Exercises 2 in Firedrake

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Contents

1 Notes

Note that all the code and simulation data can be found in the file `swekin2_oop.ipynb` in the repository.

The code is run on much smaller examples than those suggested within the exercise specification to allow for faster simulation. It can easily be changed to work for the larger examples that have been described.

2 Test Case 0

We check that the model is working correctly by simulating constant flow as a boundary condition at the upstream and constant width and depth throughout the domain. We expect that the constancy will be retained throughout the simulation.

2.1 Plot through time

See Figure 1.

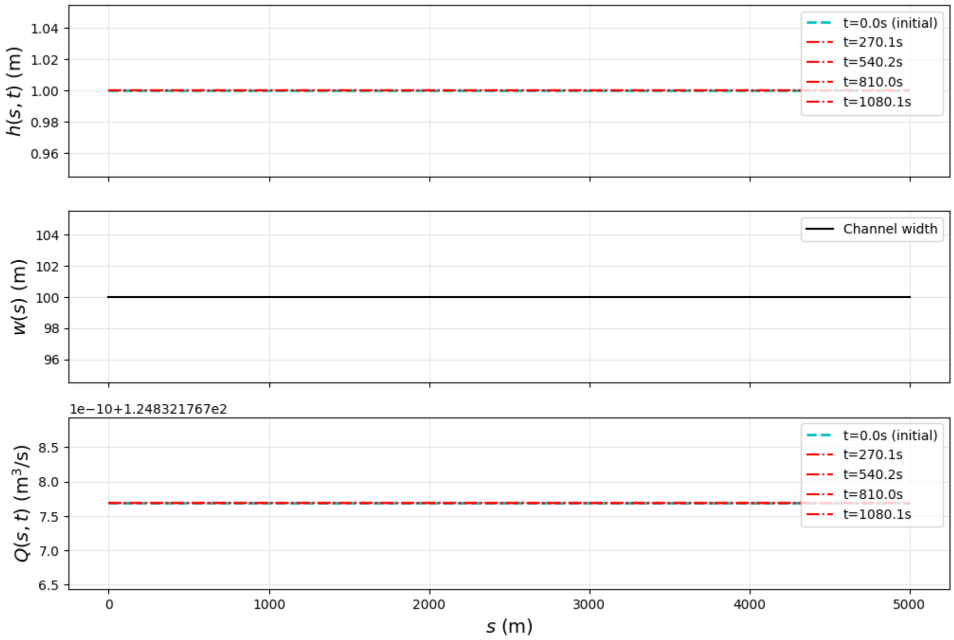

Figure 1: Plot of water depth, width and discharge through time for Test Case 0. All values remain constant as expected.

3 Test Case 1

The second test case simulates a flood hydrograph entering a channel. We expect a shock to form and propagate downstream and an eventual return to the steady state.

3.1 Plot through time

See Figure 2.

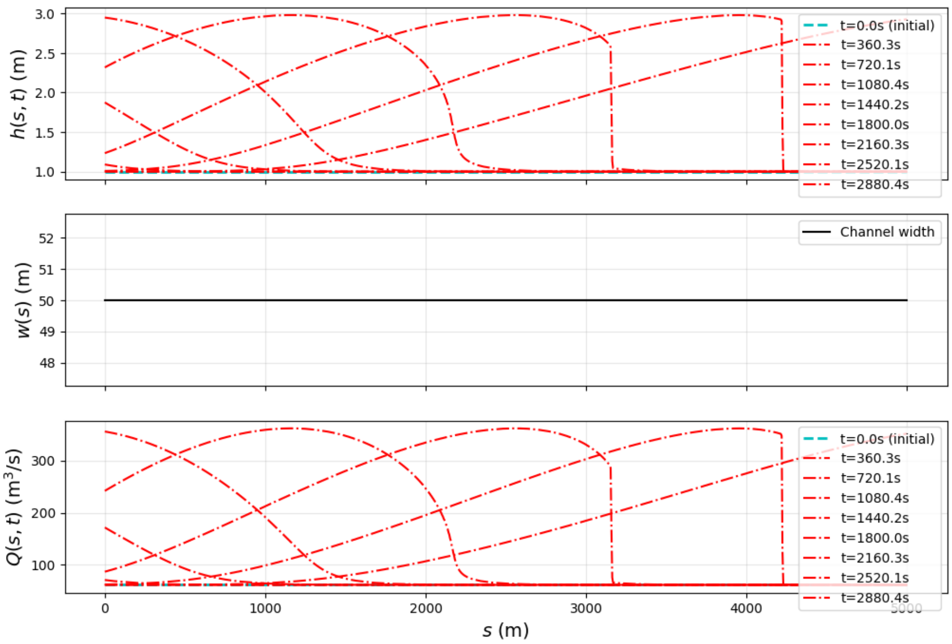

Figure 2: Plot of water depth, width and discharge through time for Test Case 1. A shock forms and propagates downstream as expected.

3.2 Shock Speed

To calculate the shock speed, we track the position of the shock at 3 time periods and calculate the speed from these positions. See the data in Table 1.

Time (s)	**Position (m)**
2016	2747
2304	3573
2592	4434

Table 1: Shock capture data

Computing the shock speed by using the differences between the positions and time we find two values for the speed of $\approx 2.99m/s$ and $\approx 2.87m/s$. These values are similar enough to assume constant shock speed.

3.3 Convergence

To compute the convergence we take some constant value of one parameter and use the finest value of the varying parameter as the "exact" solution comparing the relative error at a given timestep between this and other values of the varying parameter. We do this for both Δx and CFL.

3.3.1 Δx

See in Table 2 the percentage error at different values of Δx with constant CFL = 0.4.

Δx	**Error**
0.5	0.00251
1.0	0.00744
1.5	0.01339
2.0	0.01683
4.0	0.03870
8.0	0.08886

Table 2: Error as a function of Δx with CFL = 0.4

We see a rough halving of the error for a halving of the Δx value indicating first order convergence.

3.3.2 CFL

See in Table 3 the percentage error at different values of CFL with constant $\Delta x = 1.5$.

CFL	**Error**
0.2	0.00101
0.3	0.00225
0.4	0.00351
0.5	0.00477
0.6	0.00593
0.7	0.00608

Table 3: Error as a function of CFL with $\Delta x = 1.5$

In this case for a halving of the CFL value we see the error roughly reducing to a third of the value indicating some order between first and second. We note however that the variation is much smaller than in the Δx case suggesting the model is much closer to convergence and thus less sensitive to CFL changes.

3.3.3 Constant Discharge

A question was posed around constant discharge after the passing of the pulse. In Figure 3 we see that the discharge does return to a constant value for large times.

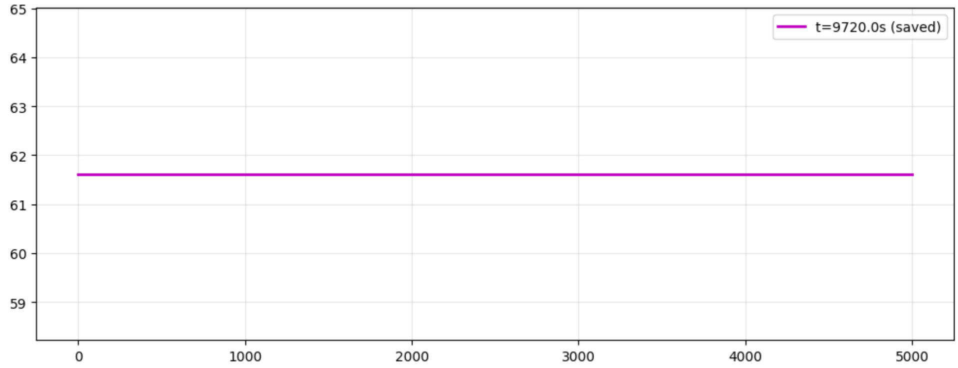

Figure 3: Plot of discharge through time for Test Case 1 at time 9720s. The discharge returns to constant as expected. Note that the left axis is discharge in m^3/s and the bottom axis is distance down the stream in m.

4 Test Case 2

The third test case simulates the same flood hydrograph as in test case 1 but against a different bed profile where flood plains are used above 4m depth. We note that the implementation of the flood plains leads to a discontinuity at this 4m depth.

4.1 Plot through time

See Figure 4 and the discontinuity forming at 4m.

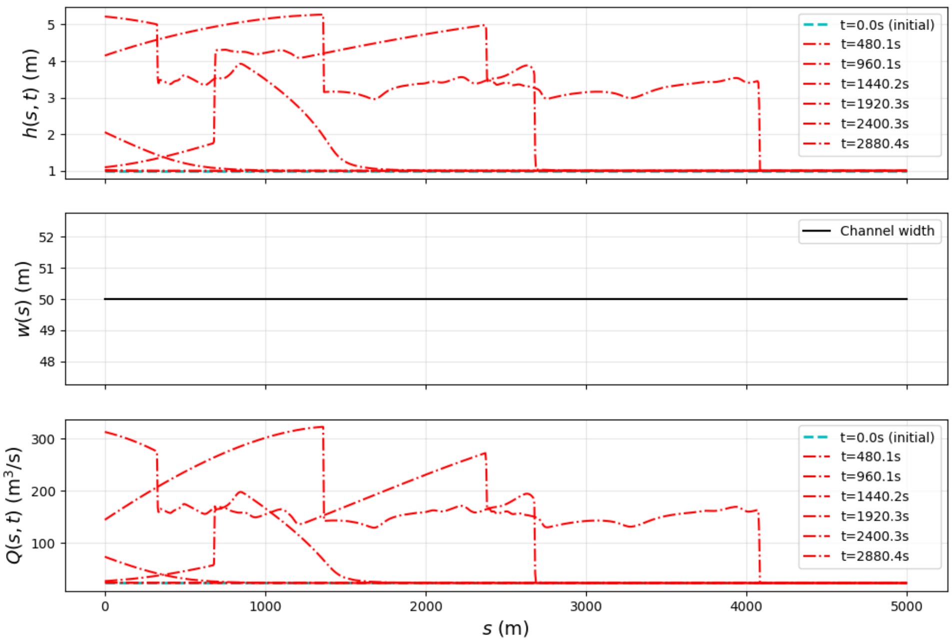

Figure 4: Plot of water depth, width and discharge through time for Test Case 2. A shock forms and propagates downstream as expected. A discontinuity is seen at 4m depth due to the floodplain implementation.

4.2 Shock Speed

We track the two shock positions as seen in Table 4.2 and seen in the plots of Figure 4.

Time (s)	Position 1 (m)	Position 2 (m)
2592	1791	3243
2736	2097	3658
2448	1466	2817

Table 4: Shock positions measured at each time step

Calculating the shock speeds as before we find speeds of $\approx 2.15 m/s$ for the higher shock on the flood plain and $\approx 2.9 m/s$ for the lower again considering consitency between 2 difference calculations.

4.3 Convergence

We calculate convergence as in test case 1.

4.3.1 Δx

See in Table 4.3.1 the percentage error at different values of Δx with constant CFL = 0.4.

Δx	**Error**
0.5	0.29634
1.0	0.94120
1.5	1.76604
2.0	2.24424
4.0	4.52196
8.0	9.45551

Table 5: Error as a function of Δx with CFL = 0.4

We again see a rough halving of the error for a halving of the Δx value indicating first order convergence.

4.3.2 CFL

See in Table 6 the percentage error at different values of CFL with constant $\Delta x = 1.5$.

CFL	**Error**
0.2	3.8829
0.3	7.0997
0.4	13.3967
0.5	16.5244
0.6	12.2315
0.7	28.4830

Table 6: Error as a function of CFL with $\Delta x = 1.5$

In this case we have a lot more inconsitent behaviour suggesting instability at the higher CFL values and generally much higher sensitivity to CFL than Δx changes. Comparing the values 0.2 and 0.4 we see roughly a third order convergence but more data would be required to confirm this. Note that smaller CFL values than those shown were not used in this case due to the computational cost of the running the simulations in this state.