UE STEOP: Introduction to Mathematics in Data Science
Problem Set 7

Problem 1. Define R on $\mathbb{Z}$ as $mRn \iff 2 \mid m^2 + m^2$ for all $m, n \in \mathbb{Z}$. Is R an equivalence relation?

Problem 2. Let R be a relation on the real numbers defined by $xRy \iff x^2 = y^2$. Is R an equivalence relation?

Problem 3. Define R on $\mathbb{Z}$ as $xRy \iff 4 \mid x + 3y$ for all $x, y \in \mathbb{Z}$. Is R an equivalence relation?

Problem 4. Prove or disprove: If R is an equivalence relation on an infinite set S, then R has infinitely many equivalence classes.

Problem 5. There are five different equivalence relations on the set $A = \{a, b, c\}$. Describe them all. (Here you can just draw a picture instead of text if you prefer).

Problem 6. Let S be the set of all triangles in the plane. Define $T_1 \sim T_2$ if T_1 is similar to T_2. Prove that $\sim$ is an equivalence relation. (Here I intentionally use an alternative notation for an equivalence relation.)

Problem 7. Consider the partition $P = \Big\{\{0\}, \{-1, 1\}, \{-2, 2\}, \{-3, 3\}, \dots\Big\}$ of $\mathbb{Z}$. Describe the equivalence relation whose equivalence classes are the elements of P.

Problem 8. A relation R on $\mathbb{R}$ is defined as $xRy \iff |x| \le |y|$. Is it a partial order? Is it a total order?

Problem 9. In a basketball tournament, each team plays every other team exactly once. Define a relation $\preccurlyeq$ on the set of all teams as: for two teams A and B, we say $A \preccurlyeq B \iff$ the final ranking of B is at least as high as that of A. Is $\preccurlyeq$ a partial order? Is it a total order?

Problem 10. A relation $\preccurlyeq$ on $\mathbb{R}^2$ is defined as

$$(x_1, y_1) \preccurlyeq (x_2, y_2) \quad \iff \quad |x_2 - x_1| \le y_2 - y_1.$$

Prove that $\preccurlyeq$ is a partial order. Is it a total order?

Problem 11. For the problems above, where there is a partial order, decide which elements are: greatest/least, and maximal/minimal.

Problem 12. If a subset of a partially ordered set has exactly one minimal element, must that element be a least element? Give either a proof or a counter example to justify your answer.

Problem 13. Let A given by

$$A = \left\{\frac{m}{n} : m, n \in \mathbb{N},\ m < n\right\}$$

be a subset of $\mathbb{R}$ with the standard order $\le$. Find $\sup A$, $\inf A$, $\max A$ and $\min A$ (if they exist).