Numerical Exercise 1

1a). $\partial_t b + \partial_z(\alpha b^3 - \beta b^3 \partial_z b) = 0$

$$= \partial_t b + 3\alpha b^2 \partial_z b - 3\beta b^2 (\partial_z b)^2 - \beta b^3 \partial_{zz} b$$

$$b = D_0 + b'$$

$$= \partial_t b' + 3\alpha (D_0 + b')^2 \partial_z b' - 3\beta b^2 (\partial_z b')^2 - \beta (D_0 + b')^3 \partial_{zz} b'$$

$$= \partial_t b' + 3\alpha (D_0^2 + 2D_0 b' + b'^2) \partial_z b' - \beta (D_0^3 + 3D_0^2 b' + 3D_0 b'^2 + b'^3) \partial_{zz} b' + O(b'^2)$$

$$\boxed{= \partial_t b' + 3\alpha D_0^2 \partial_z b' - \beta D_0^3 \partial_{zz} b' = 0}$$

(5) and (6) both contain terms that describe diffusion and convection

$\Rightarrow$ as diffusion equation is in the form: $u_t = \sigma u_{xx}$

and convection equation is in the form:
$$u_t + c u_x = 0$$

b). $\partial_t b = \dfrac{b_j^{n+1} - b_j^n}{\Delta t}$

$\partial_z(\alpha b^3)$:

Using upwind scheme, need to consider $\alpha > 0$

$$\partial_z(\alpha b^3) = \alpha\left[\frac{(b_j^n)^3 - (b_{j-1}^n)^3}{\Delta z}\right]$$

$-\beta\, \partial_z(b^3 \partial_z b)$

$$= \frac{-\beta}{\Delta z}\left[(b_{j+1/2}^n)^3\left(\frac{b_{j+1}^n - b_j^n}{\Delta z}\right) - (b_{j-1/2}^n)^3\left(\frac{b_j^n - b_{j-1}^n}{\Delta z}\right)\right]$$

$$= \frac{-\beta}{(\Delta z)^2}\left[(b_{j+1/2}^n)^3(b_{j+1}^n - b_j^n) - (b_{j-1/2}^n)^3(b_j^n - b_{j-1}^n)\right]$$

Using $b_{j+1/2}^n = \dfrac{b_{j+1}^n + b_j^n}{2}$

$$= -\frac{\beta}{2^3(\Delta z)^2}\left[(b_{j+1}^n + b_j^n)^3(b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3(b_j^n - b_{j-1}^n)\right]$$

(5) can be written as:

$$\boxed{\frac{b_j^{n+1}-b_j^n}{\Delta t}+\frac{\alpha}{\Delta z}\left[(b_j^n)^3-(b_{j-1}^n)^3\right] - \frac{\beta}{8(\Delta z)^2}\left[(b_{j+1}^n+b_j^n)^3(b_{j+1}^n-b_j^n)-(b_j^n+b_{j-1}^n)^3(b_j^n-b_{j-1}^n)\right]}$$

6 : $3\alpha D_0^2 \partial_z b'$

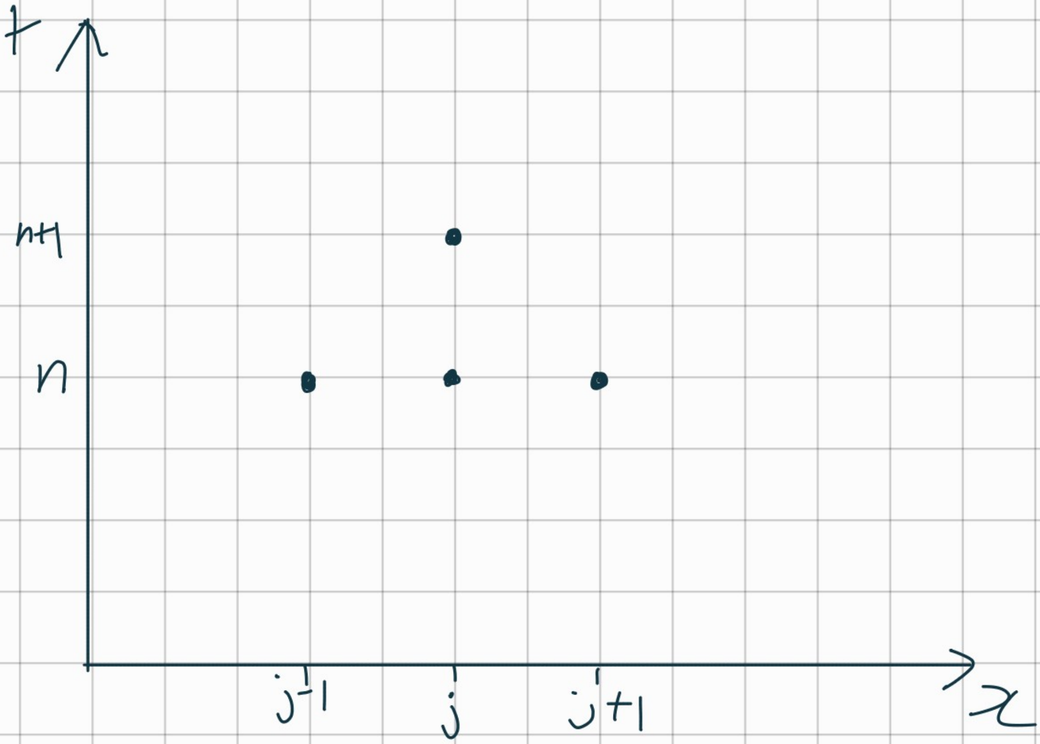
$$= 3\alpha D_0^2 \frac{b'^n_j - b'^n_{j-1}}{\Delta z}$$

$$-\beta D_0^3 \partial_{zz} b'$$

$$= -\frac{\beta D_0^3}{(\Delta z)^2}\left(b'^n_{j+1} - 2b'^n_j + b'^n_{j-1}\right)$$

6 can be written as:

$$\boxed{\frac{b'^{n+1}_j - b'^n_j}{\Delta t} + \frac{3\alpha D_0^2}{\Delta z}\left(b'^n_j - b'^n_{j-1}\right) - \frac{\beta D_0^3}{(\Delta z)^2}\left(b'^n_{j+1} - 2b'^n_j + b'^n_{j-1}\right) = 0}$$



Linearising ⑤: $(b = D_0 + b')$

$$\frac{b'^{\,n+1}_j - b'^{\,n}_j}{\Delta t} + \frac{\alpha}{\Delta z}\left[\left(D_0 + b'^{\,n}_j\right)^3 - \left(D_0 + b'^{\,n}_{j-1}\right)^3\right]$$

$$- \frac{\beta}{8(\Delta z)^2}\Big[\left(2D_0 + b'^{\,n}_{j+1} + b'^{\,n}_j\right)^3\left(b'^{\,n}_{j+1} - b'^{\,n}_j\right)$$

$$- \left(2D_0 + b'^{\,n}_j + b'^{\,n}_{j-1}\right)^3\left(b'^{\,n}_j - b'^{\,n}_{j-1}\right)\Big]$$

$$= 0$$

$$\frac{b_j'^{n+1} - b_j'^{n}}{\Delta t} + \frac{\alpha}{\Delta z}\left[D_0^3 + 3D_0^2 b_j'^n - D_0^3 - 3D_0^2 b_{j-1}'^n + O(b')^2 \right]$$

$$- \frac{\beta}{8(\Delta z)^2}\left[8D_0^3 (b_{j+1}'^n - b_j'^n) - 8D_0^3 (b_j'^n - b_{j-1}'^n) + O(b')^2 \right]$$

$$= \frac{b_j'^{n+1} - b_j'^{n}}{\Delta t} + \frac{3\alpha D_0^2}{\Delta z}(b_j'^n - b_{j-1}'^n)$$

$$- \frac{\beta D_0^3}{(\Delta z)^2}(b_{j+1}'^n - 2b_j'^n + b_{j-1}'^n)$$

$$= 6$$

Using adjoint form for diffusive term

$\downarrow$

$$\partial_t b - \beta \partial_z (b^3 \partial_z b) = 0$$

This means we can discretise the equation in its original form

Boundary conditions:

$$b(0,t) = b_B \quad , \quad b(H,t) = b_T$$

Initial conditions:

$$b(z,0) = b_i(z)$$

$$\left.\begin{aligned} b(0,t) = b_B &\rightarrow b_0^n = b_B \\ b(H,t) = b_T &\rightarrow b_{J-1}^n = b_T \end{aligned}\right\} \text{ for all values of } n$$

$$b(z,0) = b_i(z) \rightarrow b_j^0 = b_i(j\Delta z)$$

c). Letting $b_j^n = (\lambda)^n e^{ikj\Delta z}$

Sub into linearised discretisation:

$$\frac{\lambda \cancel{b_j^n} - \cancel{b_j^n}}{\Delta t} + \frac{3\alpha D_0^2}{\Delta z}\left(1 - e^{-ik\Delta z}\right)\cancel{b_j^n}$$

$$-\frac{\beta D_0^3}{(\Delta z)^2}\left(e^{ik\Delta z} - 2 + e^{-ik\Delta z}\right)\cancel{b_j^n} = 0$$

$\alpha = 0, \beta \neq 0$

$$\frac{\lambda - 1}{\Delta t} - \frac{\beta D_0^3}{(\Delta z)^2}(2i)^2 \sin^2\left(\frac{k\Delta z}{2}\right) = 0$$

$$\lambda = 1 - 4\beta D_0^3\left(\frac{\Delta t}{(\Delta z)^2}\right)\sin^2\left(\frac{k\Delta z}{2}\right)$$

Stable when $|\lambda| \leq 1$

$-4\beta D_0^3\left(\frac{\Delta t}{(\Delta z)^2}\right)\sin^2\left(\frac{k\Delta z}{2}\right)$ is always negative

so need to check if $1 - 4\beta D_0^3 \frac{\Delta t}{(\Delta z)^2} \geq -1$

$$4\beta D_0^3 \frac{\Delta t}{(\Delta z)^2} \leq 2$$

$$\frac{\Delta t}{(\Delta z)^2} \leq \frac{1}{2\beta D_0^3}$$

$$\boxed{\begin{array}{l}\text{Scheme for } \alpha = 0, \beta \neq 0 \text{ is conditionally stable} \\ \text{and } \Delta t \leq \dfrac{(\Delta z)^2}{2\beta D_0^3}\end{array}}$$

$\alpha \neq 0, \beta = 0$

$$\frac{\lambda - 1}{\Delta t} + \frac{3\alpha D_0^2}{\Delta z}\left(1 - e^{-ik\Delta z}\right) = 0$$

$$\lambda = 1 - 3\alpha D_0^2 \frac{\Delta t}{\Delta z}\left(1 - e^{-ik\Delta z}\right)$$

$$= 1 - 3\alpha D_0^2 \frac{\Delta t}{\Delta z}\left(1 - \cos(k\Delta z) + i\sin(k\Delta z)\right)$$

$$= 1 - 3\alpha D_0^2 \frac{\Delta t}{\Delta z}\left(1 - \cos(k\Delta z)\right) - i 3\alpha D_0^2 \frac{\Delta t}{\Delta z} \sin(k\Delta z)$$

let $\gamma = 3\alpha D_0^2 \frac{\Delta t}{\Delta z}$

$\downarrow$

$\lambda = 1 - \gamma(1-\cos k\Delta z) - i\gamma \sin k\Delta z$

$|\lambda|^2 = (1-\gamma(1-\cos k\Delta z))^2 + (\gamma \sin k\Delta z)^2$

$= (1-\gamma + \gamma\cos(k\Delta z))^2 + (\gamma \sin k\Delta z)^2$

$2|\lambda| \frac{\partial\lambda}{\partial(k\Delta z)} = 2(1-\gamma+\gamma\cos(k\Delta z))(-\gamma\sin(k\Delta z))$

$+ 2\gamma^2 \sin k\Delta z \cos k\Delta z \quad = 0$

when $|\lambda|$ is at its max or min.

$\rightarrow (1-\gamma)\sin(k\Delta z) = 0$

so min/max when $(k\Delta z) = n\pi$

$k\Delta z = \pi$

$|\lambda|^2 = (1-2\gamma)^2$

$1 - 2\gamma \geq -1$

$\gamma \leq 1$

$$3\alpha D_0^2 \frac{\Delta t}{\Delta z} \leq 1$$

$$\Delta t \leq \frac{\Delta z}{3\alpha D_0^2}$$

When $\alpha \neq 0, \beta = 0$, scheme is conditionally stable

for $\Delta t \leq \frac{\Delta z}{3\alpha D_0^2}$

d).
$$\frac{b_j^{n+1}-b_j^n}{\Delta t}+\frac{3\alpha D_0^2}{\Delta z}\left(b_j^n-b_{j-1}^n\right)$$
$$-\frac{\beta D_0^3}{(\Delta z)^2}\left(b_{j+1}^n-2b_j^n+b_{j-1}^n\right)=0$$

$$b_j^{n+1}=b_j^n-\frac{3\alpha D_0^2\Delta t}{\Delta z}\left(b_j^n-b_{j-1}^n\right)+\frac{\beta D_0^3\Delta t}{(\Delta z)^2}\left(b_{j+1}^n-2b_j^n+b_{j-1}^n\right)$$
$$=b_j^n\left[1-\frac{3\alpha D_0^2\Delta t}{\Delta z}-\frac{2\beta D_0^3\Delta t}{(\Delta z)^2}\right]$$
$$+b_{j+1}^n\left[\frac{\beta D_0^3\Delta t}{(\Delta z)^2}\right]$$
$$+b_{j-1}^n\left[\frac{3\alpha D_0^2\Delta t}{\Delta z}+\frac{\beta D_0^3\Delta t}{(\Delta z)^2}\right]$$

According to maximum principle, ~~sum of those coefficients = 1~~, and all coefficients must be non-negative.

$$\Rightarrow \text{so} \quad 1-\frac{3\alpha D_0\Delta t}{\Delta z}-\frac{2\beta D_0^3\Delta t}{(\Delta z)^2}\geqslant 0$$

$$\Delta t\left[\frac{2\beta D_0^3}{(\Delta z)^2}+\frac{3\alpha D_0^2}{\Delta z}\right]\leq 1$$

$$\boxed{\Delta t \leq \frac{\Delta z}{D_0^2} \cdot \frac{1}{\left(\frac{2\beta D_0}{\Delta z} + 3\alpha\right)}}$$

(and when looking at the $\alpha = 0$, $\beta \neq 0$ and the $\alpha \neq 0$, $\beta = 0$ cases from c, this gives same timestep limitation).

e).

$$\frac{B_j^{n+1}-B_j^n}{\Delta t}+\frac{\alpha}{\Delta z}\left[\left(B_j^n\right)^3-\left(B_{j-1}^n\right)^3\right]$$

$$-\frac{\beta}{8(\Delta z)^2}\Big[\left(B_{j+1}^n+B_j^n\right)^3\left(B_{j+1}^n-B_j^n\right)$$

$$-\left(B_j^n+B_{j-1}^n\right)^3\left(B_j^n-B_{j-1}^n\right)\Big]=0$$

$$B_j^{n+1}=B_j^n-\frac{\alpha\Delta t}{\Delta z}\left[\left(B_j^n\right)^3-\left(B_{j-1}^n\right)^3\right]$$

$$+\frac{\beta\Delta t}{8(\Delta z)^2}\left[\left(B_{j+1}^n+B_j^n\right)^3\left(B_{j+1}^n-B_j^n\right)-\left(B_j^n+B_{j-1}^n\right)^3\left(B_j^n-B_{j-1}^n\right)\right]$$

$$=B_j^n\Big(1-\frac{\alpha\Delta t}{\Delta z}\left(B_j^n\right)^2-\frac{\beta\Delta t}{8(\Delta z)^2}\left(B_{j+1}^n+B_j^n\right)^3$$

$$-\frac{\beta\Delta t}{8(\Delta z)^2}\left(B_j^n+B_{j-1}^n\right)^3\Big)$$

$$+B_{j-1}^n\left(\frac{\alpha\Delta t}{\Delta z}\left(B_{j-1}^n\right)^2+\frac{\beta\Delta t}{8(\Delta z)^2}\left(B_j^n+B_{j-1}^n\right)^3\right)$$

$$+B_{j+1}^n\left(\frac{\beta\Delta t}{8(\Delta z)^2}\left(B_{j+1}^n+B_j^n\right)^3\right)$$

For $B_j^{n+1} > 0$

$$1 - \frac{\alpha \Delta t}{\Delta z}(B_j^n)^2 - \frac{\beta \Delta t}{8(\Delta z)^2}\left[(B_{j+1}^n + B_j^n)^3 + (B_j^n + B_{j-1}^n)^3\right]$$

$$> 0$$

$$\Delta t < \frac{1}{\frac{\alpha (B_j^n)^2}{\Delta z} + \frac{\beta}{8(\Delta z)^2}\left[(B_{j+1}^n + B_j^n)^3 + (B_j^n + B_{j-1}^n)^3\right]}$$

Worse case scenario where $B_j^n = B_{j+1}^n = B_{j-1}^n = B_{max}$

$$\Delta t < \frac{1}{B_{max}^2\left[\frac{\alpha}{\Delta z} + \frac{2\beta B_{max}}{(\Delta z)^2}\right]}$$

5). $\partial_z(\alpha b^3)$:

Using upwind scheme :

$$\partial_z(\alpha b^3) = \alpha \left[\frac{(b_j^n)^3 - (b_{j-1}^n)^3}{\Delta z} \right]$$

Instead, using central difference method :

$$\partial_z(\alpha b^3) = 3\alpha b^2 \partial_z b$$

$$= 3\alpha (b_j^n)^2 \left[\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta z} \right]$$

Using $\quad b_{j+1/2}^n = \dfrac{b_{j+1} + b_j}{2}$

$$b_{j-1/2}^n = \frac{b_j + b_{j-1}}{2}$$

this becomes

$$\frac{3\alpha (b_j^n)^2}{2\Delta z} \left[b_{j+1} - b_{j-1} \right]$$

(5) becomes:

$$\frac{B_j^{n+1} - B_j^n}{\Delta t} + \frac{3\alpha (B_j^n)^2}{2\Delta z}\left(B_{j+1}^n - B_{j-1}^n\right)$$

$$- \frac{\beta}{8(\Delta z)^2}\Big[\left(B_{j+1}^n + B_j^n\right)^3\left(B_{j+1}^n - B_j^n\right) - \left(B_j^n + B_{j-1}^n\right)^3\left(B_j^n - B_{j-1}^n\right)\Big] = 0$$

$$B_j^{n+1} = B_j^n\left[1 - \frac{\beta\Delta t}{8(\Delta z)^2}\left(\left(B_{j+1}^n + B_j^n\right)^3 + \left(B_j^n + B_{j-1}^n\right)^3\right)\right]$$

$$+ B_{j+1}^n\left[-\frac{3\alpha (B_j^n)^2 \Delta t}{2\Delta z} + \frac{\beta\Delta t}{8(\Delta z)^2}\left(B_{j+1}^n + B_j^n\right)^3\right]$$

$$+ B_{j-1}^n\left[\frac{3\alpha (B_j^n)^2 \Delta t}{2\Delta z} + \frac{\beta\Delta t}{8(\Delta z)^2}\left(B_j^n + B_{j-1}^n\right)^3\right]$$

Coefficient of B_{j-1}^n is always ≥ 0

$\rightarrow$ but not true for B_{j+1}^n coefficient, or

B_j^n

$$B_j^n\left[1-\frac{\beta\Delta t}{8(\Delta z)^2}\left(\left(B_{j+1}^n+B_j^n\right)^3+\left(B_j^n+B_{j-1}^n\right)^3\right)\right]$$

$$+B_{j+1}^n\left[-\frac{3\alpha(B_j^n)^2\Delta t}{2\Delta z}+\frac{\beta\Delta t}{8(\Delta z)^2}\left(B_{j+1}^n+B_j^n\right)^3\right]>0$$

$$\Delta t\left[\frac{\beta B_j^n}{8(\Delta z)^2}\left(\left(B_{j+1}^n+B_j^n\right)^3+\left(B_j^n+B_{j-1}^n\right)^3\right)\right.$$

$$\left.-\frac{\beta B_{j+1}^n}{8(\Delta z)^2}\left(B_{j+1}^n+B_j^n\right)^3+\frac{3\alpha B_{j+1}^n(B_j^n)^2}{2\Delta z}\right]<B_j^n$$

$$\Delta t<\frac{1}{\left(\frac{\beta a}{8(\Delta z)^2}+\frac{3\alpha}{2\Delta z}B_{j+1}^nB_j^n-\frac{\beta\, b}{8(\Delta z)^2}\right)^{*}}$$

where $a=B_j^n\left[\left(B_{j+1}^n+B_j^n\right)^3+\left(B_j^n+B_{j-1}^n\right)^3\right]$

$$b=\frac{B_{j+1}^n}{B_j^n}\left(B_{j+1}^n+B_j^n\right)^3$$

__BUT__ this time step criterion only guarantees $B_j^{n+1}>0$ if $*$ is positive, otherwise it is always unstable.

2a). $\partial_z(\alpha b^3 - \beta b^3 \partial_z b) = 0$

$\Rightarrow \alpha b^3 - \beta b^3 \frac{db}{dz} = Q$ where Q is an integration constant.

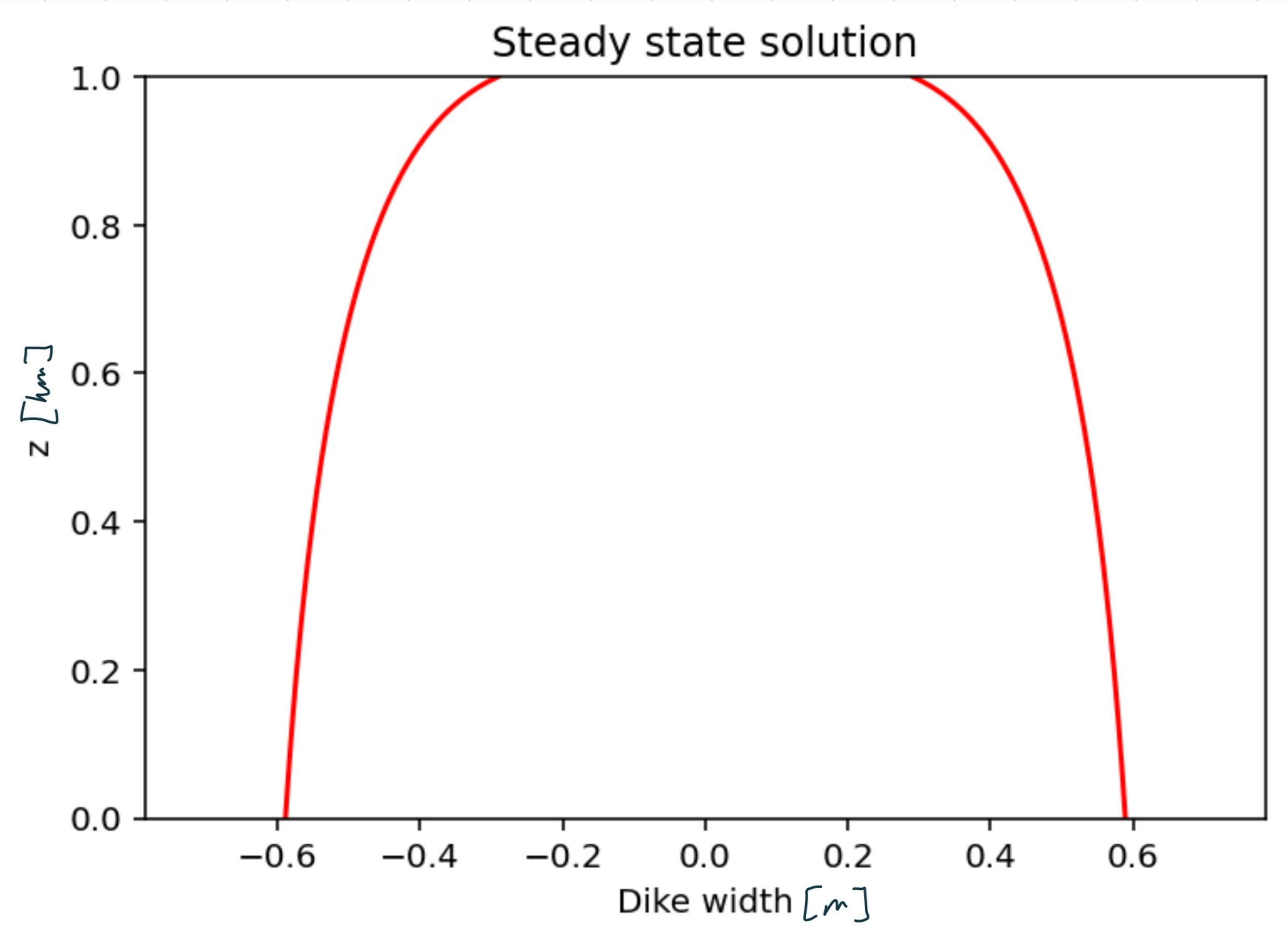
$$\boxed{\beta b^3 \frac{db}{dz} = \alpha b^3 - Q}$$

$$\beta b^3 \left(\frac{b_{j+1} - b_j}{\Delta z}\right) = \alpha b^3 - Q$$

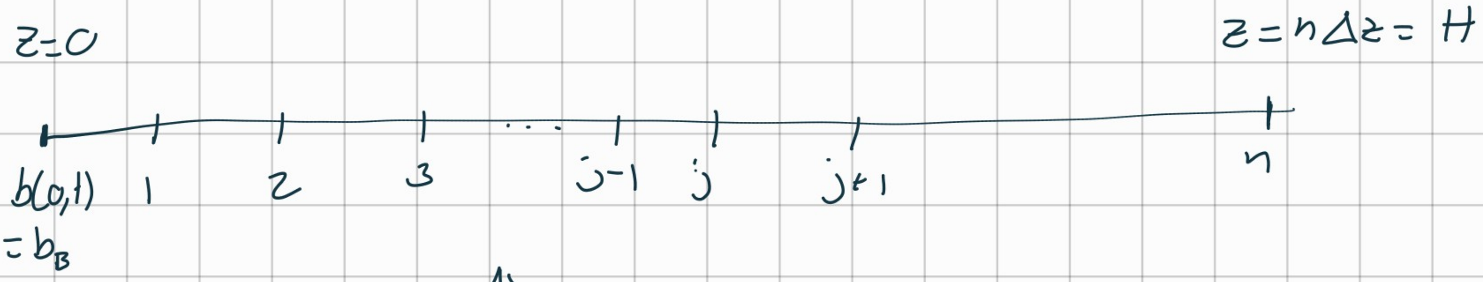
$$\beta b_j^3 \left(\frac{b_{j+1} - b_j}{\Delta z}\right) = \alpha b_j^3 - Q$$

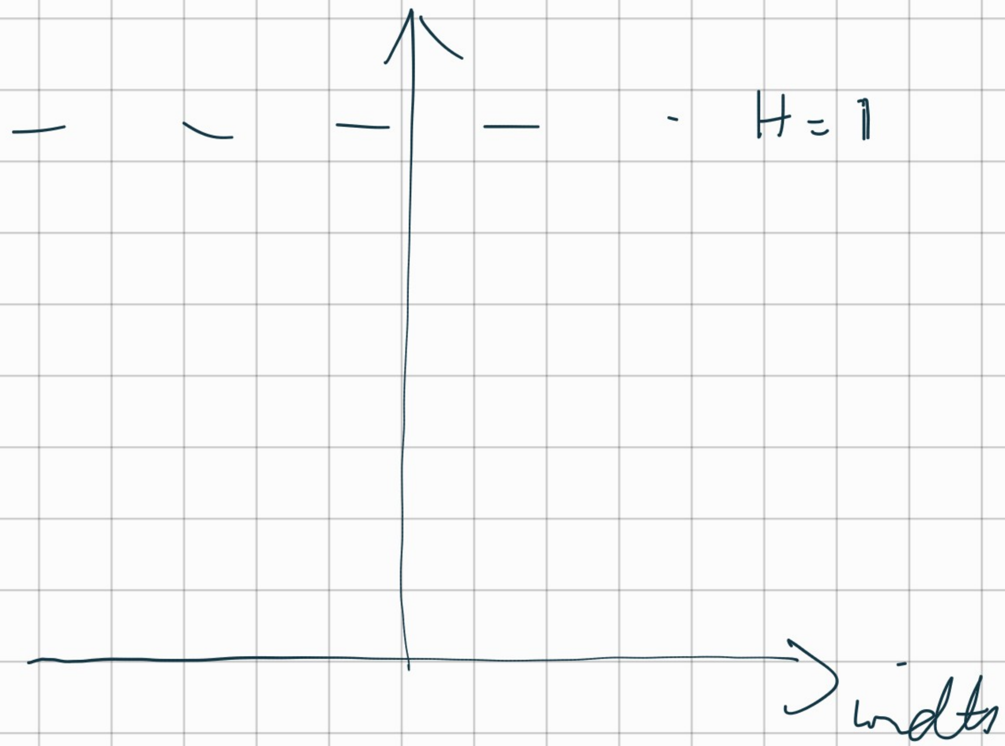
$$\frac{b_{j+1} - b_j}{\Delta z} = \frac{\alpha}{\beta} - \frac{Q}{\beta b_j^3}$$

$$\boxed{b_{j+1} = b_j + \frac{\Delta z}{\beta}\left(\alpha - \frac{Q}{b_j^3}\right)}$$



Steady state solution
1.0
0.8
0.6
0.4
0.2
0.0
z [km]
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
Dike width [m]



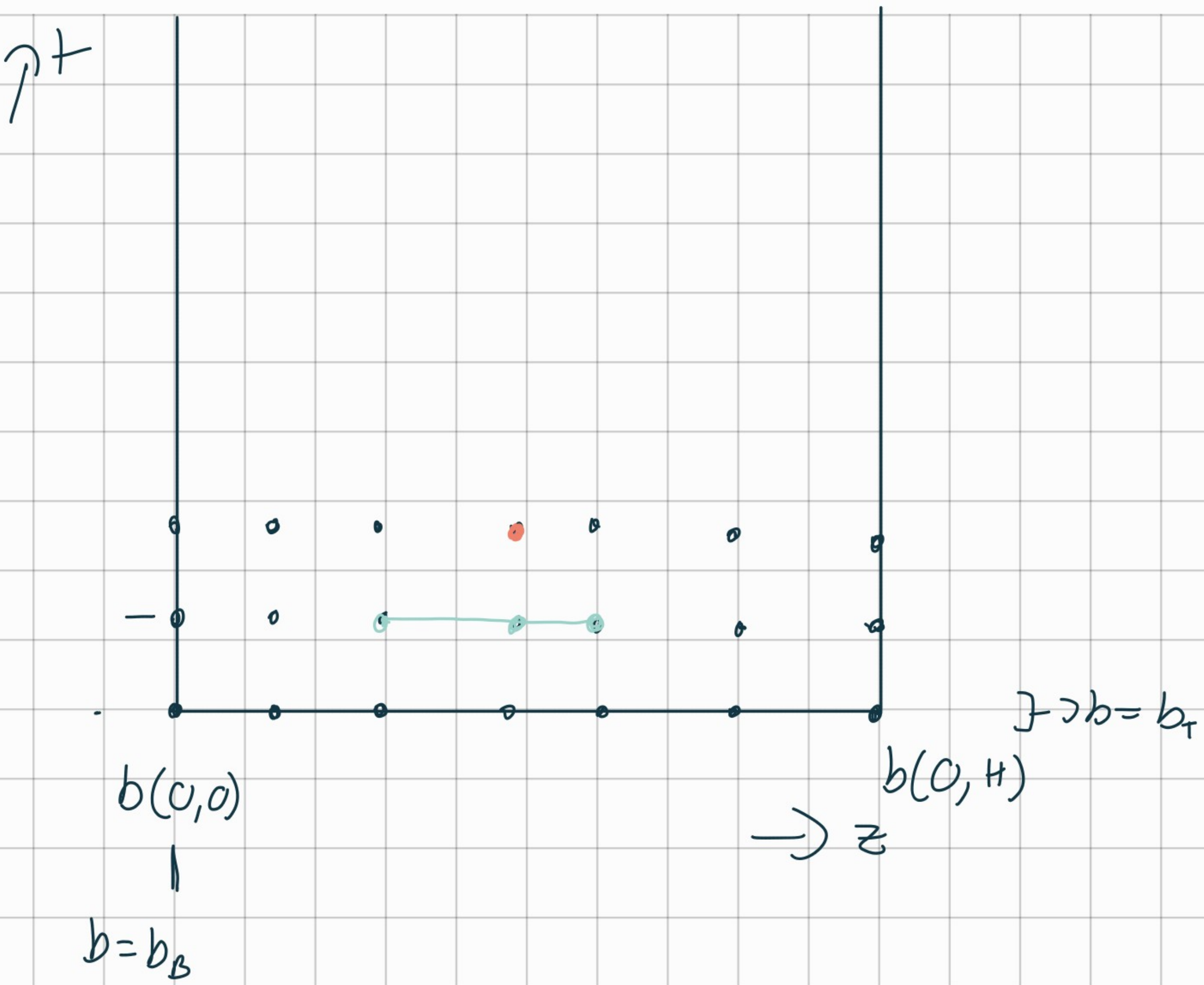


b)

$$\boxed{\frac{b_j^{n+1} - b_j^n}{\Delta t} + \frac{\alpha}{\Delta z}\left[(b_j^n)^3 - (b_{j-1}^n)^3\right] - \frac{\beta}{8(\Delta z)^2}\left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n)\right]}$$

$$b_j^{n+1} = b_j^n - \frac{\alpha \Delta t}{\Delta z}\left[(b_j^n)^3 - (b_{j-1}^n)^3\right]$$

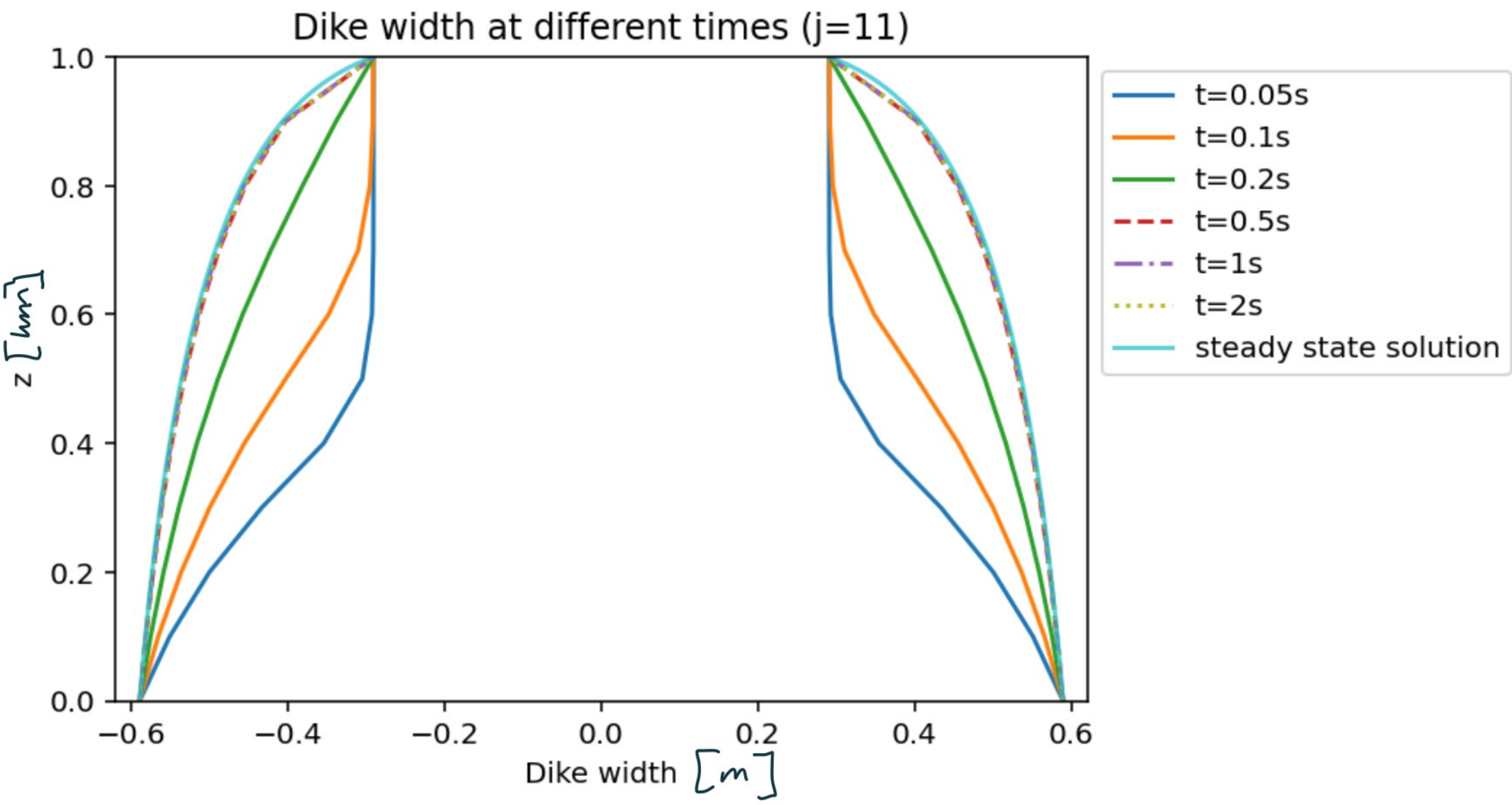
$$+ \frac{\beta \Delta t}{8(\Delta z)^2}\left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n)\right]$$



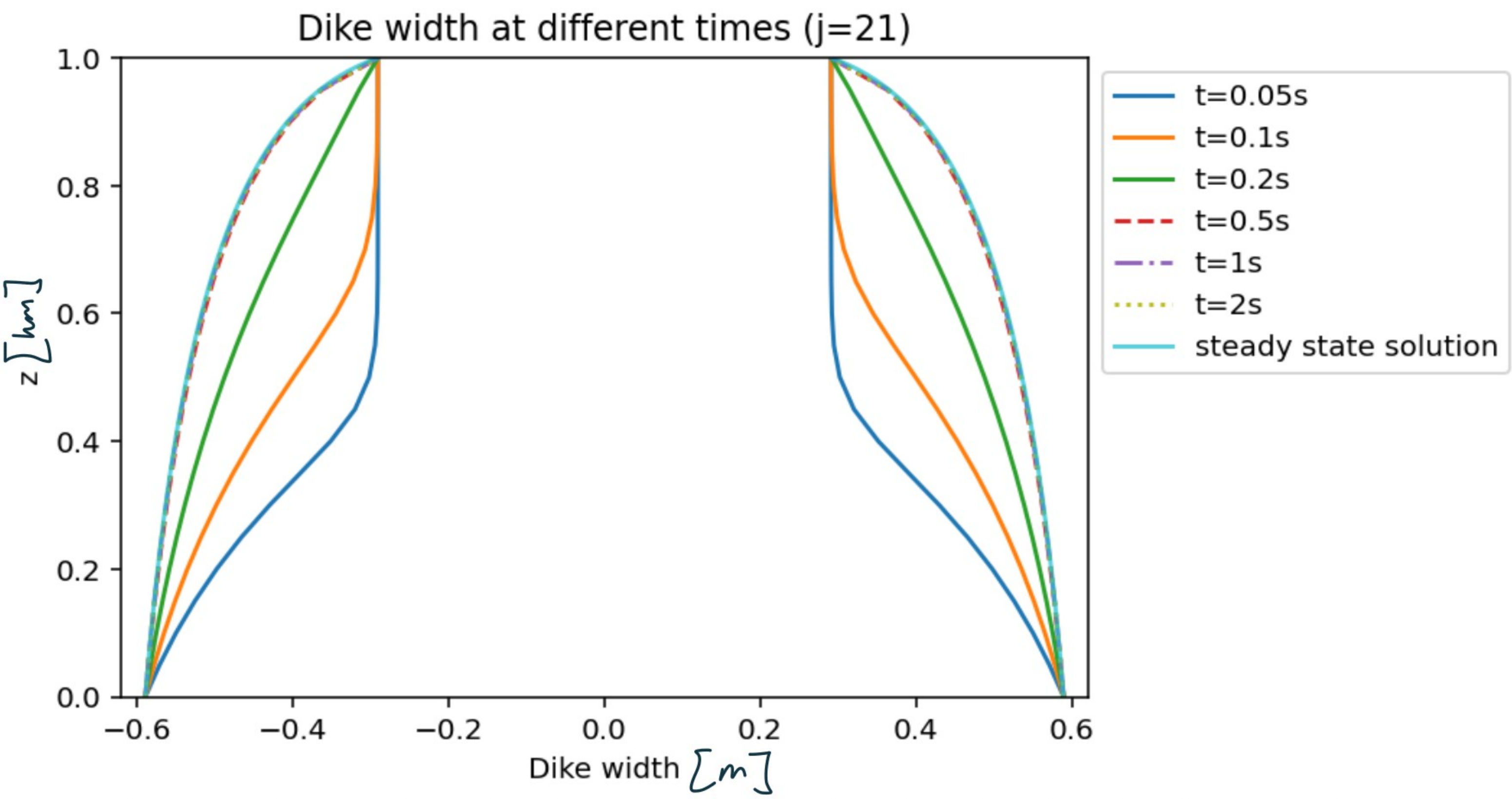
Δt chosen by finding maximum value of Δt for a stable scheme as found in earlier question

$$\Delta t < \frac{1}{B_{max}^2 \left[\frac{\alpha}{\Delta z} + \frac{2\beta B_{max}}{(\Delta z)^2} \right]}$$

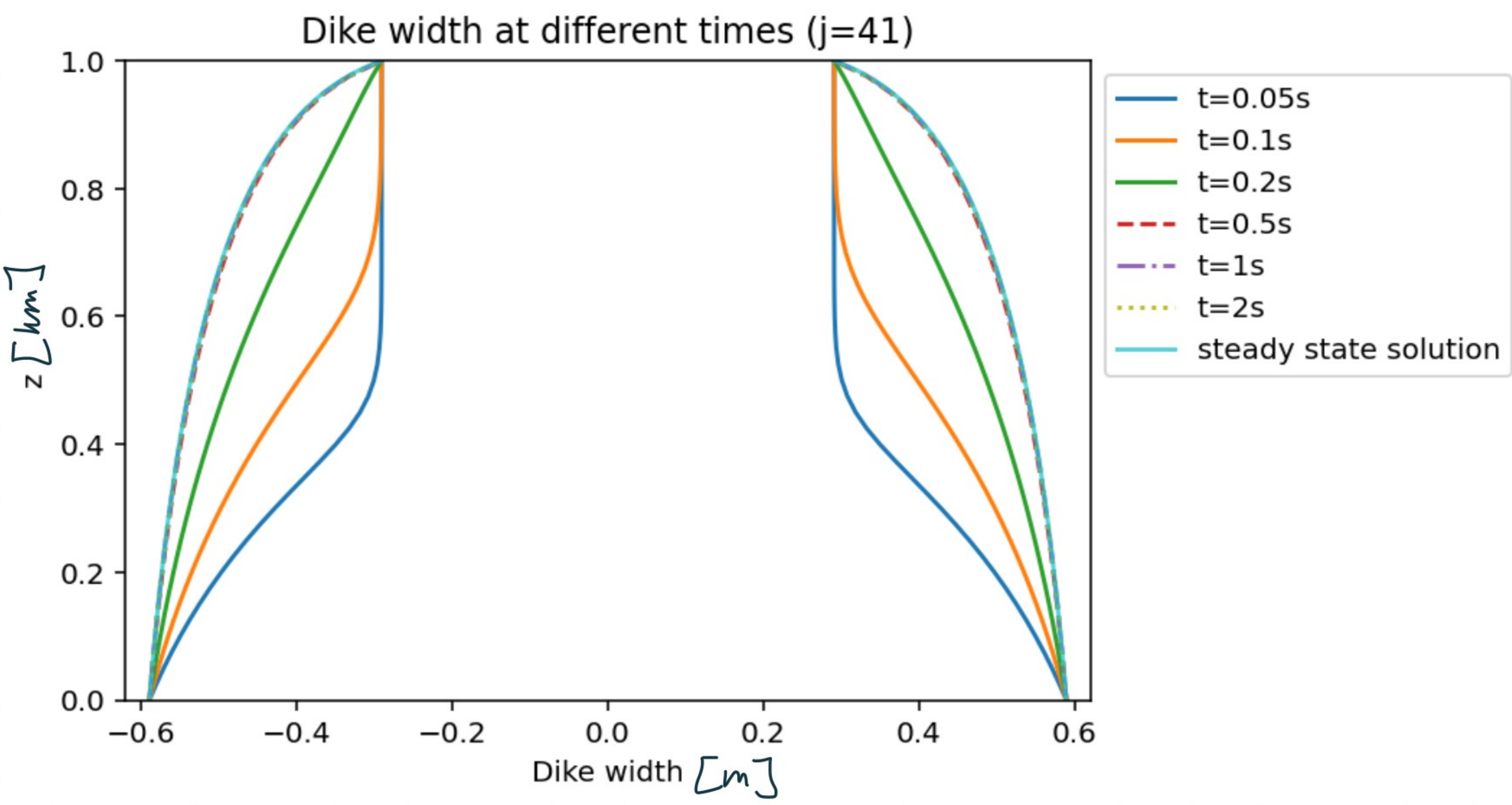
→ avoid setting Δt as value slightly less than that.



Dike width at different times (j=11)
t=0.05s
t=0.1s
t=0.2s
t=0.5s
t=1s
t=2s
steady state solution
z [km]
Dike width [m]



Dike width at different times (j=21)
t=0.05s
t=0.1s
t=0.2s
t=0.5s
t=1s
t=2s
steady state solution
z [km]
Dike width [m]


Dike width at different times (j=41)
z [km]
Dike width [m]
t=0.05s
t=0.1s
t=0.2s
t=0.5s
t=1s
t=2s
steady state solution

c). $L^2 = \sqrt{\int_0^H e^2(z,t)\,dz}$

$$= \sqrt{\frac{H}{j-1} \times \frac{1}{2}\left[e^2(0,2) + e^2(H,2) + 2\sum_{n=1}^{j-2} e^2\left(\frac{nH}{j-1}, 2\right)\right]}$$

Calculating L^2 and L^∞ using results:

j	Δz	L^2	L^∞
11	1/10	0.003633	0.005457
21	1/20	0.002212	0.003384
41	1/40	0.001235	0.001926

as Δz decreases by a factor of 2,

L^2 decrease by factor of $\sim 1.5 - 1.7$

L^∞ decreases by factor of $\sim 1.6 - 1.8$

So order of spatial discretisation $\sim 1^{st}$ order in Δz

d). $z - z_{r_0} - ct = \frac{\beta}{\alpha}\left(b(z,t) - \sqrt{\frac{c}{\alpha}} \operatorname{arctanh}\left(\sqrt{\frac{\alpha}{c}}\, b(z,t) \right) \right)$

$$b = \frac{\alpha}{\beta}\left(z - z_{r_0} - ct\right) + \sqrt{\frac{c}{\alpha}} \operatorname{arctanh}\left(\sqrt{\frac{\alpha}{c}}\, b \right)$$

$$\partial_t b = -\frac{\alpha c}{\beta} + \sqrt{\frac{c}{\alpha}} \cdot \sqrt{\frac{\alpha}{c}}\, \partial_t b \times \frac{1}{1 - \frac{b^2 \alpha}{c}}$$

$$= -\frac{\alpha c}{\beta} + \partial_t b \times \frac{1}{1 - \frac{b^2\alpha}{c}}$$

$$\partial_t b \left(1 - \frac{1}{1 - \frac{b^2 \alpha}{c}} \right) = -\frac{\alpha c}{\beta}$$

$$\partial_t b \left(\frac{-\frac{b^2 \alpha}{c}}{1 - b^2 \alpha / c} \right) = -\frac{\alpha c}{\beta}$$

$$\boxed{\partial_t b = \frac{c^2}{b^2 \beta} \times \left(1 - \frac{b^2 \alpha}{c} \right)}$$

$$\partial_z b = \frac{\alpha}{\beta} + \partial_z b \times \frac{1}{(1 - b^2\alpha/c)}$$

$$\boxed{\partial_z b = -\frac{c}{b^2\beta} \times \left(1 - \frac{b^2\alpha}{c}\right)}$$

$$\partial_z(\alpha b^3) = 3\alpha b^2 \partial_z b$$

$$= -\frac{3\alpha c}{\beta} \times \left(1 - \frac{b^2\alpha}{c}\right)$$

$$\partial_z(-\beta b^3 \partial_z b) = \partial_z\left(c b \left(1 - \frac{b^2\alpha}{c}\right)\right)$$

$$= c\,\partial_z\left(b - \frac{b^3\alpha}{c}\right)$$

$$= c\left(1 - \frac{3b^2\alpha}{c}\right)\partial_z b$$

$$= -c\left(1 - \frac{3b^2\alpha}{c}\right) \times \frac{c}{b^2\beta}\left(1 - \frac{b^2\alpha}{c}\right)$$

$$= -\frac{c^2}{b^2\beta}\left(1-\frac{3b^2\alpha}{c}\right)\left(1-\frac{b^2\alpha}{c}\right)$$

Combine

$$\partial_t b + \partial_z(\alpha b^3) - \partial_z(\beta b^3 \partial_z b)$$

$$= \frac{c^2}{b^2\beta}\left(1-\frac{b^2\alpha}{c}\right) - \frac{3\alpha c}{\beta}\left(1-\frac{b^2\alpha}{c}\right)$$

$$-\frac{c^2}{b^2\beta}\left(1-\frac{3b^2\alpha}{c}\right)\left(1-\frac{b^2\alpha}{c}\right)$$

$$= \frac{c^2}{b^2\beta} - \frac{\alpha c}{\beta} - \frac{3\alpha c}{\beta} + \frac{3\alpha^2 b^2}{\beta}$$

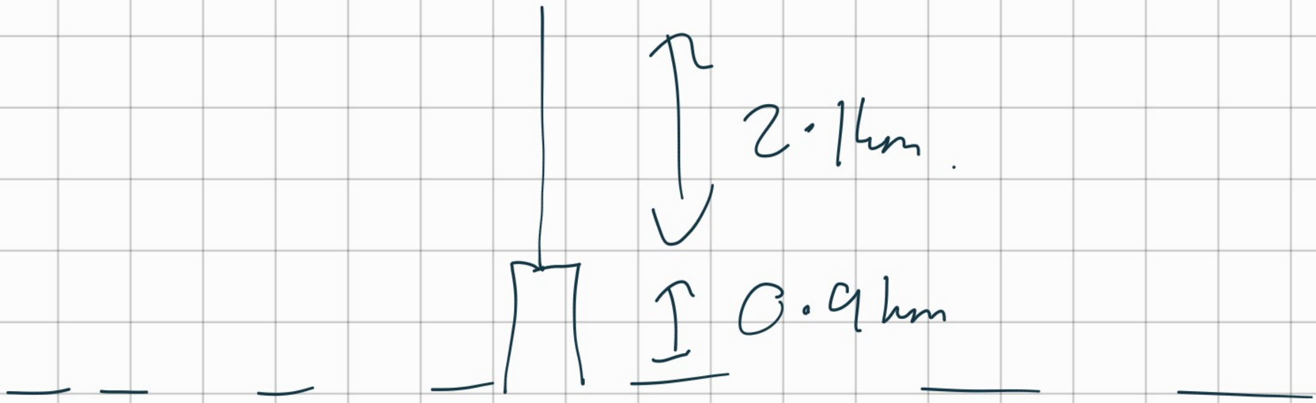
$$-\frac{c^2}{b^2\beta}\left(1-\frac{4b^2\alpha}{c}+\frac{3b^4\alpha^2}{c^2}\right)$$

$$= \frac{c^2}{b^2\beta} - \frac{4\alpha c}{\beta} + \frac{3\alpha^2 b^2}{\beta} - \frac{c^2}{b^2\beta} + \frac{4\alpha c}{\beta} - \frac{3\alpha^2 b^2}{\beta}$$

$$= 0$$

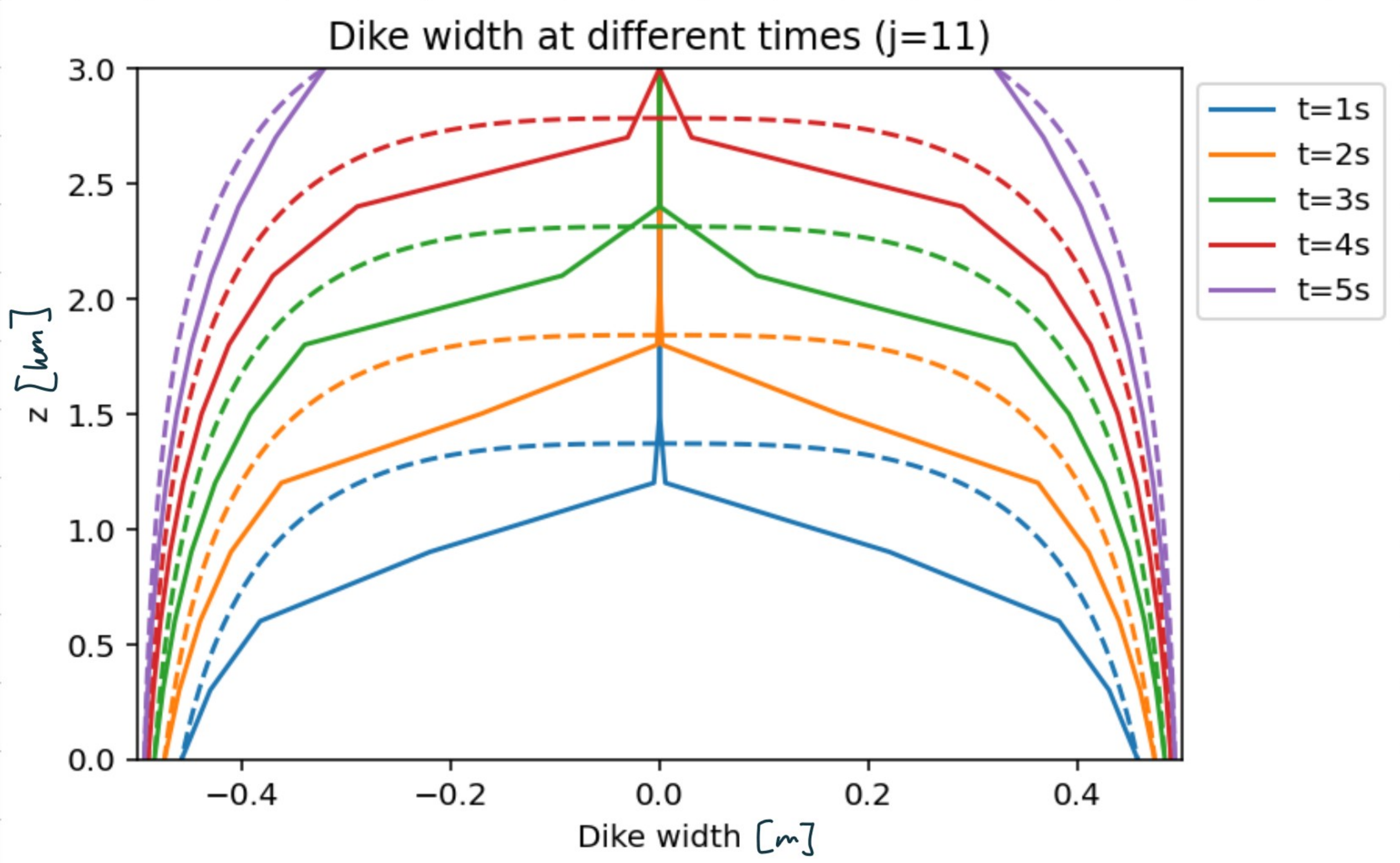
$$\boxed{\begin{gathered}\partial_t b + \partial_z\left(\alpha b^3 - \beta b^3 \partial_z b\right) = 0 \\ \text{when } z - z_{r0} - ct = \frac{\beta}{\alpha}\left(b(z,t) - \sqrt{\frac{c}{\alpha}}\,\operatorname{arctanh}\left(\sqrt{\frac{\alpha}{c}}\, b(z,t)\right)\right) \\ \text{so this is a solution to the equation.}\end{gathered}}$$

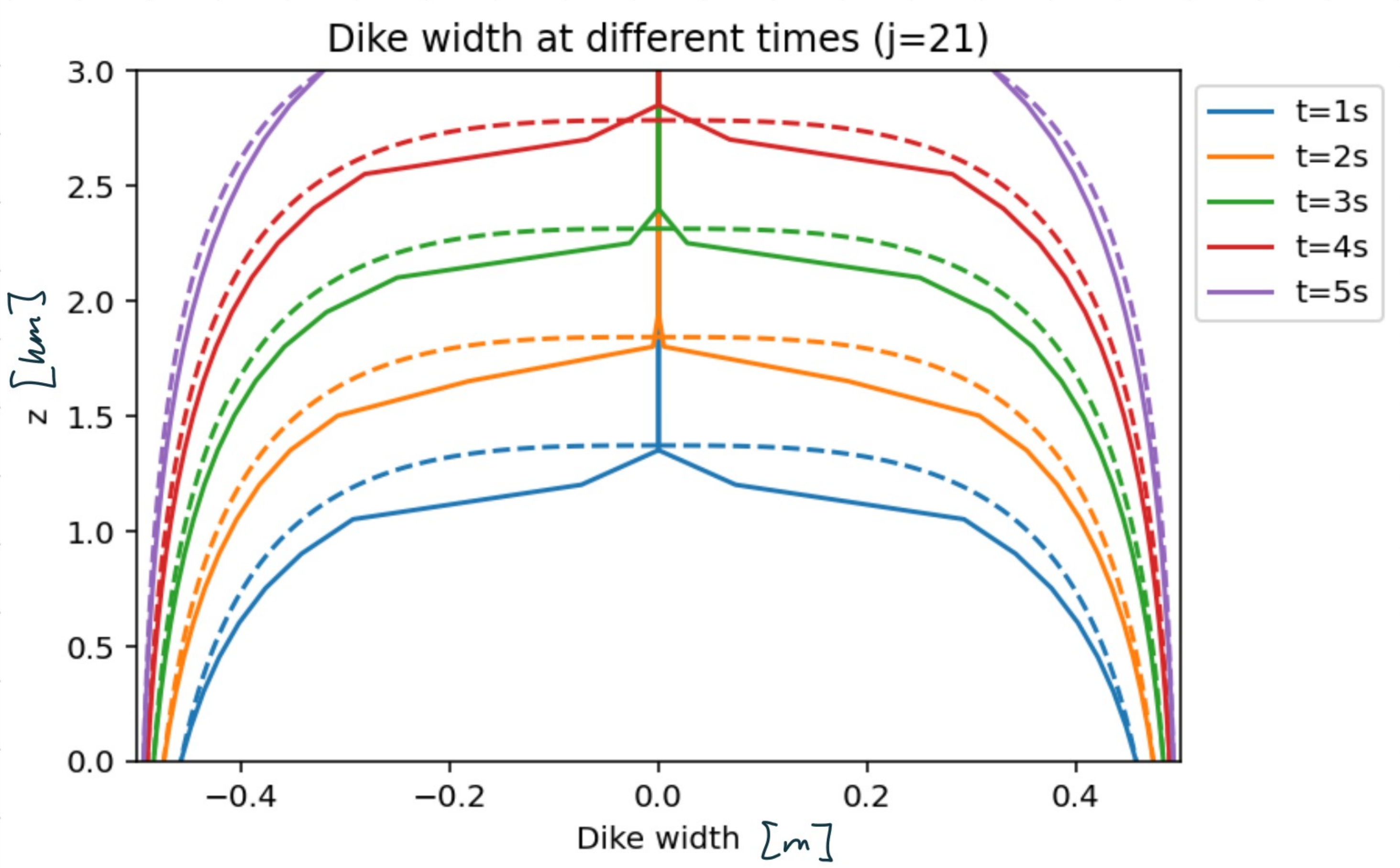
- Used maximum principle to determine maximum time step
  → where $b_{max} = 1$

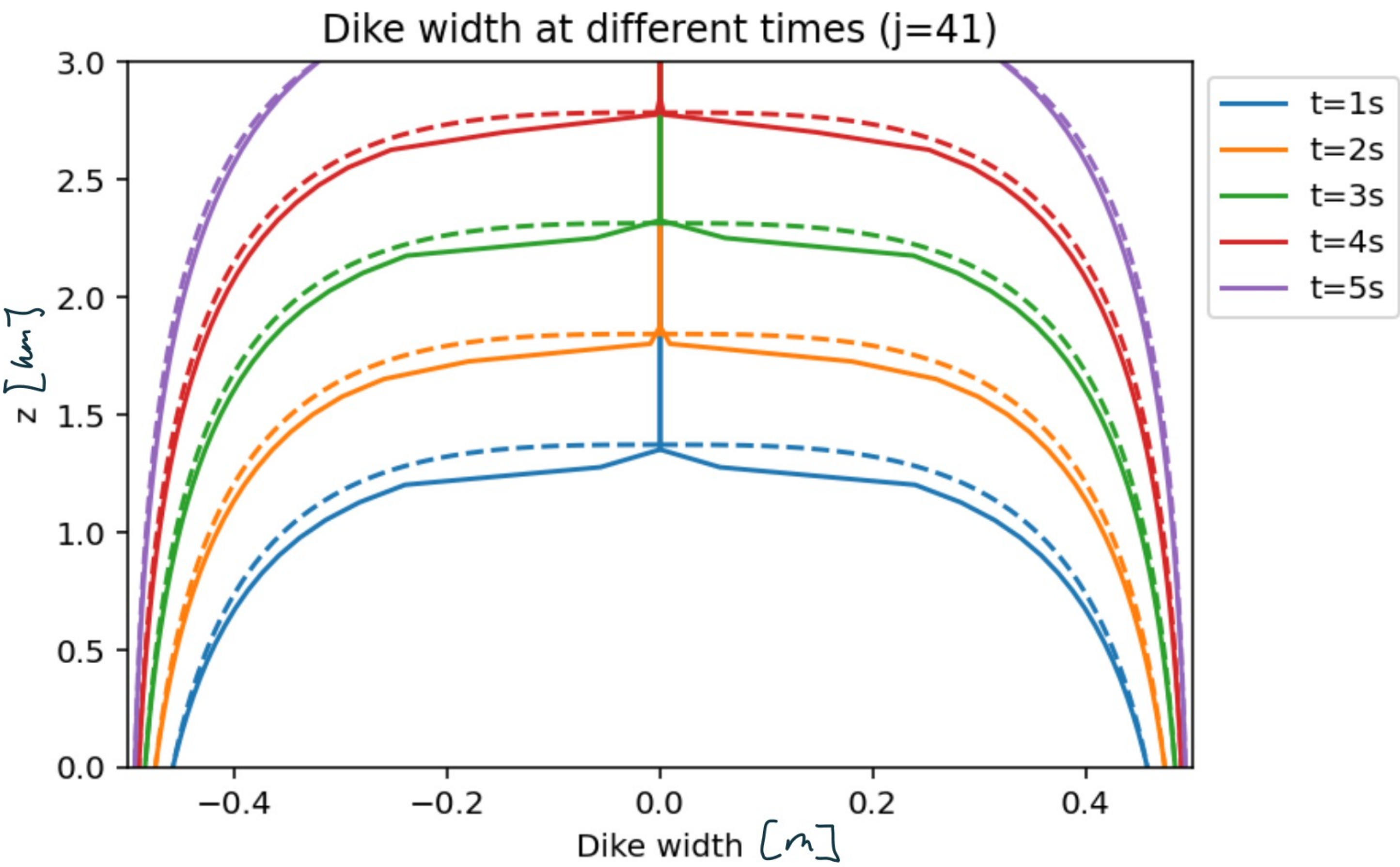


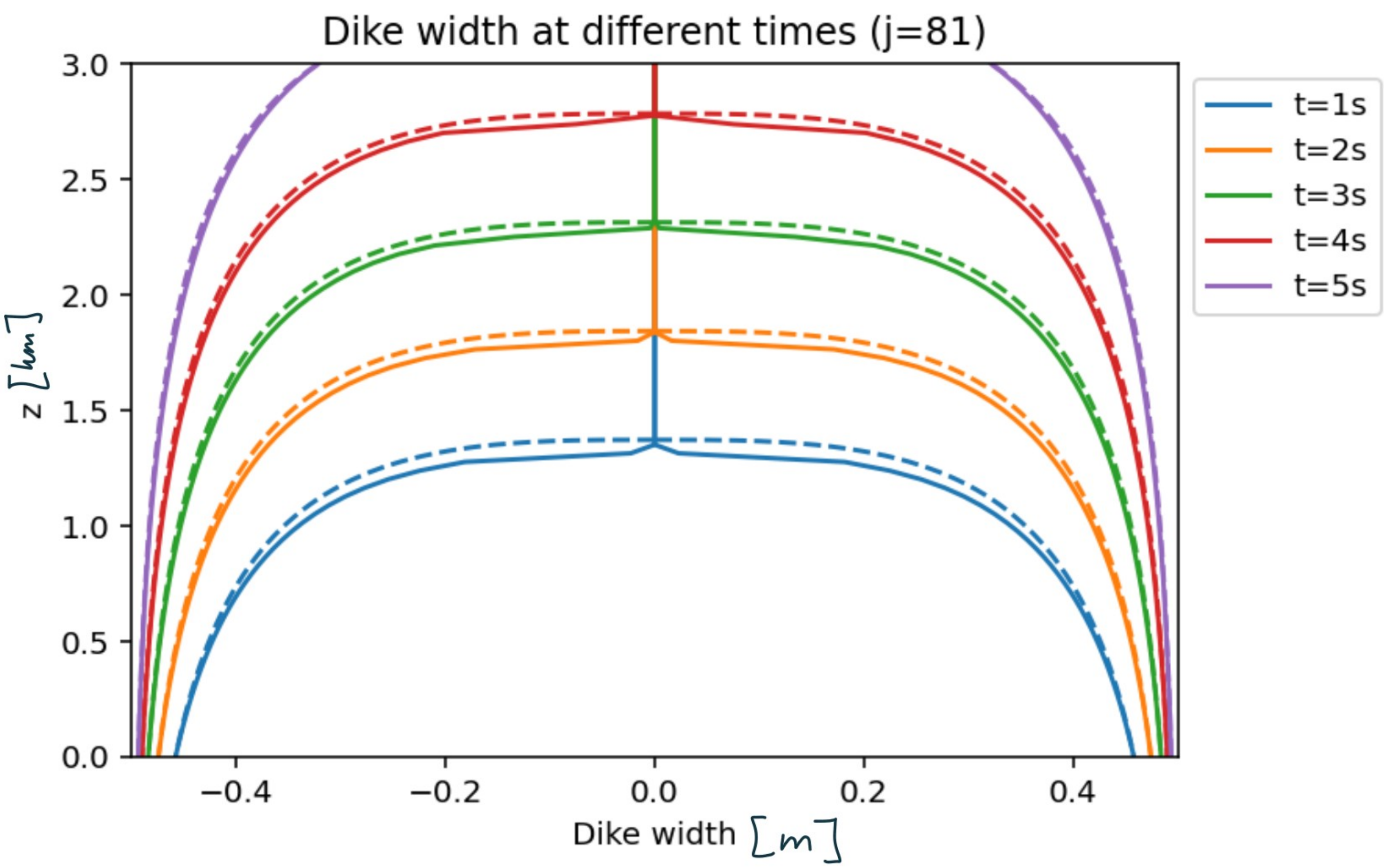
$z_{r0} = \cancel{2.1}\ 0.9$ km

For all of these plots, the dashed lines - - - correspond to the exact solution, and solid lines — correspond to the numerical solutions






Dike width at different times (j=41)
z [km]
Dike width [m]
t=1s
t=2s
t=3s
t=4s
t=5s


Dike width at different times (j=81)
z [km]
Dike width [m]
t=1s
t=2s
t=3s
t=4s
t=5s

e).

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} + \frac{\alpha}{\Delta z}\left[(b_j^n)^3 - (b_{j-1}^n)^3\right] - \frac{\beta}{8(\Delta z)^2}\left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n)\right]$$

For Crank-Nicolson scheme

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} + \frac{\alpha}{2\Delta z}\left[(b_j^{n+1})^3 - (b_{j-1}^{n+1})^3 + (b_j^n)^3 - (b_{j-1}^n)^3\right]$$

$$-\frac{\beta}{16(\Delta z)^2}\left[(b_{j+1}^{n+1} + b_j^{n+1})^3 (b_{j+1}^{n+1} - b_j^{n+1}) - (b_j^{n+1} + b_{j-1}^{n+1})^3 (b_j^{n+1} - b_{j-1}^{n+1})\right]$$

$$-\frac{\beta}{16(\Delta z)^2}\left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n)\right]$$

$= 0$

which is very complicated

let R =

$$b_j^{n+1} - b_j^n + k_1\left[(b_j^{n+1})^3 - (b_{j-1}^{n+1})^3 + (b_j^n)^3 - (b_{j-1}^n)^3\right]$$

$$- k_2\left[(b_{j+1}^{n+1} + b_j^{n+1})^3(b_{j+1}^{n+1} - b_j^{n+1}) - (b_j^{n+1} + b_{j-1}^{n+1})^3(b_j^{n+1} - b_{j-1}^{n+1})\right]$$

$$- k_2\left[(b_{j+1}^n + b_j^n)^3(b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3(b_j^n - b_{j-1}^n)\right]$$

$$= 0$$

$\Rightarrow$ note $k_1 = \dfrac{\alpha\Delta t}{2\Delta z}$, $k_2 = \dfrac{\beta\Delta t}{16(\Delta z)^2}$

$$= b_j^{n+1} - b_j^n$$

$$+ k_1\left[(b_j^{n+1})^3 - (b_{j-1}^{n+1})^3\right] + k_2\left[(b_j^{n+1} + b_{j-1}^{n+1})^3(b_j^{n+1} - b_{j-1}^{n+1})\right]$$

$$+ k_1\left[(b_j^n)^3 - (b_{j-1}^n)^3\right] + k_2\left[(b_j^n + b_{j-1}^n)^3(b_j^n - b_{j-1}^n)\right]$$

$$- k_2 \left[(b_{j+1}^{n+1} + b_j^{n+1})^3 (b_{j+1}^{n+1} - b_j^{n+1}) \right]$$

$$- k_2 \left[(b_{j+1}^{n} + b_j^{n})^3 (b_{j+1}^{n} - b_j^{n}) \right] = 0$$

Newton scheme for iteration

$$J_{ij} = \frac{\partial R_{ij}}{\partial b_{ij}}$$

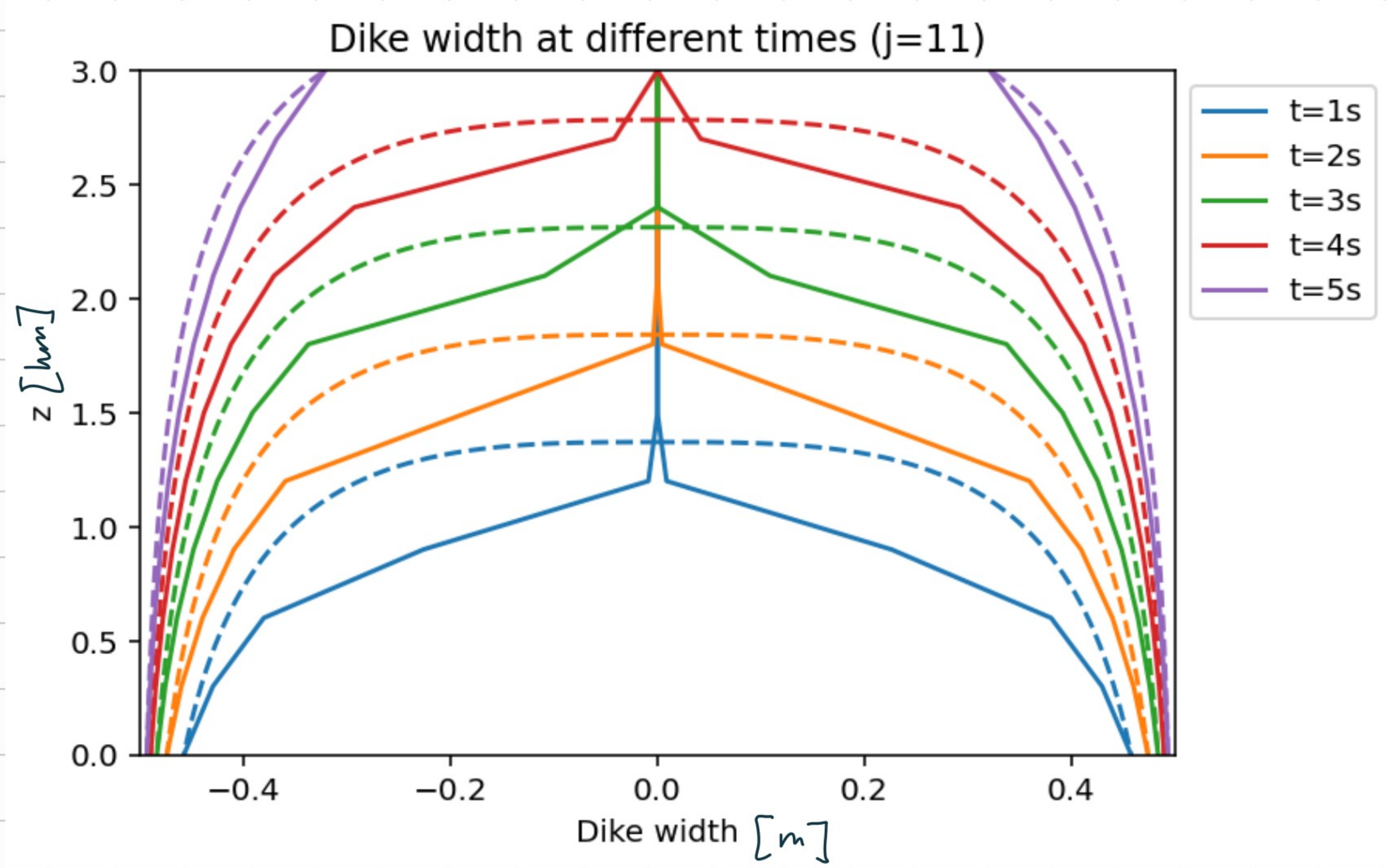
$$= \frac{R_{ij,pert} - R_{ij}}{\varepsilon}$$

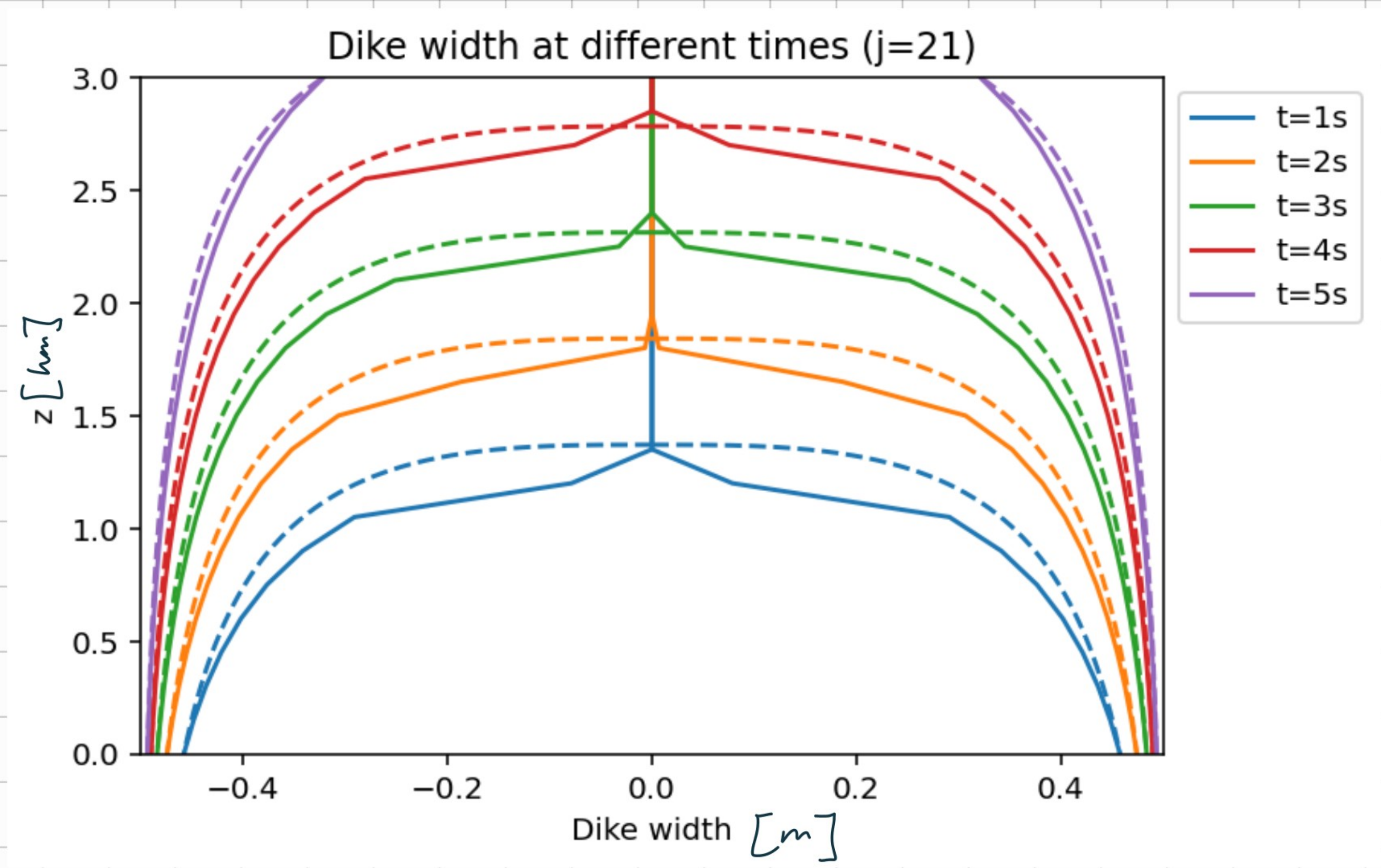
$\rightarrow$ Then solve $J\,\delta b = -R$

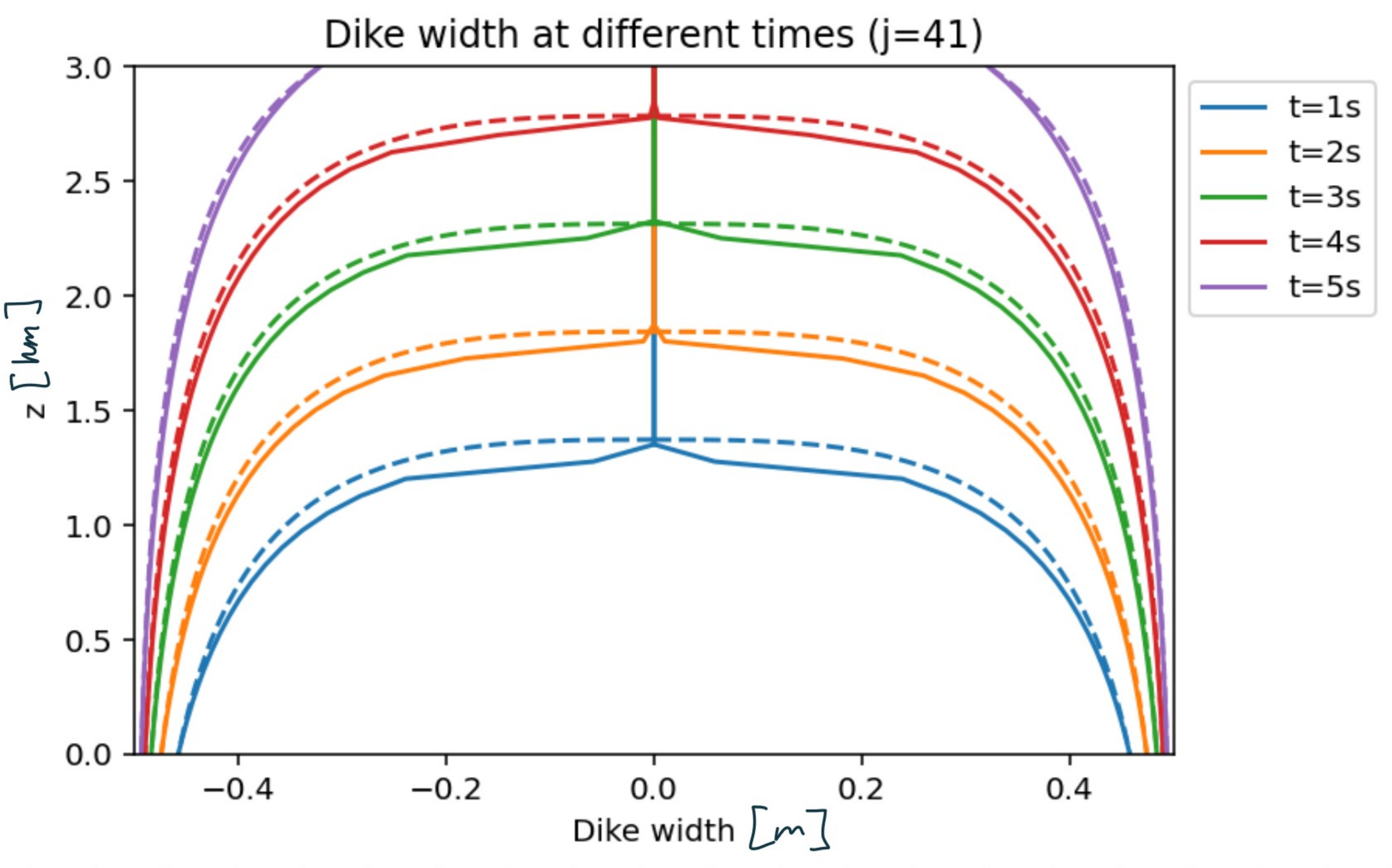
and $b_{new} = b_{old} + \delta b$

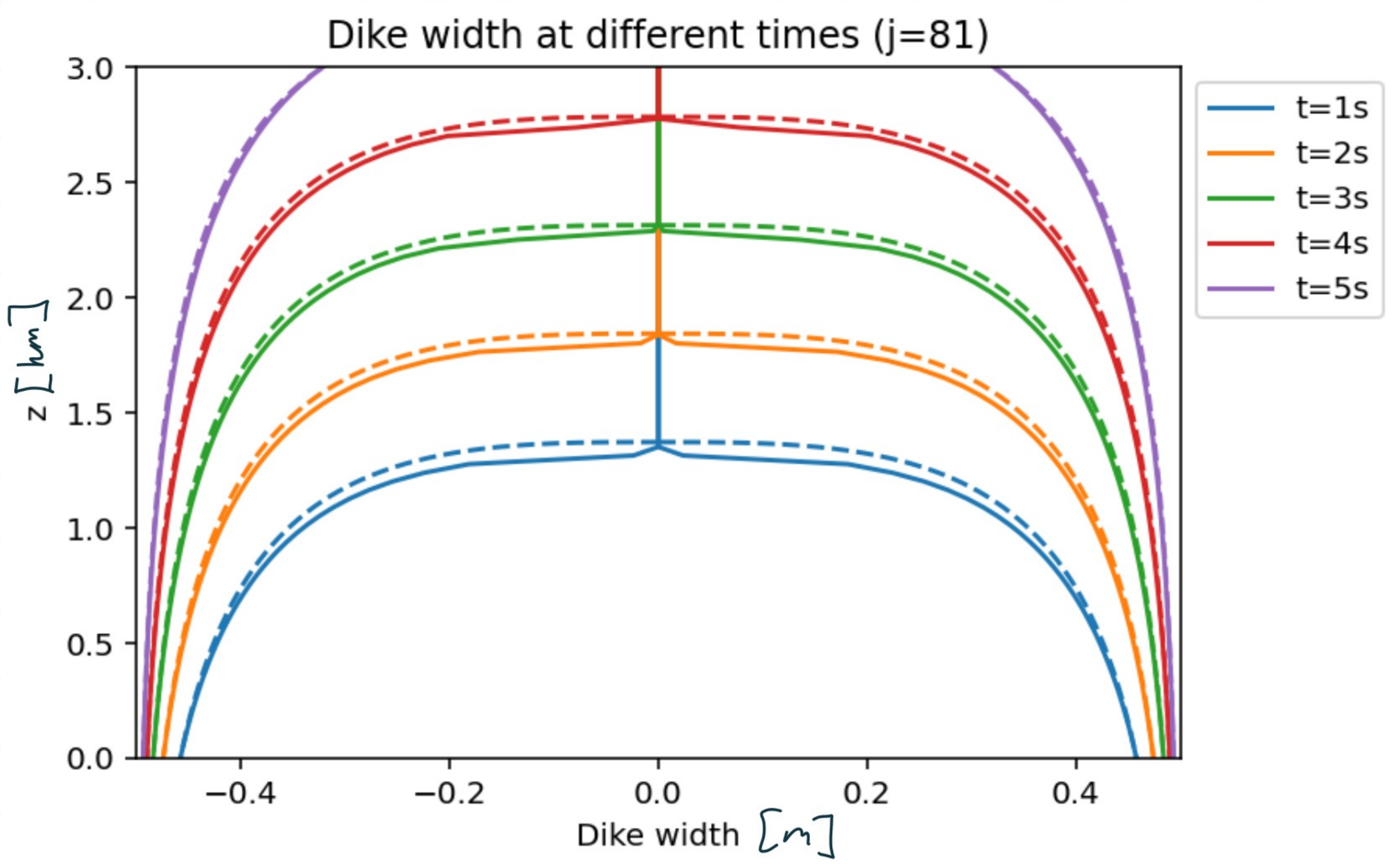
-- = exact

— = numerical






Dike width at different times (j=41)
t=1s
t=2s
t=3s
t=4s
t=5s
z [km]
Dike width [m]


Dike width at different times (j=81)
t=1s
t=2s
t=3s
t=4s
t=5s
z [km]
Dike width [m]

Scheme from	j	Time /s
d)	11	0.06304
	21	0.2115
	41	0.9744
	81	5.7097
e)	11	0.4586
	21	5.2310
	41	67.3454
	81	1028.8300

Scheme from d) is much faster than e).

(although, looking at my code, could make both schemes faster by making them more efficient when calculating for different time).

- Makes sense d) is faster, as it does not have to iterate several times to find the value of b_{n+1}