Finite Element Method HW

1). $-\nabla^2 u = f$ on $(x,y) \in [0,1]^2$ ~~$\Gamma_1 : \{x \in [0,1]\}$ $\Gamma_2 : \{y \in$~~

$f(x,y) = 2\pi^2 \sin(\pi x)\cos(\pi y)$

$u(0,y) = u(1,y) = 0$

$\partial_y u(x,y)|_{y=0} = \partial_y u(x,y)|_{y=1} = 0$

~~$I[u] = \iint_\Omega$~~ $\Gamma_1 : x = 0,1$ $\Gamma_2 : y = 0,1$.

Solution $u = u(x,y)$ minimises the functional (arguments are functions):

$$I[u] = \iint_\Omega \left(\frac{1}{2}\|\nabla u\|^2 - uf\right) d\Omega$$

Minimisation (Riesz' method) with variation

$(\delta u)(x,y) = \eta(x,y)$ a function.

$$\delta I = \frac{dI}{d\varepsilon} = \lim_{\varepsilon \to 0} \frac{I[u+\varepsilon\delta u] - I(u)}{\varepsilon} = 0.$$

~~$\therefore I(u+\varepsilon\delta u) = \frac{1}{2}\iint_\Omega (\|\nabla(u+\varepsilon\delta u)\|^2 - (u+\varepsilon\delta u)f) d\Omega$~~

$$\therefore I(u+\varepsilon\delta u) = \iint_\Omega \left(\frac{1}{2}\|\nabla(u+\varepsilon\delta u)\|^2 - (u+\varepsilon\delta u)f\right) d\Omega$$

$\to$ PTO

(2)

Expanding $\|\underline{\nabla}(u+\epsilon\delta u)\|^2$ yields.

$$\|\underline{\nabla}(u+\epsilon\delta u)\|^2 = \underline{\nabla}(u+\epsilon\delta u)\circ\underline{\nabla}(u+\epsilon\delta u)$$

$$= \|\underline{\nabla}u\|^2 + 2\epsilon(\underline{\nabla}u\circ\underline{\nabla}\delta u)\}$$

$$+ \epsilon^2\|\underline{\nabla}\delta u\|^2$$

Assuming ϵ small we get.

$$\|\underline{\nabla}(u+\epsilon\delta u)\|^2 = \|\underline{\nabla}u\|^2 + 2\epsilon(\underline{\nabla}u\circ\underline{\nabla}\delta u) + o(\epsilon).$$

$\therefore$ Neglecting $o(\epsilon)$ terms. we get

$$I(u+\epsilon\delta u) = \frac{1}{2}\iint_\Omega \left(\|\underline{\nabla}u\|^2 + 2\epsilon(\underline{\nabla}u\circ\underline{\nabla}\delta u)\right) d\Omega$$

$$- \iint_\Omega (u+\epsilon\delta u)f \, d\Omega$$

$$= \iint_\Omega \left(\frac{1}{2}\|\underline{\nabla}u\|^2 - uf\right) d\Omega$$

$$+ \epsilon\iint_\Omega \left((\underline{\nabla}u\circ\underline{\nabla}\delta u) - \delta u f\right) d\Omega$$

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(3)

$$\Rightarrow I(u+\varepsilon\delta u) = I[u] + \varepsilon \iint_\Omega \left[\nabla u \cdot \nabla \delta u - f\delta u\right] d\Omega.$$

Hence. $$\delta I = \iint_\Omega \left[\nabla u \cdot \nabla \delta u - f\delta u\right] d\Omega.$$

By definition Ritz - Galerkin principle $\Rightarrow$ variation $= 0$ $\forall \delta u$.

$$\Rightarrow \iint_\Omega \left[\nabla u \cdot \nabla \delta u - f\delta u\right] d\Omega = 0.$$

$$\cancel{\Rightarrow \iint_\Omega \nabla u \cdot \nabla \delta u = \iint_\Omega f\delta}$$

$$\Rightarrow \iint_\Omega \nabla u \cdot \nabla \delta u \, d\Omega = \iint_\Omega f\delta u \, d\Omega.$$

From the notes ~~the LHS can~~ by manipulation the LHS ~~becomes~~ becomes

(4)

Page :

Date :

$$-\iint_\Omega \delta u\, \nabla^2 u\, d\Omega \int_{\Gamma_1} \delta u\, \nabla u \cdot \underline{\bar{n}}\, d\Gamma + \int_{\Gamma_2} \delta u\, \nabla u \cdot \underline{\bar{n}}\, d\Gamma$$

$$= \iint_\Omega f\, \delta u\, d\Omega$$

where $\underline{\bar{n}}$ is the outward normal at $\partial\Omega$

& $\partial\Omega = \Gamma_1 \cup \Gamma_2$

Hence with BCs the line ~~integra~~ integrals became 0.

∴ we have.

$$-\iint_\Omega \delta u\, \nabla^2 u\, d\Omega = \iint_\Omega f\, \delta u\, d\Omega$$

Hence as this has to hold for $\forall\, \delta u$ we get

~~$\nabla^2 u = f$~~ ~~to satisfy~~

$-\nabla^2 u = f$, which ~~indeed~~ does recover the system.

The conditions for $\delta u(x,y)$ is that it must belong to the same ~~for~~ function space as the test function $w(x,y)$ AND must satisfy the form of the BCs i.e. $\delta u(0,y) = 0 = \delta u(1,y)$.

For the weak formulation let $w(x,y)$ be the test function $w(x,y)$ we multiply the system by it and integrate over Ω:

$$\iint_\Omega w(\nabla^2 u + f)\,d\Omega = 0.$$

$$\Rightarrow \iint_\Omega -\nabla w \cdot \nabla u + wf\,d\Omega$$

$$+ \int_{\Gamma_1} w \nabla u \cdot \hat{n}\,d\Gamma + \int_{\Gamma_2} w \nabla u \cdot \hat{n}\,d\Gamma = 0.$$

$\nabla u \cdot \hat{n} = 0$ on Γ_2 AND $w(x,y) = 0$ on Γ_1

Hence, we get

$$\iint_\Omega -\nabla w \cdot \nabla u + wf\,d\Omega = 0.$$

which is the same obtained through Riesz' method w/ the same condition. Hence

$w(x,y) = \delta u(x,y)$ $\square$.

(6)

2) Using Einstein summation notation as in the notes we have.

$u(x,y) \approx u_n(x,y) = u_j \varphi_j(x,y)$. ~~where φ_j is a compact~~
$w \approx w_n(x,y) = w_j \varphi_j(x,y)$
~~basis the compact basis function~~
where φ_j are compact basis functions and

u_j are nodal coefficients.

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(7)

Page

Date

Take the case $u_j = \delta_{ij} \Rightarrow w_n(x,y) = \varphi_i(x,y)$.
with $(\varphi_i(0,y) = \varphi_i(1,y) = 0)$.

Now the weak formulation states,

$$\iint_\Omega (\underline{\nabla} u_a \cdot \underline{\nabla} w - wf)\, d\Omega = 0.$$

Substituting this into the weak formulation yields.

$$\iint_\Omega (\underline{\nabla}_a u_n \cdot \underline{\nabla} w_n - w_n f)\, d\Omega = 0.$$

$$\Rightarrow \iint_\Omega (u_j \underline{\nabla} \varphi_j \cdot \nabla \varphi_i - \varphi_i f)\, d\Omega = 0,$$

as u_j ~~is constant~~ are constants.

$$\Rightarrow u_j \iint_\Omega (\underline{\nabla} \varphi_j \cdot \underline{\nabla} \varphi_i)\, d\Omega = \iint_\Omega \varphi_i f\, d\Omega.$$

Hence, this yields,

$A_{ij} u_j = b_i$ with

$$A_{ij} = \iint_\Omega \underline{\nabla} \varphi_i \cdot \underline{\nabla} \varphi_j\, d\Omega, \quad b_i = \iint_\Omega \varphi_i f\, d\Omega \; \text{D}.$$

Now consider variation of $u_n(x,y)$ w/variation δu_n

$$\delta I = \frac{dI}{d\epsilon} = \lim_{\epsilon \to 0} \left(\frac{I(u_n + \epsilon \delta u_n) - I(u_n)}{\epsilon} \right)$$

By first part we get

$$\delta I = \iint_\Omega \nabla u_n \cdot \nabla \delta u_n \, d\Omega - \iint_\Omega \delta u_n f \, d\Omega .$$

This is weak formulation provided $\delta u_n = w_n$

From the BC $\delta u_n(0,y) = \delta u_n(1,y) = 0$

$\delta u_n = w_n = \varphi_i$

$$\cancel{\delta I =} \iint_\Omega \nabla u_n \cdot \nabla \delta u_n \, d\Omega - \iint_\Omega \delta u_n f \, d\Omega -$$

$$\delta I = \iint_\Omega \left(\nabla u_n \cdot \nabla \varphi_i - \varphi_i f \right) d\Omega = 0$$

if $u_n = u_j \varphi_j(x,y)$, u_j constant weights

(9)

$$\delta I = \iint_\Omega \left(u_j \nabla \varphi_j \cdot \nabla \varphi_i - \varphi_i f \right) d\Omega = 0 .$$

~~this re save result~~

$$\Rightarrow u_j \iint_\Omega \left(\nabla \varphi_j \cdot \nabla \varphi_i \right) d\Omega = \iint_\Omega \varphi_i f \, d\Omega .$$

$$\Rightarrow A_{ij} u_j = b_i \qquad \square .$$

3. (i). Looking at L^2 error for each $\{h, p\}$ combination yields.

h	p	L^2
1/16	1	1.59×10^{-3}
1/32	1	4.01×10^{-4}
1/64	1	1.00×10^{-4}
1/128	1	2.51×10^{-5}
1/16	2	1.04×10^{-6}
1/32	2	6.52×10^{-8}
1/64	2	4.08×10^{-9}
1/128	2	2.55×10^{-10}
1/16	3	3.38×10^{-9}
1/32	3	1.06×10^{-10}
1/64	3	5.34×10^{-12}
1/128	3	2.17×10^{-12}

10

For $P=1$ halfing h reduces the error by roughly a factor of 4

Now for $\{h, P\} = \{1/16, 2\}$ the error is roughly 1.04×10^{-6} now for $\{1/128, 1\}$ the error is roughly 2.51×10^{-5} that means the error for $\{1/128, 1\}$ is 24 times greater than $\{1/16, 2\}$ $\therefore$ following the refinement pattern for $P=1$ i.e halfing h reduces error by a factor of 4. ~~Note~~

$\therefore$ Using this we need to half $1/128$ ~~2.3 ti~~ ≈ 2.3 times to achieve the same error level for $\{1/16, 2\}$ as $\log_4(24) \approx 2.3$. For the sake of simplicity let us take this to be 2. Hence, with this we get that the error level for $\{1/512, 1\}$ would be similar to $\{1/16, 2\}$.