

$$h(A, s) \omega_o(s) = A$$

$$\rightarrow \boxed{h(A, s) = \frac{A}{\omega_o(s)}}$$

$$P(A, s) = \omega_o(s) + 2h(A, s)$$

$$\boxed{P(A, s) = \omega_o(s) + \frac{2A}{\omega_o(s)}}$$

$$\partial_r A + \partial_s \left(A R^{\frac{2}{3}} \sqrt{-\partial_s b} \right)$$

$$= \partial_r A + \partial_s \left(\frac{A^{\frac{s}{3}} \sqrt{-\partial_s b}}{C_m p^{\frac{2}{3}}} \right)$$

$$= \partial_r A + \partial_s F$$

$$\text{as } P = \omega_0(s) + \frac{24}{\omega_b(s)}$$

$$F = \frac{A^{\frac{s}{3}} \sqrt{-\partial_s b}}{C_m} \times \frac{1}{\left(\omega_0 + \frac{24}{\omega_0}\right)^{\frac{2}{3}}}$$

$$\partial_r A + \partial_s F(A, s)$$

$$= \partial_r A + \frac{\partial F}{\partial A} \partial_s A + \frac{\partial F}{\partial s} = 0$$

$$\frac{\partial F}{\partial A} = \frac{5}{3} A^{\frac{s+3}{3}} \frac{\sqrt{-\partial_s b}}{C_m} \times \frac{1}{\left(\omega_0 + \frac{24}{\omega_0}\right)^{\frac{2}{3}}}$$

$$- \frac{2}{3} A^{\frac{s+3}{3}} \frac{\sqrt{-\partial_s b}}{C_m} \times \frac{2/\omega_0}{\left(\omega_0 + \frac{24}{\omega_0}\right)^{\frac{5}{3}}}$$

$$= \frac{\int -\partial_s b}{3C_m} \times \frac{(5\omega_0 + 10A/\omega_0 - 4A/\omega_0^{2/3})A^{2/3}}{(\omega_0 + 2A/\omega_0)^{5/3}}$$

$$\boxed{\frac{\partial F}{\partial A} = \frac{\int -\partial_s b}{3C_m} \frac{(5\omega_0 A^{2/3} + 6A/\omega_0^{5/3})}{(\omega_0 + 2A/\omega_0)^{5/3}} > 0}$$

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial \omega_0} \frac{d\omega_0}{ds}$$

$$= \frac{A \int \partial_s b}{C_m} \left(-\frac{2}{3} \right) \left(1 - \frac{2A}{\omega_0^2} \right) \times \frac{1}{(\omega_0 + 2A/\omega_0)^{5/3}}$$

$$\boxed{\frac{\partial F}{\partial s} = -\frac{2 \int \partial_s b}{3C_m} \frac{A^{5/3}}{(\omega_0 + 2A/\omega_0)^{5/3}} \left(1 - \frac{2A}{\omega_0^2} \right) \frac{d\omega_0}{ds}}$$

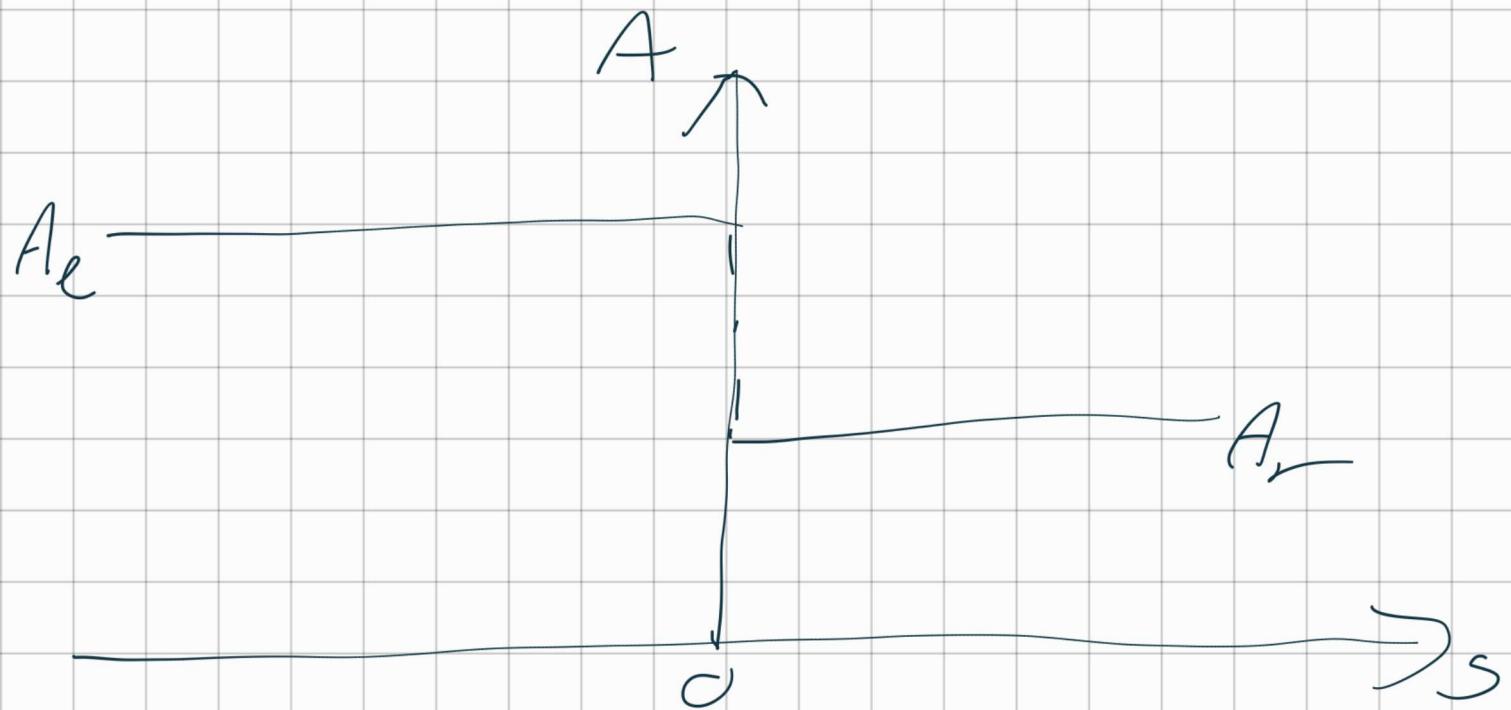
so system can be rewritten as in
(S)

$$2). \frac{d\omega_0}{ds} \sim 0$$

$$\text{so } \partial_r A + \frac{\partial F}{\partial A} \partial_s A = 0$$

$$\text{In this case, } \lambda = \frac{\partial F}{\partial A} = \sqrt{-\frac{\partial_s b}{3c_m}} \frac{(5\omega_0 A^{2/3} + 6A^{4/3}/\omega_0)}{(1 + 2A/\omega_0)^{5/3}}$$

letting $s=0$ where discontinuity is at $t=0$



$$\text{at } t=0, \lambda = \begin{cases} \lambda(A_r) & s>0 \\ \lambda(A_e) & s<0 \end{cases}$$

If $A_l > A_r$, then left side travels faster
than right side

$$\frac{ds}{dt} = \frac{\partial F}{\partial A}$$

Over discontinuity

$$\int_{A_e}^{A_r} \frac{ds}{dt} dA = \int_{A_e}^{A_r} \frac{\partial F}{\partial A} dt$$

$$\frac{ds}{dt} = \frac{F(A_r) - F(A_e)}{A_r - A_e} = u$$

For $A_l > A_r$

$$A = \begin{cases} A_e & s - ut < 0 \\ A_r & s - ut \geq 0 \end{cases}$$

$$\text{where } n = \underbrace{F(A_r) - F(A_e)}_{A_r - A_e}$$

If $A_r > A_L$, then right side travels faster
than left



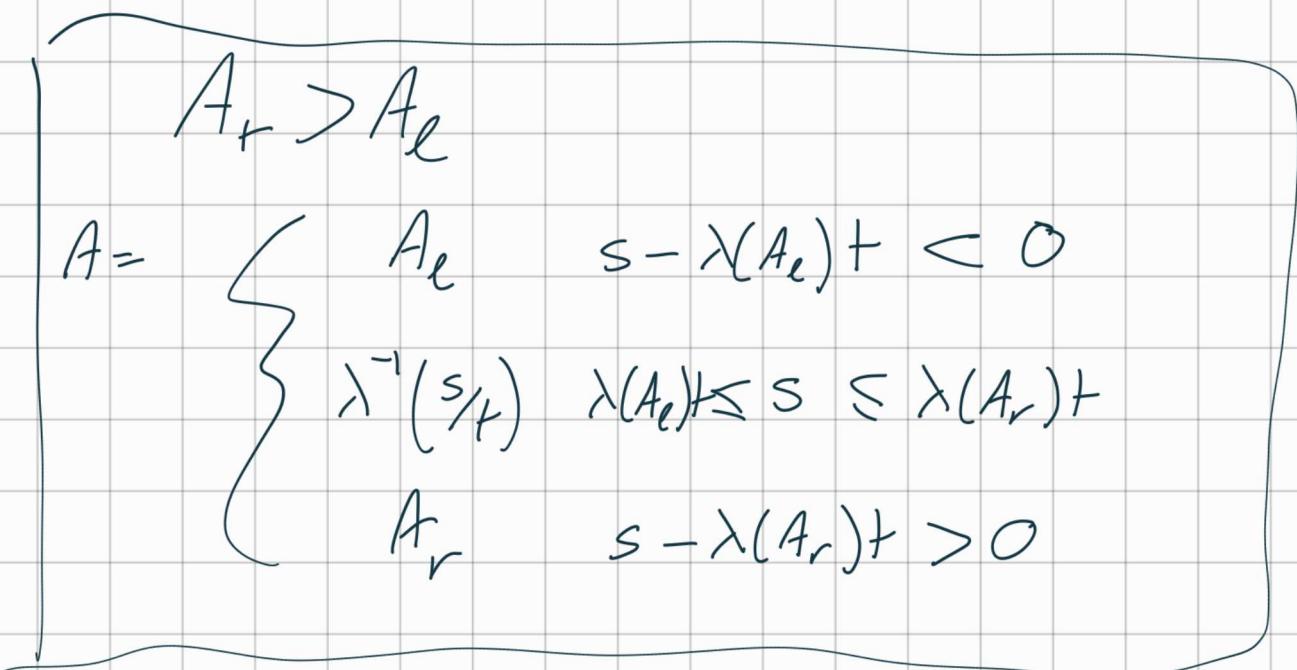
\rightarrow so discontinuity is no longer present.

A is constant in time along $\frac{ds}{dt} = \lambda(A)$

\rightarrow and so in rarefaction region, $\frac{ds}{dt} = \frac{s}{F}$ so

A can be described as a function of $\frac{s}{F}$

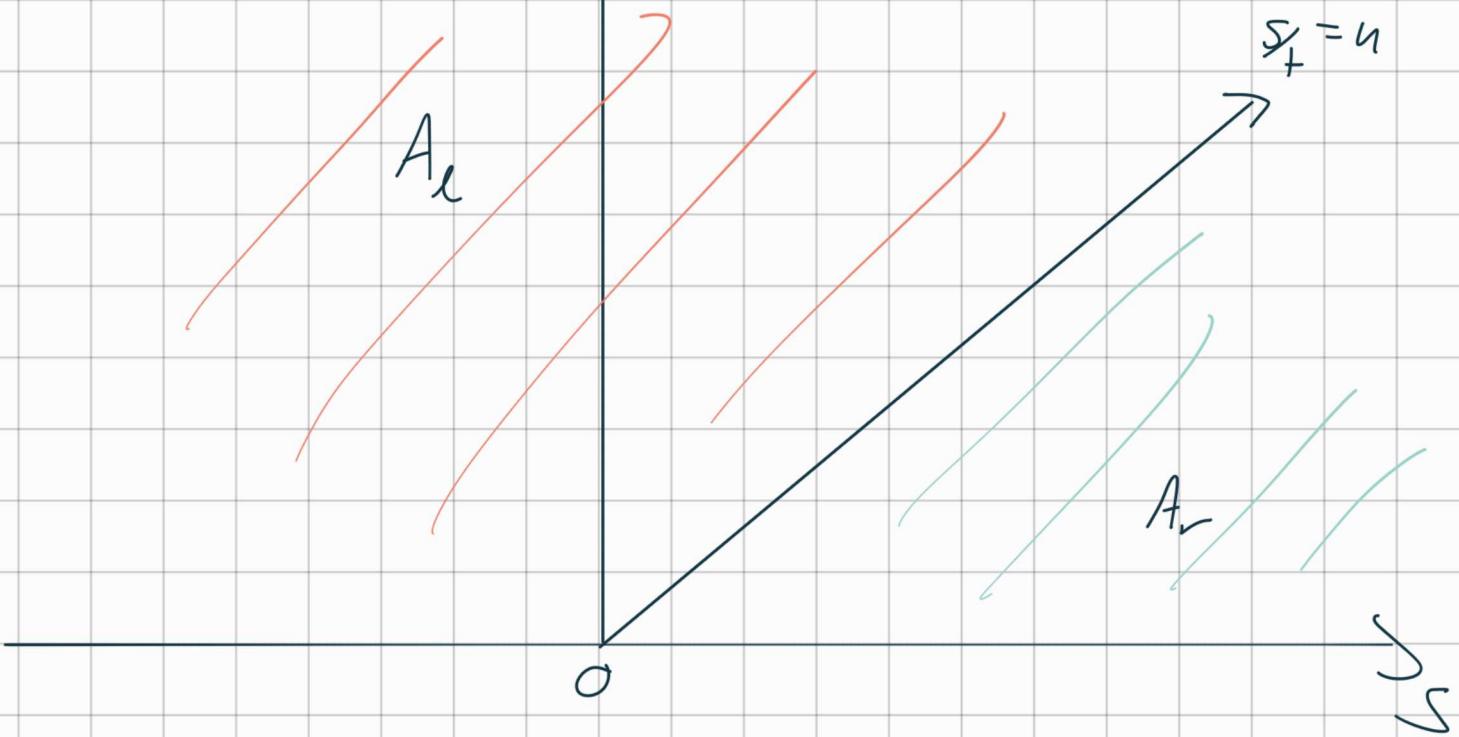
$A = \lambda^{-1}\left(\frac{s}{F}\right)$ where λ^{-1} = inverse of $\frac{\partial F}{\partial A}$ function



$$A_e > A_r$$

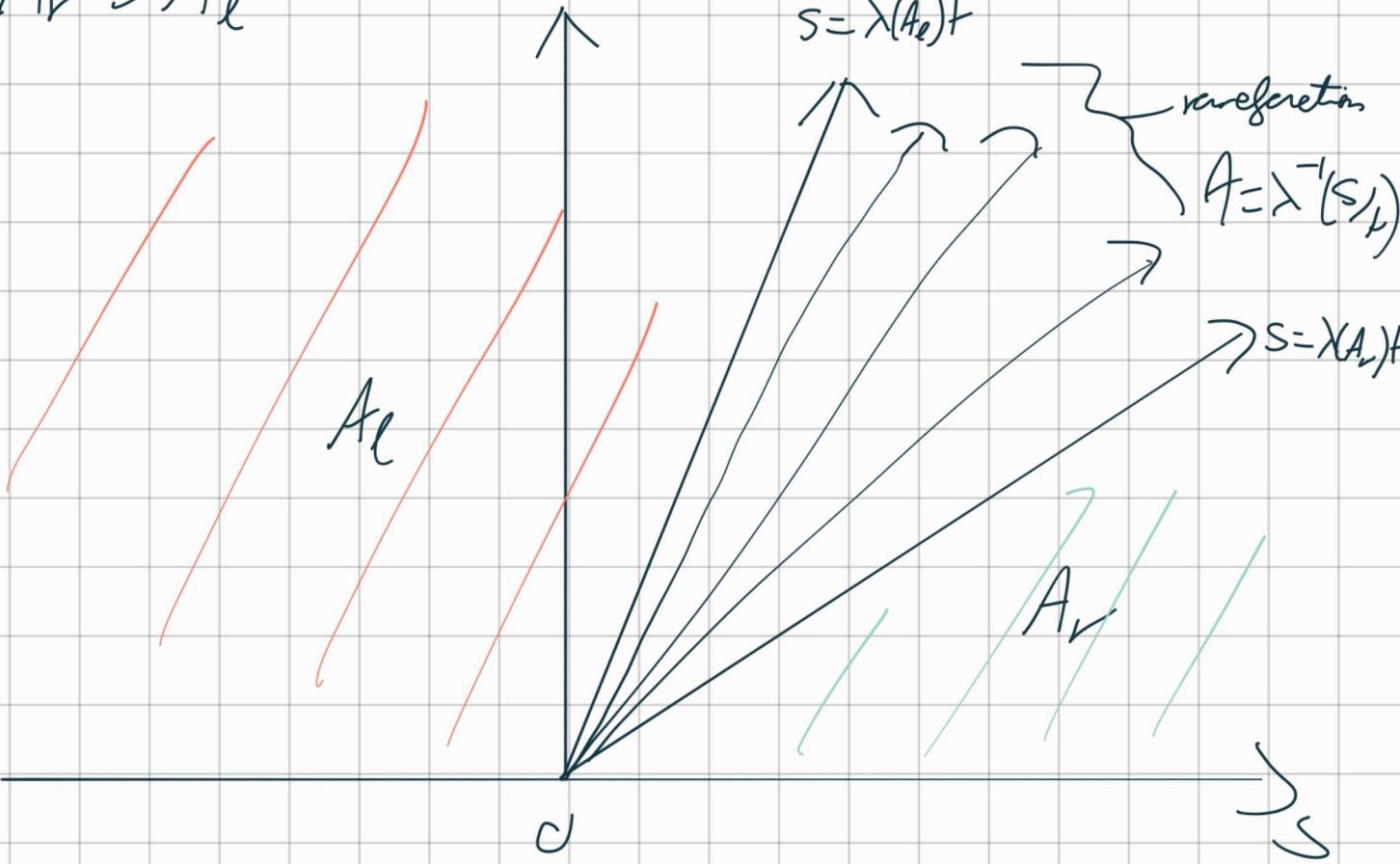
+

$$v = \frac{F(A_r) - F(A_e)}{A_r - A_e}$$



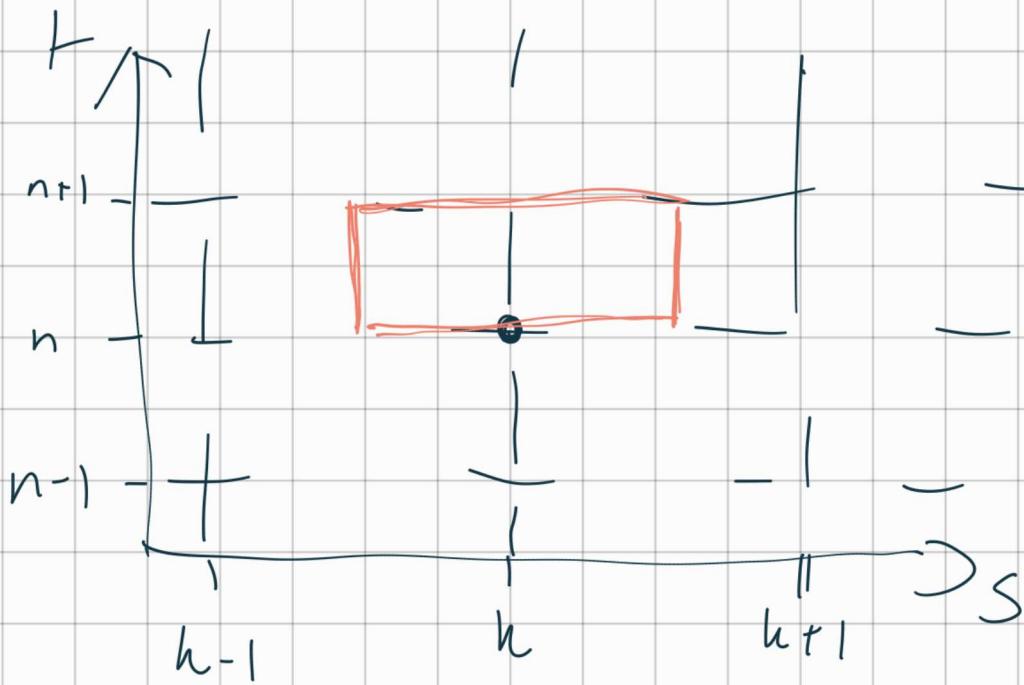
$$A_r > A_e$$

$$s = \lambda(A_e) +$$



Only considered cases where A_r , A_d are
positive, as area cannot be negative.

$$3) \partial_t A + \partial_s F(A, s) = 0$$



Integrating over \square

$$\int_{t_n}^{t_{n+1}} \int_{S_{h-1/2}}^{S_{h+1/2}} \partial_t A \, ds \, dt = - \int_{t_n}^{t_{n+1}} \int_{S_{h-1/2}}^{S_{h+1/2}} \partial_s F(A, s) \, ds \, dt$$

$$\int_{S_{h-1/2}}^{S_{h+1/2}} A^{n+1} - A^n \, ds = - \int_{t_n}^{t_{n+1}} F_{h+1/2}(A, s) - F_{h-1/2}(A, s) \, dt$$

$$\int_{S_{h-1/2}}^{S_{h+1/2}} A^n \, ds = \Delta S_h \bar{A}_h^n \text{ where } \bar{A}_h^n = \text{cell average of } A \text{ in cell } k.$$

$$\rightarrow \Delta S_n (\bar{A}_n^{n+1} - \bar{A}_n^n) = - \int_{t_n}^{t_{n+1}} F_{n+\frac{1}{2}}(A, s) - F_{n-\frac{1}{2}}(A, s) dt$$

$$\bar{A}_n^{n+1} = \bar{A}_n^n - \frac{1}{\Delta S_n} \left(\int_{t_n}^{t_{n+1}} F_{n+\frac{1}{2}}(A, s) - F_{n-\frac{1}{2}}(A, s) dt \right)$$

$$F_{n+\frac{1}{2}}(A, s) = F_{n+\frac{1}{2}}(\bar{A}_n^n, \bar{A}_{n+1}^n)$$

and $F_{n+\frac{1}{2}}(\bar{A}_n^n, \bar{A}_{n+1}^n) \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} F_{n+\frac{1}{2}}(A, s) dt$

$$\bar{A}_n^{n+1} = \bar{A}_n^n - \frac{\Delta t}{\Delta S_n} \left[F_{n+\frac{1}{2}}(\bar{A}_n^n, \bar{A}_{n+1}^n) - F_{n-\frac{1}{2}}(\bar{A}_{n-1}^n, \bar{A}_n^n) \right]$$

Godunov method relies on the solution at the previous time step to be piecewise constant, so only need to use cell average at previous time step

(e.g. A^n are constant in time for $t_n < t < t_{n+1}$).

$$4), \quad F(A_s) |_{s=s_{n+1/2}} = F_{n+1/2}(\bar{A}_n^n, \bar{A}_{n+1}^n)$$

$s_{n+1/2}$ corresponds to $s=0$

→ add on both diagrams in part 2,

$$A = A_e \text{ at } s=0$$

so Godunov flux $= F(A_e)$
 $(F_{n+1/2} = F(\bar{A}_n^n))$

This is completely upward, makes sense as

we know $\frac{\partial F}{\partial A} > 0$ always.

5)

$$\Delta t < \text{CFL} \min \left(\frac{h_n}{\lambda_n} \right)$$

$$\lambda_n = \left. \frac{\partial F}{\partial A} \right|_{A_n}$$

$$\frac{\partial F}{\partial A} = \frac{\sqrt{-\partial_s b}}{3C_m} \frac{\left(5\omega_0 A^{2/3} + 6A^{5/3}/\omega_0 \right)}{\left(\omega_0 + 2A/\omega_0 \right)^{5/3}}$$

$$\boxed{\Delta t < \text{CFL} \min \left(h_n \cdot 3C_m \left(\omega_0 + \frac{2\bar{A}_n}{\omega_0} \right)^{5/3} \right)}$$

$\sqrt{-\partial_s b} \left(5\omega_0 A_n^{2/3} + 6A_n^{5/3}/\omega_0 \right)$

6) If there is a constant discharge into the river

$$\bar{A}_{-1} \text{ needs to be set so } F(\bar{A}_{-1}) = Q_0$$

As this is an upwind scheme, \bar{A}_{N_n} would not have an effect on the flux in the river

so \bar{A}_{N_n} does not need to be set.

7). Test Case C:

• minimum time step chosen, $\Delta t = \frac{CFL \Delta x}{C_{\text{co}}}$
 $(\Delta x = 0.5m)$

$$= \frac{0.125 \times 0.5}{C_{\text{co}}}$$

• Reran for different CFL values

$$CFL = 0.125, 0.25, 0.5, 1$$

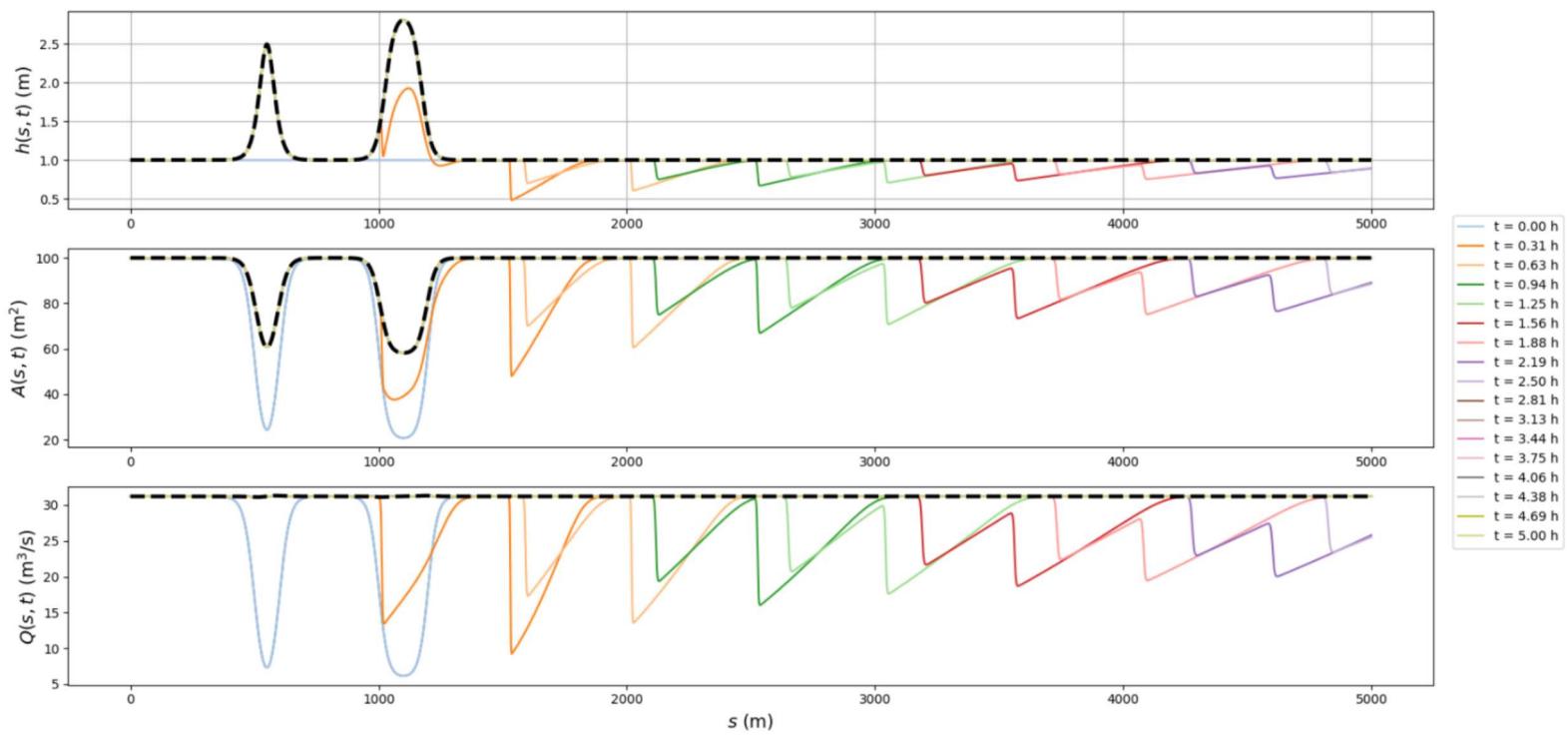
$$\rightarrow \Delta t = \Delta t_{\min}, 2\Delta t_{\min}, 4\Delta t_{\min}, 8\Delta t_{\min}$$

• Then did similar for Δx (and kept $\Delta t = \Delta t_{\min}$)

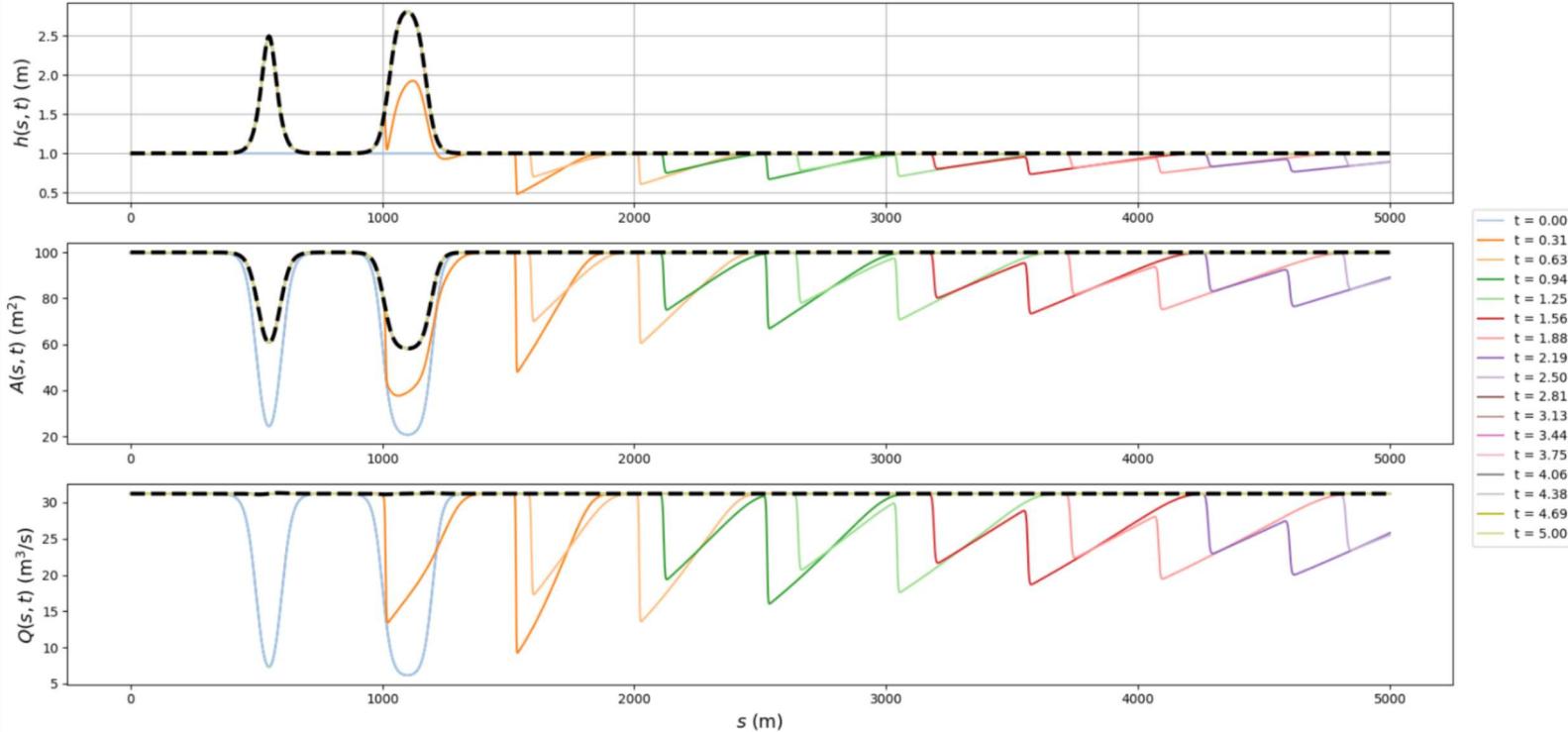
$$\Delta x_{\min} = 0.5m$$

$$\Delta x = 0.5m, 1m, 2m, 4m$$

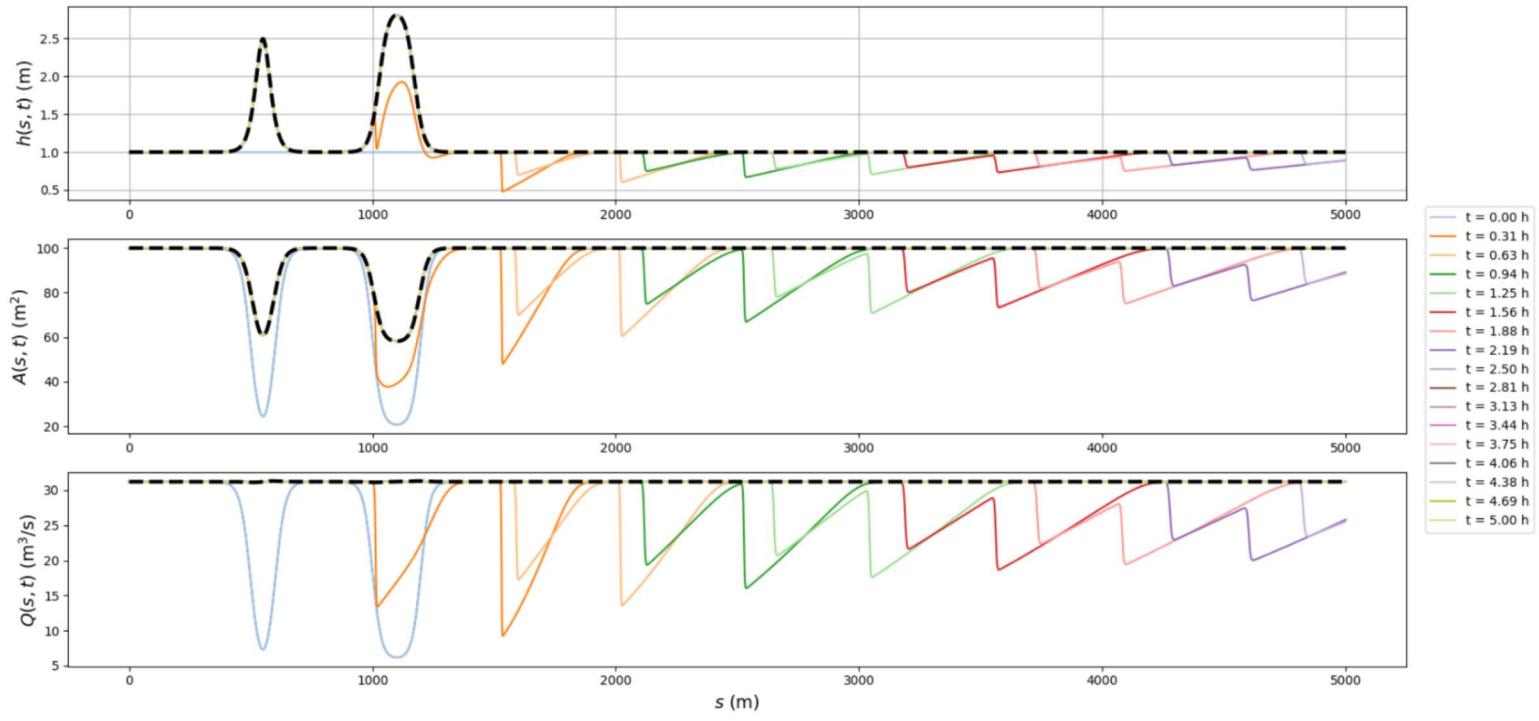
$$\Delta x = \Delta x_{\min} \quad \Delta t = \Delta t_{\min} \quad (CFL=0.125)$$



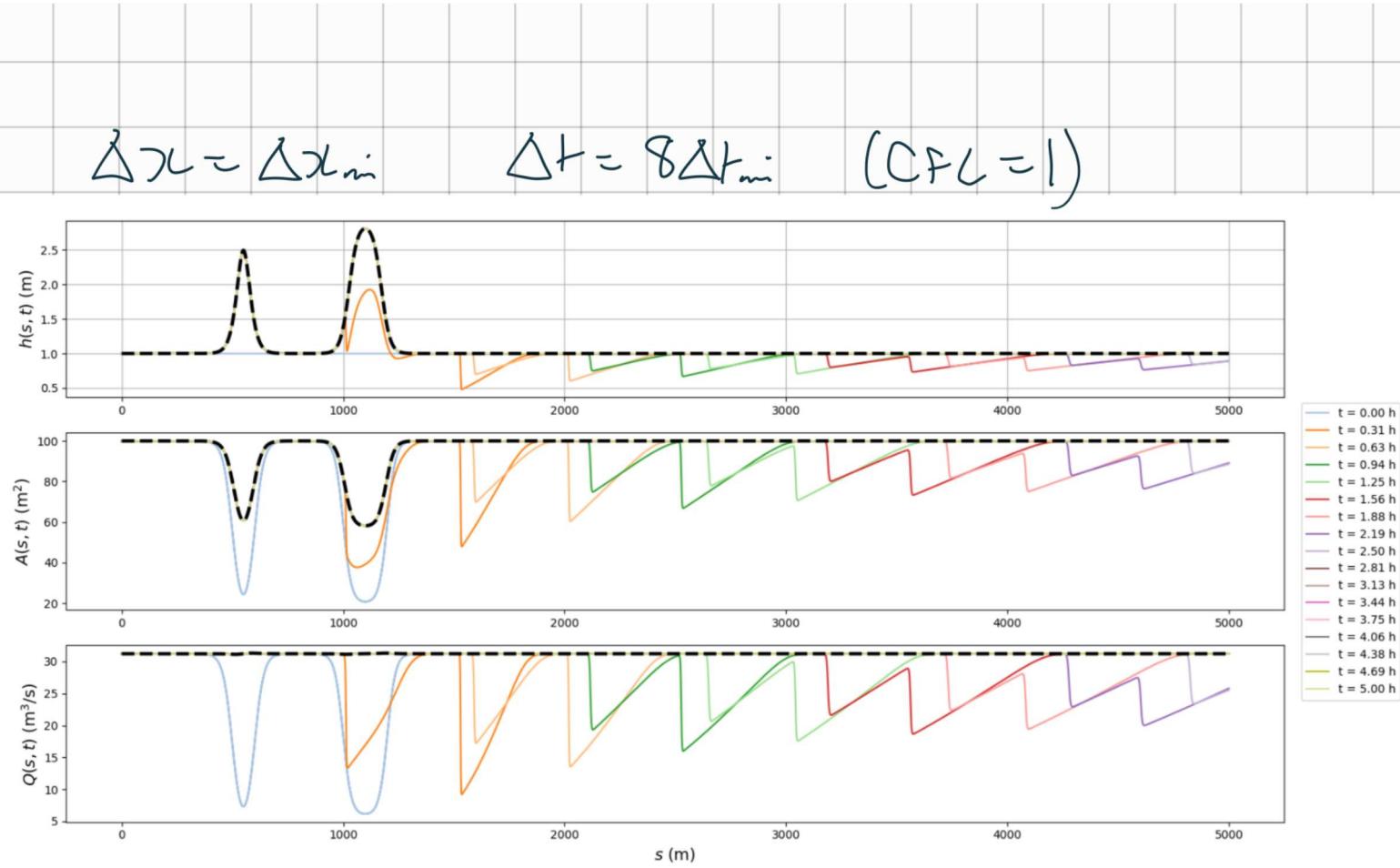
$$\Delta x = \Delta x_{\min} \quad \Delta t = 2\Delta t_{\min} \quad (CFL=0.25)$$



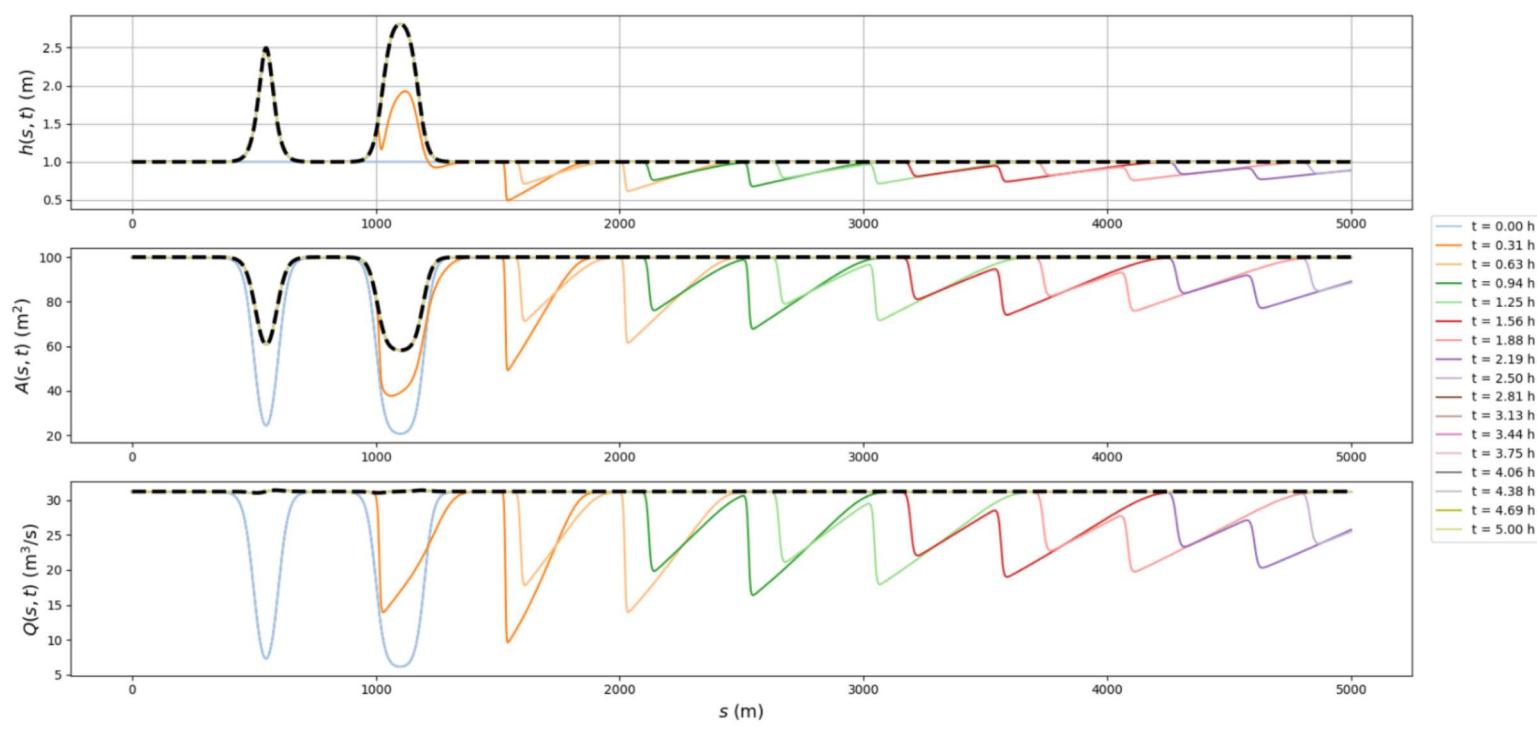
$$\Delta x = \Delta x_{\min} \quad \Delta t = 4 \Delta t_{\min} \quad (\text{CFL} = 0.5)$$



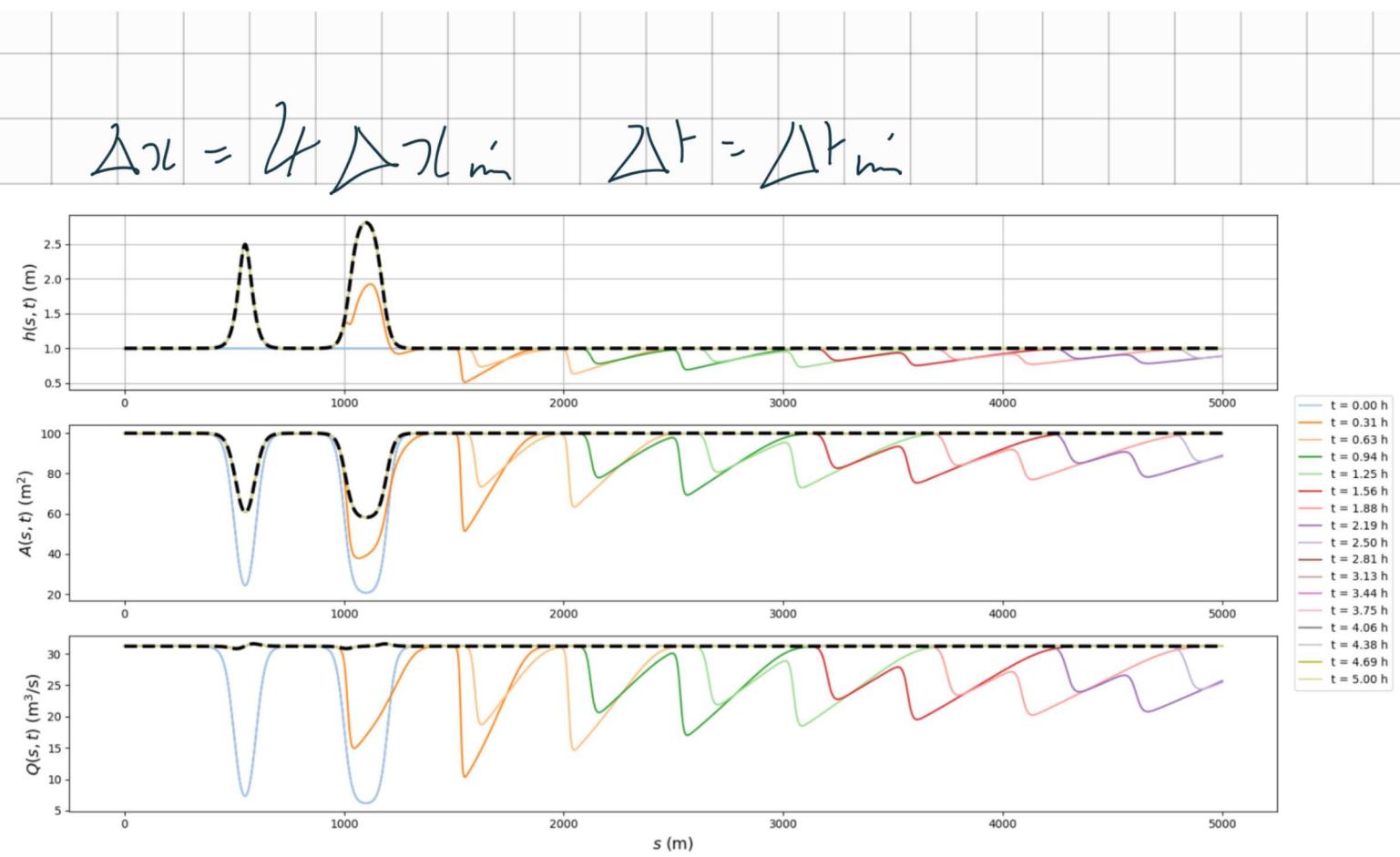
$$\Delta x = \Delta x_{\min} \quad \Delta t = 8 \Delta t_{\min} \quad (\text{CFL} = 1)$$



$$\Delta \tau = 2 \Delta \tau_{\min} \quad \Delta t = \Delta t_{\min}$$

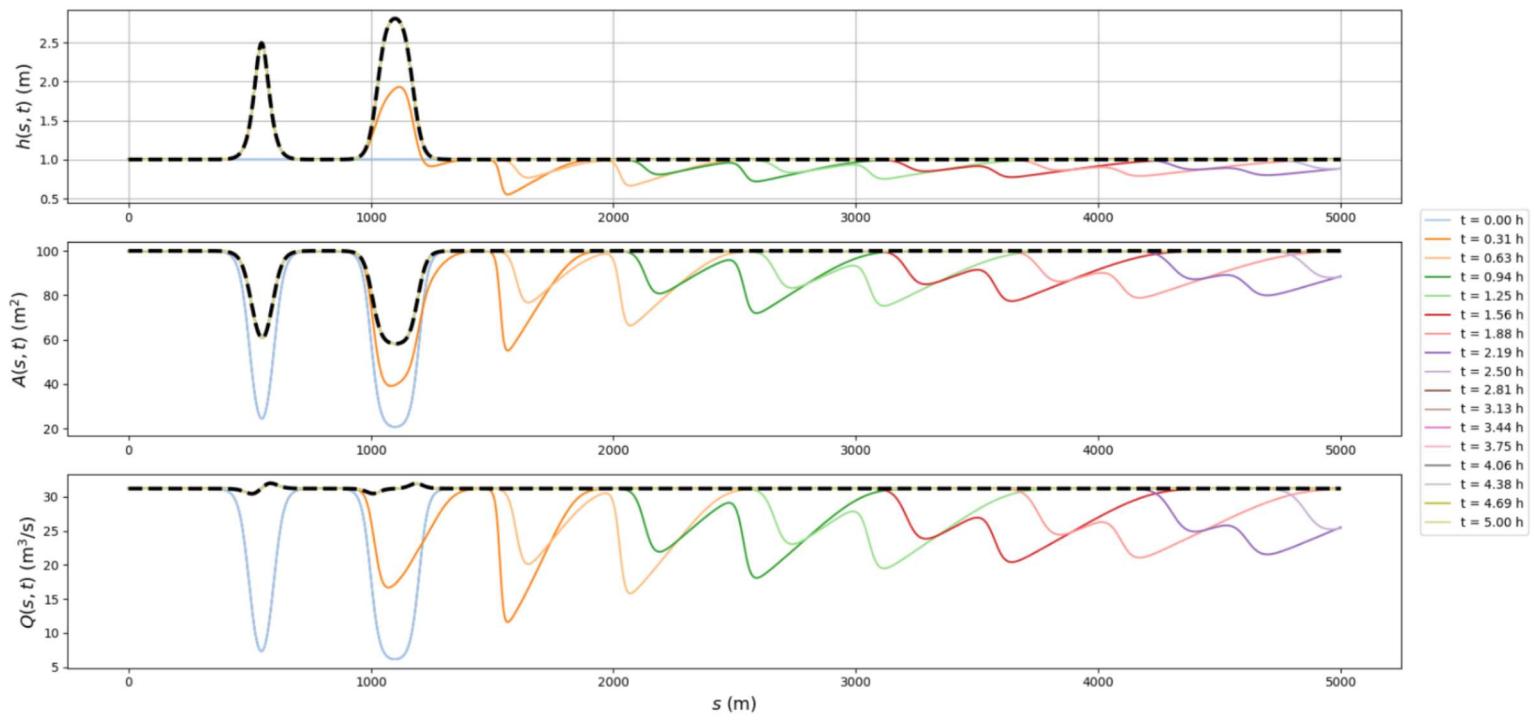


$$\Delta \tau = 4 \Delta \tau_{\min} \quad \Delta t = \Delta t_{\min}$$



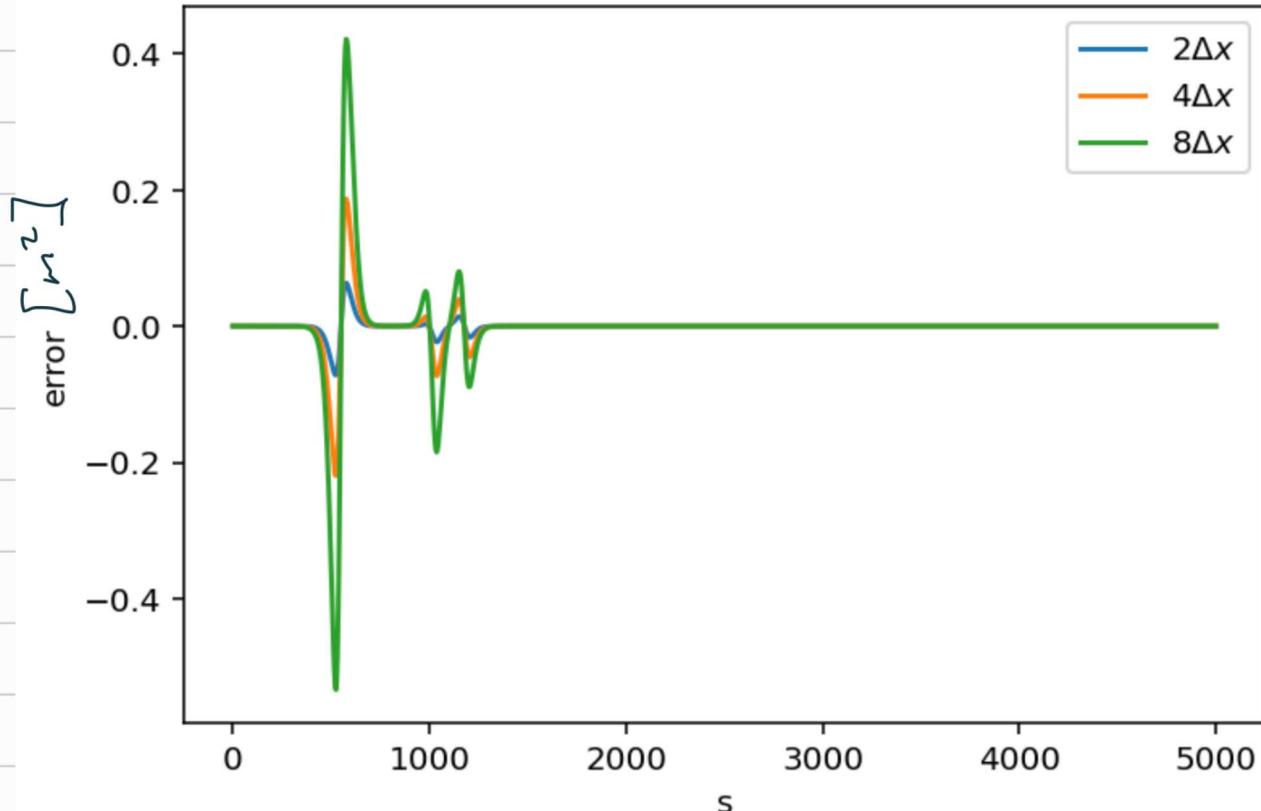
$$\Delta x = 8 \Delta x_{\min}$$

$$\Delta t = \Delta t_{\min}$$

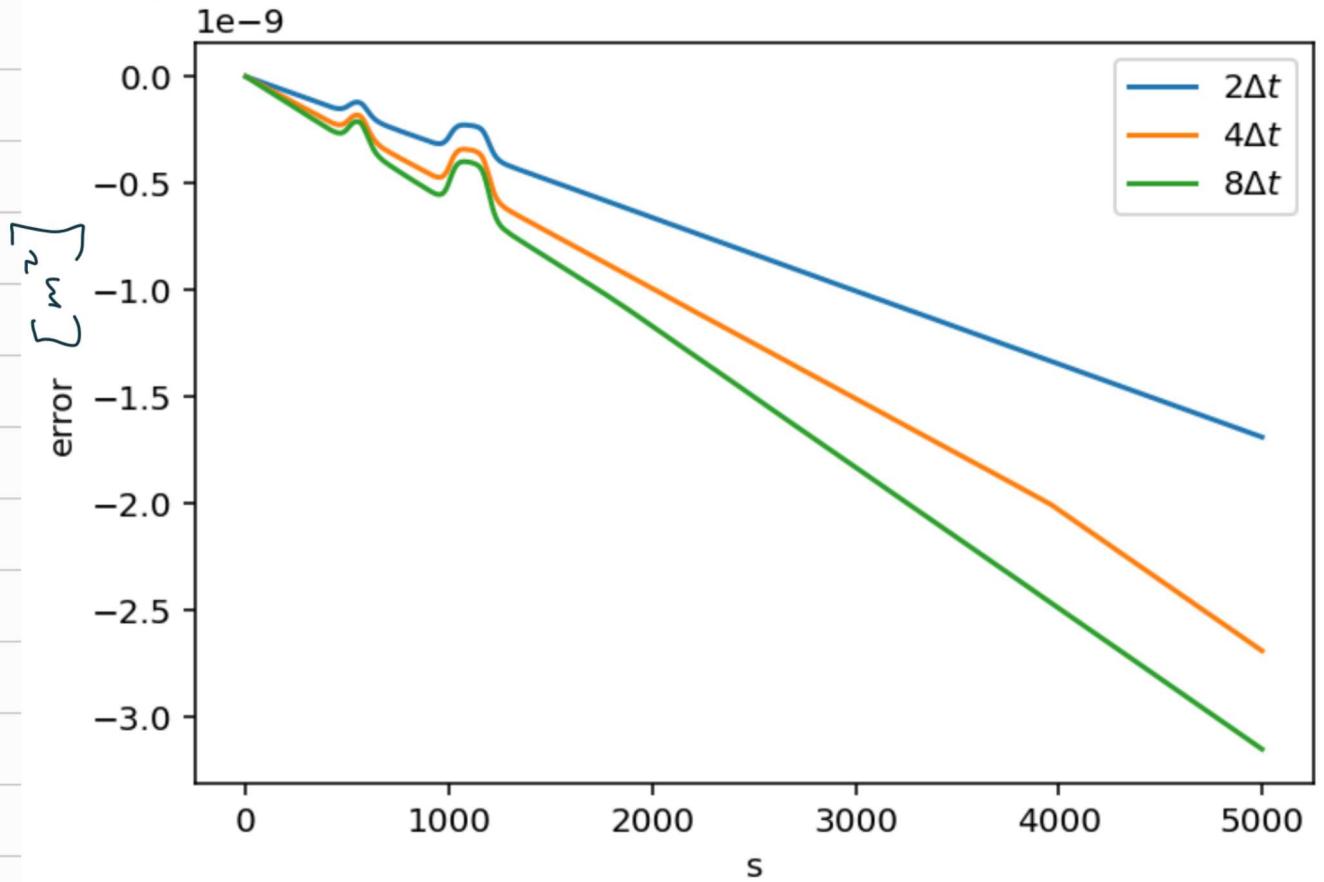


$$\text{error} = A(\Delta x = \Delta x_{\min}, \Delta t = \Delta t_{\min}) - A_{\min}$$

at fixed $\Delta t = \Delta t_{\min}$



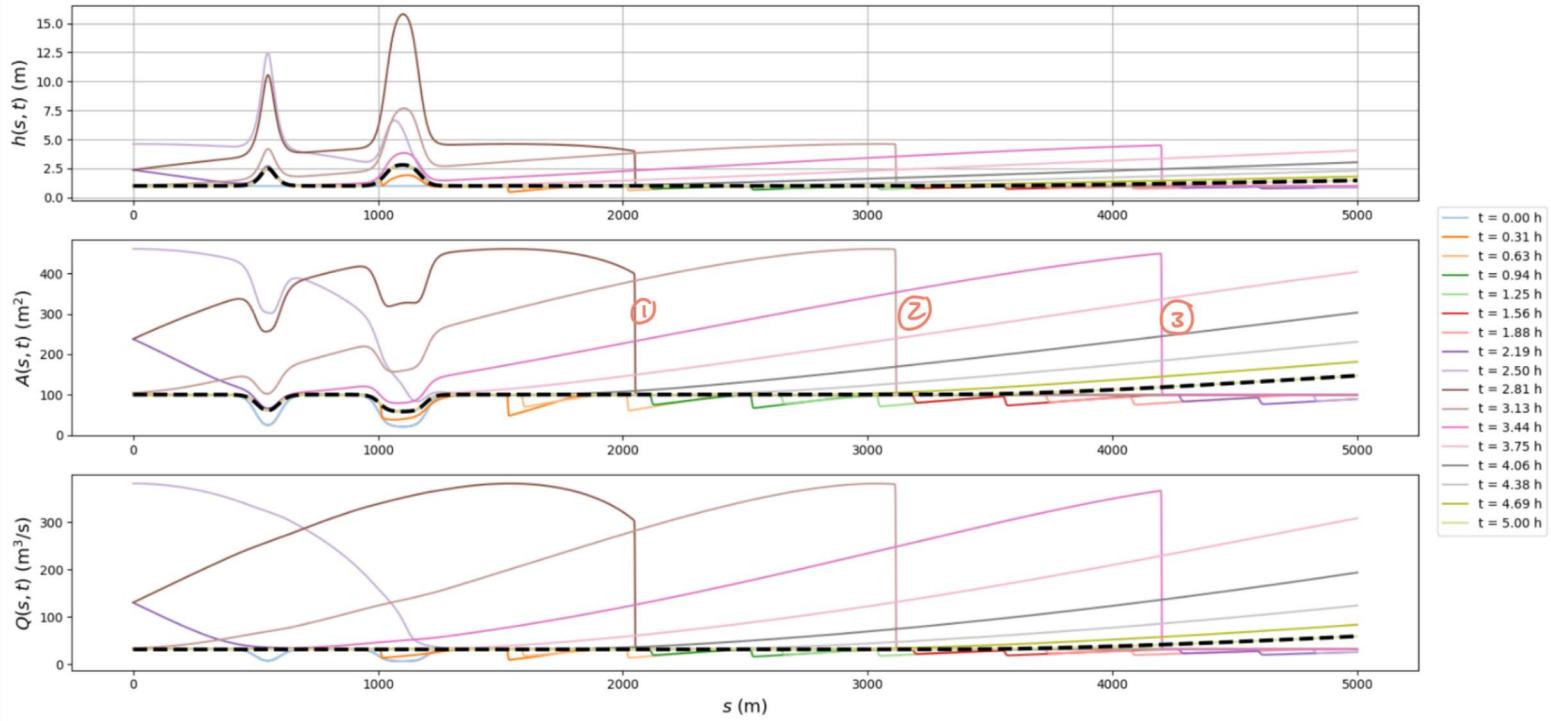
At fixed $\Delta x = \Delta x_m$



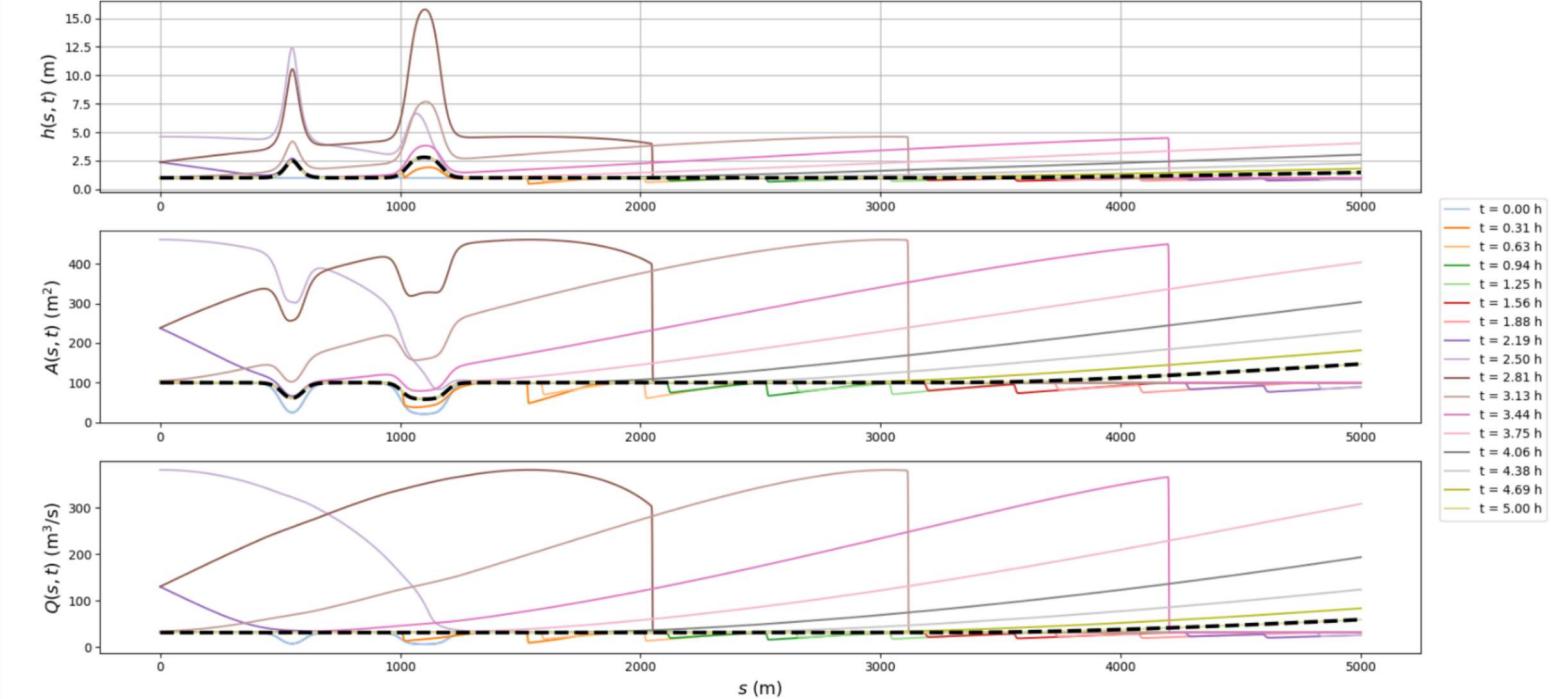
- 1st order in time and space
- Converges in space and time, as seen in error signals.
→ although comparing with a numerical solution rather than exact solution.
- More issues at points where width changes.
regarding convergence.

Did same for test case 1

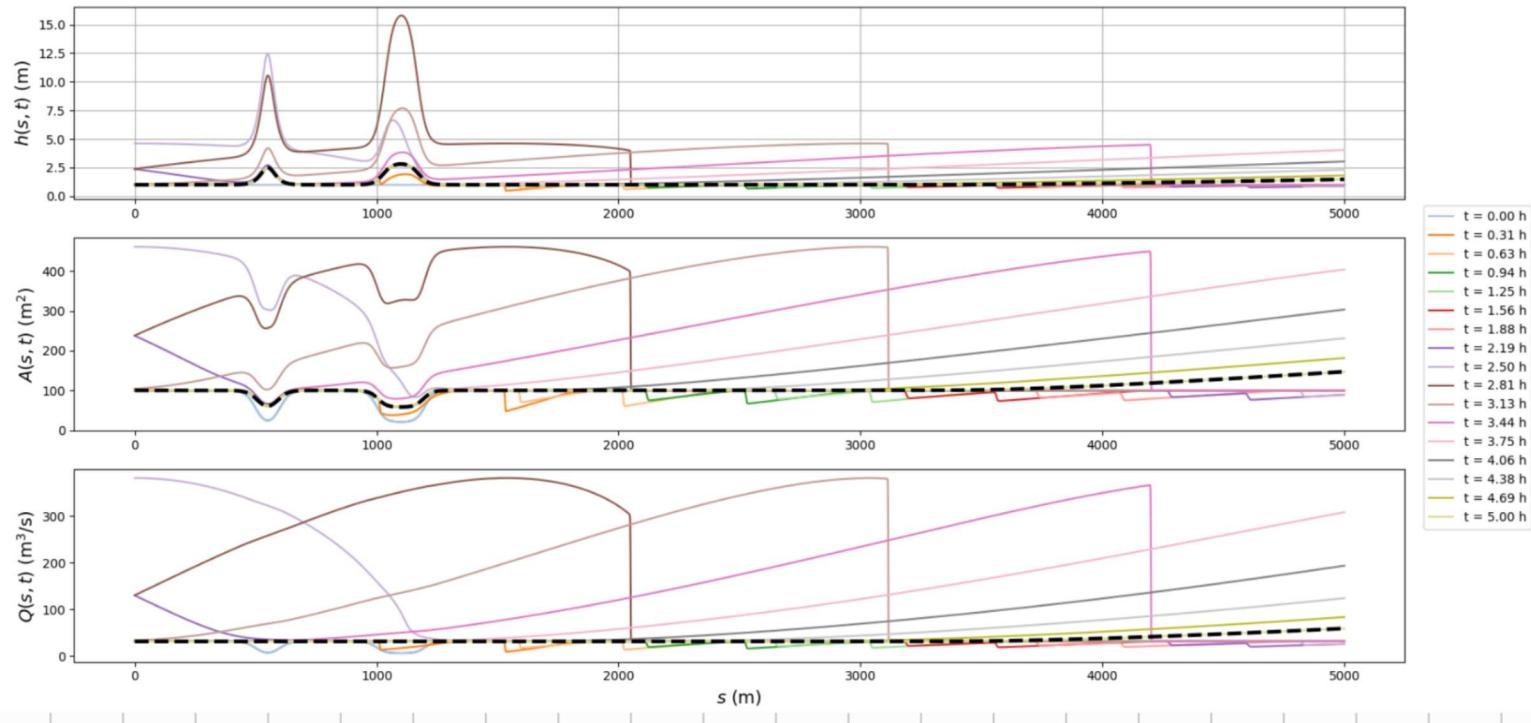
$$\Delta z = \Delta z_{\min}, \quad \Delta t = \Delta t_{\min}$$



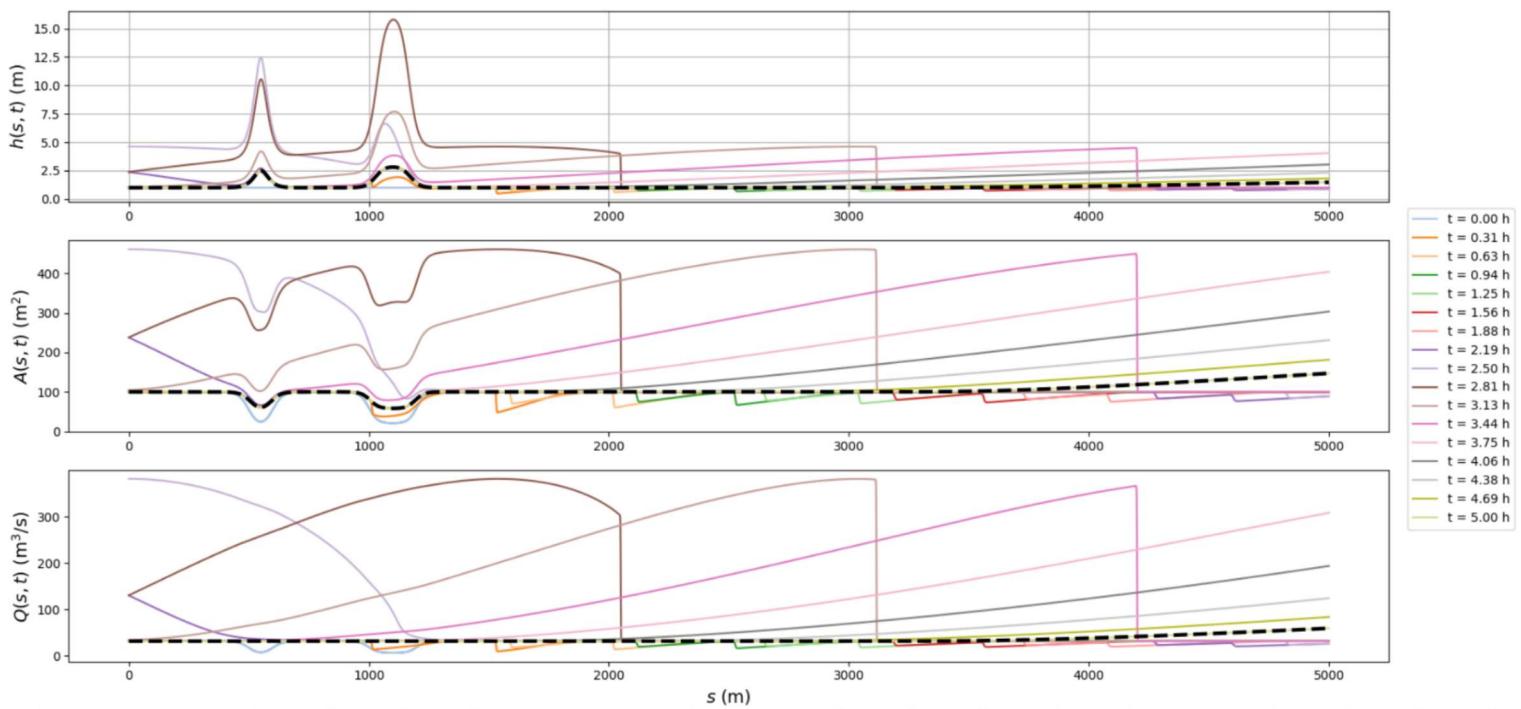
$$\Delta z = \Delta z_{\min}, \quad \Delta t = 2\Delta t_{\min}$$



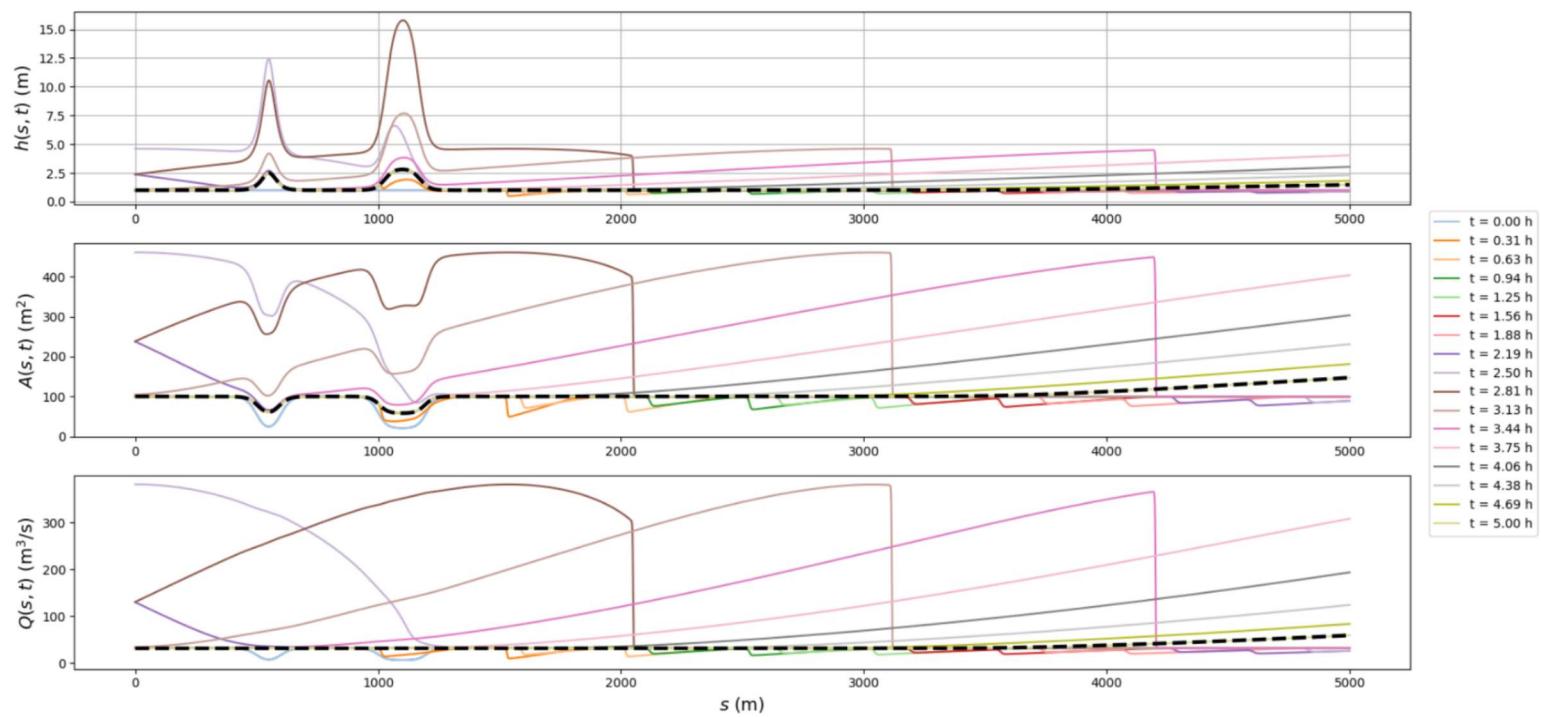
$$\Delta x = \Delta x_{\min}, \quad \Delta t = 4 \Delta t_{\min}$$



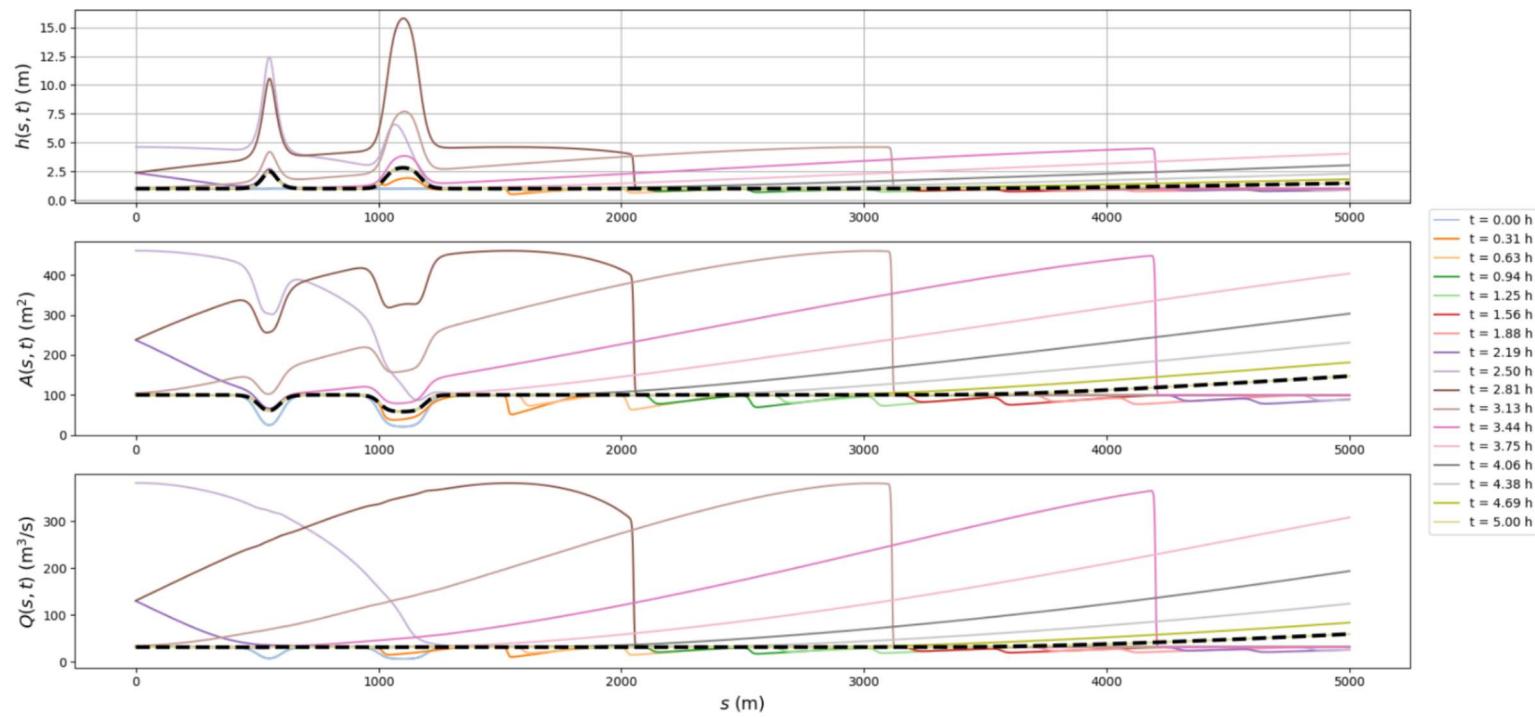
$$\Delta x = \Delta x_{\min}, \quad \Delta t = 8 \Delta t_{\min}$$



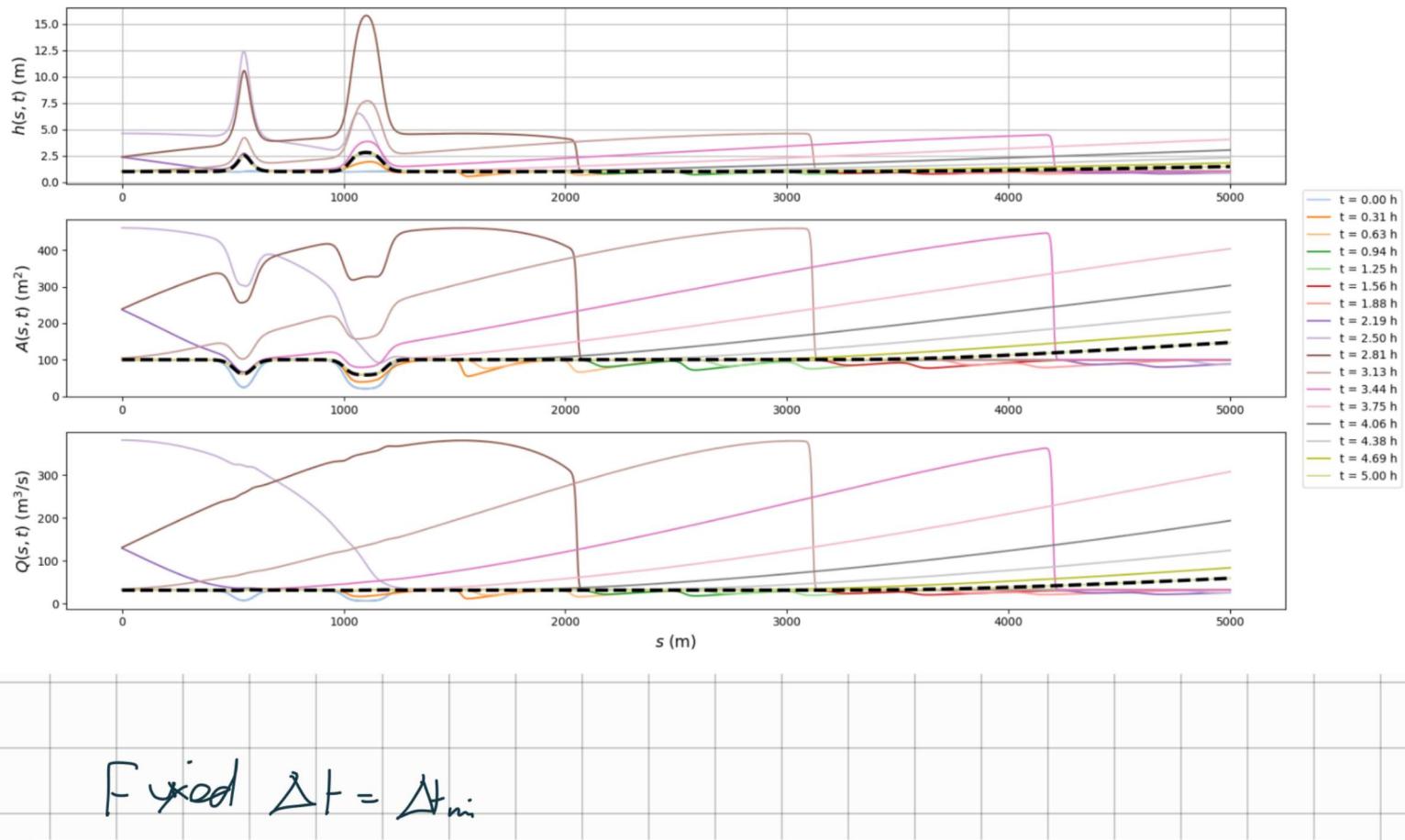
$$\Delta x = 2 \Delta x_{\min}, \quad \Delta t = \Delta t_{\min}$$



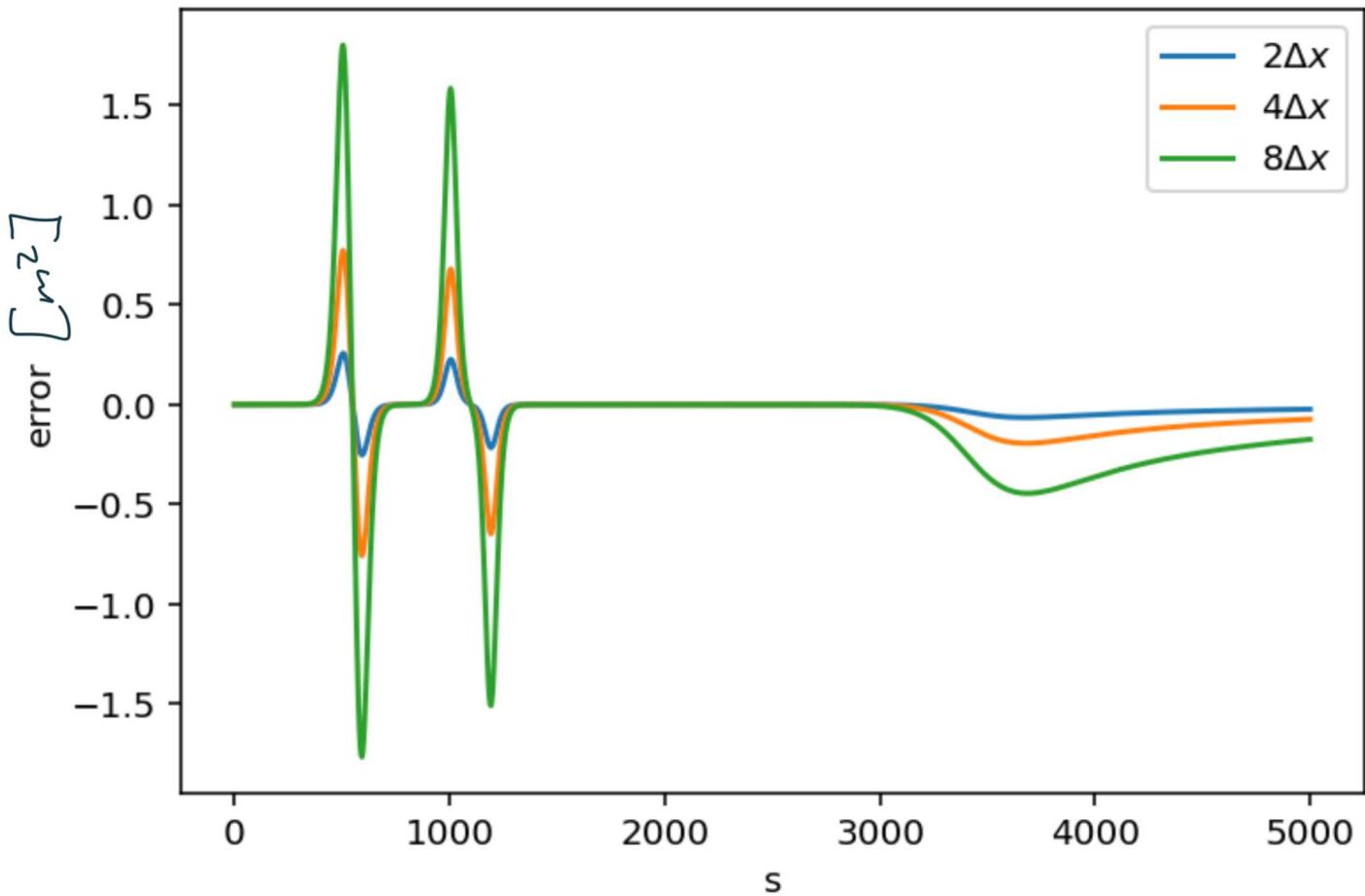
$$\Delta x = 4 \Delta x_{\min}, \quad \Delta t = \Delta t_{\min}$$



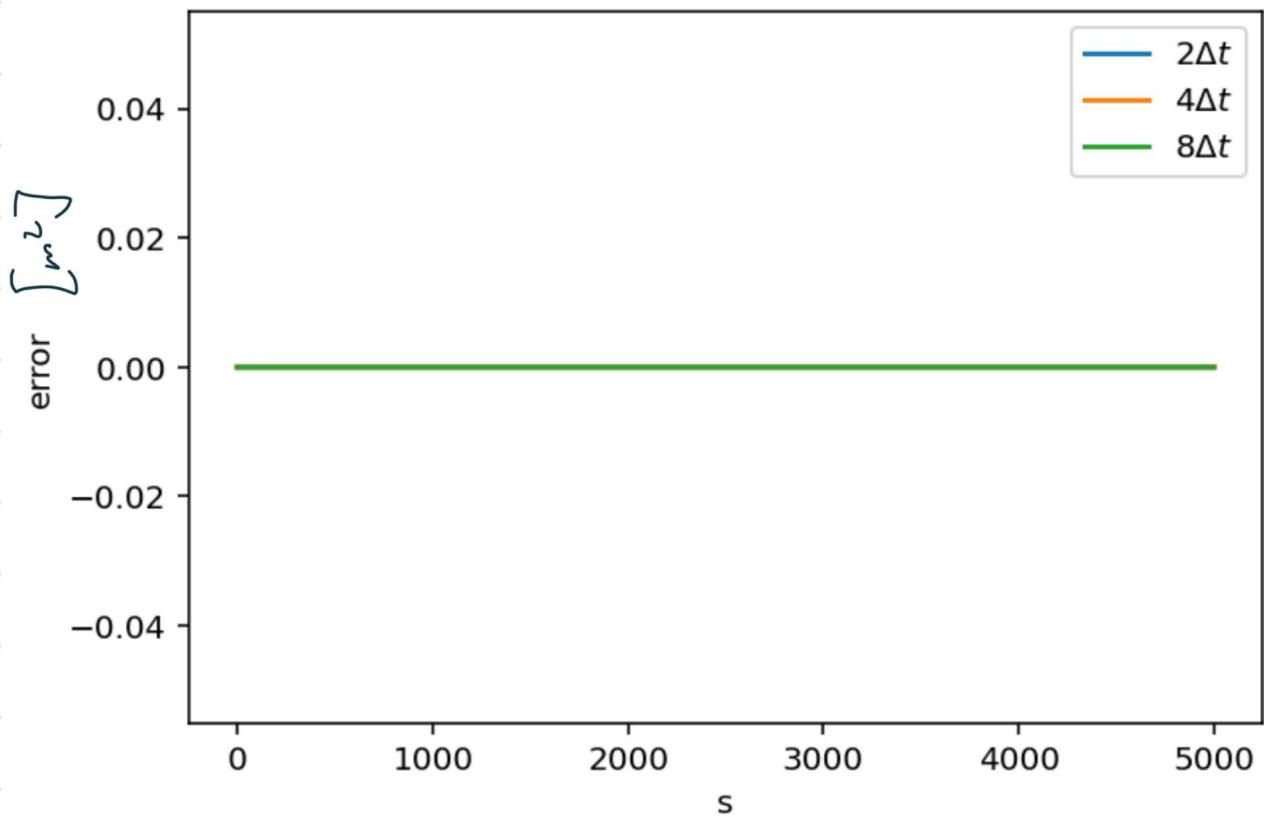
$$\Delta x = 8 \Delta x_{\text{ini}}, \Delta t = \Delta t_{\text{ini}}$$



Fixed $\Delta t = \Delta t_{\text{ini}}$



$$\text{fixed } \Delta x = \Delta x_{\min}$$



From $\Delta x = \Delta x_{\min}, \Delta t = \Delta t_{\min}$ figure

Calculated stock speed. (n_t = theoretical, n = observed)

① $t \approx 10125 \text{ s}$

$$A_L \approx 400 \text{ m}^2 \quad F(A_L) = 300$$

$$A_n \approx 100 \text{ m}^2 \quad F(A_n) = 31$$

$$s \approx 2050 \text{ m}$$

$$n_t = F(A_r) - F(A_e)$$

$$\frac{A_r - A_e}{A_r - A_e}$$

$$\textcircled{1} \rightarrow \textcircled{2}$$

$$n_t \approx 0.90 \text{ ms}^{-1}$$

$$n = \frac{3115 - 2050}{11250 - 10125} = 0.95 \text{ ms}^{-1}$$

$$\textcircled{2} \quad t = 11250 \text{ s}$$

$$A_L = 460 \text{ m}^2$$

$$A_n \approx 100 \text{ m}^2 \quad F(A_n) = 31$$

$$s \approx 3115 \text{ m}$$

$$u_f = 0.97 \text{ ms}^{-1}$$

$\textcircled{2} \rightarrow \textcircled{3}$

$$u = 0.96 \text{ ms}^{-1}$$

$$\textcircled{3} \quad t = 12375 \text{ s}$$

$$A_L \approx 450 \text{ m}^2 \quad F(A_L) = 366$$

$$A_n \approx 100 \text{ m}^2 \quad F(A_n) = 31$$

$$s \approx 4200 \text{ m}$$

$$u_f = 0.96 \text{ ms}^{-1}$$

- Theoretical shock speed \sim shock speed observed

• To have more accurate shock speed, could record data at more time intervals

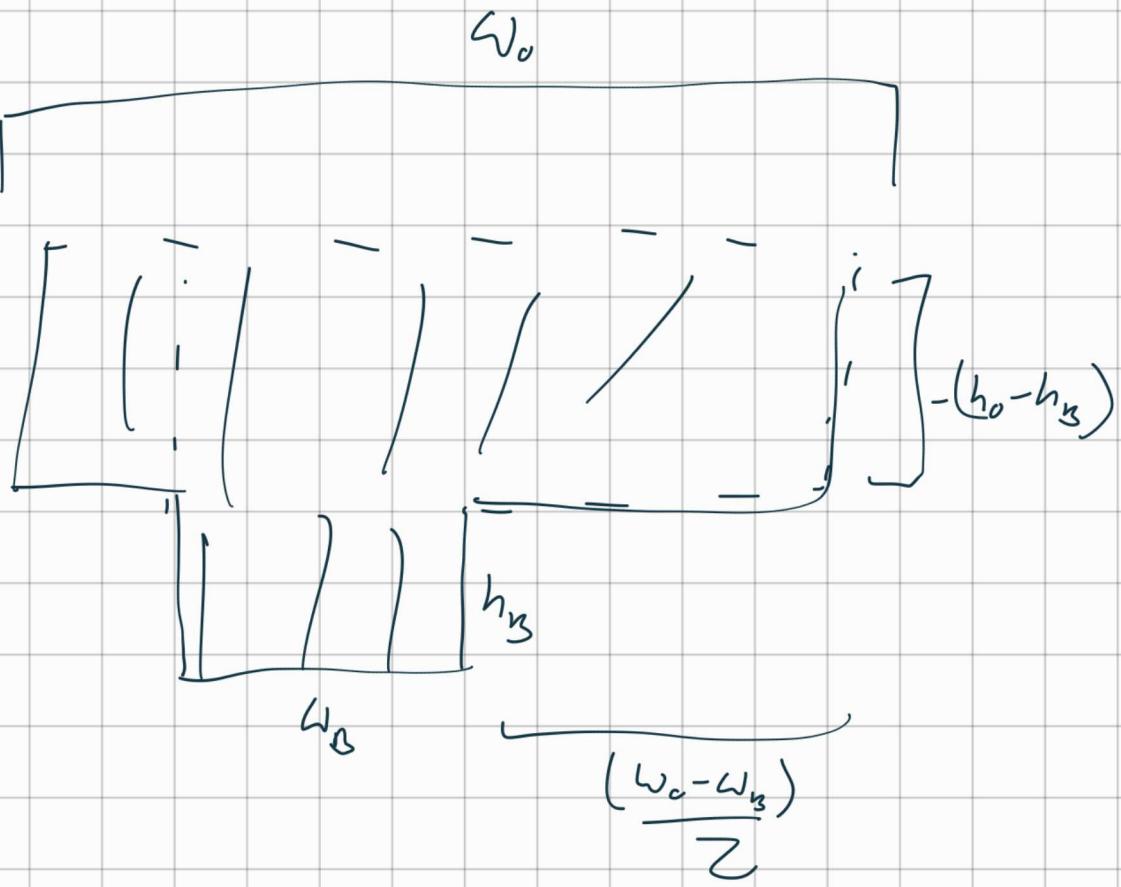
• 1st order in space and time.

• Have reached convergence in time

\rightarrow but is still converging in space.

- Interestingly, reached convergence in time for test case 1 not test case 0
→ even though errors for changing Δx steps are larger for test case 1.

2ab).



Expression for wetted perimeter will change depending as if A is $< A_B$ or $> A_B$

$$A < A_B$$

$$P = w_B + \frac{A_o}{w_B}$$

$$A > A_B$$

$$P = w_B + 2h_B + (w_o - w_B) + \frac{2(A - A_B)}{w_o}$$

For test case 0.

As constant flow is determined by

$A = H_0 \cdot w_s$ where $H_0 < h_B$, then due to the presence of the river channel, everything is constant

→ but good way to check code

2a, Constant Q :

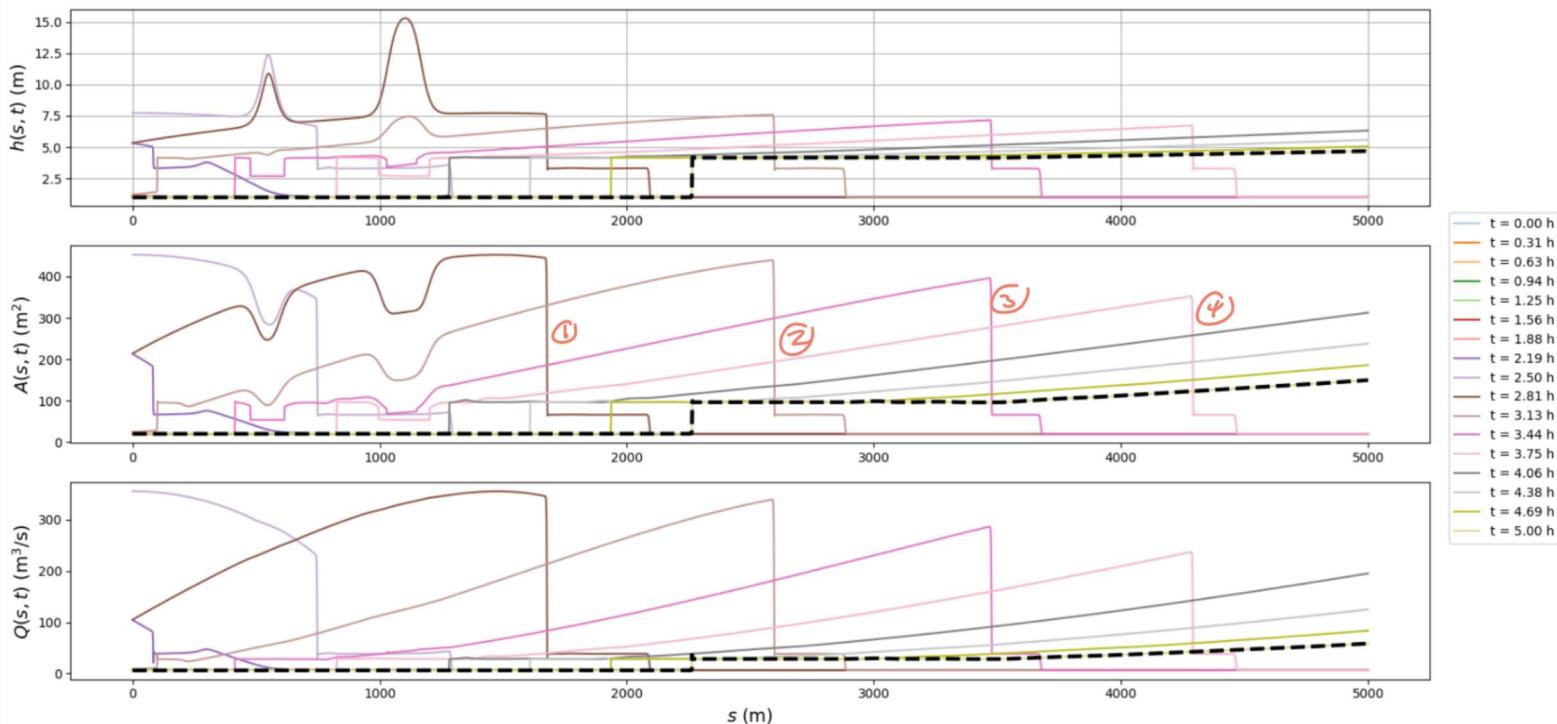


2b). Slightly different Δt_{\min} , Δx_{\min} , as this code took a lot longer to run

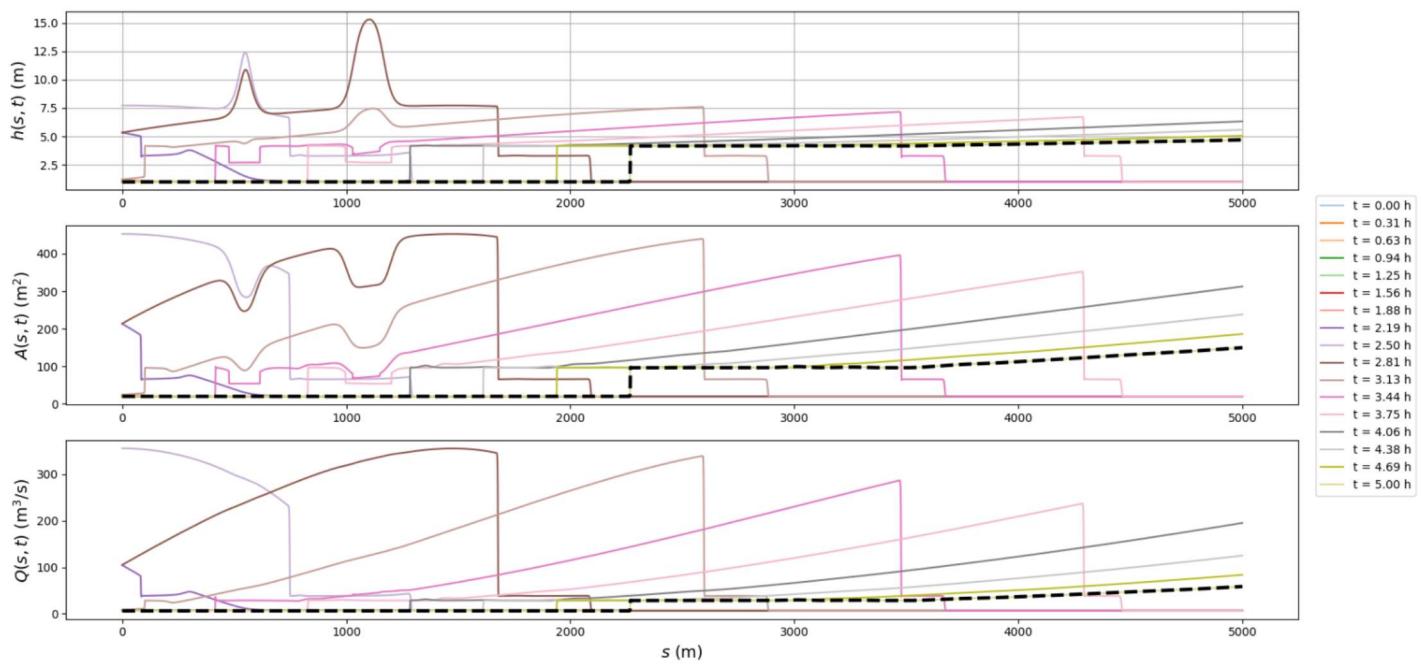
$$\rightarrow \Delta x_{\min} = 1 \text{ m}$$

$$\Delta t_{\min} = \frac{0.25 \times 1}{C_{\text{cc}}}$$

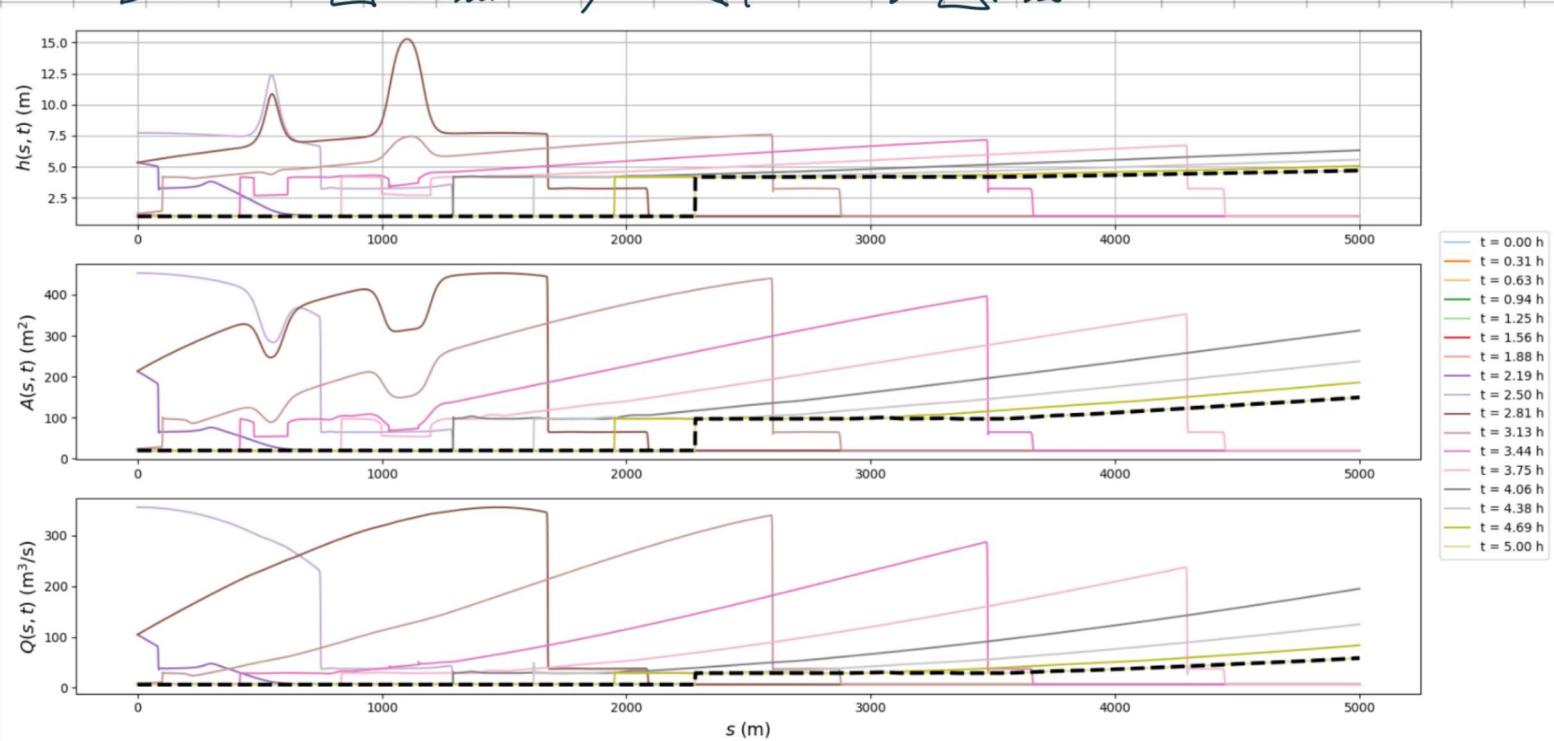
$$\Delta x = \Delta x_{\min}, \quad \Delta t = \Delta t_{\min}$$



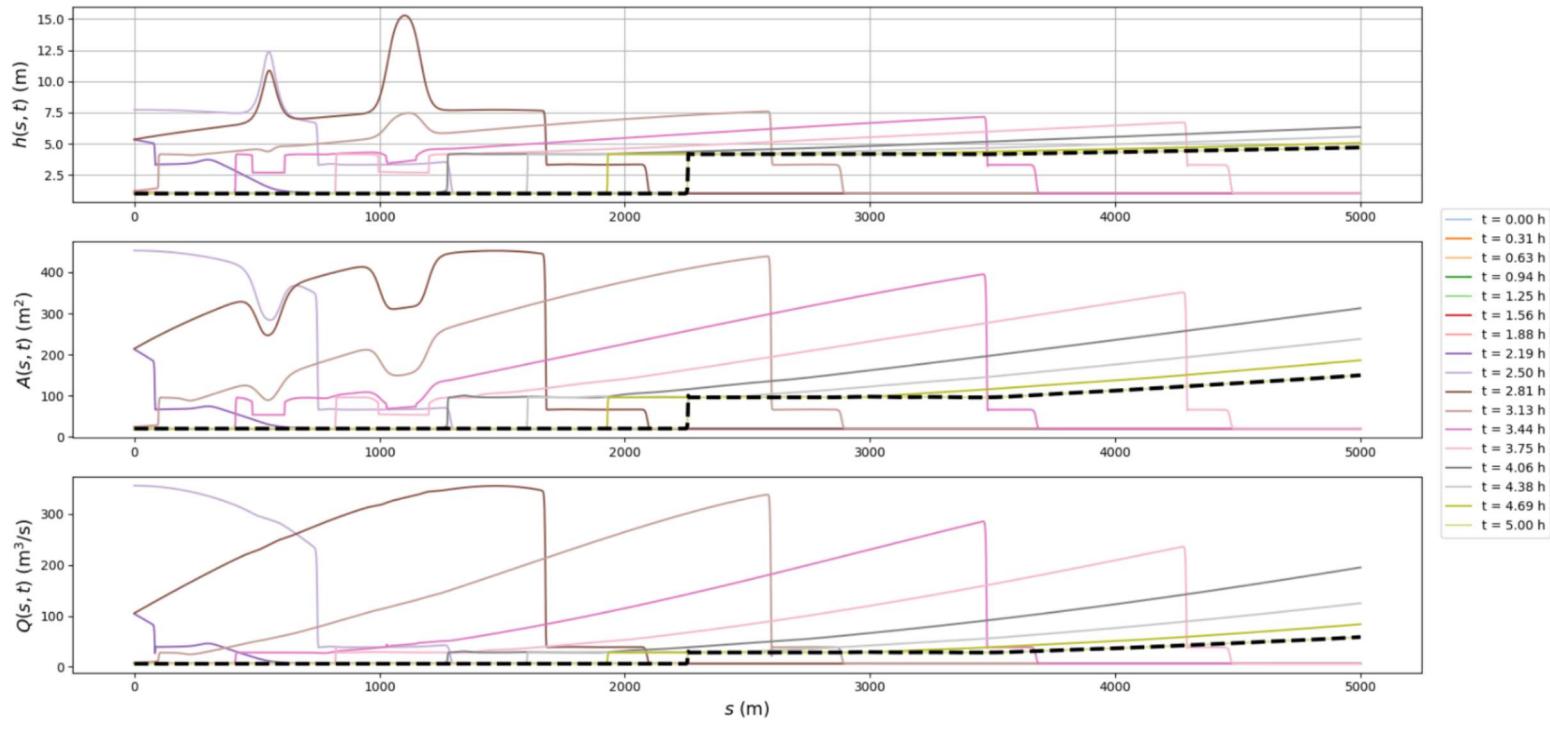
$$\Delta x = \Delta x_{\min}, \quad \Delta t = 2 \Delta t_{\min}$$



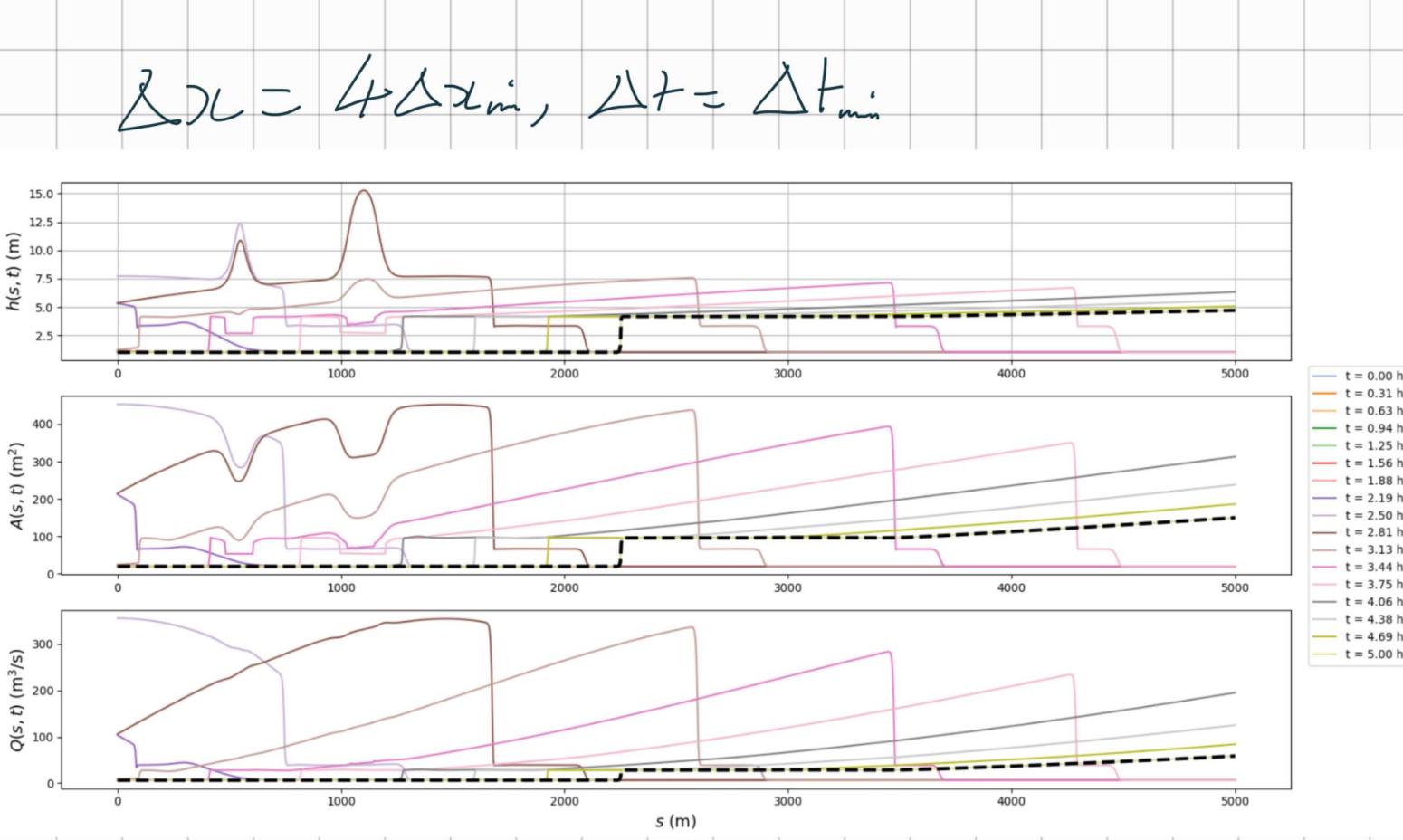
$$\Delta x = \Delta x_{\min}, \quad \Delta t = 4 \Delta t_{\min}$$

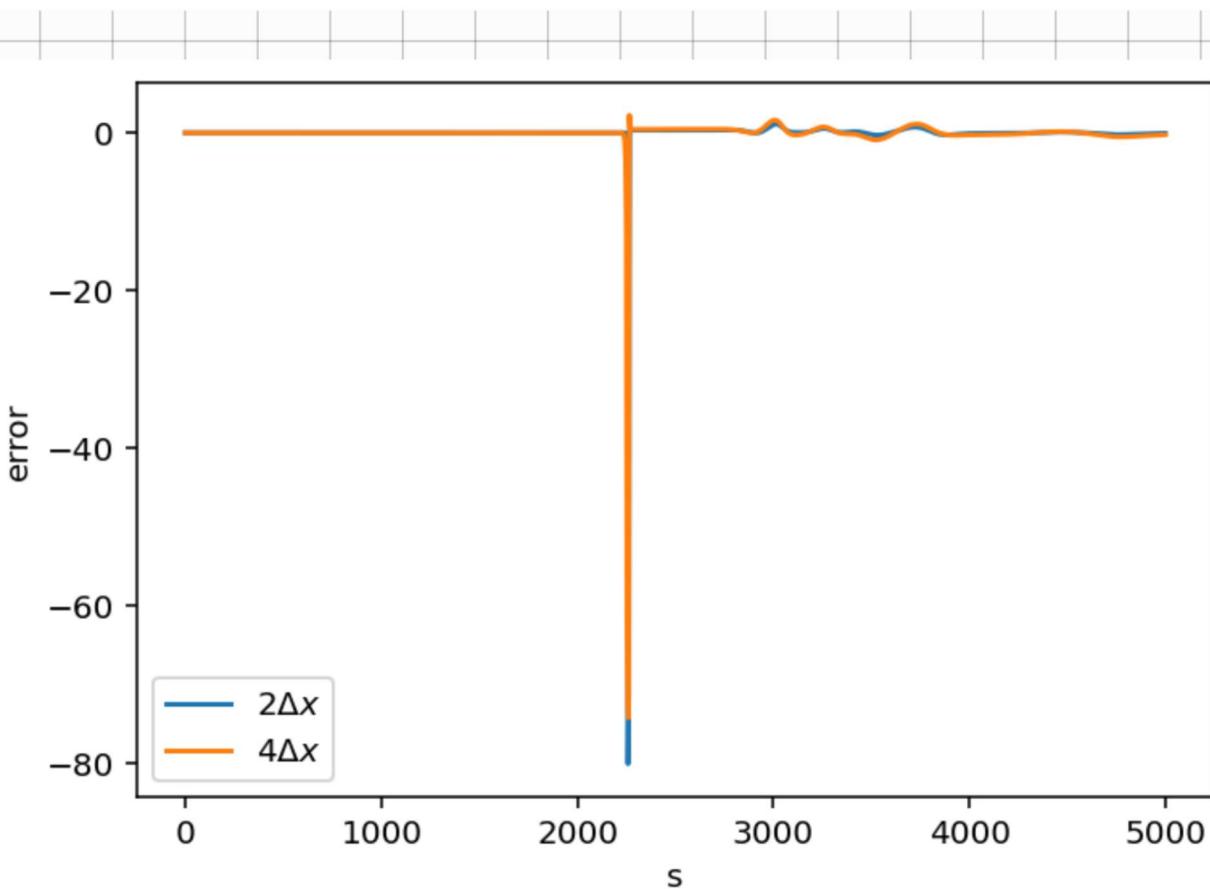
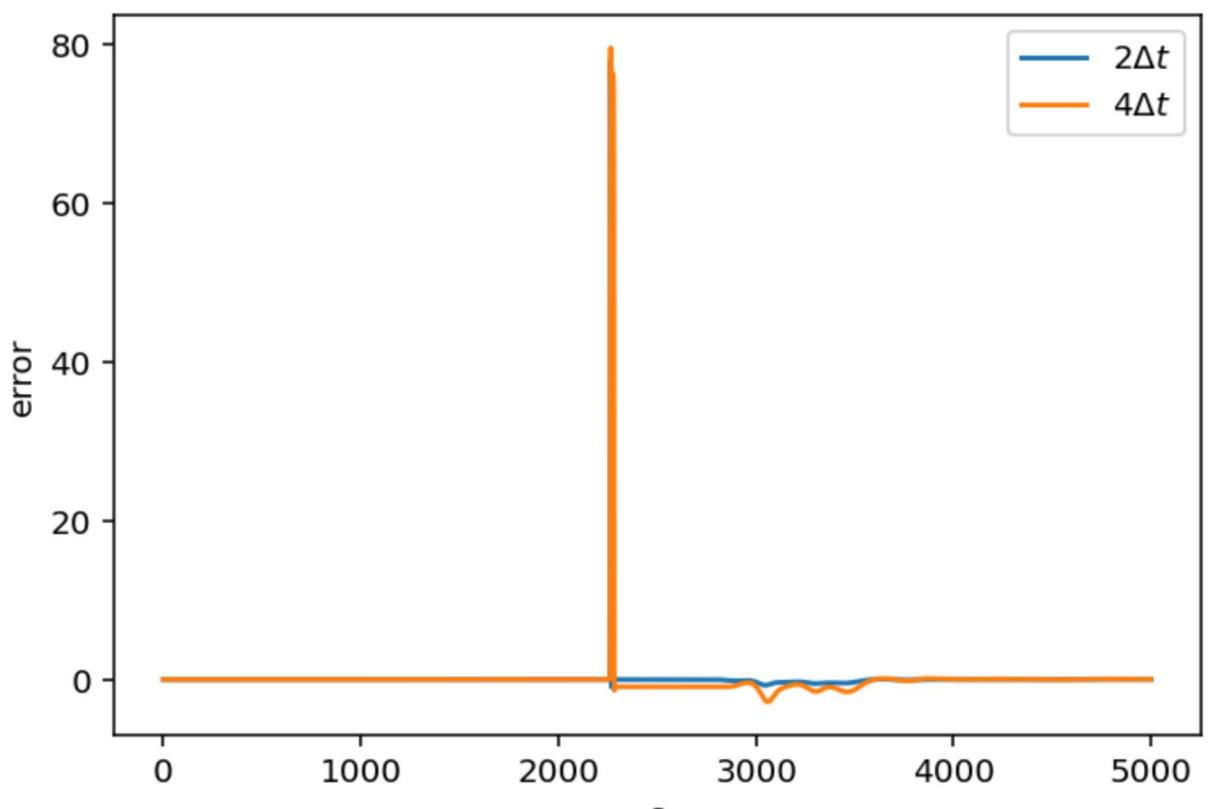


$$\Delta x = 2 \Delta x_{\min}, \quad \Delta t = \Delta t_{\min}$$



$$\Delta x = 4 \Delta x_{\min}, \quad \Delta t = \Delta t_{\min}$$





\circ 1st order in space and time

- Issues at position of the shock with convergence according to graphs
- but could be done with error calculation rather than with simulation
- But away from shock, can see that the solutions converge.

Shock speed:

$$\textcircled{1} \quad t = 1012 \text{ s}$$

$$A_L = 445 \text{ m}^2 \quad F(A_L) = 348$$

$$A_n = 65 \text{ m}^2 \quad F(A_n) = 38$$

$$s = 1677 \text{ m}$$

$\textcircled{1} \rightarrow \textcircled{2}$

$$u_+ = 0.81 \text{ ms}^{-1} \quad u = 0.82 \text{ ms}^{-1}$$

$$\textcircled{2} \quad t = 11250 \text{ s}$$

$$A_L = 441 \text{ m}^2 \quad F(A_L) = 348$$

$$A_n = 70 \text{ m}^2 \quad F(A_n) = 38$$

$$s = 2596 \text{ m}$$

$\textcircled{2} \rightarrow \textcircled{3}$

$$u_+ = 0.81 \text{ ms}^{-1} \quad u = 0.78 \text{ ms}^{-1}$$

$$\textcircled{3} \quad t = 1237 \text{ s}$$

$$A_L = 39 \text{ m}^2$$

$$A_n = 65 \text{ m}^2$$

$$s = 347 \text{ m}$$

$$F(A_L) = 237$$

$$F(A_n) = 38$$

$\textcircled{3} \rightarrow \textcircled{4}$

$$n_r = 0.75 \text{ ms}^{-1}$$

$$n = 0.72 \text{ ms}^{-1}$$

$$\textcircled{4} \quad t = 13500 \text{ s}$$

$$A_L = 352 \text{ m}^2$$

$$A_n = 65 \text{ m}^2$$

$$s = 4290 \text{ m}$$

$$F(A_L) = 237$$

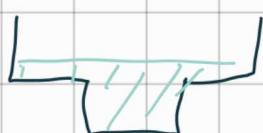
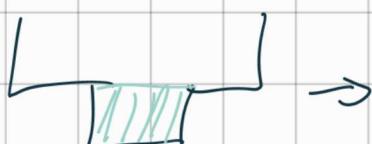
$$F(A_n) = 38$$

$$n_r = 0.69 \text{ ms}^{-1}$$

- $n_r \sim n$, but is slightly less clear when calculating for this test (could try evaluate shock speed over more times to check, as it seems to erode).

- Additional jumps due to layout

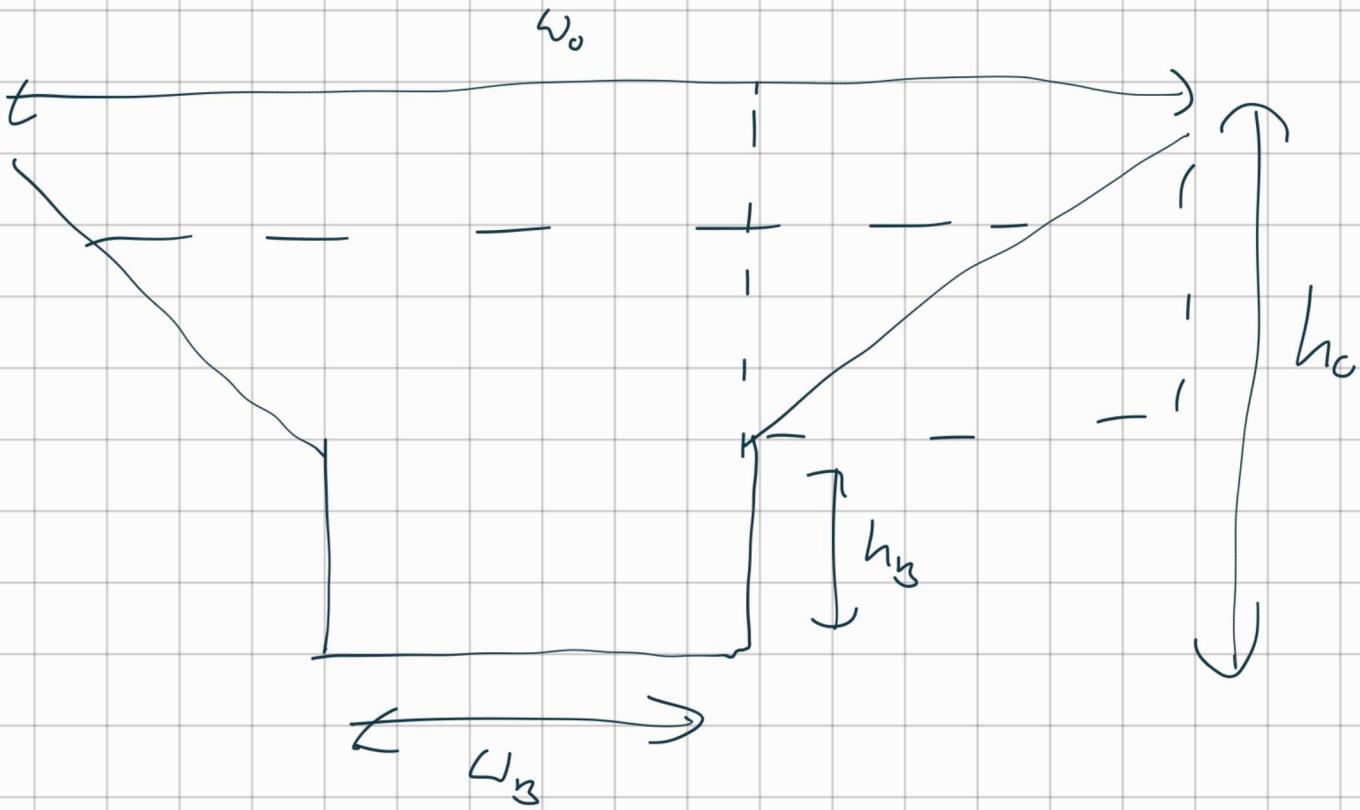
→ e.g.



Feels like there should be a sudden jump between this

z_{ab}).

Profile now looks like



Just means value of wetted perimeter
So $A \propto h_b w_b$ changes

→ But value of width of water is not fixed.

$$\rightarrow \frac{h_o - h_b}{(h_o - h_b)/2} = 0.01$$

$$w_o - w_b = \frac{2(h_o - h_b)}{0.01}$$

$$(A - h_s w_s) = \frac{1}{2} (\omega_o + \omega_s) \cdot (h_o - h_s)$$

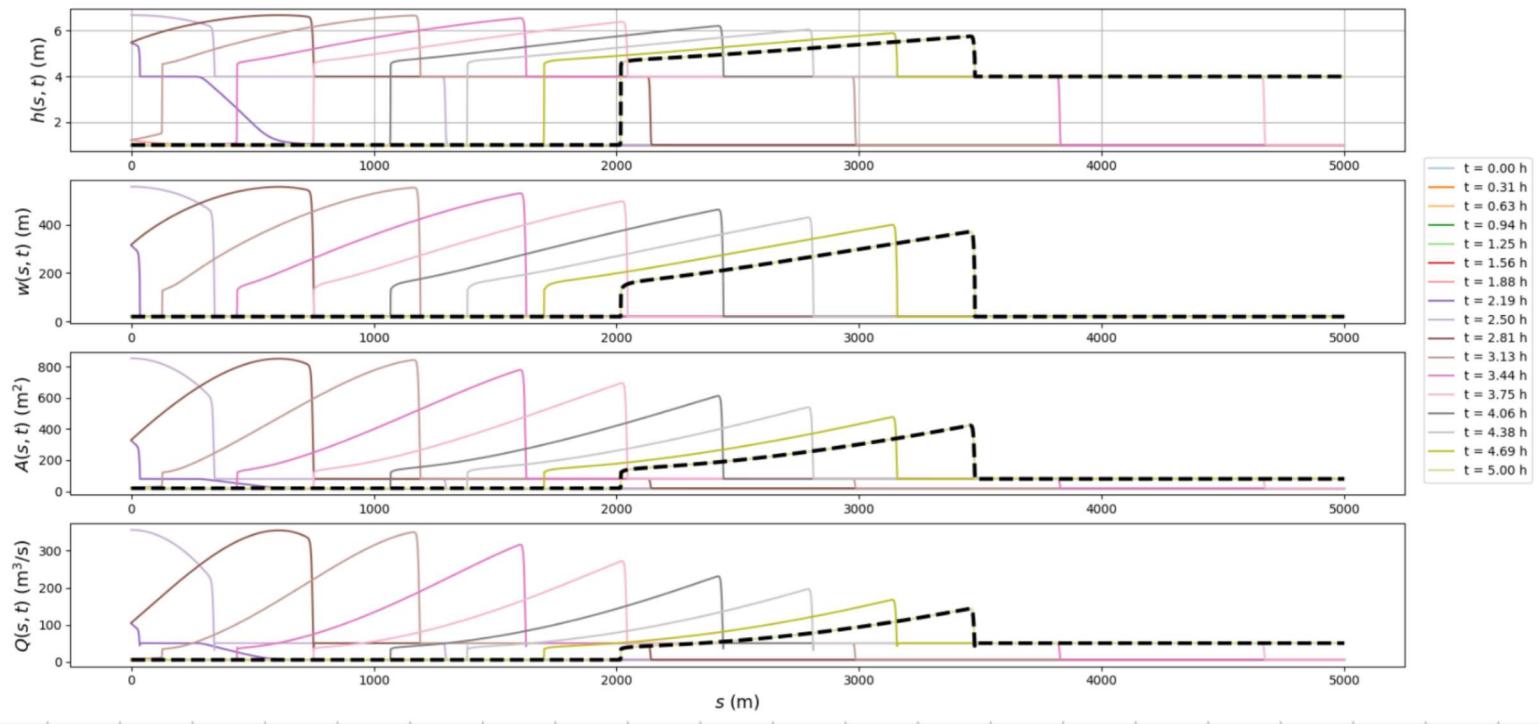
$$\begin{aligned}
 P &= 2h_s + \omega_s + 2 \sqrt{(h_o - h_s)^2 + \left(\frac{\omega_o - \omega_s}{2} \right)^2} \\
 &= 2h_s + \omega_s + 2(h_o - h_s) + \left(\frac{h_o - h_s}{0.01} \right)^2 \\
 &= 2h_s + \omega_s + 2(h_o - h_s) \sqrt{1 + 100^2} \\
 &= 2h_s + \omega_s + \frac{4(A - h_s w_s)}{\omega_o + \omega_s} \sqrt{1 + 100^2}
 \end{aligned}$$

$$A - A_s = \frac{1}{2} (\omega_o + \omega_s) \cdot \frac{(0.01)}{2} (\omega_o - \omega_s)$$

$$(\omega_o + \omega_s)(\omega_o - \omega_s) = \frac{4}{0.01} (A - A_s)$$

$$\omega_o^2 - \omega_s^2 = \frac{4}{0.01} (A - A_s)$$

$$\boxed{\omega_o = \sqrt{\omega_s^2 + 400(A - A_s)}}$$



$$\Delta x = 1\text{m}, \quad \Delta t = \frac{0.25x}{C_{00}}$$

- Could not properly check convergence, as there is issue with code

→ when $\Delta x = 1\text{m}$, but $CFL = 0.125$ solution is wrong, but solutions for larger time / space steps seem to converge to solution shown above.