

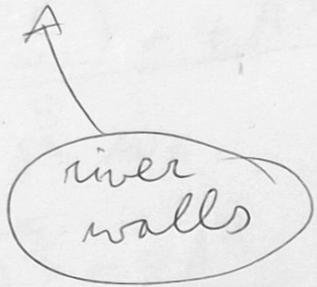
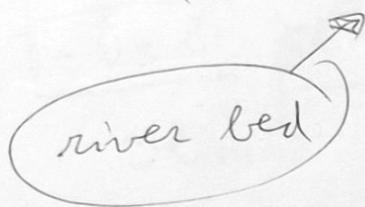
as we have rectangular shape, we get:

$$A(s,t) = w_0(s) h(s,t) \quad (*)$$

$$\Leftrightarrow h(A,s) = \frac{A(s)}{w_0(s)}.$$

Similarly, if  $P(A,s)$  is wetted perimeter,  
then it must be equal to

$$P(A,s) = w_0(s) + 2h(s,t)$$



$$\therefore P(A,s) = w_0(s) + 2 \frac{A}{w_0} \quad (**)$$

from (\*).

Yet  $S=0$  in eqn. 5:

$$\partial_t A + \partial_S \left[ A (R(A, S))^{2/3} \sqrt{-\partial_S b} C_m^{-1} \right] = 0$$

Recall that  $R(A, S) = \frac{A}{P(A, S)} = \frac{A}{(w_0^{cs}) + 2Aw_0^{cs}(S)^{-1}}$

$$\partial_t A + \partial_S \left[ \underbrace{\frac{A^{5/3} \sqrt{-\partial_S b}}{C_m (w_0 + 2Aw_0)^{2/3}}} \right] = 0$$

$F(A, S)$

$$\partial_t A + \frac{\partial F}{\partial A} \frac{\partial A}{\partial S} + \frac{\partial F}{\partial S} = 0$$

Apply product rule to get

$$\begin{aligned} \frac{\partial F}{\partial A} &= \frac{5}{3} \frac{A^{2/3} \sqrt{-\partial_S b}}{C_m (w_0 + 2Aw_0)^{2/3}} - \frac{2}{3} \frac{A^{5/3} \sqrt{-\partial_S b}}{C_m (w_0 + \frac{2A}{w_0})^{2/3}} \frac{2}{w_0} \\ &= \frac{\sqrt{-\partial_S b}}{3C_m} \left[ \frac{5A^{2/3} w_0 (w_0 + \frac{2A}{w_0}) - 4A^{5/3}}{(w_0 + \frac{2A}{w_0})^{5/3} w_0} \right] \\ &= \frac{\sqrt{-\partial_S b}}{3C_m} \left[ \frac{5A^{2/3} w_0^2 + 10A^{5/3} - 4A^{5/3}}{(w_0 + \frac{2A}{w_0})^{5/3} w_0} \right] \\ &= \frac{\sqrt{-\partial_S b}}{3C_m} \left[ \frac{5A^{2/3} w_0 + 6A^{5/3}/w_0}{(w_0 + \frac{2A}{w_0})^{5/3}} \right] > 0! \end{aligned}$$

Now, apply chain rule & treat  $F_{js}$  as constant to get:

$$\frac{\partial F}{\partial S} = \frac{\partial S}{\partial S} \left[ \frac{A^{\frac{2}{3}} \sqrt{-\partial S b}}{C_m \left( w_0 + \frac{2A}{w_0} \right)^{\frac{5}{3}}} \right] = 10$$

$$= \frac{A^{\frac{2}{3}} \sqrt{-\partial S b}}{C_m} \cdot \frac{(-2)}{3 \left( w_0 + \frac{2A}{w_0} \right)^{\frac{2}{3}}} \frac{\partial}{\partial w_0} \left[ w_0 + \frac{2A}{w_0} \right]^{\frac{5}{3}}$$

$$= -\frac{2 A^{\frac{2}{3}} \sqrt{-\partial S b}}{3 C_m \left( w_0 + \frac{2A}{w_0} \right)^{\frac{5}{3}}} \left( 1 - \frac{2A}{w_0^2} \right) \frac{\partial w_0}{\partial S}$$

as required.

2.) If  $w_0(s)$  is independent of  $s_1$

$$\frac{\partial w_0}{\partial S} = 0 \quad \& \quad \frac{\partial F}{\partial S} = 0 \text{ as a consequence.}$$

Hence, the eqn becomes

$$\partial_t A + \frac{\partial F}{\partial A} \partial_S A = 0$$

In particular, we find eigenvalue  $\lambda$  by considering

$$\frac{\partial F}{\partial A} - \lambda = 0 \Leftrightarrow \lambda = \frac{\partial F}{\partial A}(A)$$

We have  $A = A(s, \epsilon)$  and  $H(A) := \frac{\partial F}{\partial A}$  with  
the governing eqn.

$$\partial_t[A] + H(A) \partial_s[A] = 0 \quad (\textcircled{C})$$

Now, consider change of coordinates

$y = \frac{s}{t}$ , then  $A = A(y(s, \epsilon))$  and

$$\partial_t[A] = \partial_y[A] \frac{\partial y}{\partial t} = -\partial_y[A] \frac{s}{t^2}$$

~~8~~

$$\partial_s[A] = \frac{1}{t} \partial_y[A]$$

Plug this into eqn. to obtain

$$-\partial_y[A] \frac{s}{t^2} + H(A) \frac{1}{t} \partial_y[A] = 0$$

$$\Rightarrow H(A) \partial_y[A] = \frac{s}{t} \partial_y[A]$$

Hence, there are two options: 1)  $\partial_y[A] = 0$   
2)  $s = H(A)t$ .

Now, consider a method of characteristic.

Take  $A = A(s(\epsilon), \epsilon)$  &  $s = s(\epsilon)$ . We get

$$\frac{dA(s(\epsilon), \epsilon)}{d\epsilon} = \frac{\partial A}{\partial \epsilon} + \frac{\partial A}{\partial s} \frac{\partial s}{\partial \epsilon}$$

(4.)

To match (C), require that  $\frac{\partial S}{\partial t} = H(A)$ . hence,  
we have a system of eqns:

$$\frac{dA}{dt} = 0 \quad \& \quad \frac{\partial S}{\partial t} = H(A).$$

Integrating first we get A is constant,  
subject to second. eqn. Hence, we can  
integrate ~~the~~ and get

$$S = \int H(A) dt = S_0 + H(A)t.$$

Hence, A is subject to initial condition:  
 $A = \begin{cases} A_L, & \text{for } S_0 < 0 \\ A_R, & \text{for } S_0 \geq 0 \end{cases}$

and we must have  
 $A(S, t) = \begin{cases} A_L, & \text{for } S < H(A_L)t \\ A_R, & \text{for } S \geq H(A_R)t \end{cases}$

However, there is a possible region  
 $S \in [H(A_L)t, H(A_R)t]$  if  $H(A_L) < H(A_R)$ .

This is a so called rerefaction  
wave and A is no longer constant in  
this region ( $\exists t, A \neq 0$ ). Hence, we are  
in case 2) & A can be found from  $H(A) = \frac{S}{t}$

(5.)

On the other hand,  $H(A_R) > H(A_L)$  gives a shockwave. Say that position of shockwave is  $s_\theta(\epsilon)$ . Then by integrating

$$\partial_t A + \partial_s F(A) = 0$$

$$\lim_{\epsilon \rightarrow 0} \int_{s_\theta(\epsilon) - \epsilon}^{s_\theta(\epsilon) + \epsilon} \partial_t A(s, \epsilon) dx + F(A_R) - F(A_L) = 0$$

Apply Leibnitz' rule:

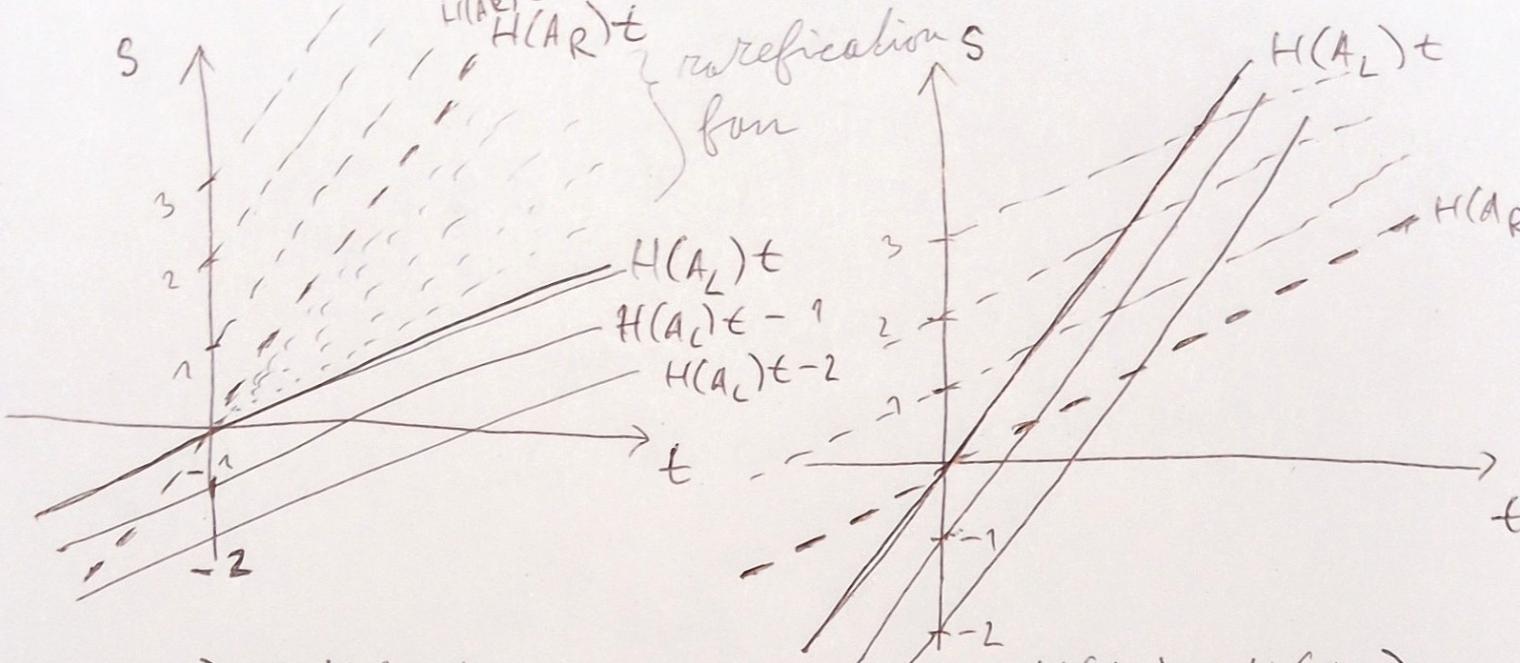
$$\lim_{\epsilon \rightarrow 0} \frac{d}{dt} \left[ \int_{s_\theta(\epsilon) - \epsilon}^{s_\theta(\epsilon) + \epsilon} A ds \right] + \frac{d s_\theta}{ds} \left( \lim_{\epsilon \rightarrow 0} -A(s_\theta + \epsilon, t) + A(s_\theta - \epsilon, t) \right) \\ = F(A_C) - F(A_R)$$

now  $\downarrow 0$  as it is ~~constant propagating~~  
the same shape propagating thru space. Then

$$\frac{ds_\theta}{ds} = \frac{F(A_R) - F(A_C)}{A_R - A_L}$$

~~$m^3/s^2$~~

gives the shock speed.

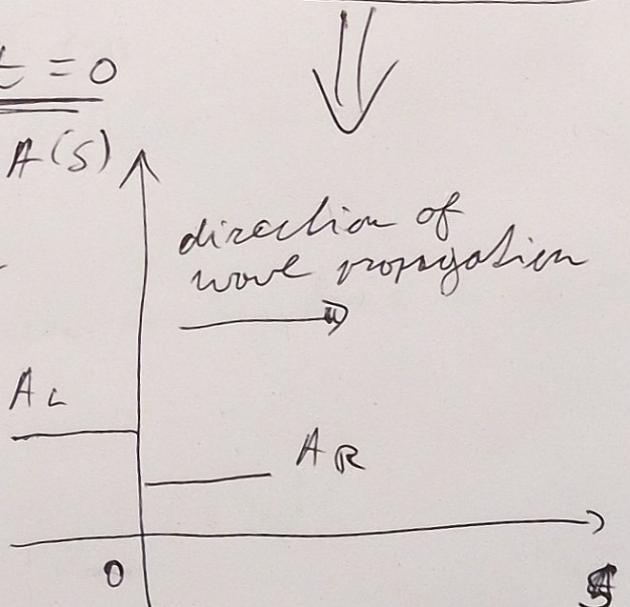
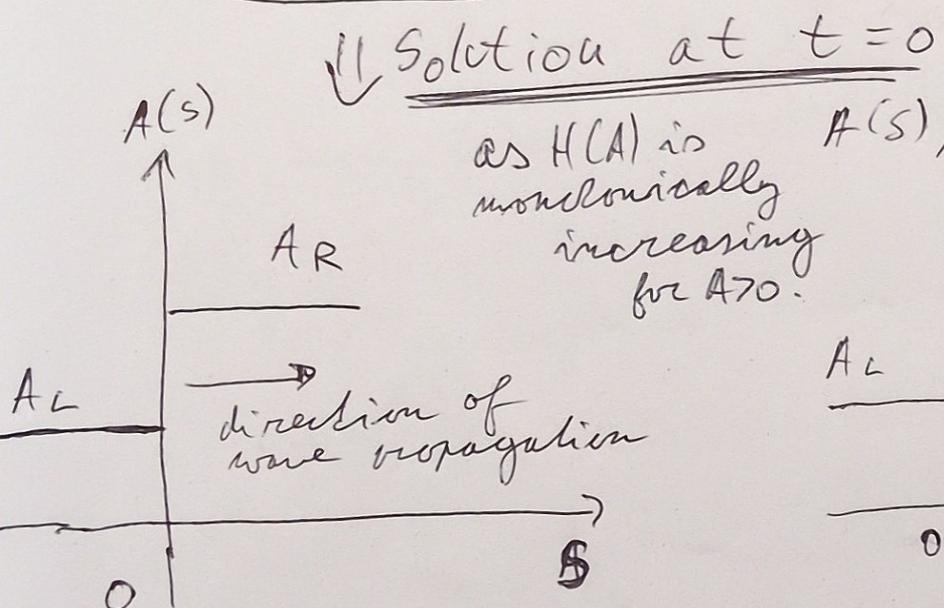


$$0 < H(A_L) < H(AR)$$

$$0 < H(AR) < H(A_L)$$

In this case  
non-characteristic  
lines are separated  
in 2 distinct  
regions

In this case we  
have crossing  
of characteristic  
lines for  $t \geq 0$



$A_L$  front is slower  
than  $A_R$  front. Hence,  
we have rarefaction  
wave.

$A_L$  front travels  
faster than  $A_R$  front  
 $\Rightarrow$  shock waves &  
discontinuity is maintained

3) For  $s=0$ , we have

$$\int_t A \neq \int_s F(A) = 0. \quad (?)$$

Now, define space discretization for  $s \in [0, L]$  into  $N$  cells as follows:

$$C_k = [s_{k-\frac{1}{2}}, s_{k+\frac{1}{2}}]$$

where  $s_{k-\frac{1}{2}}, s_{k+\frac{1}{2}}$  are cell boundaries of all  $C_k$  and  $h_k = s_{k+\frac{1}{2}} - s_{k-\frac{1}{2}}$  is the cell width.

There are  $N_t$  time intervals where

$$t \in [t_n, t_{n+1}] = I_n$$

for each time interval. Also denote

$$\Delta t_n = t_{n+1} - t_n.$$

Now, integrate (?) in each time & space

interval:

$$\int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \int_{t_n}^{t_{n+1}} \int_t A + \int_s F(A) dt ds = 0$$

$$\int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \cancel{\int_t A^{n+1}(s) - A^n(s) ds} + \int_{t_n}^{t_{n+1}} F(A(s_{k+\frac{1}{2}})) - F(A(s_{k-\frac{1}{2}})) dt$$

(6.) (7.)

Pearrange this, ~~and~~ divide by  $\Delta S_k$ , and

denote denote

$$\bar{A}_{k,n} = \frac{1}{\Delta S_k} \int_{S_k - \frac{1}{2}}^{S_k + \frac{1}{2}} A^n(s) ds = \frac{1}{\Delta S_k} \int_{C_k} A^n(s) ds$$

Hence, we get

$$\bar{A}_{k,n+1} - \bar{A}_{k,n} = - \frac{1}{\Delta S_k} \int_{I_n} F(A(s)) \Big|_{C_k} dt$$

$$\bar{A}_{k,n+1} = \bar{A}_{k,n} - \frac{1}{\Delta S_k} \int_{I_n} F(A,s) \Big|_{C_k} dt$$

as required.

Q: Why are only cell averages

at the previous time level  $n$

required in the Godunov method?

We require only cell averages, because  
the PDE was integrated and so we

avoid considering exact values.

However, it also means that we  
only obtain averages  $\bar{A}_{k,n+1}$  as the result,  
and NOT exact values.

(4) Godunov flux is given by:

$$F_{k+\frac{1}{2}}(\bar{A}_k^u, \bar{A}_{k+1}^u) = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} F_k(\bar{A}_k^u, \bar{A}_{k+1}^u)$$

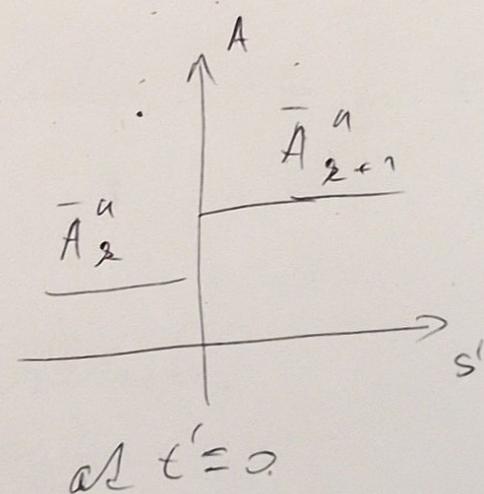
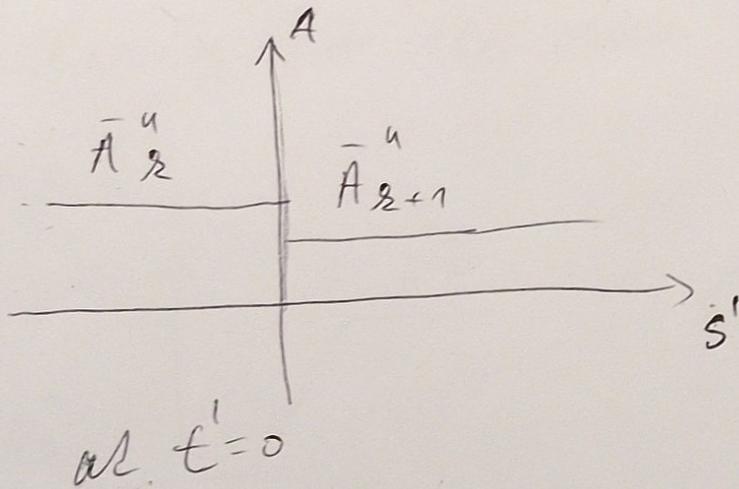
Now set up a Riemann problem in alternative coordinate system  $s' = s - \Delta t k + \frac{1}{2}$   
 $t' = t - t_n$ . We have

$$\partial_t A + \partial_s F(A) = 0$$

subject to initial condition

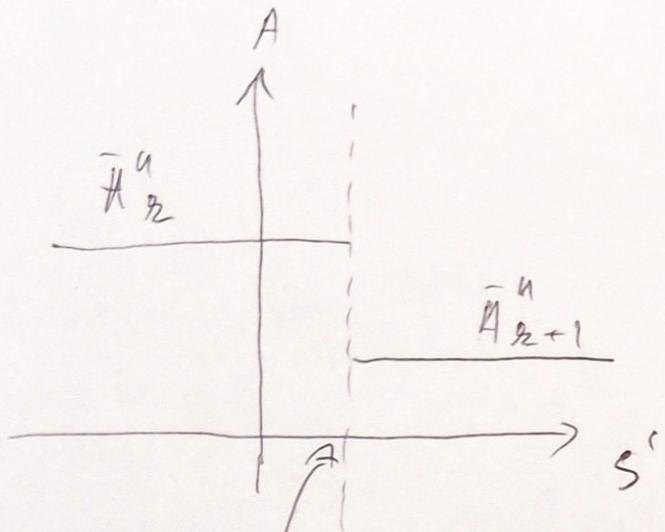
$$A(s', t'=0) = \begin{cases} \bar{A}_k^u, & \text{for } s' < 0 \\ \bar{A}_{k+1}^u, & \text{for } s' > 0 \end{cases}$$

Note that  $\bar{A}_k^u, \bar{A}_{k+1}^u$  are constants.



From exercise 2 we now that these are right traveling waves.

Hence, at  $t' = \varepsilon$  for  $\varepsilon > 0$  we get



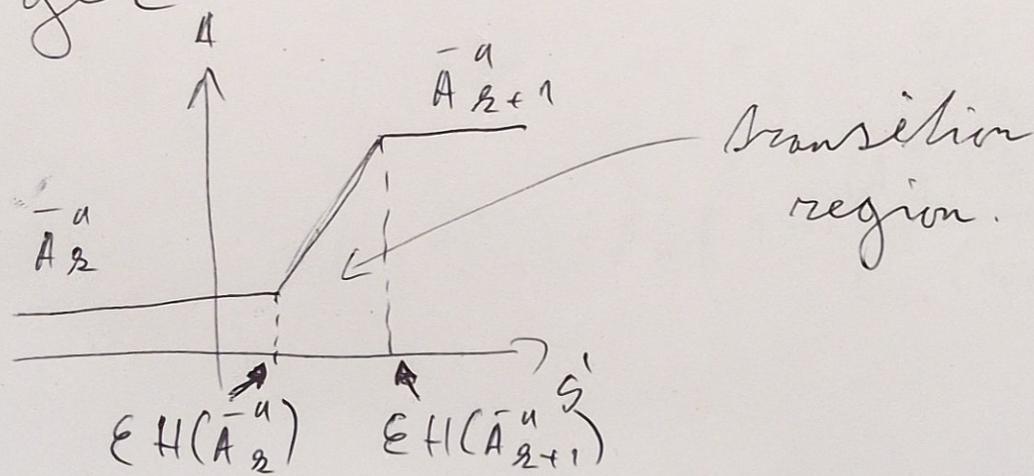
$$S = \varepsilon (H(\bar{A}_2^u) - H(\bar{A}_{2+1}^u))$$

This is the shockwave solution for  $H = \frac{\partial F}{\partial A}$  and

Gordunov flux becomes

$$\begin{aligned} \bar{F}_{2+\frac{1}{2}}(\bar{A}_2^u) &= \frac{1}{2\varepsilon} \int_{t_n}^{t_{n+1}} F(\bar{A}_2^u) dt \\ &= \frac{1}{2\varepsilon} \left( F(\bar{A}_2^u) (t_{n+1} - t_n) \right) \\ &= F(\bar{A}_2^u). \end{aligned}$$

Similarly, for the other solution  $t' = \varepsilon$  for  $\varepsilon = 0$  we get



However, at  $s' = 0$  or  $t' = \epsilon$  we get

$$\bar{F}_{s-\frac{\epsilon}{2}}(\bar{A}_s^u) = F(\bar{A}_s^u)$$

again. In both cases,

$$F(A) = \frac{A^{\frac{2}{3}} \sqrt{-\Delta s b}}{C_m (w_0 + \frac{2A}{w_0})^{\frac{2}{3}}}$$

from exercise ①.

Hence, Godunov flux

$$\bar{F}_{s-\frac{\epsilon}{2}}(\bar{A}_{s-1}^u) = F(\bar{A}_{s-1}^u).$$

Therefore the final scheme becomes

$$\bar{A}_s^{u+1} = \bar{A}_s^u - \frac{\Delta t}{\Delta s_s} (F(\bar{A}_s^u) - F(\bar{A}_{s-1}^u))$$

This is clearly an upwind scheme as ~~post~~ post information is considered.

$$\begin{aligned}\bar{A}_s^{u+1} &= F(\bar{A}_{s-1}^u) + \left( \bar{A}_s^u - \frac{\Delta t}{\Delta s_s} F(\bar{A}_s^u) \right) \\ &= F(\bar{A}_{s-1}^u) + \bar{A}_s^u \left( 1 - \frac{(\bar{A}_s^u)^{\frac{2}{3}} \sqrt{-\Delta s b}}{C_m (w_0 + \frac{2A}{w_0})^{\frac{2}{3}}} \frac{\Delta t}{\Delta s_s} \right)\end{aligned}$$

(12.)

$$5) \quad \min_{\lambda} \frac{h_k}{|\lambda_k|} = \min_{\lambda} \frac{\frac{h_k}{\sqrt{1-350}} \left( 5w_0 \bar{A}_k^{\frac{2}{3}} + 6 \bar{A}_k^{\frac{5}{3}} w_0 \right)}{3cm \left( w_0 + 2 \bar{A}_k^{\frac{2}{3}} w_0 \right)^{\frac{5}{3}}}$$

$$= \min_{\lambda} \frac{\frac{h_k}{\partial F}}{\frac{\partial F}{\partial A} \Big|_{A=\bar{A}_k^u}}$$

as  $\frac{\partial F}{\partial A}$  is monotonically increasing, we can simply pick  $\max_k \frac{\partial F}{\partial A} \Big|_{A=\bar{A}_k^u}$  & min  $h_k$ . This however could be too restrictive. In particular,

$$\gamma_k = H(A) \Big|_{\bar{A}_k^u} = \frac{\partial F}{\partial A} \Big|_{A=\bar{A}_k^u}$$

is the speed of the wave and we are trying to prevent adjacent Riemann problems from communicating with each other. Hence, impose

$$\Delta t < CFL \min_{\lambda} \frac{h_k}{|\lambda_k|}$$

6) It makes sense to impose  $\bar{A}_1 > 0$

&  $\bar{A}_{N_k} > 0$  to maintain physically.

For Godunov fluxes

$$\bar{F}_{2+\frac{1}{2}}(\bar{A}_2^u, \bar{A}_{2+1}^u) = \frac{1}{\Delta t} \int_{t_n}^{t_n + \Delta t} F_{2+\frac{1}{2}}(\bar{A}_2^u, \bar{A}_{2+1}^u) dt \\ = F(\bar{A}_2^u)$$

if and only if CFL condition is satisfied  
to prevent Riemann solutions from cells further  
away affecting the integral.

This of course depends on the wave  
propagation speed  $\gamma = H(A) = \frac{\partial F(A)}{\partial A}$  in each  
cell.

Furthermore require  $\bar{A}_1 < \bar{A}_{N_k}$   
to prevent formation of shockwaves.