1) 

$$b_t + (ub)_z = 0$$
$$u = \alpha b^2 - \beta b^2 b_z \quad \text{w/ } z \in [0,H] \qquad \alpha, \beta > 0$$

BCs: $b(0,t) = b_B$, $b(H,t) = b_T$

IC: $b(z,0) = b_i(z)$.

a) $b_t + (ub)_z = 0$, $u = \alpha b^2 - \beta b^2 b_z \implies ub = \alpha b^3 - \beta b^3 b_z$

$\implies b_t + (\alpha b^3 - \beta b^3 b_z)_z = 0 \quad (5)$

let $b = D_0 + b'$ to linearize around D_0.
$D_{0_t} = D_{0_z} = 0$ since D_0 cst.
$\implies b'_t + (\alpha(D_0+b')^3 - \beta(D_0+b')^3 b'_z)_z = 0$
b' small perturbation $\implies$ can neglect quadratic and h.o.ts in b' and derivatives.

$\implies b'_t + (\alpha(D_0^3 + 3D_0^2 b') - \beta D_0^3 b'_z)_z = 0$
$\implies b'_t + 3\alpha D_0^2 b'_z - \beta D_0^3 b'_{zz} = 0 \quad (6)$ as required.

(5) and (6) are nonlinear convection-diffusion eqs as they have both a convection term (b_z / b'_z) and a diffusive term (b_{zz} / b'_{zz}). Eq (5) is also nonlinear in b, hence being a nonlinear eq. Eq (6) is linearized as all quadratic and h.o.ts of b and its derivatives have been neglected.

b) Discretize w/ forward Euler discretization, upwind scheme for convective term, 2nd order central difference for diffusive term.

$z = j\Delta z$, $z \in [0,H]$, $j \in \{0, 1, \dots, J-1, J\}$ w/ J st. $J\Delta z = H$, $\Delta z = \frac{H}{J}$
$t = n\Delta t$, $n \in \mathbb{N}$

note: choice of J defines Δz, can choose both Δt and no. of time steps.

(5) $b_t + (\alpha b^3 - \beta b^3 b_z)_z = 0$

Forward Euler time discretization: $b_t \approx \frac{\partial}{\partial t} b(z_j, t_n)$, $b(z,t) \approx b(z_j, t_n) = b_j^n$
$b(z_j, t_n + \Delta t) = b(z_j, t_{n+1})$
Δt small (as is Δz) $\therefore$ can Taylor expand around t_n:
$$b(z_j, t_n + \Delta t) \approx b(z_j, t_n) + \Delta t \frac{\partial}{\partial t} b(z_j, t_n) + \frac{\Delta t^2}{2!} \frac{\partial^2}{\partial t^2} b(z_j, t_n) + O((\Delta t)^3)$$
$$\implies b(z_j, t_{n+1}) - b(z_j, t_n) \approx \Delta t \frac{\partial}{\partial t} b(z_j, t_n) + O((\Delta t)^2)$$
$$\therefore \frac{\partial}{\partial t} b_j^n \approx \frac{b_j^{n+1} - b_j^n}{\Delta t}$$ (neglecting h.o.ts)

Convective term: $(\alpha b^3)_z$
using the product rule obtain: $3\alpha b^2 b_z$, $3\alpha > 0 \therefore$ coefficient > 0.
upwind scheme: Taylor expand $b(z_j - \Delta z, t_n)$ about z_j
$$b(z_j - \Delta z, t_n) \approx b(z_j, t_n) - \Delta z \frac{\partial}{\partial z} b(z_j, t_n) + \frac{(\Delta z)^2}{2!} \frac{\partial^2}{\partial z^2} b(z_j, t_n) + O((\Delta z)^3)$$
$$\implies \frac{\partial b}{\partial z} \approx \frac{b(z_j, t_n) - b(z_j - \Delta z, t_n)}{\Delta z} = \frac{b_j^n - b_{j-1}^n}{\Delta z}$$
$$\therefore (\alpha b^3)_z \approx 3\alpha (b_j^n)^2 \frac{b_j^n - b_{j-1}^n}{\Delta z}$$

Diffusive term: $-\beta(b^3b_z)_z$
using adjoint method (advantages: • more stable for discretization. • more compact form.)

$$[b^3b_z]^n_{j+1/2} = (b^n_{j+1/2})^3\left(\frac{b^n_{j+1}-b^n_j}{\Delta z}\right)$$

$$[b^3b_z]^n_{j-1/2} = (b^n_{j-1/2})^3\left(\frac{b^n_j-b^n_{j-1}}{\Delta z}\right)$$

central difference $\frac{[b^3b_z]_{j+1/2}-[b^3b_z]_{j-1/2}}{\Delta z}$

$$[b^3b_z]_z \approx \frac{1}{(\Delta z)^2}\left[(b^n_{j+1/2})^3(b^n_{j+1}-b^n_j) - (b^n_{j-1/2})^3(b^n_j-b^n_{j-1})\right]$$

$$(b^n_{j+1/2})^3 = \frac{(b^n_{j+1})^3+(b^n_j)^3}{2}, \quad (b^n_{j-1/2})^3 = \frac{(b^n_j)^3+(b^n_{j-1})^3}{2}$$

then
$$-\beta(b^3b_z)_z \approx -\frac{\beta}{2(\Delta z)^2}\left[((b^n_{j+1})^3+(b^n_j)^3)(b^n_{j+1}-b^n_j) - ((b^n_j)^3+(b^n_{j-1})^3)(b^n_j-b^n_{j-1})\right]$$

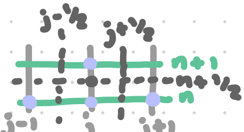
$$= -\frac{\beta}{2(\Delta z)^2}\left[(b^n_{j+1})^4 + (b^n_j)^3 b^n_{j+1} - (b^n_{j+1})^3 b^n_j - (b^n_j)^4 - (b^n_j)^4 - (b^n_{j-1})^3 b^n_j + (b^n_j)^3 b^n_{j-1} + (b^n_{j-1})^4\right]$$

$$= -\frac{\beta}{2(\Delta z)^2}\left[(b^n_j)^3[b^n_{j+1}-2b^n_j+b^n_{j-1}] + (b^n_{j+1})^3(b^n_{j+1}-b^n_j) - (b^n_{j-1})^3(b^n_j-b^n_{j-1})\right]$$

then (5) $b_t + (\alpha b^3 - b^3 b_z)_z = 0$ is approximated by the discretization:

$$\frac{b^{n+1}_j - b^n_j}{\Delta t} + 3\alpha(b^n_j)^2\frac{b^n_j-b^n_{j-1}}{\Delta z} - \frac{\beta}{2(\Delta z)^2}\left[(b^n_j)^3[b^n_{j+1}-2b^n_j+b^n_{j-1}] + (b^n_{j+1})^3(b^n_{j+1}-b^n_j) - (b^n_{j-1})^3(b^n_j-b^n_{j-1})\right] = 0$$

Grid used:



(6): $b'_t + 3\alpha D_0^2 b'_z - \beta D_0^3 b'_{zz} = 0$ is discretized to give:

$$\frac{b'^{n+1}_j - b'^n_j}{\Delta t} + \underbrace{3\alpha D_0^2 \frac{b'^n_j - b'^n_{j-1}}{\Delta z}}_{\text{upwind scheme for convective term}} - \beta D_0^3 \underbrace{\left(\frac{b'^n_{j+1}-2b'^n_j+b'^n_{j-1}}{(\Delta z)^2}\right)}_{\text{2nd order central difference for diffusive term}} = 0$$

Linearisation of discretization of (5):
$b = D_0 + b'$, neglect quadratic and h.o.t.s in b' and its derivatives:

$$\frac{b'^{n+1}_j - b'^n_j}{\Delta t} + 3\alpha D_0^2\frac{b'^n_j-b'^n_{j-1}}{\Delta z} - \frac{\beta}{2(\Delta z)^2}\left(D_0^3[b'^n_{j+1}-2b'^n_j+b'^n_{j-1}+b'^n_{j+1}-b'^n_j-b'^n_j+b'^n_{j-1}]\right) = 0$$

note: $D_{0_{j-1}} = D_{0_j} = D_{0_{j+1}}$

$$\Rightarrow \frac{b'^{n+1}_j - b'^n_j}{\Delta t} + 3\alpha D_0^2\frac{b'^n_j-b'^n_{j-1}}{\Delta z} - \frac{\beta}{2(\Delta z)^2}\left(2D_0^3[b'^n_{j+1}-2b'^n_j+b'^n_{j-1}]\right) = 0$$

∴ linearisation of discretization of (5) is the same as discretization of (6).

BCs: $b(0,t) \approx b(0,t_n) = b_B \implies b_0^n = b_B \quad \forall n$

$b(H,t) \approx b(H,t_n) = b_T \implies b_J^n = b_T \quad \forall n$

IC: $b(z,0) \approx b(z_j,0) = b_i(z_j) = b_i(j\Delta z) \implies b_j^0 = b_i(j\Delta z) \quad \forall j$

c) Use a Fourier analysis to assess stability of linearized scheme.

linearized scheme: $\dfrac{b_j'^{n+1} - b_j'^n}{\Delta t} + 3\alpha D_0^2 \dfrac{b_j'^n - b_{j-1}'^n}{\Delta z} - \beta D_0^3 \dfrac{b_{j+1}'^n - 2b_j'^n + b_{j-1}'^n}{(\Delta z)^2} = 0$

use ansatz: $b_j'^n = \lambda(k)^n e^{ikj\Delta z}$

Case 1: $\alpha = 0, \beta \neq 0$ and substitute in ansatz

$\implies \lambda(k)^n e^{ikj\Delta z}\left[\dfrac{\lambda(k)-1}{\Delta t} - \dfrac{\beta D_0^3}{(\Delta z)^2}\left(e^{ik\Delta z} - 2 + e^{-ik\Delta z}\right)\right] = 0$

$\implies \dfrac{\lambda(k)-1}{\Delta t} - \dfrac{\beta D_0^3}{(\Delta z)^2}\left(2\cos(k\Delta z) - 2\right) = 0$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \implies 2\cos(k\Delta z) - 2 = -4\sin^2\left(\frac{k\Delta z}{2}\right)$

$\implies \lambda(k) = 1 - 4\beta D_0^3 \mu \sin^2\left(\frac{k\Delta z}{2}\right) \quad , \mu := \dfrac{\Delta t}{(\Delta z)^2}$

for stability require $|\lambda(k)| \leq 1 \implies |1 - \underbrace{4\beta D_0^3 \mu \sin^2\left(\tfrac{k\Delta z}{2}\right)}_{\geq 0 \;\therefore\; |\lambda(k)| \leq 1 \text{ always.}}| \leq 1$

for stability need $1 - 4\beta D_0^3 \mu \sin^2\left(\frac{k\Delta z}{2}\right) > -1 \quad \forall k$

take most extreme case: $1 - 4\beta D_0^3 \mu > -1$ is condition for stability

$$-4\beta D_0^3 \mu > -2$$
$$4\beta D_0^3 \mu < 2$$
$$\frac{\Delta t}{(\Delta z)^2} = \mu < \frac{1}{2}\frac{1}{\beta D_0^3}$$

$\therefore$ choose Δt s.t. $\Delta t < \dfrac{1}{2}\dfrac{(\Delta z)^2}{\beta D_0^3}$ for stability

Case 2: $\alpha \neq 0, \beta = 0$

$\implies \lambda(k)^n e^{ikj\Delta z}\left(\dfrac{\lambda(k)-1}{\Delta t} + 3\alpha D_0^2\left(\dfrac{1 - e^{-ik\Delta z}}{\Delta z}\right)\right) = 0$

$\lambda(k) - 1 + 3\alpha D_0^2 \mu \Delta z (1 - e^{-ik\Delta z}) = 0$

$\lambda(k) = 1 - 3\alpha D_0^2 \mu \Delta z (1 - e^{-ik\Delta z})$

$= 1 - 3\alpha D_0^2 \mu \Delta z (1 - \cos(k\Delta z) + i\sin(k\Delta z))$

$= 1 - \mu A(1 - \cos(k\Delta z)) - i\mu A \sin(k\Delta z) \quad A = 3\alpha D_0^2 \Delta z$

$|\lambda(k)|^2 = (1 - \mu A + \mu A\cos(k\Delta z))^2 + (\mu A \sin(k\Delta z))^2$

$= (1-\mu A)^2 + 2(1-\mu A)\mu A\cos(k\Delta z) + \mu^2 A^2 \cos^2(k\Delta z) + \mu^2 A^2 \sin^2(k\Delta z)$

$= 1 - 2\mu A + 2\mu^2 A^2 + 2\mu A\cos(k\Delta z) - 2\mu^2 A^2 \cos(k\Delta z)$

$= 1 - 2\mu^2 A^2(\cos(k\Delta z) - 1) + 2\mu A(\cos(k\Delta z) - 1)$

$\cos(k\Delta z) - 1 = -2\sin^2(\frac{k\Delta z}{2})$

$$\Rightarrow |\lambda(k)|^2 = 1 + 4\mu^2 A^2 \sin^2(\tfrac{k\Delta z}{2}) - 4\mu A \sin^2(\tfrac{k\Delta z}{2})$$

$$= 1 - 4\mu A \sin^2(\tfrac{k\Delta z}{2})[1-\mu A]$$

need that $|\lambda(k)|^2 < 1 \Rightarrow 1 - 4\mu A \sin^2(\frac{k\Delta z}{2})[1-\mu A] < 1$

$$-4\mu A \sin^2(\tfrac{k\Delta z}{2})(1-\mu A) < 0$$

$$\Rightarrow 1 - \mu A > 0$$

$$\Rightarrow \mu < \frac{1}{A} \quad \text{(A +ve since } K>0\text{)}$$

$$\Rightarrow \mu < \frac{1}{3\alpha D_0^2 \Delta z}$$

$\therefore$ choose Δt s.t. $\Delta t < \frac{\Delta z}{3\alpha D_0^2}$

d) Use maximum principle to determine a stable time step for discretization of (6).

$$\frac{b_j'^{n+1} - b_j'^n}{\Delta t} + 3\alpha D_0^2 \frac{b_j'^n - b_{j-1}'^n}{\Delta z} - \beta D_0^3 \frac{b_{j+1}'^n - 2b_j'^n + b_{j-1}'^n}{(\Delta z)^2} = 0$$

suppose $b_j'^n \geq 0 \ \forall j$, want $b_j'^{n+1} \geq 0 \ \forall j$ as well

$$b_j'^{n+1} = b_j'^n - 3\alpha D_0^2 \mu \Delta z (b_j'^n - b_{j-1}'^n) + \beta D_0^3 \mu (b_{j+1}'^n - 2b_j'^n + b_{j-1}'^n)$$

$$= \beta D_0^3 \mu b_{j+1}'^n + b_{j-1}'^n (3\alpha D_0^2 \mu \Delta z + \beta D_0^3 \mu) + b_j'^n (1 - 3\alpha D_0^2 \mu \Delta z - 2\beta D_0^3 \mu)$$

$\Rightarrow$ need $1 - 3\alpha D_0^2 \mu \Delta z - 2\beta D_0^3 \mu > 0$

$$\Rightarrow \mu(3\alpha D_0^2 \Delta z + 2\beta D_0^3) < 1$$

$$\mu < \frac{1}{3\alpha D_0^2 \Delta z + 2\beta D_0^3}$$

$\therefore$ choose Δt s.t. $\Delta t < \frac{(\Delta z)^2}{3\alpha D_0^2 \Delta z + 2\beta D_0^3}$ for maximum principle to hold.

note: setting $\alpha = 0, \beta \neq 0$ or $\alpha \neq 0, \beta = 0$ recovers the conditions on Δt for stability, as derived in d).
↳ in explicit scheme $\therefore$ cond. for

e) Derive variable time step criterion for discretization of (5) for which $b_j^{n+1} > 0$.
$\Rightarrow$ width b guaranteed to be +ve

$$(5) \quad \frac{b_j^{n+1} - b_j^n}{\Delta t} + 3\alpha (b_j^n)^2 \frac{b_j^n - b_{j-1}^n}{\Delta z} - \frac{\beta}{2(\Delta z)^2}\Big[(b_j^n)^3[b_{j+1}^n - 2b_j^n + b_{j-1}^n] + (b_{j+1}^n)^3(b_{j+1}^n - b_j^n) - (b_{j-1}^n)^3(b_j^n - b_{j-1}^n)\Big] = 0$$

$$b_j^{n+1} = b_j^n - 3\alpha\mu\Delta z (b_j^n)^2 (b_j^n - b_{j-1}^n) + \frac{\beta\mu}{2}\Big[(b_j^n)^3[b_{j+1}^n - 2b_j^n + b_{j-1}^n] + (b_{j+1}^n)^3(b_{j+1}^n - b_j^n) - (b_{j-1}^n)^3(b_j^n - b_{j-1}^n)\Big]$$

$$= b_j^n + 3\alpha\mu\Delta z (b_j^n)^2 + \frac{\beta}{2}\mu\big[(b_j^n)^3 b_{j+1}^n + (b_j^n)^3 b_{j-1}^n + (b_{j+1}^n)^4 + (b_{j-1}^n)^4\big]$$

$$- 3\alpha\mu\Delta z (b_j^n)^3 - \frac{\beta}{2}\mu\big[2(b_j^n)^4 + b_j^n (b_{j+1}^n)^3 + b_j^n (b_{j-1}^n)^3\big]$$

then since $b_j^n > 0$ and $\alpha, \beta, \mu > 0$

$\Rightarrow$ require $b_j^n - 3\alpha\mu\Delta z (b_j^n)^3 - \frac{\beta\mu}{2}(b_j^n)[2(b_j^n)^3 + (b_{j-1}^n)^3 + (b_{j+1}^n)^3] > 0$

$1 - 3\alpha\mu\Delta z (b_j^n)^2 - \frac{\beta\mu}{2}[2(b_j^n)^3 + (b_{j-1}^n)^3 + (b_{j+1}^n)^3] > 0$

$\mu[3\alpha\Delta z(b_j^n)^2 + \frac{\beta}{2}(2(b_j^n)^3 + (b_{j-1}^n)^3 + (b_{j+1}^n)^3)] < 1$

$$\mu < \frac{1}{3\alpha\Delta z(b_j^n)^2 + \frac{\beta}{2}(2(b_j^n)^3 + (b_{j-1}^n)^3 + (b_{j+1}^n)^3)}$$

$\therefore$ choose time-step s.t. $$\Delta t < \frac{(\Delta z)^2}{3\alpha\Delta z(b_j^n)^2 + \frac{\beta}{2}(2(b_j^n)^3 + (b_{j-1}^n)^3 + (b_{j+1}^n)^3)}$$

f) second-order spatial discretization of (5):

consider $b(t_n, z_j + \Delta z) = b_j^n + \Delta z \frac{\partial}{\partial z} b_j^n + \frac{(\Delta z)^2}{2}\frac{\partial^2}{\partial z^2} b_j^n + O((\Delta z)^3)$

$b(t_n, z_j - \Delta z) = b_j^n - \Delta z \frac{\partial}{\partial z} b_j^n + \frac{(\Delta z)^2}{2}\frac{\partial^2}{\partial z^2} b_j^n + O((\Delta z)^3)$

$\Rightarrow b_{j+1}^n - b_{j-1}^n = 2\Delta z \frac{\partial}{\partial z} b_j^n + O((\Delta z)^3)$

$\therefore \frac{\partial b}{\partial z} \approx \frac{b_{j+1}^n - b_{j-1}^n}{2\Delta z}$ is 2nd order spatial discretization for convective term.

(5) $\mapsto$ $b_j^{n+1} = b_j^n - \frac{3}{2}\alpha\mu\Delta z (b_j^n)^2(b_{j+1}^n - b_{j-1}^n) + \frac{\beta\mu}{2}[(b_j^n)^3(b_{j+1}^n - 2b_j^n + b_{j-1}^n) + (b_{j+1}^n)^3(b_{j+1}^n - b_j^n) - (b_{j-1}^n)^3(b_j^n - b_{j-1}^n)]$

$b_j^n > 0 \Rightarrow$ require for $b_j^{n+1} > 0$ $\quad b_j^n - \frac{3}{2}\alpha\mu\Delta z(b_j^n)^2 b_{j+1}^n - \frac{\beta\mu}{2} b_j^n[2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3] > 0$

$\Rightarrow \mu[\frac{3}{2}\alpha\Delta z\, b_j^n b_{j+1}^n + \frac{\beta}{2}[2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3]] < 1$

$\therefore$ time step required for $b_j^{n+1} > 0$ if $b_j^n > 0$ is Δt s.t.:

$$\Delta t < \frac{2(\Delta z)^2}{3\alpha\Delta z\, b_j^n b_{j+1}^n + \beta[2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3]}$$

for beta=0 the scheme is unstable, which is not apparent. -0.5

2) a) Claim: steady state sol^n of (S) satisfies $\beta b^3 b_z = \alpha b^3 - Q$, Q cst of integration.

(S) $b_t + (\alpha b^3 - \beta b^3 b_z)_z = 0$

steady state sol^n is sol^n b s.t. $b_t = 0$

$\Rightarrow (\alpha b^3 - \beta b^3 b_z)_z = 0$

$\Rightarrow \alpha b^3 - \beta b^3 b_z = Q \quad \Rightarrow \beta b^3 b_z = \alpha b^3 - Q$ as required.

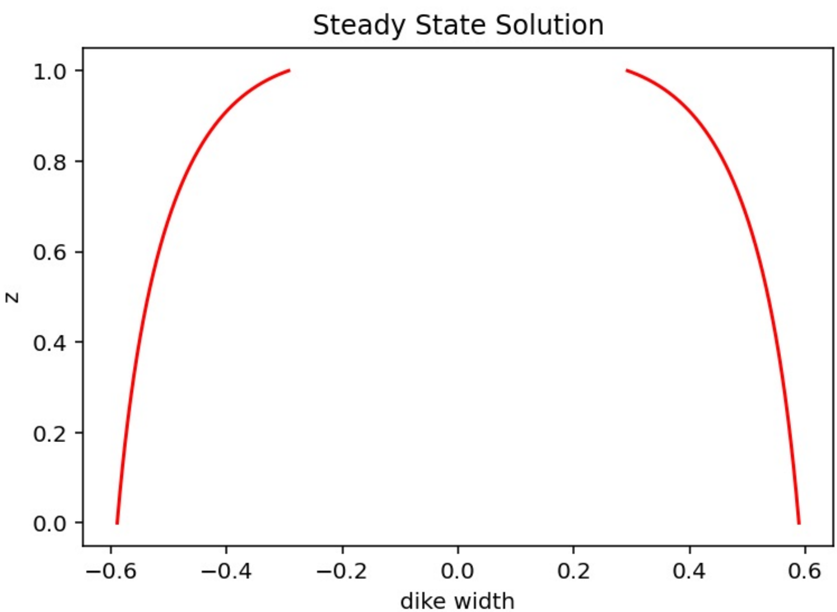
$$b_z \approx \frac{b_{j+1} - b_j}{\Delta z}$$

$\Rightarrow$ eq discretized to: $\beta (b_j)^3 \dfrac{b_{j+1} - b_j}{\Delta z} = \alpha (b_j)^3 - Q$

explicitly gives $b_{j+1} = b_j + \dfrac{\alpha \Delta z}{\beta} - \dfrac{Q \Delta z}{\beta (b_j)^3}$

w/ $b(0,t) = b_B \Rightarrow b_0 = b_B$ and then each b_{j+1} can be determined iteratively.

+ code for plotting steady state solution.



above is steady state sol^n obtained using $J = 3000$ giving $b_J = 0.58546028$ (note: b_J agrees w/ b_T given in b) to 3 sig figs)

b) Sol^n of nonlinear convection-diffusion eq (5) at times: $t = 0.05, 0.1, 0.2, 0.5, 1, 2$

$b_j^0 = b_T = b_J^n$

$A\underline{b} = \underline{f} \quad \Rightarrow \underline{b} = A^{-1}\underline{f}$

$\Delta z = \frac{H}{11}, \frac{H}{21}, \frac{H}{41}$

max. principle $\Rightarrow b_{max} = b_B = 1.17816434 3$ ($> b_T = 0.58537379 8$)

need to choose Δt s.t. $\Delta t < \dfrac{(\Delta z)^2}{3\alpha\Delta z (b_j^n)^2 + \frac{\beta}{2}\left(2(b_j^n)^3 + (b_{j-1}^n)^3 + (b_{j+1}^n)^3\right)}$

note: max. principle bound stronger than stability.

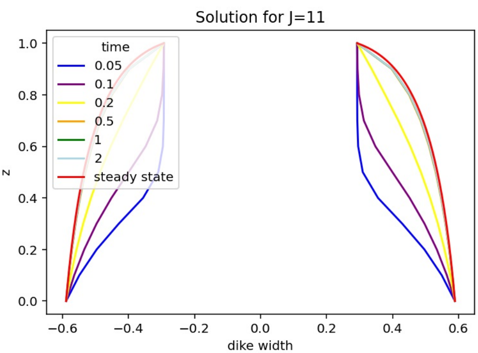
note: strictest bound on Δt comes from $b_{max} = b_B$

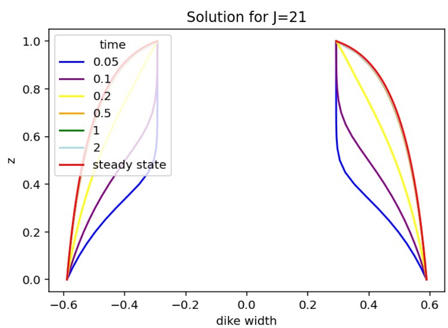
$\Rightarrow \Delta t < \dfrac{(\Delta z)^2}{3\alpha\Delta z (b_B)^2 + \frac{\beta}{2}\left(2(b_B)^3 + (b_B)^3 + (b_B)^3\right)} = \dfrac{(\Delta z)^2}{3\alpha(\Delta z) b_B^2 + 2\beta b_B^3}$

note: smallest Δz when consider 41 grid pts: $\Delta z = \frac{H}{40} = \frac{1}{40}$

$\Rightarrow \Delta t < 0.00018827$

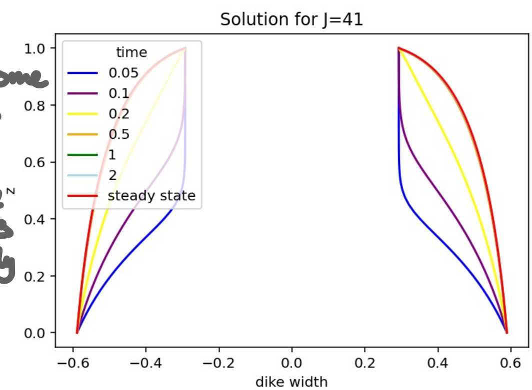
if use $\Delta t = 10^{-4}$ then satisfy stability for all scenarios





For increasing J the solutions become more smooth ~ as is expected.

For increasing t the numerical solutions approach the steady state solution.



note: this code used first order convection discretisation. I found that the 2nd order led to an over estimate for $t \geq 0.5$

c) $e(z,t) = b(z,t) - b_{numerical}(z,t)$

L^2-norm and L^∞-norm as a fnct of resolution.

note: $error(j) = b(z_j, t) - b_{numerical}(z_j, t)$, $z_j = j\Delta z$

calculated the L^2-norm using approximation:

$$L^2 \approx \sqrt{\frac{dz}{2}\left(error(0)^2 + 2\sum_{n=1}^{J-2}\left(error(n)^2\right) + error(J-1)^2\right)}$$

for $J = 11, 21, 41, 81, 161$ $(321?)$

$\Delta z = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}$ $\left(\frac{1}{320}?\right) \Rightarrow$ halving Δz

then plotting the log of the L^2 norm against the log of Δz.

find that the gradient approaches 1 for decreasing Δz.

$\Rightarrow$ linear dependence.

$\Rightarrow$ first order spatial discretization.

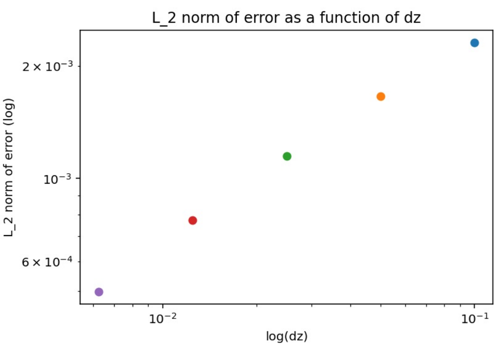
similarly for the L^∞-norm, where $L^\infty = \max_{0 \le j \le J-1}(error(j))$

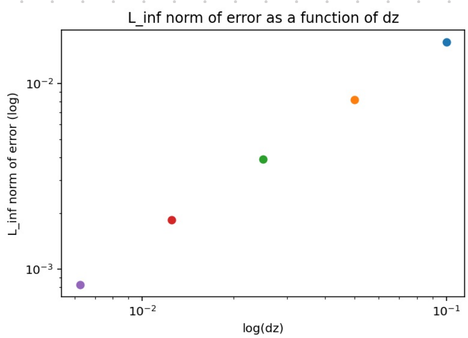
plotting the log of L^∞-norm against the log of Δz, find that the gradient approaches 1 from above.

$\Rightarrow$ first order spatial discretization.

Thus giving the expected result that the spatial discretization is first order.

Looks 1 but can you provide the slope? -0.5





d) $z - z_{r_0} - ct = \frac{\beta}{\alpha}\left[b(z,t) - \sqrt{\frac{c}{\alpha}}\,\text{atanh}\left(\sqrt{\frac{\alpha}{c}}\, b(z,t)\right)\right]$ w/ z_{r_0} reference cst

$$z = z_{r_0} + ct + \frac{\beta}{\alpha} b(z,t) - \frac{\beta}{\alpha}\sqrt{\frac{c}{\alpha}}\,\text{atanh}\left(\sqrt{\frac{\alpha}{c}}\, b(z,t)\right)$$

$$b(z,t) = \frac{\alpha}{\beta}(z - z_{r_0} - ct) + \sqrt{\frac{c}{\alpha}}\,\text{atanh}\left(\sqrt{\frac{\alpha}{c}}\, b(z,t)\right)$$

$$b_t = -\frac{\alpha c}{\beta} + \sqrt{\frac{c}{\alpha}}\left(\sqrt{\frac{\alpha}{c}}\, b_t\right)\left(\frac{1}{1 - \frac{\alpha}{c} b^2}\right) = -\frac{\alpha c}{\beta} + b_t\left(\frac{c}{c - \alpha b^2}\right)$$

$$b_t\left(1 - \frac{c}{c - \alpha b^2}\right) = -\frac{\alpha c}{\beta} \Rightarrow b_t\left(\frac{c - \alpha b^2 - c}{c - \alpha b^2}\right) = -\frac{\cancel{\alpha} c}{\beta} = b_t\left(-\frac{\cancel{\alpha} b^2}{c - \alpha b^2}\right)$$

$$b_t = -\frac{c}{\beta}\left(-\frac{c - \alpha b^2}{b^2}\right) = \frac{c^2 - \alpha c b^2}{\beta b^2}$$

$$b_z = \frac{\alpha}{\beta} + \sqrt{\frac{c}{\alpha}}\sqrt{\frac{\alpha}{c}}\, b_z\left(\frac{c}{c - \alpha b^2}\right) = \frac{\alpha}{\beta} + b_z\left(\frac{c}{c - \alpha b^2}\right)$$

$$b_z\left(-\frac{\alpha b^2}{c - \alpha b^2}\right) = \frac{\alpha}{\beta} \Rightarrow b_z = -\frac{c - \alpha b^2}{\beta b^2}$$

note: $b_t = -c b_z$

WTS $\mapsto$ $\partial_t b + \partial_z(\alpha b^3 - \beta b^3 b_z) = 0$

$$b_t + 3\alpha b^2 b_z - (\beta b^3 b_z)_z$$

$$b_t + 3\alpha b^2 b_z - \left(\beta b^3\left(-\frac{c - \alpha b^2}{\beta b^2}\right)\right)_z$$

$$b_t + 3\alpha b^2 b_z - (-cb + \alpha b^3)_z$$

$$b_t + 3\alpha b^2 b_z + c b_z - 3\alpha b^2 b_z = b_t + c b_z$$

$$= b_t - b_t = 0$$

Hence given implicit $b(z,t)$ is a sol^n of eq (5) $\Rightarrow$ sol^n of (1)

$c = \alpha$, $\beta = 1.0$
$H = 3$ km, $z_{r_0} = 0.9$

$$z = z_{r_0} + \alpha t + \frac{1}{\alpha} b - \frac{1}{\alpha}\sqrt{1}\,\text{atanh}(\sqrt{1}\, b)$$

$$z = z_{r_0} + \alpha t + \frac{1}{\alpha} b - \frac{1}{\alpha}\,\text{atanh}(b)$$

$T = 0.5, 1, 2, 3, 4, 5$, $M = 10001$ $\Delta t = \frac{T}{M-1} = \frac{1}{10000} = 10^{-4} = 0.0001$

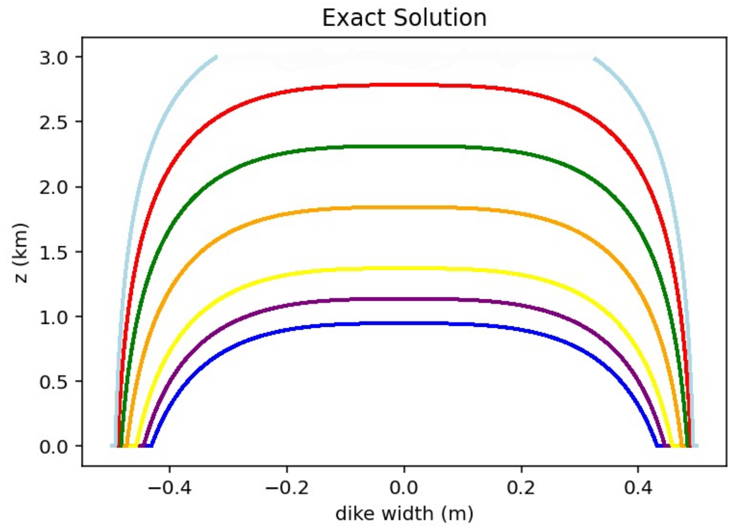
$\downarrow$ to satisfy max. principle so that $b_j^{n+1} > 0$ if $b_j^n > 0$ $\forall j$

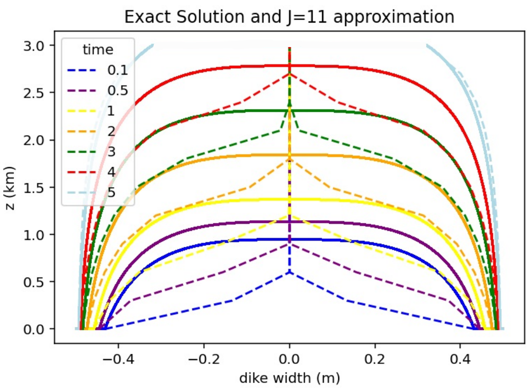
Numerical: $\underbrace{b_B = \max(b_{exact}) \quad b_T = \min(b_{exact})}_{}$

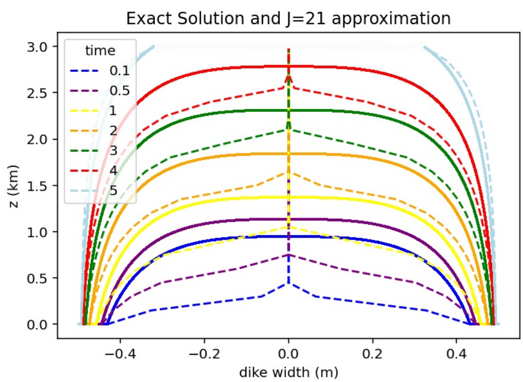
both time dependent now $\therefore$ need to be redefined each timestep.

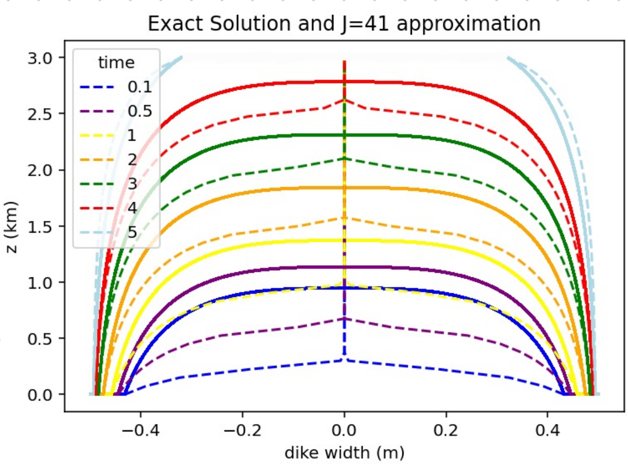
① $\partial_t b + \partial_z(ub) = 0$, $u = \alpha b^2 - \beta b^2 \partial_z b$

$$b_j^{n+1} = b_j^n - 3\alpha\mu\Delta z (b_j^n)^2(b_j^n - b_{j-1}^n) + \frac{\beta\mu}{2}\left[(b_j^n)^3\left[b_{j+1}^n - 2b_j^n + b_{j-1}^n\right] + (b_{j+1}^n)^3(b_{j+1}^n - b_j^n) - (b_{j-1}^n)^3(b_j^n - b_{j-1}^n)\right]$$









You need a much higher resolution. -0.5
18.5/20