Analytical and numerical coupling of nonlinear wave motion to buoy dynamics using an inequality constraint

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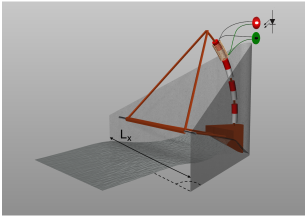
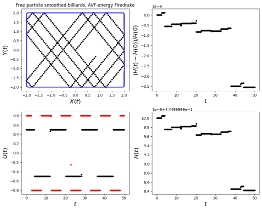
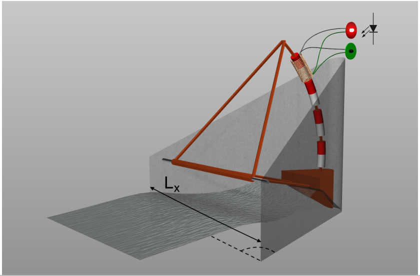




Outline

I am reporting the first near-complete (nonlinear) wave-to-wire FEM, made with Firedrake, using an inequality constaint, VP/Hamiltonian, averaged-vector-field “AVF-E” energy conservation, plus symmetric damping of the electric circuits and energy-harvesting load:

- Grand continuum variational principle (VP) entire model plus non-conservative terms.



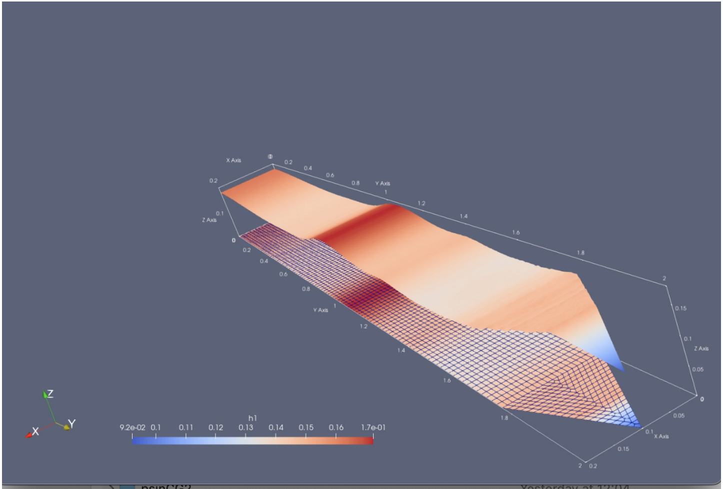
Outline

Implementation hierarchy, time-discrete AVF-E with inequality constraint for:

- a bouncing ball under gravity with $Z \geq 0$,
- a billiard ball in rectangle with $0 \leq X \leq L_x, 0 \leq Y \leq L_y$,
- water and buoy at rest, in hydrostatic balance, $h(x, y, t) - h_b(Z(t); x, y) \leq 0$,
- nonlinear water waves, buoy motion & power generation,
  $\sim$automated AVF-E, space-discretisation
  $\sim$Firedrake-automated via Hamiltonian.

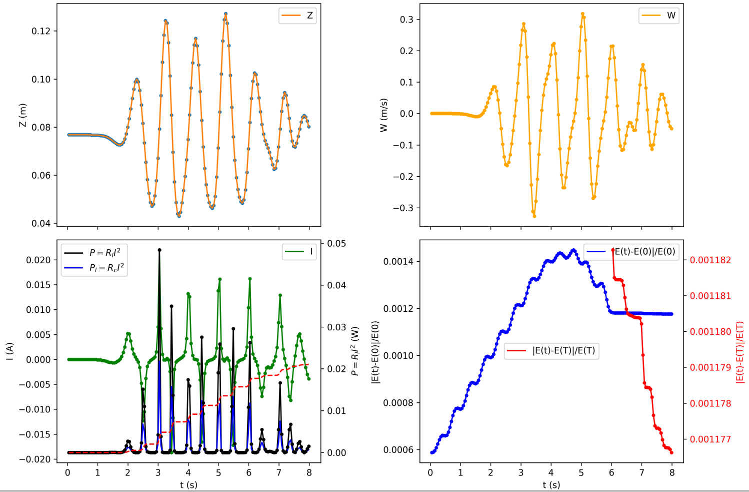
Main advance: wave-buoy dynamics

- Continuous Galerkin *CG1/CG2* [https://youtu.be/9ATf-jcHGhs & https://www.youtube.com/watch?v=p7wteDOgMGo]:



Main advance: full model single coil, CG1/CG2

- Preliminary: fully coupled wave-buoy-generator model



Grand variational principle of wave-to-wire model

Equations of motion follow from variational principle (red=waves, blue=buoy, green=EM-generator, coupling, B. et al. 2019):

$$
\begin{aligned}
0 =& \rho_0 \delta \int_0^T \int_0^{L_x} \int_{R(t)}^{l_y(x)} \int_0^h -(\partial_t \phi + \frac{1}{2}|\nabla\phi|^2)\mathrm{d}z - gh(\frac{1}{2}h - H_0) \\
& - \underline{\frac{1}{2\gamma}\Big(F_+(\gamma(h-h_b)-\lambda)^2 - \lambda^2\Big)}\mathrm{d}y\mathrm{d}x \\
& MW\dot{Z} - \frac{1}{2}MW^2 - MgZ + (L_i I - \underline{K(Z)})\dot{Q} - \frac{1}{2}L_i I^2 \mathrm{d}t
\end{aligned} \tag{1}
$$

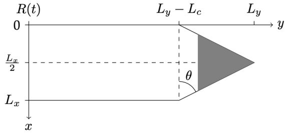
velocity $u = \nabla\phi(x,y,z,t)$, depth $h(x,y,t)$, rest depth H_0, e.g. buoy $h_b(Z,y) = Z - K_h - \tan\theta(L_y - y)$, piston $R(t)$, coupling function $\gamma_m G(Z) = K'(Z)$, buoy mass M, keel height K_h, buoy coordinate $Z(t)$, buoy velocity $W(t) = \dot{Z}$, charge $Q(t)$, current $I(t) = \dot{Q}$.

Wave-to-wire: PDEs

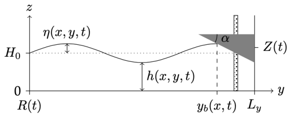
- Potential-flow water-wave dynamics (Laplace equation in interior, kinematic & Bernoulli equations at free surface):

$$
\begin{aligned}
\delta\phi: &\quad \nabla^2\phi = 0 \quad \text{in} \quad \Omega\\
(\delta\phi)|_{z=h}: &\quad \partial_t h + \nabla\phi\cdot\nabla h = \phi_z \quad \text{at} \quad z = h\\
\delta h: &\quad \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g(z - H_0) - \lambda = 0 \quad \text{at} \quad z = h.
\end{aligned}
$$

- Coupled elliptic Laplace equation to hyperbolic free-surface equations, plus a (Lagrange) **multiplier** λ.



(b) Top view of the tank and buoy, outlining the tank's dimensions and how the buoy fits the shape of the contraction.



(c) Side view at time t, with the buoy constrained to move vertically.

Wave-to-wire: inequality constraint & ODEs

- Karush-Kuhn-Tucker inequality conditions satisfied at every space-time x, y, t-position are (Burman et al. 2023):

$$\delta\lambda : \lambda = -[\gamma(h - h_b) - \lambda]_+ = -F_+(\gamma(h - h_b) - \lambda)$$
$$\Longrightarrow \underline{h(x, y, t) - h_b(Z, y) \le 0}, \lambda \le 0, \lambda(h - h_b) = 0.$$

- Add resistance R_i, R_c & Shockley load $V_s(|I|)$ to submodel:

$$\delta W: \quad \dot{Z} = W,$$
$$\delta Z: \quad M\dot{W} = -Mg \underline{-\gamma_m G(Z) I} - \rho_0 \int_0^{L_x} \int_0^{l_y(x)} \lambda \, \mathrm{d}y \mathrm{d}x$$
$$\delta I: \quad \dot{Q} = I,$$
$$\delta Q; \quad L_i \dot{I} = \underline{\gamma_m G(Z) \dot{Z}} - (R_i + R_c) I - \frac{I}{|I|} V_S(|I|).$$

Discretisation strategy: overview

- Consider Hamiltonian/variational dynamics with energy conservation due to skew-symmetric J-formulation and an auxiliary variable φ, if any, e.g. with $\mathcal{H}[h, \phi, \varphi]$:

$$\partial_t h + \frac{\delta \mathcal{H}}{\delta \phi} = 0, \quad \partial_t \phi - \frac{\delta \mathcal{H}}{\delta h} = 0, \quad \frac{\delta \mathcal{H}}{\delta \varphi} = 0.$$

- More abstractly, with (partial) time derivative, function(al) derivative, (field) variables z:

$$\dot{z} - J(z)\frac{\delta H}{\delta z} = 0.$$

Discretisation strategy: AVF-E

- Automate spatial discretisation using fd.derivative to derive weak forms from Hamiltonian: quicker & fewer errors in (water-wave) cases.
- Use Averaged Vector Field (AVF) method linear in interval time $s \in [0,1]$ to establish second-order time integration, e.g. with $h(s) = h^n + s(h^{n+1} - h^n)$, etc.

$$
\begin{aligned}
\frac{(h^{n+1} - h^n)}{\Delta t} + \int_0^1 \frac{\delta \mathcal{H}[h(s), \phi(s), \varphi(s)]}{\delta \phi(s)} \,\mathrm{d}s =& 0,\\
\frac{(\phi^{n+1} - \phi^n)}{\Delta t} - \int_0^1 \frac{\delta \mathcal{H}[h(s), \phi(s), \varphi(s)]}{\delta h(s)} \,\mathrm{d}s =& 0,\\
\int_0^1 \frac{\delta \mathcal{H}[h(s), \phi(s), \varphi(s)]}{\delta \varphi(s)} \,\mathrm{d}s =& 0.
\end{aligned}
$$

Discretisation strategy: AVF-E

- Or abstractly with $z(s) = z^n + s(z^{n+1} - z^n)$:

$$\frac{(z^{n+1} - z^n)}{\Delta t} - J(z^{n+1/2}) \int_0^1 \frac{\delta \mathcal{H}[z(s)]}{\delta z(s)} \, \mathrm{d}s = 0.$$

- Automated exact integration in s, Firedrake Legrendre-Gauss quadrature for polynomial-part of Hamiltonian.
- For weak forms, one then uses (that):

$$\int_0^1 \frac{\delta \mathcal{H}}{\delta z(s)} \delta z(s) \, \mathrm{d}s = \int_0^1 \frac{\delta \mathcal{H}}{\delta z(s)} \left((1-s)\delta z^n + s \delta z^{n+1} \right) \, \mathrm{d}s$$

$$\text{s.t.} \quad \int_0^1 ((1-s)+s) \frac{\delta \mathcal{H}}{\delta z(s)} \mathrm{d}s = \frac{\delta}{\delta z^n} \int_0^1 H[z(s)] \, \mathrm{d}s + \frac{\delta}{\delta z^{n+1}} \int_0^1 H[z(s)] \, \mathrm{d}s,$$

- wherein automated fd.derivatives of the s-integrated Hamiltonian are used wrt z^n, z^{n+1}.

Discretisation strategy: AVF-E

The discrete auxiliary equation, if any, follows from energy conservation, e.g., for a case with $\mathcal{H}[h,\phi,\varphi]$:

$$\frac{\mathcal{H}^{n+1}-\mathcal{H}^n}{\Delta t} = \iint \int_0^1 \frac{\delta\mathcal{H}}{\delta h(s)} \frac{(h^{n+1}-h^n)}{\Delta t} + \frac{\delta\mathcal{H}}{\delta\phi(s)} \frac{(\phi^{n+1}-\phi^n)}{\Delta t} + \frac{\delta\mathcal{H}}{\delta\varphi(s)} \frac{(\varphi^{n+1}-\varphi^n)}{\Delta t} \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y = 0,$$

when we take $\int_0^1 \langle \delta\mathcal{H}/\delta\varphi(s), v_\varphi\rangle^\star \,\mathrm{d}s{=}0$, since we do not have a prognostic equation $(\varphi^{n+1}-\varphi^n)=0$ (v_φ has no s) and the first two terms cancel out another.

Discretisation strategy: inequality constraint

- *IC-step-1* The starting point for $\mathsf{z} = (\mathsf{q}, \mathsf{p})$ is the augmented Lagrangian formulation

$$0 = \delta \int_0^T \mathsf{p} \cdot \dot{\mathsf{q}} - H(\mathsf{q}, \mathsf{p}) - \underline{\frac{1}{2\gamma}\Big(F_+(\gamma G(\mathsf{q})) - \lambda)^2 - \lambda^2\Big)}\,\mathrm{d}t$$

- Variation of λ leads to Karush-Kuhn-Tucker KKT relations with, recall, $F_+(Q) \equiv \max(Q, 0)$ for some argument Q, s.t.

$$\lambda = -F_+(\gamma G(\mathsf{q})) - \lambda) \Longleftrightarrow \lambda \leq 0, \quad -G(\mathsf{q}) \leq 0, \quad \lambda G(\mathsf{q}) = 0.$$

Discretisation strategy: inequality constraint

- *IC-step-2* $F_+(Q) = \max(Q, 0)$ smoothed. For some smoothed functions one can solve the Lagrange multiplier λ explicitly.
- For (well-known) smoothed functions, discretisations poor.
- For $F_+(Q) = \frac{1}{2}Q + \sqrt{b^2 + Q^2/4} \rightarrow_{b\rightarrow 0} \max(Q, 0)$, one finds $\lambda = -\frac{b^2}{\gamma G(\mathsf{q})}$.
- Conjugate equations then contain a derived penalty term:

$$\dot{\mathsf{p}} + \frac{\delta \mathcal{H}}{\delta \mathsf{q}} + \frac{\partial G(\mathsf{q})}{\partial \mathsf{q}}\lambda = \dot{\mathsf{p}} + \frac{\delta \mathcal{H}}{\delta \mathsf{q}} - \frac{\partial G(\mathsf{q})}{\partial \mathsf{q}}\frac{b^2}{\gamma G(\mathsf{q})} = 0.$$

- Augmented Hamiltonian contains penalty potential:

$$\widehat{\mathcal{H}}[\mathsf{q}, \mathsf{p}] = \mathcal{H}[\mathsf{q}, \mathsf{p}] - \frac{b^2}{\gamma}\ln G(\mathsf{q}).$$

Inequality constraint: superfunction

- *IC-step-3* Recall that the KKT-relations

$$\lambda \leq 0, \quad -G(\mathrm{q}) \leq 0, \quad \lambda G(\mathrm{q}) = 0,$$

in the $G(\mathrm{q}), \lambda$-plane, delineate fourth quadrant.

- Hence, seek a (straightforward) "superfunction" approximating this delination & $\lambda = -F_+(\gamma G(\mathrm{q}) - \lambda)$, i.e.,

$$\lambda = -\frac{1}{b} e^{-bG(\mathrm{q})}.$$

- Only an implicit ∼definition of $F_+(Q)$ can then be found:

$$Q = \frac{\gamma}{b} \ln\left(bF_+(Q)\right) + F_+(Q).$$

Inequality constraint: superfunction & AVF-E

- Superfunction leads to an augmented Hamiltonian:

$$\widehat{\mathcal{H}}[\mathrm{q},\mathrm{p}] = \mathcal{H} + \frac{1}{b^2} e^{-bG(\mathrm{q})}.$$

- *IC-step-4* For constraints linear in q, such that $G(\mathrm{q}) = c_j q_j$, the AVF-E integral becomes

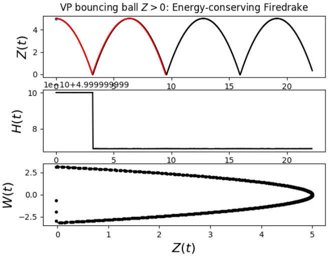
$$0 = \frac{(p_j^{n+1} - p_j^n)}{\Delta t} + \int_0^1 \frac{\delta \mathcal{H}}{\delta q_j(s)} - \frac{c_j}{b} e^{-bc_k q_k(s)} \, \mathrm{d}s,$$

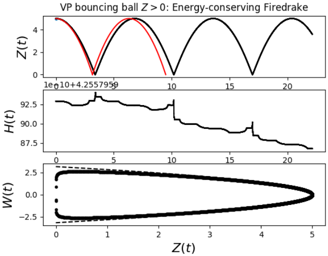
wherein the last integral becomes (fd.conditional in Firedrake)

$$-\frac{c_j}{b} e^{-\frac{1}{2} bc_l (q_l^{n+1} + q_l^n)} \frac{\sinh\left(\frac{1}{2} bc_l (q_l^{n+1} - q_l^n)\right)}{\frac{1}{2} bc_l (q_l^{n+1} - q_l^n)} \rightarrow_{b \to 0} -\frac{c_j}{b} e^{-\frac{1}{2} bc_l (q_l^{n+1} + q_l^n)}.$$

Bouncing ball under gravity $Z \geq 0$

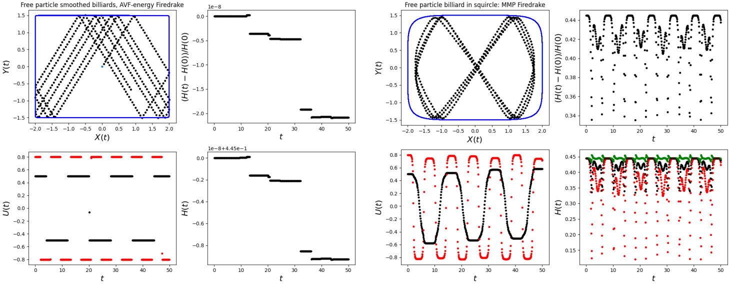
- Parameters $b = 500, \Delta t = 0.035$.
- Relationship between b and Δt.
- Solvers (best to date): nest, vinewtonrsls





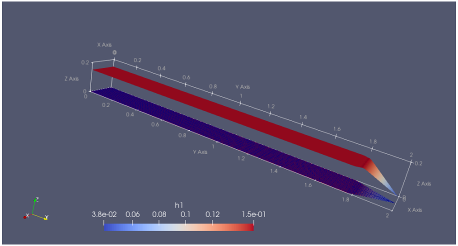
Billards in rectangle

- Parameters $b = 1000, 2000, \Delta t = 0.015, 0.0075$.
- Relationship between b and Δt.
- Solvers (best to date): nest, default



Wave-energy wave tank: initial condition rest state

- Starting point: used continuation in $b = 10, 100, 1000, 2000, 4000$ then nondimensionalisation. Rest system $\{W, \phi\}$ solved, eqns other variables $\{Z, h, \psi, I\}$ enforced to be linear & zero.
- Solvers: "bespoke" for rest state, default for dynamics.



Wave-energy wave tank: full dynamics –09-2025

- The conservative dynamics has non-canonical components in the variables $\{W, I\}$, which non-constant $J(Z)$-bracket terms, involving the nonlinear function $G(Z)$ arising from solving the 3D Maxwell's equations for axisymmetric induction coil(s), are evaluated at $Z^{n+1/2}$ (per review of Cotter 2023).
- The Variational Boussinesq Model (VBM) is used, a potential flow model with quadratic polynomial accuracy integrated out in z (e.g., Gagarina et al. 2013) with canonical variables $\{h(x,y,t), \phi(x,y,t)\}$ and interior velocity potential degree-of-freedom $\psi(x,y,t)$.

Wave-energy wave tank: full dynamics –09-2025

- Complicated yet polynomial VBM Hamiltonian ($\beta = 1$; Green-Naghdi system when $\beta = 0$), whence AVF-E automation in Firedrake:

$$\begin{aligned}\mathcal{H}[h,\phi,\psi] &= \iint \frac{1}{2}h|\bar{\mathrm{u}}|^2 + \frac{1}{6}h^3\psi^2 + \frac{1}{2}g\left((h+b)^2 - b^2\right) - ghH_0 + \frac{\beta}{90}h^5|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}y\\ &= \iint \frac{1}{2}h|\nabla\phi + h\psi\nabla h + \frac{1}{3}h^2\nabla\psi|^2 + \frac{1}{6}h^3\psi^2 + \frac{1}{2}g\left((h+b)^2 - b^2\right)\\ &\quad - ghH_0 + \frac{\beta}{90}h^5|\nabla\psi|^2 + \frac{1}{b^2}e^{-b\left(h_b(Z)-h(x,y,t)\right)}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

- Solvers used default: *CG2* `https://www.youtube.com/watch?v=p7wteDOgMGo`

Discussion

Nearly completed full numerical wave-to-wire model of the wave-energy device, 2^{nd}-order in time & spectral in space. Crucial steps have been to use, combine & integrate:

- augmented Lagrangian formalism of inequality constraints (Burman et al. 2023) with smoothed KKT relations;
- a superfunction solving and modifying the Lagrange multiplier into a derived penalty term/potential, without introducing any singularity (such as to accommodate nonlinear solvers) & with an implicit definition of the $F_+(Q) = \max(Q, 0)$-function, and
- AVF-energy conservation second-order time-integration strategy, using exact (numerical & analytical) integration.

Discussion

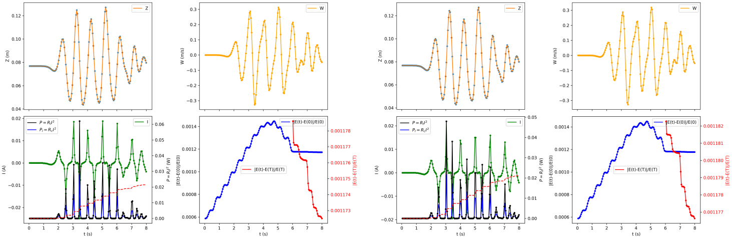
Future work, on AVF-E & λ-function(s):

- Better, faster solvers: help (-)!
- AVF-E in Irksome?
- 3D potential flow (mostly done), yielding logarithmic AVF-E singularities.
- ....
- Many thanks to Prof Harvey Thompson & Prof Colin Cotter. All errors made are mine.
- Finally, a big thank you to: Claire Savy and Deborah Clarke from LIFD, Dave Stones from SoM, and Connor and David from IC. Funding from: Firedrake, LIFD, and SoM. Thank *you*!

Nonlinear inequality constraints

Project constraint using an auxiliary variable μ by adding the following term to the energy, instead of the penalty potential

$$\delta \frac{1}{b}\left(G(\mathrm{q}) - (\mu + 1/b)\right) e^{-b\mu} = \frac{1}{b} \nabla G(\mathrm{q}) e^{-b\mu} \delta \mathrm{q} + \left(\mu - G(\mathrm{q})\right) e^{-b\mu} \delta \mu.$$


Z
Z (m)
W
W (m/s)
I (A)
t (s)
|E(t)-E(0)|/E(0)
|E(t)-E(T)|/E(T)