MATH 5453M Numerical Exercise 2

Finite volume method: linear shallow-water equation. Predicting surf height at beaches

Consider the linearised shallow-water system of eqn:-

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0 \quad \& \quad \frac{\partial u}{\partial t} + \frac{\partial (g\eta)}{\partial x} = 0 \qquad \text{— (1)}$$

for velocity $u = u(x,t)$ & free surface deviation $\eta = \eta(x,t)$, rest depth $H(x)$ & acceleration of gravity $g = 9.81\ m/sec$,

when we scale (1)

$$u = V_0 u', \quad x = L_s x', \quad t = (L_s/V_0)t'$$

$$\eta = H_{0s}\eta', \quad H = H_{0s}H'.$$

$$g' \rightarrow g H_{0s}/V_0^2,$$

Ques (i) :-

let's write the non-dimensional form of (1)

$$\frac{U_0 H_{os}}{L_s}\frac{\partial \eta'}{\partial t'} + \frac{U_0 H_{os}}{L_s}\frac{\partial (H'u')}{\partial x'} = 0 \quad \&$$

$$\frac{U_0^2}{L_s}\frac{\partial u'}{\partial t'} + \frac{U_0^2}{L_s}\frac{\partial (g'\eta')}{\partial x'} = 0$$

$$\frac{\partial \eta'}{\partial t'} + \frac{\partial (H'u')}{\partial x'} = 0 \qquad (*)$$

$$\frac{\partial u'}{\partial t'} + \frac{\partial (g'\eta')}{\partial x'} = 0 \qquad (**)$$

↳ after non-dimensionalisation also we are having same equation as (1).

Except $g' = g\frac{H_{os}}{U_0^2}$

- Since both equation (*) & (**) are conservation equation (conservation of mass & conservation of momentum respectively).

- Both are coupled because u & η influence each other.

↓

we can write these two equations in matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} \eta \\ H_0 u \end{pmatrix} = 0$$

where $A = \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix}$

$\Rightarrow c_0^2 = gH_0$; wave speed

using linear algebra. we can decouple these equation.

finding eigen values & eigen vectors of matrix A

$$\det|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ c_0^2 & -\lambda \end{vmatrix} = \lambda^2 - c_0^2 = 0$$

$\therefore$ eigen values are $\lambda_1 = C_0$, $\lambda_2 = -C_0$.

Now finding the eigen vectors

$$A v_1 = \lambda_1 v_1 \quad \& \quad A v_2 = \lambda_2 v_2$$

$$\begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = C_0 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

$$v_{12} = C_0 v_{11}$$
$$C_0^2 v_{11} = C_0 v_{12}$$
$$\downarrow$$
$$C_0^2 v_{11} = C_0 (C_0 v_{11})$$

$$\Rightarrow \quad v_1 = \frac{1}{2C_0} \begin{pmatrix} 1 \\ C_0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = -C_0 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$
$$\downarrow$$
$$v_2 = \frac{1}{2C_0} \begin{pmatrix} -1 \\ C_0 \end{pmatrix}$$

$\Rightarrow$ Therefore, Matrix B is consisting eigen vectors is,

$$B = \frac{1}{2C_0} \begin{pmatrix} 1 & -1 \\ C_0 & C_0 \end{pmatrix}$$

$$\Rightarrow \quad \text{Now} \quad B^{-1} = \frac{1}{|B|}\begin{pmatrix} C_0 & -C_0 \\ 1 & 1 \end{pmatrix}^T$$

$$= \frac{1}{|B|}\begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix}$$

$$= \frac{1}{\frac{1}{2C_0}(C_0 + C_0)}\begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix}$$

$$\& \quad AB = \begin{pmatrix} 0 & 1 \\ C_0^2 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2C_0} & \frac{-1}{2C_0} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{C_0}{2} & \frac{-C_0}{2} \end{pmatrix}$$

$$\therefore \quad B^{-1}AB = \begin{pmatrix} C_0 & 1 \\ -C_0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{C_0}{2} & -\frac{C_0}{2} \end{pmatrix}$$

$$= \begin{pmatrix} C_0 & 0 \\ 0 & -C_0 \end{pmatrix}$$

$$B^{-1}AB \Rightarrow \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \lambda I$$

$\Rightarrow$ Now, we will proceed to decoupling with new variable γ

$\hookrightarrow \quad \gamma = B^{-1} (h , H_0 u)^T$

$$\Rightarrow \gamma = B^{-1} \begin{pmatrix} h \\ H_0 u \end{pmatrix}$$

$$\Rightarrow \quad B\gamma = \begin{pmatrix} h \\ H_0 u \end{pmatrix}$$

But $\dfrac{\partial}{\partial t} \begin{pmatrix} h \\ H_0 u \end{pmatrix} = B \dfrac{\partial \gamma}{\partial t}$

& $\dfrac{\partial}{\partial x} \begin{pmatrix} h \\ H_0 u \end{pmatrix} = B \dfrac{\partial \gamma}{\partial x}$

$\downarrow$

original eqn Becomes

$$B \frac{\partial \gamma}{\partial t} + AB \frac{\partial \gamma}{\partial x} = 0 \quad - \; *** $$

But from above we have, $B^{-1}AB = \lambda I$

$\Rightarrow$ Multiplying B^{-1} in (***):

$$B^{-1}B\frac{\partial r}{\partial t} + B^{-1}AB\frac{\partial r}{\partial x} = 0$$

$$\boxed{\frac{\partial r}{\partial t} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\frac{\partial r}{\partial x} = 0}$$

$\hookrightarrow$ decoupled set of linear advection equation

where $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ & $r_1 = H_0 u + C_0 h$
$r_2 = H_0 u - C_0 h$

($r \rightarrow$ Riemann invariant)

Ques : 2 →

for piecewise constant initial data

$$r_1(x,0) = \begin{cases} r_{1l} & \text{for } x<0 \\ r_{1r} & \text{for } x\geq 0 \end{cases}, \quad r_2(x,0) \begin{cases} r_{2l} & \text{for } x<0 \\ r_{2r} & \text{for } x\geq 0 \end{cases}$$

from equation — 7

$$\frac{\partial r}{\partial t} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{\partial r}{\partial x} = 0$$

$$\rightarrow \frac{\partial r_1}{\partial t} + c_0 \frac{\partial r_1}{\partial x} = 0 \qquad \begin{pmatrix} \lambda_1 = c_0 \\ \lambda_2 = -c_0 \end{pmatrix}$$

& $$\frac{\partial r_2}{\partial t} - c_0 \frac{\partial r_2}{\partial x} = 0$$

↓

these are linear advection equation,

∴ the solution would be:

$$\begin{cases} r_1(x,t) = r_1(x - c_0 t, 0) \; \& \\ r_2(x,t) = r_2(x + c_0 t, 0) \end{cases}$$

$c_0 \rightarrow$ propagation speed.

→ r_1 const. along characteristics line $x - c_0 t = \text{const}$ &

r_2 along $x + c_0 t = \text{const}$

Based on the given initial condition

$$r_1(x,t) = \begin{cases} r_{1l} & \text{if } x - C_0 t < 0 \\ r_{1r} & \text{if } x - C_0 t \geq 0 \end{cases}$$

$$r_1(x,t) = \begin{cases} r_{1l} & \text{if } x < C_0 t \\ r_{1r} & \text{if } x \geq C_0 t \end{cases} \quad \text{— (A)}$$

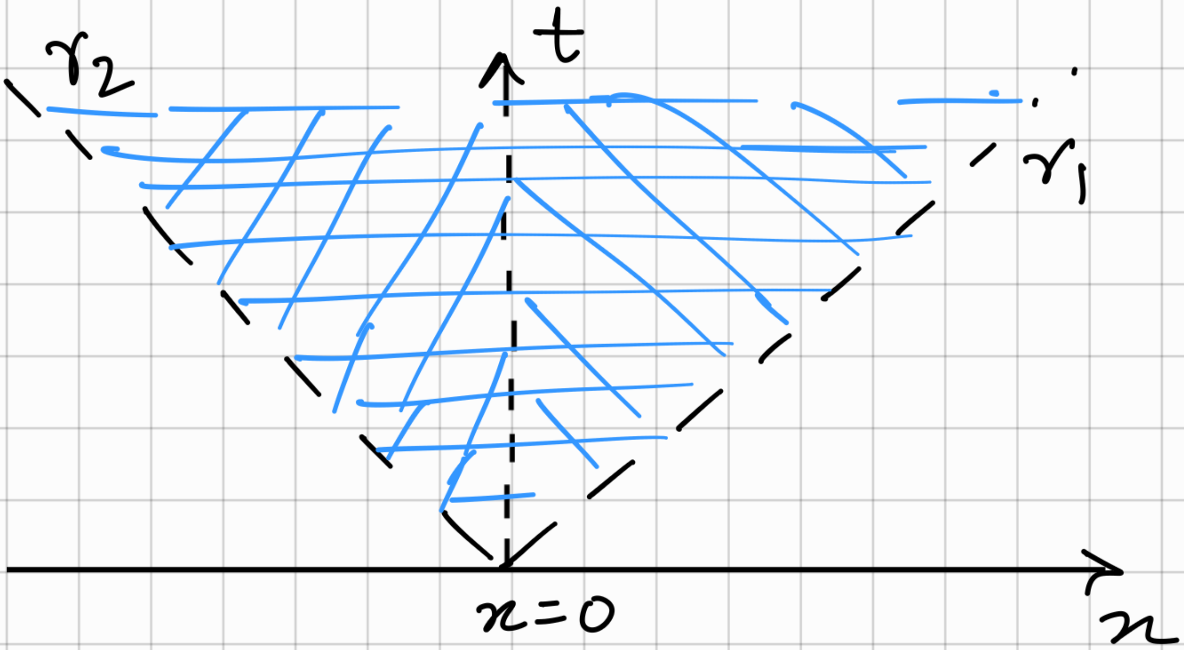
and

$$r_2(x,t) = \begin{cases} r_{2l} & \text{if } x + C_0 t < 0 \\ r_{2r} & \text{if } x + C_0 t \geq 0 \end{cases}$$

$$r_2(x,t) = \begin{cases} r_{2l} \, , & x \leq -C_0 t \\ r_{2r} \, , & x \geq -C_0 t \end{cases} \quad \text{— (B)}$$

Since the initial discontinuity is at $x=0$, @ $t=0$

if we plot the $r_1(x,t)$ & $r_2(x,t)$



<u>Now</u>:- given that $r_1 = H_0 u + c_0 \eta$

$$r_2 = H_0 u - c_0 \eta$$

and $H_0 u = \frac{1}{2}(r_1 + r_2)$,

$$\eta = \frac{1}{2}(r_1 - r_2)/c_0$$

⇒ as per r_1 & r_2 value given in (A) & (B)

(1) for $x < -c_0 t$

↳ $r_1 = r_{1l}$ & $r_2 = r_{2l}$

$$\therefore H_0 u(x,t) = \frac{1}{2}(r_{1l} + r_{2l})$$

$$= \frac{1}{2}\left(H_0 u_l + \cancel{c_0 \eta_l} + H_0 u_l - \cancel{c_0 \eta_l}\right)$$

$$H_0 u(x,t) = \frac{1}{2}(2 H_0 u_l) = H_0 u_l$$

② for the shaded part,

$-c_0 t \leq x \leq c_0 t$

$$r_1 = r_{1l} \ , \ r_2 = r_{2r}$$

$$\therefore H_0 u(x,t) = \frac{1}{2}(r_{1l} + r_{2r})$$

③ for $x > c_0 t$

$$r_1 = r_{1r} \ \& \ r_2 = r_{2r}$$

$$\therefore H_0 u(x,t) = \frac{1}{2}(r_{1r} + r_{2r})$$

$$= \frac{1}{2}(H_0 u_r + \cancel{c_0 \eta_r} + H_0 u_r - \cancel{c_0 \eta_r})$$

$$H_0 u(x,t) = H_0 u_r$$

$$\Rightarrow \quad H_0 u(x,t) = \begin{cases} H_0 u_l & x < c_0 t \\ \frac{1}{2}(r_{1l} + r_{2r}) & -c_0 t \leq x \leq c_0 t \\ H_0 u_r & x > c_0 t \end{cases}$$

Similarly for $\eta(x,t)$

① $x < -C_0 t$

$r_1 = r_{1l}$, $r_2 = r_{2l}$

$$\eta(x,t) = \frac{1}{2C_0}\left(\cancel{H_0 u_l} + C_0\eta_l - \cancel{H_0 u_l} + C_0\eta_l\right)$$

$$\eta(x,t) = \eta_l$$

② for $-C_0 t \leq x \leq C_0 t$

$r_1 = r_{1l}$ $r_2 = r_{2r}$

$$\eta(x,t) = \frac{1}{2C_0}\left(H_0 u_l + C_0\eta_l - H_0 u_r + C_0\eta_r\right)$$

③ for $x > C_0 t$

$r_1 = r_{1r}$ $r_2 = r_{2r}$

$$\eta(x,t) = \frac{1}{2C_0}\left(\cancel{H_0 u_r} + C_0\eta_r - \cancel{H_0 u_r} + C_0\eta_r\right)$$

$$\eta(x,t) = \eta_r$$

$$\Rightarrow \eta(x,t) = \frac{1}{2C_0}\left(r_1(x,t) - r_2(x,t)\right)$$

$$\Rightarrow \eta(x,t) = \begin{cases} \eta_l & x < -c_0 t \\ H_0(u_l - u_r) + c_0(\eta_l + \eta_r) & \text{for } -c_0 t \le x \le c_0 t \\ \eta_r & x > c_0 t \end{cases}$$

$\Rightarrow$ Riemann invariants

$$r_1 = H_0 u + \sqrt{gH_0}\,\eta$$
$$r_2 = H_0 u - \sqrt{gH_0}\,\eta$$

& as per above:

$$H_0 u = \frac{1}{2}(r_1 + r_2) \quad \& \quad \eta = \frac{1}{2\sqrt{gH_0}}(r_1 - r_2)$$

$\therefore$ from $\dfrac{\partial \eta}{\partial t} + H_0 \dfrac{\partial u}{\partial x} = 0$

$$\frac{\partial}{\partial t}\left(\frac{1}{2\sqrt{gH_0}}(r_1 - r_2)\right) + H_0 \frac{\partial}{\partial x}\left(\frac{1}{2H_0}(r_1 + r_2)\right) = 0$$

$$\frac{1}{2\sqrt{gH_0}}\left(\frac{\partial r_1}{\partial t} - \frac{\partial r_2}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial r_1}{\partial x} + \frac{\partial r_2}{\partial x}\right) = 0 \longrightarrow C$$

Similarly for

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2H_0}(r_1+r_2)\right) + g\frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{gH_0}}(r_1-r_2)\right) = 0$$

$$\hookrightarrow \frac{1}{2H_0}\left(\frac{\partial r_1}{\partial t} + \frac{\partial r_2}{\partial t}\right) + \frac{g}{2\sqrt{gH_0}}\left(\frac{\partial r_1}{\partial x} - \frac{\partial r_2}{\partial x}\right) = 0 \quad \text{— (D)}$$

from (C) multiply by $2\sqrt{gH_0}$

$$\left(\frac{\partial r_1}{\partial t} - \frac{\partial r_2}{\partial t}\right) + \sqrt{gH_0}\left(\frac{\partial r_1}{\partial x} + \frac{\partial r_2}{\partial x}\right) = 0$$

for $r_1 \rightarrow \dfrac{\partial r_1}{\partial t} + \sqrt{gH_0}\dfrac{\partial r_1}{\partial x} = 0$

$r_2 \rightarrow \dfrac{\partial r_2}{\partial t} - \sqrt{gH_0}\dfrac{\partial r_2}{\partial x} = 0$

↳ uncoupled

r_1 & r_2 do not depend on each other.

$\Rightarrow$ characteristic lines for r_1, would be

$$x - \sqrt{gH_0}\,t = \text{const}$$

for r_2

$$x + \sqrt{gH_0}\,t = \text{const}$$

Question 3:-

from given Given linearised shallow-water system of equations.

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0 \quad \text{and,}$$

$$\frac{\partial u}{\partial t} + \frac{\partial (g\eta)}{\partial x} = 0$$

Such that, $u = u(x,t) \rightarrow$ Variable velocity

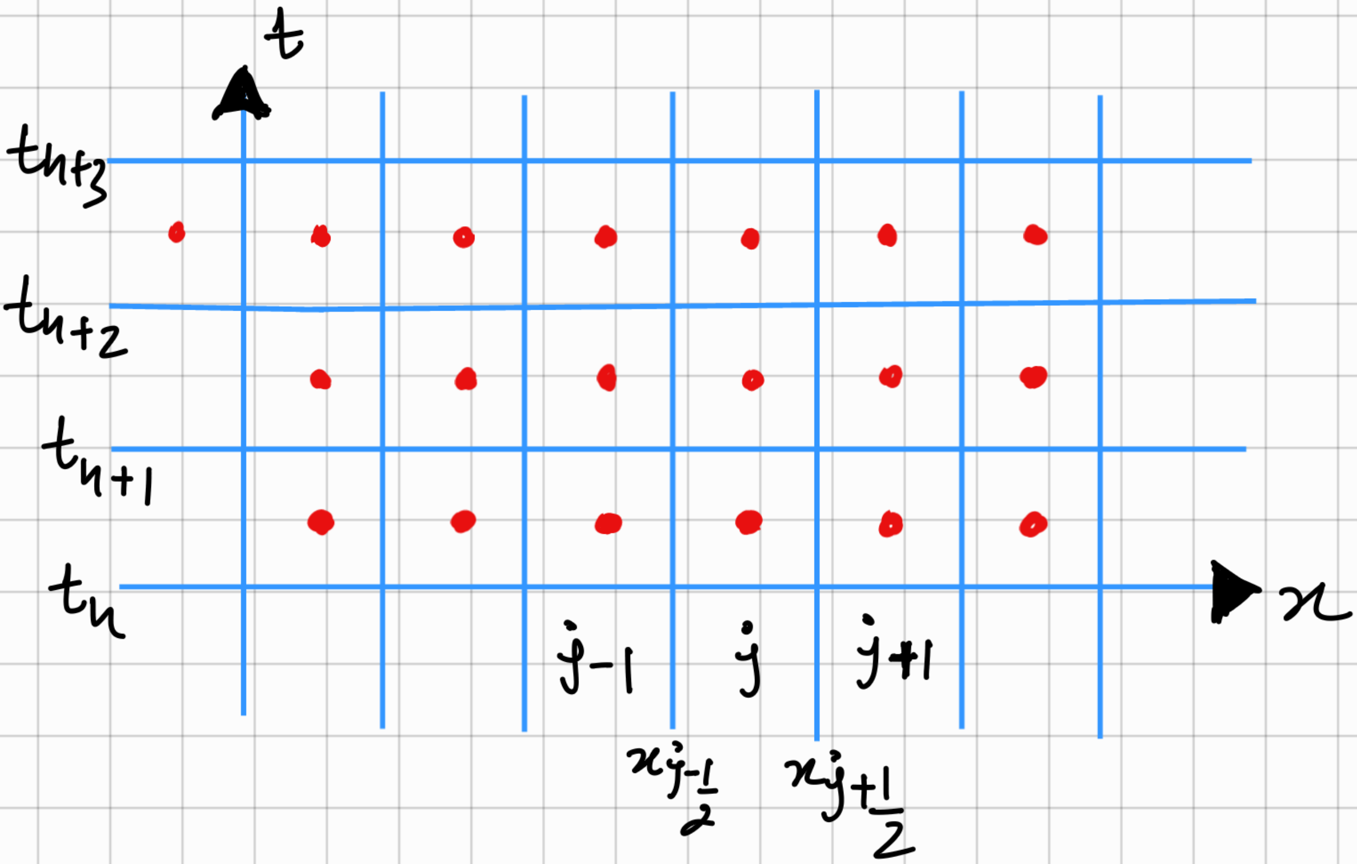
free Surface deviation, $\eta = \eta(x,t)$

for Godunov method.

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

where, $u(x,t) = \begin{pmatrix} \eta \\ u \end{pmatrix}$

& $f(u) = \begin{pmatrix} Hu \\ gh \end{pmatrix} \rightarrow$ flux



For Godunov discretization,

$$U_j^{n+1} = U_j^n - \frac{1}{h_j}\int_{t_n}^{t_{n+1}} f_{j+\frac{1}{2}}(t) - f_{j-\frac{1}{2}}(t)\,dt,$$

we have defined the mean cell average U_j in cell j.

$$U_j(t) = \frac{1}{h_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t)\,dx,$$

The flux is evaluated at the cell boundaries

$$f_{j+\frac{1}{2}}(t) = f\left(u\left(x = x_{j+\frac{1}{2}}, t\right)\right).$$

$\Rightarrow$ $U_j(t)$ & $f_{j+\frac{1}{2}}(t)$ are function of time t,
Such that:

$$U_j^n = U_j(t = t_n) \text{ So on}\ldots$$

we need to determine $f_{j+\frac{1}{2}}(t)$ over the time interval $t_n < t < t_{n+1}$ to obtain the F.V.M Scheme.

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{h_j}\left[F_{j+\frac{1}{2}}(U_j^n, U_{j+1}^n) - F_{j-\frac{1}{2}}(U_{j-1}^n, U_j^n)\right]$$

A

numerical flux approximated as,

$$F_{j+\frac{1}{2}}(U_j^n, U_{j+\frac{1}{2}}^n) \simeq \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f_{j+\frac{1}{2}}(t)\,dt$$

Assuming,

- h & H_0 are constant locally.
- Flux is evaluated using the central solution of Riemann problem at the interface

Fluxes will be;

$$f(u) = \begin{pmatrix} H_0 u \\ gh \end{pmatrix}$$

$$\Rightarrow \quad f_1(u(x_{j+\frac{1}{2}},t)) = H_0 u_{j+\frac{1}{2}}(t)$$

$$= \frac{1}{2}\left(H_0 u_{j+1} + H_0 u_j\right) + \frac{c_0}{2}\left(\eta_j - \eta_{j+1}\right)$$

or

$$= \frac{1}{2}\left((H_0 u)_r + (H_0 u)_l\right) + \frac{c_0}{2}\left(\eta_l - \eta_r\right)$$

&

$$\Rightarrow \quad f_2(u(x_{j+\frac{1}{2}},t)) = g\eta_{j+\frac{1}{2}}(t)$$

$$= \frac{1}{2} g(\eta_l + \eta_r) + \frac{1}{2c_0}\left((H_0 u)_l - (H_0 u)_r\right)$$

$\Rightarrow$ Time discretization by averaging the fluxes over time interval $[t^n, t^{n+1}]$

$$F_{j+\frac{1}{2}} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f_1(u)\,dt = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (Hu)_{j+1}\,dt = (Hu)_{j+\frac{1}{2}}$$

$$= \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f_2(u)\,dt = \frac{1}{\Delta t}\int_{t^n}^{t^{n}} (g\eta)_{j+\frac{1}{2}}\,dt = (g\eta)_{j+\frac{1}{2}}$$

using these in eqn (A);

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{h_j}\left[F_{j+\frac{1}{2}}(U_j^n, U_{j+1}^n) - F_{j-\frac{1}{2}}(U_{j-1}^n, U_j^n)\right] \quad \text{(A)}$$

$$\Rightarrow F_{1,j+\frac{1}{2}} = \frac{1}{2}\left((H_0u)_{j+1}^n + (H_0u)_j^n\right) + \frac{c_0}{2}\left(\eta_j^n - \eta_{j+1}^n\right)$$

$$F_{2,j+\frac{1}{2}} = \frac{1}{2}\left((g\eta)_j^n + (g\eta)_{j+1}^n\right) + \frac{1}{2c_0}\left((H_0u)_j^n - (H_0u)_{j+1}^n\right)$$

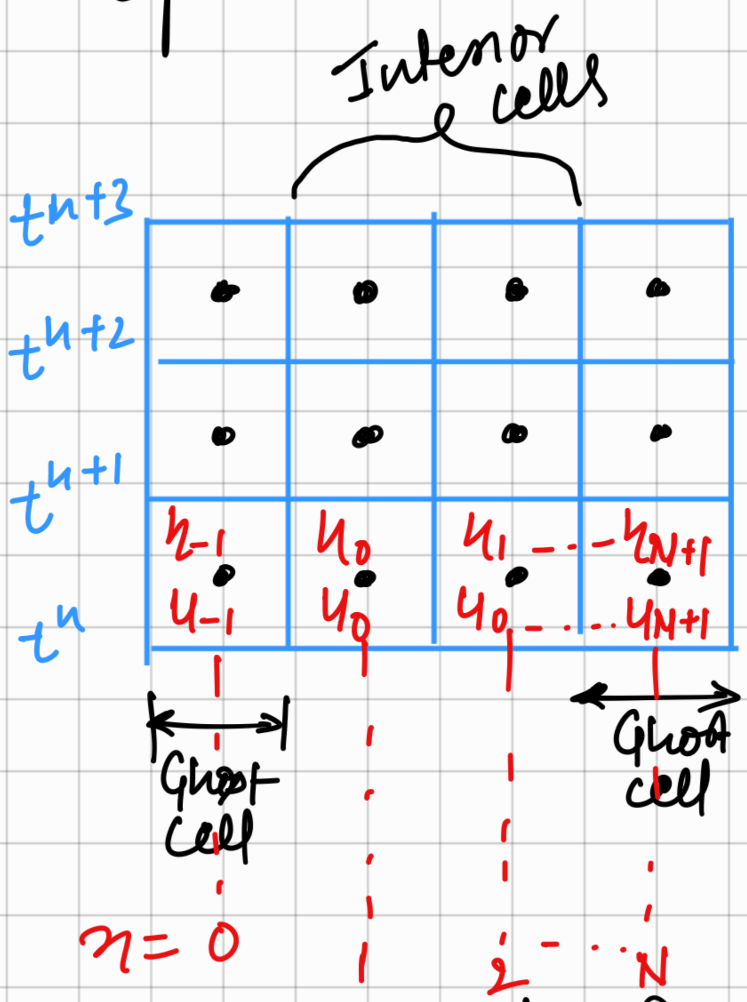
Similarly for

$$F_{1,j-\frac{1}{2}} = \frac{1}{2}\left((H_0u)_{j-1}^n + (H_0u)_j^n\right) + c_0\left(\eta_{j-1}^n - \eta_j^n\right)$$

$$F_{2,j-\frac{1}{2}} = \frac{1}{2}g\left(\eta_{j-1}^n + \eta_j^n\right) + \frac{1}{2c_0}\left((H_0u)_{j-1}^n - (H_0u)_j^n\right)$$

$\Rightarrow$ Extrapolating the Boundary condition

(1) open :-



at $x=0$, left boundary $(x=0)$

↳ $h_0 = h_{-1}$ & $u_{-1} = u_0$

& for first interior cell

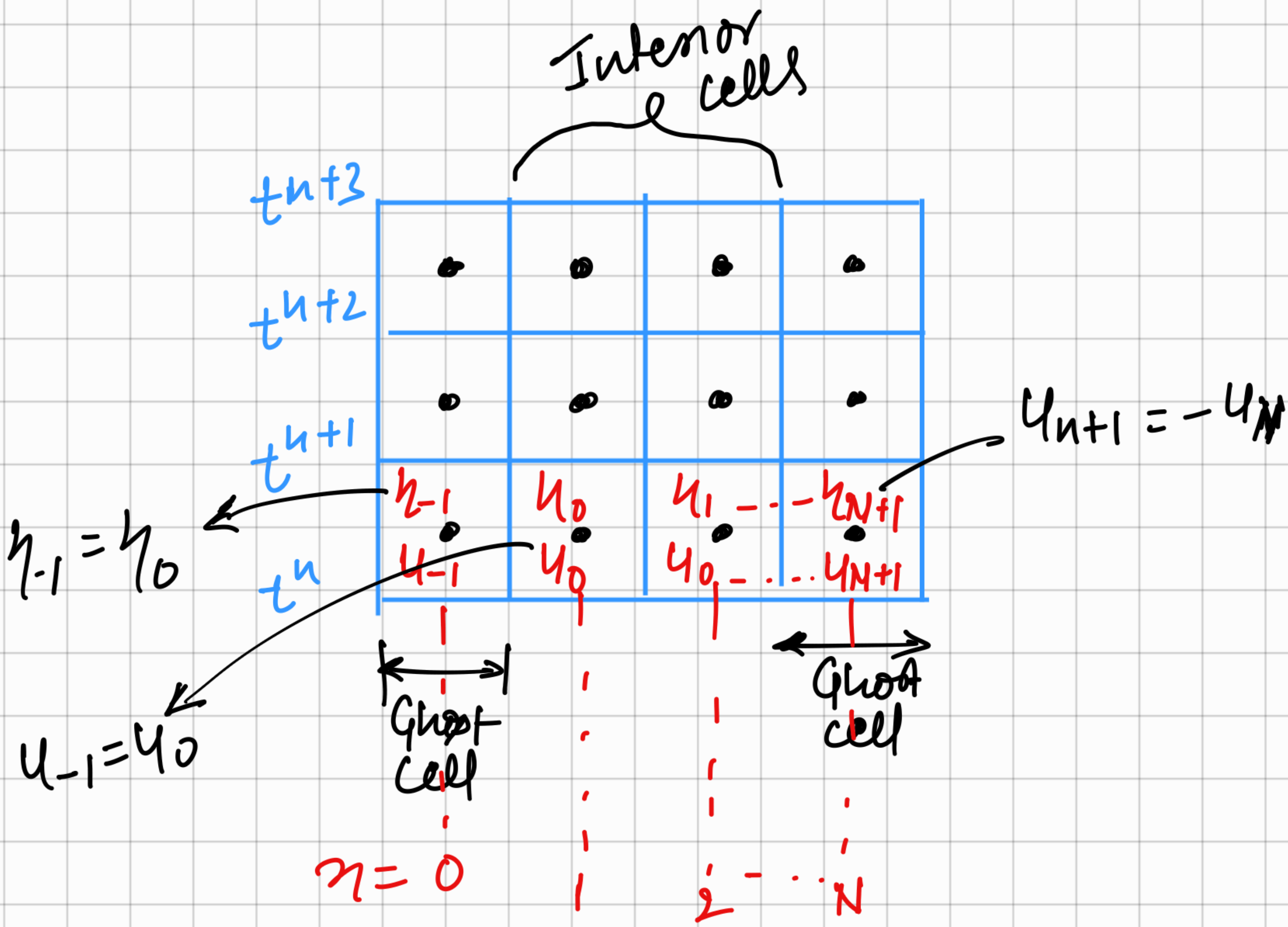
u_0 & u_0

at $x = N$ (right boundary)

$$\eta_{N+1} = \eta_N \quad \& \quad u_{N+1} = u_N$$

② closed:-

- No flux passes through boundary
- Taking $u_{-1} = -u_0$ (−ve velocity)
  $\eta_{-1} = -\eta_0$
- $f_{1/2} = 0$ & $f_{N+\frac{1}{2}} = 0$



$\Rightarrow$ for time steps estimation, restricting CFL condition

$$\Delta t \leq \frac{\Delta x}{\max|\lambda|}$$

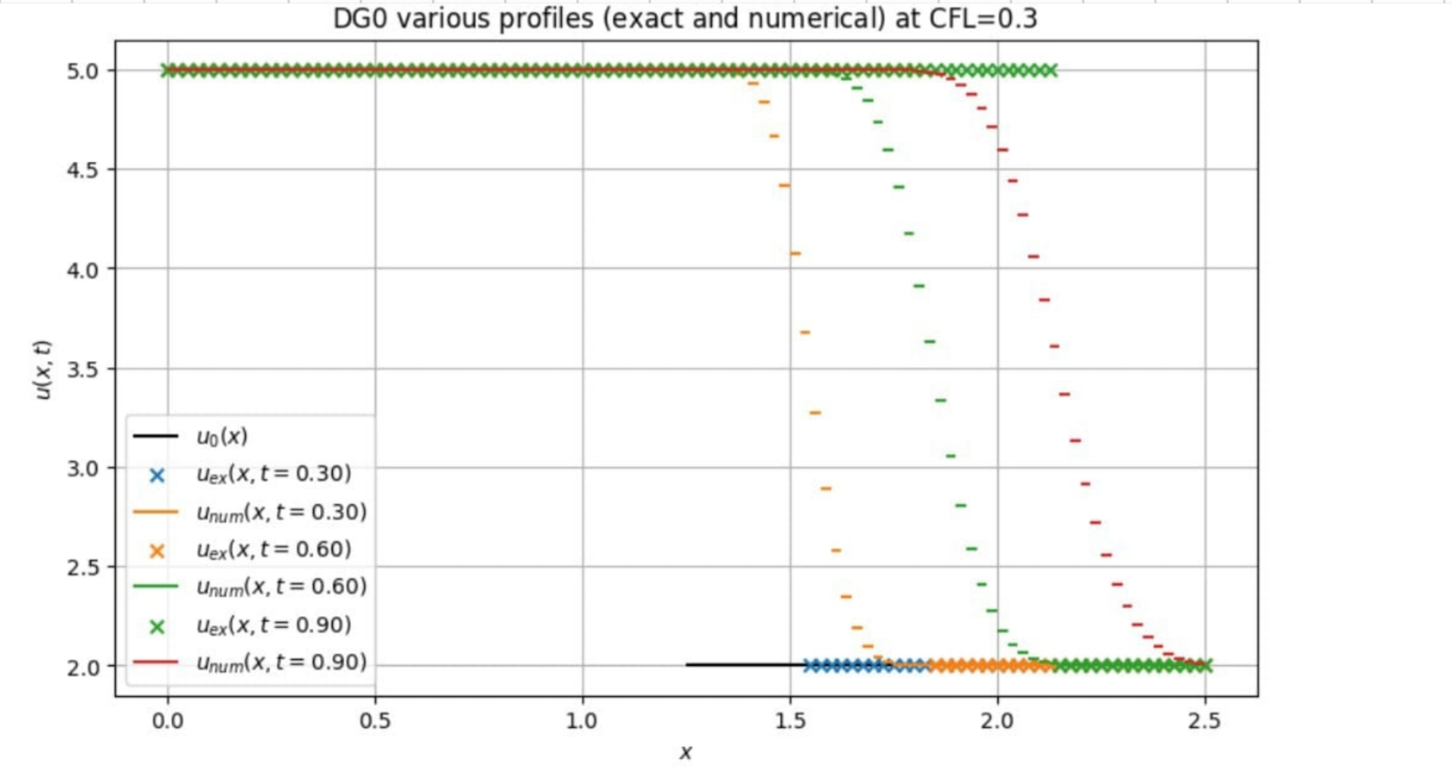
λ_1 & $\lambda_2 \rightarrow$ eigen values

- Assumption of $H(x)$ is locally constant at cell edge because,

  $\rightarrow$ The FVM scheme computes fluxes based on cell-averaged quantities, which naturally alling with the approximation of $H(x)$ as locally constant.

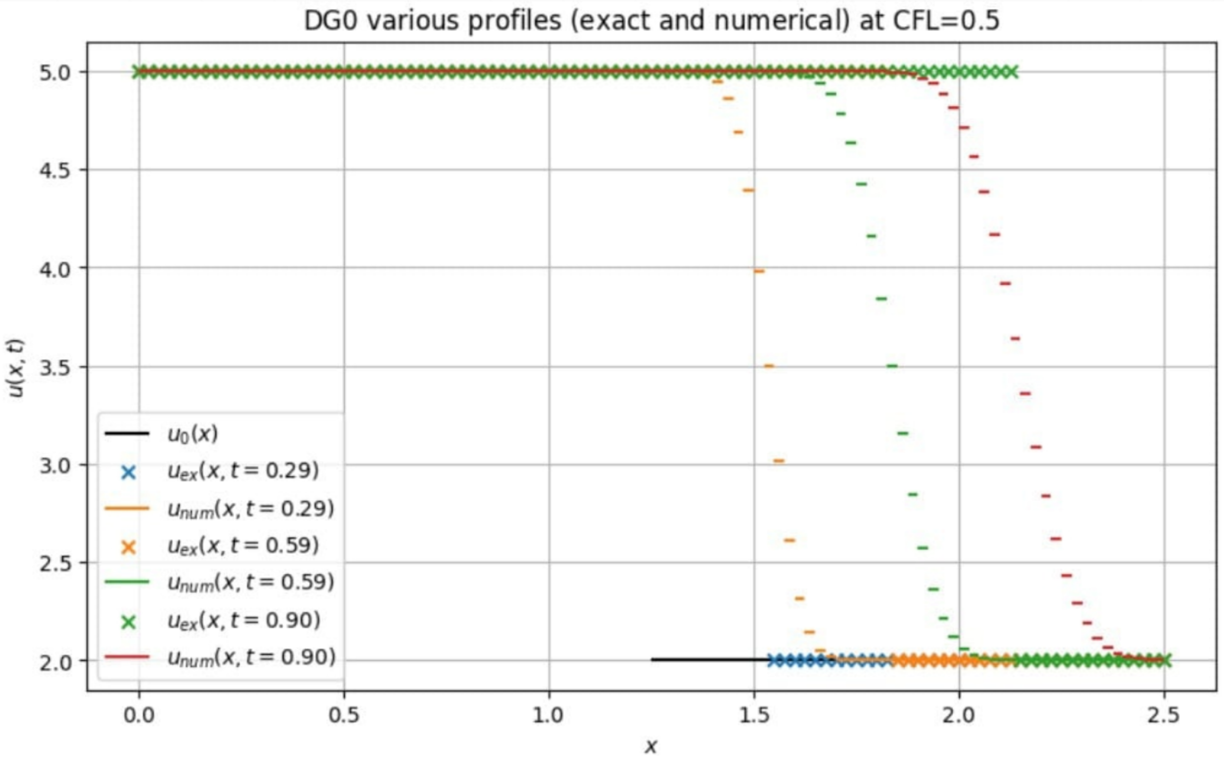
Question 4:- Implemented Godunov scheme for (1) in firedrake as DG0, showing various profiles (exact & numerical) at various CFL = 0.3, 0.5, 0.9.
used actual dimensionless time in time loop.

used $u_l = 5$ & $u_r = 2$,
a or $H_0 = 1$

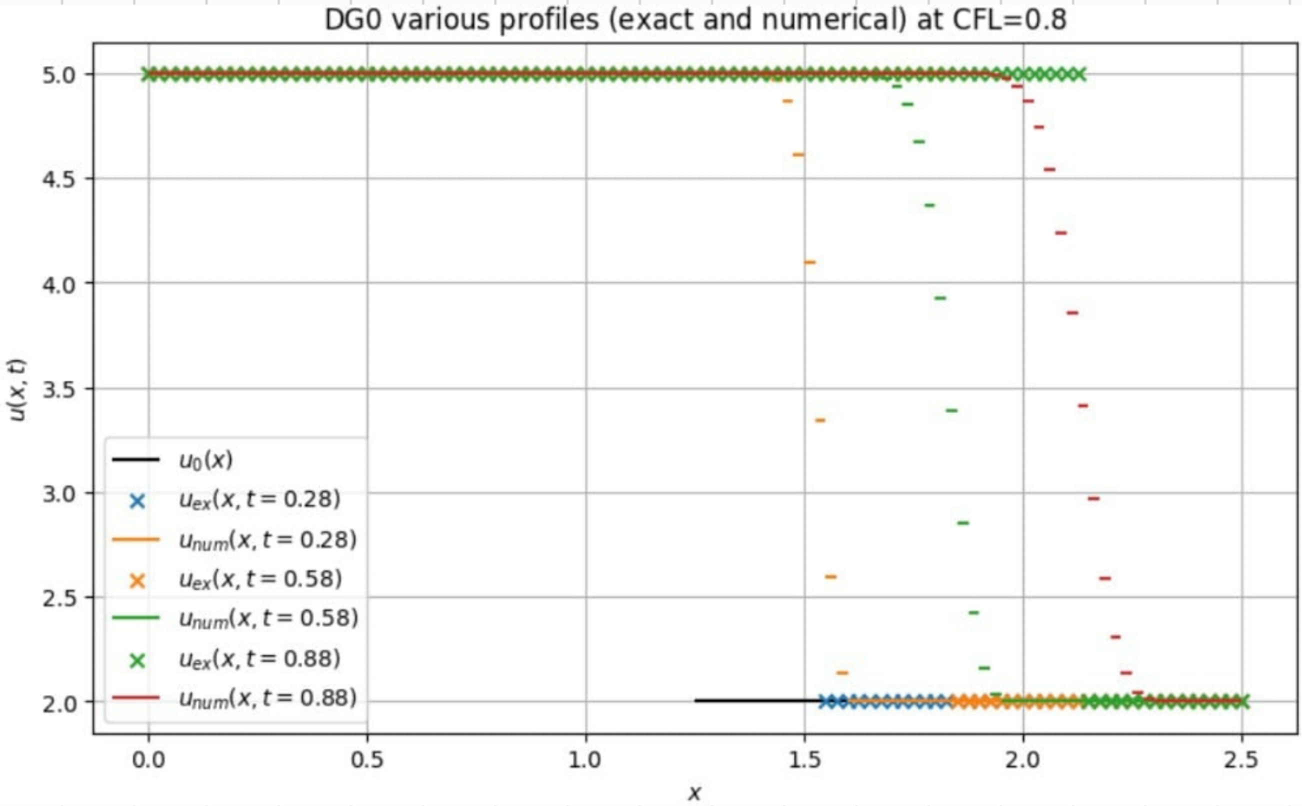


$$\Delta t = CFL \times \left(\frac{\text{mesh size}}{\text{abs}(a)} \right)$$

- Across each interface, the flux leaving one element is exactly the flux entering the next. Hence, no loss or gain of mass. Therefore FVM scheme is conservative



DG0 various profiles (exact and numerical) at CFL=0.5
$u(x,t)$
x
$u_0(x)$
$u_{ex}(x, t = 0.29)$
$u_{num}(x, t = 0.29)$
$u_{ex}(x, t = 0.59)$
$u_{num}(x, t = 0.59)$
$u_{ex}(x, t = 0.90)$
$u_{num}(x, t = 0.90)$



DG0 various profiles (exact and numerical) at CFL=0.8
$u(x,t)$
x
$u_0(x)$
$u_{ex}(x, t = 0.28)$
$u_{num}(x, t = 0.28)$
$u_{ex}(x, t = 0.58)$
$u_{num}(x, t = 0.58)$
$u_{ex}(x, t = 0.88)$
$u_{num}(x, t = 0.88)$

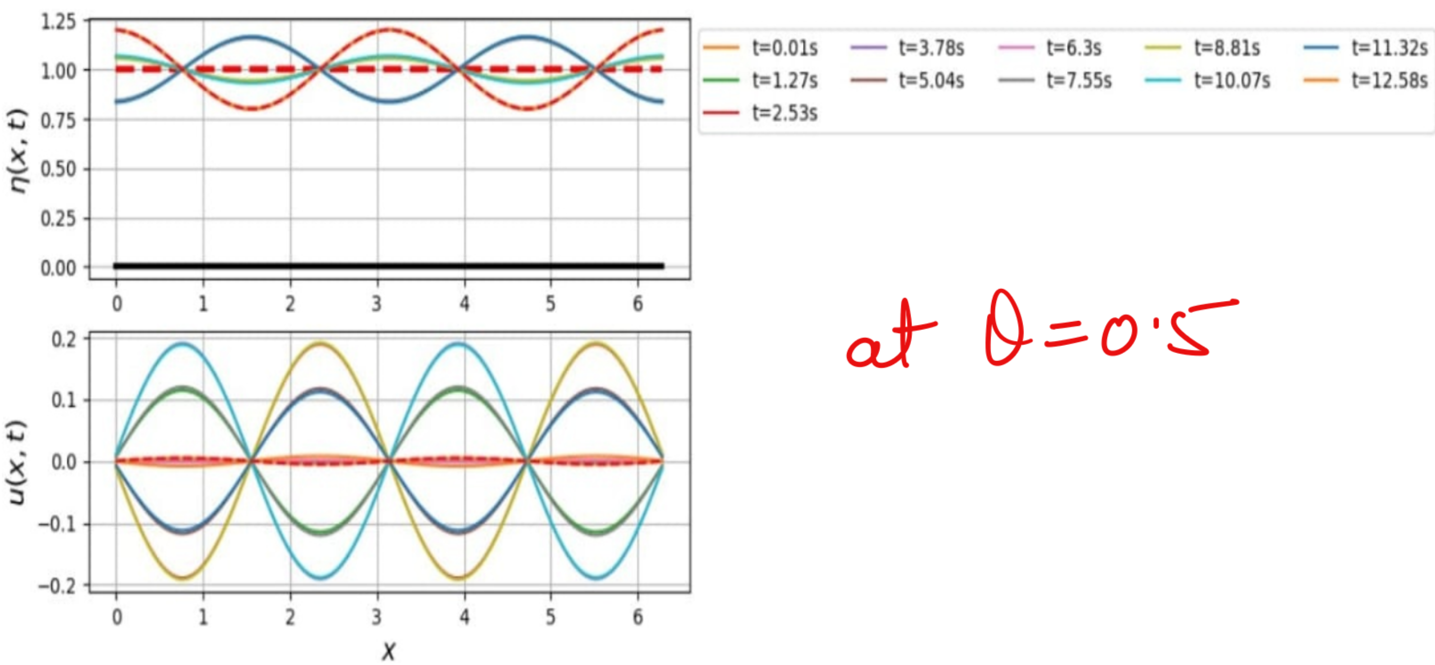
Ques 6:-

using code SweDGFV.py, Implemented the the Solid-wall boundary by considering $\theta = 0.5$

The code is provided in problem6.py file

```
Tend 12.566370614359172
energy 0 0.0
Prior to start time loop. Hallo!
Elapsed time (min): 0.06793102423350016

******** Convergence check at final time t=12.5789 ********
L2 error in eta = 3.562162e-04
L2 error in u   = 2.321674e-02
*******************************************************
```



at $\theta = 0.5$

Exact solution :-