x contains N terms. However, due to the definition of the Kronecker Delta $\delta_{x,k}$:

- For all $x \neq k$, the term is $f(x) \cdot 0 = 0$.
- For the single case $x = k$, the term is $f(k) \cdot 1 = f(k)$.

Thus, the sum collapses to a single term:

$$\text{RHS} = f(k)$$

Thus,

$$f(k) = \sum_{u=0}^{N-1} \alpha(u) C(u) \cos \left[\frac{\pi(2k+1)u}{2N} \right]$$