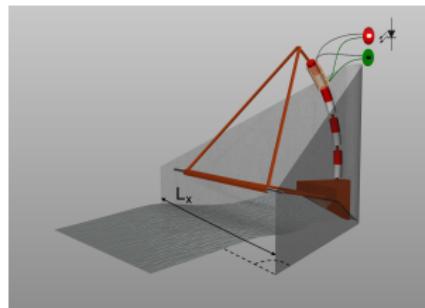


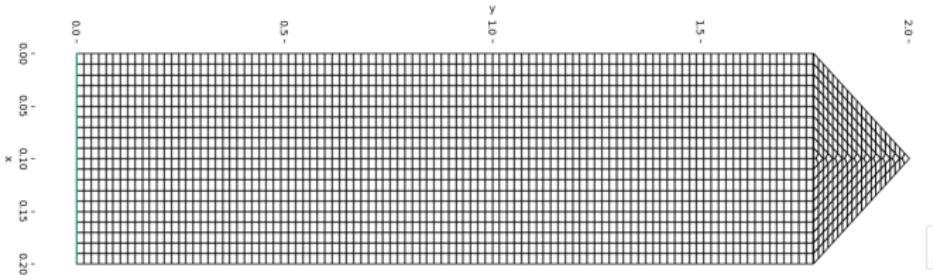
# Numerical methods for fluid dynamics

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£: CDT Fluid Dynamics



## Outline: assessment

- ▶ Attendance at practical sessions.
- ▶ Three numerical exercises (on finite difference, finite volume and finite element methods); to be handed in individually but you may work as a team.
- ▶ Example programs (for use at your own risk) will be provided in Python. Python use is recommended.



## Finite differences: $\theta$ -method

- $\theta$ -method will be explained for heat equation w. Dirichlet BCs:

$$\partial_t u = \partial_{xx} u \quad (1)$$

$$u(x, 0) = u_0(x) \quad (2)$$

$$u(0, t) = u(1, t) = 0. \quad (3)$$

- Mesh points are  $x_j = j\Delta x$ ; constant time step is used  $t_n = n\Delta t$  for  $j = 0, \dots, N_x$  and  $n = 0, 1, \dots$
  - Time step can also be varied, in which case  $\Delta t_n$  varies and  $t_n$  is sum of time steps taken.
  - Approximate values of  $u(x, t)$  on mesh points are denoted by  $U_j^n \approx u(x_j, t_n)$ .
  - Initial values are  $U_j^0 = u_0(x_j)$ ; in general exact (why could there be an issue here?).

# Finite differences: approximations

- The next issue is to find a difference approximation of the PDE (1) in terms of the approximations  $U_j^n$ .
- Time derivative is approximated in a forward manner, expressed in terms of several difference operators  $\Delta_{+t}$  and  $\delta_t$ :

$$(\partial_t u)(x_j, t_n) \approx \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t} \quad (4)$$

$$\equiv \frac{\Delta_{+t} u(x_j, t_n)}{\Delta t} \quad (5)$$

$$\equiv \frac{\delta_t u(x_j, t_{n+1/2})}{\Delta t} \quad (6)$$

$$\approx (\partial_t u)(x_j, t_{n+1/2}) \quad (7)$$

- *Exercise:* check approximations by performing suitable Taylor expansions of  $u$  around, e.g.,  $t^n = t_n$  or  $t^{n+1/2} = t_{n+1/2}$ .

## Finite differences: Taylor expansions

- 2<sup>nd</sup> spatial derivative approximated symmetrically as

$$(\partial_{xx} u)(x_j, t_n) \approx \frac{u(x_{j-1}, t_n) - 2u(x_j, t_n) + u(x_{j+1}, t_n)}{\Delta x^2} \quad (8)$$

$$= \frac{\delta_x^2 u(x_j, t_n)}{\Delta x^2} = \frac{\delta_x(\delta_x u)|_{x_j}^{t_n}}{\Delta x^2} \quad (9)$$

with  $\delta_x u(x, t) \equiv (u(x + \Delta x/2, t) - u(x - \Delta x/2, t)) / \Delta x$ .

- ▶ Exercise: check this approximation  $y$  using Taylor expansions of  $u$  around, e.g.,  $t^n$  and  $x_j$ .
  - ▶ This approximation also holds at  $t_{n+1}$

$$\begin{aligned} (\partial_{xx} u)(x_j, t_{n+1}) &\approx \frac{u(x_{j-1}, t_{n+1}) - 2u(x_j, t_{n+1}) + u(x_{j+1}, t_{n+1})}{\Delta x^2} \\ &= \frac{\delta_x^2 u(x_j, t_{n+1})}{\Delta x^2}. \end{aligned} \quad (10)$$

Finite differences:  $\theta$  scheme

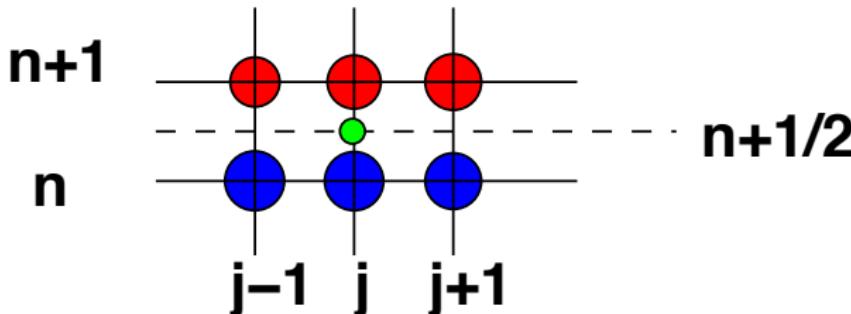
- ▶ By combining approximations with  $\mu = \Delta t / \Delta x^2$ , PDE (1) can be approximated on a 6-point stencil (see Fig.) )

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{\theta}{\Delta x^2} \delta_x^2 U_j^{n+1} + \frac{(1-\theta)}{\Delta x^2} \delta_x^2 U_j^n \quad (11a)$$

$$U_i^{n+1} = U_i^n + \mu\theta(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}) + \mu(1-\theta)(U_{i-1}^n - 2U_i^n + U_{i+1}^n). \quad (11b)$$

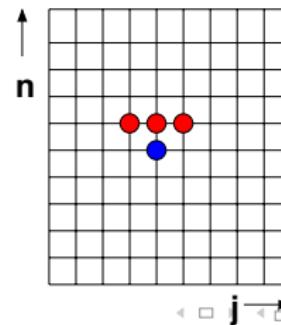
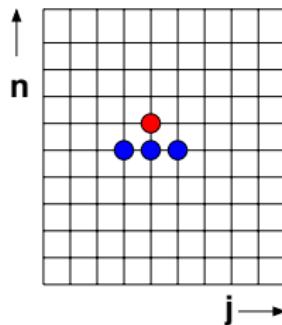
- Rewritten form with unknowns on the LHS and  $0 \leq \theta \leq 1$

$$-\mu\theta U_{i-1}^{n+1} + (1+2\mu\theta)U_i^{n+1} - \mu\theta U_{i+1}^{n+1} = (1-2\mu(1-\theta))U_i^n + \mu(1-\theta)(U_{i-1}^n + U_{i+1}^n). \quad (12)$$



## Finite differences: EF, EB, CN schemes

- When  $\theta = 0$  scheme is explicit, solved for  $U_j^{n+1}$ : *Euler forward* scheme uses stencil of 4 points with 1 point in future.
  - When  $\theta = 1$  scheme is fully implicit; *Euler backward* scheme. Uses a stencil of 4 points with 3 points in future, see Fig.
  - When  $\theta = 1/2$ , a stencil of 6 points is used: this is the classical Crank-Nicolson scheme (Crank & Nicolson 1947).
  - *Exercise  $\theta$ -scheme:* suitable space-time grid point for a Taylor expansion?



# Finite difference methods: homework

- ▶ Study Morton and Mayers (2005), Chapter 2, sections 2.2, 2.4, 2.5.
- ▶ Read Morton and Mayers (2005), Chapter 2 (intro), sections 2.1, 2.3.
- ▶ Sign up to GitHub and send login name.
- ▶ Run/study the two example codes and study the example task.
- ▶ Study and start exercise-I.

# Finite difference methods: Fourier analysis

- ▶ To assess stability of scheme, Fourier analysis will be used.
- Substitute Fourier Ansatz  $U_j^n = \lambda^n e^{ijk\Delta x}$  with imaginary number  $i$  satisfying  $i^2 = -1$ , amplification factor  $\lambda$ , and wavenumber  $k$  into discretization (11).
- ▶ In general  $\lambda$  is complex with real and imaginary parts such that  $\lambda = \Re(\lambda) + i \Im(\lambda)$ . Scheme is stable when  $|\lambda| \leq 1$ , which for complex  $\lambda$  implies that we need to take the modulus of  $\lambda$ .
- ▶ When  $|\lambda| > 1$ , approximation  $U_j^n$  will blow up over time since  $|\lambda|^n$  becomes unbounded and the scheme is unstable.

$$-\theta\mu\lambda e^{-ik\Delta x} + (1 + 2\theta\mu)\lambda - \theta\mu\lambda e^{ik\Delta x} = 1 - 2(1 - \theta)\mu + (1 - \theta)\mu(e^{ik\Delta x} + e^{-ik\Delta x}) \quad (13)$$

$$\iff \lambda + \lambda\theta\mu(2 - e^{ik\Delta x} - e^{-ik\Delta x}) = 1 - (1 - \theta)\mu(2 - e^{ik\Delta x} - e^{-ik\Delta x}) \quad (14)$$

$$\iff \lambda = \frac{1 - 4(1 - \theta)\mu \sin^2(k\Delta x/2)}{1 + 4\theta\mu \sin^2(k\Delta x/2)}. \quad (15)$$

## Finite difference methods: Fourier analysis

- ▶ Note that this scheme is a special, symmetric case for which  $\lambda$  is real. Since  $0 \leq \theta \leq 1$  and  $\mu > 0$ , we note that  $\lambda < 1$ .
- ▶ Instability can then occur only when  $\lambda < -1$ , i.e., when

$$\begin{aligned} 1 - 4(1 - \theta)\mu \sin^2(k\Delta x/2) &< -(1 + 4\theta\mu \sin^2 k(\Delta x/2)) \\ \implies 4\mu(1 - 2\theta) \sin^2(k\Delta x/2) &> 2. \end{aligned} \quad (16)$$

- ▶ Instability occurs for  $\mu(1 - 2\theta) > 1/2$  for case  $k\Delta x/2 = \pi/2$ . For  $\theta \geq 1/2$  the  $\theta$ -scheme unconditionally stable, while for  $0 \leq \theta < 1/2$  scheme conditionally stable when

$$\mu = \frac{\Delta t}{(\Delta x)^2} < \frac{1}{[2(1 - 2\theta)]}. \quad (17)$$

- ▶ Note that we ignored the boundary conditions in deriving Fourier stability because we imposed the harmonic Ansatz for periodic boundaries. Other BCs, see Morton and Mayers 2005.

# Finite difference methods: maximum principle

## Theorem

*M&M 2005: The  $\theta$ -method (11) satisfies*

$$U_{min} \leq U_j^n \leq U_{max}$$

$$U_{min} = \min(U_0^m, 0 \leq m \leq n; U_j^0, 0 \leq j \leq N_x; U_{N_x}^m, 0 \leq m \leq n)$$

$$U_{max} = \max(U_0^m, 0 \leq m \leq n; U_j^0, 0 \leq j \leq N_x; U_{N_x}^m, 0 \leq m \leq n)$$

*given the conditions  $0 \leq \theta \leq 1$  and  $\mu(1 - \theta) \leq 1/2$ .*

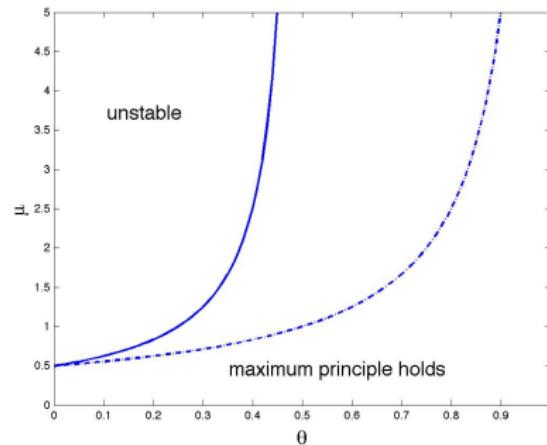
# Finite difference methods: maximum principle

- ▶ Verification of conditions required for maximum principle to hold forms a practical test for stability of numerical finite difference schemes for diffusion and advection-diffusion equations.
- ▶ Maximum principle states that value of variable  $U_j^n$  bounded between boundary values and initial values. E.g., when  $u$  is seen as a temperature, temperature values can not go above or below temperatures imposed initially or at boundaries.

# Finite difference methods: stability criteria

The Fourier stability condition (17) and condition for maximum principle, displayed in Fig.:

$$\mu = \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{[2(1 - 2\theta)]} \quad \text{and} \quad \mu(1 - \theta) \leq \frac{1}{2}. \quad (18)$$



# Finite difference methods: homework, week 2

- ▶ Study Morton and Mayers (2005), Chapter 2, sections 2.7, 2.11.
- ▶ Read Morton and Mayers (2005), Chapter 2, sections 2.8, 2.9, 2.10.
- ▶ Continue/finish exercise-I.
- ▶ Urgent: get *Firedrake* installed on your machines, asap, please!
- ▶ ... hints for Exercise-I ....

# Finite volume or Godunov method

- ▶ Finite volume methods may be most natural for hyperbolic PDEs expressed as conservation laws.
- ▶ We only consider the 1D case here:

$$\partial_t u + \partial_x(f(u)) = 0 \quad \text{or} \quad (19)$$

$$\partial_t u_i + \partial_{x_j} f_{ij} = 0 \quad \text{with} \quad j = 1 \quad (20)$$

with  $u = (u_1, u_2, \dots, u_n)$  and  $u = (f_1, f_2, \dots, f_n)$ .

# Finite volume: example conservation laws

Examples:

- ▶ Euler equations in 2D:

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E + p) \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E + p) \end{pmatrix} = 0 \quad (21)$$

with  $u$  and  $v$  velocity components in  $x$  and  $y$ ,  $p$  pressure and  $E$  total energy. EOS:  $p = (\gamma - 1)(E - \frac{1}{2}\rho(u^2 + v^2))$ .

- ▶ Shallow-water equations in 1D:

$$\partial_t h + \partial_x(hu) = 0, \partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2) = 0 \quad (22)$$

with water depth  $h(x, t)$  and depth-averaged velocity  $u(x, t)$ .

# Finite volume: examples

Examples of conservative systems with extra terms:

- Width-averaged shallow-water or St. Venant equations:

$$\partial_t A + \partial_s(Au) = S \quad (23)$$

$$\partial_t(Au) + \partial_s(Au^2 + gAh) = gh\partial_s A - gA\partial_s b - F, \quad (24)$$

with source and friction terms  $S = S(s, t)$ ,

$F = gC_m Au|u|/R(A, s)^{4/3}$ ; along-river coordinate  $s$ ;

cross-section  $A(s, t)$ ; water depth  $h = h(A, s)$ ; depth-averaged velocity  $u(s, t)$ ; river slope  $-\partial_s b$ , and, acceleration of gravity  $g$ .

## Finite volume: examples

- ▶ Limit of kinematic river equation, conservative with extra source term:

$$\partial_t A + \partial_s \left( AR(A, s)^{2/3} \sqrt{-\partial_s b} / C_m \right) = S, \quad (25)$$

with Manning coefficient  $C_m$ , hydraulic radius  $R(A, s)$  (wetted area  $A$  over wetted perimeter) and “volume”  $S(s, t)$ .

## Finite volume: overview for Burgers-advection system

- ▶ Illustrate Godunov method or finite volume discretization for simple system of hyperbolic equations.
- ▶ Step by step discretization for system of uncoupled Burgers' and linear advection equations

$$u_t + (u^2/2)_x = 0 \quad (26)$$

$$v_t + a v_x = 0 \quad (27)$$

with  $a > 0$  constant,  $u = u(x, t)$  and  $v = v(x, t)$  on  $x \in [0, L]$ ,  $(\cdot)_t = \partial_t$ , etc.

- ▶ Boundary conditions required, not specified presently.
- ▶ Initial conditions  $u(x, 0)$  and  $v(x, 0)$  are given at  $t = t_0 = 0$ .

# Finite volume method: studying

From Leveque's book *Finite volume methods*:

- ▶ Study §1.1 (conservation law) §3.1 (linear advection Eq.) till §3.1.1 & 3.2 (Burgers' Eq).
- ▶ Study §3.1, §3.3 (shock formation).

## Godunov method example: Step-1

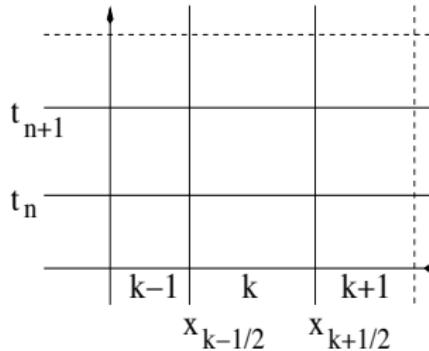
- ▶ Exposition of simple uncoupled system plus algorithm facilitates implementation for genuinely coupled fluid system, presented elsewhere.
- ▶ *Step 1:*  
System (26) is written concisely in hyperbolic form (20)

$$u_t + (f(u))_x = 0, \quad (28)$$

after identification  $u = (u, v)^T$  and flux  $f(u) = (u^2/2, a v)^T$  (transpose  $(\cdot)^T$ ).

## Godunov method example: Step-2

- ▶ *Step 2:* Define space-time mesh with  $N$  “finite volumes” on domain  $x \in [0, L]$  in time interval  $I_n = [t_n, t_{n+1}]$  (Fig. 22).
- ▶ Cell  $k$  occupies  $x_{k-1/2} < x < x_{k+1/2}$  and  $k = 1, 2, \dots, N$ .
- ▶  $N + 1$  cell boundaries  $x_{1/2}, \dots, x_{N-1/2}, x_{N+1/2}$ . Cell lengths  $h_k = x_{k+1/2} - x_{k-1/2}$  and time step  $\Delta t_n = t_{n+1} - t_n$  may vary.
- ▶ There are  $n = 0, \dots, N_t$  time intervals  $I_n$ , where  $t = t_n$  is the time after  $n$  time steps, initial conditions at  $t = t_0 = 0$ .



## Godunov method example: Step-3

- ▶ *Step 3: Integrate (28) in space-time element*  
 $x_{k-1/2} < x < x_{k+1/2}$  and  $t_n < t < t_{n+1}$ , Fig. 22.
- ▶ Via coordinate transformation  $x' = x - x_{k+1/2}$ ,  $t' = t - t_n$ , right-bottom corner becomes origin  $(x', t')^T = (0, 0)^T$ .
- ▶ After integration of (28) over space-time element:

$$U_k^{n+1} = U_k^n - \frac{1}{h_k} \int_{t_n}^{t_{n+1}} f_{k+1/2}(t) - f_{k-1/2}(t) dt, \quad (29)$$

with mean cell average  $U_k$  in cell  $k$

$$U_k(t) = \frac{1}{h_k} \int_{x_{k-1/2}}^{x_{k+1/2}} u(x, t) dx. \quad (30)$$

- ▶ Flux is at the cell boundaries:  $f_{k+1/2}(t) = f(u(x = x_{k+1/2}, t))$ .
- ▶  $U_k(t)$  in (30) and  $f_{k+1/2}(t)$  still functions of time  $t$ , and  
 $U_k^n = U_k(t = t_n)$ , etc.

## Godunov method example: Step-3

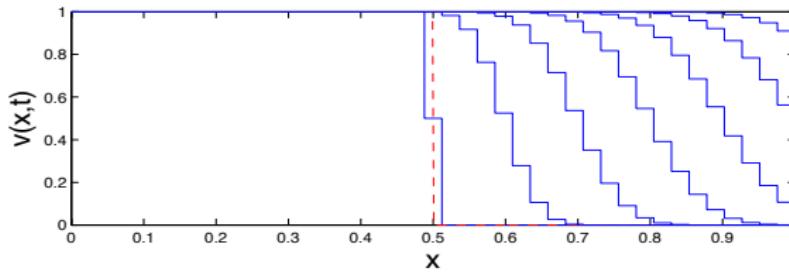
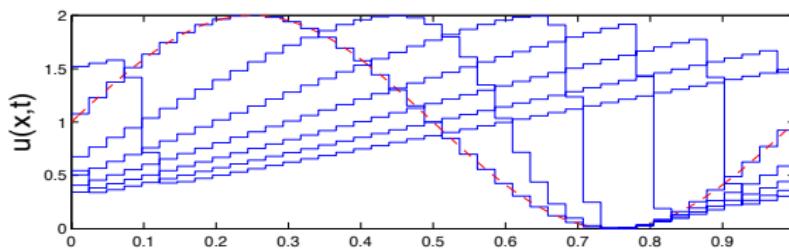
- ▶ Integral expression (29) exact provided that  $u(x, t)$  known.  
Start at  $n = 0$ , calculate  $U_k^0 = U_k(t = t_0)$  using (30).
- ▶ Graphically,  $U_k^0$  is projection of initial data on piecewise constant profiles at time  $t_0$ , cf. initial step profiles in Fig. 25.
- ▶ Determine  $f_{k+1/2}(t)$  over  $t_n < t < t_{n+1}$  in (29) to obtain

$$U_k^{n+1} = U_k^n - \frac{\Delta t}{h_k} \left( F_{k+1/2}(U_k^n, U_{k+1}^n) - F_{k-1/2}(U_{k-1}^n, U_k^n) \right) \quad (31)$$

with (approximate) numerical flux

$$F_{k+1/2}(U_k^n, U_{k+1}^n) \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f_{k+1/2}(t) dt. \quad (32)$$

## Godunov method example: Step-3



## Godunov method example: Step-4

- ▶ *Step 4:* Calculate numerical fluxes  $F_{k+1/2}(U_k^n, U_{k+1}^n)$  in (32) at all nodes  $x_{k+1/2}$  for  $k = 0, 1, \dots, N$ .
- ▶ There are various strategies to find (32), usually involving an approximation.
- ▶ Consider classical Godunov strategy based on the Riemann solution.
- ▶ Calculate  $F_{k+1/2}(t)$  in (32) exactly over  $t_n < t < t_{n+1}$  in (29), only feasible for piecewise constant approximation  $U_k^n$  at time  $t_n$  and starting with projected initial condition  $U_k^0$ , instead of smoother initial condition.
- ▶ Solution of Riemann problem in new coordinates, i.e. with  $u = u(x', t')$  and  $f = f(u(x', t'))$ , provides such exact solution.

## Godunov method example: Riemann problem

- The Riemann problem is defined as

$$\frac{\partial u}{\partial t'} + \frac{\partial f}{\partial x'} = 0 \quad (33)$$

for special initial conditions

$$u(x', t' = 0) = \begin{cases} u_{left} = u_l = U_k^n & \text{for } x' < 0 \\ u_{right} = u_r = U_{k+1}^n & \text{for } x' \geq 0 \end{cases} . \quad (34)$$

- Riemann solution such that  $u(x', t')$  constant along characteristics  $x' = x'_0 + C t'$  for some  $C$  depending on  $u_{l,r}$ .
- $f_{k+1/2}(t) = f_{k+1/2}(u(x_{k+1/2}, t))$  is constant; (32) can be integrated exactly —note that  $u(x' = 0, t') = u(x_{k+1/2}, t)$  due to coordinate change.

# Godunov method example: Riemann problem

NB:

- ▶ While formula (29) exact, numerical approximation occurs because we continue from a piecewise constant profile with value  $U_k^n$  in cell  $k$  at a certain time  $t_n$ .
- ▶ At cell boundary  $x_{k+1/2}$  between cells  $k$  and  $k + 1$ , adjacent values are  $U_k^n$  and  $U_{k+1}^n$ .
- ▶ When other cell boundaries  $x_{k-1/2}$  and  $x_{k+3/2}$  are “far away” in sense that time step  $\Delta t$  is “small enough”, we are locally dealing with a Riemann problem around cell boundary  $x_{k+1/2}$ .
- ▶ Although Riemann solution no longer piecewise constant within each cell at end of time step, we continue only with information stored in the mean values  $U_k^{n+1}$ .
- ▶ This is the approximation made.

## Godunov method example: Riemann problem

- The characteristic form of (26)–(27) is as follows

$$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = u \quad (35)$$

$$\frac{dv}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = a, \quad (36)$$

which has solution

$$x = x_{01} + ut, u = u_0(x_{01}) \quad \text{s. t.} \quad u(x, t) = u_0(x - tu(x, t))$$

$$x = x_{02} + at, v = v_0(x_{02}) \quad \text{s. t.} \quad v(x, t) = v_0(x - at).$$

- At  $t = 0$  we see that  $x = x_{01}$  or  $x = x_{02}$  and these parameterise the domain for each equation.
- Instead of two PDEs, we have four ODEs.
- For constant  $a$ , solution linear advection equation is shift of original profile to left or right, depending on sign of  $a$ .

# Godunov method: studying

From Leveque's book *Finite volume methods*:

- ▶ Study §3.5 (Riemann problem), §3.6 (shock speed).
- ▶ Study §13.2 (Godunov method), §13.5 (Godunov for scalar conservation laws); read Chap. 9.

## Godunov method example: Riemann problem

The Riemann problem for the uncoupled system (26) special:  
consists of separate Riemann problems for two equations.

- ▶ Solution of Riemann problem for linear advection equation:

$$\frac{\partial v}{\partial t'} + \frac{\partial(a v)}{\partial x'} = 0 \quad (37)$$

with constant  $a > 0$ .

- ▶ NB: piecewise constant initial conditions

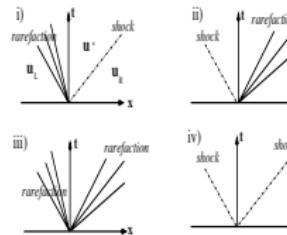
$$v(x', t' = 0) = \begin{cases} v_{left} = v_l = V_k^n & \text{for } x' < 0 \\ v_{right} = v_r = V_{k+1}^n & \text{for } x' \geq 0 \end{cases} . \quad (38)$$

- ▶ For  $a > 0$  all characteristics are  $x' = x_0 + a t'$ . Solution  
 $v(x' = 0, t') = v(x_{k+1/2}, t) = v_l^n$  with  $f(v(x_{k+1/2}, t)) = a v_l^n$   
 within a sufficiently small time interval.

Linear advection equation: Integral (32) straightforward to evaluate; Godunov scheme (31) becomes

$$V_k^{n+1} = V_k^n - \frac{\Delta t}{h_k} a (V_k^n - V_{k-1}^n). \quad (39)$$

- Correspondence of (39) with upwind finite difference discretization is clear (Chapter 4 in M& M), but  $V_k^n$  is mean value of  $v(x, t_n)$  over cell  $k$  and not a grid point value.



## Godunov method example: CFL condition

- Application of maximum principle to (39) yields, by imposing that all coefficients  $1$ ,  $(1 - \Delta t a / h_k)$  and  $a \Delta t / h_k$  of  $V_k^{n+1}$ ,  $V_k^n$ ,  $V_{k-1}^n$  are larger than zero (and  $a > 0$ ):

$$V_k^{n+1} = (1 - \frac{\Delta t}{h_k} a) V_k^n + \frac{\Delta t}{h_k} a V_{k-1}^n \quad (40)$$

$$\implies 1 - \frac{\Delta t}{h_k} a > 0 \iff \Delta t < \frac{h_k}{a}. \quad (41)$$

- The well-known Courant-Friedrichs-Lowy or CFL condition.
- When  $a < 0$ , the upwind discretization should be:

$$V_k^{n+1} = V_k^n + \frac{\Delta t}{h_k} |a| (V_{k+1}^n - V_k^n). \quad (42)$$

- For general  $a$ , the CFL condition thus reads  $\Delta t < h_k / |a|$ , which also makes sense dimensionally since  $a$  is a “wind” speed.

# Godunov method example: CFL condition

Graphically, upwinding for linear advection equation:

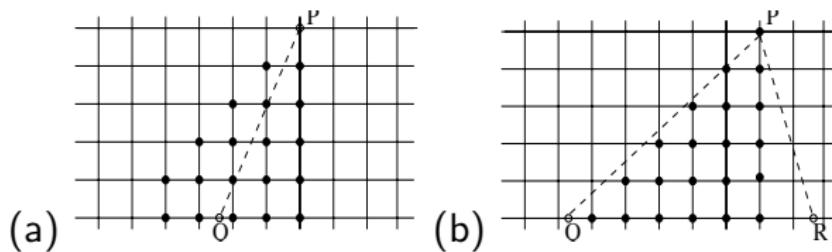


Figure: Consider the linear advection equation  $u_t + a u_x = 0$  with  $a > 0$ .  
(a) The solution  $u(x, t) = u^0(x - at)$  has a characteristic tracing through point  $P$  back to point  $Q$  satisfying the CFL condition  $\Delta t < \Delta x/|a|$ . (b) The CFL condition is violated when  $\Delta t > \Delta x/|a|$  as the information transported along the characteristic falls outside the discretization stencil. It shows the dependence of the numerical solution on the initial data.

## Godunov method example: Riemann problem

- ▶ Burgers' equation allows discontinuous or shock solutions, where  $u(x, t)$  obtains different limiting values.
- ▶ Discontinuity resides at position  $x = x_b(t)$  and moves with shock speed  $s \equiv dx_b/dt$ .
- ▶ Integrate Burgers' equation in (26) around  $x_b(t)$ , and let  $\epsilon \rightarrow 0$ , to obtain

$$\begin{aligned}
 & \lim_{\epsilon \rightarrow 0} \int_{x_b - \epsilon}^{x_b + \epsilon} \frac{\partial u(x, t)}{\partial t} dx + \int_{x_b - \epsilon}^{x_b + \epsilon} \frac{\partial (u^2/2)}{\partial x} dx = 0 \\
 \Leftrightarrow & \frac{\partial}{\partial t} \lim_{\epsilon \rightarrow 0} \int_{x_b(t) - \epsilon}^{x_b(t) + \epsilon} u(x, t) dx - \frac{dx_b(t)}{dt} \lim_{\epsilon \rightarrow 0} u(x, t) \Big|_{x_b(t) - \epsilon}^{x_b(t) + \epsilon} + [u^2/2] = 0 \quad (43) \\
 \Leftrightarrow & -\frac{dx_b(t)}{dt} [u] + [u^2/2] = 0 \Leftrightarrow s = \frac{dx_b(t)}{dt} = \frac{[u^2/2]}{[u]} = \frac{1}{2} (u_l + u_r)
 \end{aligned}$$

with  $[u] = u_r - u_l$ ,  $[u^2/2] = (u_r^2 - u_l^2)/2$ ,  $u_r = u(x, t)|_{x \downarrow x_b}$  and  $u_l = u(x, t)|_{x \uparrow x_b}$ .

- ▶ In literature, approximate numerical fluxes are found such as Roe solvers and kinetic fluxes.

# Godunov method example: Riemann problem Burgers

- ▶ *Exercise-Burgers:* (i) Derive the Riemann solution for the Riemann problem of the Burgers' equation.
- ▶ *Exercise-Burgers:* (ii) Complete the Godunov method for the Burgers' equation using this Riemann solution.
- ▶ See also

<https://www.youtube.com/watch?v=izMsj639hGI> and  
[https://www.youtube.com/watch?v=goL8\\_rET1H0](https://www.youtube.com/watch?v=goL8_rET1H0)

Provide both the actual solution in formula form as well as the solution in graphical form in the characteristic  $t,x$ -plane (or  $t',x'$ -plane).

# Godunov method example: Riemann problem Burgers

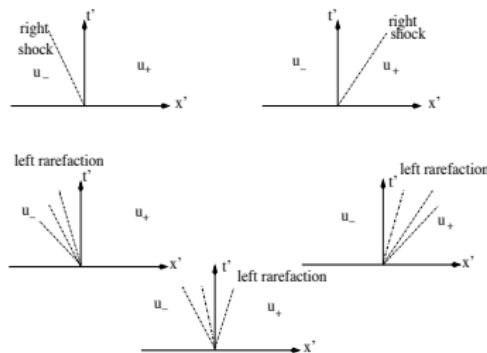


Figure: Graphical solution of Riemann problem for Burgers' equation.  
 $u_l > u_r$ : shock wave with shock speed  $s = (u_l + u_r)/2$ .  $u_l \leq u_r$ : rarefaction wave results in solution  $x'/t'$  in the interval  $u_l t' < x' < u_r t'$ .  $u_l$  and  $u_r$ : initial condition in definition Riemann problem.

# Godunov method example: Riemann problem Burgers

- ▶ Solution is constant along each characteristic by differentiating solution along the characteristic:

$$\frac{d}{dt} u(x(t), t) = \frac{\partial}{\partial t} u(x(t), t) + \frac{\partial}{\partial x} u(x(t), t) \frac{dx}{dt} = u_t + uu_x = 0.$$

- ▶ If solution is smooth then it can be constructed by backtracking characteristics to initial solution as long as they do not cross each other: For each  $(x, t)$  we can solve the following equations for  $\xi$ :

$$u = u_0(\xi) = u(\xi, 0) \quad \text{and} \quad x = \xi + u(\xi, 0)t$$

after which we can determine the solution at time  $t$ :

$$u(x, t) = u(\xi, 0).$$

- ▶ Hence, the implicit solution of Burgers' equation is  $u(x, t) = u_0(x - u(x, t)t)$ , since

$$0 = \frac{d}{dt} u(x(t), t) = \frac{\partial u(x, t)}{\partial t} + u \frac{\partial u(x, t)}{\partial x} \quad \text{and}$$

$$\partial_x u = u'_0(1 - t\partial_x u) \implies \partial_x u = \frac{u'_0}{1 + tu'_0}$$

with  $u'_0$  the derivative of  $u_0$  with respect to its argument.

# Godunov method example: Riemann problem Burgers

- ▶ The solution to the Riemann problem is either a shock wave or a rarefaction wave.
- ▶ It is a shock wave when  $u_l > u_r$  and a rarefaction wave when  $u_l < u_r$ , which follows from considering the characteristics  $dx/dt = u$  in the  $x-t$ -plane.
- ▶ The shock wave has shock speed  $s = (u_l + u_r)/2$  and its position is given by  $x' = s t'$ ; to the left of the shock  $u(x', t) = u_l$  and to the right  $u(x', t) = u_r$ .
- ▶ Since the numerical flux is evaluated at  $x' = 0$  (i.e. at  $x = x_{k+1/2}$ ), the flux  $u^2/2$  is thus either  $u_l^2/2 = (U_k^n)^2/2$  when  $s > 0$ , or  $u_r^2/2 = (U_{k+1}^n)^2/2$  when  $s < 0$  for the shock wave case.

## Godunov method example: Riemann problem Burgers

- ▶ The rarefaction wave has characteristics  $dx'/dt' = u$  on which  $u$  is constant. The tail and the head of the rarefaction wave lie at  $x' = u_l t'$  and  $x' = u_r t'$ , respectively.
- ▶ Hence the rarefaction wave solution is

$$u(x', t') = \begin{cases} u_l & x' < u_l t' \\ x'/t' & u_l t' < x' < u_r t' \\ u_r & x' > u_r t' \end{cases} . \quad (44)$$

- ▶ We deduce from this solution that  $u_l < u_r$ . So at  $x' = 0$ , or  $x = x_{j+1/2}$ , we find for the rarefaction wave case that  $u(0, t') = u_l$  when  $u_l > 0$ ,  $u(0, t') = 0$  when  $u_l < 0$  and  $u_r > 0$ , and  $u(0, t') = u_r$  when  $u_r < 0$ .
- ▶ Note that  $u(x', t') = x'/t'$  is a similarity solution of Burgers' equation.

## Godunov method example: Riemann problem Burgers

- Numerical flux function  $F$  at each face  $x_{k+1/2}$  is defined as:

$$F(U_k^n, U_{k+1}^n) = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \frac{1}{2} \left( u^\dagger(U_k^n, U_{k+1}^n) \right)^2 dt \quad (45)$$

with  $u^\dagger(U_k^n, U_{k+1}^n) = u(x_{k+1/2}, t)$  for the special piecewise constant data at time  $t_n$ .

- Finally, we need to evaluate the integral (45) by combining these shock wave and rarefaction wave solutions for  $u_l > u_r$  and  $u_l < u_r$  respectively.
- Note that the solution is constant at  $x' = x - x_{k+1/2} = 0$ , which simplifies the time integration in (45).

# Godunov method example: Riemann problem

Homework Exercise-II.