

Exercises 9
sheet 2

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The groundwater model:

$$\partial_t [w_v h_m] - \alpha g \partial_y [w_v h_m \partial_y [h_m]] = \frac{w_v R}{m_{por} \sigma e}$$

in $y \in [0, L_y]$

$$\partial_y h_m = 0 \text{ at } y = L_y$$

$$h_m(0, t) = h_{cm}(t) \text{ at } y = 0$$

$$L_c w_v \frac{dh_{cm}}{dt} = w_v m_{por} \frac{\sigma e}{2} \alpha g \partial_y [h_m^2] \Big|_{y=0} - w_v \sqrt{g} \max\left(\frac{2}{3} h_{cm}(t), 0\right)^{\frac{3}{2}}$$

Take the 1st equation & multiply it

by $\frac{q(y)}{w_v}$ and integrate. we get:

$$\int_0^{L_y} q \partial_t [h_m] - \cancel{\alpha g \frac{q}{w_v} \partial_y h_m \partial_y [h_m]} dy = \int_0^{L_y} \frac{q R}{m_{por} \sigma e} dy$$

(1.)

Note that

$$A \partial_y [q h_m \partial_y [h_m]] = A \partial_y [q] h_m \partial_y [h_m] \\ + A q \partial_y [h_m \partial_y [h_m]]$$

Hence,

$$-A q \partial_y [h_m \partial_y [h_m]] = A \partial_y [q] h_m \partial_y [h_m] \\ - A \partial_y [q h_m \partial_y [h_m]]$$

Integrate to get

$$-\int_0^{L_y} A q \partial_y [h_m \partial_y [h_m]] dy = \int_0^{L_y} A \partial_y [q] h_m \partial_y [h_m] dy \\ - A \int_0^{L_y} \partial_y [q h_m \partial_y [h_m]] dy \\ + A q h_m \partial_y [h_m] \Big|_{y=0}$$

$$\text{as } \partial_y h_m (y=L_y) = 0.$$

Returning to the equation, we get:

$$\int_0^{L_y} q \partial_t [h_m] + \alpha q \partial_y [q] h_m \partial_y [h_m] dy + \alpha g \left(q h_m \partial_y [h_m] \right) \Big|_{y=0} \\ = \int_0^{L_y} \frac{q R}{m \rho g \sigma e} dy \quad (\star)$$

Now, focus on the equation with $\frac{dh_m}{dt}$ term.
 We get; by multiplying by $\frac{1}{m_{por} \sigma_e}$;

$$\frac{L_c}{m_{por} \sigma_e} \frac{dh_m}{dt} = \frac{1}{2} \alpha g \partial_y [h_m^2] \Big|_{y=0} - \frac{\sqrt{g}}{m_{por} \sigma_e} \max\left(\frac{2}{3} h_m, 0\right)^{\frac{3}{2}}$$

(*) This is equivalent to:

$$\alpha g q(0) h_m(0) \partial_y [h_m^{(0)}] \Big|_{y=0} = \frac{L_c q(0) dh_m}{m_{por} \sigma_e dt} + \frac{\sqrt{g} q^{(0)}}{m_{por} \sigma_e} \max\left(\frac{2}{3} h_m, 0\right)^{\frac{3}{2}}$$

Plug this into (\star)

$$\int_0^{L_y} q \partial_t [h_m] + \alpha g \partial_y [q] h_m \partial_y [h_m] dy + \frac{q(0)L_c}{m_{por} \sigma_e} \frac{dh_m(0,t)}{dt} = \int_0^{L_y} \frac{qR}{m_{por} \sigma_e} dy - \frac{q(0)\sqrt{g}}{m_{por} \sigma_e} \max\left(\frac{2}{3} h_m^{(0,t)}, 0\right)^{\frac{3}{2}}$$

Now, rearrange to obtain:

(3.)

$$\int_0^{L_y} q \partial_t [h_m] dy + \frac{q(0) L_c}{m_{por} \sigma_e} \frac{dh_m(0, t)}{dt} = \int_0^{L_y} -\alpha g h_m \partial_y [q] \partial_y [h_m] dy$$

$$+ \frac{q R}{m_{por} \sigma_e} dy - \frac{q(0)}{m_{por} \sigma_e} \sqrt{g} \max\left(\frac{2}{3} h_m(0, t), 0\right)^{\frac{3}{2}}$$

Discretization

Consider the forward Euler time discretization:

$$\int_0^{L_y} q \frac{h_m^{n+1} - h_m^n}{\Delta t} dy + (\dots) = (\dots).$$

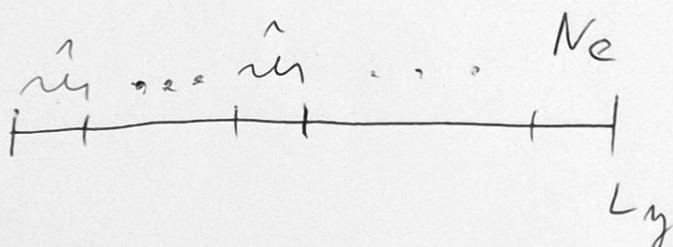
Rearrange to get:

$$\int_0^{L_y} q h_m^{n+1} dy + \frac{q(0) L_c}{m_{por} \sigma_e} h_m^{n+1} = \int_0^{L_y} q h_m^n dy + \frac{q(0) L_c h_m^n}{m_{por} \sigma_e}$$

$$+ \Delta t \int (-\alpha g h_m^n \partial_y [q] \partial_y [h_m^n] + \frac{q R^n}{m_{por} \sigma_e}) dy$$

$$- \Delta t \frac{q(0)}{m_{por} \sigma_e} \sqrt{g} \max\left(\frac{2}{3} h_m^n, 0\right)^{\frac{3}{2}} \quad (\textcircled{P})$$

Now, $h_m^{n+1} \in \mathcal{H}^1$ constrained by boundary conditions. Now, divide the domain into N_E elements:



and consider $V_{Ne} \subset \mathcal{H}$ s.t. the basis functions $\varphi_i(y)$ are linear on i^{th} elements and zero on other elements.

Then choose $q = \varphi_i(y)$ and $b_m = h_j \varphi_j(y)$ for constants h_j & $i, j \in \{1, \dots, N_e\}$

Substituting into (P) we get:

$$\int_0^{L_y} \varphi_i \varphi_j dy h_j^{n+1} + \frac{q(0)L_c}{\text{m por o}} h_1^{n+1} \delta_{ij} = \int_0^{L_y} \varphi_i \varphi_j dy h_j^n$$

$$+ \frac{q(0)L_c}{\text{m por o}} \delta_{ii} h_1^n + \Delta t \int (-\alpha g h_m^n dy \varphi_i dy h_m^n + \frac{\varphi_i R^n}{\text{m por o}})$$

$$- \Delta t \frac{q(0)}{\text{m por o}} \sqrt{g} \max \left(\frac{2}{3} h_1^n, 0 \right)^{\frac{3}{2}} \delta_{ii} \quad (\text{P})$$

where we used $h_{cm}^n = h_1^n$ and the Einstein's summation convention.

Now, define

$$M_{ij} = \int_0^{L_y} \varphi_i \varphi_j dy$$

$$\& \delta_i^n = \int_0^{L_y} \left(-\alpha g h_m^n dy \varphi_i dy h_m^n + \frac{\varphi_i R^n}{\text{m por o}} \right) dy.$$

Finally, the (QP) becomes:

$$M_{ij} h_j^{u+1} + \frac{q(0)Lc}{\text{mpore}} \delta_{ij} h_i^{u+1} = M_{ij} h_j^u + \frac{q(0)Lc}{\text{mpore}} \delta_{ij} h_j^u$$
$$+ \Delta t b_i - \Delta t \frac{\sqrt{g} q(0)}{\text{mpore}} \max\left(\frac{2}{3} h_i^u, 0\right)^{\frac{2}{3}}$$

it remains to set $q(0) = 1$.

All fine.