Casey - Exercise 3

1. $I[u] = \iint_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - uf \right) d\Omega$

Variation of the functional I is defined as:

$$\delta I(u) = \frac{\partial I}{\partial \epsilon} = \lim_{\epsilon \to 0} \frac{I(u+\epsilon\delta u) - I(u)}{\epsilon} = 0$$

$$\Rightarrow I(u+\epsilon\delta u) = \frac{1}{2}\int_{\Omega} |\nabla(u+\epsilon\delta u)|^2 dA - \int_{\Omega} f(u+\epsilon\delta u)\, dA$$

$(dA = dx\,dy)$

Expanding $|\nabla(u+\epsilon\delta u|^2$

$$= \nabla(u+\epsilon\delta u)\cdot\nabla(u+\epsilon\delta u)$$

$$= \nabla u\cdot\nabla u + 2\epsilon(\nabla u\cdot\nabla\delta u) + \epsilon^2(\nabla\delta u\cdot\nabla\delta u)$$

$$= |\nabla u|^2 + 2\epsilon(\nabla u\cdot\nabla\delta u) + O(\epsilon^2)$$

$$\Rightarrow I(u+\epsilon\delta u) = \frac{1}{2}\iint_{\Omega}\left[|\nabla u|^2 + 2\epsilon(\nabla u\cdot\nabla\delta u)\right] dA - \int_{\Omega} f(u+\epsilon\delta u)\, dA$$

$$= \left[\frac{1}{2}\int_{\Omega}|\nabla u|^2 - \int_{\Omega} uf\, dA\right] + \epsilon\left[\int_{\Omega}(\nabla u\cdot\nabla\delta u)\, dA - \int \delta u f\, dA\right]$$

$$= I[u] + \epsilon\left[\int_{\Omega}(\nabla u\cdot\nabla\delta u)\, dA - \int_{\Omega}\delta u f\, dA\right]$$

Variation $\delta I[u] = \int_{\Omega}(\nabla u\cdot\nabla\delta u)\, dA - \int \delta u f\, dA$

Ritz-Galerkin principle - variation $= 0$ for all admissable δu

$$\rightarrow \int_\Omega \nabla u \cdot \nabla \delta u \, dA - \int_\Omega \delta u f \, dA = 0$$

$$\rightarrow \int_\Omega \nabla u \cdot \nabla \delta u \, dA = \int_\Omega \delta u f \, dA$$

LHS can be written as:

$$-\int_\Omega (\nabla^2 u) \delta u \, dA + \oint_{\partial\Omega} (\nabla u \cdot \vec{n}) \delta u \, ds$$

Where $\partial\Omega$ is the boundary

This yields:

$$-\int_\Omega (\nabla^2 u) \delta u \, dA + \oint_{\partial\Omega} (\nabla u \cdot \vec{n}) \delta u \, ds = \int_\Omega f \delta u \, dA$$

$$\Rightarrow \int_\Omega (-\nabla^2 u - f) \delta u \, dA + \oint_{\partial\Omega} (\nabla u \cdot \vec{n}) \delta u \, ds = 0$$

Bit more detail on how BC's applied.

Given the boundary conditions, the boundary integral $= 0$ for all boundaries

$$\Rightarrow \int_\Omega (-\nabla^2 u - f) \delta u \, dA = 0$$

This must hold for all admissable $\delta u \Rightarrow -\nabla^2 u - f = 0$ to satisfy, recovering the system.

Conditions for $\delta u(x,y)$

- Must belong to the same function space as the test function $w(x,y)$.
- Must satisfy the form of the Boundary Conditions.

$$\delta u(0,y) = 0 \quad , \quad \delta u(1,y) = 0$$

Weak formulation for test function $w(x,y)$

$$\rightarrow \int_{\Omega} \nabla u \cdot \nabla w \, dA = \int w f \, dA$$

which will yield the same result as for δu

$$\Rightarrow w(x,y) = \delta u(x,y)$$

2. $u(x,y) \sim u_n(x,y) = \sum_{j=1}^{N} U_j \phi_j(x,y)$

where U_j are nodal coefficients

Substituting into the function I

$$I(u_n) = \frac{1}{2}\int_{\Omega} |\nabla u_n|^2 \, dA - \int_{\Omega} u_n f \, dA$$

$$I(u) = \frac{1}{2}\int_\Omega \left|\nabla\left(\sum_{j=1}^{N} U_j \phi_j\right)\right|^2 dA - \int_\Omega f\left(\sum_{j=1}^{N} U_j \phi_j\right) dA$$

Since coefficients are constant wrt integration

$$\Rightarrow I(u) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} U_i U_j \left(\int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, dA\right) - \sum_{j=1}^{N} U_j \left(\int_\Omega f \phi_j \, dA\right)$$

Ritz - Galerkin principle states solution vector $\vec{U}$ minimises the function I

Take partials of I to find minimum:

$$\frac{\partial I(\vec{U})}{\partial U_k} = 0 \qquad k = 1, \dots, N$$

LHS: $\frac{\partial}{\partial U_k}\left[\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} U_i U_j \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, dA\right] = \sum_{j=1}^{N} U_j \int_\Omega \nabla\phi_k \cdot \nabla\phi_j \, dA$

How is lHS dealt with?

RHS: $\frac{\partial}{\partial U_k}\left[\sum_{j=1}^{N} U_j F_j\right] = \int_\Omega f \phi_k \, dA$

$$\Rightarrow \frac{\partial I(U)}{\partial U_k} = \sum_{j=1}^{N} U_j \int_\Omega \nabla\phi_k \cdot \nabla\phi_j \, dA - \int_\Omega f \phi_k \, dA = 0 \quad \text{for } k=1, \dots, N$$

$$\Rightarrow \sum_{j=1}^{N} U_j \int_\Omega \nabla\phi_k \cdot \nabla\phi_j \, dA = \int_\Omega f \phi_k \, dA = 0$$

delta u_k=0 for k not k’.

Can be written as discrete algebraic system

where: $\vec{K}\vec{U} = \vec{F}$ where $\vec{K} = \int_\Omega \nabla\phi_j \cdot \nabla\phi_k \, dA$, $\vec{F} = \int_\Omega f \phi_k \, dA$

and $\vec{U}$ the vector of unknown coefficients

3.

Matrix assembly:

For element k:

α and β representative of local index

k representative of global index

$\Rightarrow i = \text{index}(k, \alpha)$

$j = \text{index}(k, \beta)$

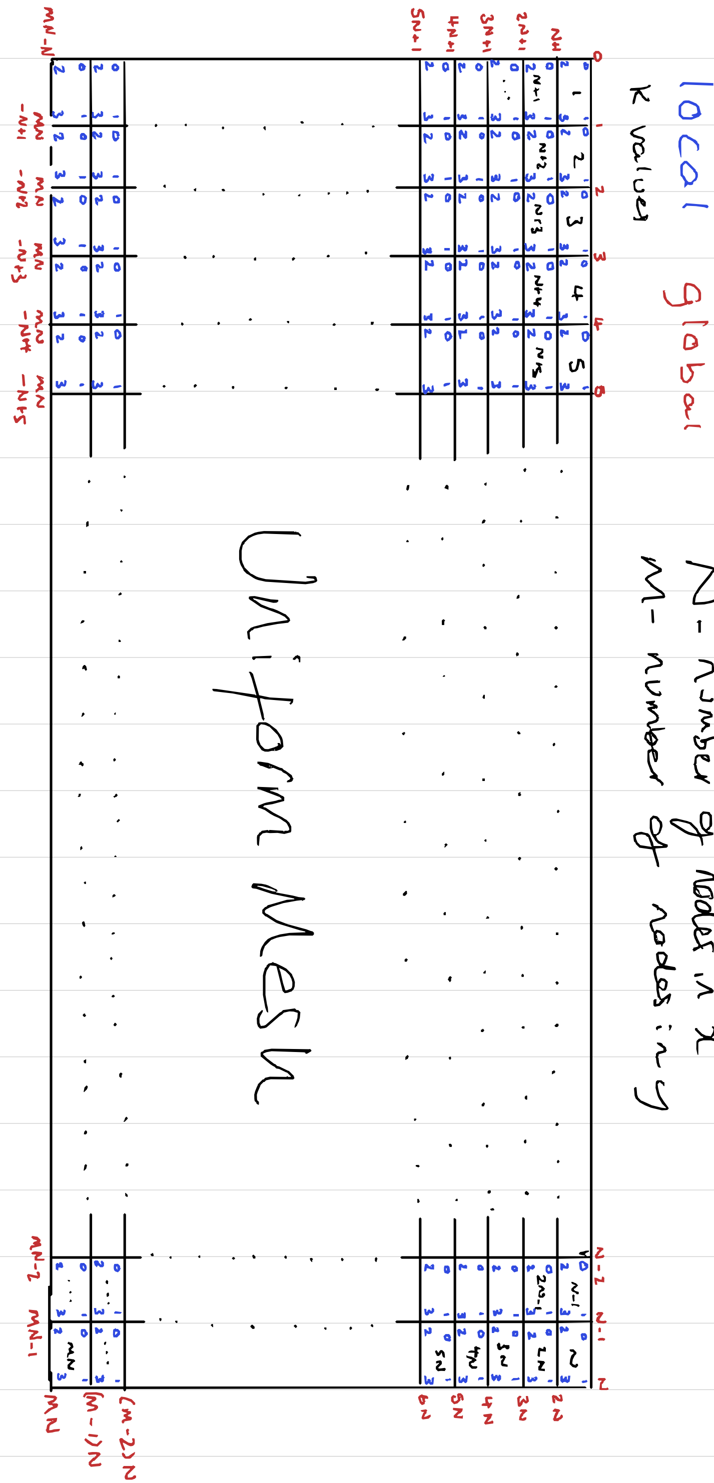
$\Rightarrow A_{ij} = A_{ij} + \hat{A}_{\alpha\beta}$ (determined by element)

+ locally determined

$b_i = b_i + \hat{b}_\alpha$

Based solely on local index.

Giving algebraic system $\overline{\overline{A}}\vec{b}$ representing the mesh element wise.



Part 2

1. Weak formulation derivation: $\times q(y)$ and integrate

$$\rightarrow \int_0^{L_y} q\,\partial_t h_m\,dy - \int_0^{L_y} q\,\alpha g\,\partial_y(h_m\,\partial_y h_m)\,dy = \int_0^{L_y} \frac{qR}{m_{por}\,\sigma_e}\,dy$$

Integrating by parts (second term)

$$\int_0^{L_y} q\,\alpha\,\partial_y(h_m\,\partial_y h_m)\,dy$$

$$\rightarrow \int_0^{L_y} \alpha g h_m\,\partial_y q\,\partial_y h_m\,dy + \left[\alpha g q h_m\,\partial_y h_m\right]_0^{L_y}$$

At $y = L_y$, $\partial_y h_m = 0 \Rightarrow$ term $\rightarrow 0$ (Eq. 12)
At $y = 0$, can be evaluated as $\alpha g q(0) h_m \partial_y h_m |_{y=0}$
which can be rewritten to give $\alpha g \frac{1}{2} \partial_y (h_m)^2 |_{y=0}$
Using canal level ODE (Eq. 14) $\rightarrow$ (Eq. 28)

Substituting back to weak form yields
(Eq. 29) the weak formulation.

Using the FEM expansion for h_m

$h_m(y,t) \approx \sum h_j(t)\varphi_i(y)$ and $q = \varphi_i(y)$

and considering time stepping: $\partial_t h = \frac{h^{n+1} - h^n}{\Delta t}$

Algebraic system is as seen (Eq. 33).

Time step restriction

Considering system as:

$$\bar{\bar{M}}\vec{h}^{n+1} = \bar{\bar{M}}\vec{h}^n - \Delta t\,\bar{\bar{K}}(\vec{h}^n)\vec{h}^n$$

Δt restriction can be derived from stability of the algebraic system

$$\rightarrow \quad \vec{h}^{n+1} = (\bar{\bar{I}} - \Delta t\,\bar{\bar{M}}^{-1}\bar{\bar{K}})\vec{h}^n$$

Eigenvalues of $(\bar{\bar{I}} - \Delta t\,\bar{\bar{M}}^{-1}\bar{\bar{K}})$ must be ≤ 1

$$\Rightarrow \quad \Delta t \leq \frac{2}{\lambda_{max}(\bar{\bar{M}}^{-1}\bar{\bar{K}})}$$

This can be expressed in terms of the parameters of the problem

$$\longrightarrow \quad \Delta t \leq \frac{M_{por}\,\sigma_e\,(\Delta y)^2}{\alpha g\, h_m}$$

where $h_m = \max(h_m)$ in the domain.