

Numerical Exercises 1 : Answers

(a) i) Simplify the system: $\partial_t b + \partial_z (ub) = 0$, $u = \alpha b^2 - \beta b^3 \partial_z b$, $\partial_t b = \frac{\partial b}{\partial t}$, $\alpha > 0$, $\beta > 0$ to a one convection-diffusion equation for $b(z, t)$

Physical context: Modeling how magma flows through a vertical dike with elastic walls. The width of the dike $b(z, t)$ evolves overtime due to pressure and electricity

- Substitute u in $\frac{\partial b}{\partial t} + \frac{\partial}{\partial z} (ub) = 0$

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial z} \left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z} \right) = 0 \quad (5)$$

ii) Linearise this equation around $b = D_0$, assuming that b' is small into (5) to get (6)

Approximating a non linear equation near a known solution by assuming small perturbations for example $f(x)$ at $x=a$, $L(x) = f(a) + f'(a)(x-a)$

- With a small perturbation of $b(z, t) = D_0 + b'(z, t)$ where $b' \ll D_0$

→ Substitute into PDE and expand using Taylor approximations.

$$b(z, t)^3 = (D_0 + b'(z, t))^3 = D_0^3 + 3D_0^2 b' + \dots$$

→ linearisation is a first order approximation so only retain terms linear in b' and neglect higher order since b' is small

$$\frac{\partial b}{\partial z} = \frac{\partial b'}{\partial z} \text{ as } D_0 \text{ is a constant}$$

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial z} \left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z} \right) = 0 \quad \text{with } b(z, t)^3 = D_0^3 + 3D_0^2 b'$$

$$\Rightarrow \frac{\partial b'}{\partial t} + \frac{\partial}{\partial z} \left((D_0^3 + 3D_0^2 b') \alpha - (D_0^3 + 3D_0^2 b') \beta \frac{\partial b'}{\partial z} \right) = 0$$

$$\Rightarrow \frac{\partial b'}{\partial t} + \frac{\partial}{\partial z} \left((\alpha D_0^3 + 3\alpha D_0^2 b') + (\beta D_0^3 \frac{\partial b'}{\partial z}) \right) = 0$$

$$\Rightarrow \frac{\partial b'}{\partial t} + 3\alpha D_0^2 \frac{\partial b'}{\partial z} + \beta D_0^3 \frac{\partial^2 b'}{\partial z^2} = 0 \quad \text{which gives equation (6) from the task sheet}$$

iii) Why are (5) and (6) called nonlinear convection-diffusion equations?

These are convection-diffusion equations as they have a transport term (convection) and a smoothing (diffusion term) which are both dependent on b

$\alpha b^3 \rightarrow$ convection term \rightarrow flow of magma driven by pressure

$-\beta b^3 \frac{\partial b}{\partial z} \rightarrow$ diffusion term

They take the general form of the sum of a convective effect $\alpha \frac{\partial b'}{\partial z}$ and a diffusive effect $\beta \frac{\partial^2 b'}{\partial z^2}$

1b ii) Discretize (5) and (6) with a forward Euler time discretisation, an upwind scheme and second order central difference scheme

- Forward Euler Time discretisation: Uses current state b_j^n to compute fluxes and explicitly step forward in time
- Upwind Scheme: Convection considers directional transport so an upwind scheme uses spatial differences based on the direction of flow. Use a backward difference whilst assuming flow is upward
- Central Difference Scheme: Approximates the spatial derivative of the diffusion term using a formula that considers the values of the function at points centred around the grid point of interest to approx $\frac{\partial^2 b}{\partial z^2}$

→ Setup for discretisation

$$z_j = j\Delta z \text{ for } j=0, 1, 2, \dots, J, t_n = n\Delta t, b_j^n \approx b(z_j, t_n), b_0^n \approx b_0(z_j, t_n)$$

→ Time Derivative (Forward Euler)

$$\frac{\partial b}{\partial t} \approx \frac{b_j^{n+1} - b_j^n}{\Delta t} \quad \text{for both linear and non linear PDE}$$

→ Convection Term (Upwind scheme)

$$\text{Non linear: } \frac{\partial}{\partial z} (\beta b^3) = \beta \frac{\partial}{\partial z} b^3 \rightarrow \approx \beta \frac{(b_j^n)^3 - (b_{j-1}^n)^3}{\Delta z} \quad \left[\begin{array}{l} \text{Assuming flow is upward} \\ \text{so doing backward difference} \\ \text{as } 3\alpha D_o^2 \text{ is a positive coefficient} \end{array} \right]$$

$$\text{Linear: } 3\beta D_o^2 \frac{\partial b_0}{\partial z} \approx 3\beta D_o^2 \frac{b_{0,j}^n - b_{0,j-1}^n}{\Delta z}$$

→ Diffusive Term (Central Difference Scheme Second Order)

Non linear: Use the adjoint form

$$\begin{aligned} b_{j+\frac{1}{2}}^n &= \frac{1}{2} (b_j^n + b_{j+1}^n), \quad \left(\frac{\partial b}{\partial z} \right)_{j+\frac{1}{2}} \approx \frac{b_{j+1}^n - b_j^n}{\Delta z} \\ \Rightarrow U_{j+\frac{1}{2}}^n &= \beta \left(b_{j+\frac{1}{2}}^n \right)^2 - \alpha \left(b_{j+\frac{1}{2}}^n \right)^3 \left(\frac{b_{j+1}^n - b_j^n}{\Delta z} \right) \\ \Rightarrow \frac{\partial}{\partial z} \left(\alpha b^3 \frac{\partial b}{\partial z} \right) &\approx \frac{U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n}{\Delta z} \end{aligned}$$

$$\text{Linear: } -\alpha D_o^3 \frac{\partial^2 b_0}{\partial z^2} \approx -\alpha D_o^3 \frac{b_{0,j+1}^n - 2b_{0,j}^n + b_{0,j-1}^n}{(\Delta z)^2}$$

→ Full discretised forms:

You needed to use the upwind scheme -1.

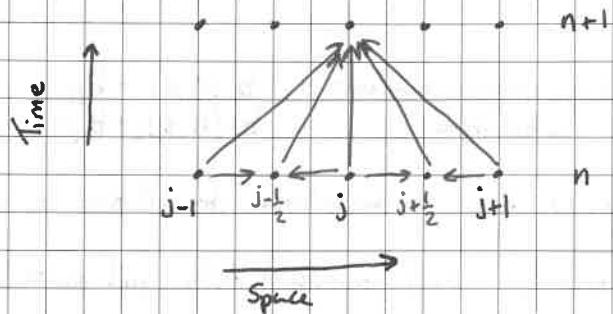
$$\text{Non linear: } b_j^{n+1} = b_j^n - \Delta t \left(\frac{U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n}{\Delta z} \right) \quad \text{where } U_{j+\frac{1}{2}}^n = \beta \left(b_{j+\frac{1}{2}}^n \right)^2 - \alpha \left(b_{j+\frac{1}{2}}^n \right)^3 \frac{b_{j+1}^n - b_j^n}{\Delta z} \quad \text{3 error}$$

$$\text{Linear: } b_{0,j}^{n+1} = b_{0,j}^n - \Delta t \left(3\beta D_o^2 \left(\frac{b_{0,j}^n - b_{0,j-1}^n}{\Delta z} \right) - \beta D_o^3 \left(\frac{b_{0,j+1}^n - 2b_{0,j}^n + b_{0,j-1}^n}{(\Delta z)^2} \right) \right)$$

$$\text{Non linear: } b_j^{n+1} = b_j^n - \Delta t \left(\alpha \frac{b_j^n - b_{j-1}^n}{\Delta z} + \beta \frac{b_{j+\frac{1}{2}}^n (b_{j+1}^n - b_j^n) - b_{j-\frac{1}{2}}^n (b_j^n - b_{j-1}^n)}{(\Delta z)^2} \right) \quad (\text{Expendded})$$

ii) Make a sketch of your grid and its numbering

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iii) Check whether the discretization of (6) is the linearized version of the discretization of (5)

- To do this substitute $b_{ij}^n = D_0 + b_{0,ij}$ into the nonlinear equation and expand the terms using Taylor expansions where D_0 is constant and $b_{0,ij}$ is small

$$b_{j+\frac{1}{2}}^n = \frac{1}{2} (D_0 + b_{0,j}^n + D_0 + b_{0,j+1}^n) = D_0 + \frac{1}{2} (b_{0,j}^n + b_{0,j+1}^n)$$

$$(b_{j+\frac{1}{2}}^n)^2 \approx D_0^2 + D_0 (b_{0,j}^n + b_{0,j+1}^n)$$

$$(b_{j+\frac{1}{2}}^n)^3 \approx D_0^3 + \frac{3}{2} D_0^2 (b_{0,j}^n + b_{0,j+1}^n)$$

$$\frac{b_{j+1}^n - b_j^n}{\Delta z} = \frac{b_{0,j+1}^n - b_{0,j}^n}{\Delta z}$$

- Take these terms and put them into $U_{j+\frac{1}{2}}^n$ expression

$$U_{j+\frac{1}{2}}^n \approx \alpha (D_0^2 + D_0 (b_{0,j}^n + b_{0,j+1}^n)) - \beta (D_0^3 + \frac{3}{2} D_0^2 (b_{0,j}^n + b_{0,j+1}^n)) \frac{b_{0,j+1}^n - b_{0,j}^n}{\Delta z}$$

- Using $\frac{U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n}{\Delta z}$ you can get a discretized expression from substituting in each of the above

$$\Rightarrow b_{0,j}^{n+1} = b_{0,j}^n - \Delta t \left[3\alpha D_0^2 \left(\frac{b_{0,j}^n - b_{0,j-1}^n}{\Delta z} \right) - \beta D_0^3 \left(\frac{b_{0,j+1}^n - 2b_{0,j}^n + b_{0,j-1}^n}{(\Delta z)^2} \right) \right]$$

This shows the discretization of the linear equation (6) is the linearised version of the discretization of the nonlinear equation 5. This means that the numerical scheme behaves consistently under small perturbations given by the assumptions.

iv) What is the advantage of using the adjoint form for the nonlinear diffusive term?

- Adjoint form conserves mass discretizing
- The adjoint form of the diffusive term in linearizing the non linear form of the equation ensures that there is mass conservation within the system

v) Clearly indicate how you implement the boundary conditions

From the tasksheet given as:

Dirichlet Conditions

↳ At the bottom of the dike, magma chamber

$$b(0, t) = b_g$$

↳ At the top of the dike, atmosphere

$$b(H, t) = b_T$$

These are the values of the solution of b at the boundaries for all time t and are set by the geometry of the dike

The initial condition is $b_{0,j} = b_0$, the second previously overleaf is a sketch of the grid on which this is calculated, showing the flow of information

c) Use a Fourier analysis to assess the stability of the linearized numerical scheme.

- Need to use Fourier analysis to determine the stability condition. Such that what timestep Δt keeps the scheme stable as in $|\lambda| \leq 1$ for two different cases

$$\alpha = 0, \beta \neq 0 \leftarrow \text{pure convection}$$

$$\alpha \neq 0, \beta = 0 \leftarrow \text{pure diffusion}$$

- Fourier mode from Morton and Mayers (2.72) $U_j^n = \lambda e^{n i k j \Delta x}$ which represents a wave-like solution where λ is the amplification factor per time step

$$U_{j+1}^n = \lambda e^{n i k (j+1) \Delta x} = U_j^n e^{i k \Delta x} = \lambda U_j^n$$

$$U_{j-1}^n = \lambda e^{n i k (j-1) \Delta x}$$

$$\text{Using, } U_j^{n+1} = U_j^n - \Delta t \left(\frac{\alpha}{3\beta D_o} \right)^2 \left(\frac{U_j^n - U_{j-1}^n}{\Delta x} \right) - \beta D_o^3 \left(\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} \right)$$

Sub in Fourier expressions to get

$$U_j^{n+1} = U_j^n \left[1 - \Delta t \left(\frac{3\alpha D_o^2}{\Delta x^2} (1 - e^{-ik\Delta x}) - \frac{\beta D_o^3}{\Delta x^2} (e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \right) \right]$$

$$\lambda U_j^{n+1} = U_j^n \left[1 - \Delta t \left(\frac{3\alpha D_o^2}{\Delta x^2} (1 - e^{-ik\Delta x}) - \frac{\beta D_o^3}{\Delta x^2} (e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \right) \right]$$

$$\lambda = 1 - \Delta t \left(\frac{3\alpha D_o^2}{\Delta x^2} (1 - e^{-ik\Delta x}) - \frac{\beta D_o^3}{\Delta x^2} (e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \right)$$

Use this expression to assess the stability for the pure convection and pure diffusion case

Pure Convection (no diffusion) ($\beta = 0$)

$$\lambda = 1 - \Delta t \left(\frac{3\alpha D_o^2}{\Delta x^2} (1 - e^{-ik\Delta x}) - \frac{\beta D_o^3}{\Delta x^2} (e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \right) \Rightarrow \lambda = 1 - \Delta t \frac{3\alpha D_o^2}{\Delta x^2} (1 - e^{-ik\Delta x})$$

- Stability is thought of as when errors from rounding or initial conditions don't grow uncontrollably as it progresses
If $|\lambda| \leq 1$, system is stable

If $|\lambda| > 1$, system is unstable and errors grow exponentially

To get the maximum possible diffusive term and therefore the mode with the highest frequency and most prone instability. If this constraint is satisfied then all lower modes will remain stable

$$1 - e^{-ik\Delta x} \leftarrow \max \text{ when } k\Delta x = \pi \text{ so that } e^{-ik\Delta x} = e^{-i\pi} = -1$$

$$\lambda = 1 - \Delta t \frac{3\alpha D_o^2}{\Delta x^2} (1 - 1) = 1 - \Delta t \frac{6\alpha D_o^2}{\Delta x^2}$$

for $|\lambda| \leq 1$ then $\left| 1 - \Delta t \frac{6\alpha D_0^2}{\Delta z} \right| \leq 1$ so

$$1 - \frac{6\alpha D_0^2 \Delta t}{\Delta z} \geq -1$$

$$-2 \leq -\frac{6\alpha D_0^2 \Delta t}{\Delta z}$$

$$\Rightarrow \Delta t \leq \frac{2\Delta z}{6\beta D_0^2} = \frac{\Delta z}{3\alpha D_0^2}$$

Pure Convection ($\alpha = 0$)

$$\lambda = 1 - \Delta t \left(\frac{3\alpha D_0^2}{\Delta z} (1 - e^{-ik\alpha z}) - \frac{3\beta D_0^3}{\Delta z^2} (e^{ik\alpha z} - 2 + e^{-ik\alpha z}) \right)$$

$$\begin{aligned} \lambda &= 1 + \Delta t \frac{3\beta D_0^3}{\Delta z^2} (e^{ik\alpha z} - 2 + e^{-ik\alpha z}) = 1 + \Delta t \frac{3\beta D_0^3}{\Delta z^2} (2 \cos(k\alpha z) - 2) \\ &= 1 + \Delta t \frac{3\beta D_0^3}{\Delta z^2} (-2(1 - \cos k\alpha z)) = 1 + \Delta t \frac{3\beta D_0^3}{\Delta z^2} (-2(2 \sin^2(\frac{k\alpha z}{2}))) \\ &= 1 - \Delta t \frac{4\beta D_0^3}{\Delta z^2} \sin^2\left(\frac{k\alpha z}{2}\right). \end{aligned}$$

Interested in $|\lambda| \leq 1$ so need to find worst-case scenario as previously

$$\sin^2\left(\frac{k\alpha z}{2}\right) \text{ with } k\alpha z = \pi \text{ so } \sin^2\left(\frac{k\alpha z}{2}\right) = 1$$

then

$$\lambda = 1 - \Delta t \frac{4\beta D_0^3}{\Delta z^2} \quad \text{For } \lambda \geq -1 \text{ to be satisfied we get}$$

the resulting expression:

$$1 - \frac{4\beta D_0^3}{\Delta z^2} \Delta t \geq -1$$

$$\frac{4\beta D_0^3}{\Delta z^2} \Delta t \leq 2$$

$$\Rightarrow \Delta t \leq \frac{\Delta z^2}{2\beta D_0^3}$$

There 2 cases are for diffusion and convection driven flows separately. The timestep that should be used needs to satisfy the stability of both convective and diffusive components, as they are summed, so a timestep less than the smallest of these two limiting timesteps found should be used.

d) Use the maximum principle to determine a stable time step for the discretisation of (6)

The maximum principle states that the value of variable U_j^n is bounded between boundary values and initial values. In this case we can't have any negative values for the dike width perturbation b_0 .

Interested in $b_{0,j}^{n+1} \geq 0$ when $b_{0,j}^n \geq 0$ so group together coefficients of $b_{0,j-1}^n, b_{0,j}^n, b_{0,j+1}^n$ to give

$$b_{0,j}^{n+1} = k_1 b_{0,j-1}^n + k_2 b_{0,j}^n + k_3 b_{0,j+1}^n$$

where

$$k_1 = \left(\frac{3\alpha D_o^2}{\Delta z} + \frac{\beta D_o^3}{\Delta z^2} \right) \Delta t, \quad k_2 = 1 - \Delta t \left(\frac{-3\alpha D_o^2}{\Delta z} - \frac{2\beta D_o^3}{\Delta z^2} \right), \quad k_3 = \Delta t \frac{\beta D_o^3}{\Delta z^2}$$

We have assumed that $b_{0,j-1}^n \geq 0, b_{0,j}^n \geq 0$ and $b_{0,j+1}^n \geq 0$ and we want $b_{0,j}^{n+1} \geq 0$ so this is guaranteed if k_1, k_2 and k_3 are all ≥ 0

From their definitions k_1 and k_3 are always greater than zero however if the timestep is too large k_2 can become negative. To ensure that $k_2 \geq 0$ we impose a stability condition as previously and rearrange for Δt

$$1 - \Delta t \left(\frac{3\alpha D_o^2}{\Delta z} + \frac{2\beta D_o^3}{\Delta z^2} \right) \geq 0$$

$$1 \geq \Delta t \left(\frac{3\alpha D_o^2}{\Delta z} + \frac{2\beta D_o^3}{\Delta z^2} \right)$$

$$\Delta t \leq \frac{1}{\frac{3\alpha D_o^2}{\Delta z} + \frac{2\beta D_o^3}{\Delta z^2}} \Rightarrow \Delta t \leq \frac{\Delta z^2}{D_o^2 (3\alpha \Delta z + 2D_o \beta)}$$

Which is the maximum allowable time step that guarantees the numerical solution is non negative and bounded given that the initial condition is also non negative which is key due to the physical spatial application of this fluid flow problem.

e) Derive a (variable) time step criterion for the discretisation of (5) for which $B_j^{n+1} > 0$

The physical application of this condition is to ensure that the non linear convection-diffusion equation (5) only considers positive values in the evolution of the dike width $b(z,t)$

Interested in $B_j^{n+1} > 0$ if $B_j^n > 0$ for the equation $b_j^n = b_j - \frac{\Delta t}{\Delta z} (U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n)$

We can find a time step for which this condition is met by considering

$$b_j^n - \frac{\Delta t}{\Delta z} (U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n) > 0$$

$$b_j^n > \frac{\Delta t}{\Delta z} (U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n)$$

$$\Delta t < \frac{\Delta z b_j^n}{U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n}$$

You were supposed to use the upwind scheme. for beta=0 your scheme is unstable -0.5

This time step means that from a change in b_j from flux across all boundaries won't exceed the current value of b_j^n which ensures positivity

To ensure this is true for all b_j^n the minimisation needs to be imposed as if this is true for the value of b_j^n giving the smallest restriction so it will be true for all other b_j^n values ensuring stability

1f) i) Derive a second order spatial discretization of (5) by changing the spatial discretization of the convective term.

To get second order accuracy of the convective term use the central difference scheme similarly to how the diffusive term was treated in the original discretisation

$$\frac{\partial}{\partial z} \alpha b^3 \approx \alpha \left(\frac{(b_{j+1}^n)^3 - (b_{j-1}^n)^3}{2\Delta z} \right)$$

Taylor expansions correct to 2nd order so additional terms only for Δz^2 and higher terms

The full discretized of (5) with second order central difference for diffusion and convection becomes

$$\begin{aligned} b_j^{n+1} &= b_j^n - \Delta t \left[\frac{\alpha}{2\Delta z} (b_{j+1}^n)^3 - b_{j-1}^n)^3 - \beta (b_j^n + b_{j+1}^n)^3 \right] \\ b_j^{n+1} &= b_j^n - \frac{\Delta t}{\Delta z} \left(\alpha \left(\frac{b_j^n + b_{j+1}^n}{2} \right)^3 - \beta \left(\frac{b_j^n + b_{j+1}^n}{2} \right)^3 \left(\frac{b_{j+1}^n - b_j^n}{\Delta z} \right) + \frac{\Delta t}{\Delta z^2} \left(\frac{\beta}{\Delta z} \left(\frac{b_j^n + b_{j+1}^n}{2} \right)^3 (b_{j+1}^n - b_j^n) \right) \dots - \left(\frac{b_{j+1}^n + b_j^n}{2} \right)^3 (b_j^n - b_{j-1}^n) \right) \dots \\ \Rightarrow b_j^{n+1} &= b_j^n - \frac{\Delta t}{2\Delta z} \alpha (b_{j+1}^n)^3 - b_{j-1}^n)^3 + \frac{\Delta t}{\Delta z^2} (U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n) \\ U_{j+\frac{1}{2}}^n &= \beta \left(\frac{b_j^n + b_{j+1}^n}{2} \right)^3 \left(\frac{b_{j+1}^n - b_j^n}{\Delta z} \right) \\ U_{j-\frac{1}{2}}^n &= \beta \left(\frac{b_{j-1}^n + b_j^n}{2} \right)^3 \left(\frac{b_j^n - b_{j-1}^n}{\Delta z} \right) \end{aligned}$$

ii) What is a step criterion to guarantee that $b_j^{n+1} > 0$ for this new discretisation

Again we are checking that $b_j^{n+1} > 0$ given that $b_j^n > 0$ so the change in b_j due to net fluxes doesn't exceed the b_j^n value with the same physical application of positive dike widths

Using the same process as before by setting,

$$b_j^n - \frac{\Delta t}{\Delta z} \alpha (b_{j+1}^n)^3 - b_{j-1}^n)^3 - \frac{1}{\Delta z^2} (U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n) > 0 \quad \text{and rearranging to find } \Delta t \text{ condition}$$

$$\Rightarrow \Delta t < \frac{\alpha}{2\Delta z} (b_{j+1}^n)^3 - b_{j-1}^n)^3 - \frac{1}{\Delta z^2} (U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n)$$

where b_j^n = current value of the dike width at grid point j

$b_{j\pm 1}^n$ = neighbouring values at gridpoints $j \pm 1$

Δz = spatial grid spacing

β = convection coefficient

α = diffusion coefficient

-0.5 this scheme is sec unstable; see chapter 4

This time step criterium ensures that a numerical update (move to neighbouring value) doesn't subtract more than the available value so the dike width will always be positive as given by physical constraints.

2a) Show that the steady state solution of (5) satisfies $\beta b^3 \frac{db}{dz} = \alpha b^3 - Q$ where Q = integration constant

Non linear convection-diffusion equation (5)

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial z} \left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z} \right) = 0$$

Steady state means the solution doesn't change in time so $\frac{\partial b}{\partial t} = 0$

$$\frac{\partial}{\partial z} \left(\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z} \right) = \frac{d}{dz} \left(\alpha b^3 - \beta b^3 \frac{db}{dz} \right) = 0$$

which implies the total flux is constant and the constant of integration is given by Q

$$\alpha b^3 - \beta b^3 \frac{db}{dz} = Q$$

$$-\beta b^3 \frac{db}{dz} = Q - \alpha b^3$$

$$\beta b^3 \frac{db}{dz} = \alpha b^3 - Q \rightarrow \text{as given by (10) on tasksheet}$$

Integrate (10) and plot the solution for example by using an Euler forward spatial discretisation

$$\beta b^3 \frac{db}{dz} = \alpha b^3 - Q$$

$$\frac{db}{dz} = \frac{\alpha b^3 - Q}{\beta b^3} \quad \leftarrow \text{First order ODE for } b(z) \text{ so need to integrate numerically}$$

$$b_{j+1} = b_j + \Delta z \frac{db}{dz} \Big|_j \quad \Rightarrow b_{j+1} = b_j + \Delta z \frac{\alpha b_j^3 - Q}{\beta b_j^3}$$

$$\frac{db}{dz} \Big|_j = \frac{\alpha b_j^3 - Q}{\beta b_j^3}$$

Figure 1 is plotted in python by the code entitled "tasksheet 1Q2a" with 1000 gridpoints and constants defined as given in the tasksheet.

It produces a value of $b_f \approx 0.593352276945$ and the figure 1 shows how the dike width between $\frac{b}{2}$ and $-\frac{b}{2}$ evolves with height in the dike

2b) Figures 2 through 4 show the evolution of the dike diameter as time progresses with 11, 21 and 41 grid points respectively

The solution of the non linear convection diffusion equation was computed numerically using the scheme as described to linearise equation (5). This was computed in python in the script entitled "tasksheet 1Q2b". To ensure the stability requirement on the timestep was met it was derived from the maximum principle and in the script its implementation was to check the coefficient αb_j of b_j was positive

Also note that in some cases the solutions for larger values of t are identical and this is due to having reached the steady state. Exact details are given in each figures caption as to which t 's included

c) Consider the L^2 -norm and the L^∞ norm for the error $e(z,t) = b(z,t) - b_{\text{numerical}}(z,t)$

The point of this question is to assess how the spatial resolution affects the accuracy of the numerical solution - to estimate the order of spatial discretization

- Need to keep the time step Δt constant across all grid sizes to isolate the effect of spatial resolution. This makes sure any change in error is due to the number of grid points, not the time stepping accuracy $\Delta t = 0.0001$
- We treat the steady state solution as an approximation to the true analytical solution which physically represents the final shape of the dike after it has settled once convection and diffusion have balanced
- For each grid size, the numerical solution is computed at $t=2$ and the error is the difference between the numerical and steady state solutions for each grid size point
- The composite trapezoidal rule approximates the integral of the squared error over the domain of $z \in (0,1)$

- To estimate

- The plot shows that the error decreases as the number of grid points increases which indicates convergence
- To estimate the order of spatial discretisation you can plot the error vs grid spacing Δz and since its a log-log plot the slope of the line gives the order of accuracy

$$L_2 = \frac{1}{\sqrt{2}} \sqrt{\frac{1}{2} (b_0 - b(0, t_c))^2 + \frac{1}{2} (b_1 - b(1, t_c))^2 + \sum_{j=1}^N (b_j^n - b_{\text{numerical}}(x_j, t_c))^2 \Delta z}$$

$$L_2 = \sqrt{\sum_{j=1}^N (b_j^n - b_{\text{numerical}}(x_j, t_c))^2 \Delta z}$$

- The first two terms in above are the boundary conditions which are equal for both exact methods so can be taken out and give a reduced L_2 .
- The error scales as $\text{Err} = K_4 (\Delta z)^\gamma \propto (N_{\text{spatial}} - 1)^{-\gamma}$ where the order γ , N_{spatial} is the number of grid points and K_4 is a constant
- The error decreases exponentially with Δz

Figure 5 is a graph that shows the approximate error in the numerical solution vs the number of grid points where the exponent $-\gamma = -1$

d) i) Verify that the following is an implicit yet explicit, time dependent travelling wave solution (12)

$$z - z_{r,0} - ct = \frac{\beta}{\alpha} \left(b(z,t) - \sqrt{\frac{c}{\alpha}} \operatorname{atanh} \left(\sqrt{\frac{\alpha}{c}} b(z,t) \right) \right)$$

with $z_{r,0}$ a reference constant and c = wave speed. b is of the form $b = b(\phi)$

Substitute $\phi = z - ct$ to give:

$$\phi - z_{r,0} = \frac{\beta}{\alpha} \left(b(\phi) - \sqrt{\frac{c}{\alpha}} \operatorname{atanh} \left(\sqrt{\frac{\alpha}{c}} b(\phi) \right) \right)$$

Differentiate wrt ϕ

$$1 = \frac{\beta}{\alpha} \left(\frac{\partial b}{\partial \phi} - \frac{\frac{\partial b}{\partial \phi}}{1 - \frac{\alpha}{c} b^2} \right) \quad \text{with } \frac{\partial b}{\partial \phi} = \frac{\infty}{\beta} \frac{\frac{\alpha}{c} b^2 - 1}{\frac{\alpha}{c} b^2} \quad \text{Eq (2)}$$

Using chain rule $\frac{\partial}{\partial z} = \frac{\partial}{\partial \phi}$ and $\frac{\partial}{\partial t} = -c \frac{\partial}{\partial \phi}$ so substituting these into $\frac{\partial b}{\partial t} + \frac{\partial}{\partial z} (\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z}) = 0$ gives that

$$\begin{aligned} & -c \frac{\partial b}{\partial \phi} + \frac{\partial}{\partial \phi} (\alpha b^3 - \beta b^3 \frac{\partial b}{\partial \phi}) \\ \Rightarrow & -c \frac{\partial b}{\partial \phi} + 3\alpha b^2 \frac{\partial b}{\partial \phi} - \frac{\partial}{\partial \phi} \left(\beta b^3 \frac{\alpha}{\beta} \frac{\frac{\alpha}{c} b^2 - 1}{\frac{\alpha}{c} b^2} \right) \\ \Rightarrow & -c \frac{\partial b}{\partial \phi} + 3\alpha b^2 \frac{\partial b}{\partial \phi} + \frac{\partial}{\partial \phi} (cb - 3\alpha b^3) \\ \Rightarrow & -c \frac{\partial b}{\partial \phi} + 3\alpha b^2 \frac{\partial b}{\partial \phi} + c \frac{\partial b}{\partial \phi} - 3\alpha b^2 \frac{\partial b}{\partial \phi} = \\ & = (-c + c + 3b^2 - 3\alpha b^2) \frac{\partial b}{\partial \phi} = 0 \end{aligned}$$

showing how the RHS of equation (2) so the given equation in the question is a solution

ii) Verify the numerical solution against the exact solution and pay attention to the required condition that $b(x,t) \geq 0$. The dike remains closed below a dimensionless depth of 0.9km in a total domain of 3km deep.

- Exact solution was defined using the implicit equation from the tasksheet and then rearranged to make $b(z,t)$ the subject to allow the initial and boundary conditions to be derived from it.
- For the numerical scheme used the forward Euler method and second order spatial discretisation then the boundary conditions were updated at each step using the time dependent exact solution. Used the maximum principle to ensure positivity of solution remained true. Also using the maximum principle created an additional function to make sure the maximum stable time step kept numerical stability.
- This simulation shows the evolution of a magma filled dike going upwards through rock. The travelling wave solution is a steady propagation of the dike front and the numerical scheme captures how the dike width evolves over time.

Figures 6 through 8 are showing how the dike width changes at different times to show its propagation upwards through the mantle. This was completed for all 3 grid sizes $j=11, 21, 41$. The numerical solutions are shown with solid lines and the dashed lines are the exact solution

e) Extend the numerics in (d) to the Crank-Nicolson scheme, and solve the nonlinear algebraic system using iteration techniques

The Crank-Nicolson method is based on the trapezoidal rule giving a second-order convergence in time. It's equation is a combination of the forward Euler method at n and the backward Euler method at $n+1$, which is implicit such that to get to the "next" value of u in time a system of algebraic equations is solved

This scheme usually improves the numerical stability as it has better stability and accuracy as an implicit method rather than explicit schemes. It would be beneficial for this scenario as it allows for larger time steps while maintaining stability so it makes it easier to simulate long term evolution

Crank-Nicolson Scheme: Averages the spatial terms between time levels n and $n+1$

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} = \frac{1}{2} [F(b^n) + F(b^{n+1})]$$

where $F(b)$ is the spatial operator applied to solution b . This includes all the spatial derivatives and non-linear terms from the convection-diffusion equation

In this case $F(b)$ would represent include convection and nonlinear diffusion so the updated relation would be

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} = \frac{1}{2} [F(b^n) + F(b^{n+1})]$$

$$\text{so with the PDE } \frac{\partial b}{\partial t} + \frac{\partial}{\partial z} (\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z}) = 0$$

$$\frac{\partial b}{\partial t} = -F(b) \quad \text{where} \quad F(b) = \frac{\partial}{\partial z} (\alpha b^3 - \beta b^3 \frac{\partial b}{\partial z})$$

The Crank-Nicolson scheme averages the spatial operators between time levels n and $n+1$

$$\frac{b_j^{n+1} - b_j^n}{\Delta t} = -\frac{1}{2} [F(b^n)_j + F(b^{n+1})_j]$$

$$\Rightarrow b_j^{n+1} = b_j^n - \frac{\Delta t}{2} [F(b^n)_j + F(b^{n+1})_j]$$

Convection term : $F_{\text{conv}}(b)_j = \alpha \frac{(b_{j+1})^3 - (b_{j-1})^3}{2 \Delta z}$

Diffusive term : $F_{\text{diff}}(b)_j = \frac{\beta}{(\Delta z)^2} \left(\left(\frac{b_j + b_{j+1}}{2} \right)^3 \frac{b_{j+1} - b_j}{\Delta z} - \left(\frac{b_{j+1} + b_j}{2} \right)^3 \frac{b_j - b_{j-1}}{\Delta z} \right)$

$$\Rightarrow F(b)_j = \alpha \frac{(b_{j+1})^3 - (b_{j-1})^3}{2 \Delta z} - \frac{\beta}{\Delta z^2} \left[\left(\frac{b_j + b_{j+1}}{2} \right)^3 (b_{j+1} - b_j) - \left(\frac{b_{j+1} + b_j}{2} \right)^3 (b_j - b_{j-1}) \right]$$

Since b_j^{n+1} appears on both sides of the equation the system has to be solved using fixed-point iteration to solve for all $j = 1, 2, \dots, J-2$ (excluding boundaries)

This code uses a Crank-Nicolson scheme to solve the non-linear convection diffusion equation modelling the evolution of a magma filled dike. It compares the numerical solution at various times with the exact travelling wave solution.

- Use the exact travelling wave solution at $t=0$ to initialise dike shape and initial conditions
- For each time t the code:

- Computes exact solution
- Initialises numerical solution
- Computes the number of timesteps and time step size

- Continues the solution using Newton Raphson iteration for the non linear system

computed
This is repeated
for each gridpoint
position

- { - Computes the residual
- Creates a linearise around guess $b^{(k)}$ and solve for
- Then update the g guess

This process is repeated until the modulus of the update is small which suggests the system has reached convergence

In the context of the magma flow:

- The dike is driven by pressure and resisted by surrounding rock
- The nonlinear PDE models this balance
- Newton Raphson ensures that the numerical solution ^{keeps} this balance at each timestep
- The plot shows how the dike evolves overtime, matching the expected travelling wave behaviour

Figures 9 through 11 are plots that show this numerical scheme implemented for the dike width profile at different times ($t = 1, 2, 3, 4, 5$)

- The solid lines represent the numerical solution obtained using Newton Raphson iteration
- The dashed lines represent the exact travelling wave solution

The Newton Raphson method continues to ensure the solutions

- Converge to a physically meaningful profile
- Preserves positivity so the dike width isn't impossibly negative
- Tracks the wave front as it propagates upward through the domain

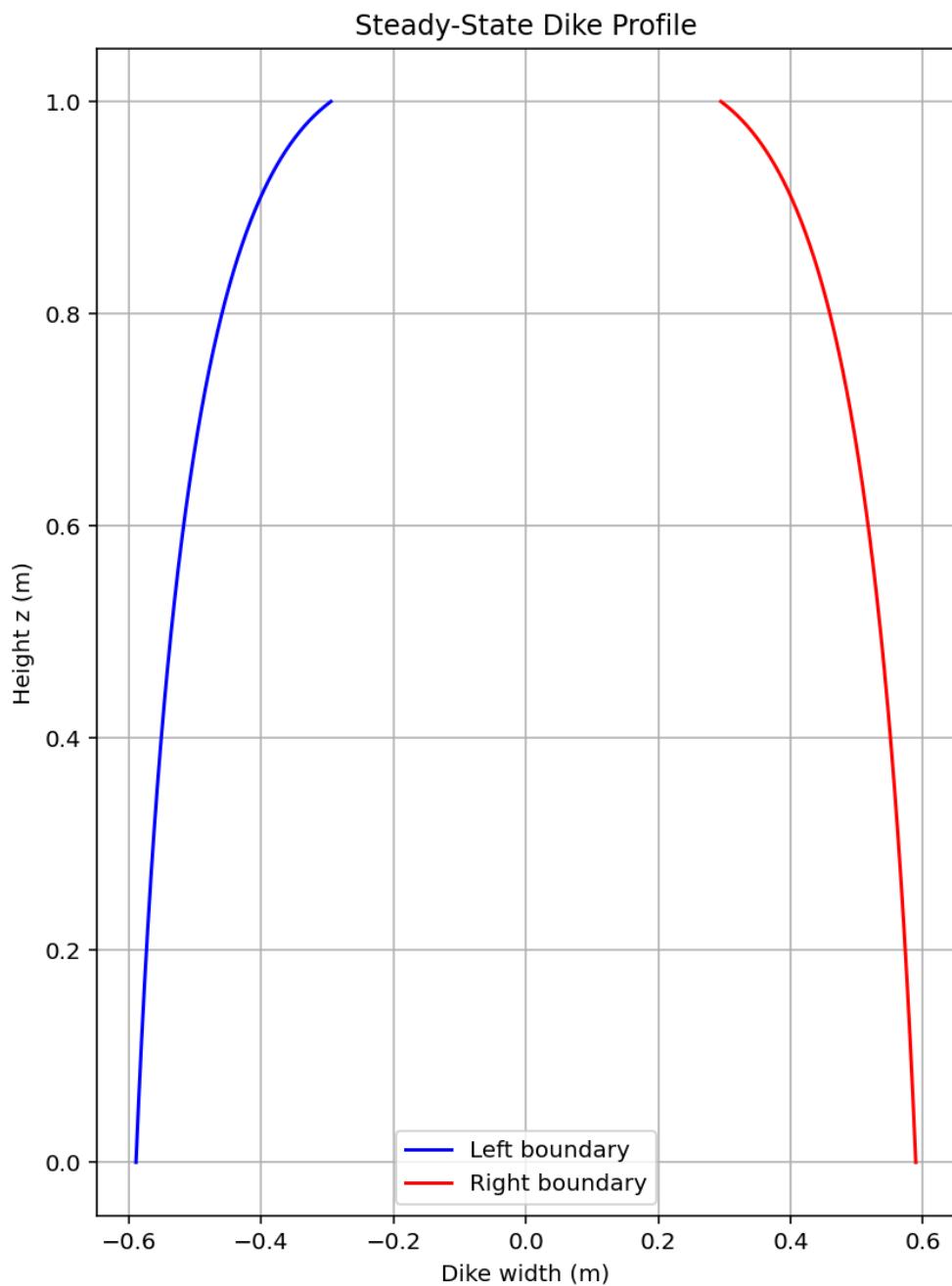


Figure 1: Dike width plotted as a function of height in the Dike for the steady state.

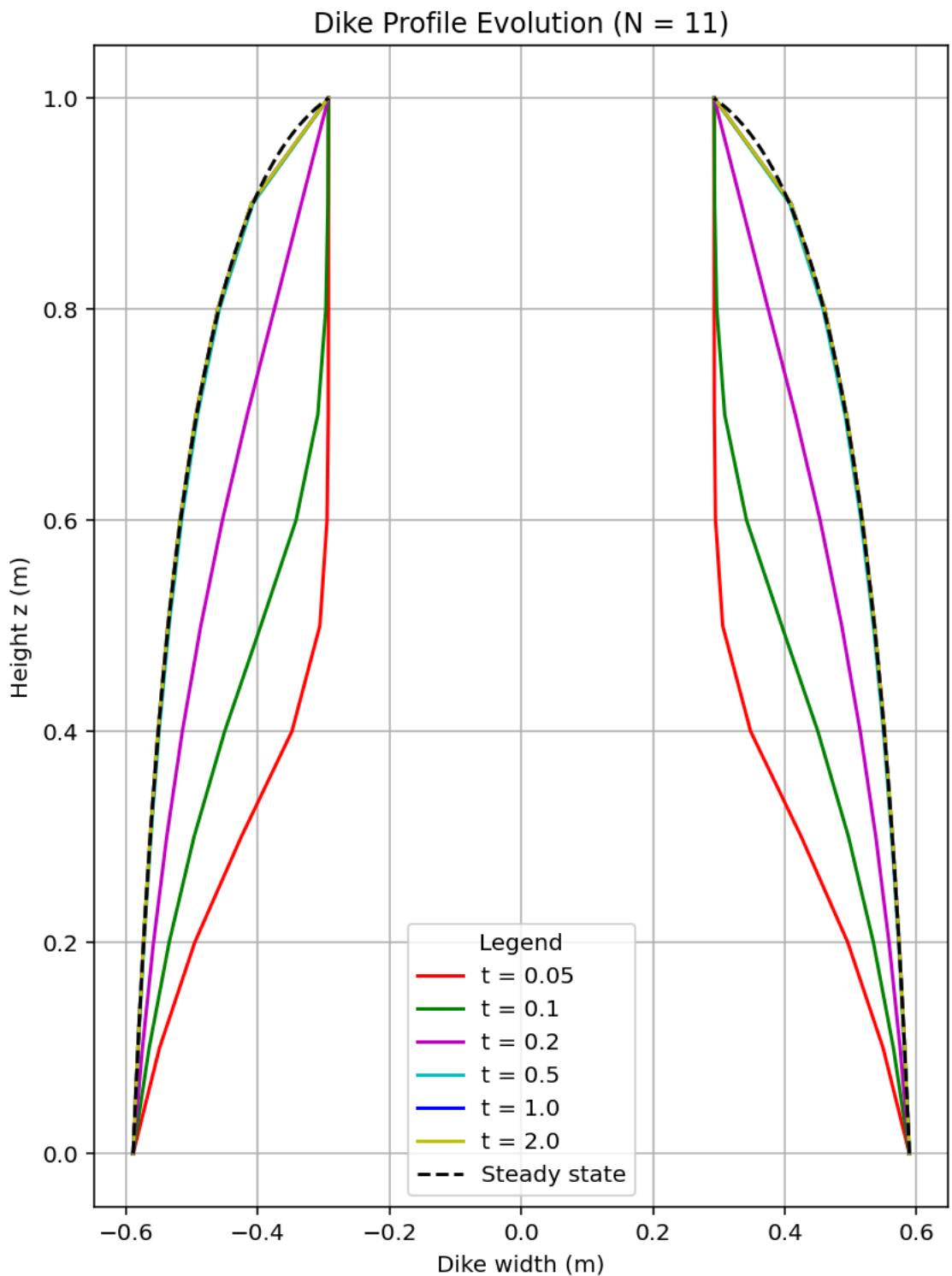


Figure 2: Graph showing the evolution of the dike diameters as time progresses with 11 spatial grid points. The solutions for $t=0.5, 1$ and 2 are the same having reached the steady state

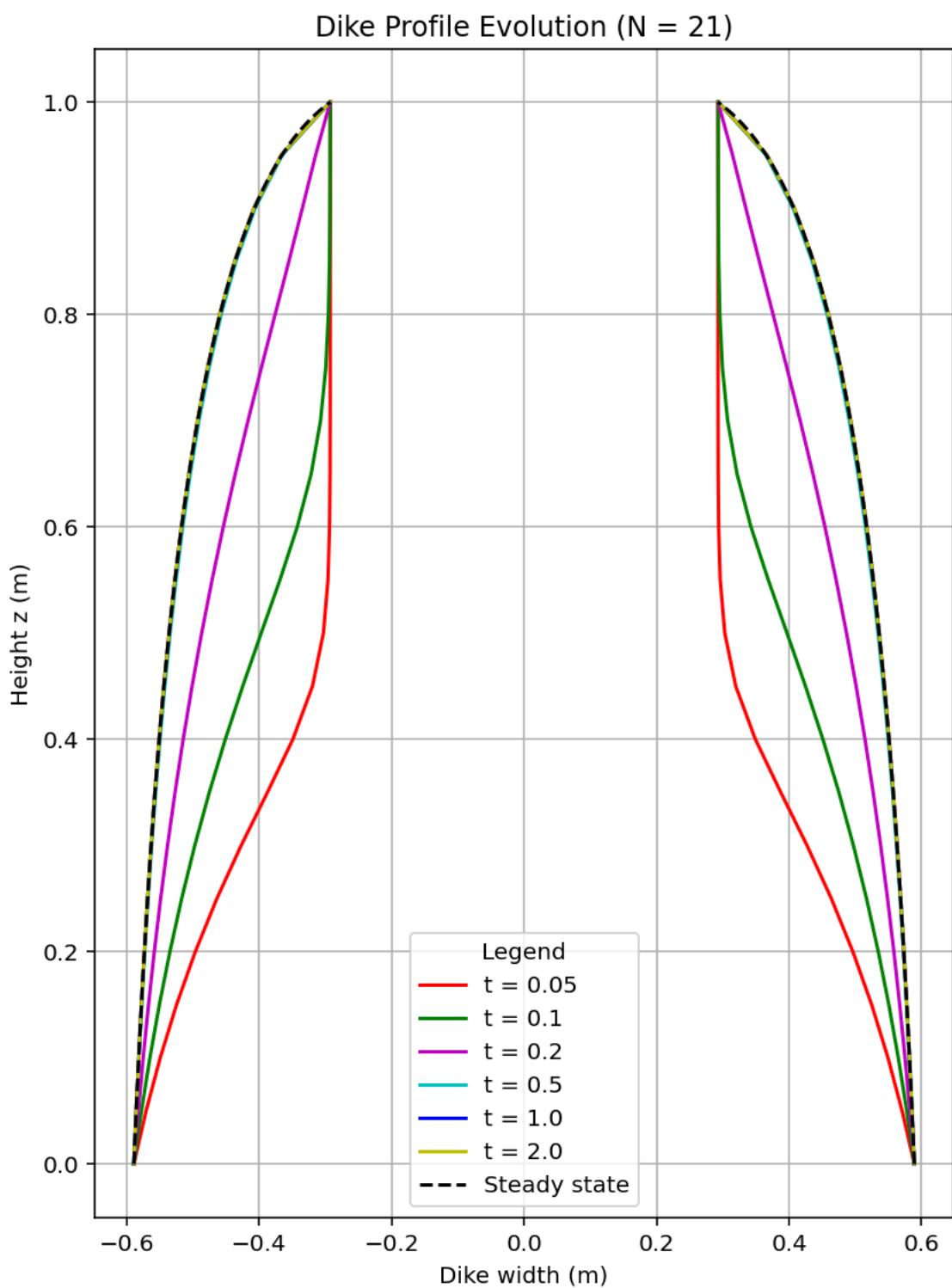


Figure 3: Graph showing the evolution of the dike diameters as time progresses with 21 spatial grid points. The solutions for $t=0.5, 1$ and 2 are the same having reached the steady state

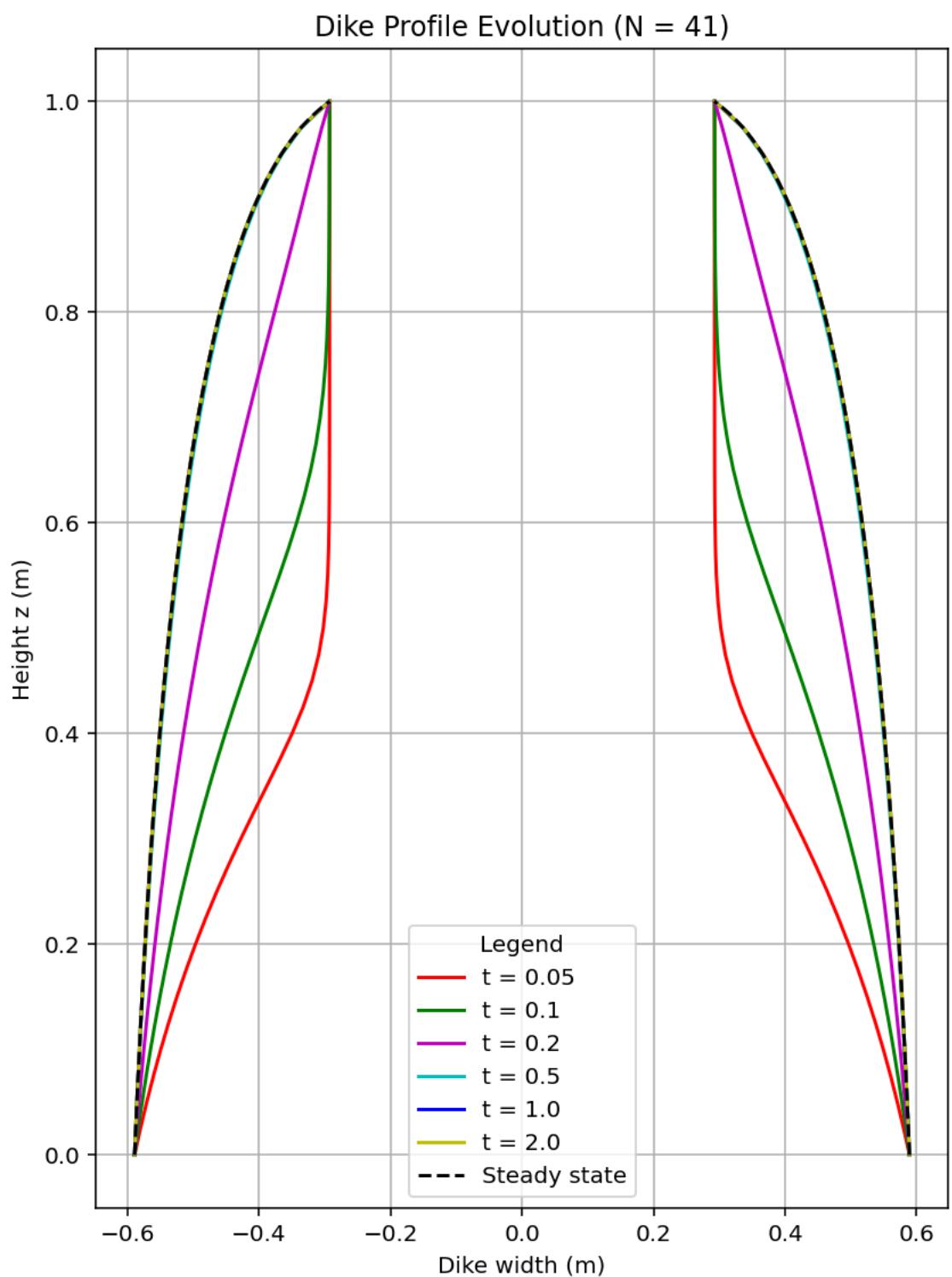


Figure 4: Graph showing the evolution of the dike diameters as time progresses with 41 spatial grid points. The solutions for $t=0.5, 1$ and 2 are the same having reached the steady state

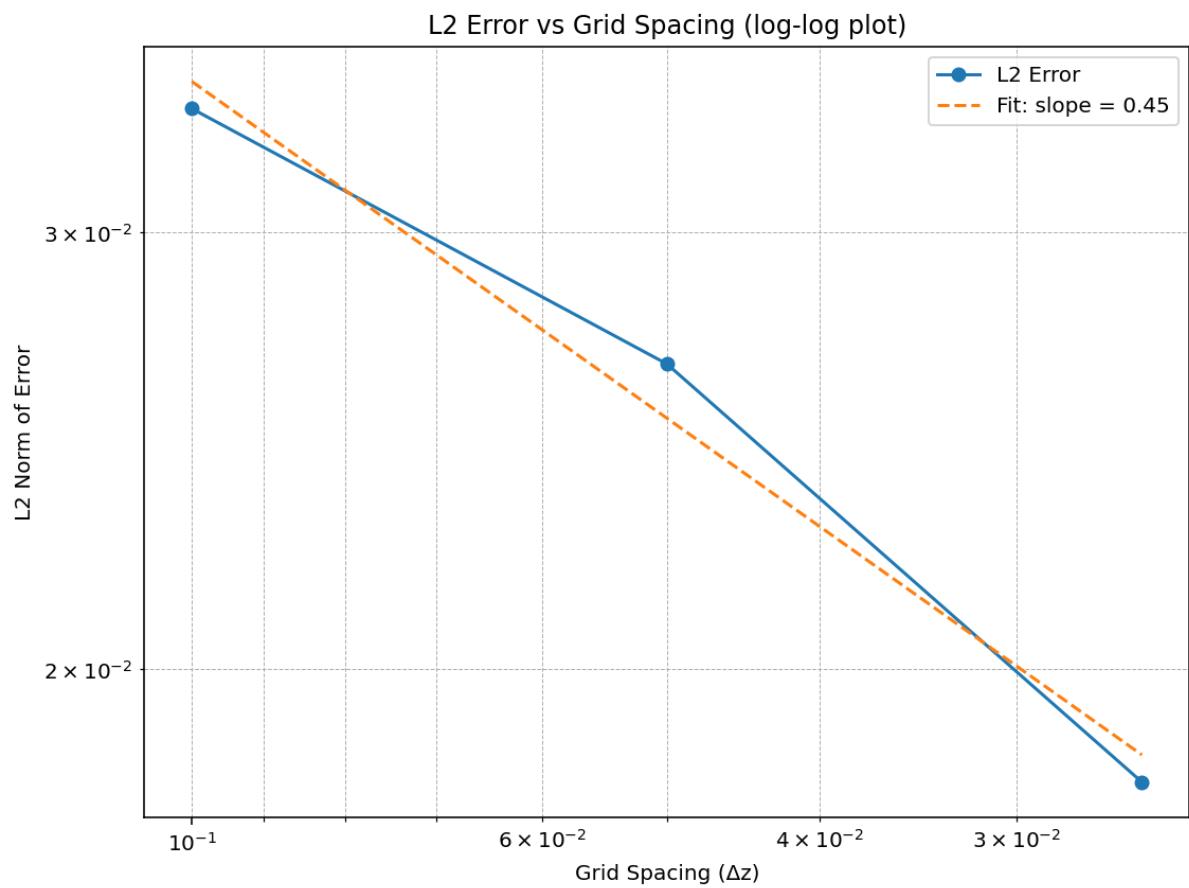


Figure 5: Graph showing the approximate error in the numerical solution vs number of grid points. This shows a linear fit on the log log plot.

slope should be one or less, so either your time step is too large and/or you are not at steady state yet. error should at least half every resolution doubling -1

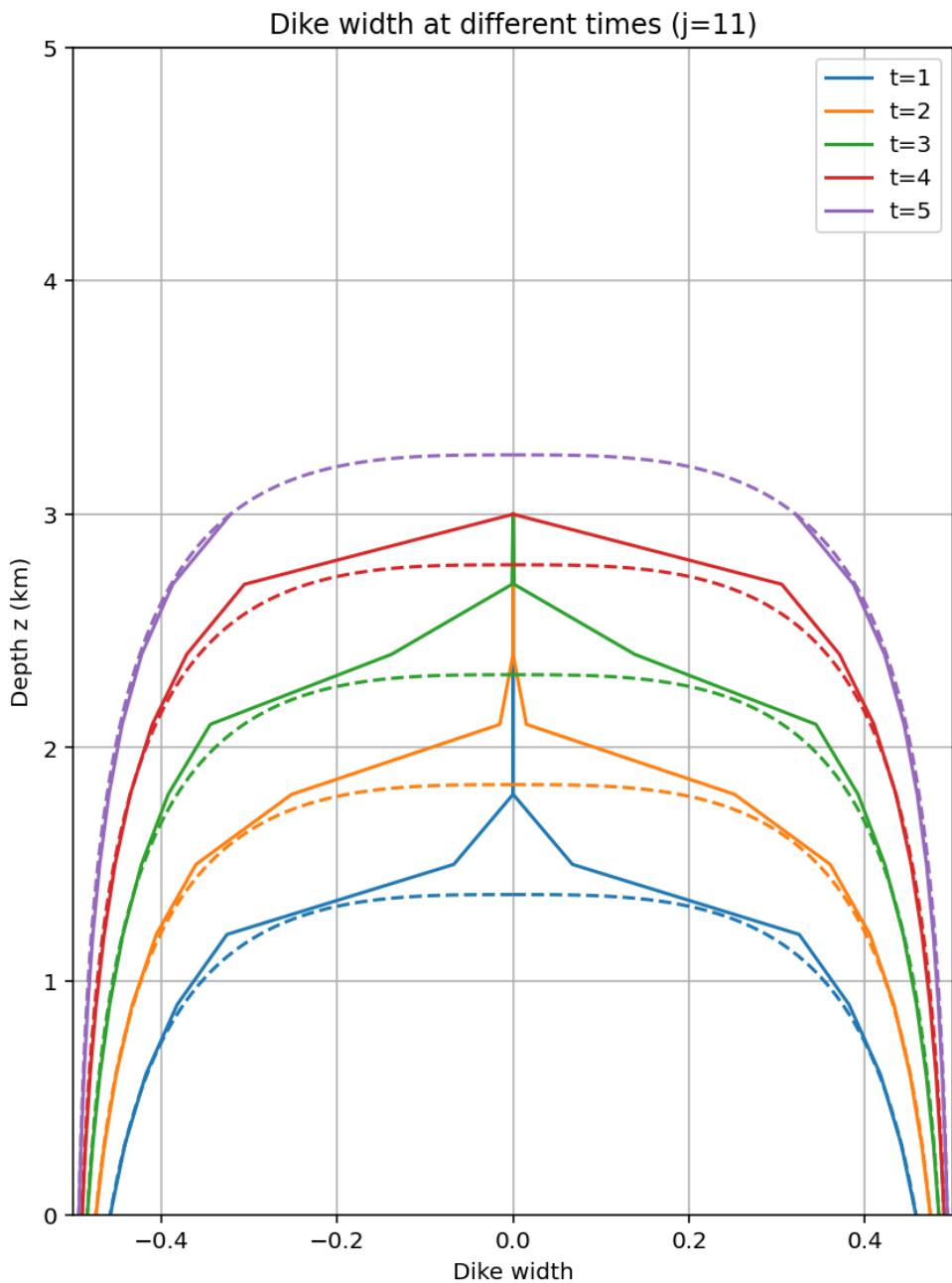


Figure 6: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 11 spatial grid points. It shows the propagation upwards through the mantle. The numerical solutions are shown with solid lines and the dashed lines are the exact solutions

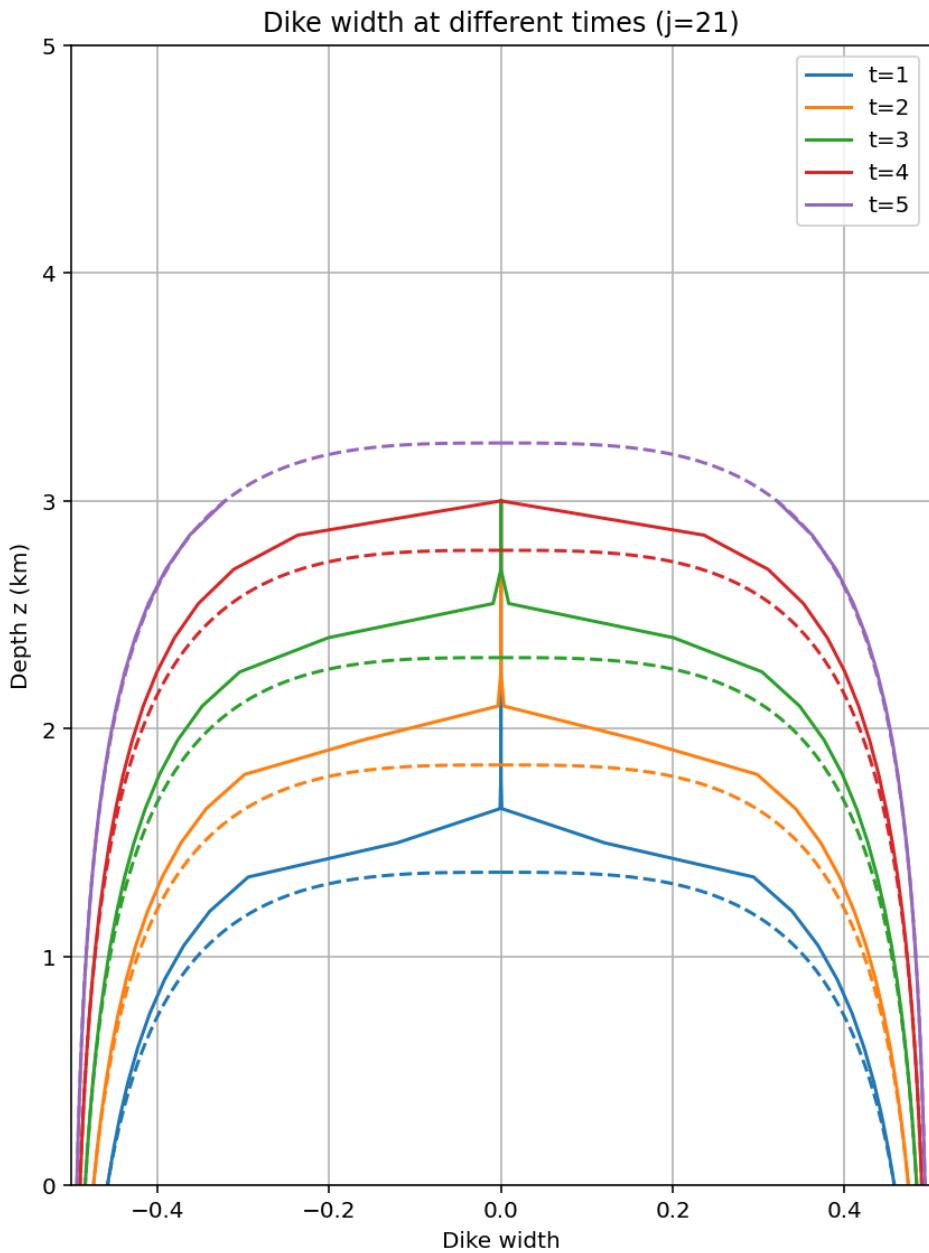


Figure 7: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 21 spatial grid points. It shows the propagation upwards through the mantle. The numerical solutions are shown with solid lines and the dashed lines are the exact solutions

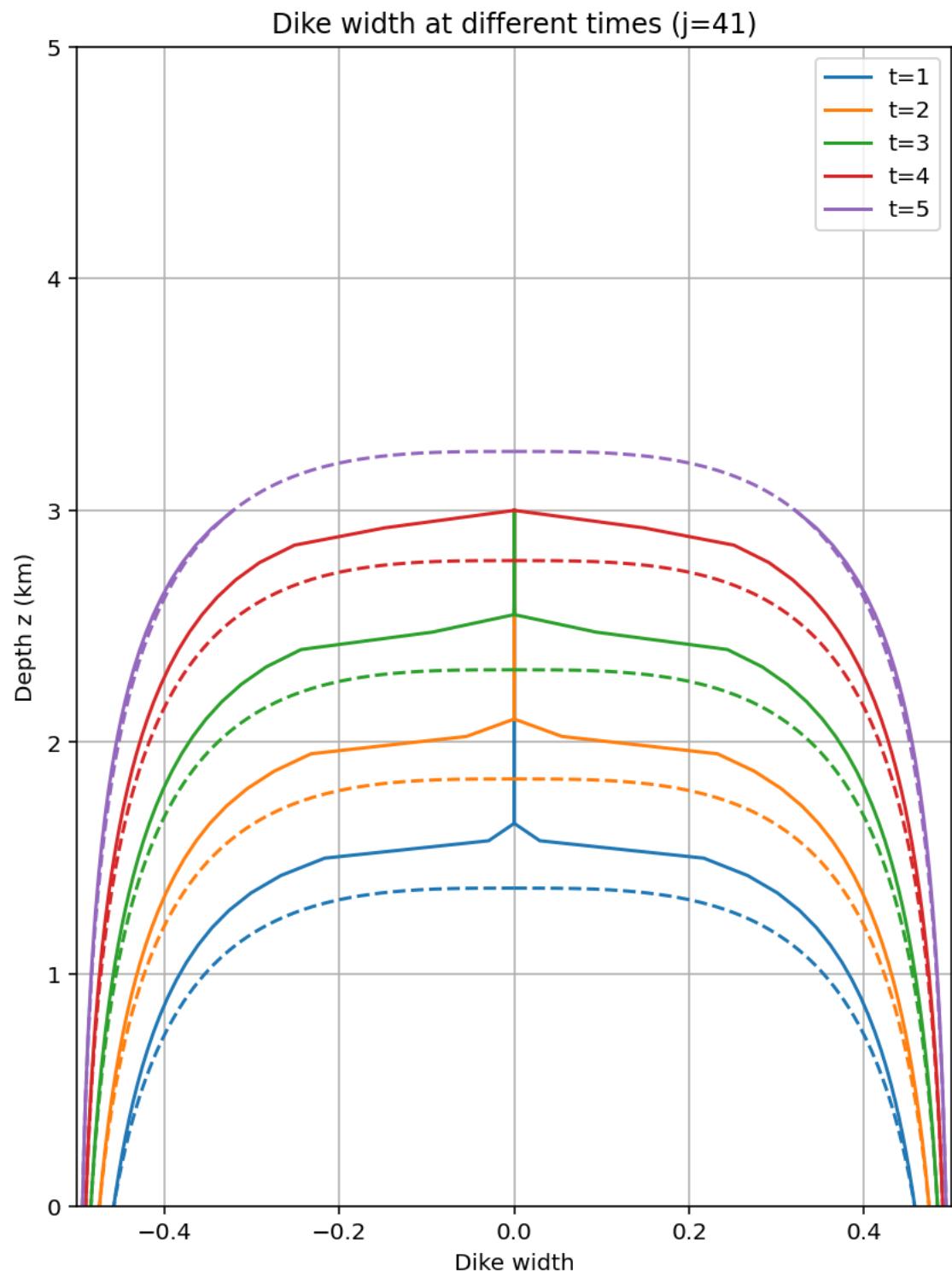


Figure 8: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 41 spatial grid points. It shows the propagation upwards through the mantle. The numerical solutions are shown with solid lines and the dashed lines are the exact solutions

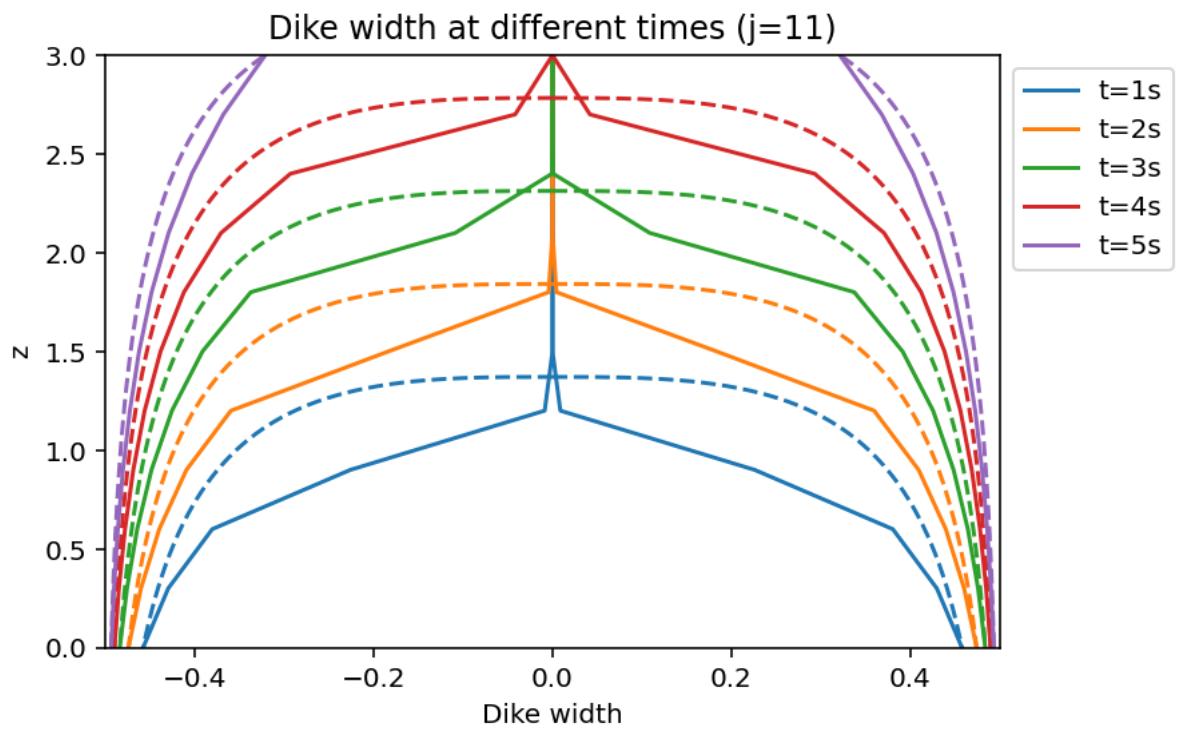


Figure 9: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 11 spatial grid points. It shows the upwards propagation through the mantle. The numerical solution is implanted using the Crank Nicholson Scheme and Newton Raphson iterations and displayed by solid lines. The dashed lines represent the exact traveling wave solution

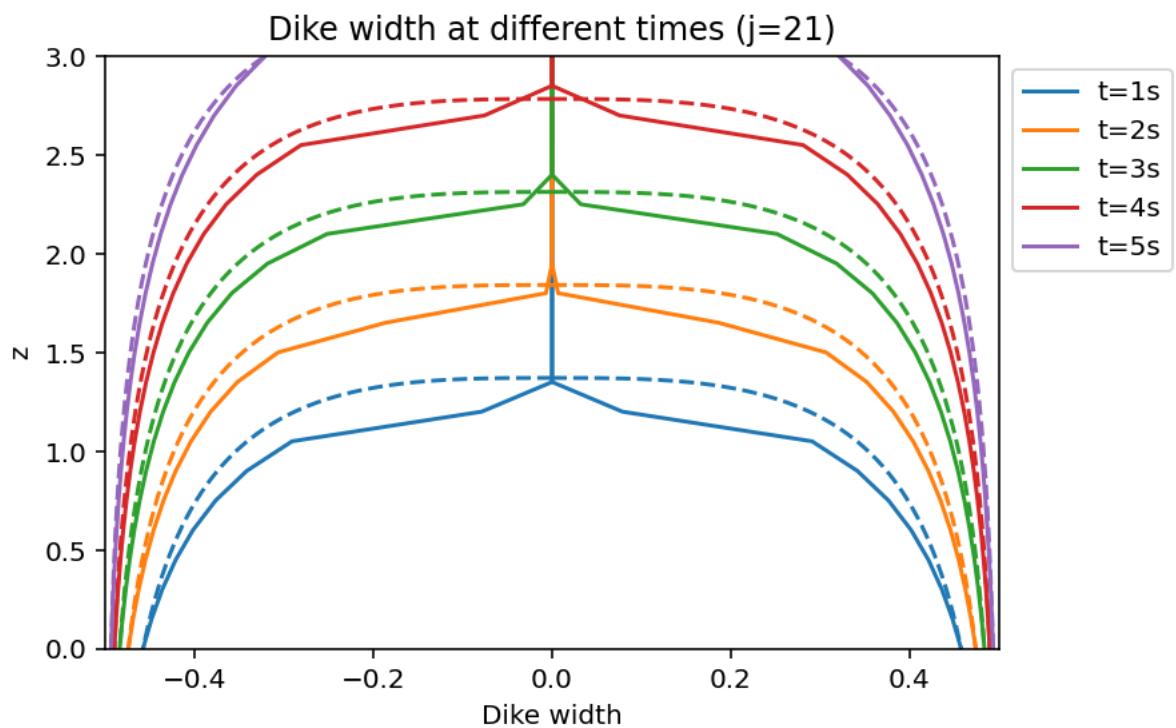


Figure 10: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 21 spatial grid points. It shows the upwards propagation through the mantle. The numerical solution is implanted using the Crank Nicholson Scheme and Newton Raphson iterations and displayed by solid lines. The dashed lines represent the exact traveling wave solution

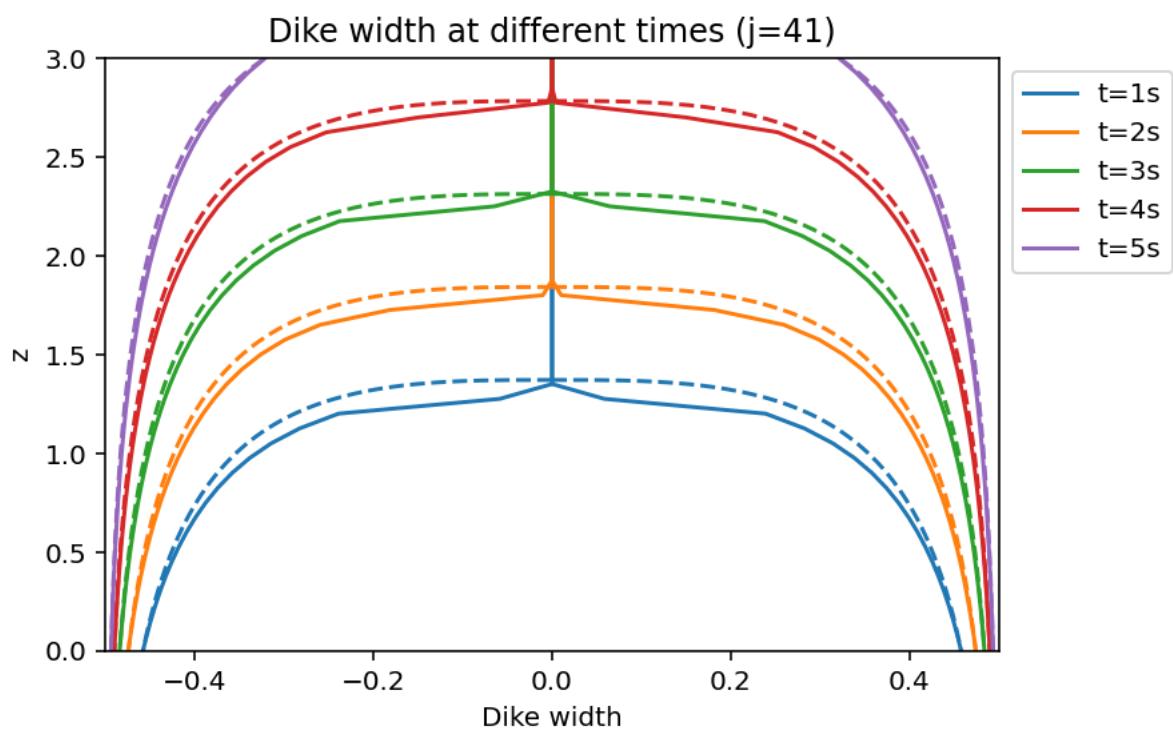


Figure 11: Graph showing the evolution of the dike diameters as time progresses from 1s to 5s with 41 spatial grid points. It shows the upwards propagation through the mantle. The numerical solution is implanted using the Crank Nicolson Scheme and Newton Raphson iterations and displayed by solid lines. The dashed lines represent the exact traveling wave solution

need much higher resolution -1
where are your codes?
16/20