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Consider the linearised shallow-water system of equations

$$\frac{\partial \eta}{\partial t}+\frac{\partial (Hu)}{\partial x}=0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}+\frac{\partial (g\eta)}{\partial x}=0 \tag{1}$$

for the variables velocity $u = u(x;t)$ and free-surface deviation $\eta = \eta(x;t)$, rest depth $H(x)$ and acceleration of gravity $g = 9.81m/s^2$. When we scale (1) as follows

$$u = U_0 u', \quad x = L_s x', \quad t = (L_s/U_0)t', \quad \eta = H_{0s}\eta', \quad H = H_{0s}H' \tag{2}$$

with primed dimensionless variables then g can be replaced by a dimensionless $g' = gH_{0s}/U_0^2$ and we can work with scaled equations. (Show this.) These latter, scaled equations look exactly the same as (1), when we drop the primes, but g' (or g once primes are dropped) can attain different (dimensionless) values depending on the choices of L_s, H_{0s} and U_{0s}; furthermore, when $U_0^2 = gH_{0s}$ we note that $g' = 1$. We usually drop the primes after the scaling.

Solution: (showing the scaling (1))

First the time and space derivative are given by

$$\frac{\partial}{\partial t}=\frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}=\frac{U_0}{L_s}\frac{\partial}{\partial t'}, \tag{3}$$

$$\frac{\partial}{\partial x}=\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}=\frac{1}{L_s}\frac{\partial}{\partial x'}. \tag{4}$$

Then for the first equation in (1)

$$\frac{\partial \eta}{\partial t}+\frac{\partial (Hu)}{\partial x}=0, \tag{5}$$

$$\frac{U_0}{L_s}\frac{\partial (H_{0s}\eta')}{\partial t'}+\frac{1}{L_s}\frac{\partial (H_{0s}H'U_0u')}{\partial x'}=0, \tag{6}$$

$$\frac{U_0H_{0s}}{L_s}\frac{\partial \eta'}{\partial t'}+\frac{H_{0s}U_0}{L_s}\frac{\partial (H'u')}{\partial x'}=0, \tag{7}$$

then by dividing both sides by U_0H_{0s}/L_s we have

$$\frac{\partial \eta'}{\partial t'}+\frac{\partial (H'u')}{\partial x'}=0. \tag{8}$$

For the second equation in (1), we have

$$\frac{\partial u}{\partial t}+\frac{\partial (g\eta)}{\partial x}=0, \tag{9}$$

$$\frac{U_0}{L_s}\frac{\partial (U_0u')}{\partial t'}+\frac{1}{L_s}\frac{\partial (gH_{0s}\eta')}{\partial x'}=0, \tag{10}$$

$$\frac{U_0^2}{L_s}\frac{\partial u'}{\partial t'}+\frac{gH_{0s}}{L_s}\frac{\partial \eta'}{\partial x'}=0, \tag{11}$$

$$\tag{12}$$

by dividing this equation by U_0^2/L_s

$$\frac{\partial u'}{\partial t'}+\frac{gH_{0s}}{U_0^2}\frac{\partial \eta'}{\partial x'}=0. \tag{13}$$

Finally, by considering $g' = gH_{0s}/U_0^2$ we can write

$$\frac{\partial u'}{\partial t'}+g'\frac{\partial \eta'}{\partial x'}=0, \tag{14}$$

$$\frac{\partial u'}{\partial t'}+\frac{\partial (g'\eta')}{\partial x'}=0. \tag{15}$$

Now, dropping the prime in Equations (8) and (15) we get

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0, \tag{16}$$

$$\frac{\partial u}{\partial t} + \frac{\partial (g\eta)}{\partial x} = 0. \tag{17}$$

a scaled equation that looks the same as (1).

QUESTION 1

Define the Riemann problem for (the scaled) system (1) and derive the characteristics $\lambda_1 = \sqrt{gH_0}$, $\lambda_2 = \cdots$ for (1) for the special case $H(x) = H_0$ constant.
Rewrite (1) in vector form by using a matrix A, as follows (with $H_0 = H$):

$$\partial_t \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + A\partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0 \tag{18}$$

with appropriate 2×2 matrix A.
Answer:

$$A = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \tag{19}$$

with $c_0^2 = gH_0$. Find the eigenvalues λ and eigenvectors$\lambda_1 = c_0$; $\lambda_2 = -c_o$ of A, i.e. determine λ in $det(A - \lambda I) = 0$ with identity matrix I. Construct the matrix of eigenvectors

$$B = \frac{1}{2c_0} \begin{pmatrix} 1 & -1 \\ c_0 & c_0 \end{pmatrix} \tag{20}$$

Now $B^{-1}B = I$ and show that

$$B^{-1}AB = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{21}$$

By multiplying (18) (with the appropriate expression) and with $\mathbf{r} = B^{-1}(\eta, H_0 u)^T$ show that we obtain a decoupled set of linear advection equations

$$\partial_t \mathbf{r} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x \mathbf{r} = 0 \tag{22}$$

with $\mathbf{r} = (r1, r2)^T$ and$r1 = H_0 u + c_0 \eta$; $r_2 = H_0 u - c_0 \eta$. By simple addition or subtraction of (1) (with one multiplication), show that we could immediately have arrived at these so-called Riemann invariants $\mathbf{r}$.

Solution:

First I will show how we can re-write the Scaled Equations (1) or Equations (16) and (17) as Equation (18). For this I start by multiplying Equation (17) by H_0

$$\frac{\partial H_0 u}{\partial t} + \frac{\partial (H_0 g \eta)}{\partial x} = 0. \tag{23}$$

Now I can write the Equation system formed by (16) and (23) as

$$\partial_t \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + \partial_x \begin{pmatrix} H_0 u \\ H_0 g \eta \end{pmatrix} = 0. \tag{24}$$

In order to write Equation (24) as Equation (18), I have to find a matrix A such that

$$A\partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = \partial_x \begin{pmatrix} H_0 u \\ H_0 g \eta \end{pmatrix}, \tag{25}$$

or

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \partial_x \eta \\ \partial_x (H_0 u) \end{pmatrix} = \begin{pmatrix} \partial_x (H_0 u) \\ \partial_x (H_0 g \eta) \end{pmatrix}, \tag{26}$$

this give us the following

$$a_1\eta + a_2 H_0 u = H_0 u, \tag{27}$$
$$a_3\eta + a_4 H_0 u = H_0 g\eta. \tag{28}$$

For this system to be satisfied we need to have that $a_1 = 0$, $a_2 = 1$, $a_3 = H_0 g = c_0^2$ and $a_4 = 0$. This give us that the matrix A is

$$A = \begin{pmatrix} 0 & 1 \\ H_0 g & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \tag{29}$$

And with that we can write the equation system in (24) as

$$\partial_t \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + A\partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0, \tag{30}$$

That is the same equation as in (18).

Next we can find the eigenvalues of A by doing $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ c_0^2 & -\lambda \end{pmatrix}, \tag{31}$$
$$\det(A - \lambda I) = \lambda^2 - c_0^2, \tag{32}$$
$$\lambda^2 = c_0^2, \tag{33}$$
$$\lambda = \pm c_0, \tag{34}$$

Therefore we have that $\lambda_1 = c_0 = \sqrt{h_0 g}$ and $\lambda_2 = -c_0 = -\sqrt{H_0 g}$. To find the eigenvectors **b1** and **b2** we have to find a vector such that $(A - \lambda_i I)\mathbf{b_i} = 0$. Firstly, for $\lambda_1 = c_0$ and $\mathbf{b_1} = (b_1, b_2)$ we have

$$(A - \lambda I)\mathbf{b_1} = \begin{pmatrix} -c_0 & 1 \\ c_0^2 & -c_0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \tag{35}$$

$$\mathbf{b_1} = b_1 \begin{pmatrix} 1 \\ c_0 \end{pmatrix} = \frac{1}{2c_0} \begin{pmatrix} 1 \\ c_0 \end{pmatrix}, \tag{36}$$

where b_1 is an scaling factor, that can be chosen to be $b_1 = 1/2c_0$.

For $\lambda_2 = -c_0$ and $\mathbf{b_2} = (b_3, b_4)$ we have

$$(A - \lambda I)\mathbf{b_2} = \begin{pmatrix} c_0 & 1 \\ c_0^2 & c_0 \end{pmatrix} \begin{pmatrix} b_3 \\ b_4 \end{pmatrix} = 0 \tag{37}$$

From this we have that $b_4 = -c_0 b_3$, and then we can write $\mathbf{b_2}$ as

$$\mathbf{b_2} = \frac{1}{2c_0} \begin{pmatrix} -1 \\ c_0 \end{pmatrix}, \tag{38}$$

where I set the scaling factor to be $b_3 = -1/2c_0$. Now we write the eigenvectors as

$$B = \frac{1}{2c_0} \begin{pmatrix} 1 & -1 \\ c_0 & c_0 \end{pmatrix} \tag{39}$$

The next step is to calculate the inverse matrix of B, B^{-1}, such that $B^{-1}B = I$. So let say that the inverse matrix B^{-1} is given by

$$B^{-1} = \begin{pmatrix} b_5 & b_6 \\ b_7 & b_8 \end{pmatrix}, \tag{40}$$

then we have that

$$\begin{pmatrix} \frac{1}{2c_0} b_5 + \frac{1}{2} b_6 & -\frac{1}{2c_0} b_5 + \frac{1}{2} b_6 \\ \frac{1}{2c_0} b_7 + \frac{1}{2} b_8 & -\frac{1}{2c_0} b_7 + \frac{1}{2} b_8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{41}$$

From this we have that

$$\frac{1}{2c_0} b_5 + \frac{1}{2} b_6 = 1 \quad \text{and} \quad -\frac{1}{2c_0} b_5 + \frac{1}{2} b_6 = 0 \tag{42}$$
$$\frac{1}{c_0} b_5 + b_6 = 2 \quad \text{and} \quad b_6 = \frac{1}{c_0} b_5 \tag{43}$$
$$b_5 = c_0 \quad \text{and} \quad b_6 = 1 \tag{44}$$
$$\tag{45}$$

and also

$$\frac{1}{2c_0}b_7 + \frac{1}{2}b_8 = 0 \quad \text{and} \quad -\frac{1}{2c_0}b_7 + \frac{1}{2}b_8 = 1 \tag{46}$$

$$b_7 = -c_0 b_8 \quad \text{and} \quad -\frac{1}{c_0}b_7 + b_8 = 2 \tag{47}$$

$$b_7 = -c_0 \quad \text{and} \quad b_8 = 1. \tag{48}$$

$$\tag{49}$$

Then can write the inverse matrix as

$$B^{-1} = \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix}. \tag{50}$$

If we multiply B^{-1} by A we get

$$B^{-1}A = \begin{pmatrix} c_0^2 & c_0 \\ c_0^2 & -c_0 \end{pmatrix}, \tag{51}$$

if we multiply this new matrix by B we gets

$$B^{-1}AB = \begin{pmatrix} \frac{c_0}{2} + \frac{c_0}{2} & -\frac{c_0}{2} + \frac{c_0}{2} \\ \frac{c_0}{2} - \frac{c_0}{2} & -\frac{c_0}{2} - \frac{c_0}{2} \end{pmatrix} = \begin{pmatrix} c_0 & 0 \\ 0 & -c_0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} I \tag{52}$$

Next we can multiply Equation (18) by B^{-1}

$$B^{-1}(18) = B^{-1}\left(\partial_t \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + A\partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0\right) \tag{53}$$

$$= \partial_t \left[B^{-1} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix}\right] + B^{-1}A\partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} \tag{54}$$

$$= \partial_t \mathbf{r} + \begin{pmatrix} c_0^2 & c_0 \\ c_0^2 & -c_0 \end{pmatrix} \partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} \tag{55}$$

$$= \partial_t \mathbf{r} + c_0 \begin{pmatrix} c_0 & 1 \\ c_0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} \tag{56}$$

$$= \partial_t \mathbf{r} + c_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix} \partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} \tag{57}$$

$$= \partial_t \mathbf{r} + \begin{pmatrix} c_0 & 0 \\ 0 & -c_0 \end{pmatrix} B^{-1}\partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} \tag{58}$$

$$\tag{59}$$

and finally I get that

$$\partial_t \mathbf{r} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x \mathbf{r} = 0. \tag{60}$$

And with that we get that the shock speed values are λ_1 and λ_2.

QUESTION 2

Now solve the Riemann problem of (7) for piecewise constant initial data

$$r_1(x,0) = \begin{cases} r_{1l} & \text{for } \ x < 0 \\ r_{1r} & \text{for } \ x \geq 0 \end{cases}, \qquad r_2(x,0) = \begin{cases} r_{2l} & \text{for } \ x < 0 \\ r_{2r} & \text{for } \ x \geq 0 \end{cases} \tag{61}$$

Show that the solution of this Riemann problem is (analyically and/or graphically -this may seem straightforward but please state the "obvious")

$$r_1(x,t) = \begin{cases} r_{1l} & \text{for } \ x < c_0 t \\ r_{1r} & \text{for } \ x \geq c_0 t \end{cases}, \qquad r_2(x,t) = \begin{cases} r_{2l} & \text{for } \ x < -c_0 t \\ r_{2r} & \text{for } \ x \geq -c_0 t \end{cases} \tag{62}$$

Using this solution solve the Riemann solution for (1), given that $r_1 = H_0 u + c_0 \eta$, $r_2 = H_0 u - c_0 \eta$ and $H_0 u = \frac{1}{2}(r_1 + r_2)$,$\eta = \frac{1}{2}(r_1 - r_2)/c_0$, i.e., use the piecewise initial data u_l, u_r, η_l, η_r. Hence, show that

$$H_0 u(x,t) = \frac{1}{2}(r_1(x,t) + r_2(x,t)) = \begin{cases} H_0 u_l & \text{for } \ x < -c_0 t \\ ... & \text{for } \ -c_0 t \leq x \leq c_0 t \\ H_0 u_r & \text{for } \ x > c_0 t \end{cases} \tag{63}$$

$$\eta(x,t) = \frac{1}{2}(r_1(x,t) - r_2(x,t))/c_0 = \begin{cases} \eta_l & \text{for } \ x < -c_0 t \\ ... & \text{for } \ -c_0 t \leq x \leq c_0 t \\ \eta_r & \text{for } \ x > c_0 t \end{cases} \tag{64}$$

Hence, we have defined the Riemann invariants $r_1 = H_0 u + \sqrt{(gH_0)}\eta$, $r_2 = ...$, of (1) (for this case with H_0 constant) and show that these are two uncoupled linear advection equations advected by the respective characteristics. By using this linear transformation from u,η to these new, Riemann variables r_1, r_2 and vice versa, solve the Riemann problem for the original system and in terms of the original variables.

Solution:

To solve the Riemman Problem in

$$\partial_t \mathbf{r} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x \mathbf{r} = 0, \tag{65}$$

using the initial conditions in

$$r_1(x,0) = \begin{cases} r_{1l} & \text{for } \ x < 0 \\ r_{1r} & \text{for } \ x \geq 0 \end{cases}, \qquad r_2(x,0) = \begin{cases} r_{2l} & \text{for } \ x < 0 \\ r_{2r} & \text{for } \ x \geq 0 \end{cases} \tag{66}$$

I will start by considering $\mathbf{u} = \mathbf{r}$ and $f(\mathbf{u}) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{r}$, we have that the shock speed is given by

$$\frac{dx}{dt} = s = \frac{f(u_l) - f(u_r)}{u_l - u_r} \tag{67}$$

so for r_1 we have

$$s_1 = \frac{\lambda_1 (r_{1l} - r_{1r})}{r_{1l} - u_{1r}} = \lambda_1 \tag{68}$$

It was shown in the last exercise that λ_1 and λ_2 are the shock speed for this problem, given that the shock wave for r_1 and r_2 are given by

$$x_1 = x_{0,1} + \lambda_1 t \tag{69}$$
$$x_2 = x_{0,2} + \lambda_2 t \tag{70}$$

using the initial conditions in Equation (66), and that the eigenvalues are $\lambda_1 = c_0$ and $\lambda_2 = -c_0$ we have

$$x_1 = c_0 t \tag{71}$$
$$x_2 = -c_0 t, \tag{72}$$

Given that, we have, for any time t

$$r_1(x_1,t) = \begin{cases} r_{1l} & \text{for } \ x_1 < c_0 t \\ r_{1r} & \text{for } \ x_1 \geq c_0 t \end{cases}, \qquad r_2(x_2,t) = \begin{cases} r_{2l} & \text{for } \ x_2 < -c_0 t \\ r_{2r} & \text{for } \ x_2 \geq -c_0 t \end{cases} \tag{73}$$

The behaviour of r_1 and r_2 can be observed in Figure (1), in the case where $r_{1l} = r_{2l}$ and $r_{1r} = r_{2r}$, we can see that the discontinuity in r_1 propagates to the right, while the discontinuity in r_2 propagates to the left. The code that produces this Figure is in `codes/Q2/Q2_shock_wave.ipynb`.

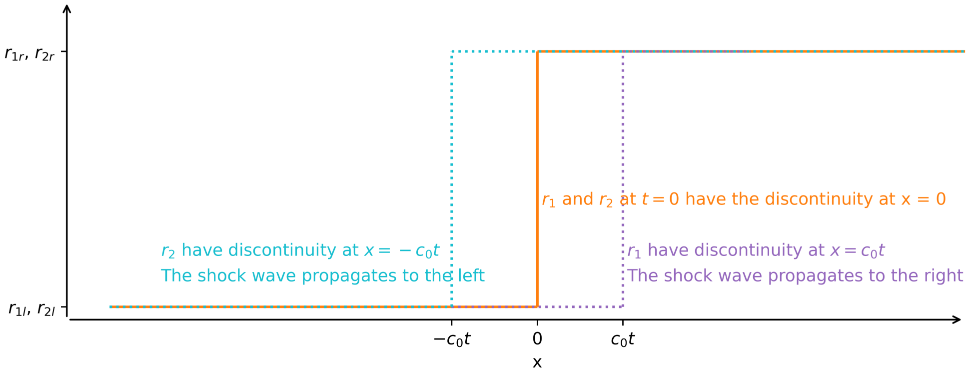

Figure 1: Sketch of what is observed in Equation (73).

Now I can use it to solve the system em equation (1). We have three cases to consider:

Case 1: we have that

$$x < -c_0 t, \qquad r_1 = r_{1l} \ \text{ and } \ r_2 = r_{2l}; \tag{74}$$

For $H_0 u$

$$H_0 u = \frac{r_1 + r_2}{2} = \frac{r_{1l} + r_{2l}}{2} = \frac{1}{2}(H_0 u_l + c_0 \eta_l + H_0 u_l - c_0 \eta_l) = H_0 u_l \tag{75}$$

For η:

$$\eta = \frac{1}{2}\frac{r_1 - r_2}{c_0} = \frac{1}{2}\frac{r_{1l} - r_{2l}}{c_0} = \frac{1}{2c_0}(H_0 u_l + c_0 \eta_l - H_0 u_l + c_0 \eta_l) = \eta_l \tag{76}$$

Case 2: we have that

$$-c_0 t \leq x < c_0 t, \qquad r_1 = r_{1l} \ \text{ and } \ r_2 = r_{2r}; \tag{77}$$

For $H_0 u$

$$H_0 u = \frac{r_1 + r_2}{2} = \frac{r_{1l} + r_{2r}}{2} = \frac{1}{2}(H_0 u_l + c_0 \eta_l + H_0 u_r - c_0 \eta_r) = \frac{H_0 u_l + H_0 u_r}{2} + \frac{c_0(\eta_l - \eta_r)}{2} \tag{78}$$

For η:

$$\eta = \frac{1}{2}\frac{r_1 - r_2}{c_0} = \frac{1}{2}\frac{r_{1l} - r_{2r}}{c_0} = \frac{1}{2c_0}(H_0 u_l + c_0 \eta_l - H_0 u_r + c_0 \eta_r) = \frac{H_0 u_l - H_0 u_r}{2c_0} + \frac{\eta_l + \eta_r}{2} \tag{79}$$

Case 3: we have that

$$x > c_0 t, \qquad r_1 = r_{1r} \ \text{ and } \ r_2 = r_{2r}; \tag{80}$$

For $H_0 u$

$$H_0 u = \frac{r_1 + r_2}{2} = \frac{r_{1r} + r_{2r}}{2} = \frac{1}{2}(H_0 u_r + c_0 \eta_r + H_0 u_r - c_0 \eta_r) = H_0 u_r \tag{81}$$

For η:

$$\eta = \frac{1}{2}\frac{r_1 - r_2}{c_0} = \frac{1}{2}\frac{r_{1r} - r_{2r}}{c_0} = \frac{1}{2c_0}(H_0 u_r + c_0 \eta_r - H_0 u_r + c_0 \eta_r) = \eta_r \tag{82}$$

We can now put everything together to write

$$H_0 u(x,t) = \begin{cases} H_0 u_l & \text{for } \ x < -c_0 t \\ \frac{1}{2}(H_0 u_l + H_0 u_r) + \frac{c_0}{2}(\eta_l - \eta_r) & \text{for } \ -c_0 t \leq x \leq c_0 t \\ H_0 u_r & \text{for } \ x > c_0 t \end{cases}, \tag{83}$$

$$\eta(x,t) = \begin{cases} \eta_l & \text{for } \ x < -c_0 t \\ \frac{1}{2c_0}(H_0 u_l - H_0 u_r) + \frac{1}{2}(\eta_l + \eta_r) & \text{for } \ -c_0 t \leq x \leq c_0 t \\ \eta_r & \text{for } \ x > c_0 t \end{cases}. \tag{84}$$

QUESTION 3

Work out the Godunov scheme for (1) and derive a time step estimate. You can either use$(\eta, H_0 u)$ as variables but with an eye on the variable $H(x)$-case use$(\eta; u)$ variables in the discretisation (simply relate these to r_1, r_2 where needed, i.e. in the flux only). Use extrapolating boundary conditions to mimic an "open" domain and use ghost values to mimic a closed domain by taking the velocity equal and opposite to the velocity in the domain while taking η to be equal on either side of the boundary. Alternatively, one may set the relevant flux to zero for solid walls at $x = 0, L$. (What should the condition on η be?) First consider the case with $H(x)$ constant, but extend the discretisation and code to variable but continuous $H(x)$. At each cell edge the Riemann solution/flux can be calculated with a "locally approximately constant" $H(x)$. Why is that reasonable?

Solution:

If we look back to Equations (16) and (17), we have

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} + \partial_x \begin{pmatrix} Hu \\ g\eta \end{pmatrix} = 0. \tag{85}$$

Therefore, for the Godunov method we have that

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \eta \\ u \end{pmatrix}, \tag{86}$$

and

$$f(\mathbf{u}) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} Hu \\ g\eta \end{pmatrix} = \begin{pmatrix} Hu_2 \\ gu_1 \end{pmatrix}. \tag{87}$$

For the Godunov discretization we have that

$$U_{i,j}^{n+1} = U_{i,j}^n - \frac{\Delta t}{\Delta x_j}(F_{i,j+1/2}(U_{i,j}^n, U_{i,j+1}^n) - F_{i,j-1/2}(U_{i,j-1}^n, U_{i,j}^n)), \tag{88}$$

with $i = 1, 2$ the indices of the velocity and flux components, $j = 1, ..., N-1$ the number of the space cell, n indicating the time step, and

$$F_{i,j+1/2}(U_{i,j}^n, U_{i,j+1}^n) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f_i(u_i(x_{j+1/2}, t))dt. \tag{89}$$

Locally u, H and η are constant, and because of that we can solve Equation (89) using the solutions in Equations (83) and (84), but shifted to be centred in $x_{j+1/2}$. And because we are evaluated it in the centre of the Riemann problem, meaning $x = x_{j+1/2}$, we will use only the central solution of $H_0 u(x, t)$ and $\eta(x, t)$. Hence, we have that

$$f_1(u_1(x_{j+1/2}, t)) = H_0 u(x_{j+1/2}, t) = (Hu)_{j+1/2}(t) = \frac{1}{2}((Hu)_l + (Hu)_r) + \frac{c_0}{2}(\eta_l - \eta_r) \tag{90}$$

and

$$f_2(u_2(x_{j+1/2}, t)) = g\eta(x_{j+1/2}, t) = g\eta_{j+1/2}(t) = g\left(\frac{1}{2c_0}((Hu)_l - (Hu)_r) + \frac{1}{2}(\eta_l + \eta_r)\right). \tag{91}$$

Using this to solve Equation (89), and considering that neither $(Hu)_{j+1/2}$ nor $\eta_{j+1/2}$ depend explicitly from time, we have that

$$F_{1,j+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (Hu)_{j+1/2} dt = (Hu)_{j+1/2}, \tag{92}$$

$$F_{2,j+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} g\eta_{j+1/2} dt = g\eta_{j+1/2} \tag{93}$$

$$. \tag{94}$$

Besides that, for $F_{i,j+1/2}$ centred in $x_{j+1/2}$, we have that $(Hu)_l = H_j^n U_{2,j}^n$, $(Hu)_r = H_{j+1}^n U_{2,j+1}^n$, $\eta_l = U_{1,j}^n$ and $\eta_r = U_{1,j+1}^n$. Given that, we have that for $F_{1,j+1/2}$

$$F_{1,j+1/2} = \frac{1}{2}(H_j^n U_{2,j}^n + H_{j+1}^n U_{2,j+1}^n + c_0(U_{1,j}^n - U_{1,j+1}^n)), \tag{95}$$

and for $F_{2,j+1/2}$

$$F_{2,j+1/2} = \frac{g}{2c_0}(H_j^n U_{2,j}^n - H_{j+1}^n U_{2,j+1}^n + c_0(U_{1,j}^n + U_{1,j+1}^n)). \tag{96}$$

And for $F_{i,j-1/2}$ centred in $x_{j-1/2}$, we have that $(Hu)_l = H_{j-1}^n U_{2,j-1}^n$, $(Hu)_r = H_j^n U_{2,j}^n$, $\eta_l = U_{1,j-1}^n$ and $\eta_r = U_{1,j}^n$. Given that, we have that for $F_{1,j-1/2}$

$$F_{1,j-1/2} = \frac{1}{2}(H_{j-1}^n U_{2,j-1}^n + H_j^n U_{2,j}^n + c_0(U_{1,j-1}^n - U_{1,j}^n)), \tag{97}$$

and for $F_{2,j-1/2}$

$$F_{2,j-1/2} = \frac{g}{2c_0}(H_{j-1}^n U_{2,j-1}^n - H_j^n U_{2,j}^n + c_0(U_{1,j-1}^n + U_{1,j}^n)). \tag{98}$$

For implementation in Python, I will use the index k in the place of $j-1/2$. So we have that k = j, with $k = 0, 1, ..., N$, $j = 0, 1, ..., N-1$, and N the maximum number of nodes. See how the mesh should look like in Figure (2). One implication of this is that now $F_{i,j-1/2} = F_{i,k}$ and $F_{i,j+1/2} = F_{i,k+1}$.

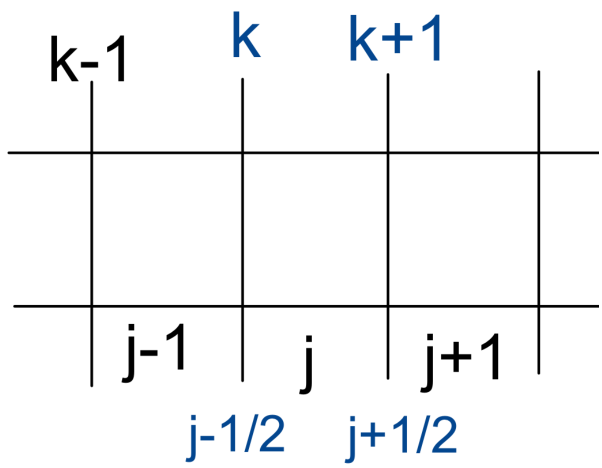

Figure 2: Mesh for implementation of the Gudonov scheme in python.

For boundary conditions, in the case of an open domain, we have for the borders that if $j = 0$, then $U_{i,-1} = U_{i,N-1}$, for $j = N-1$ we have that $U_{i,N} = U_{i,0}$. In the case of a closed domain, we can set a ghost velocity with the same intensity but contrary sing $-u$, therefore we have that for $j = 0$, $U_{2,-1} = -U_{2,0}$, and for $j = N-1$, $U_{2,N} = -U_{2,N-1}$. When we are multiply u by -1, we are also multiplying η by the same factor, in order to keep $\partial_t u + g\partial_x \eta = 0$ true. Therefore, the ghost "velocity" for the η component, has to be positive. In another words for $j = 0$, $U_{1,-1} = U_{1,0}$, and for $j = N-1$, $U_{1,N} = U_{1,N-1}$. Alternatively, we can also make the flux in the boundary walls equal to zero, as nothing can be transported to a solid wall, this mean that for $j = 0$, $F_{i,k} = 0$, and for $j = N-1$, $F_{i,N} = 0$.

To summarize, I worked the Godunov scheme for (1). I did it considering a solution where $H(x)$ is constant and extended to the case where $H(x)$ is a function of x. It is reasonable to do it because locally, for each cell, we are considering the average of u, η and Hu. As it is averaged when we integrated in x, as demonstrated in LeVeque. In practice, what really changes is the value of H in the cell $j-1$, j and $j+1$.

QUESTION 4

Numerics in Firedrake: Implement the Godunov scheme for (1) in Firedrake[1] as zeroth-order discontinuous Galerkin method and verify it against standing wave solutions (derive/state these with solid wall boundary conditions -see theory lectures), or the exact Riemann solutions (derive/state these for open boundary conditions). Why is the finite volume scheme conservative? Plot various profiles (exact and numerical) in time from $3T_p$ to $4T_p$ with the relevant period T_p and, also for $9T_p$ to $10T_p$. (Visual convergence suffices but feel free to do a formal convergence analysis.) What should the order of accuracy be, in space and in time? Try various CFL-numbers including one,$\Delta t = CFL\Delta x = \max\lambda$ with $0 \leq CFL \leq 1$. Interpret your results. Do not use a counter but use actual dimensionless time in your time-loop.

Solution:

I implemented the Godunov scheme for (1), for the Riemann solutions using open boundary conditions. The output, when using $\eta_l = 1$, $\eta_r = 0$, $u_l = 0.5$ and $u_r = 0$, $H_0 = 1$ and $g = 1$ are shown in Figure (3). The blue curve shows the initial conditions, the markers are the numerical solutions for t = 0.3 (orange) and t = 0.6 (purple). The solid lines are the exact Riemann Solutions calculated using Equations (83) and (84).

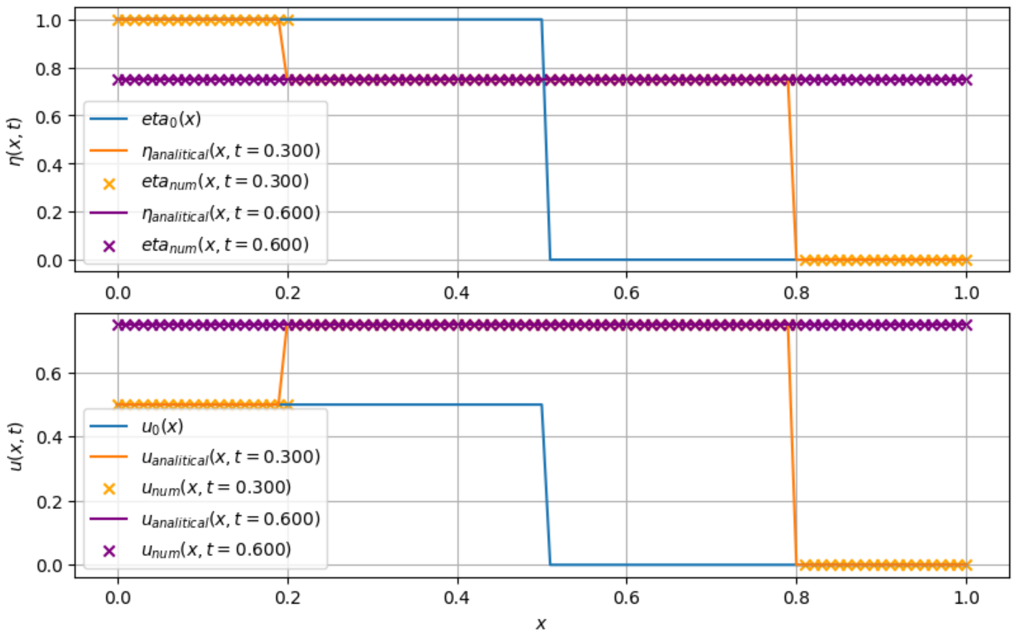

Figure 3: Solution for Equation (1) using Riemann solution in Equations (83) and (84) as initial conditions, whit $\eta_l = 1$, $\eta_r = 0$, $u_l = 0.5$ and $u_r = 0$.

This similar to a situation where a dan brake and wave is arriving in a location with standing water. When the wave arrives, the velocity of the stand water start to increase. It also increase the volume of the water, in a way such that, when the wave passes the total volume should be in something in between the volume of the wave (η_l) and the volume of the standing water (η_r). The code that produces Figure (3) is in `codes/Q4/LAE_DG0_modify_riemman.py`.

QUESTION 6

Implement the solid-wall boundary conditions using that this flux at the boundary is zero. Be careful and take θ not close to 0 or 1 at the boundaries.

Solution:

The solution for Equation (1) using alternated fluxes are in the provided code `sweDGFV.py`. Using the standing waves as initial condition, with close wall boundary, and constant $H(x) = H_0$, I tested different values for θ, as shown in Figure (4), after one period $T = \pi$. The leftmost figure is for $\theta = 0.25$, the figure in the centre is for $\theta = 0.5$ and the figure in the right is for $\theta = 0.75$. We can observe that the most stable solution is for $\theta = 0.5$, and I will use that result in the following analysis.

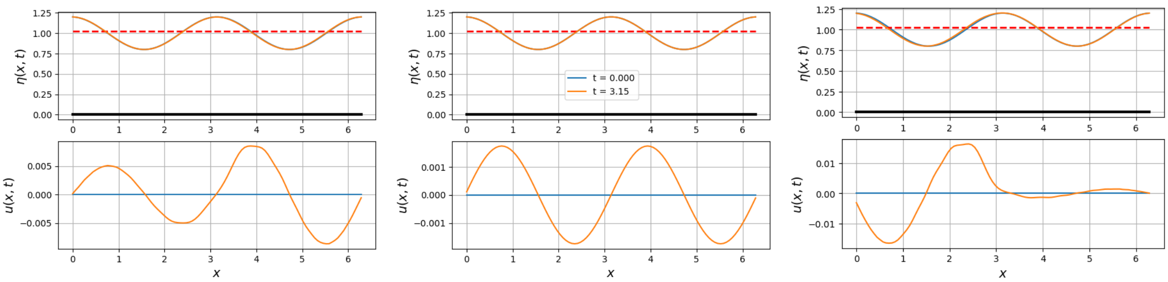

Figure 4: Solution of Equation (1) using standing waves and alternate Fluxes, with $\theta = 0.25$ (left), $\theta = 0.5$ (middle) and $\theta = 0.75$ (right).

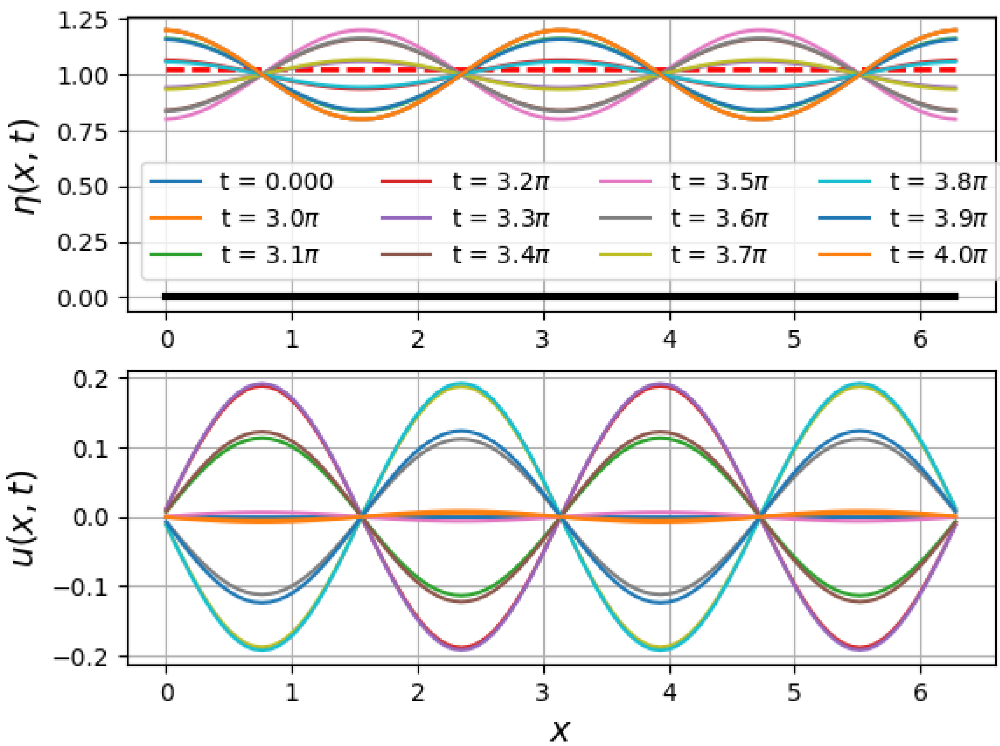

Figure 5: Solution of Equation (1) using standing waves and alternate Fluxes, for varying t between $3T < t < 4T$, with the period $T = \pi$.

Additionally, I also plotted η and u profiles for variated t between $3T < t < 4T$ (Figure (5)) and between $9T < t < 10T$ (Figure (6)). In both cases we can clearly see the formation of nodes, maximums and minimums characteristic of the standing wave, wave.

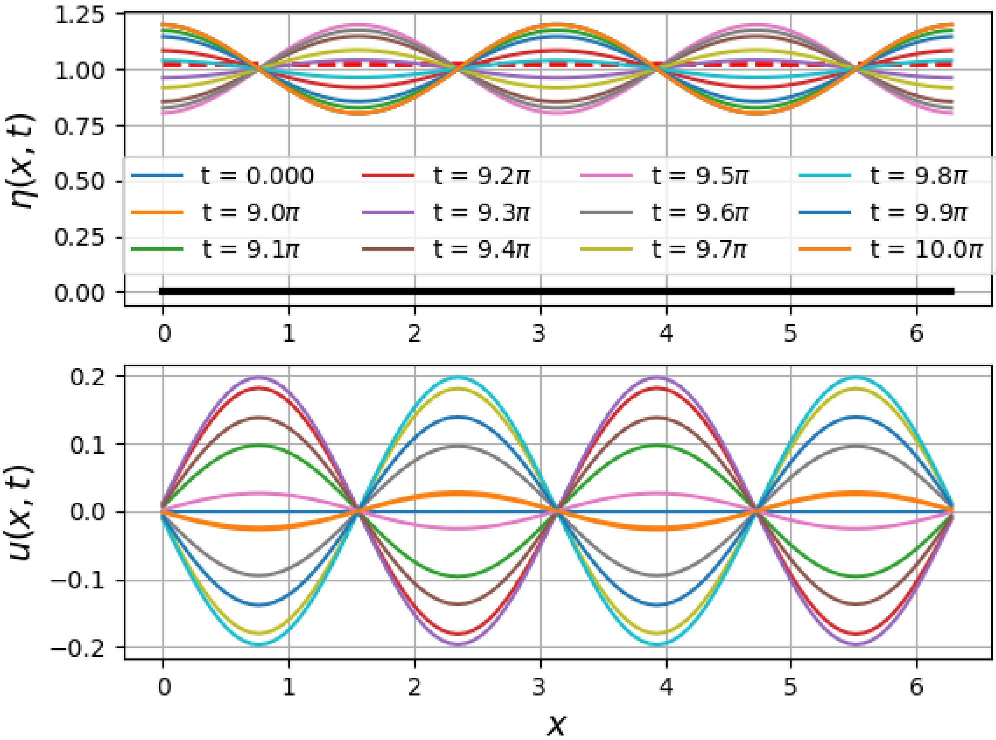

Figure 6: Solution of Equation (1) using standing waves and alternate Fluxes, for varying t between $9T < t < 10T$, with the period $T = \pi$.