COMP5454M Assignment 1B

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1 Water-wave and wind-turbine interactions

1.1 Introduction

Exercise: Reflect briefly in one or a few paragraphs, i.e., in a short essay, on the methodological modelling choices made in this chapter. Juxtapose these choices during the course against other choices made in this course on FSI. Do this exercise after completing the other exercises.

To deal with water-wave and wind-turbine interaction problem in this chapter, the strategy can be summarised as follows. First, under the Eulerian framework, the nonlinear potential flow water-wave equations are derived by using variational principle and then they are linearised around a hydro-static state of rest. Second, the nonlinear equations of motion for the hyperelastic beam are derived by using variational principle under the Lagrangian framework. And they are linearised litho-static state of rest with a transformation to the Eulerian framework. Next, the fluid dynamics is coupled with the solid dynamics via a Lagrange multiplier, whose variation yields constraint at the fluid-solid interface. Correspondingly, the combined variational principle are derived and linearised to obtain the governing equations of the system. After eliminating the Lagrange mulyiplier, the linearised equations for the coupled system can be obatined and finally they are numerically solved through finite element discretisation.

1.2 Air-water (interface) dynamics using potential flow

Exercise: Write down and discuss what the "arbitrariness of variations" means and how it is used, in a (simplified) example.

Let's consider a simple example: soap film supported by a pair of coaxial rings [2]. The free energy of the soap film is equal to twice (once for each liquid-air interface) the surface tension σ of the soap solution times the area of the film. The film can therefore minimize its free energy by minimizing its area, and the axial symmetry suggests that the minimal surface will be a surface of revolution about the x axis. We therefore seek the profile $y(x)$ that makes the area

$$S = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,\mathrm{d}x \tag{1}$$

of the surface of revolution the least among all such surfaces bounded by the circles of radii $y(x_1) = y_1$ and $y(x_2) = y_2$. Here we can use variational principle to derive the differential equation that governing $y(x)$, i.e.,

$$\begin{aligned} 0 &= \delta J[y] && \text{(2a)} \\ &= \lim_{\epsilon \to 0} \frac{J[y+\epsilon\delta y] - J[y]}{\epsilon} && \text{(2b)} \end{aligned}$$

with functional

$$J[y] = 4\pi\sigma \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,\mathrm{d}x = \int_{x_1}^{x_2} f(x, y, y')\,\mathrm{d}x. \tag{2c}$$

Then

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{J[y+\epsilon\delta y]-J[y]}{\epsilon} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{x_1}^{x_2} f(x,y+\epsilon\delta y,y'+\epsilon\delta y')-f(x,y,y')\,\mathrm{d}x \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{x_1}^{x_2}\left[\epsilon\delta y\frac{\partial f}{\partial y}+\epsilon\frac{\mathrm{d}\delta y}{\mathrm{d}x}\frac{\partial f}{\partial y'}+O(\epsilon^2)\right]\mathrm{d}x \\ &= \delta y\frac{\partial f}{\partial y'}\bigg|_{x_1}^{x_2}+\int_{x_1}^{x_2}\delta y\left[\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right]\mathrm{d}x\end{aligned} \tag{3}$$

After using fixed endpoints condition $\delta y(x_1)=\delta y(x_2)=0$, the integrated-out part vanishes. Furthermore, considering arbitrariness of variations δy, we find that the functional $J[y]$ will be stationary when

$$\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)=0 \tag{4}$$

Mathematically, arbitrariness of variations means $\forall\delta y(x), x\in(x_1,x_2)$, we always have $\delta J=0$. Physically, it means when we introduce any arbitrary small perturbations, the system can always get back to its stationary state, where it has lowest energy. In this example, the stationary state of the soap film correspond to the state where it has minimum free energy, which means the profile $y(x)$ should satisfy (4). We begin by forming the partial derivatives

$$\frac{\partial f}{\partial y}=4\pi\sigma\sqrt{1+y'^2},\qquad \frac{\partial f}{\partial y'}=\frac{4\pi\sigma yy'}{\sqrt{1+y'^2}} \tag{5}$$

and substitute them into (4) we have

$$\sqrt{1+y'^2}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{yy'}{\sqrt{1+y'^2}}\right)=0. \tag{6}$$

Finally the solution is

$$y=\kappa\,\cosh\frac{x+a}{\kappa}, \tag{7}$$

where the constants κ and a should be selected to fit the endpoints $y(x_1)=y_1$ and $y(x_2)=y_2$.

1.3 Linearisation

Exercise: Show that the corresponding variational principal for (1.3.3) reads

$$0=\delta\int_0^T\left(\iint_{\Omega_H}\phi\partial_t\eta-\frac{1}{2}g\eta^2\mathrm{d}x\mathrm{d}y-\iint_{\Omega_H}\int_0^{H_0}\frac{1}{2}\left|\nabla\phi\right|^2\mathrm{d}z\mathrm{d}x\mathrm{d}y\right)\mathrm{d}t \tag{1.3.4}$$

by proving (1.3.3) results from (1.3.4). Observe that this derivation is easier than in the non-linear case because the domain is fixed.

Considering the end-point conditions $\delta\eta|_{t=0}=\delta\eta|_{t=T}=0$, the first two terms in (1.3.4) become

$$\begin{aligned}&\delta\int_0^T\left(\iint_{\Omega_H}\phi\partial_t\eta-\frac{1}{2}g\eta^2\mathrm{d}x\mathrm{d}y\right)\mathrm{d}t \\ &=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\iint_{\Omega_H}\left[(\phi+\epsilon\delta\phi)\,\partial_t\,(\eta+\epsilon\delta\eta)-\frac{1}{2}g\,(\eta+\epsilon\delta\eta)^2\right]-\left(\phi\partial_t\eta-\frac{1}{2}g\eta^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &=\int_0^T\iint_{\Omega_H}\phi\partial_t(\delta\eta)+\delta\phi\partial_t\eta-g\eta\delta\eta\,\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &=\iint_{\Omega_H}(\phi\delta\eta)|_{t=0}^{t=T}\,\mathrm{d}x\mathrm{d}y+\int_0^T\iint_{\Omega_H}\delta\phi\partial_t\eta-\delta\eta\partial_t\phi-g\eta\delta\eta\,\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &=\int_0^T\iint_{\Omega_H}\partial_t\eta\delta\phi-(\partial_t\phi+g\eta)\,\delta\eta\,\mathrm{d}x\mathrm{d}y\mathrm{d}t.\end{aligned} \tag{8}$$

After using Gauss's theorem and considering that the outward normal at the upper surface $z = H_0$ is $\hat{\mathbf{n}} = (0,0,1)^T$ and solid wall boundary conditions $\hat{\mathbf{n}} \cdot \nabla\phi = 0$ at domain boundaries $\partial\Omega_w$, the last term in (1.3.4) becomes

$$
\begin{aligned}
\delta \int_0^T \iint_{\Omega_H} \int_0^{H_0} -\frac{1}{2}|\nabla\phi|^2 \,\mathrm{d}z\mathrm{d}x\mathrm{d}y\mathrm{d}t &= \lim_{\epsilon\to 0} -\frac{1}{2\epsilon} \int_0^T \iint_{\Omega_H} \int_0^{H_0} |\nabla(\phi+\epsilon\delta\phi)|^2 - |\nabla\phi|^2 \,\mathrm{d}z\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= \int_0^T \iint_{\Omega_H} \int_0^{H_0} -\nabla\phi\cdot\nabla(\delta\phi)\,\mathrm{d}z\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= \int_0^T \iint_{\Omega_H} \int_0^{H_0} \nabla^2\phi\delta\phi - \nabla\cdot(\delta\phi\nabla\phi)\,\mathrm{d}z\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= \int_0^T \iint_{\Omega_H} \int_0^{H_0} \nabla^2\phi\delta\phi\,\mathrm{d}z\mathrm{d}x\mathrm{d}y\mathrm{d}t - \int_0^T \iint_{\Omega_H} \partial_z\phi\delta\phi\,\mathrm{d}x\mathrm{d}y\mathrm{d}t.
\end{aligned}
\tag{9}
$$

Combining (8) and (9), the equation (1.3.4) becomes

$$
0 = \int_0^T \iint_{\Omega_H} (\partial_t\eta - \partial_z\phi)\delta\phi - (\partial_t\phi + g\eta)\,\delta\eta\,\mathrm{d}x\mathrm{d}y\mathrm{d}t + \int_0^T \iint_{\Omega_H} \int_0^{H_0} \nabla^2\phi\delta\phi\,\mathrm{d}z\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{10}
$$

Finally, using the arbitrariness of variations $\delta\phi$ and $\delta\eta$, we can derive the linearised equations of motion (1.3.3).

Exercise: Explain what kind of coordinate framework has been used, Eulerian or Lagrangian, in the above exposition, and why?

We use Eulerian framework in the above mathematical modelling, because the observer is fixed in space and each physical quantity is a function of spatial coordinates.

1.4 Hyperelastic equations for a wind-turbine mast

Exercise: Verify (1.4.6) and (1.4.8) in detail.

The variational principle reads

$$
0 = \delta \int_0^T \iiint_{\Omega_O} \rho_0\mathbf{U}\cdot\partial_t\mathbf{X} - \frac{1}{2}\rho_0|\mathbf{U}|^2 - \rho_0 gZ - W(\underline{\mathbf{E}})\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t \tag{1.4.1}
$$

After using end-point conditions $\delta\mathbf{X}(\mathbf{a},0) = \delta\mathbf{X}(\mathbf{a},T) = 0$, the first term in (1.4.1) becomes

$$
\begin{aligned}
&\delta \int_0^T \iiint_{\Omega_O} \rho_0\mathbf{U}\cdot\partial_t\mathbf{X}\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T \iiint_{\Omega_O} \rho_0(\mathbf{U}+\epsilon\delta\mathbf{U})\cdot\partial_t(\mathbf{X}+\epsilon\delta\mathbf{X}) - \rho_0\mathbf{U}\cdot\partial_t\mathbf{X}\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t \\
&= \int_0^T \iiint_{\Omega_O} \rho_0\delta\mathbf{U}\cdot\partial_t\mathbf{X} + \rho_0\mathbf{U}\cdot\partial_t(\delta\mathbf{X})\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t \\
&= \int_0^T \iiint_{\Omega_O} \rho_0\delta\mathbf{U}\cdot\partial_t\mathbf{X} - \rho_0\delta\mathbf{X}\cdot\partial_t\mathbf{U}\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t + \iiint_{\Omega_O} \rho_0\mathbf{U}\cdot\delta\mathbf{X}\big|_{t=0}^{t=T}\,\mathrm{d}a\mathrm{d}b\mathrm{d}c \\
&= \int_0^T \iiint_{\Omega_O} \rho_0\delta\mathbf{U}\cdot\partial_t\mathbf{X} - \rho_0\delta\mathbf{X}\cdot\partial_t\mathbf{U}\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t
\end{aligned}
\tag{11}
$$

The second term in (1.4.1):

$$
\begin{aligned}
\delta \int_0^T \iiint_{\Omega_O} -\frac{1}{2}\rho_0|\mathbf{U}|^2\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T \iiint_{\Omega_O} -\frac{1}{2}\rho_0|\mathbf{U}+\epsilon\delta\mathbf{U}|^2 + \frac{1}{2}\rho_0|\mathbf{U}|^2\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t \\
&= \int_0^T \iiint_{\Omega_O} -\rho_0\mathbf{U}\cdot\delta\mathbf{U}\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t
\end{aligned}
\tag{12}
$$

The third term in (1.4.1):

$$\begin{aligned}\delta\int_0^T\iiint_{\Omega_O}-\rho_0 gZ\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\iiint_{\Omega_O}-\rho_0 g\delta_{3l}(X_l+\epsilon\delta X_l)+\rho_0 g\delta_{3l}X_l\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t\\ &= \int_0^T\iiint_{\Omega_O}-\rho_0 g\delta_{3l}\delta X_l\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t\end{aligned}\tag{13}$$

Considering $E_{ij}=\frac{1}{2}(F_{ki}F_{kj}-\delta_{ij})=E_{ji}$, we have $E_{ij}(F_{ki}\delta F_{kj}+F_{kj}\delta F_{ki})=2E_{ij}F_{ki}\delta F_{kj}$. The fourth term in (1.4.1) becomes

$$\begin{aligned}&\delta\int_0^T\iiint_{\Omega_O}-W(\underline{\mathbf{E}})\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t\\ &=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\iiint_{\Omega_O}-\frac{1}{2}\lambda(E_{ii}+\epsilon\delta E_{ii})(E_{jj}+\epsilon\delta E_{jj})-\mu(E_{ij}+\epsilon\delta E_{ij})^2+\frac{1}{2}\lambda E_{ii}E_{jj}+\mu E_{ij}^2\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t\\ &=-\int_0^T\iiint_{\Omega_O}\frac{1}{2}\lambda(E_{ii}\delta E_{jj}+E_{jj}\delta E_{ii})+2\mu E_{ij}\delta E_{ij}\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t\\ &=-\int_0^T\iiint_{\Omega_O}\lambda E_{ii}\delta E_{jj}+\mu E_{ij}(F_{ki}\delta F_{kj}+F_{kj}\delta F_{ki})\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t\\ &=-\int_0^T\iiint_{\Omega_O}\lambda\mathrm{tr}(\underline{\mathbf{E}})F_{kj}\delta F_{kj}+2\mu E_{ij}F_{ki}\delta F_{kj}\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t\\ &=-\int_0^T\iiint_{\Omega_O}[\lambda\mathrm{tr}(\underline{\mathbf{E}})F_{kj}+2\mu E_{ij}F_{ki}]\frac{\partial(\delta X_k)}{\partial a_j}\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t\end{aligned}\tag{14a}$$

Next we change subscripts in (14a) and denote the stress tensor as $T_{li}=\lambda\mathrm{tr}(\underline{\mathbf{E}})F_{li}+2\mu E_{ki}F_{lk}$. After using Gauss's theorem and considering $\delta X_l=0$ at the bottom of the beam $\partial\Omega_O^b$, (14a) becomes

$$\begin{aligned}&\delta\int_0^T\iiint_{\Omega_O}-W(\underline{\mathbf{E}})\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t\\ &=-\int_0^T\iiint_{\Omega_O}[\lambda\mathrm{tr}(\underline{\mathbf{E}})F_{li}+2\mu E_{ki}F_{lk}]\frac{\partial(\delta X_l)}{\partial a_i}\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t\\ &=\int_0^T\left(\iiint_{\Omega_O}\frac{\partial T_{li}}{\partial a_i}\delta X_l\,\mathrm{d}a\mathrm{d}b\mathrm{d}c-\iint_{\partial\Omega_O/\partial\Omega_O^b}n_iT_{li}\delta X_l\,\mathrm{d}S\right)\mathrm{d}t,\end{aligned}\tag{14b}$$

where $\mathrm{d}S$ denotes a surface element on the free boundaries of the beam. Substitute (11), (12), (13) and (14b) into (1.4.1), it becomes

$$\begin{aligned}0=&\int_0^T\iiint_{\Omega_O}\rho_0(\partial_t\mathbf{X}-\mathbf{U})\cdot\delta\mathbf{U}-\rho_0\partial_t\mathbf{U}\cdot\delta\mathbf{X}-\rho_0 g\delta_{3l}\delta X_l+\frac{\partial T_{li}}{\partial a_i}\delta X_l\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\,\mathrm{d}t\\ &-\int_0^T\iint_{\partial\Omega_O/\partial\Omega_O^b}n_iT_{li}\delta X_l\,\mathrm{d}S\,\mathrm{d}t,\end{aligned}\tag{15}$$

which is exactly (1.4.6). After collecting terms, it can be written as

$$\begin{aligned}0=&\int_0^T\iiint_{\Omega_O}\rho_0(\partial_t\mathbf{X}-\mathbf{U})\cdot\delta\mathbf{U}-\left(\rho_0\partial_t U_l+\rho_0 g\delta_{3l}-\frac{\partial T_{li}}{\partial a_i}\right)\delta X_l\,\mathrm{d}a\mathrm{d}b\mathrm{d}c\,\mathrm{d}t\\ &-\int_0^T\iint_{\partial\Omega_O/\partial\Omega_O^b}n_iT_{li}\delta X_l\,\mathrm{d}S\,\mathrm{d}t\end{aligned}\tag{16}$$

By using the arbitrariness of variations on (16), we can obtain the equations of motion (1.4.8):

$$\delta\mathbf{U}: \quad \partial_t\mathbf{X} = \mathbf{U} \quad \text{on } \Omega_O \tag{17a}$$

$$\delta X_l: \quad \rho_0\partial_t U_l = -\rho_0 g\delta_{3l} + \frac{\partial T_{li}}{\partial a_i} \quad \text{on } \Omega_O \tag{17b}$$

$$\delta X_l|_{\partial\Omega_O/\partial\Omega_O^b}: \quad 0 = n_i T_{li} \quad \text{on } \partial\Omega_O/\partial\Omega_O^b, \tag{17c}$$

where stress tensor $T_{li} = \lambda \mathrm{tr}(\underline{\mathbf{E}})F_{li} + 2\mu E_{ki}F_{lk}$ and n_i is the outward normal component.

1.4.1 Linearisation

Exercise: Show that after linearisation of the domain in (1.4.10) becomes the variational principle in the fixed Eulerian domain with Ω_O now involving the use of Eulerian coordinates as well

$$0 = \delta\int_0^T \iiint_{\Omega_O} \rho_0\mathbf{U}\cdot\partial_t\tilde{\mathbf{X}} - \frac{1}{2}\rho_0|\mathbf{U}|^2 - \frac{1}{2}\lambda e_{ii}e_{jj} - \mu e_{ij}^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t \tag{1.4.11}$$

Denote functions in Eulerian coordinates by $(\cdot)^E$ so that $f(\mathbf{a}) = f^E(\mathbf{x} = \mathbf{X}(\mathbf{a},t))$. Considering $\mathbf{X} = \mathbf{a}+\tilde{\mathbf{X}}$, under small displacements ($\|\partial_\mathbf{a}\tilde{\mathbf{X}}\| \ll 1$) we have

$$\frac{\partial\tilde{\mathbf{X}}}{\partial\mathbf{a}} = \frac{\partial\mathbf{X}}{\partial\mathbf{a}}\frac{\partial\tilde{\mathbf{X}}^E}{\partial\mathbf{x}} = \left(\underline{\mathbf{I}} + \frac{\partial\tilde{\mathbf{X}}}{\partial\mathbf{a}}\right)\frac{\partial\tilde{\mathbf{X}}^E}{\partial\mathbf{x}} \approx \frac{\partial\tilde{\mathbf{X}}^E}{\partial\mathbf{x}}. \tag{18}$$

Then

$$\mathbf{e} \approx \frac{1}{2}\left[\left(\frac{\partial\tilde{\mathbf{X}}^E}{\partial\mathbf{x}}\right)^T + \left(\frac{\partial\tilde{\mathbf{X}}^E}{\partial\mathbf{x}}\right)\right] = \mathbf{e}^E \tag{19}$$

In addition, under small displacements, the determinant J is

$$J = \det(\mathbf{F}) = \left|\frac{\partial\mathbf{X}}{\partial\mathbf{a}}\right| = \left|\underline{\mathbf{I}} + \frac{\partial\tilde{\mathbf{X}}}{\partial\mathbf{a}}\right| \approx \det(\underline{\mathbf{I}}) = 1 \tag{20}$$

Consequently, the Eulerian form of the variational principle (1.4.10) is

$$0 = \delta\int_0^T \iiint_{\Omega_O} \rho_0\mathbf{U}^E\cdot\partial_t\tilde{\mathbf{X}}^E - \frac{1}{2}\rho_0|\mathbf{U}^E|^2 - \frac{1}{2}\lambda e_{ii}^E e_{jj}^E - \mu(e_{ij}^E)^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t. \tag{21}$$

After omitting the superscript, (21) becomes (1.4.11).

Exercise: Show that the resulting, linearised equations of motion, after ignoring gravity, are

$$\delta\mathbf{U}: \quad \partial_t\tilde{\mathbf{X}} = \mathbf{U} \quad \text{on } (x,y,x)\in\Omega_O \tag{1.4.13a}$$

$$\delta\tilde{X}_l: \quad \rho_0\partial_t U_l = \partial_{x_i}(\lambda e_{jj}\delta_{il} + 2\mu e_{il}) \quad \text{on } (x,y,x)\in\Omega_O \tag{1.4.13b}$$

$$\delta\tilde{X}_l\Big|_{\partial\Omega_O/\{c=0\}}: \quad n_i(\lambda e_{jj}\delta_{il} + 2\mu e_{il}) = 0 \quad \text{on } (x,y,x)\in\partial\Omega_O/\{c=0\}. \tag{1.4.13c}$$

In what sense is ignoring the gravity term inconsistent in the linearisation? How can one reconcile to a proper derivation of the above equations?

Similar to (11), using end-point conditions $\delta\tilde{\mathbf{X}}(\mathbf{x},0) = \delta\tilde{\mathbf{X}}(\mathbf{x},T) = 0$, the first term in (1.4.11) becomes

$$\begin{aligned}
&\delta\int_0^T \iiint_{\Omega_O} \rho_0\mathbf{U}\cdot\partial_t\tilde{\mathbf{X}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t \\
&= \lim_{\epsilon\to0}\frac{1}{\epsilon}\int_0^T \iiint_{\Omega_O} \rho_0(\mathbf{U}+\epsilon\delta\mathbf{U})\cdot\partial_t(\tilde{\mathbf{X}}+\epsilon\delta\tilde{\mathbf{X}}) - \rho_0\mathbf{U}\cdot\partial_t\tilde{\mathbf{X}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t \\
&= \int_0^T \iiint_{\Omega_O} \rho_0\delta\mathbf{U}\cdot\partial_t\tilde{\mathbf{X}} - \rho_0\delta\tilde{\mathbf{X}}\cdot\partial_t\mathbf{U}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t
\end{aligned} \tag{23}$$

Similar to (12), the second term in (1.4.11):

$$\begin{aligned}\delta\int_0^T\iiint_{\Omega_O}-\frac{1}{2}\rho_0|\mathbf{U}|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\iiint_{\Omega_O}-\frac{1}{2}\rho_0|\mathbf{U}+\epsilon\delta\mathbf{U}|^2+\frac{1}{2}\rho_0|\mathbf{U}|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t\\&=\int_0^T\iiint_{\Omega_O}-\rho_0\mathbf{U}\cdot\delta\mathbf{U}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t\end{aligned}\tag{24}$$

Similar to (14a) and (14b), considering $e_{ij}\approx\frac{1}{2}(\partial_{x_i}\tilde{X}_j+\partial_{x_j}\tilde{X}_i)$, we have $e_{ij}=e_{ji}$ and $e_{ij}\delta e_{ij}=e_{ij}\partial_{x_i}(\delta\tilde{X}_j)$, the rest terms in (1.4.11) becomes

$$\begin{aligned}&\delta\int_0^T\iiint_{\Omega_O}-\frac{1}{2}\lambda e_{ii}e_{jj}-\mu e_{ij}^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t\\&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\iiint_{\Omega_O}-\frac{1}{2}\lambda(e_{ii}+\epsilon\delta e_{ii})(e_{jj}+\epsilon\delta e_{jj})-\mu(e_{ij}+\epsilon\delta e_{ij})^2+\frac{1}{2}\lambda e_{ii}e_{jj}+\mu e_{ij}^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t\\&=-\int_0^T\iiint_{\Omega_O}\lambda e_{ii}\delta e_{jj}+2\mu e_{ij}\delta e_{ij}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t\\&=-\int_0^T\iiint_{\Omega_O}(\lambda e_{jj}\delta_{il}+2\mu e_{il})\frac{\partial(\delta\tilde{X}_l)}{\partial x_i}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t\\&=\int_0^T\left[\iiint_{\Omega_O}\frac{\partial(\lambda e_{jj}\delta_{il}+2\mu e_{il})}{\partial x_i}\delta\tilde{X}_l\,\mathrm{d}x\mathrm{d}y\mathrm{d}z-\iint_{\partial\Omega_O/\{c=0\}}n_i(\lambda e_{jj}\delta_{il}+2\mu e_{il})\delta\tilde{X}_l\,\mathrm{d}S\right]\mathrm{d}t\end{aligned}\tag{25}$$

After substituting (23), (24) and (25) into (1.4.11) and collecting terms, it becomes

$$\begin{aligned}0=&\int_0^T\iiint_{\Omega_O}\rho_0(\partial_t\tilde{\mathbf{X}}-\mathbf{U})\cdot\delta\mathbf{U}-\left[\rho_0\partial_t U_l-\frac{\partial(\lambda e_{jj}\delta_{il}+2\mu e_{il})}{\partial x_i}\right]\delta\tilde{X}_l\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}t\\&-\iint_{\partial\Omega_O/\{c=0\}}n_i(\lambda e_{jj}\delta_{il}+2\mu e_{il})\delta\tilde{X}_l\,\mathrm{d}S\end{aligned}\tag{26}$$

By using the arbitrariness of variations on (26), we can obtain the linearised equations of motion (1.4.13).

From the equation for the lithostatic state (1.4.8b), it can be seen that the gradient of stress tensor in the c-direction is balanced by $\rho_0 g$, and its solution is nonlinear in the material position. Under the condition of small displacement gradients, the linearised equations of motion (1.4.13) can also be derived directly from the equations of motion (17) by using the approximations $\mathrm{tr}(\underline{\mathbf{E}})F_{li}=E_{jj}F_{li}\approx e_{jj}\delta_{li}$ and $E_{ki}F_{lk}\approx e_{ik}\delta_{lk}=e_{il}$.

1.5 Finite-element discretisation

Exercise: Define the matrix, vector and tensors used in (1.5.3). Also explore whether is possible to simplify the expressions due to symmetries in the tensors. Abstractly write down the two symplectic schemes mentioned. Why do the above symplectic time-stepping schemes lead to explicit schemes? Is there a time-step restriction and, if so, what is it?

The matrix and vector used in (1.5.3) are defined as [1]

$$A_{\alpha\beta}=\iiint_{\Omega_O}\varphi_\alpha\varphi_\beta\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,\qquad C_\alpha=\iiint_{\Omega_O}\varphi_\alpha\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.\tag{27}$$

The two symplectic schemes are symplectic Euler scheme and Störmer-Verlet scheme. These two schemes lead to explicit schemes because although the model is non-linear, the sole nonlinearity lies in the internal energy term $W(\underline{\mathbf{E}})$, which is evaluated at a known time level. The time-step restriction for both scheme is

$$\Delta t\leq\frac{2\Delta x}{\pi c},\tag{28}$$

where Δx is the smallest distance between the mesh nodes and c here represents the speed of sound in water.

1.6 Coupled water-wave and wind-turbine mast

Exercise: Show that the system (1.6.4) can be obtained from the following VP, a reduction of (1.6.2) without the Lagrange multiplier.

$$0 = \delta \int_0^T \left[\int_0^{L_s} \int_0^{L_y} \rho \phi_f \partial_t \eta - \frac{1}{2} \rho g \eta^2 \, \mathrm{d}y \mathrm{d}x - \int_0^{L_s} \int_0^{L_y} \int_0^{H_0} \frac{1}{2} \rho \left| \nabla \phi \right|^2 \mathrm{d}z \mathrm{d}y \mathrm{d}x \right. \tag{1.6.5a}$$

$$+ \int_0^{L_y} \int_0^{H_0} \rho \phi_s \partial_t \tilde{X}_s \, \mathrm{d}z \mathrm{d}y \tag{1.6.5b}$$

$$\left. + \int_{L_s}^{L_x} \int_0^{L_y} \int_0^{L_z} \rho_0 \mathbf{U} \cdot \partial_t \tilde{\mathbf{X}} - \frac{1}{2} \rho_0 |\mathbf{U}|^2 - \frac{1}{2} \lambda e_{ii} e_{jj} - \mu e_{ij}^2 \, \mathrm{d}z \mathrm{d}y \mathrm{d}x \right] \mathrm{d}t \tag{1.6.5c}$$

Similar to (8), the first two terms in (1.6.5a) become

$$\begin{aligned}
&\delta \int_0^T \left(\int_0^{L_s} \int_0^{L_y} \rho \phi_f \partial_t \eta - \frac{1}{2} \rho g \eta^2 \, \mathrm{d}y \mathrm{d}x \right) \mathrm{d}t \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_0^{L_s} \int_0^{L_y} \left[\rho \left(\phi_f + \epsilon \delta \phi_f \right) \partial_t \left(\eta + \epsilon \delta \eta \right) - \frac{1}{2} \rho g \left(\eta + \epsilon \delta \eta \right)^2 \right] - \left(\rho \phi_f \partial_t \eta - \frac{1}{2} \rho g \eta^2 \right) \mathrm{d}y \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_0^{L_s} \int_0^{L_y} \rho \phi_f \partial_t (\delta \eta) + \rho \delta \phi_f \partial_t \eta - \rho g \eta \delta \eta \, \mathrm{d}y \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^{L_s} \int_0^{L_y} \left(\rho \phi_f \delta \eta \right) \Big|_{t=0}^{t=T} \mathrm{d}y \mathrm{d}x + \int_0^T \int_0^{L_s} \int_0^{L_y} \rho \delta \phi_f \partial_t \eta - \rho \delta \eta \partial_t \phi_f - \rho g \eta \delta \eta \, \mathrm{d}y \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_0^{L_s} \int_0^{L_y} \rho \delta \phi_f \partial_t \eta - \rho \delta \eta \partial_t \phi_f - \rho g \eta \delta \eta \, \mathrm{d}y \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{29}$$

Similar to (9), using Gauss's theorem and considering $\phi_f = \phi(x, y, H_0, t)$ and $\phi_s = \phi(L_s, y, z, t)$, the third term in (1.6.5a) becomes

$$\begin{aligned}
&\delta \int_0^T \int_0^{L_s} \int_0^{L_y} \int_0^{H_0} -\frac{1}{2} \rho \left| \nabla \phi \right|^2 \mathrm{d}z \mathrm{d}y \mathrm{d}x \, \mathrm{d}t \\
&= \lim_{\epsilon \to 0} -\frac{1}{2\epsilon} \int_0^T \int_0^{L_s} \int_0^{L_y} \int_0^{H_0} \rho \left| \nabla (\phi + \epsilon \delta \phi) \right|^2 - \rho \left| \nabla \phi \right|^2 \mathrm{d}z \mathrm{d}y \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_0^{L_s} \int_0^{L_y} \int_0^{H_0} -\rho \nabla \phi \cdot \nabla (\delta \phi) \, \mathrm{d}z \mathrm{d}y \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_0^{L_s} \int_0^{L_y} \int_0^{H_0} \rho \nabla^2 \phi \delta \phi - \rho \nabla \cdot (\delta \phi \nabla \phi) \, \mathrm{d}z \mathrm{d}y \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \left(\int_0^{L_s} \int_0^{L_y} \int_0^{H_0} \rho \nabla^2 \phi \delta \phi \, \mathrm{d}z \mathrm{d}y \mathrm{d}x - \int_0^{L_s} \int_0^{L_y} \rho \partial_z \phi \delta \phi_f \, \mathrm{d}y \mathrm{d}x - \int_0^{L_y} \int_0^{H_0} \rho \partial_x \phi \delta \phi_s \, \mathrm{d}z \mathrm{d}y \right) \mathrm{d}t.
\end{aligned} \tag{30}$$

After using $\delta\tilde{X}(L_s, y, z, 0) = \delta\tilde{X}(L_s, y, z, T) = 0$, and $\rho\partial_t\phi_s\delta\tilde{X}_s = \delta_{1j}\rho\partial_t\phi_s\delta\tilde{X}_j$, (1.6.5b) becomes

$$\begin{aligned}
&\delta\int_0^T\int_0^{L_y}\int_0^{H_0}\rho\phi_s\partial_t\tilde{X}_s\,\mathrm{d}z\mathrm{d}y\mathrm{d}t\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\int_0^{L_y}\int_0^{H_0}\rho(\phi_s+\epsilon\delta\phi_s)\partial_t(\tilde{X}_s+\epsilon\delta\tilde{X}_s)-\rho\phi_s\partial_t\tilde{X}_s\,\mathrm{d}z\mathrm{d}y\mathrm{d}t\\
&=\int_0^T\int_0^{L_y}\int_0^{H_0}\rho\delta\phi_s\partial_t\tilde{X}_s+\rho\phi_s\partial_t(\delta\tilde{X}_s)\,\mathrm{d}z\mathrm{d}y\mathrm{d}t\\
&=\int_0^T\int_0^{L_y}\int_0^{H_0}\rho\delta\phi_s\partial_t\tilde{X}_s-\rho\delta\tilde{X}_s\partial_t\phi_s\,\mathrm{d}z\mathrm{d}y\mathrm{d}t+\int_0^{L_y}\int_0^{H_0}\rho\phi_s\delta\tilde{X}_s\Big|_{t=0}^{t=T}\,\mathrm{d}z\mathrm{d}y\\
&=\int_0^T\int_0^{L_y}\int_0^{H_0}\rho\delta\phi_s\partial_t\tilde{X}_s-\delta_{1j}\rho\partial_t\phi_s\delta\tilde{X}_j\,\mathrm{d}z\mathrm{d}y\mathrm{d}t.
\end{aligned}\tag{31}$$

Similar to (23), using end-point conditions $\delta\tilde{\mathbf{X}}(\mathbf{x}, 0) = \delta\tilde{\mathbf{X}}(\mathbf{x}, T) = 0$, the first term in (1.6.5c) becomes

$$\begin{aligned}
&\delta\int_0^T\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z}\rho_0\mathbf{U}\cdot\partial_t\tilde{\mathbf{X}}\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\,\mathrm{d}t\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z}\rho_0(\mathbf{U}+\epsilon\delta\mathbf{U})\cdot\partial_t(\tilde{\mathbf{X}}+\epsilon\delta\tilde{\mathbf{X}})-\rho_0\mathbf{U}\cdot\partial_t\tilde{\mathbf{X}}\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z}\rho_0\delta\mathbf{U}\cdot\partial_t\tilde{\mathbf{X}}-\rho_0\delta\tilde{\mathbf{X}}\cdot\partial_t\mathbf{U}\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{32}$$

Similar to (24), the second term in (1.6.5c) becomes

$$\begin{aligned}
&\delta\int_0^T\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z}-\frac{1}{2}\rho_0|\mathbf{U}|^2\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\,\mathrm{d}t\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z}-\frac{1}{2}\rho_0|\mathbf{U}+\epsilon\delta\mathbf{U}|^2+\frac{1}{2}\rho_0|\mathbf{U}|^2\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z}-\rho_0\mathbf{U}\cdot\delta\mathbf{U}\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{33}$$

Denote linear stress tensor as $T_{jk} = \lambda e_{ii}\delta_{jk} + 2\mu e_{jk} = T_{kj}$. Similar to (25), after using Gauss's theorem, the rest terms in (1.6.5c) becomes

$$\begin{aligned}
&\delta\int_0^T\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z}-\frac{1}{2}\lambda e_{ii}e_{jj}-\mu e_{ij}^2\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\,\mathrm{d}t\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z}-\frac{1}{2}\lambda(e_{ii}+\epsilon\delta e_{ii})(e_{jj}+\epsilon\delta e_{jj})-\mu(e_{ij}+\epsilon\delta e_{ij})^2+\frac{1}{2}\lambda e_{ii}e_{jj}+\mu e_{ij}^2\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\,\mathrm{d}t\\
&=-\int_0^T\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z}\lambda e_{ii}\delta e_{jj}+2\mu e_{ij}\delta e_{ij}\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\,\mathrm{d}t\\
&=-\int_0^T\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z}(\lambda e_{ii}\delta_{jk}+2\mu e_{jk})\frac{\partial(\delta\tilde{X}_j)}{\partial x_k}\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\left(\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z}\frac{\partial T_{jk}}{\partial x_k}\delta\tilde{X}_j\,\mathrm{d}z\mathrm{d}y\mathrm{d}x-\int_0^{L_y}\int_0^{H_0}T_{1j}\delta\tilde{X}_j\,\mathrm{d}z\mathrm{d}y\right)\mathrm{d}t.
\end{aligned}\tag{34}$$

After substituting (29)–(34) into (1.6.5) and collecting terms, we have

$$
\begin{aligned}
0 = &\int_0^T \left[\int_0^{L_s}\int_0^{L_y} \rho\delta\phi_f(\partial_t\eta - \partial_z\phi) - \rho\delta\eta(\partial_t\phi_f + g\eta)\,\mathrm{d}y\mathrm{d}x + \int_0^{L_s}\int_0^{L_y}\int_0^{H_0} \rho\nabla^2\phi\delta\phi\,\mathrm{d}z\mathrm{d}y\mathrm{d}x \right]\mathrm{d}t \\
&+ \int_0^T \left[\int_0^{L_y}\int_0^{H_0} \rho\delta\phi_s\left(\partial_t\tilde{X}_s - \partial_x\phi\right) - (\delta_{1j}\rho\partial_t\phi_s + T_{1j})\,\delta\tilde{X}_j\,\mathrm{d}z\mathrm{d}y \right]\mathrm{d}t \\
&+ \int_0^T \left[\int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z} \rho_0\delta\mathbf{U}\cdot\left(\partial_t\tilde{\mathbf{X}} - \mathbf{U}\right) + \delta\tilde{X}_j\left(\nabla_k T_{jk} - \rho_0\partial_t U_j\right)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x \right]\mathrm{d}t.
\end{aligned} \tag{35}
$$

By using the arbitrariness of variations on (35), we can obtain the linearised governing equations for the coupled system (1.6.4).

References

[1] Tomasz Jakub Salwa. *On variational modelling of wave slamming by water waves*. PhD thesis, University of Leeds, 2018.

[2] Michael Stone. *Mathematics for physics: a guided tour for graduate students*. Cambridge University Press, Cambridge, 2009.