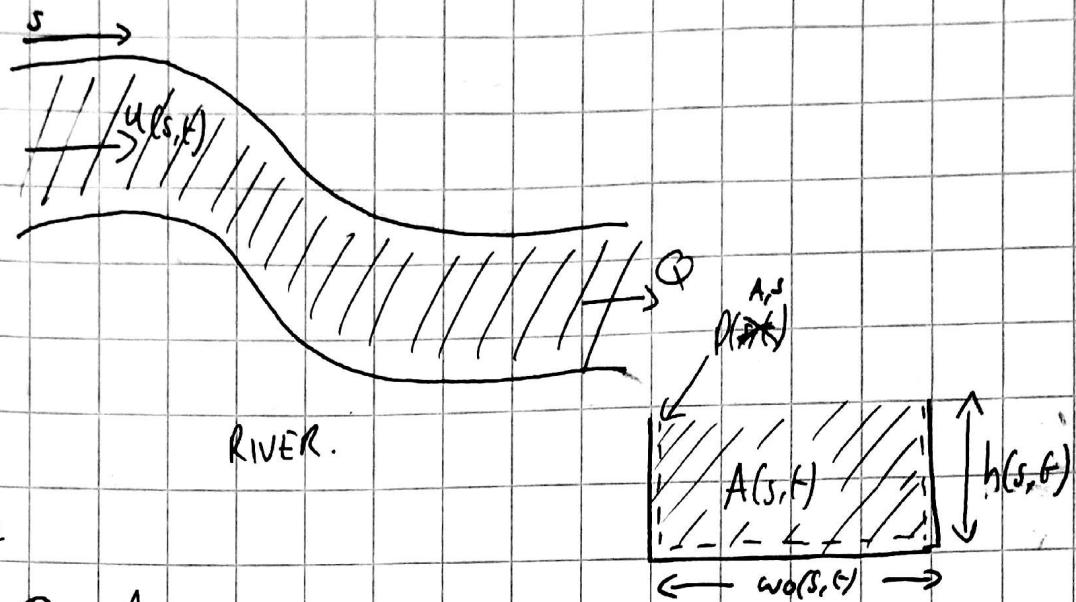


NUMERICS HOMEWORK 2.

pg 1

time, t



s , distance

$$\text{discharge } Q = Au$$

$$R = \frac{A}{P}$$

Downslope $-dsb$ $dsb \uparrow$

Manning Friction Coeff, C_m

$$u = \frac{R^{2/3} \sqrt{-dsb}}{C_m}$$

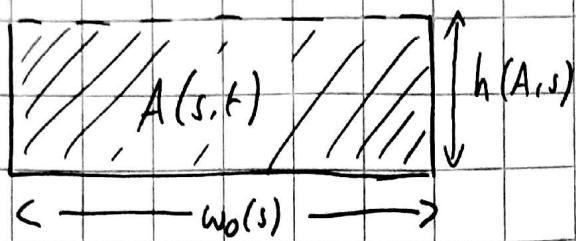
$$Q = Au = \frac{A^{5/3} \sqrt{-dsb}}{C_m P(A,s)^{2/3}}$$

$$\partial_t A + \partial_s (AR(A,s)^{2/3} \sqrt{-dsb}/C_m) = s$$

Eq 3/4

$$\Rightarrow \partial_t A + \partial_s F(A,s) = S \quad \text{where } F(A,s) = \frac{A^{5/3} \sqrt{-dsb}}{C_m P(A,s)^{2/3}}$$

1.



$$\text{Height} < \frac{\text{Area}}{\text{width}} \Rightarrow h(A,s) = \frac{A(s,t)}{w_0(s)}$$

$$\text{Perimeter : } h \underbrace{|}_{w_0(s)} |h| \Rightarrow P(A,s) = w_0(s) + 2h(A,s)$$

$$= w_0(s) + \frac{2A(s,t)}{w_0(s)}$$

$$S = 0$$

$$\textcircled{3} \quad \partial_t A + \partial_s F(A,s) = S^0$$

$$\partial_t A + \partial_s F(A,s) = 0$$

By the chain rule:

$$\partial_s [F(A(s,t), s)] = \frac{\partial F}{\partial A} \frac{\partial A}{\partial s} + \frac{\partial F}{\partial s}$$

put into equation

$$\left| \partial_t A + \frac{\partial F}{\partial A} \partial_s A + \frac{\partial F}{\partial s} = 0 \right|$$

$$\frac{\partial F}{\partial A} = \frac{\partial}{\partial A} \left[\frac{A^{6/5} \sqrt{-\partial_s b}}{C_m P(A,s)^{2/5}} \right]$$

$$\text{Sub in } P = w_0 + 2h = w_0(s) + \frac{2A(r,t)}{w_0(s)} \quad pg^2$$

$$\frac{\partial F}{\partial A} = \frac{\partial}{\partial A} \left[\frac{A^{5/3} \sqrt{-\delta b}}{C_m (w_0 + 2A/w_0)^{2/3}} \right]$$

assume $-\delta b$ is positive, small & constant

$$\Rightarrow \frac{\sqrt{-\delta b}}{C_m} \frac{\partial}{\partial A} \left[A^{5/3} \left(w_0 + \frac{2A}{w_0} \right)^{-2/3} \right]$$

$$= \frac{\sqrt{-\delta b}}{C_m} \left[\frac{5}{3} A^{2/3} \left(w_0 + \frac{2A}{w_0} \right)^{-3/3} + -A^{5/3} \frac{2}{3} \left(w_0 + \frac{2A}{w_0} \right)^{-5/3} \times \frac{2}{w_0} \right]$$

$$= \frac{\sqrt{-\delta b}}{3C_m} \left[\frac{5A^{2/3} \left(w_0 + \frac{2A}{w_0} \right)}{\left(w_0 + \frac{2A}{w_0} \right)^{5/3}} - \frac{4A^{5/3}}{w_0 \left(w_0 + \frac{2A}{w_0} \right)^{5/3}} \right]$$

$$= \frac{\sqrt{-\delta b}}{3C_m} \left[\frac{5A^{2/3} w_0 + \frac{10A^{5/3}}{w_0}}{\left(w_0 + \frac{2A}{w_0} \right)^{5/3}} - \frac{4A^{5/3}}{w_0} \right]$$

$$\frac{\partial F}{\partial A} = \frac{\sqrt{-\delta b}}{3C_m} \left[\frac{5w_0 A^{2/3} + 5A^{5/3}/w_0}{\left(w_0 + 2A/w_0 \right)^{5/3}} \right]$$

$$-\delta b > 0$$

$$w_0(s) > 0$$

$$A > 0$$

$$C_m > 0$$

$$\Rightarrow \frac{\partial F}{\partial A} > 0$$

$$\frac{\partial F}{\partial s} = \frac{\partial}{\partial s} \left[\frac{A^{5/3} \sqrt{-\Delta b}}{C_m P(A, \omega)^{2/3}} \right]$$

$$= \frac{\sqrt{-\Delta b}}{C_m} \frac{\partial}{\partial s} \left(A^{5/3} \left(\omega_0 + \frac{2A}{\omega_0} \right)^{-2/3} \right)$$

~~A(s,t)~~ $\omega_0(s)$ $A(s,t)$

~~$$= \frac{\sqrt{-\Delta b}}{C_m} \left[-\frac{2}{3} A^{5/3} \left(\omega_0 + \frac{2A}{\omega_0} \right)^{-5/3} \right]$$~~

$$= \frac{\sqrt{-\Delta b}}{C_m} \left[\frac{5}{3} A^{2/3} \frac{dA}{ds} - \frac{2}{3} A^{5/3} \left(\omega_0 + \frac{2A}{\omega_0} \right)^{-5/3} \left(1 - \frac{2A}{\omega_0^2} \right) \frac{d\omega_0}{ds} \right]$$

$$= \frac{\sqrt{-\Delta b}}{3C_m} \left[SA^{2/3} \frac{dA}{ds} - \frac{2}{3} A^{5/3} \frac{1}{\left(\omega_0 + \frac{2A}{\omega_0} \right)^{5/3}} \left(1 - \frac{2A}{\omega_0^2} \right) \frac{d\omega_0}{ds} \right]$$

$$\frac{\partial F}{\partial s} = \frac{2\sqrt{-\Delta b}}{3C_m} \frac{A^{5/3}/\omega_0}{\left(\omega_0 + \frac{2A}{\omega_0} \right)^{5/3}} \left(1 - \frac{2A}{\omega_0^2} \right) \frac{d\omega_0}{ds}$$

if $\frac{dA}{ds}$ is 0 ?

2. Take the limit in which $w_0(s)$ is independent of s

pg 3

$$\Rightarrow \frac{\partial w_0}{\partial s} = 0 \Rightarrow \frac{\partial F}{\partial s} = 0$$

$$\partial t A + \frac{\partial F}{\partial A} \partial s A + \cancel{\frac{\partial F}{\partial s}} = 0$$

$$\boxed{\partial t A + \frac{\partial F}{\partial A} \partial s A = 0}$$

Eigenvalue of equation $\lambda = \frac{\partial F}{\partial A}$

second term $\frac{\partial F}{\partial A} \partial s(A)$

can move $\frac{\partial F}{\partial A}$ inside derivative & it is constant, i.e. no longer has a dependence on s , $\frac{\partial}{\partial s} \left(\frac{\partial F}{\partial A} \right) = \frac{\partial^2 F}{\partial s \partial A} = \frac{\partial}{\partial A} \frac{\partial F}{\partial s} \text{ and } \frac{\partial F}{\partial s} = 0$

$$\partial s \left(\frac{\partial F}{\partial A} A \right) = \partial s (\lambda A)$$

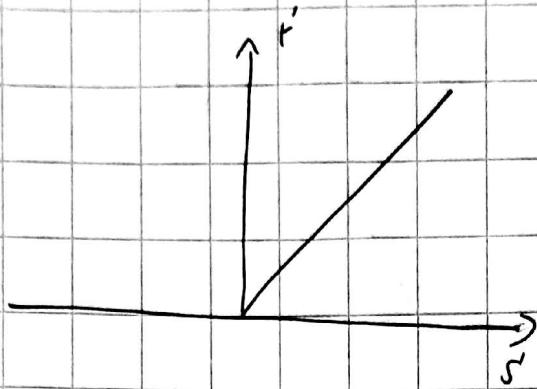
Can write equation as:

$$\partial t'(A) + \partial s'(\lambda A) = 0$$

with constant $\lambda > 0$

$$A(s', f' = 0) = \begin{cases} A_{left} = A_L & s' < 0 \\ A_{right} = A_R & s' \geq 0 \end{cases}$$

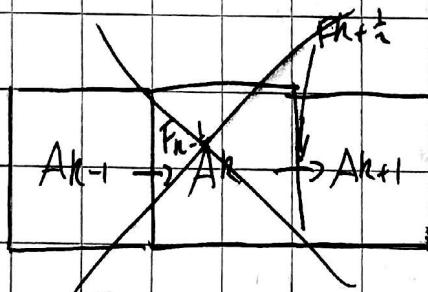
characteristics $s' = s_0 + \lambda t'$

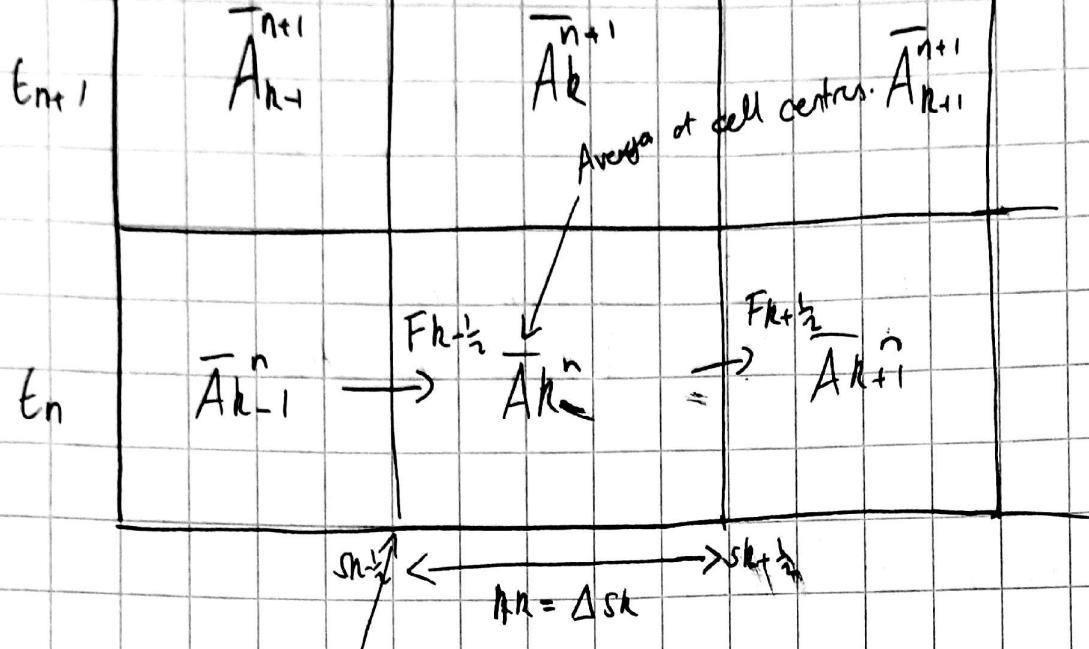


3. $\partial_t A + \partial_s (\lambda A) = 0$

$$\partial_t A + (F(A))_s = 0$$

~~Prob~~ $F(A)$ is the flux





Flux at
cell
faces.
FACES.

$$\Delta t + (F(A, s))_s = 0$$

integrate this over.

$$s_{k-\frac{1}{2}} < s < s_{k+\frac{1}{2}}$$

$$t_n < t < t_{n+1}$$

$$\Rightarrow \bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta s_h} \int_{t_n}^{t_{n+1}} [F(A, s)|_{s=s_{k+\frac{1}{2}}} - F(A, s)|_{s=s_{k-\frac{1}{2}}}] ds$$

Cell average

$$\bar{A}_k(t) = \frac{1}{\Delta s_h} \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} A(s, t) ds$$

Flux evaluated at cell boundaries.

$$F(A, s)|_{s=s_{k+\frac{1}{2}}} = F_{k+\frac{1}{2}}(\bar{A}_k^n, \bar{A}_k^{n+1})$$

2. Derive a five-step condition based on Pg 5.

Forward time method, only need information from previous time step.

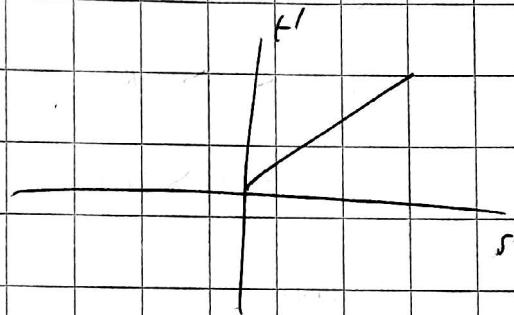
4. characteristics or $s' = s_0' + \lambda t'$ and $\lambda > 0$

Right

$$A(s', t' = 0) = \begin{cases} A_{\text{left}} = A_L = A_L^n & s' < 0 \\ A_{\text{right}} = A_R = A_{R+1}^n & s' \geq 0 \end{cases}$$

$$A(s' = 0, t') = A(s_{n+\frac{1}{2}}, t) = A_L^n$$

$$F(A(s_{n+\frac{1}{2}}, t)) = \lambda A_L^n$$



Right shock, so use information behind. Upwind.

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{\Delta t}{\Delta s_n} \lambda (\bar{A}_k^n - \bar{A}_{k-1}^n)$$

5. Derive a five-step condition based on pg 5.

$$\Delta t < \text{CFL} \min \frac{h}{|\lambda_h|}$$

For scheme

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{\Delta t}{\Delta s_k} \lambda (\bar{A}_k^n - \bar{A}_{k-1}^n)$$

Apply maximum principle, ref coefficients all > 0

$$\bar{A}_k^{n+1} = \left(1 - \frac{\Delta t}{\Delta s_k} \lambda\right) \bar{A}_k^n + \frac{\Delta t}{\Delta s_k} \lambda \bar{A}_{k-1}^n$$

$$1 - \frac{\Delta t}{\Delta s_k} \lambda > 0$$

$$\frac{\Delta t}{\Delta s_k} \lambda > 0$$

$$\Rightarrow \cancel{\Delta t} \Delta t < \frac{\Delta s_k}{\lambda}$$

$$\Delta t < \frac{\Delta s_k}{|\lambda_h|}$$

$$\lambda_h = \frac{\partial F}{\partial A}$$

$$\Delta t < \text{CFL} \frac{\Delta s_k}{|\lambda_h|}$$

Maximum stable value when $\text{CFL} = 1$

take $\cancel{\Delta t} 0 < \text{CFL} < 1$

6. What inflow/outflow conditions imposed
at $s=0$ and $s=L_e$?

constant inflow at $s=0$, depending on conditions?
outflow = inflow.