MATH5453M Numerical Exercises 1

1) The equation (1) is given as

$$\begin{cases} u_t - a(t)u_x - \epsilon u_{xx} = 0 & \text{for } x \in [L_p, L] \\ u(x,0) = u_0(x) \\ u(L_p,t) = u(L,t) = 0 \end{cases}$$

A differential equation is linear if the dependent variable and all of it's ordinary/partial derivatives which appear have a linear order (order of 1). This is clearly true for u_t, u_x and u_{xx} in our equation above.

The term $-a(t)u_x$ is called the advection term, since a simple version of the advection equation is

$$u_t = a u_x.$$

Advection is the bulk motion of a fluid.

The term $-\epsilon u_{xx}$ is called the diffusion term, ~~with~~ will small diffusion constant ϵ, since a simple version of the diffusion equation is

$$u_t = \epsilon u_{xx}.$$

Diffusion is the movement of a fluid as it spreads out, keeping the same volume but decresing concentration.

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2) Equation (2.80) in M& M is given as:

$$\delta_t u_j^{n+1/2} = u_j^{n+1} - u_j^n = \left[\Delta t u_t + \tfrac{1}{24}(\Delta t)^3 u_{ttt} + \cdots\right]_j^{n+1/2}$$

$(\cdots = \text{higher order terms})$

To verify this, we expand u_j^{n+1} and u_j^n as Taylor series around $t = t_{n+1/2}$

$$u_j^{n+1} = u_j^{n+1/2} + (t_{n+1} - t_{n+1/2})\frac{\partial}{\partial t}u_j^{n+1/2} + \frac{1}{2!}(t_{n+1} - t_{n+1/2})^2\frac{\partial^2}{\partial t^2}u_j^{n+1/2} + \frac{1}{3!}(t_{n+1} - t_{n+1/2})^3\frac{\partial^3}{\partial t^3}u_j^{n+1} + \cdots$$

Since $t_{n+1} - t_{n+1/2} = \frac{1}{2}\Delta t$, we have:

$$u_j^{n+1} = \left[u + \tfrac{1}{2}\Delta t u_t + \tfrac{1}{2}(\tfrac{1}{2}\Delta t)^2 u_{tt} + \tfrac{1}{6}(\tfrac{1}{2}\Delta t)^3 u_{ttt} + \cdots\right]_j^{n+1/2}$$

$$= \left[u + \tfrac{1}{2}\Delta t u_t + \tfrac{1}{8}(\Delta t)^2 u_{tt} + \tfrac{1}{48}(\Delta t)^3 u_{ttt} + \cdots\right]_j^{n+1/2} \quad (\dagger)$$

Similarly,

$$u_j^n = u_j^{n+1/2} + (t_n - t_{n+1/2})\frac{\partial}{\partial t}u_j^{n+1/2} + \frac{1}{2!}(t_n - t_{n+1/2})^2\frac{\partial^2}{\partial t^2}u_j^{n+1/2} + \frac{1}{3!}(t_n - t_{n+1/2})^3\frac{\partial^3}{\partial t^3}u_j^{n+1/2} + \cdots$$

Since $t_n - t_{n+1/2} = -\frac{1}{2}\Delta t$, we have:

$$u_j^n = \left[u - \tfrac{1}{2}\Delta t u_t + \tfrac{1}{2}(-\tfrac{1}{2}\Delta t)^2 u_{tt} + \tfrac{1}{6}(-\tfrac{1}{2}\Delta t)^3 u_{ttt} + \cdots\right]_j^{n+1/2}$$

$$= \left[u - \tfrac{1}{2}\Delta t u_t + \tfrac{1}{8}(\Delta t)^2 u_{tt} - \tfrac{1}{48}(\Delta t)^3 u_{ttt} + \cdots\right]_j^{n+1/2} \quad (\dagger\dagger)$$

Now, if we compute $(\dagger) - (\dagger\dagger)$, the even terms will cancel,

$$u_j^{n+1} - u_j^n = \left[\Delta t u_t + \tfrac{1}{24}(\Delta t)^3 u_{ttt} + \cdots\right]_j^{n+1/2}$$

Thus, we have proven (2.80)

Equation (2.81) in M&M is:

$$\delta_x^2 u_j^{n+1} = \left[(\Delta x)^2 u_{xx} + \frac{1}{12}(\Delta x)^4 u_{xxxx} + \frac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+1}$$

The second-order central difference, $\delta_x^2 u(x,t)$, is defined

$$\delta_x^2 u(x,t) = u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)$$

i.e. $\delta_x^2 u_j^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$

Proof:

Consider the first-order central differences of u_j^n,

$$\delta_x u_j^n = u_{j+1/2}^n - u_{j-1/2}^n$$

If we apply it once more, we get

$$\begin{aligned}\delta_x^2 u_j^n &= \delta_x u_{j+1/2}^n - \delta_x u_{j-1/2}^n \\ &= (u_{j+1}^n - u_j^n) - (u_j^n - u_{j-1}^n) \\ &= u_{j+1}^n - 2u_j^n + u_{j-1}^n\end{aligned}$$

$$\therefore \delta_x^2 u_j^{n+1} = u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}$$

We expand these three terms as Taylor series around $x = x_j$

$$u_{j+1}^{n+1} = \left[u + \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} + \frac{1}{3!}(\Delta x)^3 u_{xxx} + \frac{1}{4!}(\Delta x)^4 u_{xxxx} + \frac{1}{5!}(\Delta x)^5 u_{xxxxx} + \frac{1}{6!}(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+1}$$

$$-2u_j^{n+1} = [-2u]_j^{n+1}$$

$$u_{j-1}^{n+1} = \left[u - \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} - \frac{1}{3!}(\Delta x)^3 u_{xxx} + \frac{1}{4!}(\Delta x)^4 u_{xxxx} - \frac{1}{5!}(\Delta x)^5 u_{xxxxx} + \frac{1}{6!}(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+1}$$

Adding these together, we see that the first term and all odd terms cancel:

$$\delta_x^2 u_j^{n+1} = \left[(\Delta x)^2 u_{xx} + \frac{1}{12}(\Delta x)^4 u_{xxxx} + \frac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+1}$$

Thus, we have proven (2.81)

Equation (2·82) in M&M is given as;

$$\theta\delta_x^2 U_j^{n+1} + (1-\theta)\delta_x^2 u_j^n = \left[(\Delta x)^2 u_{xx} + \tfrac{1}{12}(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+1/2}$$
$$+ \left(\theta - \tfrac{1}{2}\right)\Delta t\left[(\Delta x)^2 u_{xxt} + \tfrac{1}{12}(\Delta x)^4 u_{xxxxt} + \cdots\right]_j^{n+1/2}$$
$$+ \tfrac{1}{8}(\Delta t)^2(\Delta x)^2\left[u_{xxtt}\right] + \cdots$$

Consider each term on the RHS of (2·81) and expand them around $t = t_{n+1/2}$

$$\left[(\Delta x)^2 u_{xx}\right]_j^{n+1} = (\Delta x)^2\left[u_{xx} + \tfrac{1}{2}\Delta t\, u_{xxt} + \tfrac{1}{2}\left(\tfrac{1}{2}\Delta t\right)^2 u_{xxtt} + \cdots\right]_j^{n+1/2}$$

$$\left[\tfrac{1}{12}(\Delta x)^4 u_{xxxx}\right]_j^{n+1} = \tfrac{1}{12}(\Delta x)^4\left[u_{xxxx} + \tfrac{1}{2}\Delta t\, u_{xxxxt} + \cdots\right]_j^{n+1/2}$$

$$\left[\tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx}\right]_j^{n+1} = \tfrac{2}{6!}(\Delta x)^6\left[u_{xxxxxx} + \cdots\right]_j^{n+1/2}$$

Adding these together, we see that

$$\delta_x^2 u_j^{n+1} = \left[(\Delta x)^2 u_{xx} + \tfrac{1}{12}(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+1/2}$$
$$+ (\Delta t)\left[\tfrac{1}{2}(\Delta x)^2 u_{xxt} + \tfrac{1}{24}(\Delta x)^4 u_{xxxxt} + \cdots\right]_j^{n+1/2} \qquad (*)$$
$$+ (\Delta t)^2\left[\tfrac{1}{8}(\Delta x)^2 u_{xxtt} + \cdots\right]_j^{n+1/2} + \cdots$$

We now do the same for $\delta_x^2 u_j^n$:

$$\left[(\Delta x)^2 u_{xx}\right]_j^n = (\Delta x)^2\left[u_{xx} - \tfrac{1}{2}\Delta t\, u_{xxt} + \tfrac{1}{2}\left(\tfrac{1}{2}\Delta t\right)^2 u_{xxtt} + \cdots\right]_j^{n+1/2}$$

$$\left[\tfrac{1}{12}(\Delta x)^4 u_{xxxx}\right]_j^n = \tfrac{1}{12}(\Delta x)^4\left[u_{xxxx} - \tfrac{1}{2}\Delta t\, u_{xxxxt} + \cdots\right]_j^{n+1/2}$$

$$\left[\tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx}\right]_j^n = \tfrac{2}{6!}(\Delta x)^6\left[u_{xxxxxx} + \cdots\right]_j^{n+1/2}$$

$$\therefore \quad \delta_x^2 u_j^n = \left[(\Delta x)^2 u_{xx} + \tfrac{1}{12}(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+\frac{1}{2}}$$
$$+ (\Delta t)\left[-\tfrac{1}{2}(\Delta x)^2 u_{xxt} - \tfrac{1}{24}(\Delta x)^4 u_{xxxxt} + \cdots\right]_j^{n+1/2} \qquad (**)$$
$$+ (\Delta t)^2\left[\tfrac{1}{8}(\Delta x)^2 u_{xxtt} + \cdots\right]_j^{n+1/2} + \cdots$$

Finally, we can substitute (*) and (**) into the θ discretisation $\theta\delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n$

$$\theta\delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n = \left[\theta(\Delta x)^2 u_{xx} + \tfrac{1}{12}\theta(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}\theta(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+1/2}$$
$$+ (\Delta t)\left[\tfrac{1}{2}\theta(\Delta x)^2 u_{xxt} + \tfrac{1}{24}\theta(\Delta x)^4 u_{xxxxt} + \cdots\right]_j^{n+1/2}$$
$$+ (\Delta t)^2\left[\tfrac{1}{8}\theta(\Delta x)^2 u_{xxtt} + \cdots\right]_j^{n+1}$$
$$+ \left[(1-\theta)(\Delta x)^2 u_{xx} + \tfrac{1}{12}(1-\theta)(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}\theta(1-\theta)(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+1/2}$$
$$+ (\Delta t)\left[-\tfrac{1}{2}(1-\theta)(\Delta x)^2 u_{xxt} - \tfrac{1}{24}(1-\theta)(\Delta x)^4 u_{xxxxt} + \cdots\right]_j^{n+1/2}$$
$$+ (\Delta t)^2 \cancel{\left[\tfrac{1}{8}(\Delta x)^2 u_{xxtt} + \cdots\right]} \left[\tfrac{1}{8}(1-\theta)(\Delta x)^2 u_{xxtt} + \cdots\right]_j^{n+1/2} + \cdots$$

By relating like terms, this can be reduced to:

$$\theta\delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n = \left[(\Delta x)^2 u_{xx} + \tfrac{1}{12}(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx} + (\theta-\tfrac{1}{2})\Delta t(\Delta x)^2 u_{xxt}\right.$$
$$\left. + \tfrac{1}{12}(\theta-\tfrac{1}{2})\Delta t(\Delta x)^4 u_{xxxxt} + \tfrac{1}{8}(\Delta t)^2(\Delta x)^2 u_{xxtt} + \cdots\right]_j^{n+1/2}$$
$$= \left[(\Delta x)^2 u_{xx} + \tfrac{1}{12}(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+1/2}$$
$$+ (\theta-\tfrac{1}{2})\Delta t\left[(\Delta x)^2 u_{xxt} + \tfrac{1}{12}(\Delta x)^4 u_{xxxxt} + \cdots\right]_j^{n+1/2}$$
$$+ \tfrac{1}{8}(\Delta t)^2(\Delta x)^2\left[u_{xxtt} + \cdots\right]_j^{n+1/2}$$
$$+ \cdots$$

Thus, we have proven (2.82)

Finally, equations (2.83) and (2.84) in M&M are given by:

$$\left\{\begin{aligned} T_j^{n+1/2} &:= \frac{\delta_t u_j^{n+1/2}}{\Delta t} - \frac{\theta\delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n}{(\Delta x)^2} \\ &= \left[u_t - u_{xx}\right]_j^{n+1/2} + \left[(\tfrac{1}{2}-\theta)\Delta t u_{xxt} - \tfrac{1}{12}(\Delta x)^2 u_{xxxx} + \cdots\right]_j^{n+1/2} \\ &+ \left[\tfrac{1}{24}(\Delta t)^2 u_{ttt} - \tfrac{1}{8}(\Delta t)^2 u_{xxtt} + \cdots\right]_j^{n+1/2} \\ &+ \left[\tfrac{1}{12}(\tfrac{1}{2}-\theta)\Delta t(\Delta x)^2 u_{xxxxt} - \tfrac{2}{6!}(\Delta x)^4 u_{xxxxxx} + \cdots\right]_j^{n+1/2} + \cdots \end{aligned}\right.$$

From equation (2.80),

$$\frac{\delta_t u_j^{n+1/2}}{\Delta t} = \left[u_t + \tfrac{1}{24}(\Delta t)^2 u_{ttt} + \cdots\right]_j^{n+1/2} \qquad (1)$$

From equation (2.82),

$$\frac{\theta\delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n}{(\Delta x)^2} = \left[u_{xx} + \tfrac{1}{12}(\Delta x)^2 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^4 u_{xxxxxx} + \cdots\right]_j^{n+1/2} \qquad (2)$$
$$+ (\theta-\tfrac{1}{2})\Delta t\left[u_{xxt} + \tfrac{1}{12}(\Delta x)^2 u_{xxxxt} + \cdots\right]_j^{n+1/2}$$
$$+ \tfrac{1}{8}(\Delta t)^2\left[u_{xxtt} + \cdots\right]_j^{n+1/2} + \cdots$$

Collecting like terms and ~~simplifying~~ simplifying, (Computing (1) − (2))

$$T_j^{n+1/2} = \left[u_t - u_{xx}\right]_j^{n+1/2} + \left[(\tfrac{1}{2}-\theta)\Delta t u_{xxt} - \tfrac{1}{12}(\Delta x)^2 u_{xxxx} + \cdots\right]_j^{n+1/2}$$
$$\cancel{+(\theta-\tfrac{1}{2})\Delta t[u_{xxt} + \tfrac{1}{12}(\Delta x}\ + \left[\tfrac{1}{24}(\Delta t)^2 u_{ttt} - \tfrac{1}{8}(\Delta t)^2 u_{xxtt} + \cdots\right]_j^{n+1/2}$$
$$+ \left[\tfrac{1}{12}(\tfrac{1}{2}-\theta)\Delta t(\Delta x)^2 u_{xxxxt} - \tfrac{2}{6!}(\Delta x)^4 u_{xxxxxx} + \cdots\right]_j^{n+1/2} + \cdots$$

Thus, we have proven (2.84)

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(3) Equation (1), along with its initial and boundary conditions, is given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = a(t)\dfrac{\partial u}{\partial x} + \epsilon \dfrac{\partial^2 u}{\partial x^2}, & x \in [L_p, L] \\ & t \geq 0 \\ u(x,0) = u_0(x) & \\ u(L_p, t) = u(L,t) = 0 & \end{cases}$$

with $\epsilon = \text{const.}$, $a(t)$, ~~given~~ $u_0(x)$ given functions.

For the θ-method, we must consider both an explicit and implicit difference scheme.

In the explicit scheme,

$$\frac{\partial u}{\partial t} \approx \frac{\Delta_{+t} U_j^n}{\Delta t} = \frac{U_j^{n+1} - U_j^n}{\Delta t}, \quad \frac{\partial u}{\partial x} = \frac{\Delta_{+x} U_j^n}{\Delta x} = \frac{U_{j+1}^n - U_j^n}{\Delta x}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\delta_x^2 U_j^n}{(\Delta x)^2} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

Putting these together,

$$U_j^{n+1} - U_j^n = a(t_n)\Delta x \mu \Delta_{+x} U_j^n + \epsilon \mu \delta_x^2 U_j^n, \quad \text{where} \quad \mu = \frac{\Delta t}{(\Delta x)^2}$$

Similarly, for the implicit scheme,

$$\frac{\partial u}{\partial t} \approx \frac{\Delta_{+t} U_j^n}{\Delta t} = \frac{U_j^{n+1} - U_j^n}{\Delta t}, \quad \frac{\partial u}{\partial x} = \frac{\Delta_{+x} U_j^{n+1}}{\Delta x}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\delta_x^2 U_j^{n+1}}{(\Delta x)^2}$$

$$\Rightarrow U_j^{n+1} - U_j^n = a(t_{n+1})\Delta x \mu \Delta_{+x} U_j^{n+1} + \epsilon \mu \delta_x^2 U_j^{n+1}$$

Putting these together, the θ-scheme is

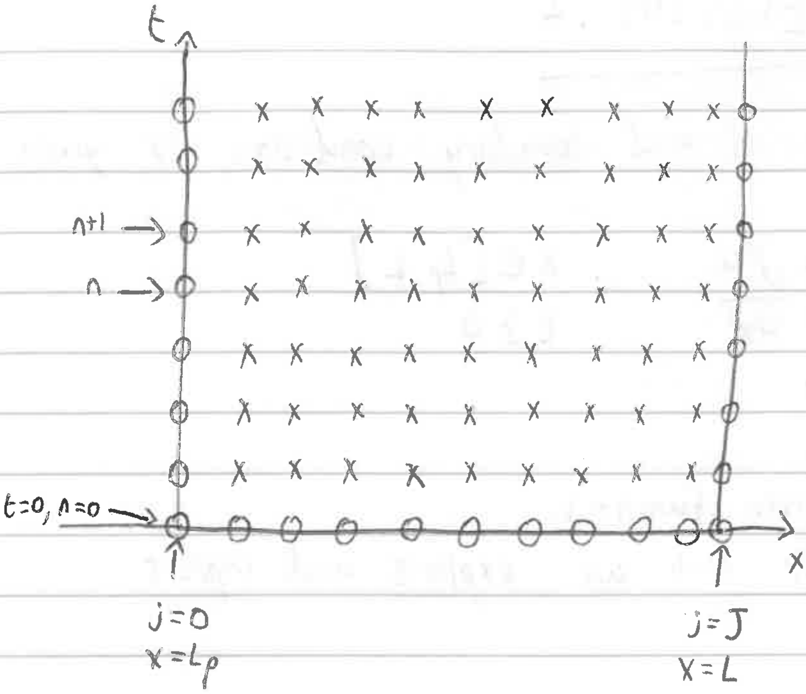
$$U_j^{n+1} - U_j^n = \theta\left[\mu a^{n+1} \Delta x \Delta_{+x} U_j^{n+1} + \mu \epsilon \delta_x^2 U_j^{n+1}\right] + (1-\theta)\left[\mu a^n \Delta x \Delta_{+x} U_j^n + \mu \epsilon \delta_x^2 U_j^n\right] \quad (t)$$

where $a^n = a(t_n)$

In these calculations, we have used

$$U_j^n \approx u(x_j, t_n), \quad x_j = L_p + j\Delta x, \quad t_n = n\Delta t, \quad j = 0, 1, \dots, J, \quad n = 0, 1, \dots$$

$$J = \frac{(L - L_p)}{\Delta x}$$



O = known points on the boundary
x = unknown points in the interior

$$u(x,0)=u_0(x) \Rightarrow U_j^0 = u_0(x_j)$$
$$u(L_p,t)=0 \Rightarrow U_0^n = 0$$
$$u(L,t)=0 \Rightarrow U_J^n = 0$$

We can rewrite the θ-scheme (†) as

$$-\theta\mu\epsilon U_{j-1}^{n+1} + (1+\theta\mu a^{n+1}\Delta x + 2\theta\mu\epsilon)U_j^{n+1} - (\theta\mu a^{n+1}\Delta x + \theta\mu\epsilon)U_{j+1}^{n+1} = [1+(\Delta x a^n \Delta_{+x} + \epsilon\delta_x^2)(1-\theta)\mu] \cdot U_j^n$$

Relating this to (2.65) in M & M, we see that

$$a_j = \theta\mu\epsilon$$
$$b_j = 1+\theta\mu a^{n+1}\Delta x + 2\theta\mu\epsilon = 1+\theta\mu(a^{n+1}\Delta x + 2\epsilon)$$
$$c_j = \theta\mu a^{n+1}\Delta x + \theta\mu\epsilon = \theta\mu(a^{n+1}\Delta x + \epsilon)$$
$$d_j = [1+(\Delta x a^n \Delta_{+x} + \epsilon\delta_x^2)(1-\theta)\mu]U_j^n$$

We now use the Thomas algorithm as described in M & M:

$$U_j^{n+1} = e_j U_{j+1}^{n+1} + f_j \qquad j=1,\dots,J-1 \text{ (interior points)} \quad (*)$$

with
$$e_j = \frac{c_j}{b_j - a_j e_{j-1}} = \frac{\theta\mu(a^{n+1}\Delta x + \epsilon)}{1+\theta\mu(a^{n+1}\Delta x + 2\epsilon) - \theta\mu\epsilon e_{j-1}}$$
$$f_j = \frac{U_j^n + (1-\theta)\mu[a^n\Delta x(U_{j+1}^n - U_j^n) + \epsilon(U_{j+1}^n - 2U_j^n + U_{j-1}^n)] + \theta\mu\epsilon f_{j-1}}{1+\theta\mu(a^{n+1}\Delta x + 2\epsilon) - \theta\mu\epsilon e_{j-1}}$$

and $e_0, f_0 = 0$

This algorithm works by solving along each row of the grid in turn, using values from the row below which are implemented into the recursive relation $f_j\ (U_{j+1}^n, U_j^n, U_{j-1}^n)$. Once a point on a row is calculated, (e.g. U_{j+1}^{n+1}), the point to its left can be evaluated using (*), so the algorithm works from $j=J-1 \to j=1$.

The first row, U_j^0, $j=1,\dots,J-1$ is already given from the ~~boundary~~ initial condition and all U_J^n are given by the boundary condition, so every point can be evaluated.

For time step $t = t_n = n\Delta t$,

$$\begin{pmatrix} U_{J-1}^n \\ U_{J-2}^n \\ \vdots \\ U_2^n \\ U_1^n \end{pmatrix} = \begin{pmatrix} e_{J-1}U_J^n + f_{J-1} \\ e_{J-2}U_{J-1}^n + f_{J-2} \\ \vdots \\ e_2 U_3^n + f_2 \\ e_1 U_2^n + f_1 \end{pmatrix}$$

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4) In M&M, we are given the problem

$$\begin{cases} u_t = u_{xx} & , \quad t>0 \quad , \quad 0<x<1 \\ u(0,t) = u(1,t) = 0 & , \quad t>0 \\ u(x,0) = u^0(x) & , \quad t=0 \quad , \quad 0 \leq x \leq 1 \end{cases}$$

with

$$u^0(x) = \begin{cases} 2x & , \quad 0 \leq x \leq \frac{1}{2} \\ 2-2x & , \quad \frac{1}{2} \leq x \leq 1 \end{cases}$$

This problem has the θ-scheme: $U_j^{n+1} - U_j^n = \mu\left[\theta \delta_x^2 U_j^{n+1} + (1-\theta)\delta_x^2 U_j^n\right]$
The explicit scheme is given by setting $\theta = 0$

$$\Rightarrow \boxed{U_j^{n+1} - U_j^n = \mu \delta_x^2 U_j^n} \qquad (\dagger)$$

Similarly to in question 3), I have applied the Thomas algorithm to this scheme and have fond that

$$U_j^{n+1} = e_j U_{j+1}^{n+1} + f_j$$

with $e_j = \dfrac{\theta\mu}{1+2\theta\mu - \theta\mu e_{j-1}}$, $f_j = \dfrac{U_j^n + \mu(1-\theta)(U_{j-1}^n - 2U_j^n + U_{j+1}^n) + \theta\mu f_{j-1}}{1+2\theta\mu - \theta\mu e_{j-1}}$

Setting $\theta = 0$, $e_j = 0$, $f_j = U_j^n + \mu(U_{j-1}^n - 2U_j^n + U_{j+1}^n)$
Thus,

$$\boxed{U_j^{n+1} = U_j^n + \mu(U_{j-1}^n - 2U_j^n + U_{j+1}^n)}$$

(the same result as if we had expanded the right hand side of $(\dagger)$)

I have implemented the general θ scheme into my Python code and set $\theta = 0$ for the explicit scheme. This has resulted in the eight graphs shown in Fig. 2.2 of M&M, along with two graphs which display multiple time-profiles. I have made it so that the value of Δt is changeable, so results can be fond for both $\Delta t = 0.0012$ and $\Delta t = 0.0013$, as in M&M.

Let us now consider the stability properties of this scheme. Clearly, our results are stable for $\Delta t = 0.0012$ and unstable for $\Delta t = 0.0013$. (with $\Delta x = 0.05$)
We will begin with a Fourier analysis.

Let $U_j^n = \lambda^n e^{ikj\Delta x}$, k is some scalar.

From this we can see that:

$$U_j^{n+1} = \lambda^{n+1} e^{ikj\Delta x} = \lambda U_j^n$$

Substituting this into our difference equation,

$$\lambda \cdot \lambda^n e^{ikj\Delta x} = \lambda^n e^{ikj\Delta x} + \mu \lambda^n \left(e^{ik(j-1)\Delta x} - 2e^{ikj\Delta x} + e^{ik(j+1)\Delta x}\right)$$

$$\Rightarrow \quad \lambda = 1 + \mu\left(e^{-ik\Delta x} - 2 + e^{ik\Delta x}\right)$$

$$= 1 + 2\mu(\cos k\Delta x - 1)$$

$$\lambda = 1 - 4\mu \sin^2\left(\tfrac{1}{2}k\Delta x\right)$$

Since $U_j^{n+1} = \lambda U_j^n$, the system is stable if $|\lambda| \leq 1$

i.e. $\left|1 - 4\mu\sin^2\left(\tfrac{1}{2}k\Delta x\right)\right| \leq 1$

$\max_k \left|1 - 4\mu\sin^2\left(\tfrac{1}{2}k\Delta x\right)\right| = |1 - 4\mu|$ which gives us the "worst" value of k,

$$k = \frac{(2m-1)\pi}{\Delta x}, \quad m \in \mathbb{Z} \quad \left(\therefore \sin^2\left(\frac{k\Delta x}{2}\right) = 1\right)$$

So, $\quad -1 \leq 1 - 4\mu \leq 1$

Since $\mu > 0$, $1 - 4\mu \leq 1$ is always true, so we only need to consider

$$-1 \leq 1 - 4\mu$$

$$\Rightarrow \boxed{\mu \leq \frac{1}{2}} \quad \text{This is our stability condition}$$

i.e. $\dfrac{\Delta t}{(\Delta x)^2} \leq \dfrac{1}{2} \Rightarrow \Delta t \leq \dfrac{(\Delta x)^2}{2}$

Since $J = 20$ (the number of sections the x domain is split into), we have used $\Delta x = 0.05$

$\therefore$ The explicit scheme is stable for $\Delta t \leq 0.00125$

This explains why $\Delta t = 0.0012$ is stable but $\Delta t = 0.0013$ is unstable.

According to the maximum principle, the θ scheme converges with $\Delta x, \Delta t \to 0$ if

$$\mu(1-\theta) \leq \frac{1}{2}$$

Since we are considering the explicit scheme, we set $\theta = 0$ and see that this is the same as the stability condition

$$\mu \leq \frac{1}{2}$$

So, it will converge for $\Delta t = 0.0012$, and not converge for $\Delta t = 0.0013$.

Finally, we extend the explicit scheme to the problem (1). Already in question 3), we computed the θ scheme to be:

$$U_j^{n+1} - U_j^n = \theta\left[\mu a^{n+1} \Delta x \Delta_{+x} U_j^{n+1} + \mu \epsilon \delta_x^2 U_j^{n+1}\right] + (1-\theta)\left[\mu a^n \Delta x \Delta_{+x} U_j^n + \mu \epsilon \delta_x^2 U_j^n\right]$$

Setting $\theta = 0$:

$$U_j^{n+1} - U_j^n = \mu a^n \Delta x \Delta_{+x} U_j^n + \mu \epsilon \delta_x^2 U_j^n$$

Expanding the differences, we get the explicit scheme:

$$\left[U_j^{n+1} = U_j^n + \mu a^n \Delta x (U_{j+1}^n - U_j^n) + \mu \epsilon (U_{j+1}^n - 2U_j^n + U_{j-1}^n)\right], \quad j = 1, 2, \dots, J-1, \quad n = 0, 1, \dots$$

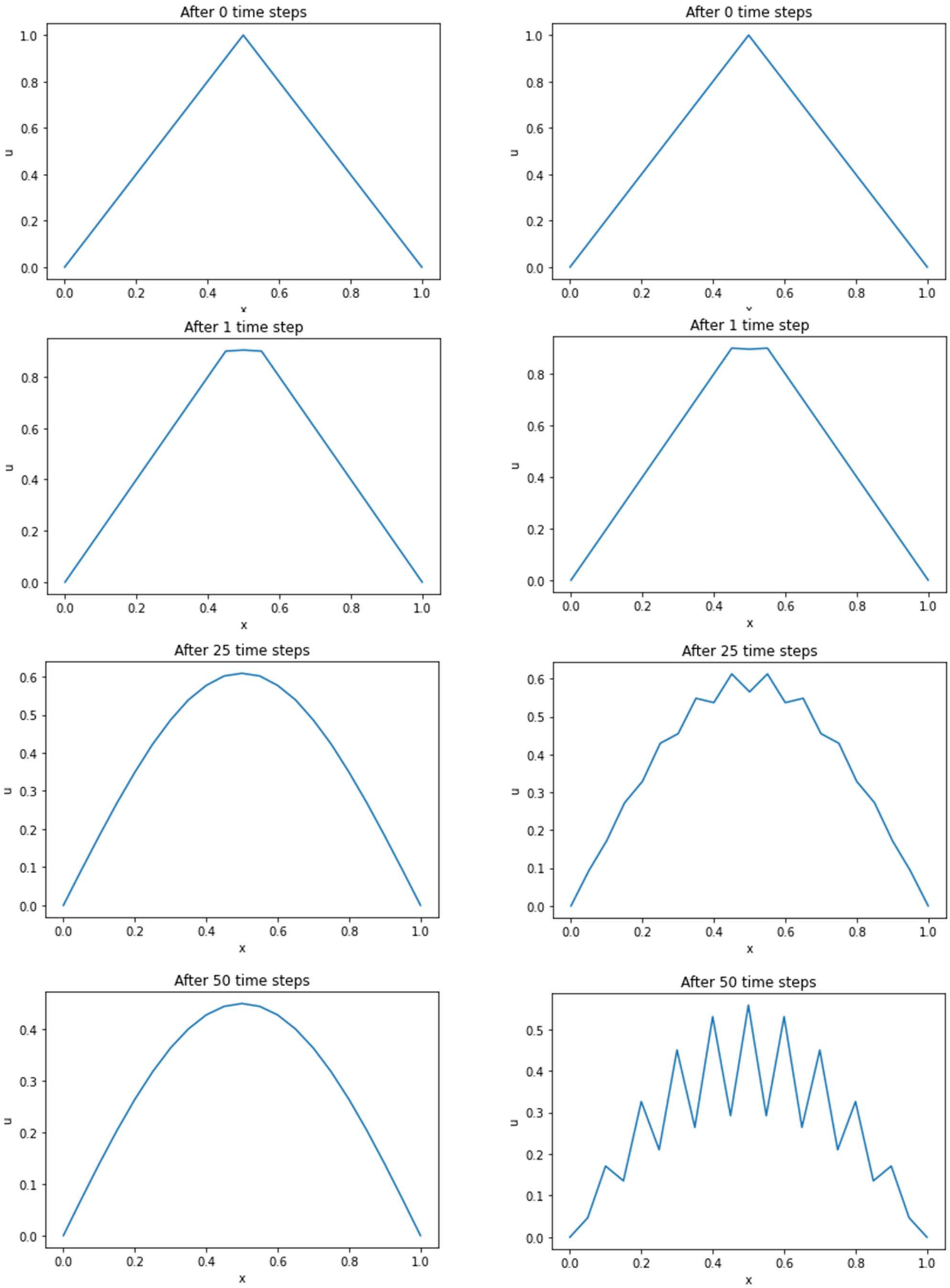

Figure 1: Recreation of Fig. 2.2 in M&M, using the explicit scheme to solve equations (2.7)-(2.9). Plots on the left show results for Δt=0.0012 and plots on the right show results for Δt=0.0013. Results produced using M&M.py

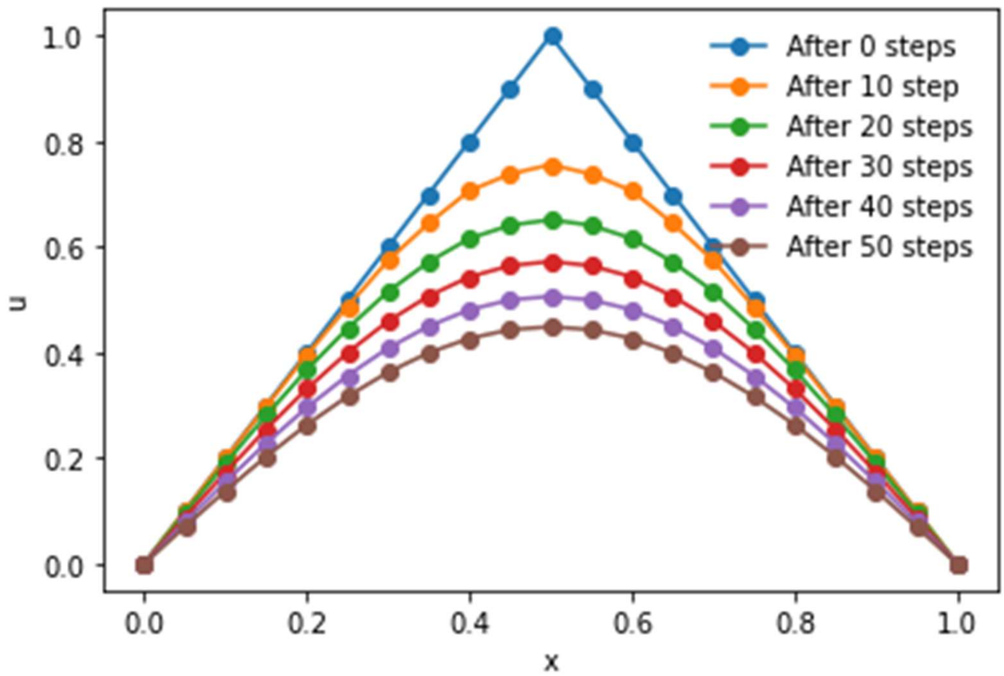

Figure 2: Results of an explicit scheme for equations (2.7)-(2.9) in M&M, with Δt=0.0012 after multiples of ten time steps. Results produced using M&M.py

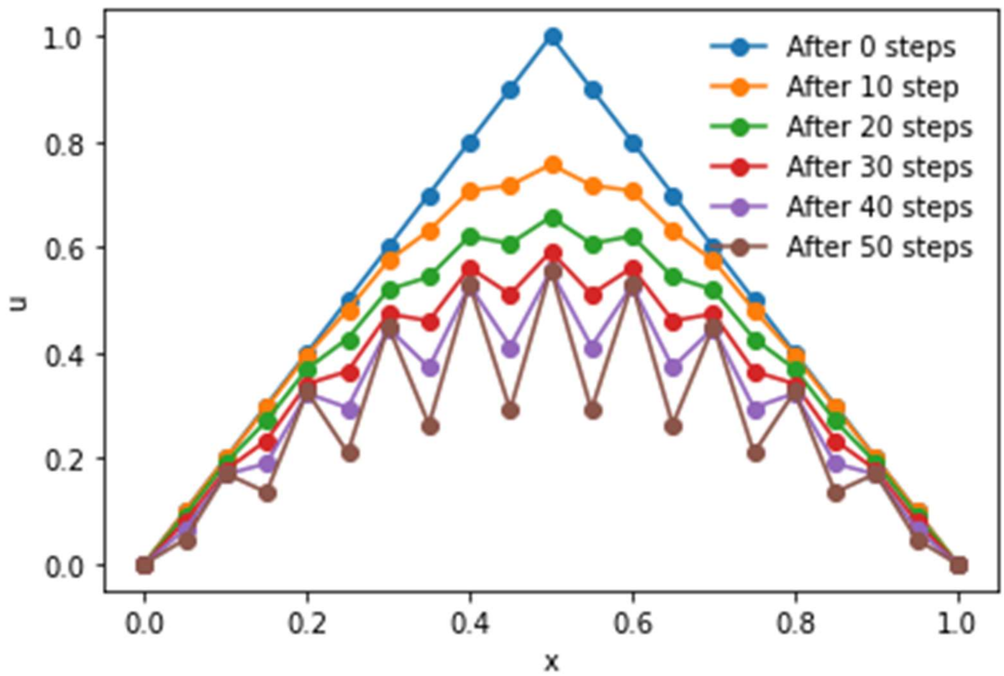

Figure 3: Results using the same scheme as Figure 2, but with Δt=0.0013. The stability of the results seems to decrease as more time steps are performed.

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5) The θ-scheme for problem (2.7)-(2.9) given in M&M is given by

$$U_j^{n+1} - U_j^n = \mu[\theta\delta_x^2 U_j^{n+1} + (1-\theta)\delta_x^2 U_j^n], \quad j = 1, 2, \cdots, J-1$$
$$\theta \in [0,1]$$

This can be expanded and rewritten as

$$-\theta\mu U_{j-1}^{n+1} + (1+2\theta\mu)U_j^{n+1} - \theta\mu U_{j+1}^{n+1} = U_j^n + (1-\theta)\mu(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

This clearly satisfies the conditions (2.67) and (2.68), which means we can use the Thomas algorithm, which gives us

$$\boxed{U_j^{n+1} = e_j U_{j+1}^{n+1} + f_j}$$

with $e_j = \dfrac{\theta\mu}{1+2\theta\mu - \theta\mu e_{j-1}}$ and $f_j = \dfrac{U_j^n + (1-\theta)\mu(U_{j-1}^n - 2U_j^n + U_{j+1}^n) + \theta\mu f_{j-1}}{1+2\theta\mu - \theta\mu e_{j-1}}$

I have implemented this scheme into my code to produce a plot of time profiles after 0, 10, 20, 30, 40 and 50 time steps.
Setting $\theta = 0$, we see that these are the same results as in question 4)
(I have also implemented the θ-scheme for problem (1) in question 6))

Let us now perform a Fourier analysis on the advection and diffusion parts of (1) in separation.
The diffusion part of the θ-scheme is given by:

$$U_j^{n+1} - U_j^n = \theta\mu\epsilon(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) + (1-\theta)\mu\epsilon(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

Let $U_j^n = \lambda^n e^{ijk\Delta x}$ for some scalar k

$$\lambda^{n+1}e^{ijk\Delta x} - \lambda^n e^{ijk\Delta x} = \theta\mu\epsilon\lambda^{n+1}(e^{i(j+1)k\Delta x} - 2e^{ijk\Delta x} + e^{i(j-1)k\Delta x})$$
$$+ (1-\theta)\mu\epsilon\lambda^n(e^{i(j+1)k\Delta x} - 2e^{ijk\Delta x} + e^{i(j-1)k\Delta x})$$

Dividing through by $\lambda^n e^{ijk\Delta x}$

$$\lambda - 1 = \theta\mu\epsilon\lambda(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) + (1-\theta)\mu\epsilon(e^{ik\Delta x} - 2 + e^{-ik\Delta x})$$

Since $e^{-ik\Delta x} + e^{ik\Delta x} - 2 = 2\cos k\Delta x - 2 = -4\sin^2(\tfrac{1}{2}k\Delta x)$, this can be rearranged to

$$\lambda = \frac{1 - 4(1-\theta)\mu\epsilon\sin^2(\tfrac{1}{2}k\Delta x)}{1 + 4\theta\mu\epsilon\sin^2(\tfrac{1}{2}k\Delta x)}$$

For stability, we require that $|\lambda| \leq 1$.
Since the numerator is always less than or equal to 1 and the denominator is always greater than or equal to 1, $\lambda \leq 1$ must always be true.

$$\therefore \quad -1 \leq \frac{1-4(1-\theta)\mu\epsilon\sin^2(\frac{1}{2}k\Delta x)}{1+4\theta\mu\epsilon\sin^2(\frac{1}{2}k\Delta x)}$$

$$\Rightarrow \quad (1-2\theta)\mu\epsilon\sin^2(\tfrac{1}{2}k\Delta x) \leq \tfrac{1}{2}$$

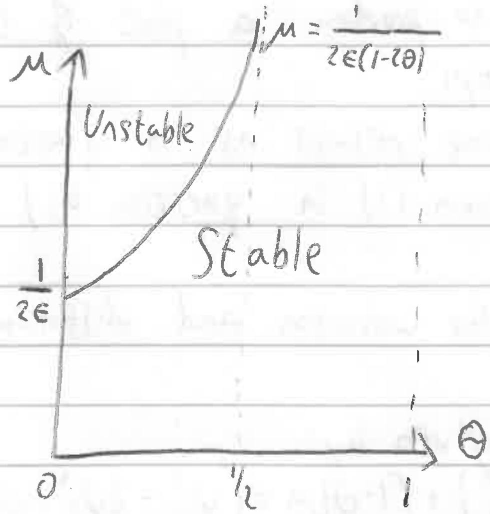
$\max_k \left|(1-2\theta)\mu\epsilon\sin^2(\frac{1}{2}k\Delta x)\right| = \cancel{(1-2\theta)}|(1-2\theta)\mu\epsilon|$ which gives the "worst" value of k

$$k = \frac{(2m-1)\pi}{\Delta x}, \quad m \in \mathbb{Z} \quad \left(\text{i.e. } \sin^2\left(\frac{k\Delta x}{2}\right) = 1\right).$$

So, $(1-2\theta)\mu\epsilon \leq \frac{1}{2}$

$$\Rightarrow \boxed{\mu \leq \frac{1}{2\epsilon(1-2\theta)}}$$

This is our stability condition for the diffusion part.



As seen on this graph, if $\theta \geq \frac{1}{2}$, then the diffusion part will be stable, independent from the choice of μ.
If $\theta = 0$, then we have stability for $\mu \leq \frac{1}{2\epsilon}$, which agrees with the example in question 4) where $\epsilon = 1$.

The advection part of the θ-scheme is given by:

$$U_j^{n+1} - U_j^n = \theta\mu a^{n+1}\Delta x(U_{j+1}^{n+1} - U_j^{n+1}) + (1-\theta)\mu a^n\Delta x(U_{j+1}^n - U_j^n)$$

Once again, let $U_j^n = \lambda^n e^{ijk\Delta x}$ for scalar k.
Substituting in and rearranging as in the diffusion case,

$$\lambda = \frac{1+(1-\theta)\mu a^n\Delta x(e^{ik\Delta x}-1)}{1-\theta\mu a^{n+1}\Delta x(e^{ik\Delta x}-1)}$$

To find the "worst" k, we want $|\lambda|$ to be as high as possible, this is achieved by finding $\max|e^{ik\Delta x}-1|$ to maximise the numerator and minimise the denominator.

Since $e^{ix} = \cos x + i\sin x$ and $|x+iy| = \sqrt{x^2+y^2}$, we can compute that

$$|e^{ik\Delta x} - 1| = \sqrt{2 - 2\cos k\Delta x}$$

which is maximal when $\cos k\Delta x = -1 \Rightarrow \max|e^{ik x} - 1| = \sqrt{4} = 2$

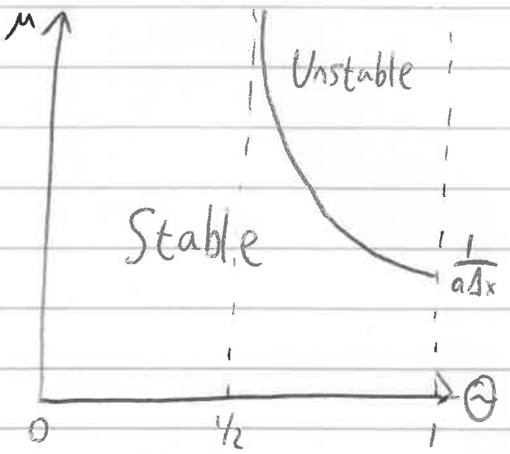
$$\therefore \quad \left| \frac{1 + 2(1-\theta)\mu a^n \Delta x}{1 - 2\theta\mu a^{n+1}\Delta x} \right| \le 1$$

Once again, the numerator ≥ 1 and the denominator ≤ 1 so $\lambda \le 1$ is always true.

$$-1 \le \frac{1 + 2(1-\theta)\mu a^n \Delta x}{1 - 2\theta\mu a^{n+1}\Delta x}$$

If we set $a = \text{const.}$ this simplifies to

$$\boxed{\mu \le \frac{1}{a\Delta x(2\theta - 1)}}$$



If $\theta \le \frac{1}{2}$, then the advection part will be stable.

If $\theta = 1$, then we have stability for $\mu \le \frac{1}{a\Delta x}$

Since $\mu = \Delta t/(\Delta x)^2$, this can be rearranged to $\frac{a\Delta t}{\Delta x} \le 1$.

This corresponds to the CFL condition with $CFL = 1$, and $\epsilon = 0$ to ignore diffusion.

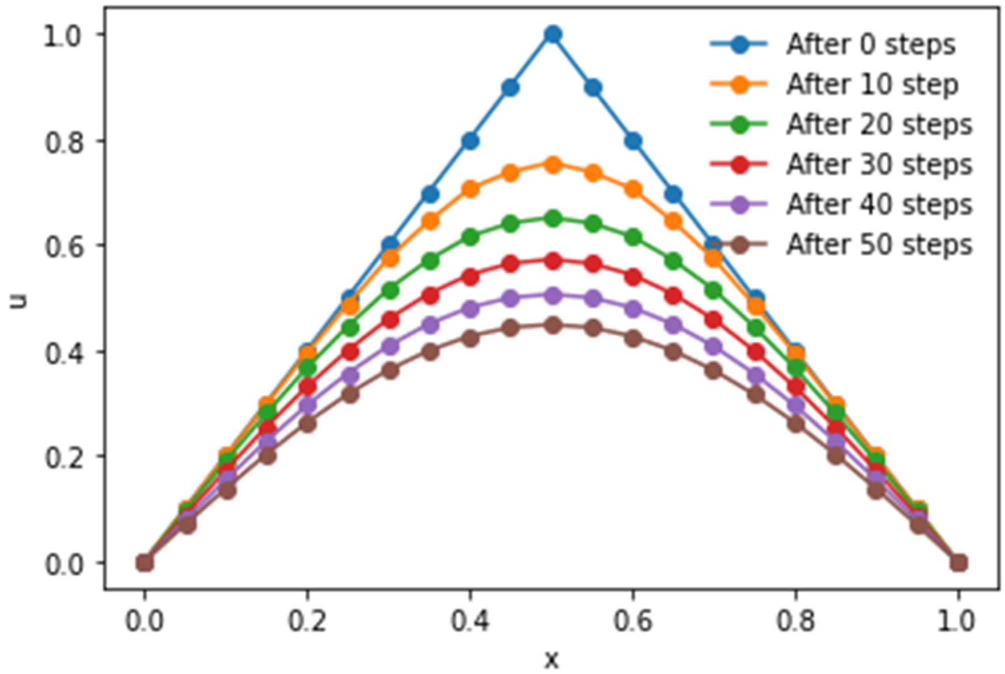
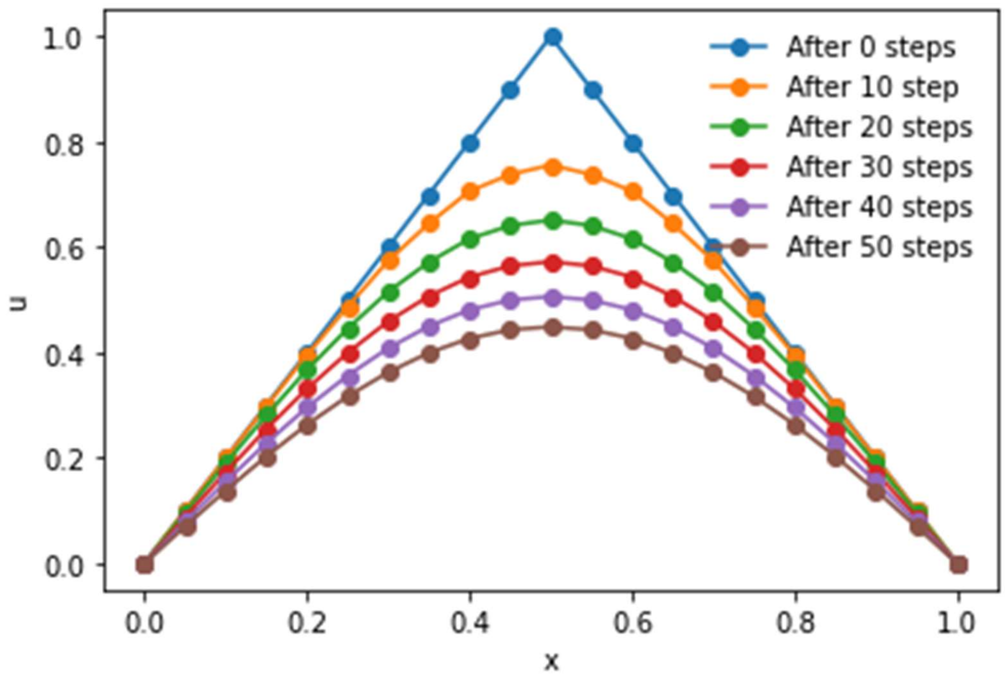


Figure 4: Results of the θ-scheme for equations (2.7)-(2.9) in M&M. Setting θ=0 and Δt=0.0012, the plot is identical to Figure 2 (shown below for reference), which is to be expected. Results produced using M&M-theta.py.



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6) I have coded the θ scheme to solve the problem

$$\begin{cases} u_t - a(t)u_x - \epsilon u_{xx} = 0 \quad , \quad x \in [L_p, L] \quad , \quad t \in [0,T] \\ u(x,0) = u_0(x) \\ u(L_p,t) = u(L,t) = 0 \end{cases}$$

with $L_p = -1$, $L = 1$, $\epsilon = 10^{-3}$, $T = 1$

In the first code, I have implemented the initial condition

$$u_0(x) = (1-x)^4(1+x)$$

and in the second, I have implemented

$$u_0(x) = (1-x)^4(1+x)\left(\sum_{k=0}^{3} b_k \phi_k(x) + C\right)$$

where ϕ_k are Legendre polynomials, b_k are uniformally distributed random coefficients in the range $[0,1)$, and $C \geq 0$ is a constant determined such that the final bracket on the right-hand side is non-negative.

For the values of b_k, I used the function 'random.seed(314)' which gave me

$b_k = [0.9168735814929675, 0.5885419136419221, 0.2650477511724620\!4,$
$0.7832053795417385]$

These values are printed when the code is ran.

In question 3), we defined an explicit scheme for equation (1), which is

$$U_j^{n+1} = U_j^n + a^n \Delta x \mu \Delta_{+x} U_j^n + \epsilon \mu \delta_x^2 U_j^n$$
$$= U_j^n + a^n \Delta x \mu (U_{j+1}^n - U_j^n) + \epsilon \mu (U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

I have also coded this scheme and for all three codes produced ~~time~~ ~~profiles~~ plots displaying time profiles for $t = 0, 0.01, 0.25, 0.5, 1$.

For the θ scheme codes, I have displayed 6 plots for each: depending on whether ~~$\mu =$~~ $\theta = 0$ or 1, and whether $\mu = 0.49$ ($J = \frac{1}{\Delta x} = 14$, $N = 100$), 0.5625 ($J = \frac{1}{\Delta x} = 15$, $N = \frac{1}{\Delta t} = 100$) or 6.25 ($J = \frac{1}{\Delta x} = 50$, $N = \frac{1}{\Delta t} = 100$).

We see for all cases that the results are stable for both $\mu < \frac{1}{2}$ and $\mu > \frac{1}{2}$, unlike in the explicit scheme. We also see smoother curves when considering a smaller spatial step ($\mu = 6.25$).

Comparing the graphs for $\theta = 0$ and the explicit scheme, we see that they are identical as assumed.

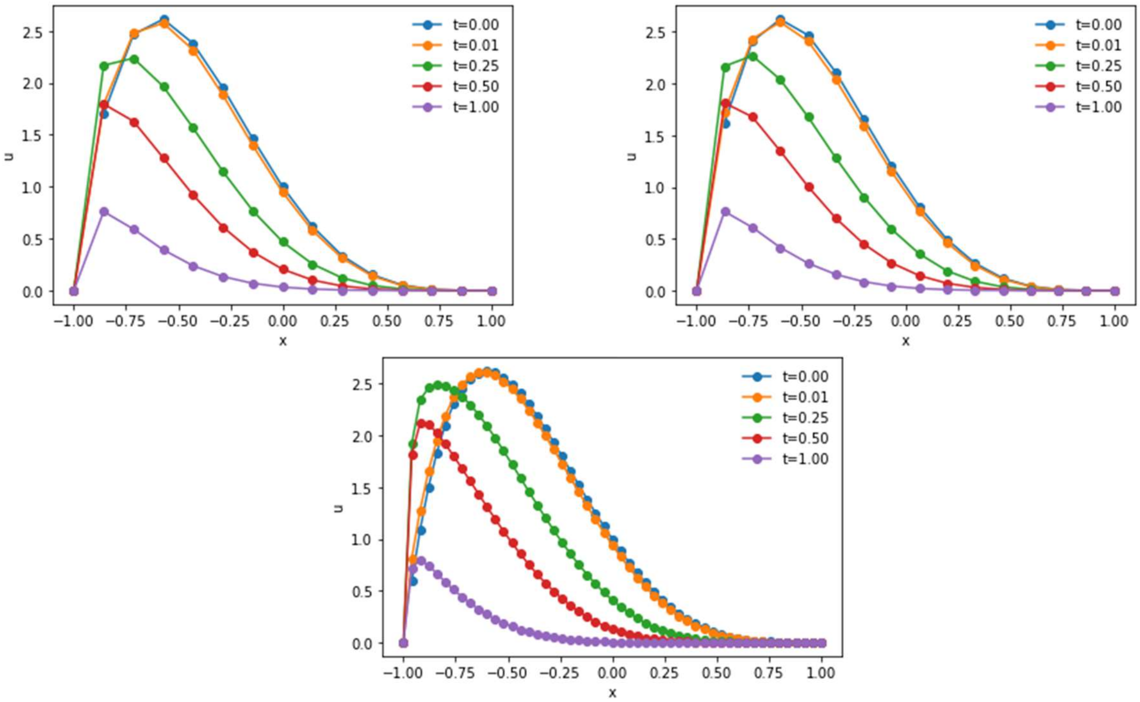

Figure 5: Results for the θ-scheme for equation (1) given in the exercise sheet, with the first initial condition given in exercise 6. Δt=0.01, θ=0 and μ=0.49, 0.5625 and 6.25. The μ value has been increased by decreasing the discretisation step Δx, which creates a smoother curve. We see that the results are stable for all choices of μ. Results produced by theta-1.py.

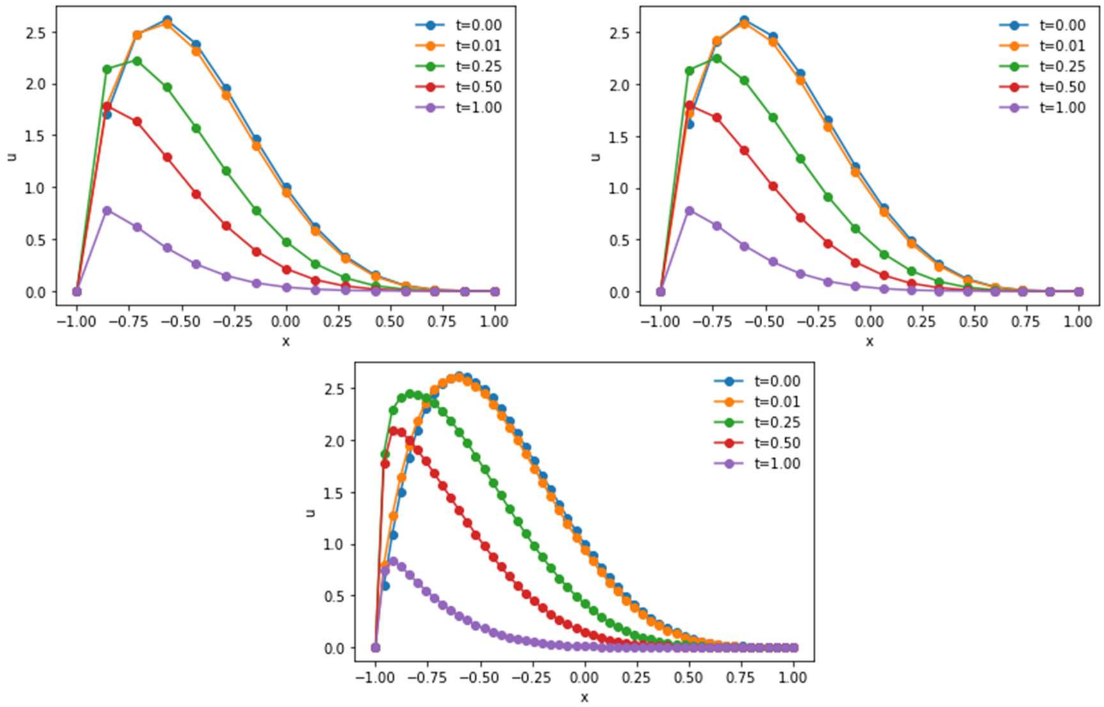

Figure 6: Results using the same initial condition and variables as Figure 5, but with θ=1. We see that the results are very similar. Results produced by theta-1.py.

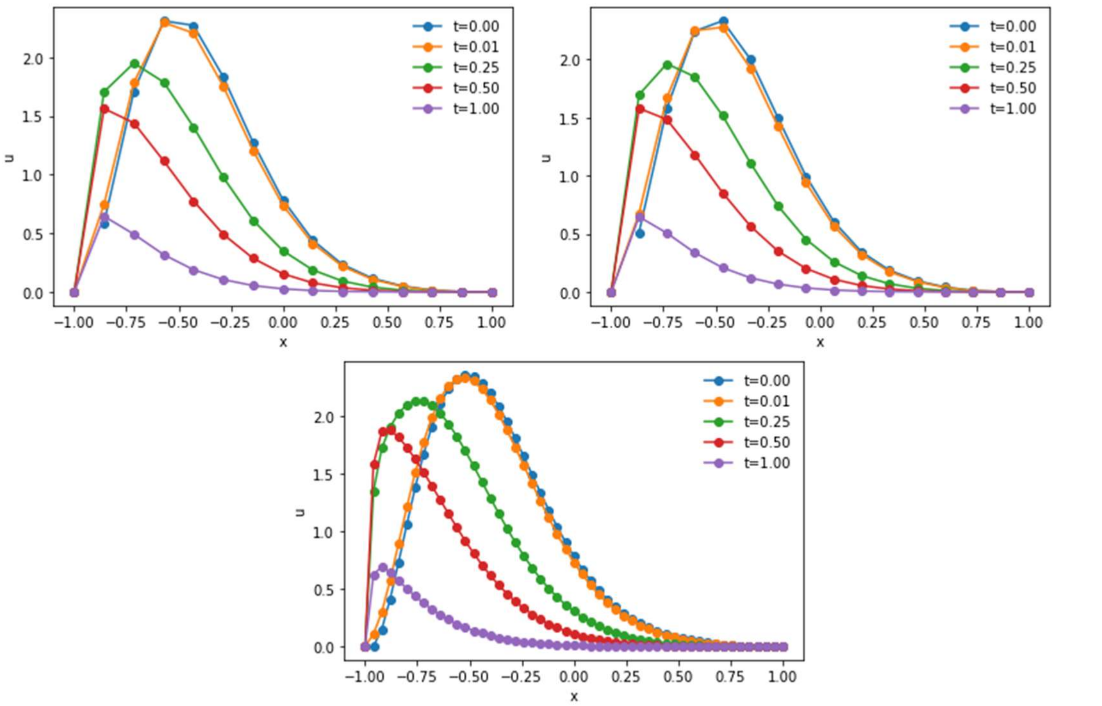

Figure 7: Results from the same variables as Figure 5, but with the second initial condition. Θ=0. Results produced by theta-2.py.

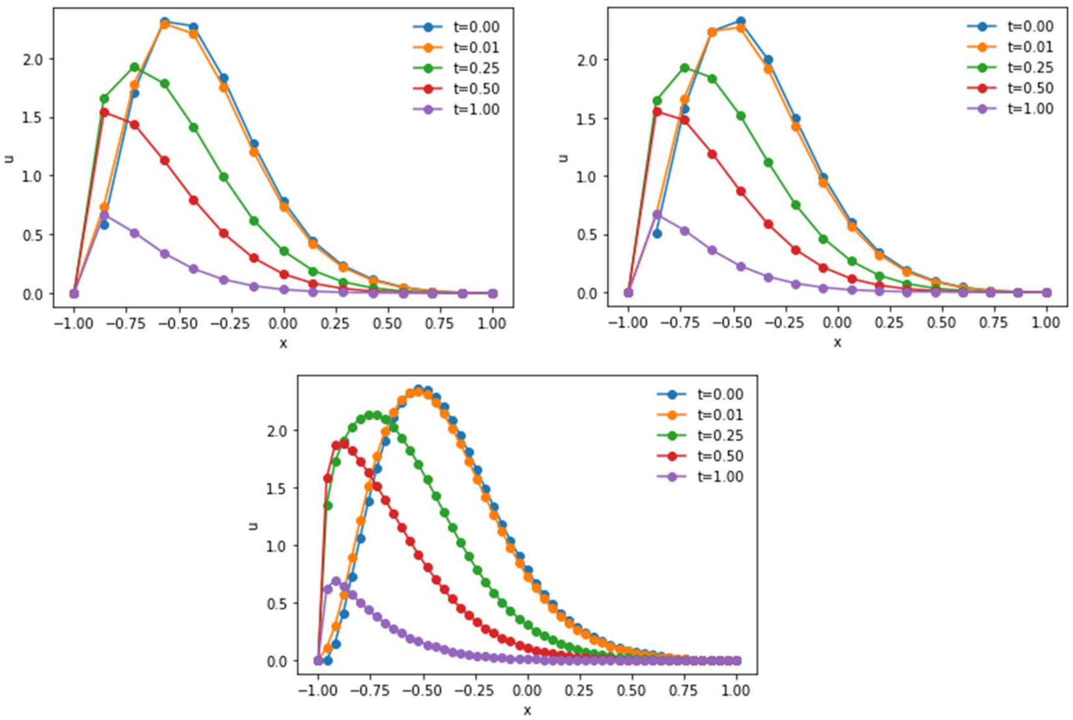

Figure 8: Results from the same variables as Figure 6, but with the second initial condition. Θ=1. Results produced by theta-2.py.

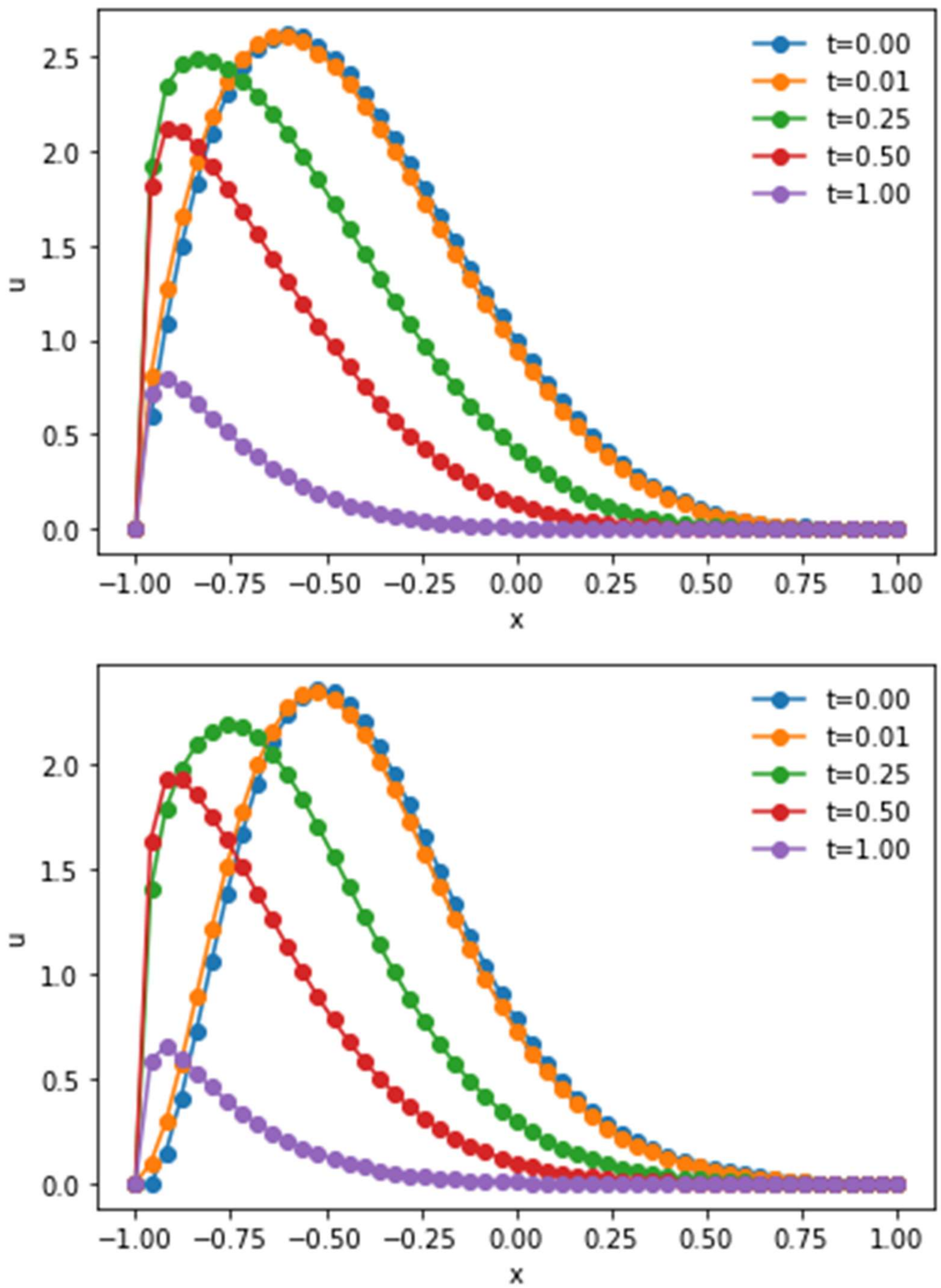

Figure 9: Results of an explicit scheme for problem (1) given in the exercise sheets. These graphs are identical to the plots with μ=6.25 in Figures 5 and 7 which each have θ=0, as expected. These graphs were produced by explicit-scheme.py.

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7) Using the code implemented in question 6), I have plotted four graphs with $\theta = 1$, refining Δx each time.

The values I have used are:

- $J = 10$, $\Delta x = 0.2$
- $J = 20$, $\Delta x = 0.1$
- $J = 40$, $\Delta x = 0.05$
- $J = 100$, $\Delta x = 0.02$

$\left(\text{since } \Delta x = \frac{L - L_p}{J} = \frac{2}{J}\right)$

From these graphs we can clearly see that the results are converging.

I have also plotted time profiles at times $t = 0.00, 0.01, 0.25, 0.50, 1.00$.

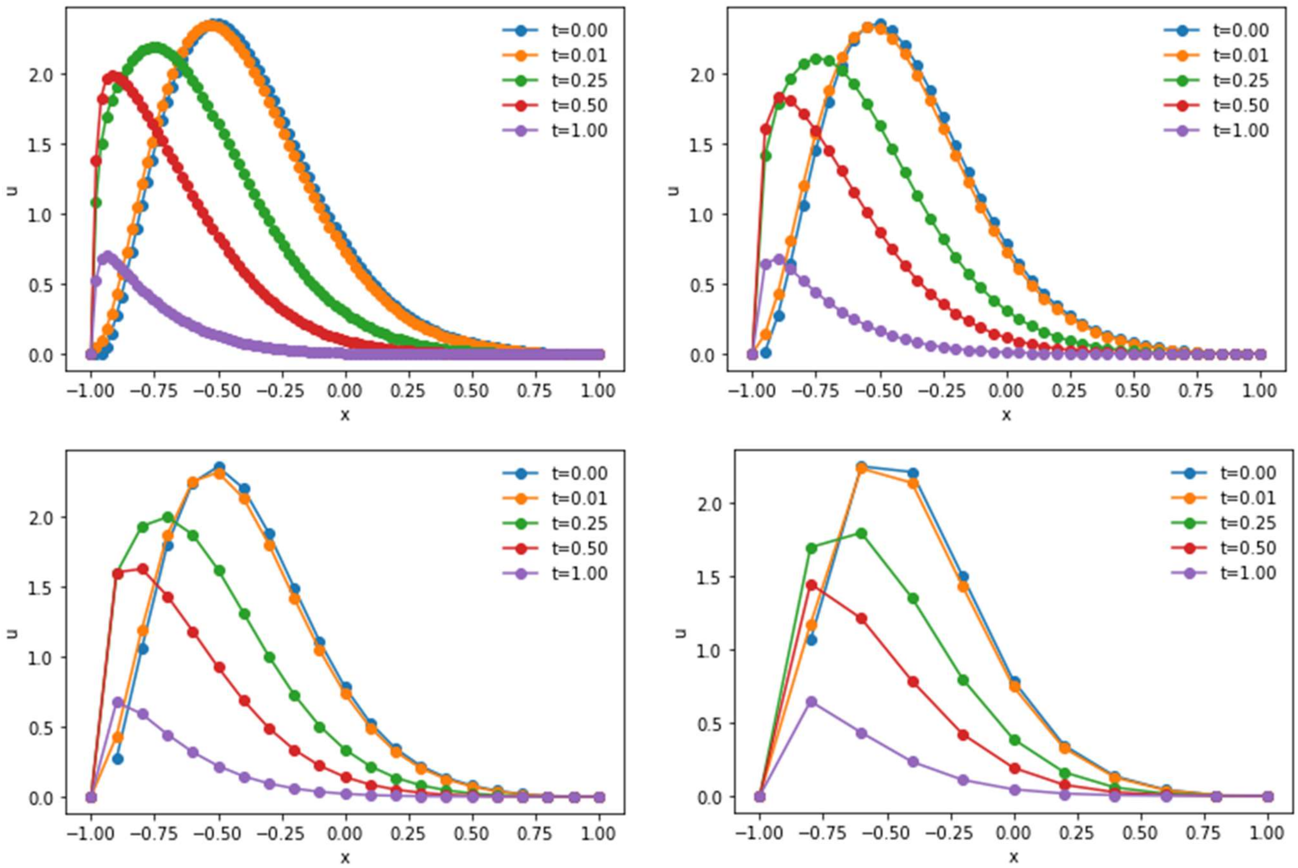

Figure 10: Results of the θ-scheme for problem (1) in the exercise sheets with the second given initial condition. In each plot, Δx has been refined so that it is equal to: 0.05, 0.02, 0.1 and 0.2. We can see that the results converge to a smooth curve as the discretisation step decreases.

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8) Finally, I have plotted at the same time steps but using the values
$\theta = 1$, $\Delta x = 0.02$, $\epsilon = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 0$
We can clearly see that, as $\epsilon \to 0$, the wave will drop less and less, which makes sense as ϵ is the diffusion constant. The fluid will diffuse less and less.

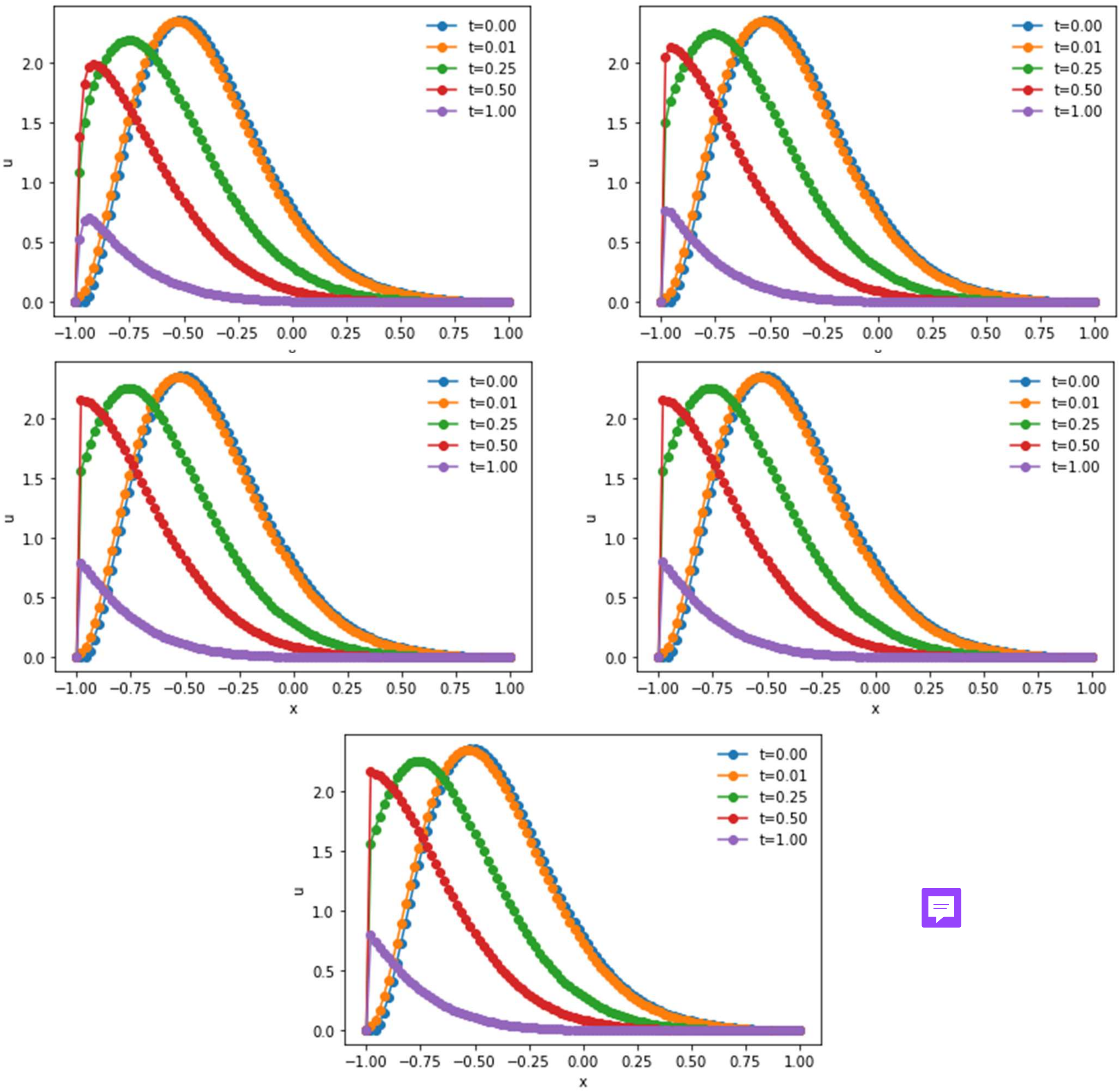

Figure 11: Results of the same scheme as in Figure 10, with θ=1, Δx=0.02 and ε=10^-3, 10^-4, 10^-5, 10^-6 and 0. With each plot, the diffusion lessens.