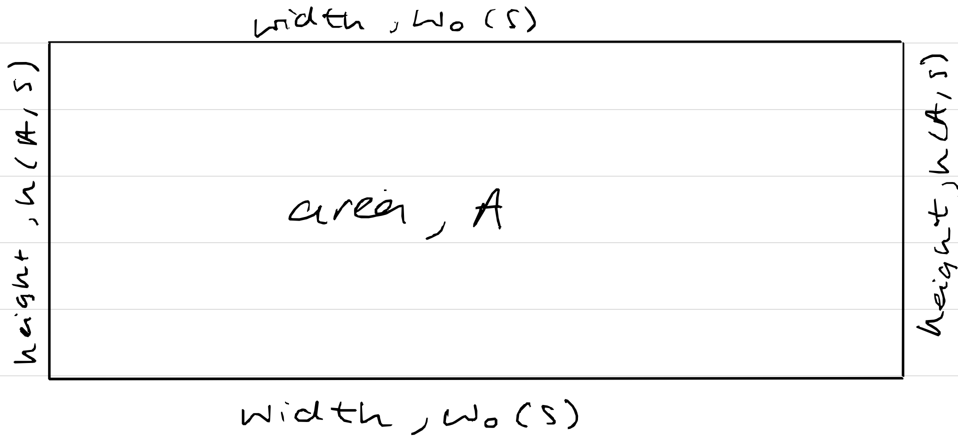
1. Rectangular cross section



Given rectangular model, Area = width × height

$\Rightarrow \quad A = w_0(S) \times h(A,S)$

$\Rightarrow \quad h(A,S) = \dfrac{A}{w_0(S)}$

Wetted perimeter, $P(A,S) = \sum$ all edges with exception of river surface

$\Rightarrow P = w_0 + h + h$

given $h(A,S) = \dfrac{A}{w_0(S)}$

$\rightarrow P(A,S) = w_0 + 2\left(\dfrac{A}{w_0(S)}\right)$

$$F(A,S) = \frac{A^{5/3}\sqrt{-\partial_s b}}{C_m P(A,S)^{2/3}}$$

$$= A^{5/3} P(A,S)^{-2/3} \cdot G(S)$$

where $G(S) \equiv \sqrt{\frac{\partial_s b}{C_m}}$ (only dependent on $S \Rightarrow$ only a constant in $\partial_A F$)

$$\frac{\partial F}{\partial A} = G(S) \cdot \frac{\partial}{\partial A}\left(A^{5/3} P^{-2/3}\right)$$

$$= G(S) \cdot \left(\frac{5}{3} A^{2/3} \cdot P^{-2/3} + A^{5/3}\left(-2/3\right) P^{-5/3} \frac{\partial P}{\partial A}\right)$$

$$= G(S) \cdot \left(\frac{5}{3} A^{2/3} P^{-2/3} - \frac{2}{3} A^{5/3} P^{-5/3} \cdot \frac{2}{\omega_0}\right)$$

$$\frac{\partial F}{\partial A} = \frac{5\sqrt{-\partial_s b}}{3\, C_m} \cdot \frac{A^{2/3}}{P^{2/3}} - \frac{2\sqrt{-\partial_s b}}{3\, C_m} \cdot \frac{2A^{5/3}}{\omega_0 P^{5/3}}$$

given $P = W_0 + 2A/W_0$

$$\Rightarrow \frac{\partial F}{\partial A} = \frac{5\sqrt{-\partial_s b}}{3\, C_m} \cdot \frac{A^{2/3}}{(W_0 + 2A/W_0)^{2/3}} - \frac{2\sqrt{-\partial_s b}}{3\, C_m} \cdot \frac{2A^{5/3}}{(W_0 + 2A/W_0)^{5/3}}$$

Can also be written as

$$\frac{\sqrt{-\partial_s b}}{3\, C_m} \cdot A^{2/3}\left(W_0 + 2A/W_0\right)^{-5/3}\left(5\left(W_0 + 2A/W_0\right) - \frac{4A}{W_0}\right)$$

$$= \frac{\sqrt{-\partial_s b}}{3 c_m} A^{2/3} (w_0 + 2A/w_0)^{-5/3} \left(5w_0 + \frac{10A}{w_0} - \frac{4A}{w_0} \right)$$

$$= \frac{\sqrt{-\partial_s b}}{3 c_m} A^{2/3} \left(w_0 + \frac{2A}{w_0}\right)^{-5/3} \left(5w_0 + \frac{6A}{w_0}\right)$$

$$= \frac{\sqrt{-\partial_s b}}{3 c_m} \cdot \frac{(5 w_0 A^{2/3} + 6A^{5/3}/w_0)}{(w_0 + 2A/w_0)^{5/3}}$$

w_0, A, c_m - real scalar quantities $\Rightarrow > 0$

$-\partial_s b$ is positive else the expressions are not valid

hence $\sqrt{-\partial_s b}$ hence all values

$$\frac{\partial F}{\partial A} > 0$$

$$\frac{\partial F}{\partial s} = \frac{\partial}{\partial s} \left(G(s) \cdot A^{5/3} p^{-2/3} \right)$$

Assuming A is independt of s

$$\rightarrow \frac{\partial F}{\partial s} = A^{5/3} \cdot \frac{\partial}{\partial s} \left(G(s) \cdot p^{-2/3} \right)$$

$$= A^{5/3} \left(G'(s) \cdot p^{-2/3} - \frac{2}{3} G(s) \cdot p^{-5/3} \frac{\partial p}{\partial s} \right)$$

$$= A^{5/3} \left[G'(s) p^{-2/3} - \frac{2}{3} G(s) p^{-5/3} \left(\frac{\partial}{\partial s} \left(w_0 + \frac{2A}{w_0} \right) \right) \right]$$

$$= A^{5/3}\left(G'(s)\,p^{-2/3} - \frac{2}{3} G(s)\, p^{-5/3}\left(\frac{\partial w_0}{\partial s} - \frac{2A}{w_0^2}\frac{\partial w_0}{\partial s}\right)\right)$$

$G'(s)$ will be much less than the second term and is considered negligible

$$\Rightarrow \frac{\partial F}{\partial s} = -\frac{2\sqrt{gb}}{3\,C_m} A^{5/3} p^{-5/3}\left(1 - \frac{2A}{w_0^2}\right)\frac{dw_0}{ds}$$

given $p = w_0 + \dfrac{2A}{w_0}$

$$\rightarrow \frac{\partial F}{\partial s} = -\frac{2\sqrt{g_s b}}{3\,C_m} \cdot \frac{A^{5/3}}{(w_0 + 2A/w_0)^{5/3}}\left(1 - \frac{2A}{w_0^2}\right)\frac{\partial w_0}{\partial s}$$

With $S = 0$, equation (3) becomes

$$\partial_t A + \partial_s F(A, s) = 0$$

which can be expressed as

$$\partial_t A + \frac{\partial F}{\partial A}\partial_s A + \frac{\partial F}{\partial s} = 0$$

2. Where $w_0(s)$ is independent of s, or varying very slowly the PDE (3):

$\partial_t A + \partial_s F(A, s) = 0$ can be treated as a scalar conservation law where

$$\lambda(A) = \frac{\partial F}{\partial A}$$

$$= \frac{1\sqrt{-\partial_s b}}{3\, C_m} \cdot \frac{5w_0 + 6A^{5/3}/w_0}{(w_0 + 2A/w_0)^{5/3}}$$

Initial data

$$A(s, 0) = \begin{cases} A_L, & s < 0 \\ A_R, & s > 0 \end{cases}$$

$\frac{df}{dA} > 0 \Rightarrow \lambda(A) > 0$ hence characteristics move downstream

Given monotonicity of $\lambda(A)$ (A, w_0 are physical scalar quantities) solutions of type:

$A_L < A_R$ characteristics diverge $\rightarrow$ rarefaction fan

$A_L > A_R$ characteristics converge $\rightarrow$ shock formation

$A_L = A_R$ constant solution

Rarefaction ($A_L < A_R$)

Self-similar solution depending on $\xi (= s/t)$

Inside a centred rarefaction fan, the solution satisfies the self-similar relation

$$\xi = \lambda(A(\xi))$$

Because λ is strictly increasing ($A_L < A_R$ and $A > 0$) relation can be inverted to give

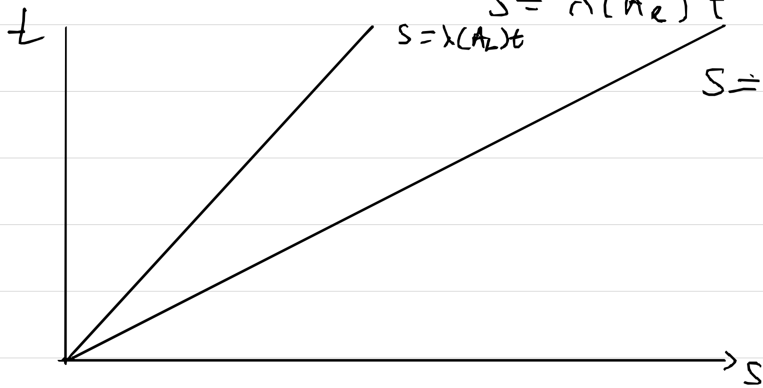
$$A(s,t) = \lambda^{-1}(s/t) \quad \text{for } \lambda(A_L) \le s/t \le \lambda(A_R)$$

Providing the Riemann solution

$$A(s,t) = \begin{cases} A_L, & s/t < \lambda(A_L) \\ \lambda^{-1}(s/t), & \lambda(A_L) \le s/t \le \lambda(A_R) \\ A_R, & s/t > \lambda(A_R) \end{cases}$$

Characteristic lines: $s = \lambda(A_L)t$ (slower wave speed)

$s = \lambda(A_R)t$ (faster wave speed)



Shock ($A_L > A_R$)

Shock moves with Rankine-Hugoniot speed:

$$\sigma = \frac{f(A_R) - f(A_L)}{A_R - A_L}$$

Entropy condition for a compressive shock enforces that

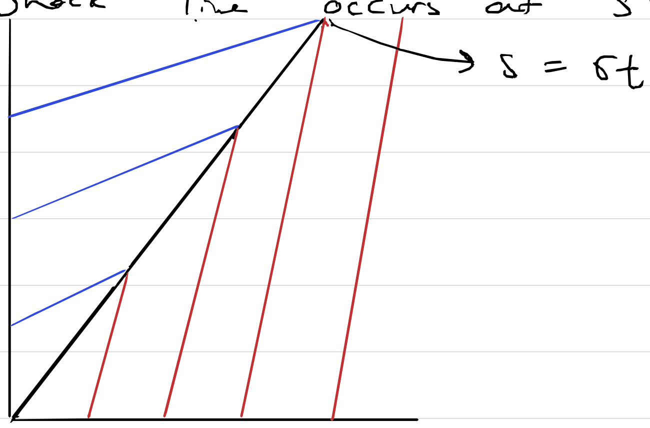
$$\lambda(A_R) < \sigma < \lambda(A_L)$$

Meaning: characteristics left of the shock impinge upon it and characteristics leave right of it.

Riemann solution:

$$A(s,t) = \begin{cases} A_L, & s < \sigma t \\ A_R, & s > \sigma t \end{cases}$$

Shock line occurs at $s = \sigma t$



— left characteristics

— right characteristics

If $w_0(s)$ is slowly varying, the solution can be treated as locally constant, therefore the Riemann solution is still valid locally. Over longer distances / times, the characteristics will curve since λ weakly depends on s due to $w_0(s)$ causing small differences in the rarefaction or shock speed over large distance / time.

3. $\partial_t A(s,t) + \partial_s F(A,s) = 0$

Integrate over cell K $(s_{K-1/2}, s_{K+1/2})$ and over time interval (t_n, t_{n+1})

$$\int_{t_n}^{t_{n+1}} \int_{s_{K-1/2}}^{s_{K+1/2}} (\partial_t A + \partial_s F)\, ds\, dt = 0$$

Swapping integrals and using divergence theorem for 1D:

$$\Rightarrow \int_{s_{K-1/2}}^{s_{K+1/2}} \left(A(s,t^{n+1}) - A(s,t^n)\right) ds + \int_{t_n}^{t_{n+1}} \left(F(A,s)\big|_{s_{K+1/2}} - F(A,s)\big|_{s_{K-1/2}} \right) dt = 0$$

Defining cell average:

$$\bar{A}_K^n = \frac{1}{\Delta s_K} \int_{s_{K-1/2}}^{s_{K+1/2}} A(s,t^n)\, ds \qquad [\text{where } \Delta s_K = \text{cell width}]$$

Then the update formula becomes

$$\bar{A}_k^{n+1} - \bar{A}_k^n = \frac{1}{\Delta S_k} \int_{t_n}^{t_{n+1}} \left(F(A,S)\big|_{S_{k+1/2}} - F(A,S)\big|_{S_{k-1/2}} \right) dt = c$$

$$\Rightarrow \bar{A}_k^{n+1} = \bar{A}_k^n + \frac{1}{\Delta S_k} \int_{t^n}^{t_{n+1}} \left(F(A,S)\big|_{S_{k+1/2}} - F(A,S)\big|_{S_{k-1/2}} \right) dt$$

In Godunov's method, the solution is represented as piecewise constant states equal to cell averages:

$$A(S,t^n) \approx \begin{cases} \bar{A}_k^n, & S \in (S_{k-1/2}, S_{k+1/2}) \\ \bar{A}_{k+1}^n, & S \in (S_{k+1/2}, S_{k+3/2}) \end{cases}$$

At each interface $S_{k+1/2}$ solve the Riemann problem with left state $\bar{A}_k^n$ and right state $\bar{A}_{k+1}^n$

$$\Rightarrow A(S,t^n) = \begin{cases} \bar{A}_k^n, & S < S_{k+1/2} \\ \bar{A}_{k+1}^n, & S > S_{k+1/2} \end{cases}$$

The Riemann solution is self-similar: $A(S,t) = A(\xi)$ with $\xi = \frac{S - S_{k+1/2}}{t - t^n}$. The state exactly at the interface corresponds to $\xi = 0$ and is determined entirely by the cell averaged values $\bar{A}_k^n, \bar{A}_{k+1}^n$

$$\Rightarrow A^*_{k+1/2} = A(0; \bar{A}_k^n, \bar{A}_{k+1}^n)$$

Since Riemann solution is self-similar, flux at the interface $t \in (t_n, t_{n+1})$ is constant in time and given by $F(A^*_{k+1/2}, S_{k+1/2})$ therefore the time integral reduces to $\Delta t \cdot F(A^*_{k+1/2}, S_{k+1/2})$

Godunov numerical flux given by

$$F_{k+1/2}(\bar{A}^n_k, \bar{A}^n_{k+1}) = f(A^*_{k+1/2}, S_{k+1/2})$$

Update becomes

$$\bar{A}^{n}_{k+1} = \bar{A}^n_k - \frac{\Delta t}{\Delta S_k}\left[F_{k+1/2}(\bar{A}^n_k, \bar{A}^n_{k+1}) - F_{k-1/2}(\bar{A}^n_{k-1}, \bar{A}^n_k)\right]$$

Cell averages are used as!

- Godunov reconstructs solution based on piecewise constant values from previous cells, defining the Riemann problem at each interface.
- Self similarity of the Riemann solution means at each interface only the left and right initial states are required.
- Explicit time stepping means $\bar{A}^{n+1}$ is not solved within each step $\Rightarrow$ fluxes are known.

4. Because for the manning flux $\lambda(A) = F'(A) > 0$ for all physically admissible $A \geq 0$, every characteristic at an interface points downstream. Therefore the Godunov flux is purely upwind.

$$F_{k+1/2} = F(A_L, S_{k+1/2}) = F(\bar{A}_k^n, S_{k+1/2})$$

$$\lambda(A) = F'(A) = \frac{1}{3}\frac{\sqrt{S_0}}{C_m} \cdot \frac{(5w_0 A^{2/3} + 6A^{5/3}/w_0)}{(w_0 + 2A/w_0)^{5/3}}$$

$\lambda(A) > 0$ for all $A \geq 0 \Rightarrow$ every characteristic travels downstream. In case of both centred reconstruction and slock, left value should be used.

Flux: $F_{k+1/2}(\bar{A}_k^n, \bar{A}_{k+1}^n) - F_{k-1/2}(\bar{A}_{k-1}^n, \bar{A}_k^n)$

5. $$\bar{A}_k^{n+1} = \bar{A}_k^n + \frac{\Delta t}{\Delta S_k}(F_{k+1/2} - F_{k-1/2})$$

cell width $\Delta S_k = h_k$

For the numerical scheme to be stable, numerical information must not travel beyond a single cell during a time step, providing the

CFL condition :

$$\max_k \frac{|\lambda_k| \Delta t}{h_k} \leq CFL \qquad 0 < CFL < 1$$

for time step, $\Delta t \leq CFL \min_k \frac{h_k}{|\lambda_k|}$

Linearized conservation law

$\rightarrow \quad \partial_t A + \lambda(A_k) \partial_s A$.

Where $\lambda(A) = \frac{dF}{dA}$

Consider $\lambda(A)$ to be a constant coefficient at an interface $\Rightarrow \lambda(A_k)$. Applying this for every cell interface allows the global condition to be enforced as would be the case for a linear system.

$$\lambda(A, s) = \frac{1\sqrt{-\partial_s b}}{3\ cm} \cdot \frac{5 w_0 A^{2/3} + 6A^{5/3}/w_0}{(w_0 + 2A/w_0)^{5/3}}$$

given $h = A/w_0$, this can be expressed as

$$\lambda(h, s) = \frac{1\sqrt{-\partial_s b}}{3cm} \cdot \frac{5 w_0^{5/3} h^{2/3} + 6h^{5/3} w_0^{2/3}}{(w_0 + 2h)^{5/3}}$$

$\Delta t \leq CFL \min_k \frac{h_k}{\lambda(h_k, s_k)}$ which can be expressed as

$$\Delta t \leq CFL \min_k \frac{h_k}{\frac{1\sqrt{-\partial_s b}}{3\ cm} \cdot \frac{5 w_0^{5/3} h^{2/3} + 6h^{5/3} w_0^{2/3}}{(w_0 + 2h)^{5/3}}}$$

Simplified to give

$$\Delta t \leq CFL \min_k \left[3 \frac{C_m}{\sqrt{-\partial_s b(s_k)}} \cdot \frac{h_k^{1/3} (W_0(s_k) + 2h_k)^{5/3}}{5 W_0(s_k)^{5/3} + 6 h_k W_0(s_k)^{2/3}} \right]$$

6. Godunov flux determines interface state from the incoming characteristic direction. Since $\partial_A F(A) > 0$ at all points in the stream, all characteristics point downstream.

- Meaning at $s = 0$, characteristics enter the domain, so inflow should be defined here by use of the left ghost cell $\bar{A}_{-1}$.
- And upstream at $s = L_c$, the solution will be determined from the interior values of the domain so no flux should be imposed.

All fine.