

MATH 5453M Numerical Exercise 2

finite volume method: linear shallow-water equation. predicting surf height at beaches

Consider the linearised shallow-water system of eqn:-

$$\frac{\partial h}{\partial t} + \frac{\partial(Hu)}{\partial x} = 0 \quad \text{&} \quad \frac{\partial u}{\partial t} + \frac{\partial(gh)}{\partial x} = 0 \quad \text{--- (1)}$$

for velocity $u = u(x, t)$ & free surface deviation

$h = h(x, t)$, rest depth $H(x)$ & acceleration of gravity $g = 9.81 \text{ m/s}^2$,

when we scale (1)

$$u = U_0 u', \quad x = L_S x', \quad t = (\omega/U_0) t'$$

$$h = H_0 s h', \quad H = H_0 H'.$$

$$g' \rightarrow g H_0 s / U_0^2,$$

Ques (1) :-

Let's write the non-dimensional form of (1)

$$\frac{U_0 H_{0s}}{L_s} \frac{\partial h'}{\partial t'} + \frac{U_0 H_{0s}}{L_s} \frac{\partial (H' h')}{\partial x'} = 0 \quad \&$$

$$\frac{U_0^2}{L_s} \frac{\partial u'}{\partial t'} + \frac{U_0^2}{L_s} \frac{\partial (g' h')}{\partial x'} = 0$$

$$\frac{\partial h'}{\partial t'} + \frac{\partial (H' h')}{\partial x'} = 0$$

(*)

$$\frac{\partial u'}{\partial t'} + \frac{\partial (g' h')}{\partial x'} = 0$$

(**)

↳ after non-dimensionalisation also we are having same equation as (1).

Except $g' = g \frac{H_{0s}}{U_0^2}$

- Since both equation (*) & (**) are conservation equation (conservation of mass & conservation of momentum respectively).

- Both are coupled, because \dot{u} & \dot{y} influence each other.



we can write these two equations in matrix form

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ H_0 u \end{pmatrix} + A \frac{\partial}{\partial x} \begin{pmatrix} u \\ H_0 u \end{pmatrix} = 0$$

where $A = \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix}$

$$\Rightarrow C_0^2 = gH_0 ; \text{ wave speed}$$

using linear algebra, we can de couple these equation,

finding eigen values & eigen vectors of matrix A

$$\det |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ C_0^2 - \lambda & 0 \end{vmatrix} = \lambda^2 - C_0^2 = 0$$

\therefore eigenvalues are $\lambda_1 = C_0$, $\lambda_2 = -C_0$.

Now finding the eigen vectors

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \quad \& \quad A\vec{v}_2 = \lambda_2 \vec{v}_2$$

$$\begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix} \begin{pmatrix} \vec{v}_{11} \\ \vec{v}_{12} \end{pmatrix} = C_0 \begin{pmatrix} \vec{v}_{11} \\ \vec{v}_{12} \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix} \begin{pmatrix} \vec{v}_{21} \\ \vec{v}_{22} \end{pmatrix} = -C_0 \begin{pmatrix} \vec{v}_{21} \\ \vec{v}_{22} \end{pmatrix}$$

$$\vec{v}_{12} = C_0 \vec{v}_{11}$$

$$C_0^2 \vec{v}_{11} = C_0 \vec{v}_{12}$$



$$C_0^2 \vec{v}_{11} = C_0 (C_0 \vec{v}_{11})$$

$$\Rightarrow \vec{v}_1 = \frac{1}{2C_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{v}_2 = \frac{1}{2C_0} \begin{pmatrix} -1 \\ C_0 \end{pmatrix}$$

\Rightarrow Therefore, Matrix B is consisting eigen vectors is,

$$B = \frac{1}{2C_0} \begin{pmatrix} 1 & -1 \\ C_0 & C_0 \end{pmatrix}$$

$$\Rightarrow \text{Now } B^{-1} = \frac{1}{|B|} \begin{pmatrix} C_0 & -C_0 \\ 1 & 1 \end{pmatrix}^T$$

$$= \frac{1}{|B|} \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix}$$

$$= \frac{1}{\frac{1}{2} (C_0 + C_0)} \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix}$$

& $AB = \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2C_0} & \frac{-1}{2C_0} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{C_0}{2} & -\frac{C_0}{2} \end{pmatrix}$

$\therefore B^{-1}AB = \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{C_0}{2} & -\frac{C_0}{2} \end{pmatrix}$

$$= \begin{pmatrix} C_0 & 0 \\ 0 & -C_0 \end{pmatrix}$$

$$B^{-1}AB \Rightarrow \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \lambda I$$

\Rightarrow Now, we will proceed to decoupling with new variable γ

$$\hookrightarrow \gamma = B^{-1} \begin{pmatrix} h \\ H_0 u \end{pmatrix}^T$$

$$\Rightarrow \gamma = B^{-1} \begin{pmatrix} h \\ \frac{h}{H_0 u} \end{pmatrix}$$

$$\Rightarrow Br = \begin{pmatrix} h \\ H_0 u \end{pmatrix}$$

But $\frac{\partial}{\partial t} \begin{pmatrix} h \\ H_0 u \end{pmatrix} = B \frac{\partial r}{\partial t}$

$$B \frac{\partial}{\partial x} \begin{pmatrix} h \\ H_0 u \end{pmatrix} = B \frac{\partial r}{\partial x}$$

\downarrow

original eqn becomes

$$B \frac{\partial r}{\partial t} + AB \frac{\partial r}{\partial x} = 0 \quad - \text{***}$$

But from above we have, $B^{-1}AB = \lambda I$

\Rightarrow Multiplying B^{-1} in ***;

$$B^{-1} \frac{\partial r}{\partial t} + B^T A B \frac{\partial r}{\partial x} = 0$$

$$\frac{\partial r}{\partial t} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{\partial r}{\partial x} = 0$$

→ decoupled set of linear advection equation

where $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ & $r_1 = H_0 u + C_0 h$
 $r_2 = H_0 u - C_0 h$

($r \rightarrow$ Riemann invariant)

Ques : 2 →

for piecewise constant initial data

$$r_1(x, 0) = \begin{cases} r_{1l} & \text{for } x < 0 \\ r_{1r} & \text{for } x \geq 0 \end{cases}, \quad r_2(x, 0) = \begin{cases} r_{2l} & \text{for } x < 0 \\ r_{2r} & \text{for } x \geq 0 \end{cases}$$

from equation - ⑦

$$\frac{\partial r}{\partial t} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{\partial r}{\partial x} = 0$$

$$\rightarrow \frac{\partial r_1}{\partial t} + c_0 \frac{\partial r_1}{\partial x} = 0 \quad (\lambda_1 = c_0, \lambda_2 = -c_0)$$

$$\& \frac{\partial r_2}{\partial t} - c_0 \frac{\partial r_2}{\partial x} = 0$$



These are linear advection equations,
∴ the solution would be:

$$\begin{cases} r_1(x, t) = r_1(x - c_0 t, 0) \\ r_2(x, t) = r_2(x + c_0 t, 0) \end{cases}$$

$c_0 \rightarrow$ propagation speed.

→ r_1 const. along characteristics
line $x - c_0 t = \text{const}$ &
 r_2 along $x + c_0 t = \text{const}$

Based on the given initial condition

$$\sigma_1(x,t) = \begin{cases} \sigma_{1l} & \text{if } x - c_0 t < 0 \\ \sigma_{1r} & \text{if } x - c_0 t \geq 0 \end{cases}$$

$$\sigma_1(x,t) = \begin{cases} \sigma_{1l} & \text{if } x < c_0 t \\ \sigma_{1r} & \text{if } x \geq c_0 t \end{cases} \quad \text{--- A}$$

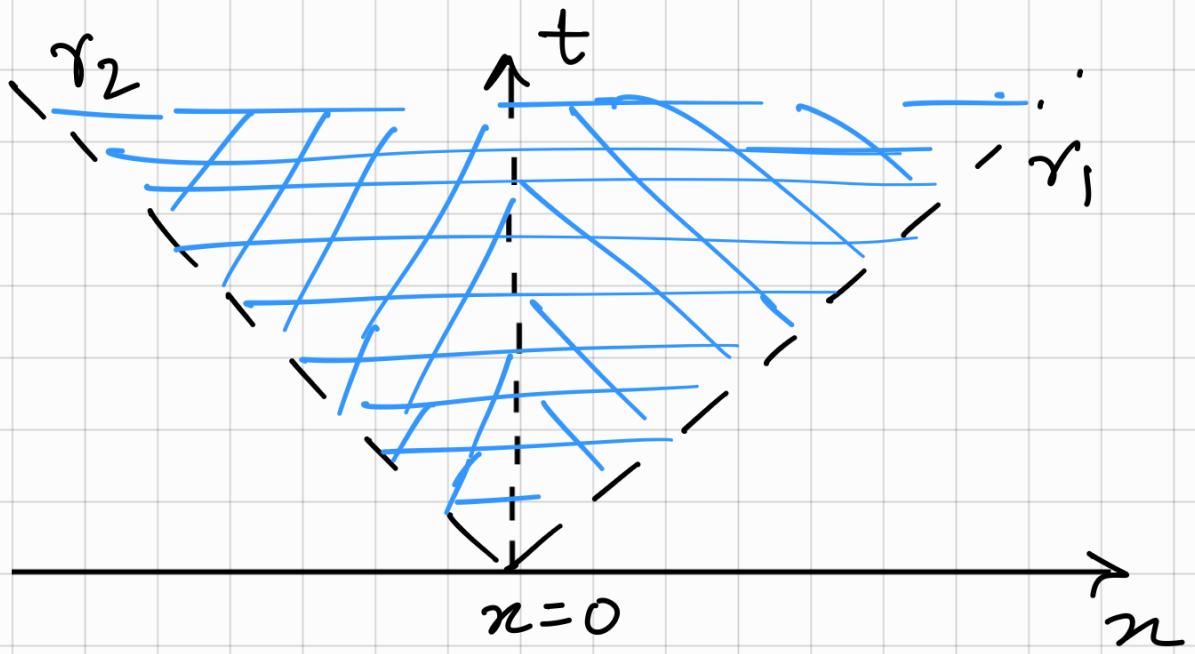
and

$$\sigma_2(x,t) = \begin{cases} \sigma_{2l} & \text{if } x + c_0 t < 0 \\ \sigma_{2r} & \text{if } x + c_0 t \geq 0 \end{cases}$$

$$\sigma_2(x,t) = \begin{cases} \sigma_{2l} , & x \leq -c_0 t \\ \sigma_{2r} , & x \geq -c_0 t \end{cases} \quad \text{--- B}$$

Since the initial discontinuity is at $x=0$, @ $t=0$

if we plot the $\sigma_1(x,t)$ & $\sigma_2(x,t)$



Now :- given that $r_1 = H_0 u + C_0 h$

$$r_2 = H_0 u - C_0 h$$

and $H_0 u = \frac{1}{2} (r_1 + r_2)$,

$$h = \frac{1}{2} (r_1 - r_2) / C_0$$

\Rightarrow as per r_1 & r_2 value given in
 (A) & (B)

(i) for $n < -C_0 t$

$\Rightarrow r_1 = r_1 l$ & $r_2 = r_2 l$

$$\therefore H_0 u(r, t) = \frac{1}{2} (r_1 l + r_2 l)$$

$$= \frac{1}{2} (H_0 u_l + C_0 h_l + H_0 u_l - C_0 h_l)$$

$$H_0 U(x, t) = \frac{1}{2} (2 H_0 U_l) = H_0 U_l$$

② for the shaded part,
 $-C_0 t \leq n \leq C_0 t$

$$\gamma_1 = \gamma_{1l}, \quad \gamma_2 = \gamma_{2s}$$

$$\therefore H_0 U(x, t) = \frac{1}{2} (\gamma_{1l} + \gamma_{2s})$$

③ for $x > C_0 t$

$$\gamma_1 = \gamma_{1s} \quad \& \quad \gamma_2 = \gamma_{2s}$$

$$\therefore H_0 U(x, t) = \frac{1}{2} (\gamma_{1s} + \gamma_{2s})$$

$$= \frac{1}{2} (H_0 U_r + C_0 h_r + H_0 U_r - C_0 h_r)$$

$$H_0 U(x, t) = H_0 U_r$$

$$\Rightarrow H_0 U(x, t) = \begin{cases} H_0 U_l & n \leq C_0 t \\ \frac{1}{2} (\gamma_{1l} + \gamma_{2s}) & -C_0 t \leq n \leq C_0 t \\ H_0 U_r & n > C_0 t \end{cases}$$

Similarly for $U(x, t)$

$$\textcircled{1} \quad n < -C_0 t$$

$$\sigma_1 = \sigma_{1,l}, \quad \sigma_2 = \sigma_{2,r}$$

$$\gamma(x,t) = \frac{1}{2C_0} (H_0 \gamma_l + C_0 h_l - H_0 \gamma_r + C_0 h_r)$$

$$\gamma(x,t) = \gamma_l$$

$$\textcircled{2} \quad \text{for } -C_0 t \leq n \leq C_0 t$$

$$\sigma_1 = \sigma_{1,l} \quad \sigma_2 = \sigma_{2,r}$$

$$\gamma(x,t) = \frac{1}{2C_0} (H_0 \gamma_l + C_0 h_l - H_0 \gamma_r + C_0 h_r)$$

$$\textcircled{3} \quad \text{for } n > C_0 t$$

$$\sigma_1 = \sigma_{1,r} \quad \sigma_2 = \sigma_{2,r}$$

$$\gamma(x,t) = \frac{1}{2C_0} (H_0 \gamma_r + C_0 h_r - H_0 \gamma_l + C_0 h_l)$$

$$\gamma(x,t) = \gamma_r$$

$$\Rightarrow \gamma(x,t) = \frac{1}{2C_0} (\sigma_1(x,t) - \sigma_2(x,t))$$

$$\Rightarrow \gamma(x, t) = \begin{cases} \gamma_d & n < -c_0 t \\ H_0(u_d - u_s) + c_0(u_d + u_s) & -c_0 t \leq n \leq c_0 t \\ \gamma_s & n > c_0 t \end{cases}$$

\Rightarrow Riemann invariant s

$$x_1 = H_0 u + \sqrt{g H_0} \gamma$$

$$x_2 = H_0 u - \sqrt{g H_0} \gamma$$

& as per above:

$$H_0 u = \frac{1}{2}(x_1 + x_2) \quad \& \quad \gamma = \frac{1}{2\sqrt{g H_0}}(x_1 - x_2)$$

so from $\frac{\partial h}{\partial t} + H_0 \frac{\partial u}{\partial x} = 0$

$$\frac{\partial}{\partial t} \left(\frac{1}{2\sqrt{g H_0}} (x_1 - x_2) \right) + H_0 \frac{\partial}{\partial x} \left(\frac{1}{2} (x_1 + x_2) \right) = 0$$

$$\frac{1}{2\sqrt{g H_0}} \left(\frac{\partial x_1}{\partial t} - \frac{\partial x_2}{\partial t} \right) + \frac{1}{2} \left(\frac{\partial x_1}{\partial x} + \frac{\partial x_2}{\partial x} \right) = 0$$

$\rightarrow C$

Similarly for

$$\frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2H_0} (\gamma_1 + \gamma_2) \right) + g \frac{\partial}{\partial x} \left(\frac{1}{2\sqrt{gH_0}} (\gamma_1 - \gamma_2) \right) = 0$$

$$\rightarrow \frac{1}{2H_0} \left(\frac{\partial \gamma_1}{\partial t} + \frac{\partial \gamma_2}{\partial t} \right) + \frac{g}{2\sqrt{gH_0}} \left(\frac{\partial \gamma_1}{\partial x} - \frac{\partial \gamma_2}{\partial x} \right) = 0$$

— (1)

from (1) multiply by $2\sqrt{gH_0}$

$$\left(\frac{\partial \gamma_1}{\partial t} - \frac{\partial \gamma_2}{\partial t} \right) + \sqrt{gH_0} \left(\frac{\partial \gamma_1}{\partial x} + \frac{\partial \gamma_2}{\partial x} \right) = 0$$

for $\gamma_1 \rightarrow \frac{\partial \gamma_1}{\partial t} + \sqrt{gH_0} \frac{\partial \gamma_1}{\partial x} = 0$

$\gamma_2 \rightarrow \frac{\partial \gamma_2}{\partial t} - \sqrt{gH_0} \frac{\partial \gamma_2}{\partial x} = 0$

↳ uncoupled

γ_1 & γ_2 do not depend on each other,

⇒ characteristic lines for σ_1 , would be

$$x - \sqrt{gH_0 t} = \text{const}$$

for σ_2

$$x + \sqrt{gH_0 t} = \text{const}$$

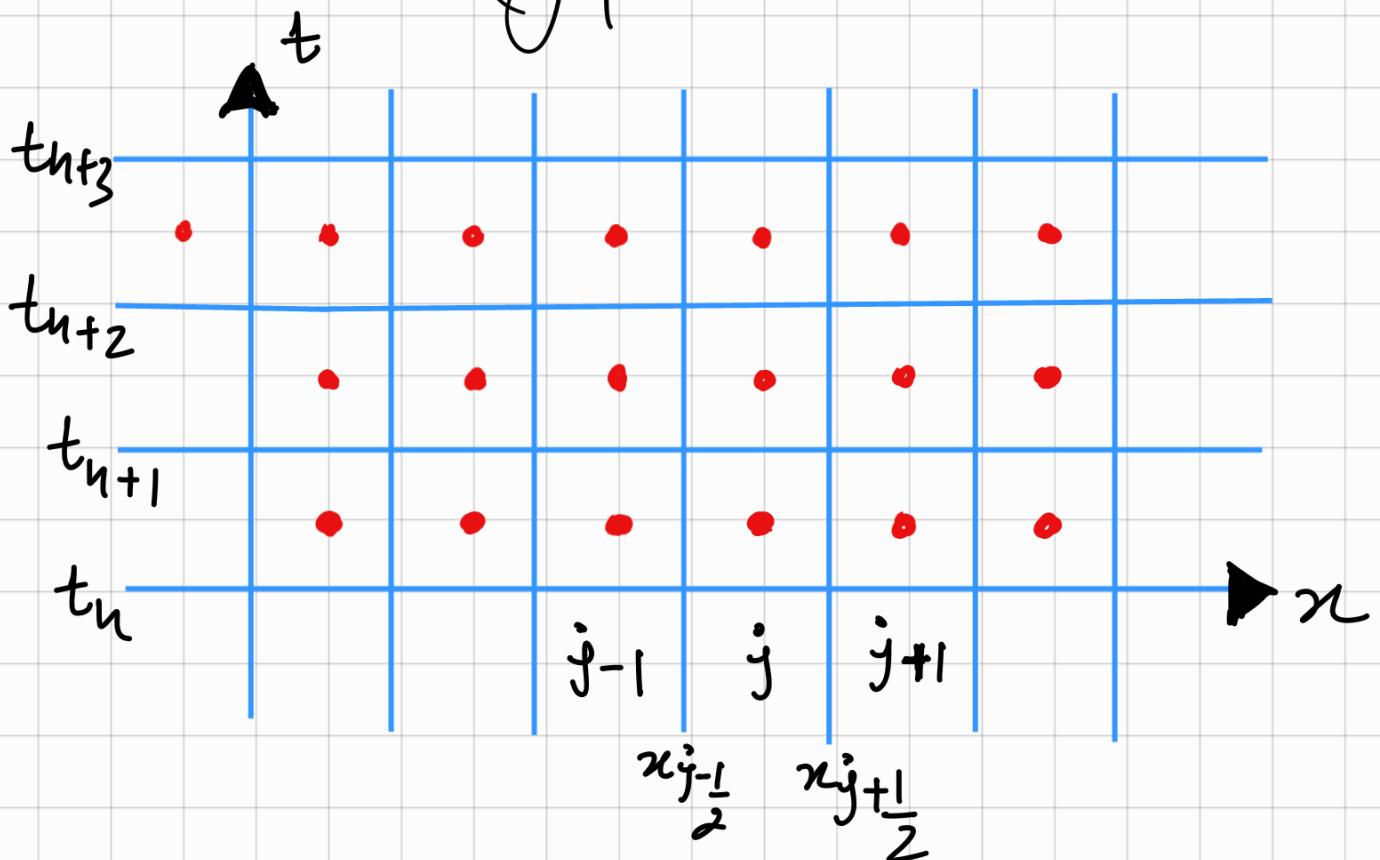
Question 3:-

Godunov scheme for question ①:-

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

where $u(x,t) = \begin{pmatrix} h \\ H_0 u \end{pmatrix}$

& $f(u) = \begin{pmatrix} Hu \\ g_h \end{pmatrix} \rightarrow \text{flux}$



- In Godunov Scheme: we compute numerical flux at cell interface
- It ensures the conservation across the

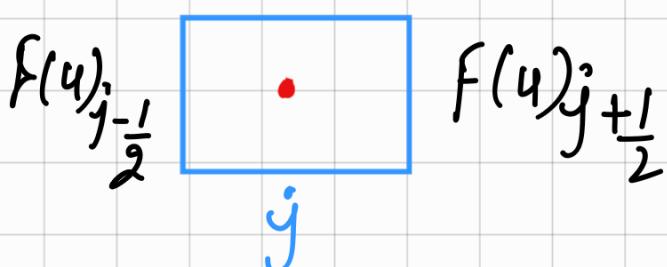
domain.

Now

$$\frac{\partial}{\partial t} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t) dx + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{\partial f(u)}{\partial x} dx = 0$$

$$\Rightarrow \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{\partial f(u)}{\partial x} dx = F(u) \Big|_{x_{j+\frac{1}{2}}} - F(u) \Big|_{x_{j-\frac{1}{2}}}$$

$$\therefore \frac{\partial}{\partial t} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t) dx + \left[F(u)_{j+\frac{1}{2}} - F(u)_{j-\frac{1}{2}} \right] = 0$$



Now :- Cell averaged quantity :-

$$U_j(t) = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t) dx$$

divided by dt (rate of change)

$$\frac{U_j(t)}{dt} = -\frac{1}{Dn} \left(f(u)_{j+\frac{1}{2}}^n - f(u)_{j-\frac{1}{2}}^n \right) \quad A$$

\downarrow represents the balance between the change in conserved quantity U_j inside the cell & the net flux across the boundaries

\Rightarrow Now, discretize the conserved quantity in time

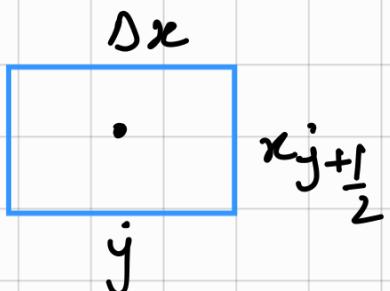
$$\frac{dU_j}{dt} \approx \frac{U_j^{n+1} - U_j^n}{Dt}$$

Substitute in eqn A

$$\frac{U_j^{n+1} - U_j^n}{Dt} = -\frac{1}{Dn} \left[f(u)_{j+\frac{1}{2}}^n - f(u)_{j-\frac{1}{2}}^n \right]$$

$$U_j^{n+1} = U_j^n - \frac{Dt}{Dn} \left[f(u)_{j+\frac{1}{2}}^n - f(u)_{j-\frac{1}{2}}^n \right]$$

→ Now to compute the fluxes at interface
 we will solve the Riemann problem at
 the interface between two cell



if we use the Riemann problem at
 the interface $x_{j+\frac{1}{2}}$ between two cell, as per
 initial condition given in ques(2)

$$\gamma_1(x, 0) = \begin{cases} \gamma_{1l} & x < 0 \\ \gamma_{1r} & x \geq 0 \end{cases}$$

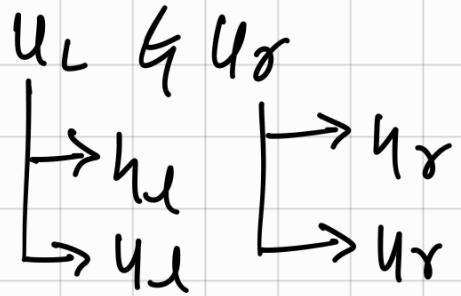
$$\gamma_2(x, 0) = \begin{cases} \gamma_{2l} & x < 0 \\ \gamma_{2r} & x \geq 0 \end{cases}$$

where

$l \rightarrow$ left side of the interface

$r \rightarrow$ right side of the interface

In general :-



- The numerical flux $f_{j+\frac{1}{2}}$ is computed from

$$f_{j+\frac{1}{2}} = f(u_L) + \sum_{j: \lambda_j^+ > 0} \lambda_j^+ w_j - \sum_{j: \lambda_j^- < 0} \lambda_j^- w_j$$

where,

$w_j \rightarrow$ wave strength
 $\lambda_j^+ \rightarrow$ eigen value

$$\Rightarrow u(x, 0) = \begin{cases} u_L & n < x_{j+\frac{1}{2}} \\ u_R & n > x_{j+\frac{1}{2}} \end{cases}$$

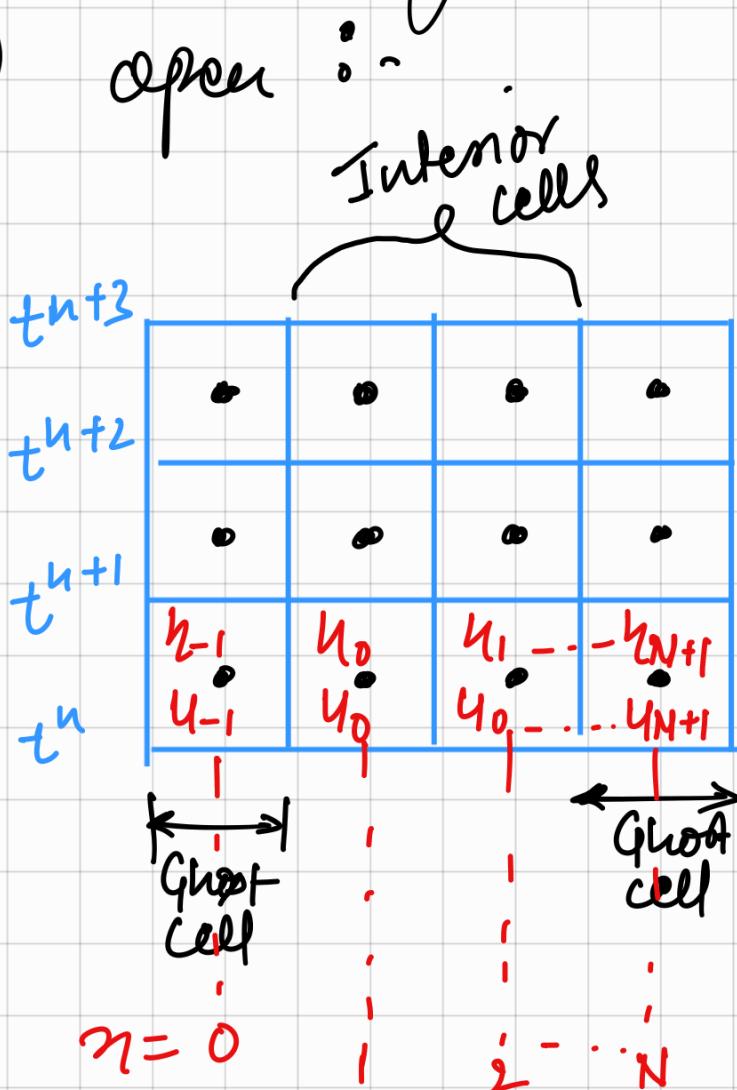
Assuming wave propagating at constant speed.

Now at interface,

$$\Delta U = U_f - U_l$$

\Rightarrow Extrapolating the Boundary condition

(1)



at $n=0$, left boundary ($n=0$)

$$\Leftrightarrow h_0 = h_{-1} \quad \& \quad u_{-1} = u_0$$

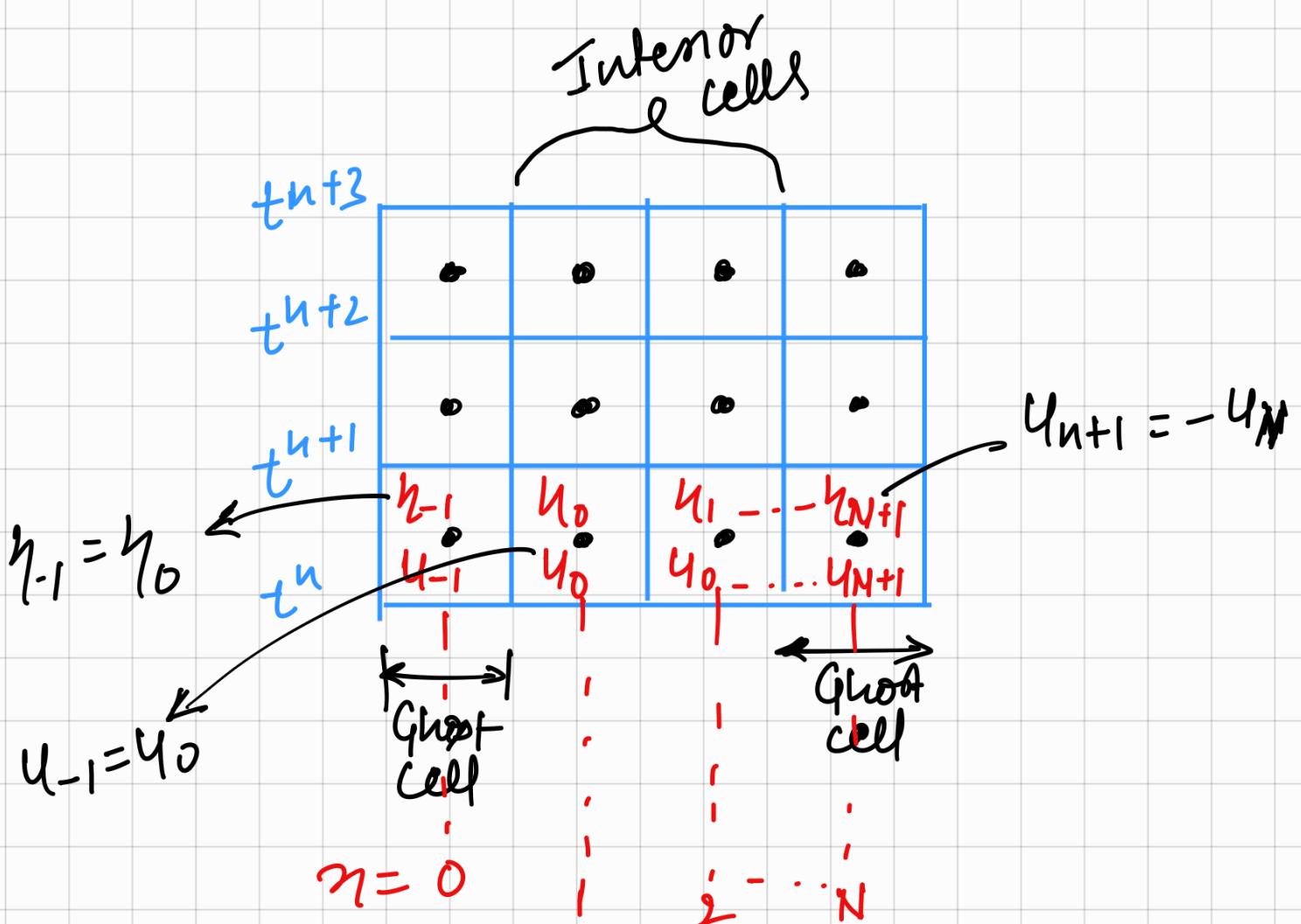
& for first interior cell
 h_0 & u_0

at $n=N$ (right boundary)

$$h_{N+1} = h_N \quad \& \quad u_{N+1} = u_N$$

② closed :-

- No flux pass through boundary
- Taking $u_{-1} = -u_0$ (-ve velocity)
 $h_{-1} = -h_0$
- $f_{1/2} = 0 \quad \& \quad f_{N+1/2} = 0$



\Rightarrow for time steps estimation,
restricting CFL condition

$$\Delta t \leq \frac{\Delta x}{\max|\lambda|}$$

$\lambda_1, \lambda_2 \rightarrow$ eigen values

\Rightarrow Now if we consider $H(x)$ is continuous
then,

To solve the Riemann problem, we will
use "locally approximately constant" $H(x)$ at
each cell edge



To achieve it \rightarrow finer grid resolution
will be required

\therefore at cell edge, $x_{j+\frac{1}{2}}$

approximating $H(x_{j+\frac{1}{2}})$ as

$$H_{j+\frac{1}{2}} = \frac{H_j + H_{j+1}}{2}$$

where H_j & $H_{j+1} \rightarrow$ depth in all j-alet cell.



Eigen values will change accordingly,

$$\lambda_1 = \sqrt{gH(x)}, \quad \lambda_2 = -\sqrt{gH(x)}$$

Similarly eigen vector will also be change



These changes will lead to flux at cell edges!

Hence, to solve the flux at $(x_{j+\frac{1}{2}})$

we have to use local $H(x_{j+\frac{1}{2}})$

⇒ Similarly, time steps will also be affected.

