Numerical Techniques Exercise 1

1a)

Simplify

$$\partial_t b + \partial_z(u\,b) = 0, \qquad u = \alpha\, b^2 - \beta\, b^2\, \partial_z b \quad \text{with} \quad z \in [0, H], \tag{1}$$

to

$$\partial_t b + \partial_z(\alpha\, b^3 - \beta\, b^3\, \partial_z b) = 0. \tag{5}$$

Working:

a) Simplyfy system

$$\partial_t b + \partial_z (u b) = 0$$

$$ub = \alpha b^3 - \beta b^3 \partial_z b$$

$$\boxed{\partial_t b + \partial_z (\alpha b^3 - \beta b^3 \partial_z b) = 0}$$

Linearize by subbing in b=D_o + b'

$$\frac{\partial}{\partial t}(D_0 + b') + \frac{\partial}{\partial z}\left(\alpha (D_0 + b')^3 - \beta (D_0 + b')^3 \partial_z (D_0 + b')\right) = 0$$

Time derivative

$\frac{\partial}{\partial t} D_0 = 0$ as constant so this term is

$$\boxed{\partial_t b'}$$

Spatial derivatives

$$\frac{\partial}{\partial z}\left(\alpha (D_0 + b')^3\right)$$

$$= \alpha \frac{\partial}{\partial z}(D_0 + b')^3$$

$$= \alpha\left[\left(3\underbrace{(D_0 + b')^2}_{\approx D_0^2}\right) \times \partial_z b'\right]$$

D_0^2 as terms small $\left(D_0^2 + \cancel{2D_0 b' + b'^2}\right)\partial_z b'$

$$= 3\alpha D_0^2 \, \partial_z b'$$

$$\frac{\partial}{\partial z}\left(-\beta (D_0 + b')^3 \, \partial_z \overset{(D_0+b')}{b}\right)$$

expand out, $b' \partial_{zz} b'$ $b'^2 \partial_{zz} b'$ and $b'^3 \partial_{zz} b'$ are ~~const.~~ small, negligible so are neglected

As b' is a small perturbation (b'<<1), b' taken to a power (b'^2,b'^3 etc..) is taken to be negligible. Its derivatives are also taken to be negligible. Therefore the equation simplifies to

$$\boxed{\partial_t b' + \underbrace{3\alpha D_0^2 \partial_z b'}_{\text{Convective term}} \underbrace{- \beta D_0^3 \partial_{zz} b'}_{\text{Diffusive term}} = 0} \tag{6}$$

These are convection-diffusion equations as have both convection and diffusion terms.

b) i) Discretize equation (6)

Taylor expansions are used to find approximations for δt, δz and δzz.

δt:

b_j^n is b at time $t = n$, $z = j$

~~dt~~ for small time step Δt, taylor series expansion

$$b(z, t+\Delta t) = b(z,t) + \Delta t \frac{\partial b}{\partial t}(z,t) + \frac{1}{2}\Delta t^2 \frac{\partial^2 b}{\partial t^2}(z,t) + \frac{1}{6}\Delta t^3 \frac{\partial^3 b}{\partial t^3}(z,t) + \dots$$

$$b(z, t+\Delta t) = b(z,t) + \Delta t \frac{\partial b}{\partial t}(z,t) + \underbrace{O(\Delta t^2)}_{\text{error}}$$

$$b(z, t+\Delta t) \approx b(z,t) + \Delta t \frac{\partial b}{\partial t}(z,t)$$

$$\frac{\partial b}{\partial t} \approx \frac{b(z, t+\Delta t) - b(z,t)}{\Delta t}$$

Changing notation to make easier to write:

for $t = n$ z varies by Δz

$t + \Delta t \Rightarrow n+1$ & $z + \Delta z = j+1$

$$\boxed{\frac{\partial b}{\partial t} \approx \frac{b_j^{n+1} - b_j^n}{\Delta t}}$$

For δz a similar method is used, scheme is upwind so information from j-1 is used

upwind scheme

$$\partial_t b + \partial_z (\alpha b^3) = 0$$

$\Downarrow$ diffn

$$\partial_t b + 3 b^2 \alpha \frac{\partial b}{\partial z} = 0$$

posit

$$\frac{\partial u}{\partial t} + 1 \times \frac{\partial u}{\partial x} = 0$$

information behind

need to use $j-1$

$$b^n_{j-1} = b^n_j - \Delta z \frac{\partial}{\partial z}\left(b^n_j\right) + \underbrace{\frac{1}{2} \Delta z^2 \frac{\partial^2}{\partial z^2}\left(b^n_j\right) + \cdots}_{O(\Delta z^2)}$$

$$\boxed{\frac{\partial b}{\partial z} \approx \frac{b^n_j - b^n_{j-1}}{\Delta z}}$$

For δzz the forward and backward differences are summed in order to find the central difference approximation central difference scheme for $\frac{\partial^2}{\partial z^2}$

Taylor expansions forward and backward

$$b_{j-1}^{n} = b_j^{n} - \Delta z \frac{\partial}{\partial z}\left(b_j^{n}\right) + \frac{1}{2}\Delta z^2 \frac{\partial^2}{\partial z^2}\left(b_j^{n}\right) + O(\Delta z)^3$$

$$b_{j+1}^{n} = b_j^{n} + \Delta z \frac{\partial}{\partial z}\left(b_j^{n}\right) + \frac{1}{2}\Delta z^2 \frac{\partial^2}{\partial z^2}\left(b_j^{n}\right) + O^3$$

$$b_{j-1}^{n} + b_{j+1}^{n} = 2b_j^{n} + \Delta z^2 \frac{\partial^2}{\partial z^2}\left(b_j^{n}\right)$$

$$\boxed{\frac{\partial^2}{\partial z^2} b \approx \frac{b_{j-1}^{n} + b_{j+1}^{n} - 2b_j^{n}}{\Delta z^2}}$$

Subbing these approximations into the equation gives

$$\frac{b_j^{n+1} - b_j^{n}}{\Delta t} + \left(3\alpha D_0^2\right)\frac{b_j^{n} - b_{j-1}^{n}}{\Delta z} - \left(\beta D_0^3\right)\frac{b_{j-1}^{n} - 2b_j^{n} + b_{j+1}^{n}}{\Delta z^2} = 0$$

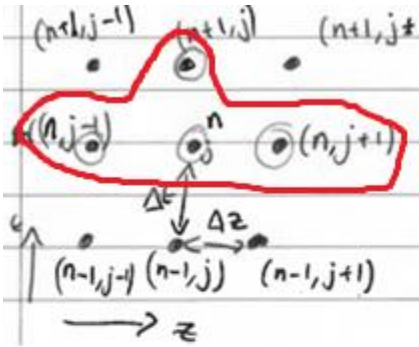
Which can be rearranged for b(n+1,j) and written in the form

$$b_j^{n+1} = b_j^{n} - \mu\left(3\alpha D_0^2 \Delta z\left(b_j^{n} - b_{j-1}^{n}\right) - \beta D_0^3\left(b_{j-1}^{n} - 2b_j^{n} + b_{j+1}^{n}\right)\right)$$

Where

$$\mu = \frac{\Delta t}{\Delta z^2}$$

This is an explicit scheme so values of b at timestep n+1 only depend on values at timestep n



explicit b_j^{n+1} depends on values of b^n

J+1 points

J = no of grid points in x

$$\Delta x = \frac{H}{J}$$

Boundary conditions:

Boundary conditions Dirichlet

$$\left.\begin{array}{l} b(0,t) = b_B \\ b(H,t)\ b_T \end{array}\right\}$$

For j = 0, j = J $b = b_T$

For j = 1 ..... J-1 use equation

initial conditions $b(z,0) = b_i(z)$

b) ii) Discretize equation 5 (non-linearised)

$$\partial_t b + \partial_z \left(\alpha b^3 - \beta b^3 \partial_z b \right) = 0$$

$$\underset{①}{\frac{\partial b}{\partial t}} + \underset{②}{\frac{\partial}{\partial z}\left(\alpha b^3\right)} - \underset{③}{\frac{\partial}{\partial z}\left(\beta b^3 \frac{\partial b}{\partial z}\right)} = 0$$

Time derivative, forward euler – same as before :

① $\frac{\partial b}{\partial t}$, as before $\Rightarrow \frac{\partial b}{\partial t} \approx \frac{b_j^{n+1} - b_j^n}{\Delta t}$

Convective term:

② $\frac{\partial}{\partial z}(\alpha b^3) = \alpha \frac{\partial}{\partial z}(b^3) = 3\alpha b^2 \frac{\partial b}{\partial z}$

$$\Rightarrow 3\alpha (b_j^n)^2 \left(\frac{b_j^n - b_{j-1}^n}{\Delta z} \right)$$

Same discretisation for $\frac{\partial b}{\partial z}$ as before.

Diffusive term:

AS GIVEN IN TEXTBOOK

Adjoint form of $\frac{\partial}{\partial z}\left(\beta b^3 \frac{\partial b}{\partial z}\right) = \beta \frac{\partial}{\partial z}\left(b^3 \frac{\partial b}{\partial z}\right)$

$\Rightarrow$ if P is the function ~~of~~ b^3 $\Rightarrow P = b^3$

then

$$\left[P \frac{\partial b}{\partial z}\right]^n_{j+\frac{1}{2}} \approx \left(P^n_{j+\frac{1}{2}}\right)\left(\frac{b^n_{j+1} - b^n_j}{\Delta z}\right)$$

time is constant

half step in space.

$$\left[P \frac{\partial b}{\partial z}\right]^n_{j-\frac{1}{2}} \approx \left(P^n_{j-\frac{1}{2}}\right)\left(\frac{b^n_j - b^n_{j-1}}{\Delta z}\right)$$

$j-\frac{1}{2}$ j $j+\frac{1}{2}$

Δz

to find $\left[P \frac{\partial b}{\partial z}\right]^n_j \approx \dfrac{\left[P\frac{\partial b}{\partial z}\right]^n_{j+\frac{1}{2}} - \left[P\frac{\partial b}{\partial z}\right]^n_{j-\frac{1}{2}}}{\Delta z}$

$$\left[P \frac{\partial b}{\partial z}\right]^n_j \approx \frac{1}{\Delta z^2}\left(\left(P^n_{j+\frac{1}{2}}\right)\left(b^n_{j+1} - b^n_j\right) - \left(P^n_{j-\frac{1}{2}}\right)\left(b^n_j - b^n_{j-1}\right)\right)$$

$P^n_{j+\frac{1}{2}} = \left(b^3\right)^n_{j+\frac{1}{2}}$ $\rightarrow$ approximate as the average of b^{3n}_j and b^{3n}_{j+1}

$$\Rightarrow \frac{\left(b^n_j\right)^3 + \left(b^n_{j+1}\right)^3}{2}$$

$$P^n_{j-\frac{1}{2}} = \left(b^3\right)^n_{j-\frac{1}{2}} \Rightarrow \frac{\left(b^n_j\right)^3 + \left(b^n_{j-1}\right)^3}{2}$$

sub this back in

$$\left[b^2 \frac{\partial b}{\partial z}\right]_j^n \approx \frac{1}{2\Delta z^2}\left[\left((b_j^n)^3 + (b_{j+1}^n)^3\right)(b_{j+1}^n - b_j^n) - \left((b_j^n)^3 + (b_{j-1}^n)^3\right)(b_j^n - b_{j-1}^n)\right]$$

Full discretization:

Full discretization

$$b_j^{n+1} = b_j^n - \Delta t\left(3\alpha (b_j^n)^2 \left(\frac{b_j^n - b_{j-1}^n}{\Delta z}\right) - \frac{\beta}{2\Delta z^2}\left[\cdots\right]\right)$$

$$b_j^{n+1} = b_j - \frac{\Delta t}{\Delta z^2}\left(\cancel{\Delta z}\, 3\alpha (b_j^n)^2 (b_j^n - b_{j-1}^n) - \frac{1}{2}\beta\left[\cdots\right]\right)$$

$$\frac{\Delta t}{\Delta z^2} = \mu$$

$$\cancel{b_j^{n+1} = b_j - 3\mu\alpha (b_j)^2 (b_j^n - b_{j-1}^n) -}$$

$$\boxed{b_j^{n+1} = b_j^n - 3\Delta z \mu \alpha (b_j^n)^2 (b_j^n - b_{j-1}^n) + \frac{1}{2}\mu\beta\left[\left((b_j^{n3} + b_{j+1}^{n3})(b_{j+1}^n - b_j^n)\right) - \left((b_j^{n3} + b_{j-1}^{n3})(b_j^n - b_{j-1}^n)\right)\right]}$$

Same bc's as before

$b(0,t) = b_B \qquad b(H,t) = b_T$

ie when $j=0 \qquad$ when $j=J$

Solve equations for middle points

$b(z,0) = b_i(z)$, ie when $t=0$

c) Fourier analysis for stability for linearised discretisation:

Form for b is subbed into the discretised equation to give

$$b_j^n = \lambda^n e^{ikj\Delta z} \quad \Rightarrow \quad b_j^{n+1} = \lambda b_j^n \quad \text{and} \quad b_{j+1}^n = e^{ik\Delta z} b_j^n$$

(sub this into discretised version of eq 6

$$b_j^{n+1} = b_j^n \mp \mu \left(3\alpha D_0^2 \Delta z \left(b_j^n - b_{j-1}^n\right) - \beta D_0^3 \left(b_{j-1}^n - 2b_j^n + b_{j+1}^n\right)\right)$$

$$\lambda b_j^n = b_j^n \mp \mu (3\alpha D_0^2 \Delta z \left(b_j^n - e^{-ik\Delta z} b_j^n\right) - \beta D_0^3$$

$$\left(b_j^n e^{-ik\Delta z} - 2b_j^n + b_j^n e^{ik\Delta x}\right)$$

$$\lambda = 1 \mp \mu \left(3\alpha D_0^2 \Delta z \left(1 - e^{-ik\Delta z}\right)\right) - \beta D_0^3 \left(-2\right.$$

$$\left. + e^{-ik\Delta z} + e^{ik\Delta x}\right)$$

$$\lambda = 1 \mp \mu \left(3\alpha D_0^2 \Delta z \left(1 - e^{-ik\Delta z}\right) - 2\beta D_0^3 \left(-1\right.\right.$$

$$\left.\left. + \cos k\Delta z\right)\right.$$

For stability the $|\lambda| < 1$ as if it is greater than 1, λ^n will increase every timestep and 'blow up' -> the solution will be unstable.

In the case where alpha = 0 and beta is non zero

* if $\alpha = 0$ and $\beta \neq 0$

$$\lambda = 1 + 2\mu\beta D_0^3 (\cos k\Delta z - 1)$$

$|\lambda| < 1$ for stability as $b_j^n = \lambda^n e^{ikj\Delta x}$

$$\left|1 + 2\mu\beta D_0^3(\cos k\Delta x - 1)\right| < 1$$

$$\sin^2\tfrac{1}{2}\theta = \tfrac{1}{2}(1-\cos\theta)$$

$$1 + 2\mu\beta D_0^3(\cos k\Delta x - 1)$$

$$= 1 - 4\mu\beta D_0^3\left(\tfrac{1}{2}(1-\cos k\Delta x)\right)$$

$$\Rightarrow \left|1 - 4\mu\beta D_0^3 \sin^2\left(\tfrac{1}{2}k\Delta x\right)\right| < 1$$

$\sin^2$ is +ve so $1 - 4\mu\beta D_0^3 \sin^2$ - - - is always < 1

want $1 - 4\mu\beta D_0^3 \sin^2\left(\tfrac{1}{2}k\Delta x\right) > -1$

$$4\mu\beta D_0^3 \sin^2\left(\tfrac{1}{2}k\Delta x\right) < 2$$

$$\mu\beta D_0^3 \sin^2\left(\tfrac{1}{2}k\Delta x\right) < \tfrac{1}{2}$$

max value of $\sin^2\left(\tfrac{1}{2}k\Delta x\right)$ is 1

in that case

$$\mu\beta D_0^3 < \tfrac{1}{2}$$

$$\mu < \frac{1}{2\beta D_0^3}$$

$$\frac{\Delta t}{\Delta z^2} < \frac{1}{2}\beta^{-1} D_0^{-3}$$

$$\Delta t < \frac{1}{2}\beta^{-1} D_0^{-3}\Delta z^2$$

Max time step $\Delta t = \frac{1}{2}\beta^{-1} D_0^{-3}\Delta z^2$

In the case that alpha is non zero and beta is 0:

$$\frac{\Delta t}{\Delta z^2} < \frac{1}{2}\beta D_0^3$$

$$\Delta t < \frac{1}{2}\beta D_0^3 \Delta z^2$$

max time step $\Delta t = \frac{1}{2}\beta D_0^3 \Delta z^2$

~~D~~ $\alpha \neq 0 \quad \beta = 0$

$$\lambda = 1 - \underbrace{3\alpha\mu D_0^2 \Delta z}_{A} \left(1 - e^{-i\overbrace{k\Delta z}^{\theta}}\right)$$

$$\lambda = 1 - A\left(1 - e^{-i\theta}\right)$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\lambda = 1 - A + A\cos\theta - iA\sin\theta$$

take modulus, as want $|\lambda| < 1$

$$\sqrt{(1 - A + A\cos\theta)^2 + (-A\sin\theta)^2} < 1$$

both brackets will be positive as they are ~~squared~~ SQUARED

$$\Rightarrow (1 - A + A\cos\theta)^2 + A^2\sin^2\theta < 1$$

Multiplying out bracket

$(1 - A + A\cos\theta)^2$

$= 1 - 2A + A^2 + 2A\cos\theta - 2A^2\cos\theta + A^2\cos^2\theta$

sub back in

$1 - 2A + A^2 + 2A\cos\theta - 2A^2\cos\theta + A^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} <$

$\cancel{1} + 2A^2 - 2A + 2A\cos\theta - 2A^2\cos\theta < \cancel{1}$

$\cancel{2}A^2 - \cancel{2}A + \cancel{2}A\cos\theta - \cancel{2}A^2\cos\theta < 0$

$A^2 - A + A\cos\theta - A^2\cos\theta < 0$

$(A^2 - A)(1 - \cos\theta) < 0$

$1 - \cos\theta = 2\sin^2\frac{1}{2}\theta$

$(A^2 - A)\,2\sin^2\frac{1}{2}\theta < 0$

if $\sin^2\frac{1}{2}\theta = 0$. trivial result

if $\sin^2\frac{1}{2}\theta = 1$

$A^2 - A < 0$

$A(A - 1) < 0$

$A < 1$

$$\mu = \frac{\Delta t}{\Delta z^2}$$

$$A = 3\alpha \mu D_0^2 \Delta z \quad \Rightarrow \quad 3 \frac{\Delta t}{\Delta z^2} \alpha D_0^2 \cancel{\Delta z}$$

$$\frac{3\Delta t}{\Delta z} \alpha D_0^2 < 1$$

$$\Delta t < \frac{1}{3\alpha D_0^2} \Delta z$$

max time step $\Delta t = \frac{1}{3\alpha D_0^2} \Delta z$

d) Use the maximum principle to determine a suitable timestep for the discretisation of the linearised equation

d) Maximum principle to determine a stable time step for (6)

Discretization of 6 =>

$$b_j^{n+1} = b_j^n - \mu\left(3\alpha D_0^2 \Delta z\left(b_j^n - b_{j-1}^n\right) - \beta D_0^3\left(b_{j-1}^n - 2b_j^n + b_{j+1}^n\right)\right)$$

Rearrange to find the weighting for each ~~term:~~ gridpoint => b_j^n, b_{j+1}^n, b_{j-1}^n

$$b_j^{n+1} = b_j^n - \mu 3\alpha D_0^2 \Delta z\, b_j^n - \mu\beta D_0^3 2 b_j^n + \mu 3\alpha D_0^2 \Delta z\, b_{j-1}^n + \mu\beta D_0^3 b_{j-1}^n + \mu\beta D_0^3 b_{j+1}^n$$

(1) Weighting for b_j^n = $1 - 3\alpha\mu D_0^2 \Delta z - 2\mu\beta D_0^3$

(2) Weighting for b_{j-1}^n = $3\alpha\mu D_0^2\Delta z + \mu\beta D_0^3$

(3) Weighting for b_{j+1}^n = $\mu\beta D_0^3$

α, μ, D_0, β are all +ve ~~values~~

From maximum principle, want b_j^{n+1} to be the maximum value. For this to hold we need the weightings to be non negative. (2) and (3) will always be positive as coefficients are non-negative.

Therefore want weighting ① > 0

$$1 - 3\alpha D_0^2 \mu \Delta z - 2\beta D_0^3 \mu > 0$$

$$3\alpha D_0^2 \mu \Delta z + 2\beta D_0^3 \mu < 1$$

$$\mu \left(3\alpha D_0^2 \Delta z + 2\beta D_0^3\right) < 1$$

$$\mu < \frac{1}{3\alpha D_0^2 \Delta z + 2\beta D_0^3}$$

$$\frac{\Delta t}{(\Delta z)^2} < \frac{1}{3\alpha D_0^2 \Delta z + 2\beta D_0^3}$$

$$\boxed{\Delta t < \frac{(\Delta z)^2}{3\alpha D_0^2 \Delta z + 2\beta D_0^3}}$$

e) Derive stable time-step for non-linear example

$$b_j^{n+1} = b_j^n - 3\Delta z \mu \alpha \underbrace{(b_j^n)^2}_{A}(b_j^n - b_{j-1}^n) + \frac{1}{2}\mu\beta \Big[\underbrace{\left((b_j^{n})^3 + (b_{j+1}^{n})^3\right)}_{B}(b_{j+1}^n - b_j^n) - \underbrace{\left((b_j^{n})^3 + (b_{j-1}^{n})^3\right)}_{C}(b_j^n - b_{j-1}^n)\Big]$$

WANT TO EXPAND DISCRETIZATION TO FIND +VE AND -VE TERMS

(A) $-3\Delta z\mu\alpha\left(b_j^n\right)^2\left(b_j^n - b_{j-1}^n\right) \Rightarrow +3\Delta z\mu\alpha\left(b_j^n\right)^2\left(b_{j-1}^n\right)$
$$- 3\Delta z\mu\alpha\left(b_j^n\right)^3$$

(B) $+\frac{1}{2}\mu\beta\left(b_j^{n3} + b_{j+1}^{n\,3}\right)\left(b_{j+1}^n - b_j^n\right)$

$$= \frac{1}{2}\mu\beta\left(b_j^{n3}b_{j+1}^n + b_{j+1}^{n\,4}\right)$$
$$- \frac{1}{2}\mu\beta b_j\left(b_j^3 + b_{j+1}^{n\,3}\right)$$

(C) $-\frac{1}{2}\mu\beta\left(b_j^{n3} + b_{j-1}^{n\,3}\right)\left(b_j^n - b_{j-1}^n\right)$

$$= -\frac{1}{2}\mu\beta\left(b_j^{n4} + b_j^n b_{j-1}^{n3}\right)$$
$$+ \frac{1}{2}\mu\beta\left(b_j^{n3}b_{j-1}^n + b_{j-1}^{n\,4}\right)$$

$$\Rightarrow -\frac{1}{2}\mu\beta b_j^n\left(b_j^{n\,3} + b_{j-1}^{n\,3}\right)$$
$$+ \frac{1}{2}\mu\beta\left(b_j^{n3}b_{j-1}^n + b_{j-1}^{n\,4}\right)$$

Full discretization $= b_j^{n+1} = b_j^n +$ (A) $+$ (B) $+$ (C)

The ~~value~~ positive terms within (A) (B) and (C)

will always be positive, as coefficients are positive
$\Delta z,\ \mu,\ \alpha \geq 0$ so if $B_j^n > 0$, these terms are positive

If b_j^n − negative terms of, (A), (B), (C) > 0

then $b_j^{n+1} > 0$ will always hold.

so =>

$$b_j^n - 3\Delta z \mu \alpha (b_j^n)^3 - \frac{1}{2}\mu\beta b_j (b_j^3 + b_{j+1}^{n\,3} + b_j^3 + b_{j-1}^3) > 0$$

÷ by b_j^n

$$1 > 3\Delta z \mu \alpha (b_j^n)^2 + \frac{1}{2}\mu\beta (2b_j^3 + b_{j+1}^{n\,3} + b_{j-1}^3)$$

$$\mu < \frac{1}{3\Delta z \alpha (b_j^n)^2 + \frac{1}{2}\beta (2b_j^3 + b_{j+1}^3 + b_{j-1}^3)}$$

$$\mu < \frac{2}{6\Delta z \alpha (b_j^n)^2 + \beta (2b_j^{n\,3} + b_{j+1}^{n\,3} + b_{j-1}^{n\,3})}$$

$$\mu = \frac{\Delta t}{\Delta z^2}$$

$$\boxed{\Delta t < \frac{2\Delta z^2}{6\Delta z \alpha (b_j^n)^2 + \beta (2b_j^{n\,3} + b_{j+1}^{n\,3} + b_{j-1}^{n\,3})}}$$

2)a) **Steady state solution of equation**

2a)

Show that steady state solution of

$$\partial_t b + \partial_z (\alpha b^3 - \beta b^3 \partial_z b) = 0$$

satisfies $$\beta b^3 \frac{db}{dz} = \alpha b^2 - Q$$

· Steady state => $\partial_t b = 0$

b is ~~a~~ solely a function of z as t = constant

so partial derivative $\partial_z \Rightarrow \frac{d}{dz}$

$$\frac{d}{dz}\left(\alpha b^3 - \beta b^3 \frac{db}{dz}\right) = 0$$

~~ingreate~~ integrate wrt z

=> $$\alpha b^3 - \beta b^3 \frac{db}{dz} = Q$$ Q = some constant

$$\beta b^3 \frac{db}{dz} = \alpha b^3 - Q$$

Euler forward spacial discretisation $\Rightarrow$

$$\frac{db}{dz} \approx \frac{b_{j+1} - b_j}{\Delta z}$$

Sub in to equation

$$\beta b_j^3 \left(\frac{b_{j+1} - b_j}{\Delta z} \right) = \alpha b_j^3 - Q$$

$$\frac{b_{j+1} - b_j}{\Delta z} = \frac{\alpha}{\beta} - \frac{Q}{\beta b_j^3}$$

$$b_{j+1} = b_j + \Delta z \left(\frac{\alpha}{\beta} - \frac{Q}{\beta b_j^3} \right).$$

This solution has been programmed in the file ‘steadystate.py’ and gives the following result for a gridsize of J=10000, delta_z =0.0001

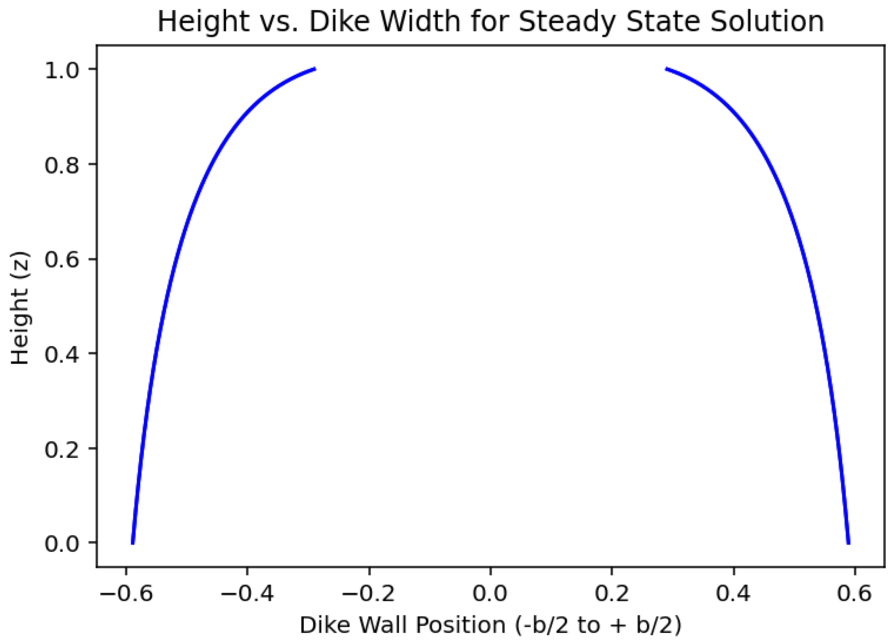


Figure 1: Height vs. dike width for steady state solution, 10000 gridpoints used

b) Non-linear solver programmed within 'nonlinearsolver.py'. The criteria from part e) was used to pick a suitable timestep for each iteration. For each iteration the maximum time step was calculated for each value of j (as maximum timestep is dependant on B(j, n), B(j+1,n) etc.) and the smallest value chosen in order to ensure stability for all points. Plots were generated for the specified grid sizes (J= number of points) and times (t):

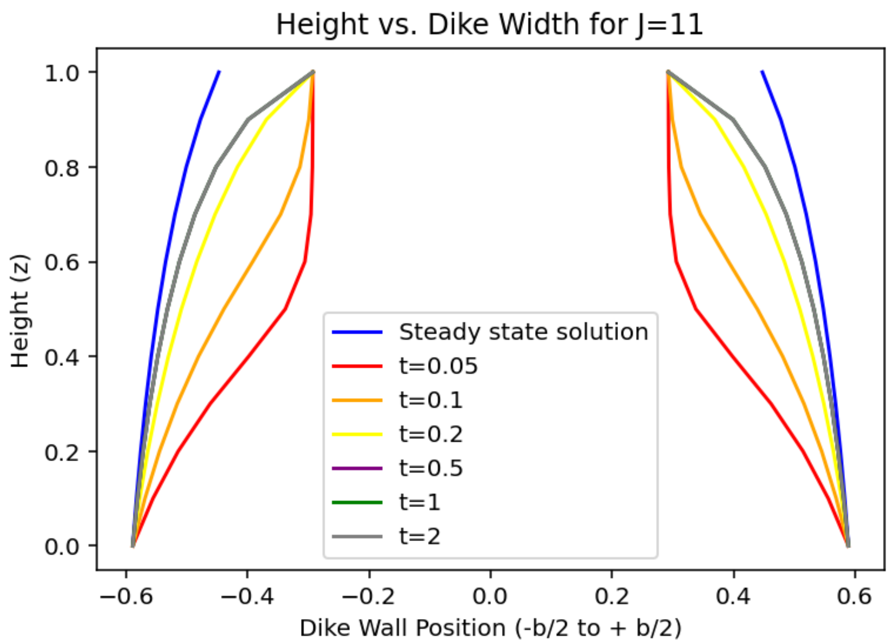


Figure 2: Height vs. dike width for steady state and time dependant solutions, J=11

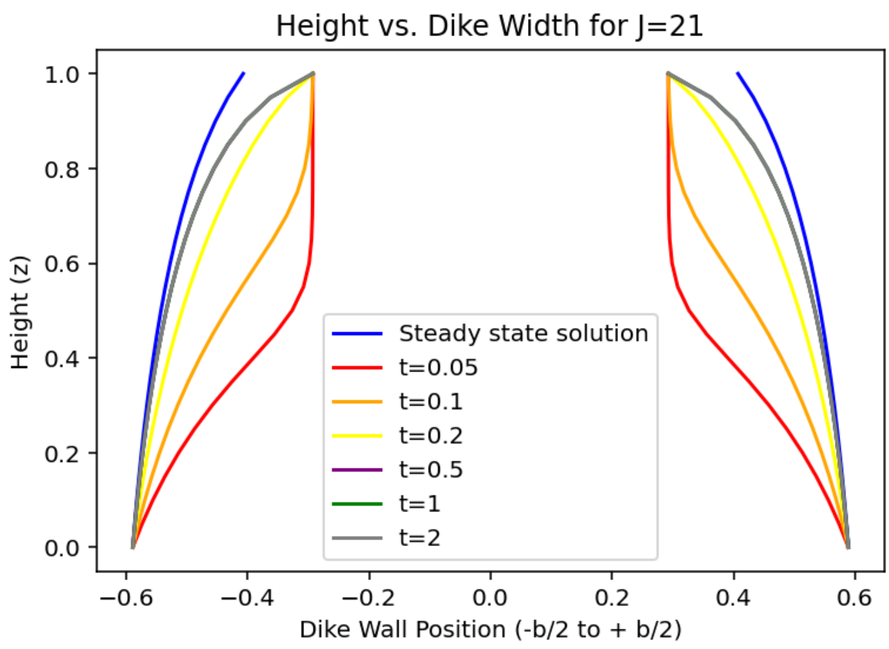


Figure 3: Height vs. dike width for steady state and time dependant solutions, J=21

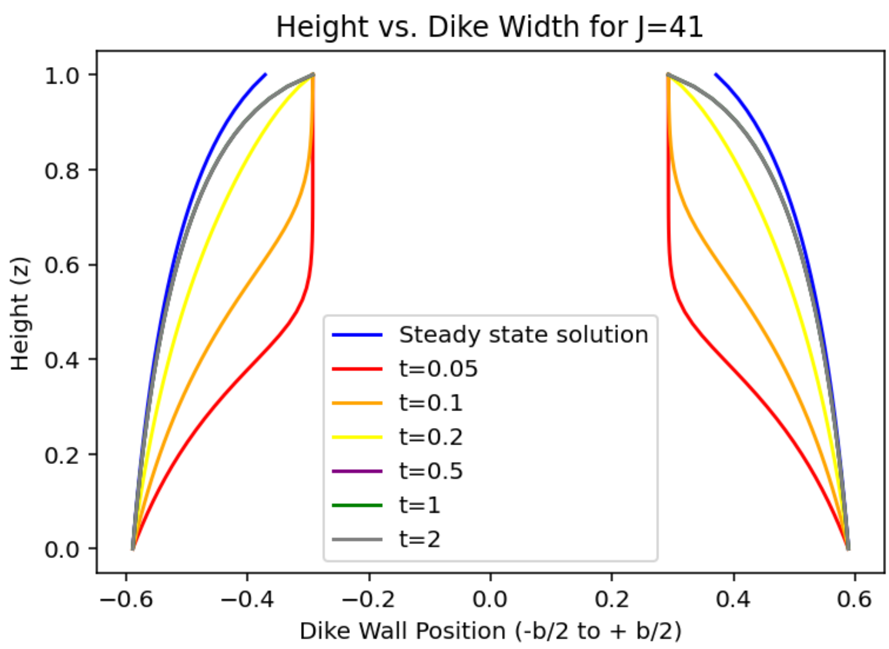


Figure 4: Height vs. dike width for steady state and time dependant solutions, J=41

As the time increases, the time dependant solution approaches the steady state solution, with results for t=0.5,1 and 2 being indistinguishable on the plot. **Figure 5** shows the results for a grid size of 41 used for the time dependant result, against a higher resolution steady state result with the number of grid points in steady state simulation (J_ss) equal to 10000, as in 2a. A higher resolution steady state solution is closer to the time dependant solution.

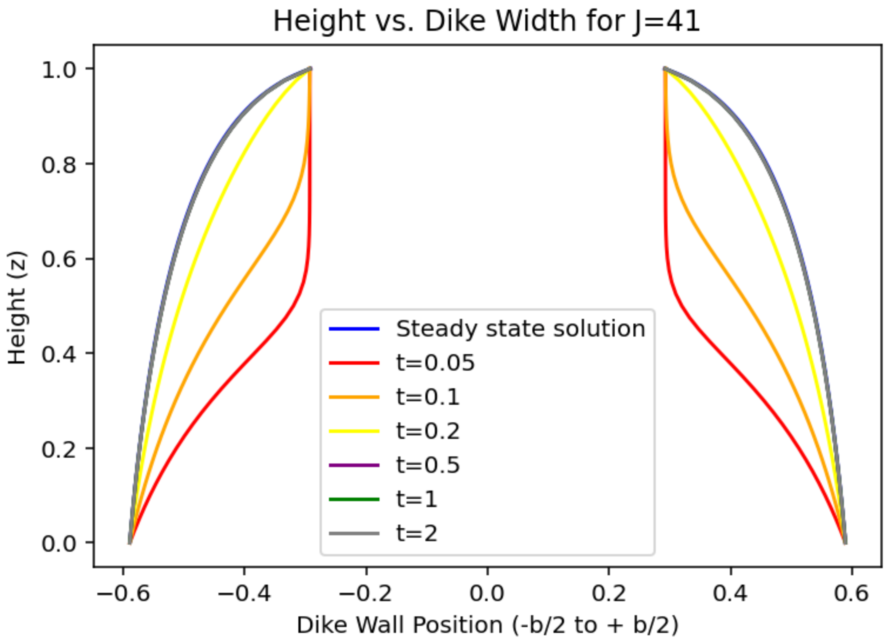


Figure 5: Height vs. dike width for steady state and time dependant solutions, with high resolution steady state solution (J_ss=10000)

c) Code used for these solutions is in the files ‘l2norm.py’ and ‘question5c.py’. Error was calculated between the time-dependant solution (calculated at t=2) and the steady-state solution, calculated using J_ss=100001 grid points. I observed that for coarser grids in the time dependant solution (J=11,21) the error decreased at the last point, whereas for finer meshes (J=41) it increased, as shown in **Figures 6 and 7.**

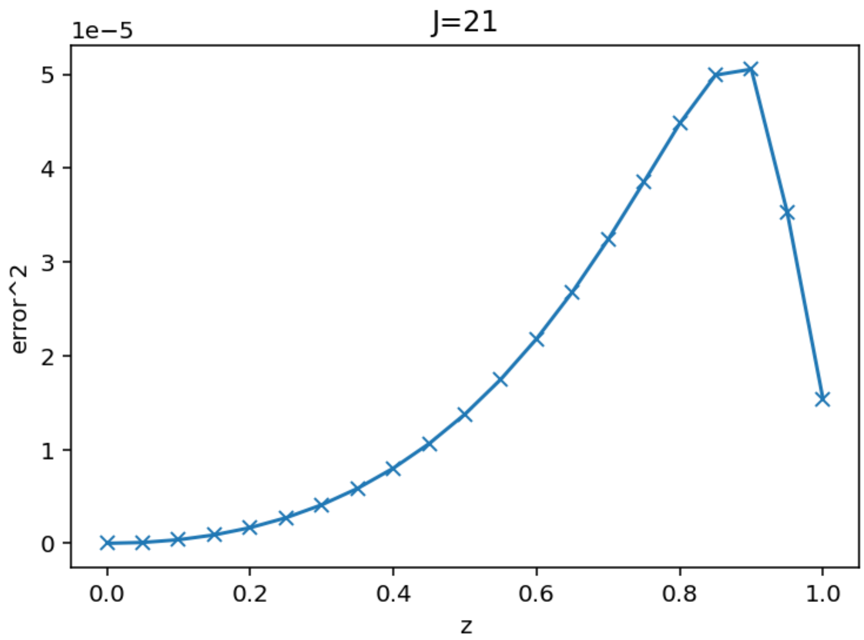


Figure 6: Error^2 vs Z for a grid of 21 points used for solution at t=2, b_t = 0.585373798

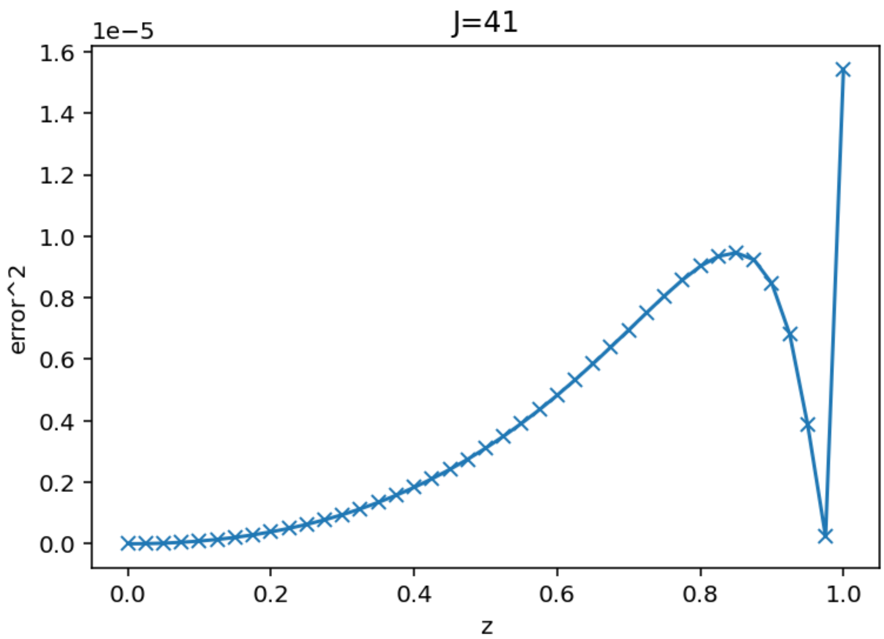


Figure 7: Error^2 vs Z for a grid of 41 points used for solution at t=2, b_t = 0.585373798

I concluded this was to do with the boundary condition imposed, b(H,t)=b_t. I therefore changed my value of b_t to the final value in my steady state solution, b_t = 0.58144799. From this I found the results shown in **Figure 7** and **8**. The graphs now follow the same behaviour: error increases with iterations (Z increasing) however decreases close to the boundary condition. However, through comparing these graphs, it can be seen that the error for earlier points has increased for the new value of b_t. I decided to calculate the L^2 norm using the revised values for b_t, as I will be more accurately able to estimate the integral of a function without jumps. This is a possible area to be refined.

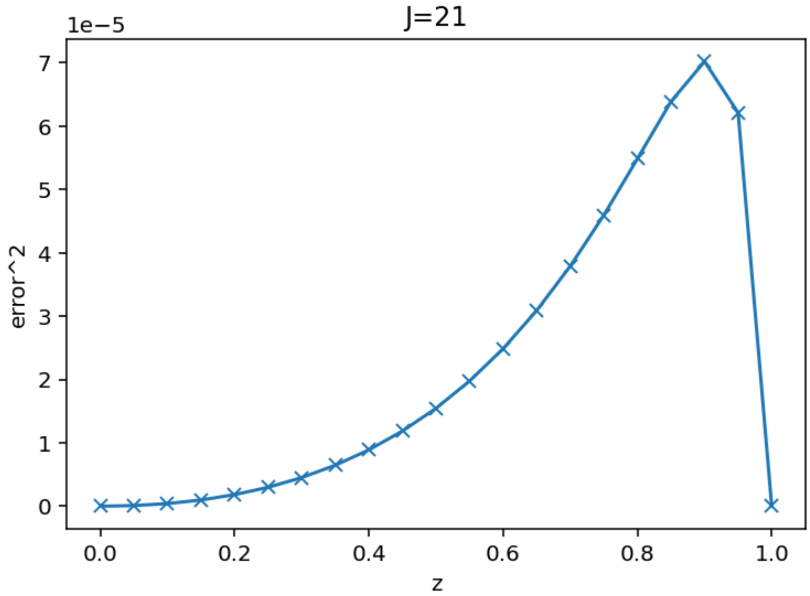


Figure 7: Error^2 vs Z for a grid of 21 points used for solution at t=2, b_t = 0.58144799

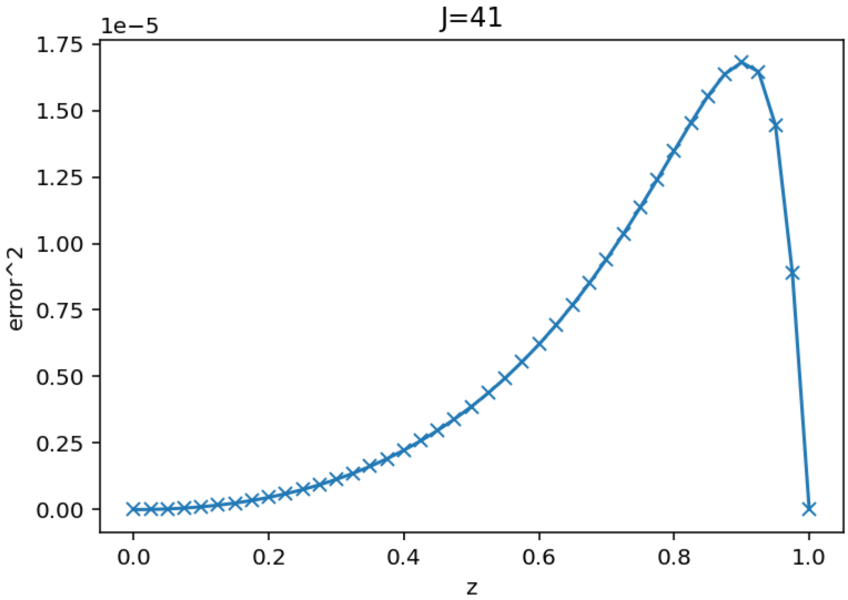
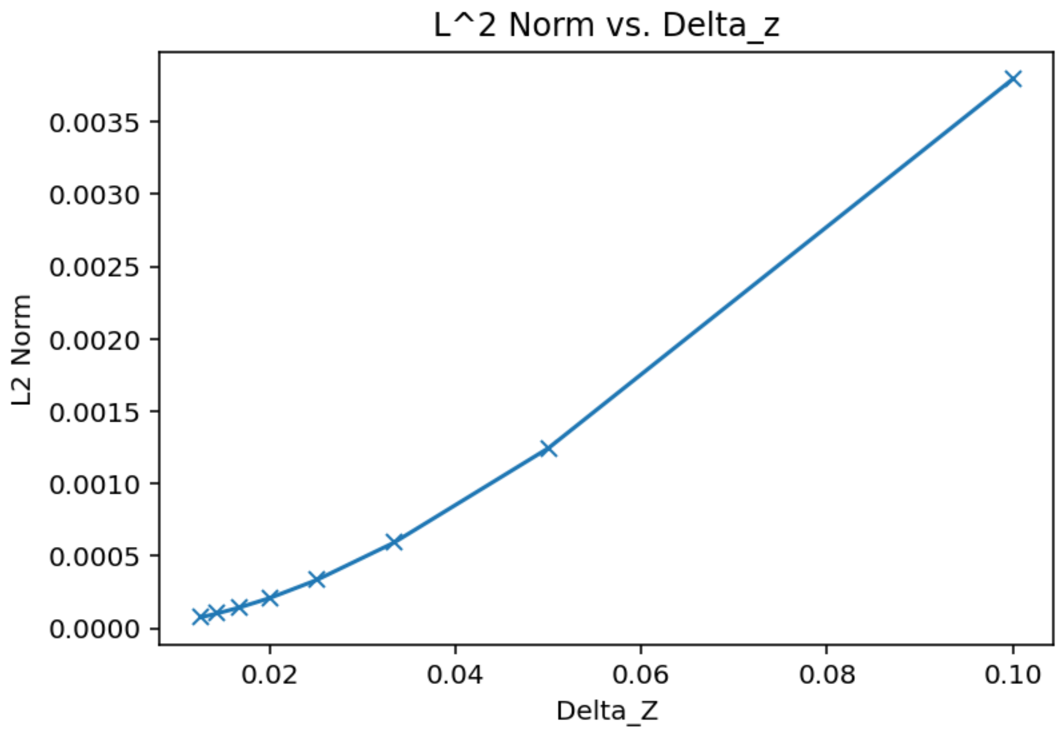


Figure 8: Error^2 vs Z for a grid of 41 points used for solution at t=2, b_t = 0.58144799

The L^2 norm was calculated using Equation 11, with the area under the graphs estimated using the trapezoid rule.

$$L^2 = \sqrt{\int_0^H e^2(z,t)\,\mathrm{d}z} \tag{11}$$

The l^2 norm was calculated for J=11,21,31,41,51,61,71,81 , corresponding to delta_z values of 0.0125 to 0.1.



So what is the order of numerical accuracy?
-1

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