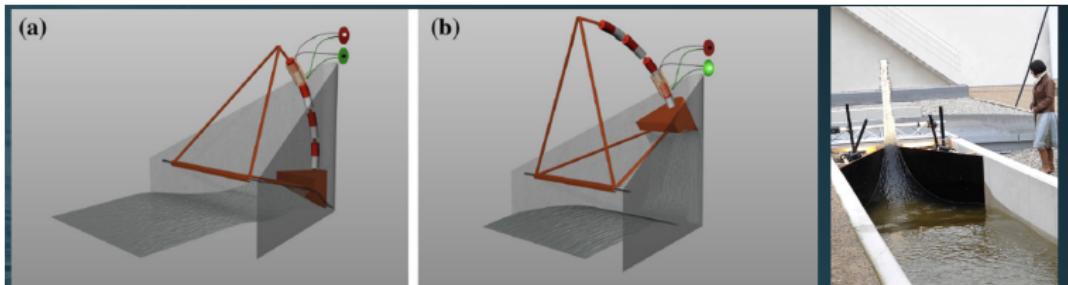


Power optimisation of a heaving buoy in a wave-enhancing contraction (+)

Onno Bokhove, P. Grieve, J. Kim, J. Lee, O. Naar, H. Thompson,
Exeter 13-06-2025

£: CDT2 Fluid Dynamics, teaching

Leeds Institute for Fluid Dynamics, UK

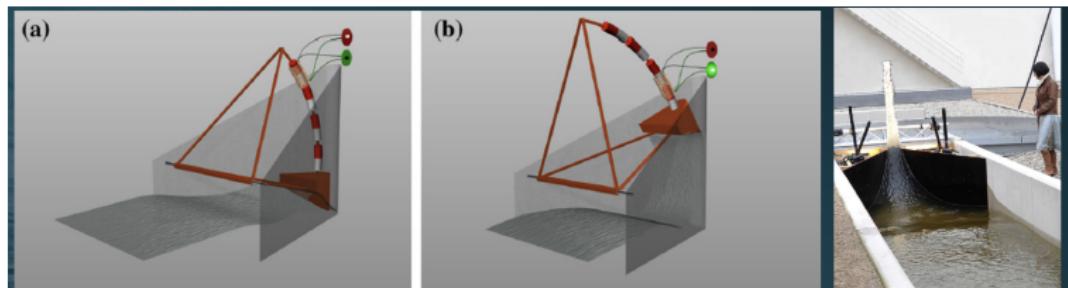


Outline

Overview of our work on the dynamics of:

- ▶ a novel **wave-energy device**, B. et al (2019, 2020, IEEE2024).
- ▶ Based on **extreme and extremely-high water waves**, [e.g., Kadomtsev & Petviashvili 1970, Benney & Luke 1964, Luke 1967, Kodama (2010, 2018), B. and Kalogirou 2016, Choi et al. (2022, 2024)].
- ▶ Inspired by 2010 bore-soliton-splash:

<https://www.youtube.com/watch?v=YSXsXNX4zW0>



Outline

Our variational, mathematical and numerical, (non)linear wave-to-wire model consists of three coupled components:

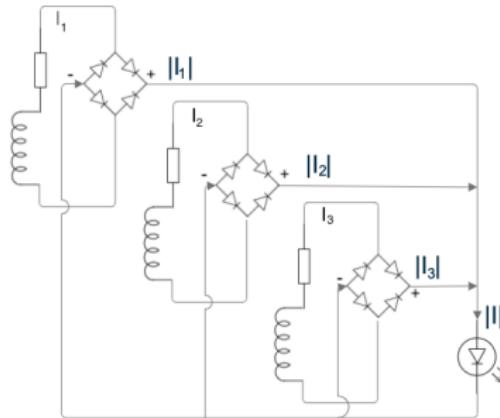
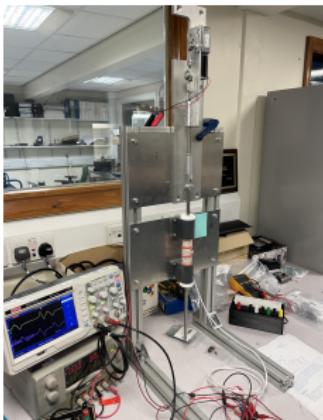
- ▶ (i) **hydrodynamics**, using nonlinear potential flow with (free) water surface,
- ▶ (ii) **buoy motion** in 1D, using (vertical) position and velocity, with magnets leading to
- ▶ (iii) **power generation**, using charge and current equations via coil(s), and a load.

I will present: variational principle (full) model, partial model results, including first generator measurements, and open alleys.

Model partitioning

It is ideally placed in a breakwater or in an array of contractions moored at sea. Model partitioning:

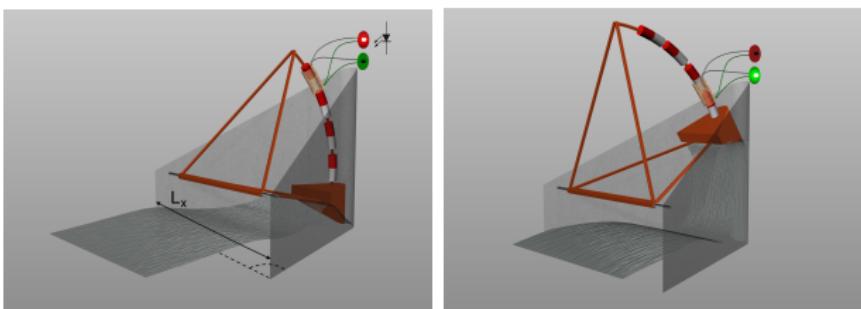
- ▶ (iii) generator model with prescribed magnet motion,
- ▶ (ii)–(iii) a buoy-generator sub-model, and
- ▶ (i)–ii) a hydrodynamic wave and buoy sub-model.



Novel wave-energy device in a breakwater contraction

Proof of principle: https://www.youtube.com/watch?v=SZhe_S0xBWo&t=254s.

Sketch wave amplification in contraction with angle θ_c :



Grand variational principle of novel wave-energy device

Equations of motion follow from variational principle (**blue**=waves, **red**=buoy, **green**=EM-generator, **coupling**):

$$0 = \delta \int_0^T \rho \int_0^{L_x} \int_{R(t)}^{l_y(x)} \int_0^h -(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2) dz - gh(\frac{1}{2} h - H_0) \\ - \frac{1}{2\gamma} (F_+(\gamma(h - h_b(Z, x)) - \lambda)^2 - \lambda^2) dy dx \\ MW \dot{Z} - \frac{1}{2} MW^2 - MgZ + (L_i I - K(Z)) \dot{Q} - \frac{1}{2} L_i I^2 dt$$

velocity $u = \nabla \phi(x, y, z, t)$, depth $h(x, y, t)$, rest depth H_0 , buoy $h_b(Z, x) = Z - K - \tan \theta(L_y - x)$, piston $R(t)$, coupling function $\gamma_m G(Z) = K'(Z)$, buoy mass M , keel height K , buoy coordinate $Z(t)$, buoy velocity $W(t) = \dot{Z}$, charge $Q(t)$, current $I(t) = \dot{Q}$.

Mathematical modelling water-buoy: PDEs

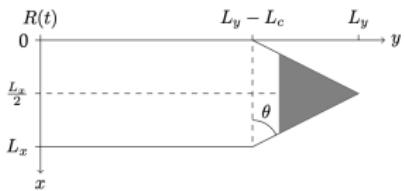
- Potential-flow water-wave dynamics (Laplace equation in interior, kinematic & Bernoulli equations at free surface):

$$\delta\phi : \nabla^2 \phi = 0 \quad \text{in } \Omega$$

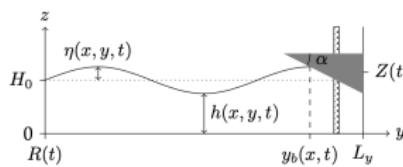
$$(\delta\phi)|_{z=h} : \partial_t h + \nabla \phi \cdot \nabla h = \phi_z \quad \text{at } z = h$$

$$\delta h : \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(z - H_0) - \lambda = 0 \quad \text{at } z = h.$$

- Coupled elliptic Laplace equation to hyperbolic free-surface equations; (Lagrange) multiplier λ (neg. pressure on buoy).



(b) Top view of the tank and buoy, outlining the tank's dimensions and how the buoy fits the shape of the contraction.



(c) Side view at time t , with the buoy constrained to move vertically.

Mathematical modelling buoy-surface: inequality constraint

- ▶ Define $F_+(q) \equiv \max(q, 0)$.
- ▶ Karush-Kuhn-Tucker inequality conditions satisfied at every space-time x, t -position are:

$$\begin{aligned} \delta\lambda : \lambda &= -[\gamma(h - h_b) - \lambda]_+ = -F_+(\gamma(h - h_b) - \lambda) \\ \implies \underline{h(x, t) - h_b(Z, x)} &\leq 0, \lambda \leq 0, \lambda(h - h_b) = 0. \end{aligned}$$

- ▶ For csts $\gamma, b > 0$ (-), approximations of F_+ include:

$$F_+(q) = \frac{1}{2}q + \sqrt{b^2 + \frac{1}{4}q^2} \xrightarrow{b \rightarrow 0} \max(q, 0) \quad \text{with}$$

$$F'_+(q) = \frac{1}{2} + \frac{\frac{1}{4}q}{\sqrt{b^2 + \frac{1}{4}q^2}} \xrightarrow{b \rightarrow 0} \Theta(q).$$

Mathematical modelling buoy-generator: ODEs

- ▶ Solve 3D Maxwell's eqns for cylindrical magnet-single-coil set-up: reduction from PDEs to 4 ODEs with coupling function $G(Z) = K'(Z)$.
- ▶ Add **resistance R_i, R_c** & **(Shockley) load $V_s(I)$** to submodel:

$$\delta W : \dot{Z} = W,$$

$$\delta Z : M\dot{W} = -Mg - \underline{\gamma_m G(Z)I} - \int_0^{L_x} \int_0^{l_y(x)} \lambda \, dy \, dx$$

$$\delta I : \dot{Q} = I,$$

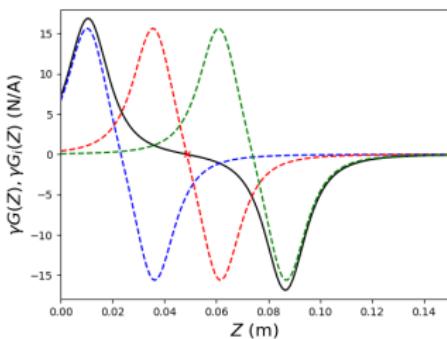
$$\delta Q : L_I \dot{I} = \underline{\gamma_m G(Z)\dot{Z}} - (R_i + R_c)I - V_s(I).$$

Mathematical model generator: multiple coils

Multiple coils $i, j = 1, \dots, N$; after ideal rectification s.t. $I_i \rightarrow |I_i|$
connected in parallel:

- ## ► Current equations:

$$L_I^{(i)} \dot{l}_i + M_{ij} \dot{l}_j = \sum_{i=1}^{N_q} \gamma_m G_i(Z) \dot{Z} - V_S^{(i)}(l_j).$$



Mathematical modelling buoy-generator: loads

- ▶ Shockley load represting parallel-connected LEDs:

$$V_S^{(i)}(I_j) = \frac{n_q V_T I_i}{|I_i|} \ln \left(1 + \sum_{k=1}^{N_i} \frac{|I_k|}{I_{sat}} \right)$$

- ▶ Butler-Volmer battery model for small β :

$$V_S^{(i)}(I_j) \approx \frac{I_i}{|I_i|} \frac{\operatorname{asinh}(|I|/I_s)}{q_v/2 + \beta q_v \frac{|I|/I_s}{\sqrt{1+(|I|/I_s)^2}}}$$

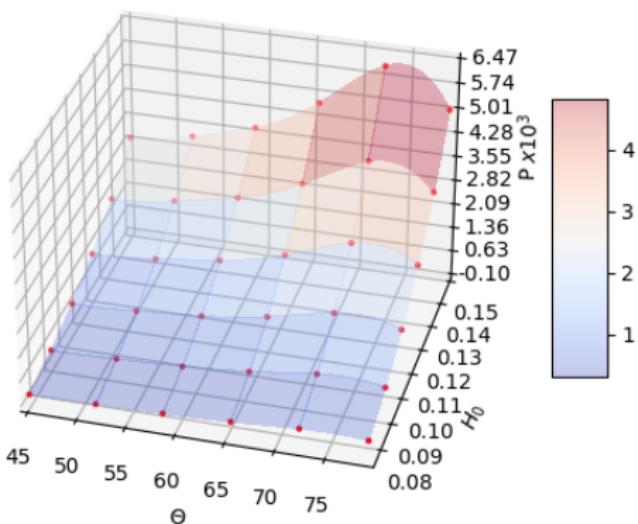
- ▶ Laboratory test set-up: resistor R after rectification $|I|$ connected in parallel

$$V_S^{(i)}(I_j) = R|I| = R \sum_{i=1}^{N_i} |I_i|.$$

Optimisation wave-energy device: geometry

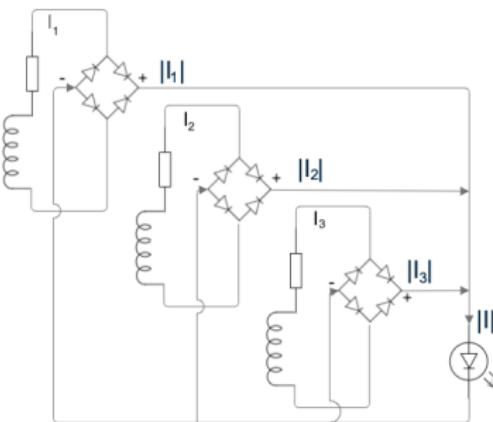
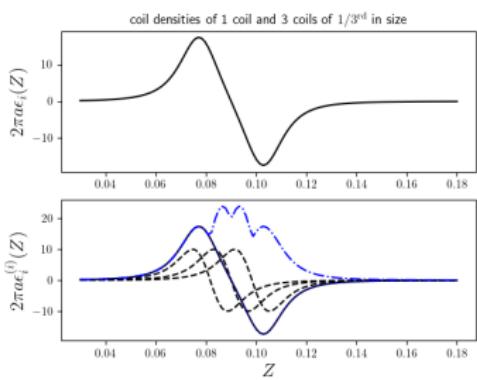
Surrogate modelling geometry angle/rest depth θ_c-H_0 :

RBF approximation of Power Output beta=2.0 and n=36



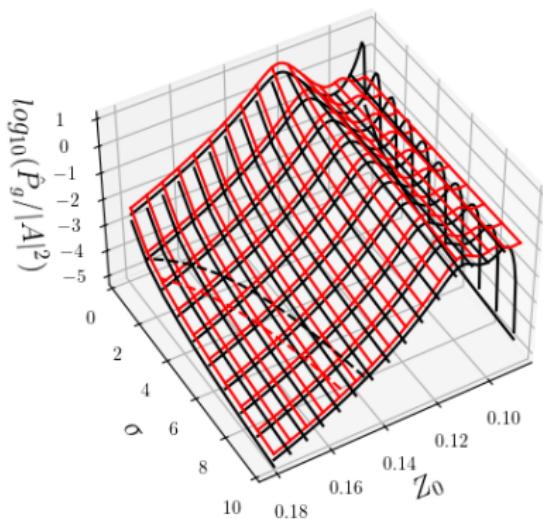
Optimisation wave-energy device: coils

Compare linearized power 1-coil vs. simple 3-coil-in-parallel with $L_I = 3L_I^{(i)}$:



Optimisation wave-energy device: coils

Linearized **1-coil** vs. simple **3-coil-in-parallel** power P_g (B. et al IEEE2024):



Optimisation wave-energy device: multiple coils

Open query: convergence (implicit mid-point/Brown) time integrators with absolute signs and multiple integrating factors.

- ▶ Two-coil model, two cases $I_1, I_2 > 0, < 0$ and $I_1 < 0, > 0; I_2 > 0, < 0$:

$$L\dot{I}_1 + M\dot{I}_2 = G_1(Z)\dot{Z} - \text{sign}(I_1)R(|I_1| + |I_2|)$$

$$M\dot{I}_1 + L\dot{I}_2 = G_2(Z)\dot{Z} - \text{sign}(I_2)R(|I_1| + |I_2|)$$

- ▶ Interim transformation with sign-factor $s_2 = \text{sign}(I_1)\text{sign}(I_2)$:

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \frac{1}{2}(I_{10} + s_2 I_{20}) \begin{pmatrix} 1 \\ s_2 \end{pmatrix} e^{\frac{-2R(t-t_j)}{(L+s_2 M)}} + \frac{1}{2}(I_{10} - s_2 I_{20}) \begin{pmatrix} 1 \\ -s_2 \end{pmatrix}$$

$$\frac{d(I_{10} + s_2 I_{20})}{dt} = \frac{(G_1(Z) + G_2(Z))\dot{Z}}{L + s_2 M} e^{\frac{2R(t-t_j)}{(L+s_2 M)}} \equiv f_1(t)$$

$$\frac{d(I_{10} - s_2 I_{20})}{dt} = \frac{(G_1(Z) - s_2 G_2(Z))\dot{Z}}{L - s_2 M} \equiv f_2(t).$$

Laboratory and model results generator

In summary:

- ▶ One-coil measurements and ideal-rectifier model agree reasonably well. Observed peak/mean values: (1.884, 0.921)V; modelled ones: (1.843, 0.736)V.
- ▶ Ideal rectified-in-parallel and three-phase rectified generator perform best.
- ▶ Three-coil model overpredicts by factor three and including mutual-coil induction further increases overpredictions. Observed peak and mean values: (2.341, 1.122)V; prediction ones (7.203, 2.504)V. Parameter values:
 $1/T_{period} = 2.6\text{Hz}$, $R = 100\text{Ohm}$, $l_2 = 0.035\text{m}$.

Analysis (numerical) of inequality constraints

Strategy on buoy-water surface coupling (work in progress):

- ▶ Goal is to solve Bernoulli & KKT inequality equations:

$$\delta h : \quad \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(z - H_0) - \lambda = 0 \quad \text{at} \quad z = h.$$

$$\delta \lambda : \lambda = -[\gamma(h - h_b) - \lambda]_+ = -F_+(\gamma(h - h_b) - \lambda)$$

$$\implies \underline{h(x, t) - h_b(Z, x) \leq 0}, \lambda \leq 0, \lambda(h - h_b) = 0.$$

- ▶ Analyse rest-flow case, when $\phi = 0$ with $h_b(Z, x) = Z - K - \tan \theta(L_y - x)$.
- ▶ Analyse “simpler” problem of a ball with position $Z(t)$ falling under gravity to a flat surface such that $Z(t) \geq 0$. Done.
- ▶ Use Firedrake’s in-built inequality-constraint solvers.

VP with Benny-Luke water waves

VP-model of Benney-Luke water waves (cf. Eq. (32) in B. & Kalogirou 2016) coupled to buoy (2D-gradients):

$$\begin{aligned}
 0 = & \delta \int_0^T \int_0^{L_x} \int_{R(t)}^{l_y(x)} -h\partial_t\phi - \frac{1}{2}\mu\nabla h \cdot \partial_t(\nabla\phi) \\
 & - \frac{1}{2}|\nabla\phi|^2 - gh\left(\frac{1}{2}h - H_0\right) \\
 & - \mu\left(\nabla q \cdot \nabla\phi - \frac{3}{4}q^2\right) - \frac{1}{2\gamma}\left(F_+(\gamma(h - h_b) - \lambda)^2 - \lambda^2\right)dydx \\
 & - MZ\dot{W} - \frac{1}{2}MW^2 - MgZdt. \tag{1}
 \end{aligned}$$

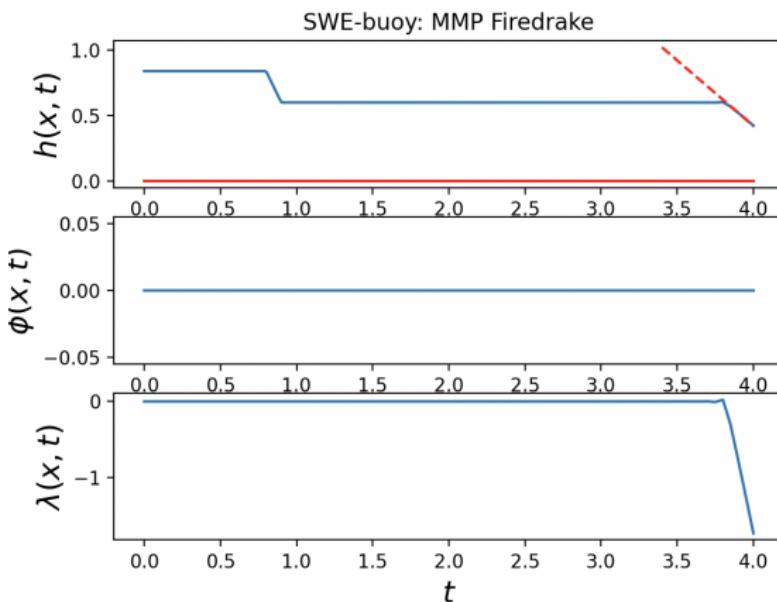
Midpoint/Brown time integrators

- ▶ Convergent solver or **modified midpoint** (MMP) time discretisation of model with CG-FEM has not been found.
- ▶ We therefore consider **Brown's midpoint scheme** (MMP') with $\{h, Z, \lambda, q\}$ at mid-time level $t^{n+1/2}$ & $\{\phi, W\}$ at entire times t^n :

$$\begin{aligned}
 0 = & \delta \sum_{n=0}^{N_t} \int_0^{L_X} \int_{R(t)}^{ly(x)} -h^{n+1/2} \frac{(\phi^{n+1} - \phi^n)}{\Delta t} - \frac{1}{2} \mu \nabla h^{n+1/2} \cdot \frac{(\nabla \phi^{n+1} - \nabla \phi^n)}{\Delta t} \\
 & - \frac{1}{2} h^{n+1/2} \left| \frac{1}{2} \nabla (\phi^{n+1} + \phi^n) \right|^2 - gh^{n+1/2} \left(\frac{1}{2} h^{n+1/2} - H_0 \right) - \mu \left(\frac{1}{2} \nabla q^{n+1/2} \cdot \nabla (\phi^{n+1} + \phi^n) \right. \\
 & \left. - \frac{3}{4} (q^{n+1/2})^2 \right) - \frac{1}{2\gamma} \left(F_+ (\gamma(h^{n+1/2} - h_b(Z^{n+1/2})) - \lambda^{n+1/2})^2 - (\lambda^{n+1/2})^2 \right) dy dx \\
 & - MZ^{n+1/2} \frac{(W^{n+1} - W^n)}{\Delta t} - \frac{1}{2} M \left(\frac{1}{2} (W^{n+1} + W^n) \right)^2 - MgZ^{n+1/2} dt,
 \end{aligned} \tag{2}$$

with variations $\{h^{m+1/2}, Z^{m+1/2}, \lambda^{m+1/2}, q^{m+1/2}, \phi^m, W^m\}$.

Rest flow



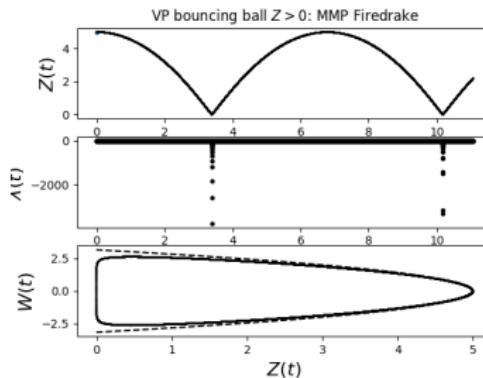
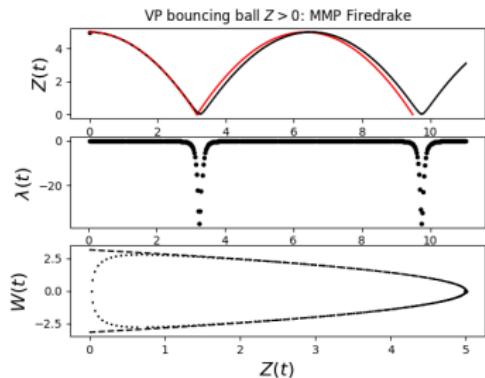
Rest flow

To date, only rest flow works with MMP' & some dynamics with MMP, noting that:

- ▶ “Soft” inequality approach with MMP cannot guarantee constraint on h^{n+1} due to extrapolation and constraining at half $n + 1/2$ time.
- ▶ Brown's MPP' can by construction.
- ▶ Explicit schemes cannot work, unless $\{h^{n+1/2}, \phi^{n+1}, Z^{n+1/2}, \lambda^{n+1/2}\}$ are intrinsically coupled, see also work by Bueler, Bueler & Farrell in Firedrake (hard-inequality solvers).
- ▶ **Open alley:** relationship between softening parameter $b, \Delta t$ and preconditioners.

Soft inequality: bouncing ball and billiards

Bouncing ball results with MMP and MMP', phase plots:



Novel wave-energy device: concluding remarks

- ▶ **Brief overview given** of wave-energy device, based on extreme-wave amplification in a contraction. Achieved modelling with VPs & geometric numerical integrators.
- ▶ Rest-flow case as stepping stone is in progress.
- ▶ For the dissipative terms in the electro-magnetic generator, either integrating factors required: backward, forward & sympletic (Euler) discretisations are unstable/wrong.
- ▶ Achieved first geometric and coil optimizations.

Novel wave-energy device: future work

- ▶ **Future work:** optimisation, real-time control using Pontryagin's principle, surrogate modelling & ML.
- ▶ Experiment soliton interactions: <https://www.youtube.com/watch?v=cBhr0DnVclU>
- ▶ A **laboratory set-up** of the wave-energy device is **in progress**, in steps:
 - a) a haning dry & driven generator, and
 - b) full hydrodynamic coupling in our 2m wave tank.
- ▶ New wave-energy idea: **test flexible electronic sheet** as surf-zone damping (Dr Jeamin Lee).

<https://www.sciencedirect.com/science/article/abs/pii/S1364032121007590>

Thanks very much for your attention ...

- ▶ B., Kalogirou, Zweers 2019: From bore-soliton-splash to a new wave-to-wire wave-energy model. *Water Waves* 1 10.1007/s42286-019-00022-9 Bore-soliton-splash:
<https://www.youtube.com/watch?v=YSXsXNX4zW0&list=FL6mc7mUa6M4Bo2VkJ970urw>
- ▶ Choi, Kalogirou, Lu, B., Kelmanson 2024: A study of extreme water waves using a hierarchy of models based on potential-flow theory. *Water Waves* <https://doi.org/10.1007/s42286-024-00084-4>
- ▶ B., Bolton, H. Thompson, Geometric power optimisation of a rogue-wave energy device in a (breakwater) contraction. 8th IEEE Conference on Control Technology & Applications (CCTA) (2024) 6 pp. Preprint
<https://eartharxiv.org/repository/view/7260/>

Aspects of wave-energy device: results to date

To date the following modelling and results have been obtained:

- ▶ Full **nonlinear wave-to-wire model** formulated for both equality and inequality constraints of the 2D buoy-free-surface water interface (B et al 2019).
- ▶ Full **numerical 2D linearised (shallow-water) model** of wave-to-wire model with equality constraint (B. et al 2019, Bolton et al. 2021, B et al IEEE2024).
- ▶ ***Optimisations*** done for the contraction geometry using full and **(Latin-hypercube) surrogate** modelling (B et al IEEE2024).

Aspects of wave-energy device: results to date

. . . following modelling and results have been obtained:

- ▶ Full **nonlinear 3D potential-flow water-wave submodel & numerical simulations** (Gidel et al 2022, Choi et al 2022/2024, Lu et al 2024), implementing model's space-time variational principle (VP). 2nd order in time, higher-order in space.
- ▶ Full **nonlinear buoy-generator submodel & numerics** based on time-discrete VP plus symmetric/consistent dissipative terms.
- ▶ Optimisation (linear) buoy-generator submodel by **analysing 1-coil and 3-coils-in-parallel generators**. Partially successful comparison measurements of generator.

Why variational principles (VPs)? Advantages:

- ▶ When (main) dynamics has a VP, multiple coupled equations are **succinctly** described by one space-time VP.
- ▶ Associated with a VP are **conservation properties** of associated PDEs/ODEs.
- ▶ Within the (*finite-element*) environment *Firedrake*, the time-discrete VP can be implemented directly, with automated generation of (complicated 3D+1D) weak forms of the equations. Adjoint solvers, optimal control.
- ▶ **Advantages:** enormous reduction in development time, efficient, flexible, higher-order spectrally-accurate space discretisations plus (automatic) preservation of discrete forms of conservation properties.