Numerical methods for fluid dynamics

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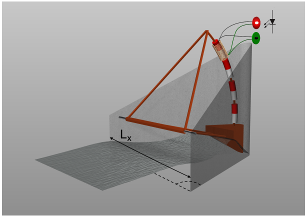
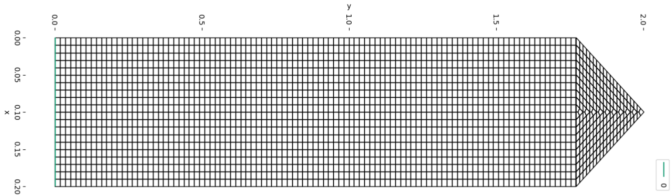
£: CDT Fluid Dynamics



Outline: assessment

- Attendance at practical sessions.
- Three numerical exercises (on finite difference, finite volume and finite element methods); to be handed in individually but you may work as a team.
- Example programs (for use at your own risk) will be provided in Python. Python use is recommended.



Finite differences: θ-method

- θ–method will be explained for heat equation w. Dirichlet BCs:

$$\partial_t u = \partial_{xx} u \quad (1)$$

$$u(x,0) = u_0(x) \quad (2)$$

$$u(0,t) = u(1,t) = 0. \quad (3)$$

- Mesh points are $x_j = j\Delta x$; constant time step is used $t_n = n\Delta t$ for $j = 0, \dots, N_x$ and $n = 0, 1, \dots$.
- Time step can also be varied, in which case Δt_n varies and t_n is sum of time steps taken.
- Approximate values of $u(x,t)$ on mesh points are denoted by $U_j^n \approx u(x_j, t_n)$.
- Initial values are $U_j^0 = u_0(x_j)$; in general exact (why could there be an issue here?).

Finite differences: approximations

- The next issue is to find a difference approximation of the PDE (1) in terms of the approximations U_j^n.
- Time derivative is approximated in a forward manner, expressed in terms of several difference operators Δ_{+t} and δ_t:

$$
\begin{aligned}
(\partial_t u)(x_j, t_n) &\approx \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t} && (4)\\
&\equiv \frac{\Delta_{+t} u(x_j, t_n)}{\Delta t} && (5)\\
&\equiv \frac{\delta_t u(x_j, t_{n+1/2})}{\Delta t} && (6)\\
&\approx (\partial_t u)(x_j, t_{n+1/2}) && (7)
\end{aligned}
$$

- *Exercise:* check approximations by performing suitable Taylor expansions of u around, e.g., $t^n = t_n$ or $t^{n+1/2} = t_{n+1/2}$.

Finite differences: Taylor expansions

- 2^{nd} spatial derivative approximated symmetrically as

$$(\partial_{xx}u)(x_j,t_n) \approx \frac{u(x_{j-1},t_n) - 2u(x_j,t_n) + u(x_{j+1},t_n)}{\Delta x^2} \tag{8}$$

$$= \frac{\delta_x^2 u(x_j,t_n)}{\Delta x^2} = \frac{\delta_x(\delta_x u)|_{x_j}^{t_n}}{\Delta x^2} \tag{9}$$

with $\delta_x u(x,t) \equiv \big(u(x+\Delta x/2,t) - u(x-\Delta x/2,t)\big)/\Delta x$.

- *Exercise:* check this approximation y using Taylor expansions of u around, e.g., t^n and x_j.
- This approximation also holds at t_{n+1}

$$(\partial_{xx}u)(x_j,t_{n+1}) \approx \frac{u(x_{j-1},t_{n+1}) - 2u(x_j,t_{n+1}) + u(x_{j+1},t_{n+1})}{\Delta x^2}$$

$$= \frac{\delta_x^2 u(x_j,t_{n+1})}{\Delta x^2}. \tag{10}$$

Finite differences: θ scheme

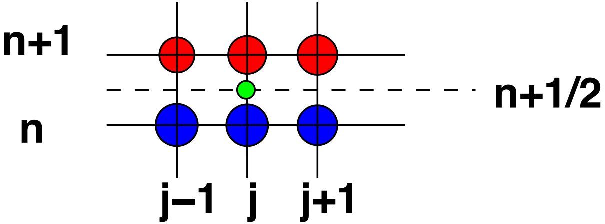
- By combining approximations with $\mu = \Delta t/\Delta x^2$, PDE (1) can be approximated on a 6-point stencil (see Fig.))

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{\theta}{\Delta x^2}\delta_x^2 U_j^{n+1} + \frac{(1-\theta)}{\Delta x^2}\delta_x^2 U_j^n \tag{11a}$$

$$U_j^{n+1} = U_j^n + \mu\theta(U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}) + \mu(1-\theta)(U_{j-1}^n - 2U_j^n + U_{j+1}^n). \tag{11b}$$

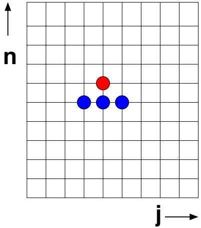
- Rewritten form with unknowns on the LHS and $0 \le \theta \le 1$

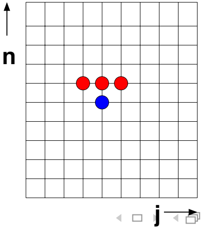
$$-\mu\theta U_{j-1}^{n+1} + (1+2\mu\theta)U_j^{n+1} - \mu\theta U_{j+1}^{n+1} = (1-2\mu(1-\theta))\,U_j^n + \mu(1-\theta)(U_{j-1}^n + U_{j+1}^n). \tag{12}$$



Finite differences: EF, EB, CN schemes

- When $\theta = 0$ scheme is explicit, solved for U_j^{n+1}: *Euler forward* scheme uses stencil of 4 points with 1 point in future.
- When $\theta = 1$ scheme is fully implicit; *Euler backward* scheme. Uses a stencil of 4 points with 3 points in future, see Fig.
- When $\theta = 1/2$, a stencil of 6 points is used: this is the classical Crank-Nicolson scheme (Crank & Nicolson 1947).
- *Exercise θ–scheme:* suitable space-time grid point for a Taylor expansion?





Finite difference methods: homework

- Study Morton and Mayers (2005), Chapter 2, sections 2.2, 2.4, 2.5.
- Read Morton and Mayers (2005), Chapter 2 (intro), sections 2.1, 2.3.
- Sign up to GitHub and send login name.
- Run/study the two example codes and study the example task.
- Study and start exercise-I.

Finite difference methods: Fourier analysis

- To assess stability of scheme, Fourier analysis will be used. Substitute Fourier Ansatz $U_j^n = \lambda^n e^{ijk\Delta x}$ with imaginary number i satisfying $i^2 = -1$, amplification factor λ, and wavenumber k into discretization (11).
- In general λ is complex with real and imaginary parts such that $\lambda = \Re(\lambda) + i\,\Im(\lambda)$. Scheme is stable when $|\lambda| \leq 1$, which for complex λ implies that we need to take the modulus of λ.
- When $|\lambda| > 1$, approximation U_j^n will blow up over time since $|\lambda|^n$ becomes unbounded and the scheme is unstable.

$$
\begin{aligned}
-\theta\mu\lambda e^{-ik\Delta x} + (1+2\theta\mu)\lambda - \theta\mu\lambda e^{ik\Delta x} &= 1 - 2(1-\theta)\mu + (1-\theta)\mu(e^{ik\Delta x} + e^{-ik\Delta x}) && (13)\\
\iff \lambda + \lambda\theta\mu(2 - e^{ik\Delta x} - e^{-ik\Delta x}) &= 1 - (1-\theta)\mu(2 - e^{ik\Delta x} - e^{-ik\Delta x}) && (14)\\
\iff \lambda &= \frac{1 - 4(1-\theta)\mu\sin^2(k\Delta x/2)}{1 + 4\theta\mu\sin^2(k\Delta x/2)}. && (15)
\end{aligned}
$$

Finite difference methods: Fourier analysis

- Note that this scheme is a special, symmetric case for which λ is real. Since $0 \leq \theta \leq 1$ and $\mu > 0$, we note that $\lambda < 1$.
- Instability can then occur only when $\lambda < -1$, i.e., when

$$
\begin{aligned}
& 1-4(1-\theta)\mu \sin^2(k\Delta x/2) < -\left(1+4\theta\mu \sin^2 k(\Delta x/2)\right) \\
\Longrightarrow\ & 4\mu(1-2\theta)\sin^2(k\Delta x/2) > 2. \qquad (16)
\end{aligned}
$$

- Instability occurs for $\mu(1-2\theta) > 1/2$ for case $k\Delta x/2 = \pi/2$. For $\theta \geq 1/2$ the θ–scheme unconditionally stable, while for $0 \leq \theta < 1/2$ scheme conditionally stable when

$$
\mu = \frac{\Delta t}{(\Delta x)^2} < \frac{1}{[2(1-2\theta)]}. \qquad (17)
$$

- Note that we ignored the boundary conditions in deriving Fourier stability because we imposed the harmonic Ansatz for periodic boundaries. Other BCs, see Morton and Mayers 2005.

Finite difference methods: maximum principle

Theorem

M&M 2005: The θ–method (11) satisfies

$$
\begin{aligned}
U_{min} &\leq U_j^n \leq U_{max} \\
U_{min} &= \min\left(U_0^m, 0 \leq m \leq n; U_j^0, 0 \leq j \leq N_x; U_{N_x}^m, 0 \leq m \leq n\right) \\
U_{max} &= \max\left(U_0^m, 0 \leq m \leq n; U_j^0, 0 \leq j \leq N_x; U_{N_x}^m, 0 \leq m \leq n\right)
\end{aligned}
$$

given the conditions $0 \leq \theta \leq 1$ and $\mu(1-\theta) \leq 1/2$.

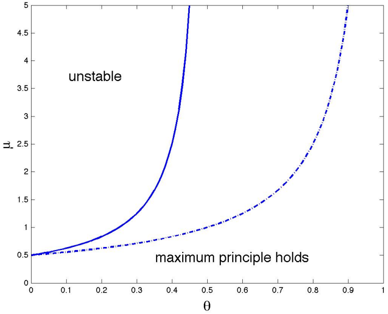
Finite difference methods: maximum principle

- Verification of conditions required for maximum principle to hold forms a practical test for stability of numerical finite difference schemes for diffusion and advection-diffusion equations.
- Maximum principle states that value of variable U_j^n bounded between boundary values and initial values. E.g., when u is seen as a temperature, temperature values can not go above or below temperatures imposed initially or at boundaries.

Finite difference methods: stability criteria

The Fourier stability condition (17) and condition for maximum principle, displayed in Fig.:

$$\mu = \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{[2(1-2\theta)]} \quad \text{and} \quad \mu(1-\theta) \leq \frac{1}{2}. \qquad (18)$$



Finite difference methods: homework, week 2

- Study Morton and Mayers (2005), Chapter 2, sections 2.7, 2.11.
- Read Morton and Mayers (2005), Chapter 2, sections 2.8, 2.9, 2.10.
- Continue/finish exercise-I.
- Urgent: get *Firedrake* installed on your machines, asap, please!
- . . . hints for Exercise-I . . . .

Finite volume or Godunov method

- Finite volume methods may be most natural for hyperbolic PDEs expressed as conservation laws.
- We only consider the 1D case here:

$$\partial_t \mathsf{u} + \partial_x(\mathsf{f}(\mathsf{u}) = 0 \qquad \text{or} \tag{19}$$

$$\partial_t \mathsf{u}_i + \partial_{x_j} \mathsf{f}_{ij} = 0 \quad \text{with} \quad j = 1 \tag{20}$$

with $\mathsf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathsf{u} = (f_1, f_2, \ldots, f_n)$.

Finite volume: example conservation laws

Examples:

- Euler equations in 2D:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u(E+p)\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ v(E+p)\end{pmatrix}=0 \tag{21}$$

  with u and v velocity components in x and y, p pressure and E total energy. EOS: $p=(\gamma-1)(E-\frac{1}{2}\rho(u^2+v^2))$.

- Shallow-water equations in 1D:

$$\partial_t h+\partial_x(hu)=0, \partial_t(hu)+\partial_x(hu^2+\frac{1}{2}gh^2)=0 \tag{22}$$

  with water depth $h(x,t)$ and depth-averaged velocity $u(x,t)$.

Finite volume: examples

Examples of conservative systems with extra terms:

- Width-averaged shallow-water or St. Venant equations:

$$\partial_t A + \partial_s (Au) = S \tag{23}$$

$$\partial_t (Au) + \partial_s \left(Au^2 + gAh \right) = gh\partial_s A - gA\partial_s b - F, \tag{24}$$

with source and friction terms $S = S(s,t)$,
$F = gC_m Au|u|/R(A,s)^{4/3}$; along-river coordinate s;
cross-section $A(s,t)$; water depth $h = h(A,s)$; depth-averaged velocity $u(s,t)$; river slope $-\partial_s b$, and, accelation of gravity g.

Finite volume: examples

- Limit of kinematic river equation, conservative with extra source term:

$$\partial_t A + \partial_s \left(AR(A,s)^{2/3} \sqrt{-\partial_s b} / C_m \right) = S, \qquad (25)$$

with Manning coefficient C_m, hydraulic radius $R(A,s)$ (wetted area A over wetted perimeter) and "volume" $S(s,t)$.

Finite volume: overview for Burgers-advection system

- Illustrate Godunov method or finite volume discretization for simple system of hyperbolic equations.
- Step by step discretization for system of uncoupled Burgers' and linear advection equations

$$\begin{aligned} u_t + \left(u^2/2\right)_x &= 0 \qquad (26) \\ v_t + a\, v_x &= 0 \qquad (27) \end{aligned}$$

with $a > 0$ constant, $u = u(x,t)$ and $v = v(x,t)$ on $x \in [0, L]$, $(\cdot)_t = \partial_t$, etc.

- Boundary conditions required, not specified presently.
- Initial conditions $u(x,0)$ and $v(x,0)$ are given at $t = t_0 = 0$.

Finite volume method: studying

From Leveque's book *Finite volume methods*:

- Study §1.1 (conservation law) §3.1 (linear advection Eq.) till §3.1.1 & 3.2 (Burgers' Eq).
- Study §3.1, §3.3 (shock formation).

Godunov method example: Step-1

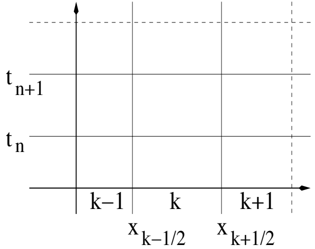
- Exposition of simple uncoupled system plus algorithm facilitates implementation for genuinely coupled fluid system, presented elsewhere.
- *Step 1*:
  System (26) is written concisely in hyperbolic form (20)

$$\mathrm{u}_t + (\mathrm{f}(\mathrm{u}))_x = 0, \tag{28}$$

  after identification $\mathrm{u} = (u, v)^T$ and flux $\mathrm{f}(\mathrm{u}) = (u^2/2, a\,v)^T$ (transpose $(\cdot)^T$).

Godunov method example: Step-2

- *Step 2:* Define space-time mesh with N "finite volumes" on domain $x \in [0, L]$ in time interval $I_n = [t_n, t_{n+1}]$ (Fig. 22).
- Cell k occupies $x_{k-1/2} < x < x_{k+1/2}$ and $k = 1, 2, \ldots, N$.
- $N + 1$ cell boundaries $x_{1/2}, \ldots, x_{N-1/2}, x_{N+1/2}$. Cell lengths $h_k = x_{k+1/2} - x_{k-1/2}$ and time step $\Delta t_n = t_{n+1} - t_n$ may vary.
- There are $n = 0, \ldots, N_t$ time intervals I_n, where $t = t_n$ is the time after n time steps, initial conditions at $t = t_0 = 0$.



Godunov method example: Step-3

- *Step 3:* Integrate (28) in space-time element $x_{k-1/2} < x < x_{k+1/2}$ and $t_n < t < t_{n+1}$, Fig. 22.
- Via coordinate transformation $x' = x - x_{k+1/2}$, $t' = t - t_n$, right-bottom corner becomes origin $(x', t')^T = (0, 0)^T$.
- After integration of (28) over space-time element:

$$\mathrm{U}_k^{n+1} = \mathrm{U}_k^n - \frac{1}{h_k} \int_{t_n}^{t_{n+1}} \mathrm{f}_{k+1/2}(t) - \mathrm{f}_{k-1/2}(t)\,\mathrm{d}t, \tag{29}$$

with mean cell average U_k in cell k

$$\mathrm{U}_k(t) = \frac{1}{h_k} \int_{x_{k-1/2}}^{x_{k+1/2}} \mathrm{u}(x, t)\,\mathrm{d}x. \tag{30}$$

- Flux is at the cell boundaries: $\mathrm{f}_{k+1/2}(t) = \mathrm{f}\left(\mathrm{u}(x = x_{k+1/2}, t)\right)$.
- $\mathrm{U}_k(t)$ in (30) and $\mathrm{f}_{k+1/2}(t)$ still functions of time t, and $\mathrm{U}_k^n = \mathrm{U}_k(t = t_n)$, etc.

Godunov method example: Step-3

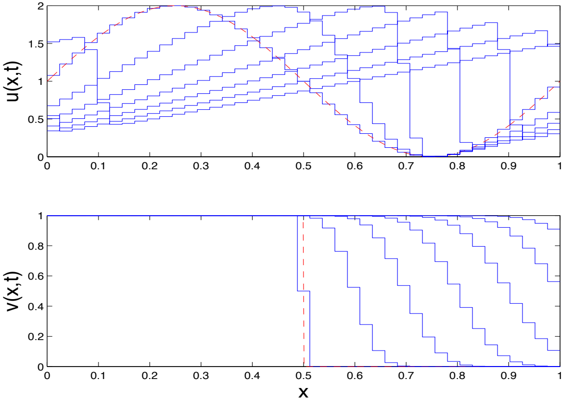
- Integral expression (29) exact provided that $\mathrm{u}(x, t)$ known. Start at $n = 0$, calculate $\mathrm{U}_k^0 = \mathrm{U}_k(t = t_0)$ using (30).
- Graphically, U_k^0 is projection of initial data on piecewise constant profiles at time t_0, cf. initial step profiles in Fig. 25.
- Determine $\mathrm{f}_{k+1/2}(t)$ over $t_n < t < t_{n+1}$ in (29) to obtain

$$\mathrm{U}_k^{n+1} = \mathrm{U}_k^n - \frac{\Delta t}{h_k}\left(\mathrm{F}_{k+1/2}(\mathrm{U}_k^n, \mathrm{U}_{k+1}^n) - \mathrm{F}_{k-1/2}(\mathrm{U}_{k-1}^n, \mathrm{U}_k^n)\right) \tag{31}$$

with (approximate) numerical flux

$$\mathrm{F}_{k+1/2}(\mathrm{U}_k^n, \mathrm{U}_{k+1}^n) \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathrm{f}_{k+1/2}(t)\,\mathrm{d}t. \tag{32}$$

Godunov method example: Step-3



Godunov method example: Step-4

- *Step 4:* Calculate numerical fluxes $\mathrm{F}_{k+1/2}(\mathrm{U}_k^n, \mathrm{U}_{k+1}^n)$ in (32) at all nodes $x_{k+1/2}$ for $k = 0, 2, \dots, N$.
- There are various strategies to find (32), usually involving an approximation.
- Consider classical Godunov strategy based on the Riemann solution.
- Calculate $\mathrm{F}_{k+1/2}(t)$ in (32) exactly over $t_n < t < t_{n+1}$ in (29), only feasible for piecewise constant approximation U_k^n at time t_n and starting with projected initial condition U_k^0, instead of smoother initial condition.
- Solution of Riemann problem in new coordinates, i.e. with $\mathrm{u} = \mathrm{u}(x', t')$ and $\mathrm{f} = \mathrm{f}\left(\mathrm{u}(x', t')\right)$, provides such exact solution.

Godunov method example: Riemann problem

- The Riemann problem is defined as

$$\frac{\partial \mathrm{u}}{\partial t'} + \frac{\partial \mathrm{f}}{\partial x'} = 0 \tag{33}$$

for special initial conditions

$$\mathrm{u}(x', t' = 0) = \begin{cases} \mathrm{u}_{left} = \mathrm{u}_l = \mathrm{U}_k^n & \text{for} \quad x' < 0 \\ \mathrm{u}_{right} = \mathrm{u}_r = \mathrm{U}_{k+1}^n & \text{for} \quad x' \geq 0 \end{cases} . \tag{34}$$

- Riemann solution such that $u(x', t')$ constant along characteristics $x' = x'_0 + C\, t'$ for some C depending on $\mathrm{u}_{l,r}$.
- $\mathrm{f}_{k+1/2}(t) = \mathrm{f}_{k+1/2}(\mathrm{u}(x_{k+1/2}, t))$ is constant; (32) can be integrated exactly —note that $\mathrm{u}(x' = 0, t') = \mathrm{u}(x_{k+1/2}, t)$ due to coordinate change.

Godunov method example: Riemann problem

NB:

- While formula (29) exact, numerical approximation occurs because we continue from a piecewise constant profile with value U_k^n in cell k at a certain time t_n.
- At cell boundary $x_{k+1/2}$ between cells k and $k+1$, adjacent values are U_k^n and U_{k+1}^n.
- When other cell boundaries $x_{k-1/2}$ and $x_{k+3/2}$ are "far away" in sense that time step Δt is "small enough", we are locally dealing with a Riemann problem around cell boundary $x_{k+1/2}$.
- Although Riemann solution no longer piecewise constant within each cell at end of time step, we continue only with information stored in the mean values U_k^{n+1}.
- This is the approximation made.

Godunov method example: Riemann problem

- The characteristic form of (26)–(27) is as follows

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \quad \text{on} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u \tag{35}$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = 0 \quad \text{on} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = a, \tag{36}$$

which has solution

$$x = x_{01} + ut, u = u_0(x_{01}) \quad \text{s. t.} \quad u(x,t) = u_0\left(x - u(x,t)\right)$$
$$x = x_{02} + at, v = v_0(x_{02}) \quad \text{s. t.} \quad v(x,t) = v_0(x - at).$$

- At $t = 0$ we see that $x = x_{01}$ or $x = x_{02}$ and these parameterise the domain for each equation.
- Instead of two PDEs, we have four ODEs.
- For constant a, solution of linear advection equation is a mere shift of original profile to left or right, depending on sign of a.

Godunov method: studying

From Leveque's book *Finite volume methods*:

- Study §3.5 (Riemann problem), §3.6 (shock speed).
- Study §13.2 (Godunov method), §13.5 (Godunov for scalar conservation laws); read Chap. 9.

Godunov method example: Riemann problem

The Riemann problem for the uncoupled system (26) special: consists of separate Riemann problems for two equations.

- Solution of Riemann problem for linear advection equation:

$$\frac{\partial v}{\partial t'} + \frac{\partial (a\, v)}{\partial x'} = 0 \tag{37}$$

  with constant $a > 0$.

- NB: piecewise constant initial conditions

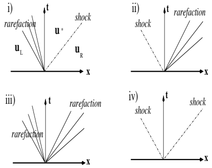
$$v(x', t' = 0) = \begin{cases} v_{left} = v_l = V_k^n & \text{for} \quad x' < 0 \\ v_{right} = v_r = V_{k+1}^n & \text{for} \quad x' \geq 0 \end{cases} . \tag{38}$$

- For $a > 0$ all characteristics are $x' = x_0 + a\, t'$. Solution $v(x' = 0, t') = v(x_{k+1/2}, t) = v_l^n$ with $\mathrm{f}\left(v(x_{k+1/2}, t)\right) = a\, v_l^n$ within a sufficiently small time interval.

Linear advection equation: Integral (32) straightforward to evaluate; Godunov scheme (31) becomes

$$V_k^{n+1} = V_k^n - \frac{\Delta t}{h_k} a \left(V_k^n - V_{k-1}^n\right). \tag{39}$$

- Correspondence of (39) with upwind finite difference discretization is clear (Chapter 4 in M& M), but V_k^n is mean value of $v(x, t_n)$ over cell k and not a grid point value.



Godunov method example: CFL condition

- Application of maximum principle to (39) yields, by imposing that all coefficients 1, $(1 - \Delta t a/h_k)$ and $a\Delta t/h_k$ of $V_k^{n+1}, V_k^n, V_{k-1}^n$ are larger than zero (and $a > 0$):

$$V_k^{n+1} = (1 - \frac{\Delta t}{h_k} a) V_k^n + \frac{\Delta t}{h_k} a V_{k-1}^n \tag{40}$$

$$\Longrightarrow 1 - \frac{\Delta t}{h_k} a > 0 \Longleftrightarrow \Delta t < \frac{h_k}{a}. \tag{41}$$

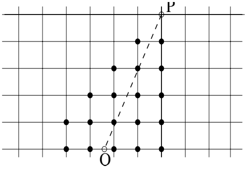
- The well-known Courant-Friedrichs-Lewy or CFL condition.
- When $a < 0$, the upwind discretization should be:

$$V_k^{n+1} = V_k^n + \frac{\Delta t}{h_k} |a| (V_{k+1}^n - V_k^n). \tag{42}$$

- For general a, the CFL condition thus reads $\Delta t < h_k/|a|$, which also makes sense dimensionally since a is a "wind" speed.

Godunov method example: CFL condition

Graphically, upwinding for linear advecion equation:

(a) 

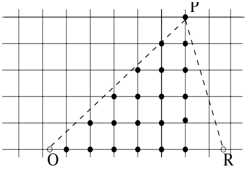
(b) 

Figure: Consider the linear advection equation $u_t + a\,u_x = 0$ with $a > 0$. (a) The solution $u(x,t) = u^0(x - at)$ has a characteristic tracing through point P back to point Q satisfying the CFL condition $\Delta t < \Delta x/|a|$. (b) The CFL condition is violated when $\Delta t > \Delta x/|a|$ as the information transported along the characteristic falls outside the discretization stencil. It shows the dependence of the numerical solution on the initial data.

Godunov method example: Riemann problem

- Burgers' equation allows discontinuous or shock solutions, where $u(x,t)$ obtains different limiting values.
- Discontinuity resides at position $x = x_b(t)$ and moves with shock speed $s \equiv \mathrm{d}x_b/\mathrm{d}t$.
- Integrate Burgers' equation in (26) around $x_b(t)$, and let $\epsilon \to 0$, to obtain

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{x_b-\epsilon}^{x_b+\epsilon}\frac{\partial u(x,t)}{\partial t}\mathrm{d}x + \int_{x_b-\epsilon}^{x_b+\epsilon}\frac{\partial (u^2/2)}{\partial x}\mathrm{d}x =& 0\\
\Leftrightarrow \frac{\partial}{\partial t}\lim_{\epsilon\to 0}\int_{x_b(t)-\epsilon}^{x_b(t)+\epsilon} u(x,t)\mathrm{d}x - \frac{\mathrm{d}x_b(t)}{\mathrm{d}t}\lim_{\epsilon\to 0} u(x,t)\big|_{x_b(t)-\epsilon}^{x_b(t)+\epsilon} + [u^2/2] =& 0 \qquad (43)\\
\Leftrightarrow -\frac{\mathrm{d}x_b(t)}{\mathrm{d}t}[u] + [u^2/2] = 0 \Leftrightarrow s = \frac{\mathrm{d}x_b(t)}{\mathrm{d}t} = \frac{[u^2/2]}{[u]} =& \frac{1}{2}(u_l + u_r)
\end{aligned}$$

  with $[u] = u_r - u_l$, $[u^2/2] = (u_r^2 - u_l^2)/2$, $u_r = u(x,t)|_{x\downarrow x_b}$ and $u_l = u(x,t)|_{x\uparrow x_b}$.
- In literature, approximate numerical fluxes are found such as Roe solvers and kinetic fluxes.

Godunov method example: Riemann problem Burgers

- *Exercise-Burgers:* (i) Derive the Rieman solution for the Riemann problem of the Burgers' equation.
- *Exercise-Burgers:* (ii) Complete the Godunov method for the Burgers' equation using this Riemann solution. See `https://www.youtube.com/watch?v=izMsj639hGI` and `https://www.youtube.com/watch?v=goL8_rET1H0`

Provide both the actual solution in formula form as well as the solution in graphical form in the characteristic t,x-plane (or t',x'-plane).

Godunov method example: Riemann problem Burgers

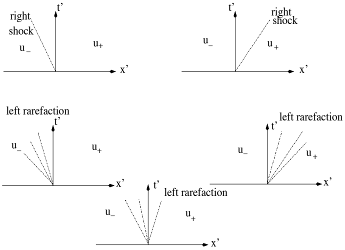


Figure: Graphical solution of Riemann problem for Burgers' equation. $u_l > u_r$: shock wave with shock spreed $s = (u_l + u_r)/2$. $u_l \leq u_r$: rarefaction wave results with solution x'/t' in the interval $u_l\, t' < x' < u_r\, t'$. u_l and u_r: initial condition in definition Riemann problem.

Godunov method example: Riemann problem Burgers

- Solution is constant along each characteristic by differentiating solution along the characteristic:

$$\frac{d}{dt}u(x(t),t) = \frac{\partial}{\partial t}u(x(t),t) + \frac{\partial}{\partial x}u(x(t),t)\frac{\mathrm{d}x}{\mathrm{d}t} = u_t + uu_x = 0.$$

- If solution is smooth then it can be constructed by backtracking characteristics to initial solution as long as they do not cross each other: For each (x,t) we can solve the following equations for ξ:

$$u = u_0(\xi) = u(\xi,0) \qquad \text{and} \qquad x = \xi + u(\xi,0)t$$

  after which we can determine the solution at time t:

$$u(x,t) = u(\xi,0).$$

- Hence, the implicit solution of Burgers' equation is $u(x,t) = u_0(x - u(x,t)\,t)$, since

$$0 = \frac{d}{dt}u(x(t),t) = \frac{\partial u(x,t)}{\partial t} + u\frac{\partial u(x,t)}{\partial x} \qquad \text{and}$$
$$\partial_x u = u_0'(1 - t\partial_x u) \Longrightarrow \partial_x u = \frac{u_0'}{1+tu_0'}$$

  with u_0' the derivative of u_0 with respect to its argument.

Godunov method example: Riemann problem Burgers

- The solution to the Riemann problem is either a shock wave or a rarefaction wave.
- It is a shock wave when $u_l > u_r$ and a rarefaction wave when $u_l < u_r$, which follows from considering the characteristics $\mathrm{d}x/\mathrm{d}t = u$ in the x–t-plane.
- The shock wave has shock speed $s = (u_l + u_r)/2$ and its position is given by $x' = s\,t'$; to the left of the shock $u(x', t) = u_l$ and to the right $u(x', t) = u_r$.
- Since the numerical flux is evaluated at $x' = 0$ (i.e. at $x = x_{j+1/2}$), the flux $u^2/2$ is thus either $u_l^2/2 = (U_j^n)^2/2$ when $s > 0$, or $u_r^2/2 = (U_{j+1}^n)^2/2$ when $s < 0$ for the shock wave case.

Godunov method example: Riemann problem Burgers

- The rarefaction wave has characteristics $dx'/dt' = u$ on which u is constant. The tail and the head of the rarefaction wave lie at $x' = u_l\, t'$ and $x' = u_r\, t'$, respectively.
- Hence the rarefaction wave solution is

$$u(x', t)' = \begin{cases} u_l & x' < u_l\, t' \\ x'/t' & u_l\, t' < x' < u_r\, t' \\ u_r & x' > u_r\, t' \end{cases} . \tag{44}$$

- We deduce from this solution that $u_l < u_r$. So at $x' = 0$, or $x = x_{j+1/2}$, we find for the rarefaction wave case that $u(0, t') = u_l$ when $u_l > 0$, $u(0, t') = 0$ when $u_l < 0$ and $u_r > 0$, and $u(0, t') = u_r$ when $u_r < 0$.
- Note that $u(x', t') = x'/t'$ is a similarity solution of Burgers' equation.

Godunov method example: Riemann problem Burgers

- Numerical flux function F at each face $x_{j+1/2}$ is defined as:

$$F(U_j^n, U_{j+1}^n) = \frac{1}{\triangle t} \int_{t_n}^{t_{n+1}} \frac{1}{2} \left(u^\dagger(U_j^n, U_{j+1}^n) \right)^2 dt \qquad (45)$$

  with $u^\dagger(U_j^n, U_{j+1}^n) = u(x_{j+1/2}, t)$ for the special piecewise constant data at time t_n.
- Finally, we need to evaluate the integral (45) by combining these shock wave and rarefaction wave solutions for $u_l > u_r$ and $u_l < u_r$ respectively.
- Note that the solution is constant at $x' = x - x_{j+1/2} = 0$, which simplifies the time integration in (45).

Godunov method example: Riemann problem

Homework Exercise-II.