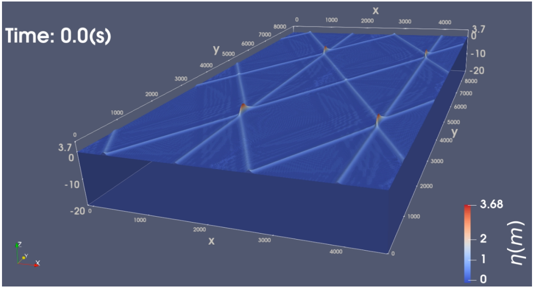
Nonlinear water-wave and wave-impact dynamics using variational principles and automated numerics in Firedrake

Onno Bokhove & Mark Kelmanson; EU-GA859983

Leeds Institute for Fluid Dynamics –TU Denmark seminar 04-09-2023



1. Motivation on modelling extremely high water waves

- Origin 2010 *bore-soliton-splash*:
- To what extent do exact but idealised extreme- or rogue-wave solutions survive in more realistic settings?
- Will such extreme waves fall apart due to dispersion or other mechanisms?
- Use exact fourfold and ninefold amplifications of interacting solitons/cnoidal waves.
- What do you think: will we be able to reach the ninefold wave amplification in more realistic calculations, using potential-flow dynamics, or in reality?

1. Motivation: statements & goals

- Nonlinear water-wave & wave-impact dynamics on wind-turbine masts succinctly described by variational principles (VPs); advantageous theoretical interconnectivity with conservation laws.
- Mathematical modelling based on VPs allows inclusion of simple dissipation, piston wavemakers and waveflaps.
- VPs can be *directly* implemented numerically into FEM framework "Firedrake", advantageous in moving domains and reduction in development time.
- Examples.

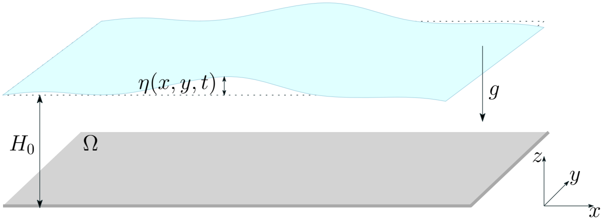
Examples

Examples:

- *Ex. 1*: 2D nonlinear potential-flow code with waveflap.
- *Ex. 2*: 3D nonlinear potential-flow code extreme-wave interactions of cnoidal/solitary waves.
- *Ex. 3*: FSI-work led to generalised coordinate-transformation asymptotics for nonlinear wind-turbine motion.
- *Ex. 4*: Brief overview of rogue-wave energy device.

2. Mathematical hierarchy: potential-flow theory

Velocity potential $\phi(x, y, z, t)$, defined by $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ (irrotational flow)



Water-wave equations (PFE)

$$\boldsymbol{\nabla}^2\phi = 0 \quad \text{in} \quad \Omega$$

$$\partial_t\eta + \nabla\phi\cdot\nabla\eta - \partial_z\phi = 0 \quad \text{at} \quad z = H_0 + \eta$$

$$\partial_t\phi + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g\eta = 0 \quad \text{at} \quad z = H_0 + \eta$$

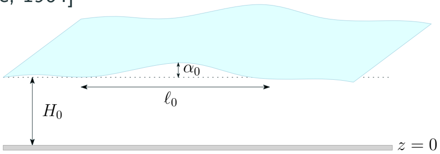
$$\mathbf{n}\cdot\nabla\phi = 0 \quad \text{on} \quad z = 0 \;\; \text{and} \;\; \partial\Omega$$

2. Mathematical hierarchy: BLE and KPE approximations

⇝ Shallow water approximation: long wave length compared to mean water depth

⇝ Boussinesq approximation: includes weak dispersive effects

- KdV equation: wave propagation in 1D [Korteweg & de Vries, 1895]
- KP equation: unidirectional propagation in 2DH [Kadomtsev & Petviashvili, 1970]
- Benney-Luke equations: bidirectional propagation in 2DH [Benney & Luke, 1964]



$\epsilon = \alpha_0/H_0 \ll 1$

$\mu = (H_0/\ell_0)^2 \ll 1$

Expansion about the sea-bed potential $\Phi(x, y, t) = \phi(x, y, z = 0, t)$, in powers of the small parameter μ [Pego & Quintero, 1999]

Benney-Luke equations (BLE)

$$
\begin{aligned}
&\partial_t \eta - \frac{\mu}{2}\partial_t \nabla^2 \eta + \nabla\cdot\big((1+\epsilon\eta)\,\nabla\Phi\big) - \frac{2\mu}{3}\nabla^4\Phi = 0 && \text{in } \Omega\\
&\partial_t \Phi - \frac{\mu}{2}\partial_t \nabla^2 \Phi + \frac{\epsilon}{2}\left|\nabla\Phi\right|^2 + \eta = 0 && \text{in } \Omega\\
&\mathbf{n}\cdot\nabla\Phi = 0 && \text{on } \partial\Omega\\
&\mathbf{n}\cdot\nabla(\nabla^2\Phi) = 0 && \text{on } \partial\Omega
\end{aligned}
$$

Total Energy

$$E(t) = \int_\Omega \left(\frac{1}{2}\eta^2 + \frac{1}{2}(1+\epsilon\eta)|\nabla\Phi|^2 + \frac{\mu}{3}(\nabla^2\Phi)^2\right)\,\mathrm{d}x\,\mathrm{d}y$$

is conserved in time due to the Hamiltonian nature of the system

[Bokhove & Kalogirou, 2016]

Kadomtsev-Petviashvili equation (KPE)

The KPE can be obtained from the BLE by introducing the formal perturbation expansions

$$\eta = \tilde{u} + \mathcal{O}(\epsilon^2), \qquad \mathit{\Phi} = \sqrt{\epsilon}\left(\tilde{\Psi} + \mathcal{O}(\epsilon^2)\right),$$

using the transformations

$$X = \sqrt{\frac{\epsilon}{\mu}}\left(\frac{3}{\sqrt{2}}\right)^{1/3}(x-t), \qquad Y = \sqrt{\epsilon}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{3}{\sqrt{2}}\right)^{2/3} y,$$

$$\tau = \epsilon\sqrt{\frac{2\epsilon}{\mu}}\, t, \qquad u = \left(\frac{3}{4}\right)^{1/3}\tilde{u},$$

and taking $\mu = \epsilon^2$, resulting in the KPE in "standard" form

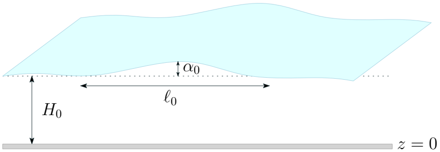
$$\boxed{\partial_X\left(4\partial_\tau u + 6u\partial_X u + \partial_{XXX} u\right) + 3\partial_{YY} u = 0}$$

This equation includes weak dispersion effects in the y-direction.

Why VPs? Compact formulation, conservation laws, numerically advantageous.

⇝ Examples:

- Linear shallow-water dynamics (pedagogial example).
- Benney-Luke equations –BLE: bidirectional propagation in 2DH, Boussinesq-type model [Benney & Luke, 1964].
- Potential-flow equations –PFE: 3D [Luke 1967].

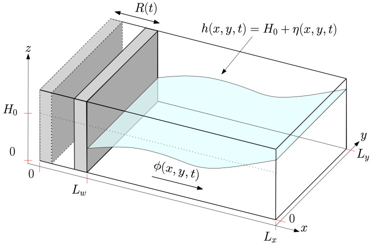


$$\epsilon = \alpha_0/H_0 \ll 1$$
$$\mu = (H_0/\ell_0)^2 \ll 1$$

- In 1D, VP for linear & shallow-water dynamics is:

$$
\begin{aligned}
0 =& \delta \mathcal{L}_{swe}[\phi, \eta] \\
=& \delta \int_0^T \int_0^L \phi \partial_t \eta - \frac{1}{2} H_0 |\partial_x \phi|^2 - \frac{1}{2} g \eta^2 \,\mathrm{d}x - H_0 \dot{R} \phi|_{x=0} \,\mathrm{d}t
\end{aligned}
$$

- Piston wavemaker at $x = R(t)$ is linearised around $x = 0$.
- *Firedrake*: directly program time-discrete version of VP.



VP for linear shallow-water dynamics

- Variations defined by:

$$\delta \mathcal{L}_{swe}[\phi, \eta] \equiv \lim_{\epsilon \to 0} \frac{\mathcal{L}_{swe}[\phi + \epsilon \delta\phi, \eta + \epsilon \delta\eta] - \mathcal{L}_{swe}[\phi, \eta]}{\epsilon}$$

- Equations of motion (continuity & Bernoulli eqns plus BCs):

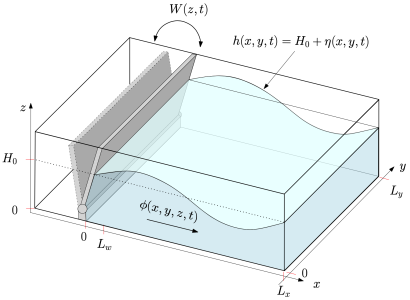
$$\begin{aligned} \partial_t h = \partial_t \eta &= -\partial_x \left(H_0 \partial_x \phi \right), \\ \partial_t \phi &= -g\eta \\ \partial_x \phi &= \dot{R} \quad \text{at} \quad x = 0, \quad \text{and} \\ \partial_x \phi &= 0 \quad \text{at} \quad x = L. \end{aligned} \tag{1}$$

VP for 2D potential-flow with waveflap

- Luke's (1967) VP –issue is parameterisation of domain:

$$
\begin{aligned}
0 =& \delta \mathcal{L}[\phi, h] && (2)\\
=& \delta \int_0^T \iiint_\Omega \partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + g(z - H_0)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. && (3)
\end{aligned}
$$

- VP quite involved/horrendous in transformed fixed domain.



VP for 2D potential-flow with waveflap

- VP quite involved/horrendous after transformation to coordinates in a fixed domain (equations really involved):

$$
\begin{aligned}
& z = \zeta h/H_0, \quad x_\xi = \frac{(L_w - W)}{L_w} + \frac{\zeta}{H_0} W_z \frac{(L_w - \xi)}{L_w} h_\xi, \\
& x_\zeta = \frac{(L_w - \xi)}{L_w} W_z \frac{h}{H_0}, z_\xi = \zeta h_\xi / H_0, \quad z_\zeta = h/H_0, \\
& W = W(\zeta h/H_0, \tau), \quad W_\tau = \partial_\tau W(z,\tau)|_{z=\zeta h/H_0}, \\
& W_z = \partial_z W(z,\tau)|_{z=\zeta h/H_0}, \quad |J| = x_\xi z_\zeta - x_\zeta z_\xi \\
0 =& \delta \int_0^T \int_0^{L_s} \Big(-(1 - W/L_w)\phi\, h_\tau + (1 - \xi/L_W) h_\xi W_\tau \phi \\
& + x_\xi g (\frac{1}{2} z^2 - H_0 z)\Big)|_{\zeta = H_0} \, \mathrm{d}\xi \\
& + \int_0^{H_0} \Big(z_\zeta W_\tau \phi + x_\zeta g (\frac{1}{2} z^2 - H_0 z)\Big)|_{\xi=0} \, \mathrm{d}\zeta \\
& + \int_0^{L_s} \int_0^{H_0} \frac{1}{2|J|} \Big((x_\zeta^2 + z_\zeta^2) |\partial_\xi \phi|^2 \\
& - 2 (x_\xi x_\zeta + z_\xi z_\zeta) \partial_\xi \phi \partial_\zeta \phi \\
& + (x_\xi^2 + z_\xi^2) |\partial_\zeta \phi|^2 \Big) \, \mathrm{d}\xi \, \mathrm{d}\zeta \, \mathrm{d}\tau.
\end{aligned}
$$

VP for 3D potential-flow dynamics in x-periodic domain

- VP in 3D quite involved after coordinate transformation into fixed domain, eqn. (8) in Gidel, Lu et al. (2022).
- Partition $\mathbf{u} = \nabla\phi$ into $\phi(x,y,z,t) = \psi(x,y,t)\hat{\phi}(z) + \varphi(x,y,z,t)$ with $\hat{\phi}|_{z=H_0} = 1$, $\varphi|_{z=H_0} = 0$; VP:

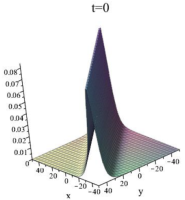
$$
\begin{aligned}
0 =&\delta \int_0^T \int_{\hat{\Omega}_{x,y}} \Bigg[\Big(- H_0\psi\partial_t h + H_0 g h(\frac{1}{2}h - H_0)\Big) \\
&+ \int_0^{H_0} \Bigg[\frac{1}{2}h\Bigg(\psi_x\hat{\phi} + \varphi_x - \frac{z h_x}{h}(\psi\hat{\phi}_z + \varphi_z)\Bigg)^2 \\
&+ \frac{1}{2}h\Bigg(\psi_y\hat{\phi} + \varphi_y - \frac{z h_y}{h}(\psi\hat{\phi}_z + \varphi_z)\Bigg)^2 \\
&\quad + \frac{1}{2}\frac{H_0^2}{h}(\psi\hat{\phi}_z + \varphi_z)^2\Bigg]\,\mathrm{d}z\Bigg]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,
\end{aligned} \tag{4}
$$

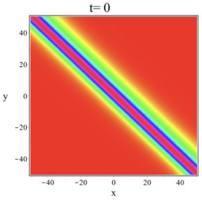
4. Exact 1-2-3 (interacting) line solitons –SP1-KPE

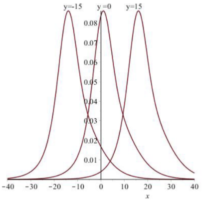
Single line solitons have $(N, M) = (1, 2)$, resulting in $K = f_1 = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}$ and the line soliton solution is

$$\begin{aligned} u(X, Y, \tau) &= 2\partial_{XX} \ln K(X, Y, \tau) = \frac{1}{2}(k_1 - k_2)^2 \mathrm{sech}^2 \frac{1}{2}(\theta_1 - \theta_2) \\ &= \frac{1}{2}(k_1 - k_2)^2 \mathrm{sech}^2 \frac{1}{2}\big((k_1 - k_2)X + (k_1^2 - k_2^2)Y - (k_1^3 - k_2^3)\tau\big). \end{aligned}$$

The soliton amplitude is $\tilde{A} = \frac{1}{2}(k_1 - k_2)^2$ and its centreline is found by setting the sech^2 argument to zero.







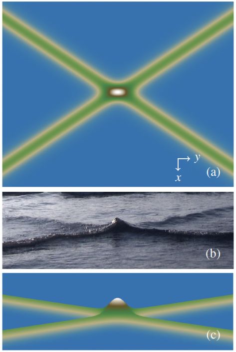
4. Example: two interacting line solitons –SP2-KPE

Two line solitons have $(N,M)=(2,4)$, also called $(2,2)$-solitons or O-solitons, obtained with functions $f_1=\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}$, $f_2=\mathrm{e}^{\theta_3}+\mathrm{e}^{\theta_4}$, and

$$K(X,Y,\tau)=(k_3-k_1)\mathrm{e}^{\theta_1+\theta_3}+(k_3-k_2)\mathrm{e}^{\theta_2+\theta_3}+(k_4-k_1)\mathrm{e}^{\theta_1+\theta_4}+(k_4-k_2)\mathrm{e}^{\theta_2+\theta_4}.$$

In the far field $Y\to\pm\infty$, we find the single line solitons

$$u_{[1,2]}(X,Y,\tau)=\frac{1}{2}(k_2-k_1)^2\mathrm{sech}^2\frac{1}{2}(\theta_1-\theta_2-\ln a),$$
$$u_{[3,4]}(X,Y,\tau)=\frac{1}{2}(k_4-k_3)^2\mathrm{sech}^2\frac{1}{2}(\theta_3-\theta_4-\ln b),$$



where a,b depend on k_j. For equal far-field soliton amplitudes $\tilde{A}=\frac{1}{2}(k_2-k_1)^2=\frac{1}{2}(k_4-k_3)^2$, the solution satisfies [Kodama, 2010]

$$2\tilde{A}\le\max_{(X,Y,\tau)}u(X,Y,\tau)\le 2\left(1+\frac{1-\sqrt{\Delta_o}}{1+\sqrt{\Delta_o}}\right)\tilde{A},$$

where $0\le\Delta_o\le 1$, hence $\boxed{2\tilde{A}\le\max u\le 4\tilde{A}}$.

4. Example: three interacting line solitons –SP3-KPE

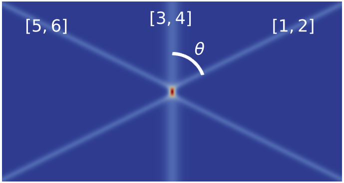
Three line solitons, known as $(3,3)$-solitons, have $(N,M)=(3,6)$ and functions $f_1=\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}$, $f_2=\mathrm{e}^{\theta_3}+\mathrm{e}^{\theta_4}$, $f_3=\mathrm{e}^{\theta_5}+\mathrm{e}^{\theta_6}$, and

$$
\begin{aligned}
K(X,Y,\tau) = {} & \underline{A_{135}}\,\mathrm{e}^{\theta_1+\theta_3+\theta_5} + \underline{\underline{A_{235}}}\,\mathrm{e}^{\theta_2+\theta_3+\theta_5} + \underbrace{A_{136}}\,\mathrm{e}^{\theta_1+\theta_3+\theta_6} + A_{236}\,\mathrm{e}^{\theta_2+\theta_3+\theta_6} \\
& + A_{145}\,\mathrm{e}^{\theta_1+\theta_4+\theta_5} + \underline{\underline{A_{245}}}\,\mathrm{e}^{\theta_2+\theta_4+\theta_5} + \underbrace{A_{146}}\,\mathrm{e}^{\theta_1+\theta_4+\theta_6} + \underline{A_{246}}\,\mathrm{e}^{\theta_2+\theta_4+\theta_6},
\end{aligned}
$$

with the following parameter ordering $k_1<k_2<k_3<0<k_4<k_5<k_6$.

In the far field $Y\to\pm\infty$, we find the single line solitons

$$
\begin{aligned}
u_{[1,2]} &\approx \frac{1}{2}(k_2-k_1)^2\,\mathrm{sech}^2\,\frac{1}{2}(\theta_1-\theta_2-\ln\tilde{a}),\\
u_{[5,6]} &\approx \frac{1}{2}(k_6-k_5)^2\,\mathrm{sech}^2\,\frac{1}{2}(\theta_5-\theta_6-\ln\tilde{b}),\\
u_{[3,4]} &\approx \frac{1}{2}(k_4-k_3)^2\,\mathrm{sech}^2\,\frac{1}{2}(\theta_3-\theta_4),
\end{aligned}
$$



with $\theta_i-\theta_j=(k_i-k_j)\Big(X+(k_i+k_j)Y-(k_i^2+k_ik_j+k_j^2)\tau\Big)$.

Example: three interacting line solitons

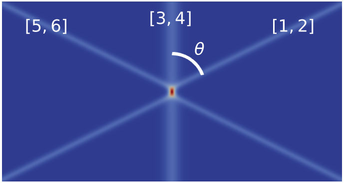
Parameters $k_1, \ldots, k_6$ are determined from

$$k_3 + k_4 = 0$$
$$k_5 + k_6 = -(k_1 + k_2) = \tan\theta$$
$$k_4 - k_3 = \sqrt{2\tilde{A}}$$
$$k_6 - k_5 = k_2 - k_1 = \sqrt{2\tilde{A}/\lambda},$$

where angle $\theta > 0$, $\tilde{A} = \frac{1}{2}(k_4 - k_3)^2$ is the amplitude of the $[3,4]$ soliton, and the outer two solitons are assumed to have amplitude $\tilde{A}/\lambda$, for $\lambda \geq 1$.

Solving the above six equations, gives

$$k_6 = -k_1 = \sqrt{\tilde{A}}\left(\sqrt{2/\lambda} + \sqrt{1/2} + \delta\right)$$
$$k_5 = -k_2 = \sqrt{\tilde{A}}\left(\sqrt{1/2} + \delta\right)$$
$$k_4 = -k_3 = \sqrt{\tilde{A}/2},$$

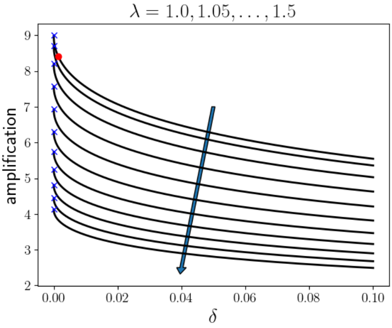


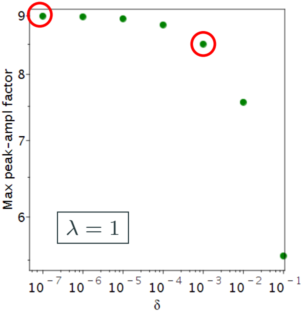
where δ is defined by

$$\delta = \frac{\tan\theta}{2\sqrt{\tilde{A}}} - \left(\sqrt{1/2\lambda} + \sqrt{1/2}\right) > 0.$$

Maximum 9-fold amplification in KPE

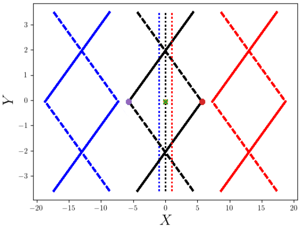
- Proof is based on a geometric argument (additional secondary proof)
- Find 5 centrelines of each of three line solitons (no phase shift at peak)
- Look for intersection points $\rightsquigarrow$ this gives two values of Y, with mean at a unique point $Y_{*\,\delta\to 0} \to -\infty$ when $\tau_* = 0$ and $X_* = 0$ for $a = b = c = 1$
- The space-time point of maximum amplification is (X_*, Y_*, τ_*)
- Amplification: $\dfrac{u(X_*, Y_*, \tau_*)}{\tilde{A}} = \dfrac{(\sqrt{\lambda}+2)^2}{\lambda} + \mathcal{O}(\sqrt{\delta}) \xrightarrow[\delta=0]{} 1 + \dfrac{4}{\lambda} + \dfrac{4}{\sqrt{\lambda}}$

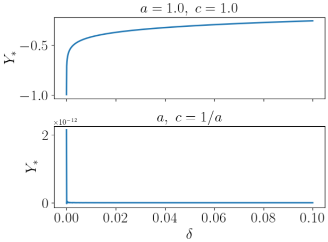




Proof of maximum 9-fold amplification in KP

- Three shift parameters $a, b = 1, c = 1/a$ can be optimised such that splash occurs at $(X^*, Y^*, \tau^*) = (0, 0, 0)$.
- Amplification
  $u(X^*, Y^*, \tau^*)/\tilde{A} = 9 - 8\sqrt{3}\sqrt[4]{2}\sqrt{\delta} + 16\sqrt{2}\delta - 19\,2^{3/4}\sqrt{3}\delta^{3/2}/3$
- Principle Minor Theorem proofs that (X^*, Y^*, τ^*) is a maximum.
- Involved and combined geometrical and analytical proofs.





Firedrake — *An automated system for the solution of PDEs using the Finite Element Method (FEM).*

Firedrake employs Unified Form Language (UFL) and linear & non-linear PETSc solvers [Rathgeber et al., 2016].

- Space-time discretisation $2^{\text{nd}}, 4^{\text{th}}$ order of VP for BLE/PFE: bounded energy oscillations, phase-space conserved.
- Continuous Galerkin (CG) FEM in space for VP, with approximations & test functions/variations $\delta\eta_h,\ \delta\Phi_h$:

$$\eta(x, y, t) \approx \eta_h(x, y, t) = \sum_k \eta_k(t) w_k(x, y),\ \ldots$$

- Symplectic Störmer-Verlet & MMP time-stepping schemes.
- Stable numerical scheme: no artificial amplitude damping ...

5. Firedrake: exciting aspects & VPs

- "**Firedrake**" offers continuous and discontinuous Galerkin FEM, with various types of basis functions, mesh types (gmsh, extruded meshes) focussed on applications in Geophysical and Geological Fluid Dynamics. Comparable with "**FEniCS**".
- Firedrake has (automated) MPI-HPC, various preconditioners and also time-integration options.
- Generally, the (time-discrete) weak forms of a system of equations are implemented.
- The exciting novel & pursued development is to implement (time-discrete) VPs directly, whereafter weak forms are generated automatically via command "*derivative*".

Firedrake: exciting aspects & VPs

- The exciting novel & pursued development is to implement (time-discrete) VPs directly via command "*derivative*".
- Rapid prototyping, reduction time-to-development; 3D potential-flow code in few days, 2D potential-flow code with waveflap in few months (by OB with JC).
- Compare with **years** it took for Floriane Gidel to develop (2015-2018) geometric potential-flow version of Bingham/Engsig-Karup's PFE model (and Gagarina's).
- New codes more versatile: horizontal mesh with spectral GLL combined with (i) vertical elements with GLL or (ii) 1 vertical element with high-order spectral GLL.

- New codes more versatile: horizontal spectral GLL with (i) vertical GLL-elements or (ii) 1 vertical high-order spectral GLL element; strategy/comparison needed.
- HPC-MPI immediately available (Junho Choi), preconditioners (Colin Cotter): hitherto speed-up of $3.7\times$ to $8.75\times$ (optimised Additive Schwarz Method-Star pre-conditioner).
- To include h or p multigrid.
- Compare approach with one with weak-forms for BLE and PFE, verifications and validations.

6. Simulation of 2D wave off waveflap –steps

Time-discrete VP using modified midpoint scheme directly typed in, as phonetic script, into Firedrake:

- time-discrete VP using modified-midpoint (MMP) scheme,
- use "derivative" command $3\times$ with respect to 3 variables $\psi^{n+1/2}, \eta^{n+1/2}, \varphi^{n+1/2}$,
- combine into one solver call with solver-parameters,
- time loop with plotting & saving data,
- perform MPI-HPC call,
- do plotting,
- (nearly) done.
- ... piston-wave limit (done), then full waveflap: verification & validation (in progress).

6. Simulation of 2D wave off waveflap –code

Time-discrete VP using modified midpoint scheme directly typed in; *“UFL is fully decomposable”* (CC):

$$VPnl = (-fd.inner(phimp, (eta\_new - eta)/dt) + fd.inner(etamp, (phiii - phi\_f)/dt)$$

$$+ Fpsi0 * (1.0 - x[0]/Lw) * (dRwavedt + FtnWM * dWflapdts) * etamp.dx(0))$$

$$* fd.ds_t(degree = vpolyp)$$

$$+ 0.5 * (1.0/FJacobian) * ((Fdxdxi3 * *2 + Fdzdxi3 * *2) * (Fdpsidxi0 * phihat + varphimp.dx(0)) *$$

$$* (Fpsi0 * phihat.dx(1) + varphimp.dx(1))$$

$$+ (Fdxdxi * *2 + Fdzdxi * *2) * (Fpsi0 * phihat.dx(1) + varphimp.dx(1)) * *2)$$

$$* fd.dx(degree = (vpoly))$$

$$+ gg * FJacobian * (Fz - H0x) * fd.dx(degree = (vpoly))$$

$$+ (Fdzdxi3 * (dRwavedt + FtnWM * dWflapdts) * (Fpsi0 * phihat + varphimp))$$

$$* fd.ds\_v(1, degree = vpolyp)$$

$$\ldots$$

$$phif\_exprnl1 = fd.derivative(VPnl, phimp, du = vvmp0)$$

$$\ldots$$

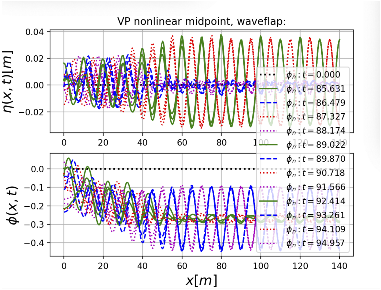
$$phi\_combonl = fd.NonlinearVariationalSolver(fd.NonlinearVariationalProblem($$

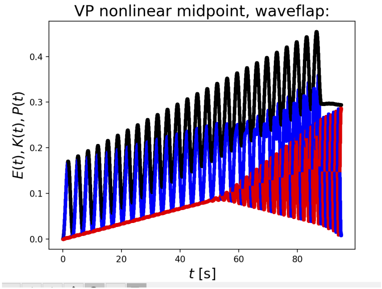
$$Fexprnl, result\_mixedmp, bcs = BC\_varphi\_mixedmp),$$

$$solver\_parameters = lines\_parameters)$$

6. Simulation of 2D wave off waveflap –Preliminary results

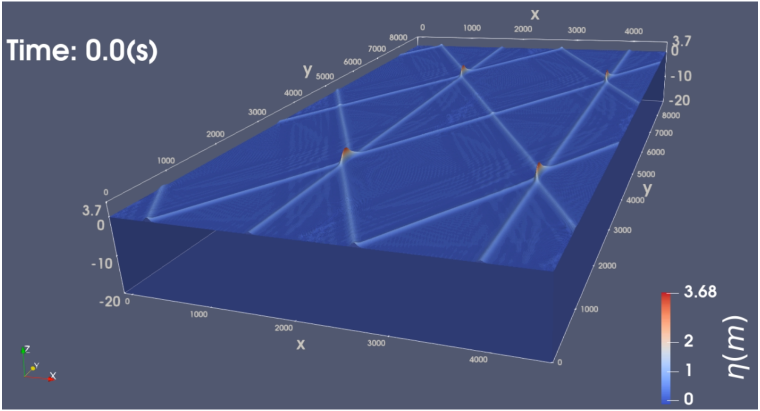
Profiles & energy, regular (standing) waves generated by waveflap switched off after $28T_p$ (Rehman et al. OMAE2023ab, new data):





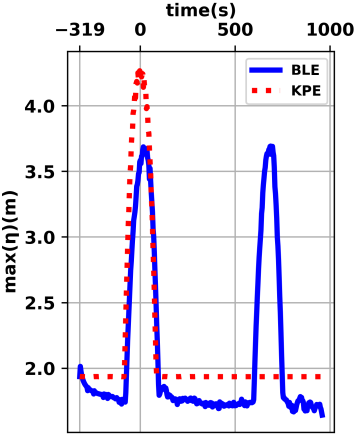
7. Simulation three-soliton interaction (dimensional)

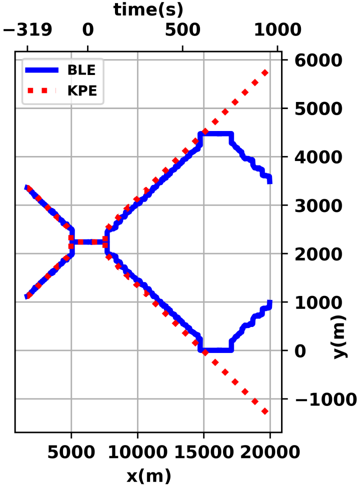
Crossing seas (BLE 4 or 8 domains combined –*YouTube*)



7. Simulation three-soliton interaction (dimensional)

Cnoidal waves with periodicity in x, y, t (max. vs. t & x–y tracks):





7. Simulation of extreme waves: initial conditions

Initial conditions: seed with 2 (SP2) or 3 (SP3) KPE web-solitons:

$$\eta_0(x,y) = \eta(x,y,t_0) = 2\left(\frac{4}{3}\right)^{1/3}\partial_{XX}\ln K(X,Y,\tau_0),$$

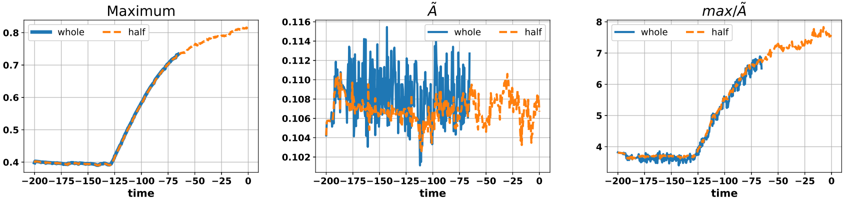
$$\Phi_0(x,y) = \Phi(x,y,t_0) = 2\sqrt{\epsilon}\left(\frac{4\sqrt{2}}{9}\right)^{1/3}\partial_X \ln K(X,Y,\tau_0).$$

Domain constructed s.t. initial conditions satisfy "x-periodic BC".

Case	L_x	L_y	L_z	T	N_x	N_y	N_z	Δ	ϵ	Δt_{BLE}
SP2	4110	16000	20	1713	124	480	2–4	-	0.05	0.005
SP3	17725	40000	20	3170	266	600	2–4	10^{-5}	0.01	0.005
				6775	355 (133)	1000 (300)				

Runs (40c HPC): SP2 721 to 2853 and SP3 396 to 2880min

- KPE solutions hold on infinite horizontal plane, so domain has to be sufficiently large to eliminate reflection at boundaries.
- Solutions can be set to become approximately periodic in sufficiently large domains.
- Transform $\Phi = U_0(y,z)x + c_0(y,z) + \tilde{\Phi}$, where $\tilde{\Phi}$ is x–periodic, then solve the BLE for η and $\tilde{\Phi}$.

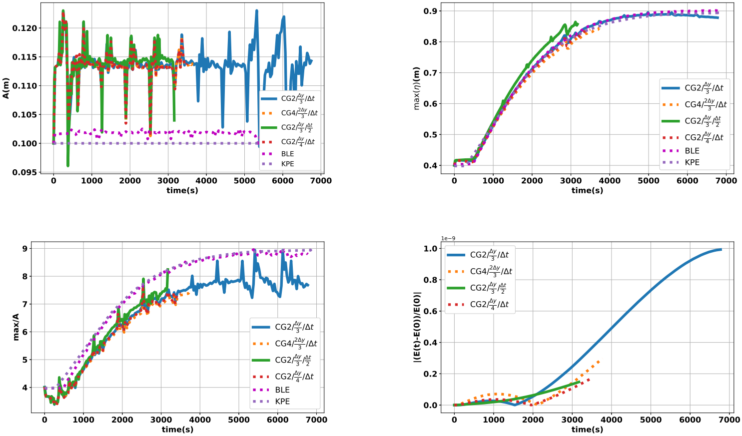


PFE-simulations SP1, SP2, SP3

VP with modified midpoint typed in phonetically; ASM preconditioner by CC:

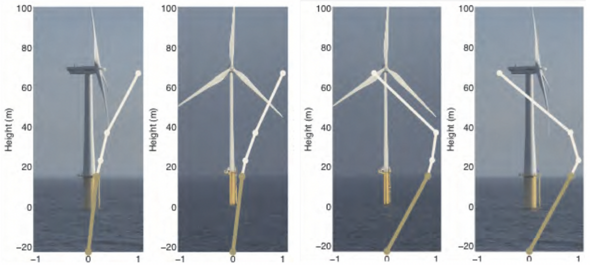
$$
\begin{aligned}
0 =& \delta \int_{\hat{\Omega}_{x,y}} \Big[\Big(- H_0 W \psi^{n+1/2} \frac{(h^{n+1} - h^n)}{\Delta t} + H_0 W \psi^{n+1} \frac{h^{n+1}}{\Delta t} - H_0 W \psi^n \frac{h^n}{\Delta t} \\
&\quad + \frac{1}{2} H_0 g W \big(h^{n+1} (\frac{1}{2} h^{n+1} - H_0) + h^n (\frac{1}{2} h^n - H_0) \big) \Big) \\
&+ \frac{1}{2} \int_0^{H_0} \Big[\frac{1}{2} \frac{L_w^2}{W} h^{n+1} \big(\psi_x^{n+1/2} \hat{\phi} + \varphi_x^{n+1/2} - \frac{1}{h^{n+1}} \big(+ z h_x^{n+1} \big) (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2}) \big)^2 \\
&+ \frac{1}{2} W h^{n+1} \Big(\psi_y^{n+1/2} \hat{\phi} + \varphi_y^{n+1/2} - \frac{1}{h^{n+1}} \big(z h_y^{n+1} \big) (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2}) \Big)^2 \\
&\quad + \frac{1}{2} W \frac{H_0^2}{h^{n+1}} (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2})^2 \Big] \, \mathrm{d}z \\
&\quad + \frac{1}{2} W \frac{H_0^2}{h^n} (\psi^{n+1/2} \hat{\phi}_z + \varphi_z^{n+1/2})^2 \Big] \, \mathrm{d}z \Big] \mathrm{d}x \, \mathrm{d}y.
\end{aligned}
$$

7. Simulation of extreme waves: BLE & PFE SP3



8. FSI asymptotic coordinate transformation for turbine mast

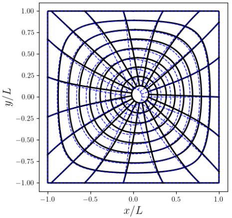
- Material deviation of centre wind-turbine mast larger in horizontal but small in vertical (Ridder et al. 2017):

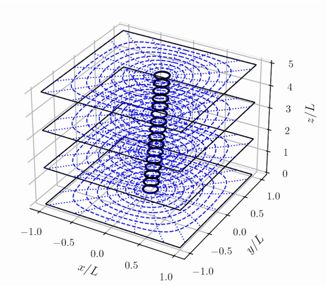


- Deviations from circular cross-section small, accumulation in horizontal larger.
- *Elliptical-grid mapping* in horizontal at every z-level for FSI coupling waves & mast.

8. FSI asymptotic coordinate transformation for wind-turbine

- *Elliptical-grid 3D-squircles mapping* in horizontal at every z-level for FSI coupling waves & mast:

(a) 

(b) 

- Monolythic VP derived & available for water waves and hyperelastic mast: rapid prototyping in Firedrake.

FSI asymptotic coordinate transformation for wind-turbine

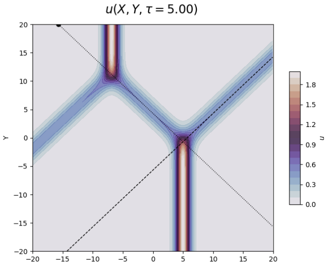
- Horizontal Lagrangian label coordinates a, b, centre or mass $X_0(z,t)$ at every z-level deviates:

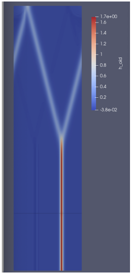
$$X_0(z,t) = \iint X(a,b,z,t)\,\mathrm{d}a\,\mathrm{d}b \quad \text{and} \quad Y_0(z,t) = \iint Y(a,b,z,t)\,\mathrm{d}a\,\mathrm{d}b$$

- Elliptical grid mapping: $\sim$circle centred at (X_0, Y_0) with radius $R(z)L$ wind turbine mast's cross-section to fixed domain $r \in [R,1], \theta \in [0, 2\pi]$ with $R(z) \ll 1$:

$$\begin{aligned}
u =& r\cos\theta, \quad v = r\sin\theta \qquad (5)\\
\frac{x(r,\theta,z)}{L} =& \frac{1}{2}\sqrt{2+u^2-v^2+2\sqrt{2}u} - \frac{1}{2}\sqrt{2+u^2-v^2-2\sqrt{2}u} + \frac{(1-r)}{(1-R)}X_0(z,t)\\
\frac{y(r,\theta,z)}{L} =& \frac{1}{2}\sqrt{2-u^2+v^2+2\sqrt{2}v} - \frac{1}{2}\sqrt{2-u^2+v^2-2\sqrt{2}v} + \frac{(1-r)}{(1-R)}Y_0(z,t),
\end{aligned}$$

- Spectral-GLL FEM via time-discrete VPs: rapid prototyping.
- PFE examples: 2D waveflap-driven & 3D extreme waves.
- Amplifications: KPE 9, BLE $[7.5, 9)$, PFE ~ 8
- To-be-improved x-periodic domain & KPE seeding; *SP2*
- 2-way asymptotic FSI-VP coupling waves & turbine mast.
- *Rogue-wave* energy device.





References

- Choi, B, Kalogirou, Kelmanson (2022) *Water Waves* **4**. `https://link.springer.com/article/10.1007/s42286-022-00059-3`
- B, Kalogirou, Zweers (2019) From bore-soliton-splash to a new wave-to-wire wave-energy model. *Water Waves* **1**.
- Gidel, B, Kalogirou (2017) Variational modelling of extreme waves through oblique interaction . . . . *Nonl. Proc. Geophys.* **24**.
- B, Kalogirou (2016) Variational Water Wave Modelling: from Continuum to Experiment. *Theory of Water Waves*, Bridges et al., *London Math. Soc.* **426**.
- Hairer, Lubich, Wanner (2006) *Geometric Numerical Integration*.
- Pego, Quintero (1999) 2D solitary waves of a BL equation. *Physica D* **132**.
- Crossing seas *YouTube movie*: `https://www.youtube.com/watch?v=EGhpQ7BM2jA`
- Korean Sejong Science project of Junho Choi & EU project w. Yang Lu & Wajiha Rehman
- Rehman et al 2023: two OMAE papers (FSI new data: `https://github.com/EAGRE-water-wave-impact-modelling/FSI_Experiments`).

Email: `O.Bokhove@leeds.ac.uk` (EU Eagre: GA859983)