On a novel wave-to-wire energy device in a breakwater contraction

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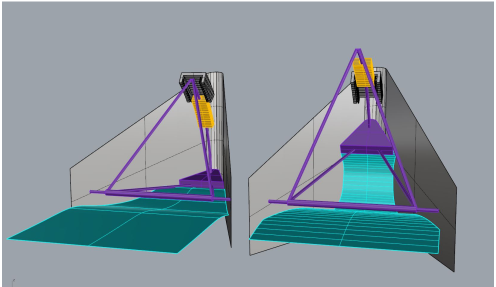


Introduction: maths wave-2-wire device

Goals & challenges:

- introduce novel wave-energy device;
- provide nonlinear model for new wave-2-wire device;
- provide linearised shallow-water model of device;
- discretise in space, in time & in space-time;
- show preliminary simulations; and,
- open questions: validation, optimisation, ...

Motivation: rogue waves

Rogue waves are anomalously high waves defined relative to a significant wave height H_s.

- Index (Kharif et al. '09, Dysthe et al. '08):

$$AI = H_{rw}/H_s > 2 \qquad \text{or} \qquad AI = \eta_{rw}/H_s > 1.25$$

- Relevance in maritime & coastal engineering
- Pyramidal rogue wave (Faulkner 2001):

Fig.1. Pyramidal wave off south Japan

Motivation: rogue waves

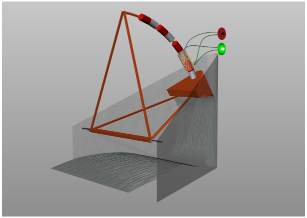
There are many causes of rogue waves, see Kharif et al. (2009):

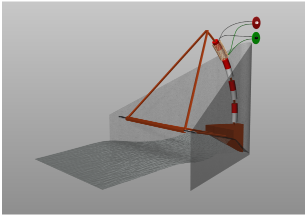
- E.g., crossing seas, nearly standing/pyramidal waves
- We made a *bore-soliton splash* rogue wave: $AI = \frac{H_{rw}}{H_s} = \frac{3.5}{0.35} \approx 10$
- It inspired our wave-2-wire wave-energy device.

Introduction: working wave-2-wire device

Design by B. & Zweers:

- Sketch with water waves, wave-activated buoy motion & electro-magnetic generator & **LED**-loads:





- *Proof-of-principle* of laboratory device (2013) `https://www.youtube.com/watch?v=SZhe_SOxBWo`

Wave-to-wire energy device

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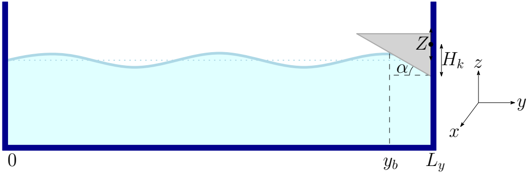
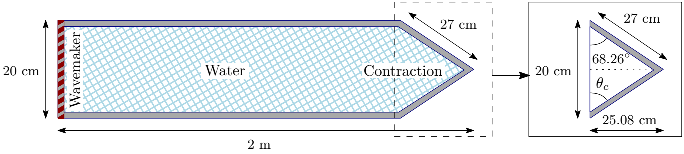
Modelling

- Wave-2-wire model in a wavetank, side view along center-line:



- Wave-2-wire model in a wavetank, top view:



Nonlinear wave-2-wire model

Dynamics of water waves $\{\phi, h\}$, wave-activated buoy motion $\{Z, W\}$ & electro-magnetic generator $\{Q, P_Q\}$ with current I, $I = \dot{Q}$ and $P_Q = L_i I - K(Z)$ & **LED**-loads:

$$\nabla^2 \phi = 0$$

$$\partial_t h + \nabla_H \phi \cdot \nabla h - \partial_z \phi = 0 \text{ on } z = h$$

$$\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + g(h - H_0) + \underline{\lambda \Theta(y - y_b)} = 0 \quad \text{on} \quad z = h$$

$$\underline{h - h_b} = 0 \quad \text{for} \quad y \geq y_b(x, t)$$

$$\dot{Z} = W$$

$$M\dot{W} = -Mg - \gamma\big(P_Q + K(Z)\big)\frac{G(Z)}{L_i} + \rho_0 \int_0^{L_x}\int_0^{l_y(x)} \underline{\lambda \Theta\left(y - y_b\right)}\, \mathrm{d}x\, \mathrm{d}y$$

$$\dot{Q} = \big(P_Q + K(Z)\big)/L_i \equiv I$$

$$\dot{P}_Q = -(R_c + R_i)I - \underline{\underline{\text{sign}(I) n_q V_T \ln\left(|I|/I_{sat} + 1\right)}}.$$

Nonlinear wave-2-wire model: hydrodynamics

Variables & parameters water waves:

- potential flow $\mathbf{u} = \nabla\phi(x, y, z, t)$
- (single-valued) water depth $h = h(x, y, t)$, surface potential $\phi_s(x, y, t) = \phi(x, y, h(x, y, t), t)$

$$\nabla^2\phi = 0$$

$$\partial_t h + \nabla_H\phi \cdot \nabla h - \partial_z\phi = 0 \text{ on } z = h$$

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g(h - H_0) + \underline{\lambda\Theta(y - y_b)} = 0 \quad \text{on} \quad z = h$$

$$\underline{h - h_b} = 0 \quad \text{for} \quad y \geq y_b(x, t)$$

- acceleration of gravity g and vertical (free-surface) rest level H_0
- (single-valued) waterline $y = y_b(x, t)$.

Nonlinear wave-2-wire model: buoy motion

Variables & parameters:

- wave-activated buoy: position $Z(t)$ & velocity $W(t)$

$$\underline{h - h_b} = 0 \quad \text{for} \quad y \geq y_b(x,t)$$

$$\dot{Z} = W$$

$$M\dot{W} = -Mg - \gamma\left(P_Q + K(Z)\right)\frac{G(Z)}{L_i} + \rho_0 \int_0^{L_x}\int_0^{l_y(x)} \underline{\lambda\Theta\left(y - y_b\right)}\,\mathrm{d}x\,\mathrm{d}y$$

- *Lagrange multiplier* $\lambda(x,y,t)$ imposes that water depth h equals wetted buoy shape h_b:

$$\underline{h(x,y,t) - h_b\left(x,y;Z(t)\right)} = 0$$

Nonlinear wave-2-wire model: EM motor

Variables & parameters:

- electromagnetic generator: charge $Q(t)$ & conjugate $P_Q(t) = L_i\dot{Q} - K(Z)$; $\dot{P}_Q(t) = \underline{L_i\dot{I}} - \gamma G(Z)\dot{Z}$
- EM-theory $K(Z) = \int^Z \gamma G(\tilde{Z})\mathrm{d}\tilde{Z}$ thin-wire & symmetric
- induction L_i, combined buoy-magnet mass M
- **two LEDs** in parallel: adapted Shockley equation.

$$\dot{Q} = \big(P_Q + K(Z)\big)/L_i \equiv I$$
$$\dot{P}_Q = -(R_c + R_i)I - \underline{\underline{\mathrm{sign}(I)n_q V_T \ln\big(|I|/I_{sat} + 1\big)}}.$$

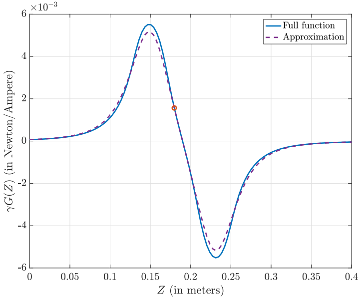
Wave-2-wire model: EM motor

- By solving the Maxwell's equations, the function

$$G(Z) = \frac{\pm 1}{\pi A_m^2 L_m a}\int_{-L/2}^{L/2}\int_0^{2\pi}\int_0^{A_m} F(z)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}q$$

- arises with $F(z) = f(-z) - f(z), z = q + \bar{Z} + \alpha_h H_m - Z$,

$$f(z) = r(a - r\cos\theta)/(r^2 + (\frac{L_m}{2} + z)^2 + a^2 - 2ra\cos\theta)^{3/2}.$$



Wave-2-wire model

- When the change of current I in time & loads small, s.t. $\dot{I} \approx 0$:

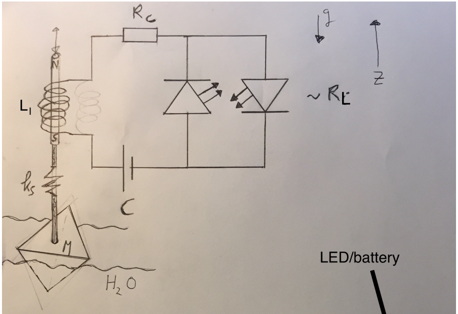
$$(R_c + R_i + \underline{R_l})I \approx \gamma G(Z)\dot{Z},$$

- then the vertical momentum equation for buoy, mast and magnet becomes approximately

$$M\dot{W} = -Mg - \frac{\gamma^2 G^2(Z)}{L_i(R_c + R_i + \underline{R_l})} W + \rho_0 \int_0^{L_x} \int_0^{l_y(x)} \lambda \Theta\left(y - y_b\right) \mathrm{d}x\, \mathrm{d}y.$$

- *Resistance and loads combined act as drag* on the buoy-mast-magnet unit in this partial linear limit.

Wave-2-wire model: circuits



$$\dot{Q} = I$$

$$\dot{P}_Q = L_I \dot{I} - \gamma G(Z)\dot{Z} = -R_c I - \frac{Q}{C} + V_{LED}(Q,I)$$

$$M\dot{Z} = MW$$

$$M\dot{W} = -Mg - k_s Z - \gamma G(Z) I + F_{hydrodynamic}$$

$$I = (P_Q + K(Z))/L_I$$

$$K(Z) = \underbrace{\int^{Z} \gamma\, G(\tilde{Z})\, d\tilde{Z}}_{} \rightarrow \text{3D symmetric EM theory}$$

Wave-2-wire model: queries

- Charge equation not needed (Bolton), unless a capacitor is added and/or a battery depending on Q, I.
- Mathematical model for battery (Li & Ke, 2011):
  - Shepherd and modified Shepherd models $V_{battery}(Q, I)$.
- Addition of spring with adjustable spring constant in motor & model for active control of resonance?
- Originally, we used schematic of Wellstead (2000), but that model does not linearise?

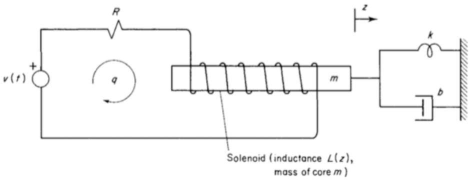


Figure 7.25

Wave-2-wire variational principle

Wave-to-wire model posited using 3 combined VPs:

- Luke (1967), Cotter & B. (2010, JEM), $1+1+1=3$:

$$0=\delta\int_0^T \mathcal{L}[D,\phi,h,\phi_s,Z,W,Q,P_Q,p,\lambda]\mathrm{d}t$$

$$\equiv\delta\int_0^T \rho_0\int_0^{L_x}\int_{R(t)}^{l_y(x)}\int_0^{h(x,y,t)} D\partial_t\phi\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x - MW\dot{Z} - P_Q\dot{Q}+\mathcal{H}\,\mathrm{d}t$$

$$\equiv\delta\int_0^T \rho_0\int_0^{L_x}\int_{R(t)}^{l_y(x)}\int_0^{h(x,y,t)} D\partial_t\phi+\frac{1}{2}D|\nabla\phi|^2+gD\left(z-H_0\right)$$

$$+p(D-1)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$+\rho_0\int_0^{L_x}\int_{R(t)}^{l_y(x)}\lambda(h-h_b)\Theta\left(y-y_b(x,t)\right)\mathrm{d}x\,\mathrm{d}y$$

$$\overline{-MW\dot{Z}-P_Q\dot{Q}+\frac{1}{2}MW^2+MgZ+\frac{1}{2}\frac{\left(P_Q+K(Z)\right)^2}{L_i}\,\mathrm{d}t}$$

Linearise wave-2-wire equations

Linearise & simplify to shallow water with only (x, y):

$$\eta - \tilde{Z} = 0 \quad \text{for} \quad y \geq L_b, \qquad \dot{R} = \partial_y \tilde{\phi} \quad \text{at} \quad y = 0$$

$$\partial_t \eta + \nabla \cdot (H \nabla \tilde{\phi}) = 0, \partial_t \tilde{\phi} + g\eta + \tilde{\lambda} \Theta(y - L_b) = 0$$

$$\dot{\tilde{Z}} = \tilde{W}$$

$$M \dot{\tilde{W}} + \gamma G(\bar{Z}) \tilde{I} - \rho_0 \int_0^{L_x} \int_0^{l_y(x)} \tilde{\lambda} \Theta(y - L_b) \, \mathrm{d}y \, \mathrm{d}x = 0$$

$$\dot{\tilde{Q}} = \tilde{I}, \quad \dot{\tilde{P}}_Q = -\left(R_c + R_i + \frac{n_q V_T}{I_{sat}}\right) \tilde{I}, \quad \tilde{I} = \frac{\left(\tilde{P}_Q + \gamma G(\bar{Z}) \tilde{Z}\right)}{L_i}$$

$$\Longrightarrow \nabla \cdot (H \nabla \tilde{\lambda}) - \frac{\rho_0}{M} \int_0^{L_x} \int_0^{l_y(x)} \tilde{\lambda} \Theta(y - L_b) \, \mathrm{d}y \, \mathrm{d}x = - \nabla \cdot (gH \nabla \eta) - \frac{\gamma}{M} G\left(\bar{Z}\right) \tilde{I} \quad \text{for} \quad y \geq L_b.$$

Discrete wave-2-wire equations

- Finite elements in space CG1, symplectic Euler in time.
- Consistent discretisation.

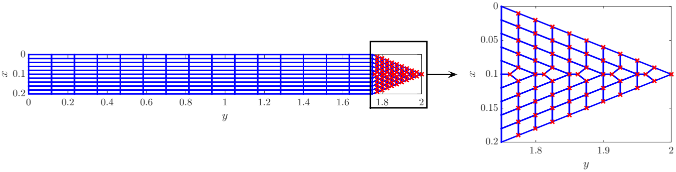
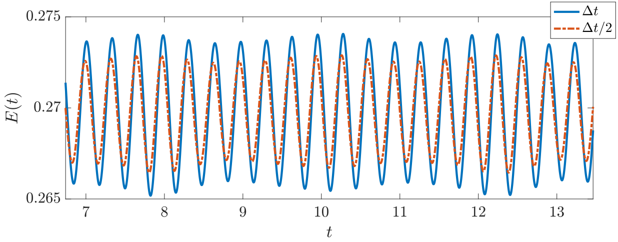


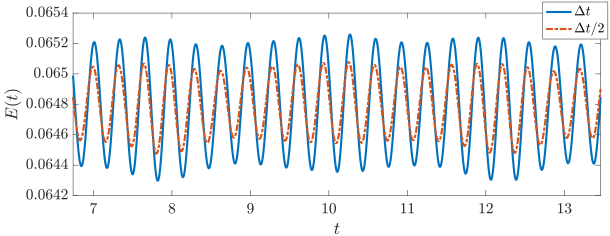
Figure: Computational mesh for $N_x = 10$, $N_y = 15$.

- For Δx & $\Delta x/2$: $O(\sqrt{\Delta t})$ & $O(\Delta t^{3/4})$ & $O(\Delta x^{1.7})$.
- Energy partitioning over subsystems consistent.

Convergence in time

For Δx & $\Delta x/2$, we find $O(\sqrt{\Delta t})$ & $O(\Delta t^{3/4})$:





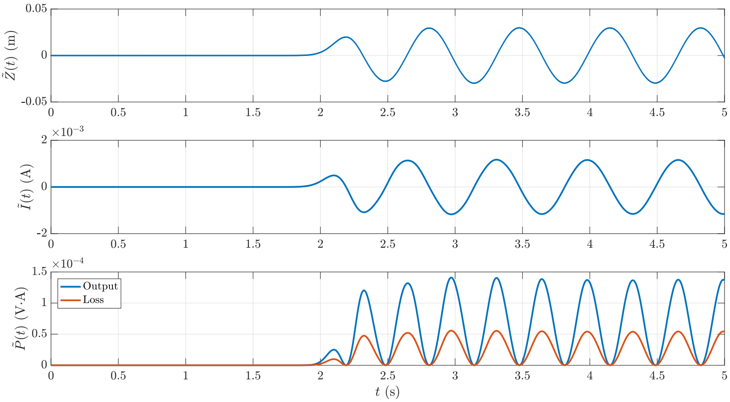
Convergence in space

Table: Convergence rates n using three different norms ($\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_\infty$) evaluated using the value of the velocity potential $\tilde{\phi}$ at the final time of the simulation, i.e. at $t = T$.

	x-el	y-el	Total el	Nodes	Rate
Symbol	N_x	N_y	N_k	N_n	n
Mesh 1	6	30	201	241	$\mathcal{L}_1$: 1.711293
Mesh 2	12	60	798	877	$\mathcal{L}_2$: 1.696554
Mesh 3	24	120	3180	3337	$\mathcal{L}_\infty$: 1.765833

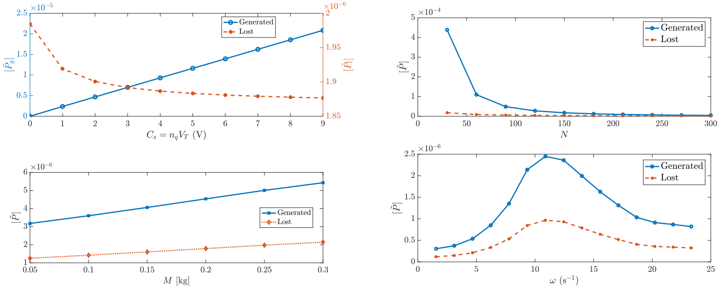
Preliminary results full model

Vertical displacement, current & total power $(R_c + R_i + \underline{n_q V_T / I_{sat}})\tilde{I}^2$:



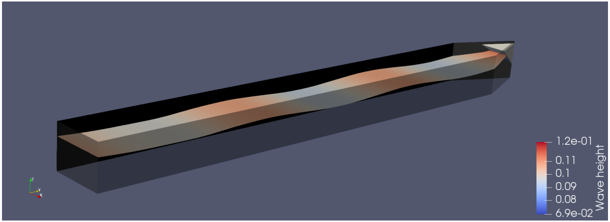
Preliminary results

Optimal harvest in linear case at resonance, maximum loads and minimum windings:



Conclusions & Future work

- Full wave-2-wire maths model from 1^{st} *principles.*
- Established space-time compatible discretisation of linearised model.
- Explored parameter space in preliminary fashion.
- Leeds' Fluid Dynamics CDT project with *J. Bolton*:
  - nonlinear Benney-Luke hydrodynamic model (2DH)
  - faster than 3D potential-flow for optimization of parameter space (geometries, mass, spring constant, loads)



Conclusions & Future work

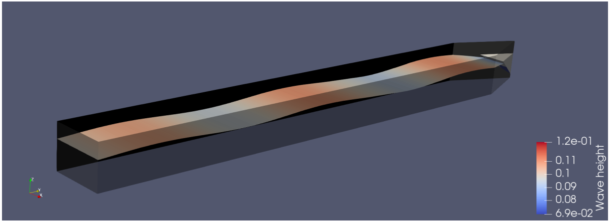
- Validation! Preferred location: breakwater.
- Discretise to higher-order in space & time.
- Numerically implement nonlinear models.
- Optimise wave-2-wire device wrt energy-output efficiency.

- B., Kalogirou & Zweers 2019: On wave-2-wire model. *Water Waves* **1**.

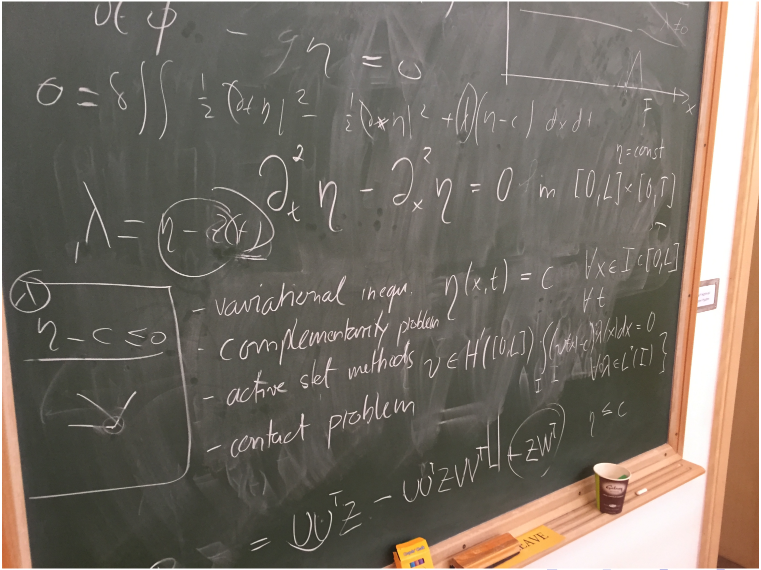
  https://link.springer.com/article/10.1007/s42286-019-00022-9

- B., Kalogirou, Henry & Thomas 2020: *Int. Marin Energy J.* **30** (EWTEC2019, Napoli)

- Bolton et al. 2022: EWTEC2021, Southampton, pending.



Thank you!


- variational inequ.
- complementarity problem
- active set methods
- contact problem

Appendix Wave-2-wire VP

Vary & get eqns of motion (conj. pairs + multipliers):

$$\delta D: \quad \partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + g(z - H_0) + p = 0 \tag{13a}$$

$$\delta \phi: \quad \partial_t D + \nabla \cdot (D \nabla \phi) = 0 \tag{13b}$$

$$\delta p: \quad D = 1 \tag{13c}$$

$$\delta \lambda: \quad h - h_b = 0 \quad \text{for} \quad y \geq y_b(x,t) \tag{13d}$$

$$\delta \phi_s: \quad \partial_t h + \nabla\phi \cdot \nabla h = \partial_z \phi \quad \text{at} \quad z = h(x,y,t) \tag{13e}$$

$$\delta h: \quad \partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + g(h - H_0) + \lambda \Theta\left(y - y_b(x,y,t)\right) = 0 \quad \text{at} \quad z = h(x,y,t) \tag{13f}$$

$$\delta \phi_R: \quad \dot{R} = \partial_y \phi \quad \text{at} \quad y = R(t) \tag{13g}$$

$$\delta W: \quad \dot{Z} = W \tag{13h}$$

$$\delta Z: \quad M\dot{W} + Mg + \frac{\gamma G(Z)}{L_i}(P_Q + K(Z)) - \rho_0 \int_0^{L_x} \int_{R(t)}^{l_y(x)} \lambda \Theta\left(y - y_b(x,t)\right) \mathrm{d}x\, \mathrm{d}y = 0 \tag{13i}$$

$$\delta P_Q: \quad \dot{Q} = \frac{(P_Q + K(Z))}{L_i} \equiv I \tag{13j}$$

$$\delta Q: \quad \dot{P}_Q = 0, \tag{13k}$$

Wave-2-wire VP

- Free surface: $z = h(x, y, t)$ for $y < y_b(x, t)$
- Constrained surface: $z = h(x, y, t)$ for $y \geq y_b(x, t)$
- Waterline motion $y = y_b(x, t)$; evaluate Bernoulli in interior at surface, compare with free-surface Bernoulli:

$$\begin{aligned} p(x, y, h(x, y, t), t) =& 0 \quad y < y_b(x, t) \\ p(x, y, h(x, y, t), t) =& \lambda(x, y, t) \quad y \geq y_b(x, t), \end{aligned}$$

- so $\lambda(x, y_b(x, t), t) = 0$ at $y = y_b$ is derived
- when $D = 1$ imposed strongly, like in Luke (1967 –WW only), then need to impose $\lambda(x, y_b, t) = 0$ a priori.

Wave-2-wire VP

- Add electrical resistance & LED loads (Shockley eqns in parallel):

$$\dot{P}_Q = -(R_c + R_i)I - \mathrm{sign}(I) n_q V_T \ln\left(\frac{|I|}{I_{sat}} + 1\right)$$

$$\text{with } I = \frac{(P_Q + K(Z))}{L_i}.$$

- Simplex buoy shape:

$$z = h_b(x, y; Z(t)) = Z(t) - H_k - \tan\alpha\,(y - L_y).$$

Appendix Discrete wave-2-wire equations

Space CG (second-order, tent functions) FEM needs to be consistent; trick: local extension around waterline $y = L_b$

$$
\begin{aligned}
N^T_{\tilde{k}l}\left(\eta_l - 1_l\tilde{Z}\right) &= 0 && \text{(28a)}\\
M_{kl}\dot{\phi}_l &= -gM_{kl}\eta_l - N_{k\hat{l}}\lambda_{\hat{l}} - N_{k\tilde{b}}\lambda_{\tilde{b}} && \text{(28b)}\\
M_{kl}\dot{\eta}_l &= S_{kl}\phi_l + T_k\dot{R} && \text{(28c)}\\
\dot{\tilde{Z}} &= \tilde{W} && \text{(28d)}\\
\dot{\tilde{W}} &= C\tilde{Q}_{\hat{l}}\lambda_{\hat{l}} + C\tilde{Q}_{\tilde{b}}\lambda_{\tilde{b}} - C_1G(\bar{Z})(\tilde{P}_Q + \gamma G(\bar{Z})\tilde{Z}) && \text{(28e)}\\
\dot{\tilde{Q}} &= \frac{(\tilde{P}_Q + \gamma G(\bar{Z})\tilde{Z})}{L_i} && \text{(28f)}\\
\dot{\tilde{P}}_Q &= -C_2(\tilde{P}_Q + \gamma G(\bar{Z})\tilde{Z}) && \text{(28g)}\\
\left(\tilde{S}_{\tilde{k}\hat{l}} + C\tilde{Q}_{\tilde{k}}\tilde{Q}_{\hat{l}}\right)\lambda_{\hat{l}} &= -gS_{\tilde{k}l}\eta_l - C\tilde{Q}_{\tilde{k}}\tilde{Q}_{\tilde{b}}\lambda_{\tilde{b}} - \tilde{S}_{\tilde{k}\tilde{b}}\lambda_{\tilde{b}}\\
&\quad + C_1\tilde{Q}_{\tilde{k}}G(\bar{Z})(\tilde{P}_Q + \gamma G(\bar{Z})\tilde{Z}) && \text{(28h)}
\end{aligned}
$$

Discrete wave-2-wire equations

- Indexing of nodes: N_n nodes $k, l = 1, \dots, N_n$; $N_n - N_p + 1$ nodes under buoy $\tilde{k}, \tilde{l} = N_p, \dots, N_n$; N_b waterline nodes $\tilde{b} = N_p, \dots, N_p + N_b - 1$
- Mass & Laplace matrices:

$$M_{kl} = \int_0^{L_x} \int_0^{l_y(x)} \varphi_k(x,y)\varphi_l(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{29a}$$

$$S_{kl} = \int_0^{L_x} \int_0^{l_y(x)} H(y)\nabla\varphi_k(x,y)\cdot\nabla\varphi_l(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{29b}$$

$$\tilde{S}_{\tilde{k}\tilde{l}} = \int_0^{L_x} \int_0^{l_y(x)} H(y)\nabla\varphi_{\tilde{k}}(x,y)\cdot\nabla\varphi_{\tilde{l}}(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{29c}$$

$$\tilde{Q}_{\tilde{k}} = \int_0^{L_x} \int_0^{l_y(x)} \varphi_{\tilde{k}}(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{29d}$$

$$N_{k\tilde{l}} = \int_0^{L_x} \int_0^{l_y(x)} \varphi_k(x,y)\varphi_{\tilde{l}}(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{29e}$$

$$T_k = \int_0^{L_x} H(0)\varphi_k(0,y)\,\mathrm{d}y. \tag{29f}$$

Consistency on discrete level (but only with trick):

The key consistency check is to ensure that the first seven equations in (28) are consistent with the last seven equations in (28). Consider the first seven equations. Take the time derivative of the primary constraint and eliminate the time derivatives by using two of the other seven equations, to obtain the secondary constraint

$$N^T_{\tilde{k}l}\left(M^{-1}_{lk}(S_{km}\phi_m + T_k\dot{R}) - 1_l\tilde{W}\right) = 0. \tag{30}$$

Now take the time derivative of this secondary constraint above and again eliminate the time derivatives by using two different equations of these seven equations, to obtain the consistency equation

$$\begin{aligned}\left(N^T_{\tilde{k}l}M^{-1}_{lk}S_{km}M^{-1}_{mn}N_{n\hat{l}} + CN^T_{\tilde{k}l}1_l\tilde{Q}_{\hat{l}}\right)\lambda_{\hat{l}} &= -gN^T_{\tilde{k}l}M^{-1}_{lk}S_{km}\eta_m \\ &- \left(N^T_{\tilde{k}l}M^{-1}_{lk}S_{km}M^{-1}_{mn}N_{n\tilde{b}} + CN^T_{\tilde{k}l}1_l\tilde{Q}_{\tilde{b}}\right)\lambda_{\tilde{b}} \\ &+ N^T_{\tilde{k}l}M^{-1}_{lk}T_k\ddot{R} + C_1N^T_{\tilde{k}l}1_lG(\bar{Z})\left(\tilde{P}_Q + \gamma G(\bar{Z})\tilde{Z}\right).\end{aligned} \tag{31a}$$

This consistency equation matches the last equation (28h) if and only if the following relations hold

$$\tilde{S}_{\tilde{k}\hat{l}} = N^T_{\tilde{k}l}M^{-1}_{lk}S_{km}M^{-1}_{mn}N_{n\hat{l}} \tag{31b}$$

$$S_{\tilde{k}m} = N^T_{\tilde{k}l}M^{-1}_{lk}S_{km} \tag{31c}$$

$$\tilde{S}_{\tilde{k}\tilde{b}} = N^T_{\tilde{k}l}M^{-1}_{lk}S_{km}M^{-1}_{mn}N_{n\tilde{b}} \tag{31d}$$

$$\tilde{Q}_{\tilde{k}} = N^T_{\tilde{k}l}1_l \tag{31e}$$

$$N^T_{\tilde{k}l}M^{-1}_{lk}T_k = 0. \tag{31f}$$

Discrete wave-2-wire equations

- Time only: symplectic Euler first order

$$\eta^{n+1} - \tilde{Z}^{n+1} = 0 \quad \text{for} \quad y \geq L_b \tag{25a}$$

$$\frac{(\tilde{\phi}^{n+1} - \tilde{\phi}^n)}{\Delta t} + g\eta^n + \tilde{\lambda}^n \Theta(y - L_b) = 0 \tag{25b}$$

$$\frac{(\tilde{W}^{n+1} - \tilde{W}^n)}{\Delta t} + C_1 G(\bar{Z})(\tilde{P}_Q^n + \gamma G(\bar{Z})\tilde{Z}^n) \\ - C \int_0^{L_x} \int_0^{l_y(x)} \tilde{\lambda}^n \Theta(y - L_b)\, \mathrm{d}y\, \mathrm{d}x = 0 \tag{25c}$$

$$\frac{(\tilde{P}_Q^{n+1} - \tilde{P}_Q^n)}{\Delta t} = -C_2(\tilde{P}_Q^{n+1} + \gamma G(\bar{Z})\tilde{Z}^n) \tag{25d}$$

$$\frac{(\tilde{Z}^{n+1} - \tilde{Z}^n)}{\Delta t} = \tilde{W}^{n+1} \tag{25e}$$

$$\frac{(\tilde{Q}^{n+1} - \tilde{Q}^n)}{\Delta t} = \frac{(\tilde{P}_Q^{n+1} + \gamma G(\bar{Z})\tilde{Z}^n)}{L_i}, \tag{25f}$$

$$\frac{(\eta^{n+1} - \eta^n)}{\Delta t} + \nabla \cdot (H \nabla \tilde{\phi}^{n+1}) = 0 \\ \text{with} \quad \partial_y \tilde{\phi}^{n+1}|_{y=l_y(x)} = 0, \qquad \partial_y \tilde{\phi}^{n+1}|_{y=0} = \dot{R}^{n+1} \tag{25g}$$

$$\nabla \cdot (H \nabla \tilde{\lambda}^n) - C \int_0^{L_x} \int_0^{l_y(x)} \tilde{\lambda}^n \Theta(y - L_b)\, \mathrm{d}y\, \mathrm{d}x = -g \nabla \cdot (H \nabla \eta^n) \\ - C_1 G(\bar{Z})(\tilde{P}_Q^n + \gamma G(\bar{Z})\tilde{Z}^n) \quad y \geq L_b \\ \text{with } \tilde{\lambda}^n(x, L_b, t) = g\left(\eta^n(x, L_b^-, t) - \tilde{Z}^n(t)\right) \text{ and } \hat{\boldsymbol{n}} \cdot \nabla \tilde{\lambda}^n|_{\partial\Omega_h, y > L_y - L_b} = 0, \tag{25h}$$