

if and only if

$$1 \geq \frac{\Delta t}{\Delta z} D_0^2 \left( 3x + 2 \frac{B}{(\Delta z)} \right)$$

$$\Delta t \leq \frac{\Delta z}{D_0^2 (3x + 2 \frac{B}{\Delta z})} = \frac{(\Delta z)^2}{D_0^2 (3x \Delta z + 2B)}$$

e) Recall discretization of (5) ~~and (6)~~:

$$\underline{b_j^{n+1}} = -3\alpha D_0^2 (b_j^n - b_{j-1}^n)$$

$$b_j^{n+1} = -3\alpha (b_j^n)^2 (b_j^n - b_{j-1}^n) \frac{\Delta t}{\Delta z}$$

$$+ \frac{B \Delta t}{2(\Delta z)^2} \left[ (b_{j+1}^n)^3 + (b_j^n)^3 \right] (b_{j+1}^n - b_j^n)$$

$$- \left[ (b_{j-1}^n)^3 + (b_j^n)^3 \right] (b_j^n - b_{j-1}^n) + b_j^n$$

assume  $b_j^n > 0$  for all  $j \in \{0, \dots, J\}$ , then

$$\underline{b_j^{n+1}} > -3\alpha (b_j^n)^2 \frac{\Delta t}{\Delta z} + \frac{\Delta t}{2(\Delta z)^2} \left[ (b_{j+1}^n)^3 + (b_j^n)^3 \right] (-b_j^n)$$

$$- \left[ (b_{j-1}^n)^3 + (b_j^n)^3 \right] \underline{b_j^n}$$

$$\text{let } A = (b_{j+1}^n)^3 + (b_j^n)^3, \quad B = (b_{j-1}^n)^3 + (b_j^n)^3,$$

we get:

$$b_j^{n+1} = 3\alpha \frac{\Delta t}{\Delta z} (b_j^n)^2 b_{j-1}^n + \frac{\beta \Delta t}{2(\Delta z)^2} \left( A b_{j+1}^n + B b_{j-1}^n \right)$$

$$+ b_j^n \left( 1 - 3\alpha \frac{\Delta t}{\Delta z} (b_j^n)^2 - \frac{\beta}{2} \frac{\Delta t}{(\Delta z)^2} (A + B) \right)$$

$> 0$  iff

~~if and only if~~

$$1 - 3\alpha \frac{\Delta t}{\Delta z} (b_j^n)^2 - \frac{\beta}{2} \frac{\Delta t}{(\Delta z)^2} (A + B) > 0$$

$$\frac{\Delta t}{\Delta z^2} \left( 3\alpha (b_j^n)^2 \Delta z + \frac{\beta}{2} (A + B) \right) < 1$$

$$\Delta t < \frac{1}{2} (\Delta z)^2 \left( 6 (b_j^n)^2 \Delta z + \beta (A + B) \right)^{-1}$$

COMMENT:  $A$  &  $B$  depend on  $n$  &  $j$ ! They are  
not constants.

f) Derivation of 2<sup>nd</sup>-order spatial discretisation of 1<sup>st</sup> order derivative.

Consider the following Taylor expansions:

$$b_j^{n+1} = b_j^n + \Delta z \partial_z b_j^n + \frac{(\Delta z)^2}{2} \partial_z^2 b_j^n + O((\Delta z)^3)$$

$$b_j^{n-1} = b_j^n - \Delta z \partial_z b_j^n + \frac{(\Delta z)^2}{2} \partial_z^2 b_j^n + O((\Delta z)^3)$$

Now consider

$$\frac{b_j^{n+1} - b_j^{n-1}}{\Delta z} = 0 + 2 \partial_z b_j^n + O + O((\Delta z)^2).$$

Hence,

$$\partial_z b_j^n \approx \frac{b_j^{n+1} - b_j^{n-1}}{2 \Delta z}$$

Changing this term in part e) we get

$$\begin{aligned} b_j^{n+1} &= -3 \alpha \frac{\Delta t}{2 \Delta z} \left( b_j^{n+1} - b_j^{n-1} \right) (b_j^n)^2 \\ &\quad + \frac{B \Delta t}{2 (\Delta z)^2} \left( A \left( b_j^{n+1} - b_j^n \right) - B \left( b_j^n - b_j^{n-1} \right) \right) + b_j^n. \end{aligned}$$

assume  $b_j^n > 0$ , then

$$b_j^{n+1} = \beta \alpha \frac{\Delta t}{2\Delta z} b_{j-1}^n + \frac{\beta \Delta t}{2(\Delta z)^2} \left( B b_{j-1}^n \right)$$

$$\frac{\Delta t}{2(\Delta z)^2} \left[ b_{j+1}^n \left( AB - \beta \alpha \Delta z (b_j^n)^2 \right) \right]$$

$$+ b_j^n \left( 1 - \frac{\beta \Delta t}{2(\Delta z)^2} (A+B) \right) > 0$$

if and only if

$$\frac{\Delta t}{2(\Delta z)^2} \left[ b_{j+1}^n \left( AB - \beta \alpha \Delta z (b_j^n)^2 \right) \right] \\ + b_j^n \left( 1 - \frac{\beta \Delta t}{2(\Delta z)^2} (A+B) \right) > 0.$$

Require two conditions:

$$\beta A - \beta \alpha \Delta z (b_j^n)^2 > 0$$

$$\Leftrightarrow \Delta z < \frac{\beta A}{\beta \alpha (b_j^n)^2}$$

and

$$1 - \frac{\beta \Delta t}{2(\Delta z)^2} (A+B) > 0$$

$$\Leftrightarrow \Delta t < \frac{2(\Delta z)^2}{\beta (A+B)}$$

Combining these we get

Scheme known to be unstable for beta=0; how can that be seen? -0.5

$$\Delta t < \frac{2 \beta A^2}{(A+B) 9 \alpha^2 (b_j^n)^4}$$

(2.) a) For steady solution, require

$b = b(z)$  only and (5) becomes:

$$0 + \partial_z [\alpha b^3 - \beta b^3 b'] = 0$$

Integrate to get

$$\alpha b^3 - \beta b^3 b' + A = 0$$

for  $A$  constant. Rearrange and set  $Q := -A$

to get:

$$\beta b^3 b' = \alpha b^3 - Q$$

for integration constant  $Q$ . Rearrange again

$$\frac{db}{dz} = \frac{\alpha}{\beta} - \frac{Q}{\beta} b^{-3}$$

Let  $b = \exp(\gamma z)$ , then

$$\beta \exp(\beta \gamma z) \lambda \exp(\lambda t) = \alpha \exp(\beta \gamma z) - Q$$

$$b^3 (\alpha - \beta b') = Q$$

$$\partial_z [\alpha, \beta] = Q = b^3 (\alpha - \beta b')$$

$$\alpha' \alpha + \alpha \alpha' = \cancel{\alpha^3} \alpha - \beta b^3 b'$$

(2.) c.) The order of spatial discretisation is  $O(\Delta z)$  as given by graph part c. pug.  
as we double the number of grid points,  
the  $L_2$  error halves.

Looking at graph part c. pug the linear relationship is clear!



log ( $L_2$  error)

To calculate  $L_2$  error, trapezoidal

rule was used:

$$L^2 = \sqrt{\int_0^H e^2(z, t) dz} \approx \sqrt{\frac{\Delta z}{2} \left[ e^2(0, t) + 2 \sum_{j=1}^{N-1} e^2(j \Delta z, t) + e^2(N \Delta z, t) \right]}$$

for  $e^2 = (b_{\text{steady}} - b_{\text{temporal}})^2$  and  $N$  - number of grid points.

d) First calculate all derivatives:

$$\begin{aligned}
 \rightarrow \partial_t [z - z_{r_0} - ct] &= -c, \\
 -c &= \frac{\beta}{\alpha} \left( \partial_t b - \sqrt{\frac{c}{\alpha}} \frac{1}{1 - \frac{\alpha}{c} b^2} \right) \sqrt{\frac{c}{\alpha}} \partial_t b \\
 &= \partial_t [b] \frac{\beta}{\alpha} \left( 1 - \frac{1}{1 - \frac{\alpha}{c} b^2} \right) \} \text{ (*) This is } -A \\
 &= \partial_t [b] \frac{\beta}{\alpha} \left( \frac{1 - \frac{\alpha}{c} b^2 - 1}{(1 - \frac{\alpha}{c} b^2)} \right) \\
 &= -\partial_t [b] \beta \left( \frac{b^2}{(c - \alpha b^2)} \right)
 \end{aligned}$$

Get  $A := \beta \frac{b^2}{(c - \alpha b^2)}$ , then

$$\partial_t [b] = \frac{c}{A}.$$

Similarly,

$$\partial_z [z - z_{r_0} - ct] = 1$$

and we get

$$\partial_z [b] = -\frac{1}{A}.$$

The calculation above is analogous to  $\partial_t [b]$

It remains to calculate

$$\partial_z^2 [b] = -\frac{1}{A^2} \partial_z [A] = \partial_z \left[ -\frac{1}{A} \right]$$

(19)

$$\begin{aligned}
 \partial_2^2 [b] &= \frac{1}{A^2} \partial_2 \left[ -\frac{\beta}{\alpha} \left( 1 - \frac{1}{1 - \frac{\alpha}{c} b^2} \right) \right] \quad \text{This is } (*) \text{ on previous page} \\
 &= -\frac{1}{A^2} \frac{\beta}{\alpha} \left( \frac{1}{(1 - \frac{\alpha}{c} b^2)^2} \left( -\frac{\alpha}{c} 2b \partial_2 b \right) \right) \\
 &= -\frac{1}{A^2} \frac{\beta}{\alpha} \left( \frac{1}{(1 - \frac{\alpha}{c} b^2)^2} \left( \frac{\alpha}{c} 2b - \frac{1}{A} \right) \right) \\
 &= -\frac{1}{A^3} \frac{\beta 2b - c}{(c - \alpha b^2)^2} \\
 &= \frac{-2}{\beta b^3 A^3} \left( \frac{\beta^2 b^2 c}{(c - \alpha b^2)^2} \right) \\
 &= \frac{-2c}{\beta b^3 A}
 \end{aligned}$$

Plug everything in to get

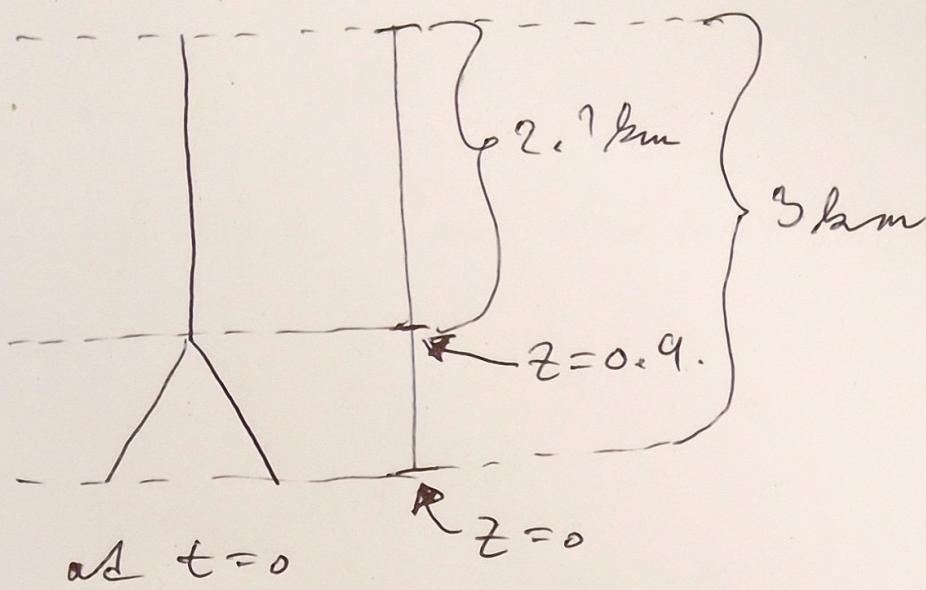
$$\begin{aligned}
 \partial_t b + \partial_2 [u b] &= 0 \\
 \partial_t b + 3\alpha b^2 \partial_2 b - 3\beta b^2 (\partial_2 b)^2 + \cancel{\frac{1}{A} \cancel{\beta b^3 \partial_2^2 b}} &= 0 \\
 \frac{c}{A} + 3\alpha b^2 \left( -\frac{1}{A} \right) - \frac{3\beta b^2}{A^2} + \beta b^3 \left( \frac{2c}{\beta b^3 A} \right) &= 0
 \end{aligned}$$

$$c + 3\alpha b^2 - 3\beta b^2 \left( \beta^{-1} \frac{(c - \alpha b^2)}{b^2} \right) + 2c = 0$$

$$c - 3\alpha b^2 - 3(c - \alpha b^2) + 2c = 0$$

$$3c - 3\alpha b^2 - 3(c - \alpha b^2) = 0$$

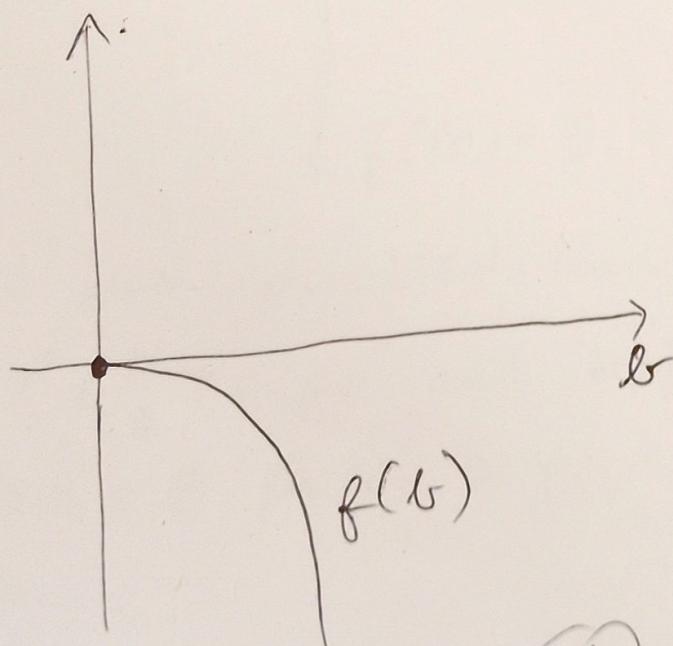
Now, arrange the initial dike shape.  
 it is closed till a dimensional depth 2.7 km  
 with  $t=3$  km.



Note that  $\operatorname{arcsinh}(b)$  restricts  $b \in [-1, 1]$ ,  
 but to enforce physicality, require  $b \in [0, 1]$ .  
 At  $t=0$  we get

$$z = \frac{\beta}{\alpha} (b - \operatorname{arcsinh}(b)) + z_0$$

Let  $f = \frac{\beta}{\alpha} (b - \operatorname{arcsinh}(b))$ , then  
 for  $b \in [0, 1]$



However, we have  $z \in [0, \infty)$  and require  
for  
that  $\forall z \in (0, 3]$ ,  $f(b) = 0$ . Note that  
as at  $t=0$  &  $z=0.9$ , we get

$$0.9 = f(0) + z_0$$

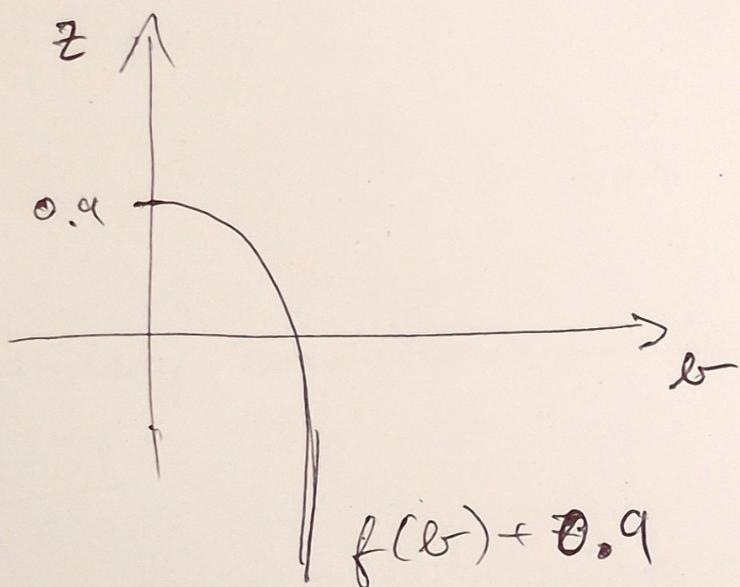
$f(0) = 0$  as  $\text{arctanh}(0) = 0$ . Then

$$z_0 = 0.9.$$

Now, plot

$$z = f(b) + \underbrace{0.9}_{z_0}$$

we get

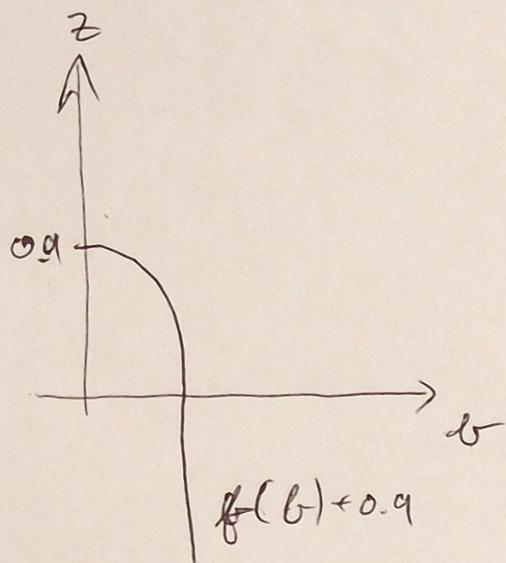


As  $z \geq 0$  and provided maximum principle  
holds ( $b_j^{\alpha} \geq 0$  for all  $(i, j)$ ) we have  
 $f(b) + 0.9 = 0$  for all  $z \geq 0.9$ .

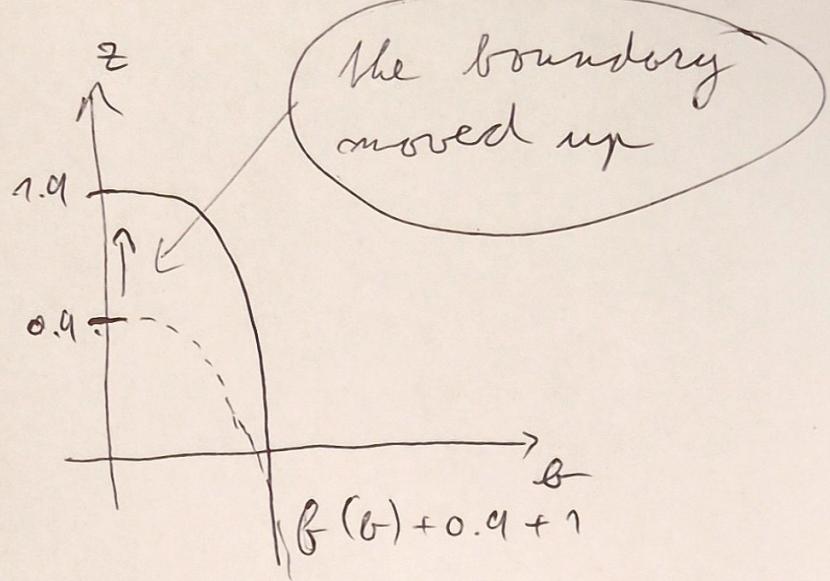
Now, note that if we reintroduce time  $t$ , we get

$$z = f(b) + 0.9 + \alpha t,$$

i.e. as time progresses, more and more values of  $z$  are allowed to be non-zero



$$t = 0$$



$$t = 1$$

For boundary conditions we have

$$b_B, z=0$$

d) Still need more points. 18.5/20

$$0 - 0.9 - t = (b_B - \operatorname{arctanh}(b_B)) \frac{b}{\alpha}$$

$$b_T, z=1$$

$$1 - 0.9 - t = 2.7 - t = (b_T - \operatorname{arctanh}(b_T)) \frac{b}{\alpha}$$

~~and we need to solve for  $b_B$  &  $b_T$  numerically~~