Numerical 2

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1 Q1

Linearized shallow water equations:

$$\frac{\partial \eta}{\partial t} + \frac{\partial Hu}{\partial x} = 0$$
$$\frac{\partial u}{\partial t} + \frac{\partial g\eta}{\partial x} = 0$$

Dimensionless form:

$$(\frac{U_0 H_{0s}}{L_s})\frac{\partial \eta^{'}}{\partial t'} + (\frac{U_0 H_{0s}}{L_s})\frac{\partial H u^{'}}{\partial x'} = 0$$
$$(\frac{U_0^2}{L_s})\frac{\partial u^{'}}{\partial t'} + (\frac{U_0^2}{L_s})\frac{\partial g^{'}\eta^{'}}{\partial x'} = 0$$

Eliminate the coefficients:

$$\frac{\partial \eta^{'}}{\partial t'} + \frac{\partial H u^{'}}{\partial x'} = 0$$
$$\frac{\partial u^{'}}{\partial t'} + \frac{\partial g^{'}\eta^{'}}{\partial x'} = 0$$

Which is exactly same as (1). Given $H = H_0$, multiply both sides of the second equation by H_0:

$$\frac{\partial \eta}{\partial t} + H_0\frac{\partial u}{\partial x} = 0$$
$$H_0\frac{\partial u}{\partial t} + H_0 g\frac{\partial \eta}{\partial x} = 0$$

Which can be written as:

$$\partial_t f + A\partial_x f = 0$$
$$f = \begin{bmatrix} \eta \\ H_0 u \end{bmatrix}$$

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$Af = \begin{bmatrix} H_0 u \\ H_0 g\eta \end{bmatrix}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \eta \\ H_0 u \end{bmatrix} = \begin{bmatrix} H_0 u \\ H_0 g\eta \end{bmatrix}$$
$$a\eta + bH_0 u = H_0 u$$
$$c\eta + dH_0 u = H_0 g\eta$$
$$a = 0, b = 1, c = H_0 g, d = 0$$

The equations can be written in this form:

$$\partial_t \begin{bmatrix} \eta \\ H_0 u \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ gH_0 & 0 \end{bmatrix} \partial_x \begin{bmatrix} \eta \\ H_0 u \end{bmatrix} = 0$$

With $c_0^2 = gH_0$

$$A = \begin{bmatrix} 0 & 1 \\ c_0^2 & 0 \end{bmatrix}$$
$$det(A - \lambda I) = 0$$
$$\begin{vmatrix} -\lambda & 1 \\ c_0^2 & -\lambda \end{vmatrix} = 0$$

Two eigenvalues:$\lambda_1 = c_0, \lambda_2 = -c_0$

$$B^{-1} = \frac{1}{det(B)} \begin{bmatrix} \frac{1}{2} & \frac{-1}{c_0} \\ -\frac{1}{2} & \frac{1}{2c_0} \end{bmatrix} = \begin{bmatrix} c_0 & 1 \\ -c_0 & 1 \end{bmatrix}$$
$$B^{-1}AB = \begin{bmatrix} c_0 & 1 \\ -c_0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ c_0^2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ c_0 & c_0 \end{bmatrix} \frac{1}{2c_0} = \begin{bmatrix} c_o & 0 \\ 0 & -c_0 \end{bmatrix}$$

Riemann invariants: $r_1 = H_0 u + c_0\eta, r_2 = H_0 u - c_0\eta$

$$\frac{\partial r_1}{\partial t} = H_0 \frac{\partial u}{\partial t} + c_0 \frac{\partial \eta}{\partial t}$$

substitute (1) in

$$\frac{\partial r_1}{\partial t} = -H_0 g \frac{\partial \eta}{\partial x} - c_0 H_o \frac{\partial u}{\partial x}$$

replace g with c_0^2/H_0

$$\frac{\partial r_1}{\partial t} = -c_0^2\frac{\partial \eta}{\partial x} - c_0 H_o \frac{\partial u}{\partial x}$$
$$\frac{\partial r_1}{\partial t} = -c_0(H_0\frac{\partial u}{\partial x} + c_0\frac{\partial \eta}{\partial x})$$
$$\frac{\partial r_1}{\partial t} + c_o\frac{\partial r_1}{\partial x} = 0$$

same way use in r_2

$$\frac{\partial r_2}{\partial t} - c_o\frac{\partial r_2}{\partial x} = 0$$

2 Q2

Piecewise constant initial condition:

$$r_1(x,0) = \begin{cases} r_{1L}, & x<0 \\ r_{1R}, & x>0 \end{cases}$$
$$r_2(x,0) = \begin{cases} r_{2L}, & x<0 \\ r_{2R}, & x>0 \end{cases}$$

The characteristic curve:

$$x = x_0 + c_0 t, x = x_0 - c_0 t$$

Which means $r_1$and r_2 propagate at constant speed $\pm c_0$, so at time t ($r_1$moves right at c_o, $r_2$moves left at -c_0)

$$r_1(x,t) = \begin{cases} r_{1L}, & x<0+c_o t \\ r_{1R}, & x>0+c_o t \end{cases}$$
$$r_2(x,t) = \begin{cases} r_{2L}, & x<0-c_0 t \\ r_{2R}, & x>0-c_0 t \end{cases}$$

With Riemann invariants $r_1 = H_0 u + c_0\eta, r_2 = H_0 u - c_0\eta$

Consider:
for $x < -c_0 t$: $r_1(x,t) = r_{1L}, r_2(x,t) = r_{2L}$
for $-c_0 t < x < c_0 t$: $r_1(x,t) = r_{1L}, r_2(x,t) = r_{2R}$
for $x > c_0 t$: $r_1(x,t) = r_{1R}, r_2(x,t) = r_{2R}$

$$H_0 u(x,t) = \frac{1}{2}(r_1(x,t) + r_2(x,t))$$

$$H_0u(x,t)=\frac{1}{2}(r_1(x,t)+r_2(x,t))=\begin{cases}H_0u_L, & x<-c_0t\\ \frac{H_0(u_L+u_R)+c_0(\eta_L-\eta_R)}{2}, & -c_0t<x<c_0t\\ H_0u_R, & x>c_0t\end{cases}$$

$$\eta(x,t)=\frac{1}{2}(r_1(x,t)-r_2(x,t))/c_0$$

$$\eta(x,t)=\frac{1}{2c_0}(r_1(x,t)-r_2(x,t))=\begin{cases}\eta_L, & x<-c_0t\\ \frac{H_0(u_L-u_R)+c_0(\eta_L+\eta_R)}{2c_0}, & -c_0t<x<c_0t\\ \eta_R, & x>c_0t\end{cases}$$

$$u(x,t)=\begin{cases}\frac{r_{1L}+r_{2L}}{2H_0}, & x<-c_0t\\ \frac{r_{1L}+r_{2R}}{2H_0}, & -c_0t<x<c_0t\\ \frac{r_{1R}+r_{2R}}{2H_0}, & x>c_0t\end{cases}$$

$$\eta(x,t)=\begin{cases}\frac{r_{1L}-r_{2L}}{2c_0}, & x<-c_0t\\ \frac{r_{1L}-r_{2R}}{2c_0}, & -c_0t<x<c_0t\\ \frac{r_{1R}-r_{2R}}{2c_0}, & x>c_0t\end{cases}$$

3 Q3

Let $u_1=\eta, F_1=H_0u, u_2=H_0u, F_2=c_0^2u_1$

$$\frac{\partial u_1}{\partial t}+\frac{\partial F_1}{\partial x}=0$$
$$\frac{\partial u_2}{\partial t}+\frac{\partial F_2}{\partial x}=0$$

Where the flux F_1 and F_2

$$F_1=H_0u, F_2=c_0^2\eta$$

The discrete update equations based on Godunov's method for a time step Δt

$$u_{1i}^{n+1}=u_{1i}^n-\frac{\Delta t}{\Delta x}(F_{1,i+1/2}-F_{1,i-1/2})$$
$$u_{2i}^{n+1}=u_{2i}^n-\frac{\Delta t}{\Delta x}(F_{2,i+1/2}-F_{2,i-1/2})$$

The time step must satisfy the CFL condition which should account for the maximum wave speed over the entire domain:

$$\frac{\Delta t}{\Delta x}\leq\frac{1}{c_0}$$

Where c_0 is the maximum wave speed, so the time step:

$$\Delta t \leq \frac{\Delta x}{c_0} = \frac{\Delta x}{\sqrt{gH_o}}$$

For and open domain(extrapolating boundary conditions):
Left boundary(at x=0):

$$\eta_{-1} = \eta_0, u_{-1} = u_0$$

Right boundary(at x=L):

$$\eta_{N+1} = \eta_N, u_{N+1} = u_N$$

For closed boundary conditions(ghost values):
Left boundary(at x=0):

$$\eta_{-1} = \eta_0, u_{-1} = -u_0$$

Right boundary(at x=L):

$$\eta_{N+1} = \eta_N, u_{N+1} = -u_N$$