Numerical Exercises 3 ~ James Dunstan

1.) We are given the 2-dimensional Poisson System

$$-\nabla^2 u = f \quad , \text{ on } \Omega = (x,y) \in [0,1]^2$$

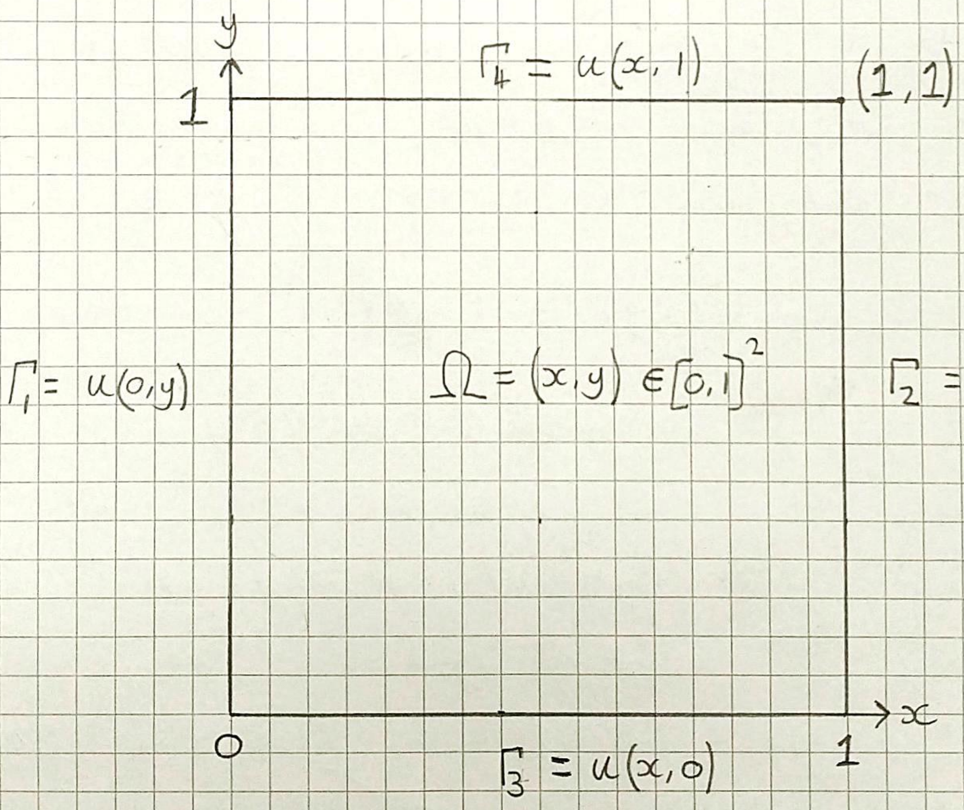
for the given function $f(x,y) = 2\pi^2 \sin(\pi x)\cos(\pi y)$ and unknown $u = u(x,y)$. Dirichlet boundary conditions are given by:

$$u(0,y) = u(1,y) = 0 \, ,$$

and Neumann boundary conditions are given by:

$$\partial_y u(x,y)\Big|_{y=0} = \partial_y u(x,y)\Big|_{y=1} = 0 \, .$$

We can sketch the problem domain as follows



and the exact solution is given by $u(x,y) = \sin(\pi x)\cos(\pi y)$ since

$$\begin{aligned}
\nabla^2 u &= -\pi^2 \sin(\pi x)\cos(\pi y) - \pi^2 \sin(\pi x)\cos(\pi y) \, , \\
&= -2\pi^2 \sin(\pi x)\cos(\pi y) \, , \\
&= -f(x,y) \, ,
\end{aligned}$$

and the boundary conditions are satisfied since there always exists a $\sin(\pi x)$ or $\sin(\pi y)$ term that is zero at $x=0$ and $x=1$ or $y=0$ and $y=1$.

Knowing the exact solution we can then use the Finite Element Method to numerically approximate this solution using the Ritz-Galerkin method. First, we derive an integral formulation of the system.

- As it turns out the given Poisson system can be derived via the minimisation problem:

$$\min_{u \in \Omega} F[u] \quad ; \quad F[u] = \int_0^1\int_0^1 \tfrac{1}{2}|\nabla u|^2 - uf \, dx\,dy$$

Where F is a functional and so not a function of x or y, and $u(x,y)$ must still follow the Dirichlet boundary conditions. We consider a class of functions

$$u(x,y) = \tilde{u}(x,y) + \varepsilon\, \delta u(x,y) \quad , \quad \text{where}$$

ε is a variable parameter, $\tilde{u}(x,y)$ is a smooth solution of the minimisation problem with some variation $\delta u(x,y)$ which is fixed. Assuming both $\tilde{u}$ and u satisfy the boundary conditions then we must also have $\delta u(0,y) = \delta u(1,y) = 0$. Now, a necessary condition for the existence of a minimal solution is that

$$\frac{d F[u]}{d\varepsilon} = 0 \quad , \quad \text{and so}$$

$$0 = \frac{d}{d\varepsilon}\left[\int_0^1\int_0^1 \tfrac{1}{2}|\nabla(\tilde{u}+\varepsilon\delta u)|^2 - (\tilde{u}+\varepsilon\,\delta u) f \, dx\,dy\right] ,$$

$$0 = \int_0^1\int_0^1 \frac{\partial}{\partial \epsilon}\left[\frac{1}{2}\left(\frac{\partial(\tilde{u}+\epsilon\delta u)}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial(\tilde{u}+\varepsilon\delta u)}{\partial y}\right)^2\right] - \frac{\partial}{\partial \varepsilon}\left[(\tilde{u}+\varepsilon\delta u) f\right] dx\,dy ,$$

$$0 = \int_0^1\int_0^1 \nabla(\tilde{u}+\varepsilon\delta u)\cdot\nabla(\delta u) - f\,\delta u \; dx\,dy ,$$

and since $u = \tilde{u} + \varepsilon\,\delta u$ implies this is minimised for $\varepsilon = 0$ we reduce the equation to

$$0 = \int_0^1\int_0^1 \nabla\tilde{u}\cdot\nabla\delta u - f\,\delta u \; dx\,dy . \qquad \langle * \rangle$$

By peforming integration by parts on the first and second (but not third!) terms with respect to x and y respectively we find

$$0 = \int_0^1 \left[\frac{\partial\tilde{u}}{\partial x}\delta u\right]_{x=0}^{x=1} - \int_0^1 \delta u\,\frac{\partial^2\tilde{u}}{\partial x^2}\,dx \quad dy$$

$$+ \int_0^1 \left[\frac{\partial\tilde{u}}{\partial y}\delta u\right]_{y=0}^{y=1} - \int_0^1 \delta u\,\frac{\partial^2\tilde{u}}{\partial y^2}\,dy \quad dx$$

$$- \int_0^1\int_0^1 f\,\delta u\; dx\,dy$$

and remembering $\frac{\partial\tilde{u}}{\partial y}(x,0) = \frac{\partial\tilde{u}}{\partial y}(x,1) = \delta u(0,y) = \delta u(1,y) = 0$ we recover

$\int_0^1\int_0^1 \delta u\left(-\nabla^2\tilde{u} - f\right) dx\,dy = 0$, which by the lemma of Dubois-Reymond must satisfy $-\nabla^2\tilde{u} = f$, proving the derivation. This derivation also gives us a weak form of the differential equation given by <*> which can also be derived using an arbitrary test function $w(x,y)$ to lower the order of the $u(x,y)$ derivates by switching them onto $w(x,y)$. Hence, let us multiply the strong form of the system by the test function and integrate over domain Ω:

$$-\int_0^1\int_0^1 w\nabla^2 u \, dy\,dx = \int_0^1\int_0^1 wf \, dy\,dx ,$$

$$-\int_0^1\int_0^1 \nabla\cdot(w\nabla u)\, dx\,dy + \int_0^1\int_0^1 \nabla w\cdot\nabla u \, dx\,dy = \int_0^1\int_0^1 wf \, dy\,dx ,$$

and by Gauss' theorem

$$\int_0^1\int_0^1 \nabla w\cdot\nabla u \, dy\,dx - \oint_\Gamma w\nabla u\cdot\hat{n}\, d\Gamma = \int_0^1\int_0^1 wf \, dy\,dx .$$

Now, looking at the $\oint w\nabla u\cdot\hat{n}\, d\Gamma$ term on the boundary and recalling our sketch from earlier we can observe for $\Gamma_1 \Rightarrow \hat{n} = (-1,0)^T$; for $\Gamma_2 \Rightarrow \hat{n} = (1,0)^T$; for $\Gamma_3 \Rightarrow \hat{n} = (0,-1)^T$ and for $\Gamma_4 \Rightarrow \hat{n} = (0,1)^T$. Hence for each case on the boundary we have

on Γ_1 : $\nabla u\cdot\hat{n} = -\dfrac{\partial u}{\partial x}$, but $u(0,y) = 0$ and therefore $\dfrac{\partial u}{\partial x} = 0$ on this boundary segment.

on Γ_2 : $\nabla u\cdot\hat{n} = \dfrac{\partial u}{\partial x} = 0$ by similarity to Γ_1.

on Γ_3 : $\nabla u \cdot \hat{n} = -\frac{\partial u}{\partial y}$, but $\partial_y \, u(x,y)\big|_{y=0} = 0$

and so $\frac{\partial u}{\partial y} = 0$ on this boundary segment.

on Γ_4 : $\nabla u \cdot \hat{n} = \frac{\partial u}{\partial y} = 0$ by similarity to Γ_3.

The boundary integral thus vanishes leaving the weak formulation

$$\int_0^1 \int_0^1 (\nabla u \cdot \nabla w - wf) \, dy \, dx = 0 \qquad \langle \square \rangle$$

where clearly $\langle \square \rangle$ is the same as $\langle * \rangle$ with the test function $w(x,y) = \delta u(x,y)$.

2.) Introducing a Finite Element method Expansion to discretise the problem we let

$$u(x,y) \approx u_h(x,y)$$

where $u_h(x,y)$ is the approximation of the unknown solution by a finite linear combination of basis functions: $u_h(x,y) = \sum_{j=1}^{N_n} \hat{u}_j \, \phi_j(x,y)$, where

$\hat{u}_j$ are coefficients and $\phi_j(x,y)$ are the proposed basis functions. Note 'N_n' is the number of nodes in the 2-dimensional space, one for each basis function. Now since we have Dirichlet boundary conditions then the coefficients on boundary nodes are zero, thus our approximation becomes

$$u_h(x,y) = \sum_{j=1}^{N_n - N_{boundary}} \hat{u}_j \, \phi_j(x,y) .$$

Returning to the minimisation problem but now in discrete form we have

$$\min_{u_h \in \Omega} F[u_h] \quad ; \quad F[u_h] = \int_0^1\int_0^1 \frac{1}{2}\sum_{j=1}^{N_n}\sum_{i=1}^{N_n} \hat{u}_j \nabla\phi_j \cdot \hat{u}_i \nabla\phi_i \, dx\,dy$$
$$- \int_0^1\int_0^1 f \sum_{j=1}^{N_n} \hat{u}_j \phi_j \, dx\,dy .$$

Again we can use a method of variations to derive a discrete weak formulation. Assuming we have a class of functions for each $\hat{u}_j$ so that

$$\hat{u}_j = \tilde{u}_j + \varepsilon\,\delta u_j \quad ,$$ then the minimisation problem becomes

$$\frac{\partial F[U_h]}{\partial \varepsilon} = 0 \quad ,$$

$$\boxed{\frac{\partial}{\partial\varepsilon}}\left(\sum_{i=1}^{N_n}\sum_{j=1}^{N_n} \frac{\hat{u}_i \hat{u}_j}{2} \int_0^1\int_0^1 \nabla\phi_i \cdot \nabla\phi_j \, dx\,dy - \sum_{j=1}^{N_n} \hat{u}_j \int_0^1\int_0^1 f\phi_j \, dx\,dy\right) = 0,$$

and so defining $A_{ij} = \int_0^1\int_0^1 \nabla\phi_i \cdot \nabla\phi_j \, dx\,dy$ and

$b_j = \int_0^1\int_0^1 f\phi_j \, dx\,dy$ we have the problem

$$\frac{\partial}{\partial\varepsilon}\left[\frac{1}{2}\sum_{i=1}^{N_n}\sum_{j=1}^{N_n} \hat{u}_i \hat{u}_j A_{ij} - \sum_{j=1}^{N_n} \hat{u}_j b_j\right] = 0 . \quad \text{where } \hat{U}_j = \tilde{u}_j + \varepsilon\delta u_j .$$

This leads to

$$\frac{\partial}{\partial\varepsilon}\left[\frac{1}{2}\sum_{i=1}^{N_n}\sum_{j=1}^{N_n} (\tilde{u}_i + \varepsilon\delta u_i)(\tilde{u}_j + \varepsilon\delta u_j) A_{ij} - \sum_{j=1}^{N_n} (\tilde{u}_j + \varepsilon\delta u_j) b_j\right] = 0,$$

$$\frac{\partial}{\partial\varepsilon}\left[\frac{1}{2}\sum_{i=1}^{N_n}\sum_{j=1}^{N_n}\left(\tilde{u}_i\tilde{u}_j + \varepsilon(\tilde{u}_i\delta u_j + \tilde{u}_j\delta u_i) + \varepsilon^2 \delta u_i \delta u_j\right) A_{ij} - \sum_{j=1}^{N_n} \tilde{u}_j b_j + \varepsilon\delta u_j b_j\right] = 0.$$

Taking the epsilon derivative and again remembering that $\varepsilon = 0$ minimises the problem, since $\hat{u}_j = \tilde{u}_j + \varepsilon \delta u_j$ and the $\tilde{u}_j$ terms are assumed to be minimal solutions already, we have

$$\left[\frac{1}{2}\sum_{i=1}^{N_n}\sum_{j=1}^{N_n}\left(\tilde{u}_i \delta u_j + \tilde{u}_j \delta u_i\right) A_{ij} - \sum_{j=1}^{N_n} \delta u_j b_j\right] = 0 .$$

Since $A_{ij} = \int_0^1\int_0^1 \nabla\phi_i \cdot \nabla\phi_j \, dx dy$ it is clear this matrix is symmetric and so $A_{ij} = A_{ji}$. The above equation can consequently be reduced to

$$\frac{1}{2}\sum_{i=1}^{N_n}\sum_{j=1}^{N_n} 2\tilde{u}_i \delta u_j A_{ij} - \sum_{j=1}^{N_n} \delta u_j b_j = 0 ,$$ and factorising out δu_j we have

$$\sum_{j=1}^{N_n}\left(\sum_{i=1}^{N_n} A_{ij}\tilde{u}_i - b_j\right)\delta u_j = 0 .$$

Since δu_j is arbitrary we can again deduce that $\sum_{i=1}^{N_n} A_{ij}\hat{u}_i = b_j$ for all $j = 1, 2, \ldots N_n$.

Now, Returning to the weak formulation $\langle \square \rangle$ and discretising we can notice that the test function $w(x,y)$ must be in the same space as the discretised $u_h(x,y) = \sum_{j=1}^{N_n} \hat{u}_j \phi_j(x,y)$ and so we assume it takes the same form $w(x,y) = \sum_{i=1}^{N_n} \hat{w}_i \phi_i(x,y)$.

Substituting this into the weak formulation gives

$$\int_0^1\int_0^1 \left(\sum_{j=1}^{N_n}\sum_{i=1}^{N_n} \hat{u}_j \nabla\phi_j \cdot \hat{w}_i \nabla\phi_i - \sum_{i=1}^{N_n} \hat{w}_i \phi_i f \right) dx\,dy = 0,$$

defining $A_{ji} = \int_0^1\int_0^1 \nabla\phi_j \cdot \nabla\phi_i \; dx\,dy$ and $b_i = \int_0^1\int_0^1 f\,\phi_i \; dx\,dy$ then we can write this as

$$\sum_{i=1}^{N_n}\sum_{j=1}^{N_n} \hat{w}_i \hat{u}_j A_{ji} - \sum_{i=1}^{N_n} \hat{w}_i b_i = 0 ,$$

$$\sum_{i=1}^{N_n} \left(A_{ji} \hat{u}_j - b_i \right) \hat{w}_i = 0 .$$ This is the same as the variational method with $\hat{w}_i = \delta u_i$.