

Numerics Exercise 3

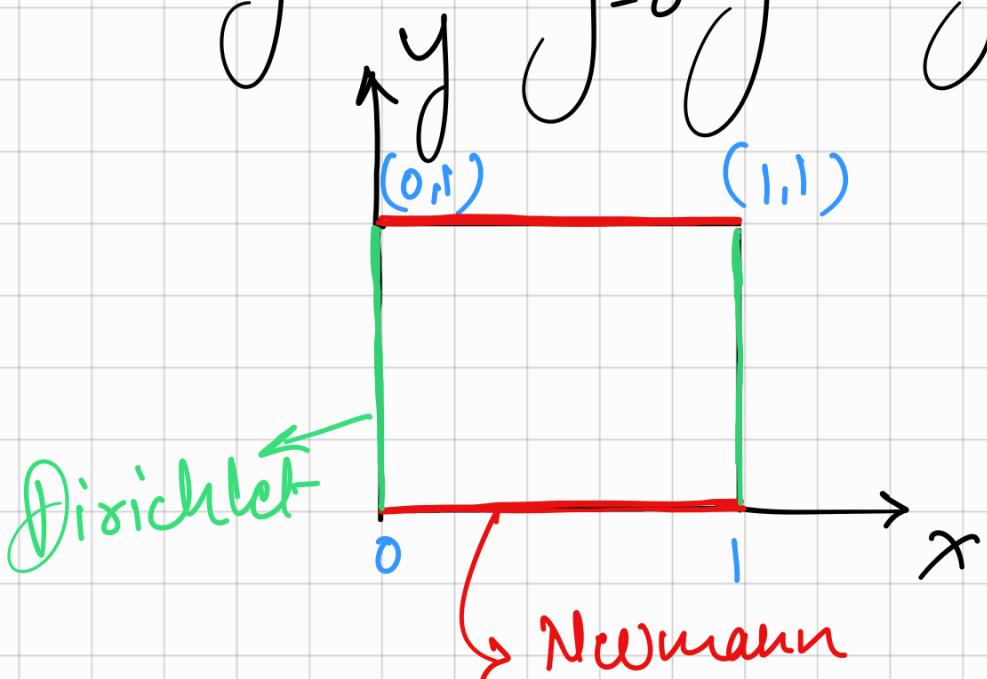
Consider Poisson system

$$-\nabla^2 u = f \text{ on } (x, y) \in [0, 1]^2$$

$$f(x, y) = 2\pi^2 \sin(\pi x) \cos(\pi y)$$

Dirichlet BC $\rightarrow u(0, y) = u(1, y) = 0$

Neumann BC $\rightarrow \frac{\partial u}{\partial y}(x, y)|_{y=0} = \frac{\partial u}{\partial y}(x, y)|_{y=1} = 0$



The exact solution is

$$u_e(x, y) = \sin(\pi x) \cos(\pi y)$$

\Rightarrow 2d - Poisson equation,

$$-\nabla^2 \phi(x, y) = f(x, y)$$

$$\Rightarrow f(x, y) = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 2\pi^2 \sin \pi x \cos \pi y$$

$\phi(x, y) \rightarrow$ unknown function

$f(x, y) \rightarrow$ given function

\Rightarrow To check the exact solution,

$$\frac{\partial u_e}{\partial x} = \pi \cos(\pi x) \sin(\pi y)$$

$$\frac{\partial^2 u_e}{\partial x^2} = -\pi^2 \sin(\pi x) \cos(\pi y)$$

$$\frac{\partial u_e}{\partial y} = -\pi \sin(\pi x) \sin(\pi y)$$

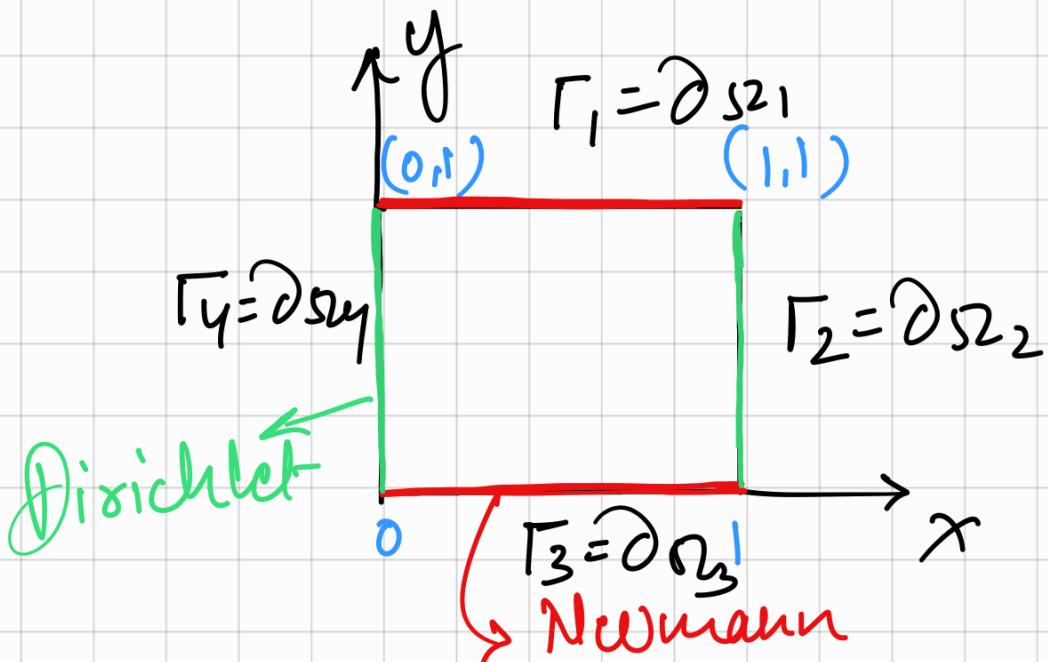
$$\frac{\partial^2 u_e}{\partial y^2} = -\pi^2 \sin(\pi x) \cos(\pi y)$$

$$\Rightarrow -\nabla^2 u_e = -\left(\frac{\partial^2 u_e}{\partial x^2} + \frac{\partial^2 u_e}{\partial y^2}\right)$$

$$= -\left(-2\pi^2 \sin \pi x \cos \pi y\right)$$

$$-\nabla^2 u_e = f$$

1.



- introduce a test function

$$w(x,y) = \underline{f} \bar{v}(x,y)$$

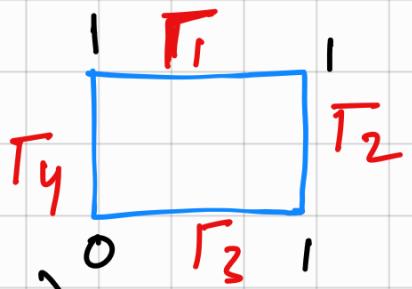
\underline{f} is a variation and

Since the variable unknown is denoted by $v(x,y)$

$$\Rightarrow - \iint_{0,0}^{1,1} w \nabla^2 v \, dx \, dy = \iint_{0,0}^{1,1} w f \, dx \, dy$$

$$\Rightarrow \iint_{0,0}^{1,1} \nabla(w) \cdot \nabla v \, ds_2 - \oint_{\Gamma} w \hat{n} \cdot \nabla v \, d\Gamma \xrightarrow{\text{over } \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4} \\ = \iint_{S^2} w f \, ds_2 \quad (\text{for Both } S^{2,4} \text{ and } S^{2,2})$$

Now :- $\oint \omega \hat{n} \cdot \nabla u d\Gamma$



$$\textcircled{1} \quad \Gamma_1 = \hat{n} \cdot \nabla u = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}^0$$

But $\partial_y u = 0$ (Neumann B.C)

$$\Rightarrow \Gamma_1 = 0$$

$$\textcircled{2} \quad \Gamma_3 = \hat{n} \cdot \nabla u = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}^0$$

But $\partial_y u = 0$ (Neumann B.C)

$$\Rightarrow \Gamma_3 = 0$$

$$\textcircled{3} \quad \Gamma_2 = \hat{n} \cdot \nabla u = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{does not work}} \cdot \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}$$

does not work

and; $u(0, y) = a, \omega(0, y) = 0$

$u(1, y) = b, \omega(1, y) = 0$

Therefore $\Gamma_2, \Gamma_4 \rightarrow 0$

\Rightarrow Hence,

$$\iint_{\Omega} \nabla(\omega) \cdot \nabla u \, dS_2 - \oint_{\partial\Omega} \omega \hat{n} \cdot \nabla u \, d\Gamma \xrightarrow{\text{for both } S_2 \text{ & } \Omega_N} 0$$
$$= \iint_{\Omega} \omega f \, dS_2$$

$$\iint_{\Omega} \nabla(\omega) \cdot \nabla u \, dx dy = \iint_{\Omega} \omega f \, dx dy$$

$$\iint_{\Omega} (\nabla u \cdot \nabla(\omega) - \omega f) \, dx dy = 0$$

\hookrightarrow weak form

Now :-

⇒ minimization of the the Ritz-Galerkin integral.

$$F(u) = \int_0^1 \int_0^1 \frac{1}{2} |\nabla u|^2 - uf dx dy = 0$$

↓ Functional

Variational ;

$$\delta F(u) = \lim_{\epsilon \rightarrow 0} \frac{F[u + \epsilon \delta u] - F(u)}{\epsilon}$$

But;

$$= \lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} F[u + \epsilon \delta u] = 0$$

$$\Rightarrow F(u + \epsilon \delta u) = \int_0^1 \int_0^1 \frac{1}{2} \left[\nabla(u + \epsilon \delta u) \right]^2 - (u + \epsilon \delta u) f dx dy$$

$$\Rightarrow \lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} F[u + \epsilon \delta u] = \int_0^1 \int_0^1 -(\delta u) f$$

$$+ \lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} \left(\frac{1}{2} |\nabla u + \epsilon \delta u|^2 \right)$$

↓

$$\lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} (\epsilon \nabla u \cdot \nabla \delta u + O(\epsilon))$$

↓

Ignoring
Squeeze
terms

$$(\nabla u \cdot \nabla \delta u) + O$$



$$\Rightarrow \lim_{\epsilon \rightarrow 0} F[u + \epsilon \delta u] = \int_0^1 \int_0^1 -(\delta u) f$$

$$+ (\nabla u \cdot \nabla \delta u) dx dy$$

$$0 = \int_0^1 \int_0^1 -(\delta u) f + (\nabla u \cdot \nabla \delta u) dx dy$$

$$0 = \int_0^1 \int_0^1 (-f + \nabla^2 u) \delta u dx dy$$

$$+ \oint \delta u \vec{n} \cdot \nabla u d\Gamma$$

$\rightarrow 0$ (Because of B.C.)

$$\Rightarrow \int_0^1 \int_0^1 (\nabla u \cdot \nabla (\delta u) - f(\delta u)) dx dy$$

\rightarrow This is exactly the Weak form

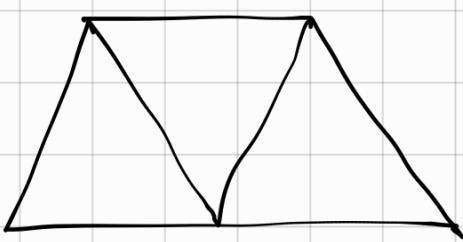
\rightarrow This shows, the test function $w(x, y)$ is same as $\delta u(x, y)$

$$w(x, y) = \delta u(x, y)$$

2.

Discrete Ritz-Galerkin

- $U(x, y)$ is continuous but not differentiable
- $U(x, y)$ is the unknown &
 $U_n(x, y) \rightarrow$ FEM Approximation



$h \rightarrow$ minimum
element
size

$\Rightarrow U(x, y) \approx U_n(x, y)$ when $h \rightarrow 0$

$$U(x, y) = \lim_{h \rightarrow 0} U_n(x, y)$$

$$U(x, y) = \sum_{j=1}^{N_n} V_j \phi_j(x, y)$$

$\hat{U}_j \rightarrow$ Coefficient
 $\phi_j(x, y) \rightarrow$ are the
 basis function.



These basis function have
 compact support for CG1.



Means, $\phi_j(x, y) = 1$ on the
 main node by \odot for neighboring
 nodes & outside the neighboring
 element

\hookrightarrow Thus compact support.

$$\begin{aligned}
 \Rightarrow U_h(x, y) &= \sum_{j=1}^{N_{Dof}} U_j \phi_j(x, y) \\
 &= \sum_{j=1}^{N_{Dof}(\text{Neumann})} U_j \phi_j(x, y) \\
 &\cancel{\sum_{j=1}^{N_{Dof}(\text{Dirichlet})} U_j \phi_j(x, y)} \rightarrow O(B.C.)
 \end{aligned}$$



$$u_n(x, y) = \sum_{j=1}^N \hat{u}_j \phi_j(x, y)$$

$$\Rightarrow \text{take } \omega(x, y) = \phi_i(x, y)$$

for $i = 1, 2, \dots, N_{\text{nodes}}$
 But $(N_m + N_{\text{nodes}})$

from energy functional:

$$F(u) = \iint_0^1 \frac{1}{2} |\nabla u|^2 - f u \, dx dy$$

↓ Functional

Substitute $u_n(x, y)$ into $F(u)$

$$F(u_n) = \iint_0^1 \frac{1}{2} |\nabla u_n|^2 - \iint_0^1 f u_n \, dx dy$$

Now:

$$F(u_h) = \frac{1}{2} \iint_0^1 \left| \sum_{j=1}^{N_h} \hat{u}_j \phi_j(x, y) \right|^2 - \iint_0^1 f \left(\sum_{j=1}^{N_h} \hat{u}_j \phi_j(x, y) \right)$$

\Rightarrow expanding $|\nabla u_h|^2$,

$$\Rightarrow |\nabla u_h| \cdot |\nabla u_h|,$$

$$\Rightarrow \sum_{j=1}^{N_h} \hat{u}_j \nabla \phi_j(x, y) \cdot \sum_{i=1}^{N_h} \hat{u}_i \nabla \phi_i(x, y),$$

\Rightarrow Substitute it in Functional $F(u_h)$

$$F(u_h) = \frac{1}{2} \iint_0^1 \sum_{j=1}^{N_h} \sum_{i=1}^{N_h} \hat{u}_j \hat{u}_i \nabla \phi_j \cdot \nabla \phi_i dx dy$$

$$- \iint_0^1 f \sum_{j=1}^{N_u} u_j \phi_j(x, y) dx dy,$$

\Rightarrow To minimize it, $\frac{\partial F(u)}{\partial u_j} = 0$

$$\Rightarrow \sum_{j=1}^{N_u} u_j \iint_0^1 \nabla \phi_j \cdot \nabla \phi_i dx dy$$

$$- \sum_{j=1}^{N_u} \iint_0^1 f \phi_j dx dy = 0,$$

u matrix form,

$$A_{ij} u_i = b_j$$

where $A_{ij} = \sum_{j=1}^{N_u} \iint_0^1 \nabla u_j \cdot \nabla \phi_i dx dy$

\Rightarrow

$$b_i = \sum_{j=1}^{N_{\text{nodes}}} \iint_0^1 f \phi_j dx dy$$

$f_{\text{ex}} \quad 2 - 1, 2, \dots \text{ Nnodes}$
 $\downarrow \quad j = 1, 2, \dots N_u$

Now for discrete weak formulation,

$$\int_0^1 \int_0^1 (\nabla U \cdot \nabla (\omega) - \omega f) dx dy = 0$$

\hookrightarrow weak form

$$\Rightarrow \int_0^1 \int_0^1 \nabla \left(\sum_{j=1}^{N_u} U_j \phi_j \right) \cdot \nabla \phi_i dx dy$$

$$= \int_0^1 \int_0^1 f \phi_i dx dy$$

$$\Rightarrow \sum_{j=1}^{N_u} \int_0^1 \int_0^1 \nabla U_j \cdot \nabla \phi_j \cdot \nabla \phi_i dx dy$$

$$= \sum_{i=1}^{N_{\text{node}}} \int_0^1 \int_0^1 f \phi_i dx dy$$

In matrix form,

$$A_{ij} \psi_j = b_i$$

where $A_{ij} = \sum_{j=1}^{N_u} \int_0^1 \int_0^1 \psi_j \nabla \phi_j$

\Rightarrow

$$b_i = \sum_{j=1}^{N_u} \int_0^1 \int_0^1 f \phi_i dxdy$$

for $i = 1, 2, \dots, N_{\text{nodes}}$

& $j = 1, 2, \dots, N_u$

↳ for both processes

the "discrete" forms are same

\Rightarrow therefore final discrete form

$$\sum_{j=1}^{N_{\text{nodes}}} V_{ij} u_j = b_i \quad \text{for } i=1, 2, \dots, N_{\text{nodes}}$$

\Rightarrow unknown Coefficient Vector

$$u_i,$$

$$U = [u_1, u_2, \dots, u_{N_{\text{nodes}}}]^T$$

\Rightarrow Now, variation in discrete energy functional.

$$F(U) = \frac{1}{2} \iint |\nabla u_n|^2 dx dy - \iint f u_n dx dy$$

Substituting

$$\nabla u_n = \nabla \left(\sum_{j=1}^{N_{\text{nodes}}} u_j \phi_j(x, y) \right)$$

taking variation of $\mathcal{S}u_n$

$$\begin{aligned}\therefore \nabla u_h &= \nabla u_n + \varepsilon \nabla \delta u_n \\ &= \sum_{j=1}^{N_h} u_j \nabla \phi_j + \varepsilon \sum_{j=1}^{N_h} \delta u_j \nabla \phi_j\end{aligned}$$

Now

$$\begin{aligned}|\nabla u_h|^2 &= |\nabla u_n|^2 + 2\varepsilon \nabla u_n \cdot \nabla \delta u_n \\ &\quad + \varepsilon^2 |\nabla \delta u_n|^2.\end{aligned}$$

Substitute in $F(u_n)$

$$F(u_n + \varepsilon \delta u_n) = \int_0^1 \int_0^1 \frac{1}{2} \left| \nabla (u_n + \varepsilon \delta u_n) \right|^2 - (u_n + \varepsilon \delta u_n) f dx dy$$

$$\lim_{\varepsilon \rightarrow 0} \frac{\partial F(u_n + \varepsilon \delta u_n)}{\partial \varepsilon} = \int_0^1 \int_0^1 -(\delta u_n) f$$

$$+ \lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} \left(\frac{1}{2} \| \nabla u_h + \epsilon \delta u_h \|^2 \right)$$

↓

neglecting linear scale

$$(\nabla u \cdot \nabla \delta u) + 0$$

(ignoring square term)

$$\Rightarrow \lim_{\epsilon \rightarrow 0} F(u_h + \epsilon \delta u_h)$$

$$= \iint_{\Omega} (-\delta u_h f + (\nabla u_h \cdot \nabla \delta u_h) b) dx dy$$

$$\Rightarrow \iint_{\Omega} (\nabla u_h \cdot \nabla \delta u_h) dx dy$$

$$- \iint_{\Omega} (\delta u_h f) dx dy$$

$$\Rightarrow \int_0^1 \int_0^1 \sum_{j=1}^{N_u} u_j \nabla \phi_j \cdot \sum_{i=1}^{N_u} \delta u_i \nabla \phi_i dx dy$$

$$- \int_0^1 \int_0^1 \sum_{i=1}^{N_u} \delta u_i \nabla \phi_i f dx dy$$

$$\Rightarrow \sum_{j=1}^{N_u} \sum_{i=1}^{N_u} u_j \delta u_i \int_0^1 \int_0^1 \nabla \phi_j \cdot \nabla \phi_i f dx dy$$

$$= \sum_{i=1}^{N_u} \delta u_i \int_0^1 \int_0^1 f \nabla \phi_i \cdot \nabla \phi_i dx dy$$

Since δu_i is arbitrary

$$\Rightarrow \sum_{j=1}^{N_u} \sum_{i=1}^{N_u} u_j u_i \int_0^1 \int_0^1 \nabla \phi_j \cdot \nabla \phi_i dx dy$$

$$= \sum_{i=1}^{N_u} u_i \int_0^1 \int_0^1 f \nabla \phi_i \cdot \nabla \phi_i dx dy$$

$$\Rightarrow \sum_{j=1}^{N_n} \sum_{i=1}^{N_n} U_j \int_0^1 \int_0^1 \nabla \phi_j \cdot \nabla \phi_i \, dx dy - \sum_{i=1}^{N_n} \int_0^1 \int_0^1 f \nabla \phi_i \cdot dx dy$$

\Rightarrow get Matrix form.

$$A_{ij} U_j = b_j$$

Hence, variation in first
yields the later,

Question 4: Contour plots

1. Figures 1,2,3,4 and 5 show the solution contour plots of $u_h(x, y)$ under different space resolution at $p = 1$.

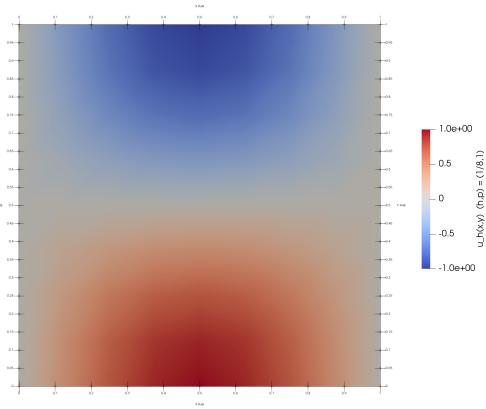


Figure 1: $u_h(x, y)$ (h, p) = (1/8, 1)

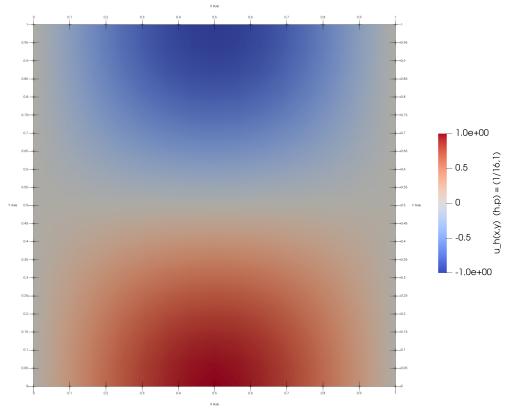


Figure 2: $u_h(x, y)$ (h, p) = (1/16, 1)

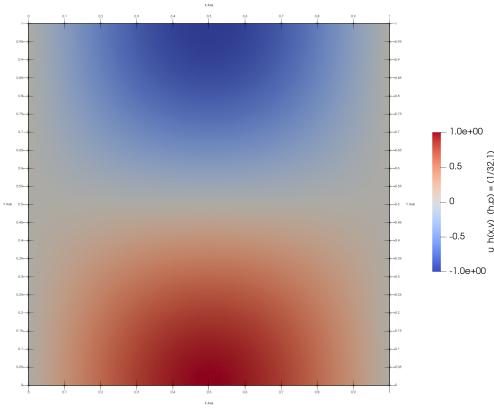


Figure 3: $u_h(x, y)$ (h, p) = (1/32, 1)

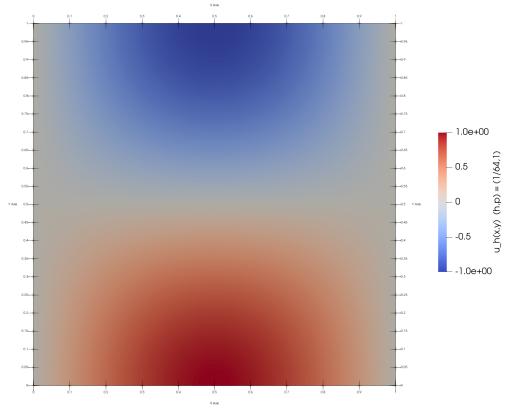


Figure 4: $u_h(x, y)$ (h, p) = (1/64, 1)

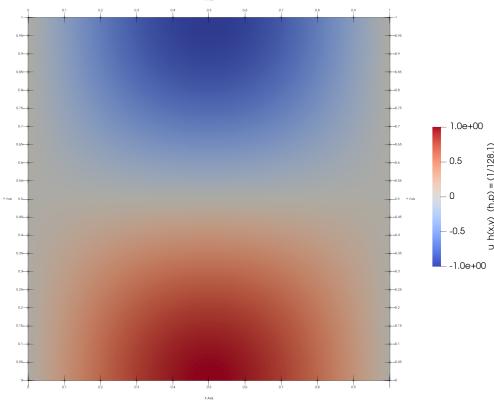


Figure 5: $u_h(x, y)$ (h, p) = (1/128, 1)

2. Figure 6 to 20 shows the Error contour of error $|u_h - u_e|$ for spatial resolution of 8×8 , 16×16 , 32×32 , 64×64 , 128×128 at $p = 1, 2, 3$ in each case.

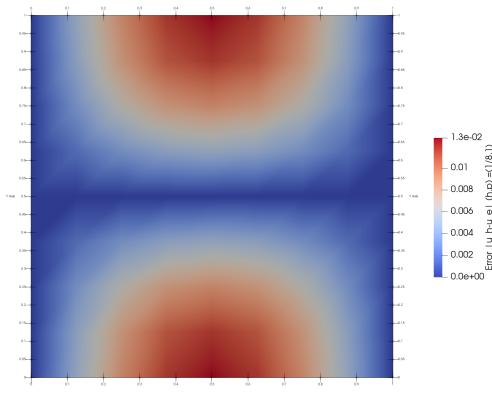


Figure 6: $\text{Error}|u_h - u_e|$ (h, p) = (1/8, 1)

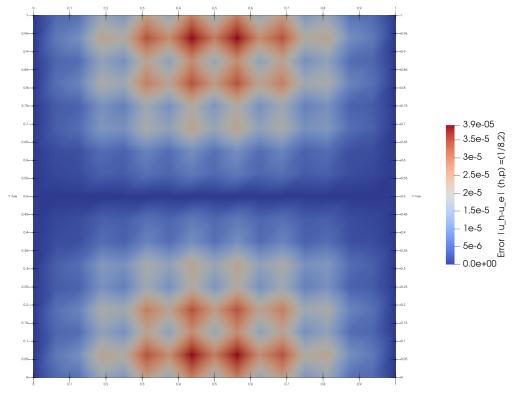


Figure 7: $\text{Error}|u_h - u_e|$ (h, p) = (1/8, 2)

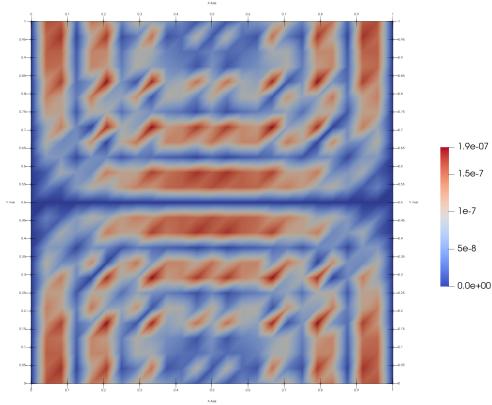


Figure 8: $\text{Error}|u_h - u_e|$ (h, p) = (1/8, 3)

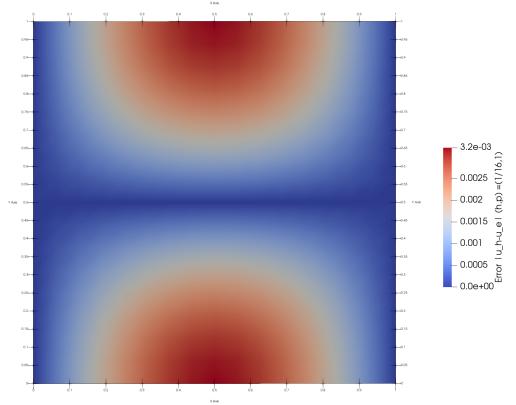


Figure 9: $\text{Error}|u_h - u_e|$ (h, p) = (1/16, 1)

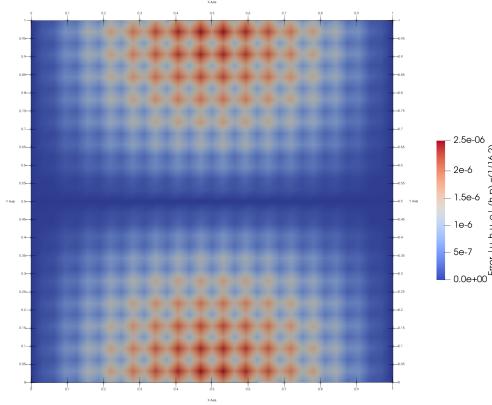


Figure 10: $\text{Error}|u_h - u_e|$ (h, p) = (1/16, 2)

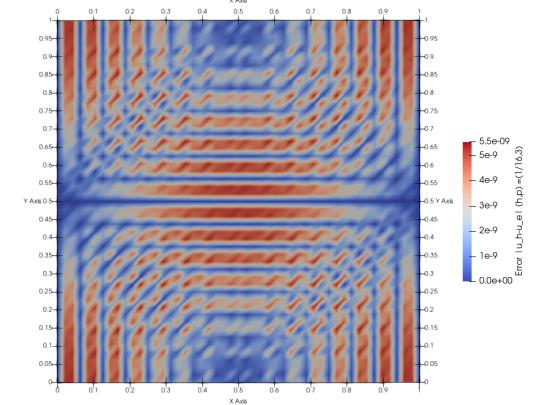


Figure 11: $\text{Error}|u_h - u_e|$ (h, p) = (1/16, 3)

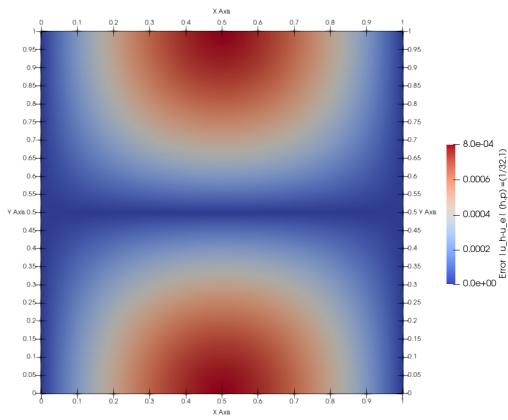


Figure 12: $Error|u_h - u_e| (h, p) = (1/32, 1)$

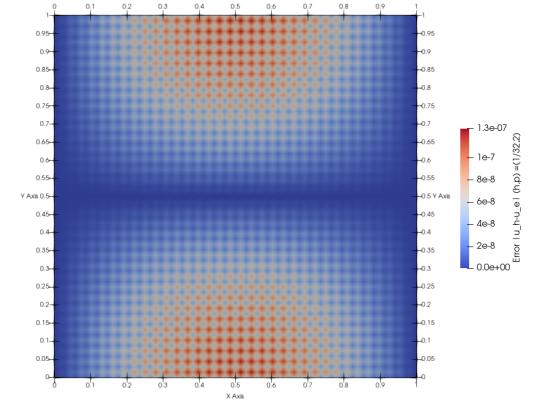


Figure 13: $Error|u_h - u_e| (h, p) = (1/32, 2)$

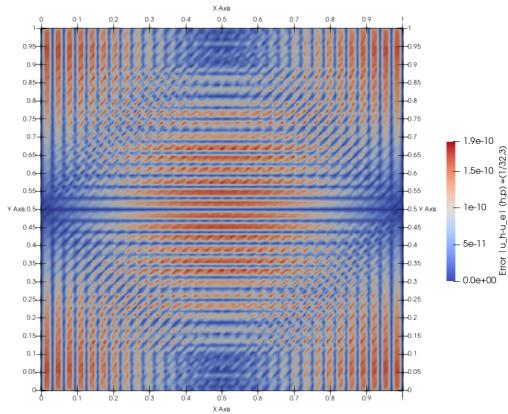


Figure 14: $Error|u_h - u_e| (h, p) = (1/32, 3)$

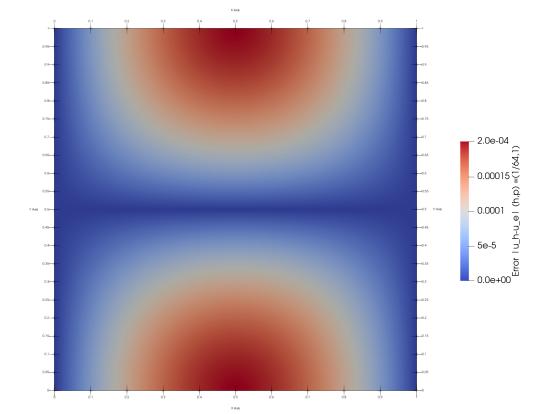


Figure 15: $Error|u_h - u_e| (h, p) = (1/64, 1)$

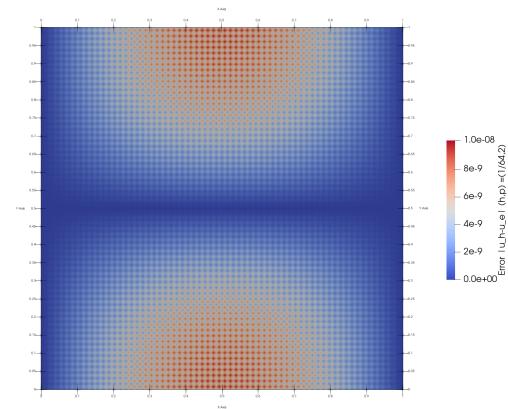


Figure 16: $Error|u_h - u_e| (h, p) = (1/64, 2)$

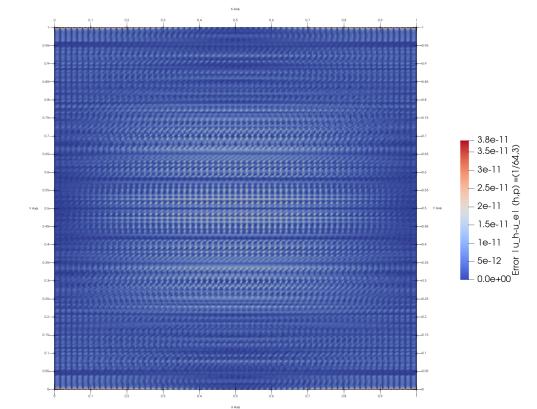


Figure 17: $Error|u_h - u_e| (h, p) = (1/64, 3)$

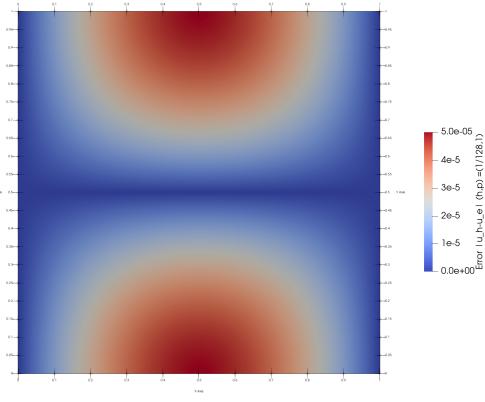


Figure 18: $Error|u_h - u_e| (h,p) = (1/128, 1)$

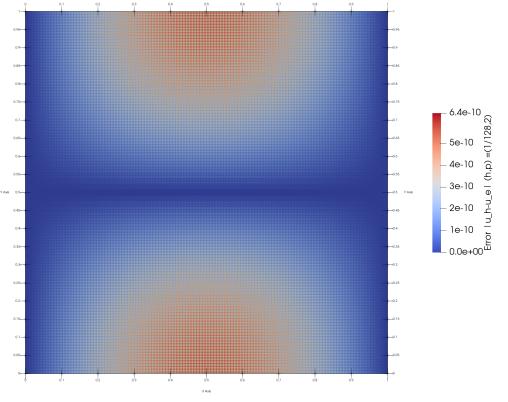


Figure 19: $Error|u_h - u_e| (h,p) = (1/128, 2)$

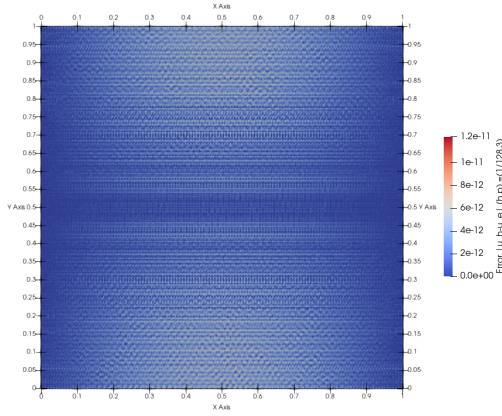


Figure 20: $Error|u_h - u_e| (h,p) = (1/128, 3)$

- From Figure 6 to 20, as the n or $1/h$ (Spatial resolution) increases, The error $|u_h - u_e|$ increase too in order of $O(h^2)$ when $p = 1$.
- When refining p , it reduces the error more effectively than refining n . Therefore, it is advisable to prioritize the refinement of p to minimize error. However, it is essential to find the optimal combination of both (h, p) to achieve the lowest error. Increasing p while keeping h at a smaller value can result in missing important features due to insufficient spatial resolution.
- From Figures 17 and 20, both exhibiting an error on the order of $O(10^{-11})$, we observe that for the combinations of $(h, p) = (1/128, 3)$ and $(h, p) = (1/64, 3)$, higher values of p result in minimized error. Additionally, the effect of increasing the value of n becomes negligible.



Question 5

I have commented in the *PoissonsEq2(Commented).py* file, which is uploaded to Github.

Question 6

As per your feedback on numerics exercises 1 and 2 I need to redo a few FVM problems again, I will redo it by next week and submit it all together.