Onno Assignment 2

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1 Question 1

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0$$

and (1)

$$\frac{\partial u}{\partial t} + \frac{\partial (g\eta)}{\partial x} = 0$$

where $[x]$ denotes dimension of x, length scale $= L$, time scale $= T$

Velocity $u = U_0 u'$, $u' = \frac{u}{U_0}$, $\frac{[u]}{[U_0]} = [u'] = 1$.
Deviation $x = L_s x'$, $x' = \frac{x}{L_s}$, $\frac{[x]}{[L_s]} = [x'] = 1$.
Time, distance over speed, $t = \frac{L_s}{U_0} t'$, $t' = \frac{tU_0}{L_s}$, $\frac{[T][X]}{[T][X]} = t' = 1$

And so forth for the other quantities. Therefore, if each of these is substituted into the coupled equation we obtain a non-dimensional version of the equations

$$\frac{\partial H_0 \eta'}{\partial \frac{L_s}{U_0} t'} + \frac{\partial H_0 H' U_0 u'}{\partial L_s x'} = 0$$

and

$$\frac{\partial U_0 u'}{\partial \frac{L_s}{U_0} t'} + \frac{\partial g' \eta}{\partial L_s x'} = 0$$

For the sake of notational simplicity we write this in the same form as equation 1) and drop the primes.

Taking the case where $H(x) = H_0$, $U_0^2 = gH_0$

$$\frac{\partial}{\partial \eta} \begin{bmatrix} \eta \\ H_0 \mu \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ gH_0 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \eta \\ H_0 u \end{bmatrix} = 0$$

Finding the eigenvectors of the matrix, which we label A,

$$\det(A - \lambda I) = 0$$
$$det(\begin{bmatrix} -\lambda & 1 \\ gH_0 & -\lambda \end{bmatrix}) = 0$$

$$(-\lambda)^2 - gH_0 = 0$$
$$\lambda^2 = gH_0$$
$$\lambda = \pm\sqrt{gH_0}$$

Defining the matrix B as,

$$B = \frac{1}{2\sqrt{gH_0}} \begin{bmatrix} 1 & -1 \\ \sqrt{gH_0} & \sqrt{gH_0} \end{bmatrix}$$

$$AB = \begin{bmatrix} \frac{\sqrt{gH_0}}{2\sqrt{gH_0}} & \frac{gH_0}{2\sqrt{gH_0}} \\ \frac{gH_0}{2\sqrt{gH_0}} & \frac{-gH_0}{2\sqrt{gH_0}} \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \sqrt{gH_0} & 1 \\ -\sqrt{gH_0} & 1 \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} \frac{gH_0}{2\sqrt{gH_0}} + \frac{gH_0}{2\sqrt{gH_0}} & \frac{gH_0}{2\sqrt{gH_0}} - \frac{gH_0}{2\sqrt{gH_0}} \\ \frac{-gH_0}{2\sqrt{gH_0}} + \frac{gH_0}{2\sqrt{gH_0}} & \frac{-gH_0}{2\sqrt{gH_0}} - \frac{gH_0}{2\sqrt{gH_0}} \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} \frac{1}{2}\sqrt{gH_0} + \frac{1}{2}\sqrt{gH_0} & \frac{1}{2}\sqrt{gH_0} - \frac{1}{2}\sqrt{gH_0} \\ \frac{-1}{2}\sqrt{gH_0} + \frac{1}{2}\sqrt{gH_0} & \frac{-1}{2}\sqrt{gH_0} - \frac{1}{2}\sqrt{gH_0} \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} \sqrt{gH_0} & 0 \\ 0 & -\sqrt{gH_0} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} I$$

Using

$$\frac{\partial H_0 \eta'}{\partial \frac{L_s}{U_0} t'} + \frac{\partial H_0 H' U_0 u'}{\partial L_s x'} = 0$$

and

$$\frac{\partial U_0 u'}{\partial \frac{L_s}{U_0} t'} + \frac{\partial g' \eta'}{\partial L_s x'} = 0$$

we obtain, by setting $U_0^2 = gH_0$

$$\frac{\partial H_0\eta'}{\partial\frac{L_s}{\sqrt{gH_0}}t'}+\frac{\partial H_0H'\sqrt{gH_0}u'}{\partial L_sx'}=0$$

and

$$\frac{\partial\sqrt{gH_0}u'}{\partial\frac{L_s}{\sqrt{gH_0}}t'}+\frac{\partial g'\eta'}{\partial L_sx'}=0$$

Multiplying by L_s

$$\frac{\partial H_0\eta'\sqrt{gH_0}}{\partial t'}+\frac{\partial H_0H'\sqrt{gH_0}u'}{\partial x'}=0$$

$$\frac{\partial 2\sqrt{gH_0}u'}{\partial t'}+\frac{\partial g'\eta'}{\partial x'}=0$$

Subtracting the bottom equation from the top equation, and noting that $g'=1$ when $U_0^2=gH_0$

$$\frac{\partial H_0\eta'\sqrt{gH_0}-\partial 2\sqrt{gH_0}}{\partial t'}+\frac{\partial H_0H'\sqrt{gH_0}u'-\partial\eta'}{\partial x'}=0$$

$$\frac{\partial\sqrt{gH_0}(H_0\eta'-2)}{\partial t'}+\frac{\partial H_0H'\sqrt{gH_0}u'-\partial\eta'}{\partial x'}=0$$

2 Question 2

We have the initial conditions

$$r_1(x,0)=\begin{cases}r_{1l} & x<0\\ r_{lr} & x\geq 0\end{cases},\, r_2(x,0)=\begin{cases}r_{2l} & x<0\\ r_{2r} & x\geq 0\end{cases}$$

assuming we are looking at a rarefaction wave at time $t=0$. Our characteristics are:

$c_0t=\sqrt{gH_0}t$ for r_1,
$-c_0t=-\sqrt{gH_0}t$ for r_2.

each wave moves at a constant velocity along the characteristics, figure 1. It is clear that if the diving line between the left and right wave is at $x=0$ for $t=0$, then it will move to c_0t for time $t=t$ in the the right moving wave r_1 and $-c_0t$ for time $t=t$ in the left moving wave r_2, yielding

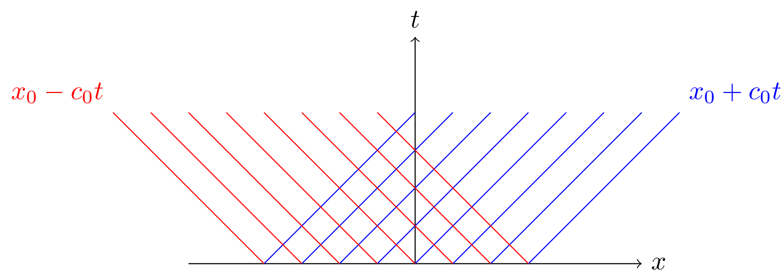

Figure 1: Characteristics.

$$r_1(x,t) = \begin{cases} r_{1l} & x < c_0t \\ r_{lr} & x \geq c_0t \end{cases}, r_2(x,t) = \begin{cases} r_{2l} & x < -c_0t \\ r_{2r} & x \geq -c_0t \end{cases}$$

Given $H_0u = \frac{1}{2}(r_1 + r_2)$, $\eta = \frac{1}{2}(r_1 - r_2)/c_0$, $r_1 = H_0u + c_0\eta$, $r_2 = H_0u - c_0\eta$: substituting our previous piecewise solution of the Reimann problem for r_1, r_2

$$H_0u(x,t) = \frac{1}{2}(r_1(x,t) + r_2(x,t))$$

$$\eta(x,t) = \frac{1}{2}(r_2(x,t) - r_1(x,t))/c_0$$

By inspection, we can see that the pure left wave will be when $x < -c_0t$ and the pure right wave when $x > c_0t$.

$$H_0u(x,t) = \frac{1}{2}(r_1(x,t) + r_2(x,t)) = \begin{cases} H_0u_l & x < -c_0t \\ H_0u_r & x > c_0t \end{cases}$$

Then

$$H_0u(x,t) = \frac{1}{2}(r_1(x,t) + r_2(x,t)) = \begin{cases} H_0u_l & x < -c_0t \\ \frac{H_0(u_l+u_r)}{2} + \frac{c_0(\eta_l-\eta_r)}{2} & -c_0t < x < c_0t \\ H_0u_r & x > c_0t \end{cases}$$

$$\eta(x,t) = \frac{1}{2}(r_1(x,t) - r_2(x,t))/c_0 = \begin{cases} \eta_l & x < -c_0t \\ \frac{H_0(u_l-u_r)}{2c_0} + \frac{(\eta_l+\eta_r)}{2} & -c_0t < x < c_0t \\ \eta_r & x > c_0t \end{cases}$$

From the initial conditions, at $r_1(x,0)$, $r_2(x,0)$, we can see we have a rarefaction wave and $u_l < u_r$.

Writing in the form

$$\partial_t \mathbf{r} + \mathbf{A}\partial_{\mathbf{x}}\mathbf{r} = \mathbf{0}$$

where $\mathbf{r} = \begin{bmatrix} H_0 u + c_0\eta \\ H_0 u - c_0\eta \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} \sqrt{gH_0} & 0 \\ 0 & -\sqrt{gH_0} \end{bmatrix}$

we obtain two equations

$$\frac{\partial H_0 u + c_0\eta}{\partial t} + \frac{\partial\sqrt{gH_0}(H_0 u + c_0\eta)}{\partial x} = 0$$

$$\frac{\partial H_0 u - c_0\eta}{\partial t} + \frac{\partial - \sqrt{gH_0}(H_0 u - c_0\eta)}{\partial x} = 0$$

which can be written in the form

$$\frac{\partial f(u)}{\partial t} + c\frac{f(u)}{\partial x} = 0$$

which is the form of the general linear advection equation where c is the characteristics of the system.

from $\mathbf{r}$,

$$r_1 = H_0 u + c_0\eta$$
$$r_2 = H_0 u - c_0\eta$$

$$r_{1_l} = H_0 u_l + c_0\eta_l$$
$$r_{1_r} = H_0 u_r + c_0\eta_r$$
$$r_{2_l} = H_0 u_l - c_0\eta_l$$
$$r_{2_r} = H_0 u_r - c_0\eta_r$$

therefore

$$r_1(x,t) = \begin{cases} H_0 u_l + c_0\eta_l & x < c_0 t \\ H_0 u_r + c_0\eta_r & x \geq c_0 t \end{cases}$$

$$r_2(x,t) = \begin{cases} H_0 u_l - c_0\eta_l & x < c_0 t \\ H_0 u_r - c_0\eta_r & x \geq c_0 t \end{cases}$$

3 Question 3

$$r_1(x,0) = H_0 u_l + c_0\eta_l, x < 0$$
$$r_1(x,0) = H_0 u_r + c_0\eta_r, x \geq 0$$
$$r_2(x,0) = H_0 u_l - c_0\eta_l, x < 0$$
$$r_2(x,0) = H_0 u_r - c_0\eta_r, x \geq 0$$

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{h}[F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)]$$

where

$$F(U_j^n, U_{j+1}^n) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{f}_{j+\frac{1}{2}}(t)dt = \mathbf{f}_{j+\frac{1}{2}}\mathbf{u}(j+\frac{1}{2}, t)$$

because the Riemann solution is constant, giving the result of the integration as $t\mathbf{f}_{j+\frac{1}{2}}\mathbf{u}(j+\frac{1}{2}, t)\frac{1}{\Delta t}$ and in the limit $\lim_{t\to 0} t = \Delta t$.

using our Riemann solution in $H_0 u(x,t)$, $\eta(x,t)$, evaluated at $x = 0$

$$\mathbf{f}_{j+\frac{1}{2}}\mathbf{u}(j+\frac{1}{2}, t) = \begin{bmatrix} \frac{H_0(u_l+u_r)}{2} + \frac{c_0(\eta_l-\eta_r)}{2} \\ \frac{H_0(u_l-u_r)}{2c_0} + \frac{(\eta_l+\eta_r)}{2} \end{bmatrix}$$

With an "open" domain we simply loop wrap around at the edges, such that we have, at the right hand edge and left hand edge respectively, where N is the number of volumes

$$U_j^{n+1} = U_N^n - \frac{\Delta t}{h}[F(U_N^n, U_0^n) - F(U_{N-1}^n, U_N^n)]$$

$$\mathbf{f}_{j+\frac{1}{2}}\mathbf{u}(j+\frac{1}{2}, t) = \begin{bmatrix} \frac{H_0(u_{r_N}+u_{r_0})}{2} + \frac{c_0(\eta_{r_N}-\eta_{r_0})}{2} \\ \frac{H_0(u_{r_N}-u_{r_0})}{2c_0} + \frac{(\eta_{r_N}+\eta_{r_0})}{2} \end{bmatrix}$$

and

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{h}[F(U_0^n, U_1^n) - F(U_N^n, U_0^n)$$

$$\mathbf{f}_{j+\frac{1}{2}}\mathbf{u}(j+\frac{1}{2},t) = \begin{bmatrix} \frac{H_0(u_{l_N}+u_{l_0})}{2c_0} - \frac{\eta_{l_N}-\eta_{l_0}}{2} \\ \frac{H_0(u_{l_N}-u_{l_0})}{2} + \frac{c_0(\eta_{l_N}+\eta_{l_0})}{2} \end{bmatrix}$$

For a closed domain we instead use ”ghost” values at the wall. At the wall we have

$$\partial_t H_0 u + \partial_x c_o^2 \eta = 0$$
$$\partial_x c_o^2 \eta = -\partial_t H_0 u$$

In order to which way around the signs are on the ghost values we can think physically about the system. In order for the free surface deviation to be unchanging it must be the same on either side of the wall, as having a negative on one side would create a discontinuity which the system would try to ”even out”. This leaves the negative to be assigned to the velocity. This also makes sense as we are canceling out the velocity in the final volume by meeting it with an opposite velocity on the wall side. This forces the solution at the wall to be constant.

Therefore, we create two volumes outside the solution domain at indices -1 and N+1 with,

$$\mathbf{u} = \begin{bmatrix} \eta_0 \\ -H_0 u_0 \end{bmatrix}$$

and

$$\mathbf{u} = \begin{bmatrix} \eta_N \\ -H_0 u_N \end{bmatrix}$$

The timestep is determined with the CFL condition $\Delta t \leq \frac{\Delta x}{|a|}$ where a is the wave speed. For a constant cell width h and the speed of the wave c_0, determined by the characteristics.

$$\Delta t \leq \frac{h}{c_0}$$

or, for a cell width which varies in each volume h_j

$$\Delta t \leq min(\frac{h_j}{c_0})$$

The scheme can be extended for varying $H(x)$ by simply taking a local, constant, approximation $H(x) = H_0$ within each volume. In this case the existing

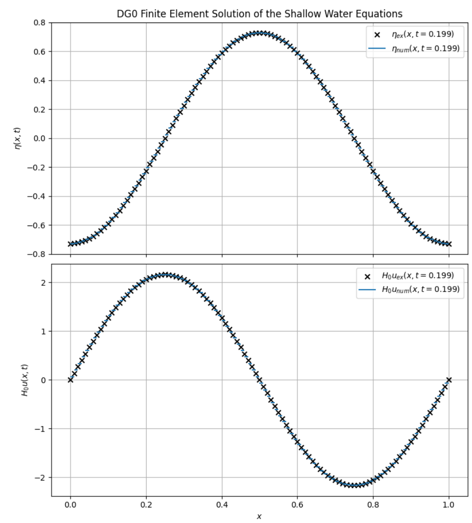

Figure 2: The visual convergence of the finite volume scheme against the exact solution for $4T_p$ at CFL=1, 100 elements

mathematics holds without modification. This is an approximation. However, it is the same kind of approximation we have already made for the velocity and free surface deviation terms, i.e we treat a continuous function as locally constant within some small limit and then sum over these local approximations. The overall logic holds that any continuous function $f(x)$ can be locally approximated as a linear function in the limit Δx. This is a natural consequence of the Taylor expansion.

4 Question 4.

The finite volume method is conservative because it calculates the next time step in each cell based on the fluxes into that cell at the current time step. This naturally conserves quantities as there is no way for new energy to be added to the system, with the exception of numerical error at the flux calculation.

Figure 2 shows, visually, the solution finite volume method for the closed boundary case against the exact solution. We could zoom in to get a better idea of the convergence, however given we wish to examine the impact of changing variables and the changes are likely to be subtle it is better to take an exact approach.

The convergence was tested by taking $MAE = \frac{\Sigma_0^n abs(u-u_a)}{n}$, i.e. the mean of the absolute error for each element. This is know as mean absolute error or MAE. Table 1 shows how the MAE changes as the grid is refined. We see that the error has only converged to the $\approx 1e^{-}2$ level. The error scales linearly with the number of elements $E \approx \frac{1}{M}$ or $E \approx \Delta x$ where M is the number of elements and $\Delta x = \frac{1}{M}$ and is the size of each element. As such, obtaining a good error would require with this scheme would require a very large mesh and a lot of computational time. For this reason, all further experiments were conducted with $M = 100$.

Elements	25	50	75	100
Mean error η	0.047023	0.027382	0.0190526	0.014579
Mean error H_0U	0.204263	0.092208	0.058988	0.04329

Table 1: MAE while changing the number of elements for time period $4t_p$, CFL=1

Table 2 shows how the error changes as the time period over which the solution is sought increases. We can see that at CFL=1 and $M = 100$ the error does not change to 6 significant figures in the interval $[4T_p, 10T_p]$. A brief test $M = 25$ showed no change in errors when changing time from $4T_p$ to $10T_p$ either (errors remained as reported in table 1). This is unexpected as we anticipate an increase in error.

Time	$4T_p$	$5T_p$	$9T_p$	$10T_p$
Mean error η	0.014579	0.014579	0.014579	0.014579
Mean error H_0U	0.043295	0.043295	0.043295	0.043295

Table 2: Changing time period for 100 elements, CFL=1

Table 3 shows how reducing the CFL number changes the error. We observe that as we decrease the CFL number the error actually increases. The reason for this is not immediately obvious. However, we observe that the system is already stable at CFL=1. Reducing the CFL number further does not add any further stability to the system as the timestep is already held to the correct size for the "flow" of information in the system. As we decrease the time step, beyond that required for stability, we begin to accumulate errors. At each time step we are making an approximation to the exact solution at each spatial point. This necessarily discards information and causes some error. As we do this more times we decreased Δt we accumulate more of these errors. We also have more opportunities to accumulate any arithmetic errors (e.g. round-off) which may be occurring.

CFL	1.0	0.5	0.2	0.1
Mean error η	0.014579	0.031347	0.047222	0.052451
Mean error H_0U	0.043295	0.093088	0.133791	0.147513

Table 3: Changing the CFL, for time period $4T_p$ and 100 elements

5

There was no question here?

6

See code, adapted from code given.

7

Figure 3 shows the experiment with alternating flux, for the same set of parameters as used in figure 2. We can immediately see a degradation in accuracy, particularly around the peaks solution. Figure 4. Shows the energy in the alternating flux solution. We can clearly see that the solution is non-conservative. Energy is increasing over time. This is the reason for the degraded accuracy, despite the scheme being ”better” on paper.

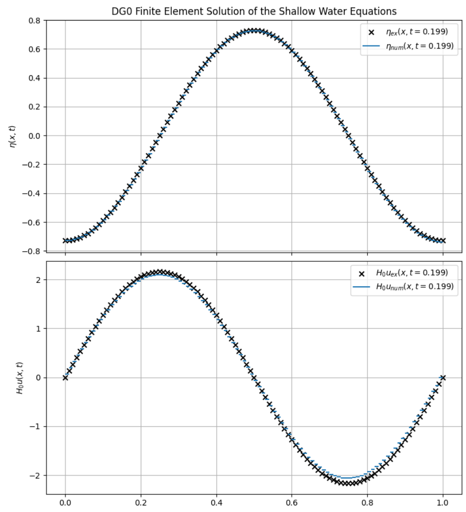

Figure 3: The visual convergence of the alternating flux scheme against the exact solution for $4T_p$ at CFL=1, 100 elements

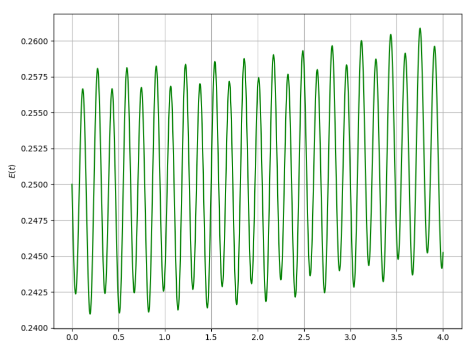

Figure 4: Energy in the alternating flux solution.