

Casey - Exercise 3

$$1. I[u] = \iint_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - uf \right) dA$$

Variation of the functional I is defined as:

$$\delta I(u) = \frac{\partial I}{\partial \epsilon} = \lim_{\epsilon \rightarrow 0} \frac{I(u + \epsilon \delta u) - I(u)}{\epsilon} = 0$$

$$\Rightarrow I(u + \epsilon \delta u) = \frac{1}{2} \int_{\Omega} |\nabla(u + \epsilon \delta u)|^2 dA - \int_{\Omega} f(u + \epsilon \delta u) dA$$

$(dA = dx dy)$

Expanding $|\nabla(u + \epsilon \delta u)|^2$

$$\begin{aligned} &= \nabla(u + \epsilon \delta u) \cdot \nabla(u + \epsilon \delta u) \\ &= \nabla u \cdot \nabla u + 2\epsilon (\nabla u \cdot \nabla \delta u) + \epsilon^2 (\nabla \delta u \cdot \nabla \delta u) \\ &= |\nabla u|^2 + 2\epsilon (\nabla u \cdot \nabla \delta u) + O(\epsilon^2) \end{aligned}$$

$$\Rightarrow I(u + \epsilon \delta u) = \frac{1}{2} \int_{\Omega} [|\nabla u|^2 + 2\epsilon (\nabla u \cdot \nabla \delta u)] dA - \int_{\Omega} f(u + \epsilon \delta u) dA$$

$$\begin{aligned} &= \left[\frac{1}{2} \int_{\Omega} |\nabla u|^2 dA - \int_{\Omega} uf dA \right] + \epsilon \left[\int_{\Omega} (\nabla u \cdot \nabla \delta u) dA - \int_{\Omega} f \delta u dA \right] \\ &= I[u] + \epsilon \left[\int_{\Omega} (\nabla u \cdot \nabla \delta u) dA - \int_{\Omega} f \delta u dA \right] \end{aligned}$$

$$\text{Variation } \delta I[u] = \int_{\Omega} (\nabla u \cdot \nabla \delta u) dA - \int_{\Omega} f \delta u dA$$

Ritz-Galerkin principle - variation = 0 for all

admissible δu

$$\rightarrow \int_{\Omega} \nabla u \cdot \nabla \delta u \, dA - \int_{\Omega} \delta u f \, dA = 0$$

$$\rightarrow \int_{\Omega} \nabla u \cdot \nabla \delta u \, dA = \int_{\Omega} \delta u f \, dA$$

LHS can be written as:

$$- \int_{\Omega} (\nabla^2 u) \delta u \, dA + \oint_{\partial\Omega} (\nabla u \cdot \vec{n}) \delta u \, ds$$

where $\partial\Omega$ is the boundary

This yields:

$$- \int_{\Omega} (\nabla^2 u) \delta u \, dA + \oint_{\partial\Omega} (\nabla u \cdot \vec{n}) \delta u \, ds = \int_{\Omega} f \delta u \, dA$$

$$\Rightarrow \int_{\Omega} (-\nabla^2 u - f) \delta u \, dA + \oint_{\partial\Omega} (\nabla u \cdot \vec{n}) \delta u \, ds = 0$$

Given the boundary conditions, the boundary integral

= 0 for all boundaries

$$\Rightarrow \int_{\Omega} (-\nabla^2 u - f) \delta u \, dA = 0$$

This must hold for all admissible $\delta u \Rightarrow -\nabla^2 u - f = 0$ to

satisfy, recovering the system.

Conditions for $\delta u(x,y)$

- Must belong to the same function space as the test function $w(x,y)$.
- Must satisfy the form of the boundary conditions.
 $\delta u(0,y) = 0, \delta u(1,y) = 0$

Weak formulation for test function $w(x,y)$

$$\rightarrow \int_{\Omega} \nabla u \cdot \nabla w \, dA = \int_{\Omega} wf \, dA$$

which will yield the same result as for δu

$$\Rightarrow w(x,y) = \delta u(x,y)$$

$$2. u(x,y) \sim u_n(x,y) = \sum_{j=1}^N U_j \phi_j(x,y)$$

where U_j are nodal coefficients

Substituting into the function I

$$I(u_n) = \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 \, dA - \int_{\Omega} u_n f \, dA$$

$$I(u) = \frac{1}{2} \int_{\Omega} \left| \nabla \left(\sum_{j=1}^N u_j \phi_j \right) \right|^2 dA - \int_{\Omega} f \left(\sum_{j=1}^N u_j \phi_j \right) dA$$

Since coefficients are constant w.r.t integration

$$\Rightarrow I(u) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N u_i u_j \left(\int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j dA \right) - \sum_{j=1}^N u_j \left(\int_{\Omega} f \phi_j dA \right)$$

Ritz-Galerkin principle states solutions vector \vec{U} minimises the functional I

Take partials of I to find minimum:

$$\frac{\partial I(\vec{U})}{\partial U_k} = 0 \quad k = 1, \dots, N$$

$$\text{LHS: } \frac{\partial}{\partial U_k} \left[\sum_{i=1}^N \sum_{j=1}^N u_i u_j \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j dA \right] = \sum_{i=1}^N u_i \int_{\Omega} \nabla \phi_k \cdot \nabla \phi_i dA$$

$$\text{RHS: } \frac{\partial}{\partial U_k} \left[\sum_{j=1}^N u_j F_j \right] = \int_{\Omega} f \phi_k dA$$

$$\Rightarrow \frac{\partial I(u)}{\partial U_k} = \sum_{j=1}^N u_j \int_{\Omega} \nabla \phi_k \cdot \nabla \phi_j dA - \int_{\Omega} f \phi_k dA = 0 \quad \text{for } k = 1, \dots, N$$

$$\Rightarrow \sum_{j=1}^N u_j \int_{\Omega} \nabla \phi_k \cdot \nabla \phi_j dA = \int_{\Omega} f \phi_k dA = 0$$

Can be written as discrete algebraic system

where: $\vec{K}\vec{U} = \vec{F}$ where $\vec{K} = \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_k dA$, $\vec{F} = \int_{\Omega} f \phi_k dA$

and \vec{U} the vector of unknown coefficients

Local Global

N - number of nodes in x
M - number of nodes in y

	0	1	2	3	4	5	6	7	8	9
0	0	-N ₁₁	-N ₁₂	-N ₁₃	-N ₁₄	-N ₁₅	-N ₁₆	-N ₁₇	-N ₁₈	-N ₁₉
1	-N ₁₁	0	-N ₂₁	-N ₂₂	-N ₂₃	-N ₂₄	-N ₂₅	-N ₂₆	-N ₂₇	-N ₂₈
2	-N ₁₂	-N ₂₁	0	-N ₃₁	-N ₃₂	-N ₃₃	-N ₃₄	-N ₃₅	-N ₃₆	-N ₃₇
3	-N ₁₃	-N ₂₂	-N ₃₁	0	-N ₄₁	-N ₄₂	-N ₄₃	-N ₄₄	-N ₄₅	-N ₄₆
4	-N ₁₄	-N ₂₃	-N ₃₂	-N ₄₁	0	-N ₅₁	-N ₅₂	-N ₅₃	-N ₅₄	-N ₅₅
5	-N ₁₅	-N ₂₄	-N ₃₃	-N ₄₂	-N ₅₁	0	-N ₆₁	-N ₆₂	-N ₆₃	-N ₆₄
6	-N ₁₆	-N ₂₅	-N ₃₄	-N ₄₃	-N ₅₂	-N ₆₁	0	-N ₇₁	-N ₇₂	-N ₇₃
7	-N ₁₇	-N ₂₆	-N ₃₅	-N ₄₄	-N ₅₃	-N ₆₂	-N ₇₁	0	-N ₈₁	-N ₈₂
8	-N ₁₈	-N ₂₇	-N ₃₆	-N ₄₅	-N ₅₄	-N ₆₃	-N ₇₂	-N ₈₁	0	-N ₉₁
9	-N ₁₉	-N ₂₈	-N ₃₇	-N ₄₆	-N ₅₅	-N ₆₄	-N ₇₃	-N ₈₂	-N ₉₁	0

Uniform Mesh

	M-1	M-2	M-3	M-4	M-5	M-6	M-7	M-8	M-9	M-10
	M-1	M-2	M-3	M-4	M-5	M-6	M-7	M-8	M-9	M-10
M-1	0	-M ₁₁	-M ₁₂	-M ₁₃	-M ₁₄	-M ₁₅	-M ₁₆	-M ₁₇	-M ₁₈	-M ₁₉
M-2	-M ₁₁	0	-M ₂₁	-M ₂₂	-M ₂₃	-M ₂₄	-M ₂₅	-M ₂₆	-M ₂₇	-M ₂₈
M-3	-M ₁₂	-M ₂₁	0	-M ₃₁	-M ₃₂	-M ₃₃	-M ₃₄	-M ₃₅	-M ₃₆	-M ₃₇
M-4	-M ₁₃	-M ₂₂	-M ₃₁	0	-M ₄₁	-M ₄₂	-M ₄₃	-M ₄₄	-M ₄₅	-M ₄₆
M-5	-M ₁₄	-M ₂₃	-M ₃₂	-M ₄₁	0	-M ₅₁	-M ₅₂	-M ₅₃	-M ₅₄	-M ₅₅
M-6	-M ₁₅	-M ₂₄	-M ₃₃	-M ₄₂	-M ₅₁	0	-M ₆₁	-M ₆₂	-M ₆₃	-M ₆₄
M-7	-M ₁₆	-M ₂₅	-M ₃₄	-M ₄₃	-M ₅₂	-M ₆₁	0	-M ₇₁	-M ₇₂	-M ₇₃
M-8	-M ₁₇	-M ₂₆	-M ₃₅	-M ₄₄	-M ₅₃	-M ₆₂	-M ₇₁	0	-M ₈₁	-M ₈₂
M-9	-M ₁₈	-M ₂₇	-M ₃₆	-M ₄₅	-M ₅₄	-M ₆₃	-M ₇₂	-M ₈₁	0	-M ₉₁
M-10	-M ₁₉	-M ₂₈	-M ₃₇	-M ₄₆	-M ₅₅	-M ₆₄	-M ₇₃	-M ₈₂	-M ₉₁	0

3.

Matrix assembly:

For element k:

α and β representation

of local index

k representation of

global index

$$\Rightarrow i = \text{index}(k, \alpha)$$

$$j = \text{index}(k, \beta)$$

$$\Rightarrow A_{ij} = A_{ij} + \hat{A}_{\alpha\beta}$$

determined by element

+ locally determined

$$b_i = b_i + \hat{b}_{\alpha}$$

Based solely on local

index

Giving algebraic system

\vec{AB} representing

the mesh

element wise

Part 2

1. Weak formulation derivation: $\times q(y)$ and integrate

$$\rightarrow \int_0^{Ly} q \frac{\partial}{\partial y} h_m dy - \int_0^{Ly} q \alpha \delta_y(h_m) dy = \int_0^{Ly} \frac{q^R}{\alpha \mu \sigma_e} dy$$

Integrating by parts (Second term)

$$\int_0^{Ly} q \alpha \delta_y(h_m) dy$$

$$\rightarrow \int_0^{Ly} \alpha g h_m \delta_y q \delta_y h_m dy + [\alpha g h_m \delta_y h_m]_0^{Ly}$$

At $y = Ly$, $\delta_y h_m = 0 \Rightarrow$ term $\rightarrow 0$ (Eq. 12)

At $y = 0$, can be evaluated as $\alpha g q(0) h_m \delta_y h_m = 0$

which can be rewritten to give $\alpha g^2 \delta_y(h_m)^2 |_{Ly} = 0$

Using canal level ODE (Eq. 14) \rightarrow (Eq. 28)

Substituting back to weak form yields
(Eq. 29) the weak formulation.

Using the FEM expansion for u_m

$u_m(y, t) \approx \sum u_i(t) \varphi_i(y)$ and $q = \varphi_i(y)$

and considering time stepping: $\Delta t h = \frac{h}{\Delta t}$

Algebraic system is as seen (Eq. 33).

Time step restrictions

Considering system as:

$$\bar{\bar{M}}\vec{u}^{n+1} = \bar{\bar{M}}\vec{u}^n - \Delta t \bar{\bar{K}}(\vec{u}^n)\vec{u}^n$$

Δt restriction can be derived from stability

of the algebraic system

$$\rightarrow \vec{u}^{n+1} = (\bar{\bar{I}} - \Delta t \bar{\bar{M}}^{-1} \bar{\bar{K}}) \vec{u}^n$$

Eigenvalues of $(\bar{\bar{I}} - \Delta t \bar{\bar{M}}^{-1} \bar{\bar{K}})$ must be ≤ 1

$$\Rightarrow \Delta t \leq \frac{2}{\lambda_{\max}(\bar{\bar{M}}^{-1} \bar{\bar{K}})}$$

This can be expressed in terms of the parameters of the problem

$$\rightarrow \Delta t \leq \frac{\text{Max } \sigma_e (\Delta y)^2}{\deg h_m}$$

where $h_m = \max(h_m)$ in the domain.

