MATH 5453M

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# Numerical Exercises 1 : Answers

1a) i) Simplify the system : $\partial_t b + \partial_z (ub) = 0$ , $u = \alpha b^2 - \beta b^2 \partial_z b$, $\partial_t b = \frac{\partial b}{\partial t}$ , $\alpha > 0$, $\beta > 0$ to a one convection - diffusion equation for $b(z,t)$

Physical context : Modeling how magma flows through a vertical dike with elastic walls. The width of the dike $b(z,t)$ evolves overtime due to pressure and elasticity

- Substitute $u$ in $\frac{\partial b}{\partial t} + \frac{\partial}{\partial z}(ub) = 0$

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial z}\left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z}\right) = 0 \qquad (5)$$

ii) Linearise this equation around $b = D_0$, assuming that $b'$ is small into (5) to get (6)

Approximating a non linear equation near a known solution by assuming small pertubations for example $f(x)$ at $x = a$, $L(x) = f(a) + f'(a)(x-a)$

- With a small pertubation of $b(z,t) = D_0 + b'(z,t)$ where $b' << D_0$

→ Substitute into PDE and expand using taylor approximations

$$b(z,t)^3 = (D_0 + b'(z,t))^3 = D_0^3 + 3D_0^2 b' + \dots$$

→ linearisation is a first order approximation so only retain terms linear in $b'$ and neglect higher order since $b'$ is small

$$\frac{\partial b}{\partial z} = \frac{\partial b'}{\partial z} \quad \text{as } D_0 \text{ is a constant}$$

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial z}\left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z}\right) = 0 \quad \text{with} \quad b(z,t)^3 = D_0^3 + 3D_0^2 b'$$

$$\Rightarrow \frac{\partial b'}{\partial t} + \frac{\partial}{\partial z}\left((D_0^3 + 3D_0^2 b')\alpha - (D_0^3 + 3D_0^2 b')\beta \frac{\partial b'}{\partial z}\right) = 0$$

$$\Rightarrow \frac{\partial b'}{\partial t} + \frac{\partial}{\partial z}\left((\alpha D_0^3 + 3\alpha D_0^2 b') + \left(\beta D_0^3 \frac{\partial b'}{\partial z}\right)\right) = 0$$

$$\Rightarrow \frac{\partial b'}{\partial t} + 3\alpha D_0^2 \frac{\partial b'}{\partial z} + \beta D_0^3 \frac{\partial^2 b'}{\partial z^2} = 0$$ which gives equation (6) from the task sheet

iii) Why are (5) and (6) called non linear convection - diffusion equations ?

These are convection-diffusion equations as they have a transport term (convection) and a smoothing (diffusion term) which are both dependent on b

$\alpha b^3 \rightarrow$ Convection term $\rightarrow$ flow of magma driven by pressure

$-\beta b^3 \frac{\partial b}{\partial z} \rightarrow$ Diffusion term

They take the general form of the sum of a convective effect $\propto \frac{\partial b'}{\partial z}$ and diffusive effect $\propto \frac{\partial b'^2}{\partial^2 z}$

1b i) Discretize (5) and (6) with a forward Euler time discretisation, an upwind scheme and second order central difference scheme

→ Forward Euler Time discretisation: Uses current state $b_j^n$ to compute fluxes and explicitely step forward in time
→ Upwind Scheme: Convection considers directional transport so an upwind scheme uses spatial differences based on the direction of flow. Use a backward difference whilst assuming flow is upward
→ Central Difference Scheme: Approximates the spatial derivative of the diffusion term using a formula that considers the values of the function at points centered around the grid point of interest to approx $\frac{\partial^2 b}{\partial z^2}$

→ Setup for discretisation

$$z_j = j\Delta z \quad \text{for } j = 0,1,2\ldots J, \quad t_n = n\Delta t, \quad b_j^n \approx b(z_j, t_n), \quad b_o^n \approx b_o(z_j, t_n)$$

→ Time Derivative (Forward Euler)

$$\frac{\partial b}{\partial t} \approx \frac{b_j^{n+1} - b_j^n}{\Delta t} \qquad \text{for both linear and non linear PDE}$$

→ Convection Term (Upwind scheme)

Non linear : $\frac{\partial}{\partial z}(\beta b^3) = \beta \frac{\partial}{\partial z} b^3 \longrightarrow \approx \beta \frac{(b_j^n)^3 - (b_{j-1}^n)^3}{\Delta z}$ [Assuming flow is upward so doing backward difference as $3\alpha D_o^2$ is a positive coefficient]

Linear : $3\beta D_o^2 \frac{\partial b_o}{\partial z} \approx 3\beta D_o^2 \frac{b_{o,j}^n - b_{o,j-1}^n}{\Delta z}$

→ Diffusive Term (Central Difference Scheme Second Order)

Non linear : Use the adjoint form

$$b_{j+\frac{1}{2}}^n = \frac{1}{2}\left(b_j^n + b_{j+1}^n\right), \quad \left(\frac{\partial b}{\partial z}\right)_{j+\frac{1}{2}}^n \approx \frac{b_{j+1}^n - b_j^n}{\Delta z}$$

$$\Rightarrow U_{j+\frac{1}{2}}^n = \beta\left(b_{j+\frac{1}{2}}^n\right)^2 - \alpha\left(b_{j+\frac{1}{2}}^n\right)^3\left(\frac{b_{j+1}^n - b_j^n}{\Delta z}\right)$$

$$\Rightarrow \frac{\partial}{\partial z}\left(\alpha b^3 \frac{\partial b}{\partial z}\right) \approx \frac{U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n}{\Delta z}$$

Linear :

$$-\alpha D_o^3 \frac{\partial^2 b_o}{\partial z^2} \approx -\alpha D_o^3 \frac{b_{o,j+1}^n - 2b_{o,j}^n + b_{o,j-1}^n}{(\Delta z)^2}$$

→ Full discretised forms :

Non linear $\quad b_j^{n+1} = b_j^n - \Delta t\left(\frac{U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n}{\Delta z}\right)$ where $U_{j+\frac{1}{2}}^n = \overset{\alpha}{\cancel{\beta}}\left(b_{j+\frac{1}{2}}^n\right)^2 - \overset{\beta}{\cancel{\alpha}}\left(b_{j+\frac{1}{2}}^n\right)^3 \frac{b_{j+1}^n - b_j^n}{\Delta z}$

Linear $\quad b_{o,j}^{n+1} = b_{o,j}^n - \Delta t\left(3\alpha D_o^2\left(\frac{b_{o,j}^n - b_{o,j-1}^n}{\Delta z}\right) - \beta D_o^3\left(\frac{b_{o,j+1}^n - 2b_{o,j}^n + b_{o,j-1}^n}{(\Delta z)^2}\right)\right)$

Non linear (Expanded) : $\quad b_j^{n+1} = b_j^n - \Delta t\left(\alpha \frac{{b_j^n}^3 - {b_{j-1}^n}^3}{\Delta z} + \beta \frac{{b_{j+\frac{1}{2}}^n}^3\left(b_{j+1}^n - b_j^n\right) - {b_{j-\frac{1}{2}}^n}^3\left(b_j^n - b_{j-1}^n\right)}{(\Delta z)^2}\right)$

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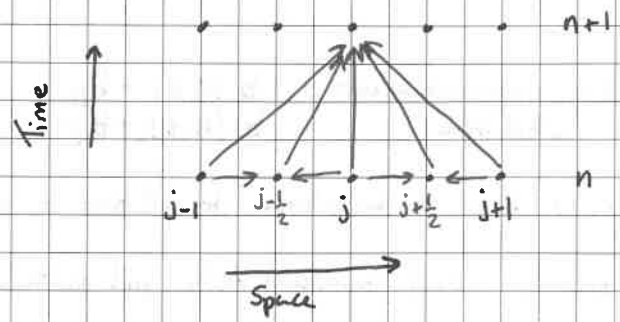
ii) Make a sketch of your grid and its numbering




iii) Check whether the discretization of (6) is the linearized version of the discretization of (5)

- To do this substitute $b_j^n = D_0 + b_{o,j}^n$ into the non linear equation and expand the terms using taylor expansions where $D_0$ is constant and $b_{o,j}^n$ is small

$$b_{j+\frac{1}{2}}^n = \frac{1}{2}\left(D_0 + b_{o,j}^n + D_0 + b_{o,j+1}^n\right) = D_0 + \frac{1}{2}\left(b_{o,j}^n + b_{o,j+1}^n\right)$$

$$\left(b_{j+\frac{1}{2}}^n\right)^2 \approx D_0^2 + D_0\left(b_{o,j}^n + b_{o,j+1}^n\right)$$

$$\left(b_{j+\frac{1}{2}}^n\right)^3 \approx D_0^3 + \frac{3}{2}D_0^2\left(b_{o,j}^n + b_{o,j+1}^n\right)$$

$$\frac{b_{j+1}^n - b_j^n}{\Delta z} = \frac{b_{o,j+1}^n - b_{o,j}^n}{\Delta z}$$

- Take these terms and put them into $U_{j+\frac{1}{2}}$ expression

$$U_{j+\frac{1}{2}}^n \approx \alpha\left(D_0^2 + D_0\left(b_{o,j}^n + b_{o,j+1}^n\right) - \beta\left(D_0^3 + \frac{3}{2}D_0^2\left(b_{o,j}^n + b_{o,j+1}\right)\right)\right)\frac{b_{o,j+1}^n - b_{o,j}^n}{\Delta z}$$

- Using $\frac{U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n}{\Delta z}$ you can get a discretized expression from substituting in each of the above

$$\Longrightarrow b_{o,j}^{n+1} = b_{o,j}^n - \Delta t\left[3\alpha D_0^2\left(\frac{b_{o,j}^n - b_{o,j-1}}{\Delta z}\right) - \beta D_0^3\left(\frac{b_{o,j+1}^n - 2b_{o,j}^n + b_{o,j-1}^n}{(\Delta z)^2}\right)\right.$$

This shows the discretisation of the linear equation (6) is the linearised version of the discretisation of the nonlinear equation 5. This means that the numerical scheme behaves consistently under small pertubations given by the assumptions.

iv) What is the advantage of using the adjoint form for the nonlinear diffusive term?

- Adjoint form conserves mass

- The adjoint form of the diffusive term in ~~linearising~~ discretising the non linear form of the equation ensures that there is mass conservation within the system

v) Clearly indicate how you implement the boundary conditions

From the tasksheet given as:

Dirichlet Conditions
↳ At the bottom of the dike, magma chamber $b(0,t) = b_B$
↳ At the top of the dike, atmosphere $b(H,t) = b_T$

These are the values of the solution of b at the boundaries for all time t and are set by the geometry of the dike

The initial condition is $b_{0,j} = b_0$, ~~the~~ ~~and~~ previously overleaf is a sketch of the grid on which this is calculated, showing the flow of information

c) Use a fourier analysis to assess the stability of the linearized numerical scheme

- Need to use fourier analysis to determine the stability condition. Such that what time step $\Delta t$ keeps the scheme stable as in $|\lambda| \leq 1$ for two different cases

$$\alpha = 0, \ \beta \neq 0 \leftarrow \text{Pure convection}$$
$$\alpha \neq 0, \ \beta = 0 \leftarrow \text{Pure diffusion}$$

- Fourier mode from Morton and Mayers (2.72) $U_j^n = \lambda^n e^{ikj\Delta x}$ which represents a wavelike solution where $\lambda$ is the amplification factor per time step

$$U_{j+1}^n = \lambda^n e^{ik(j+1)\Delta x} = U_j^n e^{ik\Delta x} = \lambda U_j^n$$
$$U_{j-1}^n = \lambda^n e^{ik(j-1)\Delta x}$$

Using,
$$U_j^{n+1} = U_j^n - \Delta t \left( 3\cancel{\beta}\alpha D_0^2 \left( \frac{U_j^n - U_{j-1}^n}{\Delta x} \right) - \beta D_0^3 \left( \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} \right) \right)$$

Sub in fourier expressions to get

$$U_j^{n+1} = U \left[ 1 - \Delta t \left( \frac{3\alpha D_0^2}{\Delta x}(1 - e^{-ik\Delta x}) - \frac{\beta D_0^3}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \right) \right]$$

$$\lambda \cancel{U_j^n} = \cancel{U_j^n} \left[ 1 - \Delta t \left( \frac{3\alpha D_0^2}{\Delta x}(1 - e^{-ik\Delta x}) - \frac{\beta D_0^3}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \right) \right]$$

$$\lambda = 1 - \Delta t \left( \frac{3\alpha D_0^2}{\Delta x}(1 - e^{ik\Delta x}) - \frac{\beta D_0^3}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{ik\Delta x}) \right)$$

Use this expression to assess the stability for the pure convection and pure diffusion case

Pure ~~Convection (α=0)~~ Diffusion ($\beta = 0$)

$$\lambda = 1 - \Delta t \left( \frac{3\alpha D_0^2}{\Delta z}(1 - e^{-ik\Delta x}) - \cancel{\frac{\beta D_0^3}{\Delta z^2}}(e^{ik\Delta x} - 2 + e^{-ik\Delta x})^{\nearrow 0} \right) \Rightarrow \lambda = 1 - \Delta t \frac{3\alpha D_0^2}{\Delta z}(1 - e^{-ik\Delta x})$$

- stability is thought of as when errors from rounding or initial conditions don't grow uncontrollably as it progresses
If $|\lambda| \leq 1$, system is stable
If $|\lambda| > 1$, system is unstable and errors grow exponentially

To get the maximum possible diffusive term and therefore the mode with the highest frequency and most prone instability. If this constraint is satisfied then all lower modes will remain stable

$$1 - e^{-ik\Delta x} \leftarrow \text{max when } k\Delta x = \pi \text{ so that } e^{-ik\Delta x} = e^{-i\pi} = -1$$

$$\lambda = 1 - \Delta t \frac{3\alpha D_0^2}{\Delta z}(1 - -1) = 1 - \Delta t \frac{6\alpha D_0^2}{\Delta z}$$

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For $|\lambda| \le 1$ then $\left| 1 - \Delta t \frac{6\alpha D_0^2}{\Delta z} \right| \le 1$ so

$$1 - \frac{6\alpha D_0^2 \Delta t}{\Delta z} \geqslant -1$$

$$-2 \le -\frac{6\alpha D_0^2 \Delta t}{\Delta z}$$

$$\Longrightarrow \quad \Delta t \le \frac{2\Delta z}{6\alpha D_0^2} = \frac{\Delta z}{3\alpha D_0^2}$$

Pure Convection $(\alpha = 0)$

$$\lambda = 1 - \Delta t \left( \cancelto{0}{\frac{3\alpha D_0^2}{\Delta z}(1 - e^{-ik\Delta x})} - \frac{\beta D_0^3}{\Delta z^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\right)$$

$$\lambda = 1 + \Delta t \frac{\beta D_0^3}{\Delta z^2}\left(e^{ik\Delta z} - 2 + e^{-ik\Delta z}\right) = 1 + \Delta t \frac{\beta D_0^3}{\Delta z^2}\left(2\cos(k\Delta x) - 2\right)$$

$$= 1 + \Delta t \frac{\beta D_0^3}{\Delta z^2}\left(-2\left(1 - \cos k\Delta x\right)\right) = 1 + \Delta t \frac{\beta D_0^3}{\Delta z^2}\left(-2\left(2\sin^2\left(\frac{k\Delta x}{2}\right)\right)\right)$$

$$= 1 - \Delta t \frac{4\beta D_0^3}{\Delta z^2} \sin^2\left(\frac{k\Delta x}{2}\right)$$

Interested in $|\lambda| \le 1$ so need to find worst case scenario as previously

$\sin^2\left(\frac{k\Delta z}{2}\right)$ with $k\Delta x = \pi$ so $\sin^2\left(\frac{k\Delta x}{2}\right) = 1$

then

$\lambda = 1 - \Delta t \frac{4\beta D_0^3}{\Delta z^2}$ For $\lambda \geqslant -1$ to be satisfied we get the resulting expression:

$$1 - \frac{4\beta D_0^3}{\Delta z^2}\Delta t \geqslant -1$$

$$\frac{4\beta D_0^3}{\Delta z^2}\Delta t \le 2$$

$$\Longrightarrow \quad \Delta t \le \frac{\Delta z^2}{2\beta D_0^3}$$

These 2 cases are for diffusion and convection driven flow seperately. The timestep that should be used needs to satisfy the stability of both convective and diffusive components, as they are summed, so a timestep less than the smallest of these two limiting timesteps found should be used

d) Use the maximum principle to determine a stable time step for the discretisation of (6)

The maximum principle states that the value of variable $U_j^n$ is bounded between boundary values and initial values. In this case we can't have any negative values for the dike width pertubation $b_0$

Interested in $b_{0,j}^{n+1} \geq 0$ when $b_{0,j}^n \geq 0$ so group together coefficients of $b_{0,j-1}^n$, $b_{0,j}^n$, $b_{0,j+1}^n$ to give

$$b_{0,j}^{n+1} = K_1 b_{0,j-1}^n + K_2 b_{0,j}^n + K_3 b_{0,j+1}^n$$

where

$$K_1 = \left(\frac{3\alpha D_0^2}{\Delta z} + \frac{\beta D_0^3}{\Delta z^2}\right)\Delta t \; ; \; K_2 = 1 - \Delta t\left(-\frac{3\alpha D_0^2}{\Delta z} - \frac{2\beta D_0^3}{\Delta z^2}\right), \; K_3 = \Delta t \frac{\beta D_0^3}{\Delta z^2}$$

We have assumed that $b_{0,j-1}^n \geq 0$, $b_{0,j}^n \geq 0$ and $b_{0,j+1}^n \geq 0$ and we want $b_{0,j}^{n+1} \geq 0$ so this is guarenteed if $K_1$, $K_2$ and $K_3$ are all $\geq 0$

From their definitions $K_1$ and $K_3$ are always greater than zero however if the timestep is too large $K_2$ can become negative. To ensure that $K_2 \geq 0$ we impose a stability condition as previously and rearrange for $\Delta t$

$$1 - \Delta t\left(\frac{3\alpha D_0^2}{\Delta z} + \frac{2\beta D_0^3}{\Delta z^2}\right) \geq 0$$

$$1 \geq \Delta t\left(\frac{3\alpha D_0^2}{\Delta z} + \frac{2\beta D_0^3}{\Delta z^2}\right)$$

$$\Delta t \leq \frac{1}{\frac{3\alpha D_0^2}{\Delta z} + \frac{2\beta D_0^3}{\Delta z^2}} \Rightarrow \Delta t \leq \frac{\Delta z^2}{D_0^2(3\alpha \Delta z + 2 D_0 \beta)}$$

Which is the maximum allowable time step that guarentees the numerical solution is non negative and bounded given that the initial condition is also non negative which is key due to the physical spatial application of this fluid flow problem.

e) Derive a (variable) time step criterion for the discretisation of (5) for which $B_j^{n+1} > 0$

The physical application of this condition is to ensure that the non linear ~~combination~~ convection diffusion equation (5) only considers positive values in the evolution of the dike width $b(z,t)$

Interested in $B_j^{n+1} > 0$ if $B_j^n > 0$ for the equation $b_j^{n+1} = b_j - \frac{\Delta t}{\Delta z}\left(U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n\right)$

We can find a time step for which this condition is met by considering

$$b_j^n - \frac{\Delta t}{\Delta z}\left(U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n\right) > 0$$

$$b_j^n > \frac{\Delta t}{\Delta z}\left(U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n\right)$$

$$\Delta t < \frac{\Delta z\, b_j^n}{U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n}$$

This time step means that from a change in $b_j$ from flux across all bounderies won't exceed the current value of $b_j^n$ which ensures positivity

To ensure this is true for all $b_j^n$ the minimisation needs to be imposed as if this is true for the value of $b_j^n$ giving the smallest restriction so it will be true for all other $b_j^n$ values ensuring stability

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1f) i) Derive a second order spatial discretisation of (5) by changing the spatial discretization of the convective term.

To get second order accuracy of the convective term use the central difference scheme similarly to how the diffusive term was treated in the original discretisation

$$\frac{\partial}{\partial z}\alpha b^3 \approx \alpha\left(\frac{(b_{j+1}^n)^3-(b_{j-1}^n)^3}{2\Delta z}\right)$$

Taylor expansions correct to 2nd order so additional terms only for $\partial z^3$ and higher terms

The full discretized of (5) with second order central difference for diffusion and convection becomes

$$b_j^{n+1} = b_j^n - \Delta t\left[\frac{\alpha}{2\Delta z}\left({b_{j+1}^n}^3 - {b_{j-1}^n}^3 - \beta(b_j^n + b_{j+1}^n)^3\right.\right.$$

$$b_j^{n+1} = b_j^n - \frac{\Delta t}{\Delta z}\left(\alpha\left(\frac{b_j^n+b_{j+1}^n}{2}\right)^3 - \beta\left(\frac{b_j^n+b_{j+1}^n}{2}\right)^3\left(\frac{b_{j+1}^n-b_j^n}{\Delta z}\right) + \frac{\Delta t}{\Delta z^2}\left(\frac{\beta}{\Delta z}\left(\frac{b_j^n+b_{j+1}^n}{2}\right)^3(b_{j+1}^n-b_j^n)\right.\right. \dots$$

$$\dots - \left(\frac{b_{j-1}+b_j}{2}\right)^3\left(b_j^n - b_{j-1}^n\right)\Big)\Big)$$

$$\Rightarrow b_j^{n+1} = b_j^n - \frac{\Delta t}{2\Delta z}\alpha\left({b_{j+1}^n}^3 - {b_{j-1}^n}^3\right) + \frac{\Delta t}{\Delta z^2}\left(U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n\right)$$

$$U_{j+\frac{1}{2}}^n = \beta\left(\frac{b_j^n+b_{j+1}^n}{2}\right)^3\left(\frac{b_{j+1}^n-b_j^n}{\Delta z}\right)$$

$$U_{j-\frac{1}{2}}^n = \beta\left(\frac{b_{j-1}^n+b_j^n}{2}\right)^3\left(\frac{b_j^n-b_{j-1}^n}{\Delta z}\right)$$

ii) What is a step criterion to guarentee that $B_j^{n+1} > 0$ for this new discretisation

Again we are checking that $b_j^{n+1} > 0$ given that $b_j^n > 0$ so the change in $b_j$ due to net fluxes doesn't exceed the $b_j^n$ value with the same physical application of positive dike widths

Using the same process as before by setting,

$$b_j^n - \frac{\Delta t}{\Delta z}\alpha\left({b_{j+1}^n}^3 - {b_{j-1}^n}^3\right) - \frac{1}{\Delta z^2}\left(U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n\right) > 0$$ and rearranging to find $\Delta t$ condition

$$\Rightarrow \Delta t < \frac{b_n^j}{\frac{\alpha}{2\Delta z}\left({b_{j+1}^n}^3 - {b_{j-1}^n}^3\right) - \frac{1}{\Delta z^2}\left(U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n\right)}$$

where $b_j^n$ = current value of the dike width at grid point $j$
$b_{j\pm1}^n$ = neighbouring values at grid points $j \pm 1$
$\Delta z$ = Spatial grid spacing
$\beta$ = convection coefficient
$\alpha$ = diffusion coefficient

This time step criterion ensures that a numerical update (move to neighbouring value) doesn't subtract more than the available value so the dike ~~wth~~ width will always be positive as given by physical constraints.

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2a) Show that the steady state solution of (5) satisfies $\beta b^3 \frac{db}{dz} = \cancel{\phi}\alpha b^3 - Q$ where $Q$ = integration constant

Non linear convection - diffusion equation (5)

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial z}\left(\overset{\alpha}{\cancel{\phi}} b^3 - \beta b^3 \frac{\partial b}{\partial z}\right) = 0$$

Steady state means the solution doesn't change in time so $\frac{\partial b}{\partial t} = 0$

$$\frac{\partial}{\partial z}\left(\overset{\alpha}{\cancel{\phi}} b^3 - \beta b^3 \frac{\partial b}{\partial z}\right) = \frac{d}{dz}\left(\alpha b^3 - \beta b^3 \frac{db}{dz}\right) = 0$$

which implies the total flux is constant and the constant of integration is given by $Q$

$$\alpha b^3 - \beta b^3 \frac{db}{dz} = Q$$

$$-\beta b^3 \frac{db}{dz} = Q - \cancel{\phi}\alpha b^3$$

$$\beta b^3 \frac{db}{dz} = \alpha b^3 - Q \longrightarrow \text{as given by (10) on tasksheet}$$

Integrate (10) and plot the solution for example by using an Euler forward spatial discretisation

$$\beta b^3 \frac{db}{dz} = \alpha b^3 - Q$$

$$\frac{db}{dz} = \frac{\alpha b^3 - Q}{\beta b^3} \quad \leftarrow \text{First order ODE for } b(z) \text{ so need to integrate numerically}$$

$$b_{j+1} = b_j + \Delta z \left.\frac{db}{dz}\right|_j \qquad \Longrightarrow b_{j+1} = b_j + \Delta z \frac{\alpha b_j^3 - Q}{\beta b_j^3}$$

$$\left.\frac{db}{dz}\right|_j = \frac{\alpha b_j^3 - Q}{\beta b_j^3}$$

Figure 1 is plotted in python by the code entitled "tasksheet 1Q2a" with 1000 gridpoints and constants defined as given in the tasksheet.

It produces a value of $b_T \approx 0.59335227694S$ and the figure 1 shows how the dike width between $\frac{b}{2}$ and $-\frac{b}{2}$ evolves with height in the dike

2b) Figures 2 through 4 show the evolution of the dike diameter as time progresses with 11, 21 and 41 grid points respectively

The solution of the non linear convection diffusion equated was computed numerically using the scheme as described to linearise equation (5). This was computed in python in the script entitled "tasksheet 1Q2b"
To ensure the stability requirement on the timestep was met it was derived from the maximum principle and in the script it's implementation was to check the coefficient ~~of~~ of $b_j^n$ was positive

Also note that in some cases the solutions for larger values of $t$ are identical and this is due to having reached the steady state. Exact details are given in each figures caption as to which t's included

c) Consider the $L^2$-norm and the $L^\infty$ norm for the error $e(z,t) = b(z,t) - b_{numerical}(z,t)$

The point of this question is to assess how the spatial resolution affects the accuracy of the numerical solution - to estimate the order of spatial discretization

- Need to keep the time step $\Delta t$ constant across all grid sizes to isolate the effect of spatial resolution. This makes sure any change in error is due to the number of grid points, not the time stepping accuracy $dt = 0.0001$

- We treat the steady state solution as an approximation to the true analytical solution which physically represents the final shape of the dike after it has settled once convection and diffusion have balanced

- For each grid size, the numerical solution is computed at $t=2$ and the error is the difference between the numerical and steady state solutions for each grid ~~size~~ point

- The composite trapezoidal rule approximates the integral of the squared error over the domain of $z \in (0,H)$

- ~~To estimate~~

- The plot shows that the error decreases as the number of grid points increases which indicates convergence

- To estimate the order of spatial discretisation you can plot the error vs grid spacing $\Delta z$ and since it's a log-log plot the slope of the line gives the order of accuracy

$$L_2 = \frac{1}{\sqrt{L}}\sqrt{\frac{1}{2}\left(b_B - b(0,t_c)\right)^2 + \frac{1}{2}\left(b_T - b(H,t_c)\right)^2 + \sum_{j=1}^{N}\left(b_j^n - b_{numerical}(x_j,t_c)\right)^2 \Delta z}$$

$$L_2 = \sqrt{\sum_{j=1}^{N}\left(b_j^n - b_{numerical}(x_j,t_c)\right)^2 \Delta z}$$

- The first two terms in above are the boundary conditions which are equal for both ~~contact~~ methods so can be taken out and give a reduced $L_2$.
- The error scales as $Err = K_4(\Delta z)^\gamma \propto (N_{spatial} - 1)^\gamma$ where the order ~~of~~ is $\gamma$, $N_{spatial}$ is the number of grid points and $K_4$ is a constant

- the error decreases exponentially with $\Delta z$

Figure 5 is a graph that shows the approximate error in the numerical solution vs the number of grid points where the exponent $-\gamma = -1$

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d) i) Verify that the following is an implicit yet explicit, time dependent travelling wave solution (12)

$$z - z_{r,0} - ct = \frac{\beta}{\alpha}\left(b(z,t) - \sqrt{(c/\alpha)}\ \operatorname{atanh}\left(\sqrt{\frac{\alpha}{c}}\ b(z,t)\right)\right)$$

with $z_{r,0}$ a reference constant and $c$ = wave speed. $b$ is of the form $b = b(z-ct)$

Substitute $\phi = z-ct$ to give:

$$\phi - z_{r,0} = \frac{\beta}{\alpha}\left(b(\phi) - \sqrt{\frac{c}{\alpha}}\ \operatorname{atanh}\left(\sqrt{\frac{\alpha}{c}}\ b(\phi)\right)\right)$$

Differentiate wrt $\phi$

$$1 = \frac{\beta}{\alpha}\left(\frac{\partial b}{\partial \phi} - \frac{\frac{\partial b}{\partial \phi}}{1-\frac{\alpha}{c}b^2}\right) \quad \text{with } \frac{\partial b}{\partial \phi} = \frac{\alpha}{\beta}\,\frac{\frac{\alpha}{c}b^2 - 1}{\frac{\alpha}{c}b^2}$$

Eq (2)

Using chain rule $\frac{\partial}{\partial z} = \frac{\partial}{\partial \phi}$ and $\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \phi}$ so substituting these into $\frac{\partial b}{\partial t} + \frac{\partial}{\partial z}\left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z}\right) = 0$

gives that

$$-c\frac{\partial b}{\partial \phi} + \frac{\partial}{\partial \phi}\left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial \phi}\right)$$

$$\Rightarrow -c\frac{\partial b}{\partial \phi} + 3\alpha b^2 \frac{\partial b}{\partial \phi} - \frac{\partial}{\partial \phi}\left(\beta b^3 \frac{\alpha}{\beta}\,\frac{\frac{\alpha}{c}b^2 - 1}{\frac{\alpha}{c}b^2}\right)$$

$$\Rightarrow -c\frac{\partial b}{\partial \phi} + 3\alpha b^2 \frac{\partial b}{\partial \phi} + \frac{\partial}{\partial \phi}\left(cb - 3\alpha b^3\right)$$

$$\Rightarrow -c\frac{\partial b}{\partial \phi} + 3\alpha b^2 \frac{\partial b}{\partial \phi} + c\frac{\partial b}{\partial \phi} - 3\alpha b^2 \frac{\partial b}{\partial \phi} =$$

$$= \left(-c + c + 3b^2 - 3\alpha b^2\right)\frac{\partial b}{\partial \phi} = 0$$

showing how the RHS of equation (2) so the given equation in the question is a solution

ii) Verify the numerical solution against the exact solution and pay attention to the required condition that $b(x,t) \geq 0$. The dike remains closed below a dimensional depth of 0.9km in a total domain of 3km deep.

- Exact solution was defined using the implicit equation from the tasksheet and then rearranged to make $b(z,t)$ the subject to allow the initial and boundary conditions to be derived from it.

- For the numerical scheme used the forward Euler method and second order spatial discretisation then the ~~b~~ boundary conditions were updated at each step using the time dependent exact solution. Used the maximum principle to ensure positivity of solution remained true. Also using the maximum principle created an additional function to make sure the maximum stable time step kept numerical stability

- This simulation shows the evolution of a magma filled dike going upwards through rock. The travelling wave solution is a steady propagation of the dike front and the numerical scheme captures how the dike width evolves over time

Figures 6 through 8 are showing how the dike width changes at different times to show it's propagation upwards through the mantle. This was completed for all 3 grid sizes $j = 11, 21, 41$. The numerical solutions are shown with solid lines and the dashed lines are the exact solution

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e) Extend the numerics in (d) to the Crank-Nicolson scheme, and solve the nonlinear algebraic system using iteration techniques

The Crank Nicholson method is based on the trapezium rule giving a second-order convergence in time. It's equation is a combination of the forward Euler method at n and the backward Euler method at n+1, which is implicit such that to get to the "next" value of u in time a system of algebraic equations is solved

This scheme usually improves the numerical stability as it has better stability and accuracy as an implicit method rather than explicit schemes. It would be beneficial for this scenario as it allows for larger time steps while maintaining stability so it makes it easier to simulate long term evolution

Crank Nicolson Scheme: Averages the spatial terms between time levels n and n+1

$$\frac{b_j^{n+1}-b_j^n}{\Delta t} = \frac{1}{2}\left[F(b^n)+F(b^{n+1})\right]$$

where $F(b)$ is the spatial operator applied to solution $b$. This includes all the spatial derivatives and non linear terms from the convection diffusion equation

In this case $F(b)$ would ~~represent~~ include convection and nonlinear diffusion so the updated version would be

$$\frac{b_j^{n+1}-b_j^n}{\Delta t} = \frac{1}{2}\left[F(b^n)+F(b^{n+1})\right]$$

So with the PDE $\frac{\partial b}{\partial t}+\frac{\partial}{\partial z}\left(\alpha b^3-\beta b^3\frac{\partial b}{\partial z}\right)=0$

$$\frac{\partial b}{\partial t} = -F(b) \quad \text{where} \quad F(b)=\frac{\partial}{\partial z}\left(\alpha b^3-\beta b^3\frac{\partial b}{\partial z}\right)$$

The Crank Nicholson scheme averages the spatial operators between time levels n and n+1

$$\frac{b_j^{n+1}-b_j^n}{\Delta t} = \frac{-1}{2}\left[F(b^n)_j+F(b^{n+1})_j\right]$$

$$\Longrightarrow \quad b_j^{n+1} = b_j^n - \frac{\Delta t}{2}\left[F(b^n)_j+F(b^{n+1})_j\right]$$

Convective term : $F_{conv}(b)_j = \alpha\dfrac{(b_{j+1})^3-(b_{j-1})^3}{2\Delta z}$

Diffusive term : $F_{diff}(b)_j = \dfrac{\beta}{(\Delta z)^2}\left(\left(\dfrac{b_j+b_{j+1}}{2}\right)^3\dfrac{b_{j+1}-b_j}{\Delta z}-\left(\dfrac{b_{j-1}+b_j}{2}\right)^3\dfrac{b_j-b_{j-1}}{\Delta z}\right)$

$$\Longrightarrow \quad F(b)_j = \alpha\frac{(b_{j+1})^3-(b_{j-1})^3}{2\Delta z} - \frac{\beta}{\Delta z^2}\left[\left(\frac{b_j+b_{j+1}}{2}\right)^3(b_{j+1}-b_j)-\left(\frac{b_{j-1}+b_j}{2}\right)^3(b_j-b_{j-1})\right]$$

Since $b_j^{n+1}$ appears on both sides of the equation the system has to be solved using fixed-point iteration to solve for all $j=1,2,\dots,J-2$ (excluding boundaries)

This code uses a crank-Nicolson scheme to solve the non linear convection diffusion equation modelling the evolution of a magma filled dike. It compares the numerical solution at various times with the exact travelling wave solution.

- Use the exact travelling wave solution at $t=0$ to initialise dike shape and initial conditions
- For each time $t$ the code:
  - Computes exact solution
  - Initialises numerical solution
  - Computes the number of timesteps and time step size
  - ~~A~~ Continues the solution using Newton Raphson iteration for the non linear system

This is ~~repeated~~ computed for each gridpoint position

- Computes the residual
- ~~Creates a~~ linearise around guess $b^{(k)}$ and solve for
- Then update the ~~sa~~ guess

This process is repeated until the modulus of the update is small which suggests the system has reached convergence

In the context of the magma flow:

- The dike is driven by pressure and resisted by surrounding rock
- The nonlinear PDE models this balance
- Newton Rhaphson ensures that the numerical solution ~~resp~~ keeps this balance at each timestep
- The plot shows how the dike evolves over time, matching the expected travelling wave behaviour

Figures 9 through 11 are plots that show this numerical scheme implemented ~~at~~ for the dike width profile at different times ($t = 1, 2, 3, 4, 6$)

- The solid lines represent the numerical solution obtained using Newton Raphson iteration
- The dashed lines represent the exact travelling wave solution

The Newton Raphson method continues to ensure the solutions

- Converge to a physically meaningful profile
- Preserves positivity so the dike width isn't impossibly negative
- Tracks the wave front as it propogates upward through the domain

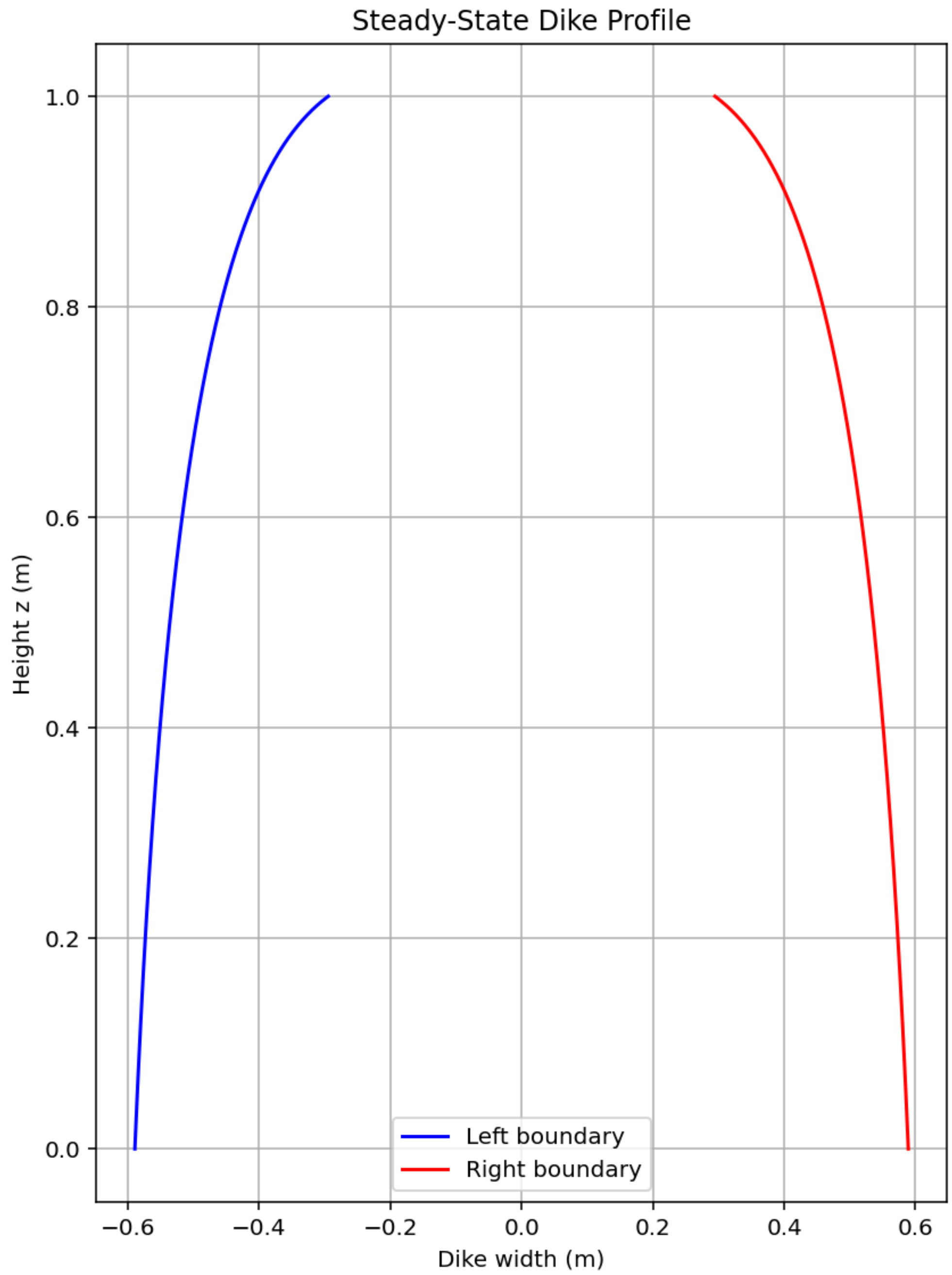


Figure 1: Dike width plotted as a function of height in the Dike for the steady state.

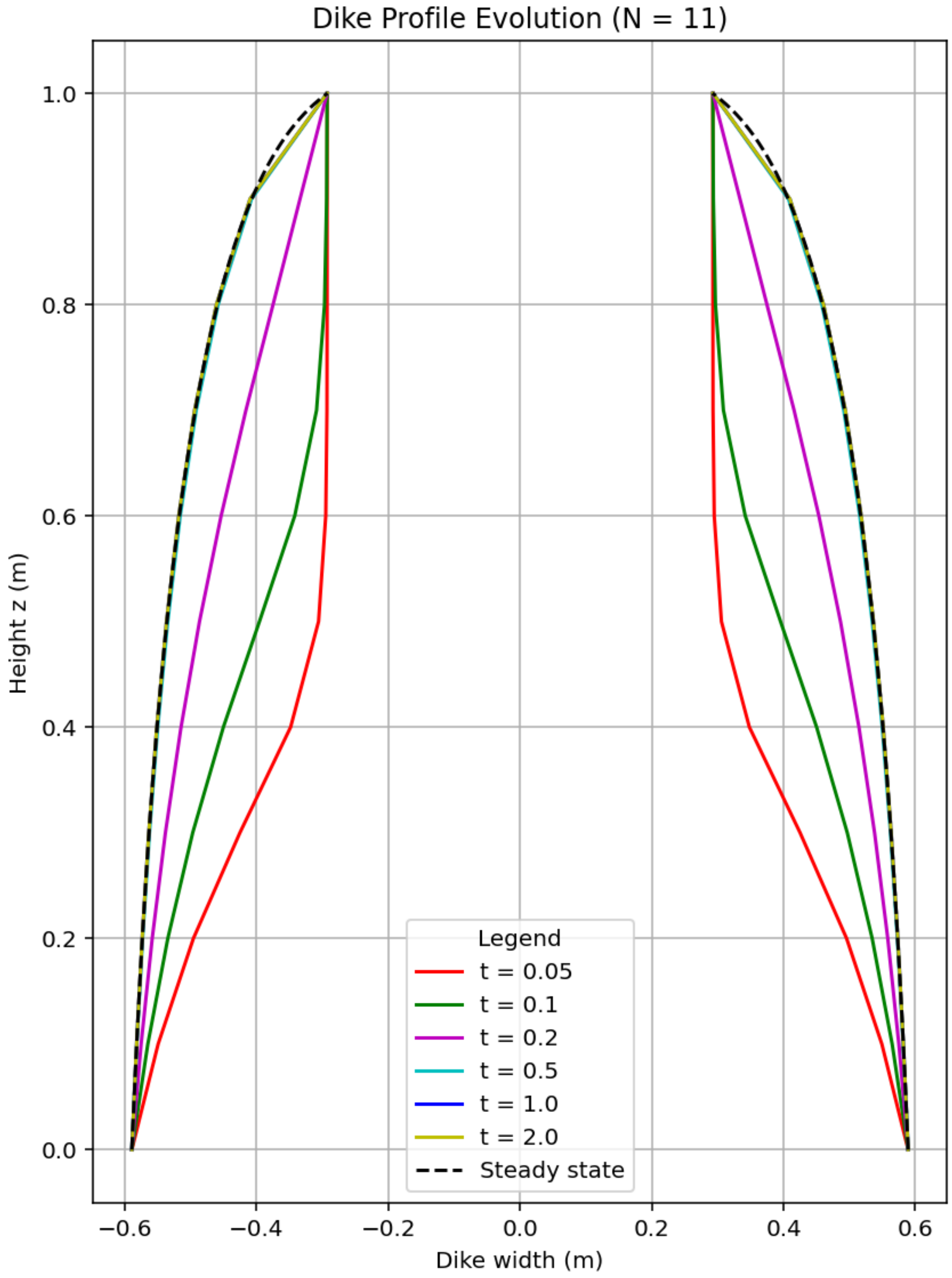


Figure 2: Graph showing the evolution of the dike diameters as time progresses with 11 spatial grid points. The solutions for t=0.5,1 and 2 are the same having reached the steady state

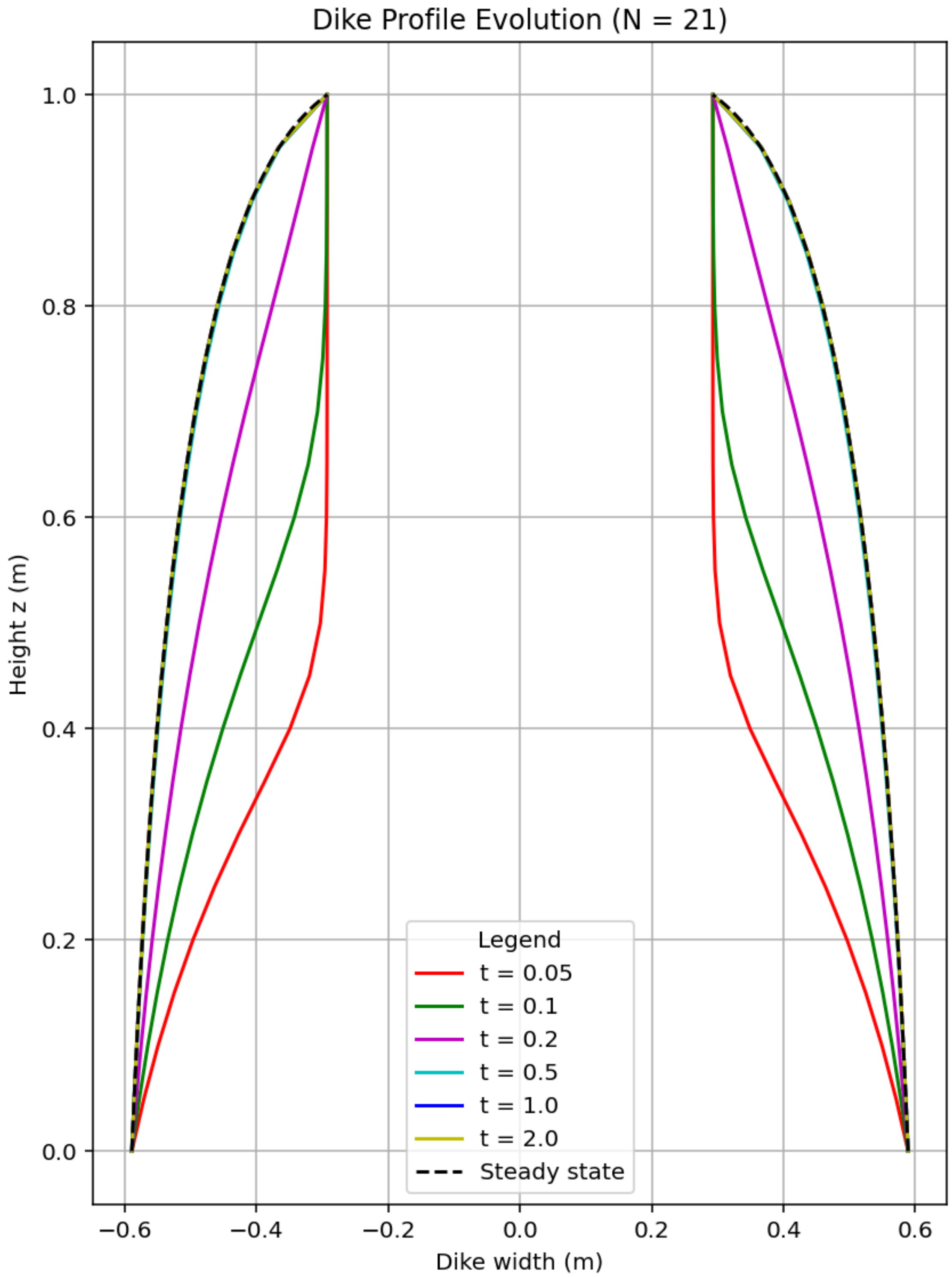


Figure 3: Graph showing the evolution of the dike diameters as time progresses with 21 spatial grid points. The solutions for t=0.5,1 and 2 are the same having reached the steady state

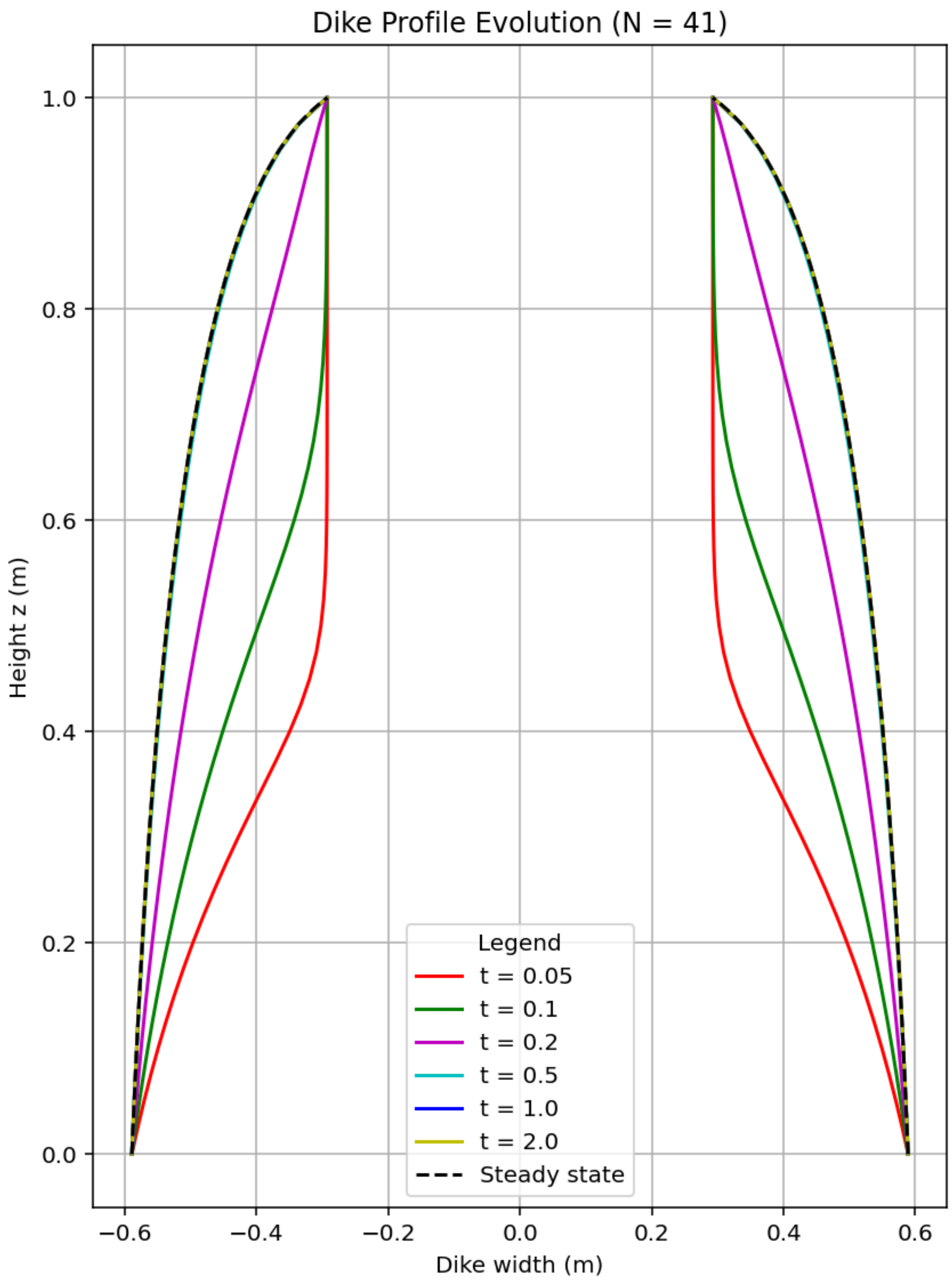


Figure 4: Graph showing the evolution of the dike diameters as time progresses with 41 spatial grid points. The solutions for t=0.5,1 and 2 are the same having reached the steady state

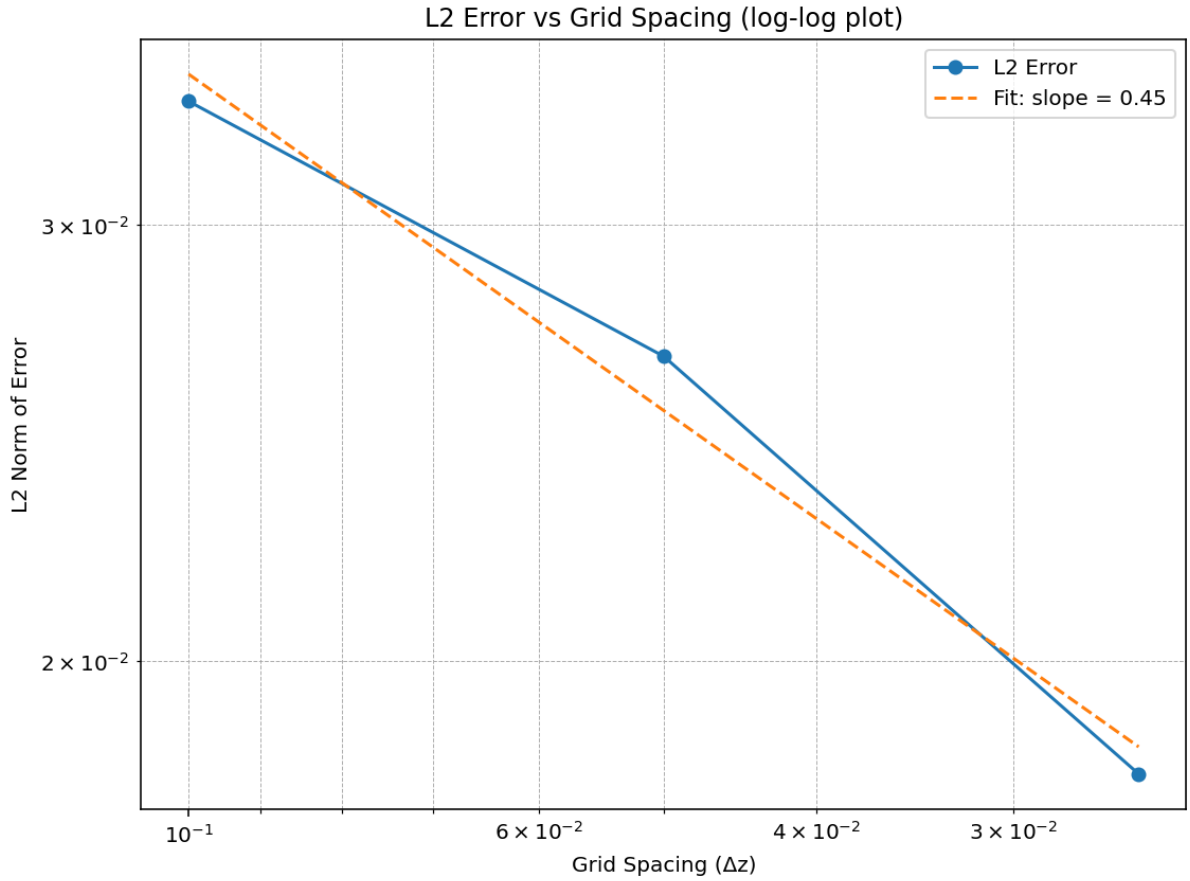


Figure 5: Graph showing the approximate error in the numerical solution vs number of grid points. This shows a linear fit on the log log plot.

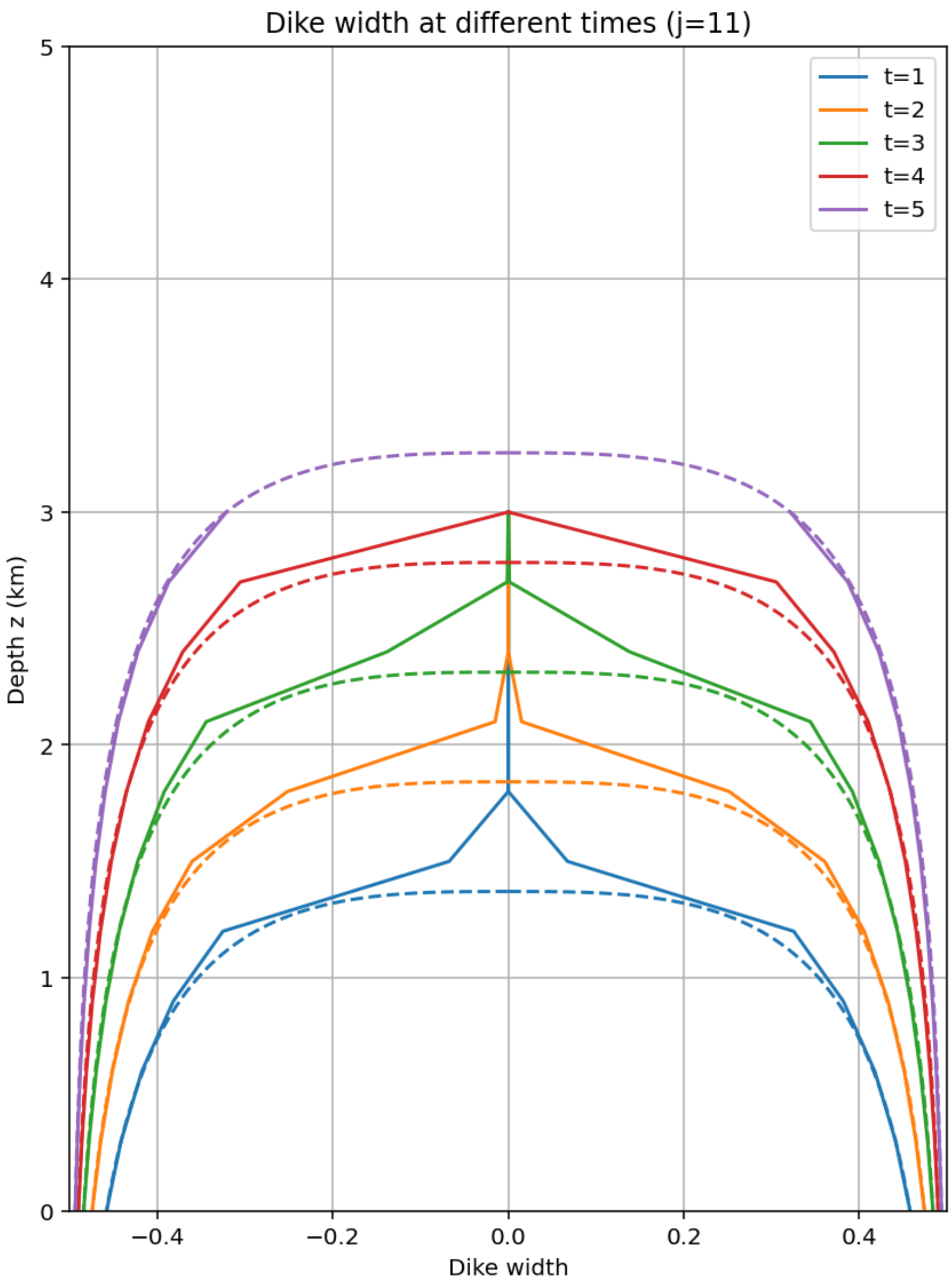


Figure 6: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 11 spatial grid points. It shows the propagation upwards through the mantle. The numerical solutions are shown with solid lines and the dashed lines are the exact solutions

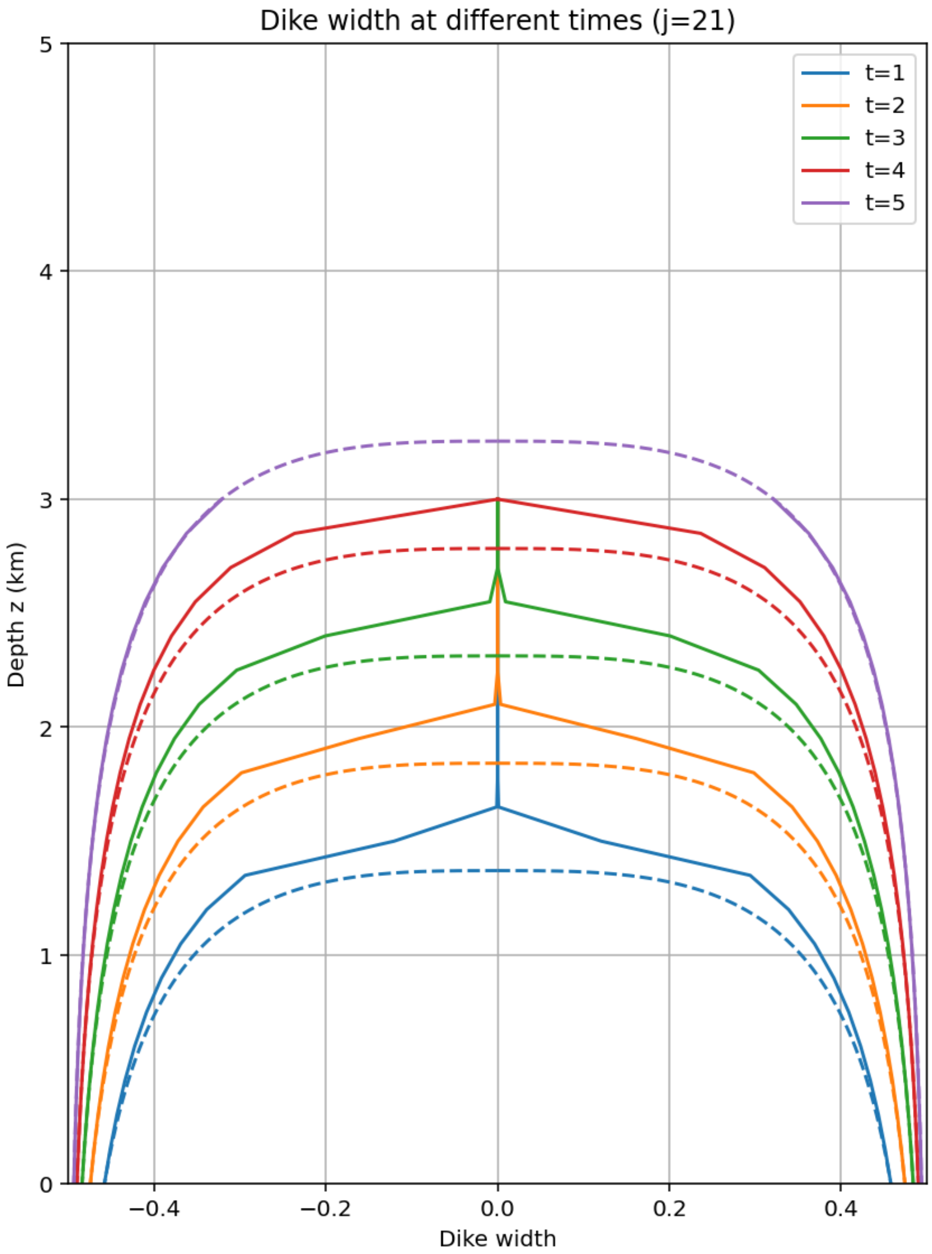


Figure 7: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 21 spatial grid points. It shows the propagation upwards through the mantle. The numerical solutions are shown with solid lines and the dashed lines are the exact solutions

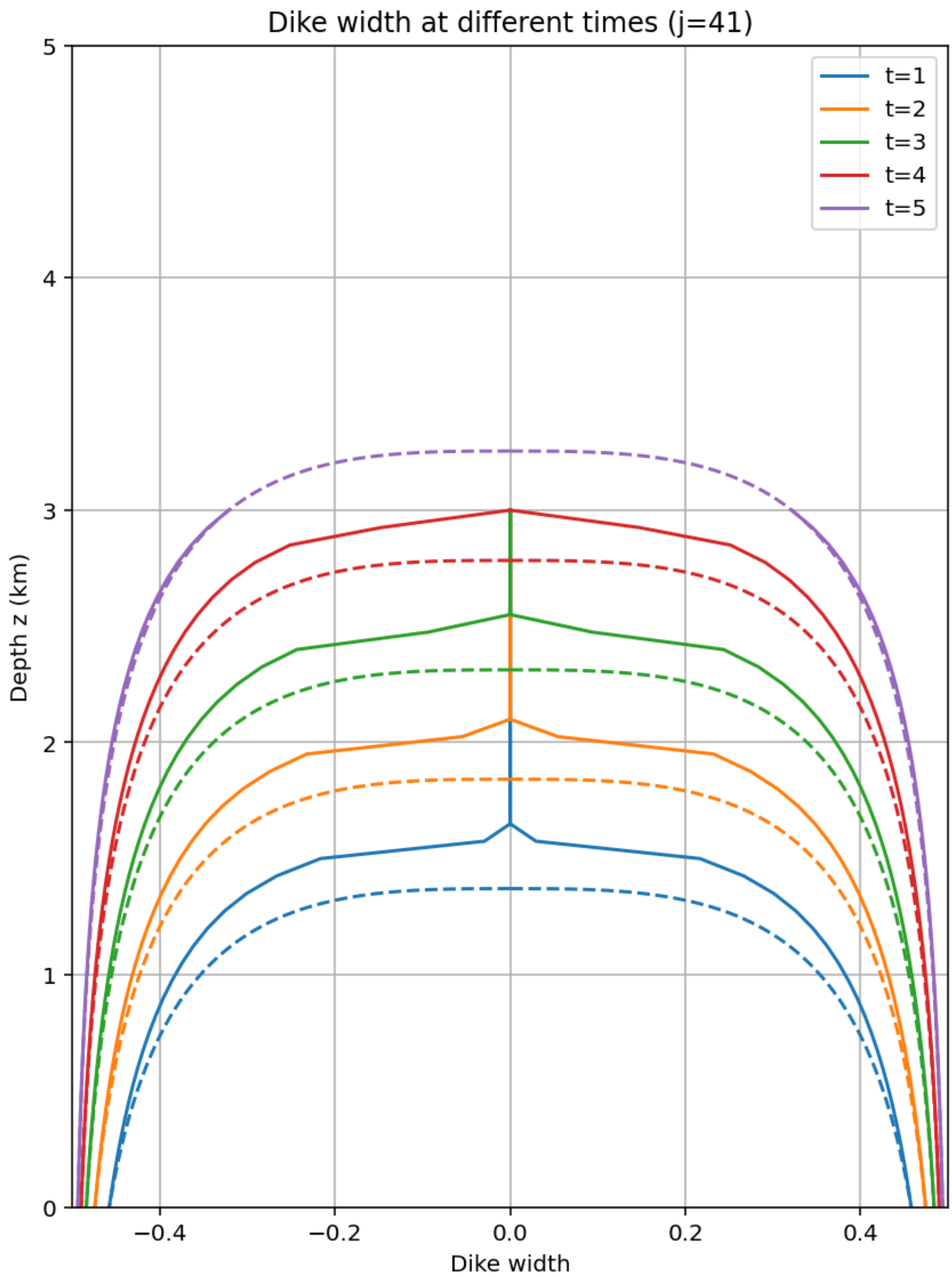


Figure 8: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 41 spatial grid points. It shows the propagation upwards through the mantle. The numerical solutions are shown with solid lines and the dashed lines are the exact solutions

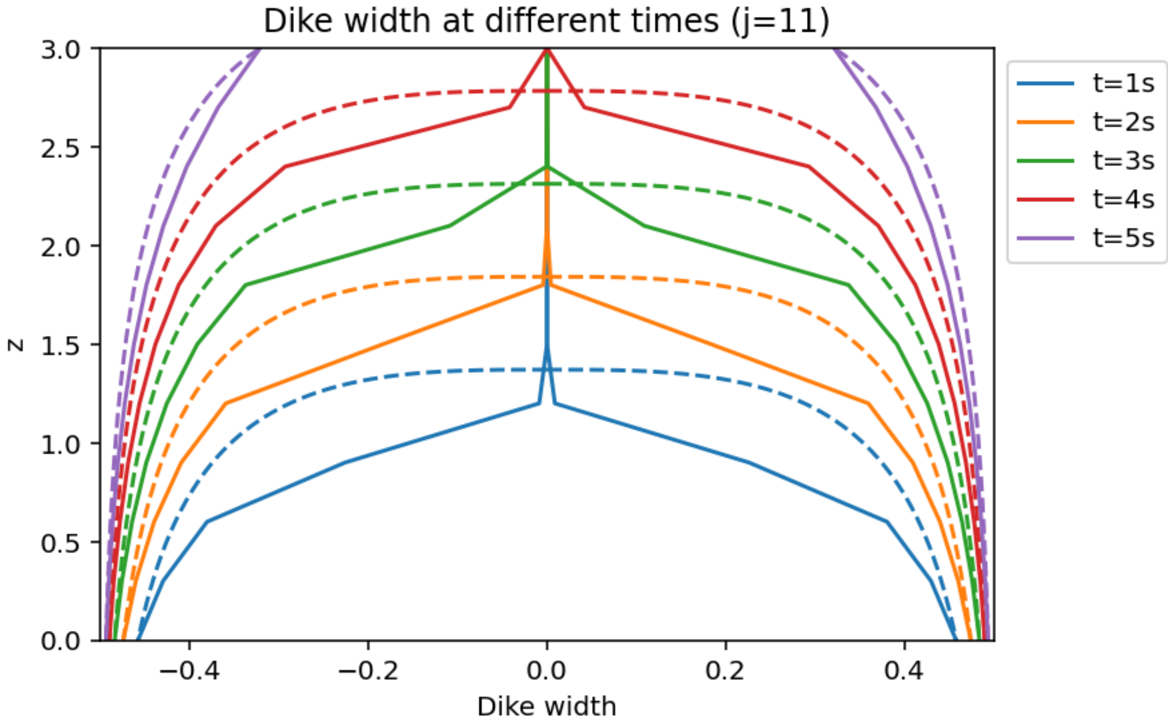


Figure 9: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 11 spatial grid points. It shows the upwards propagation through the mantle. The numerical solution is implanted using the Crank Nicholson Scheme and Newton Raphson iterations and displayed by solid lines. The dashed lines represent the exact traveling wave solution

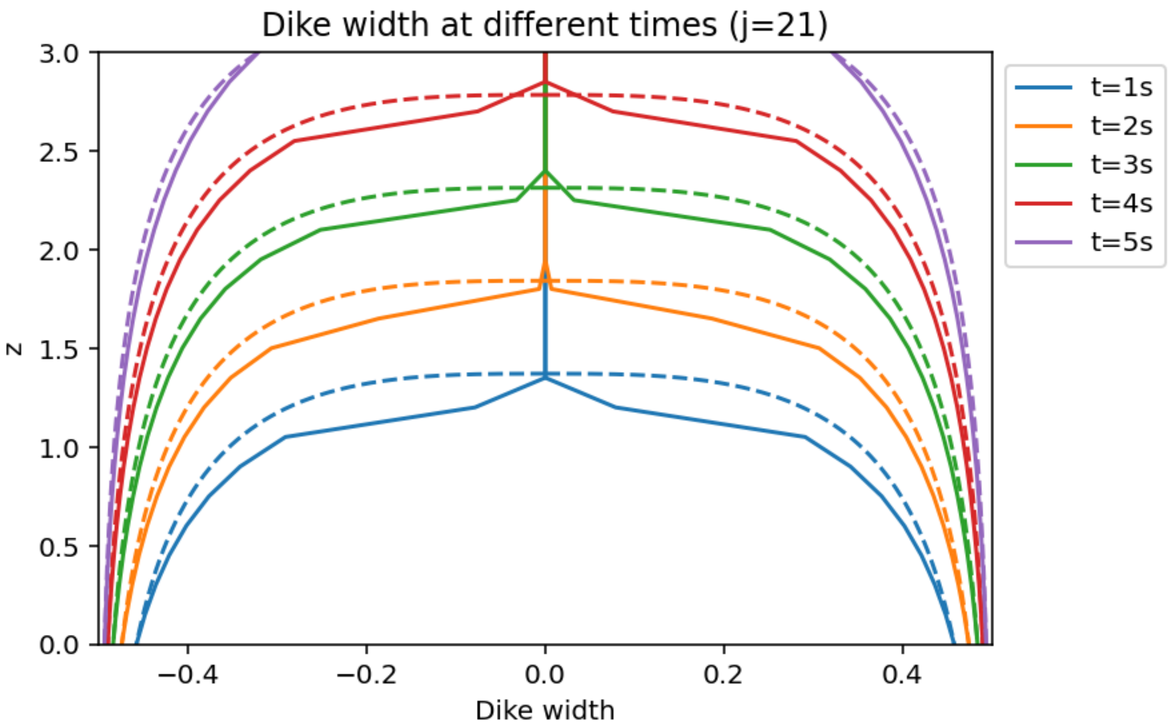


Figure 10: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 21 spatial grid points. It shows the upwards propagation through the mantle. The numerical solution is implanted using the Crank Nicholson Scheme and Newton Raphson iterations and displayed by solid lines. The dashed lines represent the exact traveling wave solution

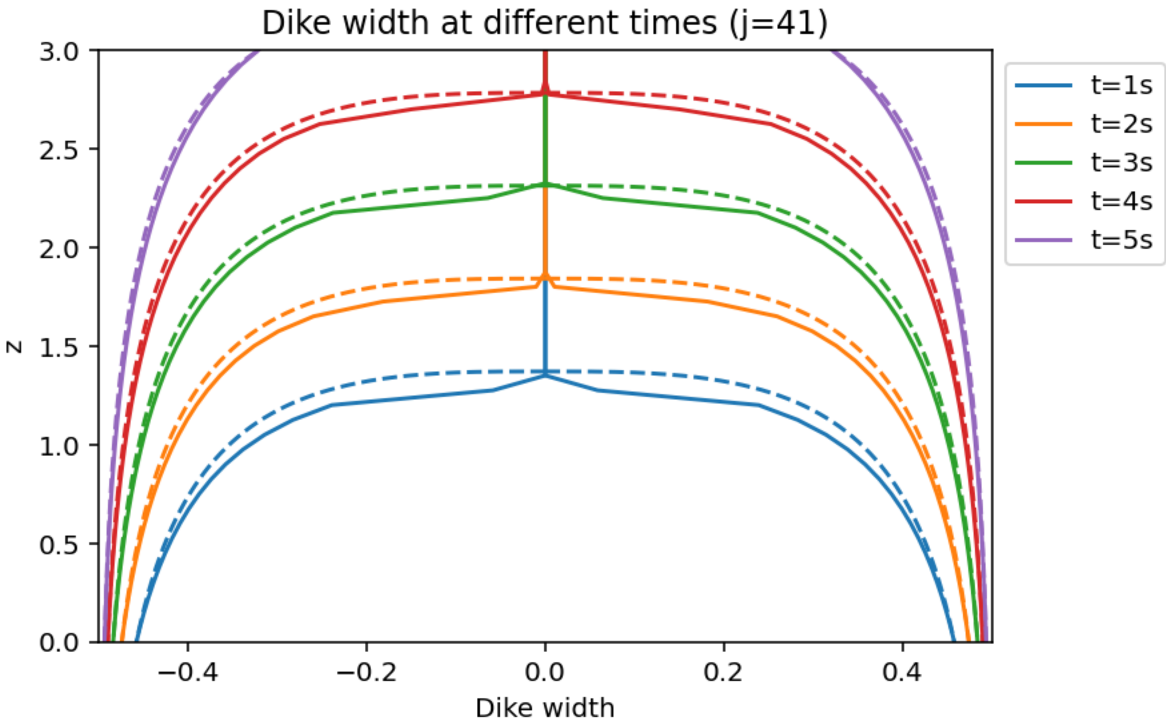


Figure 11: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 41 spatial grid points. It shows the upwards propagation through the mantle. The numerical solution is implanted using the Crank Nicholson Scheme and Newton Raphson iterations and displayed by solid lines. The dashed lines represent the exact traveling wave solution