MATH5453M Numerical Exercise 1.

Given:

$$u_t - a(t)u_x - \varepsilon u_{xx} = 0 \quad \text{for } x \in [L_p, L]$$

$$u(x,0) = u_0(x)$$

$$u(L_p,t) = u(L,t) = 0,$$

(Small) constant diffusion ε & a given function $a(t)$. The B.C. are classical homogeneous Dirichlet conditions.

Solutions

Problem 1:-

$$u_t - a(t)u_x - \varepsilon u_{xx} = 0 \quad \text{for } x \in [L_p, L]$$

- The above given equation is linear because dependent variable $u(x,t)$ and it's derivative appears linear in the above equation.

① Linear: if we look into the equation there is no term containing multiplication

There is no term containing multiplication of $u(x,t)$ & its derivatives or any power of dependent variable $u(x,t)$.

(2) <u>Advection:</u> '$-a(t)\,u_x$' represents the advection. because it is showing the advection of u_x.

$$-a(t)\frac{\partial u}{\partial x}$$

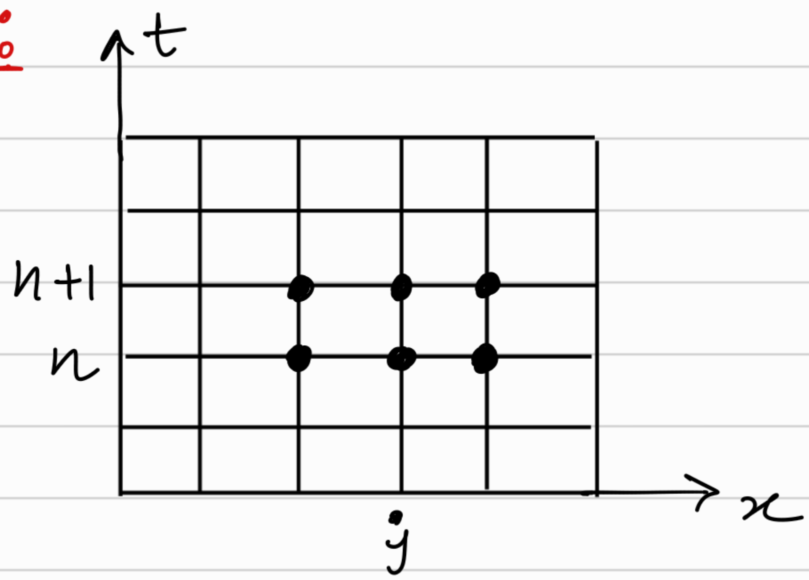
(3) <u>Diffusion:</u> 'εu_{xx}'

$$\downarrow$$

$$\varepsilon\frac{\partial^2 u}{\partial x^2} \rightarrow \text{represents}$$

the diffusion in the equation, with diffusion Coefficient of 'ε'.

Problem 2:



Taylor Series expansion of u_j^{n+1} & u_j^n about $\left(\frac{\Delta t}{2}\right)$

$$u_j = \left[u + \frac{1}{2}\Delta t\, u_t + \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} + \frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt} + \cdots\right]_j^{n+1/2}$$

& $$u_j^n = \left[u - \frac{1}{2}\Delta t\, u_t + \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} - \frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt} + \cdots\right]_j^{n+1/2}$$

Substracting these two, all odd will cancle

$$u_j^{n+1} - u_j^n = \left[\Delta t\, u_t + \frac{1}{24}(\Delta t)^3 u_{ttt} + \cdots\right]_j^{n+1/2}$$

$$u_j^{n+1} - u_j^n = \delta_t u_j^{n+1/2}$$

that means, (2.80)

$$\delta_t u_j^{n+1/2} = \left[\Delta t\, u_t + \frac{1}{24}(\Delta t)^3 u_{ttt} + \cdots\right]_j^{n+1/2} \quad \text{--- (2.80)}$$

from eqn (2.30),

$$\delta_x^2 u(x,t) = u_{xx}(\Delta x)^2 + \frac{1}{12} u_{xxxx}(\Delta x)^4 + \cdots$$

& from $$\delta_x^2 u_j^{n+1} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2}$$

Expanding

$$u_{j+1}^{n+1} = u_j + \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} + \frac{1}{6}(\Delta x)^3 u_{xxx}$$

$$u_{j-1}^{n+1} = u_j - \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} - \frac{1}{6}(\Delta x)^3 u_{xxx}$$

$$\therefore \quad \delta_x^2 u_j^{n+1} = \frac{\left(\cancel{u_j} + \cancel{\Delta x u_x} + \frac{1}{2}(\Delta x)^2 u_{xx} + \cancel{\frac{1}{6}(\Delta x)^3 u_{xxx}}\right) - \cancel{2u_j} + \left(\cancel{u_j} - \cancel{\Delta x u_x} + \frac{1}{2}(\Delta x)^2 u_{xx} - \cancel{\frac{1}{6}(\Delta x)^3 u_{xxx}}\right)}{(\Delta x)^2}$$

(2.80)

$$\delta_x^2 u_j^{n+1} = u_{xx} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \cdots \quad \text{— (2.81)}$$

- Now Considering weighted Average of Spatial derivatives

$$\theta \delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n$$

using expansion of $\delta_x^2 u_j^{n+1}$ & $\delta_x u_j^n$

$$\delta_x^2 u_j^{n+1} = \left[(\Delta x)^2 u_{xx} + \frac{1}{12}(\Delta x)^4 u_{xxxx} + \frac{1}{12}(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+1/2}$$

$$+ \frac{1}{2}\Delta t \left[(\Delta x)^2 u_{xxt} + \frac{1}{12}(\Delta x)^4 u_{xxxxt} + \cdots\right]_j^{n+\frac{1}{2}}$$

$$+\frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2\left[(\Delta t)^2 u_{xxtt}+\cdots\right]_j^{n+\frac{1}{2}}$$

Similarly :-

$$\delta_x^2 u_j^n=\left[(\Delta x)^2 u_{xx}+\frac{1}{12}(\Delta x)^4 u_{xxxx}+\cdots\right]_j^{n+\frac{1}{2}}$$
$$-\frac{1}{2}\Delta t\left[(\Delta x)^2 u_{xxt}+\frac{1}{12}(\Delta x)^4 u_{xxxxt}+\cdots\right]_j^{n+\frac{1}{2}}$$
$$+\frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2+\left[(\Delta t)^2 u_{xxtt}+\cdots\right]_j^{n+\frac{1}{2}}$$

using these two in

$$\theta\delta_x^2 u_j^{n+1}+(1-\theta)\delta_x^2 u_j^n$$
$$=\left[(\Delta x)^2 u_{xx}+\frac{1}{12}(\Delta x)^4 u_{xxxx}+\cdots\right]$$
$$+\left(\theta-\frac{1}{2}\right)\Delta t\left[(\Delta x)^2 u_{xxt}+\frac{1}{12}(\Delta x)^4 u_{xxxxt}+\cdots\right]$$
$$+\frac{1}{8}(\Delta t)^2(\Delta x)^2\left[u_{xxtt}\right]+\cdots \quad \text{---}(2.82)$$

Truncation Error:

$$T_j^{n+\frac{1}{2}}=\frac{\delta_t u_j^{n+\frac{1}{2}}}{\Delta t}-\frac{\theta\delta_x^2 u_j^{n+1}+(1-\theta)\delta_x^2 u_j^n}{(\Delta x)^2} \quad \text{---}(2.83)$$

using $\frac{\delta_t u_j^{n+1/2}}{\Delta t} = u_t + \frac{1}{24}(\Delta t)^2 u_{ttt} + \cdots$

& $\frac{\theta\delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n}{(\Delta x)^2} = u_{xx} + \frac{1}{12}(\Delta x)^2 u_{xxxxt} \cdots$

$$\therefore \; T_j^{n+\frac{1}{2}} = [u_t - u_{xx}] + \left[\left(\frac{1}{2}-\theta\right)\Delta t\, u_{xxt} - \frac{1}{12}(\Delta x)^2 u_{xxxx}\right] + \left[\frac{1}{12}(\Delta t)^2 u_{ttt} - \frac{1}{8}(\Delta t)^2\right] + \left[\frac{1}{12}\left(\frac{1}{2}-\theta\right)\Delta t(\Delta x)^2 u_{xxxxt} - \frac{2}{6!}(\Delta x)^4 u_{xxxxxx}\right] \quad \text{—} (2.84)$$

Problem 3:-

The given advection-diffusion equs

$$u_t - a(t)u_x - \varepsilon u_{xx} = 0$$

$$u_x = \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

$$-a(t)u_x = a(t)\, u_{i+1}^n - u_i^n$$

$$\frac{u_{i+1} - u_i}{\Delta x}$$

for negative $a(t)$ & $a(t) = -1$, consider the forward difference

$$u_{xx} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

θ- discretisation :-

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \theta\left(a(t) \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} + \varepsilon \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} \right)$$
$$+ (1-\theta)\left(a(t) \frac{u_{i+1}^n - u_i^n}{\Delta x} + \varepsilon \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right)$$

taking unknown to LHS:

$$u_i^{n+1} = u_i^n + \Delta t \left[\theta\left(\frac{a(t)}{\Delta x}\left(u_{i+1}^{n+1} - u_i^{n+1}\right) + \frac{\varepsilon}{\Delta x^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) \right) + (1-\theta)\left(\frac{a(t)}{\Delta x}\left(u_{i+1}^n - u_i^n\right) + \frac{\varepsilon}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \right) \right]$$

assuming $\left.\begin{array}{l} \dfrac{\Delta t}{\Delta x} = \lambda \\ \& \ \dfrac{\Delta t}{(\Delta x)^2} = \mu \end{array}\right\}$

$$u_i^{n+1} - \theta\lambda a(t)\, u_{i+1}^{n+1} + \theta\lambda a(t)\, u_i^{n+1}$$
$$- \varepsilon\mu\theta\, u_{i+1}^{n+1} + \theta\varepsilon\mu 2\, u_i^{n+1} - \varepsilon\theta\mu\, u_{i-1}^{n+1}$$
$$= (1-\theta)\left(\frac{a(t)}{\Delta x}\left(u_{i+1}^{n} - u_i^{n}\right) + \frac{\varepsilon}{\Delta x^2}\left(u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}\right)\right)$$

$$\Rightarrow \left[1 + \theta\varepsilon\mu 2 + \theta\lambda a(t)\right] u_i^{n+1} + \left[-\theta\lambda a(t) - \varepsilon\mu\theta\right] u_{i+1}^{n+1}$$
$$- \varepsilon\theta\mu\, u_{i-1}^{n+1} = (1-\theta)\left[a(t)\lambda\left(u_{i+1}^{n} - u_i^{n}\right) + \varepsilon\mu\left(u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}\right)\right]$$

$$\Rightarrow \left[1 + \theta\varepsilon\mu\times 2 + \theta\lambda a(t)\right] u_i^{n+1} + \left[-\theta\lambda a(t) - \varepsilon\mu\theta\right] u_{i+1}^{n+1}$$
$$- \varepsilon\theta\mu\, u_{i-1}^{n+1} = a(t)\lambda\, u_{i+1}^{n} - a(t)\lambda\, u_i^{n} + \varepsilon\mu\, u_{i+1}^{n}$$
$$- \varepsilon\mu 2\, u_i^{n} + \varepsilon\mu\, u_{i-1}^{n} - a(t)\theta\lambda\, u_{i+1}^{n}$$
$$+ \theta a(t)\lambda\, u_i^{n} - \varepsilon\theta\mu\, u_{i+1}^{n} + 2\varepsilon\mu\theta\, u_i^{n}$$
$$- \varepsilon\mu\theta\, u_{i-1}^{n}$$

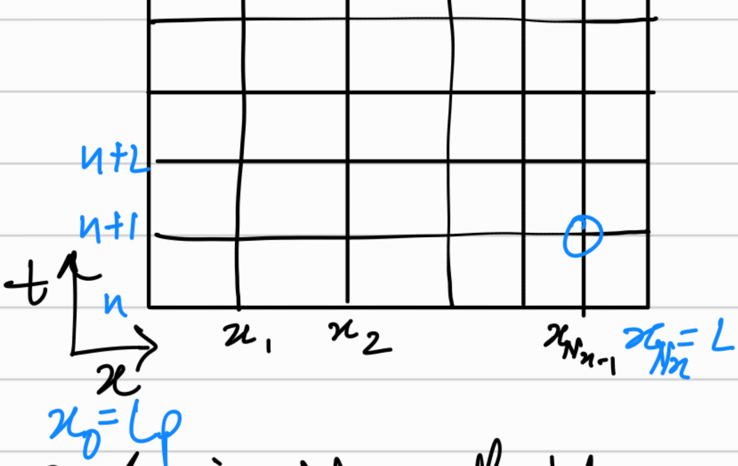
$$\Rightarrow \left[1 + \theta\varepsilon\mu\times 2 + \theta\lambda a(t)\right] u_i^{n+1} - \left[\theta\lambda a(t) + \varepsilon\mu\theta\right] u_{i+1}^{n+1}$$
$$- \varepsilon\theta u_{i-1}^{n+1} = \left[a(t)\lambda - a(t)\theta\lambda + \varepsilon\mu - \varepsilon\theta\mu\right] u_{i+1}^{n}$$
$$+ \left[-a(t)\lambda - 2\varepsilon\mu + \theta a(t)\lambda + 2\varepsilon\mu\theta\right] u_i^{n}$$
$$+ \left[\varepsilon\mu - \varepsilon\mu\theta\right] u_{i-1}^{n}$$

Now, let N_x be the number of subinterval in l_p to L domain, $x \in [l_p, L]$.
Then we have N_x+1 grid points, $i = 0, 1, \cdots N_x$
each grid point is, $x_i = l_p + i\Delta x$, $\Delta x = \dfrac{L - l_p}{N_x}$

→ In python, we store as $u[i, n]$
$i \to$ from 0 to N_x

$i \to$ from 0 to N_x
$n \to$ from n to N_t

at B.C $\to$ $u(L_p, t) = 0$ &
$u(L, t) = 0$,



the interior nodes.
$i = 1, 2, \ldots N_x - 1$,
Boundary nodes are $i = 0$ & $i = N_x$ which are zero

for first order upwind for Advection term
- near the boundary at $i = 1$, we will need $u[1]$ & $u[0]$ but $u[0] = 0$, therefore, upwind difference becomes $\frac{u[1] - 0}{\Delta x}$.

- at $i = N_x - 1$, we need $u[N_x - 1]$ & $u[N_x]$
But $u[N_x]$ is zero

Problem: 4

Given:

$$u_t = u_{xx} \quad \text{for } t > 0, \; 0 < x < 1$$

&

$$u^0(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2-2x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

use

$J = 20 \rightarrow$ Total grid points

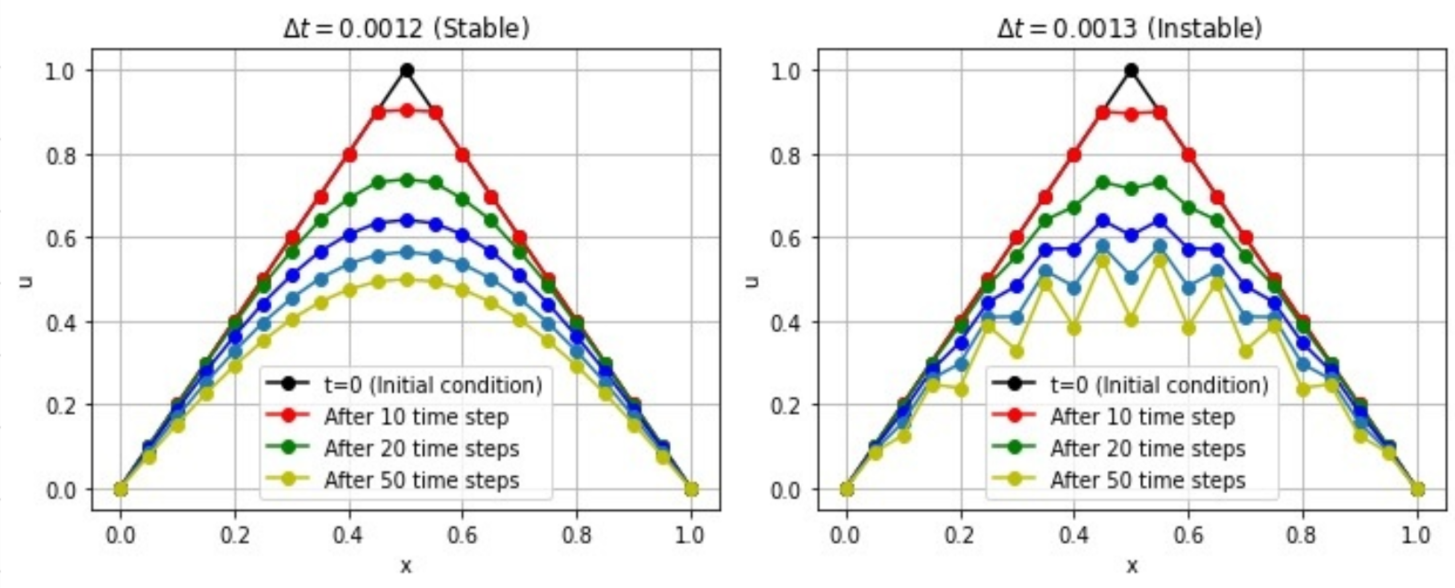
$\Delta x = 0.05 \rightarrow$ Spatial step size

$\Delta t_1 = 0.0012 \rightarrow$ First time step size

$\Delta t_2 = 0.0013 \rightarrow$ 2nd time step size

using explicit scheme, a python code written in 'Problem_4.py' file & results has been matched & compared with fig 2.2 (MGM

- for two different time steps the output profile is given below.



- The above output image clearly shows the effect of time step size & as we can see for time step size 0.0013, there is instability can be observed.

- Now we will check the stability of this scheme, with both fourier analysis and max. principle

① Fourier Analysis :-

(a) $U_j^{n+1} = \lambda^{n+1} e^{ikj\Delta x} = \lambda \lambda^n e^{ikj\Delta x}$

$\hookrightarrow U_j^n$

$$U_j^{n+1} = \lambda U_j^n$$

(b) $U_{j+1}^n = \lambda^n e^{ik(j+1)\Delta x}$

$$= \lambda^n e^{ikj\Delta x} e^{ik\Delta x}$$

$$U_{j+1}^{n} = U_j^n e^{ik\Delta x}$$

Similarly

$$U_{j-1}^{n} = U_j^n e^{-ik\Delta x}$$

Now use all these terms in explicit scheme difference equ

$$U_j^{n+1} = U_j^n + \mu\left(U_{j+1}^n - 2U_j + U_{j+1}^n\right)$$

$$\lambda U_j^n = U_j^n + \mu\left[U_j^n e^{ik\Delta x} - 2U_j + U_j^n \times e^{-ik\Delta x}\right]$$

divided by U_j^n

$$\Rightarrow \lambda = 1+\mu\left[e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right]$$

$$\lambda = 1+2\mu\left[\frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2} - 1\right]$$

$$= 1+2\mu\left[\cos(k x) - 1\right]$$

$$\lambda = 1 - 4\mu \sin^2 \frac{k\Delta x}{2}$$

$\hookrightarrow \lambda = f(k)$

for $k = m\pi$, we get

$$U_j^n = \sum_{m=-\infty}^{\infty} A_m e^{im\pi(j\Delta x)} [\lambda_{m\pi}]^n$$

$\therefore \lambda \rightarrow$ complex no.

$\hookrightarrow \lambda = Re(\lambda) + i\, Im(\lambda)$

for stability requires that,

$|\lambda| \leq 1$

$\therefore \sin^2 \frac{k\Delta x}{2} = 1$ is the worst

$\Rightarrow \quad |\lambda| \leq 1 \Rightarrow |1 - 4\mu| \leq 1$

$\mu \leq \frac{1}{2}$ is the stability criterion.

&

for $\lambda = 1 - 4\lambda < -1$

$\mu > \frac{1}{2}$ → unstable

Now, ∵ $\mu \le \frac{1}{2}$

$$\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}$$

↓

$$\Delta t \le \frac{1}{2} \Delta x^2$$

∴ (I) for $\Delta t_1 = 0.0012$
& $\Delta x = 0.05$ } Stable

$\mu = \frac{0.0012}{(0.05)^2} \Rightarrow 0.48 \le \frac{1}{2}$

(II) for $\Delta t_2 = 0.0013$ } unstable

$\Delta x = 0.05$

$$r = \frac{0.0013}{(0.05)^2} = 0.52 > \frac{1}{2}$$

(2) Max. principle :-

According to Max. principle, a difference approximation to $u_t = u_{xx}$ should possess beyond consider as $\Delta x, \Delta t \to 0$

$$\therefore \quad r(1-\theta) \leq \frac{1}{2}$$

& for explicit scheme, $\theta = 0$

$$r \leq \frac{1}{2}$$

$\to$ which will give

$\Delta t_1 = 0.0012$ stable

$\Delta t_2 = 0.0013$ instable

$\Rightarrow$ we extend this explicit scheme to 1

$$u_t - a(t) u_x - \varepsilon u_{xx} = 0$$

$$u(x,0) = u_0(x)$$

$$u(L_p,t) = u(L,t) = 0$$

$$\Rightarrow [1+\theta\varepsilon\mu\times 2+\theta\lambda a(t)] u_i^{n+1} - [\theta\lambda a(t)+\varepsilon\mu\theta] u_{i+1}^{n+1} - \varepsilon\theta u_{i-1}^{n+1} = [a(t)\lambda - a(t)\theta\lambda + \varepsilon\mu - \varepsilon\theta\mu] u_{i+1}^{n}$$

$$+ [-a(t)\lambda - 2\varepsilon\mu + \theta a(t)\lambda + 2\varepsilon\mu\theta] u_i^n$$

$$+ [\varepsilon\mu - \varepsilon\mu\theta] u_{i-1}^{n},$$

for $\theta = 0$: Explicit Scheme

$$u_i^{n+1} = [a(t)\lambda + \varepsilon\mu] u_{i+1}^{n} + [-a(t)\lambda - 2\varepsilon\mu] u_i^n + [\varepsilon\mu] u_{i-1}^{n}$$

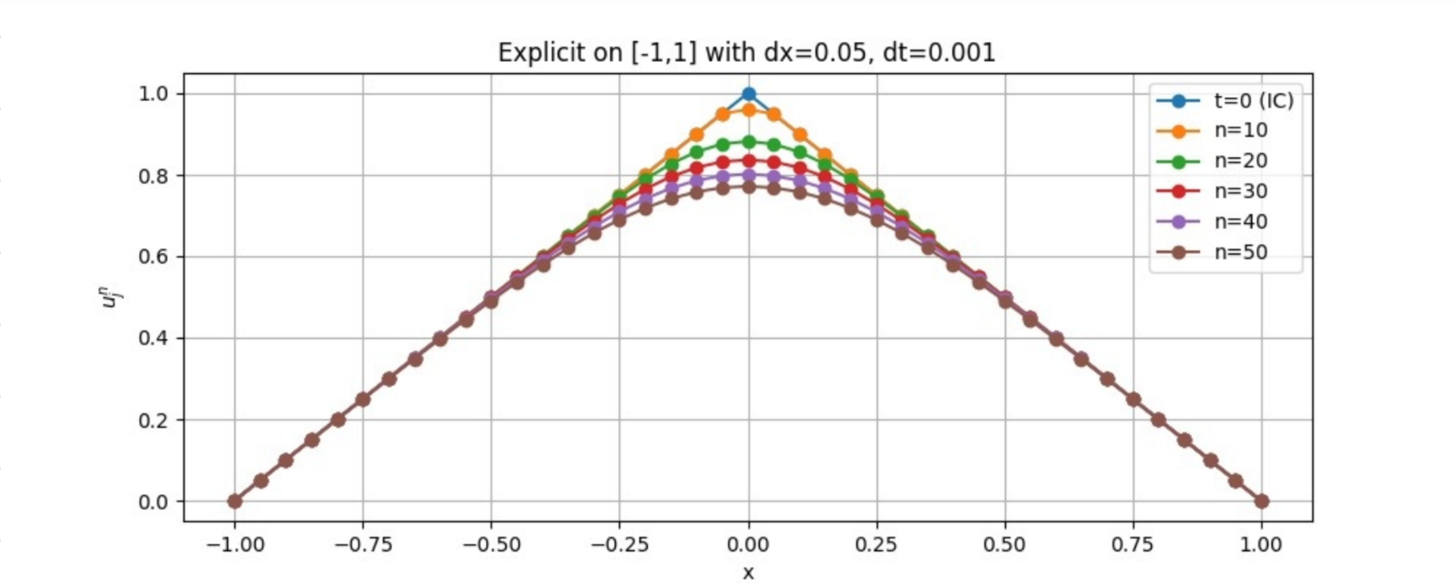
$$\mu = \frac{\Delta t}{\Delta x^2} \quad \& \quad \lambda = \frac{\Delta t}{\Delta x},$$

To extend explicit Scheme to 1, for $[1,-1]$

I.C, will be,
$$u^0(x) = \begin{cases} 1+x & \text{if } -1 \leq x \leq 0 \\ 1-x & \text{if } 0 < x \leq 1 \end{cases}$$

taken $\Delta x = 0.05$

$J = 40$ & $a = \varepsilon = 1$



Problem 5:-

Implimented θ-Method and used the linear Algebra process in Problem5.py

$\therefore$ $$u_t - a(t)u_x - \varepsilon u_{xx} = 0$$
$$u(x,0) = u_0(x)$$
$$u(L_p,t) = u(L,t) = 0$$

used $L_p = 0$ & $L = 1$ (B.Cs)

& I.C $\rightarrow$ $$u_0(x) = \begin{cases} 2x & \text{if } 0 < x \leq 0.5 \\ 2 - 2x & \text{if } 0.5 \leq x \leq 1 \end{cases}$$

$\therefore$ $$u_x = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

& $$u_{xx} = \frac{u_i^{n+1} - u_i^{n}}{(\Delta x)^2}$$

$$u_t = \frac{u_i^{n+1} - u_i^{n}}{\Delta t}$$

θ- Method implimented:-

$$\Rightarrow [1+\theta\varepsilon\mu\times 2+\theta\lambda a(t)]u_i^{n+1}-[\theta\lambda a(t)+\varepsilon\mu\theta]u_{i+1}^{n+1}$$
$$-\varepsilon\theta u_{i-1}^{n+1}=[a(t)\lambda-a(t)\theta\lambda+\varepsilon\mu-\varepsilon\theta\mu]u_{i+1}^{n}$$
$$+[-a(t)\lambda-2\varepsilon\mu+\theta a(t)\lambda+2\varepsilon\mu\theta]u_i^{n}$$
$$+[\varepsilon\mu-\varepsilon\mu\theta]u_{i-1}^{n}$$

$\Rightarrow$ Stability properties;

$$v_j^{n+1}=v_j^n+\theta[\mu a^{n+1}(u_{j+1}^{n+1}-v_j^{n+1})+$$
$$\varepsilon\mu(v_{j+1}^{n+1}-2v_j^{n+1}+v_{j-1}^{n+1})]+(1-\theta)[\mu a^n(v_{j+1}^n-v_j^n)$$
$$+\varepsilon\mu(v_{j+1}^n-2v_j^n+v_{j-1}^n)].$$

assuming $v_j^n=\lambda^n e^{ikj\Delta x}$

$$\Rightarrow \lambda^{n+1}e^{ikj\Delta x}=\lambda^n e^{ikj\Delta x}+\theta[\mu a^{n+1}(\lambda^{n+1}$$
$$e^{ik(j+1)\Delta x}-\lambda^{n+1}e^{ikj\Delta x})+\varepsilon\mu(\lambda^{n+1}e^{ik(j+1)\Delta x}-2\lambda^{n+1}e^{ikj\Delta x}$$
$$+\lambda^{n+1}e^{ik(j-1)\Delta x})]+$$
$$(1-\theta)[\mu a^n(\lambda^n e^{ik(j+1)\Delta x}-\lambda^n e^{ikj\Delta x})+\varepsilon\mu(\lambda^n e^{ik(j+1)\Delta x}$$
$$-2\lambda^n e^{ikj\Delta x}+\lambda^n e^{ik(j-1)\Delta x})]$$

$\downarrow$

$$\lambda=1+\lambda\theta[\mu a^{n+1}(e^{ik\Delta x}-1)+\varepsilon\mu(e^{ik\Delta x}-2+e^{-ik\Delta x})]$$
$$+(1-\theta)[\mu a^n(e^{ik\Delta x}-1)+\varepsilon\mu(e^{ik\Delta x}-2+e^{-ik\Delta x})]$$

$$\Rightarrow \lambda = \frac{1+(1-\theta)\left[\mu a^n(e^{ik\Delta x}-1)+\epsilon\mu(e^{ik\Delta x}-2+e^{-ik\Delta x})\right]}{1-\theta\left[\mu a^{n+1}(e^{ik\Delta x}-1)+\epsilon\mu(e^{ik\Delta x}-2+e^{-ik\Delta x})\right]}$$

$\Rightarrow$ for diffusion part, $a=0$, & $|\lambda|\leq 1$

$$\boxed{\mu \leq \frac{1}{2\epsilon(1-2\theta)}}$$

Now for pure advection case ($\epsilon=0$)

from $\lambda = \dfrac{1+(1-\theta)\mu a^n(e^{ik\Delta x}-1)}{1-\theta\mu a^{n+1}(e^{ik\Delta x}-1)}$

for $a=$ const $\Rightarrow a^{n+1}=a^n$

using $e^{i\theta}=\cos(\theta)+i\sin\theta$;

$$\lambda = \frac{1+(1-\theta)\mu a\left[\cos(k\Delta x)+i\sin(k\Delta x)-1\right]}{1-\theta\mu a\left[\cos(k\Delta x)+i\sin(k\Delta x)-1\right]}$$

Now $\cos(2\theta)-1=-2\sin^2\theta$

$$\lambda = \frac{\left[1-2(1-\theta)\mu a\sin^2(k\Delta x/2)\right]+i(1-\theta)\mu a\sin(k\Delta x)}{\left[1+2\theta\mu a\sin^2(k\Delta x/2)\right]-i\theta\mu a\sin(k\Delta x)}$$

for stability condition $|\lambda|\leq 1$

$$\Rightarrow |\lambda|^2\leq 1$$

$$|\lambda|^2 = \left(\frac{[1-2(1-\theta)\mu a \sin^2(k\Delta x/2)] + i(1-\theta)\mu a \sin(k\Delta x)}{[1+2\theta\mu a \sin^2(k\Delta x/2) - i\theta\mu a \sin(k\Delta x)} \right)$$

$$\times \left(\text{Conjugate of } \lambda \right)$$

$$|\lambda|^2 = \frac{[1-2(1-\theta)\mu a \sin^2(k\Delta x/2)]^2 + (1-\theta)^2(\mu a)^2 \sin^2(k\Delta x)}{[1+2\theta\mu a \sin^2(k\Delta x/2)]^2 + (\theta\mu a)^2 \sin^2(k\Delta x)}$$

$$\leq 1,$$

$$\Rightarrow \cancel{1} - 4(1-\theta)\mu a \sin^2(k\Delta x/2) + 4(1-\theta)^2(\mu a)^2 \sin^4(k\Delta x/2) + (1-\theta)^2(\mu a)^2 \sin^2(k\Delta x) \leq \cancel{1} + 4\theta\mu a \sin^2(k\Delta x/2) + 4(\theta\mu a)^2 \sin^4(k\Delta x/2) + (\theta\mu a)^2 \sin^2(k\Delta x),$$

dividing by μa on both side

$$-4(1-\theta)\sin^2(k\Delta x/2) + 4(1-\theta)^2\mu a \sin^4(k\Delta x/2) + (1-\theta)^2(\mu a)\sin^2(k\Delta x) \leq 4\sin^2(k\Delta x/2) + 4(\theta\mu a)\sin^4(k\Delta x/2) + (\theta\mu a)\sin^2(k\Delta x),$$

$$\Rightarrow -4(1-\theta)\sin^2(k\Delta x/2) - 4\theta\sin^2(k\Delta x/2) + 4(1-2\theta+\theta^2)(\mu a)\sin^4(k\Delta x/2) - 4(\theta\mu a)\sin^4(k\Delta x/2) + (1-2\theta+\theta^2)(\mu a)\sin^2(k\Delta x/2) - (\theta\mu a)\sin^2(k\Delta x) \leq 0$$

$$\Rightarrow -4\sin^2(k\Delta x/2) + 4(1-2\theta)\mu a \sin^4(k\Delta x/2)$$

$$+(1-2\theta)\mu a\,\sin^2(K\Delta x)\le 0,$$

can be written as,

$$-4\sin^2(K\Delta x/2)+(1-2\theta)\mu a\left[4\sin^4(K\Delta x/2)+\sin^2(K\Delta x)\right]\le 0$$

using $\sin 2\theta = 2\sin\theta\cos\theta$,

$$-4\sin^2(K\Delta x/2)+(1-2\theta)\mu a\left[4\sin^4(K\Delta x/2)+4\sin^2(K\Delta x/2)\cos^2(K\Delta x/2)\right]\le 0,$$

$$\Rightarrow -4\sin^2(K\Delta x/2)+4(1-2\theta)\sin^2(K\Delta x/2)\mu a\underbrace{\left[\sin^2(K\Delta x/2)+\cos^2(K\Delta x/2)\right]}_{1}\le 0$$

$$\Rightarrow -4\sin^2(K\Delta x/2)+4(1-2\theta)\mu a\sin^2(K\Delta x/2)\le 0$$

dividing by $4\sin^2(K\Delta x/2)$,

$$(1-2\theta)\mu a\le 1$$

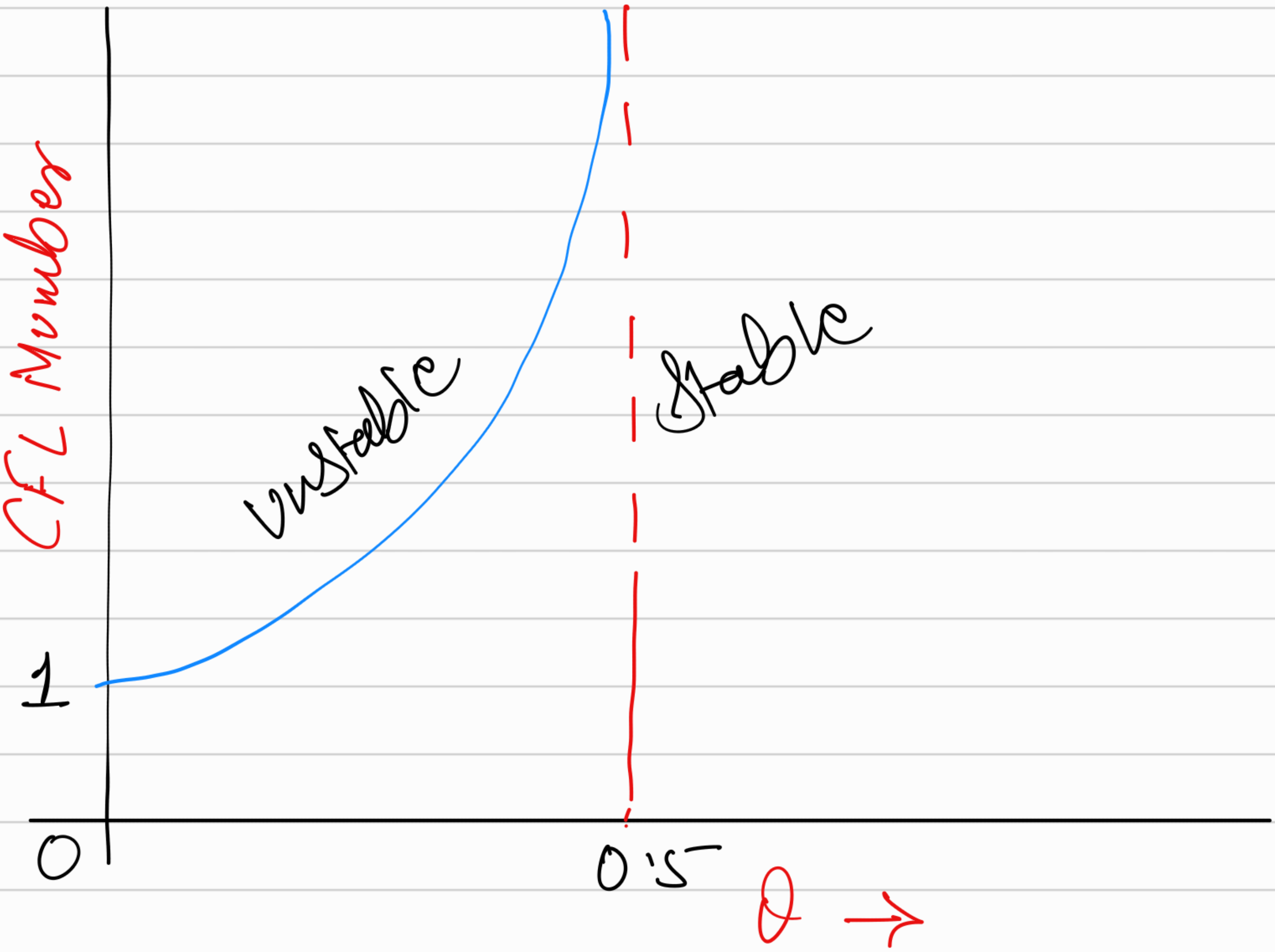
$$\boxed{\mu a\le\frac{1}{(1-2\theta)}}$$

- This is the stability condition, when a is positive constant.

here, $\mu = \frac{\Delta t}{\Delta x}$ & $a\mu = \frac{a\Delta t}{\Delta x}$

which is CFL number

$$\text{CFL Number} = \frac{a\Delta t}{\Delta x}$$



CFL = 1 for full explicit case $\to \theta = 1$

Problem 6:-

In Python code Problem6_Part1.py
Problem6_Part2.py

Case 1:-

$$u(x,0) = (1-x)^4 (1+x)$$

Case 2:-

$u(x,0)$ with random values of R

we have

$$\begin{cases} u_t - a(t)x - \varepsilon u_{xx} = 0 \\ u(x,0) = u_0 x \\ u(L_p,t) = u(L,t) = 0 \end{cases}$$

$L_p = -1$, $L = 1$, $\varepsilon = 10^{-3}$,

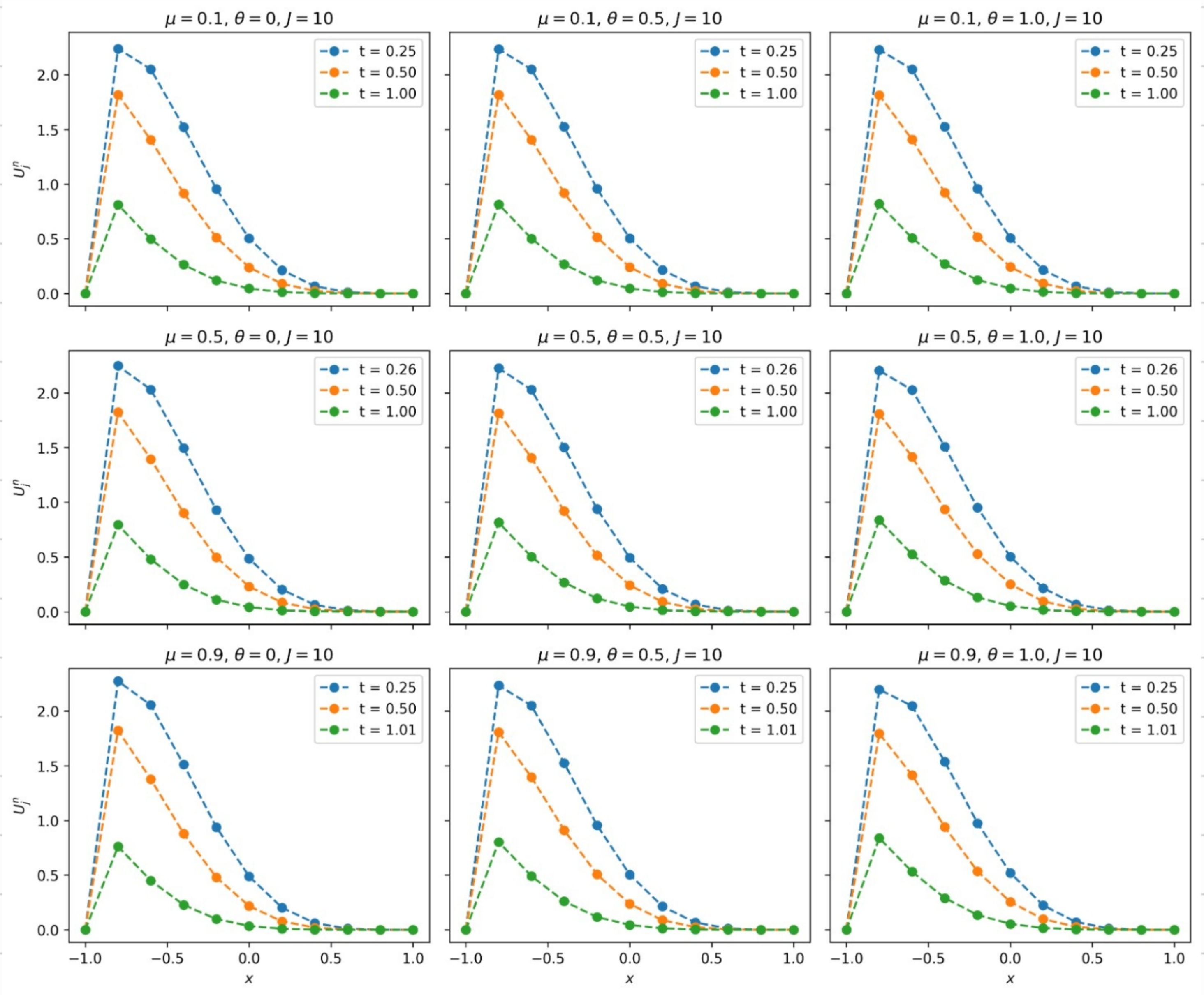
at $\theta = 0, 0.5, 1$ & $y = 0.1, 0.5, 0.9$

for case 1, three plot has been plotted for

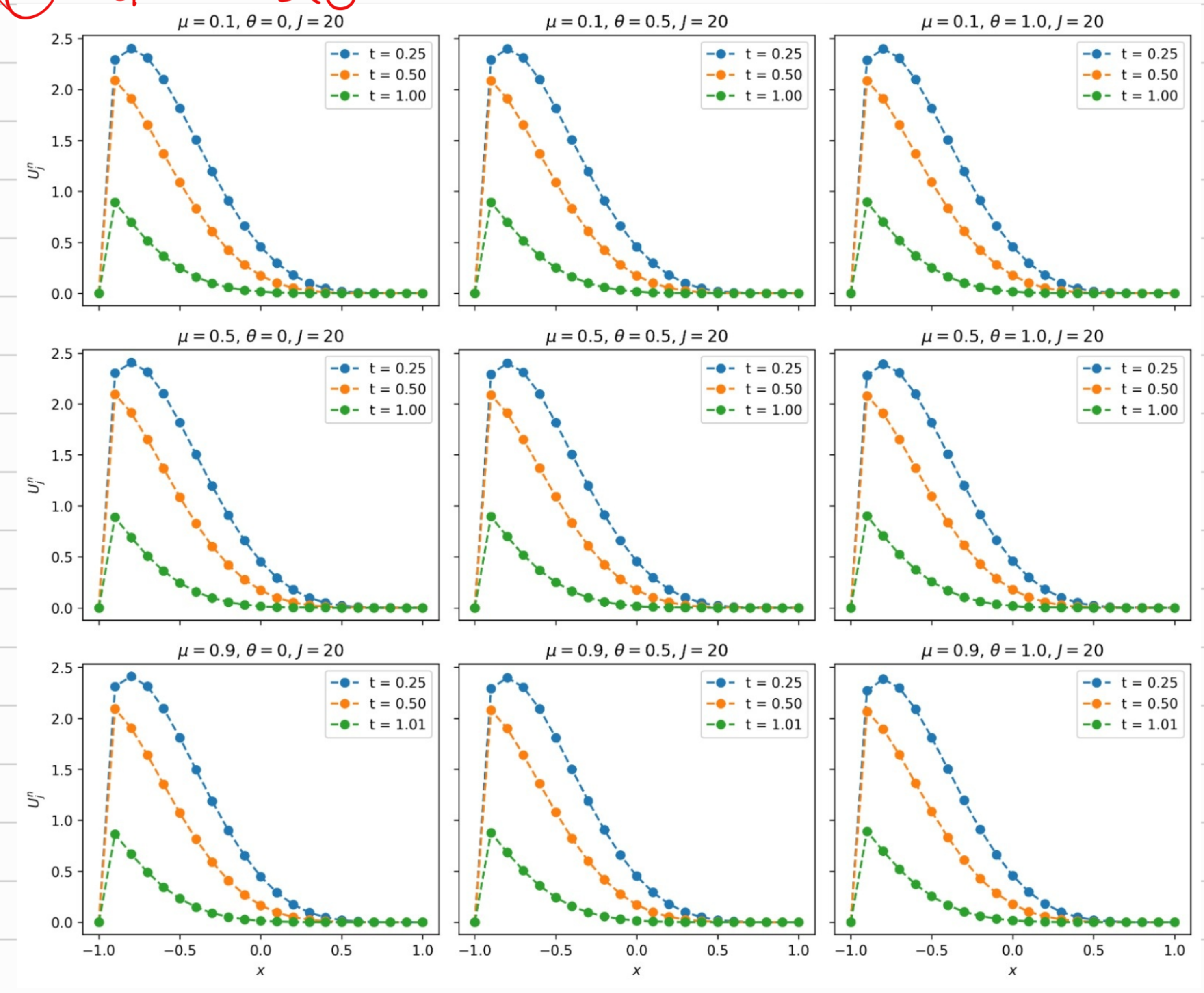
$\hookrightarrow J = 10, 20, 40.$

To check the convergence

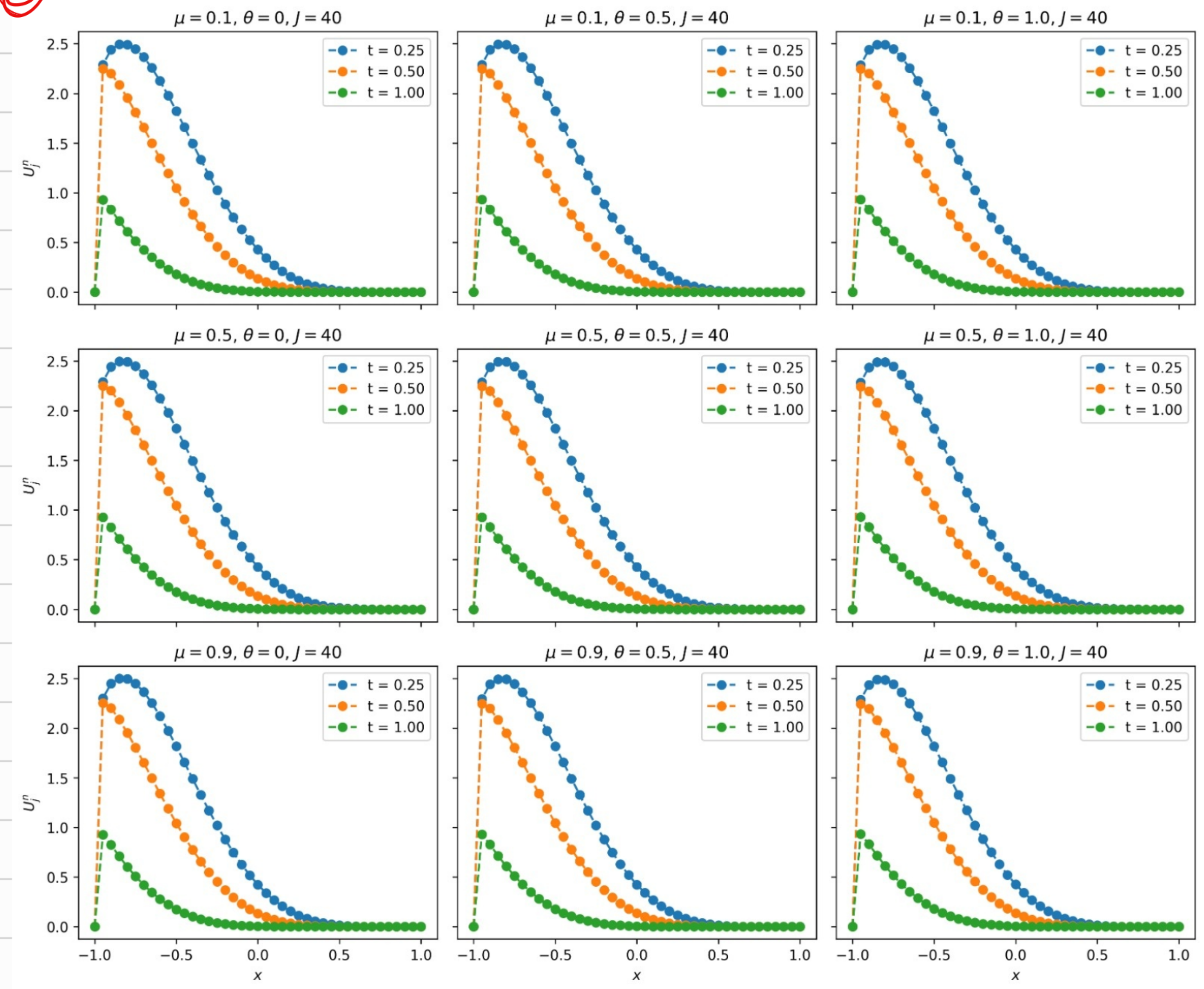
① at J=10



② at J= 20



③ at J= 40



part ② :- at $a(t)=1$ with

$$u(x,0) = (1-x)^4(1+x)\left(\sum_{k=0}^{3} b_k \phi_k(x) + C\right)$$