Modeling extreme water waves with Firedrake

Onno Bokhove with Yang Lu, Wajiha Rehman & Junho Choi

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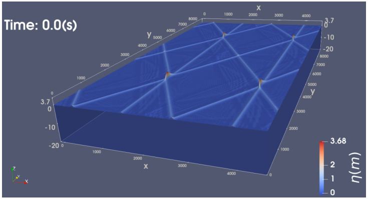


Summary with movies

Firedrake extreme wave modelling of *Benney-Luke (BL) eqns*:

- horizontally $\Phi(x, y, t), \eta(x, y, t)$: CG1, CG2-2-CG4; space-time discrete variational principle (VP),
- discrete VP: wave-amplitude & phase-space conservation,
- x-periodic mesh, 3 weak forms,
- mesh refinement needed; other element types?

Summary with movies

Firedrake modelling of (driven) potential-flow (PF) *water waves*:

- space-time discrete VP, variables $h(x, y, t), \tilde{\phi}(x, y, t) = \phi(x, y, b(x, y) + h(x, y, t), t), \phi(x, y, z, t)$ with mixed horizontal and vertical coordinates,
- transformation to fixed domain,
- now, vertical z: one element in vertical with Lagrange/Chebychev polynomials: user-arranged,
- space-time discrete VP in horizontal: CG1 polynomials; 3 to 5 weak forms.
- In progress: FD implementation of VPs for complicated moving domains in x, y, z.

Firedrake for extreme wave amplification in BL

- Origin 2010 *bore-soliton-splash*:
- To what extent do exact but idealised extreme- or rogue-wave solutions survive in more realistic settings?
- Rogue wave: $A_r \geq 2.2 \times A_{ambient}$.
- Will such extreme waves fall apart due to dispersion or other mechanisms?
- Use $4\times$ & $9\times$ KP amplifications of interacting solitons/cnoidal waves.
- What do you think: will we be able to reach the ninefold wave amplification in more realistic calculations or in reality?

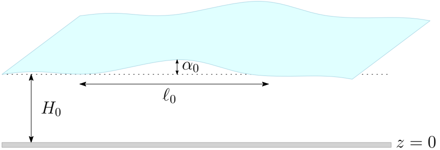
Mathematical hierarchy: BL and KP approximations

- Kadomtsev & Petviashvili (1970) eqn: unidirectional in 2DH
- Benney-Luke (1964) eqns –BL: bidirectional in 2DH

$$\partial_t \eta - \frac{\mu}{2}\partial_t \nabla^2 \eta + \nabla \cdot \left((1+\epsilon\eta)\, \nabla \Phi \right) - \frac{2\mu}{3}\nabla^4 \Phi = 0 \qquad \text{in } \Omega$$

$$\partial_t \Phi - \frac{\mu}{2}\partial_t \nabla^2 \Phi + \frac{\epsilon}{2}\left|\nabla \Phi\right|^2 + \eta = 0 \qquad \text{in } \Omega$$

$$\mathsf{n}\cdot\nabla\Phi = 0 \text{ on } \partial\Omega \quad \text{and} \quad \mathsf{n}\cdot\nabla(\nabla^2\Phi) = 0 \qquad \text{on } \partial\Omega$$

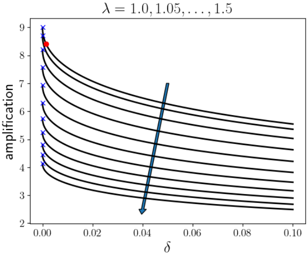


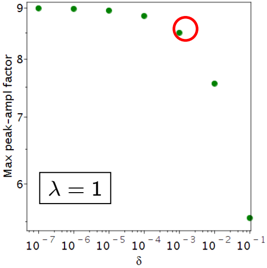
$\epsilon = \alpha_0/H_0 \ll 1$

$\mu = (H_0/\ell_0)^2 \ll 1$

Maximum 9-fold amplification in KP & BL?

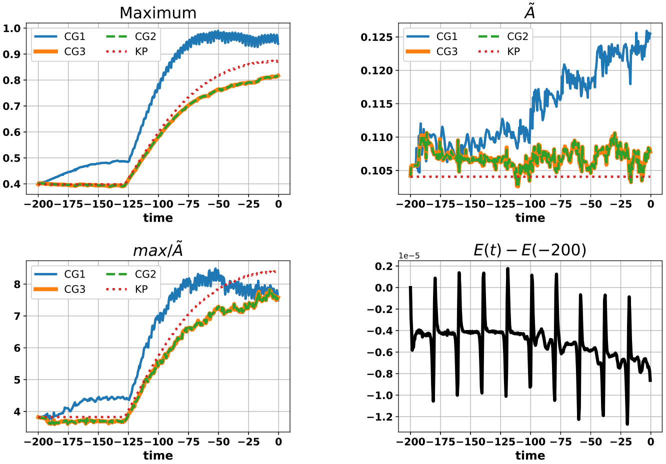
- Amplification: $\dfrac{u(X_*, Y_*, \tau_*)}{\tilde{A}} = \dfrac{(\sqrt{\lambda}+2)^2}{\lambda} + \mathcal{O}(\sqrt{\delta}) \xrightarrow[\delta=0,\lambda\to 1]{} 9$





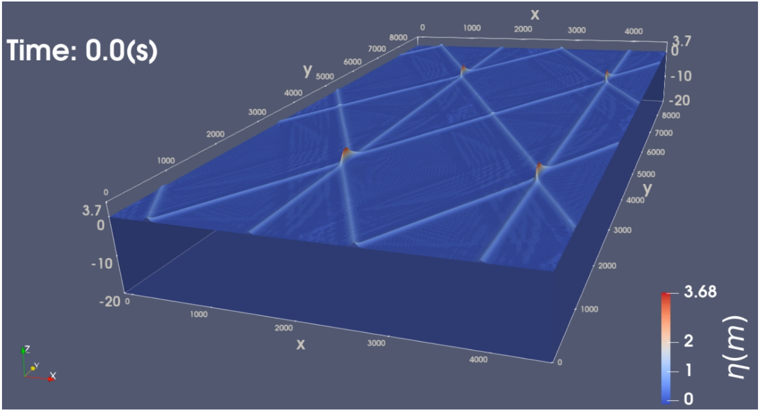
- Seed BL with KP-solution prior to maximum reached.
- Wave amplitude & phase space volume preserved via space-time discrete VP: no loss of amplitude.

Results BL-simulation three-soliton interaction



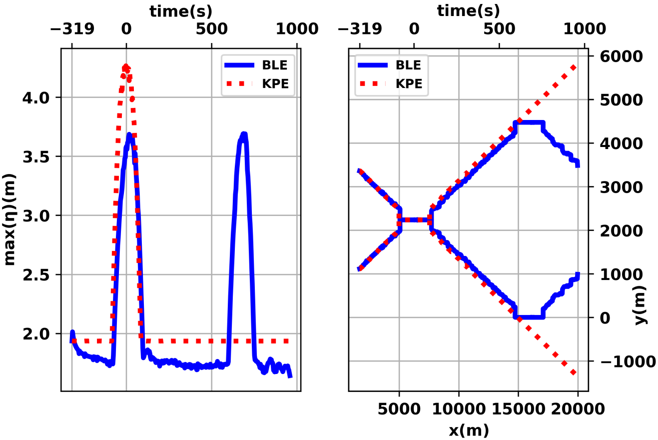
Results simulation three-soliton interaction (dimensional)

Crossing seas (4 or 8 domains combined –*YouTube*)

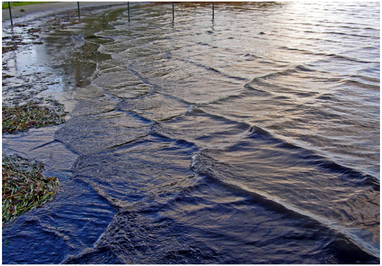


Results simulation three-soliton interaction (dimensional)

Cnoidal waves with periodicity in x, y, t (max. vs. t & x–y tracks):



Conclusion

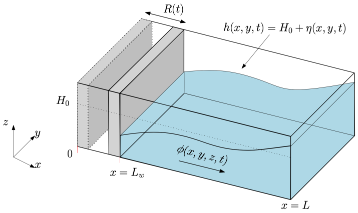


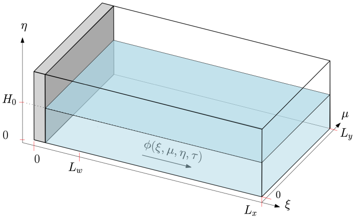
- Nine-fold soliton amplification shown theoretically, but only in limit $\delta \to 0$
- Web-soliton amplification KP≈ 8.4, simulated for BL≈ 7.8
- Amplifications achieved as simulated cnoidal crossing seas
- Local p or h mesh-refinement needed in x-periodic channel.
- Can amplifications survive in potential-flow equations?

Firedrake for potential flow water waves

Water-wave equations

$$
\begin{aligned}
\boldsymbol{\nabla}^2\phi &= 0 \quad \text{in} \quad \Omega\\
\partial_t\eta + \nabla\phi\cdot\nabla\eta - \partial_z\phi &= 0 \quad \text{at} \quad z = H_0+\eta\\
\partial_t\phi + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g\eta &= 0 \quad \text{at} \quad z = H_0+\eta\\
\mathbf{n}\cdot\nabla\phi &= 0 \quad \text{on} \quad z=0 \ \text{and} \ \partial\Omega
\end{aligned}
$$

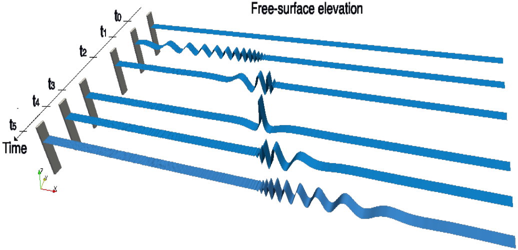




Current strategy: Firedrake in horizontal domain

Current strategy (Gidel 2018; Gidel et al. 2022/23)

- Time-dependent free surface & piston wavemaker, transformed to fixed domain.
- One element in vertical Lagrange/Chebychev $p = 4, 9$; user-arranged.
- Firedrake: space-time discrete VP in horizontal with CG1/CG2, Störmer-Verlet.
- Time-dependent VP, non-autonomous Hamiltonian $H(t)$ *(wavemaker)*.
- Good comparison with experimental data (Gidel et al 2022/23); soliton amplification similation in progress.



Future strategy: Firedrake via VPs

Goal: implement (time-discrete) VP & derive weak forms automatically.

- Why: VPs in moving domains complicated, e.g. wave tank with waveflap, wave-beam FSI, numerical mesh motion.
- Simple case; nonlinear PF in $\{x, z\}$, $\partial_y = 0, W = L_w \le L_x$, no wavemaker:

$$0 = \delta \int_0^T \int_0^{L_x} \int_0^{H_0} - \left[\frac{1}{2} \frac{L_w^2}{W} h(\phi_x + (z/h)h_x)\phi_z)^2 + \frac{1}{2} W \frac{H_0^2}{h} (\phi_z)^2 \right] \mathrm{d}z\mathrm{d}x \\ + \int_0^{L_x} -gH_0 Wh(\tfrac{1}{2}h - H_0) + H_0 W \phi|_{z=H_0} h_t \mathrm{d}x\mathrm{d}t$$

- Time-discrete version, coupled surface-interior system, use *fd.derivative*: $\delta h^{n+1}, \delta\varphi^{n+1}$, explicit fderivative ϕ^{n+1} to automatically derive weak forms:

$$0 = \quad \delta \int_0^{L_x} \int_0^{H_0} - \left[\frac{1}{2} \frac{L_w^2}{W} h^n(\phi_x^{n+1} + (z/h^n)h_x^n)\phi_z^{n+1})^2 + \frac{1}{2} W \frac{H_0^2}{h^n} (\phi_z^{n+1})^2 \right] \mathrm{d}z\mathrm{d}x \\ + \int_0^{L_x} -gH_0 Wh^n(\tfrac{1}{2}h^n - H_0) + H_0 W \phi^{n+1}|_{z=H_0} \frac{(h^{n+1} - h^n)}{\Delta t} + H_0 W \phi^n|_{z=H_0} \frac{h^n}{\Delta t} \mathrm{d}x$$

Future strategy: Firedrake for complicated VPs

Why? Complexity of eqns associated with VP for waveflap-driven water waves:

$$
\begin{aligned}
z^{n+1} &= \eta h^{n+1}/H_0, \quad x_\xi^{n+1,n} = \frac{(L_w - W^{n+1,n})}{L_w} + \frac{\eta}{H_0} W_z^{n+1,n} \frac{(L_w-\xi)}{L_w} h_\xi^{n+1}, && \text{(2.69a)}\\
x_\eta^{n+1,n} &= \frac{(L_w-\xi)}{L_w} W_z^{n+1,n} \frac{h^{n+1}}{H_0}, && \text{(2.69b)}\\
z_\xi^{n+1} &= \eta h_\xi^{n+1}/H_0, \quad z_\eta^{n+1} = h^{n+1}/H_0, && \text{(2.69c)}\\
W^{n+1,n} &= W(\eta h^{n+1}/H_0, \tau^n), \quad W_\tau^{n+1,n} = \partial_\tau W(z,\tau^n)|_{z=\eta h^{n+1}/H_0}, && \text{(2.69d)}\\
W_z^{n+1,n} &= \partial_z W(z,\tau^n)|_{z=\eta h^{n+1}/H_0}, && \text{(2.69e)}\\
\psi^* &= \psi^{n,n+1} \equiv \frac{\Pi^n}{1 - W(\eta h^{n+1}/H_0, \tau^n)/L_w}, && \text{(2.69f)}\\
J^{n+1,n} &= x_\xi^{n+1,n} z_\eta^{n+1} - x_\eta^{n+1,n} z_\xi^{n+1}, && \text{(2.69g)}\\
0 &= \delta \int_0^{L_s} \Big(\Pi^n \frac{h^{n+1}-h^n}{\Delta\tau} - \Pi^{n+1} \frac{h^{n+1}}{\Delta\tau} + (1-\xi/L_W) h_\xi^{n+1} W_\tau^{n+1,n} \psi^* \\
&\quad + x_\xi^{n+1,n} g\big(\tfrac{1}{2}(h^{n+1})^2 - H_0 h^{n+1}\big)\Big)\Big|_{\eta=H_0} \,\mathrm{d}\xi \\
&\quad + \int_0^{H_0} \Big(z_\eta^n W_\tau^{n+1,n} (\psi^* \eta/H_0 + \varphi^*)) + x_\eta^{n+1,n} g\big(\tfrac{1}{2}(z^{n+1})^2 - H_0 z^{n+1}\big)\Big)\Big|_{\xi=0} \,\mathrm{d}\eta \\
&\quad + \int_0^{L_s}\int_0^{H_0} \frac{1}{2J^{n+1,n}} \Big(((x_\eta^{n+1,n})^2 + (z_\eta^{n+1})^2) |(\eta/H_0)\partial_\xi \psi^* + \partial_\xi \varphi^*|^2 \\
&\quad - 2(x_\xi^{n+1,n} x_\eta^{n+1,n} + z_\xi^{n+1} z_\eta^{n+1})((\eta/H_0)\partial_\xi \psi^* + \partial_\xi \varphi^*)(\psi^*/H_0 + \partial_\eta \varphi^*) \\
&\quad + ((x_\xi^{n+1,n})^2 + (z_\xi^{n+1,n})^2)|\psi^*/H_0 + \partial_\eta \varphi^*|^2 \Big) \,\mathrm{d}\xi\,\mathrm{d}\eta. && \text{(2.69h)}
\end{aligned}
$$

In the above, the definitions in the first lines are short-hands for substitution in the VP in the last line. The two weak formulations following from the variations $\delta\Pi^n, \delta\varphi^*$ of (2.69h) are solved in unison, solving h^{n+1} and φ^*, followed by the explicit solver step in the weak formulation resulting from the variation δh^{n+1}, yielding Π^{n+1} of (2.69h). These variations will be undertaken automatically within *Firedrake* as well as the associated iterative solver for the coupled two weak formulations and the explicit solver for the last

Challenges Firedrake via VPs

Computational-science challenges within Firedrake?

- Coupled interior-free-surface system, two sets of mixed partially canonical variables. ($\phi(x,z,t) = \phi(x,z=H_0,t) + \varphi(x,z,t)$.)
- Weak forms automatically generated: via *fd.derivatives* wrt $\{h^{n+1}(x), \varphi^{n+1}(x,z)\}$ for $\{\phi^{n+1}(x,H_0), \varphi(x,z)^{n+1}\}$, and via
- *fd.derivative* wrt $\phi^{n+1}(x,H_0)$ for $h^{n+1}(x)$.
- Use linearised system as stepping stone, then add nonlinearity, add piston wavemaker, add waveflap wavemaker, ...
- *Solution-1:* Use extruded mesh with tensor-product spaces $CG \times R$ (cst in z-direction) and $CG \times CG$.
- *Solution-2:* Take *fd.derivative* wrt $\phi(x,H_0,t)$ and $\phi(x,z,t), z \neq H_0$ –is that possible? *Issue:* $\phi(x,z=H_0,t)$ is Dirichlet bc for $\nabla^2\phi = 0$.
- Speed –battle against special-purpose potential-flow codes (not compatible in space and time), JCP review.
- "Time-to-scheme" argument, MPI, improve preconditioning.

Firedrake via VP: linear potential flow

- Firedrake:

```
VP11 = ( fd.inner(phii, (eta_new - eta)/dt) + fd.inner(phi_f, eta/dt) - (1/2 * gg * fd.inner(eta, eta)) )* fd.ds_t \
    - ( 1/2 * fd.inner(fd.grad(phii+varphii), fd.grad(phii+varphii))  ) * fd.dx

# Step-1 and 2 must be solved in tandem: f-derivative VP wrt eta to find update of phi at free surface
# int -phi/dt + phif/dt - gg*et) delta eta ds a=0 -> (phi-phif)/dt = -gg * eta
phif_expr1 = fd.derivative(VP11, eta, du=vvp0)  # du=v_W represents perturbation # 23-12 Make du split variable

# Step-2: f-derivative VP wrt varphi to get interior phi given surface update phi
# int nabla (phi+varphi) cdot nabla delta varphi dx = 0
phi_expr1 = fd.derivative(VP11, varphii, du=vvp1)
Fexpr = phif_expr1+phi_expr1
phi_combo = fd.NonlinearVariationalSolver(fd.NonlinearVariationalProblem(Fexpr, result_mixed, bcs =
[BC_exclude_beyond_surface_mixed,BC_varphi_mixed]))

#
# Step-3: f-derivative wrt phi but restrict to free surface to find updater eta_new; only solve for eta_new by using exclude
eta_expr2 = fd.derivative(VP11, phii, du=v_R)
eta_expr = fd.NonlinearVariationalSolver(fd.NonlinearVariationalProblem(eta_expr2,eta_new,bcs=BC_exclude_beyond_surface))
```

Outlook

- Firedrake simulation for extreme 3-soliton amplifications: speed-increase needed & MPI.
- Efficiency gain via Firedrake implemenation using VPs & *fd.derivative*.
- Geometric time-stepping schemes: SE (DG0), SV (CG1-DG1), 6-step DG2 scheme, $\cdots$ can be phrased as (FEM-in-time) VPs.
- Moving meshes for water waves within VPs (B 2022), overturning & breaking waves, FSI (linear 3D case works).

References

- Gidel et al. 2022/23: *EarthArxiv link* (has GitHub link).
- B 2022: Variational water-wave modeling: from deep water to beaches. In: *Mathematics of Marine Modeling*.
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- B, Kalogirou 2016: Variational Water Wave Modelling: from Continuum to Experiment. *Theory of Water Waves, London Math. Soc.* **426**. *preprint link*.
- Choi et al 2022: Crossing seas *YouTube movie –link*.