Numerical Exercises 2

1. We are given the linearised shallow-water system of equations:

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial t} + \frac{\partial (g\eta)}{\partial x} = 0 .$$

The variables in the equations are

Velocity $:= u(x,t)$,
Free-surface deviation $:= \eta(x,t)$,
Rest Depth $:= H(x)$,
Acceleration of gravity $:= g = 9.81\,ms^{-2}$.

We can scale the system of equations using dimensionless variables as follows:

$$u = U_0 u', \quad x = L_s x', \quad t = \frac{L_s}{U_0} t', \quad \eta = H_{0s}\eta', \quad H = H_{0s} H'.$$

If we let $g' = \dfrac{g H_{0s}}{U_0^2}$ then we can

non-dimensionalise the equations as

$$\frac{U_0 H_{0s}}{L_s}\frac{\partial \eta'}{\partial t'} + \frac{U_0 H_{0s}}{L_s}\frac{\partial (H'u')}{\partial x'} = 0 \quad \text{and}$$

$$\frac{U_0^2}{L_s}\frac{\partial u'}{\partial t'} + \frac{U_0^2}{L_s}\frac{\partial (g'\eta')}{\partial x'} = 0 .$$

This gives the same equations in non-dimensional variables except now g' depends on the choices of the dimensionless variable coefficients i.e. U_0, H_{0s}, L_s.

We can write the system in vector form as

$$\frac{\partial \vec{x}}{\partial t} + A\frac{\partial \vec{x}}{\partial x} = 0, \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix},$$

$c_0^2 = gH_0$ and $\vec{x} = (\eta, H_0 u)^T$.

Using linear algebra we may decouple this system as follows:

Finding the eigenvectors of matrix A we solve

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ c_0^2 & -\lambda \end{vmatrix} = \lambda^2 - c_0^2 = 0, \quad \text{and}$$

So the eigenvalues are $\lambda_1 = c_0$, $\lambda_2 = -c_0$.
Finding the eigenvectors that must satisfy

$A\underline{\lambda_1} = \lambda_1 \underline{\lambda_1}$ and $A\underline{\lambda_2} = \lambda_2 \underline{\lambda_2}$ we have

$$\underline{\lambda_1} = \frac{1}{2c_0}\begin{pmatrix} 1 \\ c_0 \end{pmatrix} \quad \text{and} \quad \underline{\lambda_2} = \frac{1}{2c_0}\begin{pmatrix} -1 \\ c_0 \end{pmatrix}.$$

Now defining B as the matrix whose columns are the eigenvectors

$$B = \frac{1}{2c_0}\begin{pmatrix} 1 & -1 \\ c_0 & c_0 \end{pmatrix}, \quad \text{then}$$

$$AB = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2c_0} & -\frac{1}{2c_0} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{c_0}{2} & -\frac{c_0}{2} \end{pmatrix}, \quad \text{and}$$

$$B^{-1}AB = \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{c_0}{2} & -\frac{c_0}{2} \end{pmatrix} = \begin{pmatrix} c_0 & 0 \\ 0 & -c_0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Defining $\vec{r} = B^{-1}\vec{x} = \left(c_0\eta + H_0 u,\ -c_0\eta + H_0 u\right)^T$

we can manipulate the system of equations, since $A = BDB^{-1}$ where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ the equations can be written as

$$\frac{\partial \vec{x}}{\partial t} + BDB^{-1}\frac{\partial \vec{x}}{\partial x} = 0$$

and left multiplying by B^{-1} gives

$$\frac{\partial \vec{r}}{\partial t} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\frac{\partial \vec{r}}{\partial x} = 0.$$

2.) We now have a Riemann problem in terms of $\vec{r} = (r_1, r_2)^T$ with conservation laws

$$(1) \quad \frac{\partial r_1}{\partial t} + c_0 \frac{\partial r_1}{\partial x} = 0,$$

$$(2) \quad \frac{\partial r_2}{\partial t} - c_0 \frac{\partial r_2}{\partial x} = 0,$$

and initial conditions

$$r_1(x,0) = \begin{cases} r_{1l} & \text{for } x < 0 \\ r_{1r} & \text{for } x \geq 0 \end{cases}, \quad r_2(x,0) = \begin{cases} r_{2l} & \text{for } x < 0 \\ r_{2r} & \text{for } x \geq 0 \end{cases}.$$

The first conservation law is a linear advection equation with a solution $r_1(x,t) = r_0(x - c_0 t)$, which for the initial condition above gives

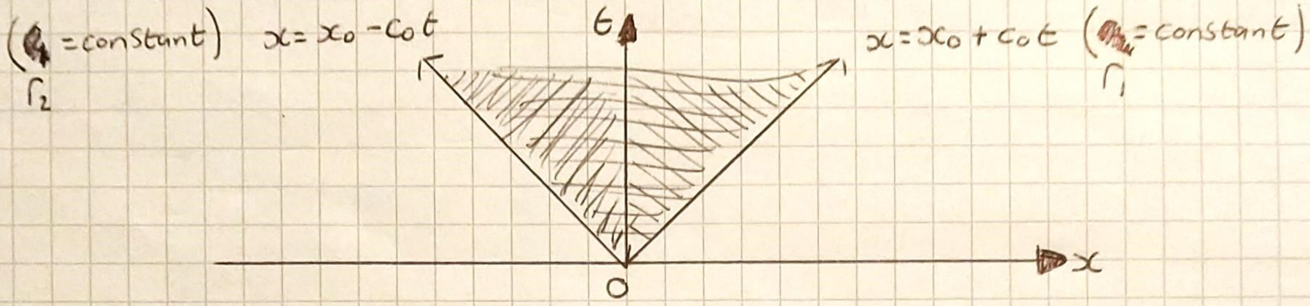
$$r_1(x,t) = \begin{cases} r_{1l} & \text{for } x - c_0 t < 0 \\ r_{1r} & \text{for } x - c_0 t \geq 0 \end{cases} = \begin{cases} r_{1l} & \text{for } x < c_0 t \\ r_{1r} & \text{for } x \geq c_0 t \end{cases}.$$

Similarly, the second conservation law has solution $r_2(x,t) = r_0(x + c_0 t)$ which for the initial condition above gives

$$r_2(x,t) = \begin{cases} r_{2l} & \text{for } x + c_0 t < 0 \\ r_{2r} & \text{for } x + c_0 t \geq 0 \end{cases} = \begin{cases} r_{2l} & \text{for } x < -c_0 t \\ r_{2r} & \text{for } x \geq -c_0 t \end{cases}.$$

Using this solution and the relation between r_1, r_2 and u, η we can calculate the solutions $u(x,t)$ and $\eta(x,t)$ to the original

System of equations. ~~Since $r_1 = H_0 u + c_0 \eta$ then~~
Since $r_1 = H_0 u + c_0 \eta$ and $r_2 = H_0 u - c_0 \eta$ then $H_0 u(x,t) = \frac{1}{2}\left(r_1(x,t) + r_2(x,t)\right)$. The characteristics for r_1 and r_2 are drawn below for $x_0 = 0$:



When we are at a part of the graph where $x < -c_0 t$ then $r_1(x < -c_0 t, t) = r_{1l} = H_0 u_l + c_0 \eta_l$ and $r_2(x < -c_0 t, t) = r_{2l} = H_0 u_l - c_0 \eta_l$. Therefore $H_0 u(x < -c_0 t, t) = \frac{1}{2}(2 H_0 u_l + c_0 \eta_l - c_0 \eta_l) = H_0 u_l$.

In the shaded fan section in the middle we have $-c_0 t \leq x < c_0 t$ and so $r_1 = r_{1l} = H_0 u_l + c_0 \eta_l$ and $r_2 = r_{2r} = H_0 u_r - c_0 \eta_r$. This gives

$$H_0 u(-c_0 t \leq x < c_0 t, t) = \frac{1}{2}\left(H_0 u_l + c_0 \eta_l + H_0 u_r - c_0 \eta_r\right).$$

Finally, in the part of the graph right of the r_1 characteristic we have

$$H_0 u(x \geq c_0 t, t) = \frac{1}{2}(r_{1r} + r_{2r}) = \frac{1}{2}(2 H_0 u_r) = H_0 u_r.$$

We follow the same procedure for $\eta(x,t) = \frac{\frac{1}{2}\left(r_1(x,t) - r_2(x,t)\right)}{c_0}$ and find $\eta(x < -c_0 t, t) = \frac{1}{2c_0}\left(H_0 u_l + c_0 \eta_l - H_0 u_l + c_0 \eta_l\right) = \ldots$
$\ldots \eta_l$. Then $\eta(-c_0 t \leq x < c_0 t, t) = \frac{1}{2c_0}(r_{1l} - r_{2r}) \ldots$
$\ldots = \frac{1}{2c_0}\left(H_0 u_l + c_0 \eta_l - H_0 u_r + c_0 \eta_r\right)$. Finally,

$$\eta(x \geq c_0 t, t) = \frac{1}{2c_0}\left(H_0 u_r + c_0 \eta_r - H_0 u_r + c_0 \eta_r\right) = \eta_r.$$

Writing these solutions in more succinct form we have

$$H_0 u(x,t) = \begin{cases} H_0 u_l & \text{for } x < -c_0 t \\ \frac{1}{2}\left(H_0(u_l+u_r) + c_0(\eta_l - \eta_r)\right) & \text{for } -c_0 t \le x < c_0 t \\ H_0 u_r & \text{for } x \geq c_0 t \end{cases},$$

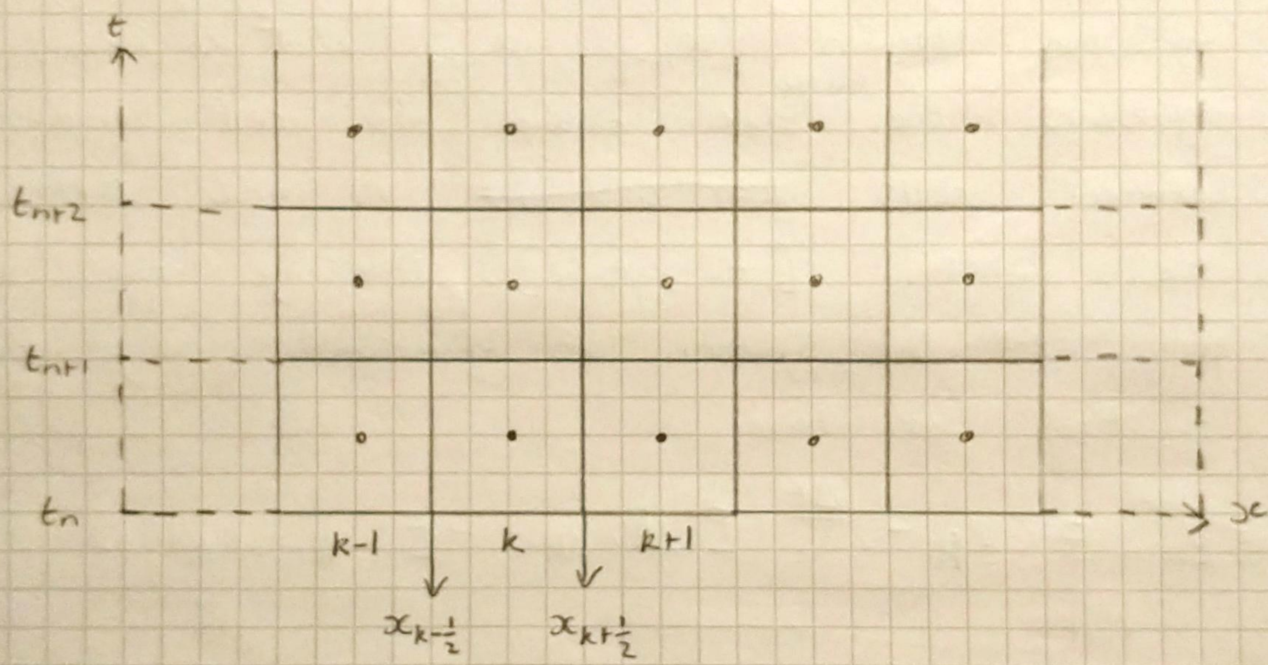
$$\eta(x,t) = \begin{cases} \eta_l & \text{for } x < -c_0 t \\ \frac{1}{2c_0}\left(H_0(u_l-u_r) + c_0(\eta_l+\eta_r)\right) & \text{for } -c_0 t \le x < c_0 t \\ \eta_r & \text{for } x > c_0 t \end{cases}.$$

3.) We now implement the Godunov numerical discretisation scheme for the system of equations from Q1, namely

$$\partial_t \underline{u} + \partial_x \underline{f}(\underline{u}) = 0,$$

where $\underline{u} = (\eta, H_0 u)^T$ and $\underline{f}(\underline{u}) = (u_2, c_0^2 u_1)^T$.

In a finite volume method we are calculating a volume average to assign to each point in the grid, ~~If we~~ where each point is ~~the~~ at the centre of the volume. We can draw this as follows:



~~In each~~ Each cell k occupies $x_{k-\frac{1}{2}} < x < x_{k+\frac{1}{2}}$ an $t_n < t < t_{n+1}$. Now integrating the system of equations in space and time for each cell we have

$$\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \int_{t_n}^{t_{n+1}} \frac{\partial \underline{u}(x,t)}{\partial t} \, dt\, dx = - \int_{t_n}^{t_{n+1}} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \frac{\partial \underline{f}(\underline{u}(x,t))}{\partial x} \, dx\, dt,$$

$$\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \underline{u}(x, t_{n+1}) - \underline{u}(x, t_n) \, dx = - \int_{t_n}^{t_{n+1}} \underline{f}(\underline{u}(x_{k+\frac{1}{2}}, t)) - \underline{f}(\underline{u}(x_{k-\frac{1}{2}}, t)) \, dt.$$

we define functions

$$U_k(t) = \frac{1}{h_k} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \underline{u}(x,t)\, dx \quad \text{and}$$

$$F(U_k^n , U_{k+1}^n) \approx \frac{1}{\Delta t} \int_{t_n}^{t^{n+1}} \underline{f}(\underline{u}(x_{k+\frac{1}{2}}, t))\, dt$$

where $\Delta t = t_{n+1} - t_n$, $h_k = x_{k+\frac{1}{2}} - x_{k-\frac{1}{2}}$. So for example U_k^n represents the ~~integ~~ averaged value of the function $\underline{u}$ at time $t = t_n$ ~~along~~ in the domain $[x_{k-\frac{1}{2}} , x_{k+\frac{1}{2}}]$. $F(U_k^n , U_{k+1}^n)$ represents the average flux through the cell boundary between cells ~~k=1 and k=~~ k and k+1 in the domain $[t_n , t_{n+1}]$. rewriting the integral form of the system of equations using these functions we have

$$\underline{U}_k^{n+1} = \underline{U}_k^n - \frac{\Delta t}{h_k}\Big(\underline{F}(\underline{U}_k^n, \underline{U}_{k+1}^n) - \underline{F}(U_{k-1}^n , U_k^n)\Big).$$

~~Based on the solutions found in Q2 we know that there is a constant solution~~

Based on the solution in Q2 we know that for the fluxes there will be a constant solution in the 'fan' so that we have

$$\underline{F}(\underline{u}_k^n , \underline{u}_{k+1}^n) \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left[H_0 u(x_{k+\frac{1}{2}}, t) ,\ c_0^2 \eta(x_{k+\frac{1}{2}}, t)\right]^T . dt$$

$$\approx \left[\frac{1}{2}\big(H_0(u_l + u_r) + c_0(\eta_l - \eta_r)\big) ,\ \frac{c_0}{2}\big(H_0(u_l - u_r) + c_0(\eta_l + \eta_r\ldots\right.$$

where $(\eta_l, H_0 u_l)^T = \underline{U}_k^n$ and

$(\eta_r, H_0 u_r)^T = \underline{U}_{k+1}^n$.

For the boundary conditions we can either set extrapolating boundaries where the value $\underline{U}_0^k$ is assigned to the left ghost cell $k = -1$ and $\underline{U}_N^k$ is assigned in the right ghost cell $k = N+1$.

Or we might mimic a closed domain by setting an equal and opposite velocity in the ghost cells – i.e $-H_0 u_0^n$ and $-H_0 u_N^n$ and an equal surface height η i.e $c_0^2 \eta_0^n$ and $c_0^2 \eta_N^n$.

For the time step estimate we follow the CFL equation

$$\Delta t \leq \min \frac{\Delta x}{c_0}.$$