Modeling extreme water waves with Firedrake

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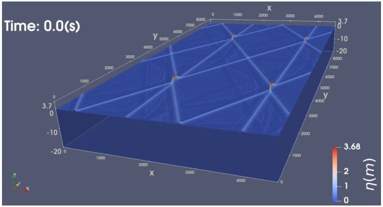
£€₩: EPSRC/STW Fast Ships, EU Surfs-Up/Eagre-Aegir, Sejong Science fellowship

Leeds Institute for Fluid Dynamics; Totnes' Firedrake Workshop 2023



Summary with movies

Firedrake extreme wave modelling of *Benney-Luke (BL) eqns*:

- horizontally $\Phi(x, y, t), \eta(x, y, t)$: CG1, CG2-2-CG4; space-time discrete variational principle (VP),
- discrete VP: wave-amplitude & phase-space conservation,
- x-periodic mesh, 3 weak forms,
- mesh refinement needed; other element types?

Summary

Firedrake modelling of (driven) potential-flow (PF) *water waves*:

- space-time discrete VP, variables $h(x, y, t), \tilde{\phi}(x, y, t) = \phi(x, y, b(x, y) + h(x, y, t), t), \phi(x, y, z, t)$ with mixed horizontal and vertical coordinates,
- transformation to fixed domain,
- now, vertical z: one vertical element with Lagrange/Chebychev polynomials: user-arranged,
- space-time discrete VP in horizontal: CG1 polynomials; 3 to 5 weak forms.
- In progress: FD implementation of VPs for complicated moving domains in x, y, z.

Firedrake for extreme wave amplification in BL

- Origin 2010 *bore-soliton-splash*:
- To what extent do exact but idealised extreme- or rogue-wave solutions survive in more realistic settings?
- Rogue wave: $A_r \geq 2.2 \times A_{ambient}$.
- Will such extreme waves fall apart due to dispersion or other mechanisms?
- Use $4\times$ & $9\times$ KP amplifications of interacting solitons/cnoidal waves.
- What do you think: will we be able to reach the ninefold wave amplification in more realistic calculations or in reality?

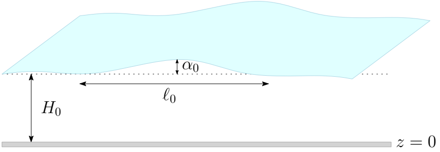
Mathematical hierarchy: BL and KP approximations

- Kadomtsev & Petviashvili (1970) eqn: unidirectional in2DH
- Benney-Luke (1964) eqns –BL: bidirectional in 2DH

$$\partial_t\eta - \frac{\mu}{2}\partial_t\nabla^2\eta + \nabla\cdot\left(\left(1+\epsilon\eta\right)\nabla\Phi\right) - \frac{2\mu}{3}\nabla^4\Phi = 0 \quad \text{in } \Omega$$

$$\partial_t\Phi - \frac{\mu}{2}\partial_t\nabla^2\Phi + \frac{\epsilon}{2}\left|\nabla\Phi\right|^2 + \eta = 0 \quad \text{in } \Omega$$

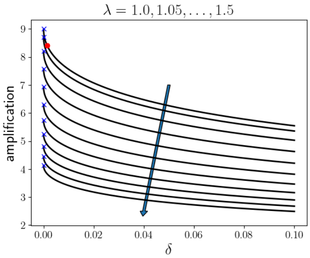
$$\mathsf{n}\cdot\nabla\Phi = 0 \text{ on } \partial\Omega \quad \text{and} \quad \mathsf{n}\cdot\nabla(\nabla^2\Phi) = 0 \quad \text{on } \partial\Omega$$

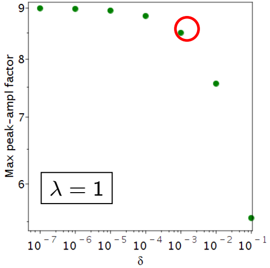


$\epsilon = \alpha_0/H_0 \ll 1$
$\mu = (H_0/\ell_0)^2 \ll 1$

Maximum 9-fold amplification in KP & BL?

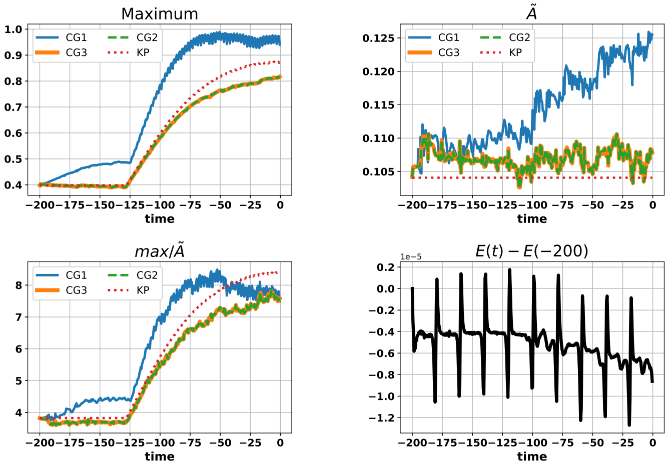
- Amplification: $\dfrac{u(X_*, Y_*, \tau_*)}{\tilde{A}} = \dfrac{(\sqrt{\lambda}+2)^2}{\lambda} + \mathcal{O}(\sqrt{\delta}) \xrightarrow[\delta=0,\lambda\to 1]{} 9$





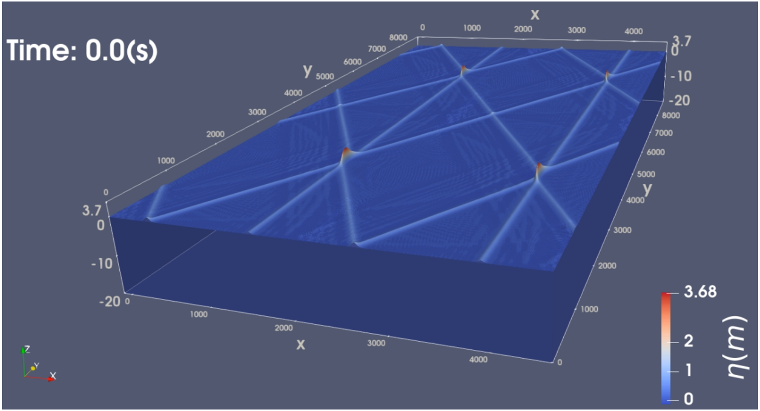
- Seed BL with KP-solution prior to maximum reached.
- Wave amplitude & phase space volume preserved via space-time discrete VP: no loss of amplitude.

Results BL-simulation three-soliton interaction



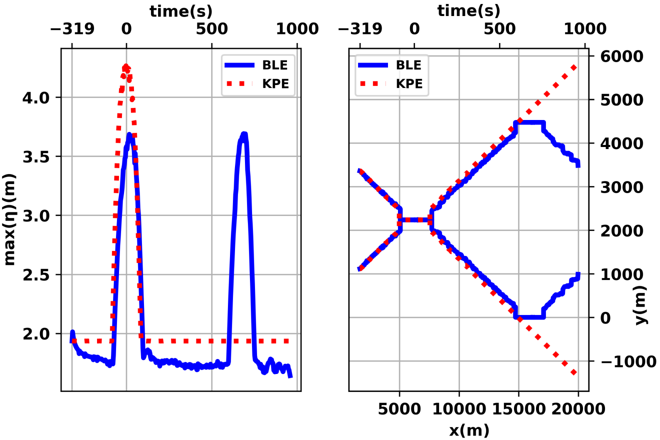
Results simulation three-soliton interaction (dimensional)

Crossing seas (4 or 8 domains combined –*YouTube*)



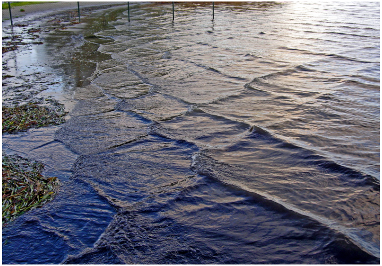
Results simulation three-soliton interaction (dimensional)

Cnoidal waves with periodicity in x, y, t (max. vs. t & x–y tracks):



Summary

- Nine-fold soliton amplification shown theoretically, but only in limit $\delta \to 0$
- Web-soliton amplification KP$\approx$ 8.4, simulated for BL$\approx$ 7.8

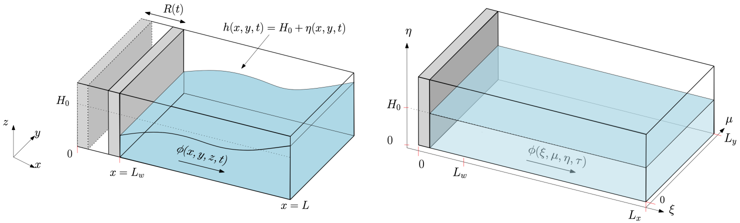


- Amplifications achieved as simulated cnoidal crossing seas
- Local p or h mesh-refinement needed in x-periodic channel.
- Can amplifications survive in potential-flow equations?

Firedrake for potential flow water waves

Water-wave equations

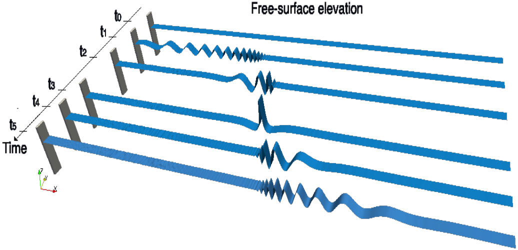
$$
\begin{aligned}
\nabla^2\phi = 0 \quad &\text{in} \quad \Omega\\
\partial_t\eta + \nabla\phi\cdot\nabla\eta - \partial_z\phi = 0 \quad &\text{at} \quad z = H_0 + \eta\\
\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0 \quad &\text{at} \quad z = H_0 + \eta\\
\mathbf{n}\cdot\nabla\phi = 0 \quad &\text{on} \quad z = 0 \ \text{and} \ \partial\Omega
\end{aligned}
$$



Current strategy: Firedrake in horizontal domain

Current strategy (Gidel 2018; Gidel et al. 2022/23)

- Time-dependent free surface & piston wavemaker, transformed to fixed domain.
- One element in vertical Lagrange/Chebychev $p = 4, 9$; user-arranged.
- Firedrake: space-time discrete VP in horizontal with CG1/CG2, Störmer-Verlet.
- Time-dependent VP, non-autonomous Hamiltonian.
- Good comparison with experimental data (Gidel et al 2022/23); soliton amplification similation in progress.



Future strategy: Firedrake via VPs

Goal: implement (time-discrete) VP & derive weak forms automatically.

- Why: VPs in moving domains complicated, e.g. wave tank with waveflap, wave-beam FSI, numerical mesh motion.
- Simple case; nonlinear PF in $\{x, z\}$, $\partial_y = 0, W = L_w \leq L_x$, no wavemaker:

$$0 = \delta \int_0^T \int_0^{L_x} \int_0^{H_0} -\Big[\tfrac{1}{2}\tfrac{L_w^2}{W} h(\phi_x + (z/h)h_x)\phi_z)^2 + \tfrac{1}{2} W \tfrac{H_0^2}{h}(\phi_z)^2\Big] \mathrm{d}z\mathrm{d}x \\ + \int_0^{L_x} -gH_0 Wh(\tfrac{1}{2}h - H_0) + H_0 W\phi|_{z=H_0} h_t \mathrm{d}x\mathrm{d}t$$

- Time-discrete version, coupled surface-interior system, use *fd.derivative*: $\delta h^{n+1}, \delta\varphi^{n+1}$, explicit fderivative ϕ^{n+1}:

$$0 = \quad \delta \int_0^{L_x} \int_0^{H_0} -\Big[\tfrac{1}{2}\tfrac{L_w^2}{W} h^n(\phi_x^{n+1} + (z/h^n)h_x^n)\phi_z^{n+1})^2 + \tfrac{1}{2} W \tfrac{H_0^2}{h^n}(\phi_z^{n+1})^2\Big] \mathrm{d}z\mathrm{d}x \\ + \int_0^{L_x} -gH_0 Wh^n(\tfrac{1}{2}h^n - H_0) + H_0 W\phi^{n+1}|_{z=H_0} \frac{(h^{n+1} - h^n)}{\Delta t} + H_0 W\phi^n|_{z=H_0} \frac{h^n}{\Delta t} \mathrm{d}x$$

Future strategy: Firedrake for complicated VPs

Why? Complexity of eqns associated with VP for waveflap-driven water waves:

$$
\begin{aligned}
z^{n+1} =& \eta h^{n+1}/H_0, \quad x_\xi^{n+1,n} = \frac{(L_w - W^{n+1,n})}{L_w} + \frac{\eta}{H_0} W_z^{n+1,n} \frac{(L_w-\xi)}{L_w} h_\xi^{n+1}, && (2.69a)\\
x_\eta^{n+1,n} =& \frac{(L_w-\xi)}{L_w} W_z^{n+1,n} \frac{h^{n+1}}{H_0}, && (2.69b)\\
z_\xi^{n+1} =& \eta h_\xi^{n+1}/H_0, \quad z_\eta^{n+1} = h^{n+1}/H_0, && (2.69c)\\
W^{n+1,n} =& W(\eta h^{n+1}/H_0, \tau^n), \quad W_\tau^{n+1,n} = \partial_\tau W(z,\tau^n)|_{z=\eta h^{n+1}/H_0}, && (2.69d)\\
W_z^{n+1,n} =& \partial_z W(z,\tau^n)|_{z=\eta h^{n+1}/H_0}, && (2.69e)\\
\psi^* =& \psi^{n,n+1} \equiv \frac{\Pi^n}{1 - W(\eta h^{n+1}/H_0, \tau^n)/L_w}, && (2.69f)\\
J^{n+1,n} =& x_\xi^{n+1,n} z_\eta^{n+1} - x_\eta^{n+1,n} z_\xi^{n+1}, && (2.69g)\\
0 =& \delta \int_0^{L_s} \Big(\Pi^n \frac{h^{n+1}-h^n}{\Delta\tau} - \Pi^{n+1}\frac{h^{n+1}}{\Delta\tau} + (1-\xi/L_W) h_\xi^{n+1} W_\tau^{n+1,n} \psi^* \\
&+ x_\xi^{n+1,n} g\big(\frac{1}{2}(h^{n+1})^2 - H_0 h^{n+1}\big)\Big)|_{\eta=H_0}\, \mathrm{d}\xi \\
&+ \int_0^{H_0} \Big(z_\eta^n W_\tau^{n+1,n}(\psi^*\eta/H_0 + \varphi^*)) + x_\eta^{n+1,n} g\big(\frac{1}{2}(z^{n+1})^2 - H_0 z^{n+1}\big)\Big)|_{\xi=0}\, \mathrm{d}\eta \\
&+ \int_0^{L_s}\int_0^{H_0} \frac{1}{2J^{n+1,n}} \Big(((x_\eta^{n+1,n})^2 + (z_\eta^{n+1})^2)|(\eta/H_0)\partial_\xi\psi^* + \partial_\xi\varphi^*|^2 \\
&- 2(x_\xi^{n+1,n} x_\eta^{n+1,n} + z_\xi^{n+1} z_\eta^{n+1})((\eta/H_0)\partial_\xi\psi^* + \partial_\xi\varphi^*)(\psi^*/H_0 + \partial_\eta\varphi^*) \\
&+ ((x_\xi^{n+1,n})^2 + (z_\xi^{n+1,n})^2)|\psi^*/H_0 + \partial_\eta\varphi^*|^2\Big)\, \mathrm{d}\xi\, \mathrm{d}\eta. && (2.69h)
\end{aligned}
$$

In the above, the definitions in the first lines are short-hands for substitution in the VP in the last line. The two weak formulations following from the variations $\delta\Pi^n, \delta\varphi^*$ of (2.69h) are solved in unison, solving h^{n+1} and φ^*, followed by the explicit solver step in the weak formulation resulting from the variation δh^{n+1}, yielding Π^{n+1} of (2.69h). These variations will be undertaken automatically within *Firedrake* as well as the associated iterative solver for the coupled two weak formulations and the explicit solver for the last

Challenges Firedrake via VPs

Computational-science challenges within Firedrake?

- Coupled interior-free-surface system, two sets of mixed partially canonical variables.
- Weak forms automatically generated: via *fd.derivatives* wrt $\{h^{n+1}(x), \varphi^{n+1}(x,z)\}$ for $\{\phi^{n+1}(x,H_0), \varphi(x,z)^{n+1}\}$, and via
- *fd.derivative* wrt $\phi^{n+1}(x,H_0)$ for $h^{n+1}(x)$.
- Use linearised system as stepping stone, then add nonlinearity, add piston wavemaker, add waveflap wavemaker, ...

Outlook

- Firedrake simulation for extreme 3-soliton amplifications: speed-increase needed & MPI.
- Efficiency gain via Firedfrake implemenation via VPs & *fd.derivative*.
- Moving meshes for water waves within VPs (B 2022), overturning & breaking waves, FSI.

References

- Gidel et al. 2022/23: *EarthArxiv link* (has GitHub link).
- Choi, B, Kalogirou, Kelmanson (2022) *Water Waves* **4** (has GitHub/Zenodo links).
- Gidel 2018: PhD thesis Leeds. `https://etheses.whiterose.ac.uk/21730/`
- B, Kalogirou 2016: Variational Water Wave Modelling: from Continuum to Experiment. *Theory of Water Waves, London Math. Soc.* **426**. *preprint link*.
- Choi et al 2022: Crossing seas *YouTube movie –link*.