Numerics Exercise 3 - Hannah Franklin

Part 2 (Ground Water):

(1) $\partial_t (w_v h_m) - \alpha g\, \partial_y (w_v h_m\, \partial_y h_m) = \dfrac{w_v R}{m_{por}\, \sigma_e}$, $h_m : h_m(y,t)$

$w_v \approx 0.1\,m$

$y \in [0, L_y]$, $L_y \approx 0.85\,m$

$R = R(t)$ rainfall

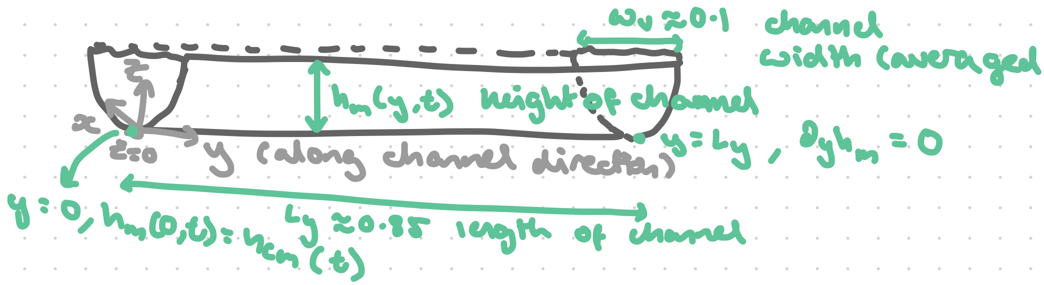
$m_{por} \in [0.1, 0.3]$, $\sigma_e \in [0.5, 1]$ is fraction of pore that can be filled w/ water due to residual air

$\alpha = \dfrac{k}{\nu\, m_{por}\, \sigma_e}$, $k \in [10^{-8}, 10^{-9}]\,m^2$ permeability, $\nu = 10^{-6}\,m^2 s^{-1}$ viscosity.

BCs : no flow at $y = L_y$ $\Rightarrow$ $\partial_y h_m = 0$

Dirichlet at $y = 0$ $\Rightarrow$ $h_m(0,t) = h_{cm}(t)$

IC : $h_m(y,0) = h_{m_0}(y)$



assumptions :
- ground water level stays underground - no surface run-off & flow is hydrostatic.
  Variations in y-direction much longer than vertical length scales.
- canal level $h_{cm}(t)$ holds in a short channel of length L_c btw $-L_c < y < 0$ w/ weir at $y = -L_c$.
  ↳ water level at weir critical $\Rightarrow$ flow speed $V_c = \sqrt{g h_c}$ critical at $y = -L_c$
- stationarity & Bernoulli's eq $\Rightarrow$ at weir:

$$g h_{cm} + \tfrac{1}{2} V_{cm}^2 \approx g h_{cm} = g h_c + \tfrac{1}{2} V_c^2 = \tfrac{3}{2} g h_c$$

$$\Rightarrow h_c = \tfrac{2}{3} h_{cm} \quad \text{s.t.} \quad Q_c = h_c V_c = \sqrt{g}\, \max\left(\tfrac{2 h_{cm}}{3}, 0\right)^{3/2}$$

assuming $V_{cm}^2 \ll g h_{cm} \approx 0$

(1) $\Rightarrow$ $\partial_t (w_v h_m) + \partial_y (v h_m) = \dfrac{w_v R}{m_{por}\, \sigma_e}$, $v = -\dfrac{k g}{\nu\, m_{por}\, \sigma_e}\, \partial_y h_m$

↳ v: Darcy velocity

Darcy flux: $Q \equiv w_v q \equiv w_v h_m v$
$$= -w_v h_m \frac{k}{\mu} \partial_y p = -w_v h_m \frac{k}{\nu \rho_0} \partial_y p$$
$$\approx -w_v h_m \frac{kg}{\nu} \partial_y h_m = -w_v h_m \frac{kg}{\nu m_{por} \sigma_e} \partial_y h_m = -w_v \alpha g \partial_y \left(\frac{h_m^2}{2}\right)$$

$\hookrightarrow \frac{\partial_y p}{\rho_0} \approx g \partial_y h_m$

if no y-dependence: $\partial_t h_m = \frac{R}{m_{por} \sigma_e}$

$\Rightarrow$ ($h_m = 0$ at $t=0$) $h_m = \frac{tR}{m_{por} \sigma_e}$ $\Rightarrow m_{por} = \sigma_e = 1$

canal level modelled by outflow at $y=-L_c$ and inflow at $y=0$:
$$L_c w_v \frac{dh_{cm}}{dt} = m_{por} \sigma_e Q_0 - Q_c \equiv w_v m_{por} \sigma_e \tfrac{1}{2} \alpha g \partial_y (h_m^2)|_{y=0} - w_v \sqrt{g}\, \max\left(\tfrac{2}{3} h_{cm}(t), 0\right)^{3/2}$$

COMPLETE MODEL:

(11) $\partial_t (w_v h_m) - \alpha g \partial_y (w_v h_m \partial_y h_m) = \frac{w_v R}{m_{por} \sigma_e}$ in $y \in [0, L_y]$

(12) $\partial_y h_m = 0$ at $y = L_y$

(13) $h_m(0,t) = h_{cm}(t)$ at $y=0$

(14) $L_c w_v \frac{dh_{cm}}{dt} = w_v m_{por} \frac{\sigma_e}{2} \alpha g \partial_y (h_m^2)|_{y=0} - w_v \sqrt{g}\, \max\left(\tfrac{2}{3} h_{cm}(t), 0\right)^{3/2}$

(+ ICs for h_m and h_{cm})

1) Discretise 11-14 using explicit finite element space. Show how flux at $y=0$ can be eliminated.

multiple eq (11) by test function $q(y)$ and integrate over the length of the channel.

$$\Rightarrow \int_0^{L_y} q(y) \left[\partial_t (w_v h_m) - \tfrac{1}{2} \alpha g \partial_y (w_v \partial_y (h_m^2))\right] dy = \int_0^{L_y} \frac{w_v R\, q(y)}{m_{por} \sigma_e} dy$$

$$\Rightarrow \int_0^{L_y} \left[q\, \partial_t (w_v h_m) - \tfrac{1}{2} \alpha g\, q\, \partial_y (w_v \partial_y (h_m^2))\right] dy = \int_0^{L_y} \frac{w_v R q}{m_{por} \sigma_e} dy$$

note: $w_v \approx 0.1$m cst as averaged width

$$\Rightarrow \int_0^{L_y} \left[q\, \partial_t h_m - \tfrac{1}{2} \alpha g\, q\, \partial_y^2 (h_m^2)\right] dy = \int_0^{L_y} \frac{Rq}{m_{por} \sigma_e} dy$$

note: (for integration by parts)

$$\partial_y (q\, \partial_y (h_m^2)) = \partial_y q\, \partial_y (h_m^2) + q\, \partial_y^2 (h_m^2)$$
$$\Rightarrow q\, \partial_y^2 (h_m^2) = \partial_y (q\, \partial_y (h_m^2)) - \partial_y q\, \partial_y (h_m^2)$$

$$\Rightarrow \int_0^{L_y} q\,\partial_t h_m\,dy + \int_0^{L_y}\left[\tfrac{1}{2}\alpha g\,\partial_y q\,\partial_y(h_m^2) - \tfrac{1}{2}\alpha g\,\partial_y\left(q\,\partial_y(h_m^2)\right)\right]dy = \int_0^{L_y}\frac{Rq}{m_{por}\sigma_c}\,dy$$

$$\Rightarrow \int_0^{L_y}\left[q\,\partial_t h_m + \tfrac{1}{2}\alpha g\,\partial_y q\,\partial_y(h_m^2)\right]dy - \left[\tfrac{1}{2}\alpha g\,q\,\partial_y(h_m^2)\right]_0^{L_y} = \int_0^{L_y}\frac{Rq}{m_{por}\sigma_c}\,dy$$

$$\left[\tfrac{1}{2}\alpha g\,q\,\partial_y(h_m^2)\right]_0^{L_y} = \left[\alpha g\,q\,h_m\,\partial_y h_m\right]_0^{L_y} = -\tfrac{1}{2}\alpha g\,q\big|_{y=0}\,\partial_y(h_m^2)\big|_{y=0}$$

since $\partial_y h_m\big|_{L_y} = 0$ (12)

$$\Rightarrow \int_0^{L_y}\left[q\,\partial_t h_m + \tfrac{1}{2}\alpha g\,\partial_y q\,\partial_y(h_m^2)\right]dy + \tfrac{1}{2}\alpha g\,q\big|_{y=0}\,\partial_y(h_m^2)\big|_{y=0} = \int_0^{L_y}\frac{Rq}{m_{por}\sigma_c}\,dy$$

(14): $$L_c w_v \frac{dh_{cm}}{dt} = w_v m_{por}\frac{\sigma_c}{2}\alpha g\,\partial_y(h_m^2)\big|_{y=0} - w_v\sqrt{g}\,\max\left(\tfrac{2}{3}h_{cm}(t),0\right)^{3/2}$$

$$\Rightarrow \tfrac{1}{2}\partial_y(h_m^2)_{y=0} = \frac{2}{w_v m_{por}\sigma_c\alpha g}\left[L_c w_v\frac{dh_{cm}}{dt} + w_v\sqrt{g}\,\max\left(\tfrac{2}{3}h_{cm}(t),0\right)^{3/2}\right]$$

$$\Rightarrow \tfrac{1}{2}\alpha g\,\partial_y(h_m^2)_{y=0} = \frac{1}{m_{por}\sigma_c}\left[L_c\frac{dh_{cm}}{dt} + \sqrt{g}\,\max\left(\tfrac{2}{3}h_{cm}(t),0\right)^{3/2}\right]$$

$$\Rightarrow \int_0^{L_y}\left[q\,\partial_t h_m + \tfrac{1}{2}\alpha g\,\partial_y q\,\partial_y(h_m^2)\right]dy + \frac{q(0)}{m_{por}\sigma_c}\left[L_c\frac{dh_{cm}}{dt} + \sqrt{g}\,\max\left(\tfrac{2}{3}h_{cm}(t),0\right)^{3/2}\right] = \int_0^{L_y}\frac{Rq}{m_{por}\sigma_c}\,dy$$

$$\Rightarrow \int_0^{L_y}\left[q\,\partial_t h_m\right]dy + \frac{q(0)L_c}{m_{por}\sigma_c}\frac{dh_{cm}}{dt} = \int_0^{L_y}\left[\frac{Rq}{m_{por}\sigma_c} - \tfrac{1}{2}\alpha g\,\partial_y q\,\partial_y(h_m^2)\right]dy - \frac{q(0)\sqrt{g}}{m_{por}\sigma_c}\max\left(\tfrac{2}{3}h_{cm}(t),0\right)^{3/2}$$

$$\Rightarrow \int_0^{L_y}\left[q\,\partial_t h_m\right]dy + \frac{q(0)L_c}{m_{por}\sigma_c}\frac{dh_{cm}}{dt} = \int_0^{L_y}\left[\frac{Rq}{m_{por}\sigma_c} - \tfrac{1}{2}\alpha g\,\partial_y q\,\partial_y(h_m^2)\right]dy - \frac{q(0)\sqrt{g}}{m_{por}\sigma_c}\max\left(\tfrac{2}{3}h_m(0,t),0\right)^{3/2}$$

this eliminates flux at $y=0$ using $h_m(0,t)=h_{cm}(t)$ at the last step to ensure continuity between the canal and the channel.

Note: test fnct $q(y)$ is unconstrained.

Explicit discretisation ($\theta = 0$):

using forward euler discretisation:

$$\partial_t h_m \approx \frac{h_m^{n+1} - h_m^n}{\Delta t}$$

$$\frac{dh_{cm}}{dt} \approx \frac{h_{cm}^{n+1} - h_{cm}^n}{\Delta t}$$

=> LHS of eq becomes:

$$\frac{1}{\Delta t}\left[\int_0^{L_y} q\, h_m^{n+1}\, dy + \frac{q(0)L_c}{m_{por}\sigma_e} h_{cm}^{n+1}\right] - \frac{1}{\Delta t}\left[\int_0^{L_y} q\, h_m^n\, dy + \frac{q(0)L_c}{m_{por}\sigma_e} h_{cm}^n\right]$$

$$\Rightarrow \int_0^{L_y} q\, h_m^{n+1}\, dy + \frac{q(0)L_c}{m_{por}\sigma_e} h_{cm}^{n+1} = \int_0^{L_y} q\, h_m^n\, dy + \frac{q(0)L_c}{m_{por}\sigma_e} h_{cm}^n$$
$$+ \Delta t \int_0^{L_y} \left[q \frac{R^n}{m_{por}\sigma_e} - \alpha g h_m^n \partial_y q\, \partial_y h_m^n\right] dy$$
$$- \Delta t \frac{q(0)\sqrt{g}}{m_{por}\sigma_e} \max\left(\tfrac{2}{3} h_{cm}^n, 0\right)^{3/2}$$

taking $q_i = \varphi_i(x)$, $h_m = h_j \varphi_j(x)$ $\forall i,j = 1, \dots, N_n$ w/ N_n no. nodes.
$h_{cm} = h_1$, $q(0) = \varphi_i \delta_{i1}$

LHS : $\int_0^{L_y} \varphi_i \varphi_j h_j^{n+1}\, dy + \frac{L_c}{m_{por}\sigma_e} h_1^{n+1} \delta_{i1}$

$M_{ij} = \int_0^{L_y} \varphi_i \varphi_j\, dy \quad \Rightarrow M_{ij} h_j^{n+1} + \frac{L_c h_1^{n+1}}{m_{por}\sigma_e} \delta_{i1}$

$$\int_0^{L_y} q\, h_m^n\, dy + \frac{q(0)L_c}{m_{por}\sigma_e} h_{cm}^n \mapsto M_{ij} h_j^n + \frac{L_c h_1^n}{m_{por}\sigma_e} \delta_{i1}$$

This leads to:

$$M_{ij} h_j^{n+1} + \frac{L_c h_1^{n+1}}{m_{por}\sigma_e} \delta_{i1} = M_{ij} h_j^n + \frac{L_c h_1^n}{m_{por}\sigma_e} \delta_{i1} + \Delta t \int_0^{L_y} \frac{\varphi_i R^n}{m_{por}\sigma_e} dy$$
$$- \Delta t \int_0^{L_y} \alpha g h_m^n\, \partial_y \varphi_i\, \partial_y h_m^n\, dy$$
$$+ \Delta t \frac{\sqrt{g}}{m_{por}\sigma_e} \max\left(\tfrac{2}{3} h_1^n, 0\right)^{3/2} \delta_{i1}$$

$$M_{ij} h_j^{n+1} + \frac{L_c h_1^{n+1}}{m_{por}\sigma_e} \delta_{i1} = M_{ij} h_j^n + \frac{L_c h_1^n}{m_{por}\sigma_e} \delta_{i1} + \Delta t\, b_i^n - \Delta t \frac{\sqrt{g}}{m_{por}\sigma_e} \max(\dots)^{3/2} \delta_{i1}$$

where $b_i^n = \int_0^{L_y} \left(\frac{\varphi_i R^n}{m_{por}\sigma_e} - \alpha g h_m^n \partial_y \varphi_i \partial_y h_m^n \right) dy$

consider again eq (11):

$$\partial_t (w_v h_m) - \alpha g \partial_y (w_v h_m \partial_y h_m) = \frac{w_v R}{m_{por}\sigma_e}$$

$$\Rightarrow \partial_t (h_m) - \alpha g \partial_y (h_m \partial_y h_m) = \frac{R}{m_{por}\sigma_e}$$

a finite difference discretisation (forward Euler):

$$\Rightarrow \frac{h_i^{n+1} - h_i^n}{\Delta t} - \frac{\alpha g}{\Delta y}\left[h_{i+1/2}^n \partial_y h_{i+1/2}^n - h_{i-1/2}^n \partial_y h_{i-1/2}^n \right] = \frac{R^n}{m_{por}\sigma_e}$$

$$\Rightarrow \frac{h_i^{n+1} - h_i^n}{\Delta t} - \frac{\alpha g}{(\Delta y)^2}\left[h_{i+1/2}^n (h_{i+1}^n - h_i^n) - h_{i-1/2}^n (h_i^n - h_{i-1}^n) \right] = \frac{R^n}{m_{por}\sigma_e}$$

$$h_{i+1/2}^n = \frac{h_{i+1}^n + h_i^n}{2}, \quad h_{i-1/2}^n = \frac{h_i^n + h_{i-1}^n}{2}$$

$$\Rightarrow \frac{h_i^{n+1} - h_i^n}{\Delta t} - \frac{\alpha g}{2(\Delta y)^2}\left[(h_{i+1}^n + h_i^n)(h_{i+1}^n - h_i^n) - (h_i^n + h_{i-1}^n)(h_i^n - h_{i-1}^n) \right] = \frac{R^n}{m_{por}\sigma_e}$$

$$\Rightarrow \frac{h_i^{n+1} - h_i^n}{\Delta t} - \frac{\alpha g}{2(\Delta y)^2}\left[(h_{i+1}^n)^2 - 2(h_i^n)^2 + (h_{i-1}^n)^2 \right] = \frac{R^n}{m_{por}\sigma_e}$$

$i \mapsto j$ for time step criterion calculation

letting $h_j^n = \lambda^n e^{ikj\Delta y}$

$$\Rightarrow \left(\frac{\lambda - 1}{\Delta t} - \frac{\alpha g}{2(\Delta y)^2}\left[e^{2ik\Delta y} - 2 + e^{-2ik\Delta y} \right] \right) h_j^n = \frac{R^n}{m_{por}\sigma_e}$$

$$\frac{\lambda - 1}{\Delta t} - \frac{\alpha g}{2(\Delta y)^2} \left(4 (i)^2 \sin^2 (k\Delta y) \right) = \frac{R^n}{m_{por}\sigma_e} \frac{1}{h_j^n}$$

$$\lambda = 1 - \frac{2\Delta t}{(\Delta y)^2} \alpha g \sin^2 (k \Delta y) + \frac{\Delta t R^n}{m_{por}\sigma_e} \frac{1}{h_j^n}$$

require $|\lambda| \leq 1$ for scheme to be stable.

$$\Rightarrow \left| (\Delta t) \left(\frac{R^n}{M_{por}\sigma_e} \frac{1}{h_j^n} - \frac{2\alpha g}{(\Delta y)^2} \sin^2(k\Delta y) \right) \right| \leq 1$$

$$\Rightarrow \Delta t \leq \left| \frac{1}{\frac{R^n}{M_{por}\sigma_e} \frac{1}{h_j^n} - \frac{2\alpha g}{(\Delta y)^2} \sin^2(k\Delta y)} \right|$$

$$\leq \left| \frac{1}{\frac{R^n}{M_{por}\sigma_e} \frac{1}{\max(h_j^n)} + \underbrace{\frac{2\alpha g}{(\Delta y)^2}}} \right|$$

this term dominates since $\Delta y >>$ vertical length scales

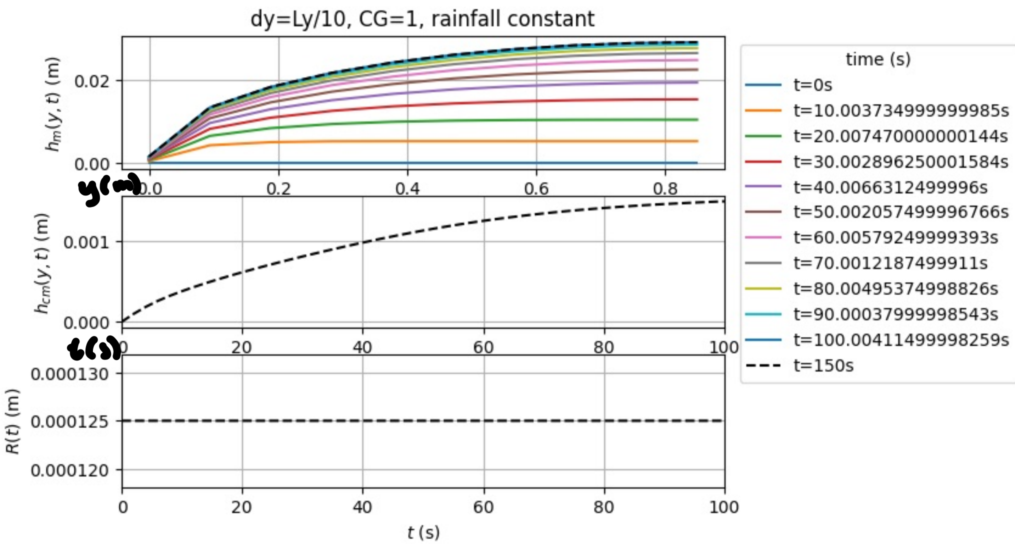
from assumptions $\Rightarrow \Delta y >> h_j^n \quad \forall j,n$

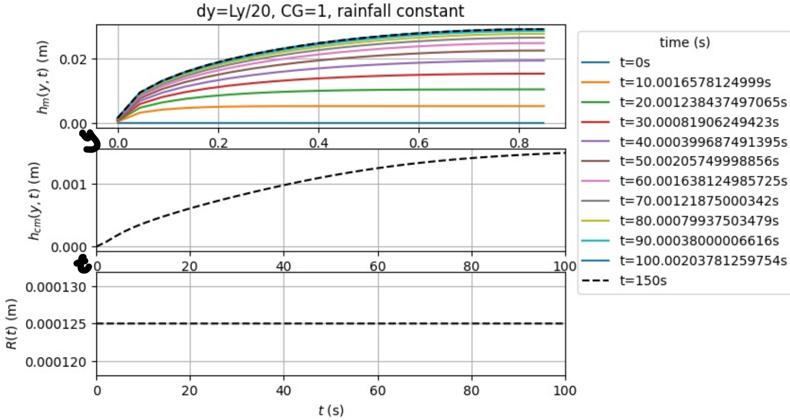
$$\Rightarrow \Delta t \lesssim \frac{(\Delta y)^2}{2\alpha g}$$

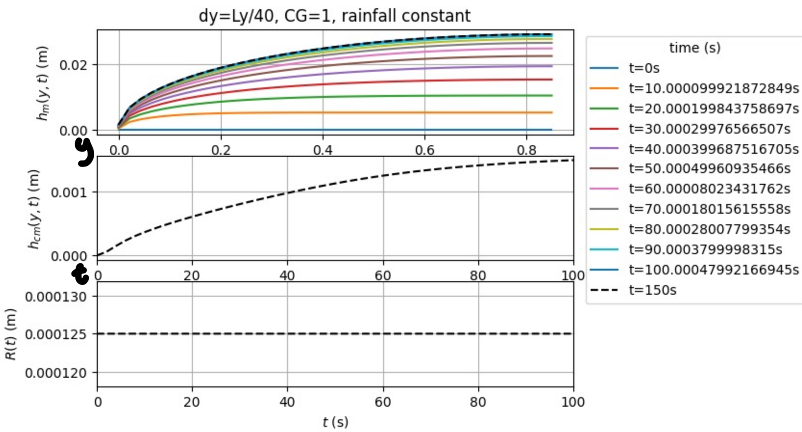
2) Numerical simulations for explicit scheme w/ parameter values:

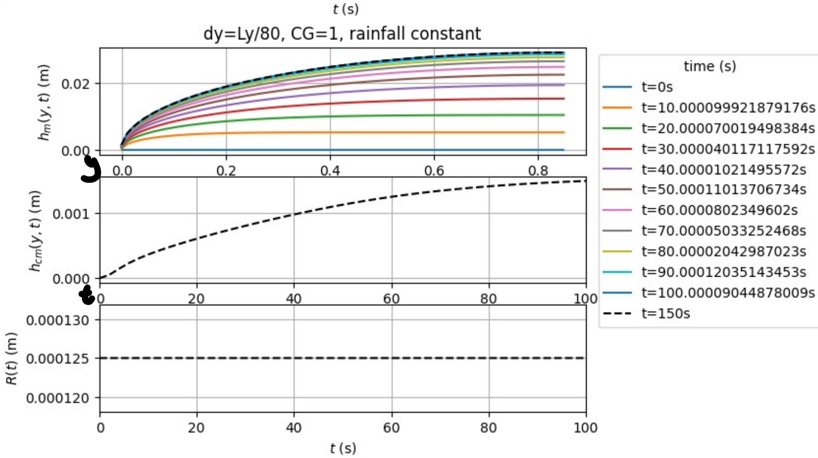
$M_{por} = 0.3$, $\sigma_e = 0.8$, $L_y = 0.85m$, $K = 10^{-8}\ m^2$, $\omega_v = 0.1m$,

$R_{max} = 0.000125\ ms^{-1}$, $L_c = 0.05m$

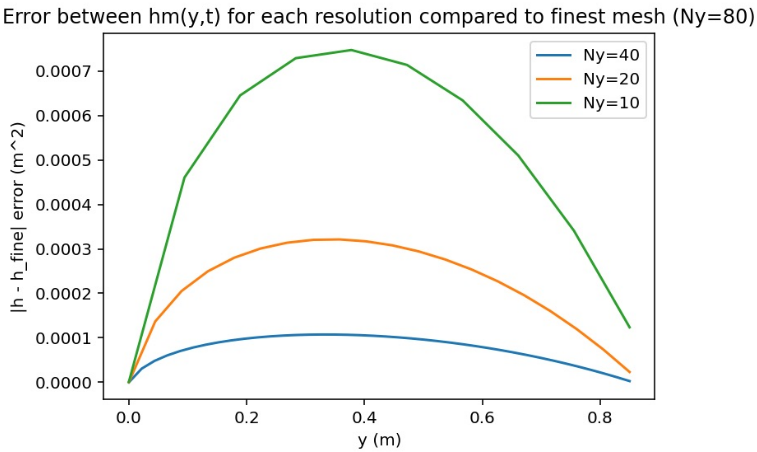


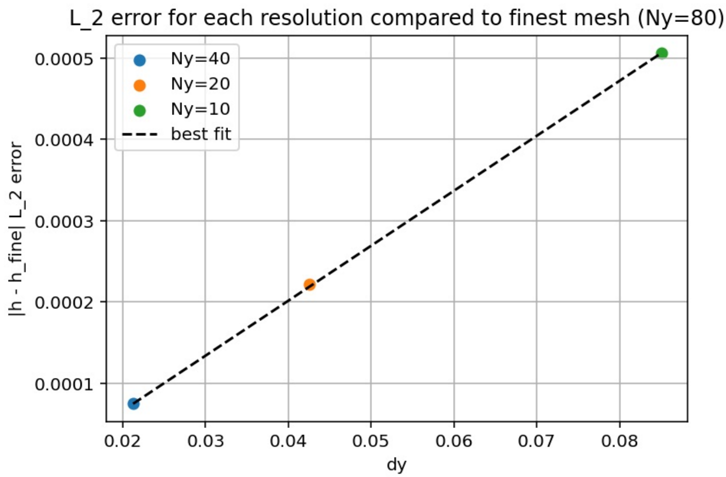






Error analysis is done using $|h_m(y,t) - h_{m\_fine}(y,t)|$ and the L_2 norm. Where h_{m_fine} is the h_m profile when $N_y = 80$/ $\Delta y = \frac{L_y}{80}$



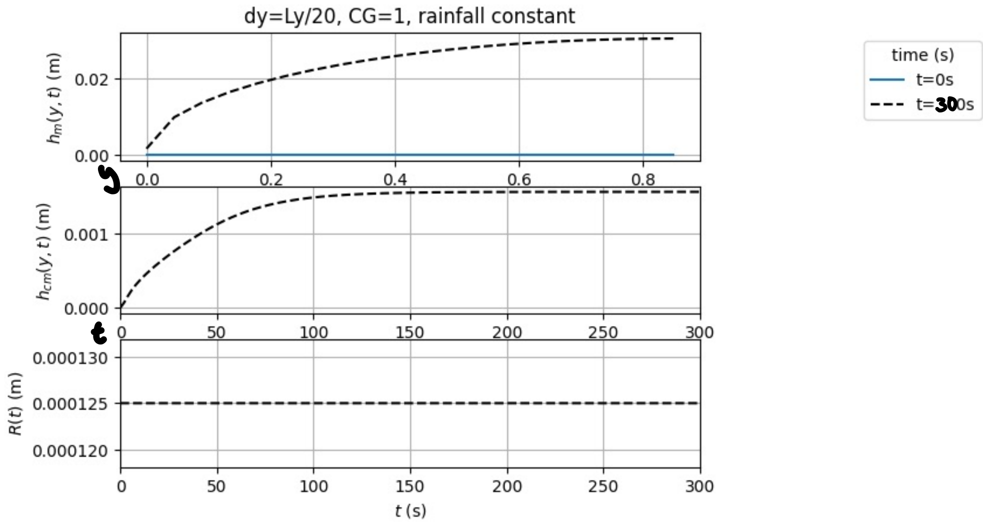


Plotting $|h_m - h_{m\_fine}|$ for each Δy at $t=100s$ shows that numerical convergence is occuring.

Plotting the L_2 error as a function of Δy again suggests numerical convergence is occuring.

Plot above for CG1 shows a linear relationship btw the L_2 error and Δy as is expected for order 1 scheme.

The system has not quite reached steady state at $t=100s$.

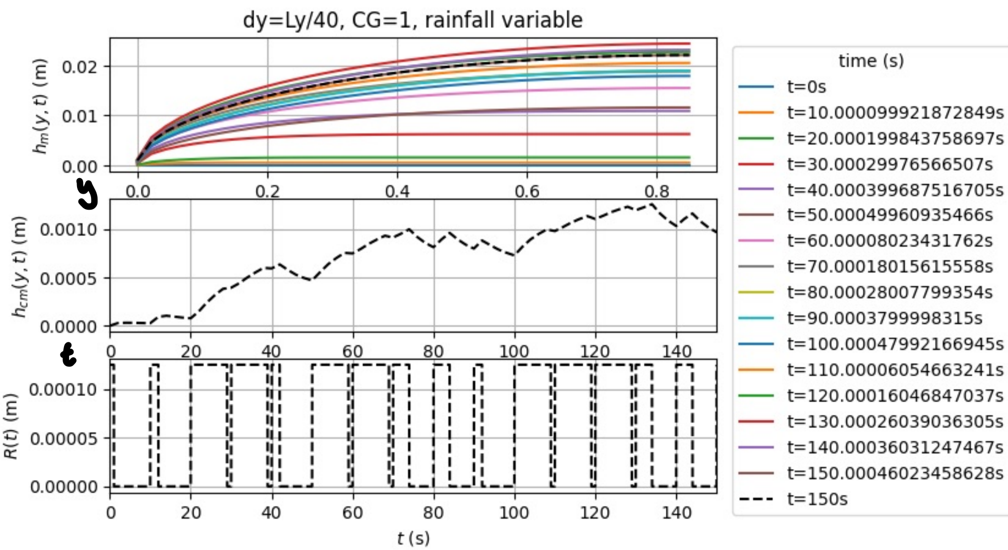


Plotting the final $h_m(y)$ profile at $t=300s$ shows the steady-state $h_m(y)$ profile.

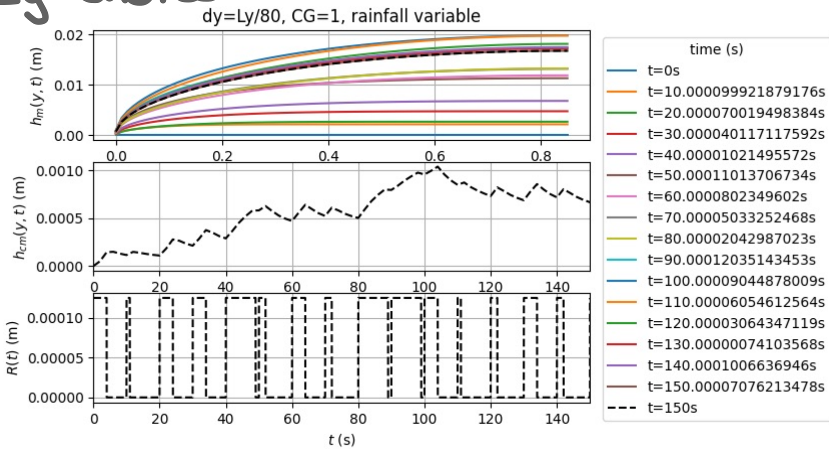
The steady state value of $h_{cm}(t)$:

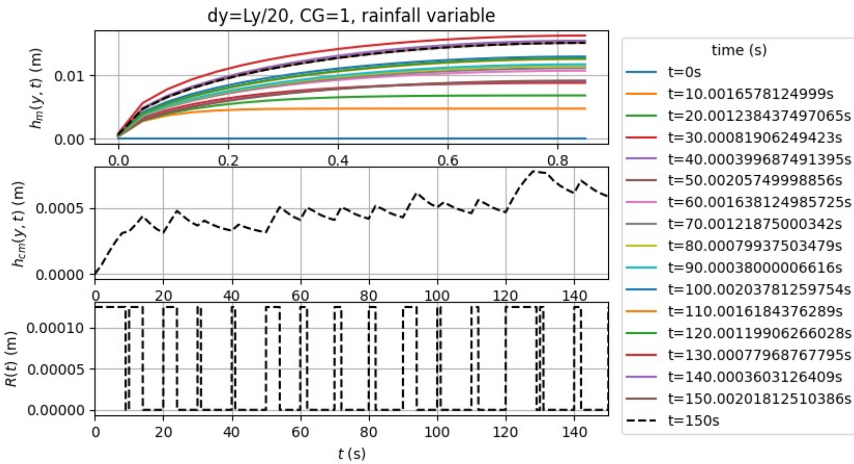
$$h_{cm}(300) = 0.00157$$
$$= h_{cm} \text{ steady state value}$$

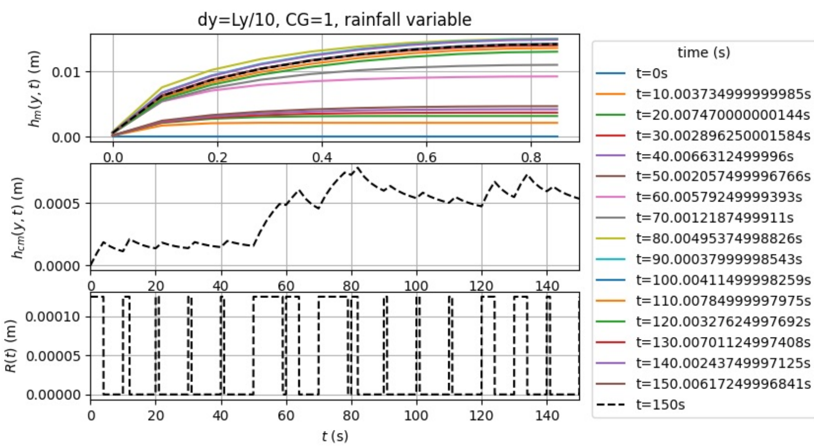
Implementing a variable rainfall function & plotting $h_m(y,t)$, $h_{cm}(t)$, $R(t)$



Other Δy choices:







Code uses randomly chooses a rain duration from 1s, 2s, 4s, 9s every 10s.

3) Crank-Nicolson scheme:

$\theta = 0.5$

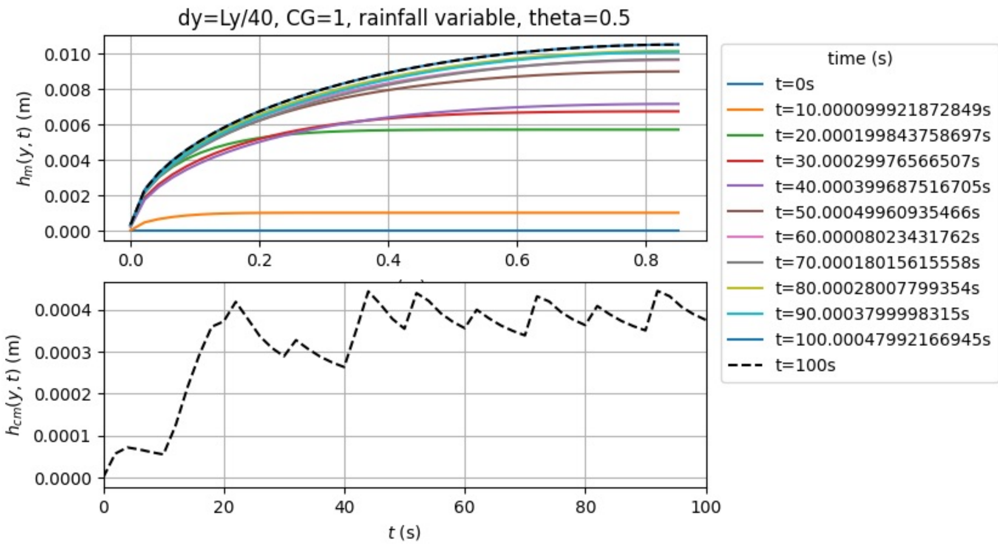
Crank-Nicolson weak formulation:

$$\int_0^{L_y} q\, h_m^{n+1}\, dy + \frac{L_c\, q(0)\, h_{cm}^{n+1}}{m_{por}\sigma_e} + \frac{1}{2}\Delta t \int_0^{L_y} \left(\alpha g\, h_m^{n+1}\, \partial_y q\, \partial_y (h_m^{n+1})\right) dy$$
$$+ \frac{1}{2}\Delta t \frac{\sqrt{g}\, q(0)}{m_{por}\sigma_e} \max\left(\frac{2}{3} h_{cm}^{n+1}, 0\right)^{3/2}$$
$$= \int_0^{L_y} q\, h_m^n\, dy + \frac{L_c\, q(0)\, h_{cm}^n}{m_{por}\sigma_e} + \frac{1}{2}\Delta t \int_0^{L_y} \left[-\alpha g\, h_m^n\, \partial_y q\, \partial_y (h_m^n) + \frac{q(R^n + R^{n+1})}{m_{por}\sigma_e}\right] dy$$
$$- \frac{1}{2}\Delta t \frac{\sqrt{g}\, q(0)}{m_{por}\sigma_e} \max\left(\frac{2}{3} h_{cm}^n, 0\right)^{3/2}$$

$q = \varphi_i, \quad h_m = h_j \varphi_j, \quad h_{cm} = h_1 \qquad q(0) = \delta_{i1}$

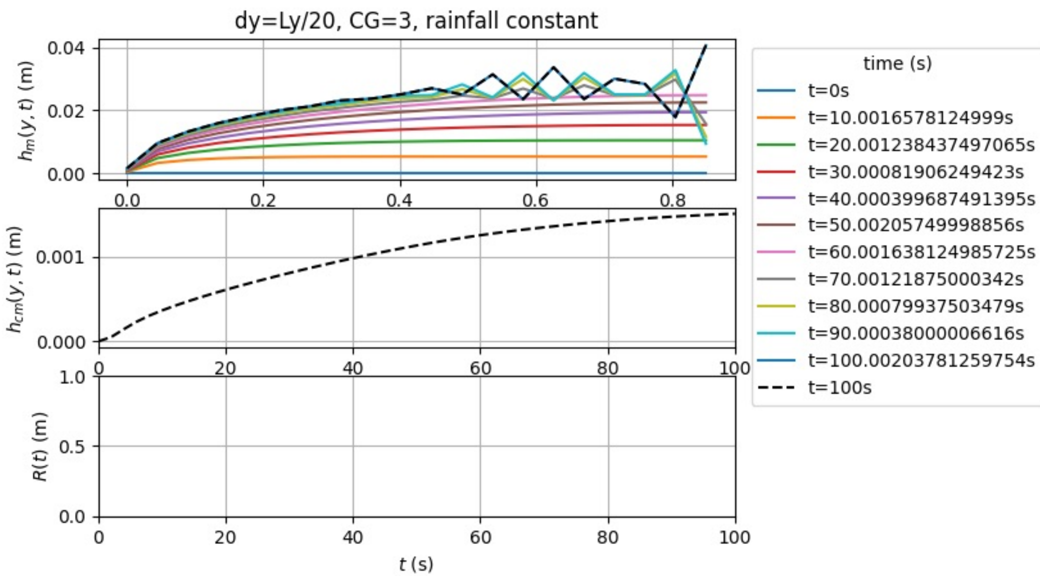
$$\Rightarrow \int_0^{L_y} \varphi_i h_j^{n+1} \varphi_j\, dy + \frac{L_c h_1^{n+1}}{m_{por}\sigma_e}\delta_{i1} + \frac{1}{2}\Delta t \int_0^{L_y} \left[\alpha g\, h_m^n\, \partial_y \varphi_i\, \partial_y h_m^n\right] dy$$
$$+ \frac{1}{2}\Delta t \frac{\sqrt{g}}{m_{por}\sigma_e} \max\left(\frac{2}{3} h_1^{n+1}, 0\right)^{3/2} \delta_{i1}$$
$$= \int_0^{L_y} \varphi_i h_j^n \varphi_j\, dy + \frac{L_c h_1^n}{m_{por}\sigma_e}\delta_{i1} + \frac{1}{2}\Delta t \int_0^{L_y} \left[-\alpha g\, h_m^n\, \partial_y \varphi_i\, \partial_y h_m^n\right] dy$$
$$- \frac{1}{2}\Delta t \frac{\sqrt{g}}{m_{por}\sigma_e} \max\left(\frac{2}{3} h_1^n, 0\right)^{3/2} \delta_{i1}$$

$$\Rightarrow M_{ij} h_j^{n+1} + \frac{L_c}{m_{por}\sigma_e} h_1^{n+1}\delta_{i1} + \frac{1}{2}\Delta t\, b_i^{n+1} + \frac{1}{2}\Delta t \frac{\sqrt{g}\,\delta_{i1}}{m_{por}\sigma_e} \max\left(\frac{2}{3} h_1^{n+1}, 0\right)^{3/2}$$
$$= M_{ij} h_j^n + \frac{L_c}{m_{por}\sigma_e} h_1^n \delta_{i1} - \frac{1}{2}\Delta t\, b_i^n - \frac{1}{2}\Delta t \frac{\sqrt{g}\,\delta_{i1}}{m_{por}\sigma_e} \max\left(\frac{2}{3} h_1^{n+1}, 0\right)^{3/2}$$



4) CG 3:

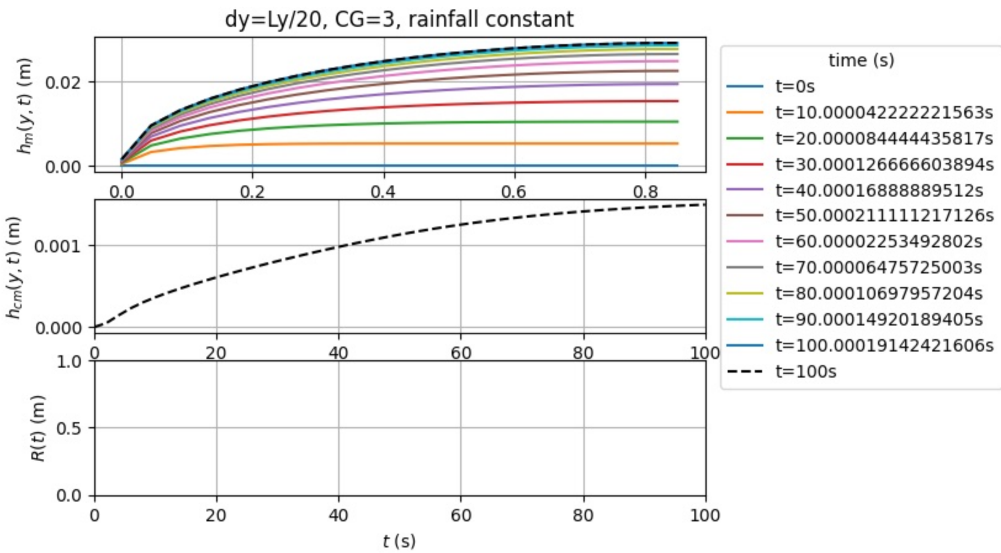
oscillations at Ly end of channel.



When CG 3 is used w/ explicit euler scheme we see oscillations at the end of the channel.

This is due to the time step used being s.t. the scheme is stable for a 1st order scheme.
However, for CFL = 2.3 the stability condition is violated at higher order solvers (such as CG = 3 example)

instead using $dt = \left(\frac{1}{3}\right)^2 \times \frac{1}{2} \times CFL \times (\Delta y)^2$:



Thus need to use a smaller CFL number for the scheme to be stable.