

# MATH5453M Numerical Exercise 1.

Given :

$$u_t - a(t)u_x - \epsilon u_{xx} = 0 \quad \text{for } x \in [l_p, L]$$

$$u(x, 0) = u_0(x)$$

$$u(l_p, t) = u(L, t) = 0,$$

(small) constant diffusion  $\epsilon$  & a given function  $a(t)$ . The B.C. are classical homogeneous Dirichlet conditions.

Solutions

Problem 1:-

$$u_t - a(t)u_x - \epsilon u_{xx} = 0 \quad \text{for } x \in [l_p, L]$$

- The above given equation is linear because dependent variable  $u(x, t)$  and its derivative appears linear in the above equation.

① Linear : if we look into the equation there is no term containing multiplication

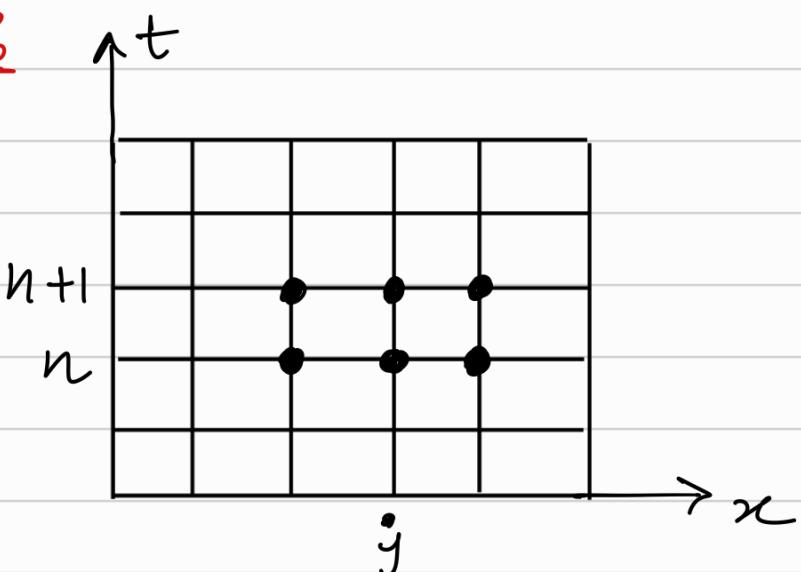
there is no term containing multiplication of  $U(x, t)$  & its derivatives or any power of dependent variable  $U(x, t)$ .

② Advection: ' $-a(t)U_x$ ' represents the advection because it is showing the advection of  $U_x$ .

$$-a(t) \frac{\partial U}{\partial x}$$

③ Diffusion: ' $\epsilon U_{xx}$ '  
 $\downarrow$   
 $\epsilon \frac{\partial^2 U}{\partial x^2}$  → represents the diffusion in the equation, with diffusion coefficient of ' $\epsilon$ '.

Problem 2:



Taylor Series expansion of  $U_j^{n+1}$  &  $U_j^n$  about  $(\frac{\Delta t}{2})$

$$U_j^{n+1} = U_j^n + \frac{1}{2} \Delta t U_t + \frac{1}{2} \left( \frac{1}{2} \Delta t \right)^2 U_{ttt} + \frac{1}{6} \left( \frac{1}{2} \Delta t \right)^3 U_{tttt}$$

$+ \dots \underbrace{\dots}_{j}^{n+1/2}$

$$\text{& } U_j^n = U_j - \frac{1}{2} \Delta t U_t + \frac{1}{2} \left( \frac{1}{2} \Delta t \right)^2 U_{ttt} - \frac{1}{6} \left( \frac{1}{2} \Delta t \right)^3 U_{tttt}$$

$+ \dots \underbrace{\dots}_{j}^{n+1/2}$

Subtracting these two, all odd will cancel

$$U_j^{n+1} - U_j^n = \left[ \Delta t U_t + \frac{1}{24} (\Delta t)^3 U_{tttt} + \dots \right]_j^{n+1/2}$$

$$U_j^{n+1} - U_j^n = \delta_t U_j^{n+1/2}$$

that means, (2.80)

$$\delta_t U_j^{n+1/2} = \left[ \Delta t U_t + \frac{1}{24} (\Delta t)^3 U_{tttt} + \dots \right]_j^{n+1/2} - (2.80)$$

from eqn (2.30),

$$\sum_x^2 U(x, t) = U_{nn} (\Delta x)^2 + \frac{1}{12} U_{nnnn} (\Delta x)^4 + \dots$$

$$\text{& from } \sum_x^2 U_j^{n+1} = \frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{(\Delta x)^2}$$

Expanding

$$U_j^{n+1} = U_j + \Delta n U_{j\bar{x}} + \frac{1}{2} (\Delta n)^2 U_{j\bar{x}\bar{x}} + \frac{1}{6} (\Delta n)^3 U_{j\bar{x}\bar{x}\bar{x}}$$

$$U_{j-1}^{n+1} = U_j - \Delta n U_{j\bar{x}} + \frac{1}{2} (\Delta n)^2 U_{j\bar{x}\bar{x}} - \frac{1}{6} (\Delta n)^3 U_{j\bar{x}\bar{x}\bar{x}}$$

$$\therefore \delta_x^2 U_j^{n+1} = \cancel{\left( U_j + \Delta n U_{j\bar{x}} + \frac{1}{2} (\Delta n)^2 U_{j\bar{x}\bar{x}} + \frac{1}{6} (\Delta n)^3 U_{j\bar{x}\bar{x}\bar{x}} \right)} \\ - \cancel{2U_j} + \cancel{\left( U_j - \Delta n U_{j\bar{x}} + \frac{1}{2} (\Delta n)^2 U_{j\bar{x}\bar{x}} \right)} \\ - \cancel{\frac{1}{6} (\Delta n)^3 U_{j\bar{x}\bar{x}\bar{x}}} \\ \underline{\hspace{10em}} \\ (\Delta n)^2$$

(2.80) ↓

$$\delta_x^2 U_j^{n+1} = U_{j\bar{x}\bar{x}} + \frac{1}{12} (\Delta n)^2 U_{j\bar{x}\bar{x}\bar{x}\bar{x}} + \dots$$

— (2.81)

- Now considering weighted Average of spatial derivatives

$$\theta \delta_x^2 U_j^{n+1} + (1-\theta) \delta_x^2 U_j^n$$

Using expansion of  $\delta_x^2 U_j^{n+1}$  &  $\delta_x^2 U_j^n$

$$\delta_x^2 U_j^{n+1} = \left[ (\Delta n)^2 U_{j\bar{x}\bar{x}} + \frac{1}{12} (\Delta n)^4 U_{j\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{1}{12} (\Delta n)^6 U_{j\bar{x}\bar{x}\bar{x}\bar{x}\bar{x}} + \dots \right]^{n+1} \\ + \frac{1}{2} \theta t \left[ (\Delta n)^2 U_{j\bar{x}\bar{x}t} + \frac{1}{12} (\Delta n)^4 U_{j\bar{x}\bar{x}\bar{x}x} + \dots \right]^{\frac{n+1}{2}}$$

$$+\frac{1}{2} \left( \frac{1}{2} \Delta t \right)^2 \int (\Delta t)^2 u_{xxx} dt + \dots \sim_j^{\frac{n+1}{2}}$$

Similarly :-

$$\begin{aligned} \delta_x^2 u_j^n &= \left[ (\Delta n)^2 u_{xx} + \frac{1}{12} (\Delta n)^4 u_{xxxx} + \dots \right]_j^{\frac{n+1}{2}} \\ &\quad - \frac{1}{2} \Delta t \left[ (\Delta n)^2 u_{xxt} + \frac{1}{12} (\Delta n)^4 u_{xxxxt} + \dots \right]_j^{\frac{n+1}{2}} \\ &\quad + \frac{1}{2} \left( \frac{1}{2} \Delta t \right)^2 + \int (\Delta t)^2 u_{xxtt} dt + \dots \end{aligned}$$

using these two in

$$\begin{aligned} \theta \delta_x^2 u_j^{n+1} + (1-\theta) \delta_x^2 u_j^n \\ = \left[ (\Delta n)^2 u_{xx} + \frac{1}{12} (\Delta n)^4 u_{xxxx} + \dots \right] \\ + \left( \theta - \frac{1}{2} \right) \Delta t \left[ (\Delta n)^2 u_{xxt} + \frac{1}{12} (\Delta n)^4 u_{xxxxt} \right. \\ \left. + \dots \right] \\ + \frac{1}{8} (\Delta t)^2 (\Delta n)^2 \int u_{xxtt} dt + \dots \\ \quad \text{--- (2.82)} \end{aligned}$$

Truncation Error :

$$T_j^{\frac{n+1}{2}} = \frac{\delta_x^2 u_j^{\frac{n+1}{2}}}{\Delta t} - \frac{\theta \delta_x^2 u_j^{n+1} + (1-\theta) \delta_x^2 u_j^n}{(\Delta n)^2}$$

--- (2.83)

$$\text{Using } \frac{\partial_t u_j}{\Delta t} = u_t + \frac{1}{2\Delta t} (\Delta t)^2 u_{ttt} + \dots$$

$$\text{L} \frac{\theta \delta_x^2 u_j^{n+1} + (1-\theta) \delta_x^2 u_j^n}{(\Delta x)^2} = u_{xx} + \frac{1}{12} (\Delta x)^2 u_{xxxx} + \dots$$

$$\begin{aligned} \therefore u_j^{n+\frac{1}{2}} &= [u_t - u_{xx}] + \left[ \left( \frac{1}{2} - \theta \right) \Delta t u_{xt} - \frac{1}{12} (\Delta x)^2 u_{xxxx} \right] \\ &\quad + \left[ \frac{1}{12} (\Delta t)^2 u_{ttt} - \frac{1}{8} (\Delta t)^2 \right] \\ &\quad + \left[ \frac{1}{12} \left( \frac{1}{2} - \theta \right) \Delta t (\Delta x)^2 u_{xxxx} - \frac{2}{6!} (\Delta x)^4 u_{xxxxxx} \right] \\ &\quad - (2.84) \end{aligned}$$

### Problem 3:

The given advection-diffusion eqns

$$u_t - a(t) u_x - \epsilon u_{xx} = 0$$

$$u_x = \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

$$-a(t) u_x = a(t) \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

for negative  $a(t)$  &  $a(+)= -1$ , consider the forward difference

$$U_{\Delta t n} = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta t^2}$$

$\theta$ -discretisation :-

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \theta \left( a(+) \frac{U_{i+1}^{n+1} - U_i^{n+1}}{\Delta t} + \epsilon \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{(\Delta t)^2} \right) + (1-\theta) \left( a(t) \frac{U_{i+1}^n - U_i^n}{\Delta t} + \epsilon \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta t^2} \right)$$

taking unknown to LHS:

$$U_i^{n+1} = U_i^n + \Delta t \left[ \theta \left( \frac{a(+)}{\Delta t} (U_{i+1}^{n+1} - U_i^{n+1}) + \frac{\epsilon}{\Delta t^2} (U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) \right) + (1-\theta) \left( \frac{a(t)}{\Delta t} (U_{i+1}^n - U_i^n) + \frac{\epsilon}{\Delta t^2} (U_{i+1}^n - 2U_i^n + U_{i-1}^n) \right) \right]$$

$$\left. \begin{aligned} \text{assuming } \frac{\Delta t}{\Delta x} = \lambda \\ \text{& } \frac{\Delta t}{(\Delta x)^2} = \mu \end{aligned} \right\}$$

$$\begin{aligned}
 u_i^{n+1} &= \theta \lambda a(t) \underline{u}_{i+1}^n + \theta \lambda a(t) \overline{u}_i^n \\
 &\quad - \epsilon M \theta \underline{u}_{i+1}^n + \theta \epsilon M 2 \underline{u}_i^n - \epsilon \theta M \overline{u}_{i-1}^n \\
 &= (1-\theta) \left( \frac{a(t)}{\Delta t} (u_{i+1}^n - u_i^n) \right) + \frac{\epsilon}{\Delta t^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n)
 \end{aligned}$$

$$\Rightarrow [1 + \theta \epsilon M 2 + \theta \lambda a(t)] \underline{u}_i^n + [-\theta \lambda a(t) - \epsilon \theta M] \overline{u}_{i+1}^n \\
 - \epsilon \theta M \overline{u}_{i-1}^n = (1-\theta) \left[ a(t) \lambda (u_{i+1}^n - u_i^n) \right. \\
 \left. + \epsilon M (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \right]$$

$$\Rightarrow [1 + \theta \epsilon M 2 + \theta \lambda a(t)] \underline{u}_i^n + [-\theta \lambda a(t) - \epsilon \theta M] \overline{u}_{i+1}^n \\
 - \epsilon \theta M \overline{u}_{i-1}^n = a(t) \lambda \underline{u}_{i+1}^n - a(t) \lambda \underline{u}_i^n + \epsilon M \underline{u}_{i+1}^n \\
 - \epsilon M 2 \underline{u}_i^n + \epsilon M \underline{u}_{i-1}^n - a(t) \theta \lambda \underline{u}_{i+1}^n \\
 + \theta a(t) \lambda \underline{u}_i^n - \epsilon \theta M \underline{u}_{i+1}^n + 2 \epsilon M \theta \underline{u}_i^n \\
 - \epsilon M \theta \underline{u}_{i-1}^n$$

$$\Rightarrow [1 + \theta \epsilon M 2 + \theta \lambda a(t)] \underline{u}_i^n - [\theta \lambda a(t) + \epsilon M \theta] \overline{u}_{i+1}^n \\
 - \epsilon \theta M \overline{u}_{i-1}^n = [a(t) \lambda - a(t) \theta \lambda + \epsilon M - \epsilon \theta M] \underline{u}_{i+1}^n \\
 + [-a(t) \lambda - 2 \epsilon M + \theta a(t) \lambda + 2 \epsilon M \theta] \underline{u}_i^n \\
 + [\epsilon M - \epsilon M \theta] \underline{u}_{i-1}^n$$

Now, let  $N_x$  be the number of subintervals in  $L_p$  to  $L$  domain,  $x \in [L_p, L]$ .

Then we have,  $N_x + 1$  grid points,  $i = 0, 1, \dots, N_x$   
 each grid point is,  $x_i = L_p + i \Delta x$ ,  $\Delta x = \frac{L - L_p}{N_x}$

→ In python, we store as  $u[i, n]$   
 $i \rightarrow$  from 0 to  $N_x$

$u \rightarrow$  from  $n$  to  $N_t$

at B.C  $\rightarrow u(L_p, t) = 0$  &  
 $u(L, t) = 0,$



the interior nodes.

$i = 1, 2, \dots, N_x - 1,$

Boundary nodes are  $i = 0$  &  $i = N_x$  which are zero

for first order upwind for Advection term

- near the boundary at  $i = 1$ , we will need  $u[1]$  &  $u[0]$  but  $u[0] = 0$ , therefore, upwind difference becomes  $\frac{u[1] - 0}{\Delta x}$ .

- at  $i = N_x - 1$ , we need  $u[N_x - 1]$  &  $u[N_x]$  but  $u[N_x]$  is zero

## Problem: 4

Given :

$$u_t = u_{xn} \quad \text{for } t > 0, \\ 0 < n < 1$$

q

$$u^0(x) = \begin{cases} 2n & \text{if } 0 \leq n \leq \frac{1}{2} \\ 2 - 2n & \text{if } \frac{1}{2} \leq n \leq 1 \end{cases}$$

use

$J = 20 \rightarrow$  Total grid points

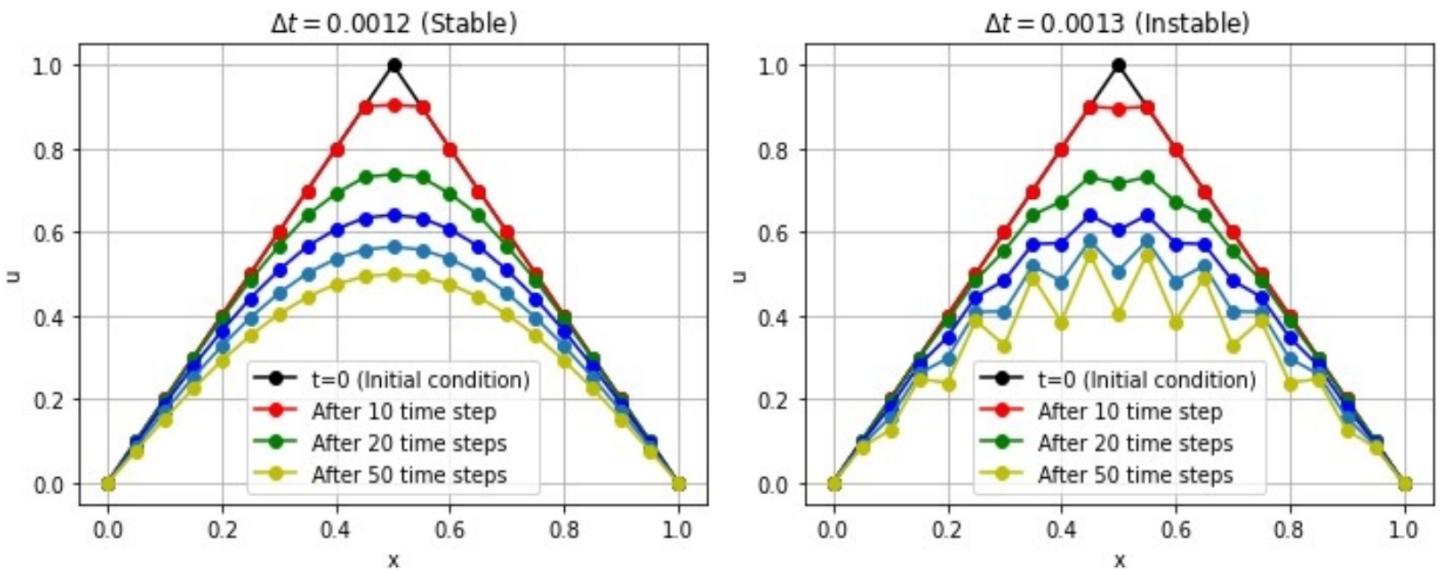
$\Delta x = 0.05 \rightarrow$  Spatial step size

$\Delta t_1 = 0.0012 \rightarrow$  First time step size

$\Delta t_2 = 0.0013 \rightarrow$  2nd time step size

using explicit scheme, a python code written in Problem4.py file & results has been matched & compared with fig 2.2 (MGMM)

• for two different time steps the output profile is given below.



- The above output image clearly shows the effect of time step size & as we can see for time step size 0.0013, there is instability can be observed.
- Now we will check the stability of this scheme, with both Fourier analysis and max. principle

### ① Fourier Analysis :-

$$(a) U_j^{n+1} = \lambda^{n+1} e^{ikj\Delta x} = \lambda \underbrace{[e^{ikj\Delta x}]}_{\rightarrow U_j^n}$$

$$U_j^n = \lambda U_j^{n-1}$$

$$(b) U_{j+1}^n = \lambda^n e^{ik(j+1)\Delta x} \\ = \underbrace{\lambda^n e^{ikj\Delta x}}_{\rightarrow U_j^n} e^{ik\Delta x}$$

$$U_j^n = U_j^0 e^{ikon}$$

Similarly

$$U_{j-1}^n = U_j^0 e^{-ikon}$$

Now use all these terms in explicit scheme difference eqn

$$U_j^{n+1} = U_j^n + \mu \left[ U_j^n - 2U_j^n + U_{j+1}^n \right]$$

$$\lambda U_j^n = U_j^n + \mu \left[ U_j^n e^{ikon} - 2U_j^n + U_{j+1}^n e^{-ikon} \right]$$

divided by  $U_j^n$

$$\Rightarrow \lambda = 1 + \mu \left[ e^{ikon} - 2 + e^{-ikon} \right]$$

$$\lambda = 1 + 2\mu \left[ \frac{e^{ikon} + e^{-ikon}}{2} - 1 \right]$$

$$= 1 + 2\mu \left[ \cos(kn) - 1 \right]$$

$$\lambda = 1 - 4\mu \sin^2 \frac{kn}{2}$$

$$\rightarrow \lambda = f(k)$$

for  $k = m\pi$ , we get

$$v_j^n = \sum_{m=-\infty}^{\infty} A_m e^{im\pi(jn)} f_{m\pi}^n$$

$\therefore \lambda \rightarrow$  complex no.

$$\rightarrow \lambda = \operatorname{Re}(\lambda) + i \operatorname{Im}(\lambda)$$

for stability requires that,

$$|\lambda| \leq 1$$



$$\therefore \sin^2 \frac{\lambda \pi n}{2} = 1 \text{ is the worst}$$

$$\Rightarrow |\lambda| \leq 1 \Rightarrow |1 - \lambda| \leq 1$$

$$\left( \lambda \leq \frac{1}{2} \right)$$

is the stability condition.

for  $\lambda = 1 - \gamma\lambda < -1$

$\mu > \frac{1}{2}$  → unstable

$$\text{Now, } \because \mu \leq \frac{1}{2}$$

$$\frac{\Delta t}{\Delta n^2} \leq \frac{1}{2}$$

$$\downarrow$$

$$\Delta t \leq \frac{1}{2} \Delta n^2$$

$$\therefore \textcircled{1} \text{ for } \begin{aligned} \Delta t_1 &= 0.0012 \\ \Delta n &= 0.05 \end{aligned} \quad \left. \begin{array}{l} \text{stable} \end{array} \right\}$$

$$\mu = \frac{0.0012}{(0.05)^2} \Rightarrow \underline{\underline{0.48}} \leq \frac{1}{2}$$

$$\textcircled{2} \text{ for } \Delta t_2 = 0.0013 \quad \left. \begin{array}{l} \text{unstable} \end{array} \right\}$$

$$\lambda = \frac{0.0013}{(0.05)^2} = 0.52 > \frac{1}{2}$$

(2) Max. principle :-

According to Max. principle, a difference approximation to  $U_t = U_{xx}$  should possess beyond condition as

$$\Delta x, \Delta t \rightarrow 0$$

$$\therefore \lambda(1-\theta) \leq \frac{1}{2}$$

& for explicit scheme,  $\theta = 0$

$$\lambda \leq \frac{1}{2}$$

$\rightarrow$  which will give

$$\Delta t_1 = 0.0012 \text{ stable}$$

$$\Delta t_2 = 0.0013 \text{ unstable}$$

$\Rightarrow$  we extend this explicit scheme

to ①

$$U_t - a(t)U_n - \varepsilon U_{n+1} = 0$$

$$U(x, 0) = U_0(x)$$

$$U(L_p, t) = U(L, t) = 0$$

$$\begin{aligned} & \Rightarrow [1 + \theta \varepsilon \lambda x_2 + \theta \lambda a(t)] U_i^{n+1} - [\theta \lambda a(t) + \varepsilon \lambda \theta] U_{i+1}^{n+1} \\ & - \varepsilon \theta U_{i-1}^{n+1} = [a(t)\lambda - a(t)\theta \lambda + \varepsilon \lambda - \varepsilon \theta \lambda] U_i^n \\ & + [-a(t)\lambda - 2\varepsilon \lambda + \theta a(t)\lambda + 2\varepsilon \lambda \theta] U_i^n \\ & + [\varepsilon \lambda - \varepsilon \lambda \theta] U_{i-1}^n, \end{aligned}$$

for  $\theta = 0$ : Explicit Scheme

$$U_i^{n+1} = [a(t)\lambda + \varepsilon \lambda] U_{i+1}^n + [-a(t)\lambda - 2\varepsilon \lambda] U_i^n + [\varepsilon \lambda] U_{i-1}^n$$

$$\lambda = \frac{\Delta t}{\Delta x^2} \quad \& \quad \theta = \frac{\Delta t}{\Delta x},$$

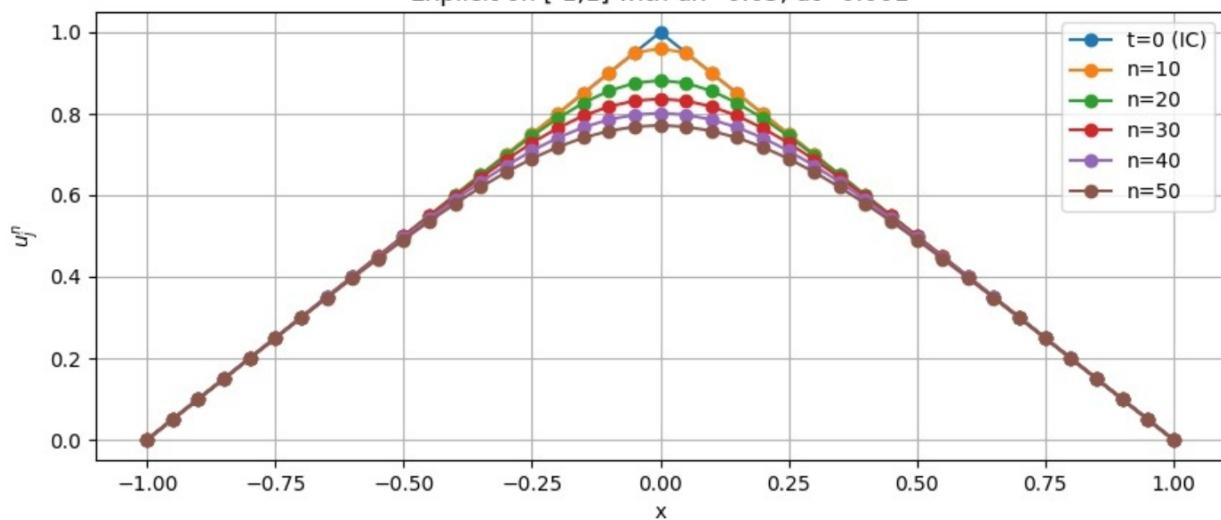
To extend explicit scheme to ①, for  $[1, -1]$   
I.C. will be,

$$U_0(x) = \begin{cases} 1+x & \text{if } -1 \leq x \leq 0 \\ 1-x & \text{if } 0 < x \leq 1 \end{cases}$$

$$\Delta x = 0.05$$

$$\Delta t = 40 \quad \& \quad \theta = \varepsilon = 1$$

Explicit on [-1,1] with  $dx=0.05$ ,  $dt=0.001$



## Problem 5 :-

Implemented F-Method and used the linear Algebra process in Problem 5.py

$$\therefore u_t - a(t) u_n - \varepsilon u_{nn} = 0$$

$$u(n, 0) = u_0(x)$$

$$u(L_p, t) = u(L, t) = 0$$

used  $\varphi = 0$  &  $L = 1$  (B.Cs)

$$\text{F.G.C} \rightarrow u_0(x) = \begin{cases} 2x & \text{if } 0 < x \leq 0.5 \\ 2 - 2x & \text{if } 0.5 \leq x \leq 1 \end{cases}$$

$\therefore$

$$u_n = \frac{u_{n+1} - u_{n-1}}{2\Delta x}$$

$$\text{if } u_{nx} = \frac{u_{n+1} - u_n}{(\Delta x)^2}$$

$$u_t = \frac{u_{n+1} - u_n}{\theta dt}$$

D-Method implemented:-

$$\Rightarrow \left[ 1 + \theta \epsilon \lambda x_2 + \theta \lambda a(t) \right] u_i^{n+1} - \left[ \theta \lambda a(t) + \epsilon M \theta \right] u_{i+1}^{n+1} \\ - \epsilon \theta u_{i-1}^{n+1} = \left[ a(t) \lambda - a(t) \theta \lambda + \epsilon M - \epsilon \theta u_i^n \right] u_{i+1}^n \\ + \left[ -a(t) \lambda - 2\epsilon M + \theta a(t) \lambda + 2\epsilon M \theta \right] u_i^n \\ + \left[ \epsilon M - \epsilon M \theta \right] u_{i-1}^n$$

$\Rightarrow$  Stability properties;

$$U_j^{n+1} = U_j^n + \theta \int M a^{n+1} (U_{j+1}^{n+1} - U_j^{n+1}) + \\ \epsilon M (U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) ] + (1-\theta) \int M a^n (U_{j+1}^n - U_j^n) \\ + \epsilon M (U_{j+1}^n - 2U_j^n + U_{j-1}^n) ].$$

assuming  $U_j^n = \lambda^n e^{ikj\Delta x}$

$$\Rightarrow \lambda^{n+1} e^{ikj\Delta x} = \lambda^n e^{ikj\Delta x} + \theta \left[ M a^{n+1} (\lambda^{n+1} e^{ik(j+1)\Delta x} - \lambda^n e^{ikj\Delta x} ) + \epsilon M (\lambda^{n+1} e^{ik(j+1)\Delta x} - 2\lambda^n e^{ikj\Delta x} + \lambda^{n+1} e^{ik(j-1)\Delta x} ) \right] + \\ (1-\theta) \left[ M a^n (\lambda^n e^{ik(j+1)\Delta x} - \lambda^n e^{ikj\Delta x}) + \epsilon M (\lambda^n e^{ik(j+1)\Delta x} - 2\lambda^n e^{ikj\Delta x} + \lambda^n e^{ik(j-1)\Delta x} ) \right]$$



$$\lambda = 1 + \lambda \theta \left[ M a^{n+1} (e^{ik\Delta x} - 1) + \epsilon M (e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \right] \\ + (1-\theta) \left[ M a^n (e^{ik\Delta x} - 1) + \epsilon M (e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \right]$$

$$\Rightarrow \lambda = \frac{1 + (1-\theta) \left[ Ma^n (e^{ik\Delta n} - 1) + \epsilon \ell \gamma (e^{ik\Delta n} - 2 + e^{-ik\Delta n}) \right]}{1 - \theta \left[ Ma^{n+1} (e^{ik\Delta n} - 1) + \epsilon \ell \gamma (e^{ik\Delta n} - 2 + e^{-ik\Delta n}) \right]}$$

$\Rightarrow$  for diffusion part,  $\alpha = 0$ , &  $| \lambda | \leq 1$

$$\mu \leq \frac{1}{2\epsilon(1-2\theta)}$$

Now for pure advection case ( $\epsilon = 0$ )

$$\text{from } \lambda = \frac{1 + (1-\theta) Ma^n (e^{ik\Delta n} - 1)}{1 - \theta Ma^{n+1} (e^{ik\Delta n} - 1)}$$

$$\text{for } \alpha = \text{const} \Rightarrow a^{n+1} = a^n$$

$$\text{using } c^{i\theta} = \cos(\theta) + i \sin(\theta);$$

$$\lambda = \frac{1 + (1-\theta) Ma \left[ \cos(k\Delta n) + i \sin(k\Delta n) - 1 \right]}{1 - \theta Ma \left[ \cos(k\Delta n) + i \sin(k\Delta n) - 1 \right]}$$

$$\text{Now } \cos(2\theta) - 1 = -2\sin^2\theta$$

$$\lambda = \frac{\left[ 1 - 2(1-\theta) Ma \sin^2(k\Delta n/2) \right] + i(1-\theta) Ma \sin(k\Delta n)}{\left[ 1 + 2\theta Ma \sin^2(k\Delta n/2) - i\theta Ma \sin(k\Delta n) \right]}$$

for stability condition  $|\lambda| \leq 1$

$$\Rightarrow |\lambda|^2 \leq 1$$

$$|\lambda|^2 = \frac{\left[ 1 - 2(1-\theta)Mq\sin^2(KOn/2) \right] + i(1-\theta)Mq\sin(KOn)}{\left[ 1 + 2\theta Mq\sin^2(KOn/2) - i\theta Mq\sin(KOn) \right]} \times \text{Conjugate of } \lambda$$

$$|\lambda|^2 = \frac{\left[ 1 - 2(1-\theta)Mq\sin^2(KOn/2) \right]^2 + (1-\theta)^2 Mq^2 \sin^4(KOn)}{\left[ 1 + 2\theta Mq\sin^2(KOn/2) \right]^2 + (\theta Mq)^2 \sin^4(KOn)} \leq 1,$$

$$\Rightarrow \cancel{1 - 4(1-\theta)Mq\sin^2(KOn/2)} + 4(1-\theta)^2(Mq)^2 \sin^4(KOn/2) \\ + (1-\theta)^2(Mq)^2 \sin^2(KOn) \leq \cancel{1 + 4\theta Mq\sin^2(KOn/2)} \\ + 4(\theta Mq)^2 \sin^4(KOn/2) + (\theta Mq)^2 \sin^2(KOn),$$

dividing by  $Mq$  on both sides

$$-4(1-\theta)\sin^2(KOn/2) + 4(1-\theta)^2 Mq \sin^4(KOn/2) \\ + (1-\theta)^2(Mq) \sin^2(KOn) \leq 4 \sin^2(KOn/2) + 4(\theta Mq) \\ \sin(KOn/2) + (\theta Mq) \sin^2(KOn),$$

$$\Rightarrow -4(1-\theta)\sin^2(KOn/2) - 4\theta \sin^2(KOn/2) + \\ 4(1-2\theta+\theta^2)(Mq) \sin^4(KOn/2) - 4(\theta Mq) \sin^4(KOn/2) \\ + (1-2\theta+\theta^2)(Mq) \sin^2(KOn/2) - (\theta Mq) \sin^2(KOn) \leq 0$$

$$\Rightarrow -4\sin^2(KOn/2) + 4(1-2\theta)Mq \sin^4(KOn/2)$$

$$+ (1-2\theta) \mu q \sin^2(k_0 n/2) \leq 0,$$

can be written as,

$$-\mu q \sin^2(k_0 n/2) + (1-2\theta) \mu q \left[ \mu q \sin^4(k_0 n/2) + \sin^2(k_0 n) \right] \leq 0$$

using  $\sin 2\theta = 2 \sin \theta \cos \theta$ ,

$$-\mu q \sin^2(k_0 n/2) + (1-2\theta) \mu q \left[ \mu q \sin^4(k_0 n/2) + \sin^2(k_0 n/2) \cos^2(k_0 n/2) \right] \leq 0,$$

$$\Rightarrow -\mu q \sin^2(k_0 n/2) + 4(1-2\theta) \sin^2(k_0 n/2) \mu q \left[ \sin^2(k_0 n/2) + \cos^2(k_0 n/2) \right] \leq 0$$

$\cancel{\sin^2(k_0 n/2) + \cos^2(k_0 n/2)} \quad \textcircled{1}$

$$\Rightarrow -\mu q \sin^2(k_0 n/2) + 4(1-2\theta) \mu q \sin^2(k_0 n/2) \leq 0$$

dividing by  $\mu q \sin^2(k_0 n/2)$ ,

$$(1-2\theta) \mu q \leq 1$$

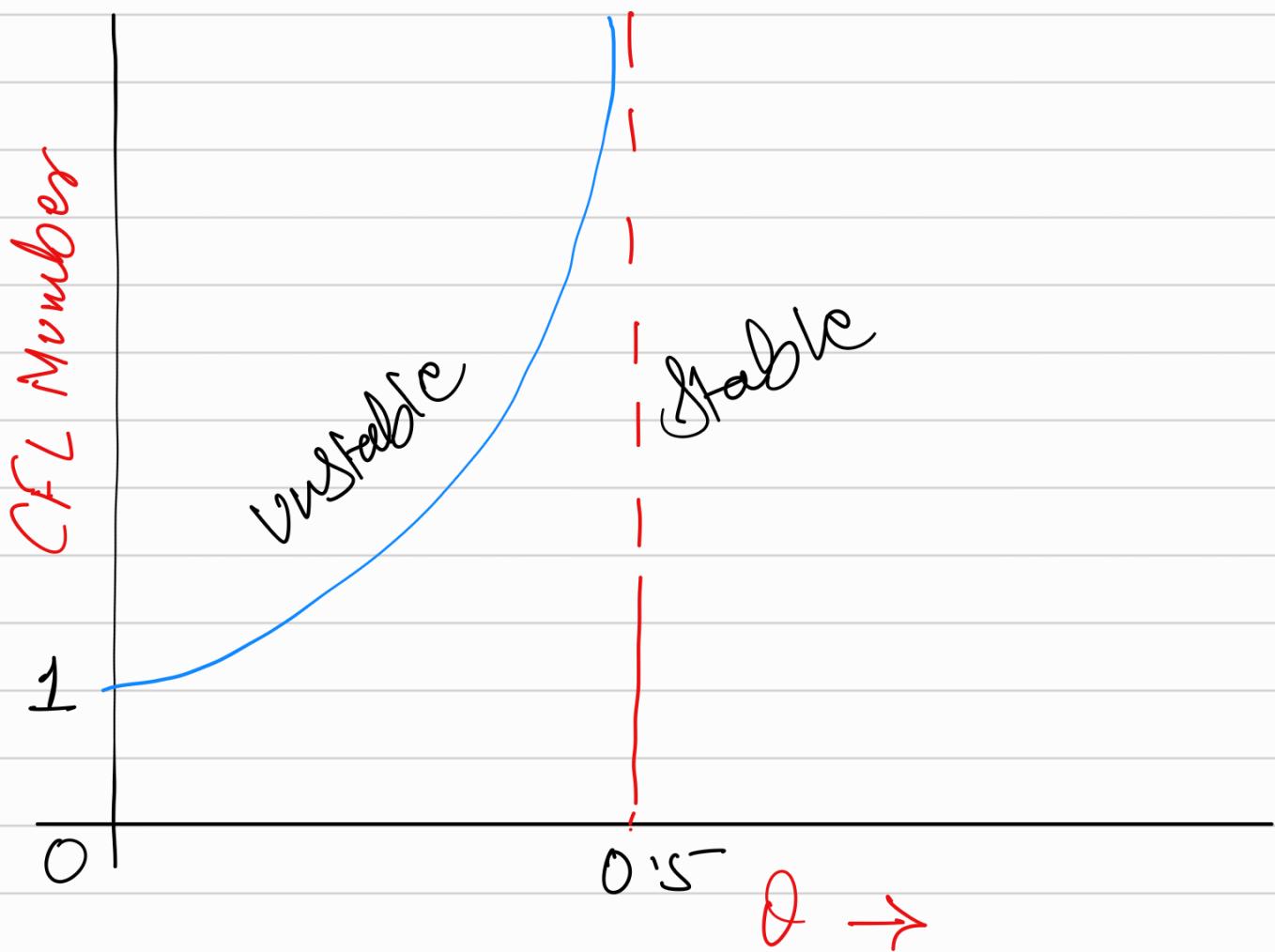
$$\mu q \leq \frac{1}{(1-2\theta)}$$

- This is the stability condition, when  $\alpha$  is positive constant.

$$\text{here, } \theta = \frac{\Delta t}{\Delta x} \quad \& \quad \text{and } \theta = \frac{a \Delta t}{\Delta x}$$

which is CFL number

$$\text{CFL Number} = \theta \frac{a \Delta t}{\Delta x}$$



$\text{CFL} = 1$  for fully explicit case  
 $\downarrow \theta = 1$

## Problem 6:-

In python code Problem 6-Part1.py  
Problem 6-Part2.py

Case 1:-

$$u(x, 0) = (1-x)^4 (1+x)$$

Case 2:-

$u(x, 0)$  with random value of  
 $\textcircled{B}$

we have

$$\left\{ \begin{array}{l} u_t - a(t)x - \varepsilon u_{xx} = 0 \\ u(x, 0) = u_0 x \\ u(l_p, t) = u(l, t) = 0 \end{array} \right.$$

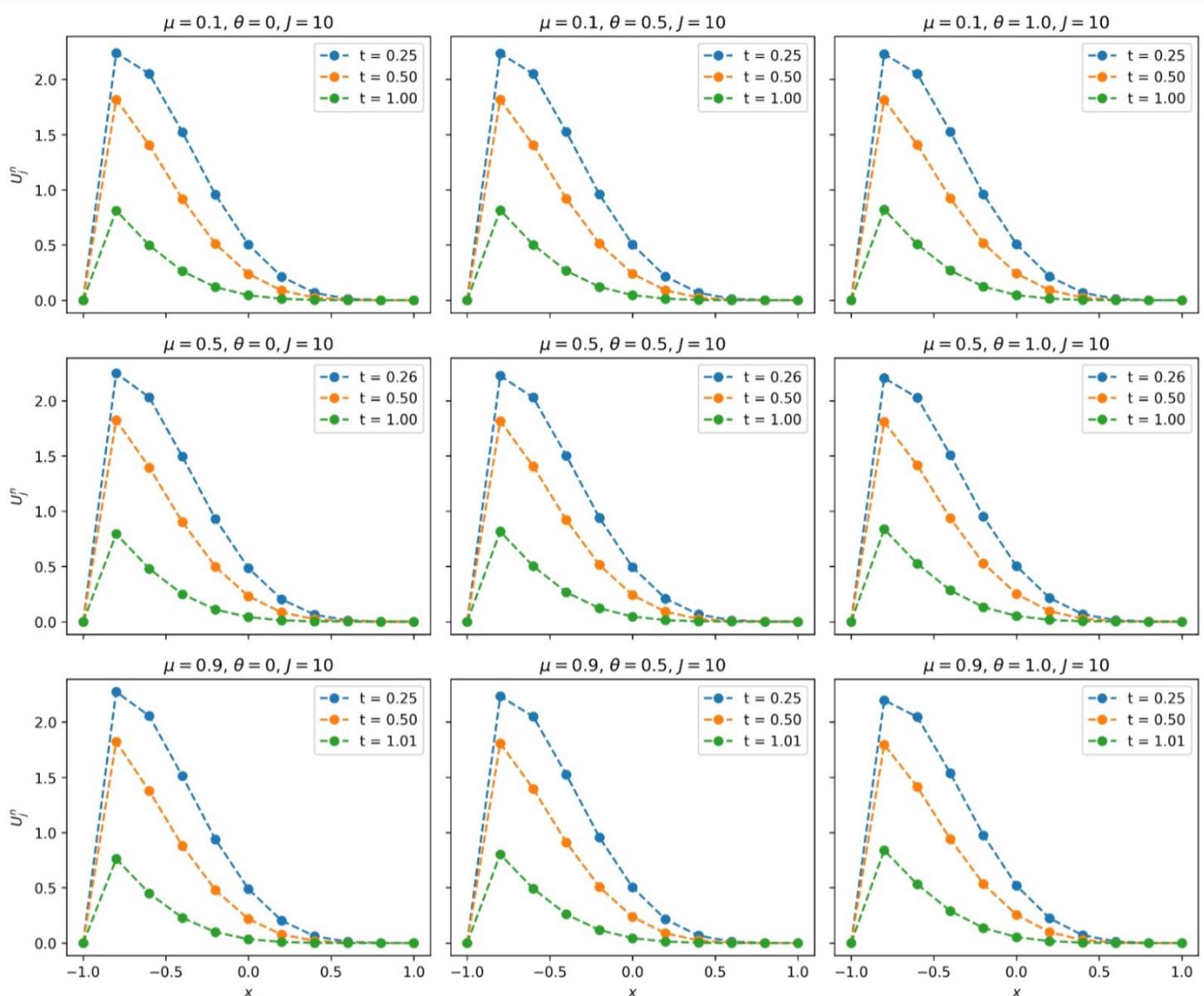
$$l_p = -1, \quad l = 1, \quad \varepsilon = 10^{-3},$$

at  $\theta = 0, 0.5, 1$  &  $u = 0.1, 0.5, 0.9$

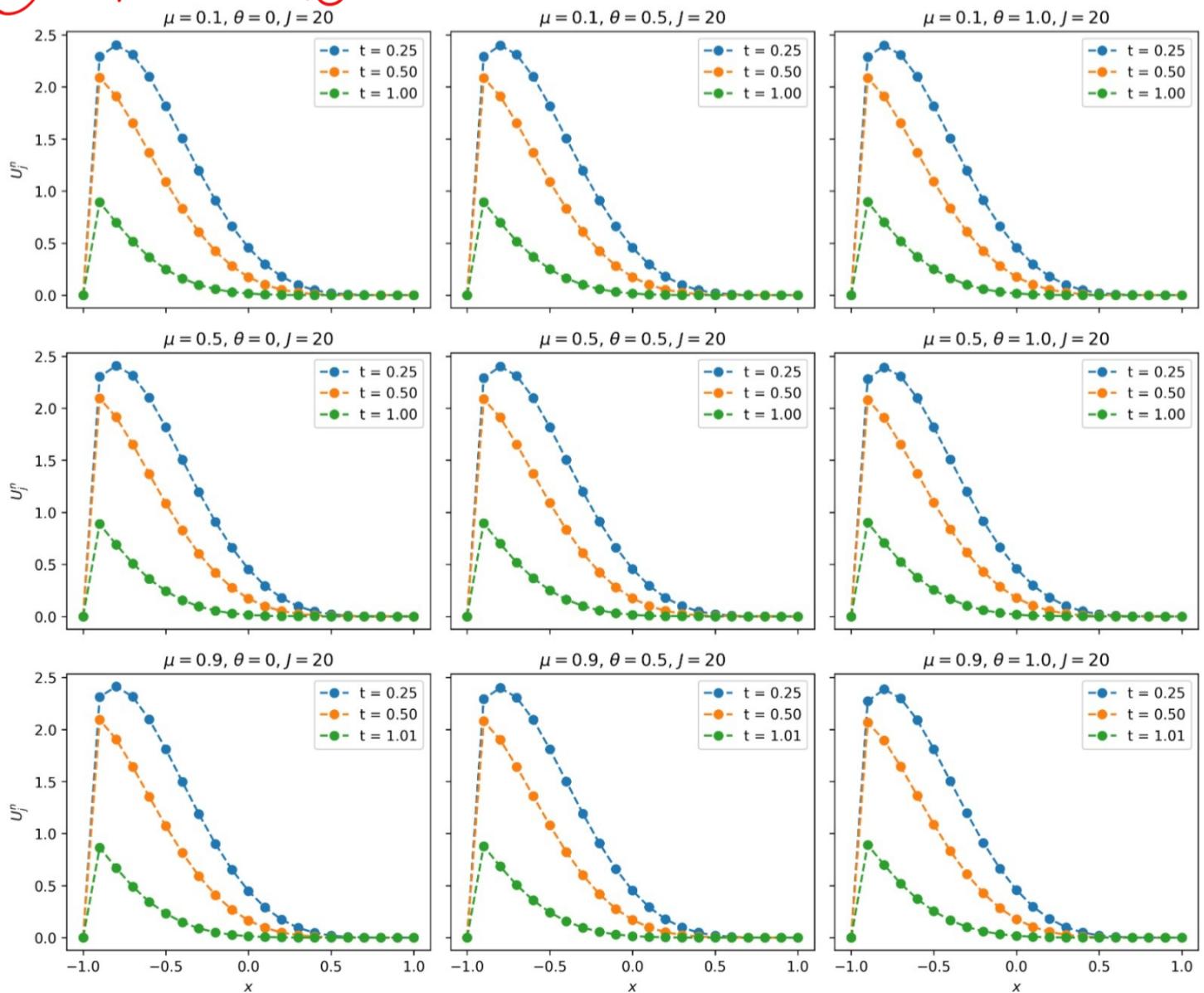
for case 0, three plot has been plotted  
 $\Rightarrow J = 10, 20, 40,$

To check the convergence

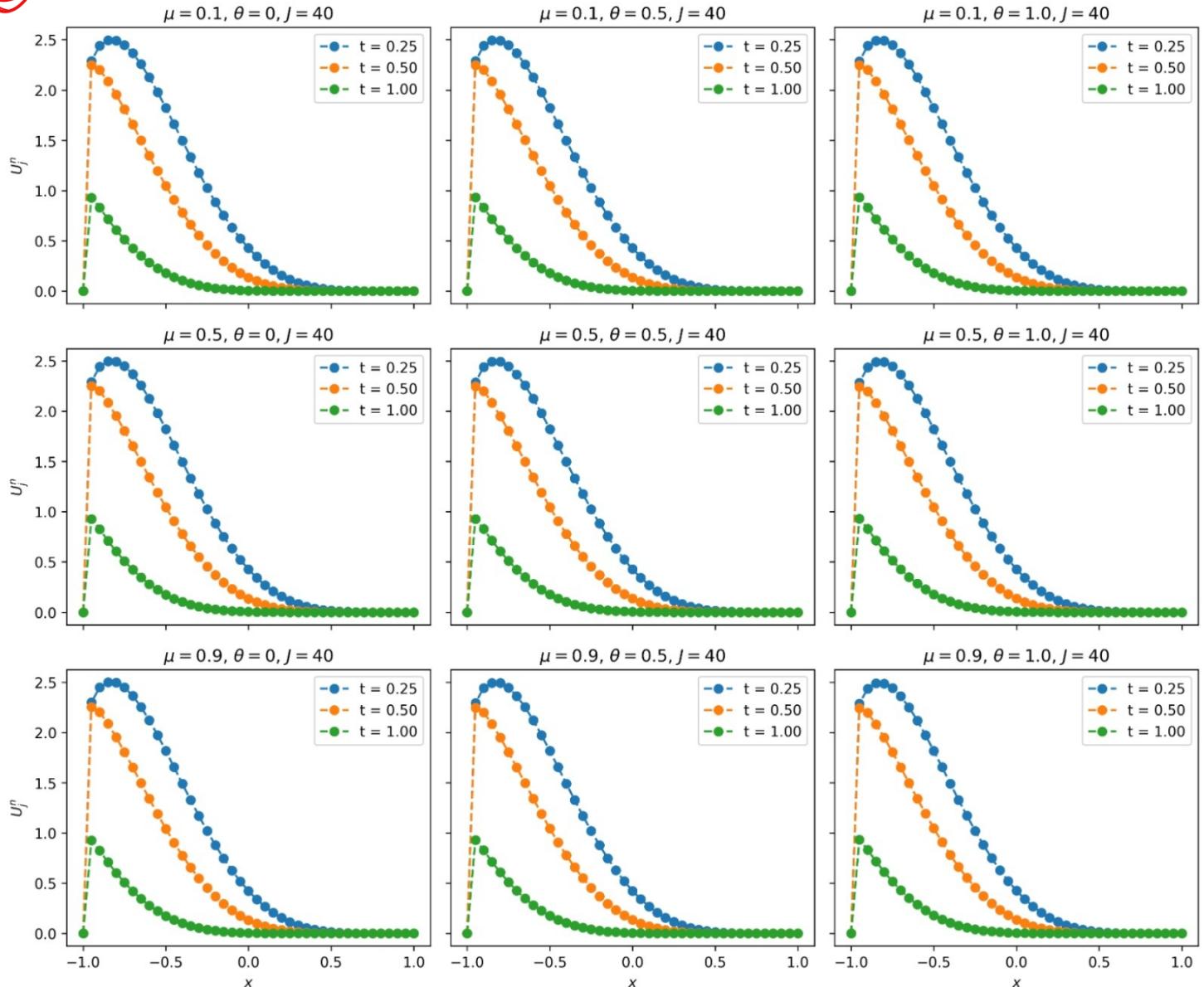
① at  $J=10$



② at  $J=20$



③ at  $J=40$



part ② :- at  $\alpha(+)=1$  with

$$u(x, 0) = (1-x)^4 (1+x) \left( \sum_{K=0}^3 b_K \phi_K(x) + C \right)$$

