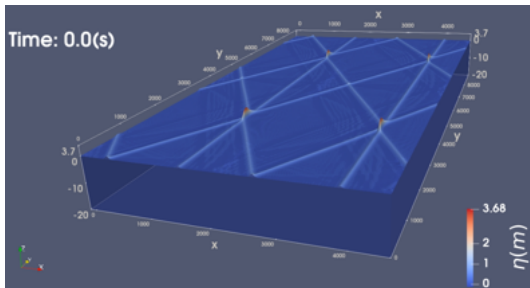


Maximum water-wave amplification of three interacting solitons in Kadomtsev-Petviashvili and potential-flow equations

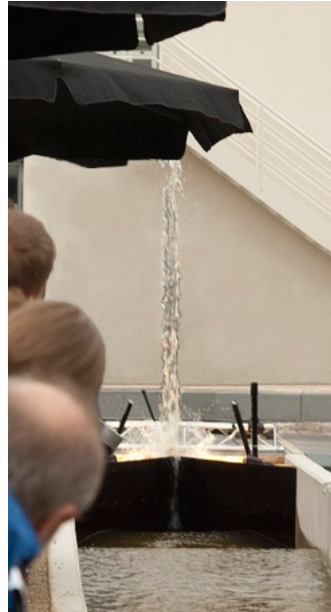
Onno Bokhove and Junho Choi; EU-GA859983

Leeds Institute for Fluid Dynamics; [ICTAM2024: ... FM17 Waves in fluids](#)



Motivation on modelling extremely high water waves

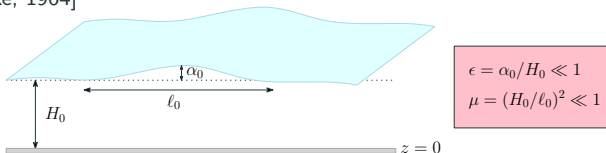
- Origin 2010 *bore-soliton-splash*:
- To what extent do exact but idealised extreme- or rogue-wave solutions survive in more realistic settings?
- Will such extreme waves fall apart due to dispersion or other mechanisms?
- Use fourfold and ninefold KP amplifications of interacting solitons/cnoidal waves.
- What do you think: will we be able to reach the ninefold wave amplification in more realistic calculations, using potential-flow dynamics, or in reality?



Mathematical hierarchy: PFE, BLE & KPE approximations

↪ Boussinesq-type approximation: includes weak dispersive effects

- KdV equation: wave propagation in 1D [Korteweg & de Vries, 1895]
- KPE equation: unidirectional propagation in 2DH [Kadomtsev & Petviashvili, 1970]
- Benney-Luke equations –BLE: bidirectional propagation in 2DH [Benney & Luke, 1964]



Expansion about the sea-bed potential $\Phi(x, y, t) = \phi(x, y, z = 0, t)$, in powers of the small parameter μ [Pego & Quintero, 1999]

- More realistic or parent potential-flow equations (PFE).

Kadomtsev-Petviashvili (KPE) equation

The KPE equation can be obtained from the Benney-Luke equations by introducing the formal perturbation expansions

$$\eta = \tilde{u} + \mathcal{O}(\epsilon^2), \quad \Phi = \sqrt{\epsilon} \left(\tilde{\Psi} + \mathcal{O}(\epsilon^2) \right),$$

using the transformations

$$X = \sqrt{\frac{\epsilon}{\mu}} \left(\frac{3}{\sqrt{2}} \right)^{1/3} (x - t), \quad Y = \sqrt{\epsilon} \sqrt{\frac{\epsilon}{\mu}} \left(\frac{3}{\sqrt{2}} \right)^{2/3} y, \\ \tau = \epsilon \sqrt{\frac{2\epsilon}{\mu}} t, \quad u = \left(\frac{3}{4} \right)^{1/3} \tilde{u},$$

with $\mu = \epsilon^2$ (in the VP), resulting in the **KPE equation** in “standard” form

$$\partial_X (4\partial_\tau u + 6u\partial_X u + \partial_{XXX} u) + 3\partial_{YY} u = 0$$

This equation includes weak effects in the y -direction.

Exact solution of the KP equation

Web and line-soliton solutions can be constructed using Hirota's transformation

$$u(X, Y, \tau) = 2\partial_{XX} \ln K(X, Y, \tau) = \frac{2\partial_{XX} K}{K} - 2\left(\frac{\partial_X K}{K}\right)^2,$$

where function $K(X, Y, \tau)$ can be obtained from the Wronskian

$$K(X, Y, \tau) = \begin{vmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(N-1)} \\ f_2 & f_2^{(1)} & \cdots & f_2^{(N-1)} \\ \vdots & \vdots & & \vdots \\ f_N & f_N^{(1)} & \cdots & f_N^{(N-1)} \end{vmatrix}.$$

Particular soliton solutions are obtained by taking [Kodama, 2010]

$$f_i = \sum_{j=1}^M a_{ij} e^{\theta_j}, \quad \text{where } \theta_j = k_j X + k_j^2 Y - k_j^3 \tau,$$

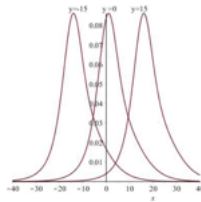
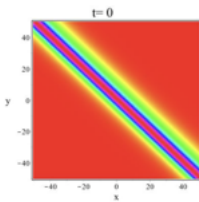
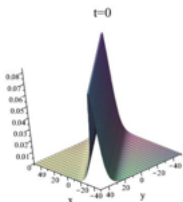
with coefficients k_j being ordered as $k_1 < k_2 < \cdots < k_M$. This solution is called a (N_-, N_+) -soliton, comprising line solitons in the far-field $Y \rightarrow \pm\infty$.

Example: single line soliton

Single line solitons have $(N, M) = (1, 2)$, resulting in $K = f_1 = e^{\theta_1} + e^{\theta_2}$ and the line soliton solution is

$$\begin{aligned} u(X, Y, \tau) &= \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1 - \theta_2) \\ &= \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2 \frac{1}{2}((k_1 - k_2)X + (k_1^2 - k_2^2)Y - (k_1^3 - k_2^3)\tau). \end{aligned}$$

The soliton amplitude is $\tilde{A} = \frac{1}{2}(k_1 - k_2)^2$ and its centreline is found by setting the sech^2 argument to zero.



Example: two interacting line solitons

Two line solitons have $(N, M) = (2, 4)$, also called $(2, 2)$ -solitons or O -solitons, obtained with functions $f_1 = e^{\theta_1} + e^{\theta_2}$, $f_2 = e^{\theta_3} + e^{\theta_4}$, and

$$K(X, Y, \tau) = (k_3 - k_1)e^{\theta_1 + \theta_3} + (k_3 - k_2)e^{\theta_2 + \theta_3} + (k_4 - k_1)e^{\theta_1 + \theta_4} + (k_4 - k_2)e^{\theta_2 + \theta_4}.$$

In the far field $Y \rightarrow \pm\infty$, we find the single line solitons

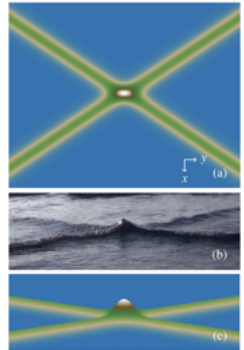
$$u_{[1,2]}(X, Y, \tau) = \frac{1}{2}(k_2 - k_1)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1 - \theta_2 - \ln a),$$

$$u_{[3,4]}(X, Y, \tau) = \frac{1}{2}(k_4 - k_3)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_3 - \theta_4 - \ln b),$$

where a, b depend on k_j . For equal far-field soliton amplitudes $\tilde{A} = \frac{1}{2}(k_2 - k_1)^2 = \frac{1}{2}(k_4 - k_3)^2$, the solution satisfies [Kodama, 2010]

$$2\tilde{A} \leq \max_{(X, Y, \tau)} u(X, Y, \tau) \leq 2 \left(1 + \frac{1 - \sqrt{\Delta_o}}{1 + \sqrt{\Delta_o}} \right) \tilde{A},$$

where $0 \leq \Delta_o \leq 1$, hence $2\tilde{A} \leq \max u \leq 4\tilde{A}$.



Example: three interacting line solitons

Three line solitons, known as (3, 3)-solitons, have $(N, M) = (3, 6)$ and functions $f_1 = e^{\theta_1} + e^{\theta_2}$, $f_2 = e^{\theta_3} + e^{\theta_4}$, $f_3 = e^{\theta_5} + e^{\theta_6}$, and

$$K(X, Y, \tau) = \underline{A_{135}} e^{\theta_1 + \theta_3 + \theta_5} + \underline{A_{235}} e^{\theta_2 + \theta_3 + \theta_5} + \underbrace{A_{136}} e^{\theta_1 + \theta_3 + \theta_6} + A_{236} e^{\theta_2 + \theta_3 + \theta_6} \\ + A_{145} e^{\theta_1 + \theta_4 + \theta_5} + \underline{A_{245}} e^{\theta_2 + \theta_4 + \theta_5} + \underbrace{A_{146}} e^{\theta_1 + \theta_4 + \theta_6} + \underline{A_{246}} e^{\theta_2 + \theta_4 + \theta_6},$$

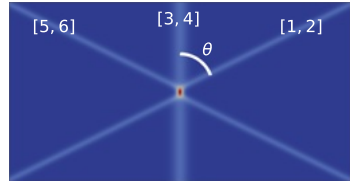
with parameter ordering $k_1 < k_2 < k_3 < 0 < k_4 < k_5 < k_6$ & $a, b = 1, c$.

In the far field $Y \rightarrow \pm\infty$, we find the single line solitons

$$u_{[1,2]} \approx \frac{1}{2}(k_2 - k_1)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1 - \theta_2 - \ln \tilde{a}),$$

$$u_{[5,6]} \approx \frac{1}{2}(k_6 - k_5)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_5 - \theta_6 - \ln \tilde{b}),$$

$$u_{[3,4]} \approx \frac{1}{2}(k_4 - k_3)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_3 - \theta_4),$$



with $\theta_i - \theta_j = (k_i - k_j) \left(X + (k_i + k_j)Y - (k_i^2 + k_i k_j + k_j^2)\tau \right)$.

Example: three interacting line solitons

Parameters k_1, \dots, k_6 are determined from

$$k_3 + k_4 = 0$$

$$k_5 + k_6 = -(k_1 + k_2) = \tan \theta$$

$$k_4 - k_3 = \sqrt{2\tilde{A}}$$

$$k_6 - k_5 = k_2 - k_1 = \sqrt{2\tilde{A}/\lambda}$$

Solving the above six equations, gives

$$k_6 = -k_1 = \sqrt{\tilde{A}} \left(\sqrt{2/\lambda} + \sqrt{1/2} + \delta \right)$$

$$k_5 = -k_2 = \sqrt{\tilde{A}} \left(\sqrt{1/2} + \delta \right)$$

$$k_4 = -k_3 = \sqrt{\tilde{A}/2}$$

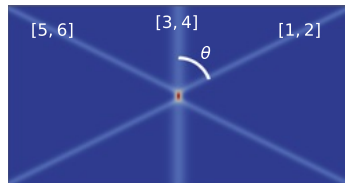
where δ is defined by

$$\delta = \frac{\tan \theta}{2\sqrt{\tilde{A}}} - \left(\sqrt{1/2\lambda} + \sqrt{1/2} \right) > 0.$$

where angle $\theta > 0$,

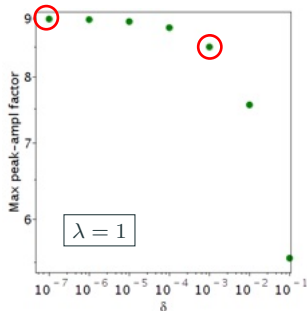
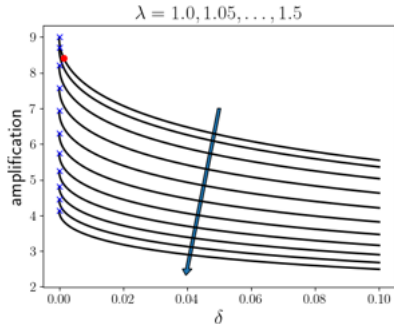
$\tilde{A} = \frac{1}{2}(k_4 - k_3)^2$ is the amplitude of the $[3, 4]$

soliton, and the outer two solitons are assumed to have amplitude \tilde{A}/λ , for $\lambda \geq 1$.



Maximum 9-fold amplification in KP

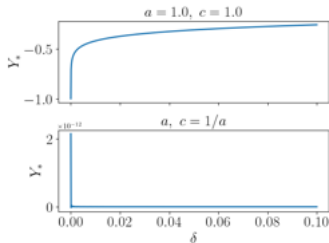
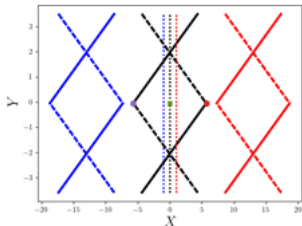
- Proof is based on a **geometric argument** (additional secondary proof)
- Find **5 centrelines** of each of three line solitons (no phase shift at peak)
- Look for intersection points \rightsquigarrow this gives two values of Y , with mean at a unique point $Y_{*\delta \rightarrow 0} \rightarrow -\infty$ when $\tau_* = 0$ and $X_* = 0$
- The space-time point of maximum amplification is (X_*, Y_*, τ_*)
- **Amplification:** $\frac{u(X_*, Y_*, \tau_*)}{\tilde{A}} = \frac{(\sqrt{\lambda} + 2)^2}{\lambda} + \mathcal{O}(\sqrt{\delta}) \xrightarrow{\delta=0} 1 + \frac{4}{\lambda} + \frac{4}{\sqrt{\lambda}}$



Proof of maximum 9-fold amplification in KP

- Three shift parameters $a, b = 1, c = 1/a$ can be optimised such that splash occurs at $(X^*, Y^*, \tau^*) = (0, 0, 0)$.
- Amplification

$$u(X^*, Y^*, \tau^*)/\tilde{A} = 9 - 8\sqrt{3}\sqrt[4]{2}\sqrt{\delta} + 16\sqrt{2}\delta - 19\cdot 2^{3/4}\sqrt{3}\delta^{3/2}/3$$
- Principle Minor Theorem proves that (X^*, Y^*, τ^*) is a maximum.
- Involved and combined geometrical and analytical proofs (WW2022-2024).



Numerical implementation



Firedrake

An automated system for the solution of PDEs using the Finite Element Method (FEM).

Firedrake employs Unified Form Language (UFL) and linear & non-linear solvers PETSc solvers [Rathgeber et al., 2016].

- Space-time discretisation 2nd order of **variational principle for BLE**: bounded energy oscillations, phase-space conserved.
- Continuous Galerkin (CG) FEM in space for VP, with approximations & test functions/variations $\delta\eta_h$, $\delta\Phi_h$:

$$\eta(x, y, t) \approx \eta_h(x, y, t) = \sum_k \eta_k(t) w_k(x, y), \dots$$

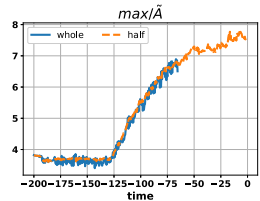
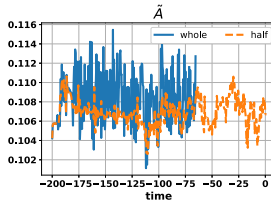
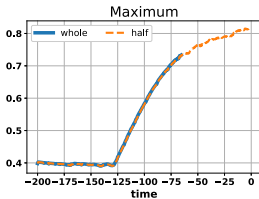
- Symplectic Störmer-Verlet & MMP time stepping schemes.
- **Stable numerical scheme**: no artificial amplitude damping ...

5. Firedrake: exciting aspects & VPs

- **Exciting novel & pursued** development is to implement (time-discrete) VPs directly via command “*derivative*”.
- Advantages: stunning **reduction time-to-development**.
- **New codes more versatile**: horizontal mesh with spectral GLL combined with (i) vertical elements with GLL or (ii) 1 vertical element with high-order spectral GLL.
- Firedrake has (automated) MPI-HPC, various preconditioners and also time-integration options.

Computational domain: \sim cnoidal waves

- KPE solutions hold on infinite horizontal plane, so domain has to be sufficiently large to eliminate reflection at boundaries.
- Solutions can be set to become **approximately periodic** in sufficiently large domains.
- Transform $\Phi = U_0(y)x + c_0(y) + \tilde{\Phi}$, where $\tilde{\Phi}$ is periodic, then solve the BLE for η and $\tilde{\Phi}$.
- Doubly or singly periodic domain?



Initial conditions and boundaries

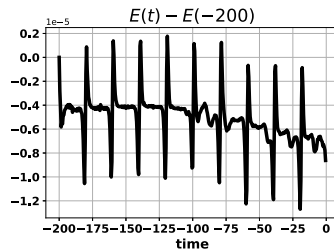
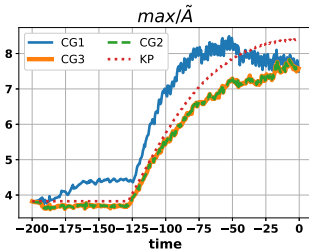
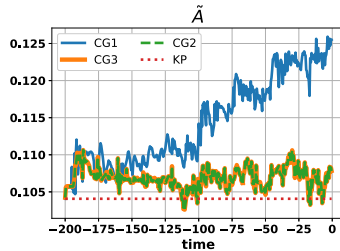
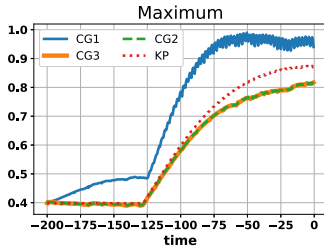
Initial condition consists of two (SP2) or three (SP3) line solitons, expressions of which are known from the KP-solution:

$$\eta_0(x, y) = \eta(x, y, t_0) = 2\left(\frac{4}{3}\right)^{1/3} \partial_{XX} \ln K(X, Y, \tau),$$
$$\Phi_0(x, y) = \Phi(x, y, t_0) = 2\sqrt{\epsilon} \left(\frac{4\sqrt{2}}{9}\right)^{1/3} \partial_X \ln K(X, Y, \tau).$$

Computational domain is constructed such that initial condition satisfies “**periodic boundary conditions**” in x -direction.

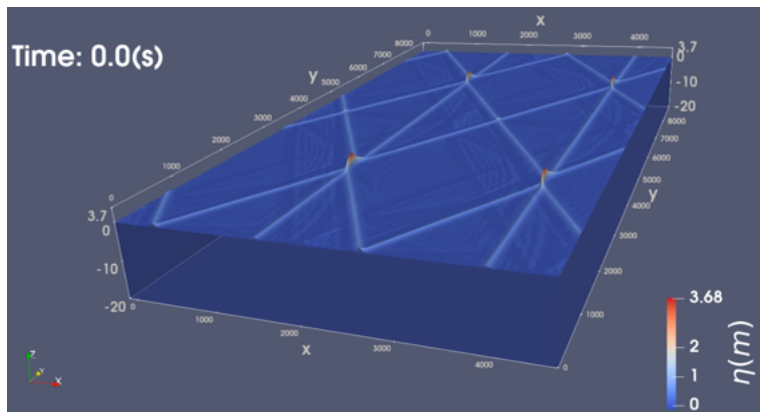
Case	L_x	L_y	T	N_x	N_y	$\Delta x = \frac{L_x}{N_x}$	$\Delta y = \frac{L_y}{N_y}$	Δt
SP2	10.3	40	50	132	480	0.0779	0.0833	0.005
SP3	20.9	47	200	252	564	0.0829	0.0833	0.005

Results BLE-simulation three-soliton interaction



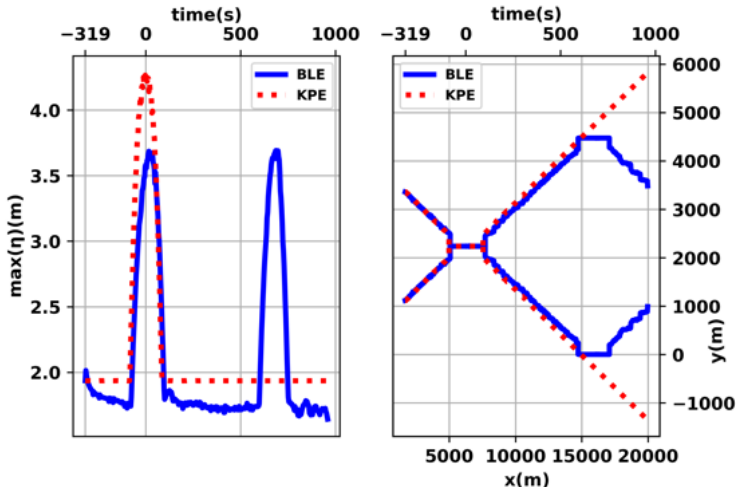
Results simulation three-soliton interaction (dimensional)

Crossing seas (4 or 8 domains combined – YouTube)



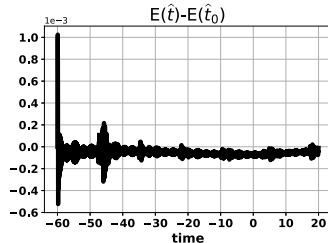
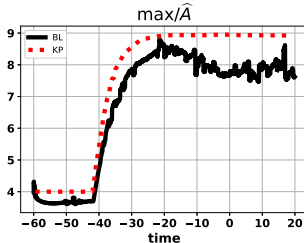
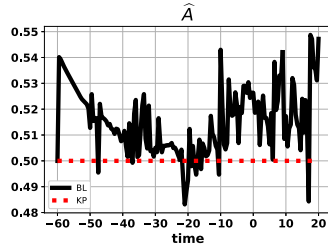
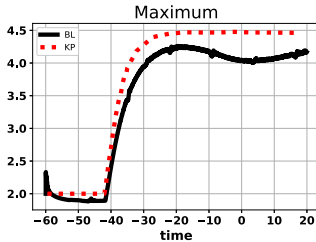
Results simulation three-soliton interaction (dimensional)

Cnoidal waves with periodicity in x, y, t (max. vs. t & x - y tracks):



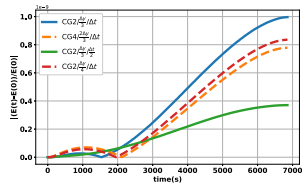
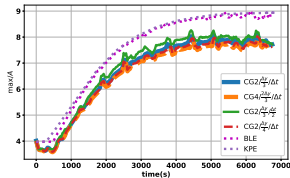
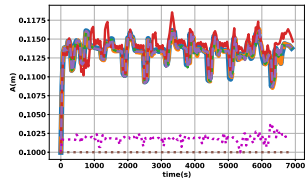
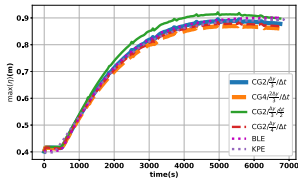
Results BLE-simulations three-soliton interactions

- KPE with $\{\epsilon = 0.05, \delta = 10^{-10}, 9^- \times\}$ seeding of BLE simulation yields **7.5 to 8.5 \times amplification** $t_{BLE} \in [-60, 20]$.

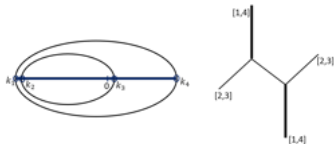


Results BLE- & PFE-simulations three-soliton interactions

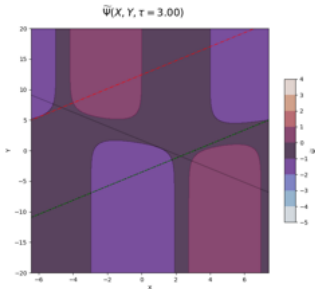
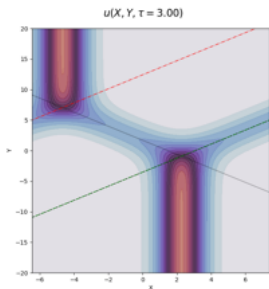
- Demanding **PFE simulations**: HPC simulation with optimised Additive Schwarz Method-Star pre-conditioner. Amplification 7.5 to 8 at low $\epsilon = 0.01, \delta = 10^{-5}$.



Potential-flow P-type two-soliton/cnoidal interactions

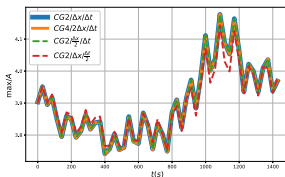
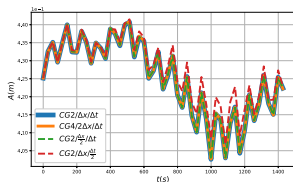
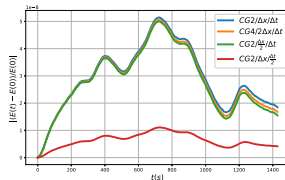
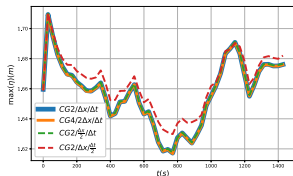


- Sketch (thanks to Prof Yuji Kodama) and exact KPE-solution.
- $K(X, Y, \tau) = (k_3 - k_1)e^{\theta_1} \left(e^{\theta_3} + \frac{k_2 - k_1}{(k_3 - k_1)} e^{\theta_2} \right) + (k_4 - k_3)e^{\theta_4} \left(e^{\theta_3} + \frac{(k_4 - k_2)}{(k_4 - k_3)} e^{\theta_2} \right)$,
wherein $\theta_i = k_i X + k_i^2 Y - k_i^3 \tau$, $k_1 = -k_4 < k_2 = k_1 + \delta < k_3 = \delta < k_4$, $\delta = 10^{-5}$.



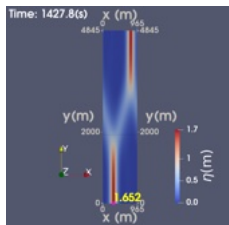
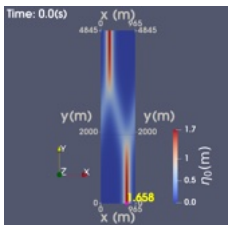
Potential-flow P-type two-soliton interactions

- Demanding **PFE simulations** with a travelling-wave *P-type web-soliton* with amplitude 4, wavelength 400m, wave height 1.6m, $\epsilon = 0.05$.



Summary

- 9-fold soliton amplification proven, when $\delta \rightarrow 0$ & $(X, Y, \tau) = (0.0, 0.0)$
- Web-soliton amplification of KPE is 9^- , seeding BLE-PFE simulations with amplifications ≈ 7.8 & 8.5
- It is open question how to reach higher amplitudes and set up three-soliton-amplification experiments (continuation).
- We used novel geometric discretisation of time-discrete VPs, automated via Firedrake, with reduction-of-time-to-development & MPI-HPC.
- Smoothness of the computational “periodisation” is suboptimal. A new P-type web-soliton yields better simulations with higher amplitudes:



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- Gidel, B, Kalogirou (2017) Variational modelling of extreme waves through oblique interaction *Nonl. Proc. Geophys.* **24**.
- B, Kalogirou (2016) Variational Water Wave Modelling: from Continuum to Experiment. *Theory of Water Waves*, Bridges et al., *London Math. Soc.* **426**.
- Crossing seas *YouTube movie*: <https://www.youtube.com/watch?v=EGhpQ7BM2jA>