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Exercises $\Rightarrow$ sheet 2

The groundwater model:

$$\partial_t[w_v h_m] - \alpha g \partial_y[w_v h_m \partial_y [h_m]] = \frac{w_v R}{m_{por}\sigma_e}$$

in $y \in [0, L_y]$

$$\partial_y h_m = 0 \text{ at } y = L_y$$

$$h_m(0,t) = h_{cm}(t) \text{ at } y = 0$$

$$L_c w_v \frac{d h_{cm}}{dt} = w_v m_{por} \frac{\sigma_e}{2} \alpha g \partial_y [h_m^2]\Big|_{y=0}$$
$$- w_v \sqrt{g} \max\left(\frac{2}{3} h_{cm}(t), 0\right)^{\frac{3}{2}}$$

Take the 1st equation & multiply it by $\frac{q(y)}{w_v}$ and integrate. We get:

$$\int_0^{L_y} q\, \partial_t[h_m] - \frac{\alpha g q}{\cancel{2}} \partial_y[h_m \partial_y [h_m]]\, dy$$
$$= \int_0^{L_y} \frac{q R}{m_{por} \sigma_e} dy$$

Note that

$$A\,\partial_y[q\, h_m\, \partial_y[h_m]] = A\,\partial_y[q]\, h_m\, \partial_y[h_m] + A q\, \partial_y[h_m\, \partial_y[h_m]]$$

Hence,

$$-Aq\,\partial_y[h_m\,\partial_y[h_m]] = A\,\partial_y[q]\, h_m\, \partial_y[h_m] - A\,\partial_y[q\, h_m\, \partial_y[h_m]]$$

Integrate to get

$$-\int_0^{L_y} Aq\,\partial_y[h_m\,\partial_y[h_m]]\,dy = \int_0^{L_y} A\,\partial_y[q]\, h_m\, \partial_y[h_m]\,dy \underbrace{- A\int_0^{L_y} \partial_y[q\, h_m\, \partial_y[h_m]]\,dy}_{+A q\, h_m\, \partial_y[h_m]\big|_{y=0}}$$

as $\partial_y h_m (y=L_y) = 0$.

Returning to the equation, we get:

$$\int_0^{L_y} q\,\partial_t[h_m] + \alpha g\, \partial_y[q]\, h_m\, \partial_y[h_m]\,dy + \alpha g \left(q\, h_m\, \partial_y[h_m]\right)\Big|_{y=0} = \int_0^{L_y} \frac{qR}{m_{por}\,\sigma_e}\,dy \quad (\star)$$

Now, focus on the equation with $\frac{\partial h_{cm}}{dt}$ term.

We get, by multiplying by $\frac{1}{w_{v}\, m_{por}\, \sigma_e}$;

$$\frac{L_c}{m_{por}\sigma_e}\frac{d h_{cm}}{dt} = \frac{1}{2}\alpha g\, \partial_y [h_m^2]\Big|_{y=0} - \frac{\sqrt{g}}{m_{por}\sigma_e}\max\left(\frac{2}{3}h_{cm}, 0\right)^{3/2}.$$

(A) This is equivalent to:

$$\alpha g\, q(0)\, h_m(0)\, \partial_y [h_m(0)]\Big|_{y=0} = \frac{L_c\, q(0)}{m_{por}\sigma_e}\frac{d h_{cm}}{dt} + \frac{\sqrt{g}\, q(0)}{m_{por}\sigma_e}\max\left(\frac{2}{3}h_{cm}, 0\right)^{3/2}.$$

Plug this into (☆)

$$\int_0^{L_y} q\, \partial_t [h_m] + \alpha g\, \partial_y [q]\, h_m \partial_y [h_m]\, dy + \frac{q(0) L_c}{m_{por}\sigma_e}\frac{d h_m(0,t)}{dt} = \int_0^{L_y} \frac{qR}{m_{por}\sigma_e}\, dy - \frac{q(0)\sqrt{g}}{m_{por}\sigma_e}\max\left(\frac{2}{3}h_m(0,t), 0\right)^{3/2}$$

Now, rearrange to obtain:

$$\int_0^{L_y} q\,\partial_t[h_m]\,dy + \frac{q(0)L_c}{m_{por}\sigma_e}\frac{dh_m(0,t)}{dt} = \int_0^{L_y} -\alpha g h_m \partial_y[q]\,\partial_y[h_m] + \frac{qR}{m_{por}\sigma_e}\,dy - \frac{q(0)}{m_{por}\sigma_e}\sqrt{g}\,\max\left(\frac{2}{3}h_m(0,t),0\right)^{\frac{3}{2}}$$

Discretisation

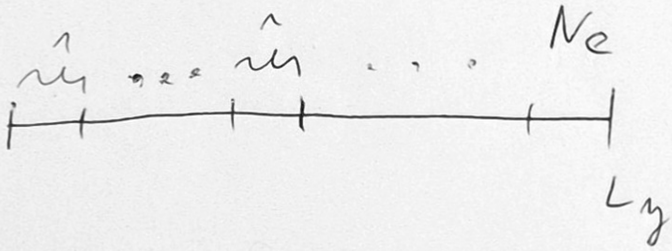
Consider the forward Euler time discretisation:

$$\int_0^{L_y} q\,\frac{h_m^{n+1}-h_m^{n}}{\Delta t}\,dy + (\ldots) = (\ldots).$$

Rearrange to get:

$$\int_0^{L_y} q\,h_m^{n+1}\,dy + \frac{q(0)L_c}{m_{por}\sigma_e}h_{cm}^{n+1} = \int_0^{L_y} q\,h_m^{n}\,dy + \frac{q(0)L_c\,h_{cm}^{n}}{m_{por}\sigma_e}$$
$$+\Delta t\int\left(-\alpha g h_m^{n}\partial_y[q]\,\partial_y[h_m^{n}] + \frac{qR^{n}}{m_{por}\sigma_e}\right)dy$$
$$-\Delta t\,\frac{q(0)}{m_{por}\sigma_e}\sqrt{g}\,\max\left(\frac{2}{3}h_{cm}^{n},0\right)^{3/2} \qquad (*)$$

Now, $h_m^{n+1}\in\mathcal{H}^1$ constrained by boundary conditions. Now, divide the domain into N_e elements:



and consider $V_{Ne} \subset \mathcal{H}^1$ s.t. the basis functions $\varphi_i(y)$ are linear on i^{th} element and zero ~~on~~ other elements.

Then choose $q = \varphi_i(y)$ and $h_m = h_j \varphi_j(y)$ for constants h_j & $i, j \in \{1, \dots, Ne\}$

Substituting into ( ) we get:

$$\int_0^{L_y} \varphi_i \varphi_j \, dy \, h_j^{n+1} + \frac{q(0) L_c}{m_{por}\sigma_e} h_1^{n+1} \delta_{1i} = \int_0^{L_y} \varphi_i \varphi_j \, dy \, h_j^n + \frac{q(0) L_c}{m_{por}\sigma_e} \delta_{1i} h_1^n + \Delta t \int \left(-\alpha g h_m^n \partial_y \varphi_i \partial_y h_m^n + \frac{\varphi_i R^n}{m_{por}\sigma_e} \right) dy - \Delta t \frac{q(0)}{m_{por}\sigma_e} \sqrt{g} \max\left(\frac{2}{3} h_1^n, 0 \right)^{\frac{3}{2}} \delta_{i1}$$

where we used $h_{cm}^n = h_1^n$ and the Einstein's summation convention.

Now, define

$$M_{ij} = \int_0^{L_y} \varphi_i \varphi_j \, dy$$

&

$$b_i^n = \int_0^{L_y} \left(-\alpha g h_m^n \partial_y \varphi_i \partial_y h_m^n + \frac{\varphi_i R^n}{m_{por}\sigma_e} \right) dy .$$

Finally, the (qp) becomes:

$$M_{ij} h_j^{n+1} + \frac{q^{(0)} L_c}{n_{por} \sigma_e} \delta_{1i} h_1^{n+1} = M_{ij} h^n{}_j + \frac{q^{(0)} L_c}{n_{por} \sigma_e} \delta_{1i} h_1^n$$

$$+ \Delta t \, b_i^n - \Delta t \frac{\sqrt{g} \, q^{(0)}}{n_{por} \sigma_e} \max\left(\frac{2}{3} h^n{}_1, 0\right)^{\frac{2}{3}}$$

it remains to set $q^{(0)} = 1$.