

MATH 5453M Numerical Exercise 2

Finite Volume Method: Linear shallow-water equation. predicting surf height at beaches

Consider the linearised shallow-water system of eqn:-

$$\frac{\partial h}{\partial t} + \frac{\partial(Hu)}{\partial x} = 0 \quad \text{&} \quad \frac{\partial u}{\partial t} + \frac{\partial(gh)}{\partial x} = 0 \quad \text{--- (1)}$$

for velocity $u = u(x, t)$ & free surface deviation

$h = h(x, t)$, rest depth $H(x)$ & acceleration of gravity $g = 9.81 \text{ m/sce}$,

when we scale (1)

$$u = U_0 u', \quad x = L_S x', \quad t = (\omega/U_0) t'$$

$$h = H_0 s h', \quad H = H_0 H'.$$

$$g' \rightarrow g H_0 s / U_0^2,$$

Ques (1) :-

Let's write the non-dimensional form of (1)

$$\frac{U_0 H_{0s}}{L_s} \frac{\partial h'}{\partial t'} + \frac{U_0 H_{0s}}{L_s} \frac{\partial (H' h')}{\partial x'} = 0 \quad \&$$

$$\frac{U_0^2}{L_s} \frac{\partial u'}{\partial t'} + \frac{U_0^2}{L_s} \frac{\partial (g' h')}{\partial x'} = 0$$

$$\frac{\partial h'}{\partial t'} + \frac{\partial (H' h')}{\partial x'} = 0 \quad (*)$$

$$\frac{\partial u'}{\partial t'} + \frac{\partial (g' h')}{\partial x'} = 0 \quad (**)$$

↳ after non-dimensionalisation also we are having same equation as (1).

Except $g' = g \frac{H_{0s}}{U_0^2}$

- Since both equation (*) & (**) are conservation equation (conservation of mass & conservation of momentum respectively).

- Both are coupled, because \dot{u} & \dot{y} influence each other.



we can write these two equations in matrix form

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ H_0 u \end{pmatrix} + A \frac{\partial}{\partial x} \begin{pmatrix} u \\ H_0 u \end{pmatrix} = 0$$

where $A = \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix}$

$$\Rightarrow C_0^2 = gH_0 ; \text{ wave speed}$$

using linear algebra, we can decouple these equation,

finding eigen values & eigen vectors of matrix A

$$\det |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ C_0^2 - \lambda & 0 \end{vmatrix} = \lambda^2 - C_0^2 = 0$$

\therefore eigenvalues are $\lambda_1 = C_0$, $\lambda_2 = -C_0$.

Now finding the eigen vectors

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \quad \& \quad A\vec{v}_2 = \lambda_2 \vec{v}_2$$

$$\begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = C_0 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = -C_0 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

$$v_{12} = C_0 v_{11}$$

$$C_0^2 v_{11} = C_0 v_{12}$$



$$C_0^2 v_{11} = C_0 (C_0 v_{11})$$

$$\Rightarrow \vec{v}_1 = \frac{1}{2C_0} \begin{pmatrix} 1 \\ C_0 \end{pmatrix}$$

$$\downarrow$$

$$\vec{v}_2 = \frac{1}{2C_0} \begin{pmatrix} -1 \\ C_0 \end{pmatrix}$$

\Rightarrow Therefore, Matrix B is consisting eigen vectors is,

$$B = \frac{1}{2C_0} \begin{pmatrix} 1 & -1 \\ C_0 & C_0 \end{pmatrix}$$

$$\Rightarrow \text{Now } B^{-1} = \frac{1}{|B|} \begin{pmatrix} C_0 & -C_0 \\ 1 & 1 \end{pmatrix}^T$$

$$= \frac{1}{|B|} \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix}$$

$$= \frac{1}{\frac{1}{2} (C_0 + C_0)} \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix}$$

& $AB = \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{C_0}{2} & -\frac{C_0}{2} \end{pmatrix}$

$\therefore B^{-1}AB = \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{C_0}{2} & -\frac{C_0}{2} \end{pmatrix}$

$$= \begin{pmatrix} C_0 & 0 \\ 0 & -C_0 \end{pmatrix}$$

$$B^{-1}AB \Rightarrow \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \lambda I$$

\Rightarrow Now, we will proceed to decoupling with new variable γ

$$\hookrightarrow \gamma = B^{-1} \begin{pmatrix} h \\ H_0 u \end{pmatrix}^T$$

$$\Rightarrow \gamma = B^{-1} \begin{pmatrix} h \\ \frac{h}{H_0 u} \end{pmatrix}$$

$$\Rightarrow B\gamma = \begin{pmatrix} h \\ H_0 u \end{pmatrix}$$

But $\frac{\partial}{\partial t} \begin{pmatrix} h \\ H_0 u \end{pmatrix} = B \frac{\partial \gamma}{\partial t}$

$$B \frac{\partial}{\partial x} \begin{pmatrix} h \\ H_0 u \end{pmatrix} = B \frac{\partial \gamma}{\partial x}$$

\downarrow

original eqn becomes

$$B \frac{\partial \gamma}{\partial t} + AB \frac{\partial \gamma}{\partial x} = 0 \quad - \text{***}$$

But from above we have, $B^{-1}AB = \lambda I$

\Rightarrow Multiplying B^{-1} in ***;

$$B^{-1} \frac{\partial r}{\partial t} + B^T A B \frac{\partial r}{\partial x} = 0$$

$$\frac{\partial r}{\partial t} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{\partial r}{\partial x} = 0$$

→ decoupled set of linear advection equation

where $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ & $r_1 = H_0 u + c_0 h$
 $r_2 = H_0 u - c_0 h$

($r \rightarrow$ Riemann invariant)

Ques : 2 →

for piecewise constant initial data

$$\gamma_1(x, 0) = \begin{cases} \gamma_{1l} & \text{for } x < 0 \\ \gamma_{1r} & \text{for } x \geq 0 \end{cases}, \quad \gamma_2(x, 0) = \begin{cases} \gamma_{2l} & \text{for } x < 0 \\ \gamma_{2r} & \text{for } x \geq 0 \end{cases}$$

from equation - ⑦

$$\frac{\partial r}{\partial t} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{\partial r}{\partial x} = 0$$

$$\rightarrow \frac{\partial \gamma_1}{\partial t} + c_0 \frac{\partial \gamma_1}{\partial x} = 0 \quad (\lambda_1 = c_0, \lambda_2 = -c_0)$$

$$\& \frac{\partial \gamma_2}{\partial t} - c_0 \frac{\partial \gamma_2}{\partial x} = 0$$



These are linear advection equations,
∴ the solution would be:

$$\left\{ \begin{array}{l} \gamma_1(x, t) = \gamma_1(x - c_0 t, 0) \\ \gamma_2(x, t) = \gamma_2(x + c_0 t, 0) \end{array} \right.$$

$c_0 \rightarrow$ propagation speed.

→ γ_1 const. along characteristics
line $x - c_0 t = \text{const}$ &
 γ_2 along $x + c_0 t = \text{const}$

Based on the given initial condition

$$\sigma_1(x,t) = \begin{cases} \sigma_{1l} & \text{if } x - c_0 t < 0 \\ \sigma_{1r} & \text{if } x - c_0 t \geq 0 \end{cases}$$

$$\sigma_1(x,t) = \begin{cases} \sigma_{1l} & \text{if } x < c_0 t \\ \sigma_{1r} & \text{if } x \geq c_0 t \end{cases} \quad \text{--- A}$$

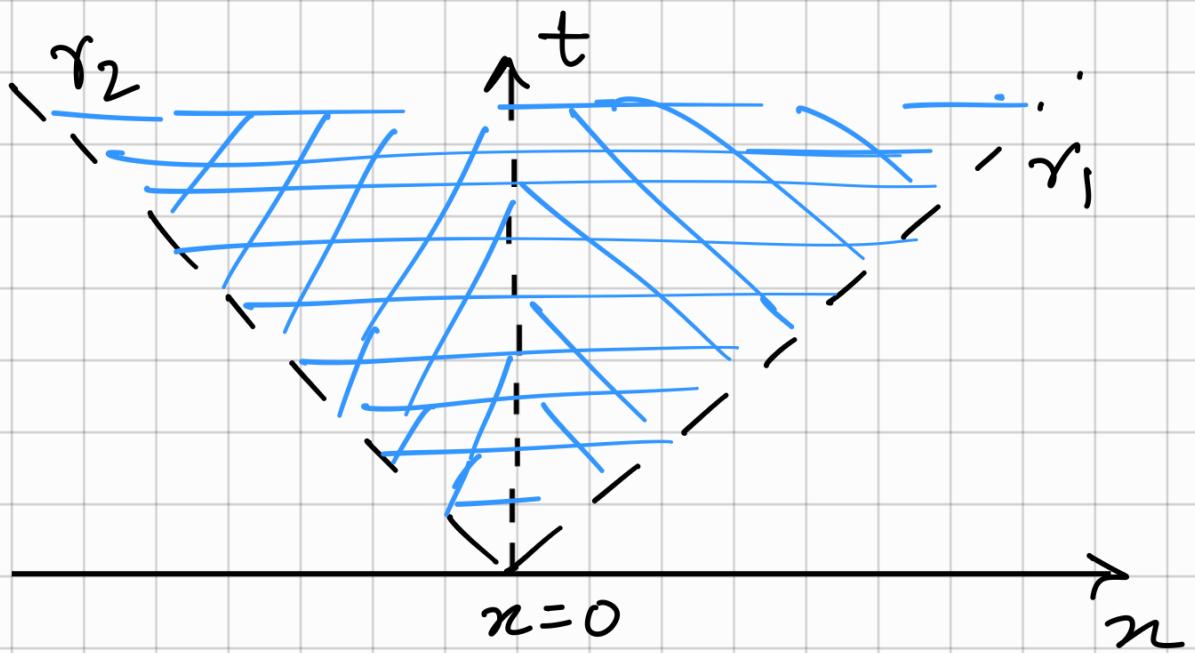
and

$$\sigma_2(x,t) = \begin{cases} \sigma_{2l} & \text{if } x + c_0 t < 0 \\ \sigma_{2r} & \text{if } x + c_0 t \geq 0 \end{cases}$$

$$\sigma_2(x,t) = \begin{cases} \sigma_{2l} & , x \leq -c_0 t \\ \sigma_{2r} & , x \geq -c_0 t \end{cases} \quad \text{--- B}$$

Since the initial discontinuity is
at $x=0$, @ $t=0$

if we plot the $\sigma_1(x,t)$ & $\sigma_2(x,t)$



Now :- given that $\gamma_1 = H_0 U + C_0 V$

$$\gamma_2 = H_0 U - C_0 V$$

and $H_0 U = \frac{1}{2} (\gamma_1 + \gamma_2)$,

$$V = \frac{1}{2} (\gamma_1 - \gamma_2) / C_0$$

\Rightarrow as per γ_1 & γ_2 value given in
 (A) & (B)

(i) for $n < -C_0 V$

$\Rightarrow \gamma_1 = \gamma_1 l$ & $\gamma_2 = \gamma_2 l$

$$\therefore H_0 U(\alpha, t) = \frac{1}{2} (\gamma_1 l + \gamma_2 l)$$

$$= \frac{1}{2} \left(H_0 U_l + C_0 V_l + H_0 U_l - C_0 V_l \right)$$

$$H_0 U(x, t) = \frac{1}{2} (2 H_0 U_l) = H_0 U_l$$

② for the sheared part,
 $-C_0 t \leq n \leq C_0 t$

$$\gamma_1 = \gamma_{1l}, \quad \gamma_2 = \gamma_{2s}$$

$$\therefore H_0 U(x, t) = \frac{1}{2} (\gamma_{1l} + \gamma_{2s})$$

③ for $n > C_0 t$

$$\gamma_1 = \gamma_{1s} \quad \& \quad \gamma_2 = \gamma_{2s}$$

$$\therefore H_0 U(x, t) = \frac{1}{2} (\gamma_{1s} + \gamma_{2s})$$

$$= \frac{1}{2} (H_0 U_r + C_0 h_r + H_0 U_r - C_0 h_r)$$

$$H_0 U(x, t) = H_0 U_r$$

$$\Rightarrow H_0 U(x, t) = \begin{cases} H_0 U_l & n \leq C_0 t \\ \frac{1}{2} (\gamma_{1l} + \gamma_{2s}) & -C_0 t \leq n \leq C_0 t \\ H_0 U_r & n > C_0 t \end{cases}$$

Similarly for $U(x, t)$

$$\textcircled{1} \quad n < -C_0 t$$

$$\gamma_1 = \gamma_{1,l}, \quad \gamma_2 = \gamma_{2,r}$$

$$\gamma(x,t) = \frac{1}{2C_0} (\cancel{H_0\gamma_l + C_0\gamma_l} - \cancel{H_0\gamma_r + C_0\gamma_r})$$

$$\gamma(x,t) = \gamma_l$$

$$\textcircled{2} \quad \text{for } -C_0 t \leq n \leq C_0 t$$

$$\gamma_1 = \gamma_{1,l} \quad \gamma_2 = \gamma_{2,r}$$

$$\gamma(x,t) = \frac{1}{2C_0} (H_0\gamma_l + C_0\gamma_l - H_0\gamma_r + C_0\gamma_r)$$

$$\textcircled{3} \quad \text{for } n > C_0 t$$

$$\gamma_1 = \gamma_{1,r} \quad \gamma_2 = \gamma_{2,r}$$

$$\gamma(x,t) = \frac{1}{2C_0} (H_0\gamma_r + C_0\gamma_r - H_0\gamma_r + C_0\gamma_r)$$

$$\gamma(x,t) = \gamma_r$$

$$\Rightarrow \gamma(x,t) = \frac{1}{2C_0} (\gamma_1(x,t) - \gamma_2(x,t))$$

$$\Rightarrow \gamma(x, t) = \begin{cases} \gamma_L & n < -c_0 t \\ H_0(u_L - u_R) + c_0(u_L + u_R) \\ \text{for } -c_0 t \leq n \leq c_0 t \\ \gamma_R & n > c_0 t \end{cases}$$

\Rightarrow Riemann invariant s

$$x_1 = H_0 u + \sqrt{g H_0} \gamma$$

$$x_2 = H_0 u - \sqrt{g H_0} \gamma$$

& as per above:

$$H_0 u = \frac{1}{2}(x_1 + x_2) \quad \& \quad \gamma = \frac{1}{2\sqrt{g H_0}}(x_1 - x_2)$$

$$\text{from } \frac{\partial h}{\partial t} + H_0 \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2\sqrt{g H_0}} (x_1 - x_2) \right) + H_0 \frac{\partial}{\partial x} \left(\frac{1}{2 H_0} (x_1 + x_2) \right) = 0$$

$$\frac{1}{2\sqrt{g H_0}} \left(\frac{\partial x_1}{\partial t} - \frac{\partial x_2}{\partial t} \right) + \frac{1}{2} \left(\frac{\partial x_1}{\partial x} + \frac{\partial x_2}{\partial x} \right) = 0$$

$\rightarrow C$

Similarly for

$$\frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2H_0} (\gamma_1 + \gamma_2) \right) + g \frac{\partial}{\partial x} \left(\frac{1}{2\sqrt{gh_0}} (\gamma_1 - \gamma_2) \right) = 0$$

$$\rightarrow \frac{1}{2H_0} \left(\frac{\partial \gamma_1}{\partial t} + \frac{\partial \gamma_2}{\partial t} \right) + \frac{g}{2\sqrt{gh_0}} \left(\frac{\partial \gamma_1}{\partial x} - \frac{\partial \gamma_2}{\partial x} \right) = 0$$

— (1)

from (1) multiply by $2\sqrt{gh_0}$

$$\left(\frac{\partial \gamma_1}{\partial t} - \frac{\partial \gamma_2}{\partial t} \right) + \sqrt{gh_0} \left(\frac{\partial \gamma_1}{\partial x} + \frac{\partial \gamma_2}{\partial x} \right) = 0$$

for $\gamma_1 \rightarrow \frac{\partial \gamma_1}{\partial t} + \sqrt{gh_0} \frac{\partial \gamma_1}{\partial x} = 0$

$\gamma_2 \rightarrow \frac{\partial \gamma_2}{\partial t} - \sqrt{gh_0} \frac{\partial \gamma_2}{\partial x} = 0$

↳ uncoupled

γ_1 & γ_2 do not depend on each other,

\Rightarrow characteristic lines for σ_1 , would be

$$x - \sqrt{g H_0 t} = \text{const}$$

for σ_2

$$x + \sqrt{g H_0 t} = \text{const}$$

Question 3 :-

from given Given linearised shallow-water system of equations.

$$\frac{\partial h}{\partial t} + \frac{\partial (H u)}{\partial x} = 0 \quad \text{and,}$$

$$\frac{\partial u}{\partial t} + \frac{\partial (g h)}{\partial x} = 0$$

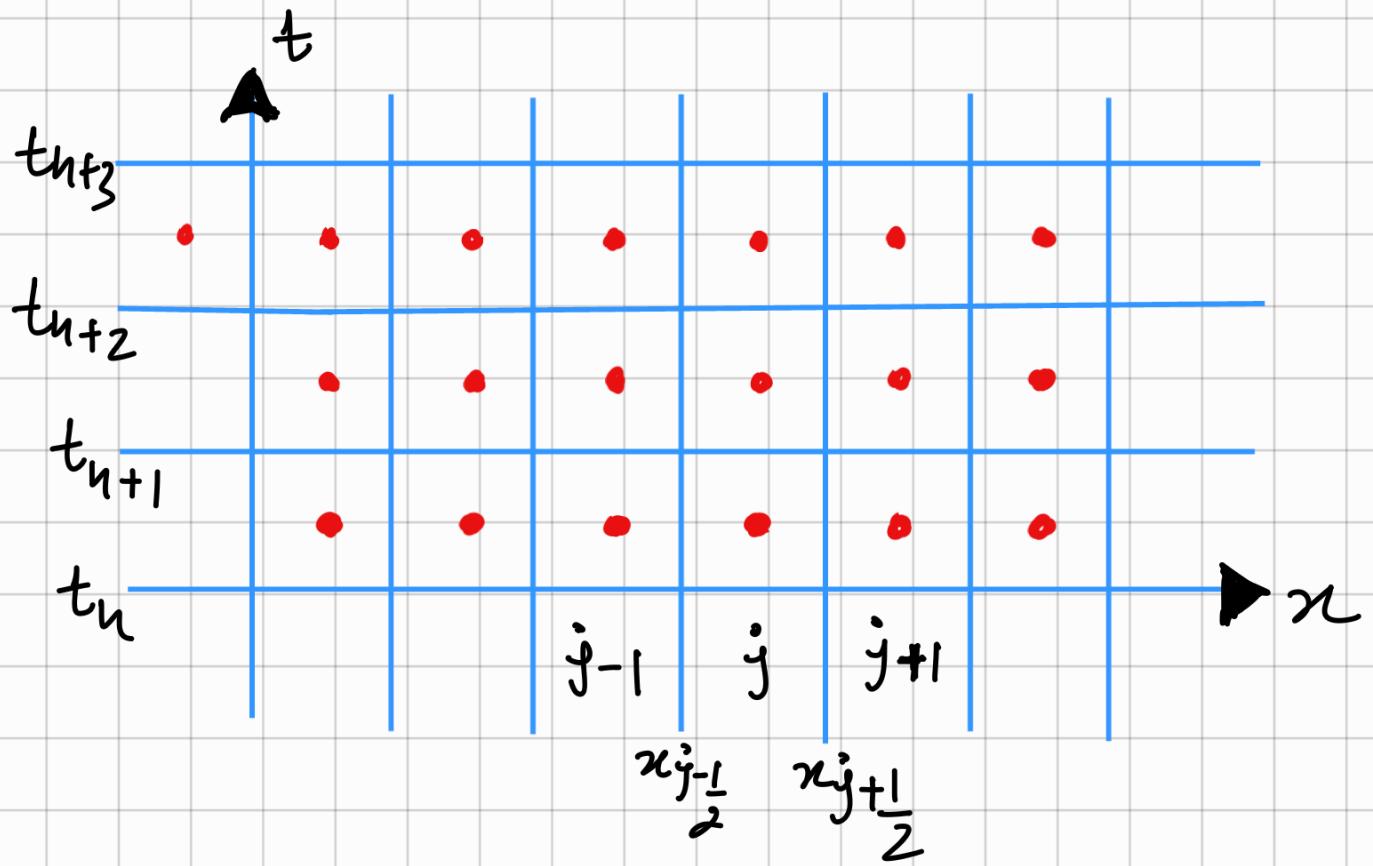
such that, $u = u(x, t) \rightarrow$ Variable velocity free surface deviation, $\gamma = h(x, t)$

for Godunov method.

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

$$\text{where, } u(x, t) = \begin{pmatrix} h \\ u \end{pmatrix}$$

$$f(u) = \begin{pmatrix} Hu \\ g_h \end{pmatrix} \rightarrow \text{flux}$$



For Godunov discretization,

$$U_j^{n+1} = U_j^n - \frac{1}{\Delta x_j} \int_{t_n}^{t_{n+1}} f_{j+\frac{1}{2}}(t) - f_{j-\frac{1}{2}}(t) dt,$$

we have defined the mean cell average
 U_j in cell j .

$$U_j(t) = \frac{1}{L_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t) dx,$$

The flux is evaluated at the cell boundaries

$$f_{j+\frac{1}{2}}(t) = f\left(u(x=x_{j+\frac{1}{2}}, t)\right).$$

$\Rightarrow U_j(t)$ & $f_{j+\frac{1}{2}}(t)$ are function of time t ,

Such that;

$$U_j^h = U_j(t=t_n) \text{ So on...}$$

we need to determine $f_{j+\frac{1}{2}}(t)$ over the
 time interval $t_n \leq t \leq t_{n+1}$ to obtain the

F.V.M Scheme.

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\ell_{ij}} \left[F_{j+\frac{1}{2}}(U_j^n, U_{j+\frac{1}{2}}^n) - F_{j-\frac{1}{2}}(U_{j-1}^n, U_j^n) \right]$$

A

numerical flux approximated as,

$$F_{j+\frac{1}{2}}(U_j^n, U_{j+\frac{1}{2}}^n) \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f_{j+\frac{1}{2}}(t) dt$$

Assuming,

- h & H_0 are constant locally.
- Flux is evaluated using the central solution of Riemann problem at the interface

Fluxes will be;

$$f(u) = \begin{pmatrix} H_0 u \\ g h \end{pmatrix}$$

$$\Rightarrow f_1(u(x_{j+\frac{1}{2}}, t)) = H_0 u_{j+\frac{1}{2}}(t) \\ = \frac{1}{2} (H_0 u_{j+1} + H_0 u_j) + \frac{C_0}{2} (h_j - h_{j+1})$$

or

$$= \frac{1}{2} ((H_0 u)_r + (H_0 u)_l) \\ + \frac{C_0}{2} (h_l - h_r)$$

g

$$\Rightarrow f_2(u(x_{j+\frac{1}{2}}, t)) = g_{j+\frac{1}{2}}(t) \\ = \frac{1}{2} g(h_j + h_r) + \frac{1}{2} C_0 ((H_0 u)_l - (H_0 u)_r)$$

\Rightarrow time discretization by averaging the fluxes over time interval $[t^n, t^{n+1}]$

$$F_{j+\frac{1}{2}} = \frac{1}{Dt} \int_{t^n}^{t^{n+1}} f_1(u) dt = \frac{1}{Dt} \int_{t^n}^{t^{n+1}} (Hu)_{j+\frac{1}{2}} dt = (Hu)_{j+\frac{1}{2}}$$

$$= \frac{1}{Dt} \int_{t^n}^{t^{n+1}} f_2(u) dt = \frac{1}{Dt} \int_{t^n}^{t^{n+1}} (gu)_{j+\frac{1}{2}} dt = (gu)_{j+\frac{1}{2}}$$

using these in eqn (A);

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{L_j} \left[F_{j+\frac{1}{2}}(U_j^n, U_{j+1}^n) - F_{j-\frac{1}{2}}(U_{j-1}^n, U_j^n) \right]$$

(A)

$$\Rightarrow F_{1, j+\frac{1}{2}} = \frac{1}{2} \left((H_0 U)_j^n + (H_0 U)_{j+1}^n \right) + C_0 \left(h_j^n - h_{j+1}^n \right)$$

$$F_{2, j+\frac{1}{2}} = \frac{1}{2} \left((g_h)_j^n + (g_h)_{j+1}^n \right) + \frac{1}{2C_0} \left((H_0 U)_j^n - (H_0 U)_{j+1}^n \right)$$

Analogously for

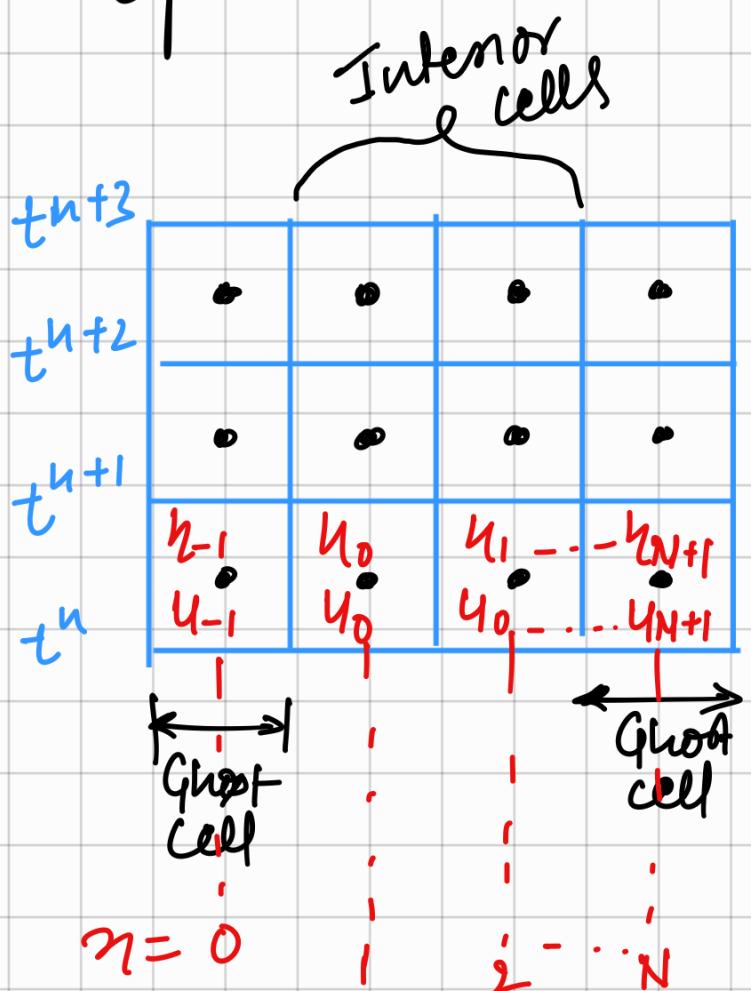
$$F_{1, j-\frac{1}{2}} = \frac{1}{2} \left((H_0 U)_{j-1}^n + (H_0 U)_j^n \right) + C_0 \left(h_{j-1}^n - h_j^n \right)$$

$$F_{2, j-\frac{1}{2}} = \frac{1}{2} \left(h_{j-1}^n + h_j^n \right) + \frac{1}{2C_0} \left((H_0 U)_{j-1}^n - (H_0 U)_j^n \right)$$

\Rightarrow Extrapolating the Boundary condition

(1)

open :-



at $n=0$, left boundary ($n=0$)

$$\Leftrightarrow h_0 = h_{-1} \quad \& \quad u_{-1} = u_0$$

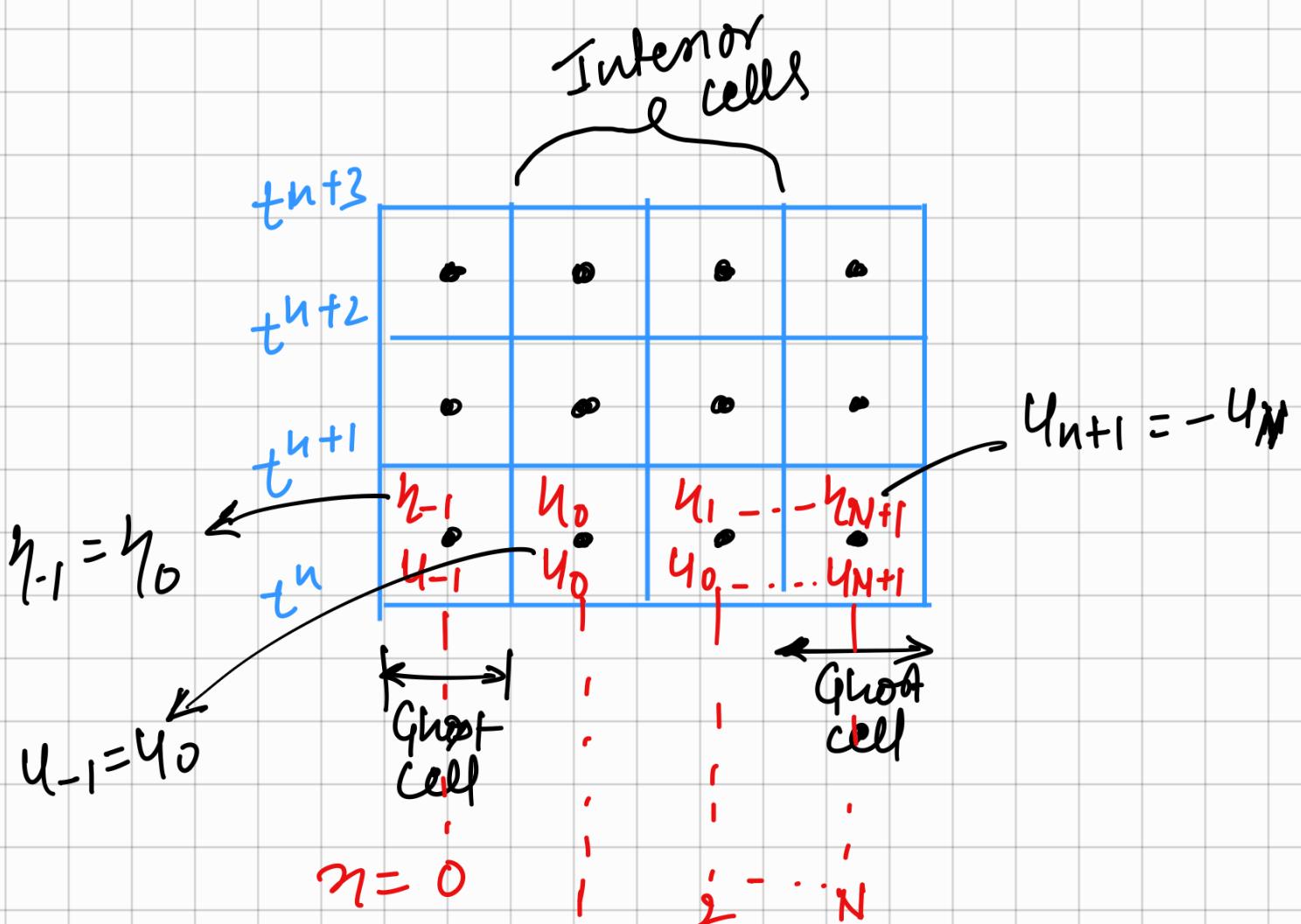
& for first interior cell
 u_0 & u_0

at $n=N$ (right boundary)

$$h_{N+1} = h_N \quad \& \quad u_{N+1} = u_N$$

(2) closed :-

- No flux pass through boundary
- Taking $u_{-1} = -u_0$ (-ve velocity)
 $h_{-1} = -h_0$
- $f_{1/2} = 0 \quad \& \quad f_{N+1/2} = 0$



\Rightarrow for time steps estimation,
restricting CFL condition

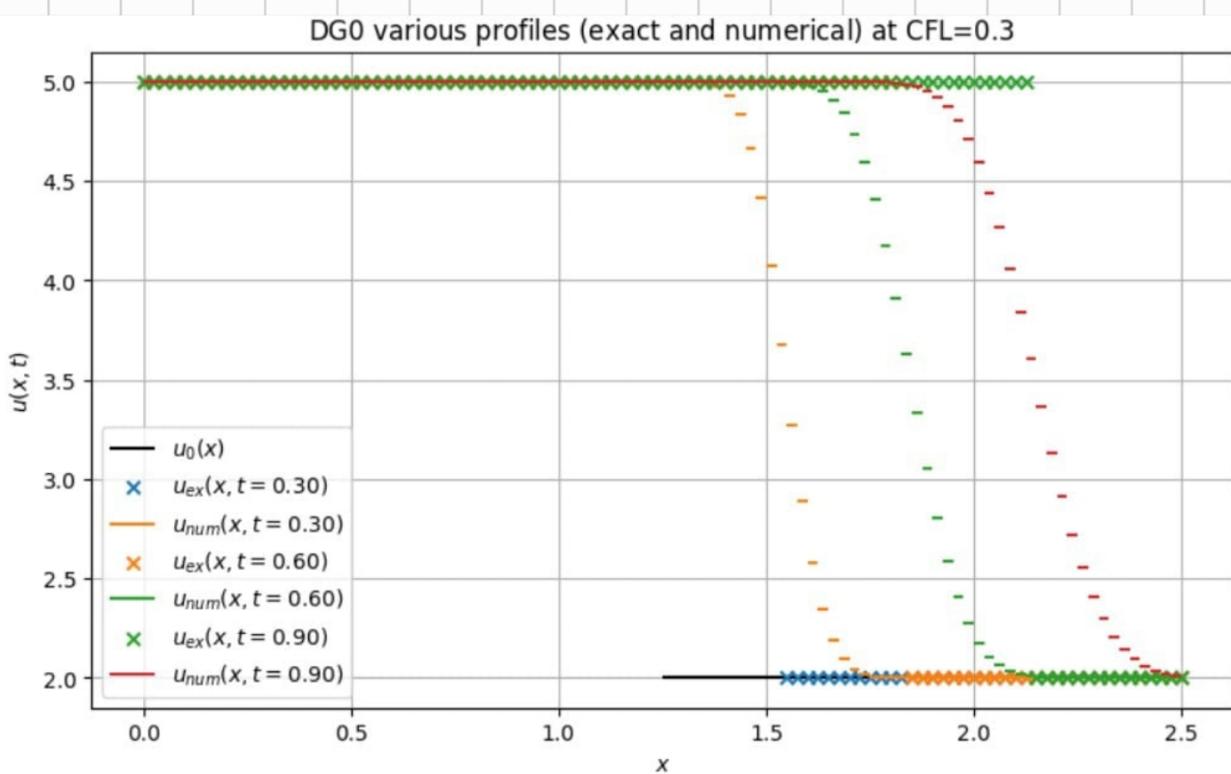
$$\Delta t \leq \frac{\Delta x}{\max|\lambda|}$$

$\lambda_1, \lambda_2 \rightarrow$ eigen values

- Assumption of $H(x)$ is locally constant at cell edge because,
 \rightarrow The FVM Scheme computes fluxes based on cell-averaged quantities, which naturally align with the approximation of $H(x)$ as locally constant.

Question 4o - Implemented Godunov Scheme for (1) in firecracker as DG0, showing various profiles (exact & numerical) at various $CFL = 0.3, 0.5, 0.9$, used actual dimensions time in three loop.

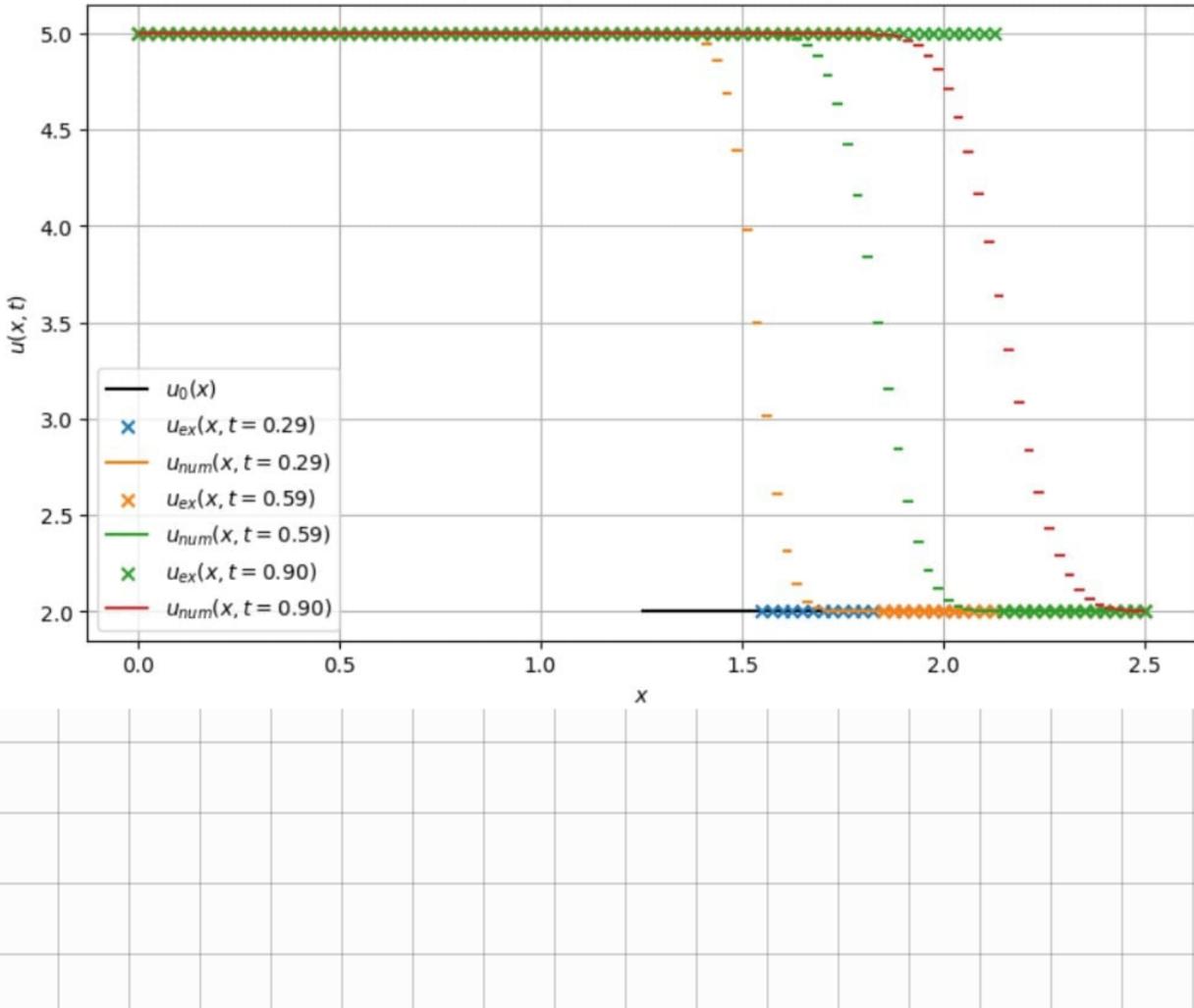
$$\text{used } U_l = 5 \text{ & } U_r = 2, \\ q \text{ or } H_0 = 1$$



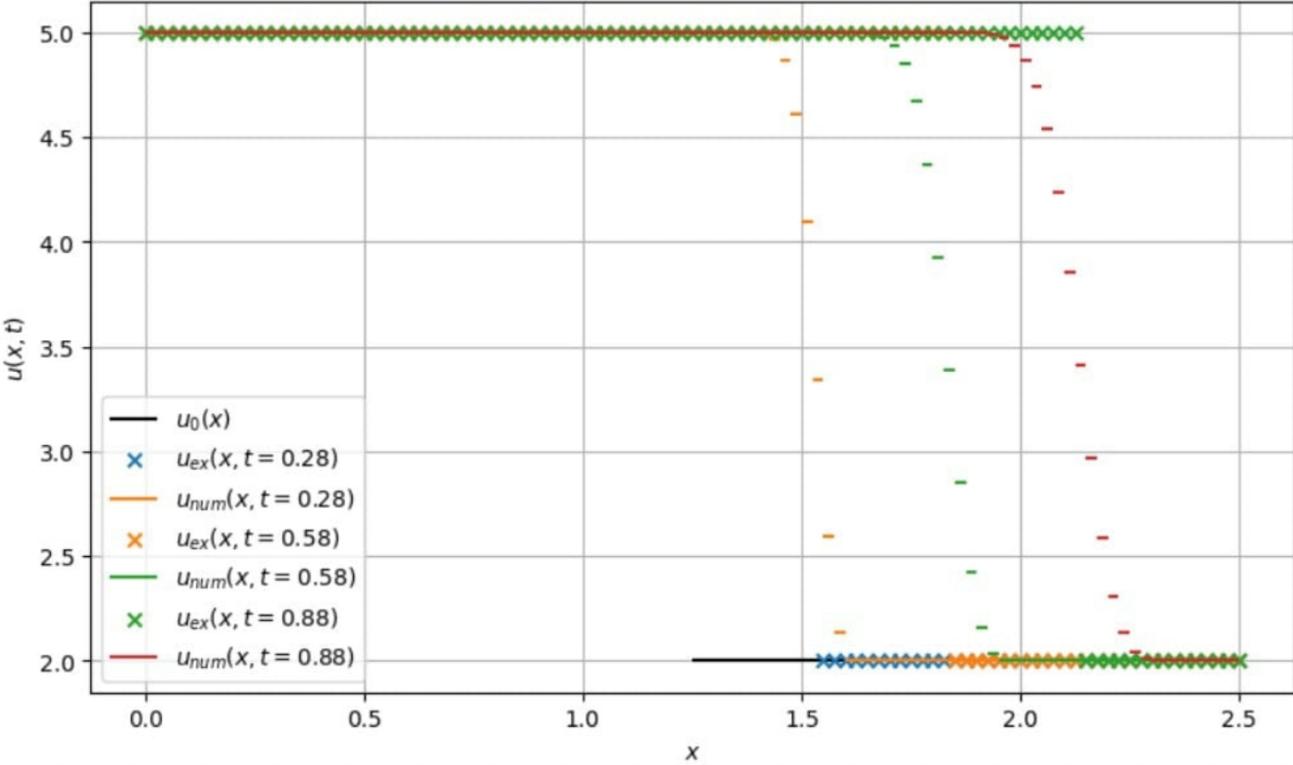
$$\Delta t = CFL \times \left(\frac{\text{mesh size}}{\text{abs}(q)} \right)$$

- Across each interface, the flux leaving one element is exactly the flux entering the next. Hence, no loss or gain occurs. Therefore FVM scheme is conservative

DG0 various profiles (exact and numerical) at CFL=0.5



DG0 various profiles (exact and numerical) at CFL=0.8



Ques 6 :-

using code SweDFV.py, implemented
the the solid-wall boundary by
considering $\theta = 0.5$

file Code is provided in problem6.py

Tend 12.566370614359172

energy 0 0.0

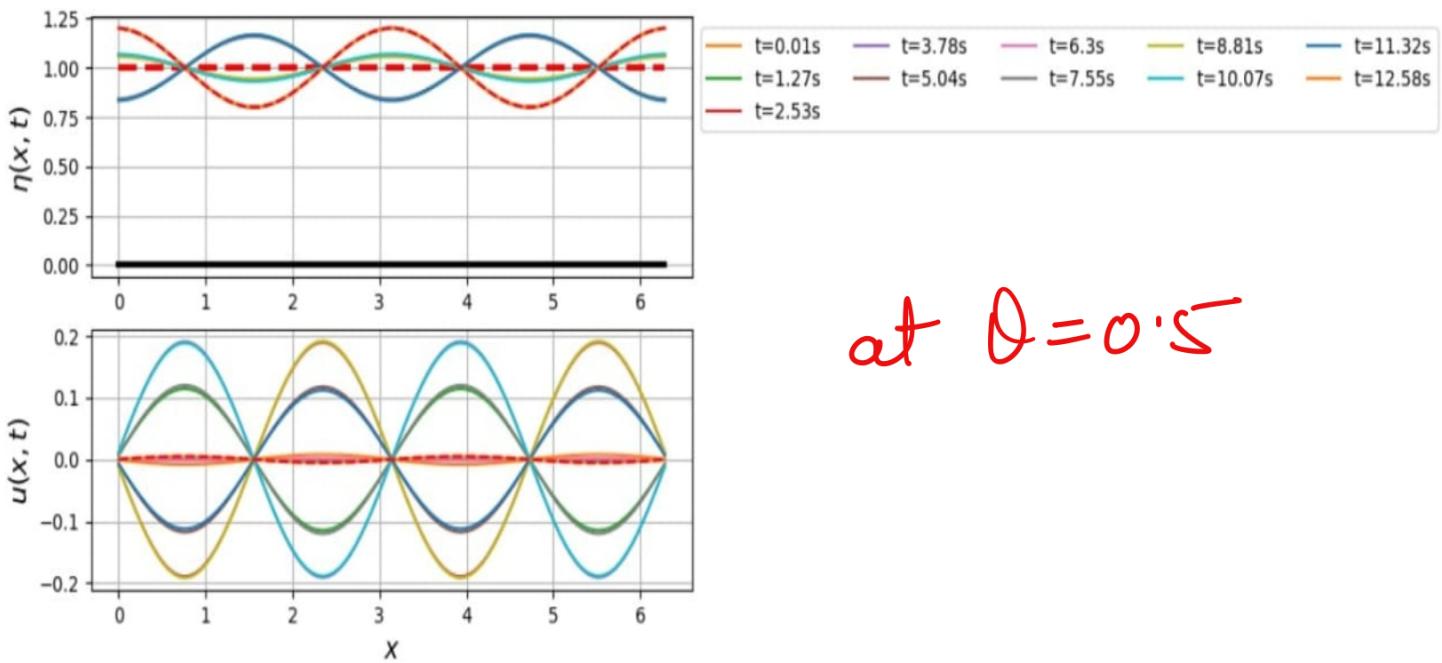
Prior to start time loop. Hallo!

Elapsed time (min): 0.06793102423350016

***** Convergence check at final time t=12.5789 *****

L2 error in eta = 3.562162e-04

L2 error in u = 2.321674e-02



at $\theta = 0.5$

Exact Solution :-

Solid wall boundary condition

