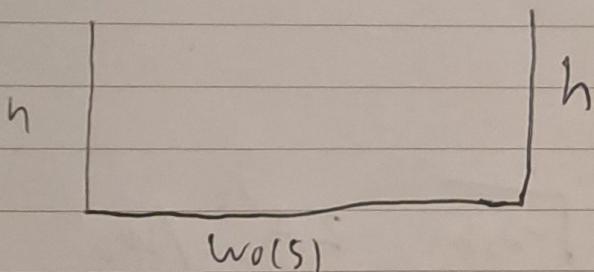


(1)

1) By defⁿ the wetted area A is the height water depth multiplied by the river width.

Hence, $A = h w_0(s) \Rightarrow h(A, s) = \frac{A}{w_0(s)}$, (clearly $w_0(s) > 0$).

For the perimeter consider the sketch



$$\therefore P = w_0(s) + 2h(A, s) = w_0(s) + \frac{2A}{w_0(s)} \quad \text{D}.$$

If $s=0$ then eqn (4) in the question sheet yields

$$\partial_e A + \partial_s F(A, s) = 0.$$

$$\therefore \partial_e A + \frac{\partial F}{\partial A} \frac{\partial A}{\partial s} + \frac{\partial F}{\partial s} = 0 \quad (\text{B, chain rule}).$$

$$\therefore \partial_e A + \frac{\partial F}{\partial A} \partial_s A + \frac{\partial F}{\partial s} = 0 \quad \text{D}.$$

$$F = \frac{A^{\frac{2}{3}} \sqrt{-\partial_s b}}{(cm P(A, s))^{\frac{2}{3}}}$$

$$\frac{\partial F}{\partial A} = \frac{2}{3} A^{-\frac{1}{3}} \cancel{\sqrt{-\partial_s b}} \quad \cancel{\frac{2}{3} P(A, s)^{\frac{2}{3}}} \quad \cancel{3 P(A, s)^{\frac{2}{3}}}$$

(7)

$$\frac{\partial F}{\partial A} = \frac{5\sqrt{-dsb}}{3(m)} - \frac{2\sqrt{-dsb}}{3(m)} \frac{\frac{\partial P}{\partial A} A^{\frac{5}{3}}}{P(A, S)^{\frac{2}{3}}} + \frac{2\sqrt{-dsb}}{3(m)} \frac{\frac{\partial P}{\partial A} A^{\frac{5}{3}}}{P(A, S)^{\frac{5}{3}}}$$

$$P = w_0(S) + \frac{2A}{w_0(S)}$$

$$\therefore \cancel{\frac{\partial P}{\partial A}} \frac{\partial P}{\partial A} = \frac{2}{w_0(S)}$$

$$\begin{aligned} \therefore \frac{\partial F}{\partial A} &= \frac{5\sqrt{-dsb}}{3(m)} \frac{A^{\frac{2}{3}}}{(w_0 + \frac{2A}{w_0})^{\frac{2}{3}}} - \frac{2\sqrt{-dsb}}{3(m)} \frac{2A^{\frac{5}{3}}/w_0}{(w_0 + \frac{2A}{w_0})^{\frac{5}{3}}} \\ &= \frac{+ \sqrt{-dsb}}{3(m)} \frac{(5w_0 A^{\frac{2}{3}} + 6A^{\frac{5}{3}}/w_0)}{(w_0 + \frac{2A}{w_0})^{\frac{5}{3}}} \geq 0. \end{aligned}$$

Similarly,

$$\frac{\partial F}{\partial S} = \frac{-A^{\frac{5}{3}} \partial ssb}{2(m) P(A, S)^{\frac{2}{3}} \sqrt{-dsb}} - \frac{2\sqrt{-dsb}}{3(m)} \frac{\bar{A}^{\frac{5}{3}}}{P(A, S)^{\frac{5}{3}}} \frac{\partial P}{\partial S}.$$

Now assuming the river slope is constant, or that the gradient difference in the slope is small we have:

$$\frac{\partial F}{\partial S} = - \frac{2\sqrt{-dsb}}{3(m)} \frac{A^{\frac{5}{3}}}{(w_0(S) + \frac{2A}{w_0(S)})^{\frac{5}{3}}} \frac{\frac{\partial P}{\partial w_0}}{\frac{dw_0}{dS}}.$$

$$= - \frac{2\sqrt{-dsb}}{3(m)} \frac{A^{\frac{5}{3}}}{(w_0 + \frac{2A}{w_0})^{\frac{5}{3}}} \left(1 - \frac{2A}{w_0^2} \right) \frac{dw_0}{dS} \quad \square.$$

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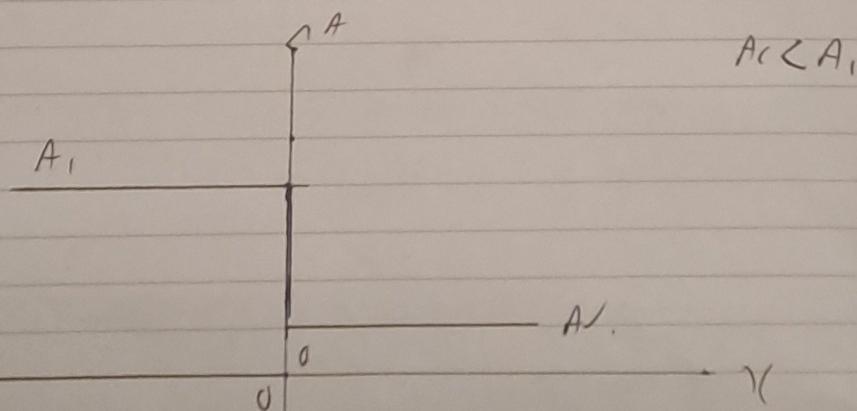
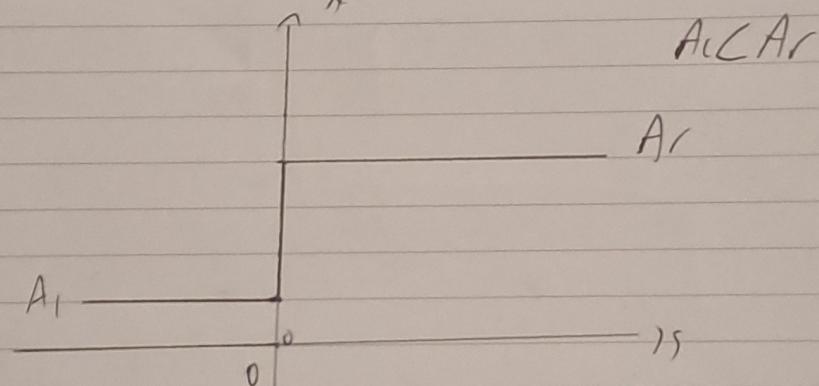
7. In the limit $w(s)$ is independent of s or varying (w_0) slowly so its dependency on s can be ignored we have $\frac{\partial F}{\partial s} = 0$.

To solve the Riemann problem, we start with the initial data

$$A(s, 0) = \begin{cases} A_L & s < 0 \\ A_R & s > 0. \end{cases}$$

Whole either $A_L < A_R$ or $A_R > A_L$.

- Initial data sketch is the following



Now, by def we have

$$\frac{ds}{dt} = \lambda_{A1} = \frac{\partial F}{\partial A} = \frac{\alpha F - \beta s A}{3cm} \frac{(5w_0 A^{\frac{2}{3}} + 6A^{\frac{5}{3}}/w_0)}{(w_0 + 2A)^{5/3}} > 0.$$

Hence, as $\lambda > 0$ we have all characteristics of the form $s = \lambda t$. Hence, the characteristics are all moving to the right i.e. we have signs travelling wrong

(4)

Now, to solve the Riemann problem we first need to determine whether the waves we are dealing with are rarefaction waves or shock waves.

Now clearly if $A_R > A_L$

$$= |\lambda(A_R)| > |\lambda(A_L)|$$

~~This discontinuity at 0 does not persist~~ Hence, by defn we have rarefaction waves

(5)

By Section 11.10 in the Finite Volume Methods for Hyperbolic Equations having by Leveque, the soln to a conservation law is a similarity soln, or a function of $\frac{s}{t}$ alone. So we have to solve

$$A(s, t) = \tilde{A}\left(\frac{s}{t}\right) \because \text{we have } \partial_t \tilde{A} + \partial_s F(\tilde{A})\left(\frac{s}{t}\right) = 0$$

$$\therefore \frac{\partial}{\partial t} \tilde{A}\left(\frac{s}{t}\right) = -\frac{\partial}{\partial s} F\left(\tilde{A}\left(\frac{s}{t}\right)\right)$$

\therefore we have

$$-\frac{s}{t^2} \tilde{A}' + \frac{1}{t} \tilde{A}' F'(\tilde{A}) = 0.$$

$$\therefore \text{Either } \tilde{A}'\left(\frac{s}{t}\right) = 0 \text{ OR } F'(\tilde{A}) = \frac{s}{t}.$$

In other words either $A = \text{constant}$ along the lines $\frac{s}{t} = \text{const.}$

OR the flux speed makes the ~~characteristic speed~~ the local speed. The local speed $\frac{s}{t}$ makes the local characteristic speed $F'(\tilde{A})$; this is the slope factor for relaxation.

$$\therefore \text{If } F'(\tilde{A}) = \frac{s}{t}$$

$$\lambda(\tilde{A}) = \frac{s}{t}$$

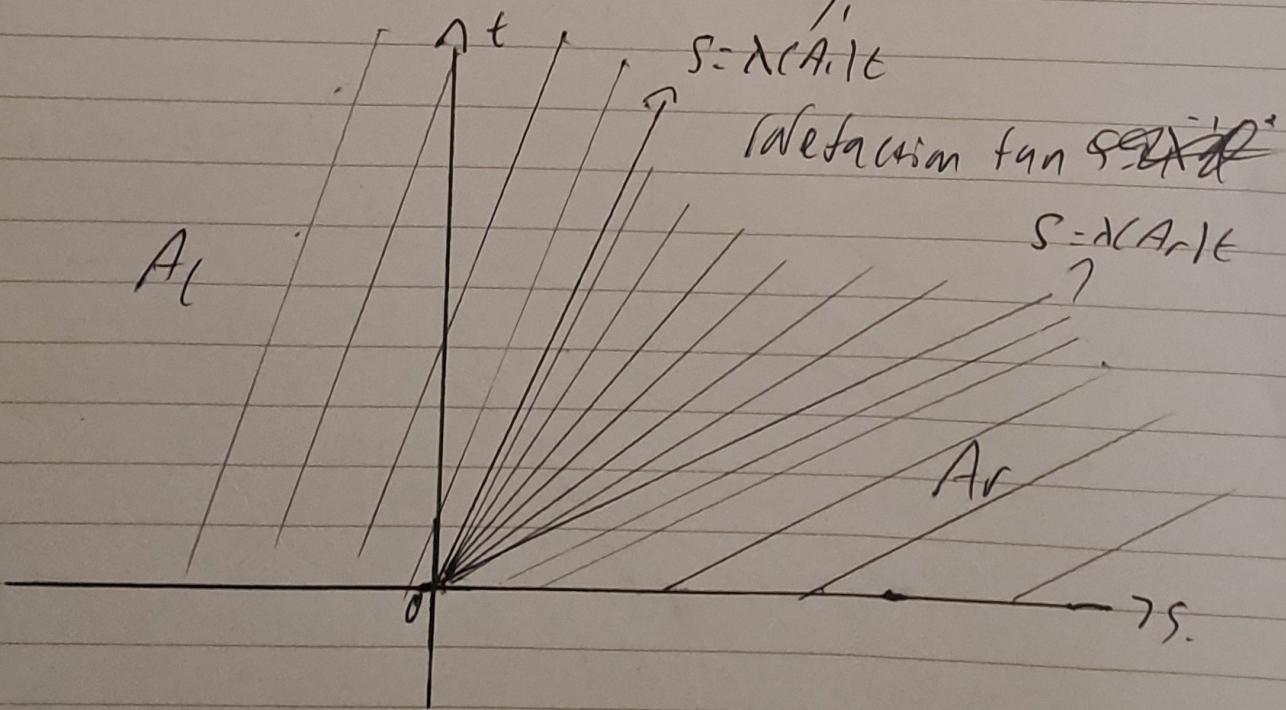
$$\therefore \tilde{A} = \lambda^{-1}\left(\frac{s}{t}\right).$$

Hence, we have the following soln

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$$\tilde{A} = \begin{cases} A_1 & \frac{s}{t} < \lambda(A_1) \\ \tilde{A}^{-1}\left(\frac{s}{t}\right) & \lambda(A_1) < \frac{s}{t} < \lambda(A_r) \\ A_r & \frac{s}{t} > \lambda(A_r). \end{cases}$$

A sketch of these characteristics are given by,



(conversely, suppose $A_r < A_c = \lambda(A_r) < \lambda(A_l)$)

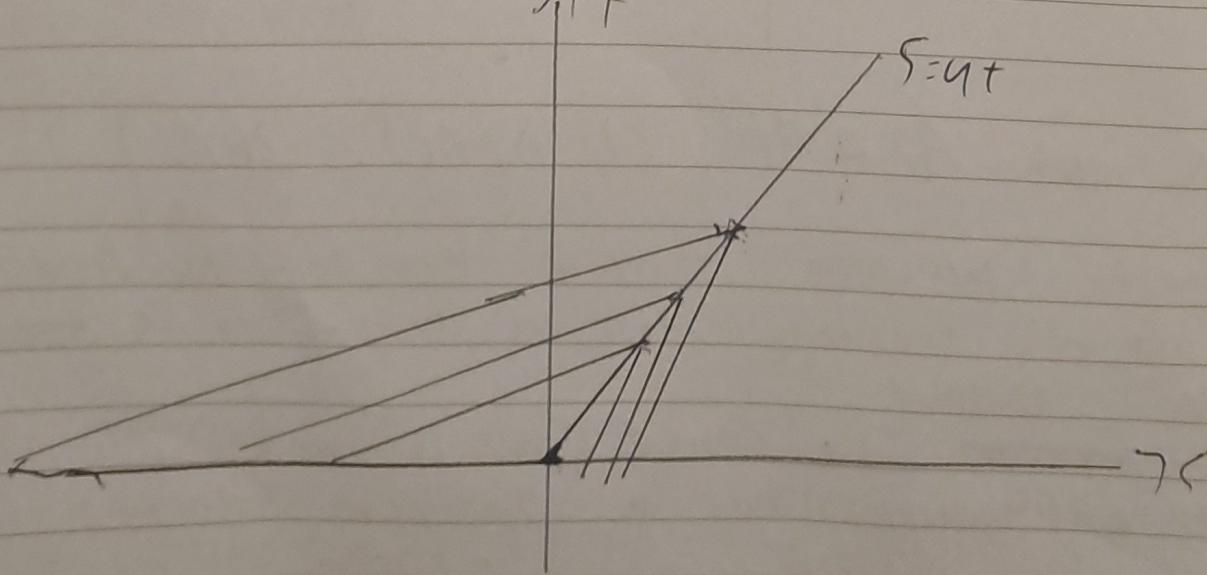
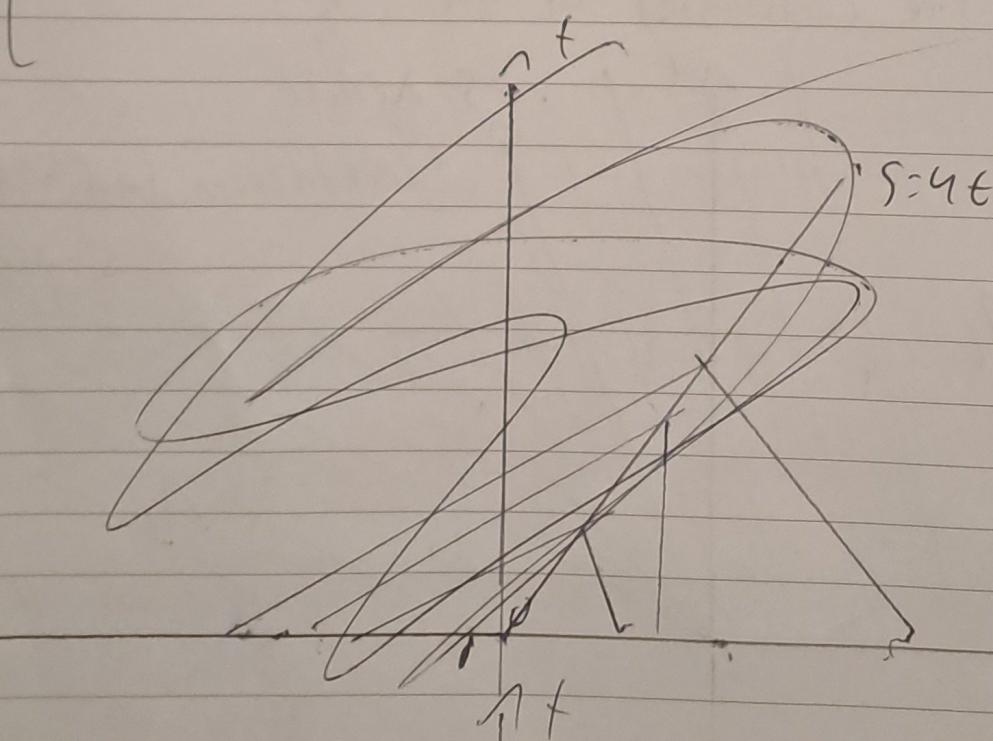
so that the left wave is moving faster than the right one and will eventually 'catch up' leading to a shock wave.

The speed of the shock speed is given by the Rankine-Hugoniot condition, which expresses

(2)

$$y = \frac{F(A_r) - F(A_c)}{A_r - A_c}$$

$$\therefore A = \begin{cases} A_c & S - y_t < 0 \\ A_r & S - y_t \geq 0 \end{cases}$$



3). For $S=0$, to derive the finite-volume Godunov Scheme for eqn (3) in the question Sheet we integrate

$\partial_t A + \partial_S F(A, S) = 0$, over the Space-time element

$$\int_{S_{k-\frac{1}{2}}}^{S_{k+\frac{1}{2}}} \int_{t_n}^{t_{n+1}} \partial_t A \, dt \, ds = - \int_{S_{k-\frac{1}{2}}}^{S_{k+\frac{1}{2}}} \int_{t_n}^{t_{n+1}} \partial_S F(A, S) \, ds \, dt$$

$$= \int_{S_{k-\frac{1}{2}}}^{S_{k+\frac{1}{2}}} (A^{n+1} - A^n) \, ds = - \int_{t_n}^{t_{n+1}} (F_{k+\frac{1}{2}} - F_{k-\frac{1}{2}}) \, dt$$

By FTC & re-switching order of integration on RHS, we get

$$A^n = A(S, t_n) \quad \& \quad F_{k+\frac{1}{2}} = F\left(\frac{1}{2}(S_{k+\frac{1}{2}} + S_{k-\frac{1}{2}}), t\right).$$

Now, let \bar{A}_k be the mean cell average in cell k . giving

$$\bar{A}_k = \frac{1}{\Delta S_k} \int_{S_{k-\frac{1}{2}}}^{S_{k+\frac{1}{2}}} A \, ds, \text{ where } \Delta S_k = S_{k+\frac{1}{2}} - S_{k-\frac{1}{2}}.$$

(9)

Subbing this in yields -

$$\Delta S_k \left[\bar{A}_{ic}^{n+1} - \bar{A}_{ic}^n \right] = - \int_{t_n}^{t_{n+1}} \left(F_{k+\frac{1}{2}} - F_{k-\frac{1}{2}} \right) dt$$

$$\Rightarrow \bar{A}_{ic}^{n+1} = \bar{A}_{ic}^n + -\frac{1}{\Delta S_{ic}} \int_{t_n}^{t_{n+1}} \left(F_{k+\frac{1}{2}} - F_{k-\frac{1}{2}} \right) dt \quad \square$$

The Godunov method essentially begins by reconstructing a piecewise constant state from the cell averages at time t_n , it then evolves the eqn with this initial data.

As required.

\therefore Solution at t_{n+1} can be constructed by piecing together the piecewise solutions, provided that the time step Δt is sufficient. Sufficiently small enough that the waves from 2 adjacent Riemann problems have not yet started to interact.

a) $\frac{\partial F}{\partial A} > 0 \quad \therefore$ All the characteristics are moving to the right \therefore If u is completely upwind as in forward propagates rightward.

\therefore As the wave is always being advected to the right we should always choose the flux w/ the left cell.

$$\text{Hence, } F(A|S=S_{k+\frac{1}{2}}) = F(\bar{A}_{ic}^n).$$

(10)

5).

$$\Delta t \leq (FL_{\min_k} \frac{h_k}{|\lambda_{1c}|})$$

$$\lambda_{1c} = \frac{\sqrt{-2s_b}}{3 \text{ cm}} \quad \frac{(5w_0 \bar{A}_{1c}^{2/3} + 6 \bar{A}_{1c}^{5/3} / w_0)}{(w_0 + 2 \bar{A}_{1c} / w_0)^{5/3}} > 0$$

$$\Delta t \leq (FL_{\min_k} \left(\frac{3h_k \text{ cm} (w_0 + 2 \bar{A}_{1c} / w_0)^{5/3}}{\sqrt{-2s_b} (5w_0 \bar{A}_{1c}^{2/3} + 6 \bar{A}_{1c}^{5/3} / w_0)} \right))$$

No coe theory fine: 9/20

6). We first add ghost cell values

$$\bar{A}_{-1} \quad \& \quad \bar{A}_{N_c} \quad \bar{A}_{-1} = \bar{A}_c \quad \& \quad \bar{A}_{N_c} = \bar{A}_R.$$

For the left ghost cell we enforce the flux condition
condition, i.e. other nodes the flux at the left cell

$E(\bar{A}_{-1})$ equals the base flux i.e.

$$F_{1c+1}(\bar{A}_{-1}) = Q_0, \text{ where } Q_0 = \text{base flux as defined}$$

\therefore simply change \bar{A}_{-1} so this is satisfied.

As aforementioned as $\lambda > 0$ the it is completely upwind

\bar{A}_{N_c} has no effect on the flux. Hence, we just
choose to extrapolate i.e. $\bar{A}_{N_c} = \bar{A}_{N_c}, \bar{A}_{N_c-1}$.