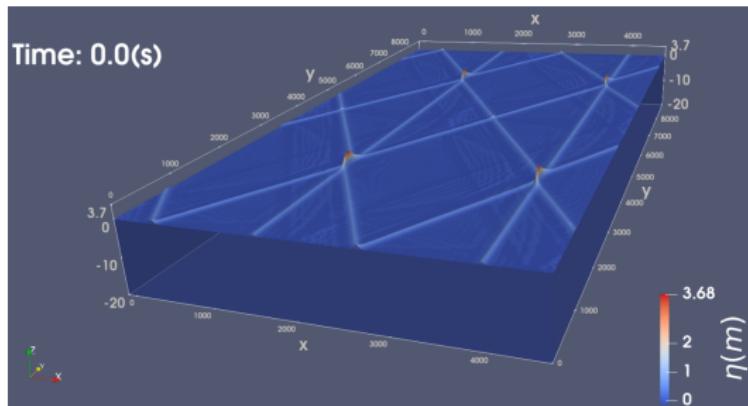


# Analysis and Simulations of Three-Soliton Interactions with Extreme Wave-Amplification in a Hierarchy of Water-Wave Models

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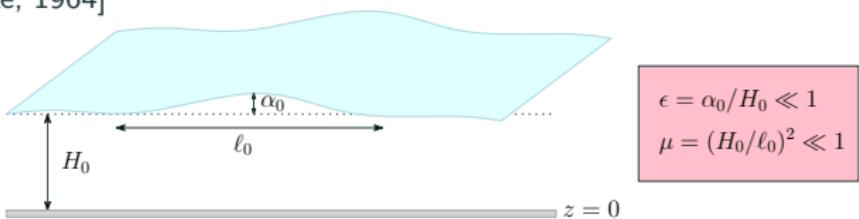
# Motivation on modelling extremely high water waves

- Origin 2010 *bore-soliton-splash*:
- To what extent do exact but idealised extreme- or rogue-wave solutions survive in more realistic settings?
- Will such extreme waves fall apart due to dispersion or other mechanisms?
- Use fourfold and ninefold KP amplifications of interacting solitons/cnoidal waves.
- What do you think: will we be **able to reach the ninefold wave amplification** in more realistic calculations, using potential-flow dynamics, or in reality?



# Mathematical hierarchy: PFE, BLE & KPE approximations

- ~> Boussinesq-type approximation: includes weak dispersive effects
  - KdV equation: wave propagation in 1D [Korteweg & de Vries, 1895]
  - **KPE equation**: unidirectional propagation in 2DH [Kadomtsev & Petviashvili, 1970]
  - **Benney-Luke equations –BLE**: bidirectional propagation in 2DH [Benney & Luke, 1964]



Expansion about the sea-bed potential  $\Phi(x, y, t) = \phi(x, y, z = 0, t)$ , in powers of the small parameter  $\mu$  [Pego & Quintero, 1999]

- More realistic or parent **potential-flow equations (PFE)**.

## Kadomtsev-Petviashvili (KPE) equation

The KPE equation can be obtained from the Benney-Luke equations by introducing the formal perturbation expansions

$$\eta = \tilde{u} + \mathcal{O}(\epsilon^2), \quad \Phi = \sqrt{\epsilon} \left( \tilde{\Psi} + \mathcal{O}(\epsilon^2) \right),$$

using the transformations

$$X = \sqrt{\frac{\epsilon}{\mu}} \left( \frac{3}{\sqrt{2}} \right)^{1/3} (x - t), \quad Y = \sqrt{\epsilon} \sqrt{\frac{\epsilon}{\mu}} \left( \frac{3}{\sqrt{2}} \right)^{2/3} y,$$
$$\tau = \epsilon \sqrt{\frac{2\epsilon}{\mu}} t, \quad u = \left( \frac{3}{4} \right)^{1/3} \tilde{u},$$

and taking  $\mu = \epsilon^2$ , resulting in the **KPE equation** in “standard” form

$$\partial_X (4\partial_\tau u + 6u\partial_X u + \partial_{XXX} u) + 3\partial_{YY} u = 0$$

This equation includes weak effects in the  $y$ -direction.

# Exact solution of the KP equation

Web and line-soliton solutions can be constructed using Hirota's transformation

$$u(X, Y, \tau) = 2\partial_{XX} \ln K(X, Y, \tau) = \frac{2\partial_{XX} K}{K} - 2\left(\frac{\partial_X K}{K}\right)^2,$$

where function  $K(X, Y, \tau)$  can be obtained from the Wronskian

$$K(X, Y, \tau) = \begin{vmatrix} f_1 & f_1^{(1)} & \dots & f_1^{(N-1)} \\ f_2 & f_2^{(1)} & \dots & f_2^{(N-1)} \\ \vdots & \vdots & & \vdots \\ f_N & f_N^{(1)} & \dots & f_N^{(N-1)} \end{vmatrix}.$$

Particular soliton solutions are obtained by taking [Kodama, 2010]

$$f_i = \sum_{j=1}^M a_{ij} e^{\theta_j}, \quad \text{where } \theta_j = k_j X + k_j^2 Y - k_j^3 \tau,$$

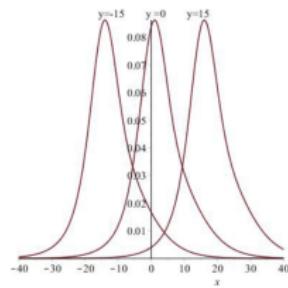
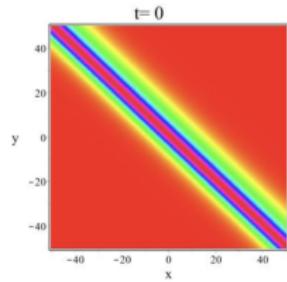
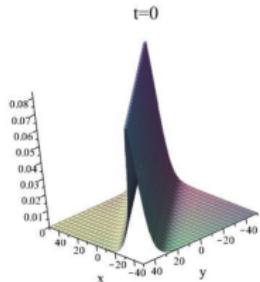
with coefficients  $k_j$  being ordered as  $k_1 < k_2 < \dots < k_M$ . This solution is called a  $(N_-, N_+)$ -soliton, comprising line solitons in the far-field  $Y \rightarrow \pm\infty$ .

## Example: single line soliton

Single line solitons have  $(N, M) = (1, 2)$ , resulting in  $K = f_1 = e^{\theta_1} + e^{\theta_2}$  and the line soliton solution is

$$\begin{aligned} u(X, Y, \tau) &= \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1 - \theta_2) \\ &= \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2 \frac{1}{2}((k_1 - k_2)X + (k_1^2 - k_2^2)Y - (k_1^3 - k_2^3)\tau). \end{aligned}$$

The soliton amplitude is  $\tilde{A} = \frac{1}{2}(k_1 - k_2)^2$  and its centreline is found by setting the  $\operatorname{sech}^2$  argument to zero.



## Example: two interacting line solitons

Two line solitons have  $(N, M) = (2, 4)$ , also called  $(2, 2)$ -solitons or  $O$ -solitons, obtained with functions  $f_1 = e^{\theta_1} + e^{\theta_2}$ ,  $f_2 = e^{\theta_3} + e^{\theta_4}$ , and

$$K(X, Y, \tau) = (k_3 - k_1)e^{\theta_1 + \theta_3} + (k_3 - k_2)e^{\theta_2 + \theta_3} + (k_4 - k_1)e^{\theta_1 + \theta_4} + (k_4 - k_2)e^{\theta_2 + \theta_4}.$$

## Example: two interacting line solitons

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In the far field  $Y \rightarrow \pm\infty$ , we find the single line solitons

$$u_{[1,2]}(X, Y, \tau) = \frac{1}{2}(k_2 - k_1)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1 - \theta_2 - \ln a),$$

$$u_{[3,4]}(X, Y, \tau) = \frac{1}{2}(k_4 - k_3)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_3 - \theta_4 - \ln b),$$

where  $a, b$  depend on  $k_j$ .

## Example: two interacting line solitons

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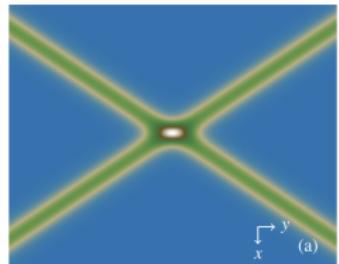
$$u_{[1,2]}(X, Y, \tau) = \frac{1}{2}(k_2 - k_1)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1 - \theta_2 - \ln a),$$

$$u_{[3,4]}(X, Y, \tau) = \frac{1}{2}(k_4 - k_3)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_3 - \theta_4 - \ln b),$$

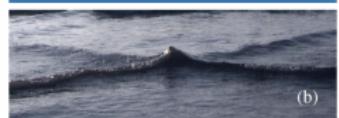
where  $a, b$  depend on  $k_j$ . For equal far-field soliton amplitudes  $\tilde{A} = \frac{1}{2}(k_2 - k_1)^2 = \frac{1}{2}(k_4 - k_3)^2$ , the solution satisfies [Kodama, 2010]

$$2\tilde{A} \leq \max_{(X, Y, \tau)} u(X, Y, \tau) \leq 2 \left( 1 + \frac{1 - \sqrt{\Delta_o}}{1 + \sqrt{\Delta_o}} \right) \tilde{A},$$

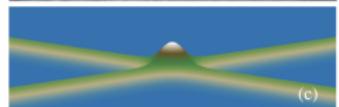
where  $0 \leq \Delta_o \leq 1$ , hence  $2\tilde{A} \leq \max u \leq 4\tilde{A}$ .



(a)



(b)



(c)

## Example: three interacting line solitons

Three line solitons, known as  $(3, 3)$ -solitons, have  $(N, M) = (3, 6)$  and functions  $f_1 = e^{\theta_1} + e^{\theta_2}$ ,  $f_2 = e^{\theta_3} + e^{\theta_4}$ ,  $f_3 = e^{\theta_5} + e^{\theta_6}$ , and

$$K(X, Y, \tau) = \underbrace{A_{135}}_{e^{\theta_1+\theta_3+\theta_5}} e^{\theta_1+\theta_3+\theta_5} + \underbrace{A_{235}}_{e^{\theta_2+\theta_3+\theta_5}} e^{\theta_2+\theta_3+\theta_5} + \underbrace{A_{136}}_{e^{\theta_1+\theta_3+\theta_6}} e^{\theta_1+\theta_3+\theta_6} + A_{236} e^{\theta_2+\theta_3+\theta_6} \\ + A_{145} e^{\theta_1+\theta_4+\theta_5} + \underbrace{A_{245}}_{e^{\theta_2+\theta_4+\theta_5}} e^{\theta_2+\theta_4+\theta_5} + \underbrace{A_{146}}_{e^{\theta_1+\theta_4+\theta_6}} e^{\theta_1+\theta_4+\theta_6} + \underbrace{A_{246}}_{e^{\theta_2+\theta_4+\theta_6}} e^{\theta_2+\theta_4+\theta_6},$$

with parameter ordering  $k_1 < k_2 < k_3 < 0 < k_4 < k_5 < k_6$  &  $a, b = 1, c$ .

## Example: three interacting line solitons

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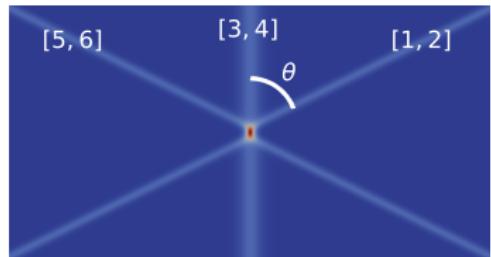
with parameter ordering  $k_1 < k_2 < k_3 < 0 < k_4 < k_5 < k_6$  &  $a, b = 1, c$ .

In the far field  $Y \rightarrow \pm\infty$ , we find the single line solitons

$$u_{[1,2]} \approx \frac{1}{2}(k_2 - k_1)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1 - \theta_2 - \ln \tilde{a}),$$

$$u_{[5,6]} \approx \frac{1}{2}(k_6 - k_5)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_5 - \theta_6 - \ln \tilde{b}),$$

$$u_{[3,4]} \approx \frac{1}{2}(k_4 - k_3)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_3 - \theta_4),$$



with  $\theta_i - \theta_j = (k_i - k_j) \left( X + (k_i + k_j)Y - (k_i^2 + k_i k_j + k_j^2)\tau \right)$ .

## Example: three interacting line solitons

Parameters  $k_1, \dots, k_6$  are determined from

$$k_3 + k_4 = 0$$

$$k_5 + k_6 = -(k_1 + k_2) = \tan \theta$$

$$k_4 - k_3 = \sqrt{2\tilde{A}}$$

$$k_6 - k_5 = k_2 - k_1 = \sqrt{2\tilde{A}/\lambda}$$

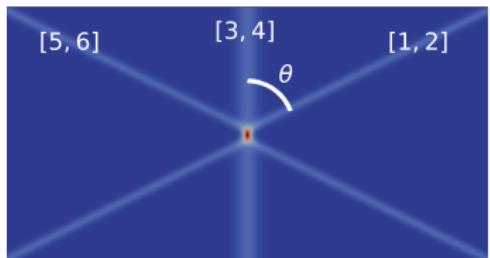
where angle  $\theta > 0$ ,  
 $\tilde{A} = \frac{1}{2}(k_4 - k_3)^2$  is the  
amplitude of the [3, 4]  
soliton, and the outer two  
solitons are assumed to have  
amplitude  $\tilde{A}/\lambda$ , for  $\lambda \geq 1$ .

Solving the above six equations, gives

$$k_6 = -k_1 = \sqrt{\tilde{A}} \left( \sqrt{2/\lambda} + \sqrt{1/2} + \delta \right)$$

$$k_5 = -k_2 = \sqrt{\tilde{A}} \left( \sqrt{1/2} + \delta \right)$$

$$k_4 = -k_3 = \sqrt{\tilde{A}/2}$$

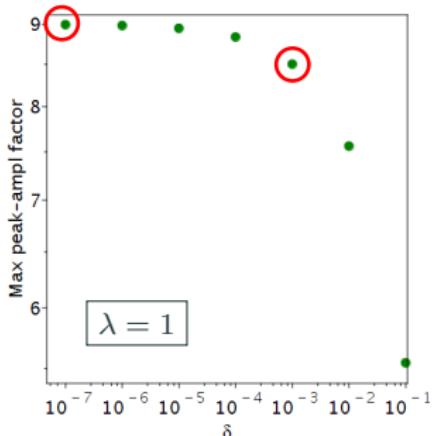
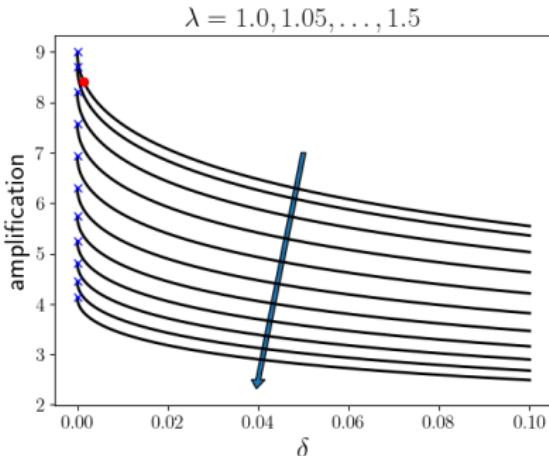


where  $\delta$  is defined by

$$\delta = \frac{\tan \theta}{2\sqrt{\tilde{A}}} - \left( \sqrt{1/2\lambda} + \sqrt{1/2} \right) > 0.$$

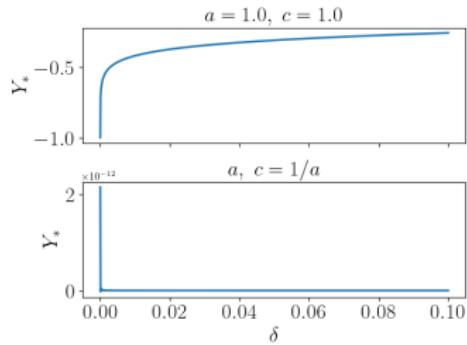
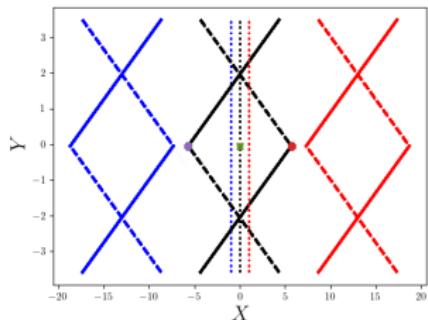
# Maximum 9-fold amplification in KP

- Proof is based on a **geometric argument** (additional secondary proof)
- Find **5 centrelines** of each of three line solitons (no phase shift at peak)
- Look for intersection points  $\rightsquigarrow$  this gives two values of  $Y$ , with mean at a unique point  $Y_{*\delta \rightarrow 0} \rightarrow -\infty$  when  $\tau_* = 0$  and  $X_* = 0$
- The space-time point of maximum amplification is  $(X_*, Y_*, \tau_*)$
- Amplification:** 
$$\frac{u(X_*, Y_*, \tau_*)}{\tilde{A}} = \frac{(\sqrt{\lambda} + 2)^2}{\lambda} + \mathcal{O}(\sqrt{\delta}) \xrightarrow[\delta=0]{} 1 + \frac{4}{\lambda} + \frac{4}{\sqrt{\lambda}}$$



# Proof of maximum 9-fold amplification in KP

- Three shift parameters  $a, b = 1, c = 1/a$  can be optimised such that splash occurs at  $(X^*, Y^*, \tau^*) = (0, 0, 0)$ .
- Amplification  
 $u(X^*, Y^*, \tau^*)/\tilde{A} = 9 - 8\sqrt{3}\sqrt[4]{2}\sqrt{\delta} + 16\sqrt{2}\delta - 192^{3/4}\sqrt{3}\delta^{3/2}/3$
- Principle Minor Theorem proofs that  $(X^*, Y^*, \tau^*)$  is a maximum.
- Involved and combined geometrical and analytical proofs.



# Numerical implementation



*Firedrake*

*An automated system for the solution of PDEs  
using the Finite Element Method (FEM).*

*Firedrake* employs Unified Form Language (UFL) and linear & non-linear solvers PETSc solvers [Rathgeber et al., 2016].

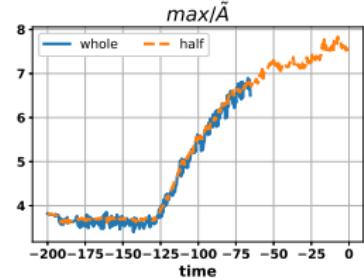
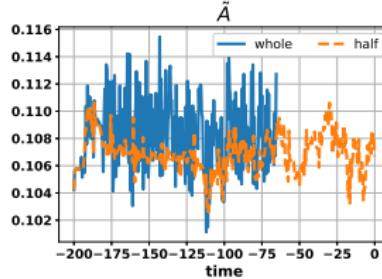
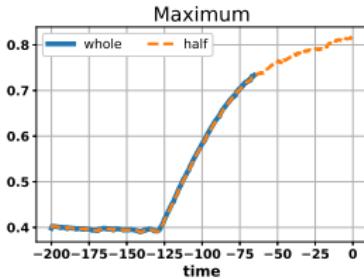
- Space-time discretisation 2nd order of **variational principle** for **BLE**: bounded energy oscillations, phase-space conserved.
- Continuous Galerkin (CG) FEM in space for VP, with approximations & test functions/variations  $\delta\eta_h$ ,  $\delta\Phi_h$ :

$$\eta(x, y, t) \approx \eta_h(x, y, t) = \sum_k \eta_k(t) w_k(x, y), \dots$$

- Symplectic Störmer-Verlet time stepping scheme.
- **Stable numerical scheme**: no artificial amplitude damping ...

## Computational domain: $\sim$ cnoidal waves

- KP solutions hold on infinite horizontal plane, so domain has to be sufficiently large to eliminate reflection at boundaries.
- Solutions can be set to become **approximately periodic** in sufficiently large domains.
- Transform  $\Phi = U_0(y)x + c_0(y) + \tilde{\Phi}$ , where  $\tilde{\Phi}$  is periodic, then solve the Benney-Luke equations for  $\eta$  and  $\tilde{\Phi}$ .
- Doubly or singly periodic domain?



## Initial conditions and boundaries

Initial condition consists of two (SP2) or three (SP3) line solitons, expressions of which are known from the KP-solution:

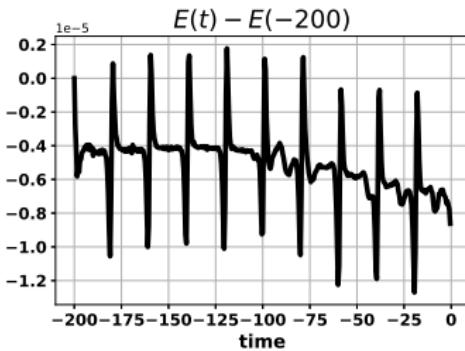
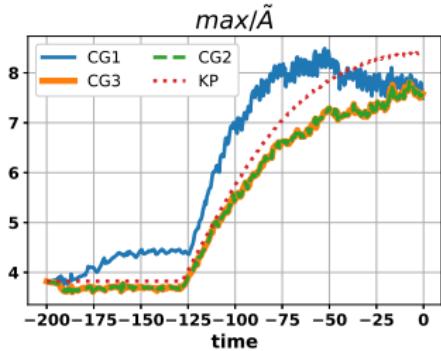
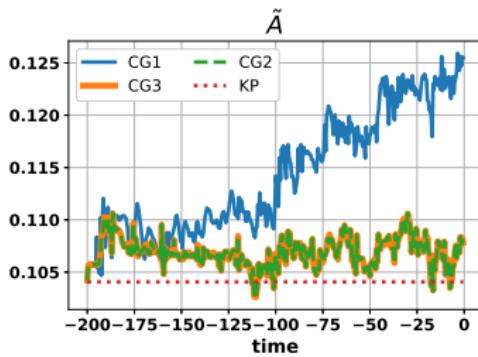
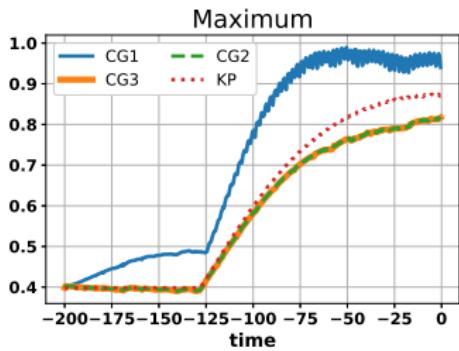
$$\eta_0(x, y) = \eta(x, y, t_0) = 2\left(\frac{4}{3}\right)^{1/3} \partial_{XX} \ln K(X, Y, \tau),$$

$$\varPhi_0(x, y) = \varPhi(x, y, t_0) = 2\sqrt{\epsilon} \left(\frac{4\sqrt{2}}{9}\right)^{1/3} \partial_X \ln K(X, Y, \tau).$$

Computational domain is constructed such that initial condition satisfies “periodic boundary conditions” in  $x$ -direction.

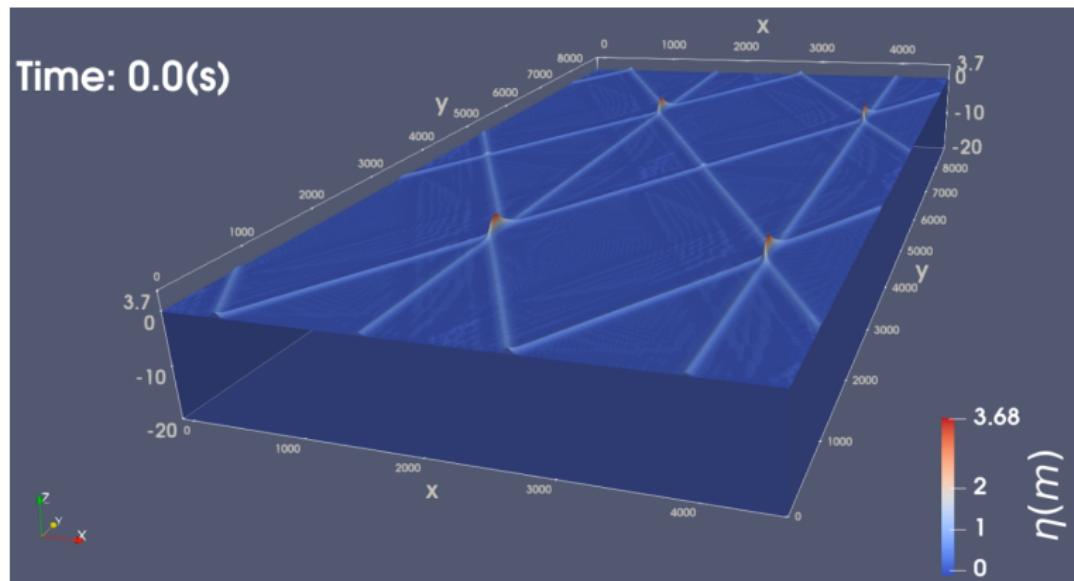
Case	$L_x$	$L_y$	$T$	$N_x$	$N_y$	$\Delta x = \frac{L_x}{N_x}$	$\Delta y = \frac{L_y}{N_y}$	$\Delta t$
SP2	10.3	40	50	132	480	0.0779	0.0833	0.005
SP3	20.9	47	200	252	564	0.0829	0.0833	0.005

# Results BLE-simulation three-soliton interaction



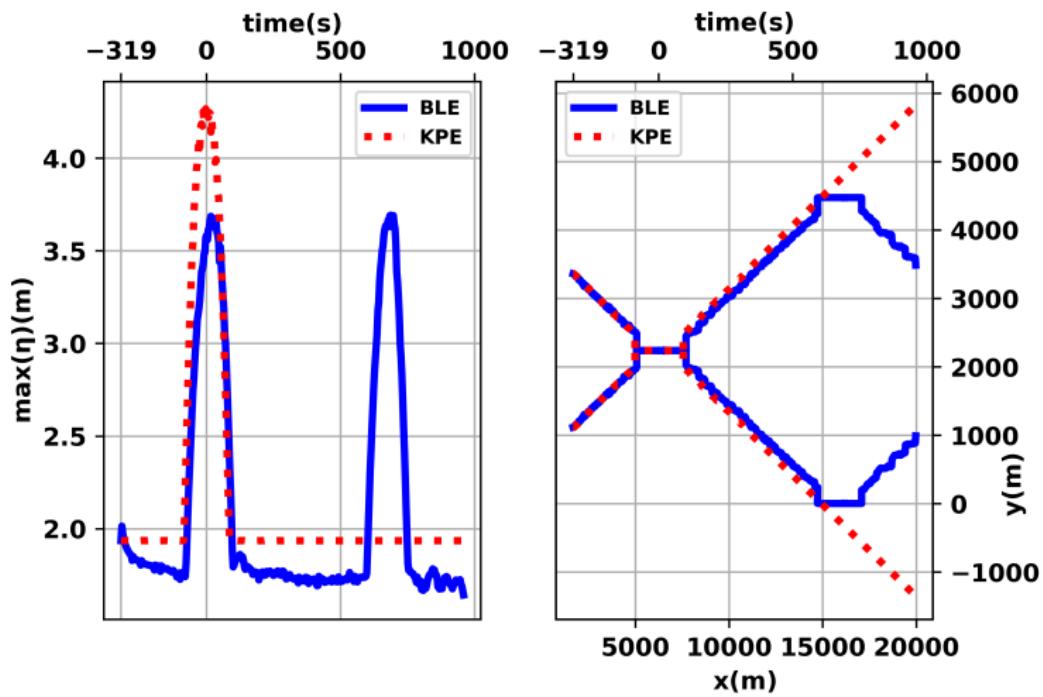
# Results simulation three-soliton interaction (dimensional)

*Crossing seas (4 or 8 domains combined – YouTube)*



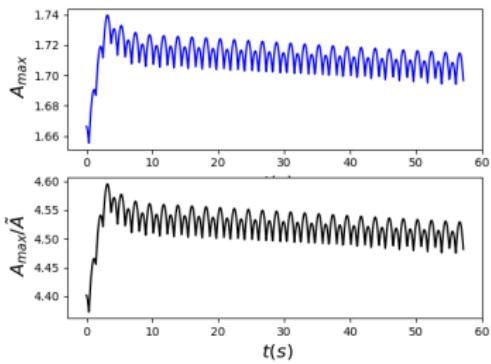
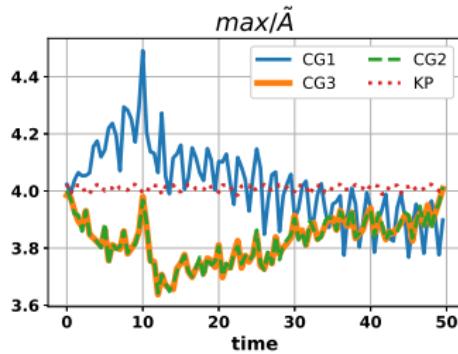
# Results simulation three-soliton interaction (dimensional)

Cnoidal waves with periodicity in  $x, y, t$  (max. vs.  $t$  &  $x-y$  tracks):



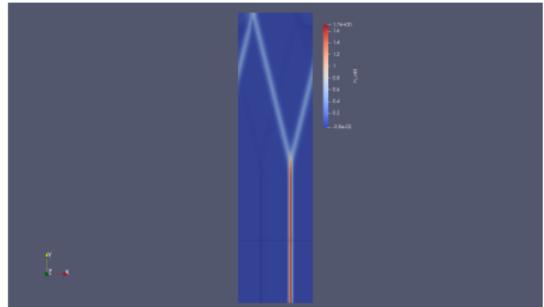
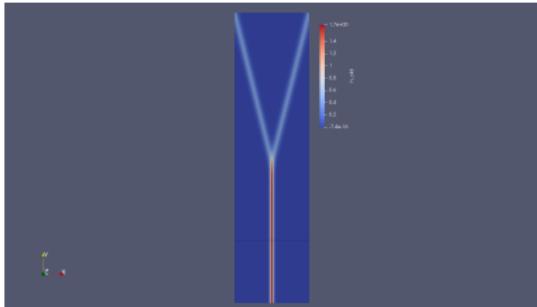
# New results BLE- & PFE-simulations three-soliton interaction

- BLE simulation seeded KPE  $\delta = 5 \times 10^{-11}$  yields **9 $\times$  amplification**  $t \in [-110, 0]$  (Junho Choi).
- PFE simulation based on 3+1D **discretisation of time-discrete variational principle (VP) in Firedrake**: robust, fast development, fewer human errors. MMP & SV time discretisations.
- Demanding **PFE simulations in progress**: HPC simulation with optimised ASM\* pre-conditioners. SP2:



# Summary

- Nine-fold soliton amplification proven, with stable limit  $\delta \rightarrow 0$
- Web-soliton amplification  $KPE \approx 8.4 \text{ & } 9$ , simulated for  $BLE \approx 7.8 \text{ & } 9$
- Amplifications achieved as simulated cnoidal crossing seas
- Rogue wave calculation:  $AI > 2 \text{ to } 4.0 = \max(\eta)/H_s$
- PFE simulations in progress, convergence not yet reached.



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