Numerics Homework 3 - Finite Element   Tash Moore.

Poisson System:

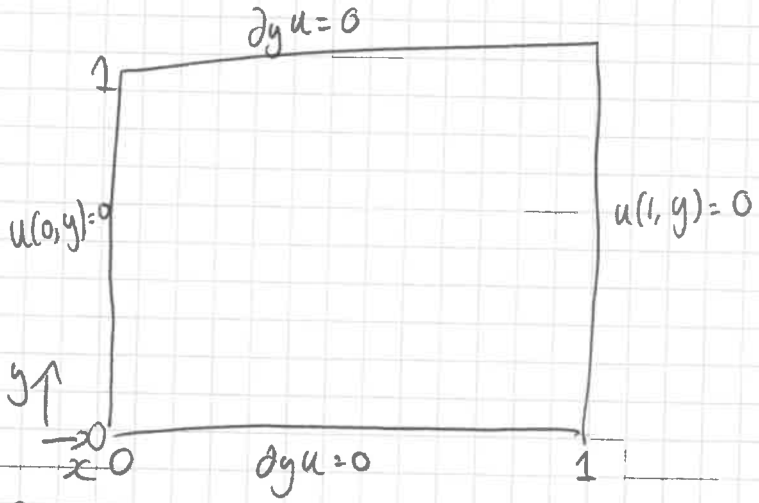
$\cancel{-\nabla^2 f}$

$$-\nabla^2 u = f \quad \text{on} \quad (x,y) \in [0,1]^2$$

$$f(x,y) = 2\pi^2 \sin(\pi x)\cos(\pi y)$$

$$u(0,y) = u(1,y) = 0 \quad (\text{Dirichlet}) \quad \Gamma_1$$

$$\partial_y u(x,y)\big|_{y=0} = \partial_y u(x,y)\big|_{y=1} = 0 \quad (\text{Neumann}) \quad \Gamma_2$$



boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ $\quad \Gamma_1 = x = 0,1 \; (\partial\Omega_D)$ $\cancel{\Gamma}$ Γ_2 : $y = 0,1 \; (\partial\Omega_N)$

exact solution is $u_e(x,y) = \sin(\pi x)\cos(\pi \cancel{x} y)$

$$\frac{\partial}{\partial x}\left(\sin(\pi x)\cos(\pi y)\right) = \pi \cancel{\sin}\cos(\pi x)\cos(\pi y)$$

$$\frac{\partial^2}{\partial x^2}\left(\sin(\pi x)\cos(\pi y)\right) = -\pi^2 \sin(\pi x)\cos(\pi y)$$

$$\frac{\partial}{\partial y}\left(\sin(\pi x)\cos(\pi y)\right) = -\pi \sin(\pi x)\cancel{\cos}\sin(\pi y)$$

$$\frac{\partial^2 u}{\partial y^2} = -\pi^2 \sin(\pi x)\cos(\pi y)$$

$$\cancel{\nabla^2 u = -\pi^2 \left(\sin(\pi x) +\right.}$$

$$-\nabla^2 u = +\pi^2\left(\cancel{\sin(\pi x)}\cos(\pi y) + \sin(\pi x)\cos(\pi y)\right)$$

$$-\nabla^2 u = 2\pi^2 \sin(\pi x)\cos(\pi y)$$

$$\Rightarrow \; -\nabla^2 u = f$$

$\boxed{\text{exact solution}}$

~~1. Step 1~~:

~~Write down the Ritz Galerkin principle and derive the weak formulation:~~

~~1. Step 1:~~

To solve poisson system numerically we want to be able to write $u(x,y)$ as a sum of weighted shape functions. $u(x,y) = \sum_{i=0}^{n} u_i N_i(x,y)$ where N_i is a shape function, 1 at node i and 0 at other nodes. (n is number of nodes). weights are u_i.

To find the weights, want to put the poisson system in a weak formulation.

(1) Step 1: Weak formulation

Find weak formulation through a test function

Method 1

$-\nabla^2 u = f$: strong form

multiply by test function $w(x,y)$, this is a arbitrary function of x and y.

$$-\nabla^2 u(x,y)\, w(x,y) = f(x,y)\, w(x,y)$$

integrate over ~~boundary~~ domain Ω

$$-\int_\Omega \nabla^2 u(x,y)\, w(x,y)\, d\Omega = \int_\Omega f(x,y)\, w(x,y)\, d\Omega \quad (1)$$

~~Use partial integration to change form of LHS.~~
want to change form of LHS.

$$-\int_\Omega \nabla^2 u\, w\, \partial\Omega \Rightarrow LHS.$$

PRODUCT RULE

$$\nabla\cdot(w\nabla u) = w\nabla^2 u + \nabla u\cdot\nabla w$$

$$\Rightarrow \quad -w\nabla^2 u = \nabla u\cdot\nabla w - \nabla\cdot(w\nabla u)$$

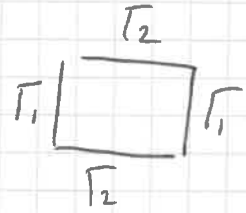
$$\Rightarrow \quad -\int_\Omega w\nabla^2 u\,\partial\Omega = -\int_\Omega \nabla\cdot(w\nabla u)\,\partial\Omega + \int_\Omega \nabla u\cdot\nabla w\,\partial\Omega$$

LHS of (1)

Use divergence theorem to rewrite $\int_\Omega \nabla\cdot(w\nabla u)\,\partial\Omega$ as

$$\cancel{\oint_{\partial\Omega}} \quad \int_{\Gamma_1} w\nabla u\cdot\hat{n}\,\partial\Gamma + \int_{\Gamma_2} w\nabla u\cdot\hat{n}\,\partial\Gamma$$

~~Fz~~ Γ_1 when $x=0,1$ $\quad \Gamma_2$ when $y=0,1$



so closed surface.

Can eliminate this term as $\nabla u = 0$ on Γ_2 and can choose w so satisfys $u=0$ on Γ_1

Therefore

$$\boxed{+\int_\Omega \nabla u\cdot\nabla w\,\partial\Omega = \int_\Omega f w\,\partial\Omega}$$

$$-\int_\Omega \nabla u\cdot\nabla w\,\cancel{\partial\Omega} + f w\,\partial\Omega = 0$$

① Finding weak Formulation through variational principle | Method 2

want to minimise the functional → type of function that maps vectors to numbers. Argument are functions

$$I[u] = \iint_{\Omega} \frac{1}{2} \| \nabla u|^2 - uf \, \partial \Omega$$

Minimise this function by finding point where $\delta I = 0$ with a variation

$$\delta I \equiv \frac{\delta I}{\delta \epsilon} = \lim_{\epsilon \to 0} \frac{I[u(x,y) + \epsilon (\delta u)(x,y)] - I[u(x,y)]}{\epsilon}$$

$$\partial I = \delta \left(\iint_{\Omega} \frac{1}{2} \| \nabla u|^2 - uf \, \delta \Omega \right)$$

$$= \iint_{\omega} 2 \times \frac{1}{2} \nabla u \cdot \nabla(\delta u) - f \delta u \, d\Omega$$

$$= \iint_{\omega} \left(\nabla u \cdot \nabla(\delta u) - f \delta u \right) d\Omega = 0$$

$$\iint_{\omega} \left(\nabla u \cdot \nabla(\delta u) - f \delta u \right) \partial\Omega = 0$$

Like before product rule =>

$$\nabla \cdot (\delta u \nabla u) = \delta u \nabla^2 u + \nabla u \cdot \nabla \delta u$$

$$\Rightarrow -\delta u \nabla^2 u = \nabla u \cdot \nabla \delta u - \nabla \cdot (\delta u \nabla u)$$

$$\Rightarrow \nabla u \cdot \nabla(\delta u) = -\delta u \nabla^2 u + \nabla \cdot (\delta u \nabla u)$$

Sub this in and use Gauss's divergence theorem.

$$-\iint_{\omega} \delta u \nabla^2 u + \delta u f \, \partial\Omega + \int_{\Gamma_1} \cancelto{0}{\delta u \nabla u \cdot \hat{\underline{n}}} \, d\Gamma + \int_{\Gamma_2} \cancelto{0}{\delta u \nabla u \cdot \hat{\underline{n}}} \, d\Gamma = 0$$

$\nabla u = 0$ on Γ_2 from boundary conditions

and choose δu such that $(\delta u)_{\Gamma_1} = 0$

$$\Rightarrow \boxed{-\iint_{\omega} (\delta u)(\nabla^2 u + f) \, d\Omega = 0}$$

Yields the same result as using the test function if $\boxed{\delta u = \omega}$

$$\boxed{\int_{\Omega} \nabla u \cdot \nabla \delta u \, \partial\Omega = \int_{\Omega} f \delta u \, \partial\Omega}$$

~~2. Step 2~~ :

(2) $\boxed{\text{Discretise WEAK FORMULATION through use of test function}}$

$$u(x,y) \approx \boxed{u_h(x,y) = u_j \varphi_j(x,y)}$$

~~u_j = weights~~ u_j = weights

φ_j = global basis function

$$w(x,y) \approx w_h(x,y) = a_j \varphi_j(x,y)$$

a_j = weights for test function, if take $a_j = \delta_{ij}$ then

$$\boxed{w_h = \varphi_i(x,y)}$$

$$u(x) = \sum_{i=0}^{N} u_j \varphi_j(x,y)$$

substitute $u_h(x,y)$ and $w_h(x,y)$ into weak formulation:

$$\int_\Omega \nabla u \cdot \nabla w \, \partial\Omega = \int_\Omega f w$$

$$\iint_\Omega \nabla u_h \cdot \nabla w_h \, \partial\Omega = \iint_\Omega f \, w_h \, d\Omega$$

$\nabla u_h = \nabla(u_j \varphi_j(x,y))$ $\quad u_j$ are constants so can take out of the derivative.

$$\nabla u_h = u_j \nabla \varphi_j(x,y)$$

$$\nabla w_h = \nabla \varphi_i(x,y)$$

$$\Rightarrow \iint_\Omega u_j \nabla\varphi_j(x,y) \cdot \nabla\varphi_i(x,y) = \iint_\Omega f \, \nabla\varphi_i(x,y) \, d\Omega$$

or. $$A_{ij} u_j = b_i$$

with
$$A_{ij} = \iint_\Omega \nabla \varphi_i(x,y) \cdot \nabla \varphi_j(x,y) \, d\Omega$$
$$b_i = \iint_\Omega \varphi_i(x,y) f(x,y) \, d\Omega$$

Can Find ~~solution~~ u_j by inverting matrix

~~or~~

~~If use compact support~~

Values are known on Dirichlet boundaries so can rewrite:

sum of solution over known values of boundary condition

$$A_{i_*' j'} u_{j'} = b_{i'} - \sum_{k=N_{nodes}+1}^{N_n} A_{i_*' k} U_k$$

N_{nodes} = non-dirichlet nodes

N_n = total nodes.

② Discretise through variational principle

$$\boxed{u(x,y) \approx uh(x,y) = u_j \varphi_j(x,y)}$$

- substitute finite element expansion for uh into the minimal principle
- introduce integrals
- taking variation with respect to $u_{j'}$

$\delta u_{j'} = \eta_{j'}$

sub into functional

$$I[u] = \iint_\Omega \frac{1}{2} \|\nabla u|^2 - uf \,\partial\Omega$$

$$I = \iint_\Omega \left(\frac{1}{2} |\nabla u_j \varphi_j|^2 - u_j \varphi_j f\right) d\Omega$$

~~m~~ set ~~vari~~ $\delta I = 0$ and sub in δu_j test function

$$\delta I \equiv \frac{\partial I}{\partial \epsilon} = \lim_{\epsilon \to 0} \frac{I[u_j\varphi_j(x,y) + \epsilon(\delta u_j)(x,y)] - I[u_j\varphi_j(x,y)]}{\epsilon}$$

It is delta u_j bvarphi_j(x.y); step below also needs more detail after this correctiob;-0.5; in addition delta u_j =0 for j not j'.

$$\partial I = \iint_\omega \left(\nabla u_j \varphi_j \cdot \nabla(\delta u_j) - f\delta u\right) d\Omega = 0$$

sub in ~~u(x,y)~~ $\delta u_j = \varphi_i(x,y)$

$\Rightarrow$

$$\iint_\Omega u_j \nabla\varphi_j(x,y) \cdot \nabla\varphi_i(x,y) = \iint_\Omega f \cancel{\nabla} \varphi_i(x,y)\, d\Omega$$

$\Rightarrow \quad A_{ij} u_j = b_i \qquad b_i = \iint_\Omega \varphi_i(x,y) f(x,y)\, d\Omega$

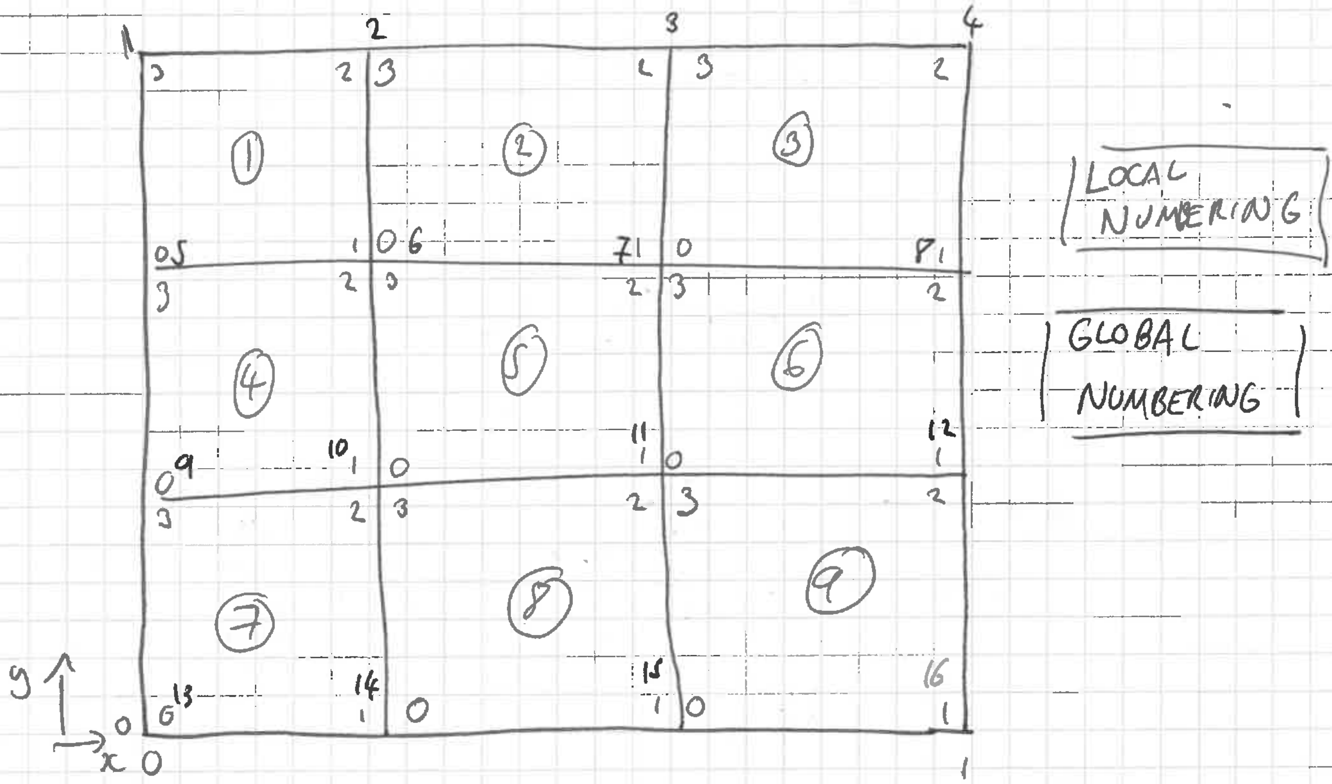
~~the A, b for as~~

$$A_{ij} = \iint_\Omega \nabla\varphi_i(x,y) \cdot \nabla\varphi_j(x,y)\, \partial\Omega$$

(3) Introduce local coordinate system and reference coordinates

Quadrilateral element

eg. 3×3 Mesh.



Element (9)

LOCAL(α)	GLOBAL
0	15
1	16
2	12
3	11

etc.

Mapping between global and local.

$$\bar{x} = F_{K_k}(\bar{\xi}) = \sum_{\alpha=0}^{N_\alpha^e-1} \bar{x}_{k\alpha} \, \chi_\alpha(\bar{\xi})$$

Define basis functions with global node number i on Element K_k

$$\varphi_\alpha(x,y) = \hat{\varphi}_\alpha(F_K^{-1}(x,y)) = \chi_\alpha(\bar{\xi})$$

α = local elmt index, for local support $w_\alpha = 1$ on global node i and 0 on other nodes.

Apply this to :

$$\boxed{\begin{aligned} A_{ij}\, u_j &= b_i \\ A_{ij} &= \iint_\Omega \nabla \varphi_i(x,y)\cdot\nabla\varphi_j(x,y)\, d\Omega \\ b_i &= \iint_\Omega \varphi_i(x,y)\, f(x,y)\, d\Omega \end{aligned}}$$

$$\Rightarrow \quad \hat{A}_{\alpha\beta} = \int_K \nabla\chi_\alpha\cdot\nabla\chi_\beta\, d\Omega$$

$$= \int_{\hat K}\left((J^T)^{-1}\begin{pmatrix}\frac{\partial\chi_\alpha}{\partial\xi_1}\\ \frac{\partial\chi_\alpha}{\partial\xi_2}\end{pmatrix}\right)\cdot\left((J^T)^{-1}\begin{pmatrix}\frac{\partial\chi_\beta}{\partial\xi_1}\\ \frac{\partial\chi_\beta}{\partial\xi_2}\end{pmatrix}\right)|\det J|\, d\bar\xi$$

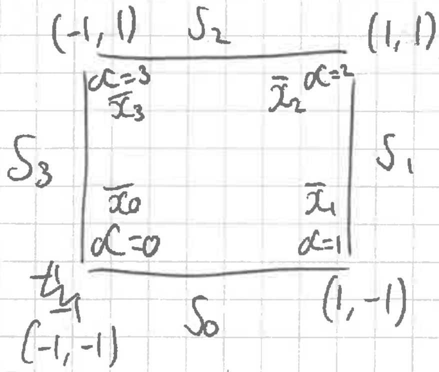
$$\hat{b}_\alpha = \int_K f\,\varphi_\alpha\, d\Omega$$

$$= \int_{\hat K} f(x(\xi_1,\xi_2), y(\xi_1,\xi_2))\,\chi_\alpha(\xi_1,\xi_2)\,|\det J(\bar\xi)|\, d\bar\xi$$

for $\alpha,\beta = 0, \dots N_n^K - 1$ on each reference cell $\hat K$

For quadrilateral $\alpha,\beta = 0, 1, 2, 3$

For reference element $\bar\xi \in (-1,1)^2$



To assemble matrix loop through α and β for the total number of nodes N_n^k where k is the number of element and n the number of nodes per element. Assign $\hat{b}_\alpha$ and $\hat{A}_{\alpha\beta}$ through:

$$\hat{A}_{\alpha\beta} = \int_K \nabla \chi_\alpha \cdot \chi_\beta \, d\Omega$$

$$\hat{b}_\alpha = \int_K f \, \varphi_\alpha \, d\Omega$$

for each element into a global matrix

$A=0$ $b=0$ $A_{ij} = b_i = 0$

for all elements K_k, $k=1$, No of elements:

for $\alpha = 1$ to N_n^k:

$i = \text{Index}(k, \alpha)$ ← set i to relevant value for kth element and $\alpha = 0, 1, 2, 3$ for that element

for $\beta = 1, N_n^k$:

$j = \text{Index}(k, \beta)$ ← set j to relevant value for ~~value~~ element k and $\beta = 0, 1, 2, 3$

$A_{ij} = A_{ij} + \hat{A}_{\alpha\beta}$ ← calculate $\hat{A}_{\alpha\beta}$ and assign to matrix

$b_i = b_i + \hat{b}_\alpha$ ← Assign to b matrix for all values of α