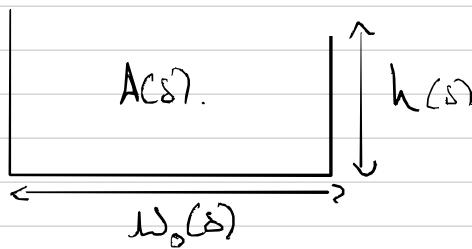


Numerics 2: Alex Parry 2018/09/27.

- Given some cross section



We define the relationship

$$A = h \cdot W_0$$

such that h is given as a function of A, W_0

$$\text{for } A/W_0(s)$$

Similarly, the wetted perimeter is given geometrically as

$$P = 2h + W_0 = W_0 + 2A/W_0$$

such that

$$P(A, s) = W_0(s) + 2A/W_0(s).$$

noting A is also a function of s .

We consider

$$\partial_r A + \partial_s F = 0$$

where we have

$$F = \frac{A^{\frac{5}{3}}}{C_m P^{\frac{2}{3}}} \sqrt{-\partial_s b}$$

the flux out δ . We can

$$\partial_s F = \partial_s A \partial_A F + \partial_s F$$

where A is considered independent of s when calculating $\partial_s F$. We compute

$$\partial_A F = \frac{5}{3} \left(\frac{A}{P} \right)^{\frac{2}{3}} \frac{\sqrt{-\partial_s b}}{C_m} - \frac{2}{3} \frac{A^{\frac{5}{3}} \sqrt{\partial_s b}}{P^{\frac{5}{3}}} \frac{\partial_A P}{C_m}$$

where $\partial_A P = 2/\omega_0$ such that

$$\partial_A F = \frac{5}{3} \left(\frac{A}{P} \right)^{\frac{2}{3}} \frac{\sqrt{-\partial_s b}}{C_m} - \frac{4}{3} \left(\frac{A}{P} \right)^{\frac{5}{3}} \frac{\sqrt{\partial_s b}}{C_m \omega_0}$$

$$= \frac{\sqrt{-\partial_s b}}{3C_m} \left(\frac{A}{P} \right)^{\frac{2}{3}} \left[5 - 4 \frac{A}{P \omega_0} \right]$$

$$= \frac{\sqrt{-\partial_s b}}{3C_m} \left(\frac{A}{P} \right)^{\frac{2}{3}} \left[\frac{5\omega_0^2 + 6A}{P \omega_0} \right]$$

$$= \frac{\sqrt{-\partial_s b}}{3C_m} \left[\frac{5A^{\frac{2}{3}}\omega_0 + 6A^{\frac{5}{3}}\omega_0}{(1 + 2A\omega_0)^{\frac{5}{3}}} \right].$$

We similarly compute

$$\partial_s F = -\frac{2}{3} \left(\frac{A}{P} \right)^{\frac{5}{3}} \frac{\sqrt{1-d_b}}{C_m} \partial_s P$$

where $\partial_s P = \partial_s \omega_0 (1 - 2A/\omega_0^2)$ such
that

$$\partial_s F = -\frac{2}{3} \frac{A}{(1 + 2A/\omega_0^2)^{\frac{5}{3}}} \frac{\sqrt{1-d_b}}{C_m} \left(1 - \frac{2A}{\omega_0^2} \right) \partial_s \omega_0$$

as required. Note that $\partial_s \omega_0$ is only dependent
on s $\partial_s \omega_0 = d\omega_0/ds$.

2) We consider a case where the variation in ω_0 is small, that is

$$\partial_S \omega_0 = 0$$

and we have

$$\partial_t A + \partial_A F \partial_S A = 0 \quad (1)$$

which has eigenvalue $\partial_A F$.

We consider the Riemann problem defined by (1) on some piecewise constant initial conditions

$$A(s, t=0) = \begin{cases} A_L & s < s_{k+\frac{1}{2}} \\ A_r & s \geq s_{k+\frac{1}{2}} \end{cases}$$

towards the computation of some fluxes on the boundary.

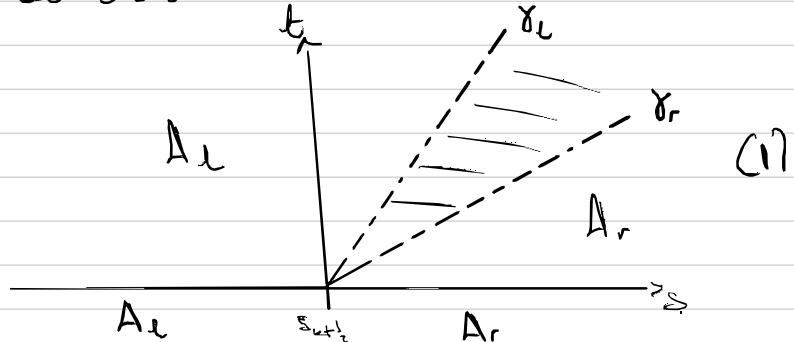
We define characteristics along which the value of A is constant. In this case with

$$\partial_t s = \partial_A F(A, s).$$

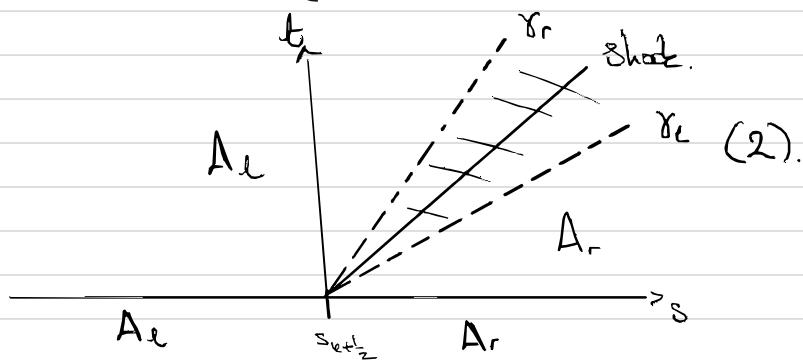
We note that $\partial_A F$ can often similarly take only two values dependent on A

$$(\partial_A F)_L = \partial_A F(A_L), (\partial_A F)_r = \partial_A F(A_r)$$

As $\partial_A F > 0$ we find the following two possible cases



(1)



(2).

where γ_l, γ_r are the lines traced from the origin with gradients $(\partial_A F)_l, (\partial_A F)_r$ respectively.

In case (1) the central fan represents an undeflected rarefaction region where A continuously decreases

$$A = \frac{s}{t}, \quad -0.5 \text{ (not complete solution)}$$

In case (2) the central region is deflected and ends to a shock front along the line.

with gradient

$$\frac{(\partial_A F)_c + (\partial_A F)_r}{2}$$

3) Given

$$\partial_t A + \partial_x F(A, \dot{A}) = 0$$

we consider the integrals in time and space to find

$$\int_{t_n}^{t_{n+1}} \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \partial_t A + \partial_x F \, ds \, dt = 0$$

$$\Rightarrow \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} [A]_{t_n}^{t_{n+1}} \, ds + \int_{t_n}^{t_{n+1}} [F]_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \, dt = 0$$

At this stage we apply the nature of the scheme where \bar{A} is the average of the cell such that we have

$$\Delta s_k [\bar{A}]_{t_n}^{t_{n+1}} + \int_{t_n}^{t_{n+1}} [F]_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \, dt = 0$$

and we have scheme

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta s_k} \int_{t_n}^{t_{n+1}} [F]_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \, dt.$$

4) We consider the flux on the boundaries given

$$F_{k+\frac{1}{2}}, F_{k-\frac{1}{2}}.$$

We note F is dependent on A and that the results of the Ricci's Problem show that the value of A on the boundary is given by

$$A_{k+\frac{1}{2}}(t) = \bar{A}_k^+$$

for $t \in (t_n, t_{n+1})$ and ev such

$$F_{k+\frac{1}{2}}(t) = F(\bar{A}_k^+)$$

on the same time restricted domain.

5. We derive a time step restriction from the fact that

$$\Delta t_n (\partial_t s)_e < \Delta s_e = h_e$$

in dictating the values on the boundary $\partial\Omega^+$. Noting the flux approximation from part 4 we have

$$\begin{aligned} (\partial_t s)_e &= (\partial_A F)_e \\ &= \partial_A F(\tilde{A}_e) \\ &= \lambda_e^\sim \end{aligned}$$

For a single restriction we must have

$$\Delta t^\sim < \min_e \frac{h_e}{|\lambda_e^\sim|}.$$

We further restrict for stability by defining $CFL \in [0, 1]$ w/

$$\Delta t^\sim < CFL \min_e \frac{h_e}{|\lambda_e^\sim|}.$$

6. As the scheme is upwind, we impose a boundary condition on area discharge at $\delta=0$ without it being necessary to impose a condition for $\delta=1$.

$$A(0, t) = A_0(t).$$