Numerical Methods - Coursework 3

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Question 4

In Figure 1 a plot the numerical results of the FEM for the Poisson system are given. In order to check the accuracy of this method we can compare this result to the exact solution found in Q1.

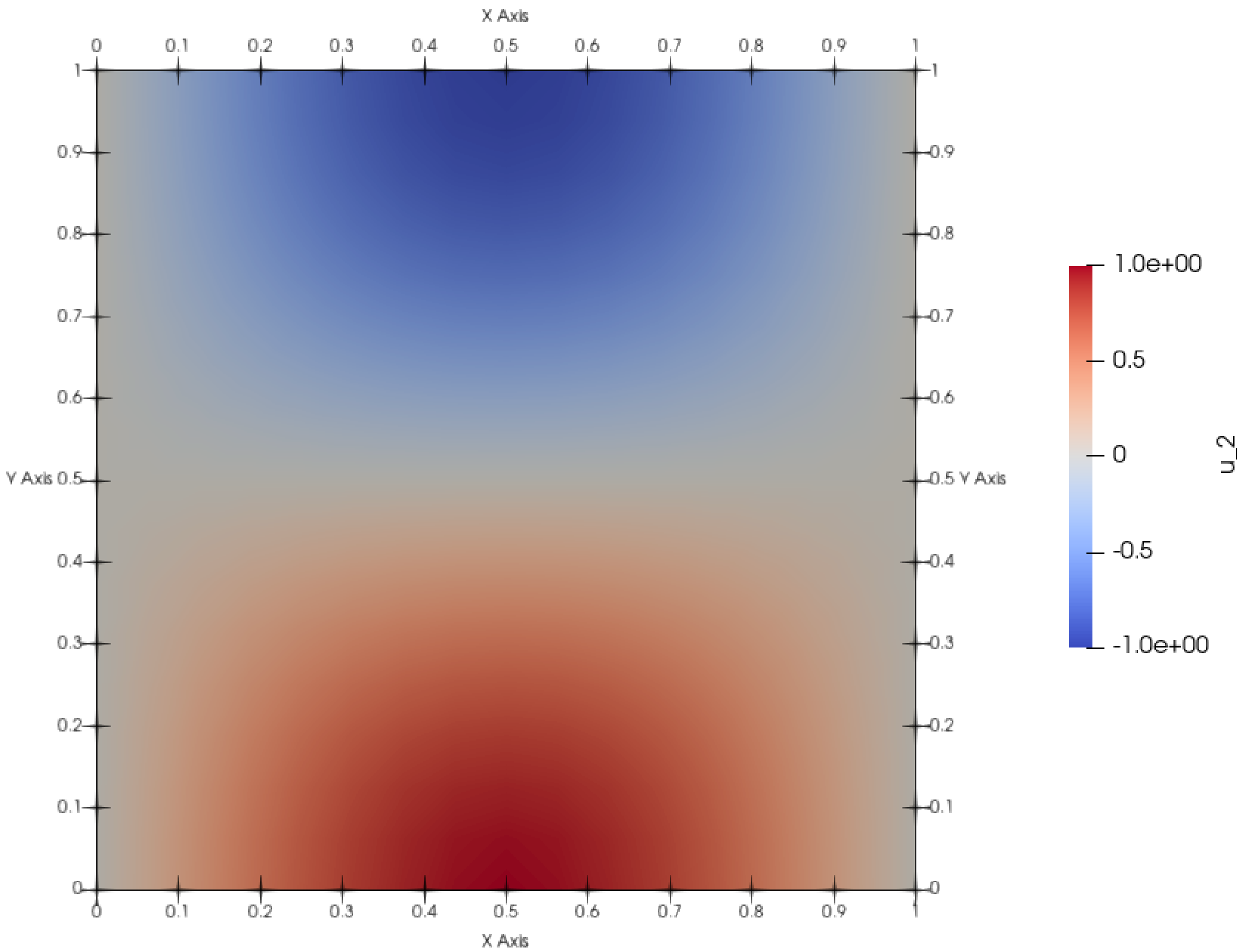

Figure 1: Solution of the FEM Poisson System on the 16*16 grid.

The error is plotted in Figure 2 on a 16*16 grid for the linear (p=1) continuous Galerkin method. We can see that the maximum error shown as the darkest red colour is on the order of 0.0032. Looking at the errors of the linear (1st order in p) Galerkin method for different mesh resolutions in Figures 3-6 we observe that the maximum error becomes smaller as the mesh resolution increases. Now, defining the mesh refinement parameter 'h' as $h = \frac{1}{(nx)^2}$ we can observe that the scaling of the error is linear. As we refine the mesh from $\frac{1}{16^2}$ to $\frac{1}{32^2}$, a division of 4, the error scales from 0.0032 to 0.0008 which is again a division by 4. Now, looking at the plots for the same mesh resolutions but

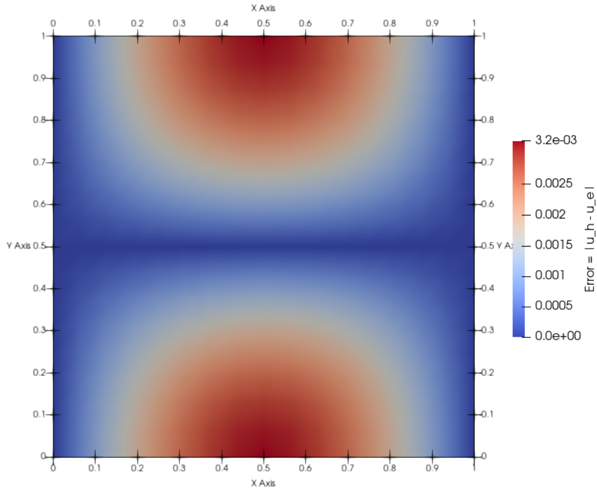

Figure 2: Error of the FEM Poisson System on the 16*16 grid for p=1

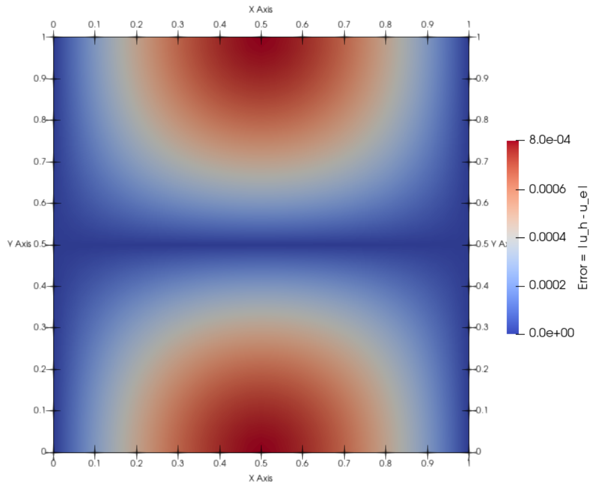

Figure 3: Error of the FEM Poisson System on the 32*32 grid for p=1.

for a quadratic Galerkin i.e $p = 2$, we observe that again as we divide h by 4, say from $\frac{1}{16^2}$ to $\frac{1}{32^2}$, the error scales from 0.0000025 to 0.00000016 or approximately a division of 16. Further investigation proves this relationship for the same changes in mesh refinement as we carried out for $p = 1$, instead of being linear this relationship seems to be quadratic. It would seem then that the error follows a scaling on the order of $O(h^p)$ since it is linear when $p = 1$ and quadratic when $p = 2$. We can then try and predict equivalent h and p combinations based on this rule. If we had $p = 3$ then a 16 by 16 grid would give an error scaling of approximately $\frac{1}{16^6} \approx 5.96 \times 10^{-08}$. Looking at Figure 9 we can see a similar error scaling and and therefore would predict the h-p pairing (h $= \frac{1}{64^2} : p = 2$) and (h $= \frac{1}{16^2} : p = 3$) would be equivalent. We can see in Figure 12 that this is the case. Similarly, the same rule would suggest that the pairing (h $= \frac{1}{128^2} : p = 2$) and (h $= \frac{1}{32^2} : p = 3$) would be equivalent since $\frac{1}{32^6} \approx 9.31 \times 10^{-10}$ and this is the same as the scale in Figure 10. Comparing Figures 10 and 13 we can see that this is approximately true.

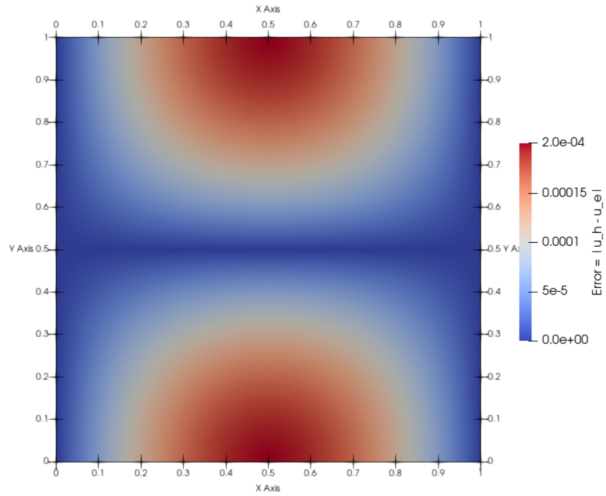

Figure 4: Error of the FEM Poisson System on the 64*64 grid for p=1.

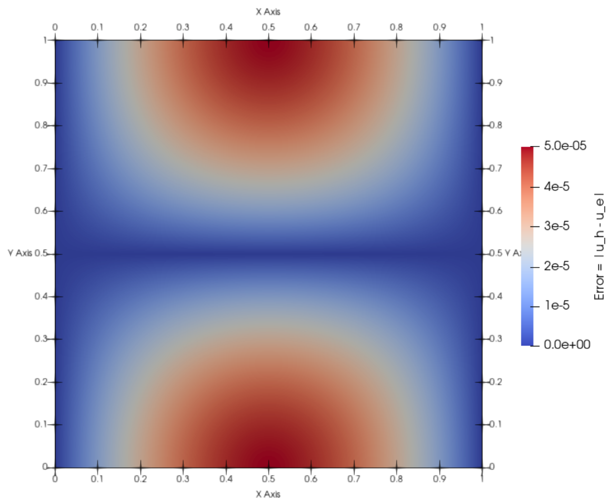

Figure 5: Error of the FEM Poisson System on the 128*128 grid for p=1.

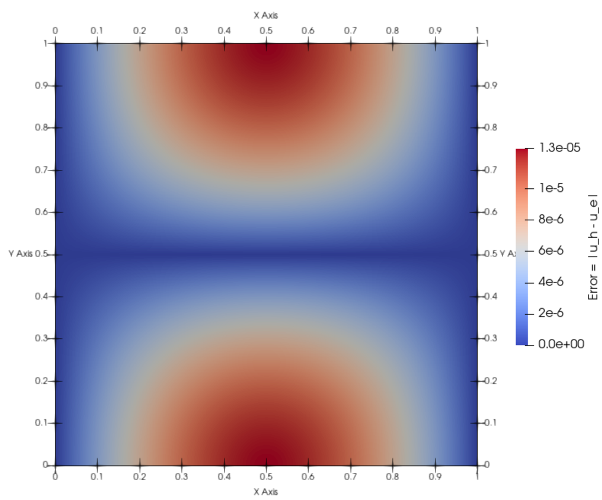

Figure 6: Error of the FEM Poisson System on the 256*256 grid for p=1.

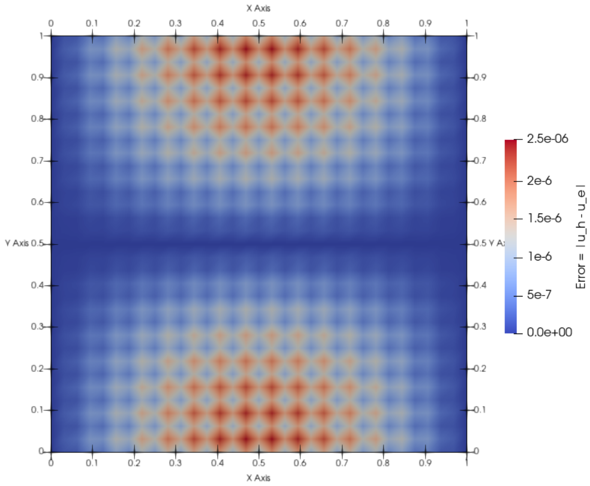

Figure 7: Error of the FEM Poisson System on the 16*16 grid for p=2.

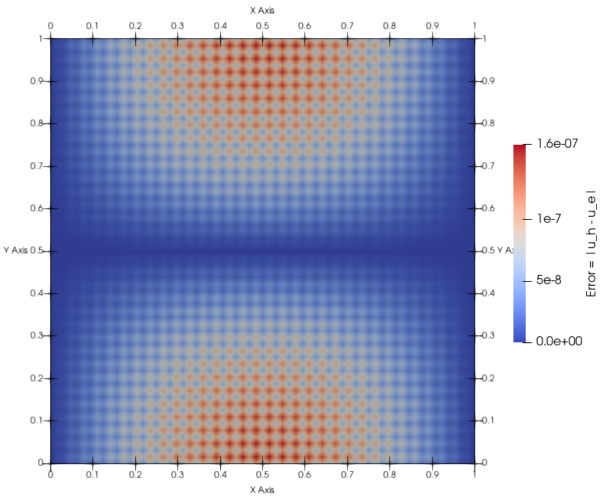

Figure 8: Error of the FEM Poisson System on the 32*32 grid for p=2.

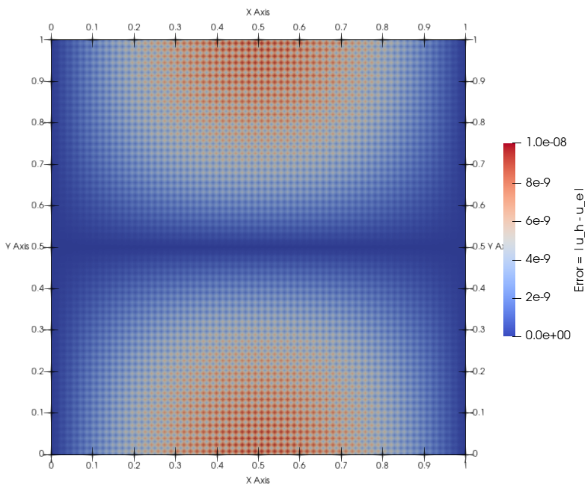

Figure 9: Error of the FEM Poisson System on the 64*64 grid for p=2.

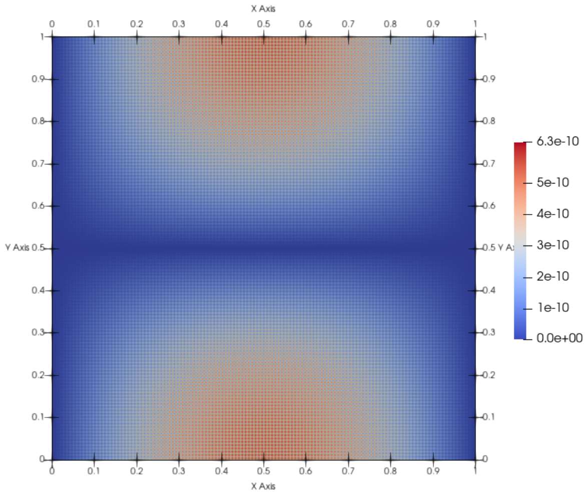

Figure 10: Error of the FEM Poisson System on the 128*128 grid for p=2.

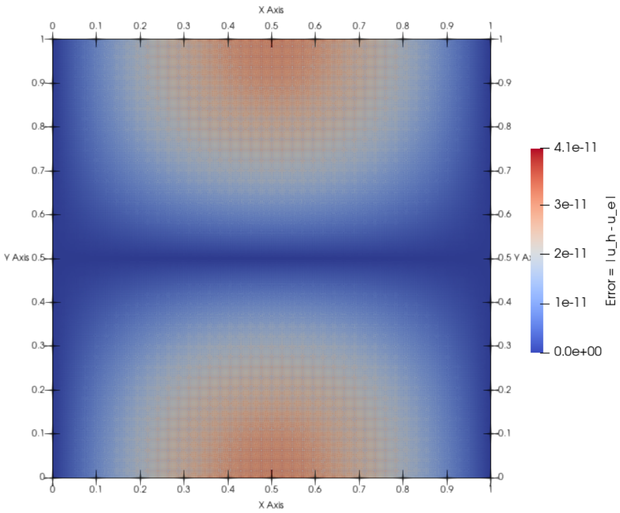

Figure 11: Error of the FEM Poisson System on the 256*256 grid for p=2.

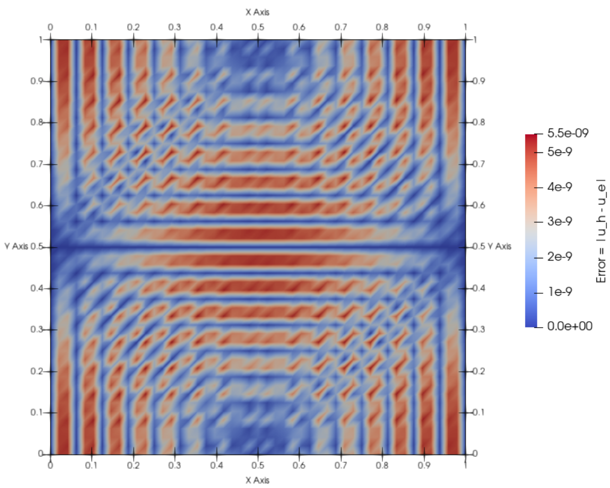

Figure 12: Error of the FEM Poisson System on the 16*16 grid for p=3.

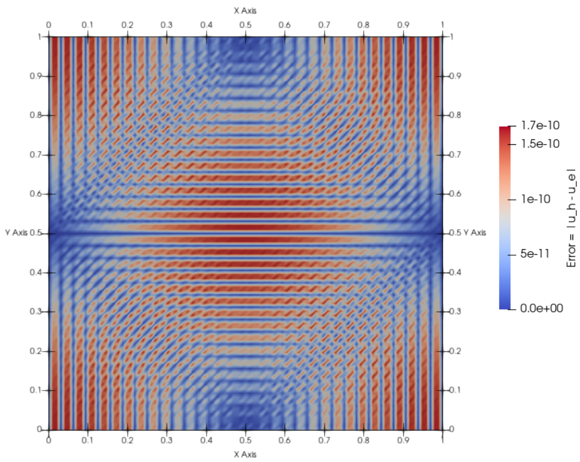

Figure 13: Error of the FEM Poisson System on the 32*32 grid for p=3.