Numerical Exercises 2

We consider the linearised shallow water system of equations

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + \dfrac{\partial (Hu)}{\partial x} = 0, \quad \dfrac{\partial u}{\partial t} + \dfrac{\partial (g\eta)}{\partial x} = 0 \\ u = u(x,t), \; \eta = \eta(x,t), \; H = H(x), \quad g = 9.81\,ms^{-2} \end{cases}$$

Begin by scaling these equations by ~~and~~ introducing primed ~~dimensionle~~ dimensionless variables:

$$u = U_0 u', \; x = L_s x', \; t = \frac{L_s t'}{U_0}, \; \eta = H_{0s}\eta', \quad H = H_{0s} H'$$

- Substituting these in, our system becomes

$$\begin{cases} \dfrac{H_{0s} U_0}{L_s} \dfrac{\partial \eta'}{\partial t'} + \dfrac{H_{0s} U_0}{L_s} \dfrac{\partial (H'u')}{\partial x'} = 0 \quad \Rightarrow \quad \dfrac{\partial \eta'}{\partial t'} + \dfrac{\partial (H'u')}{\partial x'} = 0 \\ \dfrac{U_0^2}{L_s} \dfrac{\partial u'}{\partial t'} + \dfrac{H_{0s}}{L_s} \dfrac{\partial (g\eta')}{\partial x'} = 0 \end{cases}$$

Setting $g = \dfrac{U_0^2 g'}{H_{0s}}$, the second equation becomes $\dfrac{\partial u'}{\partial t'} + \dfrac{\partial (g'\eta')}{\partial x'} = 0$

Dropping the primes, this is identical to the original equations.

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1) The Riemann problem for this system is (with $H(x) = H_0$ constant) is

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + \dfrac{\partial (H_0 u)}{\partial x} = 0 & (1) \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial (g\eta)}{\partial x} = 0 & (2) \\ u(x, t=0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases} \\ \eta(x, t=0) = \begin{cases} \eta_L, & x < 0 \\ \eta_R, & x > 0 \end{cases} \end{cases}$$

where u_L, u_R, η_L, η_R are constants.

Multiplying equation (2) by H_0, we get

$$\frac{\partial(H_0 u)}{\partial t} + H_0 g \frac{\partial \eta}{\partial x} = 0$$

Letting $C_0^2 = H_0 g$, the system in matrix form is

$$\frac{\partial}{\partial t}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Letting $A = \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix}$, the eigenvalues are given by equation

$$\det|A - \lambda I| = 0$$ where I is the identity matrix

$$\Rightarrow \left|\begin{pmatrix} -\lambda & 1 \\ C_0^2 & -\lambda \end{pmatrix}\right| = 0$$

$$\Rightarrow \lambda^2 - C_0^2 = 0$$

Thus, the eigenvalues of A are $\lambda_1 = C_0$, $\lambda_2 = -C_0$

The corresponding eigenvectors are given by $(A - \lambda_i I)\underline{v}_i = 0$, $i = 1, 2$.

$$\Rightarrow \underline{v}_1 = A\begin{pmatrix} 1 \\ C_0 \end{pmatrix} \quad \text{and} \quad \underline{v}_2 = b\begin{pmatrix} -1 \\ C_0 \end{pmatrix}$$

The normalised matrix of eigenvectors is

$$B = \frac{1}{2C_0}\begin{pmatrix} 1 & -1 \\ C_0 & C_0 \end{pmatrix}$$

The inverse of matrix B is

$$B^{-1} = \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix} \quad \text{so} \quad B^{-1}B = BB^{-1} = I$$

$$B^{-1}AB = \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2C_0} & -\frac{1}{2C_0} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} C_0 & 0 \\ 0 & -C_0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Multiplying on the right by B^{-1},

$$B^{-1}ABB^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} B^{-1}$$

$$\Rightarrow \boxed{B^{-1}A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} B^{-1}}$$

Returning to our system of equations,

$$\partial_t \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + A\, \partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiply on the ~~right~~ left by B^{-1}. Since it is made up of constants, it may "pass through" derivatives. i.e. $B^{-1}\partial_t f = \partial_t (B^{-1} f)$

$$\therefore \quad \partial_t \left[B^{-1} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} \right] + \underbrace{B^{-1} A}_{= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} B^{-1}} \partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (\lambda_1 = c_0 ,\ \lambda_2 = -c_0)$$

$$\Rightarrow \partial_t \left[B^{-1} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} \right] + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x \left[B^{-1} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} \right] = 0$$

Let $\underline{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = B^{-1} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = \begin{pmatrix} H_0 u + c_0 \eta \\ H_0 u - c_0 \eta \end{pmatrix}$

Therefore, we have ~~the~~ a decoupled set of linear advection equations:

$$\frac{\partial r_1}{\partial t} + c_0 \frac{\partial r_1}{\partial x} = 0$$

$$\frac{\partial r_2}{\partial t} - c_0 \frac{\partial r_2}{\partial x} = 0$$

with Riemann invariants $r_1 = H_0 u + c_0 \eta$, $r_2 = H_0 u - c_0 \eta$.

There is a much simpler way of deriving these invariants. Looking back at the original system:

$$\begin{cases} \partial_t \eta + \partial_x (H_0 u) = 0 & (1) \\ \partial_t u + \partial_x (g\eta) = 0 & (2) \end{cases}$$

Multiply (1) through by c_0 and (2) through by H_0 (remembering $c_0^2 = gH_0$)

$$\begin{cases} \partial_t (c_0 \eta) + \partial_x (c_0 H_0 u) = 0 & (3) \\ \partial_t (H_0 u) + \partial_x (c_0^2 \eta) = 0 & (4) \end{cases}$$

Then (4)+(3) and (4)−(3) give us

$$\partial_t(H_0 u + C_0 \eta) + C_0 \partial_x(H_0 u + C_0 \eta) = 0$$
$$\partial_t(H_0 u - C_0 \eta) - C_0 \partial_x(H_0 u - C_0 \eta) = 0$$

respectively.

2) The given piecewise constant initial data is

$$r_1(x,0) = \begin{cases} r_{1L} = H_0 u_L + C_0 \eta_L, & x < 0 \\ r_{1R} = H_0 u_R + C_0 \eta_R, & x \geq 0 \end{cases}$$
$$r_2(x,0) = \begin{cases} r_{2L} = H_0 u_L - C_0 \eta_L, & x < 0 \\ r_{2R} = H_0 u_R - C_0 \eta_R, & x \geq 0 \end{cases}$$

we want to find characteristics, on which r_1 is constant

i.e. $\frac{Dr_1}{Dt} = 0$

$$\Rightarrow \frac{\partial r_1}{\partial t} + \frac{dx}{dt}\frac{\partial r_1}{\partial x} = 0$$

From the system of equations, $\partial_t r_1 = -C_0 \partial_x r_1$

$$\therefore \quad -C_0 \frac{\partial r_1}{\partial x} + \frac{dx}{dt}\frac{\partial r_1}{\partial x} = 0$$
$$\Rightarrow \quad -C_0 + \frac{dx}{dt} = 0$$
$$\Rightarrow \quad \boxed{x - x_0 = C_0 t}$$

as $r_1 = \text{Const.}$ on this characteristic i.e. $\forall x_0 \in \mathbb{R}$

At the initial conditions, $t=0$, $x = x_0$

$$r_1(x_0, 0) = r_1(x_0 + C_0 t, t)$$
$$\therefore \quad r_1(x_0 + C_0 t, t) = \begin{cases} r_{1L}, & x_0 < 0 \\ r_{1R}, & x_0 \geq 0 \end{cases}$$

Since $x_0 = x - C_0 t$, $\boxed{r_1(x,t) = \begin{cases} r_{1L}, & x < C_0 t \\ r_{1R}, & x \geq C_0 t \end{cases}}$

Similarly, for r_2,

$$\frac{Dr_2}{Dt} = \frac{\partial r_2}{\partial t} + \frac{dx}{dt}\frac{\partial r_2}{\partial x} = c_0 \frac{\partial r_2}{\partial x} + \frac{dx}{dt}\frac{\partial r_2}{\partial x} = 0$$

$$\Rightarrow \quad \frac{dx}{dt} = -C_0$$

$$\Rightarrow \quad \boxed{X - X_0 = -C_0 t}$$ r_2 is constant on this characteristic.

$$r_2(x_0, 0) = r_2(x_0 - C_0 t, t)$$

$$\therefore r_2(X_0 - C_0 t, t) = \begin{cases} r_{2L}, & x_0 < 0 \\ r_{2R}, & x_0 \geq 0 \end{cases}$$

Since $x_0 = x + C_0 t$,

$$\boxed{r_2(x,t) = \begin{cases} r_{2L}, & x < -C_0 t \\ r_{2R}, & x \geq -C_0 t \end{cases}}$$

Since $r_1 = H_0 u + C_0 \eta$, $r_2 = H_0 u - C_0 \eta$, we can solve to find

$$H_0 u = \frac{1}{2}(r_1 + r_2)$$

$$\eta = \frac{1}{2c_0}(r_1 - r_2)$$

$$\therefore H_0 u(x,t) = \begin{cases} \frac{1}{2}(r_{1L} + r_{2L}), & x < -C_0 t \\ \frac{1}{2}(r_{1L} + r_{2R}), & -C_0 t \leq x \leq C_0 t \\ \frac{1}{2}(r_{1R} + r_{2R}), & x > C_0 t \end{cases}$$

Now, using the fact that

$r_{1L} = H_0 u_L + C_0 \eta_L$, $r_{1R} = H_0 u_R + C_0 \eta_R$, $r_{2L} = H_0 u_L - C_0 \eta_L$,
$r_{2R} = H_0 u_R - C_0 \eta_R$,

$$\boxed{H_0 u(x,t) = \begin{cases} H_0 u_L, & x < -C_0 t \\ \frac{1}{2}(H_0 u_L + C_0 \eta_L + H_0 u_R - C_0 \eta_R), & -C_0 t \leq x \leq C_0 t \\ H_0 u_R, & x > C_0 t \end{cases}}$$

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Similarly,

$$\eta(x,t) = \begin{cases} \frac{1}{2c_0}(r_{1L} - r_{2L}) & , \quad x < -c_0 t \\ \frac{1}{2c_0}(r_{1L} - r_{2R}) & , \quad -c_0 t \le x \le c_0 t \\ \frac{1}{2c_0}(\cancel{r_{1L}}\ r_{1R} - r_{2R}) & , \quad x > c_0 t \end{cases}$$

$$\Rightarrow \boxed{\eta(x,t) = \begin{cases} \eta_L & , \quad x < -c_0 t \\ \frac{1}{2c_0}(H_0 u_L + c_0 \eta_L - H_0 u_R + c_0 \eta_R) & , \quad -c_0 t \le x \le c_0 t \\ \eta_R & , \quad x > c_0 t \end{cases}}$$

$c_0 = \sqrt{gH_0}$, so

$\cancel{H_0 u(x,t) = \{ H_0 u_L}$

$$\boxed{\begin{aligned} u(x,t) &= \begin{cases} u_L & , \quad x < -c_0 t \\ \frac{1}{2}\left(u_L + u_R + \sqrt{\frac{g}{H_0}}\eta_L - \sqrt{\frac{g}{H_0}}\eta_R\right) & , \quad -c_0 t \le x \le c_0 t \\ u_R & , \quad x > c_0 t \end{cases} \\ \eta(x,t) &= \begin{cases} \eta_L & , \quad x < -c_0 t \\ \frac{1}{2}\left(\sqrt{\frac{H_0}{g}} u_L - \sqrt{\frac{H_0}{g}} u_R + \eta_L + \eta_R\right) & , \quad -c_0 t \le x \le c_0 t \\ \eta_R & , \quad x > c_0 t \end{cases} \end{aligned}}$$

Finally, let us re-dimensionalise by recalling that:

$$u' = \frac{u}{U_0}, \quad x' = \frac{x}{L_s}, \quad t' = \frac{U_0 t}{L_s}, \quad \eta' = \frac{\eta}{H_{0s}}, \quad H_0' = \frac{H_0}{H_{0s}}, \quad g' = \frac{H_{0s}}{U_0^2} g$$

$$\Rightarrow \frac{u(x,t)}{U_0} = \begin{cases} u_L/U_0 & , \quad x < -c_0 t \\ \frac{1}{2}\left(\frac{u_L}{U_0} + \frac{u_R}{U_0} + \sqrt{H_{0s}^2 g / U_0^2 H_0}\left(\frac{\eta_L}{H_{0s}} - \frac{\eta_R}{H_{0s}}\right)\right) & , \quad -c_0 t \le x \le c_0 t \\ u_R/U_0 & , \quad x > c_0 t \end{cases}$$

$$\boxed{u(x,t) = \begin{cases} u_L & , \quad x < -c_0 t \\ \frac{1}{2}\left(u_L + u_R + \sqrt{\frac{g}{H_0}}(\eta_L - \eta_R)\right) & , \quad -c_0 t \le x \le c_0 t \\ u_R & , \quad x > c_0 t \end{cases}}$$

$$\boxed{\frac{\eta(x,t)}{H_{0s}} = \begin{cases} \eta_L / H_{0s} & x < -C_0 t \\ \frac{1}{2}\left(\sqrt{V_0^2 H_0 / H_{0s}^2 g}\left(\frac{u_L}{V_0} - \frac{u_R}{V_0}\right) + \frac{\eta_L}{H_{0s}} + \frac{\eta_R}{H_{0s}}\right) & -C_0 t \leq x \leq C_0 t \\ \eta_R / H_{0s} & x > C_0 t \end{cases}}$$

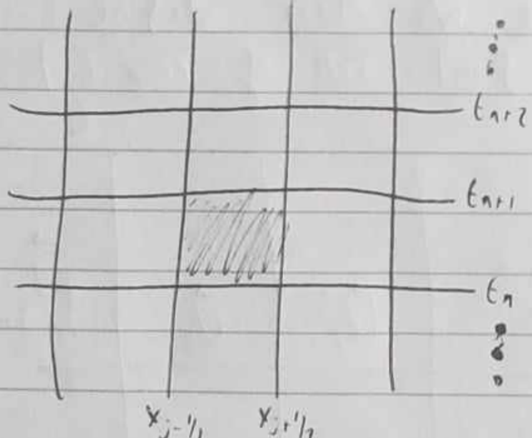
$$\Rightarrow \eta(x,t) = \begin{cases} \eta_L & x < -C_0 t \\ \frac{1}{2}\left(\sqrt{\frac{H_0}{g}}\,(u_L - u_R) + \eta_L + \eta_R\right) & -C_0 t \leq x \leq C_0 t \\ \eta_R & x > C_0 t \end{cases}$$

3) Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \eta \\ H_0 u \end{pmatrix}$ and $\vec{f}(\vec{u}) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = A\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = \begin{pmatrix} H_0 u \\ C_0^2 \eta \end{pmatrix} = \begin{pmatrix} u_2 \\ C_0^2 u_1 \end{pmatrix}$

Our system of equations is

$$\partial_t \vec{u} + \partial_x \vec{f}(\vec{u}) = 0$$

Consider cell bounded by $x = x_{j-\frac{1}{2}}$, $x = x_{j+1/2}$, $t = t_n$, $t = t_{n+1}$, $j, N \in \mathbb{N}_0$ and define the mean to be



$$\boxed{\vec{U}_j^n = \frac{1}{\Delta x_j}\int_{x_{j-1/2}}^{x_{j+1/2}} \vec{u}(x, t_n)\, dx}$$ where $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-1/2}$

and flux to be

$$\vec{F}_{j+1/2}\left(\vec{U}_j^n, \vec{U}_{j+1}^n\right) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \vec{f}\left(\vec{u}(x_{j+1/2}, t)\right) dt$$

Integrate our system over this cell:

$$\int_{t_n}^{t_{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} \partial_t \vec{u}\, dx\, dt = -\int_{t_n}^{t_{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} \partial_x \vec{f}(\vec{u})\, dx\, dt$$

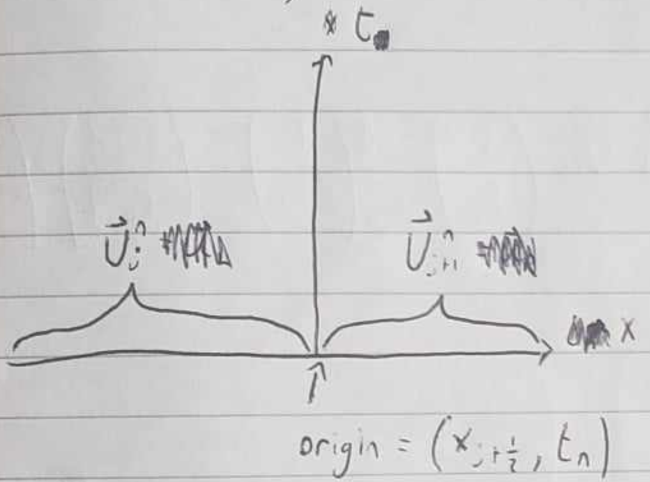
$$\Rightarrow \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\vec{u}(x, t_{n+1}) - \vec{u}(x, t_n)\right) dx = -\int_{t_n}^{t_{n+1}} \left(\vec{f}\left(\vec{u}(x_{j+1/2}, t)\right) - \vec{f}\left(\vec{u}(x_{j-1/2}, t)\right)\right) dt$$

$$\Rightarrow \Delta x_j \left(\vec{U}_j^{n+1} - \vec{U}_j^n\right) = -\Delta t\left(\vec{F}_{j+1/2}\left(\vec{U}_j^n, U_{j+1}^n\right) - \vec{F}_{j-1/2}\left(\vec{U}_{j-1}^n, \vec{U}_j^n\right)\right)$$

$$\therefore \boxed{\vec{U}_j^{n+1} = \vec{U}_j^n - \frac{\Delta t}{\Delta x_j}\left(\vec{F}_{j+1/2}\left(\vec{U}_j^n, \vec{U}_{j+1}^n\right) - \vec{F}_{j-1/2}\left(\vec{U}_{j-1}^n, \vec{U}_j^n\right)\right)}$$

This is the Godunov scheme

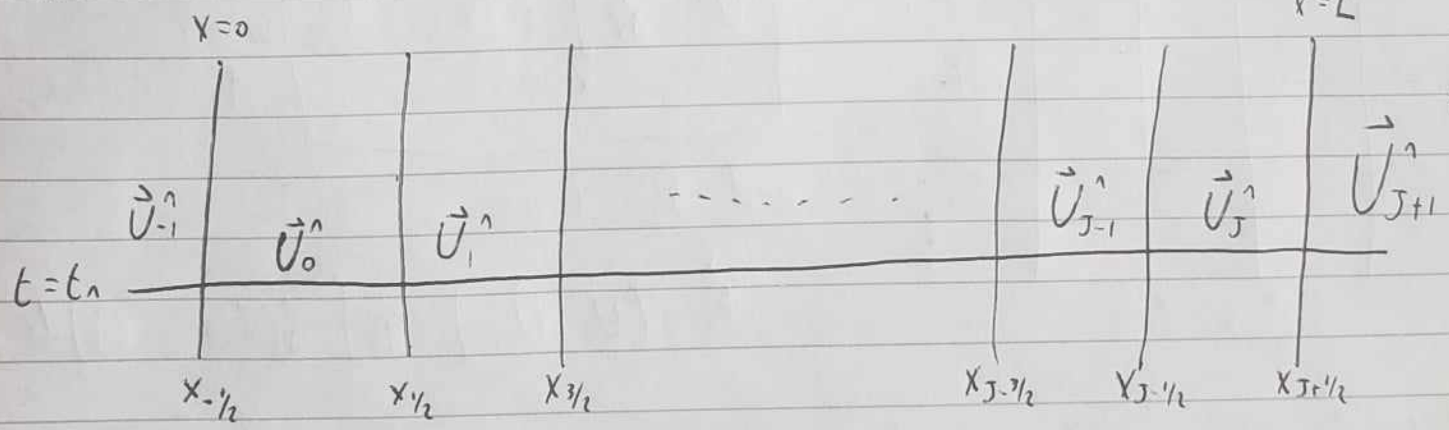
Each cell boundary can be considered as a separate Riemann problem.



$$\partial_t \vec{u} + \partial_x \vec{f}(\vec{u}) = 0, \quad t > t_n$$

$$\vec{u}(x, t_n) = \begin{cases} \vec{U}_j^n, & x < x_{j+1/2} \\ \vec{U}_{j+1}^n, & x > x_{j+1/2} \end{cases}$$

To be able to implement boundary conditions, we must define $\vec{U}_{-1}^n$ and ~~Un (where Un~~ $\vec{U}_{J+1}^n$ (where $\vec{U}_J^n$ is the final cell of row t_n)



For an "open" domain, we can use extrapolating boundary conditions.
Using a first order extrapolation, let

$$\vec{U}_0^n = \tfrac{1}{2}\left(\vec{U}_1^n + \vec{U}_{-1}^n\right), \qquad \vec{U}_J^n = \tfrac{1}{2}\left(\vec{U}_{J-1}^n + \vec{U}_{J+1}^n\right)$$

$$\Rightarrow \boxed{\vec{U}_{-1}^n = 2\vec{U}_0^n - \vec{U}_1^n} \qquad \Rightarrow \boxed{\vec{U}_{J+1}^n = 2\vec{U}_J^n - \vec{U}_{J-1}^n}$$

Alternatively, for a "closed" domain, we take velocity to be equal and opposite and free-surface deviation to be equal.

i.e. if $\vec{U}_j^n = \begin{pmatrix} \bar{\eta}_j^n \\ \bar{u}_j^n \end{pmatrix}$, then

$$\boxed{\vec{U}_{-1}^n = \begin{pmatrix} \bar{\eta}_{-1}^n \\ \bar{u}_{-1}^n \end{pmatrix} = \begin{pmatrix} \bar{\eta}_0^n \\ -\bar{u}_0^n \end{pmatrix}} \quad \& \quad \boxed{\vec{U}_{J+1}^n = \begin{pmatrix} \bar{\eta}_{J+1}^n \\ \bar{u}_{J+1}^n \end{pmatrix} = \begin{pmatrix} \bar{\eta}_J^n \\ -\bar{u}_J^n \end{pmatrix}}$$

Alternatively, for solid walls at $x=0$ and $x=L$, we can set the relavent fluxes equal to zero.

- i.e. $\vec{F}_{-1/2}(\vec{U}_{-1}^n, U_0^n) = \vec{F}_{J+1/2}(\vec{U}_J^n, \vec{U}_{J+1}^n) = 0$

$$\Rightarrow \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \vec{f}(\vec{u}(x_{-1/2}, t))\,dt = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \vec{f}(\vec{u}(x_{J+1/2}, t))\,dt = 0$$

$$\Rightarrow \int_{t_n}^{t_{n+1}} \vec{f}(\vec{u}(0,t))\,dt = \int_{t_n}^{t_{n+1}} \vec{f}(\vec{u}(L,t))\,dt = 0$$

where J is the number of cells in each row of the domain.

So far, we have only been considering the case where $H(x) = H_0$ is constant.

- We can extend the scheme by defining $\bar{H}_{j+1/2} = \bar{H}(x_{j+1/2})$, so that we consider H to be constant on each cell edge.

~~These will b~~

This makes ~~the~~ solving the local Riemann problems on each edge much simpler as the initial conditions remain constant, but scaled by $\bar{H}_{j+1/2}$.

For a suitable time step, we use the following CFL condition

$$\Delta t \leq \frac{\Delta x}{\max(\lambda)}$$

where $\lambda = \{\lambda_1, \lambda_2\}$ is the set of eigenvalues of A computed in question 1.

$$\therefore \max(\lambda) = \max(\{gH, -gH\})$$

$$\Rightarrow \quad \Delta t = CFL \cdot \frac{\Delta x}{gH} \quad \text{with} \quad CFL \in (0,1].$$

6) Let $\vec{U}_j^n = \begin{pmatrix} \bar{\eta}_j^n \\ \bar{u}_j^n \end{pmatrix} = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} \begin{pmatrix} \eta(x,t_n) \\ H(x)u(x,t_n) \end{pmatrix} dx$

Our fluxes are defined to be:

$$\vec{F} = \vec{F}_{j+1/2} = \begin{pmatrix} F_{\eta,j+1/2} \\ F_{u,j+1/2} \end{pmatrix} = \begin{pmatrix} \theta \bar{H}_{j+1/2} \bar{u}_{j+1}^n + (1-\theta) \bar{H}_{j+1/2} \bar{u}_j^n \\ (1-\theta) g \bar{\eta}_{j+1}^{n+1} + \theta g \bar{\eta}_j^{n+1} \end{pmatrix}$$

with $\theta \in [0,1]$

These are <u>alternating fluxes</u>.

Our Godunov scheme was given as

$$\boxed{\vec{U}_j^{n+1} = \vec{U}_j^n - \frac{\Delta t}{\Delta x_j}\left(\vec{F}_{j+1/2} - \vec{F}_{j-1/2}\right)}$$

Substituting in the alternating fluxes,

$$\boxed{\begin{aligned} \bar{\eta}_j^{n+1} &= \bar{\eta}_j^n - \frac{\Delta t}{\Delta x_j}\left(\theta \bar{H}_{j+1/2} \bar{u}_{j+1}^n + (1-\theta)\bar{H}_{j+1/2}\bar{u}_j^n - \theta \bar{H}_{j-1/2}\bar{u}_j^n - (1-\theta)\bar{H}_{j-1/2}\bar{u}_{j-1}^n\right) \\ \bar{u}_j^{n+1} &= \bar{u}_j^n - \frac{\Delta t}{\Delta x_j}\left((1-\theta) g \bar{\eta}_{j+1}^{n+1} + \theta g \bar{\eta}_j^{n+1} - (1-\theta) g \bar{\eta}_j^{n+1} - \theta g \bar{\eta}_{j-1}^{n+1}\right) \end{aligned}}$$

On the boundaries, $x = x_{j-1/2} = 0$ and $x = x_{J+1/2} = L$, we define the fluxes to be zero

$\therefore \bar{\eta}_0^{n+1} = \bar{\eta}_0^n, \quad \bar{\eta}_J^{n+1} = \bar{\eta}_J^n, \quad \bar{u}_0^{n+1} = \bar{u}_0^n, \quad \bar{u}_J^{n+1} = \bar{u}_J^n$