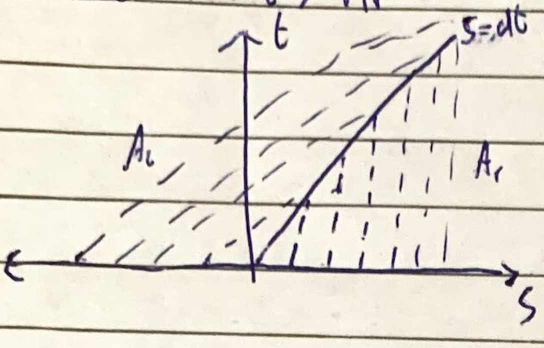
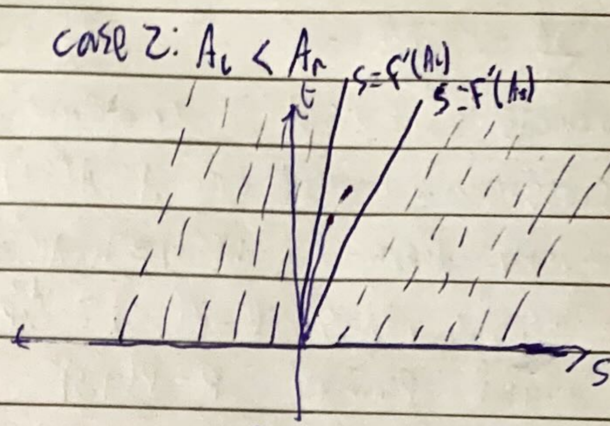
Case 1: $A_l > A_r$



case 2: $A_l < A_r$



Rarefaction region where $S=dt$

3) For $S=0$, where the conservation law

$$\frac{\partial A}{\partial t} + \frac{\partial F(A,S)}{\partial s} = 0$$

cell $k = [s_{k-\frac{1}{2}}, s_{k+\frac{1}{2}}]$, we integrate the conservation law over the cell k in s.

$$\int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \frac{\partial A}{\partial t} + \frac{\partial F(A,S)}{\partial s}\, ds = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} A\, ds + \left[F(A,S)\right]_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} = 0$$

for cell lengths $h_k = s_{k+\frac{1}{2}} - s_{k-\frac{1}{2}}$, define cell averages for cell k

$$\bar{A}_k(t) = \frac{1}{h_k}\int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} A(s,t)\, ds$$

~~Now integrate from time t_n to time t_{n+1}~~

So our equation becomes

$$\frac{\partial}{\partial t} h_k \bar{A}_k(t) + F(A,S)\big|_{s_{k+\frac{1}{2}}} - F(A,S)\big|_{s_{k-\frac{1}{2}}} = 0$$

Now we integrate this from time t_n to t_{n+1}

$$\int_{t_n}^{t_{n+1}} \frac{\partial}{\partial t} h_k \bar{A}_k(t) + F(A,S)\big|_{s_{k+\frac{1}{2}}} - F(A,S)\big|_{s_{k-\frac{1}{2}}}\, dt = 0$$

$$\Rightarrow \bar{A}_k^{t_{n+1}} - \bar{A}_k^{t_n} + \int_{t_n}^{t_{n+1}} F(A,S)\big|_{s_{k+\frac{1}{2}}} - F(A,S)\big|_{s_{k-\frac{1}{2}}}\, dt = 0$$

$$\Rightarrow \bar{A}_k^{t_{n+1}} = \bar{A}_k^{t_n} - \int_{t_n}^{t_{n+1}} F(A,S)\big|_{s_{k+\frac{1}{2}}} - F(A,S)\big|_{s_{k-\frac{1}{2}}}\, dt = 0$$