Exercise 3 : Part II

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## 1) Ground Water Model

Nonlinear diffusion equation : Describes how groundwater level evolves in time and space

$$\frac{\partial W_y h_m}{\partial t} - ag\frac{\partial}{\partial y}\left(W_y h_m \frac{\partial h_m}{\partial y}\right) = \frac{w_y R}{m_{por}\theta_e}$$

$h_m(y,t)$ = groundwater level above datum , $w_y$ = channel width ($\sim 0.1m$) , $m_{por}$ = porosity (0.1-0.3)
$\theta_e$ = effective pore fraction (0.5-1) , $R(t)$ = rainfall input , $a = \dfrac{K}{V m_{por}\theta_e}$ = coefficient with permeability k and viscosity $v$

This PDE is essentially a continuity equation : rainfall adds water, diffusion spreads it along the channel and the porosity controls storage

### BC's :

→ At the far end of channel $(y = L_y)$, no water leaves so $\frac{\partial h_m}{\partial y} = 0$
→ At the canal end $(y=0)$, groundwater levels equals canal level so $h_m(0,t) = h_{cm}(t)$

### Canal Coupling

The canal is short ($L_c \approx 0.05m$) and has a weir at its end. Flow over the weir is critical, meaning velocity and depth satisfy Bernoulli's relation

$$h_e = \frac{2}{3}h_{cm} \qquad Q_c = V_c h_e = \sqrt{g}\ \max\left(\frac{2h_{cm}}{3}, 0\right)^{3/2}$$

This shows the canal outflow depends nonlinearly on canal depth. The canal equation links inflow from groundwater at $y=0$ with outflow at the weir, ensuring mass balance

### Initial Condition

Both groundwater and canal levels start at zero

$$h_m(y,0) = 0 \ , \ h_{cm}(0) = 0$$

### Assumptions

→ Flow is hydrostatic
→ Rainfall raises groundwater directly, modulated by porosity and effective pore fraction
→ No surface runoff is considered

### Summary of Scenario

Formulation sets up a coupled PDE - ODE system
↳ PDE for groundwater diffusion along the channel
↳ ODE for canal water level evolution, linked via boundary flux at $y=0$

Questions

1) i) Write PDE-ODE system in conservative form

$$\frac{\partial}{\partial t}(W_y h_m) - ag\frac{\partial}{\partial y}\left(W_y h_m \frac{\partial}{\partial y} h_m\right) = \frac{W_y R}{m_{por}\,\theta_e}$$

Using darcy velocity and hydrostatic scaling flux form gives $-ag\frac{\partial}{\partial y}\left(h_m \frac{\partial}{\partial y} h_m\right) = \frac{-ag}{2}\,\partial y\left(\partial y\left(\cancel{\ } h_m^2\right)\right) \leftarrow$ diffusion term

BC's $\rightarrow$ $y = L_y$ $\quad \frac{\partial h_m}{\partial y} = 0$ $\leftarrow$ No flow

$\rightarrow$ $y = 0$ $\quad h_m(0,t) = h_{cm}(t)$ $\leftarrow$ Dirichlet

Canal ODE (mass balance)

Combine inflow from groundwater at $y=0$ and weir outflow at $y = -L_c$

$$\frac{d}{dt}(W_y h_{cm}) = W_y\, m_{por}\,\theta_e\, ag\frac{\partial}{\partial y} h_m\Big|_{y=0} - W_y\sqrt{g}\,\max\left(\tfrac{2}{3}h_{cm}, 0\right)^{3/2}$$

$$h_e = \frac{2}{3}h_{cm}, \quad Q_c = \sqrt{g}\,\max\left(\frac{2h_{cm}}{3}, 0\right)^{3/2}$$

This can be used to eliminate the boundary flux term that appears ~~after~~ in the PDE weak form

~~ii) Derive weak form and expose boundary flux~~

ii) Multiply by a test function and integrate by parts (adjoint form)

Scalar test function $= q(y)$ Integrate over $[0, L_y]$ integrating diffusion term by parts and using $\frac{\partial h_m}{\partial y} = 0$ at $y = L_y$

$$\int_0^{L_y} q\frac{\partial}{\partial t}(W_y h_m)\,dy - ag\int_0^{L_y} h_m \frac{\partial q}{\partial y}\frac{\partial h_m}{\partial y}\,dy + ag\,q(0)\,h_m\frac{\partial h_m}{\partial y}\Big|_{y=0}$$

$$= \int_0^{L_y} q\,\frac{W_y R}{m_{por}\,\theta_e}\,dy \quad \leftarrow \text{weak form}$$

Boundary form for flux elimination : $ag\,q(0)\,\frac{1}{2}\frac{\partial}{\partial y}(h_m^2)\Big|_{y=0}$

iii) Eliminate boundary flux using canal equation

$\rightarrow$ Solve for groundwater flux at $y=0$ and rewrite in weak form boundary term

$$ag\,q(0)\,\frac{1}{2}\frac{\partial h_m^2}{\partial y}\Big|_{y=0} \Rightarrow q(0)\left(\frac{L_c}{m_{por}\,\theta_e}h_{cm} - \frac{1}{m_{por}\,\theta_e}\sqrt{g}\,\max\left(\frac{2h_{cm}}{3}, 0\right)^{3/2}\right)$$

PDE weak form has ~~unknown boundary~~ no unknown boundary flux; the canal variable $h_{cm}(t)$ becomes the boundary datum and a coupled unknown via the ODE

iv) Continuous Galerkin finite element setup

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Basis: $\{\varphi_i(y)\}_{i=1}^{N}$ on $[0, L_y]$ continuous, piecewise linear

Field: $h_m(y,t) \approx \sum_{j=1}^{N} h_j(t)\,\varphi_j(y)$ with boundary node $h_1(t) = h_{cm}(t)$

Element wise and global forms

$$M_{ij} = \int_0^{L_y} W_y\, \varphi_i\, \varphi_j\, dy \quad \leftarrow \text{Mass matrix}$$

$$F_i[h(t)] = -ag\int_0^{L_y} \omega_y\, h_m(y,t)\, \frac{\partial \varphi_i}{\partial y}\, \frac{\partial h_m(y,t)}{\partial y}\, dy \quad \leftarrow \text{Non linear adjoint diffusion vector}$$

$$S_i(t) = \int_0^{L_y} \varphi_i\, \frac{W_y R(t)}{m_{por}\,\theta_e}\, dy \quad \leftarrow \text{Source - Rainfall}$$

$$B_i(t) = \varphi_i(0)\left(\frac{L_c}{m_{por}\,\theta_e}\, h_{cm}(t) - \frac{1}{m_{por}\,\theta_e}\sqrt{g}\,\max\left(\frac{2h_{cm}(t)}{3}, 0\right)^{3/2}\right) \leftarrow \text{Boundary coupling vector (from flux elimination)}$$

v) Forward Euler ~~to~~ Explicit time stepping

Groundwater: $M\left(h^{n+1} - h^n\right) = \Delta t\left(F[h^n] + S^n + B^n\right)$ with $h_1^{n+1} = h_{cm}^{n+1}$ at boundary node

Canal ODE: $h_{cm}^{n+1} = h_{cm}^{n} + \Delta t\,\frac{1}{L_c}\left(m_{por}\,\theta_e\, a g\left(\frac{\partial h_m^n}{\partial y}\right)\Big|_0 - \sqrt{g}\,\max\left(\frac{2h_{cm}^n}{3}, 0\right)^{3/2}\right)$

vi) Time step estimate

$D(y,t) = ag\,h_m(y,t)$ For explicit scheme on a uniform mesh with spacing $\Delta y$, heat equation stability bound through forward Euler is

$$\Delta t \leq \frac{1}{2}\frac{\Delta y^2}{D_{max}}$$

where $D(y,t) = ag\,h_m(y,t)$ and $D_{max}$ is maximum diffusivity over domain at current time level

$\hookrightarrow D_{max} = \max_y\left(a g\, h_m^n(y)\right)$

$$\Rightarrow \Delta t \leq \frac{1}{2}\frac{\Delta y^2}{\max_y\left(ag\,h_m^n(y)\right)}$$

As $h_m$ grows from rainfall, $D$ increases and the stable $\Delta t$ decreases

Consider Canal:

$$h_{cm} = \frac{1}{L_c}\left(m_{por}\,\theta_e\, a g\, \frac{\partial h_m}{\partial y}\Big|_0 - \sqrt{g}\left(\tfrac{2}{3}h_{cm}\right)^{3/2}\right.$$

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The derivative of the outflow term wrt $h_{cm}$ is

$$\frac{d}{dh_{cm}}\left[\frac{1}{L_c}\sqrt{g}\left(\frac{2}{3}h_{cm}\right)^{3/2}\right] = \frac{\sqrt{g}}{L_c}\left(\frac{2}{3}\right)^{3/2}\frac{3}{2}h_{cm}^{1/2}$$

$$\Rightarrow \quad \Delta t \leq \frac{\eta}{\frac{\sqrt{g}}{L_c}\left(\frac{2}{3}\right)^{3/2}\frac{3}{2}\max\left(\hat{h}_{cm}\right)^{1/2}}$$

For final timestep estimate take minimum of the diffusion and canal bounds

$$\Delta t = \min\left\{\frac{1}{2}\frac{\Delta y^2}{\max_y\left(ag\hat{h}_m(y)\right)},\ \frac{\eta L_c}{\sqrt{g}\left(\frac{2}{3}\right)^{3/2}\frac{3}{2}\max\left(\hat{h}_{cm}\right)^{1/2}}\right\}$$

Do you have any idea why projection works?
Who else did this? I.e.,/E.g., Amelia and Anthony have similar results but did not say they had projected. 20/20.