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1 We have a system defined by

$$\partial_t b + \partial_z u b = 0 \quad \text{and} \quad u = \alpha b^2 - \beta b^2 \partial_z b \quad (1)$$

for $z \in [0, H]$, $\alpha, \beta > 0$ and Dirichlet Boundary Conditions (BCs)

$$b(0,t) = b_B \quad \text{and} \quad b(H,t) = b_T$$

for $t > 0$. We have Initial Conditions (ICs)

$$b(z,0) = b_i(z).$$

a) We consider the system (1) and substitute the u value from the second equation to obtain

$$\partial_t b + \partial_z(\alpha b^3 - \beta b^3 \partial_z b) = 0. \quad (2)$$

We consider $b = D_0 + b'$ for some small b' and constant D_0 such that (2) becomes

$$\partial_t b' + 3\alpha(D_0 + b')^2 \partial_z b' - 3\beta(D_0+b')^2(\partial_z b')^2$$
$$- \beta(D_0+b')^3 \partial_z^2 b' = 0.$$

As b' is small (relative to D_0) and the scale of z is much larger than that of b, giving $\partial_z b \ll 1$, we have further

$$\partial_t b' + 3\alpha D_0^2 \partial_z b' - \beta(D_0 + b')^3 \partial_z^2 b' = 0. \quad (3)$$

We label (2) & (3) convection-diffusion equations as they have convective term

$$\partial_z b'$$

and diffusive term

$$\partial_z^2 b'.$$

b) We discretise (2) by considering each component. First we define Δt and Δz respectively the time step and grid spacing such that

$$t_n - t_{n-1} = \Delta t$$

$$z_j - z_{j-1} = \Delta z.$$

defining $b_j^n = b(z_j, t_n)$.

We the consider the discrete approximation

$$\partial_t b \approx \frac{b_j^{n+1} - b_j^n}{\Delta t}.$$

As we are considering a forward Euler scheme the following approximations are explicit (solely depending on data from time step n to determine data at time step $n+1$).

For the first order component we consider an upwind scheme with

$$\partial_z \alpha b^3 \approx \alpha\left[\frac{(b_j^n)^3-(b_{j-1}^n)^3}{\Delta z)}\right]$$

and for the second order component we utilise the adjoint method first defining

$$F_{j+\frac{1}{2}}^n = (b_{j+\frac{1}{2}}^n)^3 \frac{b_{j+1}^n - b_j^n}{\Delta z}$$

an approximation of $b^3 \partial_z b$ where $b_{j+\frac{1}{2}}^n = \frac{1}{2}(b_{j+1}^n + b_j^n)$. We further have that

$$\partial_z(-\beta b^3 \partial_z b) \approx -\beta \partial_z F$$

$$\approx -\beta\left(\frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta z}\right).$$

This leads to full discretization

$$\frac{b_j^{n+1}-b_j^n}{\Delta t} + \alpha\left[\frac{(b_j^n)^3-(b_{j-1}^n)^3}{\Delta z}\right]$$

$$-\beta\left[\frac{(b_{j+1}^n+b_j^n)^3(b_{j+1}^n-b_j^n)-(b_j^n+b_{j-1}^n)^3(b_j^n-b_{j-1}^n)}{8(\Delta z)^2}\right]$$

$$= 0. \qquad (4)$$

We can similarly discretise (3) considering similarly defined time and grid spacing on a field b' of perturbation such that

$$\partial_t b' \approx \frac{(b')_j^{n+1} - (b')_j^{n}}{\Delta t}.$$

The linear component is approximated

$$3\alpha D_0^2 \partial_z b' \approx 3\alpha D_0^2 \left(\frac{(b')_j^n - (b')_{j-1}}{\Delta z}\right)$$

using an upwind scheme and we have

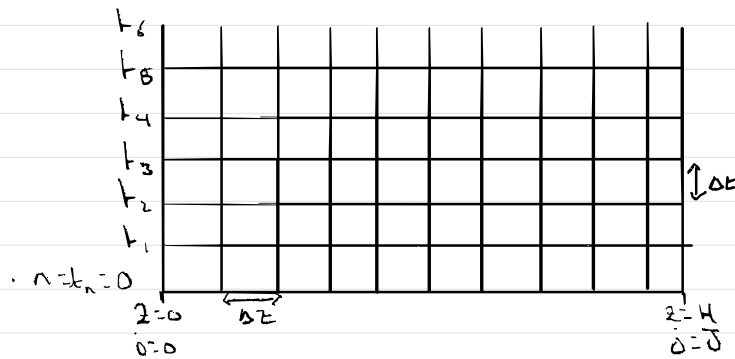
$$-\beta D_0^3 \partial_z^2 b' \approx -\beta D_0^3 \frac{\left((b')_{j+1}^n - 2(b')_j^n + (b')_{j-1}^n\right)}{(\Delta z)^2}$$

from the second order central difference. This results in

$$\frac{(b')_j^{n+1} - (b')_j^{n}}{\Delta t} + 3\alpha D_0^2 \left(\frac{(b')_j^n - (b')_{j-1}}{\Delta z}\right)$$

$$-\beta D_0^3 \frac{\left((b')_{j+1}^n - 2(b')_j^n + (b')_{j-1}^n\right)}{(\Delta z)^2} = 0. \quad (5).$$

The grid is given



If we consider $(b')_j^n = b_j^n - D_0$ then we can linearise (4) noting that the expansion

$$(b_j^n)^3 - (b_{j-1}^n)^3 \approx D_0^2((b')_j^n - b_{j-1}^n)$$

after D_0^3 terms cancel and we assume domination of the higher order D_0 term. We have also that

$$(b_j^n + b_{j+1}^n)^3 \approx (2D_0)^3$$

giving the result of linearisation on (4)

$$\frac{(b')_j^{n+1} - (b')_j^n}{\Delta t} + \frac{3\alpha D_0^2((b')_j^n - (b')_j^n)}{\Delta z}$$

$$- \frac{\beta D_0^3((b')_{j+1}^n - 2(b')_j^n + (b')_{j-1}^n)}{\Delta z}$$

corresponding with (5).

The adjoint discretisation form does not require the introduction of an extra term resulting from the chain rule which would also have to be discretised, further complicating the scheme.

The boundary conditions dictate

$$b_j^0 = b_i(z_j)$$

and for $n > 0$ that

$$b_0^n = b_B \quad \text{and} \quad b_J^n = b_T$$

where $z_J = H$ and $H/J = \Delta z$.

As the scheme is explicit the order of evaluation of spatial points is not relevant and we will only use the scheme to evaluate

$$b_j^n \quad \text{for } j = 1, \dots, J-1.$$

c) We assume that the solution $b'(z,t)$ has some Fourier Decomposition of form

$$(b')_j^n = \sum_{k=-\infty}^{\infty} c_k (\lambda(k))^n e^{ik(j\Delta z)}.$$

We consider a single mode and note that it must satisfy the linearized (5) such that

$$\frac{\lambda - 1}{\Delta t} + \frac{3\alpha D_0^2 (1 - e^{-ik\Delta z})}{\Delta z}$$

$$- \beta D_0^3 \frac{(e^{ik\Delta z} - 2 + e^{-ik\Delta z})}{(\Delta z)^2} = 0$$

after division by $(b')_j^n$. As such, we have

$$\lambda(k) = 1 - \Delta t \left[\frac{3\alpha D_0^2}{\Delta z} (1 - e^{-ik\Delta z}) - \frac{\beta D_0^3}{(\Delta z)^2} \left(e^{\frac{1}{2}ik\Delta z} - e^{-\frac{1}{2}ik\Delta z} \right)^2 \right.$$

$$= 1 - \Delta t \left[\frac{3\alpha D_0^2}{\Delta z} (1 - e^{-ik\Delta z}) + \frac{4\beta D_0^3}{(\Delta z)^2} \sin^2 \tfrac{1}{2} k \Delta z \right]$$

If $\alpha = 0$, $\beta \neq 0$ then we have

$$\lambda(k) = 1 - \frac{4\Delta t \beta D_0^3}{(\Delta z)^2} \sin^2\left(\tfrac{1}{2} k \Delta z\right). \leq 1.$$

For stability, we then only require $|\lambda(k)| \leq 1$ which holds if

$$\frac{4\Delta t}{(\Delta z)^2} \beta D_0^3 \leq 2$$

$$\Rightarrow \quad \Delta t \leq \frac{(\Delta z)^2}{2\beta D_0^3}.$$

If $\alpha \neq 0$, $\beta = 0$ then

$$\lambda(k) = 1 - \frac{3\alpha \Delta t D_0^2}{\Delta z}\left(1 - e^{-ik\Delta z}\right)$$

Considering the maximum $|\lambda(k)|$ which is either attained at $k\Delta z = 0$ or π* we have

* See next page.

$$\max |\lambda(k)| = \max\left\{1, \left|1 - 3\alpha \frac{\Delta t}{\Delta z} D_0^2 (2)\right|\right\}$$

and so we can say $|\lambda(k)| \leq 1$ if

$$1 - 6\alpha \frac{\Delta t}{(\Delta z)^2} D_0^2 \geq -1$$

$$\Leftrightarrow \quad \Delta t \leq \frac{(\Delta z)^2}{3\alpha D_0^2}.$$

Considering $|\lambda(k)| \leq 1$ we have

$$|\lambda(k)| = |1 - a(1 - e^{-ik\Delta z})|$$

for $a = 3\alpha \Delta t D_0^2 / \Delta z$. Further

$$|\lambda(k)| = |[1 - a(1 + \cos k\Delta z)] - ai\sin k\Delta z|.$$

It is sufficient that $|\lambda(k)|^2 \leq 1$ so we consider

$$|\lambda(k)|^2 = [1 - a(1 + \cos k\Delta z)]^2 + [a\sin k\Delta z]^2.$$

Letting $\theta = k\Delta z$ we have

$$\partial_\theta |\lambda(k)|^2 = 2(1 - a(1 + \cos(\theta)))a\sin\theta$$

$$+ 2a^2\sin\theta\cos\theta$$

$$= (2 - 2a)a\sin\theta$$

giving maximum/minimum values at $\theta = 0, \pi$.

d) We can derive an update scheme from (5)

$$(b')_j^{n+1} = (b')_j^n - \Delta t\left(3\alpha D_0^2\left[(b')_j^n - (b')_{j-1}^n\right]/\Delta z - \beta D_0^3\left[(b')_{j+1}^n + (b')_{j-1}^n - 2(b')_j^n\right]/(\Delta z)^2\right)$$

We consider the coefficients of each data point with

$$(b')_j^{n+1} = (b')_j^n\left(1 - \frac{\Delta t}{\Delta z}\left(3\alpha D_0^2 - \frac{2\beta D_0^3}{\Delta z}\right)\right)$$

$$+(b')_{j+1}^n\left(\frac{\beta D_0^3}{(\Delta z)^2}\right)$$

$$+(b')_{j-1}^n\left(\frac{\Delta t}{\Delta z}\left(3\alpha D_0^2 + \frac{\beta D_0^3}{\Delta z}\right)\right)$$

where the second two coefficients are clearly greater than zero and we have the sum of coefficients equal to 1. To apply the maximum principle we must then only have

$$1 - \frac{\Delta t}{(\Delta z)^2}\left(3\alpha D_0^2 \Delta z - 2\beta D_0^3\right) \geq 0$$

$$\Rightarrow \Delta t \leq \frac{(\Delta z)^2}{3\alpha D_0^2 \Delta z - 2\beta D_0^3}$$

e) We aim to derive a suitable time step criterion for $\Delta t_n = t_{n+1} - t_n$ such that $B_j^{n+1} > 0$ if $B_j^n > 0$.

We consider (4) and find the update scheme

$$b_j^{n+1} = b_j^n - \Delta t_n \left[\frac{a_n}{\Delta z} - \frac{d_n}{(\Delta z)^2} \right]$$

where

$$a_n = \alpha \left[(b_j^n)^3 - (b_{j-1}^n)^3 \right]$$

$$d_n = \frac{\beta}{8} \left[(b_{j+1}^n + b_j^n)^3 (b_{j+1}^n - b_j^n) - (b_j^n + b_{j-1}^n)^3 (b_j^n - b_{j-1}^n) \right].$$

We require

$$b_j^n - \Delta t_n [a_n / \Delta z - d_n / (\Delta z)^2] > 0$$

$$\Rightarrow \quad \Delta t_n < \frac{(\Delta z)^2 b_j^n}{a_n \Delta z - d_n}.$$

f) We consider an explicit central approximation of

$$\partial_z b_j^n \approx \frac{b_{j+1}^n - b_{j-1}^n}{2\Delta z}$$

using this to discretize (2) and finding update scheme

$$b_j^{n+1} = b_j^n - \Delta t_n \left(\frac{c_n}{\Delta z} - \frac{d_n}{(\Delta z)^2} \right)$$

where d_n is as previous and

$$c_n = \frac{b_{j+1}^n - b_{j-1}^n}{2}$$

with variable time step criterion

$$\Delta t_n < \frac{(\Delta z)^2 b_j^n}{c_n \Delta z - d_n}$$

2 (a) Given that we are at some steady state

$$\partial_t = 0$$

then we have, from (2), that

$$\partial_z(\alpha b^3 - \beta b^3 \partial_z b) = 0$$

$$\Rightarrow \quad \alpha b^3 - \beta b^3 \partial_z b = Q$$

for some $Q \in \mathbb{R}$.

d) We consider the derivatives of the given exact first with respect to time giving

$$-\alpha^2 = \partial_t b \left(1 - \frac{1}{1-b^2}\right)$$

where $\alpha = c$ and $\beta = 1$. We then have

$$\partial_t b = -\alpha^2 (1 - b^{-2}).$$

We then consider the derivative w.r.t. z finding

$$\alpha = \partial_z b \left(1 - 1/1-b^2\right)$$

$$\Rightarrow \partial_z b = \alpha (1 - b^{-2}).$$

We have

$$u = \alpha b^2 - b^2 \alpha (1 - b^{-2})$$

$$= \alpha$$

and that the full eq^n (1) gives

$$-\alpha^2 (1 - b^{-2}) + \alpha (\alpha (1 - b^{-2})) = 0$$

and the sol^n satisfies the equations.

We note that

$$\operatorname{arctanh} x \approx x + \frac{x^3}{3} + \frac{x^5}{5}$$

and that the initial condition gives, after scaling such that $H = 1$,

$$b(0.3, 0) = 0$$

$$\Rightarrow \quad z_{r0} = 0.3.$$

We can then consider values such that $b = 0$ noting the travelling wave nature results in

$$z = 0.3 + \alpha t.$$

this being the height above which the data is dead.

Further, we can use the arctanh approximation above to find

$$z - 0.3 - \alpha t \approx \frac{1}{\alpha}\left(-\frac{b^3}{3}\right)$$

$$\Rightarrow \quad b \approx \left[3\alpha(\alpha t + 0.3 - z)\right]^{\frac{1}{3}}$$

which we use to define the initial condition with

$$b(z, 0) = \left[3\alpha(0.3 - z)\right]^{2}.$$

We define boundary conditions using these same approximations such that

$$b(0,t) \approx \left[3\alpha^2\left(t + \frac{0.3}{\alpha}\right)\right]^{\frac{1}{3}}$$

$$b(1,t) \approx \left[3\alpha^2\left(t - \frac{0.7}{\alpha}\right)\right]^{\frac{1}{3}}$$

where $b(1,t)$ is valid for $t \geq 0.7/\alpha$ and is 0 for $t \in [0, 0.7/\alpha]$.

Note that in the end we did not use this approximation instead using the exact solution to create a map

$$z(b,t)$$

that could be searched for a range of values of z and thus "inverted" at high enough resolution.

2e)