MATH 5453M Numerical Exercise 2

Finite Volume Method: Linear Shallow-water equation. Predicting surf height at beaches

Consider the linearised shallow-water system of eqn:-

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0 \quad \& \quad \frac{\partial u}{\partial t} + \frac{\partial (g\eta)}{\partial x} = 0 \quad \text{--- (1)}$$

for velocity $u = u(x,t)$ & free surface deviation $\eta = \eta(x,t)$, rest depth $H(x)$ & acceleration of gravity $g = 9.81 \, m/sec$,

when we scale (1)

$$u = V_0 u', \quad x = L_s x', \quad t = (L_s/V_0)t'$$

$$\eta = H_{0s}\eta', \quad H = H_{0s}H'.$$

$$g' \rightarrow g H_{0s}/V_0^2,$$

Ques (i) :-

let's write the non-dimensional form of (1)

$$\frac{U_0 H_{0s}}{L_s}\frac{\partial \eta'}{\partial t'} + \frac{U_0 H_{0s}}{L_s}\frac{\partial (H'u')}{\partial x'} = 0 \quad \&$$

$$\frac{U_0^2}{L_s}\frac{\partial u'}{\partial t'} + \frac{U_0^2}{L_s}\frac{\partial (g'\eta')}{\partial x'} = 0$$

$$\frac{\partial \eta'}{\partial t'} + \frac{\partial (H'u')}{\partial x'} = 0 \qquad (*)$$

$$\frac{\partial u'}{\partial t'} + \frac{\partial (g'\eta')}{\partial x'} = 0 \qquad (**)$$

↳ after non-dimensionalisation also we are having same equation as (1).

Except $g' = g\frac{H_{0s}}{U_0^2}$

- Since both equation (*) & (**) are conservation equation (conservation of mass & conservation of momentum respectively).

- Both are coupled because u & η influence each other.

$\downarrow$

we can write these two equations in matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0$$

where $A = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix}$

$\Rightarrow c_0^2 = gH_0$; wave speed

using linear algebra. we can decouple these equation.

finding eigen values & eigen vectors of matrix A

$$\det|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ c_0^2 & -\lambda \end{vmatrix} = \lambda^2 - c_0^2 = 0$$

$\therefore$ eigen values are $\lambda_1 = c_0$, $\lambda_2 = -c_0$.

Now finding the eigen vectors

$$A v_1 = \lambda_1 v_1 \quad \& \quad A v_2 = \lambda_2 v_2$$

$$\begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = c_0 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

$$v_{12} = c_0 v_{11}$$
$$c_0^2 v_{11} = c_0 v_{12}$$
$$\downarrow$$
$$c_0^2 v_{11} = c_0 (c_0 v_{11})$$

$$\Rightarrow \quad v_1 = \frac{1}{2c_0} \begin{pmatrix} 1 \\ c_0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = -c_0 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$
$$\downarrow$$
$$v_2 = \frac{1}{2c_0} \begin{pmatrix} -1 \\ c_0 \end{pmatrix}$$

$\Rightarrow$ Therefore, Matrix B is consisting eigen vectors is,

$$B = \frac{1}{2c_0} \begin{pmatrix} 1 & -1 \\ c_0 & c_0 \end{pmatrix}$$

$\Rightarrow$ Now $B^{-1} = \frac{1}{|B|}\begin{pmatrix} c_0 & -c_0 \\ 1 & 1 \end{pmatrix}^T$

$$= \frac{1}{|B|}\begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix}$$

$$= \frac{1}{\frac{1}{2c_0}(c_0 + c_0)}\begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix}$$

& $AB = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2c_0} & \frac{-1}{2c_0} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{c_0}{2} & \frac{-c_0}{2} \end{pmatrix}$

$\therefore$ $B^{-1}AB = \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{c_0}{2} & \frac{-c_0}{2} \end{pmatrix}$

$$= \begin{pmatrix} c_0 & 0 \\ 0 & -c_0 \end{pmatrix}$$

$B^{-1}AB \Rightarrow \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \lambda I$

⇒ Now, we will proceed to decoupling with new variable γ

↳ $\gamma = B^{-1}(\eta, H_0 u)^T$

$$\Rightarrow \gamma = B^{-1}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix}$$

$$\Rightarrow \quad B\gamma = \begin{pmatrix} \eta \\ H_0 u \end{pmatrix}$$

But $\frac{\partial}{\partial t}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = B\frac{\partial \gamma}{\partial t}$

& $\frac{\partial}{\partial x}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = B\frac{\partial \gamma}{\partial x}$

↓

original eqn Becomes

$$B\frac{\partial \gamma}{\partial t} + AB\frac{\partial \gamma}{\partial x} = 0 \quad - \ ***$$

But from above we have, $B^{-1}AB = \lambda I$

$\Rightarrow$ Multiplying B^{-1} in (***);

$$B^{-1}B\frac{\partial r}{\partial t} + B^{-1}AB\frac{\partial r}{\partial x} = 0$$

$$\frac{\partial r}{\partial t} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\frac{\partial r}{\partial x} = 0$$

$\to$ decoupled set of linear advection equation

where $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ & $r_1 = H_0 u + C_0 \eta$
$r_2 = H_0 u - C_0 \eta$

($r \to$ Riemann invariant)

Ques : 2 →

for piecewise constant initial data

$$r_1(x,0) = \begin{cases} r_{1l} & \text{for } x<0 \\ r_{1r} & \text{for } x \geq 0 \end{cases}, \quad r_2(x,0) \begin{cases} r_{2l} & \text{for } x<0 \\ r_{2r} & \text{for } x \geq 0 \end{cases}$$

from equation – (7)

$$\frac{\partial r}{\partial t} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{\partial r}{\partial x} = 0$$

$$\frac{\partial r_1}{\partial t} + C_0 \frac{\partial r_1}{\partial x} = 0 \qquad \begin{pmatrix} \lambda_1 = C_0 \\ \lambda_2 = -C_0 \end{pmatrix}$$

& $$\frac{\partial r_2}{\partial t} - C_0 \frac{\partial r_2}{\partial x} = 0$$

these are linear advection equation,

∴ the solution would be:

$$\begin{cases} r_1(x,t) = r_1(x - C_0 t, 0) \ \& \\ r_2(x,t) = r_2(x + C_0 t, 0) \end{cases}$$

$C_0 \rightarrow$ propagation speed.

→ r_1 const. along characteristics line $x - C_0 t = \text{const}$ &

r_2 along $x + C_0 t = \text{const}$

Based on the given initial condition

$$r_1(x,t) = \begin{cases} r_{1l} & \text{if } x - C_0 t < 0 \\ r_{1r} & \text{if } x - C_0 t \geq 0 \end{cases}$$

$$r_1(x,t) = \begin{cases} r_{1l} & \text{if } x < C_0 t \\ r_{1r} & \text{if } x \geq C_0 t \end{cases} \quad \text{— (A)}$$

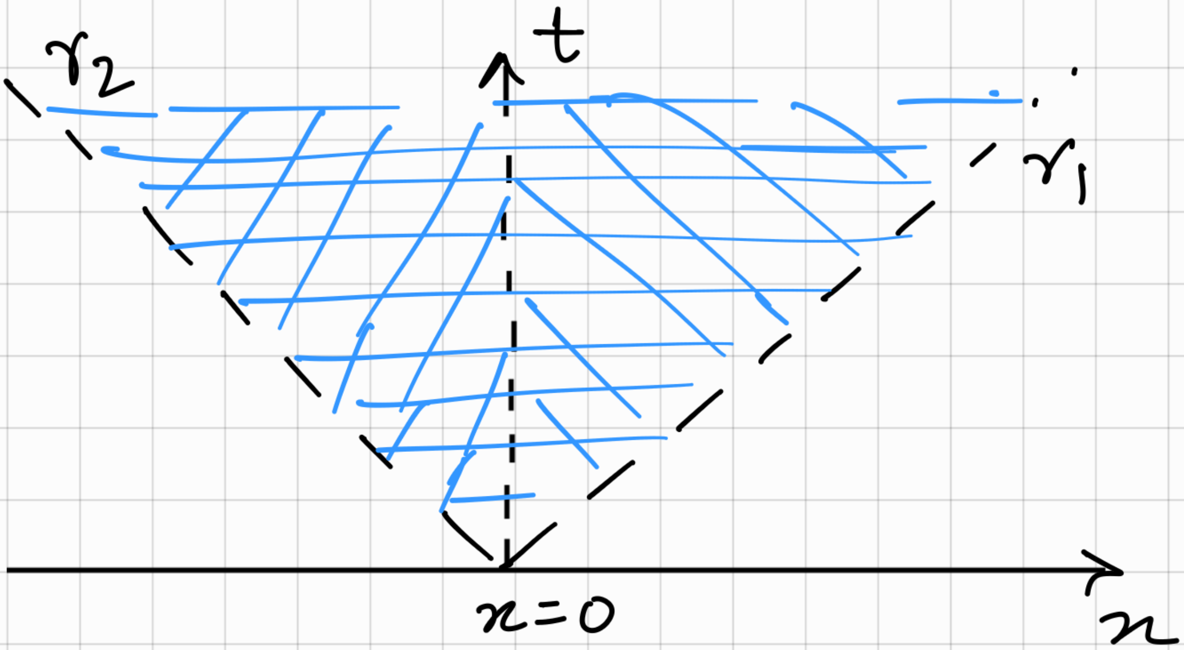
and

$$r_2(x,t) = \begin{cases} r_{2l} & \text{if } x + C_0 t < 0 \\ r_{2r} & \text{if } x + C_0 t \geq 0 \end{cases}$$

$$r_2(x,t) = \begin{cases} r_{2l}, & x \leq -C_0 t \\ r_{2r}, & x \geq -C_0 t \end{cases} \quad \text{— (B)}$$

Since the initial discontinuity is at $x=0$, @ $t=0$

if we plot the $r_1(x,t)$ & $r_2(x,t)$



Now:- given that $r_1 = H_0 u + c_0 \eta$

$$r_2 = H_0 u - c_0 \eta$$

and $H_0 u = \frac{1}{2}(r_1 + r_2)$,

$$\eta = \frac{1}{2}(r_1 - r_2)/c_0$$

$\Rightarrow$ as per r_1 & r_2 value given in Ⓐ & Ⓑ

① for $x < -c_0 t$

$\hookrightarrow$ $r_1 = r_{1l}$ & $r_2 = r_{2l}$

$$\therefore \quad H_0 u(x,t) = \frac{1}{2}(r_{1l} + r_{2l})$$

$$= \frac{1}{2}\left(H_0 u_l + \cancel{c_0 \eta_l} + H_0 u_l - \cancel{c_0 \eta_l}\right)$$

$$H_0 u(x,t) = \frac{1}{2}(2H_0 u_l) = H_0 u_l$$

(2) for the shaded part,

$-C_0 t \leq x \leq C_0 t$

$$r_1 = r_{1l}\ ,\ r_2 = r_{2r}$$

$$\therefore\ H_0 u(x,t) = \frac{1}{2}(r_{1l} + r_{2r})$$

(3) for $x > C_0 t$

$$r_1 = r_{1r}\ \&\ r_2 = r_{2r}$$

$$\therefore\ H_0 u(x,t) = \frac{1}{2}(r_{1r} + r_{2r})$$

$$= \frac{1}{2}(H_0 u_r + \cancel{C_0 h_r} + H_0 u_r - \cancel{C_0 h_r})$$

$$H_0 u(x,t) = H_0 u_r$$

$$\Rightarrow \quad H_0 u(x,t) = \begin{cases} H_0 u_l & x < C_0 t \\ \frac{1}{2}(r_{1l} + r_{2r}) & -C_0 t \leq x \leq C_0 t \\ H_0 u_r & x > C_0 t \end{cases}$$

Similarly for $\eta(x,t)$

① $x < -C_0 t$

$r_1 = r_{1l} \quad , r_2 = r_{2l}$

$$\eta(x,t) = \frac{1}{2C_0}\left(\cancel{H_0 u_l} + C_0\eta_l - \cancel{H_0 u_l} + C_0\eta_l\right)$$

$$\eta(x,t) = \eta_l$$

② for $-C_0 t \le x \le C_0 t$

$r_1 = r_{1l} \quad r_2 = r_{2r}$

$$\eta(x,t) = \frac{1}{2C_0}\left(H_0 u_l + C_0\eta_l - H_0 u_r + C_0\eta_r\right)$$

③ for $x > C_0 t$

$r_1 = r_{1r} \quad r_2 = r_{2r}$

$$\eta(x,t) = \frac{1}{2C_0}\left(\cancel{H_0 u_r} + C_0\eta_r - \cancel{H_0 u_r} + C_0\eta_r\right)$$

$$\eta(x,t) = \eta_r$$

$$\Rightarrow \eta(x,t) = \frac{1}{2C_0}\left(r_1(x,t) - r_2(x,t)\right)$$

$$\Rightarrow \eta(x,t) = \begin{cases} \eta_l & x < -c_0 t \\ H_0(u_l - u_r) + c_0(\eta_l + \eta_r) & \text{for } -c_0 t \le x \le c_0 t \\ \eta_r & x > c_0 t \end{cases}$$

$\Rightarrow$ Riemann invariants

$$r_1 = H_0 u + \sqrt{gH_0}\,\eta$$
$$r_2 = H_0 u - \sqrt{gH_0}\,\eta$$

& as per above:

$$H_0 u = \frac{1}{2}(r_1 + r_2) \quad \& \quad \eta = \frac{1}{2\sqrt{gH_0}}(r_1 - r_2)$$

$\therefore$ from $\frac{\partial \eta}{\partial t} + H_0 \frac{\partial u}{\partial x} = 0$

$$\frac{\partial}{\partial t}\left(\frac{1}{2\sqrt{gH_0}}(r_1 - r_2)\right) + H_0 \frac{\partial}{\partial x}\left(\frac{1}{2H_0}(r_1 + r_2)\right) = 0$$

$$\frac{1}{2\sqrt{gH_0}}\left(\frac{\partial r_1}{\partial t} - \frac{\partial r_2}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial r_1}{\partial x} + \frac{\partial r_2}{\partial x}\right) = 0 \longrightarrow C$$

Similarly for

$$\frac{\partial y}{\partial t} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2H_0}(r_1+r_2)\right) + g\frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{gH_0}}(r_1-r_2)\right) = 0$$

$$\hookrightarrow \frac{1}{2H_0}\left(\frac{\partial r_1}{\partial t} + \frac{\partial r_2}{\partial t}\right) + \frac{g}{2\sqrt{gH_0}}\left(\frac{\partial r_1}{\partial x} - \frac{\partial r_2}{\partial x}\right) = 0 \quad \text{— } (D)$$

from (C) multiply by $2\sqrt{gH_0}$

$$\left(\frac{\partial r_1}{\partial t} - \frac{\partial r_2}{\partial t}\right) + \sqrt{gH_0}\left(\frac{\partial r_1}{\partial x} + \frac{\partial r_2}{\partial x}\right) = 0$$

for $r_1 \rightarrow \dfrac{\partial r_1}{\partial t} + \sqrt{gH_0}\dfrac{\partial r_1}{\partial x} = 0$

$r_2 \rightarrow \dfrac{\partial r_2}{\partial t} - \sqrt{gH_0}\dfrac{\partial r_2}{\partial x} = 0$

$\hookrightarrow$ uncoupled

r_1 & r_2 do not depend on each other,

$\Rightarrow$ characteristic lines for r_1, would be

$$x - \sqrt{gH_0}\,t = \text{const}$$

for r_2

$$x + \sqrt{gH_0}\,t = \text{const}$$

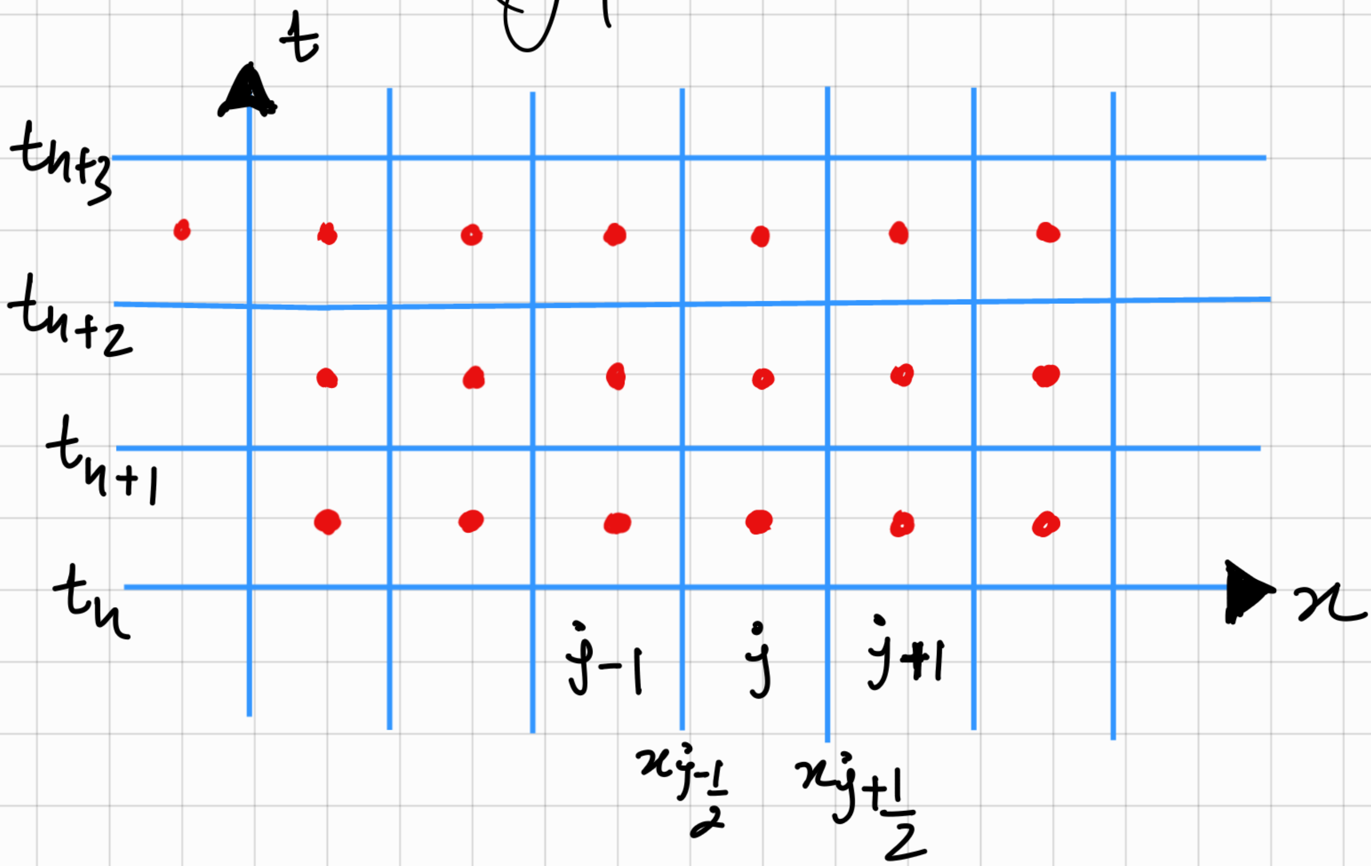
Question ③ :-

Godunov scheme for question ①:-

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0$$

where $u(x,t)=\begin{pmatrix} \eta \\ H_0 u \end{pmatrix}$

& $f(u)=\begin{pmatrix} Hu \\ g\eta \end{pmatrix} \longrightarrow$ flux



- In Godunov scheme: we compute numerical flux at cell interface
- In ensures the conservation across the

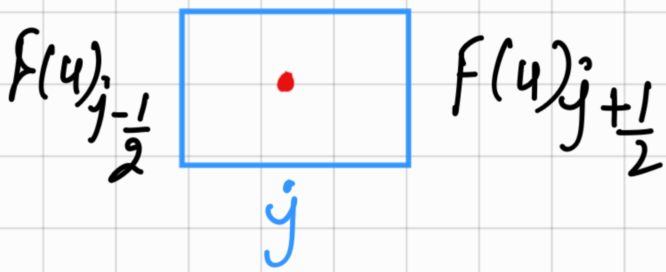
domain.

Now

$$\frac{\partial}{\partial t}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t)\,dx + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{\partial f(u)}{\partial x}\,dx = 0$$

$$\Rightarrow \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{\partial f(u)}{\partial x}\,dx = f(u)\Big|_{x_{j+\frac{1}{2}}} - f(u)\Big|_{x_{j-\frac{1}{2}}}$$

$$\therefore \frac{\partial}{\partial t}\int_{x_{j-\frac{1}{2}}}^{x_{j+1}} u(x,t)\,dx + \left[f(u)_{j+\frac{1}{2}} - f(u)_{j-\frac{1}{2}} \right] = 0$$



Now :- cell averaged quantity :-

$$u_j(t) = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+\frac{1}{2}}} u(x,t)\,dx$$

divided by dt (rate of change)

$$\frac{u_j(t)}{dt} = -\frac{1}{\Delta x}\left(f(u)_{j+\frac{1}{2}} - f(u)_{j-\frac{1}{2}}\right) \quad \text{— } A$$

↳ represents the balance between the change in conserved quantity u_j inside the cell & the net flux across the boundaries

⇒ Now discritize the conserved quantity in time

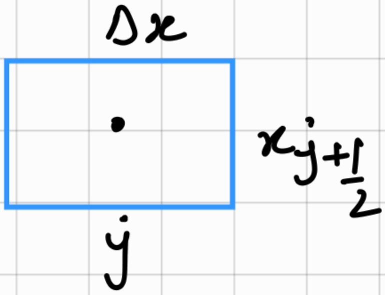
$$\frac{dU_j}{dt} \approx \frac{u_j^{n+1} - u_j^{n}}{\Delta t}$$

Substitute in eqn (A)

$$\frac{u_j^{n+1} - u_j^{n}}{\Delta t} = -\frac{1}{\Delta x}\left[f(u)_{j+\frac{1}{2}} - f(u)_{j-\frac{1}{2}}\right]$$

$$u_j^{n+1} = u_j^{n} - \frac{\Delta t}{\Delta x}\left[f(u)_{j+\frac{1}{2}} - f(u)_{j-\frac{1}{2}}\right]$$

$\Rightarrow$ Now to compute the fluxes at interface we will solve the Riemann problem at the interface between two cell



if we use the Riemann problem at the interface $x_{j+\frac{1}{2}}$ b/etween two cell, as per initial condition given in ques (2)

$$r_1(x,0) = \begin{cases} r_{1l} & x < 0 \\ r_{1r} & x \geq 0 \end{cases} \quad \&$$

$$r_2(x,0) = \begin{cases} r_{2l} & x < 0 \\ r_{2r} & x \geq 0 \end{cases}$$

where

$l \rightarrow$ left side of the interface

$r \rightarrow$ right side of the interface

In general :- u_L & u_r

$$u_L \to u_l,\ u_l \qquad u_r \to u_r,\ u_r$$

- The numerical flux $f_{j+\frac{1}{2}}$ is computed from

$$f_{j+\frac{1}{2}} = f(u_L) + \sum_{j:\lambda_j>0} \lambda_j W_j - \sum_{j:\lambda_j<0} \lambda_j W_j$$

where,

$W_j \to$ wave strength

$\lambda_j \to$ eigen value

$$\Rightarrow \quad u(x,0) = \begin{cases} u_L & x < x_{j+1/2} \\ u_r & x > x_{j+\frac{1}{2}} \end{cases}$$

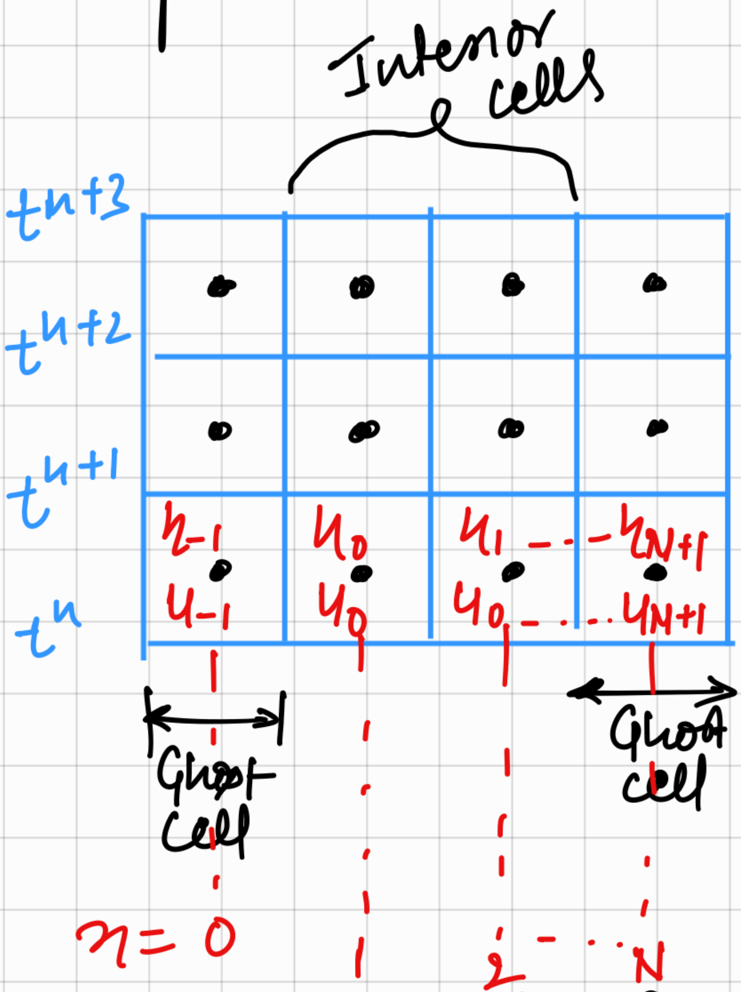
Assuming wave propagating at constant speed.

↳ Now at interface,

$$\Delta u = u_r - u_l$$

⇒ Extrapolating the Boundary condition

(1) open :-



at $x=0$, left boundary $(x=0)$

↳ $h_0 = h_{-1}$ & $u_{-1} = u_0$

& for first interior cell

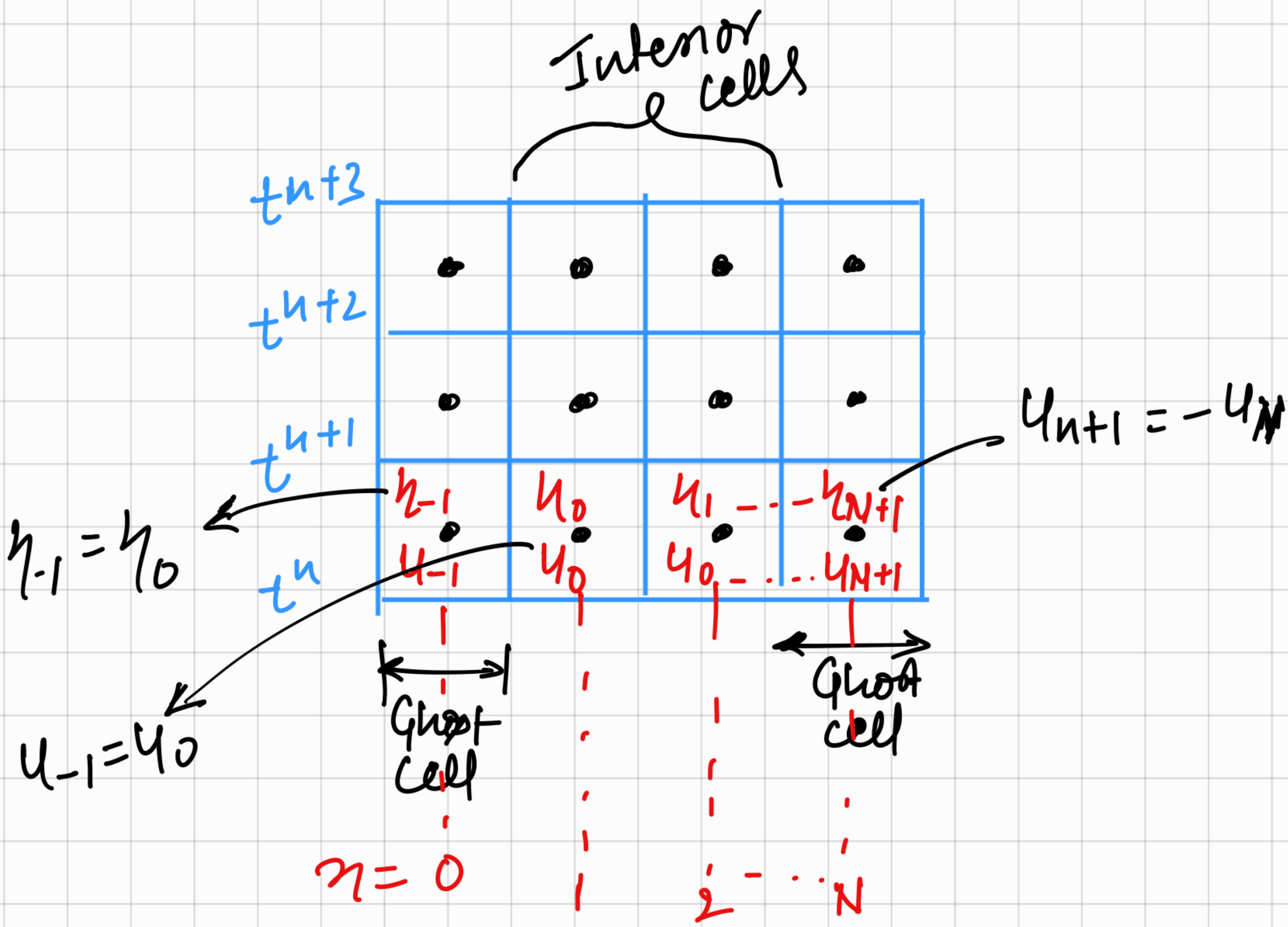
u_0 & u_0

at $x = N$ (right boundary)

$$\eta_{N+1} = \eta_N \quad \& \quad u_{N+1} = u_N$$

② closed:-

- No flux passes through boundary
- Taking $u_{-1} = -u_0$ (-ve velocity)
  $\eta_{-1} = -\eta_0$
- $f_{1/2} = 0$ & $f_{N+\frac{1}{2}} = 0$



⇒ for time steps estimation, restricting CFL condition

$$\Delta t \leq \frac{\Delta x}{\max|\lambda|}$$

λ_1 & λ_2 → eigen values

⇒ Now if we consider $H(x)$ is continuous then,

To solve the Riemann problem, we will use "locally approximately constant" $H(x)$ at each cell edge

↓

To achieve it → finer grid resolution will be required

∴ at cell edge, $x_{j+\frac{1}{2}}$

approximating $H(x_{j+\frac{1}{2}})$ as

$$H_{j+\frac{1}{2}} = \frac{H_j + H_{j+1}}{2}$$

where H_j & H_{j+1} → depth in adljacet cell.

↓ Eigen values will change accordingly,

$$\lambda_1 = \sqrt{gH(x)}, \quad \lambda_2 = -\sqrt{gH(x)}$$

Simillarly eigen vector will also be change

↓

These changes will lead to flux at cell edges,

Hence, to solve the flux at $(x_{j+\frac{1}{2}})$ we have to use local $H(x_{j+\frac{1}{2}})$

⇒ Simillarly, time steps will also be affected.