$\boxed{Q1}$

- Linear advection-diffusion eq. describes both the advection and diffusion of a system via a linear PDE.

~~linear equation~~

- Advection eq

Describes the transportation of some conserved quantity or material by bulk motion of a fluid.

ADVECTION EQUATION FOR QUANTITY $u(x,t)$: $\left[\partial u/\partial t = f(x,t) \partial u/\partial x \right.$

$f(x,t)$ is some given function.

- Diffusion eq

Describes diffusion of material/ ~~u(x,t)~~ quantity $u(x,t)$ due to macroscopic movements of particles via collisions, etc in the flow.

$$\left.\partial u/\partial t = D \frac{\partial^2 u}{\partial x^2} \right] \text{DIFFUSION EQUATION FOR } u(x,t)$$

where D is ~~some~~ the diffusion coefficient.

As the advection eq and diffusion eq are linearly independent, they can be added together to form the advection-diffusion eq:

$$\partial u/\partial t = D \frac{\partial^2 u}{\partial x^2} + f \frac{\partial u}{\partial x}$$

For the eq given in the question, $f = a(t)$ and $D = \epsilon$.

A linear PDE is a function of the form (k^{th}-order)

~~$F(x; \frac{\partial u}{\partial x}) \; F(x_1, \ldots, x_n; \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \ldots, \frac{\partial^k u}{\partial x_n} \ldots) = 0$ where F is a linear function, that is, $a_\alpha(x) \frac{\partial t}{\partial x}$~~ $|\alpha| \leq k$

$$F(\underline{x}; \underline{u}(\underline{x}); \partial \underline{u}/\partial \underline{x}, \ldots, \partial^n \underline{u}/\partial \underline{x}^n) = 0 \quad \}\text{PDE}$$

st

$$F(\underline{x}) = \sum_{|\alpha| \leq k} a_\alpha(\underline{x}) \partial^k/\partial \underline{x}^k \, \underline{u}$$

that is, $F(\underline{x})$ is a linear eq of its derivatives. Our eq clearly fits this def with coefficients $a(t)$, ϵ.

[Q2] From standard Taylor expansion in two variables to 3rd order

u_j^n: [using $t - \frac{1}{2}\Delta t$]: (about $(x_j, t_{n+1/2})$)

$$u_j^n = u + u_t(-\tfrac{1}{2}\Delta t) + \tfrac{1}{2} u_{tt} \cdot (-\tfrac{1}{2})^2 (\Delta t)^2 + \tfrac{1}{6}(-\tfrac{1}{2}\Delta t)^3 u_{ttt}$$

$$= u - \tfrac{1}{2}\Delta t\, u_t + \tfrac{1}{8}(\Delta t)^2 u_{tt} - \tfrac{1}{48}(\Delta t)^3 u_{ttt} + \dots$$

u_j^{n+1}: [using $t - \frac{1}{2}\Delta t + \Delta t$] (about $(x_j, t_{n+1/2})$)

$$u_j^{n+1} = u + \tfrac{1}{2}\Delta t\, u_t + \tfrac{1}{8}(\Delta t)^2 u_{tt} + \tfrac{1}{48}(\Delta t)^3 u_{ttt} \dots$$

Then proving (2.80):

$$\delta_t u(x, t + \tfrac{1}{2}\Delta t) = u(x, (t + \tfrac{1}{2}\Delta t) + \tfrac{1}{2}\Delta t) - u(x, (t + \tfrac{1}{2}\Delta t) - \tfrac{1}{2}\Delta t)$$
$$= u(x, t + \Delta t) - u(x, t)$$

that is, $\delta_t u_j^{n+1/2} = u_j^{n+1} - u_j^n$

using Taylor expansions:

$$\delta_t u_j^{n+1/2} = [u - u] + [\tfrac{1}{2} - (-\tfrac{1}{2})] u_t \Delta t + [\tfrac{1}{8} - \tfrac{1}{8}](\Delta t)^2 u_{tt} + [\tfrac{1}{48} - (-\tfrac{1}{48})](\Delta t)^3 u_{ttt} + \dots$$

$$= 0 + \Delta t\, u_t + \tfrac{1}{24}(\Delta t)^3 u_{ttt} + \dots$$

As required.

Proving (2.81):

using $\delta_x^2 u(x, t + \Delta t) = u(x + \Delta x, t + \Delta t) - 2u(x, t + \Delta t) + u(x - \Delta x, t + \Delta t)$

that is, $\delta_x^2 u_j^{n+1} = u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}$.

$~~u_j^{n+1} = \sum_{n=0}^{\infty} \dots~~$

[Q2]

Expansion up to $n=6$ using $f(x-\Delta x, t-\Delta t) = \sum_{n=0}^{\infty} 1/n! \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \partial^n f/\partial x^{n-k} \partial t^k (\Delta x)^{n-k} (\Delta y)^k$

u_j^{n+1} using $(1/2 \Delta t)$, $(0 \Delta x)$

All terms including x are zero so -

$u_j^{n+1} = u + 1/2 \Delta t u_t + 1/2 \cdot 1/4 (\Delta t)^2 u_{tt} + 1/3! \cdot (1/2 \Delta t)^3 u_{ttt} +$

$1/4! (1/2 \Delta t)^4 u_{tttt} + 1/5! (1/2 \Delta t)^5 u_{ttttt} + 1/6! (1/2 \Delta t)^6 u_{tttttt}$.

$u_{j+1}^{n+1} + u_{j-1}^{n+1}$: using $(1/2 \Delta t)$, u_{j+1}^{n+1} has $(-\Delta x)$, u_{j-1}^{n+1} has (Δx).

unsure of the adjustments?

Then the taylor expansion becomes -

$[\sum_{n=0}^{\infty} 1/n! \sum_{k=0}^{n} n!/k!(n-k)! \, \partial^n u/\partial x^{n-k} \partial t^k (-\Delta x)^{n-k} (1/2 \Delta t)^k] +$

$[\sum_{n=0}^{\infty} 1/n! \sum_{k=0}^{n} n!/k!(n-k)! \, \frac{\partial^n u}{\partial x^{n-k} \partial t^k} (\Delta x)^{n-k} (1/2 \Delta t)^k]$

$= \sum_{n=0}^{\infty} 1/n! \sum_{k=0}^{n} n!/k!(n-k)! \, \frac{\partial^n u}{\partial x^{n-k} \partial t^k} (1/2 \Delta t)^k [(-\Delta x)^{n-k} + (\Delta x)^{n-k}]$

Then

$n=1$

$\boxed{k=0}$ $u_x [-\Delta x + \Delta x]$ + $\boxed{k=1}$ $u_t \cdot (1/2 \Delta t)[1+1]$

$= 1/2 \Delta t u_t$

$n=2$

$\boxed{k=0}$ $1/2 [\, 2/1 \cdot 2 \, u_{xx} [(-\Delta x)^2 + (\Delta x)^2]$ + $\boxed{k=1}$ $\xrightarrow{} n-k=1$ $\frac{2}{1 \cdot 1} u_{xt} (1/2 \Delta t)[0]$

$\boxed{k=2} \to n-k=0$

$2/2 \, u_{tt} (1/2 \Delta t)^2 [2]$ $= 1/2 \, u_{xx} (\Delta x)^2 + 1/2 (\Delta t)^2 u_{tt}$

$n=3$

$1/3! [\, \boxed{k=0} \to n-k=3$ $3!/1 \cdot 1 \, u_{xxx} [0]$ + $\boxed{k=1} \to n-k=2$ $3!/2! \, u_{xxt} (1/2 \Delta t) [2(\Delta x)^2]$ + $\boxed{k=2} \to n-k=1$ $3!/2! \, u_{xtt} (1/2 \Delta t)^2 [0]$

$= 1/6 [\, 3 \cdot 1/2 \cdot 2 (\Delta t)(\Delta x)^2 u_{xxt} +] = 1/2 (\Delta t)(\Delta x)^2 u_{xxt}$ } shouldn't this be zero?

Forgot - $\boxed{k=3} \to n-k=0$ $1/3! \cdot 3!/3! \, u_{ttt} (1/2 \Delta t)^3 \cdot 2 \; = 1/2 (\Delta t)(\Delta x)^2 u_{xxt} + 1/24 u_{ttt} (\Delta t)^3$

$n=4$ $\boxed{k=0} \to n-k=4$ $\boxed{k=1} \to n-k=3$ $\boxed{k=2} \to n-k=2$

$1/4! [\, \frac{4!}{1} u_{xxxx} \cdot 2(\Delta x)^4 + \frac{4!}{3!} u_{xxxt} (1/2 \Delta t)[0] + \frac{4!}{2!\,2!} u_{xxtt} (1/2 \Delta t)^2 [2(\Delta x)^2]$

$\boxed{k=3} \to n-k=1$ $\boxed{k=4} \to n-k=0$

$\frac{4!}{3!} u_{xttt} (1/2 \Delta t)^3 [0] + \frac{4!}{4!} u_{tttt} (1/2 \Delta t)^4 \cdot 2] =$ $2(\Delta x)^4 u_{xxxx}$ $+ 6 \cdot 1/4 \cdot 2 \, u_{xxtt} (\Delta t)^2 (\Delta x)^2$ $+ 1/24 \cdot 1/16 \cdot 2 (\Delta t)^4 u_{tttt}$

$\boxed{Q2}$ Continued

$\underline{n=4}$

$\hookrightarrow 2(\Delta x)^4 u_{xxxx} + 3u_{xxtt}(\Delta t)^2(\Delta x)^2 + 1/192\,(\Delta t)^4 u_{tttt}$

shouldn't these be zero?

$\underline{n=5}$

$\boxed{k=0} \to n-k=5$, $\boxed{k=1} \to n-k=4$, $\boxed{k=2} \to n-k=3$

$\frac{1}{5!}\Big[u_{xxxxx}[0] + \frac{5!}{1!4!} u_{xxxxt}(\tfrac{1}{2}\Delta t)[2(\Delta x)^4] + \frac{5!}{2!3!} u_{xxxtt}(\tfrac{1}{2}\Delta t)^2[0]$

$\boxed{k=3} \to n-k=2$, $\boxed{k=4} \to n-k=1$

$+ \frac{5!}{3!2!} u_{xxttt}(\tfrac{1}{2}\Delta t)^3 \cdot 2(\Delta x)^2 + \frac{5!}{4!} u_{xtttt}\cdot(\tfrac{1}{2}\Delta t)^4\cdot[0]$

$\boxed{k=5} \to n-k=0$

$+ \frac{5!}{5!} u_{ttttt}(\tfrac{1}{2}\Delta t)^5 \cdot 2 \quad = \frac{1}{6!}\Big[5u_{xxxxt}(\Delta t)(\Delta x)^4 + 5/2\, u_{xxttt}(\Delta t)^3(\Delta x)^2\Big]$

$+ 1/5!\ 1/2^4\, u_{ttttt}(\Delta t)^5.$

$\underline{n=6}$ $\boxed{k=0} \to n-k=6$, $\boxed{k=1} \to n-k=5$

$\frac{1}{6!}\Big[\frac{6!}{6!} u_{xxxxxx}\, 2(\Delta x)^6 + \frac{6!}{1!5!} u_{xxxxxt}(\tfrac{1}{2}\Delta t)[0]$

$\boxed{k=2} \to n-k=4$, $\boxed{k=3} \to n-k=3$

$\frac{6!}{2!4!} u_{xxxxtt}(\tfrac{1}{2}\Delta t)^2 \cdot 2(\Delta x)^4 + \frac{6!}{3!3!} u_{xxxttt}(\tfrac{1}{2}\Delta t)^3[0] +$

$\boxed{k=4} \to n-k=2$, $\boxed{k=5} \to n-k=1$

$\frac{6!}{4!2!} u_{xxtttt}(\tfrac{1}{2}\Delta t)^4 \cdot 2(\Delta x)^2 + [0]$

$\boxed{k=6} \to n-k=0$

$\frac{6!}{6!} u_{tttttt}(\tfrac{1}{2}\Delta t)^6 \cdot 2 \quad = 2/6!\, u_{xxxxxx}(\Delta x)^6 +$

$1/4! \cdot 1/4\, u_{xxxxtt}(\Delta t)^2(\Delta x)^4$

$+ 1/4!\ 1/16\, u (\Delta t)^4(\Delta x)^2 + 2/6!\, u_{tttttt}(\tfrac{1}{2}\Delta t)^6$

$= 1/360\, u_{xxxxxx}(\Delta x)^6 + 1/96\, u_{xxxxtt}(\Delta t)^2(\Delta x)^4$

$+ 1/384\, u_{xxtttt}(\Delta t)^4(\Delta x)^2 + 1/2^6\cdot 6!\, u_{tttttt}(\Delta t)^6$

~~Present terms~~ Grouping terms:

t-derivatives –

$\Delta t u_t(1-2\cdot\tfrac{1}{2}) + (\Delta t)^2 u_{tt}(\tfrac{1}{4} - 2\cdot\tfrac{1}{2}\cdot\tfrac{1}{4}) + (\Delta t)^3 u_{ttt}(2\cdot\tfrac{1}{3!}\cdot\tfrac{3}{2^3} - 2\tfrac{1}{3!}\tfrac{1}{2^3})$

$+ \dots\dots = \underline{\underline{0}}$. As $\partial^n u/\partial t^n \Delta t - 2\,\partial^n u/\partial t^n(\Delta t)^n + \frac{\partial^n u}{\partial t^n}(\Delta t)2 = 0.$

x-derivatives –

only for n even so –

$2\Big[\tfrac{1}{2}(\Delta x)^2 u_{xx} + \tfrac{1}{4!}(\Delta x)^4 u_{xxxx} + \tfrac{1}{6!}(\Delta x)^6 u_{xxxxxx}\Big]$ expected value of $\partial x^2 u_j^{n+1}$

mixed x-t derivatives – [~~should cancel~~ $\xrightarrow{?} 0$] as seen.

$\hookrightarrow$ ~~Reasons why they may not cancel~~

$\hookrightarrow$ Error in Taylor series formula ~~by me~~, made by me

$\boxed{\text{Q2}}$ Continued

Then all we have are even n x derivatives and mixed derivatives for $u_{j+1}^{n+1} + u_{j-1}^{n+1}$

$$\partial_x^2 u_j^{n+1} = \left[(\Delta x)^2 u_{xx} + 2(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \dots \right]$$

$$+\Big[\tfrac{1}{2}(\Delta t)(\Delta x)^2 u_{xxt} + 3u_{xxtt}(\Delta t)^2(\Delta x)^2 + \tfrac{5}{5!}(\Delta t)(\Delta x)^4 u_{xxxxt}$$

$$+ \tfrac{5}{2}5!(\Delta t)^3(\Delta x)^2 u_{xxttt} + \tfrac{1}{96} u_{xxxxtt}(\Delta t)^2(\Delta x)^4 +$$

$$\tfrac{1}{384}(\Delta t)^4(\Delta x)^2 u_{xxtttt} + \dots \Big] + \text{further terms.}$$

comparing to ~~2.81~~ - & u terms are $[2u-2u]=0$

~~$(\Delta x)^4 u_{xxxx}$ coefficient incorrect, u_{xxxxt} coeff incorrect,~~

Some incorrect ~~into~~ coefficients! ~~eg~~ TO CORRECT IF TIME.

May run on into further proofs - ~~prioritising methodology~~. using given coefficients in 2.81 in further proof.

Proving 2.82

$$\partial_x^2 u_j^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$$

- will use adjustment ~~to~~ (Δt term) i.e. $(-\tfrac12\Delta t)$.
- Adjust terms from $u_{j+1}^{n+1} + u_{j-1}^{n+1}$ expansion - replacing $(\tfrac12\Delta t)$ with $(-\tfrac12\Delta t)$.

Then

n=1

$$2\cdot(-\tfrac12\Delta t)u_t \; +$$

n=2

$$\tfrac12\left[\tfrac{2!}{2!}\cdot 2(\Delta x)^2 u_{xx} + \tfrac{2!}{2!}(-\tfrac12\Delta t)^2 u_{tt}\cdot 2\right.$$

n=3

$$\tfrac{1}{3!}\left[\tfrac{3!}{2!}\cdot 2\cdot(-\tfrac12 t)(\Delta x)^2 + \tfrac{3!}{3!}\cdot 2\cdot(-\tfrac12\Delta t)^3 u_{ttt}\right.$$

n=4

$$\tfrac{1}{4!}\left[2\tfrac{4!}{4!} u_{xxxx}(\Delta x)^4 + \tfrac{4!}{2!2!}(-\tfrac12\Delta t)^2\cdot 2\cdot(\Delta x)^2 u_{xxtt} + \tfrac{4!}{4!}\cdot 2\cdot(-\tfrac12\Delta t)^4 u_{tttt}\right]$$

n=5

$$\tfrac{1}{5!}\left[\tfrac{5!}{4!}(-\tfrac12\Delta t)\cdot 2\cdot(\Delta x)^4 u_{xxxxt} + \tfrac{5!}{3!2!}(-\tfrac12\Delta t)^3\cdot 2\cdot(\Delta x)^2 u_{xxttt} + \tfrac{5!}{5!}\cdot(-\tfrac12\Delta t)^5\cdot 2\cdot u_{ttttt}\right)$$

n=6

$$\tfrac{1}{6!}\left[\tfrac{6!}{6!}\cdot 2\cdot(\Delta x)^6 u_{xxxxxx} + \tfrac{6!}{2!4!} u_{xxxxtt}(-\tfrac12\Delta t)^2\cdot 2(\Delta x)^4 + \right.$$

$$\tfrac{6!}{4!2!}(-\tfrac12\Delta t)^4\cdot 2(\Delta x)^2 u_{xxtttt} + \tfrac{6!}{6!}\cdot(-\tfrac12\Delta t)^6\cdot 2$$

|Q2 continued|

Giving

$\partial_x^2 u_j^n =$ ~~...~~

t derivatives:

$\frac{1}{n!} \frac{\partial^n u}{\partial t^n} \cdot (-\tfrac{1}{2}\Delta t)^n \cdot 2 - 2 \cdot \frac{1}{n!} \frac{\partial^n u}{\partial t^n} (-\tfrac{1}{2}\Delta t)^n = \underline{0}$.

Then remaining terms are x-derivatives of $u_{j+1}^n + u_{j-1}^n$ and also the mixed derivatives - so looking at the full expansion now -

$$\theta \partial_x^2 u_j^{n+1} + \partial_x^2 u_j^n - \theta \partial_x^2 u_j^n = \partial_x^2 u_j^n + \theta(\partial_x^2 u_j^{n+1} - \partial_x^2 u_j^n)$$

$=$

$$\theta\Big[\tfrac{1}{2} \cdot \tfrac{2!}{2!} \cdot 2(\Delta x)^2 u_{xx} + \tfrac{1}{3!} \cdot \tfrac{3!}{2!} \cdot 2 \cdot (-\tfrac{1}{2}\Delta t)(\Delta x)^2 u_{xxt}$$
$$+ 2 \cdot \tfrac{1}{4!} \cdot \tfrac{4!}{4!} (\Delta x)^4 u_{xxxx} + \tfrac{1}{4!} \tfrac{4!}{2!2!} (-\tfrac{1}{2}\Delta t)^2 \cdot 2 \cdot (\Delta x)^2 u_{xxtt}$$
$$+ \tfrac{1}{5!} \cdot \tfrac{5!}{4!} (-\tfrac{1}{2}\Delta t) \cdot 2(\Delta x)^4 u_{xxxxt} + \tfrac{1}{5!} \cdot \tfrac{5!}{3!2!} (-\tfrac{1}{2}\Delta t)^3 \cdot 2 \cdot (\Delta x)^2 u_{xxttt}$$
$$+ \tfrac{1}{6!} \cdot \tfrac{6!}{6!} \cdot 2(\Delta x)^6 u_{xxxxxx} + \tfrac{1}{6!} \cdot \tfrac{6!}{2!4!} u_{xxxxtt} (-\tfrac{1}{2}\Delta t)^2 \cdot 2 \cdot (\Delta x)^4$$
$$+ \tfrac{1}{6!} \tfrac{6!}{4!2!} (-\tfrac{1}{2}\Delta t)^4 \cdot 2 \cdot (\Delta x)^2 u_{xxtttt} + \ldots\Big]$$

$+$

$$\theta\Big[(\Delta x)^2 u_{xx} + \tfrac{1}{12}(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \tfrac{1}{24}(\Delta t)(\Delta x)^2 u_{xxxxt}$$
$$\tfrac{1}{2}(\Delta t)(\Delta x)^2 u_{xxt} + \tfrac{1}{4}(\Delta t)^2(\Delta x)^2 u_{xxtt} + \tfrac{1}{96} u_{xxxxtt} (\Delta t)^2 (\Delta x)^4 +$$
$$+ \tfrac{5}{2 \cdot 5!}(\Delta t)^3 (\Delta x)^2 u_{xxttt} + \ldots\Big]$$
$$\tfrac{1}{384}(\Delta t)^4 (\Delta x)^2 u_{xxtttt} + \ldots$$

$$= \overset{x\ \text{derivs}}{\Big[(\Delta x)^2 u_{xx}} + \tfrac{1}{12}(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx}\Big]$$
$$+ \Big[(\Delta t)(\Delta x)^4 u_{xxt} + \ldots\Big]$$

no time to write full expansion.
+ rearrange.

$\boxed{Q2}$ Contined MATH5653

Proving ~~B~~ 2.83 & 2.84

~~T~~ ~~Need~~ Have terms for ~~$\partial_t u_j^{n+1/2}$~~ $\partial_x^2 u_j^{n+1}, \partial_x^2 u_j^n$.

Need terms for $\partial_t u_j^{n+1/2}$:

$$\partial_t u(x, t+\tfrac{1}{2}\Delta t) = u(x,(t+\tfrac{1}{2}\Delta t)+\tfrac{1}{2}\Delta t) - u(x,(t+\tfrac{1}{2}\Delta t)-\tfrac{1}{2}\Delta t)$$
$$= u(x, t+\Delta t) - u(x,t)$$

so $\partial_t u_j^{n+1/2} = u_j^{n+1} - u_j^n$ } we already have their terms ~~for~~ in their full expansion from previous qus.

$$T_j^{n+1/2} = \tfrac{1}{\Delta t}(u_j^{n+1} - u_j^n) - \tfrac{1}{(\Delta x)^2}\Big[\theta\big[u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\big]$$
$$+ (1-\theta)\big[u_{j+1}^n - 2u_j^n + u_{j-1}^n\big]$$

No time to do full expansion & rearrange - all these terms have been previously calculated.

Q3.

Q3 MATHS453

Explicit scheme:

$\partial u/\partial t(x_j, x_n) \approx U_j^{n+1} - U_j^n$] Forward diff in time on U_j^n

$\partial^2 u/\partial x^2(x_j, x_n) \approx U_{j+1}^n - 2U_j^n + U_{j-1}^n$] central difference ~~in~~ in space on U_j^n.

~~$\partial u/\partial x(x_j, x_n) \approx U_j^n - U_{j-1}^n$] backward diff in space on U_j^n → upwind scheme~~

~~Explicit The scheme is with scaling~~

~~$1/\Delta t(U_j^{n+1} - U_j^n) + \frac{a(t)}{\Delta x}[U_j^n - U_{j-1}^n] - 1/(\Delta x)^2[U_{j+1}^n - 2U_j^n + U_{j-1}^n] = 0$~~

$$\partial u/\partial x(x_j, x_n) = \begin{cases} U_j^n - U_{j-1}^n, & -a < 0 \\ U_{j+1}^n - U_j^n, & -a > 0 \end{cases}$$

] backwards/forwards ~~scheme~~ difference depending on sign of $a(t)$.

not sure on these.

Explicit scheme is then -

$$1/\Delta t[U_j^{n+1} - U_j^n] = \begin{cases} \frac{a(t)}{\Delta x}[U_j^n - U_{j-1}^n] + 1/(\Delta x)^2 \epsilon [U_{j+1}^n - 2U_j^n + U_{j-1}^n] & \leftarrow \text{For } -a<0 \\ \frac{a(t)}{\Delta x}[U_{j+1}^n - U_j^n] + 1/(\Delta x)^2 \epsilon [U_{j+1}^n - 2U_j^n + U_{j-1}^n] & \leftarrow \text{For } -a>0 \end{cases}$$

Implicit ~~solution~~ scheme

$u_t \approx U_j^{n+1} - U_j^n$] ~~Forward diff~~ Backwards diff on U_j^{n+1}

$u_{xx} \approx U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}$] central diff. as before. on U_j^{n+1}

$$u_x \approx \begin{cases} U_j^{n+1} - U_{j-1}^{n+1}, & -a<0 \\ U_{j+1}^{n+1} - U_j^{n+1}, & -a>0 \end{cases}$$

} same spatial scheme as before on U_j^{n+1}.

Hence we have -

$$1/\Delta t[U_j^{n+1} - U_j^n] = \begin{cases} \frac{a(t)}{\Delta x}[U_j^{n+1} - U_{j-1}^{n+1}] + \frac{\epsilon}{(\Delta x)^2}[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^n], & -a<0 \\ \frac{a(t)}{\Delta x}[U_{j+1}^{n+1} - U_j^{n+1}] + \frac{\epsilon}{(\Delta x)^2}[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^n], & -a>0 \end{cases}$$

(~~missing~~ ~~μ~~ ~~because not sure how~~ (need to introduce some μ).

Θ-scheme

u_t remains the same. We average out ~~ut, uxx~~ u_x, u_{xx} for fw explicit & implicit schemes.
With weighting θ, $0 \le \theta \le 1$.

Case $-a<0$:

implicit scheme

$$\frac{1}{\Delta t}(U_j^{n+1}-U_j^n) = \theta\left[\frac{a(t)}{\Delta x}(U_j^{n+1}-U_{j-1}^{n+1}) + \frac{\epsilon}{(\Delta x)^2}(U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1})\right]$$

$$+(1-\theta)\left[\frac{a(t)}{\Delta x}(U_j^n-U_{j-1}^n)+\frac{\epsilon}{(\Delta x)^2}(U_{j+1}^n-2U_j^n+U_{j-1}^n)\right]$$

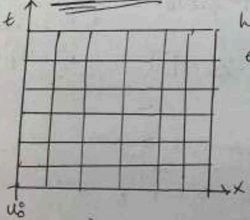
explicit scheme

Case $-a>0$

$$\frac{1}{\Delta t}(U_j^{n+1}-U_j^n) = \theta\left[\frac{a(t)}{\Delta x}(U_{j+1}^{n+1}-U_j^{n+1}) + \frac{\epsilon}{(\Delta x)^2}(U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1})\right]$$

$$+(1-\theta)\left[\frac{a(t)}{\Delta x}(U_{j+1}^n-U_j^n)+\frac{\epsilon}{(\Delta x)^2}(U_{j+1}^n-2U_j^n+U_{j-1}^n)\right]$$

Mesh and ordering



Let $M = |L - L_p|$, assuming $L, L_p \in \mathbb{R}$.
$t \in [0, t_F]$. We define J mesh points for x – $\Delta x = 1/J$.
and N mesh points for t – $\Delta t = 1/N$. J does not have to equal N, but $J = N$ is simple.

For our ordering of the mesh, we use $U_j^n = [x]_{J(n)+(j+1)}$ } may cause some issues at $J-1$, $N-1$!

Boundaries + ICs.

That is,

$$\partial\Omega = \{U_j^0,\ j=0,1,\dots,J\} \cup \{U_{J-1}^n,\ n=0,1,\dots,N-1\} \cup \{U_0^n : n=0,1,\dots,N-1\}$$

IC, BC, BC

where $U_j^0 = u(x_j,0) = u_0(x)$, $U_{J-1}^n = u(L,t) = 0 = u(L_p,t) = U_0^n$.

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$\boxed{Q3}$ Continued

Scheme at boundaries & ICs:

~~U_j^0: $-a \cdot \theta$ $1/\Delta t [U_j^1 - u_0(x)] = \theta [\frac{a(t)}{\Delta x} [U_j^1 - U_{j-1}^1]]$~~

~~U_j^n: $-a\theta$: $+ \epsilon/(\Delta x)^2 (U_{j+1}^1 - 2$~~

~~$1/\Delta t\, U_j^{n+1}$~~

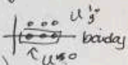
$U_j^0,\ -a \leq 0$: ~~$-1/\Delta t\, U_j^{n+1}$~~

LHS (unknown)

$1/\Delta t\, U_j^{n+1} - \frac{a\theta}{\Delta x} U_j^{n+1} - \frac{a\theta}{\Delta x} U_{j-1}^{n+1} - \epsilon/(\Delta x)^2 [U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}]$

3 known, 3 unknown

$j=1, n=0$



RHS (known)

$= 1/\Delta t\, U_j^n + (1-\theta) [\frac{a}{\Delta x} (U_j^n - U_{j-1}^n) + \epsilon/(\Delta x)^2 (U_{j+1}^n - 2U_j^n + U_{j-1}^n)]$

$U_j^0,\ -a > 0$:

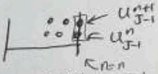
$1/\Delta t\, U_j^{n+1} + \theta [\frac{a\theta}{\Delta x} (U_{j+1}^{n+1} - U_j^{n+1}) + \epsilon/(\Delta x)^2 (U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1})] = 1/\Delta t\, U_j^n + (1-\theta) [\frac{a(t)}{\Delta x} (U_{j+1}^n - U_j^n) + \epsilon/(\Delta x)^2 (U_{j+1}^n - 2U_j^n + U_{j-1}^n)]$

$U_{J-1}^n,\ -a < 0$:

known where for $j = J-1$

2 known

4 unknown



$1/\Delta t (U_j^{n+1} - U_j^{n}) + \frac{\theta a(t)}{\Delta x} U_j^{n+1} - \frac{\theta a}{\Delta x} U_{j-1}^{n+1} + 2\epsilon/(\Delta x)^2 U_j^{n+1}$

$- \theta \epsilon/(\Delta x)^2 (U_{j+1}^{n+1} + U_{j-1}^{n+1})$

$- \epsilon/(\Delta x)^2 (1-\theta) [U_{j+1}^n + U_{j-1}^n]$

$= + (1-\theta) \frac{a}{\Delta x} [-U_j^n] + \frac{2\epsilon}{(1-\theta)(\Delta x)^2} U_j^n + 1/\Delta t\, U_j^{n+1} \pm 1/\Delta t\, U_j^{n+1}$

$U_{j-1}^n,\ -a > 0$:

~~$1/\Delta t (U_j^{n+1} - U_j^n) - \frac{\theta a}{\Delta x} U_j^{n+1}$~~ ~~as above~~ same as above.

~~$- \epsilon/(\Delta x)^2 (U_{j+1}^{n+1}$~~

$1/\Delta t (U_j^{n+1} - U_j^n) - \theta \epsilon/(\Delta x)^2 (\theta (U_{j+1}^{n+1} + U_{j-1}^{n+1}) - (1-\theta)(U_{j+1}^n + U_{j-1}^n) - \frac{\theta a}{\Delta x} (U_{j+1}^{n+1}) + (1-\theta) \frac{a}{\Delta t} (U_{j+1}^n)$

$= 1/\Delta t\, U_j^{n} + \frac{2\epsilon}{(\Delta x)^2} U_j^{n+1} + (1-\theta) (\frac{2\epsilon}{(\Delta x)^2} U_j^n) \pm 1/\Delta t\, U_j^{n+1}$

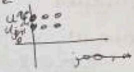
$+$

$\boxed{Q3}$ Continued

$n = u; j = 0$

2 unknown

4 unknown



u_0^n : $-a < 0$:

LHS (unknown)

$\frac{1}{\Delta t} \quad \frac{1}{\Delta t} - \frac{\epsilon}{(\Delta x)^2} u_{j+1}^{n+1} - (1-\theta)\left(\frac{\epsilon}{(\Delta x)^2}(u_{j+1}^n)\right.$

$= \frac{1}{\Delta t}(u_j^{n+1} - u_j^n) +$

$\frac{\theta a}{\Delta x}(u_j^{n+1}) - \frac{2u_j^{n+1} \cdot \epsilon}{(\Delta x)^2}$

RHS (known)

$+ (1-\theta)\left[\frac{a}{\Delta x}(u_j^n) + \frac{\epsilon}{(\Delta x)^2}(-2u_j^n)\right.$.