Homework 1

Exercise 1

a) Simplify the system (1) to one convection - diffusion equation for $b(z,t)$.

$$\partial_t b + \partial_z (ub) = 0 \quad , \quad u = \alpha b^2 - \beta b^2 \partial_z b \qquad -(1)$$

with $z \in [0, H]$.

Substituting u, we get:

$$\partial_t b + \partial_z \left(\alpha b^3 - \beta b^3 \partial_z b \right) = 0 \qquad -(5)$$

$$\Rightarrow \partial_t b + 3\alpha b^2 \frac{\partial b}{\partial z} - 3\beta b^2 \left(\frac{\partial b}{\partial z} \right)^2 - \beta b^3 \frac{\partial^2 b}{\partial z^2} = 0$$

Linearize this by substituting $b = D_0 + b'$,

$$\Rightarrow \frac{\partial (D_0 + b')}{\partial t} + 3\alpha (D_0 + b')^2 \frac{\partial (D_0 + b')}{\partial z} - 3\beta (D_0 + b')^2 \left(\frac{\partial (D_0 + b')}{\partial z} \right)^2 - \beta (D_0 + b')^3 \left(\frac{\partial^2 (D_0 + b')}{\partial z^2} \right) = 0$$

As $D_0 =$ constant, then

$$\Rightarrow \frac{\partial b'}{\partial t} + 3\alpha (D_0 + b')^2 \frac{\partial b'}{\partial z} - 3\beta (D_0 + b')^2 \left(\frac{\partial b'}{\partial z} \right)^2 - \beta (D_0 + b')^3 \frac{\partial^2 b'}{\partial z^2} = 0$$

considering b' is small, then the nonlinear terms involving b' can be neglected:

$$\Rightarrow \frac{\partial b'}{\partial t} + 3\alpha (D_0 + b')^2 \frac{\partial b'}{\partial z} - \beta (D_0 + b')^3 \frac{\partial^2 b'}{\partial z^2} = 0$$

$$\Rightarrow \frac{\partial b'}{\partial t} + 3\alpha \left(D_0^2 + 2D_0 b' + b'^2\right) \frac{\partial b'}{\partial z} - \beta \left(D_0^3 + 3D_0^2 b' + 3D_0 b'^2 + b'^3\right) \frac{\partial^2 b'}{\partial z^2} = 0 .$$

$$\Rightarrow \frac{\partial b'}{\partial t} + 3\alpha D_0^2 \frac{\partial b'}{\partial z} - \beta D_0^3 \frac{\partial^2 b'}{\partial z^2} = 0 \qquad (6)$$

The derivative of b exists in the original equation and b is also a dependent variable of b itself, that's why they are called non-linear equation.

b)