

### Numerical Exercises 3 ~ James Dunstan

1.) We are given the 2-dimensional Poisson system

$$-\nabla^2 u = f, \text{ on } \Omega = (x, y) \in [0, 1]^2$$

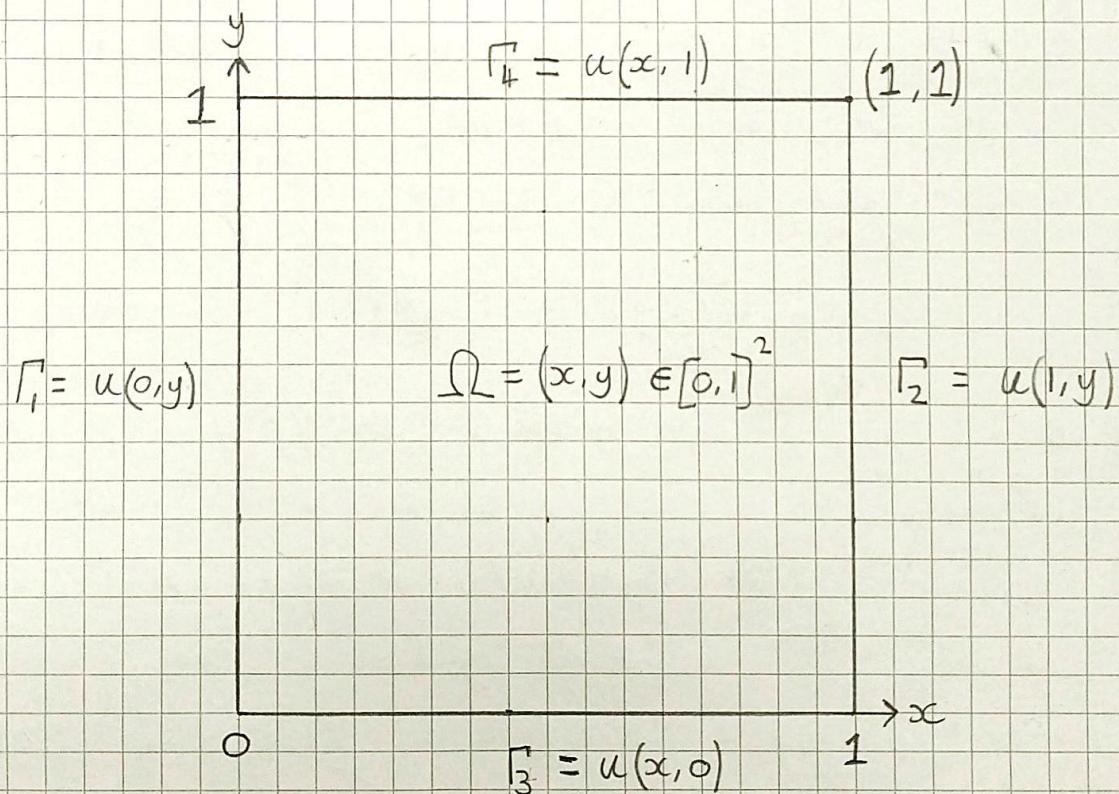
For the given function  $f(x, y) = 2\pi^2 \sin(\pi x) \cos(\pi y)$  and unknown  $u = u(x, y)$ . Dirichlet boundary conditions are given by:

$$u(0, y) = u(1, y) = 0,$$

and Neumann boundary conditions are given by:

$$\frac{\partial}{\partial y} u(x, y) \Big|_{y=0} = \frac{\partial}{\partial y} u(x, y) \Big|_{y=1} = 0.$$

We can sketch the problem domain as follows



and the exact solution is given by  $u(x, y) = \sin(\pi x) \cos(\pi y)$

$$\text{since } \nabla^2 u = -\pi^2 \sin(\pi x) \cos(\pi y) - \pi^2 \sin(\pi x) \cos(\pi y),$$

$$= -2\pi^2 \sin(\pi x) \cos(\pi y),$$

$$= -f(x, y),$$

and the boundary conditions are satisfied since there always exists a  $\sin(\pi x)$  or  $\sin(\pi y)$  term that is zero at  $x=0$  and  $x=1$  or  $y=0$  and  $y=1$ .

Knowing the exact solution we can then use the Finite Element Method to numerically approximate this solution using the Ritz-Galerkin method. First, we derive an integral formulation of the system.

- As it turns out the given Poisson system can be derived via the minimisation problem:

$$\min_{u \in \Omega} F[u] ; \quad F[u] = \iint_0^1 \frac{1}{2} |\nabla u|^2 - u f \, dx dy$$

where  $F$  is a functional and so not a function of  $x$  or  $y$ , and  $u(x,y)$  must still follow the Dirichlet boundary conditions. We consider a class of functions

$$u(x,y) = \tilde{u}(x,y) + \varepsilon \delta u(x,y), \quad \text{where}$$

$\varepsilon$  is a variable parameter,  $\tilde{u}(x,y)$  is a smooth solution of the minimisation problem with some variation  $\delta u(x,y)$  which is fixed. Assuming both  $\tilde{u}$  and  $u$  satisfy the boundary conditions then we must also have  $\delta u(0,y) = \delta u(1,y) = 0$ . Now, a necessary condition for the existence of a minimal solution is that

$$\frac{\partial F[u]}{\partial \varepsilon} = 0, \quad \text{and so}$$

$$0 = \frac{d}{d\varepsilon} \left[ \iint_0^1 \frac{1}{2} |\nabla(\tilde{u} + \varepsilon \delta u)|^2 - (\tilde{u} + \varepsilon \delta u) f \, dx dy \right],$$

$$O = \iint_{\Omega} \frac{\partial}{\partial \varepsilon} \left[ \frac{1}{2} \left( \frac{\partial (\tilde{u} + \varepsilon \delta u)}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial (\tilde{u} + \varepsilon \delta u)}{\partial y} \right)^2 \right] - \frac{\partial}{\partial \varepsilon} [(\tilde{u} + \varepsilon \delta u) f] dx dy ,$$

$$O = \iint_{\Omega} \nabla(\tilde{u} + \varepsilon \delta u) \cdot \nabla(\delta u) - f \delta u dx dy ,$$

and since  $u = \tilde{u} + \varepsilon \delta u$  implies this is minimised for  $\varepsilon = 0$  we reduce the equation to

$$O = \iint_{\Omega} \nabla \tilde{u} \cdot \nabla \delta u - f \delta u dx dy . \quad \langle * \rangle$$

By performing integration by parts on the first and second (but not third!) terms with respect to  $x$  and  $y$  respectively we find

$$\begin{aligned} O &= \int_0^1 \left[ \frac{\partial \tilde{u}}{\partial x} \delta u \right]_{x=0}^{x=1} - \int_0^1 \delta u \frac{\partial^2 \tilde{u}}{\partial x^2} dx dy \\ &\quad + \int_0^1 \left[ \frac{\partial \tilde{u}}{\partial y} \delta u \right]_{y=0}^{y=1} - \int_0^1 \delta u \frac{\partial^2 \tilde{u}}{\partial y^2} dy dx \\ &\quad - \iint_{\Omega} f \delta u dx dy \end{aligned}$$

and remembering  $\frac{\partial \tilde{u}}{\partial y}(x, 0) = \frac{\partial \tilde{u}}{\partial y}(x, 1) = \delta u(0, y) = \delta u(1, y) = 0$   
we recover

$$\iint_{\Omega} \delta u (-\nabla^2 \tilde{u} + f) dx dy = 0 , \text{ which by the lemma}$$

of Dubois-Reymond must satisfy  $-\nabla^2 \tilde{u} = f$ , proving the

derivation. This derivation also gives us a weak form of the differential equation given by  $\langle \ast \rangle$  which can also be derived using an arbitrary test function  $w(x, y)$  to lower the order of the  $u(x, y)$  derivatives by switching them onto  $w(x, y)$ . Hence, let us multiply the strong form of the system by the test function and integrate over domain  $\Omega$ :

$$-\int_0^1 \int_0^1 w \nabla^2 u \, dy \, dx = \int_0^1 \int_0^1 w f \, dy \, dx,$$

$$-\int_0^1 \int_0^1 \nabla \cdot (w \nabla u) \, dy \, dx + \int_0^1 \int_0^1 \nabla w \cdot \nabla u \, dy \, dx = \int_0^1 \int_0^1 w f \, dy \, dx,$$

and by Gauss' theorem

$$\int_0^1 \int_0^1 \nabla w \cdot \nabla u \, dy \, dx - \oint_{\Gamma} w \nabla u \cdot \hat{n} \, d\Gamma = \int_0^1 \int_0^1 w f \, dy \, dx.$$

Now, looking at the  $\oint_{\Gamma} w \nabla u \cdot \hat{n} \, d\Gamma$  term on the boundary and recalling our sketch from earlier we can observe for  $\Gamma_1 \Rightarrow \hat{n} = (-1, 0)^T$ ; for  $\Gamma_2 \Rightarrow \hat{n} = (1, 0)^T$ ; for  $\Gamma_3 \Rightarrow \hat{n} = (0, -1)^T$ ; and for  $\Gamma_4 \Rightarrow \hat{n} = (0, 1)^T$ . Hence for each case on the boundary we have

on  $\Gamma_1$ :  $\nabla u \cdot \hat{n} = -\frac{\partial u}{\partial x}$ , but  $u(0, y) = 0$  and therefore  $\frac{\partial u}{\partial x} = 0$  on this boundary segment.

on  $\Gamma_2$ :  $\nabla u \cdot \hat{n} = \frac{\partial u}{\partial x} = 0$  by similarity to  $\Gamma_1$ .

On  $\Gamma_3$  :  $\nabla u \cdot \hat{n} = -\frac{\partial u}{\partial y}$ , but  $\frac{\partial u}{\partial y}|_{y=0} = 0$

and so  $\frac{\partial u}{\partial y} = 0$  on this boundary segment.

on  $\Gamma_4$  :  $\nabla u \cdot \hat{n} = \frac{\partial u}{\partial y} = 0$  by similarity to  $\Gamma_3$ .

The boundary integral thus vanishes leaving the weak formulation

$$\oint_0^1 (\nabla u \cdot \nabla w - w f) dy dx = 0 \quad \langle \square \rangle$$

where clearly  $\langle \square \rangle$  is the same as  $\langle * \rangle$  with the test function  $w(x, y) = g u(x, y)$ .

2.) Introducing a Finite Element Expansion method to discretise the problem we let

$$u(x, y) \approx u_h(x, y)$$

where  $u_h(x, y)$  is the approximation of the unknown solution by a finite linear combination of basis functions:

$$u_h(x, y) = \sum_{j=1}^{N_h} \hat{u}_j \phi_j(x, y), \text{ where}$$

$\hat{u}_j$  are coefficients and  $\phi_j(x, y)$  are the proposed basis functions. Note  $N_h$  is the number of nodes in the 2-dimensional space, one for each basis function.

Now since we have Dirichlet boundary conditions then the coefficients on boundary nodes are zero, thus our approximation becomes

$$u_h(x, y) = \sum_{j=1}^{N_h - N_{\text{boundary}}} \hat{u}_j \phi_j(x, y).$$

Returning to the minimisation problem but now in discrete form we have

$$\min_{u_h \in \Omega} F[u_h] ; \quad F[u_h] = \iint_0^1 \frac{1}{2} \sum_{j=1}^{N_h} \sum_{i=1}^{N_h} \hat{u}_j \nabla \phi_j \cdot \hat{u}_i \nabla \phi_i dx dy - \iint_0^1 \int \sum_{j=1}^{N_h} \hat{u}_j \phi_j dx dy .$$

Again we can use a method of variations to derive a discrete weak formulation. Assuming we have a class of functions for each  $\hat{u}_j$  so that

$\hat{u}_j = \tilde{u}_j + \varepsilon \delta u_j$ , then the minimisation problem becomes

$$\frac{\partial F[u_h]}{\partial \varepsilon} = 0 ,$$

$$\boxed{\frac{\partial}{\partial \varepsilon} \left( \sum_{i=1}^{N_h} \sum_{j=1}^{N_h} \frac{\hat{u}_i \hat{u}_j}{2} \iint_0^1 \nabla \phi_i \cdot \nabla \phi_j dx dy - \sum_{j=1}^{N_h} \hat{u}_j \iint_0^1 f \phi_j dx dy \right) = 0 ,}$$

and so defining  $A_{ij} = \iint_0^1 \nabla \phi_i \cdot \nabla \phi_j dx dy$  and

$b_j = \iint_0^1 f \phi_j dx dy$  we have the problem

$$\frac{\partial}{\partial \varepsilon} \left[ \frac{1}{2} \sum_{i=1}^{N_h} \sum_{j=1}^{N_h} \hat{u}_i \hat{u}_j A_{ij} - \sum_{j=1}^{N_h} \hat{u}_j b_j \right] = 0 . \text{ where } \hat{u}_j = \tilde{u}_j + \varepsilon \delta u_j .$$

This leads to

$$\frac{\partial}{\partial \varepsilon} \left[ \frac{1}{2} \sum_{i=1}^{N_h} \sum_{j=1}^{N_h} (\tilde{u}_i + \varepsilon \delta u_i)(\tilde{u}_j + \varepsilon \delta u_j) A_{ij} - \sum_{j=1}^{N_h} (\tilde{u}_j + \varepsilon \delta u_j) b_j \right] = 0 ,$$

$$\frac{\partial}{\partial \varepsilon} \left[ \frac{1}{2} \sum_{i=1}^{N_h} \sum_{j=1}^{N_h} (\tilde{u}_i \tilde{u}_j + \varepsilon (\tilde{u}_i \delta u_j + \tilde{u}_j \delta u_i) + \varepsilon^2 \delta u_i \delta u_j) A_{ij} - \sum_{j=1}^{N_h} (\tilde{u}_j b_j + \varepsilon \delta u_j b_j) \right] = 0 .$$

Taking the epsilon derivative, and again remembering

that  $\epsilon = 0$  minimises the problem, since

$$\tilde{u}_j = \tilde{u}_j + \epsilon s_{uj} \quad \text{and the } \tilde{u}_j \text{ terms are assumed}$$

to be minimal solutions already, we have

$$\left[ \frac{1}{2} \sum_{i=1}^{N_n} \sum_{j=1}^{N_n} (\tilde{u}_i s_{uj} + \tilde{u}_j s_{ui}) A_{ij} - \sum_{j=1}^{N_n} s_{uj} b_j \right] = 0.$$

Since  $A_{ij} = \iint_0^1 \nabla \phi_i \cdot \nabla \phi_j \, dx dy$  it is clear this

matrix is symmetric and so  $A_{ij} = A_{ji}$ . The above equation can consequently be reduced to

$$\frac{1}{2} \sum_{i=1}^{N_n} \sum_{j=1}^{N_n} 2 \tilde{u}_i s_{uj} A_{ij} - \sum_{j=1}^{N_n} s_{uj} b_j = 0, \quad \text{and factorising}$$

out  $s_{uj}$  we have

$$\sum_{j=1}^{N_n} \left( \sum_{i=1}^{N_n} A_{ij} \tilde{u}_i - b_j \right) s_{uj} = 0.$$

Since  $s_{uj}$  is arbitrary we can again deduce that  $\sum_{i=1}^{N_n} A_{ij} \tilde{u}_i = b_j$  for all  $j = 1, 2, \dots, N_n$ .

Now, returning to the weak formulation  $\langle \square \rangle$  and discretising we can notice that the test function

$w(x,y)$  must be in the same space as the

discretised  $u_h(x,y) = \sum_{j=1}^{N_n} \hat{u}_j \phi_j(x,y)$  and so we

assume it takes the same form  $w(x,y) = \sum_{i=1}^{N_n} \hat{w}_i \phi_i(x,y)$ .

Substituting this into the weak formulation gives

$$\int_0^1 \int_0^1 \left( \sum_{j=1}^{N_n} \sum_{i=1}^{N_n} \hat{u}_j \nabla \phi_j \cdot \hat{w}_i \nabla \phi_i - \sum_{i=1}^{N_n} \hat{w}_i \phi_i f \right) dx dy = 0,$$

defining  $A_{ji} = \int_0^1 \int_0^1 \nabla \phi_j \cdot \nabla \phi_i dx dy$  and

$b_i = \int_0^1 \int_0^1 f \phi_i dx dy$  then we can write this as

$$\sum_{i=1}^{N_n} \sum_{j=1}^{N_n} \hat{w}_i \hat{u}_j A_{ji} - \sum_{i=1}^{N_n} \hat{w}_i b_i = 0,$$

$$\sum_{i=1}^{N_n} \left( A_{ji} \hat{u}_j - b_i \right) \hat{w}_i = 0. \text{ This is the same}$$

as the variational method with  $\hat{w}_i = \delta u_i$ .