

# Numerics Exercise 3 - Hannah Franklin

## Part 2 (Ground Water):

$$(1) \partial_t (\omega_v h_m) - \alpha g \partial_y (\omega_v h_m \partial_y h_m) = \frac{\omega_v R}{m_{\text{por}} \sigma_c}$$

$m_{\text{por}} \in [0.1, 0.3]$ ,  $\sigma_c \in [0.5, 1]$  is fraction of pore that can be filled w/ water due to residual air

$$h_m = h_m(y, t)$$

$$\omega_v \approx 0.1 \text{ m}$$

$$y \in [0, L_y], L_y \approx 0.85 \text{ m}$$

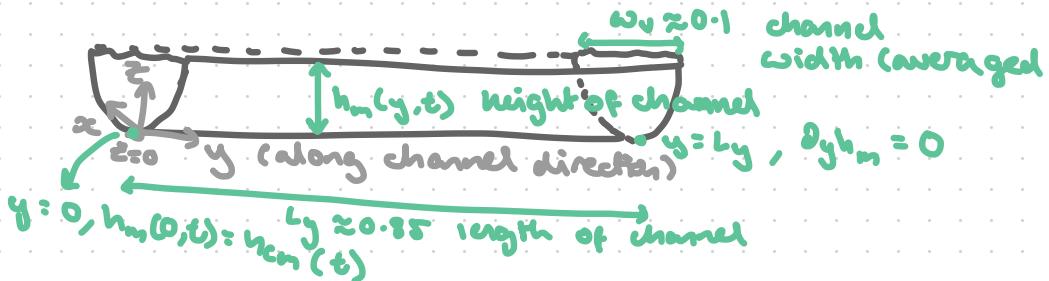
$$R = R(t) \text{ rainfall}$$

$$\alpha = \frac{k}{2m_{\text{por}} \sigma_c}, k \in [10^{-6}, 10^{-4}] \text{ m}^2 \text{ permeability}, v = 10^{-6} \text{ m}^2/\text{s} \text{ viscosity.}$$

$$\text{BCs: no flow at } y = L_y \Rightarrow \partial_y h_m = 0$$

$$\text{Dirichlet at } y = 0 \Rightarrow h_m(0, t) = h_m(t)$$

$$\text{IC: } h_m(y, 0) = h_{m_0}(y)$$



- assumptions:
  - ground water level stays underground - no surface runoff & flow is hydrostatic.
  - Variations in  $y$ -direction much longer than vertical length scales.
  - canal level  $h_m(t)$  holds in a short channel of length  $L_c$  b/c  $-L_c < y < 0$  w/ weir at  $y = -L_c$ .
  - water level at weir critical  $\Rightarrow$  flow speed  $V_c = \sqrt{gh_c}$  critical at  $y = -L_c$
  - stationarity & Bernoulli's eq  $\Rightarrow$  at weir:

$$g h_m + \frac{1}{2} V_c^2 \approx g h_m = g h_c + \frac{1}{2} V_c^2 : \frac{3}{2} g h_c$$

$$\Rightarrow h_c = \frac{2}{3} h_m \text{ s.t. } Q_c = h_c V_c = \sqrt{g} \max\left(\frac{2h_m}{3}, 0\right)^{1/2}$$

assuming  $V_c^2 \ll gh_m \approx 0$

$$(1) \Rightarrow \partial_t (\omega_v h_m) + \partial_y (V_h_m) = \frac{\omega_v R}{m_{\text{por}} \sigma_c}, V = -\frac{k g}{2m_{\text{por}} \sigma_c} \partial_y h_m$$

$\hookrightarrow V$ : Darcy velocity

$$\text{Darcy flux: } Q = \omega_v g = \omega_v h_m v = -\omega_v h_m \frac{K}{2\mu} \partial_y p = -\omega_v h_m \frac{K}{2\mu \rho_0} \partial_y p$$

$$x = \omega_v h_m \frac{Kg}{2\mu} \partial_y h_m = -\omega_v h_m \frac{Kg}{2\mu_{\text{por}} \sigma_c} \partial_y h_m = -\omega_v a g \partial_y \left( \frac{h_m^2}{2} \right)$$

$$\hookrightarrow \frac{\partial_y p}{\rho_0} \approx g \partial_y h_m$$

if no  $y$ -dependence:  $\partial_y h_m = \frac{R}{m_{\text{por}} \sigma_c}$

$$\Rightarrow (h_m = 0 \text{ at } t=0) \quad h_m = \frac{tR}{m_{\text{por}} \sigma_c} \Rightarrow m_{\text{por}} = \sigma_c = 1$$

canal level modelled by outflow at  $y = L_c$  and inflow at  $y = 0$ :

$$L_c \omega_v \frac{dh_m}{dt} = m_{\text{por}} \sigma_c Q_0 - Q_c = \omega_v m_{\text{por}} \sigma_c \frac{1}{2} a g \partial_y (h_m^2) |_{y=0} - \omega_v \sqrt{g} \max\left(\frac{2}{3} h_m(t), 0\right)^{3/2}$$

### COMPLETE MODEL:

$$(11) \quad \partial_t (\omega_v h_m) - \alpha g \partial_y (\omega_v h_m \partial_y h_m) = \frac{\omega_v R}{m_{\text{por}} \sigma_c} \quad \text{in } y \in [0, L_y]$$

$$(12) \quad \partial_y h_m = 0 \quad \text{at } y = L_y$$

$$(13) \quad h_m(0, t) = h_m(t) \quad \text{at } y = 0$$

$$(14) \quad L_c \omega_v \frac{dh_m}{dt} = \omega_v m_{\text{por}} \frac{\sigma_c}{2} \alpha g \partial_y (h_m^2) |_{y=0} - \omega_v \sqrt{g} \max\left(\frac{2}{3} h_m(t), 0\right)^{3/2}$$

(+ BC for  $h_m$  and  $h_m'$ )

1) Discretise 11-14 using explicit finite element space. Show how flux at  $y=0$  can be eliminated.

multiple eq (11) by test function  $g(y)$  and integrate over the length of the channel.

$$\Rightarrow \int_0^{L_y} g(y) [\partial_t (\omega_v h_m) - \alpha g \partial_y (\omega_v \partial_y h_m)] dy = \int_0^{L_y} \frac{\omega_v R g}{m_{\text{por}} \sigma_c} dy$$

$$\Rightarrow \int_0^{L_y} [g \partial_t (\omega_v h_m) - \frac{1}{2} \alpha g^2 \partial_y (\omega_v \partial_y h_m)] dy = \int_0^{L_y} \frac{\omega_v R g}{m_{\text{por}} \sigma_c} dy$$

note:  $\omega_v \approx 0.1m$  cst as averaged width

$$\Rightarrow \int_0^{L_y} [g \partial_t h_m - \frac{1}{2} \alpha g^2 \partial_y^2 h_m] dy = \int_0^{L_y} \frac{\omega_v R g}{m_{\text{por}} \sigma_c} dy$$

note: (for integration by parts)

$$\partial_y(g \partial_y h_m) = \partial_y g \partial_y h_m + g \partial_y^2 h_m$$

$$\Rightarrow g \partial_y^2 h_m = \partial_y(g \partial_y h_m) - \partial_y g \partial_y h_m$$

$$\Rightarrow \int_0^{L_y} g \partial_t h_m dy + \int_0^{L_y} [ \frac{1}{2} \alpha g \partial_y g \partial_y h_m^2 - \frac{1}{2} \alpha g \partial_y g \partial_y (h_m^2) ] dy = \int_0^{L_y} \frac{Rg}{m_p \sigma_c} dy$$

$$\Rightarrow \int_0^{L_y} [ g \partial_t h_m + \frac{1}{2} \alpha g \partial_y g \partial_y (h_m^2) ] dy - [ \frac{1}{2} \alpha g g \partial_y (h_m^2) ]_0^{L_y} = \int_0^{L_y} \frac{Rg}{m_p \sigma_c} dy$$

$$[ \frac{1}{2} \alpha g g \partial_y (h_m^2) ]_0^{L_y} = [ \alpha g g h_m \partial_y h_m ]_0^{L_y} = - \frac{1}{2} \alpha g g \partial_y (h_m^2) \Big|_{y=0}$$

since  $\partial_y h_m|_{y=0} = 0 \quad (12)$

$$\Rightarrow \int_0^{L_y} [ g \partial_t h_m + \frac{1}{2} \alpha g \partial_y g \partial_y (h_m^2) ] dy + \frac{1}{2} \alpha g g \partial_y (h_m^2) \Big|_{y=0} = \int_0^{L_y} \frac{Rg}{m_p \sigma_c} dy$$

$$(14): L_c w_v \frac{dh_m}{dt} = w_v m_p \sigma_c \frac{\sigma_c}{2} \alpha g \partial_y (h_m) \Big|_{y=0} - w_v \sqrt{g} \max\left(\frac{2}{3} h_m(t), 0\right)^{3/2}$$

$$\Rightarrow \frac{1}{2} \partial_y (h_m^2) \Big|_{y=0} = \frac{2}{w_v m_p \sigma_c \alpha g} \left[ L_c w_v \frac{dh_m}{dt} + w_v \sqrt{g} \max\left(\frac{2}{3} h_m(t), 0\right)^{3/2} \right]$$

$$\Rightarrow \frac{1}{2} \alpha g \partial_y (h_m^2) \Big|_{y=0} = \frac{1}{m_p \sigma_c \sigma_c} \left[ L_c \frac{dh_m}{dt} + \sqrt{g} \max\left(\frac{2}{3} h_m(t), 0\right)^{3/2} \right]$$

$$\Rightarrow \int_0^{L_y} [ g \partial_t h_m + \frac{1}{2} \alpha g \partial_y g \partial_y (h_m^2) ] dy + \frac{g(0)}{m_p \sigma_c \sigma_c} \left[ L_c \frac{dh_m}{dt} + \sqrt{g} \max\left(\frac{2}{3} h_m(t), 0\right)^{3/2} \right]$$

$$= \int_0^{L_y} \frac{Rg}{m_p \sigma_c} dy$$

$$\Rightarrow \int_0^{L_y} [ g \partial_t h_m ] dy + \frac{g(0) L_c}{m_p \sigma_c} \frac{dh_m}{dt} = \int_0^{L_y} \left[ \frac{Rg}{m_p \sigma_c} - \frac{1}{2} \alpha g \partial_y g \partial_y (h_m^2) \right] dy$$

$$- \frac{g(0) \sqrt{g}}{m_p \sigma_c} \max\left(\frac{2}{3} h_m(t), 0\right)^{3/2}$$

$$\Rightarrow \int_0^{L_y} [ g \partial_t h_m ] dy + \frac{g(0) L_c}{m_p \sigma_c} \frac{dh_m}{dt} = \int_0^{L_y} \left[ \frac{Rg}{m_p \sigma_c} - \frac{1}{2} \alpha g \partial_y g \partial_y (h_m^2) \right] dy$$

$$- \frac{g(0) \sqrt{g}}{m_p \sigma_c} \max\left(\frac{2}{3} h_m(0, t), 0\right)^{3/2}$$

This eliminates flux at  $y=0$  using  $h_m(0, t) = h_m(t)$  at the last step to ensure continuity between the canal and the channel.

Note: test fact:  $g(y)$  is unconstrained.

Explicit discretisation ( $\theta=0$ ):

using forward Euler discretisation:

$$\Delta t h_m \approx \frac{h_m^{n+1} - h_m^n}{\Delta t}$$

$$\frac{\partial h_m}{\partial t} \approx \frac{h_m^{n+1} - h_m^n}{\Delta t}$$

$\Rightarrow$  LHS of eq. becomes:

$$\frac{1}{\Delta t} \left[ \int_0^{L_y} g h_m^{n+1} dy + \frac{g(0)L_c}{m_{\text{por}}\sigma_e} h_m^{n+1} \right] - \frac{1}{\Delta t} \left[ \int_0^{L_y} g h_m^n dy + \frac{g(0)L_c}{m_{\text{por}}\sigma_e} h_m^n \right]$$

$$\Rightarrow \int_0^{L_y} g h_m^{n+1} dy + \frac{g(0)L_c}{m_{\text{por}}\sigma_e} h_m^{n+1} = \int_0^{L_y} g h_m^n dy + \frac{g(0)L_c}{m_{\text{por}}\sigma_e} h_m^n \\ + \Delta t \int_0^{L_y} \left[ g \frac{R^n}{m_{\text{por}}\sigma_e} - \alpha g h_m^n dy g dy h_m^n \right] dy \\ - \Delta t \frac{g(0) \sqrt{5}}{m_{\text{por}}\sigma_e} \max\left(\frac{2}{3} h_m^n, 0\right)^{3/2}$$

taking  $g_i := \varphi_i(x)$ ,  $h_m := h_j \varphi_j(x)$   $\forall i, j = 1, \dots, N_y$ ,  $\omega / N_y$  no. nodes.  
 $h_m = h_i$ ,  $g(0) = \varphi_i \delta_{ii}$ ,

$$\text{LHS} : \int_0^{L_y} \varphi_i \varphi_j h_j^{n+1} dy + \frac{L_c}{m_{\text{por}}\sigma_e} h_i^{n+1} \delta_{ii}$$

$$M_{ij} = \int_0^{L_y} \varphi_i \varphi_j dy \Rightarrow M_{ij} h_j^{n+1} + \frac{L_c h_i^{n+1}}{m_{\text{por}}\sigma_e} \delta_{ii}$$

$$\int_0^{L_y} g h_m^n dy + \frac{g(0)L_c}{m_{\text{por}}\sigma_e} h_m^n \mapsto M_{ij} h_j^n + \frac{L_c h_i^n}{m_{\text{por}}\sigma_e} \delta_{ii}$$

This leads to:

$$M_{ij} h_j^{n+1} + \frac{L_c h_i^{n+1}}{m_{\text{por}}\sigma_e} \delta_{ii} = M_{ij} h_j^n + \frac{L_c h_i^n}{m_{\text{por}}\sigma_e} \delta_{ii} + \Delta t \int_0^{L_y} \frac{\varphi_i R^n}{m_{\text{por}}\sigma_e} dy \\ - \Delta t \int_0^{L_y} \alpha g h_m^n dy \varphi_i dy h_m^n dy \\ + \Delta t \frac{\sqrt{5}}{m_{\text{por}}\sigma_e} \max\left(\frac{2}{3} h_i^n, 0\right)^{3/2} \delta_{ii}$$

$$M_{ij} h_j^{n+1} + \frac{L_c h_i^{n+1}}{m_{\text{por}}\sigma_e} \delta_{ii} = M_{ij} h_j^n + \frac{L_c h_i^n}{m_{\text{por}}\sigma_e} \delta_{ii} + \Delta t b_i^n - \Delta t \frac{\sqrt{5}}{m_{\text{por}}\sigma_e} \max\left(\frac{2}{3} h_i^n, 0\right)^{3/2} \delta_{ii}$$

where  $b_i^n = \int_0^{L_y} \left( \frac{\psi; R^n}{m_{\text{per}} \sigma c} - \alpha g h_m^n \partial_y \psi; \partial_y h_m^n \right) dy$

consider again eq. (11) :

$$\partial_t(\omega v h_m) - \alpha g \partial_y (\omega v h_m \partial_y h_m) = \frac{\epsilon \omega R}{m_{\text{per}} \sigma c}$$

$$\Rightarrow \partial_t(h_m) - \alpha g \partial_y(h_m \partial_y h_m) = \frac{R}{m_{\text{per}} \sigma c}$$

a finite difference discretisation (forward Euler):

$$\Rightarrow \frac{h_i^{n+1} - h_i^n}{\Delta t} - \frac{\alpha g}{\Delta y} \left[ h_{i+\frac{1}{2}}^n \partial_y h_{i+\frac{1}{2}}^n - h_{i-\frac{1}{2}}^n \partial_y h_{i-\frac{1}{2}}^n \right] = \frac{R^n}{m_{\text{per}} \sigma c}$$

$$\Rightarrow \frac{h_i^{n+1} - h_i^n}{\Delta t} - \frac{\alpha g}{2(\Delta y)} \left[ h_{i+\frac{1}{2}}^n (h_{i+1}^n - h_i^n) - h_{i-\frac{1}{2}}^n (h_i^n - h_{i-1}^n) \right] = \frac{R^n}{m_{\text{per}} \sigma c}$$

$$h_{i+\frac{1}{2}}^n = \frac{h_{i+1}^n + h_i^n}{2}, \quad h_{i-\frac{1}{2}}^n = \frac{h_i^n + h_{i-1}^n}{2}$$

$$\Rightarrow \frac{h_i^{n+1} - h_i^n}{\Delta t} - \frac{\alpha g}{2(\Delta y)^2} \left[ (h_{i+1}^n + h_i^n)(h_{i+1}^n - h_i^n) - (h_i^n + h_{i-1}^n)(h_i^n - h_{i-1}^n) \right] = \frac{R^n}{m_{\text{per}} \sigma c}$$

$$\Rightarrow \frac{h_i^{n+1} - h_i^n}{\Delta t} - \frac{\alpha g}{2(\Delta y)^2} \left[ (h_{i+1}^n)^2 - 2(h_i^n)^2 + (h_{i-1}^n)^2 \right] = \frac{R^n}{m_{\text{per}} \sigma c}$$

$i \mapsto j$  for time step criterion calculation

letting  $h_j^n = \lambda^n e^{ik_j y}$

$$\Rightarrow \left( \frac{\lambda - 1}{\Delta t} - \frac{\alpha g}{2(\Delta y)^2} \left[ e^{2ik_j y} - 2 + e^{-2ik_j y} \right] \right) h_j^n = \frac{R^n}{m_{\text{per}} \sigma c}$$

$$\frac{\lambda - 1}{\Delta t} - \frac{\alpha g}{2(\Delta y)^2} (4(\zeta_j)^2 \sin^2(k_j \Delta y)) = \frac{R^n}{m_{\text{per}} \sigma c} \frac{1}{h_j^n}$$

$$\lambda = 1 - \frac{2\Delta t}{(\Delta y)^2} \alpha g \sin^2(k_j \Delta y) + \frac{4t R^n}{m_{\text{per}} \sigma c} \frac{1}{h_j^n}$$

require  $|\lambda| \leq 1$  for scheme to be stable.

$$\Rightarrow \left| (\Delta t) \left( \frac{R^n}{m_{\text{por}} \sigma_c} \frac{i}{h_j^n} - \frac{2\alpha g}{(\Delta y)^2} \sin^2(k \Delta y) \right) \right| \leq 1$$

$$\Rightarrow \Delta t \leq \left| \frac{\frac{R^n}{m_{\text{por}} \sigma_c} \frac{i}{h_j^n} - \frac{2\alpha g}{(\Delta y)^2} \sin^2(k \Delta y)}{\frac{R^n}{m_{\text{por}} \sigma_c} \frac{i}{\max(h_j^n)}} \right|$$

$$\leq \left| \frac{\frac{R^n}{m_{\text{por}} \sigma_c} \frac{i}{\max(h_j^n)} + \frac{2\alpha g}{(\Delta y)^2}}{\frac{R^n}{m_{\text{por}} \sigma_c} \frac{i}{\max(h_j^n)}} \right|$$

from assumption this term dominates since  $\Delta y \gg$  vertical length scales

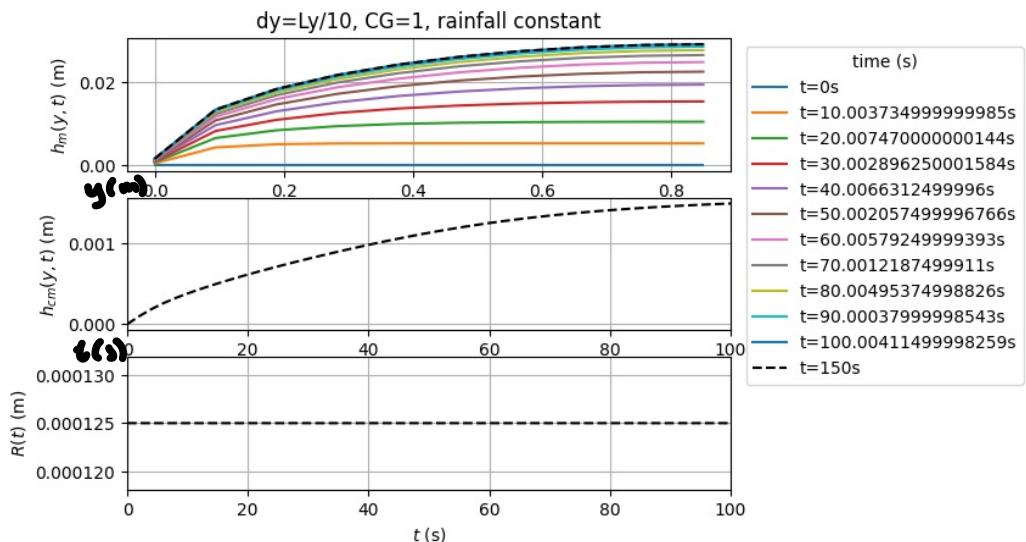
$$\Rightarrow \Delta y \gg h_j^n \forall j, n$$

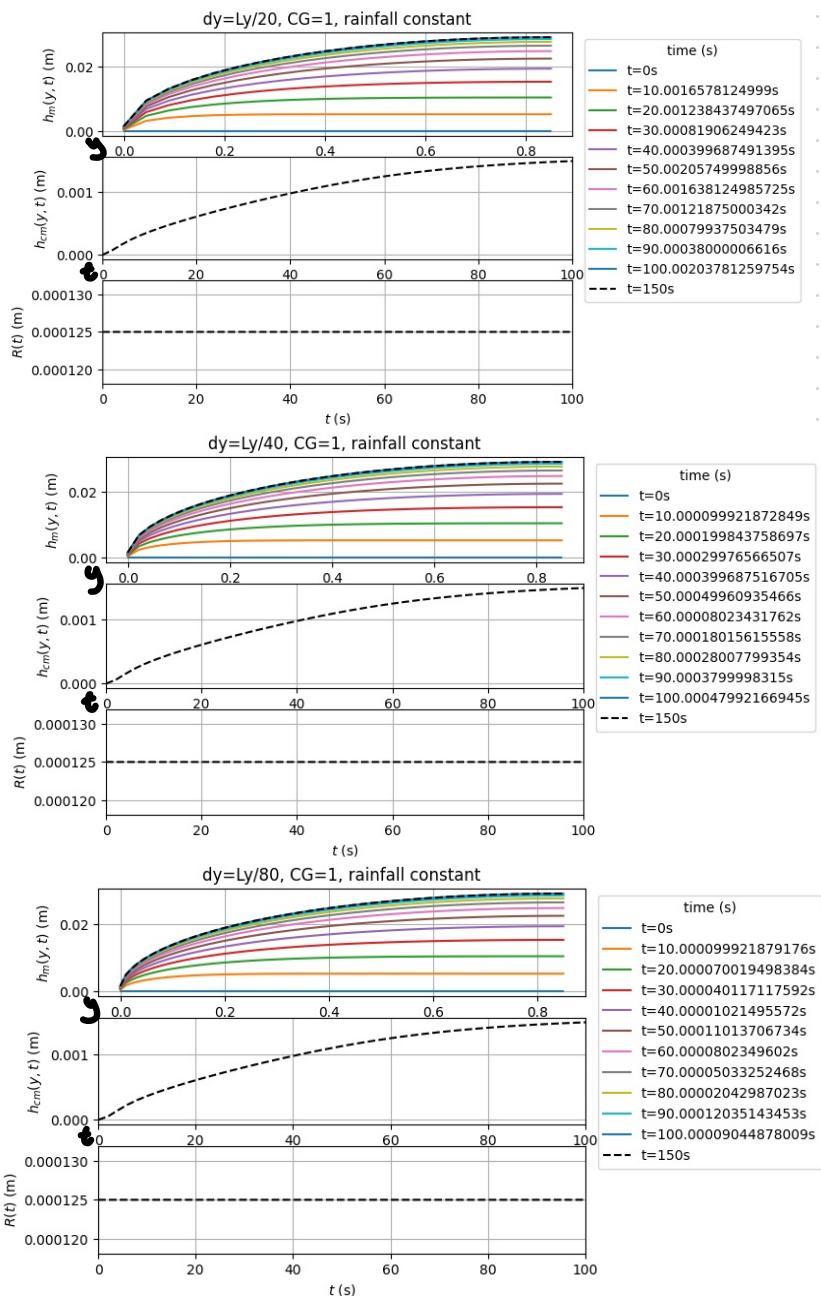
$$\Rightarrow \Delta t \leq \frac{(\Delta y)^2}{2\alpha g}$$

## 2) Numerical simulations for explicit scheme w/ parameter values:

$$m_{\text{por}} = 0.3, \sigma_c = 0.8, L_y = 0.85 \text{ m}, k = 10^{-8} \text{ m}^2, \omega_v = 0.1 \text{ m},$$

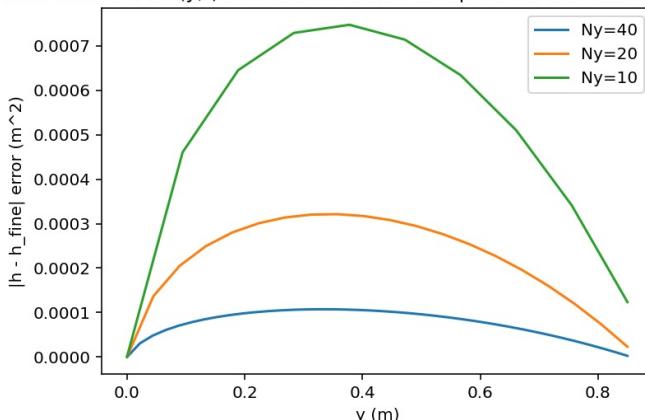
$$R_{\text{max}} = 0.000125 \text{ m s}^{-1}, L_c = 0.05 \text{ m}$$



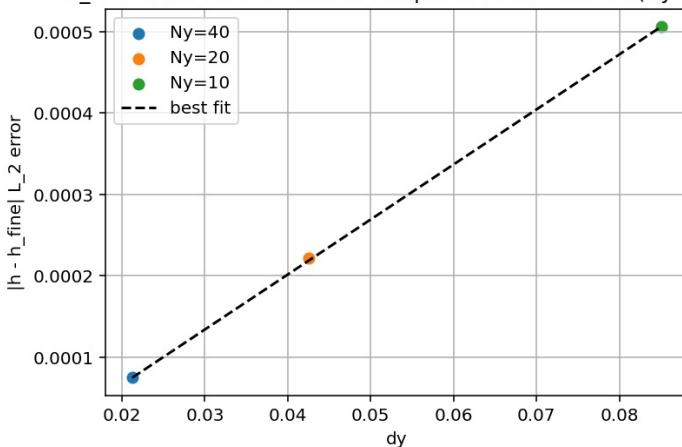


Error analysis is done using  $|h_m(y,t) - h_{m\text{ fine}}(y,t)|$  and the L<sub>2</sub> norm. Where  $h_{m\text{ fine}}$  is the  $h_m$  profile when  $N_y = 80$ .  
 $\Delta y = \frac{Ly}{80}$

Error between  $h_m(y,t)$  for each resolution compared to finest mesh ( $Ny=80$ )



$L_2$  error for each resolution compared to finest mesh ( $Ny=80$ )

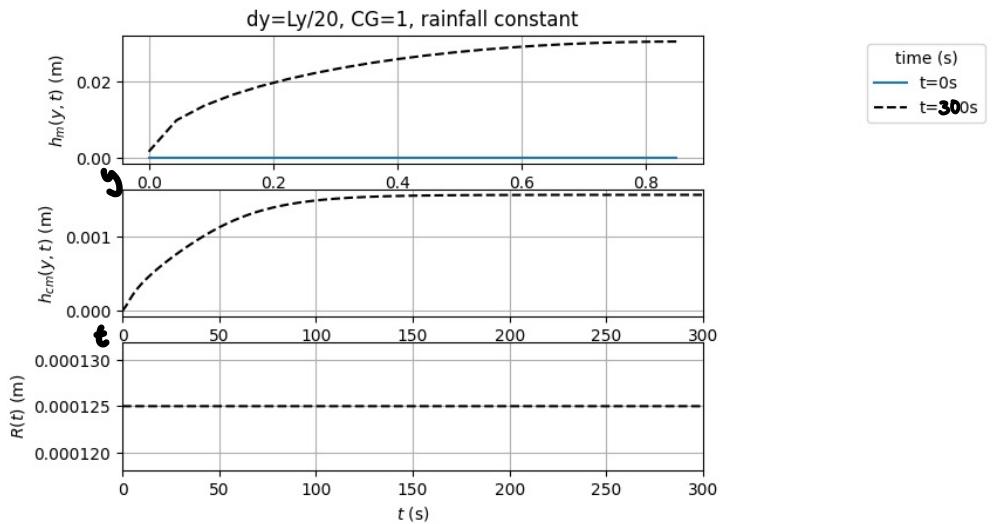


Plotting  $|h_m - h_m\text{-fine}|$  for each  $\Delta y$  at  $t=100s$  shows that numerical convergence is occurring.

Plotting the  $L_2$  error as a function of  $\Delta y$  again suggests numerical convergence is occurring.

Plot above for CG1 shows a linear relationship b/w the  $L_2$  error and  $\Delta y$  as is expected for order 1 scheme.

The system has not quite reached steady state at  $t=100s$ .



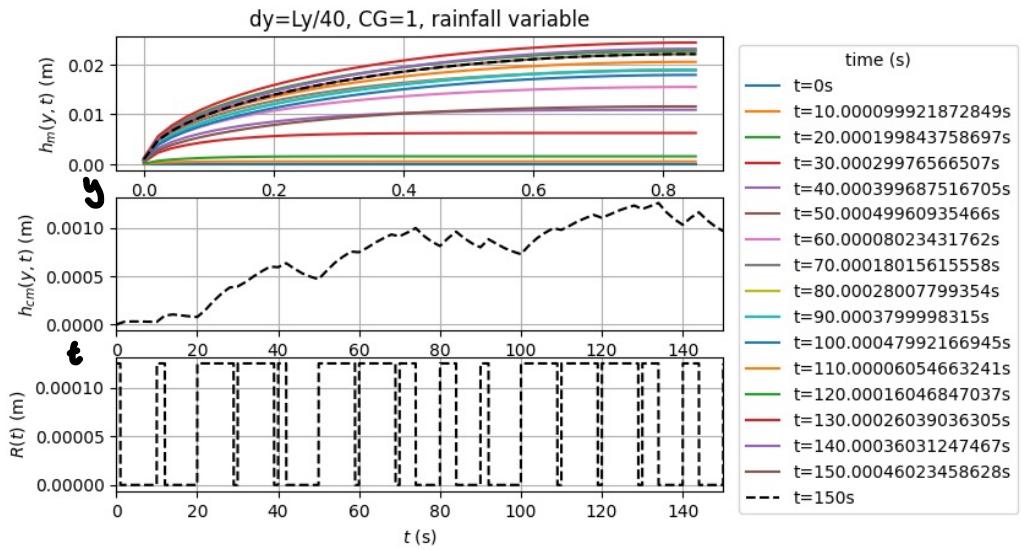
Plotting the final  $h_m(y)$  profile at  $t=300s$  shows the steady-state  $h_m(y)$  profile.

The steady state value of  $h_m(t)$ :

$$h_m(300) \approx 0.00157$$

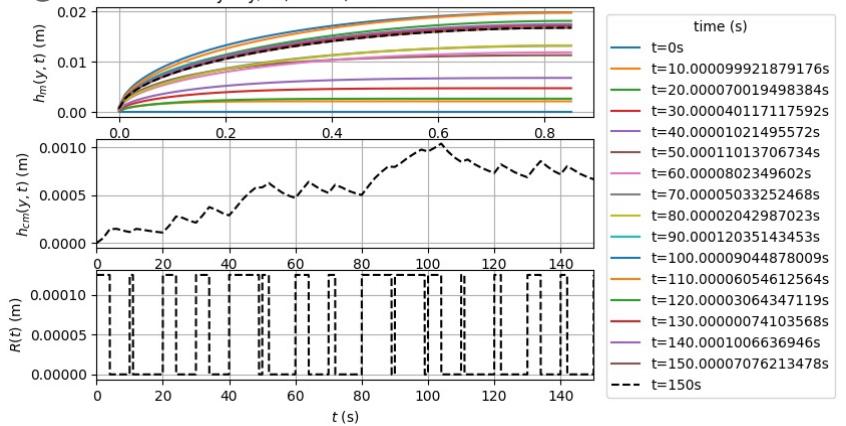
$= h_m$  steady state value

Implementing a variable rainfall function & plotting  $h_m(y, t)$ ,  $h_m(t)$ ,  $R(t)$

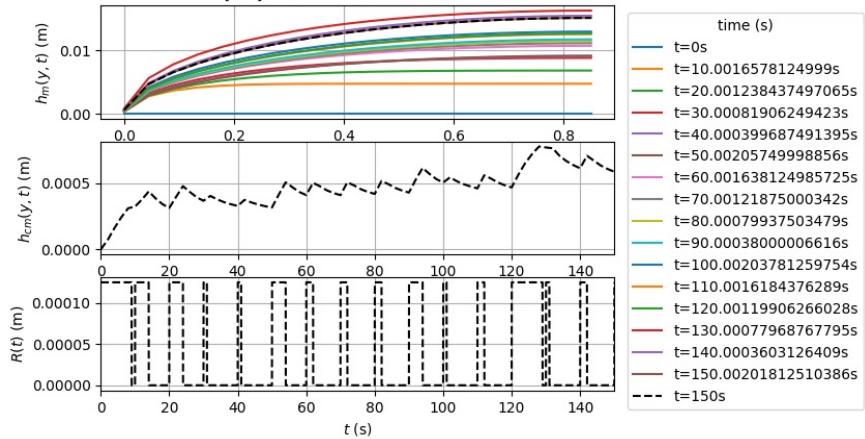


# Other dy choices:

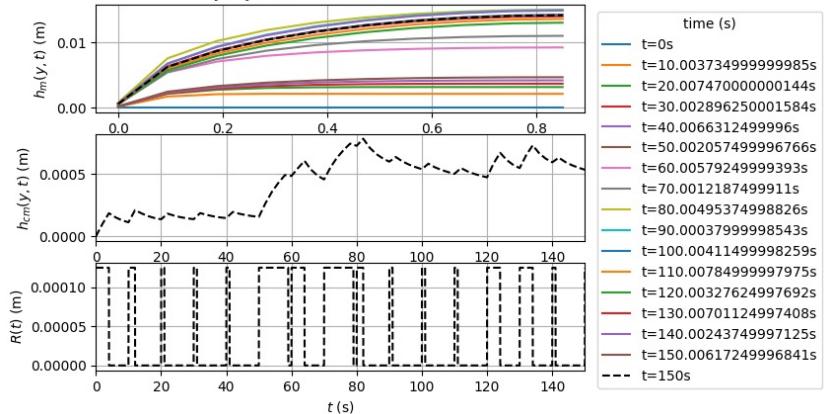
dy=Ly/80, CG=1, rainfall variable



dy=Ly/20, CG=1, rainfall variable



dy=Ly/10, CG=1, rainfall variable



Code uses randomly chooses a rain duration from 1s, 2s, 4s, 8s every 10s.

### 3) Crank-Nicolson scheme:

$$\theta = 0.5$$

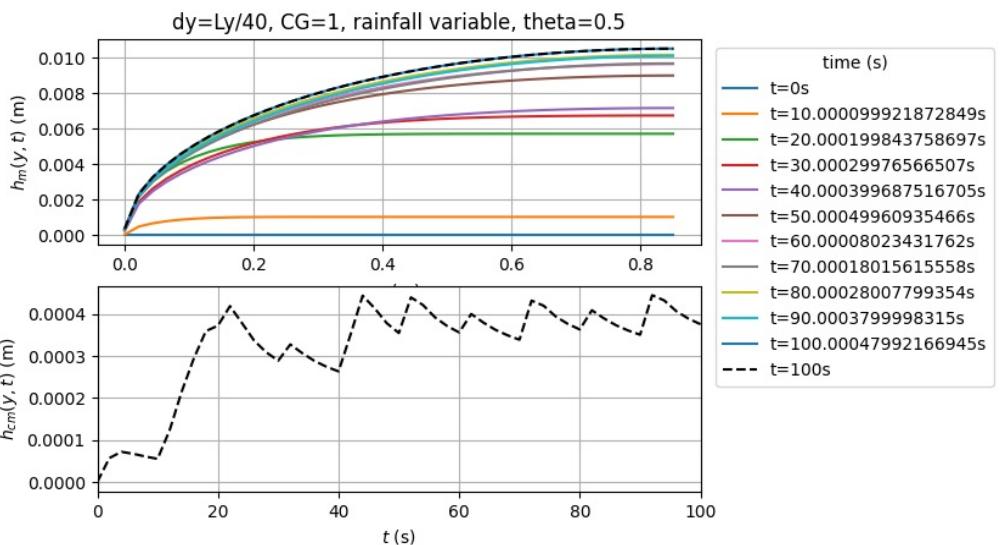
Crank-Nicolson weak formulation:

$$\begin{aligned} & \int_0^{L_y} g h_m^{n+1} dy + \frac{L_c g(0) h_m^{n+1}}{m_{\text{por}} \sigma_c} + \frac{1}{2} \Delta t \int_0^{L_y} [g h_m^{n+1} \partial_y g \partial_y (h_m^{n+1})] dy \\ & + \frac{1}{2} \Delta t \frac{\sqrt{g} g(0)}{m_{\text{por}} \sigma_c} \max\left(\frac{2}{3} h_m^n, 0\right)^{3/2} \\ & = \int_0^{L_y} g h_m^n dy + \frac{L_c g(0) h_m^n}{m_{\text{por}} \sigma_c} + \frac{1}{2} \Delta t \int_0^{L_y} [-g h_m^n \partial_y g \partial_y (h_m^n) + \frac{2(R^n + R^m)}{m_{\text{por}} \sigma_c}] dy \\ & - \frac{1}{2} \Delta t \frac{\sqrt{g} g(0)}{m_{\text{por}} \sigma_c} \max\left(\frac{2}{3} h_m^n, 0\right)^{3/2} \end{aligned}$$

$$g = \ell_i, \quad h_m = h, \quad g(0) = \delta_{ii},$$

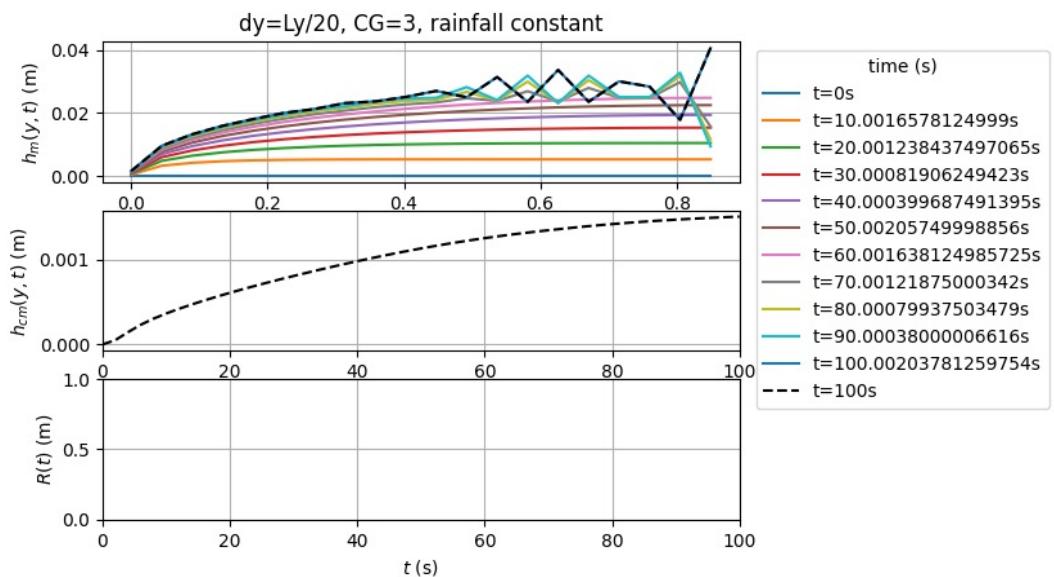
$$\begin{aligned} & \Rightarrow \int_0^{L_y} \ell_i h_j^{n+1} \ell_j dy + \frac{L_c h_i^{n+1}}{m_{\text{por}} \sigma_c} \delta_{ii} + \frac{1}{2} \Delta t \int_0^{L_y} [-g h_m^{n+1} \partial_y \ell_i \partial_y h_m^{n+1}] dy \\ & + \frac{1}{2} \Delta t \frac{\sqrt{g}}{m_{\text{por}} \sigma_c} \max\left(\frac{2}{3} h_i^{n+1}, 0\right)^{3/2} \delta_{ii}, \\ & = \int_0^{L_y} \ell_i h_j^n \ell_j dy + \frac{L_c h_i^n}{m_{\text{por}} \sigma_c} \delta_{ii} + \frac{1}{2} \Delta t \int_0^{L_y} [-g h_m^n \partial_y \ell_i \partial_y h_m^n] dy \\ & - \frac{1}{2} \Delta t \frac{\sqrt{g}}{m_{\text{por}} \sigma_c} \max\left(\frac{2}{3} h_i^n, 0\right)^{3/2} \delta_{ii} \end{aligned}$$

$$\begin{aligned} & \Rightarrow M_{ij} h_j^{n+1} + \frac{L_c}{m_{\text{por}} \sigma_c} h_i^{n+1} \delta_{ii} + \frac{1}{2} \Delta t b_i^{n+1} + \frac{1}{2} \Delta t \frac{\sqrt{g} \delta_{ii}}{m_{\text{por}} \sigma_c} \max\left(\frac{2}{3} h_i^{n+1}, 0\right)^{3/2} \\ & : M_{ij} h_j^n + \frac{L_c}{m_{\text{por}} \sigma_c} h_i^n \delta_{ii} - \frac{1}{2} \Delta t b_i^n - \frac{1}{2} \Delta t \frac{\sqrt{g} \delta_{ii}}{m_{\text{por}} \sigma_c} \max\left(\frac{2}{3} h_i^n, 0\right)^{3/2} \end{aligned}$$



4) CG 3:

oscillations at Ly end of channel.

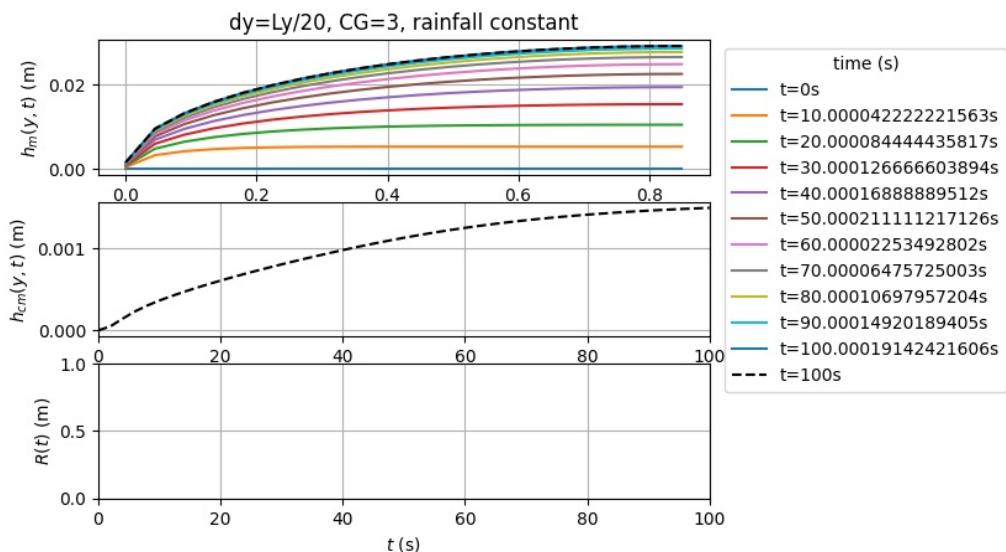


When CG 3 is used w/ explicit euler scheme we see oscillations at the end of the channel.

This is due to the time step used being s.t. the scheme is stable for a 1st order scheme.

However, for  $CFL = 2.3$  the stability condition is violated at higher order solvers (such as  $CG=3$  example)

instead using  $dt = \left(\frac{dy}{3}\right)^2 \times \frac{1}{2} \times CFL \times (\Delta y)^2$ :



Thus need to use a smaller CFL number for the scheme to be stable.

All fine: 19/20