Finite Volume

Eg. sheet 2.

Q1)

Scaling the shallow water equations:

$U = U_0 U'$ $\quad x = L_s x'$, $t = (L_s/U_0)t'$, $\eta = H_{0s}\eta'$

$H = H_{0s} H'$

eq1 $$\frac{\partial \eta}{\partial t} + \frac{\partial (HU)}{\partial x} = 0$$

* substitute scaling factors:-

$$\frac{\partial (H_{0s}\eta')}{\partial t} + \frac{\partial (H_{0s}H'U_0U')}{\partial x} = 0$$

$$\underbrace{H_{0s}\frac{\partial \eta'}{\partial t}}_{\text{term 1}} + \underbrace{H_{0s}U_0\frac{\partial (H'U')}{\partial x}}_{\text{term 2}} = 0$$

term 1: $$\frac{\partial \eta'}{\partial t} = \frac{\partial \eta'}{\partial t'}\frac{\partial t'}{\partial t}$$

and $t = (L_s/U_0)t'$

$$t' = \frac{t}{L_s/U_0}$$

$$\frac{\partial t'}{\partial t} = \frac{U_0}{L_s}$$

then

$$\frac{\partial n'}{\partial t} = \frac{\partial n'}{\partial t'} \frac{U_0}{L_s}$$

$$\Rightarrow \frac{U_0 H_{0s}}{L_s} \frac{\partial n'}{\partial t'}$$

and $x = L_s x'$

$$x' = \frac{x}{L_s}$$

$$\frac{dx'}{dx} = 1/L_s$$

term 2:

$$\frac{\partial (H'U')}{\partial x} = \frac{\partial (H'U')}{\partial x'} \frac{\partial x'}{\partial x}$$

$$= \frac{\partial (H'U')}{\partial x'} \frac{1}{L_s}$$

2 $$\frac{H_{0s} U_0}{L_s} \frac{\partial (H'U')}{\partial x'}$$

$$\frac{H_{0s} U_0}{L_s} \left(\frac{\partial n'}{\partial t'} + \frac{\partial (H'U')}{\partial x'} \right) = 0$$

$$\Rightarrow \frac{\partial n'}{\partial t'} + \frac{\partial (H'U')}{\partial x'} = 0$$ Ans.

same as eq 1

eq2

$$\frac{\partial v}{\partial t} + \frac{\partial (gn)}{\partial x} = 0$$

$$\frac{\partial (U_0 U')}{\partial t} + \frac{\partial \left((g' U_0^2 / H_{0s}) H_{0s} n' \right)}{\partial x} = 0$$

$$U_0 \frac{\partial U'}{\partial t} + \frac{U_0^2}{H_{0s}} \times H_{0s} \frac{\partial g' n'}{\partial x} = 0$$

$$U_0 \left[\frac{\partial U'}{\partial t'} \frac{\partial t'}{\partial t} \right] + U_0^2 \left[\frac{\partial g' n'}{\partial x'} \frac{\partial x'}{\partial x} \right] = 0$$

$$x = L_s x' \qquad \Rightarrow \qquad \frac{\partial x'}{\partial x} = \frac{1}{L_s}$$

$$\frac{x}{L_s} = x'$$

$$\Rightarrow U_0 \left[\frac{\partial U'}{\partial t'} \frac{U_0}{L_s} \right] + U_0^2 \left[\frac{\partial g' n'}{\partial x'} \times \frac{1}{L_s} \right] = 0$$

$$\frac{U_0^2}{L_s}\left[\frac{\partial u'}{\partial t'} + \frac{\partial g' \eta'}{\partial x'}\right] = 0$$

$$\therefore \quad \frac{\partial u'}{\partial t'} + \frac{\partial g'\eta'}{\partial x'} = 0 \quad \text{Ans}$$

same as eq 2

Rewrite (1) using matrix A as follows

$$\partial_t \begin{pmatrix} \eta \\ H_0 U \end{pmatrix} + A\, \partial_x \begin{pmatrix} \eta \\ H_0 U \end{pmatrix} = 0 \quad (3)$$

$$A = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix}$$

substitute A $\Rightarrow$

$$\partial_t \begin{pmatrix} \eta \\ H_0 U \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \partial_x \begin{pmatrix} \eta \\ H_0 U \end{pmatrix} = 0$$

$$\partial_t \begin{pmatrix} \eta \\ H_0 U \end{pmatrix} + \partial_x \begin{pmatrix} H_0 U \\ c_0^2 \eta \end{pmatrix} = 0$$

$$\partial_t \begin{pmatrix} \eta \\ H_0 U \end{pmatrix} + \partial_x \begin{pmatrix} H_0 U \\ g H_0 \eta \end{pmatrix} = 0$$

first eq $\Rightarrow \quad \frac{\partial \eta}{\partial t} + \frac{\partial H_0 U}{\partial x} = 0$ Ans

same as eq. 1

Second eq $\Rightarrow$ $$\frac{\partial H_0 U}{\partial t} + \frac{\partial H_0 \eta g}{\partial x} = 0$$

$$H_0 \left(\frac{\partial U}{\partial t} + \frac{\partial g \eta}{\partial x} \right) = 0$$

$$\frac{\partial U}{\partial t} + \frac{\partial g \eta}{\partial x} = 0$$

same as eq. 2

$\Rightarrow$ finding the Eigenvalues and Eigenvectors

$$A = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ c_0^2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 1 \\ c_0^2 & -\lambda \end{bmatrix}$$

$$\det |A - \lambda I| \Rightarrow \lambda^2 - (c_0^2)$$

$$(\lambda - c_0)(\lambda + c_0) = 0$$

$$h_1 = C_0 \qquad h_2 = -C_0$$

$$h_1 \Rightarrow \begin{pmatrix} -C_0 & 1 \\ C_0^2 & -C_0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-C_0 x + y = 0$$
$$+C_0^2 x - C_0 y = 0$$

$$y = \frac{C_0^2 x}{C_0} \qquad y = C_0 x$$

let $x = 1 \therefore y = C_0$

$$X_1 = \alpha \begin{bmatrix} 1 \\ C_0 \end{bmatrix}$$

$$h_2 \Rightarrow \begin{pmatrix} C_0 & 1 \\ C_0^2 & C_0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$C_0 x + y = 0$$
$$C_0^2 x + C_0 y = 0$$

$$-C_0 x = y$$

let $x = -1$
$y = C_0$

$$x_2 = \beta \begin{bmatrix} -1 \\ C_0 \end{bmatrix}$$

$$\therefore \quad B = \frac{1}{2c_0}\begin{bmatrix} 1 & -1 \\ c_0 & c_0 \end{bmatrix} \quad (5)$$

since α and β can be any scalar value.

Now $B^{-1}B = I$, showing

$$B^{-1}AB = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} I.$$

find B^{-1} $\Rightarrow$ $\det B = \frac{1}{2c_0}(c_0 + c_0)$

$= 1$

$$\therefore \quad B^{-1} = \frac{1}{2c_0}\begin{bmatrix} c_0 & 1 \\ -c_0 & 1 \end{bmatrix}$$

first multiply AB

$$AB = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \frac{1}{2c_0}\begin{pmatrix} 1 & -1 \\ c_0 & c_0 \end{pmatrix}$$

$$AB = \frac{1}{2c_0}\begin{pmatrix} c_0 & c_0 \\ c_0^2 & -c_0^2 \end{pmatrix}$$

then do $B^{-1}(AB)$

$$\Rightarrow \frac{1}{2c_0}\begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix} \frac{1}{2c_0}\begin{pmatrix} c_0 & c_0 \\ c_0^2 & -c_0^2 \end{pmatrix}$$

$$= \frac{1}{2c_0}\begin{pmatrix} 2c_0^2 & 0 \\ 0 & -2c_0 \end{pmatrix}$$

$$= \begin{pmatrix} c_0 & 0 \\ 0 & -c_0 \end{pmatrix}$$

which in the same as $\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \underline{I}$ since multiplying by identity matrix doesn't change the matrix.

$$\therefore \quad B^{-1}AB = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \underline{I} \quad (6)$$

$\Rightarrow$ multiplying (3) by B^{-1}

$$B^{-1}\partial_t\begin{pmatrix}\eta\\ H_0U\end{pmatrix}+B^{-1}A\,\partial_x\begin{pmatrix}\eta\\ H_0U\end{pmatrix}=0$$

Then we introduce $B^{-1}B=I$
into the second term since the identity matrix does not affect the matrix

$$\Rightarrow B^{-1}\partial_t\begin{pmatrix}\eta\\ H_0U\end{pmatrix}+\underbrace{B^{-1}ABB^{-1}}_{=\begin{pmatrix}h_1 & 0\\ 0 & h_2\end{pmatrix}}\partial_x\begin{pmatrix}\eta\\ H_0U\end{pmatrix}=0$$

$$\Rightarrow \quad \partial_t r+\begin{pmatrix}h_1 & 0\\ 0 & h_2\end{pmatrix}\partial_x r=0 \quad (7)$$

Ans

$$\Rightarrow \frac{\partial\eta}{\partial t}+\frac{\partial(H_0U)}{\partial x}=0$$

$$\sqrt{C_0^2}=gH_0$$

$$C_0=\sqrt{gH_0}$$

$$\frac{\partial u}{\partial t}+\frac{\partial(g\eta)}{\partial x}=0$$

$$\partial_t\begin{pmatrix}H_0U+C_0\eta\\ H_0U-C_0\eta\end{pmatrix}+\begin{pmatrix}0 & 1\\ C_0^2 & 0\end{pmatrix}\partial_x\begin{pmatrix}H_0U+C_0\eta\\ H_0U-C_0\eta\end{pmatrix}=0$$

Question 2

$$\partial_t r + \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \partial_x r = 0$$

$h_1 = c_0$
$h_2 = -c_0$

$$\partial_t r_1 + h_1 \partial_x r_1 = 0$$

$$\partial_t r_2 + h_2 \partial_x r_2 = 0$$

$$\left\{ \begin{array}{l} \dfrac{\partial r_2}{\partial t} + \underbrace{h_2}_{\text{constant}} \dfrac{\partial r_2}{\partial x} = 0 \\[2ex] \dfrac{\partial r_1}{\partial t} + \underbrace{h_1}_{\text{constant}} \dfrac{\partial r_1}{\partial x} = 0 \end{array} \right.$$

linear advection equations

$\Rightarrow$ the solution of the linear advection equation is simply the $u(x,t) = u_0(x-at)$ where a is the constant, and in our equations h_1 and h_2 are constants.

$\rightarrow$ given the initial conditions we can then show for (7)

$$\Rightarrow \quad r_1(x,t) = \begin{cases} r_{1l} & \text{for } (x-h_1t) < 0 \\ r_{1r} & \text{for } (x-h_2t) \geq 0 \end{cases}$$

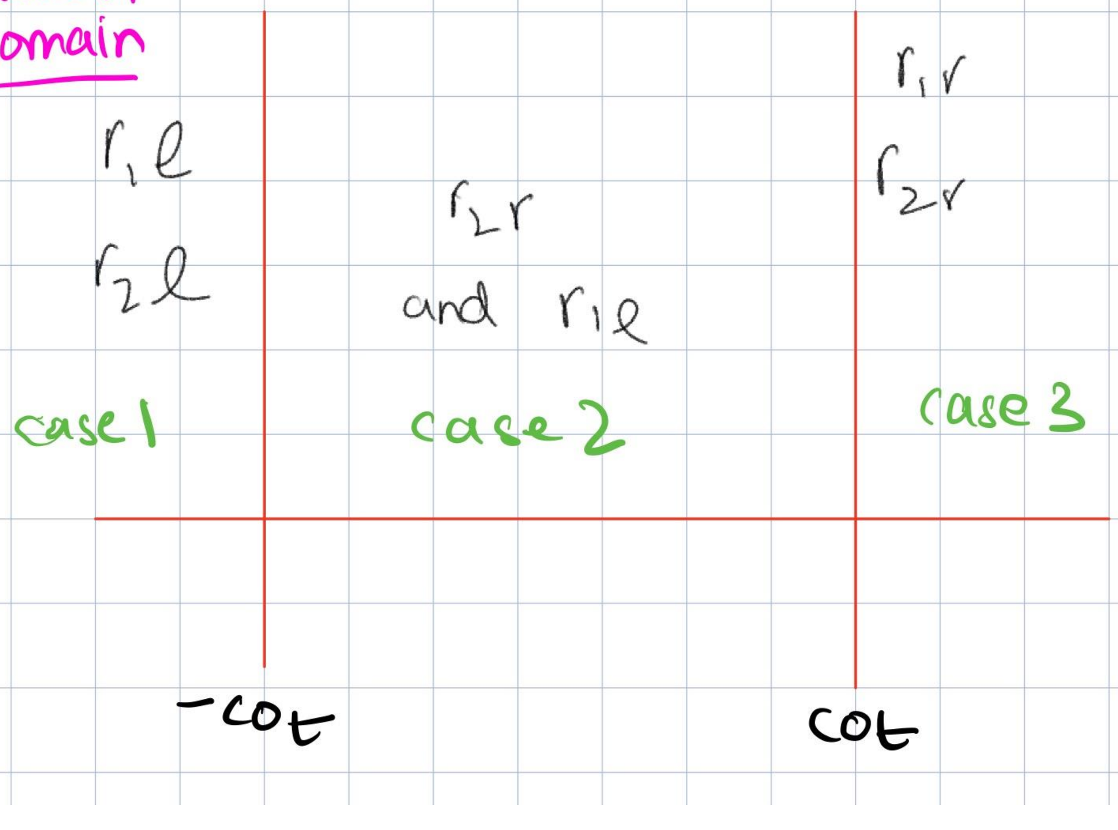
$* \; h_1 = c_0$

$$\therefore \quad r_1(x,t) = \begin{cases} r_{1l} & \text{for } x < c_0t \\ r_{2l} & \text{for } x \geq c_0t \end{cases} \qquad (9)$$

and similarly :-

$$r_2(x,t) = \begin{cases} r_{2l} & \text{for } x < -c_0t \\ r_{2r} & \text{for } x \geq -c_0t \end{cases}$$

Solution domain



for case 1

$$H_0 U = \frac{1}{2}(r_1 + r_2)$$

and $\quad r_1 = H_0 U + C_0 \eta \qquad r_2 = H_0 U - C_0 \eta$

$$r_1 = H_0 U_e + C_0 \eta_e \qquad r_2 = H_0 U_e - C_0 \eta_e$$

$$H_0 U = \frac{1}{2}\left(H_0 U_e + \cancel{C_0 \eta_e} + H_0 U_e - \cancel{C_0 \eta_e}\right)$$

$$H_0 U = \frac{1}{\cancel{2}}\left(2 H_0 U_e\right)$$

$$= H_0 U_e$$

$$\eta = \frac{1}{2C_0}\left(\cancel{H_0 U_e} + C_0 \eta_e - \cancel{H_0 U_e} - C_0 \eta_e\right)$$

$$\eta = \frac{2 C_0 \eta_e}{2 C_0}$$

$$\eta = \eta_e$$

for case 3 :

$$r_1 = H_0U_R + C_0\eta_R \qquad r_2 = H_0U_R - C_0\eta_R$$

$$\eta = \frac{1}{2}(r_1 - r_2)/C_0$$

$$H_0U = \frac{1}{2}\left(H_0U_R + \cancel{C_0\eta_R} + H_0U_R - \cancel{C_0\eta_R}\right)$$

$$H_0U = H_0U_R$$

$$\eta = \frac{1}{2C_0}\left(\cancel{H_0U_R} + C_0\eta_R - \cancel{H_0U_R} + C_0\eta_R\right)$$

$$\eta = \eta_R$$

case 3

$$H_0U(x,t) = \frac{1}{2}\left(H_0U_L + C_0\eta_L + H_0U_R - C_0\eta_R\right)$$

$$= \frac{1}{2}\left(H_0U_L + H_0U_R\right) + \frac{1}{2}\left(C_0\eta_L - C_0\eta_R\right)$$

$$\eta(x,t) = \frac{1}{2C_0}\left(H_0U_L + C_0\eta_L - H_0U_R + C_0\eta_R\right)$$

$$\eta(x,t) = \frac{H_0U_L - H_0U_R}{2C_0} + \frac{\eta_L + \eta_R}{2}$$

thus combining all cases $\Rightarrow$

$$H_0 U(x,t) = \begin{cases} H_0 U_L & \text{for } x < -c_0 t \\ \frac{1}{2}(H_0 U_L + H_0 U_R) + \frac{1}{2}(c_0 \eta_L - c_0 \eta_R) & \text{for } -c_0 t < x \leq c_0 t \\ H_0 U_R & \text{for } x > c_0 t \end{cases}$$

$$\eta(x,t) = \begin{cases} \eta_L & \text{for } x < -c_0 t \\ \frac{H_0 U_L - H_0 U_R}{2c_0} + \frac{\eta_L + \eta_R}{2} & \text{for } -c_0 t < x < c_0 t \\ \eta_R & \text{for } x > c_0 t \end{cases}$$

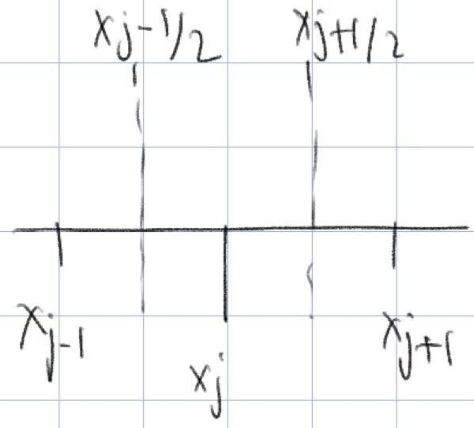
Ans.

Question 3

$\vec{U} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$

$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

equation 1



$$\partial_t \vec{U} + \partial_x \vec{f}(U) = 0$$

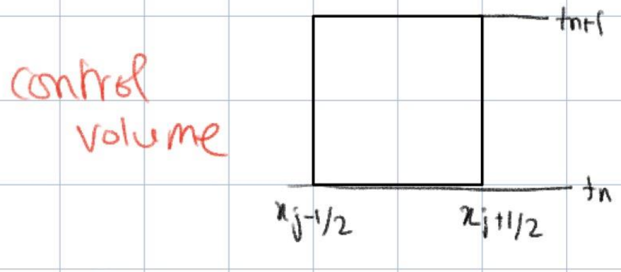
$U_1 = \eta \quad U_2 = H_0 u$

$f_1 = U_2 \quad f_2 = c_0^2 U_1$

define a mean $\Rightarrow$ $$U_j(t) = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} \bar{U}(x,t)\,dx \quad (*)$$

$\Rightarrow$ we then integrate the conservation law over a small control volume

$$\partial_t \bar{U} + \partial_x f(\bar{U}) = 0 \quad \text{over}$$



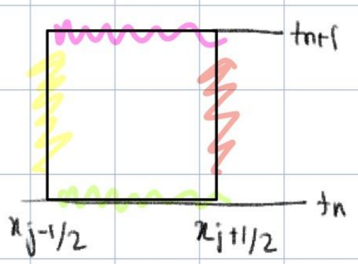
$$\Rightarrow \int_{x_{j-1/2}}^{2} \int_{t_n} \partial_t \bar{U} \, dt\, dx = - \int_t \int_x \partial_x f(U(x,t)) \, dx\, dt$$

$$\Rightarrow \int_{x_{k-1/2}}^{x_{k+1/2}} u(x, t_{n+1}) - u(x, t_n)\, dx$$

$$= -\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t)) - f(u(x_{j-1/2}, t))\, dt$$

⇒ using the defined mean equation (+)

$$\Rightarrow \Delta x_j \left(U_j^{n+1} - U_j^{n} \right) = -\int_{t_n}^{t_{n+1}} \bar{f}(U(x_{j+1/2}, t)) - f(U(x_{j-1/2}, t))\, dt$$



⇒ defining the numerical flux function as

$$\Rightarrow F(U_j^n, U_{j+1}^n) \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(U(x_{j+1/2}, t))\, dt$$

⇒ the expression above then becomes

$\therefore \quad \bar{U}_j^{n+1} = \bar{U}_j^{n} - \frac{\Delta t}{\Delta x}\left(\vec{F}_{j+1/2}\left(\bar{U}_j^n, \bar{U}_{j+1}^n\right) - F_{j-1/2}\left(\bar{U}_{j-1}, \bar{U}_j^n\right)\right)$

* the reimann problem can be solved locally here with the reimann solution found previously

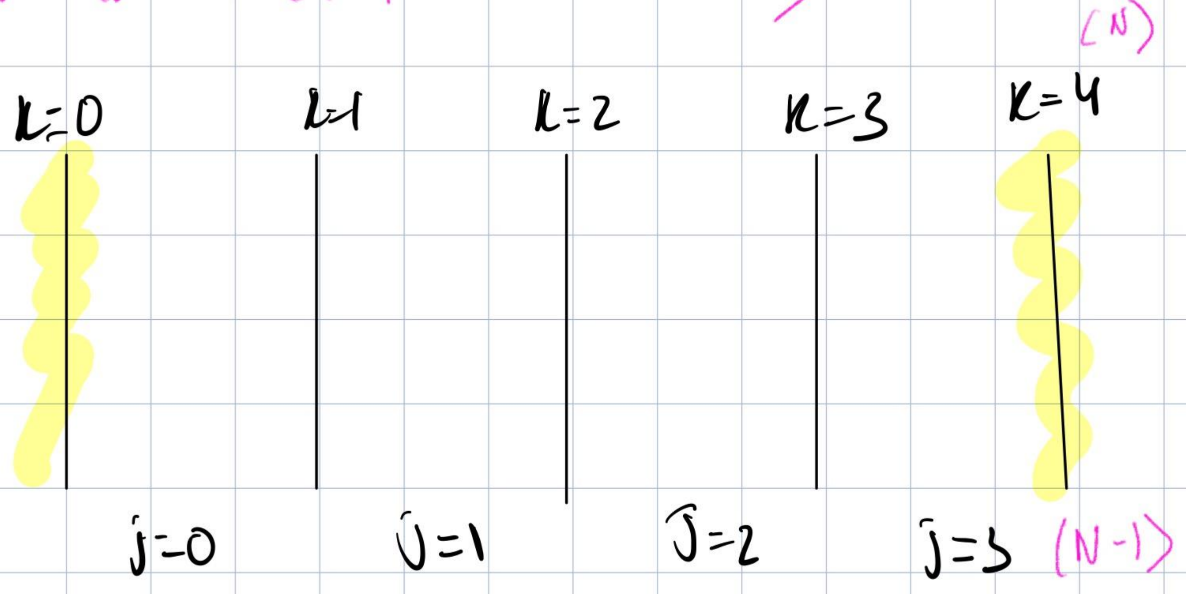
Deriving the time step estimate using the CFL condition $\Rightarrow$

$$\Delta t = CFL \frac{\Delta x_j}{c_0}$$

$$\therefore \quad \Delta t \leq \min \frac{x_j}{c_0}$$

$$CFL < 1$$

for our domain and code $\Rightarrow$



$$\vec{U}_j^{n+1} = \vec{U}_j^{n} - \frac{\Delta x_j}{\Delta t_n}\left(\vec{F}_{j+1}(U_j^n, U_{j+1}^n) - \vec{F}_j(U_j, U_{j-1})\right)$$

$j = 0, \dots, N-1$

$\Rightarrow$ using our solution to question 2, the fluxes then become

$$F_{j+1/2} = \begin{pmatrix} \frac{1}{2}\left((H_0 u_L + H_0 u_R) + (c_0 \eta_L - c_0 \eta_R)\right) \\ c_0^2 \times \frac{1}{2c_0}\left(H_0 u_L - H_0 u_R + c_0 \eta_L + c_0 \eta_R\right) \end{pmatrix}$$

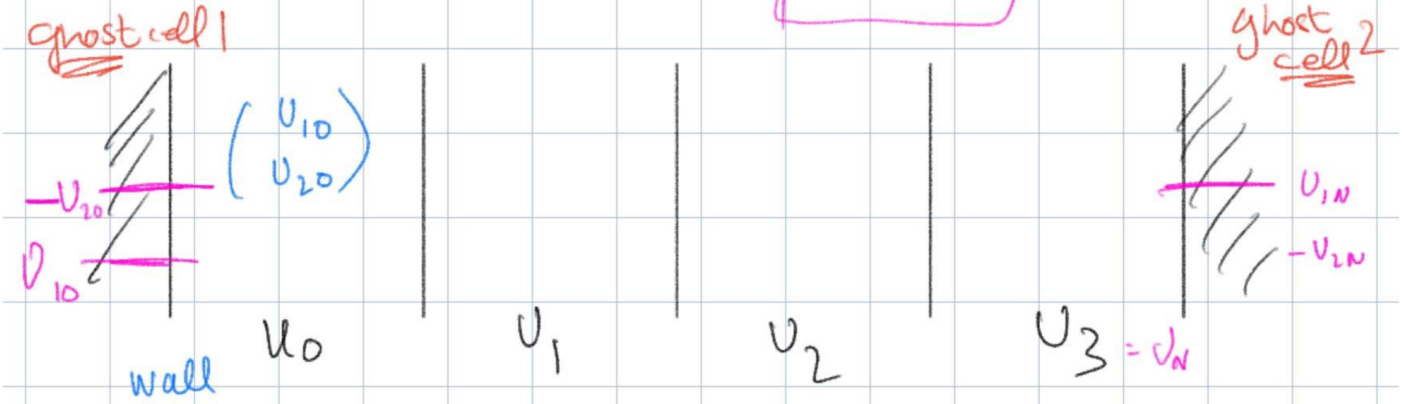
extrapolating B·Cs ("open domain")

$\Rightarrow$ for extrapolating B·Cs we simply create an open domain

$\Rightarrow$ where $U_{-1} = U_0$

closed domain

$$\boxed{\begin{aligned} U_1 &= \eta \\ U_2 &= H_0 U \end{aligned}}$$



at the wall the boundary condition are zero so at the boundary we set a ghost velocity so that it is zero such that the ghost cell

ghost cell 1 $\Rightarrow \begin{pmatrix} U_{10} \\ -U_{20} \end{pmatrix}$

ghost cell 2 $\Rightarrow \begin{pmatrix} U_{1N} \\ -U_{2N} \end{pmatrix}$

$\Rightarrow$ If we set the flux at the boundary to be equal to zero, this implies that the gradient of η at the boundary be zero too

☆ $\frac{\partial \eta}{\partial x} = 0$ at $x=0$ $x=L$

Question 6

Using the code provided, the value of "Nbc" is set to 1, which is the solid wall boundary condition. θ is also set at 0.25. The code is provided in Question6and7.py.

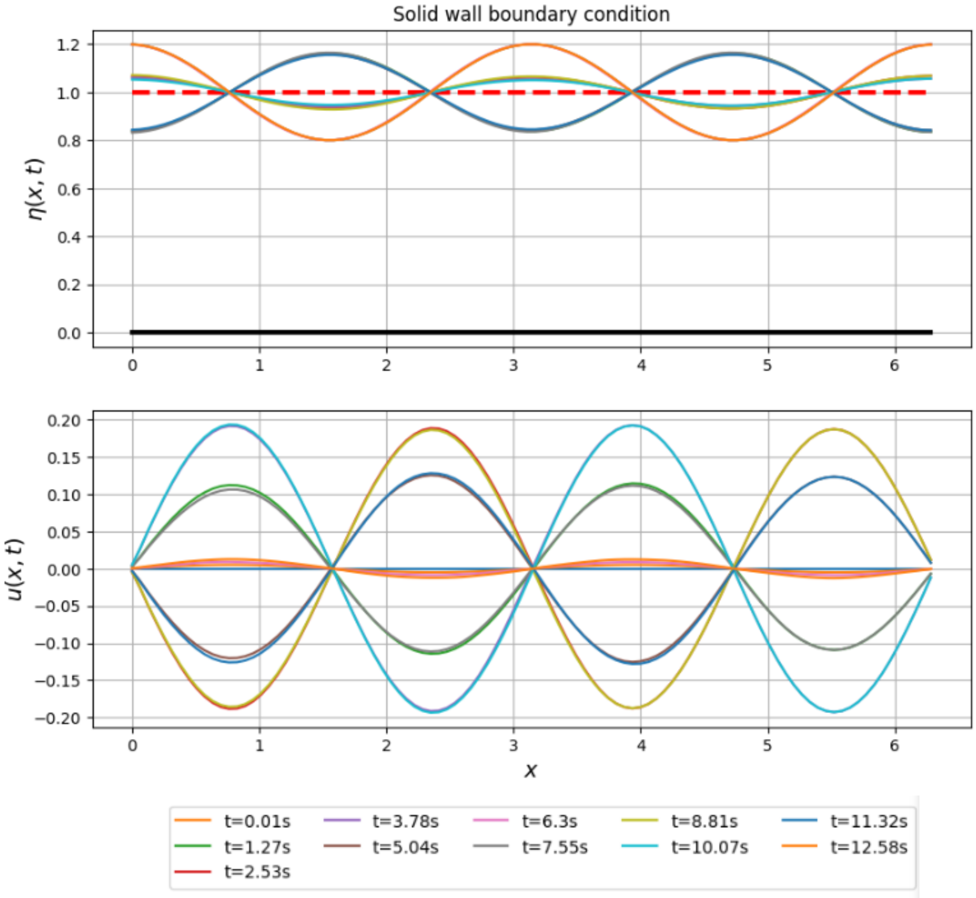


Figure 1: Solid wall boundary condition for the alternating flux scheme for $\theta = 0.25$

Question 7

The analytical values of η and u are calculated as:

$$\eta_{analytical} = ck * \cos\left(\frac{mint * \pi * x}{Ld}\right) * \cos(\omega t) \qquad (1)$$

$$where\ Ld = length\ of\ spatial\ domain = 2\pi, mint = 4\ and\ ck = 0.2$$

$$u_{analytical} = \omega * \frac{ck}{H_0} * \frac{Ld}{mint * \pi} * \sin\left(\frac{mint * \pi * x}{Ld}\right) * \sin(\omega t)$$

The numerical solutions are shown below in Figure 2, and the analytical solutions are shown in Figure 3, all plotted at the same timesteps.

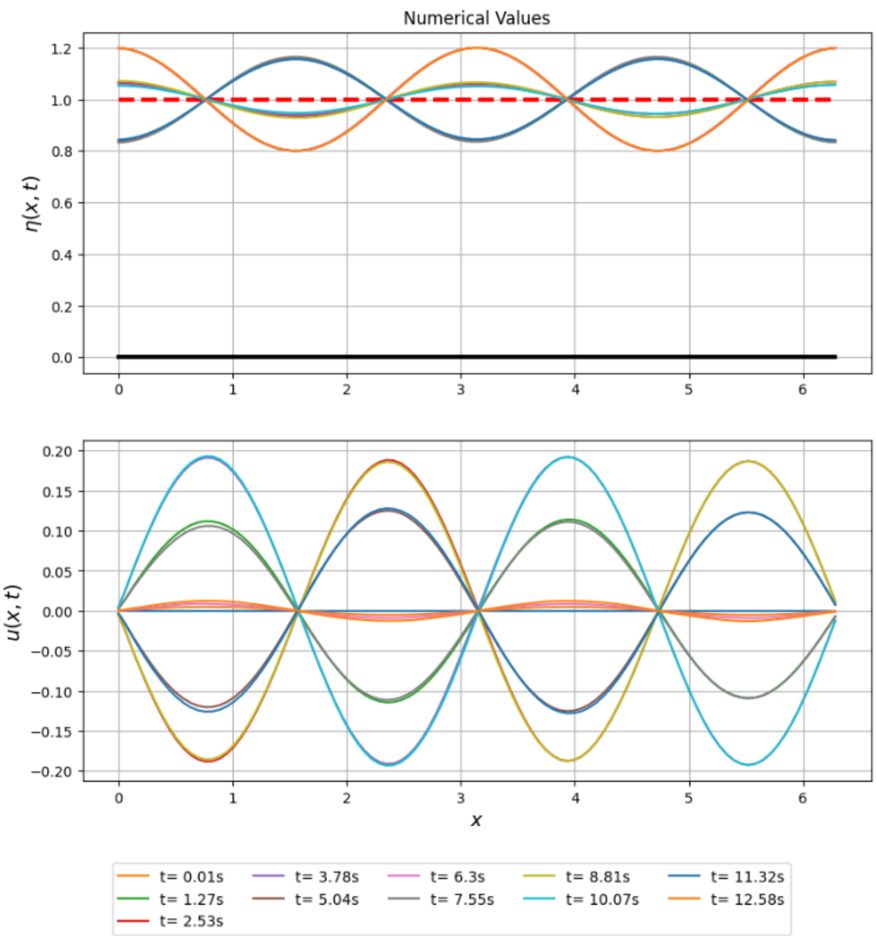


Figure 2: Numerical solutions to the alternating flux scheme with solid wall boundary conditions.

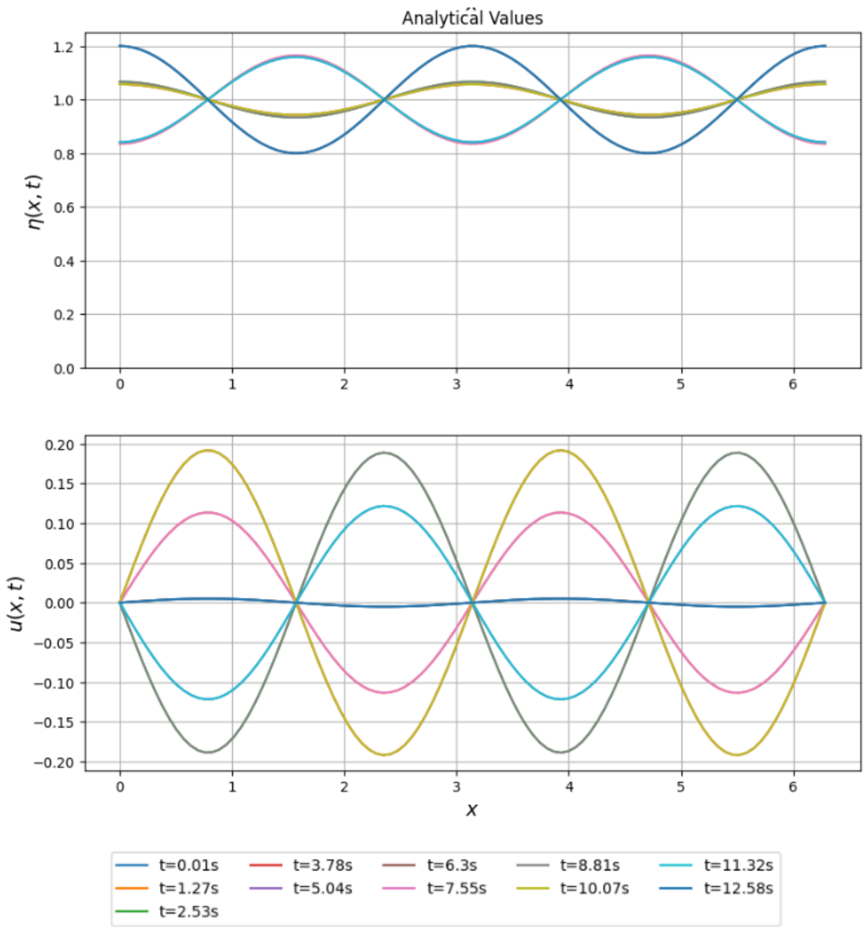


Figure 3: Analytical solutions to the alternating flux scheme with solid wall boundary conditions.