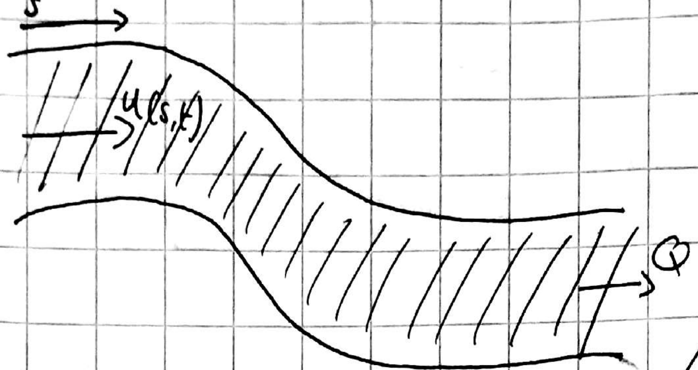
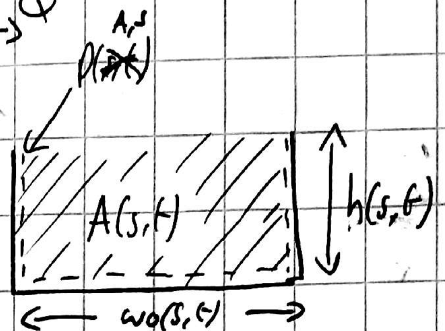
NUMERICS HOMEWORK 2.

time, t





s, distance

discharge $Q = Au$

$$R = \frac{A}{P}$$

Downslope $-\partial_s b$

Manning Friction coeff, C_m

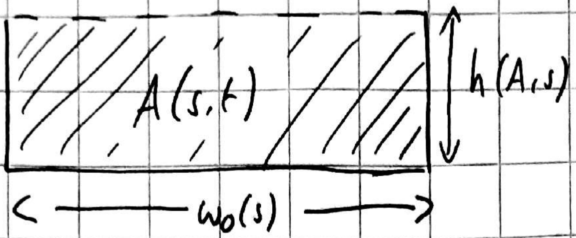
$$\boxed{u = \frac{R^{2/3}\sqrt{\partial_s b}}{C_m}} \qquad \boxed{Q = Au = \frac{A^{5/3}\sqrt{-\partial_s b}}{C_m P(A,s)^{2/3}}}$$

$$\partial_t A + \partial_s \left(A R(A,s)^{2/3} \sqrt{-\partial_s b} / C_m \right) = S$$

Eq 3/4

$$\Rightarrow \boxed{\partial_t A + \partial_s F(A,s) = S \quad \text{where } F(A,s) = \frac{A^{5/3}\sqrt{-\partial_s b}}{C_m P(A,s)^{2/3}}}$$

1.



$$\text{Height} = \frac{\text{Area}}{\text{width}} \Rightarrow h(A,s) = \frac{A(s,t)}{w_0(s)}$$

Perimeter :

$$\Rightarrow P(A,s) = w_0(s) + 2h(A,s)$$
$$= w_0(s) + \frac{2A(s,t)}{w_0(s)}$$

$S = 0$

$$③ \quad \partial_t A + \partial_s F(A,s) = \cancel{S}^{\to 0}$$

$$\partial_t A + \partial_s F(A,s) = 0$$

By the chain rule:

$$\partial_s\left[F(A(s,t),s)\right] = \frac{\partial F}{\partial A}\frac{\partial A}{\partial s} + \frac{\partial F}{\partial s}$$

$\Rightarrow$ put into equation

$$\boxed{\partial_t A + \frac{\partial F}{\partial A}\partial_s A + \frac{\partial F}{\partial s} = 0}$$

$$\frac{\partial F}{\partial A} = \frac{\partial}{\partial A}\left[\frac{A^{5/3}\sqrt{-\partial_s b}}{C_m P(A,s)^{2/3}}\right]$$

sub in $\rho = w_s + 2h = w_0(s) + \dfrac{2A(s,t)}{w_0(s)}$ pg 2

$$\frac{\partial F}{\partial A} = \frac{\partial}{\partial A}\left[\frac{A^{5/3}\sqrt{-\partial_s b}}{C_m\left(w_0 + 2A/w_0\right)^{2/3}}\right]$$

assume $-\partial_s b$ is positive, small + constant

$$\Rightarrow \frac{\sqrt{\partial_s b}}{C_m}\frac{\partial}{\partial A}\left[A^{5/3}\left(w_0 + \frac{2A}{w_0}\right)^{-2/3}\right]$$

$$= \frac{\sqrt{-\partial_s b}}{C_m}\left[\frac{5}{3}A^{2/3}\left(w_0 + \frac{2A}{w_0}\right)^{-2/3} + -A^{5/3}\frac{2}{3}\left(w_0 + \frac{2A}{w_0}\right)^{-5/3} \times \frac{2}{w_0}\right]$$

$$= \frac{\sqrt{-\partial_s b}}{3C_m}\left[\frac{5A^{2/3}\left(w_0 + \frac{2A}{w_0}\right)}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}} - \frac{4A^{5/3}}{w_0\left(w_0 + \frac{2A}{w_0}\right)^{5/3}}\right]$$

$$= \frac{\sqrt{-\partial_s b}}{3C_m}\left[\frac{5A^{2/3}w_0 + \frac{10A^{5/3}}{w_0} - \frac{4A^{5/3}}{w_0}}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}}\right]$$

$$\frac{\partial F}{\partial A} = \frac{\sqrt{-\partial_s b}}{3C_m}\left[\frac{5w_0A^{2/3} + 6A^{5/3}/w_0}{\left(w_0 + 2A/w_0\right)^{5/3}}\right]$$

$-\partial_s b > 0$

$w_0(s) > 0$

$A > 0$ $\Rightarrow$ $\dfrac{\partial F}{\partial A} > 0$

$C_m > 0$

$$\frac{\partial F}{\partial s} = \frac{\partial}{\partial s}\left[\frac{A^{5/3}\sqrt{-2sb}}{C_m\, P(A,s)^{2/3}}\right]$$

$$= \frac{\sqrt{-2sb}}{C_m}\frac{\partial}{\partial s}\left(A^{5/3}\left(w_0 + \frac{2A}{w_0}\right)^{-2/3}\right)$$

$A(\cancel{s,t})$ $\quad w_0(s) \quad A(s,t)$

$$= \frac{\sqrt{-2sb}}{C_m}\cancel{\left[-\frac{2}{3}A^{5/3}\left(w_0 + \frac{2A}{w_0}\right)^{-5/3}\right]}$$

$$= \frac{\sqrt{-2sb}}{C_m}\left[\frac{5}{3}A^{+\frac{2}{3}}\frac{dA}{ds} - \frac{2}{3}A^{5/3}\left(w_0 + \frac{2A}{w_0}\right)^{-5/3}\left(1 - \frac{2A}{w_0^2}\right)\frac{dw_0}{ds}\right]$$

$$= \frac{\sqrt{2sb}}{3C_m}\left[5A^{2/3}\frac{dA}{ds} - \frac{2A^{5/3}}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}}\left(1 - \frac{2A}{w_0^2}\right)\frac{dw_0}{ds}\right)$$

$$\frac{\partial F}{\partial s} = -\frac{2\sqrt{-2sb}}{3C_m}\,\frac{A^{5/3}/w_0}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}}\left(1 - \frac{2A}{w_0^2}\right)\frac{dw_0}{ds}$$

if $\frac{dA}{ds}$ is 0 ?

2. Take the limit in which $w_0(s)$ is independent of s

$$\Rightarrow \quad \frac{\partial w_0}{\partial s} = 0 \quad \Rightarrow \quad \frac{\partial F}{\partial s} = 0$$

$$\partial_t A + \frac{\partial F}{\partial A} \partial_s A + \cancel{\frac{\partial F}{\partial s}}^{\,0} = 0$$

$$\boxed{\partial_t A + \frac{\partial F}{\partial A} \partial_s A = 0}$$

Eigenvalue of equation $\lambda = \frac{\partial F}{\partial A}$

second term $\frac{\partial F}{\partial A} \partial_s (A)$

can move $\frac{\partial F}{\partial A}$ inside derivative as it is constant, ie. no longer has a dependence on s, $\frac{\partial}{\partial s}\left(\frac{\partial F}{\partial A}\right) = \frac{\partial^2 F}{\partial s \partial A} = \frac{\partial}{\partial A}\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial s} = 0$

$$\partial_s \left(\frac{\partial F}{\partial A} A \right) = \partial_s (\lambda A)$$

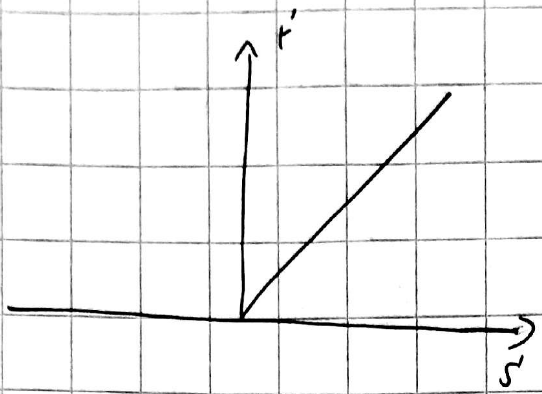
Can write equation as:

$$\partial_{t'}(A) + \partial_{s'}(\lambda A) = 0$$

with constant $\lambda > 0$

$$A(s', t' = 0) = \begin{cases} A_{left} = A_L & s' < 0 \\ A_{right} = A_R & s' \geq 0 \end{cases}$$

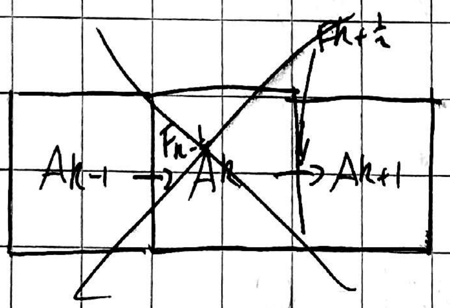
characteristics $s' = s_0 + \lambda t'$

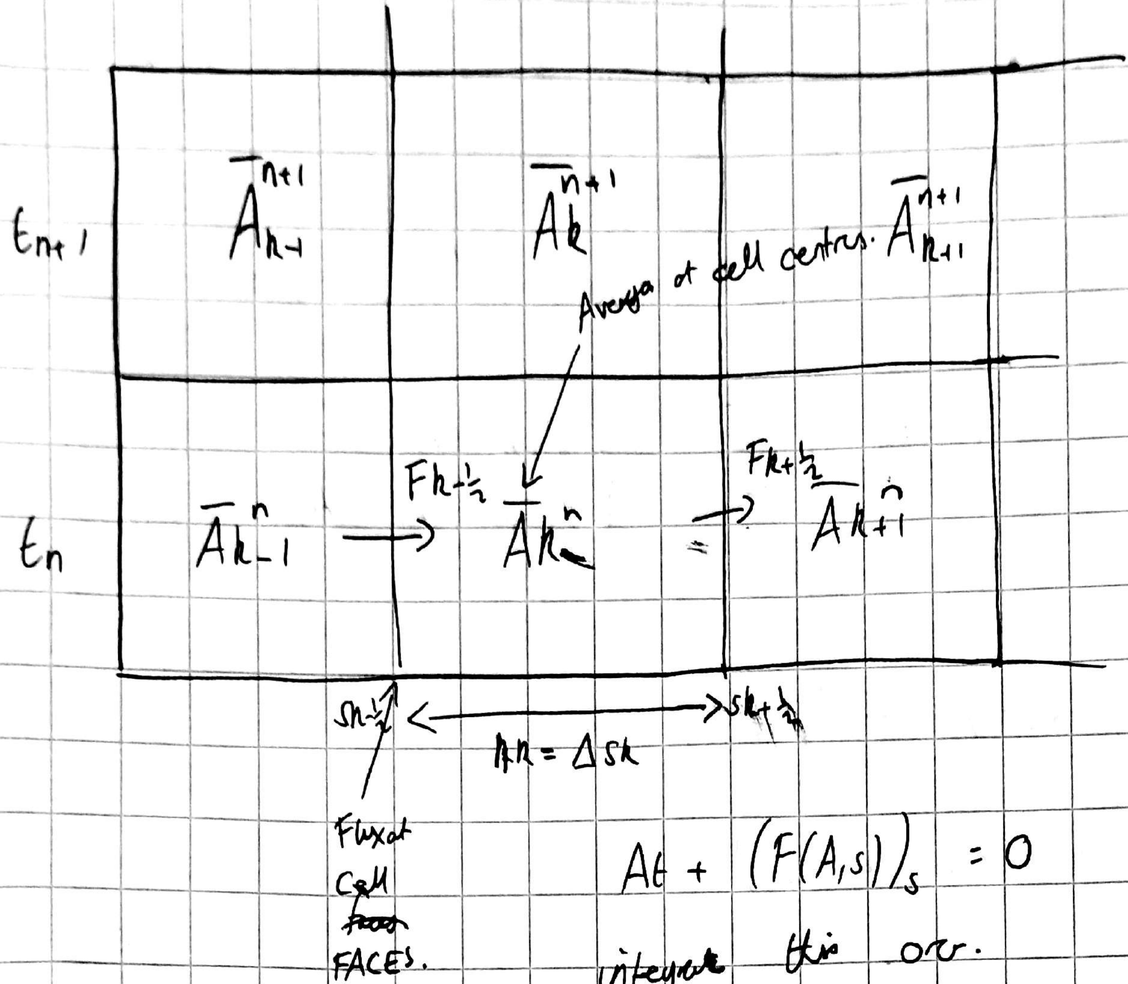


3. $$\partial_t A + \partial_s(\lambda A) = 0$$

$$A_t + (F(A))_s = 0$$

~~Flux~~ $F(A)$ is the flux





$$A_t + \left(F(A,s)\right)_s = 0$$

integrate this over.

$$s_{k-\frac{1}{2}} < s < s_{k+\frac{1}{2}}$$

$$t_n < t < t_{n+1}$$

$$\Rightarrow \quad \bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta s_k} \int_{t_n}^{t_{n+1}} \left[F(A,s)\Big|_{s=s_{k+\frac{1}{2}}} - F(A,s)\Big|_{s=s_{k+\frac{1}{2}}} \right]$$

cell average

$$\bar{A}_k(t) = \frac{1}{\Delta s_k} \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} A(s,t)\, ds$$

Flux evaluated at cell boundaries.

$$F(A,s)\Big|_{s=s_{k+\frac{1}{2}}} = F_{k+\frac{1}{2}}\left(\bar{A}_k^n, \bar{A}_k^{n+1}\right)$$

3. Derive a time-step condition based on

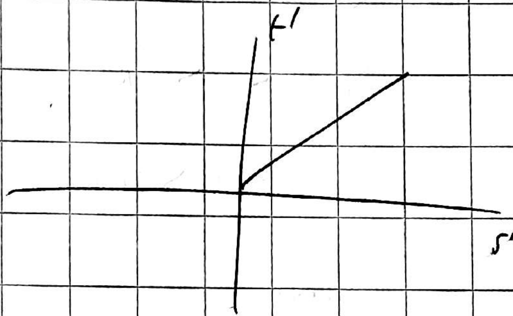
Forward time method, only need information from previous time step.

4. characteristics are $s' = s_0' + \lambda t'$ and $\lambda > 0$

$$A(s', t'=0) = \begin{cases} A_{left} = A_c = A_n^n & s' < 0 \\ A_{right} = A_r = \bar{A}_{n+1}^n & s' \geq 0 \end{cases}$$

$$A(s'=0, t') = A(s_{n+\frac{1}{2}}, t) = A_c^n$$

$$F(A(s_{n+\frac{1}{2}}, t)) = \lambda A_c^n$$



Right shock, so use information behind. Upwind.

$$\bar{A}_n^{n+1} = \bar{A}_n^n - \frac{\Delta t}{\Delta s_n} \lambda \left(\bar{A}_n^n - \bar{A}_{n-1}^n \right)$$

5. Derive a time-step condition based on

$$\Delta t < CFL \min_k \frac{h_k}{|\lambda_k|}$$

For scheme

$$\bar{A}_k^{n+1} = \bar{A}_k^{n} - \frac{\Delta t}{\Delta s_k} \lambda \left(\bar{A}_k^{n} - \bar{A}_{k-1}^{n}\right)$$

Apply maximum principle, ~~coef~~ coefficients all >0

$$\bar{A}_k^{n+1} = \left(1 - \frac{\Delta t}{\Delta s_k}\lambda\right)\bar{A}_k^{n} + \frac{\Delta t}{\Delta s_k}\lambda \bar{A}_{k-1}^{n}$$

$$1 - \frac{\Delta t}{\Delta s_k}\lambda > 0 \qquad \frac{\Delta t}{\Delta s_k}\lambda > 0$$

$$\Rightarrow \cancel{h}\ \Delta t < \frac{\Delta s_k}{\lambda}$$

$$\Delta t < \frac{\Delta s_k}{|\lambda|}$$

$$\lambda_k = \frac{\partial F}{\partial A} \qquad \Delta t < CFL \frac{\Delta s_k}{|\lambda_k|}$$

Maximum stable value when $CFL = 1$

take $0 < CFL < 1$

6. What inflow / outflow conditions imposed at $s=0$ and $s=L_e$?

constant inflow at $s=0$, depending on conditions?

outflow = inflow.

Q7/Q8: no convergence results shown.
Q9/Q010 absent: 14/20