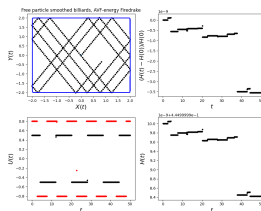
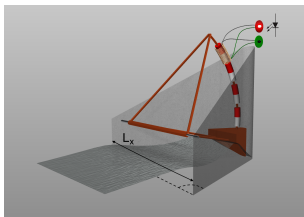


# Analytical and numerical coupling of nonlinear wave motion to buoy dynamics using an inequality constraint

Onno Bokhove, Firedrake 17-09-2025

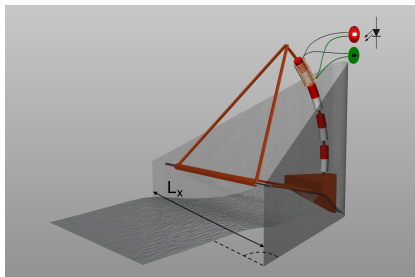
SoM, Leeds Institute for Fluid Dynamics, UK



# Outline

I am reporting the first near-complete (nonlinear) wave-to-wire FEM, made with Firedrake, using an inequality constraint, VP/Hamiltonian, averaged-vector-field “AVF-E” energy conservation, plus symmetric damping of the electric circuits and energy-harvesting load:

- Grand continuum variational principle (VP) entire model plus non-conservative terms.



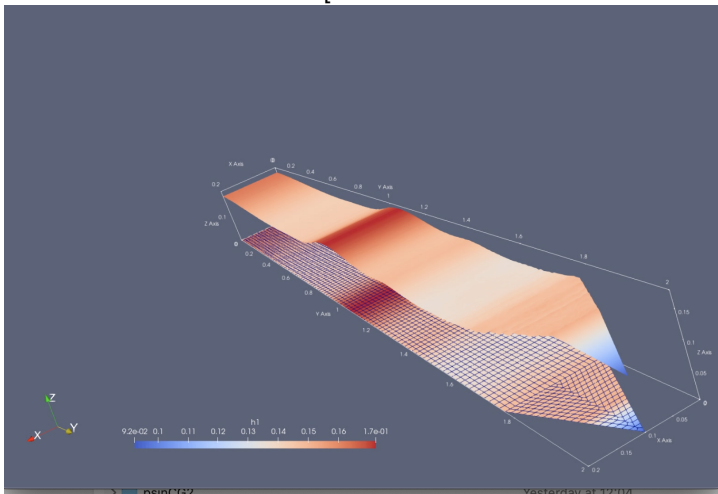
# Outline

**Implementation hierarchy**, **time-discrete AVF-E** with inequality constraint for:

- ▶ a **bouncing ball** under gravity with  $Z \geq 0$ ,
- ▶ a **billiard ball in rectangle** with  $0 \leq X \leq L_x, 0 \leq Y \leq L_y$ ,
- ▶ **water and buoy at rest**, in hydrostatic balance,  
 $h(x, y, t) - h_b(Z(t); x, y) \leq 0$ ,
- ▶ nonlinear water waves, buoy motion & power generation,  
~automated AVF-E, space-discretisation  
~Firedrake-automated via Hamiltonian.

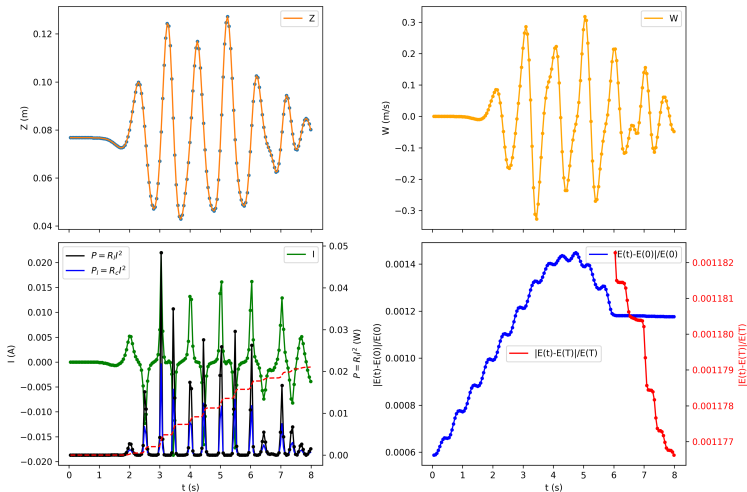
# Main advance: wave-buoy dynamics

- Continuous Galerkin *CG1* [<https://youtu.be/9ATf-jcHGhs>]:



# Main advance: full model single coil, CG1/CG2

## ► Preliminary: fully coupled wave-buoy-generator model



# Grand variational principle of wave-to-wire model

Equations of motion follow from variational principle (**red**=waves, **blue**=buoy, **green**=EM-generator, coupling, B. et al. 2019):

$$0 = \rho_0 \delta \int_0^T \int_0^{L_x} \int_{R(t)}^{l_y(x)} \int_0^h -(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2) dz - gh(\frac{1}{2} h - H_0) \\ - \frac{1}{2\gamma} \left( F_+ (\gamma(h - h_b) - \lambda)^2 - \lambda^2 \right) dy dx \\ MW\dot{Z} - \frac{1}{2} MW^2 - MgZ + (L_i I - \underline{K(Z)})\dot{Q} - \frac{1}{2} L_i I^2 dt \quad (1)$$

velocity  $u = \nabla \phi(x, y, z, t)$ , depth  $h(x, y, t)$ , rest depth  $H_0$ , e.g. buoy  $h_b(Z, y) = Z - K_h - \tan \theta (L_y - y)$ , piston  $R(t)$ , coupling function  $\gamma_m G(Z) = K'(Z)$ , buoy mass  $M$ , keel height  $K_h$ , buoy coordinate  $Z(t)$ , buoy velocity  $W(t) = \dot{Z}$ , charge  $Q(t)$ , current  $I(t) = \dot{Q}$ .

## Wave-to-wire: PDEs

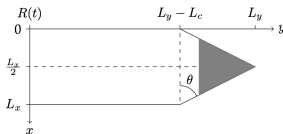
- Potential-flow water-wave dynamics (Laplace equation in interior, kinematic & Bernoulli equations at free surface):

$$\delta\phi : \quad \nabla^2\phi = 0 \quad \text{in} \quad \Omega$$

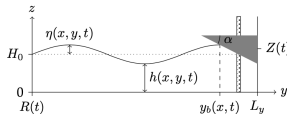
$$(\delta\phi)|_{z=h} : \quad \partial_t h + \nabla\phi \cdot \nabla h = \phi_z \quad \text{at} \quad z = h$$

$$\delta h: \quad \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(z - H_0) - \lambda = 0 \quad \text{at} \quad z = h.$$

- Coupled **elliptic Laplace equation** to **hyperbolic free-surface equations**, plus a (Lagrange) **multiplier**  $\lambda$ .



(b) Top view of the tank and buoy, outlining the tank's dimensions and how the buoy fits the shape of the contraction.



(c) Side view at time  $t$ , with the buoy constrained to move vertically.

# Wave-to-wire: inequality constraint & ODEs

- **Karush-Kuhn-Tucker inequality conditions** satisfied at every space-time  $x, y, t$ -position are (Burman et al. 2023):

$$\delta\lambda : \lambda = -[\gamma(h - h_b) - \lambda]_+ = -F_+(\gamma(h - h_b) - \lambda) \\ \implies \underline{h(x, y, t) - h_b(Z, y) \leq 0}, \lambda \leq 0, \lambda(h - h_b) = 0.$$

- Add **resistance  $R_i, R_c$  & Shockley load  $V_s(|I|)$**  to submodel:

$$\delta W : \dot{Z} = W,$$

$$\delta Z : M\dot{W} = - \underline{Mg - \gamma_m G(Z)I} - \rho_0 \int_0^{L_x} \int_0^{l_y(x)} \lambda \, dy \, dx$$

$$\delta I : \dot{Q} = I,$$

$$\delta Q : L_i \dot{I} = \underline{\gamma_m G(Z)\dot{Z}} - (R_i + R_c)I - \frac{I}{|I|} V_s(|I|).$$



# Discretisation strategy: overview

- Consider Hamiltonian/variational dynamics with energy conservation due to skew-symmetric  $J$ -formulation and an auxiliary variable  $\varphi$ , if any, e.g. with  $\mathcal{H}[h, \phi, \varphi]$  :

$$\partial_t h + \frac{\delta \mathcal{H}}{\delta \phi} = 0, \quad \partial_t \phi - \frac{\delta \mathcal{H}}{\delta h} = 0, \quad \frac{\delta \mathcal{H}}{\delta \varphi} = 0.$$

- More abstractly, with (partial) time derivative, function(al) derivative, (field) variables  $z$ :

$$\dot{z} - J(z) \frac{\delta H}{\delta z} = 0.$$

## Discretisation strategy: AVF-E

- ▶ Automate spatial discretisation using fd.derivative to derive weak forms from Hamiltonian: quicker & fewer errors in (water-wave) cases.
- ▶ Use Averaged Vector Field (AVF) method linear in interval time  $s \in [0, 1]$  to establish second-order time integration, e.g. with  $h(s) = h^n + s(h^{n+1} - h^n)$ , etc.

$$\begin{aligned}\frac{(h^{n+1} - h^n)}{\Delta t} + \int_0^1 \frac{\delta \mathcal{H}[h(s), \phi(s), \varphi(s)]}{\delta \phi(s)} ds &= 0, \\ \frac{(\phi^{n+1} - \phi^n)}{\Delta t} - \int_0^1 \frac{\delta \mathcal{H}[h(s), \phi(s), \varphi(s)]}{\delta h(s)} ds &= 0, \\ \int_0^1 \frac{\delta \mathcal{H}[h(s), \phi(s), \varphi(s)]}{\delta \varphi(s)} ds &= 0.\end{aligned}$$

# Discretisation strategy: AVF-E

- Or abstractly with  $z(s) = z^n + s(z^{n+1} - z^n)$ :

$$\frac{(z^{n+1} - z^n)}{\Delta t} - J(z^{n+1/2}) \int_0^1 \frac{\delta \mathcal{H}[z(s)]}{\delta z(s)} ds = 0.$$

- Automated exact integration in  $s$ , Firedrake Legendre-Gauss quadrature for polynomial-part of Hamiltonian.
- For weak forms, one then uses (that):

$$\begin{aligned} \int_0^1 \frac{\delta \mathcal{H}}{\delta z(s)} \delta z(s) ds &= \int_0^1 \frac{\delta \mathcal{H}}{\delta z(s)} \left( (1-s)\delta z^n + s\delta z^{n+1} \right) ds \\ \text{s.t. } \int_0^1 ((1-s) + s) \frac{\delta \mathcal{H}}{\delta z(s)} ds &= \frac{\delta}{\delta z^n} \int_0^1 H[z(s)] ds + \frac{\delta}{\delta z^{n+1}} \int_0^1 H[z(s)] ds, \end{aligned}$$

- wherein automated fd.derivatives of the  $s$ -integrated Hamiltonian are used wrt  $z^n, z^{n+1}$ .

## Discretisation strategy: AVF-E

The discrete auxiliary equation, if any, follows from energy conservation, e.g., for a case with  $\mathcal{H}[h, \phi, \varphi]$ :

$$\begin{aligned} \frac{\mathcal{H}^{n+1} - \mathcal{H}^n}{\Delta t} = & \iint \int_0^1 \frac{\delta \mathcal{H}}{\delta h(s)} \frac{(h^{n+1} - h^n)}{\Delta t} + \\ & \frac{\delta \mathcal{H}}{\delta \phi(s)} \frac{(\phi^{n+1} - \phi^n)}{\Delta t} + \frac{\delta \mathcal{H}}{\delta \varphi(s)} \frac{(\varphi^{n+1} - \varphi^n)}{\Delta t} ds dx dy = 0, \end{aligned}$$

when we take  $\int_0^1 \langle \delta \mathcal{H} / \delta \varphi(s), v_\varphi \rangle^* ds = 0$ , since we do not have a prognostic equation  $(\varphi^{n+1} - \varphi^n) = 0$  ( $v_\varphi$  has no  $s$ ) and the first two terms cancel out another.

# Discretisation strategy: inequality constraint

- *IC-step-1* The starting point for  $z = (q, p)$  is the augmented Lagrangian formulation

$$0 = \delta \int_0^T p \cdot \dot{q} - H(q, p) - \frac{1}{2\gamma} \left( F_+(\gamma G(q)) - \lambda \right)^2 - \lambda^2 \, dt$$

- Variation of  $\lambda$  leads to Karush-Kuhn-Tucker KKT relations with, recall,  $F_+(Q) \equiv \max(Q, 0)$  for some argument  $Q$ , s.t.

$$\lambda = - F_+(\gamma G(q)) - \lambda \iff \lambda \leq 0, \quad -G(q) \leq 0, \quad \lambda G(q) = 0.$$

## Discretisation strategy: inequality constraint

- ▶ *IC-step-2*  $F_+(Q) = \max(Q, 0)$  smoothed. For some smoothed functions one can solve the Lagrange multiplier  $\lambda$  explicitly.
- ▶ For (well-known) smoothed functions, discretisations poor.
- ▶ For  $F_+(Q) = \frac{1}{2}Q + \sqrt{b^2 + Q^2/4} \rightarrow_{b \rightarrow 0} \max(Q, 0)$ , one finds  $\lambda = -\frac{b^2}{\gamma G(q)}$ .
- ▶ Conjugate equations then contain a derived penalty term:

$$\dot{p} + \frac{\delta \mathcal{H}}{\delta q} + \frac{\partial G(q)}{\partial q} \lambda = \dot{p} + \frac{\delta \mathcal{H}}{\delta q} - \frac{\partial G(q)}{\partial q} \frac{b^2}{\gamma G(q)} = 0.$$

- ▶ Augmented Hamiltonian contains penalty potential:

$$\hat{\mathcal{H}}[q, p] = \mathcal{H}[q, p] - \frac{b^2}{\gamma} \ln G(q).$$

# Inequality constraint: superfunction

- *IC-step-3* Recall that the KKT-relations

$$\lambda \leq 0, \quad -G(q) \leq 0, \quad \lambda G(q) = 0,$$

in the  $G(q), \lambda$ -plane, delineate fourth quadrant.

- Hence, seek a (straightforward) “superfunction” approximating this delination &  $\lambda = -F_+(\gamma G(q) - \lambda)$ , i.e.,

$$\lambda = -\frac{1}{b}e^{-bG(q)}.$$

- Only an implicit  $\sim$ definition of  $F_+(Q)$  can then be found:

$$Q = \frac{\gamma}{b} \ln(bF_+(Q)) + F_+(Q).$$

# Inequality constraint: superfunction & AVF-E

- Superfunction leads to an augmented Hamiltonian:

$$\widehat{\mathcal{H}}[q, p] = \mathcal{H} + \frac{1}{b^2} e^{-bG(q)}.$$

- *IC-step-4* For constraints linear in  $q$ , such that  $G(q) = c_j q_j$ , the AVF-E integral becomes

$$0 = \frac{(p_j^{n+1} - p_j^n)}{\Delta t} + \int_0^1 \frac{\delta \mathcal{H}}{\delta q_j(s)} - \frac{c_j}{b} e^{-bc_k q_k(s)} ds,$$

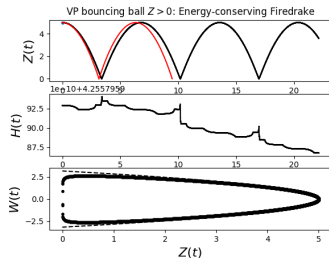
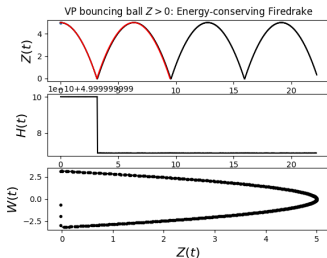
wherein the last integral becomes (fd.conditional in Firedrake)

$$- \frac{c_j}{b} e^{-\frac{1}{2} bc_l (q_l^{n+1} + q_l^n)} \frac{\sinh\left(\frac{1}{2} bc_l (q_l^{n+1} - q_l^n)\right)}{\frac{1}{2} bc_l (q_l^{n+1} - q_l^n)} \rightarrow_{b \rightarrow 0} - \frac{c_j}{b} e^{-\frac{1}{2} bc_l (q_l^{n+1} + q_l^n)}.$$



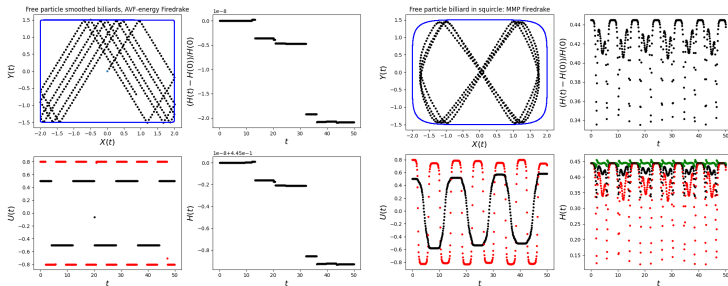
# Bouncing ball under gravity $Z \geq 0$

- Parameters  $b = 500, \Delta t = 0.035$ .
- Relationship between  $b$  and  $\Delta t$ .
- Solvers (best to date): nest, vinewtonrsls



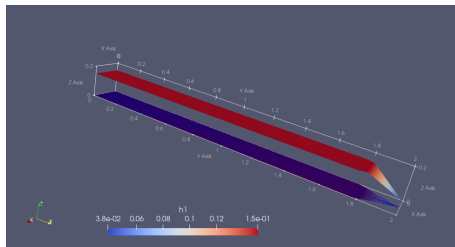
# Billards in rectangle

- Parameters  $b = 1000, 2000, \Delta t = 0.015, 0.0075$ .
- Relationship between  $b$  and  $\Delta t$ .
- Solvers (best to date): nest, default



# Wave-energy wave tank: initial condition rest state

- ▶ Starting point: used continuation in  $b = 10, 100, 1000, 2000, 4000$  then nondimensionalisation. Rest system  $\{W, \phi\}$  solved, eqns other variables  $\{Z, h, \psi, I\}$  enforced to be linear & zero.
- ▶ Solvers: “bespoke” for rest state, default for dynamics.



# Wave-energy wave tank: full dynamics –09-2025

- ▶ The conservative dynamics has non-canonical components in the variables  $\{W, I\}$ , which non-constant  $J(Z)$ -bracket terms, involving the nonlinear function  $G(Z)$  arising from solving the 3D Maxwell's equations for axisymmetric induction coil(s), are evaluated at  $Z^{n+1/2}$  (per review of Cotter 2023).
- ▶ The Variational Boussinesq Model (VBM) is used, a potential flow model with quadratic polynomial accuracy integrated out in  $z$  (e.g., Gagarina et al. 2013) with canonical variables  $\{h(x, y, t), \phi(x, y, t)\}$  and interior velocity potential degree-of-freedom  $\psi(x, y, t)$ .

# Wave-energy wave tank: full dynamics –09-2025

- Complicated yet polynomial VBM Hamiltonian ( $\beta = 1$ ; Green-Naghdi system when  $\beta = 0$ ), whence AVF-E automation in Firedrake:

$$\begin{aligned}\mathcal{H}[h, \phi, \psi] &= \iint \frac{1}{2} h |\bar{u}|^2 + \frac{1}{6} h^3 \psi^2 + \frac{1}{2} g \left( (h+b)^2 - b^2 \right) - ghH_0 + \frac{\beta}{90} h^5 |\nabla \psi|^2 \, dx \, dy \\ &= \iint \frac{1}{2} h |\nabla \phi + h\psi \nabla h + \frac{1}{3} h^2 \nabla \psi|^2 + \frac{1}{6} h^3 \psi^2 + \frac{1}{2} g \left( (h+b)^2 - b^2 \right) \\ &\quad - ghH_0 + \frac{\beta}{90} h^5 |\nabla \psi|^2 + \frac{1}{b^2} e^{-b(h_b(Z) - h(x,y,t))} \, dx \, dy.\end{aligned}$$

- Solvers used default: CG2 <https://www.youtube.com/watch?v=p7wteD0gMGo>

# Discussion

**Nearly completed full numerical wave-to-wire model** of the wave-energy device, 2<sup>nd</sup>-order in time & spectral in space.

Crucial [steps](#) have been to use, combine & integrate:

- ▶ augmented Lagrangian formalism of inequality constraints (Burman et al. 2023) with smoothed KKT relations;
- ▶ a superfunction solving and modifying the Lagrange multiplier into a derived penalty term/potential, without introducing any singularity (such as to accommodate nonlinear solvers) & with an implicit definition of the  $F_+(Q) = \max(Q, 0)$ -function, and
- ▶ AVF-energy conservation second-order time-integration strategy, using exact (numerical & analytical) integration.

# Discussion

Future work, on AVF-E &  $\lambda$ -function(s):

- ▶ Better, faster solvers: help (-)!
- ▶ AVF-E in Irksome?
- ▶ 3D potential flow (mostly done), yielding logarithmic AVF-E singularities.
- ▶ ....
- ▶ **Many thanks** to Prof Harvey Thompson & Prof Colin Cotter.  
All errors made are mine.
- ▶ **Finally, a big thank you to:** Claire Savy and Deborak Clarke from LIFD, Dave Stones from SoM, and Colin and David from IC. Funding from: **Firedrake, LIFD, and SoM**. Thank *you*!

# Nonlinear inequality constraints

Project constraint using an auxiliary variable  $\mu$  by adding the following term to the energy, instead of the penalty potential

$$\delta \frac{1}{b} (G(\mathbf{q}) - (\mu + 1/b)) e^{-b\mu} = \frac{1}{b} \nabla G(\mathbf{q}) e^{-b\mu} \delta \mathbf{q} + (\mu - G(\mathbf{q})) e^{-b\mu} \delta \mu.$$



