MATH5453M Numerical Exercise 1.

Given:

$$u_t - a(t)u_x - \varepsilon u_{xx} = 0 \quad \text{for } x \in [L_p, L]$$

$$u(x,0) = u_0(x)$$

$$u(L_p,t) = u(L,t) = 0,$$

(small) constant diffusion ε & a given function $a(t)$. The B.C. are classical homogeneous Dirichlet conditions.

Solutions

Problem 1:-

$$u_t - a(t)u_x - \varepsilon u_{xx} = 0 \quad \text{for } x \in [L_p, L]$$

- The above given equation is linear because dependent variable $u(x,t)$ and it's derivative appears linear in the above equation.

① Linear : if we look into the equation there is no term containing multiplication of $u(x,t)$ & its derivatives or any power of dependent variable $u(x,t)$.

② <u>Advection:</u> '$-a(t)u_x$' represents the advection. because it is showing the advection of u_x.

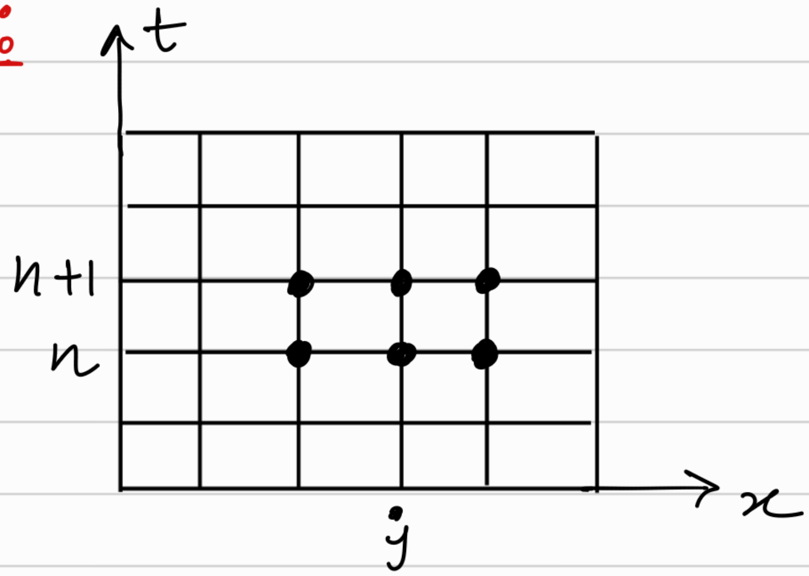
$$-a(t)\frac{\partial u}{\partial x}$$

③ <u>Diffusion:</u> 'εu_{xx}'

$$\downarrow \quad \varepsilon\frac{\partial^2 u}{\partial x^2} \rightarrow \text{represents}$$

the diffusion in the equation, with diffusion Coefficient of 'ε'.

Problem 2:



Taylor Series expansion of u_j^{n+1} & u_j^n about $\left(\frac{\Delta t}{2}\right)$

$$u_j^{n+1} = \left[u + \frac{1}{2}\Delta t\, u_t + \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} + \frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt} + \dots\right]_j^{n+1/2}$$

$$\& \quad u_j^n = \left[u - \frac{1}{2}\Delta t\, u_t + \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} - \frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt} + \dots\right]_j^{n+1/2}$$

Substracting these two, all odd will canele

$$u_j^{n+1} - u_j^n = \left[\Delta t\, u_t + \frac{1}{24}(\Delta t)^3 u_{ttt} + \dots\right]_j^{n+1/2}$$

$$u_j^{n+1} - u_j^n = \delta_t u_j^{n+1/2}$$

that means, (2.80)

$$\delta_t u_j^{n+1/2} = \left[\Delta t\, u_t + \frac{1}{24}(\Delta t)^3 u_{ttt} + \dots\right]_j^{n+1/2} \quad \text{— (2.80)}$$

from eqn (2.30),

$$\delta_x^2 u(x,t) = u_{xx}(\Delta x)^2 + \frac{1}{12} u_{xxxx}(\Delta x)^4 + \dots$$

$$\& \text{ from } \delta_x^2 u_j^{n+1} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2}$$

Expanding

$$U_{j+1}^{n+1} = u_j + \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} + \frac{1}{6}(\Delta x)^3 u_{xxx}$$

$$U_{j-1}^{n+1} = u_j - \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} - \frac{1}{6}(\Delta x)^3 u_{xxx}$$

$$\therefore \quad \delta_x^2 u_j^{n+1} = \frac{\left(\cancel{u_j} + \cancel{\Delta x u_x} + \frac{1}{2}(\Delta x)^2 u_{xx} + \cancel{\frac{1}{6}(\Delta x)^3 u_{xxx}}\right) - 2\cancel{u_j} + \left(\cancel{u_j} - \cancel{\Delta x u_x} + \frac{1}{2}(\Delta x)^2 u_{xx} - \cancel{\frac{1}{6}(\Delta x)^3 u_{xxx}}\right)}{(\Delta x)^2}$$

(2.80)

$$\delta_x^2 u_j^{n+1} = u_{xx} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \cdots \quad \text{— (2.81)}$$

- Now Considering weighted Average of Spatial derivatives

$$\theta \delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n$$

using expansion of $\delta_x^2 u_j^{n+1}$ & $\delta_x^2 u_j^n$

$$\delta_x^2 u_j^{n+1} = \left[(\Delta x)^2 u_{xx} + \frac{1}{12}(\Delta x)^4 u_{xxxx} + \frac{1}{12}(\Delta x)^6 u_{xxxxxx} + \cdots\right]_j^{n+1/2}$$

$$+ \frac{1}{2}\Delta t \left[(\Delta x)^2 u_{xxt} + \frac{1}{12}(\Delta x)^4 u_{xxxxt} + \cdots\right]_j^{n+\frac{1}{2}}$$

$$+\frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2\left[(\Delta t)^2 u_{xxtt} + \cdots\right]_j^{n+\frac{1}{2}}$$

Similarly :-

$$\delta_x^2 u_j^n = \left[(\Delta x)^2 u_{xx} + \frac{1}{12}(\Delta x)^4 u_{xxxx} + \cdots\right]_j^{n+\frac{1}{2}}$$
$$-\frac{1}{2}\Delta t\left[(\Delta x)^2 u_{xxt} + \frac{1}{12}(\Delta x)^4 u_{xxxxt} + \cdots\right]_j^{n+\frac{1}{2}}$$
$$+\frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 + \left[(\Delta t)^2 u_{xxtt} + \cdots\right]_j^{n+\frac{1}{2}}$$

using these two in

$$\theta\delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n$$
$$= \left[(\Delta x)^2 u_{xx} + \frac{1}{12}(\Delta x)^4 u_{xxxx} + \cdots\right]$$
$$+ \left(\theta - \frac{1}{2}\right)\Delta t\left[(\Delta x)^2 u_{xxt} + \frac{1}{12}(\Delta x)^4 u_{xxxxt} + \cdots\right]$$
$$+\frac{1}{8}(\Delta t)^2(\Delta x)^2\left[u_{xxtt}\right] + \cdots \quad \text{— (2.82)}$$

Truncation Error:

$$T_j^{n+\frac{1}{2}} = \frac{\delta_t u_j^{n+\frac{1}{2}}}{\Delta t} - \frac{\theta\delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n}{(\Delta x)^2} \quad \text{— (2.83)}$$

using $\dfrac{\delta_t u_j^{n+\frac{1}{2}}}{\Delta t} = u_t + \frac{1}{24}(\Delta t)^2 u_{ttt} + \cdots$

& $\dfrac{\theta \delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n}{(\Delta x)^2} = u_{xx} + \frac{1}{12}(\Delta x)^2 u_{xxxxt} \cdots$

$$\therefore \; T_j^{n+\frac{1}{2}} = [u_t - u_{xx}] + \left[\left(\frac{1}{2}-\theta\right)\Delta t\, u_{xxt} - \frac{1}{12}(\Delta x)^2 u_{xxxx}\right] + \left[\frac{1}{12}(\Delta t)^2 u_{ttt} - \frac{1}{8}(\Delta t)^2\right] + \left[\frac{1}{12}\left(\frac{1}{2}-\theta\right)\Delta t(\Delta x)^2 u_{xxxxt} - \frac{2}{6!}(\Delta x)^4 u_{xxxxxx}\right] \quad (2.84)$$

Problem 3:-

The given advection-diffusion equs

$$u_t - a(t)u_x - \varepsilon u_{xx} = 0$$

$$u_x = \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

$$-a(t)u_x = -a(t)\, u_{i+1}^n - u_i^n$$

$$\frac{u_{i+1}-u_i}{\Delta x}$$

for negative $a(t)$ & $a(t)=1$: consider the forward difference

$$u_{xx} = \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{\Delta x^2}$$

θ- discretisation :-

$$\frac{u_i^{n+1}-u_i^{n}}{\Delta t} = \theta\left(-a(t)\frac{u_{i+1}^{n+1}-u_i^{n+1}}{\Delta x} + \varepsilon\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(1-\theta)\left(-a(t)\frac{u_{i+1}^{n}-u_i^{n}}{\Delta x} + \varepsilon\frac{u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n}}{\Delta x^2}\right)$$

taking unknown to LHS:

$$u_i^{n+1} = u_i^{n} + \Delta t\Big[\theta\Big(-\frac{a(t)}{\Delta x}\left(u_{i+1}^{n+1}-u_i^{n+1}\right) + \frac{\varepsilon}{\Delta x^2}\left(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}\right)\Big)$$
$$+(1-\theta)\Big(-\frac{a(t)}{\Delta x}\left(u_{i+1}^{n}-u_i^{n}\right)+\frac{\varepsilon}{\Delta x^2}\left(u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n}\right)\Big)\Big]$$

assuming $\left.\begin{array}{l}\dfrac{\Delta t}{\Delta x} = \lambda \\ \& \ \dfrac{\Delta t}{(\Delta x)^2} = \mu\end{array}\right\}$

$$u_i^{n+1} + \theta\lambda a(t) u_{i+1}^{n+1} - \theta\lambda a(t) u_i^{n+1}$$
$$- \varepsilon\mu\theta u_{i+1}^{n+1} + \theta\varepsilon\mu 2 u_i^{n+1} - \varepsilon\theta\mu u_{i-1}^{n+1}$$
$$= (1-\theta)\left(\frac{-a(t)}{\Delta x}(u_{i+1}^n - u_i^n) + \frac{\varepsilon}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)\right)$$

$$\Rightarrow [1 + \theta\varepsilon\mu 2 - \theta\lambda a(t)] u_i^{n+1} + [\theta\lambda a(t) - \varepsilon\mu\theta] u_{i+1}^{n+1}$$
$$- \varepsilon\theta\mu u_{i-1}^{n+1} = (1-\theta)\left[-a(t)\lambda(u_{i+1}^n - u_i^n) + \varepsilon\mu(u_{i+1}^n - 2u_i^n + u_{i-1}^n)\right]$$

$$\Rightarrow [1+\theta\varepsilon\mu\times 2 - \theta\lambda a(t)] u_i^{n+1} + [\theta\lambda a(t) - \varepsilon\mu\theta] u_{i+1}^{n+1}$$
$$-\varepsilon\theta\mu u_{i-1}^{n+1} = -a(t)\lambda u_{i+1}^n + a(t)\lambda u_i^n + \varepsilon\mu u_{i+1}^n$$
$$- \varepsilon\mu 2 u_i^n + \varepsilon\mu u_{i-1}^n + a(t)\theta\lambda u_{i+1}^n$$
$$- \theta a(t)\lambda u_i^n - \varepsilon\theta\mu u_{i+1}^n + 2\varepsilon\mu\theta u_i^n$$
$$- \varepsilon\mu\theta u_{i-1}^n$$

$$\Rightarrow [1+\theta\varepsilon\mu\times 2 - \theta\lambda a(t)] u_i^{n+1} + [\theta\lambda a(t) - \varepsilon\mu\theta] u_{i+1}^{n+1}$$
$$-\varepsilon\theta u_{i-1}^{n+1} = [a(t)\theta\lambda - a(t)\lambda - \varepsilon\mu\theta] u_{i+1}^n + [a(t)\lambda - 2\varepsilon\mu$$
$$+ 2\varepsilon\mu\theta] u_i^n + [\varepsilon\mu - \varepsilon\mu\theta] u_{i-1}^n$$

Now: $x_i = L_p + i\Delta x$

$x_i = L_p + i\Delta x$ $\qquad i = 0, 1, 2 \ldots J$

for $i = 0$, $x_0 = L_p$ (left boundary)
$i = J$, $x_J = L$ (right boundary)

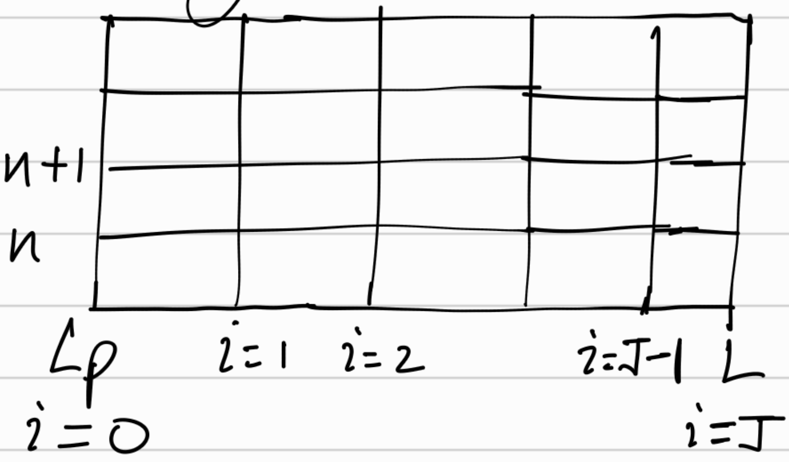
& $t^n = (n\Delta t)$, $n = 0, 1 \ldots N$

So Near left Boundary: $i = 1$
& Near right Boundary: $i = J-1$

& Mesh right Boundary:

and as given

$$U_0^n = U_J^n = 0 \quad \text{for } n = 0, 1, 2 \ldots$$



Problem: 4

Given:

$$U_t = U_{xx} \quad \text{for } t > 0, \; 0 < x < 1$$

&

$$U^0(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

use $J = 20$ → Total grid points

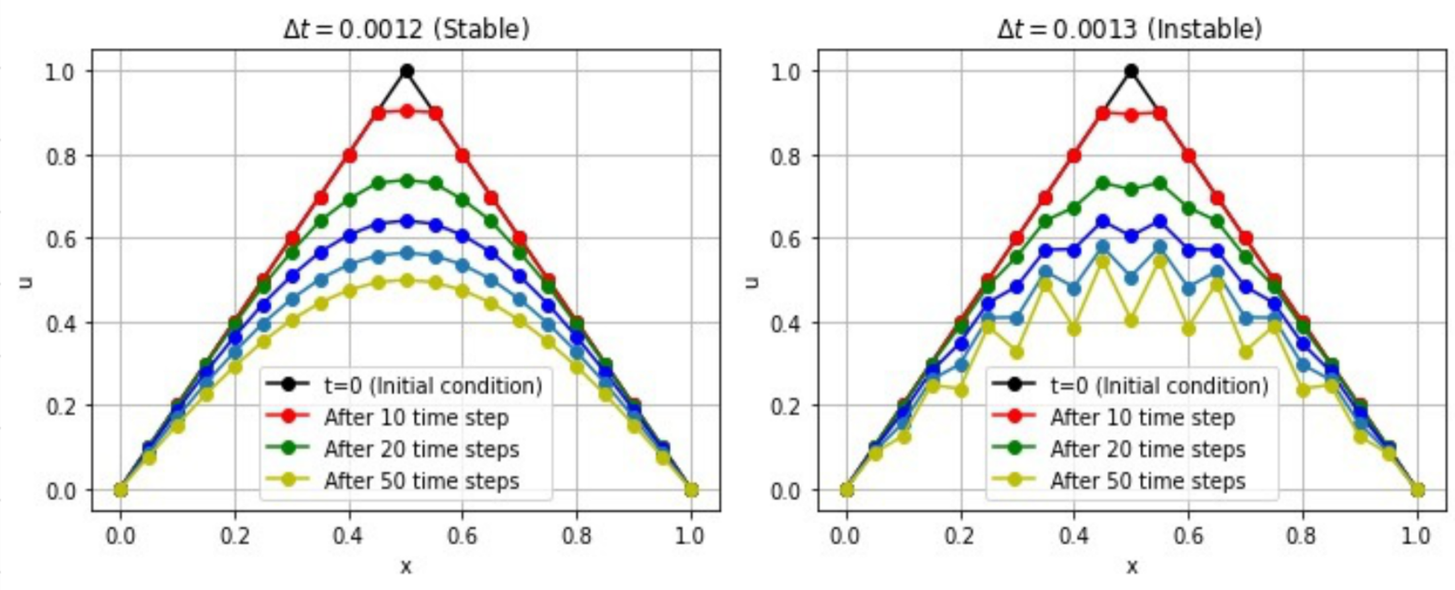
$\Delta x = 0.05$ → Spatial Step size

$\Delta t_1 = 0.0012$ → First time step size

$\Delta t_2 = 0.0013$ → 2nd time step size

using explicit scheme, a python code written in 'Problem4.py' file & results has been matched & compared with fig 2.2 (M&M)

- for two different time steps the output profile is given below:



- The above output image clearly shows the effect of time step size & as we can see for time step size 0.0013, there is instability can be observed.

- Now we will check the stability of this scheme, with both fourier analysis and max. principle

① Fourier Analysis :-

(a) $U_j^{n+1} = \lambda^{n+1} e^{ikj\Delta x} = \lambda \underbrace{\lambda^n e^{ikj\Delta x}}_{\to U_j^n}$

$$U_j^{n+1} = \lambda U_j^n$$

(b) $U_{j+1}^n = \lambda^n e^{ik(j+1)\Delta x}$

$$= \lambda^n e^{ik j\Delta x} e^{ik\Delta x}$$

$$U_{j+1}^n = U_j^n e^{ik\Delta x}$$

Similarly

$$U_{j-1}^n = U_j^n e^{-ik\Delta x}$$

Now use all these terms in explicit scheme difference equn

$$U_j^{n+1} = U_j^n + \mu\left(U_{j+1}^n - 2U_j + U_{j+1}^n\right)$$

$$\lambda U_j^n = U_j^n + \mu\left[U_j^n e^{ik\Delta x} - 2U_j + U_j^n \times e^{-ik\Delta x}\right]$$

divided by U_j^n

$$\Rightarrow \lambda = 1 + \mu\left[e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right]$$

$$\lambda = 1 + 2\mu\left[\frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2} - 1\right]$$

$$= 1 + 2\mu\left[\cos(k\Delta x) - 1\right]$$

$$\lambda = 1 - 4\mu \sin^2 k \frac{\Delta x}{2}$$

$\hookrightarrow \lambda = f(k)$

for $k = m\pi$, we get

$$U_j^n = \sum_{m=-\infty}^{\infty} A_m e^{im\pi(j\Delta x)} \left[\lambda_{m\pi}\right]^n$$

$\therefore \lambda \to$ complex no.

$\hookrightarrow \lambda = Re(\lambda) + i\,Im(\lambda)$

for stability requires that,

$|\lambda| \leq 1$

$\downarrow$

$\therefore \sin^2 k \frac{\Delta x}{2} = 1$ is the worst

$\Rightarrow \quad |\lambda| \le 1 \Rightarrow |1-4\mu| \le 1$

$$\mu \le \frac{1}{2}$$ is the stability criterion

&

for $\lambda = 1-4\lambda < -1$

$$\mu > \frac{1}{2} \longrightarrow \text{unstable}$$

Now, $\because \mu \le \frac{1}{2}$

$$\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}$$

$$\downarrow$$

$$\Delta t \le \frac{1}{2} \Delta x^2$$

$\therefore$ (1) for $\Delta t_1 = 0.0012$ & $\Delta x = 0.05$ } Stable

$$\mu = \frac{0.0012}{(0.05)^2} \Rightarrow 0.48 \le \frac{1}{2}$$

(i) for $\Delta t_2 = 0.0013$, $\Delta x = 0.05$ } unstable

$$r = \frac{0.0013}{(0.05)^2} = 0.52 > \frac{1}{2}$$

(2) Max. principle :-

According to Max. Principle, a difference approximation to $u_t = u_{xx}$ should possess beyond consider as $\Delta x, \Delta t \to 0$

$$\therefore \quad r(1-\theta) \le \frac{1}{2}$$

& for explicit scheme, $\theta = 0$

$$r \le \frac{1}{2}$$

which will give

$\Delta t_1 = 0.0012$ Stable
$\Delta t_2 = 0.0013$ instable

$\Rightarrow$ we extend this explicit Scheme to (1)

$$u_t - a(t) u_x - \varepsilon u_{xx} = 0$$

$$u(x,0) = u_0(x)$$

$$u(L_\rho, t) = u(L,t) = 0$$

$$[1 + \theta \varepsilon \mu \times 2 - \theta \lambda a(t)] u_i^{n+1} + [\theta \lambda a(t) - \varepsilon \mu \theta] u_{i+1}^{n+1} - \varepsilon \theta u_{i-1}^{n+1} = [a(t)\theta\lambda - a(t)\lambda - \varepsilon\mu\theta] u_{i+1}^{n} + [a(t)\lambda - 2\varepsilon + 2\varepsilon\mu\theta] u_i^n + [\varepsilon\mu - \varepsilon\mu\theta] u_{i-1}^n$$

for $\theta = 0$: Explicit Scheme

$$u_i^{n+1} = -a(t)\lambda u_{i+1}^n + (a(t)\lambda - 2\varepsilon) u_i^n + \varepsilon\mu u_{i-1}^n$$

$$\mu = \frac{\Delta t}{\Delta x^2} \quad \& \quad \lambda = \frac{\Delta t}{\Delta x}$$

Problem 5 :-

Implimented θ-Method and used the linear Algebra process in Problem 5.py

$$u_t - a(t)u_x - \varepsilon u_{xx} = 0$$
$$u(x,0) = u_0(x)$$
$$u(L_0,t) = u(L,t) = 0$$

used $L_0 = 0$ & $L = 1$ (B.Cs)

& I.C $\rightarrow$ $u_0(x) = \begin{cases} 2x & \text{if } 0 < x \leq 0.5 \\ 2-2x & \text{if } 0.5 \leq x \leq 1 \end{cases}$

$$u_x = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

& $$u_{xx} = \frac{u_i^{n+1} - u_i^n}{(\Delta x)^2}$$

$$u_t = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

θ-Method implimented:-

$$[1+\theta\varepsilon \mu \times 2 - \theta\lambda a(t)]u_i^{n+1} + [\theta\lambda a(t) - \varepsilon\mu\theta]u_{i+1}^{n+1} - \varepsilon\theta u_{i-1}^{n+1} = [a(t)\theta\lambda - a(t)\lambda - \varepsilon\mu\theta]u_{i+1}^n + [a(t)\lambda - 2\varepsilon\mu + 2\varepsilon\mu\theta]u_i^n + [\varepsilon\mu - \varepsilon\mu\theta]u_{i-1}^n$$

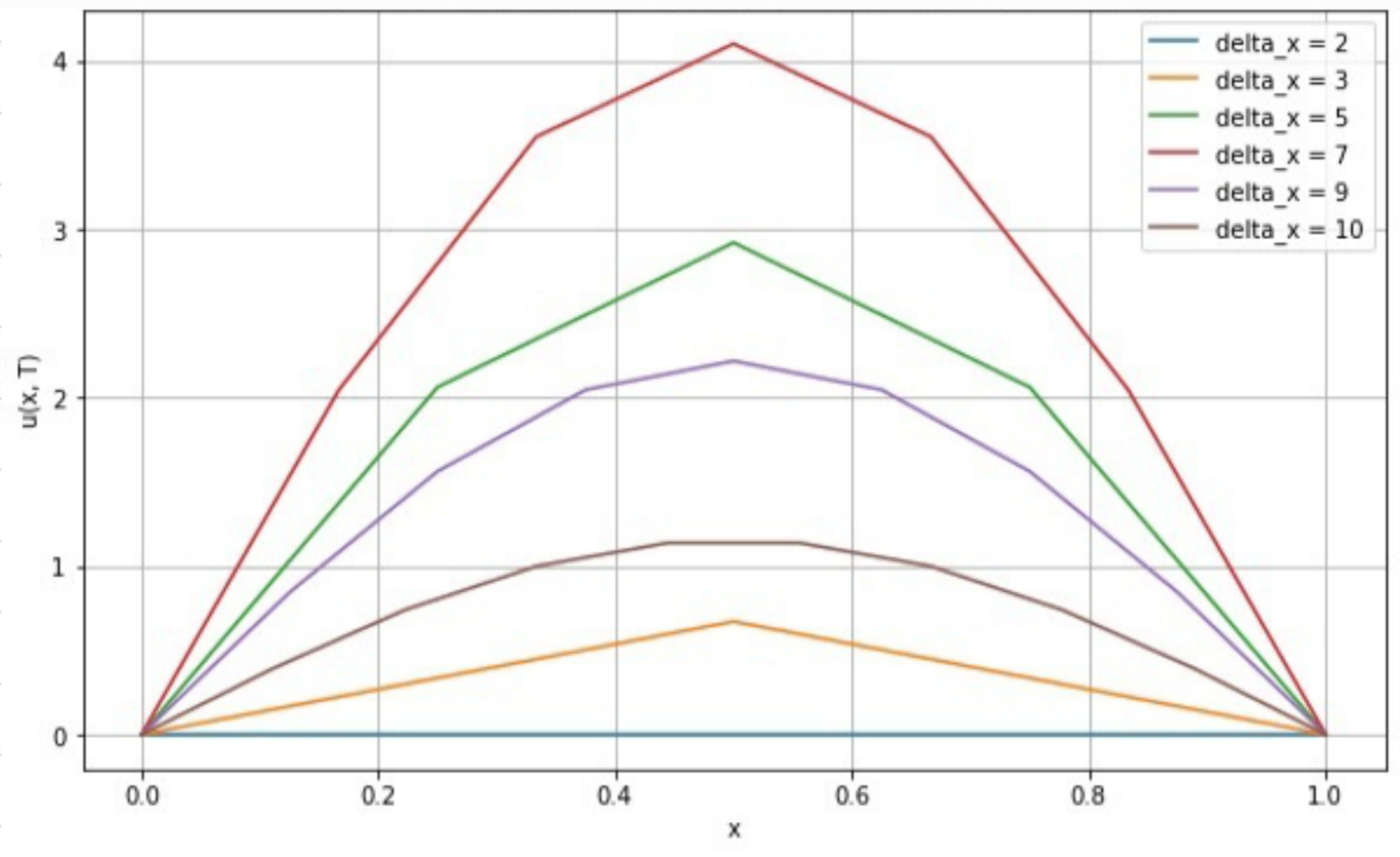
$\rightarrow$ $\theta = 0 \rightarrow$ 0 Explicit Scheme

$\theta = 1 \rightarrow$ Implicit Scheme

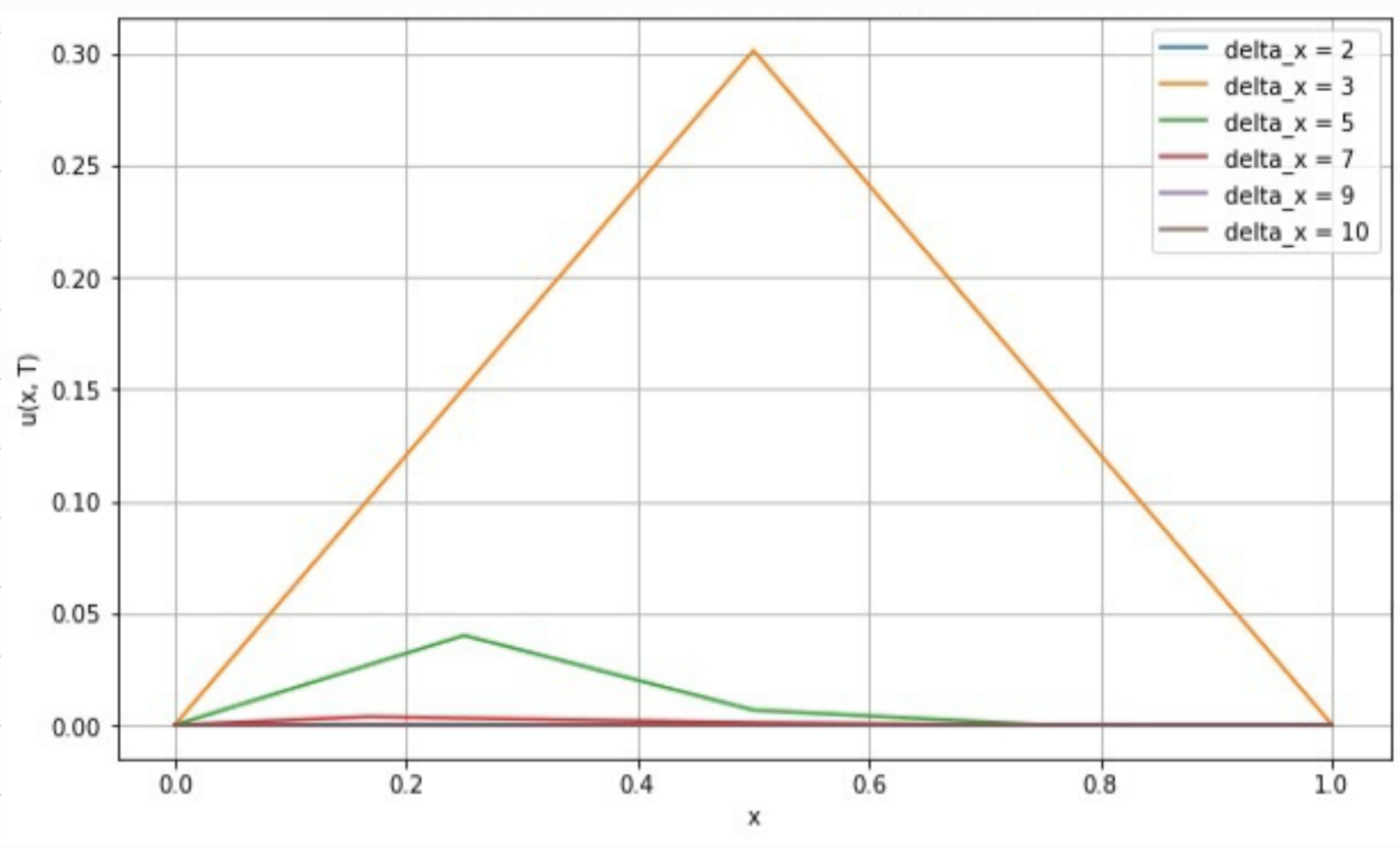
$\theta = 0.5 \rightarrow$ Crank-Nicholson

by varying θ value in Problem 5.py python file to 0, 1, 0.5, following plot were found

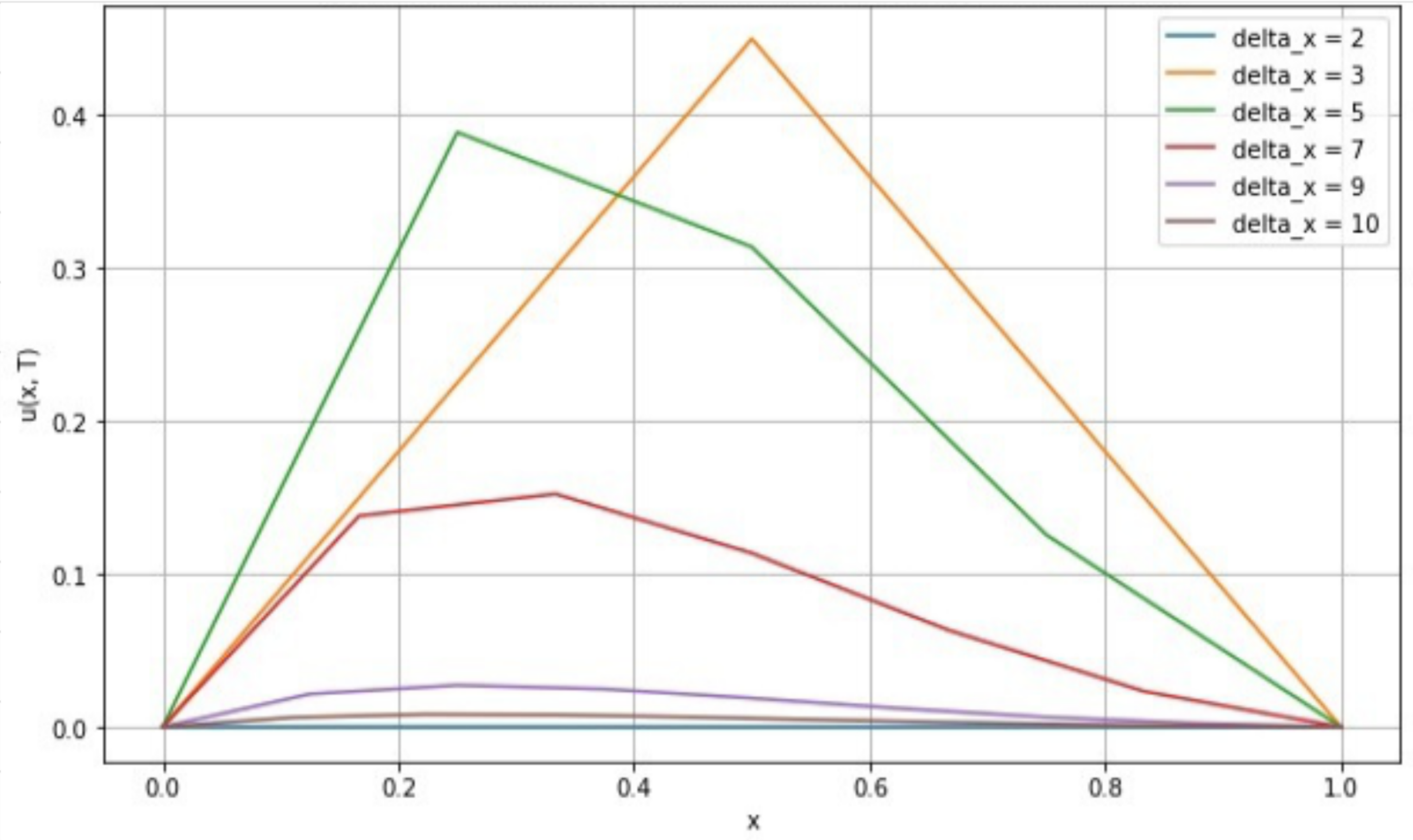
① → at $\theta = 1$, $\varepsilon = 0.1$, $\Delta t = 0.0012$



② at $\theta = 0$, $\varepsilon = 0.1$, $\Delta t = 0.0012$



(3) at $\theta = 0.5$, $\varepsilon = 0.1$, $\Delta t = 0.0012$



Now for Stability :-

$$U_j^{n+1} = U_j^n + \theta \mu \epsilon \left(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}\right)$$
$$+ (1-\theta)\mu\epsilon\left(U_{j+1}^{n} - 2U_j^{n} + U_{j-1}^{n}\right)$$

let $U_j^n = \lambda^n e^{ijk\Delta x}$

$$\lambda^{n+1} e^{ijk\Delta x} - \lambda^n e^{ijk\Delta x} = \theta\mu\epsilon\lambda^{n+1}\left(e^{i(j+1)k\Delta x}\right.$$
$$\left. - 2e^{ijk\Delta x} + e^{i(j-1)k\Delta x}\right) +$$
$$(1-\theta)\mu\epsilon\left(\lambda^n\left(e^{i(j+1)k\Delta x} - 2e^{ijk\Delta x} + e^{i(j-1)k\Delta x}\right)\right.$$

$\div$ by $\lambda^n e^{ijk\Delta x}$

$$\lambda - 1 = \theta \mu \epsilon \lambda \left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) + (1-\theta)\mu\epsilon\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

using $\dfrac{e^{ik\Delta x} + e^{-ik\Delta x}}{2} = \cos k\Delta x$

$\downarrow$

$$\lambda - 1 = 2\theta \mu \epsilon \lambda (1 - \cos k\Delta x) + (1-\theta)\mu\epsilon 2 (1 - \cos k\Delta x)$$

$\downarrow$

$$\lambda - 1 = -\theta \lambda \epsilon \mu \left(4 \sin^2 \tfrac{1}{2} k\Delta x\right) + (1-\theta)\mu\epsilon\left(-4 \sin^2 \tfrac{1}{2} k\Delta x\right)$$

$$\lambda = \frac{1 - 4(1-\theta)\mu\epsilon \sin^2\left(\frac{1}{2} k\Delta x\right)}{1 + 4\theta\epsilon\mu \sin^2\left(\frac{1}{2} k\Delta x\right)}$$

for Stability, $|\lambda| \leq 1$

$\downarrow$

$$(1-2\theta)\mu\epsilon \sin^2\left(\tfrac{1}{2} k\Delta x\right) \leq \frac{1}{2}$$

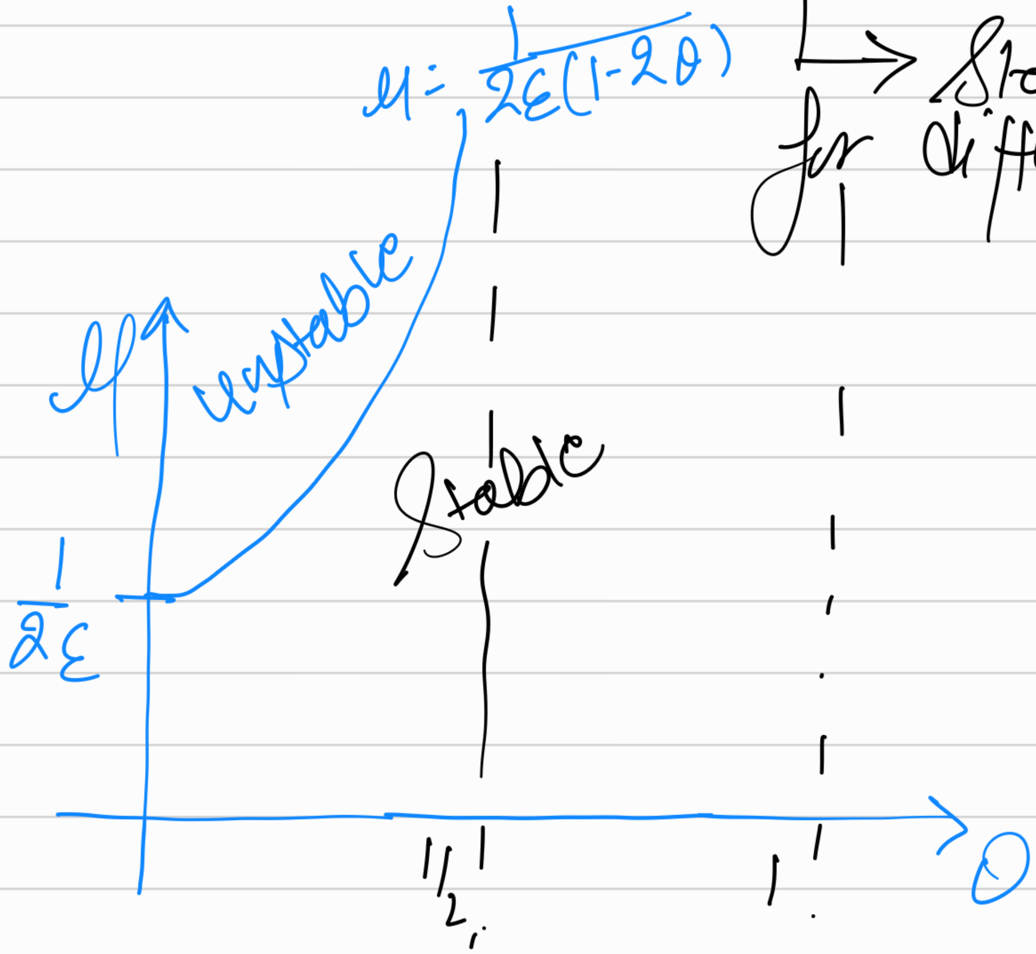
worst will be

$$\sin^2\left(\frac{k\Delta x}{2}\right) = 1$$

So $(1-2\theta)\mu\varepsilon \le \frac{1}{2}$

$$\boxed{\mu \le \frac{1}{2\varepsilon(1-2\theta)}}$$

→ Stability criterion for diffusion part



Now for advection part :-

$$\lambda = \frac{1 + (1-\theta)\mu a^4 \Delta x (e^{ik\Delta x} - 1)}{1 - \theta\mu a^{4+1} \Delta x (e^{ik\Delta x} - 1)}$$

for worst value of k, $|\lambda|$ to be highest.

$$\max|e^{ik\Delta x}-1|$$

$$e^{ik\Delta x} = \cos k\Delta x + i\sin k\Delta x$$

$\therefore$ when $\cos k\Delta x = -1$

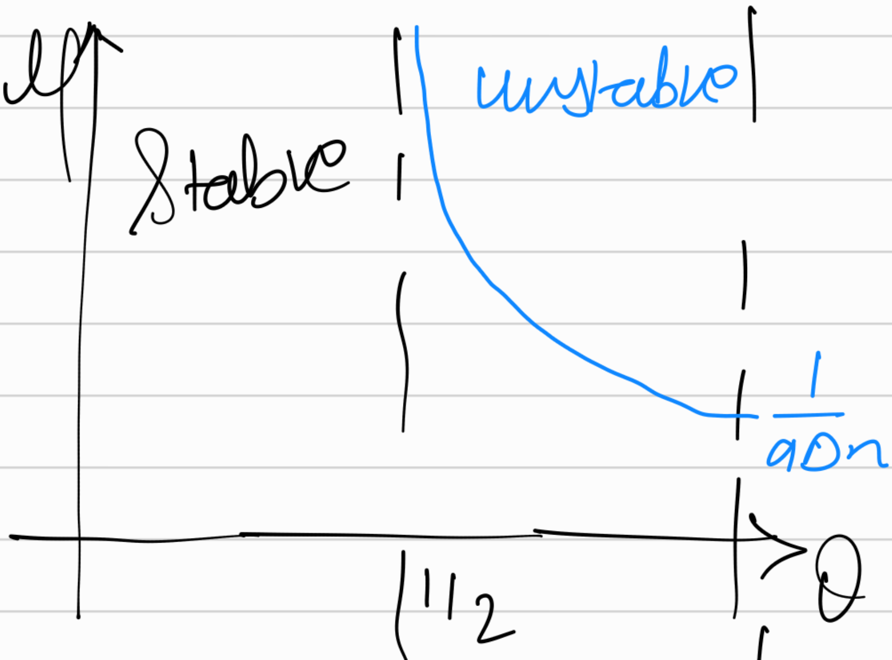
$$\max|e^{ik\Delta x}-1| = \sqrt{4} = 2$$

$$\lambda \le 1$$

$$\therefore \quad -1 \le \frac{1+2(1-\theta)\mu a^n \Delta x}{1-2\theta\mu a^{n+1}\Delta x}$$

for $a = \text{const}$

$$\boxed{\mu \le \frac{1}{a\Delta x(2\theta-1)}}$$



Problem 6:-

In python code Problem 6.py.

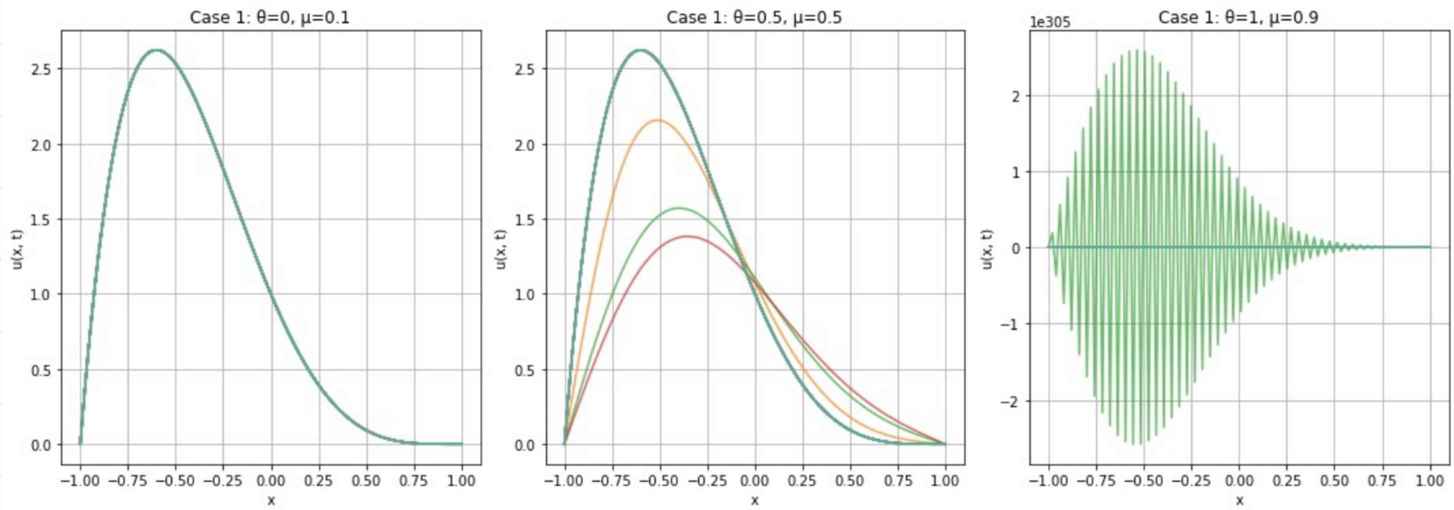
Case 1:- $u(x,0) = (1-x)^4(1+x)$

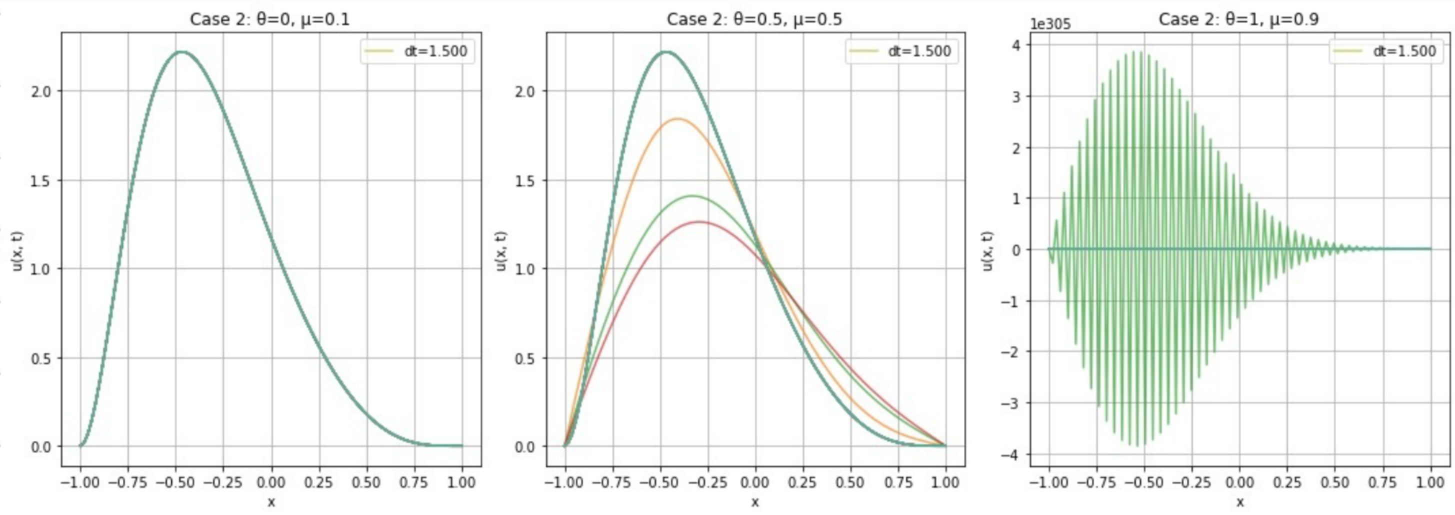
Case 2:- $u(x,0)$ with random values of $\mathbb{R}$

we have

$$\begin{cases} u_t - a(t)x - \varepsilon u_{xx} = 0 \\ u(x,0) = u_0 x \\ u(L_p,t) = u(L,t) = 0 \end{cases}$$

$L_p = -1$, $L = 1$, $\varepsilon = 10^{-3}$, $T = 1$





Case 2: θ=0, μ=0.1
dt=1.500
u(x, t)
x
Case 2: θ=0.5, μ=0.5
dt=1.500
u(x, t)
x
Case 2: θ=1, μ=0.9
1e305
dt=1.500
u(x, t)
x