

$$A_t + (Au)_s = 0, \quad u_t + uu_s = -g(h+b)_s - gC_m^2 u|u|R^{-4/3}$$

$$R(A, S) = AP^{-1}(A, S)$$

$$\text{discharge } Q = Au$$

phenomenological Manning relation, provided $-b_s > 0$:

$$u = C_m^{-1} R^{2/3} \sqrt{-b_s}, \quad Q = Au = C_m^{-1} P^{-2/3} A^{5/3} \sqrt{-b_s}$$

subbing into 1st St Venant eq: $A_t + (AR^{2/3} \sqrt{-b_s} C_m^{-1})_s = S$

$$\Rightarrow A_t + (F(A, S))_s = S$$

$$F(A, s) = C_m^{-1} P^{-2/3} A^{5/3} \sqrt{-b_s}$$

$$1) \quad \begin{array}{c} \uparrow h(A, s) \\ \hline \text{Claim: } h(A, s) = \frac{A}{\omega_0(s)}, \quad P(A, s) : \omega_0(s) + 2h(A, s) \\ \hline \omega_0(s) \end{array} \quad \begin{aligned} &: \omega_0 + 2A\omega_0^{-1} \\ \text{area of rectangle} &= \text{width} \times \text{height} \Rightarrow A = h(A, s) \times \omega_0(s) \\ &\Rightarrow h(A, s) = A \omega_0^{-1} \text{ required.} \end{aligned}$$

wetted perimeter of river = width + 2 × height

$$\Rightarrow P(A, s) = \omega_0(s) + 2h(A, s) \Rightarrow P = \omega_0 + 2A\omega_0^{-1} \text{ required.}$$

$$\text{for } S = 0, \quad A_t + (AR^{2/3} \sqrt{-b_s} C_m^{-1})_s = S \Rightarrow A_t + (AR^{2/3} \sqrt{-b_s} C_m^{-1})_s = 0$$

$$R^{2/3} : A^{2/3} P^{-2/3} \Rightarrow A_t + (A^{5/3} P^{-2/3} \sqrt{-b_s} C_m^{-1})_s = 0$$

$$\Rightarrow A_t + (F(A, S))_s = 0$$

$$F = F(A, S) \Rightarrow (F(A, S))_s = F_A A_s + F_S \quad \text{by chain rule}$$

$$F_A = \left(C_m^{-1} \sqrt{-b_s} A^{5/3} \left(\omega_0 + \frac{2A}{\omega_0} \right)^{-2/3} \right)_A$$

$$= \frac{SF}{3A} - \frac{4}{3\omega_0} F \left(\omega_0 + \frac{2A}{\omega_0} \right)^{-1} = \frac{F}{3} \left(\frac{5(\omega_0 + \frac{2A}{\omega_0})}{A(\omega_0 + \frac{2A}{\omega_0})} - \frac{4A}{A\omega_0(\omega_0 + \frac{2A}{\omega_0})} \right)$$

$$= \frac{F}{3} \left(\frac{5\omega_0 + \frac{10A}{\omega_0}}{A(\omega_0 + \frac{2A}{\omega_0})} - \frac{4A}{A(\omega_0 + \frac{2A}{\omega_0})} \right) = \frac{F}{3} \left(\frac{5\omega_0 + \frac{6A}{\omega_0}}{A(\omega_0 + \frac{2A}{\omega_0})} \right)$$

$$= \frac{1}{3} \sqrt{-b_s} C_m^{-1} \left(\frac{5\omega_0 A^{2/3} + 6A^{5/3}}{\omega_0 + \frac{2A}{\omega_0}} \right) > 0 \quad \text{as required.}$$

$$F_S = \left(\sqrt{-b_s} C_m^{-1} A^{5/3} \left(\omega_0 + \frac{2A}{\omega_0} \right)^{-2/3} \right)_s$$

$$= \sqrt{-b_s} C_m^{-1} \times -\frac{2}{3} A^{5/3} \left(\omega_0 + \frac{2A}{\omega_0} \right)^{-5/3} \left(\omega_0 + \frac{2A}{\omega_0} \right)_s \quad (\text{assuming } (b_s)_s = 0)$$

$$= -\frac{2}{3} \sqrt{-b_s} C_m^{-1} \left(\frac{A}{\omega_0 + \frac{2A}{\omega_0}} \right)^{5/3} \left(1 - \frac{2A}{\omega_0^2} \right) \frac{d\omega_0}{ds} \quad \text{as required.}$$

2) $S=0$, limit in which $\omega_0(s)$ indep. of s (or varying v. slowly).
What is eigenvalue λ for kinematic inv. eq.?

$$A_t + F_A A_s + F_s = 0 \text{ from Q1.}$$

$$\frac{d\omega_0}{ds} = 0 \text{ from limit} \Rightarrow F_s = 0 \Rightarrow A_t + F_A A_s = 0$$

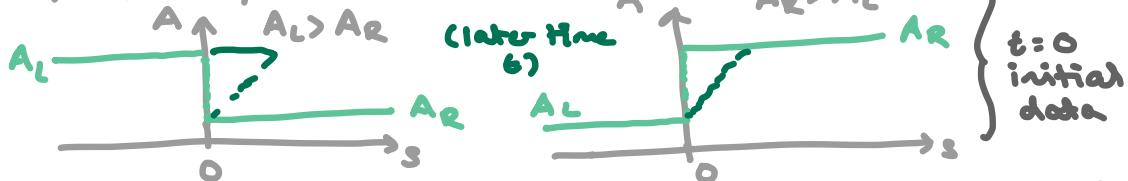
$$\lambda = F_A (> 0 \text{ from Q1.})$$

note: $\lambda_s = (F_A)_s = (F_s)_A = 0$ from commutativity of partial derivatives and limit.

$$\Rightarrow A_t + \lambda A_s = 0 = A_t + (\lambda A)_s$$

$$A_t + (\lambda A)_s = 0 \text{ w/ piecewise cst initial data } A(s,0) = \begin{cases} A_L, s < 0 \\ A_R, s \geq 0 \end{cases}$$

↳ Riemann problem



Riemann soln is s.t. $A(s,t)$ is cst along $s = s_0 + \lambda t$ characteristics

$$\Rightarrow A(s_0, 0) = \begin{cases} A_L, s_0 < 0 \\ A_R, s_0 \geq 0 \end{cases} \quad \begin{aligned} s &= s_0 + \lambda(A_L)t \\ s &= s_0 + \lambda(A_R)t \end{aligned}$$

$$\text{also: } \lambda = \begin{cases} \lambda(A_L), s_0 < 0 \\ \lambda(A_R), s_0 \geq 0 \end{cases} \text{ as } F_A \text{ defined for each } A_L, A_R$$

discretely at $s = s_n(t)$ and moves w/ shock speed $u = \frac{ds_n}{dt}$

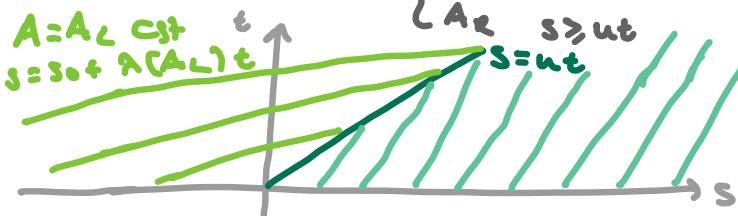
$$\therefore \lim_{\varepsilon \rightarrow 0} \left(\int_{s_n-\varepsilon}^{s_n+\varepsilon} \frac{\partial A}{\partial t} ds + \int_{s_n-\varepsilon}^{s_n+\varepsilon} (\lambda A)_s ds \right) = 0$$

$$\Leftrightarrow \frac{\partial}{\partial t} \lim_{\varepsilon \rightarrow 0} \int_{s_n-\varepsilon}^{s_n+\varepsilon} A ds - \frac{d s_n}{d t} \lim_{\varepsilon \rightarrow 0} (A(s, t))|_{s_n-\varepsilon}^{s_n+\varepsilon} + [\lambda A]|_{s_n-\varepsilon}^{s_n+\varepsilon} = 0$$

$$\Leftrightarrow \frac{d s_n}{d t} = \frac{[\lambda A]}{[A]} = \frac{\lambda(A_R) A_R - \lambda(A_L) A_L}{A_R - A_L} = u \quad (> 0 \text{ since } \lambda > 0)$$

for $A_L > A_R$: $\lambda(A_L) > \lambda(A_R) \Rightarrow$ left moving quicker than right
 $\lambda(A_R) < u < \lambda(A_L)$
∴ have a shock w/ speed u .

$$A_L > A_R: A(s, t): \begin{cases} A_L & s < ut \\ A_R & s > ut \end{cases}$$



$$A = A_R \text{ cst along } s = s_0 + \lambda(A_R)t \quad \rightarrow s = s_0 + \lambda(A_R)t \\ t = \frac{1}{\lambda(A_R)}(s - s_0)$$

for $A_R > A_L : \lambda(A_R) > \lambda(A_L) \Rightarrow$ right side moves quicker than left side
hence no longer have a shock.

rarefaction region: $\lambda(A_L)t \leq s \leq \lambda(A_R)t$ ($s=0$ since distn at $s=0$ when $t=0$)

A cst along $s = \lambda(A)t$ characteristics $\Rightarrow A = \lambda^{-1}(\frac{s}{t})$

then $A(s,t) = \begin{cases} A_L, & s < \lambda(A_L) \\ \lambda^{-1}(\frac{s}{t}), & \lambda(A_L) \leq s \leq \lambda(A_R) \\ A_R, & s > \lambda(A_R) \end{cases}$ $\rightarrow A$ varies from A_L to A_R

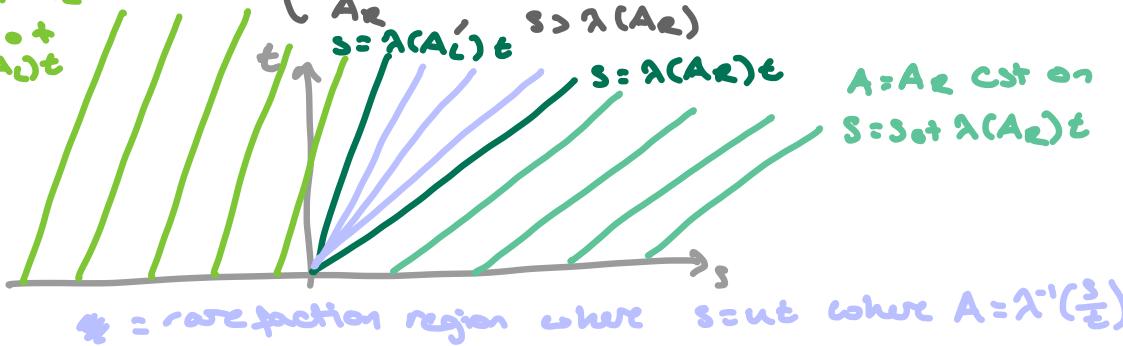
$A = A_L$ cst on

$$s = s_0 + \lambda(A_L)t$$

$$\begin{cases} A_L, & s < \lambda(A_L) \\ \lambda^{-1}(\frac{s}{t}), & \lambda(A_L) \leq s \leq \lambda(A_R) \\ A_R, & s > \lambda(A_R) \end{cases}$$

A varies from A_L to A_R

$A = A_R$ cst on
 $s = s_0 + \lambda(A_R)t$



* = rarefaction region where $s = ut$ where $A = \lambda^{-1}(\frac{s}{t})$

When $A_L = A_R$ get a constant solution for A , where $A = A_L = A_R \forall s, t$.

3) For $S=0$ derive FV/Godunov scheme:

$$\bar{A}_K^{n+1} = \bar{A}_K^n - \frac{1}{\Delta s_K} \int_{t_n}^{t_{n+1}} (F(A,s)|_{S=S_{K+\frac{1}{2}}} - F(A,s)|_{S=S_{K-\frac{1}{2}}}) dt$$

w/ flux $F(A,s)|_{S=S_{K+\frac{1}{2}}} = F_{K+\frac{1}{2}}(\bar{A}_K^n, \bar{A}_{K+1}^n)$ evaluated at $S_{K+\frac{1}{2}}$ node.

$$A_t + (F(A,s))_s = 0 \quad (1)$$

define space-time mesh.

on domain $S \in [0, L]$ in interval $I_n = [t_n, t_{n+1}]$, cell K occupies

$$S_{K-\frac{1}{2}} < S < S_{K+\frac{1}{2}} \text{ for } K=1, 2, \dots, N.$$

$$\Delta s_K = S_{K+\frac{1}{2}} - S_{K-\frac{1}{2}}$$
 is cell length.

integrate (1) over space-time element $S_{K-\frac{1}{2}} < S < S_{K+\frac{1}{2}}, t_n < t < t_{n+1}$:

$$\int_{S_{K-\frac{1}{2}}}^{S_{K+\frac{1}{2}}} (A(s,t)|_{t=t_{n+1}} - A(s,t)|_{t=t_n}) ds + \int_{t_n}^{t_{n+1}} (F(A,s)|_{S=S_{K+\frac{1}{2}}} - F(A,s)|_{S=S_{K-\frac{1}{2}}}) dt = 0$$



$$\text{define mean cell average: } \bar{A}_K = \frac{1}{\Delta s_K} \int_{S_{K-\frac{1}{2}}}^{S_{K+\frac{1}{2}}} A(s,t) ds$$

$$\Rightarrow \bar{A}_K^{n+1} - \bar{A}_K^n + \frac{1}{\Delta s_K} \int_{t_n}^{t_{n+1}} (F(A,s)|_{S=S_{K+\frac{1}{2}}} - F(A,s)|_{S=S_{K-\frac{1}{2}}}) dt = 0$$

$$\Rightarrow \bar{A}_K^{n+1} = \bar{A}_K^n - \frac{1}{\Delta s_K} \int_{t_n}^{t_{n+1}} (F(A,s)|_{S=S_{K+\frac{1}{2}}} - F(A,s)|_{S=S_{K-\frac{1}{2}}}) dt \text{ as required.}$$

$$\Rightarrow \bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta x_k} \int_{t_n}^{t_{n+1}} (F_{k+\frac{1}{2}}(\bar{A}_k^n, \bar{A}_{k+1}^n) - F_{k-\frac{1}{2}}(\bar{A}_{k-1}^n, \bar{A}_k^n)) dt$$

$$(\text{using } F(A, s)|_{S=3\text{-dep}} = F_{k+\frac{1}{2}}(\bar{A}_k^n, \bar{A}_{k+1}^n))$$

expression for \bar{A}_k^{n+1} is explicit \therefore provided cell averages at previous time level are known, \bar{A}_k^{n+1} can be calculated.
Hence only cell averages at previous time level are needed for Godunov method.

Godunov method also relies on solⁿ to be piecewise cst so only use cell averages (\bar{A} rather than A).

4) Given $A_L + (A R^{k+3} \sqrt{-b_3} C_m)_L = S$ and $A_L + F_L A_R + F_R = 0$ ($S=0$)
derive Godunov flux using Riemann solⁿ.

$$F(A, s)|_{S=3\text{-dep}} = F_{k+\frac{1}{2}}(\bar{A}_k^n, \bar{A}_{k+1}^n) : \text{Godunov flux}$$

from Q2 have solⁿ in limit of s-dependence (can use this since considering $S_{k-\frac{1}{2}} \leq S \leq S_{k+\frac{1}{2}}$ on which it can be approximated that there is no explicit s-dependence)

Q2 solⁿ \Rightarrow flux travels from left to right \therefore upwind scheme

$S_{k+\frac{1}{2}}$ corresponds to $s=0$ in Q2 $\Rightarrow A=A_L$

$$\Rightarrow F_{k+\frac{1}{2}}(\bar{A}_k^n, \bar{A}_{k+1}^n) = F_{k+\frac{1}{2}}(\bar{A}_k^n) = \lambda_k \bar{A}_k^n \quad (\lambda_k = \left. \frac{\partial F}{\partial A} \right|_{A=A_L})$$

cst by s-dependence limit

Upwind scheme:

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta x_k} \int_{t_n}^{t_{n+1}} F_{k+\frac{1}{2}} - F_{k-\frac{1}{2}} dt$$

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{1}{\Delta x_k} \int_{t_n}^{t_{n+1}} (\lambda_k \bar{A}_k^n - \lambda_{k-1} \bar{A}_{k-1}^n) dt$$

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{\Delta t}{\Delta x_k} (\lambda_k \bar{A}_k^n - \lambda_{k-1} \bar{A}_{k-1}^n) \quad (\lambda_k, \bar{A}_k^n \text{ cst by s-dependence limit / cell average})$$

5) time-step restriction / CFL cond. based on $\Delta t < \text{CFL} \min_k \frac{h_k}{|\lambda_k|}$
cell length h_k , information speed/local eigenvalue λ_k .

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{\Delta t}{h_k} (\lambda_k \bar{A}_k^n - \lambda_{k-1} \bar{A}_{k-1}^n)$$

$$= \left(1 - \frac{\Delta t}{h_k} \lambda_k\right) \bar{A}_k^n + \lambda_{k-1} \frac{\Delta t}{h_k} \bar{A}_{k-1}^n$$

Note: $\lambda_k > 0 \forall k$ since $\frac{\partial F}{\partial A} > 0$ (Q1), local eigenvalue.

maximum principle \Rightarrow require $1 - \lambda_K \frac{\Delta t}{h_K} \geq 0$

$$\Leftrightarrow \Delta t \leq \frac{h_K}{\lambda_K}$$

$$\lambda_K > 0 \Rightarrow \lambda_K = |\lambda_{K1}|.$$

require this over whole region $\Rightarrow \Delta t \leq \min_K \frac{h_K}{\lambda_K}$

hence have CFL cond.: $\Delta t < \text{CFL} \min_K \frac{h_K}{|\lambda_{K1}|}$

for $0 < \text{CFL} < 1$ this then satisfies maximum principle.

$$\lambda_K = \left. \frac{\partial F}{\partial A} \right|_{A=\bar{A}_K^*} = \frac{1}{3} \sqrt{-b_3} C_m \frac{(5w_0 \bar{A}_K^{*2/3} + 6\bar{A}_K^{*5/3} w_0')}{(w_0 + 2\bar{A}_K^* w_0')^{5/3}} > 0.$$

6) inflow/outflow cond. imposed at $s=0$ and end $s=L_e$.

require constant value inflow into the river (Q_0).

hence \bar{A}_{-1} (ghost cell) must be set s.t. $F_{N_{K-1}}(\bar{A}_{-1}) = Q_0$.

$\Rightarrow F_{-1/2}(\bar{A}_{-1}) = Q_0 = \lambda_{-1/2} \bar{A}_{-1}$, where $\lambda_{-1/2} > 0$ to justify flow moving left to right (upwind scheme).

Since upwind scheme, fixing $\bar{A}_{N_K}/F_{N_{K+1/2}}$ at $s=L_e$ would not have an effect on the flux in the river.

So don't need an outflow condition.

7) Numerics in Firedrake:

All test cases run using provided code.

Running over: $N_x = 1250, 2500, 5000, 10000$

$$CFL = 0.125, 0.25, 0.5, 1.0$$

$$\Delta x_{\min} = 0.5, CFL_{\min} = 0.125$$

finest mesh run over $N_x = 10000$ ($\Delta x_{\min} = 0.5$), $CFL = 0.125$

Δx analysis done w/ $N_x = 1250, 2500, 5000, 10000$

note: $\Delta t = CFL \frac{\Delta x}{\lambda}$ ∵ varying Δx also varies Δt

$$N_x = 10000, CFL = 1$$

$$N_x = 2500, CFL = 0.25$$

$$N_x = 5000, CFL = 0.5$$

$$N_x = 1250, CFL = 0.125$$

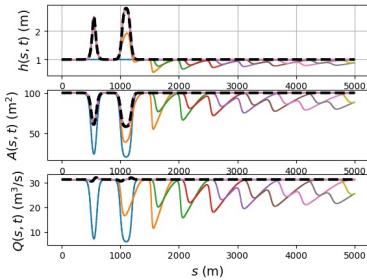
To fix Δt whilst varying Δx to just investigate effect of Δt .

$$\Delta x := 1 \quad (N_x = 5000 \text{ as base})$$

$$\Delta t := CFL \times \left(\frac{\Delta x}{\lambda} \right) \quad (N_x = 5000, CFL = 0.5 \text{ as base})$$

Test-Case 0:

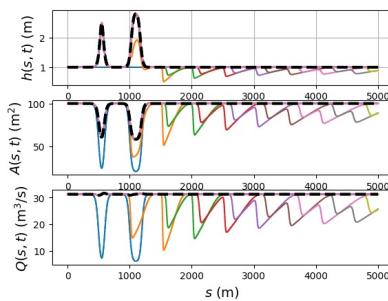
$4\Delta x$:



$$N_x = 1250 \\ CFL = 0.125$$

time (hrs)
0.00
0.31
0.63
0.94
1.25
1.56
1.88
2.19
2.50
2.81
3.13
3.44
3.75
4.06
4.38
4.69
5.00

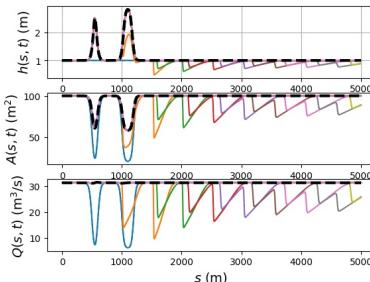
$2\Delta x$:



$$N_x = 2500 \\ CFL = 0.25$$

time (hrs)
0.00
0.31
0.63
0.94
1.25
1.56
1.88
2.19
2.50
2.81
3.13
3.44
3.75
4.06
4.38
4.69
5.00

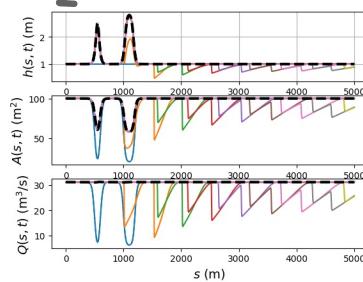
Δx :



$$N_x = 5000 \\ CFL = 0.5$$

time (hrs)
0.00
0.31
0.63
0.94
1.25
1.56
1.88
2.19
2.50
2.81
3.13
3.44
3.75
4.06
4.38
4.69
5.00

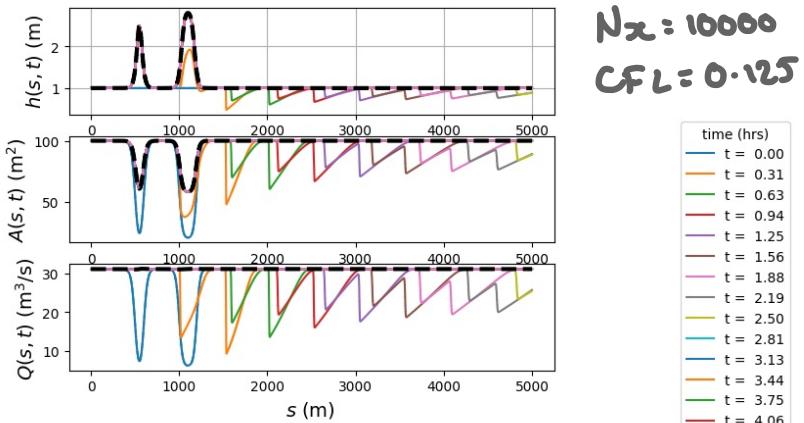
$\frac{1}{2}\Delta x$:



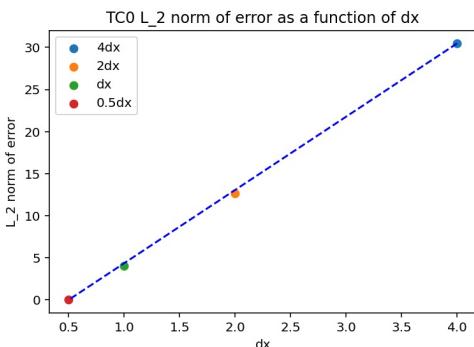
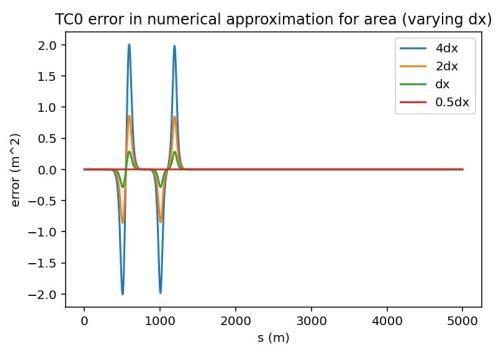
$$N_x = 10000 \\ CFL = 1$$

time (hrs)
0.00
0.31
0.63
0.94
1.25
1.56
1.88
2.19
2.50
2.81
3.13
3.44
3.75
4.06
4.38
4.69
5.00

Finest mesh:



Errors calculated using final $A(s,t)$ ($t=5\text{ hrs}$)
and using finest mesh as approximate solution.

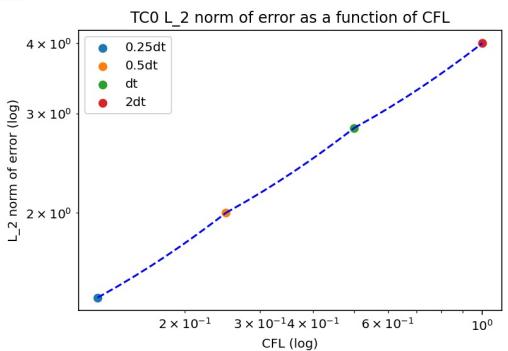
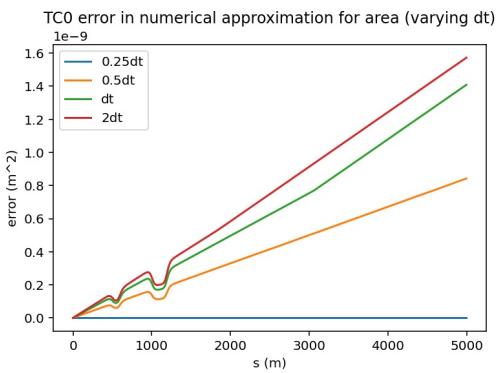
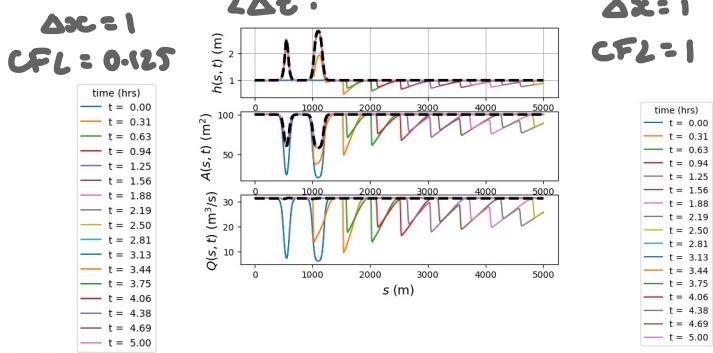
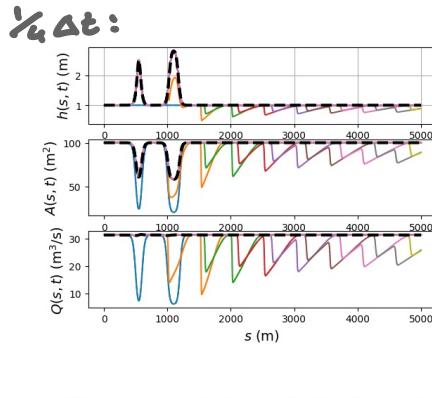
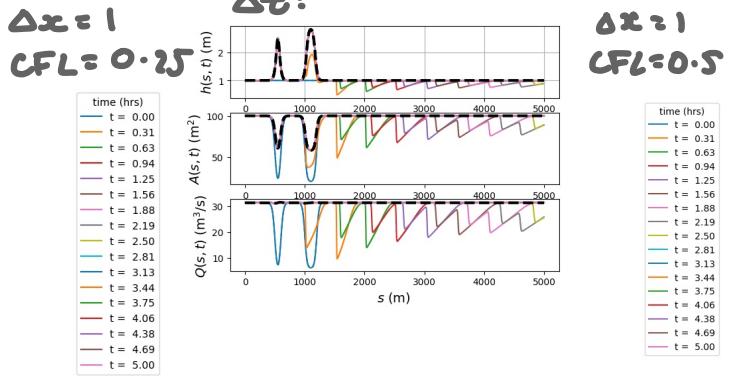
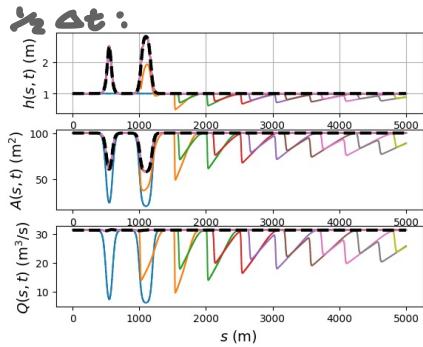


Errors plot shows that as Δx decreases we converge towards the "actual" sol! (that of finest mesh)

Plotting L_2 norm of error as a func of Δx and drawing a 'line of best fit' shows a linear relation between L_2 -norm of error and Δx .

\therefore 1st order accuracy wrt Δx .

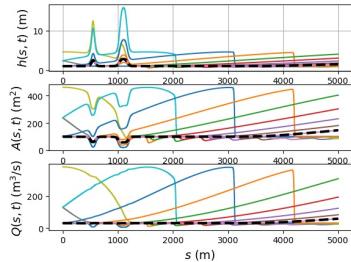
Also note that the smaller Δx considered, the closer to constant the final $Q(s,t)$.



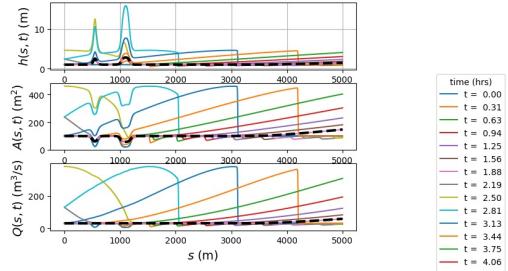
Again errors show convergence as Δt decreases.
 Plotting L_2 -norm of error as func of CFL against CFL
 doesn't show obvious linear relation.
 but plotting logs against each other leads to a
 gradient of 1.08 \Rightarrow linear relation
 \Rightarrow first order accuracy in Δt
 too.

Test - Case 1:

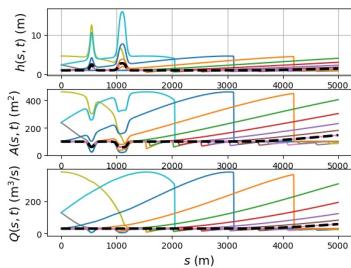
$4\Delta x$:



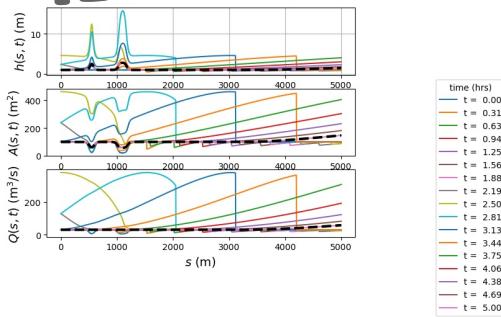
$2\Delta x$:



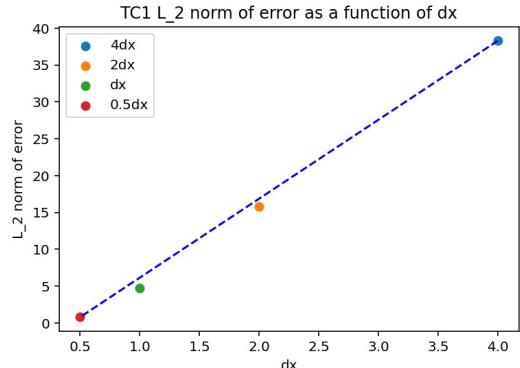
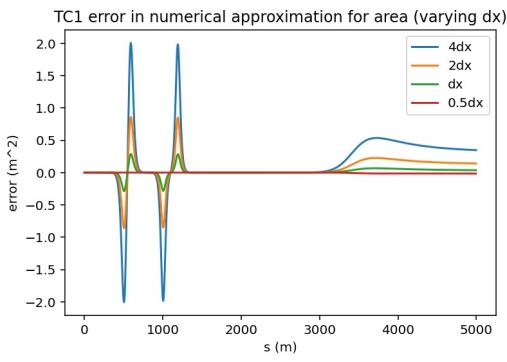
Δx :



$\frac{1}{2}\Delta x$:

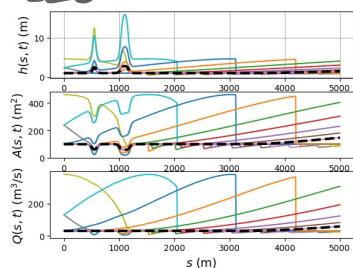


Δx error analysis:

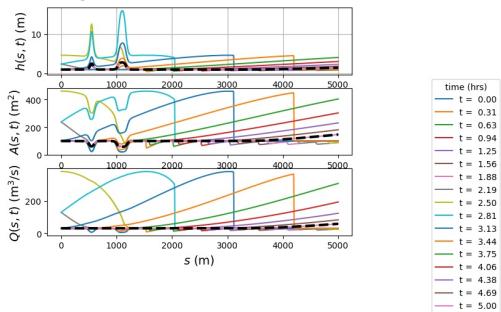


Δt error analysis:

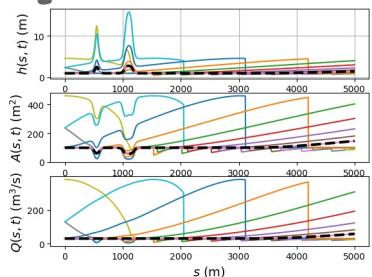
$2\Delta t$:



Δt :

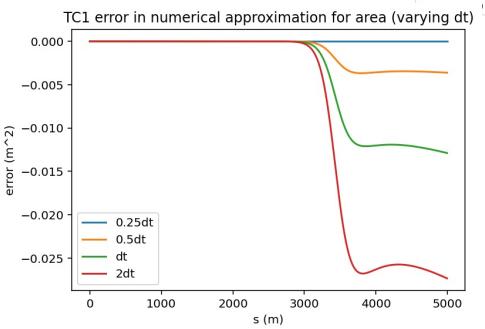
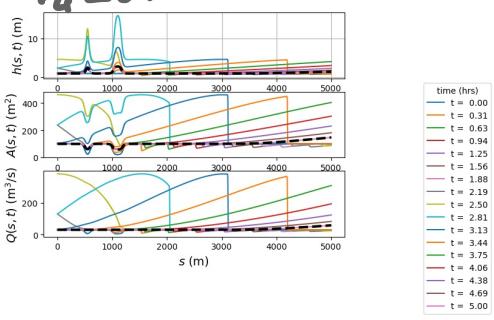


$\frac{1}{2}\Delta t$:

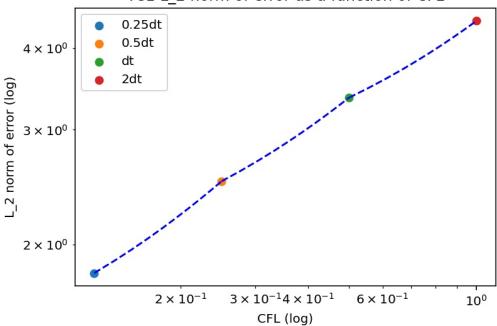


time (hrs)
 — t = 0.00
 — t = 0.31
 — t = 0.63
 — t = 0.94
 — t = 1.25
 — t = 1.56
 — t = 1.88
 — t = 2.19
 — t = 2.50
 — t = 2.81
 — t = 3.13
 — t = 3.44
 — t = 3.75
 — t = 4.06
 — t = 4.38
 — t = 4.69
 — t = 5.00

$\frac{1}{4}\Delta t$:



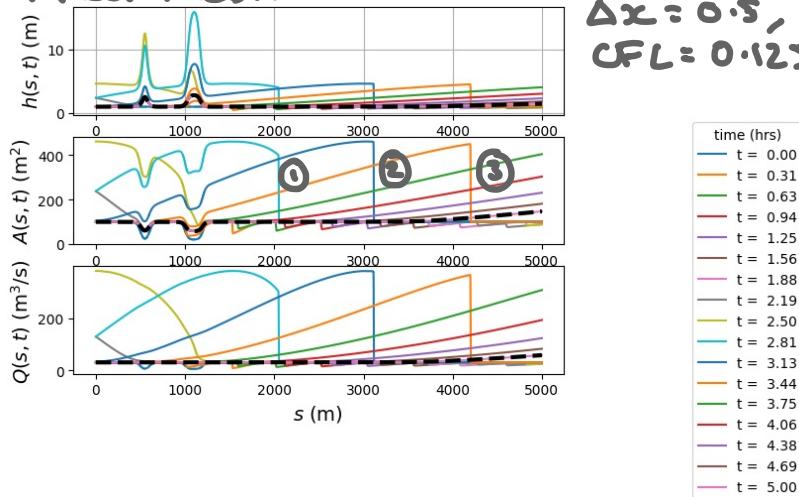
TC1 L_2 norm of error as a function of CFL



Analysing error results for Δx and Δt in the same way as TCO : the same results are found.
 \therefore have convergence and first order accuracy.

Shock speed:

Finest mesh:



Can follow the shock over $t = 2.81 \text{ hrs}, 3.13 \text{ hrs}, 3.44 \text{ hrs}$ to compare observed shock speed to actual shock speed.

① $t = 2.81 \text{ hrs} = 10125 \text{ s}$ (from counter in code)

$$A_L = 400, A_R = 100 \quad F(A_L) = Q(A_L) = 300, F(A_R) = Q(A_R) = 3 \\ S = 2050$$

$$\text{numerical shock speed: } u = \frac{F(A_R) - F(A_L)}{A_R - A_L} \approx 0.90 \text{ m/s}$$

② $t = 3.13 \text{ hrs} = 11250 \text{ s}$

$$A_L = 460, A_R = 100 \quad Q(A_L) = 380, Q(A_R) = 31 \\ S = 3115$$

$$\text{numerical shock speed: } u = \frac{F(A_R) - F(A_L)}{A_R - A_L} \approx 0.97 \text{ m/s}$$

③ $t = 3.44 \text{ hrs} = 12375 \text{ s}$

$$A_L = 450, A_R = 100 \quad Q(A_L) = 366, Q(A_R) = 31 \\ S = 4200$$

$$\text{numerical shock speed: } u = \frac{F(A_R) - F(A_L)}{A_R - A_L} \approx 0.96 \text{ m/s}$$

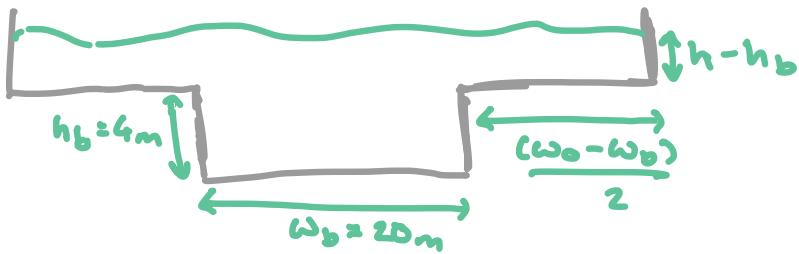
$$\textcircled{1} \rightarrow \textcircled{2} \quad \frac{3115 - 2050}{11250 - 10125} \approx 0.95 \text{ m/s}$$

$$\textcircled{1} \rightarrow \textcircled{3} \quad \frac{4200 - 3115}{12375 - 11250} \approx 0.96 \text{ m/s} \text{ observed shock speed}$$

\therefore Observed shock speed \approx numerical shock speed.

$$\Delta x = 0.5, N_x = 10000 \\ CFL = 0.125$$

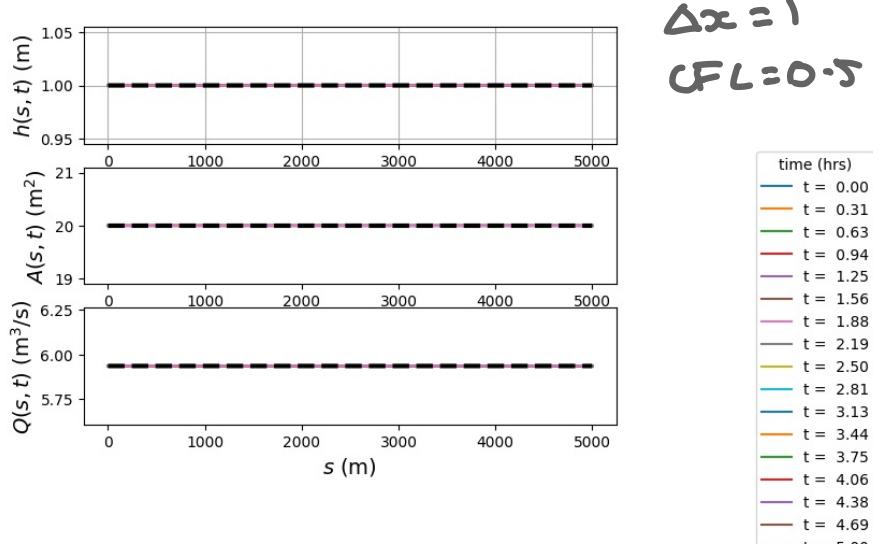
Test - Case 2:



$$\text{area: } A = w_b h_b + w_0(z) (h - h_b)$$

$$\text{wetted perimeter: } p = w_b + 2h_b + w_0 + \frac{2}{w_0} (A - w_b h_b)$$

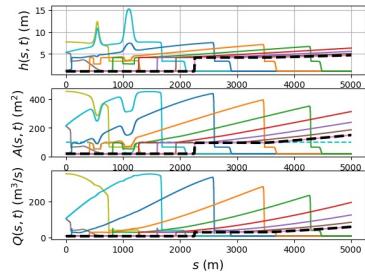
Test - Case 2a:



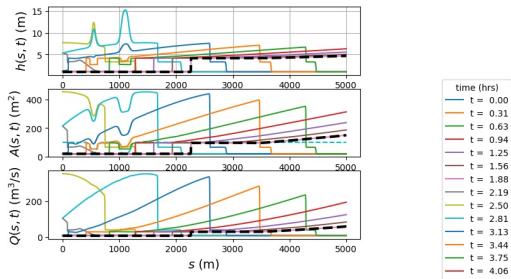
Due to cst rectangular river channel
 everything is cst when cst flux.

Test - Case 2b:

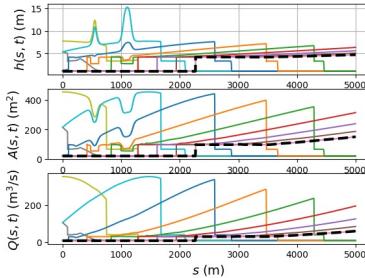
$4\Delta x$:



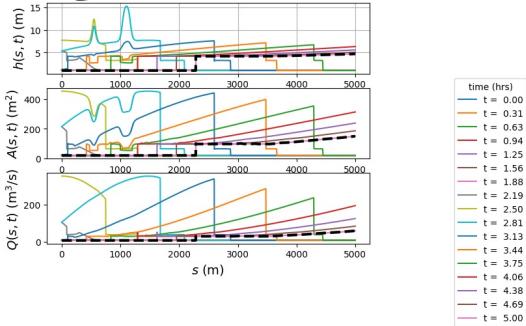
$2\Delta x$:



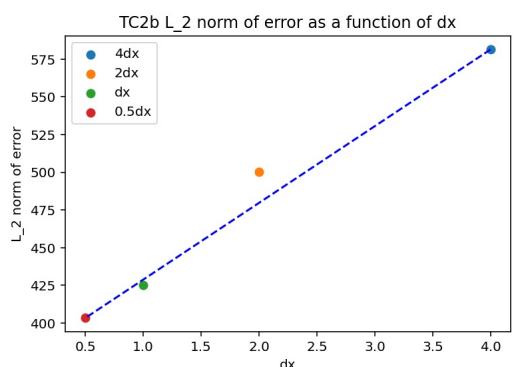
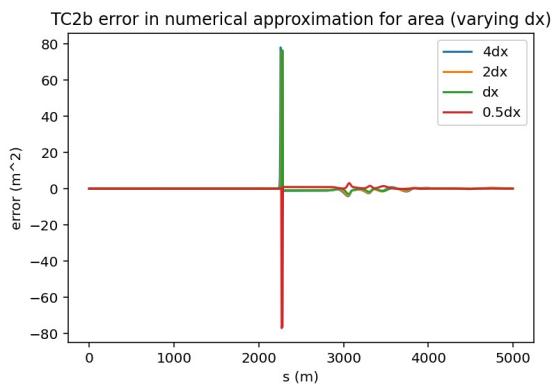
Δx :



$\frac{1}{2}\Delta x$:

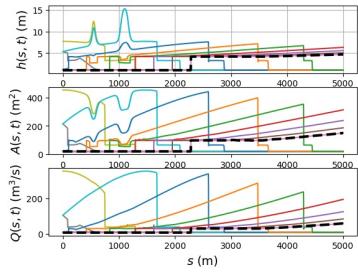


Δx error analysis:



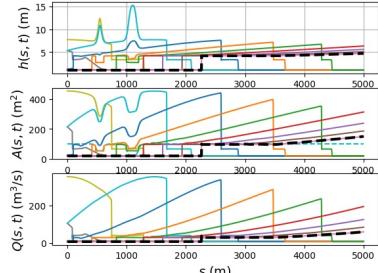
Error plot shows that plots are still converged in Δx . Need a finer mesh for numerical solution to converge. However, plotting L₂-norm of error against Δx find a linear relation \Rightarrow first order accuracy.

Δt :



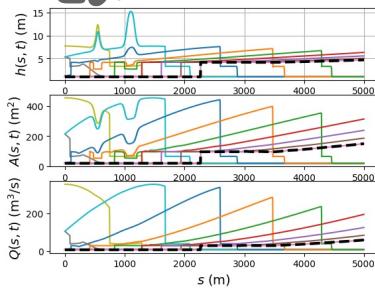
time (hrs)
 — t = 0.00
 — t = 0.31
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 — t = 1.56
 — t = 1.88
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 — t = 2.81
 — t = 3.13
 — t = 3.44
 — t = 3.75
 — t = 4.06
 — t = 4.38
 — t = 4.69
 — t = 5.00

$\frac{1}{2} \Delta t$:



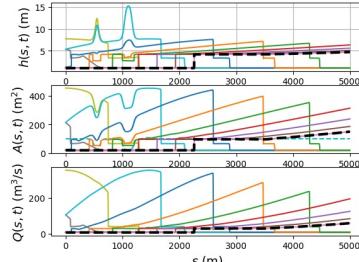
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 — t = 4.69
 — t = 5.00

Δt :



time (hrs)
 — t = 0.00
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 — t = 0.63
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 — t = 4.69
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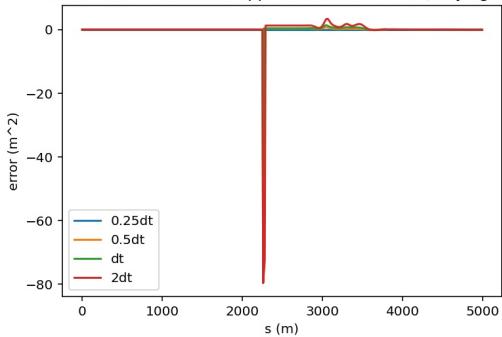
$\frac{1}{4} \Delta t$:



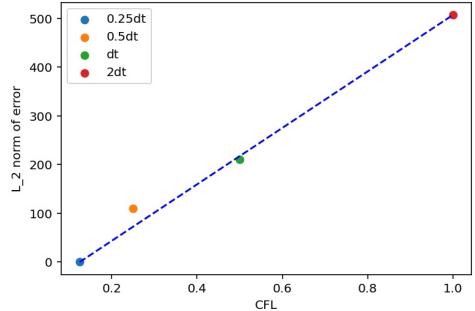
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 — t = 4.06
 — t = 4.38
 — t = 4.69
 — t = 5.00

Δt error analysis:

TC2b error in numerical approximation for area (varying Δt)



TC2b L_2 norm of error as a function of CFL

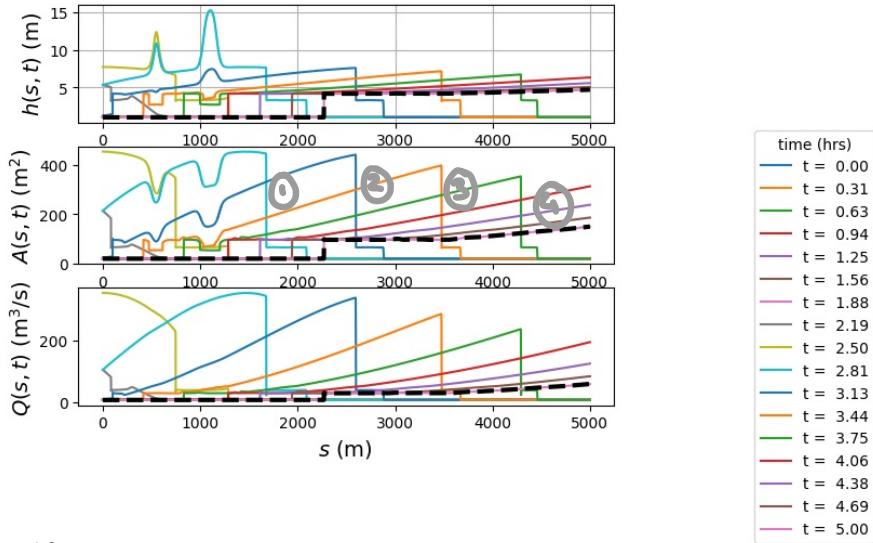


Error plot shows still converging in Δt , similarly to Δx . Also similarly to Δx find linear relationship btw Δt and L_2 -norm of error \Rightarrow first order accuracy.

Error analysis \Rightarrow need much finer mesh for numerical solⁿ to converge in TC2b than TC1/TC0.

Shock speed:

Finest mesh:



$$\textcircled{1} \quad t = 2.81 \text{ hrs} = 10125 \text{ s}$$

$$A_L = 445, A_R = 65 \quad Q(A_L) = 346, Q(A_R) = 38 \\ s = 1677$$

numerical shock speed: $u = \frac{F(A_R) - F(A_L)}{A_R - A_L} \approx 0.81 \text{ ms}^{-1}$ ($F = Q$)

$$\textcircled{2} \quad t = 3.13 \text{ hrs} = 11250 \text{ s}$$

$$A_L = 441, A_R = 70 \quad Q(A_L) = 340, Q(A_R) = 38 \\ s = 2596 \\ u \approx 0.81 \text{ ms}^{-1}$$

$$\textcircled{3} \quad t = 3.44 \text{ hrs} = 12375 \text{ s}$$

$$A_L = 395, A_R = 65 \quad Q(A_L) = 287, Q(A_R) = 38 \\ s = 3475 \\ u \approx 0.75 \text{ ms}^{-1}$$

$$\textcircled{4} \quad t = 3.75 \text{ hrs} = 13500 \text{ s}$$

$$A_L = 352, A_R = 65 \quad Q(A_L) = 237, Q(A_R) = 38 \\ s = 4290$$

$$u \approx 0.69 \text{ ms}^{-1}$$

$$\textcircled{1} \rightarrow \textcircled{2} \quad \text{observed speed} = \frac{2596 - 1677}{11250 - 10125} = 0.82 \text{ ms}^{-1}$$

$$\textcircled{2} \rightarrow \textcircled{3} \quad \text{observed } u = 0.78 \text{ ms}^{-1}$$

③ → ⑤ observed $u = 0.72 \text{ ms}^{-1}$

find that averaged btw shock speeds are approx.
the observed shock speeds.