

1) Ground Water Model

Nonlinear diffusion equation : Describes how groundwater level evolves in time and space

$$\frac{\partial W_y h_m}{\partial t} - a g \frac{\partial}{\partial y} \left(W_y h_m \frac{\partial h_m}{\partial y} \right) = \frac{W_y R}{m_{\text{por}} \theta_e}$$

$h_m(y, t)$ = groundwater level above datum , W_y = channel width ($\sim 0.1\text{m}$) , m_{por} = porosity (0.1-0.3)

θ_e = effective pore fraction (0.5-1) , $R(t)$ = rainfall input , $a = \frac{K}{V m_{\text{por}} \theta_e}$ = coefficient with permeability K and viscosity V

This PDE is essentially a continuity equation : rainfall adds water, diffusion spreads it along the channel and the porosity controls storage

BC's :

- At the far end of channel ($y = L_y$) , no water leaves so $\frac{\partial h_m}{\partial y} = 0$
- At the canal end ($y=0$) , groundwater level equals canal level so $h_m(0, t) = h_{cm}(t)$

Canal Coupling

The canal is short ($L_c \approx 0.05\text{m}$) and has a weir at its end. Flow over the weir is critical, meaning velocity and depth satisfy Bernoulli's relation

$$h_e = \frac{2}{3} h_{cm} \quad Q_c = V_c h_e = \sqrt{g} \max \left(\frac{2 h_{cm}}{3}, 0 \right)^{3/2}$$

This shows the canal outflow depends nonlinearly on canal depth. The canal equation links inflow from groundwater at $y=0$ with outflow at the weir, ensuring mass balance

Initial Condition

Both groundwater and canal levels "at zero" $\stackrel{\text{start}}{=}$

$$h_m(y, 0) = 0, h_{cm}(0) = 0$$

Assumptions

- Flow is hydrostatic
- Rainfall raises groundwater directly, modulated by porosity and effective pore fraction
- No surface runoff is considered

Summary of Scenario

Formulation sets up a coupled PDE - ODE system

- ↳ PDE for groundwater diffusion along the channel
- ↳ ODE for canal water level evolution, linked via boundary flux at $y=0$

Questions

- I) i) Write PDE-ODE system in conservative form

$$\frac{\partial}{\partial t} (W_y h_m) - a g \frac{\partial}{\partial y} (W_y h_m \frac{\partial}{\partial y} h_m) = \frac{W_y R}{m_{\text{por}} \theta_e}$$

Using darcy velocity and hydrostatic scaling flux form gives $-a g \frac{\partial}{\partial y} (h_m \frac{\partial}{\partial y} h_m) = -\frac{a g}{2} \frac{\partial y}{\partial y} (\partial y (h^2))$ ^{diffusion term}

BC's $\rightarrow y = L_y \quad \frac{\partial h_m}{\partial y} = 0 \quad \leftarrow \text{No flow}$
 $\rightarrow y = 0 \quad h_m(0, t) = h_{cm}(t) \quad \leftarrow \text{Dirichlet}$

Canal ODE (mass balance)

Combine inflow from groundwater at $y=0$ and weir outflow at $y=L_c$

$$\frac{d}{dt} (W_y h_{cm}) = W_y m_{\text{por}} \theta_e a g \frac{\partial}{\partial y} h_m|_{y=0} + -W_y \sqrt{g} \max\left(\frac{2}{3} h_{cm}, 0\right)^{3/2}$$

$$h_c = \frac{2}{3} h_{cm}, \quad Q_c = \sqrt{g} \max\left(\frac{2 h_{cm}}{3}, 0\right)^{3/2}$$

This can be used to eliminate the boundary flux term that appears after in the PDE weak form

iii) Derive weak form and express boundary flux

ii) Multiply by a test function and integrate by parts (adjoint form)

scalar test function $= q_1(y)$ Integrate over $[0, L_y]$ integrating diffusion term by parts and using $\frac{\partial h_m}{\partial y} = 0$ at $y=L_y$

$$\int_0^{L_y} q_1 \frac{\partial}{\partial t} (W_y h_m) dy - a g \int_0^{L_y} h_m \frac{\partial q_1}{\partial y} \frac{\partial h_m}{\partial y} dy + a g q_1(0) h_m \frac{\partial h_m}{\partial y}|_{y=0}$$

$$= \int_0^{L_y} q_1 \frac{W_y R}{m_{\text{por}} \theta_e} dy \quad \leftarrow \text{weak form}$$

$$\text{Boundary form for flux elimination : } a g q_1(0) \frac{1}{2} \frac{\partial}{\partial y} (h^2)|_{y=0}$$

iii) Eliminate boundary flux using canal equation

\rightarrow Solve for groundwater flux at $y=0$ and rewrite in weak form boundary form

$$a g q_1(0) \frac{1}{2} \frac{\partial h_m^2}{\partial y}|_{y=0} \Rightarrow q_1(0) \left(\frac{L_c}{m_{\text{por}} \theta_e} h_{cm} - \frac{1}{m_{\text{por}} \theta_e} \sqrt{g} \max\left(\frac{2 h_{cm}}{3}, 0\right)^{3/2} \right)$$

PDE weak form has ~~unknown boundary~~ no unknown boundary flux; the canal variable $h_{cm}(t)$ becomes the boundary datum and a coupled unknown via the ODE

iv) Continuous Galerkin finite element setup

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Basis : $\{\varphi_i(y)\}_{i=1}^N$ on $[0, L_y]$ continuous, piecewise linear

Field : $h_m(y, t) \approx \sum_{j=1}^N h_j(t) \varphi_j(y)$ with boundary node $h_1(t) = h_{cm}(t)$

Element wise and global forms

$$M_{ij} = \int_0^{L_y} w_y \varphi_i \varphi_j dy \quad \leftarrow \text{Mass matrix}$$

$$F_i[h(t)] = -ag \int_0^{L_y} w_y h_m(y, t) \frac{\partial \varphi_i}{\partial y} \frac{\partial h_m}{\partial y} dy \quad \leftarrow \text{Nonlinear adjoint diffusion vector}$$

$$S_i(t) = \int_0^{L_y} \varphi_i \frac{w_y R(t)}{m_{per} \theta_e} dy \quad \leftarrow \text{Source-Rankine}$$

$$\beta_i(t) = \varphi_i(0) \left(\frac{L_c}{m_{per} \theta_e} h_{cm}(t) - \frac{1}{m_{per} \theta_e} \sqrt{g} \max\left(\frac{2h_m(t)}{3}, 0\right)^{3/2} \right) \quad \leftarrow \begin{array}{l} \text{Boundary coupling vector} \\ (\text{form flux elimination}) \end{array}$$

v) forward Euler & Explicit time stepping

Groundwater : $M(h^{n+1} - h^n) = \Delta t (F[h^n] + S^n + \beta^n)$ with $h_i^{n+1} = h_{cm}^{n+1}$ at boundary node

$$\text{Canal ODE : } h_{cm}^{n+1} = h_{cm}^n + \Delta t \frac{1}{L_c} \left(m_{per} \theta_e ag \left(\frac{\partial h_m}{\partial y} \right) \Big|_0 - \sqrt{g} \max\left(\frac{2h_m}{3}, 0\right)^{3/2} \right)$$

vi) Time step estimate

$D(y, t) = ag h_m(y, t)$ for explicit scheme on uniform mesh with spacing Δy , heat equation stability bound through forward Euler is

$$\Delta t \leq \frac{1}{2} \frac{\Delta y^2}{D_{max}} \quad \text{where } D(y, t) = ag h_m(y, t) \text{ and } D_{max} \text{ is maximum diffusivity over domain at current time level}$$

$$\Rightarrow \Delta t \leq \frac{1}{2} \frac{\Delta y^2}{\max(ag h_m(y))} \quad \hookrightarrow D_{max} = \max_y (ag h_m(y))$$

As h_m grows from rainfall, D increases and the stable Δt decreases

Consider Canal

$$h_{cm} = \frac{1}{L_c} \left(m_{per} \theta_e ag \frac{\partial h_m}{\partial y} \Big|_0 - \sqrt{g} \left(\frac{2}{3} h_{cm} \right)^{3/2} \right)$$

The derivative of the outflow term wrt h_{out} is

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$$\frac{d}{dh_{\text{out}}} \left[\frac{1}{L_c} \sqrt{g} \left(\frac{2}{3} h_{\text{out}} \right)^{3/2} \right] = \frac{\sqrt{g}}{L_c} \left(\frac{2}{3} \right)^{3/2} \frac{3}{2} h_{\text{out}}^{1/2}$$

$$\Rightarrow \Delta t \leq \frac{n}{\frac{\sqrt{g}}{L_c} \left(\frac{2}{3} \right)^{3/2} \frac{3}{2} \max(h_{\text{out}})^{1/2}}$$

for final timestep estimate take minimum of the diffusion and cfl bounds

$$\Delta t = \min \left\{ \frac{1}{2} \frac{\Delta y^2}{\max_y(\alpha g h_m(y))}, \frac{n L_c}{\sqrt{g} \left(\frac{2}{3} \right)^{3/2} \frac{3}{2} \max(h_{\text{out}})^{1/2}} \right\}$$