

Finite Volume

Eg. sheet 2 -

Q1)

Scaling the shallow water equations:

$$U = U_0 U' \quad x = L_s x' \quad t = (L_s/U_0)t' \quad \eta = H_{0s} n'$$

$$H = H_{0s} H'$$

~~eq 1~~ $\frac{dn}{dt} + \frac{d(HU)}{dx} = 0$

* substitute scaling factors:-

$$\frac{d(H_{0s} n')}{dt} + \frac{d(H_{0s} H' U_0 U')}{dx} = 0$$

term 1 $H_{0s} \frac{dn'}{dt}$ + term 2 $H_{0s} U_0 \frac{d(H' U')}{dx} = 0$

term 1: $\frac{dn'}{dt} = \frac{\partial n'}{\partial t'} \frac{dt'}{dt}$ { and } $t = (L_s/U_0)t'$
 and $t' = \frac{t}{L_s/U_0}$
 $\frac{dt'}{dt} = \frac{U_0}{L_s}$

then

$$\frac{\partial n'}{\partial t} = \frac{\partial n'}{\partial t'} \frac{V_0}{L_S}$$

} and $\lambda = L_S x'$
 $x' = \frac{x}{L_S}$

$$\frac{\partial x'}{\partial x} = \frac{1}{L_S}$$

$$\Rightarrow \frac{V_0}{L_S} \frac{\partial n'}{\partial t'}$$

term 2: $\frac{\frac{\partial(H'U')}{\partial x'}}{2\lambda} = \frac{\frac{\partial(H'U')}{\partial x'}}{\frac{\partial x'}{\partial x}} \frac{\partial n'}{\partial x}$

$$= \frac{\frac{\partial(H'U')}{\partial x'}}{\frac{\partial n'}{\partial t'}} \frac{1}{L_S}$$

② $\frac{H_0 V_0}{L_S} \frac{\frac{\partial(H'U')}{\partial x'}}{\frac{\partial x'}{\partial x}}$

$$\frac{H_0 V_0}{L_S} \left(\frac{\frac{\partial n'}{\partial t'}}{\frac{\partial x'}{\partial t'}} + \frac{\frac{\partial(H'U')}{\partial x'}}{\frac{\partial n'}{\partial t'}} \right) = 0$$

$$\Rightarrow \frac{\partial n'}{\partial t'} + \frac{\partial(H'U')}{\partial x'} = 0 \text{ Ans.}$$

↑ same as eq 1

eq2

$$\frac{\partial U}{\partial t} + \frac{\partial (gn)}{\partial x} = 0$$

$$\frac{\partial (U_0 U')}{\partial t} + \frac{\partial ((g' U_0^2 / H_{0S}) H_{0S} n')}{\partial x} = 0$$

$$U_0 \frac{\partial U'}{\partial t} + \frac{U_0^2}{H_{0S}} x H_{0S} \frac{\partial g' n'}{\partial x} = 0$$

$$U_0 \left[\frac{\partial U'}{\partial t'} \frac{\partial t'}{\partial t} \right] + U_0^2 \left[\frac{\partial g' n'}{\partial x'} \frac{\partial x'}{\partial x} \right] = 0$$

$$x = L_S x' \Rightarrow \frac{\partial x'}{\partial x} = \frac{1}{L_S}$$
$$\frac{x}{L_S} = x'$$

$$\Rightarrow U_0 \left[\frac{\partial U'}{\partial t'} \frac{U_0}{L_S} \right] + U_0^2 \left[\frac{\partial g' n'}{\partial x'} \times \frac{1}{L_S} \right] = 0$$

$$\frac{U_0^2}{L_s} \left[\frac{\partial U'}{\partial t'} + \frac{\partial g' n'}{\partial x'} \right] = 0$$

$$\therefore \frac{\partial U'}{\partial t'} + \frac{\partial g' n'}{\partial x'} = 0 \quad \text{Ans}$$

\uparrow same as eq 2

Rewrite (1) using matrix A as follows

$$2t \begin{pmatrix} n \\ H_0 U \end{pmatrix} + A dx \begin{pmatrix} n \\ H_0 U \end{pmatrix} = 0 \quad (3)$$

$$A = \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix}$$

Substitute $A \Rightarrow$

$$2t \begin{pmatrix} n \\ H_0 U \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix} dx \begin{pmatrix} n \\ H_0 U \end{pmatrix} = 0$$

$$2t \begin{pmatrix} n \\ H_0 U \end{pmatrix} + 2x \begin{pmatrix} H_0 U \\ C_0^2 n \end{pmatrix} = 0$$

$$2t \begin{pmatrix} n \\ H_0 U \end{pmatrix} + 2x \begin{pmatrix} H_0 U \\ gH_0 n \end{pmatrix} = 0$$

first eq $\Rightarrow \frac{\partial n}{\partial t} + \frac{\partial H_0 U}{\partial x} = 0$ \cancel{Ans}

\uparrow same as eq. 1

Second eq $\Rightarrow \frac{\partial H_0 U}{\partial t} + \frac{\partial H_0 g}{\partial x} = 0$

$$H_0 \left(\frac{\partial U}{\partial t} + \frac{\partial g n}{\partial x} \right) = 0$$

$$-\frac{\partial U}{\partial t} + \frac{\partial g n}{\partial x} = 0 \quad \text{p.}$$

↑ same as eq. 2

\Rightarrow finding the Eigenvalues and Eigenvectors

$$A = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix}$$

$$A - hI = \begin{bmatrix} 0 & 1 \\ c_0^2 & 0 \end{bmatrix} - \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}$$

$$= \begin{pmatrix} -h & 1 \\ c_0^2 & -h \end{pmatrix}$$

$$\det |A - hI| \Rightarrow h^2 - (c_0^2) (h - c_0)(h + c_0) = 0$$

$$h_1 = C_0 \quad h_2 = -C_0$$

$$h_1 \Rightarrow \begin{pmatrix} -C_0 & 1 \\ C_0^2 & -C_0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -C_0 x + y &= 0 \\ +C_0^2 x - C_0 y &= 0 \end{aligned}$$

$$y = \frac{C_0^2 x}{C_0} \quad y = C_0 x$$

$$\text{let } x=1 \quad \therefore \quad y = C_0$$

$$x_1 = \alpha \begin{bmatrix} 1 \\ C_0 \end{bmatrix}$$

$$h_2 \Rightarrow \begin{pmatrix} C_0 & 1 \\ C_0^2 & C_0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} C_0 x + y &= 0 \\ C_0^2 x + C_0 y &= 0 \end{aligned}$$

$$-C_0 x = y$$

$$x_2 = \beta \begin{bmatrix} -1 \\ C_0 \end{bmatrix}$$

$$\begin{aligned} \text{let } x = -1 \\ y = C_0 \end{aligned}$$

$$\therefore B = \frac{1}{2c_0} \begin{bmatrix} 1 & -1 \\ c_0 & c_0 \end{bmatrix} \quad (5)$$

since α and β can be
any scalar value.

Now $B^{-1}B = I$, showing

$$B^{-1}AB = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} I.$$

$$\text{Find } B^{-1} \Rightarrow \det B = \frac{1}{2c_0} (c_0 + c_0) = 1$$

$$\therefore B^{-1} = \frac{1}{2c_0} \begin{bmatrix} c_0 & 1 \\ -c_0 & 1 \end{bmatrix}$$

First multiply AB

$$AB = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \frac{1}{2c_0} \begin{pmatrix} 1 & -1 \\ c_0 & c_0 \end{pmatrix}$$

$$AB = \frac{1}{2c_0} \begin{pmatrix} c_0 & c_0 \\ c_0^2 & -c_0^2 \end{pmatrix}$$

then do $B^{-1}(AB)$

$$\Rightarrow \frac{1}{2c_0} \begin{pmatrix} (0 & 1) \\ (-c_0 & 1) \end{pmatrix} \frac{1}{2c_0} \begin{pmatrix} (0 & 0) \\ (c_0^2 - c_0^2) \end{pmatrix}$$

$$= \frac{1}{2c_0} \begin{pmatrix} 2c_0^2 & 0 \\ 0 & -2c_0 \end{pmatrix}$$

$$= \begin{pmatrix} c_0 & 0 \\ 0 & -c_0 \end{pmatrix}$$

* which is the same

as $\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}^T$ since multiplying

by identity matrix doesn't change
the matrix.

$$\therefore B^{-1}AB = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}^T \quad (6)$$

\Rightarrow multiplying (3) by B^{-1}

$$B^{-1} \frac{dt}{(H_{0U})} + B^{-1} A \frac{dx}{(H_{0U})} = 0$$

then we introduce $B^{-1}B = I$
 into the second term since the identity
 matrix does not affect the matrix

$$\Rightarrow B^{-1} \frac{dt}{(H_{0U})} + \underbrace{B^{-1} A B B^{-1}}_{= \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}} \frac{dx}{(H_{0U})} = 0$$

$$\Rightarrow dt r + \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} dx r = 0 \quad (7)$$

Ans

$$\Rightarrow \frac{dn}{dt} + \frac{\partial (H_U)}{\partial x} = 0$$

$$\sqrt{C_0^2} = g H_0$$

$$\frac{du}{dt} + \frac{\partial (gn)}{\partial x} = 0$$

$$C_0 = \sqrt{g H_0}$$

$$\frac{dt}{\frac{H_{0U} + C_0 n}{H_{0U} - C_0 n}} + \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix} \frac{dx}{\frac{H_{0U} + C_0 n}{H_{0U} - C_0 n}} = 0$$

Question 2

$$\partial_t r + \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \partial_x r = 0$$

$$h_1 = C_0$$
$$h_2 = -C_0$$

$$\partial_t r_1 + h_1 \partial_x r_1 = 0$$

$$\partial_t r_2 + h_2 \partial_x r_2 = 0$$

$$\left\{ \begin{array}{l} \frac{\partial r_2}{\partial t} + \underbrace{h_2 \frac{\partial r_2}{\partial x}}_{\text{constant}} = 0 \\ \frac{\partial r_1}{\partial t} + \underbrace{h_1 \frac{\partial r_1}{\partial x}}_{\text{constant}} = 0 \end{array} \right.$$

linear
advection
equation

⇒ The solution of the linear advection equation is

simply $u(x, t) = U_0 (x - at)$

where a is the constant, and in our equations h_1 and h_2 are constants.

→ Given the initial condition we can then show for (7)

$$\Rightarrow r_1(x,t) = \begin{cases} r_{1L} & \text{for } (x-h_1 t) < 0 \\ r_{1R} & \text{for } (x-h_1 t) \geq 0 \end{cases}$$

$\star h_1 = c_0$

$$\therefore r_1(x,t) = \begin{cases} r_{1L} & \text{for } x < c_0 t \\ r_{1R} & \text{for } x \geq c_0 t \end{cases}$$

and similarly :-

$$r_2(x,t) = \begin{cases} r_{2L} & \text{for } x < -c_0 t \\ r_{2R} & \text{for } x \geq -c_0 t \end{cases}$$

Solution Domain

case 1
 r_{1L}
 r_{2L}

case 2
 r_{2R}
 and r_{1L}

case 3
 r_{1R}
 r_{2R}

$-c_0 t$

$c_0 t$

for case 1

$$H_{00} = \frac{1}{2} (r_1 + r_2)$$

$$\text{and } r_1 = H_{00} + \text{core} \quad r_2 = H_{00} - \text{core}$$

$$r_1 = H_0 U_e + C_0 \eta_e \quad r_2 = H_0 U_e - C_0 \eta_e$$

$$H_{00} = \frac{1}{2} (H_0 U_e + \cancel{\text{core}} + H_0 U_e - \cancel{\text{core}})$$

$$H_{00} = \frac{1}{2} (2 H_0 U_e)$$

$$= H_0 U_e$$

$$\eta = \frac{1}{2C_0} (H_0 U_e + \text{core} - H_0 U_e - \text{core})$$

$$\eta = \frac{2 \text{core}}{2 C_0}$$

$$\eta = \eta_e$$

for case 3:

$$r_1 = H_{OUR} + C_{oN_R} \quad r_2 = H_{OUR} - C_{oN_R}$$

$$\eta_L = \frac{1}{2} (r_1 - r_2) / C_0$$

$$H_{oU} = \frac{1}{2} (H_{OUR} + C_{oN_R} + H_{oUR} - C_{oN_R})$$

$$H_{oU} = H_{oUR}$$

$$\eta = \frac{1}{2C_0} (H_{OUR} + C_{oN_R} - H_{oUR} - C_{oN_R})$$

$$\eta = \eta_R$$

case 3

$$H_{oU}(x,t) = \frac{1}{2} (H_{oUL} + C_{oN_L} + H_{oUR} - C_{oN_R}) \\ - \frac{1}{2} (H_{oUL} + H_{oUR}) + \frac{1}{2} (C_{oN_L} - C_{oN_R})$$

$$\eta(x,t) = \frac{1}{2C_0} (H_{oUL} + C_{oN_L} - H_{oUR} + C_{oN_R})$$

$$\eta(x,t) = \frac{H_{oUL} - H_{oUR}}{2C_0} + \frac{\eta_L + \eta_R}{2}$$

thus combining all cases \Rightarrow

$$H_{LV}(x,t) = \begin{cases} H_{VL} & \text{for } x < -C_0 t \\ \frac{1}{2}(H_{VL} + H_{VR}) + \frac{1}{2}(C_0 t - C_0 t) & \text{for } -C_0 t \leq x \leq C_0 t \\ H_{VR} & \text{for } x > C_0 t \end{cases}$$

$$\eta(x,t) = \begin{cases} \eta_L & \text{for } x < -C_0 t \\ \frac{H_{VL} - H_{VR}}{2C_0} + \frac{\eta_L + \eta_R}{2} & \text{for } -C_0 t \leq x \leq C_0 t \\ \eta_R & \text{for } x > C_0 t \end{cases}$$

Anm -

Question 3

equation 1

$$\partial_t \bar{U} + \partial_x f(\bar{U}) = 0$$

$$U_1 = R \quad U_2 = H_0 v$$

$$f_1 = U_1 \quad f_2 = C_0^2 U_1$$

$$\vec{U} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

$$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

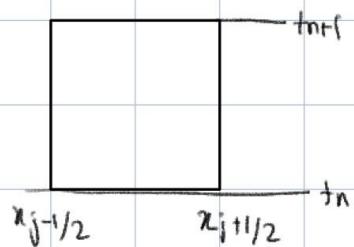


define a mean $\Rightarrow U_{j|t} = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} \bar{U}(x, t) dx$ (1)

we then integrate the conservation law over a small control volume

$$\partial_t \bar{U} + \partial_x f(\bar{U}) = 0 \text{ over}$$

control volume



$$\Rightarrow \iint_{\substack{x_{j-1/2} \\ t_{n-1}}}^{\substack{x_{j+1/2} \\ t_n}} \partial_t \bar{U} dt dx = - \iint_{\substack{x_{j-1/2} \\ t_n}}^{\substack{x_{j+1/2} \\ t_n}} \partial_x f(U_{j|t}) dx dt$$

$$\Rightarrow \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1}) - v(x, t_n) dx$$

$$= - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t)) - f(v(x_{j-1/2}, t)) dt$$

\Rightarrow using the defined mean equation (+)

$$\Rightarrow \partial x_j (v_j^{n+1} - v_j^n) = - \int_{t_n}^{t_{n+1}} \bar{f}(u(x_{j+1/2}, t)) - f(v(x_{j+1/2}, t)) dt$$

\Rightarrow defining the numerical flux function as

$$\Rightarrow F(v_j^n, v_{j+1}^n) \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(v(x_{j+1/2}, t)) dt$$

\Rightarrow the expression above then becomes

$$\therefore \bar{U}_j^{n+1} = \bar{U}_j^n - \frac{\Delta t}{\Delta x} \left(\bar{F}_{j+1/2}(\bar{U}_j^n, \bar{U}_{j+1}^n) - \bar{F}_{j-1/2}(\bar{U}_{j-1}, \bar{U}_j^n) \right)$$

* The Riemann problem can be solved locally here with the Riemann solution found previously

Deriving the time step estimate using the CFL condition =)

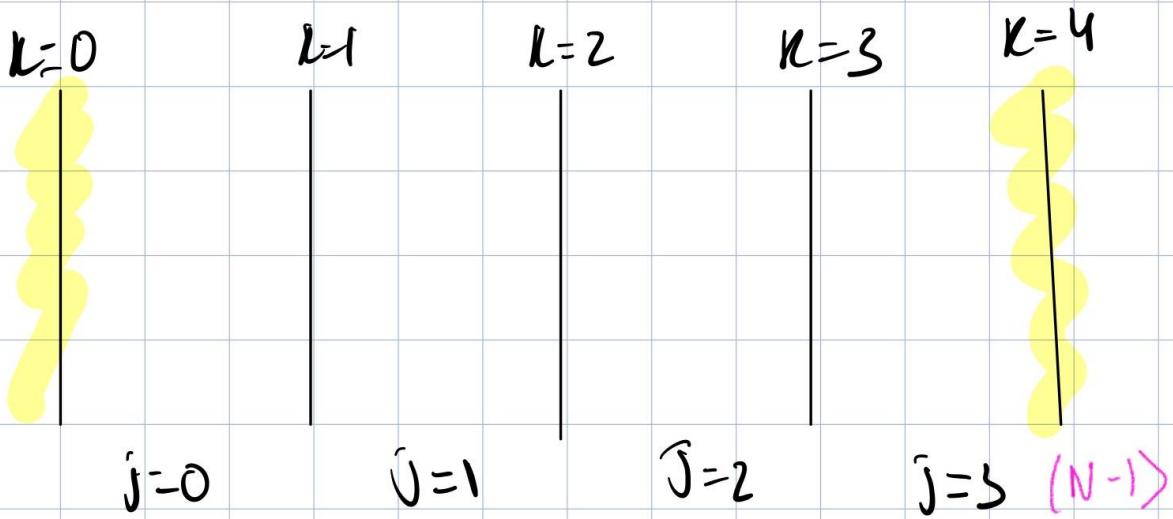
$$\Delta t = \text{CFL} \frac{\Delta x_j}{c_0}$$

$$\therefore \Delta t \leq \min \frac{x_j}{c_0}$$

$$\text{CFL} < 1$$

for our domain and code \Rightarrow

(N)



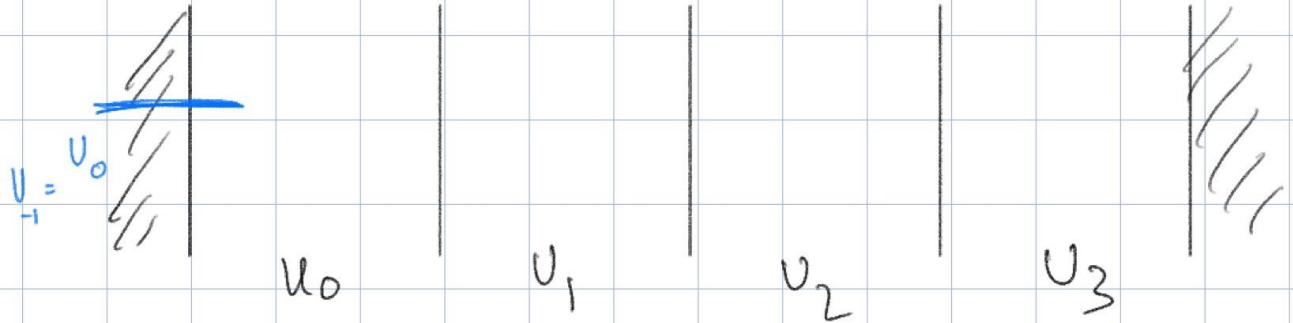
$$\vec{U}_j^{n+1} = \vec{U}_j^n - \frac{\Delta x_j}{D t_n} \left(\vec{F}_{j+1/2}^n(U_j^n, U_{j+1}^n) - \vec{F}_{j-1/2}^n(U_j^n, U_{j-1}^n) \right)$$

$j = 0, \dots, N-1$

\Rightarrow using our solution to question 2, the fluxes then become

$$F_{j+1/2} = \left(\begin{array}{l} \frac{1}{2} (H_0 V_L + H_0 V_R) + (C_{0L} - C_{0R}) \\ \frac{C_0^2 \times 1}{2 C_0} (H_0 V_L - H_0 V_R + C_{0L} + C_{0R}) \end{array} \right)$$

extrapolating B.Cs ("open domain")

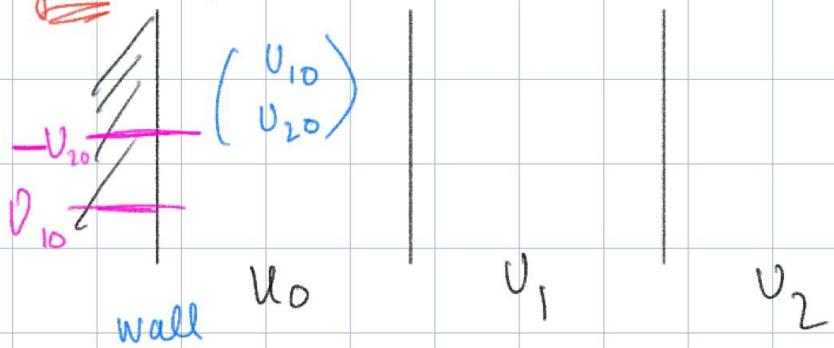


\Rightarrow for extrapolating B.Cs we simply create an open domain

\Rightarrow where $U_{-1} = U_0$

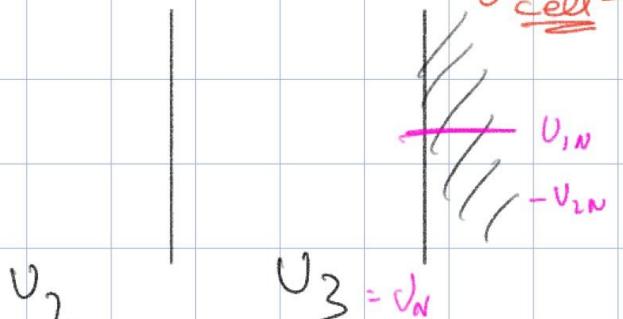
closed domain

ghost cell 1



$$\boxed{U_1 = L \\ U_2 = H_0 U}$$

ghost cell 2



at the wall the boundary condition are zero
so at the boundary we set a ghost

velocity so that it is zero
such that the ghost cell

$$\text{ghost cell } 1 \Rightarrow \begin{pmatrix} v_{10} \\ -v_{20} \end{pmatrix}$$

$$\text{ghost cell } 2 \Rightarrow \begin{pmatrix} v_{1N} \\ -v_{2N} \end{pmatrix}$$

\Rightarrow If we set the flux at the boundary to be equal to zero, this implies that the gradient of η at the boundary be zero too

$$\star \quad \frac{\partial \eta}{\partial x} = 0 \text{ at } x=0 \quad x=L$$

Question 6

Using the code provided, the value of “Nbc” is set to 1, which is the solid wall boundary condition. θ is also set at 0.25. The code is provided in Question6and7.py.

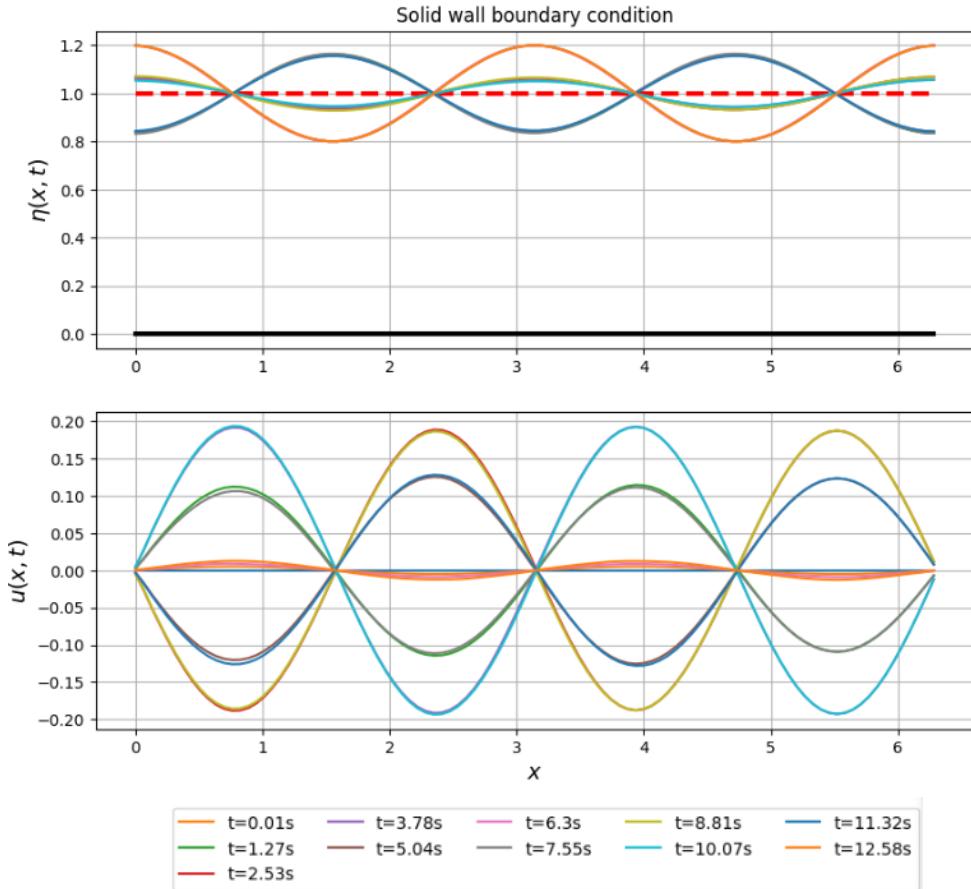


Figure 1: Solid wall boundary condition for the alternating flux scheme for $\theta = 0.25$

Question 7

The analytical values of η and u are calculated as:

$$\eta_{analytical} = ck * \cos\left(\frac{mint * \pi * x}{Ld}\right) * \cos(\omega t) \quad (1)$$

where $Ld = \text{length of spatial domain} = 2\pi$, $mint = 4$ and $ck = 0.2$

$$u_{analytical} = \omega * \frac{ck}{H_0} * \frac{Ld}{mint * \pi} * \sin\left(\frac{mint * \pi * x}{Ld}\right) * \sin(\omega t)$$

The numerical solutions are shown below in Figure 2, and the analytical solutions are shown in Figure 3, all plotted at the same timesteps.

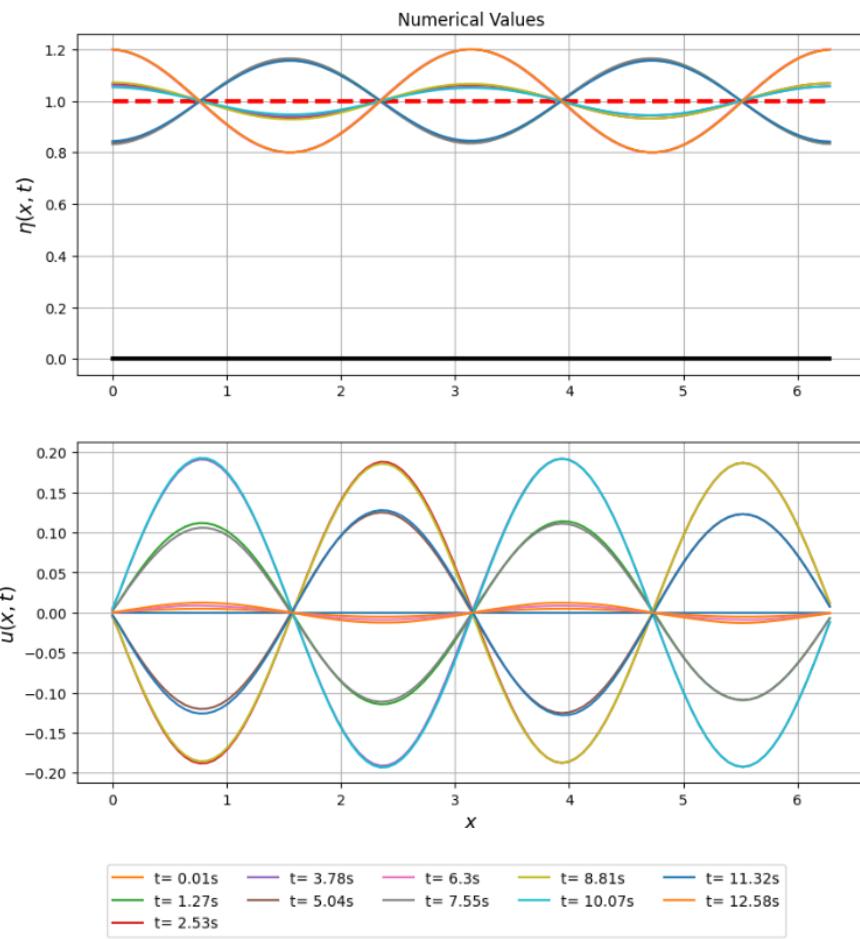


Figure 2: Numerical solutions to the alternating flux scheme with solid wall boundary conditions.

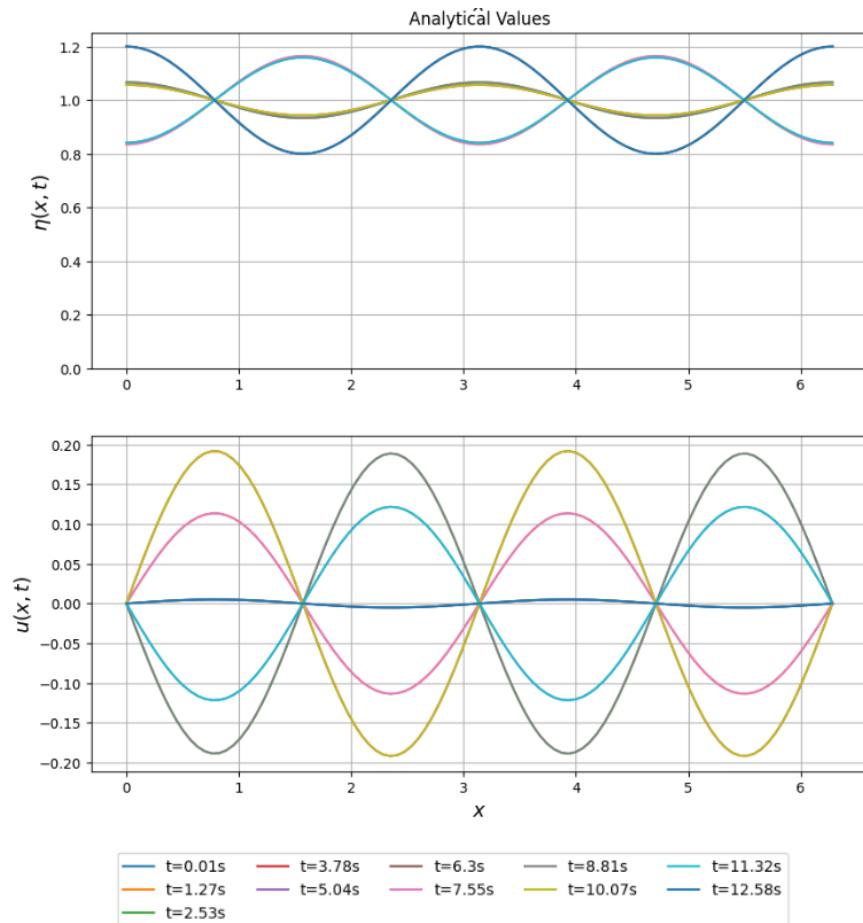


Figure 3: Analytical solutions to the alternating flux scheme with solid wall boundary conditions.