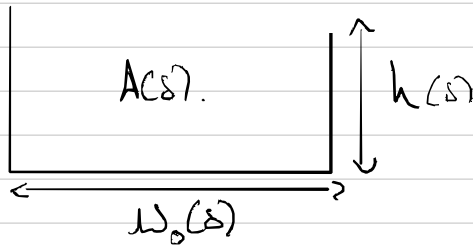


Numerics 2: Alex Carey 201810977.

1. Given some cross section



we define the relationship

$$A = h \cdot w_0$$

such that h is given as a function of A, w_0

$$\text{with } A/w_0(s)$$

Similarly the wetted perimeter is given geometrically as

$$P = 2h + w_0 = w_0 + 2A/w_0$$

such that

$$P(A, s) = w_0(s) + 2A/w_0(s).$$

noting A is also a function of s .

We consider

$$\partial_t A + \partial_s F = 0$$

where we have

$$F = \frac{A^{5/3}}{C_m \rho^{2/3}} \sqrt{-\partial_s b}$$

the flux at s . We can

$$\partial_s F = \partial_s A \partial_A F + \partial_s F$$

where A is considered independent of s when calculating $\partial_s F$. We compute

$$\partial_A F = \frac{5}{3} \left(\frac{A}{\rho} \right)^{2/3} \frac{\sqrt{-\partial_s b}}{C_m} - \frac{2}{3} \frac{A^{5/3} \sqrt{-\partial_s b}}{\rho^{2/3}} \frac{\partial_A \rho}{C_m}$$

where $\partial_A \rho = 2/W_0$ such that

$$\partial_A F = \frac{5}{3} \left(\frac{A}{\rho} \right)^{2/3} \frac{\sqrt{-\partial_s b}}{C_m} - \frac{4}{3} \left(\frac{A}{\rho} \right)^{5/3} \frac{\sqrt{-\partial_s b}}{C_m W_0}$$

$$= \frac{\sqrt{-\partial_s b}}{3 C_m} \left(\frac{A}{\rho} \right)^{2/3} \left[5 - 4 \frac{A}{\rho W_0} \right]$$

$$= \frac{\sqrt{-\partial_s b}}{3 C_m} \left(\frac{A}{\rho} \right)^{2/3} \left[\frac{5 W_0^2 + 6 A}{\rho W_0} \right]$$

$$= \frac{\sqrt{-\partial_s b}}{3 C_m} \left[\frac{5 A^{2/3} W_0 + 6 A^{5/3} W_0}{(W_0 + 2 A / W_0)^{5/3}} \right]$$

We similarly compute

$$\partial_\delta F = -\frac{2}{3} \left(\frac{A}{P} \right)^{\frac{5}{3}} \frac{\sqrt{-\partial_\delta b}}{C_m} \partial_\delta P$$

where ~~that~~ $\partial_\delta P = \partial_\delta \omega_0 (1 - 2A/\omega_0^2)$ such

$$\partial_\delta F = -\frac{2}{3} \frac{A}{(\omega_0 + 2b)^{\frac{5}{3}}} \frac{\sqrt{-\partial_\delta b}}{C_m} \left(1 - \frac{2A}{\omega_0^2} \right) \partial_\delta \omega_0$$

as required. Note that as ω_0 is only dependent on δ $\partial_\delta \omega_0 = d\omega_0/d\delta$.

2) We consider a case where the variation on W_0 is small, then as

$$\partial_s W_0 = 0$$

and so have

$$\partial_t A + \partial_A F \partial_s A = 0 \quad (1)$$

which has eigenvalue $\partial_A F$.

We consider the Riemann problem defined by (1) on some piecewise constant initial conditions

$$A(s, t=0) = \begin{cases} A_L & s < s_{0+\frac{1}{2}} \\ A_R & s \geq s_{0+\frac{1}{2}} \end{cases}$$

towards the computation of some flux on the boundary.

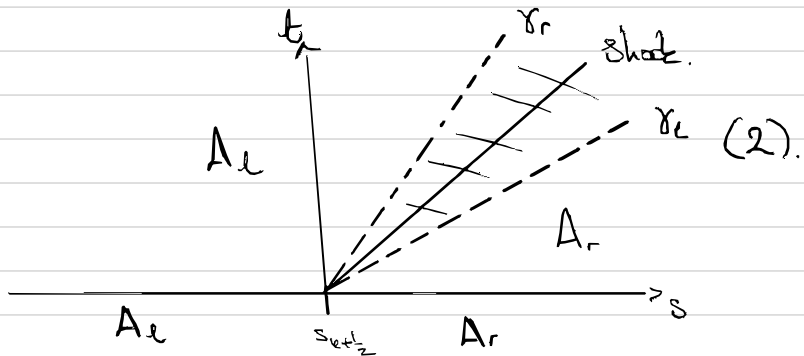
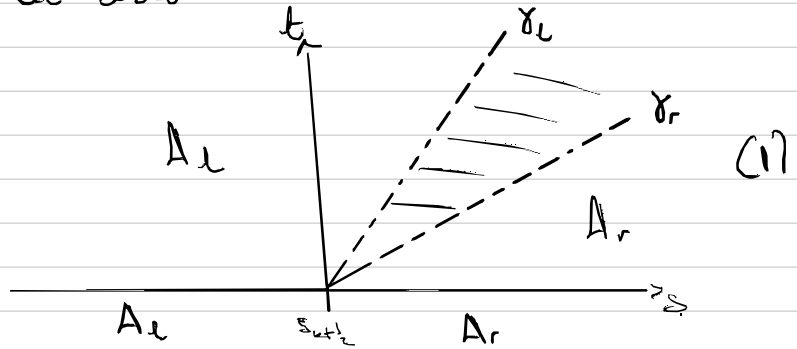
We define characteristics along which the value of A is constant in this case with

$$\partial_t s = \partial_A F(A, s).$$

We note that $\partial_A F$ can then similarly take only two values dependent on A

$$(\partial_A F)_L = \partial_A F(A_L), (\partial_A F)_R = \partial_A F(A_R)$$

As $\partial_A F > 0$ we find the following two possible cases



where γ_L, γ_R are the lines traced from the origin with gradients $(\partial_A F)_L, (\partial_A F)_R$ respectively.

In case (1) the central fan represents an undefined solution region where A defⁿ continuously as

$$A = s_{ut/2}^{-0.5} \text{ (not complete solution)}$$

In case (2) the central region is doubly defⁿ and so a shock forms along the line.

with gradient

$$\frac{(\partial_A F)_L + (\partial_A F)_R}{2}.$$

3) Given

$$\partial_t A + \partial_s F(A, \phi) = 0$$

we consider the integrals in time and space to find

$$\int_{t_n}^{t_{n+1}} \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \partial_t A + \partial_s F \, ds \, dt = 0$$

$$\Rightarrow \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} [A]_{t_n}^{t_{n+1}} \, ds + \int_{t_n}^{t_{n+1}} [F]_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \, dt = 0$$

At this stage we apply the nature of the scheme where \bar{A} is the average of the cell such that we have

$$\Delta s_k [\bar{A}_k]_{t_n}^{t_{n+1}} + \int_{t_n}^{t_{n+1}} [F]_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \, dt = 0$$

and we have scheme

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta s_k} \int_{t_n}^{t_{n+1}} [F]_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \, dt.$$

4) We consider the flux on the boundaries given

$$F_{k+\frac{1}{2}}, F_{k-\frac{1}{2}}.$$

We note F is dependent on A and that the results of the Riemann Problem state that the value of A on the boundary is given by

$$A_{k+\frac{1}{2}}(t) = \bar{A}_k^+$$

for $t \in (t_n, t_{n+1})$ and ev such

$$F_{k+\frac{1}{2}}(t) = F(\bar{A}_k^+)$$

on the same k restricted domain.

5. We derive a time step restriction from the fact that

$$\Delta t_n (\partial_t S)_k < \Delta S_k = h_k$$

in distating the values on the boundary $\partial_{\Gamma^{\pm}} S$. Noting the flux approximation from part 4 we have

$$\begin{aligned} (\partial_t S)_k &= (\partial_A F)_k \\ &= \partial_A F(\tilde{A}_k) \\ &= \tilde{\lambda}_k \end{aligned}$$

For a single restriction we must have

$$\Delta t^* < \min_k \frac{h_k}{|\tilde{\lambda}_k|}.$$

We further restrict for stability by defining $CFL \in (0, 1]$ w/

$$\Delta t^* < CFL \min_k \frac{h_k}{|\tilde{\lambda}_k|}.$$

6. As the scheme is upwind, we impose a boundary condition on local discharge at $s=0$ without it being necessary to impose a condition for $s=1$.

$$A(0, t) = A_0(t).$$