Numerical 3

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1 STEP 1

The given Poisson system:

$$-\Delta u = f \tag{1}$$

on a 2D area $\Omega, (x, y) \in [0, 1]^2$ where:

$$f(x, y) = 2\pi^2 sin(\pi x)cos(\pi y) \tag{2}$$

with Dirichlet boundary conditions on left and right boundaries:

$$u(0, y) = u(1, y) = 0 \tag{3}$$

and Neumann boundary conditions on upper and lower boundaries:

$$\partial_y u(x, y)|_{y=0} = \partial_y u(x, y)|_{y=1} = 0 \tag{4}$$

The equation can be written as:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f \tag{5}$$

Ritz-Galerkin principle of the system:

$$\delta F(u) = \delta \int_\Omega \frac{1}{2}|\nabla u|^2 - fudxdy = 0 \tag{6}$$

$$\int_\Omega \nabla u \cdot \nabla \delta u - f\delta udxdy = 0 \tag{7}$$

In the Ritz-Galerkin method, the test function $w(x, y)$ is chosen to be the variation $\delta u(x, y)$. The weak form of the problem is derived by multiplying the equation with the test function $w(x, y)$ and integrating over the whole domain:

$$-\int_\Omega w\Delta udxdy = \int_\Omega wfdxdy \tag{8}$$

Integrate the left side by part, first consider chain rule of multiplication:

$$\begin{aligned} \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}w) = \frac{\partial^2 u}{\partial x^2}w + \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} \\ \frac{\partial}{\partial y}(\frac{\partial u}{\partial y}w) = \frac{\partial^2 u}{\partial y^2}w + \frac{\partial u}{\partial y}\frac{\partial w}{\partial y} \end{aligned} \tag{9}$$

Add these equations:

$$w(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}w) + \frac{\partial}{\partial y}(\frac{\partial u}{\partial y}w) - (\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial w}{\partial y}) \tag{10}$$

Substitute into the left side of the integral:

$$-\int_\Omega \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}w) + \frac{\partial}{\partial y}(\frac{\partial u}{\partial y}w) - (\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial w}{\partial y})dxdy = \int_\Omega wfdxdy \tag{11}$$

$$-\int_\Omega \nabla \cdot (w\nabla u) - \nabla u \cdot \nabla wdxdy = \int_\Omega wfdxdy \tag{12}$$

$$-\int_\Omega \nabla \cdot (w\nabla u)dxdy + \int_\Omega \nabla u \cdot \nabla wdxdy = \int_\Omega wfdxdy \tag{13}$$

According to the divergence theorem(2D):

$$\int_\Omega \nabla \cdot pdxdy = \int_\Gamma p \cdot ndS \tag{14}$$

Then we have:

$$-\int_\Gamma w\nabla u \cdot ndS + \int_\Omega \nabla u \cdot \nabla wdxdy = \int_\Omega wfdxdy \tag{15}$$

In left hand side:

$$-\int_{\Gamma-Dirichlit} w\nabla u \cdot ndS - \int_{\Gamma-Neumann} w\nabla u \cdot ndS + \int_\Omega \nabla u \cdot \nabla wdxdy \tag{16}$$

In Dirichlet boundary condition w=0, in Neumann boundary condition $\nabla u \cdot n = 0$. So the final weak form:

$$\int_\Omega \nabla u \cdot \nabla wdxdy = \int_\Omega wfdxdy \tag{17}$$

2 STEP 2

Suppose u_h illustrates the approximation of the unknown:

$$u_h = \sum_{i=1}^{n} \phi_i(x,y)u_i \tag{18}$$

where ϕ_i is the shape function(basis function) which is piecewise continuous, u_i is the coefficient. The shape function satisfy that take the value one at its corresponding node P_i and is zero everywhere else, which can be expressed as:

$$\phi_i(P_j) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise} \end{cases} \tag{19}$$

Substitute the expanding u_h into the weak form and choosing $w = \phi_j$:

$$\int_\Omega \nabla(\sum_{i=1}^{n} \phi_i u_i) \cdot \nabla\phi_j dxdy = \sum_{j=1}^{n} \int_\Omega f\phi_j dxdy \tag{20}$$

Extract the coefficient:

$$\sum_{i=1}^{n} u_i \int_\Omega \nabla\phi_i \cdot \nabla\phi_j dxdy = \sum_{j=1}^{n} \int_\Omega f\phi_j dxdy \tag{21}$$

which can be written in a matrix form:

$$Au = F \tag{22}$$

where $A_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j dxdy$ is the stiffness matrix; $F_j = \int_\Omega f\phi_j dxdy$ is the load vector.

3 STEP 3

We consider a quadrilateral element in reference coordinate and local coordinate. The coordinate of the point is $P(\xi, \eta)$ in reference coordinate and $p(x, y)$ in local coordinate. The four nodes of the reference element are $P_1(\xi_1, \eta_1) = (-1, -1)$, $P_2(\xi_2, \eta_2) = (1, -1)$, $P_3(\xi_3, \eta_3) = (1, 1)$, $P_4(\xi_4, \eta_4) = (-1, 1)$. The four nodes of the local element are $p_1(x_1, y_1)$, $p_2(x_2, y_2)$, $p_3(x_3, y_3)$, $p_4(x_4, x_4)$, they have the transformation relationship:

$$(x, y) = \phi(\xi, \eta) \tag{23}$$

Construct a bilinear approximation of x in terms of ξ and η:

$$\begin{aligned} x &= \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta \\ y &= \beta_1 + \beta_2\xi + \beta_3\eta + \beta_4\xi\eta \end{aligned} \tag{24}$$

If we substitute the coordinates of P_{1-4} in:

$$\begin{aligned} x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \\ y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} &= \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} \end{aligned} \tag{25}$$

x and y have the same coefficient matrix so take x as example. So the matrix α can be expressed as:

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} = \begin{bmatrix}1 & -1 & -1 & 1\\ 1 & 1 & -1 & -1\\ 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\end{bmatrix}^{-1} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \tag{26}$$

Substitute it into equation:

$$x = \begin{bmatrix}1 & \xi & \eta & \xi\eta\end{bmatrix} \begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} \tag{27}$$

After solving the inverse of the equation we have:

$$x = \frac{1}{4}\begin{bmatrix}1 & \xi & \eta & \xi\eta\end{bmatrix} \begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 1 & 1 & -1\\ -1 & -1 & 1 & 1\\ 1 & -1 & 1 & -1\end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \tag{28}$$

$$x = \phi_1 x_1 + \phi_2 x_2 + \phi_3 x_3 + \phi_4 x_4 \tag{29}$$

Shape functions can be calculated:

$$\begin{aligned}\phi_1 &= \frac{1}{4}(1-\xi-\eta+\xi\eta) = \frac{1}{4}(1-\xi)(1-\eta)\\ \phi_2 &= \frac{1}{4}(1+\xi-\eta-\xi\eta) = \frac{1}{4}(1+\xi)(1-\eta)\\ \phi_3 &= \frac{1}{4}(1+\xi+\eta+\xi\eta) = \frac{1}{4}(1+\xi)(1+\eta)\\ \phi_4 &= \frac{1}{4}(1-\xi+\eta-\xi\eta) = \frac{1}{4}(1-\xi)(1+\eta)\end{aligned} \tag{30}$$

Obviously the sum of ϕ_{1-4} equals to one, and take the value of one in their corresponding node while take the value of zero on everywhere else.
The Jacobian of the transformation is:

$$J = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix} \tag{31}$$

Take the first element as example:

$$\frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}\begin{bmatrix}\phi_1 & \phi_2 & \phi_3 & \phi_4\end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \tag{32}$$

The Jacobian can be calculated by:

$$J = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \xi} x_i & \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \eta} x_i & \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \eta} y_i \end{bmatrix} \tag{33}$$

The Nabla operator of the two coordinates have the relationship:

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \tag{34}$$

$$\nabla_{(x,y)} = J^{-1} \nabla_{(\xi,\eta)} \tag{35}$$

The transformation between integral:

$$\int_{\Omega} dxdy = \int_{-1}^{1} \int_{-1}^{1} det|J| d\xi d\eta \tag{36}$$

So the left hand side of the equation:

$$\int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j dxdy = \int_{-1}^{1} \int_{-1}^{1} J^{-1} \nabla \phi_i \cdot J^{-1} \nabla \phi_j det|J| d\xi d\eta \tag{37}$$

Right hand side:

$$\int_{\Omega} f \phi_i dxdy = \int_{-1}^{1} \int_{-1}^{1} \phi_j f(\xi, \eta) det|J| d\xi d\eta \tag{38}$$

4 STEP 4

First take the case that mesh resolutions 16×16 and the degree of order 1 as example, the result contour of u_1 calculated by Firedrake is shown in Figure 1. From the plot, it appears as a smooth sinusoidal wave with positive values near $x = 0.5, y = 0$ and negative values near $x = 0.5, y = 1$, as the color gradient transitions smoothly from positive (red) to negative (blue).
Meanwhile, the contour of error can also be plotted by putting the value of absolute difference of numerical and exact solutions $u_h - u_e$ in the same function space V as u_h and u_e, as Figure 2 illustrates.
If we increase the order of accuracy or the resolutions of mesh, as shown in Figure 3 and 4, the error decreases obviously as the maximum error in Figure 2 is around 3.2e-03 and the maximum error in Figure 3 and 4 are 2.5e-06 and 2.0e-04.
If we keep the order unchanged, a plot of Δx vs. error can be obtained(Figure 5), the green line illustrates that as Δx decreases, the error converges quadratically. On the contrary, we can also draw a plot of order vs. error(Figure 6). These two plots show that increasing mesh resolution and raising the order both significantly improve the accuracy at the beginning. However, when the mesh resolution and order increase to a enough point, continue to increase brings little gain in accuracy but instead increases the amount of computation, which does not seem reasonable.

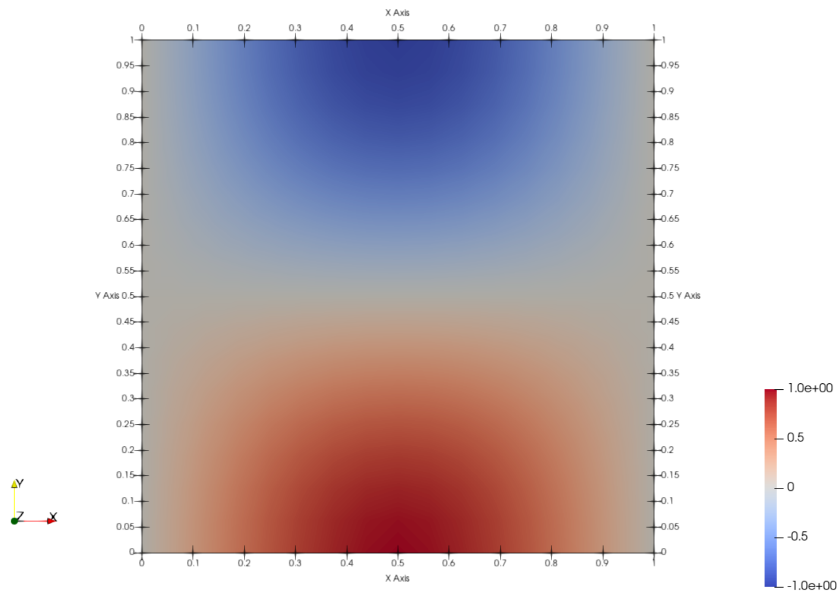

Figure 1: Contour of u_1(16×16 mesh, order 1)

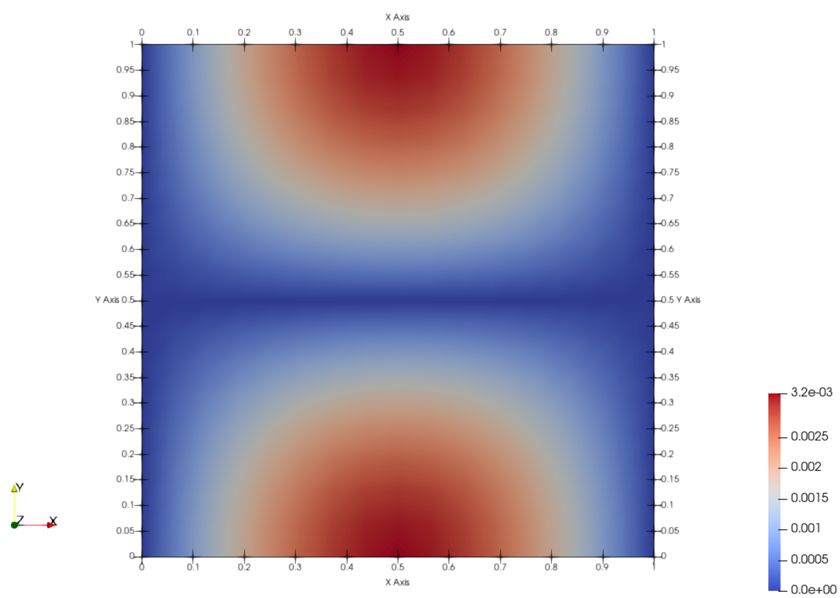

Figure 2: Contour of error(16×16 mesh, order 1)

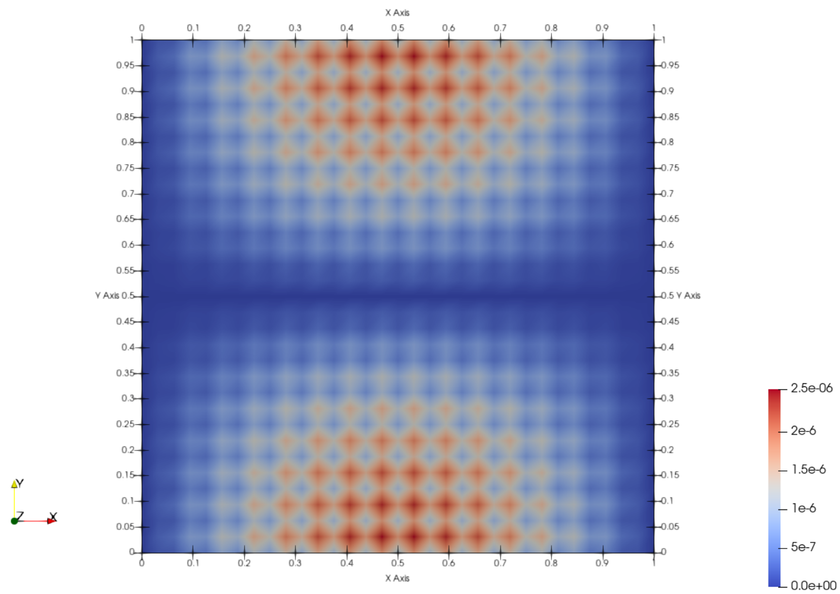

Figure 3: Contour of error(16×16 mesh, order 2)

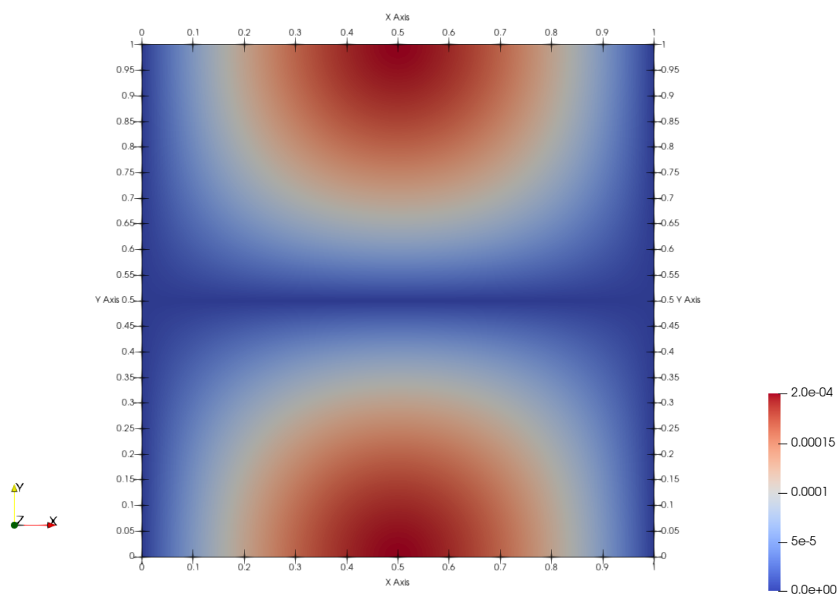

Figure 4: Contour of error(64×64 mesh, order 1)

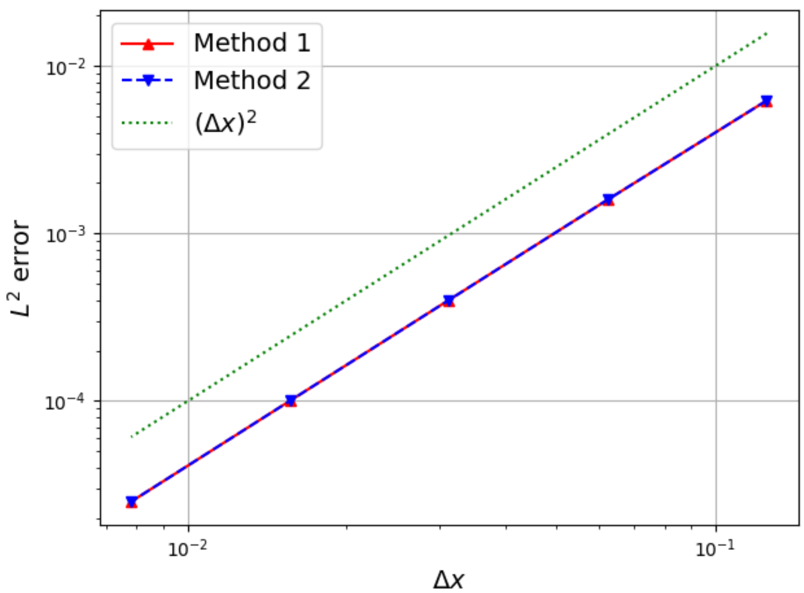

Figure 5: Error varies with mesh resolution

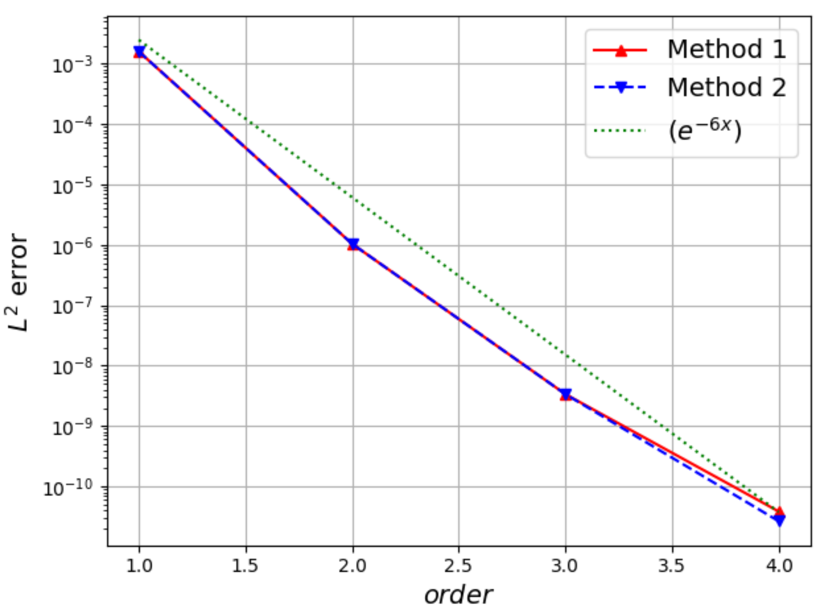

Figure 6: Error varies with order