

1) 
 $b_z + (ub)_z = 0$
 $u = \alpha b^2 - \beta b^3 b_z \quad \omega / z \in [0, H] \quad \alpha, \beta > 0$
 $BCs: b(0, t) = b_0, b(H, t) = b_T$
 $IC: b(z, 0) = b_i(z).$

a) $b_z + (ub)_z = 0, u = \alpha b^2 - \beta b^3 b_z \Rightarrow ub = \alpha b^3 - \beta b^3 b_z$

$\Rightarrow b_z + (\alpha b^3 - \beta b^3 b_z)_z = 0 \quad (5)$

Let $b = D_0 + b'$ to linearize around D_0 .

$D_{0z} = D_{0zz} = 0 \text{ since } D_0 \text{ cst.}$

$\Rightarrow b'_z + ((D_0 + b')^3 - \beta (D_0 + b')^3 b'_z)_z = 0$

b' small perturbation, can neglect quadratic and h.o.t.s in b' and derivatives.

$\Rightarrow b'_z + (\alpha (D_0^3 + 3D_0^2 b') - \beta D_0^3 b'_z)_z = 0$

$\Rightarrow b'_z + 3\alpha D_0 b'_z - \beta D_0 b'_z = 0 \quad (6) \text{ as required.}$

(5) and (6) are nonlinear convection-diffusion eqs as they have both a convection term (b_z / b'_z) and a diffusive term (b_{zz} / b'_{zz}). Eq (5) is also nonlinear in b , hence being a nonlinear eq. Eq (6) is linearized as all quadratic and h.o.t.s of b and its derivatives have been neglected.

- b) Discretize w/ forward Euler discretization, upwind scheme for convective term, 2nd order central difference for diffusive term.

$z = j \Delta z, z \in [0, H], j \in \{0, 1, \dots, J-1, J\} \quad \omega / J \approx \pi. \quad \Delta z = H, \Delta z = \frac{H}{J}$

$t = n \Delta t, n \in \mathbb{N}$

note: choice of J defines Δz , can choose both Δt and no. of time steps.

$(5) \quad b_z + (\alpha b^3 - \beta b^3 b_z)_z = 0$

Forward Euler time discretization: $b_z \approx \frac{\partial}{\partial z} b(z_j, t_n)$, $b(z, t) \approx b(z_j, t_n) = b_j^n$

Δt small (Δz is Δz) \therefore can Taylor expand around t_n :

$b(z_j, t_n + \Delta t) \approx b(z_j, t_n) + \Delta t \frac{\partial}{\partial z} b(z_j, t_n) + \frac{\Delta t^2}{2!} \frac{\partial^2}{\partial z^2} b(z_j, t_n) + O((\Delta t)^3)$

$\Rightarrow b(z_j, t_{n+1}) - b(z_j, t_n) \approx \Delta t \frac{\partial}{\partial z} b(z_j, t_n) + O((\Delta t)^2)$

$\therefore \frac{\partial}{\partial z} b_j^n \approx \frac{b_j^{n+1} - b_j^n}{\Delta z} \text{ (neglecting h.o.t.s)}$

Convective term: $(ab^3)_z$

using the product rule obtain: $3\alpha b^2 b_z, 3\alpha > 0 \therefore \text{coefficient} > 0$.

upwind scheme: Taylor expand $b(z_j - \Delta z, t_n)$ about z_j

$b(z_j - \Delta z, t_n) \approx b(z_j, t_n) - \Delta z \frac{\partial}{\partial z} b(z_j, t_n) + \frac{(\Delta z)^2}{2!} \frac{\partial^2}{\partial z^2} b(z_j, t_n) + O((\Delta z)^3)$

$\Rightarrow \frac{\partial b}{\partial z} \approx \frac{b(z_j, t_n) - b(z_j - \Delta z, t_n)}{\Delta z} = \frac{b_j^n - b_{j-1}^n}{\Delta z}$

$\therefore (ab^3)_z \approx 3\alpha (b^n)^2 \frac{b_j^n - b_{j-1}^n}{\Delta z}$

Dissipative term: $-\beta(b^3 b_z)_z$

using adjoint method (advantages: • more stable for discretization.) • more compact form.

$$[b^3 b_z]_{j+\frac{1}{2}} = (b_j^n)^3 \left(\frac{b_{j+1}^n - b_j^n}{\Delta z} \right)$$

$$[b^3 b_z]_{j-\frac{1}{2}} = (b_j^n)^3 \left(\frac{b_j^n - b_{j-1}^n}{\Delta z} \right)$$

$$[b^3 b_z]_z \approx \frac{1}{2(\Delta z)^2} [(b_{j+\frac{1}{2}}^n)^3 (b_{j+1}^n - b_j^n) - (b_{j-\frac{1}{2}}^n)^3 (b_j^n - b_{j-1}^n)]$$

$$(b_{j+\frac{1}{2}}^n)^3 = \frac{(b_{j+1}^n)^3 + (b_j^n)^3}{2}, \quad (b_{j-\frac{1}{2}}^n)^3 = \frac{(b_j^n)^3 + (b_{j-1}^n)^3}{2}$$

$$\text{then } -\beta(b^3 b_z)_z \approx \frac{\beta}{2(\Delta z)^2} [(b_{j+\frac{1}{2}}^n)^3 + (b_j^n)^3)(b_{j+1}^n - b_j^n) - ((b_j^n)^3 + (b_{j-1}^n)^3)(b_j^n - b_{j-1}^n)]$$

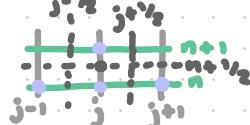
$$= -\frac{\beta}{2(\Delta z)^2} [(b_{j+1}^n)^4 + (b_j^n)^3 b_{j+1}^n - (b_{j+1}^n)^3 b_j^n - (b_j^n)^4 - (b_{j-1}^n)^3 b_j^n + (b_j^n)^3 b_{j-1}^n + (b_{j-1}^n)^4]$$

$$= -\frac{\beta}{2(\Delta z)^2} [(b_j^n)^2 [b_{j+1}^n - 2b_j^n + b_{j-1}^n] + (b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_{j-1}^n)^3 (b_j^n - b_{j-1}^n)]$$

then (5) $b_t + (\alpha b^3 \cdot b^3 b_z)_z = 0$ is approximated by the dissipation:

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} + 3\alpha(b_j^n)^2 \frac{b_j^n - b_{j-1}^n}{\Delta z} - \frac{\beta}{2(\Delta z)^2} [(b_j^n)^3 [b_{j+1}^n - 2b_j^n + b_{j-1}^n] + (b_{j+1}^n)^3 (b_{j+1}^n - b_j^n) - (b_{j-1}^n)^3 (b_j^n - b_{j-1}^n)] = 0$$

Grid used:



(6): $b'_t + 3\alpha D_0^2 b'_z - \beta D_0^2 b'_{zz} = 0$ is discretized to give:

$$\frac{b_j'^{n+1} - b_j'^n}{\Delta t} + 3\alpha D_0^2 \frac{b_j'^n - b_{j-1}'^n}{\Delta z} - \beta D_0^2 \underbrace{\left(\frac{b_{j+1}'^n - 2b_j'^n + b_{j-1}'^n}{(\Delta z)^2} \right)}_{\text{upwind scheme for convective term}} = 0$$

$\underbrace{\text{upwind scheme for convective term}}$ $\underbrace{2\text{nd order central difference for dissipative term}}$

Linearisation of discretization of (5):

$b = D_0 + b'$, neglect quadratic and h.o.t.s in b' and its derivatives:

$$\frac{b_j'^{n+1} - b_j'^n}{\Delta t} + 3\alpha D_0^2 \frac{b_j'^n - b_{j-1}'^n}{\Delta z} - \frac{\beta}{2(\Delta z)^2} (D_0^2 [b_{j+1}'^n - 2b_j'^n + b_{j-1}'^n + b_{j+1}'^n - b_j'^n - b_j'^n - b_{j-1}'^n]) = 0$$

Note: $D_{0,j-1} = D_{0,j} = D_{0,j+1}$

$$\Rightarrow \frac{b_j'^{n+1} - b_j'^n}{\Delta t} + 3\alpha D_0^2 \frac{b_j'^n - b_{j-1}'^n}{\Delta z} - \frac{\beta}{2(\Delta z)^2} (2D_0^2 [b_{j+1}'^n - 2b_j'^n + b_{j-1}'^n]) = 0$$

∴ linearisation of discretization of (5) is the same as discretization of (6).

$$BCs: b(0, t) \approx b(0, t_n) = b_0 \Rightarrow b_0^n = b_0 \quad \forall n$$

$$b(H, t) \approx b(H, t_n) = b_T \Rightarrow b_T^n = b_T \quad \forall n$$

$$IC: b(z, 0) \approx b(z_j, 0) = b_j(z_j) = b_j(j\Delta z) \Rightarrow b_j^0 = b_j(j\Delta z) \quad \forall j$$

c) Use a Fourier analysis to assess stability of linearized scheme.

$$\text{linearized scheme: } \frac{b_j^{n+1} - b_j^n}{\Delta t} + 3\alpha D_0 \frac{b_j^n - b_{j-1}^n}{\Delta z} - \beta D_0^3 \frac{b_{j+1}^n - 2b_j^n + b_{j-1}^n}{(\Delta z)^2} = 0$$

$$\text{use ansatz: } b_j^n = \lambda(k)^n e^{ikz_j \Delta z}$$

Case 1: $\alpha=0, \beta \neq 0$ and substitute in ansatz

$$\Rightarrow \lambda(k)^n e^{ikz_j \Delta z} \left[\frac{\lambda(k)-1}{\Delta t} - \frac{\beta D_0^3}{(\Delta z)^2} (e^{ikz_j \Delta z} - 2 + e^{-ikz_j \Delta z}) \right] = 0$$

$$\Rightarrow \frac{\lambda(k)-1}{\Delta t} - \frac{\beta D_0^3}{(\Delta z)^2} (2\cos(k\Delta z) - 2) = 0$$

$$\cos 2z = \cos^2 z - \sin^2 z = 1 - 2\sin^2 z \Rightarrow 2\cos(k\Delta z) - 2 = -4\sin^2\left(\frac{k\Delta z}{2}\right)$$

$$\Rightarrow \lambda(k) = 1 - 4\beta D_0^3 M \sin^2\left(\frac{k\Delta z}{2}\right), M := \frac{\Delta t}{(\Delta z)^2}$$

$$\text{for stability require } |\lambda(k)| \leq 1 \Rightarrow \underbrace{|1 - 4\beta D_0^3 M \sin^2\left(\frac{k\Delta z}{2}\right)|}_{\geq 0} \leq 1$$

$\Rightarrow |\lambda(k)| \leq 1 \text{ always.}$

$$\text{for stability need } 1 - 4\beta D_0^3 M \sin^2\left(\frac{k\Delta z}{2}\right) > -1 \quad \forall k$$

take most extreme case: $1 - 4\beta D_0^3 M > -1$ is condition for stability

$$-4\beta D_0^3 M > -2$$

$$4\beta D_0^3 M < 2$$

$$\frac{\Delta t}{(\Delta z)^2} = M < \frac{1}{2} \frac{1}{\beta D_0^3}$$

\therefore choose Δt s.t. $\Delta t < \frac{1}{2} \frac{(\Delta z)^2}{\beta D_0^3}$ for stability

Case 2: $\alpha \neq 0, \beta = 0$

$$\Rightarrow \lambda(k)^n e^{ikz_j \Delta z} \left(\frac{\lambda(k)-1}{\Delta t} + 3\alpha D_0 \left(\frac{1 - e^{-ikz_j \Delta z}}{\Delta z} \right) \right) = 0$$

$$\lambda(k) - 1 + 3\alpha D_0 \Delta z (1 - e^{-ik\Delta z}) = 0$$

$$\lambda(k) = 1 - 3\alpha D_0 \Delta z (1 - e^{-ik\Delta z})$$

$$= 1 - 3\alpha D_0 \Delta z (1 - \cos(k\Delta z) + i\sin(k\Delta z))$$

$$= 1 - \mu A (1 - \cos(k\Delta z)) - i\mu A \sin(k\Delta z) \quad A = 3\alpha D_0 \Delta z$$

$$|\lambda(k)|^2 = (-\mu A + \mu A \cos(k\Delta z))^2 + (\mu A \sin(k\Delta z))^2$$

$$= (1 - \mu A)^2 + ((1 - \mu A)\mu A \cos(k\Delta z) + \mu^2 A^2 \cos^2(k\Delta z) + \mu^2 A^2 \sin^2(k\Delta z))$$

$$= 1 - 2\mu A + 2\mu^2 A^2 + 2\mu A \cos(k\Delta z) - 2\mu^2 A^2 \cos(k\Delta z)$$

$$= 1 - 2\mu^2 A^2 (\cos(k\Delta z) - 1) + 2\mu A (\cos(k\Delta z) - 1)$$

$$\cos(k\Delta z) - 1 = -2 \sin^2\left(\frac{k\Delta z}{2}\right)$$

$$\Rightarrow |2(k)|^2 = 1 + 4\mu^2 A^2 \sin^2\left(\frac{k\Delta z}{2}\right) - 4\mu A \sin^2\left(\frac{k\Delta z}{2}\right)$$

$$= 1 - 4\mu A \sin^2\left(\frac{k\Delta z}{2}\right) [1 - \mu A]$$

need that $|2(k)|^2 < 1 \Rightarrow 1 - 4\mu A \sin^2\left(\frac{k\Delta z}{2}\right) [1 - \mu A] < 1$

$$-4\mu A \sin^2\left(\frac{k\Delta z}{2}\right) (1 - \mu A) < 0$$

$$\Rightarrow 1 - \mu A > 0$$

$$\Rightarrow \mu < \frac{1}{A} \quad (A \text{ tve since } k > 0)$$

$$\Rightarrow \mu < \frac{1}{3\alpha D_0^2 \Delta z}$$

$$\therefore \text{choose } \Delta t \text{ s.t. } \Delta t < \frac{\Delta z}{3\alpha D_0^2}$$

d) Use maximum principle to determine a stable time step for discretization of (6).

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} + 3\alpha D_0^2 \frac{b_j^n - b_{j-1}^n}{\Delta z} - \beta D_0^3 \frac{b_{j+1}^n - 2b_j^n + b_{j-1}^n}{(\Delta z)^2} = 0$$

suppose $b_j^n \geq 0 \forall j$, want $b_j^{n+1} \geq 0 \forall j$ as well

$$b_j^{n+1} = b_j^n - 3\alpha D_0^2 \mu \Delta z (b_j^n - b_{j-1}^n) + \beta D_0^3 \mu (b_{j+1}^n - 2b_j^n + b_{j-1}^n)$$

$$= \beta D_0^3 \mu b_{j+1}^n + b_{j-1}^n (3\alpha D_0^2 \mu \Delta z + \beta D_0^3 \mu) + b_j^n (1 - 3\alpha D_0^2 \mu \Delta z - 2\beta D_0^3 \mu)$$

$$\Rightarrow \text{need } 1 - 3\alpha D_0^2 \mu \Delta z - 2\beta D_0^3 \mu > 0$$

$$\Rightarrow \mu (3\alpha D_0^2 \Delta z + 2\beta D_0^3) < 1$$

$$\mu < \frac{1}{3\alpha D_0^2 \Delta z + 2\beta D_0^3}$$

$$\therefore \text{choose } \Delta t \text{ s.t. } \Delta t < \frac{(\Delta z)^2}{3\alpha D_0^2 \Delta z + 2\beta D_0^3} \text{ for maximum principle to hold.}$$

note: setting $\alpha = 0, \beta \neq 0$ or $\alpha \neq 0, \beta = 0$ recovers the conditions on Δt for stability, as derived in d).

In explicit scheme \therefore cond. for

e) Derive variable time step criterion for discretization of (7) for which $b_j^{n+1} > 0$.
 \Rightarrow width to be tve

$$(7) \frac{b_j^{n+1} - b_j^n}{\Delta t} + 3\alpha (b_j^n)^2 \frac{b_j^n - b_{j-1}^n}{\Delta z} - \frac{\mu}{2(\Delta z)^2} [(b_j^n)^3 [b_{j+1}^n - 2b_j^n + b_{j-1}^n] + (b_{j+1}^n)^3 (b_{j+1}^n - b_j^n) - (b_{j-1}^n)^3 (b_j^n - b_{j-1}^n)] = 0$$

$$b_j^{n+1} = b_j^n - 3\alpha \mu \Delta z (b_j^n)^2 (b_j^n - b_{j-1}^n) + \frac{\mu M}{2} [(b_j^n)^3 [b_{j+1}^n - 2b_j^n + b_{j-1}^n] + (b_{j+1}^n)^3 (b_{j+1}^n - b_j^n) - (b_{j-1}^n)^3 (b_j^n - b_{j-1}^n)]$$

$$= b_j^n + 3\alpha \mu \Delta z (b_j^n)^2 + \frac{\mu M}{2} [(b_j^n)^3 b_{j+1}^n + (b_{j+1}^n)^3 b_{j+1}^n + (b_{j-1}^n)^3 + (b_{j-1}^n)^3] - 3\alpha \mu \Delta z (b_j^n)^3 - \frac{\mu M}{2} [2(b_j^n)^4 + b_j^n (b_{j+1}^n)^3 + b_j^n (b_{j-1}^n)^3]$$

then since $b_j^n > 0$ and $\alpha, \beta, \mu > 0$

$$\Rightarrow \text{require } b_j^n - 3\alpha\mu\Delta z(b_j^n)^2 - \frac{\mu\mu}{2}(b_j^n) [2(b_j^n)^3 + (b_{j-1}^n)^3 + (b_{j+1}^n)^3] > 0$$

$$1 - 3\alpha\mu\Delta z(b_j^n)^2 - \frac{\mu\mu}{2}[2(b_j^n)^3 + (b_{j-1}^n)^3 + (b_{j+1}^n)^3] > 0$$

$$\mu[3\alpha\Delta z(b_j^n)^2 + \frac{\mu}{2}(2(b_j^n)^3 + (b_{j-1}^n)^3 + (b_{j+1}^n)^3)] < 1$$

$$\mu < \frac{1}{3\alpha\Delta z(b_j^n)^2 + \frac{\mu}{2}(2(b_j^n)^3 + (b_{j-1}^n)^3 + (b_{j+1}^n)^3)}$$

$$\therefore \text{choose time-step s.t. } \Delta t < \frac{(\Delta z)^2}{3\alpha\Delta z(b_j^n)^2 + \frac{\mu}{2}(2(b_j^n)^3 + (b_{j-1}^n)^3 + (b_{j+1}^n)^3)}$$

f) second-order spatial discretization of (5):

$$\text{consider } b(t_n, z_j + \Delta z) = b_j^n + \Delta z \frac{\partial}{\partial z} b_j^n + \frac{(\Delta z)^2}{2} \frac{\partial^2}{\partial z^2} b_j^n + O((\Delta z)^3)$$

$$b(t_n, z_j - \Delta z) = b_j^n - \Delta z \frac{\partial}{\partial z} b_j^n + \frac{(\Delta z)^2}{2} \frac{\partial^2}{\partial z^2} b_j^n + O((\Delta z)^3)$$

$$\Rightarrow b_{j+1}^n - b_{j-1}^n = 2\Delta z \frac{\partial}{\partial z} b_j^n + O((\Delta z)^3)$$

$\therefore \frac{\partial b}{\partial z} \approx \frac{b_{j+1}^n - b_{j-1}^n}{2\Delta z}$ is 2nd order spatial discretization for convective term.

$$(5) \rightarrow b_j^{n+1} = b_j^n - \frac{3}{2}\alpha\mu\Delta z(b_j^n)^2(b_{j+1}^n - b_{j-1}^n) + \frac{\mu\mu}{2}[b_j^n]^3(b_{j+1}^n - 2b_j^n + b_{j-1}^n) + (b_{j+1}^n)^3(b_{j+1}^n - b_j^n) - (b_{j-1}^n)^3(b_j^n - b_{j-1}^n)]$$

$$b_j^n > 0 \Rightarrow \text{require } b_j^n - \frac{3}{2}\alpha\mu\Delta z(b_j^n)^2 b_{j+1}^n - \frac{\mu\mu}{2} b_j^n [2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3] > 0$$

$$\Rightarrow \mu[\frac{3}{2}\alpha\Delta z b_j^n b_{j+1}^n + \frac{\mu}{2}[2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3]] < 1$$

.. time step required for $b_j^{n+1} > 0$ if $b_j^n > 0$ is Δt s.t.:

$$\Delta t < \frac{2(\Delta z)^2}{3\alpha\Delta z b_j^n b_{j+1}^n + \mu[2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3]}$$

2) a) Claim: steady state sol² of (5) satisfies $\beta b^3 b_2 = \alpha b^3 - Q$, Q cst of integration.

$$(5) \quad b_t + (\alpha b^3 - \beta b^3 b_2)_z = 0$$

steady state sol² is sol² b s.t. $b_t = 0$

$$\Rightarrow (\alpha b^3 - \beta b^3 b_2)_z = 0$$

$$\Rightarrow \alpha b^3 - \beta b^3 b_2 = Q \Rightarrow \beta b^3 b_2 = \alpha b^3 - Q \text{ as required.}$$

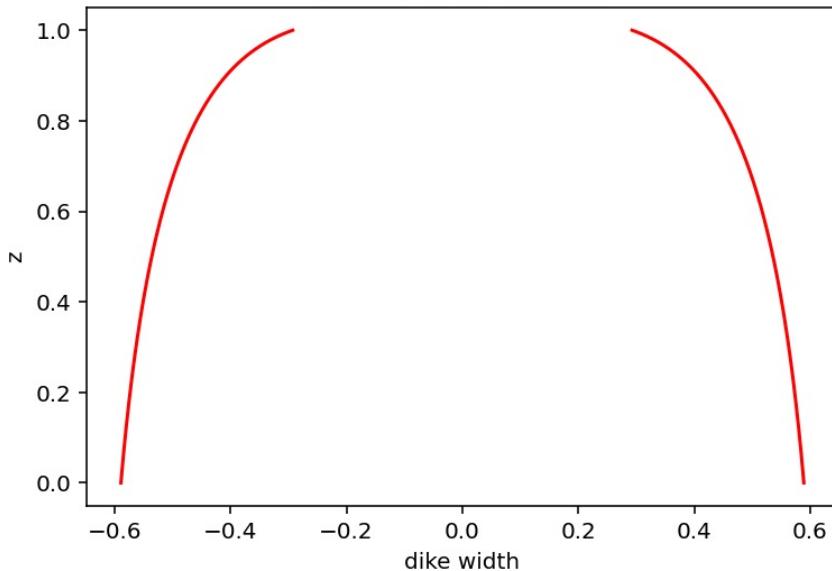
$$b_2 \approx \frac{b_{j+1} - b_j}{\Delta z}$$

$$\Rightarrow e.g. \text{ discretized to: } \beta(b_j)^3 \frac{b_{j+1} - b_j}{\Delta z} = \alpha(b_j)^3 - Q$$

$$\text{explicitly gives } b_{j+1} = b_j + \frac{\alpha \Delta z}{\beta} - \frac{Q \Delta z}{\beta(b_j)^3}$$

w/ $b(0,t) = b_0 \Rightarrow b_0 = b_g$ and then each b_{j+1} can be determined iteratively.
+ code for plotting steady state solution.

Steady State Solution



above is steady state sol² obtained using $J=3000$

giving $b_g = 0.58546028$ (note: b_g agrees w/ b_T given in b) to 3 sig figs)

b) Solⁿ of nonlinear convection-diffusion eq (5) at times:
 $t = 0.05, 0.1, 0.2, 0.5, 1, 2$

$$b_j^0 : b_T = b_J^n$$

$$A b = f \Rightarrow b = A^{-1} f$$

$$\Delta z = \frac{H}{11}, \frac{H}{21}, \frac{H}{41}$$

$$\text{max. principle} \Rightarrow b_{\max} = b_0 = 1.178164343 \quad (> b_T = 0.585373798)$$

need to choose Δt s.t. $\Delta t <$

$$\text{note: Max. principle bound} \quad 3\alpha\Delta z(b_J^n)^2 + \beta(2(b_J^n)^2 + (b_{J-1}^n)^2 + (b_{J+1}^n)^2)$$

stronger than stability.

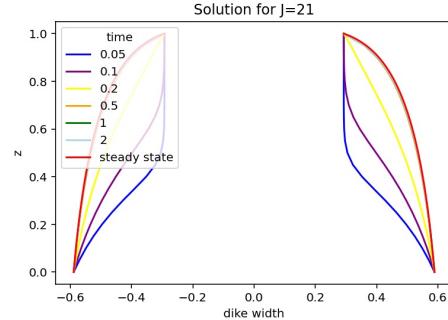
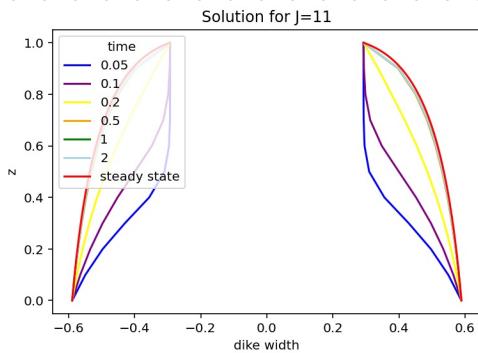
Note: Strictest bound on Δt comes from $b_{\max} = b_0$

$$\Rightarrow \Delta t < \frac{(\Delta z)^2}{3\alpha\Delta z(b_0)^2 + \beta(2(b_0)^2 + (b_0)^2 + (b_0)^2)} = \frac{(\Delta z)^2}{3\alpha(\Delta z)b_0^2 + 2\beta b_0^2}$$

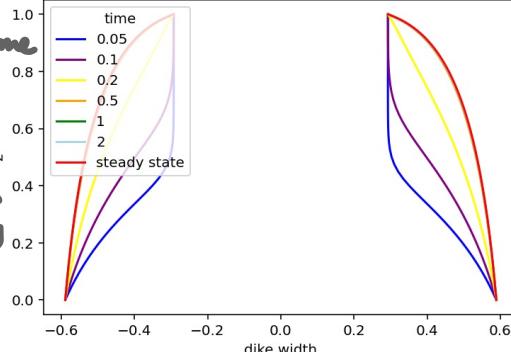
$$\text{note: smallest } \Delta z \text{ when consider 41 grid pts: } \Delta z = \frac{H}{40} = \frac{1}{40}$$

$$\Rightarrow \Delta t < 0.00018827$$

if use $\Delta t = 10^{-4}$ then satisfy stability for all scenarios



Solution for $J=41$



For increasing J
the solutions become
more smooth ~ as
is expected.

For increasing t the
numerical solutions
approach the steady
state solution.

note: this code
used first order
convection
discretisation.

I found that the
2nd order led
to an over
estimate for
 $t \geq 0.5$

c) $e(z, t) = b(z, t) - \text{numerical}(z, t)$
 L^2 -norm and L^∞ -norm as a function of resolution.
 note: $\text{error}(j) = b(z_j, t) - \text{numerical}(z_j, t)$, $z_j = j\Delta z$
 calculated the L^2 -norm using approximation:

$$L^2 \approx \sqrt{\frac{dz}{2} (\text{error}(0)^2 + 2 \sum_{j=1}^{J-1} (\text{error}(j)^2) + \text{error}(J-1)^2)}$$

for $J = 11, 21, 41, 81, 161$ (321?)

$$\Delta z = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160} \quad (\frac{1}{320}?) \Rightarrow \text{halving } \Delta z$$

then plotting the log of the L^2 norm against the log of Δz .

find that the gradient approaches 1 for decreasing Δz .
 \Rightarrow linear dependence.

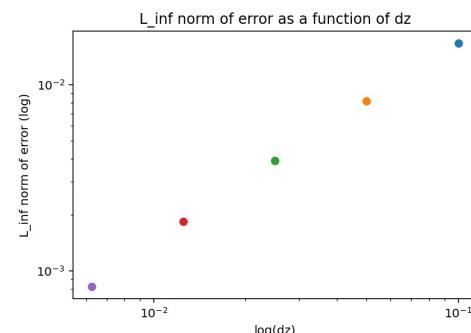
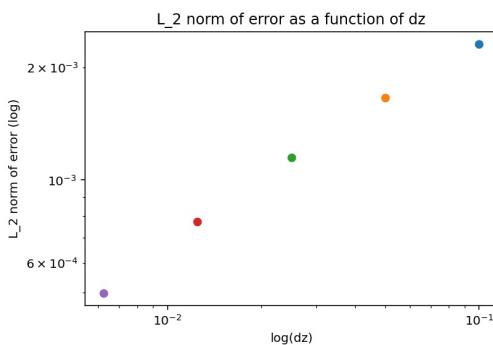
\Rightarrow first order spatial discretization.

similarly for the L^∞ -norm, where $L^\infty = \max_{0 \leq j \leq J-1} (\text{error}(j))$

plotting the log of L^∞ -norm against the log of Δz , find that the gradient approaches 1 from above.

\Rightarrow first order spatial discretization.

Thus giving the expected result that the spatial discretization is first order.



$$d) z - z_{r_0} - ct = \frac{\mu}{\alpha} [b(z, t) - \sqrt{\frac{c}{\alpha}} \operatorname{atanh}(\sqrt{\frac{c}{\alpha}} b(z, t))] \quad w/ z_{r_0} \text{ reference cst}$$

$$\bar{z} = z_{r_0} + ct + \frac{\mu}{\alpha} b(z, t) - \frac{\mu}{\alpha} \sqrt{\frac{c}{\alpha}} \operatorname{atanh}(\sqrt{\frac{c}{\alpha}} b(z, t))$$

$$b(z, t) = \frac{\mu}{\rho} (\bar{z} - z_{r_0} - ct) + \sqrt{\frac{c}{\alpha}} \operatorname{atanh}(\sqrt{\frac{c}{\alpha}} b(z, t))$$

$$b_t = -\frac{\alpha c}{\rho} + \sqrt{\frac{c}{\alpha}} (\sqrt{\frac{c}{\alpha}} b_0) \left(\frac{1}{1 - \frac{c}{\alpha} b^2} \right) = -\frac{\alpha c}{\rho} + b_0 \left(\frac{c}{c - \alpha b^2} \right)$$

$$b_0 \left(1 - \frac{c}{c - \alpha b^2} \right) = -\frac{\alpha c}{\rho} \Rightarrow b_0 \left(\frac{c - \alpha b^2 - c}{c - \alpha b^2} \right) = -\frac{\alpha c}{\rho} \Rightarrow b_0 \left(-\frac{\alpha b^2}{c - \alpha b^2} \right)$$

$$b_0 = -\frac{c}{\rho} \left(-\frac{c - \alpha b^2}{\alpha b^2} \right) = \frac{c^2 - \alpha c b^2}{\rho b^2}$$

$$b_z = \frac{c}{\rho} + \sqrt{\frac{c}{\alpha}} \sqrt{\frac{c}{\alpha}} b_0 \left(\frac{c}{c - \alpha b^2} \right) = \frac{c}{\rho} + b_0 \left(\frac{c}{c - \alpha b^2} \right)$$

$$b_0 \left(-\frac{\alpha b^2}{c - \alpha b^2} \right) = \frac{c}{\rho} \Rightarrow b_0 = -\frac{c - \alpha b^2}{\rho b^2} \quad \text{note: } b_0 = -cb_z$$

$$WTS \mapsto \partial_t b + \partial_z (\alpha b^2 - \beta b^2 b_z) = 0$$

$$b_t + 3\alpha b^2 b_z - (\alpha b^2 b_z)_z$$

$$b_t + 3\alpha b^2 b_z - (\alpha b^2 \left(-\frac{c - \alpha b^2}{\rho b^2} \right))_z$$

$$b_t + 3\alpha b^2 b_z - (-Cb + \alpha b^3)_z$$

$$b_t + 3\alpha b^2 b_z + Cb_z - 3\alpha b^2 b_z = b_t + Cb_z$$

$$= b_t - b_0 = 0$$

Hence given implicit $b(z, t)$ is a soln of eq (5) \Rightarrow soln of (1)

$$C = \alpha, \beta = 1.0$$

$$H = 3 \text{ Km}, z_{r_0} = 0.9$$

$$z = z_{r_0} + ct + \frac{1}{\alpha} b - \frac{1}{\alpha} \sqrt{\frac{c}{\alpha}} \operatorname{atanh}(\sqrt{\frac{c}{\alpha}} b)$$

$$z = z_{r_0} + ct + \frac{1}{\alpha} b - \frac{1}{\alpha} \operatorname{tanh}(b)$$

$$T = 0.5, 1, 2, 3, 4, 5, M = 10001, \Delta t = \frac{T}{M-1} = \frac{1}{10000} = 10^{-4} = 0.0001$$

$$\text{Numerical: } b_0 = \max(b_{\text{exact}})$$

$$\underbrace{b_T = \min(b_{\text{exact}})}$$

both time dependent

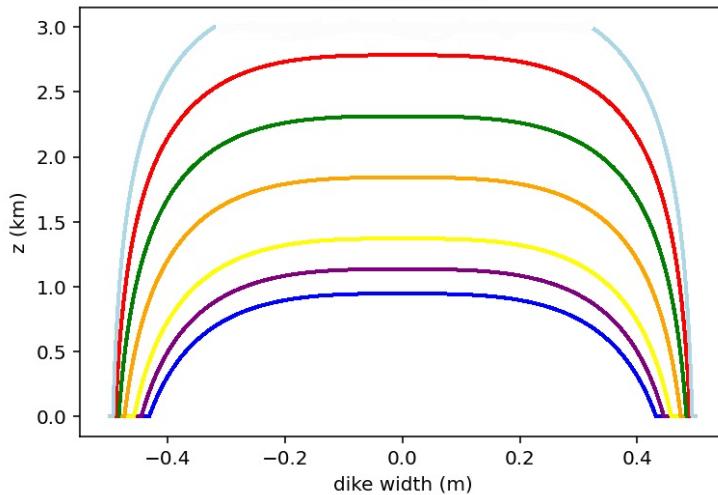
now \therefore need to be redefined
each timestep.

$$\textcircled{1} \quad \partial_t b + \partial_z (ub) = 0, u = \alpha b^2 - \beta b^2 \partial_z b$$

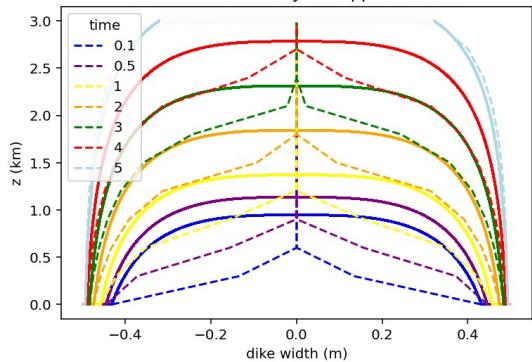
\downarrow
to satisfy max.
principle so that
 $b_j^{un} > 0$ if $b_j^n > 0 \forall j$

$$b_j^{un} = b_j^n - 3\alpha \mu \Delta z (b_j^n)^2 (b_j^n - b_{j-1}^n) + \frac{\beta \mu M}{2} [(b_j^n)^3 (b_{j+1}^n - 2b_j^n + b_{j-1}^n) + (b_{j+1}^n)^3 (b_{j+1}^n - b_j^n)] - (b_{j-1}^n)^2 (b_j^n - b_{j-1}^n)]$$

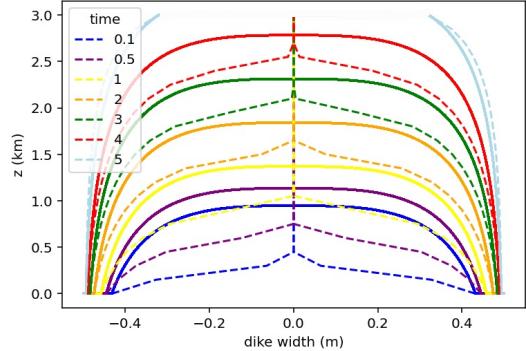
Exact Solution



Exact Solution and $J=11$ approximation



Exact Solution and $J=21$ approximation



Exact Solution and $J=41$ approximation

