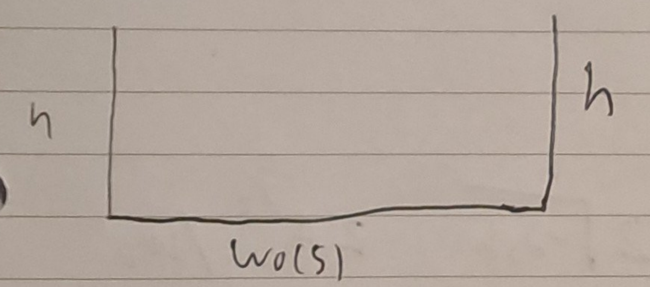
1) By def^n the wetted area A is the ~~height~~ water depth multiplied by the river width.

Hence, $A = h w_0(s) \Rightarrow h(A,s) = \frac{A}{w_0(s)}$, (Clearly $w_0(s) > 0$).

For the perimeter consider the sketch



$\therefore P = w_0(s) + 2h(A,s) = w_0(s) + \frac{2A}{w_0(s)}$. $\square$.

If $S=0$ ~~(3)~~ eq^n ~~(4)~~ in the question sheet yields

$\partial_t A + \partial_s F(A,s) = 0.$

$\Rightarrow \partial_t A + \frac{\partial F}{\partial A}\frac{\partial A}{\partial s} + \frac{\partial F}{\partial s} = 0$ (By chain rule).

$\Rightarrow \partial_t A + \frac{\partial F}{\partial A}\partial_s A + \frac{\partial F}{\partial s} = 0$ $\square$.

$F = \frac{A^{\frac{5}{3}}\sqrt{-\partial_s b}}{C_m P(A,s)^{\frac{2}{3}}}$

$\cancel{\frac{\partial F}{\partial A} = \frac{5A}{3} \frac{5}{3P(A,s)^{\frac{2}{3}}}}$

(2)

$$\frac{\partial F}{\partial A} = \frac{5\sqrt{-\partial_s b}}{3C_m} \frac{A^{2/3}}{P^{2/3}(A,s)} - \frac{2\sqrt{-\partial_s b}}{3C_m} \frac{\frac{\partial P}{\partial A} A^{5/3}}{P(A,s)^{5/3}}$$

$$P = w_0(s) + \frac{2A}{w_0(s)}$$

$$\Rightarrow \frac{\partial P}{\partial A} = \frac{2}{w_0(s)}$$

$$\therefore \frac{\partial F}{\partial A} = \frac{5\sqrt{-\partial_s b}}{3C_m} \frac{A^{2/3}}{\left(w_0 + \frac{2A}{w_0}\right)^{2/3}} - \frac{2\sqrt{-\partial_s b}}{3C_m} \frac{2A^{5/3}/w_0}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}}$$

$$= \frac{\sqrt{-\partial_s b}}{3C_m} \frac{\left(5w_0 A^{2/3} + 6A^{5/3}/w_0\right)}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}} > 0.$$

Similarly,

$$\frac{\partial F}{\partial S} = \frac{-A^{5/3}\partial_{ss} b}{2C_m P(A,s)^{2/3}\sqrt{-\partial_s b}} - \frac{2\sqrt{-\partial_s b}}{3C_m} \frac{A^{5/3}}{P(A,s)^{5/3}} \frac{\partial P}{\partial S}.$$

Now, assuming the river slope is constant, or that the gradient difference in the slope is small we have:

$$\frac{\partial F}{\partial S} = -\frac{2\sqrt{-\partial_s b}}{3C_m} \frac{A^{5/3}}{\left(w_0(s) + \frac{2A}{w_0(s)}\right)^{5/3}} \frac{\partial P}{\partial w_0} \frac{dw_0}{dS}.$$

$$= \frac{-2\sqrt{-\partial_s b}}{3C_m} \frac{A^{5/3}}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}} \left(1 - \frac{2A}{w_0^2}\right) \frac{dw_0}{ds} \quad \square$$

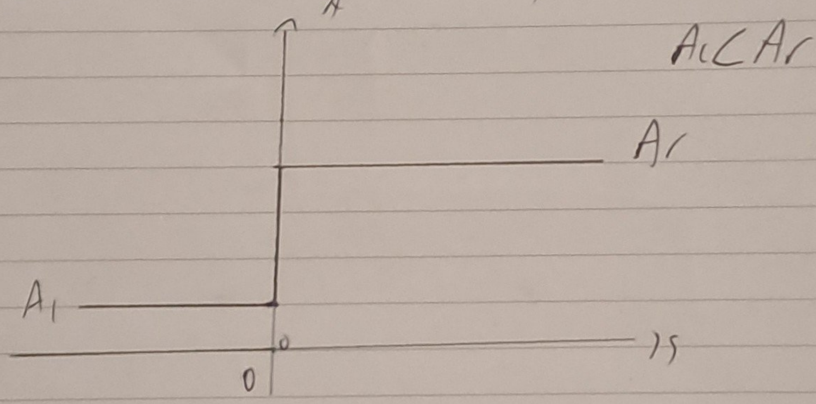
7/. In the limit $w_0(S)$ is independent of S or $w_0 \sin(v_0 S)$ slowly s.t its dependence on S can be ignored we have $\frac{\partial F}{\partial S} = 0$.

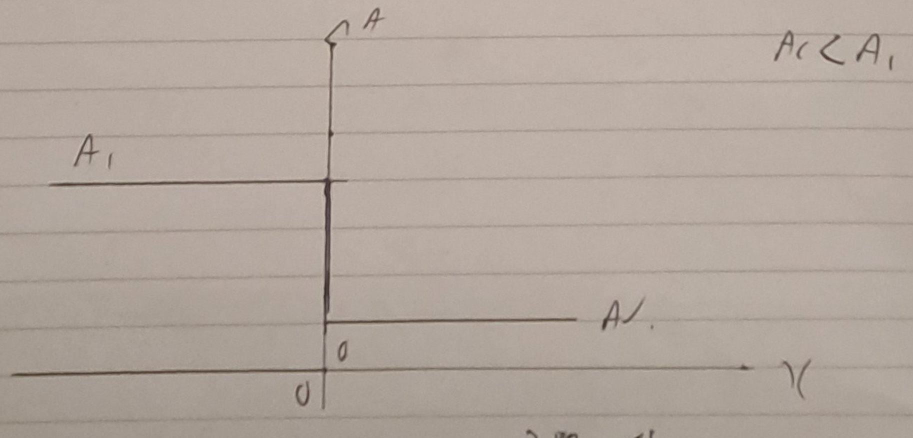
To solve the Riemann problem, we start with the initial data

$$A(S,0) = \begin{cases} A_l & S < 0 \\ A_r & S > 0. \end{cases}$$

Where either $A_l < A_r$ OR $A_r > A_l$.

∴ Initial data sketch is of the following





Now, by def^n we have $\frac{dS}{dt} = \lambda(A) = \frac{\partial F}{\partial A}$

$$\frac{dS}{dt} = \lambda(A) = \frac{\partial F}{\partial A} = \frac{\sqrt{-\partial_S b}}{3C_m} \frac{(5w_0 A^{2/3} + 6A^{5/3}/w_0)}{\left(w_0 + \frac{2A}{w_0}\right)^{5/3}} > 0.$$

Hence, as $\lambda > 0$ we have all characteristics of the form $S = \lambda t$. Hence, the characteristics are all moving to the right i.e we have right travelling waves

Now, to solve the Riemann problem we first need to determine whether the waves we are dealing with are rarefaction waves or shock waves.

~~Now~~ Clearly if $A_R > A_L$

$$\Rightarrow \lambda(A_R) > \lambda(A_L)$$

~~This discontinuity at 0 does not persist~~ Hence, by defn we have rarefaction waves

$\rightarrow$

5

By Section 11.10 in the Finite Volume Methods for Hyperbolic Equations book by Leveque, the sol^n to a conservation law is a similarity sol^n, ~~of~~ a function of $\frac{s}{t}$ alone s.t we have the sol^n

$$A(s,t) = \tilde{A}\left(\frac{s}{t}\right) \therefore \text{we have } \partial_t \tilde{A} + \partial_s F\left(\tilde{A}\left(\tfrac{s}{t}\right)\right) = 0$$

$$\cancel{\Rightarrow \tilde{A}' = -\frac{s}{t^2}\tilde{A} + F'(\tilde{A})}$$

$\therefore$ we have

$$-\frac{s}{t^2}\tilde{A}' + \frac{1}{t}\tilde{A}' F'(\tilde{A}) = 0.$$

$$\therefore \text{Either } \tilde{A}'\left(\frac{s}{t}\right) = 0 \text{ OR } F'(\tilde{A}) = \frac{s}{t}.$$

In other words either A = constant along the lines $\frac{s}{t}$ = const.

OR ~~the flux speed matches the characteristic speed~~. the ray ~~the~~ speed $\frac{s}{t}$ matches the local characteristic speed $F'(A)$, this is the rarefaction fan relation.

$$\therefore \text{ If } F'(\tilde{A}) = \frac{s}{t}$$

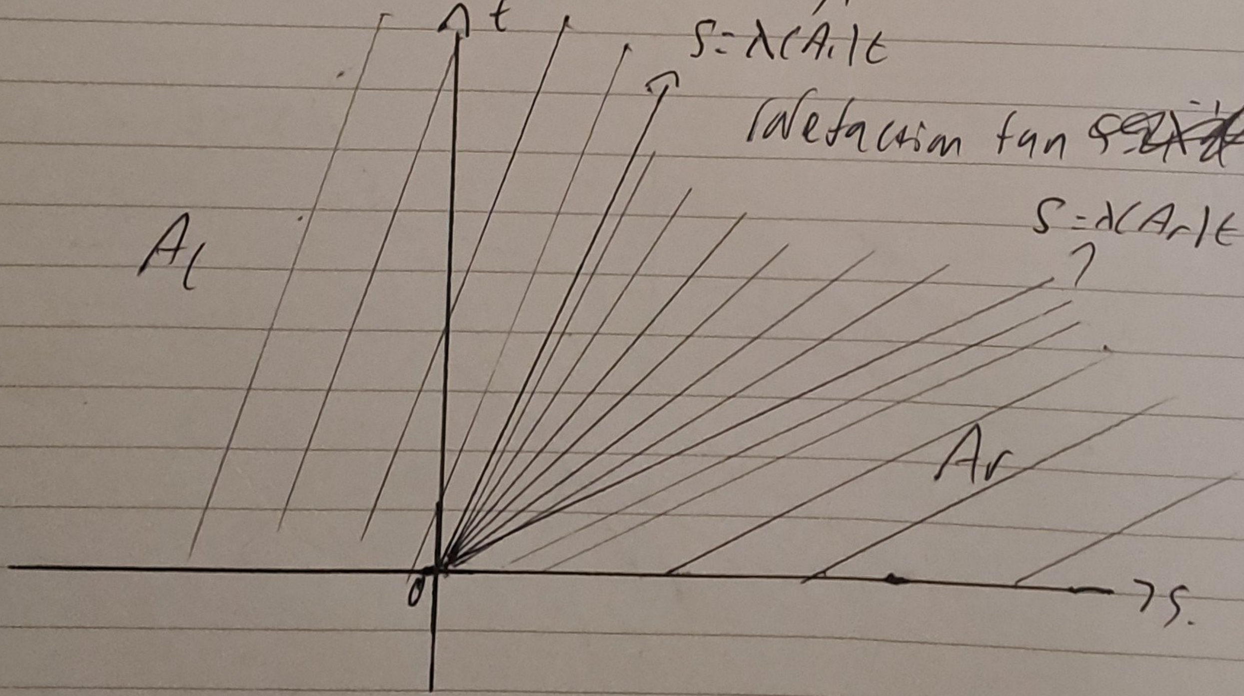
$$\lambda(\tilde{A}) = \frac{s}{t}$$

$$\Rightarrow \tilde{A} = \lambda^{-1}\left(\frac{s}{t}\right).$$

Hence, we have the following sol^n

$$\tilde{A} = \begin{cases} A_l & \frac{s}{t} < \lambda(A_l) \\ \lambda^{-1}\left(\frac{s}{t}\right) & \lambda(A_l) < \frac{s}{t} < \lambda(A_r) \\ A_r & \frac{s}{t} > \lambda(A_r). \end{cases}$$

A sketch of these characteristics are given by



Conversely, suppose $A_r < A_l \Rightarrow \lambda(A_r) < \lambda(A_l)$

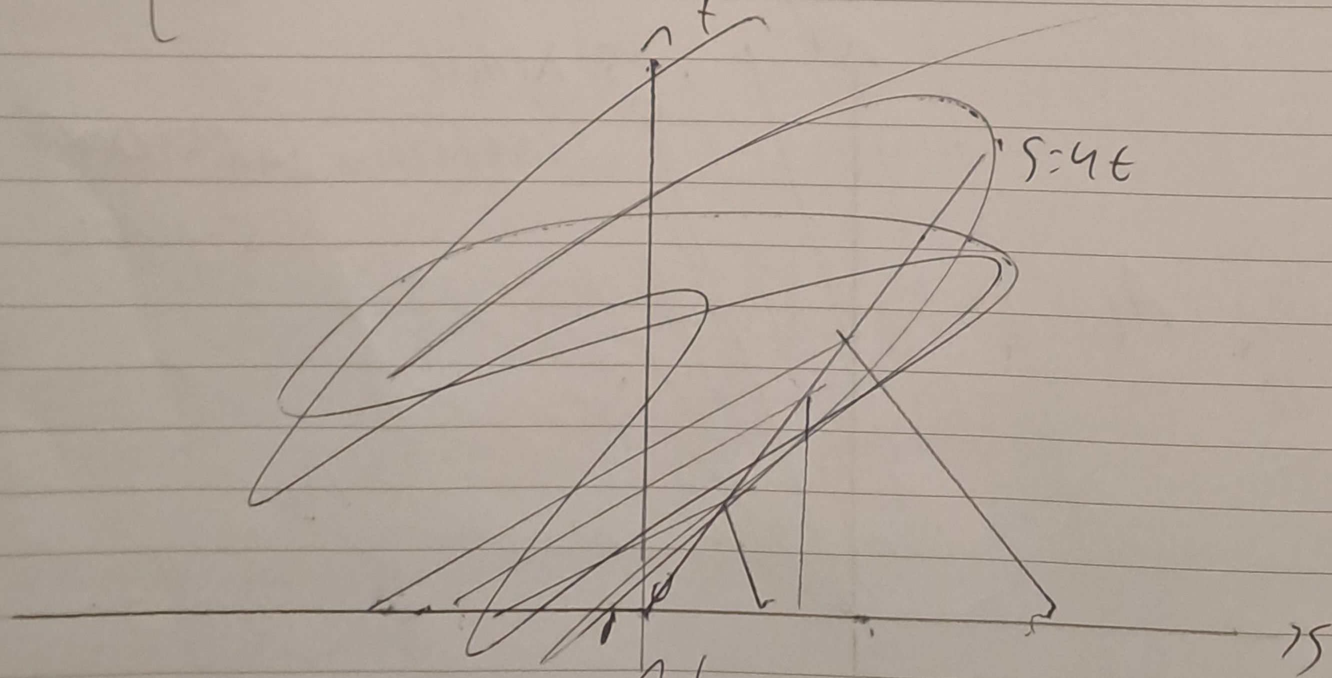
$\Rightarrow$ that the left wave is moving faster than the right one and will eventually 'catch up' leading to a ~~shock~~ 'shock' wave.

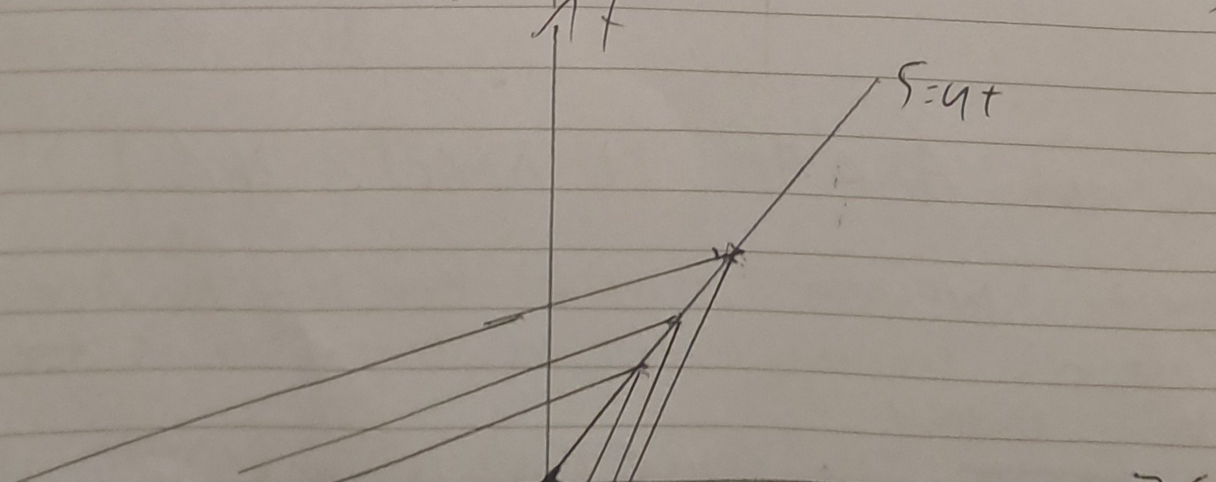
~~the speed of~~ The shock speed is given by the Rankine-Hugoniot condition, which states

7

$$u = \frac{F(A_r) - F(A_l)}{A_r - A_l}$$

$$\therefore A = \begin{cases} A_l & s - ut < 0 \\ A_r & \cancel{s-ut}\ s - ut > 0. \end{cases}$$





3). For $S=0$, to derive the finite-volume Godunov Scheme for eq^n (3) in the question sheet we integrate

$\partial_t A + \partial_s F(A,s) = 0$, over the space-time element

$s_{k-\frac{1}{2}} < s < s_{k+\frac{1}{2}}$, $t_n < t < t_{n+1}$, which yields.

$$\int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \int_{t_n}^{t_{n+1}} \partial_t A \, dt \, ds = - \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \int_{t_n}^{t_{n+1}} \partial_s F(A,s) \, dt \, ds$$

$$= \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \left(A^{n+1} - A^n\right) ds = - \int_{t_n}^{t_{n+1}} \left(F_{k+\frac{1}{2}} - F_{k-\frac{1}{2}}\right) dt.$$

By FTC & switching order of integration on RHS, where

$A^n = A(s, t_n)$ & $F_{k+\frac{1}{2}} = F(s_{k+\frac{1}{2}}, t)$.

Now, let $\bar{A}_k$ be the mean cell average in cell k. giving

$$\bar{A}_k = \frac{1}{\Delta s_k} \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} A \, ds, \quad \text{where } \Delta s_k = s_{k+\frac{1}{2}} - s_{k-\frac{1}{2}}.$$

Subbing this in yields -

$$\Delta S_k \left[\bar{A}_k^{n+1} - \bar{A}_k^{n} \right] = - \int_{t_n}^{t_{n+1}} \left(F_{k+\frac{1}{2}} - F_{k-\frac{1}{2}} \right) dt$$

$$\Rightarrow \bar{A}_k^{n+1} = \bar{A}_k^{n} - \frac{1}{\Delta S_k} \int_{t_n}^{t_{n+1}} \left(F_{k+\frac{1}{2}} - F_{k-\frac{1}{2}} \right) dt \quad \square$$

As required.

The Godunov method essentially begins by reconstructing a piecewise constant state from the cell averages at time t_n, it then evolves the eqn with this initial data.

$\therefore$ Solution at t_{n+1} can be constructed by piecing together the piecewise solutions, provided that the time step, Δt, is ~~sufficient~~ ~~small~~ sufficiently small enough that the waves from 2 adjacent Riemann-problems have not yet started to interact.

a) $\frac{\partial F}{\partial A} > 0$ $\therefore$ All the characteristics are moving to the right $\therefore$ It ~~sho~~ is completely upwind as information propagates rightward.

$\therefore$ As the wave is always being advected to the right we should always choose the flux at the left cell,

Hence, $F(A,S)|_{S=S_{k+\frac{1}{2}}} = F(\bar{A}_k^{n})$.

5).

$$\Delta t < CFL \min_k \frac{h_k}{|\lambda_k|}.$$

$$\lambda_k = \frac{\sqrt{-2sb}}{3C_m} \frac{\left(5w_0 \bar{A}_k^{2/3} + 6\bar{A}_k^{5/3}/w_0\right)}{\left(w_0 + 2\bar{A}_k/w_0\right)^{5/3}} > 0.$$

$$\Delta t < CFL \min_k \left\{ \frac{3h_k C_m \left(w_0 + 2\bar{A}_k/w_0\right)^{5/3}}{\sqrt{-2sb}\left(5w_0 \bar{A}_k^{2/3} + 6\bar{A}_k^{5/3}/w_0\right)} \right\}.$$

6). We first add ghost cell ~~values~~

~~$\bar{A}_{-1}$ & $\bar{A}_{N_k}$~~ $\bar{A}_{-1} = \bar{A}_L$ & $\bar{A}_{N_k} = \bar{A}_R$.

For the left ghost cell we enforce the flux ~~condition~~ condition, in other words the flux at the left cell ~~$F(\bar{A}_{-1})$~~ equals the base flow i.e.

$F_{k+1}(\bar{A}_{-1}) = Q_0$, where Q_0 = base flow as defined.

$\therefore$ simply change $\bar{A}_{-1}$ s.t. this is satisfied.

As aforementioned as $\lambda > 0$ then it is completely upwind ∴ $\bar{A}_{N_k}$ has no effect on the sol. Hence, we just change to extrapolate i.e $\bar{A}_{N_k} =$ ~~$\bar{A}_{N_{k-1}}$~~ $\bar{A}_{N_{k-1}}$.