FINITE ELEMENT METHODS

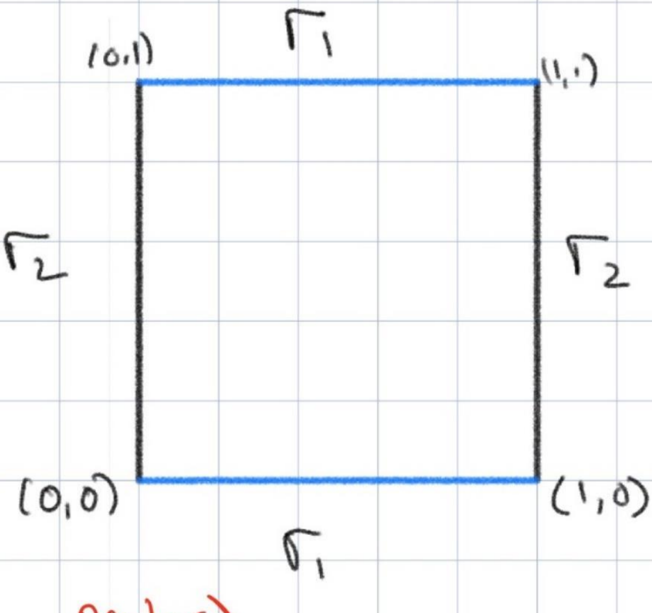
$$-\nabla^2 u = f \quad \text{on} \quad (x,y) \in [0,1]^2 \quad (+)$$

$$f(x,y) = 2\pi^2 \sin(\pi x)\cos(\pi y)$$

$$u(0,y) = u(1,y) = 0$$

$$\partial_y u(x,y)\Big|_{y=0} = \partial_y u(x,y)\Big|_{y=1} = 0$$

diagram



STEP 1 part a)

$\Rightarrow$ we first introduce a test function, which is here denoted as:

$w(x,y)$

$\Rightarrow$ we then multiply (+) by the test function and integrate by parts over the entire domain (Ω)

$$-\int_0^1\int_0^1 w(x,y)\,\nabla^2 u \,dx\,dy = \int_0^1\int_0^1 w(x,y)\,f(x,y)\,dx\,dy$$

$\rightarrow$ integrating the L.H.S by parts then gives

$$\int_0^1\int_0^1 \nabla(w)\cdot\nabla u \,d\Omega - \oint w\,\hat{n}\cdot\nabla u \,d\Gamma = \int_0^1\int_0^1 w f \,d\Omega$$

if we then look at the boundary conditions provided

$$\Gamma_2: \ [\hat{n}\cdot\nabla u] = \begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}\partial_x u\\ \partial_y u\end{pmatrix} = \partial_y u$$

$\star$ from B.Cs $\partial_y u(x,y)\big|_{y=0} = \partial_y u(x,y)\big|_{y=1} = 0$

(neumann)

such that $\partial_y u$ at $\Gamma_2 = 0$

and then

$$\Gamma_1 : \left[\hat{n}\cdot\nabla u\right] = \underbrace{\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}\partial_x u\\ \partial_y u\end{pmatrix}}_{\text{does not work}} = \delta v\,\partial_x u = 0$$

and $\quad u(0,y) = a, \quad w(0,y) = 0$

$\quad\quad\;\; u(1,y) = b, \quad w(1,y) = 0$

so $\quad \Gamma_1 \to 0$

$$\int_0^1\!\!\int_0^1 \nabla(w)\cdot\nabla u\, d\Omega - \oint \cancelto{0}{w\,\hat{n}\cdot\nabla u\, d\Gamma} = \int_0^1\!\!\int_0^1 v\, f\, d\Omega$$

and the weak formulation becomes:

$$\int_0^1\!\!\int_0^1 \nabla(w)\cdot\nabla u\, dx\,dy = \int_0^1\!\!\int_0^1 \delta u\, f\, dx\,dy$$

✓ weak form

$$\Rightarrow \boxed{\iint \left(\nabla(w)\cdot\nabla u - w f\right) dx\,dy = 0}$$

$$\therefore \quad \delta \iint \frac{1}{2}|\nabla u|^2 - f u\; dx\,dy = 0 \quad (**)$$

Step 1 part b)

=) the minimisation problem

$$F[u] = \iint \frac{1}{2} |\nabla u|^2 - uf \, dx dy$$

=) consider a class of functions

$$u(x) = u(x) + \varepsilon \delta u(x,y)$$

=) which is the variation around $u(x)$

=) The condition for the existance of aminimum

$$\frac{d F[u]}{d\varepsilon} = 0$$

=) plugging the class of functions into the minimization problem and differentiating with respect to ε gives us the weak formulation

$$0 = \frac{d}{d\varepsilon}\left[\int_0^1 \int_0^1 \frac{1}{2} |\nabla(u + \varepsilon \delta u)|^2 - f(u + \varepsilon \delta u) \, dx dy \right]$$

$$0 = \int_0^1\int_0^1 \left[\frac{d}{d\epsilon}\left(\left(\frac{1}{2}\frac{\partial(u+\varepsilon \delta u)}{dx}\right)^2 + \left(\frac{1}{2}\frac{\partial(u+\varepsilon \delta u)}{\partial y}\right)^2\right) - \frac{\partial}{\partial \varepsilon}(u+\varepsilon \delta u) f \right] dx\,dy$$

which then reduces to

$$0 = \int_0^1\int_0^1 \frac{\partial(u+\varepsilon \delta u)}{dx}\frac{\partial(\delta u)}{dx} + \frac{\partial(u+\varepsilon \delta u)}{\partial y} + \frac{\partial(\delta y)}{dy} - f\,\delta u \; dx\,dy$$

and that becomes

$$\boxed{\int_0^1\int_0^1 \nabla u \cdot \nabla \delta u - f\,\delta u \; dx\,dy = 0}$$

this is the same as the weak form derivided in step 1a with $w(x,y) = \delta u\,(x,y)$

STEP 2

⇒ The solution is approximated by a finite dimensional subset of the global basis functions ($\phi_j(x,y)$) such that

$$u_h = \sum_{j=1}^{N} \hat{u}_j \phi_j(x,y)$$

where $\hat{u}_j$'s are the coefficients and $\phi_j(x,y)$ are the global basis functions. These basis fcns have compact support.

⇒ U_h is further split by considering the boundary conditions such that

$$u_h(x,y) = \sum_{j=1}^{N_h} U_j \phi_j(x,y) = \sum_{j'}^{N_{dof}-N_{dirichlet}} U_j \phi_j(x,y) + \cancel{\sum^{N_{dirichlet}} U_0 \phi_j(x,y)} \quad (*)$$

⇒ from the B.Cs we know that $U_0 = 0$ and that term cancels out we then substitute this approximation into the R.G principle defined in step 1.

$$\delta \iint \frac{1}{2} |\nabla \bar{U}|^2 - fu\,dx\,dy = 0$$

$$\delta \iint \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} u_i \nabla \phi_i \circ u_j \nabla \phi_j - f \sum u_j \phi_j \, dx\,dy = 0$$

$$\Rightarrow \sum_{j=1}^{N} \sum_{i=1}^{N} u_i u_j \frac{1}{2} \iint \nabla \phi_i(x,y) \cdot \nabla \phi_j(x,y)\, dx\,dy$$

$$- \sum_{j=1}^{N} u_j \iint f(x,y)\, \phi_j(x,y)\, dx\,dy = 0$$

if we then take

$$A_{ij} = \iint \phi_i(x,y) \cdot \phi_j(x,y)\, dx\,dy$$

and $b_j = \iint f \phi_j(x,y)\, dx\,dy$

we can rewrite the R.G principle as

$$\delta \sum_{i=1}^{N} \sum_{j=1}^{N} u_i u_j \frac{1}{2} A_{ij} - \sum_{j=1}^{N} u_j b_j = 0$$

variations around v_j and v_i

$$\boxed{\begin{aligned} u_j &= v_j + \epsilon \delta v_j \\ u_i &= v_i + \epsilon \delta v_i \end{aligned}}$$

$$\frac{d}{d\epsilon}\left[\sum \sum (v_j + \epsilon \delta v_j)(v_i + \epsilon \delta v_i) \frac{1}{2} A_{ij} - \sum (v_j + \epsilon \delta v_j) b_j\right] = 0$$

$$\frac{d}{d\epsilon}\left[\sum \sum \left[v_j v_i + v_j \epsilon \delta v_i + \epsilon \delta v_j v_i + \epsilon^2 \delta v_i \delta v_j\right] \frac{A_{ij}}{2}\right.$$

$$\left. - \sum (u_j + \epsilon \delta v_j) b_j \right] = 0$$

then we differenciate $\Rightarrow$

$$\sum \sum \left[u_j \delta v_i + v_i \delta v_j + \cancelto{0}{2 \epsilon \delta v_i \delta v_j} \right] \frac{A_{ij}}{2} - \sum \delta v_j b_j = 0$$

since the variations show that the equation reaches a minimum when $\epsilon = 0$

$$\sum \sum \left(v_j \delta v_i + v_i \delta v_j \right) \frac{A_{ij}}{2} - \sum \delta v_j b_j = 0$$

$u_j \delta u_i$ = $i \leftrightarrow j$ $u_i \delta u_j$

$$\sum_{i=1} u_i \delta u_j A_{ij} = \sum_{j=1} \delta u_j b_j$$

$$\therefore \quad \sum \underset{\text{arbitrary}}{\delta u_j} \left(\sum_{i=1} u_i A_{ij} - b_j \right) = 0$$

$$\therefore \quad \sum_{i=1}^{N} A_{ij} u_i - b_j = 0$$

(similar to $ku = F$)

<u>Discrete weak formulation</u>

substituting $\quad u_h(x,y) = \sum_{i=1}^{N} u_i \phi_i(x,y)$

into the weak formulation derived in step 1:

$$\int_0^1 \int_0^1 \nabla u \cdot \nabla \delta u - f \, \delta u \; dx\,dy = 0$$

$$\Rightarrow \int_0^1 \int_0^1 \nabla \left(\sum_{j=1}^{N} \delta u_j \phi_j \right) \cdot \nabla \left(\sum_{i=1} u_i \phi_i \right) dx\,dy$$

$$= \int_0^1\int_0^1 \sum_{j=1}^{N} \delta v_j \phi_j f \, dx\,dy$$

$$L.H.S \Rightarrow \quad \sum_{j=1}^{N}\sum_{i=1}^{N} \delta u_j v_i \int_0^1\int_0^1 \nabla\phi_j \cdot \nabla\phi_i \, dx\,dy$$

$$R.H.S \Rightarrow \quad \sum_{j=1}^{N} \delta v_j \int_0^1\int_0^1 \phi_j f \, dx\,dy$$

we can then factor out δv_j

$$\Rightarrow \sum_{j=1}^{N} \delta v_j \left(\sum_{i=1}^{N} u_i \int_0^1\int_0^1 \nabla\phi_j \cdot \nabla\phi_i \, dx\,dy - \int_0^1\int_0^1 \phi_j f \, dx\,dy \right) = 0$$

* since δv_j is arbitrary, the terms in the bracket must vanish for all j

$$\sum_{i=1}^{N} u_i \int_0^1\int_0^1 \nabla\phi_j \cdot \nabla\phi_i \, dx\,dy = \int_0^1\int_0^1 \phi_j f \, dx\,dy$$

we then define

$$A_{ij} = \iint \nabla\phi_j \cdot \nabla\phi_i \, dx\,dy$$ symmetric

$$b_j = \iint \phi_j f \, dx\,dy$$

so the equations become

$$\sum_{i=1}^{N} \delta u_i \, A_{ij} - b_j = 0$$

$$\Rightarrow \quad A_{ij} u_i - b_j = 0$$

(similar to $Ku = F$)

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Step 4

The solution to the system using the provided Firedrake code is depicted in Figure 1 for a value of $nx = 16$ $(h = 1/16)$.

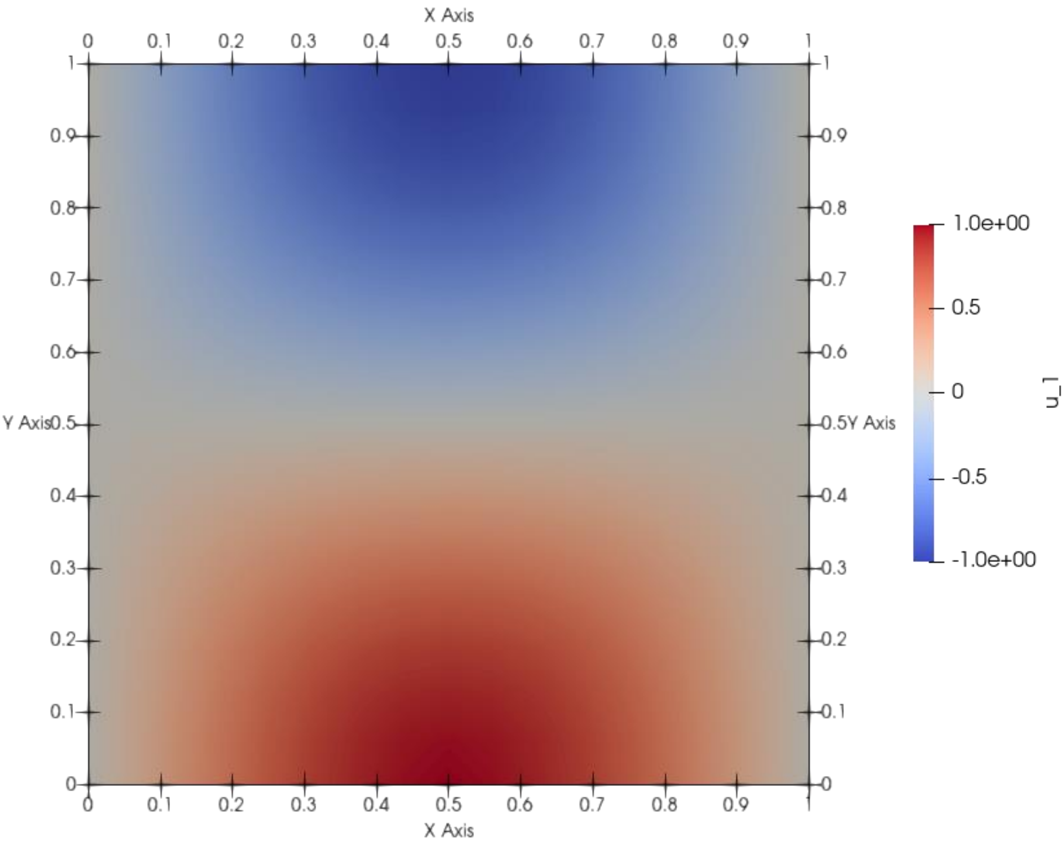

Figure 1: Numerical results for $u_h(x, y)$ for $h = 1/16$.

The difference $|u_h(x, y) - u_e(x, y)|$ is then plotted for varying h −refinements. The values of nx, ny are varied as 16, 32, 64, 128, and 256. This was all done for $p = 1$, so the order of accuracy for these is $O(\Delta x^2)$.

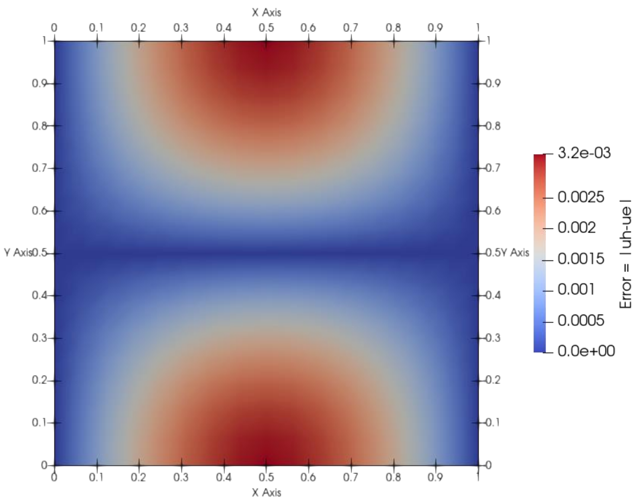

Figure 2: $p = 1, h = 1/16$

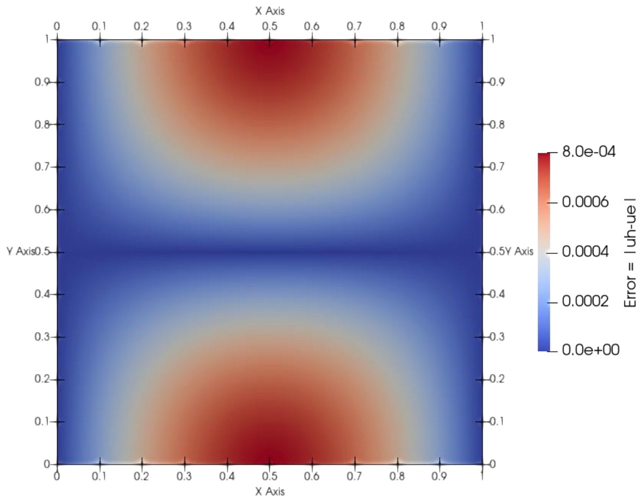

Figure 3: $p = 1, h = 1/32$

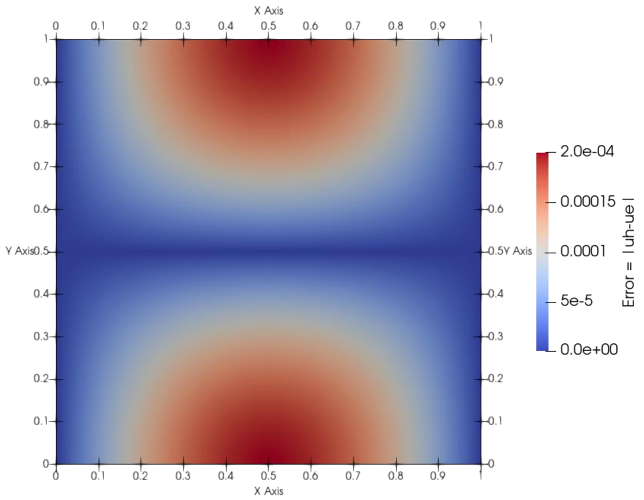

Figure 4: $p = 1, h = 1/64$

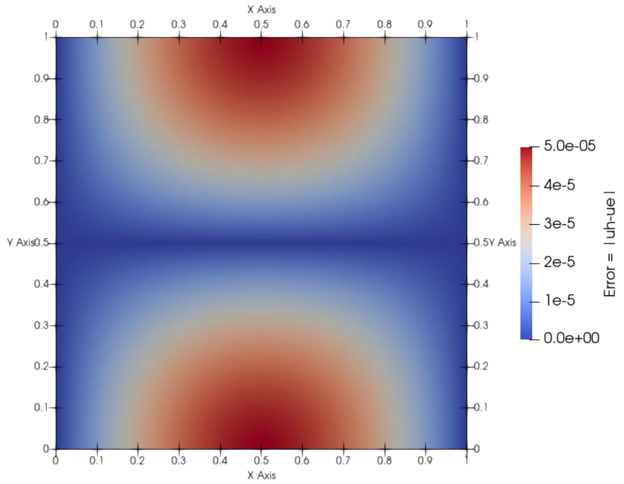

Figure 5: $p = 1, h = 1/128$

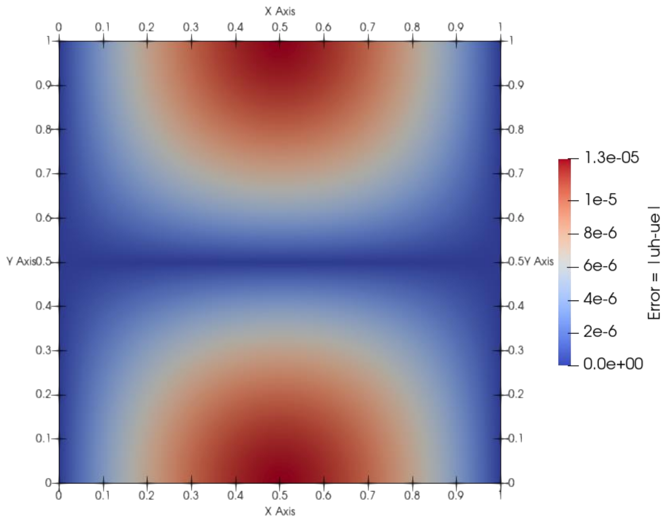

Figure 6: $p = 1, h = 1/256$

From these figures, it is clear that as the value of h is halved, the error decreases by approximately a factor of four each time, suggesting the relationship:

$$Error \sim h^2$$

If we utilize the same colour map, we can confirm convergence occurs after $h = 1/128$. This is shown below, indicated by the error tending to zero as we decrease the element size.

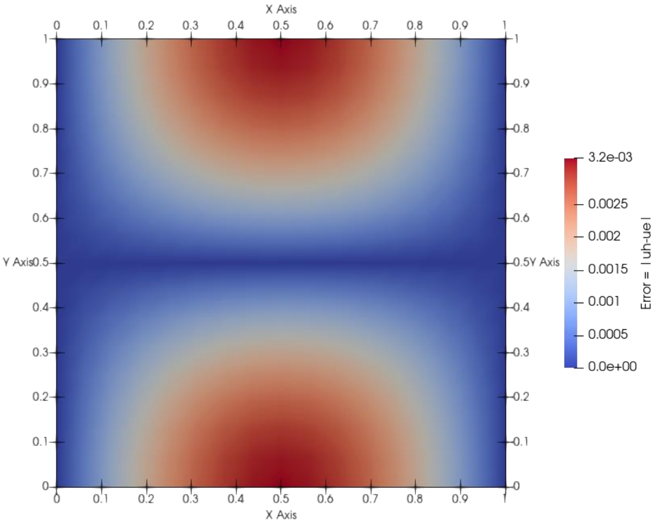

Figure 7: $p = 1, h = 1/16$

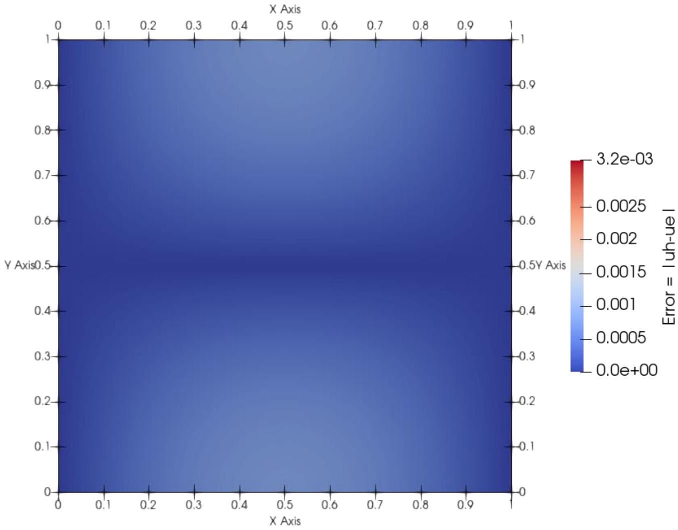

Figure 8: $p = 1, h = 1/32$

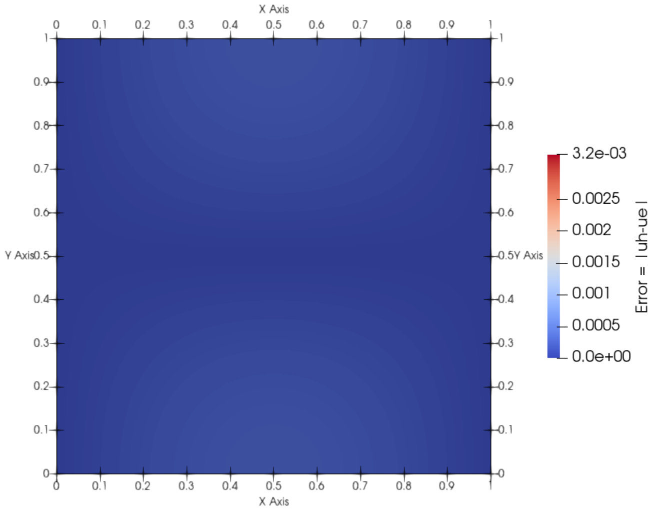

Figure 9: $p = 1, h = 1/64$

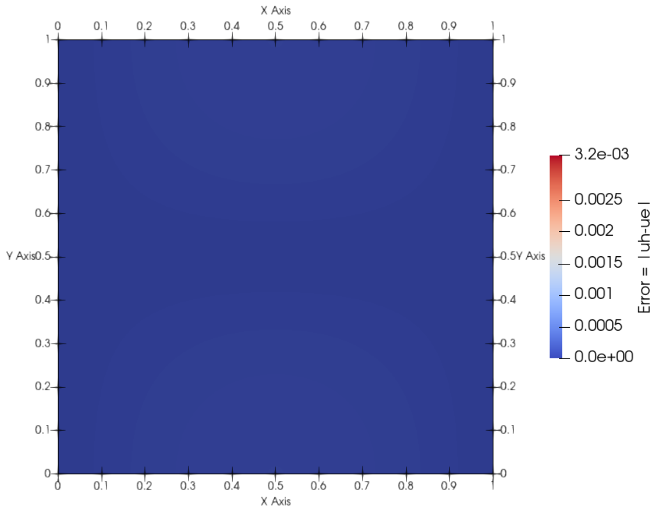

Figure 10: $p = 1, h = 1/128$

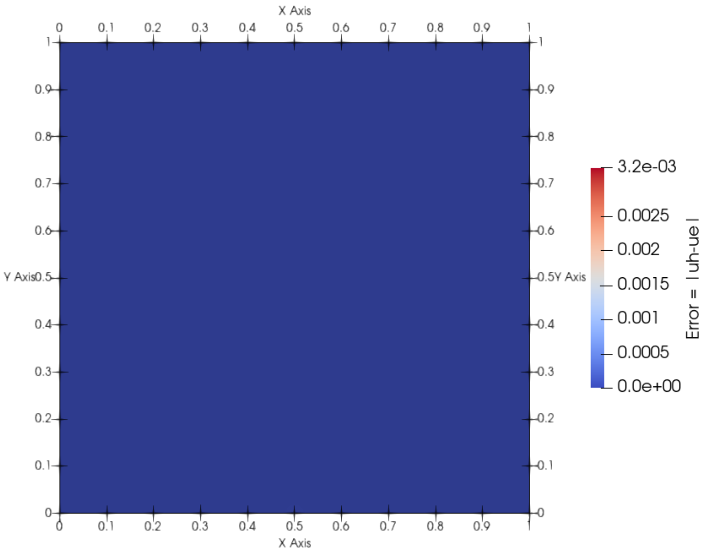

Figure 11: $p = 1, h = 1/256$

The value of p can then be altered to change the polynomial function and the changes in error can be observed. Generally, as p increases the order of accuracy scales as follows:

$$Order\ of\ accuracy \sim O\Delta x^{p+1}$$

The order of accuracy for different p-refinements is summarized below in Table 1.

Table 1: p vs. order of accuracy.

p	**Order of accuracy**
1	$O(\Delta x^2)$
2	$O(\Delta x^3)$
3	$O(\Delta x^4)$
4	$O(\Delta x^{5)}$

To investigate the effect of changing the degree polynomial (p), the value of p is altered whilst the value of h is kept constant in a p-refinement study. The value of h chosen is $h = 1/128$ $(nx = 128)$ and p is varied from 1 till 4. This is shown in the figures below.

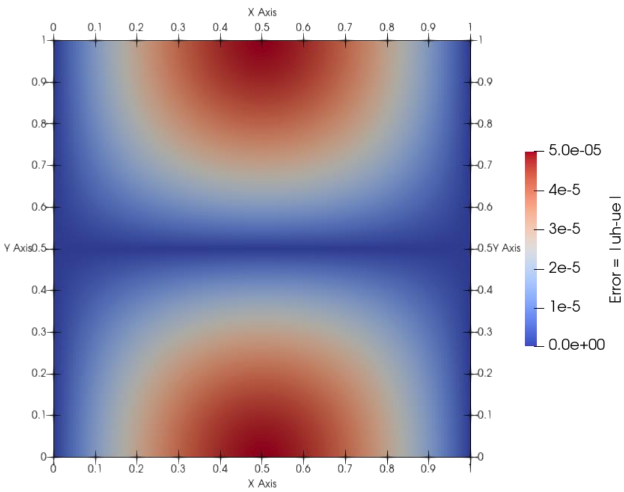

Figure 12: $p = 1\ (O\Delta(x^2)), h = 1/128$

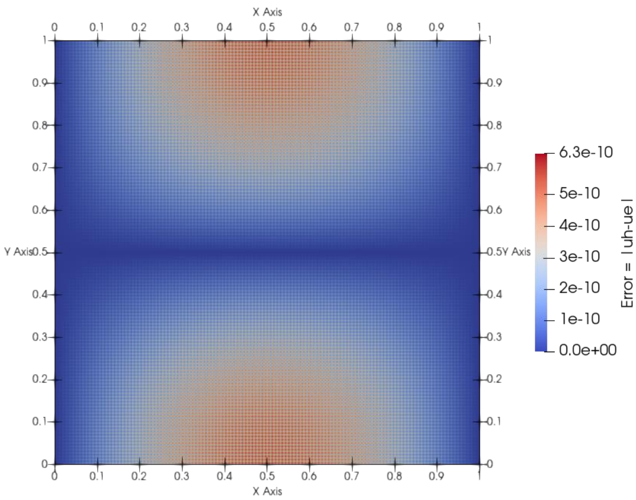

Figure 13: $p = 2\ (O\Delta(x^3)), h = 1/128$

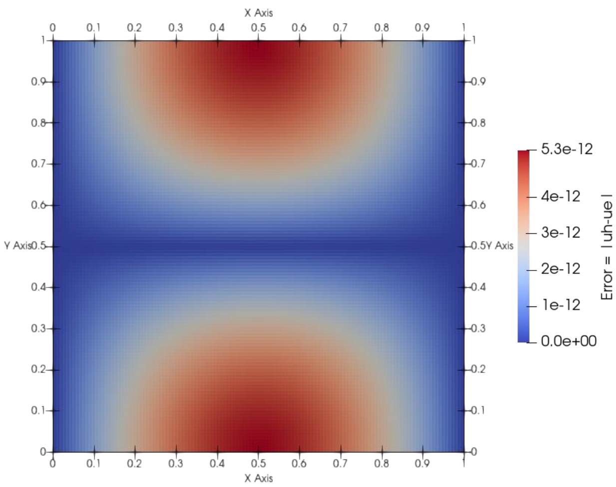

Figure 13: $p = 3(O\Delta(x^4)), h = 1/128$

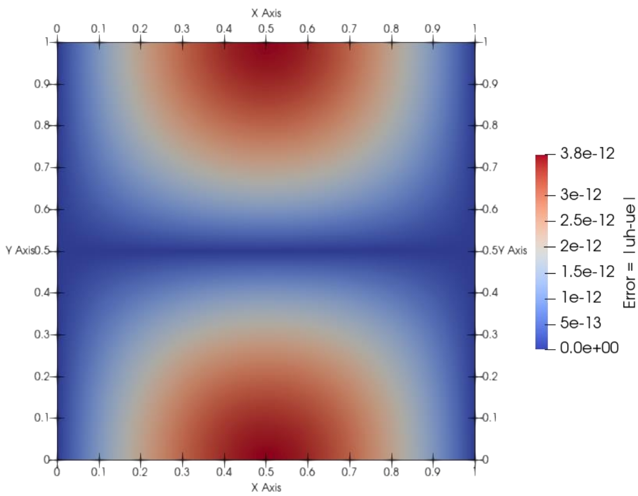

Figure 14: $p = 4, (O\Delta(x^4))\ h = 1/128$

From these graphs, we can see that as p increases, the error reduces by a much larger magnitude than just refining h. Generally, the error for these systems scales as:

$$Error \sim O(h^{p+1})$$

For some $\{h, p\}$ refinements, the errors are approximately the same. Three observed cases with similar errors are shown below.

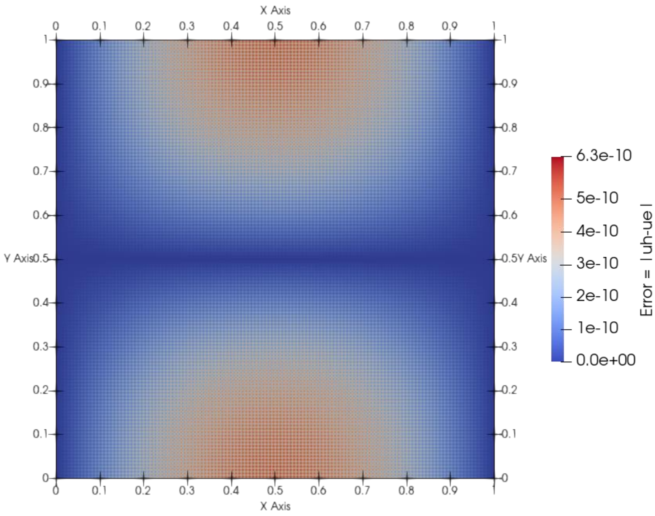

Figure 15: $p = 2(O\Delta(x^3)), h = 1/128$

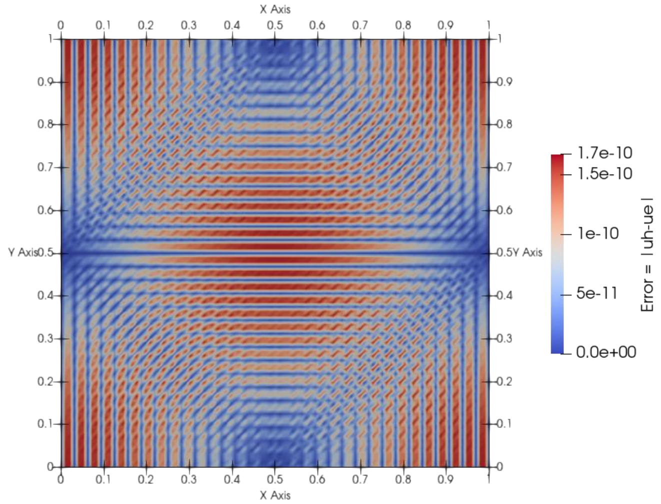

Figure 16: $p = 3(O\Delta(x^4)), h = 1/32$

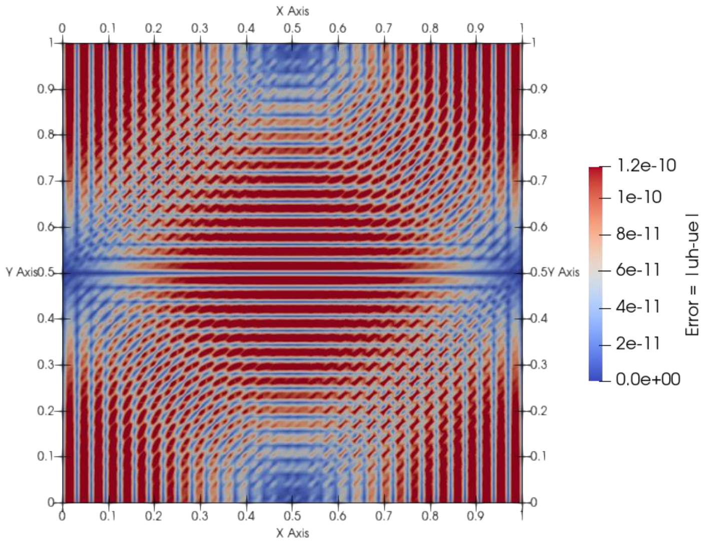

Figure 17: $p = 4(O\Delta(x^5)), h = 1/16$

The reason these errors are similar is because their order of accuracy is also quite similar:

$$\left\{p = 2, h = \frac{1}{126}\right\} \to Error \sim O(\frac{1}{126})^3 \sim 4.99 \times 10^{-7}$$

$$\left\{p = 3, h = \frac{1}{32}\right\} \to Error \sim O(\frac{1}{32})^4 \sim 9.53 \times 10^{-7}$$

$$\left\{p = 4, h = \frac{1}{16}\right\} \to Error \sim O(\frac{1}{16})^5 \sim 9.54 \times 10^{-7}$$