$\boxed{Q1}$

- Linear advection-diffusion eq describes both the advection and diffusion of a system via a linear PDE.

- ~~linear equation~~

- <u>Advection eq</u>

  Describes the transportation of some conserved quantity or material by bulk motion of a fluid.

  ADVECTION EQUATION FOR QUANTITY $u(x,t)$ $\left[\partial u / \partial t = f(x,t) \partial u / \partial x \right.$

  $f(x,t)$ is some given function.

- <u>Diffusion eq</u>

  Describes diffusion of material / quantity $u(x,t)$ due to macroscopic movements of particles via collisions, etc in the flow.

  $$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \quad \Big] \text{ DIFFUSION EQUATION FOR } u(x,t)$$

  where D is the diffusion coefficient.

As the advection eq and diffusion eq are linearly independent, they can be added together to form the advection-diffusion eq:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f \frac{\partial u}{\partial x}$$

For the eq given in the question, $f = a(t)$ and $D = \epsilon$.

A linear PDE is a function of the form (k^{th}-order)

$$F(\underline{x}; \underline{u}(\underline{x}); \partial u/\partial \underline{x}, \dots, \partial^k u/\partial \underline{x}^k) = 0 \quad \Big\} \text{PDE}$$

st

$$F(\underline{x}) = \sum_{|\alpha| \le k} a_\alpha(\underline{x}) \, \partial^k / \partial \underline{x}^k \, \underline{u}$$

that is, $F(\underline{x})$ is a linear eq of its derivatives. Our eq clearly fits this def with coefficients $a(t)$, ϵ.

[Q2] From standard Taylor expansion in two variables to 3rd order

u_j^n: [using $t - \frac{1}{2}\Delta t$]: (about $(x_j, t_{n+1/2})$)

$$u_j^n = u + u_t\left(-\tfrac{1}{2}\Delta t\right) + \tfrac{1}{2} u_{tt} \cdot \left(-\tfrac{1}{2}\right)^2 (\Delta t)^2 + \tfrac{1}{6}\left(-\tfrac{1}{2}\Delta t\right)^3 u_{ttt}$$

$$= u - \tfrac{1}{2}\Delta t\, u_t + \tfrac{1}{8}(\Delta t)^2 u_{tt} - \tfrac{1}{48}(\Delta t)^3 u_{ttt} + \dots$$

u_j^{n+1} [using $t - \frac{1}{2}\Delta t + \Delta t$] (about $(x_j, t_{n+1/2})$)

$$u_j^{n+1} = u + \tfrac{1}{2}\Delta t\, u_t + \tfrac{1}{8}(\Delta t)^2 u_{tt} + \tfrac{1}{48}(\Delta t)^3 u_{ttt} \dots$$

Then proving (2.80):

$$\delta_t u(x, t + \tfrac{1}{2}\Delta t) = u\left(x, (t + \tfrac{1}{2}\Delta t) + \tfrac{1}{2}\Delta t\right) - u\left(x, (t + \tfrac{1}{2}\Delta t) - \tfrac{1}{2}\Delta t\right)$$

$$= u(x, t + \Delta t) - u(x, t)$$

that is, $\delta_t u_j^{n+1/2} = u_j^{n+1} - u_j^n$

using Taylor expansions:

$$\delta_t u_j^{n+1/2} = [u - u] + \left[\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)\right] u_t \Delta t + \left[\tfrac{1}{8} - \tfrac{1}{8}\right](\Delta t)^2 u_{tt} + \left[\tfrac{1}{48} - \left(-\tfrac{1}{48}\right)\right](\Delta t)^3 u_{ttt} + \dots$$

$$= 0 + \Delta t\, u_t + \tfrac{1}{24}(\Delta t)^3 u_{ttt} + \dots$$

As required.

Proving (2.81):

using $\delta_x^2 u(x, t+\Delta t) = u(x + \Delta x, t + \Delta t) - 2u(x, t + \Delta t) + u(x - \Delta x, t + \Delta t)$

that is, $\delta_x^2 u_j^{n+1} = u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}$.

[Q2]

Expansion up to $n=6$ using $f(x-\Delta x, t-\Delta t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{\partial^n f}{\partial x^{n-k} \partial t^k} (\Delta x)^{n-k} (\Delta y)^k$

u_j^{n+1} : using $(\frac{1}{2}\Delta t)$, $(0\Delta x)$

All terms including x are zero so –

$u_j^{n+1} = u + \frac{1}{2}\Delta t\, u_t + \frac{1}{2} \cdot \frac{1}{4}(\Delta t)^2 u_{tt} + \frac{1}{3!} \cdot (\frac{1}{2}\Delta t)^3 u_{ttt} +$

$\frac{1}{4!}(\frac{1}{2}\Delta t)^4 u_{tttt} + \frac{1}{5!}(\frac{1}{2}\Delta t)^5 u_{ttttt} + \frac{1}{6!}(\frac{1}{2}\Delta t)^6 u_{tttttt}$.

$u_{j+1}^{n+1} + u_{j-1}^{n+1}$: using $(\frac{1}{2}\Delta t)$, u_{j+1}^{n+1} has $(-\Delta x)$, u_{j-1}^{n+1} has (Δx).

* unsure of the adjustments?

Then the taylor expansion becomes –

$[\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{\partial^n u}{\partial x^{n-k} \partial t^k} (-\Delta x)^{n-k} (\frac{1}{2}\Delta t)^k] +$

$[\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{\partial^n u}{\partial x^{n-k} \partial t^k} (\Delta x)^{n-k} (\frac{1}{2}\Delta t)^k]$

$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{\partial^n u}{\partial x^{n-k} \partial t^k} (\frac{1}{2}\Delta t)^k [(-\Delta x)^{n-k} + (\Delta x)^{n-k}]$

Then

$n=1$

$\boxed{k=0}$ $u_x [-\Delta x + \Delta x]$ $+$ $\boxed{k=1}$ $u_t \cdot (\frac{1}{2}\Delta t)[1+1]$

$= \frac{1}{2}\Delta t\, u_t$

$n=2$

$\boxed{k=0}$ $\frac{1}{2}[\frac{2}{1 \cdot 2} u_{xx} [-(\Delta x)^2 + (\Delta x)^2]$ $+$ $\boxed{k=1} \to n-k=1$ $\frac{2}{1 \cdot 1} u_{xt} (\frac{1}{2}\Delta t)[0]$

$\boxed{k=2} \to n-k=0$ $\frac{2}{2} u_{tt} (\frac{1}{2}\Delta t)^2 [2]$ $= \frac{1}{2} u_{xx} (\Delta x)^2 + \frac{1}{2}(\Delta t)^2 u_{tt}$

$n=3$

$\frac{1}{3!}[$ $\boxed{k=0} \to n-k=3$ $\frac{3!}{1 \cdot 1} u_{xxx}[0]$ $+$ $\boxed{k=1} \to n-k=2$ $\frac{3!}{2!} u_{xxt} (\frac{1}{2}\Delta t)[2(\Delta x)^2]$ $+$ $\boxed{k=2} \to n-k=1$ $\frac{3!}{2!} u_{xtt} (\frac{1}{2}\Delta t)^2 [0]$

$= \frac{1}{6}[3 \cdot \frac{1}{2} \cdot 2(\Delta t)(\Delta x)^2 u_{xxt} +] = \frac{1}{2}(\Delta t)(\Delta x)^2 u_{xxt}$ } shouldn't this be zero?

Forgot – $\boxed{k=3} \to n-k=0$ $\frac{1}{3!} \cdot \frac{3!}{3!} u_{ttt} (\frac{1}{2}\Delta t)^3 \cdot 2$ $= \frac{1}{2}(\Delta t)(\Delta x)^2 u_{xxt} + \frac{1}{24} u_{ttt} (\Delta t)^3$

$n=4$

$\frac{1}{4!}[$ $\boxed{k=0} \to n-k=4$ $\frac{4!}{1} u_{xxxx} \cdot 2(\Delta x)^4 +$ $\boxed{k=1} \to n-k=3$ $\frac{4!}{3!} u_{xxxt} (\frac{1}{2}\Delta t)[0]$ $+$ $\boxed{k=2} \to n-k=2$ $\frac{4!}{2!\,2!} u_{xxtt} (\frac{1}{2}\Delta t)^2 [2(\Delta x)^2]$

$\boxed{k=3} \to n-k=1$ $\frac{4!}{3!} u_{xttt} (\frac{1}{2}\Delta t)^3 [0]$ $+$ $\boxed{k=4} \to n-k=0$ $\frac{4!}{4!} u_{tttt} (\frac{1}{2}\Delta t)^4 \cdot 2]$ $=$ $\frac{1}{12} \cdot 2(\Delta x)^4 u_{xxxx}$ $+ 6 \cdot \frac{1}{4} \cdot 2\, u_{xxtt} (\Delta t)^2 (\Delta x)^2$ $+ \frac{1}{24} \cdot \frac{1}{16} \cdot 2(\Delta t)^4 u_{tttt}$

MATHS453

$\boxed{Q2}$ Continued

$\underline{n=4}$

$\hookrightarrow 2(\Delta x)^4 u_{xxxx} + 3u_{xxtt}(\Delta t)^2(\Delta x)^2 + \frac{1}{192}(\Delta t)^4 u_{tttt}$.

shouldn't these be zero?

$\underline{n=5}$

$\boxed{k=0} \to n-k=5$, $\boxed{k=1} \to n-k=4$, $\boxed{k=2} \to n-k=3$

$$\frac{1}{5!}\left[u_{xxxxx}[0] + \frac{5!}{1!4!} u_{xxxxt}\left(\tfrac{1}{2}\Delta t\right)\left[2(\Delta x)^4\right] + \frac{5!}{2!3!} u_{xxxtt}\left(\tfrac{1}{2}\Delta t\right)^2[0]\right.$$

$\boxed{k=3} \to n-k=2$, $\boxed{k=4} \to n-k=1$

$$+ \frac{5!}{3!2!} u_{xxttt}\left(\tfrac{1}{2}\Delta t\right)^3 \cdot 2(\Delta x)^2 + \frac{5!}{4!} u_{xtttt}\cdot\left(\tfrac{1}{2}\Delta t\right)^4\cdot[0]$$

$\boxed{k=5} \to n-k=0$

$$+ \frac{5!}{5!} u_{ttttt}\left(\tfrac{1}{2}\Delta t\right)^5 \cdot 2 = \frac{1}{6!}\left[5u_{xxxxt}(\Delta t)(\Delta x)^4 + \tfrac{5}{2}u_{xxttt}(\Delta t)^3(\Delta x)^2\right] + \frac{1}{5!}\frac{1}{2^4}u_{ttttt}(\Delta t)^5.$$

$\underline{n=6}$ $\boxed{k=0} \to n-k=6$, $\boxed{k=1} \to n-k=5$

$$\frac{1}{6!}\left[\frac{6!}{6!} u_{xxxxxx}\, 2(\Delta x)^6 + \frac{6!}{1!5!} u_{xxxxxt}\left(\tfrac{1}{2}\Delta t\right)[0]\right.$$

$\boxed{k=2} \to n-k=4$, $\boxed{k=3} \to n-k=3$

$$\frac{6!}{2!4!} u_{xxxxtt}\left(\tfrac{1}{2}\Delta t\right)^2 \cdot 2(\Delta x)^4 + \frac{6!}{3!3!} u_{xxxttt}\left(\tfrac{1}{2}\Delta t\right)^3[0] +$$

$\boxed{k=4} \to n-k=2$, $\boxed{k=5} \to n-k=1$

$$\frac{6!}{4!2!} u_{xxtttt}\left(\tfrac{1}{2}\Delta t\right)^4 \cdot 2(\Delta x)^2 + [0]$$

$\boxed{k=6} \to n-k=0$

$$\frac{6!}{6!} u_{tttttt}\left(\tfrac{1}{2}\Delta t\right)^6 \cdot 2 \;.\; = \frac{2}{6!} u_{xxxxxx}(\Delta x)^6 + \frac{1}{4!}\cdot\frac{1}{4} u_{xxxxtt}(\Delta t)^2(\Delta x)^4 + \frac{1}{4!}\frac{1}{16}u(\Delta t)^4(\Delta x)^2 + \frac{2}{6!}u_{tttttt}\left(\tfrac{1}{2}\Delta t\right)^6$$

$$= \frac{1}{360}u_{xxxxxx}(\Delta x)^6 + \frac{1}{96}u_{xxxxtt}(\Delta t)^2(\Delta x)^4 + \frac{1}{384}u_{xxtttt}(\Delta t)^4(\Delta x)^2 + \frac{1}{2^6\cdot 6!}u_{tttttt}(\Delta t)^6$$

~~Present terms~~ Grouping terms:

t-derivatives –

$$\Delta t\, u_t\left(1 - 2\cdot\tfrac{1}{2}\right) + (\Delta t)^2 u_{tt}\left(\tfrac{1}{4} - 2\cdot\tfrac{1}{2}\cdot\tfrac{1}{4}\right) + (\Delta t)^3 u_{ttt}\left(2\cdot\tfrac{1}{3!}\cdot\tfrac{1}{2^3} - 2\tfrac{1}{3!}\tfrac{1}{2^3}\right) + \dots = \underline{\underline{0}}.$$

As $\frac{\partial^n u}{\partial t^n}\Delta t - 2\frac{\partial^n u}{\partial t^n}(\Delta t)^n + \frac{\partial^n u}{\partial t^n}(\Delta t)2 = 0$.

x-derivatives –

only for n even so –

$$2\left[\tfrac{1}{2}(\Delta x)^2 u_{xx} + \tfrac{1}{4!}(\Delta x)^4 u_{xxxx} + \tfrac{1}{6!}(\Delta x)^6 u_{xxxxxx}\right]$$

expected value of $\partial x^2 u_j^{n+1}$

mixed x–t derivatives – [~~should cancel~~ $\overset{?}{0}$] as seen.

↳ ~~reasons why they may not cancel~~

↳ Error in Taylor series formula ~~by me~~, made by me

$\boxed{Q2}$ Continued

Then all we have are even n x derivatives and mixed derivatives for $u_{j+1}^{n+1} + u_{j-1}^{n+1}$

$$\partial_x^2 u_j^{n+1} = \left[(\Delta x)^2 u_{xx} + 2(\Delta x)^4 u_{xxxx} + \tfrac{3}{6!}(\Delta x)^6 u_{xxxxxx} + \dots \right]$$

$$+\left[\tfrac{1}{2}(\Delta t)(\Delta x)^2 u_{xxt} + 3u_{xxtt}(\Delta t)^2(\Delta x)^2 + \tfrac{5}{8!}(\Delta t)(\Delta x)^4 u_{xxxxt} \right.$$

$$+ \tfrac{5}{2}5!(\Delta t)^3(\Delta x)^2 u_{xxttt} + \tfrac{1}{96} u_{xxxxtt}(\Delta t)^2(\Delta x)^4 +$$

$$\left. \tfrac{1}{384}(\Delta t)^4(\Delta x)^2 u_{xxtttt} + \dots \right] + \text{further terms.}$$

~~comparing to 2.81 -~~ & u terms are $[2u-2u]=0$

~~$(\Delta x)^4 u_{xxxx}$ coefficient incorrect, u_{xxxxt} coeff incorrect,~~

Some incorrect ~~into~~ coefficients! ~~eg~~ TO CORRECT IF TIME.

May mean into further proofs -

~~prioritising methodology~~. using given coefficients in 2.81 in further proof.

Proving 2.82

$$\partial_x^2 u_j^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$$

- will use adjustment ~~to~~ (Δt term) is $(-\tfrac{1}{2}\Delta t)$.
- Adjust terms from $u_{j+1}^{n+1} + u_{j-1}^{n+1}$ expansion - replacing $(\tfrac{1}{2}\Delta t)$ with $(-\tfrac{1}{2}\Delta t)$.

Then

n=1

$$2\cdot(-\tfrac{1}{2}\Delta t)u_t \; +$$

n=2

$$\tfrac{1}{2}\left[\tfrac{2!}{2!}\cdot 2(\Delta x)^2 u_{xx} + \tfrac{2!}{2!}(-\tfrac{1}{2}\Delta t)^2 u_{tt}\cdot 2 \right.$$

n=3

$$\tfrac{1}{3!}\left[\tfrac{3!}{2!}\cdot 2\cdot(-\tfrac{1}{2}t)(\Delta x)^2 + \tfrac{3!}{3!}\cdot 2\cdot(-\tfrac{1}{2}\Delta t)^3 u_{ttt} \right.$$

n=4

$$\tfrac{1}{4!}\left[2\tfrac{4!}{4!} u_{xxxx}(\Delta x)^4 + \tfrac{4!}{2!2!}(-\tfrac{1}{2}\Delta t)^2\cdot 2\cdot(\Delta x)^2 u_{xxtt} + \tfrac{4!}{4!}\cdot 2\cdot(-\tfrac{1}{2}\Delta t)^4 u_{tttt}\right]$$

n=5

$$\tfrac{1}{5!}\left[\tfrac{5!}{4!}(-\tfrac{1}{2}\Delta t)\cdot 2\cdot(\Delta x)^4 u_{xxxxt} + \tfrac{5!}{3!2!}(-\tfrac{1}{2}\Delta t)^3\cdot 2\cdot(\Delta x)^2 u_{xxttt} + \tfrac{5!}{5!}\cdot(-\tfrac{1}{2}\Delta t)^5\cdot 2\cdot u_{ttttt} \right)$$

n=6

$$\tfrac{1}{6!}\left[\tfrac{6!}{6!}\cdot 2\cdot(\Delta x)^6 u_{xxxxxx} + \tfrac{6!}{2!4!}u_{xxxxtt}(-\tfrac{1}{2}\Delta t)^2\cdot 2(\Delta x)^4 + \right.$$

$$\tfrac{6!}{4!2!}(-\tfrac{1}{2}\Delta t)^4\cdot 2(\Delta x)^2 u_{xxtttt} + \tfrac{6!}{6!}\cdot(-\tfrac{1}{2}\Delta t)^6\cdot 2$$

Q2 Continued

Giving:

$\partial_x^2 u_j^n =$

t derivatives:

$$\frac{1}{n!}\frac{\partial^n u}{\partial t^n}\cdot\left(-\tfrac{1}{2}\Delta t\right)^n\cdot 2 - 2\cdot\frac{1}{n!}\frac{\partial^n u}{\partial t^n}\left(-\tfrac{1}{2}\Delta t\right)^n = \underline{\underline{0}}$$

Then remaining terms are x-derivatives of $u_{j+1}^n + u_{j-1}^n$ and also the mixed derivatives - so looking at the full expansion now -

$$\theta \partial_x^2 u_j^{n+1} + \partial_x^2 u_j^n - \theta\partial_x^2 u_j^n = \partial_x^2 u_j^n + \theta\left(\partial_x^2 u_j^{n+1} - \partial_x^2 u_j^n\right)$$

$=$

$$\left[\tfrac{1}{2}\cdot\frac{2!}{2!}\cdot 2(\Delta x)^2 u_{xx} + \tfrac{1}{3!}\cdot\frac{3!}{2!}\cdot 2\cdot\left(-\tfrac{1}{2}\Delta t\right)(\Delta x)^2 u_{xxt}\right.$$
$$+ \tfrac{2}{4!}\cdot\tfrac{4!}{4!}(\Delta x)^4 u_{xxxx} + \tfrac{1}{4!}\cdot\frac{4!}{2!2!}\left(-\tfrac{1}{2}\Delta t\right)^2\cdot 2\cdot(\Delta x)^2 u_{xxtt}$$
$$+ \tfrac{1}{5!}\cdot\tfrac{5!}{4!}\left(-\tfrac{1}{2}\Delta t\right)\cdot 2(\Delta x)^4 u_{xxxxt} + \tfrac{1}{5!}\cdot\frac{5!}{3!2!}\left(-\tfrac{1}{2}\Delta t\right)^3\cdot 2\cdot(\Delta x)^2 u_{xxttt}$$
$$+ \tfrac{1}{6!}\cdot\frac{6!}{6!}\cdot 2(\Delta x)^6 u_{xxxxxx} + \tfrac{1}{6!}\,\tfrac{6!}{2!4!} u_{xxxxtt}\left(-\tfrac{1}{2}\Delta t\right)^2\cdot 2\cdot(\Delta x)^4$$
$$\left. + \tfrac{1}{6!}\,\tfrac{6!}{4!2!}\left(-\tfrac{1}{2}\Delta t\right)^4\cdot 2\cdot(\Delta x)^2 u_{xxtttt} + \dots\right]$$

$+$

$$\theta\left[(\Delta x)^2 u_{xx} + \tfrac{1}{12}(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \tfrac{1}{24}(\Delta t)(\Delta x)^2 u_{xxxxt}\right.$$
$$\tfrac{1}{2}(\Delta t)(\Delta x)^2 u_{xxt} + \tfrac{1}{4}(\Delta t)^2(\Delta x)^2 u_{xxtt} + \tfrac{1}{96} u_{xxxxtt}(\Delta t)^2(\Delta x)^4 +$$
$$+ \frac{5}{2\cdot 5!}(\Delta t)^3(\Delta x)^2 u_{xxttt} + \dots \Big]$$
$$\tfrac{1}{384}(\Delta t)^4(\Delta x)^2 u_{xxtttt} +$$

$$= \overset{x\ \text{derivs}}{\left[(\Delta x)^2 u_{xx}\right.}\ \ \tfrac{1}{12}(\Delta x)^4 u_{xxxx} + (\tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx}\Big]$$
$$+\left[\ (\Delta t)(\Delta x)^4 u_{xxt} + \dots\right]$$

no time to write full expansion.

+ rearrange.

Proving ~~Eq~~ 2.83 & 2.84

~~Proof~~ ~~Need~~ Have terms for ~~$\partial_t u_j^{n+1/2}$~~ $\partial_x^2 u_j^{n+1}, \partial_x^2 u_j^n$.

Need terms for $\partial_t u_j^{n+1/2}$:

$$\partial_t u(x, t+\tfrac{1}{2}\Delta t) = u(x, (t+\tfrac{1}{2}\Delta t)+\tfrac{1}{2}\Delta t) - u(x, (t+\tfrac{1}{2}\Delta t)-\tfrac{1}{2}\Delta t)$$
$$= u(x, t+\Delta t) - u(x,t)$$

so $\partial_t u_j^{n+1/2} = u_j^{n+1} - u_j^n$ { we already have these terms ~~for~~ in their full expansion for previous qus.

$$T_j^{n+1/2} = \tfrac{1}{\Delta t}(u_j^{n+1} - u_j^n) - \tfrac{1}{(\Delta x)^2}\Big[\theta\big[u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\big] + (1-\theta)\big[u_{j+1}^n - 2u_j^n + u_{j-1}^n\big]$$

No time to do full expansion & rearrange - all these terms have been previously calculated.

Q3.

Explicit scheme:

$\partial u/\partial t(x_j,x_n) \approx U_j^{n+1} - U_j^n$] Forward diff in time on U_j^n

$\partial^2 u/\partial x^2(x_j,x_n) \approx U_{j+1}^n - 2U_j^n + U_{j-1}^n$] central difference in space on U_j^n.

~~$\partial u/\partial x(x_j,x_n) \approx U_j^n - U_{j-1}^n$] backward diff in space on U_j^n ↳ upwind scheme~~

~~Explicit The x scheme is with scaling~~

~~$1/\Delta t(U_j^{n+1} - U_j^n) + \frac{a(t)}{\Delta x}[U_j^n - U_{j-1}^n] - 1/(\Delta x)^2 [U_{j+1}^n - 2U_j^n + U_{j-1}^n] = 0$~~

$$\partial u/\partial x(x_j,x_n) = \begin{cases} U_j^n - U_{j-1}^n, & -a < 0 \\ U_{j+1}^n - U_j^n, & -a > 0 \end{cases}$$

] forwards/backwards/upwinds difference depending on sign of $a(t)$.

not sure on these.

Explicit scheme is then –

$$1/\Delta t[U_j^{n+1} - U_j^n] = \begin{cases} -\frac{a(t)}{\Delta x}[U_j^n - U_{j-1}^n] + 1/(\Delta x)^2 \epsilon[U_{j+1}^n - 2U_j^n + U_{j-1}^n] & \leftarrow \text{For } -a < 0 \\ \frac{a(t)}{\Delta x}[U_{j+1}^n - U_j^n] + 1/(\Delta x)^2 \epsilon[U_{j+1}^n - 2U_j^n + U_{j-1}^n] & \leftarrow \text{For } -a > 0 \end{cases}$$

Implicit ~~solution~~ scheme

$u_t \approx U_j^{n+1} - U_j^n$] ~~Forward diff~~ Backwards diff on U_j^{n+1}

$u_{xx} \approx U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}$] central diff. as before. on U_j^{n+1}

$$u_x \approx \begin{cases} U_j^{n+1} - U_{j-1}^{n+1}, & -a < 0 \\ U_{j+1}^{n+1} - U_j^{n+1}, & -a > 0 \end{cases}$$ same spatial scheme as before on U_j^{n+1}.

Hence we have –

$$1/\Delta t[U_j^{n+1} - U_j^n] = \begin{cases} \frac{a(t)}{\Delta x}[U_j^{n+1} - U_{j-1}^{n+1}] + \frac{\epsilon}{(\Delta x)^2}[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^n], & -a < 0 \\ \frac{a(t)}{\Delta x}[U_{j+1}^{n+1} - U_j^{n+1}] + \frac{\epsilon}{(\Delta x)^2}[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^n], & -a > 0 \end{cases}$$

(~~missing a because of the ... new~~ (need to introduce some μ).

Q3 continued

θ-scheme

u_t remains the same. We average out ~~ut, utt~~ u_x, u_{xx} for explicit & implicit schemes. With weighting $0 \le \theta \le 1$.

Case $-a < 0$:

implicit scheme

$$\tfrac{1}{\Delta t}\left(u_j^{n+1} - u_j^n\right) = \theta\left[\frac{a(t)}{\Delta x}\left(u_j^{n+1} - u_{j-1}^{n+1}\right) + \frac{\epsilon}{(\Delta x)^2}\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right)\right]$$

$$+ (1-\theta)\left[\frac{a(t)}{\Delta x}\left(u_j^n - u_{j-1}^n\right) + \epsilon/(\Delta x)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)\right]$$

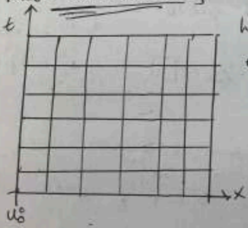
explicit scheme

Case $-a > 0$

$$\tfrac{1}{\Delta t}\left(u_j^{n+1} - u_j^n\right) = \theta\left[\frac{a(t)}{\Delta x}\left(u_{j+1}^{n+1} - u_j^{n+1}\right) + \epsilon/(\Delta x)^2\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right)\right]$$

$$+ (1-\theta)\left[\frac{a(t)}{\Delta x}\left(u_{j+1}^n - u_j^n\right) + \epsilon/(\Delta x)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)\right]$$

Mesh and ordering



Let $M = |L - L_p|$, assuming $L, L_p \in \mathbb{R}$. $t \in [0, t_F]$. We define ~~J Δx points~~ J mesh points for x – $\Delta x = 1/J$. and N mesh points for t – $\Delta t = 1/N$. J does not have to equal N, but $J = N$ is simple.

For our indexing of the mesh, use $u_j^n = [x]_{J(n)+(j+1)}$ } may cause some issues at $J-1$, $N-1$!

Boundaries + ICs

That is,

$$\partial\Omega = \{u_j^0,\ j = 0, 1, \dots, J-1\} \cup \{u_{J-1}^n : n = 0, 1, \dots, N-1\}$$

$$\cup \{u_0^n : n = 0, 1, \dots, N-1\}$$

IC, BC, BC

where $u_j^0 = u(x_j, 0) = u_0(x)$, $u_{J-1}^n = u(L, t) = 0 = u(L_p, t) = u_0^n$.

MATHS453

$\boxed{Q3}$ Continued

Scheme at boundaries & ICs:

~~U_j^0: $-a \ldots \theta$ $1/\Delta t[U_j^1 - u_0(x)] = \theta[\frac{a(t)}{\Delta x}[U_j^1 - U_{j-1}^1]$~~

~~U_j^n: $+ \epsilon/(\Delta x)^2 (U_{j+1}^1 - 2$~~

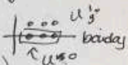
U_j^0, $-a \leq 0$:

LHS (unknown)

$1/\Delta t\, U_j^{n+1} - \frac{a\theta}{\Delta x} U_j^{n+1} - \frac{a\theta}{\Delta x} U_{j-1}^{n+1} + \epsilon/(\Delta x)^2 [U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}]$

3 known, 3 unknown

$j=1, n=0$



RHS (known)

$= 1/\Delta t\, U_j^n + (1-\theta)[\frac{a}{\Delta x}(U_j^n - U_{j-1}^n) + \epsilon/(\Delta x)^2 (U_{j+1}^n - 2U_j^n + U_{j-1}^n)]$

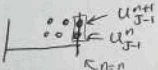
U_j^0, $-a \geq 0$:

$1/\Delta t\, U_j^{n+1} + \theta[\frac{a\theta}{\Delta x}(U_{j+1}^{n+1} - U_j^{n+1}) + \epsilon/(\Delta x)^2 (U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1})$ $=$ $1/\Delta t\, U_j^n + (1-\theta)[\frac{a(t)}{\Delta x}(U_{j+1}^n - U_j^n) + \epsilon/(\Delta x)^2 (U_{j+1}^n - 2U_j^n + U_{j-1}^n)]$

U_{J-1}^n, $-a < 0$:

known where for $j = J-1$

2 known, 4 unknown



$1/\Delta t(U_j^{n+1} - U_j^{n+1}) + \frac{\theta a(t)}{\Delta x} U_j^{n+1} - \frac{\theta a}{\Delta x} U_{j-1}^{n+1} + 2\epsilon/(\Delta x)^2 U_j^{n+1}$

$-\theta\epsilon/(\Delta x)^2 (U_{j+1}^{n+1} + U_{j-1}^{n+1})$

$-\epsilon/(\Delta x)^2 (1-\theta)[U_{j+1}^n + U_{j-1}^n]$

$= + (1-\theta)\, a/\Delta x[-U_j^n] + \frac{2\epsilon}{(1-\theta)(\Delta x)^2} U_j^n + 1/\Delta t\, U_j^{n+1} \pm 1/\Delta t\, U_j^{n+1}$

U_{j-1}^n, $-a > 0$:

~~$1/\Delta t(U_j^{n+1} - U_j^n) - \frac{\theta a}{\Delta x} U_j^{n+1}$~~ ~~as above~~ same as above.

~~$-\epsilon/(\Delta x)^2 (U_{j+1}^{n+1}$~~

$1/\Delta t(U_j^{n+1} - U_j^n) - \theta\,\epsilon/(\Delta x)^2$

$(\theta(U_{j+1}^{n+1} + U_{j-1}^{n+1}) - (1-\theta)(U_{j+1}^n + U_{j-1}^n)$

$- \frac{\theta a}{\Delta x}(U_{j+1}^{n+1}) + (1-\theta)\, a/\Delta t(U_{j+1}^n)$

$= 1/\Delta t\, U_j^{n} + \frac{2\epsilon}{(\Delta x)^2} U_j^{n+1} + (1-\theta)(\frac{2\epsilon}{(\Delta x)^2} U_j^n) \pm 1/\Delta t\, U_j^{n+1}$

$+$

$\boxed{Q3}$ Continued

2 unknown
4 unknown

u_0^n : $-a<0$:

LHS (unknown)

$$-\frac{\epsilon\theta}{(\Delta x)^2} u_{j+1}^{n+1} - (1-\theta)\left(\frac{\epsilon}{(\Delta x)^2}\left(u_{j+1}^n\right)\right.$$

$$= \frac{1}{\Delta t}\left(u_j^{n+1} - u_j^n\right) + \frac{\theta a}{\Delta x}\left(u_j^{n+1}\right) - \frac{2u_j^{n+1}\cdot\epsilon}{(\Delta x)^2}$$

$$+ (1-\theta)\left[\frac{a}{\Delta x}\left(u_j^n\right) + \frac{\epsilon}{(\Delta x)^2}\left(-2u_j^n\right)\right.$$

RHS (known)