substitution

$$\tilde{\phi}^{n+1} = 2\tilde{\phi}^{n+1/2} - \tilde{\phi}^n \tag{60a}$$
$$h^{n+1} = 2h^{n+1/2} - h^n \tag{60b}$$
$$W^{n+1} = 2W^{n+1/2} - W^n \tag{60c}$$
$$Z^{n+1} = 2Z^{n+1/2} - Z^n \tag{60d}$$

is made such that the coupled system is solved in terms of the (five) midpoint variables. Once these are solved, the update to (discrete) time level $n+1$ is made using the above relations (60). To numerically solve the system at rest in order to establish the initial condition, the following VP can be employed

$$0 = \delta\Big[\int_0^{L_x} g h^{n+1/2} V^{n+1/2}\Big(\frac{1}{2}h^{n+1/2} - H_0\Big) - \frac{1}{2} g V^{n+1/2} H_0^2 + V^{n+1/2} F\left(\frac{h_b(Z^{n+1/2},x) - h^{n+1/2}}{\alpha}\right) \mathrm{d}x + mgZ^{n+1/2}\Big] \tag{61}$$

for $n = 0$ and with $V^{n+1/2} = L_w$. Variational derivatives with respect to $\{h^{n+1/2}, Z^{n+1/2}\}$ will establish the weak formulations to be solved.

For reference, Achimedes' law yields the following based on geometric considerations. The position of the keel is $Z - K$. Assuming a rest water level $H_0$, the submerged buoy depth at $x = L_x$ is $H_0 + K - Z$. The width of the buoy at the water line is then

$$L_x - L_b = (H_0 + K - Z)/\tan\theta. \tag{62}$$

Hence, the submerged volume is $V_b = \frac{1}{2} L_y (H_0 + K - Z)^2/\tan\theta$. The displaced water mass equals the mass of the buoy. Hence, the rest reference level $Z$ is

$$\rho V_b = \rho \frac{1}{2} L_y (H_0 + K - Z)^2/\tan\theta = M$$
$$\Longrightarrow Z = H_0 + K - \sqrt{2m\tan\theta} \tag{63}$$

with water density $\rho$, which results can be used as reference for the numerical solution using the soft-boundary method. For example, when $L_y = H_0 = Z = 1$m, $K = 0.5$m and $\tan\theta = 1$ we find that $m = (1/8)$m such that the mass of the buoy $M = 125$kg.

Alternatively, the VP using the exact constraint $h - h_b(Z,x) = 0$ between free surface and buoy reads

$$0 = \delta\int_0^T\int_0^{L_x} \phi\partial_t h - \frac{1}{2}h(\partial_x\phi)^2 - gh(\frac{1}{2}h - H_0) + \frac{1}{2}gH_0^2 + \lambda\left(h_b(Z,x) - h\right)\Theta(x - x_p)\,\mathrm{d}x + mW\dot{Z} - \frac{1}{2}mW^2 - mgZ\,\mathrm{d}t. \tag{64}$$

with Lagrange multiplier $\lambda$ and Heaviside function $\Theta(\cdot)$. Its variations are

$$0 = \int_0^T\int_0^{L_x} \delta\phi\left(\partial_t h + \partial_x(h\partial_x\phi)\right) - \delta h\left(\partial_t\phi + \frac{1}{2}(\partial_x\phi)^2 + g(h - H_0) + \lambda\Theta(x - x_p)\right)\mathrm{d}x + m\delta W(\dot{Z} - W) - \delta Z(m\dot{W} + mg - \int_{x_b}^{L_x}\lambda\,\mathrm{d}x)\,\mathrm{d}t \tag{65}$$

with $x_p$ the dynamic waterline point.

Hence, at rest, the following system needs to be solved

$$g(h - H_0) = -\lambda\Theta(x - L_b) \tag{66a}$$
$$h - h_b(Z,x) = 0 \tag{66b}$$
$$h_b(Z,x) = Z - K + \theta(L_X - x) \tag{66c}$$
$$mg = \int_{L_b}^{L_x}\lambda\,dx \tag{66d}$$

where at rest $x_b = L_b$ is as before the function of $Z$ and $H_0$ in (62). An interim step in the solution of the above system is $\lambda = g\left(H_0 - h_b(Z,x)\right)$, which substitution into the last equation while using the expression for $h_b(Z,x)$ again yields (63).

Comparison of the two VPs (53) and (64) provides an interpretation of the buffer potential $\mu F(s)$, in that

$$\lambda\left(h_b(Z,x) - h\right)\Theta(x - x_b) \approx -\mu F\left(\frac{h_b(Z,x) - h}{\alpha}\right) \tag{67a}$$
$$\lambda\Theta(x - x_b) \approx -\frac{\mu}{\alpha}F'\left(\frac{h_b(Z,x) - h}{\alpha}\right). \tag{67b}$$

We can therefore attempt to tune $\alpha, \beta, \mu$ for sufficiently small values towards the exact rest-state solution in order to improve the numerical solution of the rest state using the buffer potential $\mu F(s)$ instead of the constraint and its Lagrange multiplier.

## APPENDIX G. 3D POTENTIAL FLOW VIA VP

The VP (a simplification of expression (8) in[21]) for 3D potential flow without wavemaker (in an $x$–periodic) domain is

$$0 = \delta\int_0^T\Big\{\int_{\hat{\Omega}_{x,y}}\Big[\int_0^{H_0}\Big[\frac{1}{2}\frac{L_w^2}{W}h(\phi_x - \frac{1}{h}(H_0 b_x + z h_x)\phi_z)^2 + \frac{1}{2}Wh\Big(+\phi_y - \frac{1}{h}(H_0 b_y + z h_y)\phi_z\Big)^2 + \frac{1}{2}W\frac{H_0^2}{h}(\phi_z)^2\Big]\,\mathrm{d}z + H_0\Big(gWh(\frac{1}{2}h - H_0) - \phi W h_t\Big)_{z=H_0}\Big]\mathrm{d}x\,\mathrm{d}y\Big\}\,\mathrm{d}t \tag{68}$$

with

$$W = L_w, \tag{69}$$

noting that $W$ is more complicated when there is a piston wavemaker present. In addition, topography $b$ needs to be periodic in the $x$–direction. Implemented and seems to be working on initialisation with linear waves; soliton in periodic channel next, then SP2 and SP3; then wavebreaking. parameterisation.

Upon making the splitting $\phi(x,y,z,t) = \psi(x,y,t)\hat{\phi}(z) + \varphi(x,y,z,t)$ with $\hat{\phi}(H_0 = 1)$ and $\varphi(x,y,H_0,t) = 0$, the MMP time discretisation of (69) reads

$$
\begin{aligned}
0 = \int_{\hat{\Omega}_{x,y}} \Bigg[ \Big( & - H_0 W \psi^{n+1/2} \frac{(h^{n+1} - h^n)}{\Delta t} + H_0 h^{n+1/2} \frac{(W\psi^{n+1} - W\psi^n)}{\Delta t} \\
& + H_0 g W h^{n+1/2} (\frac{1}{2} h^{n+1/2} - H_0) \Big) \\
+ \int_0^{H_0} & \Bigg[ \frac{1}{2} \frac{L_w^2}{W} h^{n+1/2} (\psi_x^{n+1/2} \hat{\phi} + \varphi_x^{n+1/2} - \frac{1}{h^{n+1/2}} (H_0 b_x + z h_x^{n+1/2}) (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2}))^2 \\
+ \frac{1}{2} W h^{n+1/2} & \Big( \psi_y^{n+1/2} \hat{\phi} + \varphi_y^{n+1/2} - \frac{1}{h^{n+1/2}} (H_0 b_y + z h_y^{n+1/2}) (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2}) \Big)^2 \\
& + \frac{1}{2} W \frac{H_0^2}{h^{n+1/2}} (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2})^2 \Bigg] \mathrm{d}z \Bigg] \mathrm{d}x \, \mathrm{d}y. \qquad (70)
\end{aligned}
$$

The time-discrete VP corresponding to SV reads

$$
\begin{aligned}
0 = \int_{\hat{\Omega}_{x,y}} \Bigg[ \Big( & - H_0 W \psi^{n+1/2} \frac{(h^{n+1} - h^n)}{\Delta t} + H_0 W \psi^{n+1} \frac{h^{n+1}}{\Delta t} - H_0 W \psi^n \frac{h^n}{\Delta t} \\
& + \frac{1}{2} H_0 g W \big( h^{n+1} (\frac{1}{2} h^{n+1} - H_0) + h^n (\frac{1}{2} h^n - H_0) \big) \Big) \\
+ \frac{1}{2} \int_0^{H_0} & \Bigg[ \frac{1}{2} \frac{L_w^2}{W} h^{n+1} (\psi_x^{n+1/2} \hat{\phi} + \varphi_x^{n+1/2} - \frac{1}{h^{n+1}} (H_0 b_x + z h_x^{n+1}) (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2}))^2 \\
+ \frac{1}{2} W h^{n+1} & \Big( \psi_y^{n+1/2} \hat{\phi} + \varphi_y^{n+1/2} - \frac{1}{h^{n+1}} (H_0 b_y + z h_y^{n+1}) (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2}) \Big)^2 \\
& + \frac{1}{2} W \frac{H_0^2}{h^{n+1}} (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2})^2 \Bigg] \mathrm{d}z \\
+ \frac{1}{2} \int_0^{H_0} & \Bigg[ \frac{1}{2} \frac{L_w^2}{W} h^n (\psi_x^{n+1/2} \hat{\phi} + \varphi_x^{n+1/2} - \frac{1}{h^n} (H_0 b_x + z h_x^n) (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2}))^2 \\
+ \frac{1}{2} W h^n & \Big( \psi_y^{n+1/2} \hat{\phi} + \varphi_y^{n+1/2} - \frac{1}{h^n} (H_0 b_y + z h_y^n) (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2}) \Big)^2 \\
& + \frac{1}{2} W \frac{H_0^2}{h^n} (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2})^2 \Bigg] \mathrm{d}z \Bigg] \mathrm{d}x \, \mathrm{d}y. \qquad (71)
\end{aligned}
$$

Variations herein are taken with respect to $\{h^n, \varphi^{n+1/2}\}$ to update $\{\psi^{n+1/2}, \varphi^{n+1/2}\}$ in unison, then $\psi^{n+1/2}$ to update $h^{n+1}$, and finally $h^{n+1}$ to update to $\psi^{n+1}$.