if and only if

$$1 \geq \frac{\Delta t}{\Delta z} D_0^2 \left(3\alpha + 2 \frac{\beta}{(\Delta z)} \right)$$

$$\Delta t \leq \frac{\Delta z}{D_0^2 \left(3\alpha + 2\frac{\beta}{\Delta z}\right)} = \frac{(\Delta z)^2}{D_0^2 (3\alpha \Delta z + 2\beta)}$$

---

e) Recall discretization of (5) ~~(9)~~:

$$\cancel{b_j'^{n+1} = -3\alpha D_0^2 \left(b_j'^n - b_{j-1}'^n\right.}$$

$$b_j^{n+1} = -3\alpha \left(b_j^n\right)^2 \left(b_j^n - b_{j-1}^n\right) \frac{\Delta t}{\Delta z}$$

$$+ \frac{\beta \Delta t}{2(\Delta z)^2} \Big[\left(\left(b_{j+1}^n\right)^3 + \left(b_j^n\right)^3\right)\left(b_{j+1}^n - b_j^n\right)$$

$$- \left(\left(b_{j-1}^n\right)^3 + \left(b_j^n\right)^3\right)\left(b_j^n - b_{j-1}^n\right) \Big] + b_j^n$$

assume $b_j^n > 0$ for all $j \in \{0, \dots, J\}$, then

$$\cancel{b_j^{n+1} > -3\alpha \left(b_j^n\right)^2 \frac{\Delta t}{\Delta z} + \frac{\Delta t}{2(\Delta z)^2} \Big[\left(\left(b_{j+1}^n\right)^3 + \left(b_j^n\right)^3\right)\left(-b_j^n\right)}$$

$$\cancel{- \left(\left(b_{j-1}^n\right)^3 + \left(b_j^n\right)^3\right) b_j^n \Big]}$$

let $A = (b^n_{j+1})^3 + (b^n_j)^3$, $B = (b^n_{j-1})^3 + (b^n_j)^3$,

we get:

$$b^{n+1}_j = 3\alpha \frac{\Delta t}{\Delta z}(b^n_j)^2 b^n_{j-1} + \frac{\beta \Delta t}{2(\Delta z)^2}\left(A b^n_{j+1} + B b^n_{j-1}\right)$$

$$+ b^n_j\left(1 - 3\alpha\frac{\Delta t}{\Delta z}(b^n_j)^2 - \frac{\beta}{2}\frac{\Delta t}{(\Delta z)^2}(A+B)\right)$$

> 0 ~~iff~~

if ~~and only if~~

$$1 - 3\alpha\frac{\Delta t}{\Delta z}(b^n_j)^2 - \frac{\beta}{2}\frac{\Delta t}{(\Delta z)^2}(A+B) > 0$$

$$\frac{\Delta t}{\Delta z^2}\left(3\alpha(b^n_j)^2\Delta z + \frac{\beta}{2}(A+B)\right) < 1$$

$$\Delta t < \frac{1}{2}\cancel{\beta}(\Delta z)^2\left(6(b^n_j)^2\Delta z + \beta(A+B)\right)^{-1}$$

COMMENT: A & B depend on n & j! They are **not** constants.

f) Derivation of 2nd-order spatial discretisation of 1st order derivative.

Consider the following Taylor expansions:

$$b^n_{j+1} = b^n_j + \Delta z\, \partial_z b^n_j + \frac{(\Delta z)^2}{2} \partial_z^2 b^n_j + O((\Delta z)^3)$$

$$b^n_{j-1} = b^n_j - \Delta z\, \partial_z b^n_j + \frac{(\Delta z)^2}{2} \partial_z^2 b^n_j + O((\Delta z)^3)$$

Now consider

$$\frac{b^n_{j+1} - b^n_{j-1}}{\Delta z} = 0 + 2\partial_z b^n_j + 0 + O((\Delta z)^2).$$

Hence,

$$\partial_z b^n_j \approx \frac{b^n_{j+1} - b^n_{j-1}}{2\Delta z}$$

Changing this term in part e) we get

$$b^{n+1}_j = -3\alpha \frac{\Delta t}{2\Delta z}\left(b^n_{j+1} - b^n_{j-1}\right)\left(b^n_j\right)^2 + \frac{\beta \Delta t}{2(\Delta z)^2}\left(A\left(b^n_{j+1} - b^n_j\right) - B\left(b^n_j - b^n_{j-1}\right)\right) + b^n_j.$$

Assume $b^n_j > 0$, then

$$b_j^{n+1} = 3\alpha \frac{\Delta t}{2\Delta z} b_{j-1}^n (b_j^n)^2 + \frac{\beta \Delta t}{2(\Delta z)^2}\left(B b_{j-1}^n\right)$$

$$\frac{\Delta t}{2(\Delta z)^2}\left[b_{j+1}^n\left(A\beta - 3\alpha \Delta z (b_j^n)^2\right)\right]$$

$$+ b_j^n\left(1 - \frac{\beta\Delta t}{2(\Delta z)^2}(A+B)\right) > 0$$

if ~~and only if~~

$$\frac{\Delta t}{2(\Delta z)^2}\left[b_{j+1}^n\left(A\beta - 3\alpha \Delta z (b_j^n)^2\right)\right]$$

$$+ b_j^n\left(1 - \frac{\beta\Delta t}{2(\Delta z)^2}(A+B)\right) > 0.$$

Require two conditions:

$$\beta A - 3\alpha \Delta z (b_j^n)^2 > 0$$

$$\Leftrightarrow \quad \Delta z < \frac{\beta A}{3\alpha (b_j^n)^2}$$

and

$$1 - \frac{\beta \Delta t}{2(\Delta z)^2}(A+B) > 0$$

$$\Leftrightarrow \quad \Delta t < \frac{2(\Delta z)^2}{\beta(A+B)}.$$

Combining these we get

$$\Delta t < \frac{2\beta A^2}{(A+B)\,9\alpha^2 (b_j^n)^4}.$$

2. a) For steady solution, require

$b = b(z)$ only and (5) becomes:

$$0 + \partial_z[\alpha b^3 - \beta b^3 \partial_z b] = 0$$

Integrate to get

$$\alpha b^3 - \beta b^3 b' + A = 0$$

for A constant. Rearrange and set $Q := -A$

to get:

$$\beta b^3 b' = \alpha b^3 - Q$$

for integration constant Q. Rearrange again

$$\cancel{\frac{db}{dz} = \frac{\alpha}{\beta} - Q\frac{1}{\beta b^3}} \quad \frac{db}{dz} = \frac{\alpha}{\beta} - \frac{Q}{\beta} b^{-3}$$

Let $b = \exp(\lambda z)$, then

$$\beta \exp(3\lambda z)\, \lambda \exp(\lambda z) = \alpha \exp(3\lambda z) - Q$$

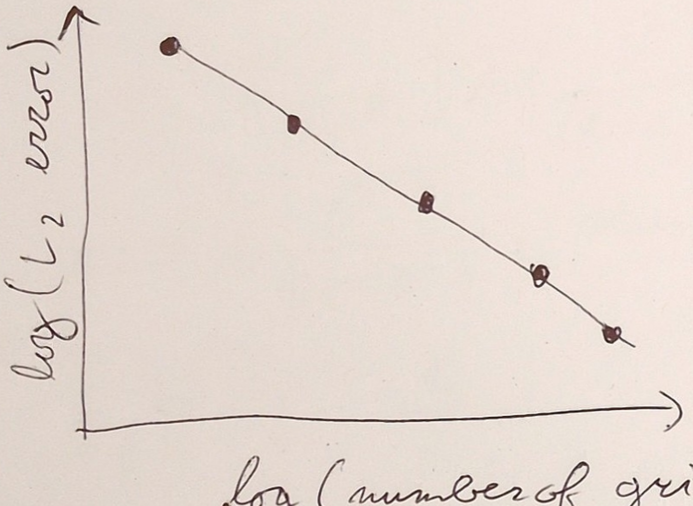
$$b^3(\alpha - \beta b') = Q$$

$$\partial_z[a, \cancel{d}] = Q \cancel{A} = b^3(\alpha - \beta b')$$

$$a' d + a d' = \cancel{b}^3 \alpha - \beta b^3 b'$$

2\. c.) The order of spatial discretisation is $O(\Delta z)$ as given by graph part c.png. As we double the number of grid points, the L2 error halves.

Looking at graph part c.png the linear relationship is clear!



To calculate L_2 error, trapezoidal rule was used:

$$L^2 = \sqrt{\int_0^H e^2(z,t)\,dz} \approx \sqrt{\frac{\Delta z}{2}\left[e^2(0,t) + \left(2\sum_{j=1}^{N-1} e^2(j\Delta z,t)\right) + e^2(N\Delta z,t)\right]}$$

for $e^2 = (b_{steady} - b_{temporal})^2$ and N - number of grid points.

d) First calculate all derivatives:

$$\to \partial_t [z - z_{r0} - ct] = -c$$

$$-c = \frac{\beta}{\alpha}\left(\partial_t b - \cancel{\sqrt{\frac{c}{\alpha}}}\, \frac{1}{1-\frac{\alpha}{c} b^2}\, \cancel{\sqrt{\frac{\alpha}{c}}}\, \partial_t b\right)$$

$$= \partial_t [b]\, \frac{\beta}{\alpha}\left(1 - \frac{1}{1-\frac{\alpha}{c} b^2}\right) \Big\} (*) \text{ This is } -A$$

$$= \partial_t [b]\, \frac{\beta}{\alpha}\left(\frac{\cancel{1} - \frac{\alpha}{c} b^2 - \cancel{1}}{(1-\frac{\alpha}{c} b^2)}\right)$$

$$= -\partial_t [b]\, \beta \left(\frac{b^2}{(c - \alpha b^2)}\right)$$

Let $A := \beta \frac{b^2}{(c-\alpha b^2)}$, then

$$\partial_t [b] = \frac{c}{A}.$$

Similarly,

$$\partial_z [z - z_{r0} - ct] = 1$$

and we get

$$\partial_z [b] = -\frac{1}{A}.$$

The calculation above is analogous to $\partial_t [b]$

It remains to calculate

$$\partial_z^2 [b] = \frac{1}{A^2}\, \partial_z [A] = \partial_z \left[-\frac{1}{A}\right]$$

$$\partial_z^2 [b] = \frac{1}{A^2} \, \partial_z \left[-\frac{\beta}{\alpha} \left(1 - \frac{1}{1 - \frac{\alpha}{c} b^2} \right) \right]$$ } this is (*) on previous page

$$= -\frac{1}{A^2} \frac{\beta}{\alpha} \left(\frac{1}{(1 - \frac{\alpha}{c} b^2)^2} \left(-\frac{\alpha}{c} 2 b \, \partial_z b \right) \right)$$

$$= -\frac{1}{A^2} \frac{\beta}{\alpha} \left(\frac{1}{(1 - \frac{\alpha}{c} b^2)^2} \left(\frac{\alpha}{c} 2 b \frac{1}{A} \right) \right)$$

$$= -\frac{1}{A^3} \frac{\beta \, 2 b \, c}{(c - \alpha b^2)^2}$$

$$= \frac{-2}{\beta b^3 A^3} \left(\frac{\beta^2 b^2 c}{(c - \alpha b^2)^2} \right)$$

$$= \frac{-2c}{\beta b^3 A}$$

Plug everything in to get

$$\partial_t b + \partial_z [u \, b] = 0$$

$$\partial_t b + 3 \alpha b^2 \partial_z b - 3 \beta b^2 (\partial_z b)^2 + \beta b^3 \partial_z^2 b = 0$$

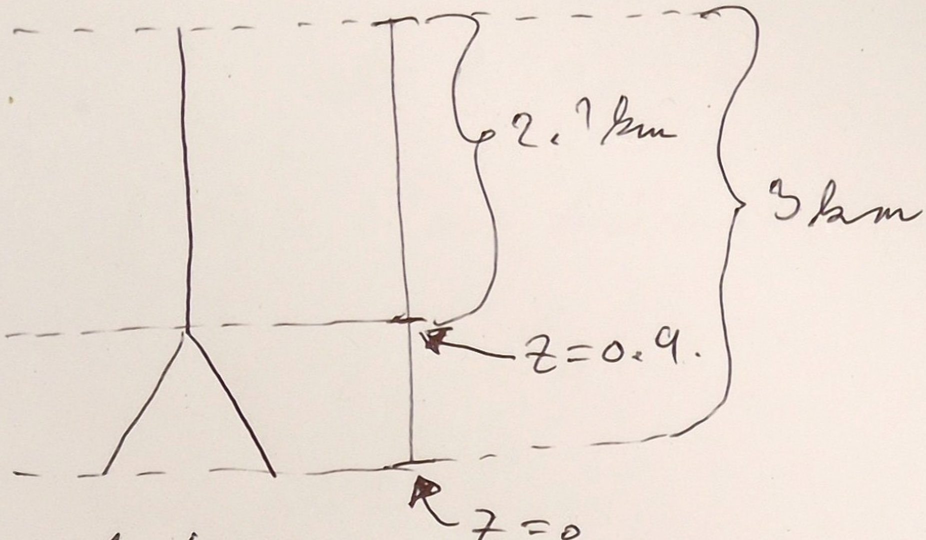
$$\frac{c}{A} + 3 \alpha b^2 \left(-\frac{1}{A} \right) - \frac{3 \beta b^2}{A^2} + \beta b^3 \left(\frac{2c}{\beta b^3 A} \right) = 0$$

$$c - 3 \alpha b^2 - 3 \beta b^2 \left(\beta^{-1} \frac{(c - \alpha b^2)}{b^2} \right) + 2c = 0$$

$$c - 3 \alpha b^2 - 3 (c - \alpha b^2) + 2c = 0$$

$$3c - 3 \alpha b^2 - 3 (c - \alpha b^2) = 0$$

Now, arrange the initial dike shape s.t. it is closed till a dimensional depth 2.1 km with H = 3 km:



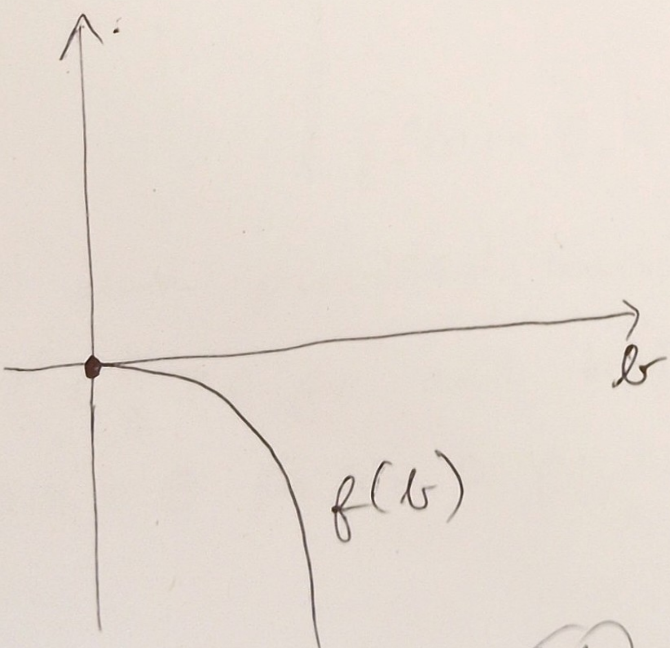
at t = 0

Note that arctanh (b) restricts $b \in [-1, 1]$, but to enforce physicality, require $b \in [0, 1]$.

~~at~~ t = 0 we get

$$z = \frac{\beta}{\alpha}\left(b - \operatorname{arctanh}(b)\right) + z_0$$

Let $f = \frac{\beta}{\alpha}\left(b - \operatorname{arctanh}(b)\right)$, then for $b \in [0, 1]$



However, we have $z \in [0, \infty)$ and require that for $\forall z \in (0.9, 3]$, $f(b) = 0$. Note that at at $t = 0$ & $z = 0.9$, we get

$$0.9 = f(0) + z_0$$

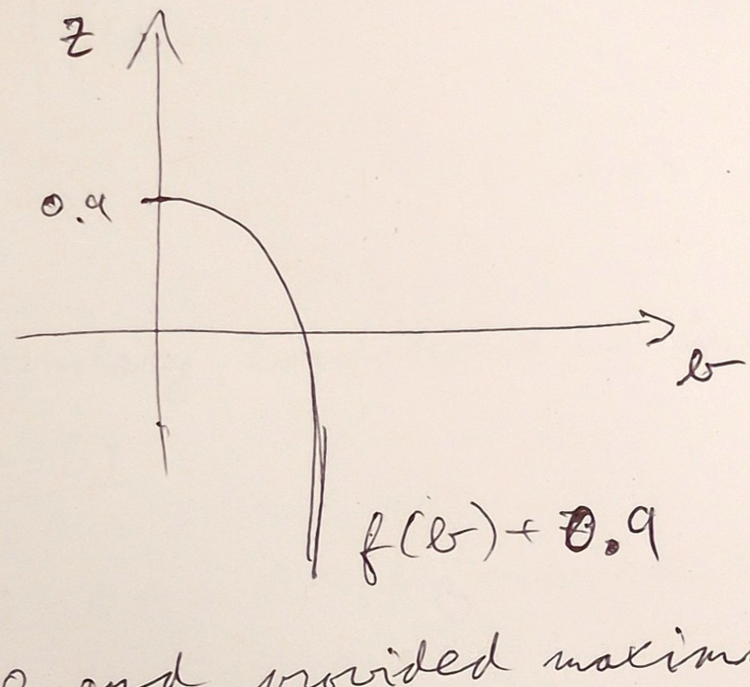
$f(0) = 0$ as $\operatorname{arctanh}(0) = 0$. Hence

$$z_0 = 0.9.$$

Now, plot

$$z = f(b) + \underbrace{0.9}_{z_0}$$

we get



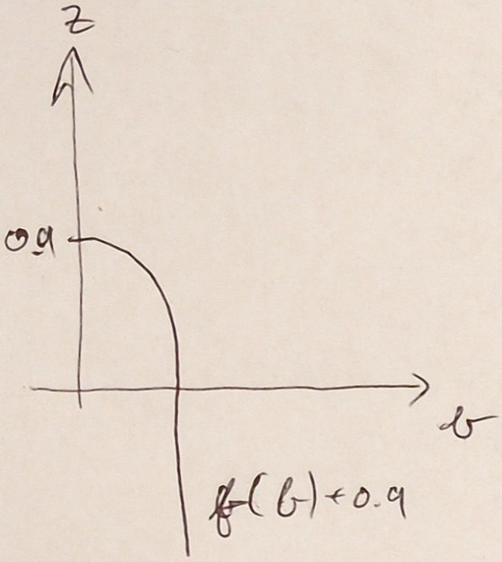
As $z \geq 0$ and provided maximum principle holds ($b_j^n \geq 0$ for all n, j) we have $f(b) + 0.9 = \cancel{z}$ ~~for~~ is 0 for all $z \geq 0.9$.

Now, note that if we reintroduce time t,

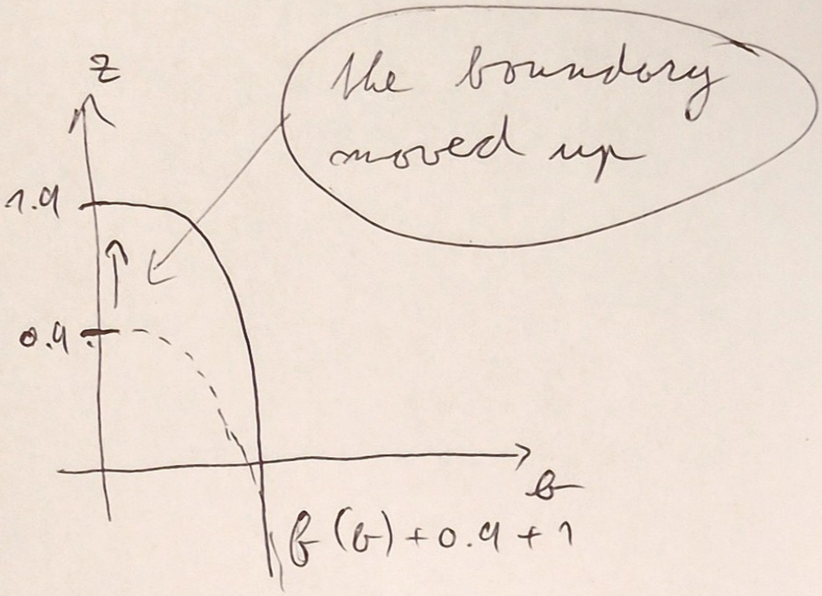
we get

$$z = f(b) + 0.9 + t,$$

i.e. as time progresses, more and more values of z are allowed to be non-zero



$t = 0$



$t = 1$

For boundary conditions we have

<u>b_B, $z=0$:</u>

$$0 - 0.9 - t = (b_B - \operatorname{arctanh}(b_B))\frac{\beta}{\alpha}$$

<u>b_T, $z=3$:</u>

$$3 - 0.9 - t = 2.1 - t = (b_T - \operatorname{arctanh}(b_T))\frac{\beta}{\alpha}$$

~~And we need to solve for b_B & b_T numerically~~