Numerics Ex 3

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1). $I(u) = \iint_\Omega \frac{1}{2}|\underline{\nabla} u|^2 - uf \, d\Omega = 0$

is minimised for solution $u = u(x,y)$

Via's variation of δu

$$\Rightarrow \delta I \equiv \frac{dI}{d\varepsilon} = \lim_{\varepsilon \to 0} \frac{I(u+\varepsilon\delta u) - I(u)}{\varepsilon} = 0$$

$$\Rightarrow \iint \frac{1}{2} \frac{|\underline{\nabla}(u+\varepsilon\delta u)|^2 - \frac{1}{2}|\underline{\nabla} u|^2 - f(u+\varepsilon\delta u) + fu}{\varepsilon} \, d\Omega$$

$$= \iint_\Omega \frac{\cancel{\frac{1}{2}|\underline{\nabla} u|^2} + \underline{\nabla} u \cdot \varepsilon \nabla \delta u + \frac{\varepsilon^2}{2}|\underline{\nabla}\delta u|^2 - \cancel{\frac{1}{2}|\underline{\nabla} u|^2} - f\varepsilon\delta u}{\varepsilon} \, d\Omega$$

$$= \iint_\Omega \underline{\nabla} u \cdot \underline{\nabla}\delta u - f\delta u \, d\Omega \quad (\text{as } \varepsilon \to 0)$$

$$\iint_\Omega \underline{\nabla} u \cdot \underline{\nabla}\delta u - f\delta u \, d\Omega = 0$$

$\underline{\nabla} \cdot (\delta u \underline{\nabla} u)$

$= \underline{\nabla} u \cdot \underline{\nabla} \delta u + \delta u \underline{\nabla}^2 u$

so integral can be rewritten as

$$-\iint \delta u (\underline{\nabla}^2 u + f) \, d\Omega + \iint \underline{\nabla} \cdot (\delta u \underline{\nabla} u) \, d\Omega$$

Defining the boundary as

$\Gamma_1 : x = 0, 1$ for all y

$\Gamma_2 : y = 0, 1$ for all x

so $\iint \underline{\nabla} \cdot (\delta u \underline{\nabla} u) \, d\Omega$

$$= \int_{\Gamma_1} \delta u (\underline{\nabla} u \cdot \underline{\hat{n}}) \, d\Gamma + \int_{\Gamma_2} \delta u (\underline{\nabla} u \cdot \underline{\hat{n}}) \, d\Gamma$$

Using boundary conditions ; $u(0, y) = u(1, y) = 0$

$\rightarrow$ so over Γ_1, $\delta u = 0$

; $\partial_y u|_{y=0} = \partial_y u|_{y=1} = 0$

so over Γ_2, $\underline{\nabla} u \cdot \underline{\hat{n}} = 0$

This leaves us with

$$\iint \delta u \left(\nabla^2 u + f \right) d\Omega = 0$$

as δu is arbitrary, then $\nabla^2 u + f = 0$ for this to always be true

$$\boxed{\text{so} \quad -\nabla^2 u = f \quad \text{on } (x,y) \in [0,1]^2}$$

$$\boxed{\delta u(0,y) = \delta u(1,y) = 0}$$

Weak Formulation

$\underline{\nabla}^2 u + f = 0$

$w \underline{\nabla}^2 u + wf = 0$

Integrating over Ω

$$\iint w \underline{\nabla}^2 u + wf \, d\Omega = 0$$

Using $\underline{\nabla} \cdot (w \underline{\nabla} u) = w \underline{\nabla}^2 u + \underline{\nabla} w \underline{\nabla} u$
again

$$= \iint \underline{\nabla} \cdot (w \underline{\nabla} u) - \underline{\nabla} w \cdot \underline{\nabla} u + wf \, d\Omega$$

$$= \iint_\Omega - \underline{\nabla} w \cdot \underline{\nabla} u + wf \, d\Omega + \int_{\Gamma_1} w \underline{\nabla} u \cdot \underline{\hat{n}} \, d\Gamma + \int_{\Gamma_2} w \underline{\nabla} u \cdot \underline{\hat{n}} \, d\Gamma$$

like before $w(0,y) = w(1,y) = 0$
and $\partial_y u = 0$ at $y = 0, 1$

$$\Rightarrow \boxed{\iint - \underline{\nabla} w \cdot \underline{\nabla} u + wf \, d\Omega = 0 \quad \text{weak formulation}}$$

As for variational principle

$$\iint \underline{\nabla}\delta u \cdot \underline{\nabla} u - f\delta u \, d\Omega = 0$$

$$\boxed{\text{Then} \quad w(x,y) = \delta u(x,y)}$$

2). $u(x,y) \approx u_h(x,y) = u_j \varphi_j(x,y)$

$$I(u) = \iint_\Omega \frac{1}{2} |\nabla u|^2 - uf \, d\Omega$$

$$\boxed{I = \iint \frac{1}{2} \left| \nabla(u_j \varphi_j) \right|^2 - u_j \varphi_j f \, d\Omega}$$

$u \approx u_h = u_j \varphi_j$

$w \approx w_h = w_j \varphi_j$

Let $w_j = \delta_{ij}$

$\Rightarrow w_h = \varphi_i$

Weak formulation:

$$\iint - \nabla w \cdot \nabla u + wf \, d\Omega = 0$$

$$\iint \nabla w \cdot \nabla u \, d\Omega = \iint wf \, d\Omega$$

$$\iint \nabla \varphi_i \cdot \nabla(u_j \varphi_j) \, d\Omega = \iint \varphi_i f \, d\Omega$$

$\rightarrow A_{ij} u_j = b_i$

where $A_{ij} = \iint \nabla \varphi_i \cdot \nabla \varphi_j \, d\Omega$

$b_i = \iint \varphi_i f \, d\Omega$

Need to enforce Dirichlet boundary conditions where $\varphi_i = 0$ $(x = 0, 1)$

So actually $A_{ij} u_j = b_j$

can be split

$\rightarrow A_{i'j'} u_{j'} + A_{i'k} u_k = b_{i'}$

where i', j' are nodes not on Dirichlet boundary and k are the Dirichlet nodes

In this scenario, $u = 0$ on boundary so $u_k = 0$

$$\rightarrow \boxed{A_{i'j'} u_{j'} = b_{i'}}$$

$$I = \iint \frac{1}{2} \left| \nabla (u_j \varphi_j) \right|^2 - u_j \varphi_j f \; d\Omega$$

As before we showed $\delta u = \omega$

$\Rightarrow \delta u \simeq \delta u_n = \varphi_i$

$$\delta I = \frac{1}{\varepsilon} \iint \Big[\frac{1}{2} \left| \nabla (u_j \varphi_j + \varepsilon \varphi_i) \right|^2 - (u_j \varphi_j + \varepsilon \varphi_i) f$$

$$- \frac{1}{2} \left| \nabla (u_j \varphi_j) \right|^2 + u_j \varphi_j f \Big] d\Omega$$

$$= \frac{1}{\varepsilon} \iint \varepsilon u_j \nabla \varphi_j \nabla \varphi_i + \frac{1}{2} \varepsilon^2 \left| \nabla \varphi_i \right|^2 - \varepsilon \varphi_i f \, d\Omega$$

as $\varepsilon \to 0$

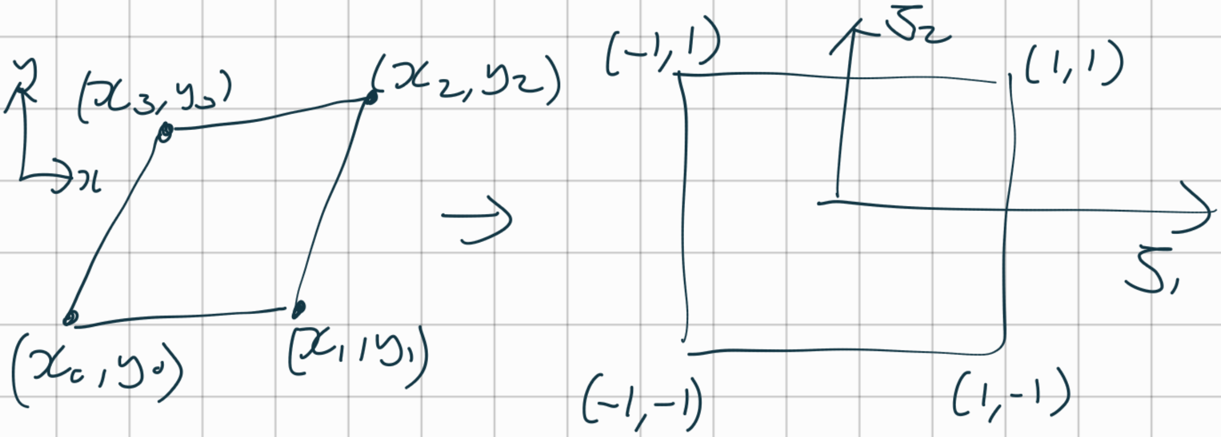
$$\delta I \simeq \iint u_j \nabla \varphi_j \cdot \nabla \varphi_i - \varphi_i f \, d\Omega = 0$$

$$\iint \nabla \varphi_i \cdot \nabla \varphi_j \, d\Omega \, u_j = \iint \varphi_i f \, d\Omega$$

so like before (when applying $u=0$ at Dirichlet boundary)

$$\boxed{A_{i',j'} u_{j'} = b_{i'}}$$

3). Using coordinate system for quadrilateral elements



want it to map kind of like this.

Introducing reference element

$\hat{K} \in [-1,1]^2$ with reference coordinates
$\underline{\xi} = (\xi_1, \xi_2)^T$

This means local coordinate of a node of element k, $\underline{x}_{k,\alpha}$, can be mapped onto reference element.

$$\Rightarrow \underline{x} = \sum_{\alpha=0}^{3} x_{k,\alpha} \chi_\alpha(\underline{\xi})$$

as we want $\underline{x}_{k,0} \Rightarrow (-1,-1)^T$
$\underline{x}_{k,1} \Rightarrow (1,-1)^T$
$\vdots$
etc.

Then can use shape functions.

$$\chi_0 = \frac{(1-\xi_1)(1-\xi_2)}{4}$$

$$\chi_1 = \frac{(1+\xi_1)(1-\xi_2)}{4}$$

$$\chi_2 = \frac{(1+\xi_1)(1+\xi_2)}{4}$$

$$\chi_3 = \frac{(1-\xi_1)(1+\xi_2)}{4}$$

→ and these are essentially the basis functions in the reference system

We want to transform

$$A_{ij} = \iint \nabla \varphi_i \cdot \nabla \varphi_j \, d\Omega$$

$$b_i = \iint \varphi_i f \, d\Omega$$

into reference coordinate system.

$$\hat{A}_{\alpha,\beta} = \int \nabla \chi_\alpha \cdot \nabla \chi_\beta \, d\Omega$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} \end{pmatrix}$$

$$\underline{\xi} = J \underline{x}$$

$$\underline{x} = J^{-1} \underline{\xi}$$

so $$\underline{\nabla} = (J^T)^{-1} \begin{pmatrix} \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \end{pmatrix}$$

and $$d\Omega = |\det(J)|\, d\underline{\xi}$$

$$\hat{A}_{\alpha,\beta} = \int_{\hat{K}} \left((J^T)^{-1} \begin{pmatrix} \frac{\partial \varkappa_\alpha}{\partial \xi_1} \\ \frac{\partial \varkappa_\alpha}{\partial \xi_2} \end{pmatrix} \right) \cdot \left((J^T)^{-1} \begin{pmatrix} \frac{\partial \varkappa_\beta}{\partial \xi_1} \\ \frac{\partial \varkappa_\beta}{\partial \xi_2} \end{pmatrix} \right) |\det J|\, d\underline{\xi}$$

and $$\hat{b}_\alpha = \int_{\hat{K}} f(x(\underline{\xi}), y(\underline{\xi}))\, \varkappa_\alpha\, |\det(J)|\, d\underline{\xi}$$

For quadrilaterals

$$\underline{x} = \begin{pmatrix} x_0 \psi_0 + x_1 \psi_1 + x_2 \psi_2 + x_3 \psi_3 \\ y_0 \psi_0 + y_1 \psi_1 + y_2 \psi_2 + y_3 \psi_3 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$ so $$\frac{\partial x}{\partial \xi_1} = -\frac{x_0}{4}(1-\xi_2) + \frac{x_1}{4}(1-\xi_2)$$

$$+ \frac{x_2}{4}(1+\xi_2) - \frac{x_3}{4}(1+\xi_2)$$

$$= \frac{1}{4}\left[(1-\xi_2)(x_1 - x_0) + (1+\xi_2)(x_2 - x_3)\right]$$

This can be repeated for other terms.

$J^T =$

$$\frac{1}{4}\begin{pmatrix} (1-\xi_2)(x_1-x_0)+(1+\xi_2)(x_2-x_3) & (1-\xi_2)(y_1-y_0)+(1+\xi_2)(y_2-y_3) \\ (1-\xi_1)(x_3-x_0)+(1+\xi_1)(x_2-x_1) & (1-\xi_1)(y_3-y_0)+(1+\xi_1)(y_2-y_1) \end{pmatrix}$$

$(J^T)^{-1}$ can be found from this, and so can find complete expression for $\hat{A}_{\alpha,\beta}$, $\hat{b}_\alpha$

Matrix assembly:

Initially set all $A_{ij}, b_i = 0$

Assembling global matrix

① for $k = 1, N_{elements}$:

② for $\alpha = 0, 3$:

③ $i = \text{Index}(k, \alpha)$:

④ for $\beta = 0, 3$:

⑤ $j = \text{Index}(k, \beta)$

⑥ $A_{ij} = A_{ij} + \hat{A}_{\alpha\beta}$

⑦ $b_i = b_i + \hat{b}_\alpha$

① Looping over every element in mesh.

② Looping over every node in reference element.

③ mapping the reference node α of element k to the global coordinate system i.

④ Looping over every node in reference element.

⑤ mapping the reference node β of element k to the global coordinate system, j

⑥ Local matrix, $\hat{A}_{\alpha,\beta}$ added to global matrix A_{ij}

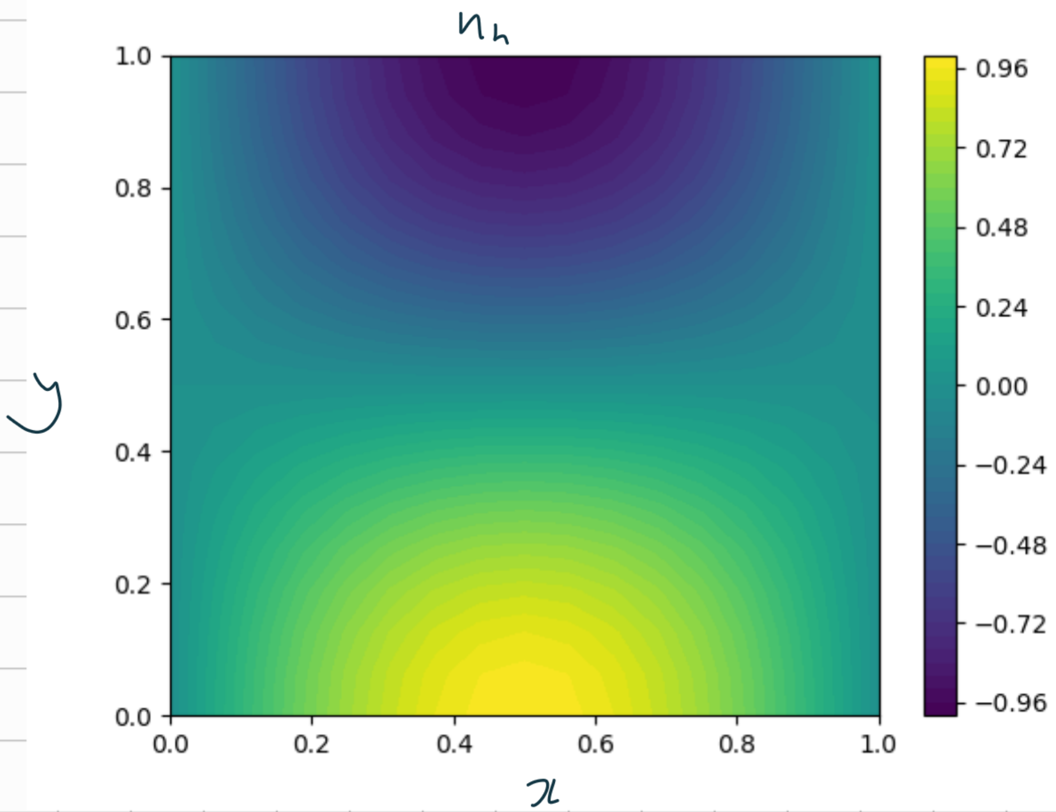
→ because we know where the reference nodes correspond to in global system.

⑦ $\hat{b}_\alpha$ added to b_i

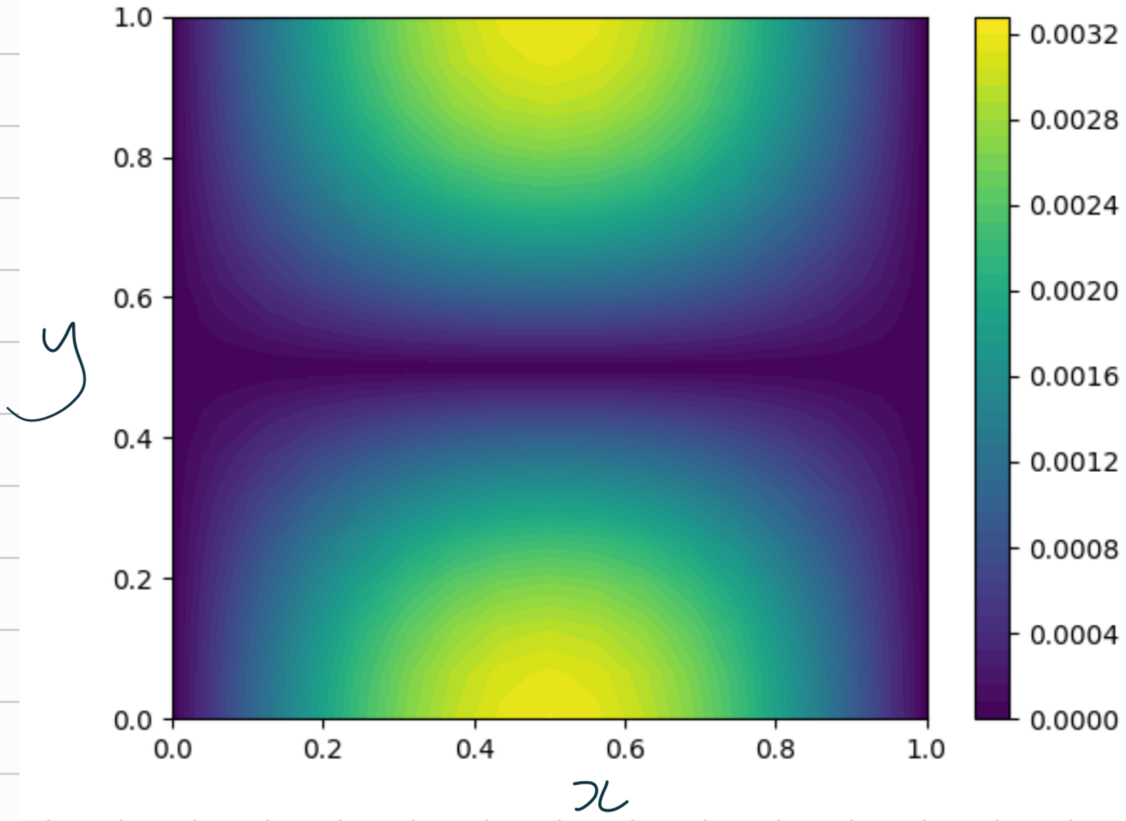
4).

Contour Plot for $\boxed{16 \times 16 \text{ grid, order } 1}$

$$\{h, p\} = \{\tfrac{1}{16}, 1\}$$

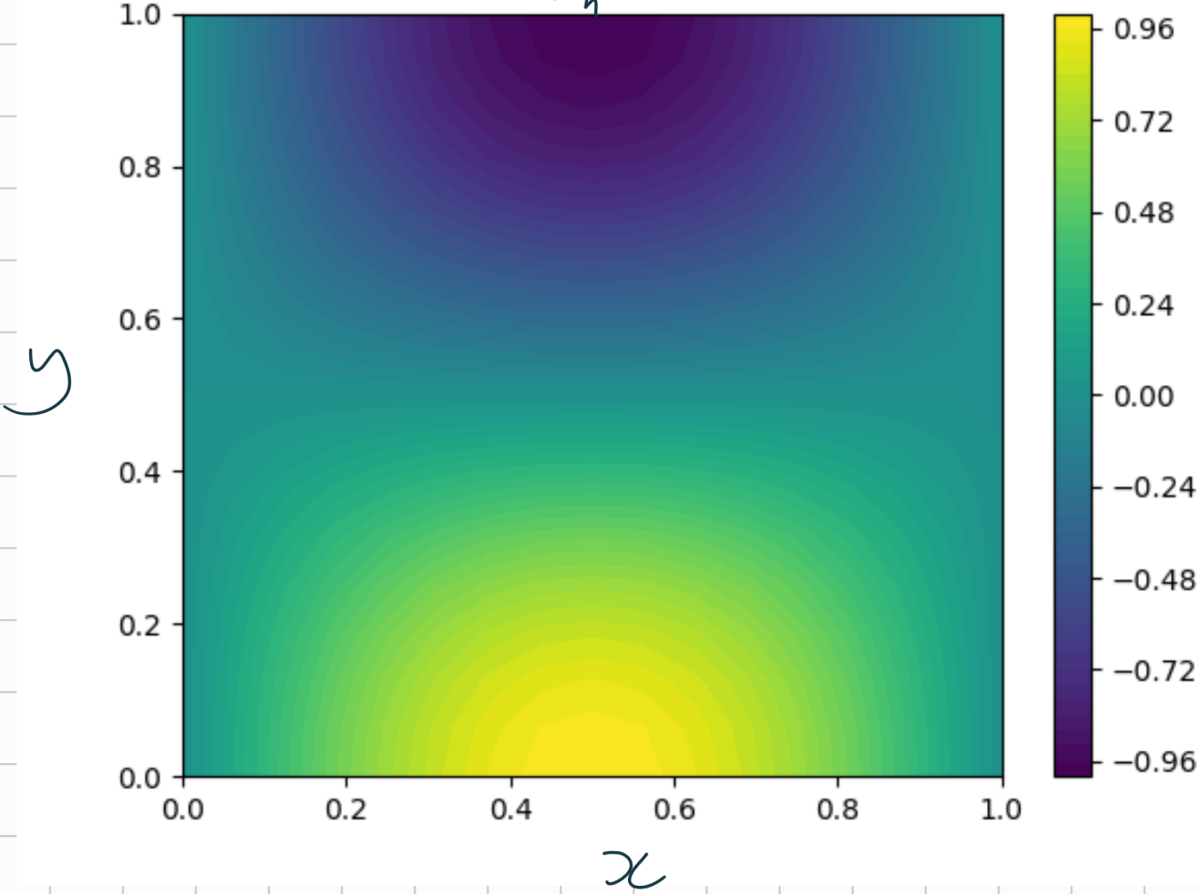


Difference between numerical and exact solution

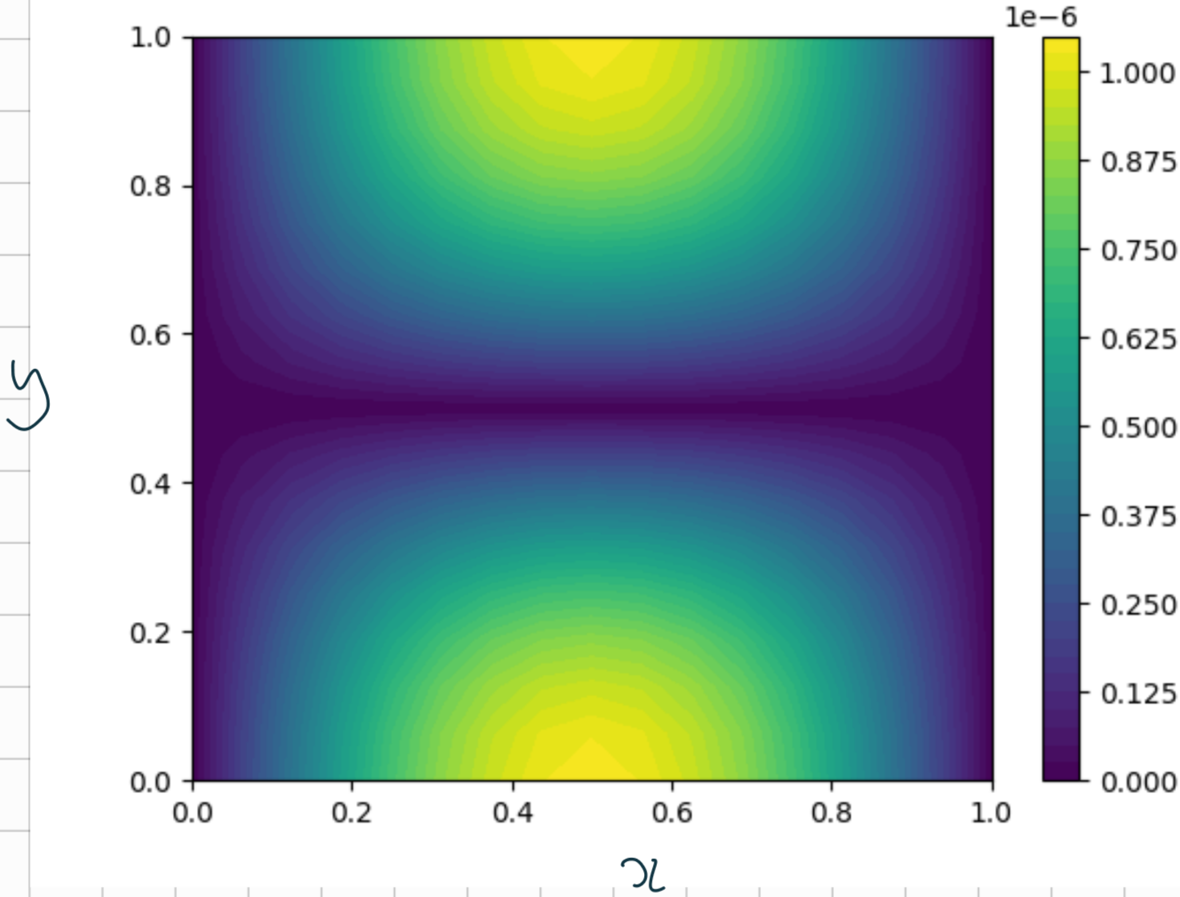


$\boxed{16\times 16 \text{ mesh} \quad p = 2}$ $\{h, p\} = \{\frac{1}{16}, 2\}$

u_h



$|u_h - u_{exact}|$



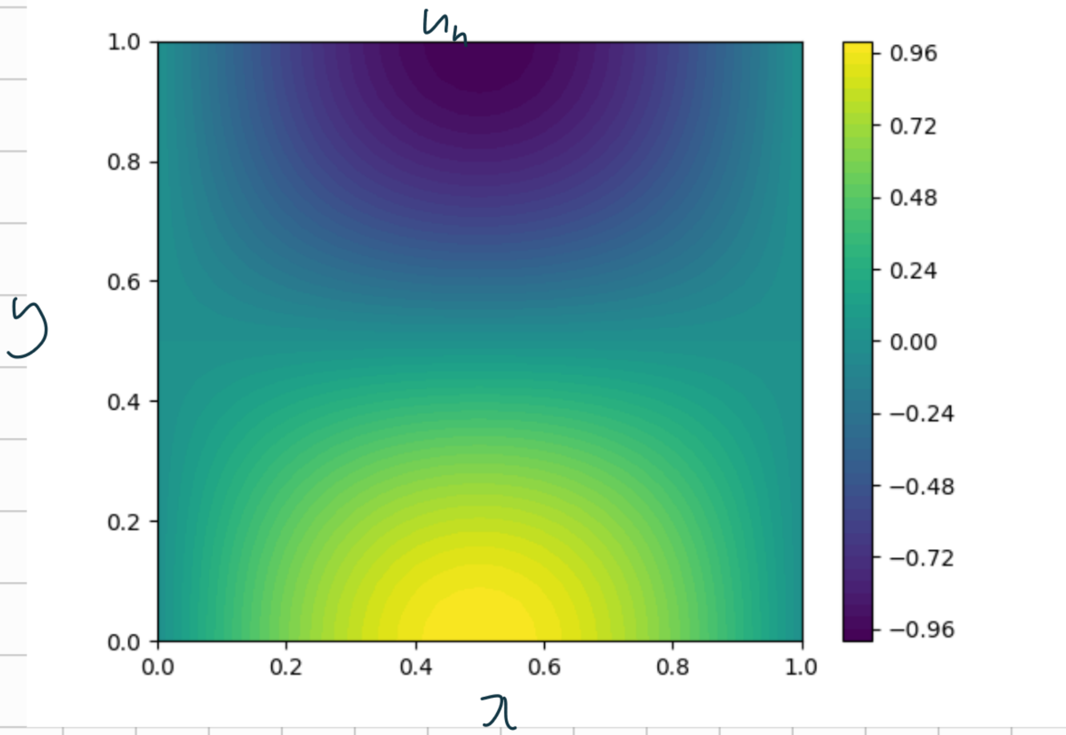
→ Increasing the order of accuracy from 1 to 2, significantly changes difference between numerical and exact solution

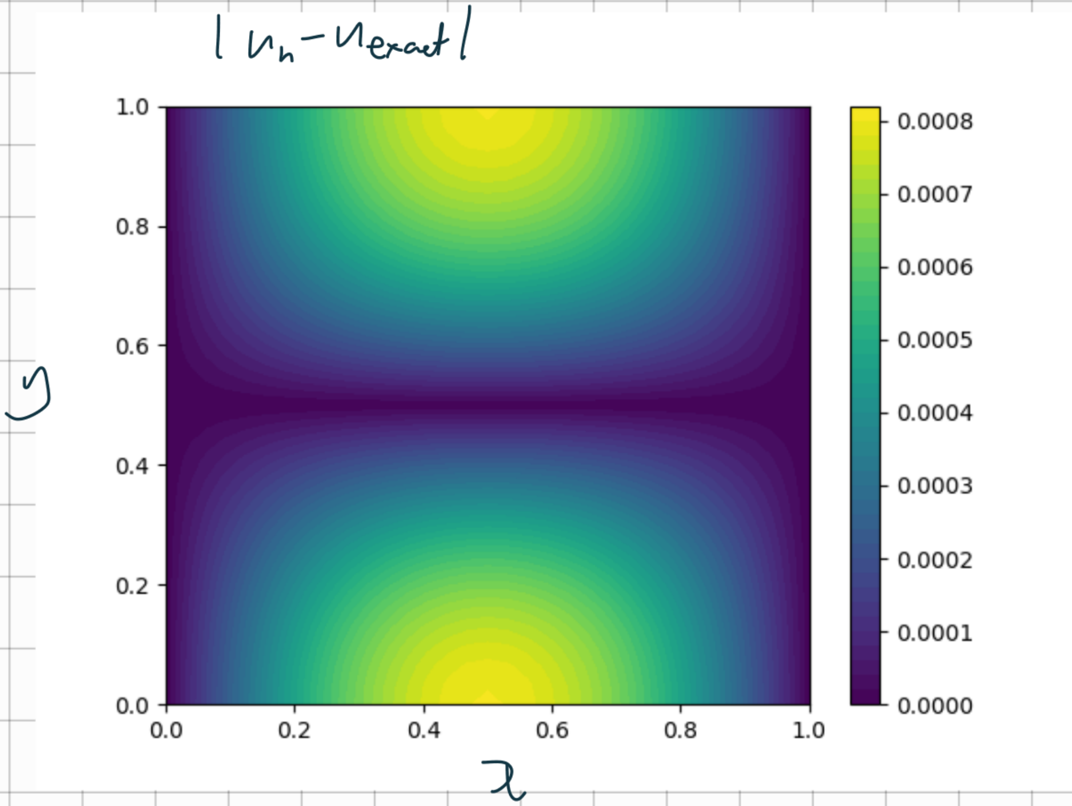
e.g. $|u_n - u_{exact}|_{max} \sim 10^{-3}$ for $p=1$

$|u_n - u_{exact}|_{max} \sim 10^{-6}$ for $p=2$

→ so now looking at how mesh refinement changes it

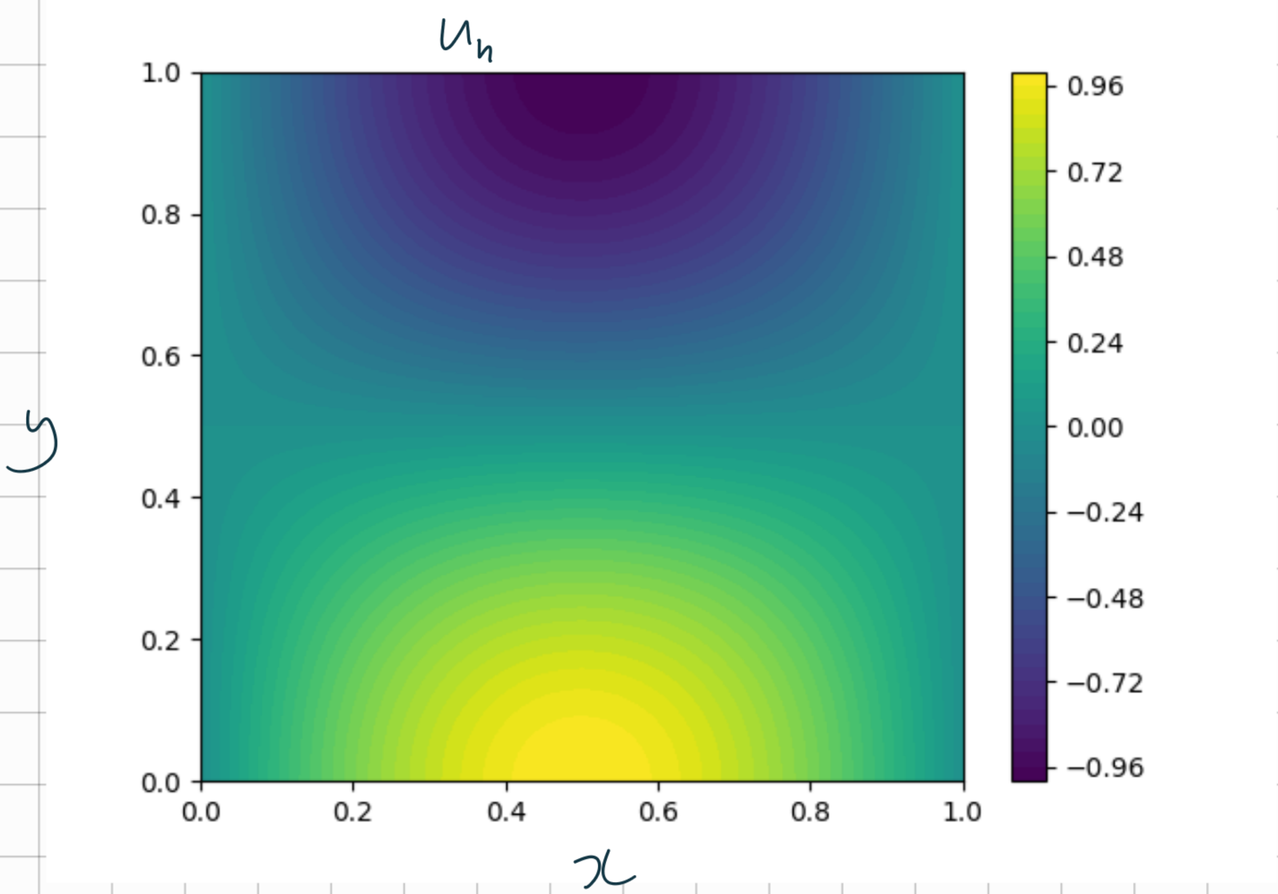
$\boxed{32\times 32, \ p=1}$ $\{h, p\} = \{\frac{1}{32}, 1\}$

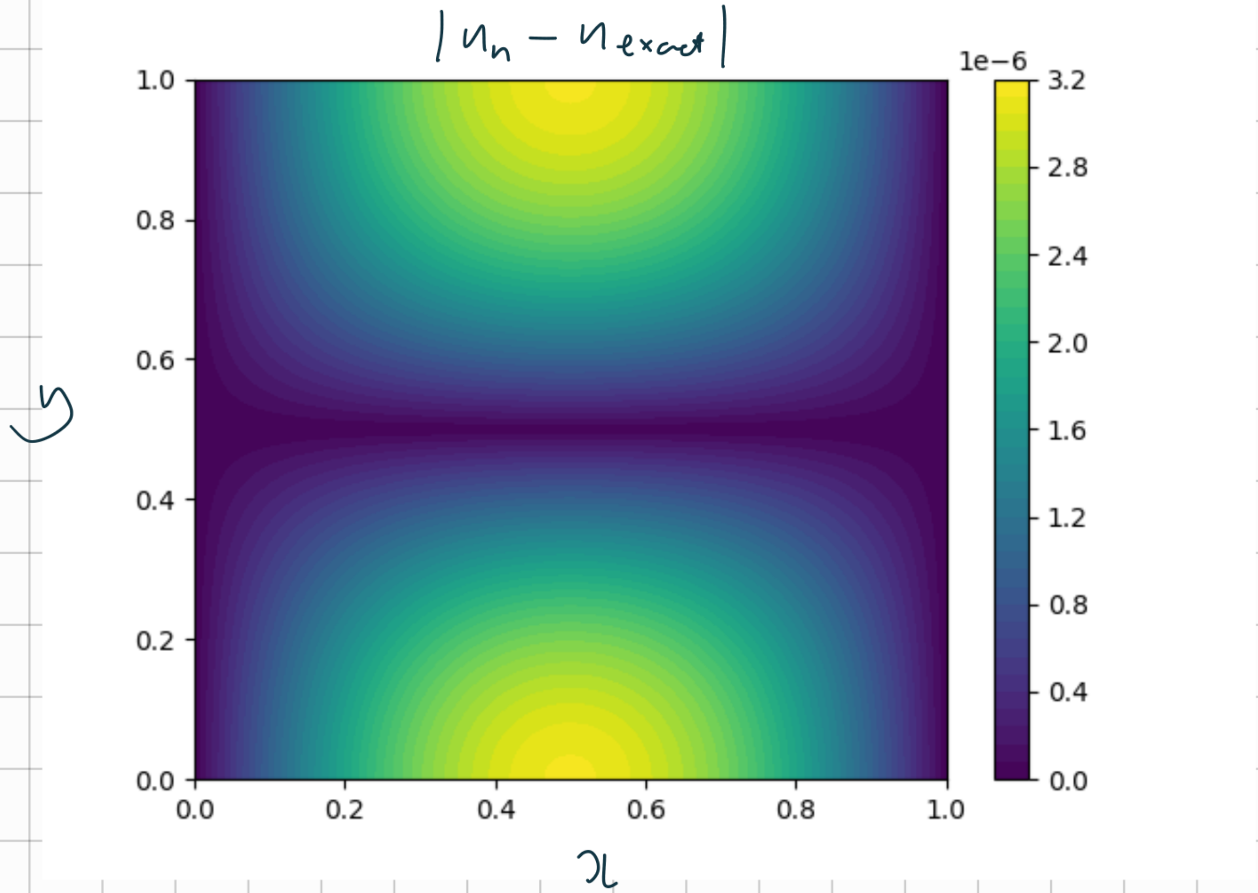




Decreasing element Δx by 2, decreases the difference from $\sim 10^{-3}$ to 10^{-4}

$\boxed{512 \times 512, \ p = 1}$ $\{h, p\} = \{\frac{1}{512}, 1\}$





$|u_h - u_{exact}|$ is approximately the same for a grid of 16×16, $p=2$
and of 512×512, $p=1$

$\Rightarrow$ Instead will look at L^2 error for each

h	p	L^2
$1/16$	1	2×10^{-3}
$1/16$	2	1×10^{-6}
$1/16$	3	4×10^{-9}
$1/32$	1	4×10^{-4}
$1/64$	1	1×10^{-4}
$1/128$	1	3×10^{-5}
$1/32$	2	7×10^{-8}
$1/64$	2	4×10^{-9}

100

For $p=1$, as h decreases by a factor of 2,
L^2 decreases by ~ 4

$p=2$ h decreases by
L^2 decreases by ~ 16

For $h = \frac{1}{16}$ as p increases

L^2 decreases by 16^2

$h = \frac{1}{32}$, L^2 decreases by $\sim 32^2$

so $O(\text{error}) \sim h^{2p}$

$$\boxed{\text{\{h,p\} combinations with similar values of } h^{2p} \text{ are roughly equivalent}}$$

5).- Step 1)

```
# This method is the same as STEP 1 creating function and test function
u = TrialFunction(V) # The unknown or variable u(x,y)
v = TestFunction(V)  # The testfunction of u, which may be better called delu or deltau
```

```
#Step 1 creating weak form using function and test function
a = (inner(grad(u),grad(v)))*dx # Step 2/3: The weak form first term
L = (f*v)*dx # Step 2/3: The weak form second term; dx is the infinitesimal piece in the damain here: dx*dy=dA with area A.
# Weak form generated
```

Step 2

```
#creates discrete function space (step 2)
V = FunctionSpace(mesh, 'CG', 1) # Piecewise linear continuous Galerkin function space or polynomials
```

```
#discrete u function
u_1 = Function(V, name='u_1') # Name of solution for first method
```

Step 3+4

```
#If it wasn't automated, this is where step 3 would be
# and local, reference coordinate system would be introduced

#Firedrake Implementation Step 4
solve(a == L, u_1, solver_parameters={'ksp_type': 'cg', 'pc_type': 'none'}, bcs=[bc_x0,bc_x1]) # Step 4: the solution assigned to u1
```

All these can be found in code

poisson_eq_AH_ex3.py

6). $f(x,y) = 5$

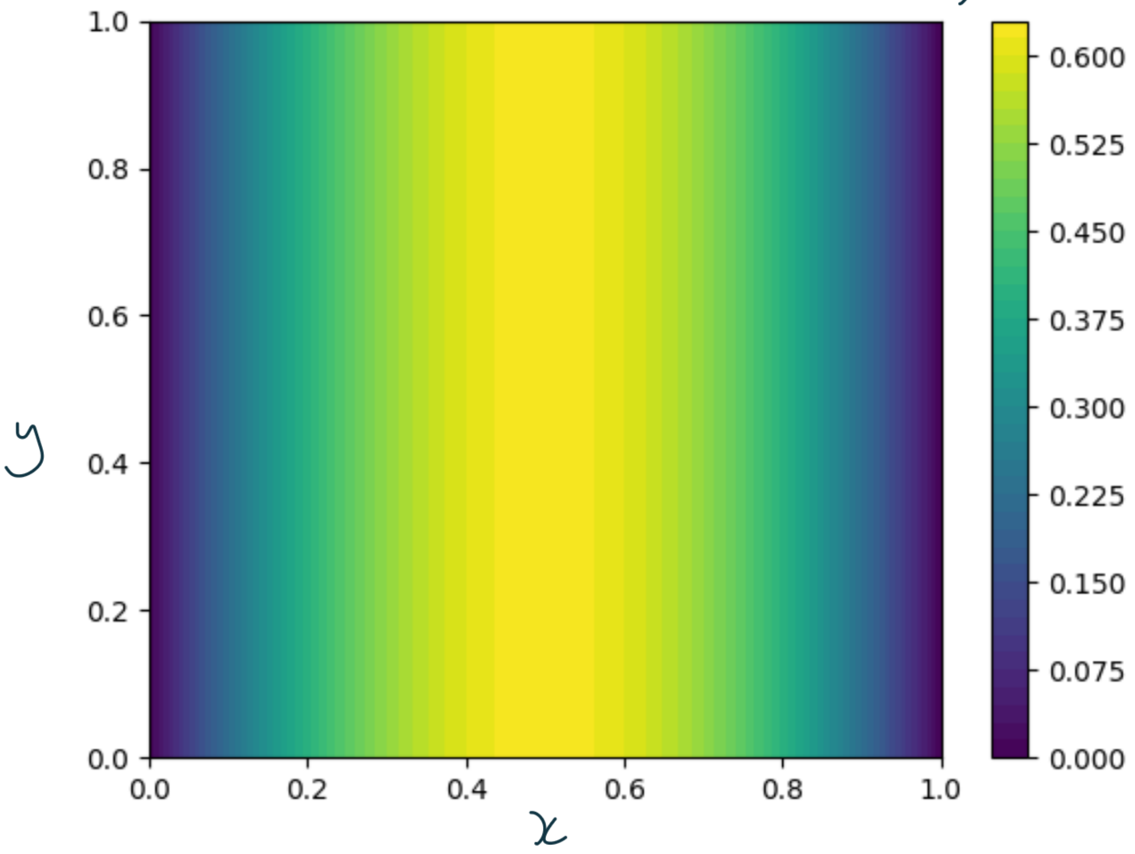
$$u = \frac{5}{2} x(1-x)$$

BCs same as before

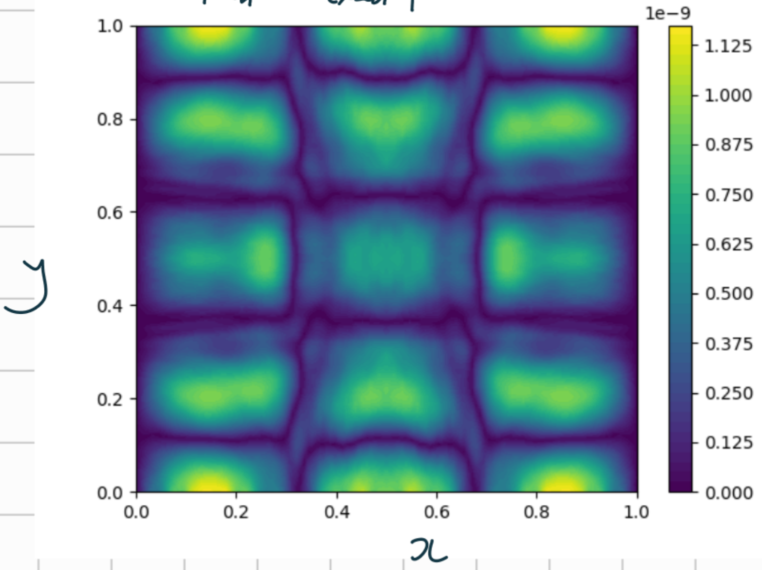
$u(0,y) = u(1,y) = 0$

$\partial_y u|_{y=0} = \partial_y u|_{y=1} = 0$

$u_h \qquad \{h,p\} = \{\frac{1}{128}, 1\}$



$|u_h - u_{exact}|$

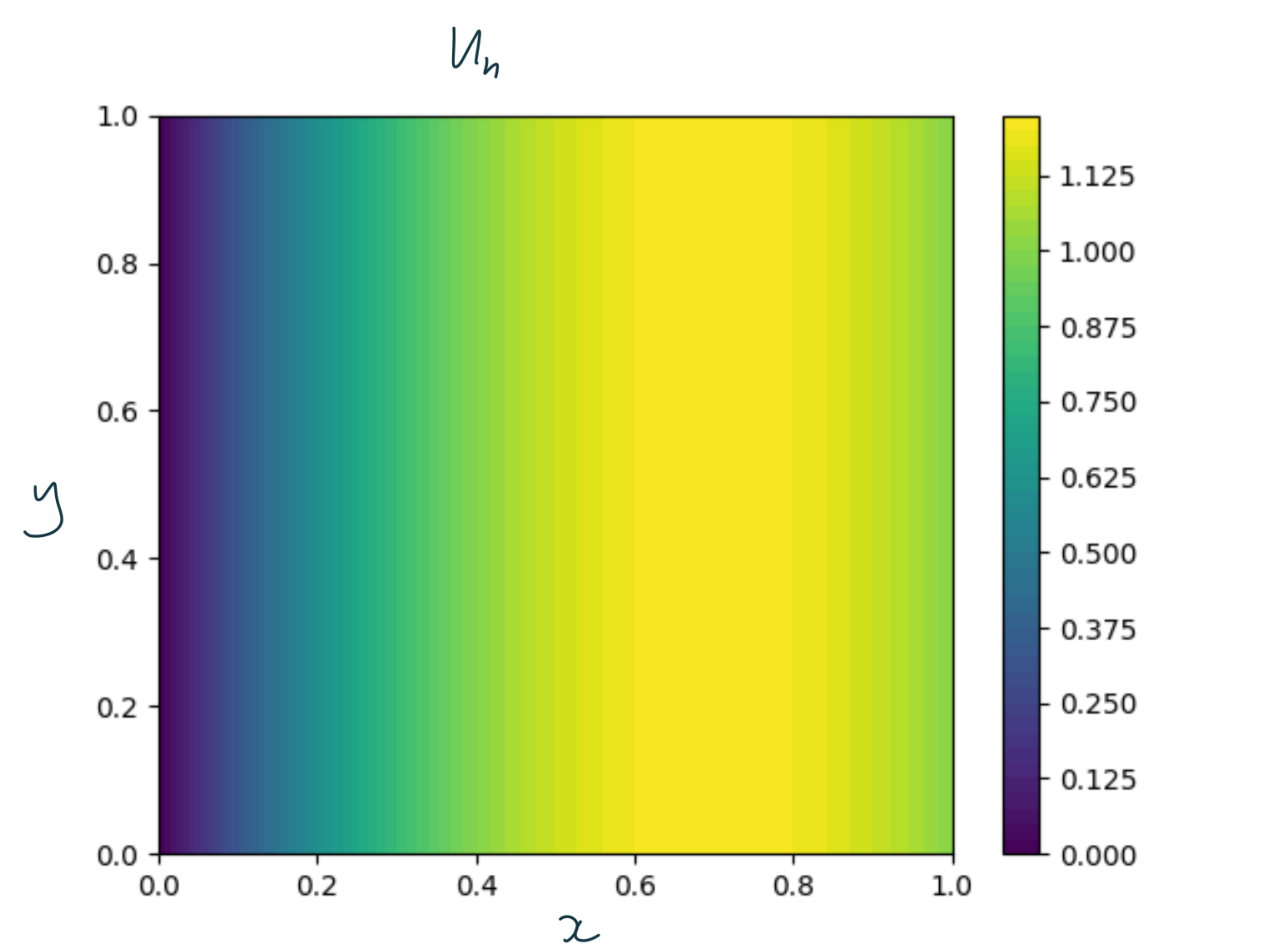


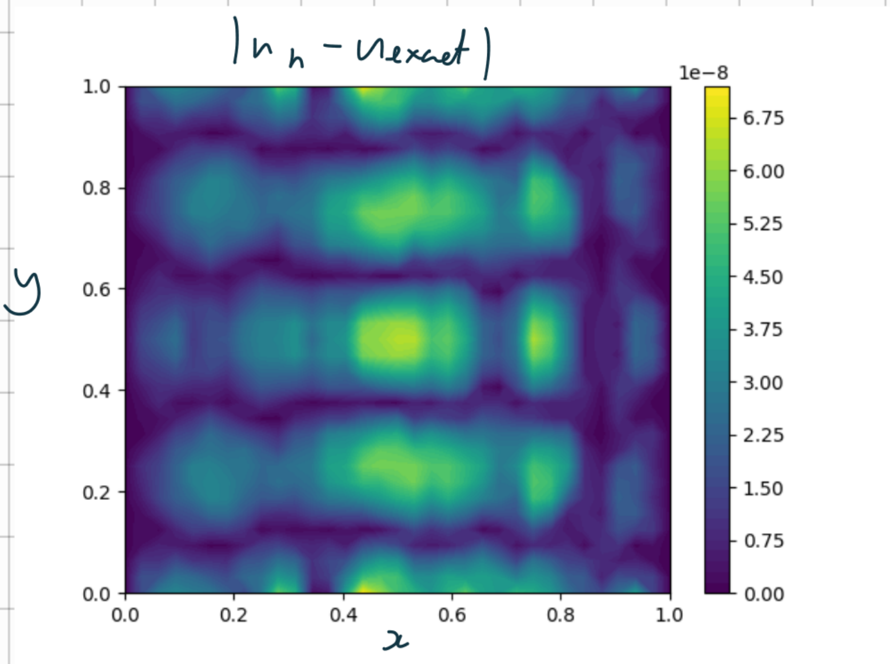
$$f(x,y) = 5$$

$$u(0,y) = 0, \quad u(1,y) = 1$$

$$\partial_y u\big|_{y=0} = \partial_y u\big|_{y=1} = 0$$

$$u = \frac{x}{2}(7 - 5x)$$





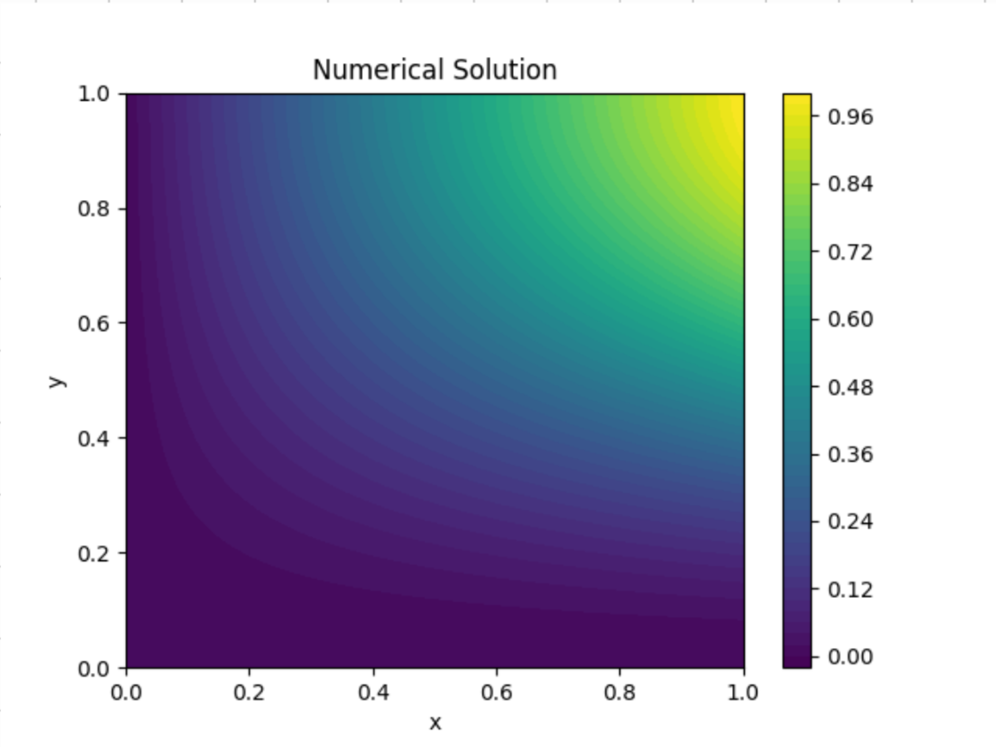
$$f(x) = -6(1-2y)x$$

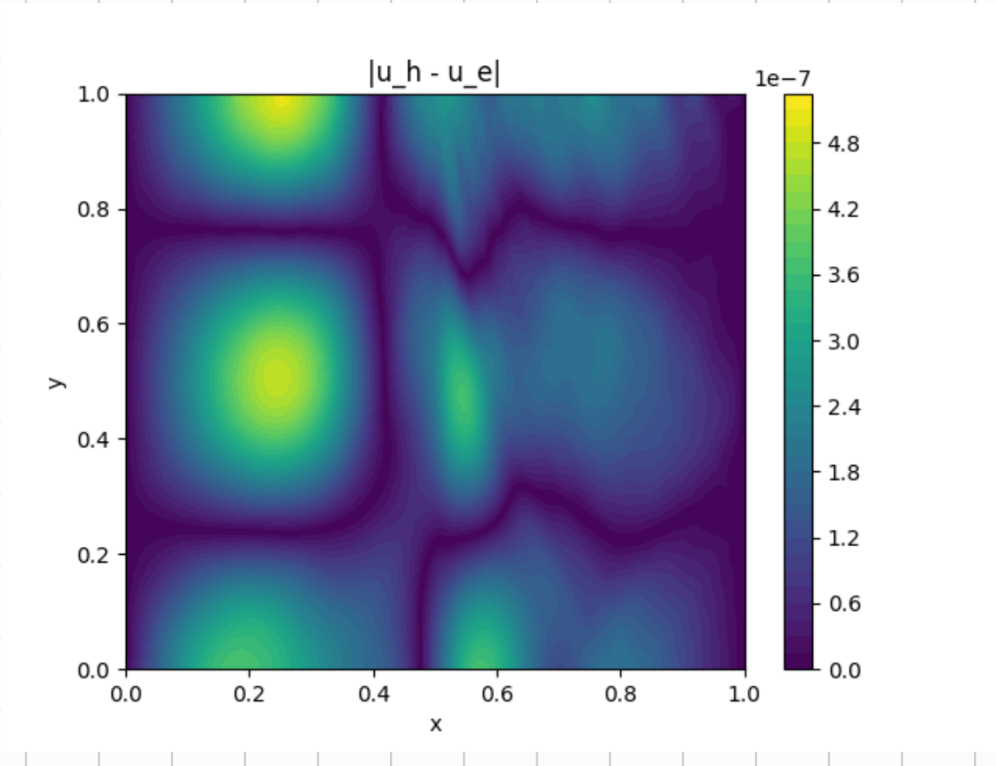
$$u(0,y) = 0 \qquad u(1,y) = y^2(3-2y)$$

$$\partial_y u\big|_{y=0} = \partial_y u\big|_{y=1} = 0$$

$$u = y^2(3-2y)x$$

$$\frac{\partial u}{\partial y} = 2y(3-2y)x - 2y^2 x$$





To implement (All have Neumann BCs)

$f = 5$, $u(0,y) = u(1,y) = 0$
set f_choice = 1

$f = 5$, $u(0,y) = 0$, $u(1,y) = 1$
set f_choice = 2

$f(x,y) = -6y(1-2y)x$, $u(0,y) = 0$, $u(1,y) = y^2(3-2y)$
set f_choice = 3

→ All these seem to be successfully implemented as have checked $|u_h - u_e|$

→ So could then add functions which don't have nice exact solutions

Part 2

1). Multiply ins equation (11) by $q = q(y)$ where $q(y) =$ a test function.

Then integrate it over y

$$\to \int_0^{L_y} q\left[\partial_t(w_v h_m) - \alpha g \partial_y(w_v h_m \partial_y h_m)\right] dy = \int_0^{L_y} \frac{q w_v R}{m_{por}\sigma_e}$$

$$\int_0^{L_y} \left(q \partial_t(w_v h_m) - \alpha g q \partial_y(w_v h_m \partial_y h_m)\right) dy = \int_0^{L_y} \frac{q w_v R}{m_{por}\sigma_e}$$

Using $\partial_y(q w_v h_m \partial_y h_m)$

$$= (\partial_y q)(w_v h_m \partial_y h_m) + q \partial_y(w_v h_m \partial_y h_m)$$

This can be rewritten as

$$\int_0^{L_y} \left(q \partial_t h_m + \alpha g(\partial_y q)(h_m \partial_y h_m) - \alpha g \partial_y(q h_m \partial_y h_m)\right) dy$$

$$= \int_0^{L_y} \frac{q R}{m_{por}\sigma_e} dy$$

$$\int_0^{L_y} \xi\, \partial_t h_m + \alpha g h_m \partial_y \xi\, \partial_y h_m \; dy - \alpha g \left[\xi h_m \partial_y h_m \right]_0^{L_y}$$

$$= \int_0^{L_y} \frac{\xi R}{m_{por} \sigma_e} \, dy$$

As $\partial_y h_m = 0$ at $y = L_y$

Then $-\alpha g \left[\xi h_m \partial_y h_m \right]_0^{L_y}$

$$= -\alpha g\, \xi(0,t) h_m \partial_y h_m \big|_{y=0}$$

$$= -\alpha g\, \xi(0) h_m \partial_y h_m \big|_{y=0}$$

$$\int_0^{L_y} \xi\, \partial_t h_m + \alpha g h_m \partial_y \xi\, \partial_y h_m \, dy - \alpha g \xi(0) h_m \partial_y h_m \big|_{y=0}$$

$$= \int_0^{L_y} \frac{\xi R}{m_{por} \sigma_e} \, dy$$

Using equation (14):

$$L_c \frac{dh_{cm}}{dt} = m_{por} \frac{\sigma_e}{2} \alpha g \, \partial_y (h_m^2) \big|_{y=0} - \sqrt{g} \max\left(\frac{2}{3} h_{cm}(t), 0 \right)^{3/2}$$

$$\alpha g h_m \partial_y h_m \big|_{y=0} = \frac{1}{m_{por} \sigma_e} \left[L_c \frac{dh_{cm}}{dt} + \sqrt{g} \max\left(\frac{2 h_{cm}}{3}, 0 \right)^{3/2} \right]$$

Eliminate $h_m \partial_y h_m |_{y=0}$ from equations

$$\int_0^{L_y} \xi \partial_t h_m + \alpha g h_m \partial_y \xi \partial_y h_m \, dy$$

$$- \frac{\xi(0)}{m_{par}\sigma_e} \left[L_c \frac{d h_m(0,t)}{dt} + \sqrt{g} \max\left(\frac{2 h_m(0,t)}{3}, 0\right)^{3/2} \right]$$

$$= \int_0^{L_y} \frac{\xi \mathcal{U}}{m_{par}\sigma_e} \, dy$$

Using Forward Euler $(\theta = 0)$

$$\int_0^{L_y} \xi \frac{\left(h_m^{n+1} - h_m^n\right)}{\Delta t} + \alpha g h_m^n \partial_y \xi \partial_y h_m^n \, dy$$

$$- \frac{\xi(0)}{m_{par}\sigma_e} \left[\frac{L_c \left(h_{cm}^{n+1} - h_{cm}^n\right)}{\Delta t} + \sqrt{g} \max\left(\frac{2}{3} h_{cm}^n, 0\right) \right]$$

$$= \int_0^{L_y} \frac{\xi \mathcal{U}^n}{m_{par}\sigma_e} \, dy$$

$$\int_0^{L_y} \eta h_m^{n+1} dy + \frac{L_c h_{cm}^{n+1}}{m_{por}\sigma_e} \eta(0)$$

$$= \int_0^{L_y} \eta h_m^n dy + \frac{L_c h_{cm}^n}{m_{por}\sigma_e} \eta(0)$$

$$+ \Delta t \int_0^{L_y} \left(-\alpha g h_m^n \partial_y \eta \partial_y h_m^n + \frac{\eta K^n}{m_{por}\sigma_e} \right) dy$$

$$- \Delta t \frac{\sqrt{g}}{m_{por}\sigma_e} \max\left(\frac{2 h_{cm}^n}{3}, 0 \right)^{3/2} \eta(0)$$

Let $\eta = \varphi_i(y)$, $h_m = h_j \varphi_j(y)$

for $i, j = 1, \ldots N_n$ where N_n = no. of nodes

$\rightarrow$ and $h_1 = h_{cm}$ $\rightarrow$ this means $\varphi_1 = 1$ (e.g. $\eta(0) = 1$)

$$\int_0^{L_y} \varphi_i \varphi_j dy \, h_j^{n+1} + \frac{L_c h_1^{n+1}}{m_{por}\sigma_e} \delta_{i1}$$

$$= \int_0^{L_y} \varphi_i \varphi_j dy \, h_j^n + \frac{L_c h_1^n}{m_{por}\sigma_e} \delta_{i1}$$

$$+ \Delta t \int_0^{L_y} \left(-\alpha g h_m^n \partial_y \varphi_i \partial_y h_m^n + \frac{\varphi_i K^n}{m_{por}\sigma_e} dy \right.$$

$$- \Delta t \frac{\sqrt{g}}{m_{por}\sigma_e} \max\left(\frac{2 h_1^n}{3}, 0 \right)^{3/2}$$

$$\Rightarrow M_{ij} h_j^{n+1} + \frac{L_c h_1^{n+1}}{m_{por}\sigma_e}\delta_{i1}$$

$$= M_{ij} h_j^{n} + \frac{L_c h_1^{n}}{m_{por}\sigma_e}\delta_{i1} + \Delta t\, b_i^n - \frac{\Delta t \sqrt{S}}{m_{por}\sigma_e}\max\left(\frac{2h_1^n}{3}, 0\right)^{3/2}$$

where $M_{ij} = \int_0^{L_y} \varphi_i \varphi_j \, dy$

$$b_i = \int_0^{L_y}\left(-\alpha g h_m^n \partial_y \varphi_i \partial_y h_m^n + \frac{\varphi_i R^n}{m_{por}\sigma_e}\right) dy$$

In finite difference case

$$\partial_t h_m - \alpha g \partial_y (h_m \partial_y h_m) = \frac{R}{m_{por}\sigma_e}$$

becomes

$$\frac{h_j^{n+1} - h_j^n}{\Delta t} - \frac{\alpha g}{\Delta y}\left[h_{j+1/2}^n \partial_y h_{j+1/2}^n - h_{j-1/2}^n \partial_y h_{j-1/2}^n\right] = \frac{R}{m_{por}\sigma_e}$$

$$\frac{h_j^{n+1} - h_j^n}{\Delta t} - \frac{\alpha g}{2(\Delta y)^2}\left[(h_{j+1}^n + h_j^n)(h_{j+1}^n - h_j^n) - (h_j^n + h_{j-1}^n)(h_j^n - h_{j-1}^n)\right]$$

$$= \frac{R}{m_{por}\sigma_e}$$

letting $h_j^n = \lambda^n e^{ikj\Delta x}$

$$\frac{\lambda - 1}{\Delta t} - \frac{\alpha g h_j^n}{2(\Delta y)^2}\left[(e^{ik\Delta x}+1)(e^{ik\Delta x}-1) - (1+e^{-ik\Delta x})(1-e^{-ik\Delta x})\right]$$

$$= \frac{K}{m_{poro}\sigma_e h_j^n}$$

$$\frac{\lambda - 1}{\Delta t} - \frac{\alpha g h_j^n}{2(\Delta y)^2}\left[(e^{2ik\Delta x}-1) - (1-e^{-2ik\Delta x})\right] = \frac{K}{m_{poro}\sigma_e h_j^n}$$

$$\lambda - 1 = \frac{h_j^n \alpha g \,\Delta t}{2(\Delta y)^2}\left(e^{2ik\Delta x} - 2 + e^{-2ik\Delta x}\right) + \frac{K}{m_{poro}\sigma_e h_j^n}\Delta t$$

$$= \frac{\alpha g \Delta t\, 2 i^2 h_j^n}{(\Delta y)^2}\sin^2(k\Delta x) + \frac{K}{m_{poro}\sigma_e h_j^n}\Delta t$$

$$= \left(\frac{K}{m_{poro}\sigma_e h_j^n} - \frac{2\alpha g h_j^n}{(\Delta y)^2}\sin^2(k\Delta x)\right)\Delta t$$

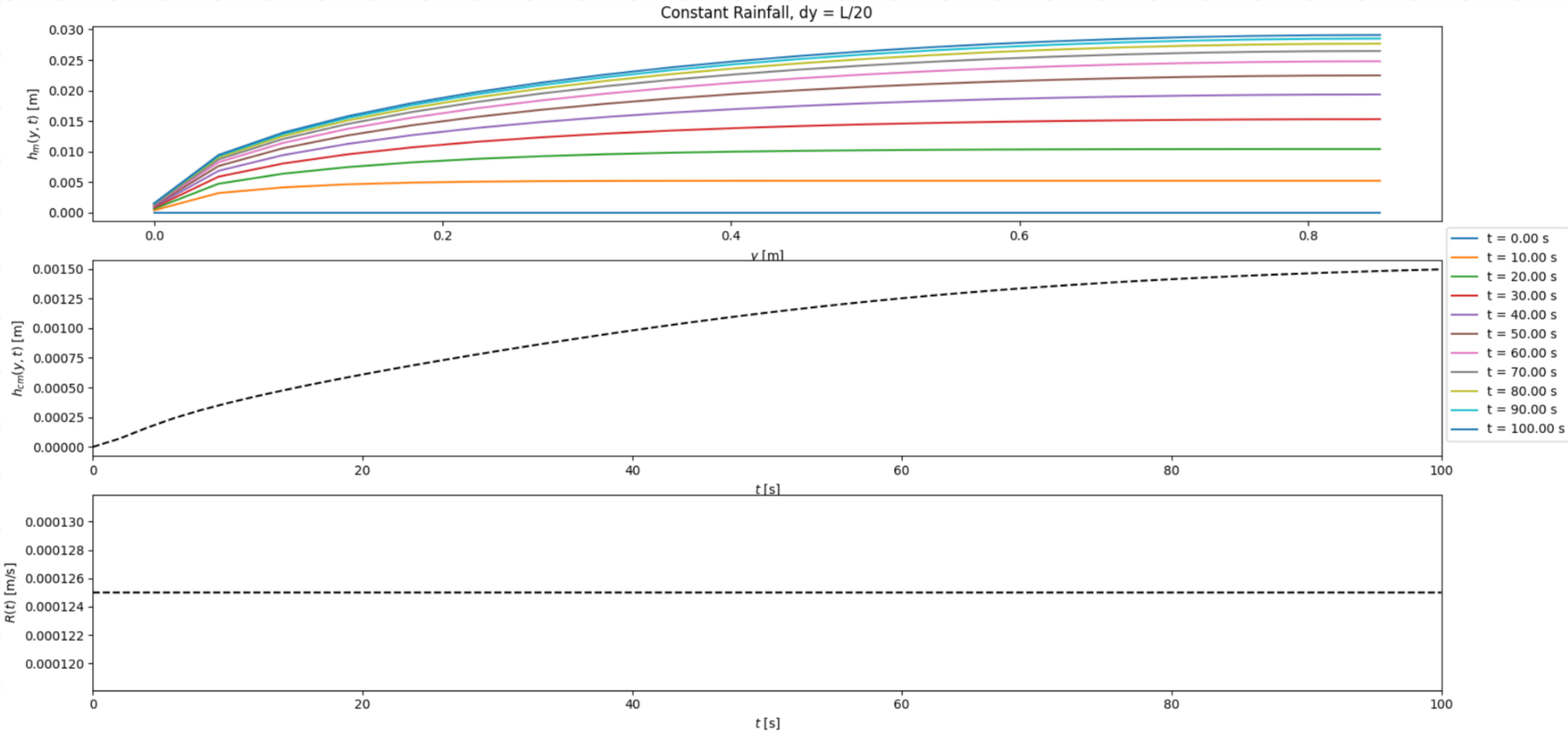
$|\lambda| \leq 1$ to be stable

$\rightarrow$ so

$$\Delta t \leq \left|\frac{1}{\frac{K}{m_{poro}\sigma_e h_j^n} - \frac{2\alpha g h_j^n}{(\Delta y)^2}}\right|$$

$$\leq \left| \frac{1}{\frac{\eta \nu}{k h_j^n} - \underbrace{\frac{2 g h_j^n}{(\Delta y)^2}}_{\text{This term dominates}}} \right| \times \frac{1}{\alpha}$$

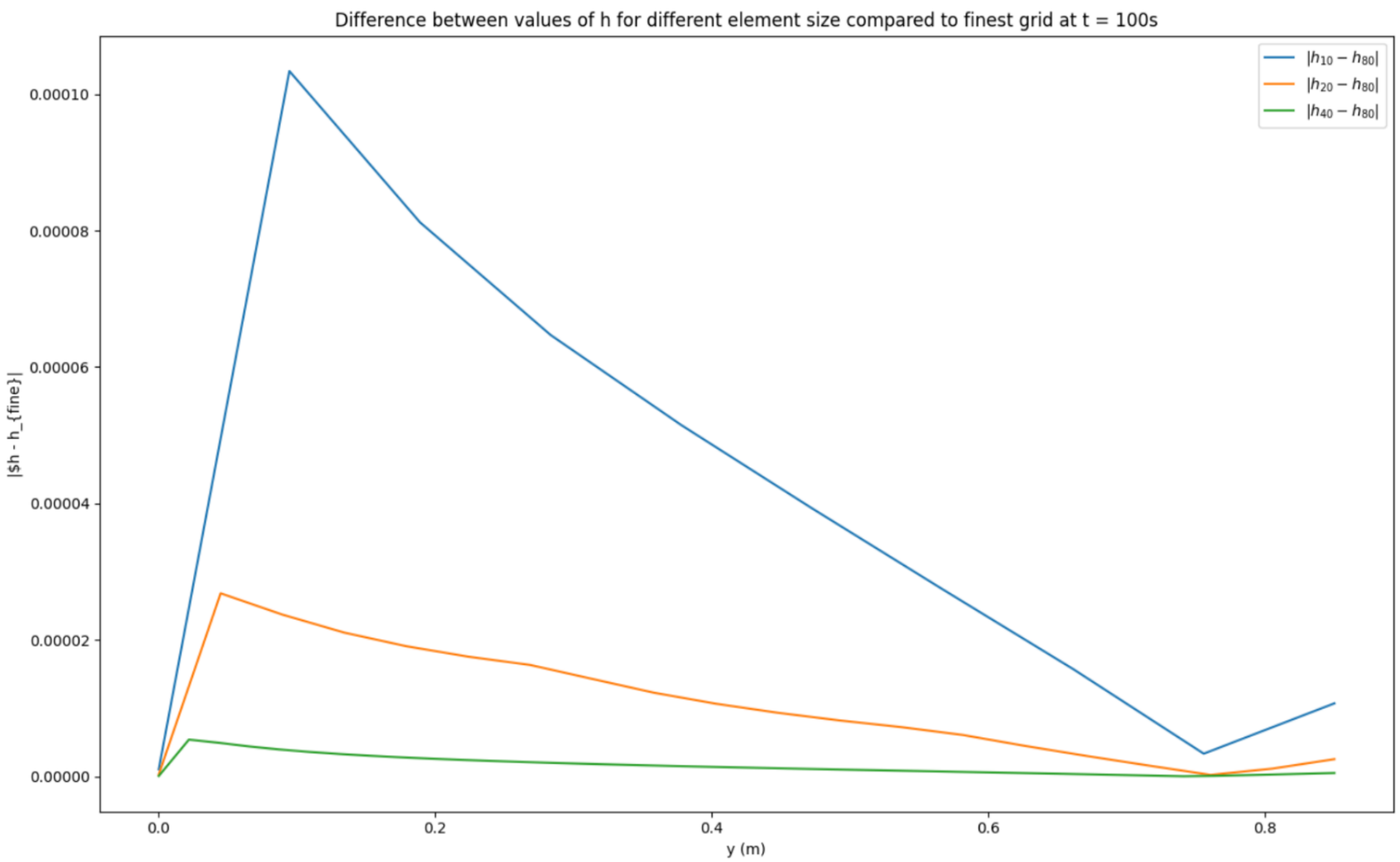
$$\boxed{\text{so} \quad \Delta t \lesssim \frac{(\Delta y)^2}{2 \alpha g |h_{max}|}}$$



• Profiles for $dy = \frac{L_y}{20}$

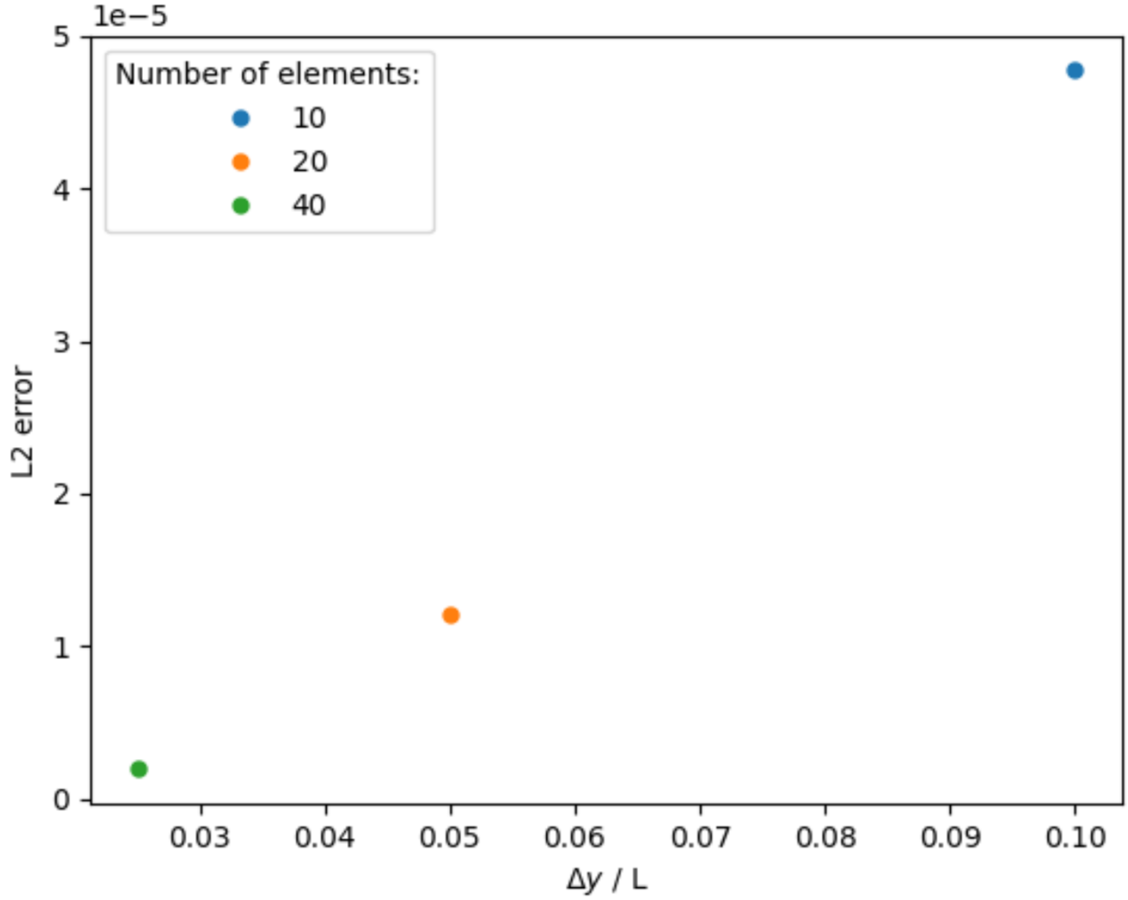
Repeated this for different dy, and then compared the solutions.

$$dy = \frac{L_y}{10}, \frac{L_y}{20}, \frac{L_y}{40}, \frac{L_y}{80}$$



- Difference in solution clearly decreases as dy decreases $\Rightarrow$ so numerical solution converges.

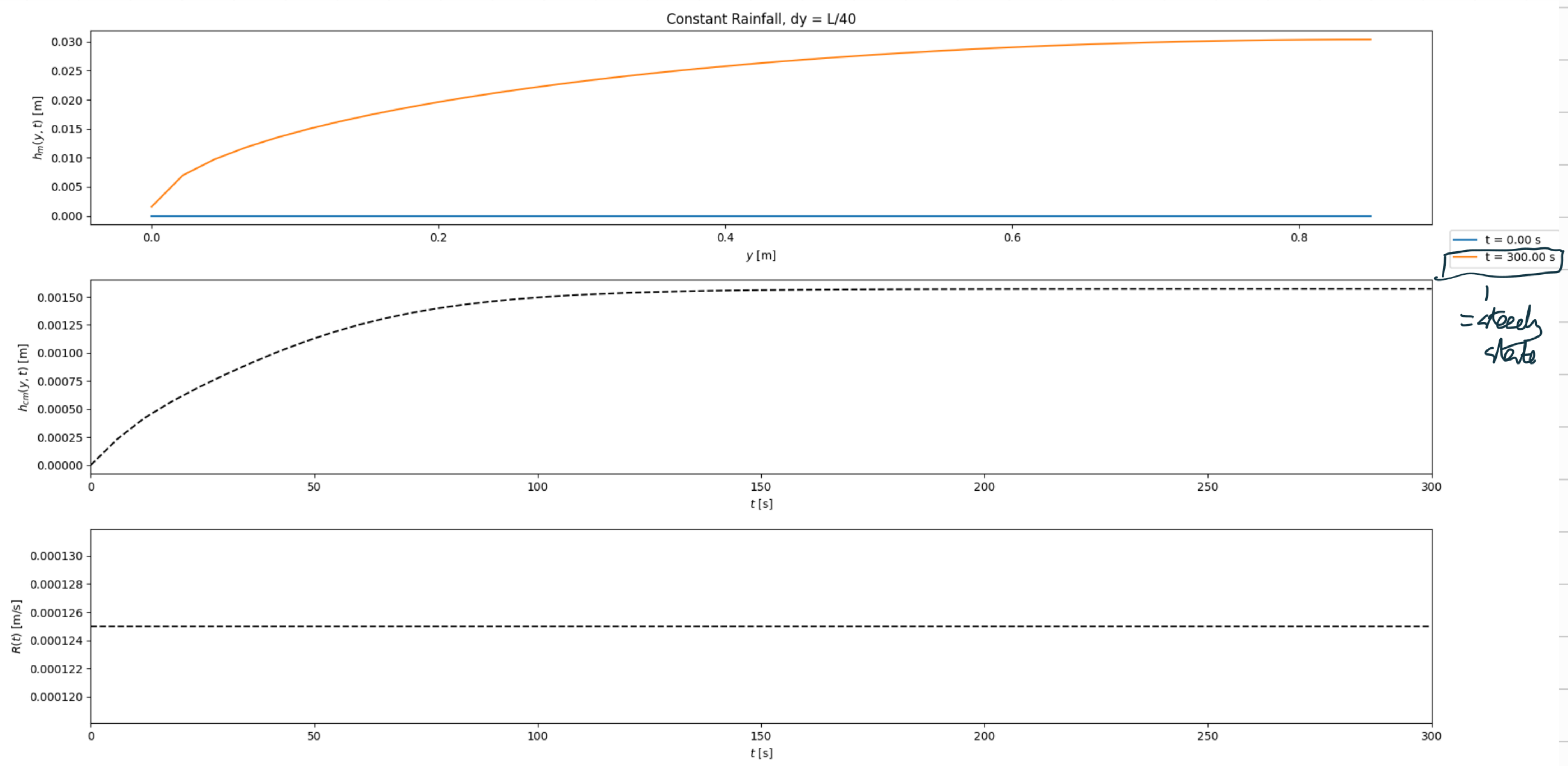
L^2 error between solution for $dy = \frac{L_y}{80}$ and other dy



→ L^2 decreases, once again showing solution is converging.

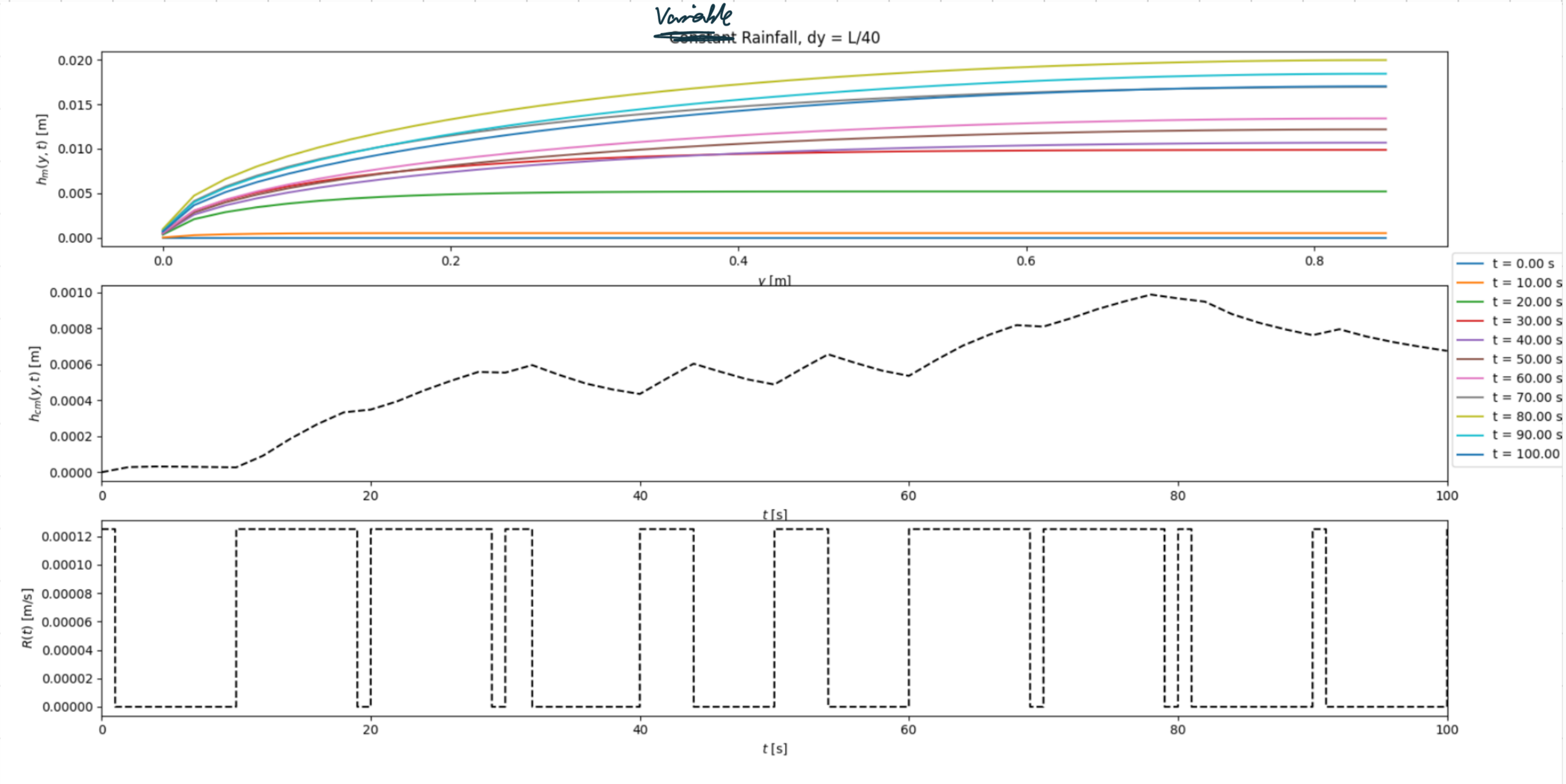
- System has not yet reached steady state.

→ will find by increasing t_{max}



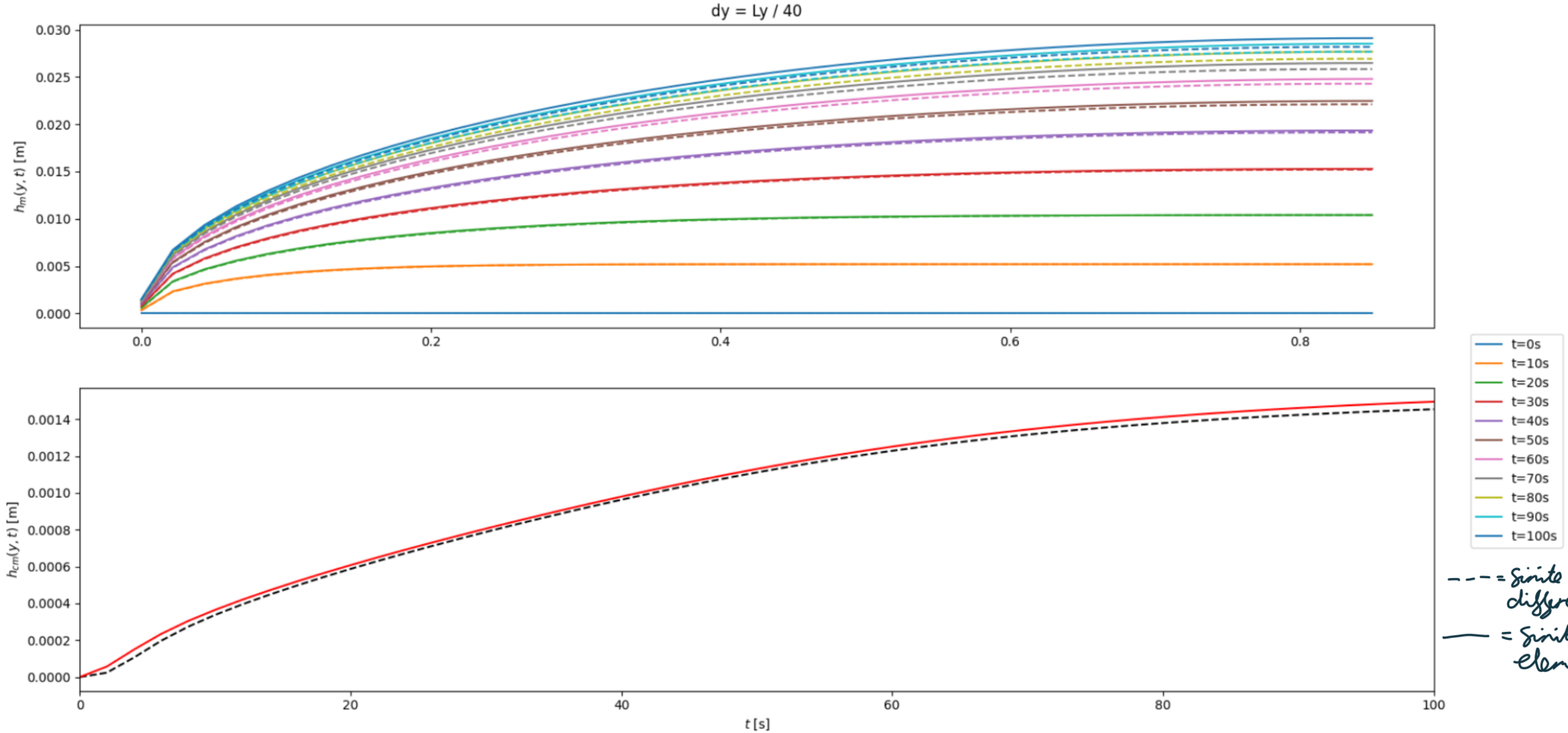
h_{cm} at steady state : $h_{cm} \approx 0.00157\,m$

$t = 300s$ shows steady state h_m



In code, change Rvar from 0 to 1 to set variable rain

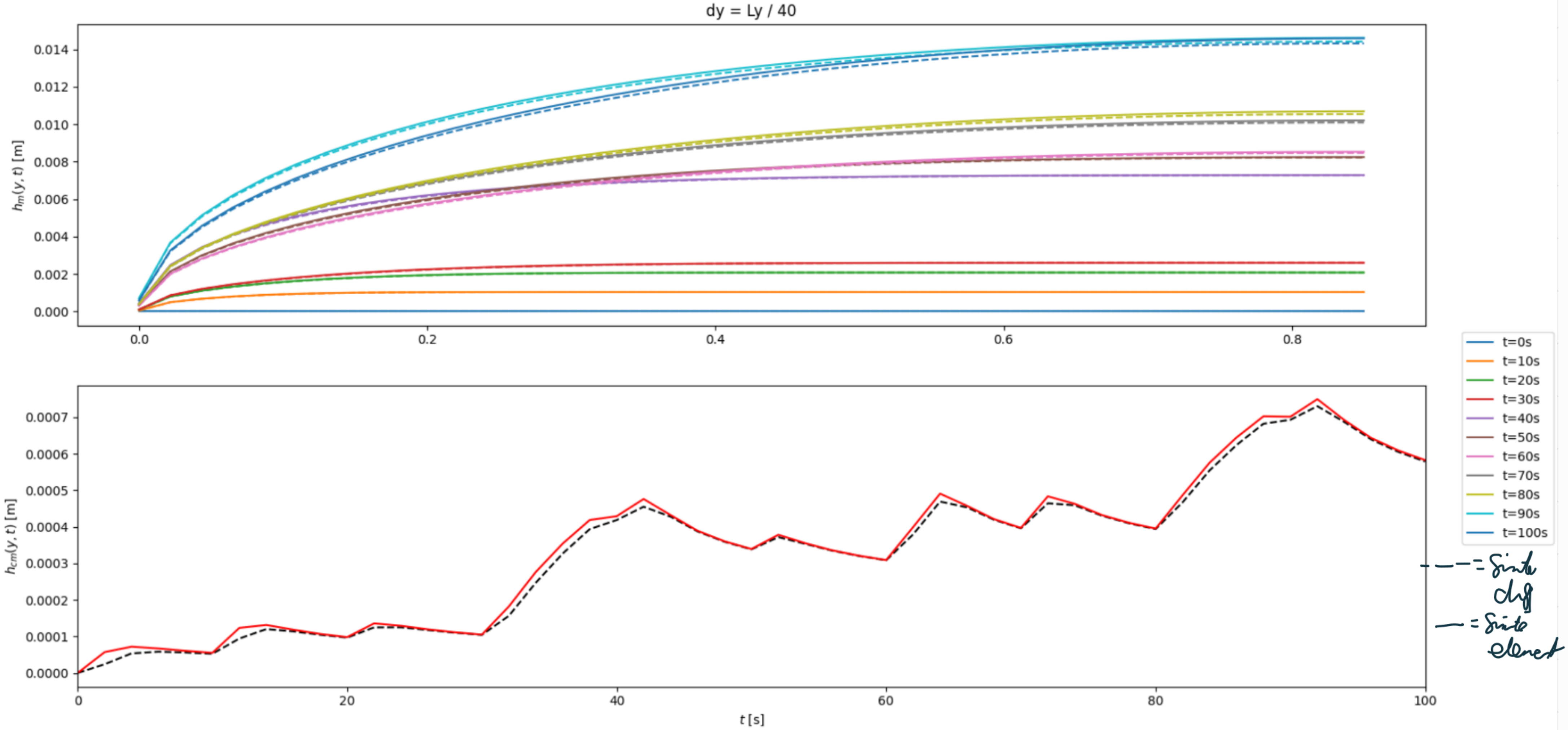
Constant Rain



- - - = finite difference

— = finite element

Variable rain



- - = finite diff

— = finite element

- Finite difference and finite element do not match up completely

  → and this is worse for variable rainfall

- This difference might be due to how finite difference handles h_{cm} seperately to h_m.

3) Applying Crank–Nicholson to this:

$$\int_0^{L_y} \xi\, \partial_t h_m + \alpha g\, h_m\, \partial_y \xi\, \partial_y h_m \, dy$$

$$- \frac{\xi(0)}{m_{por}\sigma_e}\left[L_c \frac{d h_m(0,t)}{dt} + \sqrt{g}\, \max\left(\frac{2h_m(0,t)}{3}, 0\right)^{3/2} \right]$$

$$= \int_0^{L_y} \frac{\xi R}{m_{por}\sigma_e}\, dy$$

$\rightarrow$ e.g. $h_m = \dfrac{h_m^{n+1} + h_m^{n}}{2}$

$$\int_0^{L_y} \xi h_m^{n+1}\, dy + \frac{L_c h_{cm}^{n+1}}{m_{por}\sigma_e}\xi(c) + \frac{\Delta t}{2}\int_c^{L_y} \alpha g\, h_m^{n+1}\, \partial_y \xi\, \partial_y h_m^{n+1}\, dy$$

$$+ \frac{\Delta t}{2}\frac{\sqrt{g}\,\xi(c)}{m_{por}\sigma_e}\max\left(\frac{2h_{cm}^{n+1}}{3}, 0\right)^{3/2}$$

$$= \int_0^{L_y} \xi h_m^{n}\, dy + \frac{L_c h_{cm}^{n}}{m_{por}\sigma_e}\xi(0) + \frac{\Delta t}{2}\int_c^{L_y} \alpha g\, h_m^{n}\, \partial_y \xi\, \partial_y h_m^{n}\, dy$$

$$+ \frac{\Delta t}{2}\int_c^{L_y} \xi\frac{(R^n + R^{n+1})}{m_{por}\sigma_e}\, dy - \frac{\Delta t}{2}\frac{\sqrt{g}\,\xi(c)}{m_{por}\sigma_e}\max\left(\frac{2h_{cm}^{n}}{3}, 0\right)^{3/2}$$

$$q = \varphi_i \,,\; h = h_j \varphi_j$$

$$\rightarrow h_{cm} = h_1$$

$$\int_0^{L_y} \varphi_i \varphi_j \, dy \; h_j^{n+1} + \frac{L_c h_1^{n+1}}{m_{por} \sigma_e} \delta_{i1}$$

$$+ \frac{\Delta t}{2} \int_0^{L_y} \alpha g \, h_m^{n+1} \partial_y \varphi_i \, \partial_y h_m^{n+1} \, dy$$

$$+ \frac{\Delta t}{2} \frac{\sqrt{g}}{m_{por} \sigma_e} \delta_{i1} \max\left(\frac{2h_1^{n+1}}{3}, 0\right)^{3/2}$$

$$= \int_0^{L_y} \varphi_i \varphi_j \, dy \; h_j^n + \frac{L_c h_1^n}{m_{por} \sigma_e} \delta_{i1}$$

$$+ \frac{\Delta t}{2} \int_0^{L_y} \varphi_i \frac{(R^n + R^{n+1})}{m_{por} \sigma_e} \, dy - \frac{\Delta t}{2} \frac{\sqrt{g}}{m_{por} \sigma_e} \delta_{i1} \max\left(\frac{2h_1^n}{3}, 0\right)^{3/2}$$

$$- \frac{\Delta t}{2} \int_0^{L_y} \alpha g \, h_m^{n+1} \partial_y \varphi_i \, \partial_y h_m^{n+1} \, dy$$

$$\boxed{\begin{aligned} &M_{ij} h_j^{n+1} + \frac{L_c h_1^{n+1}}{m_{por}\sigma_e}\delta_{i1} - \frac{\Delta t}{2} b_i^{n+1} + \frac{\Delta t}{2}\frac{\sqrt{g}\,\delta_{i1}}{m_{por}\sigma_e}\max\left(\frac{2h_1^{n+1}}{3}, 0\right)^{3/2} \\ &= M_{ij} h_j^{n} + \frac{L_c h_1^{n}\delta_{i1}}{m_{por}\sigma_e} + \frac{\Delta t}{2} b_i^{n} - \frac{\Delta t}{2}\frac{\sqrt{g}\,\delta_{i1}}{m_{por}\sigma_e}\max\left(\frac{2h_1^{n}}{3}, 0\right)^{3/2} \\ &\text{where } M_{ij} = \int_0^{L_y} \varphi_i \varphi_j \, dy \\ &\qquad b_i^n = \int_0^{L_y}\left(-\alpha g h_m^n \, \partial_y \varphi_i \, \partial_y h_m^n + \frac{\varphi_i R^n}{m_{por}\sigma_e}\right) dy \end{aligned}}$$

- Can let this expression $= R_i h_j$

And define $J_{ij} = \dfrac{\partial R_i}{\partial h_j}$

If we let $h_{new} = h_{old} + \delta h$

$$\Rightarrow R_i(h+\delta h) \simeq R_i(h) + \frac{\partial R_i}{\partial h_j}\delta h_j$$

We want to solve this so it $R_i(h+\delta h) = 0$

Solve for $J_{ij}(h)\,\delta h_j = -R_i(h)$

Then set $h_{new} = h + \delta h$

Repeat until $|\delta h|$ is within a tolerance

This can be done to solve the non-linear algebraic system.

In the code, 'hsp_rtol': 1e-14 is set

→ as residual is set, it means the code iterates until this is reached

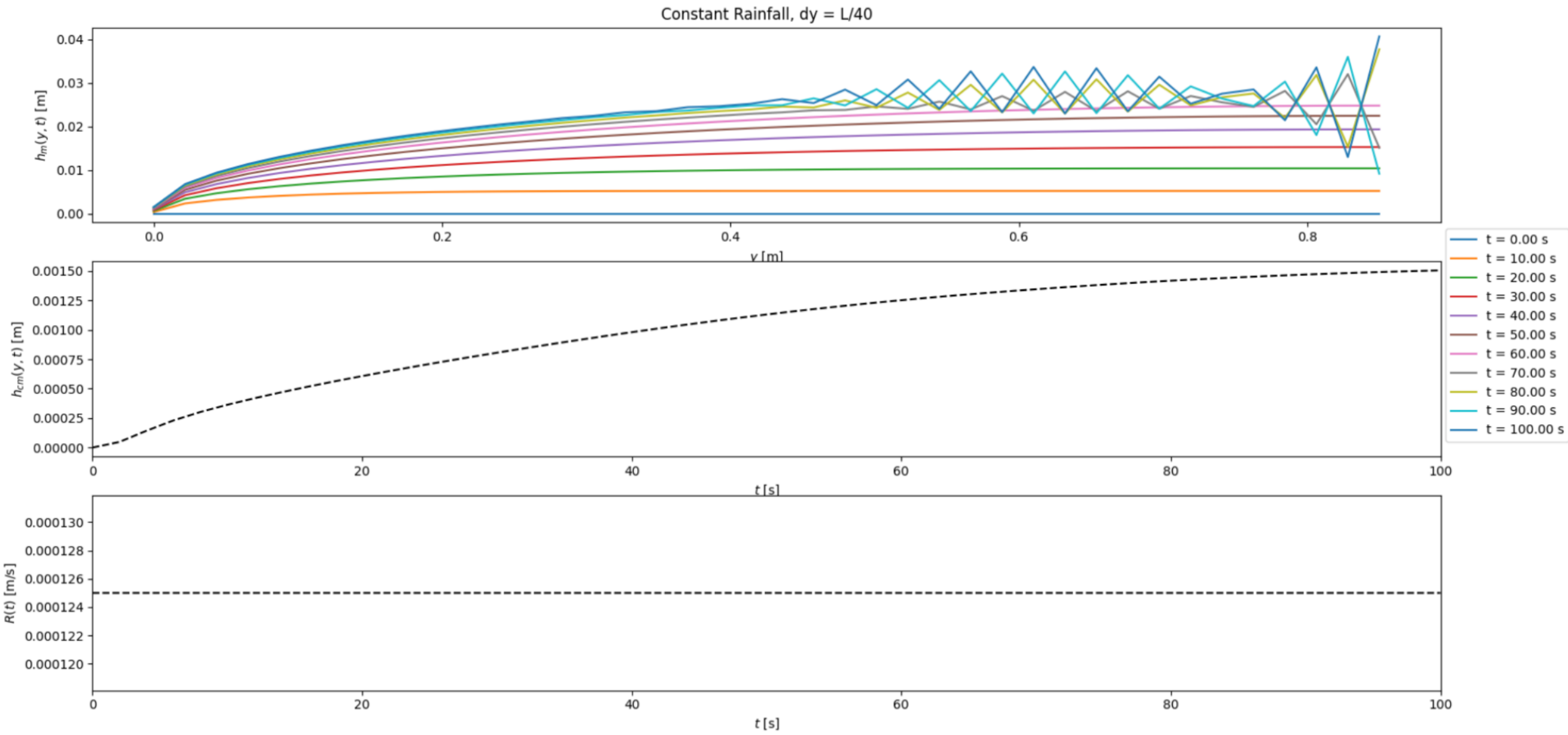
→ as code runs and gives a solution, this must mean the iteration converges.

4). CG1 = piecewise linear
CG2 = piecewise quadratic

- Find that at higher orders cause oscillations near $y = L_y$ (not $y=0$ like question asked).

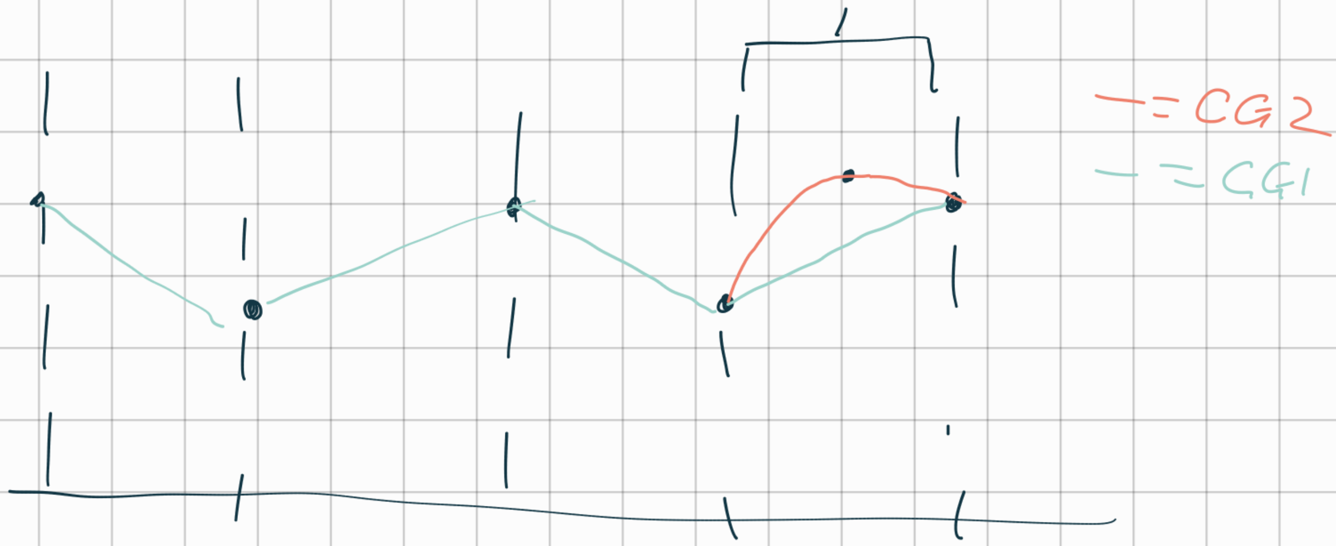
- Neute BC at $y = L_y$ $(\partial_y h = 0)$

CG3



$$\Delta t = 2 \cdot 3 \times 0.5 \times (\Delta y)^2$$

For a 1D line



more nodes within element for higher orders

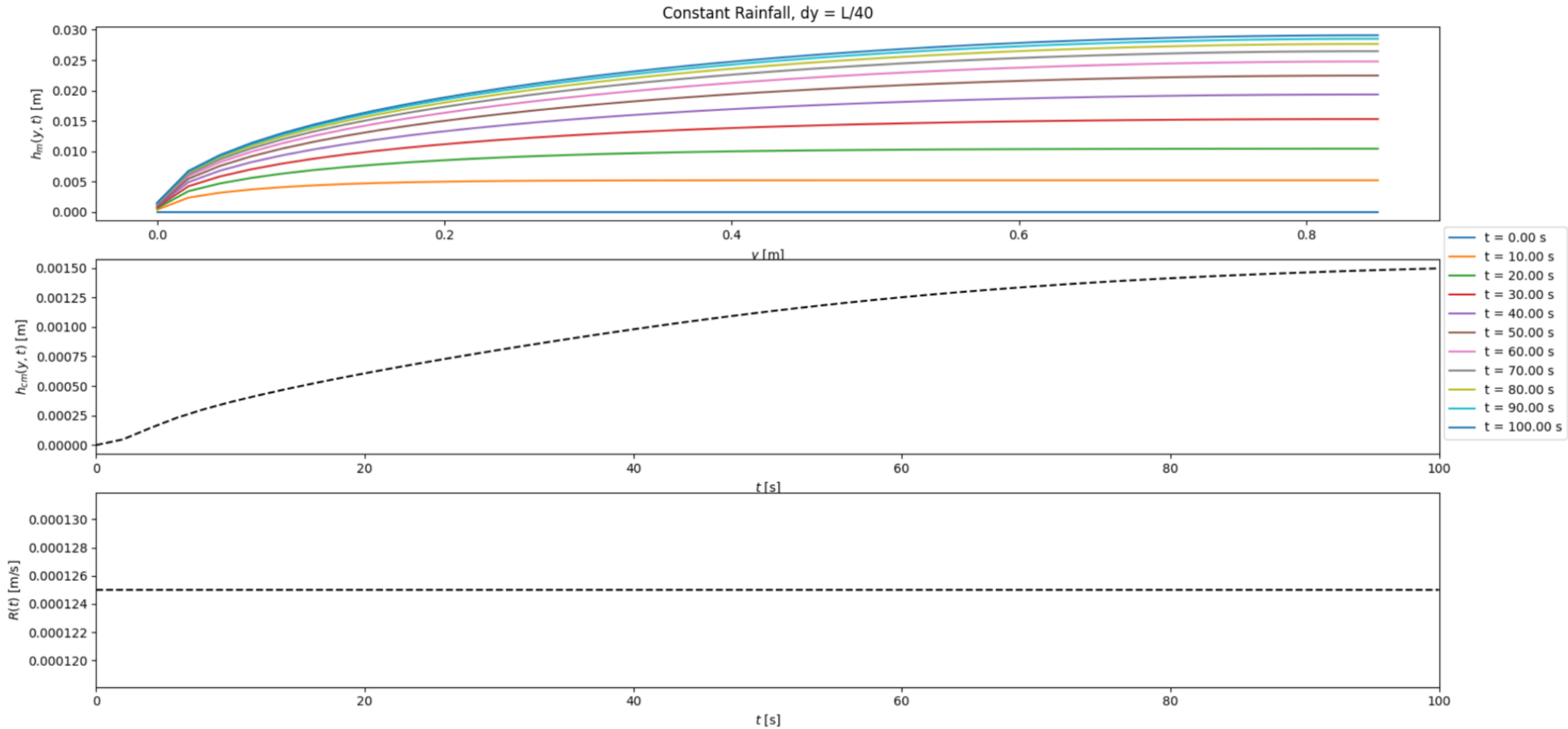
→ This effectively means that $\Delta t < k(\Delta y)^2$ should be replaced by $\Delta t < k\left(\frac{\Delta y}{nCG}\right)^2$ where nCG = orde

It likely appears unstable at $y = L_y$, as the BC at $y = 0$ is strongly enforced

→ whereas the derivative for higher orders being enforced still means they can have high curvature within the element.

→ Changing timestep gets rid of oscillations

CG3



$\Rightarrow \Delta t = 2.3 \times 0.5 \times \left(\frac{\Delta y}{3}\right)^2$

$\Rightarrow$ Oscillations are gone!