Continuous-Galerkin Finite Element Method (CGFEM) and its implementation in `Firedrake`

A short introduction

Yang Lu

School of Mathematics, University of Leeds

mmyl@leeds.ac.uk

November 24, 2023

Table of Contents

Table of Contents

CGFEM procedure

1. Derive the weak formulation
   - Multiply a PDE by an (arbitrary) test function
   - Integrate over the entire domain or
     integrate over each element and then take the sum
   - Integration by parts and apply the boundary conditions

CGFEM procedure

1. Derive the weak formulation
   - Multiply a PDE by an (arbitrary) test function
   - Integrate over the entire domain or
     integrate over each element and then take the sum
   - Integration by parts and apply the boundary conditions
2. Discretise the weak formulation and build the algebraic system
   - Expand the dependent variables with a series of basis functions
   - Discretise the test function with the same choice of basis functions
   - Substitute into the weak formulations $\Rightarrow$ an algebraic system

CGFEM procedure

1. Derive the weak formulation
   - Multiply a PDE by an (arbitrary) test function
   - Integrate over the entire domain or
     integrate over each element and then take the sum
   - Integration by parts and apply the boundary conditions
2. Discretise the weak formulation and build the algebraic system
   - Expand the dependent variables with a series of basis functions
   - Discretise the test function with the same choice of basis functions
   - Substitute into the weak formulations $\Rightarrow$ an algebraic system
3. Determine the matrices
   - Evaluate the integrals for each element (locally)
   - Assemble the matrix (globally)

CGFEM procedure

1. Derive the weak formulation
   - Multiply a PDE by an (arbitrary) test function
   - Integrate over the entire domain or
     integrate over each element and then take the sum
   - Integration by parts and apply the boundary conditions
2. Discretise the weak formulation and build the algebraic system
   - Expand the dependent variables with a series of basis functions
   - Discretise the test function with the same choice of basis functions
   - Substitute into the weak formulations $\Rightarrow$ an algebraic system
3. Determine the matrices
   - Evaluate the integrals for each element (locally)
   - Assemble the matrix (globally)
4. Solve the algebraic system

Poisson's equation in a unit square

Example

For a given function f, seek u such that in $\Omega = [0,1] \times [0,1]$

$$-\nabla^2 u = f, \tag{1a}$$

$$u(0,y) = u(1,y) = 0, \tag{1b}$$

$$\partial_y u(x,0) = \partial_y u(x,1) = 0. \tag{1c}$$

Table of Contents

Firedrake implementation

"Firedrake is an automated system for the solution of partial differential equations using the finite element method (FEM). Firedrake uses *sophisticated code generation* to provide mathematicians, scientists, and engineers with a very high productivity way to create sophisticated high performance simulations."

Firedrake implementation

"Firedrake is an automated system for the solution of partial differential equations using the finite element method (FEM). Firedrake uses *sophisticated code generation* to provide mathematicians, scientists, and engineers with a very high productivity way to create sophisticated high performance simulations."

Example

Choose f to be

$$f = 2\pi^2 \sin(\pi x) \cos(\pi y), \tag{2}$$

which yields the exact solution:

$$u_{\text{ex}} = \sin(\pi x) \cos(\pi y). \tag{3}$$

Demonstration

- Mathematical and numerical modelling: **CGFEM**
- Computational modelling: **Firedrake**
  https://www.firedrakeproject.org/
- Post-Processing
  - Visualising the results: **ParaView**
    https://www.paraview.org/
  - Verification and validation
    e.g. convergence analysis based on the L^2 error:

$$L^2(\Delta x) = \sqrt{\sum_i \left(u_{h,i} - u_{\text{ex},i}\right)^2}$$

Table of Contents

Differential equations and minimisation problems

Poisson's equation in a unit square

For a given function f, consider the PDE with homogeneous boundary conditions in $\Omega = [0,1] \times [0,1]$:

$$-\nabla^2 u = f, \tag{4a}$$

$$u|_{\Gamma_1} = 0, \tag{4b}$$

$$\nabla u \cdot \mathbf{n}|_{\Gamma_2} = 0, \tag{4c}$$

where the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$.

Then the solution u minimises the functional

$$I[u] = \iint_\Omega \frac{1}{2}|\nabla u|^2 - uf \, \mathrm{d}\Omega \tag{5}$$

over the space $\Sigma = \{u \text{ smooth} \mid u|_{\Gamma_1=0}\}$.

On the other hand, if u minimises (5) then u satisfies (4).

From the minimisation problem to the PDE[1]

Let $\hat{u}$ minimise (5), consider a set of functions $u(x,y) = \hat{u}(x,y) + \epsilon\eta(x,y)$, where ϵ is a parameter and η is an arbitrary function satisfying $\eta|_{\Gamma_1} = 0$. A necessary condition for the existence of a minimum of (5) at $\epsilon = 0$ is

$$\delta I \equiv \left.\frac{\mathrm{d}I}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \lim_{\epsilon\to 0}\frac{I[\hat{u}(x,y) + \epsilon\,\eta(x,y)] - I[\hat{u}(x,y)]}{\epsilon} = 0. \qquad (6)$$

$$\Rightarrow \qquad \iint_\Omega \nabla\hat{u} \cdot \nabla\eta - \eta f \,\mathrm{d}\Omega = 0 \qquad (7)$$

Using integration by parts and Gauss' theorem:

$$\Rightarrow \quad -\iint_\Omega \eta\left(\nabla^2\hat{u} + f\right)\mathrm{d}\Omega + \int_{\Gamma_1}\cancelto{0}{\eta\nabla\hat{u}\cdot\mathbf{n}\,\mathrm{d}\Gamma} + \int_{\Gamma_2}\cancelto{0}{\eta\nabla\hat{u}\cdot\mathbf{n}\,\mathrm{d}\Gamma} = 0 \quad (8)$$

Instead of solving (4), alternatively, we can solve the weak formulation (7).

---

[1] J. van Kan (2014) *Numerical methods in Scientific Computing*. Delft Academic Press, The Netherlands. Chapter 5.

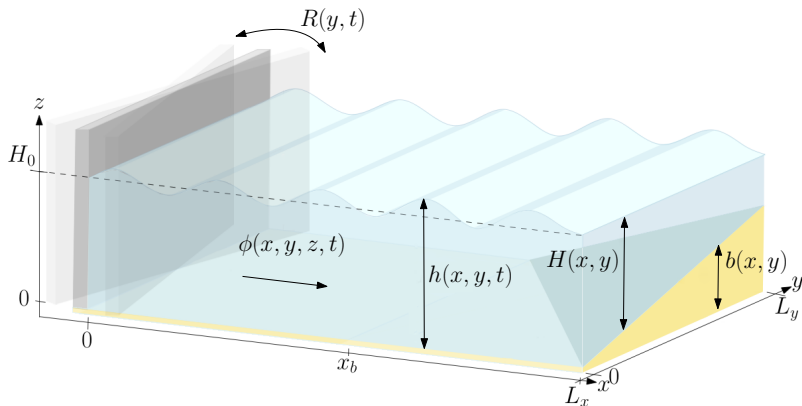# Table of Contents

Mathematical model

Figure: Schematic of the numerical wave tank. Waves are generated by a vertical piston wavemaker oscillating horizontally at $x = R(y, t)$ around $x = 0$. The depth at rest $H(x, y)$ varies in space due to the nonuniform seabed topography $b(x, y)$.

Mathematical model

In this study, the nonlinear potential-flow equations (PFE)

$$\delta\phi: \ \nabla^2\phi = 0, \quad \text{in } \Omega, \tag{9a}$$

$$(\delta\phi)|_{z=b+h}: \ \partial_t h + \nabla(h+b)\cdot\nabla\phi - \partial_z\phi = 0, \quad \text{at } z = b+h, \tag{9b}$$

$$\delta h: \ \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g(b+h-H_0) = 0, \quad \text{at } z = b+h, \tag{9c}$$

$$(\delta\phi)|_{x=R}: \ \partial_x\phi - \partial_y\phi\,\partial_y R = \partial_t R, \quad \text{at } x = R, \tag{9d}$$

are obtained from Luke's variational principle [2]:

$$0 = \delta \int_0^T \iint_{\Omega_h} \int_{b(x,y)}^{b(x,y)+h(x,y,t)} \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g(z-H_0)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \tag{10}$$

---

[2]Luke, J. (1967). A variational principle for a fluid with a free surface. Journal of Fluid Mechanics, 27(2), 395-397.
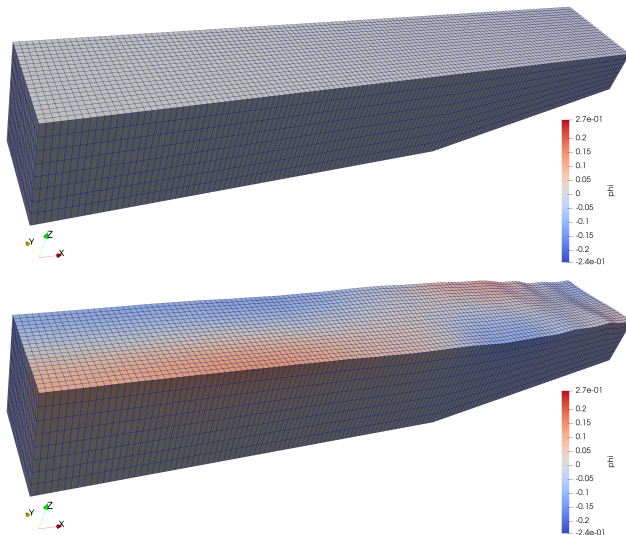
Numerical results

Figure: Velocity potential fields at $t = t_0$ (top) and $t = t_{\text{end}}$ (bottom).

Thank you!
Questions?