Foundations of Fluids Numerical Exercises 2

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We consider the linearised shallow water equations

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial t} + \frac{\partial (g\eta)}{\partial x} = 0 \tag{1}$$

where $u = u(x,t)$ is the velocity, $\eta = \eta(x,t)$ is the free surface deviation, $H(x)$ is the rest depth and g is the acceleration due to gravity. We introduce the scales

$$u = U_0 u' \qquad x = L_s x' \qquad t = \left(\frac{L_s}{U_0}\right) t' \qquad \eta = H_{0s}\eta' \qquad H = H_{0s} H' \tag{2}$$

After having dropped the primes for convenience, the first equation remains unchanged. The second equation reads

$$\frac{\partial u}{\partial t} + \frac{H_{0s}}{U_0^2}\frac{\partial (g\eta)}{\partial x} = 0 \tag{3}$$

where we have once again dropped primes for convenience. If we introduce the scale

$$g = \frac{U_0^2}{H_{0s}} g'$$

then we obtain identical equations to (1) which are now dimensionless. Our Riemann problem will consist of equation (1) along with the piecewise-constant initial conditions

$$u(x,0) = \begin{cases} u_l & \text{for } x < 0 \\ u_r & \text{for } x \geq 0 \end{cases} \quad \text{and} \quad \eta(x,0) = \begin{cases} \eta_l & \text{for } x < 0 \\ \eta_r & \text{for } x \geq 0 \end{cases} \tag{4}$$

We now assume that $H(x) = H_0$ is constant and write our system in the form

$$\frac{\partial}{\partial t}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0 \tag{5}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix}$$

and $c_0^2 = gH_0$. It is easy to verify that equations (1) and (5) are identical. We now calculate the eigenvalues of A by noting that

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ c_0^2 & -\lambda \end{vmatrix} = \lambda^2 - c_0^2$$

Setting this equal to zero yields $\lambda_1 = c_0$ and $\lambda_2 = -c_0$. Having found the eigenvalues, we now calculate the corresponding eigenvectors. For λ_1 we have the eigenvector $\mathbf{v} = (v_1, v_2)^T$ where $v_1 = \frac{v_2}{c_0}$. For λ_2, we have $v_1 = -\frac{v_2}{c_0}$. We are free to choose v_2 as $\frac{1}{2}$ in which case we have the matrix of right eigenvalues

$$\mathbf{B} = \frac{1}{2c_0}\begin{pmatrix} 1 & -1 \\ c_0 & c_0 \end{pmatrix} \tag{6}$$

By performing some elementary row operations we obtain the inverse

$$\mathbf{B^{-1}} = \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix} \tag{7}$$

From linear algebra principles, we would expect that $\mathbf{B^{-1}AB} = \Lambda$ where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{8}$$

Let us demonstrate this. We have

$$\mathbf{B^{-1}A} = \begin{pmatrix} c_0 & 1 \\ -c_0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ c_0^2 & 1 \end{pmatrix} = \begin{pmatrix} c_0^2 & c_0 \\ c_0^2 & -c_0 \end{pmatrix} \tag{9}$$

Then

$$\mathbf{B^{-1}AB} = \frac{1}{2c_0} \begin{pmatrix} c_0^2 & c_0 \\ c_0^2 & -c_0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ c_0 & c_0 \end{pmatrix} = \frac{1}{2c_0} \begin{pmatrix} 2c_0^2 & 0 \\ 0 & -2c_0^2 \end{pmatrix} \tag{10}$$

which clearly shows this to be true. We now define a vector $\mathbf{r} = \mathbf{B^{-1}} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix}$ where we note that the components of $\mathbf{r}$ read $r_1 = c_0\eta + H_0 u$ and $r_2 = H_0 u - c_0\eta$. Making use of the identity $\mathbf{B^{-1}B} = \mathbf{I}$, we write equation (5) as

$$\mathbf{B^{-1}} \frac{\partial}{\partial t} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + \mathbf{B^{-1}ABB^{-1}} \frac{\partial}{\partial x} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0 \tag{11}$$

This simplifies to give

$$\frac{\partial}{\partial t}\mathbf{r} + \Lambda \frac{\partial}{\partial x}\mathbf{r} = 0 \tag{12}$$

Since Λ is diagonal, we have a pair of linear advection equations, the first of which reads

$$\frac{\partial r_1}{\partial t} + \lambda_1 \frac{\partial r_1}{\partial x} = 0 \tag{13}$$

which we solve subject to the initial condition

$$r_1(x, 0) = \begin{cases} r_{1l} & \text{for } x < 0 \\ r_{1r} & \text{for } x \geq 0 \end{cases} \tag{14}$$

Geometrically, equation (13) asserts that, in the (x, t) plane, the directional derivative of r_1 is zero in the direction of the vector $(1, \lambda_1)^T$. In other words, r_1 is constant along the characteristic lines satisfying $\frac{dx}{dt} = \lambda_1$. That is, r_1 is constant along the lines $x - \lambda_1 t = C$ where C represents a constant. Hence, advection equations such as (13) simply translate the initial condition and we have

$$r_1(x, t) = r_1(x - \lambda_1 t, 0) \tag{15}$$

In our case, we have

$$r_1(x, t) = \begin{cases} r_{1l} & \text{for } x < c_0 t \\ r_{1r} & \text{for } x \geq c_0 t \end{cases} \tag{16}$$

Analogous working holds for our other linear advection equation which comes from (12) and we obtain

$$r_2(x, t) = \begin{cases} r_{2l} & \text{for } x < -c_0 t \\ r_{2r} & \text{for } x \geq -c_0 t \end{cases} \tag{17}$$

We now use these expressions to solve our original problem. By noting that $u = \frac{1}{2}(r_1 + r_2)/H_0$ and $\eta = \frac{1}{2}(r_1 - r_2)/c_0$, we have

$$H_0 u(x,t) = \frac{1}{2}(r_1(x,t) + r_2(x,t)) = \begin{cases} u_l & \text{for } x < -c_0 t \\ \frac{1}{2}\left[H_0(u_l + u_r) + c_0(\eta_l - \eta_r)\right] & \text{for } -c_0 t \leq x \leq c_0 t \\ u_r & \text{for } x > c_0 t \end{cases} \tag{18}$$

$$\eta(x,t) = \frac{1}{2c_0}(r_1(x,t) - r_2(x,t)) = \begin{cases} \eta_l & \text{for } x < -c_0 t \\ \frac{1}{2}\left[\frac{H_0}{c_0}(u_l - u_r) + (\eta_l + \eta_r)\right] & \text{for } -c_0 t \leq x \leq c_0 t \\ \eta_r & \text{for } x > c_0 t \end{cases} \tag{19}$$

1 Godunov's Method

We will now describe the Godunov method which applies to the linear shallow water equations (1). We can write these equations in the form

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0 \tag{20}$$

where we have $\mathbf{u} = \begin{pmatrix} \eta \\ u \end{pmatrix}$ and $\mathbf{f}(\mathbf{u}) = \begin{pmatrix} Hu \\ g\eta \end{pmatrix}$ where $H = H(x)$ is no longer necessarily constant. We integrate (20) over $x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}$ and $t_n < t < t_{n+1}$. We obtain

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{u}(x, t_{n+1}) dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{u}(x, t_n) dx + \int_{t_n}^{t_{n+1}} \mathbf{u}(x_{j-\frac{1}{2}}, t) dt - \int_{t_n}^{t_{n+1}} \mathbf{u}(x_{j+\frac{1}{2}}, t) dt = 0 \tag{21}$$

We define the cell average

$$\mathbf{U}_j^n = \frac{1}{h_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{u}(x, t_n) dx$$

Dividing equation (21) by $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ and rearranging gives us

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n + \frac{1}{h_j}\left[\int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}}, t)) dt - \int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t)) q dt\right] \tag{22}$$

If we define the approximate numerical flux as

$$F(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n) = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \mathbf{u}(x_{j+\frac{1}{2}}, t) dt \tag{23}$$

We obtain

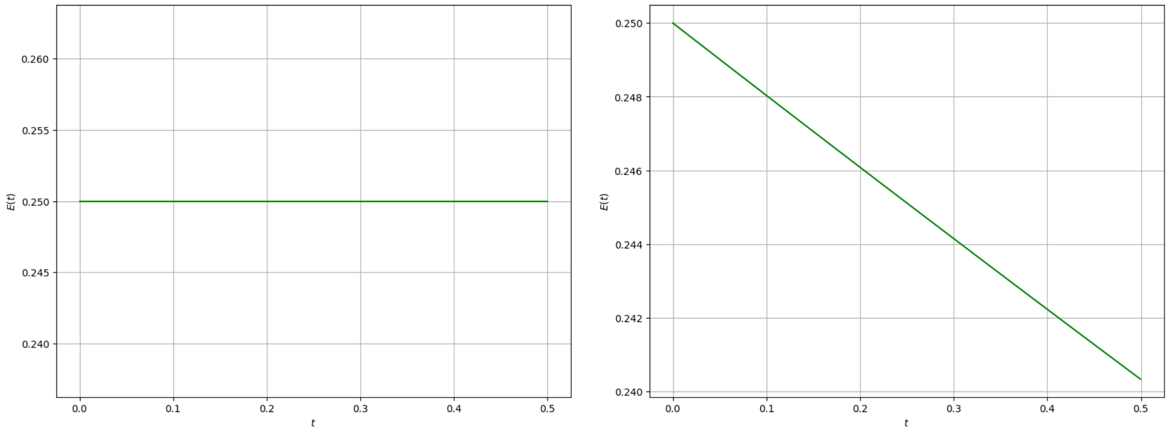
$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n + \frac{\Delta t}{h_j}\left[F(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n) - F(\mathbf{U}_{j-1}^n, \mathbf{U}_j^n)\right] \tag{24}$$

Of course, this represents a system of two equations which will evolve η and u respectively. Care must be taken when defining the numerical fluxes. At each cell edge, we can use a locally approximate constant $H(x)$ since we are solving the local Riemann problems and we choose a small enough time step that the solutions of such problems have not begun to interact. Namely, we have the time step estimate :

$$\Delta t \leq \frac{\text{CFL}\Delta x}{c_0}$$

2 Implementation

We will now use the given code to investigate the firedrake implementation. Assuming that the topography depth is constant, we have the following energy plots



(a) CFL=1 (b) CFL=0.9

Figure 1: Effects of changing the CFL number on the energy of the system.

We also have the following plot.

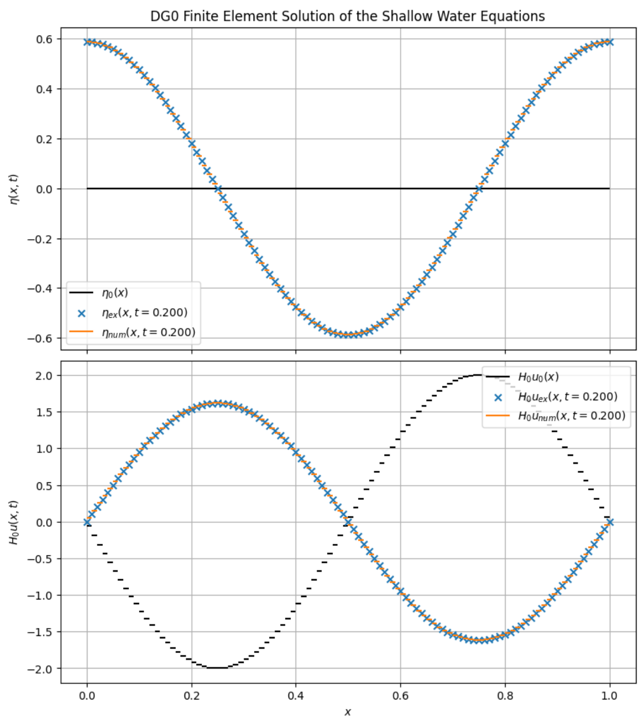


Figure 2: Godunov flux method

We cam also use the given code to investigate the solution for an alternating flux. We have

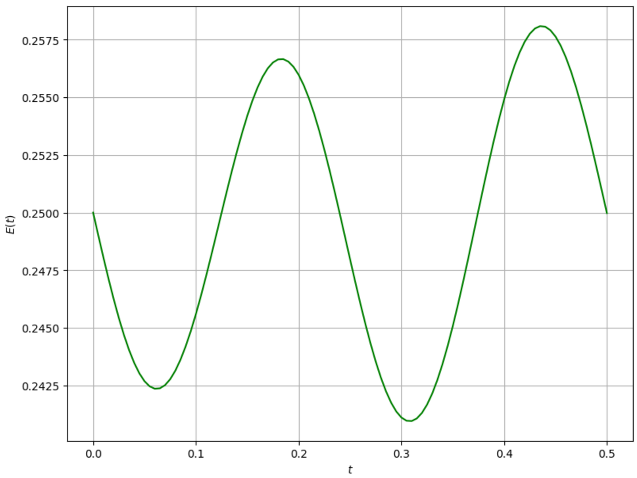


Figure 3: Energy proile when using alternating flux.