

# Mathematical Calculations

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#LATEX:		

# 1 Discrete Cosine Transform

$$C(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) \cos\left[\frac{\pi(2x+1)u}{2N}\right]$$

# 2 Inverse Discrete Cosine Transform

## 2.1 Definition:

$$C(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) \cos\left[\frac{\pi(2x+1)u}{2N}\right]$$

$$\alpha(u) = \begin{cases} \sqrt{\frac{1}{N}} & \text{if } u = 0 \\ \sqrt{\frac{2}{N}} & \text{if } u \neq 0 \end{cases}$$

## 2.2 The Core Lemma:

$$\sum_{u=0}^{N-1} \alpha^2(u) \cos\left[\frac{\pi(2x+1)u}{2N}\right] \cos\left[\frac{\pi(2k+1)u}{2N}\right] = \delta_{x,k}$$

where  $\delta_{x,k} = 1$  if  $x = k$  and  $\delta_{x,k} = 0$  otherwise.

## 2.3 Derivation

We multiply both sides of the forward equation by the basis function corresponding to index  $k$ , weighted by  $\alpha(u)$ , and sum over the frequency domain  $u = 0$  to  $N - 1$ .

$$\begin{aligned} & \sum_{u=0}^{N-1} \alpha(u) C(u) \cos\left[\frac{\pi(2k+1)u}{2N}\right] \\ &= \sum_{u=0}^{N-1} \alpha(u) \left( \alpha(u) \sum_{x=0}^{N-1} f(x) \cos\left[\frac{\pi(2x+1)u}{2N}\right] \right) \cos\left[\frac{\pi(2k+1)u}{2N}\right] \end{aligned}$$

Now rearranging.

$$\text{RHS} = \sum_{x=0}^{N-1} f(x) \left[ \sum_{u=0}^{N-1} \alpha^2(u) \cos\left[\frac{\pi(2x+1)u}{2N}\right] \cos\left[\frac{\pi(2k+1)u}{2N}\right] \right]$$

The inside term bracket is exactly the  $\delta_{x,k}$  from *Definition*:

$$\left[ \sum_{u=0}^{N-1} \alpha^2(u) \cos\left[\frac{\pi(2x+1)u}{2N}\right] \cos\left[\frac{\pi(2k+1)u}{2N}\right] \right] = \delta_{x,k}$$

Therefore, the equation simplifies to:

$$\text{RHS} = \sum_{x=0}^{N-1} f(x) \cdot \delta_{x,k}$$

The summation over  $x$  contains  $N$  terms. However, due to the definition of the Kronecker Delta  $\delta_{x,k}$ :

- For all  $x \neq k$ , the term is  $f(x) \cdot 0 = 0$ .
- For the single case  $x = k$ , the term is  $f(k) \cdot 1 = f(k)$ .

Thus, the sum collapses to a single term:

$$\text{RHS} = f(k)$$

Thus,

$$f(k) = \sum_{u=0}^{N-1} \alpha(u) C(u) \cos \left[ \frac{\pi(2k+1)u}{2N} \right]$$