1a) $\partial_t b + \partial_z(\alpha b^3 - \beta b^3 \partial_z b) = 0$

$\Rightarrow \partial_t b + 3\alpha b^2 \partial_z - \beta b^3 \partial_{zz} b \quad = 0$

$\qquad\qquad (1) \qquad\qquad\quad (2)$

Linearize: $\quad b = D_0 + b'$

$\partial_t b' + 3\alpha (D_0 + b')^2 \partial_z b' - \beta (D_0 + b')^3 \partial_{zz} b' = 0$

b' terms sufficiently small $b' \times b' \sim 0$

$\Rightarrow \quad \partial_t b' + 3\alpha D_0^2 \partial_z b' - \beta D_0^3 \partial_{zz} b' = 0$

$\qquad\qquad (1) \qquad\qquad\quad (2)$

The equations are referred to as the convection-diffusion equations as the (1) terms represent a convective flux, and the (2) terms represent a diffusive flux.

$$\text{b) } \underbrace{\frac{b_j^{n+1} - b_j^{n}}{\Delta t}}_{\partial_t b} + \underbrace{3\alpha (b_j^{\wedge})^2 \left(\frac{b_j^{\wedge} - b_{j-1}^{\wedge}}{\Delta z} \right)}_{\partial_z b \ (\text{upwind})} \qquad \text{⑤}$$

$$\underbrace{- \beta \left(q_{j+1/2}^{\wedge} - q_{j-1/2}^{\wedge} \right) \cdot \frac{1}{\Delta z^2}}_{\partial_{zz} b \ (2^{nd} \text{ order } CD)} = 0$$

where $\quad q = b^3 \, \partial_{zz} b$

$$\Rightarrow \quad q_{j+1/2}^{\wedge} = (b_{j+1/2}^{\wedge})^3 (b_{j+1}^{\wedge} - b_j^{\wedge})$$

$$q_{j-1/2}^{\wedge} = (b_{j-1/2}^{\wedge})^3 (b_j^{\wedge} - b_{j-1}^{\wedge})$$

$$b_{j+1/2}^{\wedge} = \frac{1}{2}(b_{j+1}^{\wedge} + b_j^{\wedge})$$

$$b_{j-1/2}^{\wedge} = \frac{1}{2}(b_j^{\wedge} + b_{j-1}^{\wedge})$$

Complete discretization

$$\frac{b_j^{n+1} - b_j^{n}}{\Delta t} + 3\alpha (b_j^{\wedge})^2 \left(\frac{b_j^{\wedge} - b_{j-1}^{\wedge}}{\Delta z} \right)$$

$$- \beta \left(\frac{(b_{j+1/2}^{\wedge})^3 (b_{j+1}^{\wedge} - b_j^{\wedge}) - (b_{j-1/2}^{\wedge})^3 (b_j^{\wedge} - b_{j-1}^{\wedge})}{(\Delta z)^2} \right)$$

$$= 0$$

$$\frac{b_j^{n+1} - b_j^{n}}{\Delta t} + 3\alpha D_0^2 \left(\frac{b_j^{\wedge} - b_{j-1}^{\wedge}}{\Delta z} \right) \qquad \text{⑥}$$

$$- \beta D_0^3 \left(\frac{b_{j+1}^{\wedge} - 2 b_j^{\wedge} + b_{j-1}^{\wedge}}{(\Delta z)^2} \right) = 0$$

$b_j^{1,2,3 \cdots N}$

$b_{1,2,3 \cdots J}^n$

t axis, z axis. Grid with labels $n+1$, n, $n-1$ on vertical; $j-1$, j, $j+1$ on horizontal. Points marked: b_j^{n+1}, b_{j-1}^n, b_j^n, b_{j+1}^n.
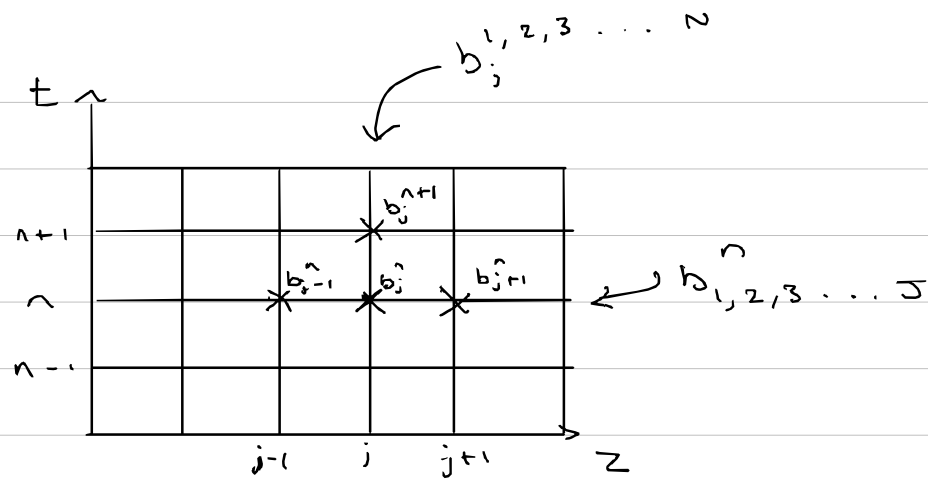
Linearizing ⑤ : $b = D_0 + b' \rightarrow b_j^n = D_0 + b_j'^n$

Time Derivative unchanged $\Rightarrow \dfrac{b_j^{n+1} - b_j^n}{\Delta t}$

Convective term

$3 \dfrac{d}{\Delta z} (b_j^n)^2 (b_j^n - b_{j-1}^n)$

$\rightarrow 3 \dfrac{d}{\Delta z} (D_0 + b_j'^n)^2 \left((D_0 + b_j'^n) - (D_0 + b_{j-1}'^n) \right)$

$= 3 \dfrac{d}{\Delta z} (D_0^2 + 2 D_0 b_j'^n + b_j'^n)(b_j'^n - b_{j-1}'^n)$

$2 D_0 b_j'^n \cdot b_j'^n$ or $\cdot b_{j-1}'^n \sim 0$ leaving:

$3 \dfrac{d}{\Delta z} D_0^2 (b_j'^n - b_{j-1}'^n) \Rightarrow$ Same as

the discretization of ⑥ .

Diffusive term

Considering term in term $-\frac{\beta}{\Delta z}\left(\hat{q}_{j+1/2} - \hat{q}_{j-1/2}\right)$

$\hat{q}_{j+1/2} = \frac{1}{\Delta z}\left(b_{j+1/2}^n\right)^3 \left(\hat{b}_{j+1} - \hat{b}_j\right)$

$\left(b_{j+1/2}^n\right)^3 \Rightarrow \left(D_0 + b'_{j+1/2}\right)^3$

$= D_0^3 + 3D_0^2 b'_{j+1/2} + 3D_0\underbrace{\left(b'_{j+1/2}\right)^2}_{\sim 0} + \underbrace{\left(b'_{j+1/2}\right)^3}_{\sim 0}$

$\underbrace{\sim 0 \text{ when}}_{\text{multiplied by } \left(b'_{j+1} - b'_j\right)}$

$\Rightarrow \hat{q}_{j+1/2} = \frac{1}{\Delta z} D_0^3 \left(b'_{j+1} - b'_j (+D_0 - D_0)\right)$

$\Rightarrow \hat{q}_{j-1/2} = \frac{1}{\Delta z} D_0^3 \left(b'_j - b'_{j-1} (+D_0 - D_0)\right)$

Diffusive term $\Rightarrow -\frac{\beta}{(\Delta z)^2} D_0^3 \left(b'_{j+1} - 2b'_j + b'_{j-1}\right)$

$\Rightarrow$ The linearization of the discretization
of ⑤ yields the same as the
discretization of the linearized ⑥.

The adjoint method is conservative, retaining
the original form of the PDE, hence is more
stable, meaning it's advantageous to use.

The boundary conditions are implemented such that

$$b_j^o = b_i(z_j)$$

$\Rightarrow$ for $n > 0$

$$b_o^n = b_B \quad \text{and} \quad b_J^n = b_T$$

where $z_J = H \Rightarrow \Delta z = \frac{H}{J}$

1c) Fourier analysis: $\alpha = 0, \beta \neq 0$

$\Rightarrow$ equation becomes

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} - \beta D_o^3 \left(\frac{b_{j+1}^n - 2b_j^n + b_{j-1}^n}{(\Delta z)^2} \right) = 0$$

which can also be written as:

$$b_j^{n+1} = b_j^n + \nu \left(b_{j+1}^n - 2b_j^n + b_{j-1}^n \right)$$

where $\nu = \beta D_o^3 \frac{\Delta t}{(\Delta z)^2}$

$$b_j^n = \lambda^n e^{i k_j \Delta z}$$

$\Rightarrow \lambda = 1 + \nu \left(-2 + e^{i k \Delta z} + e^{-i k \Delta z} \right)$

$\Rightarrow \lambda = 1 + \nu \left(-4 \sin^2 \left(\frac{k \Delta z}{2} \right) \right)$

Scheme stable where $|\lambda|^n \leq 1$

$\nu > 0 \Rightarrow \lambda > -1 / \lambda \leq 1$

$$1 + \nu(-4\sin^2(\frac{k\Delta z}{2})) > -1$$

$$\nu(-4\sin^2(\frac{k\Delta z}{2})) > -2$$

$$\Rightarrow \quad \nu < \frac{2}{4\sin^2(\frac{k\Delta z}{2})}, \quad \nu < \frac{1}{2\sin^2(\frac{k\Delta z}{2})}$$

worst possible case: $\sin^2(k) = 1$

$$\Rightarrow \quad \Delta t < \frac{(\Delta z)^2}{2\beta D_0^3}$$

where $\alpha \neq 0, \beta = 0$

$$\frac{1}{\Delta t}(b_j^{n+1} - b_j^{\wedge}) = -3\alpha D_0^2 (b_j^{\wedge} - b_{j-1}^{\wedge}) \frac{1}{\Delta z}$$

$$\mu = \frac{\Delta t}{\Delta z}(-3\alpha D_0^2)$$

$$\Rightarrow \quad b_j^{n+1} = b_j^{\wedge} + \mu(b_j^{\wedge} - b_{j-1}^{\wedge})$$

$$\lambda - 1 = \mu(1 - e^{-ik\Delta z})$$

$$= \mu(1 - (\cos k\Delta z - i\sin k\Delta z))$$

$$\lambda_R = 1 - \mu(1 - \cos k\Delta z)$$

$$\lambda_I = \mu \sin k\Delta z$$

$$|\lambda|^2 = |\lambda_R|^2 + |\lambda_I|^2$$

$$|\lambda_R|^2 = [(1-\mu) + \mu\cos k\Delta z]^2 + [\mu\sin k\Delta z]^2$$

$$= (1-\mu)^2 + \mu^2 + 2\mu(1-\mu)\cos k\Delta z$$

$$= 1 - 2\mu(1-\mu)(1 - \cos k\Delta z)$$

$$= 1 - 4\mu(1-\mu)\sin^2(\frac{k\Delta z}{2})$$

$$|\lambda|^{\wedge} < 1$$

$$\lambda < 1 / \lambda > -1$$

$$1 - 4(1 - \mu) \sin^2\left(\frac{K\Delta z}{2}\right) > -1$$

$$4(1 - \mu) \sin^2\left(\frac{K\Delta z}{2}\right) > -2$$

worst possible case Text$\sin^2(K) = 1$

$$\Rightarrow \qquad 4(1 - \mu) < 2$$

$$1 - \mu < \frac{1}{2} \qquad \text{mu<=1 -0.5}$$

$$\mu > \frac{1}{2}$$

$$\frac{\Delta t}{\Delta z^2}(-3\alpha D_o^2) > \frac{1}{2}$$

$$\Rightarrow \qquad \Delta t < \frac{(\Delta z)^2}{6\alpha D_o^2}$$

incorrect

$$\Rightarrow \qquad \Delta t < \min\left(\frac{\Delta z^2}{2\beta D_o^3}, \quad \frac{(\Delta z)^2}{6\alpha D_o^3}\right)$$

1d) Considering update scheme for b_j^{n+1}

$$b_j^{n+1} = b_j^{\wedge} - \frac{\Delta t}{\Delta z}\left(3\alpha D_o^2 [b_j^{\wedge} - b_{j-1}^{\wedge}]\right)$$
$$+ \frac{\Delta t}{(\Delta z)^2}\beta D_o^3\left(b_{j+1}^{\wedge} + b_{j-1}^{\wedge} - 2b_j^{\wedge}\right)$$

which can also be written as

$$b_j^{n+1} = b_j^{\wedge}\left(1 - \frac{\Delta t}{\Delta z} 3\alpha D_o^2 - \frac{\Delta t}{\Delta z^2} 2\beta D_o^3\right)$$
$$+ b_{j+1}^{\wedge}\left(\frac{\Delta t}{\Delta z^2}\beta D_o^3\right) + b_{j-1}^{\wedge}\left(\frac{\Delta t}{\Delta z}3\alpha D_o^2 + \frac{\Delta t}{\Delta z^2}\beta D_o^3\right)$$

To satisfy the maximum principle, the coefficients of discrete b values must be > 0. For $b_{j,+1}^{\wedge}$ and $b_{j,-1}^{\wedge}$, the values are all real and positive. Provides the condition for $b_j^{\wedge}$:

$$1 - \left(\frac{\Delta t}{\Delta z} \, 3 \alpha D_0^2 + \frac{\Delta t}{(\Delta z)^2} \, 2 \beta D_0^3 \right) > 0$$

$$\Delta t < \frac{(\Delta z)^2}{3 \alpha D_0^2 \Delta z + 2 \beta D_0^3}$$

1e) To derive variable time step, consider update scheme such that

$$b_j^{\wedge n+1} = b_j^{\wedge} - \Delta t \left[\frac{B_{con}}{\Delta z} - \frac{B_{dif}}{(\Delta z)^2} \right]$$

where $B_{dif} = \beta \left[(b_{j+1/2}^{\wedge})^3 (b_{j+1}^{\wedge} - b_j^{\wedge}) - (b_{j-1/2}^{\wedge})^3 (b_j^{\wedge} - b_{j-1}^{\wedge}) \right]$

and $B_{con} = 3 \alpha (b_j^{\wedge})^2 (b_j^{\wedge} - b_{j-1}^{\wedge})$

Given condition $b_j^{\wedge n+1} > 0$

$$b_j^{\wedge} - \Delta t \left[\frac{B_{con}}{\Delta z} - \frac{B_{dif}}{(\Delta z)^2} \right] > 0$$

$$\Rightarrow \quad \Delta t < \frac{b_j^{\wedge} (\Delta z^2)}{B_{con} \cdot \Delta z - B_{dif}}$$

1f) Consider the convection approximation

using a central approximation

$\Rightarrow \quad \partial_z b_i^n \approx \dfrac{b_{i+1}^n - b_{i-1}^n}{2 \Delta z}$

Similar to previously

$b_j^{n+1} = b_j^n - \Delta t \left(\dfrac{B_{2con}}{\Delta z} - \dfrac{B_{dif}}{(\Delta z)^2} \right)$

where $B_{2con} = 3\alpha \, (b_j^n)^2 \, (b_{j+1}^n - b_{j-1}^n) \cdot \dfrac{1}{2 \Delta z}$

$\Rightarrow \Delta t < \dfrac{b_j^n (\Delta z)^2}{B_{2con} \Delta z - B_{dif}}$

For beta=0 this is unstable;
with no stable time step;
which is not visible. -0.5

2a) $\partial_t b + \partial_z (\alpha b^3 - \beta b^3 \partial_z b) = 0$ ⑤

steady state $\Rightarrow \partial_t b = 0$

$\partial_z (\alpha b^3 - \beta b^3 \partial_z b) = 0$

integrating $\rightarrow \quad \alpha b^3 - \beta b^3 \partial_z b = Q$ (integration constant

$\Rightarrow \beta b^3 \dfrac{\partial b}{\partial z} = \alpha b^3 - Q$

$\dfrac{db}{dz} = \dfrac{\alpha}{\beta} - \dfrac{Q}{\beta} b^{-3}$

$\Rightarrow b_{j+1} = b_j + \Delta z \left(\dfrac{\alpha}{\beta} - \dfrac{Q}{\beta} b_j^{-3} \right)$

2c) Through variation g spatial step (Δz) the Solution $b(z,t)$ can be compared with the Solution to the high resolution Solution to the steady state Solution. The rate g change g error with respect to the change in the spatial step can be used as the order g spatial accuracy $\Rightarrow$ here it is 1st order accurate. (plots for L^2, L^∞ in code)

2d) For travelling wave $b(z,t) = b(s)$

where $s = z - z_{ro} - ct$

$\partial_t b = \frac{db}{ds} \partial_t s = -cb'(s), \quad \partial_z b = b'(s)$

$\rightarrow$ substituting into PDE.

$-cb' + \alpha b^3 - \beta b^3 b' = Q$ (constant g integration)

$\beta b^3 b' = \alpha b^3 - cb - Q$ [1st order PDE for $b(s)$]

$\Rightarrow$ any $b(s)$ satisfying equation provides travelling wave solution.

Where $Q = 0$

$\times b^{-3}$; $\beta b^3 b' = \alpha b^3 - cb$

$\beta b' = \alpha - cb^{-2}$

seperating variables : $\dfrac{db}{\alpha - cb^{-2}} = \dfrac{1}{\beta} ds$

$\Rightarrow \displaystyle\int \dfrac{b^2}{\alpha b^2 - c} \, db = \dfrac{1}{\beta} s + k_1$

$\left(\dfrac{b^2}{\alpha b^2 - c} = \dfrac{\alpha b^2 - c + c}{\alpha (\alpha b^2 - c)} = \dfrac{1}{\alpha} + \dfrac{c}{\alpha} \dfrac{1}{\alpha b^2 - c} \right)$

$\displaystyle\int \left(\dfrac{1}{\alpha} + \dfrac{c}{\alpha} \cdot \dfrac{1}{\alpha b^2 - c} \right) db = \dfrac{1}{\beta} s + k_1$

$\dfrac{b}{\alpha} - \dfrac{1}{\alpha} \sqrt{\dfrac{c}{\alpha}} \, \text{atanh} \left(\sqrt{\dfrac{\alpha}{c}} \, b \right) + k_2 = \dfrac{1}{\beta} s + k_1$

$\Rightarrow s = \dfrac{\beta}{\alpha} \left(b - \sqrt{\dfrac{c}{\alpha}} \, \text{atanh} \left(\sqrt{\dfrac{\alpha}{c}} \, b \right) \right) + k_3$

$(= k_2 - k_1)$

Constant absorbed by zero $\Rightarrow$ (12) provides an

exact solution to the PDE.

I get a division by 0 error in your exact solution;
also you need to show solutions and 41 still not sufficient. 17.5/20

2e) Applying crank-nicholson scheme to

problem

$\Rightarrow b_j^{n+1} = b_j^n - \alpha \dfrac{\Delta t}{2 \Delta z} \left[\left(b_j^n \right)^3 \left(b_j^n - b_{j-1}^n \right) + \left(b_j^{n+1} \right)^3 \left(b_j^{n+1} - b_{j-1}^{n+1} \right) \right]$

$\qquad + \beta \dfrac{\Delta t}{16 (\Delta z)^2} \left[b_{j,j+1/2}^n \left(b_{j,j+1}^n - y_j^n \right) - \left(b_{j,j-1/2}^n \right)^3 \left(b_j^n - b_{j-1}^n \right) \right]$

can be written as

$$b_j^{n+1} = b_j^n - C_1 \left[(b_j^n)^3 - (b_{j-1}^n)^3 + (b_j^{n+1})^3 - (b_{j-1}^{n+1})^3 \right]$$

$$+ C_2 \left[(b_{j+1}^n + b_{j-1}^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n) \right]$$

$$+ C_2 \left[(b_{j+1}^{n+1} - b_{j-1}^{n+1})^3 (b_{j+1}^{n+1} - b_j^{n+1}) - (b_j^{n+1} + b_{j-1}^{n+1})^3 (b_j^{n+1} - b_{j-1}^{n+1}) \right]$$

where $\quad C_1 = \alpha \dfrac{\Delta t}{2 \Delta z} \quad$ and $\quad C_2 = \beta \dfrac{\Delta t}{16 (\Delta z)^2}$

$$\overline{J_{ij}} = \frac{\partial R_{ij}}{\partial b_{ij}} = \frac{R_{ij+pert} - R_{ij}}{\varepsilon}$$

Solve for $\quad J \delta b = - R$

$$\Rightarrow b^{n+1} = b^n + \delta b$$

- Crank-nicholson takes longer to compile
  given more calculations at each step
  of the calculation.
- Slightly more accurate solution achieved.