Onno Assignment 2

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1 Question 1

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0$$

and (1)

$$\frac{\partial u}{\partial t} + \frac{\partial (g\eta)}{\partial x} = 0$$

where $[x]$ denotes dimension of x, length scale $= L$, time scale $= T$

Velocity $u = U_0 u'$, $u' = \frac{u}{U_0}$, $\frac{[u]}{[U_0]} = [u'] = 1$.
Deviation $x = L_s x'$, $x' = \frac{x}{L_s}$, $\frac{[x]}{[L_s]} = [x'] = 1$.
Time, distance over speed, $t = \frac{L_s}{U_0} t'$, $t' = \frac{tU_0}{L_s}$, $\frac{[T][X]}{[T][X]} = t' = 1$

And so forth for the other quantities. Therefore, if each of these is substituted into the coupled equation we obtain a non-dimensional version of the equations

$$\frac{\partial H_0 \eta'}{\partial \frac{L_s}{U_0} t'} + \frac{\partial H_0 H' U_0 u'}{\partial L_s x'} = 0$$

and

$$\frac{\partial U_0 u'}{\partial \frac{L_s}{U_0} t'} + \frac{\partial g' \eta}{\partial L_s x'} = 0$$

For the sake of notational simplicity we write this in the same form as equation 1) and drop the primes.

Taking the case where $H(x) = H_0$, $U_0^2 = gH_0$

$$\frac{\partial}{\partial \eta} \begin{bmatrix} \eta \\ H_0 \mu \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ gH_0 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \eta \\ H_0 u \end{bmatrix} = 0$$

Finding the eigenvectors of the matrix, which we label A,

$$\det(A - \lambda I) = 0$$

$$det(\begin{bmatrix} -\lambda & 1 \\ gH_0 & -\lambda \end{bmatrix}) = 0$$

$$(-\lambda)^2 - gH_0 = 0$$

$$\lambda^2 = gH_0$$

$$\lambda = \pm\sqrt{gH_0}$$

Defining the matrix B as,

$$B = \frac{1}{2\sqrt{gH_0}} \begin{bmatrix} 1 & -1 \\ \sqrt{gH_0} & \sqrt{gH_0} \end{bmatrix}$$

$$AB = \begin{bmatrix} \frac{\sqrt{gH_0}}{2\sqrt{gH_0}} & \frac{gH_0}{2\sqrt{gH_0}} \\ \frac{gH_0}{2\sqrt{gH_0}} & \frac{-gH_0}{2\sqrt{gH_0}} \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \sqrt{gH_0} & 1 \\ -\sqrt{gH_0} & 1 \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} \frac{gH_0}{2\sqrt{gH_0}} + \frac{gH_0}{2\sqrt{gH_0}} & \frac{gH_0}{2\sqrt{gH_0}} - \frac{gH_0}{2\sqrt{gH_0}} \\ \frac{-gH_0}{2\sqrt{gH_0}} + \frac{gH_0}{2\sqrt{gH_0}} & \frac{-gH_0}{2\sqrt{gH_0}} - \frac{gH_0}{2\sqrt{gH_0}} \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} \frac{1}{2}\sqrt{gH_0} + \frac{1}{2}\sqrt{gH_0} & \frac{1}{2}\sqrt{gH_0} - \frac{1}{2}\sqrt{gH_0} \\ \frac{-1}{2}\sqrt{gH_0} + \frac{1}{2}\sqrt{gH_0} & \frac{-1}{2}\sqrt{gH_0} - \frac{1}{2}\sqrt{gH_0} \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} \sqrt{gH_0} & 0 \\ 0 & -\sqrt{gH_0} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} I$$

Using

$$\frac{\partial H_0 \eta'}{\partial \frac{L_s}{U_0} t'} + \frac{\partial H_0 H' U_0 u'}{\partial L_s x'} = 0$$

and

$$\frac{\partial U_0 u'}{\partial \frac{L_s}{U_0} t'} + \frac{\partial g' \eta'}{\partial L_s x'} = 0$$

we obtain, by setting $U_0^2 = gH_0$

$$\frac{\partial H_0\eta'}{\partial \frac{L_s}{\sqrt{gH_0}}t'} + \frac{\partial H_0 H'\sqrt{gH_0}u'}{\partial L_s x'} = 0$$

and

$$\frac{\partial \sqrt{gH_0}u'}{\partial \frac{L_s}{\sqrt{gH_0}}t'} + \frac{\partial g'\eta'}{\partial L_s x'} = 0$$

Multiplying by L_s

$$\frac{\partial H_0\eta'\sqrt{gH_0}}{\partial t'} + \frac{\partial H_0 H'\sqrt{gH_0}u'}{\partial x'} = 0$$

$$\frac{\partial 2\sqrt{gH_0}u'}{\partial t'} + \frac{\partial g'\eta'}{\partial x'} = 0$$

Subtracting the bottom equation from the top equation, and noting that $g' = 1$ when $U_0^2 = gH_0$

$$\frac{\partial H_0\eta'\sqrt{gH_0} - \partial 2\sqrt{gH_0}}{\partial t'} + \frac{\partial H_0 H'\sqrt{gH_0}u' - \partial\eta'}{\partial x'} = 0$$

$$\frac{\partial \sqrt{gH_0}(H_0\eta' - 2)}{\partial t'} + \frac{\partial H_0 H'\sqrt{gH_0}u' - \partial\eta'}{\partial x'} = 0$$

2 Question 2

We have the initial conditions

$$r_1(x,0) = \begin{cases} r_{1l} & x < 0 \\ r_{lr} & x \geq 0 \end{cases}, r_2(x,0) = \begin{cases} r_{2l} & x < 0 \\ r_{2r} & x \geq 0 \end{cases}$$

assuming we are looking at a rarefaction wave at time $t = 0$. Our characteristics are:

$c_0 t = \sqrt{gH_0}t$ for r_1,
$-c_0 t = -\sqrt{gH_0}t$ for r_2.

each wave moves at a constant velocity along the characteristics, figure 1. It is clear that if the diving line between the left and right wave is at $x = 0$ for $t = 0$, then it will move to $c_0 t$ for time $t = t$ in the the right moving wave r_1 and $-c_0 t$ for time $t = t$ in the left moving wave r_2, yielding

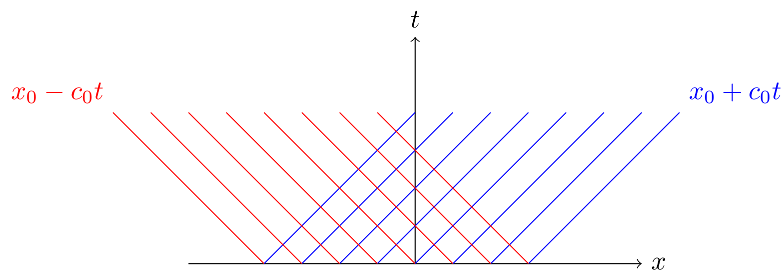

Figure 1: Characteristics.

$$r_1(x,t) = \begin{cases} r_{1l} & x < c_0 t \\ r_{lr} & x \geq c_0 t \end{cases}, r_2(x,t) = \begin{cases} r_{2l} & x < -c_0 t \\ r_{2r} & x \geq -c_0 t \end{cases}$$

Given $H_0 u = \frac{1}{2}(r_1 + r_2)$, $\eta = \frac{1}{2}(r_1 - r_2)/c_0$, $r_1 = H_0 u + c_0 \eta$, $r_2 = H_0 u - c_0 \eta$: substituting our previous piecewise solution of the Reimann problem for r_1, r_2

$$H_0 u(x,t) = \frac{1}{2}(r_1(x,t) + r_2(x,t))$$

$$\eta(x,t) = \frac{1}{2}(r_2(x,t) - r_1(x,t))/c_0$$

By inspection, we can see that the pure left wave will be when $x < -c_0 t$ and the pure right wave when $x > c_0 t$.

$$H_0 u(x,t) = \frac{1}{2}(r_1(x,t) + r_2(x,t)) = \begin{cases} H_0 u_l & x < -c_0 t \\ H_0 u_r & x > c_0 t \end{cases}$$

Then

$$H_0 u(x,t) = \frac{1}{2}(r_1(x,t) + r_2(x,t)) = \begin{cases} H_0 u_l & x < -c_0 t \\ \frac{H_0(u_l + u_r)}{2} + \frac{c_0(\eta_l - \eta_r)}{2} & -c_0 t < x < c_0 t \\ H_0 u_r & x > c_0 t \end{cases}$$

$$\eta(x,t) = \frac{1}{2}(r_1(x,t) - r_2(x,t))/c_0 = \begin{cases} \eta_l & x < -c_0 t \\ \frac{H_0(u_l - u_r)}{2c_0} + \frac{(\eta_l + \eta_r)}{2} & -c_0 t < x < c_0 t \\ \eta_r & x > c_0 t \end{cases}$$

From the initial conditions, at $r_1(x,0)$, $r_2(x,0)$, we can see we have a rarefaction wave and $u_l < u_r$.

Writing in the form

$$\partial_t \mathbf{r} + \mathbf{A}\partial_{\mathbf{x}}\mathbf{r} = \mathbf{0}$$

where $\mathbf{r} = \begin{bmatrix} H_0 u + c_0\eta \\ H_0 u - c_0\eta \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} \sqrt{gH_0} & 0 \\ 0 & -\sqrt{gH_0} \end{bmatrix}$

we obtain two equations

$$\frac{\partial H_0 u + c_0\eta}{\partial t} + \frac{\partial \sqrt{gH_0}(H_0 u + c_0\eta)}{\partial x} = 0$$

$$\frac{\partial H_0 u - c_0\eta}{\partial t} + \frac{\partial - \sqrt{gH_0}(H_0 u - c_0\eta)}{\partial x} = 0$$

which can be written in the form

$$\frac{\partial f(u)}{\partial t} + c\frac{f(u)}{\partial x} = 0$$

which is the form of the general linear advection equation where c is the characteristics of the system.

from $\mathbf{r}$,

$$r_1 = H_0 u + c_0\eta$$
$$r_2 = H_0 u - c_0\eta$$

$$r_{1_l} = H_0 u_l + c_0\eta_l$$
$$r_{1_r} = H_0 u_r + c_0\eta_r$$
$$r_{2_l} = H_0 u_l - c_0\eta_l$$
$$r_{2_r} = H_0 u_r - c_0\eta_r$$

therefore

$$r_1(x,t) = \begin{cases} H_0 u_l + c_0\eta_l & x < c_0 t \\ H_0 u_r + c_0\eta_r & x \geq c_0 t \end{cases}$$

$$r_2(x,t) = \begin{cases} H_0 u_l - c_0\eta_l & x < c_0 t \\ H_0 u_r - c_0\eta_r & x \geq c_0 t \end{cases}$$

3 Question 3

$$r_1(x,0) = H_0 u_l + c_0\eta_l, x < 0$$
$$r_1(x,0) = H_0 u_r + c_0\eta_r, x \geq 0$$
$$r_2(x,0) = H_0 u_l - c_0\eta_l, x < 0$$
$$r_2(x,0) = H_0 u_r - c_0\eta_r, x \geq 0$$

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{h}[F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)]$$

where

$$F(U_j^n, U_{j+1}^n) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{f}_{j+\frac{1}{2}}(t)dt = \mathbf{f}_{j+\frac{1}{2}}\mathbf{u}(j+\frac{1}{2}, t)$$

because the Riemann solution is constant, giving the result of the integration as $t\mathbf{f}_{j+\frac{1}{2}}\mathbf{u}(j+\frac{1}{2}, t)\frac{1}{\Delta t}$ and in the limit $\lim_{t\to 0} t = \Delta t$.

using our Riemann solution in $H_0 u(x,t)$, $\eta(x,t)$, evaluated at $x = 0$

$$\mathbf{f}_{j+\frac{1}{2}}\mathbf{u}(j+\frac{1}{2}, t) = \begin{bmatrix} \frac{H_0(u_l+u_r)}{2} + \frac{c_0(\eta_l-\eta_r)}{2} \\ \frac{H_0(u_l-u_r)}{2c_0} + \frac{(\eta_l+\eta_r)}{2} \end{bmatrix}$$

With an "open" domain we simply loop wrap around at the edges, such that we have, at the right hand edge and left hand edge respectively, where N is the number of volumes

$$U_j^{n+1} = U_N^n - \frac{\Delta t}{h}[F(U_N^n, U_0^n) - F(U_{N-1}^n, U_N^n)]$$

$$\mathbf{f}_{j+\frac{1}{2}}\mathbf{u}(j+\frac{1}{2}, t) = \begin{bmatrix} \frac{H_0(u_{r_N}+u_{r_0})}{2} + \frac{c_0(\eta_{r_N}-\eta_{r_0})}{2} \\ \frac{H_0(u_{r_N}-u_{r_0})}{2c_0} + \frac{(\eta_{r_N}+\eta_{r_0})}{2} \end{bmatrix}$$

and

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{h}[F(U_0^n, U_1^n) - F(U_N^n, U_0^n)$$

$$\mathbf{f}_{j+\frac{1}{2}}\mathbf{u}(j+\frac{1}{2},t) = \begin{bmatrix} \frac{H_0(u_{l_N}+u_{l_0})}{2c_0} - \frac{\eta_{l_N}-\eta_{l_0}}{2} \\ \frac{H_0(u_{l_N}-u_{l_0})}{2} + \frac{c_0(\eta_{l_N}+\eta_{l_0})}{2} \end{bmatrix}$$

For a closed domain we instead use "ghost" values at the wall. At the wall we have

$$\partial_t H_0 u + \partial_x c_o^2 \eta = 0$$
$$\partial_x c_o^2 \eta = -\partial_t H_0 u$$

In order to which way around the signs are on the ghost values we can think physically about the system. In order for the free surface deviation to be unchanging it must be the same on either side of the wall, as having a negative on one side would create a discontinuity which the system would try to "even out". This leaves the negative to be assigned to the velocity. This also makes sense as we are canceling out the velocity in the final volume by meeting it with an opposite velocity on the wall side. This forces the solution at the wall to be constant.

Therefore, we create two volumes outside the solution domain at indices -1 and N+1 with,

$$\mathbf{u} = \begin{bmatrix} \eta_0 \\ -H_0 u_0 \end{bmatrix}$$

and

$$\mathbf{u} = \begin{bmatrix} \eta_N \\ -H_0 u_N \end{bmatrix}$$

The timestep is determined with the CFL condition $\Delta t \leq \frac{\Delta x}{|a|}$ where a is the wave speed. For a constant cell width h and the speed of the wave c_0, determined by the characteristics.

$$\Delta t \leq \frac{h}{c_0}$$

or, for a cell width which varies in each volume h_j

$$\Delta t \leq min(\frac{h_j}{c_0})$$

The scheme can be extended for varying $H(x)$ by simply taking a local, constant, approximation $H(x) = H_0$ within each volume. In this case the existing mathematics holds without modification. This is an approximation. However, it is the same kind of approximation we have already made for the velocity and free surface deviation terms, i.e we treat a continuous function as locally constant within some small limit and then sum over these local approximations. The overall logic holds that any continuous function $f(x)$ can be locally approximated as a linear function in the limit Δx. This is a natural consequence of the Taylor expansion.

4 Question 4.

The finitie volume method is conservative because it calulates the next time step in each cell based on the fluxes into that cell at the current time step. This naturally conserves quantities as there is no way for new energy to be added to the system, with the exception of numerical error at the flux calculation.

Onno - I put a lot of time into trying to get Firedrake working but my code isn't right. I've uploaded what I have. I'm not certain whether my theory is wrong or it's my implementation. I ran out of time to keep trying with it.