$\boxed{Q1}$.

- Linear advection-diffusion eq. describes both the advection and diffusion of a system via a linear PDE.

* ~~Linear equation~~

  - Advection eq
    Describes the transportation of some conserved quantity or material by bulk motion of a fluid.

    ADVECTION EQUATION FOR QUANTITY $u(x,t)$. $\left[\partial u / \partial t = f \, \partial u / \partial x \right.$

    $f(x,t)$ is some given function.

  - Diffusion eq
    Describes diffusion of material / ~~quantity~~ quantity $u(x,t)$ due to macroscopic movements of particles via collisions, etc in the flow.

    $$\left. \partial u / \partial t = D \frac{\partial^2 u}{\partial x^2} \right] \text{DIFFUSION EQUATION FOR } u(x,t)$$

    where D is ~~some~~ the diffusion coefficient.

As the advection eq and diffusion eq are linearly independent, they can be added together to form the advection-diffusion eq:

$$\partial u/\partial t = D \frac{\partial^2 u}{\partial x^2} + f \, \partial u / \partial x$$

For the eq given in the ~~e~~question, $f = a(t)$ and $D = \epsilon$.

A linear PDE is a function of the form (n^{th}-order)

~~$F(x; \partial^k u / \partial x) = F(x_1, \dots, x_n; \partial u/\partial x_1, \dots, \partial u/\partial x_n, \dots, \partial^k u/\partial x_n \dots) = 0$ where F is a linear function, that is, $\sum_{|\alpha| \leq k} a_\alpha(x) \frac{\partial^k}{\partial x}$~~

$$F(\underline{x}; \underline{u}(\underline{x}); \partial^k \underline{u}/\partial \underline{x}, \dots, \partial^n \underline{u}/\partial \underline{x}^n) = 0 \quad \Big\} \text{PDE}$$

or

$$F(\underline{x}) = \sum_{|\alpha| \leq k} a_\alpha(\underline{x}) \, \partial^k / \partial \underline{x}^k \, \underline{u}$$

that is, $F(\underline{x})$ is a ~~polynomial of~~ linear eq of its derivatives. Our eq clearly fits this def with coefficients $a(t)$, ϵ.

[Q2] From standard Taylor expansion in two variables to 3rd order

u_j^n: [using $t - \frac{1}{2}\Delta t$]: (about $(x_j, t_{n+1/2})$)

$$u_j^n = u + u_t(-\tfrac{1}{2}\Delta t) + \tfrac{1}{2}u_{tt}\cdot(-\tfrac{1}{2})^2(\Delta t)^2 + \tfrac{1}{6}(-\tfrac{1}{2}\Delta t)^3 u_{ttt}$$

$$= u - \tfrac{1}{2}\Delta t\, u_t + \tfrac{1}{8}(\Delta t)^2 u_{tt} - \tfrac{1}{48}(\Delta t)^3 u_{ttt} + \dots$$

u_j^{n+1}: [using $t - \frac{1}{2}\Delta t + \Delta t$]: (about $(x_j, t_{n+1/2})$)

$$u_j^{n+1} = u + \tfrac{1}{2}\Delta t\, u_t + \tfrac{1}{8}(\Delta t)^2 u_{tt} + \tfrac{1}{48}(\Delta t)^3 u_{ttt} \dots$$

Then proving (2.80):

$$\partial_t u(x, t+\tfrac{1}{2}\Delta t) = u(x, (t+\tfrac{1}{2}\Delta t)+\tfrac{1}{2}\Delta t) - u(x, (t+\tfrac{1}{2}\Delta t) - \tfrac{1}{2}\Delta t)$$
$$= u(x, t+\Delta t) - u(x, t)$$

that is, $\partial_t u_j^{n+1/2} = u_j^{n+1} - u_j^n$

using Taylor expansions:

$$\partial_t u_j^{n+1/2} = [u - u] + [\tfrac{1}{2} - (-\tfrac{1}{2})]u_t\Delta t + [\tfrac{1}{8} - \tfrac{1}{8}](\Delta t)^2 u_{tt} + [\tfrac{1}{48} - (-\tfrac{1}{48})](\Delta t)^3 u_{ttt} + \dots$$
$$= 0 + \Delta t\, u_t + \tfrac{1}{24}(\Delta t)^3 u_{ttt} + \dots$$

As required.

Proving (2.81):

Using $\partial_x^2 u(x, t+\Delta t) = u(x+\Delta x, t+\Delta t) - 2u(x, t+\Delta t) + u(x-\Delta x, t+\Delta t)$

that is, $\partial_x^2 u_j^{n+1} = u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}$.

[Q2]

Expansion up to $n=6$ using $f(x-\Delta x, t-\Delta t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{\partial^n f}{\partial x^{n-k} \partial t^k} (\Delta x)^{n-k} (\Delta y)^k$

u_j^{n+1}: using $(\frac{1}{2}\Delta t)$, $(0\Delta x)$.
All terms including x are zero so -

$$u_j^{n+1} = u + \tfrac{1}{2}\Delta t u_t + \tfrac{1}{2}\cdot\tfrac{1}{4}(\Delta t)^2 u_{tt} + \tfrac{1}{3!}\cdot(\tfrac{1}{2}\Delta t)^3 u_{ttt} + \tfrac{1}{4!}(\tfrac{1}{2}\Delta t)^4 u_{tttt} + \tfrac{1}{5!}(\tfrac{1}{2}\Delta t)^5 u_{ttttt} + \tfrac{1}{6!}(\tfrac{1}{2}\Delta t)^6 u_{tttttt}.$$

$u_{j+1}^{n+1} + u_{j-1}^{n+1}$: using $(\frac{1}{2}\Delta t)$, u_{j+1}^{n+1} has $(-\Delta x)$, u_{j-1}^{n+1} has (Δx).
* unsure of the adjustments?

Then the taylor expansion becomes -

$$\left[\sum_{n=0}^{\infty} \tfrac{1}{n!} \sum_{k=0}^{n} \tfrac{n!}{k!(n-k)!} \tfrac{\partial^n u}{\partial x^{n-k}\partial t^k} (-\Delta x)^{n-k} (\tfrac{1}{2}\Delta t)^k\right] +$$
$$\left[\sum_{n=0}^{\infty} \tfrac{1}{n!} \sum_{k=0}^{n} \tfrac{n!}{k!(n-k)!} \tfrac{\partial^n u}{\partial x^{n-k}\partial t^k} (\Delta x)^{n-k} (\tfrac{1}{2}\Delta t)^k\right]$$
$$= \sum_{n=0}^{\infty} \tfrac{1}{n!} \sum_{k=0}^{n} \tfrac{n!}{k!(n-k)!} \tfrac{\partial^n u}{\partial x^{n-k}\partial t^k} (\tfrac{1}{2}\Delta t)^k \left[(-\Delta x)^{n-k} + (\Delta x)^{n-k}\right]$$

Then

$\underline{n=1}$

$\boxed{k=0}$ $u_x [-\Delta x + \Delta x]$ $+$ $\boxed{k=1}$ $u_t \cdot (\frac{1}{2}\Delta t)[1+1]$

$= \frac{1}{2}\Delta t u_t$

$\underline{n=2}$

$\frac{1}{2}[$ $\boxed{k=0}$ $\frac{2}{1\cdot 2} u_{xx} [-(\Delta x)^2 + (\Delta x)^2]$ $+$ $\boxed{k=1}\xrightarrow{} n-k=1$ $\frac{2}{1\cdot 1} u_{xt} (\frac{1}{2}\Delta t)[0]$

$\boxed{k=2} \to n-k=0$ $\frac{2}{2} u_{tt} (\frac{1}{2}\Delta t)^2 [2]$ $= \frac{1}{2} u_{xx} \cdot (\Delta x)^2 + \frac{1}{2}(\Delta t)^2 u_{tt}$

$\underline{n=3}$

$\frac{1}{3!}[$ $\boxed{k=0}$ $- n-k=3$ $\frac{3!}{1\cdot 1} u_{xxx} [0]$ $+$ $\boxed{k=1}\to n-k=2$ $\frac{3!}{2!} u_{xxt} (\frac{1}{2}\Delta t)[2(\Delta x)^2]$ $+$ $\boxed{k=2}\to n-k=1$ $\frac{3!}{2!} u_{xtt} (\frac{1}{2}\Delta t)^2 [0]$

$= \frac{1}{6}[3 \cdot \frac{1}{2} \cdot 2(\Delta t)(\Delta x)^2 u_{xxt}] = \frac{1}{2}(\Delta t)(\Delta x)^2 u_{xxt}$ } shouldn't this be zero?

Forgot - $\boxed{k=3}\to n-k=0$ $\frac{1}{3!} \cdot \frac{3!}{3!} u_{ttt} (\frac{1}{2}\Delta t)^3 \cdot 2$ $= \frac{1}{2}(\Delta t)(\Delta x)^2 u_{xxt} + \frac{1}{24} u_{ttt} (\Delta t)^3$

$\underline{n=4}$

$\frac{1}{4!}[$ $\boxed{k=0}\to n-k=4$ $\frac{4!}{1} u_{xxxx} \cdot 2(\Delta x)^4 +$ $\boxed{k=1}\to n-k=3$ $\frac{4!}{3!} u_{xxxt} (\frac{1}{2}\Delta t)[0]$ $+$ $\boxed{k=2}\to n-k=2$ $\frac{4!}{2!\,2!} u_{xxtt} (\frac{1}{2}\Delta t)^2 [2(\Delta x)^2]$

$\boxed{k=3}\to n-k=1$ $\frac{4!}{3!} u_{xttt} (\frac{1}{2}\Delta t)^3 [0]$ $+$ $\boxed{k=4}\to n-k=0$ $\frac{4!}{4!} u_{tttt} (\frac{1}{2}\Delta t)^4 \cdot 2]$ $=$ $2(\Delta x)^4 u_{xxxx} + 6\cdot\frac{1}{4}\cdot 2\, u_{xxtt} (\Delta t)^2 (\Delta x)^2 + \frac{1}{24}\cdot\frac{1}{16}\cdot 2(\Delta t)^4 u_{tttt}$

$\boxed{Q32}$ Continued

$\underline{n=4}$

$\hookrightarrow 2(\Delta x)^4 u_{xxxx} + 3u_{xxtt}(\Delta t)^2(\Delta x)^2 + \frac{1}{192}(\Delta t)^4 u_{tttt}$.

shouldn't those be zero?

$\underline{n=5}$

$\boxed{k=0} \to n-k=5$, $\boxed{k=1} \to n-k=4$, $\boxed{k=2} \to n-k=3$

$\frac{1}{5!}\left[u_{xxxxx}\right][0] + \frac{5!}{1!4!} u_{xxxxt}\left(\tfrac{1}{2}\Delta t\right)\left[2(\Delta x)^4\right] + \frac{5!}{2!3!} u_{xxxtt}\left(\tfrac{1}{2}\Delta t\right)^2 [0]$

$\boxed{k=3} \to n-k=2$, $\boxed{k=4} \to n-k=1$

$+ \frac{5!}{3!2!} u_{xxttt}\left(\tfrac{1}{2}\Delta t\right)^3 \cdot 2(\Delta x)^2 + \frac{5!}{4!} u_{xtttt}\cdot\left(\tfrac{1}{2}\Delta t\right)^4\cdot[0]$

$\boxed{k=5} \to n-k=0$

$+ \frac{5!}{5!} u_{ttttt}\left(\tfrac{1}{2}\Delta t\right)^5 \cdot 2 \quad = \frac{1}{5!}\left[5u_{xxxxt}(\Delta t)(\Delta x)^4 + \tfrac{5}{2}u_{xxttt}(\Delta t)^3(\Delta x)^2\right] + \frac{1}{5!}\cdot\frac{1}{2^4}u_{ttttt}(\Delta t)^5$.

$\underline{n=6}$ $\boxed{k=0} \to n-k=6$, $\boxed{k=1} \to n-k=5$

$\frac{1}{6!}\left[\frac{6!}{6!} u_{xxxxxx}\, 2(\Delta x)^6 + \frac{6!}{1!5!} u_{xxxxxt}\left(\tfrac{1}{2}\Delta t\right)[0]\right.$

$\boxed{k=2} \to n-k=4$, $\boxed{k=3} \to n-k=3$

$\frac{6!}{2!4!} u_{xxxxtt}\left(\tfrac{1}{2}\Delta t\right)^2\cdot 2(\Delta x)^4 + \frac{6!}{3!3!} u_{xxxttt}\left(\tfrac{1}{2}\Delta t\right)^3[0] +$

$\boxed{k=4} \to n-k=2$, $\boxed{k=5} \to n-k=1$

$\frac{6!}{4!2!} u_{xxtttt}\left(\tfrac{1}{2}\Delta t\right)^4\cdot 2(\Delta x)^2 + [0]$

$\boxed{k=6} \to n-k=0$

$\frac{6!}{6!} u_{tttttt}\left(\tfrac{1}{2}\Delta t\right)^6 \cdot 2 \quad = \frac{2}{6!}u_{xxxxxx}(\Delta x)^6 + \frac{1}{4!}\cdot\frac{1}{4}u_{xxxxtt}(\Delta t)^2(\Delta x)^4 + \frac{1}{4!}\cdot\frac{1}{16}u(\Delta t)^4(\Delta x)^2 + \frac{2}{6!}u_{tttttt}\left(\tfrac{1}{2}\Delta t\right)^6$

$= \frac{1}{360}u_{xxxxxx}(\Delta x)^6 + \frac{1}{96}u_{xxxxtt}(\Delta t)^2(\Delta x)^4 + \frac{1}{384}u_{xxtttt}(\Delta t)^4(\Delta x)^2 + \frac{1}{2^6\cdot 6!}u_{tttttt}(\Delta t)^6$

~~Present terms~~ : Grouping terms :

t-derivatives –

$\Delta t u_t(1-2\cdot\tfrac{1}{2}) + (\Delta t)^2 u_{tt}\left(\tfrac{1}{4} - 2\cdot\tfrac{1}{2}\cdot\tfrac{1}{4}\right) + (\Delta t)^3 u_{ttt}\left(2\cdot\tfrac{1}{3!}\cdot\tfrac{1}{2^3} - 2\tfrac{1}{3!}\tfrac{1}{2^3}\right)$

$+ \ldots = \underline{\underline{0}}$. As $\cancel{\partial^n u/\partial t^n \Delta t} - 2\,\partial^n u/\partial t^n(\Delta t)^n + \frac{\partial^n u}{\partial t^n}(\Delta t)2 = 0$.

x-derivatives –

only for n even so –

$2\left[\tfrac{1}{2}(\Delta x)^2 u_{xx} + \tfrac{1}{4!}(\Delta x)^4 u_{xxxx} + \tfrac{1}{6!}(\Delta x)^6 u_{xxxxxx}\right]$ expected value of $\partial x^2 u_j^{n+1}$

mixed x-t derivatives – [~~should cancel?~~] as seen.

$\hookrightarrow$ ~~Reasons why they may not cancel~~

$\hookrightarrow$ Error in Taylor series formula ~~by me~~, made by me.

$\boxed{Q2}$ Continued

Then all we have are $\overset{\text{even } n}{\vee}$ x derivatives and mixed derivatives from $u_{j+1}^{n+1} + u_{j-1}^{n+1}$

$$\partial_x{}^2 u_j^{n+1} = \left[(\Delta x)^2 u_{xx} + 2(\Delta x)^4 u_{xxxx} + 3/6!(\Delta x)^6 u_{xxxxxx} + \dots \right]$$

$$+\left[\tfrac{1}{2}(\Delta t)(\Delta x)^2 u_{xxt} + 3u_{xxtt}(\Delta t)^2(\Delta x)^2 + \tfrac{5}{5!}(\Delta t)(\Delta x)^4 u_{xxxxt} \right.$$

$$+ \tfrac{5}{2\cdot 5!}(\Delta t)^3(\Delta x)^2 u_{xxttt} + \tfrac{1}{96} u_{xxxxtt}(\Delta t)^2(\Delta x)^4 +$$

$$\left. \tfrac{1}{384}(\Delta t)^4(\Delta x)^2 u_{xxtttt} + \dots \right] + \text{further terms}.$$

~~comparing to 2.81 -~~ $\quad$ & u terms are $[2u - 2u] = 0$

~~$(\Delta x)^4 u_{xxxx}$ coeff incorrect, u_{xxxxt} coeff incorrect,~~

Some incorrect ~~into~~ coefficients! ~~eg~~ $\quad$ TO CORRECT IF TIME.

May run ~~as~~ into further proofs - ~~prioritising methodology~~. using given coefficients in 2.81 in further proof.

Proving 2.82

$$\partial_x{}^2 u_j^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$$

- will use adjustment ~~to~~ (Δt term) is $(-\tfrac{1}{2}\Delta t)$.
- Adjust terms from $u_{j+1}^{n+1} + u_{j-1}^{n+1}$ expansion - replacing $(\tfrac{1}{2}\Delta t)$ with $(-\tfrac{1}{2}\Delta t)$.

Then

$n=1$

$$2\cdot(-\tfrac{1}{2}\Delta t)u_t \; +$$

$n=2$

$$\tfrac{1}{2}\left[\tfrac{2!}{2!}\cdot 2(\Delta x)^2 u_{xx} + \tfrac{2!}{2!}(-\tfrac{1}{2}\Delta t)^2 u_{tt}\cdot 2 \right.$$

$n=3$

$$\tfrac{1}{3!}\left[\tfrac{3!}{2!}\cdot 2\cdot(-\tfrac{1}{2}t)(\Delta x)^2 + \tfrac{3!}{3!}\cdot 2\cdot(-\tfrac{1}{2}\Delta t)^3 u_{ttt} \right]$$

$n=4$

$$\tfrac{1}{4!}\left[2\cdot\tfrac{4!}{4!} u_{xxxx}(\Delta x)^4 + \tfrac{4!}{2!2!}(-\tfrac{1}{2}\Delta t)^2\cdot 2\cdot(\Delta x)^2 u_{xxtt} + \tfrac{4!}{4!}\cdot 2\cdot(-\tfrac{1}{2}\Delta t)^4 u_{tttt} \right]$$

$n=5$

$$\tfrac{1}{5!}\left[\tfrac{5!}{4!}(-\tfrac{1}{2}\Delta t)\cdot 2\cdot(\Delta x)^4 u_{xxxxt} + \tfrac{5!}{3!2!}(-\tfrac{1}{2}\Delta t)^3\cdot 2\cdot(\Delta x)^2 u_{xxttt} + \tfrac{5!}{5!}\cdot(-\tfrac{1}{2}\Delta t)^5\cdot 2\cdot u_{ttttt} \right)$$

$n=6$

$$\tfrac{1}{6!}\left[\tfrac{6!}{6!}\cdot 2\cdot(\Delta x)^6 u_{xxxxxx} + \tfrac{6!}{2!4!} u_{xxxxtt}(-\tfrac{1}{2}\Delta t)^2\cdot 2(\Delta x)^4 + \right.$$

$$\tfrac{6!}{4!2!}(-\tfrac{1}{2}\Delta t)^4\cdot 2(\Delta x)^2 u_{xxtttt} + \tfrac{6!}{6!}\cdot(-\tfrac{1}{2}\Delta t)^6\cdot 2$$

|Q2 Continued|

Giving:

$$\partial_x^2 u_j^n = \cancel{\ldots}$$

t derivatives:

$$\tfrac{1}{n!}\partial^n u/\partial t^n \cdot (-\tfrac{1}{2}\Delta t)^n \cdot 2 - 2 \cdot \tfrac{1}{n!}\tfrac{\partial^n u}{\partial t^n}(-\tfrac{1}{2}\Delta t)^n = \underline{0}.$$

Then remaining terms are x-derivatives of $u_{j+1}^n + u_{j-1}^n$ and also the mixed derivatives - so looking at the full expansion now -

$$\theta \partial_x^2 u_j^{n+1} + \partial_x^2 u_j^n - \theta \partial_x^2 u_j^n = \partial_x^2 u_j^n + \theta(\partial_x^2 u_j^{n+1} - \partial_x^2 u_j^n)$$

$$=$$

$$\theta\Big[\tfrac{1}{2}\cdot\tfrac{2!}{2!}\cdot 2(\Delta x)^2 u_{xx} + \tfrac{1}{3!}\cdot\tfrac{3!}{2!}\cdot 2\cdot(-\tfrac{1}{2}\Delta t)(\Delta x)^2 u_{xxt}$$

$$+ 2\cdot\tfrac{1}{4!}\cdot\tfrac{4!}{4!}(\Delta x)^4 u_{xxxx} + \tfrac{1}{4!}\cdot\tfrac{4!}{2!\,2!}(-\tfrac{1}{2}\Delta t)^2\cdot 2\cdot(\Delta x)^2 u_{xxtt}$$

$$+ \tfrac{1}{5!}\cdot\tfrac{5!}{4!}(-\tfrac{1}{2}\Delta t)\cdot 2(\Delta x)^4 u_{xxxxt} + \tfrac{1}{5!}\cdot\tfrac{5!}{3!\,2!}(-\tfrac{1}{2}\Delta t)^3\cdot 2\cdot(\Delta x)^2 u_{xxttt}$$

$$+ \tfrac{1}{6!}\cdot\tfrac{6!}{6!}\cdot 2(\Delta x)^6 u_{xxxxxx} + \tfrac{1}{6!}\,6!/2!\,4!\; u_{xxxxtt}(-\tfrac{1}{2}\Delta t)^2\cdot 2\cdot(\Delta x)^4$$

$$+ \tfrac{1}{6!}\cdot 6!/4!\,2!\,(-\tfrac{1}{2}\Delta t)^4\cdot 2\cdot(\Delta x)^2 u_{xxtttt} + \ldots\Big]$$

$$+$$

$$\theta\Big[(\Delta x)^2 u_{xx} + \tfrac{1}{12}(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \tfrac{1}{24}(\Delta t)(\Delta x)^2 u_{xxxxt}$$

$$\tfrac{1}{2}(\Delta t)(\Delta x)^2 u_{xxt} + \tfrac{1}{4}(\Delta t)^2(\Delta x)^2 u_{xxtt} + \tfrac{1}{96}u_{xxxxtt}(\Delta t)^2(\Delta x)^4 +$$

$$+ \tfrac{5}{2\cdot 5!}(\Delta t)^3(\Delta x)^2 u_{xxttt}$$

$$\tfrac{1}{384}(\Delta t)^4(\Delta x)^2 u_{xxtttt} + \ldots\Big]$$

$$= \overset{x\ \text{devs}}{\Big[(\Delta x)^2 u_{xx}} + \tfrac{1}{12}(\Delta x)^4 u_{xxxx} + \tfrac{2}{6!}(\Delta x)^6 u_{xxxxxx}\Big]$$

$$+\Big[(\Delta t)(\Delta x)^4 u_{xxt} + \ldots\Big]$$

no time to write full expansion.

+ rearrange.

Proving ~~2.~~ 2.83 & 2.84

~~Need~~ Have terms for $\partial_x^2 u_j^{n+1}, \partial_x^2 u_j^n$.

Need terms for $\partial_t u_j^{n+1/2}$:

$$\partial_t u(x, t+1/2\Delta t) = u(x,(t+1/2\Delta t)+1/2\Delta t) - u(x,(t+1/2\Delta t)-1/2\Delta t)$$
$$= u(x, t+\Delta t) - u(x,t)$$

so $\delta_t u_j^{n+1/2} = u_j^{n+1} - u_j^n$ } we already have these terms in their full expansion for previous qus.

$$T_j^{n+1/2} = 1/\Delta t(u_j^{n+1} - u_j^n) - 1/(\Delta x)^2\left[\theta\left[u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right]\right.$$
$$\left. + (1-\theta)\left[u_{j+1}^n - 2u_j^n + u_{j-1}^n\right]\right]$$

No time to do full expansion & rearrange - all these terms have been previously calculated.

Q3.

Explicit scheme:

$\partial u/\partial t(x_j,x_n) \approx U_j^{n+1} - U_j^n$] Forward diff in time on U_j^n

$\partial^2 u/\partial x^2(x_j,x_n) \approx U_{j+1}^n - 2U_j^n + U_{j-1}^n$] central difference in space on U_j^n.

~~$\partial u/\partial x(x_j,x_n) \approx U_j^n - U_{j-1}^n$] backward diff in space on U_j^n → upwind scheme.~~

~~The scheme is — with scaling —~~

~~$1/\Delta t(U_j^{n+1} - U_j^n) + \frac{a(t)}{\Delta x}[U_j^n - U_{j-1}^n] - 1/(\Delta x)^2 \epsilon [U_{j+1}^n - 2U_j^n + U_{j-1}^n] = 0$~~

$$\partial u/\partial x(x_j,x_n) = \begin{cases} U_j^n - U_{j-1}^n, & -a < 0 \\ U_{j+1}^n - U_j^n, & -a > 0 \end{cases}$$] backwards/forwards difference depending on sign of $a(t)$.

not sure on these.

Explicit scheme is then —

$$1/\Delta t[U_j^{n+1} - U_j^n] = \begin{cases} \frac{a(t)}{\Delta x}[U_j^n - U_{j-1}^n] + 1/(\Delta x)^2 \epsilon [U_{j+1}^n - 2U_j^n + U_{j-1}^n] & \leftarrow \text{For } -a<0 \\ \frac{a(t)}{\Delta x}[U_{j+1}^n - U_j^n] + 1/(\Delta x)^2 \epsilon [U_{j+1}^n - 2U_j^n + U_{j-1}^n] & \leftarrow \text{For } -a>0. \end{cases}$$

Implicit ~~solution~~ scheme

$u_t \approx U_j^{n+1} - U_j^n$] ~~Forward diff~~ Backwards diff on U_j^{n+1}

$u_{xx} \approx U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}$] central diff. as before. on U_j^{n+1}

$$u_x \approx \begin{cases} U_j^{n+1} - U_{j-1}^{n+1}, & -a<0 \\ U_{j+1}^{n+1} - U_j^{n+1}, & -a>0 \end{cases}$$ same spatial scheme as before on U_j^{n+1}.

Hence we have —

$$1/\Delta t[U_j^{n+1} - U_j^n] = \begin{cases} \frac{a(t)}{\Delta x}[U_j^{n+1} - U_{j-1}^{n+1}] + \frac{\epsilon}{(\Delta x)^2}[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^n], & -a<0 \\ \frac{a(t)}{\Delta x}[U_{j+1}^{n+1} - U_j^{n+1}] + \frac{\epsilon}{(\Delta x)^2}[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^n], & -a>0 \end{cases}$$

(~~missing μ because not sure how~~ (need to introduce some μ).

θ-scheme

u_t remains the same. We average out ~~the u_t, the~~ u_x, u_{xx} for explicit & implicit schemes. With weighting $0 \le \theta \le 1$.

Case $-a<0$:

implicit scheme

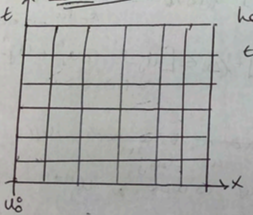
$$\frac{1}{\Delta t}(U_j^{n+1}-U_j^n) = \theta\left[\frac{a(t)}{\Delta x}(U_j^{n+1}-U_{j-1}^{n+1}) + \frac{\epsilon}{(\Delta x)^2}(U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1})\right]$$
$$+(1-\theta)\left[\frac{a(t)}{\Delta x}(U_j^n-U_{j-1}^n)+\frac{\epsilon}{(\Delta x)^2}(U_{j+1}^n-2U_j^n+U_{j-1}^n)\right]$$

explicit scheme

Case $-a>0$:

$$\frac{1}{\Delta t}(U_j^{n+1}-U_j^n) = \theta\left[\frac{a(t)}{\Delta x}(U_{j+1}^{n+1}-U_j^{n+1}) + \frac{\epsilon}{(\Delta x)^2}(U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1})\right]$$
$$+(1-\theta)\left[\frac{a(t)}{\Delta x}(U_{j+1}^n-U_j^n)+\frac{\epsilon}{(\Delta x)^2}(U_{j+1}^n-2U_j^n+U_{j-1}^n)\right]$$

Mesh and ordering



Let $M=|L-L_p|$ ~~the~~, assuming $L, L_p\in\mathbb{R}$.

$t\in[0,t_F]$. We define ~~J Δx points~~ J mesh points for x — $\Delta x = 1/J$.
and N mesh points for t — $\Delta t = 1/N$. J does not have to equal N, but J = N is simple.

For our indexing of the mesh, we $U_j^n = [x]_{J(n)+(j+1)}$ } may cause some issues at J-1, N-1!

Boundaries + ICs.

That is,

$$\partial\Omega = \overset{IC}{\{U_j^0 : j=0,1,\dots,J-1\}} \cup \overset{BC}{\{U_{J-1}^n : n=0,1,\dots,N-1\}}$$
$$\cup \overset{BC}{\{U_0^n : n=0,1,\dots,N-1\}}$$

where $U_j^0 = u(x_j,0) = u_0(x)$, $U_{J-1}^n = u(L,t) = 0 = u(L_p,t) = U_0^n$.

MATHS453

$\boxed{Q3}$ Continued

Scheme at boundaries & ICs:

~~u_j^0: $-a \geq 0$ $\frac{1}{\Delta t}[u_j^1 - u_0(x)] = \theta[\frac{a(t)}{\Delta x}[u_j^1 - u_{j-1}^1]$~~

~~u_j^0: $-a \leq 0$: $+ \epsilon/(\Delta x)^2 (u_{j+1}^1 - 2$~~

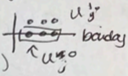
u_j^0, $-a \leq 0$:

LHS (unknown)

$\frac{1}{\Delta t} u_j^{n+1} - \frac{a\theta}{\Delta x} u_j^{n+1} - \frac{a\theta}{\Delta x} u_{j-1}^{n+1} - \epsilon/(\Delta x)^2 [u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}]$

$j=1$, $n=0$

3 known, 3 unknown



RHS (known)

$= \frac{1}{\Delta t} u_j^n + (1-\theta)[\frac{a}{\Delta x}(u_j^n - u_{j-1}^n) + \frac{\epsilon}{(\Delta x)^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)]$

u_j^0, $-a > 0$:

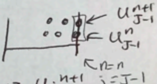
$\frac{1}{\Delta t} u_j^{n+1} + \theta[\frac{a\theta}{\Delta x}(u_{j+1}^{n+1} - u_j^{n+1}) + \frac{\epsilon}{(\Delta x)^2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})$ $=$ $\frac{1}{\Delta t} u_j^n + (1-\theta)[\frac{a(t)}{\Delta x}(u_{j+1}^n - u_j^n) + \frac{\epsilon}{(\Delta x)^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)]$

u_{J-1}^n, $-a < 0$:

known where for $j = J-1$

2 known

4 unknown



$\frac{1}{\Delta t}(u_j^{n+1}) - \frac{\theta a(t)}{\Delta x} u_j^{n+1} - \frac{\theta a}{\Delta x} u_{j-1}^{n+1} + \frac{2\epsilon}{(\Delta x)^2} u_j^{n+1}$

$- \theta \epsilon/(\Delta x)^2 (u_{j+1}^{n+1} + u_{j-1}^{n+1})$

$- \epsilon/(\Delta x)^2 (1-\theta)[u_{j+1}^n + u_{j-1}^n]$

$= + (1-\theta) \frac{a}{\Delta x}[-u_j^n] + \frac{2\epsilon}{(1-\theta)(\Delta x)^2} u_j^n + \frac{1}{\Delta t} u_j^{n+1} \pm \frac{1}{\Delta t} u_j^{n+1}$

u_{j-1}^n $-a > 0$:

~~$\frac{1}{\Delta t}(u_j^{n+1} - u_j^n) - \frac{\theta a}{\Delta x} u_j^{n+1}$~~ ~~eq~~ same as above.

~~$- \epsilon/(\Delta x)^2 (u_{j+1}^{n+1}$~~

$\frac{1}{\Delta t}(u_j^{n+1}) - \theta \epsilon/(\Delta x)^2$

$(\theta(u_{j+1}^{n+1} + u_{j-1}^{n+1}) - (1-\theta)(u_{j+1}^n + u_{j-1}^n)$

$- \frac{\theta a}{\Delta x}(u_{j+1}^{n+1}) + (1-\theta)\frac{a}{\Delta t}(u_{j+1}^n)$

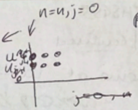
$= \frac{1}{\Delta t} u_j^{n} + \frac{2\epsilon}{(\Delta x)^2} u_j^{n+1} + (1-\theta) (\frac{2\epsilon}{(\Delta x)^2} u_j^n) \pm \frac{1}{\Delta t} u_j^{n+1}$

$+$

$\boxed{Q3}$ Continued

2 unknown

BC unknown



u_0^n : $-a<0$:

LHS (unknown)

$$1/\Delta t \quad 1/\Delta t \quad -\frac{\theta\epsilon}{(\Delta x)^2} u_{j+1}^{n+1} - (1-\theta)\left(\frac{\epsilon}{(\Delta x)^2}(u_{j+1}^{n})\right)$$

$$= 1/\Delta t\,(u_j^{n+1} - u_j^n) + \frac{\theta a}{\Delta x}(u_j^{n+1}) - \frac{2u_j^{n+1}\cdot\epsilon}{(\Delta x)^2}$$

RHS (known)

$$+ (1-\theta)\left[a/\Delta x\,(u_j^n) + \frac{\epsilon}{(\Delta x)^2}(-2u_j^n)\right].$$