

# Fluid Dynamics — Numerical Techniques

## MATH5453M FEM Numerical Exercises 3, 2023

Due date: December 2024

Consider the Poisson system

$$-\nabla^2 u = f \quad \text{on} \quad (x, y) \in [0, 1]^2 \quad (1a)$$

$$f(x, y) = 2\pi^2 \sin(\pi x) \cos(\pi y) \quad (1b)$$

$$u(0, y) = u(1, y) = 0 \quad (1c)$$

$$\partial_y u(x, y)|_{y=0} = \partial_y u(x, y)|_{y=1} = 0 \quad (1d)$$

with variable or unknown  $u(x, y)$ , given function  $f(x, y)$ , and with Dirichlet and Neumann boundary conditions. The exact solution is  $u_e(x, y) = \sin(\pi x) \cos(\pi y)$ , please check.

1. *Step 1:* Write down the Ritz-Galerkin principle for the above Poisson system and show that variation thereof yields the system. What are the conditions on the variation  $\delta u(x, y)$ ? Derive the weak formulation for the above system. Show that the test function  $w(x, y)$ , say, used is the same as  $w(x, y) = \delta u(x, y)$ .
2. *Step 2:* Write down the algebraic or discrete Ritz-Galerkin principle after introducing a FEM expansion  $u_h(x, y)$  for  $u(x, y) \approx u_h(x, y)$  in terms of global basis functions. Write down the algebraic or discrete weak formulation after introducing a FEM expansion in terms of global basis functions. Show that the variation of the former yields the latter.
3. *Step 3:* Although not needed in Firedrake, feel free to introduce a local coordinate system and reference coordinates, and explain the matrix assembly involved in getting the system in *Step 2*. Use quadrilateral elements. (Optional.)
4. *Step 4:* Solve the system in Firedrake with the provided or other Firedrake codes. Plot the numerical results for  $u_h(x, y)$  with Paraview as a contour plot (with clear labelling/indicating of the values used in that plot). Plot the difference  $|u_h(x, y) - u_e(x, y)|$  (of numerical and exact solutions) as a contour

plot in Firedrake for a few suitable resolutions ( $-$ ). Mention the function spaces used and the order of accuracy. Explore different order ( $p$ -refinement) and mesh resolutions ( $h$ -refinements). Explain and show which  $\{h, p\}$  combinations are (roughly) equivalent and why? Provide clear figure captions with information on resolution, etc., such as  $\{h, p\}$ .

5. Explain how the above first two or three steps are implemented in Firedrake, also by adding clear comments to your code.
6. TBD: revisit the numerics of the FV/DG0 exercise.