

Q1

Scaling the system -

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(hu) = \frac{1}{(L_s/u_0)} \cdot H_{0s} \frac{\partial u'}{\partial x'} + \frac{u_0 \cdot H_{0s}}{L_s} \left( u' \frac{\partial h'}{\partial x'} + h' \frac{\partial u'}{\partial x'} \right)$$

Then -

$$0 \approx \frac{u_0 H_{0s}}{L_s} \left( \frac{\partial u'}{\partial x'} + \frac{\partial}{\partial x'}(h'u') \right)$$

Dropping the primes  $\frac{u_0 H_{0s}}{L_s}$  gives -

$$0 \approx \frac{\partial u'}{\partial t} + \frac{\partial}{\partial x'}(hu')$$

$$\frac{\partial u}{\partial t} + \frac{\partial(gu)}{\partial x} \approx \frac{u_0}{(L_s/u_0)} \frac{\partial u'}{\partial x'} + g \frac{H_{0s}}{L_s} \frac{\partial u'}{\partial x'}$$

$$0 \approx \frac{u_0^2}{L_s} \frac{\partial u'}{\partial t} + \frac{g H_{0s}}{L_s} \frac{\partial u'}{\partial x'} \\ \approx \frac{1}{L_s} \frac{\partial u'}{\partial t} + \frac{1}{L_s} \frac{g H_{0s}}{u_0^2} \frac{\partial u'}{\partial x'}$$

Setting dimensionless  $g' = \frac{g H_{0s}}{u_0^2}$ :

$$0 \approx \frac{1}{L_s} \left( \frac{\partial u'}{\partial t} + g' \frac{\partial u'}{\partial x'} \right)$$

Dropping prime  $\frac{1}{L_s}$  gives -

$$0 \approx \frac{\partial u'}{\partial t} + g' \frac{\partial u'}{\partial x'}$$

Defining the Riemann ProblemWe want  $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$  st  $\frac{\partial q}{\partial t} + A \frac{\partial q}{\partial x} = 0$  for some matrix  $A$ .

Then we will have equations:

$$\begin{pmatrix} q_1 t + a_{11} q_1 x + a_{12} q_2 x \\ q_2 t + a_{21} q_1 x + a_{22} q_2 x \end{pmatrix} = \begin{pmatrix} 1_t + H_0 u x \\ u x + g n x \end{pmatrix}$$

For  $H(x) = H_0$  const,  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Clearly  $q_1 = \eta$ ,  $a_{11} = 0$ ,  $a_{22} = 0$ . We have the choice of :

$$\begin{aligned} q_2 &= u, a_{12} = Mo, a_{21} = g. \quad \text{or} \\ q_2 &= Mo\eta, a_{12} = 1, a_{21} = gMo \end{aligned}$$

To simplify the matrix, choose  $q_2 = Mo\eta$ , so we have -

$$\partial_t \begin{pmatrix} \eta \\ Mo\eta \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ gMo & 0 \end{pmatrix} \partial_x \begin{pmatrix} \eta \\ Mo\eta \end{pmatrix} = 0$$

When  $Mo\partial_t \eta + gMo\eta_x = Mo(\eta_t + g\eta_x) = 0$  so  $\eta_t + g\eta_x = 0$  as desired.

Let  $c_0 = \sqrt{gMo}$  where  $c_0$  is phase speed.

Then we have the Neumann problem -

$$\partial_t q + \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \partial_x q = 0$$

to be coupled with constant initial conditions. This is a system of coupled PDEs, so we can now decouple the equations.

### Eigenvalues and eigenvectors of A

$$\text{Det}(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ c_0^2 & -\lambda \end{vmatrix} = (-\lambda)^2 - c_0^2 = 0$$

gives eigenvalues  $\lambda = \pm c_0$ , that is,  $\lambda_1 = c_0 = \sqrt{gMo}$ ,  $\lambda_2 = -c_0 = -\sqrt{gMo}$ . Then we have eigenvectors -

$$\lambda_1 = c_0;$$

$$\begin{pmatrix} -c_0 & 1 \\ c_0^2 & -c_0 \end{pmatrix} \begin{pmatrix} r_{11} \\ r_{12} \end{pmatrix} = \begin{pmatrix} -c_0 r_{11} + r_{12} \\ c_0^2 r_{11} - c_0 r_{12} \end{pmatrix} = 0 \quad \text{then} \quad r_{12} = c_0 r_{11}$$

$$\text{let } r_{11} = 1, \text{ then } r_{12} = c_0 \text{ for eigenvector } \Sigma_1 = \begin{pmatrix} 1 \\ c_0 \end{pmatrix}$$

$$\lambda_2 = -c_0;$$

$$\begin{pmatrix} c_0 & 1 \\ c_0^2 & c_0 \end{pmatrix} \begin{pmatrix} r_{21} \\ r_{22} \end{pmatrix} = \begin{pmatrix} c_0 r_{21} + r_{22} \\ c_0^2 r_{21} + c_0 r_{22} \end{pmatrix} = 0 \quad \text{then} \quad r_{22} = -c_0 r_{21}$$

$$\text{let } r_{21} = 1, \text{ then } r_{22} = -c_0. \text{ for eigenvector } \Sigma_2 = \begin{pmatrix} 1 \\ -c_0 \end{pmatrix}$$

[These are both clearly scalar multiples of given eigenvectors]

$$\frac{1}{2\omega} \begin{pmatrix} 1 \\ \omega \end{pmatrix} \text{ and } -\frac{1}{2\omega} \begin{pmatrix} 1 \\ -\omega \end{pmatrix} \quad ]$$

Our matrix of eigenvectors is:  $B = \begin{pmatrix} 1 & 1 \\ \omega & -\omega \end{pmatrix}$  with  $B^{-1} = -\frac{1}{2\omega} \begin{pmatrix} -\omega & 1 \\ \omega & 1 \end{pmatrix}$ .

Then

$$B^{-1}B = \begin{pmatrix} -\omega & 1 \\ \omega & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \omega & -\omega \end{pmatrix} \cdot -\frac{1}{2\omega} = \begin{pmatrix} -2\omega & 0 \\ 0 & -2\omega \end{pmatrix} \cdot -\frac{1}{2\omega} = I$$

and

$$\begin{aligned} B^{-1}AB &= -\frac{1}{2\omega} \begin{pmatrix} -\omega & 1 \\ \omega & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \omega & -\omega \end{pmatrix} = -\frac{1}{2\omega} \begin{pmatrix} -\omega & 1 \\ \omega & 1 \end{pmatrix} \begin{pmatrix} \omega & -\omega \\ \omega^2 & \omega^2 \end{pmatrix} \\ &= -\frac{1}{2\omega} \begin{pmatrix} -2\omega^2 & 0 \\ 0 & 2\omega^2 \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix} I. \end{aligned}$$

Let

$$\underline{v} = B^{-1}\underline{q} \quad \text{ie} \quad \underline{v} = -\frac{1}{2\omega} \begin{pmatrix} -\omega & 1 \\ \omega & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \text{Mo}u \end{pmatrix} = -\frac{1}{2\omega} \begin{pmatrix} -\text{co}\eta + \text{Ho}u \\ -\text{co}\eta + \text{Ho}u \end{pmatrix}$$

$$\begin{pmatrix} \text{co} & 0 \\ 0 & -\text{co} \end{pmatrix} \underline{v} = \frac{1}{2\omega} \begin{pmatrix} \text{co}\eta + \text{Ho}u \\ \text{co}\eta - \text{Mo}u \end{pmatrix} \quad \left. \right\} \text{slightly different to given } \Sigma \text{ due to using different (but correct) eigenvectors.}$$

Then  $\underline{q} = B\underline{v}$  and we have:

$$B_{\partial_t} \underline{v} + AB_{\partial_x} \underline{v} = 0$$

as our equations. Then multiplying by  $B^{-1}$ , we have:

$$B^{-1}B_{\partial_t} \underline{v} + B^{-1}AB_{\partial_x} \underline{v} = 0, \quad \text{then}$$

$$\partial_t \underline{v} + \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix} \partial_x \underline{v} = 0 \quad \text{ie} \quad \frac{1}{2\omega} \begin{bmatrix} (\text{co}\partial_t \eta + \text{Ho}\partial_t u) \\ (\text{co}\partial_t \eta - \text{Mo}\partial_t u) \end{bmatrix} + \begin{bmatrix} \text{co}^2 \partial_x \eta + \text{co} \text{Ho} \partial_x u \\ -\text{co}^2 \eta + \text{co} \text{Ho} \partial_x u \end{bmatrix} = 0$$

Giving decoupled equations:

Eq 1

$$\text{co} \partial_t \eta + \text{Ho} \partial_t u + \text{co}^2 \partial_x \eta + \text{co} \text{Ho} \partial_x u = 0$$

Eq 2

$$\text{co} \partial_t \eta - \text{Ho} \partial_t u - \text{co}^2 \partial_x \eta + \text{co} \text{Ho} \partial_x u = 0$$

## Rearranging original equations

From the original equations  $\partial_t \eta + \partial_x (H_0 u) = 0$ ,  $\partial_t u + \partial_x (g\eta) = 0$  (with  $H(x) = H_0$ ), we have -

$$c_0 (\partial_t \eta + H_0 \partial_x u) = 0, \quad H_0 (\partial_t u + g \partial_x \eta) = 0$$

$$\text{Then: } 0 + 0 = c_0 (\partial_t \eta + H_0 \partial_x u) + H_0 (\partial_t u + g \partial_x \eta),$$

$$0 = c_0 \partial_t \eta + c_0 H_0 \partial_x u + H_0 \partial_t u + c_0^2 \partial_x \eta$$

That is, it equals Eq 1 as required.

And

$$0 - 0 = c_0 (\partial_t \eta + H_0 \partial_x u) - H_0 (\partial_t u + g \partial_x \eta)$$

$$0 = c_0 \partial_t \eta + c_0 H_0 \partial_x u - H_0 \partial_t u - c_0^2 \partial_x \eta$$

That is, we have Eq 2 as required.

(Q2)

Using the given  $\Sigma$ ,  $\partial_t \Sigma + (c_0 - c_1) \partial_x \Sigma = 0$  gives us the two linear advection equations:

$$\underline{\text{Eq 1}} : \partial_t (r_1) + c_0 \partial_x (r_1) = 0$$

$$\underline{\text{Eq 2}} : \partial_t (r_2) - c_0 \partial_x (r_2) = 0 \quad \text{where } \Sigma = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} H_0 u + c_0 \eta \\ H_0 u - c_0 \eta \end{pmatrix}$$

For Eq 1:

As a linear advection eq, evidently the characteristics are defined by:

$$\frac{dx}{dt} = c_0 \quad [= \sqrt{g H_0}]$$

and  $r_1(x, t)$  is constant along these characteristics as:

$$\begin{aligned} \frac{dr_1}{dt} &= \frac{\partial r_1}{\partial x} \frac{dx}{dt} + \frac{\partial r_1}{\partial t} = \frac{\partial r_1}{\partial x} \cdot \frac{dx}{dt} + (-c_0 \frac{\partial r_1}{\partial x}) \\ &= \frac{\partial r_1}{\partial x} \left( \frac{dx}{dt} - c_0 \right) = \frac{\partial r_1}{\partial x} \cdot 0 = 0 \end{aligned}$$

For  $t=0$ ,  $r_1(0) = \xi$ , then characteristic curves are given by

$$x(t) = c_0 t + [\text{const}]$$

$$r_1(t) = [\text{const}] = \xi$$

Then  $x(t) = \cot t + \xi$ . (and  $\xi = x - \cot t$ ).  
 There are straight lines with  $x$ -intercept  $\xi$ , slope  $k_c = c_0$ . Note  $r_i$  is constant along these lines, then

$$r_i(x,t) = r_i(x - \cot t, 0) = r_i(\xi, 0) = \begin{cases} r_{il}, & \xi < 0 \\ r_{ir}, & \xi \geq 0 \end{cases}$$

as the characteristics.

Then as  $\xi = x - \cot t$ , then we can adjust the conditions:

$$\begin{aligned}\xi < 0 &\Rightarrow x - \cot t < 0 \Rightarrow x < \cot t \\ \xi \geq 0 &\Rightarrow x - \cot t \geq 0 \Rightarrow x \geq \cot t\end{aligned}$$

giving solution:

$$r_i(x,t) = \begin{cases} r_{il}, & x < \cot t \\ r_{ir}, & x \geq \cot t \end{cases}$$

as required.

For Eq 2:

We can find the solution for Eq 2 in a similar way, with characteristics defined by

$$dx/dt = -c_0 \quad [= -\sqrt{M_0}]$$

where we have

$$\begin{aligned}d\tau/dt &= \partial\tau^2/\partial t + \partial\tau^2/\partial x \frac{dx}{dt} = c_0 \frac{\partial\tau^2}{\partial x} + \frac{\partial\tau^2}{\partial x} \frac{dx}{dt} = \frac{\partial\tau^2}{\partial x} (c_0 + dx/dt) \\ &= \frac{\partial\tau^2}{\partial x} = 0 = 0\end{aligned}$$

that is,  $r_2$  is constant on the characteristics.

Integrating and using  $\tau(0) = \xi$  for some  $\xi$ , we have characteristic curves:

$$x(t) = -\cot t + \xi, \quad \xi = x + \cot t$$

As  $r_2$  is constant on these characteristic curves -

$$r_2(x,t) = r_2(x + \cot t, 0) = r_2(\xi, 0) = \begin{cases} r_{2l}, & \xi < 0 \\ r_{2r}, & \xi \geq 0 \end{cases}$$

Adjusting conditions:

$$\begin{aligned}\xi < 0 &\Rightarrow x + \cot t < 0 \Rightarrow x < -\cot t \\ \xi \geq 0 &\Rightarrow x + \cot t \geq 0 \Rightarrow x \geq -\cot t\end{aligned}$$

Then we have:

$$r_2(x,t) = \begin{cases} r_{2L}, & x < -\cot \\ r_{2R}, & x \geq -\cot \end{cases}$$

on the characteristics as required.

Solving for  $u(x,t)$ ,  $\eta(x,t)$ :

Evidently, we have that:

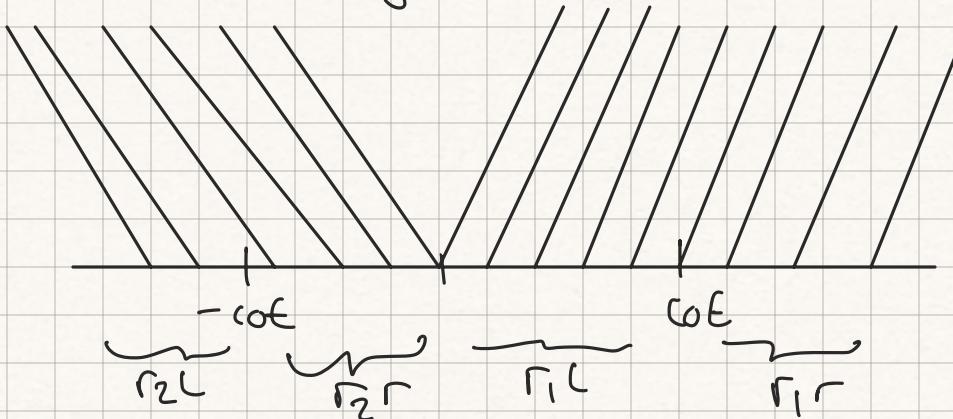
$$Hu = r_1 - \cot \eta = r_1 + r_2 - Hu \Rightarrow 2Hu = r_1 + r_2, Hu = \frac{1}{2}(r_1 + r_2)$$

and

$$\eta = \frac{r_1 - Hu}{\cot} = \frac{r_1 - (r_2 + \cot)}{\cot} = \frac{r_1 - r_2 - \cot}{\cot} \Rightarrow 2\cot\eta = r_1 - r_2$$

$$\text{i.e. } \eta = \frac{1}{2}\cot(r_1 - r_2).$$

Sketching characteristics for  $\lambda_1 = \cot$  (corresponding to  $r_1$ ) and  $\lambda_2 = -\cot$  (corresponding to  $r_2$ ), so far -



Using that  $\lambda_1, \lambda_2$  are constant and do not depend on  $u$  or  $\eta$ .

So we are seeking solutions for  $(Hu, \eta)$  in the intervals:

- $x < -\cot$  : corresponds to  $(Hu_L, \eta_L)$
- $-\cot \leq x < \cot$  : corresponds to  $(Hu_f, \eta_f)$
- $x \geq \cot$  : corresponds to  $(Hu_R, \eta_R)$

where for left and right states, we can use initial data of  $r_1, r_2$ , that is,

$$r_{1L} = Hu_L + \cot\eta_L, r_{2L} = Hu_L - \cot\eta_L$$

$$r_{1R} = Hu_R + \cot\eta_R, r_{2R} = Hu_R - \cot\eta_R$$

denote define

$$(Hu_L, \eta_L) = \left( \frac{1}{2}(r_{1L} + r_{2L}), \frac{1}{2}\cot(r_{1L} - r_{2L}) \right)$$

$$(t_{\text{hour}}, n_r) = \left(\frac{1}{2}(r_{1r} + r_{2r}), \frac{x}{2e}(n_r - n_{1r})\right).$$

$(t_{\text{hour}}, n_f)$  defines the solution in the middle region, which we need to find. Clearly, these characteristics must be in a fan.

holding at characteristics  $\lambda_i = c_0$  in the fan, they come from the origin at slope  $\nu = x/e$ , ie -

$$\lambda_1 = c_0 = \frac{x}{e}, \quad \lambda_2 = -c_0 = -\frac{x}{e}$$

Then on  $-c_0 t < x < c_0 t$

$$r_{2r} = t_{\text{hour}} + \frac{x}{e} n_r \quad \text{const on } -c_0 = \lambda_2.$$

$$= t_{\text{hour}} + \frac{x}{e} n$$

$$n_{1r} = t_{\text{hour}} + \frac{x}{e} n_l \quad \text{const on } c_0 = \lambda_1.$$

$$= t_{\text{hour}} + \frac{x}{e} n_L$$

\* I think the crossed out solutions are technically correct, but their dependence on  $x/e$  complicates things later on.

Then:

$$t_{\text{hour}} = \frac{1}{2}(r_{1r} + r_{2r}) = \frac{1}{2}\left(t_{\text{hour}} + \frac{x}{e} n_r + t_{\text{hour}} + \frac{x}{e} n_L\right)$$

$$= \frac{t_{\text{hour}}}{2}(u_r + u_L) + \frac{x}{2e}(n_r + n_L)$$

$$n_f = \frac{1}{2}(r_{1r} - r_{2r}) = \frac{1}{2}(t_{\text{hour}} - \frac{x}{e} n_L - (t_{\text{hour}} - \frac{x}{e} n_r))$$

$$= \frac{t_{\text{hour}}}{2}(u_r - u_L) + \frac{x}{2e}(n_r - n_L)$$

giving :

$$t_{\text{hour}}(k, t) = \begin{cases} t_{\text{hour}}, & x < -c_0 t \\ \frac{t_{\text{hour}}}{2}(u_r + u_L) + \frac{x}{2e}(n_r + n_L), & -c_0 t < x < c_0 t \\ t_{\text{hour}}, & x > c_0 t \end{cases}$$

$$n(k, t) = \begin{cases} n_L, & x < -c_0 t \\ \frac{t_{\text{hour}}}{2}(u_r - u_L) + \frac{x}{2e}(n_r - n_L), & -c_0 t < x < c_0 t \\ n_r, & x > c_0 t \end{cases}$$

\* May be some sign errors.

Then making sure we have to substitute in for  $x/c$ :

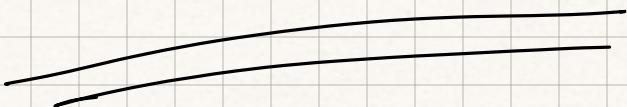
$$\begin{aligned} \text{Houf} &= \frac{1}{2}(r_{1c} + r_{2r}) = \frac{1}{2}(\text{Hou}_L + \text{cou}_L) + \frac{1}{2}(\text{Hou}_R - \text{cou}_R) \\ &= \frac{\text{Ho}}{2}(u_L + u_R) + \frac{\text{co}}{2}(n_L - n_R) \end{aligned}$$

$$\begin{aligned} \eta_f &= \frac{1}{2}(r_{1L} - r_{2R}) = \frac{1}{2}(\text{Hou}_L + \text{cou}_L) - \frac{1}{2}(\text{Hou}_R - \text{cou}_R) \\ &= \frac{\text{Ho}}{2}(u_L - u_R) + \frac{\text{co}}{2}(n_L + n_R) \end{aligned}$$

giving:

$$\text{Hou}(k, t) = \begin{cases} \text{Hou}_L, & x < -\cot \\ \frac{\text{Ho}}{2}(u_R + u_L) + \frac{\text{co}}{2}(n_L - n_R), & -\cot < x < \cot \\ \text{Hou}_R, & x > \cot \end{cases}$$

$$\eta(k, t) = \begin{cases} n_L, & x < -\cot \\ \frac{\text{Ho}}{2}(u_L - u_R) + \frac{\text{co}}{2}(n_L + n_R), & -\cot < x < \cot \\ n_R, & x > \cot \end{cases}$$



Q3

We're using:

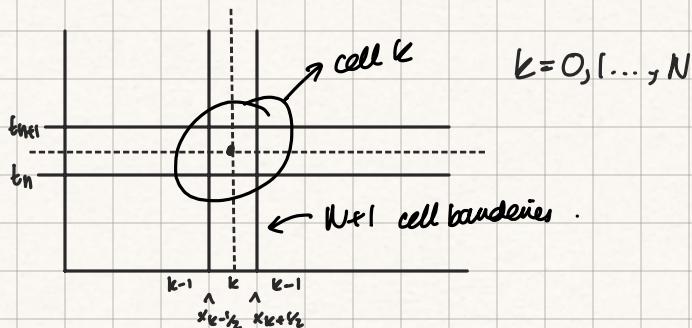
( for  $H(k) = \text{const.} = H_0$  )

$$q_t + f(q)_x = 0$$

where

$$q = \begin{pmatrix} \eta \\ u \end{pmatrix}, \quad f(q) = \begin{pmatrix} Mu \\ qu \end{pmatrix} \quad \left. \begin{array}{l} \frac{\partial}{\partial t} \\ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(fu) \\ \frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(qu) \end{array} \right\} = 0,$$

Diagram of cell arrangement in this method:



### Douglas Crooknes Scheme

For cell  $x_{k-1/2} < x_{k+1/2}$ :

$$Q_k(t) = \frac{1}{h_k} \int_{x_{k-1/2}}^{x_{k+1/2}} q(x, t) dx$$

where  
 $h_k = x_{k+1/2} - x_{k-1/2}$   
 (not assumed uniform cell size)

where  $Q_k(t)$  is a vector st

$$Q_k(t) = \frac{1}{h_k} \begin{pmatrix} \int_{x_{k-1/2}}^{x_{k+1/2}} \eta(x, t) dx \\ \int_{x_{k-1/2}}^{x_{k+1/2}} u(x, t) dx \end{pmatrix}$$

As

$$0 = q_t + f(q)_x$$

Then

$$\int_{t_n}^{t_{n+1}} \int_{x_{k-1/2}}^{x_{k+1/2}} \frac{\partial q}{\partial t} + \frac{\partial}{\partial x} f(q) dx dt =$$

$$\int_{t_n}^{t_{n+1}} \int_{x_{k-1/2}}^{x_{k+1/2}} 0 dx dt = 0$$

Then:

$$\begin{aligned}
0 &= \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \int_{t_n}^{t_{n+1}} \frac{\partial}{\partial t} q \, dt \, dx + \int_{t_n}^{t_{n+1}} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \frac{\partial}{\partial x} f(q) \, dx \, dt \\
&= \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} q(x, t_{n+1}) \, dx - \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} q(x, t_n) \, dx + \int_{t_n}^{t_{n+1}} f(x_{k+\frac{1}{2}}, t) \, dt \\
&\quad - \int_{t_n}^{t_{n+1}} f(x_{k-\frac{1}{2}}, t) \, dt.
\end{aligned}$$

where  
 $f(x_{k+\frac{1}{2}}, t) = \begin{pmatrix} M(x_{k+\frac{1}{2}}, t) u(x_{k+\frac{1}{2}}, t) \\ g\eta(x_{k+\frac{1}{2}}, t) \end{pmatrix}$ ,  
 $f(x_{k-\frac{1}{2}}, t) = \begin{pmatrix} M(x_{k-\frac{1}{2}}, t) u(x_{k-\frac{1}{2}}, t) \\ g\eta(x_{k-\frac{1}{2}}, t) \end{pmatrix}$

Dividing by  $\Delta x$ :

$$\begin{aligned}
0 &= \frac{1}{\Delta x} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} q(x, t_{n+1}) \, dx - \frac{1}{\Delta x} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} q(x, t_n) \, dx + \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} f(x_{k+\frac{1}{2}}, t) \, dt \\
&\quad - \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} f(x_{k-\frac{1}{2}}, t) \, dt. \\
&= Q_k^n(t_{n+1}) - Q_k^n(t_n) + \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} f(x_{k+\frac{1}{2}}, t) \, dt - \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} f(x_{k-\frac{1}{2}}, t) \, dt.
\end{aligned}$$

Writing: where  $Q_k^n$  is constant,

$$Q_k^n - Q_k^{n+1} = \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} f(x_{k+\frac{1}{2}}, t) \, dt - \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} f(x_{k-\frac{1}{2}}, t) \, dt$$

We define approximate numerical flux for next time  $\Delta t$ .

$$F(Q_k^n, Q_{k+1}^n) = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(x_{k+\frac{1}{2}}, t) \, dt = \left( \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} M(x_{k+\frac{1}{2}}, t) u(x_{k+\frac{1}{2}}, t) \, dt \right) - \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} g\eta(x_{k+\frac{1}{2}}, t) \, dt$$

Hence:

$$Q_k^n - Q_k^{n+1} = \frac{\Delta t}{\Delta x} \left[ F(Q_k^n, Q_{k+1}^n) - F(Q_{k-1}^n, Q_k^n) \right]$$

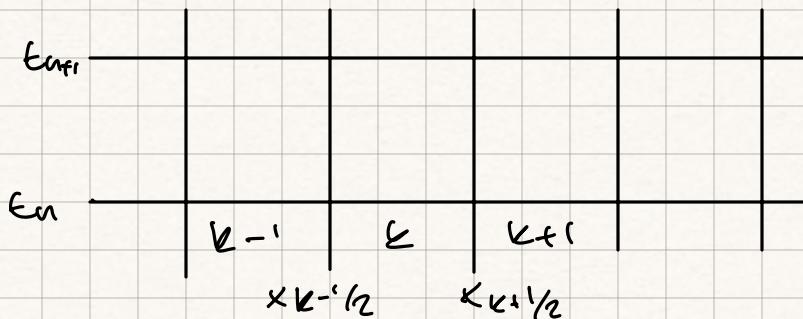
Our Crank-Nicolson Scheme.

(bearing in mind that :

$$Q_k^n = \frac{1}{h} \int_{k-\frac{1}{2}}^{x_{k+\frac{1}{2}}} q(x, t_n) dx = \left( \begin{array}{l} \frac{1}{h} \int_{k-\frac{1}{2}}^{x_{k+\frac{1}{2}}} \eta(x, t_n) dx \\ \frac{1}{h} \int_{k-\frac{1}{2}}^{x_{k+\frac{1}{2}}} u(x, t_n) dx \end{array} \right)$$

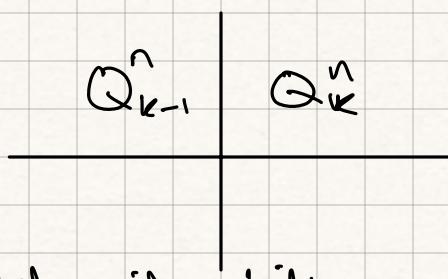
i.e.  $Q_k^n, f$  are vector values)

Moving out selection domain:

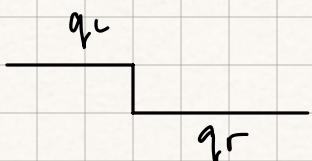


where

$Q_k^n$  is constant in its cell - each cell has their own Reissner problem and at faces

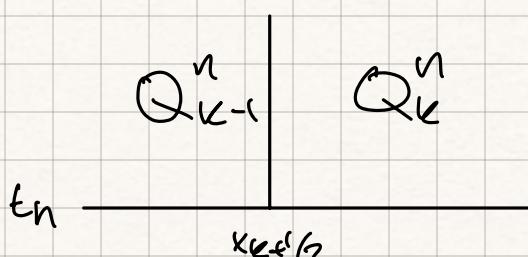


we can model it like a Reissner problem.



where  $q_L = Q_{k-1}^n$ ,  $q_R = Q_k^n$

Let us look at cell edge  $x_{k+\frac{1}{2}}$



Referring to our original Reissner problem:

We have co-ordinate shift  $(\sigma, \eta) \mapsto (x_j + v_2, t_n)$   
so  $(k, t) \mapsto (k + k_j + v_2, t + t_n)$

Note

$$Q_k^n = \begin{pmatrix} E_k^n \\ U_k^n \end{pmatrix} \quad \text{for } E_k^n \text{ corresponding to } \eta, \\ U_k^n \text{ corresponding to } u.$$

Then

$$Q_{k+1}^n = \begin{pmatrix} E_{k+1}^n \\ U_{k+1}^n \end{pmatrix} \text{ corresponds to } \begin{pmatrix} U_L \\ U_C \end{pmatrix}$$

$$Q_{k+1}^n = \begin{pmatrix} E_{k+1}^n \\ U_{k+1}^n \end{pmatrix} \text{ corresponds to } \begin{pmatrix} U_R \\ U_T \end{pmatrix}$$

for

$$\underline{Q_k^n \mid Q_{k+1}^n}$$

So we have : Neumann problems -

$$M_{kl} = \begin{cases} M_0 U_k^n, & x < c_0 t_n \\ \frac{M_0}{2} (U_{k+1}^n + U_k^n) + \frac{c_0}{2} (E_k^n - E_{k+1}^n), & -c_0 t_n < x < c_0 t_n \\ M_0 U_{k+1}^n, & x > c_0 t_n \end{cases}$$

$$\eta = \begin{cases} E_k^n, & x < -c_0 t_n \\ \frac{M_0}{2} (U_k^n - U_{k+1}^n) + \frac{x c_0 t_n}{2 t_n} (E_{k+1}^n + E_k^n), & -c_0 t_n < x < c_0 t_n \\ E_{k+1}^n, & x > c_0 t_n \end{cases}$$

As we use constants, we can drop the  $M_k = f_k$  constant case by taking  $M_C, M_T$  corresponding to  $U_C, U_T$  respectively in the above, set

$$U_k^n U_{k+1}^n = M_C U_C, \quad U_{k+1}^n U_{k+1}^n = M_T U_T,$$

and

$$M_0 (U_{k+1}^n + U_k^n), M_0 (U_k^n - U_{k+1}^n), M_0 U_k^n, M_0 U_{k+1}^n \text{ correspond to} - \\ (M_{k+1}^n U_{k+1}^n + M_k^n U_k^n), (M_k^n U_k^n - M_{k+1}^n U_{k+1}^n), M_k^n U_k^n, M_{k+1}^n U_{k+1}^n.$$

Time Step

$CFL < 1$  Then

$$\Delta t = \frac{c_{\text{viser}}}{k} \frac{h c}{c_0}$$

where  $CFL \approx 1$ .

### Boundary Conditions

At  $E_i^n$ , set  $E_i^n = E_0^n = r_c \quad \left. \right\} \begin{matrix} \text{Extrapolated} \\ \text{BCs.} \end{matrix}$   
 At  $U_i^n$ , set  $U_i^n = -U_0^n = -U_c \quad \left. \right\} \begin{matrix} \text{Extrapolated} \\ \text{BCs.} \end{matrix}$   
 for  $\forall n$ .

And

$$\begin{aligned} E_{J-1}^n &= E_J^n = r_r \quad \left. \right\} \text{At right boundary} \\ U_{J-1}^n &= U_J^n = -U_r \quad \left. \right\} \text{Right edge of domain.} \end{aligned}$$