1) i) Consider a rectangular river cross section with varying river width $w_0(s)$. Show that

$$h(A,s) = \frac{A}{w_0(s)} \quad \text{and} \quad P(A,s) = w_0(s) + 2h(A,s) = w_0(s) + \frac{2A}{w_0(s)}$$

→ By assuming the river cross section is rectangular with horizontal width $w_0(s)$ and vertical depth $h(s,t)$

→ Cross sectional wetted area is the area of water in that rectangle

Rectangular Channel area = width × depth

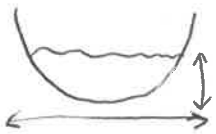
$$A(s,t) = w_0(s) \times h(s,t)$$

Depth of channel, $h(s,t) = \dfrac{A(s,t)}{w_0(s)}$ for any given s that may vary down the channel

For $P(A,s)$ wetted perimeter

↳ Sum of the bottom width and two vertical sides in contact with the water

$$P = \text{width} + 2(\text{Depth})$$



Using $P = w_0(s) + 2\left(\dfrac{A(s,t)}{w_0(s)}\right) = w_0(s) + \dfrac{2A(s,t)}{w_0(s)}$

For a fixed A by increasing the local width $w_0(s)$ it makes the depth smaller and increases P

ii) Given the continuity equation $\dfrac{\partial A}{\partial t} + \dfrac{\partial F(A,s)}{\partial s} = 0$ (as $s=0$) then the flux can be written as $F(A,s) = A \cdot u(A,s)$. We use the given Manning relation for velocity $u(A,s) = \dfrac{R^{2/3}\sqrt{-g\frac{\partial b}{\partial s}}}{C_m}$,

$$R = \frac{A}{P(A,s)} \implies F(A,s) = \frac{A^{5/3}\sqrt{-g\frac{\partial b}{\partial s}}}{C_m\, P(A,s)^{2/3}}$$

From part (i) the wetted perimeter is written only as a function of (A,s) so F can be expressed in terms of A and $w_0(s)$

$$F(A,s) = \frac{A^{5/3}\sqrt{-g\frac{\partial b}{\partial s}}}{C_m\left(w_0(s) + \frac{2A}{w_0(s)}\right)^{2/3}}$$

Non linear function of A, dependent on S

↳ To understand how this solution evolves in space and time can differentiate

F is dependent on A and S, so using chain rule $\dfrac{\partial F}{\partial s} = \dfrac{\partial F}{\partial A}\cdot\dfrac{\partial A}{\partial s} + \dfrac{\partial F}{\partial s}$ and substituting back into the continuity equation the PDE becomes

$$\frac{\partial A}{\partial t} + \underbrace{\frac{\partial F}{\partial A}\frac{\partial A}{\partial s}}_{\text{Advective term}} + \underbrace{\frac{\partial F}{\partial s}}_{\text{Geometric Source term}} = 0$$

Advective term ↑ How changes in A propogate

Geometric Source term ↑ how channel shape and slope affect the flow

∂A : How fast information travels through the system

∂s : Changes in geometry like channel shape and bed shape affect flow

By computing $\frac{\partial F}{\partial A}$ we can determine the direction and speed changes in A propogate through the domain

- This allows us to determine the direction of information flow (downstream vs upstream) and so deciding which numerical scheme to use

· Going from $\frac{\partial A}{\partial t} + \frac{\partial F(A,s)}{\partial s} = 0$ to $\frac{\partial A}{\partial t} + \frac{\partial F}{\partial A}\frac{\partial A}{\partial s} + \frac{\partial F}{\partial s} = 0$

For $\frac{\partial F}{\partial A}$: $F(A,s) = A^{5/3} Q(s)\, P(A,s)^{-2/3}$ where $Q(s) = \frac{\sqrt{-g\frac{\partial b}{\partial s}}}{C_m}$ $P(A,s) = \omega_o(s) + \frac{2A}{\omega_o(s)}$

$$\frac{\partial F}{\partial A} = Q(s)\left[\frac{5}{3}A^{2/3}\left(-\frac{2}{3}P^{-5/3}\left(\frac{2}{\omega_o(s)}\right)\right)\right] = \cancel{Q(s)A\,P^{5/3}}$$

$$\frac{\partial F}{\partial A} = \qquad u = A^{5/3} \quad v = P^{-2/3} = \left(\omega_o(s) + \frac{2A}{\omega_o(s)}\right)^{-\frac{2}{3}}$$

$$u' = \frac{5}{3}A^{2/3} \quad v' = -\frac{2}{3}P^{-5/3} =$$

$$= -\frac{2}{3}P^{-5/3}A^{5/3} + \frac{5}{3}A^{2/3}P^{-2/3}$$

$$= Q\omega_o\left(\frac{5A^{2/3}}{3}\left(\omega_o + \frac{2A}{\omega_o}\right)^{-\frac{2}{3}} + \left(-\frac{4\omega^{2/3}}{3(2A+\omega_o^2)^{5/3}}\right)A^{5/3}\right)$$

$$= (Q\omega_o)\left(\frac{5A^{2/3}\omega^{8/3} + 6A^{5/3}\omega^{2/3}}{3(\omega^2+2A)^{5/3}}\right)$$

with $Q = \frac{\sqrt{-g\frac{\partial b}{\partial s}}}{C_m}$ then $\frac{\partial F}{\partial A}$ matches given equation 5b

$$\frac{\partial F}{\partial A} = \frac{\sqrt{-g\frac{\partial b}{\partial s}}}{3C_m}\,\frac{5A^{2/3}\omega_o^{8/3} + 6A^{5/3}\omega_o^{2/3}}{(\omega_o^2+2A)^{5/3}} = \frac{\sqrt{-g\frac{\partial b}{\partial s}}}{3C_m}\,\frac{5A^{2/3}\omega_o + \frac{6A^{5/3}}{\omega_o}}{\left(\omega_o + \frac{2A}{\omega_o}\right)^{5/3}}$$

For $\frac{\partial F}{\partial s}$; $F(A,s) = \frac{A^{5/3}\sqrt{-g\frac{\partial b}{\partial s}}}{C_m\left(\omega_o(s) + \frac{2A}{\omega_o(s)}\right)^{2/3}}$

$$\frac{\partial F}{\partial s} = A^{5/3}\left[\frac{d}{ds}\left(\frac{\sqrt{-g\frac{\partial b}{\partial s}}}{C_m}\right)\left(\omega_o + \frac{2A}{\omega_o}\right)^{-2/3} + \frac{\sqrt{-g\frac{\partial b}{\partial s}}}{C_m}\underbrace{\frac{d}{ds}\left(\omega_o + \frac{2A}{\omega_o}\right)^{-2/3}}_{-\frac{2}{3}\left(\omega_o + \frac{2A}{\omega_o}\right)^{-\frac{5}{3}}\frac{d}{ds}\left(\omega_o + \frac{2A}{\omega_o}\right)}\right]$$

$$\hookrightarrow \omega_o'(s)\left(1 - \frac{2A}{\omega_o(s)^2}\right)$$

$$\Rightarrow \frac{\partial F}{\partial s} = A^{5/3}\left(\omega_o + \frac{2A}{\omega_o}\right)^{-5/3}\left[\left(\frac{d}{ds}\frac{\sqrt{-g\frac{\partial b}{\partial s}}}{C_m}\right)\left(\omega_o + \frac{2A}{\omega_o}\right) - \frac{2}{3}\left(\frac{\sqrt{-g\frac{\partial b}{\partial s}}}{C_m}\omega_o'(s)\left(1 - \frac{2A}{\omega_o(s)^2}\right)\right)\right.$$

In order to get this into the form given by the tasksheet as 5C we assume $\cancel{\frac{\partial b}{\partial s}}$ b is constant

$$\Longrightarrow \cancel{\frac{\partial F}{\partial s} \left(A^{5/3} \left(w_0 + \frac{2A}{w_0} \right)^{-5/3} - \frac{2}{3} \sqrt{-g} \right.}$$

This term was the slope derivatives and is neglected under the assumption that the varying bed slope is slow

$$\frac{\partial F}{\partial s} = -\frac{2\sqrt{-g\frac{\partial b}{\partial s}}}{3C_m} \frac{A^{5/3}}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}} \left(1 - \frac{2A}{w_0^2}\right) w_0'(s)$$

$$\Longrightarrow \frac{\partial F}{\partial s} = \frac{-2\sqrt{-\partial b/\partial s}}{3C_m} \frac{A^{5/3}}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}} \left(1 - \frac{2A}{w_0^2}\right) \frac{dw_0(s)}{ds}$$

2) i) Kinematic limit with $S = 0$ and width & $w_0(s)$ independent of S, rewrite PDE in quasilinear form and use the eigenvalue to identify the local wave speed

$$\lambda(A) = \frac{\partial F}{\partial A} = \frac{\sqrt{-\frac{\partial b}{\partial s}}}{3C_m} \frac{\left(5w_0 A^{2/3} + 6\frac{A^{5/3}}{w_0}\right)}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}} > 0 \quad \text{as given by 5b}$$

ii) Solve the Riemann problem for (3) in limit of no explicit S-dependence

$w_0(s)$ and $\frac{\partial b}{\partial s}$ are constant the PDE becomes

$$\frac{\partial A}{\partial t} + \frac{\partial F(A)}{\partial s} = 0 \quad \text{with } F(A) = \frac{A^{5/3}\sqrt{-g\frac{\partial b}{\partial s}}}{C_m\left(w_0 + \frac{2A}{w_0}\right)^{2/3}}$$

Required piecewise constant initial data A_l and A_r

What is a Riemann problem?

- Simplest initial value problem for a conservation law.
- Take a PDE of the form $\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial t} = 0$ and ~~so~~ decide on piecewise constant initial data with a single Jump point

$$u(x,0) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}$$

- How does this discontinuity evolve in time ~~g~~ under the given PDE

Conservation law is $\frac{\partial A}{\partial t} + \frac{\partial F}{\partial s}(A) = 0$ with Flux $F(A) = \frac{A^{5/3}\sqrt{g\frac{\partial b}{\partial s}}}{C_m\left(w_0 + \frac{2A}{w_0}\right)^{2/3}}$

So A is the conserved variable which is the wetted area and dependent on $s, t \Rightarrow A(s,t)$

Riemann problem becomes

$$A(s,0) = \begin{cases} A_L, & s < 0 \\ A_R, & s > 0 \end{cases} \quad \text{with the given piecewise constant initial data}$$

Given the eigenvalue $\lambda(A) = \frac{\partial F}{\partial A}$, $A > 0$, $w_0 > 0$ and $-\frac{\partial b}{\partial s} > 0$ for all physical states

$\Rightarrow$ all characteristics move downstream (to the right)

If $A_L < A_R$: Rarefaction fan spreading to the right
If $A_L > A_R$: Shock moving right at speed

The solution is either a shock or a rarefaction, depending on whether the left state is bigger or smaller than the right

In this river equation because the eigenvalue is always positive, everything moves downstream and $F(A)$ increases with A causing this convex behaviour giving standard shock / rarefaction structure

For Rarefaction $A_L < A_R$

- An increase in area moves downstream ; because F is strictly increasing in A, characteristic speeds increase with A. Left characteristics ~~A~~ A_L are slower than right characteristics A_R → This causes the characteristics to spread apart forming a continuous expanding fan

- We can introduce ξ as a variable ~~*~~ which $\xi = \frac{s}{t}$ and is a self similar variable. The solutions are self similar which means it depends on the ratio of space to time, not s and t seperately and that's the behaviour ξ captures. It represents the slope of a line through the origin in the (t, s) plane

- For $\xi < \lambda(A_L)$ → state remains A_L For $\xi > \lambda(A_R)$ → state remains A_R

$\Rightarrow$ For $\lambda(A_L) \leq \xi \leq \lambda(A_R)$ → state varies smoothly since $\xi = \lambda(A(\xi))$

For Shock $A_L > A_R$

- Characteristics from the left state are faster than those from the right. They collide and compress into a discontinuity (shock). This shock moves downstream as $\lambda(A) > 0$
- The shock speed is given by the Rankine-Hugoniot condition $C_s = \dfrac{F(A_r) - F(A_L)}{A_r - A_L}$
- Since F is increasing and $A_r < A_L$ both numerator and denominator are negative so $C_s > 0$
- $\lambda(A_L) > C_s > \lambda(A_r)$ ensures shock is physically admissable ???

So the solution structure is that the piecewise constant is seperated by the shock line

$$A(s,t) = \begin{matrix} A_L, & s < C_s t \\ A_r, & s > C_s t \end{matrix}$$

3) $S=0$

i) Deriving finite volume / Godunov scheme

Cell average : $A_k(t) = \frac{1}{h_k}\int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} A(s,t)\,ds$

With $S=0$ and conservation law $\frac{\partial A}{\partial t} + \frac{\partial F}{\partial s} = 0$ integrate PDE over cell A_k

$$\frac{d}{dt}\int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} A\,ds = -\int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \frac{\partial F}{\partial s}\,ds = -\Big[F(A,s)\Big]_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} = -\left(F_{k+\frac{1}{2}}(t) - F_{k-\frac{1}{2}}(t)\right)$$

$$\frac{d}{dt}A_k(t) = \frac{-1}{h_k}\left(F_{k+\frac{1}{2}}(t) - F_{k-\frac{1}{2}}\right)$$

→ $F_{k\pm\frac{1}{2}}(t)$ = Fluxes at cell faces $s_{k\pm\frac{1}{2}}$

→ For Godunov's method, these aren't from a pointwise value but from a solution of the local Riemann problem between the adjacent cell averages

Forward Euler in time

By considering a timestep from $t^n \to t^{n+1} = t^n + \Delta t$ then

$$A_k^{n+1} = A_k^n - \frac{\Delta t}{h_k}\left(F^n_{k+\frac{1}{2}} - F^n_{k-\frac{1}{2}}\right)$$

For explicit spatial locations

$$A_k^{n+1} = A_k^n - \frac{\Delta t}{h_k}\left(F(A,s)\Big|^n_{s=s_{k+\frac{1}{2}}} - F(A,s)\Big|^n_{s=s_{k-\frac{1}{2}}}\right)$$

The face fluxes are explicitly at the spatial location so that each interface flux is computed by a Godunov scheme using the ~~[illegible]~~ adjacent states at time level n

→ At interface $s_{k+\frac{1}{2}}$ the piecewise constant initial data ~~[illegible]~~ is $A(s,0) = \begin{cases} A_L = A_k^n, & s < s_{k+\frac{1}{2}} \\ A_r = A_{k+1}^n, & s > s_{k-\frac{1}{2}} \end{cases}$

and evolves by $\frac{\partial A}{\partial t} + \frac{\partial F(A,s)}{\partial s} = 0$

→ The Godunov flux at an interface is the physical flux of the exact Riemann solution evaluated at the interface in the limit of vanishing time

$$F^n_{k+\frac{1}{2}} = F(s = s_{k+\frac{1}{2}}, t\to 0)$$ and depends only on (A_L, A_r) at time n

ii) Why only old-time cell averages are needed?

→ Riemann solution is self similar and determined only by data at the instant t^n

→ The interface state as $t \to 0$ is only a function of A_L and A_R

This is why we don't need ~~[illegible]~~ future information because the Godunov flux at t^n uses the cell averages at t^n

The finite volume update (6) follows from integrating the conservation law over each cell and replacing face fluxes by Godunov fluxes obtained from the local riemann problem between the left/right cell averages at time n because the Riemann solution at $t \to 0$ depends only on these left/right states, the Godunov method uses only old time cell averages.

4) Derive what the Godunov flux is using Riemann Solution, given (3) and (5) for $S=0$

$\to$ Use kinematic river equation in ~~case~~ conservation form with $S=0$: $\frac{\partial A}{\partial t} + \frac{\partial F(A,s)}{\partial s} = 0$ and Riemann framework at each interface $k+\frac{1}{2}$ with $A_L = A_k^n$, $A_r = A_{k+1}^n$

$\to$ No explicit s-dependence so $F(A,s) \to F(A)$, so local Riemann problem is scalar conservation law

$\to$ In general case (slowly varying), flux is evaluated with local geometry and then the source is treated separately

General Case

$$A(s,0) = \begin{cases} A_L & s < S_{k+\frac{1}{2}} \\ A_r & s > S_{k+\frac{1}{2}} \end{cases}$$

Riemann problem

Godunov flux definition

$$F_{k+\frac{1}{2}} = F(A^*) \quad \text{where } A^* = \lim A(S_{k+\frac{1}{2}}, t)$$

where A^* depends only on A_L and A_r. For a scalar law with only increasing flux $F(A) = \lambda(A) > 0$, all characteristics move to the right, so $A^* = A_L$. $\Leftarrow$ Purely upwind to the right

Specific to Kinematic river equation

$$\Rightarrow F_{k+\frac{1}{2}} = F(A_L)$$

interface state immediately after $t=0$ is taken from the left

~~flux at interface $S_{k+\frac{1}{2}}$: $F(A, S_{k+\frac{1}{2}})$~~ =

Using equation (5) the flux is given by

$$F(A,s) = \frac{A^{\frac{5}{3}} \sqrt{-g \frac{\partial b(s)}{\partial s}}}{C_m \left(w_o(s) + \frac{2A}{w_o(s)}\right)^{\frac{2}{3}}}$$

Use equation 5b with the eigenvalue to determine wave direction

$$\lambda(A) = \frac{\partial F}{\partial A} > 0 \quad \text{for all physical states } \left(A>0,\ w_o>0,\ -\frac{\partial b}{\partial s}>0\right)$$

this means all characteristics move downstream (to the right

Because $\lambda(A) > 0$, the state at the interface immediately after $t=0$ is the left state A_L

$$\Rightarrow \text{Godunov flux}: F(A_L; S_{k+\frac{1}{2}}) = F(A_k^n; S_{k+\frac{1}{2}})$$

This is purely upwind flux to the right

Flux expression

$$\Rightarrow F_{k+\frac{1}{2}}^n = \frac{(A_k^n)^{5/3} \sqrt{-g \frac{\partial b(S_{k+\frac{1}{2}})}{\partial s}}}{C_m \left(w_o(S_{k+\frac{1}{2}}) + \frac{2A_k^n}{w_o(S_{k+\frac{1}{2}})}\right)^{2/3}}$$

This flux is used in the finite volume to compute the net flow in/out of each cell and is consistent with the downstream propagation of information implied by the positive eigenvalue

5) Derive a timestep restriction or CFL condition based on $\Delta t < CFL \min_k \frac{h_k}{|\lambda_k|}$

By continuing considering the assumptions of

→ Forward-Euler in time with upwind Godunov flux

→ Homogenous conservation law $\frac{\partial A}{\partial t} + \frac{\partial F(A,s)}{\partial s} = 0$

→ $A > 0$, $\omega_0 > 0$, $-\frac{\partial b}{\partial s} > 0 \Rightarrow$ downhill slope, the eigenvalue

In the finite volume Godunov scheme for kinematic river equation, stability of explicit time integration requires timestep restriction

$$\frac{\partial A}{\partial t} + \frac{\partial F(A,s)}{\partial s} = 0 \qquad F(A,s) = \frac{A^{5/3}\sqrt{-g\frac{\partial b(s)}{\partial s}}}{C_m\left(\omega_0(s) + \frac{2A}{\omega_0(s)}\right)^{2/3}}$$

Local characteristic speed (eigenvalue) gotten by differentiating F with respect to A

$$\lambda(A,s) = \frac{\partial F}{\partial A} = \frac{\sqrt{-g\frac{\partial b(s)}{\partial s}}}{3C_m} \frac{5\omega_0(s)A^{2/3} + \frac{6A^{5/3}}{\omega_0(s)}}{\left(\omega_0(s) + \frac{2A}{\omega_0(s)}\right)^{5/3}}$$

For physical states the eigenvalue is positive so all data information propagates to the right

CFL condition states that in an explicit scheme, information must not travel more than one cell length in a single time step so $\Delta t \leq \frac{h_k}{\lambda_k}$ where λ_k is the local eigenvalue at the cell average A_k and position

To ensure stability, use minimum over all cells and $0 < CFL < 1$ so that

$$\Delta t \leq CFL \min_k \frac{h_k}{\lambda_k}$$

This is with the characteristic speed approximated as $c = \sqrt{gh}$, shallow water instead using exact eigenvalue (equation 6b) and $h = \frac{A}{\omega_0}$, $\omega_0 + \frac{2A}{\omega_0} = \omega_0\left(1 + \frac{2h}{\omega_0}\right)$ which means that

λ scales with $\sqrt{-g\frac{\partial b}{\partial s}}$ so the improved CFL;

$$\Delta t \leq CFL \min_k \frac{h_k}{|\lambda_k|} \quad \text{where } \lambda_k = \left.\frac{\partial F}{\partial A}\right|_{A=A_k,\, s=s_k}$$

As the wavespeed bound becomes $a_{k+\frac{1}{2}} = \max(\lambda_L, \lambda_R, |c_s|)$ from the local Riemann problem

So the CFL condition becomes

$$\Delta t \leq CFL \min_k \frac{h_k}{\max\left(a_{k-\frac{1}{2}}, a_{k+\frac{1}{2}}\right)}$$

which incorporates Manning flux law and geometry

6) Boundary Conditions

→ Determine how information enters and leaves computational domain

→ Eigenvalue (equation 5b) positive so all characteristics propagate downstream

~~[illegible]~~ Inflow : informations enters from the left since $\lambda > 0$
$S = 0$

$$A^n_{-1} = A^n_{in} \implies F^n_{\frac{1}{2}} = F(A^n_{-1}, S_{\frac{1}{2}})$$

Outflow : right boundary cannot send information upstream since $\lambda > 0$
$S = L$

So extrapolate from last interior state

$$A^n_N = A^n_{N-1} \implies F^n_{N-\frac{1}{2}} = F(A^n_{N-1}, S_{N-\frac{1}{2}})$$

Evaluated W_0 and $\frac{\partial b}{\partial s}$ at $(S_{\frac{1}{2}}, S_{N_0-\frac{1}{2}})$

The outflow should be a boundary that doesn't allow waves to exit without reflection

$\lambda(A)$ is positive so all characteristics propagate downstream. Godonuv flux upwind to the right so inflow boundary is at the upstream end and outflow boundary is non reflecting

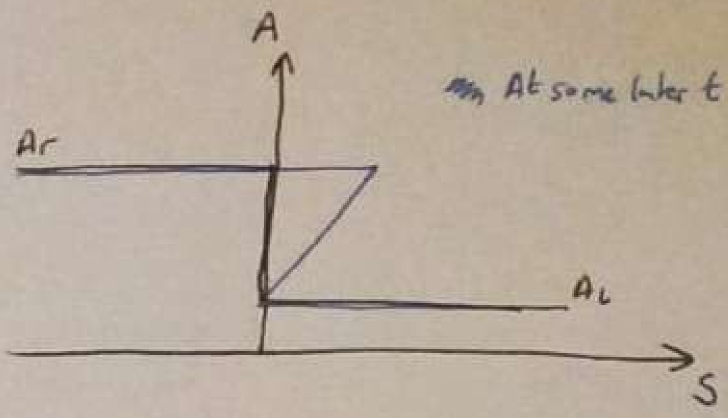
↳ At upstream boundary inflow set by depth/area or discharge and inverting flux ~~[illegible]~~ relation. At downstream boundary, extrapolation of last interior state ensures wave exit without reflection freely

↳ BC's evaluated with local geometry at faces, consistent with the flux definition, then source terms treated seperately for stability and consistency sketches

2) $t=0$ initial data:

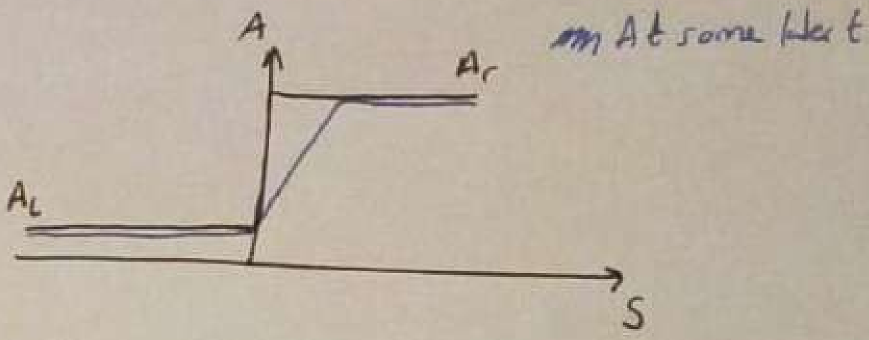
For $A_L > A_r$

$$u = \frac{F(A_r) - F(A_L)}{A_r - A_L}$$

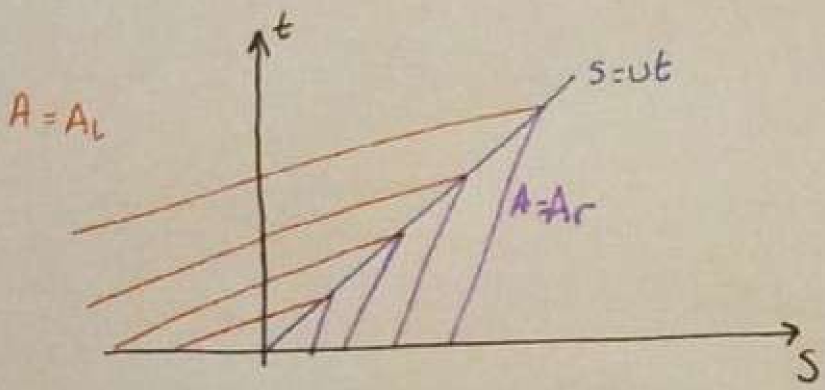


$$A_r > A_L$$

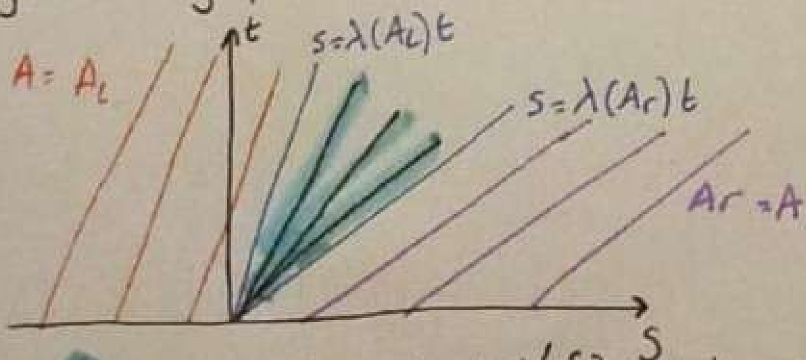
$$u = \frac{F(A_L) - F(A_r)}{A_L - A_r}$$



Left moving quicker than right



Right moving quicker than left



$s = ut$ where $A = \lambda^{-1}\left(\frac{s}{t}\right)$ which gives rarefaction region

7 - Numerics in Firedrake

Test Case 0:

Completed using the code provided with minor adjustments to adjust plotting graphics

For determining the order of accuracy in space and time convergence tests were conducted for various Nx and CFL.

Spatial Convergence

For spatial convergence Nx was varied and to keep the timestep constant then CFL was also adjusted according to the relation Nx $\propto$ CFL.

	Run 1	Run 2	Run 3	Run 4
Nx	1 000	2 500	5 000	10 000
CFL	0.2	0.5	1	2

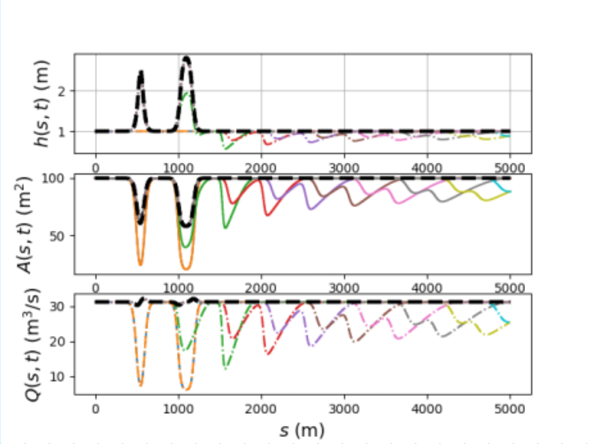


Figure - Spatial Nx = 1000; CFL=0.2

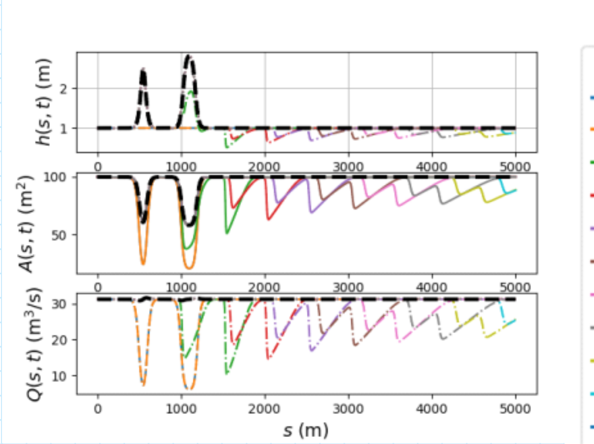
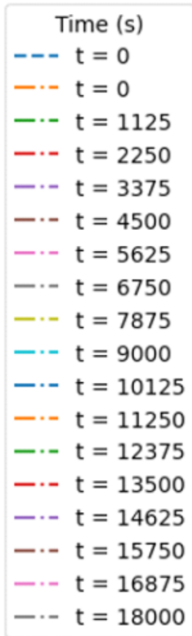


Figure - Spatial Nx = 2500; CFL = 0.5



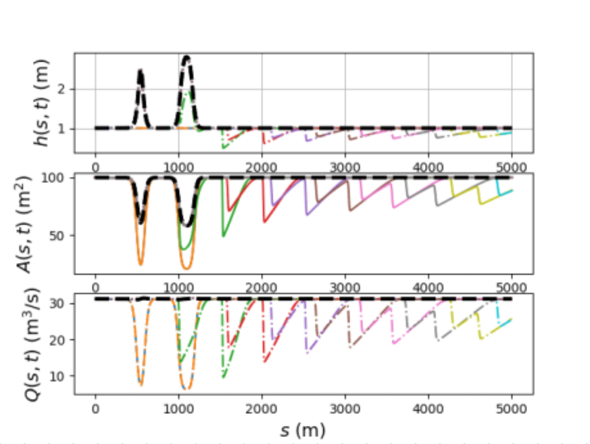


Figure - Spatial Nx = 5000; CFL = 1

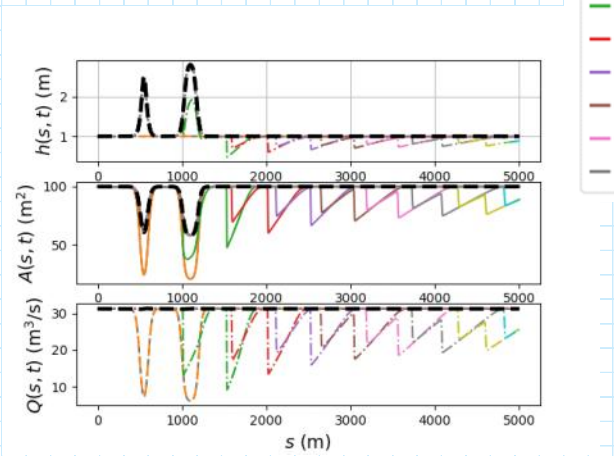
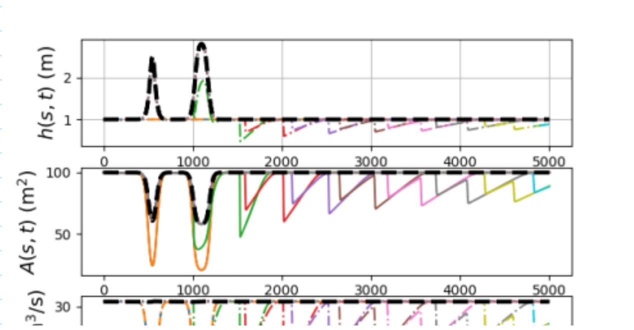


Figure - Spatial Nx = 10 000; CFL = 2

We propose the approximate final numerical solution to be using a finest mesh of Nx = 10 000 and CFL = 0.125



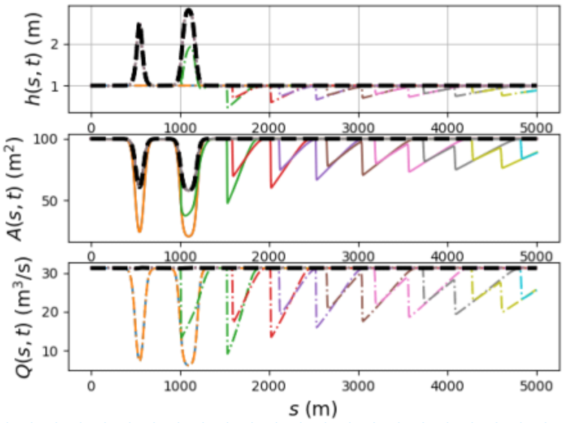
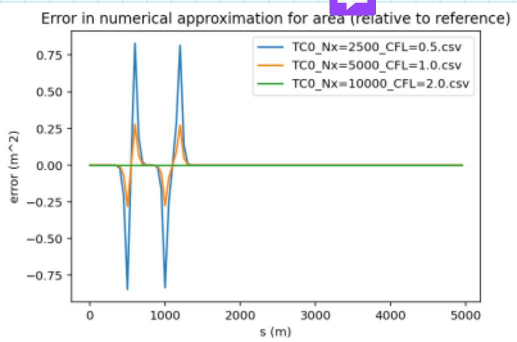
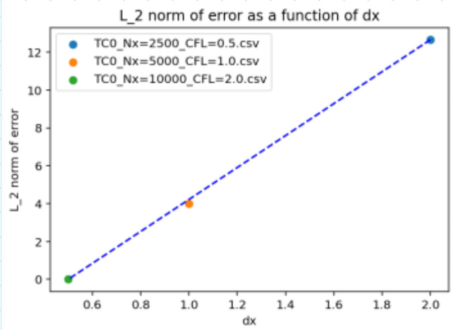


Figure 5 : Finest Mesh Nx= 10 000 ; CFL=0.125

By considering the final iteration of A(s,t) for t = 18000s the errors between each run case and the approximate solution can be considered. The error plots show that as the spatial step (dx) decreases it converges towards that of the finest mesh (actual solution). The L_2 norm of the error as a function of spatial step shows a best fit of a linear relation between the two variables. This shows that the smaller dx considered, the closer to constant the final Q(s,t) is and therefore there is first order accuracy with respect to dx.





Timestep Convergence

To alter the timestep needed to use the relation derived in part 5 that gives $\Delta t = \frac{\Delta x}{\lambda_k}$ which means a range of CFL values need to be run for a constant Nx. This was chosen to be 5000, a decision based on a balance between accuracy and runtime of the code.

	Run 1	Run 2	Run 3	Run 4
CFL	0.125	0.25	1	2

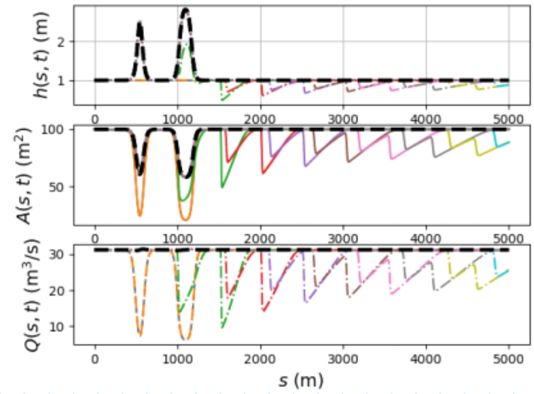


Figure - Spatial Nx = 5 000 ; CFL = 0.125

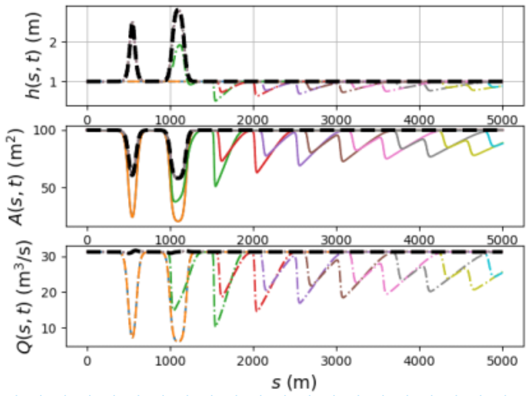
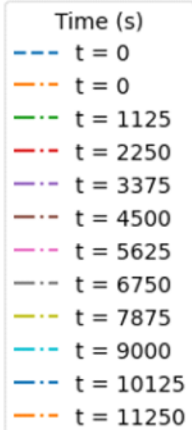
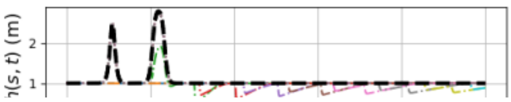
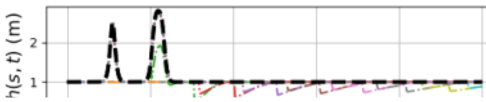
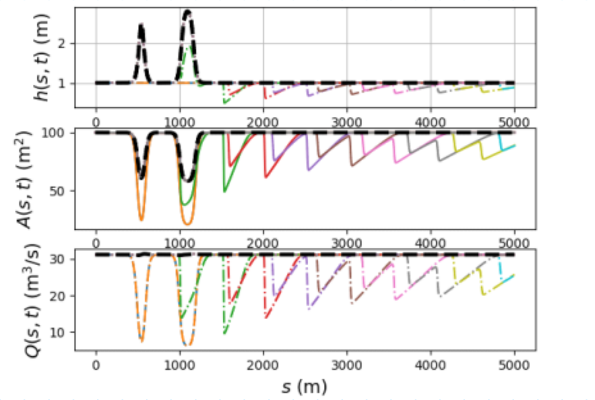


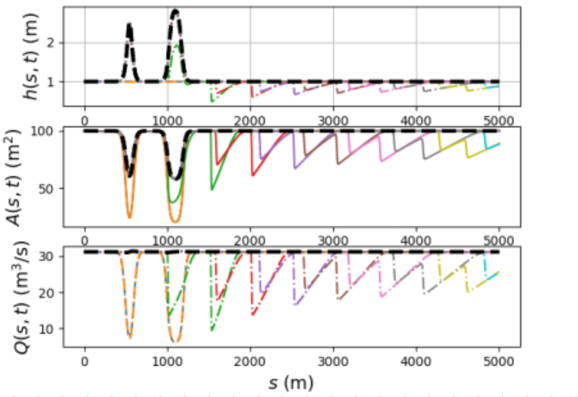
Figure - Spatial Nx = 5 000 ; CFL = 0.25











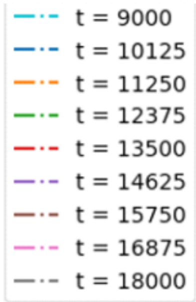
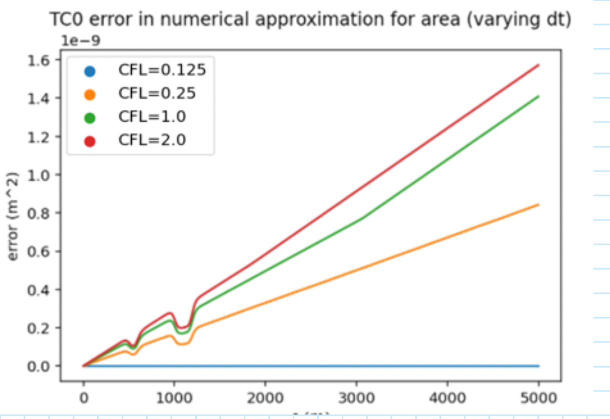
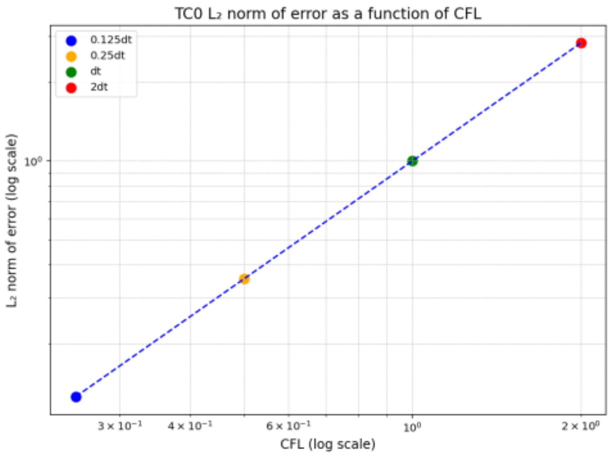


Figure - Spatial Nx = 5 000 ; CFL = 1

Figure - Spatial Nx = 5 000 ; CFL = 2

By considering the final iteration of A(s,t) for t = 18000s the errors between each run case and the approximate solution can be considered. The error plots show that as the spatial step (dx) decreases it converges towards that of the finest mesh (actual solution). The L_2 norm of the error as a function of time step shows a best fit of a linear relation between the two variables when plotted on a log log axes. This shows that the smaller dt considered, the closer to constant the final Q(s,t) is and therefore there is first order accuracy with respect to dt, as well as the previously established ds.

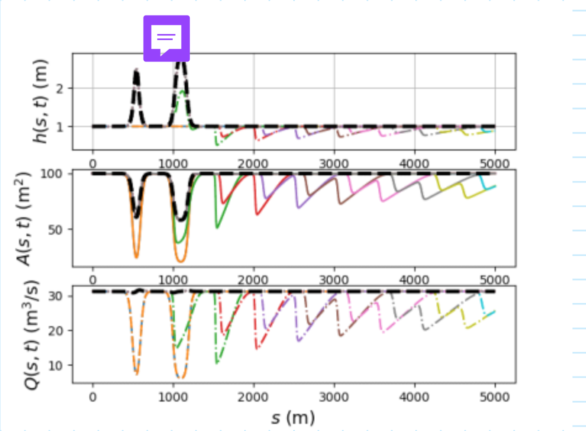


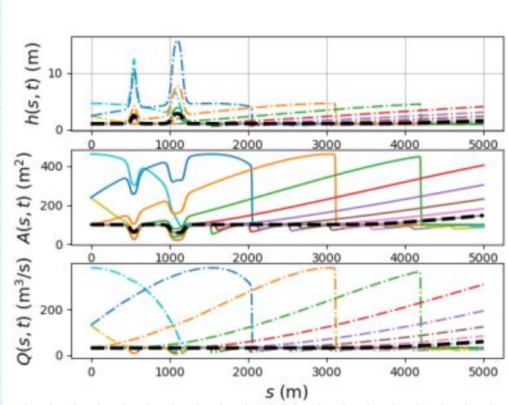


Test Case 1:

Again ran with provided code and with setting Q max = 1*350 and a few minor plotting adjustments.
This was the same spatial and time convergence tests and error analysis

Spatial Convergence:





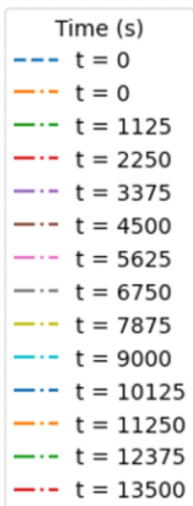
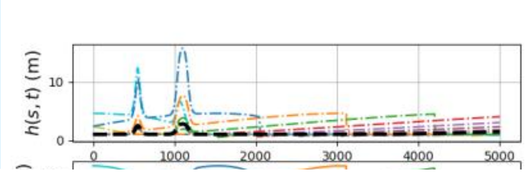
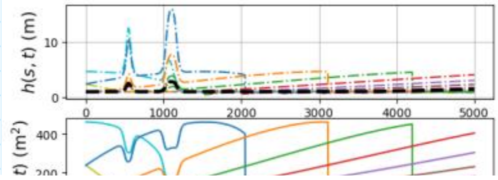


Figure - Spatial Nx = 1000 ; CFL = 0.2

Figure - Spatial Nx = 2500 ; CFL = 0.5





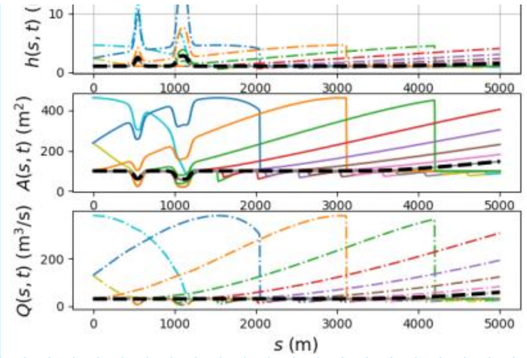


Figure - Spatial Nx = 5000 ; CFL = 1

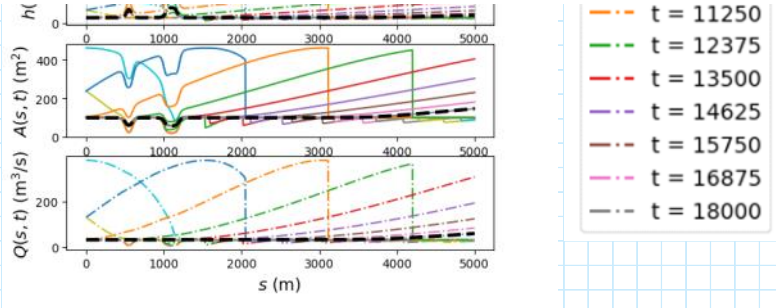
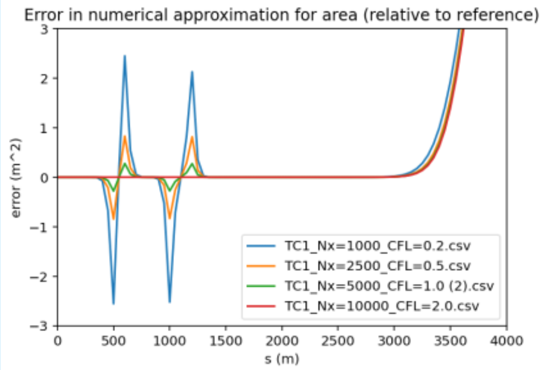
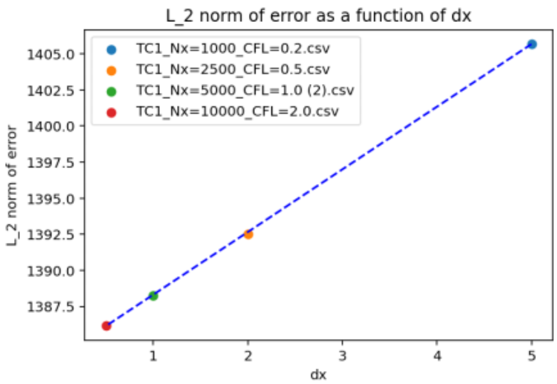


Figure - Spatial Nx = 10 000 ; CFL = 2





Timestep Convergence:

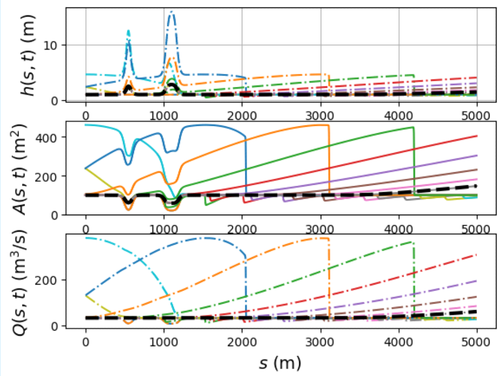


Figure - Timestep Nx = 5000 ; CFL = 0.125

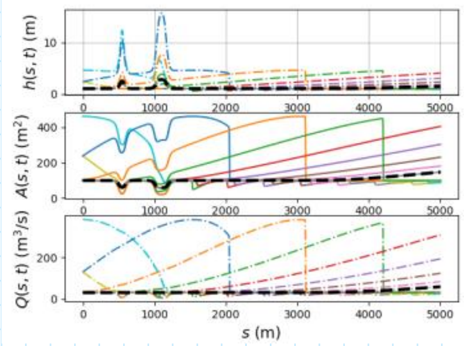


Figure - Timestep Nx = 5000 ; CFL = 0.25

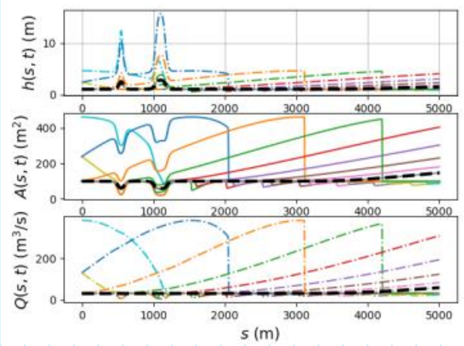


Figure - Timestep Nx = 5000 ; CFL = 0.25

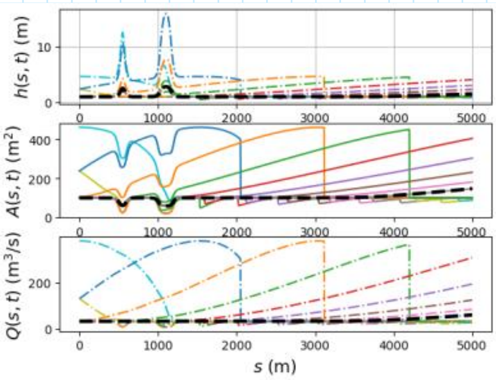
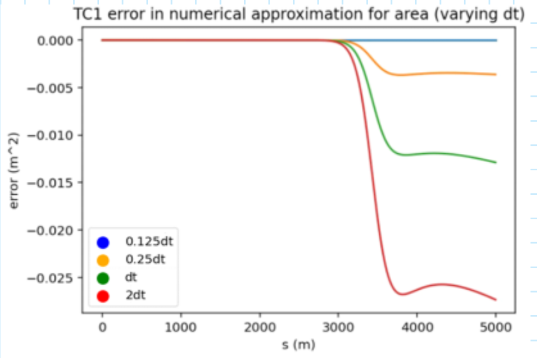
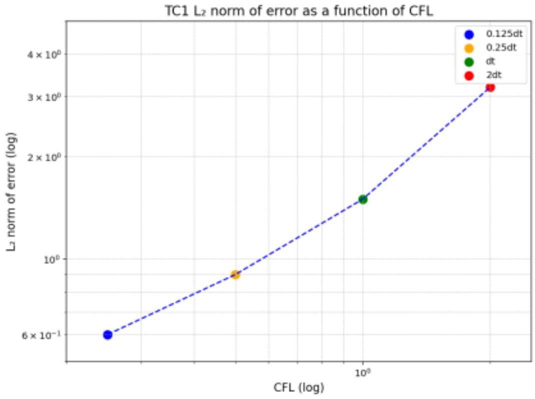


Figure - Timestep Nx = 5000 ; CFL = 0.25

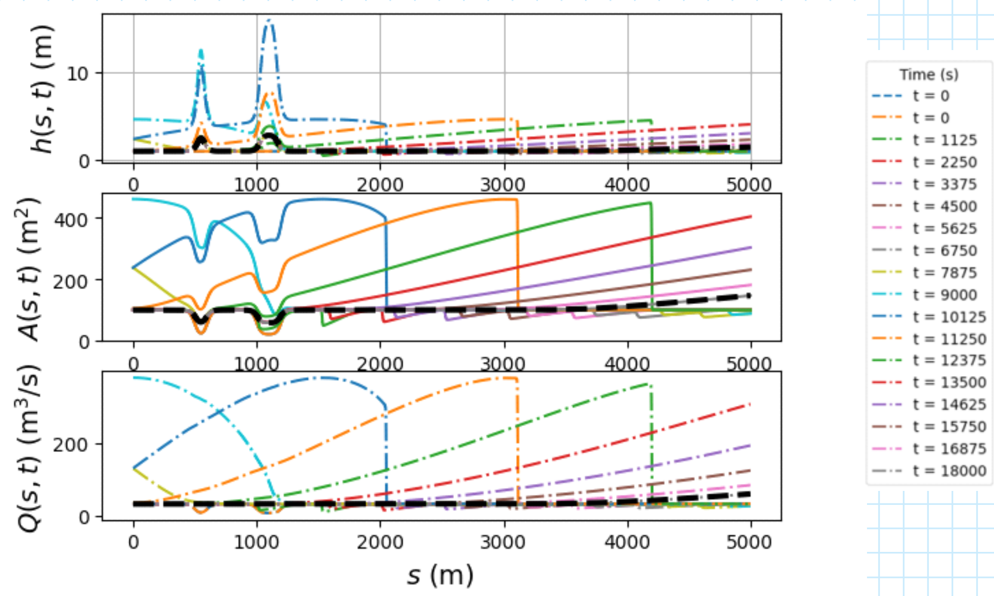




By considering the final iteration of A(s,t) for t = 18000s the errors between each run case and the approximate solution can be considered for changes in spatial and time steps. The error plots show that as the spatial step (dx) and timestep (dt) decreases it converges towards that of the finest mesh (actual solution). The L_2 norm of the error as a function of time step and spatial shows a best fit of a linear relation between the two variables when plotted on a log log axes. This gives the same results as TCO : they have convergence and first order accuracy.

Shock Speed

This plot shows how for times 10125s, 11250s, 12375s there is evidence of shocks and these can be used to compare to the actual shock speed



Numerical shock speed:

$$U = \frac{F(A_R) - F(A_L)}{A_R - A_L}$$

For $10125s = t$

$A_L = 400$, $A_R = 100$, $S = 2050$

$F(A_L) = Q(A_L) = 300$, $F(A_r) = Q(A_r) = 31$

$\Rightarrow U \approx 0.90 ms^{-1}$

For $t = 11250s$

$A_L = 460$, $A_r = 100$, $S = 3115$

$F(A_L) = Q(A_L) = 380$, $F(A_r) = Q(A_r) = 31$

$\Rightarrow U \approx 0.969 ms^{-1}$

For $t = 12375s$

$A_L = 450$, $A_r = 100$, $S = 4200$

$F(A_L) = Q(A_L) = 366$, $F(A_r) = Q(A_r) = 31$

$\Rightarrow U \approx 0.96 ms^{-1}$

Observed Shock Speeds

$10125s \longrightarrow 11250s$

$$\frac{3115 - 2050}{11250 - 10126} \approx 0.96 ms^{-1}$$

$10125s \longrightarrow 12375s$

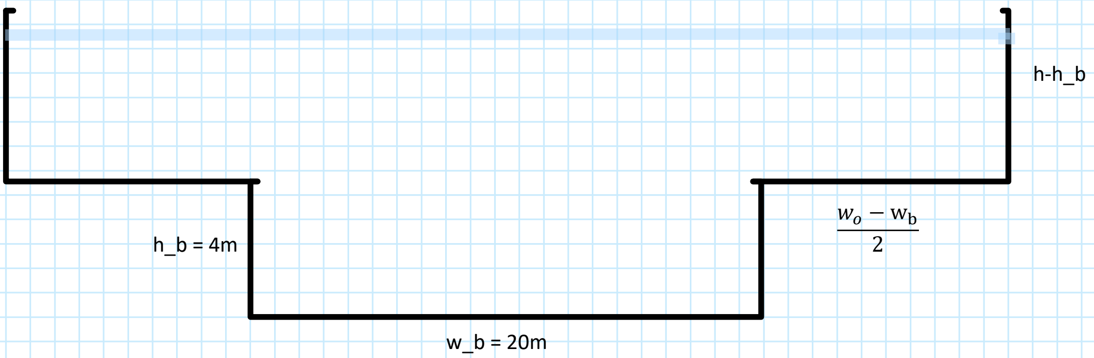
$$\frac{4200 - \cancel{3115}\ 2050}{12375 - 10125} \approx 0.96 ms^{-1}$$

$11250s \longrightarrow 12375s$

$$\frac{\cancel{2050 - 4200}\ 4200 - 3115}{12375 - 11250} \approx 0.96 ms^{-1}$$

These show that the numerical shock speed $\simeq$ observed shock speed

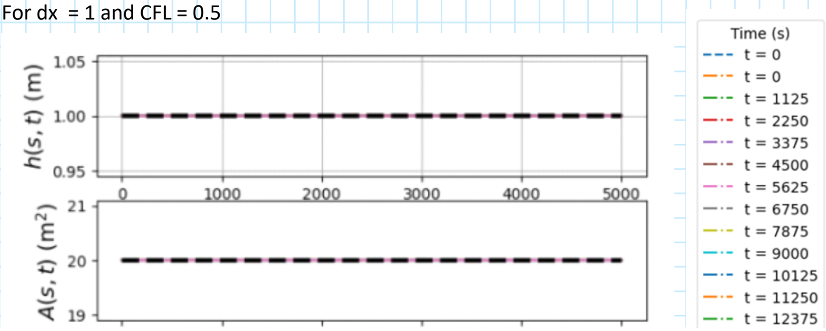
Test Case 2a:

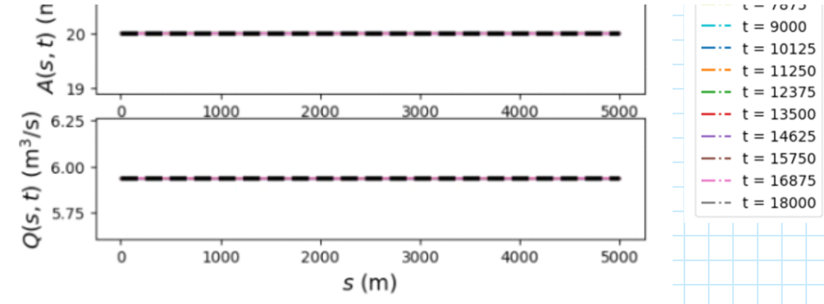


Area: $A = w_b h_b + w_0(s)(h - h_b)$

Wetted perimeter: $p = w_b + 2h_b + w_0 + 2/w_0(A - w_b h_b)$

For dx = 1 and CFL = 0.5

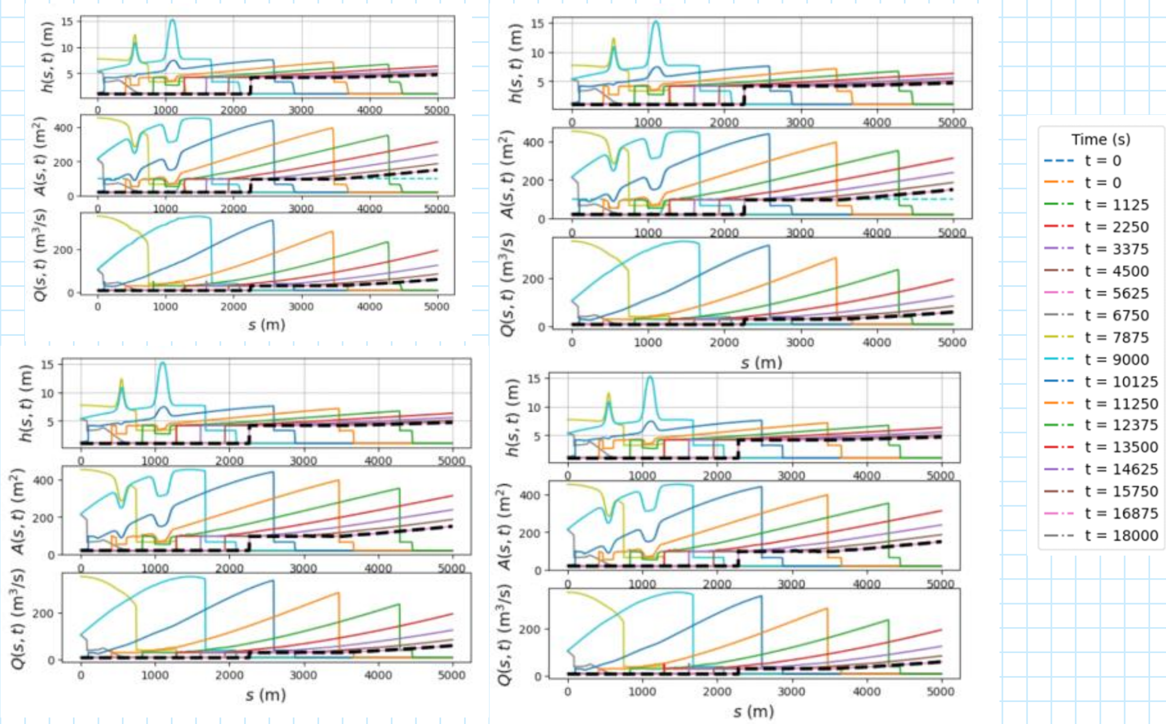




The river channel is constantly rectangular so everything is constant when there is a constant flux. This is shown by running the code in test case 0 and showing constant profiles

Test Case 2b:

Spatial Convergence: Repeating same runs as test case 0 we get:



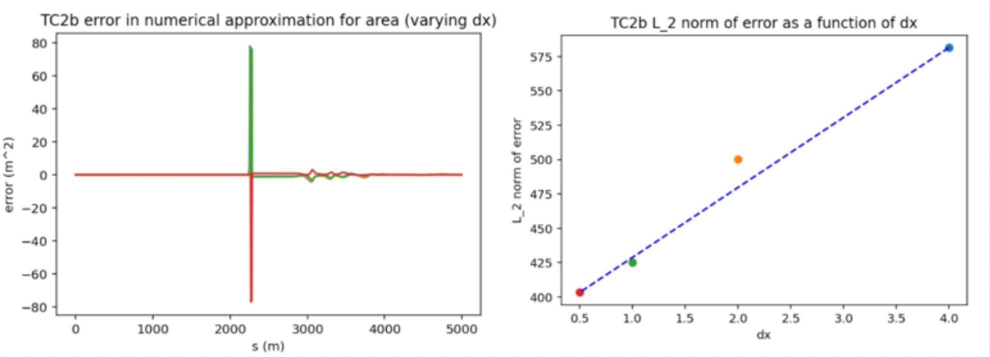
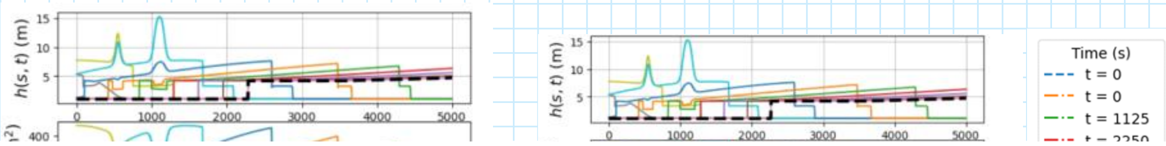
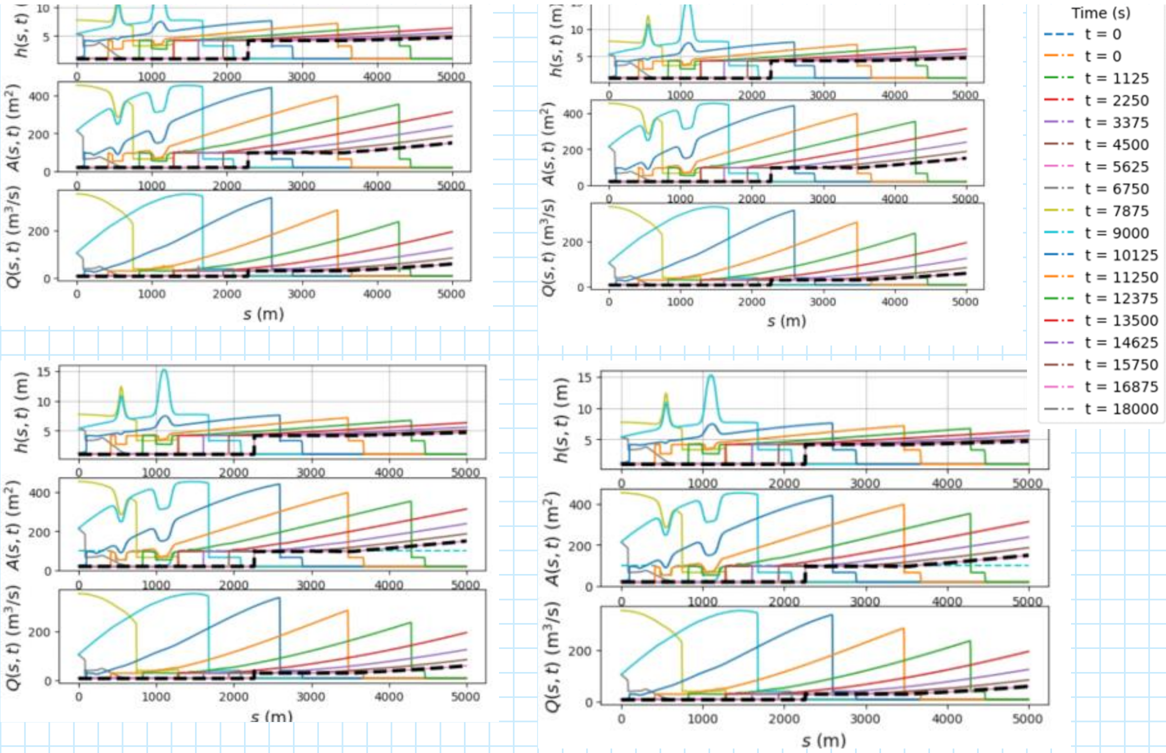


Figure : Spatial error analysis (Blue = 2dx, orange = dx, green = 0.5dx, red = 0.2dx

The error plots still show the same result that the solutions are converged in dx. The l2-norm shows there is still a mostly linear relation indicating a first order accuracy. However need a finer mesh for the numerical solution to converge.

Timestep Convergence: Repeating same runs as test case 0 we get:





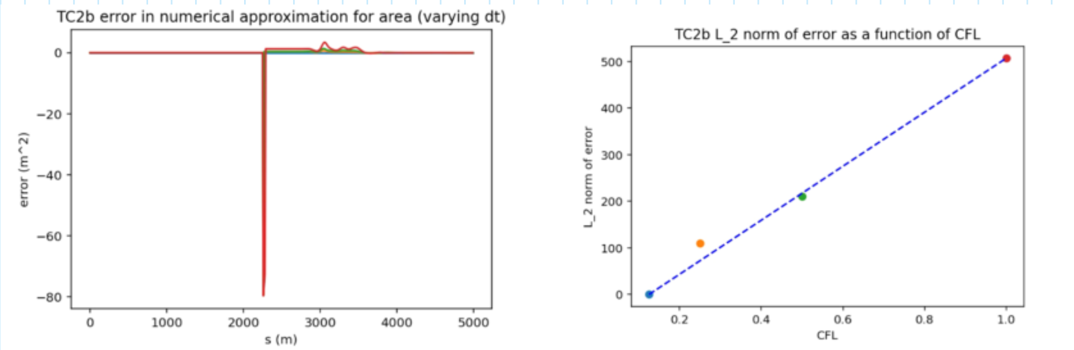
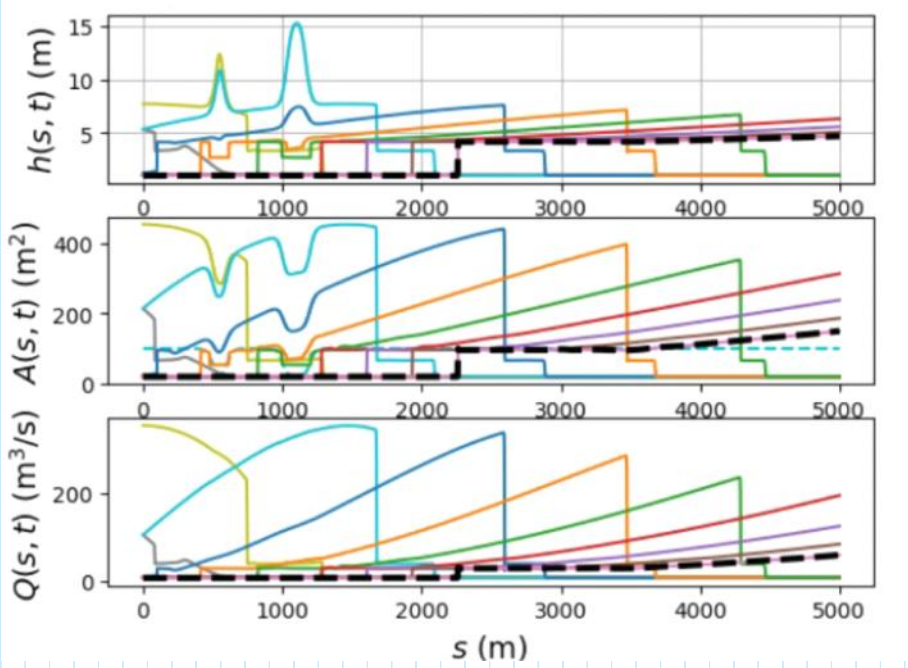


Figure : Spatial error analysis (red = 2dt, green = dt, orange = 0.5dt, blue = 0.2dt

The error plots still show the same result that the solutions are converged in dt. The l2-norm shows there is still a mostly linear relation indicating a first order accuracy. However need a finer mesh for the numerical solution to converge. There is the requirement for a finer mesh for convergence in test case 2b than there was in test case 0 or test case 1.

Shock Speed

This plot is the finest mesh considered to test case 2b. It shows how for times 10125s, 11250s, 12375s and 13500s there is evidence of shocks and these can be used to compare to the actual shock speed



Numerical Shock Speed:

$$U = \frac{F(A_r) - F(A_L)}{A_r - A_L}$$

For $t = 10125s$ ①
$A_L = 445, A_r = 65, s = 1677$
$(F = Q)$ $Q(A_L) = 346, Q(A_r) = 33$
$\Rightarrow U \simeq 0.81ms^{-1}$

For $t = 11250s$ ②
$A_L = 441, A_r = 70, s = 2596$
$Q(A_L) = 340, Q(A_r) = 35$
$\Rightarrow U \approx 0.81ms^{-1}$

For $t = 12375s$ ③
$A_L = 395$ $A_r = 65, s = 3475$
$Q(A_L) = 287, Q(A_r) = 38$
$\Rightarrow U \approx 0.75ms^{-1}$

For $t = 13500s$ ④
$A_L = 352, A_r = 65, s = 4290$
$Q(A_L) = 237$ $Q(A_r) = 38$
$\Rightarrow U \approx 0.69ms^{-1}$

Observed Shock Speeds:

From $1 \to 2$ observed shock speed $= \frac{2596 - 1677}{11250 - 10125} = 0.82ms^{-1}$

From $2 \to 3$ Observed shock speed $= \frac{3475 - 2596}{12375 - 11250} = 0.78ms^{-1}$

From $3 \to 4$ Observed shock speed $= \frac{4290 - 3475}{13500 - 12375} = 0.72ms^{-1}$

The average shock speed observed is $\approx$ numerical shock speed however with larger spread than in test case 1

All fine: 20/20.