Variational principle for a novel wave-energy device

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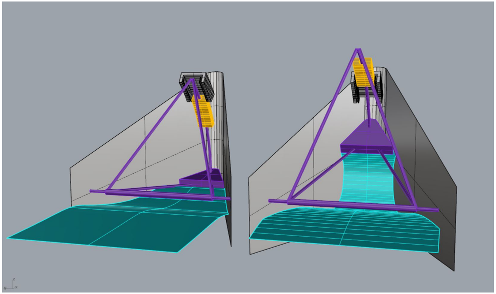


1 Introduction

2 Modelling & VP

3 Discretise wave-2-wire model

4 Preliminary results

5 Conclusions

Introduction: maths wave-2-wire device

References and the beginning:

- B., Kalogirou & Zweers 2019: On wave-2-wire model. *Water Waves* **1.**

  `https://link.springer.com/article/10.1007/s42286-019-00022-9`

- B., Kalogirou, Henry & Thomas 2020: *Int. Marine Energy J.* **30** (EWTEC2019, Napoli).
- Rogue waves are anomalously high waves defined relative to a significant wave height H_s.

Motivation: rogue waves

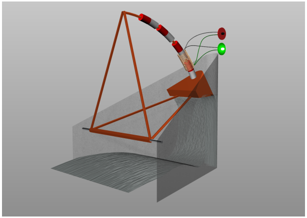
There are many causes of rogue waves, see Kharif et al. (2009):

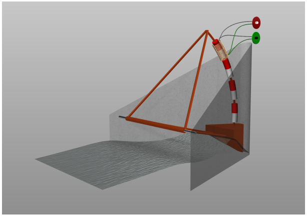
- E.g., crossing seas, nearly standing/pyramidal waves
- We made a *bore-soliton splash* rogue wave:
$AI = \frac{H_{rw}}{H_s} = \frac{3.5}{0.35} \approx 10 \gg 2!$
- It inspired our wave-2-wire wave-energy device.
- `https://www.youtube.com/watch?v=YSXsXNX4zW0`

Introduction: working wave-2-wire device

Design by B. & Zweers within breakwater:

- Sketch with water waves, wave-activated buoy motion & electro-magnetic generator & **LED**-loads:





- *Proof-of-principle* of laboratory device (2013)
  `https://www.youtube.com/watch?v=SZhe_SOxBWo`

Wave-2-wire variational principle

Wave-to-wire model posited using 3 combined VPs:

- Luke (1967), Cotter & B. (2010, JEM), $1+1+1=3$:

$$0 = \delta \int_0^T \mathcal{L}[D, \phi, h, \phi_s, Z, W, Q, P_Q, p, \lambda] \mathrm{d}t$$

$$\equiv \delta \int_0^T \rho_0 \int_0^{L_x} \int_{R(t)}^{l_y(x)} \int_0^{h(x,y,t)} D\partial_t\phi \, dz\, dy\, dx - MW\dot{Z} - P_Q\dot{Q} + \mathcal{H}\, \mathrm{d}t$$

$$\equiv \delta \int_0^T \rho_0 \int_0^{L_x} \int_{R(t)}^{l_y(x)} \int_0^{h(x,y,t)} D\partial_t\phi + \frac{1}{2}D|\nabla\phi|^2 + gD\,(z - H_0) + p(D-1)\, dz\, dy\, dx$$

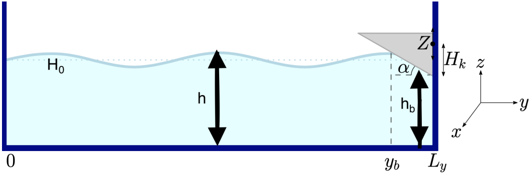
$$+ \rho_0 \int_0^{L_x} \int_{R(t)}^{l_y(x)} \lambda(h - h_b)\Theta\,(y - y_b(x,t))\, dx\, dy$$

$$\underline{}$$

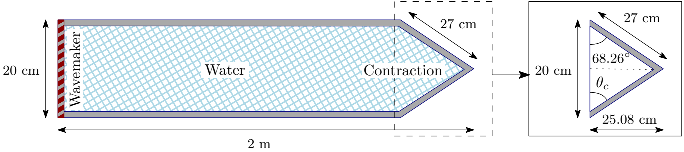
$$-MW\dot{Z} - P_Q\dot{Q} + \frac{1}{2}MW^2 + MgZ + \frac{1}{2}\frac{\left(P_Q + K(Z)\right)^2}{L_i}\, \mathrm{d}t$$

Modelling & VP

- Wave-2-wire model in a wavetank, side view along center-line:



- Wave-2-wire model in a wavetank, top view:



Nonlinear wave-2-wire model

Dynamics of water waves $\{\phi, h\}$, wave-activated buoy motion $\{Z, W\}$ & electro-magnetic generator $\{Q, P_Q\}$ with current I, $I = \dot{Q}$ and $P_Q = L_i I - K(Z)$ & **LED**-loads:

$$\nabla^2 \phi = 0$$

$$\partial_t h + \nabla_H \phi \cdot \nabla h - \partial_z \phi = 0 \text{ on } z = h$$

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(h - H_0) + \underline{\lambda \Theta(y - y_b)} = 0 \quad \text{on} \quad z = h$$

$$\underline{h - h_b} = 0 \quad \text{for} \quad y \geq y_b(x,t)$$

$$\dot{Z} = W$$

$$M\dot{W} = -Mg - \gamma\big(P_Q + K(Z)\big)\frac{G(Z)}{L_i} + \rho_0 \int_0^{L_x}\int_0^{l_y(x)} \underline{\lambda \Theta\left(y - y_b\right)}\, \mathrm{d}x\, \mathrm{d}y$$

$$\dot{Q} = \big(P_Q + K(Z)\big)/L_i \equiv I$$

$$\dot{P}_Q = -(R_c + R_i)I - \underline{\underline{\mathrm{sign}(I) n_q V_T \ln\big(|I|/I_{sat} + 1\big)}} \quad \text{absent in VP.}$$

Nonlinear wave-2-wire model: hydrodynamics

Variables & parameters water waves:

- potential flow $\mathbf{u} = \nabla\phi(x, y, z, t)$
- (single-valued) water depth $h = h(x, y, t)$, free-surface potential $\phi_s(x, y, t) = \phi(x, y, h(x, y, t), t)$

$$\nabla^2\phi = 0 \quad \text{in 3D domain}$$

$$\partial_t h + \nabla_H\phi \cdot \nabla h - \partial_z\phi = 0 \text{ on } z = h$$

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g(h - H_0) + \underline{\lambda\Theta(y - y_b)} = 0 \quad \text{on} \quad z = h$$

$$\underline{h - h_b} = 0 \quad \text{for} \quad y \geq y_b(x, t)$$

- acceleration of gravity g and vertical (free-surface) rest level H_0
- (single-valued) waterline $y = y_b(x, t)$.

Nonlinear wave-2-wire model: buoy motion

Variables & parameters:

- wave-activated buoy: position $Z(t)$ & velocity $W(t)$

$$\underline{h - h_b} = 0 \quad \text{for} \quad y \geq y_b(x,t)$$

$$\dot{Z} = W$$

$$M\dot{W} = -Mg - \gamma\big(P_Q + K(Z)\big)\frac{G(Z)}{L_i} + \rho_0 \int_0^{L_x}\int_0^{l_y(x)} \underline{\lambda\Theta\left(y - y_b\right)}\, \mathrm{d}x\, \mathrm{d}y$$

- *Lagrange multiplier* $\lambda(x,y,t)$ imposes that water depth h equals wetted buoy shape h_b:

$$\underline{h(x,y,t) - h_b\left(x,y;Z(t)\right)} = 0$$

Nonlinear wave-2-wire model: EM motor

Variables & parameters:

- electromagnetic generator: charge $Q(t)$ & conjugate $P_Q(t) = L_i\dot{Q} - K(Z)$; $\dot{P}_Q(t) = \underline{L_i\dot{I}} - \gamma G(Z)\dot{Z}$
- EM-theory $K(Z) = \int^Z \gamma G(\tilde{Z})\mathrm{d}\tilde{Z}$ thin-wire & symmetric
- induction L_i, combined buoy-magnet mass M
- **two LEDs** in parallel: $\underline{\underline{\text{adapted Shockley equation}}}$.

$$\dot{Q} = \big(P_Q + K(Z)\big)/L_i \equiv I$$
$$\dot{P}_Q = -(R_c + R_i)I - \underline{\underline{\mathrm{sign}(I)n_q V_T \ln\big(|I|/I_{sat} + 1\big)}}$$

absent in VP; damping and loads.

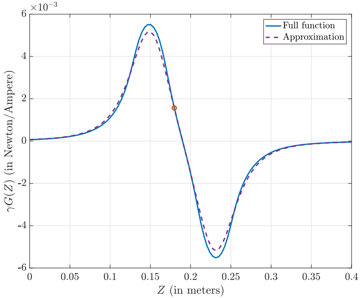
Wave-2-wire model: EM motor

- By solving the Maxwell's equations, the function

$$G(Z) = \frac{\pm 1}{\pi A_m^2 L_m a} \int_{-L/2}^{L/2} \int_0^{2\pi} \int_0^{A_m} F(z)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}q$$

- arises with $F(z) = f(-z) - f(z), z = q + \bar{Z} + \alpha_h H_m - Z,$

$$f(z) = r(a - r\cos\theta)/(r^2 + (\frac{L_m}{2} + z)^2 + a^2 - 2ra\cos\theta)^{3/2}.$$



Wave-2-wire model

- When the change of current I in time & loads small, s.t. $\dot{I} \approx 0$:

$$(R_c + R_i + \underline{R_l})I \approx \gamma G(Z)\dot{Z},$$

- then the vertical momentum equation for buoy, mast and magnet becomes approximately

$$M\dot{W} = -Mg - \frac{\gamma^2 G^2(Z)}{L_i(R_c + R_i + \underline{R_l})} W + \rho_0 \int_0^{L_x} \int_0^{l_y(x)} \lambda\Theta\left(y - y_b\right) \mathrm{d}x\, \mathrm{d}y.$$

- *Resistance and loads combined act as drag* on the buoy-mast-magnet unit in this partial linear limit.

Discrete wave-2-wire linear SW-equations

- Finite elements in space CG1, symplectic Euler in time.
- Consistent geometrically compatible discretisation.

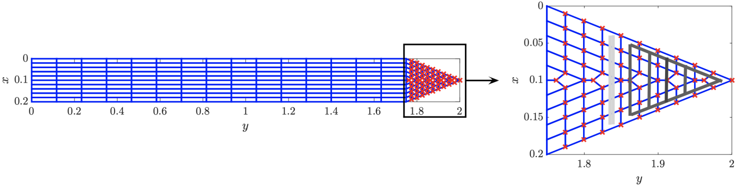
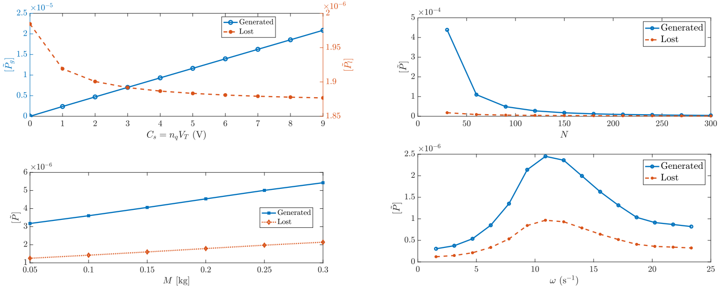


Figure: Computational mesh for $N_x = 10$, $N_y = 15$.

- For Δx & $\Delta x/2$: $O(\sqrt{\Delta t})$ & $O(\Delta t^{3/4})$ & $O(\Delta x^{1.7})$.
- Energy partitioning over subsystems consistent.

Preliminary results linear SWE model

Optimal harvest in linear case at resonance, maximum loads and minimum windings:



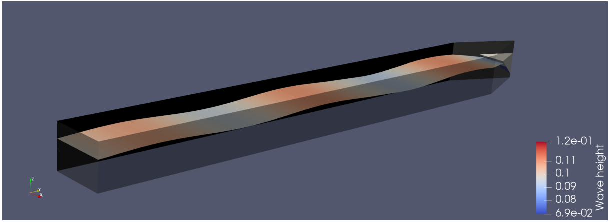
Conclusions

- Full wave-2-wire maths from 1^{st} *variational principles.*
- Established space-time compatible discretisation of linearised model.
- Validation! Preferred location: breakwater.
- Optimise wave-2-wire device wrt energy-output efficiency.

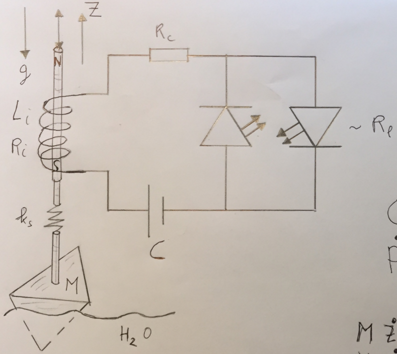
- B., Kalogirou & Zweers 2019: On wave-2-wire model. *Water Waves* **1.**

  https://link.springer.com/article/10.1007/s42286-019-00022-9

- B., Kalogirou, Henry & Thomas 2020: *Int. Marine Energy J.* **30** (EWTEC2019, Napoli)



Wave-2-wire model: circuits



$$\dot{Q} = (P_Q + K(Z))/L_i = I$$

$$\dot{P}_Q = L_i \dot{I} - \gamma G(Z)\dot{Z} = -(R_c + R_i) I - \frac{Q}{C} + V_{LED}(Q, I)$$

BATTERY

$$M\dot{Z} = M W$$

$$M\dot{W} = -Mg - k_s Z - \gamma G(Z) I + F_{HYDRODYNAMIC}$$

$$K(Z) = \int^{Z} \gamma G(\tilde{Z})\, d\tilde{Z}$$

= 3D symmetric EM THEORY

Conclusions: queries

- Charge equation not needed (Bolton), unless a capacitor is added and/or a battery depending on Q, I.
- Mathematical model for battery (Li & Ke, 2011):
  - Shepherd and modified Shepherd models $V_{battery}(Q, I)$.
- Addition of spring with adjustable spring constant $k_s(t)$ in motor & model for active control of resonance?
- Originally, we used schematic of Wellstead (2000), but that model does not linearise?

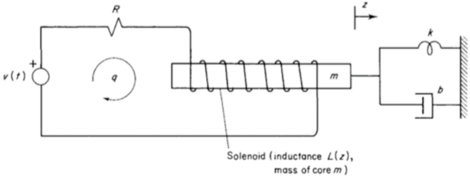
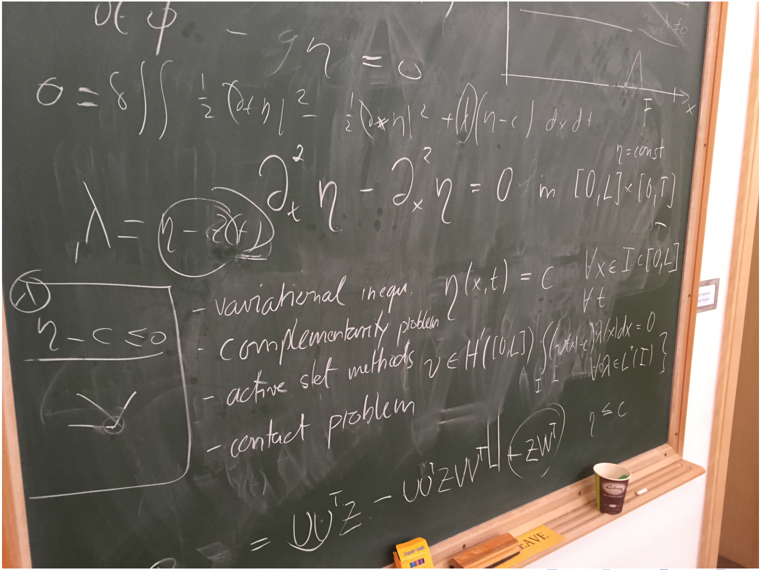


Figure 7.25

Thank you!


- variational inequ.
- complementarity problem
- active set methods
- contact problem

Appendix Wave-2-wire VP

Vary & get eqns of motion (conj. pairs + multipliers):

$$
\begin{aligned}
\delta D: &\quad \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g(z-H_0) + p = 0 && \text{(13a)}\\
\delta\phi: &\quad \partial_t D + \nabla\cdot(D\nabla\phi) = 0 && \text{(13b)}\\
\delta p: &\quad D = 1 && \text{(13c)}\\
\delta\lambda: &\quad h - h_b = 0 \quad \text{for} \quad y \geq y_b(x,t) && \text{(13d)}\\
\delta\phi_s: &\quad \partial_t h + \nabla\phi\cdot\nabla h = \partial_z\phi \quad \text{at} \quad z = h(x,y,t) && \text{(13e)}\\
\delta h: &\quad \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g(h-H_0) + \lambda\Theta\left(y - y_b(x,y,t)\right) = 0 &&\\
&\qquad\qquad \text{at} \quad z = h(x,y,t) && \text{(13f)}\\
\delta\phi_R: &\quad \dot{R} = \partial_y\phi \quad \text{at} \quad y = R(t) && \text{(13g)}\\
\delta W: &\quad \dot{Z} = W && \text{(13h)}\\
\delta Z: &\quad M\dot{W} + Mg + \frac{\gamma G(Z)}{L_i}(P_Q + K(Z)) &&\\
&\qquad - \rho_0\int_0^{L_x}\int_{R(t)}^{l_y(x)} \lambda\Theta\left(y - y_b(x,t)\right)\,\mathrm{d}x\,\mathrm{d}y = 0 && \text{(13i)}\\
\delta P_Q: &\quad \dot{Q} = \frac{(P_Q + K(Z))}{L_i} \equiv I && \text{(13j)}\\
\delta Q: &\quad \dot{P}_Q = 0, && \text{(13k)}
\end{aligned}
$$

Wave-2-wire VP

- Free surface: $z = h(x, y, t)$ for $y < y_b(x, t)$
- Constrained surface: $z = h(x, y, t)$ for $y \geq y_b(x, t)$
- Waterline motion $y = y_b(x, t)$; evaluate Bernoulli in interior at surface, compare with free-surface Bernoulli:

$$
\begin{aligned}
p(x, y, h(x, y, t), t) =& 0 \quad y < y_b(x, t) \\
p(x, y, h(x, y, t), t) =& \lambda(x, y, t) \quad y \geq y_b(x, t),
\end{aligned}
$$

- so $\lambda(x, y_b(x, t), t) = 0$ at $y = y_b$ is derived
- when $D = 1$ imposed strongly, like in Luke (1967 –WW only), then need to impose $\lambda(x, y_b, t) = 0$ a priori.

Wave-2-wire VP

- Add electrical resistance & LED loads (Shockley eqns in parallel):

$$\dot{P}_Q = -(R_c + R_i)I - \mathrm{sign}(I) n_q V_T \ln\left(\frac{|I|}{I_{sat}} + 1\right)$$

$$\text{with } I = \frac{(P_Q + K(Z))}{L_i}.$$

- Simplex buoy shape:

$$z = h_b(x, y; Z(t)) = Z(t) - H_k - \tan\alpha\,(y - L_y).$$

Linearise wave-2-wire equations

Linearise & simplify to shallow water with only (x, y):

$$\eta - \tilde{Z} = 0 \quad \text{for} \quad y \geq L_b, \qquad \dot{R} = \partial_y \tilde{\phi} \quad \text{at} \quad y = 0$$

$$\partial_t \eta + \nabla \cdot (H \nabla \tilde{\phi}) = 0, \partial_t \tilde{\phi} + g\eta + \tilde{\lambda}\Theta(y - L_b) = 0$$

$$\dot{\tilde{Z}} = \tilde{W}$$

$$M\dot{\tilde{W}} + \gamma G(\bar{Z})\tilde{I} - \rho_0 \int_0^{L_x}\int_0^{l_y(x)} \tilde{\lambda}\Theta(y - L_b)\,\mathrm{d}y\,\mathrm{d}x = 0$$

$$\dot{\tilde{Q}} = \tilde{I}, \quad \dot{\tilde{P}}_Q = -\Big(R_c + R_i + \frac{n_q V_T}{I_{sat}}\Big)\tilde{I}, \quad \tilde{I} = \frac{\left(\tilde{P}_Q + \gamma G(\bar{Z})\tilde{Z}\right)}{L_i}$$

$$\Longrightarrow \nabla \cdot (H \nabla \tilde{\lambda}) - \frac{\rho_0}{M}\int_0^{L_x}\int_0^{l_y(x)} \tilde{\lambda}\Theta(y - L_b)\,\mathrm{d}y\,\mathrm{d}x =$$
$$- \nabla \cdot (gH\nabla\eta) - \frac{\gamma}{M} G\left(\bar{Z}\right)\tilde{I} \quad \text{for} \quad y \geq L_b.$$

Appendix Discrete wave-2-wire equations

Space CG (second-order, tent functions) FEM needs to be consistent; trick: local extension around waterline $y = L_b$

$$
\begin{aligned}
N^T_{\hat{k}l}\left(\eta_l - 1_l\tilde{Z}\right) &= 0 && (28a)\\
M_{kl}\dot{\phi}_l &= -gM_{kl}\eta_l - N_{k\hat{l}}\lambda_{\hat{l}} - N_{k\bar{b}}\lambda_{\bar{b}} && (28b)\\
M_{kl}\dot{\eta}_l &= S_{kl}\phi_l + T_k\dot{R} && (28c)\\
\dot{\tilde{Z}} &= \tilde{W} && (28d)\\
\dot{\tilde{W}} &= C\tilde{Q}_{\hat{l}}\lambda_{\hat{l}} + C\tilde{Q}_{\bar{b}}\lambda_{\bar{b}} - C_1G(\bar{Z})(\tilde{P}_Q + \gamma G(\bar{Z})\tilde{Z}) && (28e)\\
\dot{\tilde{Q}} &= \frac{(\tilde{P}_Q + \gamma G(\bar{Z})\tilde{Z})}{L_i} && (28f)\\
\dot{\tilde{P}}_Q &= -C_2(\tilde{P}_Q + \gamma G(\bar{Z})\tilde{Z}) && (28g)\\
\left(\tilde{S}_{\hat{k}\hat{l}} + C\tilde{Q}_{\hat{k}}\tilde{Q}_{\hat{l}}\right)\lambda_{\hat{l}} &= -gS_{\hat{k}l}\eta_l - C\tilde{Q}_{\hat{k}}\tilde{Q}_{\bar{b}}\lambda_{\bar{b}} - \tilde{S}_{\hat{k}\bar{b}}\lambda_{\bar{b}}\\
&\quad + C_1\tilde{Q}_{\hat{k}}G(\bar{Z})(\tilde{P}_Q + \gamma G(\bar{Z})\tilde{Z}) && (28h)
\end{aligned}
$$

Discrete wave-2-wire equations

- Indexing of nodes: N_n nodes $k, l = 1, \ldots, N_n$; $N_n - N_p + 1$ nodes under buoy $\tilde{k}, \tilde{l} = N_p, \ldots, N_n$; N_b waterline nodes $\tilde{b} = N_p, \ldots, N_p + N_b - 1$
- Mass & Laplace matrices:

$$M_{kl} = \int_0^{L_x} \int_0^{l_y(x)} \varphi_k(x,y)\varphi_l(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{29a}$$

$$S_{kl} = \int_0^{L_x} \int_0^{l_y(x)} H(y)\boldsymbol{\nabla}\varphi_k(x,y)\cdot\boldsymbol{\nabla}\varphi_l(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{29b}$$

$$\tilde{S}_{\tilde{k}\tilde{l}} = \int_0^{L_x} \int_0^{l_y(x)} H(y)\boldsymbol{\nabla}\varphi_{\tilde{k}}(x,y)\cdot\boldsymbol{\nabla}\varphi_{\tilde{l}}(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{29c}$$

$$\tilde{Q}_{\tilde{k}} = \int_0^{L_x} \int_0^{l_y(x)} \varphi_{\tilde{k}}(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{29d}$$

$$N_{k\tilde{l}} = \int_0^{L_x} \int_0^{l_y(x)} \varphi_k(x,y)\varphi_{\tilde{l}}(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{29e}$$

$$T_k = \int_0^{L_x} H(0)\varphi_k(0,y)\,\mathrm{d}y. \tag{29f}$$

Discrete wave-2-wire equations

Consistency on discrete level (but only with trick):

The key consistency check is to ensure that the first seven equations in (28) are consistent with the last seven equations in (28). Consider the first seven equations. Take the time derivative of the primary constraint and eliminate the time derivatives by using two of the other seven equations, to obtain the secondary constraint

$$N^T_{\hat{k}l}\left(M^{-1}_{lk}(S_{km}\phi_m + T_k\dot{R}) - 1_l\tilde{W}\right) = 0. \tag{30}$$

Now take the time derivative of this secondary constraint above and again eliminate the time derivatives by using two different equations of these seven equations, to obtain the consistency equation

$$\begin{aligned}\left(N^T_{\hat{k}l}M^{-1}_{lk}S_{km}M^{-1}_{mn}N_{n\hat{l}} + CN^T_{\hat{k}l}1_l\tilde{Q}_{\hat{l}}\right)\lambda_{\hat{l}} &= -gN^T_{\hat{k}l}M^{-1}_{lk}S_{km}\eta_m \\ &- \left(N^T_{\hat{k}l}M^{-1}_{lk}S_{km}M^{-1}_{mn}N_{n\tilde{b}} + CN^T_{\hat{k}l}1_l\tilde{Q}_{\tilde{b}}\right)\lambda_{\tilde{b}} \\ &+ N^T_{\hat{k}l}M^{-1}_{lk}T_k\ddot{R} + C_1N^T_{\hat{k}l}1_lG(Z)\left(\tilde{P}_Q + \gamma G(Z)\tilde{Z}\right).\end{aligned} \tag{31a}$$

This consistency equation matches the last equation (28h) if and only if the following relations hold

$$\tilde{S}_{\hat{k}\hat{l}} = N^T_{\hat{k}l}M^{-1}_{lk}S_{km}M^{-1}_{mn}N_{n\hat{l}} \tag{31b}$$

$$S_{\hat{k}m} = N^T_{\hat{k}l}M^{-1}_{lk}S_{km} \tag{31c}$$

$$\tilde{S}_{\hat{k}\tilde{b}} = N^T_{\hat{k}l}M^{-1}_{lk}S_{km}M^{-1}_{mn}N_{n\tilde{b}} \tag{31d}$$

$$\tilde{Q}_{\hat{k}} = N^T_{\hat{k}l}1_l \tag{31e}$$

$$N^T_{\hat{k}l}M^{-1}_{lk}T_k = 0. \tag{31f}$$

These relations have been verified to hold up to machine precision. To date, we have not been able to verify these relations analytically. Finally, we find the consistent space-time discretisation by logically combining the time-discrete and space-discrete approaches derived in (25) and (28).

Discrete wave-2-wire equations

- Time only: symplectic Euler first order

$$\eta^{n+1} - \tilde{Z}^{n+1} = 0 \quad \text{for} \quad y \geq L_b \tag{25a}$$

$$\frac{(\tilde{\phi}^{n+1} - \tilde{\phi}^n)}{\Delta t} + g\eta^n + \tilde{\lambda}^n \Theta(y - L_b) = 0 \tag{25b}$$

$$\frac{(\tilde{W}^{n+1} - \tilde{W}^n)}{\Delta t} + C_1 G(Z)(\tilde{P}_Q^{n+1} + \gamma G(Z)\tilde{Z}^n) - C\int_0^{L_x}\int_0^{l_y(x)} \tilde{\lambda}^n \Theta(y - L_b)\, \mathrm{d}y\, \mathrm{d}x = 0 \tag{25c}$$

$$\frac{(\tilde{P}_Q^{n+1} - \tilde{P}_Q^n)}{\Delta t} = -C_2(\tilde{P}_Q^{n+1} + \gamma G(\bar{Z})\tilde{Z}^n) \tag{25d}$$

$$\frac{(\tilde{Z}^{n+1} - \tilde{Z}^n)}{\Delta t} = \tilde{W}^{n+1} \tag{25e}$$

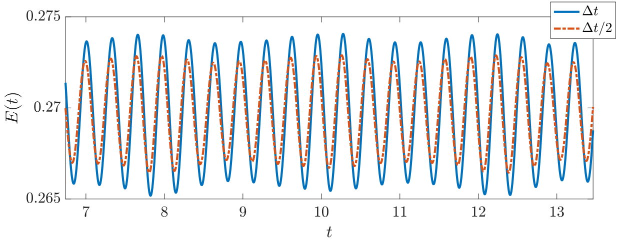
$$\frac{(\tilde{Q}^{n+1} - \tilde{Q}^n)}{\Delta t} = \frac{(\tilde{P}_Q^{n+1} + \gamma G(Z)\tilde{Z}^n)}{L_i}, \tag{25f}$$

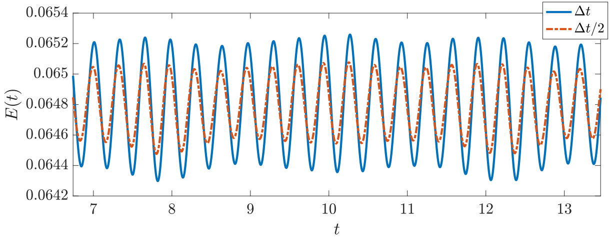
$$\frac{(\eta^{n+1} - \eta^n)}{\Delta t} + \nabla\cdot(H\nabla\tilde{\phi}^{n+1}) = 0 \quad \text{with} \quad \partial_y\tilde{\phi}^{n+1}|_{y=l_y(x)} = 0, \qquad \partial_y\tilde{\phi}^{n+1}|_{y=0} = \dot{R}^{n+1} \tag{25g}$$

$$\nabla\cdot(H\nabla\tilde{\lambda}^n) - C\int_0^{L_x}\int_0^{l_y(x)} \tilde{\lambda}^n\Theta(y - L_b)\,\mathrm{d}y\,\mathrm{d}x = -g\nabla\cdot(H\nabla\eta^n) - C_1 G(\bar{Z})(\tilde{P}_Q^{n+1} + \gamma G(\bar{Z})\tilde{Z}^n) \quad y \geq L_b$$
$$\text{with } \tilde{\lambda}^n(x, L_b, t) = g\left(\eta^n(x, L_b^-, t) - \tilde{Z}^n(t)\right) \text{ and } \hat{n}\cdot\nabla\tilde{\lambda}^n|_{\partial\Omega_h, y>L_y-L_b} = 0, \tag{25h}$$

Convergence in time

For Δx & $\Delta x/2$, we find $O(\sqrt{\Delta t})$ & $O(\Delta t^{3/4})$:





Convergence in space

Table: Convergence rates n using three different norms ($\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_\infty$) evaluated using the value of the velocity potential $\tilde{\phi}$ at the final time of the simulation, i.e. at $t = T$.

	x-el	y-el	Total el	Nodes	Rate
Symbol	N_x	N_y	N_k	N_n	n
Mesh 1	6	30	201	241	$\mathcal{L}_1$: 1.711293
Mesh 2	12	60	798	877	$\mathcal{L}_2$: 1.696554
Mesh 3	24	120	3180	3337	$\mathcal{L}_\infty$: 1.765833

Preliminary results full model

Vertical displacement, current & total power $(R_c + R_i + \underline{n_q V_T / I_{sat}})\tilde{I}^2$: