Fluid Dynamics: Numerics Exercise 2

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1)

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0 \qquad \frac{\partial u}{\partial t} + \frac{\partial (g\eta)}{\partial x} = 0$$

$$u = U_0 u', \quad x = L_s x', \quad t = \frac{L_s}{U_0} t', \quad \eta = H_{os}\eta', \quad H = H_{os} H'$$

$$\frac{\partial (H_{os}\eta')}{\partial \left(\frac{L_s}{U_0} t'\right)} + \frac{\partial (H_{os} H' U_0 u')}{\partial (L_s x')} = 0$$

$$\frac{U_0 H_{os}}{L_s} \frac{\partial \eta'}{\partial t'} + \frac{H_{os} U_0}{L_s} \frac{\partial (H' u')}{\partial x'} = 0$$

$$\div \frac{U_0 H_{os}}{L_s}$$

$$\frac{\partial \eta'}{\partial t'} + \frac{\partial (H' u')}{\partial x'} = 0$$

$$\frac{\partial (U_0 u')}{\partial \left(\frac{L_s}{U_0} t'\right)} + \frac{\partial (g H_{os} \eta')}{\partial (L_s x')} = 0$$

$$\frac{U_0^2}{L_s} \frac{\partial u'}{\partial t'} + \frac{g H_{os}}{L_s} \frac{\partial \eta'}{\partial x'} = 0$$

$$\div \frac{U_0^2}{L_s}$$

$$\frac{\partial u'}{\partial t'} + \frac{g H_{os}}{U_0^2} \frac{\partial \eta'}{\partial x'} = 0$$

$$g' = \frac{g H_{os}}{U_0^2}$$

$$\frac{\partial u'}{\partial t'} + g' \frac{\partial \eta'}{\partial x'} = 0$$

1) continued)

$\therefore$ the scaled equations are (dropping primes)

$$① \quad \frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0$$

$$② \quad \frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x'} = 0 \qquad \text{where } g = \frac{g_0 H_{0s}}{U_0^2}$$

$9.81 ms^{-2}$

<u>Riemann Problem</u>: ~~$\partial_t W(x,t) + \partial_x (f(W(x,t)) = 0$~~

$$\partial_t q(x,t) + \partial_x (f(q(x,t))) = 0$$

~~eqn ①: $W = u$,~~

$$\partial_t \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + A\partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0$$

let $A = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix}$ where $c_0^2 = gH_0$

$$\partial_t \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + \partial_x \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0$$

$$\partial_t \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + \partial_x \begin{pmatrix} H_0 u \\ c_0^2 \eta \end{pmatrix} = 0$$

$$\partial_t \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + \partial_x \begin{pmatrix} H_0 u \\ gH_0 \eta \end{pmatrix} = 0$$

$\therefore$ we recover equation ① & ② [where ② has been multiplied by H_0]

1) continued)

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda, & 1 \\ c_0^2, & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - c_0^2 = 0 \Rightarrow \lambda^2 = c_0^2$$

$$\lambda = \pm c_0$$

$\underline{\lambda_2 = -c_0}$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$A\vec{w} = \lambda_1 \vec{w}$$

$$(A + c_0)\vec{w} = 0$$

$$\begin{bmatrix} c_0, & 1 \\ c_0^2, & c_0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\div R_2$ by c_0 elim R_2

$$\begin{bmatrix} c_0, & 1 \\ 0, & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_0 w_1 = -w_2$$

$$\therefore w_1 = -1, \quad w_2 = c_0$$

$\underline{\lambda_1 = c_0}$

$$\vec{v} = \begin{matrix} v_1 \\ v_2 \end{matrix}$$

$$A\vec{v} - \lambda_2 \vec{v} = 0$$

$$\begin{bmatrix} -c_0, & 1 \\ c_0^2, & -c_0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

① $\times R_2$ by c_0
② $R_2 + R_1$

$$\begin{bmatrix} -c_0, & 1 \\ 0, & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-c_0 v_1 + v_2 = 0$$

$$c_0 v_1 = v_2$$

$$\therefore v_1 = 1, \quad v_2 = c_0$$

$$\vec{v} = \begin{pmatrix} 1 \\ c_0 \end{pmatrix}$$

use this to build: $B = \frac{1}{2c_0}\begin{pmatrix} 1, & -1 \\ c_0, & c_0 \end{pmatrix}$

factor so $B^{-1}B = I$

1 continued)

$$\det(B) = \left(\frac{1}{2c_0}\right)^2 \cdot \begin{vmatrix} 1, & -1 \\ c_0, & c_0 \end{vmatrix} = \frac{c_0 - -c_0}{(2c_0)^2} = \frac{\cancel{2c_0}}{(2c_0)^{\cancel{2}}}$$

$$= 1/2c_0$$

$$\therefore\ B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} d, & -b \\ -c, & a \end{bmatrix} \qquad B = \begin{bmatrix} a, & b \\ c, & d \end{bmatrix}$$

$$= \cancel{2c_0} \times \frac{1}{\cancel{2c_0}} \times \begin{bmatrix} c_0, & 1 \\ -c_0, & 1 \end{bmatrix} = \begin{bmatrix} c_0, & 1 \\ -c_0, & 1 \end{bmatrix}$$

<u>verification:</u>

$$B^{-1}B = \begin{bmatrix} c_0, & 1 \\ -c_0, & 1 \end{bmatrix} \begin{bmatrix} 1, & -1 \\ c_0 & c_0 \end{bmatrix} 1/2c_0$$

$$= \begin{bmatrix} c_0 + c_0, & -c_0 + c_0 \\ -c_0 + c_0, & --c_0 + c_0 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{2c_0}, & 0 \\ 0, & \cancel{2c_0} \end{bmatrix} 1/\cancel{2c_0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ this justifies prefactor of $1/2c_0$

1 continued)

$$B^{-1}AB = \begin{bmatrix} c_0, & 1 \\ -c_0, & 1 \end{bmatrix} \begin{bmatrix} 0, & 1 \\ c_0^2, & 0 \end{bmatrix} \begin{bmatrix} 1, & -1 \\ c_0, & c_0 \end{bmatrix} \cdot 1/2c_0$$

$$= \begin{bmatrix} 0 + c_0^2, & c_0 + 0 \\ 0 + c_0^2, & -c_0 \end{bmatrix} \begin{bmatrix} 1, & -1 \\ c_0, & c_0 \end{bmatrix} \cdot 1/2c_0$$

$$= \begin{bmatrix} c_0^2 + c_0^2, & -c_0^2 + c_0^2 \\ c_0^2 - c_0^2, & -c_0^2 - c_0^2 \end{bmatrix} \cdot 1/2c_0$$

$$= \begin{bmatrix} \cancel{2}c_0^{\cancel{2}}, & 0 \\ 0, & -\cancel{2}c_0^{\cancel{2}} \end{bmatrix} \cdot 1/\cancel{2c_0}$$

$$= \begin{bmatrix} c_0, & 0 \\ 0, & -c_0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1, & 0 \\ 0, & \lambda_2 \end{bmatrix}$$ where $\lambda_1 = c_0$, $\lambda_2 = -c_0$ are the eigenvalues of A

$$\partial_t \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + A\,\partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0$$

$\times I = B^{-1}B$

$\times \cancel{B}\, B^{-1}$

$$B^{-1} \cancel{B^{-1}B}\, \partial_t \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + B^{-1}B^{-1}AB\, \partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0$$

$$\partial_t \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + B^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x \begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0$$

$$\partial_t \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \begin{pmatrix} \lambda_1, & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0$$

1) continued)

$\therefore$ we get :

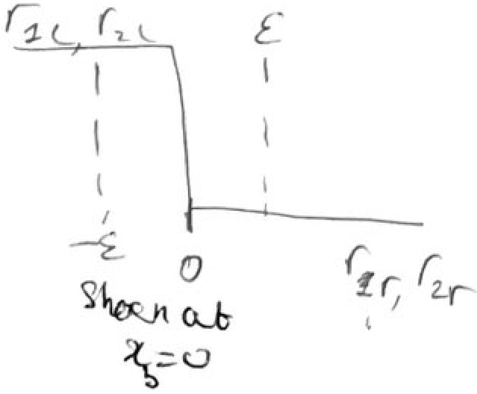
$$\partial_t \vec{r} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x \vec{r} = 0 \qquad \text{where } \vec{r} = B^{-1}(\eta, H_0 u)^T$$

$$\vec{r} = \begin{pmatrix} c_0, & 1 \\ -c_0, & 1 \end{pmatrix} \begin{pmatrix} \eta \\ H_0 u \end{pmatrix}$$

$$= \begin{pmatrix} c_0 \eta + H_0 u \\ -c_0 \eta + H_0 u \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

2) $r_1(x,0) = \begin{cases} r_{1l} & x<0 \\ r_{1r} & x \geq 0 \end{cases}$ $\quad r_2(x,0) = \begin{cases} r_{2l}, & x<0 \\ r_{2r}, & x \geq 0 \end{cases}$

$$\partial_t \vec{r} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x (\vec{r}) = 0$$



$$\partial_t \vec{r} + \partial_x \left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \vec{r} \right) = 0$$

$$\int_{-\varepsilon}^{\varepsilon} \partial_t \vec{r}\, dx + \int_{-\varepsilon}^{\varepsilon} \partial_x \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \vec{r}\, dx = 0$$

$$\int_{-\varepsilon}^{\varepsilon} \partial_t \vec{r}\, dx + \begin{bmatrix} f(\varepsilon) - f(-\varepsilon) \\ f(\varepsilon) - f(-\varepsilon) \end{bmatrix} = 0$$

~~disappears~~ ~~$\varepsilon \to 0$~~

$$\int_{-\varepsilon}^{\varepsilon} \partial_t \vec{r}\, dx + \begin{bmatrix} f(r_{1r}) - f(r_{1l}) \\ f(r_{2r}) - f(r_{2l}) \end{bmatrix} = 0$$

pos^n of shock

$$\cancel{\partial_t \int_{-\varepsilon}^{\varepsilon} u\, dx}^{=0} - \begin{pmatrix} r_{1r} - r_{1l} \\ r_{2r} - r_{2l} \end{pmatrix} \frac{dx_s}{dt} + \begin{bmatrix} f(r_{1r}) - f(r_{1l}) \\ f(r_{2r}) - f(r_{2l}) \end{bmatrix} = 0$$

$$\begin{bmatrix} f(r_{1r}) - f(r_{1l}) \\ f(r_{2r}) - f(r_{2l}) \end{bmatrix} = \begin{pmatrix} r_{1r} - r_{1l} \\ r_{2r} - r_{2l} \end{pmatrix} \frac{dx_s}{dt}$$

2 cont)

Shock speed: $\vec{S} = \frac{d\vec{x_s}}{dt} = \begin{bmatrix} \frac{f(r_{1r}) - f(r_{1l})}{r_{1r} - r_{1l}} \\ \frac{f(r_{2r}) - f(r_{2l})}{r_{2r} - r_{2l}} \end{bmatrix}$

s

in this case: $f(\vec{r}) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$

$\left[u(x,t) = \begin{cases} -, & x < st \\ -, & x > st \end{cases} \right]$ $= \begin{pmatrix} \lambda_1 r_1 \\ \lambda_2 r_2 \end{pmatrix}$

$$\therefore \vec{S} = \begin{bmatrix} \lambda_1 \frac{(r_{1r} - r_{1l})}{(r_{1r} - r_{1l})} \\ \lambda_2 \frac{\cancel{(r_{2r} - r_{2l})}}{\cancel{(r_{2r} - r_{2l})}} \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} c_0 \\ -c_0 \end{bmatrix}$$

$$\therefore \vec{r}(x,t) = \begin{bmatrix} r_1(x,t) \\ r_2(x,t) \end{bmatrix} = \begin{bmatrix} \begin{cases} r_{1l}, & x < c_0 t \\ r_{1r}, & x \geq c_0 t \end{cases} \\ \begin{cases} r_{2l}, & x < -c_0 t \\ r_{2r}, & x \geq -c_0 t \end{cases} \end{bmatrix}$$ Solutions

~~$r_1(x,t)$~~

~~$H_0 u + c_0 \eta =$~~

2 cont >

$$H_0 u = \frac{1}{2}(r_1 + r_2)$$

$$= \frac{1}{2}\left(\begin{cases} r_{1l}, & x < c_0 t \\ r_{1r}, & x \geq c_0 t \end{cases} + \begin{cases} r_{2l}, & x < -c_0 t \\ r_{2r}, & x \geq -c_0 t \end{cases} \right)$$

$$= \frac{1}{2} \begin{cases} r_{1l} + r_{2l}, & x < -c_0 t \\ r_{1l} + r_{2r}, & -c_0 t \leq x \leq c_0 t \\ r_{1r} + r_{2r}, & x > c_0 t \end{cases}$$

$$= \begin{cases} \frac{r_{1l} + r_{2l}}{2}, & x < -c_0 t \\ \frac{r_{1l} + r_{2r}}{2}, & -c_0 t \leq x \leq c_0 t \\ \frac{r_{1r} + r_{2r}}{2}, & x > c_0 t \end{cases}$$

$H_0 u = \frac{1}{2}(r_1 + r_2)$

use this

$$= \begin{cases} H_0 u_l, & x < -c_0 t \\ \frac{r_{1l} + r_{2r}}{2}, & -c_0 t \leq x \leq c_0 t \\ H_0 u_r, & x > c_0 t \end{cases}$$

~~If I can simplify~~

$$\frac{H_0 u_l + H_0 u_r}{2} = \frac{r_{1l} + r_{2l} + r_{1r} + r_{2r}}{2}$$

2 cont)

$$\eta(x,t) = \frac{1}{2c_0}\left(r_1(x,t) - r_2(x,t) \right)$$

$$= \frac{1}{2c_0}\left[\begin{cases} r_{1l}, & x < c_0 t \\ r_{1r}, & x \geq c_0 t \end{cases} - \begin{cases} r_{2l}, & x < -c_0 t \\ r_{2r}, & x \geq -c_0 t \end{cases} \right]$$

$$= \begin{cases} \dfrac{r_{1l} - r_{2l}}{2}, & x < -c_0 t \\ \dfrac{r_{1l} - r_{2r}}{2}, & -c_0 t \leq x \leq c_0 t \\ \dfrac{r_{1r} - r_{2r}}{2}, & x > c_0 t \end{cases}$$

$$= \begin{cases} \eta_l, & x < -c_0 t \\ \dfrac{(r_{1l} - r_{2r})}{2c_0}, & -c_0 t \leq x \leq c_0 t \\ \eta_r, & x > c_0 t \end{cases}$$

look at later: middle region rarefaction wave?

2 cont)

simplify H_0U:

$$\frac{r_{1l} + r_{2r}}{2}$$

$$\cancel{H_0} \ H_0U_l = \frac{1}{2}(r_{1l} + r_{2l})$$

$$H_0U_r = \frac{1}{2}(r_{1r} + r_{2r})$$

(1) $2H_0U_l = r_{1l} + r_{2l}$

(2) $2H_0U_r = r_{1r} + r_{2r}$

$\cancel{2H_0} \ 2\eta_l = \eta_{1l} + \eta_{2l}$

$2\eta_r = \eta_{1r} + \eta_{2r}$

(3) $c_0 2\eta_l = (r_{1l} - r_{2l}) \cancel{c_0}$

(4) $c_0 2\eta_r = (r_{1r} - r_{2r}) \cancel{c_0}$

~~(1) − (4) : $r_{1l} - r_2 \quad 2H_0$~~

(1) + (3) : $2H_0U_l + 2\eta_l = 2r_{1l} \Rightarrow r_{1l} = H_0U_l + \eta_l c_0$

~~(2) + (4) :~~

(2) − (4) : $2H_0U_r - 2c_0\eta_r = 2r_{2r}$

$r_{2r} = H_0U_r - \cancel{2} c_0\eta_r$

$$\therefore \frac{r_{1l} + r_{2r}}{2} = \frac{H_0(U_l + U_r) + c_0(\eta_l - \eta_r)}{2}$$

2 cont)

$$\therefore \; H_0 u = \begin{cases} H_0 u_l &, \; x < -C_0 t \\ \dfrac{H_0(u_l + u_r) + C_0(\eta_l - \eta_r)}{2} &, \; -C_0 t \le x \le C_0 t \\ H_0 u_r &, \; x > C_0 t \end{cases}$$

For η:

$$\frac{r_{1l} - r_{2r}}{2C_0} = \frac{H_0(u_l - u_r) + C_0(\eta_l + \eta_r)}{2C_0}$$

$$\therefore \; \eta = \left\{ \begin{array}{ll} \eta_l \, , & x < -C_0 t \\ \dfrac{H_0(u_l - u_r) + C_0(\eta_l + \eta_r)}{2C_0} \, , & -C_0 t \le x \le C_0 t \\ \eta_r \, , & x > C_0 t \end{array} \right\}$$

2 continued)

$\underline{\text{OUR EQN}}$: $\partial_t \vec{r} + \partial_x \left[\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \vec{r} \right] = 0$

$$\partial_t \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + \partial_x \begin{bmatrix} \lambda_1 r_1 \\ \lambda_2 r_2 \end{bmatrix} = 0$$

1 $\partial_t r_1 + \partial_x (\lambda_1 r_1) = 0$

in form: $u_t + a u_x = 0$ } • linear advection equation
• λ_1 scalar

~~same for λ_2~~

$$\cancel{\partial_t \left(H_0 u + \sqrt{gH_0}\,\eta \right) + \partial_x \left(\lambda_1 H_0 u + \lambda_1 \sqrt{gH_0}\,\eta \right) = 0}$$

adv. equation
1

2 $\partial_t r_2 + \partial_x (\lambda_2 r_2) = 0$

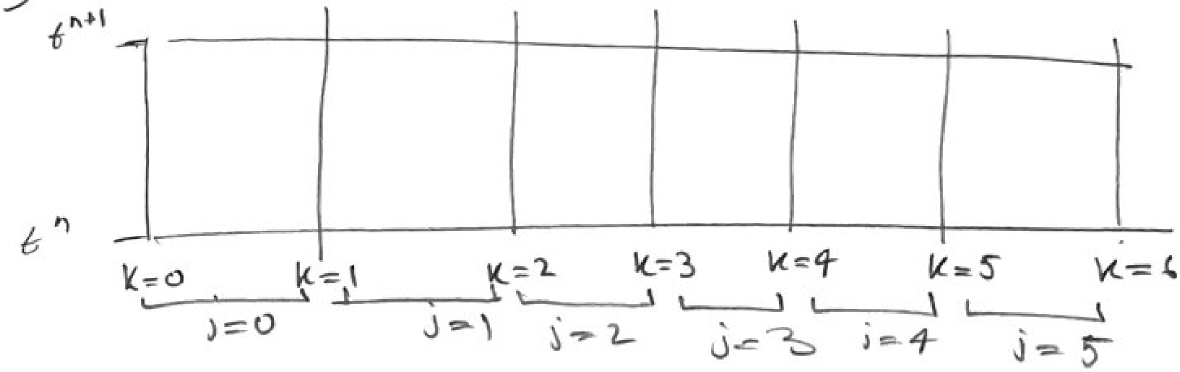
in form: $u_t + a u_x = 0$ } linear advection equation

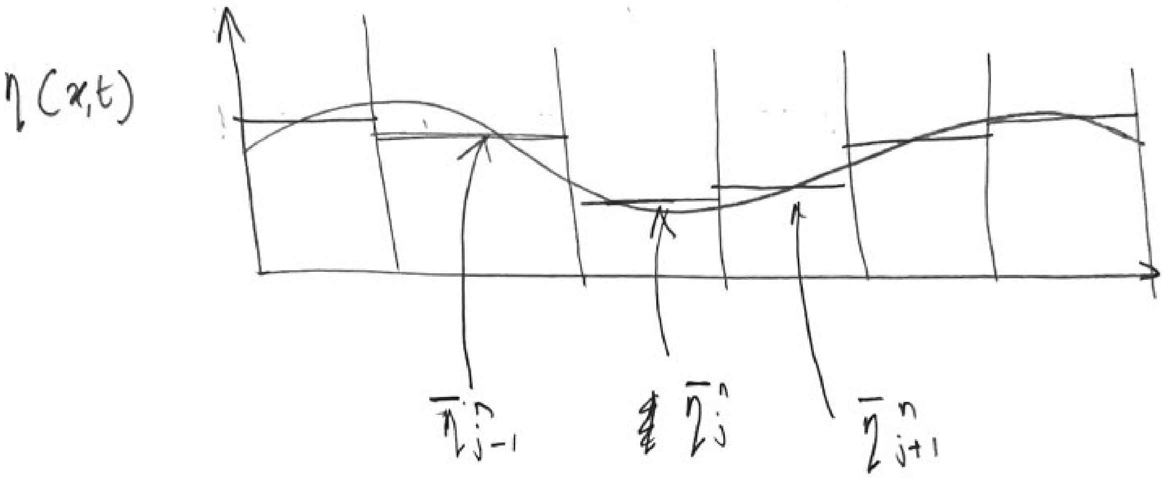
solve problem for original problem in terms of original variables * do later

↳ I think I have already done this?

$\cancel{r}$ =1

3)



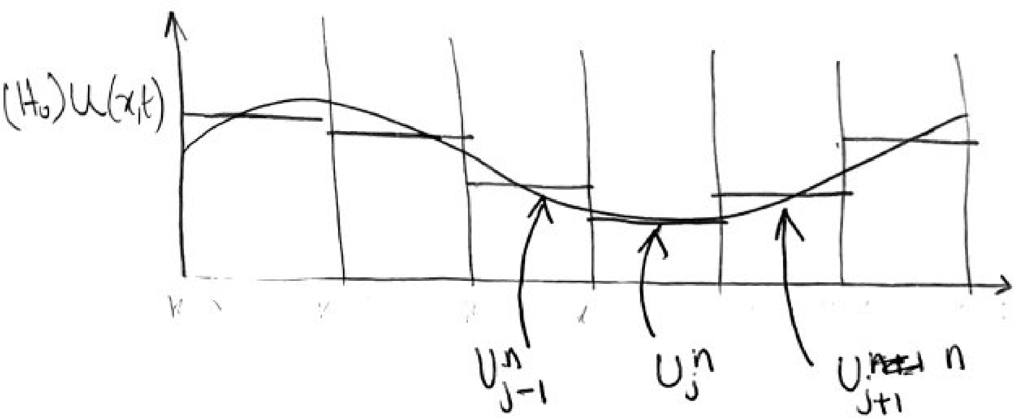


mean of η ⟶ $\bar{\eta}_j^n$ = ~~piew~~ piecewise approx of η over $x_{j-1/2} < x < x_{j+1/2}$

$$\partial_t \begin{bmatrix} \eta \\ H(x)u \end{bmatrix} + \partial_x \left[A \cancel{\partial_x} \begin{bmatrix} \eta \\ H(x)u \end{bmatrix} \right] = 0$$

(at each boundary

$$\partial_t \begin{bmatrix} \eta \\ H_0 u \end{bmatrix} + \partial_x \left[A \begin{bmatrix} \eta \\ \cancel{H(x)} H_0 u \end{bmatrix} \right] = 0$$



3 continued)

~~locally~~: $\partial_t(\eta) + \partial_x(H_0 u) = 0$ $\quad\quad \partial_t \eta + \partial_x(Hu) = 0$

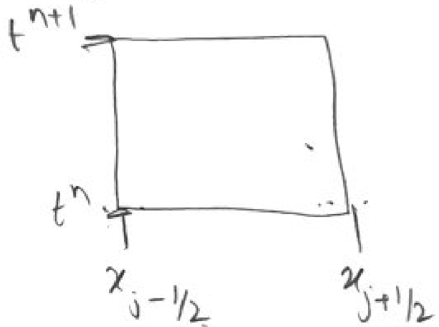
$\partial_t(H_0 u) + \partial_x(c_0^2 \eta) = 0$ $\quad\quad \partial_t u + \partial_x(\underline{g\eta}) = 0$

$\partial_t(\vec{u}) + \partial_x(\vec{f}(\vec{u})) = 0$ $\quad\quad c_0^2 = H$

where $u_1 = \eta$, $u_2 = H_0 u$ $\quad \cancel{u_3 =}\ f_1 = H_0 u = u_2$

$f_2 = c_0^2 \eta$

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \partial_x \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0$$



(integrate over cell

swap

$$\int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} dx\,dt = - \int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \partial_x \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} dx\,dt$$

LHS: $$\int_{x_{j-1/2}}^{x_{j+1/2}} \begin{bmatrix} u_1(x, t^{n+1}) - u_1(x, t^n) \\ u_2(x, t^{n+1}) - u_2(x, t^n) \end{bmatrix} dx = \Delta x_j \begin{bmatrix} \bar{u}_1{}_j^{n+1} - \bar{u}_1{}_j^{n} \\ \bar{u}_2{}_j^{n+1} - \bar{u}_2{}_j^{n} \end{bmatrix}$$

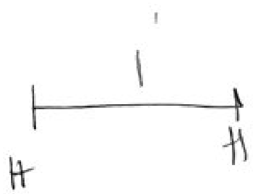
Means: $$\bar{u}_1{}_j^n = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} \eta(x, t_n)\, dx$$

$$\bar{u}_2{}_j^n = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} \cancel{H\, u(x, t_n)\, dx}$$

$H\, u(x, t_n)\, dx$

Globally: use u, η

(Riemann problem: zoom in, use $H_0 u$



$$\left.\begin{aligned} \partial_t \eta + \partial_x(Hu) &= 0 \\ \partial_t u + \partial_x(g\eta) &= 0 \end{aligned}\right\} \text{form to use globally}$$

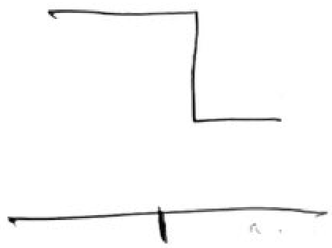
3 continued)

$$\partial_t (\eta) + \partial_x (Hu) = 0$$

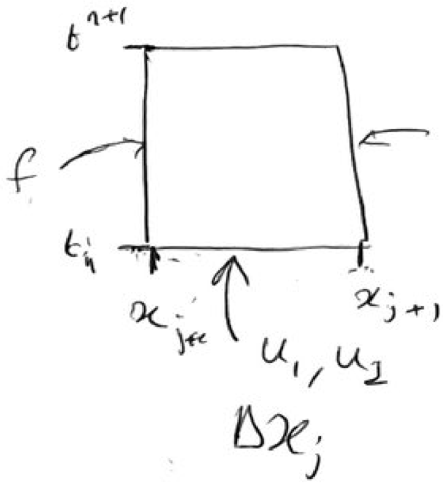
$$\partial_t (u) + \partial_x (g\eta) = 0$$

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \partial_x \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0$$

f_i !



$H = H_0$



$$u_1 = \eta, \quad u_2 = u, \quad f_1 = Hu, \quad f_2 = g\eta$$

$$\int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} dx\,dt = - \int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \partial_x \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} dx\,dt$$

LHS: $$\int_{x_{j-1/2}}^{x_{j+1/2}} \begin{bmatrix} u_1(x,t^{n+1}) - u_1(x,t^n) \\ u_2(x,t^{n+1}) - u_2(x,t^n) \end{bmatrix} dx = \Delta x_j \begin{bmatrix} \bar{u}_{1\,j}^{\,n+1} - \bar{u}_{1\,j}^{\,n} \\ \bar{u}_{2\,j}^{\,n+1} - \bar{u}_{2\,j}^{\,n} \end{bmatrix}$$

<u>Means</u>: $$\bar{u}_{1\,j}^{\,n} = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} \eta(x,t_n)\,dx = \bar{\eta}_j^n$$

$$\bar{u}_{2\,j}^{\,n} = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)\,dx = U_j^n$$

capital U to denote average

$\therefore$ LHS: $$\Delta x_j \begin{bmatrix} \bar{\eta}_j^{n+1} - \bar{\eta}_j^n \\ U_j^{n+1} - U_j^n \end{bmatrix}$$

3) continued

RHS: $\cancel{\int_{t^n}^{t^{n+1}} \vec{f}(\vec{u}(x_{j+1/2}, t))}$

$$\int_{t^n}^{t^{n+1}} \vec{f}(\vec{u}(x_{j+1/2}, t)) - \vec{f}(\vec{u}(x_{j-1/2}, t))\, dt$$

lets denote: $\vec{Q}_j^n = \begin{bmatrix} \eta_j^n \\ U_j^n \end{bmatrix}$

$$\vec{Q}_j^{n+1} = \vec{Q}_j^n - \frac{\Delta t}{\Delta x_j}\left(\vec{F}_{j+1/2}(\vec{Q}_j^n, \vec{Q}_{j+1}^n) - F_{j-1/2}(\vec{Q}_{j-1}^n, \vec{Q}_j^n)\right)$$

where $\vec{F}_{j+1/2}(\vec{Q}_j^n, \vec{Q}_{j+1}^n) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \vec{f}(\vec{u}(x_{j+1/2}, t))\, dt$

NOTE: we can use a diff $\vec{u}$ and $\vec{f}$ at this boundary

at the boundaries:

$\cancel{\partial_t \vec{V} + \partial_x \vec{f}(\vec{u}}$

$$\partial_t \vec{U} + \partial_x f(\vec{U}) = 0$$

Shifted origin Riemann problem

~~Two Riemann problems: η + HU~~

at boundary: $H = H_0$ ie it is a constant (continuous boundary)

~~For η: $f_2 = g\eta$~~

earlier solution?

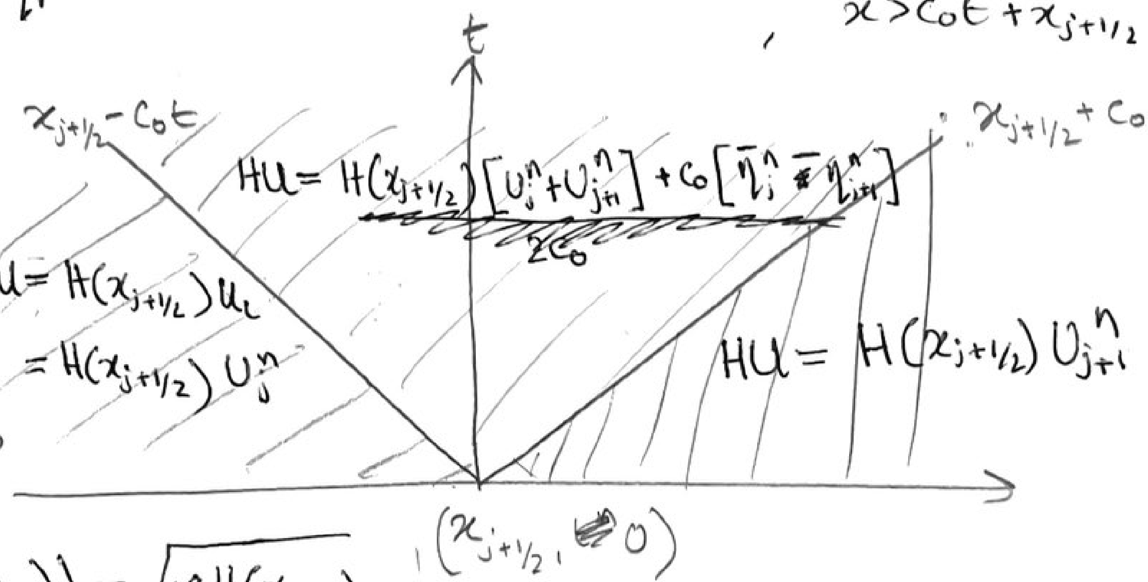
at boundaries: $H = H_0$ $\therefore$ we can use our previous solution for the Riemann problems

3) continued)

~~$H(x_{K+t})H$~~ ~~$c_0=\sqrt{g}$~~

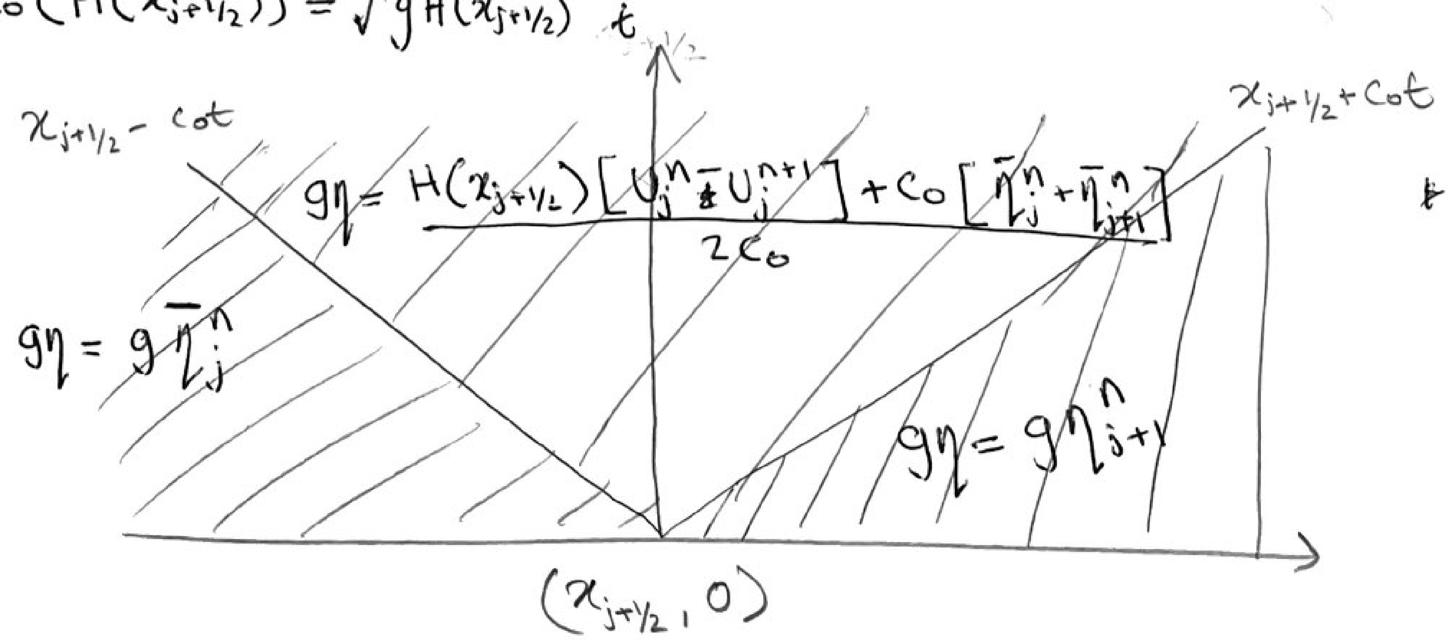
$$H(x_{j+1/2})\,U(x_j) = \begin{cases} H(x_j)\,U_l, & x < -c_0 t + x_{j+1/2} \\ H(x_j)\,[U_l + U_r] + c_0(\eta_l - \eta_r), & -c_0 t + x_{j+1/2} \le x \le c_0 t + x_{j+1/2} \\ H(x_j)\,U_r, & x > c_0 t + x_{j+1/2} \end{cases}$$

$$\eta = \begin{cases} \eta_l, & x < -c_0 t + x_{j+1/2} \\ \dfrac{H(x_j)\,[U_l - U_r] + c_0(\cancel{A}\,\eta_l + \eta_r)}{2c_0}, & -c_0 t + x_{j+1/2} \le x \le c_0 t + x_{j+1/2} \\ \eta_r, & x > c_0 t + x_{j+1/2} \end{cases}$$



it is a function but its a constant at a point (continuous)

$$c_0(H(x_{j+1/2})) = \sqrt{g\,H(x_{j+1/2})}$$



NB: g is a dimensionless gravity $g = \dfrac{g' H_{0s}}{V_0^2}$ where $g' = 9.81\,\mathrm{ms}^{-2}$

3) continued)

timestep estimate: $\Delta t \le \min \frac{\Delta x_j}{c_0}$ } ie what is the minimum speed c_0 - tune timestep based on that

$\Delta t = CFL \min \frac{\Delta x_j}{c_0}$

$CFL < 1$

how do we estimate this?

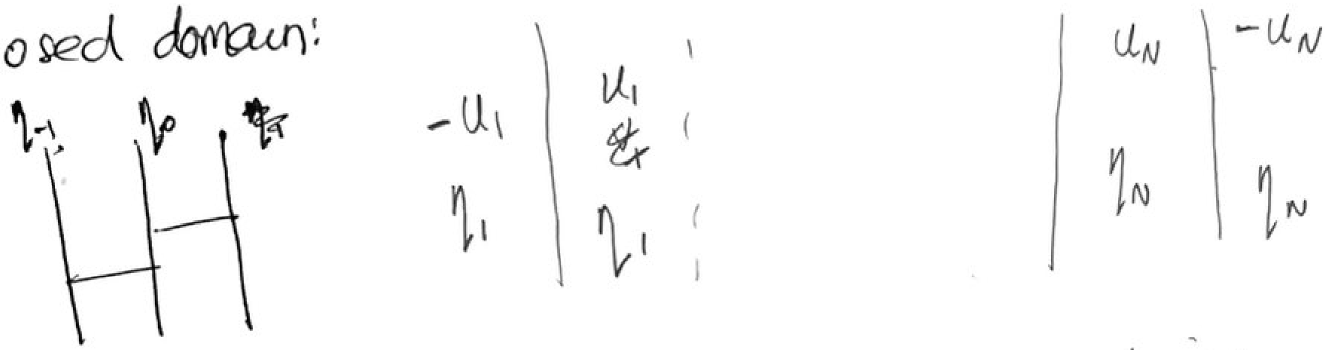
↳ $g = \frac{g' H_{0s}}{U_0^2}$

scale factor for ~~the~~ H

$$|c_0| = \sqrt{g H_0} = \frac{\sqrt{g' H_{0s} H_0}}{U_0}$$

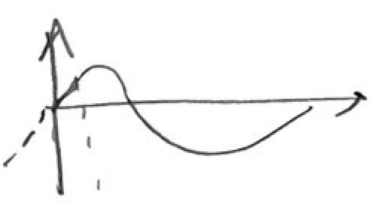
Boundary conditions

closed domain:



we can assume that ~~the~~ $H(x) = H_0$ at the boundaries (where H_0 is a constant which is defined for each boundary) as $H(x)$ is a continuous function 4 ∴ doesn't change at the boundaries.

* do other parts later



3) continued)

Boundary Conditions:

Open domain: • fluid is allowed to enter or leave the computational domain.

• non-zero flux

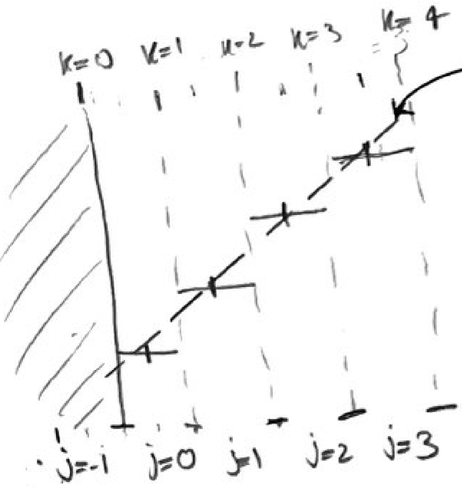
Closed domain: no fluid allowed to enter or leave computational domain

Solid wall: no flux through wall

Flux: $\vec{F}_{j+1/2} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \vec{f}(\vec{u}(x_{j+1/2}, t))\, dt$

Open domain boundary

$k = j + 1/2$ $\quad$ $k=0$: LHS boundary

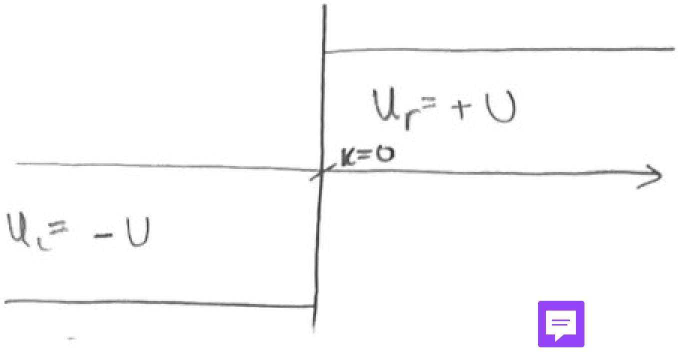


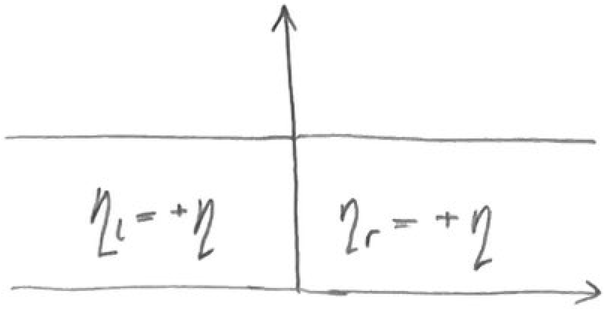
use linear extrapolation to get $k=0$ from $j = 1, 2, 3$ etc (points near boundary)

• exactly the same on other side but for $j = N$ and $k = N+1$

3) continued)

Closed domain:





$$H_0 u = \begin{cases} H_0 u_l & , x < -c_0 t \\ \dfrac{H_0 (u_l + u_r) + c_0 (\eta_l - \eta_r)}{2c_0} & , -c_0 t \le x \le c_0 t \\ H_0 u_r & , x > c_0 t \end{cases}$$

$$= \begin{cases} H_0 u_l & , x < -c_0 t \\ \dfrac{H_0 (U - U) + c_0 (\eta - \eta)}{2c_0} & , -c_0 t \le x \le c_0 t \\ H_0 u_r & , x > c_0 t \end{cases}$$

$$= \begin{matrix} H_0 u_l & , x < -c_0 t \\ 0 & , -c_0 t \le x \le c_0 t \\ H_0 u_r & , x > c_0 t \end{matrix}$$

c_0 & H_0 only uniquely defined at boundary when $H(x) \neq$ constant

$\therefore$ only middle case applies

$\therefore$ velocity flux is zero at boundary

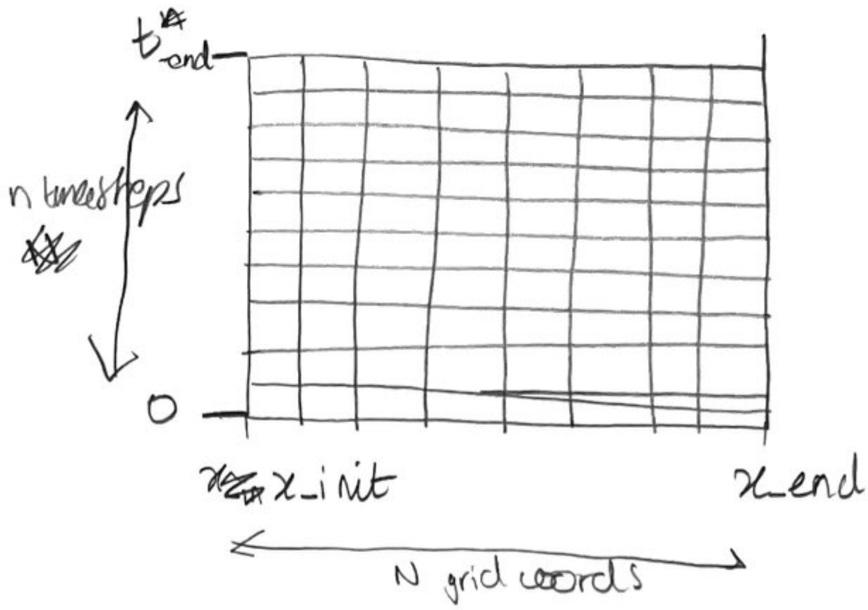
To set η flux to zero at boundary: use $u_l = U_r$, $\eta_l = -\eta_r$

We can assume that $H(x)$ is a constant H_0 at the boundaries (where the value H_0 is defined uniquely for each boundary) as $H(x)$ is a continuous function. Therefore, we know that it must have a fixed value at a point. This value can also be locally approximated ~~at~~ as a constant through a Taylor expansion.

3) continued)

code:

mesh:



variables: g (~~discrete~~ fixed)

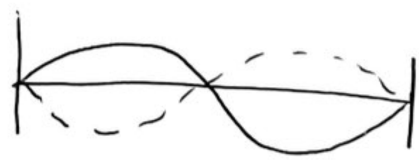
$H(x)$: continuous

$\eta(x,t)$:
$u(x,t)$: } Riemann variables

create grid/mesh defining ~~a sensible~~ timestep

What BCs should I use?

↳ standing wave solutions :



3) continued)

For linear advection equation:

$$\partial_t u + \partial_x a u = 0$$

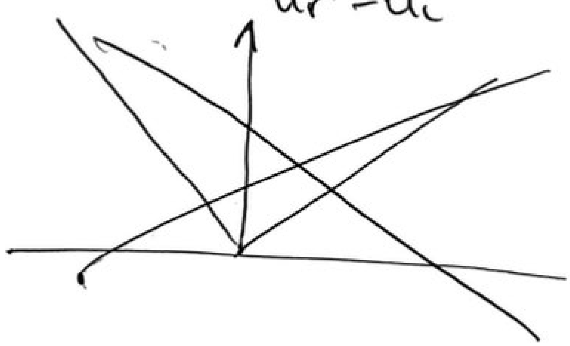
$$\int_{x_b-\varepsilon}^{x_b+\varepsilon} \partial_t u \, dx + \int_{x_b(t)-\varepsilon}^{x_b(t)+\varepsilon} \partial_x f(u)\, dx = 0$$

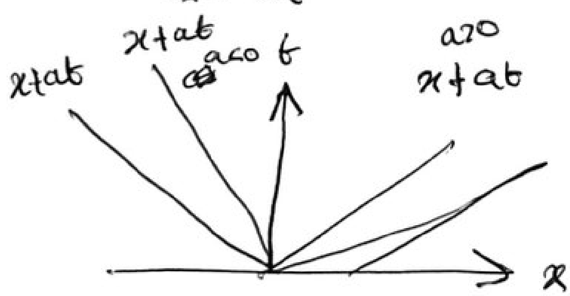
$$\cancel{\partial_t \int_{x_b-\varepsilon}^{x_b+\varepsilon} u \, dx} - (u_r - u_l)\frac{dx_b}{dt} + \int_{x_b-\varepsilon}^{x_b+\varepsilon} \partial_x f(u)\, dx = 0$$

$$f(u_r) - f(u_l) = (u_r - u_l)\frac{dx_b}{dt}$$

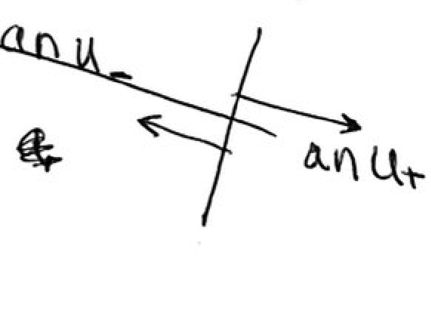
$$\frac{dx_b}{dt} = \frac{f(u_r) - f(u_l)}{u_r - u_l}$$

$$= \frac{a u_r - a u_l}{u_r - u_l} = \frac{a\,\cancel{(u_r - u_l)}}{\cancel{u_r - u_l}} = a$$





$$u = \begin{cases} u_l, & x < at \\ u_r, & x > at \end{cases}$$



LAE 3) continued

From him: $U_k^{n+1} = U_k^n - \frac{a\Delta t}{\Delta x}\left(U_k^n - U_{k-1}^n\right)$

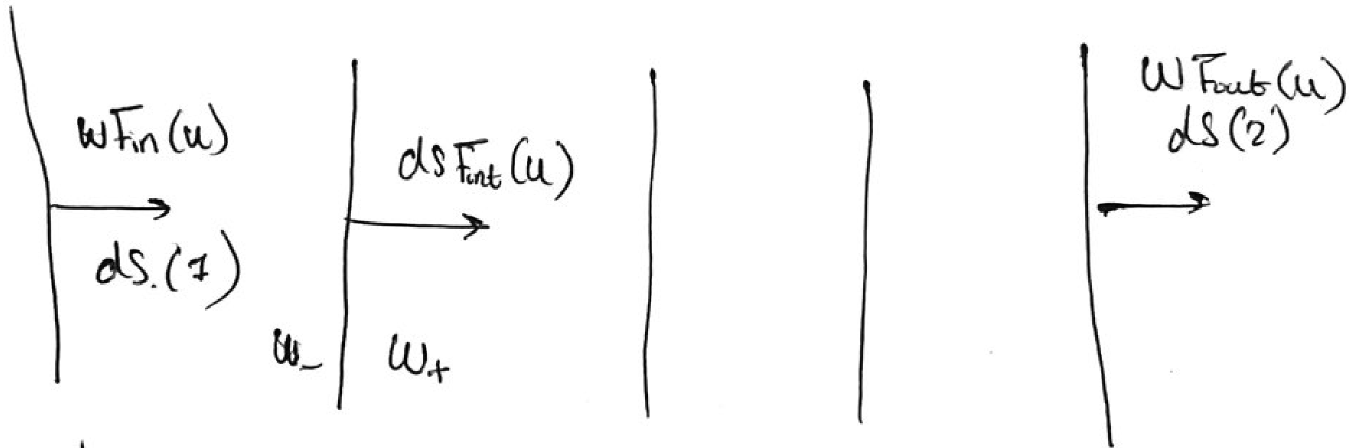
~~Fluxes~~

Weak form: $\sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \frac{\partial u}{\partial t} w_k - \overbrace{\cancel{au \frac{\partial w_k}{\partial x}}}^{=0} \, dx + w_k a u\Big|_{x_k} - w_k a u\Big|_{x_k} = 0$

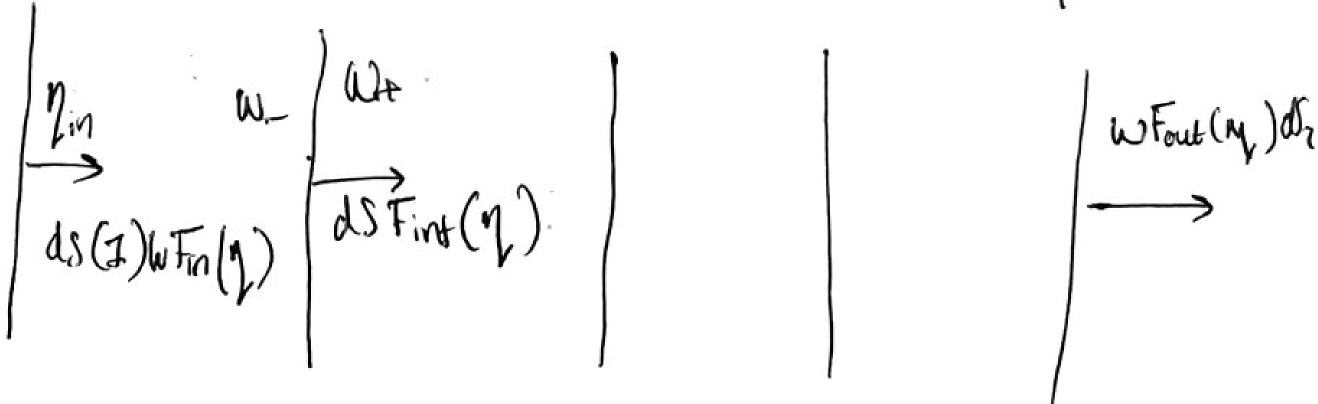
dbc: constant timestep?

For us:

u



η:



3) continued)

Code implementation:

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \partial_x \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0 \qquad \vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} Hu \\ g\eta \end{pmatrix}$$

set: g = constant
H = constant (initially)

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \eta \\ u \end{pmatrix}$$

$\cancel{f_1}$

$$\hat{F}_\eta (U_+, U_-, \hat{n}_+) = \theta H(x) u^+ + (1-\theta) H(x) u^+$$

$$\hat{F}_u (U_+, U_-, \hat{n}_+) = (1-\theta) g \eta^+ + \theta g \eta^-$$

$$\theta \in [0,1]$$

Weak formulations:

viscosity equation

$$\sum_K w_\eta \, \partial_t \eta \, dx = - \sum_\Gamma \hat{n}^+ \cdot F_\eta (U_+^*, U_-, \hat{n}_+) (w_\eta^+ - w_\eta^-) ds$$

$$- \sum_\Gamma \hat{\eta}^+ \cdot \hat{F}_\eta (U_+, U_-, \hat{n}_+) \, w_\eta^+ ds \, (1)$$

$$- \sum_\Gamma \hat{n}^+ \cdot F_\eta (U_+, U_-, \hat{n}_+) \, w_\eta^+ ds \, (2)$$

$\hat{n}$: outward normal

= flux on interior faces

= inlet flux

= outlet flux

velocity equation:

$$\sum_K \underline{w}_u \cdot \partial_t \underline{u} = - \sum_\Gamma \hat{F}_u (U_+, U_-, \hat{n}_+) \, \hat{n}_+ \cdot (\underline{w}_u^+ - \underline{w}_u^-) ds$$

$$- \sum_\Gamma \hat{F}_u (U_+, U_-, \hat{n}_+) \, \hat{n}^+ \cdot w^+ ds \, (1)$$

$$- \sum_\Gamma \hat{F}_u (U_+, U_-, \hat{n}_+) \, \hat{n}^+ \cdot \vec{w}_u^+ ds \, (2)$$

Finite volume: $w_k = \begin{cases} 1, & \text{in cell } k \\ 0, & \text{not in cell } k \end{cases}$

Q6

I have implemented the solid wall boundary conditions for a standing wave (with the numerical flux) using the code provided. To do so, the code essentially creates a jet at the boundary $x = 0$ which balances the velocity such that $U_l = U_r$ (vice versa at $x = L$) and the free surface deviation η remains invariant i.e. $\eta_l = \eta_r$.

Q7

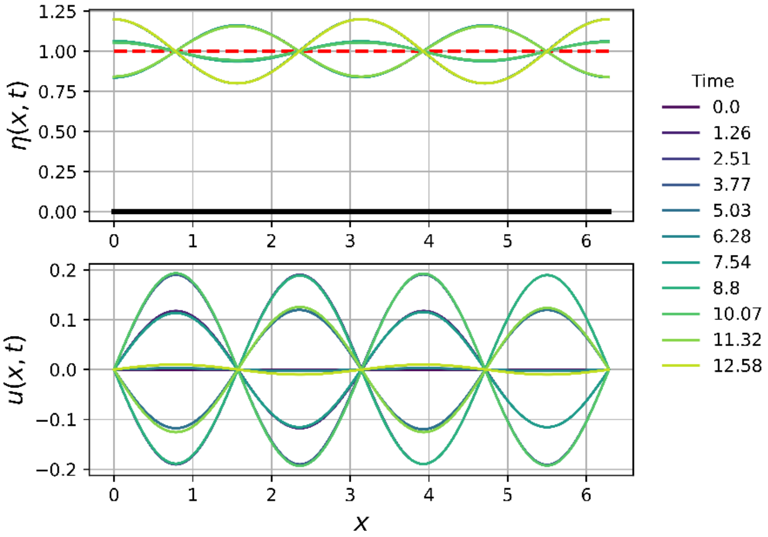


Figure 1: The standing wave solution implemented within the alternating flux scheme in dimensionless units. The black line shows the topography of the bottom surface, whilst $\eta(x,t)$ is translated by $H0$ (red dotted line) to show how the surface of the water moves and $u(x,t)$ shows how the velocity profile of the fluid varies spatially and temporally.

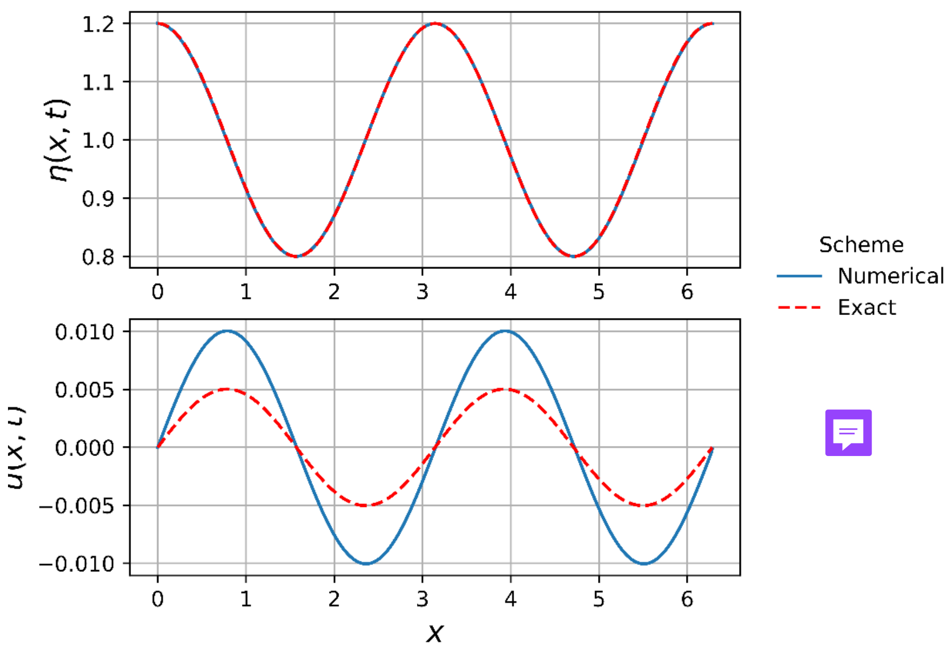


Figure 2: A comparison of the exact and numerical solutions for the standing waves when $\theta = 0$ and $N_{cells} = 500$.

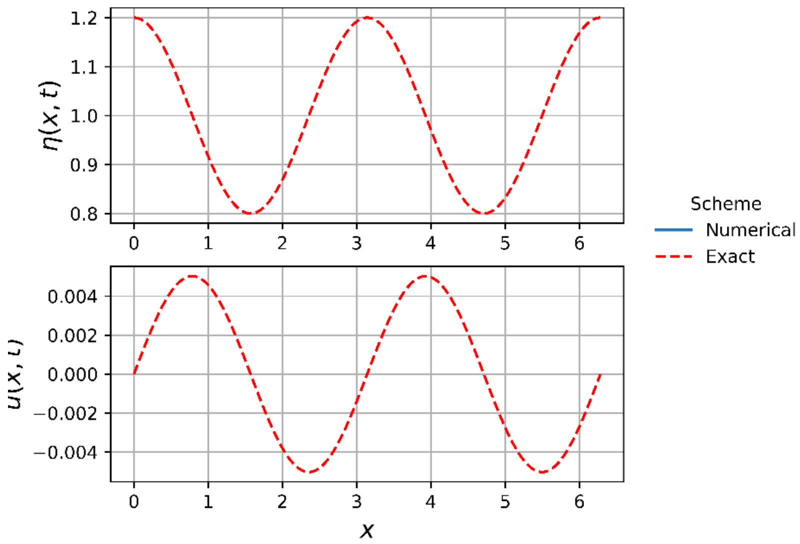


Figure 3: A comparison of the exact and numerical solutions for the standing waves when $\theta = 0$ and $N_{cells} = 500$.

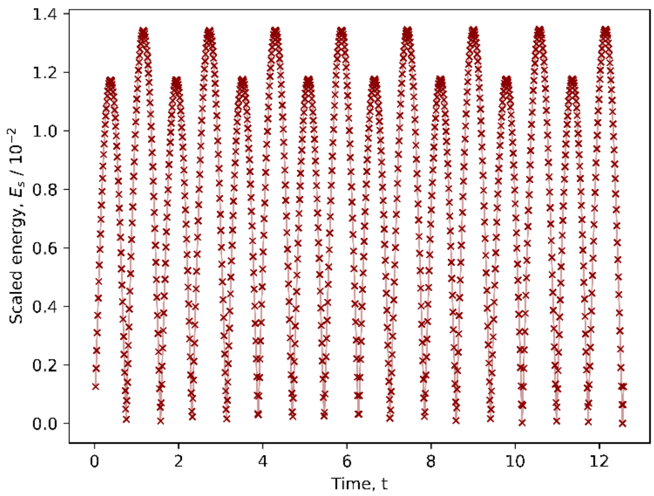


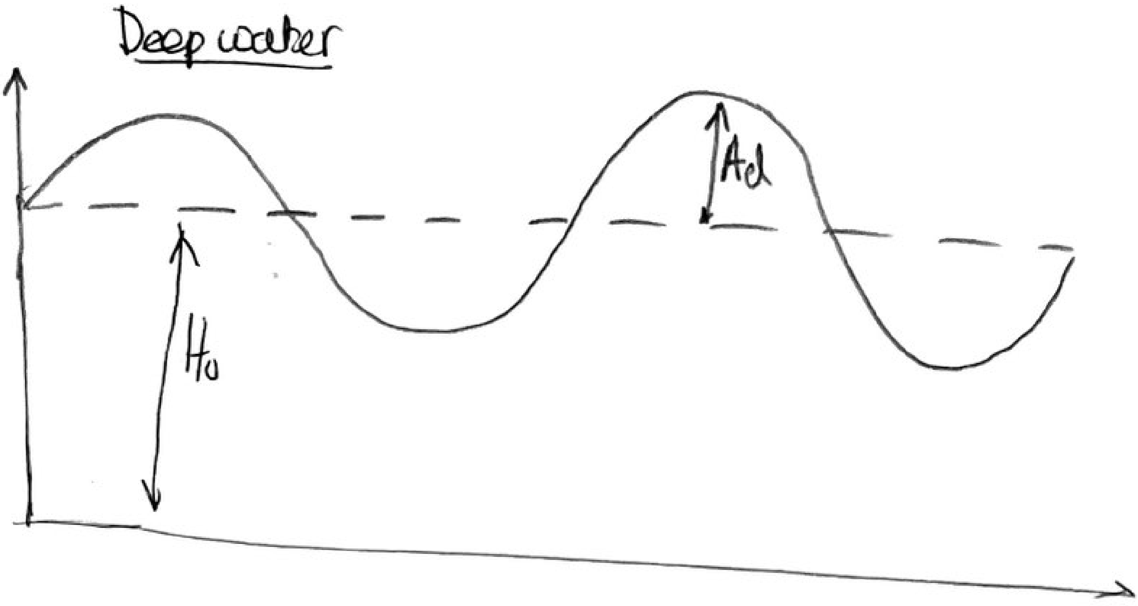
Figure 4: The scaled energy of the system as a function of time, highlighting the temporal periodicity of the system. The magnitude of E_s when $\frac{dE_s}{dt} = 0$ oscillates periodically, highlighting that no energy is escaping from the system.

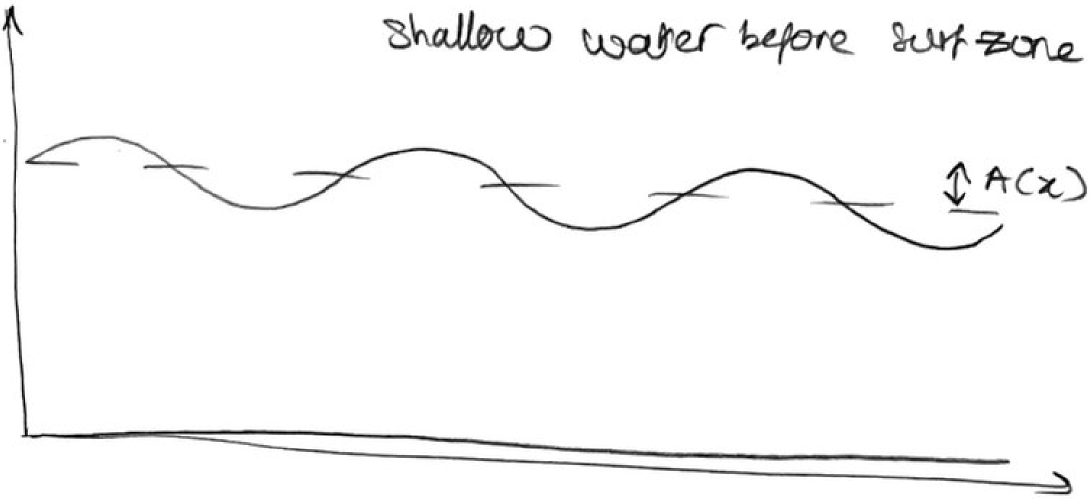
As observed visually above, the code clearly converges to the exact solution when you increase the number of elements in the spatial domain. The scaled energy is given by $E_s = \frac{E - E_0}{E_0}$, where E is the absolute energy and is given by:

$$E = \frac{1}{2}\left(H(x)u(x,t)^2 + \eta(x,t)^2\right).$$

Therefore, as E_0 is the energy of the system at $t = 0$, the scaled energy provides a means of quantifying how the energy of the system fluctuates as a function of time. When $E_s = 0$, $E = E_0$ and the system has returned to its original state. Since this is observed within our graph, it suggests that we have correctly coded standing waves within our simulation. We also observe two different types of local maxima which alternate periodically in time. Perhaps these features are symmetric and antisymmetric modes where the two waveforms are either in phase or partially out of phase.

8)





$A(x) < A_d$: $A(x) = A_d \left(\frac{H_0}{H(x)} \right)^{1/4}$ H_0: offshore depth

$$\omega = \sqrt{g\,H(x)}\;k(x)$$

↳ $\lambda(x)$ decreases as $H(x)$ decreases

↳ waves: · shorter f

· higher amplitude

8) continued)

$$H(x) = \begin{cases} H_0, & 0 < x < W \\ H_0 - S(x-W), & W \le x \le L_1 \\ H_b, & L_1 < x < L \end{cases} \qquad S = \frac{H_0 - H_b}{L_1 - W}$$

$W = 2$, $H_0 = 1$, $H_b = 0.1$, $L = 10$, $L > L_1$

let $L_1 = \cancel{8}\,10$, $L = 12$

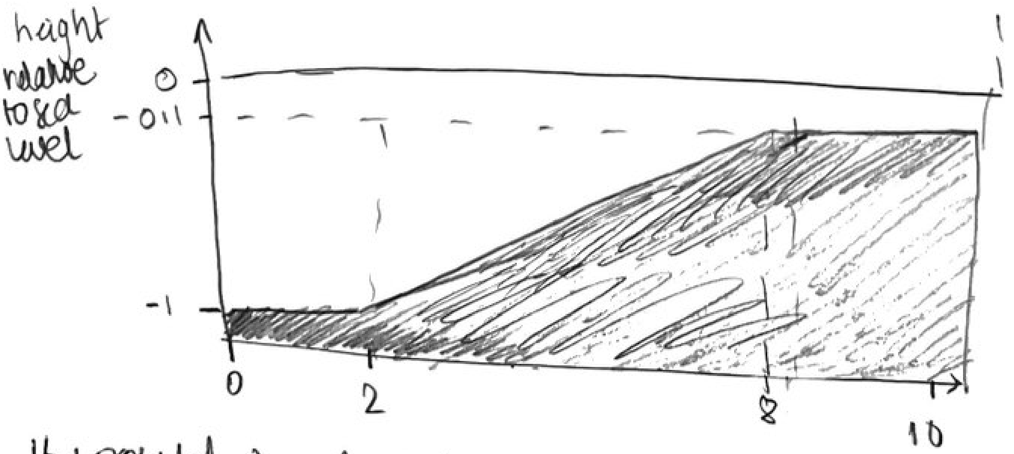
$$\cancel{S = \frac{1-0.1}{8-2} = \frac{0.9}{6} = 0.15}$$

$$\therefore H(x) = \begin{cases} 1, & 0 < x < W \\ 1 - \overset{S}{\cancel{0.15}}\,(x-2), & 2 \le x \le \cancel{8}\,10 \\ 0.1, & \underset{10}{\cancel{8}} < x < \cancel{L}\,12 \end{cases}$$

$$S = \frac{1-0.1}{8} = 0.1125$$

$$= \begin{cases} 1, & 0 < x < W \\ 1 - \overset{0.1125}{\cancel{0.15}}(x-2), & 2 \le x \le 8 \\ 0.1, & \cancel{8 < x < 10}\ 10 < x < 12 \end{cases}$$

extrapolating BC:
$U_L = U_r$
$\eta_L = \eta_r$



It would be advantageous to use a non-uniform mesh in this case as the frequency of the waves increases as H(x) decreases. Therefore, you would ideally ~~at~~ add more spatial cells ~~to resolve the~~ at lower values of H(x) to resolve these high frequency oscillations. Otherwise, you run the risk that you aren't sampling their maxima, ~~introduce causing noise to appear within your simulations~~, and observing the expected increase in their amplitude.

8) continued)

Non uniform ~~timestep~~ dx:

min timestep

Could use: $dx = dx_0 + \frac{\alpha}{H(x)}$ ~~where α is x~~

~~let dx_0~~

$dx = \alpha H(x)$ where α is the timestep when $H(x) = 1 = H_0$

~~idk~~ how to implement this in Firedrake ~~though~~?

I decided to use a uniform dx for simplicity

$U_{0S} = (g \times H_{0S})^{1/2}$ $\quad g = 9.81\,ms^{-2}$

$= (9.81\,ms^{-2} \times 40\,m)^{1/2}$ $\quad H_{0S} = 40\,m$

$= 19.81\,ms^{-1}$

$T_S = \frac{L_S}{U_{0S}} = \frac{223\,m}{19.81\,ms^{-1}} \simeq 11.36\,s$

$\therefore$ $N_p = 45$ ← number of periods

$T_{PS} = \cancel{12}\ 6\,s$ ← lets say each period is 6s

$T_p = \frac{6\,s}{11.36\,s} \simeq \frac{1.06}{2} \simeq 0.53$ } dimensionless units

$\omega = \frac{2\pi}{T_p} \simeq \frac{5.95}{2} \simeq 2.97$

$T_{end} = N_{per} \times T_p = \frac{47.53}{2} \simeq 23.75$ } total simulation time

~~I split the simulation up into 2000 timesteps and 4000 spatial cells~~

8) continued)

I ~~do~~ split the ~~set~~ spatial domain into 8000 cells ← $dx = \frac{L_x}{N_x}$ ← length of domain.
and used the following timestep

$$\cancel{dt = \frac{CFL \times dx}{|C_0|} \times \frac{1}{|C_0|}}$$

$$dt = \min \frac{CFL\, dx}{|C_0|} = 0.0015 \quad (CFL = 1)$$

As observed (within Figure), the results appear to have converged for most timesteps as the waves form an amplitude envelope. At later timesteps, they diverge near the end of the domain ($x > 2500$m), as the ~~timestep~~ cell size is not sufficiently to resolve this region. They appear very similar in form to the results obtained within the question sheet. ← [Figures]

I then ~~doubled the~~ increased N_x (number of spatial cells) to 16000 which halved the timestep. ~~This didn't~~ The results still diverged at the end of the domain, suggesting that ~~a high~~ even higher cell count is ~~necess higher~~ necessary within this region.

I then ~~plotted a steeper~~ ran the same procedure with a steeper beach topography

$$H(x) = \begin{cases} 1, & 0 < x < 4 \\ 1 - S, & 4 < x < 8 \\ 0.1, & 8 < x < 12 \end{cases} \qquad S = \frac{1 - 0.1}{4} = \frac{0.9}{4} = \underline{\underline{0.225}}$$

This ~~is~~ result is shown within Figure (with $N_{cells} = 8000$). As expected, ~~here the rate at which the amplitude of the~~ the amplitude of the waves increased more rapidly as a function of x due to the increased gradient of the beach. ~~Finally,~~ I explored a ~~shallow topography~~ where $H(x)$ ~~increases~~ linearly ~~within for all x in Figure~~ ,

Q8)

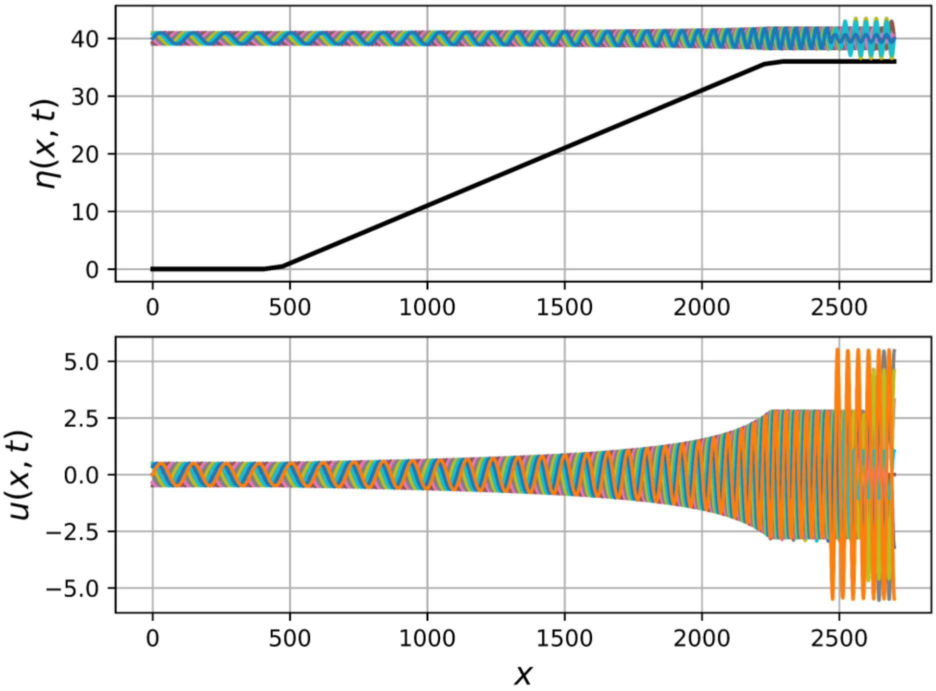


Figure 5: The amplitude envelope of the surface waves and their velocity plotted as a function of x when $N_{cells} = 8000$.

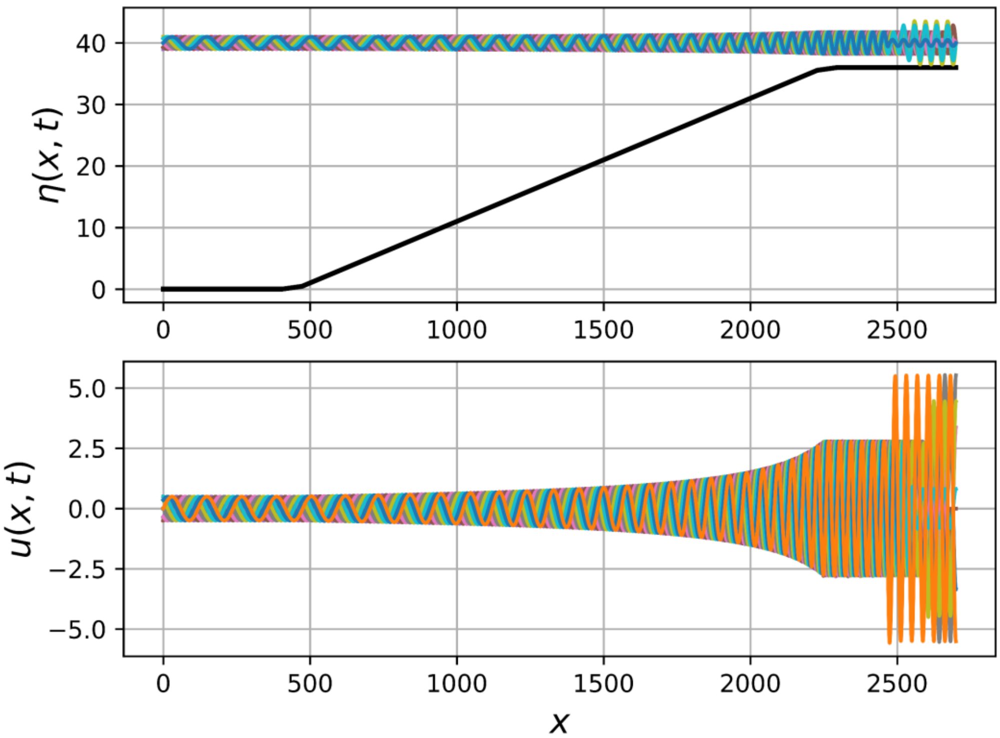


Figure 6: The amplitude envelope of the surface waves and their velocity plotted as a function of x when $N_{cells} = 16000$.

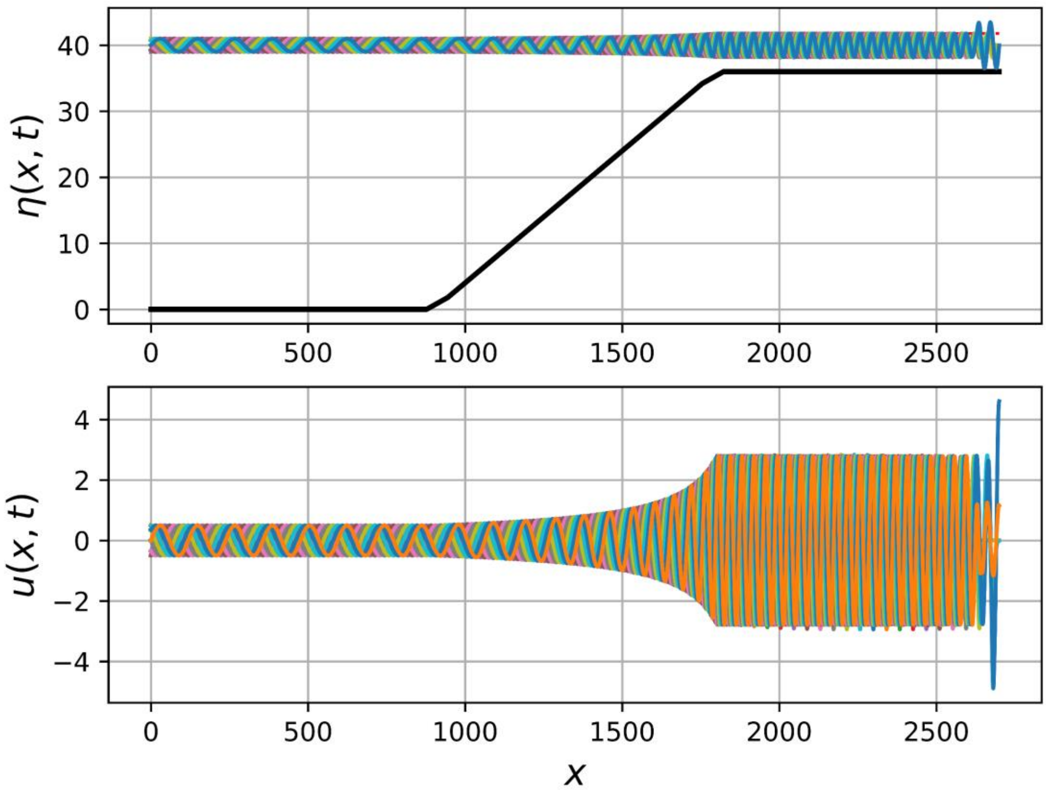


Figure 7: The amplitude envelope of the surface waves and their velocity plotted as a function of x when $N_{cells} = 8000$ for a steeper beach topography.