

### Numerics Ex 3

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$$1). \quad I(u) = \iint_{\Omega} \frac{1}{2} |\nabla u|^2 - uf \, d\Omega = 0$$

is minimised for solution  $u = u(x, y)$

This variation of  $\delta u$

$$\Rightarrow \delta I \equiv \frac{dI}{d\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{I(u + \varepsilon \delta u) - I(u)}{\varepsilon} = 0$$

$$\Rightarrow \iint_{\Omega} \frac{1}{2} \frac{|\nabla(u + \varepsilon \delta u)|^2 - |\nabla u|^2}{\varepsilon} - f(u + \varepsilon \delta u) + fu \, d\Omega$$

$$= \iint_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \nabla u \cdot \varepsilon \nabla \delta u + \frac{\varepsilon^2}{2} |\nabla \delta u|^2 - \frac{1}{2} |\nabla u|^2 - f \varepsilon \delta u \right) \, d\Omega$$

$$= \iint_{\Omega} \nabla u \cdot \nabla \delta u - f \delta u \, d\Omega \quad (\text{as } \varepsilon \rightarrow 0)$$

$$\iint_{\Omega} \nabla u \cdot \nabla \delta u - f \delta u \, d\Omega = 0$$

$$\nabla \cdot (\delta u \nabla u)$$

$$= \nabla u \cdot \nabla \delta u + \delta u \nabla^2 u$$

so integral can be rewritten as

$$-\iint \delta u (\nabla^2 u + f) d\Omega + \iint \nabla \cdot (\delta u \nabla u) d\Omega$$

Defining the boundary as

$$\Gamma_1 : x=0, 1 \text{ for all } y$$

$$\Gamma_2 : y=0, 1 \text{ for all } x$$

$$\text{so } \iint \nabla \cdot (\delta u \nabla u) d\Omega$$

$$= \int_{\Gamma_1} \delta u (\nabla u \cdot \hat{n}) d\Gamma + \int_{\Gamma_2} \delta u (\nabla u \cdot \hat{n}) d\Gamma$$

Using boundary conditions:  $u(0, y) = u(1, y) = 0$   
 $\rightarrow$  so over  $\Gamma_1$ ,  $\delta u = 0$

$$; \partial_y u|_{y=0} = \partial_y u|_{y=1} = 0$$

$$\text{so over } \Gamma_2, \nabla u \cdot \hat{n} = 0$$

This leaves us with

$$\iint \delta_n(\nabla^2 u + f) d\Omega = 0$$

as  $\delta_n$  is arbitrary, then  $\nabla^2 u + f = 0$  for this to always be true

so  $-\nabla^2 u = f$  on  $(x, y) \in [0, 1]^2$

$u(0, y) = u(1, y) = 0$

## Weak Formulation

$$\nabla^2 u + f = 0$$

$$w \nabla^2 u + wf = 0$$

Integrating over  $\Omega$

$$\iint_{\Omega} w \nabla^2 u + wf \, d\Omega = 0$$

Using  $\nabla \cdot (\underline{w} \nabla u) = \underline{w} \nabla^2 u + \nabla \underline{w} \cdot \nabla u$   
again

↓

$$= \iint_{\Omega} \nabla \cdot (\underline{w} \nabla u) - \nabla \underline{w} \cdot \nabla u + wf \, d\Omega$$

$$= \iint_{\Omega} -\nabla \underline{w} \cdot \nabla u + wf \, d\Omega + \int_{\Gamma_1} w \nabla u \cdot \hat{n} \, d\Gamma + \int_{\Gamma_2} w \nabla u \cdot \hat{n} \, d\Gamma$$

like before  $w(c, y) = w(1, y) = 0$   
and  $\partial_y u = c$  at  $y=c, 1$

$$\rightarrow \boxed{\iint_{\Omega} -\nabla \underline{w} \cdot \nabla u + wf \, d\Omega = 0 \quad \text{weak formulation}}$$

As for variational principle

$$\iint T \delta u \cdot \nabla u - \int \delta u \, d\Omega = 0$$

| Then  $w(x, y) = \delta u(x, y)$

$$2). \quad u(x, y) \approx u_n(x, y) = u_j \varphi_j(x, y)$$

$$I(u) = \iint_{\Omega} \frac{1}{2} |\nabla u|^2 - u f \, d\Omega$$

$$\boxed{I = \iint_{\Omega} \frac{1}{2} |\nabla(u_j \varphi_j)|^2 - u_j \varphi_j f \, d\Omega}$$

$$u \approx u_n = u_j \varphi_j$$

$$w \approx w_n = w_j \varphi_j$$

$$\text{Let } w_j = \delta_{ij}$$

$$\rightarrow w_i = \varphi_i$$

Weak Formulation:

$$\iint -\nabla w \cdot \nabla u + wf \, d\Omega = 0$$

$$\iint \nabla w \cdot \nabla u \, d\Omega = \iint wf \, d\Omega$$

$$\iint \nabla \varphi_i \cdot \nabla(u_j \varphi_j) \, d\Omega = \iint \varphi_i f \, d\Omega$$

$$\rightarrow A_{ij} u_j = b_i$$

$$\text{where } A_{ij} = \iint \mathbf{T}\varphi_i \cdot \mathbf{T}\varphi_j d\Omega$$

$$b_i = \iint \varphi_i f d\Omega$$

Need to enforce Dirichlet boundary conditions  
where  $\varphi_i = 0$  ( $x=0, 1$ )

$$\hookrightarrow \text{actually } A_{ij} u_j = b_i$$

can be split

$$\rightarrow A_{i'j'} u_{j'} + A_{i'n} u_n = b_{i'}$$

where  $i', j'$  are nodes not on Dirichlet boundary  
and have the Dirichlet nodes

In this scenario,  $n=0$  on boundary  $\Rightarrow u_n=0$

$$\rightarrow \boxed{A_{i'j'} u_{j'} = b_{i'}}$$

$$I = \iint \frac{1}{2} |\nabla(u; \varphi_j)|^2 - u_j \varphi_j f d\Omega$$

As before we showed  $\delta u = \omega$

$$\rightarrow \delta u \approx \delta u_i = \varphi_i$$

$$\begin{aligned} \delta I &= \frac{1}{\varepsilon} \iint \left[ \frac{1}{2} |\nabla(u_j + \varepsilon \varphi_i)|^2 - (u_j \varphi_i + \varepsilon \varphi_i f) \right. \\ &\quad \left. - \frac{1}{2} |\nabla(u_j \varphi_i)|^2 + u_j \varphi_i f \right] d\Omega \end{aligned}$$

you need a delta  $u_{-i}$  varphi  $_{-i}$  (delta  $u_{-i}$  only exist for  $i=i'$ ); set-up not right. -0.25

$$= \frac{1}{\varepsilon} \iint \varepsilon u_j \nabla \varphi_i \cdot \nabla \varphi_i + \frac{1}{2} \varepsilon^2 |\nabla \varphi_i|^2 - \varepsilon \varphi_i f d\Omega$$

as  $\varepsilon \rightarrow 0$

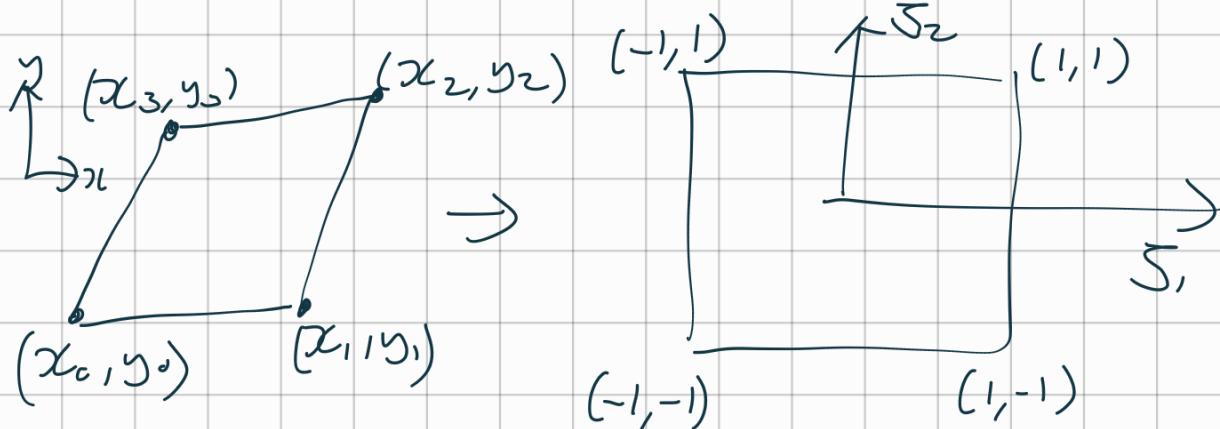
$$\delta I \approx \iint u_j \nabla \varphi_i \cdot \nabla \varphi_i - \varphi_i f d\Omega = 0$$

$$\iint \nabla \varphi_i \cdot \nabla \varphi_i d\Omega u_j = \iint \varphi_i f d\Omega$$

so like before (when apply is  $u=0$  at Dirichlet boundary)

$$\boxed{A_{i,j}, u_j = b_i}$$

3). Using coordinate systems for quadrilateral elements



want it to map kind of like this.

Introducing Reference element

$$\vec{\xi} \in [-1, 1]^2 \text{ with reference coordinates } \xi = (\xi_1, \xi_2)^T$$

This means local coordinates of a node of element \$k\$, \$x\_{k,\alpha}\$, can be mapped onto reference element.

$$\rightarrow x = \sum_{\alpha=0}^3 x_{k,\alpha} \chi_\alpha(\xi)$$

$$\text{as we want } x_{k,0} \rightarrow (-1, -1)^T \\ x_{k,1} \rightarrow (1, -1)^T$$

etc.

Then can use shape functions.

$$K_0 = \frac{(1-\xi_1)(1-\xi_2)}{4}$$

$$K_1 = \frac{(1+\xi_1)(1-\xi_2)}{4}$$

$$K_2 = \frac{(1+\xi_1)(1+\xi_2)}{4}$$

$$K_3 = \frac{(1-\xi_1)(1+\xi_2)}{4}$$

→ and these are essentially the basis functions in the reference system

We want to transform

$$A_{ij} = \iint \nabla \varphi_i \cdot \nabla \varphi_j d\Omega$$

$$b_i = \iint \varphi_i f d\Omega$$

into reference coordinates system.

$$\hat{A}_{\alpha/\beta} = \int \nabla K_\alpha \cdot \nabla K_\beta d\Omega$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} \end{pmatrix}$$

$$\zeta = J \underline{x}$$

$$\underline{x} = J^{-1} \zeta$$

$$\text{so } D = (J^T)^{-1} \begin{pmatrix} \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \end{pmatrix}$$

$$\text{and } d\Omega = |\det(J)| d\zeta$$

$$\hat{A}_{\alpha, \beta} = \int_{\hat{K}} \left( (J^T)^{-1} \begin{pmatrix} \frac{\partial x_\alpha}{\partial \xi_1} \\ \frac{\partial x_\alpha}{\partial \xi_2} \end{pmatrix} \right) \cdot \left( (J^T)^{-1} \begin{pmatrix} \frac{\partial x_\beta}{\partial \xi_1} \\ \frac{\partial x_\beta}{\partial \xi_2} \end{pmatrix} \right) \sqrt{|\det J|} d\zeta$$

$$\text{and } \hat{b}_\alpha = \int_{\hat{K}} f(x(\zeta), y(\zeta)) x_\alpha |\det(J)| d\zeta$$

For quadrilaterals

$$\underline{x} = \begin{pmatrix} x_0 K_0 + x_1 K_1 + x_2 K_2 + x_3 K_3 \\ y_0 K_0 + y_1 K_1 + y_2 K_2 + y_3 K_3 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\rightarrow \text{so } \frac{\partial x}{\partial S_1} = -\frac{x_0}{4}(1-S_2) + \frac{x_1}{4}(1-S_2)$$

$$+ \frac{x_2}{4}(1+S_2) - \frac{x_3}{4}(1+S_2)$$

$$= \frac{1}{4} \left[ (1-S_2)(x_1 - x_0) + (1+S_2)(x_2 - x_3) \right]$$

This can be repeated for other terms.

$$J^T =$$

$$\frac{1}{4} \begin{pmatrix} (1-S_2)(x_1 - x_0) + (1+S_2)(x_2 - x_3) & (1-S_2)(y_1 - y_0) + (1+S_2)(y_2 - y_3) \\ (1-S_1)(x_3 - x_0) + (1+S_1)(x_2 - x_1) & (1-S_1)(y_3 - y_0) + (1+S_1)(y_2 - y_1) \end{pmatrix}$$

$(J^+)^{-1}$  can be found from this, and we can find complete expression for  $\hat{A}_{\alpha\beta}$ ,  $\hat{b}_\alpha$

### Matrix assembly :

Initially set all  $A_{ij}, b_i = 0$

Assembling global matrix

① For  $h = 1, N_{elements}$  :

② For  $\alpha = 0, 3$  :

③  $i = \text{Index}(h, \alpha)$  :

④ For  $\beta = 0, 3$  :

⑤  $j = \text{Index}(h, \beta)$

⑥  $A_{ij} = A_{ij} + \hat{A}_{\alpha\beta}$

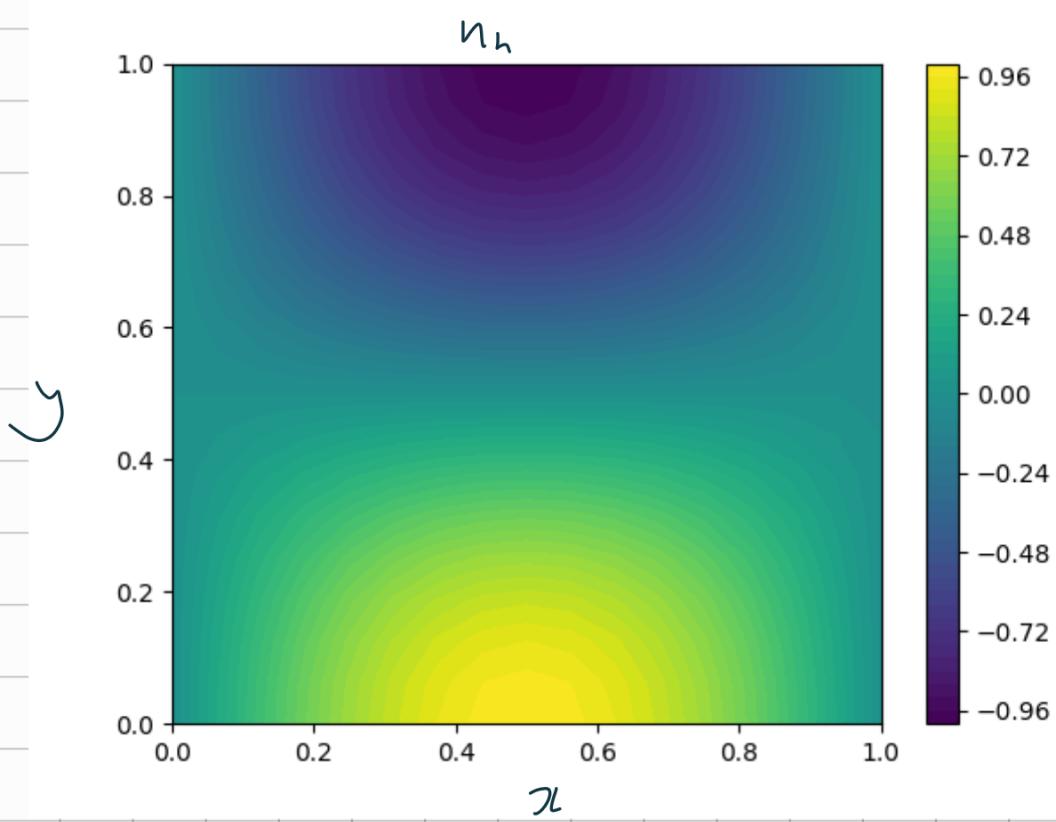
⑦  $b_i = b_i + \hat{b}_\alpha$

- ① Looping over every element in mesh.
- ② Looping over every node in reference element.
- ③ mapping the reference node  $\alpha$  of element  $k$  to the global coordinate system  $i$ .
- ④ Looping over every node in reference element.
- ⑤ mapping the reference node  $\beta$  of element  $k$  to the global coordinate system  $j$
- ⑥ Local matrix,  $\hat{A}_{\alpha\beta}$  added to global matrix  $A_{ij}$   
→ because we know where the reference nodes correspond to in global system.
- ⑦  $\hat{b}_\alpha$  added to  $b_i$

4).

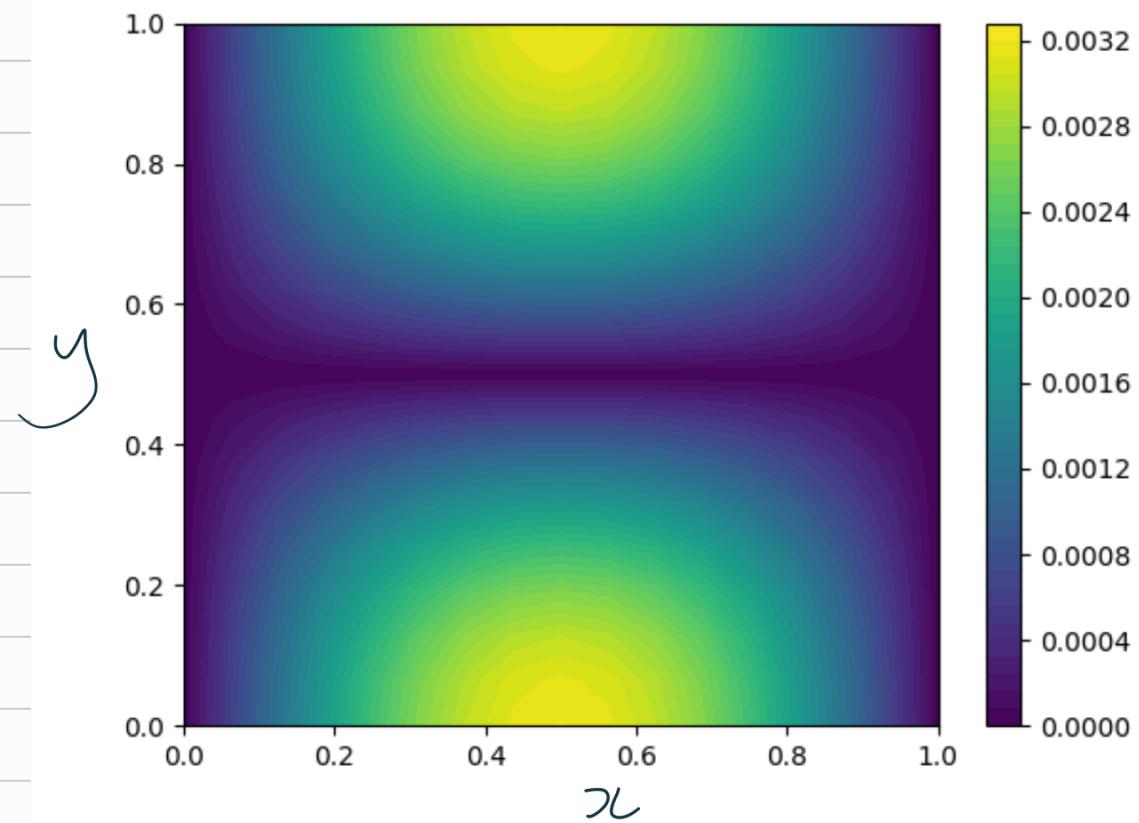
Contour Plot for

16x16 grid, order 1



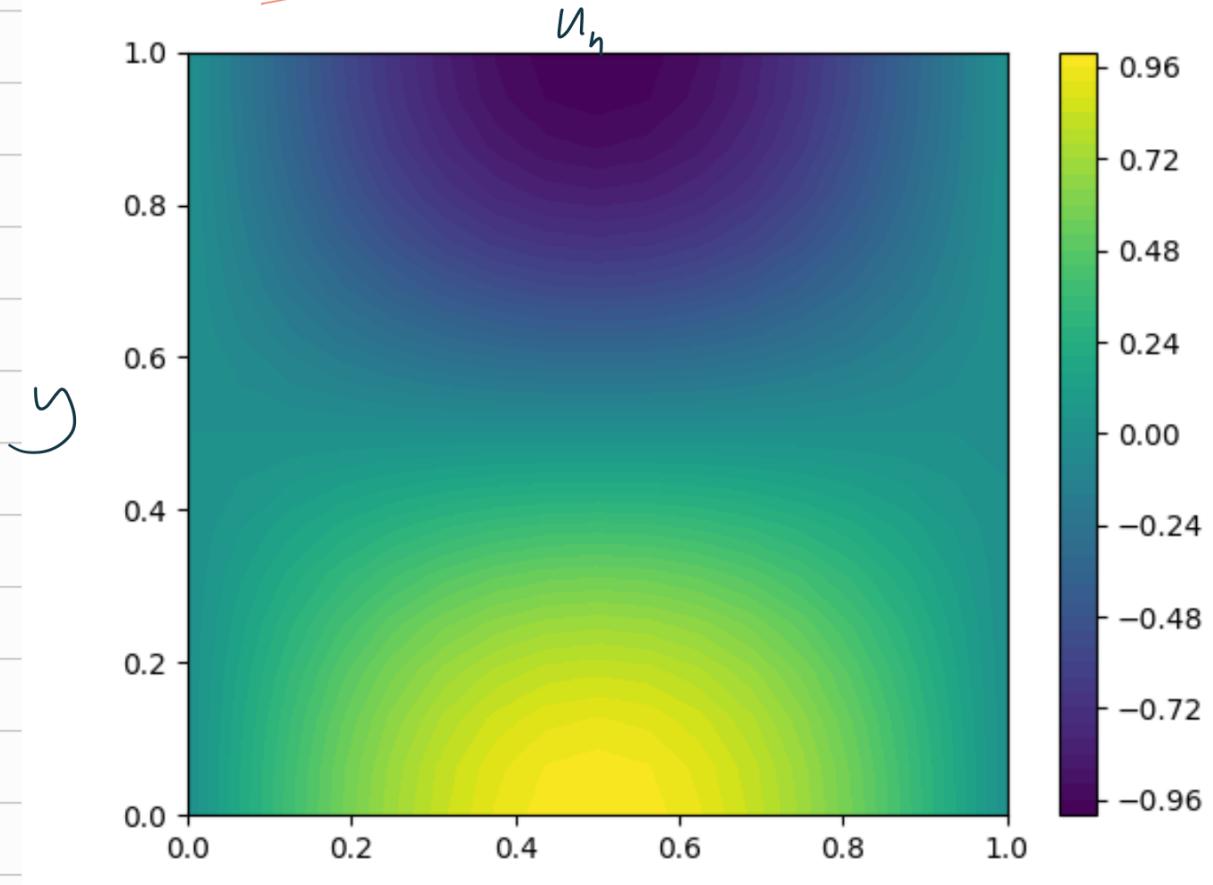
$$\begin{aligned} & \left\{ h, p \right\} \\ & = \left\{ \frac{1}{16}, 1 \right\} \end{aligned}$$

Difference between numerical and exact  
solution

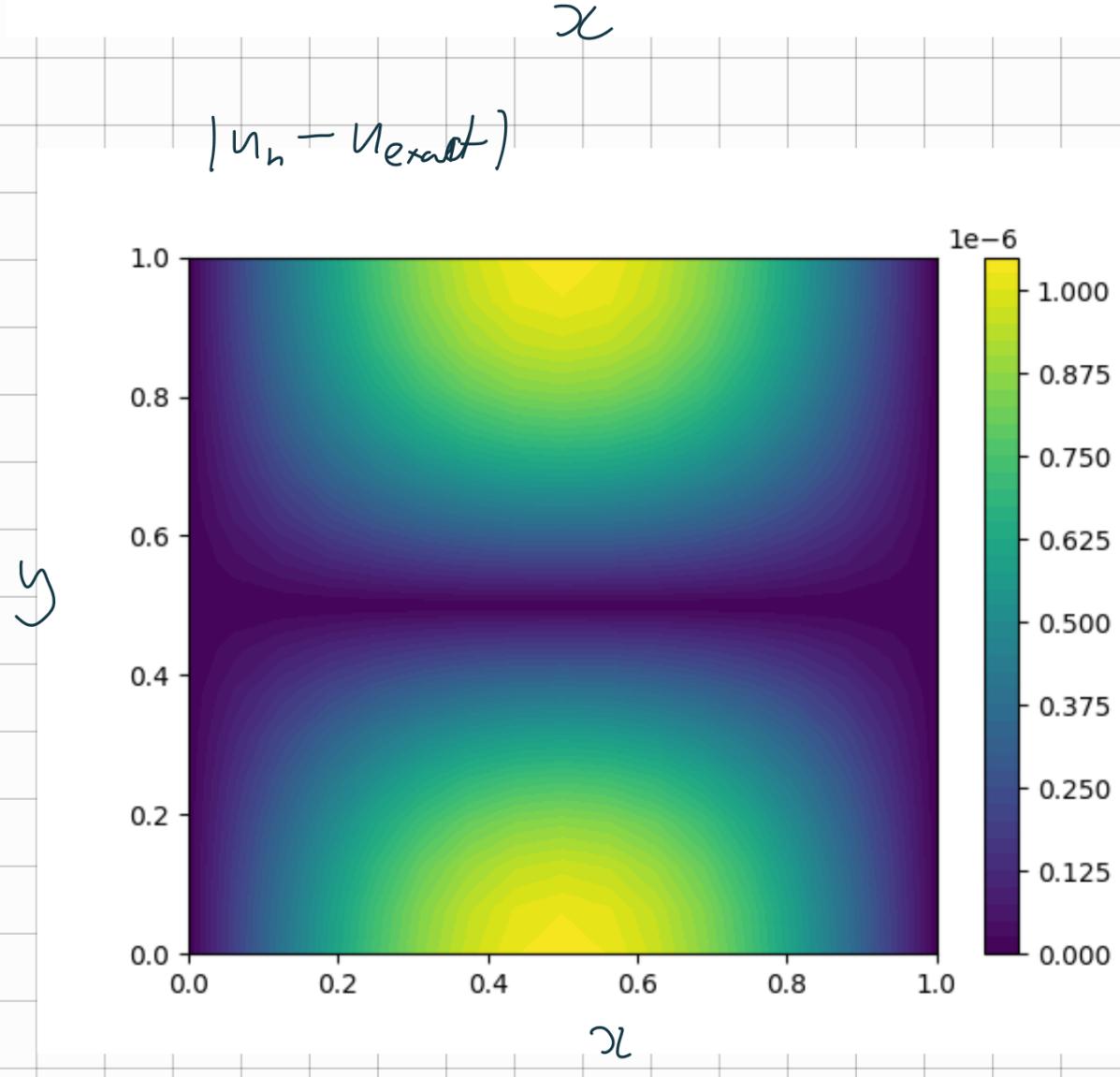


$16 \times 16$  mesh  $p = 2$

$$\{h, p\} = \left\{\frac{1}{16}, 2\right\}$$



$$|u_h - u_{exact}|$$



→ Increasing the order of accuracy from 1 to 2, significantly changes difference between numerical and exact solution

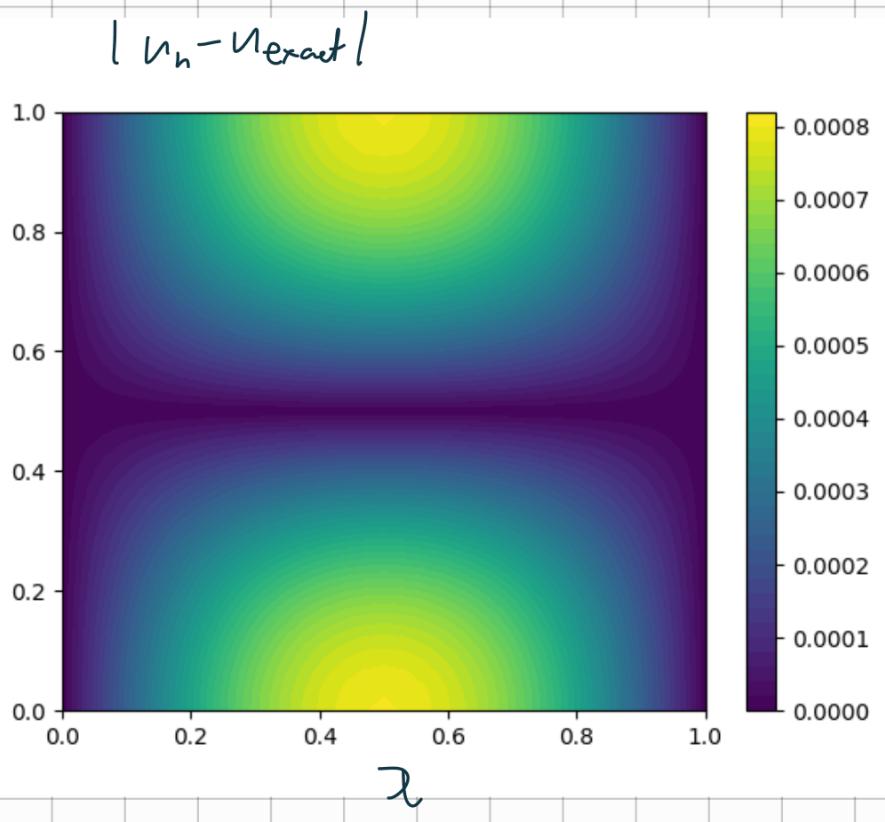
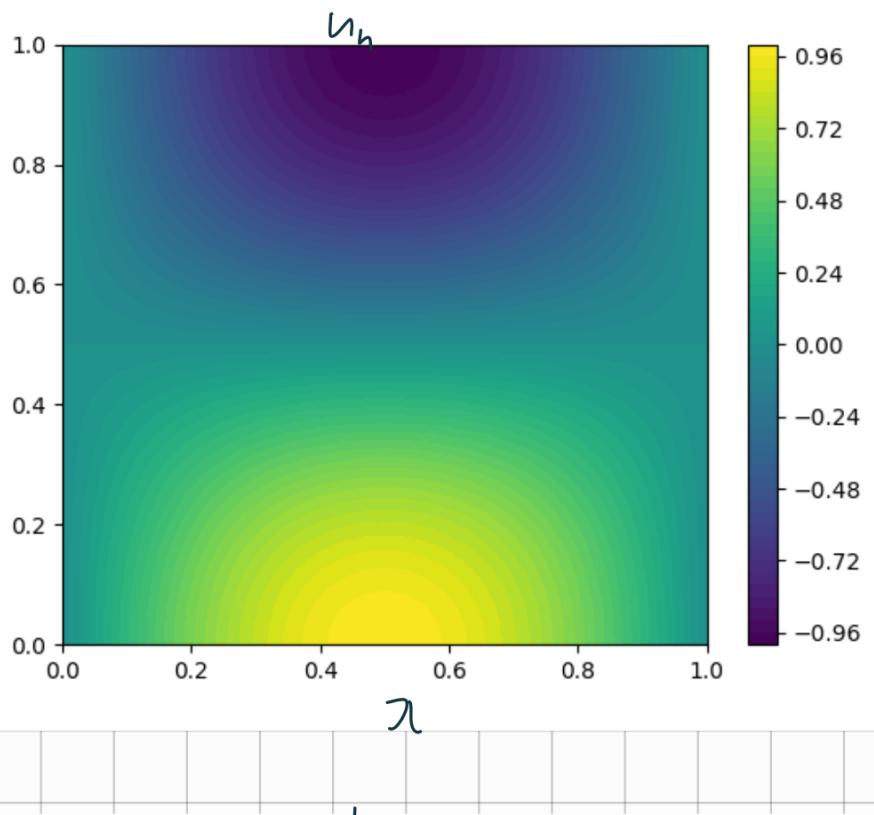
$$\text{e.g. } |u_n - u_{\text{exact}}|_{\max} \sim 10^{-3} \text{ for } p=1$$

$$|u_n - u_{\text{exact}}|_{\max} \sim 10^{-6} \text{ for } p=2$$

→ so now looking at how mesh refinement changes it

$32 \times 32, P=1$

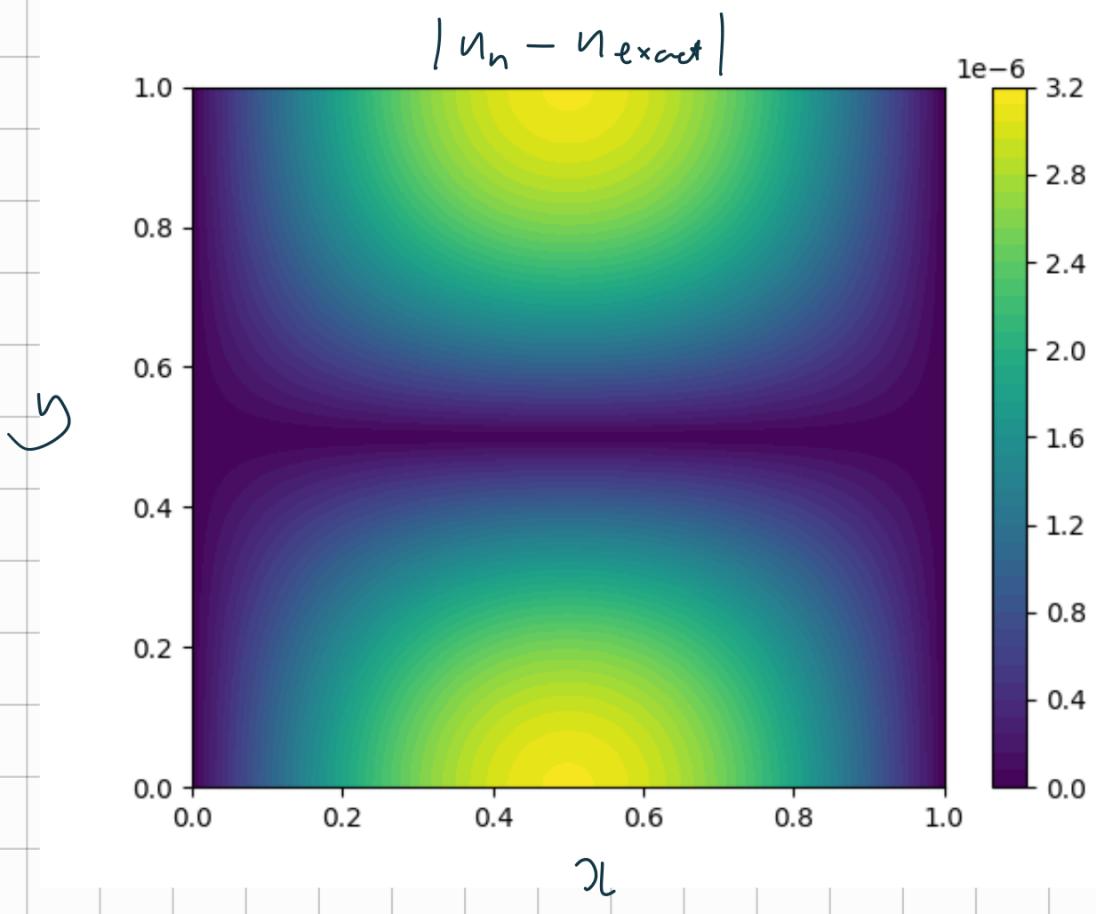
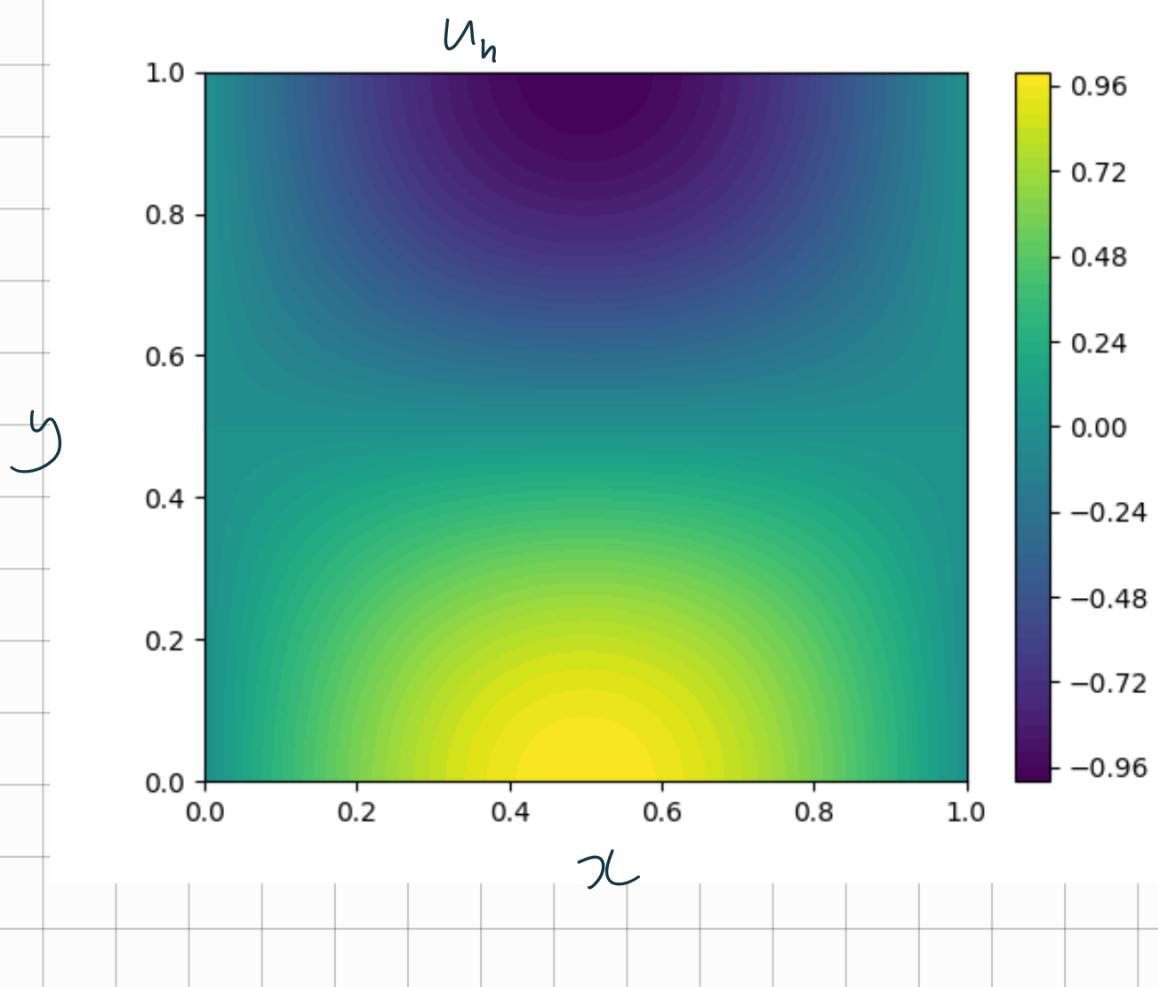
$$\{h, p\} = \left\{ \frac{1}{32}, 1 \right\}$$



Decreasing element size  $\Delta x$  by 2, decreases the difference from  $\sim 10^{-3}$  to  $10^{-4}$

$512 \times 512, \beta = 1$

$$\{h, p\} = \left\{ \frac{1}{512}, \beta \right\}$$



$|u_n - u_{\text{exact}}|$  is approximately the same  
 over a grid of  $16 \times 16$ ,  $p=2$   
 and of  $512 \times 512$ ,  $p=1$

→ Fnestbed will look at  $L^2$  error for each

$h$	$p$	$L^2$
$\gamma_{16}$	1	$2 \times 10^{-3}$
$\gamma_{16}$	2	$1 \times 10^{-6}$
$\gamma_{16}$	3	$4 \times 10^{-9}$
$\gamma_{32}$	1	$4 \times 10^{-4}$
$\gamma_{64}$	1	$1 \times 10^{-4}$
$\gamma_{128}$	1	$3 \times 10^{-5}$
$\gamma_{32}$	2	$7 \times 10^{-8}$
$\gamma_{64}$	2	$4 \times 10^{-9}$
		700

For  $p=1$ , as  $h$  decreases by a factor of 2,  
 $L^2$  decreases by  $\sim 4$

$p=2$      $h$  decreases by  
 $L^2$  decreases by  $\sim 16$

For  $h = \frac{1}{16}$  as p varies

$L^2$  decreases by  $18^2$

$h = \frac{1}{32}$ ,  $L^2$  decreases by  $\sim 32^2$

$\Rightarrow O(\text{error}) \sim h^{2p}$

$\left\{ h, p \right\}$  combinations with similar values of  $h^{2p}$   
are roughly equivalent

## S)- Step 1)

```
# This method is the same as STEP 1 creating function and test function
u = TrialFunction(V) # The unknown or variable u(x,y)
v = TestFunction(V) # The testfunction of u, which may be better called delu or deltau
```

```
#Step 1 creating weak form using function and test function
a = (inner(grad(u),grad(v)))*dx # Step 2/3: The weak form first term
L = (f*v)*dx # Step 2/3: The weak form second term; dx is the infinitesimal piece in the domain here: dx*dy=dA with area A.
# Weak form generated
```

## Step 2

```
#creates discrete function space (step 2)
V = FunctionSpace(mesh, 'CG', 1) # Piecewise linear continuous Galerkin function space or polynomials
```

```
#discrete u function
u_1 = Function(V, name='u_1') # Name of solution for first method
```

## Step 3 + 4

```
#If it wasn't automated, this is where step 3 would be
# and local, reference coordinate system would be introduced
```

```
#Firedrake Implementation Step 4
solve(a == L, u_1, solver_parameters={'ksp_type': 'cg', 'pc_type': 'none'}, bcs=[bc_x0, bc_x1]) # Step 4: the solution assigned to u1
```

All these can be found in code

} poisson\_eq\_Alt\_ex3.py

$$6). f(x, y) = 5$$

$$n = \frac{5}{2}x(1-x)$$

bit easy!

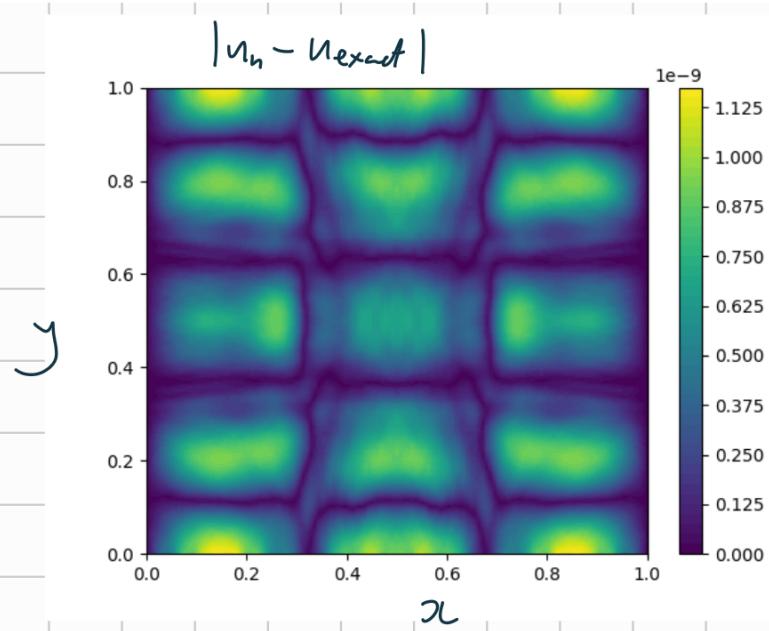
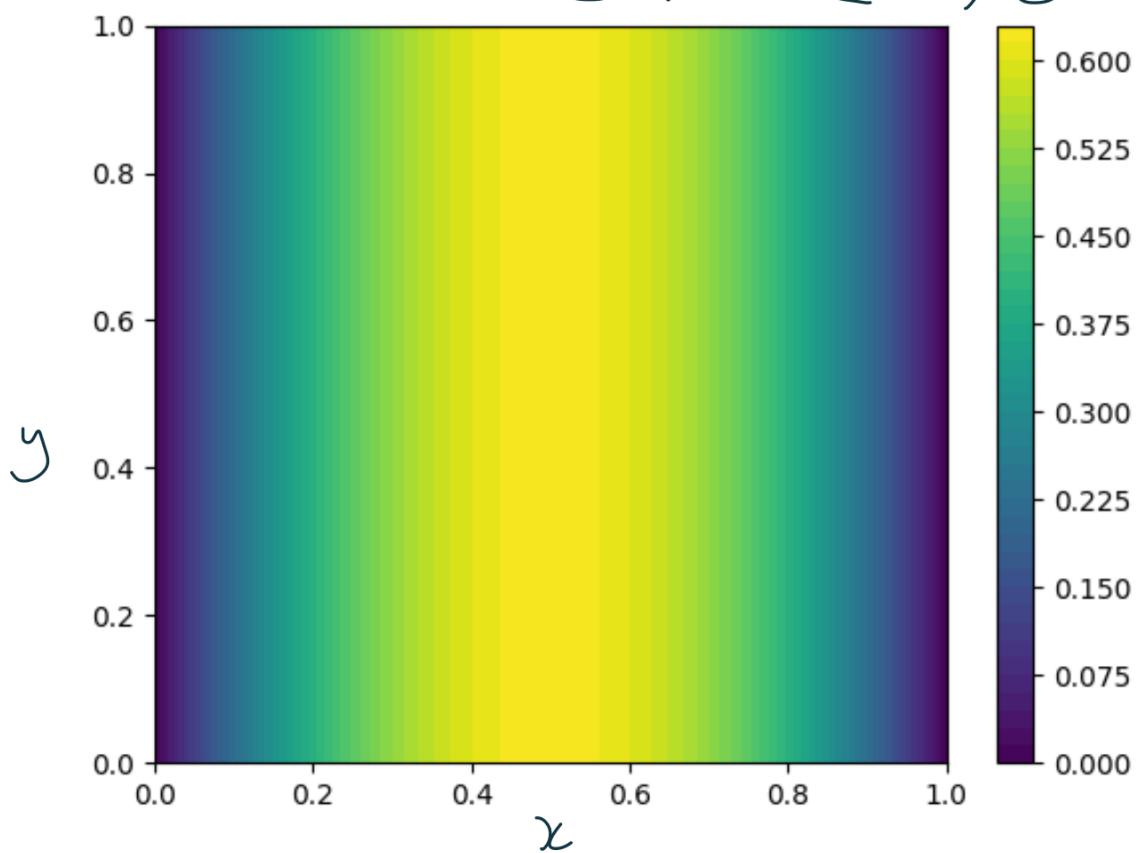
BCs same as before

$$u(0, y) = u(1, y) = 0$$

Asked to change thee as well! -0.5

$$\partial_y u \Big|_{y=0} = \partial_y u \Big|_{y=1} = 0$$

$$u_n \quad \{h, p\} = \left\{ \frac{1}{128}, 1 \right\}$$



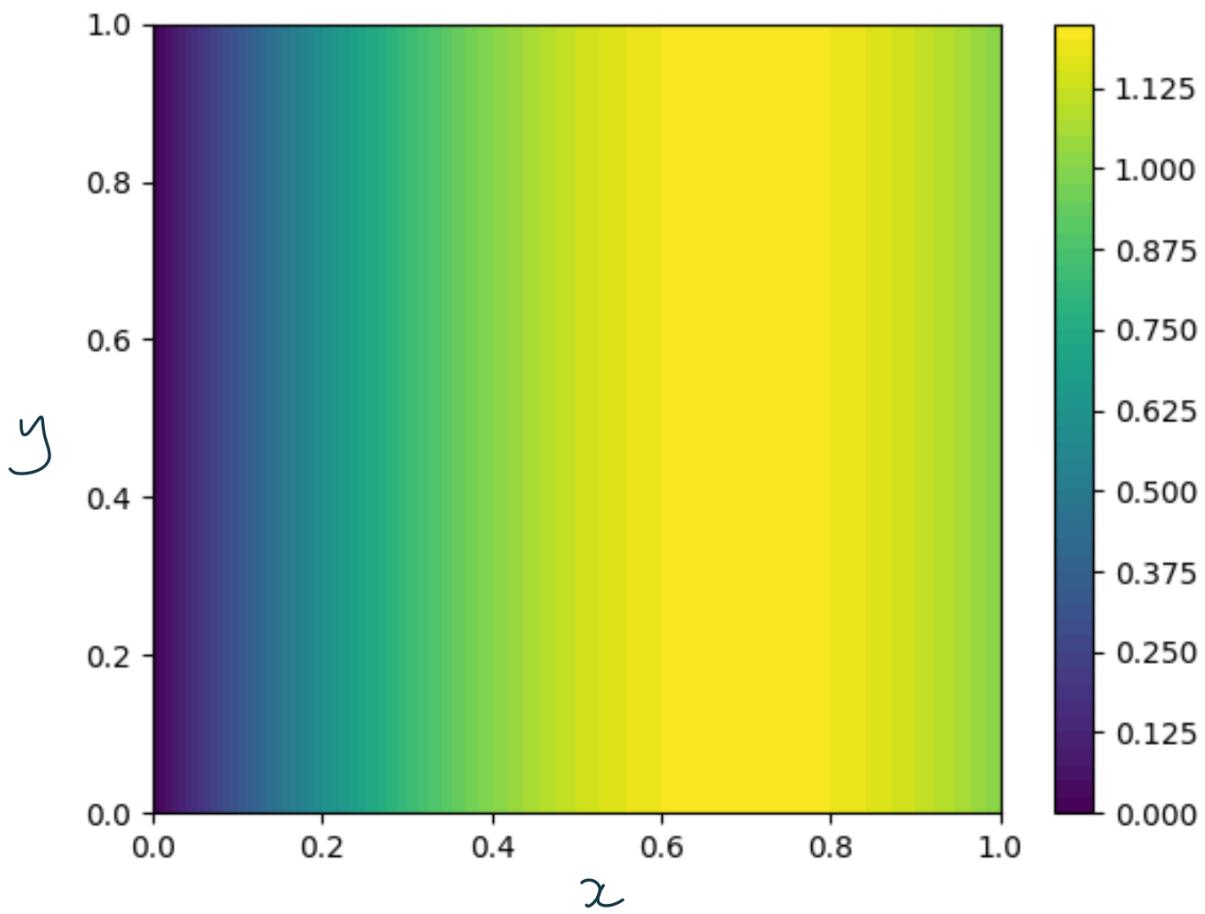
$$f(x, y) = 5$$

$$u(0, y) = 0, u(1, y) = 1$$

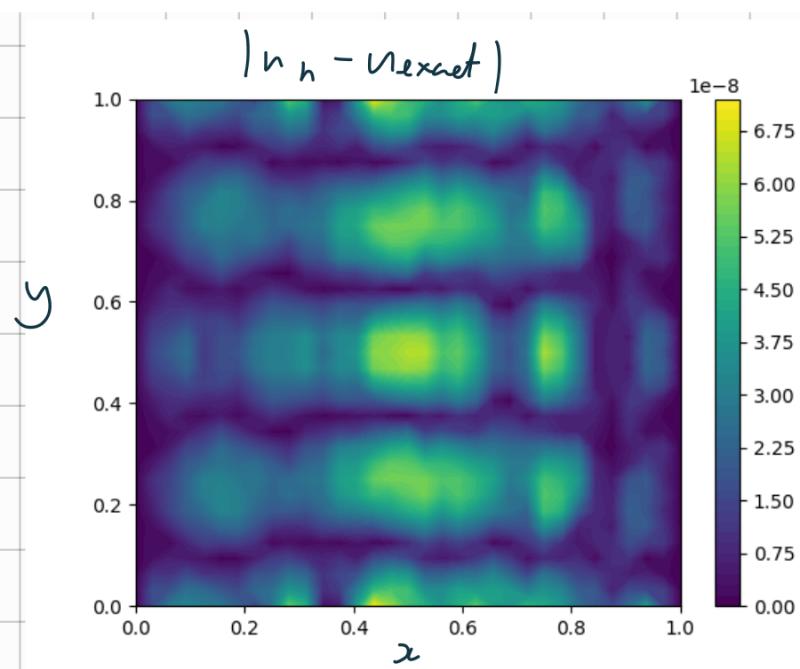
$$\partial_y u|_{y=0} = \partial_y u|_{y=1} = 0$$

$$u = \frac{x}{2}(7 - 5x)$$

$u_h$



$|u_h - u_{exact}|$



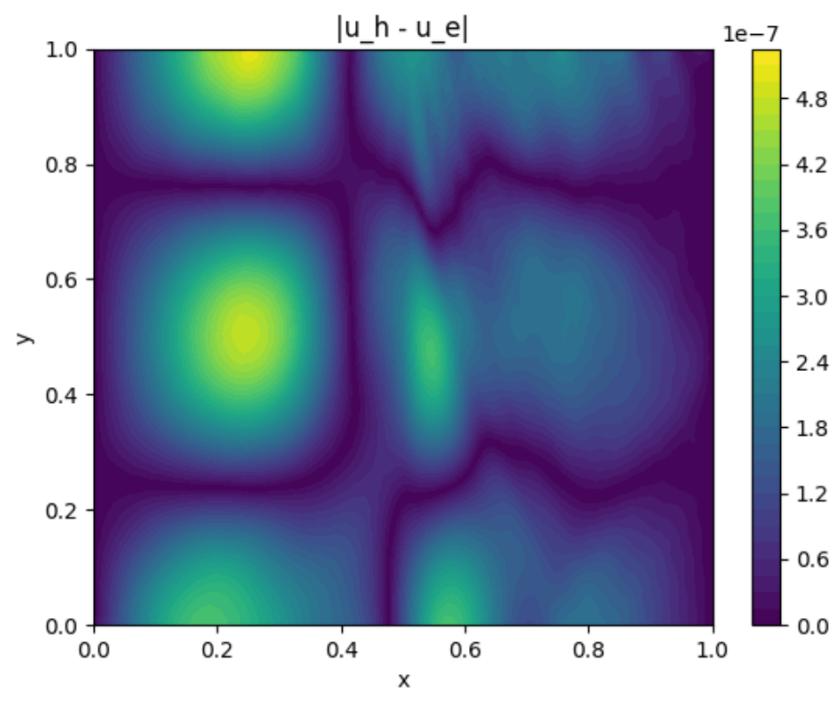
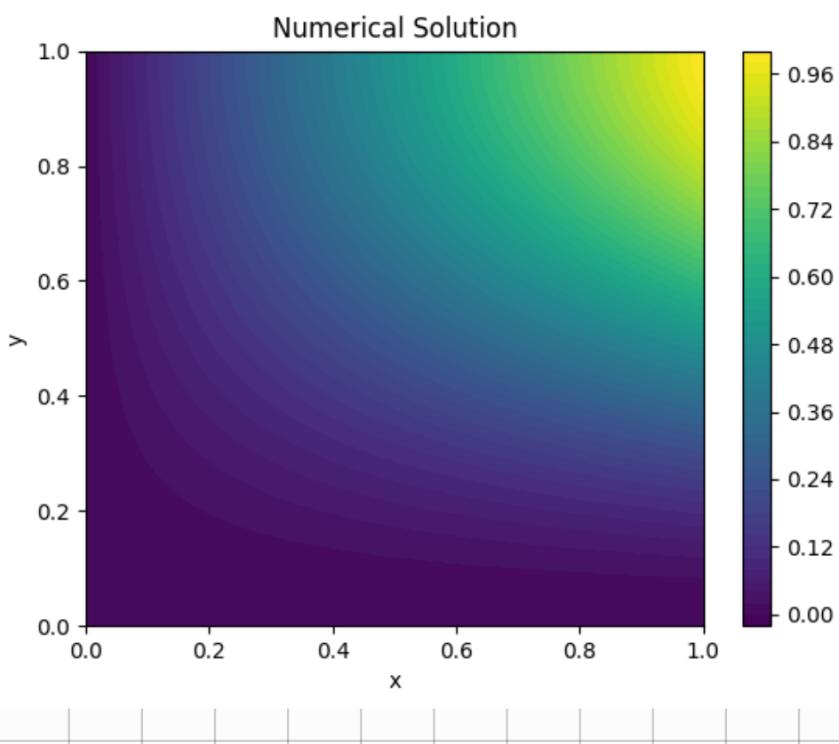
$$f(x) = -6(1-2y)x$$

$$u(0, y) = 0 \quad u(1, y) = y^2(3-2y)$$

$$\partial_y u \Big|_{y=0} = \partial_y u \Big|_{y=1} = 0$$

$$u = y^2(3-2y)x$$

$$\frac{\partial u}{\partial y} = 2y(3-2y)x - 2y^2x$$



To implement (All homogeneous BCs)

$$f = S, u(0, y) = u(1, y) = 0$$

set f-choice = 1

$$f = 5, u(0, y) = 0, u(1, y) = 1$$

set f-choice = 2

$$f(x, y) = -6y(1-2y)x, u(0, y) = 0, u(1, y) = y^2(3-2y)$$

set f-choice = 3

→ All those seem to be successfully implemented  
as have checked  $|u_h - u_e|$

→ so could then add geometries which don't  
have nice exact solutions

## Part 2

1). Multiply this equation ⑪ by  $g = g(y)$   
where  $g(y) = \text{a test function}$

Then integrate it over  $y$

$$\rightarrow \int_0^{L_y} g \left[ \partial_x (w_v h_m) - 2g \partial_y (w_v h_m \partial_y h_m) \right] dy = \int_0^{L_y} \frac{\sum w_v K}{m_{\text{probe}}} dy$$

$$\int_0^{L_y} \left( \sum \partial_x (w_v h_m) - 2g \sum \partial_y (w_v h_m \partial_y h_m) \right) dy = \int_0^{L_y} \frac{\sum w_v K}{m_{\text{probe}}} dy$$

Using  $\partial_y (\sum w_v h_m \partial_y h_m)$

$$= (\partial_y g)(w_v h_m \partial_y h_m) + g \partial_y (w_v h_m \partial_y h_m)$$

This can be rewritten as

$$\int_0^{L_y} \left( \sum \partial_x h_m + 2g (\partial_y \sum) (h_m \partial_y h_m) - 2g \partial_y (g h_m \partial_y h_m) \right) dy$$

$$= \int_0^{L_y} \frac{g K}{m_{\text{probe}}} dy$$

$$\int_0^{L_y} \left[ \zeta \partial_t h_m + \alpha g h_m \partial_y \zeta \partial_y h_m \right] dy - \alpha g \left[ \zeta h_m \partial_y h_m \right] \Big|_0^{L_y}$$

$$= \int_0^{L_y} \frac{g R}{m_{par} \sigma_e} dy$$

As  $\partial_y h_m = 0$  at  $y = L_y$

$$\text{Then } -\alpha g \left[ \zeta h_m \partial_y h_m \right] \Big|_0^{L_y}$$

$$= -\alpha g g(0, t) h_m \partial_y h_m \Big|_{y=0}$$

$$= -\alpha g g(0) h_m \partial_y h_m \Big|_{y=0}$$

$$\int_0^{L_y} \left[ \zeta \partial_t h_m + \alpha g h_m \partial_y \zeta \partial_y h_m \right] dy - \alpha g g(0) h_m \partial_y h_m \Big|_{y=0}$$

$$= \int_0^{L_y} \frac{g R}{m_{par} \sigma_e} dy$$

Using equation (14):

$$L_c \frac{dh_m}{dt} = m_{par} \frac{\sigma_e}{2} \alpha g \partial_y (h_m^2) \Big|_{y=0} - \sqrt{g} \max \left( \frac{2}{3} h_m(t), 0 \right)^{3/2}$$

$$\alpha g h_m \partial_y h_m \Big|_{y=0} = \frac{1}{m_{par} \sigma_e} \left[ L_c \frac{dh_m}{dt} + \sqrt{g} \max \left( \frac{2}{3} h_m, 0 \right)^{3/2} \right]$$

Eliminate  $h_m \partial_y h_m \Big|_{y=0}$  from equations

$$\int_0^{L_y} g \partial_t h_m + \alpha g h_m \partial_y g \partial_y h_m dy$$

$$-\frac{g(0)}{m_{par} \sigma_e} \left[ L_c \frac{dh_m(0,t)}{\Delta t} + \sqrt{g} \max \left( \frac{2h_m(0,t)}{3}, 0 \right)^{3/2} \right]$$

$$= \int_0^{L_y} \frac{g n}{m_{par} \sigma_e} dy$$

Using forward Euler ( $\theta = C$ )

$$\int_0^{L_y} g \left( \frac{h_m^{n+1} - h_m^n}{\Delta t} \right) + \alpha g h_m^n \partial_y g \partial_y h_m^n dy$$

$$-\frac{g(0)}{m_{par} \sigma_e} \left[ \frac{L_c (h_{cm}^{n+1} - h_{cm}^n)}{\Delta t} + \sqrt{g} \max \left( \frac{2h_{cm}^n}{3}, 0 \right) \right]$$

$$= \int_0^{L_y} \frac{g n}{m_{par} \sigma_e} dy$$

$$\int_0^{L_y} g h_m^{n+1} dy + \frac{L_c h_m^{n+1}}{m_{proto}} g(0)$$

$$= \int_0^{L_y} g h_m^n dy + \frac{L_c h_m^n}{m_{proto}} g(0)$$

$$+ \Delta t \int_0^{L_y} \left( -\alpha g h_m^n \partial_y g \partial_y h_m^n + \frac{g k^n}{m_{proto}} \right) dy$$

$$- \Delta t \frac{\int g}{m_{proto}} \max\left(\frac{2h_m^n}{3}, 0\right)^{3/2} g(0)$$

Let  $q = \ell_i(y)$ ,  $h_m = h_j \ell_j(y)$

for  $i, j = 1, \dots, N_n$  where  $N_n$  = no. of nodes

$\rightarrow$  and  $h_i = h_{cm} \rightarrow$  this means  $\ell_i = 1$  (e.g.  $g(0) = 1$ )

$$\int_0^{L_y} \ell_i \ell_j dy h_j^{n+1} + \frac{L_c h_i^{n+1}}{m_{proto}} s_{ij}$$

$$= \int_0^{L_y} \ell_i \ell_j dy h_j^n + \frac{L_c h_i^n}{m_{proto}} s_{ij}$$

$$+ \Delta t \int_0^{L_y} \left( -\alpha g h_m^n \partial_y \ell_i \partial_y h_m^n + \frac{\ell_i k^n}{m_{proto}} \right) dy$$

$$- \Delta t \frac{\int g}{m_{proto}} \max\left(\frac{2h_i^n}{3}, 0\right)^{3/2}$$

$$\rightarrow M_{ij} h_j^{n+1} + \frac{L_c h_i^{n+1}}{m_{\text{parte}}} \delta_{ii}$$

$$= M_{ij} h_j^n + \frac{L_c h_i^n}{m_{\text{parte}}} \delta_{ii} + \Delta t b_i^n - \frac{\Delta t \int_S \rho \left( \frac{2h_i^n}{3} \right)}{m_{\text{parte}}}$$

where  $M_{ij} = \int_0^L \varrho_i \varrho_j dy$

$$b_i = \int_0^L \left( -\alpha g h_m^n \partial_y \varrho_i \partial_y h_m^n + \frac{\varrho_i V^n}{m_{\text{parte}}} \right) dy$$

In finite difference case

$$\partial_t h_m - \alpha g \partial_y (h_m \partial_y h_m) = \frac{V}{m_{\text{parte}}}$$

becomes

$$\frac{h_j^{n+1} - h_j^n}{\Delta t} - \frac{\alpha g}{\Delta y} \left[ h_{j+1/2}^n \partial_y h_{j+1/2}^n - h_{j-1/2}^n \partial_y h_{j-1/2}^n \right] = \frac{V}{m_{\text{parte}}}$$

$$\frac{h_j^{n+1} - h_j^n}{\Delta t} - \frac{\alpha g}{2(\Delta y)^2} \left[ (h_{j+1}^n + h_j^n)(h_{j+1}^n - h_j^n) - (h_j^n + h_{j-1}^n)(h_j^n - h_{j-1}^n) \right]$$

$$= \frac{V}{m_{\text{parte}}}$$

$$\text{letting } h_j^n = \lambda^n e^{ik_j \Delta x}$$

$$\frac{\lambda - 1}{\Delta t} - \frac{\alpha g h_j^n}{2(\Delta y)^2} \left[ (e^{ik \Delta x} + 1)(e^{ik \Delta x} - 1) - (1 + e^{-ik \Delta x})(1 - e^{-ik \Delta x}) \right] \\ = \frac{\kappa}{m_{\text{poro}} e h_j^n}$$

$$\frac{\lambda - 1}{\Delta t} - \frac{\alpha g h_j^n}{2(\Delta y)^2} \left[ (e^{2ik \Delta x} - 1) - (1 - e^{-2ik \Delta x}) \right] = \frac{\kappa}{m_{\text{poro}} e h_j^n}$$

$$\lambda - 1 = \frac{h_j^n \alpha g \Delta t}{2(\Delta y)^2} \left( e^{2ik \Delta x} - 2 + e^{-2ik \Delta x} \right) + \frac{\kappa}{m_{\text{poro}} e h_j^n} \Delta t$$

$$= \frac{\alpha g \Delta t 2 i^2 h_j^n \sin^2(k \Delta x)}{(\Delta y)^2} + \frac{\kappa}{m_{\text{poro}} e h_j^n} \Delta t$$

$$= \left( \frac{\kappa}{m_{\text{poro}} e h_j^n} - \frac{2 \alpha g h_j^n \sin^2(k \Delta x)}{(\Delta y)^2} \right) \Delta t$$

$|\lambda| \leq 1$  to be stable

$\Rightarrow \Delta t$

$$\Delta t \leq \left| \frac{1}{\frac{\kappa}{m_{\text{poro}} e h_j^n} - \frac{2 \alpha g h_j^n \sin^2(k \Delta x)}{(\Delta y)^2}} \right|$$

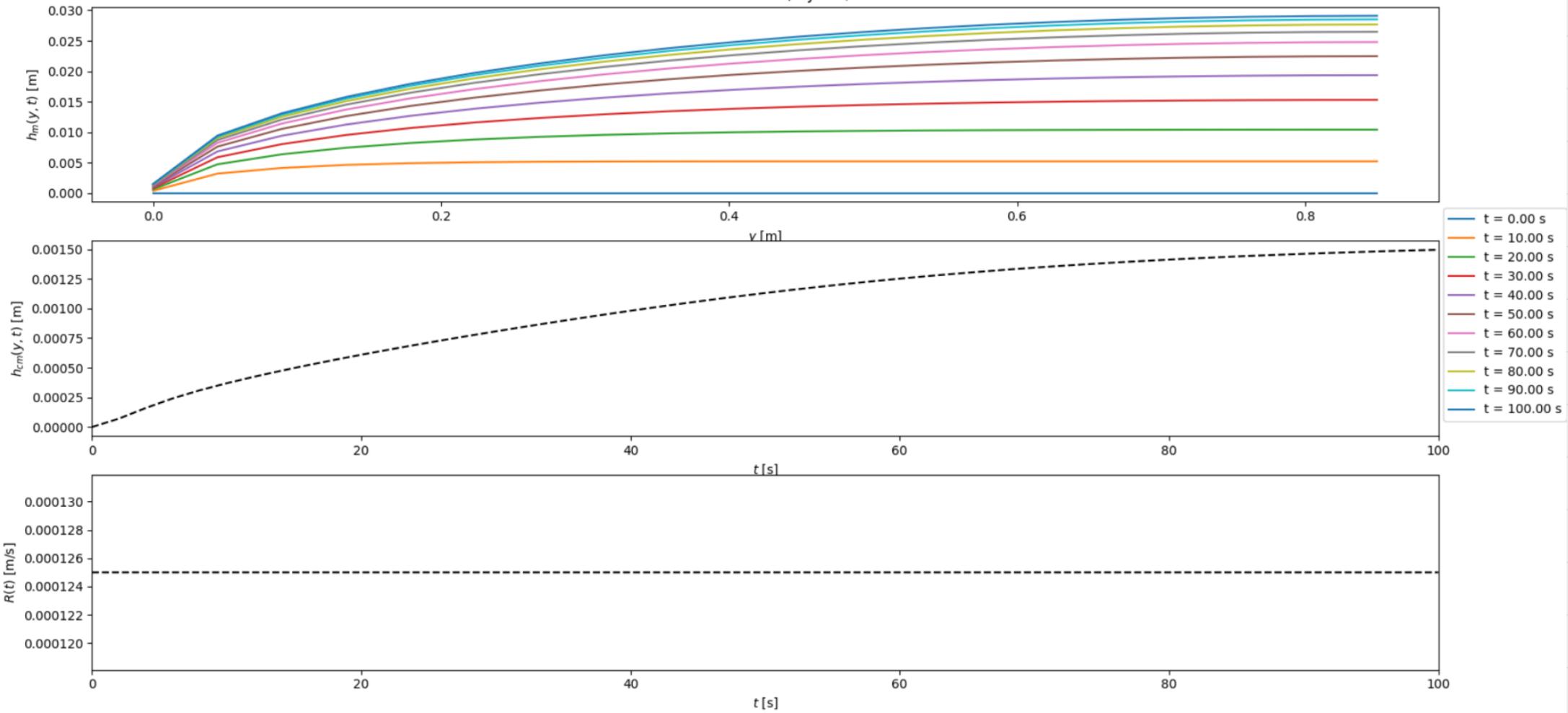
$$\leq \left| \frac{Nv}{kh_j^n} - \frac{2gh_j^n}{(\Delta y)^2} \right| \times \frac{l}{\Delta}$$

This term dominates

$$sc \Delta t \lesssim \frac{(\Delta y)^2}{2\Delta g h_{max}}$$

Nice.

Constant Rainfall,  $dy = L/20$

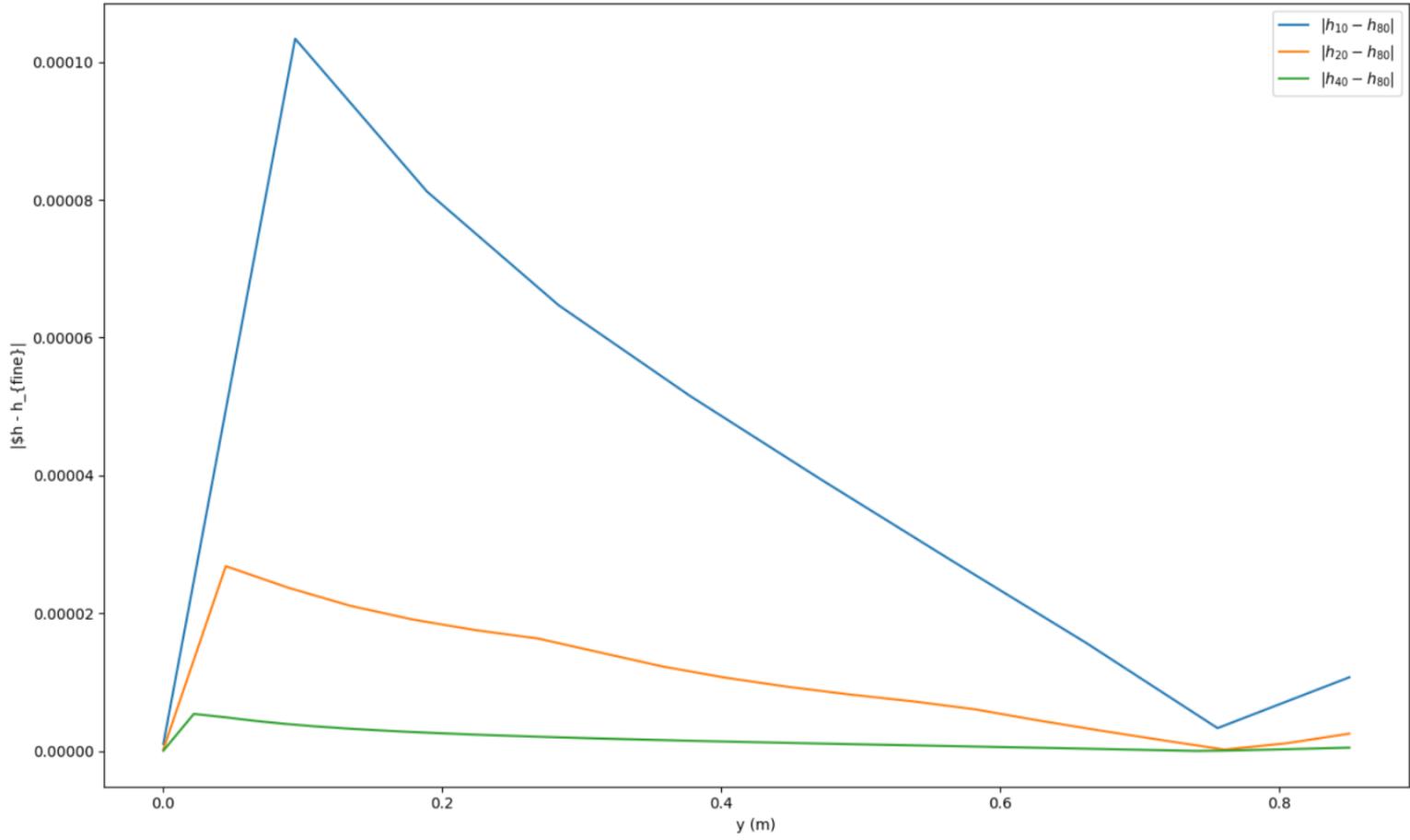


• Profilées Sur  $dy = \frac{L_y}{20}$

Repeated this for different  $\Delta y$ , and then compared the solutions.

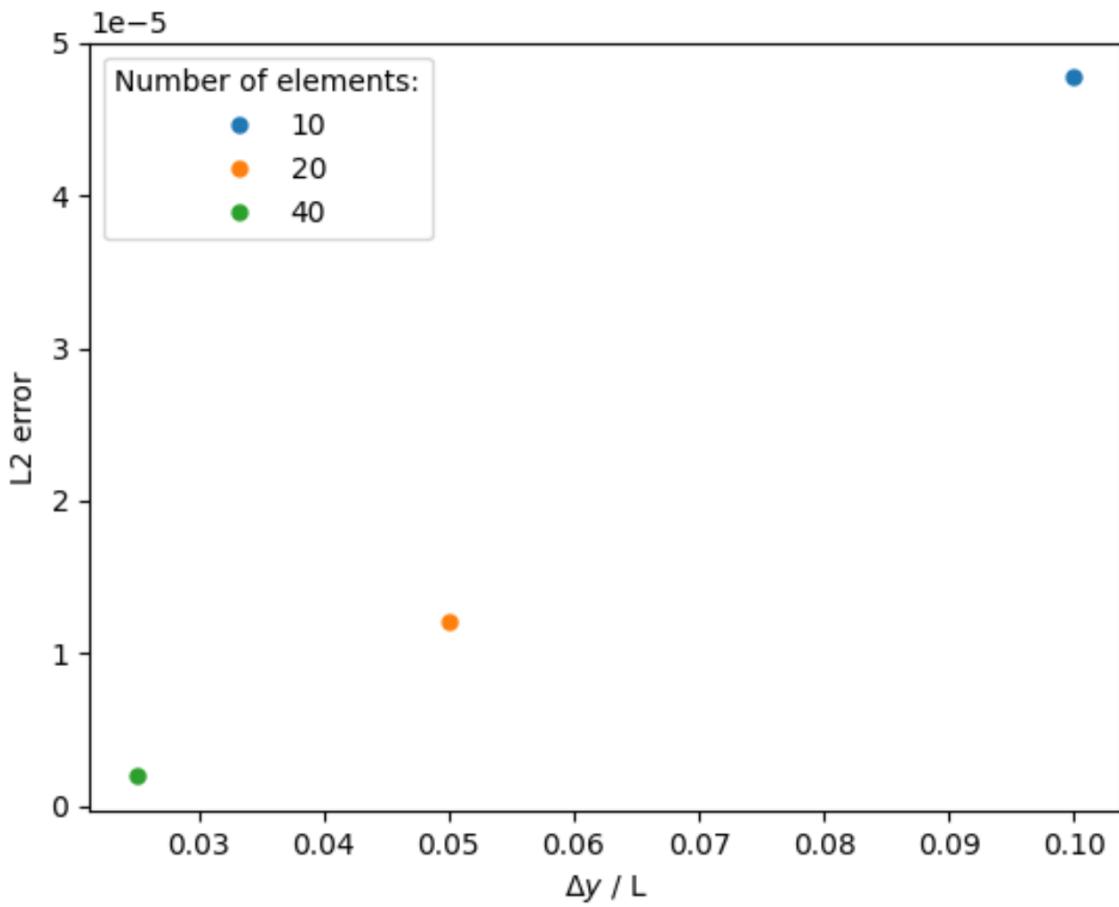
$$\Delta y = \frac{L_y}{10}, \frac{L_y}{20}, \frac{L_y}{40}, \frac{L_y}{80}$$

Difference between values of  $h$  for different element size compared to finest grid at  $t = 100s$



- Difference in solution clearly decreases as  $\Delta y$  decreases,  $\rightarrow$  so numerical solution converges.

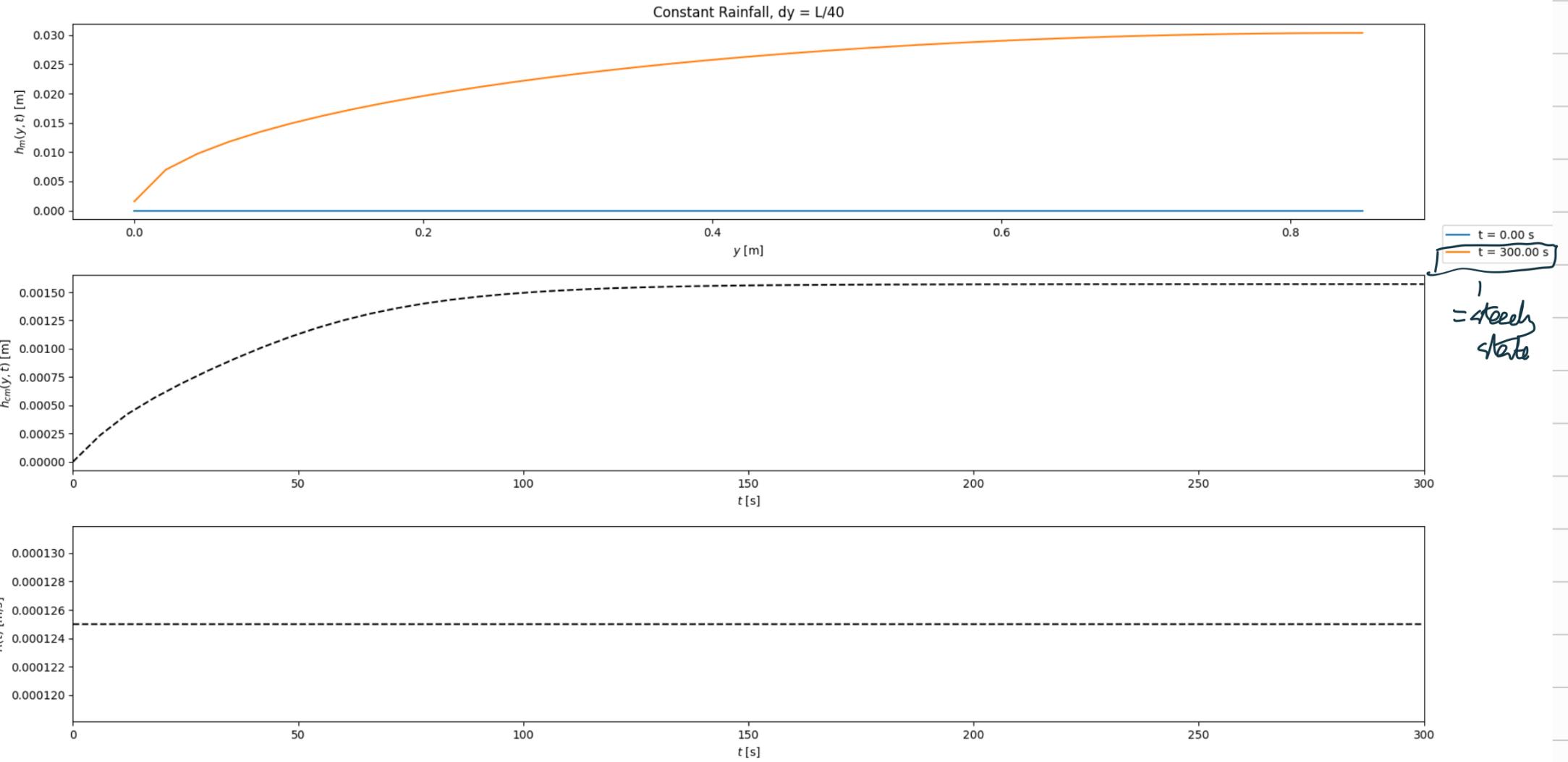
$L^2$  error between solution Sov d<sub>5</sub> =  $\frac{L_5}{80}$  and other d<sub>5</sub>



$\rightarrow L^2$  decreases, once again showing  
solution is converging.

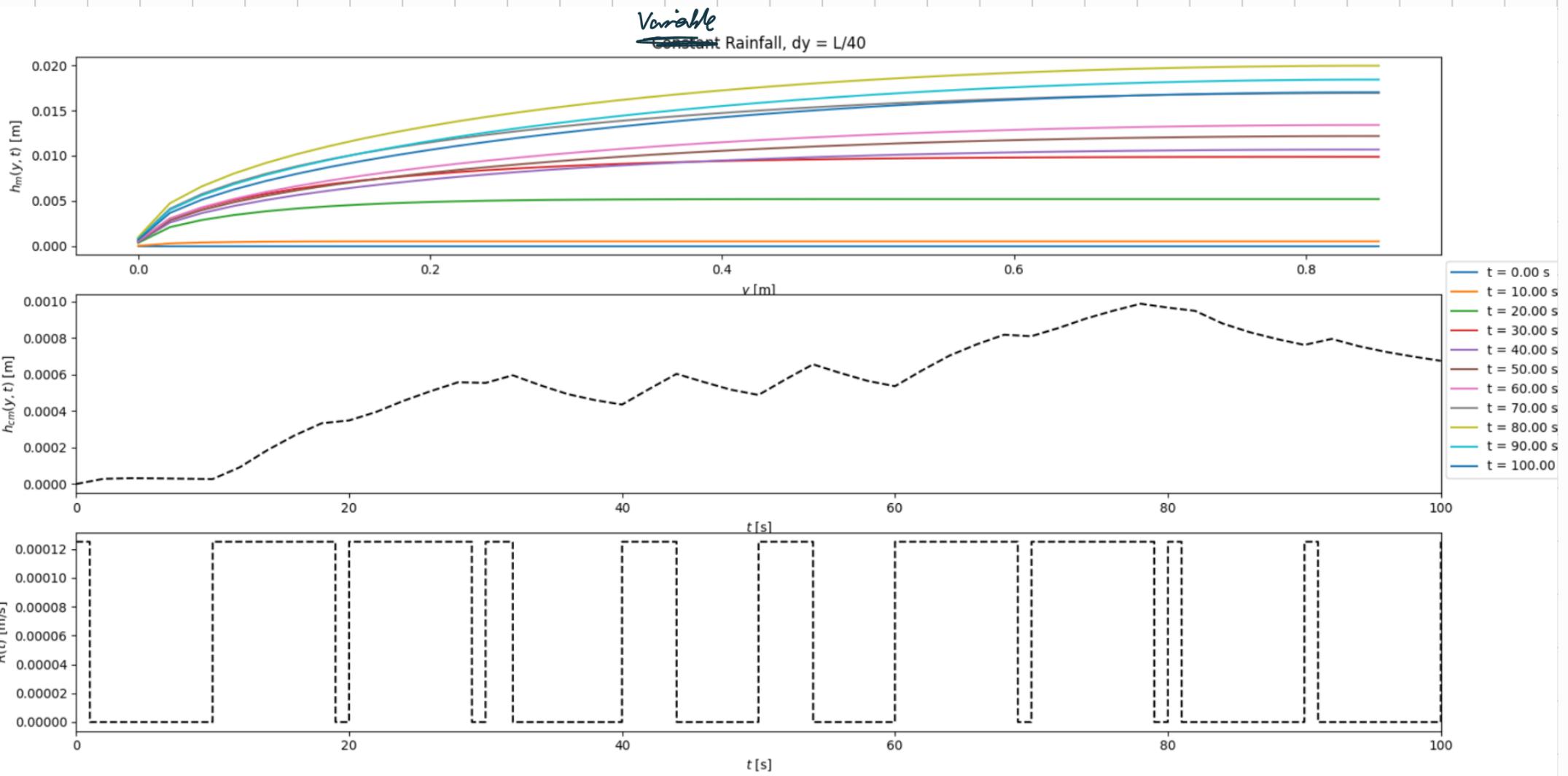
• System has not yet reached steady state.

$\rightarrow$  will find by increasing  $t_{\text{av}}$



$h_{cm}$  at steady state:  $h_{cm} \approx 0.00157 \text{ m}$

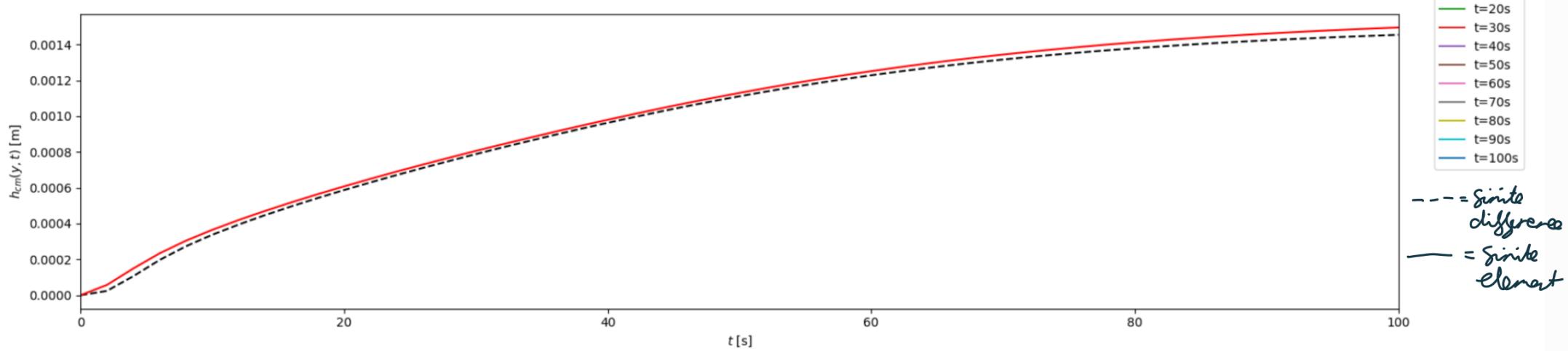
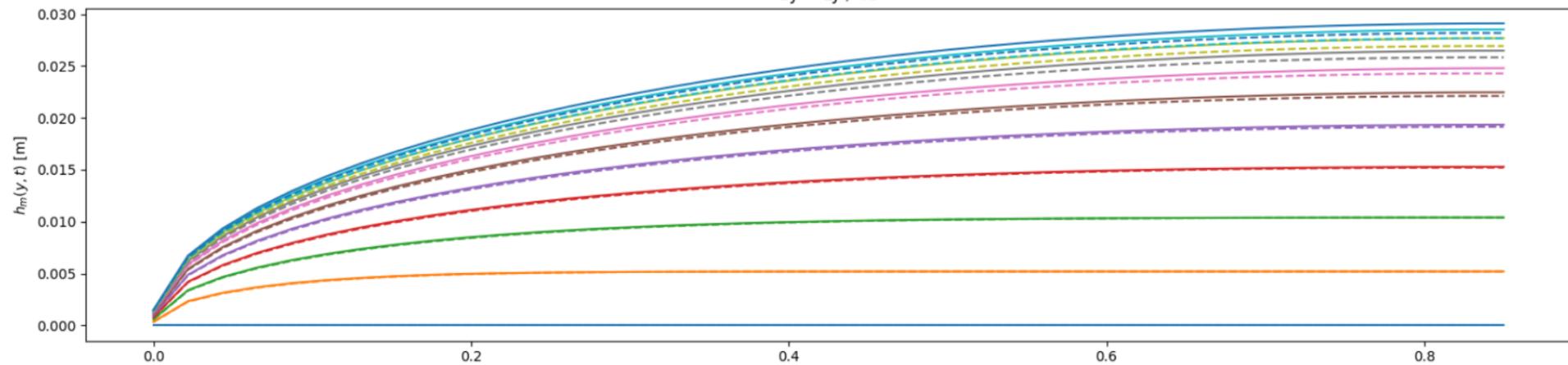
$t = 300 \text{ s}$  shows steady state  $h_m$



In code, change Rvar from 0 to 1 to set variable rain

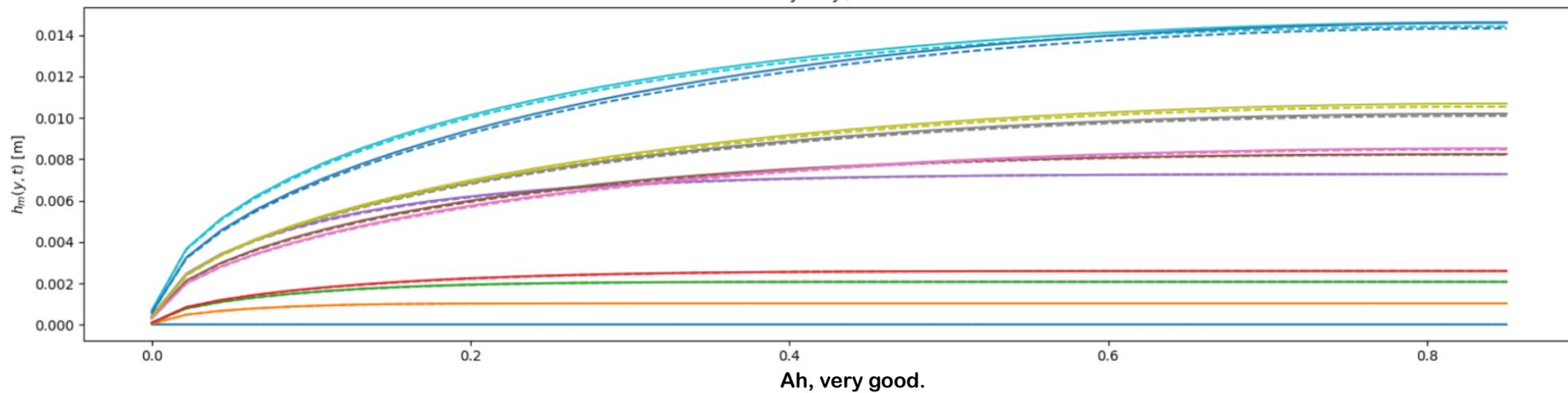
# Constant Rain

$$dy = Ly / 40$$

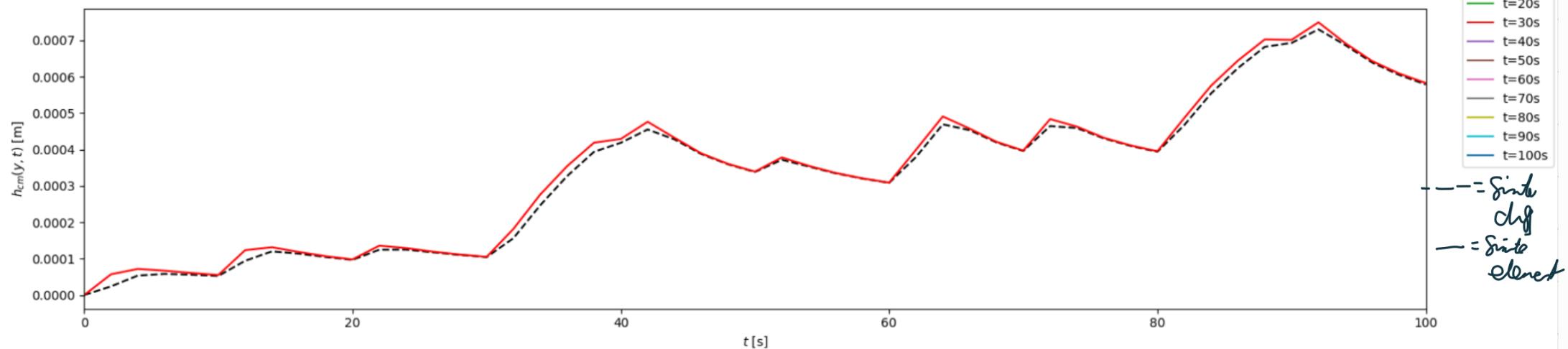


Variable rain

$$dy = Ly / 40$$



Ah, very good.



- Finite differences and finite element do not match up completely  
→ and this is worse for variable rainfall
- This difference might be due to how finite differences handles  $h_m$  separately to  $h_m$ .

3) Applying Crank-Nicholson to this:

$$\int_0^{L_y} g \partial_t h_m + \alpha g h_m \partial_y g \partial_y h_m dy$$

$$- \frac{g(0)}{m_{\text{parte}}} \left[ L_c \frac{dh_m(0,t)}{\Delta t} + \sqrt{g} \max \left( \frac{2h_m(0,t)}{3}, 0 \right)^{3/2} \right]$$

$$= \int_c^{L_y} \frac{g^n}{m_{\text{parte}}} dy$$

$$\rightarrow \text{e.g. } h_m = \frac{h_m^{n+1} + h_m^n}{2}$$

$$\int_0^{L_y} g h_m^{n+1} dy + \frac{L_c h_m^{n+1} g(c)}{m_{\text{parte}}} + \frac{\Delta t}{2} \int_c^{L_y} \alpha g h_m^{n+1} \partial_y g \partial_y h_m^{n+1} dy$$

$$+ \frac{\Delta t}{2} \sqrt{g} g(c) \max \left( \frac{2h_m^{n+1}}{3}, 0 \right)^{3/2}$$

$$= \int_0^{L_y} g h_m^n dy + \frac{L_c h_m^n g(0)}{m_{\text{parte}}} + \frac{\Delta t}{2} \int_0^{L_y} \alpha g h_m^n \partial_y g \partial_y h_m^n dy$$

$$+ \frac{\Delta t}{2} \int_c^{L_y} g \frac{(h_m^n + h_m^{n+1})}{m_{\text{parte}}} dy - \frac{\Delta t}{2} \sqrt{g} g(c) \max \left( \frac{2h_m^n}{3}, 0 \right)^{3/2}$$

$$q = \varrho_i, h = h_i, \varphi_i$$

$$\rightarrow h_{cm} = h_i$$

$$\int_0^{L_y} \varrho_i \varrho_i dy h_i^{n+1} + \frac{L_c h_i^{n+1}}{m_{proto}} \delta_{ii}$$

$$+ \frac{\Delta t}{2} \int_0^{L_y} \Delta S h_m^{n+1} \partial_y \varrho_i \partial_y h_m^{n+1} dy$$

$$+ \frac{\Delta t}{2} \int_S \delta_{ii} \max\left(\frac{2h_i^{n+1}}{3}, 0\right)^{3/2}$$

$$= \int_0^{L_y} \varrho_i \varrho_i dy h_i^n + \frac{L_c h_i^n}{m_{proto}} \delta_{ii}$$

$$+ \frac{\Delta t}{2} \int_C \varrho_i \frac{(h^n + h^{n+1})}{m_{proto}} dy - \frac{\Delta t}{2} \int_S \delta_{ii} \max\left(\frac{2h_i^n}{3}, 0\right)^{3/2}$$

$$- \frac{\Delta t}{2} \int_0^{L_y} \Delta S h_m^{n+1} \partial_y \varrho_i \partial_y h_m^{n+1} dy$$

$$M_{ij} h_j^{n+1} + \frac{L_c h_i^{n+1}}{m_{proto}} \delta_{ii} - \frac{\Delta t b_i^{n+1}}{2} + \frac{\Delta t \int_S \delta_{ii}}{2 m_{proto}} \max\left(\frac{2h_i^{n+1}}{3}, 0\right)^3$$

$$= M_{ij} h_j^n + \frac{L_c h_i^n}{m_{proto}} \delta_{ii} + \frac{\Delta t b_i^n}{2} - \frac{\Delta t \int_S \delta_{ii}}{2 m_{proto}} \max\left(\frac{2h_i^n}{3}, 0\right)^3$$

where  $M_{ij} = \int_0^L \varrho_i \varrho_j dy$

$$b_i^n = \int_0^L \alpha S h_m^n \partial_y \varrho_i \partial_y h_m^n + \frac{\varrho_i R_i^n}{m_{proto}} dy$$

• Can let this expression =  $n_i h_i$

And define  $\mathcal{R}_{ij} = \frac{\partial R_i}{\partial h_j}$

If we let  $h_{new} = h_{old} + \delta h$

$$\rightarrow R_i(h + \delta h) \approx R_i(h) + \frac{\partial R_i}{\partial h_j} \delta h_j$$

We want to solve this so if  $R_i(h + \delta h) = 0$

Solve for  $\delta h_j$ :  $J_{i,j}(h) \delta h_j = -K_i(h)$

Then set  $h_{\text{new}} = h + S\delta h$

Repeat until  $|S\delta h|$  is within a tolerance

This can be done to solve the non-linear algebraic system.

In the code, 'ksp\_rtol': 1e-14 is set

→ as residual is set, it means the code iterates until this is reached

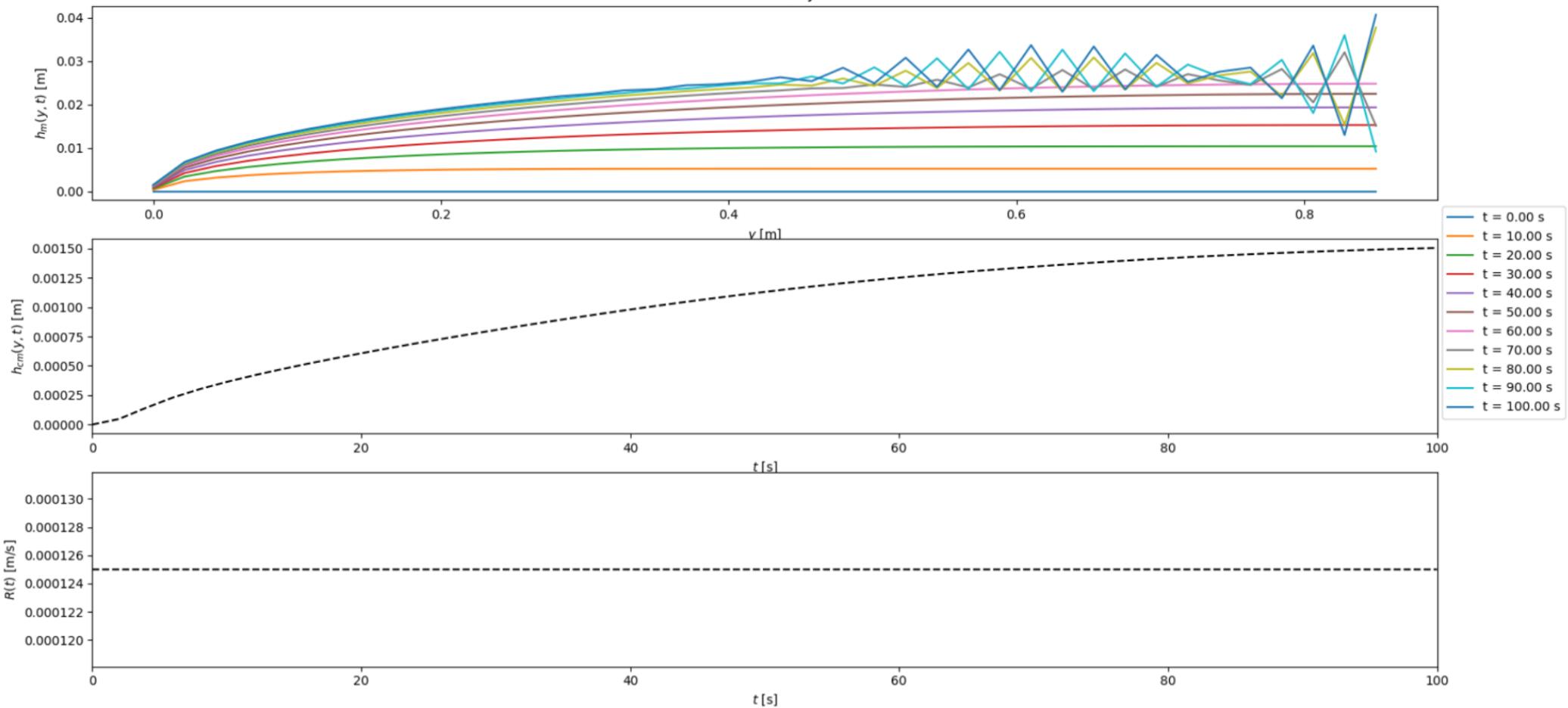
→ as code runs and gives a solution, this must mean the iteration converges.

4). CG1 = piecewise linear  
CG2 = piecewise quadratic

- Find that at higher orders cause oscillations near  $y = L_y$  (not  $y=0$  like question asked).
- Due to BC at  $y = L_y$  ( $\partial_y h = 0$ )

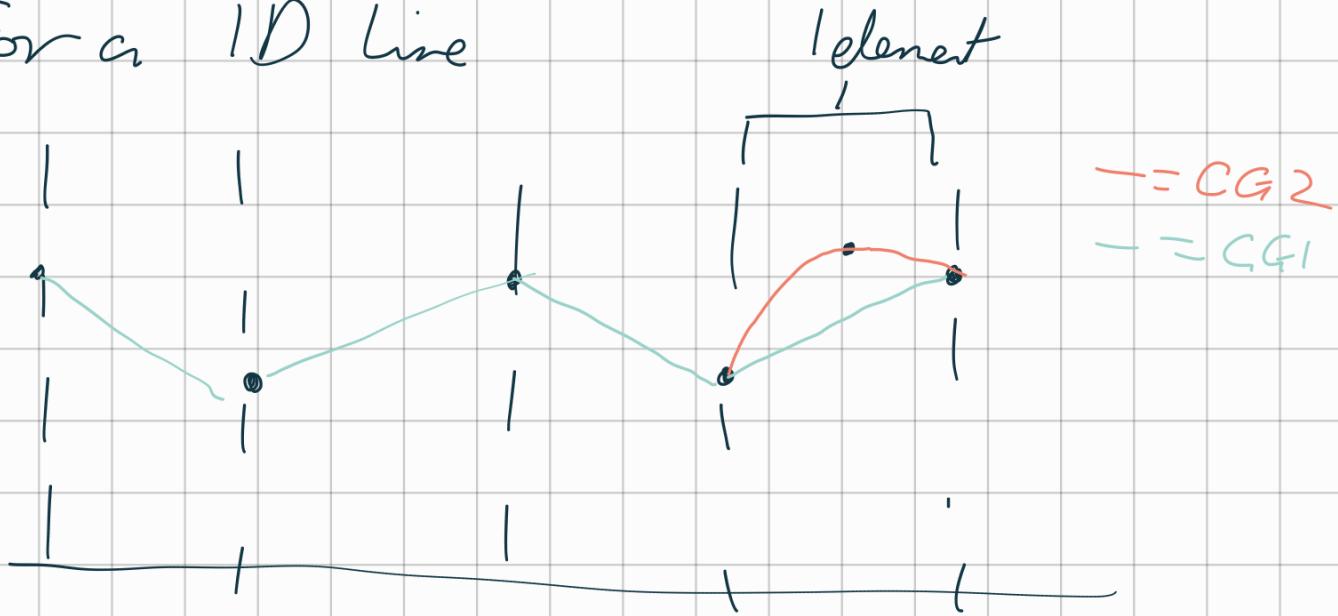
CG3

Constant Rainfall,  $dy = L/40$



$$\Delta t = 2 \cdot 3 \cdot 0.5 \times (\Delta y)^2$$

For a 1D line



more nodes within element for higher orders

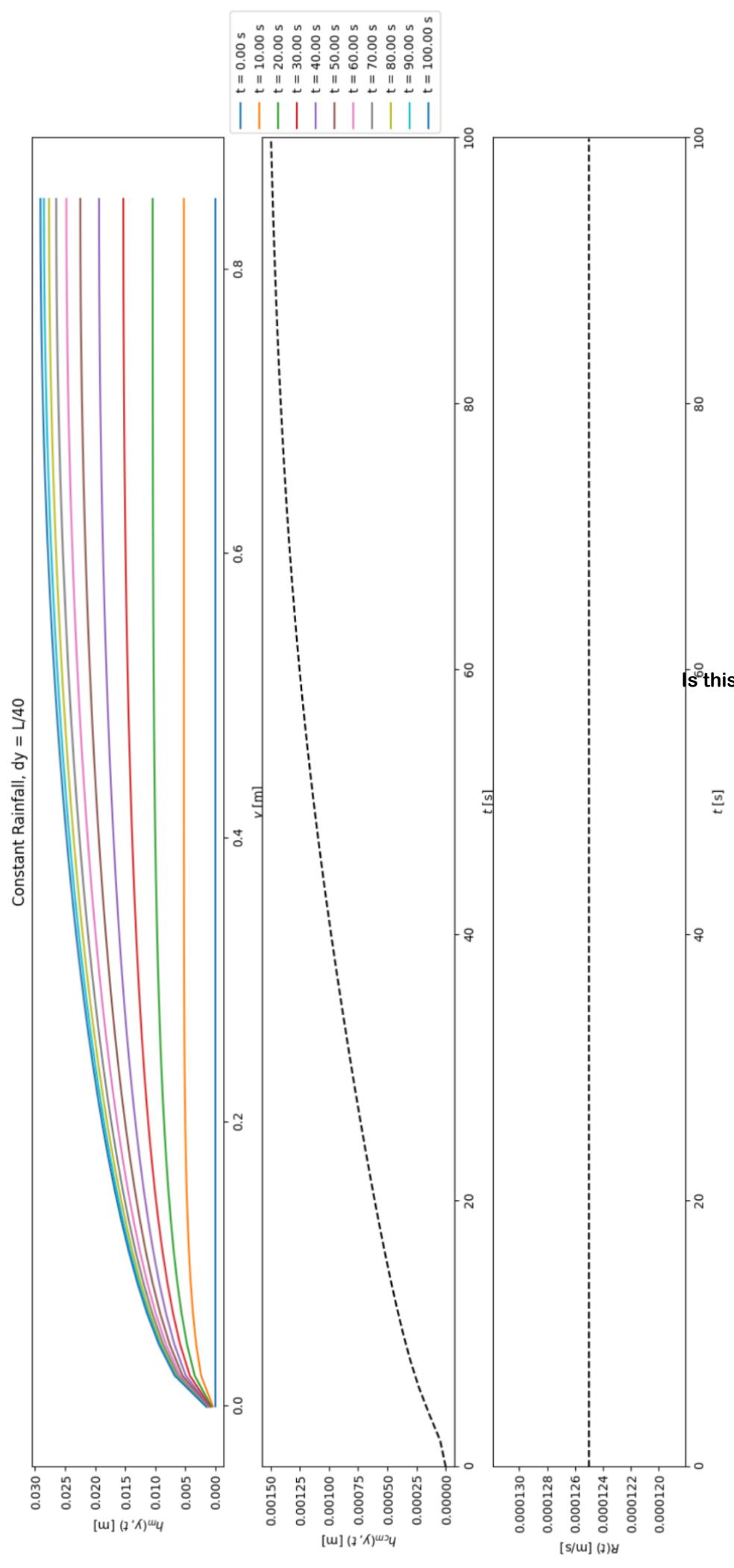
→ This effectively means that  $\Delta t < h(\Delta y)^2$  should be replaced by  $\Delta t < h\left(\frac{\Delta y}{n_{CG}}\right)^2$  where  $n_{CG}$  = order

It likely appears unstable at  $y=L_y$ , as the BC at  $y=0$  is strongly enforced

→ Increases the derivative for higher orders being enforced still means they can have high curvature within the element.

→ Changing timestep gets rid of oscillations

Ch3



Is this with theta=0.5 or other theta?  
18/20

$$\rightarrow \Delta = 2 \cdot 3 \cdot 0.5 \cdot \left(\frac{\Delta y}{3}\right)^2$$

→ oscillations are gone!