

1) Ground Water Model

Nonlinear diffusion equation: Describes how groundwater level evolves in time and space

$$\frac{\partial W_y h_m}{\partial t} - a g \frac{\partial}{\partial y} \left(W_y h_m \frac{\partial h_m}{\partial y} \right) = \frac{W_y R}{m_{\text{por}} \theta_e}$$

$h_m(y, t)$ = groundwater level above datum, W_y = Channel width ($\sim 0.1\text{m}$), m_{por} = porosity (0.1-0.3)

θ_e = effective pore fraction (0.5-1), $R(t)$ = rainfall input, $a = \frac{K}{\nu m_{\text{por}} \theta_e}$ = coefficient with permeability K and viscosity ν

This PDE is essentially a continuity equation: rainfall adds water, diffusion spreads it along the channel and the porosity controls storage

BC's:

- At the far end of channel ($y = L_y$), no water leaves so $\frac{\partial h_m}{\partial y} = 0$
- At the canal end ($y = 0$), groundwater level equals canal level so $h_m(0, t) = h_{cm}(t)$

Canal Coupling

The canal is short ($L_c \approx 0.05\text{m}$) and has a weir at its end. Flow over the weir is critical, meaning velocity and depth satisfy Bernoulli's relation

$$h_e = \frac{2}{3} h_{cm} \quad Q_c = V_c h_e = \sqrt{g} \max\left(\frac{2 h_{cm}}{3}, 0\right)^{3/2}$$

This shows the canal outflow depends nonlinearly on canal depth. The canal equation links inflow from groundwater at $y=0$ with outflow at the weir, ensuring mass balance

Initial Condition

Both groundwater and canal levels ^{start} at zero
 $h_m(y, 0) = 0, h_{cm}(0) = 0$

Assumptions

- Flow is hydrostatic
- Rainfall raises groundwater directly, modulated by porosity and effective pore fraction
- No surface runoff is considered

Summary of Scenario

Formulation sets up a coupled PDE-ODE system

- PDE for groundwater diffusion along the channel
- ODE for canal water level evolution, linked via boundary flux at $y=0$

Questions

1) i) Write PDE-ODE system in conservative form

$$\frac{\partial}{\partial t} (w_y h_m) - a g \frac{\partial}{\partial y} (w_y h_m \frac{\partial}{\partial y} h_m) = \frac{w_y R}{m_{por} \theta_c}$$

Using Darcy velocity and hydrostatic scaling flux form gives $-a g \frac{\partial}{\partial y} (h_m \frac{\partial}{\partial y} h_m) = \frac{-a g}{2} \frac{\partial}{\partial y} (\frac{\partial}{\partial y} h_m^2) \leftarrow \text{diffusion term}$

BCs $\rightarrow y = L_y \quad \frac{\partial h_m}{\partial y} = 0 \leftarrow \text{No flow}$

$\rightarrow y = 0 \quad h_m(0, t) = h_{cm}(t) \leftarrow \text{Dirichlet}$

Canal ODE (mass balance)

Combine inflow from groundwater at $y=0$ and weir outflow at $y = -L_c$

$$\frac{\partial}{\partial t} (w_y h_{cm}) = w_y m_{por} \theta_c a g \frac{\partial}{\partial y} h_m \Big|_{y=0} - w_y \sqrt{g} \max\left(\frac{2}{3} h_{cm}, 0\right)^{3/2}$$

$$h_e = \frac{2}{3} h_{cm}, \quad Q_c = \sqrt{g} \max\left(\frac{2 h_{cm}}{3}, 0\right)^{3/2}$$

This can be used to eliminate the boundary flux term that appears above in the PDE weak form

~~ii) Derive weak form and expose boundary flux~~

ii) Multiply by a test function and integrate by parts (adjoint form)

scalar test function = $q(y)$ Integrate over $[0, L_y]$ integrating diffusion term by parts and using $\frac{\partial h_m}{\partial y} = 0$ at $y = L_y$

$$\int_0^{L_y} q \frac{\partial}{\partial t} (w_y h_m) dy - a g \int_0^{L_y} h_m \frac{\partial q}{\partial y} \frac{\partial h_m}{\partial y} dy + a g q(0) h_m \frac{\partial h_m}{\partial y} \Big|_{y=0}$$

$$= \int_0^{L_y} q \frac{w_y R}{m_{por} \theta_c} dy \leftarrow \text{weak form}$$

Boundary form for flux elimination: $a g q(0) \frac{1}{2} \frac{\partial}{\partial y} (h_m^2) \Big|_{y=0}$

iii) Eliminate boundary flux using canal equation

\rightarrow Solve for groundwater flux at $y=0$ and rewrite in weak form boundary form

$$a g q(0) \frac{1}{2} \frac{\partial h_m^2}{\partial y} \Big|_{y=0} \Rightarrow q(0) \left(\frac{L_c}{m_{por} \theta_c} h_{cm} - \frac{1}{m_{por} \theta_c} \sqrt{g} \max\left(\frac{2 h_{cm}}{3}, 0\right)^{3/2} \right)$$

PDE weak form has ~~unknown boundary~~ no unknown boundary flux; the canal variable $h_{cm}(t)$ becomes the boundary datum and a coupled unknown via the ODE

Basis: $\{\varphi_i(y)\}_{i=1}^N$ on $[0, L_y]$ continuous, piecewise linear

Field: $h_m(y, t) \approx \sum_{j=1}^N h_j(t) \varphi_j(y)$ with boundary node $h_1(t) = h_{cm}(t)$

Element wise and global forms

$$M_{ij} = \int_0^{L_y} \omega_y \varphi_i \varphi_j dy \leftarrow \text{Mass matrix}$$

$$F_i[h(t)] = -ag \int_0^{L_y} \omega_y h_m(y, t) \frac{\partial \varphi_i}{\partial y} \frac{\partial h_m(y, t)}{\partial y} dy \leftarrow \text{Non linear adjoint diffusion vector}$$

$$S_i(t) = \int_0^{L_y} \varphi_i \frac{\omega_y R(t)}{m_{por} \theta_e} dy \leftarrow \text{source - rainfall}$$

$$B_i(t) = \varphi_i(0) \left(\frac{L_c}{m_{por} \theta_e} h_{cm}(t) - \frac{1}{m_{por} \theta_e} \sqrt{g} \max\left(\frac{2h_{cm}(t)}{3}, 0\right)^{3/2} \right) \leftarrow \text{Boundary coupling vector (from flux elimination)}$$

v) Forward Euler ~~the~~ Explicit time stepping

$$\text{Groundwater: } M(h^{n+1} - h^n) = \Delta t (F[h^n] + S^n + B^n) \text{ with } h_1^{n+1} = h_{cm}^{n+1} \text{ at boundary node}$$

$$\text{Canal ODE: } h_{cm}^{n+1} = h_{cm}^n + \Delta t \frac{1}{L_c} \left(m_{por} \theta_e ag \left(\frac{\partial h_m^n}{\partial y} \right) \Big|_0 - \sqrt{g} \max\left(\frac{2h_{cm}^n}{3}, 0\right)^{3/2} \right)$$

vi) Time step estimate

$D(y, t) = ag h_m(y, t)$ For explicit scheme on a uniform mesh with spacing Δy , heat equation stability bound through forward Euler is

$$\Delta t \leq \frac{1}{2} \frac{\Delta y^2}{D_{max}} \quad \text{where } D(y, t) = ag h_m(y, t) \text{ and } D_{max} \text{ is maximum diffusivity over domain at current time level}$$

$$\hookrightarrow D_{max} = \max_y (ag h_m^n(y))$$

$$\Rightarrow \Delta t \leq \frac{1}{2} \frac{\Delta y^2}{\max_y (ag h_m^n(y))} \quad \text{As } h_m \text{ grows from rain fall, } D \text{ increases and the stable } \Delta t \text{ decreases}$$

Consider Canal:

$$h_{cm} = \frac{1}{L_c} \left(m_{por} \theta_e ag \frac{\partial h_m}{\partial y} \Big|_0 - \sqrt{g} \left(\frac{2}{3} h_{cm} \right)^{3/2} \right)$$

The derivative of the outflow term wrt h_{cm} is

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$$\frac{d}{dh_{cm}} \left[\frac{1}{L_c} \sqrt{g} \left(\frac{2}{3} h_{cm} \right)^{3/2} \right] = \frac{\sqrt{g}}{L_c} \left(\frac{2}{3} \right)^{3/2} \frac{3}{2} h_{cm}^{1/2}$$

$$\Rightarrow \Delta t \leq \frac{\eta}{\frac{\sqrt{g}}{L_c} \left(\frac{2}{3} \right)^{3/2} \frac{3}{2} \max(h_{cm}^n)^{1/2}}$$

For final timestep estimate take minimum of the diffusion and convective bounds

$$\Delta t = \min \left\{ \frac{1}{2} \frac{\Delta y^2}{\max_y (a g h_m^n(y))}, \frac{\eta L_c}{\sqrt{g} \left(\frac{2}{3} \right)^{3/2} \frac{3}{2} \max(h_{cm}^n)^{1/2}} \right\}$$