Building a wave-to-wire finite-element wave-energy model in Firedrake

Onno Bokhove [et al.], Oxford, Firedrake 18-09-2024
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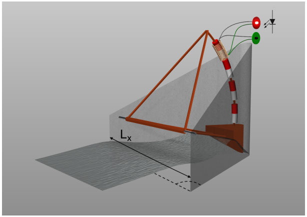
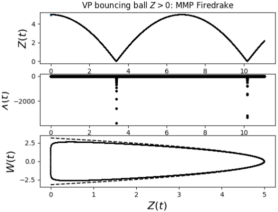
SoM, Leeds Institute for Fluid Dynamics, UK





Outline

Ongoing build-up to complete (nonlinear) wave-to-wire FEM within Firedrake using (time-discrete) VPs with nontrivial damping of the electric circuits and energy-harvesting load:

- Grand continuum variational principle (VP) entire model plus non-conservative terms.

Implementation hierarchy:

- time-discrete VP for (hanging and driven) nonlinear buoy-generator model
- (time-discrete VP with inequality constraint for bouncing ball under gravity with $Z \geq 0$)
- time-discrete VP for water and buoy at rest, in hydrostatic balance
- time-discrete VP for water waves and buoy in motion
- then **full coupled model should work** ...?

Why variational principles (VPs)? Advantages:

- When the (main) dynamics has a VP, multiple coupled equations are succintly described by one space-time VP.
- Associated with a VP are conservation properties of the resulting PDEs/ODEs.
- Within the *(finite-element) environment Firedrake*, the time-discrete VP can be implemented directly, with automated generation of (complicated 3D+1D) weak forms of the equations.
- Advantages: enormous reduction in development time, efficient, flexible, higher-order spectrally-accurate space discretisations plus (automatic) preservation of discrete forms of conservation properties.
- VP for 3D+1D nonlinear water waves as potential flow: implemented and tested (Choi et al. 2024, Lu et al. 2024).

Grand VP of wave-to-wire model

Equations of motion follow from variational principle (red=waves, blue=buoy, green=EM-generator, coupling, B. et al. 2019):

$$
\begin{aligned}
0 =& \delta \int_0^T \int_0^{L_x} \int_{R(t)}^{l_y(x)} \int_0^h -(\partial_t \phi + \frac{1}{2}|\nabla\phi|^2)\mathrm{d}z - gh(\frac{1}{2}h - H_0) \\
& \underline{- \frac{1}{2\gamma}\Big(F_+(\gamma(h-h_b)-\lambda)^2 - \lambda^2\Big)}\mathrm{d}y\mathrm{d}x \\
& MW\dot{Z} - \frac{1}{2}MW^2 - MgZ + (L_i I - \underline{K(Z)})\dot{Q} - \frac{1}{2}L_i I^2 \mathrm{d}t \qquad (1)
\end{aligned}
$$

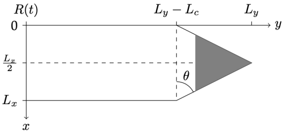
velocity $\mathrm{u} = \nabla\phi(x,y,z,t)$, depth $h(x,y,t)$, rest depth H_0, buoy $h_b(Z,y) = Z - K_h - \tan\theta(L_y - y)$, piston $R(t)$, coupling function $\gamma_m G(Z) = K'(Z)$, buoy mass M, keel height K_h, buoy coordinate $Z(t)$, buoy velocity $W(t) = \dot{Z}$, charge $Q(t)$, current $I(t) = \dot{Q}$.

Grand VP: PDEs

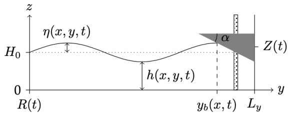
- Potential-flow water-wave dynamics (Laplace equation in interior, kinematic & Bernoulli equations at free surface):

$$
\begin{aligned}
\delta\phi: &\quad \nabla^2\phi = 0 \quad \text{in} \quad \Omega \\
(\delta\phi)|_{z=h}: &\quad \partial_t h + \nabla\phi\cdot\nabla h = \phi_z \quad \text{at} \quad z = h \\
\delta h: &\quad \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g(z - H_0) - \lambda = 0 \quad \text{at} \quad z = h.
\end{aligned}
$$

- Coupled elliptic Laplace equation to hyperbolic free-surface equations, plus a (Lagrange) **multiplier** λ.



(b) Top view of the tank and buoy, outlining the tank's dimensions and how the buoy fits the shape of the contraction.



(c) Side view at time t, with the buoy constrained to move vertically.

Grand VP: inequality constraint & ODEs

- Karush-Kuhn-Tucker inequality conditions satisfied at every space-time x, y, t-position are (Burman et al. 2023):

$$\delta\lambda : \lambda = -[\gamma(h-h_b)-\lambda]_+ = -F_+(\gamma(h-h_b)-\lambda)$$
$$\Longrightarrow \underline{h(x,y,t)-h_b(Z,y)\le 0}, \lambda \le 0, \lambda(h-h_b)=0.$$

- Add resistance R_i, R_c & Shockley load $V_s(|I|)$ to submodel:

$$\delta W : \ \dot{Z} = W,$$
$$\delta Z : \ M\dot{W} = -Mg\underline{-\gamma_m G(Z) I} - \int_0^{L_x}\int_0^{l_y(x)} \lambda \, \mathrm{d}y\mathrm{d}x$$
$$\delta I : \ \dot{Q} = I,$$
$$\delta Q; \ \ L_i \dot{I} = \underline{\gamma_m G(Z)\dot{Z}} - (R_i + R_c) I - \frac{I}{|I|} V_S(|I|).$$

VP buoy-motor

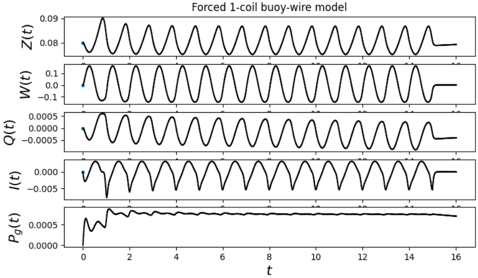
The full modified midpoint time-discrete variational principle for the single-coil model reads

$$
\begin{aligned}
0 = \delta \Bigg(& M W^{n+1/2} \frac{(Z^{n+1} - Z^n)}{\Delta t} - M Z^{n+1/2} \frac{(W^{n+1} - W^n)}{\Delta t} \\
& - \frac{1}{2} M \left(W^{n+1/2} \right)^2 - \underline{M W^{n+1/2} \dot{f}^{n+1/2}} - \frac{1}{2} k (Z^{n+1/2} - \bar{Z} - \underline{f^{n+1/2})^2} \\
& + \left(L_i I^{n+1/2} - K(Z^{n+1/2}) \right) \frac{(Q^{n+1} - Q^n)}{\Delta t} \\
& - Q^{n+1/2} \left(L_i \frac{(I^{n+1} - I^n)}{\Delta t} - \frac{K(Z^{n+1}) - K(Z^n)}{\Delta t} \right) - \frac{1}{2} L_i \left(I^{n+1/2} \right)^2 \Bigg)
\end{aligned}
$$

Terms stemming from $K(Z)$ (implementation ito VP does not work in FD) implemented into two weak equations.

VP buoy-motor

- Variations taken wrt $\{Z^{n+1/2}, Q^{n+1/2}, W^{n+1/2}\}$, augmented with $Z^{n+1} = Z^{n+1/2} - Z^n, W^{n+1} = 2W^{n+1/2} - W^n, I^{n+1} = 2I^{n+1/2} - I^n$ ("replace") plus $Q^{n+1} = Q^n + \Delta t I^{n+1/2}$, (Gagarina 2014, Choi et al. 2024).
- So, $\{Z^{n+1}, W^{n+1}, I^{n+1}\}$ eliminated after variations have been taken using mid-point definitions. SE & MMP same:



VP code buoy-motor

```
VPnl = (1/Lx)*( Mm*W12*(Z1-Z0)/dt - Mm*Z12*(W1-W0)/dt - Mm*W12*force12dt \
       - 0.5*Mm*W12**2 - rfac*0.5*kspring*(Z12-Zbar-force12)**2 \
       + (Li*Ic12)*(Q1-Q0)/dt - Q12*Li*(I1-I0)/dt - 0.5*Li*Ic12**2 )*fd.dx(degree=vpolyp
Z_expr = fd.derivative(VPnl, W12, du=vvmpc2) # du=v_R eqn for Z1
Z_expr = fd.replace(Z_expr, {Z1: 2*Z12-Z0})
W_expr = fd.derivative(VPnl, Z12, du=vvmpc0) # du=v_R eqn for W1
W_expr = W_expr + (1/Lx)*( -vvmpc0*(gamma*GZZ12*Ic12 ) )*fd.dx(degree=vpolyp) # Add buoy
W_expr = fd.replace(W_expr, {W1: 2*W12-W0}) # W1 = 2*Wh12-W0

I_expr = fd.derivative(VPnl, Q12, du=vvmpc1) # du=v_R eqn for I1
I_expr = I_expr - (1/Lx)*( vvmpc1*((Rc+Ri+lfac*Rl)*Ic12 + tfac*fd.conditional(AbsI<small
) ) *fd.dx(degree=vpolyp)
I_expr = fd.replace(I_expr, {Z1: 2*Z12-Z0})
I_expr = fd.replace(I_expr, {I1: 2*Ic12-I0})

Fexpr = Z_expr+W_expr+I_expr

solver_parameters6 = {'sns_type': 'newtonls','sns_atol': 1e-19,'mat_type': 'aij'}
solve1coil_l = fd.NonlinearVariationalSolver(fd.NonlinearVariationalProblem(Fexpr, \
                result_mixedmpc),solver_parameters=solver_parameters6)
```

Vertically-falling ball under gravity with $Z(t) \geq 0$

Continuous in time:

- VP of falling ball with unit mass $M = 1$ without constraint:

$$
\begin{aligned}
0 =&\delta\mathcal{F} = \delta \int_0^T L(Z,W)\,\mathrm{d}t \equiv \delta \int_0^T W\dot{Z} - \frac{1}{2}W^2 - Z\,\mathrm{d}t,\\
&\equiv \lim_{\epsilon\to 0} \int_0^T \frac{L(Z+\epsilon\delta Z, W+\epsilon\delta W) - L(Z,W)}{\epsilon}\,\mathrm{d}t
\end{aligned}
$$

  time t, acceleration $g = 1$, kinetic & potential energy MgZ.
- Minimisation problem with *virtual* changes $z_p = \delta Z$ and $w_p = \delta W$, i.e. $\delta Z(0) = \delta Z(T) = 0$.
- Newton's equations for position $Z(t)$ and velocity $W(t) \equiv \dot{Z}$:

$$
0 = \int_0^T (\dot{Z} - W)\delta W - (\dot{W}+1)\delta Z\,\mathrm{d}t : \dot{Z} = W, \quad \dot{W} = -1.
$$

Vertically-falling ball under gravity with $Z(t) \geq 0$

Continuous in time:

- VP of falling ball with inequality constraint:

$$\begin{aligned} 0 =&\delta \int_0^T L(Z, W)\,\mathrm{d}t \\ \equiv&\delta \int_0^T W\dot{Z} - \frac{1}{2}W^2 - Z - \frac{1}{2\gamma}\left(F_+(-\gamma Z - \lambda)^2 - \lambda^2\right)\,\mathrm{d}t, \end{aligned}$$

- Resulting equations:

$$\begin{aligned} \delta W: \quad & \dot{Z} = W \\ \delta Z: \quad & \dot{W} = -1 - \lambda \\ \delta \lambda: \quad & \lambda = -F_+(-\gamma Z - \lambda)F'_+(-\gamma Z - \lambda) \\ & \quad = -F_+(-\gamma Z - \lambda) \Longleftrightarrow \underline{-Z \leq 0}, \lambda \leq 0, \lambda Z = 0. \end{aligned}$$

Vertically-falling ball: smoothing

For $b > 0$, approximations of F_+ include:

$$
\begin{aligned}
F_+(q) =& \frac{1}{2}q + \sqrt{b^2 + \frac{1}{4}q^2} \rightarrow_{b\rightarrow 0} \max(q,0) \quad \text{with}\\
F_+'(q) =& \frac{1}{2} + \frac{\frac{1}{4}q}{\sqrt{b^2 + \frac{1}{4}q^2}} \rightarrow_{b\rightarrow 0} \Theta(q) \quad \text{or}\\
F_+(q) =& \ln(1+e^{bq})/b \rightarrow_{b\rightarrow 0} \max(q,0). \qquad (2)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\lambda =& -F_+(-\gamma Z - \lambda) = -\left(-\frac{1}{2}(\gamma Z + \lambda) + \sqrt{b^2 + (\gamma Z + \lambda)^2/4}\right)\\
\iff & \frac{1}{2}(\lambda - \gamma Z) = -\sqrt{b^2 + (\gamma Z + \lambda)^2/4} \implies\\
-\gamma Z \lambda =& b^2 \quad \text{for} \quad Z \geq 0 \iff \lambda = -\frac{b^2}{\gamma Z} \quad \text{for} \quad Z \geq 0.
\end{aligned}
$$

Vertically-falling ball: phase plot

Therefore, equations become as follows and can be solved

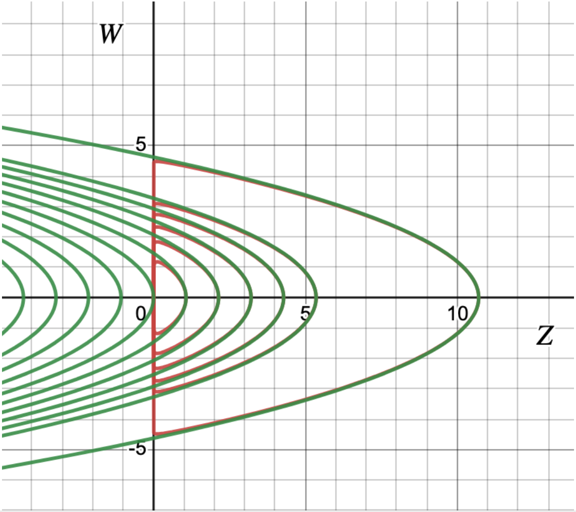
$$\dot{Z} = W, \dot{W} = -1 + \frac{b^2}{\gamma Z} \Longleftrightarrow \ddot{Z} = -1 + \frac{b^2}{\gamma Z} \Longrightarrow$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}W^2 + Z - \frac{b^2}{\gamma}\ln Z\right) = 0 \Longleftrightarrow$$

$$\frac{1}{2}W^2 + Z - \frac{b^2}{\gamma}\ln Z = H_0$$

with integration constant/energy $H_0 = H(t)$. When $W = 0$ maximum $Z = Z_{max}$ satisfies $Z_{max} - b^2/(\gamma Z_{max}) = H_0$ with as first approximation $Z_{max} \approx H_0 = H_1 - b^2/(\gamma)\ln H_1$, where $\frac{1}{2}W^2 + Z = H_1$. Make phase plot in (Z, W)-plane:

Vertically-falling ball: phase plot



Vertically-falling ball under gravity with $Z(t) \geq 0$

Time discrete VP:

- 2nd-order modified mid-point VP of falling ball with constraint:

$$0 = \delta\Big(W^{n+1/2}\frac{Z^{n+1}-Z^n}{\Delta t} - Z^{n+1/2}\frac{W^{n+1}-W^n}{\Delta t} - \frac{1}{2}(W^{n+1/2})^2 - Z^{n+1/2} - \frac{1}{2\gamma}\Big(F_+(-\gamma Z^{n+1/2}-\lambda)^2 - \lambda^2\Big)\Big)$$

with additional relations

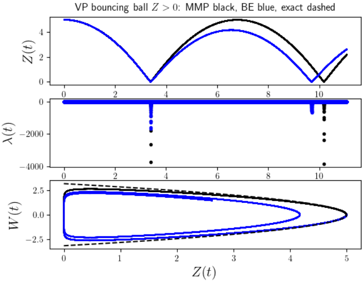
$$Z^{n+1} = 2Z^{n+1/2} - Z^n \text{ and } W^{n+1} = 2W^{n+1/2} - W^n.$$

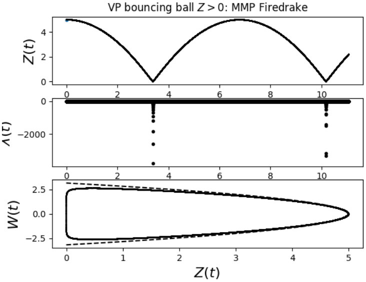
Vertically-falling ball under gravity with $Z(t) \geq 0$

- Resulting time-discrete equations (variations wrt $Z^{n+1/2}, W^{n+1/2}$):

$$\begin{aligned}
\delta W^{n+1/2}:\ \ Z^{n+1} =& Z^n + \Delta t W^{n+1/2} \Longrightarrow \\
& Z^{n+1/2} = Z^n + \frac{1}{2}\Delta t W^{n+1/2} \\
\delta Z^{n+1/2}:\ \ W^{n+1} =& W^n - \Delta t(1+\lambda) \\
\Longrightarrow \frac{4(Z^{n+1/2} - Z^n)}{\Delta t} =& 2W^n - \Delta t(1+\lambda) \\
\delta\lambda:\ \ \lambda = -F_+(-\gamma Z^{n+1/2} - \lambda)&F_+'(-\gamma Z^{n+1/2} - \lambda).
\end{aligned}$$

Vertically-falling ball under gravity with $Z(t) \geq 0$





VP buoy and water at rest

Strategy on rest-flow buoy-water surface coupling:

- Goal is to solve Bernoulli & KKT inequality equations:

$$\delta h: \quad g(z - H_0) - \lambda = 0 \quad \text{at} \quad z = h.$$

$$\delta Z: \quad \int_0^{L_x} \int_0^{l_y(x)} \lambda \mathrm{d}y\mathrm{d}x + Mg = 0$$

$$\delta\lambda: \lambda = -[\gamma(h - h_b(Z,y)) - \lambda]_+ = -F_+(\gamma(h - h_b(Z,y)) - \lambda)$$

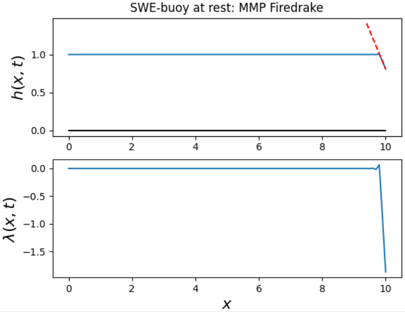
$$\Longrightarrow \underline{h(y,t) - h_b(Z,y) \leq 0}, \lambda \leq 0, \lambda(h - h_b(Z,y)) = 0.$$

with $h_b(Z,y) = Z - K_h - \tan\theta(L_y - y)$.

VP buoy and water at rest

- Solve VP for $h(x,y), Z$ and λ-equation with $F_+(q) = \ln(1+e^{aq})/b$ (buoy of finite extent L_w):

$$
\begin{aligned}
0 =& \delta \left(\int_0^{L_x} \int_0^{l_y(x)} gh(\frac{1}{2}h - H_0) + \lambda(h_b(Z,y) - h)\,\mathrm{d}y\mathrm{d}x + MgZ \right)\\
\delta\lambda : 0 =& \int_0^{L_x} \int_0^{l_y(x)} \left(\lambda + \begin{cases} F_+(\gamma(h-h_b)-\lambda) & L_y - L_w < y < L_y \\ 0 & y \le L_y - L_w \end{cases} \right) \delta\lambda\,\mathrm{d}y\mathrm{d}x
\end{aligned}
$$



VP dynamic buoy and water motion (BLE for dispersion)

Strategy on buoy-water surface coupling (work in progress):

- Goal is to solve Bernoulli & KKT inequality equations:

$$\delta h: \quad \partial_t \phi + \cdots + \frac{1}{2}|\nabla \phi|^2 + g(z - H_0) - \lambda = 0 \quad \text{at} \quad z = h.$$

$$\delta \phi: \quad \partial_t h + \cdots + \nabla \cdot (h \nabla \phi) = 0, \ldots$$

$$\delta Z: \quad M\dot{W} + \int_0^{L_x} \int_0^{l_y(x)} \lambda \mathrm{d}y \mathrm{d}x + Mg = 0$$

$$\delta \lambda: \lambda = -[\gamma(h - h_b(Z, y)) - \lambda]_+ = -F_+(\gamma(h - h_b(Z, y)) - \lambda)$$

$$\Longrightarrow \underline{h(x, y, t) - h_b(Z, y) \leq 0}, \lambda \leq 0, \lambda(h - h_b(Z, y)) = 0.$$

with $h_b(Z, y) = Z - K_h - \tan\theta(L_y - y)$.

VP dynamic buoy and water motion (BLE for dispersion)

Strategy on buoy-water surface coupling (work in progress):

- Solve VP for $h(x,y), Z$ and imposed, solved, separated λ-solution (buoy has been given finite extent L_w; $R(t)=0$):

$$
\begin{aligned}
0 =& \delta \Bigg(\int_0^{L_x} \int_0^{l_y(x)} \int_0^{h^{n+1/2}} -\frac{1}{2}|\nabla \phi^{n+1/2}|^2 \mathrm{d}z + \phi_s^{n+1/2} \frac{(h^{n+1}-h^n)}{\Delta t} - h^{n+1/2} \frac{(\phi_s^{n+1}-\phi_s^n)}{\Delta t} + \dots \\
& - g h^{n+1/2} (\frac{1}{2} h^{n+1/2} - H_0) - \lambda (h_b(Z^{n+1/2}, y) - h^{n+1/2}) \,\mathrm{d}y\mathrm{d}x \\
& + M W^{n+1/2} \frac{(Z^{n+1}-Z^n)}{\Delta t} - M Z^{n+1/2} \frac{(W^{n+1}-W^n)}{\Delta t} - \frac{1}{2} M (W^{n+1/2})^2 - M g Z^{n+1/2}
\end{aligned}
$$

$$
\delta\lambda : 0 = \int_0^{L_x} \int_0^{l_y(x)} \left(\lambda + \begin{cases} F_+(\gamma(h^{n+1/2} - h_b(y, Z^{n+1/2})) - \lambda)) & L_y - L_w < y < L_y \\ 0 & y \le L_y - L_w \end{cases} \right) \delta\lambda \,\mathrm{d}y\mathrm{d}x
$$

with $F_+(q) = \ln(1+e^{aq})/b$. Rest flow in dynamic case stays rest flow!

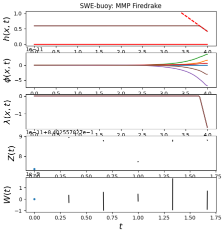
VP code buoy-waves

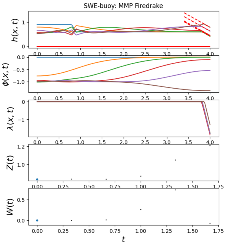
```
VPnl = ( facs*phi12*(h1-h0)/dt-facs*h12*(phi1-phi0)/dt -0.5*h12*(phi12.dx(0))**2 \
       - gg*h12*(0.5*h12-H0) + 0.5*gg*H0**2 -(0.5/gamm)*(Fplus**2-lamb12**2) ) \
       *fd.dx(degree=vpolyp)
VPnl = VPnl - facs*( 0.5*muu*h12.dx(0)*(phi1.dx(0)-phi0.dx(0))/dt \
       -0.5*muu*phi12.dx(0)*(h1.dx(0)-h0.dx(0))/dt )*fd.dx(degree=vpolyp)
VPnl = VPnl - ( muu*(q12.dx(0)*phi12.dx(0)-0.75*q12**2) )*fd.dx(degree=vpolyp)
VPnl = VPnl + (1/Lx)*( facs2*mm*W12*(Z1-Z0)/dt-facs2*mm*Zh12*(W1-W0)/dt \
       -0.5*mm*W12**2-mm*gg*Zh12)*fd.dx(degree=vpolyp)
phi_expr = fd.derivative(VPnl, h12, du=vvmpc0) #
phi_expr = fd.replace(phi_expr, {phi1: 2*phi12-phi0})
h_expr = fd.derivative(VPnl, phi12, du=vvmpc1) #
h_expr = fd.replace(h_expr, {h1: 2*h12-h0})
W_expr = fd.derivative(VPnl, Zh12, du=vvmpc3) #
W_expr = fd.replace(W_expr, {W1: 2*W12-W0}) #
Z_expr = fd.derivative(VPnl, W12, du=vvmpc4) #
Z_expr = fd.replace(Z_expr, {Z1: 2*Zh12-Z0}) #
Fexpr = h_expr+phi_expr+lamb_expr+Z_expr+W_expr

solvelamb_nl0 = fd.NonlinearVariationalSolver(fd.NonlinearVariationalProblem(Fexpr0, res
                solver_parameters=solver_parameters1s)
solvelamb_nl = fd.NonlinearVariationalSolver(fd.NonlinearVariationalProblem(Fexpr, resul
```

To do/Firedrake Questions

- time-discrete VP for water waves and buoy in motion
- then **then full coupled model should work** . . . ?
- [redacted]: Use the in-build Firedrake inequality solvers? Not geometric in time. Other (non-[redacted]) time integrators?
- How should one manage and deal with these solver-parameter settings?





Thank you very much for your attention ...

- B., Zweers 2013: Proof of principle 2013 https://www.youtube.com/watch?v=SZhe_SOxBWo&t=254s
- B., Kalogirou, Zweers 2019: From bore-soliton-splash to a new wave-to-wire wave-energy model. *Water Waves* **1** 10.1007/s42286-019-00022-9 Bore-soliton-splash: https://www.youtube.com/watch?v=YSXsXNX4zW0&list=FL6mc7mUa6M4Bo2VkD970urw
- Choi, Kalogirou, Lu, B., Kelmanson 2024: A study of extreme water waves using a hierarchy of models based on potential-flow theory. *Water Waves* https://doi.org/10.1007/s42286-024-00084-4
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- Lu, Gidel, Choi, B., Kelmanson 2024: Submitted. *J. Comp. Phys.*.