Fluid Dynamics — Numerical Techniques

MATH5453M Numerical Exercises 2, 2025

Due date: November 14^{th} 2025

Leading-order DG0-FEM, finite-volume or Godunov method

Finite Volume Method: nonlinear kinematic equation for river flow

Consider the non-linear St. Venant or width-averaged shallow-water equations

$$\frac{\partial A}{\partial t}+\frac{\partial (Au)}{\partial s}=0 \quad \text{and} \quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial s}=-g\frac{\partial (h+b)}{\partial s}-gC_m^2 u|u|/R^{4/3}, \tag{1}$$

for the variables velocity $u=u(s,t)$ with coordinate s and time t, the river cross-sectional wetted area $A=A(s,t)$ and water depth $h=h(s,t)\equiv h\left(A(s,t),s\right)$, as well as acceleration of gravity $g=9.81\text{m/s}^2$, hydraulic radius $R=R(A,s)$ and Manning friction coefficient C_m. The hydraulic radius is defined as the wetted area A over the wetted perimeter $P(A,s)$, such that $R(A,s)=A/P(A,s)$. Since the shape and dimensions of a river cross-section can change as function of the curvilinear along-river coordinate s, both water depth and wetted perimeter are generally also explicit functions of s.

When the last two terms in (1) are dominant and in approximate balance, one finds the so-called phenomenological Manning relation for the velocity u and discharge $Q=Au$, with the following and resulting expressions valid provided the river (down)slope $-\partial_s b$ is positive,

$$u=\frac{R^{2/3}\sqrt{-\partial_s b}}{C_m}, \quad Q=Au=\frac{A^{5/3}\sqrt{-\partial_s b}}{C_m P(A,s)^{2/3}}. \tag{2}$$

After substitution of one of these Manning relations into the continuity equation in (1), one finds that

$$\partial_t A+\partial_s\left(AR(A,s)^{2/3}\sqrt{-\partial_s b}/C_m\right)=S \Longrightarrow \tag{3}$$

$$\partial_t A+\partial_s F(A,s)=S \quad \text{with} \quad F(A,s)=\frac{A^{5/3}\sqrt{-\partial_s b}}{C_m P(A,s)^{2/3}} \tag{4}$$

with Manning coefficient C_m, hydraulic radius $R(A,s)$ (wetted area A over wetted perimeter) and "volume" source $S(s,t)$.

1. Consider a rectangular river cross-section with varying river width $w_0(s)$. Show that $h(A,s)=A/w_0(s)$

and $P(A,s) = w_0(s) + 2h(A,s) = w_0(s) + 2A/w_0(s)$. For $S = 0$, rewrite the system (3) as

$$\partial_t A + \frac{\partial F}{\partial A}\partial_s A + \frac{\partial F}{\partial s} = 0, \quad \text{with} \tag{5a}$$

$$\begin{aligned}\frac{\partial F}{\partial A} =& \frac{5\sqrt{-\partial_s b}}{3C_m}\frac{A^{2/3}}{(w_0+2A/w_0)^{2/3}} - \frac{2\sqrt{-\partial_s b}}{3C_m}\frac{2A^{5/3}/w_0}{(w_0+2A/w_0)^{5/3}}\\ =& \frac{\sqrt{-\partial_s b}}{3C_m}\frac{(5w_0A^{2/3} + 6A^{5/3}/w_0)}{(w_0+2A/w_0)^{5/3}} > 0\end{aligned} \tag{5b}$$

$$\frac{\partial F}{\partial s} = -\frac{2\sqrt{-\partial_s b}}{3C_m}\frac{A^{5/3}}{(w_0+2A/w_0)^{5/3}}\left(1 - \frac{2A}{w_0^2}\right)\frac{\mathrm{d}w_0}{\mathrm{d}s}. \tag{5c}$$

2. For $S = 0$, take the limit in which $w_0(s)$ is independent of s or varying (very) slowly such that its dependence on s can be approximately ignored. What is the "eigenvalue" λ for the kinematic river equation? At least conceptually, solve the Riemann problem for (3), in that limit of no or slowly varying explicit s-dependence; provide matching sketches of the solution with characteristics in the t, s–plane. Use the required piecewise constant initial data A_l, A_r.

3. For $S = 0$, derive the following finite-volume or Godunov scheme for (3), before using the solution to the Riemann problem,

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta s_k}\int_{t^n}^{t_{n+1}} F(A,s)|_{s=s_{k+1/2}} - F(A,s)|_{s=s_{k-1/2}}\,\mathrm{d}t \tag{6}$$

with the flux $F(A,s)|_{s=s_{k+1/2}} = F_{k+1/2}(\bar{A}_k^n, \bar{A}_{k+1}^n)$ evaluated at the node $s_{k+1/2}$ separating the cells k and $k+1$ with cell averages $\bar{A}_k^n, \bar{A}_{k+1}^n$. Why are only cell verages at the previous or old time level n required in the Godunov method? The (varying) cell width $h_k = \Delta s_k = s_{k+1/2} - s_{k-1/2}$.

4. Given (3) and (5) (i.e., for $S = 0$ hereafter), derive what the Godunov flux should be using your Riemann solution. (Hint: is it either completely upwind or downwind and if so, why?)

5. Derive a time-step restriction or CFL condition based on

$$\Delta t < \mathsf{CFL}\min_k \frac{h_k}{|\lambda_k|} \tag{7}$$

with (irregular) finite volume cells indexed by k, with its cell length h_k, and information speed or local eigenvalue λ_k. Define λ_k (again). Generally, one takes $0 < \mathsf{CFL} < 1$.

6. What inflow/outflow conditions should be imposed at $s = 0$ and river channel end $s = L_e$? I.e. what $\bar{A}_{-1} = \bar{A}_L$ (outside the domain at $s = 0$) should be imposed and what $\bar{A}_{N_k} = \bar{A}_R$ (outside the domain at $s = L_e$) should be imposed in the numerical flux for N_k irregular finite volume cells with $k = 0, \ldots, N_k - 1$ (Python counting).

7. *Numerics in Firedrake:*

   Implement (and/check) the Godunov scheme (provided) for (3). Verify/modify the implementation provided via a series of test cases.

   Test-Case-0: Impose $h = H_0$ for constant $w_0 = W_0$ (such that $A = W_0 H_0$) also at the inflow boundary and as initial condition; check that nothing happens. Take $L_e = 5000\text{m}, W_0 = 100\text{m}, -\partial_s b = 0.001, g = 9.81\text{m/s}^2, C_m = 0.1$ ($C_m \in [0.05, 0.15]$). Plot $A(s,t)$, water depth $h(s,t)$ and discharge $Q(s,t) = A(s,t)u(s,t) = F(A,s)$ for some times. Discharge should be constant $Q = Q_0$. Why is the discharge not constant after the pulse has passed?

   Test-Case-1: Use the same initial conditions and river bed but impose a flood hydrograph at $s = 0$ with $Q(t) \geq Q_0$, starting and returning to the base flow Q_0. E.g., be inspired by a River Aire Boxing Day flood hydrograph with floods for 32hrs and maximum discharge of circa $Q_{max} = 350\text{m}^3/\text{s}$. Translate that to an inflow condition for Q in the numerical flux at $s = 0$. E.g., take $Q(t) = Q_{base} + Q_{max} e^{-\gamma(t-t_{max})^2}$ with $t_{max} = T_{ned}/2, T_{end} = 5 \times 3600\text{s}, \gamma = 0.00001\text{s}^2$ with Q_{base} the basic discharge in Test-Case-0. Plot $A(s,t)$, water depth $h(s,t)$ and discharge $Q(s,t) = A(s,t)u(s,t) = F(A,s)$ for the span of time for which it flows in and outside the domain. Why is the discharge not constant after the pulse has passed? Verify the shock speed observed in an analysis of the numerics with the exact shock speed relation, using approximations for the limiting values upstream and downstream of the observed shock.

   Test-case-2: Define a cross-section with a central rectangular river section of fixed width 20m and fixed depth 4m connected to flood plains of width $w_0(s)$. Modify $R(A,s), P(A,s), h(A,s)$ accordingly. Redo the simulations of Test-case-1. *Updates in progress.*

   For all test cases: What us the order of accuracy in space and in time? Show that you have reached convergence in space and time by showing results for at least two spatial and two temporal resolutions. Try various CFL-numbers including one. Interpret your results. Do not only use a time counter but use the actual time in your time-loop.

References

Ambati, V.R. and Bokhove, O. 2007: Space-time finite element shallow water flows. *J. Comp. Appl. Math.* **204**(2), 452–462.

Bercea, G.-T., McRae, A.T.T., Ham, D.A., Mitchell, L., Rathgeber, F., Nardi, L., Luporini, F., and Kelly, P.H.J. 2016: A structure-exploiting numbering algorithm for finite elements on extruded meshes, and its performance evaluation in Firedrake. *Geoscientific Model Development* **9**-10, 3803–3815.

Bokhove, O., Hicks, T, Kent, T. 2020: Wetropolis extreme rainfall and flood demonstrator: from mathematical design to outreach. *Hydrol. and Earth System Sci.* **24**, 2483–2503. `https://hess.copernicus.org/articles/24/2483/2020/`

W. Kristina, B. van Groesen, and O. Bokhove 2012: Effective Coastal Boundary Conditions for Dispersive Tsunami Propagation. `https://ris.utwente.nl/ws/portalfiles/portal/5093369/memo1983.pdf`

Morton, K.W. and Mayers, D.F. (2005) *Numerical solution of partial differential equations.* Cambridge University Press, 227, 278 pp.

Internet.

W. Kristina (2014) *Effective coastal boundary conditions for tsunami simulations.* PhD Thesis, University of Twente. Section 2.3.1.

A Firedrake Discontinuous Galerkin FEM: weak forms

A discontinuous Galerkin finite element method (DGFEM) is derived for the kinematic equation. Consider the river equation in flux form with $x = s$:

$$\partial_t A + \partial_x F = 0 \quad \text{with} \quad F(A, x) = \frac{A^{5/3}\sqrt{-\partial_x b}}{C_m P(A, x)^{2/3}}, \tag{8}$$

wherein $(-\partial_x b)$ the negative river slope and Manning coefficient C_m are given parameters and wetted river cross-section perimeter $P(A, x)$ is a given function of variable A and coordinate x. Multiply this equation by a compact test function w_K locally nonzero on each element K, integrate by parts over each element K and sum over all elements. One obtains

$$\sum_K \int_K w_K \partial_t A - F \partial_x w_K \mathrm{d}x + \sum_K \int_{\partial K} w \hat{n} F \mathrm{d}\Gamma = 0, \tag{9}$$

in which we will drop the integral signs hereafter as in Firedrake, used outward normal $\hat{n}$ and inside trace evaluation on the boundary ∂K of element K. For DG0 or a leading-order finite volume method w_K is one inside element K and zero outside element K. For higher-order DGn, w_K remains zero outside element K but involves higher-order test functions within element K. Note that the sum over each face/node $\int_{\partial K}$ of an element (in one dimension these boundary integrals reduce to points) can be transferred to a sum over all faces/nodes Γ (cf. Ambati and Bokhove 2007). This transfer leads to two contributions: one from the inside of that element and from the adjacent element to that face

$$\sum_K \int_{\partial K} w\hat{n}F\mathrm{d}\Gamma = \sum_\Gamma \int_\Gamma \hat{n}_l F^l w^l + \hat{n}_r F^r w^r \,\mathrm{d}\Gamma \tag{10a}$$

$$= \sum_\Gamma \int_\Gamma (\alpha F^l + \beta F^r)(\hat{n}_l w^l + \hat{n}_r w^r) + (\hat{n}_l F^l + \hat{n}_r F^r)(\beta w^l + \alpha w^r)\,\mathrm{d}\Gamma \tag{10b}$$

$$= \sum_\Gamma \int_\Gamma (\alpha F^l + \beta F^r)(\hat{n}_l w^l + \hat{n}_r w^r)\,\mathrm{d}\Gamma \tag{10c}$$

$$\approx \sum_\Gamma \hat{n}^l \hat{F}(U_l, U_r, \hat{n}_l)(w^l - w^r)\,\mathrm{d}\Gamma \tag{10d}$$

given that $\hat{n}^l = -\hat{n}^r$ and the flux is continuous $F^l = F^r$ such that $\hat{n}_l F^l = -\hat{n}_r F^r$, wheerein, $\alpha + \beta = 1$. The notation $(\cdot)^{l,r}$ is arbitrary also in one dimension, since each face is assigned a "left" and "right" or "$\pm$" side. It is easiest to derive the above (10) going backwards. Note that w^r is the test function on the element on the other side of the face/node. A numerical flux $\hat{F}(U_l, U_r, \hat{n}_l)$ replaces the linear combination of fluxes. For DG0, $U_l = \bar{A}_l, U_r = \bar{A}_r$.

(a) In one dimension, consider the case with $(\cdot)^{l,r}$ on the left and right. Then $\hat{n}^l = 1$ and we take $\hat{n}^l\hat{F}(w^l - w^r) = \hat{F}(w^l - w^r) = F(\bar{A}_l)(w^l - w^r)$. (b) In one dimension, consider the case with $(\cdot)^{l,r}$ on the right and left. Then $\hat{n}^l = -1$ and we take $\hat{n}^l\hat{F}(w^l - w^r) = -\hat{F}(w^l - w^r) = F(\bar{A}_r)(w^l - w^r)$. Hence, in the general case, we take $\hat{n}^l\hat{F}(w^l - w^r) = \hat{n}^l F(\tilde{A})(w^l - w^r)$ with

$$\tilde{A} = \max(\hat{n}_l, 0)\bar{A}_l + \max(\hat{n}_r, 0)\bar{A}_r = \mathsf{conditonal}(\hat{n}_l > 0, \bar{A}_l, \bar{A}_r). \tag{11}$$

More generally, we can include $\partial F/\partial A$, but since it is positive that is not necessary,

$$A_m = \frac{1}{2}(A_l + A_r) \tag{12}$$

$$\tilde{A} = \max\left(\frac{\partial F(A_m)}{\partial A_m}\hat{n}_l, 0\right)\bar{A}_l + \max\left(\frac{\partial F(A_m)}{\partial A_m}\hat{n}_r, 0\right)\bar{A}_r = \mathsf{conditonal}\left(\frac{\partial F(A_m)}{\partial A_m}\hat{n}_l > 0, \bar{A}_l, \bar{A}_r\right). \tag{13}$$

In summary, we thus obtain the follwing weak formulation with in Firedrake the notation $\pm$ being arbitary, and that suitable integrals are taken is implied:

$$\sum_K w\partial_t A\mathrm{d}x = F\partial_x w\mathrm{d}x - \sum_\Gamma \hat{n}^+ F(\tilde{A})(w^+ - w^-)\,\mathrm{d}S$$
$$- \sum_\Gamma \hat{n}\cdot\hat{F}(A_{in})w^+\,\mathrm{d}s(1) - \sum_\Gamma \hat{n}F(\tilde{A})w^+\,\mathrm{d}s(2), \tag{14}$$

with boundary-types $\mathrm{d}s(1)$ and $\mathrm{d}s(2)$, interior faces $\mathrm{d}S$ and $A_{in} = A(0,t)$ the inflow boundary value.

For zeroth-order DGFEM or finite volume, the interior integrals with the gradients on the test function of course disappear in the above weak forms. For the numerical implementation, we used the above weak forms and the DGFEM examples at `https://www.firedrakeproject.org/documentation.html`.

A.1 Numerics of boundary conditions

The boundary conditions are incorporated by specification of the correct ghost values in the numerical flux. Inward coming waves at $s = 0$ can be specified. For outflow an extrapolting boundary condition is used by copying over the interior value. See Kristina et al. (2012) for outflow conditions for shallow-water systems.