Poison System:

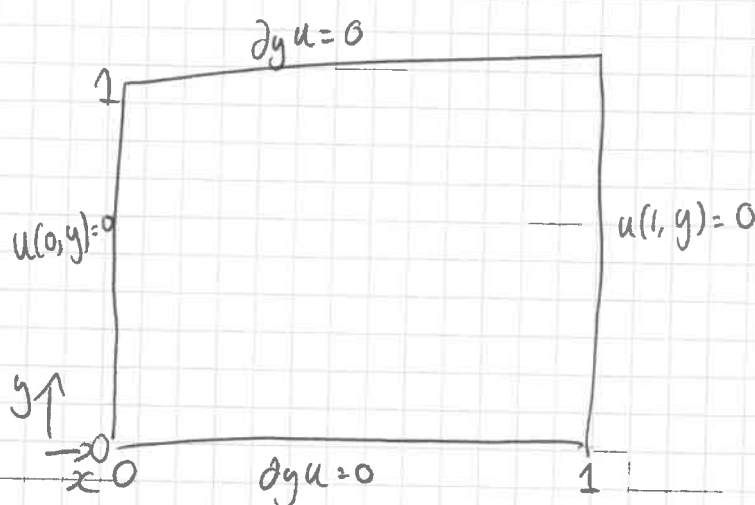$$-\cancel{\nabla^2 f}$$

$$-\nabla^2 u = f \quad \text{on} \quad (x,y) \in [0,1]^2$$

$$f(x,y) = 2\pi^2 \sin(\pi x)\cos(\pi y)$$

$$u(0,y) = u(1,y) = 0 \qquad (\text{Dirichlet}) \quad \Gamma_1$$

$$\partial_y u(x,y)|_{y=0} = \partial_y u(x,y)|_{y=1} = 0 \quad (\text{Neumann}) \quad \Gamma_2$$

boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \qquad \Gamma_1 = x = 0, 1 (\partial\Omega_D) \not{} \Gamma_2 : y = 0, 1 (\partial\Omega_N)$

exact solution is $u_e(x,y) = \sin(\pi x)\cos(\pi x)$

$$\frac{\partial}{\partial x}\left(\sin(\pi x)\cos(\pi y)\right) = \overset{\cos}{\pi \sin(\pi x)}\cos(\pi y)$$

$$\frac{\partial^2}{\partial x^2}\left(\sin(\pi x)\cos(\pi y)\right) = -\pi^2 \sin(\pi x)\cos(\pi y)$$

$$\frac{\partial}{\partial y}\left(\sin(\pi x)\cos(\pi y)\right) = -\pi \sin(\pi x)\overset{\sin}{\cos(\pi y)}$$

$$\frac{\partial^2 u}{\partial y^2} = -\pi^2 \sin(\pi x)\cos(\pi y)$$

$$\nabla^2 u = -\pi^2 \left(\sin(\pi y) + \right.$$

$$-\nabla^2 u = +\pi^2 \left(\sin(\pi x) \cos(\pi y) + \sin(\pi x) \cos(\pi y) \right)$$

$$-\nabla^2 u = 2\pi^2 \sin(\pi x) \cos(\pi y)$$

$$\Rightarrow \quad -\nabla^2 u = f$$

exact solution

1. ~~Step 1:~~
~~Write down the~~
~~Ritz-Galerkin principle~~ ad dem the week

~~formulation:~~

1. ~~Step 1:~~

To solve poisson system numerically we want to
be able to write $u(x,y)$ as a sum of weighted
shape functions. $u(x,y) = \sum_{c=0}^{n} u_i N_i(x,y)$ where N is
a shape function, 1 at node i and 0 at other nodes.
(n is number of nodes). Weights are u_i.

To find the weights, want to put the poisson
system in a weak formulation.

(1)   step 1:   Weak formulation

| Find weak formulation through a test function |

Method 1

$$-\nabla^2 u = f \qquad : \text{strong form}$$

multiply by test function $w(x,y)$, this is a
arbitrary function of x and y.

$$-\nabla^2 u(x,y)\, w(x,y) = f(x,y)\, w(x,y)$$

integrate over ~~boundary~~ domain Ω

$$-\int_{\Omega} \nabla^2 u(x,y)\, w(x,y)\, d\Omega = \int_{\Omega} f(x,y)\, w(x,y)\, d\Omega \quad \text{①}$$

~~Use partial integration to change form of LHS.~~
   want to change form of LHS.

$$-\int_{\Omega} \nabla^2 u\, w\, \partial\Omega \quad => \text{LHS.}$$

PRODUCT RULE

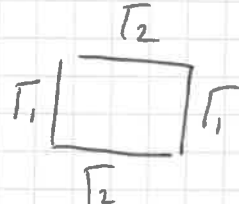$$\nabla \cdot (w \nabla u) = w \nabla^2 u + \nabla u \cdot \nabla w$$

$$\Rightarrow -w \nabla^2 u = \nabla u \cdot \nabla w - \nabla \cdot (w \nabla u)$$

$$\Rightarrow -\int_\Omega w \nabla^2 u \, d\Omega = -\int_\Omega \nabla \cdot (w \nabla u) d\Omega + \int_\Omega \nabla u \cdot \nabla w \, d\Omega$$

$\nearrow$ LHS of ①

Use divergence theorem to rewrite $\int_\Omega \nabla \cdot (w \nabla u) \, d\Omega$ as

$$\oint_{\Gamma} \int_{\Gamma_1} w \nabla u \cdot \hat{n} \, d\Gamma + \int_{\Gamma_2} w \nabla u \hat{n} \, d\Gamma$$

Γ_1 when $x = 0, 1$ Γ_2 when $y = 0, 1$

so closed surface.

Can eliminate this term as $\nabla u = 0$ on Γ_2 and can choose w so satisfys $u = 0$ on Γ_1

Therefore

$$+ \int_\Omega \nabla u \cdot \nabla w \, d\Omega = \int_\Omega f w \, d\Omega$$

$$- \int_\Omega \nabla u \cdot \nabla w \, d\Omega + f w \, d\Omega = 0$$

want to minimise the functional → type of function that maps vectors to numbers. Argument are functions

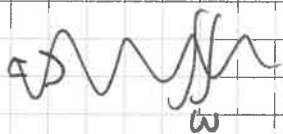$$I[u] = \iint_{\Omega} \frac{1}{2} \| \nabla u |^2 - u f \, d\Omega$$

Minimise this function by finding point where $\delta I = 0$ with a variation

$$\delta I \equiv \frac{\delta I}{\delta \epsilon} = \lim_{\epsilon \to 0} \frac{I[u(x,y) + \epsilon(\delta u)(x,y)] - I[u(x,y)]}{\epsilon}$$

$$\partial I = \delta \left(\iint_{\Omega} \frac{1}{2} \| \nabla u |^2 - u f \, \delta\Omega \right)$$

$$= \iint_{\omega} 2 \times \frac{1}{2} \nabla u \cdot \nabla(\delta u) - f \delta u \, d\Omega$$

$$= \iint_{\omega} \left(\nabla u \cdot \nabla(\delta u) - f \delta u \right) d\Omega = 0$$

$$\iint_{\omega} (\nabla u \cdot \nabla(\delta u) - f\,\delta u)\, \delta\Omega = 0$$

Using before product rule =>

$$\nabla \cdot (\delta u\, \nabla u) = \delta u\, \nabla^2 u + \nabla u \cdot \nabla \delta u$$

$$\Rightarrow -\delta u\, \nabla^2 u = \nabla u \cdot \nabla \delta u - \nabla \cdot (\delta u\, \nabla u)$$

$$\Rightarrow \nabla u \cdot \nabla(\delta u) = -\delta u\, \nabla^2 u + \nabla \cdot (\delta u\, \nabla u)$$

sub this in and use Gauss's divergence theorm.

$$-\iint_{\omega} \delta u\, \nabla^2 u + \delta u\, f\, \delta\Omega + \int_{\Gamma_1} \delta u\, \nabla u \cdot \hat{\underline{n}}\, d\Gamma + \int_{\Gamma_2} \delta u\, \nabla u \cdot \hat{\underline{n}}\, d\Gamma = 0$$

$\nabla u = 0$ on Γ_2 from boundary conditions

and choose δu such that $(\delta u)_{\Gamma_1} = 0$

$$\Rightarrow \quad \boxed{-\iint_{\omega} (\delta u)(\nabla^2 u + f)\, d\Omega = 0}$$

Yields the same result as using the
test function it $\boxed{\delta u = \omega}$

$$\boxed{\int_{\Omega} \nabla u \cdot \nabla \delta u\, \delta\Omega = \int_{\Omega} f\, \delta u\, \delta\Omega}$$

2. Step 2: | Discretise WEAK FORMULATION though use of test function |

(2)

$$u(x,y) \approx \boxed{uh(x,y) = u_j \ell_j(x,y)}$$

$u_j = \text{weights}$

$u_j = \text{weights}$
$\ell_j = \text{global basis function}$

$$w(x,y) \approx wh(x,y) = a_j \ell_j(x,y)$$

$\underline{a_j} = \text{weights for test function,}$ if take $a_j = \delta_{ij}$ then

$$\boxed{wh = \ell_i(x,y)}$$

$$u(x) = \sum_{i=0}^{N} u_j \ell_j(x,y)$$

| substitute $\quad uh(x,y)$ and $wh(x,y)$ into weak formulation : |

$$\int_{\Omega} \nabla u \cdot \nabla w \, d\Omega = \int_{\Omega} f w$$

$$\iint_{\Omega} \nabla uh \cdot \nabla wh \, d\Omega = \iint_{\Omega} f \, wh \, d\Omega$$

$$\nabla uh = \nabla(u_j \ell_j(x,y)) \qquad u_j \text{ are constants so can take out of the derivate.}$$

$$\nabla uh = u_j \nabla \ell_j(x,y)$$

$$\nabla wh = \nabla \ell_i(x,y)$$

$$\Rightarrow \iint_{\Omega} u_j \nabla \ell_j(x,y) \cdot \nabla \ell_i(x,y) = \iint_{\Omega} f \nabla \ell_i(x,y) \, d\Omega$$

or. $\qquad A_{ij} \, u_j = b_i$

with

$$A_{ij} = \iint_{\Omega} \nabla \ell_i (x,y) \cdot \nabla \ell_j (x,y) \, d\Omega$$

$$b_i = \iint_{\Omega} \ell_i (x,y) f(x,y) \, d\Omega$$

Can Find ~~solution~~ u_j by inverty matrix

or

~~If use compact support~~

Values are known on Drichlet boundaris so
can rewrite:

$$A_{i'j'} \, u_{j'} = b_{i'} - \sum_{k=N_{nodes}+1}^{N_n} A_{i'k} \, U_k$$

sum of
solution over
known values of boundary
condition

N_{nodes} = non-dirichlet nodes

N_n = total nodes.

② | Discretise through variational principle |

$$\boxed{u(x,y) \approx u_h(x,y) = u_j \varphi_j (x,y)}$$

Sub into functional

$$I[u] = \iint_\Omega \frac{1}{2} \| \nabla u \|^2 - u f \, \partial \Omega$$

$$I = \iint_\Omega \left(\frac{1}{2} | \nabla u_j \varphi_j |^2 - u_j \varphi_j f \right) d\Omega$$

Im set $\delta I = 0$ and sub in δu_j test function

$$\delta I \equiv \frac{\partial I}{\partial \epsilon} = \lim_{\epsilon \to 0} \frac{I[u_j \varphi_j (x,y) + \epsilon (\delta u_j)(x,y)] - I[u_j \varphi_j (x,y)]}{\epsilon}$$

$$\partial I = \iint_\omega \left(\nabla u_j \varphi_j \cdot \nabla (\delta u_j) - f \delta u \right) d\Omega = 0$$

Sub in $\delta u_j = \varphi_i(x,y)$

$\Rightarrow$

$$\iint_\Omega u_j \nabla \varphi_j (x,y) \cdot \nabla \varphi_i (x,y) = \iint_\Omega f \, \varphi_i (x,y) \, d\Omega$$

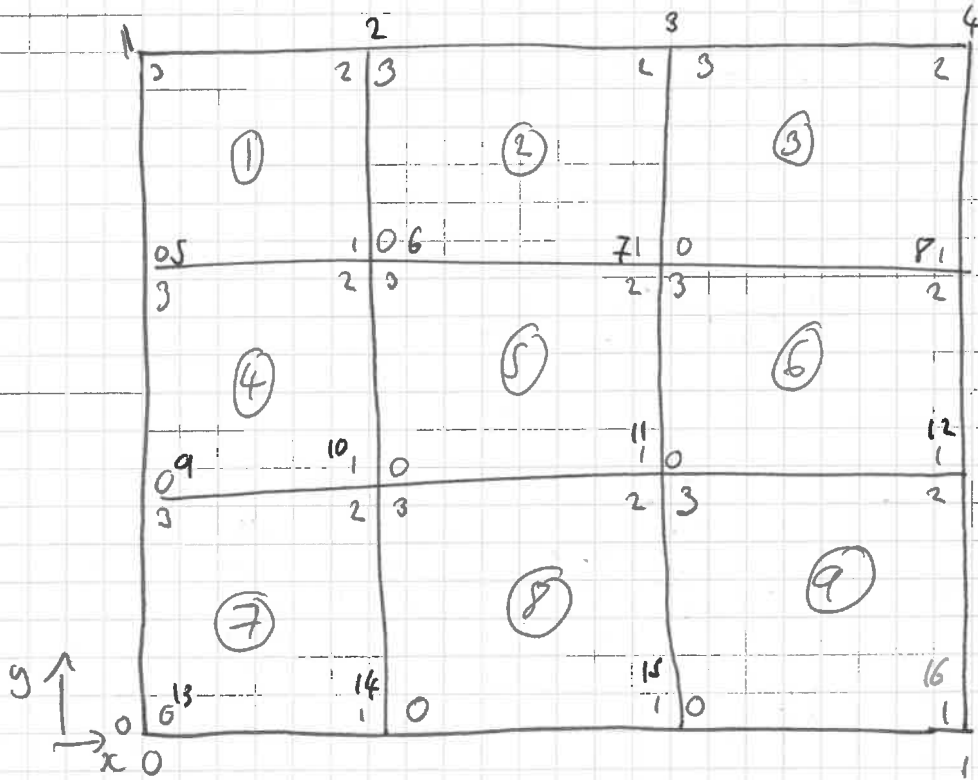$$\Rightarrow \quad A_{ij} u_j = b_i \qquad b_i = \iint_\Omega \varphi_i (x,y) f(x,y) \, d\Omega$$

with A, b for as

$$A_{ij} = \iint_\Omega \nabla \varphi_i (x,y) \cdot \nabla \varphi_j (x,y) \, d\Omega$$

(3) Introduce local coordinate system and reference coordinates

Quadrilateral element

eg. 3×3 Mesh.

LOCAL NUMBERING

GLOBAL NUMBERING

Element (9)

LOCAL (α)	GLOBAL
0	15
1	16
2	12
3	11

etc.

Mapping between global and local.

$$\bar{x} = F_{K_k}(\bar{\xi}) = \sum_{\alpha=0}^{N_{\alpha}^k-1} \bar{x}_{k\alpha}\, X_\alpha(\bar{\xi})$$

Define basis functions with global node number i on Element K_k

$$\varphi_\alpha(x,y) = \hat{\varphi}_\alpha(F_K^{-1}(x,y)) = X_\alpha(\bar{\xi})$$

α = local element index, for local support $\varphi_\alpha = 1$ on global node i and 0 on other nodes.

Apply this to :

$$A_{ij} u_j = b_i$$

$$A_{ij} = \iint_\Omega \nabla \ell_i (x,y) \cdot \nabla \ell_j (x,y) \, d\Omega$$

$$b_i = \iint_\Omega \ell_i (x,y) \, f(x,y) \, d\Omega$$

$$\Rightarrow \quad \hat{A}_{\alpha\beta} = \int_K \nabla \chi_\alpha \cdot \nabla \chi_\beta \, d\Omega$$

$$= \int_{\hat{K}} \left((J^T)^{-1} \begin{pmatrix} \frac{\partial \chi_\alpha}{\partial \ell_1} \\ \frac{\partial \chi_\alpha}{\partial \ell_2} \end{pmatrix} \right) \cdot \left((J^T)^{-1} \begin{pmatrix} \frac{\partial \chi_\beta}{\partial \xi_1} \\ \frac{\partial \chi_\beta}{\partial \ell_2} \end{pmatrix} \right) |\det J| \, d\bar{\xi}$$
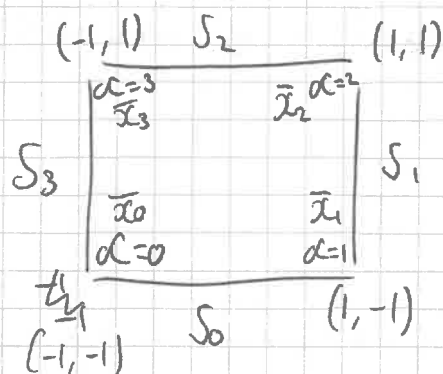
$$\hat{b}_\alpha = \int_K f \, \chi_\alpha \, d\Omega$$

$$= \int_{\hat{K}} f\left(x(\xi_1, \xi_2), y(\xi_1, \xi_2)\right) \chi_\alpha(\xi_1, \xi_2) \, |\det J(\bar{\xi})| \, d\bar{\xi}$$

for $\alpha, \beta = 0, \dots N_n^K - 1$ on each reference cell $\hat{K}$

For quadrilateral $\alpha, \beta = 0, 1, 2, 3$

For reference element $\bar{q} \in (-1, 1)^2$

To assemble matrix loop though α and β
for the total number of nodes N_n^k
where k is the number of elements and n the
number of nodes per element. Assign $\hat{b}_\alpha$ and
$\hat{A}_{\alpha\beta}$ though.

$$\hat{A}_{\alpha\beta} = \int_k \nabla \chi_\alpha \cdot \chi_\beta \, d\ell$$

$$\hat{b}_\alpha = \int_k f \, \varphi_\alpha \, d\ell$$

for each element into a global matrix

$A = 0 \quad b = 0 \quad A_{ij} = b_i = 0$

for all elements K_k, $k = 1$, No of elements:

for $\alpha = 1$ to N_n^k:

$\quad i = \text{Index}(k, \alpha) \quad \longleftarrow$ set i to relevant value for
k^{th} element and $\alpha = 0, 1, 2, 3$
for that element

$\quad$ for $\beta = 1$, N_n^k:

$\quad\quad j = \text{Index}(k, \beta) \quad \longleftarrow$ set j to relevant value
for node element

$\quad\quad A_{ij} = A_{ij} + \hat{A}_{\alpha\beta}$

Read $\beta = 0, 1, 2, 3$

$\quad b_i = b_i + \hat{b}_\alpha$

calculate
$\hat{A}_{\alpha\beta}$ and assign
to matrix

Assign to b matrix
for all values of α