

MATH5453M Numerical Exercises 1

1) The equation (1) is given as

$$\left\{ \begin{array}{l} u_t - a(t)u_x - \epsilon u_{xx} = 0 \quad \text{for } x \in [L_p, L] \\ u(x, 0) = u_0(x) \\ u(L_p, t) = u(L, t) = 0 \end{array} \right.$$

A differential equation is linear if the dependent variable and all of its ordinary/partial derivatives which appear have a linear order (order of 1). This is clearly true for u_t , u_x and u_{xx} in our equation above.

The term $-a(t)u_x$ is called the advection term, since a simple version of the advection equation is

$$u_t = a u_x.$$

Advection is the bulk motion of a fluid.

The term $-\epsilon u_{xx}$ is called the diffusion term, with small diffusion constant ϵ , since a simple version of the diffusion equation is

$$u_t = \epsilon u_{xx}.$$

Diffusion is the movement of a fluid as it spreads out, keeping the same volume but decreasing concentration.

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2) Equation (2.80) in M&M is given as:

$$\delta_t u_j^{n+\frac{1}{2}} = u_j^{n+1} - u_j^n = \left[\Delta t u_t + \frac{1}{24} (\Delta t)^3 u_{ttt} + \dots \right]_j^{n+\frac{1}{2}}$$

$\dots = \text{higher order terms}$

To verify this, we expand u_j^{n+1} and u_j^n as Taylor series around $t = t_{n+\frac{1}{2}}$

$$u_j^{n+1} = u_j^{n+\frac{1}{2}} + (t_{n+1} - t_{n+\frac{1}{2}}) \frac{\partial u_j^{n+\frac{1}{2}}}{\partial t} + \frac{1}{2!} (t_{n+1} - t_{n+\frac{1}{2}})^2 \frac{\partial^2 u_j^{n+\frac{1}{2}}}{\partial t^2} + \frac{1}{3!} (t_{n+1} - t_{n+\frac{1}{2}})^3 \frac{\partial^3 u_j^{n+\frac{1}{2}}}{\partial t^3} + \dots$$

Since $t_{n+1} - t_{n+\frac{1}{2}} = \frac{1}{2} \Delta t$, we have:

$$\begin{aligned} u_j^{n+1} &= \left[u + \frac{1}{2} \Delta t u_t + \frac{1}{2} \left(\frac{1}{2} \Delta t \right)^2 u_{tt} + \frac{1}{6} \left(\frac{1}{2} \Delta t \right)^3 u_{ttt} + \dots \right]_j^{n+\frac{1}{2}} \\ &= \left[u + \frac{1}{2} \Delta t u_t + \frac{1}{8} (\Delta t)^2 u_{tt} + \frac{1}{48} (\Delta t)^3 u_{ttt} + \dots \right]_j^{n+\frac{1}{2}} \end{aligned} \quad (†)$$

Similarly,

$$u_j^n = u_j^{n+\frac{1}{2}} + (t_n - t_{n+\frac{1}{2}}) \frac{\partial u_j^{n+\frac{1}{2}}}{\partial t} + \frac{1}{2!} (t_n - t_{n+\frac{1}{2}})^2 \frac{\partial^2 u_j^{n+\frac{1}{2}}}{\partial t^2} + \frac{1}{3!} (t_n - t_{n+\frac{1}{2}})^3 \frac{\partial^3 u_j^{n+\frac{1}{2}}}{\partial t^3} + \dots$$

Since $t_n - t_{n+\frac{1}{2}} = -\frac{1}{2} \Delta t$, we have:

$$\begin{aligned} u_j^n &= \left[u - \frac{1}{2} \Delta t u_t + \frac{1}{2} \left(-\frac{1}{2} \Delta t \right)^2 u_{tt} + \frac{1}{6} \left(-\frac{1}{2} \Delta t \right)^3 u_{ttt} + \dots \right]_j^{n+\frac{1}{2}} \\ &= \left[u - \frac{1}{2} \Delta t u_t + \frac{1}{8} (\Delta t)^2 u_{tt} - \frac{1}{48} (\Delta t)^3 u_{ttt} + \dots \right]_j^{n+\frac{1}{2}} \end{aligned} \quad (††)$$

Now, if we compute (†) - (††), the even terms will cancel,

$$u_j^{n+1} - u_j^n = \left[\Delta t u_t + \frac{1}{24} (\Delta t)^3 u_{ttt} + \dots \right]_j^{n+\frac{1}{2}}$$

Thus, we have proven (2.80)

Equation (2.81) in M&M is:

$$\delta_x^2 u_j^{n+1} = [(\Delta x)^2 u_{xx} + \frac{1}{12} (\Delta x)^4 u_{xxxx} + \frac{2}{6!} (\Delta x)^6 u_{xxxxxx} + \dots]_j^{n+1}$$

The second-order central difference, $\delta_x^2 u(x, t)$, is defined

$$\delta_x^2 u(x, t) = u(x+\Delta x, t) - 2u(x, t) + u(x-\Delta x, t)$$

$$\text{i.e. } \delta_x^2 u_j^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$$

Proof:

Consider the first-order central differences of u_j^n ,

$$\delta_x u_j^n = u_{j+1}^n - u_{j-1}^n$$

If we apply it once more, we get

$$\begin{aligned}\delta_x^2 u_j^n &= \delta_x u_{j+1}^n - \delta_x u_{j-1}^n \\ &= (u_{j+1}^n - u_j^n) - (u_j^n - u_{j-1}^n) \\ &= u_{j+1}^n - 2u_j^n + u_{j-1}^n \\ \therefore \delta_x^2 u_j^{n+1} &= u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\end{aligned}$$

We expand these three terms as Taylor series around $x=x_j$:

$$\begin{aligned}u_{j+1}^{n+1} &= [u + \Delta x u_x + \frac{1}{2} (\Delta x)^2 u_{xx} + \frac{1}{3!} (\Delta x)^3 u_{xxx} + \frac{1}{4!} (\Delta x)^4 u_{xxxx} + \frac{1}{5!} (\Delta x)^5 u_{xxxxx} + \frac{1}{6!} (\Delta x)^6 u_{xxxxxx} + \dots]_j^{n+1} \\ - 2u_j^{n+1} &= [-2u]_j^{n+1} \quad (+ \dots)\end{aligned}$$

$$u_{j-1}^{n+1} = [u - \Delta x u_x + \frac{1}{2} (\Delta x)^2 u_{xx} - \frac{1}{3!} (\Delta x)^3 u_{xxx} + \frac{1}{4!} (\Delta x)^4 u_{xxxx} - \frac{1}{5!} (\Delta x)^5 u_{xxxxx} + \frac{1}{6!} (\Delta x)^6 u_{xxxxxx} + \dots]_j^{n+1};$$

Adding these together, we see that the first term and all odd terms cancel:

$$\delta_x^2 u_j^{n+1} = [(\Delta x)^2 u_{xx} + \frac{1}{12} (\Delta x)^4 u_{xxxx} + \frac{2}{6!} (\Delta x)^6 u_{xxxxxx} + \dots]_j^{n+1}$$

Thus, we have proven (2.81)

Equation (2.82) in M&M is given as:

$$\begin{aligned}\Theta \delta_x^2 u_j^{n+1} + (1-\Theta) \delta_x^2 u_j^n &= \left[(\Delta x)^2 u_{xx} + \frac{1}{12} (\Delta x)^4 u_{xxxx} + \frac{2}{6!} (\Delta x)^6 u_{xxxxxx} + \dots \right]_j^{n+\frac{1}{2}} \\ &\quad + (\Theta - \frac{1}{2}) \Delta t \left[(\Delta x)^2 u_{xxt} + \frac{1}{12} (\Delta x)^4 u_{xxxxt} + \dots \right]_j^{n+\frac{1}{2}} \\ &\quad + \frac{1}{8} (\Delta t)^2 (\Delta x)^2 [u_{xxtt}] + \dots\end{aligned}$$

Consider each term on the RHS of (2.81) and expand them around $t = t_{n+\frac{1}{2}}$

$$[(\Delta x)^2 u_{xx}]_j^{n+1} = (\Delta x)^2 \left[u_{xx} + \frac{1}{2} \Delta t u_{xxt} + \frac{1}{2} \left(\frac{1}{2} \Delta t \right)^2 u_{xxtt} + \dots \right]_j^{n+\frac{1}{2}}$$

$$\left[\frac{1}{12} (\Delta x)^4 u_{xxxx} \right]_j^{n+1} = \frac{1}{12} (\Delta x)^4 \left[u_{xxxx} + \frac{1}{2} \Delta t u_{xxxxt} + \dots \right]_j^{n+\frac{1}{2}}$$

$$\left[\frac{2}{6!} (\Delta x)^6 u_{xxxxxx} \right]_j^{n+1} = \frac{2}{6!} (\Delta x)^6 \left[u_{xxxxxx} + \dots \right]_j^{n+\frac{1}{2}}$$

Adding these together, we see that

$$\begin{aligned}\delta_x^2 u_j^{n+1} &= \left[(\Delta x)^2 u_{xx} + \frac{1}{12} (\Delta x)^4 u_{xxxx} + \frac{2}{6!} (\Delta x)^6 u_{xxxxxx} + \dots \right]_j^{n+\frac{1}{2}} \\ &\quad + (\Delta t) \left[\frac{1}{2} (\Delta x)^2 u_{xxt} + \frac{1}{24} (\Delta x)^4 u_{xxxxt} + \dots \right]_j^{n+\frac{1}{2}} \quad (*) \\ &\quad + (\Delta t)^2 \left[\frac{1}{8} (\Delta x)^2 u_{xxtt} + \dots \right]_j^{n+\frac{1}{2}} + \dots\end{aligned}$$

We now do the same for $\delta_x^2 u_j^n$:

$$[(\Delta x)^2 u_{xx}]_j^n = (\Delta x)^2 \left[u_{xx} - \frac{1}{2} \Delta t u_{xxt} + \frac{1}{2} \left(\frac{1}{2} \Delta t \right)^2 u_{xxtt} + \dots \right]_j^{n+\frac{1}{2}}$$

$$\left[\frac{1}{12} (\Delta x)^4 u_{xxxx} \right]_j^n = \frac{1}{12} (\Delta x)^4 \left[u_{xxxx} - \frac{1}{2} \Delta t u_{xxxxt} + \dots \right]_j^{n+\frac{1}{2}}$$

$$\left[\frac{2}{6!} (\Delta x)^6 u_{xxxxxx} \right]_j^n = \frac{2}{6!} (\Delta x)^6 \left[u_{xxxxxx} + \dots \right]_j^{n+\frac{1}{2}}$$

$$\begin{aligned}\therefore \delta_x^2 u_j^n &= \left[(\Delta x)^2 u_{xx} + \frac{1}{12} (\Delta x)^4 u_{xxxx} + \frac{2}{6!} (\Delta x)^6 u_{xxxxxx} + \dots \right]_j^{n+\frac{1}{2}} \\ &\quad + (\Delta t) \left[-\frac{1}{2} (\Delta x)^2 u_{xxt} - \frac{1}{24} (\Delta x)^4 u_{xxxxt} + \dots \right]_j^{n+\frac{1}{2}} \quad (**) \\ &\quad + (\Delta t)^2 \left[\frac{1}{8} (\Delta x)^2 u_{xxtt} + \dots \right]_j^{n+\frac{1}{2}} + \dots\end{aligned}$$

Finally, we can substitute (*) and (**) into the Θ discretisation $\Theta \delta_x^2 u_j^{n+1} + (1-\Theta) \delta_x^2 u_j^n$

$$\begin{aligned} \Theta \delta_x^2 u_j^{n+1} + (1-\Theta) \delta_x^2 u_j^n &= \left[\Theta (\Delta x)^2 u_{xx} + \frac{1}{12} \Theta (\Delta x)^4 u_{xxxx} + \frac{2}{6!} \Theta (\Delta x)^6 u_{xxxxxx} + \dots \right]_j^{n+\frac{1}{2}} \\ &\quad + (\Delta t) \left[\frac{1}{2} \Theta (\Delta x)^2 u_{xxt} + \frac{1}{24} \Theta (\Delta x)^4 u_{xxxxt} + \dots \right]_j^{n+\frac{1}{2}} \\ &\quad + (\Delta t)^2 \left[\frac{1}{8} \Theta (\Delta x)^2 u_{xxtt} + \dots \right]_j^{n+\frac{1}{2}} \\ &\quad + [(1-\Theta)(\Delta x)^2 u_{xx} + \frac{1}{12}(1-\Theta)(\Delta x)^4 u_{xxxx} + \frac{2}{6!}(1-\Theta)(\Delta x)^6 u_{xxxxxx} + \dots]_j^{n+\frac{1}{2}} \\ &\quad + (\Delta t) \left[-\frac{1}{2}(1-\Theta)(\Delta x)^2 u_{xxt} - \frac{1}{24}(1-\Theta)(\Delta x)^4 u_{xxxxt} + \dots \right]_j^{n+\frac{1}{2}} \\ &\quad + (\Delta t)^2 \left[\frac{1}{8} (1-\Theta)(\Delta x)^2 u_{xxtt} + \dots \right]_j^{n+\frac{1}{2}} + \dots \end{aligned}$$

By relating like terms, this can be reduced to:

$$\begin{aligned} \Theta \delta_x^2 u_j^{n+1} + (1-\Theta) \delta_x^2 u_j^n &= \left[(\Delta x)^2 u_{xx} + \frac{1}{12} (\Delta x)^4 u_{xxxx} + \frac{2}{6!} (\Delta x)^6 u_{xxxxxx} + (\Theta - \frac{1}{2}) \Delta t (\Delta x)^2 u_{xxt} \right. \\ &\quad \left. + \frac{1}{12} (\Theta - \frac{1}{2}) \Delta t (\Delta x)^4 u_{xxxxt} + \frac{1}{8} (\Delta t)^2 (\Delta x)^2 u_{xxtt} + \dots \right]_j^{n+\frac{1}{2}} \\ &= \left[(\Delta x)^2 u_{xx} + \frac{1}{12} (\Delta x)^4 u_{xxxx} + \frac{2}{6!} (\Delta x)^6 u_{xxxxxx} + \dots \right]_j^{n+\frac{1}{2}} \\ &\quad + (\Theta - \frac{1}{2}) \Delta t \left[(\Delta x)^2 u_{xxt} + \frac{1}{12} (\Delta x)^4 u_{xxxxt} + \dots \right]_j^{n+\frac{1}{2}} \\ &\quad + \frac{1}{8} (\Delta t)^2 (\Delta x)^2 [u_{xxtt} + \dots]_j^{n+\frac{1}{2}} \\ &\quad + \dots \end{aligned}$$

Thus, we have proven (2.82)

Finally, equations (2.83) and (2.84) in M&M are given by:

$$\left\{ \begin{array}{l} T_j^{n+\frac{1}{2}} := \underbrace{\delta_t \delta_x u_j^{n+\frac{1}{2}}}_{\Delta t} - \underbrace{\Theta \delta_x^2 u_j^{n+1} + (1-\Theta) \delta_x^2 u_j^n}_{(\Delta x)^2} \\ = [u_t - u_{xx}]_j^{n+\frac{1}{2}} + [(\frac{1}{2}-\Theta) \Delta t u_{xxt} - \frac{1}{12} (\Delta x)^2 u_{xxxx} + \dots]_j^{n+\frac{1}{2}} \\ \quad + [\frac{1}{24} (\Delta t)^2 u_{ttt} - \frac{1}{8} (\Delta t)^2 u_{xxtt} + \dots]_j^{n+\frac{1}{2}} \\ \quad + [\frac{1}{12} (\frac{1}{2}-\Theta) \Delta t (\Delta x)^2 u_{xxxxt} - \frac{2}{6!} (\Delta x)^4 u_{xxxxxx} + \dots]_j^{n+\frac{1}{2}} + \dots \end{array} \right.$$

From equation (2.80),

$$\frac{\delta_t u_j^{n+\frac{1}{2}}}{\Delta t} = [u_t + \frac{1}{24} (\Delta t)^2 u_{ttt} + \dots]_j^{n+\frac{1}{2}} \quad (1)$$

From equation (2.82),

$$\frac{\Theta \delta_x^2 u_j^{n+1} + (1-\Theta) \delta_x^2 u_j^n}{(\Delta x)^2} = [u_{xx} + \frac{1}{12} (\Delta x)^2 u_{xxxx} + \frac{2}{6!} (\Delta x)^4 u_{xxxxxx} + \dots]_j^{n+\frac{1}{2}} \quad (2)$$

$$\quad + (\Theta - \frac{1}{2}) \Delta t [u_{xxt} + \frac{1}{12} (\Delta x)^2 u_{xxxxt} + \dots]_j^{n+\frac{1}{2}}$$

$$\quad + \frac{1}{8} (\Delta t)^2 [u_{xxtt} + \dots]_j^{n+\frac{1}{2}} + \dots$$

Collecting like terms and simplifying, (computing (1)-(2))

$$\begin{aligned} T_j^{n+\frac{1}{2}} &= [u_t - u_{xx}]_j^{n+\frac{1}{2}} + [(\frac{1}{2}-\Theta) \Delta t u_{xxt} - \frac{1}{12} (\Delta x)^2 u_{xxxx} + \dots]_j^{n+\frac{1}{2}} \\ &\quad + (\Theta - \frac{1}{2}) \Delta t [u_{xxt} + \frac{1}{12} (\Delta x)^2 u_{xxxxt} - \frac{1}{8} (\Delta t)^2 u_{xxtt} + \dots]_j^{n+\frac{1}{2}} \end{aligned}$$

$$+ [\frac{1}{12} (\frac{1}{2}-\Theta) \Delta t (\Delta x)^2 u_{xxxxt} - \frac{2}{6!} (\Delta x)^4 u_{xxxxxx} + \dots]_j^{n+\frac{1}{2}} + \dots \quad \text{Thus, we have proven (2.84)}$$

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(3) Equation (1), along with its initial and boundary conditions, is given by

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = a(t) \frac{\partial u}{\partial x} + \epsilon \frac{\partial^2 u}{\partial x^2}, \quad x \in [l_p, L] \\ u(x, 0) = u_0(x) \\ u(l_p, t) = u(L, t) = 0 \end{array} \right.$$

with $\epsilon = \text{const.}$, $a(t)$, ~~given~~ $u_0(x)$ given functions.

For the θ -method, we must consider both an explicit and implicit difference scheme.

In the explicit scheme,

$$\frac{\partial u}{\partial t} \approx \frac{\Delta_t U_j^n}{\Delta t} = \frac{U_j^{n+1} - U_j^n}{\Delta t}, \quad \frac{\partial u}{\partial x} \approx \frac{\Delta_x U_j^n}{\Delta x} = \frac{U_{j+1}^n - U_j^n}{\Delta x}, \quad \frac{\partial^2 u}{\partial x^2} \approx \frac{\delta_x^2 U_j^n}{(\Delta x)^2} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

Putting these together,

$$U_j^{n+1} - U_j^n = a(t_n) \Delta_x \mu \Delta_{tx} U_j^n + \epsilon \mu \delta_x^2 U_j^n, \quad \text{where } \mu = \frac{\Delta t}{(\Delta x)^2}$$

Similarly, for the implicit scheme,

$$\begin{aligned} \frac{\partial u}{\partial t} &\approx \frac{\Delta_t U_j^n}{\Delta t} = \frac{U_j^{n+1} - U_j^n}{\Delta t}, \quad \frac{\partial u}{\partial x} \approx \frac{\Delta_x U_j^{n+1}}{\Delta x}, \quad \frac{\partial^2 u}{\partial x^2} \approx \frac{\delta_x^2 U_j^{n+1}}{(\Delta x)^2} \\ \Rightarrow U_j^{n+1} - U_j^n &= a(t_{n+1}) \Delta_x \mu \Delta_{tx} U_j^{n+1} + \epsilon \mu \delta_x^2 U_j^{n+1} \end{aligned}$$

Putting these together, the θ -scheme is

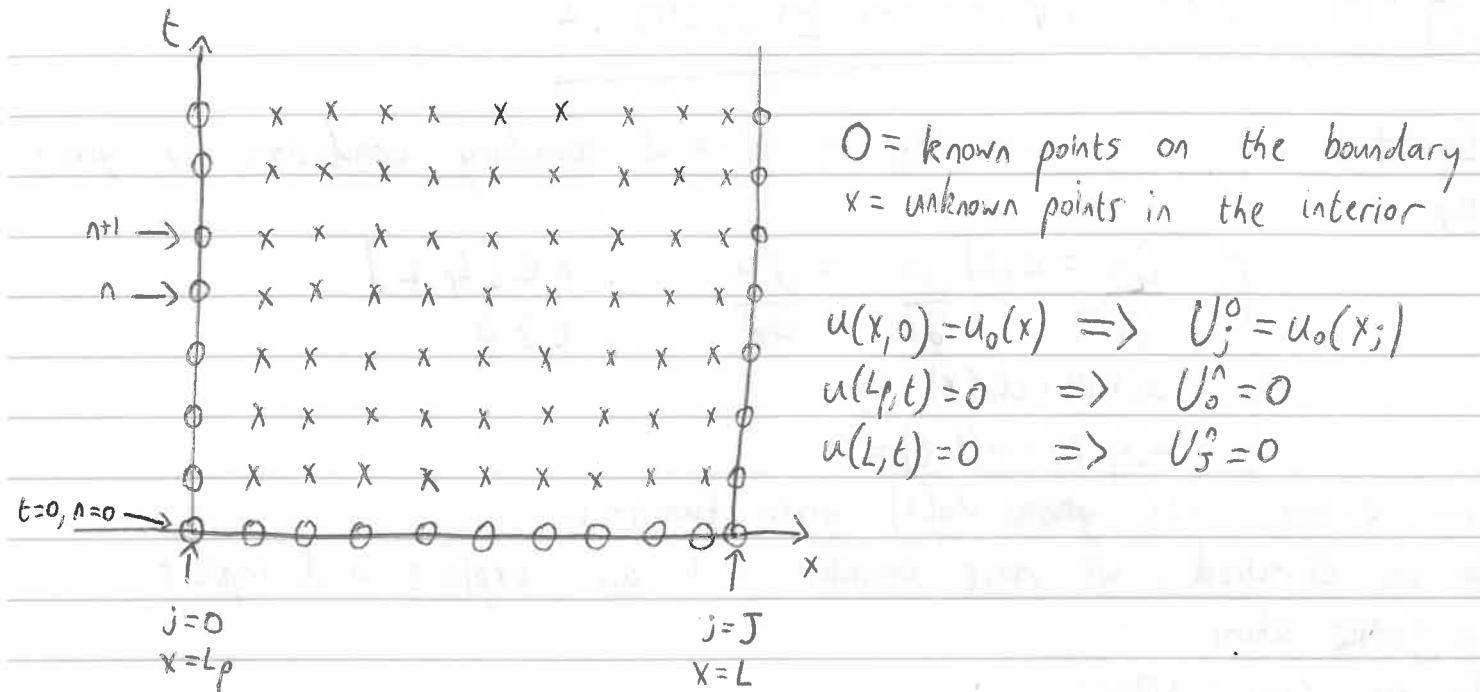
$$U_j^{n+1} - U_j^n = \theta [\mu a^{n+1} \Delta_x \Delta_{tx} U_j^{n+1} + \mu \epsilon \delta_x^2 U_j^{n+1}] + (1-\theta) [\mu a^n \Delta_x \Delta_{tx} U_j^n + \mu \epsilon \delta_x^2 U_j^n] (t)$$

where $a^n = a(t_n)$

In these calculations, we have used

$$U_j^n \approx u(x_j, t_n), \quad x_j = l_p + j \Delta x, \quad t_n = n \Delta t, \quad j = 0, 1, \dots, J, \quad n = 0, 1, \dots, N$$

$$J = \frac{(L - l_p)}{\Delta x}$$



We can rewrite the θ -scheme (†) as

$$-\theta\mu \in U_{j-1}^{n+1} + (1+\theta\mu a^{n+1}\Delta x + 2\theta\mu \epsilon) U_j^{n+1} - (\theta\mu a^{n+1}\Delta x + \theta\mu \epsilon) U_{j+1}^{n+1} = [1 + (\Delta x a^n \Delta_{tx} + \epsilon \delta_x^2)(1-\theta)\mu] U_j^n$$

Relating this to (2.65) in M&M, we see that

$$a_j = \theta\mu$$

$$b_j = (1 + \theta\mu a^{n+1}\Delta x + 2\theta\mu \epsilon) = 1 + \theta\mu(a^{n+1}\Delta x + 2\epsilon)$$

$$c_j = \theta\mu a^{n+1}\Delta x + \theta\mu \epsilon = \theta\mu(a^{n+1}\Delta x + \epsilon)$$

$$d_j = [1 + (\Delta x a^n \Delta_{tx} + \epsilon \delta_x^2)(1-\theta)\mu] U_j^n$$

We now use the Thomas algorithm as described in M&M:

$$U_j^{n+1} = e_j; U_{j+1}^{n+1} = f_j \quad j=1, \dots, J-1 \text{ (interior points)} \quad (*)$$

$$\text{with } e_j = \frac{c_j}{b_j - a_j e_{j-1}} = \frac{\theta\mu(a^{n+1}\Delta x + \epsilon)}{1 + \theta\mu(a^{n+1}\Delta x + 2\epsilon) - \theta\mu \epsilon e_{j-1}}$$

$$f_j = \frac{U_j^n + (1-\theta)\mu [a^n \Delta x (U_{j+1}^n - U_j^n) + \epsilon (U_{j+1}^n - 2U_j^n + U_{j-1}^n)] + \theta\mu \epsilon f_{j-1}}{1 + \theta\mu(a^{n+1}\Delta x + 2\epsilon) - \theta\mu \epsilon e_{j-1}}$$

$$\text{and } e_0, f_0 = 0$$

This algorithm works by solving along each row of the grid in turn, using values from the row below which are implemented into the recursive relation f_j ($U_{j+1}^n, U_j^n, U_{j-1}^n$). Once a point on a row is calculated, (e.g. U_{j+1}^n), the point to its left can be evaluated using (*), so the algorithm works from $j=J-1 \rightarrow j=1$.

The first row, U_j^0 , $j=1, \dots, J-1$ is already given from the boundary condition and all U_j^n are given by the boundary condition, so every point can be evaluated.

For time step $t = t_n = n\Delta t$,

$$\begin{pmatrix} U_{J-1}^n \\ U_{J-2}^n \\ \vdots \\ U_2^n \\ U_1^n \end{pmatrix} = \begin{pmatrix} e_{J-1} U_J^n + f_{J-1} \\ e_{J-2} U_{J-1}^n + f_{J-2} \\ \vdots \\ e_2 U_3^n + f_2 \\ e_1 U_2^n + f_1 \end{pmatrix}$$

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4) In M&M, we are given the problem

$$\left\{ \begin{array}{l} u_t = u_{xx}, \quad t > 0, \quad 0 < x < 1 \\ u(0, t) = u(1, t) = 0, \quad t > 0 \\ u(x, 0) = u^0(x), \quad t = 0, \quad 0 \leq x \leq 1 \end{array} \right.$$

With

$$u^0(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

This problem has the Θ -Scheme: $U_j^{n+1} - U_j^n = \mu [\Theta \delta_x^2 U_j^{n+1} + (1-\Theta) \delta_x^2 U_j^n]$

The explicit scheme is given by setting $\Theta = 0$

$$\Rightarrow [U_j^{n+1} - U_j^n = \mu \delta_x^2 U_j^n] \quad (t)$$

Similarly to in question 3), I have applied the Thomas algorithm to this scheme and have found that

$$U_j^{n+1} = e_j U_{j+1}^{n+1} + f_j$$

$$\text{with } e_j = \frac{\Theta \mu}{1 + 2\Theta \mu - \Theta \mu e_{j-1}}, \quad f_j = \frac{U_j^n + \mu(1-\Theta)(U_{j-1}^n - 2U_j^n + U_{j+1}^n) + \Theta \mu f_{j-1}}{1 + 2\Theta \mu - \Theta \mu e_{j-1}}$$

$$\text{Setting } \Theta = 0, \quad e_j = 0, \quad f_j = U_j^n + \mu(U_{j-1}^n - 2U_j^n + U_{j+1}^n)$$

Thus,

$$U_j^{n+1} = U_j^n + \mu(U_{j-1}^n - 2U_j^n + U_{j+1}^n)$$

(the same result as if we had expanded the right hand side of (t))

I have implemented the general Θ scheme into my Python code and set $\Theta = 0$ for the explicit scheme. This has resulted in the eight graphs shown in Fig. 2.2 of M&M, along with two graphs which display multiple time-profiles. I have made it so that the value of Δt is changeable, so results can be found for both $\Delta t = 0.0012$ and $\Delta t = 0.0013$, as in M&M.

Let us now consider the stability properties of this scheme. Clearly, our results are stable for $\Delta t = 0.0012$ and unstable for $\Delta t = 0.0013$. (with $\Delta x = 0.05$) We will begin with a Fourier analysis.

Let $U_j^n = \lambda^n e^{ikj\Delta x}$ k is some scalar.

From this we can see that:

$$U_j^{n+1} = \lambda^{n+1} e^{ikj\Delta x} = \lambda U_j^n$$

Substituting this into our difference equation,

$$\begin{aligned} \lambda \cdot \lambda^n e^{ikj\Delta x} &= \lambda^n e^{ikj\Delta x} + \mu \lambda^n (e^{ikh(j-1)\Delta x} - 2e^{ikhj\Delta x} + e^{ikh(j+1)\Delta x}) \\ \Rightarrow \lambda &= 1 + \mu (e^{-ikh\Delta x} - 2 + e^{ikh\Delta x}) \\ &= 1 + 2\mu (\cosh k\Delta x - 1) \\ \lambda &= 1 - 4\mu \sin^2(\frac{1}{2}k\Delta x) \end{aligned}$$

Since $U_j^{n+1} = \lambda U_j^n$, the system is stable if $|\lambda| \leq 1$

$$\text{i.e. } |1 - 4\mu \sin^2(\frac{1}{2}k\Delta x)| \leq 1$$

$$\max_k |1 - 4\mu \sin^2(\frac{1}{2}k\Delta x)| = |1 - 4\mu| \text{ which gives us the "worst" value of } k, \\ k = \frac{(2m-1)\pi}{\Delta x}, m \in \mathbb{Z} \quad \left(\because \sin^2\left(\frac{h\Delta x}{2}\right) = 1 \right)$$

$$\text{So, } -1 \leq 1 - 4\mu \leq 1$$

Since $\mu > 0$, $1 - 4\mu \leq 1$ is always true, so we only need to consider

$$\Rightarrow \boxed{\mu \leq \frac{1}{2}} \quad \text{This is our stability condition}$$

$$\text{i.e. } \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2} \Rightarrow \Delta t \leq \frac{(\Delta x)^2}{2}$$

Since $J=20$ (the number of sections the x domain is split into), we have used $\Delta x = 0.05$

\therefore The explicit scheme is stable for $\Delta t \leq 0.00125$

This explains why $\Delta t = 0.0012$ is stable but $\Delta t = 0.0013$ is unstable.

According to the maximum principle, the Θ scheme converges with $\Delta x, \Delta t \rightarrow 0$ if

$$\mu(1-\theta) \leq \frac{1}{2}$$

Since we are considering the explicit scheme, we set $\theta=0$ and see that this is the same as the stability condition

$$\mu \leq \frac{1}{2}$$

So, it will converge for $\Delta t = 0.0012$, and not converge for $\Delta t = 0.0013$.

Finally, we extend the explicit scheme to the problem (1). Already in question 3), we computed the Θ scheme to be:

$$U_j^{n+1} - U_j^n = \theta [\mu a^n \Delta x \Delta_{xx} U_j^{n+1} + \mu \epsilon \delta_x^2 U_j^{n+1}] + (1-\theta) [\mu a^n \Delta x \Delta_{xx} U_j^n + \mu \epsilon \delta_x^2 U_j^n]$$

Setting $\theta=0$:

$$U_j^{n+1} - U_j^n = \mu a^n \Delta x \Delta_{xx} U_j^n + \mu \epsilon \delta_x^2 U_j^n$$

Expanding the differences, we get the explicit scheme:

$$\boxed{U_j^{n+1} = U_j^n + \mu a^n \Delta x (U_{j+1}^n - U_j^n) + \mu \epsilon (U_{j+1}^n - 2U_j^n + U_{j-1}^n)}, \quad j=1, 2, \dots, J-1 \\ n=0, 1, \dots$$

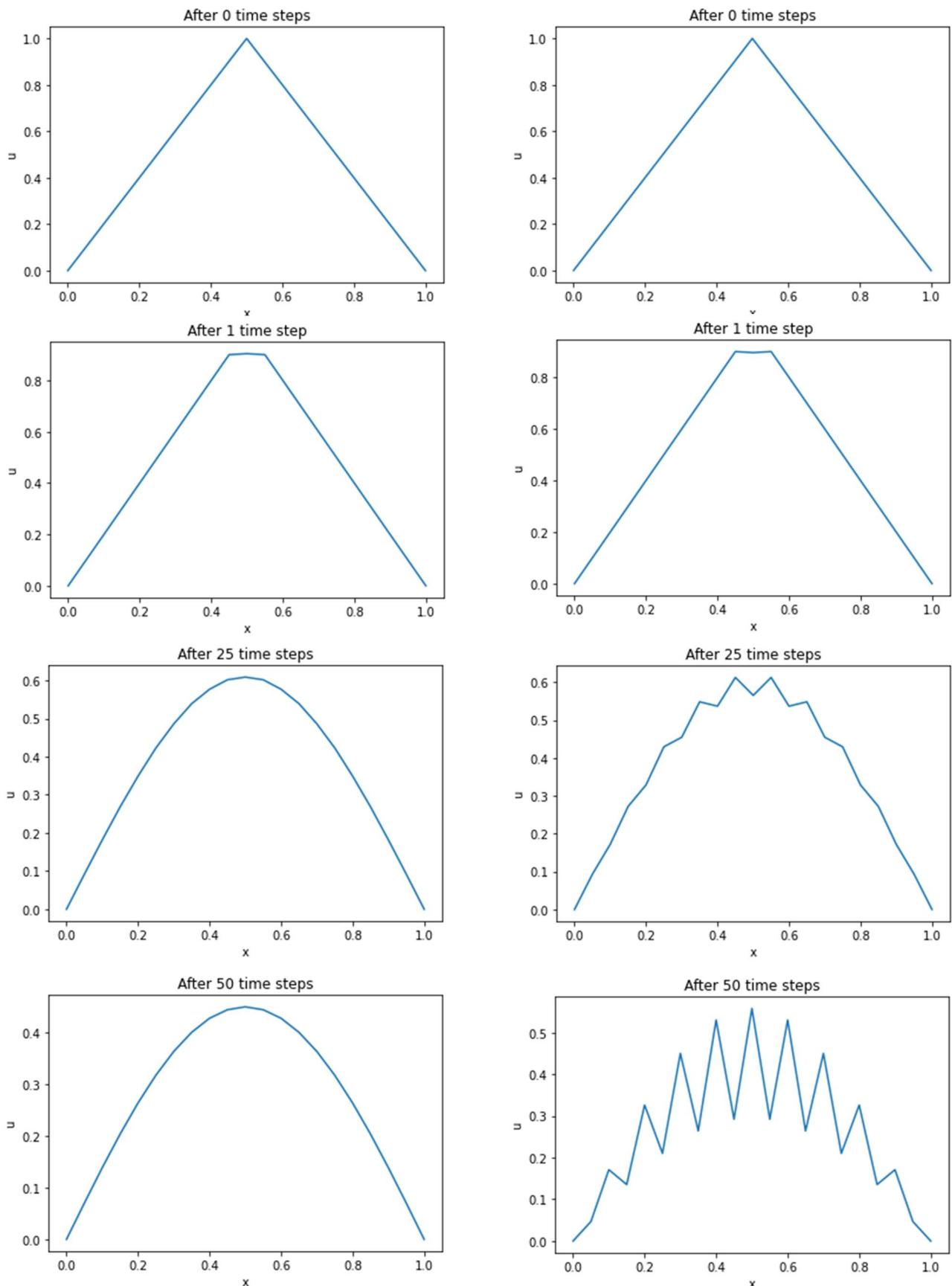


Figure 1: Recreation of Fig. 2.2 in M&M, using the explicit scheme to solve equations (2.7)-(2.9). Plots on the left show results for $\Delta t=0.0012$ and plots on the right show results for $\Delta t=0.0013$. Results produced using M&M.py

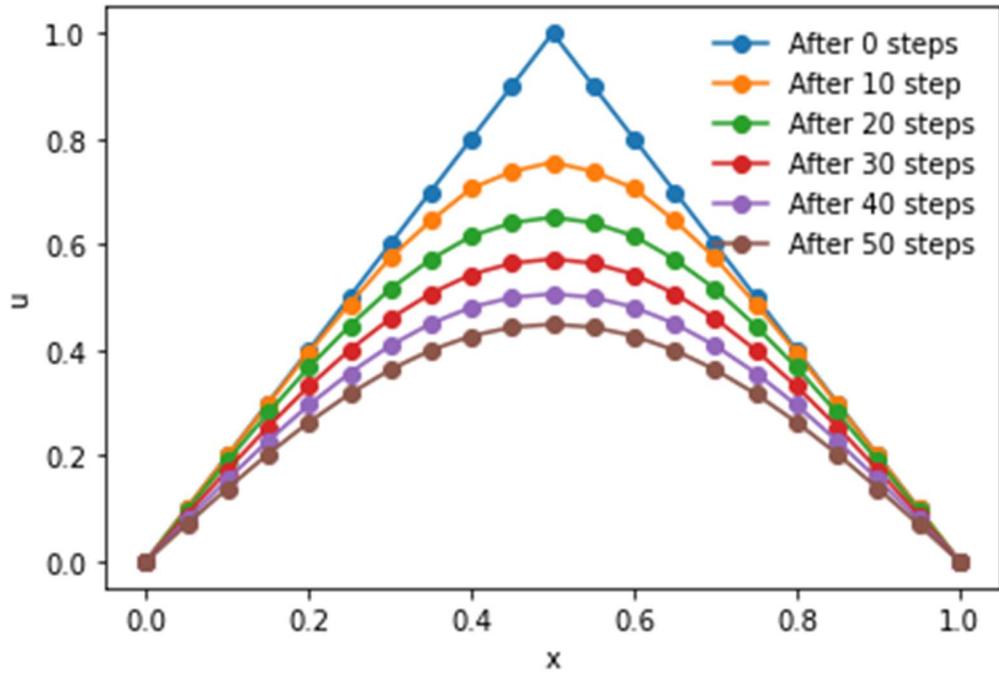


Figure 2: Results of an explicit scheme for equations (2.7)-(2.9) in M&M, with $\Delta t=0.0012$ after multiples of ten time steps. Results produced using M&M.py

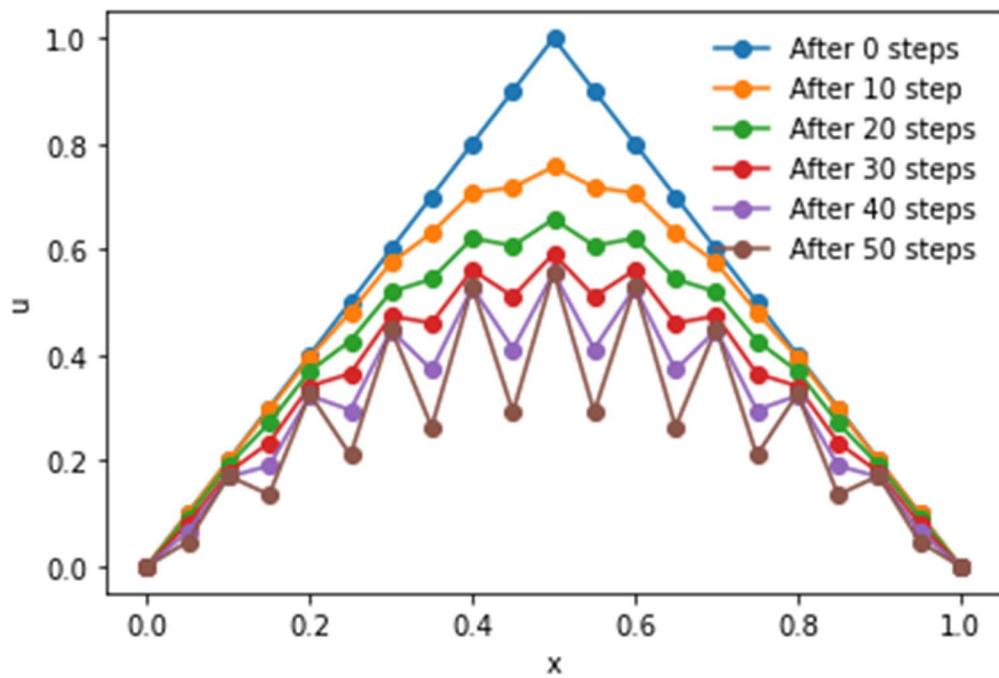


Figure 3: Results using the same scheme as Figure 2, but with $\Delta t=0.0013$. The stability of the results seems to decrease as more time steps are performed.

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5) The Θ -scheme for problem (2.7)-(2.9) given in M&M is given by

$$U_j^{n+1} - U_j^n = \mu [\theta \delta_x^2 U_j^{n+1} + (1-\theta) \delta_x^2 U_j^n], \quad j=1, 2, \dots, J-1$$

$$\theta \in [0, 1]$$

This can be expanded and rewritten as

$$-\Theta \mu U_{j-1}^{n+1} + (1+2\Theta\mu) U_j^{n+1} - \Theta \mu U_{j+1}^{n+1} = U_j^n + (1-\theta)\mu(U_{j-1}^n - 2U_j^n + U_{j+1}^n)$$

This clearly satisfies the conditions (2.67) and (2.68), which means we can use the Thomas algorithm, which gives us

$$U_j^{n+1} = e_j U_{j+1}^{n+1} + f_j$$

$$\text{with } e_j = \frac{\Theta\mu}{1+2\Theta\mu-\Theta\mu e_{j-1}} \quad \text{and} \quad f_j = \frac{U_j^n + (1-\theta)\mu(U_{j-1}^n - 2U_j^n + U_{j+1}^n) + \Theta\mu f_{j-1}}{1+2\Theta\mu-\Theta\mu e_{j-1}}$$

I have implemented this scheme into my code to produce a plot of time profiles after 0, 10, 20, 30, 40 and 50 time steps.

Setting $\Theta=0$, we see that these are the same results as in question 4)
(I have also implemented the Θ -scheme for problem (1) in question 6))

Let us now perform a Fourier analysis on the advection and diffusion parts of (1) in separation.

The diffusion part of the Θ -scheme is given by:

$$U_j^{n+1} - U_j^n = \Theta \mu \epsilon (U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) + (1-\theta)\mu \epsilon (U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

Let $U_j^n = \lambda^n e^{ikAx}$ for some scalar k

$$\lambda^{n+1} e^{ikAx} - \lambda^n e^{ikAx} = \Theta \mu \epsilon \lambda^{n+1} (e^{i(c+1)kAx} - 2e^{ikAx} + e^{i(c-1)kAx})$$

$$+ (1-\theta)\mu \epsilon (\lambda^n (e^{i(c+1)kAx} - 2e^{ikAx} + e^{i(c-1)kAx}))$$

Dividing through by $\lambda^n e^{ikAx}$

$$\lambda - 1 = \Theta \mu \epsilon \lambda (e^{ikAx} - 2 + e^{-ikAx}) + (1-\theta)\mu \epsilon (e^{ikAx} - 2 + e^{-ikAx})$$

Since $e^{-ikAx} + e^{ikAx} - 2 = 2 \cos kAx - 2 = -4 \sin^2(\frac{1}{2}kAx)$, this can be rearranged to

$$\lambda = \frac{1 - 4(1-\theta)\mu \epsilon \sin^2(\frac{1}{2}kAx)}{1 + 4\Theta\mu \epsilon \sin^2(\frac{1}{2}kAx)}$$

For stability, we require that $|\lambda| \leq 1$.

Since the numerator is always less than or equal to 1 and the denominator is always greater than or equal to 1, $|\lambda| \leq 1$ must always be true.

$$\therefore -1 \leq \frac{1-4(1-\theta)\mu\epsilon \sin^2(\frac{1}{2}k\Delta x)}{1+4\theta\mu\epsilon \sin^2(\frac{1}{2}k\Delta x)}$$

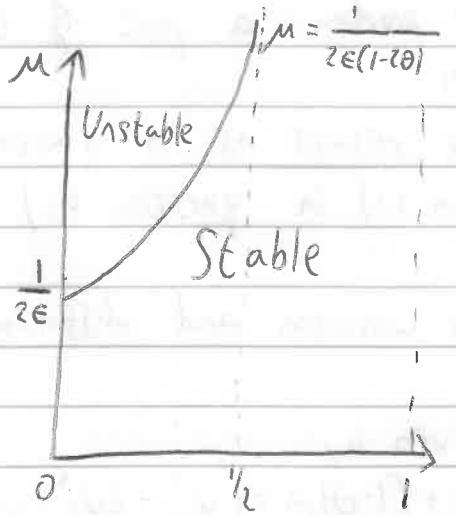
$$\Rightarrow (1-2\theta)\mu\epsilon \sin^2(\frac{1}{2}k\Delta x) \leq \frac{1}{2}$$

$$\max_k |(1-2\theta)\mu\epsilon \sin^2(\frac{1}{2}k\Delta x)| = (\lambda \theta) |(1-2\theta)\mu\epsilon| \text{ which gives the "worst" value of } k$$

$$k = \frac{(2m-1)\pi}{\Delta x}, m \in \mathbb{Z} \quad (\text{i.e. } \sin^2\left(\frac{h\Delta x}{2}\right) = 1)$$

$$\text{So, } (1-2\theta)\mu\epsilon \leq \frac{1}{2}$$

$$\Rightarrow \boxed{\mu \leq \frac{1}{2\epsilon(1-2\theta)}} \quad \text{This is our stability condition, for the diffusion part.}$$



As seen on this graph, if $\theta \geq \frac{1}{2}$, then the diffusion part will be stable, independent from the choice of μ .

If $\theta=0$, then we have stability for $\mu \leq \frac{1}{2\epsilon}$, which agrees with the example in question 4) where $\epsilon=1$.

The advection part of the θ -scheme is given by:

$$U_j^{n+1} - U_j^n = \theta \mu a^n \Delta x (U_{j+1}^n - U_j^n) + (1-\theta) \mu a^n \Delta x (U_{j+1}^n - U_j^n)$$

Once again, let $U_j^n = \lambda^n e^{ik\Delta x}$ for scalar k .

Substituting in and rearranging as in the diffusion case,

$$\lambda = \frac{1 + (1-\theta)\mu a^n \Delta x (e^{ik\Delta x} - 1)}{1 - \theta \mu a^n \Delta x (e^{ik\Delta x} - 1)}$$

To find the "worst" k , we want $|\lambda|$ to be as high as possible, this is achieved by finding $\max|e^{ik\Delta x} - 1|$ to maximise the numerator and minimise the denominator.

Since $e^{ikx} = \cos kx + i \sin kx$ and $|x+iy| = \sqrt{x^2+y^2}$, we can compute that

$$|e^{ik\Delta x} - 1| = \sqrt{2 - 2 \cos k\Delta x}$$

which is maximal when $\cos k\Delta x = -1 \Rightarrow \max |e^{ik\Delta x} - 1| = \sqrt{4} = 2$

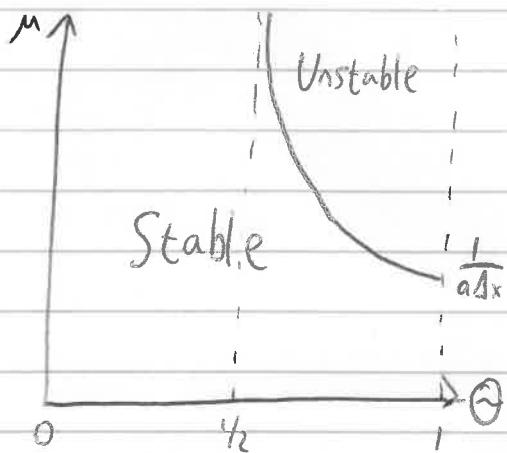
$$\therefore \left| \frac{1 + 2(1-\theta)ma^n\Delta x}{1 - 2\theta ma^{n+1}\Delta x} \right| \leq 1$$

Once again, the numerator ≥ 1 and the denominator ≤ 1 so that $\lambda \leq 1$ is always true.

$$-1 \leq \frac{1 + 2(1-\theta)ma^n\Delta x}{1 - 2\theta ma^{n+1}\Delta x}$$

If we set $a = \text{const.}$, this simplifies to

$$\boxed{\mu \leq \frac{1}{a\Delta x(2\theta-1)}}$$



If $\theta \leq \frac{1}{2}$, then the advection part will be stable.

If $\theta = 1$, then we have stability for $\mu \leq \frac{1}{a\Delta x}$.

Since $\mu = \Delta t / (\Delta x)^2$, this can be rearranged to $\frac{a\Delta t}{\Delta x} \leq 1$.

This corresponds to the CFL condition with $CFL = 1$, and $\epsilon = 0$ to ignore diffusion.

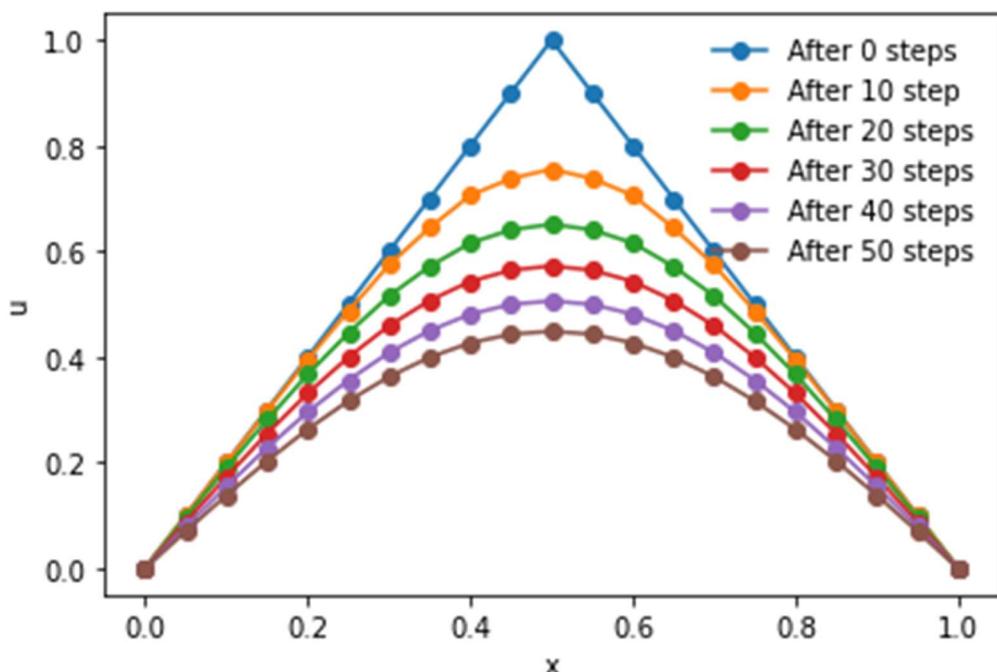
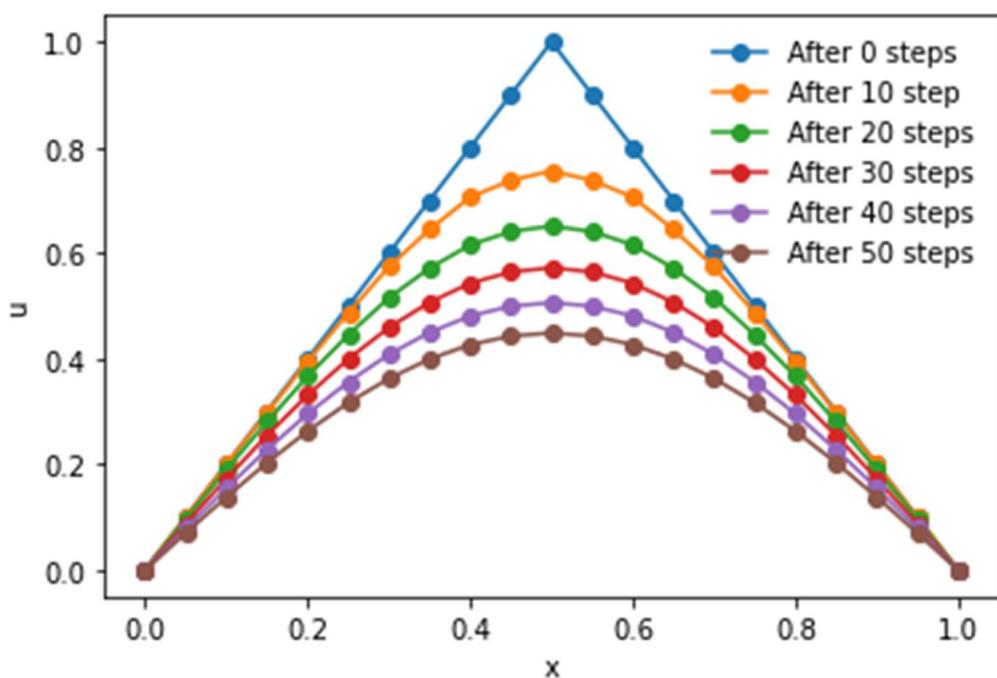


Figure 4: Results of the θ -scheme for equations (2.7)-(2.9) in M&M. Setting $\theta=0$ and $\Delta t=0.0012$, the plot is identical to Figure 2 (shown below for reference), which is to be expected. Results produced using M&M-theta.py.



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6) I have coded the Θ Scheme to solve the problem

$$\begin{cases} u_t - a(t)u_x - \epsilon u_{xx} = 0, & x \in [L_p, L], t \in [0, T] \\ u(x, 0) = u_0(x) \\ u(L_p, t) = u(L, t) = 0 \end{cases}$$

with $L_p = -1$, $L = 1$, $\epsilon = 10^{-3}$, $T = 1$

In the first code, I have implemented the initial condition

$$u_0(x) = (1-x)^4(1+x)$$

and in the second, I have implemented

$$u_0(x) = (1-x)^4(1+x)\left(\sum_{k=0}^3 b_k \phi_k(x) + C\right)$$

where ϕ_k are Legendre polynomials, b_k are uniformly distributed random coefficients in the range $[0, 1]$, and $C \geq 0$ is a constant determined such that the final bracket on the right-hand side is non-negative.

For the values of b_k , I used the function 'random.seed(314)' which gave me

$$b_k = [0.9168735814929675, 0.5885419136419221, 0.26504775117246204, 0.7832053795417385]$$

These values are printed when the code is ran.

In question 3), we defined an explicit scheme for equation (1), which is

$$\begin{aligned} U_j^{n+1} &= U_j^n + \alpha^2 \Delta x \mu \Delta_x U_j^n + \epsilon \mu \delta_x^2 U_j^n \\ &= U_j^n + \alpha^2 \Delta x \mu (U_{j+1}^n - U_j^n) + \epsilon \mu \delta_x^2 (U_{j+1}^n - 2U_j^n + U_{j-1}^n) \end{aligned}$$

I have also coded this scheme and for all three codes produced ~~time~~ plots displaying time profiles for $t = 0, 0.01, 0.25, 0.5, 1$.

For the Θ scheme codes, I have displayed 6 plots for each: depending on whether $\Theta = 0$ or 1 , and whether $\mu = 0.49$ ($J = \frac{1}{\Delta x} = 14$, $N = 100$), 0.5625 ($J = \frac{1}{\Delta x} = 15$, $N = \frac{1}{\Delta t} = 100$) or 6.25 ($J = \frac{1}{\Delta x} = 50$, $N = \frac{1}{\Delta t} = 100$).

We see for all cases that the results are stable for both $\mu < \frac{1}{2}$ and $\mu > \frac{1}{2}$, unlike in the explicit scheme. We also see smoother curves when considering a smaller spatial step ($\mu = 6.25$).

Comparing the graphs for $\theta=0$ and the explicit scheme, we see that they are identical as assumed.

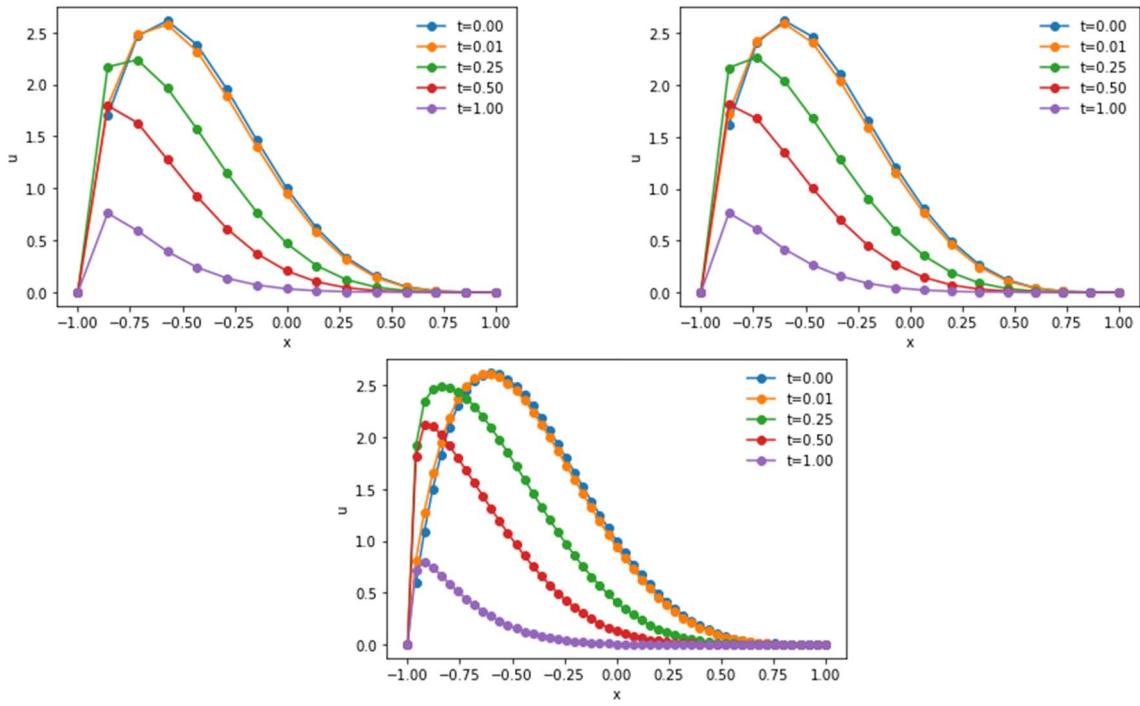


Figure 5: Results for the θ -scheme for equation (1) given in the exercise sheet, with the first initial condition given in exercise 6. $\Delta t=0.01$, $\theta=0$ and $\mu=0.49$, 0.5625 and 6.25 . The μ value has been increased by decreasing the discretisation step Δx , which creates a smoother curve. We see that the results are stable for all choices of μ . Results produced by theta-1.py.

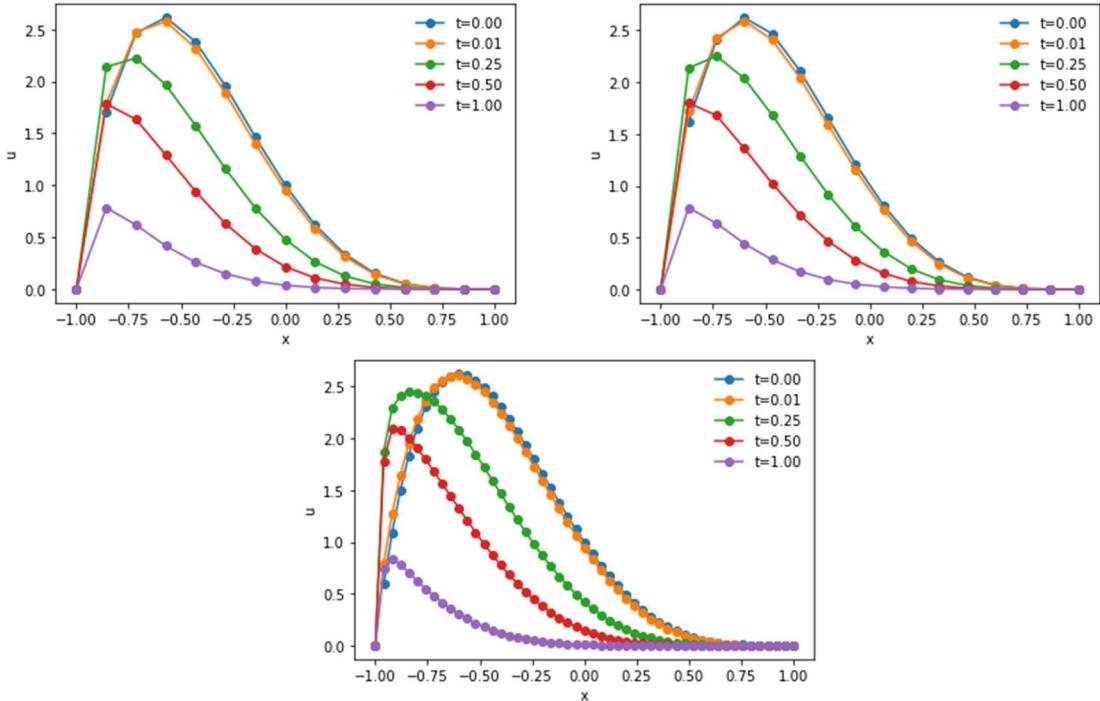


Figure 6: Results using the same initial condition and variables as Figure 5, but with $\theta=1$. We see that the results are very similar. Results produced by theta-1.py.

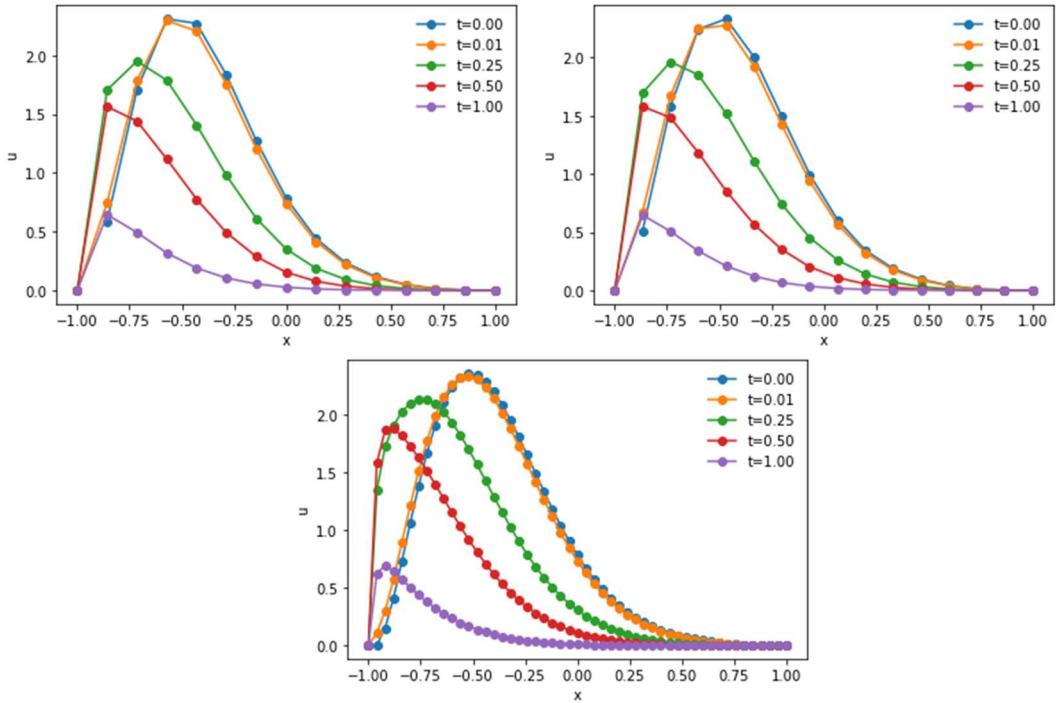


Figure 7: Results from the same variables as Figure 5, but with the second initial condition. $\Theta=0$.
Results produced by theta-2.py.

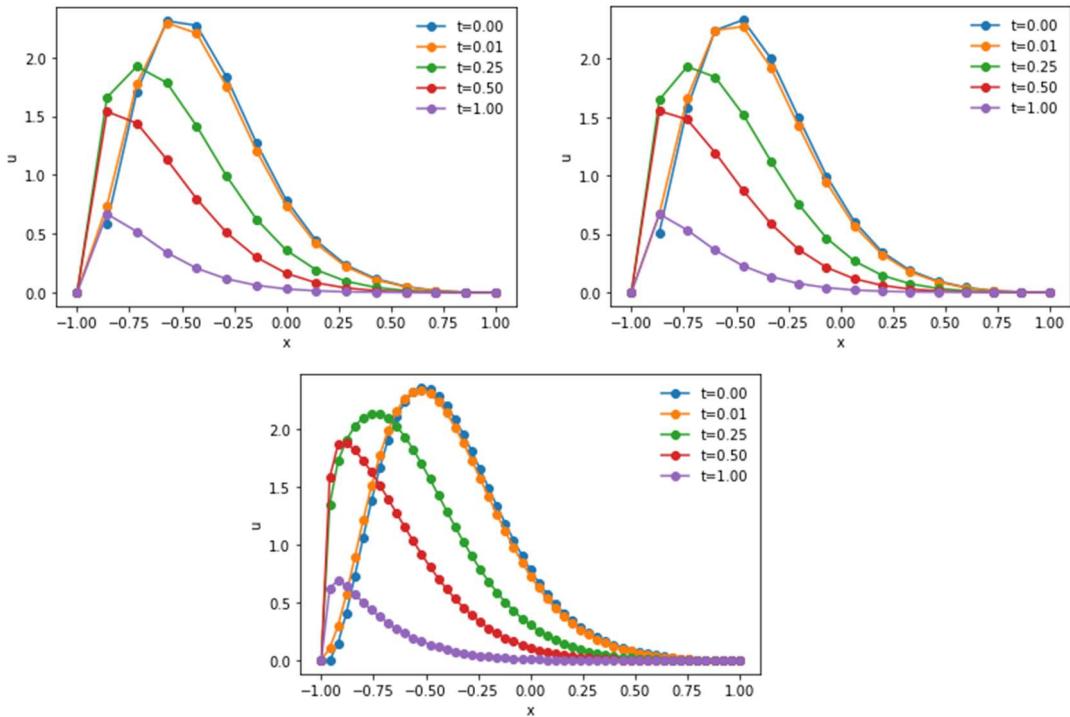


Figure 8: Results from the same variables as Figure 6, but with the second initial condition. $\Theta=1$.
Results produced by theta-2.py.

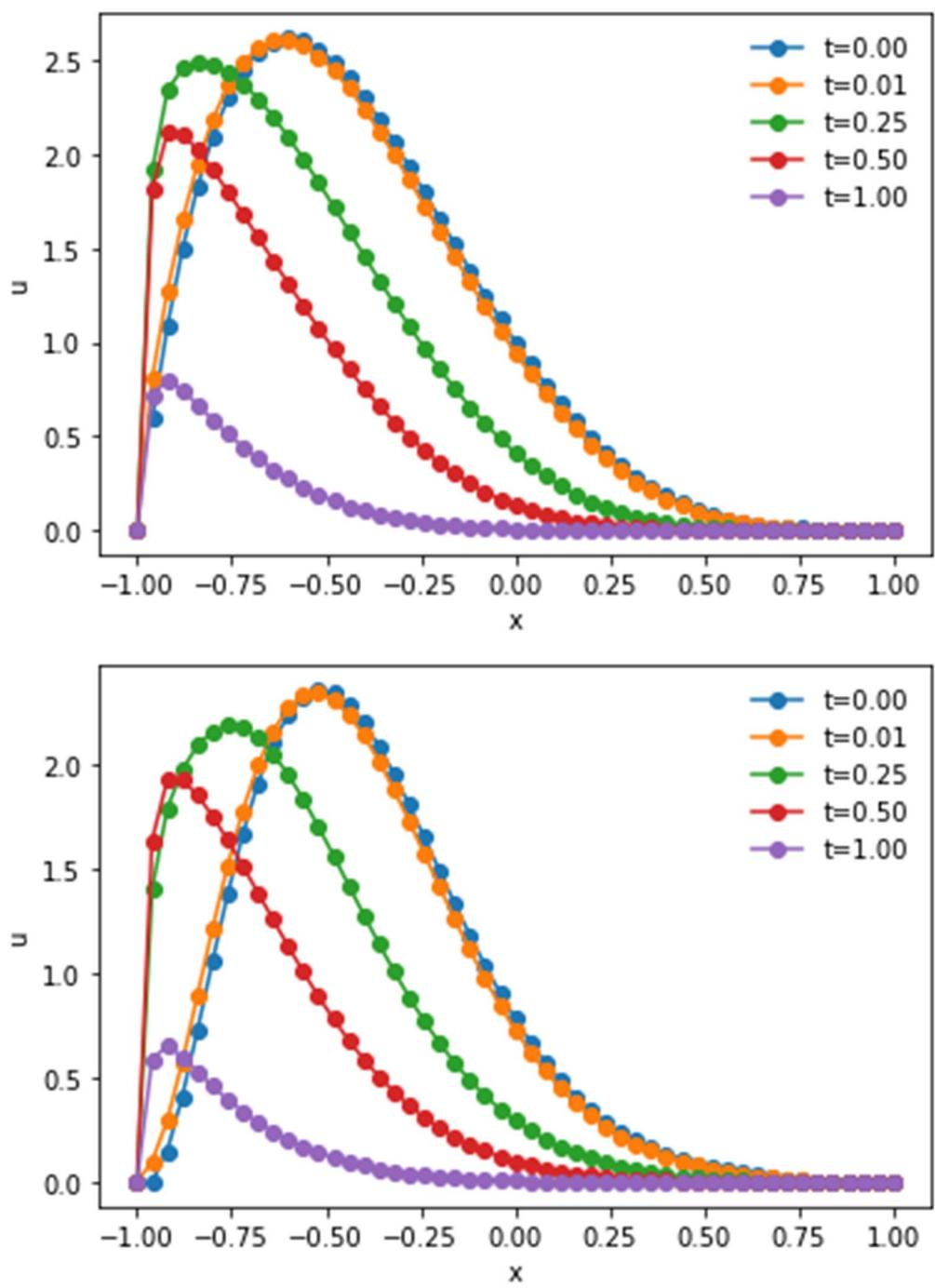


Figure 9: Results of an explicit scheme for problem (1) given in the exercise sheets. These graphs are identical to the plots with $\mu=6.25$ in Figures 5 and 7 which each have $\theta=0$, as expected. These graphs were produced by explicit-scheme.py.

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7) Using the code implemented in question 6), I have plotted four graphs with $\Theta=1$, refining Δx each time.

The values I have used are:

- $J=10, \Delta x=0.2$
- $J=20, \Delta x=0.1$ (since $\Delta x = \frac{L-L_p}{J} = \frac{2}{J}$)
- $J=40, \Delta x=0.05$
- $J=100, \Delta x=0.02$

From these graphs we can clearly see that the results are converging.

I have also plotted time profiles at times $t=0.00, 0.01, 0.25, 0.50, 1.00$.

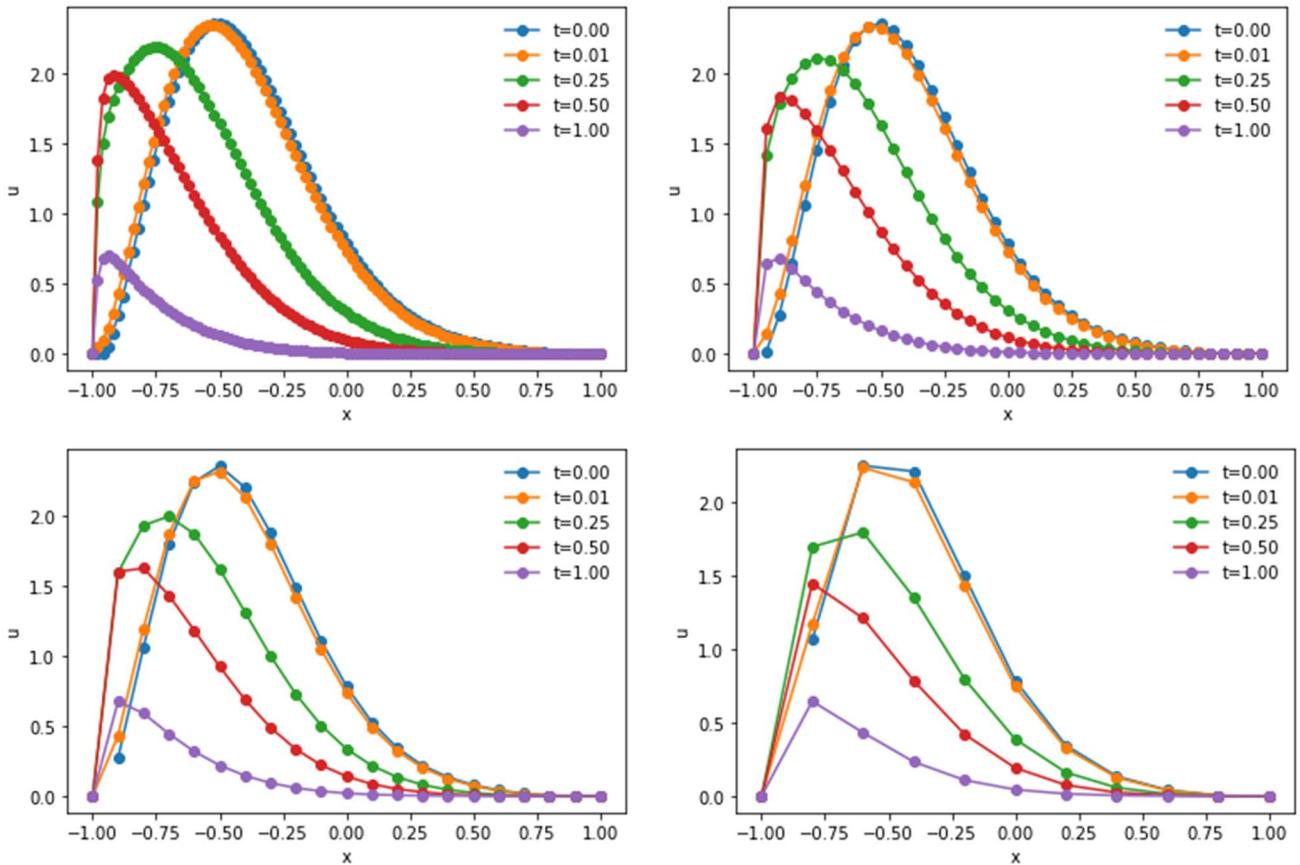


Figure 10: Results of the θ -scheme for problem (1) in the exercise sheets with the second given initial condition. In each plot, Δx has been refined so that it is equal to: 0.05, 0.02, 0.1 and 0.2. We can see that the results converge to a smooth curve as the discretisation step decreases.

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8) Finally, I have plotted at the same time steps but using the values
 $\theta=1$, $\Delta x=0.02$, $\epsilon=10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 0$
we can clearly see that, as $\epsilon \rightarrow 0$, the wave will drop less and less, which makes sense as ϵ is the diffusion constant. The fluid will diffuse less and less.

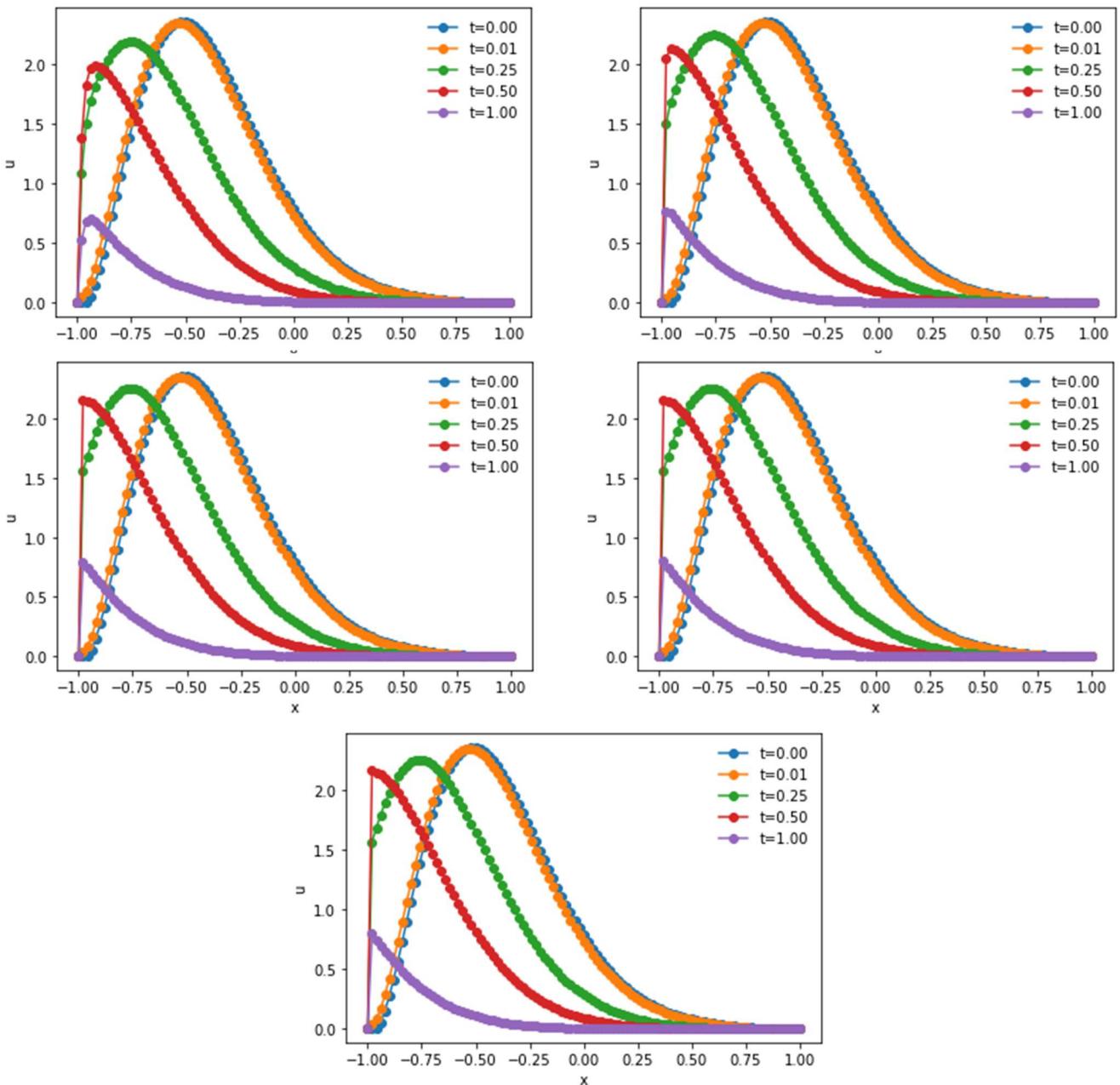


Figure 11: Results of the same scheme as in Figure 10, with $\theta=1$, $\Delta x=0.02$ and $\varepsilon=10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$ and 0. With each plot, the diffusion lessens.