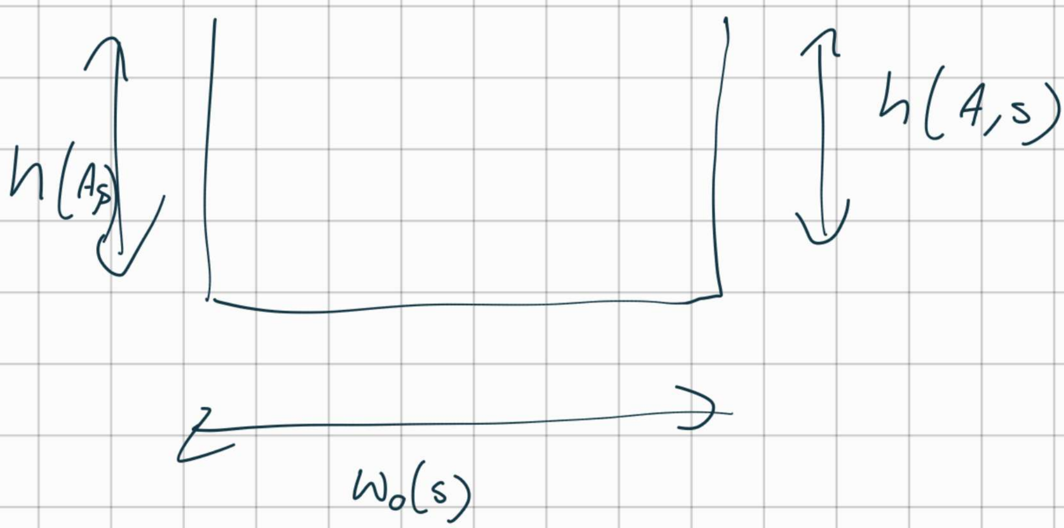
Numerics Ex 2

1).



$$h(A,s)\,w_0(s) = A$$

$$\rightarrow \boxed{h(A,s) = \frac{A}{w_0(s)}}$$

$$P(A,s) = w_0(s) + 2h(A,s)$$

$$\boxed{P(A,s) = w_0(s) + \frac{2A}{w_0(s)}}$$

$$\partial_t A + \partial_s\left(AR^{2/3}\frac{\sqrt{-\partial_s b}}{C_m}\right)$$

$$= \partial_t A + \partial_s\left(\frac{A^{5/3}\sqrt{-\partial_s b}}{C_m P^{2/3}}\right)$$

$$= \partial_t A + \partial_s F$$

as $P = w_0(s) + \frac{2A}{w_0(s)}$

$$F = \frac{A^{5/3}}{C_m}\sqrt{-\partial_s b} \times \frac{1}{\left(w_0 + \frac{2A}{w_0}\right)^{2/3}}$$

$$\partial_t A + \partial_s F(A,s)$$

$$= \partial_t A + \frac{\partial F}{\partial A}\partial_s A + \frac{\partial F}{\partial s} = 0$$

$$\frac{\partial F}{\partial A} = \frac{5}{3}A^{2/3}\frac{\sqrt{-\partial_s b}}{C_m} \times \frac{1}{\left(w_0 + \frac{2A}{w_0}\right)^{2/3}}$$

$$- \frac{2}{3}A^{5/3}\frac{\sqrt{-\partial_s b}}{C_m} \times \frac{2/w_0}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}}$$

$$= \frac{\sqrt{-\partial_s b}}{3C_m} \times \frac{\left(5\omega_0 + 10A/\omega_0 - 4\frac{A}{\omega_0}\right)A^{2/3}}{\left(\omega_0 + \frac{2A}{\omega_0}\right)^{5/3}}$$

$$\boxed{\frac{\partial F}{\partial A} = \frac{\sqrt{-\partial_s b}}{3C_m} \frac{\left(5\omega_0 A^{2/3} + 6A^{5/3}/\omega_0\right)}{\left(\omega_0 + 2A/\omega_0\right)^{5/3}} > 0}$$

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial \omega_0}\frac{d\omega_0}{ds}$$

$$= \frac{A^{5/3}\sqrt{-\partial_s b}}{C_m}\left(\frac{-2}{3}\right)\left(1 - \frac{2A}{\omega_0^2}\right) \times \frac{1}{\left(\omega_0 + \frac{2A}{\omega_0}\right)^{5/3}}$$

$$\boxed{\frac{\partial f}{\partial s} = -\frac{2\sqrt{\partial_s b}}{3C_m} \frac{A^{5/3}}{\left(\omega_0 + \frac{2A}{\omega_0}\right)^{5/3}} \left(1 - \frac{2A}{\omega_0^2}\right)\frac{d\omega_0}{ds}}$$

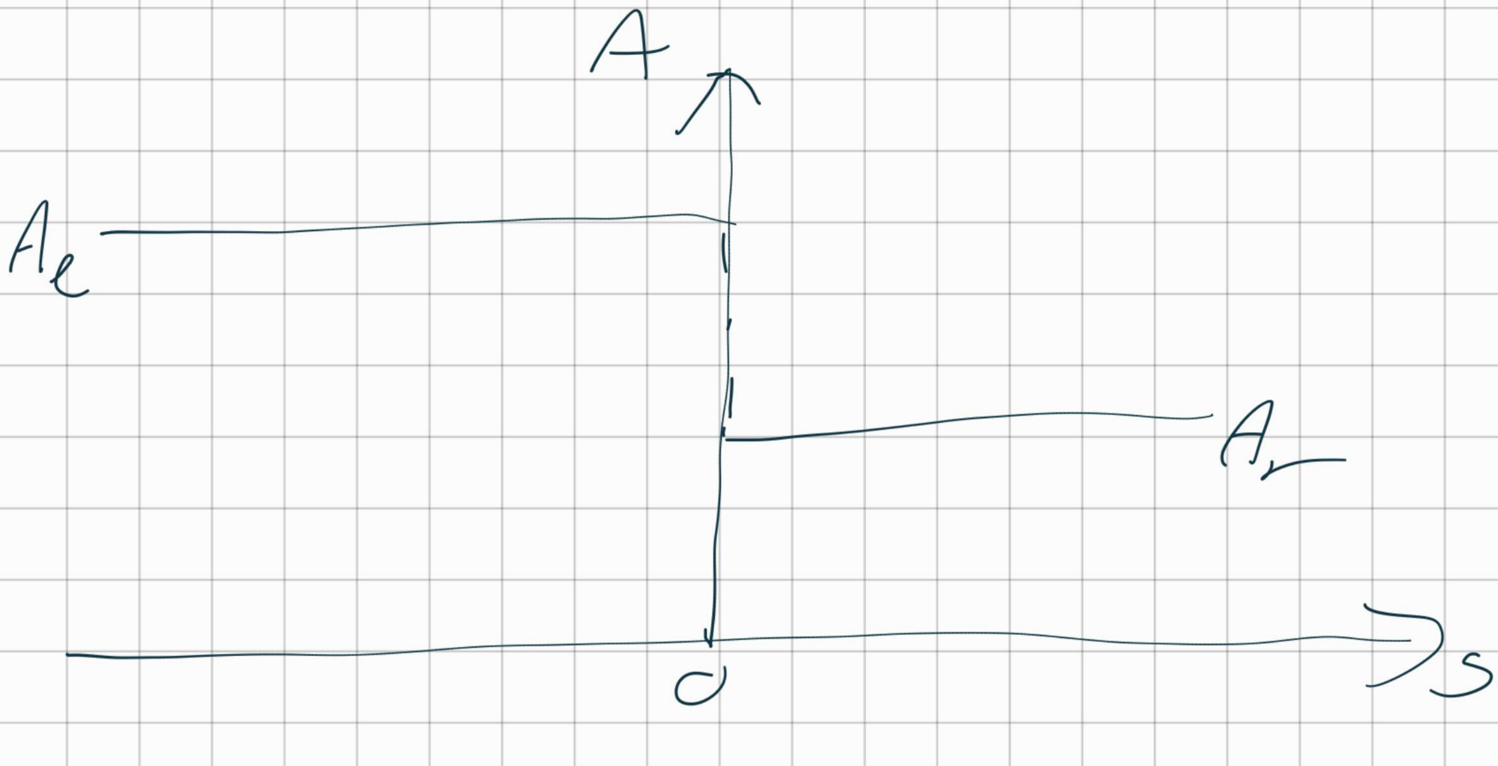
so system can be rewritten as in (S).

2). $\frac{dw_0}{ds} \sim 0$

so $\partial_t A + \frac{\partial F}{\partial A} \partial_s A = 0$

In this case, $\lambda = \frac{\partial F}{\partial A} = \sqrt{\frac{-\partial_s b}{3C_m}} \frac{\left(5w_0 A^{2/3} + 6A^{5/3}/w_0\right)}{\left(w_0 + 2A/w_0\right)^{5/3}}$

Letting $s=0$ where discontinuity is at $t=0$



at $t=0$, $\lambda = \begin{cases} \lambda(A_r) & s>0 \\ \lambda(A_l) & s<0 \end{cases}$

If $A_l > A_r$, then left side travels faster than right side

$$\frac{ds}{dt} = \frac{\partial F}{\partial A}$$

Over discontinuity

$$\int_{A_l}^{A_r} \frac{ds}{dt}\, dA = \int_{A_l}^{A_r} \frac{\partial F}{\partial A}\, dA$$

$$\frac{ds}{dt} = \frac{F(A_r) - F(A_l)}{A_r - A_l} = u$$

For $A_l > A_r$

$$A = \begin{cases} A_l & s - ut < 0 \\ A_r & s - ut \geq 0 \end{cases}$$

where $u = \dfrac{F(A_r) - F(A_l)}{A_r - A_l}$

If $A_r > A_l$, then right side travels faster than left

→ so discontinuity is no longer present.

A is constant in time along $\frac{ds}{dt} = \lambda(A)$

→ and so in rarefaction region, $\frac{ds}{dt} = \frac{s}{t}$ so

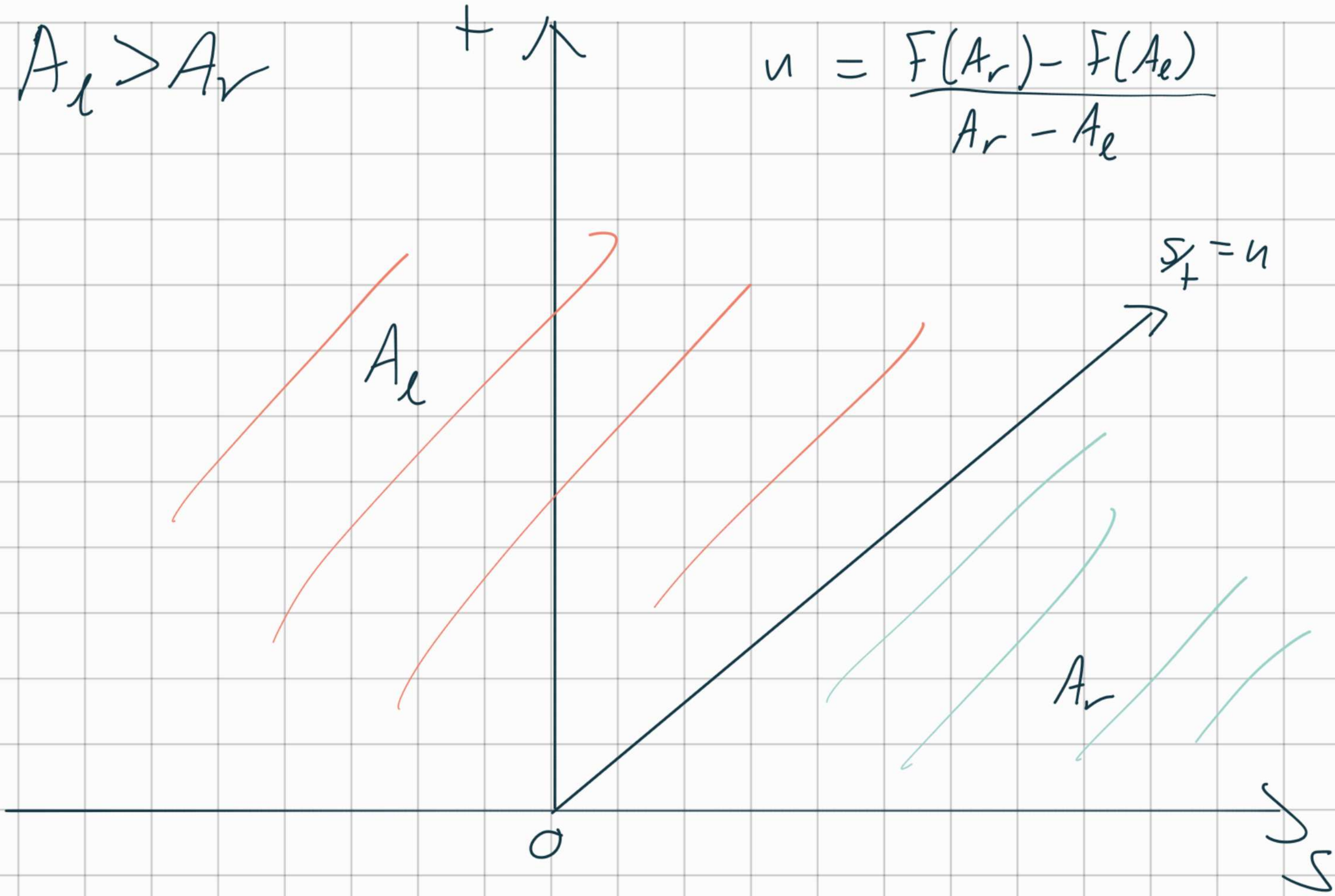
A can be described as a function of $\frac{s}{t}$

$A = \lambda^{-1}(s/t)$ where λ^{-1} = inverse of $\frac{\partial F}{\partial A}$ function

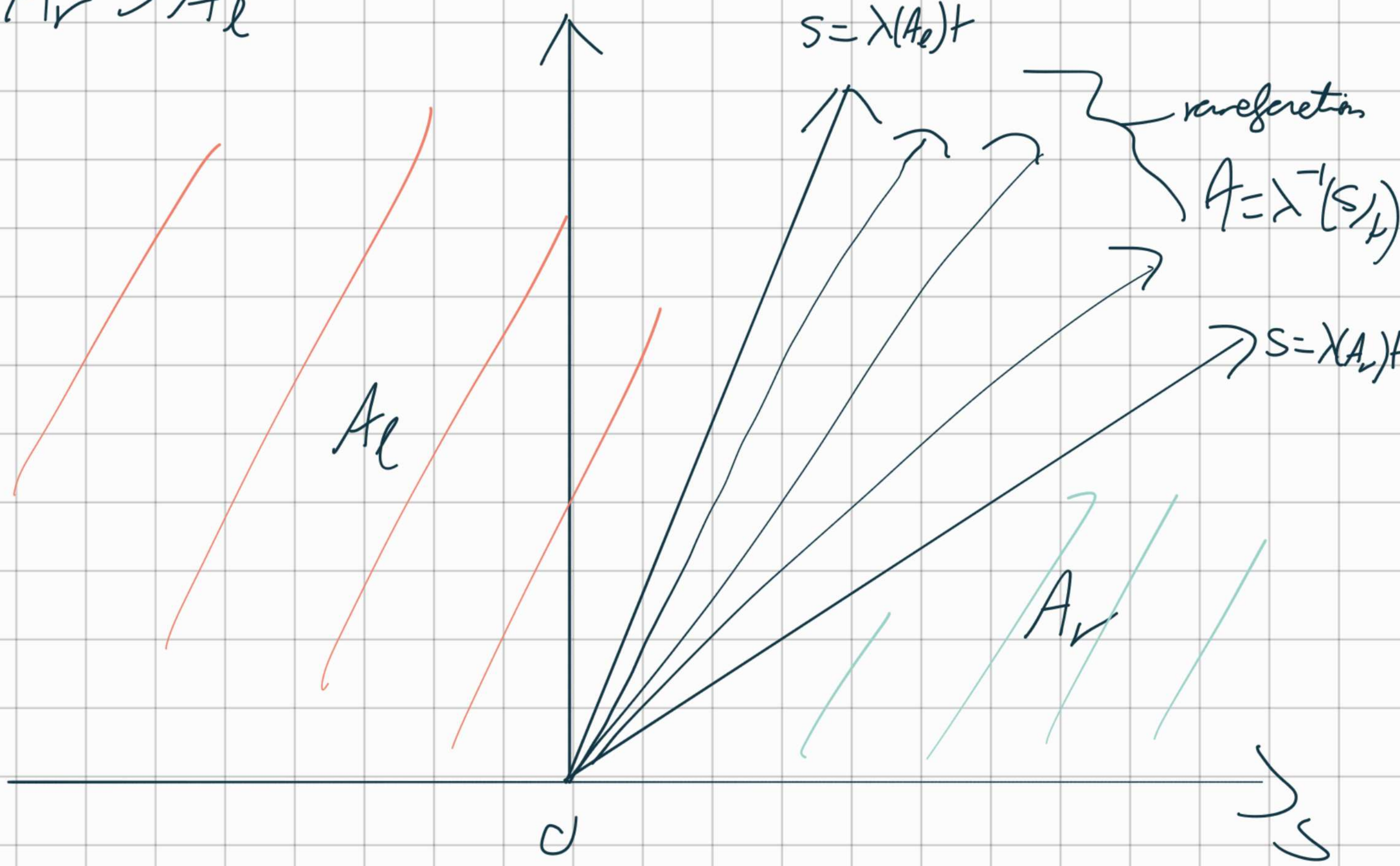
$$\boxed{\begin{gathered} A_r > A_l \\ A = \begin{cases} A_l & s - \lambda(A_l)t < 0 \\ \lambda^{-1}(s/t) & \lambda(A_l)t \le s \le \lambda(A_r)t \\ A_r & s - \lambda(A_r)t > 0 \end{cases} \end{gathered}}$$

$A_l > A_r$

$$u = \frac{F(A_r) - F(A_l)}{A_r - A_l}$$

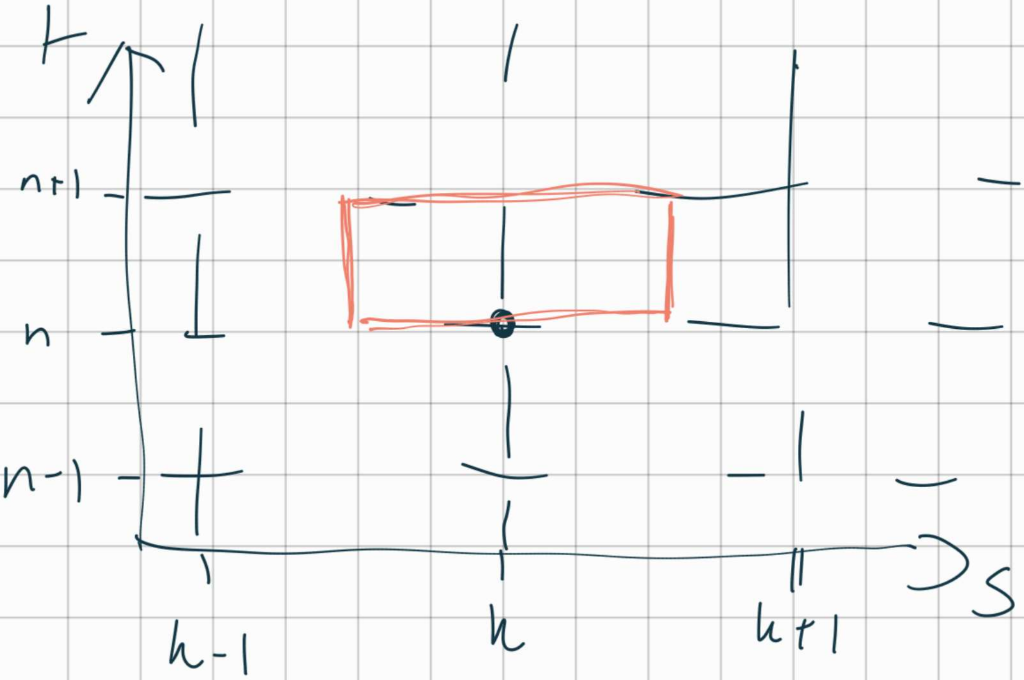


$A_r > A_l$



Only considered cases where A_r, A_l are positive, as area cannot be negative.

3) $\partial_t A + \partial_s F(A,s) = 0$



Integrating over $\square$

$$\int_{t_n}^{t_{n+1}} \int_{s_{k-1/2}}^{s_{k+1/2}} \partial_t A \, ds\, ds = - \int_{t_n}^{t_{n+1}} \int_{s_{k-1/2}}^{s_{k+1/2}} \partial_s F(A,s)\, ds\, dt$$

$$\int_{s_{k-1/2}}^{s_{k+1/2}} A^{n+1} - A^n \, ds = - \int_{t_n}^{t_{n+1}} F_{k+1/2}(A,s) - F_{k-1/2}(A,s)\, dt$$

$\int_{s_{k-1/2}}^{s_{k+1/2}} A^n \, ds = \Delta s_k \bar{A}_k^n$ where $\bar{A}_k^n$ = cell average of A in cell k.

$$\rightarrow \Delta s_k \left(\bar{A}_k^{n+1} - \bar{A}_k^n \right) = - \int_{t_n}^{t_{n+1}} F_{k+1/2}(A,s) - F_{k-1/2}(A,s)\, dt$$

$$\boxed{\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta s_k} \int_{t_n}^{t_{n+1}} F_{k+1/2}(A,s) - F_{k-1/2}(A,s)\, dt}$$

$$F_{k+1/2}(A,s) = F_{k+1/2}\left(\bar{A}_k^n, \bar{A}_{k+1}^n\right)$$

and $F_{k+1/2}\left(\bar{A}_k^n, \bar{A}_{k+1}^n\right) \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} F_{k+1/2}(A,s)\, dt$

$$\boxed{\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{\Delta t}{\Delta s_k} \left[F_{k+1/2}\left(\bar{A}_k^n, \bar{A}_{k+1}^n\right) - F_{k-1/2}\left(\bar{A}_{k-1}^n, \bar{A}_k^n\right) \right]}$$

Godunov method relies on the solution at the previous time step to be piecewise constant, so only need to use cell average at previous time step (e.g. A^n are constant in time for $t_n < t < t_{n+1}$).

4). $F(A,s)\big|_{s=s_{n+1/2}} = F_{n+1/2}(\bar{A}^n_n, \bar{A}^n_{n+1})$

$s_{n+1/2}$ corresponds to $s=0$

$\rightarrow$ and on both diagrams in part 2,

$A = A_e$ at $s=0$

So Godunov flux $= F(A_e)$
$(F_{n+1/2} = F(\bar{A}^n_n))$

This is completely upwind, makes sense as we know $\frac{\partial F}{\partial A} > 0$ always.

5)

$$\Delta t < CFL \min\left(\frac{h_k}{\lambda_k}\right)$$

$$\lambda_k = \left.\frac{\partial F}{\partial A}\right|_{A_k}$$

$$\frac{\partial F}{\partial A} = \frac{\sqrt{-\partial_s b}}{3C_m} \frac{\left(5w_0 A^{2/3} + 6A^{5/3}/w_0\right)}{\left(w_0 + 2A/w_0\right)^{5/3}}$$

$$\boxed{\Delta t < CFL \min\left(\frac{h_k \cdot 3C_m \left(w_0 + \frac{2\bar{A}_k}{w_0}\right)^{5/3}}{\sqrt{-\partial_s b}\,\left(5w_0 A_k^{2/3} + 6A^{5/3}/w_0\right)}\right)}$$

6) If there is a constant discharge into the river

$\bar{A}_{-1}$ needs to be set so $F(\bar{A}_{-1}) = Q_0$

As this is an upwind scheme, $\bar{A}_{N_n}$ would not have an effect on the flux in the river

so $\bar{A}_{N_n}$ does not need to be set.

7). Test Case C:

- minimum time step chosen, $\Delta t = \frac{CFL\,\Delta x}{c_{00}}$
  ($\Delta x = 0.5m$)

$$= \frac{0.125 \times 0.5}{c_{00}}$$

- Reran for different CFL values

$CFL = 0.125,\ 0.25,\ 0.5,\ 1$

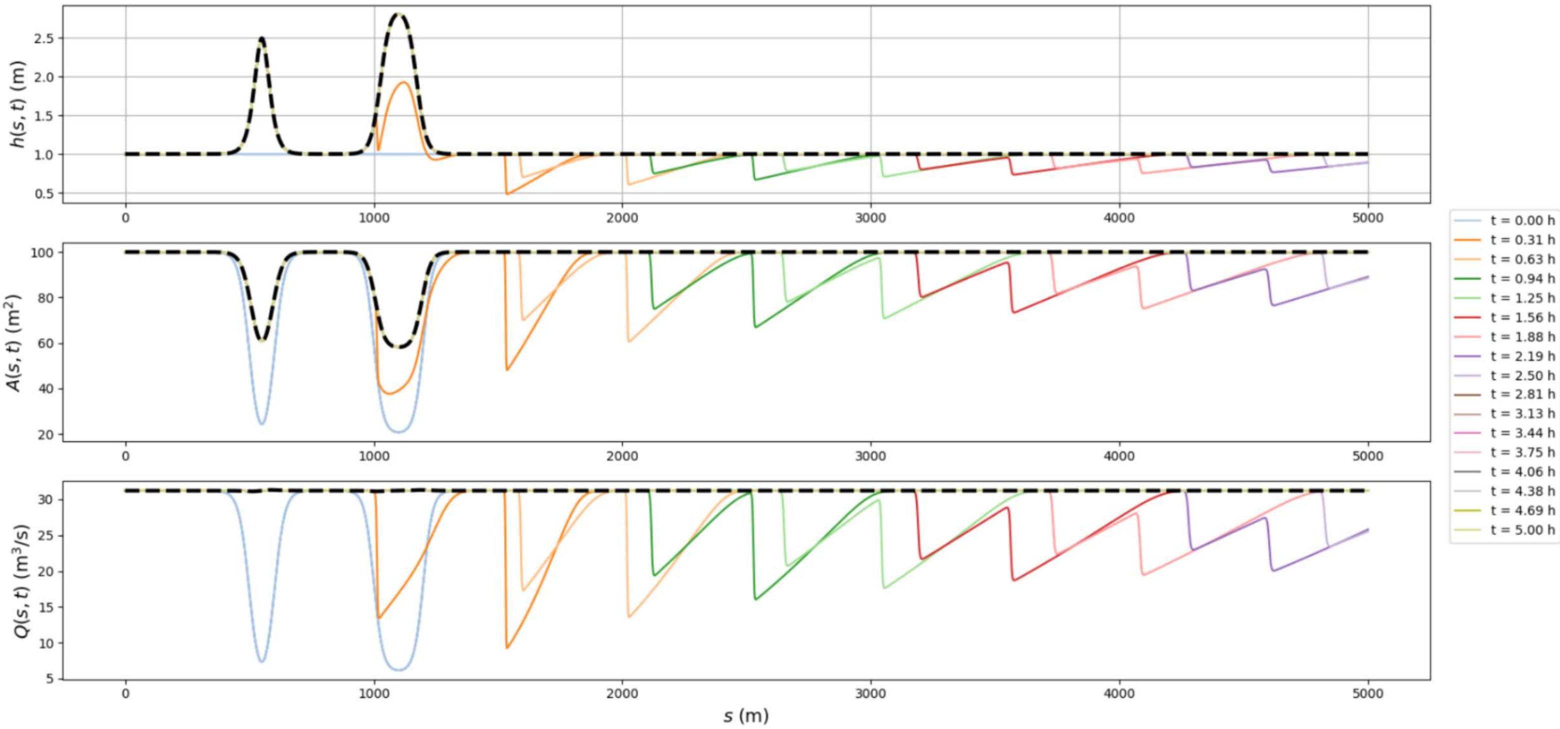
$\rightarrow \Delta t = \Delta t_{min},\ 2\Delta t_{min},\ 4\Delta t_{min},\ 8\Delta t_{min}$

- Then did similar for Δx (and kept $\Delta t = \Delta t_{min}$)

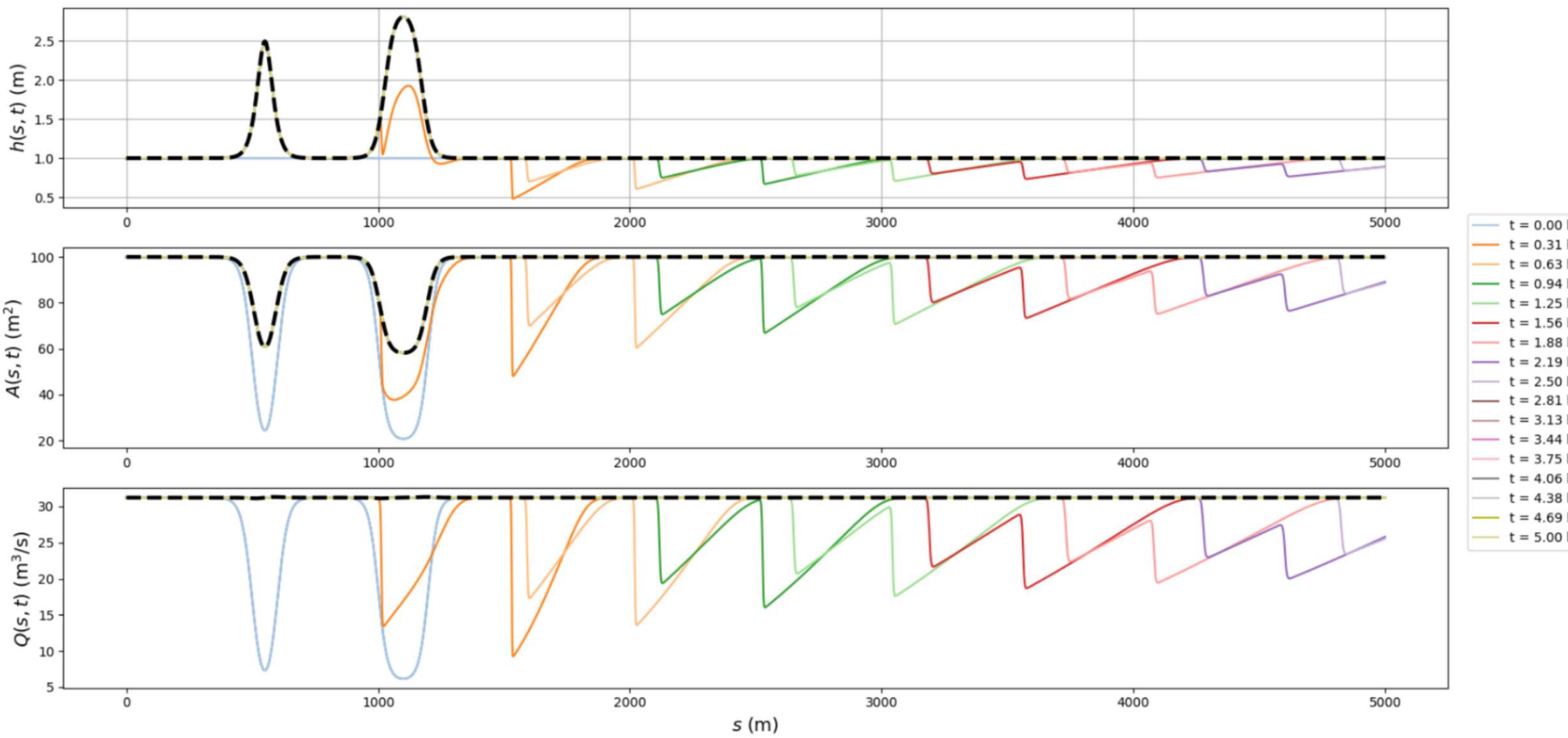
$\Delta x_{min} = 0.5m$

$\Delta x = 0.5m,\ 1m,\ 2m,\ 4m$

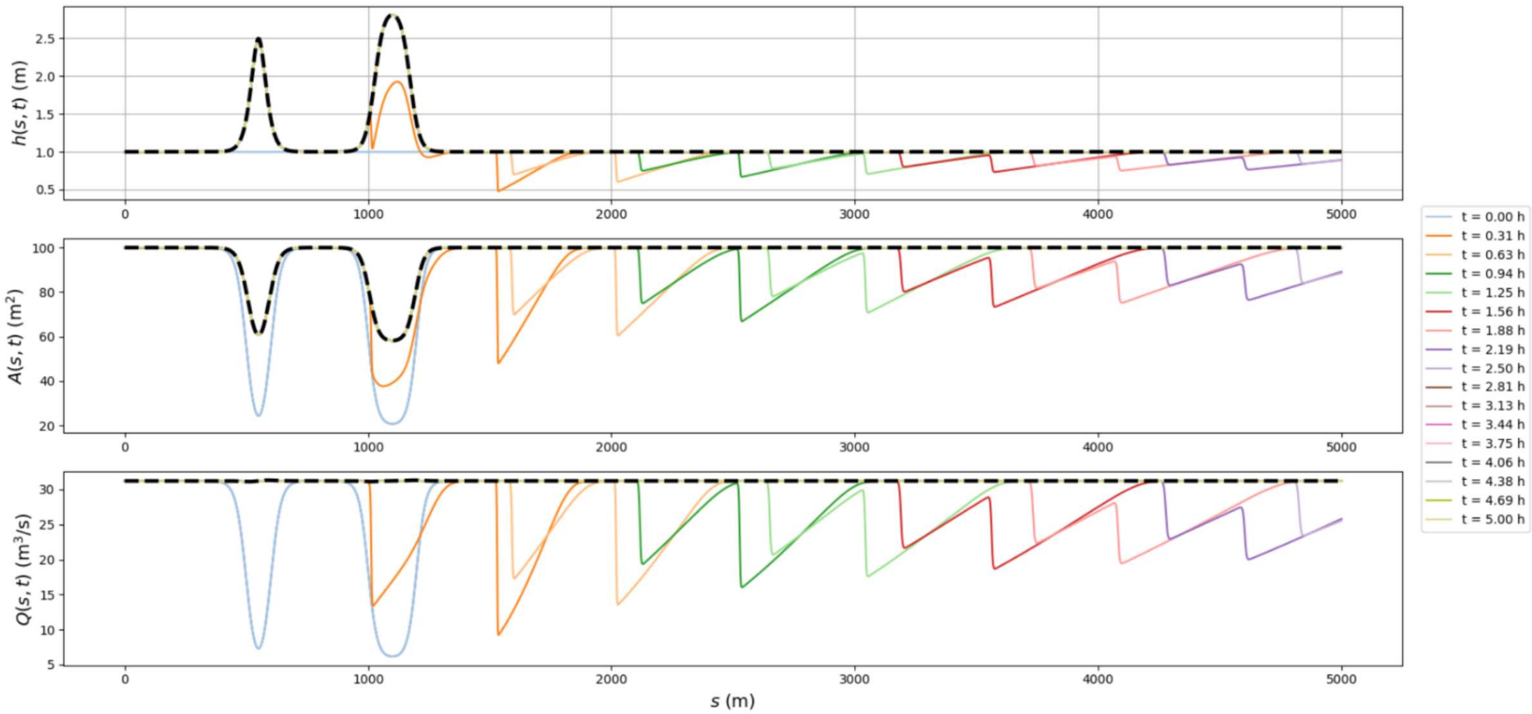
$\Delta x = \Delta x_{min}$ $\Delta t = \Delta t_{min}$ (CFL = 0.125)



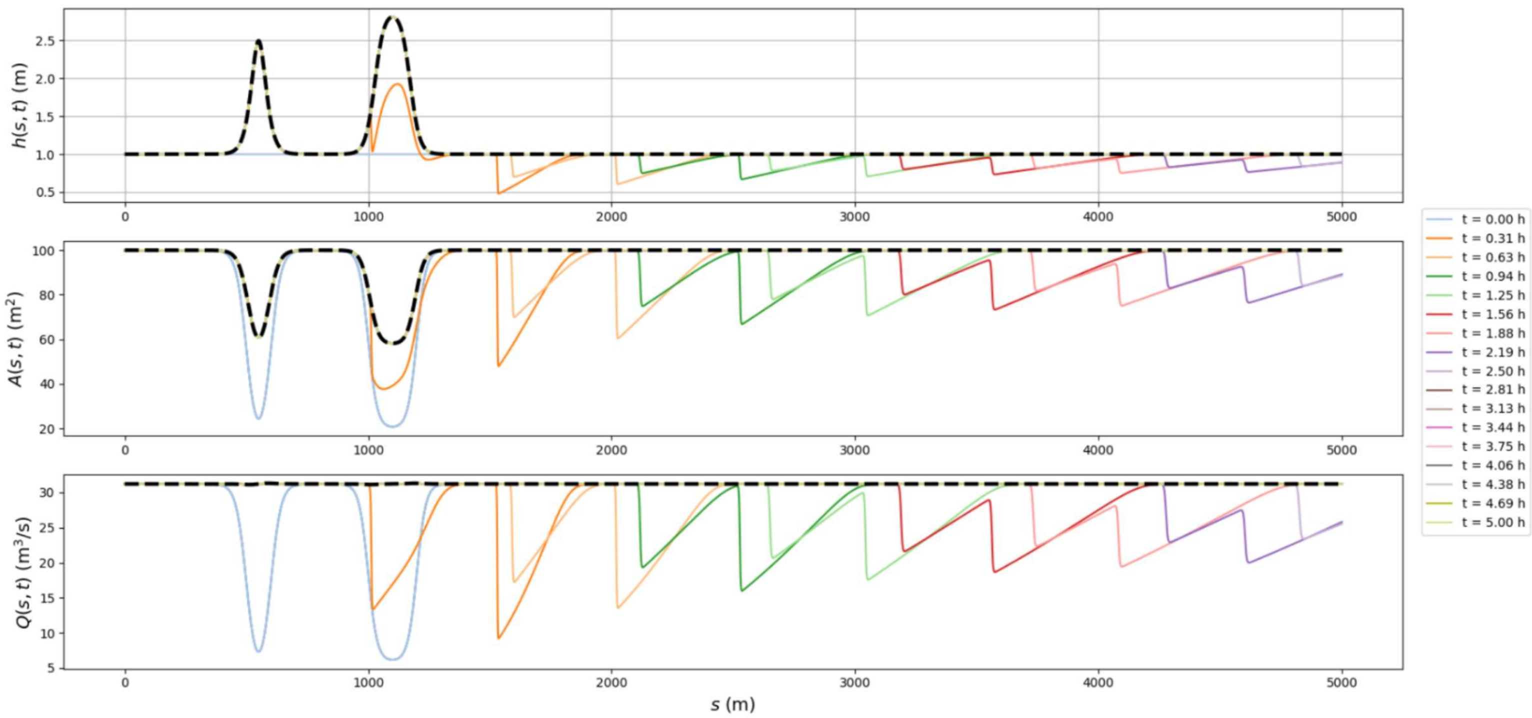
$\Delta x = \Delta x_{min}$ $\Delta t = 2\Delta t_{min}$ (CFL = 0.25)



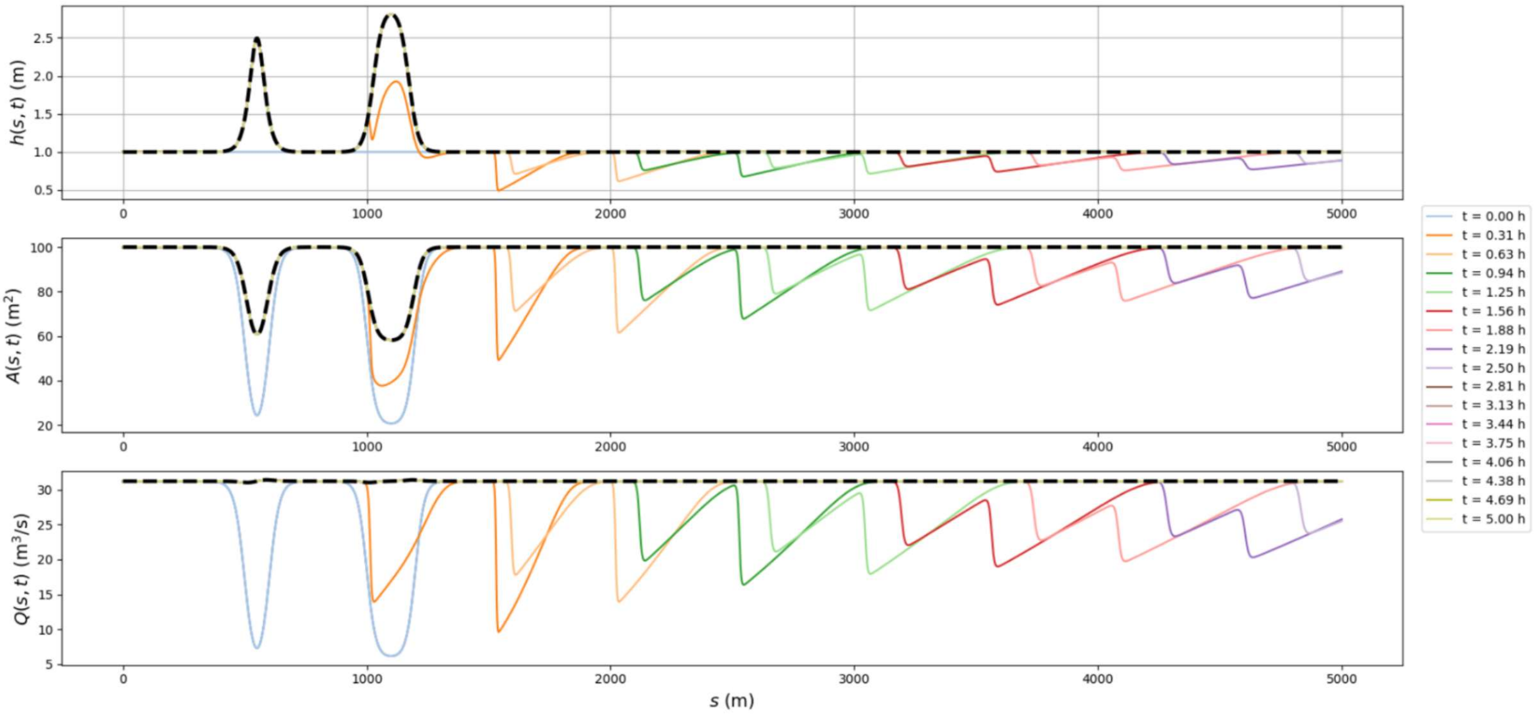
$\Delta x = \Delta x_{min}$ $\Delta t = 4\Delta t_{min}$ (CFL = 0.5)



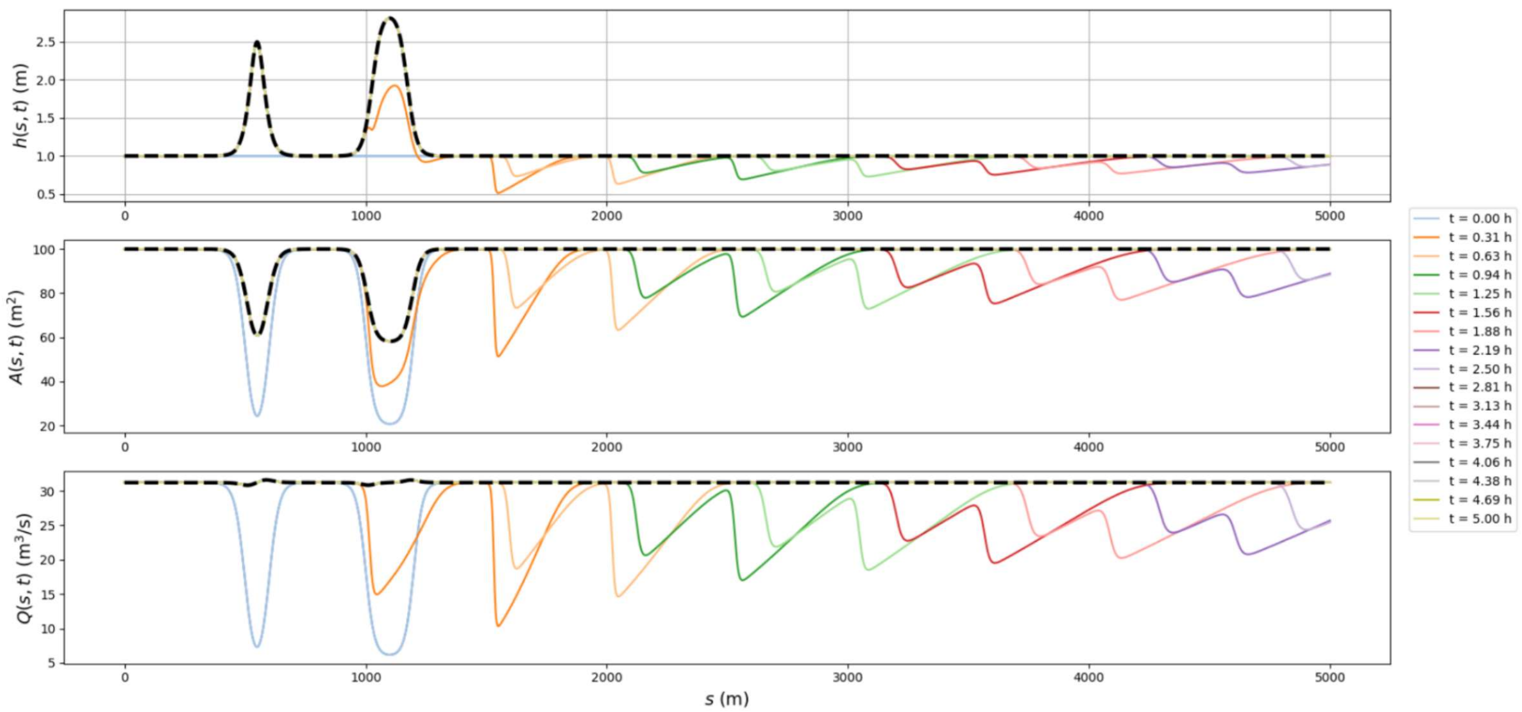
$\Delta x = \Delta x_{min}$ $\Delta t = 8\Delta t_{min}$ (CFL = 1)



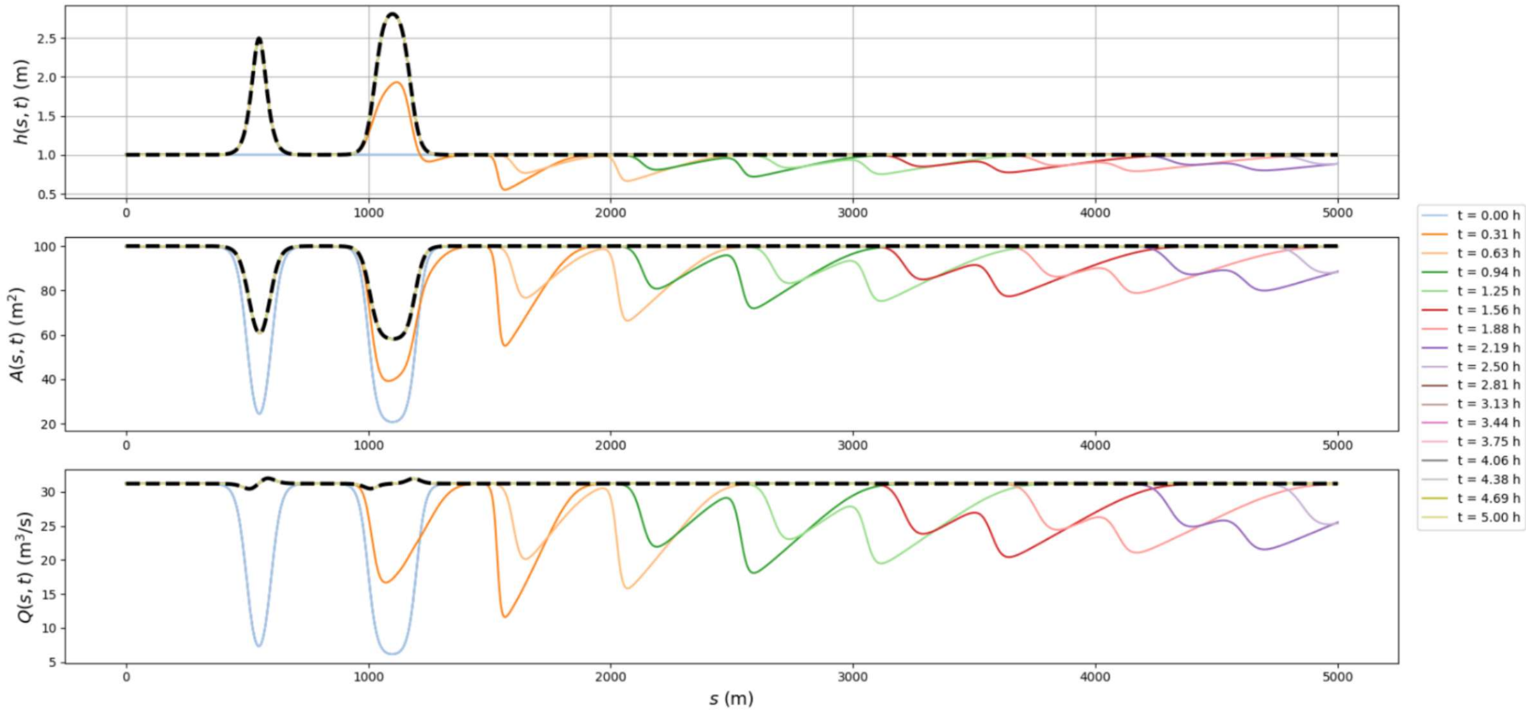
$\Delta x = 2\,\Delta x_{min} \qquad \Delta t = \Delta t_{min}$



$\Delta x = 4\,\Delta x_{min} \qquad \Delta t = \Delta t_{min}$

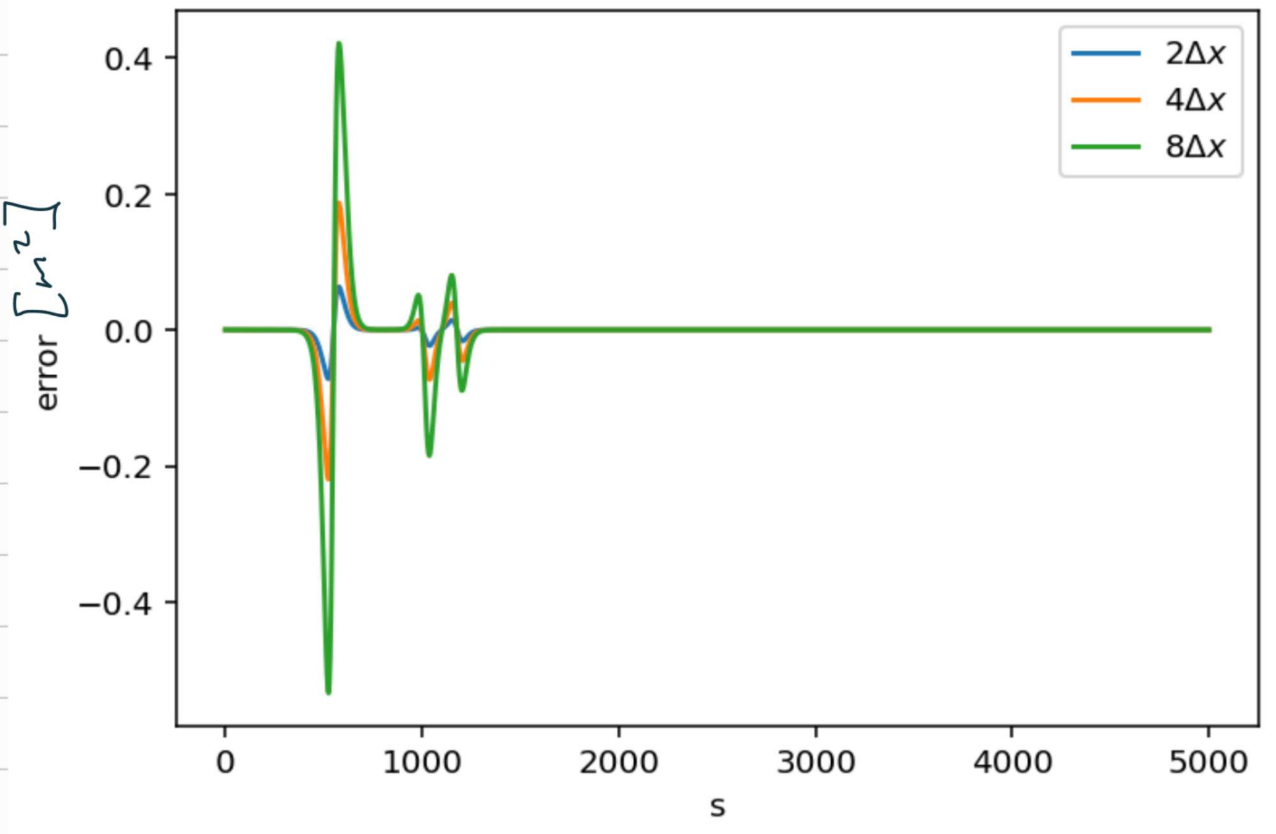


$\Delta x = 8\Delta x_{min}$ $\Delta t = \Delta t_{min}$

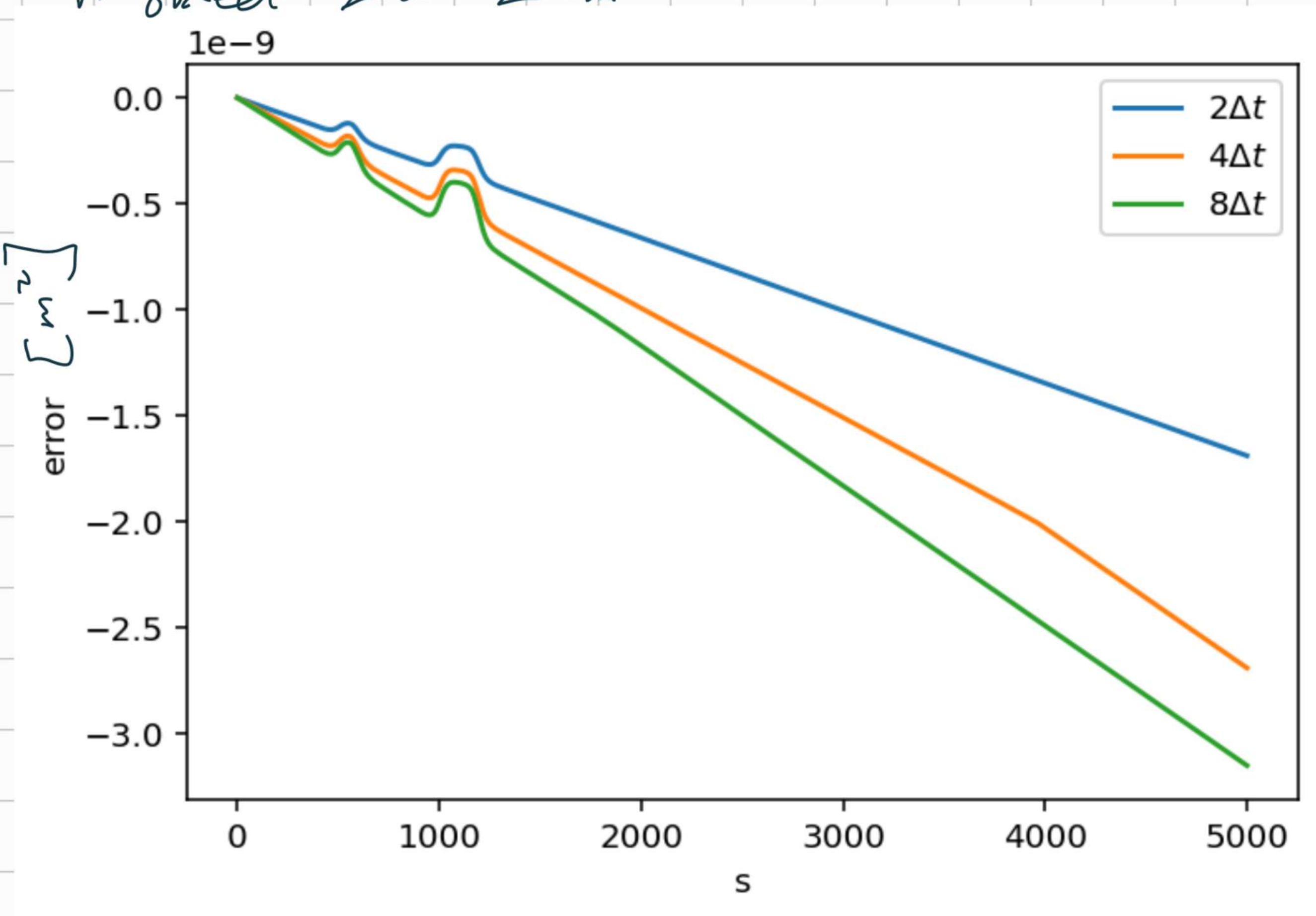


error $= A(\Delta x = \Delta x_{min}, \Delta t = \Delta t_{min}) - A_{num}$

at fixed $\Delta t = \Delta t_{min}$



At fixed $\Delta x = \Delta x_{min}$

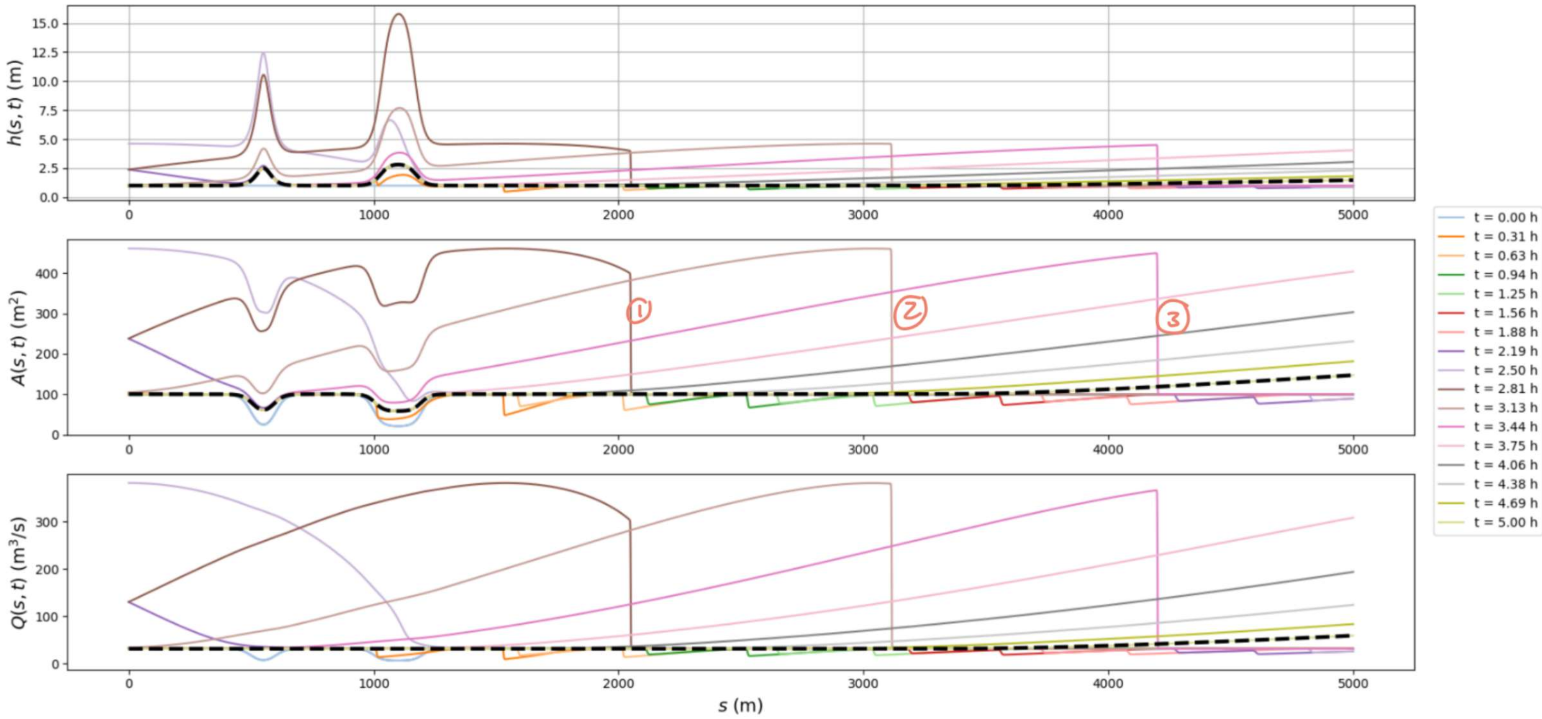


- 1st order in time and space
- Converges in space and time, as seen in error signals.
  → although comparing with a numerical solution rather than exact solution.

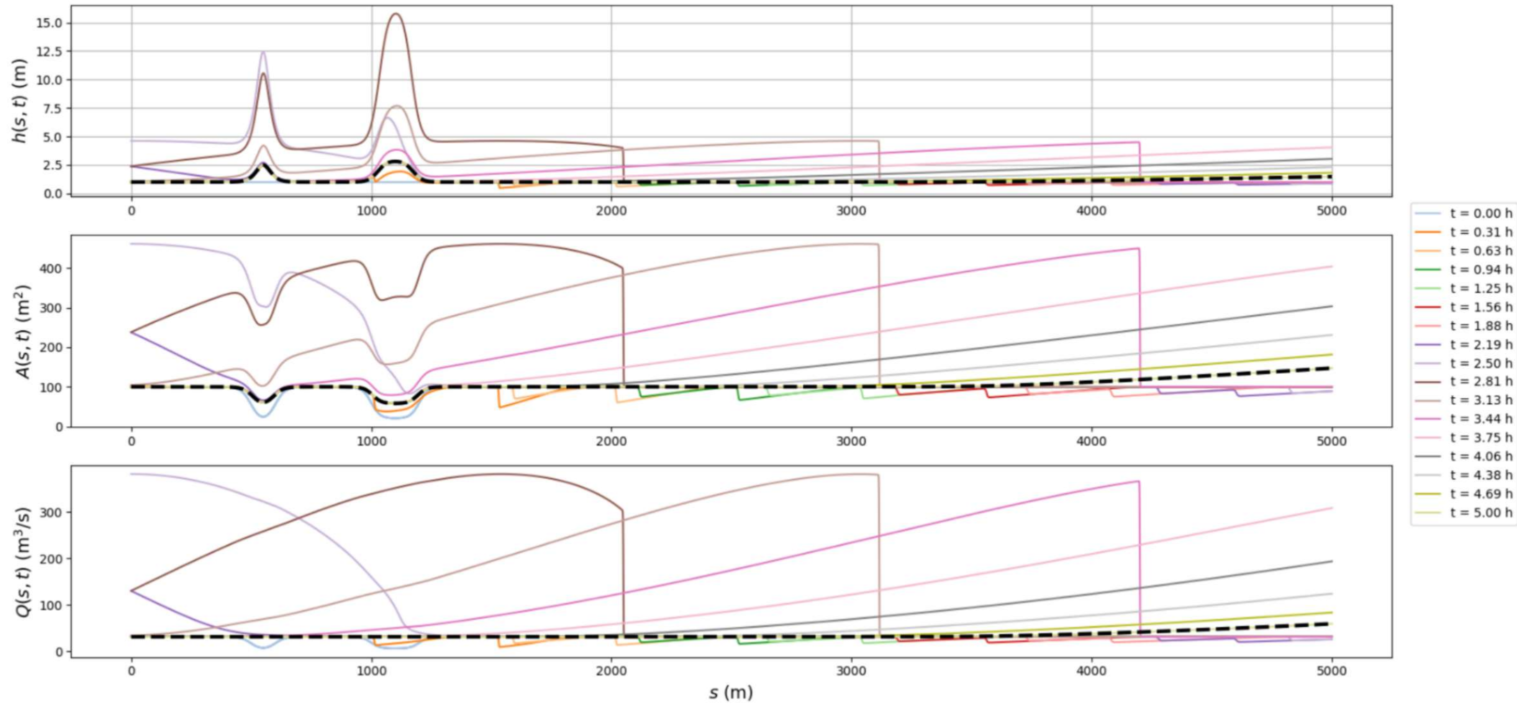
- More issues at points where width changes regarding convergence.

Did same for test case 1

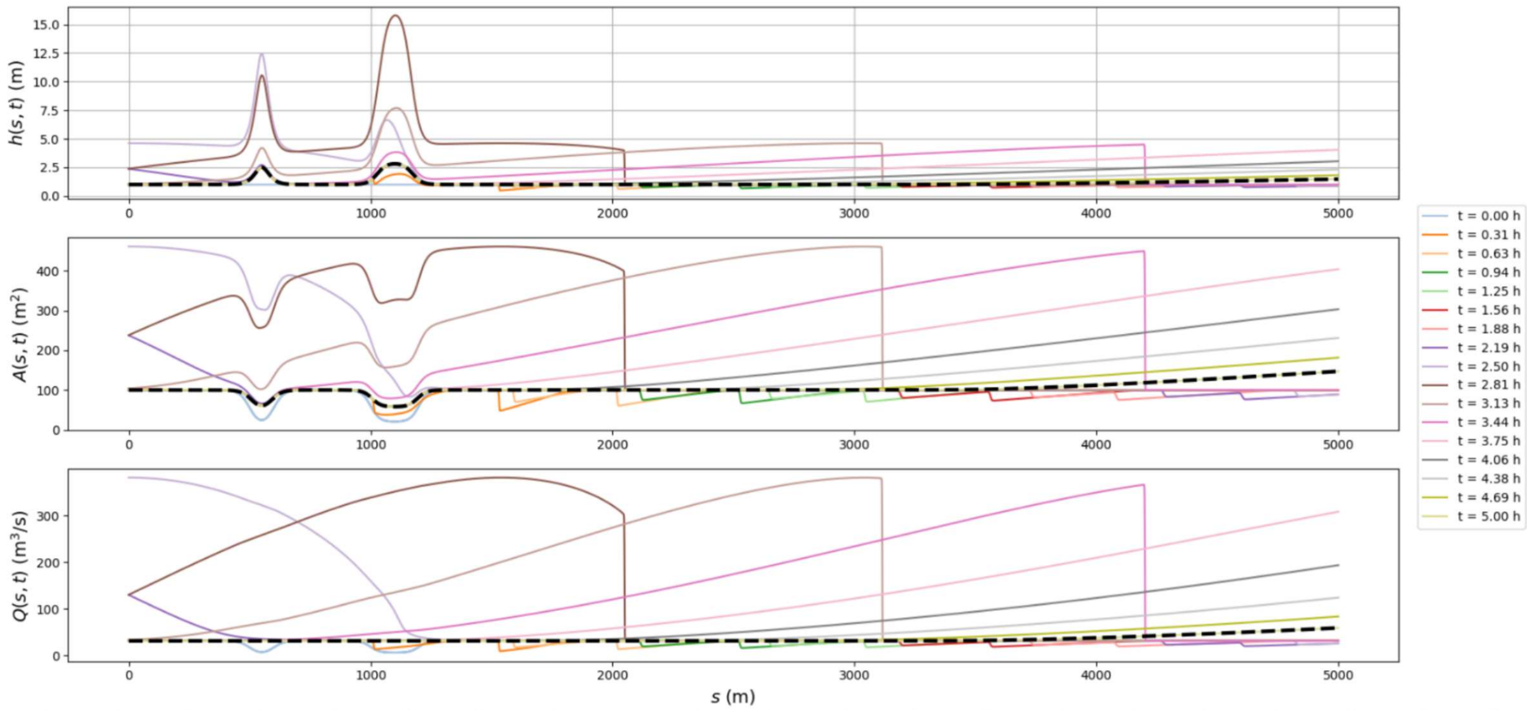
$\Delta x = \Delta x_{min}$ $\Delta t = \Delta t_{min}$



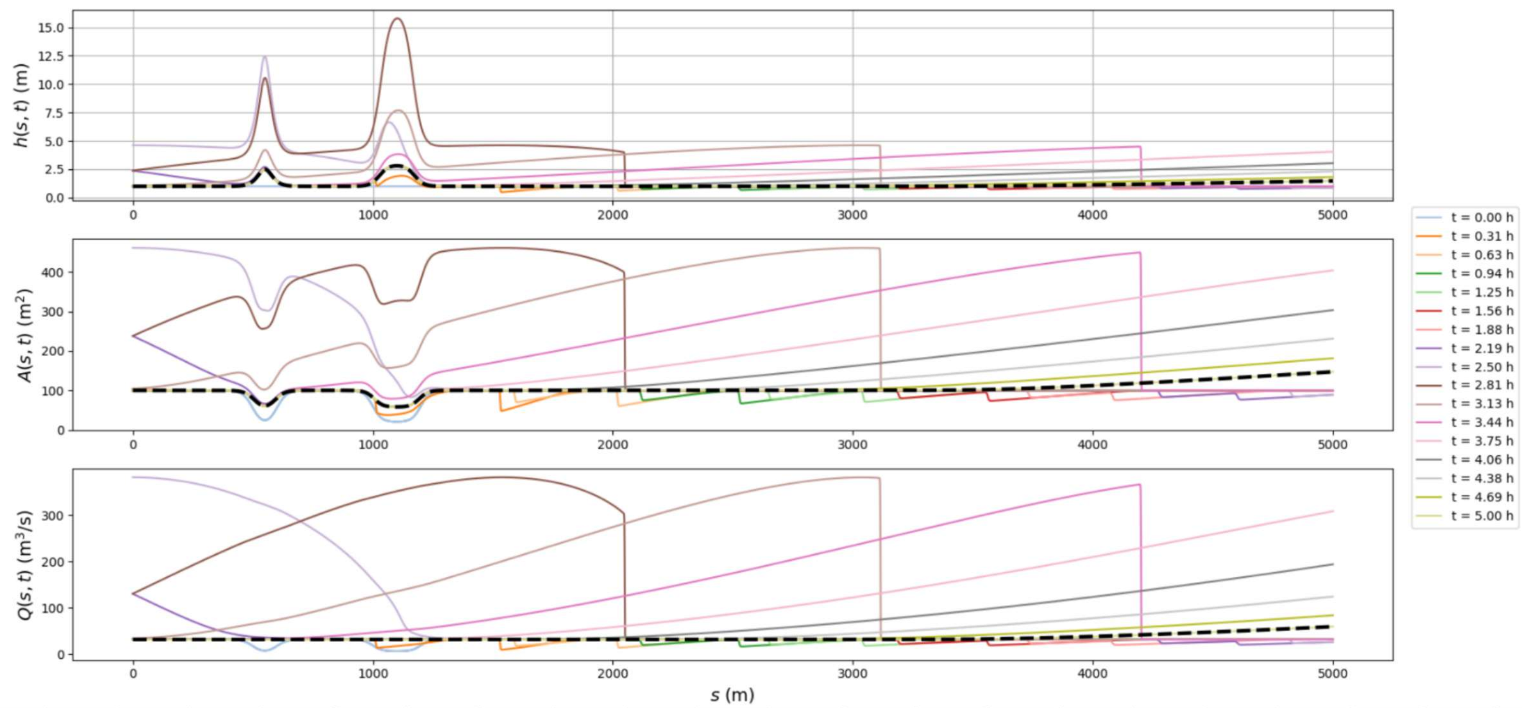
$\Delta x = \Delta x_{min}$, $\Delta t = 2\Delta t_{min}$



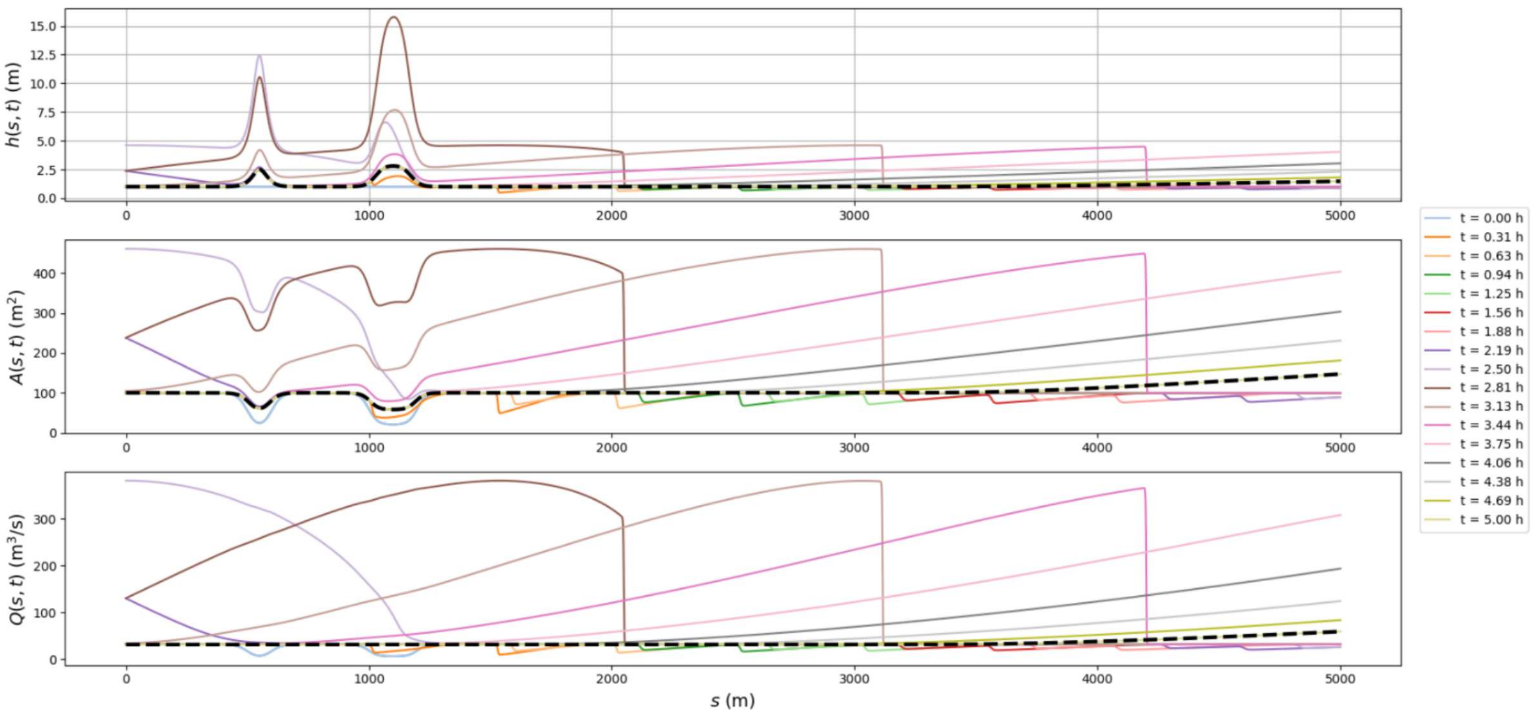
$\Delta x = \Delta x_{min}$, $\Delta t = 4 \Delta t_{min}$



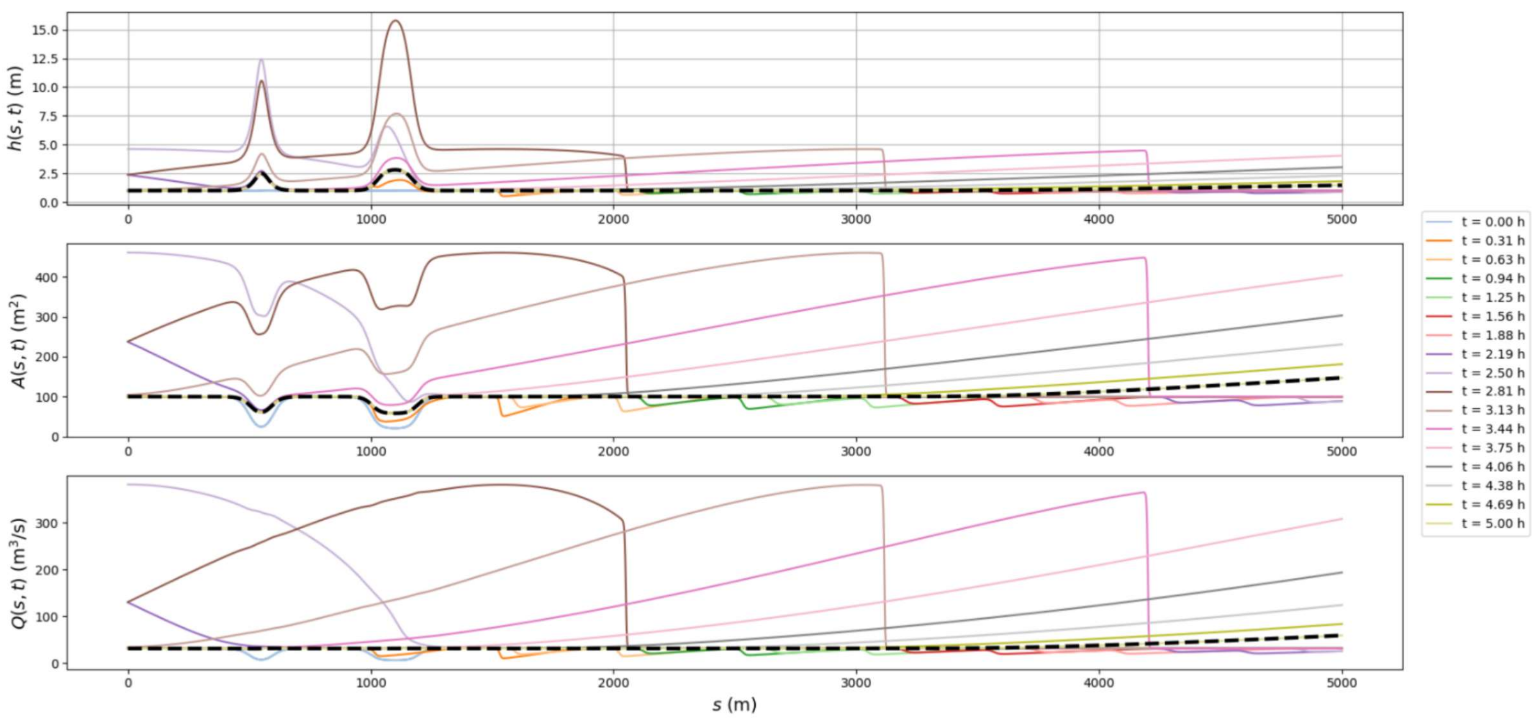
$\Delta x = \Delta x_{min}$, $\Delta t = 8 \Delta t_{min}$



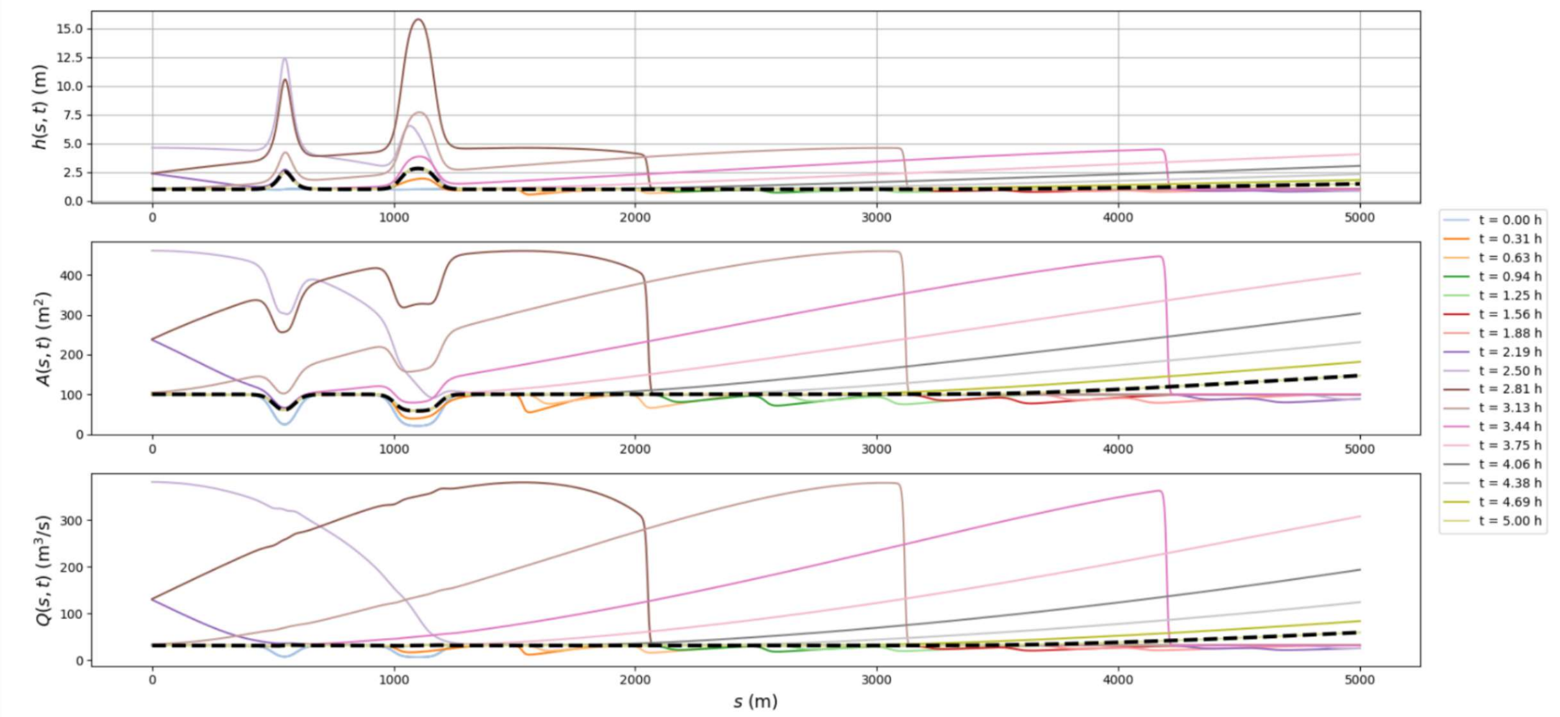
$\Delta x = 2\Delta x_{min}$, $\Delta t = \Delta t_{min}$



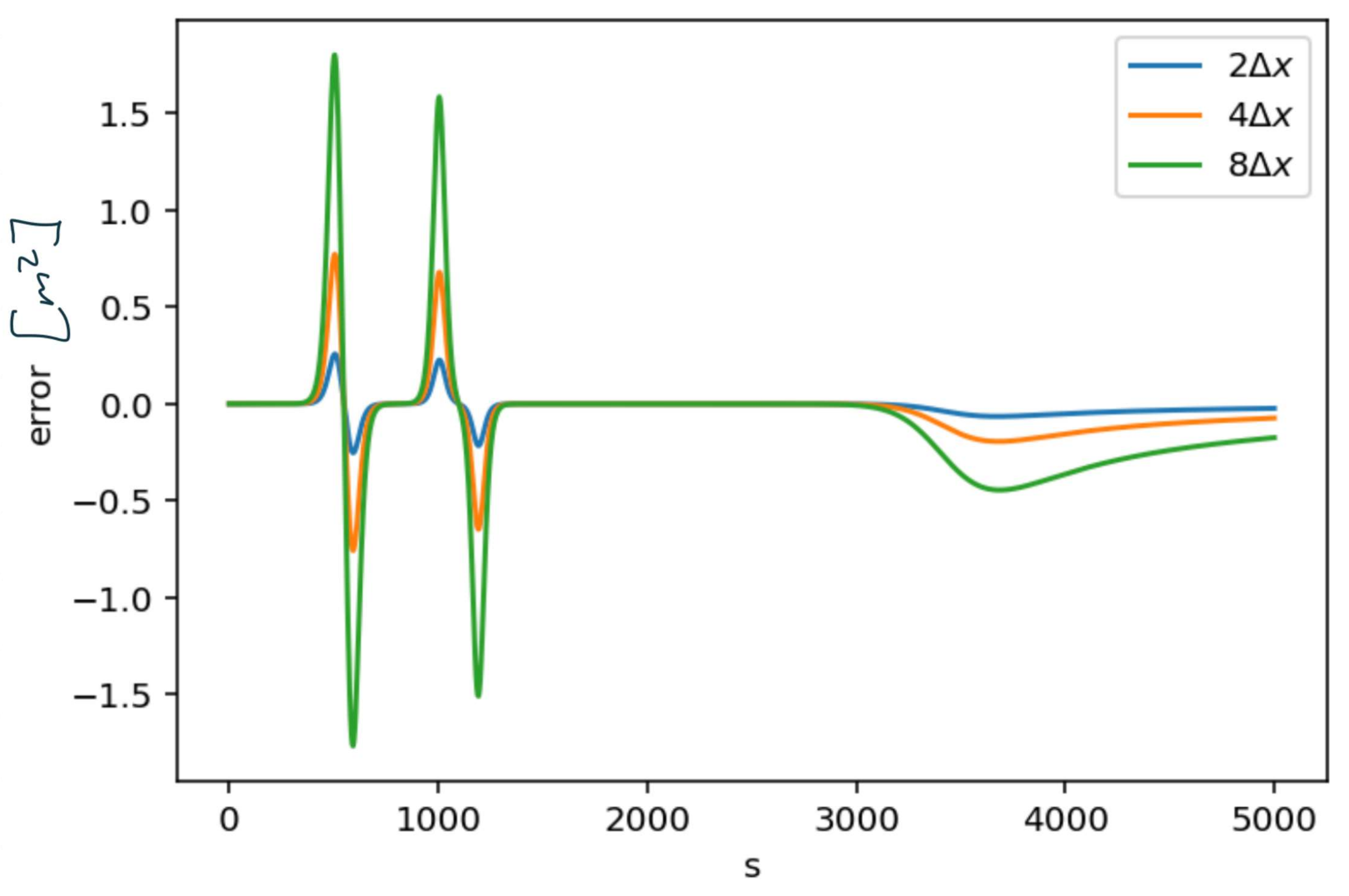
$\Delta x = 4\Delta x_{min}$, $\Delta t = \Delta t_{min}$



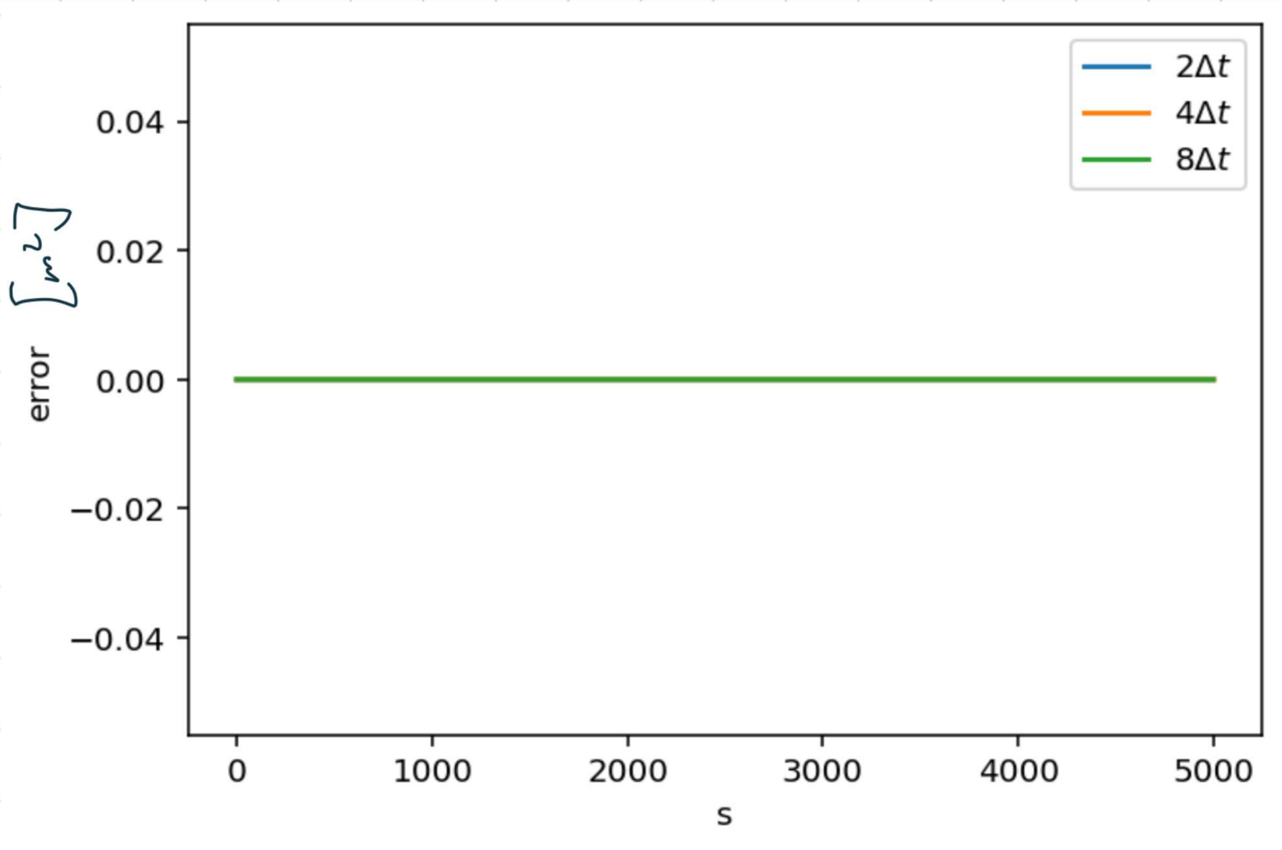
$\Delta x = 8\,\Delta x_{min}$, $\Delta t = \Delta t_{min}$



Fixed $\Delta t = \Delta t_{min}$



Fixed $\Delta x = \Delta x_{min}$



From $\Delta x = \Delta x_{min}$, $\Delta t = \Delta t_{min}$ figure

Calculated shock speed (u_t = theoretical, u = observed)

1 $t \approx 10125$ s

$A_L \approx 400$ m^2 $\quad F(A_L) = 300$

$A_R \approx 100$ m^2 $\quad F(A_R) = 31$

$s \approx 2050$ m

$$u_t = \frac{F(A_r) - F(A_l)}{A_r - A_l}$$

1 → 2

$u_t \approx 0.90 \ ms^{-1}$ $\quad u = \frac{3115 - 2050}{11250 - 10125} = 0.95 \ ms^{-1}$

(2) $t = 11250\,s$

$A_L \approx 460\,m^2$ $F(A_L) = 380$

$A_n \approx 100\,m^2$ $F(A_n) = 31$

$s \approx 3115\,m$

$u_t = 0.97\,ms^{-1}$

(2) → (3)

$u = 0.96\,ms^{-1}$

(3) $t = 12375\,s$

$A_L \approx 450\,m^2$ $F(A_L) = 366$

$A_n \approx 100\,m^2$ $F(A_n) = 31$

$s \approx 4200\,m$

$u_t = 0.96\,ms^{-1}$

- Theoretical shock speed ~ shock speed observed

- To have more accurate shock speed, could record data at more time intervals

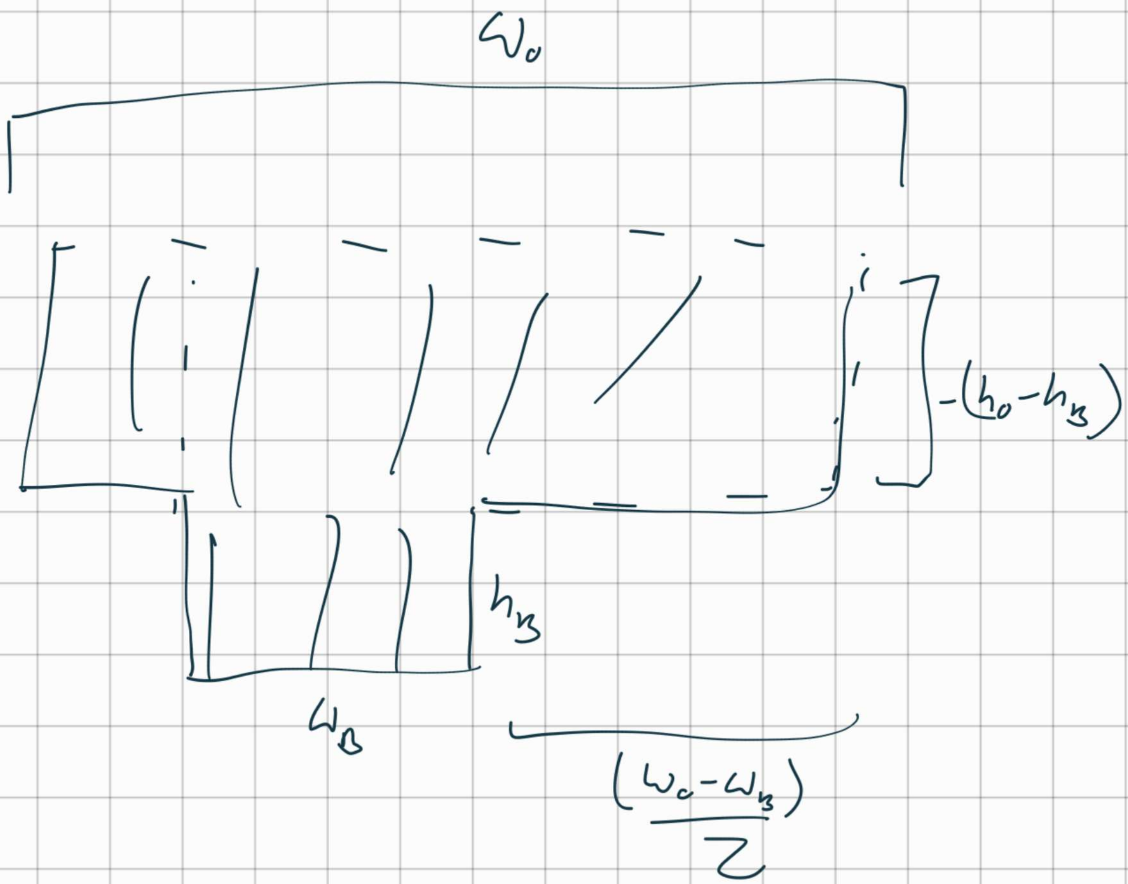
- 1st order in space and time.
- Have reached convergence in time

→ but is still converging in space.

- Interestingly, reached convergence in time for test case 1 not test case 0

→ even though errors for changing Δx steps are larger for ~~test~~ case 1.

2ab).



Expression for wetted perimeter will change depending on if A is $< A_B$ or $> A_B$

$A < A_B$

$$P = w_B + \frac{A_o}{w_B}$$

$A > A_B$

$$P = w_B + 2h_B + (w_o - w_B) + \frac{2(A - A_B)}{w_o}$$

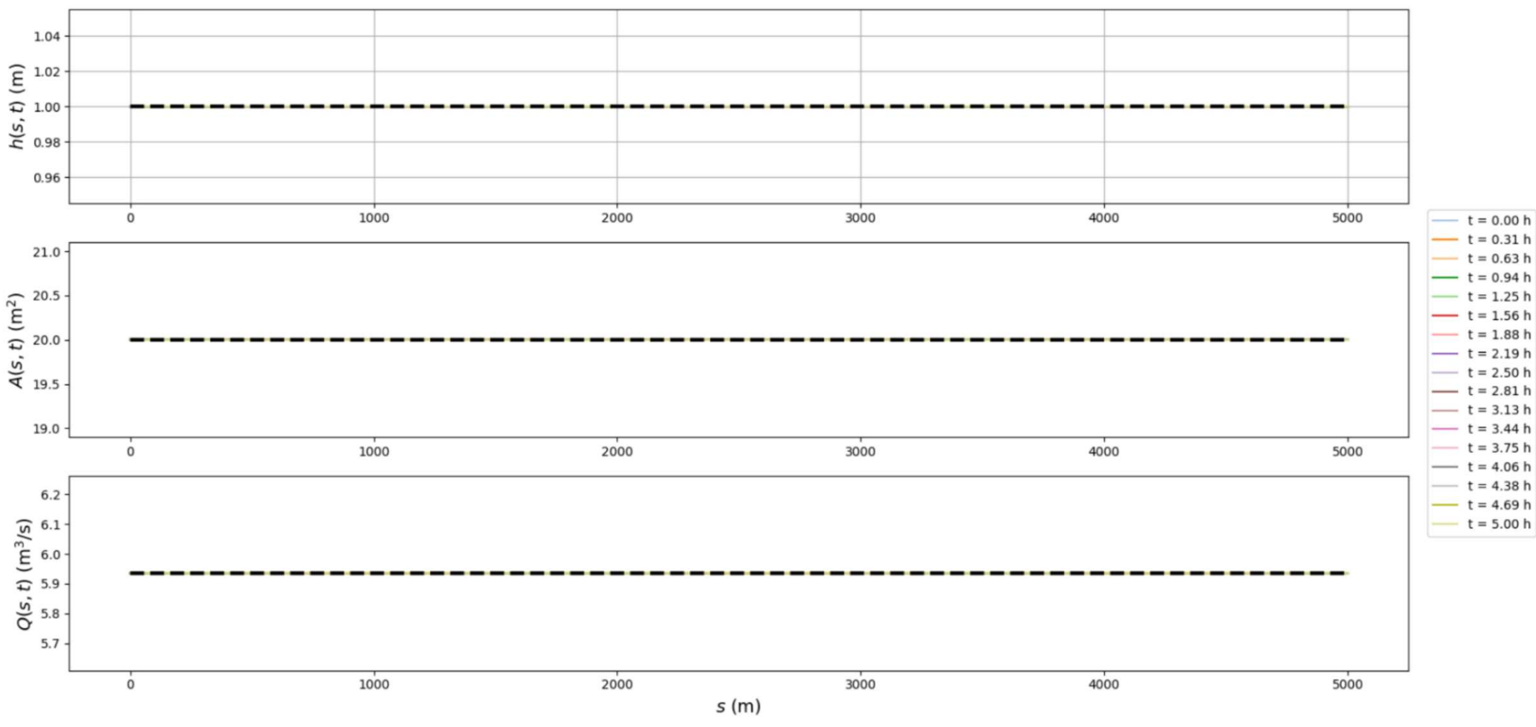
For test case 0:

As constant flux is determined by

$A = H_0 \cdot w_B$ where $H_0 < h_B$, then due to the presence of the river channel, everything is constant

→ but good way to check code

2a, Constant Q:

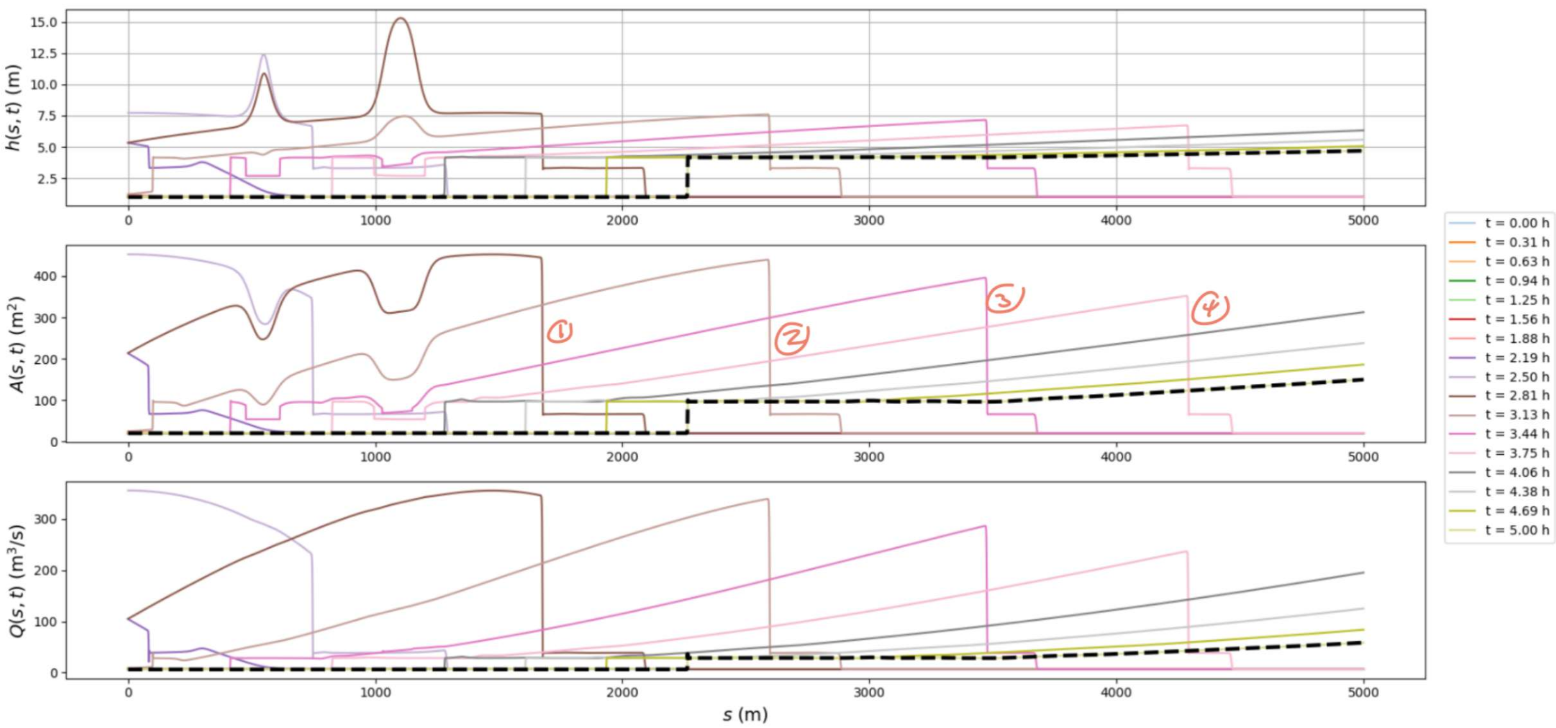


2b). Slightly different Δt_{min}, Δx_{min}, as this code took a lot longer to run

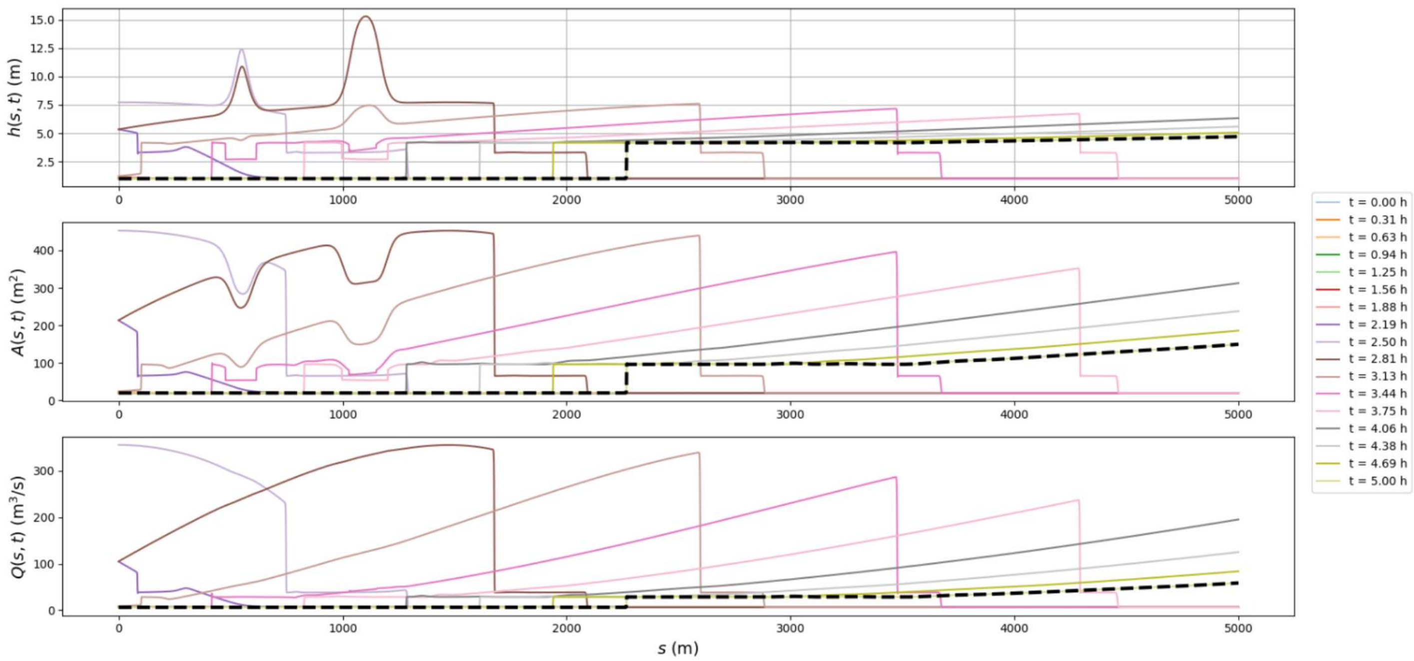
$\rightarrow \Delta x_{min} = 1m$

$$\Delta t_{min} = \frac{0.25 \times 1}{C_{oc}}$$

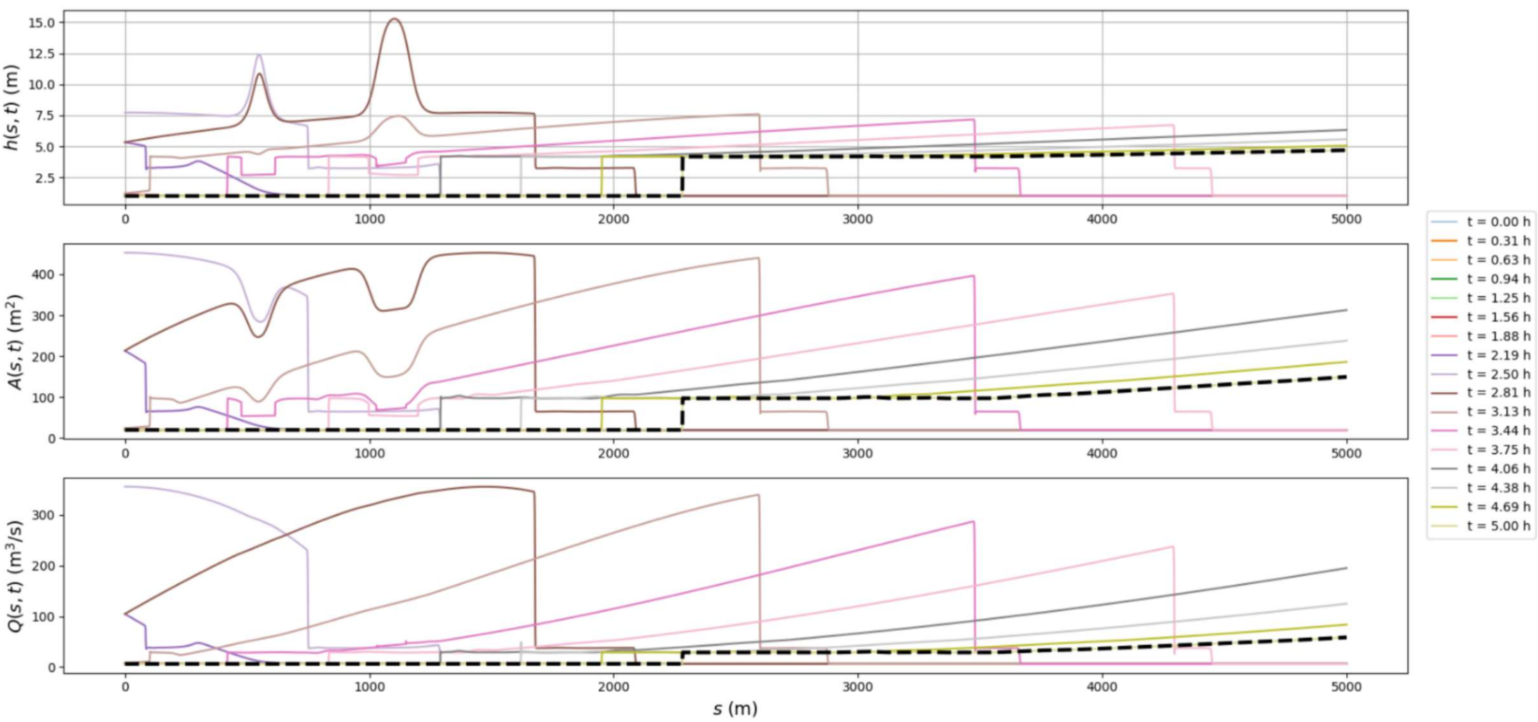
$\Delta x = \Delta x_{min}$, $\Delta t = \Delta t_{min}$



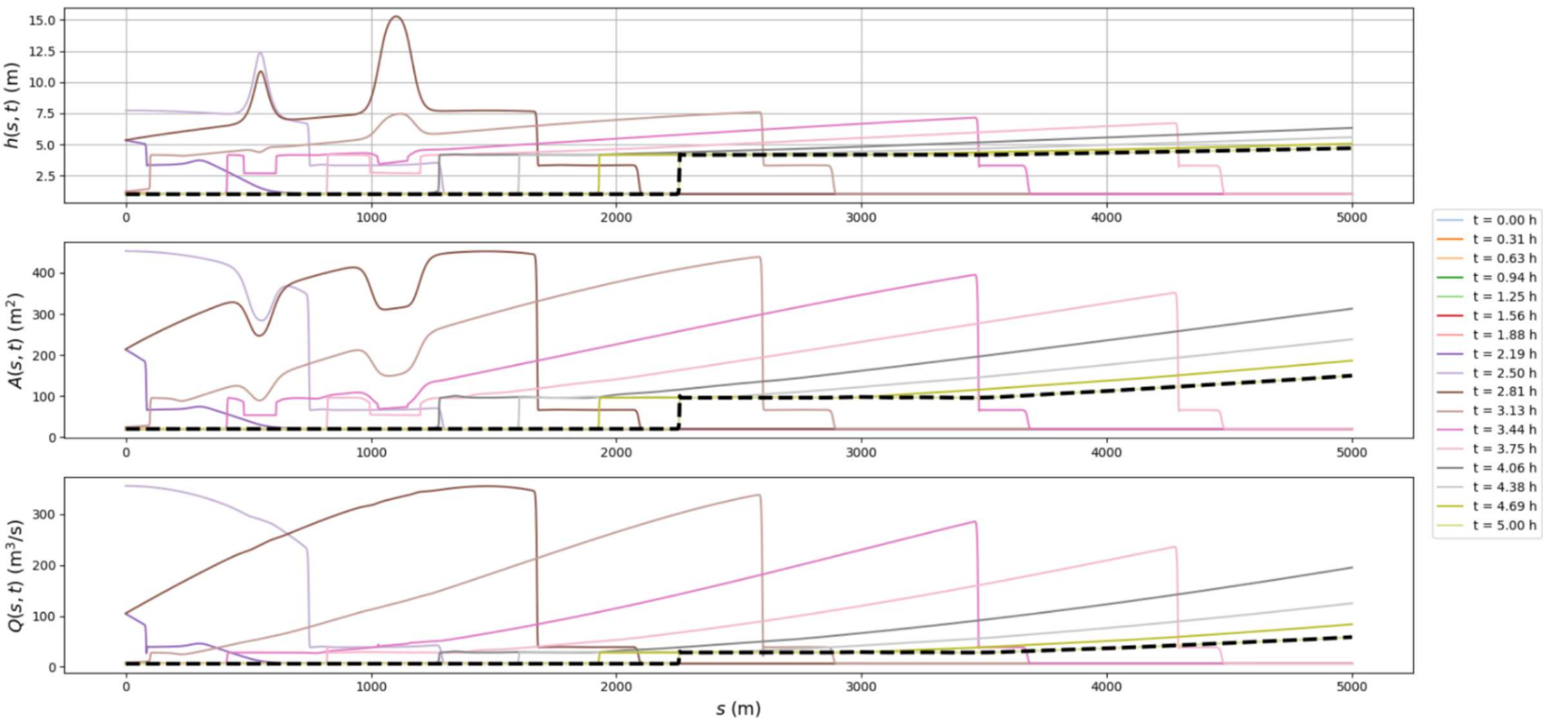
$\Delta x = \Delta x_{min}$, $\Delta t = 2\Delta t_{min}$



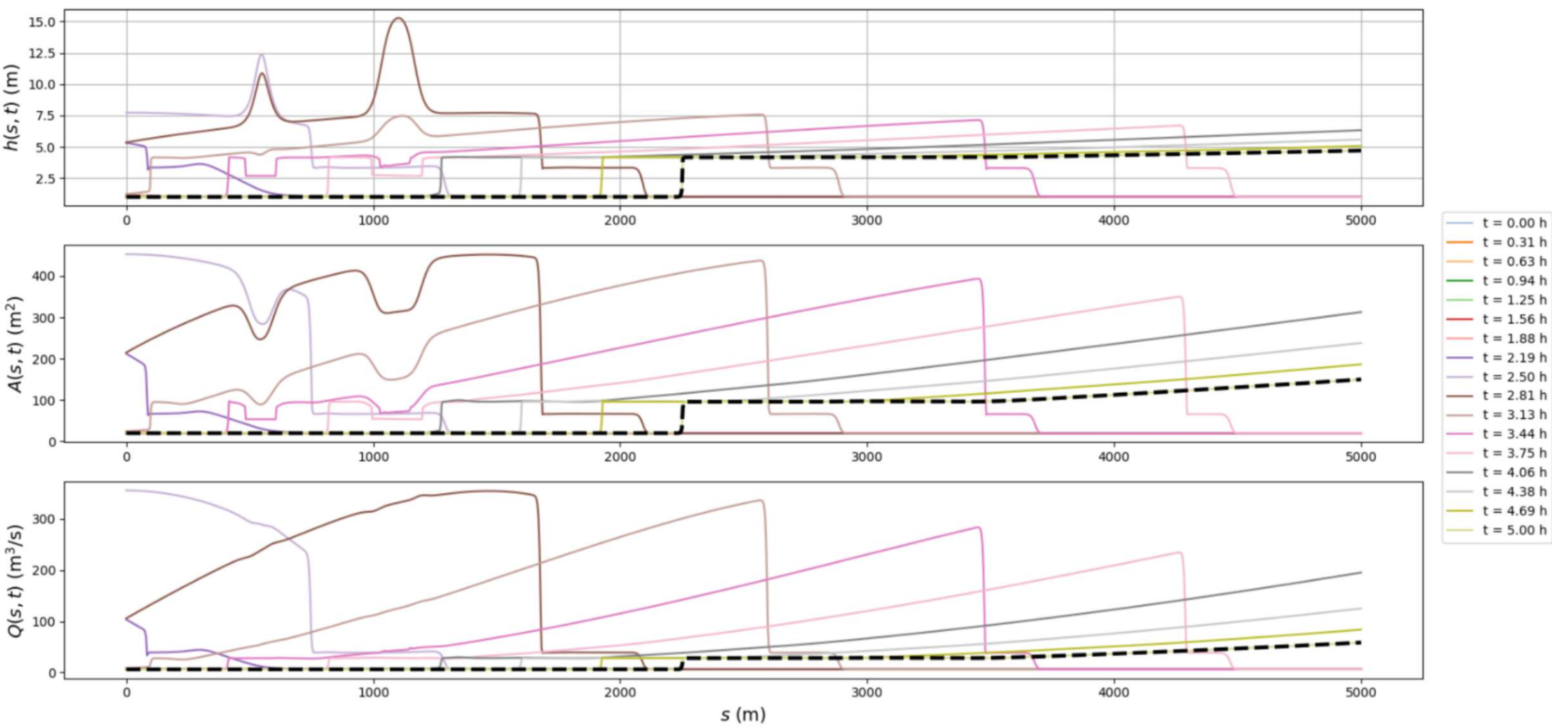
$\Delta x = \Delta x_{min}$, $\Delta t = 4\Delta t_{min}$

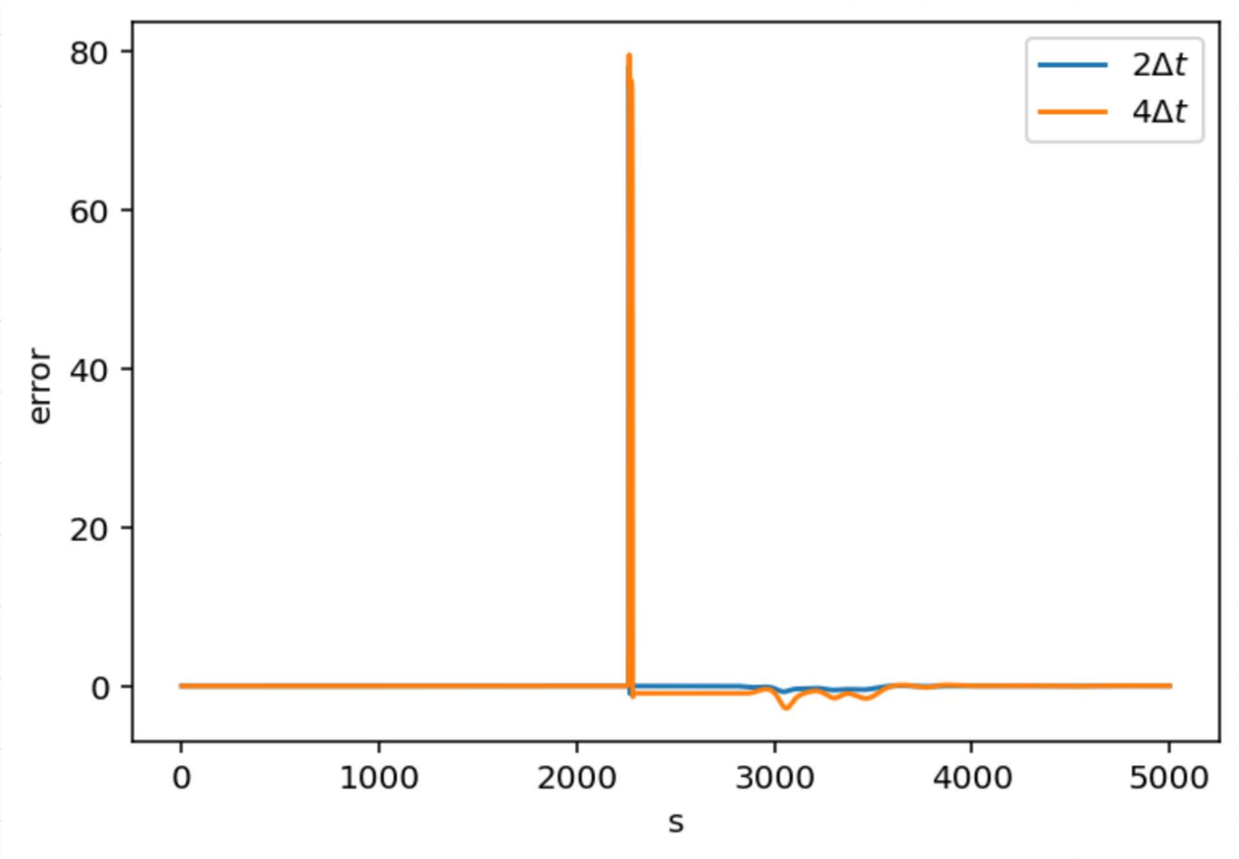


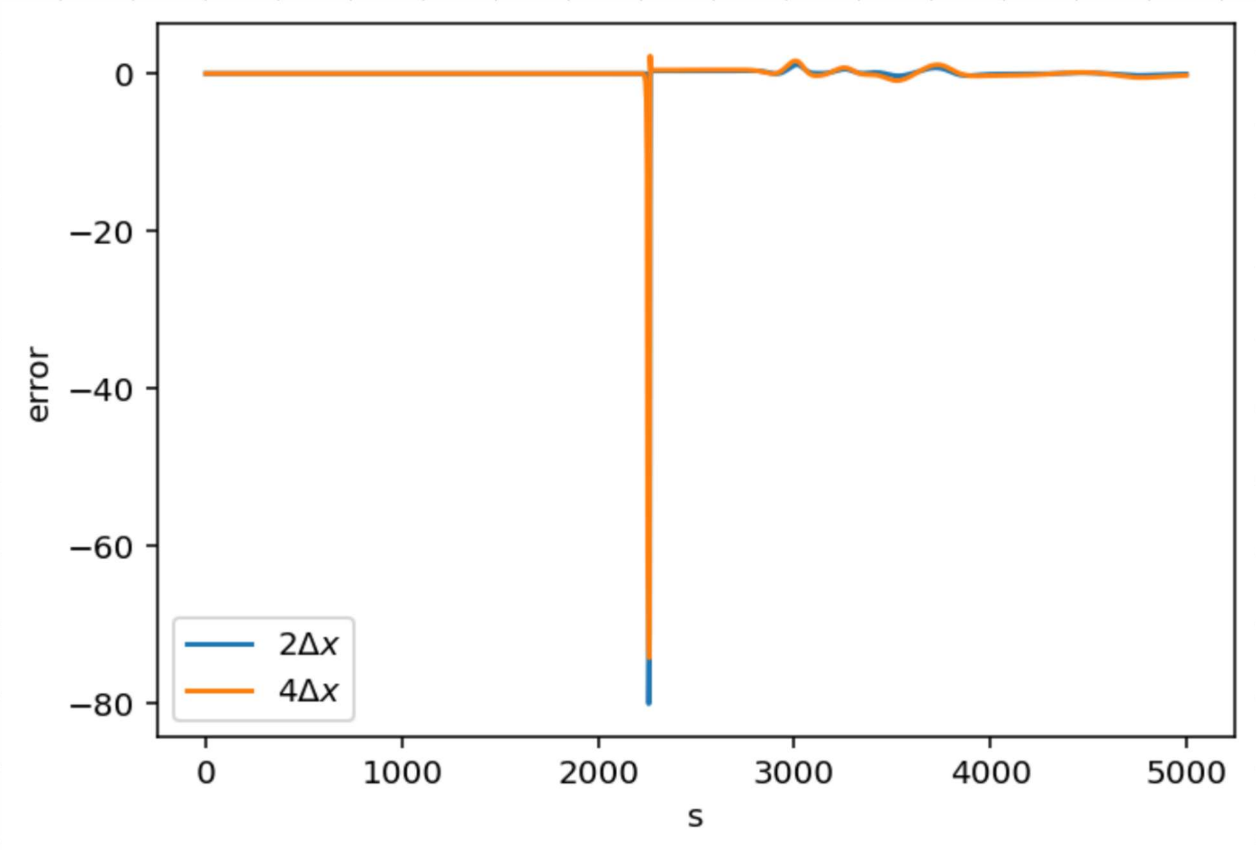
$\Delta x = 2\Delta x_{min}$, $\Delta t = \Delta t_{min}$



$\Delta x = 4\Delta x_{min}$, $\Delta t = \Delta t_{min}$







- 1st order in space and time

- Issues at position of the shock with convergence according to graphs
  → but could be issue with error calculation rather than with simulation
- But away from shock, can see that the solutions converge.

Shock speed:

1 $t = 10125\,s$
$A_L = 445\,m^2$ $F(A_L) = 346$
$A_R = 65\,m^2$ $F(A_R) = 38$
$s = 1677\,m$

1 → 2

$u_t = 0.81\,ms^{-1}$ $u = 0.82\,ms^{-1}$

2 $t = 11250\,s$
$A_L = 441\,m^2$ $F(A_L) = 348$
$A_R = 70\,m^2$ $F(A_R) = 38$
$s = 2596\,m$

2 → 3

$u_F = 0.81\,ms^{-1}$ $u = 0.78\,ms^{-1}$

(3) $t = 12375\,s$

$A_L = 395\,m^2$ $F(A_L) = 287$

$A_n = 65\,m^2$ $F(A_n) = 38$

$s = 3475\,m$

(3) → (4)

$u_t = 0.75\,ms^{-1}$ $u = 0.72\,ms^{-1}$

(4) $t = 13500\,s$

$A_L = 352\,m^2$ $F(A_L) = 237$

$A_n = 65\,m^2$ $F(A_n) = 38$

$s = 4290\,m$

$u_t = 0.69\,ms^{-1}$

- $u_t \sim u$, but is slightly less clear than calculating for this last (could try evaluate shock speed over more times to check, as it seems to erode).

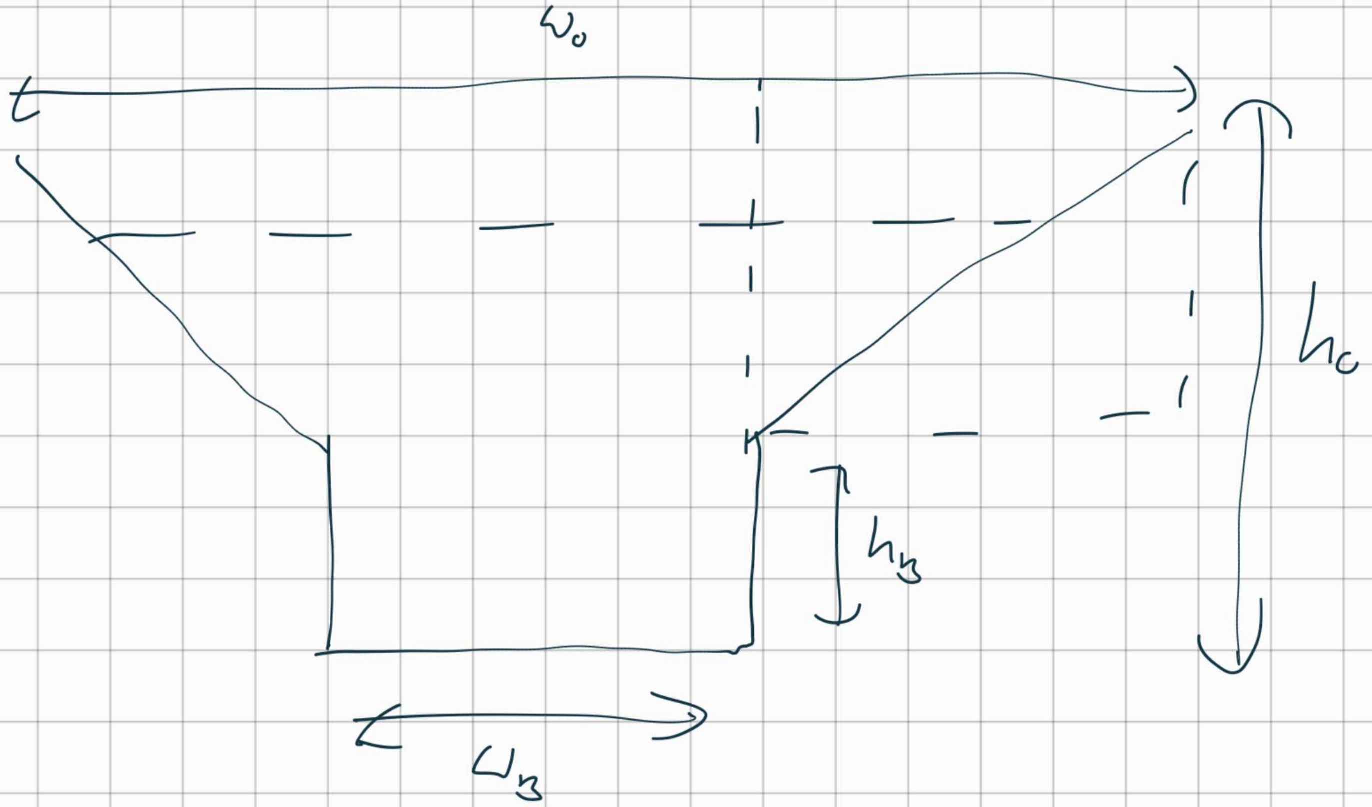
- Additional jumps due to layout

→ e.g.

Feels like there should be a sudden jump between this

3ab).

Profile now looks like



Just means value of wetted perimeter for $A > h_B w_B$ changes

→ but value of widths of water is not fixed.

→ $\dfrac{h_0 - h_B}{(w_0 - w_B)/2} = 0.01$

$$\boxed{w_0 - w_B = \frac{2(h_0 - h_B)}{0.01}}$$

$$(A - h_B w_B) = \frac{1}{2}(w_0 + w_B)\cdot(h_0 - h_B)$$

$$P = 2h_B + w_B + 2\sqrt{(h_0-h_B)^2 + \left(\frac{w_0 - w_B}{2}\right)^2}$$

$$= 2h_B + w_B + 2\sqrt{(h_0-h_B)^2 + \left(\frac{h_0 - h_B}{0.01}\right)^2}$$

$$= 2h_B + w_B + 2(h_0 - h_B)\sqrt{1 + 100^2}$$

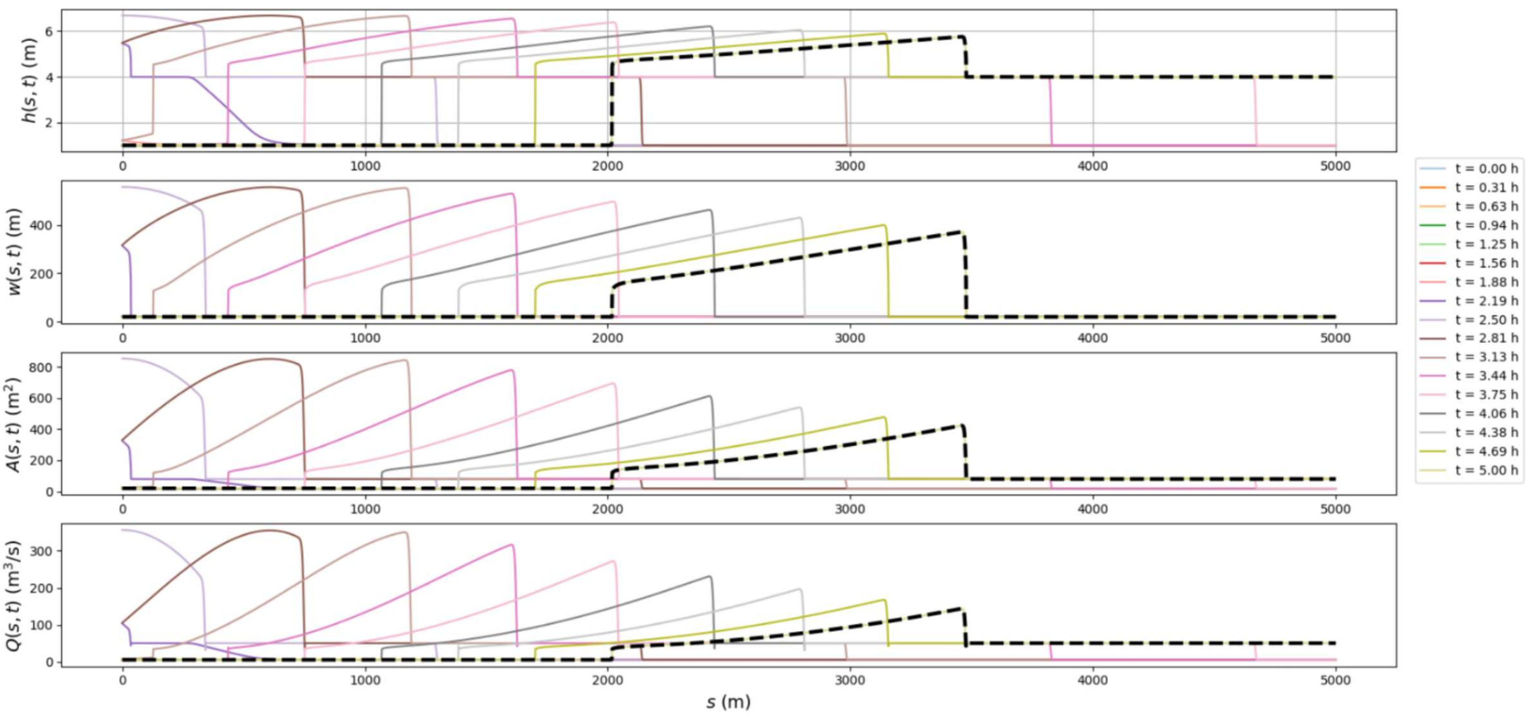
$$= 2h_B + w_B + \frac{4(A - h_B w_B)}{w_0 + w_B}\sqrt{1 + 100^2}$$

$$A - A_B = \frac{1}{2}(w_0 + w_B)\cdot\left(\frac{0.01}{2}\right)(w_0 - w_B)$$

$$(w_0 + w_B)(w_0 - w_B) = \frac{4}{0.01}(A - A_B)$$

$$w_0^2 - w_B^2 = \frac{4}{0.01}(A - A_B)$$

$$\boxed{w_0 = \mathrm{sqrt}\left(w_B^2 + 400(A - A_B)\right)}$$



$$\Delta x = 1\,\text{m}, \quad \Delta t = \frac{0.25 \times 1}{C_{\infty}}$$

- Could not properly check convergence, as there is issue with code

→ when $\Delta x = 1\,\text{m}$, but CFL $= 0.125$ solution is wrong, but solutions for larger time/space steps seem to converge to solution shown above.