$A_t + (Au)_s = 0$, $u_t + uu_s = -g(h+b)_s - gC_m^2 u|u| R^{-4/3}$

$R(A,s) = AP^{-1}(A,s)$

discharge $Q = Au$

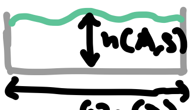
phenomenological Manning relation, provided $-b_s > 0$:

$u = C_m^{-1} R^{2/3} \sqrt{-b_s}$, $Q = Au = C_m^{-1} P^{-2/3} A^{5/3} \sqrt{-b_s}$

subbing into 1st St Venant eq: $A_t + (AR^{2/3}\sqrt{-b_s}\, C_m^{-1})_s = S$

$\Rightarrow A_t + (F(A,s))_s = S$

$F(A,s) = C_m^{-1} P^{-2/3} A^{5/3} \sqrt{-b_s}$

1) 

Claim: $h(A,s) = \frac{A}{\omega_0(s)}$, $P(A,s) = \omega_0(s) + 2h(A,s)$
$= \omega_0 + 2A\omega_0^{-1}$

area of rectangle = width × height $\Rightarrow A = h(A,s) \times \omega_0(s)$

$\Rightarrow h(A,s) = A\omega_0^{-1}$ as required.

wetted perimeter of river = width + 2 × height

$\Rightarrow P(A,s) = \omega_0(s) + 2h(A,s)$ $\Rightarrow P = \omega_0 + 2A\omega_0^{-1}$ as required.

for $S = 0$, $A_t + (AR^{2/3}\sqrt{-b_s}\,C_m^{-1})_s = S \Rightarrow A_t + (AR^{2/3}\sqrt{-b_s}\,C_m^{-1})_s = 0$

$R^{2/3} = A^{2/3}P^{-2/3}$ $\Rightarrow A_t + (A^{5/3}P^{-2/3}\sqrt{-b_s}\,C_m^{-1})_s = 0$

$\Rightarrow A_t + (F(A,s))_s = 0$

$F = F(A,s) \Rightarrow (F(A,s))_s = F_A A_s + F_s$ by chain rule

$F_A = \left(C_m^{-1}\sqrt{-b_s}\, A^{5/3}\left(\omega_0 + \frac{2A}{\omega_0}\right)^{-2/3}\right)_A$

$= \frac{5F}{3A} - \frac{4}{3\omega_0} F\left(\omega_0 + \frac{2A}{\omega_0}\right)^{-1} = \frac{F}{3}\left(\frac{5(\omega_0 + \frac{2A}{\omega_0})}{A(\omega_0 + \frac{2A}{\omega_0})} - \frac{4\frac{A}{\omega_0}}{A\omega_0(\omega_0 + \frac{2A}{\omega_0})}\right)$

$= \frac{F}{3}\left(\frac{5\omega_0 + \frac{10A}{\omega_0} - \frac{4A}{\omega_0}}{A(\omega_0 + \frac{2A}{\omega_0})}\right) = \frac{F}{3}\left(\frac{5\omega_0 + \frac{6A}{\omega_0}}{A(\omega_0 + \frac{2A}{\omega_0})}\right)$

$= \frac{1}{3}\sqrt{-b_s}\,C_m^{-1}\left(\frac{5\omega_0 A^{2/3} + 6\frac{A^{5/3}}{\omega_0}}{(\omega_0 + \frac{2A}{\omega_0})^{5/3}}\right) > 0$ as required.

$F_s = \left(\sqrt{-b_s}\,C_m^{-1} A^{5/3}\left(\omega_0 + \frac{2A}{\omega_0}\right)^{-2/3}\right)_s$

$= \sqrt{-b_s}\,C_m^{-1} \times -\frac{2}{3} A^{5/3}\left(\omega_0 + \frac{2A}{\omega_0}\right)^{-5/3}\left(\omega_0 + \frac{2A}{\omega_0}\right)_s$ (assuming $(b_s)_s = 0$)

$= -\frac{2}{3}\sqrt{-b_s}\,C_m^{-1}\left(\frac{A}{\omega_0 + \frac{2A}{\omega_0}}\right)^{5/3}\left(1 - \frac{2A}{\omega_0^2}\right)\frac{d\omega_0}{ds}$ as required.

2) $S=0$, limit in which $w_0(s)$ indep. of s (or varying v. slowly).
What is eigenvalue λ for kinematic wave eq.?

$A_t + F_A A_s + F_s = 0$ from Q1.

$\frac{dw_0}{ds} = 0$ from limit $\Rightarrow F_s = 0$ $\Rightarrow A_t + F_A A_s = 0$

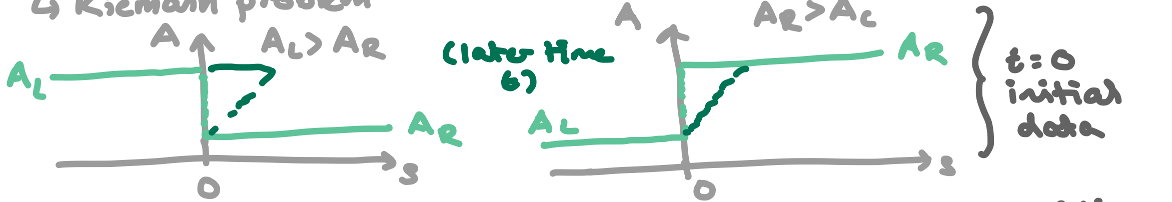
$\lambda = F_A$ (>0 from Q1.)

note: $\lambda_s = (F_A)_s = (F_s)_A = 0$ from commutivity of partial derivatives and limit.

$\Rightarrow A_t + \lambda A_s = 0 = A_t + (\lambda A)_s$

$A_t + (\lambda A)_s = 0$ w/ piecewise cst initial data $A(s,0) = \begin{cases} A_L, & s<0 \\ A_R, & s\geq 0 \end{cases}$

↳ Riemann problem



Riemann sol^n is s.t. $A(s,t)$ is cst along $s = s_0 + \lambda t$ characteristics

$\Rightarrow A(s_0, 0) = \begin{cases} A_L, & s_0 < 0 \\ A_R, & s_0 \geq 0 \end{cases}$

$s = s_0 + \lambda(A_L)t$
$s = s_0 + \lambda(A_R)t$

also: $\lambda = \begin{cases} \lambda(A_L), & s_0 < 0 \\ \lambda(A_R), & s_0 \geq 0 \end{cases}$ as F_A defined for each A_L, A_R

discty at $s = s_u(t)$ and moves w/ shock speed $u = \frac{ds_u}{dt}$

$\Rightarrow \lim_{\varepsilon\to 0}\left(\int_{s_u-\varepsilon}^{s_u+\varepsilon} \frac{\partial A}{\partial t}\,ds + \int_{s_u-\varepsilon}^{s_u+\varepsilon} (\lambda A)_s\,ds\right) = 0$

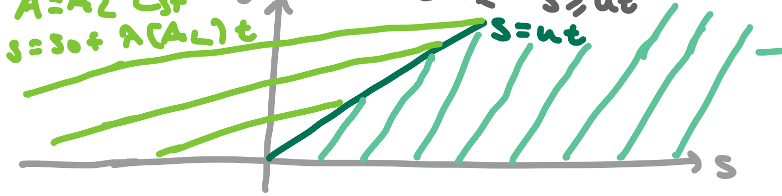
$\Leftrightarrow \frac{\partial}{\partial t}\lim_{\varepsilon\to 0}\int_{s_u-\varepsilon}^{s_u+\varepsilon} A\,ds - \frac{ds_u}{dt}\lim_{\varepsilon\to 0}(A(s,t))\Big|_{s_u-\varepsilon}^{s_u+\varepsilon} + [\lambda A]_{s_u-\varepsilon}^{s_u+\varepsilon} = 0$

$\Leftrightarrow \frac{ds_u}{dt} = \frac{[\lambda A]}{[A]} = \frac{\lambda(A_R)A_R - \lambda(A_L)A_L}{A_R - A_L} = u$ (>0 since $\lambda>0$)

for $A_L > A_R$: $\lambda(A_L) > \lambda(A_R)$ $\Rightarrow$ left moving quicker than right
$\lambda(A_R) < u < \lambda(A_L)$ $\therefore$ have a shock w/ speed u.

$A_L > A_R$: $A(s,t) = \begin{cases} A_L & s < ut \\ A_R & s \geq ut \end{cases}$

$A = A_L$ cst
$s = s_0 + \lambda(A_L)t$



$A = A_R$ cst along
$\rightarrow$ $s = s_0 + \lambda(A_R)t$
$t = \frac{1}{\lambda(A_R)}(s - s_0)$

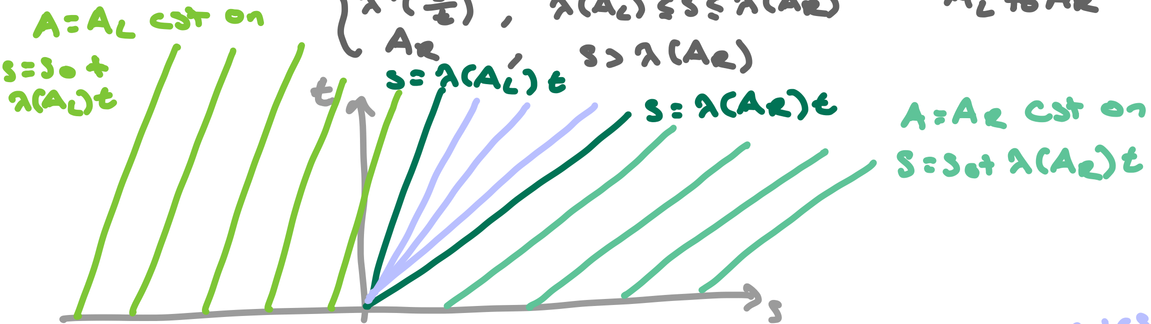
for $A_R > A_L$: $\lambda(A_R) > \lambda(A_L)$ => right side moves quicker than left side

hence no longer have a shock.

rarefaction region : $\lambda(A_L)t \leq s \leq \lambda(A_R)t$ ($s_0 = 0$ since discty at $s=0$ when $t=0$)

A cst along $s = \lambda(A)t$ characteristics => $A = \lambda^{-1}\left(\frac{s}{t}\right)$ → A varies from A_L to A_R

then
$$A(s,t) = \begin{cases} A_L, & s < \lambda(A_L) \\ \lambda^{-1}\left(\frac{s}{t}\right), & \lambda(A_L) \leq s \leq \lambda(A_R) \\ A_R, & s > \lambda(A_R) \end{cases}$$



* = rarefaction region where $s = ut$ where $A = \lambda^{-1}\left(\frac{s}{t}\right)$

When $A_L = A_R$ get a constant solution for A, where $A = A_L = A_R$ $\forall s, t$.

3) For $S=0$ derive FV/Godunov scheme:

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta s_k}\int_{t_n}^{t_{n+1}} \left(F(A,s)|_{s=s_{k+1/2}} - F(A,s)|_{s=s_{k-1/2}}\right) dt$$

w/ flux $F(A,s)|_{s=s_{k+1/2}} = F_{k+1/2}(\bar{A}_k^n, \bar{A}_{k+1}^n)$ evaluated at $s_{k+1/2}$ node.

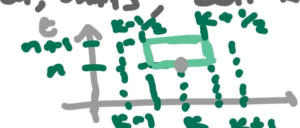
$A_t + (F(A,s))_s = 0$ ①

define space-time mesh.

on domain $s \in [0, L]$ in interval $I_n = [t_n, t_{n+1}]$, cell k occupies

$s_{k-1/2} < s < s_{k+1/2}$ for $k = 1, 2, \dots, N$.

$\Delta s_k = s_{k+1/2} - s_{k-1/2}$ is cell length.



integrate ① over space-time element $s_{k-1/2} < s < s_{k+1/2}$, $t_n < t < t_{n+1}$:

$$\int_{s_{k-1/2}}^{s_{k+1/2}} \left(A(s,t)|_{t=t_{n+1}} - A(s,t)|_{t=t_n}\right) ds + \int_{t_n}^{t_{n+1}} \left(F(A,s)|_{s=s_{k+1/2}} - F(A,s)|_{s=s_{k-1/2}}\right) dt = 0$$

define mean cell average : $\bar{A}_k = \frac{1}{\Delta s_k}\int_{s_{k-1/2}}^{s_{k+1/2}} A(s,t)\, ds$

$$\Rightarrow \bar{A}_k^{n+1} - \bar{A}_k^n + \frac{1}{\Delta s_k}\int_{t_n}^{t_{n+1}} \left(F(A,s)|_{s=s_{k+1/2}} - F(A,s)|_{s=s_{k-1/2}}\right) dt = 0$$

$$\Rightarrow \bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta s_k}\int_{t_n}^{t_{n+1}} \left(F(A,s)|_{s=s_{k+1/2}} - F(A,s)|_{s=s_{k-1/2}}\right) dt$$ as required.

$\Rightarrow \bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta s_k}\int_{t_n}^{t_{n+1}} (F_{k+1/2}(\bar{A}_k^n, \bar{A}_{k+1}^n) - F_{k-1/2}(\bar{A}_{k-1}^n, \bar{A}_k^n)\, dt$

(using $F(A,s)|_{s=s_{k+1/2}} = F_{k+1/2}(\bar{A}_k^n, \bar{A}_{k+1}^n)$)

expression for $\bar{A}_k^{n+1}$ is explicit $\therefore$ provided cell averages at previous time level are known, $\bar{A}_k^{n+1}$ can be calculated.
Hence only cell averages at previous time level are needed for Godunov method.
Godunov method also relies on sol^n to be piecewise cst so only use cell averages ($\bar{A}$ rather than A).

4) Given $A_t + (A R^{2r_3} \sqrt{-b_3} C_m^{-1})_s = S$ and $A_t + F_A A_s + F_s = 0$ ($S=0$)
derive Godunov flux using Riemann sol^n.

$F(A,s)|_{s=s_{k+1/2}} = F_{k+1/2}(\bar{A}_k^n, \bar{A}_{k+1}^n)$: Godunov flux

from Q2 have sol^n in limit of s-dependence (can use this since considering $s_{k-1/2} \le s \le s_{k+1/2}$ on which it can be approximated that there is no explicit s-dependence)

Q2 sol^n $\Rightarrow$ flux travels from left to right $\therefore$ upwind scheme
$s_{k+1/2}$ corresponds to $s=0$ in Q2 $\Rightarrow$ $A = A_L$

$\Rightarrow F_{k+1/2}(\bar{A}_k^n, \bar{A}_{k+1}^n) = F_{k+1/2}(\bar{A}_k^n) = \lambda_k \bar{A}_k^n$ $\left(\lambda_k = \frac{\partial F}{\partial A}\Big|_{A=A_k}\right)$ cst by s-depend limit

upwind scheme:

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta s_k}\int_{t_n}^{t_{n+1}} F_{k+1/2} - F_{k-1/2}\, dt$$

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta s_k}\int_{t_n}^{t_{n+1}} (\lambda_k \bar{A}_k^n - \lambda_{k-1}\bar{A}_{k-1}^n)\, dt$$

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{\Delta t}{\Delta s_k}(\lambda_k \bar{A}_k^n - \lambda_{k-1}\bar{A}_{k-1}^n)$$

($\lambda_k, \bar{A}_k^n$ cst by s-dependence limit / cell average)

5) time-step restriction / CFL cond. based on $\Delta t < CFL \min_k \frac{h_k}{|\lambda_k|}$
cell length h_k, information speed / local eigenvalue λ_k.

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{\Delta t}{h_k}(\lambda_k \bar{A}_k^n - \lambda_{k-1}\bar{A}_{k-1}^n)$$

$$= \left(1 - \frac{\Delta t}{h_k}\lambda_k\right)\bar{A}_k^n + \lambda_{k-1}\frac{\Delta t}{h_k}\bar{A}_{k-1}^n$$

note: $\lambda_k > 0 \ \forall k$ since $\frac{\partial F}{\partial A} > 0$ (Q1), local eigenvalue.

maximum principle => require $1 - \lambda_K \frac{\Delta t}{h_K} \geq 0$

$\iff \Delta t \leq \frac{h_K}{\lambda_K}$

$\lambda_K > 0 \Rightarrow \lambda_K = |\lambda_K|$.

require this over whole region $\Rightarrow \Delta t \leq \min_K \frac{h_K}{\lambda_K}$

hence have CFL cond. : $\Delta t < CFL \min_K \frac{h_K}{|\lambda_K|}$

for $0 < CFL < 1$ this then satisfies maximum principle.

$$\lambda_K = \frac{\partial F}{\partial A}\Big|_{A=\bar{A}_K^n} = \frac{1}{3}\sqrt{-b_s}\,C_m^{-1} \frac{(5w_0 \bar{A}_K^{n\,2/3} + 6\bar{A}_K^{n\,5/3} w_0^{-1})}{(w_0 + 2\bar{A}_K^n w_0^{-1})^{5/3}} > 0.$$

6) inflow/outflow cond. imposed at $s=0$ and end $s=L_e$.

require constant value inflow into the river (Q_0).
hence $\bar{A}_{-1}$ (ghost cell) must be set s.t. $F_{K+1/2}(\bar{A}_{-1}) = Q_0$
$\Rightarrow F_{-1/2}(\bar{A}_{-1}) = Q_0 = \lambda_{-1/2}\bar{A}_{-1}$ where $\lambda_{-1/2} > 0$ to justify flow moving left to right (upwind scheme).
Since upwind scheme, fixing $\bar{A}_{N_K} / F_{N_K+1/2}$ at $s = L_e$ would not have an effect on the flux in the river.
So don't need an outflow condition.

7) Numerics in Firedrake:

All test cases run using provided code.

Running over: $N_x = 1250, 2500, 5000, 10000$

$CFL = 0.125, 0.25, 0.5, 1.0$

$\Delta x_{min} = 0.5$, $CFL_{min} = 0.125$

finest mesh run over $N_x = 10000$ ($\Delta x_{min} = 0.5$), $CFL = 0.125$

Δx analysis done w/ $N_x = 1250, 2500, 5000, 10\,000$

note: $\Delta t = CFL \frac{\Delta x}{\lambda}$ $\therefore$ varying Δx also varies Δt

$N_x = 10\,000$, $CFL = 1$ $\quad$ $N_x = 2500$, $CFL = 0.25$

$N_x = 5000$, $CFL = 0.5$ $\quad$ $N_x = 1250$, $CFL = 0.125$

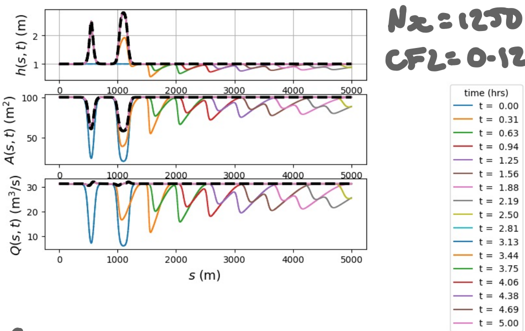
To fix Δt whilst varying Δx to just investigate effect of Δt.

$\Delta x := 1$ ($N_x = 5000$ as base)

$\Delta t := CFL \times (\frac{\Delta x}{\lambda})$ ($N_x = 5000$, $CFL = 0.5$ as base)

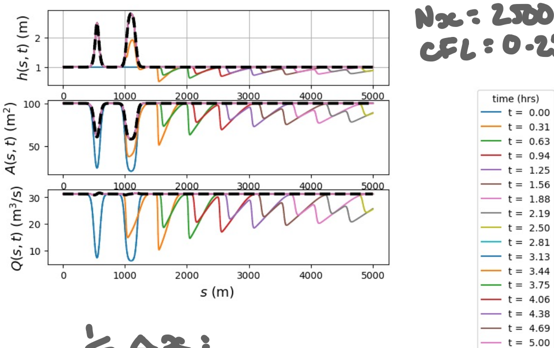
Test-Case 0:

$4\Delta x$:



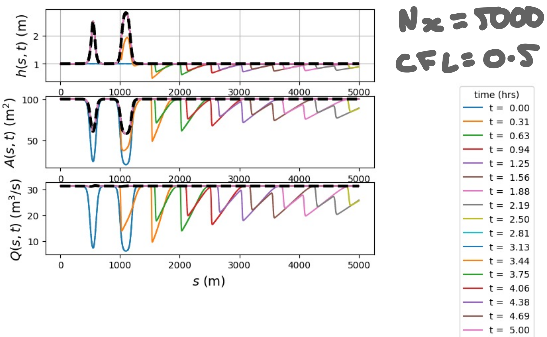
$N_x = 1250$
$CFL = 0.125$

$2\Delta x$:



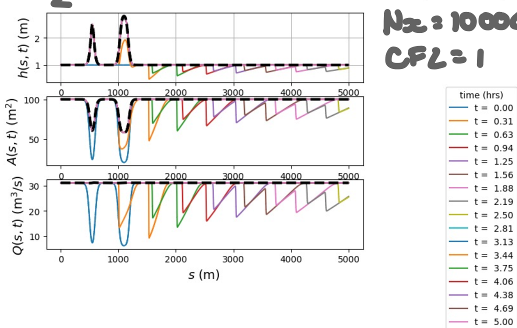
$N_x = 2500$
$CFL = 0.25$

Δx:



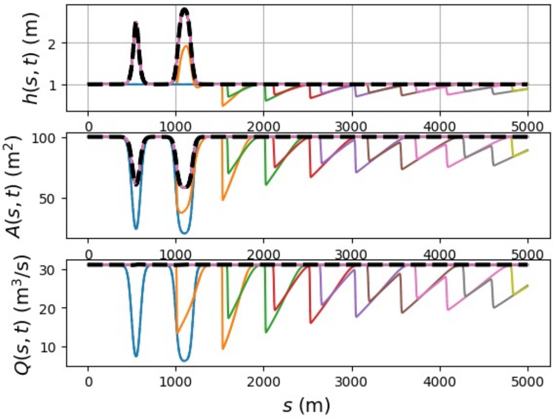
$N_x = 5000$
$CFL = 0.5$

$\frac{1}{2}\Delta x$:



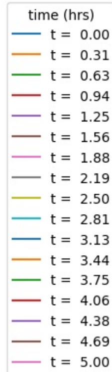
$N_x = 10000$
$CFL = 1$

Finest mesh:

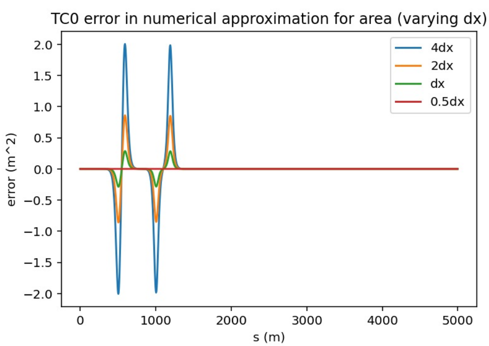


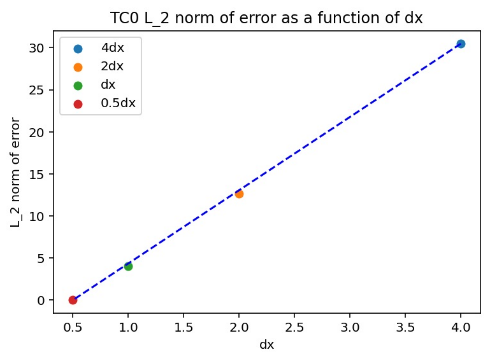
$N_x = 10000$

$CFL = 0.125$



Errors calculated using final $A(s,t)$ ($t = 5$ hrs) and using finest mesh as approximate solution.





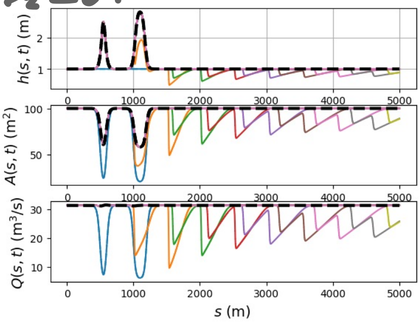
Errors plot shows that as Δx decreases we converge towards the "actual" sol^n (that of finest mesh)

Plotting L_2 norm of error as a fnct of Δx and drawing a 'line of best fit' shows a linear relation between L_2-norm of error and Δx.

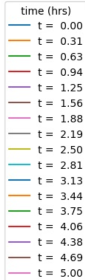
$\therefore$ 1st order accuracy wrt Δx.

Also note that the smaller Δx considered, the closer to constant the final $Q(s,t)$.

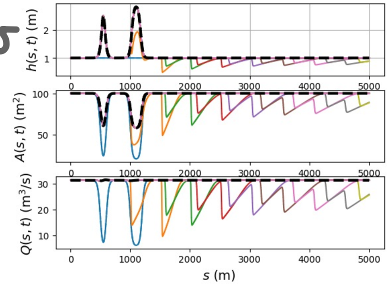
½ Δt:



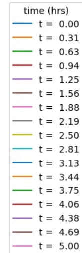
Δx = 1
CFL = 0.25



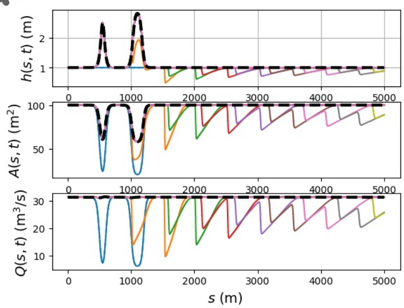
Δt:



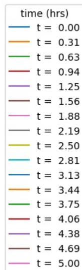
Δx = 1
CFL = 0.5



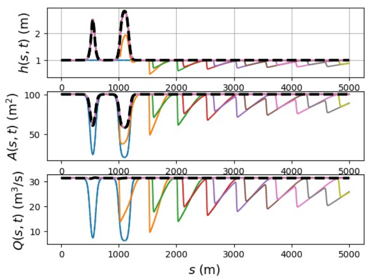
¼ Δt:



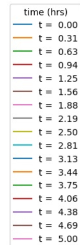
Δx = 1
CFL = 0.125

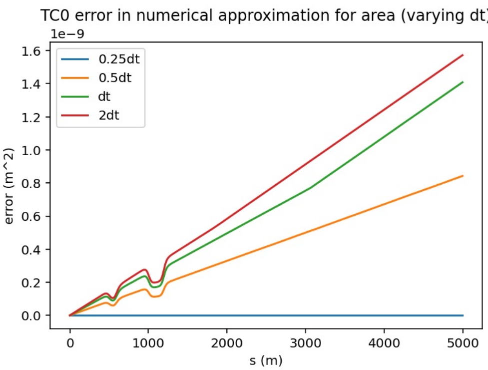


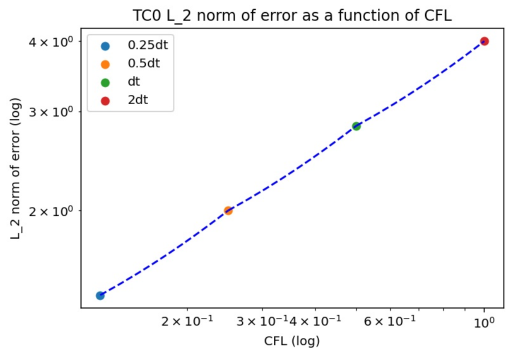
2Δt:



Δx = 1
CFL = 1



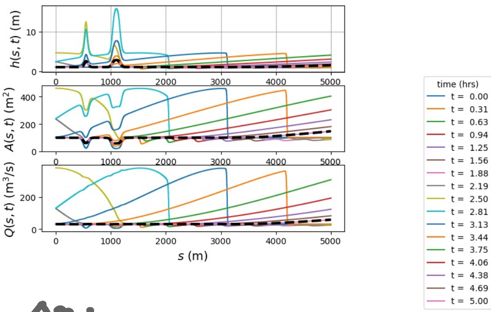




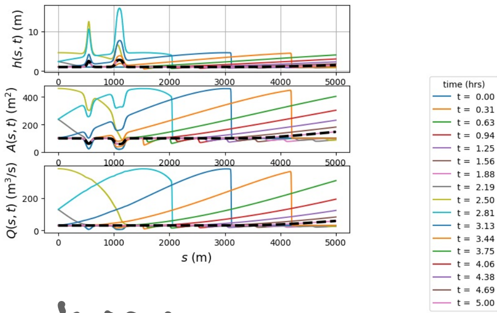
Again errors show convergence as Δt decreases.
Plotting L_2-norm of error as fnct of CFL against CFL doesn't show obvious linear relation.
but plotting logs against each other leads to a gradient of 1.08 => linear relation
=> first order accuracy in Δt too.

Test - Case 1:

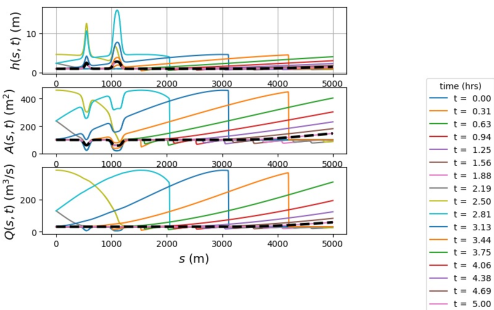
$4\Delta x$:



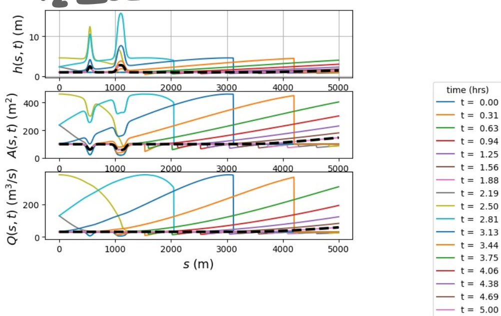
$2\Delta x$:



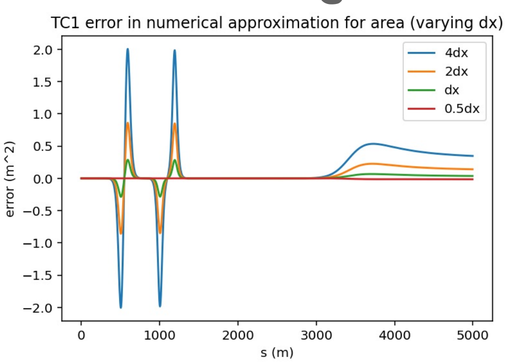
Δx:

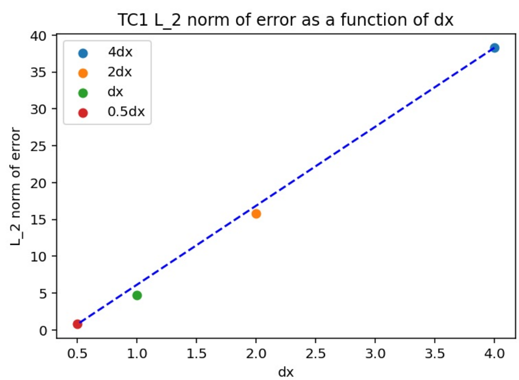


$\frac{1}{2}\Delta x$:



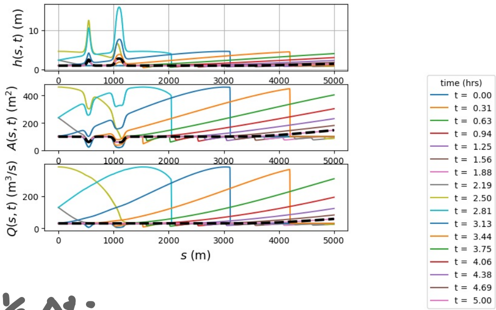
Δx error analysis:



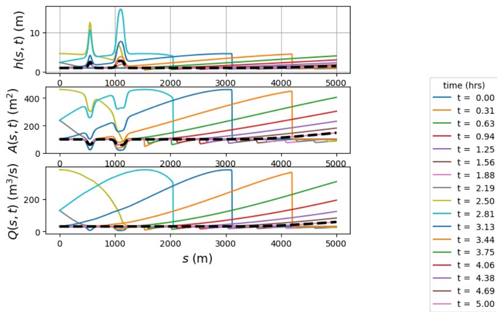


Δt error analysis:

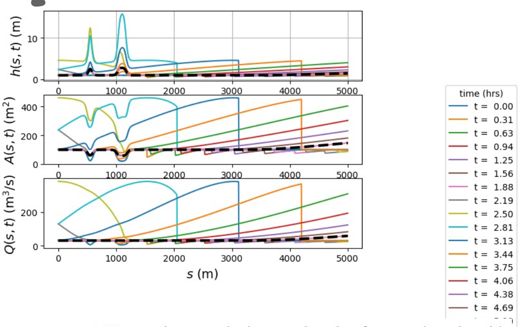
2Δt:



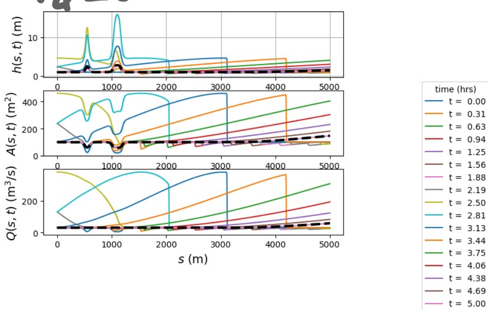
Δt:

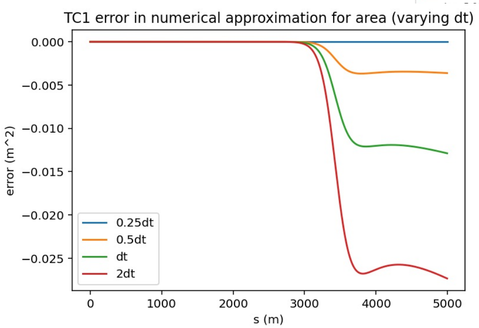


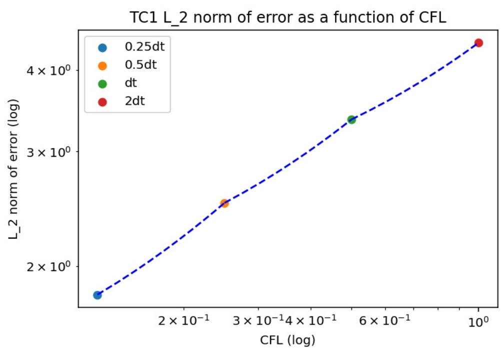
½ Δt:



¼ Δt:



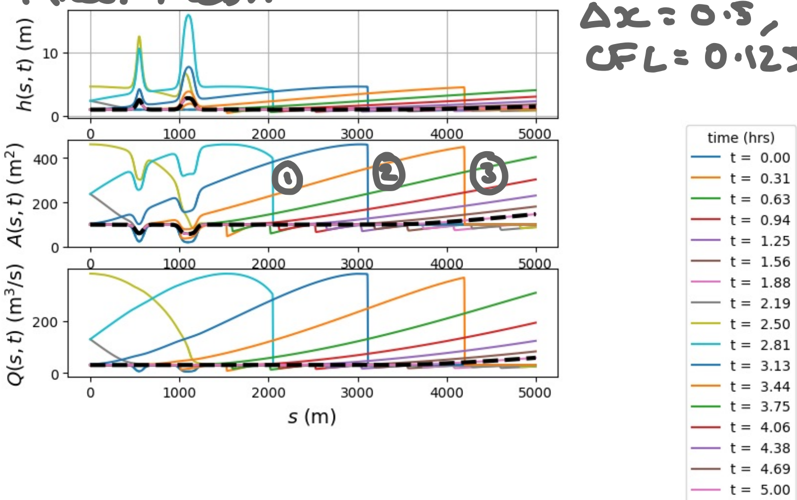




Analysing error results for Δx and Δt in the same way as TC0: the same results are found. ∴ have convergence and first order accuracy.

Shock speed:

Finest mesh:



$\Delta x = 0.5$, $N_x = 10000$
$CFL = 0.125$

Can follow the shock over $t = 2.81$ hrs, 3.13 hrs, 3.44 hrs to compare observed shock speed to actual shock speed.

(1) $t = 2.81$ hrs $= 10125$ s (from counter in code)
$A_L = 400$, $A_R = 100$ $\quad F(A_L) = Q(A_L) = 300$, $F(A_R) = Q(A_R) = 31$
$s = 2050$

numerical shock speed: $u = \dfrac{F(A_R) - F(A_L)}{A_R - A_L} \approx 0.90\,ms^{-1}$

(2) $t = 3.13$ hrs $= 11250$ s
$A_L = 460$, $A_R = 100$ $\quad Q(A_L) = 380$, $Q(A_R) = 31$
$s = 3115$

numerical shock speed: $u = \dfrac{F(A_R) - F(A_L)}{A_R - A_L} \approx 0.97\,ms^{-1}$

(3) $t = 3.44$ hrs $= 12375$ s
$A_L = 450$, $A_R = 100$ $\quad Q(A_L) = 366$, $Q(A_R) = 31$
$s = 4200$

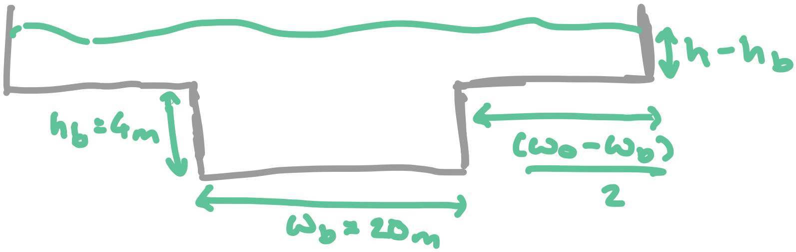
numerical shock speed: $u = \dfrac{F(A_R) - F(A_L)}{A_R - A_L} \approx 0.96\,ms^{-1}$

(1) $\mapsto$ (2) $\quad \dfrac{3115 - 2050}{11250 - 10125} \approx 0.95\,ms^{-1}$

(1) $\mapsto$ (3) $\quad \dfrac{4200 - 3115}{12375 - 11250} \approx 0.96\,ms^{-1}$ observed shock speed

$\therefore$ observed shock speed $\approx$ numerical shock speed.

Test-Case 2:



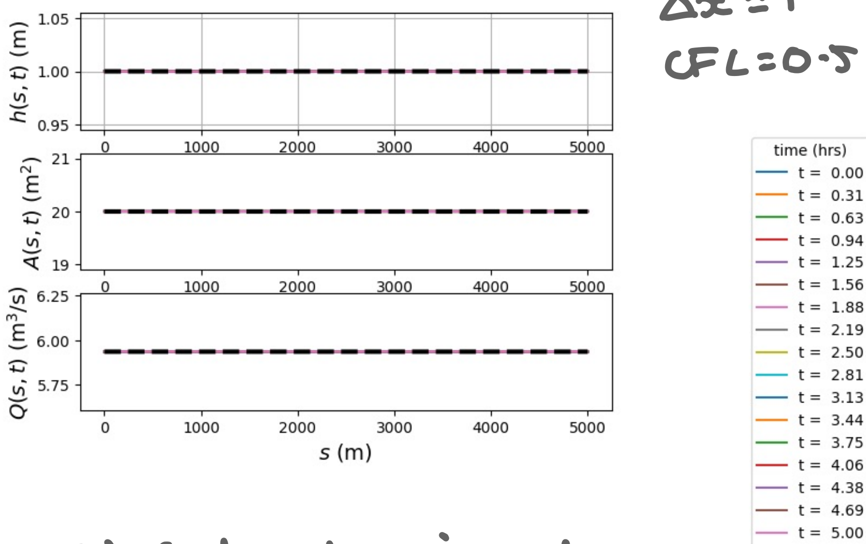
area: $A = w_b h_b + w_o(s)(h - h_b)$

wetted perimeter: $P = w_b + 2h_b + w_o + \frac{2}{w_o}(A - w_b h_b)$

Test-Case 2a:

$\Delta x = 1$

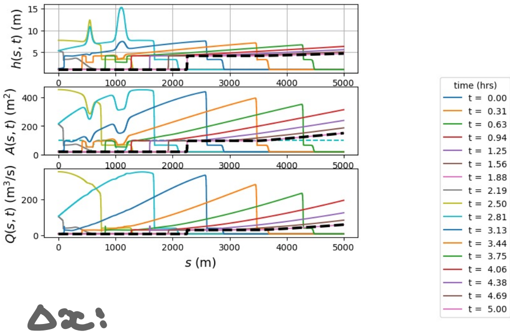
$CFL = 0.5$



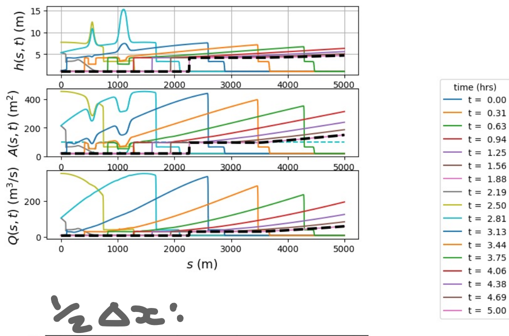
Due to cst rectangular river channel everything is cst when cst flux.

Test - Case 2b:

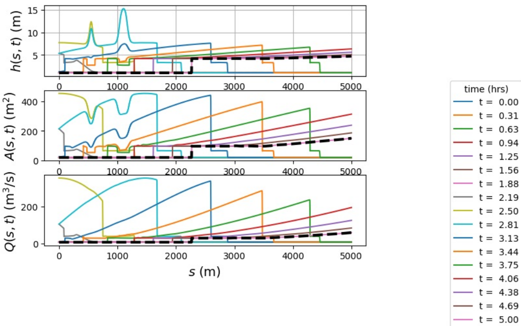
$4\Delta x$:



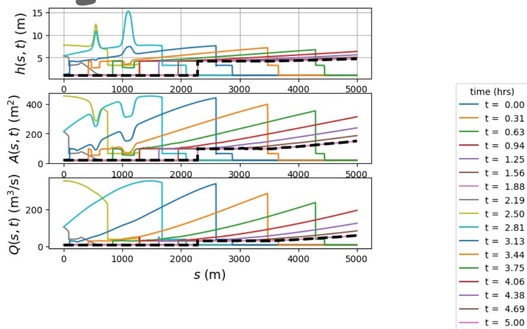
$2\Delta x$:



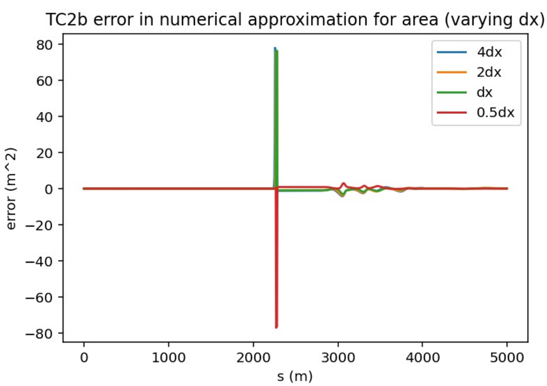
Δx:

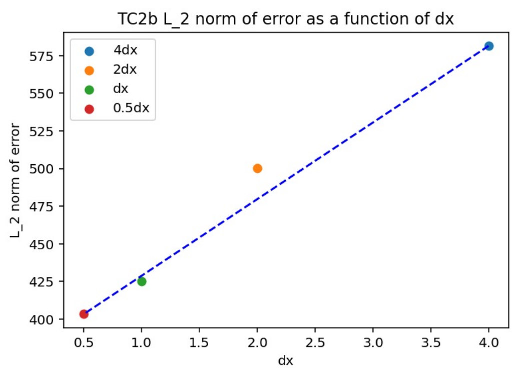


$\frac{1}{2}\Delta x$:



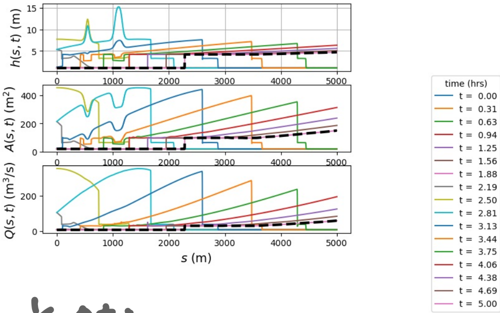
Δx error analysis:



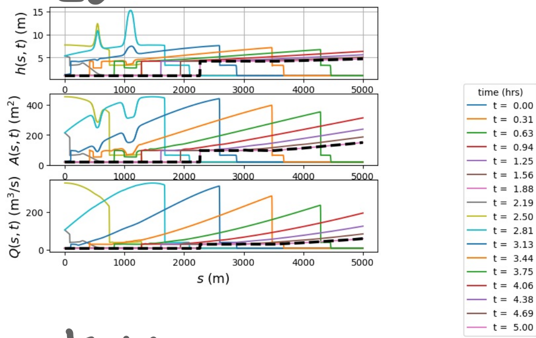


Error plot shows that plots are still converged in Δx. Need a finer mesh for numerical solution to converge. However, plotting L_2-norm of error against Δx find a linear relation => first order accuracy.

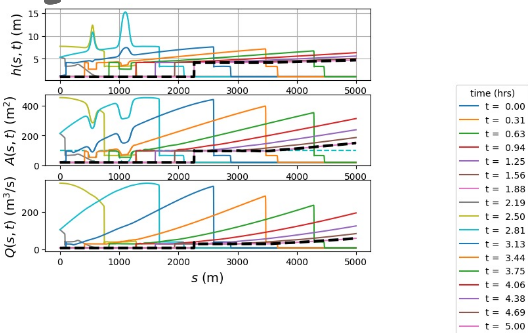
$2\Delta t$:



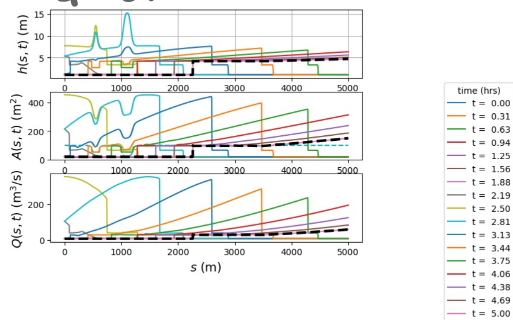
Δt:



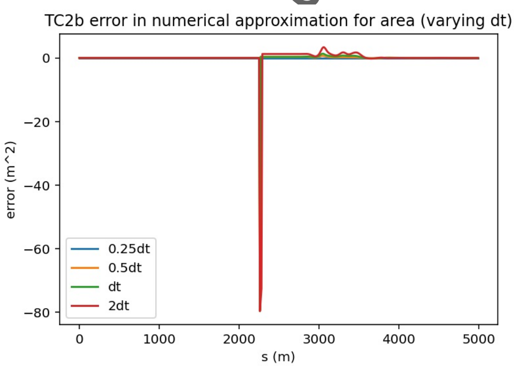
$\frac{1}{2}\Delta t$:

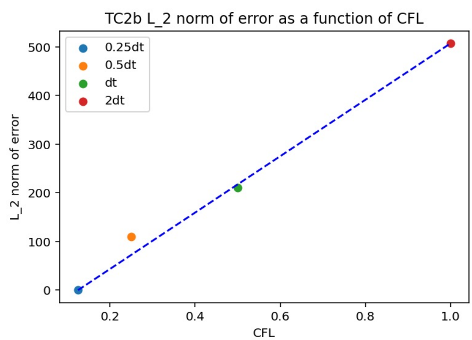


$\frac{1}{4}\Delta t$:



Δt error analysis:



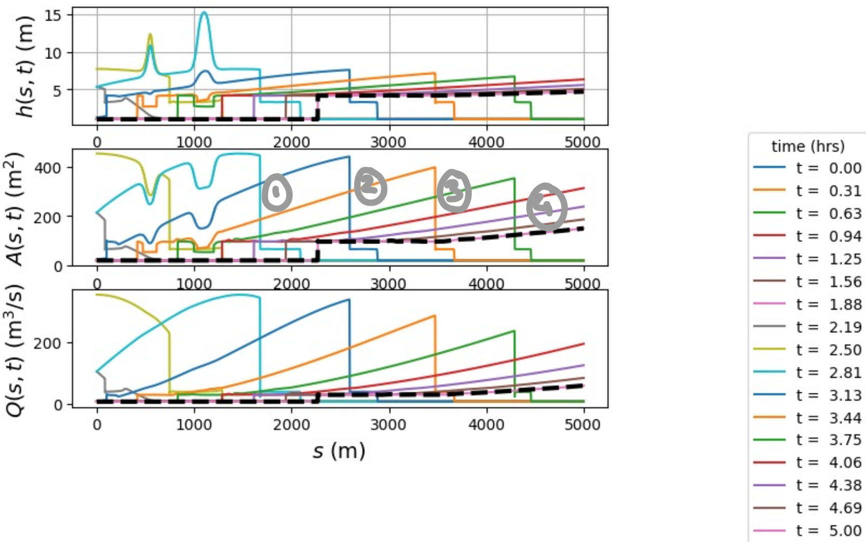


Error plot shows still converging in Δt, similarly to Δx. Also similarly to Δx find linear relationship btw Δt and L_2-norm of error => first order accuracy.

Error analysis => need much finer mesh for numerical sol^{n} to converge in TC2b than TC1/TC0.

Shock speed:

Finest mesh:



① $t = 2.81$hrs $= 10125$s
$A_L = 445$, $A_R = 65$ $\quad Q(A_L) = 346$, $Q(A_R) = 38$
$s = 1677$

numerical shock speed: $\quad u = \dfrac{F(A_R) - F(A_L)}{A_R - A_L} \approx 0.81\,\text{ms}^{-1}$ $\quad (F = Q)$

② $t = 3.13$hrs $= 11250$s
$A_L = 441$, $A_R = 70$ $\quad Q(A_L) = 340$, $Q(A_R) = 38$
$s = 2596$
$u \approx 0.81\,\text{ms}^{-1}$

③ $t = 3.44$hrs $= 12375$s
$A_L = 395$, $A_R = 65$ $\quad Q(A_L) = 287$, $Q(A_R) = 38$
$s = 3475$
$u \approx 0.75\,\text{ms}^{-1}$

④ $t = 3.75$hrs $= 13500$s
$A_L = 352$, $A_R = 65$ $\quad Q(A_L) = 237$, $Q(A_R) = 38$
$s = 4290$
$u \approx 0.69\,\text{ms}^{-1}$

① $\mapsto$ ② $\quad$ observed speed $= \dfrac{2596 - 1677}{11250 - 10125} = 0.82\,\text{ms}^{-1}$

② $\mapsto$ ③ $\quad$ observed $u = 0.78\,\text{ms}^{-1}$

③ ↦ ④ observed $u = 0.72\,ms^{-1}$

find that averages btw shock speeds are approx. the observed shock speeds.

All fine: 20/20