

Exercise sheet 3
Part I.

① Consider the Poisson system

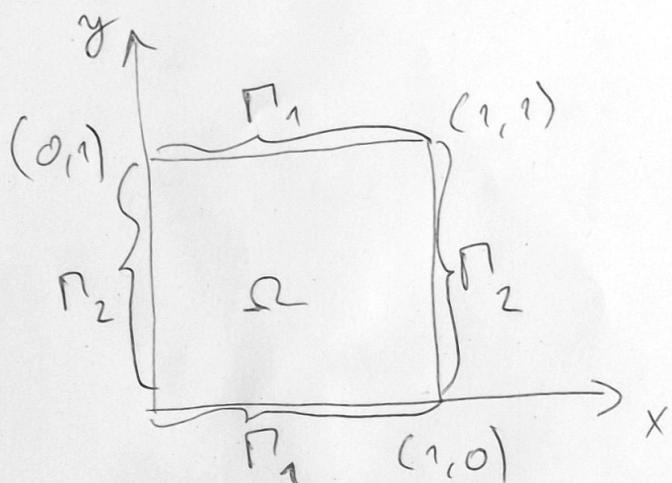
$$-\nabla^2 u = f \text{ on } (x, y) \in [0, 1]^2$$

$$f(x, y) = 2\pi^2 \sin(\pi x) \cos(\pi y)$$

$$u(0, y) = u(1, y) = 0$$

$$\partial_y[u(x, y)] \Big|_{y=0} = \partial_y[u(x, y)] \Big|_{y=1} = 0.$$

Visualise this domain



$$\text{Then } u|_{\Gamma_2} = 0 \text{ & } (\nabla u) \cdot n_1 \Big|_{\Gamma_1} = 0.$$

Now, consider the functional

$$I[u] = \int_{\Omega} \frac{1}{2} |\nabla u|^2 - u f \, d\Omega.$$

①

Introduce variation in u denoted δu

$$s.t. \quad \delta u|_{\partial \Omega} = 0, \text{ where } \partial \Omega = \Gamma_1 \cup \Gamma_2.$$

Then the variation in the functional become

$$\delta I[u] = \frac{dI}{d\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{I[u + \epsilon \delta u] - I[u]}{\epsilon}.$$

Hence,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left(\frac{1}{2} \int_{\Omega} \left(|\nabla [u + \epsilon \delta u]|^2 - |\nabla u|^2 \right) - (u + \epsilon \delta u) f + u f d\Omega \right) \\ &= \lim_{\epsilon \rightarrow 0} \left(\frac{1}{2} \int_{\Omega} \left(|\nabla u|^2 - |\nabla \delta u|^2 + 2\epsilon \nabla \delta u \cdot \nabla u + \epsilon^2 |\nabla \delta u|^2 \right) - \epsilon \delta u f d\Omega \right) \\ &= \int_{\Omega} \nabla \delta u \cdot \nabla u - \delta u f d\Omega \end{aligned}$$

Now, note that

$$\begin{aligned} \nabla \cdot [a \nabla b] &= \nabla a \cdot \nabla b + a \nabla^2 b \\ \Rightarrow \nabla a \cdot \nabla b &= \nabla \cdot [a \nabla b] - a \nabla^2 b. \end{aligned}$$

Hence, the last expression becomes:

$$\begin{aligned} & \int_{\Omega} \nabla \cdot [\delta u \nabla u] - \delta u \nabla^2 u - \delta u f d\Omega \\ &= - \int_{\Omega} \delta u (\nabla^2 u + f) d\Omega + \int_{\Gamma_1} \delta u \nabla u \cdot n_1 d\Gamma \\ &\quad + \int_{\Gamma_2} \delta u \nabla u \cdot n_2 d\Gamma \end{aligned}$$

But we have $\nabla u \cdot \mathbf{n}_1$ is 0 on Γ_1 , and $\delta u = 0$ on Γ_2 and the last two integrals vanish.

Require that the variation of the functional I vanishes to minimize it & we get

$$\int_{\Omega} \delta u (\nabla^2 u + f) d\Omega = 0$$

$\Leftrightarrow -\nabla^2 u = f$ as δu is arbitrary on the interior of Ω .

We required that $\delta u|_{\partial\Omega} = 0$, but it is sufficient to require that $\delta u|_{\Gamma_2} = 0$.

Now derive weak formulation for

$$-\nabla^2 u = f.$$

We have

$$\nabla^2 u + f = 0.$$

Now multiply by $w(x, y)$ & integrate over Ω :

$$\int_{\Omega} w(\nabla^2 u + f) d\Omega = 0$$

Note that

$$\begin{aligned}\nabla \cdot [w \nabla u] &= \nabla w \cdot \nabla u + w \nabla^2 u \\ \Rightarrow w \nabla^2 u &= \nabla \cdot [w \nabla u] - \nabla w \cdot \nabla u.\end{aligned}$$

Hence we get

$$\int_{\Omega} -\nabla w \cdot \nabla u + f w + \nabla \cdot [w \nabla u] d\Omega = 0$$

$$\begin{aligned}\int_{\Omega} \nabla w \cdot \nabla u - f w d\Omega - \int_{\Gamma_1} w \nabla u \cdot \underline{n}_1 d\Gamma \\ - \int_{\Gamma_2} w \nabla u \cdot \underline{n}_2 d\Gamma = 0\end{aligned}$$

For the last two integrals to vanish
we need $w|_{\Gamma_2} = 0$ as $\nabla u \cdot \underline{n}_1|_{\Gamma_1} = 0$.

Hence, $w(x, y) = \delta(x, y)$ and the
weak formulation is

$$\boxed{\int_{\Omega} \nabla w \cdot \nabla u - f w d\Omega = 0.}$$

(4.)

② Consider the Hilbert space

$$\mathcal{H}_0^1(\Omega) := \left\{ u \in L^2(\Omega) \mid \exists x^u, \exists y^u \in L^2(\Omega) \right. \\ \left. \text{such that } u|_{\Gamma_2} = 0 \right\}$$

Note that $L^2(\Omega) := \left\{ u \mid \int_{\Omega} |u|^2 dx < \infty \right\}$.

Now consider a subspace $V_n \subset \mathcal{H}_0^1$ with

basis e_1, e_2, \dots, e_n i.e. V_n is an n -dimensional subspace of \mathcal{H}_0^1 .

Hence, $u \in \mathcal{H}_0^1$ can be approximated with

$$u \approx u_n = \sum_{j=1}^n u_j e_j$$

where u_j are constants and $u_n \in V_n$.

Recall the Ritz-Galerkin principle

$$I[u] = \int_{\Omega} \frac{1}{2} |\nabla u|^2 - u f \, dx$$

Hence,

$$I[u_n] = \int_{\Omega} \frac{1}{2} \left| \sum_{j=1}^n u_j \nabla e_j \right|^2 - \sum_{j=1}^n u_j e_j f \, dx$$

$$= \sum_{j=1}^n u_j \int_{\Omega} \frac{1}{2} |\nabla e_j|^2 - e_j f \, dx.$$

No, incorrect indexing as said/warned
and $u_i * u_j$ should emerge;

-0.5

(5.)

The weak formulation is

$$\int_{\Omega} \nabla w \cdot \nabla u - f w \, d\Omega = 0$$

where $w|_{\Gamma_2} = 0$. Similarly, pick $w \in V_h^{s,1}$ where $w|_{\Gamma_2} = 0$. Similarly, pick $w \in V_h^{s,1}$ as $w = e_i$ for $i \in \{1, \dots, n\}$ as ~~this~~ it is arbitrary. Hence, we get

$$\begin{aligned} & \int_{\Omega} \nabla e_i \cdot \nabla \left(\sum_{j=1}^n u_j e_j \right) - f e_i \, d\Omega = 0 \\ \Leftrightarrow & \sum_{j=1}^n u_j \int_{\Omega} \nabla e_i \cdot \nabla e_j \, d\Omega = \int_{\Omega} f e_i \, d\Omega \end{aligned}$$

Now denote

$$A_{ij} := \int_{\Omega} \nabla e_i \cdot \nabla e_j \, d\Omega$$

$$b_i := \int_{\Omega} e_i f \, d\Omega$$

and apply the Einstein's summation convention to get

$$A_{ij} u_j = b_i.$$

Recall that

no, δu_n no $e_i(x,y)$ as it has been integrated out;
also $\delta u_n = 0$ for $n \neq n'$.

$$\delta I[u_n] = \frac{dI}{d\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{I[u_n + \epsilon e_i] - I[u_n]}{\epsilon}.$$

Now,

$$\begin{aligned} & \frac{1}{\epsilon} \int_{\Omega} \sum_{j=1}^n u_j \left(\frac{1}{2} |\nabla(e_j + \epsilon e_i)|^2 - \frac{1}{2} |\nabla e_j|^2 - (e_j + \epsilon e_i) f \right. \\ & \quad \left. + \epsilon e_i f \right) d\Omega \\ &= \sum_{j=1}^n \frac{u_j}{\epsilon} \int_{\Omega} \frac{1}{2} \left(|\nabla e_j|^2 - |\nabla e_j|^2 + 2\epsilon \nabla e_j \cdot \nabla e_i + \epsilon^2 |\nabla e_i|^2 \right) \\ & \quad - \epsilon e_i f d\Omega \end{aligned}$$

incorrect: -0.25

$$= \sum_{j=1}^n u_j \int_{\Omega} \cancel{\nabla e_j \cdot \nabla e_i} + \frac{\epsilon}{2} |\nabla e_i|^2 - e_i f d\Omega$$

Let $\epsilon \rightarrow 0$ and minimize $\delta I[u_n]$ to get

$$\sum_{j=1}^n u_j \int_{\Omega} \nabla e_i \cdot \nabla e_j - e_i f d\Omega = 0$$

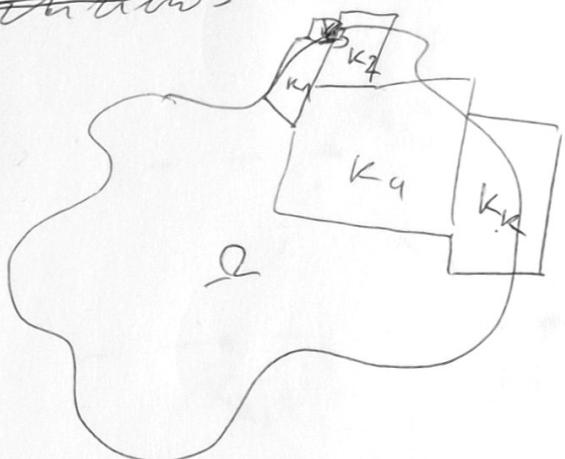
as required.

Now, consider a set

$$\mathcal{K}_h = \left\{ K_k \mid \bigcup_{k=1}^{Nel} K_k = \bar{\Omega} \text{ s.t. } K_k \cap K_{k'} \neq \emptyset \right\}$$

for $k, k' \in \{1, \dots, Nel\}\}$.

This ~~partition~~^{covers} domain Ω into elements K_k :

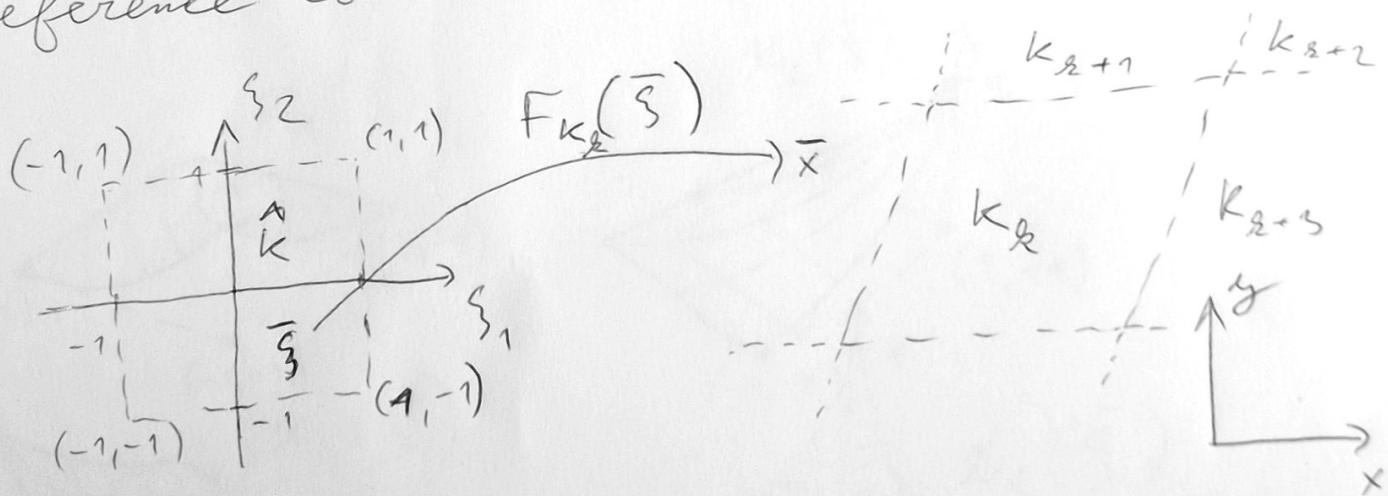


where $\bar{\Omega}$ is closure of Ω .

Now introduce a reference element \hat{K}

and a map $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that maps \hat{K} to K_h .

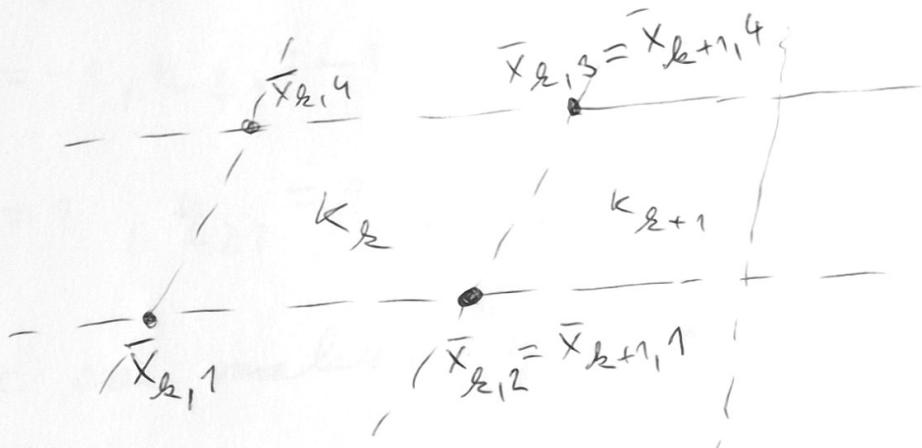
For reference element \hat{K} introduce a reference coordinate system $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2)$:



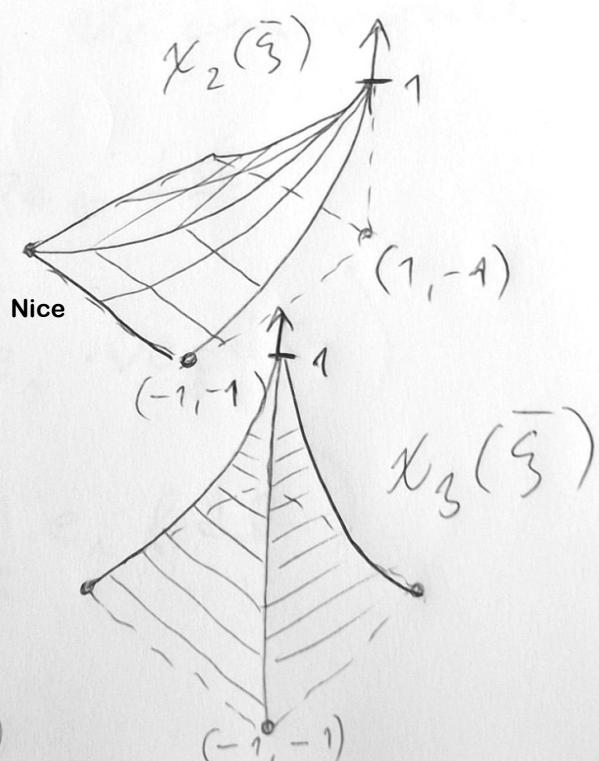
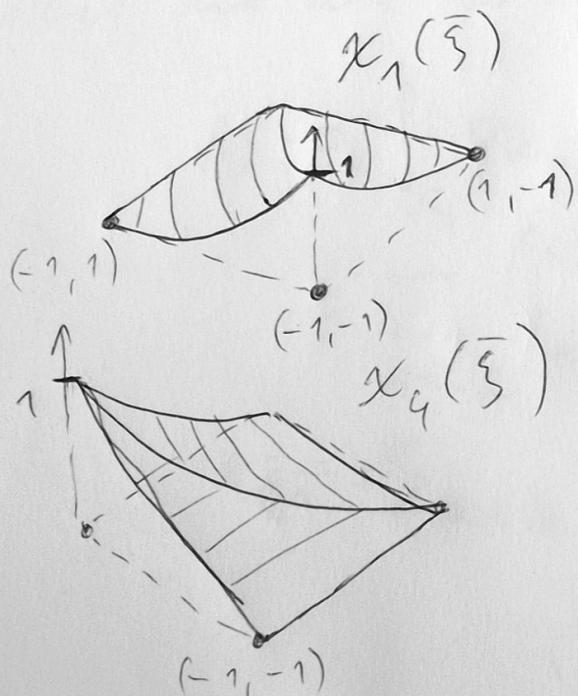
Hence, we have the relation

$$\bar{x} = F_{k_k}(\bar{\xi})$$

In particular, we can index nodes of each quadrilateral element k_k as follows:



where k - is the index of the element & $x \in \{1, 2, 3, 4\}$
refers to local index within each element k_k
Now, introduce a set of functions on k_k
 $\chi_\alpha(\bar{\xi})$ n. 1.



These functions are given by

$$x_\alpha = (1 + \gamma_{\alpha 1} \xi_1) (1 + \gamma_{\alpha 2} \xi_2)^{\frac{1}{4}}$$

where $\gamma_{\alpha i}$ for $\alpha \in \{1, 2, 3, 4\}$, $i \in \{1, 2\}$ is defined by

$$\gamma_{11} = -1, \gamma_{41} = -1, \gamma_{12} = -1, \gamma_{22} = -1$$

$$\gamma_{31} = 1, \gamma_{21} = 1, \gamma_{32} = 1, \gamma_{42} = 1.$$

Hence, we can make a specific choice for

$$F_{K_h}(\bar{\xi}) :$$

$$F_{K_h}(\bar{\xi}) = \sum_{\alpha=1}^4 \bar{x}_{k,\alpha} \chi_\alpha(\bar{\xi})$$

Now, recall the system

$$A_{ij} u_j = b_i$$

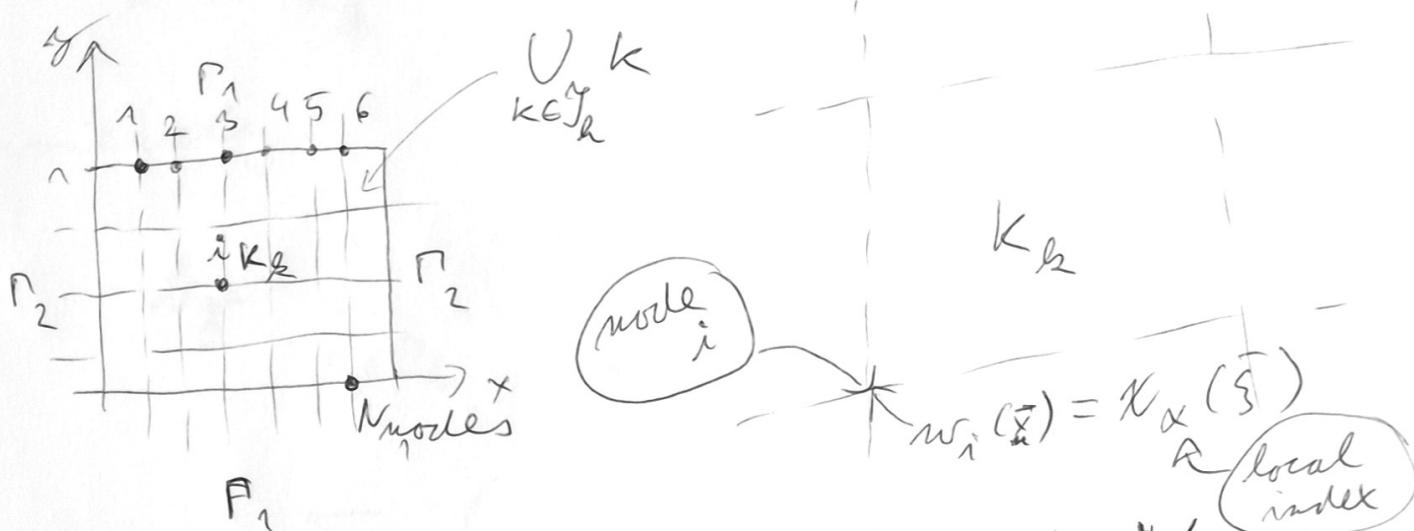
that we are solving. We can write

$$\begin{aligned} A_{ij} &= \int_{\Omega} \nabla e_i \cdot \nabla e_j d\Omega \\ &= \sum_{K \in \mathcal{T}_h} \left(\int_K \nabla e_i \cdot \nabla e_j d\Omega \right) \end{aligned}$$

$$b_i = \sum_{K \in \mathcal{T}_h} \left(\int_K e_i f d\Omega \right)$$

At this point it becomes necessary to choose $V_h \subset \mathcal{H}_0^1(\Omega)$ and specify the basis functions.

If we consider only nodes with unknown values u_j , then we get



for $i \in \{1, 2, \dots, N_{\text{nodes}}\}$. Then $V_h = \bigcup_{N_{\text{nodes}}}$ and there is one basis function for each node with global index i .

In particular, on each element K_2 choose

$$\phi_i(x, y) = \hat{\phi}_\alpha(F^{-1}_{K_2}(x, y)) = X_\alpha(\bar{s}).$$

Hence, we obtain

$$\begin{aligned} A_{\alpha\beta} &= \sum_i \int_{K_2 \in \mathcal{T}_h} \nabla \phi_i \cdot \nabla \phi_\beta \, d\Omega \\ &= \sum_i \int_{K_2 \in \mathcal{T}_h} \nabla X_\alpha \cdot \nabla X_\beta \, d\Omega \quad (11.) \end{aligned}$$

$$= \sum_{K \in \mathcal{E}_h} \int_{\hat{K}} \left((\mathbf{J}^T)^{-1} \frac{\partial \mathbf{x}_B}{\partial \xi_i} \right) d\xi$$

$$= \sum_{K \in \mathcal{E}_h} \int_{\hat{K}} \left((\mathbf{J}^T)^{-1} \right)_{ij} \frac{\partial \mathbf{x}_B}{\partial \xi_j} \Big|_{\hat{K}} \left. \frac{\partial \mathbf{x}_B}{\partial \xi_m} \right|_{\hat{K}} \det(\mathbf{J}) d\xi$$

Similarly,

$$b_{\bar{x}} = \sum_{K \in \mathcal{E}_h} \left(\int_{\hat{K}} e_i f d\xi \right)$$

$$= \sum_{K \in \mathcal{E}_h} \left(\int_{\hat{K}} \mathbf{g} \cdot \mathbf{x}_B f d\xi \right)$$

$\det(\mathbf{J})$

Note that once $\int_{\hat{K}} \cdot d\xi$ are evaluated on the reference coordinate system, they must be linked back to appropriate entries of A_{ij} , b_i , by leveraging mapping $(k, \alpha) \rightarrow i$ & $(k, \alpha) \rightarrow j$.

The Jacobian \mathbf{J} is given by

$$J_{ij} = \frac{\partial \bar{x}_i}{\partial \xi_j} = \frac{\partial}{\partial \xi_j} \left[\sum_{\alpha=1}^4 \bar{x}_{i,\alpha} x_{\alpha}(\xi) \right]$$

↑ position
of the node of element
 k with local coordinate α .

$$\mathcal{J}_{ij} = \sum_{\alpha=1}^4 \bar{x}_{i,\alpha} \frac{\partial}{\partial \xi_j} \left[(1 + \gamma_{x_1} \xi_1) (1 + \gamma_{x_2} \xi_2) \frac{1}{q} \right]$$

$$= \frac{1}{q} \sum_{\alpha=1}^4 \bar{x}_{i,\alpha} \left(u_{x_1} \delta_{j1} (1 + \gamma_{x_2} \xi_2) + u_{x_2} \delta_{j2} (1 + \gamma_{x_1} \xi_1) \right)$$

where δ_{ij} is kronecker delta & u_{x_i} was defined previously.

ASSEMBLY MISSING

Q4: you are comparing
 u_1 and u_2 difference not difference between exact and u_1 ;
; see convergence further down.

Q6: you have not adapted the BC's to match your exact solution! -1.25