Analysis and Simulations of Three-Soliton Interactions with Extreme Wave-Amplification in a Hierarchy of Water-Wave Models

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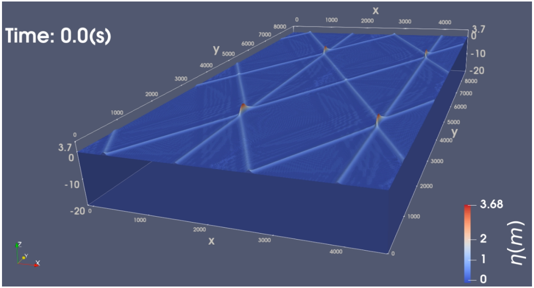
Leeds Institute for Fluid Dynamics; CFC2023-MS303B: . . . dispersive waves



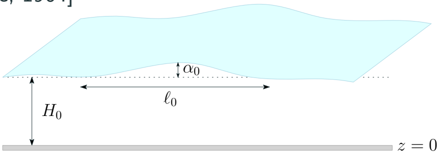
Motivation on modelling extremely high water waves

- Origin 2010 *bore-soliton-splash*:
- To what extent do exact but idealised extreme- or rogue-wave solutions survive in more realistic settings?
- Will such extreme waves fall apart due to dispersion or other mechanisms?
- Use fourfold and ninefold KP amplifications of interacting solitons/cnoidal waves.
- What do you think: will we be able to reach the ninefold wave amplification in more realistic calculations, using potential-flow dynamics, or in reality?

Mathematical hierarchy: PFE, BLE & KPE approximations

⇝ Boussinesq-type approximation: includes weak dispersive effects

- KdV equation: wave propagation in 1D [Korteweg & de Vries, 1895]
- KPE equation: unidirectional propagation in 2DH [Kadomtsev & Petviashvili, 1970]
- Benney-Luke equations –BLE: bidirectional propagation in 2DH [Benney & Luke, 1964]



$\epsilon = \alpha_0/H_0 \ll 1$

$\mu = (H_0/\ell_0)^2 \ll 1$

Expansion about the sea-bed potential $\Phi(x, y, t) = \phi(x, y, z = 0, t)$, in powers of the small parameter μ [Pego & Quintero, 1999]

- More realistic or parent potential-flow equations (PFE).

Kadomtsev-Petviashvili (KPE) equation

The KPE equation can be obtained from the Benney-Luke equations by introducing the formal perturbation expansions

$$\eta = \tilde{u} + \mathcal{O}(\epsilon^2), \qquad \Phi = \sqrt{\epsilon}\left(\tilde{\Psi} + \mathcal{O}(\epsilon^2)\right),$$

using the transformations

$$X = \sqrt{\frac{\epsilon}{\mu}}\left(\frac{3}{\sqrt{2}}\right)^{1/3}(x-t), \qquad Y = \sqrt{\epsilon}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{3}{\sqrt{2}}\right)^{2/3} y,$$

$$\tau = \epsilon\sqrt{\frac{2\epsilon}{\mu}}\, t, \qquad u = \left(\frac{3}{4}\right)^{1/3}\tilde{u},$$

and taking $\mu = \epsilon^2$, resulting in the KPE equation in "standard" form

$$\boxed{\partial_X\left(4\partial_\tau u + 6u\partial_X u + \partial_{XXX} u\right) + 3\partial_{YY} u = 0}$$

This equation includes weak effects in the y-direction.

Exact solution of the KP equation

Web and line-soliton solutions can be constructed using Hirota's transformation

$$u(X,Y,\tau) = 2\partial_{XX} \ln K(X,Y,\tau) = \frac{2\partial_{XX}K}{K} - 2\left(\frac{\partial_X K}{K}\right)^2,$$

where function $K(X,Y,\tau)$ can be obtained from the Wronskian

$$K(X,Y,\tau) = \begin{vmatrix} f_1 & f_1^{(1)} & \dots & f_1^{(N-1)} \\ f_2 & f_2^{(1)} & \dots & f_2^{(N-1)} \\ \vdots & \vdots & & \vdots \\ f_N & f_N^{(1)} & \dots & f_N^{(N-1)} \end{vmatrix}.$$

Particular soliton solutions are obtained by taking [Kodama, 2010]

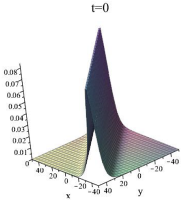
$$f_i = \sum_{j=1}^{M} a_{ij}\, \mathrm{e}^{\theta_j}, \quad \text{where } \theta_j = k_j X + k_j^2 Y - k_j^3 \tau,$$

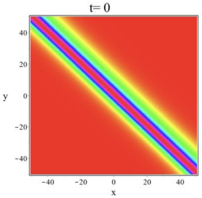
with coefficients k_j being ordered as $k_1 < k_2 < \cdots < k_M$. This solution is called a (N_-, N_+)-soliton, comprising line solitons in the far-field $Y \to \pm\infty$.

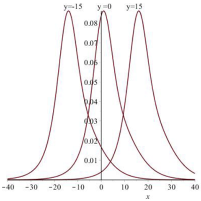
Single line solitons have $(N, M) = (1, 2)$, resulting in $K = f_1 = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}$ and the line soliton solution is

$$
\begin{aligned}
u(X, Y, \tau) &= \frac{1}{2}(k_1 - k_2)^2 \mathrm{sech}^2 \frac{1}{2}(\theta_1 - \theta_2) \\
&= \frac{1}{2}(k_1 - k_2)^2 \mathrm{sech}^2 \frac{1}{2}\big((k_1 - k_2)X + (k_1^2 - k_2^2)Y - (k_1^3 - k_2^3)\tau\big).
\end{aligned}
$$

The soliton amplitude is $\tilde{A} = \frac{1}{2}(k_1 - k_2)^2$ and its centreline is found by setting the sech^2 argument to zero.







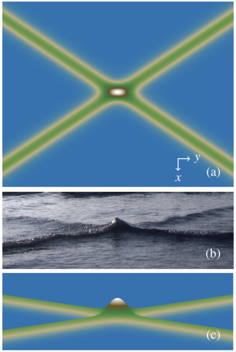
Example: two interacting line solitons

Two line solitons have $(N, M) = (2,4)$, also called $(2,2)$-solitons or O-solitons, obtained with functions $f_1 = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}$, $f_2 = \mathrm{e}^{\theta_3} + \mathrm{e}^{\theta_4}$, and

$$K(X,Y,\tau) = (k_3 - k_1)\mathrm{e}^{\theta_1+\theta_3} + (k_3-k_2)\mathrm{e}^{\theta_2+\theta_3} + (k_4-k_1)\mathrm{e}^{\theta_1+\theta_4} + (k_4-k_2)\mathrm{e}^{\theta_2+\theta_4}.$$

In the far field $Y \to \pm\infty$, we find the single line solitons

$$u_{[1,2]}(X,Y,\tau) = \frac{1}{2}(k_2-k_1)^2 \mathrm{sech}^2 \frac{1}{2}(\theta_1 - \theta_2 - \ln a),$$
$$u_{[3,4]}(X,Y,\tau) = \frac{1}{2}(k_4-k_3)^2 \mathrm{sech}^2 \frac{1}{2}(\theta_3 - \theta_4 - \ln b),$$



where a, b depend on k_j. For equal far-field soliton amplitudes $\tilde{A} = \frac{1}{2}(k_2-k_1)^2 = \frac{1}{2}(k_4-k_3)^2$, the solution satisfies [Kodama, 2010]

$$2\tilde{A} \le \max_{(X,Y,\tau)} u(X,Y,\tau) \le 2\left(1 + \frac{1-\sqrt{\Delta_o}}{1+\sqrt{\Delta_o}}\right)\tilde{A},$$

where $0 \le \Delta_o \le 1$, hence $\boxed{2\tilde{A} \le \max u \le 4\tilde{A}}$.

Example: three interacting line solitons

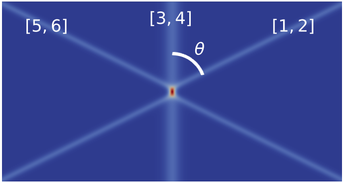
Three line solitons, known as $(3,3)$-solitons, have $(N,M)=(3,6)$ and functions $f_1=\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}$, $f_2=\mathrm{e}^{\theta_3}+\mathrm{e}^{\theta_4}$, $f_3=\mathrm{e}^{\theta_5}+\mathrm{e}^{\theta_6}$, and

$$\begin{aligned}K(X,Y,\tau)=&\ \underline{A_{135}}\,\mathrm{e}^{\theta_1+\theta_3+\theta_5}+\underline{\underline{A_{235}}}\,\mathrm{e}^{\theta_2+\theta_3+\theta_5}+\underbrace{A_{136}}\,\mathrm{e}^{\theta_1+\theta_3+\theta_6}+A_{236}\,\mathrm{e}^{\theta_2+\theta_3+\theta_6}\\ &+A_{145}\,\mathrm{e}^{\theta_1+\theta_4+\theta_5}+\underline{\underline{A_{245}}}\,\mathrm{e}^{\theta_2+\theta_4+\theta_5}+\underbrace{A_{146}}\,\mathrm{e}^{\theta_1+\theta_4+\theta_6}+\underline{A_{246}}\,\mathrm{e}^{\theta_2+\theta_4+\theta_6},\end{aligned}$$

with parameter ordering $k_1<k_2<k_3<0<k_4<k_5<k_6$ & $a,b=1,c$.

In the far field $Y\to\pm\infty$, we find the single line solitons

$$u_{[1,2]}\approx\frac{1}{2}(k_2-k_1)^2\,\mathrm{sech}^2\,\frac{1}{2}(\theta_1-\theta_2-\ln\tilde{a}),$$
$$u_{[5,6]}\approx\frac{1}{2}(k_6-k_5)^2\,\mathrm{sech}^2\,\frac{1}{2}(\theta_5-\theta_6-\ln\tilde{b}),$$
$$u_{[3,4]}\approx\frac{1}{2}(k_4-k_3)^2\,\mathrm{sech}^2\,\frac{1}{2}(\theta_3-\theta_4),$$



with $\theta_i-\theta_j=(k_i-k_j)\Big(X+(k_i+k_j)Y-(k_i^2+k_ik_j+k_j^2)\tau\Big)$.

Example: three interacting line solitons

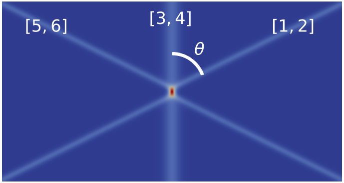
Parameters $k_1, \ldots, k_6$ are determined from

$$k_3 + k_4 = 0$$
$$k_5 + k_6 = -(k_1 + k_2) = \tan\theta$$
$$k_4 - k_3 = \sqrt{2\tilde{A}}$$
$$k_6 - k_5 = k_2 - k_1 = \sqrt{2\tilde{A}/\lambda}$$

where angle $\theta > 0$, $\tilde{A} = \frac{1}{2}(k_4 - k_3)^2$ is the amplitude of the $[3,4]$ soliton, and the outer two solitons are assumed to have amplitude $\tilde{A}/\lambda$, for $\lambda \geq 1$.

Solving the above six equations, gives

$$k_6 = -k_1 = \sqrt{\tilde{A}}\left(\sqrt{2/\lambda} + \sqrt{1/2} + \delta\right)$$
$$k_5 = -k_2 = \sqrt{\tilde{A}}\left(\sqrt{1/2} + \delta\right)$$
$$k_4 = -k_3 = \sqrt{\tilde{A}/2}$$

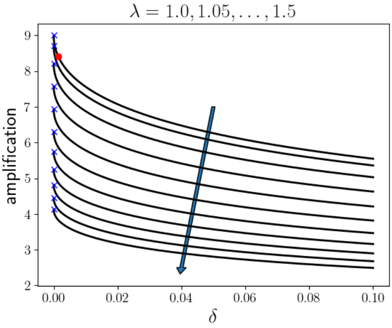


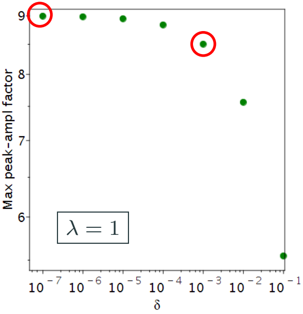
where δ is defined by

$$\delta = \frac{\tan\theta}{2\sqrt{\tilde{A}}} - \left(\sqrt{1/2\lambda} + \sqrt{1/2}\right) > 0.$$

Maximum 9-fold amplification in KP

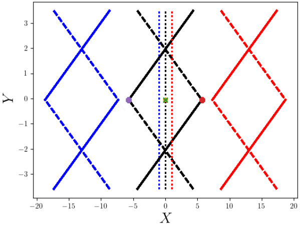
- Proof is based on a geometric argument (additional secondary proof)
- Find 5 centrelines of each of three line solitons (no phase shift at peak)
- Look for intersection points $\rightsquigarrow$ this gives two values of Y, with mean at a unique point $Y_{*\,\delta\to 0} \to -\infty$ when $\tau_* = 0$ and $X_* = 0$
- The space-time point of maximum amplification is (X_*, Y_*, τ_*)
- Amplification: $\dfrac{u(X_*, Y_*, \tau_*)}{\tilde{A}} = \dfrac{(\sqrt{\lambda}+2)^2}{\lambda} + \mathcal{O}(\sqrt{\delta}) \xrightarrow[\delta=0]{} 1 + \dfrac{4}{\lambda} + \dfrac{4}{\sqrt{\lambda}}$

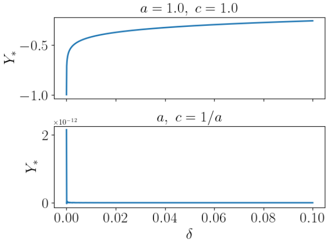




Proof of maximum 9-fold amplification in KP

- Three shift parameters $a, b = 1, c = 1/a$ can be optimised such that splash occurs at $(X^*, Y^*, \tau^*) = (0, 0, 0)$.
- Amplification
  $u(X^*, Y^*, \tau^*)/\tilde{A} = 9 - 8\sqrt{3}\sqrt[4]{2}\sqrt{\delta} + 16\sqrt{2}\delta - 19\,2^{3/4}\sqrt{3}\delta^{3/2}/3$
- Principle Minor Theorem proofs that (X^*, Y^*, τ^*) is a maximum.
- Involved and combined geometrical and analytical proofs.





Numerical implementation

Firedrake *An automated system for the solution of PDEs using the Finite Element Method (FEM).*

Firedrake employs Unified Form Language (UFL) and linear & non-linear solvers PETSc solvers [Rathgeber et al., 2016].

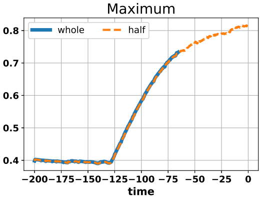
- Space-time discretisation 2nd order of variational principle for BLE: bounded energy oscillations, phase-space conserved.
- Continuous Galerkin (CG) FEM in space for VP, with approximations & test functions/variations $\delta\eta_h,\ \delta\Phi_h$:

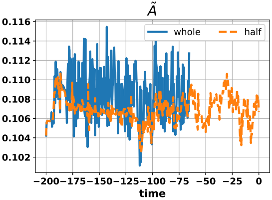
$$\eta(x,y,t) \approx \eta_h(x,y,t) = \sum_k \eta_k(t) w_k(x,y),\ \dots$$

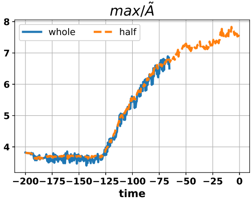
- Symplectic Störmer-Verlet time stepping scheme.
- Stable numerical scheme: no artificial amplitude damping ...

Computational domain: $\sim$cnoidal waves

- KPE solutions hold on infinite horizontal plane, so domain has to be sufficiently large to eliminate reflection at boundaries.
- Solutions can be set to become approximately periodic in sufficiently large domains.
- Transform $\Phi = U_0(y)x + c_0(y) + \tilde{\Phi}$, where $\tilde{\Phi}$ is periodic, then solve the BLE for η and $\tilde{\Phi}$.
- Doubly or singly periodic domain?







Initial conditions and boundaries

Initial condition consists of two (SP2) or three (SP3) line solitons, expressions of which are known from the KP-solution:

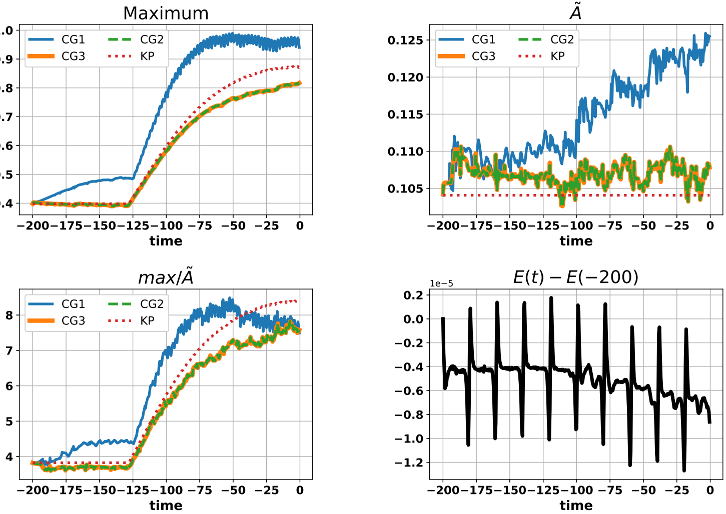
$$\eta_0(x,y) = \eta(x,y,t_0) = 2\Big(\frac{4}{3}\Big)^{1/3}\partial_{XX}\ln K(X,Y,\tau),$$

$$\Phi_0(x,y) = \Phi(x,y,t_0) = 2\sqrt{\epsilon}\,\Big(\frac{4\sqrt{2}}{9}\Big)^{1/3}\partial_{X}\ln K(X,Y,\tau).$$

Computational domain is constructed such that initial condition satisfies "periodic boundary conditions" in x–direction.

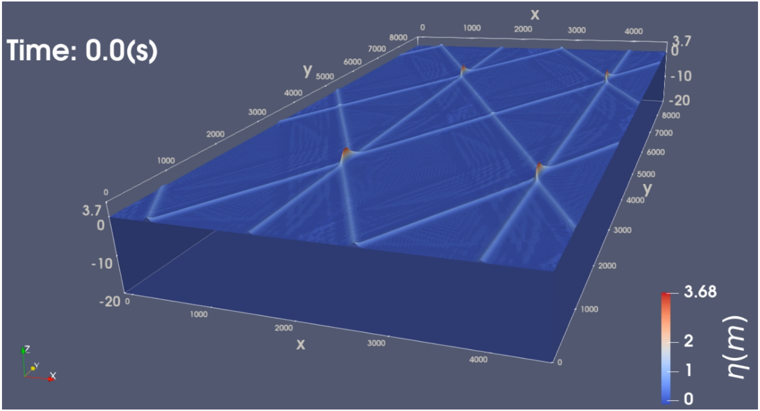
Case	L_x	L_y	T	N_x	N_y	$\Delta x = \frac{L_x}{N_x}$	$\Delta y = \frac{L_y}{N_y}$	Δt
SP2	10.3	40	50	132	480	0.0779	0.0833	0.005
SP3	20.9	47	200	252	564	0.0829	0.0833	0.005

Results BLE-simulation three-soliton interaction



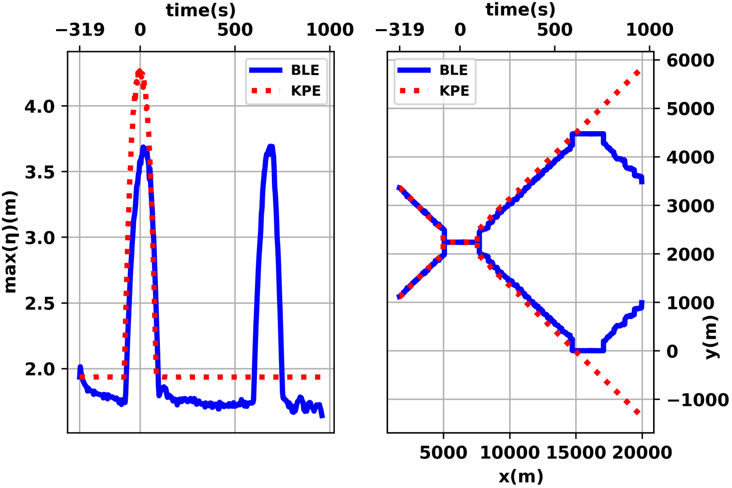
Results simulation three-soliton interaction (dimensional)

Crossing seas (4 or 8 domains combined –*YouTube*)

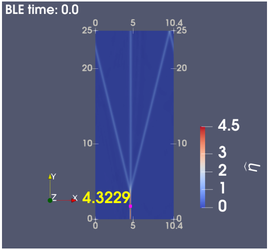


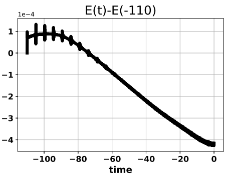
Results simulation three-soliton interaction (dimensional)

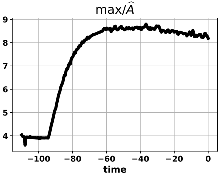
Cnoidal waves with periodicity in x, y, t (max. vs. t & x–y tracks):



- KPE with $\{\delta = 10^{-10}, 9^{-}\times\}$ seeding of BLE simulation yields $8.5\times$ amplification $t_{BLE} \in [-110, 0]$ (Junho Choi).



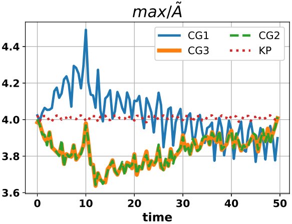




- PFE simulation based on 3+1D discretisation of time-discrete variational principle (VP) in Firedrake: robust, fast development, fewer human errors. MMP & SV time discretisations.

New results BLE- & PFE-simulations three-soliton interaction

- Demanding PFE simulations in progress: HPC simulation with optimised Aadditive Schwarz Method-Star pre-conditioner. *SP2-PFE*:



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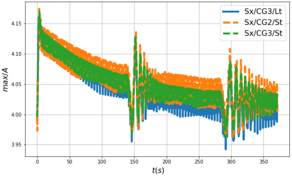
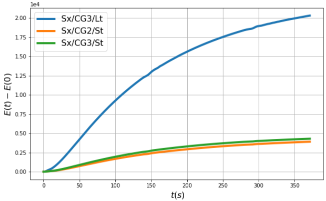
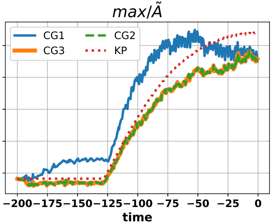




New results BLE- & PFE-simulations three-soliton interaction

- Demanding PFE simulations in progress: HPC simulation with optimised Aadditive Schwarz Method-Star pre-conditioner. *SP3-PFE*:



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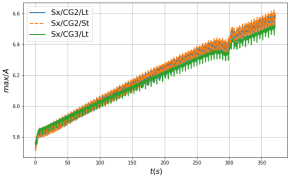
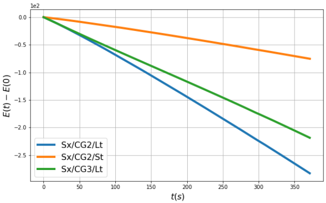
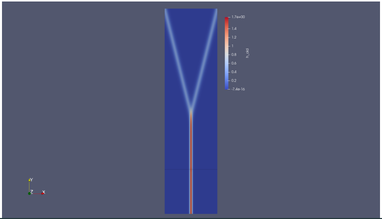
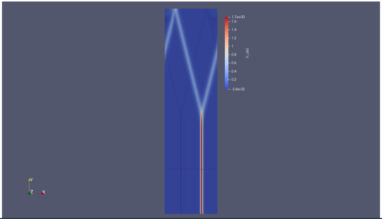




Summary

- Nine-fold soliton amplification proven, with stable limit $\delta \to 0$
- Web-soliton amplification KPE≈ 8.4 & 9, BLE simulation≈ 7.8 & 8.5
- Amplifications achieved as simulated cnoidal crossing seas
- Rogue wave calculation: $\mathrm{AI} > 2$ to $4.0 = \max(\eta)/H_s$
- PFE (and BLE) simulations, convergence checks in progress.
- Extreme waves suitable for experimental validation (design).





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- Crossing seas *YouTube movie*: https://www.youtube.com/watch?v=EGhpQ7BM2jA

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