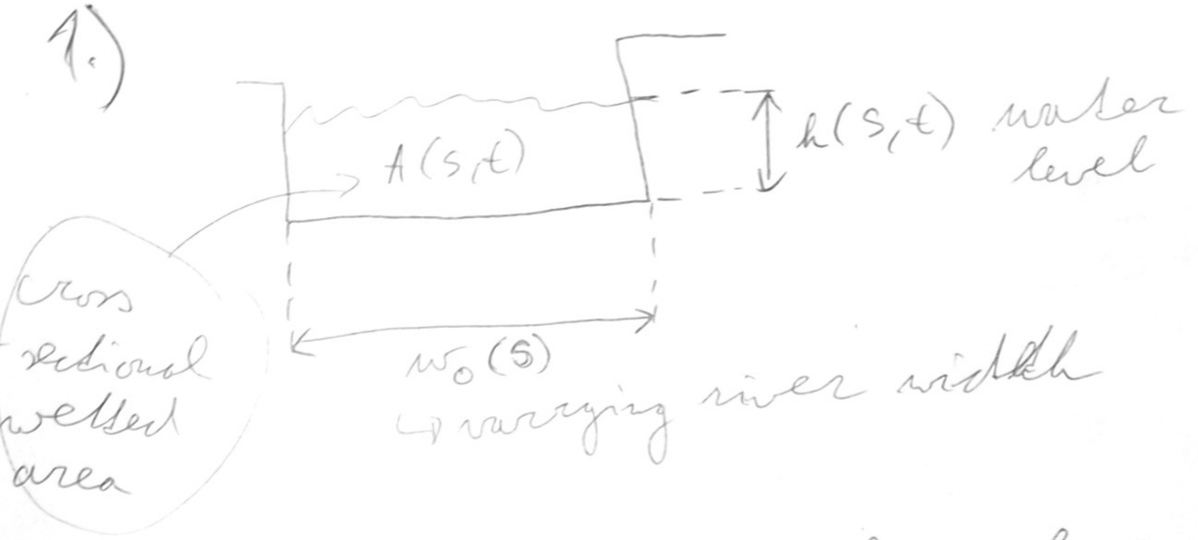
1.)



as we have rectangular shape, we get:

$$A(s,t) = w_0(s)\, h(s,t)$$

$$\Leftrightarrow \quad h(A,s) = \frac{A(s)}{w_0(s)} \qquad (*)$$

Similarly, if $P(A,s)$ is wetted parameter, then it must be equal to

$$P(A,s) = w_0(s) + 2h(s,t)$$

river bed

river walls

$$\therefore\ P(A,s) = w_0(s) + 2\frac{A}{w_0} \qquad (**)$$

from $(*)$.

Set $s=0$ in eqn. 5:

$$\partial_t A + \partial_s\left[A\left(R(A,s)\right)^{\frac{2}{3}} \sqrt{-\partial_s b}\; C_m^{-1}\right] = 0$$

Recall that $R(A,s) = \dfrac{A}{P(A,s)} = \dfrac{A}{\left(w_0(s) + 2A\, w_0(s)^{-1}\right)}$

$$\partial_t A + \partial_s\Bigg[\underbrace{\frac{A^{\frac{5}{3}}\sqrt{-\partial_s b}}{C_m\left(w_0 + \frac{2A}{w_0}\right)^{\frac{2}{3}}}}_{F(A,s)}\Bigg] = 0$$

$$\partial_t A + \frac{\partial F}{\partial A}\frac{\partial A}{\partial s} + \frac{\partial F}{\partial s} = 0$$

Apply product rule to get

$$\frac{\partial F}{\partial A} = \frac{5}{3}\,\frac{A^{\frac{2}{3}}\sqrt{-\partial_s b}}{C_m\left(w_0 + \frac{2A}{w_0}\right)^{\frac{2}{3}}} - \frac{2}{3}\,\frac{A^{\frac{5}{3}}\sqrt{-\partial_s b}}{C_m\left(w_0 + \frac{2A}{w_0}\right)^{\frac{5}{3}}}\,\frac{2}{w_0}$$

$$= \frac{\sqrt{-\partial_s b}}{3C_m}\left[\frac{5A^{\frac{2}{3}} w_0\left(w_0 + \frac{2A}{w_0}\right) - 4A^{\frac{5}{3}}}{\left(w_0 + \frac{2A}{w_0}\right)^{\frac{5}{3}} w_0}\right]$$

$$= \frac{\sqrt{-\partial_s b}}{3C_m}\left[\frac{5A^{\frac{2}{3}} w_0^2 + 10A^{\frac{5}{3}} - 4A^{\frac{5}{3}}}{\left(w_0 + \frac{2A}{w_0}\right)^{\frac{5}{3}} w_0}\right]$$

$$= \frac{\sqrt{-\partial_s b}}{3C_m}\left[\frac{5A^{\frac{2}{3}} w_0 + \frac{6A^{\frac{5}{3}}}{w_0}}{\left(w_0 + \frac{2A}{w_0}\right)^{\frac{5}{3}}}\right] > 0\,!$$

Now, apply chain rule & treat $\sqrt{-\partial_s b}$ as constant to get:

$$\frac{\partial F}{\partial s} = \partial_s \left[\frac{A^{5/3}\sqrt{-\partial_s b}}{C_m \left(w_0 + \frac{2A}{w_0}\right)^{2/3}} \right] \cancel{\neq 0}$$

$$= \frac{A^{5/3}\sqrt{-\partial_s b}}{C_m} \cdot \frac{(-2)}{3\left(w_0 + \frac{2A}{w_0}\right)^{5/3}} \frac{\partial}{\partial w_0}\left[w_0 + \frac{2A}{w_0}\right] \frac{\partial w_0}{\partial s}$$

$$= -\frac{2A^{5/3}\sqrt{-\partial_s b}}{3C_m\left(w_0 + \frac{2A}{w_0}\right)^{5/3}} \left(1 - \frac{2A}{w_0^2}\right) \frac{\partial w_0}{\partial s}$$

as required.

2.) If $w_0(s)$ is independent of s,

$\frac{\partial w_0}{\partial s} = 0$ & $\frac{\partial F}{\partial s} = 0$ as a consequence.

Hence, the eqn becomes

$$\partial_t A + \frac{\partial F}{\partial A} \partial_s A = 0$$

In particular, we find eigenvalue λ by cosidering

$$\frac{\partial F}{\partial A} - \lambda = 0 \iff \lambda = \frac{\partial F}{\partial A}(A)$$

We have $A = A(s,t)$ and $H(A) := \frac{\partial F}{\partial A}$ with

the governing equ.

$$\partial_t [A] + H(A) \, \partial_s [A] = 0 \qquad (C)$$

Now, consider change of coordinates

$\eta = \frac{s}{t}$, then $A = A(\eta(s,t))$ and

$$\partial_t [A] = \partial_\eta A \, \frac{\partial \eta}{\partial t} = - \partial_\eta [A] \frac{s}{t^2}$$

&

$$\partial_s [A] = \frac{1}{t} \partial_\eta A.$$

Plug this into equ. to obtain

$$- \partial_\eta [A] \frac{s}{t^2} + H(A) \frac{1}{t} \partial_\eta [A] = 0$$

$$\Leftrightarrow \quad H(A) \, \partial_\eta [A] = \frac{s}{t} \partial_\eta [A].$$

Hence, there are two options: 1) $\partial_\eta [A] = 0$

2) $s = H(A) t$.

Now, consider a method of characteristic.

Take $A = A(s,t)$ & $s = s(t)$. We get

$$\frac{d A(s(t),t)}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial s} \frac{\partial s}{\partial t}$$

To match (C), require that $\frac{\partial S}{\partial t} = H(A)$. Hence,
we have a system of eqns:

$$\frac{dA}{dt} = 0 \quad \& \quad \frac{\partial S}{\partial t} = H(A).$$

Integrating first we get A is constant, subject to second. ~~eqn~~. Hence, we can integrate ~~the~~ and get

$$S = \int H(A)dt = S_0 + H(A)t.$$

Hence, A is subject to initial condition:

$$A = \begin{cases} A_L, & \text{for } S_0 < 0 \\ A_R, & \text{for } S_0 \geq 0 \end{cases}$$

and we must have

$$A(S,t) = \begin{cases} A_L, & \text{for } S < H(A_L)t \\ A_R, & \text{for } S \geq H(A_R)t \end{cases}$$

However, there is a possible region $S \in [H(A_L)t, H(A_R)t)$ if $H(A_L) < H(A_R)$.

This is a so called rerefication wave and A is no longer constant in that region ($\partial_\eta A \neq 0$). Hence, we are in case 2) & A can be found from $H(A) = \frac{S}{t}$

On the other hand, $H(A_R) > H(A_L)$ gives a shockwave. Say that position of shockwave is $s_b(t)$. Then by integrating

$$\partial_t A + \partial_s F(A) = 0$$

$$\lim_{\varepsilon \to 0} \int_{s_b(t)-\varepsilon}^{s_b(t)+\varepsilon} \partial_t A(s,t)\, dx + F(A_R) - F(A_L) = 0$$

Apply Leibnitz' rule:

$$\lim_{\varepsilon \to 0} \frac{d}{dt}\left[\int_{s_b(t)-\varepsilon}^{s_b(t)+\varepsilon} A \, ds \right] + \frac{ds_b}{\partial s}\left(\lim_{\varepsilon \to 0} -A(s_b+\varepsilon, t) + A(s_b-\varepsilon, t) \right)$$

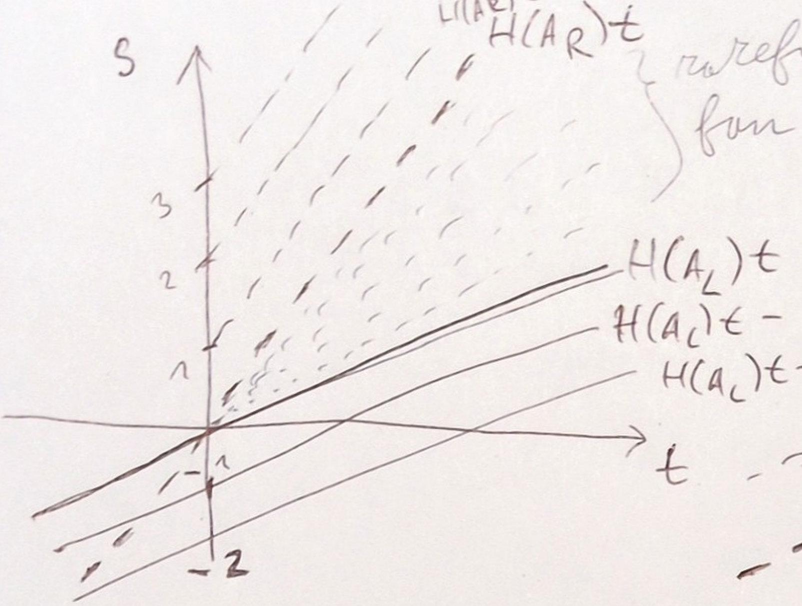
$$= F(A_L) - F(A_R)$$

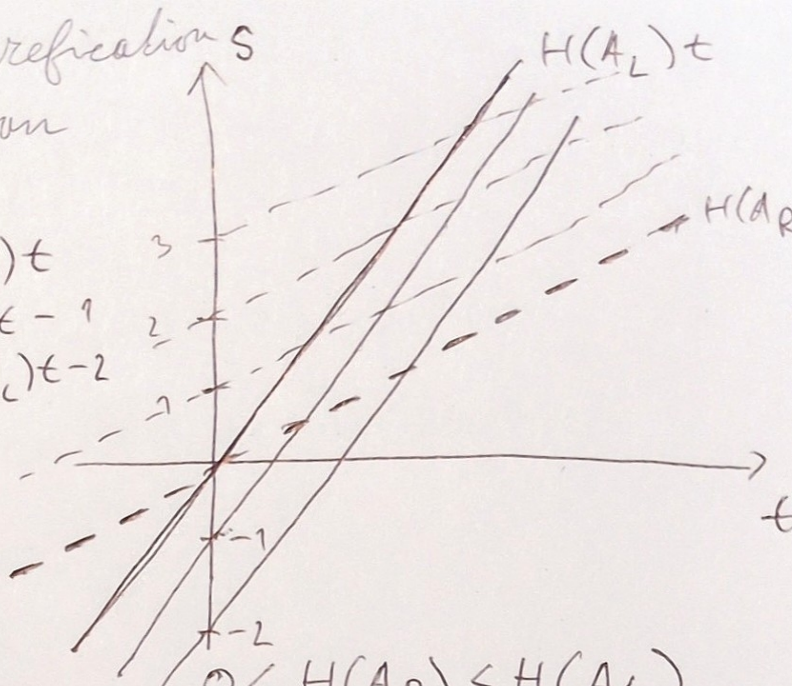
now 0 as it is ~~constant~~ ~~propagating~~ the same shape propagating thru space. Then

$$\frac{ds_b}{ds} = \frac{F(A_R) - F(A_L)}{A_R - A_L}.$$

$\cancel{m^3/s\ m^2}$

gives the shock speed.





$0 < H(A_L) < H(A_R)$

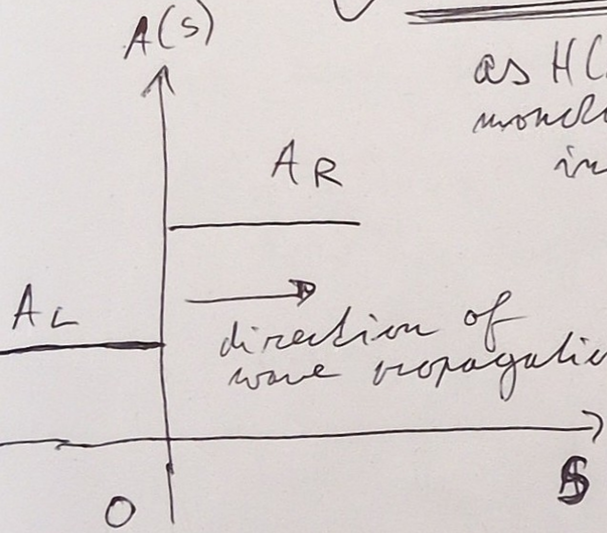
$0 < H(A_R) < H(A_L)$

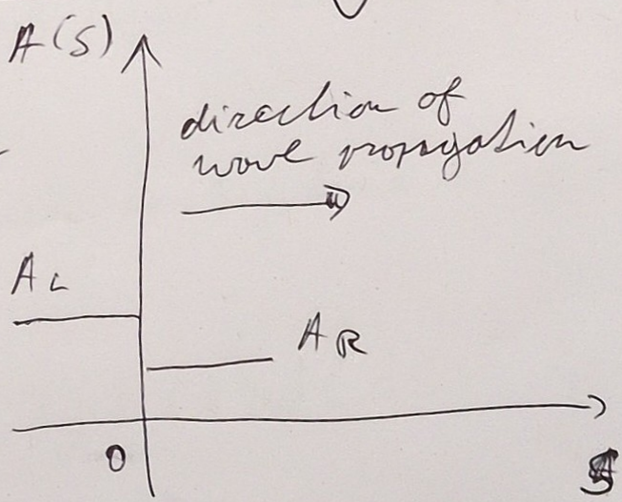
In this case ~~regi~~ characteristic lines are separated in 2 distinct regions

In this case we have crossing of characteristic lines for $t \geq 0$

$\Downarrow$ Solution at $t=0$ $\Downarrow$

as $H(A)$ is monotonically increasing for $A>0$.





A_L front is slower than A_R front. Hence, we have rarefication wave.

A_L front travels faster than A_R front $\Rightarrow$ shock waves & discontinuity is maintained

3) For $s=0$, we have

$$\partial_t A + \partial_s F(A) = 0. \quad (\heartsuit)$$

Now, define space discretization for $s \in [0, L]$ into N cells as follows:

$$C_k = [s_{k-\frac{1}{2}}, s_{k+\frac{1}{2}}]$$

where $s_{k-\frac{1}{2}}, s_{k+\frac{1}{2}}$ are cell boundaries of cell C_k and $h_k = s_{k+\frac{1}{2}} - s_{k-\frac{1}{2}}$ is the cell width.

There are N_t time intervals where

$$t \in [t_n, t_{n+1}] = I_n$$

for each time interval. Also denote $\Delta t_n = t_{n+1} - t_n$.

Now, integrate $\heartsuit$ in each time & space interval:

$$\int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} \int_{t_n}^{t_{n+1}} \partial_t A + \partial_s F(A) \, dt \, ds = 0$$

$$\int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} A^{n+1}(s) - A^n(s) \, ds + \int_{t_n}^{t_{n+1}} F(A(s_{k+\frac{1}{2}})) - F(A(s_{k-\frac{1}{2}})) \, dt$$

Rearrange this, ~~and~~ divide by Δs_k, and

~~denote~~ denote

$$\bar{A}_k^{n} = \frac{1}{\Delta s_k} \int_{s_{k-\frac{1}{2}}}^{s_{k+\frac{1}{2}}} A^n(s)\,ds = \frac{1}{\Delta s_k} \int_{C_k} A^n(s)\,ds$$

Hence, we get

$$\bar{A}_k^{n+1} - \bar{A}_k^{n} = -\frac{1}{\Delta s_k} \int_{I_n} F(A(s))\Big|_{C_k} dt$$

$$\bar{A}_k^{n+1} = \bar{A}_k^{n} - \frac{1}{\Delta s_k} \int_{I_n} F(A, s)\Big|_{C_k} dt$$

as required.

Q: Why are only cell averages at the previous time level n required in the Godunov method?

We require only cell averages, because the PDE was integrated and so we avoid considering exact values.

However, it also means that we only obtain averages $\bar{A}_k^{n+1}$ as the result, and <u>NOT</u> exact values.

4) Godunov Flux is given by:

$$\bar{F}_{k+\frac{1}{2}}(\bar{A}_k^n, \bar{A}_{k+1}^n) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F_{k+\frac{1}{2}}(\bar{A}_k^n, \bar{A}_{k+1}^n)\, dt$$

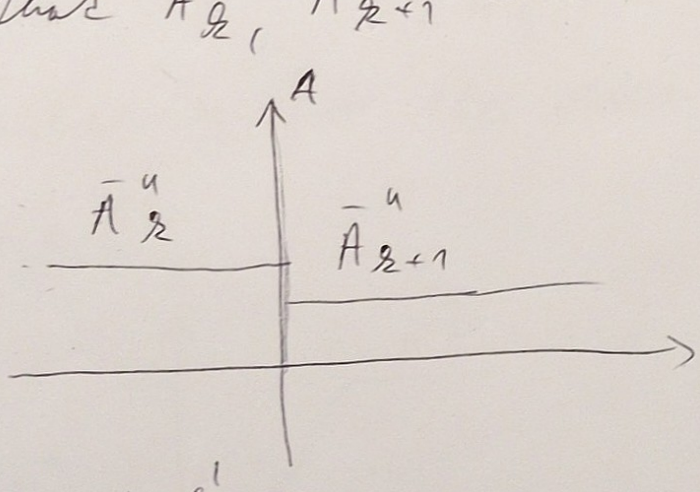
Now set up a Riemann problem in alternative coordinate system $s' = s - s_{k+\frac{1}{2}}$ $t' = t - t_n$. We have

$$\partial_{t'} A + \partial_{s'} F(A) = 0$$

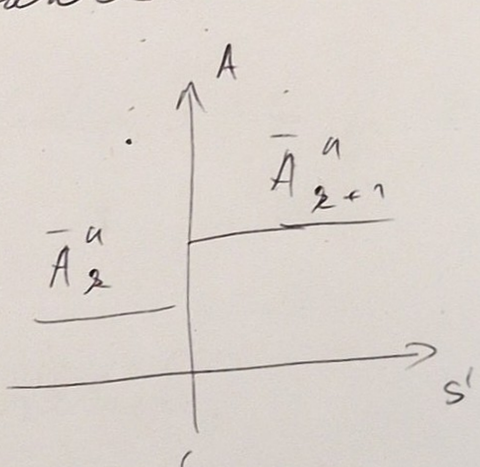
subject to initial condition

$$A(s', t'=0) = \begin{cases} \bar{A}_k^n, & \text{for } s' < 0 \\ \bar{A}_{k+1}^n & \text{for } s' > 0 \end{cases}$$

note that $\bar{A}_k^n$, $\bar{A}_{k+1}^n$ are constants.



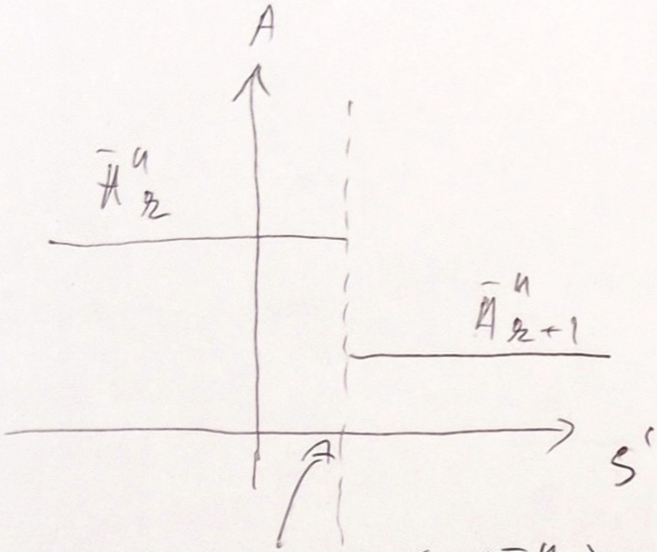
at $t' = 0$



at $t' = 0$

From exercise 2 we now that these are right traveling waves.

Hence, at $t' = \varepsilon$ for $\varepsilon > 0$ we get



$$s = \varepsilon\left(H(\bar{A}^n_r) - H(\bar{A}^n_{r+1})\right)$$

this is the shockwave solution for $H = \frac{\partial F}{\partial A}$ and

Godunov flux becomes

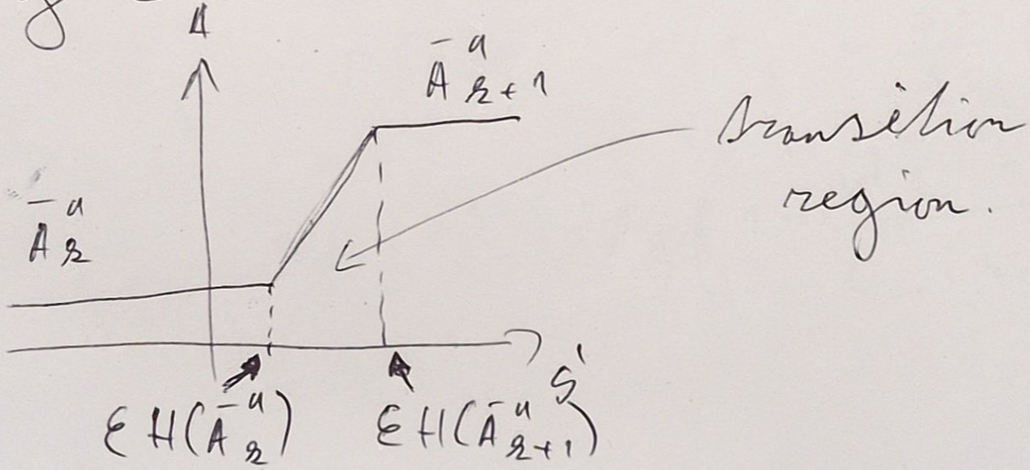
$$\bar{F}_{r+\frac{1}{2}}(\bar{A}^n_r) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F(\bar{A}^n_r)\, dt$$

$$= \frac{1}{\Delta t}\left(F(\bar{A}^n_r)(t_{n+1} - t_n)\right)$$

$$= F(\bar{A}^n_r).$$

Similarly, for the other solution $t' = \varepsilon$ for $\varepsilon = 0$

we get



However, at $s' = 0$ at $t' = t$ we get

$$\bar{F}_{k+\frac{1}{2}}(\bar{A}_k^n) = F(\bar{A}_k^n)$$

again. In both cases,

$$F(A) = \frac{A^{\frac{5}{3}}\sqrt{-\partial_s b}}{C_m\left(w_0 + \frac{2A}{w_0}\right)^{\frac{2}{3}}}$$

from exercise 1.

Hence, Godunov flux

$$\bar{F}_{k-\frac{1}{2}}(\bar{A}_{k-1}^n) = F(\bar{A}_{k-1}^n).$$

Therefore the final scheme becomes

$$\bar{A}_k^{n+1} = \bar{A}_k^n - \frac{\Delta t}{\Delta s_k}\left(F(\bar{A}_k^n) - F(\bar{A}_{k-1}^n)\right)$$

This is clearly an <u>upwind</u> scheme as past information is considered.

$$\bar{A}_k^{n+1} = F(A_{k-1}^n) + \left(\bar{A}_k^n - \frac{\Delta t}{\Delta s_k} F(\bar{A}_k^n)\right)$$

$$= F(\bar{A}_{k-1}^n) + \bar{A}_k^n\left(1 - \frac{(\bar{A}_k^n)^{\frac{2}{3}}\sqrt{-\partial_s b}}{C_m\left(w_0 + \frac{2A}{w_0}\right)^{\frac{2}{3}}}\frac{\Delta t}{\Delta s_k}\right)$$

5) $$\min_k \frac{h_k}{|\lambda_k|} = \min_k \frac{h_k}{\dfrac{\sqrt{-\partial_s \sigma}\,\left(5 w_0 \bar{A}_k^{2/3} + 6 \bar{A}_k^{5/3}/w_0\right)}{3 c_m \left(w_0 + 2\bar{A}_k^n / w_0\right)^{5/3}}}$$

$$= \min_k \frac{h_k}{\left.\dfrac{\partial F}{\partial A}\right|_{A=\bar{A}^n_k}}$$

as $\frac{\partial F}{\partial A}$ is monotonically increasing, we can simply pick $\max_k \left.\frac{\partial F}{\partial A}\right|_{A=\bar{A}^n_k}$ & $\min h_k$. This however could be too restrictive.

In particular,

$$\lambda_k = H(A)\Big|_{\bar{A}^n_k} = \left.\frac{\partial F}{\partial A}\right|_{A=\bar{A}^n_k}$$

is the speed of the wave and we are trying to prevent adjacent Riemann problems from communicating with each other. Hence, impose

$$\Delta t < CFL \min_k \frac{h_k}{|\lambda_k|}$$

6) It makes sense to impose $\bar{A}_{-1} > 0$

& $\bar{A}_{N_k} > 0$ to maintain physicality.

For Godunov fluxes

$$\bar{F}_{k+\frac{1}{2}}\left(\bar{A}^n_k, \bar{A}^n_{k+1}\right) = \frac{1}{\Delta t}\int_{t_n}^{t_n+\Delta t} F_{k+\frac{1}{2}}\left(\bar{A}^n_k, \bar{A}^n_{k+1}\right) dt$$

$$= F\left(\bar{A}^n_k\right)$$

if and only if CFL condition is satisfied to prevent Riemann solutions from cells further away affecting the integral.

This of course depends on the wave propagation speed $\lambda = H(A) = \frac{\partial F(A)}{\partial A}$ in each cell.

~~Furthermore require $\bar{A}_{-1} < \bar{A}_{N_k}$ to prevent formation of shockwaves.~~

15/20 (explanations of numerics missing)