NumericsExercise1

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October 2025

1 Problem 1

1.1 1 a)

We have the following PDE:

$$\frac{\partial b}{\partial t} + \frac{\partial ub}{\partial z} = 0 \tag{1}$$

with

$$u = \alpha b^2 - \beta b^2 \frac{\partial b}{\partial z} \tag{2}$$

To make the equation a function of b only, we substitute the expression for u to get:

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial z}\left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z}\right) = 0 \tag{3}$$

We have $b = D_0 + b'$, and to linearize the equation in (3), we first expand it by using the product rule to get:

$$\frac{\partial b}{\partial t} + 3\alpha b^2 \frac{\partial b}{\partial z} - 3\beta b^2 \left(\frac{\partial b}{\partial z}\right)^2 - \beta b^3 \frac{\partial^2 b}{\partial z^2} = 0 \tag{4}$$

Next, we substitute in our expression for b to get:

$$\frac{\partial (D_0 + b')}{\partial t} + 3\alpha (D_0 + b')^2 \frac{\partial (D_0 + b')}{\partial z} - 3\beta (D_0 + b')^2 \left(\frac{\partial (D_0 + b')}{\partial z}\right)^2 - \beta (D_0 + b')^3 \frac{\partial^2 (D_0 + b')}{\partial z^2} = 0 \tag{5}$$

Since D_0 is constant, we can write the equation as

$$\frac{\partial b'}{\partial t} + 3\alpha (D_0 + b')^2 \frac{\partial b'}{\partial z} - 3\beta (D_0 + b')^2 \left(\frac{\partial b'}{\partial z}\right)^2 - \beta (D_0 + b')^3 \frac{\partial^2 b'}{\partial z^2} = 0 \tag{6}$$

Firstly, since we assume that b' is small, we will neglect all terms that are nonlinear in b', i.e terms that involve small terms multiplied by small terms, derivatives of b' multiplied by itself or b'. That being said, we can eliminate the following term:

$$-3\beta(D_0 + b')^2 \left(\frac{\partial b'}{\partial z}\right)^2 \tag{7}$$

because it involves the derivative of b' squared. The equation then reduces to:

$$\frac{\partial b'}{\partial t} + 3\alpha(D_0 + b')^2 \frac{\partial b'}{\partial z} - \beta(D_0 + b')^3 \frac{\partial^2 b'}{\partial z^2} = 0 \tag{8}$$

When we expand the quadratic term, we get:

$$3\alpha \left(D_0^2 + 2D_0 b' + b'^2\right) \frac{\partial b'}{\partial z} \tag{9}$$

Clearly, we see that the terms $2D_0 b' \partial b'_z$ and $b' \partial b'_z$ are nonlinear since they involve the multiplication of $\partial b'_z$ by b' and b'^2.

Our equation then reduces to

$$\frac{\partial b'}{\partial t} + 3\alpha D_0^2 \frac{\partial b'}{\partial z} - \beta(D_0 + b')^3 \frac{\partial^2 b'}{\partial z^2} = 0 \tag{10}$$

Lastly, expanding the third term, we get :

$$\beta \left(D_0^3 + 3D_0^2 b' + 3D_0 (b')^2 + (b')^3\right) \frac{\partial^2 b'}{\partial z^2} \tag{11}$$

Using the same principles we applied before, we can see that the only remaining term is $\beta D_0^3 \partial b'_{zz}$. This then yields the linearized equation:

$$\frac{\partial b'}{\partial t} + 3\alpha D_0^2 \frac{\partial b'}{\partial z} - \beta D_0^3 \frac{\partial^2 b'}{\partial b^2} \tag{12}$$

In both cases, a quantity is being transported (convected or advected) and smeared out (diffused). Hence, they are called convection-diffusion equations. In its original form, the equation is nonlinear, since the transport and diffusion of b also depends on (b) itself.

1.2 1 b)

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial z}\left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z}\right) = 0 \tag{13}$$

Discretizing the time derivative of the equation above using forward Euler, we get:

$$\frac{\partial b}{\partial t} = \frac{b_j^{n+1} - b_j^n}{\Delta t} \tag{14}$$

Given that α in the convective part is larger than 1, we can discretize it using backwards differences, yielding:

$$\alpha \partial z(b^3) \approx 3\alpha (b_j^n)^2 \frac{({b_j}^n) - ({b_{j-1}}^n)}{\Delta z} \tag{15}$$

Lastly, we can approximate the diffusive term by making the following approximation:

$$\beta b^3 \frac{\partial b}{\partial z} \approx (b^n_{j+\frac{1}{2}})^3 \frac{b^n_{j+1} - b^n_j}{\Delta z} \tag{16}$$

where

$$b^n_{j+\frac{1}{2}} = \frac{b^n_j + b^n_{j+1}}{2} \tag{17}$$

Applying forward differences again, we obtain:

$$(b^n_{j+\frac{1}{2}})^3 \frac{b^n_{j+1} - b^n_j}{\Delta z^2} - (b^n_{j-\frac{1}{2}})^3 \frac{b^n_j - b^n_{j-1}}{\Delta z^2} \tag{18}$$

$$\begin{aligned} b^{n+1}_j = b^n_j &- \frac{3\Delta t\, \alpha}{\Delta z}(b^n_j)^2 \Big((b^n_j) - (b^n_{j-1})\Big) \\ &+ \frac{\Delta t\, \beta}{\Delta z^2}\left[\left(\frac{(b^n_j)^3 + (b^n_{j+1})^3}{2}\right)(b^n_{j+1} - b^n_j) - \left(\frac{(b^n_{j-1})^3 + (b^n_j)^3}{2}\right)(b^n_j - b^n_{j-1})\right] \end{aligned} \tag{19}$$

When discretizing the linear equation, the time derivative can be discretized as previously, and the convective part becomes:

$$3\alpha D_0^2 \frac{\partial b'}{\partial z} = 3\alpha D_0^2 \frac{(b_j{}^n) - (b_{j-1}{}^n)}{\Delta z} \tag{20}$$

Lastly, the diffusive part becomes:

$$-\beta D_0^3 \frac{\partial^2 b'}{\partial z^2} = -\beta D_0^3 \frac{b_{j+1}{}^n - 2b_j{}^n + b_{j-1}{}^n}{\Delta z^2} \tag{21}$$

Finally, the discretized linear equation can be written as:

$$b^{n+1}_j = b^n_j - 3\alpha D_0^2 \frac{\Delta t}{\Delta z}(b_j{}^n - b_{j-1}{}^n) + \beta D_0^3 \frac{\Delta t}{\Delta z^2}(b_{j+1}{}^n - 2b_j{}^n + b_{j-1}{}^n) \tag{22}$$

1.2.1 Checking linearization

We start by looking at the convective term:

$$C = 3\alpha (b^n_j)^2 \frac{b^n_j - b^n_{j-1}}{\Delta z}. \tag{23}$$

We substitute in $b^n_j = D_0 + b'^n_j$, which yields:

$$C = 3\alpha (D_0 + b'^n_j)^2 \frac{(D_0 + b'^n_j) - (D_0 + b'^n_{j-1})}{\Delta z} \tag{24}$$

This can be further expanded to give:

$$3\alpha\left(D_0^2 + 2D_0 b_j'^n + (b_j'^n)^2\right)\frac{b_j'^n - b_{j-1}'^n}{\Delta z} \tag{25}$$

Every term except D_0^2 contains a perturbed quantity, which will vanish upon multiplication with $b_j'^n$ and $b_{j-1}'^n$, yielding the convective term in the linearized equation.

We repeat the same for the diffusive part. Note that only the term involving $b_j^n - b_{j-1}^n$ is not expanded since it will yield the same results.

$$\begin{aligned}
\beta\frac{(b_j^n)^3 + (b_{j+1}^n)^3}{2}\frac{(b_{j+1}^n - b_j^n)}{\Delta z^2} &= \beta\frac{(D_0 + b_j')^3 + (D_0 + b_{j+1}')^3}{2}\frac{((D_0 + b_{j+1}') - (D_0 + b_j'))}{\Delta z^2} \\
&= \beta\Bigg[D_0^3 + \frac{3D_0^2}{2}(b_j' + b_{j+1}') \\
&+ \frac{3D_0}{2}\left((b_j')^2 + (b_{j+1}')^2\right) + \frac{(b_j')^3 + (b_{j+1}')^3}{2}\Bigg]\frac{(b_{j+1}' - b_j')}{\Delta z^2}
\end{aligned} \tag{26}$$

Again, D_0^3 is the only quantity that remains since it is not multiplied by a perturbed quantity, resulting in:

$$D_0^3\frac{(b_{j+1}'^n - b_j'^n)}{\Delta z^2} \tag{27}$$

Subtracting

$$D_0^3\frac{(b_j'^n - b_{j-1}'^n)}{\Delta z^2}$$

which can be obtained from expanding the second term, we get the linearized diffusion term.

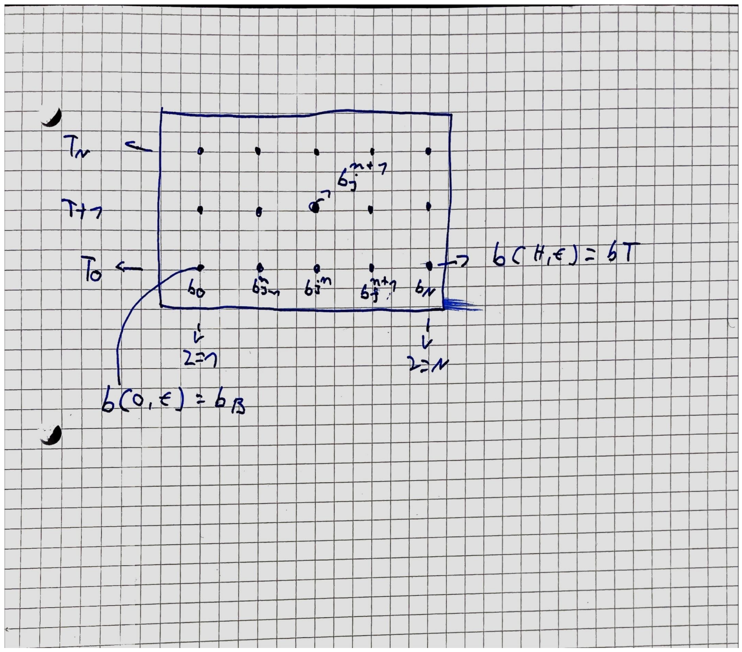

Figure 1: Grid labelling

Generally speaking, keeping the diffusion term in its self-adjoint form is preferable because it keeps the PDE in its original form. Furthermore, it makes the scheme more stable and accurate.

1.3 1 c) Fourier Analysis

When β =0, the linearized equation reduces to :

$$b_j^{n+1} = b_j^n - 3\alpha D_0^2 \frac{\Delta t}{\Delta z}({b_j}^n - {b_{j-1}}^n) \tag{28}$$

Substituting our fourier mode in, we get:

$$\lambda^{n+1} e^{ikj\Delta x} = \lambda^n e^{ikj\Delta x} - \mu(\lambda^n e^{ikj\Delta x} - \lambda^n e^{ik(j-1)\Delta x}) \tag{29}$$

with $\mu = 3\alpha D_0^2 \frac{\Delta t}{\Delta x}$. We divide by $\lambda^n e^{ikj\Delta x}$ to get:

$$\lambda = 1 - (1 - e^{-ik\Delta x}) \tag{30}$$

$$\lambda = 1 - \mu(1 - cos\theta + isin\theta) \tag{31}$$

We now take $|\lambda^2|$:

$$|\lambda^2| = (1 - \mu(1 - cos\theta))^2 + (\mu sin\theta)^2 \tag{32}$$

$$|\lambda^2| = 1 - 2\mu(1 - cos\theta) + \mu^2(1 - 2cos\theta)$$

$$|\lambda^2| = 1 - 2\mu(1 - \mu)(1 - cos\theta)$$

For stability, it is required that $|\lambda^2| \leq 1$. To find a stable μ, we consider the following inequality:

$$-2\mu(1 - \mu)(1 - cos\theta) \leq 0 \tag{33}$$

Clearly, we see that this inequality can only be true if $\mu \geq 0$ and $\mu \leq 1$.
Therefore,we need to consider the following expression:

$$0 \leq 3\alpha D_0^2 \frac{\Delta t}{\Delta z} \leq 1 \tag{34}$$

A time step criterion is therefore:

$$\Delta t \leq \frac{\Delta z}{3\alpha D_0^2}$$

When $\alpha = 0$, the equation reduces to :

$$b_j^{n+1} = b_j^n + \beta D_0^3 \frac{\Delta t}{\Delta z^2}({b_{j+1}}^n - 2{b_j}^n + {b_{j-1}}^n) \tag{35}$$

and we will write $\mu = \beta D_0^3 \frac{\Delta t}{\Delta z^2}$.

$$\lambda^{n+1} e^{ikj\Delta x} = \lambda^n e^{ikj\Delta x} + \mu(\lambda^n e^{ik(j+1)\Delta x} - 2\lambda^n e^{ikj\Delta x} + \lambda^n e^{ik(j-1)\Delta x}) \tag{36}$$

As before, we divide by $\lambda^n e^{ikj\Delta x}$ to get:

$$\lambda = 1 + \mu(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \tag{37}$$

We can then simplify this expression to:

$$\lambda = 1 - 4\mu sin^2(\frac{\theta}{2})$$

Again, we require $|\lambda| \leq 1$. The worst case in terms of stability is $\frac{\theta}{2} = \frac{\pi}{2}$ since it could lead to $|\lambda| \geq 1$

Hence, our criterion becomes

$$1 - 4\mu sin^2(\frac{\theta}{2}) \geq -1 \tag{38}$$

Taking $\frac{\theta}{2} = \frac{\pi}{2}$, we get:

$$4\beta D_0^3 \frac{\Delta t}{\Delta z^2} \leq 2 \tag{39}$$

Hence,

$$\Delta t < \frac{\Delta z^2}{2\beta D_0^3} \tag{40}$$

1.4 1 d)

$$b_j^{n+1} = b_j^n - 3\alpha D_0^2 \frac{\Delta t}{\Delta z}({b_j}^n - {b_{j-1}}^n) + \beta D_0^3 \frac{\Delta t}{\Delta z^2}({b_{j+1}}^n - 2{b_j}^n + {b_{j-1}}^n) \tag{41}$$

Firstly, we can group the terms to give the following equation:

$$b_j^{n+1} = \left(1 - 3\alpha D_0^2 \frac{\Delta t}{\Delta z} - 2\beta D_0^3 \frac{\Delta t}{\Delta z^2}\right) b_j^n + \beta D_0^3 \frac{\Delta t}{\Delta z^2}\, b_{j+1}^n + \left(3\alpha D_0^2 \frac{\Delta t}{\Delta z} + \beta D_0^3 \frac{\Delta t}{\Delta z^2}\right) b_{j-1}^n \tag{42}$$

We can see that the equation can be written as:

$$b_j^{n+1} = Ab_j^n + Bb_{j+1}^n + Cb_{j-1}^n \tag{43}$$

If we assume that α,β, Δt and Δz are positive, B and C will be positive. However, by inspection, we can clearly see that A can go negative.

Clearly, the requirement is that A ≥ 0. Hence, in order to conform to the maximum principle, the time step required is:

$$\Delta t \leq \frac{1}{3\alpha D_0^2/\Delta z \; + \; 2\beta D_0^3/\Delta z^2} \tag{44}$$

1.5 e)

$$\begin{aligned} b_j^{n+1} = b_j^n - \frac{3\Delta t\, \alpha}{\Delta z}(b_j^n)^2\Big((b_j^n) - (b_{j-1}^n)\Big) \\ + \frac{\Delta t\, \beta}{\Delta z^2}\left[\left(\frac{(b_j^n)^3 + (b_{j+1}^n)^3}{2}\right)(b_{j+1}^n - b_j^n) - \left(\frac{(b_{j-1}^n)^3 + (b_j^n)^3}{2}\right)(b_j^n - b_{j-1}^n)\right] \end{aligned} \tag{45}$$

We can expand the equation to get:

$$\begin{aligned} b_j^{n+1} &= b_j^n - \frac{3\Delta t\, \alpha}{\Delta z}(b_j^n)^3 + \frac{3\Delta t\, \alpha}{\Delta z}(b_j^n)^2(b_{j-1}^n) \\ &+ \frac{\Delta t\, \beta}{2\,\Delta z^2}\Big[(b_j^n)^3 b_{j+1}^n - 2(b_j^n)^4 + (b_{j+1}^n)^4 - (b_{j+1}^n)^3 b_j^n \\ &\quad - (b_{j-1}^n)^3 b_j^n + (b_{j-1}^n)^4 + (b_j^n)^3 b_{j-1}^n\Big]. \end{aligned} \tag{46}$$

We can see that there are negative terms that could cause b_j^n to become negative, and we only need to consider those. When we group the negative terms, we get:

$$b_j^{n+1} = b_j^n - \frac{3\Delta t\, \alpha}{\Delta z}\Big((b_j^n)^3\Big) + \frac{\Delta t\, \beta}{2\Delta z^2}\Big(-2(b_j^n)^4 - (b_{j+1}^n)^3 b_j^n - (b_{j-1}^n)^3 b_j^n\Big) \tag{47}$$

$$b_j^{n+1} = b_j^n\Big[1 - \frac{3\Delta t\, \alpha}{\Delta z}\Big((b_j^n)^2\Big) - \frac{\Delta t\, \beta}{2\Delta z^2}\Big(2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3\Big) \tag{48}$$

The criterion is that the equation (41) must be larger than 0. Hence, our inequality becomes:

$$1 - \frac{3\Delta t\,\alpha}{\Delta z}\Big((b_j^n)^2)\Big) - \frac{\Delta t\,\beta}{2\Delta z^2}\Big(2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3\Big) \geq 0 \tag{49}$$

We can rearrange the equation to find that the time step must be:

$$\Delta t \leq \frac{1}{\frac{3\alpha}{\Delta z}\Big((b_j^n)^2)\Big) + \frac{\beta}{2\Delta z^2}\Big(2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3\Big)} \tag{50}$$

1.6 f)

We can derive a second-order scheme for the convective term by considering the Taylor expansion:

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \Delta x^2 \frac{f''(x)}{2} \tag{51}$$

$$f(x-\Delta x) = f(x) - \Delta x f'(x) + \Delta x^2 \frac{f''(x)}{2} \tag{52}$$

We subtract these two equations to get:

$$f(x+\Delta x) - f(x-\Delta x) = 2\Delta x f'(x) + \mathcal{O}(\Delta x^2) \tag{53}$$

$$\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = f'(x) \tag{54}$$

The discretization for the non-linear equation then becomes:

$$\begin{aligned} b_j^{n+1} &= -\frac{3\Delta t\,\alpha}{2\Delta z}(b_j^n)^2\Big((b_{j+1}^n) - (b_{j-1}^n)\Big) \\ &+ \frac{\Delta t\,\beta}{\Delta z^2}\left[\left(\frac{(b_j^n)^3 + (b_{j+1}^n)^3}{2}\right)(b_{j+1}^n - b_j^n) - \left(\frac{(b_{j-1}^n)^3 + (b_j^n)^3}{2}\right)(b_j^n - b_{j-1}^n)\right] \end{aligned} \tag{55}$$

We can factor out the negative parts as before, which yields:

$$b_j^{n+1} = b_j^n\Big[1 - \frac{3\Delta t\,\alpha}{2\Delta z}\Big(b_j^n b_{j+1}^n\Big) - \frac{\Delta t\,\beta}{2\Delta z^2}\Big(2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3\Big)\Big] \tag{56}$$

The resulting inequality is:

$$1 - \frac{3\Delta t\,\alpha}{2\Delta z}\Big(b_j^n b_{j+1}^n\Big) - \frac{\Delta t\,\beta}{2\Delta z^2}\Big(2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3\Big) \geq 0 \tag{57}$$

The time-step criterion then becomes:

$$\Delta t \leq \frac{1}{\frac{3\alpha}{2\Delta z}\Big(b_j^n b_{j+1}^n\Big) + \frac{\beta}{2\Delta z^2}\Big(2(b_j^n)^3 + (b_{j+1}^n)^3 + (b_{j-1}^n)^3\Big)} \tag{58}$$

1.7 2

1.7.1 2 a)

The steady state solution is given by:

$$\beta b^3 \frac{db}{dz} = \alpha b^3 - Q \tag{59}$$

$$\frac{db}{dz} = \frac{\alpha}{\beta} - \frac{Q}{\beta b^3}$$

We can then discretize the equation to yield:

$$\frac{b_{j+1} - b_j}{\Delta z} = \frac{\alpha}{\beta} - \frac{Q}{\beta b_j^3} \tag{60}$$

$$b_{j+1} = b_j + \Delta z \Big(\frac{\alpha}{\beta} - \frac{Q}{\beta b_j^3} \Big)$$

We can confirm our steady state solution by starting with the non-linear convection-diffusion equation:

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial z} \left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z} \right) = 0 \tag{61}$$

At steady-state, $\partial t = 0$, giving:

$$\frac{\partial}{\partial z} \left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z} \right) = 0 \tag{62}$$

We then integrate the equation:

$$\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z} = Q \tag{63}$$

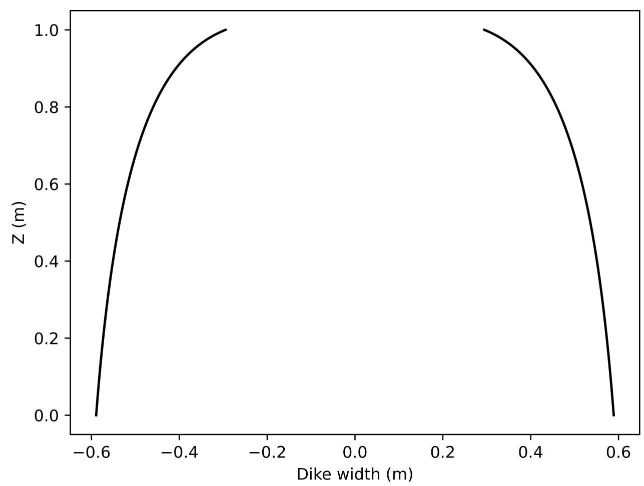

Figure 2: Steady-state plot

1.8 2 b)

Firstly, we see that the results from the transient simulations look smoother as the number of grid points increases. Secondly, the time-step required for stable time-integration needs to be reduced by a factor of 10 when the number of grid points is doubled.

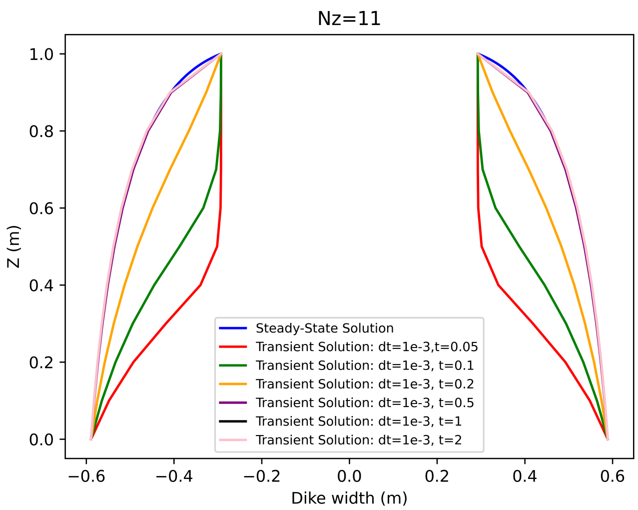

Figure 3: Transient solution with Nz=11

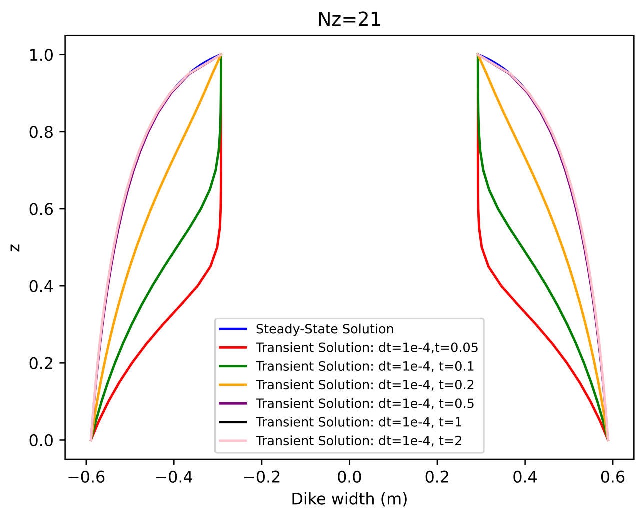

Figure 4: Transient solution with Nz=21

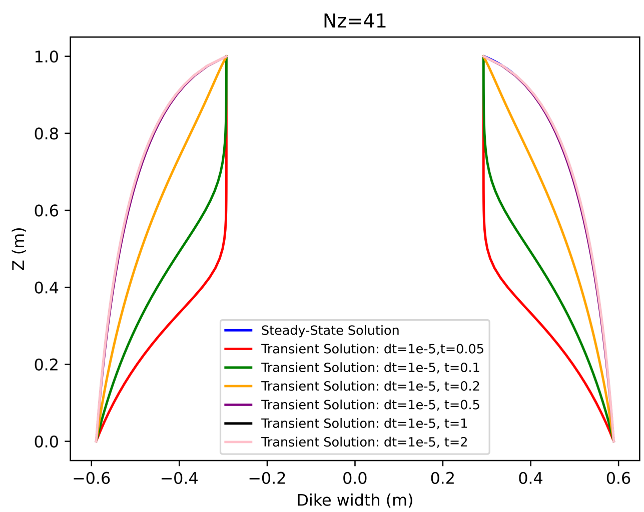

Figure 5: Transient solution with Nz=41

1.9 2 c)

We are asked to calculate the L2 norm as defined by :

$$L^2 = \sqrt{\int_0^1 e^2(z,t)\,dz} \tag{64}$$

We can rewrite it as:

$$L^2 \approx \sqrt{\frac{h}{2}\Big(e^2(z_0,t) + 2\sum_{i=1}^{N_z-1} e^2(z_i,t) + e^2(z_{N_z},t)\Big)} \tag{65}$$

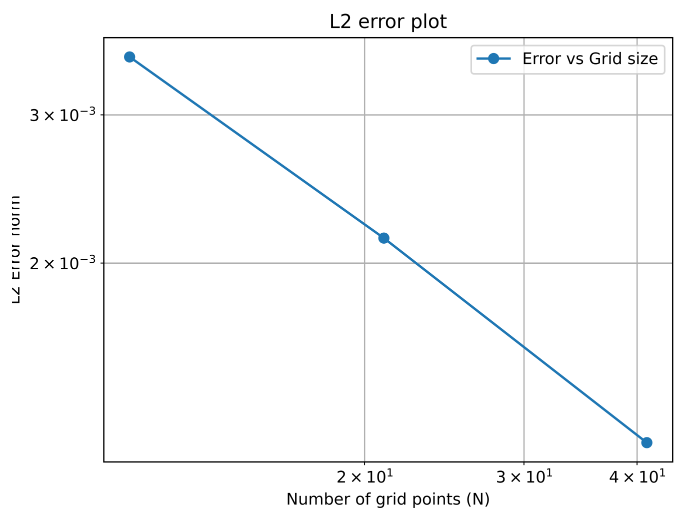

Figure 6: L2 norm for grid numbers 11,21 and 41

Clearly, the gradient is linear, and it was calculated to be 0.8 (see jupyter notebook). This strongly suggests that the scheme's order of discretization is 1. The convective term is first order, meaning it dominates the truncation error.

1.10 2 d)

The exact, implicit solution to the convection diffusion equation is given by:

$$z - zr_0 - ct = \frac{\alpha}{\beta}\Big(b(z,t) - \sqrt{\frac{c}{\alpha}}atanh[\sqrt{\frac{c}{\alpha}}b(z,t)]\Big) \tag{66}$$

We first introduce a new variable $\eta(z - ct)$ and define $f(\eta)$
We plug f into our PDE and obtain:

$$-c\frac{df}{d\eta} + \frac{d}{d\eta}\Big(\alpha f^3 - \beta f^3\frac{df}{d\eta}\Big) = 0 \tag{67}$$

After integrating, we get:

$$-cf + \alpha f^3 - \beta f^3\frac{df}{d\eta} = C \tag{68}$$

We note that the flux, C, is 0. Therefore, the integration constant vanishes and the equation becomes:

$$-cf + \alpha f^3 - \beta f^3\frac{df}{d\eta} = 0 \tag{69}$$

or in simpler notation:

$$-cf + \alpha f^3 - \beta f^3 f' = 0 \tag{70}$$

We can rearrange the equation for f'

$$f' = -\frac{c}{\beta f^2} + \frac{\alpha}{\beta} \tag{71}$$

$$f' = \frac{\alpha f^2 - c}{\beta f^2} \tag{72}$$

$$\frac{df}{d\eta} = \frac{\alpha f^2 - c}{\beta f^2} \tag{73}$$

$$df\frac{\beta f^2}{\alpha f^2 - c} = d\eta \tag{74}$$

$$d\eta = \frac{\beta}{\alpha}\frac{f^2}{f^2 - c/a}df \tag{75}$$

where we define λ^2 as c/a.

$$d\eta = \frac{\beta}{\alpha}\Big(1 + \frac{\lambda^2}{f^2 - \lambda^2}\Big)df \tag{76}$$

$$d\eta = \frac{\beta}{\alpha} + \frac{\beta\lambda^2}{\alpha(f^2 - \lambda^2)}df \tag{77}$$

We can integrate this expression:

$$d\eta = \int \frac{\beta}{\alpha} + \frac{\beta}{\alpha}\frac{\lambda^2}{(f^2-\lambda^2)}df \tag{78}$$

$$\eta = \frac{\beta}{\alpha}f + \frac{\beta\lambda^2}{\alpha}\int \frac{1}{(f^2-\lambda^2)}df \tag{79}$$

$$\eta = \frac{\beta}{\alpha}f + \frac{\beta\lambda}{2\alpha}ln\frac{f-\lambda}{(f+\lambda)} \tag{80}$$

We may also write this as:

$$\eta = \frac{\beta}{\alpha}f - \frac{\beta\lambda}{2\alpha}ln\frac{f+\lambda}{(f-\lambda)} \tag{81}$$

$$\eta = \frac{\beta}{\alpha}f - \frac{\beta\lambda}{\alpha}atanh(f/\lambda) \tag{82}$$

$$\eta = \frac{\beta}{\alpha}\left(f - \sqrt{\frac{c}{\alpha}}\,\mathrm{atanh}\left(f\sqrt{\frac{\alpha}{c}}\right)\right) \tag{83}$$

We can compare our numerical simulation to the exact solution. However, bT and bB are now time-dependent. We can find these two BCs from the exact solution and use them in the numerical simulation.

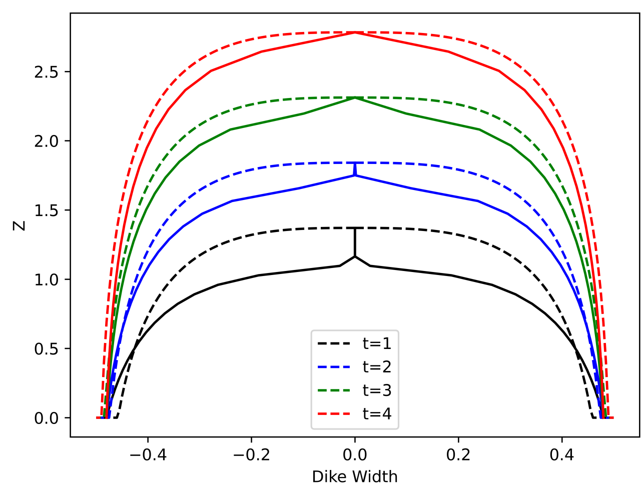

Figure 7: Exact solution (dashed line) and numerical solution (full line) at different times. Results are for Nz=21

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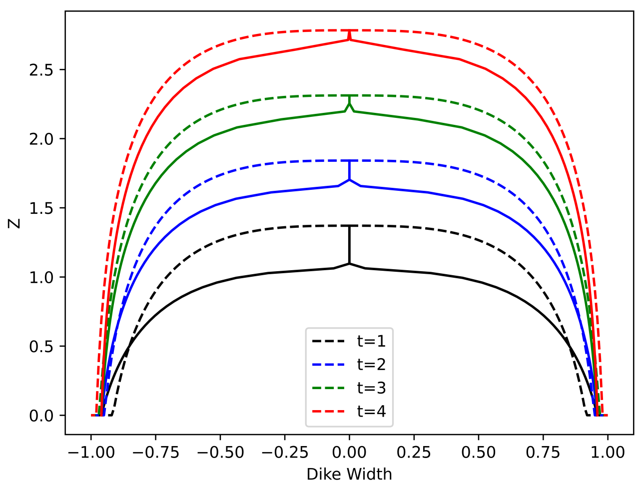

Figure 8: Exact solution (dashed line) and numerical solution (full line) at different times. Results are for Nz=41