# Checking exact solution

$u_e(x,y) = \sin(\pi x)\cos(\pi y)$

$$-\partial^2/\partial x^2\, u_e - \partial^2/\partial y^2\, u_e = -\left[-\pi^2\sin(\pi x)\cos(\pi y) - \pi^2\sin(\pi x)\cos(\pi y)\right]$$
$$= 2\pi^2\sin(\pi x)\cos(\pi y)$$
$$= f$$

So $u_e$ satisfies Poisson's eq.

$u_e(0,y) = \sin(0)\cos(\pi y) = 0$, $u_e(1,y) = \sin(\pi)\cos(\pi y) = 0$

So $u_e$ satisfies Dirichlet BCs.

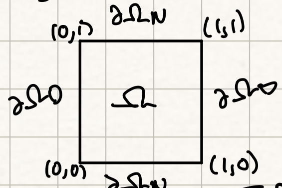
$\partial u_e/\partial y = -\pi\sin(\pi x)\sin(\pi y)$. Then

$$\partial u_e/\partial y\,|_{y=0} = -\pi\sin(\pi x)\sin(0) = 0$$
$$\partial u_e/\partial y\,|_{y=1} = -\pi\sin(\pi x)\sin(\pi) = 0$$

So $u_e$ satisfies Neumann BCs.

$\boxed{\text{Q1}}$ Step 1 of FEM: Derive the weak formulation using variations.

- Domain $\Omega = [0,1]\times[0,1]$.
- Dirichlet boundary conditions $u(0,y) = u(1,y) = 0$ on $\partial\Omega_D = \{0\}\times[0,1] \cup \{1\}\times[0,1]$
- Neumann boundary conditions $\partial u/\partial y|_{y=0} = \partial u/\partial y|_{y=1} = 0$ on $\partial\Omega_N = [0,1]\times\{0\} \cup [0,1]\times\{1\}$
- $\partial\Omega = (\partial\Omega_D \cup \partial\Omega_N - \partial\Omega_D \cap \partial\Omega_N)$ and $\partial\Omega_D \cap \partial\Omega_N = \{(0,0), (0,1), (1,0), (1,1)\}$.




Poisson's equation is in the form $I[u(x,y)] = \int_\Omega -\nabla^2 u - f \, d\Omega$.
We seek to find the extremum of $I$, that is, when $u(x,y)$ is a $(x,y)$ stationary function of $I$ (minimalisation). Consider $\hat{u}$ st
$$\hat{u}(x,y) = u(x,y) + \delta u(x,y)$$
where $\hat{u}$ is close to stationary function $u$ by small parameter $\delta u$, the variation.

We use the variation $\delta u(x,y)$, where $\delta u = \partial u/\partial \epsilon|_{\epsilon=0}\,\epsilon$ for some small parameter $\epsilon$, st. $\delta u$ is very close to $u$.
We have also that $\delta u(0,y) = \delta u(1,y) = 0$, ie $\delta u$ satisfies Dirichlet BCs.

Converting $-\nabla^2 u = f$ to its variational form by multiplying by $\delta u$ and integrating over $\Omega$:
$$-\int_\Omega \delta u \nabla^2 u \, d\Omega = \int_\Omega \delta u f \, d\Omega$$

ie

$$-\int_0^1\int_0^1 \delta u \nabla^2 u \, dx dy = \int_0^1\int_0^1 \delta u f \, dx dy$$

where $$-\int_\Omega \delta u \nabla^2 u \, d\Omega = -\int_\Omega \nabla\cdot(\delta u \nabla u) - \nabla\delta u \cdot \nabla u \, d\Omega$$

$$= -\int_{\partial\Omega} \hat{n}\cdot\delta u \nabla u \, d\Omega - \int_\Omega \nabla\delta u\cdot\nabla u \, d\Omega \; :$$

$$= -\int_{\partial\Omega_D} \hat{n}\cdot\delta u\nabla u \, d\Omega - \int_{\partial\Omega_N} \hat{n}\cdot\delta u\nabla u \, d\Omega + \int_{\partial\Omega_N\cap\partial\Omega_D} \hat{n}\cdot\delta u\nabla u \, d\Omega - \int_\Omega \nabla\delta u\cdot\nabla u \, d\Omega$$

And $\int_{\partial\Omega_D} \hat{n}\cdot\delta u\nabla u \, d\Omega = 0$ as $\delta u \equiv 0$ on $\partial\Omega_D$,

$\int_{\partial\Omega_N} \hat{n}\cdot\delta u\nabla u \, d\Omega = 0$ as $\hat{n}\cdot\nabla u = \partial u/\partial y \equiv 0$ on $\partial\Omega_N$ by def of the Neumann BC. where $\hat{n}$ is the unit normal.

$\int_{\partial\Omega_D\cap\partial\Omega_N} \hat{n}\cdot\delta u\nabla u \, d\Omega = 0$ as $\hat{n}\cdot\nabla u \equiv 0$, $\delta u \equiv 0$ on $\partial\Omega_D \cap \partial\Omega_N$.

Hence we have reduced the order of the derivative for the <u>weak form</u>:

$$-\int_\Omega \nabla\delta u\cdot\nabla \underline{u} \, d\Omega = \int \delta u f \, d\Omega$$

And hence the variational form, ie the <u>Ritz-Galerkin principle</u>, becomes; as $\delta(\nabla u\cdot\nabla u) = 2\nabla\delta u\cdot\nabla u$,

$$-\int_\Omega \nabla\delta u\cdot\nabla u - \delta(uf) \, d\Omega = -\int_\Omega \tfrac{1}{2}\delta(\nabla u\cdot\nabla u) - \delta(uf) \, d\Omega$$

$$= -\delta\int_\Omega \tfrac{1}{2}(\nabla u\cdot\nabla u) - uf \, d\Omega$$

$$= -\delta\int_\Omega \tfrac{1}{2}|\nabla u|^2 - uf \, d\Omega$$

$$= -\delta\int_\Omega \tfrac{1}{2}|\nabla u|^2 - u(2\pi^2\sin(\pi x)\cos(\pi y)) \, d\Omega$$

where $d\Omega = dx dy$.

$$= \delta[I] = 0$$

for functional $I$.

$\underline{w(x,y) = \partial u(x,y)}$

In the finite element method, we choose our test function $w$ from the Hilbert space:

$$H_0^1(\Omega) = \{ v \in L_2(\Omega) : \partial v/\partial x, \partial v/\partial y \in L^2(\Omega) \wedge v|_{\partial\Omega_D} = 0 \}$$

$L^2(\Omega)$ is the Lebesgue space defined over norm $\|\cdot\|_{H_0^1}$ where

$$\langle u, v \rangle = \int_\Omega u(x,y)\, v(x,y)\, d\Omega$$

is the inner product on $H_0^1$, ie

$$L^2(\Omega) = \{ v \in \Omega : \int_\Omega |v|^2 \, d\Omega < \infty \}$$

of square integrable $v \in \Omega$.

As $\partial u(x)$ and its derivatives are square integrable from its derivation from $u$, and $\partial u(x,y) \equiv 0$ on $\partial\Omega_D$ by definition,

$$\partial u(x,y) \in H_0^1.$$

As we choose test function $w$ arbitrarily from $H_0^1$, we can choose $w(x,y) = \partial u(x,y)$. Then the weak form becomes:

$$-\int_\Omega \nabla w \cdot \nabla u \, d\Omega = \int w f \, d\Omega$$

$\boxed{Q2}$ Step 2 of FEM: Form a discretised system by expanding variables in the domain using basis functions.

For $\Omega = [0,1] \times [0,1]$ we will partition $\Omega$ into $N$ elements, where $N$ is chosen depending on the fineness of the mesh. Each element is of the form:

$$K_k = \{x : x \in (x_k, x_{k+1})\}$$

where $k = 1, \ldots, N+1$. Clearly $K_k \cap K_{k'} = \emptyset$ for all $k \neq k'$, $k, k' = 1, \ldots, N+1$, and $\bar{\Omega} = \bigcup_k \bar{K_k}$ by construction. Then we have tessellation $\mathcal{T}_h$ st

$$\mathcal{T}_h = \{K_k : k = 1, \ldots, N+1\}$$ (not 100% sure on the indices)

We approximate $u(x)$ using global basis functions $\phi_i$ st:

$$u(x) \approx u_h(x) = \sum_{i=0}^{M} c_i \phi_i(x,y)$$

where $M$ is the number of nodes per element (that is, $M=3$ for quadrilateral meshes, $M=2$ for triangular meshes).

The requirements for $\phi_i$ are that $\phi_i \in L^2(\Omega)$, and $\phi_i(0,y) = \phi_i(1,y) = 0$, that is, they are homogeneous on the Dirichlet boundary. $\phi_i$ must also have compact support, that is, $\phi_i \equiv 0$ on neighbouring elements.

* Note that $\phi_i$ are usually polynomials — in later questions, the degree of these polynomials corresponds to $p$.

* We will not go into the definition of compactness — for

$$\text{Supp}\, \phi_i = \overline{\{(x,y) \in \Omega : \phi_i(x,y) \neq 0\}}$$

compactness of Supp $\phi_i$ very, very generally means supp $\phi_i$ is closed and bounded.

* Qu2 continued on following page

# Discretised Ritz-Galerkin Principle

We introduce $u \approx u_h$ into the Ritz-Galerkin principle to discretise it:

$$0 = -\partial \int_\Omega \tfrac{1}{2}\left|\nabla \sum_{i=0}^{M} c_i \phi_i\right|^2 - \sum_{i=0}^{M} c_i \phi_i \, 2\pi^2 \sin(\pi x)\cos(\pi y)\, d\Omega$$

$$= -\partial \int_\Omega \nabla \sum_{i=0}^{M} c_i \phi_i \cdot \nabla \sum_{i=0}^{M} c_i \phi_i - \sum_{i=0}^{M} c_i \phi_i \, 2\pi^2 \sin(\pi x)\cos(\pi y)\, d\Omega$$

$$= -\partial \int_\Omega \sum_{i=0}^{M} c_i^2 \nabla \phi_i \cdot \nabla \phi_i - \sum_{i=0}^{M} c_i \phi_i \, 2\pi^2 \sin(\pi x)\cos(\pi y)\, d\Omega$$

(Division by $c_i$) $$= -\partial \sum_{i=0}^{M} \int_\Omega c_i \nabla \phi_i \cdot \nabla \phi_i - \phi_i \, 2\pi^2 \sin(\pi x)\cos(\pi y)\, d\Omega$$

$$= \sum_{i=0}^{M} \int_\Omega c_i \nabla \partial \phi_i \cdot \nabla \phi_i - \partial \phi_i \, 2\pi^2 \sin(\pi x)\cos(\pi y)\, d\Omega$$

★ Unsure:

Then we can choose weight functions equal to $\partial \phi_i$ and $\phi_i$, as $\partial\phi_i, \phi_i \in L^2(\Omega)$ and $\partial\phi_i, \phi_i \in H_0^1$ by choice of $\phi_i$ and definition of the variation. Then

$$\partial \phi_i = w_i, \qquad \phi_i = w_j$$

where $w_i$ is not necessarily equal to $w_j$. So we have:

**Eq 1** $$0 = \sum_{K \in T_h} c_i \int_\Omega \nabla w_i \cdot \nabla w_j - w_j \, 2\pi^2 \sin(\pi x)\cos(\pi y)\, d\Omega.$$

where the summation has changed to include summation over $w_j$ & over the elements of the mesh.

Then Eq 1 can be given as

$$c_i A_{ij} = b_j$$

★ Not sure about indices?

where matrix $A_{ij} = \sum_{K \in T_h} \int_\Omega \nabla w_i \cdot \nabla w_j \, d\Omega$,

vector $b_j = \sum_{K=} \int_\Omega w_j \, 2\pi^2 \sin(\pi x)\cos(\pi y)\, d\Omega$.

and we can find the coefficients $c_i$ by solving $c_i A_{ij} = b_j$

## Discretised Weak Form

Again, we introduce $u \approx u_h$ into the weak form given in Q1. That is,

$$-\int_\Omega \nabla w_i \nabla \sum_{j=0}^{M} c_j \phi_j \, d\Omega = \int_\Omega w_i \, 2\pi^2 \sin(\pi x)\cos(\pi y) \, d\Omega$$

$$\Rightarrow \quad -\int_\Omega \sum_{j=0}^{M} \nabla w_i \cdot \nabla c_j \phi_j \, d\Omega = \int_\Omega w_i \, 2\pi^2 \sin(\pi x)\cos(\pi y) \, d\Omega$$

$$\Rightarrow \quad -\sum_{K \in T_h} \int_\Omega \nabla w_i \cdot \nabla c_j \phi_j \, d\Omega = \sum_{K \in T_h} \int w_i \, 2\pi^2 \sin(\pi x)\cos(\pi y) \, d\Omega .$$

where $\sum_{K \in T_h}$ sums over $c_j \phi_j$ and the $w_i$. Then as we can choose $\phi_j = w_j$ where $w_j$ is a weight function not necessarily equal to $w_i$. Then:

$$-\sum_{K \in T_h} c_j \int_\Omega \nabla w_i \cdot \nabla w_j \, d\Omega = \sum_{K \in T_h} \int w_i \, 2\pi^2 \sin(\pi x)\cos(\pi y) \, d\Omega$$

ie

$$\sum_{K \in T_h} \int_\Omega -(c_j \nabla w_i \cdot \nabla w_j) - w_i \, 2\pi^2 \sin(\pi x)\cos(\pi y) \, d\Omega$$

$= \underline{\underline{Eq\ 1}}$

so the variation of the weak form (ie the Ritz-Galerkin principle) yields the same equation as the weak form.

As in the discretised Ritz-Galerkin principle, we have

$$c_j A_{ij} = b_i$$

indices?

where

$$A_{ij} = \sum_{K \in T_h} -\int_\Omega \nabla w_i \cdot \nabla w_j \, d\Omega$$

$$b_i = \sum_{K \in T_h} \int_\Omega w_i \, 2\pi^2 \sin(\pi x)\cos(\pi y) \, d\Omega .$$

and we can solve this for coefficients $c_j$.