Fluid Dynamics — Numerical Techniques

MATH5453M Numerical Exercises 2, 2024
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Task 1

We begin with the linearized shallow-water system of equations:

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0, \frac{\partial u}{\partial t} + \frac{\partial (g\eta)}{\partial x} = 0, \tag{1}$$

where we let $u = u(x,t)$ be the velocity, $\eta = \eta(x,t)$ be the free-surface deviation, H_0 be the constant rest depth, and g be the gravitational acceleration.

The variables are scaled as follows:

$$u = U_0 u', \quad x = L_s x', \quad t = \frac{L_s}{U_0} t', \quad \eta = H_{0s}\eta', \quad H = H_{0s}H',$$

where U_0 is a reference velocity, L_s is a characteristic length, and H_{0s} is a characteristic water depth. The dimensionless gravity g' is given by:

$$g' = \frac{gH_{0s}}{U_0^2}.$$

Now, we are substituting the scaled variables into the equations (1), for first equation:

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0, \tag{2}$$

the time derivative of η becomes:

$$\frac{\partial \eta}{\partial t} = \frac{U_0}{L_s} H_{0s} \frac{\partial \eta'}{\partial t'},$$

and the spatial derivative of Hu becomes:

$$\frac{\partial(Hu)}{\partial x} = H_{0s}U_0\left(\frac{\partial H'u'}{\partial x'}\right)\frac{1}{L_s}.$$

Thus the first equation (2) becomes:

$$\frac{H_{0s}U_0}{L_s}\left(\frac{\partial\eta'}{\partial t'} + \frac{\partial H'u'}{\partial x'}\right) = 0. \tag{3}$$

For second equation of (1):

$$\frac{\partial u}{\partial t} + \frac{\partial(g\eta)}{\partial x} = 0, \tag{4}$$

The time derivative of u becomes:

$$\frac{\partial u}{\partial t} = \frac{U_0^2}{L_s}\frac{\partial u'}{\partial t'},$$

and the spatial derivative of $g\eta$ becomes:

$$\frac{\partial(g\eta)}{\partial x} = \frac{g'U_0^2H_{0s}}{L_s}\frac{\partial\eta'}{\partial x'}.$$

Thus, the second equation (4) becomes:

$$\frac{U_0^2}{L_s}\left(\frac{\partial u'}{\partial t'} + g'\frac{\partial\eta'}{\partial x'}\right) = 0. \tag{5}$$

And finally, we got the scaled (dimensionless) form of the shallow-water equations as:

$$\frac{\partial\eta'}{\partial t'} + \frac{\partial H'u'}{\partial x'} = 0, \tag{6}$$

and

$$\frac{\partial u'}{\partial t'} + g'\frac{\partial\eta'}{\partial x'} = 0. \tag{7}$$

As we can see clearly, the scaled form look the same as the original equation (1). Therefore from this point we drop the primes.

Now for a special case where $H(x) = H_o$ constant, we are given the scaled, linearized shallow-water equations:

$$\frac{\partial\eta}{\partial t} + \frac{\partial(H_0u)}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + \frac{\partial(g\eta)}{\partial x} = 0. \tag{8}$$

It will be easier to rewrite the equation (8) in the vector form as:

$$\frac{\partial}{\partial t}\begin{pmatrix}\eta\\ u\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}H_0u\\ g\eta\end{pmatrix}=0. \tag{9}$$

Then, we multiply the second equation from (8) by H_0 so we get:

$$\frac{\partial}{\partial t}\begin{pmatrix}\eta\\ H_0u\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}H_0u\\ H_0g\eta\end{pmatrix}=0. \tag{10}$$

Since our goal is in this form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\eta\\ H_0u\end{pmatrix}+A\frac{\partial}{\partial x}\begin{pmatrix}\eta\\ H_0u\end{pmatrix}=0, \tag{11}$$

where A is a 2×2 matrix that, when multiplied by $\begin{pmatrix}\eta\\ H_0u\end{pmatrix}$, produces the vector $\begin{pmatrix}H_0u\\ H_0g\eta\end{pmatrix}$.

To find A, we set up the matrix equation:

$$A\begin{pmatrix}\eta\\ H_0u\end{pmatrix}=\begin{pmatrix}H_0u\\ H_0g\eta\end{pmatrix}.$$

Let's assume:

$$A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}.$$

Then:

$$A\begin{pmatrix}\eta\\ H_0u\end{pmatrix}=\begin{pmatrix}a_{11}\eta+a_{12}H_0u\\ a_{21}\eta+a_{22}H_0u\end{pmatrix}.$$

For the first component, we want $a_{11}\eta+a_{12}H_0u=H_0u$. This implies:

$$a_{11}=0 \quad\text{and}\quad a_{12}=1.$$

For the second component, we want $a_{21}\eta+a_{22}H_0u=H_0g\eta$. This implies:

$$a_{21}=H_0g \quad\text{and}\quad a_{22}=0.$$

Thus, we have:

$$A=\begin{pmatrix}0 & 1\\ H_0g & 0\end{pmatrix}.$$

Finally, we can rewrite the original equation as:

$$\frac{\partial}{\partial t}\begin{pmatrix}\eta\\ H_0u\end{pmatrix}+A\frac{\partial}{\partial x}\begin{pmatrix}\eta\\ H_0u\end{pmatrix}=0, \tag{12}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ H_0 g & 0 \end{pmatrix},$$

or we can write A as:

$$A = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix}, \tag{13}$$

with $c_0 = \sqrt{H_0 g}$, the wave speed.

To find the eigenvalues of A, we solve the characteristic equation:

$$\det(A - \lambda I) = 0. \tag{14}$$

Substituting $A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ c_0^2 & -\lambda \end{pmatrix}$, we have

$$\det(A - \lambda I) = \lambda^2 - c_0^2 = 0. \tag{15}$$

Thus, the eigenvalues are:

$$\lambda_1 = c_0, \quad \lambda_2 = -c_0. \tag{16}$$

Now, we calculate the eigenvectors associated with $\lambda_1 = c_0$ and $\lambda_2 = -c_0$. For $\lambda_1 = c_0$, we solve:

$$\begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ c_0 \end{pmatrix} = c_0 \begin{pmatrix} 1 \\ c_0 \end{pmatrix}. \tag{17}$$

For $\lambda_2 = -c_0$, we solve:

$$\begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -c_0 \end{pmatrix} = -c_0 \begin{pmatrix} 1 \\ -c_0 \end{pmatrix}. \tag{18}$$

Thus, The eigenvector for λ_1 is $\begin{pmatrix} 1 \\ c_0 \end{pmatrix}$ and the eigenvector for λ_2 is $\begin{pmatrix} 1 \\ -c_0 \end{pmatrix}$.

Then we construct the matrix B of eigenvectors:

$$B = \begin{pmatrix} 1 & 1 \\ c_0 & -c_0 \end{pmatrix}. \tag{19}$$

To make B normalized or simplify calculations in later steps, we apply a scaling factor of $\frac{1}{2c_0}$. This gives:

$$B = \frac{1}{2c_0} \begin{pmatrix} 1 & -1 \\ c_0 & c_0 \end{pmatrix}. \tag{20}$$

The scaling factor doesn't change the eigenvalues or the directions of the eigenvectors, but it helps make B and its inverse B^{-1} easier to work with. The inverse of B will now have a simpler form, which can simplify calculations when we later compute $B^{-1}AB$.

Now we look for B^{-1}, note that the determinan of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by:

$$\det(B) = ad - bc. \tag{21}$$

For our matrix B (20), we have:

$$a = \frac{1}{2c_0}, \quad b = \frac{-1}{2c_0}, \quad c = \frac{c_0}{2c_0} = \frac{1}{2}, \quad d = \frac{c_0}{2c_0} = \frac{1}{2}.$$

Thus, the determinant of B is:

$$\det(B) = \left(\frac{1}{2c_0}\right) \cdot \left(\frac{1}{2}\right) - \left(\frac{-1}{2c_0}\right) \cdot \left(\frac{1}{2}\right),$$

$$\det(B) = \frac{1}{4c_0} + \frac{1}{4c_0} = \frac{2}{4c_0} = \frac{1}{2c_0}.$$

So the inverse of B is:

$$B^{-1} = \frac{1}{\frac{1}{2c_0}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2c_0} \\ -\frac{1}{2} & \frac{1}{2c_0} \end{pmatrix},$$

$$B^{-1} = 2c_0 \begin{pmatrix} \frac{1}{2} & \frac{1}{2c_0} \\ -\frac{1}{2} & \frac{1}{2c_0} \end{pmatrix},$$

$$B^{-1} = \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix}. \tag{22}$$

We already have (13), (20), and (22). To diagonalize A, we compute:

$$\begin{aligned} B^{-1}AB &= \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \left[\frac{1}{2c_0} \begin{pmatrix} 1 & -1 \\ c_0 & c_0 \end{pmatrix} \right], \\ &= \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2c_0} & \frac{-1}{2c_0} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \\ &= \begin{pmatrix} c_0^2 & c_0^2 \\ c_0 & -c_0 \end{pmatrix} \begin{pmatrix} \frac{1}{2c_0} & \frac{-1}{2c_0} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \\ &= \begin{pmatrix} c_0 & 0 \\ 0 & -c_0 \end{pmatrix}. \end{aligned} \tag{23}$$

Now, we define the Riemann invariants r_1 and r_2:

$$r = (r_1, r_2)^T = B^{-1} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix}. \tag{24}$$

The definiton (24) gives us:

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix}. \tag{25}$$

Therefore we get:

$$r_1 = H_0 u + c_0 \eta, \quad r_2 = H_0 u - c_0 \eta. \tag{26}$$

To show that the system can be rewritten as a decoupled set of linear advection equations, we proceed as follows.

First we already have matrix form of (13) and (22), then we rewrite the system using (24). We let $\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = Br$, where $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$, so we have:

$$\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = B \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \tag{27}$$

Next we differentiate with respect to t. Taking the time derivative of both sides, we get:

$$\frac{\partial}{\partial t} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = B \frac{\partial r}{\partial t}. \tag{28}$$

Then, we substitute (28) into the original equation (12) and we get:

$$B \frac{\partial r}{\partial t} + AB \frac{\partial r}{\partial x} = 0. \tag{29}$$

Multiplying both sides by B^{-1} and we get:

$$\frac{\partial r}{\partial t} + B^{-1} AB \frac{\partial r}{\partial x} = 0. \tag{30}$$

Now, we subtitute $B^{-1}AB$ from (23) and we get the equation in decoupled form as:

$$\frac{\partial r}{\partial t} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{\partial r}{\partial x} = 0. \tag{31}$$

This is a decoupled set of linear advection equations for r_1 and r_2, with advection speeds $\lambda_1 = c_0$ and $\lambda_2 = -c_0$.

Task 1: additional

Recall that we are given the scaled, linearized shallow-water equations (8), now we will let the first equation be the continuity equation as:

$$\frac{\partial \eta}{\partial t} + \frac{\partial H_0 u}{\partial x} = 0, \tag{32}$$

and the second equation be the momentum equation as:

$$\frac{\partial u}{\partial t} + \frac{\partial g\eta}{\partial x} = 0. \tag{33}$$

We then multiply the continuity equation (32) by $c_0 = \sqrt{gH_0}$:

$$c_0 \frac{\partial \eta}{\partial t} + c_0 \frac{\partial H_0 u}{\partial x} = 0, \tag{34}$$

and multiply the momentum equation (33) by H_0:

$$H_0 \frac{\partial u}{\partial t} + H_0 \frac{\partial g\eta}{\partial x} = 0. \tag{35}$$

Adding (34) to the momentum equation (35) gives:

$$\left(c_0 \frac{\partial \eta}{\partial t} + c_0 \frac{\partial H_0 u}{\partial x} \right) + \left(H_0 \frac{\partial u}{\partial t} + H_0 \frac{\partial g\eta}{\partial x} \right) = 0, \tag{36}$$

Rearrange (36) be:

$$\left(H_0 \frac{\partial u}{\partial t} + c_0 \frac{\partial \eta}{\partial t} \right) + \left(c_0 \frac{\partial H_0 u}{\partial x} + H_0 \frac{\partial g\eta}{\partial x} \right) = 0,$$

$$\left(\frac{\partial H_0 u}{\partial t} + \frac{\partial c_0 \eta}{\partial t} \right) + \left(c_0 \frac{\partial H_0 u}{\partial x} + H_0 g \frac{\partial \eta}{\partial x} \right) = 0.$$

Recall that $c_0 = \sqrt{gH_0}$ so we get:

$$\left(\frac{\partial H_0 u}{\partial t} + \frac{\partial c_0 \eta}{\partial t} \right) + \left(c_0 \frac{\partial H_0 u}{\partial x} + {c_0}^2 \frac{\partial \eta}{\partial x} \right) = 0.$$

$$\left(\frac{\partial H_0 u}{\partial t} + \frac{\partial c_0 \eta}{\partial t} \right) + c_0 \left(\frac{\partial H_0 u}{\partial x} + \frac{\partial c_0 \eta}{\partial x} \right) = 0.$$

Finally we got the form of equation (31) as :

$$\frac{\partial}{\partial t} \left(H_0 u + c_0 \eta \right) + c_0 \frac{\partial}{\partial x} \left(H_0 u + c_0 \eta \right) = 0, \tag{37}$$

that define the first Riemann invariant:

$$r_1 = H_0 u + c_0 \eta. \tag{38}$$

For the second Rienmann invariant, we recall the equation (32) and (33), we then multiply the continuity equation (32) by $-c_0$ and get:

$$-c_0 \frac{\partial \eta}{\partial t} - c_0 \frac{\partial H_0 u}{\partial x} = 0, \tag{39}$$

and multiply the momentum equation (33) by H_0:

$$H_0 \frac{\partial u}{\partial t} + H_0 \frac{\partial g \eta}{\partial x} = 0. \tag{40}$$

Adding (39) to the equation (40) gives:

$$\left(-c_0 \frac{\partial \eta}{\partial t} - c_0 \frac{\partial H_0 u}{\partial x}\right) + \left(H_0 \frac{\partial u}{\partial t} + H_0 \frac{\partial g \eta}{\partial x}\right) = 0, \tag{41}$$

Rearrange (41) be:

$$\left(H_0 \frac{\partial u}{\partial t} - c_0 \frac{\partial \eta}{\partial t}\right) + \left(-c_0 \frac{\partial H_0 u}{\partial x} + H_0 \frac{\partial g \eta}{\partial x}\right) = 0,$$

$$\left(\frac{\partial H_0 u}{\partial t} - \frac{\partial c_0 \eta}{\partial t}\right) + \left(-c_0 \frac{\partial H_0 u}{\partial x} + H_0 g \frac{\partial \eta}{\partial x}\right) = 0.$$

Recall that $c_0 = \sqrt{gH_0}$ so we get:

$$\left(\frac{\partial H_0 u}{\partial t} - \frac{\partial c_0 \eta}{\partial t}\right) + \left(-c_0 \frac{\partial H_0 u}{\partial x} + {c_0}^2 \frac{\partial \eta}{\partial x}\right) = 0.$$

Factoring out $-c_0$ on second term and we get:

$$\left(\frac{\partial H_0 u}{\partial t} - \frac{\partial c_0 \eta}{\partial t}\right) - c_0 \left(\frac{\partial H_0 u}{\partial x} - \frac{\partial c_0 \eta}{\partial x}\right) = 0.$$

Thus, we got the form of equation (31) as :

$$\frac{\partial}{\partial t}\left(H_0 u - c_0 \eta\right) - c_0 \frac{\partial}{\partial x}\left(H_0 u - c_0 \eta\right) = 0, \tag{42}$$

that define the second Riemann invariant as:

$$r_2 = H_0 u - c_0 \eta. \tag{43}$$

Task 2

We already got the updated system (31), where the matrix of eigenvalues

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

represents two independent characteristics for the system, with $\lambda_1 = c_0$ and $\lambda_2 = -c_0$. We also have the piecewise constant initial data for r_1 and r_2:

$$r_1(x,0) = \begin{cases} r_{1l} & \text{for } x < 0, \\ r_{1r} & \text{for } x \geq 0, \end{cases} \quad r_2(x,0) = \begin{cases} r_{2l} & \text{for } x < 0, \\ r_{2r} & \text{for } x \geq 0. \end{cases} \tag{44}$$

This setup describes a typical Riemann problem where there is a discontinuity at $x = 0$ for both r_1 and r_2, and our task is to determine the solution $r(x,t)$ at later times. To solve this system, we use the method of characteristics. Since the system is diagonal, we treat each component independently.

From (37), the equation for r_1 becomes:

$$\frac{\partial r_1}{\partial t} + c_0 \frac{\partial r_1}{\partial x} = 0, \tag{45}$$

and from (42) the equation for r_2 becomes:

$$\frac{\partial r_2}{\partial t} - c_0 \frac{\partial r_2}{\partial x} = 0. \tag{46}$$

These are standard linear advection equations with the respective speeds of propagation.

We solve these equations along the characteristic curves. For r_1, the characteristic curves are given by:

$$x = c_0 t + x_0, \tag{47}$$

where x_0 is the initial position. The solution for r_1 remains constant along these characteristics, so:

$$r_1(x,t) = \begin{cases} r_{1l} & \text{for } x < c_0 t, \\ r_{1r} & \text{for } x \geq c_0 t. \end{cases} \tag{48}$$

Similarly, for r_2, the characteristics are:

$$x = -c_0 t + x_0, \tag{49}$$

and the solution remains constant along these lines. Hence,

$$r_2(x,t) = \begin{cases} r_{2l} & \text{for } x < -c_0 t, \\ r_{2r} & \text{for } x \geq -c_0 t. \end{cases} \tag{50}$$

Next, we express the solutions in terms of the physical variables u and η. The Riemann invariants r_1 and r_2 are related to these physical variables by the equation (17). We can solve for u and η from these equations:

$$H_0 u = \frac{1}{2}(r_1 + r_2), \tag{51}$$

$$\eta = \frac{1}{2c_0}(r_1 - r_2). \tag{52}$$

Thus, to find the solution for $H_0 u(x,t)$ and $\eta(x,t)$, we substitute the solutions for $r_1(x,t)$ and $r_2(x,t)$ into these formulas.

For $x < -c_0 t$, both r_1 and r_2 are equal to their initial values on the left. Therefore,

$$H_0 u(x,t) = H_0 u_l, \quad \eta(x,t) = \eta_l, \tag{53}$$

where

$$H_0 u_l = \frac{1}{2}(r_{1l} + r_{2l}), \quad \eta_l = \frac{1}{2c_0}(r_{1l} - r_{2l}).$$

For $-c_0 t \leq x \leq c_0 t$, in this region, r_1 and r_2 are given by r_{1l} and r_{2r}, respectively. Thus, the solution is constant:

$$H_0 u(x,t) = \frac{1}{2}(r_{1l} + r_{2r}), \quad \eta(x,t) = \frac{1}{2c_0}(r_{1l} - r_{2r}). \tag{54}$$

For $x > c_0 t$, both r_1 and r_2 are equal to their initial values on the right. Hence,

$$H_0 u(x,t) = H_0 u_r, \quad \eta(x,t) = \eta_r, \tag{55}$$

where

$$H_0 u_r = \frac{1}{2}(r_{1r} + r_{2r}), \quad \eta_r = \frac{1}{2c_0}(r_{1r} - r_{2r}).$$

Thus, we can write the final solution for $H_0 u(x,t)$ and $\eta(x,t)$ as:

$$H_0 u(x,t) = \frac{1}{2}(r_1(x,t) + r_2(x,t)) = \begin{cases} H_0 u_l & \text{for } x < -c_0 t, \\ \frac{1}{2}(r_{1l} + r_{2r}) & \text{for } -c_0 t \leq x \leq c_0 t, \\ H_0 u_r & \text{for } x > c_0 t, \end{cases} \tag{56}$$

$$\eta(x,t) = \frac{1}{2c_0}(r_1(x,t) - r_2(x,t)) = \begin{cases} \eta_l & \text{for } x < -c_0 t, \\ \frac{1}{2c_0}(r_{1l} - r_{2r}) & \text{for } -c_0 t \le x \le c_0 t, \\ \eta_r & \text{for } x > c_0 t. \end{cases} \tag{57}$$

If we define the Riemann invariants as:

$$r_1 = H_0 u + \sqrt{gH_0}\,\eta, \quad r_2 = H_0 u - \sqrt{gH_0}\,\eta,$$

then we can write the final solutions for $H_0 u(x,t)$ and $\eta(x,t)$ as:

$$H_0 u(x,t) = \begin{cases} H_0 u_l & \text{for } x < -c_0 t, \\ \frac{1}{2}\left(H_0 u_l + \sqrt{gH_0}\,\eta_l + H_0 u_r - \sqrt{gH_0}\,\eta_r\right) & \text{for } -c_0 t \le x \le c_0 t, \\ H_0 u_r & \text{for } x > c_0 t, \end{cases} \tag{58}$$

$$\eta(x,t) = \begin{cases} \eta_l & \text{for } x < -c_0 t, \\ \frac{1}{2\sqrt{gH_0}}\left(H_0 u_l + \sqrt{gH_0}\,\eta_l - H_0 u_r + \sqrt{gH_0}\,\eta_r\right) & \text{for } -c_0 t \le x \le c_0 t, \\ \eta_r & \text{for } x > c_0 t. \end{cases} \tag{59}$$

If we simplify the final solution further, then we get:

$$H_0 u(x,t) = \begin{cases} H_0 u_l & \text{for } x < -c_0 t, \\ \frac{1}{2}\left(H_0(u_l + u_r) + \sqrt{gH_0}(\eta_l - \eta_r)\right) & \text{for } -c_0 t \le x \le c_0 t, \\ H_0 u_r & \text{for } x > c_0 t, \end{cases} \tag{60}$$

$$\eta(x,t) = \begin{cases} \eta_l & \text{for } x < -c_0 t, \\ \frac{1}{2\sqrt{gH_0}}\left(H_0(u_l - u_r) + \sqrt{gH_0}(\eta_l - \eta_r)\right) & \text{for } -c_0 t \le x \le c_0 t, \\ \eta_r & \text{for } x > c_0 t. \end{cases} \tag{61}$$

Therefore, the solution demonstrates that $H_0 u(x,t)$ and $\eta(x,t)$ are piecewise continuous, consisting of three distinct regions. The values remain constant in the regions $x < -c_0 t$ and $x > c_0 t$, corresponding to the initial left and right states. In the intermediate region $-c_0 t \le x \le c_0 t$, the solution is determined by a combination of the Riemann invariants from the left and right states, leading to a transition zone between $H_0 u_l, \eta_l$ and $H_0 u_r, \eta_r$. This structure reflects the propagation of characteristics with speeds $\pm c_0$, as expected for the given hyperbolic system.

Task 3

To apply the Godunov scheme to the system of equations (8) and derive a time step estimate, we need to follow the steps of the Godunov method, which is based on solving the Riemann problem at each grid interface. The solution to the Riemann problem provides the numerical fluxes, which are then used to update the variables at each grid point.

We are working with the following system of equations:

$$\frac{\partial \eta}{\partial t} + \frac{\partial (H_0 u)}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + \frac{\partial (g\eta)}{\partial x} = 0.$$

In Godunov's method, we first discretize both space and time. Denote the grid points as x_i with grid spacing Δx, and the time levels as t^n with time step Δt. Let η_i^n and u_i^n represent the discrete values of $\eta(x_i, t^n)$ and $u(x_i, t^n)$ at grid point x_i and time step t^n.

The Godunov scheme requires us to compute the flux at each cell interface. For our system, we define the flux for the η equation as:

$$f_\eta(x, t) = H_0 u, \tag{62}$$

and the flux for the u equation as:

$$f_u(x, t) = g\eta. \tag{63}$$

Note that in the Godunov scheme, the flux functions are computed at the interfaces between adjacent cells, not at the cell centers. For example, at the interface between cell i and cell $i+1$, the fluxes $f_\eta^{i+1/2}$ and $f_u^{i+1/2}$ represent the numerical fluxes for η and u, respectively. These fluxes are computed using the values of η and u at the cell interfaces, typically by solving the Riemann problem between adjacent cells.

We need to solve the Riemann problem at the interfaces, which means computing the flux at the boundary between two cells based on the initial values at the interfaces.

At each time step, the update for η and u are given by:

$$\eta_i^{n+1} = \eta_i^n - \frac{\Delta t}{\Delta x}\left(f_\eta(x_{i+\frac{1}{2}}, t^n) - f_\eta(x_{i-\frac{1}{2}}, t^n)\right),$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(f_u(x_{i+\frac{1}{2}}, t^n) - f_u(x_{i-\frac{1}{2}}, t^n)\right),$$

where $f_\eta(x_{i+\frac{1}{2}}, t^n)$ and $f_u(x_{i+\frac{1}{2}}, t^n)$ are the fluxes computed at the interfaces $x_{i+\frac{1}{2}}$ and $x_{i-\frac{1}{2}}$, which depend on the solutions to the Riemann problem between the adjacent cells. These fluxes are determined by the initial values of η and u at the cell interfaces, taking into account the boundary conditions and the propagation speed.

At the interface $x_{i+\frac{1}{2}}$, we compute the values of η and u by solving the Riemann problem between the left and right states:

$$f_\eta(x_{i+\frac{1}{2}}, t^n) = \frac{1}{2}\left(H_0 u_l + H_0 u_r - |c|(\eta_r - \eta_l)\right), \tag{64}$$

$$f_u(x_{i+\frac{1}{2}}, t^n) = \frac{1}{2}\left(g\eta_l + g\eta_r - |c|(u_r - u_l)\right), \tag{65}$$

where η_l and u_l are the left states, η_r and u_r are the right states, and c is the wave speed, which can be determined by the characteristics of the system.

The stability of the Godunov scheme requires that the time step Δt satisfies the CFL condition:

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{|c|}, \tag{66}$$

where $|c|$ is the speed of the fastest wave in the system. For this system, the wave speeds are determined by the characteristics of the system.

To estimate $|c|$, we observe that the characteristics for r_1 and r_2 (which are related to η and u) have speeds c_0 and $-c_0$, so we take $|c| = c_0$, and the CFL condition becomes:

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{c_0}. \tag{67}$$

Thus, the time step estimate is:

$$\Delta t \leq \frac{\Delta x}{c_0}. \tag{68}$$

To implement boundary conditions in the Godunov scheme, we use the following methods for an "open" domain and a "closed" domain. For the open domain, we use extrapolating boundary conditions. These conditions allow the solution to propagate freely across the boundary, as if the domain extends indefinitely. For the velocity u at the boundary, the boundary value is set equal to the velocity inside the domain, continued outward:

$$u_{\text{ext}}(x_0, t) = u_{\text{dom}}(x_0, t), \tag{69}$$

where x_0 is the boundary point and u_{dom} is the velocity inside the domain. For the surface elevation η at the boundary, we extrapolate the value using a forward approximation, ensuring that η at the boundary is the same as the last value inside the domain:

$$\eta_{\text{ext}}(x_0, t) = \eta_{\text{dom}}(x_0, t). \tag{70}$$

Thus, in the open domain, both u and η are freely propagated outward.

For a closed domain, we use ghost values to enforce boundary conditions. A ghost cell is a fictitious cell outside the physical domain used to enforce boundary conditions. For the velocity u, we enforce that the velocity inside the domain and outside the domain are equal but opposite in sign, which ensures no flux across the boundary:

$$u_{\text{ghost}}(x_0, t) = -u_{\text{dom}}(x_0, t), \tag{71}$$

This reflects a "closed" boundary condition, where the velocity is mirrored across the boundary. For the surface elevation η, we set the ghost value equal to the value inside the domain to maintain continuity across the boundary:

$$\eta_{\text{ghost}}(x_0, t) = \eta_{\text{dom}}(x_0, t). \tag{72}$$

Thus, the boundary is closed, with η being continuous and u mirrored across the boundary.

In the Godunov scheme, the fluxes at the boundary are computed using either the extrapolated values for an open domain or the ghost values for a closed domain.

For the open domain, the boundary fluxes are:

$$f_{\eta,\text{ext}} = H_0 u_{\text{ext}}, \quad f_{u,\text{ext}} = g\eta_{\text{ext}}. \tag{73}$$

For the closed domain, the boundary fluxes are:

$$f_{\eta,\text{ghost}} = H_0 u_{\text{ghost}}, \quad f_{u,\text{ghost}} = g\eta_{\text{ghost}}. \tag{74}$$

These conditions ensure the correct handling of boundaries in the scheme.

For solid wall boundary conditions at $x = 0$ and $x = L$, we enforce no flux through the boundary, meaning the normal flux is set to zero. The velocity u is set to zero at the boundary, reflecting no motion of the fluid at the boundary:

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0. \tag{75}$$

The surface elevation η is continuous at the boundary:

$$\eta(0, t) = \eta_{\text{boundary}} \quad \text{and} \quad \eta(L, t) = \eta_{\text{boundary}}, \tag{76}$$

where η_{boundary} is the value of η at the boundary, either determined based on the solution inside the domain or specified as a boundary condition.

In the Godunov scheme, we enforce a no-flux boundary by setting the relevant fluxes to zero at the boundaries. Specifically, the fluxes of $H_0 u$ and $g\eta$ at the boundary should be zero:

$$f_\eta(0,t) = 0, \quad f_\eta(L,t) = 0, \tag{77}$$

$$f_u(0,t) = 0, \quad f_u(L,t) = 0. \tag{78}$$

This ensures that no material flows through the boundary, representing a solid wall where the surface is static.

First, consider the case where $H(x)$ is constant. To extend the discretization to handle a variable but continuous $H(x)$, we treat $H(x)$ as piecewise constant over each computational cell. Specifically, at each cell edge x_i, we approximate $H(x)$ by a constant value H_i (which could be the value of $H(x)$ at the cell center or an interpolated value between neighboring cells).

At each cell edge, the Riemann solution and flux can be calculated by assuming that $H(x)$ is locally constant. For each cell, the fluxes are computed using the Riemann solver for the advection equations, taking into account the locally averaged values of $H(x)$ at the cell edges.

To estimate the time step, we use the CFL condition:

$$\Delta t = \frac{\Delta x}{\max\left(|u| + \sqrt{gH(x)}\right)}, \tag{79}$$

where u is the velocity and $H(x)$ is the surface elevation. The maximum is taken over all cells in the domain. This ensures that the time step is small enough to resolve the wave propagation within each cell.

Thus, by treating $H(x)$ as locally constant, we can extend the Godunov scheme to handle varying $H(x)$, making the scheme applicable to more complex scenarios with heterogeneous topography.