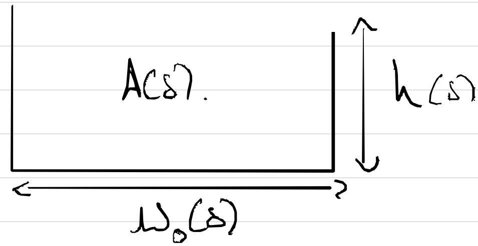
Numerics 2: Alex Carey 201810977.

1. Given some cross section



we define the relationship

$$A = h \cdot w_0$$

such that h is given as a function of A, w_0

$$h(s) = A / w_0(s)$$

Similarly the wetted perimeter is given geometrically as

$$P = 2h + w_0 = w_0 + 2A / w_0$$

such that

$$P(A, s) = w_0(s) + 2A / w_0(s).$$

noting A is also a function of s.

We consider

$$\partial_t A + \partial_\delta F = 0$$

where we have

$$F = \frac{A^{5/3}}{C_m P^{2/3}} \sqrt{-\partial_\delta b}$$

the flux at δ. We can

$$\partial_\delta F = \partial_\delta A \, \partial_A F + \partial_\delta F$$

where A is considered independent of δ when calculating $\partial_\delta F$. We compute

$$\partial_A F = \frac{5}{3}\left(\frac{A}{P}\right)^{\frac{2}{3}} \frac{\sqrt{-\partial_\delta b}}{C_m} - \frac{2}{3} \frac{A^{\frac{5}{3}}\sqrt{\partial_\delta b}}{P^{\frac{5}{3}}} \frac{\partial_A P}{C_m}$$

where $\partial_A P = 2/W_0$ such that

$$\partial_A F = \frac{5}{3}\left(\frac{A}{P}\right)^{\frac{2}{3}} \frac{\sqrt{-\partial_\delta b}}{C_m} - \frac{4}{3}\left(\frac{A}{P}\right)^{\frac{5}{3}} \frac{\sqrt{\partial_\delta b}}{C_m W_0}$$

$$= \frac{\sqrt{-\partial_\delta b}}{3C_m}\left(\frac{A}{P}\right)^{\frac{2}{3}}\left[5 - 4\frac{A}{PW_0}\right]$$

$$= \frac{\sqrt{-\partial_\delta b}}{3C_m}\left(\frac{A}{P}\right)^{\frac{2}{3}}\left[\frac{5W_0^2 + 6A}{PW_0}\right]$$

$$= \frac{\sqrt{-\partial_\delta b}}{3C_m}\left[\frac{5A^{\frac{2}{3}}W_0 + 6A^{\frac{5}{3}}/W_0}{(W_0 + 2A/W_0)^{\frac{5}{3}}}\right].$$

We similarly compute

$$\partial_s F = -\frac{2}{3}\left(\frac{A}{P}\right)^{\frac{5}{3}} \frac{\sqrt{-\partial_s b}}{C_m} \partial_s P$$

where $\partial_s P = \partial_s w_0 (1 - 2A / w_0^2)$ such that

$$\partial_s F = -\frac{2}{3} \frac{A}{(w_0 + 2(\ell b))^{5/3}} \frac{\sqrt{-\partial_s b}}{C_m}\left(1 - \frac{2A}{w_0^2}\right) \partial_s w_0$$

as required. Note that as w_0 is only dependent on s $\partial_s w_0 = dw_0/ds$.

2) We consider a case where the variation in W_0 is small, that is

$$\partial_s W_0 = 0$$

and so have

$$\partial_t A + \partial_A F \partial_s A = 0 \qquad (1)$$

which has eigenvalue $\partial_A F$.

We consider the Riemann problem defined by (1) on some piecewise constant initial conditions

$$A(s, t=0) = \begin{cases} A_l & s < s_{k+\frac{1}{2}} \\ A_r & s \geq s_{k+\frac{1}{2}} \end{cases}$$

towards the computation of some flux on the boundary.

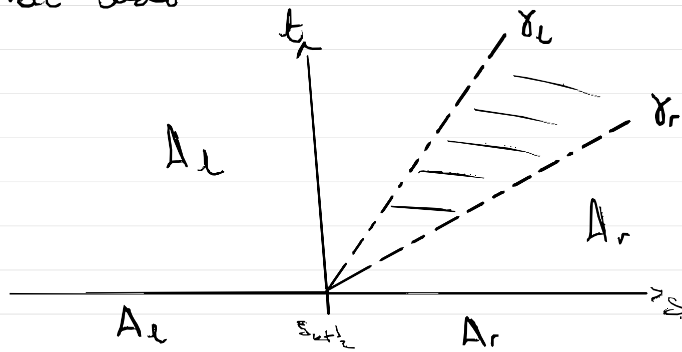
We define characteristics along which the value of A is constant in this case with

$$\partial_t s = \partial_A F(A, s).$$

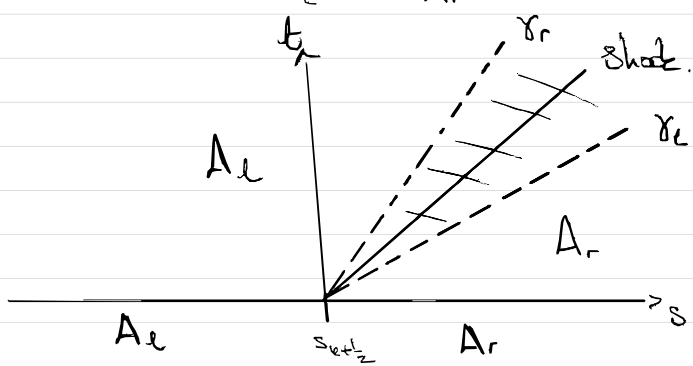
We note that $\partial_A F$ can then similarly take only two values dependent on A

$$(\partial_A F)_l = \partial_A F(A_l), \ (\partial_A F)_r = \partial_A F(A_r)$$

As $\partial_A F > 0$ we find the following two possible cases



(1)



(2).

Where γ_l, γ_r are the lines traced from the origin with gradients $(\partial_A F)_l, (\partial_A F)_r$ respectively.

In case (1) the central fan represents an undepenyed rarefaction region where we def^n A continuously as

$$A = s/t .$$

In case (2) the central region is doubly def^n and so a shock forms along the line.

with gradient

$$\frac{(\partial_A F)_l + (\partial_A F)_r}{2}.$$

3) Given

$$\partial_t A + \partial_s F(A,s) = 0$$

we consider the integrals in time and space to find

$$\int_{t_n}^{t_{n+1}} \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \partial_t A + \partial_s F \, ds \, dt = 0$$

$$\Rightarrow \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \left[A\right]_{t_{n}}^{t_{n+1}} ds + \int_{t_n}^{t_{n+1}} \left[F\right]_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} dt = 0$$

At this stage we apply the nature of the scheme where $\bar{A}$ is the average of the cell such that we have

$$\Delta s_k \left[\bar{A}_k\right]_{t_n}^{t_{n+1}} + \int_{t_n}^{t_{n+1}} \left[F\right]_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} dt = 0$$

and we have scheme

$$\bar{A}_k^{n+1} = \bar{A}_k^{n} - \frac{1}{\Delta s_k} \int_{t_n}^{t_{n+1}} \left[F\right]_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} dt.$$

4) We consider the flux on the boundaries given

$$F_{k+\frac{1}{2}}, \quad F_{k-\frac{1}{2}}.$$

We note F is dependent on A and that the results of the Riemann Problem state that the value of A on the boundary is given by

$$A_{k+\frac{1}{2}}(t) = \hat{\bar{A}}_k$$

for $t \in (t_n, t_{n+1})$ and as such

$$F_{k+\frac{1}{2}}(t) = F(\hat{\bar{A}}_k)$$

on the same t restricted domain.

5. We derive a time step restriction from the fact that

$$\Delta t_n (\partial_t S)_k < \Delta S_k = h_k$$

in dictating the value on the boundary $S_{k+\frac{1}{2}}$. Noting the flux approximation from part 4 we have

$$(\partial_t S)_k = (\partial_A F)_k$$

$$= \partial_A F(\bar{A}_k^{\hat{}})$$

$$= \hat{\lambda}_k$$

For a single restriction we must have

$$\Delta t^{\hat{}} < \min_k \frac{h_k}{|\hat{\lambda_k}|}.$$

We further restrict for stability by defining $CFL \in [0, 1]$ w/

$$\Delta t^{\hat{}} < CFL \min_k \frac{h_k}{\hat{\lambda_k}}.$$

6. As the scheme is upwind, we impose a boundary condition on canal discharge at $s=0$ without it being necessary to impose a condition for $s=L$.

$$A(0,t) = A_0(t).$$