Kane Tutty

1) PDE and Domain: $-\nabla^2 u = f$ in $\Omega = [0,1]^2$ with $f(x,y) = 2\pi^2 \sin(\pi x)\cos(\pi y)$

Boundary Conditions: Dirichlet on vertical sides $\rightarrow u(0,y) = u(1,y) = 0$

Neumann on horizontal sides $\rightarrow \frac{\partial u}{\partial y}(x,y)|_{y=0} = \frac{\partial u}{\partial y}(x,y)|_{y=1} = 0$

Exact Solution: $u_e(x,y) = \sin(\pi x)\cos(\pi y)$

i) Write down Ritz-Galerkin Principle

- Instead of solving PDE directly the Ritz principle wants to find the function that minimizes a certain energy functional. The functional is chosen so that its Euler-Lagrange equation is the PDE given

- Admissable space V
  ↳ restrict to functions that satisfy Dirichlet boundary conditions $V = \{u \in H^1(\Omega) | u(0,y) = u(1,y) = 0\}$

- Functional to minimise

$$I[u] = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right) dx$$

$\frac{1}{2}|\nabla u|^2$ = Energy stored in gradients and the variation produces the laplacian operator

fu = Work done by the forcing and the variation produces the source term

This is the Ritz-Galerkin for the given Poisson system

ii) Conditions on the variation $\delta u(x,y)$

- To find a minimser, we perturb u slightly and specify that the functional doesn't change to first order. This is called the variation: $u_\epsilon = u + \epsilon \delta u$,

BCs on δu:

Dirichlet sides: Since u is fixed to 0, variations must vanish $\delta u(0,y) = \delta u(1,y) = 0$

Neumann sides: Unrestricted on δu and will appear from the variational calculus

The conditions on δu are that it belongs to $H^1(\Omega)$, vanishes on Dirichlet boundaries and is unrestricted on the Neumann boundaries

iii) Variation there of yields the system

The first variation gives $\frac{d}{d\epsilon} I[u+\epsilon\delta u]|_{\epsilon=0} = \int_\Omega (\nabla u \cdot \nabla \delta u - f\delta u) dx$ and this expression must equal zero for all admissable δu so that it satisfies the PDE and boundary conditions

$$\int_\Omega \nabla u \cdot \nabla \delta u \, dx = -\int_\Omega (\nabla^2 u)\delta u \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} \delta u \, ds \quad \leftarrow \text{Using Greens identity}$$

for $x = 0,1 \rightarrow \delta u = 0$ so boundary terms = 0 from Dirichlet

For $y = 0,1 \rightarrow \delta u$ arbitrary so for boundary terms = 0 then $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial y} = 0$ from Neumann

$\Rightarrow$ Volume term

$$\int_\Omega (-\nabla^2 u - f)\delta u \, dx = 0 \quad \text{for all } \delta u$$

- If the integral of $(-\nabla^2 u - f) = 0$ for every δu, then the coefficient $(-\nabla^2 u - f) = 0$ everywhere in Ω. So $\int_\Omega g(x)w(x)dx = 0$ for all test functions w in space, then $g(x) = 0$ almost everywhere

In this case $g(x) = -\nabla^2 u(x) - f(x) \Rightarrow -\nabla^2 u = f$ in Ω

So then varying the functional reproduces the PDE and boundary conditions.

iv) Derive Weak formulation

The weak form is what gets discretised in Finite Element Methods. It avoids second derivatives and expresses the problem as integrals.

Multiply the PDE by arbitrary test function and integrate (w)

$$\int_\Omega -(\nabla^2 u) w \, dx = \int_\Omega f w \, dx$$

Using integration by parts with Green's identity

$$\int_\Omega (-\nabla^2 u) w \, dx = \int_\Omega \nabla u \cdot \nabla w \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} w \, ds \quad \text{where } \frac{\partial u}{\partial n} = \nabla u \cdot n$$

$$\int_\Omega \nabla u \cdot \nabla w \, dx - \int_{\partial\Omega} \left(\frac{\partial u}{\partial n}\right) w \, ds = \int_\Omega f w \, dx$$

On $x = 0, 1 \rightarrow w(0, y) = w(1, y) = 0 \rightarrow \int_{x=0} \frac{\partial u}{\partial n} w \, ds + \int_{x=1} \frac{\partial u}{\partial n} w \, ds = 0$ ← Dirichlet

On $y = 0, 1 \rightarrow n = (0, -1)$ for $y = 0$, $n = (0, +1)$ for $y = 1$ so $\frac{\partial u}{\partial n} = \pm \frac{\partial u}{\partial y}$ ← Neumann

↳ $\frac{\partial u}{\partial y}\Big|_{y=0} = \frac{\partial u}{\partial y}\Big|_{y=1} = 0$

$\Rightarrow$ boundary integral becomes $\int_{\partial\Omega} \frac{\partial u}{\partial n} w \, ds = 0$

Having got rid off the boundary term the weak formulation gives

$$\int_\Omega \nabla u \cdot \nabla w = \int_\Omega f w \, dx \quad \text{for all } w \text{ in admissable space } V$$

LHS = Energy inner product of u and test function w } this equation must hold for all test functions w in the admissable space

RHS = Forcing applied to w

v) test function = variation

- The test function used in the weak form is the admissible variation from the Ritz principle because the direction δu perturbs is arbitrary and only subjected to the boundary conditions
- In Galerkin methods the test function w is arbitrary in the same space

$$w(x,y) = \delta u(x,y)$$

2)

i) Discrete Ritz-Galerkin principle with FEM expansion

- Replace the infinite-dimensional function space by a finite-dimensional one, so PDE becomes linear algeb
- Choose global basis (shape) functions that span a finite element space and write the approximate solution as a linear combination of basis functions
- Minimising energy functional over finite space gives a quadratic optimisation where stationarity, a linear system

Discrete space: Chose conforming FE space V_h that fits Dirichlet boundary condition

$$\phi_i(0,y) = \phi_i(1,y) = 0 \quad \text{for global basis function } \phi_i$$

$$\hookrightarrow u_h(0,y) = u_h(1,y) = 0$$

The approximate solution: $u_h(x,y) = \sum_{j=1}^{N} U_j \phi_j(x,y)$ where U_j are what get solved for

With Ritz functional $I[u] = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right) dx$ then sub u_h to get functional of coefficients U

$$I_h(U) = \int_\Omega \left(\frac{1}{2}|\nabla u_h|^2 - f u_h\right) dx$$

This the 'energy' of the discrete function u_h. Minimising over all coefficients U encodes PDE and BCs in the finite-dimensional setting

ii) Discrete weak (Galerkin) formulation with a FEM expansion

- The weak formulation is the integral identity that must hold for all test functions. Enforce it on test functions from the same finite element space V_h. This creates the variational statement into a finite set of linear equations - one per basis function

Find $u \in V$ so that $\int_\Omega \nabla u \cdot \nabla w \, dx = \int_\Omega f w \, dx$ for all $w \in V$

↳ take basis functions set $\{\phi_i\}_{i=1}^N$ and $w_h = \phi_i$ so

$$\sum_{j=1}^{N} U_j \int_\Omega \nabla \phi_j \cdot \nabla \phi_i \, dx = \int_\Omega f \phi_i \, dx \quad i = 1, 2, \dots N$$

BCs : Basis functions ϕ_i vanish at $x=0$ and $x=1$ so $u_h(0,y) = u_h(1,y) = 0$ ← Dirichlet

Produce no extra boundary integrals so stiffness matrix and load vector are defined by volume ← Neumann integrals

iii) Show that the variation of the discrete Ritz functional gives the discrete weak form

- Trying to show an equivalency between minimise $I_h(u)$ = satisfy discrete weak form
- Ritz minimisation equals Galerkin projection whenever the bilinear form is symmetric positive definite (Poisson in this case)

~~Discrete functional~~

- K is the stiffness matrix and is defined by $K_{ij} = \int_\Omega \nabla \phi_j \cdot \nabla \phi_i \, dx$ and it considers how the gradients of the two basis functions interact over the domain Ω. Because of poisson giving the form $a(u,w) = \int_\Omega \nabla u \cdot w$ as it is symmetric and positive definite matrix
- F is the load vector and is defined by $F_i = \int_\Omega f \phi_i \, dx$ and it measures how $f(x,y)$ projects onto the basis function ϕ_i

- These two definitions can be used to rewrite the Discrete functional as

$$I_h(U) = \frac{1}{2} U^T K U - U^T F$$

for each i, $\frac{\partial I_h}{\partial U_i} = \frac{1}{2} \sum_j (K_{ij} + K_{ji}) U_j - F_i = \sum_j K_{ij} U_j - F_i$

When stationary $\frac{\partial I_h}{\partial U_i} = 0 \implies \sum_j K_{ij} U_j = F_i$ for all i

- Discrete Weak Equations
Like the stationary condition giving $(KU)_i = F_i$ we can test the weak form with $w_h = \phi_i$

$$\sum_j U_j \int_\Omega \nabla \phi_j \cdot \nabla \phi_i \, dx = \int_\Omega f \phi_i \, dx$$

These two linear systems are the same one from the variation (Ritz) and the discrete weak formulation (Galerkin). Hence, minimising the functional over V_h = satisfy weak form in V_h

The symmetric nature of the problem being Poisson allows the Ritz view = Galerkin view and they lead to the same algebraic system $KU = F$ where U = vector of unknown coefficients $u_h(x,y) = \sum_{j=1}^{N} U_j \phi_j$
K = stiffness matrix = $\int_\Omega \nabla \phi_j \cdot \nabla \phi_i \, dx$
F = load vector = $\int_\Omega f \phi_i \, dx$

The finite element discretisation of the poisson problem

3 i) Introduce a local coordinate system and reference coordinates

- In FEM each element can have different sizes and shapes in the physical domain. To avoid rederiving shape functions for every element they can be defined once on a reference element in a simple local coordinate system. Then the reference element can be mapped to each physical element using a transformation (with a Jacobian matrix). This makes the integration and assembly uniform. Integrate over the reference element and then correct with the Jacobian.

~~Triangular Elements~~:

~~Reference Element = Standard triangle in local coordinates (ξ, η) and vertices at $(0,0), (1,0), (0,1)$~~

- In step 2 we derived the global system $KU = F$. To compute K and F, work element by element. Each one is mapped from the reference coordinate system (ξ, η) to the physical mesh. One the reference quadrilateral we define bilinear shape functions then using the Jacobian transform gradients and integrals to physical coordinates.
- This gives local stiffness matrices and load vectors which are then assembled into the global system. So the introduction of local coordinates and reference elements makes the computation of K and F practical

Shape Functions: simple polynomials defined on a reference element (triangle or square) that interpolates the solution between the elements nodes. They ensure if you know the solution values at the element's vertices you can reconstruct the solution everywhere inside the element.

For Triangle: $N_1(\xi, \eta) = 1 - \xi - \eta$, $N_2(\xi, \eta) = \xi$, $N_3(\xi, \eta) = \eta$

For Quadrilateral: $N_1(\xi, \eta) = \frac{1}{4}(1-\xi)(1-\eta)$, $N_2 = \frac{1}{4}(1+\xi)(1-\eta)$, $N_3 = \frac{1}{4}(1+\xi)(1+\eta)$, $N_4 = \frac{1}{4}(1-\xi)(1+\eta)$

↳ interpolate values at four corners of the square $[-1, 1]^2$

Map to physical elements: Real mesh elements not unit triangles or squares they can be skewed, rotated or stretched so the shape functions need to be transformed to actual geometry. The geometry of each physical element is interpolated

using the shape functions (Isoparametric concept) $\longrightarrow$ $x(\xi, \eta) = \sum_a x_a N_a(\xi, \eta)$ x_a are the coordinates of the elements vertices

Jacobian: Accounts for the change of variables in integrals and it converts $dx\,dy \to |J|\,d\xi\,d\eta$

$$J = \frac{\partial x}{\partial(\xi, \eta)}$$ ← Jacobian for matrix mapping

$$dx\,dy = |J|\,d\xi\,d\eta$$ ← Area scaling from determinant

$$\nabla_x N_a = J^{-T} \nabla_\xi N_a$$ ← Gradient transform from inverse transpose

Gradients of shape functions: Obtained by applying the chain rule with the Jacobian

Local Stiffness Matrix: For element $K_1 = \int_{k_1} \nabla N_b \cdot \nabla N_a \, dx$ from the weak form of the PDE, then from mapping

$$K'_{ab} = \int_k \nabla_\xi N_b^T \left(J^{-1} J^{-T}\right) \nabla_\xi N_a |J| \, d\xi\, d\eta$$

Local load vector: for element k_1: $F_a^{(e)} = \int_{k_1} f N_a \, dx \xrightarrow{\text{Mapping}} F_a^{(1)} = \int_k f(x(\xi, \eta)) N_a(\xi, \eta) |J| \, d\xi\, d\eta$

Each local node a then corresponds to a global degree of freedom index. All the local contributions are added up into the global system $KU = F$

Triangular Elements : Local coordinates, mapping and matrix assembly

- Reference triangle and local shape functions

  ↳ Reference element: Standard triangle in local coordinates (ξ, η) with vertices $(0,0), (1,0), (0,1)$

  ↳ Shape functions : These interpolate nodal values at the three vertices

  Vertex 1 $(0,0)$ $\quad N_1(\xi,\eta) = 1 - \xi - \eta$

  Vertex 2 $(1,0)$ $\quad N_2(\xi,\eta) = \xi$

  Vertex 3 $(0,1)$ $\quad N_3(\xi,\eta) = \eta$

  They are polynomials that satisfy the nodal property $N_a = 1$ at node a and 0 at the other nodes, giving a linear approximation with compact support per element

- Mapping to physical triangle

  ↳ A triangle K_e with vertices $x_a = (x_a, y_a)$, $a = 1,2,3$

  ↳ Geometry interpolation $x(\xi,\eta) = \sum_{a=1}^{3} x_a N_a(\xi,\eta)$

  ↳ Jacobian : J captures the local stretching / rotation from reference to physical coordinates

$$J = \frac{\partial x}{\partial(\xi,\eta)} = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}$$

$$dx\,dy = |J|\,d\xi\,d\eta \quad \text{with} \quad |J| = \det(J)$$

  ↳ Converting derivatives from reference coordinates (ξ, η) to physical coordinates (x, y)

$$\nabla_x N_a = J^{-T} \nabla_\xi N_a \,, \quad \nabla_\xi = \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$$

Reference gradients $\quad \nabla_\xi N_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \; \nabla_\xi N_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \nabla_\xi N_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$K_{ab} = \int \nabla N_b \cdot \nabla N_a \, dx = \int_{\hat{k}} \nabla_\xi N_b^T \left(J^{-1}J^{-T}\right) \nabla_\xi N_a \, |J| \, d\xi\,d\eta = \left(\nabla_\xi N_b^T \left(J^{-1}J^{-T}\right) \nabla_\xi N_a\right) |J| |\hat{k}|$$

$$F_a = \int_{k_e} f N_a \, dx = \int_{\hat{k}} f(x(\xi,\eta)) \, N_a(\xi,\eta) \, |J| \, d\xi\,d\eta = f_e A_e \frac{1}{3} \quad \text{where } f(x,y) \equiv f_e \text{ is constant}$$

- Assembly : Put the 3×3 matrix K^e and 3×1 vector F^e into the global arrays using index map $g(e,a)$

Quadrilateral Elements: Local coordinates, mapping, and matrix assembly

- Reference element: Standard square in local coordinates (ξ, η) with domain $\hat{K} = [-1,1] \times [-1,1]$

Vertex 1: $(-1,-1)$ $\quad N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$

Vertex 2: $(1,-1)$ $\quad N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$

Vertex 3: $(1,1)$ $\quad N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$

Vertex 4: $(-1,1)$ $\quad N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$

These shape functions interpolate nodal values at the four corners

- Reference Gradients:

$$\partial_\xi N_1 = -\frac{1}{4}(1-\eta), \quad \partial_\eta N_1 = -\frac{1}{4}(1-\xi)$$

$$\partial_\xi N_2 = \frac{1}{4}(1-\eta), \quad \partial_\eta N_2 = -\frac{1}{4}(1+\xi)$$

$$\partial_\xi N_3 = \frac{1}{4}(1+\eta), \quad \partial_\eta N_3 = \frac{1}{4}(1+\xi)$$

$$\partial_\xi N_4 = \frac{1}{4}(1+\eta), \quad \partial_\eta N_4 = \frac{1}{4}(1-\xi)$$

These satisfy nodal property $N_a = 1$ at node a and 0 at the others giving a bilinear approximation with compact support per element

Physical element: A quadrilateral K_e with vertices $x_a = (x_a, y_a)$ $a = 1,2,3,4$

Geometry interpolation: $x(\xi,\eta) = \sum_{a=1}^{4} x_a N_a(\xi,\eta)$

Jacobian: $J(\xi,\eta) = \frac{\partial x}{\partial(\xi,\eta)} = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} = \sum_{a=1}^{4} \begin{bmatrix} x_a & y_a \end{bmatrix} \otimes \begin{bmatrix} \partial_\xi N_a \\ \partial_\eta N_a \end{bmatrix}^T$

J varies with (ξ,η) Area scaling $dx\,dy = |J(\xi,\eta)|\,d\xi\,d\eta$

Gradient transformation:

$$\nabla_x N_a(\xi,\eta) = J(\xi,\eta)^{-T} \nabla_\xi N_a(\xi,\eta) \quad \nabla\xi = \begin{bmatrix} \partial_\xi \\ \partial_\eta \end{bmatrix}$$

This converts derivatives from reference coordinates to physical coordinates at each quadrature point

Element stiffness matrix for poisson:

$$K_{ab}^{(e)} = \int_{k_e} \nabla N_b \cdot \nabla N_a \, dx$$

$$= \int_{\hat{k}} \nabla_\xi N_b^T \left(J^{-1} J^T \right) \nabla_\xi N_a \, |J| \, d\xi d\eta$$

$J^{-1}J^T$ corrects the gradient inner product to physical space, $|J|$ rescales area

Per quadrature

$$J_q = J(\xi_q, \eta_q)$$

$$|J_q| = G_q = J_q^{-1} J_q^{-T}$$

then $\nabla_\xi N_a (\xi_q, \eta_q)$ and transform $\nabla_x N_a = J_q^{-T} \nabla_\xi N_a$

Accumulated then so $K_{ab}^{(e)} \; + = (\nabla_x N_b \cdot \nabla_x N_a) \, |J_q| \, W_q$

Element load vector:

$$f_a^{(e)} = \int_{k_e} f N_a \, dx$$

$$f_a^{(e)} = \int_{\hat{k}} f(x) N_a \, |J| \, d\xi d\eta$$

Use the same Gauss points as stiffness then per quadrature point

Map to physical: $x_q = x(\xi_q, \eta_q)$

Evaluate $f(x_q)$ and $N_a(\xi_q, \eta_q)$

Accumulate: $f_a^{(e)} \; + = f(x_q) \, N_a (\xi_q, \eta_q) \, |J_q| \, W_q$

Assembly into global $KU = F$ system

index map: let $i = g(e,a)$ be the global DOF index for local node a on element e

stiffness: For each pair $(a,b) \longrightarrow K_{g(e,a), g(e,b)} \; + = K_{ab}^{(e)}$

Load: for each a, $\longrightarrow F_{g(e,a)} \; + = F_a^{(e)}$

BCs

Dirichlet: Prefer spaces with zero traces on Dirichlet boundaries, enforce after assembly by modifying rows/columns and RHS to impose values ($u=0$, $x=0,1$)

Neumann: No explicit boundary term, weak form already accounts for it

Insert resulting 4×4 $K^{(e)}$ and 4×1 $F^{(e)}$ into global arrays via $g(e,a)$