Fluid Dynamics — Numerical Techniques

MATH5453M Numerical Exercises 1, 2025

Due date: Oct 17$^{\text{th}}$ (provisional)

Keywords: convection-diffusion equation, explicit forward Euler time discretization, central differences and diffusion, upwind discretization, numerical stability, Fourier analysis, maximum principle, verification/comparison "exact" and numerical solutions, errors, θ–scheme.
Sources: Lecture Notes, Chapter 2 of Morton and Mayers (2005), Internet.

1. Consider the flow of incompressible lava in a magma dike between two vertical rock walls, see figure 1 and figure 2. These rock walls yield more or less elastically under sufficient fluid pressure. The width $b(z,t)$ of this magma dike is therefore a function of the vertical coordinate z and time t. The fluid velocity $u(z,t)$ is given as a function of b and its partial derivative $\partial_z b = \partial b/\partial z$. The dimensionless mass balance and velocity equations between the magma chamber at $z = 0$ and the atmosphere at $z = H > 0$ are

$$\partial_t b + \partial_z (u\, b) = 0, \qquad u = \alpha\, b^2 - \beta\, b^2\, \partial_z b \quad \text{with} \quad z \in [0, H], \tag{1}$$

with $\partial_t b = \partial b/\partial t$, $\alpha > 0$, and $\beta > 0$. The Dirichlet boundary conditions are

$$b(0,t) = b_B \qquad \text{and} \qquad b(H,t) = b_T \tag{2}$$

and the initial condition is

$$b(z,0) = b_i(z). \tag{3}$$

The dimensionless parameters are defined by

$$\alpha(z) = g[\kappa\rho_r(z) - \rho_m]/(\gamma\mu_m) \quad \text{and} \quad \beta = 1/(\gamma\mu_m\lambda) \geq 0 \tag{4}$$

with typical dimensional values $H = 3\,\text{km}$, $b = 1\,\text{m}$, magma density $\rho_m = 2500\,\text{kg/m}^3$, geometric factors $\gamma = 12$ and $\kappa \approx 0.95$, rock density $\rho_r = 2400 - 2800\,\text{kg/m}^3$ (taken

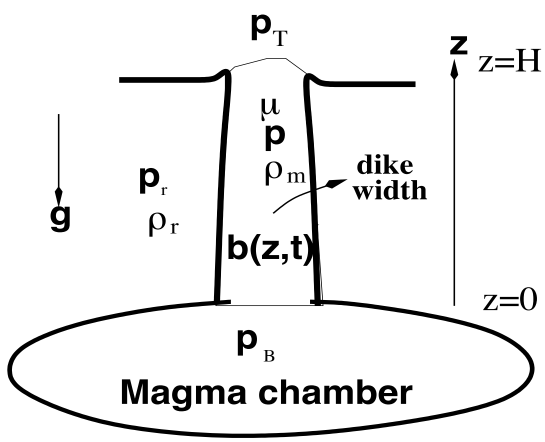

Figure 1: Configuration sketch of the geophysical application with a magma chamber, dike of aperture $b(z)$ and lava dome. The constant density of the magma and rock is ρ_m and ρ_r, respectively. The lithostatic pressure in the rock is p_r, the magma pressure at the exit of the magma chamber is p_B, and μ is the viscosity of the magma. The ambient pressure p_T is atmospheric. The gravitational acceleration is g.

to be constant here), acceleration of gravity $g = 9.8\mathrm{m/s}^2$, atmospheric pressure p_T, the pressure in the magma chamber $p_B = 25 - 100\,\mathrm{MPa}$, magma viscosity $\mu_m = 100$–$1000\mathrm{Pa\,s}$, and the elasticity of the host medium $\lambda = a(1-\nu)/G$ expressed in terms of Poisson ratio $\nu \approx 0.25$ and rigidity $G \approx 1.1250^9\mathrm{Pa}$. These have been used to scale the equations.

a) Simplify the system (1) to one convection-diffusion equation for $b(z,t)$

$$\partial_t b + \partial_z(\alpha\, b^3 - \beta\, b^3\, \partial_z b) = 0. \tag{5}$$

Linearize this equation around $b = D_0$ with D_0 constant by substituting $b = D_0 + b'$ into (5) and assuming that b' is small. Show that we obtain

$$\partial_t b' + 3\,\alpha\, D_0^2\, \partial_z b' - \beta\, D_0^3 \partial_{zz} b' = 0 \tag{6}$$

with perturbation width b'. Why are (5) and (6) called (nonlinear) convection-diffusion equations?

b) Discretize (5) and (6) with an forward Euler time discretization, an upwind scheme for the convective term, and a second-order central difference scheme for the diffusive term. Make a sketch of your grid and its numbering. Check whether the discretization of (6) is the linearized version of the discretization of (5). What is the advantage of using the adjoint form for the (nonlinear) diffusive term, cf. the discretization in the notes and equation (2.153) in Morton and Mayers (2005)? Clearly indicate how you implement the boundary conditions. I.e. for $j = 1, \dots, J-1$ and $\alpha = 0$ (for general $\alpha > 0$ upwinding is better):

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} + \frac{u_{j+1/2}^n b_{j+1/2}^n - u_{j-1/2}^n b_{j-1/2}^n}{\Delta z} = 0 \tag{7}$$

$$u_{j+1/2}^n = \dots \quad \text{(express ito)} \quad b_{j+1/2}^n, b_j^n, b_{j+1}^n \tag{8}$$

$$b_{j+1/2}^n = (b_{j+1}^n + b_j^n)/2 \quad \text{etc.} \tag{9}$$

c) Use a Fourier analysis to assess the stability of the linearized numerical scheme. Consider (only) the limits $\alpha = 0, \beta \neq 0$ and $\alpha \neq 0, \beta = 0$. What time step should be used?

d) Use the maximum principle to determine a stable time step for the discretization of (6).

e) Derive a (variable) time step criterion for the discretization of (5) for which $B_j^{n+1} > 0$. This implies that the width b is guaranteed to be positive, provided that $B_j^n > 0$.

f) Derive a second-order spatial discretization of (5) by changing the spatial discretization of the convective term. What is a time step criterion to guarantee that $B_j^{n+1} > 0$ for this new discretization?

2. a) Show that the steady state solution of (5) satisfies

$$\beta\, b^3\, db/dz = \alpha\, b^3 - Q \tag{10}$$

with integration constant Q. Integrate (10) and plot the solution for example by simply using an Euler forward spatial discretization (or another routine available). Use $Q = 0.99, \alpha = 0.4709, \beta = 1.0, H = 1, b_B = 1.178164343$ and a very high number of grid points. Make dimensional plots with the dike width $(-b/2, b/2)$ on the horizontal axis and z on the vertical axis.

b) Plot the solution of the nonlinear convection-diffusion equation (5) at times

$$t = 0.05, 0.1, 0.2, 0.5, 1, 2$$

for 11, 21, and 41 grid points in three plots (in total) for a time step which leads to stable integrations. Use the initial condition $B_j^0 = b_T$. Indicate the time step used. Plot the steady state solution in these figures as well. Use the above parameter values and the appropriate $b_T \approx 0.585373798$ (check whether this value corresponds with the steady state solution). Compare the results. Make dimensional plots with the dike width $(-b/2, b/2)$ on the horizontal axis and z on the vertical axis.

c) Consider the L^2–norm and the L^∞–norm for the error $e(z,t) = b(z,t) - b_{numerical}(z,t)$, that is,

$$L^2 = \sqrt{\int_0^H e^2(z,t)\,\mathrm{d}z} \tag{11}$$

as function of the resolution. Use your high-resolution steady state calculation as "exact" but steady solution at sufficiently large times, *e.g.* $t = 2$. Use the composite trapezoidal rule to approximate the integral. What is the order of the spatial discretization based on your numerical results? Explain clearly how you obtain your answer.

d) Verify that the following is an implicit yet exact, time-dependent (travelling-wave) solution

$$z - z_{r0} - ct = \frac{\beta}{\alpha}\left(b(z,t) - \sqrt{(c/\alpha)}\,\mathrm{atanh}\big[\sqrt{(\alpha/c)}\,b(z,t)\big]\right); \tag{12}$$

with z_{r0} a reference constant and c the wave speed (de Boer et al. 2005). A travelling wave solution is a solution of (1) of the form $b = b(z - ct)$ with phase speed c. Use the values $\alpha = 0.4709, c = \alpha$ and $\beta = 1.0$. Verify the numerical solution against this exact solution, and pay attention to the required condition that $b(x,t) \geq 0$, i.e., use the idea of the maximum principle to ensure that $B_j^{n+1} \geq 0$. Arrange the initial dike shape such that the dike is closed till a dimensional depth of 2.1km for a total depth of $H = 3$km. Note that b_B and b_T are now time dependent, and this time-dependence needs to be defined using the exact solution. (Advanced question.)

e) Extend the above numerics in (d) to the Crank-Nicolson scheme, and solve the nonlinear algebraic system using iteration techniques. The results of the explicit scheme can be used as verification before addressing the question which technique is faster? (Advanced open-ended question.)

Figure 2: Shiprock Volcano in New Mexico, U.S.A. The frozen remains of a magma dike are seen. Such a dike is typically a few kilometers long and deep and about 1 to 2 m wide.

References

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