

Numerical Exercise 1

$$1a). \partial_t b + \partial_z (\alpha b^3 - \beta b^3 \partial_z b) = 0$$

$$= \partial_t b + 3\alpha b^2 \partial_z b - 3\beta b^2 (\partial_z b)^2 - \beta b^3 \partial_{zz} b$$

$$(b = D_o + b')$$

$$= \partial_t b' + 3\alpha (D_o + b')^2 \partial_z b' - 3\beta b^2 (\partial_z b')^2 - \beta (D_o + b')^3 \partial_{zz} b'$$

$$= \partial_t b' + 3\alpha (D_o^2 + 2D_o b' + b'^2) \partial_z b' - \beta (D_o^3 + 3D_o^2 b' + 3D_o b'^2 + b'^3) \partial_{zz} b' + O(b')$$

$$= \partial_t b' + 3\alpha D_o^2 \partial_z b' - \beta D_o^3 \partial_{zz} b' = 0$$

⑤ and ⑥ both contain terms that

describe diffusion and convection

→ as diffusion equation is in the form: $u_t = \sigma u_{xx}$

and convection equation is in the form:

$$u_t + c u_x = 0$$

$$b). \quad \partial_t b = \frac{b_j^{n+1} - b_j^n}{\Delta t}$$

$$\partial_z (\alpha b^3) :$$

Using caynird scheme, need to consider $\alpha > 0$

$$\partial_z (\alpha b^3) = \alpha \frac{\left[(b_j^n)^3 - (b_{j-1}^n)^3 \right]}{\Delta z}$$

$$-\beta \partial_z (b^3 \partial_z b)$$

$$= -\frac{\beta}{\Delta z} \left[(b_{j+1/2}^n)^3 \left(\frac{b_{j+1}^n - b_j^n}{\Delta z} \right) - (b_{j-1/2}^n)^3 \left(\frac{b_j^n - b_{j-1}^n}{\Delta z} \right) \right]$$

$$= -\frac{\beta}{(\Delta z)^2} \left[(b_{j+1/2}^n)^3 (b_{j+1}^n - b_j^n) - (b_{j-1/2}^n)^3 (b_j^n - b_{j-1}^n) \right]$$

$$\text{Using } b_{j+1/2}^n = \frac{b_{j+1}^n + b_j^n}{2}$$

$$= -\frac{\beta}{2^3 (\Delta z)^2} \left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n) \right]$$

⑤ can be written as:

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} + \frac{\alpha}{\Delta z} \left[(b_j^n)^3 - (b_{j-1}^n)^3 \right]$$

$$-\frac{\beta}{8(\Delta z)^2} \left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n) \right]$$

$$⑥ : 3\alpha D_o^2 \partial_z b'$$

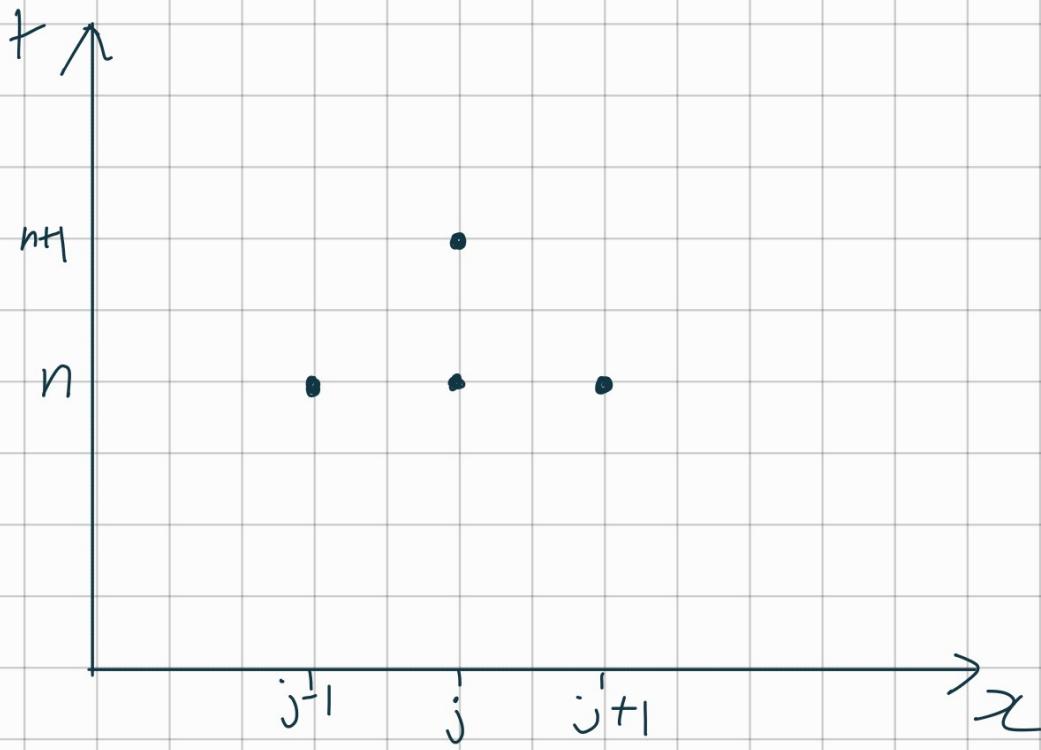
$$= 3\alpha D_o^2 \frac{b_j^{in} - b_{j-1}^{in}}{\Delta z}$$

$$-\beta D_o^3 \partial_{zz} b'$$

$$= -\frac{\beta D_o^3}{(\Delta z)^2} (b_{j+1}^{in} - 2b_j^{in} + b_{j-1}^{in})$$

⑥ can be written as:

$$\left. \begin{aligned} & \frac{b_j^{in+1} - b_j^{in}}{\Delta t} + \frac{3\alpha D_o^2}{\Delta z} (b_j^{in} - b_{j-1}^{in}) \\ & - \frac{\beta D_o^3}{(\Delta z)^2} (b_{j+1}^{in} - 2b_j^{in} + b_{j-1}^{in}) = 0 \end{aligned} \right\}$$



Linienssing (5): $(b = D_o + b')$

$$\frac{b_j^{(n+1)} - b_j^{(n)}}{\Delta t} + \frac{\alpha}{\Delta z} \left[(D_o + b_j^{(n)})^3 - (D_o + b_{j-1}^{(n)})^3 \right]$$

$$- \frac{\rho}{8(\Delta z)^2} \left[(2D_o + b_{j+1}^{(n)} + b_j^{(n)})^3 (b_{j+1}^{(n)} - b_j^{(n)}) \right. \\ \left. - (2D_o + b_j^{(n)} + b_{j-1}^{(n)})^3 (b_j^{(n)} - b_{j-1}^{(n)}) \right]$$

$$= 0$$

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} + \frac{\alpha}{\Delta z} \left[D_o^3 + 3D_o^2 b_j^n - D_o^3 - 3D_o^2 b_{j-1}^n + O(b')^2 \right]$$

$$- \frac{\beta}{8(\Delta z)^2} \left[8D_o^3 (b_{j+1}^n - b_j^n) - 8D_o^3 (b_j^n - b_{j-1}^n) + O(b')^2 \right]$$

$$= \frac{b_j^{n+1} - b_j^n}{\Delta t} + \frac{3\alpha D_o^2}{\Delta z} (b_j^n - b_{j-1}^n)$$

$$- \frac{\beta D_o^3}{(\Delta z)^2} (b_{j+1}^n - 2b_j^n + b_{j-1}^n)$$

= ⑥

Using adjoint form for diffusion term

↓

$$\partial_t b - \beta \partial_z (b^3 \partial_z b) = 0$$

This means we can discretise the equation in its original form

Boundary conditions:

$$b(0,t) = b_s, \quad b(H,t) = b_r$$

Initial conditions:

$$b(z,0) = b_i(z)$$

$$\left| \begin{array}{l} b(0,t) = b_s \rightarrow b_0^n = b_s \\ b(H,t) = b_r \rightarrow b_{J-1}^n = b_r \end{array} \right. \begin{array}{l} \text{for all values of } n \\ \text{Don't see your numbering defined; was J not the last point.} \\ \text{Where is } dz \text{ defined? -0.5} \end{array}$$

$$b(z,0) = b_i(z) \rightarrow b_j^0 = b_i(\cdot) \Delta z$$

c). Letting $b_j^n = (\lambda)^n e^{ikj\Delta z}$

Sub into linearised discretisation:

$$\frac{\lambda b_j^n - b_j^{n-1}}{\Delta t} + \frac{32 D_o^2}{\Delta z} (1 - e^{-ik\Delta z}) b_j^n - \frac{\beta D_o^3}{(\Delta z)^2} \left(e^{ik\Delta z} - 2 + e^{-ik\Delta z} \right) b_j^n = 0$$

$$\omega = 0, \beta \neq 0$$

$$\frac{\lambda - 1}{\Delta t} - \frac{\beta D_o^3}{(\Delta z)^2} (z_i)^2 \sin^2\left(\frac{k\Delta z}{2}\right) = 0$$

$$\lambda = 1 - 4\beta D_o^3 \left(\frac{\Delta t}{(\Delta z)^2} \right) \sin^2\left(\frac{k\Delta z}{2}\right)$$

stable when $|\lambda| \leq 1$

$-4\beta D_o^3 \left(\frac{\Delta t}{(\Delta z)^2} \right) \sin^2\left(\frac{k\Delta z}{2}\right)$ is always negative

so need to check if $1 - 4\beta D_o^3 \frac{\Delta t}{(\Delta z)^2} \geq -1$

$$4\beta D_o^3 \frac{\Delta t}{(\Delta z)^2} \leq 2$$

$$\frac{\Delta t}{(\Delta z)^2} \leq \frac{1}{2\beta D_o^3}$$

Scheme for $\alpha=0, \beta \neq 0$ is conditionally stable

and $\Delta t \leq \frac{(\Delta z)^2}{2\beta D_o^3}$

$$\alpha \neq 0, \beta = 0$$

$$\frac{\lambda - 1}{\Delta t} + \frac{3\alpha D_o^2}{\Delta z} \left(1 - e^{-ik\Delta z} \right) = 0$$

$$\lambda = 1 - 3\alpha D_o^2 \frac{\Delta t}{\Delta z} \left(1 - e^{-ik\Delta z} \right)$$

$$= 1 - 3\alpha D_o^2 \frac{\Delta t}{\Delta z} \left(1 - \cos(k\Delta z) + i\sin(k\Delta z) \right)$$

$$= 1 - 3\alpha D_o^2 \frac{\Delta t}{\Delta z} \left(1 - \cos(k\Delta z) \right) - i3\alpha D_o^2 \frac{\Delta t}{\Delta z} \sin(k\Delta z)$$

$$\text{let } \gamma = 3\alpha D_o^2 \frac{\zeta t}{\Delta z}$$



$$\lambda = 1 - \gamma(1 - \cos k\Delta z) - i\gamma \sin k\Delta z$$

$$\begin{aligned} |\lambda|^2 &= (1 - \gamma(1 - \cos k\Delta z))^2 + (\gamma \sin k\Delta z)^2 \\ &= (1 - \gamma + \gamma \cos(k\Delta z))^2 + (\gamma \sin k\Delta z)^2 \end{aligned}$$

$$2|\lambda| \frac{\partial \lambda}{\partial (k\Delta z)} = 2(1 - \gamma + \gamma \cos(k\Delta z))(-\gamma \sin(k\Delta z)) + 2\gamma^2 \sin(k\Delta z) \cos(k\Delta z) = 0$$

when $|\lambda|$ is at its max or min.

$$\rightarrow (1 - \gamma) \sin(k\Delta z) = 0$$

so min/max when $(k\Delta z) = n\pi$

$$k\Delta z = \pi$$

$$|\lambda|^2 = (1 - 2\gamma)^2$$

$$1 - 2\gamma \geq -1$$

$$\gamma \leq 1$$

$$3\alpha D_o^2 \frac{\Delta t}{\Delta z} \leq 1$$

$$\Delta t \leq \frac{\Delta z}{3\alpha D_o^2}$$

When $\alpha \neq 0, \beta = 0$, scheme is conditionally stable

$$\text{So } \Delta t \leq \frac{\Delta z}{3\alpha D_o^2}$$

$$d), \quad \frac{b_j^{n+1} - b_j^n}{\Delta t} + \frac{3\alpha D_o^2}{\Delta z} (b_j^n - b_{j-1}^n)$$

$$- \frac{\beta D_o^3}{(\Delta z)^2} (b_{j+1}^n - 2b_j^n + b_{j-1}^n) = 0$$

$$b_j^{n+1} = b_j^n - \frac{3\alpha D_o^2 \Delta t}{\Delta z} (b_j^n - b_{j-1}^n) + \frac{\beta D_o^3 \Delta t}{(\Delta z)^2} (b_{j+1}^n - 2b_j^n + b_{j-1}^n)$$

$$= b_j^n \left[1 - \frac{3\alpha D_o^2 \Delta t}{\Delta z} - \frac{2\beta D_o^3 \Delta t}{(\Delta z)^2} \right]$$

$$+ b_{j+1}^n \left[\frac{\beta D_o^3 \Delta t}{(\Delta z)^2} \right]$$

$$+ b_{j-1}^n \left[\frac{3\alpha D_o^2 \Delta t}{\Delta z} + \frac{\beta D_o^3 \Delta t}{(\Delta z)^2} \right]$$

According to maximum principle, ~~so that~~
~~these coefficients~~, all coefficients must be non-negative.

$$\Rightarrow \Delta t \left[1 - \frac{3\alpha D_o^2}{\Delta z} - \frac{2\beta D_o^3}{(\Delta z)^2} \right] \geq 0$$

$$\Delta t \left[\frac{2\beta D_o^3}{(\Delta z)^2} + \frac{3\alpha D_o^2}{\Delta z} \right] \leq 1$$

$$\Delta t \leq \frac{\Delta z}{D_0^2} \cdot \frac{1}{\left(\frac{2\beta D_0}{\Delta z} + 3\alpha \right)}$$

(and when looking at the $\alpha=0, \beta \neq 0$ and the $\alpha \neq 0, \beta=0$ cases from c, this gives same timestep limitation).

$$e). \frac{B_j^{n+1} - B_j^n}{\Delta t} + \frac{\alpha}{\Delta z} \left[(B_j^n)^3 - (B_{j-1}^n)^3 \right] \\ - \frac{\beta}{8(\Delta z)^2} \left[(B_{j+1}^n + B_j^n)^3 (B_{j+1}^n - B_j^n) - (B_j^n + B_{j-1}^n)^3 (B_j^n - B_{j-1}^n) \right] = 0$$

$$B_j^{n+1} = B_j^n - \frac{\alpha \Delta t}{\Delta z} \left[(B_j^n)^3 - (B_{j-1}^n)^3 \right] \\ + \frac{\beta \Delta t}{8(\Delta z)^2} \left[(B_{j+1}^n + B_j^n)^3 (B_{j+1}^n - B_j^n) - (B_j^n + B_{j-1}^n)^3 (B_j^n - B_{j-1}^n) \right] \\ = B_j^n \left(1 - \frac{\alpha \Delta t (B_j^n)^2}{\Delta z} - \frac{\beta \Delta t}{8(\Delta z)^2} (B_{j+1}^n + B_j^n)^3 \right. \\ \left. - \frac{\beta \Delta t}{8(\Delta z)^2} (B_j^n + B_{j-1}^n)^3 \right) \\ + B_{j-1}^n \left(\frac{\alpha \Delta t (B_{j-1}^n)^2}{\Delta z} + \frac{\beta \Delta t}{8(\Delta z)^2} (B_j^n + B_{j-1}^n)^3 \right) \\ + B_{j+1}^n \left(\frac{\beta \Delta t}{8(\Delta z)^2} (B_{j+1}^n + B_j^n)^3 \right)$$

For $B_j^{n+1} > 0$

$$1 - \frac{\alpha \Delta t}{\Delta z} (B_j^n)^2 - \frac{\beta \Delta t}{8(\Delta z)^2} \left[(B_{j+1}^n + B_j^n)^3 + (B_j^n + B_{j-1}^n)^3 \right] > 0$$

$$\Delta t <$$

$$\frac{\alpha (B_j^n)^2}{\Delta z} + \frac{\beta}{8(\Delta z)^2} \left[(B_{j+1}^n + B_j^n)^3 + (B_j^n + B_{j-1}^n)^3 \right]$$

Worse case scenario where $B_j^n = B_{j+1}^n = B_{j-1}^n = B_{\max}$

$$\Delta t <$$

$$B_{\max}^2 \left[\frac{\alpha}{\Delta z} + \frac{2\beta B_{\max}}{(\Delta z)^2} \right]$$

$$5). \quad \partial_z(\alpha b^3) :$$

Using upwind scheme:

$$\partial_z(\alpha b^3) = \alpha \frac{[(b_j^n)^3 - (b_{j-1}^n)^3]}{\Delta z}$$

Instead, using central differences method:

$$\begin{aligned} \partial_z(\alpha b^3) &= 3\alpha b^2 \partial_z b \\ &= 3\alpha (b_j^n)^2 \left[\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta z} \right] \end{aligned}$$

$$\text{Using } b_{j+1/2}^n = \frac{b_{j+1} + b_j}{2}$$

$$b_{j-1/2}^n = \frac{b_j + b_{j-1}}{2}$$

This becomes

$$\frac{3\alpha (b_j^n)^2}{2\Delta z} [b_{j+1} - b_{j-1}]$$

(5) becomes:

$$\frac{B_j^{n+1} - B_j^n}{\Delta t} + \frac{3\alpha(B_j^n)^2}{2\Delta z} (B_{j+1}^n - B_{j-1}^n) - \frac{\beta}{8(\Delta z)^2} \left\{ (B_{j+1}^n + B_j^n)^3 (B_{j+1}^n - B_j^n) - (B_j^n + B_{j-1}^n)^3 (B_j^n - B_{j-1}^n) \right\} = 0$$

$$B_j^{n+1} = B_j^n \left[1 - \frac{\beta \Delta t ((B_{j+1}^n + B_j^n)^3 + (B_j^n + B_{j-1}^n)^3)}{8(\Delta z)^2} \right]$$

$$+ B_{j+1}^n \left[- \frac{3\alpha(B_j^n)^2 \Delta t}{2\Delta z} + \frac{\beta \Delta t (B_{j+1}^n + B_j^n)^3}{8(\Delta z)^2} \right]$$

$$+ B_{j-1}^n \left[\frac{3\alpha(B_j^n)^2 \Delta t}{2\Delta z} + \frac{\beta \Delta t (B_j^n + B_{j-1}^n)^3}{8(\Delta z)^2} \right]$$

Coefficient of B_{j-1}^n is always ≥ 0

\rightarrow but what about coefficient of B_{j+1}^n

$$B_j^n$$

$$B_j^n \left[1 - \frac{\beta \Delta t}{8(\Delta z)^2} \left((B_{j+1}^n + B_j^n)^3 + (B_j^n + B_{j-1}^n)^3 \right) \right]$$

$$+ B_{j+1}^n \left[-\frac{3\alpha (B_j^n)^2 \Delta t}{2\Delta z} + \frac{\beta \Delta t}{8(\Delta z)^2} (B_{j+1}^n + B_j^n)^3 \right] > 0$$

$$\Delta t \left[\frac{\beta B_j^n}{8(\Delta z)^2} \left((B_{j+1}^n + B_j^n)^3 + (B_j^n + B_{j-1}^n)^3 \right) - \frac{\beta B_{j+1}^n}{8(\Delta z)^2} (B_{j+1}^n + B_j^n)^3 + \frac{3\alpha B_{j+1}^n (B_j^n)^2}{2\Delta z} \right] < B_j^n$$

$$\Delta t < \frac{1}{\left(\frac{\beta a}{8(\Delta z)^2} + \frac{3\alpha B_{j+1}^n B_j^n}{2\Delta z} - \frac{\beta b}{8(\Delta z)^2} \right) *}$$

$$\text{where } a = B_j^n \left[(B_{j+1}^n + B_j^n)^3 + (B_j^n + B_{j-1}^n)^3 \right]$$

$$b = \frac{B_{j+1}^n}{B_j^n} (B_{j+1}^n + B_j^n)^3$$

-0.5 for beta = 0 scheme is linearly unstable so more analysis is needed.

BUT this time step criterion only guarantees $B_j^{n+1} > 0$ if * is positive, otherwise it is always unstable.

$$2a), \partial_z (\alpha b^3 - \beta b^3 \partial_z b) = 0$$

$$\Rightarrow \alpha b^3 - \beta b^3 \frac{db}{dz} = Q \text{ where } Q \text{ is an integration const.}$$

$$\int \beta b^3 \frac{db}{dz} = \alpha b^3 - Q$$

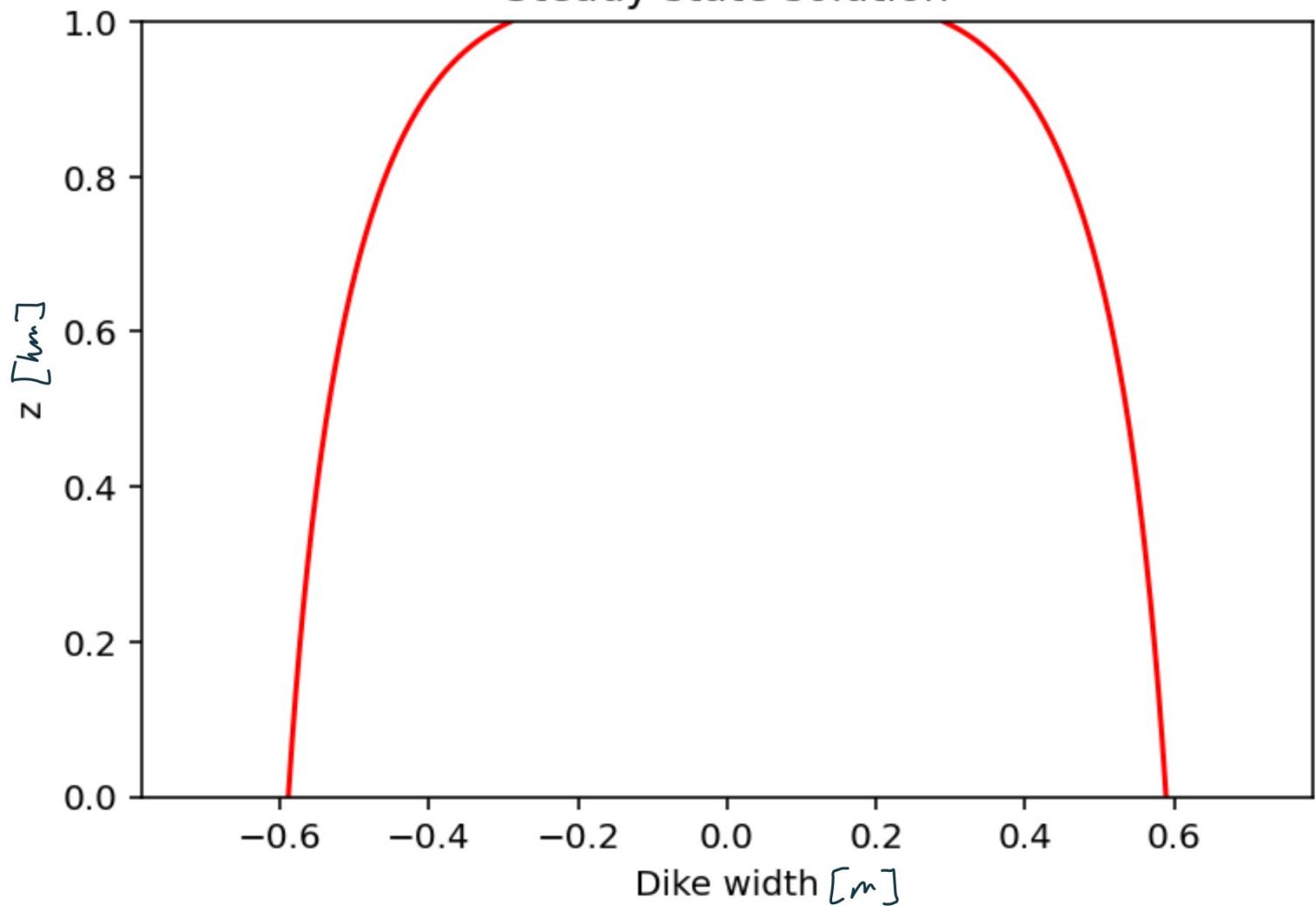
$$\beta b^3 \left(\frac{b_{j+1} - b_j}{\Delta z} \right) = \alpha b^3 - Q$$

$$\beta b_j^3 \left(\frac{b_{j+1} - b_j}{\Delta z} \right) = \alpha b_j^3 - Q$$

$$\frac{b_{j+1} - b_j}{\Delta z} = \frac{\alpha}{\beta} - \frac{Q}{\beta b_j^3}$$

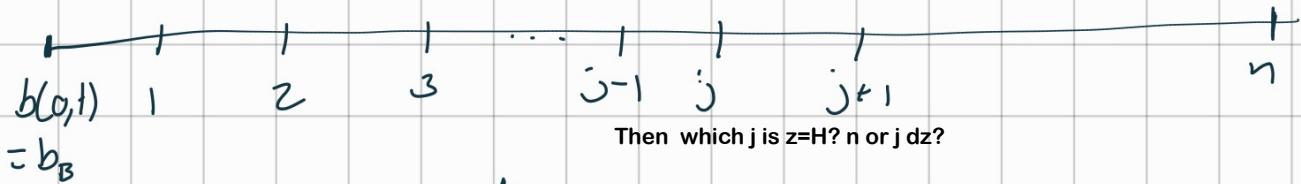
$$b_{j+1} = b_j + \frac{\Delta z}{\beta} \left(\alpha - \frac{Q}{b_j^3} \right)$$

Steady state solution



$$z=0$$

$$z = n \Delta z = H$$



$$H = 1$$

widths

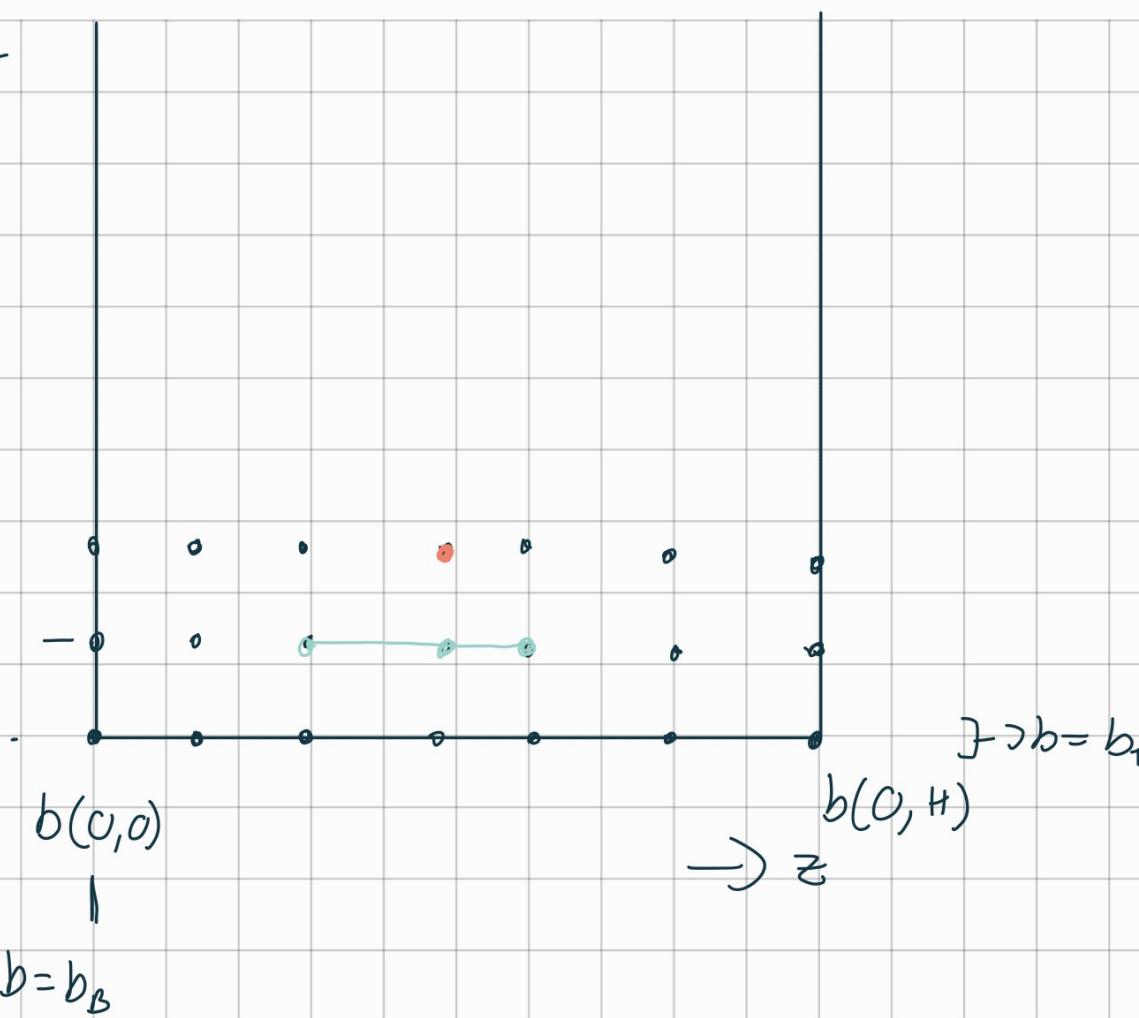
b)

$$\boxed{\frac{b_j^{n+1} - b_j^n}{\Delta t} + \frac{\alpha}{\Delta z} \left[(b_j^n)^3 - (b_{j-1}^n)^3 \right] - \frac{\beta}{8(\Delta z)^2} \left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n) \right]}$$

$$b_j^{n+1} = b_j^n - \frac{\alpha \Delta t}{\Delta z} \left[(b_j^n)^3 - (b_{j-1}^n)^3 \right]$$

$$+ \frac{\beta \Delta t}{8(\Delta z)^2} \left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n) \right]$$

γ^+

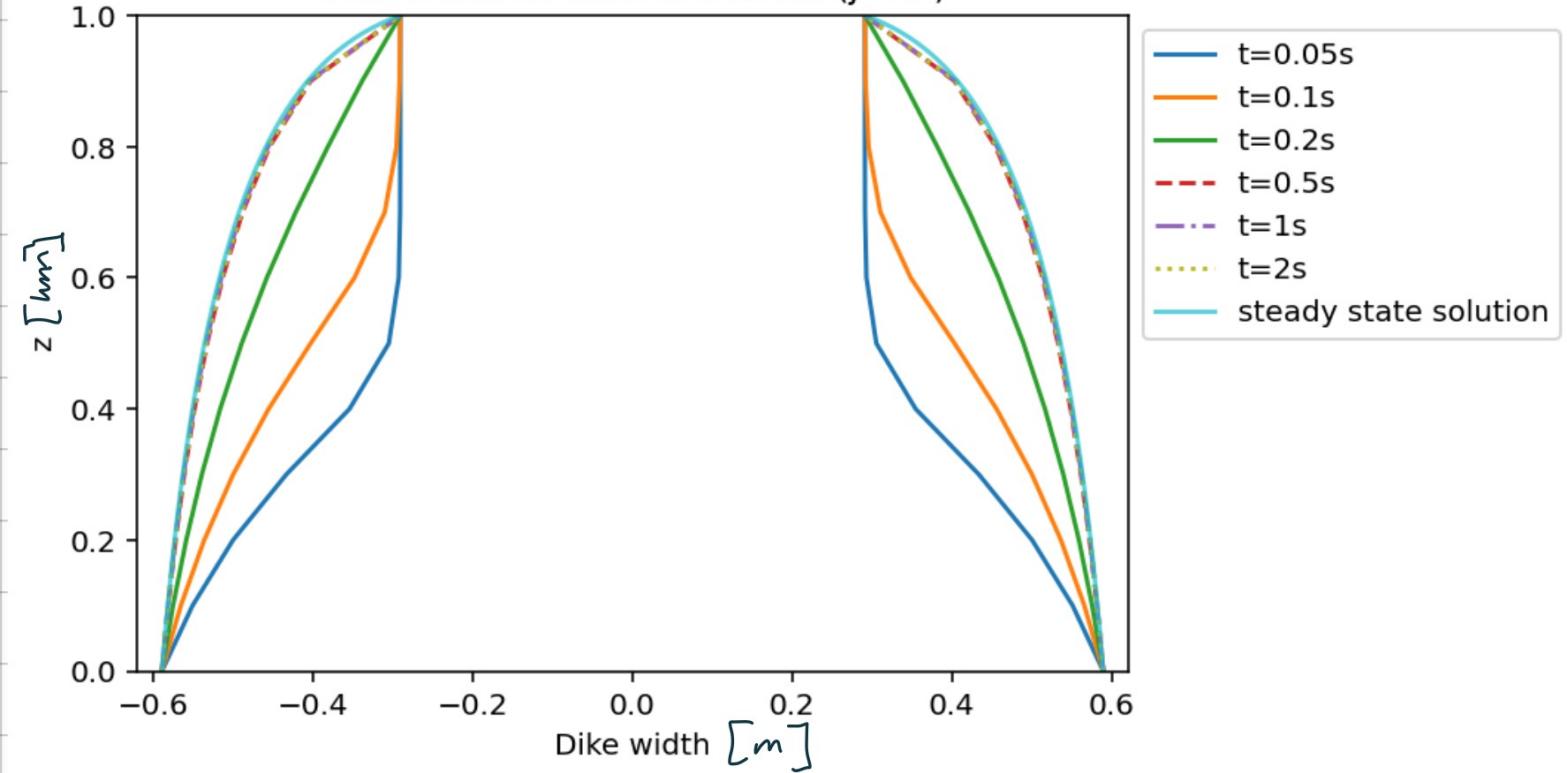


Δt chosen by finding maximum value of Δt for a stable scheme as done in earlier question

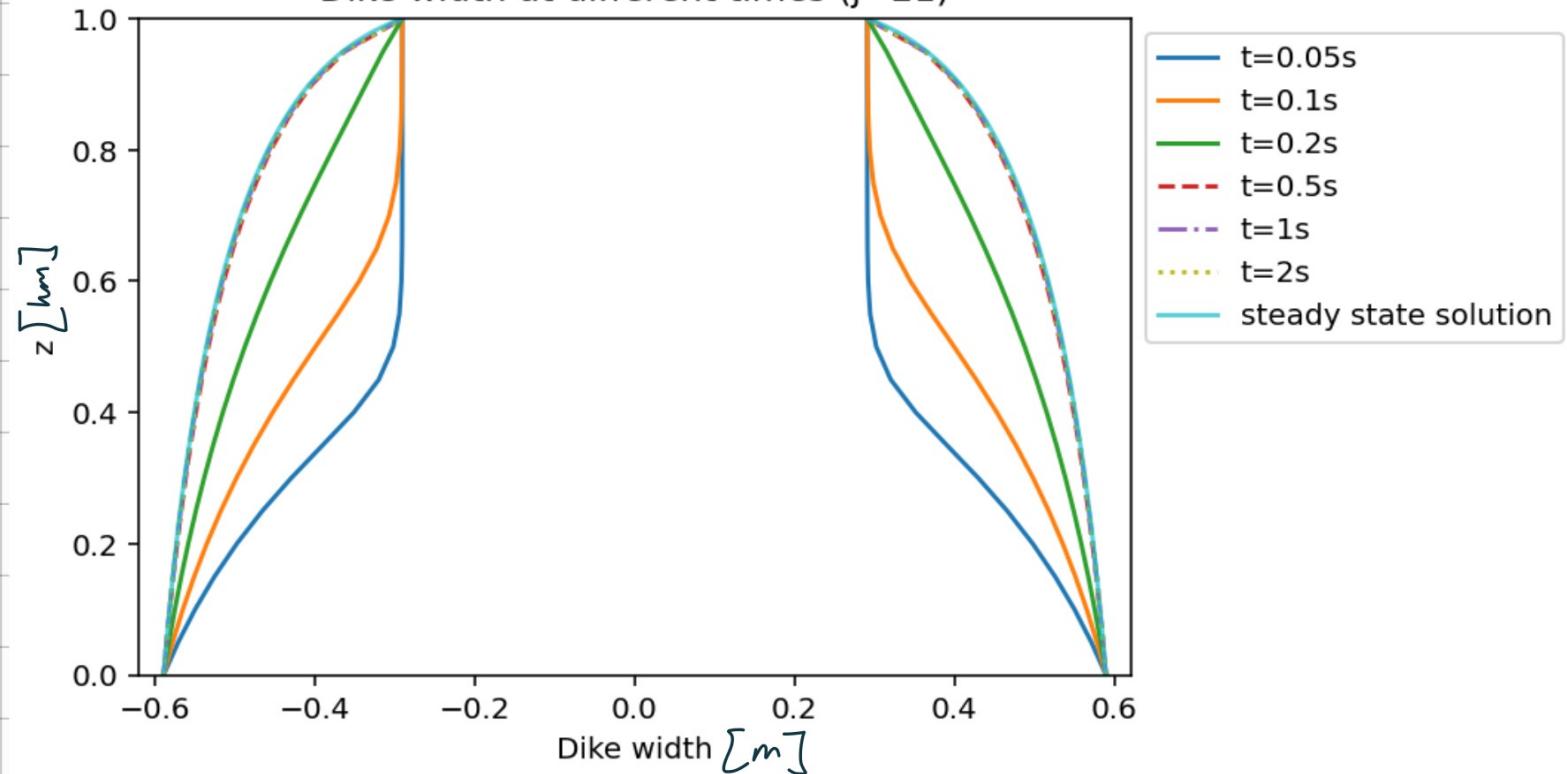
$$\Delta t < \frac{1}{B_{\max}^2 \left[\frac{\alpha}{\Delta z} + \frac{2\beta B_{\max}}{(\Delta z)^2} \right]}$$

\rightarrow avoid setting Δt as value slightly less than that.

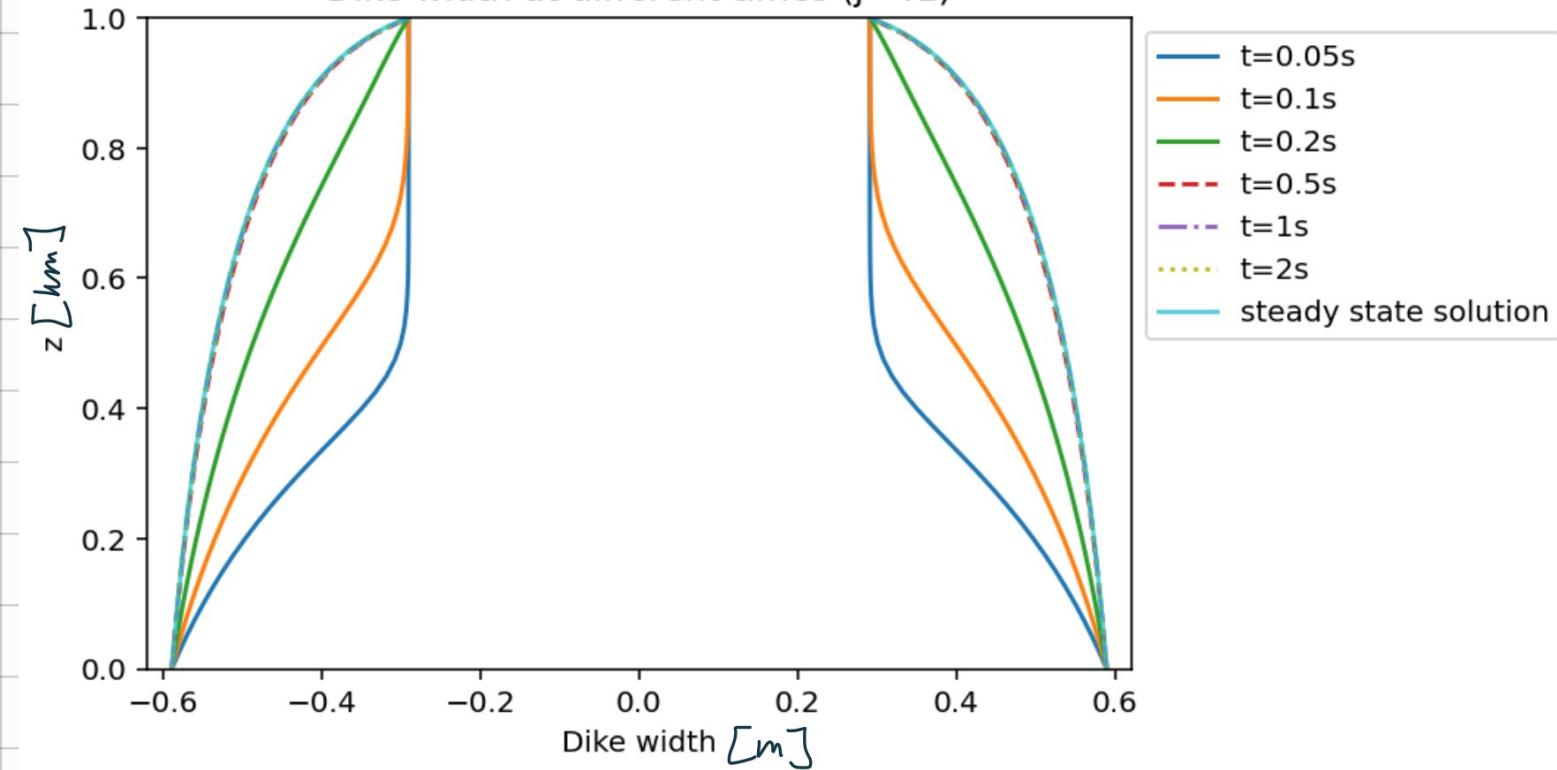
Dike width at different times ($j=11$)



Dike width at different times ($j=21$)



Dike width at different times ($j=41$)



$$\begin{aligned}
 c), L^2 &= \int_0^H e^z(z,t) dz \\
 &= \int_{j-1}^H \frac{1}{\Delta z} \left[e^z(0,z) + e^z(H,z) + 2 \sum_{n=1}^{j-2} e^z\left(\frac{nH}{j-1}, z\right) \right]
 \end{aligned}$$

Calculating L^2 and L^∞ using results:

j	Δz	L^2	L^∞
11	$\frac{1}{10}$	0.003633	0.005457
21	$\frac{1}{20}$	0.002212	0.003384
41	$\frac{1}{40}$	0.001235	0.001926

as Δz decreases by a factor of 2,

L^2 decreases by factor of $\sim 1.5 - 1.7$

L^∞ decreases by factor of $\sim 1.6 - 1.8$

So order of spatial discretisation $\sim 1^{\text{st}}$ order in Δz

should scale with factor Δz so factor 2; either your dt is too large or you have not reached steady state; -1.

$$d). z - z_{r_0} - ct = \frac{c}{\alpha} \left(b(z, t) - \sqrt{\frac{c}{\alpha}} \operatorname{arctanh} \left(\sqrt{\frac{\alpha}{c}} b(z, t) \right) \right)$$

$$b = \frac{\alpha}{\beta} (z - z_{r_0} - ct) + \sqrt{\frac{c}{\alpha}} \operatorname{arctanh} \left(\sqrt{\frac{\alpha}{c}} b \right)$$

$$\partial_t b = -\frac{\alpha c}{\beta} + \sqrt{\frac{c}{\alpha}} \cdot \sqrt{\frac{\alpha}{c}} \partial_t b \cdot \frac{1}{1 - \frac{b^2 \alpha}{c}}$$

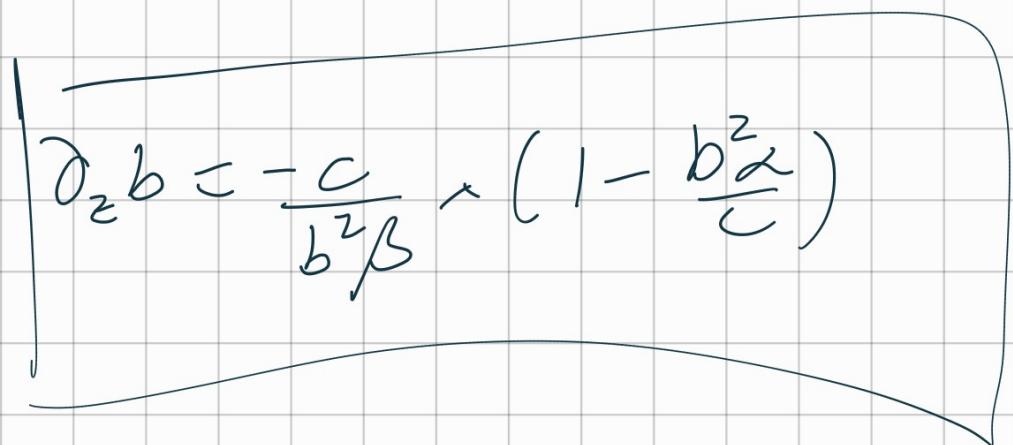
$$= -\frac{\alpha c}{\beta} + \partial_t b \cdot \frac{1}{1 - \frac{b^2 \alpha}{c}}$$

$$\partial_t b \left(1 - \frac{1}{1 - \frac{b^2 \alpha}{c}} \right) = -\frac{\alpha c}{\beta}$$

$$\partial_t b \left(\frac{-b^2 \alpha}{1 - \frac{b^2 \alpha}{c}} \right) = -\frac{\alpha c}{\beta}$$

$$\boxed{\partial_t b = \frac{c^2}{b \beta} \times \left(1 - \frac{b^2 \alpha}{c} \right)}$$

$$\partial_z b = \frac{2}{\beta} + \partial_z b + \frac{1}{(1 - b^2 \alpha)} \quad (1)$$



$$\partial_z (\alpha b^3) = 3 \alpha b^2 \partial_z b$$

$$= -\frac{3 \alpha c}{\beta} \left(1 - \frac{b^2 \alpha}{c}\right)$$

$$\partial_z (-\beta b^3 \partial_z b) = \partial_z \left(c b \left(1 - \frac{b^2 \alpha}{c}\right) \right)$$

$$= c \partial_z \left(b - \frac{b^3 \alpha}{c} \right)$$

$$= c \left(1 - \frac{3b^2 \alpha}{c}\right) \partial_z b$$

$$= -c \left(1 - \frac{3b^2 \alpha}{c}\right) \times \frac{c}{b^2 \beta} \left(1 - \frac{b^2 \alpha}{c}\right)$$

$$= -\frac{c^2}{b^2 \beta} \left(1 - \frac{3b^2 \alpha}{c}\right) \left(1 - \frac{b^2 \alpha}{c}\right)$$

Combine

$$\partial_t b + \partial_z (\alpha b^3) - \partial_z (\beta b^3 \partial_z b)$$

$$= \frac{c^2}{b^2 \beta} \left(1 - \frac{b^2 \alpha}{c}\right) - \frac{3\alpha c}{\beta} \left(1 - \frac{b^2 \alpha}{c}\right)$$

$$-\frac{c^2}{b^2 \beta} \left(1 - \frac{3b^2 \alpha}{c}\right) \left(1 - \frac{b^2 \alpha}{c}\right)$$

$$= \frac{c^2}{b^2 \beta} - \frac{\alpha c}{\beta} - \frac{3\alpha c}{\beta} + \frac{3\alpha^2 b^2}{\beta}$$

$$-\frac{c^2}{b^2 \beta} \left(1 - \frac{4b^2 \alpha}{c} + \frac{3b^4 \alpha^2}{c^2}\right)$$

$$= \frac{c^2}{b^2 \beta} - \frac{4\alpha c}{\beta} + \frac{3\alpha^2 b^2}{\beta} - \frac{c^2}{b^2 \beta} + \frac{4\alpha c}{\beta} - \frac{3\alpha^2 b^2}{\beta}$$

$$= 0$$

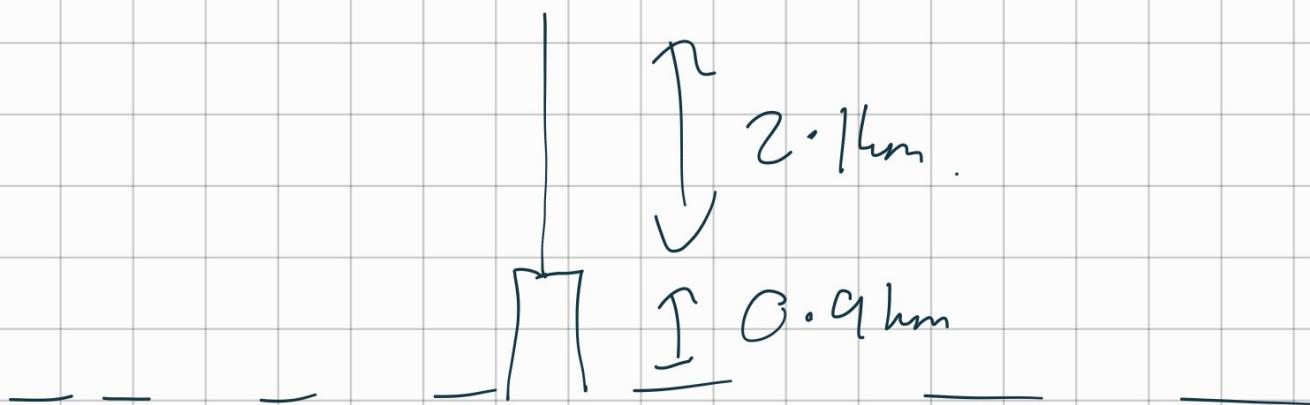
$$\partial_t b + \partial_z (\alpha b^3 - \beta b^3 \partial_z b) = 0$$

$$\text{when } z - z_{r_0} - ct = \frac{\beta}{\alpha} \left(b(z, t) - \sqrt{\frac{\alpha}{\beta}} \arctan \left(\sqrt{\frac{\alpha}{\beta}} b(z, t) \right) \right)$$

so this is a solution to the equation.

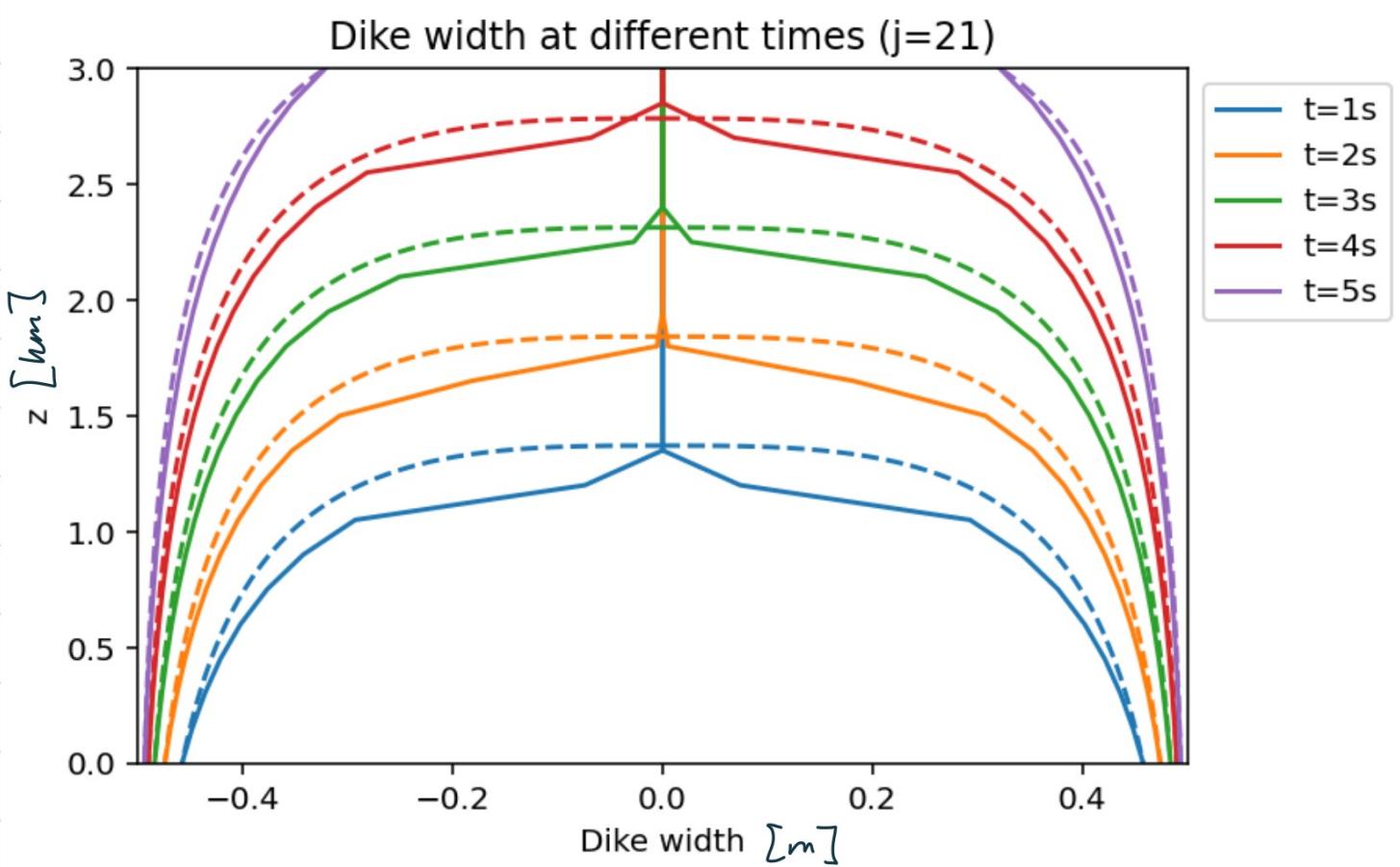
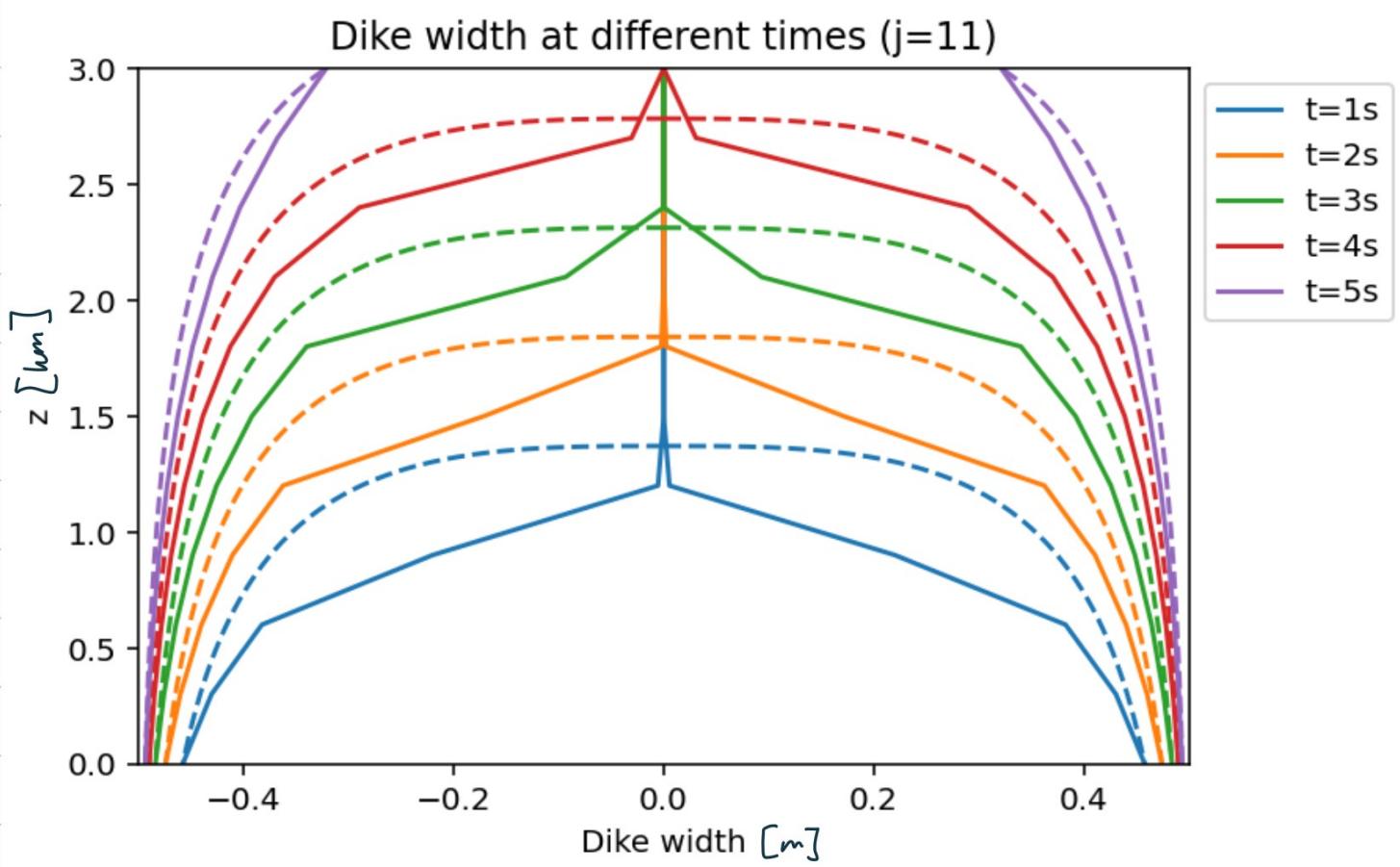
- Used maximum principle to determine maximum time step

→ where $b_{\max} = 1$

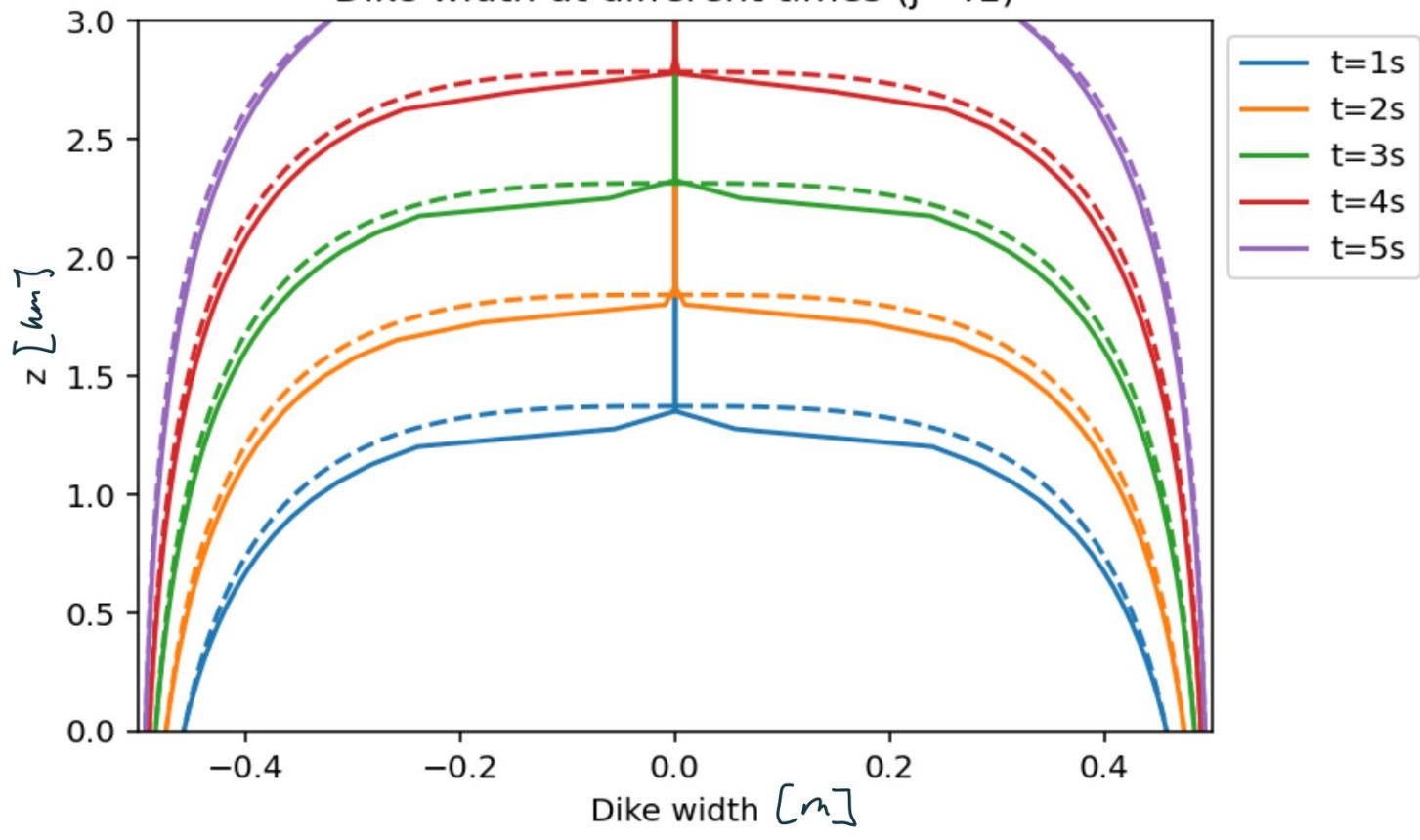


$$z_{r_0} = 2.1 - 0.9 \text{ km}$$

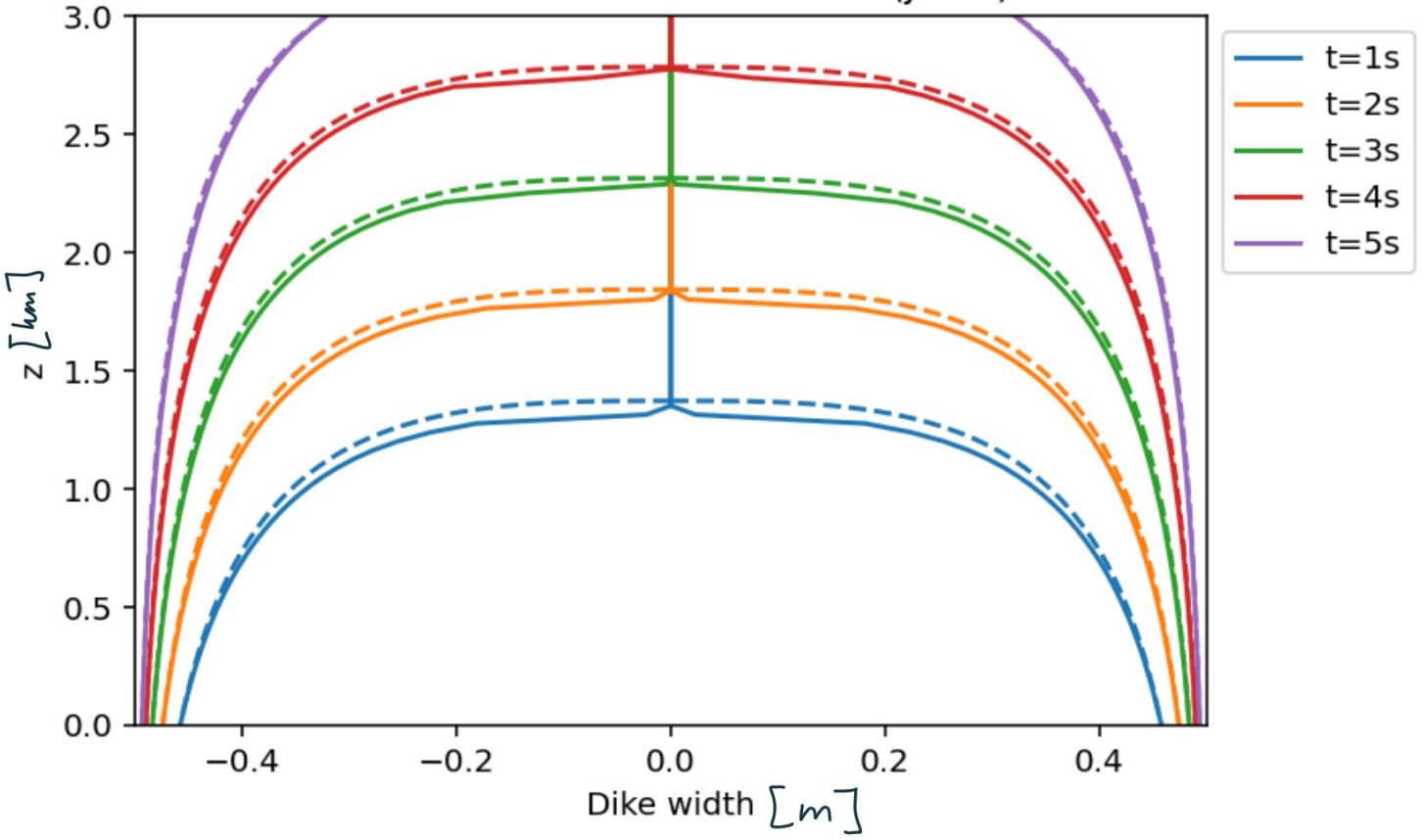
For all of these plots, the dashed lines
 --- correspond to the exact solution, and
 solid lines — correspond to the numerical solution



Dike width at different times ($j=41$)



Dike width at different times ($j=81$)



Just keep going; you need more points and perhaps your dt is too large. -0.5. 18/20

e).

$$\left| \frac{b_j^{n+1} - b_j^n}{\Delta t} + \frac{\alpha}{\Delta z} \left[(b_j^n)^3 - (b_{j-1}^n)^3 \right] \right.$$
$$\left. - \frac{\beta}{8(\Delta z)^2} \left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n) \right] \right|$$



Solve Crank-Nicolson scheme.

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} + \frac{\alpha}{2\Delta z} \left[(b_j^{n+1})^3 - (b_{j+1}^{n+1})^3 + (b_j^n)^3 - (b_{j-1}^n)^3 \right]$$

$$- \frac{\beta}{16(\Delta z)^2} \left[(b_{j+1}^{n+1} + b_j^{n+1})^3 (b_{j+1}^{n+1} - b_j^{n+1}) - (b_j^{n+1} + b_{j-1}^{n+1})^3 (b_j^{n+1} - b_{j-1}^{n+1}) \right]$$

$$- \frac{\beta}{16(\Delta z)^2} \left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n) \right]$$

$$= 0$$

which is very complicated

let $R =$

$$b_j^{n+1} - b_j^n + k_1 \left[(b_j^{n+1})^3 - (b_{j-1}^{n+1})^3 + (b_j^n)^3 - (b_{j-1}^n)^3 \right]$$

$$- h_2 \left[(b_{j+1}^{n+1} + b_j^{n+1})^3 (b_{j+1}^{n+1} - b_j^{n+1}) - (b_j^{n+1} + b_{j-1}^{n+1})^3 (b_j^{n+1} - b_{j-1}^{n+1}) \right]$$

$$- k_2 \left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n) \right]$$

$$= 0$$

$$\rightarrow \text{note } k_1 = \frac{\alpha \Delta t}{2 \Delta z}, \quad k_2 = \frac{\beta \Delta t}{16 (\Delta z)^2}$$

$$= b_j^{n+1} - b_j^n$$

$$+ h_1 \left[(b_j^{n+1})^3 - (b_{j-1}^{n+1})^3 \right] + h_2 \left[(b_j^{n+1} + b_{j-1}^{n+1})^3 (b_j^{n+1} - b_{j-1}^{n+1}) \right]$$

$$+ h_1 \left[(b_j^n)^3 - (b_{j-1}^n)^3 \right] + h_2 \left[(b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n) \right]$$

$$-k_2 \left[(b_{j+1}^{n+1} + b_j^{n+1})^3 (b_{j+1}^{n+1} - b_j^{n+1}) \right]$$

$$-k_2 \left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) \right] = 0$$

Newton scheme for iterations

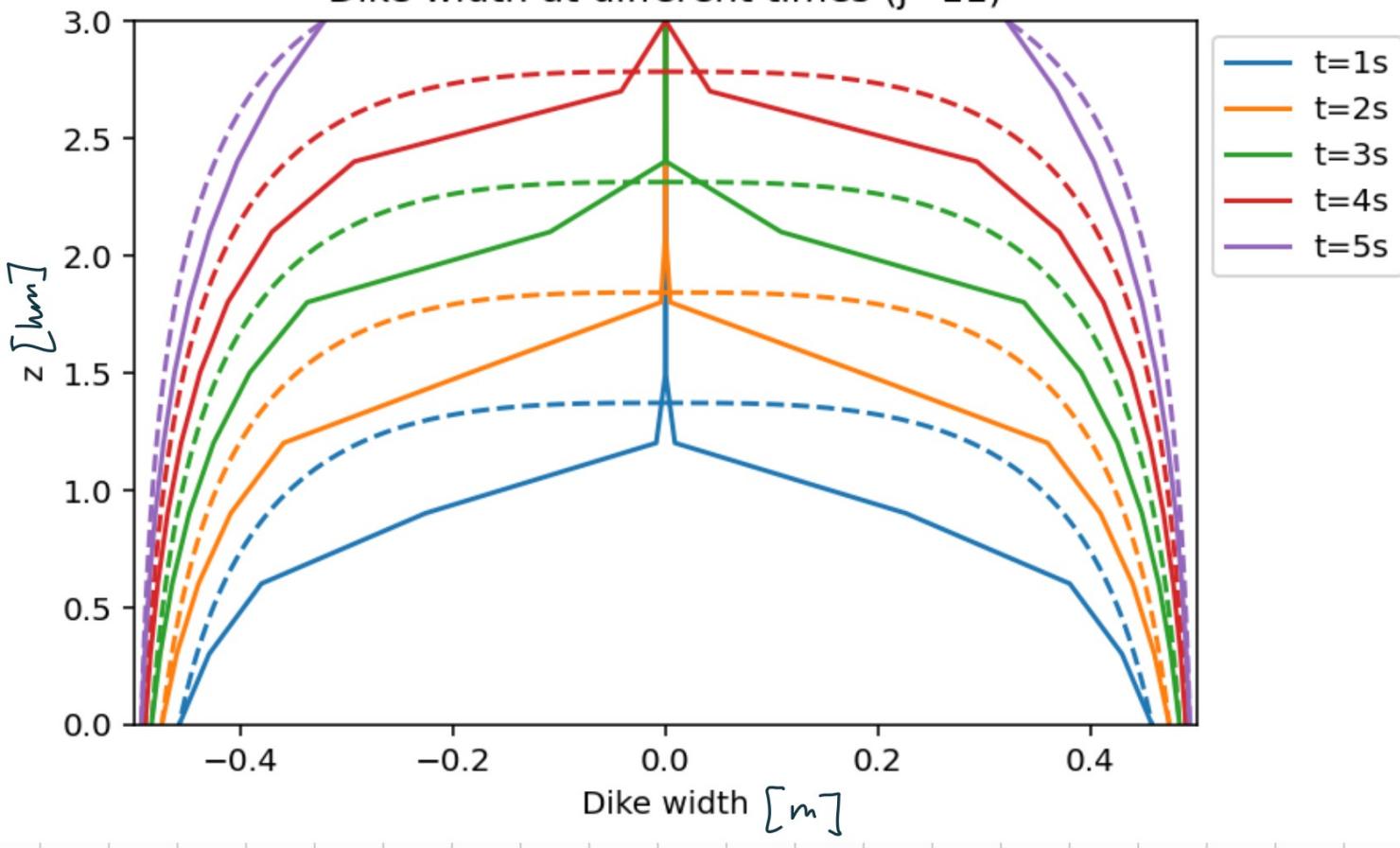
$$\begin{aligned} J_{ij} &= \frac{\partial R_{ij}}{\partial b_{ij}} \\ &= \frac{R_{ij,\text{pert}} - R_{ij}}{\epsilon} \end{aligned}$$

$$\rightarrow \text{Then solve } J\delta b = -R$$

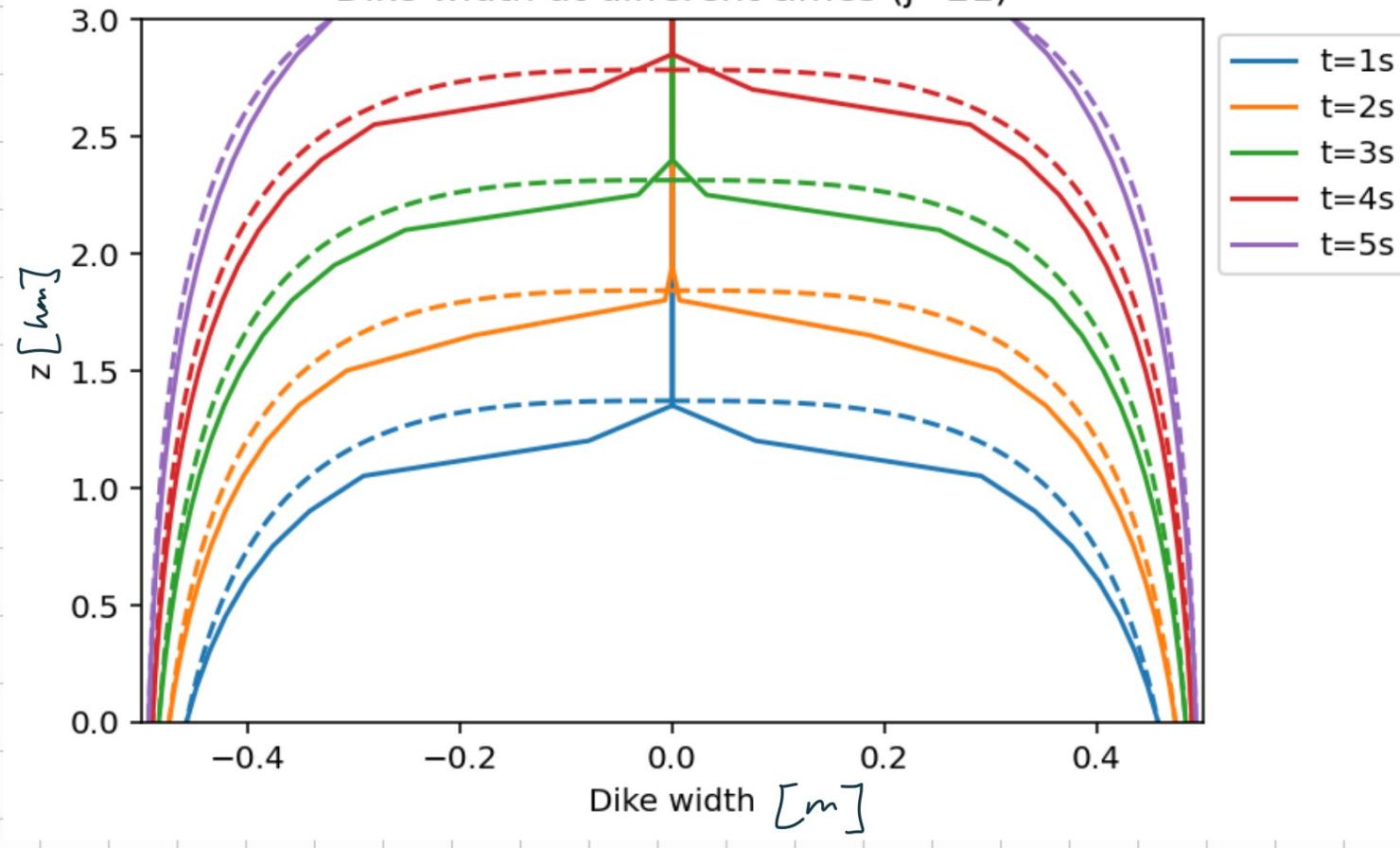
$$\text{and } b_{\text{new}} = b_{\text{old}} + \delta b$$

- - = exact
 — = numerical

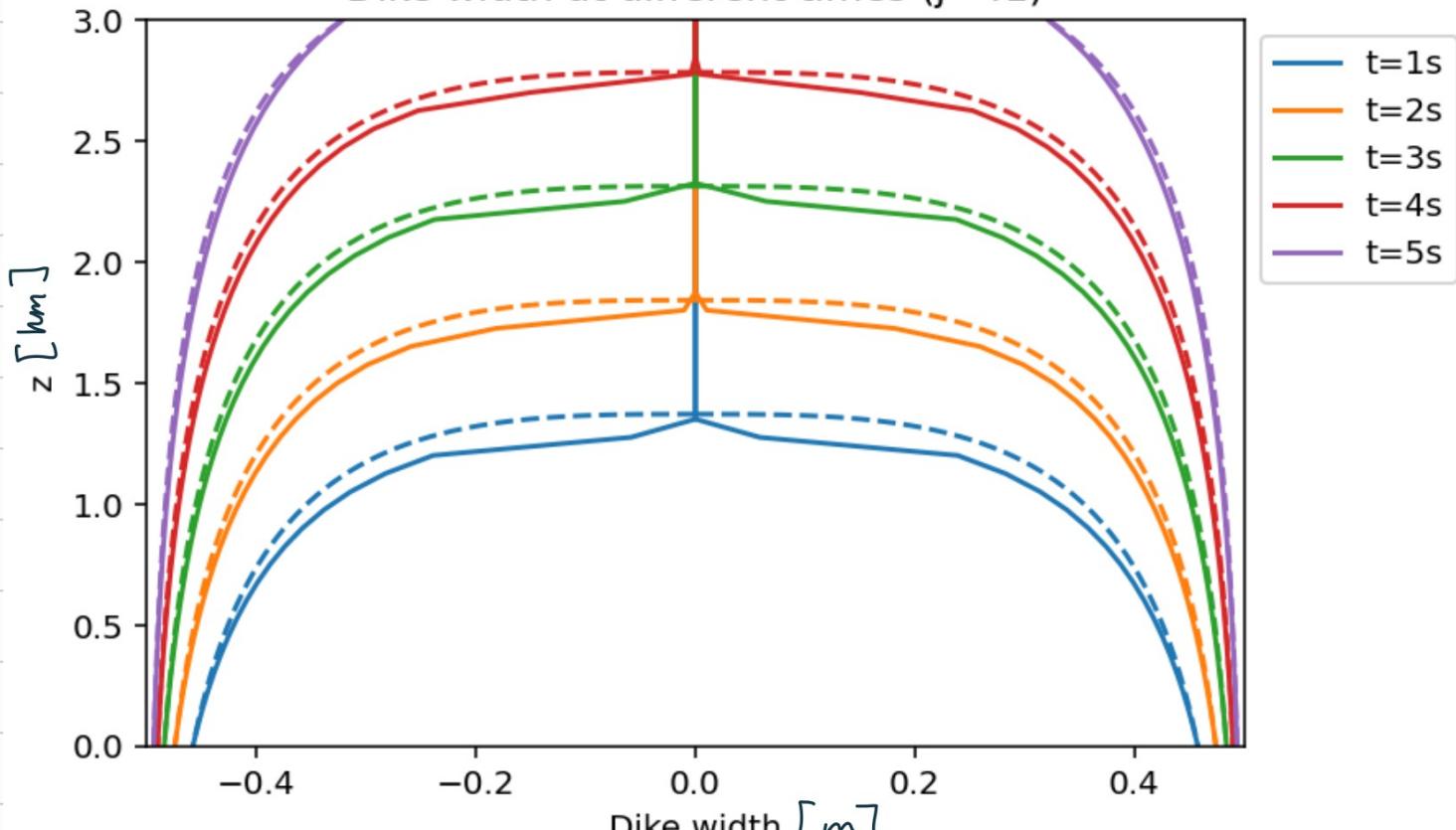
Dike width at different times ($j=11$)



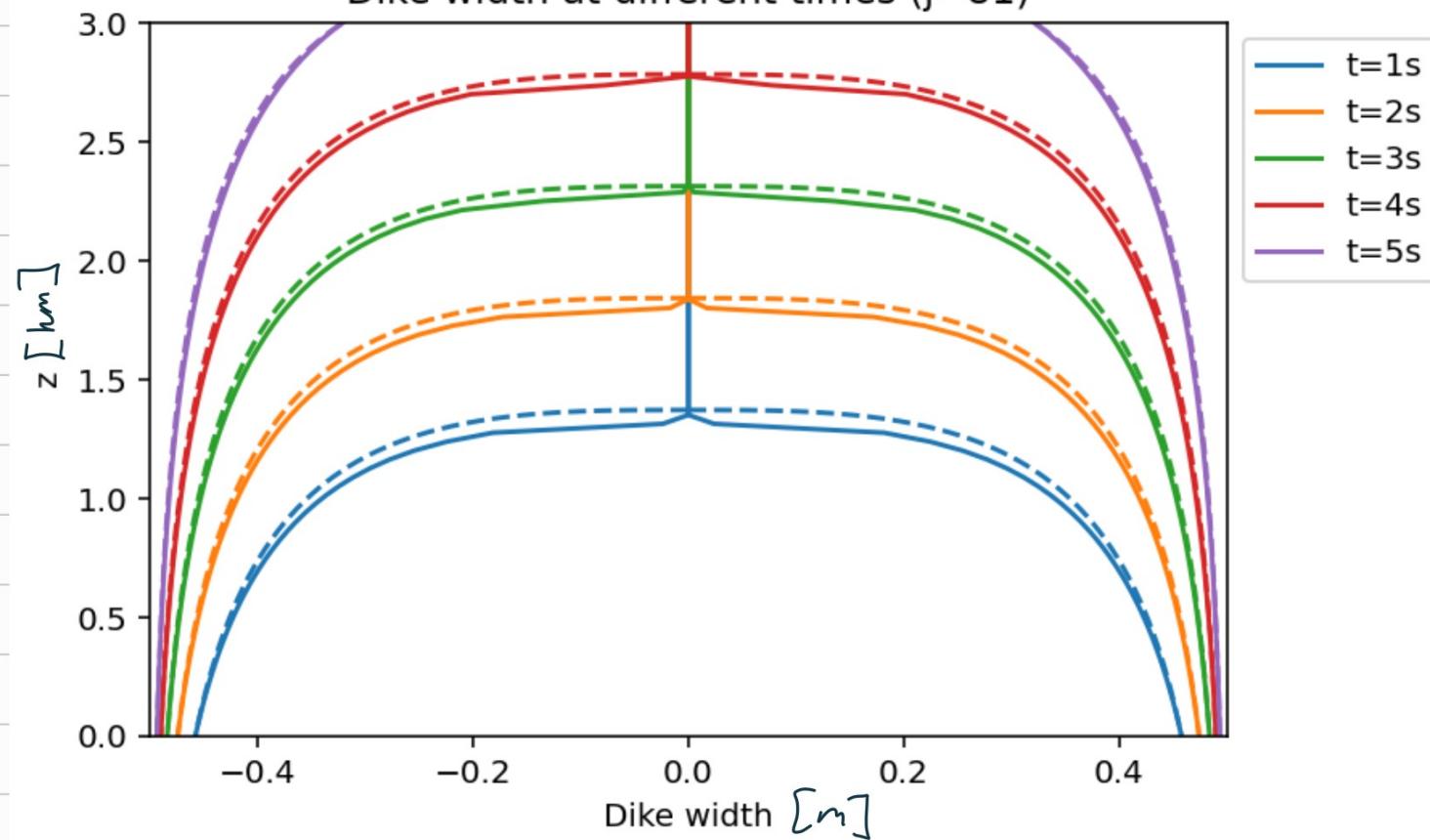
Dike width at different times ($j=21$)



Dike width at different times ($j=41$)



Dike width at different times ($j=81$)



Scheme Span

j

Time /s

d)

11
21
41
810.06304
0.2115
0.9744
5.7097

e)

11
21
41
810.4586
5.2310
67.3454
1028.8300

Scheme Span d) is much faster than e).

(although, looking at my code, could make both schemes faster by making them more efficient when calculating for different time).

- Makes sense d) is faster, as it does not have to iterate several times to find the value of b_{nn}