Fluid Dynamics — Numerical Techniques

MATH5453M FEM Numerical Exercise 3, 2025

Due date: December 2025

Consider the Poisson system

$$-\nabla^2 u = f \quad \text{on} \quad (x, y) \in [0, 1]^2 \tag{1a}$$

$$f(x, y) = 2\pi^2 \sin(\pi x) \cos(\pi y) \tag{1b}$$

$$u(0, y) = u(1, y) = 0 \tag{1c}$$

$$\partial_y u(x, y)|_{y=0} = \partial_y u(x, y)|_{y=1} = 0 \tag{1d}$$

with variable or unknown $u(x, y)$, given function $f(x, y)$, and with Dirichlet and Neumann boundary conditions. The exact solution is $u_e(x, y) = \sin(\pi x) \cos(\pi y)$, please check.

1. *Step 1:* Write down the Ritz-Galerkin principle for the above Poisson system and show that variation thereof yields the system. What are the conditions on the variation $\delta u(x, y)$? Derive the weak formulation for the above system. Show that the test function $w(x, y)$, say, used is the same as $w(x, y) = \delta u(x, y)$.

2. *Step 2:* Write down the algebraic or discrete Ritz-Galerkin principle after introducing a FEM expansion $u_h(x, y)$ for $u(x, y) \approx u_h(x, y)$ in terms of global basis functions. Write down the algebraic or discrete weak formulation after introducing a FEM expansion in terms of global basis functions. Show that the variation of the former yields the latter.

3. *Step 3:* Although not needed in Firedrake, introduce a local coordinate system and reference coordinates (for trianglar and/or quadrilateral elements), and explain the matrix assembly involved in getting the system in *Step 2*. Use quadrilateral elements.

4. *Step 4:* Solve the system in Firedrake with the provided or other Firedrake codes. Plot the numerical results for $u_h(x, y)$ with Paraview as a contour plot (with clear labelling/indicating of the values used in that plot). Plot the difference $|u_h(x, y) - u_e(x, y)|$ (of numerical and exact solutions) as a contour

plot in Firedrake for a few suitable resolutions (–). Mention the function spaces used and the order of accuracy. Explore different order (p-refinement) and mesh resolutions (h-refinements). Explain and show which $\{h, p\}$ combinations are (roughly) equivalent and why? Provide clear figure captions with information on resolution, etc., such as $\{h, p\}$.

5. Explain how the above first four steps are implemented in Firedrake, also by adding clear comments to your code.

6. Change and implement the boundary conditions for a different function $f(x, y)$ and exact solution $u(x, y)$. Test it with clear instructions how to reproduce your results. Test it for use from a terminal.