Exercise sheet 3

Part I.

(1.) Consider the Poisson system

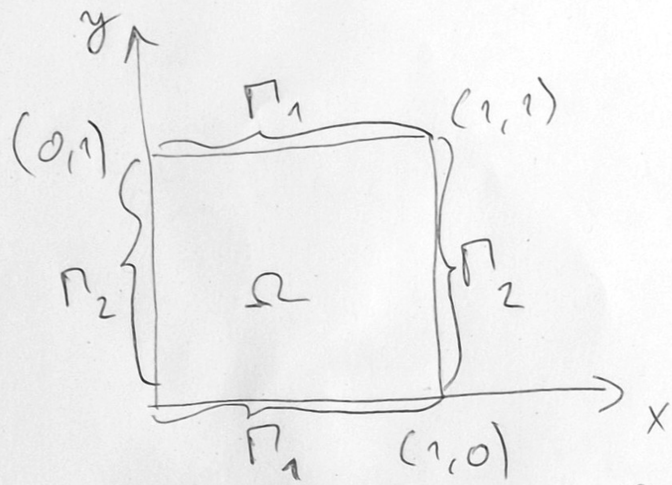
$$-\nabla^2 u = f \text{ on } (x,y) \in [0,1]^2$$

$$f(x,y) = 2\pi^2 \sin(\pi x)\cos(\pi y)$$

$$u(0,y) = u(1,y) = 0$$

$$\partial_y [u(x,y)]\Big|_{y=0} = \partial_y [u(x,y)]\Big|_{y=1} = 0.$$

Visualaise this domain



Then $u\Big|_{\Gamma_2} = 0$ & $(\nabla u)\cdot n_1\Big|_{\Gamma_1} = 0$.

Now, consider the functional

$$I[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - uf \, d\Omega.$$

Introduce variation in u denoted δu

s. t. $\delta u|_{\partial\Omega} = 0$, where $\partial\Omega = \Gamma_1 \cup \Gamma_2$.

Then the variation in the functional becomes

$$\delta I[u] = \frac{dI}{d\varepsilon} = \lim_{\varepsilon\to 0} \frac{I[u+\varepsilon\delta u] - I[u]}{\varepsilon}.$$

Hence,

$$\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\right)\int_\Omega \frac{1}{2}\left(|\nabla[u+\varepsilon\delta u]|^2 - |\nabla u|^2\right) - (\cancel{u} + \varepsilon\delta u)f + \cancel{uf}\, d\Omega$$

$$= \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\right)\int_\Omega \frac{1}{2}\left(\cancel{|\nabla u|^2} + 2\varepsilon\nabla\delta u\nabla u + \varepsilon^2|\nabla\delta u|^2\right) - \varepsilon\delta u f\, d\Omega$$

$$= \int_\Omega \nabla\delta u\nabla u - \delta u f\, d\Omega$$

Now, note that

$$\nabla\cdot[a\nabla b] = \nabla a\cdot\nabla b + a\nabla^2 b$$

$$\Rightarrow \nabla a\cdot\nabla b = \nabla\cdot[a\nabla b] - a\nabla^2 b.$$

Hence, the last expression becomes:

$$\int_\Omega \nabla\cdot[\delta u\nabla u] - \delta u\nabla^2 u - \delta u f\, d\Omega$$

$$= -\int_\Omega \delta u\left(\nabla^2 u + f\right)d\Omega + \int_{\Gamma_1}\delta u\nabla u\cdot n_1\, d\Gamma + \int_{\Gamma_2}\delta u\nabla u\cdot \underline{n}_2\, d\Gamma$$

But we have $\nabla \underline{u} \cdot \underline{n}_1$ is 0 on Γ_1 and $\delta u = 0$ on Γ_2 and the last two integrals vanish.

Require that the variation of the functional I vanishes to minimise it & we get

$$\int_\Omega \delta u \left(\nabla^2 u + f\right) d\Omega = 0$$

$$\Leftrightarrow -\nabla^2 u = f$$ as δu is arbitrary on the interior of Ω.

We required that $\delta u\big|_{\partial\Omega} = 0$, but it is sufficient to require that $\delta u\big|_{\Gamma_2} = 0$!

Now derive weak formulation for

$$-\nabla^2 u = f.$$

We have

$$\nabla^2 u + f = 0.$$

Now multiply by $w(x, y)$ & integrate over Ω:

$$\int_\Omega w\left(\nabla^2 u + f\right) d\Omega = 0$$

Note that

$$\nabla \cdot [w \nabla u] = \nabla w \cdot \nabla u + w \nabla^2 u$$

$$\Rightarrow \quad w \nabla^2 u = \nabla \cdot [w \nabla u] - \nabla w \cdot \nabla u.$$

Hence we get

$$\int_\Omega -\nabla w \cdot \nabla u + f w + \nabla \cdot [w \nabla u] \, d\Omega = 0$$

$$\int_\Omega \nabla w \cdot \nabla u - f w \, d\Omega - \int_{\Gamma_1} w \nabla u \cdot \underline{n}_1 \, d\Gamma$$

$$- \int_{\Gamma_2} w \nabla u \cdot \underline{n}_2 \, d\Gamma = 0$$

For the last two integrals to vanish we need $w\big|_{\Gamma_2} = 0$ as $\nabla u \cdot \underline{n}_1\big|_{\Gamma_1} = 0$.

Hence, $w(x,y) = \delta(x,y)$ and the weak formulation is

$$\boxed{\int_\Omega \nabla w \cdot \nabla u - f w \, d\Omega = 0.}$$

(2.) Consider the Hilbert space

$$\mathcal{H}^1_0(\Omega) := \{ u \in \mathcal{L}^2(\Omega) \mid \partial_x u, \partial_y u \in \mathcal{L}^2(\Omega) \; \& \; u|_{\Gamma_2} = 0 \}$$

Note that $\mathcal{L}^2(\Omega) := \{ u \mid \int_\Omega |u|^2 \, d\Omega < \infty \}$.

Now consider a subspace $V_n \subset \mathcal{H}^1_0$ with basis $e_1, e_2, \ldots, e_n$ i.e. V_n is an n-dimensional subspace of $\mathcal{H}^1_0$.

Hence, $u \in \mathcal{H}^1_0$ can be approximated with

$$u_n = \sum_{j=1}^{n} u_j e_j$$

where u_j are constants and $u_n \in V_n$.

Recall the Ritz-Galerkin principle

$$I[u] = \int_\Omega \frac{1}{2} |\nabla u|^2 - uf \, d\Omega$$

Hence,

$$I[u_n] = \int_\Omega \frac{1}{2} \left| \sum_{j=1}^{n} u_j \nabla e_j \right|^2 - \sum_{j=1}^{n} u_j e_j f \, d\Omega$$

$$= \sum_{j=1}^{n} u_j \int_\Omega \frac{1}{2} |\nabla e_j|^2 - e_j f \, d\Omega.$$

The weak formulation is

$$\int_\Omega \nabla w \cdot \nabla u - f w \, d\Omega = 0$$

where $w|_{\Gamma_2} = 0$. Similarly, pick $w \in V_h$ s.t. $w = e_i$ for $i \in \{1, \dots, n\}$ as it is arbitrary. Hence, we get

$$\int_\Omega \nabla e_i \cdot \nabla\left(\sum_{j=1}^{n} u_j e_j\right) - f e_i \, d\Omega = 0$$

$$\Leftrightarrow \sum_{j=1}^{n} u_j \int_\Omega \nabla e_i \cdot \nabla e_j \, d\Omega = \int_\Omega f e_i \, d\Omega$$

Now denote

$$A_{ij} := \int_\Omega \nabla e_i \cdot \nabla e_j \, d\Omega$$

$$b_i = \int_\Omega e_i f \, d\Omega$$

and apply the Einstein's summation convention to get

$$A_{ij} u_j = b_i.$$

Recall that

$$\delta I[u_n] = \frac{dI}{d\varepsilon} = \lim_{\varepsilon \to 0} \frac{I[u_n + \varepsilon e_i] - I[u_n]}{\varepsilon}.$$

Now,

$$\frac{1}{\varepsilon} \int_\Omega \sum_{j=1}^{n} u_j \left(\frac{1}{2} \left| \nabla [e_j + \varepsilon e_i] \right|^2 - \frac{1}{2} |\nabla e_j|^2 - (\cancel{e_j} + \varepsilon e_i) f \right.$$
$$\left. + \cancel{e_j f} \right) d\Omega$$

$$= \sum_{j=1}^{n} \frac{u_j}{\varepsilon} \int_\Omega \frac{1}{2} \left(\cancel{|\nabla e_j|^2} - \cancel{|\nabla e_j|^2} + 2\varepsilon \nabla e_j \cdot \nabla e_i + \varepsilon^2 |\nabla e_i|^2 \right)$$
$$- \varepsilon e_i f \, d\Omega$$

$$= \sum_{j=1}^{n} u_j \int_\Omega \nabla e_j \cdot \nabla e_i + \frac{\varepsilon}{2} |\nabla e_i|^2 - e_i f \, d\Omega$$

Let $\varepsilon \to 0$ and minimize $\delta I[u_n]$ to get

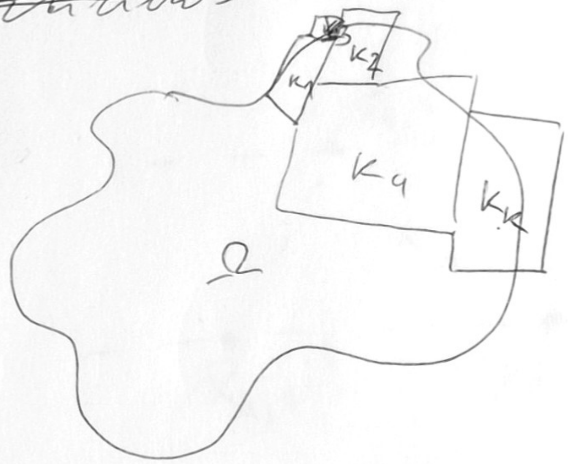
$$\sum_{j=1}^{n} u_j \int_\Omega \nabla e_i \nabla e_j - e_i f \, d\Omega = 0$$

as required.

Now, consider a set

$$\mathcal{T}_h = \left\{ K_k \;\middle|\; \bigcup_{k=1}^{N_{el}} \bar{K}_k = \bar{\Omega} \text{ s.t. } K_k \cap K_{k'} \neq \phi \text{ for } k, k' \in \{1, \dots, N_{el}\} \right\}.$$

This ~~partitions~~ covers domain Ω ~~into~~ into elements K_k:

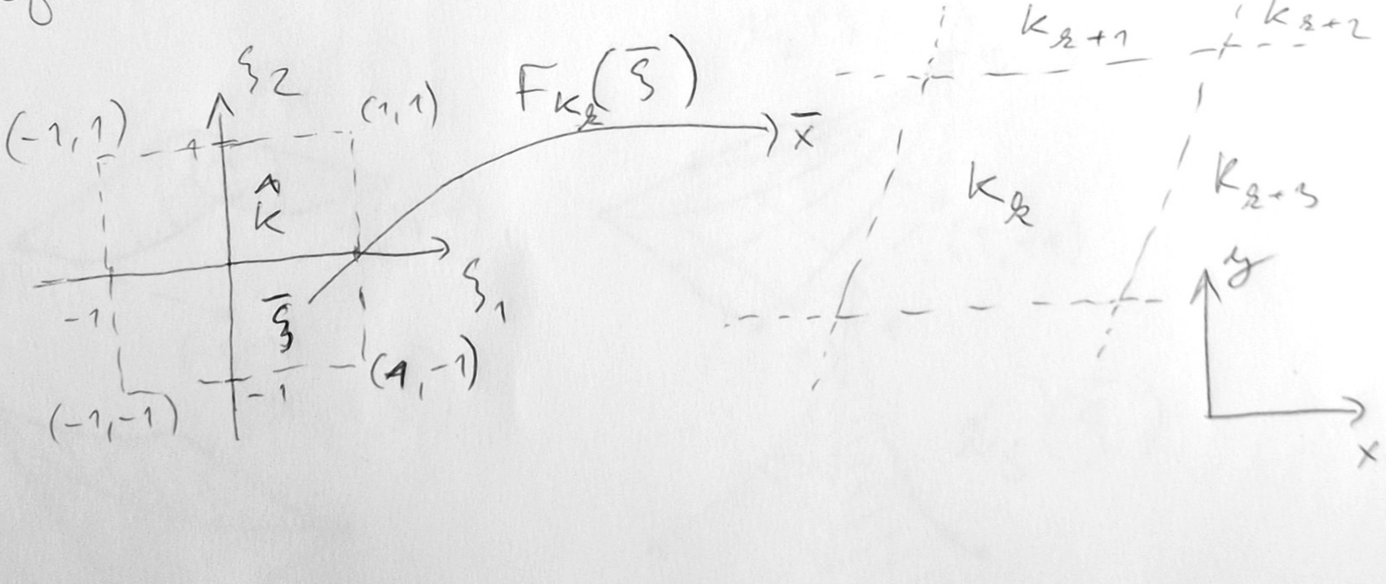


where $\bar{\Omega}$ is closure of Ω.

Now introduce a reference element $\hat{K}$

and a map $F_{K_k}: \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\hat{K}$ to K_k.

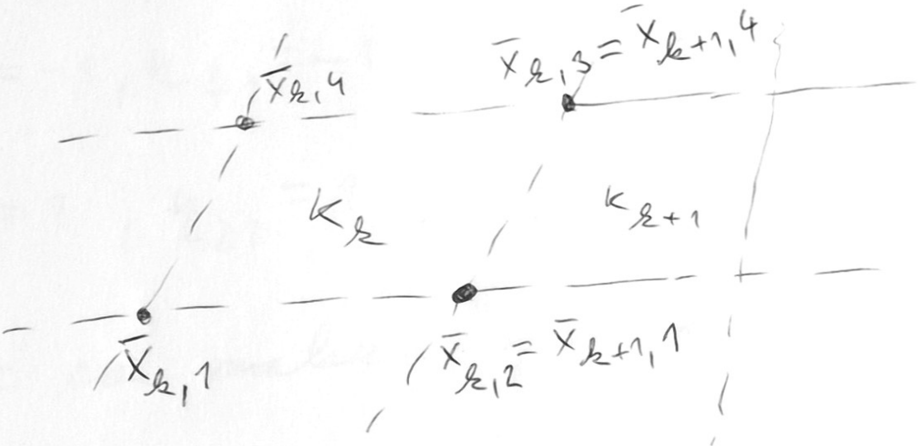
For reference element $\hat{K}$ introduce a reference coordinate system $\bar{\xi} = (\xi_1, \xi_2)$:



Hence, we have the relation

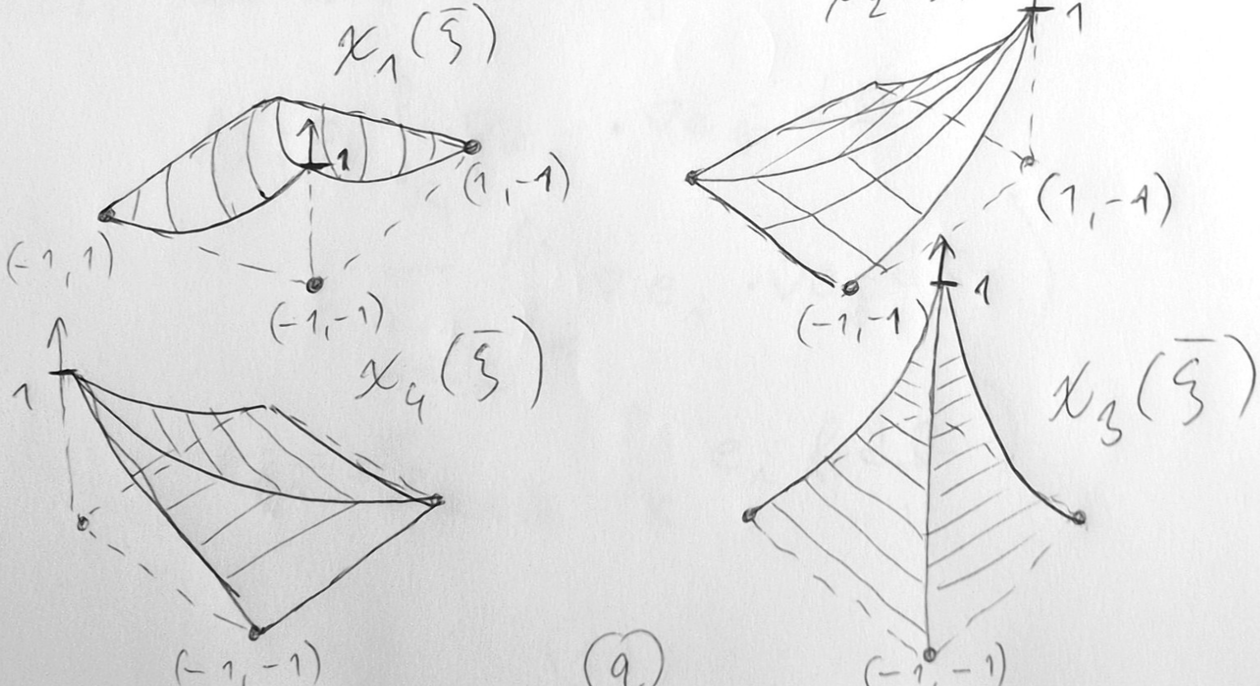
$$\bar{x} = F_{K_k}(\bar{\xi})$$

In particular, we can index nodes of each quadrilateral element K_k as follows:



where k - is the index of the element & $\alpha \in \{1,2,3,4\}$ refers to local index within each element K_k

Now, introduce a set of functions on $\hat{K}$ $\chi_\alpha(\bar{\xi})$ s.t.



These functions are given by

$$\chi_\alpha = (1+\eta_{\alpha 1}\xi_1)(1+\eta_{\alpha 2}\xi_2)\frac{1}{4}$$

where $\eta_{\alpha i}$ for $\alpha \in \{1,2,3,4\}$, $i \in \{1,2\}$ is defined by

$$\eta_{11} = -1,\ \eta_{41} = -1,\ \eta_{12} = -1,\ \eta_{22} = -1$$

$$\eta_{31} = 1,\ \eta_{21} = 1,\ \eta_{32} = 1,\ \eta_{42} = 1.$$

Hence, we can make a specific choice for $F_{k_k}(\bar{\xi})$:

$$F_{k_k}(\bar{\xi}) = \sum_{\alpha=1}^{4} \bar{x}_{k,\alpha}\, \chi_\alpha(\bar{\xi})$$

Now, recall the system

$$A_{ij} u_j = b_i$$

that we are solving. We can write

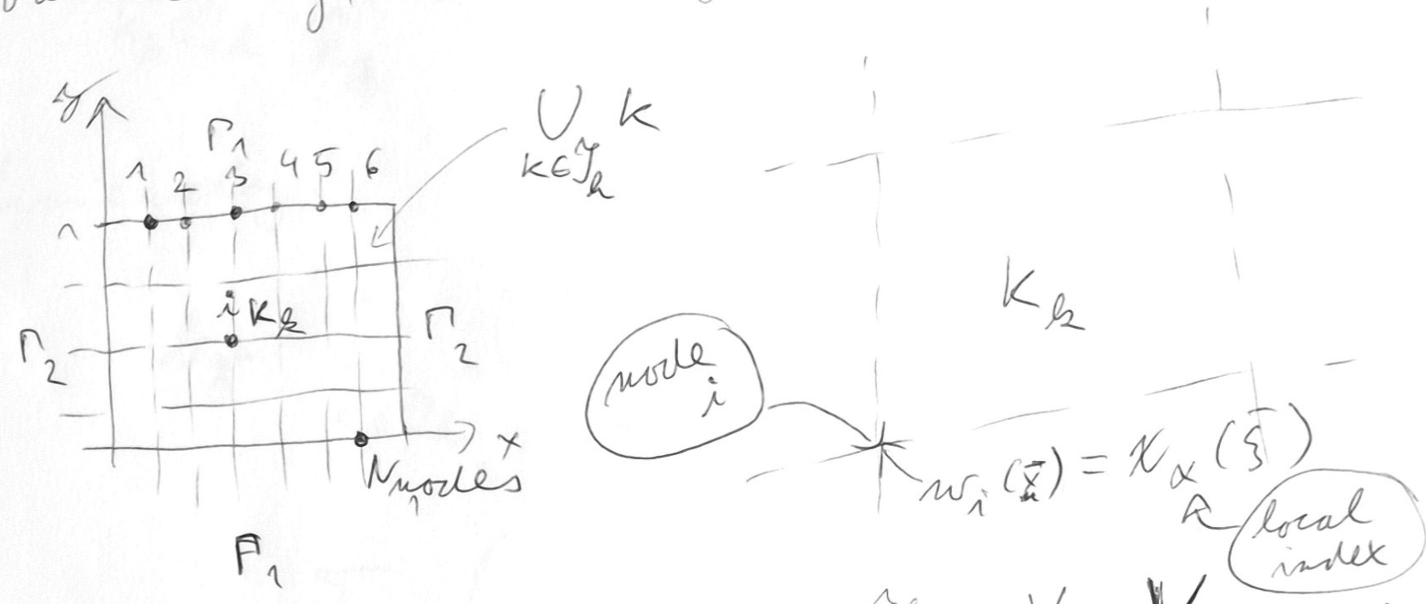
$$A_{ij} = \int_\Omega \nabla e_i \cdot \nabla e_j \, d\Omega$$

$$= \sum_{K\in\mathcal{T}_h} \left(\int_K \nabla e_i \cdot \nabla e_j \, d\Omega \right)$$

$$b_i = \sum_{K\in\mathcal{T}_h} \left(\int_K e_i f \, d\Omega \right)$$

At this point it becomes necessary to choose $V_h \subset \mathcal{H}^1_0(\Omega)$ and specify the basis functions.

If we consider, only nodes with unknown values u_j, then we get



for $i \in \{1, 2, \dots, N_{nodes}\}$. Then $V_h = V_{N_{nodes}}$ and so there is one basis function for each node with global index i.

In particular, on each element K_k choose

$$e_i(x,y) = \hat{e}_\alpha(F^{-1}_{K_k}(x,y)) = \chi_\alpha(\xi).$$

Hence, we obtain

$$A_{rq} = \sum_{K_k \in \mathcal{T}_h}' \int_{K_k} \nabla e_r \cdot \nabla e_q \, d\Omega$$

$$= \sum_{K_k \in \mathcal{T}_h}' \int_{K_k} \nabla \chi_\alpha \cdot \nabla \chi_\beta \quad d\Omega$$

$$\cancel{= \sum_{K_k \in T_h} \int_{K_k} (J^T)^{-1} \frac{\partial x_\alpha}{\partial \xi_i} \partial x_\beta}$$

$$= \sum_{K_k \in T_h} \int_{\hat{K}} \left(\left[(J^T)^{-1}\right]_{ij} \frac{\partial x_\alpha}{\partial \xi_j} \left[(J^T)^{-1}\right]_{im} \frac{\partial x_\beta}{\partial \xi_m} \right) \left(\det(J)\right) d$$

Similarly,

$$b_j = \sum_{K_k \in T_h} \left(\int_{K_k} e_i f \, d\Omega \right)$$

$$= \sum_{K_k \in T_h} \left(\int_{\hat{k}} x_\alpha f \det(J) \, d\xi \right)$$

Note that once $\int_{\hat{k}} \cdot \, d\xi$ are evaluated on the reference coordinate system, they must be linked back to appropriate entries of A_{ij}, b_i, by leveraging mapping $(k,\alpha) \to i$ & $(k,\alpha) \to j$.

The Jacobian J is given by

$$J^k_{ij} = \frac{\partial \bar{x}_i}{\partial \bar{\xi}_j} = \frac{\partial}{\partial \bar{\xi}_j} \left[\sum_{\alpha=1}^{4} \bar{x}_{i,\alpha} \, x_\alpha(\bar{\xi}) \right]$$

position of the node of element k with local coordinate α

$$J_{ij} = \sum_{\alpha=1}^{4} \bar{x}_{i,\alpha} \frac{\partial}{\partial \xi_j} \left[(1 + \eta_{\alpha 1} \xi_1)(1 + \eta_{\alpha 2} \xi_2) \frac{1}{4} \right]$$

$$= \frac{1}{4} \sum_{\alpha=1}^{4} \bar{x}_{i,\alpha} \left(\eta_{\alpha 1} \delta_{j1} (1 + \eta_{\alpha 2} \xi_2) + \eta_{\alpha 2} \delta_{j2} (1 + \eta_{\alpha 1} \xi_1 \right.$$

where δ_{ij} is kronecker delta & $\eta_{\alpha i}$ was defined previously.